From symmetries of the modular tower of genus zero real stable curves to an Euler class for the dyadic circle

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Abstract

We build actions of Thompson group \( V \) (related to the Cantor set) and of the so-called “spheromorphism” group of Neretin, on “towers” of moduli spaces of genus zero real stable curves. The latter consist of inductive limits of spaces which are the real parts of the Grothendieck-Knudsen compactification of the usual moduli spaces of punctured Riemann spheres. By a result of M. Davis, T. Januszkiewicz and R. Scott, these spaces are aspherical cubical complexes, whose fundamental groups, the “pure quasi-braid groups”, are some analogues of the classical pure braid groups. By lifting the actions of Thompson and Neretin groups to the universal covers of the towers, we get new extensions of both groups by an infinite pure quasi-braid group, and construct what we call an “Euler class” for Neretin group, justifying the terminology by exhibiting an Euler-type cocycle. Further, after introducing the infinite (non-pure) quasi-braid group, we show that both infinite (non-pure and pure) quasi-braid groups provide new examples of groups whose classifying spaces, after plus-construction, are loop spaces.

The aim of this work is to relate geometrically some discrete groups, such as Thompson group \( V \) (acting on the Cantor set) and the so-called “spheromorphism” group of Neretin \( N \) (the dyadic analogue of the diffeomorphism group of the circle, acting on the boundary of the dyadic regular tree \( T_2 \), cf. \([25, 17, 18, 19]\)), to the moduli spaces of genus zero curves. More explicitly, let \( M_{0,n+1}(\mathbb{C}) \) denote the moduli space \( (\mathbb{C}P^1)^{n+1}/\Delta \mathbb{P}L(2,\mathbb{C}) \), where \( \mathbb{C}P^1 \) is the complex projective line, \( \Delta \) the thick diagonal. The Grothendieck-Knudsen-Mumford

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compactification $\overline{M}_{0,n+1}(\mathbb{C})$ has a concrete realization as an iterated blow-up along the irreducible components of a hyperplane arrangement in $\mathbb{C}P^{n-2}$. In [20] and [21], the interest of the real part $\overline{M}_{0,n+1}(\mathbb{R})$ was revealed, and its topology has recently been studied in [3], [8], and [10]. The relevance of $\overline{M}_{0,n+1}(\mathbb{R})$ with respect to Thompson or Neretin groups comes through the common role played by planar trees. $\overline{M}_{0,n+1}(\mathbb{R})$ is a stratified space, whose strata are labeled by planar trees, while dyadic planar trees occur in symbols defining elements of Thompson and Neretin groups.

In [14], P. Greenberg builds a classifying space for Thompson group $F$, the smallest of Thompson groups, whose total space is an inductive limit of “bracelet” spaces $B_n$, shown to be isomorphic, as stratified spaces, to the famous Stasheff associahedra $K_n$’s. It happens however that this space is too small for supporting an action of Thompson group $V$. So, the idea to proceed to an analogous construction for Thompson group $V$ originated in the observation that a two-sheeted covering space $\overline{M}_{0,n+1}(\mathbb{R})$ of $M_{0,n+1}(\mathbb{R})$ was tiled by $n!$ copies of the associahedron $K_n$, and that its stabilization had the adequate size of a space which might be dynamically related to $V$.

Essentially our construction consists in building two spaces (Proposition 3), $\overline{M}_{0,2\infty}(\mathbb{R})$ and $\overline{M}_{0,3,2\infty}(\mathbb{R})$ – which we call the “towers” of moduli spaces of genus zero real curves, the second tower appearing as a real version of a stabilized moduli space considered by Kapranov in [22] –, defined as inductive limits of spaces $\overline{M}_{0,n}(\mathbb{R})$ and $\overline{M}_{0,n}(\mathbb{R})$ respectively, and making act cellularly on them not only Thompson group $V$, but also the much larger spheromorphism group $N$ (Theorem 1). This result may have some interest for two reasons:

– First, $V$ and $N$ are usually described as groups almost-acting on trees, or acting on their boundaries, which are totally disconnected spaces; though there exist some simplicial complexes on which they do act (cf. [3], [4]), they are often “abstract” topological spaces. Here however, $V$ and $N$ cellularly act on a very geometric object (it is an ind-variety of the real algebraic geometry).

– Second, $V$ or $N$, as groups of symmetries of the towers of real moduli spaces, reveal the $p$-adic (with $p = 2$) nature of the latter: indeed, $N$ contains the automorphism group $Aut(T_2)$ of the regular dyadic tree, and in particular the group $PGL(2,\mathbb{Q}_2)$, as well as the group of locally analytic transformations of the projective line $\mathbb{Q}_2P^1$, seen as the boundary of the dyadic tree (cf. [5], [9]).

One of the towers considered in this text, $\overline{M}_{0,3,2\infty}(\mathbb{R})$, involves the pieces of a cyclic operad $\{\mathcal{A}(n) = \overline{M}_{0,n+1}(\mathbb{R}), \ n \geq 1\}$, in the sense of Ginzburg-Kapranov, cf. [13]. Now between Thompson group $F$, acting on the interval $[0,1]$, Thompson group $V$, acting on the Cantor set, stands Thompson group $T$, related to cyclicity through its realization as a homeomorphism group of the circle (cf. [12]). We show that $T$ acting on the “cyclic” tower $\overline{M}_{0,3,2\infty}(\mathbb{R})$ stabilizes a fundamental tile, isomorphic to an infinite associahedron $K_\infty$, making the other tiles turn around $K_\infty$. We prove (Theorem 2) that the stabilizer of $K_\infty$
under the action of $N$ is the semi-direct product $Out(T) \triangleright T$, where $Out(T)$, the exterior automorphism group of $T$, is known to be the cyclic order 2 group. It is remarkable that passing from one tile ($K_\infty$) to the whole tower $\mathcal{M}_{0,3,2\infty}(\mathbb{R})$, the size of the symmetry group, from countable ($T$), becomes uncountable ($N$).

Further, by lifting the actions to the universal covering spaces, we deduce the existence of non-trivial extensions of $V$ and $N$: by an infinite “pure quasi-braid group” – denoted $PJ_{2\infty}$ through this text – , an inductive limit of the fundamental groups $PJ_n$ of the spaces $\mathcal{M}_{0,n+1}(\mathbb{R})$, for the first action, and by a group $Q_{3,2\infty}$ for the second one, an inductive limit of the fundamental groups $Q_n$ of the spaces $\mathcal{M}_{0,n}(\mathbb{R})$ (In fact, $PJ_{2\infty}$ and $Q_{3,2\infty}$ happen to be isomorphic). In their definitions, though not in essence, they are somewhat analogous to the pure braid groups $P_n$, this explains the terminology ([9], [10]). In particular, the resulting extension:

\[ (*) \quad 1 \to PJ_{2\infty} \to \hat{A}_N \to N \to 1 \]

leads to a new non-trivial central extension of $N$ with kernel $\mathbb{Z}/2\mathbb{Z}$ (Theorem 3), after a certain abelianization process of the kernel. Viewing the boundary $\partial T_2 \cong \mathbb{Q}_2 P^1$ as the “dyadic circle”, we suggest thinking of the resulting class as the analogue for $N$ of the Euler class of the homeomorphism group of the circle $\text{Homeo}^+(S^1)$, or of Thompson group $T$ (cf. [12]), and strengthen the analogy by exhibiting an Euler-type cocycle related to the central extension (Theorem 4).

The first section is devoted to the description of the real moduli spaces $\mathcal{M}_{0,n+1}(\mathbb{R})$, especially the combinatorics of their stratifications. Section 2 introduces the infinite moduli space $\mathcal{M}_{0,2\infty}(\mathbb{R})$ and $\mathcal{M}_{0,3,2\infty}(\mathbb{R})$ (the “towers”, cf. Proposition 3). After proving that Neretin group $N$ (and so Thompson group $V$) acts upon both towers (Theorem 1) and computing the stabilizer of a maximal tile (Theorem 2), we define the resulting extensions by the infinite pure quasi-braid groups $PJ_{2\infty}$ and $Q_{3,2\infty}$. In section 3, the infinite (non-pure) quasi-braid group $J_{2\infty}$ is introduced (Proposition 4), coming in a short exact sequence (a restriction of the extension $(*)$ above):

\[ 1 \to PJ_{2\infty} \to J_{2\infty} \to \Sigma_{2\infty} \to 1, \]

where $\Sigma_{2\infty}$ is an inductive limit of symmetric groups. A “stable” length is defined on $J_{2\infty}$, related to a modified word metric on the quasi-braid groups, and is the key for building the Euler class of $N$ (Theorem 3) and defining the associated 2-cocycle (Theorem 4). In section 4 we claim that the Euler class of $N$ restricted to $\text{PGL}(2,\mathbb{Q}_2)$ is trivial (Theorem 5); we show finally that the classifying spaces of the infinite quasi-braid groups are homologically equivalent to loop spaces.

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1 The compactified real moduli space $\overline{M}_{0,n+1}(\mathbb{R})$

1.1 The space of real $(n+1)$-stable curves $\overline{M}_{0,n+1}(\mathbb{R})$

**Definition 1** Let $k$ be the field $\mathbb{C}$ or $\mathbb{R}$. A genus zero $(n+1)$-stable curve is an algebraic curve $C$ on the field $k$ with $(n+1)$ punctures $x_0, \ldots, x_n$ such that

1. each irreducible component of $C$ is isomorphic to a projective line $\mathbb{P}_k^1$, and each double point of $C$ is ordinary;
2. the graph of $C$ is a tree;
3. each component of $C$ has at least three points, double or marked.

The graph of a curve $(C, x_0, \ldots, x_n)$ is defined as follows: The leaves (or 1-valent vertices) $A_0, \ldots, A_n$ are in correspondence with the marked points of the curve, $x_0, \ldots, x_n$, whereas the internal vertices $v_1, \ldots, v_k$ are in bijection with the irreducible components $C_1, \ldots, C_k$. There is an internal edge $[v_i, v_j]$ if $C_i$ and $C_j$ contain the same double point. Terminal edges $[A_i, v_j]$ correspond to pairs $(x_i, C_j)$ with $x_i \in C_j$.

From now on, we shall follow the convention of [13]: edges will be oriented in such a way that $[A_0 v_j]$ is the output edge, and $[A_i v_j], i = 1, \ldots, n$ are the input edges.

1.2 Explicit construction of $\overline{M}_{0,n+1}(\mathbb{R})$

We find it useful to recall the construction of $\overline{M}_{0,n+1}(\mathbb{R})$, using mixed sources, namely [3], [13], [20], [21] and [13]:

(1) $\overline{M}_{0,n+1}(\mathbb{R}) = \frac{\mathbb{R}P^1)^{n+1} \setminus \Delta}{\text{Aff}(2,\mathbb{R})} \cong \mathbb{R}^n \setminus \tilde{\Delta} \hookrightarrow \mathbb{P}^{n-2} := \mathbb{P}\{(a_1, \ldots, a_n) \in \mathbb{R}^n : \sum_{i=1}^n a_i = 0\}$, where $\Delta$ (resp. $\tilde{\Delta}$) is the thick diagonal of $(\mathbb{R}P^1)^{n+1}$ (resp. $\mathbb{R}^n$), and $\text{Aff}(2,\mathbb{R})$ is the affine group acting on $\mathbb{R}^2$. The first isomorphism is obvious; as for the embedding in the $(n-2)$-dimensional projective space, it is induced by the map sending $(x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \Delta$ to $(a_1, \ldots, a_n)$, with $a_i = x_i - \frac{1}{n} \sum_{j=1}^n x_j$.

The image of $\overline{M}_{0,n+1}(\mathbb{R})$ in $\mathbb{P}^{n-2}$ is the complementary of the union of hyperplanes $H_{i,j} : a_i = a_j$. This hyperplane arrangement is called the braid arrangement. So $\overline{M}_{0,n+1}(\mathbb{R})$ is disconnected into $\frac{n!}{2}$ connected components which are the projective Weyl chambers $W^\sigma : a_{\sigma(1)} < a_{\sigma(2)} < \ldots < a_{\sigma(n)}$, where $\sigma$ belongs to the symmetric group $\Sigma_n$. Projectivisation introduces an identification of the

\footnote{Note the difference with the complex case: $\overline{M}_{0,n+1}(\mathbb{C})$ is connected}
chambers $W^\sigma$ and $W^{\sigma\omega}$, where $\omega$ is the permutation $\begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}$.

(2) Denote by $\mathcal{B}_0$ the set of $n$ points $p_i$: $a_1 = \ldots = a_i = \ldots = a_n$, $\mathcal{B}_1$ the set of lines $(p_ip_j)$: $a_1 = \ldots = a_i = \ldots = a_j = \ldots = a_n$, and more generally, by $\mathcal{B}_k$ the set of $k$-planes $a_1 = \ldots = a_{i_1} = \ldots = a_{i_k+1} = \ldots = a_n$, $i_1 < \ldots < i_{k+1}$, for $k = 0, \ldots, n-3$ (they are the irreducible components, in the sense of Coxeter groups; this will become clear in §1.3). Along components of $\mathcal{B}_k$, hyperplanes $H_{i,j}$ do not meet transversely. So, the construction is the following: points of $\mathcal{B}_0$ are first blown-up, and we get the blow-down map $X_1 \xrightarrow{\pi_1} \mathbb{P}^{n-2}$. The proper transforms of the lines $(p_ip_j)$ (i.e. the closures of $\pi_1^{-1}((p_ip_j) \setminus \{p_i,p_j\})$) become transverse in $X_1$, consequently they can be blown-up in any order to produce $X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_3} \mathbb{P}^{n-2}$; again the proper transforms of the planes $(p_ip_jp_k)$ become transverse in $X_2$, and are blown-up in any order. Finally we get the composition of blow-down maps

$$\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) := X_{n-2} \xrightarrow{\pi_{n-2}} \ldots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{P}^{n-2},$$

where the last blow-up is useless, since blowing-up along hypersurfaces does not change the manifold. We shall denote by $p : \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \rightarrow \mathbb{P}^{n-2}$ the composition of the iterated blow-ups.

(3) Each blow-up along a smooth algebraic submanifold produces a smooth exceptional divisor in the new manifold, which is isomorphic to the projective normal bundle over the submanifold. We denote by $\mathcal{B}_k$ the set of proper transforms in $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ of the exceptional divisors produced by blowing-up the (proper transforms in $X_k$ of the) components of $\mathcal{B}_k$: they are smooth irreducible hypersurfaces of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, and meet transversely. So the union $\bigcup_{k=0}^{n-3}\mathcal{B}_k$ is the set of irreducible components of a normal crossing divisor $\mathring{D}$.

