Integral structures on the finite part $H_f^1(K, V)$ of a crystalline representation

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Abstract
We study integral structures of crystalline representations over an unramified extension $K/Q_p$ with the help of an auxiliary ring $A_{\text{exp}}$. This ring has the nice property that it contains the fundamental period (and its inverse) of $p$-adic Hodge theory, up to powers of $p$. We establish an exact sequence using $A_{\text{exp}}$ and Frobenii on its filtration, give a link to Fontaine-Laffaille modules and the Bloch-Kato fundamental exact sequence and finally compute the integral finite part of a lattice of a crystalline representation, giving a connection to the local $L$-function of $V$.

1 Introduction
In their fundamental work [1], Bloch and Kato used and developed many techniques of what is now usually called $p$-adic Hodge theory, initiated before in large parts by Fontaine. Bloch and Kato’s focus was the development of a general conjecture concerning special values of $L$-functions, which culminated in their formulation of a version of the Tamagawa number conjecture.

Working locally at a fixed prime $p$ and a fixed finite extensions $K/Q_p$ with absolute Galois group $G_K$, we take a closer look at the computations done in sections 3 and 4 in loc.cit., which depend in certain situations on the property that the $p$-adic representation $V$ of $G_K$ under consideration is “in the Fontaine-Laffaille range”. This is a condition on the filtration of the filtered $\varphi$-module associated to $V$.

We introduce an auxiliary integral ring $A_{\text{exp}}$, which, after inverting $p$, computes the module $D_{\text{cris}}(V)$ of a crystalline representation, if one fixes a $G_K$-equivariant lattice $T \subset V$. A nice property of this integral version is that it contains already (up to some $p$-powers) the inverses of the fundamental period $t$, so that no awkward twisting to a positive representation is necessary. Note that simply inverting $t$ in for example $A_{\text{cris}}$ implies that $p$ is then also already inverted, which leaves the integral world.

Using this ring, we show that one can construct a finite rank Fontaine-Laffaille module $D_{\text{exp}}(T)$ out of $T$, which is used to connect the $p$-adic valuation of the special value at $s = 0$ of the local $L$-function $P(V, p^{-s})$ to a certain measure on this Fontaine-Laffaille module (via Bloch-Kato’s fundamental exact sequence), without any condition on the filtration range of $V$:
Theorem 1.1. Assume $K/Q_p$ is unramified and let $V$ be a crystalline representation. Fix a $G_K$-equivariant lattice $T$ in $V$ and assume further that $P(V,1) \neq 0$. Then:

a) $H^1(D_{\text{exp}}(T))[1/p] \cong H^1_c(K,V)$.

b) $H^1(D_{\text{exp}}(T)) \cong H^1_c(K,T)$.

c) $\exp_e : D_{dR}(V)/D^0_{dR}(V) \to H^1(K,V)$ coincides with the composite map

$$\exp_e : (D_{\text{exp}}(T)/D^0_{\text{exp}}(T))[1/p] \rightarrow H^1(K,V)$$

$$= H^1(D_{\text{exp}}(T))[1/p] \cong H^1_c(K,V),$$

where the last canonical identification is explained in the proof.

Here, $H^1_c(K,-)$ denotes the exponential part of $H^1(K,-)$, that is, the image of the Bloch-Kato exponential map. As a corollary, we obtain:

Corollary 1.2. Let $\mu$ be the Haar-measure on the finite-dimensional $K$-vector space $H^1(K,V)$ such that the image of the lattice

$$D_{\text{exp}}(T)/D^0_{\text{exp}}(T) \subset D_{dR}(V)/D^0_{dR}(V) \sim H^1_c(K,V) = H^1_c(K,V)$$

has measure $1$. Then

$$\mu(H^1_c(K,T)) = |P(V,1)|_p^{-1}.$$  

2 Basic concepts from $p$-adic Hodge theory

Fix a prime number $p$. Let $K/Q_p$ be a finite extension of the $p$-adic numbers, and denote by $G_K$ the absolute Galois group of $K$. Let $K_0$ be the maximal unramified subextension of $K/Q_p$. Usually, $V$ will denote a $p$-adic representation, that is, a finite dimensional $Q_p$-vector space equipped with a continuous and linear $G_K$-action. Similarly, $T$ will usually denote a $G_K$-stable $\mathbb{Z}_p$-lattice in $V$. Such lattices always exist. One is interested in the classification of such $V$ and $T$, and Fontaine’s rings have proven to be a powerful tool for this. We refer to [3] as a basic reference.

Let $O_{C_p}$ be the ring of integers of the completion of an algebraic closure $C_p$ of $Q_p$. Let $\hat{E}^+ := \lim_{\leftarrow n} O_{C_p}/p^n$, which is a ring of characteristic $p$, equipped with a Frobenius $\varphi : x \mapsto x^p$, and a Galois action of $\sigma \in G_K$ via $(x_n) \mapsto (\sigma(x_n))$. If $x = (x_n) \in \hat{E}^+$, let $x^{(0)} = \lim_{m \to \infty} \hat{x}_m^{p^n}$, where $\hat{x}_m \in O_{C_p}$ is any lift of $x_m \in O_{C_p}/p$. This defines a non-archimedean valuation $v : x \mapsto v_p(x^{(0)})$ on $\hat{E}^+$.

