Exponential Riesz bases, multi-tiling and condition numbers in finite abelian groups

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Abstract

Motivated by the open problem of exhibiting a subset of Euclidean space which has no exponential Riesz basis, we focus on exponential Riesz bases in finite abelian groups. We show that every subset of a finite abelian group has such a basis, removing interest in the existence question in this context. We then define tightness quantities for subsets to measure the conditioning of Riesz bases; for normalized tightness quantities, a value of one corresponds to an orthogonal basis, and a value of infinity corresponds to nonexistence of a basis. As an application, we obtain the first weak evidence in favor of the open problem by giving a sequence of subsets of finite abelian groups whose tightness quantities go to infinity in the limit. We also prove that every cylinder set has the same tightness quantities as its base. Lastly, under an additional hypothesis, explicit bounds are given for tightness quantities in terms of a subset’s lowest multi-tiling level by a subgroup and its geometric configuration. This establishes a quantitative link between discrete geometry and harmonic analysis in this setting.

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1 Preliminaries and notations

For a vector $x \in \mathbb{C}^d$, we denote the $\ell_2$ norm by $\|x\|$ and define it by $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$. The operator norm or induced-$\ell_2$ norm of a matrix $A$ is defined by

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^d, \|x\| = 1\}.$$  

**Definition 1.1** (Condition number). The condition number of a matrix $A$ is given by $\text{cond}(A) = \|A\| \|A^{-1}\|$. We define this number to be $\infty$ if $A$ is not invertible.

It is clear that the condition number of a matrix is at least 1, as $1 = \|AA^{-1}\| \leq \text{cond}(A)$ for any invertible matrix $A$. Observe also that $\text{cond}(A) = \sigma_{\max}(A)/\sigma_{\min}(A)$, where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ denote the maximum and minimum of the set of singular values of $A$, respectively. This also justifies that $\text{cond}(A) \geq 1$ and $\text{cond}(A) = \infty$ if $A$ is not invertible.

1.1 Exponential Riesz bases

Let $m$ and $d$ be positive integers, and consider $\mathbb{Z}_m^d$. Define $\chi(t) = \exp(2\pi it)$. For nonempty $E, B \subseteq \mathbb{Z}_m^d$ of equal size, we call $(E, B)$ an equal-size pair. For any $b \in B$, we define exponential function $\chi_b : E \to S^1$ by $\chi_b(x) := \chi(\langle x, b \rangle / m)$, $x \in E$. Here, $\langle x, b \rangle$ is the usual inner product in $\mathbb{Z}_m^d$.

**Definition 1.2.** We call $(E, B)$ a basis pair if the exponential functions $\chi(\langle \cdot, b \rangle / m)$, $b \in B$, form a basis of $L^2(E)$.

It is known that in finite-dimensional space, every basis is a Riesz basis. Nevertheless, we are interested in the Riesz constants of subfamilies of the functions $\chi(\langle \cdot, b \rangle / m)$, $b \in B$, when they form a basis. This leads to the following definition.

**Definition 1.3.** Let $(E, B)$ be an equal-size pair with $|E| = n$. Let $B = \{b_1, \ldots, b_n\}$, and $f_i := \chi(\langle \cdot, b_i \rangle / m)$ be functions defined on $E$. We say $(E, B)$ is a Riesz basis pair if there are positive and non-zero constants $A_1$ and $A_2$ such that the following holds for all finite sequences $\{c_i\}_{i=1}^n \subseteq \mathbb{C}^n$

$$A_1 \sum_{i=1}^n |c_i|^2 \leq \left\| \sum_{i=1}^n c_i f_i \right\|_{L^2(E)}^2 \leq A_2 \sum_{i=1}^n |c_i|^2. \quad (1.1)$$

In this case, we say the exponentials $\{\chi(\langle \cdot, b \rangle / m) : b \in B\}$ is a Riesz basis for $L^2(E)$ or $E$ has a exponential Riesz basis.

We define the **optimal lower Riesz constant** $L_E(B)$ and **optimal upper Riesz constant** $U_E(B)$ to be, respectively, the supremum of all constants $A_1$ and the infimum of all constants $A_2$ such that the inequality (1.1) holds for $\{c_i\}_{i=1}^n \subseteq \mathbb{C}^n$. 


Notice that $0 \leq L_E(B) \leq U_E(B) < \infty$. If $(E, B)$ is a basis pair, then since the corresponding basis is Riesz, $L_E(B) > 0$ as well.

An interesting question to ask here is if any set $E$ has an exponential Riesz basis. Later we will prove that the answer to this question is affirmative and for every $E$, there is a $B$ such that $(E, B)$ is a basis pair. It is easy to see that the optimal upper and lower Riesz constants are identical if in addition the orthogonality holds. First we have a definition.

**Definition 1.4.** We call $(E, B)$ a **spectral pair** if the exponential functions $\chi(\langle \cdot, b \rangle / m)$, $b \in B$, are orthogonal on $E$.

If $(E, B)$ is a spectral pair with $|E| = n$, then since the functions $f_i$ are orthogonal and have norm $\sqrt{n}$, $L_E(B) = U_E(B) = n$. It is known that not every set $E \subseteq \mathbb{Z}_m^d$ is a spectral set. For example, the spectral sets in $\mathbb{Z}_2^p$ are of size $1, p, p^2$ [6]. We are interested in such a set $E$, given by the following definition.

**Definition 1.5.** A set $E$ is a **spectral set** if there is a $B$ such that $(E, B)$ is a spectral pair. In this case, $B$ is called a **spectrum set** for $E$.

### 1.2 Quantities of interest

We define quantities that will help us to understand the behaviors of the Riesz constants $L_E(B)$ and $U_E(B)$. Let $E = \{e_1, \ldots, e_n\}$ and $B = \{b_1, \ldots, b_n\}$, indexed according to the lexicographic order in $\mathbb{Z}_m^d$. That is, $i < j$ implies there exists $k$ such that $(e_i)_\ell = (e_j)_\ell$ for $\ell < k$ and $(e_i)_k < (e_j)_k$, and similarly for the elements of $B$. (We could pick any other order, as the quantities we are going to define do not depend on the orderings of $E, B$.)

**Definition 1.6** (Evaluation matrix). Let $(E, B)$ be an equal-size pair with $|E| = n$. The evaluation matrix of $(E, B)$, denoted by $T(E, B) = (t_{i,j})$, is the $n \times n$ matrix with the entries $t_{i,j} := \chi(\langle e_i, b_j \rangle / m)$, $1 \leq i, j \leq n$.

Note that the $j$th column of $T(E, B)$ is the function $f_j = \chi(\langle \cdot, b_j \rangle / m)$ applied to each element of $E$. With this, we can write

$$T(E, B) = \begin{pmatrix} \vdots & \vdots & \vdots \\ f_1 & f_2 & \cdots & f_n \\ \vdots & \vdots & \vdots \end{pmatrix}.$$  

The following result presents the exact values of optimal Riesz constants of a pair in terms of operator norm of evaluation matrix.

**Proposition 1.7.** Let $(E, B)$ be an equal-size pair. Let $T = T(E, B)$ be the associated evaluation matrix. Then

$$L_E(B) = \|T^{-1}\|^{-2}, \quad \text{and} \quad U_E(B) = \|T\|^2,$$

where $L_E(B) = 0$ if $T$ is not invertible. Moreover,

$$U_E(B) = \text{cond}(T)^2 L_E(B). \quad (1.2)$$
Proof. Notice the equation (1.1) can be written as
\[ A_1 \|c\|^2 \leq \|Tc\|^2 \leq A_2 \|c\|^2 \]
for all \( c \in \mathbb{C}^n \). Thus, the minimum possible \( A_2 \) is
\[ \sup_{c \neq 0} \frac{\|Tc\|^2}{\|c\|^2} = \|T\|^2. \]
On the other hand, if \( T \) is not invertible, then the maximum possible \( A_1 \) is zero. Otherwise, this value is

\[ \inf_{c \neq 0} \frac{\|T^{-1}c\|^2}{\|c\|^2} = \left( \sup_{c \neq 0} \frac{\|T^{-1}c\|^2}{\|c\|^2} \right)^{-1} = \|T^{-1}\|^{-2}. \]

The equation (1.2) now holds from the relation
\[ \sqrt{U_E(B)/L_E(B)} = \|T\| \|T^{-1}\| = \text{cond}(T). \]
This completes the proof.

Notice, according to (1.2) the quantity \( \text{cond}(T) \) captures how “tight” the Riesz basis is, with the tightest possible basis having \( L_E(B) = U_E(B) \). This will only hold if \( \text{cond}(T) = 1 \) or \( \|T\| = \|T^{-1}\|^{-1} \). Later, in Proposition 3.7, we will prove that this is only possible when the pair is a spectral pair, i.e., the orthogonality also holds.

We define a number measuring this tightness as follows.

**Definition 1.8.** The condition number of \( B \) with respect to \( E \) is \( \text{cond}_E(B) = \text{cond}(T(E, B)) \).

We are also interested in the absolute value of the determinant of \( T(E, B) \).

**Definition 1.9.** The absolute determinant of \( B \) with respect to \( E \) is \( D_E(B) = |\det T(E, B)| \).

As \( T(E, B) \) becomes more singular, \( \text{cond}_E(B) \) goes to infinity, while \( D_E(B) \) goes to zero. Thus, \( D_E(B) \) offers another way to understand how singular \( T(E, B) \) is, and this corresponds to how tight the Riesz basis is, with less tight bases having more singular \( T(E, B) \). Because \( D_E(B) \) is the absolute value of a linear combination of \( m \)th roots of unity, it is sometimes easier to work with than \( \text{cond}_E(B) \).

Later, we will show that the quantities \( L_E(B), U_E(B), \text{cond}_E(B), D_E(B) \) each provide a way of quantifying how “close” to being spectral a pair \( (E, B) \) is. By looking at the optimizing partner \( B \), we can define the following quantities for a set \( E \) which, as we later show, capture the notion of \( E \) being “close” to spectral.

**Definition 1.10.**
- The lower Riesz constant of \( E \) is \( L(E) = \max_B L_E(B) \).
- The upper Riesz constant of \( E \) is \( U(E) = \min_B U_E(B) \).
- The condition number of \( E \) is \( \text{cond}(E) = \min_B \text{cond}_E(B) \).
- The absolute determinant of \( E \) is \( D(E) = \max_B D_E(B) \).
1.3 Existence of a basis pair

As mentioned earlier, for every $E$, there is a $B$ such that $(E, B)$ is a basis pair. Here, we prove that fact, by first proving a more general proposition.

**Proposition 1.13.** Let $S$ be a finite set, and let $m = |S|$. If $\{f_1, \ldots, f_m\}$ is a basis for $L^2(S)$, then, given a nonempty subset $E$ of $S$, there exists a subset of $\{f_1, \ldots, f_m\}$ of size $|E|$ which is a basis for $L^2(E)$.

**Proof.** Observe that $\dim(L^2(S)) = m$, and let $n = |E|$. Write $E = \{x_1, \ldots, x_n\}$ and write $S = \{x_1, \ldots, x_m\}$, so $m \geq n$. Let $N = (n_{i,j})_{1 \leq i,j \leq m}$ be the $m \times m$ matrix given by $n_{i,j} = f_i(x_j)$. As $\{f_1, \ldots, f_m\}$ is a basis for $L^2(S)$, the rank of $N$ is $m$. Hence, the $m$ columns of $N$ are linearly independent. In particular, the first $n$ columns of $N$ are linearly independent. Thus, the rank of the $m \times n$ matrix $\tilde{N} = (n_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ consisting of the first $n$ columns of $N$ is $n$. As column rank equals row rank, there exist $n$ linearly independent rows of $\tilde{N}$; say $\{(f_i(x_1), \ldots, f_i(x_n))\}_{j=1}^n$ is such a set of $n$ linearly independent rows. Then $\{f_{i_1}, \ldots, f_{i_n}\}$, when restricted to $E$, is a basis for $L^2(E)$. \hfill $\Box$

**Corollary 1.12.** Let $E \subseteq \mathbb{Z}_m^d$. Then there is a $B$ such that $(E, B)$ is a basis pair.

**Proof.** The exponential functions $\chi((\cdot, b)/m)$, $b \in \mathbb{Z}_m^d$, form a basis of $L^2(\mathbb{Z}_m^d)$. The result now follows from Proposition 1.11. \hfill $\Box$

We now give an alternative proof that for every $E$, there is a $B$ such that $(E, B)$ is a basis pair.

**Proposition 1.13.** Every nonempty subset $E$ of $\mathbb{Z}_m^d$ has a (Riesz) exponential basis.

**Proof.** Let $n = |E|$. Then, $L^2(E) := \{f : E \to \mathbb{C}\}$ is $n$-dimensional, so we seek $n$ exponentials to serve as a basis. That is, we want to find $a_1, \ldots, a_n$ in $\mathbb{Z}_m^d$ such that, given $f \in L^2(E)$, there exist unique $c_1, \ldots, c_n \in \mathbb{C}$ such that

$$f(x) = \sum_{j=1}^n c_je_{a_j}(x)$$

for all $x \in E$, where $e_{a_j}(x) = e^{2\pi i (a_j \cdot x)/m}$. Writing $E$ as $E = \{x_1, \ldots, x_n\}$, we see that this is equivalent to seeking $a_1, \ldots, a_n$ such that the system of $n$ equations

$$f(x_r) = \sum_{j=1}^n c_je_{a_j}(x_r),$$

with $1 \leq r \leq n$, in the $n$ unknowns $c_1, \ldots, c_n$, has a unique solution for all $(f(x_1), \ldots, f(x_n)) \in \mathbb{C}^n$. This, in turn, is equivalent to the existence of $a_1, \ldots, a_n$ such that the matrix of coefficients of the $c_j$, denoted $M = (m_{j,r})_{1 \leq j,r \leq n}$ with $m_{j,r} = e_{a_j}(x_r)$, satisfies $\det(M) \neq 0$. If, however, no such $a_1, \ldots, a_n$ in $\mathbb{Z}_m^d$ exist, then for all $a = (a_1, \ldots, a_n) \in \mathbb{Z}_m^d$, we have

$$0 = \det(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)e(a_1 \cdot x_{\sigma(1)}) \cdots e(a_n \cdot x_{\sigma(n)})$$

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Equivalently, letting \( x_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \mathbb{Z}_m^{dn} \), we have
\[
0 = \sum_{\sigma \in S_n} \text{sgn}(\sigma)e(a \cdot x_\sigma)
\]
for all \( a \in \mathbb{Z}_m^{dn} \), where \( e(a \cdot x_\sigma) = e^{2\pi i (a \cdot x_\sigma) / m} \), contradicting the fact that the family of functions \( \{ a \mapsto e(a \cdot x_\sigma) \}_{\sigma \in S_n} \) is linearly independent, because it is a family of \( n! \) distinct orthogonal characters of \( \mathbb{Z}_m^{dn} \).

\[ \Box \]

## 2 General setting of a finite abelian group

The following is a natural setting for some of our results, specifically some invariance properties in Section 3.2.

Let \( G \) be a finite abelian group. We call a homomorphism from an abelian group \( H \) to an abelian group \( G \) a \textit{Z-linear transformation}. Thus, an invertible \( \mathbb{Z} \)-linear transformation is the same as a group isomorphism. Call a composition of a \( \mathbb{Z} \)-linear transformation followed by a translation an \textit{affine transformation}.