(4) The real algebraic variety $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is stratified in the following way: the closures of the codimension $k$ strata – or closed strata – are the non-empty intersections of $k$ irreducible components of the divisor $\mathring{D}$, with one of the $\mathfrak{d}_k^1$ closures of the preimages $p^{-1}(W^\sigma)$. A stratum – or open stratum, though it is not an open set – is then defined as the complementary of a all closed strict substra in a given closed stratum. With this terminology, an open (resp. closed) stratum is a smooth manifold (resp. with boundary) and an open (resp. closed) cell.

The construction of the complex algebraic variety $\overline{\mathcal{M}}_{0,n+1}(\mathbb{C})$ is similar, but the complex codimension $k$ closed strata are simply the non-empty intersections of $k$ irreducible components of the divisor $\mathring{D}$, and because the divisor $\mathring{D}$ is normal, they are indeed smooth closed manifolds (without boundary), but not cells.

(5) Each codimension $k$ stratum $\mathcal{M}$ is coded by a planar tree that we now define: If $\mathcal{M} = p^{-1}(W^\sigma) \cap D_1 \cap \ldots \cap D_k$ is non-empty, then all the components $D_\alpha$, $\alpha = 1, \ldots, k$ are produced by a blow-up along a set which must be of the form
\[ a_{\sigma(i)} = a_{\sigma(i+1)} = \ldots = a_{\sigma(j)}, \text{ with } 1 \leq i < j \leq n. \]

Let \( T(\sigma, (i, j)) \) denote the planar rooted tree with one output edge and \( n \) input edges, labeled from the left to the right by \( \sigma(1), \ldots, \sigma(n) \), with a unique internal edge:

![Figure 1](image_url)

Define a *contraction* of a tree as the operation consisting in collapsing an internal edge on a single vertex. Introduce the partial order on the set of \( n \)-planar trees: \( T \leq T' \) if \( T' \) is obtained from \( T \) by a sequence of contractions.

**Definition 2** The tree attached to the closed stratum \( \overline{\mathcal{M}} = p^{-1}(W^\sigma) \cap D_1 \cap \ldots \cap D_k \) (or to the stratum \( \mathcal{M} \)) is defined to be

\[
T(\sigma) = \inf_{\alpha = 1, \ldots, k} T(\sigma, (i_\alpha, j_\alpha)).
\]

**Notation:** We shall use two different notations, not to be confused: \( T(\sigma) \), with \( \sigma \in \Sigma_n \), will denote an \( n \)-planar tree with leaves labeled from \( \sigma(1) \) to \( \sigma(n) \), from the left to the right, whereas \( (T, \sigma) \) will refer to the same tree, but with the canonical labeling of the leaves, from 1 (on the left) to \( n \) (on the right), coupled with the same permutation \( \sigma \).

From the preceding description, the following facts are obvious:

**Fact 1:** The stratum \( \overline{\mathcal{M}} = p^{-1}(W^\sigma) \cap D_1 \cap \ldots \cap D_k \) is non-empty if and only if the collection of sets \( S_\alpha = \{i_\alpha, i_\alpha + 1, \ldots, j_\alpha\} \), attached to each \( D_\alpha \) as explained above, for \( \alpha = 1, \ldots, k \), is nested in the following sense:

\[
\forall \alpha, \beta, \text{ either } S_\alpha \cap S_\beta = \emptyset, S_\alpha \subset S_\beta \text{ or } S_\beta \subset S_\alpha.
\]
Fact 2: Denoting the stratum by $\mathcal{M}(T, \sigma)$, we observe that $\text{codim } \mathcal{M}(T, \sigma) = \text{card } \{\text{internal edges}\} = \text{card } \Vert(T) \setminus \{\text{root}\}$. Indeed, the set of internal edges of $T$ is in one-to-one correspondence with the codimension 1 components $D_1, \ldots, D_k$ containing $\mathcal{M}(T, \sigma)$.

Remark: In view of §1.1, it is enlightening to understand each stratum $\mathcal{M}(T, \sigma)$ as the set of stable curves whose associated planar tree is $T(\sigma)$. Then Fact 1 becomes completely clear.

1.3 Coxeter group formulation, following Davis, Januszkiewicz and Scott

Following [9], we now formulate the condition guaranteeing a collection is nested in group theoretic terms, namely in terms of the symmetric group $\Sigma_n$. Denote by $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ the set of canonical Coxeter generators of $\Sigma_n$: $\sigma_i$ is the transposition $(i, i+1)$. Let $(T, \sigma)$ be a labeled planar $n$-tree: to each vertex $v$ except the root of $T$, a proper subset $T_v = \{\sigma_i, \sigma_{i+1}, \ldots, \sigma_{j-1}\}$ of $S$ is associated, corresponding to a connected subgraph $G_{T_v}$ of the Coxeter graph of $\Sigma_n$: $i, i+1, \ldots, j$ are the labels of the leaves descending from $v$. The collection $\mathcal{T} = \{T_v, v \in \Vert(T) \setminus \{\text{root}\}\}$ is a nested collection in the following sense:

Definition 3 ([9]) A collection $\mathcal{T}$ of proper subsets of $S$ will be called nested if the following conditions hold:

1. The Coxeter subgraph $G_T$ is connected for all $T \in \mathcal{T}$.
2. For any $T, T' \in \mathcal{T}$, either $T \subset T'$, $T' \subset T$, or $G_{T \cup T'}$ is not connected.

It is now clear that there is a bijection $T \leftrightarrow \mathcal{T}$ between the set of planar $n$-trees and the set of nested collections of the symmetric group $\Sigma_n$. We shall later use the induced correspondence $(T, \sigma) \leftrightarrow (\mathcal{T}, \sigma)$.

1.4 Combinatorics of the stratification of a $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ and of its two-sheeted cover $\widehat{\mathcal{M}}_{0,n+1}(\mathbb{R})$

1.4.1 Each $K^\sigma = \overline{p^{-1}(W^\sigma)}$ is combinatorially isomorphic to the Stasheff associahedron $K_n$ (cf. [20]). This is so because the stratifications of both objects are the same. Of course, strata labeled by different planar trees may coincide in the space $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$. In [14], the rules of identification of faces are expressed in the language of polygons, which is well-adapted because the operad $\{\mathcal{M}(n) = \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}), n > 1\}$ is cyclic in the sense of Getzler-Kapranov (cf. [11], the action of $\Sigma_n$ on $\mathcal{M}(n)$ extends in an obvious way to $\Sigma_{n+1}$). However, we prefer translating them in terms of trees, because of their relevance with respect to Thompson groups. Moreover, instead of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, we shall consider a two-sheeted covering $\widehat{\mathcal{M}}_{0,n+1}(\mathbb{R})$, that we now define:
Consider the two-sheeted covering \( S^{n-2} \rightarrow \mathbb{P}^{n-2} \), where \( S^{n-2} \) is the \((n-2)\)-dimensional unit sphere of \( \{ (a_1, \ldots, a_n) \in \mathbb{R}^n : \sum_{i=1}^n a_i = 0 \} \).

Apply now the process of iterated blow-ups described in §1.1 to \( S^{n-2} \) with its lifted braid arrangement, and denote by \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \) the resulting space. This yields a commutative diagram

\[
\begin{align*}
\tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) & \rightarrow \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \\
S^{n-2} & \rightarrow \mathbb{P}^{n-2}
\end{align*}
\]

where the horizontal arrows are the obvious two-sheeted covering maps, and the vertical ones are the blow-down maps. Since \( S^{n-2} \) is tiled by \( n! \) Weyl chambers, the covering \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \) will be tiled by \( n! \) copies of the associahedron \( K_n \).

### 1.4.2 Combinatorics of the stratification of \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \).

Let \( T(\sigma) \) be an \( n \)-planar tree, and \( v \) an internal vertex other than the root. The tree disconnects in \( v \) into 2 subtrees, the “upper” one (with the same output edge) and the “lower” one (with root \( v \)). Proceed now to a reflection of the lower tree – so that the left-to-right labeling of its input edges is inversed – before glueing both pieces back together to form a new planar \( n \)-tree, denoted \( \nabla_v T(\sigma) \). We shall also denote by \( \nabla_v (T, \sigma) \) the couple \((T', \sigma')\) such that \( T'(\sigma') = \nabla_v T(\sigma) \) (cf. figure 2 for an example).

**Proposition 1 (following [10])** Two strata \( \mathcal{M}(T, \sigma) \) and \( \mathcal{M}(T', \sigma') \) in the double cover \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \) coincide if and only if \((T, \sigma) \sim (T', \sigma')\), where \( \sim \) is the equivalence relation generated by \((T, \sigma) \sim \nabla_v(T, \sigma)\) for any vertex distinct from the root. In other words, \((T, \sigma) \sim (T', \sigma')\) if and only if \( \exists v_1 \in \text{Vert}(T), \exists v_i \in \text{Vert}(\nabla_{v_{i-1}} \cdots \nabla_{v_1} T(\sigma)) \), \( i = 2, \ldots, r \), such that \((T', \sigma') = \nabla_{v_r} \cdots \nabla_{v_1} (T, \sigma) \).

Remark: It is easy to see that it can be supposed that \( v_1, \ldots, v_r \) are vertices of the same initial tree \( T \), and if \( v_i \) is a descendent of \( v_j \), then \( i \leq j \).

Proof: The rule \((T, \sigma) \sim \nabla_v (T, \sigma)\) comes from the projectivisation of the normal bundle over the blown-up components. Indeed, suppose we blow-up the component \( a_{\sigma(i)} = a_{\sigma(i+1)} = \ldots = a_{\sigma(j)} \); the equations of the blow-up are:

\[
\lambda_k(a_{\sigma(i)} - a_{\sigma(i)}) = \lambda_l(a_{\sigma(k)} - a_{\sigma(i)}), \quad k, l = i + 1, \ldots, j
\]

\[
[\lambda_{i+1} : \ldots : \lambda_j] \in \mathbb{R}P^{j-i-1}, \quad (a_1, \ldots, a_n) \in S_R^{n-2},
\]

and because \([\lambda_{i+1} : \ldots : \lambda_j] = [-\lambda_{i+1} : \ldots : -\lambda_j]\), it follows that a point of the exceptional divisor close to the cell

\[
a_{\sigma(1)} < a_{\sigma(2)} < \ldots < a_{\sigma(i)} < \ldots < a_{\sigma(j)} < \ldots < a_{\sigma(n)}
\]
is close also to the cell

\[ a_{\sigma(1)} < a_{\sigma(2)} < \ldots < a_{\sigma(j)} < \ldots < a_{\sigma(i)} < \ldots < a_{\sigma(n)}. \]

This gives the identification rules for codimension 1 strata, and since the other strata are intersections of codimension 1 strata, the complete rule follows. □

1.4.3 Combinatorics of the stratification of \( \overline{M}_{0,n+1}(\mathbb{R}) \).

Note from the proof above that if we relax the condition “distinct from the root” in Proposition 1, we get the combinatorics of the stratification of \( \overline{M}_{0,n+1}(\mathbb{R}) \) instead of its double cover \( \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \) \( (S^{n-2} \text{ must be replaced by } \mathbb{P}^{n-2}_{\mathbb{R}} \text{ in the proof, and cells labeled by } \sigma \text{ or } \sigma\omega \text{ must be identified (cf. §1.2, (1))), where } \omega \text{ is the permutation } \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix} \). In this case, there is no need to use rooted trees: strata of \( \overline{M}_{0,n+1}(\mathbb{R}) \) may now be labeled by \( (n+1) \)-unrooted planar trees, \( T(\sigma) \), with a cyclic labeling of their leaves: we mean that the leaves are labeled by \( \sigma(0), \ldots, \sigma(n) \), \( \sigma \in \Sigma_{n+1} \), following the trigonometric order of the plane. We may also consider couples \( (T, \sigma) \), where \( T \) is an unrooted planar tree, with a given cyclic labeling of its leaves, from 0 to \( n+1 \).

Instead of the \( \nabla_v \) moves on trees, two distinct moves \( \nabla_{v,up} \) and \( \nabla_{v,down} \) are introduced in [10], consisting in reflecting either the “upper” or the “lower” subtree (this distinction is arbitrary) disconnecting \( T \) in \( v \). Then the equivalence relation generated by \( T(\sigma) \sim \nabla_{v,up} T(\sigma) \sim \nabla_{v,down} T(\sigma) \) gives the combinatorics of \( \overline{M}_{0,n+1}(\mathbb{R}) \).

1.5 Translation of the identification rules in terms of nested collections and Coxeter groups

Let \( T \) be a planar rooted \( n \)-tree, \( v \) a vertex of \( T \), \( T_v = \{ \sigma_i, \sigma_{i+1}, \ldots, \sigma_{j-1} \} \) the corresponding subset of generators of \( \Sigma_n \) (cf. §1.3). Denote by \( \omega_{T_v} \) or
\(\omega_v\) the involution \(\begin{pmatrix} i & i+1 & \cdots & j \end{pmatrix}\). It is the longest element of \(\Sigma_{T_v}\) (the symmetric group generated by \(T_v\)) for the word metric.

If \(T\) is a nested collection (corresponding to an \(n\)-planar tree \(T\)), then for each \(T_v, T_w\) in \(T\), define \(j_{T_v} T_w\) the connected subset of \(S = \{\sigma_1, \ldots, \sigma_{n-1}\}\) by:

\[
j_{T_v} T_w = \begin{cases} \omega_v T_w \omega_v & \text{if } T_w \subset T_v, \text{ or equivalently, } w \text{ is a descendent of } v, \\ T_w & \text{if not.} \end{cases}
\]

**Proposition 2** Under the correspondence \((T, \sigma) \leftrightarrow (T', \sigma')\) (cf. §1.3), it holds \((T', \sigma') = \nabla_v (T, \sigma)\) if and only if \(T' = j_{T_v} T\) and \(\sigma' = \sigma \omega_{T_v}\), where \(j_{T_v} T\) is the nested collection \(\{j_{T_v} T_w \in T\}\).

