A concavity inequality for symmetric norms

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Abstract.

We review some recent convexity results for Hermitian matrices and we add a new one to the list: Let \( A \) be semidefinite positive, let \( Z \) be expansive, \( Z^*Z \geq I \), and let \( f : [0, \infty) \to [0, \infty) \) be a concave function. Then, for all symmetric norms

\[
\|f(Z^*AZ)\| \leq \|Z^*f(A)Z\|.
\]

This inequality complements a classical trace inequality of Brown-Kosaki.

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Introduction

A good part of Matrix Analysis consists in establishing results for Hermitian operators considered as generalized real numbers. In particular several results are matrix versions of inequalities for convex functions \( f \) on the real line, such as

\[
f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}
\]

for all reals \( a, b \) and

\[
f(za) \leq zf(a)
\]

for convex functions \( f \) with \( f(0) \leq 0 \) and scalars \( a \) and \( z \) with \( 0 < z < 1 \).

In this brief note we first review some recent matrix versions of (1), (2) and next we give the matrix version of the companion inequality of (2):

\[
f(za) \leq zf(a)
\]

for concave functions \( f \) with \( f(0) \geq 0 \) and scalars \( a \) and \( z \) with \( 1 < z \).

Capital letters \( A, B \ldots, Z \) mean \( n \times n \) complex matrices, or operators on a finite dimensional Hilbert space \( \mathcal{H} \); \( I \) stands for the identity. When \( A \) is positive semidefinite, resp. positive definite, we write \( A \geq 0 \), resp. \( A > 0 \).
1. Some known convexity results

The following are well-known trace versions of elementary inequalities (1) and (2).

1.1. von Neuman’s Trace Inequality: For convex functions $f$ and Hermitians $A$, $B$,

$$\text{Tr} \, f \left( \frac{A + B}{2} \right) \leq \text{Tr} \, \frac{f(A) + f(B)}{2}$$

(4)
equivalently $\text{Tr} \circ f$ is convex on the set of Hermitians.

1.2. Brown-Kosaki’s Trace Inequality [5]: Let $f$ be convex with $f(0) \leq 0$ and let $A$ be Hermitian. Then, for all contractions $Z$,

$$\text{Tr} \, f(Z^*AZ) \leq \text{Tr} \, Z^*f(A)Z.$$  

(5)

1.3. Hansen-Pedersen’s Trace Inequality [7]: Let $f$ be convex and let $\{A_i\}_{i=1}^n$ be Hermitians. Then, for all isometric columns $\{Z_i\}_{i=1}^n$,

$$\text{Tr} \, f(\sum_i Z_i^*A_iZ_i) \leq \text{Tr} \, \sum_i Z_i^*f(A_i)Z_i.$$  

Here isometric column means that $\sum_i Z_i^*Z_i = I$. Hansen-Pedersen’s result contains (4) and (5).

When $f$ is convex and monotone, we showed [2] that the above trace inequalities can be extended to operator inequalities up to a unitary congruence. Equivalently we have inequalities for eigenvalues. Let us give the precise statements corresponding to von Neumann and Brown-Kosaki trace inequalities.

1.4. Let $A$, $B$ be Hermitians and let $f$ be a monotone convex function. Then, there exists a unitary $U$ such that

$$f \left( \frac{A + B}{2} \right) \leq U \cdot \frac{f(A) + f(B)}{2} \cdot U^*$$

(6)

1.5. Let $A$ be a Hermitian, let $Z$ be a contraction and let $f$ be a monotone convex function. Then, there exists a unitary $U$ such that

$$f(Z^*AZ) \leq UZ^*f(A)ZU^*$$

(7)

Statements 1.4 and 1.5 can break down when the monotony assumption is dropped. But we recently obtained [4] substitutes involving the mean of two unitary congruences. Let us recall the precise result corresponding to inequalities (1) and (6).
1.6. Let $f$ be a convex function, let $A, B$ be Hermitians and set $X = f(\{A + B\}/2)$ and $Y = \{f(A) + f(B)\}/2$. Then, there exist unitaries $U, V$ such that

$$X \leq \frac{U Y U^* + V Y V^*}{2}.$$  

Another substitute of (6) for general convex functions $f$ would be a positive answer to the following still open problem [2]: Given Hermitians $A, B$, can we find unitaries $U, V$ such that

$$f \left( \frac{A + B}{2} \right) \leq \frac{U f(A) U^* + V f(B) V^*}{2}.$$  

We turn to a Brown Kosaki type inequality involving expansive operators $Z$, that is $Z^* Z \geq I$. We showed the following trace version of the elementary inequality (3):

