Model-Free Reinforcement Learning: from Clipped Pseudo-Regret to Sample Complexity

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Abstract

In this paper we consider the problem of learning an \( \epsilon \)-optimal policy for a discounted Markov Decision Process (MDP). Given an MDP with \( S \) states, \( A \) actions, the discount factor \( \gamma \in (0, 1) \), and an approximation threshold \( \epsilon > 0 \), we provide a model-free algorithm to learn an \( \epsilon \)-optimal policy with sample complexity \( \tilde{O}(\frac{S A \ln(1/p)}{\epsilon^2(1-\gamma)^4}) \) and success probability \( (1 - p) \). For small enough \( \epsilon \), we show an improved algorithm with sample complexity \( \tilde{O}(\frac{S A \ln(1/p)}{\epsilon^3(1-\gamma)^3}) \). While the first bound improves upon all known model-free algorithms and model-based ones with tight dependence on \( S \), our second algorithm beats all known sample complexity bounds and matches the information theoretic lower bound up to logarithmic factors.

1 Introduction

Reinforcement learning (RL) [5] studies the problem of how to make sequential decisions to learn and act in unknown environments (which is usually modeled by a Markov Decision Process (MDP)) and maximize the collected rewards. There are mainly two types of algorithms to approach the RL problems: model-based algorithms and model-free algorithms. Model-based RL algorithms keep explicit description of the learned model and make decisions based on this model. In contrast, model-free algorithms only maintain a group of value functions instead of the complete model of the system dynamics. Due to their space- and time-efficiency, model-free RL algorithms have been getting popular in a wide range of practical tasks (e.g., DQN [16], TRPO [17], and A3C [15]).

In RL theory, model-free algorithms are explicitly defined to be the ones whose space complexity is always sublinear relative to the space required to store the MDP parameters [12]. For tabular MDPs (i.e., MDPs with finite number of states and actions, usually denoted by \( S \) and \( A \) respectively), this requires that the space complexity to be \( o(S^2 A) \). Motivated by the empirical effectiveness of model-free algorithms, the intriguing question of whether model-free algorithms can be rigorously proved to perform as well as the model-based ones has attracted much attention and been studied in the settings such as regret minimization for episodic MDPs [2,12,23].

In this work, we study the PROBABLY-APPROXIMATELY-CORRECT-RL (PAC-RL) problem, i.e., to designing an algorithm for learning an approximately optimal policy. We will focus on designing

\[1\] In this work, the notation \( \tilde{O}(\cdot) \) hides poly-logarithmic factors of \( S, A, 1/(1-\gamma) \), and \( 1/\epsilon \).

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the model-free algorithms, and under the model of discounted tabular MDPs with a discount factor $\gamma$. The RL algorithm runs for infinitely many time steps. At each time step $t$, the RL agent learns a policy $\pi_t$ based on the information collected before time $t$, observes the current state $s_t$, makes an action $a_t = \pi_t(s_t)$, receives the reward $r_t$, and transits to the next state $s_{t+1}$ according to the underlying environments. The goal of the agent is to learn the policy $\pi_t$ at each time $t$ so as to maximize the $\gamma$-discounted accumulative reward $V^\pi_t(s_t)$. More concretely, we wish to minimize the sample complexity for the agent to learn an $\epsilon$-optimal policy, which is defined to be the number of time steps that $V^\pi_t(s_t) < V^*(s_t) - \epsilon$, where $V^*$ is the optimal discounted accumulative reward that starts with $s_t$, and the formal definitions of both $V^\pi$ and $V^*$ can be found in Section 2.

The PAC-RL addresses the important problem about how many trials are required to learn a good policy. We also note that in the PAC-RL definition, the exploration at each time step has to align with the learned policy (i.e., $a_t = \pi_t(s_t)$). This is stronger than the usual PAC learning definition in other online learning settings such as multi-armed bandits (see, e.g., 2) and PAC-RL with a simulator (see Section 2), where the exploration actions can be arbitrary and may incur a large regret compared to the optimum.

Quite a few algorithms have been proposed over the past nearly two decades for the PAC-RL problem. For model-based algorithms, MoRmax [21] achieves the $\tilde{O}(\frac{S^2 A \ln(1/p)}{\epsilon^2 (1-\gamma)^p})$ sample complexity, and UCRL-\(\gamma\) [14] achieves $\tilde{O}(\frac{S^2 A \ln(1/p)}{\epsilon^2 (1-\gamma)^p})$. It is also worthwhile to mention that R-max [4] was designed for learning the more general stochastic games and achieves the $\tilde{O}(\frac{S^2 A \ln(1/p)}{\epsilon^2 (1-\gamma)^p})$ sample complexity in our setting (as analyzed in [13]). Unfortunately, none of these algorithms matches the information theoretical lower bound $\Omega(\frac{S^2 A}{\epsilon^2 (1-\gamma)^p})$ proved by [14]. On the model-free side, known bounds are even less optimal – the delayed $Q$-learning algorithm proposed by [20] achieves the sample complexity of $\tilde{O}(\frac{S A \ln(1/p)}{\epsilon (1-\gamma)^p})$, and recent work [22] made an improvement to $\tilde{O}(\frac{S A \ln(1/p)}{\epsilon^2 (1-\gamma)^p})$ via a more carefully designed $Q$-learning variant.

1.1 Our Results

We design a model-free algorithm that achieves asymptotically optimal sample complexity, as follows.

**Theorem 1.** We present a model-free algorithm UCB-MULTISTAGE-ADVANTAGE, such that given a discounted MDP with $S$ states, $A$ actions, and the discount factor $\gamma$, for any approximation threshold $\epsilon \in (0, 1/poly(S, A, 1/(1-\gamma)))$ and failure probability parameter $p$, with probability $(1-p)$, the sample complexity to learn an $\epsilon$-optimal policy with UCB-MULTISTAGE-ADVANTAGE is bounded by $\tilde{O}(\frac{S A \ln(1/p)}{\epsilon^2 (1-\gamma)^p})$.

In the theorem statement, $poly(S, A, 1/(1-\gamma))$ stands for a universal polynomial that is independent of the MDP. Our UCB-MULTISTAGE-ADVANTAGE algorithm uses only $O(SA)$ space, and its time complexity per time step is $O(1)$. For asymptotically small $\epsilon$, the sample complexity of UCB-MULTISTAGE-ADVANTAGE matches the information theoretic lower bound, and improves upon all known algorithms in literature, even including the model-based ones. In Appendix A, we present a tabular view of the comparison between our algorithms and the previous works.

To prove Theorem 1 we make two main technical contributions. The first one is a novel relation between sample complexity and the so-called clipped pseudo-regret, which can also be viewed as the clipped Bellman error of the learned value function and policy at each time step. This relation enables us to reduce the sample complexity analysis to bounding the clipped pseudo-regret. Our second technique is a multi-stage update rule, where the visits to each state-action pair are partitioned according to two types of stages. An update to the $Q$-function is triggered only when a stage of either type has concluded. The lengths of the two types of stages are set by different choices of parameters so that we can reduce the clipped pseudo-regret while still maintaining a decent rate to learn the value function. Finally, we also spend much technical effort to incorporate the variance reduction technique for RL via reference-advantage decomposition introduced in the recent work [23].

A more detailed overview of our techniques is available in Section 3. Since the proof of Theorem 1 is rather involved, we will first provide a proof of the following weaker statement, and defer the full proof of Theorem 1 to Appendix D.
Theorem 2. We present a simpler model-free algorithm UCB-MULTISTAGE, such that for any approximation threshold \( \epsilon \in (0, \frac{1}{1-\gamma}) \) and any failure probability parameter \( p \), with probability \( 1 - p \), the sample complexity to learn an \( \epsilon \)-policy with UCB-MULTISTAGE is bounded by \( \tilde{O}(\frac{SA\ln(1/p)}{\epsilon^5(1-\gamma)^{5/2}}) \).

Note that the sample complexity bound in Theorem 2 holds for every possible \( \epsilon \). Although the dependency on \( \gamma \) becomes \( (1-\gamma)^{-5.5} \), UCB-MULTISTAGE still beats all known model-free algorithms and model-based algorithms with tight dependence on \( S \) and \( A \). The proof of Theorem 2 does not rely on the variance reduction technique based on reference-advantage decomposition, but is sufficient to illustrate both of our main technical contributions.

1.2 Additional Related Works

The PAC-RL problem has also been extensively studied under the setting of finite-horizon episodic MDPs \[10, 7, 8\], where the sample complexity is defined as the number of episodes in which the policy is not \( \epsilon \)-optimal. Assuming \( H \) is the length of an episode, the optimal sample complexity bound is \( \tilde{O}(\frac{SAH^2\ln(1/p)}{\epsilon^2}) \), proved by \[8\].

Much effort has also been made to study the PAC learning problem for discounted infinite-horizon MDPs, with the access to a generative model (a.k.a., a simulator). In this problem, the agent can query the simulator to draw a sample \( s' \sim P(\cdot|s,a) \) for any state-action pair \( (s,a) \), and the goal is to output an \( \epsilon \)-optimal policy (with probability \( 1-p \)) at the end of the algorithm. This problem has been studied in \[10, 1, 2, 19, 18\], and \[18\] achieves the almost tight sample complexity \( \tilde{O}(\frac{SA\ln(1/p)}{\epsilon^5(1-\gamma)^{5/2}}) \).

2 Preliminaries

A discounted Markov Decision Process is given by the five-tuple \( M = \langle S, A, P, r, \gamma \rangle \), where \( S \times A \) is the state-action space, \( P \) is the transition probability matrix, \( r \) is the deterministic reward function, and \( \gamma \in (0, 1) \) is the discount factor. The RL agent interacts with the environment for infinite number of times. At the \( t \)-th time step, the agent observes \( s_t \), executes \( a_t = \pi_t(s_t) \), receives the reward \( r(s_t, a_t) \), and then transits to \( s_{t+1} \) according to \( P(\cdot|s_t, a_t) \).

Given a deterministic stationary policy \( \pi : S \to A \), the value function and \( Q \) function are defined as

\[
V^\pi(s) = \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, \pi(s_t)) \mid s_1 = s, s_{t+1} \sim P(\cdot|s_t, \pi(s_t)) \right]
\]

\[
Q^\pi(s, a) = r(s, a) + \gamma P(\cdot|s, a)^T V^\pi = r(s, a) + P_{s,a} V^\pi,
\]

where we use \( xy \) to denote \( x^T y \) for \( x \) and \( y \) of the same dimension and use \( P_{s,a} \) to denote \( P(\cdot|s, a) \) for simplicity. The optimal value function is given by \( V^*(s) = \sup_a V^\pi(s) \) and the optimal \( Q \)-function is defined to be \( Q^*(s, a) = r(s, a) + P_{s,a} V^* \) for any \( (s, a) \in S \times A \). The goal is to minimize the sample complexity with approximation threshold parameter \( \epsilon \), which is defined as follows.

Definition 1 (Sample complexity). Given an algorithm \( \mathcal{G} \) and \( \epsilon \in (0, \frac{1}{1-\gamma}) \), the sample complexity to learn an \( \epsilon \)-optimal policy with \( \mathcal{G} \) is \( \sum_{t=1}^{\infty} \mathbb{P}[V^*(s_t) - V^\pi_t(s_t) > \epsilon] \).
Reducing Sample Complexity to Bounding the Clipped Pseudo-Regret. For any time \( t \), define the pseudo-regret vector \( \phi_t \) to be the vector such that \( \phi_t(s) = V_t(s) - (r(s, \pi_t(s)) + \gamma P_{s, \pi_t(s)} V_t) \).

We now outline our first technical idea that the sample complexity can be bounded by the total clipped pseudo-regret, approximately in the form of (2) (up to a \( \epsilon^{-1} \) factor and an additive error term).

Note that \( \phi_t \) can also be viewed as the Bellman error vector of the value function \( V_t \) and the policy \( \pi_t \). Let \( P_{\pi_t} \) be the matrix such that \( P_{\pi_t}(s) = P_{s, \pi_t(s)} \) for any \( s \in S \). By Bellman equation we have that

\[
V_t - V^{\pi_t} = \gamma P_{\pi_t}(V_t - V^{\pi_t}) + \phi_t = (\gamma P_{\pi_t})^2(V_t - V^{\pi_t}) + \gamma P_{\pi_t} \phi_t + \phi_t = \cdots = \sum_{i=0}^{\infty} (\gamma P_{\pi_t})^i \phi_t.
\]

Therefore, if \( V_t(s_t) - V^{\pi_t}(s_t) > \epsilon \), then by an averaging argument we have that for any \( M > 1 \), \( \mathbf{1}_{s_t}^T \sum_{i=0}^{\infty} (\gamma P_{\pi_t})^i \text{clip}(\phi_t, \epsilon (1-\gamma)/M) > \frac{(M-1)\epsilon}{M} \), where \( \mathbf{1}_{s_t} \) is the unit vector with the only non-zero entry at \( s_t \), and we define \( \text{clip}(x, y) = xI[x \geq y] \) for \( x, y \in \mathbb{R} \) and \( \text{clip}(x, y) = [\text{clip}(x_1, y), \ldots, \text{clip}(x_n, y)]^T \) for \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \). For any \( H = \Theta(\ln((1-\gamma)/\epsilon)) \), it then follows that

\[
\mathbb{E}[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon] \leq O\left( H \cdot \sum_{t \geq 1} \text{clip}(\phi_t(s_t), \epsilon (1-\gamma)/M) \right).
\]

We now sum up (1) over all time steps \( t \). If we can carefully design the algorithm so that \( \pi_t, V_t \) (and therefore \( \phi_t \)) do not change frequently, we have \( \pi_t = \pi_{t+i} \) and \( \phi_t = \phi_{t+i} \) for small enough \( i \) and most \( t \), and we can upper bound \( \sum_{t \geq 1} \mathbb{E}[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon] \) by the order of

\[
\sum_{t \geq 1} \mathbb{E}[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon] \leq O(H) \cdot \sum_{t \geq 1} \text{clip}(\phi_t(s_t), \epsilon (1-\gamma)/M),
\]

where the approximation (2) also uses the assumption that \( \pi_t = \pi_{t+i} \) and \( \phi_t = \phi_{t+i} \) hold for most \( t \) and \( i \). In Lemma 4, we formalize this intuition and show that if we set \( M = 8H(1-\gamma) \), the sample complexity \( \sum_{t \geq 1} \mathbb{E}[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon] \) can be upper bounded by \( O(H/\epsilon) \cdot \sum_{t \geq 1} \text{clip}(\phi_t(s_t), \epsilon (1-\gamma)/M) \) (plus an additive error), and therefore we only need to upper bound the total clipped pseudo-regret.

The Multi-Stage Update Rule. We propose a multi-stage update rule for the value and Q-function. For each state-action pair \((s, a)\), the samples are partitioned into consecutive stages. When a stage is filled, we update \( Q(s, a) \) and \( V(s) \) according to the samples in the stage via the usual value iteration method. The most interesting aspect about our method is that two types of stages, namely the type-I and type-II stages, are introduced. More concretely, the length of the \( j \)-th type-I stage is roughly \( \tilde{\epsilon}_j \approx H(1 + 1/H)^{1/B} \) and the length of the \( j \)-th type-II stage is roughly \( \tilde{\epsilon}_j \approx H(1 + 1/H)^j \), where the more precise definition and detailed description of how these stages are incorporated in the algorithm are provided in Section 4 and \( B \geq 1 \) will be set later. (Also note that throughout the paper we will use ‘\( \cdot \)’ to denote the quantities related to the type-I stage, and use ‘\( \cdot \)’ to denote the quantities related to the type-II stage.)

We note that the recent work [23] designed a (single-)stage-based model-free RL algorithm for regret minimization. Our type-II stage is similar to their work, and its goal is to make sure that the value function is learned at a decent rate. In contrast, our type-I stage is new: it is shorter than the type-II stage, so that triggers more frequent updates and helps to reduce the difference between the value functions learned in neighboring type-I stages. The two types of stages work together to reduce the clipped pseudo-regret, and therefore achieve low sample complexity.

To better explain the intuition and motivate the type-I stage, let us consider a fixed state-action pair \((s, a)\). Suppose at time step \((t - 1)\), \((s, a)\) is visited and the visit number reaches the end of a type-I stage, then the following update is triggered:

\[
Q_t(s, a) \leftarrow \min \{ r(s, a) + \tilde{b} + \frac{\gamma}{n} \sum_{i=1}^{n} V_t(s_{t_i} + 1), Q_{t-1}(s, a) \},
\]
We first focus on the second term (5) becomes much bigger. In the next subsection, we discuss how to deal with this problem via the variance reduction method, which leads to the asymptotically near-optimal bound in Theorem 1.