Let \( \hat{G} \) be the Pontryagin dual of \( G \), that is, the set of all homomorphisms from \( G \) to the circle group \( \mathbb{S}^1 \). We have \( G \cong \hat{\hat{G}} \), but there is no canonical isomorphism. However, if we pick generators so that \( G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \), then an element \( b \in \hat{G} \) acts on each \( (x_1, x_2, \ldots, x_k) \in G \), \( x_i \in \mathbb{Z}_{n_i} \), by
\[
b(x_1, x_2, \ldots, x_k) = \chi \left( \frac{x_1 b_1}{n_1} + \frac{x_2 b_2}{n_2} + \cdots + \frac{x_k b_k}{n_k} \right)
\]
for some \( b_i \in \mathbb{Z}_{n_i} \). So we can identify \( b \in \hat{G} \) with \( (b_1, b_2, \ldots, b_k) \in G \).

For the double dual \( \hat{\hat{G}} \), there is a canonical isomorphism \( G \cong \hat{\hat{G}} \) that identifies \( x \in G \) with the evaluation map \( b \mapsto b(x) \).

For nonempty \( E \subseteq G \) and \( B \subseteq \hat{\hat{G}} \) of equal size, we call \( (E, B) \) an \textit{equal-size pair}. If \( G = \mathbb{Z}_m^d \) and we identify \( \mathbb{Z}_m^d \) with \( \mathbb{Z}_m^{dn} \) as above, we recover our previous definition in \( \mathbb{Z}_m^d \).

Call an equal-size pair \( (E, B) \) a \textit{basis pair} if the elements of \( B \), after restriction to \( E \), form a basis of \( L^2(E) \). Call it a \textit{spectral pair} if these elements are orthogonal on \( E \). (Recall that \( f_1, f_2 \in L^2(E) \) are orthogonal if \( \sum_{x \in E} f_1(x) f_2(x) = 0 \).) In such a case, call \( E \) a \textit{spectral set} and \( B \) a \textit{spectral set for \( E \)}. These coincide with our previous definitions for \( \mathbb{Z}_m^d \). Similarly, we can define \( L_E(B) \) and \( U_E(B) \) by modifying Definition 1.3 in the same way.

Let \( T(E, B) \) be the linear operator from \( L^2(B) \) to \( L^2(E) \) defined by
\[
T(E, B)h = \sum_{b \in B} h(b)b \in L^2(E), \quad h \in L^2(B).
\]

Let \( E = \{x_1, x_2, \ldots, x_n\} \) and \( B = \{b_1, b_2, \ldots, b_n\} \). If we identify \( f \in L^2(E) \) with the vector \( (f(x_1), f(x_2), \ldots, f(x_n)) \in \mathbb{C}^n \) and \( h \in L^2(B) \) with the vector \( (h(b_1), h(b_2), \ldots, h(b_n)) \in \mathbb{C}^n \), then \( T(E, B) \) is the matrix whose \((i, j)\)-th entry is \( b_j(x_i) \). So this definition of \( T(E, B) \) coincides with our previous definition in the case \( G = \mathbb{Z}_m^d \). It is easy to check that Proposition 1.7 still holds. Moreover, \( \text{cond}_E(B) \) and \( D_E(B) \) can be defined as before.
Definition 1.10 depends on what ambient group $B$ is allowed to sit inside. For example, if $E$ is a subset of of a subgroup $H$ of $G$, and we optimize over $B \subseteq \widehat{H}$, then we write $L(E; H)$ for the optimal lower Riesz constant. In general, if the ambient group $G$ containing $E$ needs to be specified, we will write $L(E; G)$ instead of $L(E)$, and similarly for the other tightness quantities.

It is known that the characters of $G$ are linearly independent and $|G| = |\widehat{G}|$, so $\widehat{G}$ is a basis of $L^2(G)$. By the basis restriction proposition 1.11, for every $E \subseteq G$, there is $B \subseteq \widehat{G}$ such that $(E, B)$ is a basis pair. Therefore, all results from Section 1 can be generalized to this setting.

Similarly, results in Section 3.1 hold in this setting. We give some remarks on duality. If $(E, B)$ is an equal-size pair, then $(B, E)$ is also an equal-size pair, where we think of $E$ as an element of $\widehat{G}$ under the identification $\widehat{G} \cong G$. To show that $T(E, B) = T(B, E)^t$, we observe that the entries $b_j(x_i) = x_i(b_j)$. Alternatively, we can use Equation (2.2) to show that
\[
(T(E, B)h, f)_{L^2(E)} = (h, T(B, E)f)_{L^2(B)}
\]
for every $f \in L^2(E)$ and $h \in L^2(B)$, where $(\cdot, \cdot)$ is the natural pairing $(f_1, f_2)_{L^2(E)} = \sum_{x \in E} f_1(x)f_2(x)$.

## 3 Properties of tightness quantities

In what follows, we investigate properties of $L_E(B)$, $U_E(B)$, $\text{cond}_E(B)$, and $D_E(B)$ for an equal-size pair $(E, B)$, and $L(E)$, $U(E)$, $\text{cond}(E)$, and $D(E)$ for a set $E$.

### 3.1 Basic properties

We begin with immediate properties, whose proofs we omit.

**Proposition 3.1 (Duality).** $T(E, B) = T(B, E)^t$, $L_E(B) = L_B(E)$, $U_E(B) = U_B(E)$, $\text{cond}_E(B) = \text{cond}_B(E)$, and $D_E(B) = D_B(E)$.

**Proposition 3.2 (Basis pair).** Let $(E, B)$ be an equal-size pair. The following are equivalent:

- $(E, B)$ is a basis pair;
- $(B, E)$ is a basis pair;
- $T(E, B)$ is invertible;
- $L_E(B) > 0$;
- $\text{cond}_E(B) < \infty$;
- $D_E(B) > 0$.

The next result shows that all quantities in Proposition 3.2 have nice expressions in terms of the singular values of the matrix $T(E, B)$. These will prove very useful in understanding their behaviors.

**Proposition 3.3.** For an equal-size pair $(E, B)$ with $|E| = n$, let $0 \leq \sigma_1 \leq \sigma_2 \cdots \leq \sigma_n$ be the singular values of the matrix $T(E, B)$. Then $L_E(B) = \sigma_1^2$, $U_E(B) = \sigma_n^2$, $\text{cond}_E(B) = \sigma_n/\sigma_1$, and $D_E(B) = \sigma_1\sigma_2\ldots\sigma_n$. Moreover, we have
\[
\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2 = n^2.
\]
Proof. Put \( T := T(E, B) \). We apply Proposition 1.7. Then \( \|T\| = \sigma_n \). If \( \sigma_1 > 0 \), then \( T \) is invertible and the singular values of \( T^{-1} \) are given by \( 1/\sigma_1 \geq 1/\sigma_2 \geq \cdots \geq 1/\sigma_n \). Therefore \( \|T^{-1}\| = 1/\sigma_1 \). Since \( \sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2 \) are the eigenvalues of \( T^*T \),

\[
|\det T|^2 = \det T^*T = \sigma_1^2 \sigma_2^2 \cdots \sigma_n^2.
\]

Note that all the diagonal entries of \( T^*T \) are equal to \( n \), therefore

\[
n^2 = \text{tr} T^*T = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2.
\]

These give all our desired results.

The following result provides upper and lower bounds for \( L_E(B), U_E(B), \text{cond}_E(B) \), and \( D_E(B) \) in terms of the size of \( E \).

**Proposition 3.4.** Let \( (E, B) \) be an equal-size pair with \( |E| = n \). Then \( 0 \leq L_E(B) \leq n \leq U_E(B) \leq n^2 \), \( 1 \leq \text{cond}_E(B) \leq \infty \), and \( 0 \leq D_E(B) \leq n^{n/2} \). Moreover, each of these bounds is attained for \( E \) and \( B \) of arbitrarily large sizes.

**Proof.** The estimations for \( L_E(B), U_E(B) \) and \( \text{cond}_E(B) \) are straightforward by Proposition 3.3. For the inequality \( D_E(B) \leq n^{n/2} \), we use Proposition 3.3 and the quadratic mean-geometric mean inequality:

\[
D_E(B)^{1/n} = (\sigma_1 \sigma_2 \cdots \sigma_n)^{1/n} \leq \sqrt[2n]{\frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}{n}} = \sqrt{n}.
\]

Example 3.5 below shows that these bounds can be attained for \( E \) and \( B \) of arbitrarily large sizes.

**Example 3.5.** Let \( G = \mathbb{Z}_m^2 \), and take

\[
E = \{(0, 0), (1, 0), \ldots, (m - 1, 0)\} \quad \text{and} \quad B = \{(0, 0), (0, 1), \ldots, (0, m - 1)\}.
\]

It is easily verified that for this pair \( L_E(B) = 0 \), \( U_E(B) = n^2 \), \( \text{cond}_E(B) = \infty \) and \( D_E(B) = 0 \). If \( E = B = \mathbb{Z}_m^2 \), we obtain \( L_E(B) = U_E(B) = n \), \( \text{cond}_E(B) = 1 \) and \( D_E(B) = n^{n/2} \). To show this, notice \( T(E, B)/\sqrt{n} \) is a unitary matrix, so all the singular values of \( T(E, B) \) are equal to \( \sqrt{n} \).

**Remark 3.6.** The result that \( D_E(B) \leq n^{n/2} \) in Proposition 3.4 can be given an alternative proof using Hadamard’s inequality. This inequality states that for a given matrix \( M \) with \( n \) columns \( \{v_i\}_{i=1}^n \),

\[
|\det M| \leq \prod_{i=1}^n \|v_i\|.
\]

Applying Hadamard’s inequality to the \( n \times n \) matrix \( T \), and noting that \( \|v_i\|^2 = n \), we obtain \( |\det T| \leq n^{n/2} \). The traditional proof of Hadamard’s inequality is similar to Proposition 3.4’s proof.

We can now characterize a spectral pair in terms of our defined quantities.
Proposition 3.7 (Spectral pair). For a given equal-size pair \((E, B)\), the following are equivalent:

- \((E, B)\) is a spectral pair;
- \((B, E)\) is a spectral pair;
- \(T(E, B)/\sqrt{n}\) is a unitary matrix;
- \(L_E(B) = U_E(B)\);
- \(L_E(B) = n\);
- \(U_E(B) = n\);
- \(\text{cond}_E(B) = 1\);
- \(D_E(B) = n^{n/2}\).

Proof. By Proposition 3.3, the last five conditions are all equivalent to the statement that all singular values of \(T(E, B)\) are the same and are equal to \(\sqrt{n}\). This, in turn, is equivalent to \(T(E, B)/\sqrt{n}\) being a unitary matrix. By definition, this is equivalent to \((E, B)\) being a spectral pair. From this, we obtain duality between \(E\) and \(B\).

From Propositions 3.4 and 3.7, notice that a spectral pair \((E, B)\) has the largest \(L_E(B)\), smallest \(U_E(B)\), smallest \(\text{cond}_E(B)\), and largest \(D_E(B)\) among all pairs of the same sizes. Thus, we can think of a pair \((E, B)\) with large \(L_E(B)\), small \(U_E(B)\), small \(\text{cond}_E(B)\), and large \(D_E(B)\) as being “close to spectral.”

Recall that the definition of a spectral pair means that \(\chi(⟨·, b⟩/m), b ∈ B\), are orthogonal on \(E\). The following informal proposition assists with intuition regarding “being close to spectral.”

Proposition 3.8. Fix \(n\), and let \((E, B)\) be an equal-size pair with \(|E| = n\). Any of the following statements implies all others:

- The exponential functions \(\chi(⟨·, b⟩/m), b ∈ B\), are “close” to being orthogonal on \(E\);
- \(L_E(B)\) is “close” to \(n\);
- \(U_E(B)\) is “close” to \(n\);
- \(\text{cond}_E(B)\) is “close” to \(1\);
- \(D_E(B)\) is “close” to \(n^{n/2}\).

More precisely, define the angle \(θ ∈ [0, π/2]\) between two nonzero vectors \(c, d ∈ \mathbb{C}^n\) by

\[
\cos(θ) = \frac{\sum_{i=1}^n c_i d_i}{\|c\| \|d\|}.
\]

For a matrix \(A\) with nonzero columns, define \(\text{ortho}(A)\) to be the maximum of \(\pi/2 − θ\) over all angles \(θ\) between two distinct columns of \(A\). Then, if any of the following quantities is “close” to zero, then all others must also be “close” to zero: \(\text{ortho}(T(E, B)), L_E(B) − n, U_E(B) − n, \text{cond}_E(B) − 1, \) and \(D_E(B) − n^{n/2}\).

Proof. Let \(T = T(E, B)\). Proposition 3.3 implies that each of \(L_E(B) − n, U_E(B) − n, \text{cond}_E(B) − 1, \) and \(D_E(B) − n^{n/2}\) is close to zero exactly when all singular values of \(T(E, B)\) are close to \(\sqrt{n}\). This proves the equivalence between these statements.
Now if ortho($T$) is close to zero, then $T^*T$ is close to $nI$. By continuity, all singular values of $T$ are close to $\sqrt{n}$. Conversely, suppose that ortho($T$) is not close to zero. Specifically, assume that the angle $\theta$ between columns $i$ and $j$ is not close to $\pi/2$. Let $e_1,e_2,\ldots,e_n$ be the standard basis of $\mathbb{C}^n$. Pick $\omega \in \mathbb{C}$ of norm one such that
\[
\omega \sum_{k=1}^n (Te_i)_k (Te_j)_k = \cos \theta \|Te_i\|\|Te_j\|,
\]
that is, $\omega$ should “dephase” the inner product. Then
\[
\|T(\omega e_i + e_j)\|^2 = \|Te_i\|^2 + \|Te_j\|^2 + 2 \cos \theta \|Te_i\|\|Te_j\| = 2n + 2n \cos \theta,
\]
while $\|\omega e_i + e_j\| = \sqrt{2}$. So $\|T\| \geq \sqrt{n(1 + \cos \theta)}$. Because $\cos \theta$ is not close to zero, $\|T\|$ is not close to $\sqrt{n}$, so $U_E(B)$ is not close to $n$.

We can now put bounds on $L(E)$, $U(E)$, $\text{cond}(E)$, and $D(E)$ and characterize spectral sets in a similar way.

**Proposition 3.9.** Let $|E| = n$. Then $0 < L(E) \leq n$, $n \leq U(E) < n^2$ if $n > 1$, $\sqrt{U(E)/L(E)} \leq \text{cond}(E) < \infty$, and $0 < D(E) \leq n^{n/2}$.

**Proof.** Straightforward by the definitions and Proposition 3.4. The strict inequalities follow from the fact that for every $E$, there is a $B$ such that $(E,B)$ is a basis pair (Cor. 1.12).

The proposition below shows that a spectral set has the largest $L(E)$, smallest $U(E)$, smallest $\text{cond}(E)$, and largest $D(E)$ among all sets of the same sizes. We omit its proof.

**Proposition 3.10 (Spectral set).** Let $|E| = n$. The following are equivalent:

- $E$ is spectral;
- $L(E) = U(E)$;
- $L(E) = n$;
- $U(E) = n$;
- $\text{cond}(E) = 1$;
- $D(E) = n^{n/2}$.

### 3.2 Invariance properties

Our defined quantities also have invariance properties which we can exploit. We begin with translational invariance.

**Proposition 3.11 (Translational invariance).**

- $L_E(B)$, $U_E(B)$, $\text{cond}_E(B)$, and $D_E(B)$ are invariant under translations of $E$ and $B$.
- $L(E)$, $U(E)$, $\text{cond}(E)$, and $D(E)$ are invariant under translations of $E$.