1.7. Let $f$ be convex with $f(0) \leq 0$ and let $A \geq 0$. Then, for all expansive operators $Z$,

$$\text{Tr} \ f(Z^* AZ) \geq \text{Tr} \ Z^* f(A) Z.$$  

(8)

It is interesting to note [2] that, contrarily to the contractive case (5), the assumption $A \geq 0$ can not be dropped. Also, still contrarily to (5), this result can not be extended to eigenvalues inequalities like (7). Nevertheless, we have:

1.8. Let $f$ be nonnegative convex with $f(0) = 0$, let $A \geq 0$ and let $Z$ be expansive. Then, for all symmetric norms

$$\|f(Z^* AZ)\| \geq \|Z^* f(A) Z\|.$$  

(9)

Here, by symmetric norm we mean a unitarily invariant one, that is $\|A\| = \|UAV\|$ for all operators $A$ and all unitaries $U, V$.

2. A new concavity result

Of course if $f$ is concave with $f(0) \geq 0$ then inequality (8) is reversed and provides an extension of its scalar version (3). Assuming furthermore $f$ nonnegative we tried to extend it to all symmetric norms but, besides the trace norm, we only got the operator norm case. Here we may state:

**Theorem 2.1.** Let $f : [0, \infty) \to [0, \infty)$ be a concave function. Let $A \geq 0$ and let $Z$ be expansive. Then, for all symmetric norms

$$\|f(Z^* AZ)\| \leq \|Z^* f(A) Z\|.$$
Proof. It suffices to prove the theorem for the Ky Fan $k$-norms $\| \cdot \|_k$ (cf. [1]). This shows, since $Z$ is expansive, that we may assume that $f(0) = 0$. Note that $f$ is necessarily nondecreasing. Hence, there exists a rank $k$ spectral projection $E$ for $Z^*AZ$, corresponding to the $k$-largest eigenvalues $\lambda_1(Z^*AZ), \ldots, \lambda_k(Z^*AZ)$ of $Z^*AZ$, such that

$$
\|f(Z^*AZ)\|_k = \sum_{j=1}^k \lambda_j(Z^*AZ) = \text{Tr} Ef(Z^*AZ)E.
$$

Therefore, using a well-known property of Ky Fan norms, it suffices to show that

$$
\text{Tr} Ef(Z^*AZ)E \leq \text{Tr} EZ^*f(A)ZE.
$$

This is the same as requiring that

$$
\text{Tr} EZ^*g(A)ZE \leq \text{Tr} Eg(Z^*AZ)E \quad (10)
$$

for all convex functions $g$ on $[0, \infty)$ with $g(0) = 0$. Any such function can be approached by a combination of the type

$$
g(t) = \lambda t + \sum_{i=1}^n \alpha_i(t - \beta_i)_+ \quad (11)
$$

for a scalar $\lambda$ and some nonnegative scalars $\alpha_i$ and $\beta_i$. Here $(x)_+ = \max\{0, x\}$. By using the linearity of the trace it suffices to show that (10) holds for $g_{\beta}(t) = (t-\beta)_+$, $\beta \geq 0$. We claim that there exists a unitary $U$ such that

$$
Z^*g_{\beta}(A)Z \leq Ug_{\beta}(Z^*AZ)U^*. \quad (12)
$$

This claim and a basic property of the trace then show that (10) holds for $g_{\beta}$. Indeed, we then have

$$
\text{Tr} EZ^*g_{\beta}(A)ZE = \sum_{j=1}^k \lambda_j(EZ^*g_{\beta}(A)ZE) \\
\leq \sum_{j=1}^k \lambda_j(Z^*g_{\beta}(A)Z) \\
\leq \sum_{j=1}^k \lambda_j(g_{\beta}(Z^*AZ)) \quad \text{(by 12)} \\
= \sum_{j=1}^k \lambda_j(Eg_{\beta}(Z^*AZ)E) \\
= \text{Tr} Eg_{\beta}(Z^*AZ)E
$$

