Understanding the second quantization of fermions in Clifford and in Grassmann space

New way of second quantization of fermions

Part I

N.S. Mankoč Borštnik$^1$ and H.B.F. Nielsen$^2$

$^1$ University of Ljubljana, Slovenia

$^2$Niels Bohr Institute, Denmark

Both algebras, Clifford and Grassmann, offer the second quantization of fermions [5–7] without postulating the second quantization conditions proposed by Dirac [1–3], offering correspondingly the understanding of the Dirac’s postulates. But while fermions with the internal degrees of freedom described by the Clifford algebras manifest the half integer spins — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the Grassmann "fermions" manifest integer spins. In Part I properties of the second quantized integer spins "fermions" in Grassmann space are presented, defined by creation operators on the vacuum state and on the Hilbert space of infinite number of "Slater determinants" with all the possibilities of empty and occupied "fermion states". We demonstrate the appearance of the anticommutation relations among the creation and annihilation operators in the Grassmann case. In Part II the conditions are discussed under which the Clifford algebras offer the appearance of the second quantized fermions and the family quantum numbers. In both parts, Part I and Part II, the relation between the Dirac way and our way of the second quantization is presented.

I. INTRODUCTION

It is demonstrated in this paper how do the Grassmann algebra — in Part I — and the two kinds of the Clifford algebras — in Part II — if used to describe the internal degrees of freedom of "fermions", take care of the second quantization of fermions without postulating anticommutation relations [1–3]. Either the Grassmann algebra or the Clifford algebra offer namely the appearance of the creation and their Hermitian conjugated annihilation operators, which fulfill the anticommutation relations postulated by Dirac for fermions, if they apply on the corresponding vacuum state, Eq. (7), defined by the sum of products of all the annihilation times the corresponding Hermitian conjugated creation operators.

In $d$-dimensional Grassmann space of anticommuting coordinates $\theta^a$’s, $i = (0, 1, 2, 3, 5, \ldots, d)$,
there are $2^d$ operators ("vectors"), which are superposition of products of $\theta^a$. One can arrange them into irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to $\theta^a$'s, $\frac{\partial}{\partial \theta^a}$'s, taken in Ref. [7] as, up to a sign, Hermitian conjugated to $\theta^a$'s, $(\theta^a \dagger = \eta^{aa} \frac{\partial}{\partial \theta^a}$, $\eta^{ab} = \text{diag}\{1, -1, -1, \ldots, -1\}$), which form again $2^d$ operators ("vectors"). Grassmann space offers correspondingly $2 \cdot 2^d$ degrees of freedom.

There are two kinds of the Clifford operators ("vectors"), which are expressible with $\theta^a$ and $\frac{\partial}{\partial \theta^a}$: $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a})$, $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a})$ [6, 8, 9]. They are, up to $\eta^{aa}$, Hermitian operators. Each of these two kinds of the Clifford algebra objects has $2^d$ operators ("vectors"), together again $2 \cdot 2^d$ degrees of freedom.

The Grassmann and each of the two Clifford algebras split into odd and even parts with respect to the odd and even number of $\theta^a$'s, $\frac{\partial}{\partial \theta^a}$'s, $\gamma^a$'s, $\tilde{\gamma}^a$'s. There is the odd algebra in all three cases which, if used to generate the creation and annihilation operators for "fermions", leads to the Hilbert space of second quantized "fermions" obeying the anticommutation relations of Dirac [1] without postulating these relations.

Let us present steps which lead to the second quantized fermions:

i. The internal space of a "fermion" is described by either Clifford or Grassmann algebra of an odd Clifford character (superposition of an odd number of Clifford "coordinates" $\gamma^a$'s) or an odd Grassmann character (superposition of an odd number of Grassmann "coordinates" $\theta^a$'s).

ii. The eigenvectors of the (chosen) Cartan subalgebra of the corresponding Lorentz algebra of an odd (either an odd number of Clifford $\gamma^a$'s or an odd number of Grassmann $\theta^a$'s) character are used to define the "basis vectors" in the internal space of fermions. The application of this "basis vectors" on the corresponding vacuum state (either Clifford $|\psi_{oc}\rangle$, defined in Eq. (10) of Part II, or Grassmann $|\phi_{og}\rangle$, Eq. (7), which is just an identity) form the "basis states", which so far concern only the internal space of fermions. The members of the "basis vectors" form together with their Hermitian conjugated partners creation and annihilation operators, which anticommute, Eq. (11) in Part I and Eq. (11) in Part II.

iii. The plane wave solutions of the corresponding Weyl equations (either Clifford, Eq. (24) or Grassmann, Eq. (20)) for free massless "fermions" are the superposition of the members of the "basis vectors", with the coefficients of the superposition depending on a chosen momentum $\vec{p}$, with $|p^0| = |\vec{p}|$.

iv. These superposition of "basis states" have the properties of the creation operators, defining, when applied on the vacuum state, the single "fermion" states. Their Hermitian conjugated partners are annihilation operators.
v. The second quantized Hilbert space $\mathcal{H}_{\vec{p}}$ is correspondingly for a particular $\vec{p}$ defined by "Slater determinants" with no single particle state occupied (with no creation operators applying on the vacuum state), with one single particle state occupied (with one creation operator applying on the vacuum state), with two single particle states occupied (with two creation operator applying on the vacuum state), and so on.

vi. In the Grassmann case the creation operators solving the equations of motion and their Hermitian conjugated partners annihilation operators, applied on $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations postulated by Dirac for the second quantized fermions. In the Clifford case this happens only after "freezing out" half of the Clifford space, as it is shown in Part II, Sect. 2.2, what brings besides the correct anticommutation relations also the "family" quantum number to each irreducible representation of the Lorentz group of the remaining internal space.

vii. The total Hilbert space is the infinite tensor product ($\otimes_N$) of $\mathcal{H}_{\vec{p}}$: $\mathcal{H} = \prod_1^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$. The notation $\otimes_N$ is to point out that superposition of either an odd product of Grassmann anticommuting $\theta^a$'s ($\{\theta^a, \theta^b\}_+ = 0$) or an odd product of Clifford anticommuting $\gamma^a$'s ($\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$) "keep knowing" that they anticommutate no matter for which $\vec{p}$ they define the orthonormalized superposition of "basis vectors", solving the equations of motion. The creation and annihilation operators having an odd Clifford or Grassmann character can not change their oddness when they define solutions of the equation of motion for different choices of $\vec{p}$.

viii. Since the "parts of the basis" in the momentum space belonging to different $\vec{p}$ satisfy the "orthogonality" relations, the creation and annihilation operators determined with $\vec{p}$ anticommute with the creation and annihilation operators determined with any other $\vec{p}'$. No postulates for the second quantized fermions are needed.

ix. Correspondingly the creation and annihilation operators with the internal space described by either odd Clifford or odd Grassmann algebra, since fulfilling the anticommutation relations required for the second quantized fermions without postulates, explain the Dirac’s postulates for the second quantized fermions.

We present in Sect. II properties of the Grassmann odd (and, for our study of anticommuting "fermions" not important Grassmann even) algebra and the corresponding chosen "basis vectors" for even dimensional space-time, $d = (d - 1) + 1$, and illustrate anticommuting "basis vectors" on the case of $d = (5 + 1)$, Subsect. II.A chapter A.b. We define the action for the integer spin "fermions" in Subsect. II.B Solutions of the corresponding equations of motion define the creation operators depending on $\vec{p}$ in $d$-dimensional space-time. We illustrate the corresponding
superposition of "basis vectors", solving the equation of motion in $d = (5 + 1)$ in chapter B.a..

We present in Sect. [III] the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$ as "Slater determinants" with no "fermion state" occupied with "fermions", with one "fermion state" occupied, with two "fermion states" occupied, up to the "Slater determinant" with all possible "fermion states" for a particular $\vec{p}$ occupied. The total Hilbert space $\mathcal{H}$ is then the tensor product $\prod_{\infty} \otimes_{N} \mathcal{H}_{\vec{p}}$. On $\mathcal{H}$ the creation and annihilation operators (solving the equations of motion for free massless fermions) manifest the anticommutation relations of second quantized "fermions" without any postulates. These second quantized "fermion" fields, manifesting in the Grassmann case an integer spin, offer in $d$-dimensional space, $d > (3 + 1)$, the description of the corresponding charges in adjoint representations. We follow in this paper to some extent Ref. [7].

In Subsect. [III C] relation between the by Dirac postulated creation and annihilation operators and the creation and annihilation operators presented in this Part I — for integer spins "fermions" — are discussed.

In Sect. [IV] we comment what we have learned from the second quantized "fermion" fields with integer spin when internal degrees of freedom are described in Grassmann space and compare these recognitions with the recognitions, which the Clifford algebra is offering, discussions on which appear in Part II.

In Part II we present in equivalent sections properties of the two kinds of the Clifford algebras and discuss conditions under which operators of the two Clifford algebras demonstrate the anticommutation relations required for the second quantized fermion fields on the Hilbert space $\mathcal{H} = \prod_{\infty} \otimes_{N} \mathcal{H}_{\vec{p}}$, this time with the half integer spin, offering in $d$-dimensional space, $d > (3 + 1)$, the description of charges, as well as the appearance of families of fermions [7], both needed to describe the properties of the observed quarks and leptons and antiquarks and antileptons, appearing in families.

In Part II we discuss relations between the Dirac way of second quantization with postulates and our way using Clifford algebra.

This paper is a part of the project named the spin-charge-family theory of one of the authors (N.S.M.B.), so far offering the explanation for all the assumptions of the standard model, with the appearance of the scalar fields included.