With the example of figure 2, \(\sigma = id, \sigma' = \omega_{T_v} = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 \end{pmatrix}\), \(T_v = \{\sigma_2, \sigma_3, \sigma_4\}, T_w = \{\sigma_2\}\). The only subset changed under \(j_{T_v}\) is \(T_w\):

\[
j_{T_v} T_w = \omega_{T_v} T_w \omega_{T_v} = \{\sigma_4\}.
\]

A reformulation of the results of §1.4.2 and §1.4.3 is

**Theorem A** (Davis-Januszkiewicz-Scott, [9]) The two-sheeted covering space \(\tilde{M}_{0, n+1}\) is \(\Sigma_n\)-equivariantly homeomorphic to the geometric realization \(\left|\Sigma_n N\right|\) of the following poset \(\Sigma_n N\):

- Its elements are the equivalence classes of pairs \((T, \sigma)\), \(T\) a nested collection, \(\sigma \in \Sigma_n\), for the equivalence relation \((T, \sigma) \sim (T', \sigma')\) if and only if there exists a subset \(T'' \subset T\) such that \(\sigma' = \sigma \omega_{T''}\) and \(T' = j_{T''} T\).

Here \(\omega_{T''} = \omega_{T_1} \cdots \omega_{T_r}\) if \(T'' = \{T_1, \ldots, T_r\}\), with \(i \leq j\) if \(T_i \subset T_j\), and \(j_{T''} = j_{T_r} \cdots j_{T_1}\).

- The partial order is \([T, \sigma] \leq [T', \sigma']\) if and only if there exists some \(T''\) such that \(\sigma' = \sigma \omega_{T''}\) and \(j_{T''} T \leq T'\), i.e. \(T' \subset j_{T''} T\).

Moreover, the free involution \(\tilde{a} : \tilde{M}_{0, n+1} \to \tilde{M}_{0, n+1}\) (lifted from the antipodal involution of \(S^{n-2}\)), is combinatorial, and given on the poset \(\Sigma_n N\) by \([T, \sigma] \mapsto [j_{T} \sigma \omega_{S}],\) with \(S = \{\sigma_1, \ldots, \sigma_{n-1}\}\). Since \(\bar{M}_{0, n+1} = \tilde{M}_{0, n+1}/\tilde{a}\), \(\bar{M}_{0, n+1}\) inherits a natural cell decomposition with poset \(\Sigma_n N/\tilde{a}\).

## 2 The towers of genus zero real stable curves and action of Tompson and Neretin groups

### 2.1 The towers of genus zero real stable curves

\[\bar{M}_{0, 2^\infty}(\mathbb{R}) = \lim_{n \to \infty} \bar{M}_{0, 3, 2^n}(\mathbb{R})\] and \[\bar{M}_{0, 2^\infty}(\mathbb{R}) = \lim_{n \to \infty} \bar{M}_{0, 2^{n+1}}(\mathbb{R})\]

There is an obvious way to embed \(\bar{M}_{0, n+1}(\mathbb{R})\) into \(\bar{M}_{0, 2^{n+1}}(\mathbb{R})\):

If \(C(x_0, x_1, \ldots, x_n) \in \bar{M}_{0, n+1}(\mathbb{R})\) is a stable curve, then graft a new circle at
any marked point \( x_i, i \geq 0 \), with two marked points on it, \( y_{2i-1} \) and \( y_{2i} \) for \( i \neq 0 \), and \( y_0 \) and \( y_{2n+1} \) for \( i = 0 \). We get a new stable \( 2(n+1) \)-curve, and it is
uniquely defined, since \( PGL(2, \mathbb{R}) \) identifies the configurations of any triple of
points on a component.

If now \( C(x_0, x_1, \ldots, x_n) \in \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \), there are two distinct configurations on the
grafted circle with marked points \( y_0, y_{2n+1} \), and double point \( x_0 \). So we
may decide to expand unambiguously all the points \( x_i \) except \( x_0 \), to get a curve
\( C(x_0, y_1, y_2, \ldots, y_{2n-1}, y_{2n}) \) in \( \mathcal{M}_{0,2n+1}(\mathbb{R}) \).

In fact, the map \( C(x_0, x_1, \ldots, x_n) \mapsto C(x_0, x_1, \ldots, x_{i-1}, y_i, y_{i+1}, x_{i+1}, \ldots, x_n) \) is
a section of a forgetting map \( \overline{\mathcal{M}}_{0,n+2}(\mathbb{R}) \to \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \), and is called “stabilization” (at least in the complex case) by Knudsen (cf. [23]). It is a smooth
map.

**Proposition 3 (Dyadic expansion principle, or Stabilization)** The embedding \( \exp_n : \overline{\mathcal{M}}_{0,n}(\mathbb{R}) \hookrightarrow \overline{\mathcal{M}}_{0,2n}(\mathbb{R}) \) is a morphism of stratified spaces. The
inductive limit \( \overline{\mathcal{M}}_{0,2^n}(\mathbb{R}) = \lim_{\rightarrow} \overline{\mathcal{M}}_{0,2^n}(\mathbb{R}) \) inherits a locally non-finite CW-
complex structure.

The same is true when the moduli spaces are replaced by their two-sheeted covering spaces \( \mathcal{M}_{0,n+1}(\mathbb{R}) \), with embeddings \( \exp_n : \mathcal{M}_{0,n+1}(\mathbb{R}) \hookrightarrow \mathcal{M}_{0,2n+1}(\mathbb{R}) \),
and there is a tower \( \mathcal{M}_{0,2^n}(\mathbb{R}) = \lim_{\rightarrow} \mathcal{M}_{0,2^n+1}(\mathbb{R}) \).

We may see both towers as pointed spaces, with based point \( * \) represented by the
unique point of \( \overline{\mathcal{M}}_{0,3}(\mathbb{R}) = \mathcal{M}_{0,2+1}(\mathbb{R}) \).

**Proof:** We make the proof for the covering spaces \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \). If \( \mathcal{M}(T, \sigma) \)
is a stratum of \( \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \), then \( \exp_n(\mathcal{M}(T, \sigma)) \) is the stratum \( \mathcal{M}(\exp_n(T, \sigma)) \),
where we define \( \exp_n(T, \sigma) = (\exp_n(T), \tau) \) to be the planar rooted tree obtained
from \( (T, \sigma) \) by expanding each of the leaf by two new edges, which become the
input edges of \( \exp_n(T, \sigma) \). More precisely, if the input edges of \( T(\sigma) \) are labeled
from the left to the right by \( \sigma(1), \ldots, \sigma(n) \), then the input edges of \( \exp_n(T, \sigma) \)
will be enumerated by \( \tau \in \Sigma_{2n} \) defined by \( \tau(2i-1) = 2\sigma(i)-1, \tau(2i) = 2\sigma(i) \),
\( i = 1, \ldots, n \). By the way, what we have just defined is a group homomorphism
\[
\exp_n : \Sigma_{n} \to \Sigma_{2n} : \sigma \mapsto \tau = \exp_n(\sigma),
\]
called the “expansion morphism”, and we have \( \exp_n(T, \sigma) = (\exp_n(T), \exp_n(\sigma)) \).

Clearly, if \( (T, \sigma) \sim (T', \sigma') \), then \( \exp_n(T, \sigma) \sim \exp_n(T', \sigma') \). Moreover, though
it is not true that \( (T, \sigma) \leq (T', \sigma') \Rightarrow \exp_n(T, \sigma) \leq \exp_n(T', \sigma') \), this becomes
true by replacing \( \exp_n(T, \sigma) \) by an equivalent tree, labeling the same stratum
(using Proposition 1). So \( \exp_n \) is cellular. \( \square \)
2.2 Action of Thompson and Neretin groups on the towers \( \mathcal{M}_{0,2\approx}(\mathbb{R}) \) and \( \mathcal{M}_{0,3,2\approx}(\mathbb{R}) \)

2.2.1 Thompson group \( V \) and Neretin group \( N \)

**Thompson group.** Recall that Thompson group \( V \) (cf. eg. [2], [10]) is the group whose elements are represented by symbols \((\alpha_1, \alpha_0, \sigma)\), where \(\alpha_1\) and \(\alpha_0\) are binary planar trees with the same number of leaves, say \(n\), and \(\sigma \in \Sigma_n\) indicates how the leaves of \(\alpha_0\) are mapped to those of \(\alpha_1\): explicitly, if \(e_1, \ldots, e_n\) (resp. \(f_1, \ldots, f_n\)) are the leaves or input edges of \(\alpha_0\) (resp. \(\alpha_1\)), then \(\sigma\) prescribes the correspondence \(e_i \mapsto f_{\sigma(i)}\).

A symbol can be expanded in the following way: choose an input edge \(e_i\), expand its terminal leaf by two edges \(e_{i,l}\) (the left one), \(e_{i,r}\) (the right one), do the same with \(f_{\sigma(i)}\), and define \(\tau \in \Sigma_{n+1}\) describing the correspondence \(e_{i,l} \mapsto f_{\sigma(i),l}\), \(e_{i,r} \mapsto f_{\sigma(i),r}\), \(e_j \mapsto f_{\sigma(j)}\), \(j \neq i\), after re-labeling the new sets of input edges from 1 to \(n+1\). We get a new symbol \((\beta_0, \beta_1, \tau)\), and we consider the equivalence relation generated by \((\alpha_1, \alpha_0, \sigma) \sim (\beta_0, \beta_1, \tau)\).

The group structure on the equivalence classes of symbols is defined by

\[
\begin{align*}
(i) \quad [(\alpha_2, \alpha_1, \sigma)](\alpha_1, \alpha_0, \sigma)] &= [(\alpha_2, \alpha_1, \tau \circ \sigma)] \\
(ii) \quad [(\alpha_1, \alpha_0, \sigma)]^{-1} &= [(\alpha_0, \alpha_1, \sigma^{-1})] \\
(iii) \quad e &= [(\alpha, \alpha, id_n)], \text{ for any } n\text{-planar tree } \alpha,
\end{align*}
\]

where we used the fact that a given pair of symbols can be replaced by a pair of equivalent ones, so that the source tree of the first one coincides with the target tree of the second one. We have denoted by \(e\) the neutral element.

Observe that if we expand all the leaves of both trees in a symbol \((\alpha_1, \alpha_0, \sigma)\), we get a symbol \((\exp_n(\alpha_1), \exp_n(\alpha_0), \exp_n(\sigma))\), with the notations of the proof of Proposition 3. So denote the expanded symbol by \(\exp_n(\alpha_1, \alpha_0, \sigma)\).

**Neretin group \( N \).** Recall from [25] or [13] the definition of Neretin “sphero-morphism” group: Let \(\alpha\) be an \(n\)-planar dyadic rooted tree, and see each leaf of \(\alpha\) as the root of an (infinite) dyadic complete planar tree \(T_i^\alpha\), \(i = 1, \ldots, n\). Thus, \(\alpha \cup T_1^\alpha \cup \ldots \cup T_n^\alpha\) itself is a dyadic complete planar rooted tree.

Now elements of Neretin group \( N \) are represented by symbols \((\alpha_1, \alpha_0, q_\sigma)\), where again \(\alpha_0, \alpha_1\) are \(n\)-trees for some \(n\), \(\sigma \in \Sigma_n\), and \(q_\sigma\) is a collection of tree isomorphisms \(q_i : T_i^{\alpha_0} \to T_{\sigma(i)}^{\alpha_1}\), \(i = 1, \ldots, n\).

The notion of expansion is natural: if \(T_i^{\alpha_0}\) is replaced by its two halves \(T_{i,l}^{\alpha_0}\) and \(T_{i,r}^{\alpha_0}\), then \(q_i\) is replaced by its restrictions to the halves, \(q_{i,l}\) and \(q_{i,r}\), inducing tree isomorphisms onto the corresponding halves of \(T_{\sigma(i)}^{\alpha_1}\). But contrary to the Thompson case, there is no reason that the left (resp. right) half of \(T_i^{\alpha_0}\) should be sent onto the left (resp. right) half of \(T_{\sigma(i)}^{\alpha_1}\).

The expansion procedure generates an equivalence relation in the set of symbols, and equivalence classes of symbols composition is induced by the usual composition of tree isomorphisms. This defines the group structure on Neretin.
group $N$.
Clearly, $N$ contains $V$ as a subgroup.

2.2.2 The cyclic variant. The use of rooted trees is standard in the definition of Thompson groups, however we can treat trees occurring in symbols $(\alpha_1, \alpha_0, \sigma)$ as unrooted $(n+1)$-trees, with $\sigma \in \Sigma_{n+1}$ (instead of $\Sigma_n$). They are given with a cyclic labeling of their leaves, from 0 to $n+1$. Now define the notion of expansion as before. We get a group $V$, isomorphic to $V$.

An analogous presentation for defining Neretin group is relevant, which uses unrooted dyadic planar trees: we get a group $N$, isomorphic to $N$. In this way, $N$ (and $V$) appear as subgroups of the homeomorphism group of the boundary $\partial T_2$ of the regular dyadic tree $T_2$ (without root). In particular, $N$ contains as a subgroup the full automorphism group of the tree, $\text{Aut}(T_2)$, and so, the $p$-adic group $\text{PGL}(2, \mathbb{Q}_2)$. This illustrates how much bigger $N$ is than $V$. This corresponds also to the original presentation for $N$ (cf. [25]).

2.2.3 Thompson and Neretin groups acting on the infinite moduli spaces

On the tower $\hat{\mathcal{M}}_{0,2\infty}(\mathbb{R})$: Let $g = [(\alpha_1, \alpha_0, q_\sigma)]$ be in $N$, $[\mathcal{M}(T, \tau)]$ be a stratum of $\hat{\mathcal{M}}_{0,2\infty}(\mathbb{R})$, represented in some $\hat{\mathcal{M}}_{0,2\infty+1}(\mathbb{R})$.