Let $\hat{A}^+ = W(\hat{E}^+)$, the ring of Witt vectors of $\hat{E}^+$. This makes sense since $\hat{E}^+$ is perfect, since it is a perfect of the non-perfect ring $O_{C_p}/p$. $\hat{A}^+$ is a ring of characteristic $0$ with Frobenius $\varphi : \sum_{n \geq 0} x_n p^n \mapsto \sum_{n \geq 0} [\varphi(x_n)] p^n$, and an action of $G_K$ that is defined analogously.

One has the important ring homomorphism

$$\theta : \hat{A}^+ \to O_{C_p}, \quad \sum_{n \geq 0} x_n p^n \mapsto \sum_{n \geq 0} x_n^{(0)} p^n,$$
which arises conceptually in Fontaine’s theory of universal thickenings.

We fix a system of $p^n$-th roots of unity $\epsilon^{(n)} \in \mathbb{C}_p$ with $\epsilon^{(0)} = 1, \epsilon^{(1)} \neq 1$ and $(\epsilon^{(n+1)})^p = \epsilon^n$. Then $\epsilon = (\epsilon^{(n)}) \in \tilde{\mathbb{E}}^+$, where $\mathfrak{p}$ means reduction mod $p$. Let $\pi := \epsilon - 1$ (this notation is slightly unfortunate, but standard). One can show that $v(\pi) = \frac{\mathfrak{p}}{p\mathfrak{p}}$.

Let $\tilde{\pi} := [\pi] - 1 \in \tilde{\mathbb{A}}^+$. Observe that $\pi \approx \pi$ mod $p$, but $\pi \neq [\pi]$.

Further let $\tilde{\rho} \in \tilde{\mathbb{E}}^+$ with $\tilde{\rho}^{(0)} = -p$. Set $\xi = [\tilde{\rho}] + p \in \tilde{\mathbb{A}}^+$, i.e. $\theta(\xi) = -p + p = 0$. One can show that $\ker \theta = \langle \xi \rangle$, since $\ker \theta \subset \langle \xi, p \rangle$, using the fact that $\tilde{\mathbb{A}}^+$ is $p$-adically complete and that $\mathcal{O}_{\mathbb{C}_p}$ does not have any $p$-torsion. More generally, if $\xi' \in \tilde{\mathbb{A}}^+$ such that $\theta(\xi') = 0$ and $v(\xi') = 1$, then $\ker \theta = \langle \xi' \rangle$.

One defines $B_{\text{dR}}^+ = \lim\limits_{\leftarrow n} \tilde{\mathbb{A}}^+[1/p]/(\ker \theta)$. $\theta$ extends to $B_{\text{dR}}^+$, where ker $\theta = \langle \xi \rangle$ still holds. Let

$$t = \log(1 + \pi) = \sum_{n \geq 1} (-1)^{n+1} \frac{\pi^n}{n} \in B_{\text{dR}}^+,$$

the fundamental period of $p$-adic Hodge theory. $t$ only depends on the choice of a compatible system of $p^n$-th roots of unity. Interestingly, one has ker $\theta = \langle t \rangle$. This shows that $B_{\text{dR}}^+$ is a complete discrete valuation ring with maximal ideal $(t)$. $B_{\text{dR}} = B_{\text{dR}}^+[1/t]$ possesses the filtration $\text{Fil}^n B_{\text{dR}} = t^n B_{\text{dR}}, i \in \mathbb{Z}$. Since $B_{\text{dR}}^+/t \cong \mathbb{C}_p$, induced by the map $\theta$, one has (algebraically, non-canonically) $B_{\text{dR}} \cong \mathbb{C}_p[\![[t]]\!]$. But observe that the topology on $B_{\text{dR}}^+$ is defined via the inverse limit topology and the topology on $\tilde{\mathbb{A}}^+$, which is induced via the Witt-construction by the valuation topology on $\tilde{\mathbb{E}}^+$. With this topology, one still has a continuous action of $G_K$, but the action of $\varphi$ does not extend to $B_{\text{dR}}^+$.

This being the case one considers the ring $A_{\text{cris}}$, which is defined as the $p$-adic completion of the divided power envelope of $\tilde{\mathbb{A}}^+$ with respect to the ideal $\ker \theta$, i.e.

$$\tilde{\mathbb{A}}^+[\frac{a^n}{m^n}; a \in \ker \theta] = \tilde{\mathbb{A}}^+[\frac{\epsilon^n}{m^n}] \subset B_{\text{dR}}^+.$$