The Clifford algebra offers in even $d$-dimensional spaces, $d \geq (13 + 1)$ indeed, the description of the internal degrees of freedom for the second quantized fermions with the half integer spins, explaining all the assumptions of the standard model: The appearance of charges of the observed quarks and leptons and their families, as well as the appearance of the corresponding
gauge fields, the scalar fields, explaining the Higgs scalar and the Yukawa couplings, and in addition the appearance of the dark matter, of the matter/antimatter asymmetry, offering several predictions [5, 6, 10–16].

II. PROPERTIES OF GRASSMANN ALGEBRA IN EVEN DIMENSIONAL SPACES

In Grassmann $d$-dimensional space there are $d$ anticommuting operators $\theta^a$, $\{\theta^a, \theta^b\} = 0$, $a = (0, 1, 2, 3, 5, \ldots, d)$, and $d$ anticommuting derivatives with respect to $\theta^a$, $\frac{\partial}{\partial \theta^a}$, $\{\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\} = 0$, offering together $2 \cdot 2^d$ operators, the half of which are superposition of products of $\theta^a$ and another half corresponding superposition of $\frac{\partial}{\partial \theta^a}$.

\[
\{\theta^a, \theta^b\} = 0, \quad \frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\} = 0,
\]
\[
\{\theta^a, \frac{\partial}{\partial \theta^b}\} = \delta_{ab}, (a, b) = (0, 1, 2, 3, 5, \cdots, d).
\]

Defining [7]

\[
(\theta^a)^\dagger = \eta^{aa} \frac{\partial}{\partial \theta^a},
\]

it follows

\[
(\frac{\partial}{\partial \theta^a})^\dagger = \eta^{aa} \theta^a.
\]

The identity is the self adjoint member. The signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$ is assumed.

One can arrange $2^d$ products of $\theta^a$ into irreducible representations with respect to the Lorentz group with the generators [6]

\[
S^{ab} = i (\theta^a \frac{\partial}{\partial \theta^b} - \theta^b \frac{\partial}{\partial \theta^a}), \quad (S^{ab})^\dagger = \eta^{aa} \eta^{bb} S^{ab}.
\]

$2^{d-1}$ members of the representations have an odd Grassmann character (those which are superposition of odd products of $\theta^a$’s). All the members of any particular odd irreducible representation follow from any starting member by the application of $S^{ab}$’s.

If we exclude the self adjoint identity there is $(2^{d-1} - 1)$ members of an even Grassmann character, they are even products of $\theta^a$’s. All the members of any particular even representation follow from any starting member by the application of $S^{ab}$’s.

The Hermitian conjugated $2^{d-1}$ odd partners of odd representations of $\theta^a$’s and $(2^{d-1} - 1)$ even partners of even representations of $\theta^a$’s are reachable from odd and even representations, respectively, by the application of Eq. (2).
It is useful to make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra. We make the (usual) choice
\[ S^{03}, S^{12}, S^{56}, \ldots, S^{d-1\,d}, \] (4)
and choose the members of the irreducible representations of the Lorentz group to be the eigenvectors of all the members of the Cartan subalgebra of Eq. (4)
\[ S^{ab} \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b) = k \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b), \]
\[ S^{ab} \frac{1}{\sqrt{2}} (1 + \frac{i}{k} \theta^a \theta^b) = 0, \]
or rather
\[ S^{ab} \frac{1}{\sqrt{2}} \frac{i}{k} \theta^a \theta^b = 0, \] (5)
with \( k^2 = \eta^{aa}_{\,ik} \eta^{bb}_{\,ik} \). The eigenvector \( \frac{1}{\sqrt{2}} (\theta^0 + \theta^3) \) of \( S^{03} \) has the eigenvalue \( k = \pm i \), the eigenvalues of all the other eigenvectors of the rest of the Cartan subalgebra members, Eq. (4), are \( k = \pm 1 \).

"Basis vectors" are correspondingly products of odd \( \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b) \) and the superposition of even objects \( \frac{i}{k} \theta^a \theta^b \), with eigenvalues \( k = \pm i \) or \( k = \pm 1 \) and 0, respectively.

Let us check how does \( S^{ac} = i(\theta^a \frac{\partial}{\partial \theta^c} - \theta^c \frac{\partial}{\partial \theta^a}) \) transform the product of two "nilpotents" \( \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b) \) and \( \frac{1}{\sqrt{2}} (\theta^c + \eta^{cc}_{\,ik} \theta^d) \). Taking into account Eq. (3) one finds that \( S^{ac} \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b) \frac{1}{\sqrt{2}} (\theta^c + \eta^{cc}_{\,ik} \theta^d) = -\eta^{aa}_{\,ik} \eta^{cc}_{\,ik} \frac{1}{2k} \theta^a \theta^b + \frac{k}{k} \theta^c \theta^d \). \( S^{ac} \) transforms the product of two Grassmann odd eigenvectors of the Cartan subalgebra into the superposition of two Grassmann even eigenvectors.

"Basis vectors" have an odd or an even Grassmann character, if their products contain an odd or an even number of "nilpotents", \( \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b) \), respectively. "Basis vectors" are normalized, up to a phase, in accordance with Eq. (A1) of A.

The Hermitian conjugated representations of (either an odd or an even) products of \( \theta^a \)'s can be obtained by taking into account Eq. (2) for each "nilpotent"
\[ \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{\,ik} \theta^b)^\dagger = \eta^{aa}_{\,ik} \frac{1}{\sqrt{2}} (\partial + \eta_{\,ik}) \frac{\partial}{\partial \theta_a} \frac{\partial}{\partial \theta_b}, \]
\[ (\frac{i}{k} \theta^a \theta^b)^\dagger = \frac{i}{k} \frac{\partial}{\partial \theta_a} \frac{\partial}{\partial \theta_b}. \] (6)

Making a choice of the identity for the vacuum state,
\[ |\phi_{\text{vac}} > = |1 >, \] (7)
we see that algebraic products — we shall use a dot , , or without a dot for an algebraic product of eigenstates of the Cartan subalgebra forming "basis vectors" and \( *_A \) for the algebraic product
of "basis vectors" — of different \( \theta^a \)'s, if applied on such a vacuum state, give always nonzero contributions,

\[
(\theta^0 \mp i\theta^3) \cdot (\theta^1 \pm i\theta^2) \cdots (\theta^{d-1} \mp i\theta^d) |1 > \neq 0,
\]

(this is true also, if we substitute any of nilpotents \( \frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik} \theta^b) \) or all of them with the corresponding even operator \( \ell^{ab} \) \( \theta^a \theta^b \)), in the case of odd Grassmann irreducible representations an odd number of nilpotents, at least one nilpotent, must remain), while the corresponding Hermitian conjugated partners give zero when applying on \(|1 >\)

\[
(\frac{\partial}{\partial \theta_0} \mp \frac{\partial}{\partial \theta_3}) \cdot (\frac{\partial}{\partial \theta_1} \pm i \frac{\partial}{\partial \theta_2}) \cdots (\frac{\partial}{\partial \theta_{d-1}} \pm i \frac{\partial}{\partial \theta_d}) |1 > = 0.
\]

Let us notice the properties of the odd products \( \theta^a \)'s and of their Hermitian conjugated partners:

i. Superposition of products of different \( \theta^a \)'s, applied on the vacuum state \(|1 >\), give nonzero contribution. We make a choice of the "basis vectors", which are the eigenvectors of all the Cartan subalgebra, Eq. (4), creating on the vacuum state the corresponding "fermion" states.

ii. The Hermitian conjugated partners of the "basis vectors", they are products of derivatives \( \frac{\partial}{\partial \theta_a} \)'s, give, when applied on the vacuum state \(|1 >\), Eq. (7), zero. Each annihilation operator annihilates the corresponding creation operator.

iii. The algebraic product, \( *_A \), of a "basis vector" by itself gives zero, the algebraic anticommutator of any two "basis vectors" of an odd Grassmann character (superposition of an odd products of \( \theta^a \)'s) gives zero ("basis vectors" of the two decuplets in Table I and the "basis vector" of Eq. (13) \( \frac{1}{2}(\theta^0 \pm i\theta^3) \), for example, demonstrate this property).

iv. The algebraic application of any annihilation operator on the corresponding Hermitian conjugated "basis vector" gives identity, on all the rest of "basis vectors" gives zero. Correspondingly the algebraic anticommutators of the creation operators and their Hermitian conjugated partners, applied on the vacuum state, give identity, all the rest anticommutators of creation and annihilation operators applied on the vacuum state, give zero.

v. Correspondingly the "basis vectors" and their Hermitian conjugated partners, applied on the vacuum state \(|1 >\), Eq. (7), fulfill the properties of creation and annihilation operator, respectively, for the second quantized "fermions" on the level of one "fermion" state.
\section{Grassmann "basis vectors"}

We construct \(2^{d-1}\) Grassmann odd "basis vectors" and \(2^{d-1} - 1\) (we skip self adjoint identity, which we use to describe the vacuum state \(|1\rangle\)) Grassmann even "basis vectors" as superposition of odd and even products of \(\theta^a\)'s, respectively. Their Hermitian conjugated \(2^{d-1}\) odd and \(2^{d-1} - 1\) even partners are, according to Eqs. (2, 6), determined by the corresponding superposition of odd and even products of \(\frac{\partial}{\partial \theta^a}\)'s, respectively \cite{17}.