At the price of making an expansion of the symbol defining $g$, it can be supposed that the trees $\alpha_i$, $i = 0, 1$, are $2^n$-planar trees. Represent the stratum by a symbol $(T_{2^n}, T, \tau)$, where $T_{2^n}$ is the dyadic tree:

```
  (α₁, α₀ = T_{2^n}, q_σ)(T_{2^n}, T, τ) = (α₁, T, σ ◦ τ),
```

and making expansions, replace the resulting symbol by an equivalent one, to get a symbol of the form $(\text{exp}(\alpha_1) = T_{2^m}, \text{exp}(T), \text{exp}(\sigma ◦ \tau))$, for some $m \geq n \in \mathbb{N}$, where exp is the appropriate expansion map or morphism, and $\text{exp}(\sigma ◦ \tau) \in \Sigma_{2^m}$. Finally define

$$g[\mathcal{M}(T, \tau)] := [\mathcal{M}(\text{exp}(T), \text{exp}(\sigma ◦ \tau))].$$

On the cyclic tower $\hat{\mathcal{M}}_{0,3,2\infty}(\mathbb{R})$: We now use the presentation of $N$ (and $V$) given in the variant above, to make symbols defining group elements act
on symbols labeling strata. As can be guessed, the tree $T_{2^n}$ must be replaced here by the unrooted tree $T_{3,2^n}$. Trees involved in symbols defining elements of $\mathcal{N}$ or $\mathcal{V}$ must be represented with a marked vertex, corresponding to the central vertex of the tree $T_{3,2^n}$. If now $\overline{\mathcal{M}}(T(\tau))$ is a stratum of $\overline{\mathcal{M}}_{0,3,2^n}(\mathbb{R})$, $g = [(\alpha_1, \alpha_0 = T_{3,2^n}, q_\sigma)] \in \mathcal{N}$, with $\sigma \in \Sigma_{3,2^n}$, then compose the symbols: $(\alpha_1, \alpha_0 = T_{3,2^n}, q_\sigma). (T_{3,2^n}, T(\tau)) = (\alpha_1, T(\sigma \circ \tau))$, and expand $\alpha_1$ to get a tree of the form $T_{3,2^n} = \overline{\exp}(\alpha_1)$ (the marked vertex is useful to perform this procedure), and correspondingly expand $T(\sigma \circ \tau)$. Then define $g. [\overline{\mathcal{M}}(T(\tau))] = [\overline{\mathcal{M}}(\overline{\exp}(T(\sigma \circ \tau))).$

**Theorem 1** Denoting by $\text{Strat} \left( \overline{\mathcal{M}}_{0,2^n}(\mathbb{R}) \right)$ the set of strata of the tower $\overline{\mathcal{M}}_{0,2^n}(\mathbb{R})$, the map

$$N \times \text{Strat} \left( \overline{\mathcal{M}}_{0,2^n}(\mathbb{R}) \right) \rightarrow \text{Strat} \left( \overline{\mathcal{M}}_{0,2^n}(\mathbb{R}) \right)$$

defined above is well-defined, and induces a cellular left action of Neretin group $N$ on the tower $\overline{\mathcal{M}}_{0,2^n}(\mathbb{R})$: $\tilde{\gamma} : N \rightarrow \text{Homeo}_{cell}(\overline{\mathcal{M}}_{0,2^n}(\mathbb{R}))$,

which is faithful.

In particular, $\tilde{\gamma}$ restricts to a cellular action of Thompson group $V$.

There is an analogous statement with $\overline{\mathcal{M}}_{0,2^n}(\mathbb{R})$ replaced by $\overline{\mathcal{M}}_{0,3,2^n}(\mathbb{R})$, and $\tilde{\gamma}$ by $\overline{\gamma} : \mathcal{N} \rightarrow \text{Homeo}_{cell}(\overline{\mathcal{M}}_{0,3,2^n}(\mathbb{R}))$.

**Proof:** We prove the statement for the action on $\overline{\mathcal{M}}_{0,2^n}(\mathbb{R})$:

(1) We first need to check

$$(T, \tau) \sim (T', \tau') \implies (\exp(T), \exp(\sigma \circ \tau)) \sim (\exp(T'), \exp(\sigma \circ \tau')) :$$

It is clear we can restrict to the case where $T' = j_{T_v} T$ and $\tau' = \tau \omega_v$, for some $v \in \text{Vert}(T)$ (we use the correspondence between planar trees and nested families, cf. §1.3). For simplicity suppose $\omega_v$ is of the form

$$\omega_v = \left( \begin{array}{cccc} 1 & 2 & \cdots & k \\ k & k-1 & \cdots & 1 \end{array} \right) = \omega_{(1, \ldots, k)}$$

Using the well-defined homomorphism $\exp : \Sigma_{2^n} \rightarrow \Sigma_{2^n}$ it follows that $\exp(\sigma \circ \tau') = \exp(\sigma \circ \tau) \exp(\omega_v)$. Suppose for simplicity again that $m = n + 1$; then with the labeling of $\exp(T)$, it holds

$$\exp(\omega_v) = \left( \begin{array}{cccc} 1 & 2 & \cdots & 2l-1 \\ 2k-1 & 2k & \cdots & 2k-2l+1 \\ 2k-2l+2 & \cdots & 2k-l & 2k-2l & \cdots & 2k-l & 2k \end{array} \right).$$

Denoting by $\tilde{v}$ the vertex $v$ seen in the expanded tree $\exp(T)$, $\omega_v$ differs from

$$\omega_{\tilde{v}} = \left( \begin{array}{cccc} 1 & 2 & \cdots & 2l-1 \\ 2k & 2k-1 & \cdots & 2k-2l+2 \\ 2k-2l+2 & \cdots & 2k-l+1 & 2k \end{array} \right) = \omega_{(1, \ldots, 2k)}$$

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by the product of the $k$ transpositions $\omega \tilde{v}_1 \ldots \omega \tilde{v}_k$, where $\omega \tilde{v}_i$ is simply $\omega_i = (2i - 1, 2i)$, and $\tilde{v}_i$ is the $i$th leaf of $T$ but seen in $\exp(T)$: each $\tilde{v}_i$ has itself two descendents. It follows that

\[
\begin{cases}
\exp(T') = j_{\tilde{v}_1} \ldots j_{\tilde{v}_k} \exp(T), \\
\exp(\sigma \circ \tau') = \exp(\sigma \circ \tau) \omega_1 \omega_2 \ldots \omega_k,
\end{cases}
\]

where in fact the operations $j_{\tilde{v}_i}$ have no effect. The expected equivalence is proved.

(2) We must check also that

\[
\exp_n(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma) \exp_n(T_{2^n}, T, \tau) \sim \exp_n(\alpha_1, T, \sigma \circ \tau).
\]

Now $\exp_n(\alpha_1, T_{2^n}, q_\sigma) = (\exp_n(\alpha_1), T_{2^{n+1}}, \tilde{q}_\sigma)$, where $\tilde{\sigma}$ may differ from $\exp_n(\sigma)$ (because $g$ sits in $N$, not necessarily in $V$) by a product of transpositions $\omega_i = (2i - 1, 2i)$: $\tilde{\sigma} = \exp(\sigma) \omega_1 \ldots \omega_i$. So,

\[
(*) \quad \exp_n(\alpha_1, T_{2^n}, q_\sigma) \exp_n(T_{2^n}, T, \tau) = (\exp_n(\alpha_1), \exp_n(T), \exp_n(\sigma) \omega_1 \ldots \omega_i, \exp_n(\tau)).
\]

But $\exp(\tau) \omega_i = \omega_{\tau(i)} \exp(\tau)$, so that

\[
(*) = (\exp_n(\alpha_1), \exp_n(T), \exp_n(\sigma) \exp_n(\tau) \omega_{\tau^{-1}(i_1)} \ldots \omega_{\tau^{-1}(i_k)})
\]

\[
\sim (\exp_n(\alpha_1), \exp_n(T), \exp_n(\sigma \circ \tau)).
\]

(3) Faithfulness is obvious, or may be proven by invoking the simplicity of the group $N$ (\[\text{[S]}\]).

(4) Translated in terms of $\nabla_v$ moves, the proof above can be now easily adapted to the case of the action on the tower $M_{0,3,2^n}(\mathbb{R})$ (but using $\nabla^{up}_v$ and $\nabla^{down}_v$ moves). \[\square\]

**Two complementary remarks:**

1. **Pointwise description of the action.** Let $(a_i)_{i=1,\ldots,n}$ be the affine coordinates of a point $p \in \mathcal{M}(T, \tau)$ (cf. proof of Proposition 1, where we give the coordinates for a codimension 1 stratum). If $v$ is an internal vertex of $T$, and $i_1 < \ldots < i_k$ is the collection of the one-valent descendents (“sons”) of $v$ (recall the labeling of $T$ is canonical, from 1 (on the left) to $n$ (on the right)), this means that

\[
a_{\tau(i_1)} < \ldots < a_{\tau(i_k)}.
\]

Then $g(p)$, with $g = [(\alpha_1, \alpha_0, q_\sigma)]$ in $N$, corresponds to an adequate expansion in $\mathcal{M}(\exp(T), \exp(\sigma \circ \tau))$, of the point of $\mathcal{M}(T, \sigma \circ \tau)$ with coordinates $(b_i)_{i=1,\ldots,n}$ such that $b_{\sigma(i)} = a_i$.  

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2. **Action on the complex tower.** As was pointed out to the author by Yu. Neretin, we could define also the pointwise action of \( N \) on the complex tower \( \mathcal{M}_{0,2\infty}(\mathbb{C}) \) by the same formula as above, replacing \( \mathcal{M}(T,\tau) \) by the complex stratum \( \mathcal{M}(T) \) (the strata are indeed labeled by bare \( n \)-rooted trees, isotopically realized in \( \mathbb{R}^3 \)) and the real coordinates by complex ones. This is an action of stratified spaces, and Theorem 1 shows that the restriction of the action to the real part respects the cell decomposition in associahedra of the latter. We mention also that in the cyclic version of the action, the complex action of \( \mathcal{N} \) extends the action of \( \text{Aut}(T_2) \) that was observed by Kapranov in the note [23].

2.3 **Thompson group \( T \) acting on the cyclic tower \( \mathcal{M}_{0,3,2\infty}(\mathbb{R}) \)**

Thompson group \( T \) is the subgroup of elements of \( \mathcal{V} \) represented by symbols \((\alpha_1,\alpha_0,\sigma)\) (in the second description) where \( \sigma \) lies in the cyclic subgroup \( \mathbb{Z}/(n+1)\mathbb{Z} \) of \( \Sigma_{n+1} \) (when \( \alpha_1 \) and \( \alpha_0 \) are unrooted \((n+1)\)-trees).

On the other hand, since the expansion morphism \( \exp_n: \Sigma_n \to \Sigma_{2n} \) maps \( \mathbb{Z}/n\mathbb{Z} \) into \( \mathbb{Z}/2n\mathbb{Z} \), there is a distinguished subcomplex in the tower \( \mathcal{M}_{0,3,2\infty}(\mathbb{R}) \), which is the union of strata \( \mathcal{M}(T(\tau)) \) with \( \tau \) in the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) (if \( T(\tau) \) has \( n \) leaves). This subcomplex is simply an infinite associahedron \( K_\infty \), an inductive limit of associahedra \( K_{3,2^{n-1}} \). The cyclic tower \( \mathcal{M}_{0,3,2\infty}(\mathbb{R}) \) is tiled by an infinite number of copies of the associahedron \( K_\infty \).

**Theorem 2** The stabilizer of the distinguished associahedron \( K_\infty \) under the action of \( \mathcal{N} \) is the semi-direct product \( \text{Out}(T) \rtimes T \), where \( \text{Out}(T) \), the exterior automorphism group of \( T \), is known to be the order two cyclic group.

Remark: \( \text{Out}(T) \rtimes T \) as a symmetry group of \( K_\infty \) is the infinite-dimensional analogue of \( D_n = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z} \), the dihedral group, as the symmetry group of the associahedron \( K_{3,2^{n-1}} \).

Proof: A cell of \( K_\infty \) is represented in some \( \mathcal{M}_{0,3,2\infty}(\mathbb{R}) \) by some \( \mathcal{M}(T(\tau)) \), with \( \tau \in \mathbb{Z}/3.2^n\mathbb{Z} \). If \( g \in T \) is represented by \((\alpha_1,\alpha_0 = T_{3,2^n},\sigma)\), with \( \sigma \in \mathbb{Z}/3.2^n\mathbb{Z} \), then \( g.[\mathcal{M}(T(\tau))] = [\mathcal{M}(\exp(T(\sigma \circ \tau)))] \), where \( \sigma \circ \tau \in \mathbb{Z}/3.2^n\mathbb{Z} \), and so \( \exp(T(\sigma \circ \tau)) \) is a tree \( T'(\sigma') \), with \( \sigma' \) in some \( \mathbb{Z}/3.2^n\mathbb{Z} \). This proves that \( g.[\mathcal{M}(T(\tau))] \) is a cell of \( K_\infty \).

Now, if \( g = [(\alpha_1,\alpha_0 = T_{3,2^n},q_{n_0})] \) lies in \( \mathcal{N} \) and stabilizes \( K_\infty \), then at the price of composing \( g \) by an element of \( T \) it may be supposed that \( \alpha_1 = \alpha_0 = T_{3,2^n} \), and that \( \sigma_0(0) = 0 \). Now the the maximal cell of \( \mathcal{M}_{0,3,2\infty}(\mathbb{R}) \cap K_\infty \), labeled by the stared tree \( T(id_{3,2^n}) \), is mapped by \( g \) to the cell \( \mathcal{M}(\exp(T(\sigma_0))) \). Using \( \nabla \) moves, we see that this cell still lies in \( K_\infty \) if and only if \( \sigma_0 = id_{3,2^n} \) or there is \( \omega(1,...,3.2^n) \) (using Theorem 1). Repeating the argument in \( \mathcal{M}_{0,3,2^{n+1}}(\mathbb{R}) \) with \( \sigma_1 \), we find that \( \sigma_1 = id_{3,2^{n+1}} \) or \( \sigma_1 = \omega(1,...,3.2^{n+1}) \). But \( \sigma_1 \) is the expansion of \( \sigma_0_1 \), possibly twisted by elementary transpositions, so \( \sigma_0 = id_{3,2^n} \Rightarrow \sigma_1 = id_{3,2^{n+1}} \) and \( \sigma_0 = \omega(1,...,3.2^n) \Rightarrow \sigma_1 = \omega(1,...,3.2^{n+1}) \). Then we see that \( g \) acts like the involution \( \text{Inv} \) defined by \( \sigma_0 = \omega(1,...,3.2^n) \), \( \sigma_1 = \omega(1,...,3.2^{n+1}) \), etc. Under the isomorphism \( T \cong \text{PSL}(2,\mathbb{Z}) \) (cf. [14]), \( \text{Inv} \) corresponds to the non-preserving
orientation inversion \( x \in \mathbb{R}P^1 \mapsto \frac{1}{x} \in \mathbb{R}P^1 \). And it is known to generate the exterior automorphism group of \( \mathcal{T} \), Out(\( \mathcal{T} \)) \( \cong \mathbb{Z}/2\mathbb{Z} \) (cf. [4]). \( \square \)

2.4 Extensions of Thompson and Neretin groups by infinite pure quasi-braid groups \( P J_2^\infty \) and \( Q_{3,2^\infty} \)

It is shown in [8] that \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \) (or \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \)) is an aspherical space: its universal covering \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \) is contractible, so that its cellular homology coincides with the group homology of its fundamental group. The universal covering of \( \mathcal{M}_{0,2^\infty}(\mathbb{R}) \) is the inductive limit of the coverings \( \mathcal{M}_{0,2n+1}(\mathbb{R}) \), and will be denoted \( \tilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) \). Let us denote by \( PJ_n \) the fundamental group of \( \mathcal{M}_{0,2^\infty}(\mathbb{R}) \): \( PJ_n = \pi_1(\mathcal{M}_{0,n+1}(\mathbb{R})) \). Since \( \mathcal{M}_{0,n+1}(\mathbb{R}) \) is a double-cover of \( \tilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \), there is an extension \( 1 \to PJ_n \to Q_{n+1} \to \mathbb{Z}/2\mathbb{Z} \to 0 \), where we set \( Q_{n+1} := \pi_1(\mathcal{M}_{0,n+1}(\mathbb{R})) \).