Bounding the first term of (5). We first focus on the second term \(P_{s,a}(V_i - V_T)\) in (5). For each \(j\), let \(t_j = t_j(s,a)\) be the start time of the j-th stage of \((s,a)\). The total contribution of the first term in (5) is bounded by the order of

\[
\sum_{s,a} \sum_j \hat{\epsilon}_j P_{s,a}(V_{t_{j-1}(s,a)} - V_{t_{j+1}(s,a)}), (1-\gamma)/(2M)).
\]  

We now discuss how to deal with the two terms in (5), and how the parameter \(B\) affects the bounds. **Bounding the second term of (5)**. We first focus on the second term \(P_{s,a}(V_i - V_T, (1-\gamma)/(2M))\) in (5). For each \(j\), let \(t_j = t_j(s,a)\) be the start time of the j-th stage of \((s,a)\). The total contribution of the second term in (5) is bounded by the order of

\[
\sum_{s,a} \sum_j \hat{\epsilon}_j P_{s,a}(V_{t_{j-1}(s,a)} - V_{t_{j+1}(s,a)}), (1-\gamma)/(2M)).
\]  

Thanks to the updates triggered by the type-II stages, \(V_i\) converges to \(V^*\) at a rate that is independent of \(B\). Increasing \(B\) will shorten the length of the type-I stages, making \(V_{t_{j-1}(s,a)}\) closer to \(V_{t_{j+1}(s,a)}\), and reduce the magnitude of (6). In Lemma 6 we formalize this intuition and show that when \(M = 8H(1-\gamma)\), (6) can be upper bounded by \(\hat{O}(SAH^5 \ln(1/p)/(\epsilon B))\). Therefore, choosing a large enough \(B\) will eliminate the \(H\) factors in the numerator.

Bounding the first term of (5). On the other hand, however, a larger \(B\) means smaller number of samples in the type-I stages, leads to a bigger estimation variance, and therefore forces us to choose a greater exploration bonus \(\hat{b}\). More precisely, using the design of \(\hat{b}\) defined in Algorithm 1, the total contribution of the first term in (5) is \(\hat{O}(SAB \ln(1/p)/(\epsilon(1-\gamma)^4))\). We have to choose \(B = \Theta(\sqrt{H})\) to achieve the optimal balance between the two terms in (5). Together with the \(H\) factor in (2), this leads to the \((1-\gamma)^{-5.5}\) factor in Theorem 1.

To utilize the full power of our multi-stage update rule, we would like to set \(B = \Theta(H^3)\), so that (6) can be upper bounded by \(\hat{O}(SAH^2 \ln(1/p)/\epsilon)\) (plus lower order terms). However, the first term in (5) becomes much bigger. In the next subsection, we discuss how to deal with this problem via the variance reduction method, which leads to the asymptotically near-optimal bound in Theorem 1.

**Variance Reduction via Reference-Advantage Decomposition.** As discussed above, when \(B\) is set large, we suffer bigger estimation variance, as fewer samples are allowed in the type-I stages. In model-free regret minimization tasks, similar problem arises where the algorithm (e.g., [12]) can only use the recent tiny fraction of the samples and incurs sub-optimal dependency on the episode length. Recent work [23] resolves this problem via the reference-advantage decomposition technique.

The high-level idea is that, assuming we have a \(\delta\)-accurate estimation of \(V^*\), namely the reference value function \(V^\text{ref}\), such that \(\|V^\text{ref} - V^*\|_\infty \leq \delta\), we only need to use the samples to estimate the difference \(V^\text{ref} - V^*\), which is called the advantage. Therefore, the estimation error (incurred in places such as (3)) will be much smaller when \(\delta\) is small. Choosing \(\delta = 1/\sqrt{B}\), and together with the Bernstein-type exploration bonus (see, e.g., [4][12]), we are able to bound the total contribution of the first term in (5) by \(\hat{O}(SA/(\epsilon(1-\gamma)^2))\), which (together with the \(H\) factor in (2)) aligns with \(^4\)More precisely, we refer to the total contribution related to the exploration bonus, which is actually in a different form from the first term in (5). This is because \(b\) has to be re-designed using the Bernstein-type exploration bonus technique and evolves to a more complex expression. Please refer to Appendix [12] for more explanation.
Algorithm 1 UCB-MultiStage

\begin{algorithm}
\textbf{Initialize:} \(V(s, a) \in \mathcal{S} \times \mathcal{A}: Q(s, a) \leftarrow \frac{1}{1-\gamma}, N(s, a), \overline{N}(s, a), \overline{\mu}(s, a), \overline{\mu}(s, a) \leftarrow 0;\)
\begin{algorithmic}
\For {\(t = 1, 2, 3, \ldots\)}
\State Observe \(s_t;\)
\State Take action \(a_t = \arg \max_a Q(s_t, a)\) and observe \(s_{t+1};\)
\Comment{Maintain the statistics}
\State \((s, a, s') \leftarrow (s_t, a_t, s_{t+1});\)
\State \(n := N(s, a) \leftarrow N(s, a) + 1;\)
\State \(\overline{n} := \overline{N}(s, a) \leftarrow \overline{N}(s, a) + 1, \quad \overline{\mu} := \overline{\mu}(s, a) \leftarrow \overline{\mu}(s, a) + V(s');\)
\State \(\overline{n} := \overline{N}(s, a) \leftarrow \overline{N}(s, a) + 1, \quad \overline{\mu} := \overline{\mu}(s, a) \leftarrow \overline{\mu}(s, a) + V(s');\)
\Comment{Update triggered by a type-I stage}
\If {\(n \in \mathcal{L}\)}
\State \(\bar{b} \leftarrow \min \{2\sqrt{H^2/\overline{n}}, 1/(1-\gamma)\};\)
\State \(Q(s, a) \leftarrow \min \{r(s, a) + \gamma (\overline{\mu}/\overline{\mu} + \bar{b}), Q(s, a)\};\)
\State \(N(s, a) \leftarrow 0; \quad \overline{\mu}(s, a) \leftarrow 0; \quad V(s) \leftarrow \max_a Q(s, a);\)
\EndIf
\Comment{Update triggered by a type-II stage}
\If {\(n \in \mathcal{L}\)}
\State \(\bar{b} \leftarrow \min \{2\sqrt{H^2/\overline{n}}, 1/(1-\gamma)\};\)
\State \(Q(s, a) \leftarrow \min \{r(s, a) + \gamma (\overline{\mu}/\overline{\mu} + \bar{b}), Q(s, a)\};\)
\State \(N(s, a) \leftarrow 0; \quad \overline{\mu}(s, a) \leftarrow 0; \quad V(s) \leftarrow \max_a Q(s, a);\)
\EndIf
\EndFor
\end{algorithmic}
\end{algorithm}

the \((1-\gamma)^{-3}\) factor in the bound of Theorem 1. The discussion till now is based on the access of the reference value function \(V^\text{ref}\). In reality, however, we need to learn the reference value function on the fly. This will incur an additive warm-up cost that polynomially depends on \(1/\delta\). However, since \(\delta\) is independent of \(\epsilon\), the extra cost is only a lower-order term. This technique is only used in the proof of Theorem 1 which is deferred to Appendix D due to space constraints.

4 The UCB-MultiStage Algorithm

In this section, we introduce the UCB-MultiStage algorithm. The algorithm takes \(\mathcal{S}, \mathcal{A}, \gamma, \epsilon\), sets \(H = \max \{\frac{\ln(8/(1-\gamma))}{\ln(1/\gamma)}, \frac{1}{1-\gamma}\}\) and \(B = \sqrt{H}\). Throughout the paper, we set \(\epsilon = \ln(2/p)\). The algorithm is described in Algorithm 1 where a few related notations are explained as follows.

The precise definition of the stages. Let \(d_1 = H, d_{j+1} = \lfloor (1 + \frac{1}{J})d_j \rfloor\) for all \(j \geq 1\). The sizes of the \(j\)-th type-I and type-II stage are given by \(\bar{e}_j = \frac{d_j}{J}\) and \(\bar{e}_j = d_j\) respectively. Let \(N_0 = c_1 \cdot \overline{A}^2 \overline{H}^2 \ln(4H^2 S/\epsilon)\) for some large enough constant \(c_1\). We stop updating \(Q(s, a)\) if the number of visits to \((s, a)\) is greater than \(N_0\), since the value functions will be sufficiently learned by that time. Therefore, the time steps when an update is triggered by the type-I and type-II stages are respectively given by \(\bar{L} = \{\sum_{i=1}^{j} \bar{e}_i | 1 \leq j \leq \bar{J}\}\) and \(\bar{L} = \{\sum_{i=1}^{J} \bar{e}_i | 1 \leq j \leq J\}\), where \(\bar{J} = \max \{j | \sum_{i=1}^{j-1} \bar{e}_i < N_0\}\) and \(J = \max \{j | \sum_{i=1}^{j-1} \bar{e}_i < N_0\}\).

The statistics. We maintain the following statistics during the algorithm: for each \((s, a)\), we use \(N(s, a), \overline{N}(s, a), \overline{N}(s, a)\) to respectively denote the total visit number, the visit number in the current type-I stage and the visit number in the current type-II stage of \((s, a)\). We also maintain \(\overline{\mu}(s, a)\) and \(\overline{\mu}(s, a)\), which are respectively the accumulators for state values \(V(s')\) (where \(s'\) is the next state observed after \((s, a)\)) during the current type-I and type-II stages.
5 Analysis of Sample Complexity

In this section, we prove Theorem 2 for UCB-MULTI STAGE. We start with a few notations: we use $N_t(s,a), \tilde{N}_t(s,a), \hat{N}_t(s,a), Q_t(s,a), V_t(s)$ to denote respectively the values of $N(s,a), \tilde{N}(s,a), \hat{N}(s,a), Q(s,a), V(s)$ before the $t$-th time step. Let $\tilde{n}_t(s,a), \hat{n}_t(s,a)$ and $\hat{b}_t(s,a)$ be the values of $\tilde{n}(s,a), \hat{n}(s,a)$ and $\hat{b}(s,a)$ (respectively) in the latest type-I update of $Q(s,a)$ before the $t$-th time step. In other words, $\tilde{n}_t(s,a)$ is the length of the type-I stage immediately before the current type-I stage with respect to $(s,a)$; $\hat{n}_t(s,a) = \min\{2H^2/\tilde{n}_t(s,a), 1/(1 - \gamma)\}$; and

$$V_t(s) - V^*_{\pi_t}(s) \leq \tilde{b}_t(s,a) + \frac{\gamma}{\tilde{n}_t(s,a)} \sum_{\iota=1}^{\tilde{n}_t(s,a)} V_{t-\iota,u}(s,a)(s_{t-\iota,u}(s,a)+1) - \gamma P_{s,a} V^*_t$$

$$\leq 2\tilde{b}_t(s,a) + \gamma P_{s,a} \left(\frac{1}{\tilde{n}_t(s,a)} \sum_{\iota=1}^{\tilde{n}_t(s,a)} V_{t-\iota,u}(s,a) - V^*_t\right)$$

$$\leq 2\tilde{b}_t(s,a) + \gamma P_{s,a} (V_{\rho_t(s,a)} - V^*)$$

where Inequality (10) is due to the concentration inequality, which is part of the successful event $E_1$ defined in (28), and Inequality (11) holds because $\rho_t(s,a) \leq \tilde{l}_u$ for any $1 \leq u \leq \tilde{n}_t$ and the fact $V_t$ is non-increasing in $t$ (Proposition 3).

Iterating (12) for $H$ times, we obtain that

$$V^*(s_t) - V^*_{\pi_t}(s_t) \leq \sum_{s,a} w_t(s,a) \left(2\tilde{b}_t(s,a) + \gamma P_{s,a} (V_{\rho_t(s,a)} - V_t)\right) + \frac{\epsilon}{8}$$

$$\leq \sum_{s,a} w_t(s,a) \left(2\text{clip}(\tilde{b}_t(s,a), \frac{\epsilon}{8H}) + \gamma P_{s,a} \text{clip}(V_{\rho_t(s,a)} - V_t, \frac{\epsilon}{8H})\right) + \frac{\epsilon}{2}$$

where $w_t(s,a) = I[\pi_t(s) = a] \cdot \sum_{t=0}^{H-1} 1_{s_t}(\gamma P_{t,s}) \cdot 1_s$ is the expected discounted visit number of $(s,a)$ in the next $H$ steps following $\pi_t$ (recall that $P_{t,s}$ is the matrix such that $P_{t,s} = P_{s,\pi_t(s)}$ for
any $s \in S$); and Inequality (14) is due to an averaging argument and the fact that $\sum_{s,a} w_t(s,a) \leq H$. Let
\[
\beta_t := \sum_{s,a} w_t(s,a) \left( 2\text{clip}(\hat{b}_t(s,a), \frac{\epsilon}{8H}) + \gamma P_{s,a}\text{clip}(V_{\hat{\pi}}(s,a) - V_t, \frac{\epsilon}{8H}) \right),
\]
and let $T = \{ t \geq 1 | \beta_t > \frac{1}{2} \epsilon \}$. By (14) we have that the sample complexity of UCB-MULTISTAGE is bounded by
\[
\sum_{t \geq 1} \mathbb{I} [V^*(s_t) - V^{\pi_t}(s_t) > \epsilon] \leq \sum_{t \geq 1} \left[ \beta_t > \frac{1}{2} \epsilon \right] = |T|.
\]
To bound $|T|$, we consider bounding $\sum_{t \in T} \beta_t$ instead, since $\sum_{t \in T} \beta_t \geq \frac{|T| \epsilon}{2}$ and therefore $|T| \leq (2/\epsilon) \cdot \sum_{t \in T} \beta_t$. Let
\[
\tilde{\beta}_t := 2\text{clip}(\hat{b}_t(s_t, a_t), \frac{\epsilon}{8H}) + P_{s_t,a_t}\text{clip}(V_{\hat{\pi}}(s_t, a_t) - V_t, \frac{\epsilon}{8H}),
\]
and if $\pi_t$ does not change very frequently, we have the approximation that $\beta_t \approx \sum_{i=0}^{H-1} \tilde{\beta}_{t+i}$. More formally, we prove the following statement.

**Lemma 4.** For any $K \geq 1$, it holds that
\[
\mathbb{P}\left[ \sum_{t \in T} \beta_t \geq 12KH^3 \epsilon + 24SAH^4B \ln(N_0) \right. \left. \text{and } \sum_{t \geq 1} \tilde{\beta}_t < 3KH^2 \epsilon \right] \leq H \epsilon.
\]

By Lemma 4 and the discussion above, if we are able to bound $\sum_{t \geq 1} \tilde{\beta}_t \leq X$ (for $X \geq 3H^2 \epsilon$), then with high probability, the sample complexity of UCB-MULTISTAGE is bounded by roughly $O(H/\epsilon) \cdot X$.

### 5.2 Bounding the Clipped Pseudo-Regret

We now turn to bound $\sum_{t \geq 1} \tilde{\beta}_t$. For the first term in the definition (16) of $\tilde{\beta}_t$, we have the following lemma.

**Lemma 5.** Conditioned on the successful event $E_1$ defined in (28), it holds that
\[
\sum_{t \geq 1} \text{clip}(\tilde{b}_t(s_t, a_t), \frac{\epsilon}{8H}) \leq O\left( \frac{SAH^3 \ln(\frac{4H}{\epsilon})}{\epsilon B} + SABH^3 + SAH \ln(N_0) \right).
\]

For the second term in the definition of $\tilde{\beta}_t$, let $\alpha_t = P_{s_t,a_t}\text{clip}(V_{\hat{\pi}}(s_t, a_t) - V_t, \frac{\epsilon}{8H})$ for short. By a baseline result for learning the value function (see Lemma 12), we have that

**Lemma 6.** With probability $1 - (1 + SAH(\bar{J} + \bar{J}))p$, it holds that
\[
\sum_{t \geq 1} \alpha_t \leq O\left( \frac{SAH^3 \ln(\frac{4H}{\epsilon})}{\epsilon B} + SABH^3 + SAH \ln(N_0) \right).
\]

Combining Lemma 5 and Lemma 6 and by the definition of $\tilde{\beta}_t$, we have that

**Lemma 7.** With probability $1 - (1 + 2SAH(\bar{J} + \bar{J}))p$, it holds that
\[
\sum_{t \geq 1} \tilde{\beta}_t \leq O\left( \frac{SAH^3 \ln(\frac{4H}{\epsilon})}{\epsilon B} + \frac{SAH^3 \ln(\frac{4H}{\epsilon})}{\epsilon B} + SABH^3 \ln(N_0) \right).
\]

### 5.3 Putting Everything Together

Invoking Lemma 4 with $K = \frac{\epsilon}{\epsilon B} \left( \frac{SAH^3 \ln(\frac{4H}{\epsilon})}{\epsilon B} + \frac{SAH^3 \ln(\frac{4H}{\epsilon})}{\epsilon B} + SABH^3 \ln(N_0) \right) \geq 1$ for some large enough universal constant $c_2$, we have that conditioned on the successful event $E_1$,
\[
\mathbb{P}\left[ \sum_{t \in T} \beta_t \geq 12KH^3 \epsilon + 24SAH^4B \ln(N_0) \right]
\]
\[
\begin{align*}
&\leq P \left[ \sum_{t \in T} \beta_t \geq 12KH^3t + 24SAH^4B \ln(N_0), \sum_{t \geq 1} \beta_t < 3KH^2t \right] + P \left[ \sum_{t \geq 1} \beta_t \geq 3KH^2t \right] \\
&\leq 2SAH(\bar{J} + \bar{J}) + H + 2)p, 
\end{align*}
\]
where the second term in (17) bounded due to Lemma 7. Combining Proposition 3 with (18), we obtain that with probability \(1 - (3SA(\bar{J} + \bar{J}) + (H + 2))p\), it holds that

\[
\left| \tau_t \right| \leq \sum_{t \in T} \beta_t \leq O \left( \frac{SAH^5t}{\epsilon^2} + \frac{SAH^6 \ln \left( \frac{4H}{\epsilon} \right)}{\epsilon^2B} + \frac{SAH^4B \ln(N_0)}{\epsilon} \right).
\]

Noting that \(B = \sqrt{H}\), we conclude that the number of \(\varepsilon\)-suboptimal steps is bounded by

\[
O \left( \frac{SAH^5 \ln \left( \frac{4H}{\epsilon} \right)}{\epsilon^2} + \frac{SAH^4 \ln(N_0)}{\epsilon} \right) \leq O \left( \frac{SAH^5.5 \ln \left( \frac{4H}{\epsilon} \right)(\ln(N_0) + t)}{\epsilon^2} \right)
\]
for any \(\epsilon \in (0, \frac{1}{12\gamma})\). Noting that \(H = \hat{O}(\frac{1}{\epsilon^2})\), \(\bar{J} = O(SAH \ln(N_0))\) and \(\bar{J} = O(SAH B \ln(N_0))\), we finish the proof of Theorem 2 by replacing \(p\) with \(\frac{p}{3SA(\bar{J} + \bar{J}) + H + 2}\).

