On an iteration leading to a $q$-analogue of the Digamma function

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Abstract

We show that the $q$-Digamma function $\psi_q$ for $0 < q < 1$ appears in an iteration studied by Berg and Durán. In addition we determine the probability measure $\nu_q$ with moments $1/\sum_{k=1}^{n+1}(1-q)/(1-q^k)$, which are $q$-analogues of the reciprocals of the harmonic numbers.

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1 Introduction

For a measure $\mu$ on the unit interval $[0,1]$ we consider its Bernstein transform

$$B(\mu)(z) = \int_0^1 \frac{1-t^z}{1-t} d\mu(t), \quad \Re z > 0,$$  \hspace{1cm} (1)

as well as its Mellin transform

$$M(\mu)(z) = \int_0^1 t^z d\mu(t), \quad \Re z > 0.$$  \hspace{1cm} (2)

These functions are clearly holomorphic in the right half-plane $\Re z > 0$.

The two integral transformations are combined in the following theorem from [3] about Hausdorff moment sequences, i.e., sequences $(a_n)_{n \geq 0}$ of the form

$$a_n = \int_0^1 t^n d\mu(t),$$  \hspace{1cm} (3)

for a positive measure $\mu$ on the unit interval.

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Theorem 1.1 Let \((a_n)_{n \geq 0}\) be a Hausdorff moment sequence as in (3) with \(\mu \neq 0\). Then the sequence \((T(a_n))_{n \geq 0}\) defined by \(T(a_n) = 1/(a_0 + \ldots + a_n)\) is again a Hausdorff moment sequence, and its associated measure \(\hat{T}(\mu)\) has the properties

\[
\mathcal{B}(\mu)(z + 1)\mathcal{M}(\hat{T}(\mu))(z) = 1 \quad \text{for} \quad \Re z > 0.
\]

(4)

This means that the measure \(\hat{T}(\mu)\) is determined as the inverse Mellin transform of the function \(1/\mathcal{B}(\mu)(z + 1)\).

It follows by Theorem 1.1 that \(T\) maps the set of normalized Hausdorff moment sequences (i.e., \(a_0 = 1\)) into itself. By Tychonoff’s extension of Brouwer’s fixed point theorem, \(T\) has a fixed point \((m_n)\). Furthermore, it is clear that a fixed point \((m_n)\) is uniquely determined by the equations

\[
(1 + m_1 + \ldots + m_n)m_n = 1, \quad n \geq 1.
\]

(5)

Therefore

\[
m_{n+1}^2 + \frac{m_{n+1}}{m_n} - 1 = 0,
\]

(6)

giving

\[
m_1 = \frac{-1 + \sqrt{5}}{2}, \quad m_2 = \frac{\sqrt{22 + 2\sqrt{5}} - \sqrt{5} - 1}{4}, \ldots.
\]

Similarly, \(\hat{T}\) maps the set \(M^1_\mu([0,1])\) of probability measures on \([0,1]\) into itself. It has a uniquely determined fixed point \(\omega\) and

\[
m_n = \int_0^1 t^n \, d\omega(t), \quad n = 0, 1, \ldots
\]

(7)

Berg and Durán studied this fixed point in [4,5], and it was proved that the Bernstein transform \(f = \mathcal{B}(\omega)\) is meromorphic in the whole complex plane and characterized by a functional equation and a log-convexity property in analogy with Bohr-Mollerup’s characterization of the Gamma function. Let us also mention that \(\omega\) has an increasing and convex density with respect to Lebesgue measure \(m\) on the unit interval.

An important step in the proof is to establish that \(\omega\) is an attractive fixed point so that in particular the iterates \(\hat{T}^n(\delta_1)\) converge weakly to \(\omega\). Here and in the following \(\delta_n\) denotes the Dirac measure with mass 1 concentrated in \(a \in \mathbb{R}\).

It is easy to see that \(\hat{T}(\delta_1) = m\), because

\[
T(1,1,\ldots)_n = \frac{1}{n+1} = \int_0^1 t^n \, dt.
\]

It is well-known that the Bernstein transform of Lebesgue measure \(m\) on \([0,1]\) is related to the Digamma function \(\psi\), i.e., the logarithmic derivative of the
Gamma function, since
\[ \int_0^1 \frac{1 - t^z}{1 - t} \, dt = \psi(z + 1) + \gamma = \sum_{n=1}^\infty \frac{z}{n(n + z)}, \quad \Re z > 0, \quad (8) \]
cf. [9, 8.36]. Here \( \gamma = -\psi(1) \) is Euler’s constant.