where the fourth equality follows from the fact that $g_{\beta}$ is nondecreasing and hence $E$ is also a spectral projection of $g_{\beta}(Z^*AZ)$ corresponding to the $k$ largest eigenvalues.
The inequality (12) has been established in [2] in order to prove (8). Let us recall the proof of (12): We will use the following simple fact. If \( B \) is a positive operator with \( \text{Sp}B \subset \{0\} \cup (x, \infty) \), then we also have \( \text{Sp}Z^*BZ \subset \{0\} \cup (x, \infty) \). Indeed \( Z^*BZ \) and \( B^{1/2}Z^*ZB^{1/2} \) (which is greater than \( B \)) have the same spectrum.

Let \( P \) be the spectral projection of \( A \) corresponding to the eigenvalues strictly greater than \( \beta \) and let \( A_\beta = AP \). Since \( Z^*(A - \beta I)_+ Z = Z^*(A_\beta - \beta I)_+ Z \) we may then assume that \( A = A_\beta \). Now, the above simple fact implies

\[
(Z^*A_\beta Z - \beta I)_+ = Z^*A_\beta Z - \beta Q
\]

where \( Q = \text{supp}Z^*A_\beta Z \) is the support projection of \( Z^*A_\beta Z \). Therefore, using \( (A_\beta - \beta I)_+ = A_\beta - \beta P \), it suffices to show the existence of a unitary operator \( W \) such that

\[
Z^*A_\beta Z - \beta Q \geq W Z^*(A_\beta - \beta P)ZW^* = WZ^*A_\beta ZW^* - \beta WZ^*PZW^*.
\]

But, here we can take \( W = I \). Indeed, we have

\[
\text{supp}Z^*PZ = Q \quad \text{and} \quad \text{Sp}Z^*PZ \subset \{0\} \cup [1, \infty) \quad (**)
\]

where \( (**) \) follows from the above simple fact and the identity \( (*) \) from the observation below with \( X = P \) and \( Y = A_\beta \).

**Observation.** If \( X, Y \) are two positive operators with \( \text{supp}X = \text{supp}Y \), then for every operator \( Z \) we also have \( \text{supp}Z^*XZ = \text{supp}Z^*YZ \).

To check this, we establish the corresponding equality for the kernels,

\[
\ker Z^*XZ = \{h : Z h \in \ker X^{1/2}\} = \{h : Z h \in \ker Y^{1/2}\} = \ker Z^*YZ.
\]

In the above proof, the simple idea of approaching convex functions as in (11) was fruitful. It is also useful to prove (see [2]) the Rotfel’d Trace Inequality: For concave functions \( f \) with \( f(0) \geq 0 \) and \( A, B \geq 0 \),

\[
\text{Tr} f(A + B) \leq \text{Tr} f(A) + \text{Tr} f(B).
\]

If \( f \) is convex with \( f(0) \leq 0 \) the reverse inequality holds, in particular we have McCarthy’s inequality

\[
\text{Tr} (A + B)^p \geq \text{Tr} A^p + \text{Tr} B^p
\]

for all \( p > 1 \).

**Remark 2.2.** Though scalars inequalities (2), (3) or their concave analogous hold for a more general class than convex or concave functions, the corresponding trace inequalities need the convexity or concavity assumption (cf. [2]). A fortiori, Theorem 2.1 needs the concavity assumption.
Remark 2.3. When $f$ is operator monotone, Theorem 2.1 extends to an operator inequality which can be rephrased for contractions as follows: For nonnegative operator monotone functions $f$ on $[0, \infty)$, contractions $Z$ and $A \geq 0$, 

$$Z^* f(A) Z \leq f(Z^* A Z)$$

This is the famous Hansen’s inequality [4]. Similarly when $f$ is operator convex, Hansen-Pedersen’s Trace Inequality can be extended to an operator inequality [7] (see also [3]).

Extensions of Theorem 2.1 to infinite dimensional spaces will be considered in a forthcoming work.

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