**Proof.** Translation changes $T = T(E,B)$ to $D_1 T D_2$, where $D_i$ are diagonal with diagonal entries of absolute value 1. Notice that the $D_i$ are unitary, so the singular values of $D_1 T D_2$ and $T$ are equal. By Proposition 3.3, the results for pairs $(E,B)$ follow, and these imply the corresponding results for sets $E$. 


The rest of the invariance results are best stated in the setting of a finite abelian group $G$ (see Sec. 2), so we will do so. First, Proposition 3.11 holds in this setting with the same proof.

Recall that a group homomorphism in this setting is called a $\mathbb{Z}$-linear transformation. For $G = \mathbb{Z}_p^d$, $p$ a prime, this is a linear transformation in the sense of a vector space.

**Proposition 3.12.** Let $E \subseteq G$. The quantities $L(E)$, $U(E)$, $\text{cond}(E)$, and $D(E)$ are invariant under invertible $\mathbb{Z}$-linear transformations.

**Proof.** Follows because these quantities can be defined without choosing generators for $G$. \hfill $\square$

In fact, we can say more. Notice that a $\mathbb{Z}$-linear transformation $A : G \to G$ induces the transpose $A^t : \widehat{G} \to \widehat{G}$ defined by $A^t(b) = b \circ A$. For $G = \mathbb{Z}_p^d$, $p$ a prime, this is the usual transpose of a linear transformation. If $A$ is invertible, we can check that $A^t$ is invertible with $(A^t)^{-1} = (A^{-1})^t$; denote this by $A^{-t}$. We have the following.

**Proposition 3.13.** Let $(E, B)$ be an equal-size pair. For an invertible $\mathbb{Z}$-linear transformation $A : G \to G$, let $(E', B') = (AE, A^{-t}B)$. Then $L_{E'}(B') = L_E(B)$, $U_{E'}(B') = U_E(B)$, $\text{cond}_{E'}(B') = \text{cond}_E(B)$, and $D_{E'}(B') = D_E(B)$.

**Proof.** The entries of the evaluation matrices are the same:

$$A^{-t}b_j(Ax_i) = b_j(A^{-1}(Ax_i)) = b_j(x_i),$$

so $T(E, B)$ and $T(E', B')$ are essentially the same linear transformation.

Alternatively, we can use Equation (2.2) to check this fact in a coordinate-free way. Specifically, we can show that for every $h \in L^2(B')$,

$$T(E, B)(h \circ A^{-t}) \circ A^{-1} = T(E', B')h.$$

\hfill $\square$

Recall that an affine transformation is the composition of a $\mathbb{Z}$-linear transformation followed by a translation.

**Corollary 3.14.** Let $E \subseteq G$. The quantities $L(E)$, $U(E)$, $\text{cond}(E)$, and $D(E)$ are invariant under invertible affine transformations.

**Proof.** Follows from Propositions 3.11 and 3.12. \hfill $\square$

Recall also the notation $L(E; G)$, etc., which means $L(E)$, etc., when the group $G$ needs to be specified.

**Proposition 3.15** (Affine restriction). Suppose that $H$ is a direct summand of $G$, that is, there is a subgroup $K \subseteq G$ such that $G = H \oplus K$. Let $E$ be contained in the coset $H+a$, thought of as a group with the group structure inherited from $H$. Then $L(E; H+a) = L(E; G)$, $U(E; H+a) = U(E; G)$, $\text{cond}(E; H+a) = \text{cond}(E; G)$, and $D(E; H+a) = D(E; G)$. 
Proof. Let $Q$ denote any of the quantities $-L$, $U$, cond, and $-D$. By Proposition 3.11, we can assume that $a = 0$. Indeed, if $E = E' + a$ where $E' \subseteq H$, then $Q(E; H + a) = Q(E'; H)$ and $Q(E; G) = Q(E' + a; G) = Q(E'; G)$. So suppose that $a = 0$. Our proof is in the notation of dual groups.

Recall that $\hat{G} \cong \hat{H} \times \hat{K}$, canonically, with the following identification. For $\hat{h} \in \hat{H}$ and $\hat{k} \in \hat{K}$, let $(\hat{h}, \hat{k})$ act as an element of $G$ by $(\hat{h}, \hat{k})(h, k) = h(h)k(k)$ for $h \in H$ and $k \in K$. We define the projection $p(\hat{h}, \hat{k}) = (\hat{h}, 1)$, where “1” denotes the trivial homomorphism.

Let $B \subseteq \hat{G}$ be such that $(E, B)$ is a basis pair.

Claim 1. $|p(B)| = |B|$. Proof of Claim 1. Clearly $|p(B)| \leq |B|$. Hence, if $|p(B)| \neq |B|$, there exist $\hat{g}_1, \hat{g}_2 \in B$, $\hat{g}_1 \neq \hat{g}_2$, with $p(\hat{g}_1) = p(\hat{g}_2)$. Set $(\hat{h}_1, 1) = p(\hat{g}_1)$ and $(\hat{h}_2, 1) = p(\hat{g}_2)$. Given $x \in E$, as $x = x + 0$ with $x \in H$ and $0 \in K$, we have, for $(\hat{h}_1, \hat{k}_1) = \hat{g}_1$, that $\hat{g}_1(x) = (\hat{h}_1(x), \hat{k}_1(0))$. Hence, $\hat{g}_1(x) = (\hat{h}_1(x), 1) = p(\hat{g}_1)(x)$. Similarly, we have $p(\hat{g}_2)(x) = (\hat{h}_2(x), 1) = \hat{g}_2(x)$.

As $p(\hat{g}_1) = p(\hat{g}_2)$, it follows that $\hat{g}_1(x) = \hat{g}_2(x)$ for all $x \in E$, so $\{E \ni x \mapsto \hat{g}(x)\}_{\hat{g} \in B}$ is linearly dependent, a contradiction with $(E, B)$ being a basis pair. Thus, $|p(B)| = |B|$.

Claim 2. $T(E, B)$ and $T(E, p(B))$ are equal up to a permutation of the columns. Proof of Claim 2. From the proof of Claim 1, we have the equality of matrices $[\hat{g}(x)]_{x \in E, \hat{g} \in B} = [p(\hat{g})(x)]_{x \in E, p(\hat{g}) \in p(B)}$, up to a choice of ordering.

Claim 3. $(E, p(B))$ is a basis pair, and $Q_E(B) = Q_E(p(B))$. Proof of Claim 3. Follows directly from Claim 2.

Claim 4. We have $Q(E; H) = Q(E; G)$. Proof of Claim 4. First, as we can identify $\hat{H}$ with $\hat{H} \times \{1\} \subseteq \hat{G}$, we see that

$$Q(E; H) = \min_{B \subseteq \hat{H}} Q_E(B) = \min_{B \subseteq \hat{H} \times \{1\}} Q_E(B) \geq \min_{B \subseteq \hat{G}} Q_E(B) = Q(E; G).$$

Moreover, given $B \subseteq \hat{G}$ where $(E, B)$ is a basis pair, we have $p(B) \subseteq \hat{H} \times \{1\}$, so $Q_E(B) = Q_E(p(B)) \geq \min_{B \subseteq \hat{H} \times \{1\}} Q_E(B) = Q(E; H)$, whence $Q(E; G) = \min_{B \subseteq \hat{G}} Q_E(B) \geq Q(E; H)$. We conclude that $Q(E; H) = Q(E; G)$.

Proposition 3.15 has two important corollaries. The first corollary is for the case $G = \mathbb{Z}_p^d$, $p$ a prime, where $G$ has the extra structure of a vector space.

Corollary 3.16. Let $V$ be a vector space over $\mathbb{Z}_p$, $W$ an affine subspace of $V$ viewed as a vector space where we take any element to be the zero element, and $E \subseteq W$. Then $L(E; W) = L(E; V)$, $U(E; W) = U(E; V)$, cond($E; W) = \text{cond}(E; V)$, and $D(E; W) = D(E; V)$.

For $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ and $S \subseteq \{1, 2, \ldots, k\}$, define the coordinate subspace of $G$ with respect to $S$ to consist of elements $x$ of $G$ where for any $i \in S$, the $i$th coordinate of $x$ is zero. A coordinate affine subspace of $G$ is a translate of some coordinate subspace of $G$.

The second corollary of Proposition 3.15 is the following. It is actually equivalent to Proposition 3.15 because we can pick generators for $H$ and $K$ in that proposition.

Corollary 3.17. Let $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, $H$ be a coordinate affine subspace of $G$ viewed as a group where we take any element to be the zero element, and $E \subseteq H$. Then $L(E; H) = L(E; G)$, $U(E; H) = U(E; G)$, cond($E; H) = \text{cond}(E; G)$, and $D(E; H) = D(E; G)$. 

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From the proof of Proposition 3.15, we also obtain the following. If $H$ is a direct summand of $G$ and $E \subseteq H$, in order for $(E, B)$ to be a basis pair, no two elements of $B$ can act on $H$ in the same way. We can generalize this statement somewhat to the case where $E$ is “almost” contained in $H$, as shown below.

**Proposition 3.18.** Let $H$ be a direct summand of $G$. Define an equivalence relation $\sim$ on $\hat{G}$ as follows: $b_1 \sim b_2$ if the restrictions of $b_1$ and $b_2$ to $H$ are equal. Let $E \subseteq G$ be such that $\ell$ elements of $E$ are not contained in $H$, and let $(E, B)$ is a basis pair. For each equivalence class $C \in \hat{G}/\sim$, let $\text{cnt}(C)$ be the number of elements of $B$ in $C$. Then

$$
\sum_{C \in \hat{G}/\sim, \text{cnt}(C) \geq 1} (\text{cnt}(C) - 1) \leq \ell.
$$

**Proof.** Suppose that the sum in question is greater than $\ell$, and we show that $T(E, B)$ must be singular. Let $E = \{x_1, x_2, \ldots, x_p\}$, where $x_1, x_2, \ldots, x_{n-\ell} \in H$. View $T(E, B)$ as a matrix whose row $i$ corresponds to $x_i$.

Consider a $C \in \hat{G}/\sim$ with $\text{cnt}(C) \geq 1$. Let $c = \text{cnt}(C)$ and let $b_1, \ldots, b_c \in B \cap C$. Then the columns corresponding to $b_1, \ldots, b_c$ all have the same values in the first $n - \ell$ rows. If we subtract the column corresponding to $b_1$ from the columns corresponding to $b_2, \ldots, b_c$, we obtain $c - 1$ columns whose first $n - \ell$ entries are zero, and the resulting matrix is singular if and only if $T(E, B)$ is singular. Now repeat this process for all $C \in \hat{G}/\sim$ with $\text{cnt}(C) \geq 1$ to obtain a matrix whose $\sum_{C \in \hat{G}/\sim, \text{cnt}(C) \geq 1} (\text{cnt}(C) - 1) > \ell$ columns have zeros in their first $n - \ell$ entries. Since the nonzero entries of these columns can only be in the last $\ell$ entries, the columns must be linearly dependent. Therefore, $T(E, B)$ is singular. \qed

**Example 3.19.** We exhibit a family of subsets of $\mathbb{Z}_p^2$, one for each prime $p$, whose condition numbers tend to infinity as $p \to \infty$. Let $G = \mathbb{Z}_p^2$, let $H = \mathbb{Z}_p \times \{0\}$, and let $K = \{0\} \times \mathbb{Z}_p$, so $G = H \oplus K$. Let $E = H \cup \{(0, 1)\}$.

Let $B$ be such that $(E, B)$ is a basis pair. We first show that, under $\sim$ from Proposition 3.18, exactly one equivalence class contains two elements from $B$, while any other equivalence class contains one element from $B$. We have $\hat{G}/\sim = \cup_{t=0}^{p-1} \{(t, 0)\}$. Notice that $\ell = 1$ element of $E$ is not contained in $H$. By Proposition 3.18, we also have $\sum_{t=0 \mid |B \cap \{(t, 0)\}| \geq 1} (|B \cap \{(t, 0)\}| - 1) \leq 1$. If $|B \cap \{(t, 0)\}| = 0$ for some $t$, then since $|E| = p + 1$, the pigeonhole principle implies that the above inequality is violated. So, for all $0 \leq t \leq p - 1$, we have $|B \cap \{(t, 0)\}| \geq 1$. Hence, $|B \cap \{(s, 0)\}| = 2$ for a unique $s$ with $0 \leq s \leq p - 1$, and $|B \cap \{(t, 0)\}| = 1$ for any other $0 \leq t \leq p - 1$.

Write $B = \{b_1, b_2, \ldots, b_{p+1}\}$, with $b_1, b_2$ in $\{(s, 0)\}$. Then, $b_1$ and $b_2$ agree on $H$. Writing $E$ as $E = \{x_1, \ldots, x_p, x_{p+1}\}$ with $x_i = (i - 1, 0)$ for $1 \leq i \leq p$ and $x_{p+1} = (0, 1)$, we see that $(b_1(x_i))_{i=1}^{p+1}$ and $(b_2(x_i))_{i=1}^{p+1}$ differ only in their last entry. So, $\langle b_1, b_2 \rangle = p + z$, for some $z \in \mathbb{C}$ with $|z| = 1$. In particular, $\|\langle b_1, b_2 \rangle\| \geq p - 1$. We also observe that $\|\langle b_1(x_i) \rangle_{i=1}^{p+1}\| = \sqrt{p + 1} = \|\langle b_2(x_i) \rangle_{i=1}^{p+1}\|$. Recall that

$$
L_E(B) = \inf_{(c_i) \neq 0} \frac{\|\sum_{i=1}^{p+1} c_i b_i\|_{L^2(E)}}{\sum_{i=1}^{p+1} |c_i|^2}.
$$
Writing $\langle b_1, b_2 \rangle = re^{i\theta}$ with $r \geq 0$ and $\theta \in \mathbb{R}$, we take $(c_i)_{i=1}^{p+1}$ in $\mathbb{C}^{p+1}$ such that $c_1 = -e^{-i\theta}$, $c_2 = 1$, and $c_i = 0$ for $i > 2$. Then we have

$$L_E(B) \leq \frac{\|c_1b_1 + c_2b_2\|_{L^2(E)}^2}{|c_1|^2 + |c_2|^2}.$$ 

Writing

$$\frac{\|c_1b_1 + c_2b_2\|_{L^2(E)}^2}{|c_1|^2 + |c_2|^2} = \frac{\langle c_1b_1 + c_2b_2, c_1b_1 + c_2b_2 \rangle}{2},$$

we obtain

$$L_E(B) \leq \frac{\|(b_1(x_i))_{i=1}^{p+1}\|_2^2 + \|(b_2(x_i))_{i=1}^{p+1}\|_2^2 + 2\text{Re}(\langle c_1b_1, c_2b_2 \rangle)}{2},$$

so

$$L_E(B) \leq \frac{2(p+1) - 2|\langle b_1, b_2 \rangle|}{2} \leq (p+1) - (p-1) = 2.$$ 

Thus, $L(E) \leq 2$. As $p+1 \leq U(E)$, it follows that $\text{cond}(E) \geq \sqrt{(p+1)/2} \to \infty$ as $p \to \infty$.

**Remark 3.20.** In Example 3.19, we actually need not use Proposition 3.18. Indeed, by the Pigeonhole principle, there are distinct $b_1, b_2 \in B$ that are in the same equivalence class. Then the argument can proceed as in the last paragraph.

The corollary below is Proposition 3.18 but with generators of $G$ already chosen.