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Appendices

A Comparison with Previous Works

Table 1: Comparisons of PAC-RL algorithms for discounted MDPs

| Algorithm                          | Sample complexity                                      | Space complexity |
|-----------------------------------|--------------------------------------------------------|------------------|
| Model-based                       |                                                        |                  |
| R-max [4,13]                      | $\tilde{O}\left(\frac{S^2A \ln(1/\delta)}{\epsilon^3(1-\gamma)^6}\right)$ | $O(SA)$         |
| MoRmax [21]                       | $\tilde{O}\left(\frac{SA \ln(1/\delta)}{\epsilon^2(1-\gamma)^3}\right)$ |                  |
| UCRL-\gamma [14]                 | $\tilde{O}\left(\frac{S^2A \ln(1/\delta)}{\epsilon^2(1-\gamma)^3}\right)$ |                  |
| Delayed Q-learning [20]           | $\tilde{O}\left(\frac{SA \ln(1/\delta)}{\epsilon^2(1-\gamma)^3}\right)$ |                  |
| Infinite Q-learning with UCB [22] | $\tilde{O}\left(\frac{SA \ln(1/\delta)}{\epsilon^2(1-\gamma)^3}\right)$ |                  |
| UCB-MULTISTAGE-ADVANTAGE (Theorem 1) | $\tilde{O}\left(\frac{SA \ln(1/\delta)}{\epsilon^2(1-\gamma)^3}\right)$ | $O(SA)$         |
| UCB-MULTISTAGE (Theorem 2)        | $\tilde{O}\left(\frac{SA \ln(1/\delta)}{\epsilon^2(1-\gamma)^3}\right)$ |                  |
| Lower bound                       | $\Omega\left(\frac{SA}{\epsilon^2(1-\gamma)^3}\right)$ [14] |                  |

B Technical Lemmas

Lemma 8. Let $M_1, M_2, ..., M_k, ...$ be a series of random variables which range in $[0, 1]$ and \{\mathcal{F}_k\}_{k \geq 0}$ be a filtration such that $M_k$ is measurable with respect to $\mathcal{F}_k$ for $k \geq 1$. Define $\mu_k := \mathbb{E}[M_k | \mathcal{F}_{k-1}]$.

For any $p \in (0, 1)$ and $c \geq 1$, it holds that

$$\mathbb{P} \left[ \exists n, \sum_{k=1}^{n} \mu_k \geq 4cl, \sum_{k=1}^{n} M_k \leq cl \right] \leq p.$$  

Proof. Let $\lambda < 0$ be fixed. Let $M$ be a random variable taking values in $[0, 1]$ with mean $\mu$. By convexity of $e^{\lambda x}$ in $x$, we have that $\mathbb{E}\left[e^{\lambda M}\right] \leq \mu e^\lambda + (1 - \mu) = 1 + \mu (e^\lambda - 1) \leq e^{\mu (e^\lambda - 1)}$. Then we obtain that for any $k \geq 1$

$$\mathbb{E}\left[e^{\lambda M_k -(e^\lambda - 1)\mu_k} | \mathcal{F}_{k-1}\right] \leq 1,$$

which means $\{Y_k := e^{\lambda \sum_{i=1}^{k} M_i -(e^\lambda - 1) \sum_{i=1}^{k} \mu_i}\}_{k \geq 0}$ is a super-martingale with respect to \{\mathcal{F}_k\}_{k \geq 0}.

Let $\tau$ be the least $n$ with $\sum_{k=1}^{n} \mu_k \geq 4cl$. It is easy to verify that $|Y_{\min(\tau,n)}| \leq e^{(1-e^\lambda)(4cl+1)}$ for any $n$. By the optional stopping theorem, we have that $\mathbb{E}[Y_\tau] \leq 1$. Then

$$\mathbb{P} \left[ \exists n, \sum_{k=1}^{n} \mu_k \geq 4cl, \sum_{k=1}^{n} M_k \leq cl \right]$$

$$\leq \mathbb{P} \left[ \sum_{k=1}^{\tau} M_k \leq cl \right]$$

$$\leq \frac{1}{e^{(1-e^\lambda)(4cl+1)}}.$$  

By setting $\lambda = -\frac{1}{2}$, we obtain that $\frac{1}{e^{(1-e^\lambda)(4cl+1)}} \leq \frac{1}{e^{\epsilon^2(c)}} = \left(\frac{e}{2}\right)^c \leq p$. The proof is completed. $\square$
Lemma 9 (Freeman’s Inequality, Theorem 1.6 of [1]). Let \((M_n)_{n \geq 0}\) be a martingale such that \(M_0 = 0\) and \(|M_n - M_{n-1}| \leq c\). Let \(\text{Var}_n = \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]\) for \(n \geq 0\), where \(\mathcal{F}_k = \sigma(M_0, M_1, M_2, \ldots, M_k)\). Then, for any positive \(x\) and for any positive \(y\),

\[
\mathbb{P}\left[ \exists n : M_n \geq x \text{ and } \text{Var}_n \leq y \right] \leq \exp\left( -\frac{x^2}{2(y + cx)} \right). \tag{21}
\]

Lemma 10. Let \((M_n)_{n \geq 0}\) be a martingale such that \(M_0 = 0\) and \(|M_n - M_{n-1}| \leq c\) for some \(c > 0\) and any \(n \geq 1\). Let \(\text{Var}_n = \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]\) for \(n \geq 0\), where \(\mathcal{F}_k = \sigma(M_1, M_2, \ldots, M_k)\). Then for any positive integer \(n\), and any \(\epsilon, p > 0\), we have that

\[
\mathbb{P}\left[ |M_n| \geq 2\sqrt{2\epsilon \log\left(\frac{1}{p}\right) + 2\sqrt{\epsilon \log\left(\frac{1}{p}\right) + 2c\log\left(\frac{1}{p}\right)}} \right] \leq \left( \frac{2nc^2}{\epsilon} + 2 \right) p. \tag{22}
\]

Proof. For any fixed \(n\), we apply Lemma 9 with \(y = 2^i \epsilon\) and \(x = \pm(2\sqrt{y \log\left(\frac{1}{p}\right) + 2c\log\left(\frac{1}{p}\right)})\). For each \(i = 0, 1, 2, \ldots, \log_2(\frac{nc^2}{\epsilon})\), we get that

\[
\mathbb{P}\left[ |M_n| \geq 2\sqrt{2\epsilon \log\left(\frac{1}{p}\right) + 2\sqrt{\epsilon \log\left(\frac{1}{p}\right) + 2c\log\left(\frac{1}{p}\right)}} \right] \leq 2p. \tag{23}
\]

Then via a union bound, we have that

\[
\mathbb{P}\left[ \sum_{i=1}^{\log_2(\frac{nc^2}{\epsilon})} \mathbb{P}\left[ |M_n| \geq 2\sqrt{2\epsilon \log\left(\frac{1}{p}\right) + 2\sqrt{\epsilon \log\left(\frac{1}{p}\right) + 2c\log\left(\frac{1}{p}\right)}} \right] \right] \leq 2p. \tag{24}
\]

Then via a union bound, we have that

\[
\sum_{i=1}^{\log_2(\frac{nc^2}{\epsilon})} \mathbb{P}\left[ |M_n| \geq 2\sqrt{2\epsilon \log\left(\frac{1}{p}\right) + 2\sqrt{\epsilon \log\left(\frac{1}{p}\right) + 2c\log\left(\frac{1}{p}\right)}} \right] \leq 2p. \tag{25}
\]

\[\square\]

C Missing Proofs in Section 5

C.1 Proof of Proposition 3

Proof of Proposition 3 Let \((s, a)\) be fixed. Let \(\tilde{t}_i\) be the time when the \(i\)-th visit in the \(j\)-th type-I stage of \((s, a)\) occurs. Define \(\tilde{b}^{(j)} = \min\{2, \frac{H_j}{\epsilon_j} \frac{1}{1-\gamma}\}\) for \(j \geq 2\). By Azuma’s inequality, we obtain that for any \(1 \leq j \leq \tilde{J}\) and \((s, a)\), with probability \(1 - p\), it holds that

\[
\frac{1}{\epsilon_j} \sum_{i=1}^{\epsilon_j} V^* (s_{\tilde{t}_i(s, a)+1}) + \tilde{b}^{(j)} \geq P_{s,a} V^*. \tag{26}
\]

Similarly, letting \(\tilde{t}_i(s, a)\) be the time when the \(i\)-th visit in the \(j\)-th type-II stage of \((s, a)\) occurs, and defining \(\tilde{b}^{(j')} = \min\{2, \frac{H_j}{\epsilon_j} \frac{1}{1-\gamma}\}\) for \(j' \geq 1\), we have that for any \(1 \leq j' \leq \tilde{J}\) and \((s, a)\), with probability \(1 - p\), it holds that
\[
T \quad W \text{e split}\n\]
\[
T \quad W \text{e define}\n\]
\[
T \quad p \quad Q
\]
\[
T \quad t
\]
\[
T \quad p
\]
\[
T \quad \bar{E}^{(j)}(s, a)
\]
\[
T \quad \bar{E}^{(j')}(s, a)
\]
Let
\[
T \quad E_1 = (\cap_{s, a, 1 \leq j < j} \bar{E}^{(j)}(s, a)) \cap (\cap_{s, a, 1 \leq j' < j} \bar{E}^{(j')}(s, a)).
\]
Then \(P[E_1] \geq 1 - SA(\bar{j} + \bar{j'})p\). We will prove by induction conditioned on this event.

For \(t = 1\), \(Q_1(s, a) = \frac{1}{t} \geq Q^*(s, a)\) for any \((s, a)\). For \(t \geq 2\), assume \(Q_{t-1}(s, a) \geq Q^*(s, a)\) for \(1 \leq t' < t\) and all \((s, a)\) pairs. If there exists \((j, s, a)\) such that the \(j\)-th type-I update of \((s, a)\) happens at the \((t-1)\)-th step, by (26) we have that
\[
T \quad Q_t(s, a) = \min \{r(s, a) + \gamma \sum_{i = 1}^{\bar{t}} V_{s_i}(s, a) (s_{i+1}^{(s, a)}(s, a) + \bar{b}(s, a), Q_{t-1}(s, a)) \}
\]
\[
T \quad \geq \min \{r(s, a) + \gamma \sum_{i = 1}^{\bar{t}} V_{s_i}(s, a) (s_{i+1}^{(s, a)}(s, a) + \bar{b}(s, a), Q_{t-1}(s, a)) \}
\]
\[
T \quad \geq \min \{r(s, a) + \gamma P_{s, a} V^*, Q_{t-1}(s, a) \}
\]
\[
T \quad \geq Q^*(s, a).
\]
In a similar way, if there exists \((j, s, a)\) such that the \(j\)-th type-I update of \((s, a)\) happens at the \((t-1)\)-th step, then (26) holds for \((s, a)\). Otherwise, \(Q_{t-1}(s, a) \geq Q^*(s, a)\) for any \((s, a)\). The proof is completed.

C.2 Proof of Lemma 4

We split \(T\) into \(H\) separate subsets by define \(V_k = \{t \in T : t \mod H = k\}\) for \(k = 0, 1, 2, \ldots, H - 1\). We will prove Lemma 4 by showing that for each \(k\), it holds that
\[
P \left[ \sum_{t \in V_k} \beta_t \geq 12KH^2 + 24SAH^3 \ln(N_0), \sum_{t \geq 1} \hat{\beta}_t < 3KH^2 \right] \leq p.
\]

If (29) holds for each \(k\), then we have
\[
P \left[ \sum_{t \in T} \beta_t \geq 12KH^3 + 24SAH^4 \ln(N_0), \sum_{t \geq 1} \hat{\beta}_t < 3KH^2 \right]
\]
\[
\leq \sum_{k=0}^{H-1} P \left[ \sum_{t \in V_k} \beta_t \geq 12KH^2 + 24SAH^3 \ln(N_0), \sum_{t \geq 1} \hat{\beta}_t < 3KH^2 \right]
\]
\[
\leq Hp.
\]
Let
\[
U_t = 1 \left[ \exists t' \in \{t, t + 1, \ldots, t + H - 1\} \text{ and } (s, a) \text{ such that } Q_{t'+1}(s, a) \neq Q_{t'}(s, a) \right].
\]
We define
\[
\hat{\beta}_t := 3H^2 U_t + (1 - U_t) \sum_{i=0}^{H-1} \gamma^i \left( 2\text{clip}(b_i(s_{t+i}, a_{t+i}), \frac{e}{8H}) + \gamma P_{a_{t+i}, a_{t+i}}, \text{clip}(V_{t, s_{t+i}, a_{t+i}}(s_{t+i}, a_{t+i}) - V_t, \frac{e}{8H}) \right).
\]
For fixed \(k \in \{0, 1, 2, \ldots, H - 1\}\), we let
\[
\hat{\beta}^k_t := \hat{\beta}_{tH+k} \mathbb{I}[tH+k \in T].
\]
Noting that \( \hat{\beta}_t^k \in [0, 1] \) is measurable with respect to \( \mathcal{F}_t^k := \mathcal{F}_{(t+1)H+k-1} \) and \( \mathbb{E} \left[ \beta_t^k | \mathcal{F}_{t-1}^k \right] \geq \beta_t^k := \beta_{H+k-1}^{H+k-1} \), by Lemma 8 we obtain that for any \( K \geq 1 \),

\[
P \left[ \exists n, \sum_{t=1}^n \beta_t^k \geq 4Kt + 16SABH \ln(N_0), \quad \sum_{t=1}^n \beta_t^k \leq Kt + 4SABH \ln(N_0) \right] \leq p,
\]

which is equivalent to

\[
P \left[ \exists n, \sum_{t=1}^n \beta_t^k \geq 12K H^2 t + 24SABH^3 \ln(N_0), \quad \sum_{t=1}^n \beta_t^k \leq 3K H^2 t + 6SABH^3 \ln(N_0) \right] \leq p.
\]  

(31)

By definition of \( \hat{\beta}_t \), and noting that if \( U_t = 0 \), \( \bar{y}(s_{t+i}, a_{t+i}) = \bar{y}_{t+i}(s_{t+i}, a_{t+i}) \) and \( V_{\beta_t^k}(s_{t+i}, a_{t+i}) = V_{\hat{\beta}_{t+i}}(s_{t+i}, a_{t+i}) \) for any \( 0 \leq i \leq H - 1 \), we have

\[
\hat{\beta}_t = 3H^2 U_t + (1 - U_t) \sum_{i=0}^{H-1} \gamma^i \left( 2\text{clip}(\hat{b}_t(s_{t+i}, a_{t+i}), \frac{e}{8H}) + \gamma P_{s_{t+i}, a_{t+i}, \text{clip}(V_{\hat{\beta}_t}(s_{t+i}, a_{t+i}) - V_t, \frac{e}{8H})) \right) \\
\leq 3H^2 U_t + (1 - U_t) \sum_{i=0}^{H-1} \left( 2\text{clip}(\hat{b}_t(s_{t+i}, a_{t+i}), \frac{e}{8H}) + \gamma P_{s_{t+i}, a_{t+i}, \text{clip}(V_{\hat{\beta}_t}(s_{t+i}, a_{t+i}) - V_t, \frac{e}{8H})) \right) \\
\leq 3H^2 U_t + \sum_{i=0}^{H-1} \left( 2\text{clip}(\hat{b}_{t+i}(s_{t+i}, a_{t+i}), \frac{e}{8H}) + \gamma P_{s_{t+i}, a_{t+i}, \text{clip}(V_{\hat{\beta}_{t+i}}(s_{t+i}, a_{t+i}) - V_t, \frac{e}{8H})) \right).
\]

Then it follows that

\[
\sum_{t \in \mathcal{V}_k} \hat{\beta}_t \leq \sum_{t \in \mathcal{V}_k} \sum_{i=0}^{H-1} \left( 2\text{clip}(\hat{b}_{t+i}(s_{t+i}, a_{t+i}), \frac{e}{8H}) + \gamma P_{s_{t+i}, a_{t+i}, \text{clip}(V_{\hat{\beta}_{t+i}}(s_{t+i}, a_{t+i}) - V_t, \frac{e}{8H})) \right) \\
+ 3H^2 \sum_{t \in \mathcal{V}_k} U_t \\
\leq \sum_{t \geq 1} \left( 2\text{clip}(\hat{b}_t(s_t, a_t), \frac{e}{8H}) + \gamma P_{s_t, a_t, \text{clip}(V_{\hat{\beta}_t}(s_t, a_t) - V_t, \frac{e}{8H})) \right) + 6SABH^3 \ln(N_0) \\
= \sum_{t \geq 1} \hat{\beta}_t + 6SABH^3 \ln(N_0).
\]  

(32)

(33)

Here Inequality (32) holds because for each update, there is at most one element \( t \in \mathcal{T}' \), such that \( U_t = 1 \) due to this update.