Let us now consider the group \( PJ_2^\infty = \pi_1(\mathcal{M}_{0,2^\infty}(\mathbb{R})) \), so that

\[
PJ_2^\infty = \varprojlim_n PJ_n,
\]

as well as the group \( Q_{3,2^\infty} = \pi_1(\mathcal{M}_{0,3,2^\infty}(\mathbb{R})) = \varprojlim_n Q_{3,2^\infty} \). Now each transformation \( \tilde{\gamma}(g) : \tilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) \to \tilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) \), with \( g \in \mathbb{N} \), can be lifted -- non-uniquely -- to the universal covering \( \tilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) \), and similarly for the transformation \( \tilde{\gamma}(g) \).

**Definition 4** Let \( G \) denote Thompson group \( V \) or Neretin group \( N \). The group of lifted transformations \( \tilde{\gamma}(g) \) (resp. \( \tilde{\gamma}(g) \)) \( G \) is denoted \( \tilde{A}_G \) (resp. \( \tilde{A}_G \)), and is a subgroup of the cellular homeomorphism group of \( \mathcal{M}_{0,2^\infty}(\mathbb{R}) \) (resp. \( \tilde{\mathcal{M}}_{0,3,2^\infty}(\mathbb{R}) \)).

By construction we get epimorphisms \( \tilde{A}_G \to G \) and \( \tilde{A}_G \to G \). The kernel of the first epimorphism is the automorphism of the universal covering map \( \mathcal{M}_{0,2^\infty}(\mathbb{R}) \to \mathcal{M}_{0,2^\infty}(\mathbb{R}) \): it coincides with the fundamental group \( PJ_2^\infty \). Similarly, the kernel of the second one \( Q_{3,2^\infty} \). We are now ready for introducing the announced extensions of Thompson and Neretin groups:

**Definition 5 (Quasi-braid extensions of Thompson and Neretin groups)**

Let \( G \) (resp. \( \mathcal{G} \)) denote Thompson group \( V \) (resp. \( V \)) or Neretin group \( N \) (resp. \( N \)). The construction above provides extensions \( 1 \to PJ_2^\infty \to \tilde{A}_G \to G \to 1 \) and \( 1 \to Q_{3,2^\infty} \to \tilde{A}_G \to \mathcal{G} \to 1 \), and the natural embedding \( G \to \mathcal{G} \) induces a morphism between both extensions

\[
1 \to PJ_2^\infty \to \tilde{A}_G \to G \to 1,
\]

\[
1 \to Q_{3,2^\infty} \to \tilde{A}_G \to \mathcal{G} \to 1,
\]

where all the vertical arrows are injective morphisms.
The embedding $G \hookrightarrow G$ is induced by the embeddings of the rooted planar trees $T_{2^n}$ as subtrees of the unrooted planar tree $T_{3,2^n}$. The latter induce also the cellular embeddings $\overline{M}_{0,2^n+1} \hookrightarrow \overline{M}_{0,3,2^n}$, and morphisms $Q_{2^n+1} \hookrightarrow Q_{3,2^n}$ (they are injective, cf. Theorem 4). Composing $PJ_{2^n} \hookrightarrow Q_{2^n+1}$ with $Q_{2^n+1} \hookrightarrow Q_{3,2^n}$ gives $PJ_{2^n} \hookrightarrow Q_{3,2^n}$, and so $PJ_{2^n} \hookrightarrow Q_{3,2^n}$. 

3. An analogue of the Euler class for Neretin Spheromorphism group

Our first task is to improve our understanding of the infinite quasi-braid groups.

3.1 Quasi-braid groups $J_n$

3.1.1 Let $J_n$ be the group defined in [3] by generators and relations, with generators $\alpha_T$ for each subset of $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ such that the corresponding graph $G_T$ is connected, with the following relations:

- $\alpha_T^2 = 1$ for each $T$
- $\alpha_T \alpha_{T'} = \alpha_{j_T T'} \alpha_T$ if $T' \subset T$
- $\alpha_T \alpha_{T'} = \alpha_{T'} \alpha_T$ if $G_{T \cup T'}$ is not connected.

These relations are those verified by the involutions $\omega_T$. So there is a well-defined homomorphism $\phi : J_n \rightarrow \Sigma_n$, $\alpha_T \mapsto \omega_T$, surjective since for $T_i = \{\sigma_i = (i, i + 1)\}$, $\phi(\alpha_T) = \sigma_i$.

Theorem B (Davis-Januszkiewicz-Scott, [3]) The universal cover $\overline{M}_{0,n+1}$ of the two-sheeted cover $\overline{M}_{0,2n+1}$ is $J_n$-equivariantly homeomorphic to the geometric realization $|J_n N|$ of the poset $J_n N$:

- Its elements are the equivalence classes of pairs $(T, \alpha)$, $T$ a nested collection, $\alpha \in J_n$, with the equivalence relation $(T, \alpha) \sim (T', \alpha')$ if and only if there exists a subset $T'' \subset T$ such that $\alpha' = \alpha \alpha_{T''}$ and $T' = j_{T''} T$. Here $\alpha_{T''} = \alpha_T \ldots \alpha_T$, if $T'' = \{T_1, \ldots, T_r\}$, with $i \leq j$ if $T_i \subset T_j$, and $j_{T''} = j_{T_1} \ldots j_{T_r}$.

- The partial order is $[T, \alpha] \leq [T', \alpha']$ if and only if there exists some $T''$ such that $\alpha' = \alpha \alpha_{T''}$ and $j_{T''} T \leq T'$, i.e. $T' \subset j_{T''} T$.

Moreover, there is a natural $J_n$-left-equivariant map $J_n N \rightarrow \Sigma_n N$ given by $[T, \alpha] \mapsto [T, \phi(\alpha)]$, the $J_n$-action on $\Sigma_n N$ being defined by $\alpha : [T, \sigma] \mapsto [T, \phi(\alpha) \sigma]$.

The kernel $PJ_n := \text{Ker} \phi$ is the fundamental group of $M_{0,n+1}$, and there is a short exact sequence

$$1 \rightarrow PJ_n \rightarrow J_n \rightarrow \Sigma_n \rightarrow 1.$$
As for the fundamental group of $\overline{M}_{0,n+1}$, it is generated by $\text{Ker } \phi = PJ_n$ and a lift $\hat{a}$ of the free involution map $\hat{a} : [T, \sigma] \to [jS, \sigma \omega_S]$ (cf. Theorem A), that we now describe:

- Let $\hat{a} := \alpha_{T_1} \ldots \alpha_{T_r}$ be any lift in $J_n$ of $\omega_S$. We choose the unique lift $\hat{a}$ such that $\hat{a}[0, 1] = [0, \hat{a}]$.

- Let $[\emptyset, \beta]$ be a maximal cell of $|J_nN|$, with $\beta = \alpha_{U_1} \ldots \alpha_{U_k}$. Choose an edge path $[0, 1] \to [0, \alpha_{U_1}] \to [0, \alpha_{U_1} \alpha_{U_2}] \to \ldots \to [0, \alpha_{U_1} \ldots \alpha_{U_k} = \beta]$. Then $\hat{a}[0, 1]$ is an edge path in the dual complex of $|J_nN|$. Then $\hat{a}[0, \beta]$ is of the form $\hat{a}[0, \beta] = [0, \hat{a} \alpha_{j_{s_{U_1}}} \ldots \alpha_{j_{s_{U_k}}}].$

It is easy to check that the correspondence $jS : \alpha_T \mapsto \alpha_{j_{s_{U_1}}} \ldots \alpha_{j_{s_{U_k}}}$ induces a well-defined automorphism of the group $J_n$ (so that $jS\beta = \alpha_{j_{s_{U_1}}} \ldots \alpha_{j_{s_{U_k}}}$ does not depend on the way of writing $\beta$). Now $\hat{a}^2$ is the map such that

$$\hat{a}^2[0, \beta] = [0, \hat{a} jS(\hat{a}) \alpha_{U_1} \ldots \alpha_{U_k}].$$

In other words, $\hat{a}^2$ is the pure quasi-braid $\hat{a} jS(\hat{a}) = \alpha_{T_1} \ldots \alpha_T \alpha_{j_{s_{U_1}}} \ldots \alpha_{j_{s_{U_k}}}$, acting on the left on the cells. This was of course predictable from the short exact sequence $1 \to PJ_n \to Q_{n+1} \to \mathbb{Z}/2\mathbb{Z} \to 1$.

### 3.1.2 Describing the morphism $PJ_n \to PJ_{2n}$ and defining $J_n \to J_{2n}$

The formalism contained in Theorem B enables us to describe the morphism $PJ_n \to PJ_{2n}$ induced by the embedding $\exp_n : \overline{M}_{0,n+1} \to \overline{M}_{0,2n+1}$: Each $\alpha = \alpha_{T_1} \ldots \alpha_{T_r}$ in $PJ_n$ projects onto $\omega_{T_1} \ldots \omega_{T_r} = 1$ in $\Sigma_n$. We interpret $\alpha$ as the homotopy class of the edge loop $\gamma = (id_n \to \omega_{T_1} \to \omega_{T_1} \omega_{T_2} \to \ldots \to \omega_{T_r} \omega_{T_2} \ldots \omega_{T_r} = id_n)$ in the dual cell complex of $|\Sigma_nN| \cong \overline{M}_{0,n+1}$.

By definition of the embedding $\exp_n : \overline{M}_{0,n+1} \to \overline{M}_{0,2n+1}$, the loop $\gamma$ is mapped to the loop $\exp(\gamma) = (id_{2n} \to \exp(\omega_{T_1}) \to \exp(\omega_{T_1}) \exp(\omega_{T_2}) \to \ldots \to \exp(\omega_{T_r}) \exp(\omega_{T_2}) \ldots \exp(\omega_{T_r}) = id_{2n})$. We need however to precise what a path of the form $id_{2n} \to \exp(\omega_T)$ is:

Suppose for simplicity that $\omega_T$ is of the form

$$\omega_T = \begin{pmatrix} 1 & 2 & \ldots & k \\ k & k-1 & \ldots & 1 \end{pmatrix},$$

so that $\exp(\omega_T) = \begin{pmatrix} 1 & 2 & \ldots & 2k-1 & 2k \\ 2k-1 & 2k & \ldots & 1 & 2 \end{pmatrix}$ is the product $\omega_{\exp(T)}\omega(1,2)$ \ldots $\omega(2k-1,2k)$, with $\omega(2i-1,2i)$ the transposition $\sigma_{2i-1}$. The path $id_n \to \omega_T$ once embedded in $|\Sigma_{2n}N^0|$, and after a suitable translation to make its extremities coincide with the barycenters of the cells $id_{2n}$, and $\exp(\omega_T)$ (which are adjacent because $\exp$ is cellular, meeting along a codimension $n + 1$ stratum), becomes the straight line joining the barycenters. We claim this line is homotopic to the edge path

$$id_{2n} \to \omega_{\exp(T)} \to \omega_{\exp(T)}\omega(1,2) \to \ldots \to \omega_{\exp(T)}\omega(1,2) \ldots \omega(2k-1,2k).$$
Indeed, the path above passes through cells which all share a same codimension
$n + 1$ stratum, and the line $\text{id}_{2n} \to \exp(\omega_T)$ crosses the same stratum.

Now $\alpha_T$ may be lifted in $J_{2n}$ to

$$\exp(\alpha_T) := \alpha_{\exp(T)} \alpha_{(1, 2)} \cdots \alpha_{(2k - 1, 2k)},$$

where $\alpha_{(2i - 1, 2i)} := \alpha_{T_i}$, with $T_i = \{\sigma_{2i-1}\}$. Finally define $\exp(\alpha)$ as the product $\exp(\alpha_T_1) \cdots \exp(\alpha_T_n)$. We now claim:

**Proposition 4** 1. The map $J_n \to J_{2n} : \alpha \mapsto \exp(\alpha)$, is a well-defined

group homomorphism. More generally, each dyadic expansion map (not

morphism) $\Sigma_n \to \Sigma_{n+*}$ has a canonical lift $J_n \to J_{n+*}$.

2. Its restriction to $PJ_n$ is the morphism $(\exp_n)_* : PJ_n \to PJ_{2n}$ induced at

the fundamental group level by the embedding $\exp_n : \tilde{M}_{0,n+1} \hookrightarrow \tilde{M}_{0,2n+1}$.

3. The morphisms $(\exp_n)_*$ are injective for all $n \geq 2$. In particular, the

group $PJ_{2n}^\infty = \operatorname{lim}_{\longleftarrow} PJ_n$ is the limit of an inductive system of embed-
dings. The same injectivity property holds for the homology maps $(\exp_n)_* : H_*(PJ_n, \mathbb{Z}) \to H_*(PJ_{2n}, \mathbb{Z})$.

4. There are commutative diagrams

$$
\begin{array}{cccc}
1 & \to & PJ_n & \to & J_n & \to & \Sigma_n & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & PJ_{2n} & \to & J_{2n} & \to & \Sigma_{2n} & \to & 1
\end{array}
$$

where all the vertical maps are injective morphisms, producing the short

exact sequence

$$1 \to PJ_{2n} \to J_{2n} \to \Sigma_{2n} \to 1,$$

where $J_{2n}^\infty = \operatorname{lim}_{\longleftarrow} J_{2n}$ and $\Sigma_{2n}^\infty = \operatorname{lim}_{\longleftarrow} \Sigma_{2n}$. It may be obtained by restricting

the extension $1 \to PJ_{2n} \to \tilde{A}_V \to V \to 1$ to the subgroup $\Sigma_{2n}^\infty \subset V$.