**Corollary 3.21.** Let $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, $S$ and $S'$ be a partition of $\{1, 2, \ldots, k\}$, and $H$ and $K$ be the coordinate subspaces of $G$ with respect to $S$ and $S'$, respectively. Let $E \subseteq G$ be such that $\ell$ elements of $E$ are not contained in $H$, and let $(E, B)$ be a basis pair. Then

$$\sum_{x \in H, |B \cap (K+x)| \geq 1} (|B \cap (K+x)| - 1) \leq \ell.$$

We end with results on scale invariance. Consider a $k \in \mathbb{Z}$ such that multiplication by $k$ is an invertible $\mathbb{Z}$-linear transformation. For example, if $G = \mathbb{Z}_p^d$, $p$ a prime, then any $1 \leq k \leq p-1$ will do. By [1, Cor. 4.3], we know that in $\mathbb{Z}_p^d$, if $(E, B)$ is a spectral pair, then $(E, kB)$ is also a spectral pair. This statement generalizes to any finite abelian group $G$ as shown below.

**Proposition 3.22.** Let $k \in \mathbb{Z}$ be such that multiplication by $k$ is an invertible $\mathbb{Z}$-linear transformation on $G$. If $(E, B)$ is a spectral pair, then $(E, kB)$ is also a spectral pair.

**Proof.** View $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ and identify $\hat{G}$ with $G$ as in Equation (2.1). Then $k$ must be relatively prime to $n_1, n_2, \ldots, n_\ell$. Let $\omega = \chi(1/(n_1n_2\cdots n_\ell))$ be a primitive $n_1n_2\cdots n_\ell$-th root of unity. Then the entries of the matrix $T(E, B)$ are powers of $\omega$. Specifically, let $E = \{x_1, x_2, \ldots, x_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. If $M(x)$ is the $n \times n$ matrix whose $(i, j)$ entry is

$$x\left(\frac{\langle x_1, b_1 \rangle_1 \chi_1(n_1) + \langle x_2, b_2 \rangle_2 \chi_2(n_2) + \cdots + \langle x_\ell, b_\ell \rangle_\ell \chi_\ell(n_\ell)}{n_1n_2\cdots n_\ell}\right)^{n_1n_2\cdots n_\ell},$$

then $T(E, B) = M(\omega)$.
Notice that \( T(E, kB) = M(\omega^k) \). For \( x \in S^1 \), let \( M_i(x) \) be column \( i \) of \( M(x) \), and let \( P_{ij}(x) = M_i(x)^T M_j(x^{-1}) \in \mathbb{Z}[x, x^{-1}] \) be the inner product between columns \( i \) and \( j \) of \( M(x) \).

The fact that \( (E, B) \) is a spectral pair means that \( P_{ij}(\omega) = 0 \) for all \( i \neq j \). Because \( k \) is relatively prime to \( n_1 n_2 \ldots n_t \), this implies that \( P_{ij}(\omega^k) = 0 \) for all \( i \neq j \). Hence, the columns of \( T(E, kB) \) are orthogonal, and so \( (E, kB) \) is a spectral pair.

By Proposition 3.22, if \( (E, B) \) is a spectral pair, then our defined quantities on \( (E, B) \) are conserved when \( B \) is multiplied by \( k \). However, this result does not generalize beyond spectral pairs, as the following example shows.

**Example 3.23.** Let \( m > 2 \). Consider \( E = B = \{0, 1\} \in \mathbb{Z}_m^d \), and let \( k \) be relatively prime to \( m \). Then \( (E, B) \) is a non-spectral pair. We have

\[
T(E, B) = \begin{pmatrix} 1 & 1 \\ 1 & \chi(1/m) \end{pmatrix}, \quad T(E, kB) = \begin{pmatrix} 1 & 1 \\ 1 & \chi(k/m) \end{pmatrix},
\]

so none of the quantities \( L_E(B), U_E(B), \text{cond}_E(B), \) and \( D_E(B) \) is conserved.

### 4 Bounds

In this section, we obtain bounds on how far a basis pair \( (E, B) \) and a set \( E \) can be from being spectral. The quantities we will use to measure this are \( \text{cond}_E(B) \) and \( \text{cond}(E) \), where these quantities being close to one and \( \infty \) translate to \( (E, B) \) or \( E \) being close to and far from being spectral, respectively.

Our main findings are the following.

- In the special case of \( |E| = 2 \) and \( E \subseteq \mathbb{Z}_p^d \), \( p \) a prime, as \( p \to \infty \), \( \text{cond}(E) \to 1 \) independent of \( d \). However, this result cannot be generalized beyond \( G = \mathbb{Z}_p^d \) or beyond \( |E| = 2 \). (See Sec. 4.1.)
- For \( G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_t} \), let \( M = \text{lcm}(n_1, n_2, \ldots, n_t) \). Then for \( E \subseteq G \), \( \text{cond}_E(B) \) is bounded if \( |E| = n \) is fixed and \( M \) is bounded. In particular, for \( E \subseteq \mathbb{Z}_m^d \), \( \text{cond}_E(B) \) is bounded for \( |E| = n \) and \( m \) fixed, independent of \( d \). Moreover, the condition that \( M \) is bounded cannot be removed. (See Sec. 4.3.)
- Surprisingly, \( \text{cond}(E) \) is bounded for fixed \( |E| = n \), completely independent of \( G \). This shows that in those cases where results of Section 4.1 do not apply and we do not have \( \text{cond}(E) \to 1 \), \( \text{cond}(E) \) is still always bounded. (See Sec. 4.4.)

In summary, if one wants to construct a sequence of \( (E, B) \) such that \( \text{cond}_E(B) \to \infty \), one cannot fix \( |E| \) and \( M \). If one wants to construct a sequence of \( E \) such that \( \text{cond}(E) \to \infty \), one cannot fix \( |E| \).

#### 4.1 Sets of fixed set getting close to being spectral

In this subsection, we investigate the following question. If we fix the size of the set \( E \) and let the ambient space \( \mathbb{Z}_m^d \) “grow to infinity,” does \( E \) always become closer and closer to being spectral? Recall from Proposition 3.8 that the statement that a set \( E \) is close to being spectral can be formalized as \( \text{cond}(E) \) being close to one.
Proposition 4.1 shows this to be the case for any sequence \( E_p \subseteq \mathbb{Z}_p^d \), \( p \) being primes, with \(|E_p| = 2\). We might interpret this fact as follows. As \( p \to \infty \), there is more space, so it is becoming easier to find a partner \( B_p \) such that \((E_p, B_p)\) is close to being spectral.

Nevertheless, we also show in Examples 4.2 and 4.3 that this result does not hold when the size of \( E_p \) changes to 3 or when we consider a sequence \( E_m \subseteq \mathbb{Z}_m^d \), \( m \geq 1 \) is not necessarily prime. Thus, the phenomenon of getting close to being spectral seems to be special to the case of sets of size two in \( \mathbb{Z}_p^d \), \( p \) a prime.

Later, in Section 4.4, we show that the above intuition of having more space still holds in some weaker sense. Namely, for \( E \) of fixed size, \( \text{cond}(E) \) is bounded independent of the ambient space \( G \), where \( G \) is any finite abelian group. So \( E \) may not get closer and closer to being spectral as \( G \) grows, but it also cannot get arbitrarily far away.

**Proposition 4.1.** For any sequence of sets \( E_p \subseteq \mathbb{Z}_p^d \) indexed by primes \( p \) with \(|E_p| = 2\), we have \( \text{cond}(E_p) \to 1 \) as \( p \to \infty \).

**Proof.** By Corollary 3.16, we can assume that \( E_p \subseteq \mathbb{Z}_p \). By a further affine transformation (Cor. 3.14), we can assume that \( E_p = \{0, 1\} \).

For each \( p \geq 3 \), choose \( B_p = \{0, (p - 1)/2\} \). Then

\[
\text{cond}_{E_p}(B_p) = \text{cond}\left( \begin{array}{cc} 1 & 1 \\ 1 & \chi((p - 1)/2p) \end{array} \right).
\]

By the continuity of the condition number,

\[
\text{cond}_{E_p}(B_p) \to \text{cond}\left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) = 1,
\]

as \( p \to \infty \). Therefore, \( \text{cond}(E_p) \to 1 \) as \( p \to \infty \). \( \square \)

As mentioned at the start of the subsection, Proposition 4.1 cannot be generalized beyond \( G = \mathbb{Z}_p^d \), \( p \) a prime, or beyond sets of size two, as the following two examples demonstrate.

**Example 4.2.** We present a sequence \( E_m \subseteq \mathbb{Z}_m^d \), \( m \geq 1 \), with \(|E_m| = 2\), such that \( \text{cond}(E_m) \) does not tend to 1 as \( m \to \infty \).

Take \( E_m = \{0, m/3\} \subseteq \mathbb{Z}_m \) for \( m \) that are multiples of 3, and let \( B_m \) be such that \( \text{cond}_{E_m}(B_m) \) is minimized. By translational invariance (Prop. 3.11), we can assume that 0 \( \in B_m \). Then

\[
\text{cond}_{E_m}(B_m) = \text{cond}\left( \begin{array}{cc} 1 & 1 \\ 1 & \chi(k_m/3) \end{array} \right)
\]

for some \( k_m \in \{0, 1, 2\} \). None of the \( k_m \) makes the right-hand side one, so \( \text{cond}_{E_m}(B_m) \) cannot converge to one.

**Example 4.3.** We present a sequence \( E_p \subseteq \mathbb{Z}_p^d \), \( p \) being primes, with \(|E_p| = 3\), such that \( \text{cond}(E_p) \) does not tend to 1 as \( p \to \infty \).

Take \( E_p = \{0, 1, 3\} \subseteq \mathbb{Z}_p \), and let \( B_p \) be such that \( \text{cond}_{E_p}(B_p) \) is minimized. Again we can assume that 0 \( \in B_p \). Suppose to the contrary that \( \text{cond}_{E_p}(B_p) \to 1 \) as \( p \to \infty \). Notice that

\[
T(E_p, B_p) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & x_p & y_p \\ 1 & x_p^2 & y_p^2 \end{pmatrix},
\]

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for some \( x_p, y_p \in S^1 \). For large \( p \), since \( \text{cond}_{E_p}(B_p) \) is close to 1, Proposition 3.8 implies that the columns of \( T(E, B) \) are close to being orthogonal. Hence \( x_p + x_p^3 \) is close to \(-1\). It is easily verified that the sum of two elements \( a, b \in S^1 \) is close to \(-1\) if and only if \((a, b)\) is close to \((\chi(1/3), \chi(2/3))\) or \((\chi(2/3), \chi(1/3))\). Because \((x_p, x_p^3)\) cannot be close these pairs, we obtain a contradiction. So \( \text{cond}_{E_p}(B_p) \) cannot tend to 1 as \( p \to \infty \).

### 4.2 Bounds relating tightness quantities for \((E, B)\)

In this section, we introduce quantitative bounds that relate the tightness quantities to one another. Our main tool is Proposition 3.3. Recall from Proposition 3.8 that, for a fixed \(|E| = n\), any of the following statements implies all others: \( L_E(B) \) is close to \( n \), \( U_E(B) \) is close to \( n \), \( \text{cond}_E(B) \) is close to 1, and \( D_E(B) \) is close to \( n^{n/2} \). Thus, results in this section can be thought of as a more precise version of that proposition.

Apart from obtaining bounds when \((E, B)\) is close to being spectral, we will also use these results to derive estimates on how far \((E, B)\) and \( E \) can be from being spectral in Sections 4.3 and 4.4. Specifically, we first bound \( D_E(B) \) away from zero, this task being the easiest since \( D_E(B) \) is the absolute value of an integer linear combination of roots of unity. Then, estimates on other tightness quantities will follow.

Denote \( L_E(B) \) by \( L \), etc. We first describe how \( L \), \( U \), and \( C \) can be related to one another. Observe that

\[
(n-1)L + U = (n-1)\sigma_1^2 + \sigma_n^2 \leq \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2 = n^2.
\]

Similarly, \( L + (n-1)U \geq n^2 \). From these two inequalities, we can deduce a lower and upper bound of \( L \) in terms of \( U \), and vice versa. Moreover, these inequalities become equalities when \( \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} \) and \( \sigma_2 = \sigma_3 = \cdots = \sigma_n \), respectively, so these bounds are the best we can get from Proposition 3.3.

By writing \( C^2 = U/L \), the above gives lower and upper bounds of any of \( L \), \( U \), and \( C \) in terms of any other.

Bounds involving \( D \) are more complicated. We only derive upper bounds of \( D \) in terms of \( L \), \( U \), and \( C \) in Proposition 4.4, and not the corresponding lower bounds, as only the upper bounds will be used in later sections. Then we invert these results to get a lower bound of \( L \), an upper bound of \( U \), and an upper bound of \( C \) in terms of \( D \) in Proposition 4.5. The bounds in Proposition 4.4 are tight in the sense that an equality condition exists in terms of the singular values, while we lose some tightness in inverting them in Proposition 4.5. Bounds in Proposition 4.5 will be very useful in Sections 4.3 and 4.4.

**Proposition 4.4.** Let \((E, B)\) be a basis pair with \(|E| = n > 1\). Let \( 0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n \) be the singular values of \( T(E, B) \). Let \( L = L_E(B) \), \( U = U_E(B) \), \( C = \text{cond}_E(B) \), and \( D = D_E(B) \). Then

\[
D \leq \sqrt{L} \left( \frac{n^2 - L}{n - 1} \right)^{\frac{n-1}{2}};
\]

where equality holds if and only if \( \sigma_2 = \sigma_3 = \cdots = \sigma_n \);

\[
D \leq \sqrt{U} \left( \frac{n^2 - U}{n - 1} \right)^{\frac{n-1}{2}};
\]

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where equality holds if and only if \( \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} \); and

\[
D \leq \frac{2C}{C^2 + 1} n^{n/2},
\]

where equality holds if and only if \( \sigma_2 = \sigma_3 = \cdots = \sigma_{n-1} = \sqrt{n} \).

**Proof.** By the quadratic mean-geometric mean inequality,

\[
D = \sigma_1 \sigma_2 \cdots \sigma_n \leq \sigma_1 \left( \frac{\sigma_2^2 + \sigma_3^2 + \cdots + \sigma_n^2}{n-1} \right)^{\frac{n-1}{2}} = \sqrt{L} \left( \frac{n^2 - L}{n-1} \right)^{\frac{n-1}{2}},
\]

with the equality case as described. Similarly,

\[
D = \sigma_1 \sigma_2 \cdots \sigma_n \leq \left( \frac{\sigma_2^2 + \sigma_3^2 + \cdots + \sigma_n^2}{n-1} \right)^{\frac{n-1}{2}} \sigma_n = \sqrt{U} \left( \frac{n^2 - U}{n-1} \right)^{\frac{n-1}{2}},
\]

with the desired equality case.

The last inequality is more complicated. We claim that if \( C = \sigma_n/\sigma_1 \) is fixed and the singular values satisfy \( \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2 = n^2 \), then the maximum value of \( D \) is given by the desired expression. The case of \( n = 2 \) is easily worked out, so assume that \( n \geq 3 \). First notice that if we fix \( \sigma_1 \) and \( \sigma_n \), then the maximum of \( D \) is achieved when \( \sigma_2 = \sigma_3 = \cdots = \sigma_{n-1} \) by the quadratic mean-geometric mean inequality. So suppose that this is the case. Let \( \sigma_1 = x \), \( \sigma_n = Cx \), and the rest of the singular values be

\[
\sigma_i = \sqrt{\frac{n^2 - (C^2 + 1)x^2}{n-2}}.
\]

Then

\[
D^2 = C^2 x^4 \left( \frac{n^2 - (C^2 + 1)x^2}{n-2} \right)^{n-2}.
\]

By the arithmetic mean-geometric mean inequality,

\[
D^2 = \frac{4C^2}{(C^2 + 1)^2} \left[ \frac{(C^2 + 1)x^2}{2} \cdot \frac{(C^2 + 1)x^2}{2} \cdot \left( \frac{n^2 - (C^2 + 1)x^2}{n-2} \right)^{n-2} \right] \leq \frac{4C^2}{(C^2 + 1)^2} n^n,
\]

which implies the desired result. One can check that the equality case is \( \sigma_2 = \sigma_3 = \cdots = \sigma_{n-1} = \sqrt{n} \).