If $x \in A_{\text{cris}}$, then we may write (non-uniquely) $x = \sum n a_n \frac{\epsilon^n}{m^n}$ with $a_n \in \tilde{\mathbb{A}}^+$ and $a_n \to 0$ $p$-adically. The map $\theta$ and the $\varphi$ and $G_K$-action extend to $A_{\text{cris}}$. Further, $t \in A_{\text{cris}}$, since $\pi = b\xi$ ($\theta(\pi) = 0$) and

$$\frac{\pi^n}{n} = (n-1)! b^n \frac{\epsilon^n}{m^n}, \quad (n-1)! \to 0.$$

Let $B_{\text{cris}}^+ = A_{\text{cris}}[1/p], B_{\text{cris}} = B_{\text{cris}}^+[1/t] \subset B_{\text{dR}}$. $B_{\text{cris}}^+ = K_0$. Two facts about $A_{\text{cris}}$ are: $\varphi(\xi) \in pA_{\text{cris}}$, and $t^p - 1 \in pA_{\text{cris}}$, hence $B_{\text{cris}} = A_{\text{cris}}[1/t]$. Two caveats about $B_{\text{cris}}$ are: it has a funny topology, since one can show that the topology induced on $B_{\text{cris}}^+$ by $B_{\text{cris}}$ (which comes equipped with the locally convex final topology) is not the natural topology on $B_{\text{cris}}^+$. Furthermore, $B_{\text{cris}}^+ \not\subset \text{Fil}^0 B_{\text{cris}}$.

Let $V$ now be a $p$-adic representation. Let $D_{\text{dR}}(V) = (B_{\text{dR}} \otimes \mathbb{Q}_p V)^{G_K}$. This is a $K$-vector space, and the injectivity of the canonical map

$$\alpha : B_{\text{dR}} \otimes_K D_{\text{dR}}(V) \to B_{\text{dR}} \otimes_{\mathbb{Q}_p} V, \quad b \otimes (\sum b_n \otimes v_n) \mapsto \sum b b_n \otimes v_n$$
shows that dim_K D_{\text{dR}}(V) \leq \dim_{Q_p} V. If equality holds, we call V \textbf{de Rham}. D_{\text{dR}}(V) comes equipped with a separated and exhaustive K-vector space filtration, given by Fil^i D_{\text{dR}}(V) = (\text{Fil}^i B_{\text{dR}} \otimes V)^{G_K}.

Similarly, we let D_{\text{cris}}(V) := (B_{\text{cris}} \otimes_{Q_p} V)^{G_K}. This is a K_0-vector space, and the injectivity of the analogous \alpha-map for B_{\text{cris}} shows that dim_{K_0} D_{\text{cris}}(V) \leq \dim_{Q_p} V. If equality holds, we call V \textbf{crystalline}. D_{\text{cris}}(V) comes equipped with a \K_0-linear \varphi\text{-action. Further, } K \otimes_{K_0} D_{\text{cris}}(V) \text{ comes equipped with a } K\text{-vector space filtration. If } V \text{ is crystalline then } V \text{ is de Rham.}

One fundamental theorem of Colmez and Fontaine states: the assignment \(V \mapsto D_{\text{cris}}(V)\) induces an equivalence of categories between the crystalline representations of \(G_K\) and the category of \(K\)-filtered admissible (i.e. \(t_H(D) = t_N(D)\) and \(t_H(D') \leq t_N(D')\) for all subobjects \(D' \subset D\), where \(t_H\) resp. \(t_N\) are the Hodge number resp. the Newton number) \varphi\text{-modules. This equivalence heavily uses the fact that the map } \alpha \text{ above in the } B_{\text{cris}}\text{-case is actually an isomorphism.}

3 The period ring \(A_{\exp}\)

\textbf{Definition 3.1.} Let \(\tilde{A}^+ \left\{ \frac{\pi}{p} \right\} := \tilde{A}^+[X]/(pX - \pi)\), where \(A\{X\}\) denotes the p-adic completion of \(A[X]\) for any ring \(A\), equipped with quotient topology.

If \(x \in \tilde{A}^+ \{\pi/p\}\), we may write (non-uniquely) \(x = \sum_{n \geq 0} a_n(\pi/p)^n\) in \(\tilde{A}^+[1/p]\). The natural actions of \(\varphi\) and \(G_K\) extend to actions on \(\tilde{A}^+ \{\pi/p\}\).

\textbf{Lemma 3.2.} In \(\tilde{A}^+ \{\pi/p\} \subset B_{\text{dR}}^+\) one has the relation \(t/p = \pi/p \cdot v\) mit \(v \in \tilde{A}^+ \{\pi/p\}\), i.e. \(\tilde{A}^+ \{\pi/p\} = \tilde{A}^+ \{t/p\}\). In particular, \(t \in \tilde{A}^+ \{\pi/p\}\).

\textbf{Proof.} First, we observe that

\[
\frac{t}{p} = \sum_{n \geq 1} (-1)^{n+1} \cdot \frac{\pi^n}{p \cdot n} = \frac{\pi}{p} \cdot \left( \sum_{n \geq 1} a_n \left( \frac{\pi}{p} \right)^n \right) = \pi/p \cdot v,
\]

with \(a_n \to 0\) p-adically.