By (31) and (32), we have that

\[
P \left[ \sum_{t \in \mathcal{V}_k} \beta_t \geq 12C H^2 t + 24SABH^3 \ln(N_0), \quad \sum_{t \geq 1} \hat{\beta}_t < 3C H^2 t \right] \\
\leq P \left[ \sum_{t \in \mathcal{V}_k} \beta_t \geq 12C H^2 t + 24SABH^3 \ln(N_0), \quad \sum_{t \geq 1} \hat{\beta}_t < 3C H^2 t + 6SABH^3 \ln(N_0) \right] \\
\leq p.
\]

The proof is completed.

C.3 Proof of Lemma 5

Proof of Lemma 5 Recall that \( \hat{y}_t(s_t, a_t) = 2\sqrt{\frac{H^2}{n_t(s_t, a_t)}}, \) so \( \text{clip}(\hat{b}_t(s_t, a_t), \frac{e}{8H}) \leq 2\sqrt{\frac{H^2}{n_t(s_t, a_t)}} \right) \leq 2\sqrt{\frac{H^2}{n_t(s_t, a_t)}} [\hat{n}_t < 256 \frac{H^2}{e^2}] \). Noting that \( \hat{n}_t \geq \frac{H^2}{2e^2} \), we obtain that

\[
\sum_{t \geq 1} \text{clip}(\hat{b}_t(s_t, a_t), \frac{e}{8H}) \leq SAH^2 + \sum_{t \geq 1} 2\sqrt{\frac{2H^3 B_t}{n_t(s_t, a_t)}} [n_t < 512 \frac{H^5 B_t}{e^2}] 
\]
We next define \( s_{t+1} \) for \( t \) and therefore \( \rho_t(s,a) = \rho(j + 1, s,a) \). (The definitions of \( \rho, \rho_t \) and \( \rho_t \) are at the beginning of Section 5.)

For \( j \geq 2 \), by the definition of \( \alpha_t \) and the fact \( V_i \) is non-increasing in \( t \), we obtain that

\[
\sum_{t \in T(j,s,a)} \alpha_t [(s_t,a_t) = (s,a)] \leq \tilde{c}_j P_{s,a} \left( \text{clip}(V_{\rho(j-1,s,a)} - V_{\rho(j+1,s,a)}, \frac{\epsilon}{8H}) \right),
\]

and therefore

\[
\alpha(s,a) \leq H \sum_{i=1}^{HB} \tilde{c}_i + \sum_{HB + 1 < j < j_{X}(s,a)} \tilde{c}_j P_{s,a} \left( \text{clip}(V_{\rho(j-1,s,a)} - V_{\rho(j+1,s,a)}) \frac{\epsilon}{8H} \right). \tag{34}
\]

Here also recall that \( j_{X}(s,a) \) is defined to be \( \max_{t \geq 1} j_{t}(s,a) \leq \tilde{J} \).

We next define

\[
j(s,a,s',\epsilon') := \max\{ j \leq j_{X}(s,a) | V_{\rho(j,s,a)}(s') - V^*(s') > \epsilon' \}
\]

and

\[
\tilde{\tau}(s,a,s',\epsilon') := \sum_{i=1}^{j(s,a,s',\epsilon')} \tilde{c}_i
\]

for \( s' \in \mathcal{S} \) and \( \epsilon' > 0 \). Let \( \epsilon_i = \frac{2^{i-1}}{k} \) for \( i = 0, 1, 2, \ldots, k \) where \( k = \lfloor \log_2 \left( \frac{H}{\epsilon} \right) \rfloor \). By \( \tilde{\tau} \), we have that

\[
\alpha(s,a) \leq O(BH^2 \tilde{c}_1) + \sum_{s'_{i=1}}^{k} \sum_{HB + 1 < j < j(s,a,s',\epsilon_i,HB)} \tilde{c}_j P_{s,a}(s') \left( V_{\rho(j-1,s,a)}(s') - V_{\rho(j+1,s,a)}(s') \right)
\]

\[
\leq O(BH^2 \tilde{c}_1) + \sum_{s'_{i=1}}^{k} \sum_{HB + 1 < j < j(s,a,s',\epsilon_i,HB)} \tilde{c}_j P_{s,a}(s') \left( V_{\rho(j-1,s,a)}(s') - V_{\rho(j+1,s,a)}(s') \right)
\]

\[
\leq O(BH^2 \tilde{c}_1) + \sum_{s'_{i=1}}^{k} \frac{2 \tilde{\tau}(s,a,s',\epsilon_i-1)}{HB} P_{s,a}(s') \psi(s,a,s',i)
\]

\[
\leq O(BH^2 \tilde{c}_1) + \frac{4}{HB} \sum_{i=1}^{k} \tilde{\tau}(s,a,s',\epsilon_i-1) P_{s,a}(s') \epsilon_i, \tag{36}
\]

where

\[
\theta(s,a,s',j) := V_{\rho(j,s,a)}(s') - V_{\rho(j+2,s,a)}(s'),
\]

\[
\psi(s,a,s',i) := \theta(s,a,s',j) \leq 2 \epsilon_i.
\]

Here Inequality \( \tilde{\tau} \) is by the fact \( \tilde{c}_j \leq \frac{2 \tilde{\tau}(s,a,s',\epsilon_i-1)}{HB} \sum_{i=1}^{j} \tilde{c}_i \) for \( j \geq HB \) and Inequality \( \tilde{\tau} \) is by the definition of \( j(s,a,s',\epsilon_i) \).

In the next subsection, we will prove the following lemma.
Lemma 11. For any $\epsilon > 0$, with probability $1 - (1 + SA(\bar{J} + \bar{J}))p$ it holds that

$$\sum_{s,a,s'} \tilde{\tau}(s, a, s', \epsilon)P_{s,a}(s') \leq O \left( \frac{SAH^5 \ln\left(\frac{4H}{\epsilon}t\right)}{\epsilon^2} + SAHB \ln(N_0) \right).$$

Now, by (36) and Lemma 11 we have that

$$\sum_{t \geq 1} \alpha_t = \sum_{s,a} \alpha(s, a) \leq \sum_{s,a} \left( BH^2 \tilde{e}_1 + \frac{4}{HB} \sum_{s'} \sum_{i=1}^k \tilde{\tau}(s, a, s', \epsilon_{i-1})P_{s,a}(s') \epsilon_i \right) \leq O(SABH^3) + O \left( \frac{4}{HB} \sum_{i=1}^k \left( \frac{SAH^5 \ln\left(\frac{4H}{\epsilon}t\right)}{\epsilon_{i-1}} + SAHB \ln(N_0) \right) \epsilon_i \right) \leq O(SABH^3) + O \left( \frac{1}{HB} \frac{SAH^6 \ln\left(\frac{4H}{\epsilon}t\right)}{\epsilon} + \frac{SA \ln(N_0)}{1 - \gamma} \right) \leq O \left( \frac{SAH^5 \ln\left(\frac{4H}{\epsilon}t\right)}{\epsilon B} + SAHB^3 + SAH \ln(N_0) \right).$$

The proof is completed.

C.5 Proof of Lemma 11

We first state the following auxiliary lemma, which implies that we can learn the value function efficiently. The lemma is similar to Lemma 5 in [23], and is proved using the type-II updates. The proof of Lemma 12 will be presented immediately after this subsection.

Lemma 12. Conditioned on the successful event of $E_1$ defined in (28), for any $\epsilon_1 \in \left[ \epsilon, \frac{1}{1 - \gamma} \right]$ it holds that

$$\sum_{t=1}^{\infty} \mathbb{I} \left[ V_t(s_t) - V^*(s_t) \geq \epsilon_1 \right] \leq O \left( \frac{SAH^5 \ln\left(\frac{4H}{\epsilon}t\right)}{\epsilon_1^2} \right).$$

(38)

With the help of Lemma 12, we prove Lemma 11 as follows.

Proof of Lemma 11 We start with defining

$$\tau(s, a, s', \epsilon) := \sum_{t \geq 1} \mathbb{I} \left[ (s_t, a_t) = (s, a), V_t(s') - V^*(s') > \epsilon \right].$$

Recalling that $\tilde{\tau}(s, a, s', \epsilon) = \sum_{i=1}^{j(s,a,s',\epsilon)} \tilde{\epsilon}_i$, we have

$$\tilde{\tau}(s, a, s', \epsilon) = \sum_{i=1}^{j(s,a,s',\epsilon)} \tilde{\epsilon}_i \leq H + (1 + \frac{2}{H}) \sum_{i=1}^{j(s,a,s',\epsilon)-1} \tilde{\epsilon}_i \leq H + (1 + \frac{2}{H}) \tau(s, a, s', \epsilon).$$

So it suffices to prove that

$$\sum_{s,a,s'} \tau(s, a, s', \epsilon)P_{s,a}(s') \leq O \left( \frac{SAH^5 \ln\left(\frac{4H}{\epsilon}t\right)}{\epsilon^2} + SAHB \ln(N_0) \right).$$

(39)

To prove (39), we define $\lambda_t$ to be the vector such that $\lambda_t(s) = \mathbb{I} \left[ V_t(s) - V^*(s) > \epsilon \right]$. Note that

$$\sum_{s,a,s'} \tau(s, a, s', \epsilon)P_{s,a}(s') = \sum_{t \geq 1} \lambda_t(s_t) \lambda_t$$

and due to the infrequent updates, we have that

$$\sum_{t \geq 1} (\lambda_t(s_{t+1}) - \lambda_{t+1}(s_{t+1})) \leq \sum_{t \geq 1} \mathbb{I} \left[ V_t(s_{t+1}) \neq V_{t+1}(s_{t+1}) \right] \leq 2SAHB \ln(N_0).$$
For $C$ a large enough constant, we obtain that

$$
\begin{align*}
\mathbb{P}\left[ \sum_{s,a,s'} \tau(s,a,s',\epsilon)P_{s,a}(s') \geq 4C \frac{SAH^5 \ln \left( \frac{4H}{\epsilon^2} \right) t}{\epsilon^2} + 8SAHB \ln(N_0) \right] \\
= \mathbb{P}\left[ \sum_{t \geq 1} P_{s_t,a_t} \lambda_t \geq 4C \frac{SAH^5 \ln \left( \frac{4H}{\epsilon^2} \right) t}{\epsilon^2} + 8SAHB \ln(N_0) \right] \\
\leq \mathbb{P}\left[ \sum_{t \geq 1} P_{s_t,a_t} \lambda_t \geq 4C \frac{SAH^5 \ln \left( \frac{4H}{\epsilon^2} \right) t}{\epsilon^2} + 8SAHB \ln(N_0), \sum_{t \geq 1} \lambda_t(s_{t+1}) \leq C \frac{SAH^5 \ln \left( \frac{4H}{\epsilon^2} \right) t}{\epsilon^2} + 2SAHB \ln(N_0) \right] \\
+ \mathbb{P}\left[ \sum_{t \geq 1} \lambda_t(s_{t+1}) > C \frac{SAH^5 \ln \left( \frac{4H}{\epsilon^2} \right) t}{\epsilon^2} + 2SAHB \ln(N_0) \right] \\
\leq p + \mathbb{P}\left[ \sum_{t \geq 1} \lambda_t(s_t) \geq C \frac{SAH^5 \ln \left( \frac{4H}{\epsilon^2} \right) t}{\epsilon^2} \right] \\
\leq p + \mathbb{P}[E_1] \\
\leq p + SA(\bar{J} + \bar{J})p, \\
\leq p + SA(J + \bar{J} + \bar{J})p
\end{align*}
$$

where Inequality (40) is by Lemma 8 with $M_k = \lambda_k(s_{k+1})$ and $T_k = \sigma(s_1, a_1, ..., s_k, a_k, s_{k+1})$ for $k \geq 1$, Inequality (41) is by Lemma 12 and Inequality (42) is by Proposition 3. The proof is completed.

C.6 Proof of Lemma 12

The proof of Lemma 12 uses similar techniques as presented in Appendix B of [22] and Appendix B.2 of [23]. However, it requires more twists since the $Q$ function is only updated by at most $SA(\bar{J} + \bar{J})$ times for each state-action pair.

We first introduce a few simplified notations. Define $\delta^i := V_i(s_t) - V^*(s_t)$. Throughout this subsection, we use $\bar{n}^i$, $\bar{b}^i$ and $\bar{l}^i$ as short hands of $\bar{n}_t(s_t, a_t)$, $\bar{b}_t(s_t, a_t)$ and $\bar{l}_t(s_t, a_t)$ respectively. Conditioned on $E_1$ defined in (28), we note that (26) and (27) hold for any $j \geq 1$ and $j' \geq 1$ respectively. We will use these inequalities without additional explanation.

Let $T_1 := \{ t \geq 1 | n_t(s_t, a_t) \geq N_0 \}$. We then have the following lemma.

Lemma 13. Conditioned on successful event $E_1$ defined in (28), it holds that for any $t \in T_1$ (if $T_1$ is not empty)

$$
V_i(s_t) - V^*(s_t) \leq \frac{\epsilon}{2H}.
$$

Proof. For each $i = 1, 2, ..., S$, if there are at least $i$ states with total visit number greater or equal to $N_0$, we let $s^{(i)}$ be the $i$-th such state (sorted in the order of time to reach $N_0$) and let $T_i$ be the corresponding time (i.e., $n_{T_i}(s^{(i)}) = N_0$ and $s_{T_i} = s^{(i)}$). Otherwise we let $s^{(i)}$ be a random state in $S \setminus \{ s^{(1)}, ..., s^{(i-1)} \}$ and set $T_i = \infty$.

It suffices to prove that $V_{T_i}(s^{(i)}) - V^*(s^{(i)}) \leq \frac{\epsilon}{2H}$ for $s^{(i)}$ with finite $T_i$. We prove this by applying induction on $i$ to prove the stronger statement that $V_{T_i}(s^{(i)}) - V^*(s^{(i)}) \leq \frac{\epsilon i}{2H^i S}$.

Base case ($i = 1$): Note that for any $t \notin T_1$, we have following inequality by the update rule (8) and event $E_1$,

$$
\begin{align*}
\delta^i &= V_i(s_t) - V^*(s_t) \\
&\leq Q_i(s_t, a_t) - Q^*(s_t, a_t) \\
&\leq \frac{\| \bar{n}^i = 0 \|}{1 - \gamma} + \left( \bar{b}^i + \frac{\gamma}{\bar{n}^i} \sum_{i=1}^{\bar{n}^i} V_{T_i}(s_{T_i+1}) - P_{s, a, V^*} \right) \\
&\leq \frac{\| \bar{n}^i = 0 \|}{1 - \gamma} + \left( 2\bar{b}^i + \frac{\gamma}{\bar{n}^i} \sum_{i=1}^{\bar{n}^i} \left( V_{T_i}(s_{T_i+1}) - V^*(s^i_{T_i+1}) \right) \right)
\end{align*}
$$
where we define $\theta_{t+1}^i + 1 := V_t^n(s_{t+1}^i) - V_{t+1}(s_{t+1}^i)$.