\subsection*{A.a. Grassmann anticommuting "basis vectors" with integer spins}

Let us choose in \(d = 2(2n + 1)\)-dimensional space-time, \(n\) is a positive integer, the starting Grassmann odd "basis vector" \(\hat{b}^{\theta_1 \dagger}\), which is the eigenvector of the Cartan subalgebra of Eq. (4) with the eigenvalues \((+i, +1, +1, \cdots , +1)\), respectively, and has the Hermitian conjugated partner equal to \((\hat{b}^{\theta_1 \dagger})^\dagger = \hat{b}^{\theta_1}\).

\begin{align}
\hat{b}^{\theta_1 \dagger} &= \left(\frac{1}{\sqrt{2}}\right)^{d-3} (\theta^0 - \theta^1)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \\
&\vdots (\theta^{d-1} + i\theta^d) \\
\hat{b}^{\theta_1} &= \left(\frac{1}{\sqrt{2}}\right)^{d-3} \left(\frac{\partial}{\partial \theta^0} - i \frac{\partial}{\partial \theta^1}\right) \cdots \left(\frac{\partial}{\partial \theta^{d-1}} - i \frac{\partial}{\partial \theta^d}\right). \tag{8}
\end{align}

In the case of \(d = 4n\), \(n\) is a positive integer, the corresponding starting Grassmann odd "basis vector" can be chosen as

\begin{align}
\hat{b}^{\theta_1 \dagger} &= \left(\frac{1}{\sqrt{2}}\right)^{d-3} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \\
&\vdots (\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} - i\theta^d). \tag{9}
\end{align}

All the rest of "basis vectors", belonging to the same irreducible representation of the Lorentz group, follow by the application of \(S^{abc}\)'s.

We denote the members of this starting representation by \(\hat{b}^{\theta_k \dagger}\) and their Hermitian conjugated partners by \(\hat{b}^{\theta_k}\), with \(k = 1\).

"Basis vectors", belonging to different irreducible representations \(k\), will be denoted by \(\hat{b}^{\theta_k \dagger}\) and their Hermitian conjugated partners by \(\hat{b}^{\theta_k} = (\hat{b}^{\theta_k \dagger})^\dagger\).

\(S^{abc}\)'s, which do not belong to the Cartan subalgebra, transform step by step the two by two "nilpotents", no matter how many "nilpotents" are between the chosen two, as follows:

\[
S^{abc} \frac{1}{\sqrt{2}} (\theta^a + \frac{\eta^{ao}}{ik} \theta^b) \cdots \frac{1}{\sqrt{2}} (\theta^c + \frac{\eta^{co}}{ik} \theta^d) \propto -\frac{\eta^{ao} \eta^{co}}{2ik} (\theta^a \theta^b + k \theta^a \theta^d) \cdots
\]

leaving at each step at least one "nilpotent" unchanged, so that the whole irreducible representation remains odd.
The application of creation operators on the vacuum state for second quantized fermions. The Grassmann "fermion" states correspondingly follow by the anticommutation relations among creation and annihilation operators required by Dirac \cite{1}.

Therefore we can start another odd representation with the "basis vector" $b_2^{\Theta \dagger}$ as follows

$$\hat{b}_2^{\Theta \dagger} = \left(\frac{1}{\sqrt{2}}\right)^4 (\theta^0 + \theta^3) (\theta^1 + i\theta^2) (\theta^5 + i\theta^6) \cdots (\theta^{d-1} + i\theta^d),$$

$$\langle \hat{b}_2^{\Theta \dagger} \rangle = b_2^{\Theta} = \left(\frac{1}{\sqrt{2}}\right)^4 \left( \frac{\partial}{\partial \theta^{d-1}} - i \frac{\partial}{\partial \theta^d} \right) \cdots \left( \frac{\partial}{\partial \theta^0} - i \frac{\partial}{\partial \theta^d} \right).$$

(10)

The application of $S^{\alpha 3}$'s determines the whole second irreducible representation $\hat{b}_j^{\Theta \dagger}$.

One finds that each of these two irreducible representations has $\frac{1}{2} d(d+1)$ members, Ref. \cite{7}.

Taking into account Eq. (11) it follows that odd products of $\theta^a$'s anticommute and so do the odd products of $\frac{\partial}{\partial \theta^a}$'s. One also sees that $\frac{\partial}{\partial \theta^a} \theta^b = \eta^{ab}$, while $\frac{\partial}{\partial \theta^a} |1\rangle = 0$ and $\theta^a |1\rangle = \theta^a |1\rangle$. We can therefore conclude

$$\{ \hat{b}_i^{\Theta k}, \hat{b}_j^{\Theta \dagger} \}_{*A} + |1\rangle = \delta_{ij} \delta^{kl} |1\rangle,$$

$$\{ \hat{b}_i^{\Theta k}, \hat{b}_j^{\Theta \dagger} \}_{*A} + |1\rangle = 0 \cdot |1\rangle,$$

$$\{ \hat{b}_i^{\Theta k}, \hat{b}_j^{\Theta \dagger} \}_{*A} + |1\rangle = 0 \cdot |1\rangle,$$

$$\hat{b}_j^{\Theta k} *_A |1\rangle = 0 \cdot |1\rangle,$$

(11)

where $\{ \hat{b}_i^{\Theta k}, \hat{b}_j^{\Theta \dagger} \}_{*A}$ is meant. These anticommutation relations manifest the anticommutation relations among creation and annihilation operators required by Dirac \cite{1} for second quantized fermions. The Grassmann "fermion" states correspondingly follow by the application of creation operators on the vacuum state $|1\rangle$

$$|\phi_{\alpha i}^k \rangle = \hat{b}_i^{\Theta k} |1\rangle.$$  

(12)

But Grassmann "fermions" have an integer spin, Eq. (5), and not half integer spin as it is the case for the so far observed fermions.

**A.b. Illustration of anticommuting "basis vectors" in $d = (5 + 1)$-dimensional space**

Let us illustrate properties of Grassmann odd representations with the case that $d = (5+1)$-dimensional space.

Table II represents two decuplets, which are "eigenvectors" of the Cartan subalgebra $(S^{03}, S^{12}, S^{56})$, Eq. (4), of the Lorentz algebra $S^{ab}$. The two decuplets represent two Grassmann odd irreducible representations of $SO(5,1)$.

One can read on the same table, from the first to the third and from the fourth to the sixth line of both decuplets, two Grassmann even triplet representations of $SO(3,1)$, if paying attention on the eigenvectors.
TABLE I: The two decuplets, the odd eigenvectors of the Cartan subalgebra, Eq. (4), (S⁰³, S¹², S⁵⁶), for SO(5,1) of the Lorentz algebra in Grassmann (5 + 1)-dimensional space, forming two irreducible representations, are presented. Table is partly taken from Ref. [7]. The "basis vectors" within each decuplet are reachable from any member by S⁰⁶'s and are decoupled from another decuplet. The two operators of handedness, Γ⁰⁶⁶, d = (6, 4), are invariants of the Lorentz algebra, Eq. (B1), Γ⁵⁶⁵ for the whole decuplet, Γ³³⁴ for the "triplets" and "fourplets".

| Decuplet of eigenvectors | S⁰³ | S¹² | S⁵⁶ | Γ⁵⁶⁵ | Γ³³⁴ |
|--------------------------|-----|-----|-----|------|------|
| 1                        | 1   | 1   | 1   | 1    | 1    |
| 2                        | 0   | 0   | 0   | 0    | 0    |
| 3                        | 0   | 0   | 0   | 0    | 0    |
| 4                        | 0   | 0   | 0   | 0    | 0    |
| 5                        | 0   | 0   | 0   | 0    | 0    |
| 6                        | 0   | 0   | 0   | 0    | 0    |
| 7                        | 0   | 0   | 0   | 0    | 0    |
| 8                        | 0   | 0   | 0   | 0    | 0    |
| 9                        | 0   | 0   | 0   | 0    | 0    |
| 10                       | 0   | 0   | 0   | 0    | 0    |

of S⁰³ and S¹² alone, while the eigenvector of S⁵⁶ has, as a "spectator", the eigenvalue either +1 (the first triplet in both decuplets) or −1 (the second triplet in both decuplets). Each of the two decuplets contains also one "fourplet" with the "charge" S⁵⁶ equal to zero ((7⁷, 8⁴, 9⁴, 1⁰) lines in each of the two decuplets (Table II in Ref. [II]).

Paying attention on the eigenvectors of S⁰³ alone one recognizes as well even and odd representations of SO(1,1): θ⁰θ³ and θ⁰ ± θ³, respectively.

The Hermitian conjugated "basis vectors" follow by using Eq. (6) and is for the first "basis vector" of Table I equal to (−)²(1/2³)(δ/δθ³ − i δ/δθ⁴)(δ/δθ⁴ − i δ/δθ³)(δ/δθ⁵ + i δ/δθ⁶). One correspondingly finds that when (1/2³)(δ/δθ³ − i δ/δθ⁴)(δ/δθ⁴ − i δ/δθ³)(δ/δθ⁵ + i δ/δθ⁶) applies on (1/2³)(θ³ − θ⁴)(θ³ − θ⁴)(θ⁵ + iθ⁶) the result is identity. Application of (1/2³)(δ/δθ³ − i δ/δθ⁴)(δ/δθ⁴ − i δ/δθ³)(δ/δθ⁵ + i δ/δθ⁶) on all the rest of "basis vectors" of the decuplet I as well as on all the "basis vectors" of the decuplet II gives zero. "Basis vectors" are orthonormalized with respect to Eq. (A1). Let us notice that δ/δθ³ on a "state" which is just an identity, |1⟩, gives zero, δ/δθ³ |1⟩ = 0, while θ³ |1⟩, or any superposition of products of θα's, applied on |1⟩, gives the "vector"
One easily sees that application of products of superposition of $\theta^a$’s on $|1>$ gives nonzero contribution, while application of products of superposition of $\frac{\partial}{\partial \theta^a}$’s on $|1>$ gives zero.