We now seek to invert the inequalities in Proposition 4.4 to bound \( L \), \( U \), and \( C \) in terms of \( D \). The bounds obtained are not sharp, but they are sharp within constant factors.

**Proposition 4.5.** Let \((E, B)\) be a basis pair with \(|E| = n > 1\). Let \( 0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n \) be the singular values of \( T(E, B) \). Let \( L = L_E(B) \), \( U = U_E(B) \), \( C = \text{cond}_E(B) \), and \( D = D_E(B) \). Then

\[
L > \left( \frac{n-1}{n^2} \right)^{n-1} D^2,
\]

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where for \( \sigma_2 = \sigma_3 = \cdots = \sigma_n \), the two sides are within a factor of \((1 + \frac{1}{n-1})^{n-1} \) of each other;

\[
n^2 - U > (n - 1) \left( \frac{D}{n} \right)^\frac{2}{n-1},
\]

where for \( \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} \), the two sides are within a factor of \( n^{\frac{1}{n-1}} \) of each other; and

\[
C < \frac{2n^{n/2}}{D^2},
\]

where for \( \sigma_2 = \sigma_3 = \cdots = \sigma_{n-1} = \sqrt{n} \), the two sides are within a factor of \( 1 + 1/C^2 \) of each other.

**Proof.** Apply Proposition 4.4. Notice that \( n^2 - L < n^2 \) and the two sides are within a factor of \( n/(n - 1) \) of each other; \( \sqrt{U} < n \) and the two sides are within a factor of \( \sqrt{n} \) of each other; and \( C^2 + 1 > C^2 \).

### 4.3 Upper bounds for looseness of \((E, B)\)

We know that for a basis pair \((E, B)\), \( L(E)(B) > 0 \), \( U(E)(B) < n^2 \), \( \text{cond}_E(B) < \infty \), and \( D_E(B) > 0 \). In this subsection, we give quantitative bounds on how close these quantities can get to these extremes without collapsing to 0, \( n^2 \), or \( \infty \). This can be interpreted as how far \((E, B)\) can be from being spectral without ceasing to be a basis pair.

The estimates obtained have to depend on both \(|E| = n\) and the ambient space \( G \), as Example 4.9 demonstrates. However, Proposition 4.8 shows that they depend on \( G \) in an interesting way. Specifically, they depend on which prime powers are present in the \( n_i \) in the decomposition \( G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t} \), but not the number of times the prime powers appear. For example, for \( G = \mathbb{Z}_m^d \), the estimates do not depend on \( d \) at all.

This phenomenon may be partly explained by the affine restriction property (Prop. 3.15). Indeed, for \( E \subseteq \mathbb{Z}_m^d \), as \( d \) grows but \(|E| = n\) and \( m \) stay fixed, it is likely that \( E \) will lie in a coset of a “small” direct summand of \( G \) whose size depends on \( n \) and \( m \) but not \( d \). If this is true, then by the affine restriction property, \( d \) plays no role in the growth of \( \text{cond}_E(B) \).

Miraculously, in Section 4.4, we will show that for quantities \( L(E), U(E), \text{cond}(E), \) and \( D(E) \), the dependence on the group \( G \) can be completely eliminated, so that estimates on these quantities only depend on \(|E| = n \). The intuition behind this may be the more-space intuition mentioned in Section 4.1. As \( G \) grows, the individual pair \((E, B)\) may get further from being spectral, but there are more choices of \( B \) to choose from, so in the end the set \( E \) itself is not so far from being spectral.

We now outline the strategy of proofs in this section. Bounding \( \text{cond}_E(B) \) away from infinity (or \( L_E(B) \) away from zero, or \( U_E(B) \) away from \( n^2 \)) is a difficult task. However, as \( D_E(B) \) is the absolute value of an integer combination of roots of unity, it is easier to bound this quantity away from zero. Specifically, we use a standard argument by permuting roots of unity in Lemma 4.6. This directly gives an estimate on \( D_E(B) \). Then we translate this into estimates on other tightness quantities via Proposition 4.5.

Let \( \varphi \) be Euler’s totient function, that is, \( \varphi(n) \) counts the number of integers \( 1 \leq k \leq n \) that are relatively prime to \( n \).
Lemma 4.6. Let $m$ be a positive integer, and $\omega$ be a primitive $m$-th root of unity. Let $P(x) \in \mathbb{Z}[x]$ be such that $P(\omega) \neq 0$. Suppose that $|P(\omega^k)| \leq C$ for all $1 \leq k \leq m$ that are relatively prime to $m$. Then $|P(\omega)| \geq C^{1-\varphi(m)}$.

Proof. Because $P(\omega) \neq 0$, for any $k$ that is relatively prime to $m$, $P(\omega^k) \neq 0$. Thus

$$S = \prod_{1 \leq k \leq m, \gcd(k,m)=1} P(\omega^k) \neq 0.$$ Notice that $S$ is the value of a symmetric polynomial evaluated at $\omega^k$, $1 \leq k \leq m$ with $\gcd(k,m) = 1$. Since these $\omega^k$ are roots of a monic polynomial with integer coefficients, $S$ must be an integer. Hence $|S| \geq 1$. It follows that

$$|P(\omega)| = \frac{|S|}{\prod_{2 \leq k \leq m, \gcd(k,m)=1} |P(\omega^k)|} \geq \frac{1}{C^{\varphi(m)-1}}.$$

Remark 4.7. Lemma 4.6 is a generalization of a standard argument used to find a lower bound for a nonzero sum of $2m$-th roots of unity. To see the connection, notice that a nonzero sum of $N$ $2m$-th roots of unity can be written as $P(\omega)$ for some $P(x) \in \mathbb{Z}[x]$, where the sum of coefficients of $P(x)$ is at most $N$. Then $|P(\omega^k)| \leq N$. So the lemma implies that this sum has absolute value at least $N^{1-\varphi(m)}$. See [9] for further results on this problem.

Let $G$ be a finite group, written in multiplicative notation. Following [10, p. 202], the minimal exponent of $G$ is the smallest positive integer $m$ such that $g^m = 1$ for all $g \in G$. In other words, it is the least common multiple of the orders of all the elements of $G$. If $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$, then the minimal exponent of $G$ is the least common multiple of $n_1, \ldots, n_t$. Hence, the minimal exponent of a group is the number $M$ in the proposition below, but this number can be defined without reference to the specific way that $G$ is decomposed into a direct product of cyclic groups.

Proposition 4.8. Let $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$ and $M = \text{lcm}(n_1, n_2, \ldots, n_t)$. Let $E \subseteq G$ with $|E| = n$. If $(E, B)$ is a basis pair, then $D_E(B) \geq n^{n(1-\varphi(M))/2}$. Consequently,

$$\text{cond}_E(B) < 2n^{n\varphi(M)/2}.$$

Proof. The results for $n = 1$ are obvious, so assume that $n > 1$. In a similar manner to the proof of Proposition 3.22, let $M(x)$ be the $n \times n$ matrix whose $(i, j)$ entry is

$$x^{\left(\frac{(x_1)_1(k_1)_1}{n_1} + \frac{(x_1)_2(k_1)_2}{n_2} + \cdots + \frac{(x_1)_t(k_1)_t}{n_t}\right)}.$$

(This is slightly different from (3.1).) Let $P(x) = \det M(x) \in \mathbb{Z}[x]$. If $\omega = \chi(1/M)$, then for every $1 \leq k \leq M$ that is relatively prime to $M$, $T(E, kB) = M(\omega^k)$. Hence

$$D_E(kB) = |\det M(\omega^k)| = |P(\omega^k)|.$$ By Proposition 3.4, $|P(\omega^k)| \leq n^{n/2}$, so Lemma 4.6 yields the desired bound for $D_E(B)$. Finally, we apply Proposition 4.5 to obtain the bound for $\text{cond}_E(B)$. \qed
From Proposition 4.8, we can also derive a lower bound for $L_E(B)$ and an upper bound for $U_E(B)$ via Proposition 4.5.

The crucial point is that the bounds obtained in Proposition 4.8 depend on $G$ even for fixed $|E| = n$. The example below shows that this must be the case.

Example 4.9. Let $E = B = \{0, 1\} \subseteq \mathbb{Z}_p$, $p$ a prime. Then

$$T(E, B) = \begin{pmatrix} 1 & 1 \\ 1 & \chi(1/p) \end{pmatrix}.$$  

It is easily checked that $L_E(B) \to 0$, $U_E(B) \to 4$, $\text{cond}_E(B) \to \infty$, and $D_E(B) \to 0$ as $p \to \infty$. Therefore, $(E, B)$ gets arbitrarily far away from being spectral as $p \to \infty$.

4.4 Upper bounds for looseness of $E$

Finally, in this subsection, we derive a bound on how far from being spectral a set $E$ can be. The arguments are similar to those used in Section 4.3, but the new idea is the following. We will demonstrate that, if $B$ is carefully picked, then we can make $(E, B)$ close to being spectral independent of the ambient group $G$, even though an individual pair $(E, B)$ may be far from being spectral. Specifically, we consider $kB$ for all $k$ such that multiplication by $k$ is an invertible $\mathbb{Z}$-linear transformation on $G$. These pairs average each other out, and one of $(E, kB)$ must be rather close to being spectral, as shown in Lemma 4.10. Then estimates on tightness quantities follow as before.

The results of this section show that a weaker sense of the more-space intuition in Section 4.1 holds true. That is, for $|E| = n$ fixed and as $G$ grows, there are more choices of $B$ to choose from, so one of $(E, B)$ must be quite close to being spectral. In this case, it turns out to suffice to pick a random $B$ and consider $kB$ over all $k$.

The takeaway of these results is the following. To get a sequence of sets $E$ that are further and further away from being spectral, the size of $E$ has to increase to infinity, no matter which sequence of $G$ we choose.

We begin with a variant of Lemma 4.6.

Lemma 4.10. Let $m$ be a positive integer, and $\omega$ be a primitive $m$th root of unity. Let $P(x) \in \mathbb{Z}[x]$ be such that $P(\omega) \neq 0$. Then there is a $1 \leq k \leq m$ that is relatively prime to $m$ such that $|P(\omega^k)| \geq 1$.

Proof. Consider the quantity $S$ defined in Lemma 4.6. We have $|S| \geq 1$. This implies that there is a $1 \leq k \leq m$, $\gcd(k, m) = 1$, such that $|P(\omega^k)| \geq 1$. \hfill $\Box$

The next proposition shows that for any basis pair $(E, B)$, one of $(E, kB)$ is rather close to being spectral.

Proposition 4.11. Let $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell}$ and $M = \text{lcm}(n_1, n_2, \ldots, n_\ell)$. Let $E \subseteq G$ with $|E| = n > 1$. Suppose that $(E, B)$ is a basis pair. Then there is a $1 \leq k \leq M$ that is
relatively prime to $M$ such that $D_E(kB) \geq 1$. Consequently, for this $k$,
\[
L_E(kB) > \left( \frac{n - 1}{n^2} \right)^{n-1} > \frac{1}{en^{n-1}},
\]
\[
U_E(kB) < n^2 - \frac{n - 1}{n^{\frac{n-1}{2}}} \leq n^2 - \frac{n - 1}{4},
\]
\[
\text{cond}_E(kB) < 2n^{n/2}.
\]

Proof. An analogous argument to the proof of Proposition 4.8, but with Lemma 4.10 in place of Lemma 4.6, gives the estimate $D_E(kB) \geq 1$. Now apply Proposition 4.5 to obtain the rest of the estimates. □

Notice that in Proposition 4.11, the second steps in the bounds for $L_E(kB)$ and $U_E(kB)$ lose only at most constant factors.

We now present our main result on estimates on tightness quantities of $E$, which are independent of the group $G$.

Corollary 4.12. Let $E \subseteq G$ with $|E| = n > 1$. Then
\[
L(E) > \frac{1}{en^{n-1}}, \quad U(E) < n^2 - \frac{n - 1}{4}, \quad \text{cond}(E) < 2n^{n/2}, \quad \text{and} \quad D(E) \geq 1.
\]
In particular, these estimates are independent of the group $G$.

Proof. Follows directly from the existence of a basis pair (Cor. 1.12) and Proposition 4.11. □

5 Dimensional reduction

Let $G = H \oplus K$. In this section, we investigate what happens when a set $E \subseteq H$ has its “dimension” increased to become the set $E \times K \subseteq G$. An interesting conclusion of Section 5.1 is that $E \times K$ is not further from being spectral than $E$. Specifically, $\text{cond}(E \times K) \leq \text{cond}(E)$, and similarly for other tightness quantities. Even more strikingly, it will later turn out in Section 5.3 that $\text{cond}(E \times K) = \text{cond}(E)$, and similarly for other tightness quantities. Thus, $E$ and $E \times K$ are exactly as far from being spectral as one another.

We first need some preliminary definitions.

Definition 5.1. For an equal-size pair $(E, B)$, define the scaled tightness quantities as follows:
\[
\bar{L}_E(B) = n/L_E(B),
\]
\[
\bar{U}_E(B) = U_E(B)/n,
\]
\[
\bar{\text{cond}}_E(B) = \text{cond}_E(B),
\]
\[
\bar{D}_E(B) = \sqrt{n/D_E(B)}^{1/n}.
\]
Notice that the scaled version of $\text{cond}$ is itself.
Let $Q$ denote any tightness quantity and let $\tilde{Q}$ be its scaled version. Then $\tilde{Q}_E(B) \geq 1$ and it is one if and only if $(E, B)$ is a spectral pair. Accordingly, define

$$\tilde{Q}(E) = \min_B \tilde{Q}_E(B).$$

Notice that $\tilde{Q}(E) = 1$ if and only if $E$ is a spectral set. The relationship between $Q(E)$ and $\tilde{Q}(E)$ is the same as the relationship between $Q_E(B)$ and $\tilde{Q}_E(B)$ in Definition 5.1.

### 5.1 Cartesian products

In this subsection, we study basic properties of $(E, B)$ when $E$ and $B$ are Cartesian products. Specifically, let $H$ and $K$ be finite abelian groups, and let $G = H \oplus K$. For $E_1 \subseteq H$ and $E_2 \subseteq K$, we consider the Cartesian product $E = E_1 \times E_2 \subseteq G$. Notice that this is also the Minkowski sum $E = E_1 + E_2$ under the usual identification. Later on in Section 5.2, we will also examine more generalized Minkowski sums where $E_1$ is not necessarily contained in $H$.

Recall that $\hat{G} \cong \hat{H} \times \hat{K}$, where for any $\hat{h} \in \hat{H}$ and $\hat{k} \in \hat{K}$, $(\hat{h}, \hat{k})$ acts as an element of $\hat{G}$ by $(\hat{h}, \hat{k})(h, k) = \hat{h}(h)\hat{k}(k)$ for any $h \in H$ and $k \in K$. So for any $B_1 \subseteq \hat{H}$ and $B_2 \subseteq \hat{K}$, we can identify $B = B_1 \times B_2$ with a subset of $\hat{G}$, and we will do so without further comments. Notice that since both $B$ and the set $E$ from above are Cartesian products, there is duality between $E$ and $B$ in our situation.