It is obvious that $t \notin T_1$ if $t < T_1$. Then for any non-negative weights $\{w_t\}_{t \geq 1}$, we have that

$$\sum_{t < T_1} w_t \delta t \leq \sum_{t < T_1} w_t \left[ \frac{\bar{t}^i = 0}{1 - \gamma} + 2 \bar{b}^t + \gamma \sum_{i=1}^{\bar{t}^i} (\delta_{i+1}^t + \theta_{i+1}^t) \right],$$

(43)

where

$$w_t' = \sum_{u < T_1} \frac{1}{\bar{t}^u} \sum_{i=1}^{\bar{t}^i} I[t = \bar{t}^u + 1].$$

(45)

If we choose a sequence of non-negative weights $\{w_t\}_{t \geq 1}$ such that $\sup_{t < T_1} w_t \leq C$ and $\sum_{t < T_1} w_t \leq W$ for two positive constants $C$ and $W$, then for all $t \geq 1$, we have that

$$w_t' \leq \gamma \left( 1 + \frac{1}{H} \right) C \leq \left( 1 - \frac{1}{2H} \right) C,$$

(46)

and

$$\sum_{t < T_1} w_t' \leq \gamma \left( 1 + \frac{1}{H} \right) W \leq \left( 1 - \frac{1}{2H} \right) W.$$

(47)

**Lemma 14.** Let $\{w_t\}_{t \geq 1}$ be a sequence of non-negative weights such that $0 \leq w_t \leq C$ for any $t \notin T_1$ and $\sum_{t \notin T_1} w_t \leq W$, then it holds that

$$\sum_{t \notin T_1} w_t \left[ \frac{\bar{t}^i = 0}{1 - \gamma} \leq \frac{CSAH}{1 - \gamma} \leq CSAH^2, \right.$$

(48)

$$2 \sum_{t \notin T_1} w_t \bar{b}^t \leq 40 \left( 1 + \frac{1}{H} \right) \sqrt{SAH}^3 WC \leq 60 \sqrt{SAH}^3 WC,$$

(49)

$$\sum_{t \notin T_1} w_t \theta_t \leq \frac{SAC}{1 - \gamma} \leq SCH.$$

(50)

**Proof.** The first inequality holds because $\sum_{t \geq 1} \left[ \bar{t}^i = 0 \right] \leq SAH$, and the third inequality holds because $\sum_{t \geq 1} \left[ s_t = s \right] \theta_t \leq 1/(1 - \gamma)$. For the second inequality, we note that $\bar{b}^t \leq 2\sqrt{H^2/w_t}$, it then follows that

$$\sum_{t \notin T_1} w_t \bar{b}^t \leq 2\sqrt{H^2} \sum_{t \notin T_1} w_t \sqrt{1/w_t}$$

$$= 2\sqrt{H^2} \sum_{t \notin T_1} \sum_{s,a} \frac{1}{\bar{t}^u} \sum_{i=1}^{\bar{t}^i} I[(s_t, a_t) = (s, a)] w_t \sqrt{1/w_t}.$$

Let $\tilde{w}(s, a) = \sum_{t \notin T_1} w_t \left[ \bar{t}^u \left( s_t, a_t \right) = (s, a) \right]$. We fix $\tilde{w}(s, a)$ and consider to maximize

$$\sum_{t \notin T_1} \left[ \bar{t}^u \left( s_t, a_t \right) = (s, a) \right] w_t \sqrt{1/w_t}.$$

Define $\tilde{T}(j, s, a) := \left\{ t \geq 1 \left| (s_t, a_t) = (s, a), \sum_{i=1}^{j-1} \tilde{e}_{j_i} \leq N_t(s, a) \leq \sum_{i=1}^{j} \tilde{e}_j \right. \right\}$. Note that for each $j \geq 2$, $\sum_{t \notin T_1, t \in \tilde{T}(j, s, a)} w_t \leq \left( 1 + \frac{1}{H} \right) C \tilde{e}_{j-1}$. By rearrangement inequality we have that

$$\sum_{t \notin T_1} \left[ \bar{t}^u \left( s_t, a_t \right) = (s, a) \right] w_t \sqrt{1/w_t} \leq C \left( 1 + \frac{1}{H} \right) \sum_{j \geq 1} \sqrt{\tilde{e}_{j-1}} \left[ \sum_{i=1}^{j-1} C \tilde{e}_i \leq \tilde{w}(s, a) \right].$$

(51)
By Cauchy-Schwartz inequality, we obtain that

$$\sum_{t \in T_1} w_t \delta^t \leq 20(1 + \frac{1}{H}) \sqrt{HC \bar{w}(s, a)} \leq 20(1 + \frac{1}{H}) \sqrt{SAH^3 W C t}.$$ 

The proof is completed.  

By Lemma [14] we derive that

$$\sum_{t < T_1} w_t \delta^t \leq \sum_{t < T_1} w_t' \delta^t + 2SACH^2 + 60 \sqrt{SAH^3 W C t}. \quad (51)$$

By iteratively unrolling \([51]\) for \(2H \ln(\frac{AH^2 S}{\epsilon})\) times and setting the initial weights by \(w_t = I[s_t = s^{(1)}]\) so that \(C = 1\) and \(W = N_0\), we have

$$\sum_{t \in T_1} [s_t = s^{(1)}] \delta^t \leq 2H \ln(\frac{AH^2 S}{\epsilon}) \left(2SAH^2 + 60 \sqrt{SAH^3 N_0 t} \right) + \epsilon \sum_{t < T_1} I[s_t = s^{(1)}] \quad (52)$$

If \(V_{T_1}(s^{(1)}) - V^*(s^{(1)}) < \frac{k \epsilon}{2HS}\), then \([s_t = s^{(1)}] \delta^t > \frac{\epsilon}{2HS} \) for \(t < T_1\) due to the fact that \(V_t\) is non-increasing in \(t\), which implies that

$$\epsilon N_0 < 2H \ln(\frac{AH^2 S}{\epsilon}) (2SAH^2 + 60 \sqrt{SAH^3 N_0 t}), \quad (53)$$

which contradicts the definition of \(N_0 \) \(N_0 = \frac{SAH^3 S^2 \ln(\frac{AH^2 S}{\epsilon} \gamma^2)}{\epsilon^2} \). As a result, we have that \(V_{T_1}(s^{(1)}) \leq V^*(s^{(1)}) + \frac{k \epsilon}{2HS}\).

**Induction step:** Now suppose that \(V_{T_i}(s^{(i)}) - V^*(s^{(i)}) \leq \frac{k \epsilon}{2HS} \) holds for all \(1 \leq i \leq k\) for some \(k \geq 1\). We will prove that \(V_{T_{k+1}}(s^{(k+1)}) - V^*(s^{(k+1)}) \leq \frac{k \epsilon}{2HS} \) assuming that \(T_{k+1} \neq \infty\).

Note that if \(t < T_{k+1}\) and \(T \in T_1\), \(\delta^t \leq \frac{k \epsilon}{2HS}\). It then follows that for non-negative weights \(\{w_t\}_{t \geq 1}\) such that \(\sup_{t < T} \frac{w_t}{w_t' + 1} \leq C\) and \(\sum_{t < T} w_t \leq W\),

$$\sum_{t < T_{k+1}} w_t \delta^t \leq \sum_{t < T_{k+1}, t \neq T_1} w_t \delta^t + \sum_{t < T_{k+1}, t \neq T_1} \frac{w_t' k \epsilon}{2HS} \leq 2SACH^2 + 60 \sqrt{SAH^3 W_1} + \sum_{t < T_{k+1}} w_t' \delta^t + \sum_{t < T_{k+1}, t \neq T_1} \frac{w_t' k \epsilon}{2HS} \quad (54)$$

$$\leq 2SACH^2 + 60 \sqrt{SAH^3 W_1} + \sum_{t < T_{k+1}} w_t' \delta^t + \sum_{t < T_{k+1}, t \neq T_1} \frac{(W - W_1) k \epsilon}{2HS}, \quad (55)$$

$$\leq 2SACH^2 + 60 \sqrt{SAH^3 W_1} + \sum_{t < T_{k+1}} w_t' \delta^t + \frac{(W - W_1) k \epsilon}{2HS}, \quad (56)$$

where \(W_1 = \sum_{t < T_{k+1}, t \neq T_1} w_t\) and \(w_t' = \gamma \sum_{t < T_{k+1}, t \neq T_1} \frac{1}{n_t} \sum_{i=1}^{n_t} I[t = t_i \uparrow 1] \). Here, Inequality \(55\) is by Lemma [14]. Because \(w_t' \leq (1 - \frac{1}{H}) \frac{1}{C}, \forall t \geq 1\) and \(\sum_{t < T_{k+1}, t \neq T_1} w_t' \leq (1 - \frac{1}{H}) W_1\), by iteratively applying \(56\) for \(2H \ln(\frac{AH^2 S}{\epsilon})\) times, we have that

$$\sum_{t < T_{k+1}} w_t \delta^t \leq 2H \ln(\frac{AH^2 S}{\epsilon}) \left(2SAH^2 + 60 \sqrt{SAH^3 N_0 t} \right) + \frac{W k \epsilon}{2HS} + \frac{W \epsilon}{4HS}. \quad (57)$$

If \(V_{T_{k+1}}(s^{(k+1)}) - V^*(s^{(k+1)}) > \frac{k \epsilon}{2HS}\), choosing \(w_t = I[s_t = s^{(k+1)}, t < T_{k+1}]\) so that \(C = 1\) and \(W = N_0\) in \(57\), we obtain that

$$\frac{N_0 (k + 1) \epsilon}{2HS} \leq 2H \ln(\frac{AH^2 S}{\epsilon}) \left(2SAH^2 + 60 \sqrt{SAH^3 N_0 t} \right) + \frac{N_0 k \epsilon}{2HS} + \frac{N_0 \epsilon}{4HS}.$$
which again contradicts to the definition of \( N_0 \). Therefore we have proved that 
\[
V^*(s^{(k+1)}) - V_{T_k+1}(s^{(k+1)}) \leq \frac{(k+1)\epsilon}{2H}\frac{t}{s}. 
\]

Proof of Lemma 12 Let \( \epsilon_1 \in \left[ \frac{1}{t}, \frac{1}{2t} \right) \) be fixed. Let \( \{w_t\}_{t \geq 1} \) be a non-negative sequence such that \( \sup_{t \geq 1} w_t \leq C \) and \( \sum_{t \geq 1} w_t \leq W \). Following the derivation of (51) we have that
\[
\sum_{t \geq 1} w_t \delta^t = \sum_{t \geq 1, t \notin T_1} w_t \delta^t + \sum_{t \geq 1, t \in T_1} w_t \delta^t 
\leq \sum_{t \geq 1, t \notin T_1} w_t \delta^t + \frac{W_1 \epsilon}{2H} 
\leq \sum_{t \geq 1} w_t \delta^t + 2SAH^2 + 60\sqrt{SAH^3WCt} + \frac{W_1 \epsilon}{2H}. 
\]
where \( \{w_t\}_{t \geq 1} = \gamma \sum_{u \geq 1, u \notin T_1} \frac{1}{H} \sum_{i=1}^{\tilde{t}_u} \mathbb{I}[t = \tilde{t}_u + 1] \) and \( W_1 = \sum_{t \in T_1} w_t \). Similarly, it holds that \( w_t \leq (1 - \frac{1}{2H})C, \forall t \geq 1 \) and \( \sum_{t \geq 1} w_t \leq (1 - \frac{1}{2H})(W - W_1) \). Here Inequality (58) holds by Lemma 13 and Inequality (59) holds by Lemma 14. Again by applying (59) iteratively for \( 2H \ln \left( \frac{4H}{\epsilon} \right) \) times, we have that
\[
\sum_{t \geq 1} w_t \delta^t \leq 2H \ln \left( \frac{4H}{\epsilon} \right) \left( 2SAH^2 + 60\sqrt{SAH^3WCt} \right) + \frac{W_1 \epsilon}{2H} + \frac{W \epsilon}{4}. 
\]
By choosing \( w_t = \mathbb{I}[\delta^t > \epsilon_1] \) so that \( C = 1 \) and \( W = N(\epsilon_1) := \sum_{t \geq 1} \mathbb{I}[\delta^t > \epsilon_1] \) into (60), we obtain that
\[
\frac{N(\epsilon_1) \epsilon_1}{2} \leq 2H \ln \left( \frac{4H}{\epsilon} \right) \left( 2SAH^2 + 60\sqrt{SAH^3N(\epsilon_1) \epsilon_1} \right),
\]
which means that \( N(\epsilon_1) \leq O \left( \frac{SAH^3 \ln (\frac{4H}{\epsilon})}{\epsilon_1} \right) \). The proof is completed.

D Achieving Asymptotically Near-Optimal Sample Complexity

As mentioned in Section 3 in the UCB-MULTISTAGE-ADVANTAGE algorithm, we set \( B \) to be a much larger value (indeed, \( B = H^3 \)), employ the reference-advantage decomposition variance reduction technique [23], and re-design the exploration bonus \( \tilde{b} \) to incorporate the Bernstein-type variance estimation. To prove Theorem 1 (the sample complexity bound for UCB-MULTISTAGE-ADVANTAGE), in the analysis we split the error incurred due to the exploration bonus into two parts: the bandit loss \( b^*_r(\alpha, \beta) \) (defined in (63)) and the rest part that is due to the estimation variance of the real bandit loss. While the second part can be dealt with the variance reduction technique (Lemma 13), the bandit loss contributes the main \( \tilde{O}(SAH^3/t^2) \) term in the sample complexity (Lemma 17).

The rest of this section is organized as follows. In Appendix D.1 we present the details of the UCB-MULTISTAGE-ADVANTAGE algorithm. In Appendix D.2 we prove Theorem 1 while the proofs of all technical lemmas are deferred to Appendix D.3.

D.1 The UCB-MULTISTAGE-ADVANTAGE Algorithm

The UCB-MULTISTAGE-ADVANTAGE algorithm (Algorithm 2) has almost the same updating structure as UCB-MULTISTAGE. More specifically, the stopping condition and update triggers of UCB-MULTISTAGE-ADVANTAGE are the same as that of UCB-MULTISTAGE. The main difference between these two algorithms is 1) that UCB-MULTISTAGE-ADVANTAGE utilized a more delicate exploration bonus with the help of a reference value function in the type-I updates; 2) we set \( B = H^3 \) in UCB-MULTISTAGE-ADVANTAGE.

The Statistics. Besides the statistics maintained in UCB-MULTISTAGE, we let \( \mu^r \) and \( \sigma^r \) be the accumulators of the reference value function and square of the reference value function respectively. Different from UCB-MULTISTAGE, in UCB-MULTISTAGE-ADVANTAGE we use \( \mu \) and \( \sigma \) denote respectively the accumulator of the advantage function and square of the advantage function in the current type-I stage.
Algorithm 2 UCB-MULTISTAGE-ADVANTAGE

Initialize: \( \forall (s, a) \in S \times A: Q(s, a), Q^{\text{ref}}(s, a) \leftarrow \frac{1}{1 - \gamma}, N(s, a), \bar{N}(s, a), \bar{N}(s, a), \bar{\mu}(s, a), \hat{\mu}(s, a) \), \( \bar{\mu}(s, a) \leftarrow 0; \)

for \( t = 1, 2, 3, \ldots \) do

1. \( \text{Observe } s_t; \)
2. \( \text{Take action } a_t = \arg \max_{a} Q(s_t, a) \text{ and observe } s_{t+1}; \)
3. \( \text{Maintain the statistics} \)
   \( (s, a, s') \leftarrow (s_t, a_t, s_{t+1}); \)
   \( n := N(s, a) \leftarrow 1; \quad \bar{n} := \bar{N}(s, a) \leftarrow 1; \quad \bar{n} := \bar{N}(s, a) \leftarrow 1; \)
   \( \mu := \bar{\mu}(s, a) \leftarrow 0; \quad \bar{\mu} := \bar{\mu}(s, a) \leftarrow 0; \)
   \( \tilde{\mu} := \tilde{\mu}(s, a) \leftarrow 0; \quad \bar{\mu} := \bar{\mu}(s, a) \leftarrow 0; \)
   \( \tilde{\bar{\mu}} := \tilde{\bar{\mu}}(s, a) \leftarrow 0; \quad \bar{\mu} := \bar{\mu}(s, a) \leftarrow 0; \)

4. \( \text{Update triggered by a type-I stage} \)
   if \( n \in \mathcal{L} \) then
   \( \bar{b} := \min\{2\sqrt{2} \left( \sqrt{\frac{2H^2 t}{n} + \frac{\ln n}{n}} \right) + 7 \left( \frac{\ln n}{n} + \frac{\ln n}{n} \right) + 4 \left( \frac{\bar{H}^2}{n} + \frac{\bar{H}^2}{n} \right), \frac{1}{1 - \gamma}\}; \)
   \( Q(s, a) \leftarrow \min\{r(s, a) + \gamma (\tilde{\mu} / \bar{n} + \bar{\mu} / n + \bar{b}), Q(s, a)\} \)
   \( \bar{N}(s, a) \leftarrow 0; \quad \bar{\mu}(s, a) \leftarrow 0; \quad V(s) \leftarrow \max_{a} Q(s, a); \)

5. \( \text{Update triggered by a type-II stage} \)
   if \( n \in \mathcal{L} \) then
   \( \bar{b} := \min\{2\sqrt{2} \left( \sqrt{\frac{2H^2 t}{n} + \frac{\ln n}{n}} \right) + 7 \left( \frac{\ln n}{n} + \frac{\ln n}{n} \right) + 4 \left( \frac{\bar{H}^2}{n} + \frac{\bar{H}^2}{n} \right), \frac{1}{1 - \gamma}\}; \)
   \( Q(s, a) \leftarrow \min\{r(s, a) + \gamma (\tilde{\mu} / \bar{n} + \bar{b}), Q(s, a)\}; \)
   \( \bar{N}(s, a) \leftarrow 0; \quad \bar{\mu}(s, a) \leftarrow 0; \quad V(s) \leftarrow \max_{a} Q(s, a); \)

6. \( \text{if } \sum_{a'} N(s, a') = N_1 \text{ then } V^{\text{ref}}(s) \leftarrow V(s); \{\text{Learn the reference value function}\} \)

end for

D.2 Proof of Theorem 1

We start from showing that the Q function is optimistic and non-increasing.

Proposition 15. With probability \( \left( 1 - SA \left( 4\bar{J}(2 \log_2(N_0 H) + 1) + \bar{J} \right) p \right) \), it holds that \( Q_t(s, a) \geq Q^*(s, a) \) and \( Q_{t+1}(s, a) \leq Q_t(s, a) \) for any \( t \geq 1 \) and \( (s, a) \in S \times A \).