The two by $S^{ab}$ decoupled Grassmann decuplets of Table I are the largest two irreducible representations of odd products of $\theta^a$’s. There are 12 additional Grassmann odd “vectors”, arranged into irreducible representations of six singlets and one sixplet

$$\left( \frac{1}{2} (\theta^0 + \theta^3), \frac{1}{2} (\theta^1 + i\theta^2), \frac{1}{2} (\theta^5 + i\theta^0), \frac{1}{2} (\theta^1 + i\theta^2) \theta^0 \theta^2 \theta^6, \frac{1}{2} (\theta^5 + i\theta^0) \theta^0 \theta^3 \theta^2 \right).$$  \tag{13}

The algebraic application of products of superposition of $\frac{\partial}{\partial \theta^a}$’s on the corresponding Hermitian conjugated partners, which are products of superposition of $\theta^a$’s, leads to the identity for either even or odd Grassmann character [18].

Besides 32 Grassmann odd eigenvectors of the Grassmann Cartan subalgebra, Eq. (4), there are $(32 - 1)$ Grassmann ”basis vectors”, which we arrange into irreducible representations, which are superposition of even products of $\theta^a$’s. The even self adjoint operator identity (which is indeed the normalized product of all the annihilation times, $*A_1$, creation operators) is used to represent the vacuum state.

It is not difficult to see that Grassmann ”basis vectors” of an odd Grassmann character anticommute among themselves and so do odd products of superposition of $\theta^a$’s, while equivalent even products commute.

The Grassmann odd algebra (as well as the two odd Clifford algebras) offer the description of the anticommuting second quantized fermion fields, as postulated by Dirac. But the Grassmann ”fermions” carry the integer spins, while the observed fermions — quarks and leptons — carry half integer spin.

A.c. Grassmann commuting ”basis vectors” with integer spins

Grassmann even ”basis vectors” manifest the commutation relations, and not the anticommutation ones as it is the case for the Grassmann odd ”basis vectors”. Let us use in the Grassmann even case, that is the case of superposition of an even number of $\theta^a$’s in $d = 2(2n + 1)$, the notation $\hat{a}_j^{\theta^0}$, again chosen to be eigenvectors of the Cartan subalgebra, Eq. (4), and let us start with one representative

$$\hat{a}_j^{\theta^0} : = \left( \frac{1}{\sqrt{2}} \right)^{d-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^0) \quad \cdots (\theta^0 - \theta^3) \theta^{d-1} \theta^d. \tag{14}$$

The rest of ”basis vectors”, belonging to the same Lorentz irreducible representation, follow by the application of $S^{ab}$. The Hermitian conjugated partner of $\hat{a}_j^{\theta^0}$ is $\hat{a}_j^{\theta^0} = (\hat{a}_j^{\theta^0})^i$.

$$\hat{a}_1^{\theta^0} : = \left( \frac{1}{\sqrt{2}} \right)^{d-1} \frac{\partial}{\partial \theta^d} \frac{\partial}{\partial \theta^{d-1}} (\theta^0 - \theta^3) \quad \cdots \left( \frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3} \right). \tag{15}$$
If $\hat{a}^\dagger_{\theta k} j$ represents a Grassmann even creation operator, with index $k$ denoting different irreducible representations and index $j$ denoting a particular member of the $k^{th}$ irreducible representation, while $\hat{a}^\dagger_{\theta k} j$ represents its Hermitian conjugated partner, one obtains by taking into account Sect. II, the relations

$$\{\hat{a}^\dagger_{\theta k} i, \hat{a}^\dagger_{\theta k'} i\} \ast |1\rangle = \delta_{kk'} \delta_{ij} |1\rangle,$$
 $$\{\hat{a}^\dagger_{\theta k} i, \hat{a}^\dagger_{\theta k'} j\} \ast |1\rangle = 0 \cdot |1\rangle,$$
 $$\{\hat{a}^\dagger_{\theta k} i, \hat{a}^\dagger_{\theta k'} j\} \ast |1\rangle = 0 \cdot |1\rangle,$$
 $$\hat{a}^\dagger_{\theta k} i \ast |1\rangle = 0 \cdot |1\rangle,$$
 $$\hat{a}^\dagger_{\theta k} i \ast |1\rangle = |\phi^k_i\rangle .$$

(16)

Equivalently to the case of Grassmann odd “basis vectors” also here $\{\hat{a}^\dagger_{\theta k} i, \hat{a}^\dagger_{\theta l} j\} \ast = \hat{a}^\dagger_{\theta k} i \ast \hat{a}^\dagger_{\theta l} j - \hat{a}^\dagger_{\theta l} j \ast \hat{a}^\dagger_{\theta k} i$ is meant.

B. Action for free massless Grassmann ”fermions” with integer spin [7]

In the Grassmann case the ”basis vectors” of an odd Grassmann character, chosen to be the eigenvectors of the Cartan subalgebra of the Lorentz algebra in Grassmann space, Eq. (4), manifest the anticommutation relations of Eqs. (11, 12).

To compare the properties of creation and annihilation operators with the creation and annihilation operators postulated by Dirac for the second quantized fermions, we need to define the superposition of ”basis vectors” which correspond to particular momentum $p^a = (p^0, p^1, p^2, p^3, p^5, \cdots, p^d)$ in ordinary space, relating $p^0$ and $\vec{p}$.

In the Grassmann case we need to propose the Lorentz invariant action for a free massless ”Grassmann fermions”. We follow here the suggestion of one of us (N.S.M.B.) from Ref. [7].

$$\mathcal{A}_G = \int d^dx \ d^d\theta \ \omega \ \{\phi^\dagger \gamma^0_G \frac{1}{2} \theta^a p_a \phi\} + h.c.,$$

$$\omega = \prod_{k=0}^{d} (\delta_{kk} + \theta^k),$$

with $\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial \theta^a})$, $(\gamma^a_G)^\dagger = \gamma^a_G$, for each $a = (0, 1, 2, 3, 5, \cdots, d)$. We use the integral over $\theta^a$ coordinates with the weight function $\omega$ from Eq. (A1, A2). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator $\gamma^0_G$, which takes care of the Lorentz invariance. Namely

$$S^{ab} (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) = (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) S^{ab},$$

$$S^\dagger (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) = (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) S^{-1},$$

$$S = e^{-\frac{i}{2} \omega_{ab} (L^{ab} + S^{ab})},$$

(18)
while $\theta^a$, $\frac{\partial}{\partial \theta^a}$, and $p^a$ transform as Lorentz vectors.

The Lagrange density is up to the surface term equal to

$$L_G = \frac{1}{2} \phi^\dagger \gamma_0^0 (\theta^a - \frac{\partial}{\partial \theta^a}) (\hat{p}_a \phi)$$

$$= \frac{1}{4} \left\{ \phi^\dagger \gamma_0^0 (\theta^a - \frac{\partial}{\partial \theta^a}) \hat{p}_a \phi - (\hat{p}_a \phi^\dagger) \gamma_0^0 (\theta^a - \frac{\partial}{\partial \theta^a}) \phi \right\} ,$$

(19)

leading to the equations of motion

$$\frac{1}{2} \gamma_0^0 (\theta^a - \frac{\partial}{\partial \theta^a}) \hat{p}_a \phi = 0 ,$$

(20)

as well as the Klein-Gordon equation,

$$(\theta^a - \frac{\partial}{\partial \theta^a}) \hat{p}_a (\theta^b - \frac{\partial}{\partial \theta^b}) \hat{p}_b \phi = 0 = \hat{p}_a \hat{p}_a^a \phi .$$

The eigenstates $\phi$ of equations of motion for free massless "fermions", Eq. (20), can be found as superposition of $2^{d-1}$ "basis vectors" $\hat{b}_k^{\theta k \dagger}$, applied on the vacuum state $|1\rangle$. Let us remind the reader that the "basis vectors" are the "eigenstates" of the Cartan subalgebra, Eq. (4), fulfilling the anticommutation relations of Eq. (11).

The coefficients, determining the superposition, depend on momentum $p^a$, $a = (0, 1, 2, 3, 5, \ldots, d)$, $(p^0)^2 = (\vec{p})^2$, of the plane wave solution $e^{-ip^a x^a}$. Let us define the new creation operators and the corresponding states

$$\hat{b}_k^{\theta k \dagger} (\vec{p}) = \sum_i c_i^{ks} \hat{b}_i^{\theta k \dagger}, \quad |p^0| = |\vec{p}|, \quad |\phi_0^{ks} (\vec{p}) \rangle = \hat{b}_k^{\theta k \dagger} (\vec{p}) |1\rangle, \quad |p^0| = |\vec{p}|,$$

(21)

with $s$ representing different solutions of the equations of motion and $k$ different irreducible representations of the Lorentz group, $\vec{p}$ denotes the chosen vector $(p^0, \vec{p})$ in momentum space.

One has further

$$|\phi_0^{ks}(t, \vec{x}) \rangle = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(2\pi)^{d-1}} \hat{b}_k^{\theta k \dagger} (\vec{p}) e^{-ip^a x^a} |1\rangle, \quad |p^0| = |\vec{p}| .$$

(22)

If orthogonalized, the states $|\phi_0^{ks}(\vec{p}) \rangle$ fulfill the relation

$$< \phi_0^{ks}(\vec{p}) | \phi_0^{k's'} (\vec{p'}) > = \delta^{kk'} \delta_{ss'} \delta_{pp'} , \quad |p^0| = |\vec{p}| ,$$

$$< \phi_0^{k's'} (\vec{x'}) | \phi_0^{ks} (\vec{x}) > = \delta^{kk'} \delta_{ss'} \delta_{\vec{x'} \vec{x}} , \quad |p^0| = |\vec{p}| ,$$

(23)

where we assumed the discretization of momenta $\vec{p}$ and coordinates $\vec{x}$.
There are in even dimensional spaces \((d = 2(2n + 1)\) and \(4n\) \(2^{d-1}\) Grassmann odd superposition of ”basis vectors”, which belong to different irreducible representations, among them twice \(\frac{d!}{2 \cdot \frac{d!}{2} \cdot \frac{d!}{2}}\) of the kind presented in Eqs. \([8] [9]\) and discussed in the chapter \(A.b\). of the subsect. \(A.A\) and in Table \(I\) for a particular case \(d = (5 + 1)\). The illustration for the superposition \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p})\) is presented, again for \(d = (5 + 1)\), in chapter \(B.a\).