Now, since \(v \mod p \in \tilde{E}^+[X]/(\pi)\) is -1, hence a unit, we have that \(v \in \tilde{A}^+ \{\pi/p\}\). Hence the claim. \(\square\)

\textbf{Definition 3.3.} Let \(A\) be a subring of \(B_{\text{dR}}\), such that the Frobenius \(\varphi\) acts on \(A\) (e.g. \(A_{\text{cris}}\)). Set

\(\text{Fil}^i_p A := \{ x \in \text{Fil}^i A | \varphi(x) \in p^i A \}\),

where \((\text{Fil}^i A)_{i \in \Z}\) is the filtration induced by \(B_{\text{dR}}\).
Definition 3.4. Let $\mathbf{A}_{\text{exp}} := \tilde{\mathbf{A}}^+\{\pi/p\}[t/p]$. The Frobenius $\varphi$ on $\tilde{\mathbf{A}}^+\{\pi/p\}$ extends to $\mathbf{A}_{\text{exp}}$. We equip $\mathbf{A}_{\text{exp}}$ with the filtration given by

$$
\text{Fil}^k \mathbf{A}_{\text{exp}} := \bigcup_{i+k \geq 0} \left( \frac{p}{t} \right)^k \text{Fil}^k \tilde{\mathbf{A}}^+ \{ \pi/p \}.
$$

Since $\text{Fil}^k \tilde{\mathbf{A}}^+ \{ \pi/p \} = \tilde{\mathbf{A}}^+ \{ \pi/p \}$, this filtration is separated and exhaustive.

Proposition 3.5. For every $k \geq 0$ we have the exact $G_K$-equivariant sequence

$$
0 \longrightarrow \left( \frac{t}{p} \right)^k \cdot \mathbb{Z}_p \longrightarrow \text{Fil}_p^k \tilde{\mathbf{A}}^+ \{ \pi/p \} \longrightarrow (1-p^{-k}\varphi) \tilde{\mathbf{A}}^+ \{ \pi/p \} \longrightarrow 0,
$$

which admits a continuous (not necessarily $G_K$-equivariant) splitting $\tilde{\mathbf{A}}^+ \{ \pi/p \} \rightarrow \text{Fil}_p^k \tilde{\mathbf{A}}^+ \{ \pi/p \}$.

Proof. Obviously,

$$
\left( \frac{t}{p} \right)^k \cdot \mathbb{Z}_p \subset \ker(1-p^{-k}\varphi).
$$

On the other hand, if $x \in \ker(1-p^{-k}\varphi)$ then $x = \sum_n a_n(t/p)^n$ (see Lemma 3.2), with $\tilde{\mathbf{A}}^+ \ni a_n \rightarrow 0$ $p$-adically. For any $n \in \mathbb{N}$ we have $(p^{-k}\varphi)^n(x) \equiv \varphi^n(a_n)(t/p)^k \mod p\tilde{\mathbf{A}}^+ \{ \pi/p \}$, hence $x = y(t/p)^k$, with $y \in \tilde{\mathbf{A}}^+$ and $\varphi(y) = y$, that is, $y \in \mathbb{Z}_p$ as is well-known.

We now prove that $\text{Fil}_p^k \mathbf{A}_{\text{exp}}$ is the $p$-adic closure of the module

$$
\tilde{\mathbf{A}}^+[\xi^i \cdot (t/p)^j; i+j \geq k],
$$

which we denote by $N$. If $i+j \geq k$ one has

$$
\varphi \left( \xi^i \cdot \left( \frac{t}{p} \right)^j \right) = p^{i+j} \cdot \left( 1 + \frac{\pi_0}{p} \right)^i \cdot \left( \frac{t}{p} \right)^j,
$$

where $\pi_0$ is the trace from $K((t))$ to $K((t^p))$ of $\pi$. Here we recall that $\pi = \exp t - 1 \in K((t))$. Obviously $(t/p)^k \cdot \mathbb{Z}_p \subset N$, so we have to prove that for any $y \in \tilde{\mathbf{A}}^+ \{ \pi/p \}$ there exists an $x \in \text{Fil}_p^k \tilde{\mathbf{A}}^+ \{ \pi/p \}$ with $(1-p^{-k}\varphi)(x) = y$. Since $N$ and $\tilde{\mathbf{A}}^+ \{ \pi/p \}$ are separated and complete with respect to the $p$-adic topology, it suffices to show this result mod $p$. If $y = \sum_{n>0} a_n(t/p)^n$ then $x = -y$ will do the job.

Thus it remains to show that if $y \in \tilde{\mathbf{A}}^+$ and $j \leq k$ there exists an $x \in N$ such that

$$
(1-p^{-k}\varphi)(x) - y \left( \frac{t}{p} \right)^j \in \left( p, \left( \frac{t}{p} \right)^i ; i > n \right) \subseteq \tilde{\mathbf{A}}^+ \{ \pi/p \}.
$$

One checks that

$$
x = y' q^k - j \left( \frac{t}{p} \right)^i,
$$
where \( y' \in \tilde{A}^+ \) is a solution of
\[
\varphi(y') - q^{k-j} y' = b
\]
in \( \tilde{A}^+ \), satisfies this property (recall that \( q' = \varphi^{-1}(q) \)). \( \square \)

**Corollary 3.6.** Dividing out \((t/p)^k\) and taking the the direct limit over the sequence in Proposition 3.5 we obtain an exact sequence
\[
0 \rightarrow \mathbb{Z}_p \rightarrow \text{Fil}^0 A_{exp} \xrightarrow{1 - \varphi} A_{exp} \rightarrow 0,
\]
where \( \varphi \) is the extension of the \( \varphi \) on \( \tilde{A}^+ \).