In the proof of Proposition 15 in Appendix D.3.1, we introduce the desired event \( E_2 \) by (12). Moreover, we use \( E_2 \) to denote the complement event of \( E_2 \). As will be shown later in (75), we have

\[ \mathbb{P}[E_2] \geq \left( 1 - SA \left( 4\bar{J}(2 \log_2(N_0 H) + 1) + \bar{J} \right) p \right), \]

and thus

\[ \mathbb{P}[\overline{E_2}] \leq SA \left( 4\bar{J}(2 \log_2(N_0 H) + 1) + \bar{J} \right) p. \]

The analysis will be done assuming the successful event \( E_2 \) throughout the rest of this section.

Since the type-II stages in UCB-MULTISTAGE-ADVANTAGE are exactly the same as that in UCB-MULTISTAGE, using the same way as in the proof of Lemma 12, we can prove the following lemma (and the proof is omitted).

Lemma 16. Conditioned on \( E_2 \), for any \( \epsilon_1 \in \left[ \epsilon, \frac{1}{1 - \gamma} \right] \), it holds that

\[ \sum_{t=1}^{\infty} I[V_t(s_t) - V^*(s_t) \geq \epsilon_1] \leq O\left( \frac{SAH^5 \ln(\frac{4H^5}{\epsilon_1})}{\epsilon_1} \right). \]

We now define the bandit loss

\[ b_t^*(s, a) := \min\{2\sqrt{2} \left( \sqrt{\frac{\Psi(P_{s,a})}{n_t(s, a)}} \right), \frac{1}{1 - \gamma}\}. \]
Similarly to the way to derive (13), we can show that

\[ V_t(s) - V^\pi_t(s) \leq \sum_{s,a} w_t(s, a) \left( 2b_t(s, a) + \gamma P_{s,a}(V_{t-1}^\pi(s,a) - V_t) \right) + \frac{\epsilon}{8} \]

\[ = 2 \sum_{s,a} w_t(s, a)b_t^*(s, a) + 2 \sum_{s,a} w_t(s, a)(\tilde{b}_t(s, a) - b_t^*(s, a)) \]

\[ + \gamma \sum_{s,a} w_t(s, a)P_{s,a}(V_{t-1}^\pi(s,a) - V_t) + \frac{\epsilon}{8} \]

\[ \leq 2 \sum_{s,a} w_t(s, a)b_t^*(s, a) + 2 \sum_{s,a} w_t(s, a)\text{clip}(\tilde{b}_t(s, a) - b_t^*(s, a), \frac{\epsilon}{8H}) \]

\[ + \gamma \sum_{s,a} w_t(s, a)P_{s,a}\text{clip}(V_{t-1}^\pi(s,a) - V_t, \frac{\epsilon}{8H}) + \frac{\epsilon}{2}, \]  

(66)

where we re-define the following notations,

\[ \alpha_t := P_{s_t, a_t}\text{clip}(V_{t-1}^\pi(s_t, a_t) - V_t, \frac{\epsilon}{8H}), \]

\[ \beta_t := \sum_{s,a} w_t(s, a) \left( 2\text{clip}(\tilde{b}_t(s, a) - b_t^*(s, a), \frac{\epsilon}{8H}) + \gamma P_{s,a}\text{clip}(V_{t-1}^\pi(s,a) - V_t, \frac{\epsilon}{8H}) \right), \]

\[ \bar{\beta}_t := 2\text{clip}(\tilde{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{8H}) + P_{s_t, a_t}\text{clip}(V_{t-1}^\pi(s_t,a_t) - V_t, \frac{\epsilon}{8H}). \]

To handle the first term in RHS of (66), we prove that

**Lemma 17.** Define \( \Lambda = \left[ \log_2(\frac{256H^4}{\epsilon}) \right] \). With probability \( (1 - 2\Lambda_\epsilon) \), it holds that

\[ \sum_{t \geq 1} \sum_{s,a} w_t(s, a)b_t^*(s, a) > \frac{\epsilon}{8} \leq O \left( \frac{SAH^3\Lambda^3}{\epsilon^2} + \frac{SAH^4BA^2}{\epsilon} \ln(N_0) \right). \]

We remark that our proof of Lemma 17 is quite similar to the method of knownness in (13), in the sense that both methods rely on an argument based on the partition of the states. However, our way of partitioning seems to be simpler as we divide the states into different subsets only according to their numbers. The detailed proof is presented in Appendix D.3.2.

For the second term, we prove the pseudo-regret bounds as below.

**Lemma 18.** If we choose \( B = H^3 \), with probability \( 1 - SA\bar{J}(2\mathbb{P}[E_2] + 4p) \) it holds that

\[ \sum_{t \geq 1} \text{clip}(\tilde{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{8H}) \]

\[ \leq O \left( \frac{SAH^2}{\epsilon} \right) + \bar{O} \left( \frac{S^{3/2} A^{3/2} H^{7/2}}{\epsilon^{1/2}} + \frac{SAH^{59/12}}{\epsilon^{1/3}} + \frac{S^{5/4} A^{5/4} H^{21/8}}{\epsilon^{1/4}} + S^2 A^2 H^8 \right). \]

At last, following the same arguments as the proof of Lemma 6 for the third term we show that (the proof is omitted)

**Lemma 19.** With probability \( 1 - (\mathbb{P}[E_2] + p) \) it holds that

\[ \sum_{t \geq 1} \alpha_t \leq O \left( \frac{SAH^5 \ln(\frac{4H}{\epsilon B})}{\epsilon B} + SAH^3 B + SAH \ln(N_0) \right). \]

By Lemma 18 and 19, we obtain that

**Lemma 20.** With probability \( 1 - (SA\bar{J}(2\mathbb{P}[E_2] + 4p) + \mathbb{P}[E_2] + p) \), it holds that

\[ \sum_{t \geq 1} \bar{\beta}_t \leq O \left( \frac{SAH^2 \ln(\frac{4H}{\epsilon})}{\epsilon} \right) + \bar{O} \left( \frac{S^2 A^2 H^{59/12}}{\epsilon^{1/2}} \right). \]
Following the same arguments in Section 5.3, we obtain that with probability \( 1 - (SA \bar{J}(2P [E_2] + 4p) + P [E_2] + 2p) \), it holds that
\[
\sum_{i \geq 1} 1 \left[ \beta_i > \frac{\epsilon}{4} \right] \leq O \left( \frac{SAH^2 \ln(4H)}{\epsilon^2} \right) + \tilde{O} \left( \frac{S^2 A^2 H^{59/12}}{\epsilon^{3/2}} \right).
\] (67)

By Proposition 15 (restated) and (67), we conclude that with probability \( 1 - (SA \bar{J}(2P [E_2] + 4p) + 2P [E_2] + 2H \Lambda p + 2p) \), it holds that
\[
\sum_{i \geq 1} 1 \left[ V^* (s_i) - V^\pi_i (s_i) > \epsilon \right] \leq O \left( \frac{SAH^3 \Lambda^2 \ln(4H)}{\epsilon^2} \right) + O \left( \frac{SAH^7 \Lambda^2 \ln(N)}{\epsilon} \right) + \tilde{O} \left( \frac{S^2 A^3 H^{59/12}}{\epsilon^{3/2}} \right).
\]

The proof is finished by replacing \( p \) with \( \frac{p}{32SA^2J^2 \ln(2N_0H) + 4H \Lambda} \).

D.3 Missing Proofs in Appendix D.2

D.3.1 Proof of Proposition 15

Proposition 15 (restate). With probability \( 1 - SA \left( 4J(2 \log_2(N_0H) + 1) + \bar{J} \right) p \), it holds that \( Q_t(s, a) \geq Q^*(s, a) \) and \( Q_{t+1}(s, a) \leq Q_t(s, a) \) for any \( t \geq 1 \) and \( (s, a) \in S \times A \). The rest of this subsection is devoted to the proof of Proposition 15.

Let \( (s, a, j) \) be fixed. Let \( \mu_{ref}, \mu, \sigma_{ref} \), \( \sigma \) and \( \bar{b} \) be the values of \( \mu_{ref}, \mu, \sigma_{ref}, \sigma \) and \( \bar{b} \) in (63) in the \( j \)-th update type-I. Define \( \bar{c}_i \) to be the time when the \( i \)-th visit of \((s, a, j) \) occurs and \( \bar{l}_i \) to be the time the \( i \)-th visit of \((s, a) \) occurs respectively. Let \( \bar{n} \) and \( \bar{n} \) be the shorthands of \( \bar{c}_j \) and \( \sum_{i=1}^{\bar{n}} \bar{c}_i \) respectively.

We consider the events:
\[
\tilde{E}_1^{(j)} (s, a) := \chi_1^{(j)}(s, a) := \frac{1}{n} \sum_{i=1}^{\bar{n}} \left( V^{ref}_i (s_i, a) - P_{s, a} V^{ref}_i \right) \leq 2 \sqrt{2} \left( \frac{\sigma_{ref} / n - (\mu_{ref} / n)^2}{n} \right) t + \frac{7H \bar{c}_3}{n} + \frac{4H \bar{c}_3}{n}.
\]
and
\[
\tilde{E}_2^{(j)} (s, a) := \chi_2^{(j)}(s, a) := \frac{1}{n} \sum_{i=1}^{\bar{n}} \left( W_i (s_i, a) - P_{s, a} W_i \right) \leq 2 \sqrt{2} \left( \frac{\sigma / \bar{n} - (\mu / \bar{n})^2}{\bar{n}} \right) t + \frac{7H \bar{c}_3}{\bar{n}^{3/4}} + \frac{4H \bar{c}_3}{\bar{n}}.
\]

where \( W_i = V_i - V_i^{ref} \). If both \( \tilde{E}_1^{(j)} (s, a) \) and \( \tilde{E}_2^{(j)} (s, a) \) occur, then we have that
\[
\begin{align*}
&\ r(s, a) + \frac{\mu_{ref}}{n} - \frac{\mu}{\bar{n}} + \bar{b} \\
&\ = \frac{1}{n} \sum_{i=1}^{\bar{n}} V_i^{ref} + P_{s, a} \left( \frac{1}{n} \sum_{i=1}^{\bar{n}} V_i^{ref} \right) + \frac{1}{n} \sum_{i=1}^{\bar{n}} \left( V_i^{ref} - V_i^{ref} \right) + \chi_1^{(j)}(s, a) + \chi_2^{(j)}(s, a) + \bar{b} \\
&\ \geq r_k(s, a) + P_{s, a} \left( \frac{1}{n} \sum_{i=1}^{\bar{n}} V_i \right) + \frac{1}{n} \sum_{i=1}^{\bar{n}} \left( V_i^{ref} - V_i^{ref} \right) + \chi_1^{(j)}(s, a) + \chi_2^{(j)}(s, a) + \bar{b} \\
&\ \geq r_k(s, a) + P_{s, a} \left( \frac{1}{n} \sum_{i=1}^{\bar{n}} V_i \right) ,
\end{align*}
\]

where Inequality (68) holds by the fact \( V_i^{ref} \) is non-increasing in \( t \) and Inequality (69) follows by the definition of \( \bar{b} \).
On the other hand, for the $j'$-th type-II update, we consider the following same events as in the proof of Proposition 3:

$$E^{(j')}(s, a) = \left\{ \frac{1}{\bar{e}^{j'}} \sum_{i=1}^{\bar{e}^{j'}} V^*(s_{i_{1}+1}) + \bar{b}^{(j')} \geq P_{s,a} V^* \right\}. \quad (70)$$

Assuming $E^{(j')}(s, a)$ holds, we then have

$$r(s, a) + \gamma \bar{e}^{j'} \sum_{i=1}^{\bar{e}^{j'}} V_{i}^*(s_{i_{1}+1}) + \bar{b}^{(j')} \geq r(s, a) + \gamma P_{s,a} V^* + \gamma \left( \frac{1}{\bar{e}^{j'}} \sum_{i=1}^{\bar{e}^{j'}} (V_{i}^*(s_{i_{1}+1}) - V^*(s_{i_{1}+1})) \right). \quad (71)$$

Let

$$E_2 = (\cap_{s,a,j} E_1^{(j)}(s, a)) \cap (\cap_{s,a,j} E_2^{(j)}(s, a)) \cap (\cap_{s,a,j} E^{(j')}(s, a)). \quad (72)$$

Assuming $E_2$ holds, by the update rule (63) and (64) and noting that $V_i$ is non-increasing, for any $t \geq 2$ and $(s, a)$, it holds either $Q_i(s, a) = Q_{t-1}(s, a)$ or

$$Q_t(s, a) \geq r_{s,a} + \gamma P_{s,a} V^* + \sum_{\nu' \leq t} v_{\nu'} (V_{\nu'} - V^*)$$

for some non-negative $S$-dimensional vectors $v_1, v_2, \ldots, v_{t-1}$. Noting that $Q_1(s, a) = \frac{1}{1-\gamma} \geq Q^*(s, a)$ for any $(s, a)$, the conclusion follows easily by induction.

Therefore, it suffices to bound $P[E_2]$.

**Lemma 21.** For any $(s, a, j)$, $P[\bar{E}_1^{(j)}(s, a)] \geq 1 - 2(\log_2(N_0 H) + 1)p$.

**Proof.** Define $\forall(x, y) = xy^2 - (xy)^2$ for two vectors with the same dimension. Noticing that $s_{t+1}$ is independent of $V_{t+1}^{\text{ref}}$ conditioned on $F_{t-1}$, by Lemma 10 with $\epsilon = H$, we have that with probability $(1 - 2 \log_2(nH))p$, it holds that

$$\chi_1^{(j)}(s, a) = \frac{1}{n} \sum_{i=1}^{n} (V_{i}^{\text{ref}}(s_{i_{1}+1}) - P_{s,a} V_{i}^{\text{ref}})$$

$$\leq 2\sqrt{2} \sqrt{\left( \sum_{i=1}^{n} \frac{\forall(P_{s,a}, V_{i}^{\text{ref}}))}{n^2} \right)^{\frac{1}{2}} + \frac{2Ht}{n} + \frac{2Ht}{n}$$

$$\leq 2\sqrt{2} \sqrt{\left( \sum_{i=1}^{n} \frac{\forall(P_{s,a}, V_{i}^{\text{ref}}))}{n^2} \right)^{\frac{1}{2}} + \frac{4Ht}{n}. \quad (73)$$

By definition of $\mu^{\text{ref}}$ and $\mu^{\text{ref}}$, we have that

$$\sum_{i=1}^{n} \forall(P_{s,a}, V_{i}^{\text{ref}}) = \sum_{i=1}^{n} (P_{s,a}(V_{i}^{\text{ref}})^2 - (P_{s,a} V_{i}^{\text{ref}})^2)$$

$$= \sum_{i=1}^{n} (V_{i}^{\text{ref}}(s_{i+1}))^2 - \frac{1}{n} \left( \sum_{i=1}^{n} V_{i}^{\text{ref}}(s_{i+1}) \right)^2 + \chi_3 + \chi_4 + \chi_5$$

$$= \mu^{\text{ref}} - \frac{1}{n} (\mu^{\text{ref}})^2 + \chi_3 + \chi_4 + \chi_5,$$

where

$$\chi_3 := \sum_{i=1}^{n} (P_{s,a}(V_{i}^{\text{ref}})^2 - (V_{i}^{\text{ref}}(s_{i+1}))^2)$$

$$\chi_4 := \frac{1}{n} \left( \sum_{i=1}^{n} V_{i}^{\text{ref}}(s_{i+1}) \right)^2 - \frac{1}{n} \left( \sum_{i=1}^{n} P_{s,a} V_{i}^{\text{ref}} \right)^2$$

$$\chi_5 := \frac{1}{n} \left( \sum_{i=1}^{n} P_{s,a} V_{i}^{\text{ref}} \right)^2 - \frac{1}{n} \left( \sum_{i=1}^{n} P_{s,a} V_{i}^{\text{ref}} \right)^2.$$

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The proof is completed.

Via a union bound over $\chi_5 = \frac{1}{n} \left( \sum_{i=1}^{n} P_{s,a} V_{i}^{\text{ref}} \right)^2 - \sum_{i=1}^{n} (P_{s,a} V_{i}^{\text{ref}})^2$.