The algebraic anticommutation relations among creation, \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p})\), and annihilation, \(\hat{b}_{tot}^{\theta k s}(\vec{p})\), operators, all of the odd Grassmann character, are

\[
\{\hat{b}_{tot}^{\theta k s}(\vec{p}), \hat{b}_{tot}^{\theta k s^\dagger}(\vec{p'})\}_{\ast A} + |1> = \delta^{kk'} \delta_{ss'} \delta(\vec{p} - \vec{p'}) |1>,
\]

\[
\{\hat{b}_{tot}^{\theta k s}(\vec{p}), \hat{b}_{tot}^{\theta k s^\dagger}(\vec{p'})\}_{\ast A} + |1> = 0 \cdot |1>,
\]

\[
\{\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p}), \hat{b}_{tot}^{\theta k s^\dagger}(\vec{p'})\}_{\ast A} + |1> = 0 \cdot |1>,
\]

\[
\hat{b}_{tot}^{\theta k s}(\vec{p}) \ast_A |1> = 0 \cdot |1>,
\]

\[
|p^0| = |\vec{p}|.
\]

(24)

\(k\) labels different irreducible representations of Grassmann odd “basis vectors”, \(s\) labels different — orthogonal and normalized — solutions of equations of motion and \(p\) represent different momenta fulfilling the relation \((p^0)^2 = (\vec{p})^2\). Here we allow continuous momentum and take into account that

\[
< 1|\hat{b}_{tot}^{\theta k s}(\vec{p}) \ast_A \hat{b}_{tot}^{\theta k s^\dagger}(\vec{p'}) |1> = \delta^{kk'} \delta_{ss'} \delta(\vec{p} - \vec{p'}) ,
\]

(25)

in the case of continuous values of \(\vec{p}\) in even \(d\)-dimensional space.

For each momentum \(\vec{p}\) there are \(2^{d-1}\) members of the odd Grassmann character, belonging to different irreducible representations. The plane wave solutions, belonging to different \(\vec{p}\), are orthogonal, defining correspondingly \(\infty\) many degrees of freedom for each of \(2^{d-1}\) “fermion” states, defined by \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p})\), when applying on the vacuum state \(|1>\), Eq. \((7)\).

To define the Hilbert space of all possible ”Slater determinants” of all possible occupied and empty states and the application of \(\hat{b}_{tot}^{\theta k s}(\vec{p})\ast_T\) and \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p})\ast_T\) on ”Slater determinants”, we make the tensor products, \(\ast_T\), of all possible creation \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p})\) and annihilation operators \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p'})\ast_T\). For two creation operators we have

\[
\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p}) \ast_T \hat{b}_{tot}^{\theta k s^\dagger}(\vec{p'}).
\]

(26)

Since the creation operators \(\hat{b}_{tot}^{\theta k s^\dagger}(\vec{p})\) and their annihilation partners, having an odd Grassmann character, manifest the algebraic anticommutation relations presented in Eq. \((24)\), while the even
creation operators manifest the commutation relations, presented in Eq. (16), we keep for the tensor product of Grassmann od two creation operators and for creation and annihilation operators their odd properties, which require

\[ \hat{b}_{\theta k s}^{\dagger}(\vec{p}) * T \hat{b}_{\theta k' s'}^{\dagger}(\vec{p}') = -\hat{b}_{\theta k' s'}^{\dagger}(\vec{p}') * T \hat{b}_{\theta k s}^{\dagger}(\vec{p}), \]

if at least one of \((k, s, \vec{p})\) distinguishes from \((k', s', \vec{p}')\).

\[ \hat{b}_{\theta k s}^{\dagger}(\vec{p}) * T \hat{b}_{\theta k' s'}(\vec{p}') = 0, \]

\[ \hat{b}_{\theta k' s'}^{\dagger}(\vec{p}') * T \hat{b}_{\theta k s}^{\dagger}(\vec{p}) = 0, \]

if at least one of \((k, s, \vec{p})\) distinguishes from \((k', s', \vec{p}')\).

\[ \hat{b}_{\theta k s}(\vec{p}) * T \hat{b}_{\theta k' s'}(\vec{p}') = -\hat{b}_{\theta k' s'}(\vec{p}') * T \hat{b}_{\theta k s}(\vec{p}), \]

if at least one of \((k, s, \vec{p})\) distinguishes from \((k', s', \vec{p}')\).

\[ \hat{b}_{\theta k s}(\vec{p}) * T \hat{b}_{\theta k s}^{\dagger}(\vec{p}) |1> = |1>, \]

\[ \hat{b}_{\theta k s}^{\dagger}(\vec{p}) |1> = 0. \] (27)

This tensor product determines the application of the creation and annihilation operators on "Slater determinants": i. The creation operator \(\hat{b}_{\theta k s}^{\dagger}(\vec{p})\) must always jump over the creation operator defining the occupied state of another kind (distinguishing from the jumping creation one in any of the internal quantum numbers \((k, s)\) or \(\vec{p}\) up to the last step when it comes to its own empty one with the quantum numbers \((k, s)\) and \(\vec{p}\), and occupies this empty state, or if this state is already occupied gives zero. Whenever \(\hat{b}_{\theta k s}^{\dagger}(\vec{p})\) jumps over the occupied state the sign is changed. ii. The annihilation operator changes sign whenever jumping over the occupied state carrying different internal quantum numbers \((k, s)\) or \(\vec{p}\), unless it comes to the occupied state with its own all the internal quantum numbers \((k, s)\) and its own \(\vec{p}\), emptying this state or if this state is empty gives zero.

Since the Grassmann odd objects \(\hat{b}_{\theta k s}^{\dagger}(\vec{p})\), solving equations of motion, are superposition of \(\hat{b}_{\theta i}^{\dagger}\) fulfilling the anticommutation relations of Eqs. (11, 12), while the Grassmann even objects fulfill the commutation relations of Eq. (16), it seems the only meaningful choice that the tensor products of creation operators \(\hat{b}_{\theta k s}^{\dagger}(\vec{p})\) keep the odd character, fulfilling correspondingly the relations of Eq. (27). Correspondingly the Grassmann odd objects explain for integer spin "fermions" postulates of Dirac for the second quantized fermions.

We show in Part II that the Clifford odd "basis vectors" describe fermions with the half integer spin, offering as well the corresponding anticommutation relations, explaining Dirac’s postulates.
for second quantized fermions. In the next section we define the Hilbert space with undetermined number of Grassmann “fermions” of integer spins, first for a particular momentum \( \vec{p} \) with definite degrees of freedom and then for all possible momenta.

**B.a. Plane wave solutions of equations of motion, Eq. (20), in \( d = (5 + 1) \)-dimensional space**

One of such plane wave massless solutions of the equations of motion in \( d = (5 + 1) \)-dimensional space for momentum \( p^a = (p^0, p^1, p^2, p^3, 0, 0) \), \( p^0 = |\vec{p}| \), is the superposition of “basis vectors”, presented in Table I among the first three members of the first decuplet, \( k = I \). This particular solution \( \hat{b}^{011\dagger}_{\text{tot}}(\vec{p}) \) carries the spin \( S_{12} = 1 \) (“up”) and the “charge” \( S_{56} = 1 \) (both from the point of view of \( d = (3 + 1) \))

\[
\hat{b}^{011\dagger}_{\text{tot}}(\vec{p}) = \beta \left( \frac{1}{\sqrt{2}} \right)^2 \left\{ \frac{1}{\sqrt{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2) \right. \\
- \frac{2(|p^0| - |p^3|)}{p^1 - ip^2} (\theta^0\theta^3 + i\theta^1\theta^2) \\
\left. - \frac{(p^1 + ip^2)^2}{(|p^0|^2 + |p^3|^2)^2} \frac{1}{\sqrt{2}} (\theta^0 + \theta^3)(\theta^1 - i\theta^2) \right\} \\
\times (\theta^5 + i\theta^6) |p^0| = |\vec{p}|, \tag{28}
\]

\( \beta \) is the normalization factor. The notation \( \hat{b}^{011\dagger}_{\text{tot}}(\vec{p}) \) means that the creation operator represents the plane wave solution of the equations of motion, Eq. (20), for a particular \( |p^0| = |\vec{p}| \).

Applied on the vacuum state the creation operator defines the second quantized single particle state of particular momentum \( \vec{p} \). States, carrying different \( \vec{p} \), are orthogonal (due to the orthogonality of the plane waves of different momenta and due to the orthogonality of \( \hat{b}^{0k's\dagger}_{\text{tot}}(\vec{p}) \) and \( \hat{b}^{0k's}_{\text{tot}}(\vec{p}) \) with respect to \( k \) and \( s \), Eq. (24)).

More solutions can be found in [7] and the references therein.