Proof: 1) We must check that $\exp$ preserves the relations of the group $J_n$: For simplicity, suppose $G_T = (1, \ldots, j)$ and compute $\exp(\alpha_T)^2$: $\exp(\alpha_T)^2 = 

\alpha_{\exp(T)} \alpha_{(1, 2)} \cdots \alpha_{(2j - 1, 2j)} \alpha_{\exp(T)} \alpha_{(1, 2)} \cdots \alpha_{(2j - 1, 2j)}$. Now observe that in $J_{2n}$,

for all $i \leq j$, $\alpha_{\exp(T)} \alpha_{(2i-1,2i)} = \alpha_{2(j-i+1)-1,2(j-i+1)} \alpha_{\exp(T)}$. This fact joint to the commutation property of $\alpha_{(1, 2)}, \ldots, \alpha_{(2j-1, 2j)}$ among themselves allows to write $\exp(\alpha_T) = \alpha_{(1, 2)} \cdots \alpha_{(2j-1, 2j)} \alpha_{\exp(T)}$, and it comes easily $\exp(\alpha_T)^2 = 1.$

Then let $T' \subset T$ be such that $G_{T'} = (1, \ldots, i)$, $i \leq j$, and check the relation $\alpha_T \alpha_{T'} \alpha_T = \alpha_{j'T'}$ is preserved by $\exp$. Compute

$$\exp(\alpha_T) \exp(\alpha_{T'}) \exp(\alpha_T)$$

$$= [\alpha_{(1, 2)} \cdots \alpha_{(2j-1, 2j)} \alpha_{\exp(T)}] [\alpha_{\exp(T')} \alpha_{(1, 2)} \cdots \alpha_{(2i-1, 2i)}].$$

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which is the composition of the forgetting maps . . .

It ends at exp(1,...,j−i)+1,2j−i)−1,2j−i), so that

\[ \alpha_{j_{\exp(T), \exp(T')}}[\alpha(1,2) \ldots \alpha(2(j−i)−1,2(j−i))] = \]

\[ \alpha_{j_{\exp(T), \exp(T')}}[\alpha(1,2) \ldots \alpha(2(j−i)−1,2(j−i))], \]

and finally,

\[ (\ast) = \alpha(2(j−i+1)−1,2(j−i+1)) \ldots \alpha(2(j−2)−1,2(j−2)) \]

\[ \exp(\alpha_{j_{\exp(T), \exp(T')}}) = \exp(\alpha_{j_{\exp(T), \exp(T')}}), \]

which ends the proof of the first assertion of 1).

If now one performs, say, one simple expansion from the \(i^{th}\) label, corresponding to the expansion map \(\Sigma_n \to \Sigma_{n+1}\), there exists a lift \(J_n \to J_{n+1}\): let \(\alpha = \alpha_{T_1} \ldots \alpha_{T_n} \in J_n\), then (supposing \(\alpha_{T_1} = \alpha_{(1,...,j)}\) to simplify the notations), define first \(\exp(\alpha_{T_1}) = \alpha_{(1,...,j},i+1)\alpha(i,i+1)\) if \(i\) belongs to the support of \(T_1\) (if not, don’t modify \(\alpha_{T_1}\)), then define similarly \(\exp(\alpha_{T_2})\) by expanding the \(\omega_{T_1}(i)^{th}\) label, and so on. Finally you get \(\exp(\alpha) := \exp(\alpha_{T_1}) \ldots \exp(\alpha_{T_n}) \in J_{n+1}\), which projects onto \(\Sigma_{n+1}\) on the expansion (from the \(i^{th}\) label) of the permutation \(\omega = \omega_{T_1} \ldots \omega_{T_n} \omega_{T_1} \in \Sigma_n\). Again, it can be checked that the relations in the groups \(J_n\) and \(J_{n+1}\) are preserved by this expansion map, which proves it is well-defined.

2) Let \(\alpha = \alpha_{T_1} \alpha_{T_2} \ldots \alpha_{T_n} \in PJ_n = Ker \phi\), \(\gamma\) the combinatorial loop attached to \(\alpha\), based at \(id_n\). We claim that loop \(\exp(\gamma)\) lifts to the path \((1 \to \exp(\alpha_{T_1}) \to \ldots \to \exp(\alpha_{T_n}) \ldots \exp(\alpha_{T_n}))\), where \(1 \to \exp(\alpha_{T})\) is defined to be

\[ 1 \to \alpha_{\exp(T)} \to \alpha_{\exp(T)} \alpha(1,2) \to \ldots \to \alpha_{\exp(T)} \alpha(1,2) \ldots \alpha(2k−1,2k). \]

Indeed, applying \(\phi\) to this path gives precisely the loop \(\exp_n(\gamma)\), as described in the preliminary of Theorem 4, for the embedding \(\exp_n : \widetilde{M}_{0,n+1} \to \widetilde{M}_{0,2n+1}\).

It ends at \(\exp(\alpha_{T_1}) \ldots \exp(\alpha_{T_n}) = \exp(\alpha)\), so \((\exp_n)_*(\alpha) = \exp(\alpha)\).

3) We use the fact that the embedding \(\exp_n\) has a retraction

\[ r_n : \widetilde{M}_{0,2n+1}(\mathbb{R}) \to \widetilde{M}_{0,n+1}(\mathbb{R}), \]

which is the composition of the forgetting maps \(\widetilde{M}_{0,2n+1}(\mathbb{R}) \to \widetilde{M}_{0,2n}(\mathbb{R}) \to \ldots \to \widetilde{M}_{0,n+1}(\mathbb{R})\) (cf. [23]: the maps \(\overline{M}_{0,n+1}(\mathbb{C}) \to \overline{M}_{0,n}(\mathbb{C})\) is a universal family of \(n\)-pointed stable curves).

Recall also that \(\widetilde{M}_{0,n+1}(\mathbb{R})\) are aspherical, so their singular homology coincides
On the other hand, if $\alpha$ is a well-defined group homomorphism. Notice however that the retraction $(r_n)_n : PJ_{2n} \rightarrow PJ_n$ can not be extended to a retraction of $J_n \rightarrow J_{2n}$ (it would induce a retraction of $\Sigma_n \rightarrow \Sigma_{2n}$, which does not exist). □

3.2 Description of the extended groups $\tilde{A}_V$ and $\tilde{A}_N$

The group $\tilde{A}_V$ has a description very similar to the group $V$, by replacing $\Sigma_n$ by the quasi-braid group $J_n$ in the definition of $V$ given at the beginning of §4.1. Thus, an element of $\tilde{A}_V$ may be represented by a symbol $(\alpha_1, \alpha_0, \sigma)$, where $\alpha_0$, $\alpha_1$ are binary planar trees with $n$ leaves (for some $n$), and $\sigma$ belongs to $J_n$. In the process of dyadic expansion for a symbol, which may be used when composing them, the expansion maps $\exp : J_n \rightarrow J_{n+*}$ must be used (cf. Theorem 4.1).

As for the group $\tilde{A}_N$, its elements are represented by symbols $(\alpha_0, \alpha_1, q_\sigma)$ (cf. the definition of $N$) where $\sigma$ belongs to some $J_n$, and $q_\sigma$ is a collection of tree “quasi-braid” isomorphisms $q_i : T_i^{\alpha_0} \rightarrow T_i^{\alpha_1}$ (where $\bar{\alpha} \in \Sigma_n$ is the projection of $\sigma$): together with $\sigma \in J_n$ there is a family $(\sigma_k)_{k \in \mathbb{N}}$ of the product $\prod_{k \in \mathbb{N}} J_{2k}$, such that $\sigma_0 = \sigma$ and $\exp(\sigma_k)$ may differ from $\sigma_{k+1}$ by a product of some quasi-braid transpositions $\alpha_{(2i, 1, 2)}$.

3.3 Stable length, and a central extension for $N$

Let $\alpha = \alpha_{T_1} \ldots \alpha_{T_r}$ be in the free monoid freely generated by the generators of $J_n$. Define its length to be $\ell_n(\alpha) = r + |T_1| + \ldots + |T_r|$, where $T_i$ is the length of the graph $G_{T_i}$.

**Proposition 5 (stable length)** The length $\ell_n$ induces a well-defined group homomorphism $\ell_n : J_n \rightarrow \mathbb{Z}/2\mathbb{Z}$, $\ell_n(\alpha) = r + |T_1| + \ldots + |T_r| \mod 2$. Moreover, the collection $\{\ell_n, n \geq 1\}$ is compatible with the direct system $\{J_n, \exp_n\}$, and induces a stable length $\ell_\infty : J_{2\infty} \rightarrow \mathbb{Z}/2\mathbb{Z}$. More generally, the length is compatible with the dyadic expansion maps $J_n \rightarrow J_{n+*}$ (cf. Theorem 4.1). The restriction of $\ell_\infty$ to the infinite pure braid group $PJ_{2\infty}$ is still non-trivial. Finally, the stable length $\ell_\infty$ can be extended to $\tilde{A}_V$, but not to to the whole group $\tilde{A}_N$.

Proof: The last two relations in the presentation of $J_n$ preserve the length $\ell_n$. The first one ($\alpha_2^2 = 1$) preserves the length mod 2 only. So $\ell_n : J_n \rightarrow \mathbb{Z}/2\mathbb{Z}$ is a well-defined group homomorphism. On the other hand, if $\alpha_T = \alpha_{(1, \ldots, k)} \in J_n$ and one performs a simple expansion from the first leaf (to simplify the notations), then $\exp(\alpha_T) = \alpha_{(1, \ldots, k+1)} \alpha_{(1, 2)} \in J_{n+1}$, and $\ell_n(\alpha_T) = 1 + k \mod 2$, $\ell_{n+1}(\exp(\alpha_T)) = k + 1 + 1 + 2 + 1 = \ell_n(\alpha_T) \mod 2$. 

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Finally observe that the pure braid \( p = \alpha_{(1,2)}\alpha_{(2,3)}\alpha_{(1,2)}\alpha_{(1,2,3)} \) in \( PJ_2 \) has a stable length equal to \( 1 \ mod \ 2 \).

Let now \( g = [(\alpha_1, \alpha_0 = T_2, \sigma)] \) be in \( \tilde{A}_N \), with \( \sigma \in J_2 \). Then \( \ell_\infty(g) := \ell_\infty(\bar{\sigma}) \) is well-defined, since changing of symbols for \( g \) would replace \( \sigma \) by some expansion of it. On the contrary, if \( g = [(\alpha_1, \alpha_0 = T_2, \sigma)] \) is in \( \tilde{A}_N \), an expansion of the symbol replaces \( \sigma \) by some expansion of it, possibly twisted by some quasi-braid transpositions \( \alpha_{(2i-1,2i)} \): but the stable length of such a quasi-braid transposition is \( 1 \ mod \ 2 \), and there is no possible definition for \( \ell_\infty(g) \) (we will confirm in the next theorem this heuristic assertion) . \( \square \)

Let now \( \text{Ker} \ell_\infty \) be the kernel of the restriction of \( \ell_\infty \) to \( PJ_2 \).

**Theorem 3 (Analogue of the Euler class for \( N \))** The quasi-braid extension
\[
1 \to PJ_2 \to \tilde{A}_N \to N \to 1
\]
induces a non-trivial central extension
\[
1 \to \mathbb{Z}/2\mathbb{Z} \cong PJ_2/\text{Ker} \ell_\infty \to \tilde{A}_N/\text{Ker} \ell_\infty \to N \to 1,
\]
which defines a non-trivial cohomology class \( Eu \in H^2(N, \mathbb{Z}/2\mathbb{Z}) \).

Proof: Let \( g = [(\alpha_1, \alpha_0 = T_2, q)] \) be in \( \tilde{A}_N \) (\( \sigma \in J_2 \)), and \( p \in PJ_2 \), represented by \( [(\alpha_1, \alpha_1, p)] \), with \( p_1 \in PJ_2 \). It follows that \( g^{-1}pg \) is represented in \( PJ_2 \) by \( \sigma^{-1}p_1\sigma \), and \( \ell_\infty(g^{-1}pg) = \ell_\infty(\sigma^{-1}p_1\sigma) = -\ell_\infty(\sigma) + \ell_\infty(p_1) \). This proves in particular that \( \tilde{A}_N \) is normal in \( \tilde{A}_N \), and the extension is central.

Suppose the extension is trivial: then the embedding \( i : \mathbb{Z}/2\mathbb{Z} \to \tilde{A}_N \) would admit a retraction \( r \). But then, the composition \( PJ_2 \to \tilde{A}_N \to \tilde{A}_N/\text{Ker} \ell_\infty \to PJ_2/\text{Ker} \ell_\infty \cong \mathbb{Z}/2\mathbb{Z} \) would be \( \ell_\infty \), proving that the composition of the last two morphisms is a prolongation to \( \tilde{A}_N \) of the stable length morphism, which is impossible, admitting the assertion of the preceding proposition.

However, we now give another proof of the non-triviality independent of the assertion on the stable length: the method consists in writing the generator of the kernel \( \mathbb{Z}/2\mathbb{Z} \cong PJ_2/\text{Ker} \ell_\infty \) as a product of commutators in \( \tilde{A}_N/\text{Ker} \ell_\infty \), i. e. finding a pure quasi-braid with length \( 1 \ mod \ 2 \) as a product of commutators in \( \tilde{A}_N \) (which definitely proves that \( \ell_\infty \) can not be extended to \( \tilde{A}_N \)).

Let \( \tau \in V \subset N \) be the transposition defined by the symbol \( a \ p \ b \), the leaves \( a \) and \( b \) being permuted.

Let \( \alpha \in Aut(T_2) \subset N \) be defined by the symbol \( 1 \ p \ 2 \) (the permutations of the leaves indicated by the arrows must be read from bottom to top). Set
\( \gamma := \tau \alpha \). Then \( \gamma = \ldots \), and it appears that \( \gamma \) and \( \alpha \) are conjugated by the “translation” \( \delta = \ldots \). Precisely, we have \( \gamma = \delta \alpha \delta^{-1} \), or equivalently, \( \tau = [\delta, \alpha] \).

We now lift \( \tau, \delta \) and \( \alpha \) in \( \tilde{A}_N \) in an obvious way: \( \tau \) is lifted in \( \tilde{\tau} \) (same symbol coupled with \( \alpha_0 = 1, \alpha_1 = 1, \alpha_2 = \alpha_{(12)} \),
\[
\alpha_{k+1} = \exp(\alpha_k)\alpha_{(12)}, \quad k \in \mathbb{N}:
\]
Clearly, the same relation as in \( N \) holds: \( \tilde{\tau} = [\delta, \alpha] \in \tilde{A}_N \).