If $(E_1, B_1)$ and $(E_2, B_2)$ are equal-size pairs, then $(E, B)$ is also an equal-size pair. Our goal in this subsection is to show that the three pairs are closely related. We start with the next proposition, which relates their evaluation matrices.

**Proposition 5.2.** Let $(E_1, B_1) \subseteq H \times \hat{H}$ and $(E_2, B_2) \subseteq K \times \hat{K}$ be equal-size pairs. Let $E = E_1 \times E_2$ and $B = B_1 \times B_2$. Then, up to an ordering of rows and columns,

$$T(E, B) = T(E_1, B_1) \otimes T(E_2, B_2),$$

where $\otimes$ is the Kronecker product.

**Proof.** The entry of $T(E, B)$ in the column corresponding to $(b_1, b_2)$, $b_i \in B_i$, and the row corresponding to $(x_1, x_2)$, $x_i \in E_i$, is given by

$$(b_1, b_2)(x_1, x_2) = b_1(x_1)b_2(x_2).$$

On the other hand, the entry of $T(E_i, B_i)$ in the column corresponding to $b \in B_i$ and the row corresponding to $x \in E_i$ is $b(x)$. So the result follows from the definition of the Kronecker product.

Let $A$ and $B$ be square matrices, and let $\{\sigma_i\}_{i=1}^n$ and $\{\tau_j\}_{j=1}^m$ be the singular values of $A$ and $B$, respectively, counting multiplicities. It is known (see [5, Thm. 4.2.15]) that the singular values of the Kronecker product $A \otimes B$ are exactly $\{\sigma_i\tau_j\}_{1 \leq i \leq n, 1 \leq j \leq m}$, counting multiplicities. Using this, the tightness quantities of the three pairs can be related to one another, as the proposition below shows.
Proposition 5.3. Let \((E_1, B_1) \subseteq H \times \hat{H}\) and \((E_2, B_2) \subseteq K \times \hat{K}\) be equal-size pairs. Let \(E = E_1 \times E_2\) and \(B = B_1 \times B_2\). Then, for any scaled tightness quantity \(\tilde{Q}\),

\[
\tilde{Q}_E(B) = \tilde{Q}_{E_1}(B_1)\tilde{Q}_{E_2}(B_2).
\]

Proof. By Proposition 5.2, the singular values of \(T(E, B)\) are products of singular values of \(T(E_1, B_1)\) and \(T(E_2, B_2)\), counting multiplicities. The result now follows by writing \(\tilde{Q}\) in terms of singular values using Proposition 3.3.

Proposition 5.3 allows us to bound tightness quantities of \(E\) in terms of the corresponding quantities of \(E_1\) and \(E_2\), as shown in Corollary 5.4 below. In that corollary, \(E_1 \subseteq H \subseteq G\), but by the affine restriction property (Prop. 3.15), \(\tilde{Q}(E_1; H) = \tilde{Q}(E_1; G)\), so we need not specify the ambient group. A similar remark applies to \(E_2\).

Corollary 5.4. Let \(E_1 \subseteq H, E_2 \subseteq K\), and \(E = E_1 \times E_2\). Then, for any scaled tightness quantity \(\tilde{Q}\),

\[
\tilde{Q}(E) \leq \tilde{Q}(E_1)\tilde{Q}(E_2).
\]

In particular, if \(E_1\) and \(E_2\) are spectral, then \(E\) is spectral.

Proof. Pick \(B_1 \subseteq \hat{H}\) and \(B_2 \subseteq \hat{K}\) such that \(\tilde{Q}_E(B_i) = \tilde{Q}(E_i)\). By Proposition 5.3, \(\tilde{Q}_E(B_1 \times B_2) = \tilde{Q}(E_1)\tilde{Q}(E_2)\) with \(B_1 \times B_2 \subseteq \hat{G}\), so that \(\tilde{Q}(E) \leq \tilde{Q}(E_1)\tilde{Q}(E_2)\).

Corollary 5.4 provides a nice generalization of the well-known fact that a Cartesian product of spectral sets is spectral; an elementary proof of this is obtained by multiplying the relevant characters to form an orthogonal basis. An important special case of the corollary occurs below when \(E_2\) is spectral, for example, when \(E_2\) is the whole group \(K\).

Corollary 5.5. Let \(E_1 \subseteq H\) and \(E_2 \subseteq K\), where \(E_2\) is spectral. Let \(E = E_1 \times E_2\). Then \(E\) is not further from being spectral than \(E_1\). Specifically, for any scaled tightness quantity \(\tilde{Q}\),

\[
\tilde{Q}(E) \leq \tilde{Q}(E_1).
\]

In particular, \(\text{cond}(E_1 \times E_2) \leq \text{cond}(E_1)\) when \(E_2\) is spectral.

As an application, we construct in Example 5.6 below large sets \(E\) that do not tile but are very close to being spectral. The significance of such an example is the following. It is known that a set \(E \subseteq G\) that tiles by a direct summand of \(G\) is spectral since this implies that \(E\) tiles by a lattice. The proof that tiling by a lattice implies that \(E\) is spectral is given in [2]. We will also prove that a set \(E\) is spectral if it tiles by a direct summand in Proposition 5.13. Recall, from Corollary 4.12, that small sets cannot be very far from being spectral. Thus, the example below demonstrates that good spectral behavior can occur apart from the cases of being small and tiling by a direct summand. Later, we will discuss more about the relationship between tiling and being close to spectral.

Example 5.6. Let \(p\) be a prime and let \(G = \mathbb{Z}_p^2\). Let \(E_p = \{0, 1\} \times \mathbb{Z}_p\). By Corollary 5.5 and Proposition 4.1,

\[
\text{cond}(E_p) \leq \text{cond}(\{0, 1\}; \mathbb{Z}_p) \to 1
\]

as \(p \to \infty\). So \(E_p\) is close to being spectral for large \(p\).
An interesting question is whether the inequality in Corollary 5.4 can be strict. In the special case where $E$ is spectral, this asks whether it is necessary that $E_1$ and $E_2$ are also spectral. This problem about spectral sets is open, although it has been proved true in special cases. For the question about the inequality’s strictness, however, the answer is false for all $\tilde{Q}$ except perhaps $\tilde{D}$, as illustrated by the following example. We do not know the answer to this question for $Q = \tilde{D}$.

**Example 5.7.** Let $H = K = \mathbb{Z}_3$, $E_1 = E_2 = \{0, 1\} \subseteq \mathbb{Z}_3$, and $E = E_1 \times E_2$. It is easily checked that

$$
\tilde{L}(E_1) = 2, \quad \tilde{U}(E_1) = \frac{3}{2}, \quad \tilde{\text{cond}}(E_1) = \sqrt{3}, \quad \tilde{D}(E_1) = \frac{\sqrt{2}}{\sqrt{3}},
$$

while it can be checked by a computer that

$$
\tilde{L}(E) \doteq 3.490711985, \quad \tilde{U}(E) \doteq \frac{3}{2}, \quad \tilde{\text{cond}}(E) \doteq \frac{3 + \sqrt{5}}{2}, \quad \tilde{D}(E) \doteq \frac{2}{\sqrt{3}},
$$

where $\doteq$ means the equalities only hold numerically, i.e., up to small roundoff errors. Thus, the inequality in Corollary 5.4 can be strict for $E$ and all $\tilde{Q}$ except perhaps $\tilde{D}$.

Finally, we remark that Propositions 5.2 and 5.3 and Corollary 5.4 can be easily generalized to the case of $n$-fold Cartesian products for any $n$.

### 5.2 Minkowski sums where one summand is a partial graph, version 2

In this subsection, we investigate sets $E$ that are more general than Cartesian products. Let $G$ be a finite abelian group, let $H$ be a subgroup of $G$, and let $K := G/H$. Define a *partial graph over $K$* to be any subset of $G$ whose intersection with each coset of $H$ has at most one element. For $E_1 \subseteq H$, and $P$ a partial graph over $K$, we are interested in the Minkowski sum $E := E_1 + P$. We show that such sets $E$ behave well spectrally compared to $E_1$ and the projection $\pi(P)$ of $P$ into $K$ (Cor. 5.12), in an analogous way to Corollary 5.4. Therefore, apart from Cartesian products, this provides us with another way to construct large sets that do not tile and are not far from being spectral.

We start with a set $E$ as above. Then, there exists an integer $\ell$ such that, whenever $E$ has a nonempty intersection with a coset of $H$, the intersection contains $\ell$ elements. Moreover, there exists a positive integer $m$ such that $\pi(E)$ has $m$ elements, where $\pi : G \to G/H$ is the canonical projection.

Let $B_H \subseteq \hat{H}$ have $\ell$ elements, and let $B_K \subseteq \hat{K}$ have $m$ elements. Under the canonical isomorphism $\hat{H} \cong \hat{G}/\hat{H}^\perp$, we associate, to each $\varphi \in B_H$, an element $\hat{\varphi} \in \hat{g}H^\perp$, where $\hat{g}H^\perp$ is the coset corresponding to $\varphi$ under the canonical isomorphism. Similarly, under the canonical isomorphism $\hat{G}/\hat{H} \cong H^\perp$, we associate, to each element $\psi \in B_K$, a canonical element $\hat{\psi} \in H^\perp$, where $\hat{\psi}$ is the element corresponding to $\psi$ under the canonical isomorphism. Let $B = \{\hat{\varphi} \cdot \hat{\psi}\}_{(\varphi, \psi) \in B_H \times B_K} \subseteq \hat{G}$. For convenience, let us also denote $\hat{B}_H = \{\hat{\varphi}\}_{\varphi \in B_H}$ and $\hat{B}_K = \{\hat{\psi}\}_{\psi \in B_K}$. Then $(E, B)$ is an equal-size pair, and the proposition below concerns such pairs.
**Proposition 5.8.** Let $E$ and $B$ be as above. Then, for any scaled tightness quantity $\tilde{Q}$ except $\text{cond}$,

$$
\tilde{Q}_E(B) \leq \max_{p \in P} \tilde{Q}_{E_1+p}(\tilde{B}_H) \cdot \tilde{Q}_{\pi(P)}(B_K).
$$

For $\text{cond}$, we have instead the inequality

$$
\text{cond}_E(B) \leq \sqrt{\max_{p \in P} \tilde{L}_{E_1+p}(\tilde{B}_H) \max_{p \in P} \tilde{U}_{E_1+p}(\tilde{B}_H)} \cdot \text{cond}_{\pi(P)}(B_K).
$$

Moreover, for $\tilde{D}$, the inequality can be strengthened to the equality

$$
\tilde{D}_E(B) = \left( \prod_{p \in P} \tilde{D}_{E_1+p}(\tilde{B}_H) \right)^{1/m} \cdot \tilde{D}_{\pi(P)}(B_K).
$$

Compare this result with Proposition 5.3.

**Proof.** Write $\tilde{B}_K = \{\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_m\}$ and write $P = \{p_1, \ldots, p_m\}$. With an appropriate reordering of rows and columns, $T(E, B)$ is an $m \times m$ block matrix of $\ell \times \ell$ blocks, where the $(i, j)$ block corresponds to $\{\hat{q} \cdot \hat{k}_j\}_{\ell \in B_H}$ acting on $(E_1 + p_i)$. So the $(i, j)$ block of $T(E, B)$ is $T(E_1 + p_i, \tilde{B}_H)k_j(p_i)$. Thus, we can factor

$$
T(E, B) = \begin{pmatrix}
T(E_1 + p_1, \tilde{B}_H) & & \\
& \ddots & \\
& & T(E_1 + p_m, \tilde{B}_H)
\end{pmatrix}
\times
\begin{pmatrix}
\hat{k}_1(p_1)I_{\ell} & \hat{k}_2(p_1)I_{\ell} & \ldots & \hat{k}_m(p_1)I_{\ell} \\
\hat{k}_1(p_2)I_{\ell} & \hat{k}_2(p_2)I_{\ell} & \ldots & \hat{k}_m(p_2)I_{\ell} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{k}_1(p_m)I_{\ell} & \hat{k}_2(p_m)I_{\ell} & \ldots & \hat{k}_m(p_m)I_{\ell}
\end{pmatrix}. \quad (5.1)
$$

Call the first factor $T_1$ and the second factor $T_2$.

For the analysis of $T_1$, observe that $T_1$’s singular values are the union of the singular values of the $T(E_1 + p_i, \tilde{B}_H)$, counting multiplicities. So

$$
\|T_1\| = \max_{p \in P} \|T(E_1 + p, \tilde{B}_H)\|, \quad \|T_1^{-1}\| = \max_{p \in P} \|T(E_1 + p, \tilde{B}_H)^{-1}\|, \quad \det T_1 = \prod_{p \in P} \det T(E_1 + p, \tilde{B}_H).
$$

Furthermore, up to a permutation of rows and columns, $T_2$ is an $\ell \times \ell$ block diagonal matrix with blocks of size $m \times m$. Specifically, arrange the rows and columns in the order

$$
1, \ell + 1, \ldots, (m - 1)\ell + 1, \ 2, \ell + 2, \ldots, (m - 1)\ell + 2, \ldots, \ell, 2\ell, \ldots, m\ell.
$$

Then all diagonal blocks equal $T(\pi(P), B_K) = [\hat{k}_j(k_i)]_{ij}$. Hence

$$
\|T_2\| = \|T(\pi(P), B_K)\|, \quad \|T_2^{-1}\| = \|T(\pi(P), B_K)^{-1}\|, \quad \det T_2 = (\det T(\pi(P), B_K))^\ell.
$$

Finally, we have bounds $\|T\| \leq \|T_1\| \|T_2\|$ and $\|T^{-1}\| \leq \|T_1^{-1}\| \|T_2^{-1}\|$ and the equality $\det T = \det T_1 \det T_2$. These together with the basic characterizations of tightness quantities (Prop. 1.7) imply the desired result. \qed
Remark 5.9. In his proof that a bounded set $\Omega$ that multi-tiles $\mathbb{R}^d$ by a lattice $\Lambda$ has the property that $L^2(\Omega)$ has a Riesz exponential basis, Kolountzakis [7] gives the factorization

$$
\left[ e\left( a_j \cdot (x - \lambda_r(x)) \right) \right]_{1 \leq j, r \leq k} = \left[ e\left( - a_j \cdot \lambda_r(x) \right) \right]_{1 \leq j, r \leq k} \text{diag}\left[ e(a_1 \cdot x), \ldots, e(a_k \cdot x) \right]
$$

(cf. Equation (14) in his proof of Lemma 2). This is analogous to our factorization of $T(E, B)$ in the proof of Proposition 5.8 into $T_1$ and $T_2$. In fact, quantitatively calculating the constant $C_2$ of Kolountzakis’s Lemma 2 yields

$$
C_2 = kA_2 = k \cdot \max_{\text{distinct } N(x)} \| N(x)^{-1} \|,
$$

analogous to our above calculation $\| T_1^{-1} \| = \max_{1 \leq i \leq m} \| T(E_i, B_K)^{-1} \|$. A similar remark applies for his constant $C_1$.

Results of Proposition 5.8 in the special case where $(E_K, B_K)$ is a spectral pair can be strengthened to equalities as follows.