Therefore \( \nu_1 := \hat{T}(m) = \hat{T}^{\circ 2}(\delta_1) \) is determined by
\[ \mathcal{M}(\nu_1)(z) = \frac{1}{\mathcal{B}(m)(z + 1)} = \frac{1}{\psi(z + 2) + \gamma}. \]

The measure \( \nu_1 = \hat{T}(m) \) is given explicitly in [3] as
\[ \nu_1 = \left( \sum_{n=0}^\infty \alpha_n t^{\xi_n} \right) dt, \quad (9) \]
where \( \xi_0 = 0, \xi_n \in (n, n + 1), n = 1, 2, \ldots \) is the solution to \( \psi(1 - \xi_n) = -\gamma \) and \( \alpha_n = 1/\psi'(1 - \xi_n) \). The moments of the measure \( \nu_1 \) are the reciprocals of the harmonic numbers, i.e.,
\[ \int_0^1 t^n \, d\nu_1(t) = \frac{1}{\mathcal{H}_{n+1}} = \left( \sum_{k=1}^{n+1} \frac{1}{k} \right)^{-1}. \quad (10) \]

The purpose of this paper is to study the first elements of the sequence \( \hat{T}^{\circ n}(\delta_q) \), where \( 0 < q < 1 \) is fixed. The reason for excluding \( q = 0 \) is that \( \hat{T}(\delta_0) = \delta_1 \).

Since \( \omega \) is an attractive fixed point, we know that the sequence converges weakly to \( \omega \).

The first step in the iteration is easy:
\[ \hat{T}(\delta_q) = (1 - q) \sum_{k=0}^\infty q^k \delta_{q^k}, \quad (11) \]
because
\[ \int_0^1 t^z \, d\hat{T}(\delta_q)(t) = \frac{1 - q}{1 - q^{z+1}} = (1 - q) \sum_{k=0}^\infty q^k q^{kz}. \quad (12) \]

This shows that \( \hat{T}(\delta_q) \) is the Jackson \( dq \)-measure on \([0, 1] \) used in the theory of \( q \)-integrals, cf. [8]. It is a \( q \)-analogue of Lebesgue measure in the sense that \( dq \rightarrow m \) weakly for \( q \rightarrow 1 \).

It is therefore to be expected that \( \nu_q := \hat{T}(dq) = \hat{T}^{\circ 2}(\delta_q) \) is a \( q \)-analogue of the measure \( \nu_1 \), and we are going to determine \( \nu_q \) as closely as possible. We have
\[ \mathcal{M}(\nu_q)(z) = \frac{1}{f_q(z + 1)}, \quad (13) \]
where \( f_q \) is defined as the Bernstein transform of \( d_q t \):

\[
f_q(z) = \int_0^1 \frac{1-t^z}{1-t} \, d_q t = (1-q) \left( z + \sum_{k=1}^{\infty} q^k \frac{1-q^{kz}}{1-q^k} \right). \tag{14}
\]

This formula is a \( q \)-analogue of (8). The moments of \( \nu_q \) are \( q \)-analogues of (10)

\[
\int_0^1 t^n \, d\nu_q(t) = \left( \sum_{k=0}^{n} \frac{1-q}{1-q^{k+1}} \right)^{-1}. \tag{15}
\]

Our main result is the following (note that the Haar measure on the multiplicative group \( ]0, \infty[ \) is \( dt/t \)):

**Theorem 1.2** The measure \( \nu_q \) has a continuous density \( \tilde{\nu}_q(t) \) with respect to \( dt/t \) on \( ]0, 1[ \). It is \( C^\infty \) on each of the open intervals \( ]q^k, q^k[ \), \( k = 0, 1, \ldots \) with jump of the derivative of size \( q^k/(1-q^k)(1-q) \) at the point \( q^k, k = 1, 2, \ldots \). Furthermore, \( \lim_{t \to 0} \tilde{\nu}_q(t) = 0 \).

**Remark 1.3** It follows that the behaviour of \( \tilde{\nu}_q(t) \) is oscillatory, and therefore quite different from that of \( \lim_{q \to 1} \tilde{\nu}_q(t) \), which is increasing and convex. In fact, it follows from (16) that

\[
\lim_{q \to 1} \tilde{\nu}_q(t) = \sum_{n=0}^{\infty} \alpha_n t^{\xi n+1}, \quad 0 < t \leq 1.
\]

See Figure 1 and 2 which shows the graph of \( \tilde{\nu}_q(e^{-t}) \) for \( q = 0.5 \) and \( q = 0.9 \).