**III. HILBERT SPACE OF ANTICOMMUTING INTEGER SPIN “FERMIONS”**

The \( 2^{d-1} \) Grassmann odd creation operators \( \hat{b}^{0k's\dagger}_{\text{tot}}(\vec{p}) \) of particular momentum \( (\vec{p}) \), solving the equations of motion, Eq. (20), fulfill together with their Hermitian conjugated annihilation operators \( \hat{b}^{0k's}_{\text{tot}}(\vec{p}) \) the anticommutation relations of Eq. (24).

All the \( 2^{d-1} \) Grassmann odd creation operators of particular momentum \( \vec{p} \), if applied on the vacuum state \( |1> \), Eq. (7), define \( 2^{d-1} \) states. The Hilbert space \( \mathcal{H}_{\vec{p}} \) of a particular momentum \( \vec{p} \) consists correspondingly of \( 2^{2^{d-1}} \) “Slater determinants”, namely the one with no occupied state, those with one occupied state, those with two occupied states, up to the one with all \( 2^{d-1} \) states occupied.
The total Hilbert space $\mathcal{H}$ consists of infinite many ”Slater determinants”, due to infinite many degrees of freedom in the momentum space.

A. Hilbert space of anticommuting integer spin “fermions” of particular $\vec{p}$

Let us write down explicitly these $2^{2d-1}$ contributions to the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$, using the notation that $\hat{b}^{k}_{s\vec{p}}$ represents the unoccupied state $\hat{b}^{k}_{s\vec{p}} |1\rangle >$ (of the $s^{th}$ solution of the equations of motion belonging to the $k^{th}$ irreducible representation), while $\hat{1}^{k}_{s\vec{p}}$ represents the corresponding occupied state.

The number operator is according to Eq. (11) and Eq. (27) equal to

$$N_{\vec{p}}^{\theta k s} = \hat{b}^{\theta k s\dagger}_{\vec{p}} *_{T} \hat{b}^{\theta k s}_{\vec{p}},$$

$$N_{\vec{p}}^{\theta k s} *_{T} \hat{b}^{k}_{s\vec{p}} = 0, \ N_{\vec{p}}^{\theta k s} *_{T} \hat{1}^{k}_{s\vec{p}} = 1. \quad (29)$$

Let us simplify the notation so that we count for $k = 1$ empty states as $0_{r\vec{p}}$, and occupied states as $1_{r\vec{p}}$, with $r = (1, \ldots, s_{max}^{1})$, for $k = 2$ we count $r = s_{max}^{1} + 1, \ldots, s_{max}^{1} + s_{max}^{2}$ up to $r = 2^{d-1}$. Correspondingly we can represent $\mathcal{H}_{\vec{p}}$ as follows

$$|0_{1\vec{p}}, 0_{2\vec{p}}, 0_{3\vec{p}}, \ldots, 0_{2^{d-1}\vec{p}} >, |1_{1\vec{p}}, 0_{2\vec{p}}, 0_{3\vec{p}}, \ldots, 0_{2^{d-1}\vec{p}} >,$$

$$|0_{1\vec{p}}, 1_{2\vec{p}}, 0_{3\vec{p}}, \ldots, 0_{2^{d-1}\vec{p}} >, |0_{1\vec{p}}, 0_{2\vec{p}}, 1_{3\vec{p}}, \ldots, 0_{2^{d-1}\vec{p}} >,$$

$$\vdots$$

$$|1_{1\vec{p}}, 1_{2\vec{p}}, 0_{3\vec{p}}, \ldots, 0_{2^{d-1}\vec{p}} >, |1_{1\vec{p}}, 0_{2\vec{p}}, 1_{3\vec{p}}, \ldots, 0_{2^{d-1}\vec{p}} >,$$

$$\vdots$$

$$|1_{1\vec{p}}, 1_{2\vec{p}}, 1_{3\vec{p}}, \ldots, 1_{2^{d-1}\vec{p}} > \quad (30)$$

with a part with none of states occupied ($N_{r\vec{p}} = 0$ for all $r = 1, \ldots, 2^{d-1}$), with a part with only one of states occupied ($N_{r\vec{p}} = 1$ for a particular $r = 1, \ldots, 2^{d-1}$ while $N_{r'\vec{p}} = 0$ for all the others $r' \neq r$), with a part with only two of states occupied ($N_{r\vec{p}} = 1$ and $N_{r'\vec{p}} = 1$, where $r$ and $r'$ run from $1, \ldots, 2^{d-1}$), and so on. The last part has all the states occupied.

Taking into account Eq. (27) is not difficult to see that the creation operator $\hat{b}^{\theta k s\dagger}_{\vec{p}}$ and the annihilation operators $\hat{b}^{\theta k s}_{\vec{p}}$, when applied on this Hilbert space $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations for the second quantized “fermions”.

$$\{\hat{b}^{\theta k s}_{\vec{p}}, \hat{b}^{\theta k' s'\dagger}_{\vec{p}} \} *_{T} + \mathcal{H}_{\vec{p}} = \delta^{kk'} \delta_{ss'} \mathcal{H}_{\vec{p}},$$

$$\{\hat{b}^{\theta k s}_{\vec{p}}, \hat{b}^{\theta k' s'\dagger}_{\vec{p}} \} *_{T} + \mathcal{H}_{\vec{p}} = 0 \cdot \mathcal{H}_{\vec{p}},$$

$$\{\hat{b}^{\theta k s\dagger}_{\vec{p}}, \hat{b}^{\theta k' s'\dagger}_{\vec{p}} \} *_{T} + \mathcal{H}_{\vec{p}} = 0 \cdot \mathcal{H}_{\vec{p}}. \quad (31)$$
The proof for the above relations easily follows if taking into account that when ever the creation or annihilation operator jumps over an odd products of occupied states the sign changes. Then one sees that the contribution of the application of $\hat{b}_{\theta k s}^{\dagger} (\vec{p}^t) \ast T \hat{b}_{\theta k s}^\dagger (\vec{p}) \mathcal{H}_{\vec{p}}$ has the opposite sign than the contribution of $\hat{b}_{\theta k s}^{\dagger} (\vec{p}^t) \ast T \hat{b}_{\theta k s} (\vec{p}) \mathcal{H}_{\vec{p}}$.

If the creation and annihilation operators are Hermitian conjugated to each other, the result of

$$\left\{ \hat{b}_{\theta k s} (\vec{p}) \ast T \hat{b}_{\theta k s}^{\dagger} (\vec{p}) + \hat{b}_{\theta k s}^{\dagger} (\vec{p}) \ast T \hat{b}_{\theta k s} (\vec{p}) \right\} \mathcal{H}_{\vec{p}} = \mathcal{H}_{\vec{p}}$$

is the whole $\mathcal{H}_{\vec{p}}$ back. Each of the two summands operate on their own half of $\mathcal{H}_{\vec{p}}$. Jumping together over even number of occupied states $\hat{b}_{\theta k s} (\vec{p})$ and $\hat{b}_{\theta k s}^{\dagger} (\vec{p})$ do not change the sign of particular “Slater determinant”. (Let us add that $\hat{b}_{\theta k s} (\vec{p})$ reduces for particular $k$ and $s$ the Hilbert space $\mathcal{H}_{\vec{p}}$ for a factor $\frac{1}{2}$, and so does $\hat{b}_{\theta k s}^{\dagger} (\vec{p})$. The sum of both, applied on $\mathcal{H}_{\vec{p}}$, reproduces the whole $\mathcal{H}_{\vec{p}}$.)

**B. Hilbert space of anticommuting integer spin “fermions”**

The total Hilbert space of anticommuting “fermions” is the product $\otimes_N$ of the Hilbert spaces of particular $\vec{p}$

$$\mathcal{H} = \prod_{\vec{p}} \otimes_N \mathcal{H}_{\vec{p}}.$$

(32)

The notation $\otimes_N$ is to point out that creation operators $\hat{b}_{\theta k s}^{\dagger} (\vec{p})$, which origin in superposition of odd number of $\theta^a$’s, keep their odd character also in the tensor products of creation operators $\hat{b}_{\theta k s}^{\dagger} (\vec{p})$ and their Hermitian conjugated annihilation operators no matter for which $\vec{p}$ they define the orthonormalized superposition of “basis vectors”, solving the equations of motion. For “plane wave solutions” of equations of motion in a box the momentum $\vec{p}$ is discretized, otherwise is continuous. The number of “Slater determinants” in the Hilbert space $\mathcal{H}$ in $d$-dimensional space is infinite.

$$N_{\mathcal{H}} = \prod_{\vec{p}} 2^{2^{d-1}}.$$

(33)

Since the creation operators $\hat{b}_{\theta k s}^{\dagger} (\vec{p})$ and the annihilation operators $\hat{b}_{\theta k s} (\vec{p}')$ fulfill for particular $\vec{p}$ the anticommutation relations on $\mathcal{H}_{\vec{p}}$, Eq. (31), and since the momentum plane wave solutions are orthogonalized, ($< p^i | p'^i > = \delta (p^i - p'^i)$, for each component $p^i$), Eq. (25), the anticommutation
relations follows also for $\mathcal{H}$

$$\{\hat{b}_{\theta k s_{\text{tot}}}(\vec{p}), \hat{b}^\dagger_{\theta k s'_{\text{tot}}}(\vec{p}')\}_{*T+\mathcal{H}} = \delta^{kk'} \delta_{ss'} \delta(\vec{p} - \vec{p}') \mathcal{H},$$

$$\{\hat{b}_{\theta k s_{\text{tot}}}(\vec{p}), \hat{b}^\dagger_{\theta k s'_{\text{tot}}}(\vec{p}')\}_{*T+\mathcal{H}} = 0 \cdot \mathcal{H},$$

$$\{\hat{b}_{\theta k s_{\text{tot}}}(\vec{p}), \hat{b}^\dagger_{\theta k' s'_{\text{tot}}}(\vec{p}')\}_{*T+\mathcal{H}} = 0 \cdot \mathcal{H}. \quad (34)$$

Taking into account Eq. (34) it follows

$$\{\phi_{k's'_{\text{tot}}}(\vec{x}'), \phi_{ks_{\text{tot}}}(\vec{x})\} + = \delta^{kk'} \delta_{ss'} \delta(\vec{x} - \vec{x}'). \quad (35)$$

Creation operators, $\hat{b}^\dagger_{\theta k s_{\text{tot}}}(\vec{p})$, operating on a vacuum state, as well as on the whole Hilbert space, define the second quantized fermion states.