**Proposition 3.7.** \((A_{exp})^{G_K} = \mathcal{O}_{K_0}\)

*Proof.* Obviously, \( \mathcal{O}_{K_0} \subset (A_{exp})^{G_K} \), since \((\tilde{A}^+)^{G_K} = \mathcal{O}_{K_0}\). We have the inclusions
\[
\varphi \left( \tilde{A}^+ \left\{ \frac{\pi}{p} \right\} \right) = \tilde{A}^+ \left\{ \frac{\pi^p}{p} \right\} \subset A_{\text{cris}} \subset \tilde{A}^+ \left\{ \frac{\pi}{p} \right\},
\]
which leads to
\[
\varphi(A_{exp}[1/p]) \subset B_{\text{cris}} \subset A_{exp}[1/p].
\]
Since \((\varphi(B_{\text{cris}}))^{G_K} = (B_{\text{cris}})^{G_K} = K_0\), we have that \((\varphi(A_{exp}[1/p]))^{G_K} = K_0\). Since \( \varphi \) is injective on \( A_{exp} \), and hence on \( A_{exp}[1/p] \), we obtain \((A_{exp}[1/p])^{G_K} = K_0\).

Note that by the above exact sequence, \( 1/p \not\in A_{exp} \): otherwise \( 1/p^n \in A_{exp} \) for all \( n \geq 0 \). But taking \( G_{Q_p} \)-invariants gives an injection
\[
A_{exp} \hookrightarrow H^1(G_{Q_p}, \mathbb{Z}_p) = \text{Hom}_{cts}(G_{Q_p}, \mathbb{Z}_p).
\]
The \( \mathbb{Z}_p \)-module on the right hand side is finitely generated, which would lead to a contradiction.

Alternatively, one can use the exact sequence of 3.5, the filtration on \( A_{exp} \) and a limit argument to proceed as in the proof of the statement
\[
H^0(K, \text{Fil}^i B_{dR}/\text{Fil}^j B_{dR}) = K,
\]
if \( i \leq 0 < j \). \( \square \)

**Definition 3.8.** Let \( T \) be a full \( \mathbb{Z}_p \)-lattice of \( V \) that is invariant under the action of \( G_K \) (such lattices always exist). We define the modules
\[
D_{exp}(T) := (A_{exp} \otimes_{\mathbb{Z}_p} T)^{G_K}
\]
and
\[
D^0_{exp}(T) := (\text{Fil}^0 A_{exp} \otimes_{\mathbb{Z}_p} T)^{G_K}.
\]

**Proposition 3.9.** If \( T \) is as before, \( D_{exp}(T) \) is free \( \mathcal{O}_{K_0} \)-module of finite rank less or equal to \( \text{rk}_{\mathbb{Z}_p} T \).
Proof. This is a variation of the proof one usually encounters in Fontaine’s theory of $B$-admissible rings. We outline the idea:

Let $B$ be a topological integral domain, equipped with a continuous action of a topological group $G$. Set $C = \text{Frac}(B)$ and $S = B^G$, which is again an integral domain, and fix a closed subring $R \subset S$. Assume $T$ is a finite free $R$-module with continuous $G$-action, so that $V = \text{Frac}(R) \otimes_R T$ is a finite-dimensional $\text{Frac}(R)$-vector space. Set $D_B(T) = (B \otimes_R T)^G$, $D_C(V) = (C \otimes_{\text{Frac}(R)} V)^G$ and assume $C^G = \text{Frac}(S)$. We want to prove the injectivity of the map

$$B \otimes_S D_B(T) \to B \otimes_R T.$$

The inclusion $B \hookrightarrow C$ and the freeness of $T$ gives a diagram

$$\begin{array}{ccc}
B \otimes_S D_B(T) & \to & B \otimes_R T \\
\downarrow & & \downarrow \\
B \otimes_S D_C(V) & \to & C \otimes_{\text{Frac}(S)} D_C(V) \\
\downarrow & & \downarrow \\
C \otimes_{\text{Frac}(R)} D_C(V) & \to & C \otimes_{\text{Frac}(R)} V
\end{array}$$

so that we are reduced to the case where all the rings are fields. Now one proceeds exactly as in [3], Theorem 2.13.

The above situation applies with $B = A_{\exp}$, $R = \mathbb{Z}_p \subset \mathcal{O}_{K_0} = S$. The injectivity of the above map implies, by using the above notation and going to the quotient field $C$, that $D_B(T)$ is of $S$-rank smaller or equal than the $R$-rank of $T$. Since we these latter rings are discrete valuation rings, we are done.

**Proposition 3.10.** If $V$ and $T$ are as above, we have

$$D_{\exp}(T)[1/p] = D_{\exp}(V) = (A_{\exp}[1/p] \otimes_{\mathbb{Q}_p} V)^{G_K} = D_{\text{cris}}(V),$$

and this identification is compatible with the $K$-filtrations and the action of the Frobenius $\varphi$.