## 2 Proofs

Jackson’s \( q \)-analogue of the Gamma function is defined as

\[
\Gamma_q(z) = \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1-q)^{1-z},
\]

cf. [8], and its logarithmic derivative

\[
\psi_q(z) = \frac{d}{dz} \log \Gamma_q(z) = -\log(1-q) + \log q \sum_{k=0}^{\infty} \frac{q^{k+z}}{1-q^k} \tag{16}
\]

has been proposed in [11] as a \( q \)-analogue of the Digamma function \( \psi \). See also the recent paper [12]. We define the \( q \)-analogue of Euler’s constant as

\[
\gamma_q = -\psi_q(1) = \log(1-q) - \log q \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}. \tag{17}
\]
The Bernstein transform $f_q$ of $d_q t$ is given in (14), hence

$$
\frac{f_q(z)}{1 - q} = z + \sum_{k=1}^{\infty} q^k \sum_{n=0}^{\infty} q^{kn} (1 - q^{kz})
$$

$$
= z + \sum_{n=0}^{\infty} \left( \sum_{k=1}^{\infty} (q^{k(n+1)} - q^{k(n+1+z)}) \right)
$$

$$
= z + \frac{1}{\log(1/q)} (\gamma_q + \psi_q(z + 1)),
$$

which shows the close relationship with the $q$-Digamma function. We will be using another expression for $f_q(z)/(1 - q)$ derived from (14), namely

$$
\frac{f_q(z)}{1 - q} = z + c_q - \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} q^{kz},
$$

(18)

with

$$
c_q = \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k}.
$$

(19)

Clearly, $q/(1 - q) < c_q < q/(1 - q)^2$ for $0 < q < 1$ and $q \mapsto c_q$ is a strictly increasing map of $]0, 1[$ onto $]0, \infty[$. We mention two other expressions

$$
c_q = \sum_{n=1}^{\infty} d(n) q^n = \sum_{n=1}^{\infty} (1 - (q^n; q)_\infty),
$$

where $d(n)$ is the number of divisors in $n$, see [7, p. 14].

In order to replace the Mellin transformation by the Laplace transformation we introduce the probability measure $\tau_q$ on $[0, \infty[$ which has $\nu_q$ as image measure under $t \mapsto e^{-t}$, hence

$$
\mathcal{L}(\tau_q)(z) = \int_{0}^{\infty} e^{-tz} d\tau_q(t) = \frac{1}{f_q(z + 1)}.
$$

The analogue of Theorem 1.2 about the measure $\tau_q$ is given in the next theorem, which we shall prove first.

**Theorem 2.1** The measure $\tau_q$ has a continuous density also denoted $\tau_q$ with respect to Lebesgue measure on $[0, \infty[$. It is $C^\infty$ in each of the open intervals $]n \log(1/q), (n + 1) \log(1/q)[, n = 0, 1, \ldots$ with jump of the derivative of size

$$
J_n = \frac{q^{2n}}{(1 - q^n)(1 - q)}
$$

(20)

at the point $n \log(1/q), n = 1, 2, \ldots$. Furthermore, $\lim_{t \to \infty} \tau_q(t) = 0$. 

5
Proof of Theorem 2.1} Introducing the discrete measure

\[ \mu = \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^k} \delta_{k \log(1/q)} \]

of finite total mass

\[ ||\mu||_1 = c_q - q/(1-q) < c_q, \quad (21) \]

we can write

\[ \frac{f_q(z + 1)}{1-q} = 1 + c_q + z - \mathcal{L}(\mu)(z), \]

hence

\[ \frac{1-q}{f_q(z + 1)} = \left( (1 + c_q + z)(1 - \frac{\mathcal{L}(\mu)(z)}{1+c_q+z}) \right)^{-1} = \sum_{n=0}^{\infty} \frac{(\mathcal{L}(\mu)(z))^n}{(1+c_q+z)^{n+1}}. \quad (22) \]

Let \( \rho_q \) denote the following exponential density restricted to the positive half-line

\[ \rho_q(t) = \exp(-(1+c_q)t)Y(t), \]

where \( Y \) is the usual Heaviside function equal to 1 for \( t \geq 0 \) and equal to zero for \( t<0 \). Its Laplace transform is given as

\[ \int_0^{\infty} e^{-tz} \rho_q(t) \, dt = (1+c_q+z)^{-1}, \]

but this shows that (22) is equivalent to the following convolution equation

\[ (1-q)\tau_q = \rho_q * \sum_{n=0}^{\infty} (\mu * \rho_q)^*n = \sum_{n=0}^{\infty} \rho_q^{*(n+1)} * \mu^*n. \quad (23) \]

This equation expresses a factorization of \( (1-q)\tau_q \) as the convolution of the exponential density \( \rho_q \) and an elementary kernel \( \sum_{0}^{\infty} N^{*n} \) with \( N = \mu * \rho_q \). For information about the basic notion of elementary kernels in potential theory, see [6, p.100]. All three measures in question \( \tau_q, \rho_q \) and \( \sum_{0}^{\infty} (\mu * \rho_q)^*n \) are potential kernels on \( \mathbb{R} \) in the sense of [6].