C. Relations between creation operators $\hat{b}^\dagger_{\theta k s_{\text{tot}}}(\vec{p})$ in the Grassmann odd algebra and the creation operators postulated by Dirac

Creation operators, $\hat{b}^\dagger_{\theta k s_{\text{tot}}}(\vec{p})$ define the second quantized fermion states.

While the second quantized Dirac fermions have the half integer spin, the "fermions", the internal degrees of which is described by the Grassmann odd algebra, have the integer spin. We leave therefor the detailed comparison of the creation and annihilation operators for fermions with half integer spins between those postulated by Dirac and the ones following from the Clifford odd algebra presented in Part II to Subsect. 3.4 of Part II.

Here we discuss only the appearance of the creation and annihilation operators offered by the Grassmann odd algebra and those postulated by Dirac. In both cases we treat only $d = (3+1)$-dimensional space, that is we solve the equations of motion for $p^a = (p^0, p^1, p^2, p^3)$ (in the case that $d > 4$ the rest of space demonstrates the charges in $d = (3+1)$, when $p^a = (p^0, p^1, p^2, p^3, 0, 0, \ldots, 0)$).

It is pointed out in what follows that both internal spaces — either the internal space postulated by Dirac or the internal space offered by the Grassmann algebra — are finite dimensional, as also the internal space offered by the Clifford algebra is finite dimensional. The creation and annihilation operators, defined on the whole Hilbert space, with the single particle states defined in the internal space and the momentum space, form the continuously infinite set.

In the Dirac case the second quantized states are in $d = (3+1)$ dimensions postulated as follows

$$\psi^s(\vec{x}) = \sum_{i, \vec{p}_k} \hat{a}^\dagger_i(\vec{p}_k) v_i^s(\vec{p}_k) e^{i\vec{p}_k \cdot \vec{x}}. \quad (36)$$
\(v^s(\vec{p}_k)\) are two left handed \((\Gamma^{(3+1)} = -1)\) and two right handed \((\Gamma^{(3+1)} = 1, \text{Eq. (B.3)})\) two-component column matrix, representing the four solutions \(s\) of the Weyl equation for free massless fermions of particular momentum \(|\vec{p}_k| = p^0_k\) (\cite{2}, Eqs. (20-49) - (20-51)), the factor \(\varepsilon = \pm 1\) depends on the product of handedness and spin.

\(\hat{a}_\uparrow^*(\vec{p}_k)\) are by Dirac postulated creation operators, which together with annihilation operators \(\hat{a}_i(\vec{p}_k)\), fulfill the anticommutation relations (\cite{2}, Eqs. (20-49) - (20-51)),

\[
\{\hat{a}_i^\dagger(\vec{p}_k), \hat{a}_j^\dagger(\vec{p}_l)\}_{\gamma T} = 0 = \{\hat{a}_i(\vec{p}_k), \hat{a}_j(\vec{p}_l)\}_{\gamma T},
\]

\[
\{\hat{a}_i(\vec{p}_k), \hat{a}_j(\vec{p}_l)\}_{\gamma T} = \delta_{ij}\delta_{\vec{p}_k\vec{p}_l},
\]

(37)
in the case of discretized momenta for a fermion in a box. Creation operators \(\hat{a}_\uparrow^\dagger(\vec{p}_k)\) have on the Hilbert space of all "Slater determinants" these commutation properties. Correspondingly we use the tensor anticommutation relations.

To be able to relate the creation operators of Dirac \(\hat{a}_\uparrow^\dagger(\vec{p}_k)\) with \(\hat{b}_{\text{tot}}^{\theta k s \uparrow}(\vec{p}_k)\) from Eq. (34), let us remind the reader that \(\hat{b}_{\text{tot}}^{\theta k s \uparrow}(\vec{p}_k)\) is a superposition of basic vectors \(\hat{b}_{\gamma T}^{\theta k s \uparrow}\) with the coefficients \(c_{\gamma T}^{\theta k s \uparrow}\), which depend on the momentum \(\vec{p}\), Eq. (21), so that \(\hat{b}_{\text{tot}}^{\theta k s \uparrow}(\vec{p}_k)\) solves the equation of motion for free massless fermions for plane waves, while \(|p^0| = |\vec{p}|\).

We treat in this subsection in the Grassmann case \((3+1)\)-dimensional space only, without taking care on different irreducible representations \(k\) as well as on charges, in order to be able to relate the creation and annihilation operators in Grassmann space with the Dirac’s ones. In this case the odd Grassmann creation operators are expressible with the “basic vectors”, which are fourplets, presented in Table I on the 7th up to the 10th lines, the same on both decuplets, neglecting \(\theta^5\theta^6\) contribution.

Let us introduce in the Dirac case first the two creation operators \(c_{\uparrow}^{h \uparrow}\) and \(c_{\downarrow}^{h \uparrow}\) of handedness \(h\) and spin up and down \((\uparrow, \downarrow)\), related to \(\hat{a}_{\uparrow}^{h \uparrow}\) and \(\hat{a}_{\downarrow}^{h \uparrow}\) as follows

\[
\hat{a}_{\uparrow}^{h \uparrow} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot c_{\uparrow}^{h \uparrow}, \quad \hat{a}_{\downarrow}^{h \uparrow} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot c_{\downarrow}^{h \uparrow},
\]

for the two handedness, \(\pm 1\). Then the superposition of both creation operators, \(\hat{a}_{\text{tot}}^{h \uparrow}(\vec{p}_k) := \alpha^{h \uparrow}_{\uparrow}(\vec{p}_k) \hat{a}_{\uparrow}^{h \uparrow} + \alpha^{h \uparrow}_{\downarrow}(\vec{p}_k) \hat{a}_{\downarrow}^{h \uparrow}\) — with coefficients \(\alpha^{h \uparrow}_{\uparrow}(\vec{p}_k)\) and \(\alpha^{h \uparrow}_{\downarrow}(\vec{p}_k)\) chosen so that the superposition solves the equation of motion for a plane wave \(e^{i\varepsilon \vec{p}_k \cdot \vec{x}}\) for \(|\vec{p}_k| = |p^0|\) — determine \(\hat{a}_i^\dagger(\vec{p}_k)v_i^s(\vec{p}_k)\) in the Dirac case.

\[
\hat{a}_{\text{tot}}^{h \uparrow}(\vec{p}_k) := \alpha^{h \uparrow}_{\uparrow}(\vec{p}_k) \hat{a}_{\uparrow}^{h \uparrow} + \alpha^{h \uparrow}_{\downarrow}(\vec{p}_k) \hat{a}_{\downarrow}^{h \uparrow} = \sum_{i(h)} \hat{a}_i^\dagger(\vec{p}_k)v_i^s(\vec{p}_k),
\]
where the summation runs over $i(h)$ representing the chosen handedness. Anticommutation relations of Eq. (37), postulated by Dirac, ensure the equivalent anticommutation relations also for $\hat{a}_{tot}^{hs\dagger}(\vec{p}_k)$ and $\hat{a}_{tot}^{hs}(\vec{p}_k)$. It follows also

$$\psi^{hs}(\vec{x}, t) = \sum_{\vec{p}_k} \hat{a}_{tot}^{hs\dagger}(\vec{p}_k) e^{i\vec{p}_k \cdot \vec{x}}, \quad |\vec{p}_k| = |p_0^k|.$$  (39)

These new creation operators $\hat{a}_{tot}^{hs\dagger}(\vec{p}_k)$ carry now besides 2 (for two handedness) times $2^d - 1 = 2 \times 2$ internal degrees of freedom the index $\vec{p}_k$. But this index does not mean that the corresponding internal space is infinite dimensional, it only means, that the coefficients $\alpha_{\uparrow \downarrow}^{hs}(\vec{p}_k)$ change with the choice of the momentum $\vec{p}_k$ of the ”plane waves in a box” with $|\vec{p}_k| = |p_0^k|$, what brings the infinite degrees of freedom to fermions.

The creation and annihilation operators of Dirac fulfill obviously the anticommutation relations of Eq. (34). To see this we only have to replace $\hat{b}_{tot}^{\theta \dagger}(\vec{p})$ by $\hat{a}_{tot}^{hs\dagger}(\vec{p})$ by taking into account relation of Eq. (38).

Creation and annihilation operators of the Dirac second quantized fermions with half integer spins are in Part II, in Subsect. III.D, related to the corresponding ones, offered by the Clifford algebra. Relating the creation and annihilation operators offered by the Clifford algebra objects with the Dirac’s ones ensures us that the Clifford odd algebra explains the Dirac’s postulates.

IV. CONCLUSIONS

We learn in this Part I paper, that in $d$-dimensional space the superposition of odd products of $d\ \theta^a$’s exist, Eqs. (8, 10, 9), chosen to be the eigenvectors of the Cartan subalgebra, Eq. (5), and arranged to be solutions of the equation of motion for free massless “fermions”, Eq. (20), which together with their Hermitian conjugated partners, odd products of $\frac{\partial}{\partial \theta^a}$’s, Eqs. (2, 8, 6), fulfill all the requirements for the anticommutation relations for the Dirac’s fermions on the vacuum state $|\phi_o> = |1>$, Eq. (24), as well as on the whole Hilbert space of ”Slater determinants” of (continuous) infinite number of momenta $\times 2^{d-1}$ “Slater determinants” (for each momentum $\vec{p}$, Eq. (34)).