**Proposition 5.10.** Let $E$ and $B$ be as in Proposition 5.8, and suppose that $(E_K, B_K)$ is a spectral pair. Then, for $\tilde{Q} = \tilde{L}$ or $\tilde{Q} = \tilde{U}$,

$$
\tilde{Q}_{E_i}(B_H) = \max_{1 \leq i \leq m} \tilde{Q}_{E_i}(B_H).
$$

For cond,

$$
\text{cond}_{E_i}(B) = \sqrt{\max_{1 \leq i \leq m} \tilde{L}_{E_i}(B_H) \max_{1 \leq i \leq m} \tilde{U}_{E_i}(B_H)} \geq \max_{1 \leq i \leq m} \text{cond}_{E_i}(B_H).
$$

For $\tilde{D}$,

$$
\tilde{D}_{E_i}(B) = \left( \prod_{i=1}^{m} \tilde{D}_{E_i}(B_H) \right)^{1/m}.
$$

**Proof.** In the proof of Proposition 5.8, notice that, since $(E_K, B_K)$ is a spectral pair, $T_2$ is a multiple of a unitary matrix. So $\| T \| = \| T_1 \| \| T_2 \|$ and $\| T_1^{-1} \| = \| T_1^{-1} \| \| T_2^{-1} \|$, and equalities are attained in the bounds.

We now consider the special case where all $E_i$ are translates of one another. This happens exactly when there is a partial graph $P$ over $K$ such that $E$ is the Minkowski sum $E_1 + P$. In this case, Proposition 5.8 simplifies as follows.

**Corollary 5.11.** Let $(E_1, B_H) \subseteq H \times \tilde{H}$ and $(E_K, B_K) \subseteq K \times \tilde{K}$ be equal-size pairs. Let $P \subseteq G$ be a partial graph over $K$ that projects onto $E_K$ along $H$. Let $E = E_1 + P$ and $B = B_H \times B_K$. Then, for any scaled tightness quantity $\tilde{Q}$,

$$
\tilde{Q}_{E}(B) \leq \tilde{Q}_{E_1}(B_H)\tilde{Q}_{E_K}(B_K).
$$

Moreover, if either $\tilde{Q} = \tilde{D}$ or at least one of $(E_1, B_H)$ and $(E_K, B_K)$ is a spectral pair, then equality holds.
Also compare this result with Proposition 5.3.

Proof. By translational invariance (Prop. 3.11), \( \tilde{Q}_{E_i}(B_H) \) are all equal. So the desired inequality follows from Proposition 5.8. If \((E_1, B_H)\) or \((E_K, B_K)\) is spectral, then \(T_1\) or \(T_2\) is a multiple of a unitary matrix, respectively. So we can apply the reasoning in Proposition 5.10 to conclude that equality holds.

We now provide the following generalization of Corollary 5.4 to the case where \(E\) is a Minkowski sum of a subset of \(H\) and a partial graph over \(K\).

Corollary 5.12. Let \(E_1 \subseteq H\) and let \(P \subseteq G\) be a partial graph over \(K\) that projects onto \(E_K \subseteq K\) along \(H\). Let \(E = E_1 + P\). Then, for any scaled tightness quantity \(\tilde{Q}\),

\[
\tilde{Q}(E) \leq \tilde{Q}(E_1)\tilde{Q}(E_K).
\]

In particular, if \(E_K\) is spectral, then \(\tilde{Q}(E) \leq \tilde{Q}(E_1)\); if \(E_1\) is spectral, then \(\tilde{Q}(E) \leq \tilde{Q}(E_K)\).

In Corollary 5.12, by taking \(E_1 = \{0\}\) and \(E_K = K\) (that is, \(P\) is a full graph over \(K\)), we obtain the following known result, which is reproved here.

Proposition 5.13. Let \(H\) be a direct summand of a group \(G\). Then any set \(E \subseteq G\) that tiles with tiling partner \(H\) is spectral.

Proof. Decompose \(G = H \oplus K\). The hypothesis that \(E\) tiles with tiling partner \(H\) is equivalent to \(E\) having exactly one element in each coset of \(H\). This is further equivalent to \(E\) being a full graph over \(K\) along \(H\). So with the decomposition \(E = \{0\} + E\), we can apply Corollary 5.12 to conclude that \(E\) is spectral.

As an application of results of this subsection, we will construct in Example 5.14 large sets that do not tile and are also not Cartesian products, but are very close to being spectral. Compare this with Example 5.6.

Example 5.14. Let \(p \geq 3\) be a prime and let \(G = \mathbb{Z}_p^2\). Let

\[
E_p = (\{0, 1\} \times (\mathbb{Z}_p \setminus \{0\})) \cup \{(1, 0), (2, 0)\}.
\]

By Corollary 5.12 and Proposition 4.1,

\[
\text{cond}(E_p) \leq \text{cond}(\{0, 1\}; \mathbb{Z}_p) \to 1
\]

as \(p \to \infty\). So \(E_p\) is close to being spectral for large \(p\).

5.3 Cartesian products with a direct summand

Let \(G = H \oplus K\). In this subsection, we study sets \(E\) of the form \(E_1 \times K\) where \(E_1 \subseteq H\). We know that such a set \(E\) is not further from being spectral than \(E_1\): by Corollary 5.5, \(\tilde{Q}(E) \leq \tilde{Q}(E_1)\) for any scaled tightness quantity \(\tilde{Q}\). We now prove the surprising converse: \(\tilde{Q}(E) = \tilde{Q}(E_1)\), that is, they are exactly as far from being spectral as one another (Prop. 5.17).
Results of this type are important for the following reason. The product question for spectral sets asks: if $E_1 \times E_2$ is spectral, are $E_1$ and $E_2$ necessarily spectral? This question is still open in general. Our Proposition 5.17 implies that for any $E_1 \subseteq H$, $E_1$ is spectral if and only if $E_1 \times K$ is spectral. Therefore, in the case of finite abelian groups, that proposition resolves this question when one factor is the whole group $K$. In the case of $\mathbb{R}^d$, a similar result was obtained by Greenfeld and Lev [4]. Specifically, they proved that if $I \subseteq \mathbb{R}$ is an interval and $\Sigma \subseteq \mathbb{R}^{d-1}$, then $\Sigma$ is spectral if and only if $I \times \Sigma$ is spectral.

Recall that we have the isomorphism $\hat{G} \cong \hat{H} \oplus \hat{K}$, so we can think of $\hat{H}$ as a direct summand of $\hat{G}$. The strategy for proving our result is as follows. Let $E = E_1 \times K$, $E_1 \subseteq H$. We first show that for any $B \subseteq \hat{G}$ such that $(E, B)$ is a basis pair, $B$ has the same number of elements in each coset of $\hat{H}$ (Prop. 5.15). This allows us to apply Proposition 5.10 to the pair $(B, E)$; note that $E$ and $B$ appear here in the reverse of the usual order. The conclusion is that the intersection of $B$ with some coset of $\hat{H}$, properly translated, is a spectrally good pairing for $E_1$. From this, we can obtain the desired result.

First, we prove the following Proposition 5.15 on the structure of $B$ for which $(E, B)$ is a basis pair.

**Proposition 5.15.** Let $E_1 \subseteq H$ and $E = E_1 \times K$. Let $(E, B) \subseteq G \times \hat{G}$ be a basis pair. Then $B$ has exactly $|E_1|$ elements in each coset of $\hat{H}$.

**Proof.** The proof is in the spirit of the proof of Proposition 3.18. Suppose otherwise. Let $|E_1| = m$. Then $B$ has at least $m + 1$ elements in some coset $\hat{H} + k$, $k \in \hat{K}$. Let these elements be $\hat{h}_i + k$, $1 \leq i \leq m + 1$. We claim that these elements are linearly dependent on $E$.

Because $|E_1| = m$, there are $c_i \in \mathbb{C}$ such that $\sum_{i=1}^{m+1} c_i \hat{h}_i(h) = 0$ for all $h \in E_1$. So for any $(h, k) \in E_1 \times K$,

$$\sum_{i=1}^{m+1} c_i (\hat{h}_i + \hat{k})(h, k) = \left( \sum_{i=1}^{m+1} c_i \hat{h}_i(h) \right) \hat{k}(k) = 0.$$  

Thus, $\{\hat{h}_i + \hat{k}\}_{i=1}^{m+1}$ are linearly dependent on $E$, so that $B$ is not a basis for $E$. This contradicts the hypothesis that $(E, B)$ is a basis pair. $\Box$

**Remark 5.16.** For a general $E = E_1 \times E_2$, the same argument as in the proof of Proposition 5.15 implies that for any basis pair $(E, B)$, $B$ has at most $|E_1|$ elements in each coset of $\hat{H}$. But if $E_2 \neq K$, “at most” here cannot be improved to “exactly.”

**Proposition 5.17.** Let $E_1 \subseteq H$ and $E = E_1 \times K$. Then, for any scaled tightness quantity $\tilde{Q}$, $\tilde{Q}(E) = \tilde{Q}(E_1)$.

**Proof.** By Corollary 5.5, $\tilde{Q}(E) \leq \tilde{Q}(E_1)$. It remains to prove the reverse inequality. Let $B \subseteq \hat{G}$ be such that $\tilde{Q}(E) = \tilde{Q}_E(B)$. Our goal is to construct $B_H \subseteq \hat{H}$ such that $\tilde{Q}(E) \geq \tilde{Q}_{E_1}(B_H)$, which will imply that $\tilde{Q}(E) \geq \tilde{Q}(E_1)$.

Observe that $(E, B)$ is a basis pair. Let $|E_1| = \ell$. By Proposition 5.15, $B$ has exactly $\ell$ elements in each coset of $\hat{H}$. Let $\hat{K} = \{\hat{k}_1, \ldots, \hat{k}_m\}$, and let $B_i = (B - \hat{k}_i) \cap \hat{H} \subseteq \hat{H}$. We can now apply Proposition 5.10, where the roles of $E$ and $B$ here are switched from that
proposition. Notice that since $E_K = K$ and $B_K = \hat{K}$ in the proposition, $(E_K, B_K)$ is indeed a spectral pair. We conclude that

$$\bar{Q}_B(E) \geq \min_{1 \leq i \leq m} \bar{Q}_{B_i}(E_1).$$

(For $\bar{Q} \neq \bar{D}$, we can strengthen the inequality by replacing the minimum on the right-hand side with the maximum.) So there is some $1 \leq i \leq m$ for which $\bar{Q}_B(E) \geq \bar{Q}_{B_i}(E_1)$. By duality (Prop. 3.1), $\bar{Q}_E(B) \geq \bar{Q}_{E_i}(B_i)$. This yields $Q(E) \geq \bar{Q}(E_1)$, as desired. \qed

Remark 5.18. If $E = E_1 \times E_2$ where $E_2 \subseteq K$ is not necessarily $K$, then the proof of Proposition 5.17 does not apply. Specifically, we do not have an analogue in this case of Proposition 5.15. On the other hand, for $\bar{Q} = \bar{D}$, but not for other scaled tightness quantities, the part of the argument that uses Proposition 5.10 can be replaced by the analogous argument using Proposition 5.8.

6 Multi-tiling at low level and with few fibers

Let $G$ be a finite abelian group, and $H \subseteq G$ be a subgroup. Let $K := G/H = \{k_1, \ldots, k_m\}$. Let $E \subseteq G$ multi-tile $G$ with partner $H$ at level $\ell$. Let $E_i := E \cap k_i$. Then this multi-tiling property is equivalent to $|E_i| = \ell$ for every $i$. Call $E_i$ the fibers of $E$ with respect to $H$. For any choice of $g_i \in k_i$, we say that $E_i - g_i \subseteq H$ are the translated fibers of $E$ with respect to $H$, which are unique up to translation in $H$.

6.1 Simultaneous basis

Let $G$ be a finite abelian group. We first investigate simultaneous bases for families of subsets of $H$, defined below. This concept is crucial to our main result Proposition 6.5.

Definition 6.1. For a family of subsets $E_1, E_2, \ldots, E_m \subseteq G$ of equal size, we say that $B \subseteq \hat{G}$ is a simultaneous basis for this family if $(E_i, B)$ is a basis pair for all $i$.

By the proof of Corollary 1.12, any family of one subset has a simultaneous basis. The following example shows that not every family of subsets of $G$ of equal size has a simultaneous basis. The $\mathbb{Z}_2^2$ case is due to Kolountzakis [8].

Example 6.2. Consider the three subsets of $\mathbb{Z}_2^2$

$$E_1 = \{(0,0), (0,1)\}, \quad E_2 = \{(0,0), (1,0)\}, \quad E_3 = \{(0,0), (1,1)\}.$$

Note that these sets are subspaces of $\mathbb{Z}_2^2$. Suppose that $B$ is a simultaneous basis for all the $E_i$. By translational invariance (Prop. 3.11), assume that $B = \{(0,0), b\}$ for some $b \in \mathbb{Z}_2^2$. By Proposition 3.18, $b$ cannot be in $E_i^\perp$ for any $i$. But $\bigcup_{i=1}^3 E_i^\perp = \mathbb{Z}_2^2$, a contradiction. So the $E_i$ do not have a simultaneous basis.

In general, let $p$ be a prime and consider $G = \mathbb{Z}_p^2$. Let

$$d_1 = (1,0), \quad d_2 = (0,1), \quad d_3 = (1,1), \quad d_4 = (1,2), \quad \ldots, \quad d_{p+1} = (1,p-1)$$

be all $p+1$ directions in $G$. For $1 \leq i \leq p+1$, set $E_i = \{nd_i : 0 \leq n \leq p-1\}$. Since $\bigcup_{i=1}^{p+1} E_i^\perp = G$, by an argument analogous to above, the $E_i$ do not have a simultaneous basis.
We can ask: under what circumstances is a simultaneous basis guaranteed to exist for a family of subsets of $G$ of equal size? Kolountzakis [8] observed that if the group $G$ is cyclic, then such existence is guaranteed, as the proposition below shows.

**Proposition 6.3.** Let $G = \mathbb{Z}_m$ for some $m \geq 1$. Then any family of subsets of $G$ of equal size has a simultaneous basis.

**Proof.** Let $B = \{0, 1, \ldots, k - 1\} \subseteq \hat{G}$, under the canonical identification. We show that for any $E \subseteq G$ of size $k$, $(E, B)$ is a basis pair. Writing $E = \{x_0, x_1, \ldots, x_{k-1}\}$, $T(E, B)$ is the Vandermonde matrix

$$[\exp(2\pi i/m \cdot x_ab)]_{0 \leq a, b \leq k-1}.$$ 

Because $\exp(2\pi i/m \cdot x_a)$, $0 \leq a \leq k - 1$, are distinct, the Vandermonde determinant of this matrix is nonzero, so that $(E, B)$ is a basis pair. Hence, $B$ is a simultaneous basis for the family of all subsets of $G$ of size $k$.

The condition for when a family of subsets admits a simultaneous basis may be worth studying further, but we will not pursue this in this paper.

### 6.2 Main result

In this subsection, we prove our main result relating the multi-tiling level of a set to how close the set is to being spectral. Our main result, Proposition 6.5, states the following. Let $G$ be a finite abelian group and $H \subseteq G$ a subgroup. If a subset $E \subseteq G$ has the properties:

1. $E$ multi-tiles $G$ with partner $H$ at a low level;
2. there are a few distinct translated fibers of $E$ with respect to $H$ up to translation in $H$; and
3. these translated fibers admit a simultaneous basis in $\hat{H}$,

then $E$ is not far from being spectral. Specifically, all scaled tightness quantities of $E$ have upper bounds in terms of only the multi-tiling level and the number of distinct translated fibers of $E$.

This result generalizes the result that if $E$ tiles $G$ with partner $H$ (multi-tiles at level one and so has a unique fiber up to translation in $H$), then $E$ is spectral.

We do not know if the second (few fibers) and third (simultaneous basis) hypotheses in the above can be omitted. However, we conjecture that these hypotheses are necessary.