The measure \( \mu^*n, n \geq 1 \) is a discrete measure concentrated in the points \( k \log(1/q), k = n, n+1, \ldots \). The convolution powers of \( \rho_q \) are Gamma densities

\[ \rho_q^{*(n+1)}(t) = \frac{t^n}{n!} e^{-(1+c_q)t} Y(t), \]

as is easily seen by Laplace transformation.

Clearly, \( \rho_q * \mu \) is a bounded integrable function with

\[ ||\rho_q * \mu||_{\infty} \leq ||\rho_q||_{\infty} ||\mu||_1 < c_q, \quad ||\rho_q * \mu||_1 = ||\rho_q||_1 ||\mu||_1 < \frac{c_q}{1+c_q}, \quad (24) \]
and then $\rho_q \ast (\rho_q \ast \mu)^n$, $n \geq 1$ is a continuous integrable function on $\mathbb{R}$, vanishing for $t \leq n \log(1/q)$ and for $t \to \infty$. Furthermore,

$$||\rho_q \ast (\rho_q \ast \mu)^n||_\infty < (c_q/(1 + c_q))^n,$$

and this shows that the right-hand side of (23) converges uniformly on $[0, \infty[$, so $(1-q)^{\tau_q}$ has a continuous density on $[0, \infty[$ tending to 0 at infinity.

For $n \geq 1$ and $x \in [n \log(1/q), \infty[$ we get

$$\rho_q^{(n+1)} \ast \mu^n(x) = \int_0^x \frac{(x - t)^n}{n!} e^{-(1+c_q)(x-t)} Y(x - t) d\mu^n(t) = e^{-(1+c_q)x} \sum_{k=n}^{\infty} \frac{(x - k \log(1/q))^n}{n!} q^{-k(1+c_q)} Y(x - k \log(1/q)) \mu^n(k \log(1/q))$$

which is a finite sum, and

$$\mu^n(k \log(1/q)) = \sum_{p_1, \ldots, p_n = k} \prod_{j=1}^n \frac{q^{2p_j}}{1 - q^{p_j}}, \quad k = n, n+1, \ldots.$$  

In particular,

$$\mu^n(n \log(1/q)) = \left(\frac{q^2}{1 - q}\right)^n.$$  

For $n \geq 0$ and $0 \leq x < (n + 1) \log(1/q)$ we then get

$$(1-q)^{\tau_q}(x) = e^{-(1+c_q)x} \sum_{j=0}^{\infty} q^{-j(1+c_q)} Y(x - j \log(1/q)) \sum_{k=0}^{j} \frac{(x - j \log(1/q))^k}{k!} \mu^k(j \log(1/q)).$$

On $[0, \log(1/q)[$ it is equal to $\exp(-(1+c_q)x)$, on $[\log(1/q), 2\log(1/q)[$ it is equal to

$$\exp(-(1+c_q)x) \left(1 + \frac{q^{1-c_q}}{1 - q}(x - \log(1/q))\right),$$

on $[2\log(1/q), 3\log(1/q)[$ it is equal to

$$\exp(-(1+c_q)x) \left(1 + \frac{q^{1-c_q}}{1 - q}(x - \log(1/q)) + \frac{q^{2(1-c_q)}}{1-q^2}(x - 2 \log(1/q)) + \frac{q^{2(1-c_q)}}{2(1-q)^2}(x - 2 \log(1/q))^2\right),$$

e tc.

Using the expression (25) it is possible to calculate the derivative of $(1-q)^{\tau_q}$ from the right and from the left at the point $x = n \log(1/q), n \geq 1$. The difference
between the right and the left derivative equals \( q^{2n}/(1 - q^n) \) and this gives the jump \( J_n \) of (20). □

It is straightforward to transfer the results of Theorem 2.1 to give Theorem 1.2 using that \( \nu_q \) is the image measure of \( \tau_q \) under \( t \mapsto e^{-t} \), hence \( \tilde{\nu}_q(t) = \tau_q(\log(1/t)), \) 0 < \( t \leq 1 \).