Since the creation and annihilation operators, which are superposition of odd products of $\theta^a$’s and $\frac{\partial}{\partial \theta^a}$’s, respectively, anticommute, Eq. (11, 12), while the corresponding even products $\theta^a$’s and $\frac{\partial}{\partial \theta^a}$’s, respectively, commute, Eq. (16), the conclusion is that also tensor products of creation and annihilation operators fulfill the anticommutation relations of Eq. (34). The use of the Grassmann
odd algebra to describe the internal degrees of freedom of "fermions" offers the anticommutation relations without postulating them.

Let us add that either \( \hat{b}_{\theta^k} \) or \( \hat{b}_{\theta^k}^\dagger(\vec{p}) \) are together with their Hermitian conjugated partners operators. The internal "basis states" are chosen in the way that resemble the properties of usual Dirac’s creation and annihilation operators of second quantized fermions, but in the Grassmann case for integer spins fermions.

These Grassmann "fermions" carry the integer spin and charges, originated in \( d \geq 5 \) in the adjoint representations. No families appear in this case, that means that there is no available operators, which would connect different irreducible representations of the Lorentz group (without breaking symmetries).

No elementary "fermions" with these properties have been observed, and since the observed quarks and leptons and anti-quarks and anti-leptons have half integer spins, charges in the fundamental representations and appear in families, there is no possibility for observing the integer spin elementary "fermions", at least not from the point of view of Eq. (20) in Part II, telling that the reduction of space in Clifford case, needed for the appearance of second quantized half integer fermions, reduces also the Grassmann space, leaving no place for second quantized “fermions” with integer spin.

In Part II two kinds of operators are studied; There are namely two kinds of the Clifford algebra objects, \( \gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a}) \) and \( \tilde{\gamma}^a = i (\theta^a - \frac{\partial}{\partial \theta^a}) \), which anticommute, \( \{\gamma^a, \tilde{\gamma}^a\}_+ = 0 \) \( \{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} \) \( \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \), and offer correspondingly two kinds of independent representations.

Each of these two kinds of independent representations can be arranged into irreducible representations with respect to the two Lorentz generators \( - S^{ab} = \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a) \) and \( \tilde{S}^{ab} = \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) \). All the Clifford irreducible representations of any of the two kinds of algebras are independent and disconnected.

The two Dirac’s actions in \( d \)-dimensional space for free massless fermions \( (A = \int d^d x \frac{i}{2} (\psi^\dagger \gamma^0 \gamma^a p_a \psi) + h.c. \) and \( \tilde{A} = \int d^d x \frac{i}{2} (\psi^\dagger \tilde{\gamma}^0 \tilde{\gamma}^a p_a \psi) + h.c. \) lead to the equations of motion, which have the solutions in both kinds of algebras for an odd Clifford character (they are superposition of an odd products of \( \gamma^a \)'s and \( \tilde{\gamma}^a \)'s, respectively), forming the creation and annihilation operators, which only "almost" anticommute, while the Grassmann odd creation and annihilation operators do anticommute. Although "vectors" of one irreducible representation of an odd Clifford algebra character, anticommute among themselves and so do their Hermitian conjugated partners in each of the two kinds of the Clifford algebras, the anticommutation relations among creation and annihilation operators in each of the two Clifford algebras separately, do not
fulfill the requirement, that only the anticommutator of a creation operator and its Hermitian conjugated partner gives a nonzero contribution.

The decision, the postulate, that only one kind of the Clifford algebra objects — we make a choice of $\gamma^a$ — describes the internal space of fermions, while the second kind — $\tilde{\gamma}^a$ in this case — does not, and consequently determine “family” quantum numbers which distinguish among irreducible representations of $S^{ab}$, solves the problems: a. Creation operators and their Hermitian conjugated partners, which are odd products of superpositions of $\gamma^a$, fulfill all the requirements, which Dirac postulated for fermions. b. Different irreducible representations with respect to $S^{ab}$ carry now different ”family” quantum numbers determined by $\frac{d}{2}$ commuting operators among $\tilde{S}^{ab}$. c. Operators $\tilde{S}^{ab}$, which do not belong to the Cartan subalgebra, connect different irreducible representations of $S^{ab}$.

The above mentioned decision obviously reduces the degrees of freedom of the odd (and even) Clifford algebra, while opening the possibility for the appearance of ”families” and offering the explanation for the Dirac’s second quantized postulates. This decision, reducing as well the degrees of freedom of Grassmann algebra, disables the existence of the integer spin ”fermions” in Grassmann odd algebra, Eq. (20) in Part II.

**Appendix A: Norms in Grassmann space and Clifford space**

Let us define the integral over the Grassmann space [6] of two functions of the Grassmann coordinates $< B|\theta > < C|\theta >$, $< B|\theta > = < \theta |B > ^\dagger$,

$$ < b|\theta > = \sum_{k=0}^{d} b_{a_1...a_k} \theta^{a_1} ... \theta^{a_k}, $$

by requiring

$$ \{d\theta^a, \theta^b\}_+ = 0, \quad \int d\theta^a = 0, \quad \int d\theta^a \theta^a = 1, \quad \int d^d\theta \theta^1 ... \theta^d = 1, $$

$$ d^d\theta = d\theta^d ... d\theta^0, \quad \omega = \prod_{k=0}^{d} \left( \frac{\partial}{\partial \theta_k} + \theta^k \right), \quad (A1) $$
with $\frac{\partial}{\partial \theta^a} \theta^c = \eta^{ac}$. We shall use the weight function $\omega = \prod_{k=0}^d \left( \frac{\partial}{\partial \theta_k} + \theta_k \right)$ to define the scalar product in Grassmann space $\langle B|C \rangle$

$$\langle B|C \rangle = \int d^d \theta \omega < B|\theta> <\theta|C >$$

$$= \sum_{k=0}^d \int b_{b_1...b_k} c_{b_1...b_k}.$$  \hspace{1cm} (A2)

To define norms in Clifford space Eq. (A1) can be used as well.

### Appendix B: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group $S^{ab}$, defined as

$$\Gamma^{(d)} = \alpha \varepsilon_{a_1a_2...a_{d-1}a_d} S^{a_1a_2} : S^{a_3a_4} \ldots S^{a_{d-1}a_d},$$ \hspace{1cm} (B1)

with $\alpha$, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case $S^{ab}$ is defined in Eq. (3), while in the Clifford case Eq. (B1) simplifies, if we take into account that $S^{ab}|_{a \neq b} = \frac{i}{2} \gamma^a \gamma^b$ and $\tilde{S}^{ab}|_{a \neq b} = \frac{i}{2} \tilde{\gamma}^a \tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} : = (i)^{d/2} \prod_a \left( \sqrt{\eta^{aa}} \gamma^a \right), \hspace{1cm} \text{if } d = 2n.$$ \hspace{1cm} (B2)

### Acknowledgments

The author N.S.M.B. thanks Department of Physics, FMF, University of Ljubljana, Society of Mathematicians, Physicists and Astronomers of Slovenia, for supporting the research on the spin-charge-family theory, the author H.B.N. thanks the Niels Bohr Institute for being allowed to staying as emeritus, both authors thank DMFA and Matjaž Breskvar of Beyond Semiconductor for donations, in particular for sponsoring the annual workshops entitled "What comes beyond the standard models" at Bled.

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[17] Relations among operators and their Hermitian conjugated partners in both kinds of the Clifford algebra objects are more complicated than in the Grassmann case, where the Hermitian conjugated operators follow by taking into account Eq. (2). In the Clifford case \( \frac{1}{2}(\gamma^a + \eta a \gamma^b) \) is proportional to \( \frac{1}{\sqrt{2}}(1 + i \tilde{\gamma}^a \gamma^b) \) are self adjoint. This is the case also for representations in the sector of \( \tilde{\gamma}^a \)'s.

[18] We shall see in Part II that the vacuum states are in the Clifford case, similarly as in the Grassmann case, for both kinds of the Clifford algebra objects, \( \gamma^a \)'s and \( \tilde{\gamma}^a \)'s, sums of products of the annihilation \( \times \) its Hermitian conjugated creation operators, and correspondingly self adjoint operators, but they are not the identity.

[19] Taking into account the relations \( \gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a}), \tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a}), \gamma^0_G = -i\eta^a_\alpha \gamma^a \) the Lagrange density can be rewritten as \( L_G = -\frac{1}{2} \phi^\dagger \gamma^0_G \tilde{\gamma}^a (\hat{p}_a \phi) = \frac{1}{4} \left\{ \phi^0 \gamma^0_G \tilde{\gamma}^a \hat{p}_a \phi - \hat{p}_a \phi^0 \gamma^0_G \tilde{\gamma}^a \phi \right\} \).

[20] Varying the action with respect to \( \phi^\dagger \) and \( \phi \) it follows: \( \frac{\partial L_G}{\partial \phi^\dagger} - \hat{p}_a \frac{\partial L_G}{\partial (\hat{p}_a \phi)} = 0 = \frac{1}{2} \phi^\dagger \gamma^0_G \hat{p}_a \phi, \) and \( \frac{\partial L_G}{\partial \phi} - \hat{p}_a \frac{\partial L_G}{\partial (\hat{p}_a \phi)} = 0 = \frac{1}{2} \hat{p}_a \phi^\dagger \gamma^0_G \tilde{\gamma}^a \).