**Proof.** The proof is similarly as in Proposition 3.7. We have inclusions

$$\varphi(A_{\exp} \otimes T)^{G_K} \subset (A_{\text{cris}} \otimes T)^{G_K} \subset (A_{\exp} \otimes T)^{G_K},$$

and since $\varphi$ is bijective on $D_{\text{cris}}(V)$ and injective on $A_{\exp}$, we conclude as before $(A_{\exp}[1/t] \otimes T)^{G_K} = D_{\text{cris}}(V)$.

The compatibility with filtration and Frobenius can be checked by the construction. \qed
4 Categories in integral $p$-adic Hodge theory

Let $K/\mathbb{Q}_p$ be unramified for this section.

**Definition 4.1.** A **Fontaine-Laffaille module** over $\mathcal{O}_K$ is a triple $(M, (M^i)_{i \in \mathbb{Z}}, (\varphi^i)_{i \in \mathbb{Z}})$, which we also denote simply by $M$, consisting of

- an $\mathcal{O}_K$-module $M$,
- an exhausting and separated decreasing filtration (of $\mathcal{O}_K$-modules) $M^i$ of $M$,
- a family of $\sigma$-semilinear maps $\varphi^i : M^i \to M$ with the property $\varphi^i|_{M^i+1} = p \cdot \varphi^{i+1}$.

A morphism of Fontaine-Laffaille modules $f : M \to N$ is an $\mathcal{O}_K$-linear map $f$ such that $f(M^i) \subset N^i$ and $f \circ \varphi^i_M = \varphi^i_N \circ f$. We denote by $\text{MF}_{\mathcal{O}_K}$ the exact category of all Fontaine-Laffaille modules over $\mathcal{O}_K$.

**Definition 4.2.** A **filtered Dieudonné module** over $\mathcal{O}_K$ is a Fontaine-Laffaille module $M$ such that

- $M$ is of finite type over $\mathcal{O}_K$,
- $M^i = M$ for $i \ll 0$ and $M^j = 0$ for $j \gg 0$,
- $M_\sigma = \sum_{i \in \mathbb{Z}} \varphi^i(M^i)$.

We denote by $\text{MF}_{\mathcal{O}_K}^{fd}$ (fortement divisible) the category of all filtered Dieudonné modules.

Here, $M_\sigma$ denotes the underlying module $M$, where $W$ acts via $\sigma$. Note that contrary to the usual convention we allow our Dieudonné modules to contain torsion.

**Example 4.3.** We have $A_{\text{exp}} = (A_{\text{exp}}, (\text{Fil}^i A_{\text{exp}})_{i \in \mathbb{Z}}, (\varphi^i)_{i \in \mathbb{Z}}) \in \text{MF}_{\mathcal{O}_K}$, where $A_{\text{exp}}$ and the filtration are given as before, and $\varphi^i = 1/p^i \cdot \varphi$, with $\varphi$ induced from the Frobenius on $\tilde{A}^+$.

**Theorem 4.4.** The category $\text{MF}_{\mathcal{O}_K}^{fd}$ is abelian.

**Proof.** This follows from the fact that $\text{MF}_{\mathcal{O}_K}^{fd, \text{tor}}$, the subcategory of all torsion $\mathcal{O}_K$-modules ([2], Proposition 1.8), is abelian, and completeness: let $f : M \to N$ be a map in $\text{MF}_{\mathcal{O}_K}^{fd}$ with $M, N \in \text{MF}_{\mathcal{O}_K}^{fd}$. This gives us, for any $n \in \mathbb{N}$, a map $f_n : M/p^n \to N/p^n$ in $\text{MF}_{\mathcal{O}_K}^{fd, \text{tor}}$, since $M$ and $N$ are of finite type, hence kernel and cokernel of $f_n$ exist.

Since $M = \varprojlim M/p^n$, in a compatible way with the filtration and the Frobenii, we obtain, by going to the limit, the kernel and the cokernel of the map $f$. The normality of mono- and epimorphisms is an easy consequence again of the property that $\text{MF}_{\mathcal{O}_K}^{fd, \text{tor}}$ is abelian.

**Definition 4.5.** A **filtered $\varphi$-module** over $K$ is a triple $(D, (D^i)_{i \in \mathbb{Z}}, \varphi)$, consisting of
• a $K$-vector space $D$,

• an exhausting and separated decreasing filtration (of $K$-vector spaces) $D^i$ of $D$,

• a $\sigma$-semilinear map $\varphi : D \to D$.

A morphism of filtered $\varphi$-modules is, similarly as before, a morphism of $K$-vector spaces compatible with the filtration and Frobenius $\varphi$. We denote by $\text{MF}_K$ the category of all $\varphi$-modules.

A finite-dimensional filtered $\varphi$-module is called admissible if $t_N(D) = t_H(D)$ and $t_N(D') \leq t_H(D')$ for all subobjects $D' \subset D$ in $\text{MF}_K$, where $t_N$ resp. $t_H$ are the Newton resp. Hodge number of $D$ (see [3], 6.4.2.).

Example 4.6. If $M \in \text{MF}_K^{fd}$, one can naturally associate a finite-dimensional $\varphi$-module $D$ to $M$, namely $D := K \otimes_{\mathcal{O}_K} M$, with the filtration induced by $M^i$, and Frobenius $\varphi := 1/p^n \cdot \varphi^n$ for $n \ll 0$. We call $M$ admissible if $D$ is admissible.