The strategy to proving our main result is as follows. We start by using ideas of Section 4.4 to show that, given a family of subsets $E_i$ of $G$ with a simultaneous basis, we can “loop” this basis around to produce another simultaneous basis that has good spectral behavior with respect to all the $E_i$ (Prop. 6.4).

We first prove the proposition below that derives from a simultaneous basis of a family $E_i$ another simultaneous basis that has good spectral behavior with respect to all the $E_i$.

**Proposition 6.4.** Let $G$ be a finite abelian group and $M$ be the minimal exponent of $G$. Let $E_1, \ldots, E_m \subseteq G$ have equal size $n > 1$, and let $B \subseteq \hat{G}$ be a simultaneous basis for the $E_i$. Then there is a $1 \leq k \leq M$ that is relatively prime to $M$ such that

$$\prod_{i=1}^{m} D_{E_i}(kB) \geq 1.$$
Consequently, for this $k$,

\[
\begin{align*}
\min_{1 \leq i \leq m} L_{E_i}(kB) &> \left( \frac{n-1}{n} \right)^{n-1} \frac{1}{n^{m-1}} > \frac{1}{en^{m-1}}, \\
\max_{1 \leq i \leq m} U_{E_i}(kB) &< n^{2} - \frac{n-1}{n^{m-1} + \frac{m+1}{n-1}} \leq n^2 - \frac{n-1}{2^{m+1}n^{m-1}}, \\
\max_{1 \leq i \leq m} \text{cond}_{E_i}(kB) &< 2n^{mn/2}.
\end{align*}
\]

Proof. By the arguments in Proposition 4.8 and Lemma 4.10, for each $i$,

\[
\prod_{1 \leq k \leq M, \gcd(k,M)=1} D_{E_i}(kB) \geq 1.
\]

Taking the product for all $i$, we obtain

\[
\prod_{1 \leq k \leq M, \gcd(k,M)=1} \left( \prod_{i=1}^{m} D_{E_i}(kB) \right) \geq 1,
\]

whence the first result.

Now let $k$ be such that $\prod_{i=1}^{m} D_{E_i}(kB) \geq 1$. By Proposition 3.4, $D_{E_i}(kB) \leq n^{n/2}$ for each $1 \leq i \leq m$. Therefore, for each $i$,

\[
D_{E_i}(kB) \geq \frac{1}{\prod_{1 \leq j \leq m, j \neq i} D_{E_j}(kB)} \geq n^{n(1-m)/2}.
\]

Using Proposition 4.5 and calculating as in Proposition 4.11, we obtain the desired bounds.

We now turn to the proof of our main result below.

**Proposition 6.5.** Let $H \subseteq G$ be a subgroup. Let $E \subseteq G$ multi-tile $G$ with partner $H$ at level $\ell > 1$. Suppose that, up to translation in $H$, there are $k$ distinct translated fibers of $E$ with respect to $H$. Assume further that these translated fibers have a simultaneous basis in $\hat{H}$. Then

\[
\tilde{L}(E) < e^{\ell^{k\ell}}, \quad \tilde{U}(E) < \ell - \frac{\ell - 1}{2^{k+1}\ell-1}, \quad \text{cond}(E) < \sqrt{e^{(k\ell+1)/2}}, \quad \tilde{D}(E) \leq \sqrt{\ell}.
\]

In particular, all scaled tightness quantities have upper bounds that depend only on $\ell$ and $k$.

Proof. Let $E_1, E_2, \ldots, E_k \subseteq H$ be the distinct translated fibers of $E$ with respect to $H$. Let $B \subseteq \hat{H}$ be their simultaneous basis. By Proposition 6.4, there is an $s$ such that

\[
\prod_{i=1}^{k} D_{E_i}(sB) \geq 1, \quad \min_{1 \leq i \leq k} L_{E_i}(sB) > \frac{1}{e^{\ell^{k\ell-1}}}, \quad \max_{1 \leq i \leq k} U_{E_i}(sB) < \ell^2 - \frac{\ell - 1}{2^{k+1}\ell-1}.
\]

\[\square\]
Remark 6.6. In Proposition 6.5, in the case \( \ell = 1 \), which is not covered by the proposition, \( E \) is spectral by Proposition 5.13.

Remark 6.7. By taking \( H = G \) in Proposition 6.5, so that \( \ell = |E| \) and \( k = 1 \), we recover the results of Corollary 4.12 that the tightness quantities of \( E \) are bounded by expressions depending only on \( |E| \). We get exactly the same results for all tightness quantities except \( \text{cond} \), in which case the result of Corollary 4.12 is stronger.

Proposition 6.5 allows us to show that the set in the example below has good spectral behavior, a result which is not possible to derive from results in our previous sections.

Example 6.8. also works for \( \mathbb{Z}_m^2 \), \( m \) composite

Let \( p \geq 3 \) be a prime and let \( G = \mathbb{Z}_p^2 \). Let

\[
E_p = (\mathbb{Z}_p \setminus \{0\}) \times \{0, 1\} \cup \{(0, 0), (2, 0)\}.
\]

With \( H = \mathbb{Z}_p \times \{0\} \), \( E_p \) multi-tiles \( G \) with partner \( H \) at level \( \ell = 2 \). Moreover, \( E_p \) has \( k = 2 \) translated fibers \( \{(0, 0), (1, 0)\}, \{(0, 0), (2, 0)\} \subseteq H \). By Proposition 6.3, the two translated fibers have a simultaneous basis in \( \hat{H} \). Therefore, Proposition 6.5 yields

\[
\text{cond}(E_p) < 4\sqrt{2e}.
\]

It follows that the sets \( E_p \) have good spectral behavior independent of \( p \).

Compare Example 6.8 with Examples 5.6 and 5.14.

In Proposition 6.5, the hypothesis that \( E \) multi-tiles \( G \) with partner \( H \) at a low level has to be supplemented by the hypotheses that there are few translated fibers up to translation and that the translated fibers have a simultaneous basis. Thus, one may wonder whether the additional hypotheses are necessary.

Example 6.9. Let \( p \geq 3 \) be a prime, \( G_p = \mathbb{Z}_p^2 \times \mathbb{Z}_p \), and \( H = \mathbb{Z}_p^2 \times \{0\} \). Let \( E_i \subseteq H \) be the three subsets in Example 6.2, under the appropriate identification. Let

\[
F_p = (E_1 \times \{0\}) \cup (E_2 \times \{1\}) \cup (E_3 \times \{2, 3, \ldots, p - 1\}).
\]

Then \( F_p \) multi-tiles \( G_p \) with partner \( H \) at level \( \ell = 2 \) and has \( k = 3 \) distinct translated fibers \( E_i \), but the translated fibers do not admit a simultaneous basis in \( \hat{H} \). Thus, Proposition 6.5 does not apply to show that \( \text{cond}(F_p) \) is bounded independent of \( p \). We actually do not know if this statement is true.

Example 6.10. Let \( p \geq 3 \) be a prime, \( G_p = \mathbb{Z}_p^2 \), and \( H_p = \mathbb{Z}_p \times \{0\} \). Set

\[
E_p = \{(0, x) \in \mathbb{Z}_p \times \{0\} \cup \{(1, 0)\} \cup \{(x, x) : 1 \leq x \leq p - 1\}.
\]

Then \( E_p \) multi-tiles \( G_p \) with partner \( H_p \) at level \( \ell = 2 \), but has \( k = (p - 1)/2 \) distinct translated fibers. By Proposition 6.3, the translated fibers of \( E_p \) admit a simultaneous basis. So we can apply Proposition 6.5, but the result does not tell us whether \( \text{cond}(E_p) \) is bounded.
7 Continuity of the condition number

In this subsection, we investigate continuity properties of the condition number. The motivation for this is the potential lifting of our results to construct a subset \( E \) of \( \mathbb{R}^d \) with no Riesz basis. Specifically, we might aim to construct a sequence of sets \( E_i \subseteq \mathbb{R}^d \) with large condition numbers that converge in some sense to a set \( E \subseteq \mathbb{R}^d \). (We define the condition number \( \text{cond}_E(B) \) for a pair \((E, B)\) to be the square root of the ratio between the optimal lower and upper Riesz constants, and \( \text{cond}(E) = \inf_B \text{cond}_E(B) \).) If \( \text{cond}(E_i) \to \infty \), then we might hope that this will imply that \( \text{cond}(E) = \infty \), i.e., \( E \) has no Riesz basis.

In order to run this argument, the condition number must at least be upper semi-continuous in the sense that \( \text{cond}(E) \geq \limsup_{i \to \infty} \text{cond}(E_i) \). Nevertheless, we will show that, for a specific example in the context of finite abelian groups, the condition number is instead lower semi-continuous at points of discontinuity. Thus, the continuity seems to be “going the wrong way.” If similar behaviors occur in \( \mathbb{R}^d \), then there is a potential difficulty in using approximation to construct subsets of \( \mathbb{R}^d \) with no Riesz basis.

Because our setting of finite abelian groups is discrete, we will first extend the notion of condition number to subsets with density, called generalized subsets. Then we will compute condition numbers of generalized subsets of \( \mathbb{Z}_2 \) and show that the condition number is lower semi-continuous in this case.

Let \( G \) be a finite abelian group. A generalized subset \( E \) of \( G \) is a function \( E : G \to \mathbb{R}_{\geq 0} \). This can be viewed as the subset \( \text{supp} \, E \subseteq G \) where each element \( x \in \text{supp} \, E \) has “density” \( E(x) \). An ordinary subset \( E \subseteq G \) corresponds to the generalized subset \( 1_E \). A generalized subset \( E \) of \( G \) induces the measure \( \mu_E \) on \( G \) defined by \( \mu_E(F) = \sum_{x \in F} E(x) \) for any \( F \subseteq G \). Let \( L^2(E) \) be the space of functions \( f : G \to \mathbb{C} \) with the norm

\[
\|f\|_{L^2(E)}^2 = \int_G |f|^2 \, d\mu_E = \sum_{x \in G} |f(x)|^2 E(x),
\]

where we identify two functions that agree a.e. Hence, \( L^2(E) \) is a vector space of dimension \( |\text{supp} \, E| \).

A generalized subset \( B \) of \( \widehat{G} \) can be analogously defined. Let \( E \) and \( B \) be generalized subsets of \( G \) and \( \widehat{G} \), respectively. We say that \((E, B)\) is an equal-size pair if \( L^2(E) \) and \( L^2(B) \) have equal dimension, that is, \( | \text{supp} \, E | = | \text{supp} \, B | \). Analogously to Equation (2.2), define \( T(E, B) : L^2(B) \to L^2(E) \) to be the linear operator

\[
T(E, B)c = \sum_{\hat{g} \in \widehat{G}} c(\hat{g}) B(\hat{g}) \hat{g} \in L^2(E), \quad c \in L^2(B).
\]

We call \((E, B)\) a basis pair if \( T(E, B) \) is invertible. Notice that a basis pair is always an equal-size pair.

Let \((E, B)\) be an equal-size pair. Analogously to Definition 1.8, define the condition number of \( B \) with respect to \( E \) to be

\[
\text{cond}_E(B) := \text{cond}(T(E, B)) = \|T(E, B)\| \|T(E, B)^{-1}\|,
\]

where the norm is the induced operator norm and this quantity is defined to be \( \infty \) when \( T(E, B) \) is not invertible. (Similarly, we can also define \( L_E(B) \) and \( U_E(B) \), but we will not
use these quantities.) Define the condition number of $E$ to be $\text{cond} E := \inf_B \text{cond}_E(B)$, where the infimum ranges over all $B$ such that $(E, B)$ is an equal-size pair.

We now investigate condition numbers of generalized subsets of $\mathbb{R}_2$ in order to gain more insight into continuity properties of the condition number.

Let $E$ be a nonzero generalized subset of $\mathbb{R}_2$, and let $x = E(0)$ and $y = E(1)$, where $(x, y) \neq (0, 0)$. If $x = 0$ or $y = 0$, then $|\text{supp } E| = 1$. So if $(E, B)$ is an equal-size pair in this case, then $|\text{supp } B| = 1$. We can check that $\text{cond}_E(B) = 1$, and so $\text{cond } E = 1$.

Suppose now that $x, y \neq 0$. Let $\hat{\mathbb{R}}_2 = \{0, 1\}$, where $\hat{i}(j) = (-1)^{ij}$ for $i, j \in \{0, 1\}$. Let $B$ be a generalized subset of $\hat{\mathbb{R}}_2$, where $a = B(0)$ and $b = B(1)$ are both nonzero. Let $c \in L^2(B)$ with $c_0 = c(0)$ and $c_1 = c(1)$. We can compute

$$f(c_0, c_1) := \frac{\|T(E, B)c\|^2_{L^2(E)}}{\|c\|^2_{L^2(B)}} = \frac{x(c_0a + c_1b)^2 + y(c_0a - c_1b)^2}{ac_0^2 + bc_1^2}.$$ 

Thus, $\text{cond}_E(B)$ is the square root of the ratio of the supremum and infimum of $f(c_0, c_1)$ over all nonzero $(c_0, c_1) \in \mathbb{C}^2$.

By scale invariance, it suffices to consider the case $ac_0^2 + bc_1^2 = 1$. Let $\sqrt{a}c_0 = \cos \theta$ and $\sqrt{b}c_1 = \sin \theta$, for some $\theta \in \mathbb{R}$. Then we must maximize and minimize

$$f(\theta) = x(\sqrt{a}\cos \theta + \sqrt{b}\sin \theta)^2 + y(\sqrt{a}\cos \theta - \sqrt{b}\sin \theta)^2 = \frac{(x + y)(a + b)}{2} + \frac{(x + y)(a - b)}{2} \cos 2\theta + (x - y)\sqrt{ab}\sin 2\theta.$$ 

The maximum and minimum are $(x + y)(a + b)/2 \pm \sqrt{((x + y)(a - b)/2)^2 + (x - y)^2ab}$. So

$$\text{cond}_E(B) = \sqrt{\frac{1 + \sqrt{q}}{1 - \sqrt{q}}}$$

where

$$q = 1 - \frac{4ab}{(a + b)^2} \frac{4xy}{(x + y)^2}.$$ 

To minimize $\text{cond}_E(B)$, we must minimize $q$ over all nonzero $(a, b) \in \mathbb{C}^2$. This occurs when $a = b$, whence

$$\text{cond}(E) = \sqrt{\frac{\max(x, y)}{\min(x, y)}}.$$ 

So in the $(x, y)$-plane, $\text{cond}(E) = \sqrt{\cot \tau}$, where $\tau$ is the angle that the line from the origin to $(x, y)$ makes with the closer of the $x$-axis and the $y$-axis.

We observe the following. On the line $y = x$, $\text{cond}(E)$ is one. As we rotate the line closer to the $x$-axis or the $y$-axis, $\text{cond}(E)$ increases and approaches infinity. However, exactly on the $x$-axis and the $y$-axis, $\text{cond}(E)$ once again becomes one. Thus, the condition number is discontinuous at points where $|\text{supp } E|$ changes, in a way that shows $\text{cond}(E)$ is not upper semi-continuous. On the other hand, the above calculations prove $\text{cond}(E)$ is lower semi-continuous, i.e., $\text{cond}(E) \leq \liminf_{E_0 \to E} \text{cond}(E_0)$.

As stated above, the fact that the condition number is only lower semi-continuous in this example suggests a potential difficulty in using approximation to prove that a set in $\mathbb{R}^d$ has no exponential Riesz basis. However, if the lower semi-continuity persists in $\mathbb{R}^d$, then this leaves open the possibility of constructing interesting limit sets which have an exponential Riesz basis but do not multi-tile.
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