On the other hand, \( \tau = \ldots \) may also be written as the product \( \tau = \tau_1 \tau_2 \), where \( \tau_1 \) exchanges the leaves 1 and 3 (keeping 2 and 4 fixed) and \( \tau_2 \) exchanges the leaves 2 and 4 (keeping 1 and 3 fixed). We note abusively \( \tau_1 = (13), \tau_2 = (24) \). Introducing \( \sigma \in V \) defined by \( \sigma = (12)(34) \), we have \( \tau_2 = \sigma \tau_1 \sigma \), and \( \tau = [\tau_1, \sigma] \).

We then lift \( \tau_1 \) and \( \sigma \) in \( \tilde{A}_N \) by \( \tilde{\tau}_1 = \alpha_{(123)}, \tilde{\sigma} = \alpha_{(12)}\alpha_{(34)} \). Now \( [\tilde{\tau}_1, \tilde{\sigma}] \) differs from \( \tilde{\tau} = [\delta, \alpha] \) by a pure quasi-braid
\[
p = [\tilde{\tau}_1, \tilde{\sigma}][\delta, \alpha] = [\alpha_{(123)}, \alpha_{(12)}\alpha_{(34)}] \exp(\alpha_{(12)})
\]
\[
= [\alpha_{(123)}, \alpha_{(12)}\alpha_{(34)}] \alpha_{(12)}\alpha_{(34)}\alpha_{(123)} = \alpha_{(123)}\alpha_{(12)}\alpha_{(34)}\alpha_{(123)}\alpha_{(1234)}.
\]
The miracle is that \( \ell_\infty(p) = 1 mod 2 \) as desired. \( \square \)

**Corollary 1** The 2-cycle \( \omega \) defined by the relation \([\tau_1, \sigma][\alpha, \delta] = 1 \in N \) is non-trivial and verifies \( (Eu, [\omega]) = 1 \), where \( Eu \in H^2(N, \mathbb{F}_2) \) is the cohomology class of the extension of \( N \).

This is an immediate application of the following
Lemma 1 Let $G$ be a perfect group, $A \rightarrow \hat{G} \rightarrow G$ a central extension of $G$ with kernel an abelian group $A$, $c \in H^2(G, A)$ the associated cohomology class. If $\omega$ is a 2-cycle of $G$ associated to a relation $1 = \prod_1[g_i, h_i]$ in $G$, then $(c, [\omega]) = a \in A$, where $a$ is computed as $a = \prod_1[\hat{g}_i, \hat{h}_i]$, for any choices of lifts $\hat{g}_i, \hat{h}_i$ of $g_i, h_i$.

The proof is easy by describing the extension $A \rightarrow \hat{G} \rightarrow G$ as a push-out of the universal central extension $H_2(G) \rightarrow G^{univ} \rightarrow G$ as defined by Hopf Theorem. Note that the map $H_2(G) \rightarrow A$ defining the push-out corresponds to the cohomology class $c$ under the isomorphism $H^2(G, A) \cong Hom(H_2(G), A)$.

So, in our case, $(Eu, [\omega]) = \ell_\infty(p) = 1 \bmod 2$. □

3.4 Euler extension for the model $\mathcal{N}$ of Neretin group

We may now prefer considering the analogous construction for Neretin group $\mathcal{N}$ related to the unrooted regular dyadic tree $T_2$, since it naturally contains the tree automorphism group $\text{Aut}(T_2)$ and the group $\text{PSL}(2, \mathbb{Q})$.

Lemma 2 1. The subgroup $\text{Ker } \ell_n$ of $PJ_n$ is normal in $Q_{n+1}$, and the central extension $0 \rightarrow \mathbb{Z}/2\mathbb{Z} = PJ_n/\text{Ker } \ell_n \rightarrow Q_{n+1}/\text{Ker } \ell_n \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ is trivial. Consequently, the length $\ell_n$ may be extended to a length $\hat{\ell}_{n+1}$ on $Q_{n+1}$.

2. Let $\exp_n : Q_n \rightarrow Q_{2n}$ be the morphism induced by the embedding $\overline{\mathcal{M}}_{0,n}(\mathbb{R}) \hookrightarrow \overline{\mathcal{M}}_{0,2n}(\mathbb{R})$. Then $\exp_n$ maps $\hat{a} \in Q_n$ to a pure quasi-braid in $PJ_{2n-1}$. Consequently, the group $Q_{3,2\infty}$ is isomorphic to $PJ_{2\infty}$, and there is a well-defined stable length $\ell_\infty$ on $Q_{3,2\infty}$.

Proof: 1. As we have observed, there is a pure quasi-braid $p = \hat{a}j_S(\hat{a})$ such that $\hat{a}^{-1}p^{-1}\hat{a}$ is the pure quasi-braid equal to $\hat{a}j_S(p^{-1})j_S(q)j_S(\hat{a})$. Since $\ell_n(p) = 0$, we get $\ell_n(\hat{a}^{-1}q\hat{a}) = \ell_n(q)$, and the central extension is well-defined. It is trivial, since $\hat{a}^2 = p$ belongs to $\text{Ker } \ell_n$. Consequently, we may set $\hat{\ell}_{n+1}(\hat{a}) = 0$, this extends $\ell_n$, as a morphism, from $PJ_n$ to $Q_{n+1}$.

2. Let us choose $\hat{a} = (\alpha_{1,2},(\alpha_{1,3}\alpha_{1,2}),(\alpha_{3,4}\alpha_{2,3}\alpha_{1,2}),\ldots,(\alpha_{n-2,n-1}\ldots\alpha_{1,2})) \in J_n$, to define $\hat{a} \in Q_{n+1}$. If we proceed to a simple expansion from the last puncture, we get an embedding $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \hookrightarrow \overline{\mathcal{M}}_{0,n+2}(\mathbb{R})$ and an induced morphism $Q_{n+1} \rightarrow Q_{n+2}$. It is easy to see that this morphism maps $\hat{a}$ on the pure quasi-braid $\hat{a}\alpha_{1,\ldots,n} \in PJ_{n+1}$. This of course generalizes to an arbitrary number of expansions. Consequently, the stable length $\hat{\ell}_\infty$ is well-defined on $Q_{3,2\infty}$, since each element of $Q_{3,2\infty}$ may be represented by a pure quasi-braid (we just need the definition of the $\ell_n$’s on the pure-braid groups, not the $\hat{\ell}_n$’s). □
**Definition 6 (Euler class of \( N \))** The extension \( Q_{3,2^*} \to \tilde{A}_N \to N \) induces the central extension

\[
1 \to \mathbb{Z}/2\mathbb{Z} \cong Q_{3,2^*}/\text{Ker} \ell_\infty \to \tilde{A}_N/\text{Ker} \ell_\infty \to N \to 1
\]

which we call the Euler class of \( N \).

### 3.5 Euler cocycle

Let \( \mathcal{R} \) be the ring of \( \mathbb{Z}/2\mathbb{Z} \)-valued sequences, divided by the ideal of almost zero sequences: \( \mathcal{R} = (\mathbb{Z}/2\mathbb{Z})^{[\mathbb{N}]}/(\mathbb{Z}/2\mathbb{Z})^{(0)} \). Denote by \( 1_\mathcal{R} \) its unit.

For each \( \tilde{f} \in \tilde{A}_N \) defined by a symbol of the form \( \left( \sigma_0, \sigma_1 = T_{2^*}, q_{\sigma} \right) \) (cf. §3.2), there is a family \( (\sigma_k)_{k \geq n}, \sigma_k \in J_{2^k} \), with \( \sigma_n = \sigma \) and \( \sigma_{k+1} \) differing from \( \sigma_k \) by a product of quasi-braid transpositions. So there is a well-defined function

\[
\tilde{\ell} : \tilde{A}_N \to \mathcal{R}, \quad \tilde{f} \mapsto \tilde{\ell}(\tilde{f}),
\]

where \( \tilde{\ell}(\tilde{f}) \) is represented by the sequence \( \tilde{\ell}(\tilde{f})_k = \ell_\infty(\sigma_k) \in \mathbb{Z}/2\mathbb{Z} \) for \( k \geq n \) and, say, \( \tilde{\ell}(\tilde{f})_k = 0 \) for \( k = 0, \ldots, n - 1 \).

Denote by \( j : \mathbb{Z}/2\mathbb{Z} = PJ_{2^*}/\text{Ker} \ell_\infty \to \mathcal{R} \) the natural embedding.

**Lemma 3** For all \( \tilde{f} \) in \( \tilde{A}_N \) and \( p \in PJ_{2^*}, \tilde{\ell}(p\tilde{f}) = \tilde{\ell}(p) + \tilde{\ell}(\tilde{f}) = \tilde{\ell}(\tilde{fp}) \), and \( \tilde{\ell} \) induces a function \( \ell : A_N = \tilde{A}_N/\text{Ker} \ell_\infty \to \mathcal{R} \) such that for all \( f \in A_N, \)

\[
\ell(T.f) = 1_\mathcal{R} + \ell(f),
\]

where \( T = j(1) \).

Proof: Choose \( n \) such that \( p \) may be represented in \( PJ_{2^m} \), and \( \tilde{f} \) represented by a symbol \( \left( T_1, T_2, q_{\sigma} \right) \). Write \( p = [(T_2, T_2, \alpha)] = [(T_1, T_1, \beta)] \), where \( \alpha \) and \( \beta \) possess a common expansion in some \( PJ_{2^m}, m \geq n \). Then \( \tilde{fp} = [(T_1, T_2, q_{\sigma})] \) with \( \tau = \tau_n = \sigma \alpha, \tau_k = \sigma_k \exp_2(\alpha) \) for \( k \geq n, \tilde{fp} = [(T_1, T_2, q_{\sigma})] \) with \( v = v_n = \beta \sigma, v_k = \exp_2(\beta) \sigma_k \). Since \( \ell_\infty(\exp_2(\alpha)) = \ell_\infty(\exp_2(\beta)) = \ell_\infty(p) \forall k \geq n \), the proof is done. Note that \( \tilde{\ell}(p) = j(\ell_\infty(p)) \). \( \square \)

Since \( N \) is perfect, the injection \( j : \mathbb{Z}/2\mathbb{Z} \to \mathcal{R} \) induces an injective morphism \( j_* : H^2(N, \mathbb{Z}/2\mathbb{Z}) \to H^2(N, \mathcal{R}) \).

**Theorem 4** The image by \( j_* \) of the Euler class \( Eu \in H^2(N, \mathbb{Z}/2\mathbb{Z}) \) is the cohomology class of the well-defined cocycle \( c : N \times N \to \mathcal{R} \) defined by

\[
c(f, g) = \ell(\tilde{fg}) - \ell(\tilde{f}) - \ell(\tilde{g})
\]

where \( \tilde{f}, \tilde{g} \) are any lifts to \( A_N \) of \( f \) and \( g \) respectively.

Proof: First the fact that the cocycle \( c \) is well-defined follows from the equivariant relation of Lemma 3.
Let $\omega$ be a 2-cycle of $N$. It is associated to a relation $\prod_{i=1}^{p} [f_i, g_i] = 1 \in N$, and may be written

$$\omega = \sum_{i=1}^{p} (f_i, g_i) - (g_i, f_i) - (g_i f_i, (g_i f_i)^{-1}) + (f_i g_i, (g_i f_i)^{-1})$$

$$+ \sum_{i=1}^{p} ([f_1, g_1] \cdots [f_i, g_i], [f_{i+1}, g_{i+1}]),$$

and it follows that $([c], [\omega]) = \ell(\prod_{i=1}^{p} [\tilde{f}_i, \tilde{g}_i])$. Now $\prod_{i=1}^{p} [\tilde{f}_i, \tilde{g}_i] = \alpha \mod \text{Ker} \ell_\infty$, for some $\alpha \in PJZ_\infty$, and $\ell(\prod_{i=1}^{p} [\tilde{f}_i, \tilde{g}_i]) = \ell(\alpha) = j(\ell_\infty(\alpha))$. But as mentioned in §3.3, Lemma 1, $\ell_\infty(\alpha) = (Eu, [\omega])$, so that $([c], [\omega]) = j((Eu, [\omega])) = (j, Eu, [\omega])$, which proves $[c] = j, Eu$, since $H^2(N,R) = Hom(H_2(N), R)$. □

### 3.6 The analogy with the Euler class of homeomorphism groups of the circle

1. First (but a little naive) evidence to think of this new class on $N$ as the analogue of the Euler class of Thompson group $T$ (cf. [12]):

   The latter is obtained by lifting to $\mathbb{R}$ the action of $T$ on the circle. It is simply the restriction to $T$ of the Euler class of the group $\widehat{\text{Homeo}}^+(S^1)$ of orientation-preserving homeomorphisms of the circle, namely the class of the extension $0 \to \mathbb{Z} \to \widehat{\text{Homeo}}^+(S^1) \to \text{Homeo}^+(S^1) \to 1$, where $\widehat{\text{Homeo}}^+(S^1)$, the universal cover of the group $\text{Homeo}^+(S^1)$, is the group of $\mathbb{Z}$-equivariant homeomorphisms of $\mathbb{R}$ (viewing $S^1$ as $\mathbb{R}/\mathbb{Z}$). At first sight, one should be tempted to accept the boundary $\partial T_2$ of the dyadic infinite tree, on which $\mathcal{N}$ continuously acts, as the $p$-adic analogue of the circle (since $\partial T_2 \cong \mathbb{Q}_2 P^1$ and $S^1 \cong \mathbb{R} P^1$).

   But $\mathbb{Q}_2 P^1$ being totally disconnected, this kills any hope to do some topology. At this point, the modular tower reveals to us as the appropriate space related to $\mathcal{N}$, and the non-triviality of its homotopy type generates a non-trivial cohomology class for $\mathcal{N}$—just as the homotopy type of the circle generates Thompson group $T$’s Euler class.

   However it could be objected (and it was objected by V. Sergiescu to the author) that the discrete Godbillon-Vey type class $gv$ of Thompson group $T$ (cf. [12]) may also be derived from a topological extension of $T$, namely the Greenberg-Sergiescu braid extension $1 \to B_\infty \to A \to T \to 1$ (cf. [13]), by abelianization of the kernel. So we need a more convincing argument.

2. The more serious evidence relies on the relation between the Euler class of $\text{Homeo}^+(S^1)$ and bounded cohomology.