Proposition 4.7. Let $D \in \text{MF}_K$ be admissible. Then an $\mathcal{O}_K$-lattice $M$, equipped with a filtration $M^i$ such that $M^i[1/p] = D^i$ can be considered as an object of $\text{MF}_K^{fd}$ if and only if $\varphi(M^i) \subset p^i M$ (that is, one puts $\varphi^i := p^{-i} \varphi$).

Proof. The only thing we have to check is that the condition on $\varphi$ holds, then $D$ is already in $\text{MF}_K^{fd}$. This can be inferred by the proof of [2], Theorem 3.2, after reducing to the case where all weights are $\geq 0$ (see also [4], Proposition 7.8).

Proposition 4.8. If $V$ is crystalline and $T$ as above, then $D_{\exp}(T) \in \text{MF}_K^{fd}$.

Proof. We know from proposition 3.9 that $D_{\exp}(T)$ is a free $\mathcal{O}_K$-module of finite rank and that $D_{\exp}(T)[1/p] = D_{\cris}(V)$ (3.10) is admissible, since $V$ is crystalline. Since $D_{\exp}(T)[1/p] = \text{Fil}^i D_{\cris}(V)$ and $\varphi^i(D_{\exp}(T)) \subset D_{\exp}(T)$, the requirements of proposition 4.7 are fulfilled, hence the claim.

5 Computation of $H^1_c(K, T)$

Assume again that $K/\mathbb{Q}_p$ is unramified. We collect some facts from sections 3 and 4 of [1].

Proposition 5.1. Let $M \in \text{MF}_K^{fd}$ and put

$$H^0(M) = \ker((1 - \varphi^0) : M^0 \to M), \quad H^1(M) = \coker((1 - \varphi^0) : M^0 \to M).$$

and $H^i(M) = 0$ for $i \geq 2$. Then $(H^i)_{i \in \mathbb{N}}$ is a cohomological $\delta$-functor.

Proof. This is abundantly clear by the snake lemma.
Recall ([1], Proposition 1.17) the Bloch-Kato fundamental exact sequences
\[
0 \rightarrow \mathbb{Q}_p \rightarrow \mathbb{B}_{\text{cris}}^{\varphi = 1} \oplus \mathbb{B}_{\text{dR}}^+ \overset{f}{\rightarrow} \mathbb{B}_{\text{dR}} \rightarrow 0,
\]
where \( f(x, y) = x - y \), and
\[
0 \rightarrow \mathbb{Q}_p \rightarrow \mathbb{B}_{\text{cris}} \oplus \mathbb{B}_{\text{dR}}^+ \overset{f}{\rightarrow} \mathbb{B}_{\text{dR}} \rightarrow 0,
\]
where \( f(x, y) = ((1 - \varphi)(x), x - y) \). If \( V \) is a \( p \)-adic representation, we call the maps
\[
\exp_e : \mathcal{D}_{\text{dR}}(V) \overset{\delta}{\rightarrow} H^1(K, V), \quad \exp_f : \mathcal{D}_{\text{cris}}(V) \oplus \mathcal{D}_{\text{dR}}(V) \overset{\delta}{\rightarrow} H^1(K, V)
\]
the **Bloch-Kato exponential maps**, which are induced by the connecting homomorphism of Galois cohomology. We set
\[
H^1_e(K, V) := \text{Im}(\exp_e), \quad H^1_f(K, V) := \text{Im}(\exp_f).
\]
If \( T \subset V \) is a \( G_K \)-equivariant \( \mathbb{Z}_p \)-lattice in \( V \), denote by \( \iota : H^1_e(K, T) \rightarrow H^1_e(K, V) \) the canonical map. Set
\[
H^1_f(K, T) := \text{Im}(\exp_f).
\]
Recall that the local \( L \)-function in the case \( p = l \) is defined as
\[
P(V, X) := \det_K(1 - f \cdot X| \mathcal{D}_{\text{cris}}(V)),
\]
where \( f \) denotes the \( K \)-linear map \( \varphi^{[K:Q_p]} \).

Assume now that \( P(V, 1) \neq 0 \). Then (cf. the first lines in the proof of Theorem 4.1, [1]) \( H^1_f(K, V) = H^1_e(K, V) \).

**Theorem 5.2.** Let \( V \) be a crystalline representation, fix a \( G_K \)-equivariant lattice \( T \) in \( V \), and assume \( P(V, 1) \neq 0 \). Then:

a) \( H^1(\mathcal{D}_{\text{exp}}(T))[1/p] \cong H^1_e(K, V) \).

b) \( H^1(\mathcal{D}_{\text{exp}}(T)) \cong H^1_e(K, T) \).

c) \( \exp_e : \mathcal{D}_{\text{dR}}(V)/\mathcal{D}_{\text{dR}}^0(V) \rightarrow H^1(K, V) \) coincides with the composite map
\[
(\mathcal{D}_{\text{exp}}(T)/\mathcal{D}_{\text{exp}}^0(T))[1/p] \overset{1 - \varphi}{\rightarrow} (\mathcal{D}_{\text{exp}}(T)/(1 - \varphi^0)\mathcal{D}_{\text{exp}}(T))[1/p] \overset{\delta}{\rightarrow} H^1(\mathcal{D}_{\text{exp}}(T))[1/p] \cong H^1(K, V),
\]
where the last canonical identification is explained in the proof.