By Azuma’s inequality, we have that

$$\mathbb{P}\left[|\chi_3| > H^2 \sqrt{2n} \right] \leq p$$

and

$$\mathbb{P}\left[|\chi_4| > 2H^2 \sqrt{2n} \right] \leq \mathbb{P}\left[2H \cdot \left| \sum_{i=1}^{n} (V_{i}^{\text{ref}}(s_{i+1}) - P_{s,a} V_{i}^{\text{ref}}) \right| > 2H^2 \sqrt{2n} \right] \leq p.$$ 

On the other hand, by Cauchy-Schwartz inequality, we have $\chi_5 \leq 0$. It then follow that

$$\mathbb{P}\left[ \sum_{i=1}^{n} \chi(P_{s,a}, V_{i}^{\text{ref}}) > \sigma^{\text{ref}} - \frac{1}{n} (\mu^{\text{ref}})^2 + 5H^2 \sqrt{\frac{n}{2}} \right] \leq 2p.$$ 

(74)

Combining (73) and (74), we have that

$$\mathbb{P}\left[ \hat{E}_1^{(j)}(s,a) \right] \geq 1 - \mathbb{P}\left[ \chi^{(j)}(s,a) > 2H \sqrt{2} \left( \frac{\sum_{i=1}^{n} \chi(P_{s,a}, V_{i}^{\text{ref}}))t}{n} + \frac{4Ht}{n} \right) \right]$$

$$\mathbb{P}\left[ \sum_{i=1}^{n} \chi(P_{s,a}, V_{i}^{\text{ref}}) > \sigma^{\text{ref}} - \frac{1}{n} (\mu^{\text{ref}})^2 + 5H^2 \sqrt{\frac{n}{2}} \right] \right]$$

$$\geq 1 - 2(\log_2(nH) + 1)p$$

$$\geq 1 - 2(\log_2(N_0H) + 1)p.$$ 

Following similar arguments as above, we can prove that $\mathbb{P}\left[ \hat{E}_1^{(j)}(s,a) \right] \geq 1 - 2(\log_2(N_0H) + 1)p$ for any $1 \leq j \leq \hat{J}$. At last, by Azuma’s inequality, $\mathbb{P}\left[ \hat{E}(s,a) \right] \geq 1 - p$ for any $j'$ and $(s,a)$.

Via a union bound over $1 \leq j \leq \hat{J}$ and $1 \leq j' \leq \hat{J}$, we obtain that

$$\mathbb{P}\left[ E_2 \right] \geq 1 - 4SAJ(\log_2(N_0H) + 1)p - SAJp.$$ 

(75)

The proof is completed.

D.3.2 Proof of Lemma 17

Lemma 17 (restated). Define $\Lambda = \left[ \log_2 \left( \frac{256H^4}{\epsilon^2} \right) \right]$. With probability $(1 - 2H\Lambda p)$, it holds that

$$\sum_{t \geq 1} \mathbb{E} \left[ \sum_{s,a} w_t(s,a)b_t^*(s,a) > \frac{\epsilon}{8} \right] \leq O \left( \frac{SAH^2\Lambda^3}{\epsilon^2} + \frac{SAH^4BA^2 \ln(N_0)}{\epsilon} \right).$$

The rest of this subsection is devoted to the proof of Lemma 17.

Define $\mathcal{S}_{t,0} := \{(s,a)|n_t(s,a) < t\}$, $\mathcal{S}_{t,u} := \{(s,a)|2^{u-1}t \leq n_t(s,a) < 2^u t\}$ for $u = 1, 2, \ldots, \Lambda = \left[ \log_2 \left( \frac{256H^4}{\epsilon^2} \right) \right]$ and $\mathcal{S}_t := \{(s,a)|n_t(s,a) > \frac{H^4}{\epsilon^2} \}$. Furthermore, we define

$$\beta_t^u := \sum_{(s,a) \in \mathcal{S}_{t,u}} w_t(s,a)b_t^*(s,a)$$

and

$$\beta_t^u := \sum_{u} \beta_t^u = \sum_{(s,a) \in \mathcal{S}_t} w_t(s,a)b_t^*(s,a).$$
By the definition of $b^*_t(s, a)$, we obtain that for $1 \leq u \leq \Lambda$, 

$$
\beta^*_t = \sum_{(s, a) \in S_t, u} w_t(s, a) b^*_t(s, a)
$$

$$
\leq 2\sqrt{\epsilon_t} \sum_{(s, a) \in S_t, u} w_t(s, a) \sqrt{\frac{\nu_t(s, a) \cdot V(P_{s, a}, V^*)}{n_t(s, a)}}
$$

$$
\leq 2 \sqrt{\frac{2}{2u-1}} \sum_{(s, a) \in S_t, u} w_t(s, a) \sqrt{\nu_t(s, a) \cdot V(P_{s, a}, V^*)}
$$

$$
\leq 2 \sqrt{\frac{2}{2u-1}} \sum_{(s, a) \in S_t, u} w_t(s, a) \cdot \sqrt{\sum_{(s, a) \in S_t, u} \nu_t(s, a) \cdot V(P_{s, a}, V^*)}, \tag{76}
$$

and for $0 \leq u \leq \Lambda$, 

$$
\beta^*_t \leq \frac{1}{1 - \gamma} \sum_{(s, a) \in S_t, u} w_t(s, a).
$$

Define $w_{t, u} := \sum_{(s, a) \in S_t, u} w_t(s, a)$ and $\nu_t = \sum_{s, a} w_t(s, a) \cdot V(P_{s, a}, V^*)$. Note that 

$$
\nu_t = \sum_{s, a} w_t(s, a) (P_{s, a}(V^*)^2 - (P_{s, a}V^*)^2)
$$

$$
= \sum_{s, a} w_t(s, a) P_{s, a}(V^*)^2 - \frac{1}{\gamma^2} \sum_{s, a} w_t(s, a) (Q^*(s, a) - r(s, a))^2
$$

$$
\leq \sum_{s, a} w_t(s, a) P_{s, a}(V^*)^2 - \sum_{s, a} w_t(s, a) (Q^*(s, a) - r(s, a))^2
$$

$$
= \sum_{s, a} w_t(s, a) (P_{s, a}(V^*)^2 - (V^*(s))^2) + \sum_{s, a} w_t(s, a) ((V^*(s))^2 - (Q^*(s, a))^2) + \frac{2H}{1 - \gamma}
$$

$$
\leq \sum_{s, a} w_t(s, a) (P_{s, a}(V^*)^2 - (V^*(s))^2) + \frac{2}{1 - \gamma} \sum_{s, a} w_t(s, a) (V^*(s) - Q^*(s, a)) + \frac{2H}{1 - \gamma}
$$

$$
\leq \frac{1}{(1 - \gamma)^2} + \frac{2}{1 - \gamma} \sum_{s, a} w_t(s, a) (V^*(s) - Q^*(s, a)) + \frac{2H}{1 - \gamma}
$$

$$
\leq \frac{1}{(1 - \gamma)^2} + \frac{2}{(1 - \gamma)} (V^*(s_t) - V^{\pi_t}(s_t)) + \frac{2H}{1 - \gamma} \tag{77}
$$

$$
\leq 5H^2. \tag{79}
$$

Here Inequality (77) holds by the fact that 

$$
\sum_{s, a} w_t(s, a) (P_{s, a} - 1_s)(V^*)^2 = \sum_{s, a} (\mathbb{1}[a = \pi_t(s)] \sum_{i=0}^{H-1} \mathbb{1}_{s_t}(\gamma P_{s_t})^i 1_s) \cdot (P_{s, a} - 1_s)(V^*)^2
$$

$$
= \sum_{s, a} \mathbb{1}[a = \pi_t(s)] \left( 1_{s_t}^T (\gamma P_{s_t})^H 1_s - \mathbb{1}[s = s_t] \right) (V^*(s))^2
$$

$$
\leq \frac{1}{(1 - \gamma)^2},
$$

and Inequality (78) is due to the bound on the following telescoping sum, 

$$
V^*(s_t) - V^{\pi_t}(s_t) = \sum_{s, a} (\mathbb{1}[a = \pi_t(s)] \sum_{i=0}^{H-1} \mathbb{1}_{s_t}(\gamma P_{s_t})^i 1_s) \cdot (V^*(s) - Q^*(s, a))
$$

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We will bound the number of steps in which there exists \( u \) which means 
\[
\sum_{s,a} w_t(s,a)(V^*(s) - Q^*(s,a)).
\]

Combining (79) with the fact that \( \sum_{(s,a) \in S} w_t(s,a)b^*_t(s,a) \leq \frac{\epsilon}{10} \), we obtain that, if \( \beta^*_t > \frac{\epsilon}{8} \), there exists \( u \) such that \( \beta^*_t > \frac{\epsilon}{16\Lambda} \), which implies that \( w_t,u > \max\{ \frac{1}{10240} \cdot \frac{2^{u-1}\epsilon^2}{H^2\Lambda^2} \cdot \frac{\epsilon(1-\gamma)}{16\Lambda} \} \).

We will bound the number of steps in which there exists \( u \) satisfying \( w_t,u > \max\{ \frac{1}{10240} \cdot \frac{2^{u-1}\epsilon^2}{H^2\Lambda^2} \cdot \frac{\epsilon(1-\gamma)}{16\Lambda} \} \) by following lemma.

**Lemma 22.** For any \( k \in \{1, 2, \ldots, H\} \) and \( u \in \{1, 2, \ldots, \Lambda\} \), with probability 1 \( - p \),
\[
\sum_{t \geq 0} \mathbb{I}\{ w_t,H+k,u > \frac{1}{10240} \cdot \frac{2^{u-1}\epsilon^2}{H^2\Lambda^2} \} \leq O\left( \frac{SABH^4\Lambda^2\ln(N_0)}{2^{u-1}\epsilon^2} + \frac{SAH^2\Lambda^2}{\epsilon^2} \right). \tag{80}
\]

Moreover, for any \( u \geq 0 \), with probability 1 \( - p \),
\[
\sum_{t \geq 0} \mathbb{I}\{ w_t,H+k,u > \frac{1}{16\Lambda} \} \leq O\left( \frac{HA}{\epsilon} \left( SAH^2B\ln(N_0) + SAH + 2^{u+2}SA \right) \right). \tag{81}
\]

**Proof.** Define
\[
\tilde{U}_{t,u} = \mathbb{I}\{ \exists(s,a), i \in \{1, 2, \ldots, H-1\} \text{ such that } S_{i+1,u} \neq S_{i,u} \text{ or } Q_{i+1}(s,a) \neq Q_{i}(s,a) \},
\]
and
\[
\tilde{w}_t(s,a) = (1 - \tilde{U}_{t,u}) \sum_{i=0}^{H-1} \mathbb{I}\{ (s_{i+1}, a_{i+1}) \in S_{i+1,u} \} + H\tilde{U}_{t,u}.
\]

Note that \( \tilde{w}_t,H+k \) is measurable with respect to \( \mathcal{F}_t = \mathcal{F}_{(t+1)H+k-1} \) and \( E[\tilde{w}_t,H+k | \mathcal{F}_t] = w_t,H+k \), we then have that by Lemma[11]
\[
P\left( \sum_{t \geq 0} w_t,H+k > 8SAH^2B\ln(N_0) + 8SAH + 2^{u+2}SA \right) \leq \sum_{t \geq 0} \tilde{w}_t,H+k \leq 2SAH^2B\ln(N_0) + 2SAH + 2^uSA \leq p. \tag{82}
\]

On the other hand, we have that
\[
\sum_{t \geq 0} \tilde{w}_t,H+k \leq H \sum_{t \geq 0} \tilde{U}_{t,H+k} + \sum_{t \geq 1} \mathbb{I}\{ (s_t, a_t) \in S_{t,u} \} \leq 2SAH^2B\ln(N_0) + 2SAH + \sum_{t \geq 1} \mathbb{I}\{ (s_t, a_t) \in S_{t,u} \} \leq 2SAH^2B\ln(N_0) + 2SAH + 2^uSA, \tag{83}
\]
where Inequality (83) is because \( S_{t,u} \) changes at most \( 2SA \) times in \( t \), and Inequality (84) is by the fact that \( 2^{u-1} \leq n_t(s,a) < 2^u \) implies that \( 2^u \leq n_t(s,a) < 2^{u+1} \). It then follows that
\[
P\left( \sum_{t \geq 0} w_t,H+k > 8SAH^2B\ln(N_0) + 8SAH + 2^{u+2}SA \right) \leq p,
\]
which means
\[
P\left[ \sum_{t \geq 0} \mathbb{I}\{ w_t,H+k,u > \frac{1}{10240} \cdot \frac{2^{u-1}\epsilon^2}{H^2\Lambda^2} \} \right] > 10240 \left( \frac{16SAH^4\Lambda^2\ln(N_0)}{2^{u-1}\epsilon^2} + \frac{8SAH^2\Lambda^2}{\epsilon^2} \right) \leq p
\]
and
\[
P\left[ \sum_{t \geq 0} \mathbb{I}\{ w_t,H+k,u > \frac{\epsilon(1-\gamma)}{16\Lambda} \} \right] > \frac{16HA}{\epsilon} \left( 8SAH^2B\ln(N_0) + 8SAH + 2^{u+2}SA \right) \leq p.
\]

The proof is completed. \( \square \)
For $u$ such that $2^u \leq \frac{B H^2 \ln(N_0)}{\epsilon}$ or $u = 0$, we plug $u$ and $k = 1, 2, \ldots, H$ into (81) and obtain that with probability $1 - Hp$,

$$\sum_{t \geq 1} \mathbb{I} \left[ w_{t,u} > \frac{\epsilon(1 - \gamma)}{16\Lambda} \right] \leq O \left( \frac{SAH^4 B \Lambda \ln(N_0)}{\epsilon} \right).$$

(85)

For $u$ such that $2^u > \frac{B H^2 \ln(N_0)}{\epsilon}$, we plug $u$ and $k = 1, 2, \ldots, H$ into (80) and obtain that with probability $1 - Hp$,

$$\sum_{t \geq 1} \mathbb{I} \left[ w_{t,u} > \frac{2^{u-1} \epsilon^2}{H^2 \Lambda^2} \right] \leq O \left( \frac{SAH^4 \Lambda^2 t}{\epsilon^2} \right).$$

(86)

Via a union bound over $u$, we have that with probability $1 - 2H\Lambda p$, it holds that

$$\sum_{t \geq 1} \mathbb{I} \left[ \beta_t^u > \frac{\epsilon}{8} \right] \leq \sum_{t \geq 1} \mathbb{I} \left[ \exists u, w_{t,u} > \max \left\{ \frac{1}{10240} \frac{2^{u-1} \epsilon^2}{H^2 \Lambda^2}, \frac{\epsilon(1 - \gamma)}{8\Lambda} \right\} \text{ and } w_{t,0} > \frac{\epsilon(1 - \gamma)}{8\Lambda} \right] \leq O \left( \frac{SAH^3 \Lambda^3 t}{\epsilon^2} + \frac{SAH^4 B \Lambda^2 \ln(N_0)}{\epsilon} \right).$$

(87)

D.3.3 Proof of Lemma [18]

**Lemma [18] (restated).** With probability $1 - SAJ(2^{\mathbb{F}[\mathbb{F}_2]} + 4p)$, it holds that

$$\sum_{t \geq 1} \text{clip} \left( \hat{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{8H} \right) \leq O \left( \frac{SAH^2 t}{\epsilon} \right) + O \left( \frac{S^{3/2} A^{3/2} H^{7/2} t}{\epsilon^{1/2}} + \frac{SAH^{50/12} t}{\epsilon^{3/5}} + \frac{S^{5/4} A^{5/4} H^{21/8} t}{\epsilon^{3/4}} + S^2 A^2 H^8 t \right).$$

The rest of this subsection is devoted to the proof of Lemma [18].

Let $s, a, j$ be fixed. We follow the notations in Appendix [D.3.1]. For $t$ in the $(j+1)$-th type-I stage of $(s, a)$, recalling the definition

$$\hat{b}_t(s_t, a_t) = \min \left\{ 2\sqrt{2} \left( \sqrt{\frac{\sigma + \bar{n} - \hat{n}}{\hat{n}}} - \frac{(\mu_{\text{ref}}/n)^2}{\hat{n}} \right)_t + \sqrt{\frac{\sigma_{\text{ref}}/n - (\mu_{\text{ref}}/n)^2}{n}}_t, \frac{\epsilon}{32H} \right\},$$

we have that

$$\text{clip} \left( \hat{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{8H} \right) \leq 4\text{clip} \left( 2\sqrt{2} \left( \sqrt{\frac{\sigma_{\text{ref}}/n - (\mu_{\text{ref}}/n)^2}{n}}_t - \sqrt{\frac{\sigma_{\text{ref}}/n - (\mu_{\text{ref}}/n)^2}{n}}_t, \frac{\epsilon}{32H} \right) + 4\text{clip} \left( 2\sqrt{2} \sqrt{\frac{\bar{n}}{\hat{n}} - \frac{(\mu_{\text{ref}}/n)^2}{\hat{n}}}, \frac{\epsilon}{32H} \right),$$

$$+ 4\text{clip} \left( \frac{H_t^{3/4}}{n^{3/4}} + \frac{H_t^{3/4}}{n^{3/4}}, \frac{\epsilon}{32H} \right),$$

$$+ 4\text{clip} \left( 5 \left( \frac{H_t}{n} + \frac{H_t}{n} \right), \frac{\epsilon}{32H} \right),$$

(89)

and the trivial bound

$$\text{clip} \left( \hat{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{8H} \right) \leq \frac{1}{1 - \gamma}.$$

(90)

Here, (89) is because $\text{clip}(a + b, 2\epsilon) \leq 2\text{clip}(a, \epsilon) + 2\text{clip}(b, \epsilon)$ for any non-negative $a, b, \epsilon$.  