   Indeed, recall from [2] that there is a cocycle $\text{ord}$ (“cyclic order”) on $\text{Homeo}^+(S^1)$ whose class is twice the class of $0 \to \mathbb{Z} \to \widehat{\text{Homeo}}^+(S^1) \to \text{Homeo}^+(S^1) \to 1$, such that $\text{ord}(f,g) = \Phi_{\text{ord}}(\tilde{f} \circ \tilde{g}) - \Phi_{\text{ord}}(\tilde{f}) - \Phi_{\text{ord}}(\tilde{g})$, for any lifts $\tilde{f}, \tilde{g}$ in $\widehat{\text{Homeo}}^+(S^1)$ of $f$ and $g$ respectively, and $\Phi_{\text{ord}}(\tilde{f}) = 2E(\tilde{f}(0)) \in \mathbb{Z}$, where

27
is a certain modified integer part function. And the embedding of coefficients $\mathbb{Z} \to \mathbb{R}$ maps the integral Euler class to the class of the real cocycle $eu(f, g) = \tau(f \circ g) - \tau(f) - \tau(g)$, where $\tau(f) = \lim_{n \to +\infty} \frac{f^n(0)}{n}$ is the translation number of Poincaré. The cocycles $ord$ and $eu$ are induced by the boundary of an unbounded function on $\text{Homeo}^+(S^1)$, and the class of $eu$ stands in the bounded cohomology group $H^2_b(\text{Homeo}^+(S^1), \mathbb{R})$. We think the analogy with our class $Eu$ is very suggestive, replacing $\mathbb{Z}$, $\mathbb{R}$ and $\tau$ (or $\Phi_{ord}$) by $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{R}$ and $\ell$ respectively.

3. In [19] we have introduced a so-called analogue of the Virasoro extension of $\text{Diff}^+(S^1)$, the orientation-preserving diffeomorphism group of the circle, for the discrete group $N$. It is defined as follows: Let $PAut(T_2)$ be the group of bijections on the set of vertices $T_2^0$ of the regular dyadic tree $T_2$, which induce a simplicial action outside some finite subtree of $T_2$. Each bijection in $PAut(T_2)$ induces an element of the group $N$ by forgetting the action on finite subtrees, and there is a short exact sequence

$$1 \to \Sigma_\infty \to PAut(T_2) \to N \to 1,$$

where $\Sigma_\infty$ is the group of finitely supported permutations on the set $T_2^0$. Dividing by the alternating group $\mathfrak{A}_\infty$ now provides the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \cong \Sigma_\infty / \mathfrak{A}_\infty \to PAut(T_2)/\mathfrak{A}_\infty \to N \to 1,$$

which is non-trivial (cf. [19]). Denote by $Gv$ the cohomology class of this extension. It is certainly different from the Euler class $Eu$ we have just defined (a rigorous proof would require the construction of a non-trivial 2-cycle which gives different values once evaluated on $Eu$ and $Gv$, but we are not able to find it). Though the analogy with Thompson group $T$ is striking – indeed, $H^2(T, \mathbb{Z}) = \mathbb{Z}g_v \oplus \mathbb{Z}eu$, cf. [2], there is no natural relation between the cohomology classes on Neretin group and Thompson group, since the embedding $T \hookrightarrow N$ factors through Thompson group $V$ related to the Cantor set, which has no cohomology in degree 2.

4 Further results

In this section we give further results on the groups concerned with this article, with sketch of proofs only, to avoid to lengthen the material too much.

4.1 Vanishing of the Euler class on the subgroup $PGL(2, \mathbb{Q}_2)$. We cannot avoid the natural question of understanding the Euler class on the subgroup $PGL(2, \mathbb{Q}_2)$ of the automorphism group of the tree. Though the non-central extension of $PGL(2, \mathbb{Q}_2)$ by the pure quasi-braid group does not seem to be trivial, we prove that the induced central extension is, without however exhibiting an "obvious" section or retraction. In particular, our Euler class is not related
with the Euler cocycle of Barge constructed on $PSL(2, k)$, for every field $k$, with values in the Witt group $W(k)$ (cf. [24]). This negative result means that the nature of our class is not arithmetic.

**Theorem 5** The Euler class restricted to the subgroup $PGL(2, \mathbb{Q}_2)$ vanishes.

Proof (sketch): The proof first exploits the knowledge of $H_2(GL(2, \mathbb{Q}_2), \mathbb{Z}) = K_2(\mathbb{Q}_2) \oplus H_2(\mathbb{Q}_2^\infty, \mathbb{Z})$, where $K_2(\mathbb{Q}_2)$ is Milnor’s $K_2$, and the fact that the pullback of $Eu$ on $GL(2, \mathbb{Q}_2)$ gives zero once evaluated on $H_2(GL(2, \mathbb{Q}_2), \mathbb{Z})$; second, it relies on the evaluation of the restriction of the class $Eu$ on $PGL(2, \mathbb{Q}_2)$ on a 2-cycle associated to a certain way of writing $-I_2 \in GL(2, \mathbb{Q}_2)$ as a product of two commutators. The result of the evaluation, which requires a very heavy computation, is zero again; finally we check that $Eu$ does not come from a class of $Ext(H_1(PGL(2, \mathbb{Q}_2)), \mathbb{Z}/2\mathbb{Z})$ either. □

### 4.2. Topological monoids related to the short exact sequence $1 \to PJ_1 \to J_2 \to \Sigma_2^\infty \to 1$.

The aim of this section is essentially to mention that the groups $PJ_n$ and $J_n$ provide after stabilisation new examples of groups, namely $PJ_2^\infty$ and $J_2^\infty$, which are homologically equivalent to loop spaces. Recall from [24] that the group $\Sigma_2^\infty$ of finitely supported permutation on a countable set is homologically equivalent to one connected component $Q$ of the infinite loop space $\Omega^\infty S^\infty$ (Barratt-Priddy-Quillen theorem), whereas $\Sigma_2^\infty$ is homologically equivalent to the localisation $Q[2^{-1}]$, because the expansion map $B\Sigma_2^k \to B\Sigma_2^{k+1}$ (at the classifying space level) corresponds to multiplying by 2 in the $H$-space structure of $Q$. We first recall the construction of a telescope related to the group $\Sigma_2^\infty$, inspired from [24]:

Consider the group morphisms $\Sigma_2^k \times \Sigma_2^l \to \Sigma_2^{k+l}$, $(\sigma, \tau) \mapsto \sigma(\tau, \ldots, \tau)$, where $\sigma(\tau, \ldots, \tau)$ denotes the wreath-product of $\sigma$ with $2^k$ copies of $\tau$. It may be seen as a restriction of the map (not a morphism) $\Sigma_r \times \Sigma_n_1 \times \ldots \times \Sigma_n_r \to \Sigma_{n_1+\ldots+n_r}$, $(\sigma_1, \ldots, \sigma_r) \mapsto \sigma(\tau_1, \ldots, \tau_r)$ (with $r = 2^k$, $n_1 = \ldots = n_r = 2^l$).

They provide the disjoint union $M_\Sigma := \bigsqcup_{k \in \mathbb{N}} B\Sigma_2^k$ with a structure of topological monoid. Here, $B$ denotes the classifying space functor. The unit of the monoid $\pi_0(M_\Sigma) \cong \mathbb{N}$ of connected components of $M_\Sigma$ comes from the identity $id_2$ in $\Sigma_2$. Now form the telescope

$$M_\Sigma \xrightarrow{id_2} M_\Sigma \xrightarrow{id_2} \ldots,$$

where $id_2$ is right multiplication induced by $id_2$. The crucial point is that the right multiplication by $id_2$ is equivalent to the dyadic expansion process. So, the inductive limit of the telescope, $(M_\Sigma)_\infty$, is isomorphic to $\mathbb{Z} \times B\Sigma_2^\infty$.

We now lift the preceding construction to the quasi-braid groups.

**Proposition 6** The group morphisms $\Sigma_2^k \times \Sigma_2^l \to \Sigma_2^{k+l}$ can be lifted to the quasi-braid groups as morphisms

$$J_2^k \times J_2^l \to J_2^{k+l},$$
and there is a corresponding topological monoid $M_J := \coprod_{k \in \mathbb{N}} BJ_{2^k}$, as well as a telescope $(M_J)_\infty \cong \mathbb{Z} \times BJ_{2^\infty}$.

Proof: We define the morphism $J_{2^k} \times J_{2^l} \to J_{2^{k+l}}$, by 2 commuting morphisms $J_{2^k} \to J_{2^{k+i}}$ and $J_{2^l} \to J_{2^{k+i}}$. The first one is the dyadic expansion morphism of Theorem 3 (iterated $l$ times), $\exp^l$; as for the second one, let $\alpha_T$ be a generator of $J_{2^l}$, where $T$ is thought of as a $2^l$-labeled tree with a unique internal edge. Let $T_{2^k}$ be the star with $2^k$ leaves. Graft the tree $T$ to the $i$th leaf of $T_{2^k}$, and the star $T_{2^l}$ to the other leaves. Then contract all internal edges of the resulting tree, except the internal edge of $T$. We have thus obtained $2^k$ trees $T_i, i = 1, \ldots, 2^k$, with $2^{k+l}$ leaves and one internal edge, which define generators $\alpha_{T_i}$ of $J_{2^{k+l}}$. Finally the second morphism is defined by sending $\alpha_T$ to the product $\alpha_{T_1} \cdots \alpha_{T_{2^k+l}}$ (the factors commute among themselves). We omit the proof that this induces a well-defined homomorphism. Moreover, it can be checked that $\exp^l(\alpha_S)$ (where $\alpha_S$ is a generator of $J_{2^k}$) commutes with $\alpha_{T_1} \cdots \alpha_{T_{2^{k+l}}}$, so that the two morphisms commute. Indeed, notice that $\exp^l(\alpha_S) = \alpha_{\exp^l(S)^{\alpha_{(1,2^l)}}} \alpha_{(2^{l+1},2^{l+1})} \cdots \alpha_{(2^{l+r-1}+1,2^{l+r})}$, where for simplicity we have assumed that $S$ corresponds to the labels $(1, \ldots, r)$, with $r \leq 2^k$, so that $\exp^l(S)$ corresponds to the labels $(1, \ldots, r^2)$. Then

$$
\exp^l(\alpha_S) \alpha_{T_1} = \alpha_{\exp^l(S)^{\alpha_{(1,2^l)}} \alpha_{(2^{l+1},2^{l+1})} \cdots \alpha_{(2^{l+r-1}+1,2^{l+r})}} \alpha_{T_1}
$$

$$
= \alpha_{\exp^l(S)^{\alpha_{(1,2^l)}} \alpha_{T_1} \alpha_{(2^{l+1},2^{l+1})} \cdots \alpha_{(2^{l+r-1}+1,2^{l+r})}},
$$

but

$$
\alpha_{\exp^l(S)^{\alpha_{(1,2^l)}}} \alpha_{T_1} = \alpha_{\exp^l(S)^{\alpha_{(1,2^l)}} \alpha_{T_1} \alpha_{(1,2^l)}} = \alpha_{T_1} \alpha_{\exp^l(S)^{\alpha_{(1,2^l)}}}.
$$

Since $\alpha_{T_1}$'s commute among themselves, we finally get

$$
\exp^l(\alpha_S) \alpha_{T_1} \cdots \alpha_{T_{2^{k+l}}} = \alpha_{T_1} \cdots \alpha_{T_{2^{k+l}}} \exp^l(\alpha_S). \quad \square
$$

Let now $M_J := \coprod_{k \in \mathbb{N}} BJ_{2^k}$ be the associated topological monoid, and, $(M_J)_\infty \cong \mathbb{Z} \times BJ_{2^\infty}$ be the inductive limit of the telescope induced by the right multiplication $J_{2^k} \to J_{2^{k+1}}$ by $1 \in J_2$.

As for the spaces $\tilde{M}_{0,n+1}$, they are suitable models for the classifying spaces $B(PJ_n)$, and from the operadic structure maps

$$
(*) \quad \tilde{M}_{0,r+1} \times \tilde{M}_{0,n+1} \times \cdots \times \tilde{M}_{0,n+1} \to \tilde{M}_{0,n_1+\cdots+n_r+1}
$$

we get the composition laws $\tilde{M}_{0,2^i+1} \times \tilde{M}_{0,2^i+1} \to \tilde{M}_{0,2^{i+1}+1}$ and form the topological monoid $M_{P_J} := \coprod_{k \in \mathbb{N}} \tilde{M}_{0,2^{k+1}}$. In the associated telescope, right multiplication by the point $\tilde{M}_{0,2^1}$ is the expansion map $\tilde{M}_{0,2^1+1} \to \tilde{M}_{0,2^{k+1}+1}$. Similarly to the previous cases, we have $(M_{P_J})_\infty \cong \mathbb{Z} \times B(PJ_2) = \mathbb{Z} \times \tilde{M}_{0,2^\infty}$.

**Consequences:** According to the “Group-Completion” theorem of Quillen, the canonical map $(M_{P_J})_\infty \to \Omega BM_{P_J}$ of $H$-spaces (where $\Omega X$ denotes the based loop
space of a pointed topological space $X$, and $?\rightarrow$ must be replaced by $PJ$, $J$ or $\Sigma$) is a strong homology equivalence (i.e. induces homology isomorphisms with any local system of abelian coefficients). In particular, the commutator subgroup of $\pi_1((M_j)_\infty)$ is perfect. To conclude, we have the

**Corollary 2 (Delooping $BPJ_2\sim \cong \tilde{M}_0,\infty$ and $BJ_2\sim$)** There are homological equivalences $BPJ_2\sim \to (\Omega BM_{PJ})^0$, $BJ_2\sim \to (\Omega BM_J)^0$ (where $(\cdot)^0$ denotes the functor taking the connected component of the unit of a monoid), and the commutator subgroups $[PJ_2\sim, PJ_2\sim]$ and $[J_2\sim, J_2\sim]$ are perfect.

### 5 Concluding remarks

The result of the last section may surely be improved: it must be considered as a first step in delooping the towers and the quasi-braid groups. A related question would be to detect the algebras over the graded operad $\{H_\ast(PJ_n, \mathbb{Z}), n \geq 0\}$, having in mind that when considering the classical pure braid groups $P_n$, the algebras over the operad $\{H_\ast(P_n, \mathbb{Z}), n \geq 0\}$ are the Gerstenhaber algebras, of great interest in physical mathematics.

Concerning the rest of this paper, many problems are open. The central one concerns the relative natures of the three groups concerned: the diffeomorphism group of the circle $\text{Diff}^+(S^1)$, Thompson group $T$, and Neretin Sphero-morphism group $N$, whose cohomologies resemble each other (the continuous cohomology is considered for $\text{Diff}^+(S^1)$, and the $\mathbb{Z}/2\mathbb{Z}$ coefficients are natural for the group $N$ – since anyway it is $\mathbb{Q}$-acyclic, cf. [13]). This can not be a coincidence, and we would like to find a unified way to understand this triangle of groups.

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