**Remark 2.2** The representation (22) and Theorem 2.1 show that \( 1 - q \tau_q \) is a standard \( p \)-function in the terminology from the theory of regenerative phenomena, cf. [10].

![Figure 1: The graph of \((1 - q)\tau_q\) on \([0, 3 \log(1/q)]\) for \( q = 0.5 \)](image)

**3 Further properties of \( \tau_q \)**

Formally, by Fourier inversion we get that

\[
\tau_q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \frac{dy}{f_q(1 + iy)}.
\]
The function $1/f_q(1 + iy)$ is a non-integrable $L^2$-function, so the formula holds in the $L^2$-sense. To see this we notice that

$$f_q(z) = 1 - q + \int_0^\infty e^{-tz}h_q(t)\,dt, \quad \Re z > 0,$$

where

$$h_q(t) = (1 - q) \sum_{k>t/\log(1/q)} \frac{q^k}{1 - q^k}.$$  

(26)

In particular

$$f_q(1 + iy) = 1 - q + \int_0^\infty e^{-ity}e^{-t}h_q(t)\,dt,$$

and since $e^{-t}h_q(t)$ is integrable, it follows from the Riemann-Lebesgue Lemma that we get the asymptotic behaviour

$$f_q(1 + iy) \sim (1 - q)(1 + iy), \quad |y| \to \infty.$$  

(27)

Furthermore, we notice that

$$\Re f_q(1 + iy) = 1 - q + (1 - q) \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} (1 - q^k \cos(ky\log(q))),$$

where

Figure 2: The graph of $(1 - q)\tau_q$ on $[0, 3\log(1/q)]$ for $q = 0.9$
hence

\[ 1 \leq \Re f_q(1 + iy) \leq 1 - q + \sum_{k=1}^{\infty} q^k(1 + q^k), \]

showing that \( \Re f_q(1 + iy) \) is bounded below and above. It follows that the symmetrized density

\[
\varphi_q(x) = \begin{cases} 
\tau_q(x) & \text{if } x \geq 0, \\
\tau_q(-x) & \text{if } x < 0,
\end{cases}
\]

is the Fourier transform of the non-negative integrable function

\[
\frac{2\Re f_q(1 + iy)}{|f_q(1 + iy)|^2},
\]

and therefore \( \varphi_q(x) \) is continuous and positive definite, so \( \tau_q \) is the restriction to \([0, \infty[\) of such a function.

**Remark 3.1** The function \( f_q \) defined in (14) is a Bernstein function in the sense of [6], but not a complete Bernstein function in the sense of [13], because \( f_q(z)/z \) is not a Stieltjes function as shown by formula (26). This is in contrast to

\[
\lim_{q \to 1} f_q(z) = \psi(z + 1) + \gamma,
\]

which is a complete Bernstein function, cf. [3].

### 4 Relation to other work

The transformation \( T \) can be extended from normalized Hausdorff moment sequences to the set \( \mathcal{K} = [0, 1]^N \) of sequences \( (x_n) = (x_n)_{n \geq 1} \) of numbers from the unit interval \([0, 1]\). This was done in [2], where \( T : \mathcal{K} \to \mathcal{K} \) is defined by

\[
(T(x_n))_n = \frac{1}{1 + x_1 + \ldots + x_n}, \quad n \geq 1.
\]

The connection is that a normalized Hausdorff moment sequence \( (a_n)_{n \geq 0} \) is considered as the element \( (a_n)_{n \geq 1} \in \mathcal{K} \).

Since \( T \) is a continuous transformation of the compact convex set \( \mathcal{K} \) in the space \( \mathbb{R}^N \) of real sequences equipped with the product topology, it has a fixed point by Tychonoff’s theorem, and this is \( (m_n)_{n \geq 1} \).

There is no reason a priori that the fixed point \( (m_n) \) of (29) should be a Hausdorff moment sequence, but as we have seen above, the motivation for the study of \( T \) comes from the theory of Hausdorff moment sequences.
Although $T$ is not a contraction on $\mathcal{K}$ in the natural metric
\[
d((a_n), (b_n)) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|, \quad (a_n), (b_n) \in \mathcal{K},
\]

it was proved in [2] that $T$ maps $\mathcal{K}$ into the compact convex subset
\[
\mathcal{C} = \{(a_n) \in \mathcal{K} \mid a_1 \geq \frac{1}{2}\},
\]

and the restriction of $T$ to $\mathcal{C}$ is a contraction. It is therefore possible to infer that
$(m_n)$ is an attractive fixed point from the fixed point theorem of Banach.

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