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Let $V_{\text{ref}}^t$ be the value of $V_{\text{ref}}^t$ immediately before the beginning of the $t$-th step and $V_{\text{REF}}^t := \lim_{t \to \infty} V_{\text{ref}}^t$ (by the update rule of Algorithm 2, this limit exists). Furthermore, we define to $\lambda_t$ be the vector such that $\lambda_t(s) = \|\sum_{i=1}^t N_{i}(s, a) < N_1\|$ where $N_1 := c_{10}SHB\ln\left(\frac{4H}{\epsilon}\right)$ for some large enough constant $c_{10}$. By Lemma 16 with $\epsilon_1 = \omega := \frac{1}{\sqrt{V}}$ (assuming $\epsilon \leq \frac{1}{\sqrt{V}}$), we have that

$$
\mathbb{P}\left[|V_t^* - V_{\text{ref}}(s_t) - V_t^*(s_t)| \leq H\lambda_t(s_t) + \omega\right] \geq \mathbb{P}[E_2].
$$

(91)

We will deal with the four terms in RHS of (89) separately.

The $1$ term To handle this term, we introduce a lemma to bound $\frac{\sigma_{\text{ref}}^2}{n} - \left(\frac{\mu_{\text{ref}}^2}{n}\right)^2 - \mathbb{V}(P_{s,a}, V^*)$.

**Lemma 23.** With probability $1 - (\mathbb{P}[E_2] + 4p)$, it holds that

$$
\frac{\sigma_{\text{ref}}^2}{n} - \left(\frac{\mu_{\text{ref}}^2}{n}\right)^2 - \mathbb{V}(P_{s,a}, V^*) \leq 9\sqrt{2H^3} \frac{t}{n} + \frac{1}{n} \left(2H^2SA(J + J) + 10H^2SN_1\right) + 4H\omega.
$$

(92)

**Proof.** Note that

$$
\frac{\sigma_{\text{ref}}^2}{n} - \left(\frac{\mu_{\text{ref}}^2}{n}\right)^2 - \mathbb{V}(P_{s,a}, V^*) = \frac{1}{n}(\chi_6 + \chi_7 + \chi_8 + \chi_9),
$$

where

$$
\chi_6 := \sum_{i=1}^n \left((V_{i+1}^t(s_{i+1}))^2 - P_{s,a}(V_i^t)^2\right),
$$

$$
\chi_7 := \frac{1}{n} \left(\sum_{i=1}^nP_{s,a}V_i^t\right)^2 - \frac{1}{n} \left(\sum_{i=1}^n V_i^t(s_{i+1})\right)^2,
$$

$$
\chi_8 := \sum_{i=1}^n (P_{s,a}V_i^t)^2 - \frac{1}{n} \left(\sum_{i=1}^nP_{s,a}V_i^t\right)^2,
$$

$$
\chi_9 := \sum_{i=1}^n \mathbb{V}(P_{s,a}, V_i^t) - n\mathbb{V}(P_{s,a}, V^*).
$$

According to Azuma’s inequality, with probability $(1 - 2p)$ it holds that

$$
|\chi_6| \leq H^2\sqrt{2nt},
$$

(93)

$$
|\chi_7| \leq 2H \left|\sum_{i=1}^n (V_{i+1}^t(s_{i+1}) - P_{s,a}V_i^t)\right| \leq 2H^2\sqrt{2nt}.
$$

(94)

On the other hand, by direct computation, we have that

$$
\chi_8 = \sum_{i=1}^n (P_{s,a}V_i^t)^2 - \frac{1}{n} \left(\sum_{i=1}^nP_{s,a}V_i^t\right)^2
$$

$$
\leq \sum_{i=1}^n (P_{s,a}V_i^t)^2 - \frac{1}{n} \left(\sum_{i=1}^nP_{s,a}V_i^t\right)^2
$$

$$
= \sum_{i=1}^n ((P_{s,a}V_i^t)^2 - (P_{s,a}V_{\text{REF}})^2)
$$

$$
\leq 2H^2 \sum_{i=1}^n P_{s,a}\lambda_i
$$

(95)

$$
\leq 2H^2 \sum_{i=1}^n \lambda_i(s_{i+1}) + \sum_{i=1}^n (P_{s,a} - 1_{s_{i+1}})\lambda_i
$$

$$
= 2H^2 \sum_{i=1}^n (\lambda_i(s_{i+1}) - \lambda_i(1_{s_{i+1}})) + 2H^2 \sum_{i=1}^n \lambda_i(s_{i+1}) + 2H^2 \sum_{i=1}^n (P_{s,a} - 1_{s_{i+1}})\lambda_i
$$

(96)
\[ \leq 2H^2SA(\bar{J} + \bar{J}) + 2H^2SN_1 + 2H^2 \sum_{i=1}^{n} (P_{s,a} - 1_{s_{i+1}}) \lambda_{i}, \quad (97) \]

where Inequality (95) is by the fact that the number of updates of \( \lambda_t \) is by the definition of \( \lambda_t \) and Inequality (97) holds because \( \lambda_t \neq \lambda_{t+1} \) implies an update occurs at the \( t \)-th step and \( \sum_{i \geq 1} \lambda_t(s_i) \leq SN_1 \). Therefore, by Azuma’s inequality it holds that

\[ P \left[ \chi_8 > 2H^2SA(\bar{J} + \bar{J}) + 2H^2SN_1 + 2H^3\sqrt{2n} \right] \leq P \left[ 2H^2 \sum_{i=1}^{n} (P_{s,a} - 1_{s_{i+1}}) \lambda_{i} > 2H^3\sqrt{2n} \right] \leq p. \quad (98) \]

At last, the term \( \chi_9 \) could be bounded by

\[ \chi_9 = \sum_{i=1}^{n} \nabla(P_{s,a}, V^*_t) - n \nabla(P_{s,a}, V^*) \]

\[ \leq 4H \sum_{i=1}^{n} P_{s,a}(V^*_t - V^*) \]

\[ = 4H \sum_{i=1}^{n} (V^*_t(s_{i+1}) - V^*_t(s_{i+1}) + V^*_t(s_{i+1}) - V^*(s_{i+1})) + 4H \sum_{i=1}^{n} (P_{s,a} - 1_{s_{i+1}})(V^*_t - V^*) \]

\[ \leq 4H^2S + 4H \sum_{i=1}^{n} (V^*_t(s_{i+1}) - V^*(s_{i+1})) + 4H \sum_{i=1}^{n} (P_{s,a} - 1_{s_{i+1}})(V^*_t - V^*), \quad (99) \]

where Inequality (99) is by the fact that the number of updates of \( V^*_t \) is at most \( S \). Similarly, we have that

\[ P \left[ \chi_9 > 4H^2S + 4H^2SN_1 + 4Hn\omega + 4H^2\sqrt{2n} \right] \]

\[ \leq P \left[ \sum_{i=1}^{n} (V^*_t(s_{i+1}) - V^*(s_{i+1})) > \sum_{i=1}^{n} (H\lambda_t(s_{i+1}) + \omega) \right] + P \left[ \sum_{i=1}^{n} (P_{s,a} - 1_{s_{i+1}})(V^*_t - V^*) > H\sqrt{2n} \right] \]

\[ \leq P[E_2] + p, \quad (100) \]

where (100) holds by (91).

Combining (92), (93), (94), (98) and (100), with probability \( 1 - (P[E_2] + 5p) \) it holds that

\[ \frac{\mu^*}{n} - \frac{\mu^*}{n} \leq \nabla(P_{s,a}, V^*) \]

\[ \leq \frac{1}{n} \left( 3H^2\sqrt{2n} + 2H^2SA(\bar{J} + \bar{J}) + 2H^2SN_1 + 2H^3\sqrt{2n} + 4H(2H^2SN_1) + 4H^2\sqrt{2n} + 4H \omega \right) \]

\[ \leq 9\sqrt{2H^3} \sqrt{\frac{1}{n} + \frac{1}{n} \left( 2H^2SA(\bar{J} + \bar{J}) + 10H^2SN_1 \right) + 4H \omega}. \]

\[ \square \]

By Lemma 23 with probability \( 1 - (P[E_2] + 4p) \) it holds that

\[ \left( \frac{\sigma^*/n - (\mu^*/n)^2}{n} - \nabla(P_{s,a}, V^*) \right) \]

\[ \leq \frac{9\sqrt{2H^3}e^{3/2}}{n^{3/2}} + \frac{(2H^2SA(\bar{J} + \bar{J}) + 10H^2SN_1)e}{n^2} + \frac{4H \omega e}{n}. \quad (101) \]

As a result, for \( n > N_2 := c_3 \frac{H^3\omega e}{H^2} + c_4 \frac{H^{10/3}e}{\epsilon^2} + c_5 \frac{H}{\epsilon^2} \left( \frac{H^2SA(\bar{J} + \bar{J}) + H^2SN_1}{\epsilon} \right) \) with sufficient large constants \( c_4 \) and \( c_5 \), it holds that

\[ 2\sqrt{2} \left( \frac{\sigma^*/n - (\mu^*/n)^2}{n} - \nabla(P_{s,a}, V^*) \right) \leq \frac{\epsilon}{32H}. \quad (102) \]
The 2 term Direct computation gives that

$$\frac{\hat{\sigma}/\hat{n} - (\hat{\mu}/\hat{n})^2}{\hat{n}} \leq \frac{1}{\hat{n}^2} \sum_{i=1}^{\hat{n}} \left( V_i(s_{i+1}) - V_i^{\text{ref}}(s_{i+1}) \right)^2 \leq \frac{1}{\hat{n}^2} \sum_{i=1}^{\hat{n}} \left( V_i^{\text{ref}}(s_{i+1}) - V^*(s_{i+1}) \right)^2. \tag{103}$$

Also note that

$$\left| \sum_{i=1}^{\hat{n}} \left( V_i^{\text{ref}}(s_{i+1}) - V^*(s_{i+1}) \right)^2 - \left( V_{i+1}^{\text{ref}}(s_{i+1}) - V^*(s_{i+1}) \right)^2 \right| \leq 2H \cdot \left\| \sum_{i=1}^{\hat{n}} \left( V_i^{\text{ref}}(s_{i+1}) - V_{i+1}^{\text{ref}}(s_{i+1}) \right) \right\| \leq 2H^2(SA(\bar{J} + \bar{J})). \tag{104}$$

It then follows that

$$\mathbb{P}\left[ \frac{\hat{\sigma}/\hat{n} - (\hat{\mu}/\hat{n})^2}{\hat{n}} > \frac{H^2(2SN_1 + 2SA(\bar{J} + \bar{J}))}{\hat{n}^2} + \frac{2\omega^2}{\hat{n}} \right] \leq \mathbb{P}\left[ \sum_{i=1}^{\hat{n}} \left( V_i^{\text{ref}}(s_{i+1}) - V^*(s_{i+1}) \right)^2 > H^2(2SN_1 + 2SA(\bar{J} + \bar{J})) + 2\omega^2 \hat{n} \right] \leq \mathbb{P}\left[ \sum_{i=1}^{\hat{n}} \left( V_i^{\text{ref}}(s_{i+1}) - V^*(s_{i+1}) \right)^2 > \sum_{i=1}^{\hat{n}} (H\lambda_i(s_{i+1}) + \omega)^2 \right] \leq \mathbb{P}[\mathcal{E}_2],$$

where the last inequality is due to (91). Therefore, we have that

$$\mathbb{P}\left[ \frac{\hat{\sigma}/\hat{n} - (\hat{\mu}/\hat{n})^2}{\hat{n}} > \sqrt{\frac{H^2(2SN_1 + 2SA(\bar{J} + \bar{J}))}{\hat{n}^2} + \frac{2\omega^2}{\hat{n}}} \right] \leq \mathbb{P}[\mathcal{E}_2]. \tag{105}$$

Note that $\hat{n} \geq \frac{N_3}{2H^2}$. For $n > N_3 = c_6 \frac{H^2B_3}{\epsilon^2} + c_7 \frac{\sqrt{H^2B_3N_3}}{\epsilon^2}$ with large enough constants $c_6$ and $c_7$, we have that the following inequality holds with probability at least $1 - \mathbb{P}[\mathcal{E}_2]$,

$$2\sqrt{2} \cdot \frac{\hat{\sigma}/\hat{n} - (\hat{\mu}/\hat{n})^2}{\hat{n}} < \frac{\epsilon}{32H}. \tag{106}$$

The 3 term For $n > N_4 := c_8 \frac{H^{13/3}B_3}{\epsilon^2}$ with large enough constant $c_8$, we have

$$7 \left( \frac{H_0^{3/4}}{n^{3/4}} + \frac{H^{3/4}}{\hat{n}^{3/4}} \right) < \frac{\epsilon}{32H}. \tag{107}$$

The 4 term For $n > N_5 := c_9 \frac{H^{3/2}B_3}{\epsilon}$ with large enough constant $c_9$, we have

$$5 \left( \frac{H}{n} + \frac{H}{\hat{n}} \right) < \frac{\epsilon}{32H}. \tag{108}$$

Combining (89) with the bounds (101), (102), (105), (106), (107) and (108), using the trivial bound $\text{clip}(\hat{b}_i(s_t, a_t) - b^*_t(s_t, a_t), \frac{\epsilon}{3H}) \leq 1/(1 - \gamma)$ for early stages, and summing over all possible $s, a, j$ with a union bound, we obtain that with probability $1 - SAJ(2\mathbb{P}[\mathcal{E}_2] + 4p)$,

$$\sum_{i > 1} \text{clip}(\hat{b}_i(s_t, a_t) - b^*_t(s_t, a_t), \frac{\epsilon}{8H}) \leq O(M_1 + M_2 + M_3 + M_4), \tag{109}$$
where (noting that $\tilde{n} \geq n/(2HB)$ in (105), (107) and (108))

\[
\mathcal{M}_1 = \sum_{s, a} \left( H_t + \sum_{n = \max\{l, 1\}}^{N_2} \sqrt{\frac{9\sqrt{2}H^3\varsigma^{3/2}}{n^{3/2}} + \frac{(2H^2SA(\tilde{J} + \tilde{\tilde{J}}) + 10H^2SN_1)t}{n^2} + \frac{4H\omega_t}{n}} \right),
\]

\[
\mathcal{M}_2 = \sum_{s, a} \left( H_t + \sum_{n = \max\{l, 1\}}^{N_3} \sqrt{\frac{H^4B^2(2SN_1 + 2SA(\tilde{J} + \tilde{\tilde{J}}))}{n^2} + \frac{2HB\omega^2}{n}} \right),
\]

\[
\mathcal{M}_3 = \sum_{s, a} \left( H_t + \sum_{n = \max\{l, 1\}}^{N_4} \left( \frac{H_t^{3/4}}{n^{3/4}} + \frac{H^7/4B^3/4t^{3/4}}{n^{3/4}} \right) \right),
\]

\[
\mathcal{M}_4 = \sum_{s, a} \left( H_t + \sum_{n = \max\{l, 1\}}^{N_5} \left( \frac{H_t}{n} + \frac{H^2B_t}{n} \right) \right).
\]

Straightforward calculation shows that

\[
\mathcal{M}_1 \leq SA \cdot O \left( H_t + N_2^{1/4}H^{3/2,3/4} + \ln\left(\frac{N_2}{t}\right)\sqrt{H^2SA\tilde{J} + H^2SN_1} + \sqrt{N_2H\omega_t} \right)
\]

\[
\leq O \left( \frac{SAH^{5/4}}{\epsilon} \right) + \tilde{O} \left( \frac{SAH^{17/12}}{\epsilon^{2/3}} + \frac{(S^{3/2}A^{3/2}H^{7/4} + S^{3/2}A^{5/4}H^2 + SAH^{15/8})t}{\epsilon^{1/2}} \right)
\]

\[
+ \frac{SAH^{7/3}}{\epsilon^{1/3}} + \frac{(S^{5/4}A^{5/4}H^{5/2} + S^{5/4}A^{9/8}H^{21/8})t}{\epsilon^{1/4}} + S^2A^2H^{3/2}t + S^2A^{3/2}H^{7/2}t, \right)
\]

\[
\mathcal{M}_2 \leq SA \cdot O \left( H_t + \ln\left(\frac{N_3}{l}\right)\sqrt{H^2B^2(2SN_1 + H^2SA\tilde{J}) + \sqrt{N_3H\omega_t^2}} \right)
\]

\[
\leq O \left( \frac{SAH^{2}l}{\epsilon} \right) + \tilde{O} \left( \frac{S^{3/2}A^{5/4}H^{7/2}t}{\epsilon^{1/2}} + S^2A^{3/2}H^{15/2}t + S^2A^{2}H^{7}t \right), \right)
\]

\[
\mathcal{M}_3 \leq SA \cdot O \left( H_t + N_4^{1/4}H^{7/4}B^{3/4,3/4} \right) \leq O \left( \frac{SAH^{59/12}l}{\epsilon^{1/3}} + SAHl \right)
\]

\[
\mathcal{M}_4 \leq SA \cdot O \left( H_t + \ln\left(\frac{N_5}{l}\right)H^2B_t \right) \leq \tilde{O} \left( SAH^5l \right).
\]

Finally, together with (109), we conclude that

\[
\sum_{t \geq 1} \text{clip}(\hat{b}_t(s_t, a_t) - \hat{b}^*_t(s_t, a_t), \frac{\epsilon}{SH})
\]

\[
\leq O \left( \frac{SAH^{2}l}{\epsilon} \right) + \tilde{O} \left( \frac{S^{3/2}A^{5/2}H^{7/2}t}{\epsilon^{1/2}} + \frac{SAH^{59/12}l}{\epsilon^{1/3}} + \frac{S^{5/4}A^{5/4}H^{21/8}t}{\epsilon^{1/4}} + S^2A^{2}H^{8}t \right).
\]