**Proof.** The proof is similar to [1], Lemma 4.5. We have the following commutative diagram:
\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}_p & \rightarrow & \text{Fil}^0\mathbb{A}_{\text{exp}} & \rightarrow & \mathbb{A}_{\text{exp}} & \rightarrow & 0 \\
& & \downarrow & & \downarrow x \rightarrow (x, x) & & \downarrow x \rightarrow (x, 0) \\
0 & \rightarrow & \mathbb{Q}_p & \rightarrow & \mathbb{A}_{\text{exp}}[1/p] \oplus \mathbb{B}_{\text{dR}}^+ & \rightarrow & \mathbb{A}_{\text{exp}}[1/p] \oplus \mathbb{B}_{\text{dR}} & \rightarrow & 0
\end{array}
\]
Inverting $p$, tensoring with $V$ over $\mathbb{Q}_p$ and taking $G_K$-cohomology, we obtain a diagram

$$
0 \rightarrow H^0(D_{\exp}(T)[1/p] \rightarrow D_{\exp}^0(T)[1/p] \rightarrow D_{\exp}(T)[1/p] \rightarrow H^1(D_{\exp}(T))[1/p] \rightarrow 0
$$

$$
0 \rightarrow H^0(K, V) \rightarrow D_{\text{cris}}(V) \oplus D_{\text{dR}}^+(V) \rightarrow D_{\text{cris}}(V) \oplus D_{\text{dR}}(V) \rightarrow H^1(K, V)
$$

with exact rows. Since $D_{\exp}(T)[1/p] = D_{\text{cris}}(V) = D_{\text{dR}}(V)$ and $\text{Im}(\exp_f) = \text{Im}(\exp_e)$, we have the claimed identification $H^1(D_{\exp}(T))[1/p] \cong H^1_f(K, V)$.

The exact sequence

$$
0 \rightarrow D_{\exp}(T) \rightarrow D_{\exp}(T) \rightarrow D_{\exp}(T)/p \rightarrow 0
$$

(which exists in $\text{MF}_{\text{fd}}^0(K)$, since $D_{\exp}(T) \in \text{MF}_{\text{fd}}^0(K)$) induces a sequence

$$
H^0(D_{\exp}(T)/p) \rightarrow H^1(A \otimes T) \rightarrow H^1(A \otimes T) \rightarrow H^1(A \otimes T/p).
$$

Recall that $H^1_f(K, T) = \iota^{-1}(H^1_f(K, V))$, where $\iota : H^1(K, T) \rightarrow H^1(K, V)$. Since $H^1(D_{\exp}(T)) \subset H^1(K, T)$ it suffices to show that the cokernel of this inclusion does not have any $p$-torsion. But this follows from the commutative diagram

$$
\begin{array}{cccccc}
H^0(D_{\exp}(T)/p) & \rightarrow & H^1(A \otimes T) & \rightarrow & H^1(A \otimes T) & \rightarrow & H^1(A \otimes T/p). \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \\
H^0(T/p) & \rightarrow & H^1(T) & \rightarrow & H^1(T) & \rightarrow & H^1(T/p)
\end{array}
$$

\[
\square
\]

**Corollary 5.3.** Let $\mu$ be the Haar-measure on the finite-dimensional $K$-vector space $H^1(K, V)$ such that the image of the lattice

$$
D_{\exp}(T)/D_{\exp}^0(T) \subset D_{\text{dR}}(V)/D_{\text{dR}}^0(V) \rightarrow H^1_e(K, V) = H^1_f(K, V)
$$

has measure 1. Then

$$
\mu(H^1_f(K, T)) = |P(V, 1)|_p^{-1}.
$$

**Proof.** This follows from the definition of $P(V, X)$ and the b), c) from the previous theorem.

\[
\square
\]

**References**

[1] Pierre (ed.) Cartier, Luc (ed.) Illusie, Nicholas M. (ed.) Katz, Gérard (ed.) Laumon, Yuri I. (ed.) Manin, and Ken A. (ed.) Ribet. *The Grothendieck Festschrift. A collection of articles written in honor of the 60th birthday of Alexander Grothendieck. Volume I. Reprint of the 1990 edition.* Modern Birkhäuser Classics. Basel: Birkhäuser. xx, 498 p., 2007.
[2] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations $p$-adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608 (1983), 1982.

[3] Jean-Mark Fontaine and Yi Ouyang. Theory of $p$-adic Galois representations. [http://staff.ustc.edu.cn/~yiouyang/galoisrep.pdf](http://staff.ustc.edu.cn/~yiouyang/galoisrep.pdf). Book in preparation.

[4] Guy Laffaille. Construction de groupes $p$-divisibles. Le cas de dimension 1. In *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978)*, Vol. III, volume 65 of *Astérisque*, pages 103–123. Soc. Math. France, Paris, 1979.

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