Abstract: An attempt to put together various theoretical, mathematical, or experimental results recently developed in apparently unrelated subjects. Namely Ruelle's approach to turbulence, [1], the body of Nosé–Hoover type of molecular dynamics experiments, [2],[3],[4], mathematical results on Lyapunov exponents (the pairing rule, [5],[6]) and experimental results on them ([7],[8]), theoretical as well as mathematical results on fluctuations ([9] (multifractality), and [10] (chaotic hypothesis)). The key idea that we try to clarify is that of "dynamical ensembles", as a generalization of the classical "equilibrium ensembles", arguing that they should be identified with the SRB distributions and that they share several properties with the classical ensembles. Most of the results invoked here did not deal directly with the Navier Stokes equations and yet they seem to have a lot to do with them (as we shall argue): here the discussion will focus on the Navier Stokes and dissipative Euler equations with the aim of proposing several experiments apt to test the equivalence of dynamical ensembles and the chaotic hypothesis. The ideas developed, to a great extent, from the efforts put in interpreting the experimental results in [4]. An Erratum has been added before the references.

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§1 Reversible dissipation in Euler and Navier Stokes equations.

In [11] one finds an analysis leading to a conjecture on the equivalence between the irreversible NS equation and a reversible equation that was called GNS ("gaussian NS equation").

The ideas discussed in [11] are, however, much more general and it is worth pointing out some other applications, as well as several possible tests that seem under reach of present day experimental (numerical and "real") techniques.

The paper freely uses mixed results available in the literature. Therefore the reader who, after a first reading, has still some interest in the matter may find it difficult to distinguish between my conjectures or theorems, and other’s conjectures, results and/or theorems, and he may want to avoid a second reading and checking of the references to find out. Therefore I have added a concluding section where this is spelled out again in a concise form. The concluding section can also be read, by a reader familiar with the subject, right after seeing the equations (1.1): this will provide an informal but detailed overview of what is discussed in the paper, and of its relations with other works.

I shall focus on fluid mechanics problems considering a fluid that:

1) is enclosed in a periodic box Ω with side L, possibly with a few disks ("obstacles") removed so that no infinite straight path can be found in Ω that avoids the obstacles,

2) is incompressible with density ρ.

I shall consider four distinct evolution equations for this fluid, all of dissipative nature.
In the case \( \Omega \) contains obstacles a "no friction" boundary condition will be imposed on \( \partial \Omega \), i.e. \( u \cdot n = 0 \) if \( n \) is the normal to \( \partial \Omega \). The first equation is the well known Navier Stokes equation with \( \nu \) being the viscosity.

The second equation, introduced in [11] and called GNS, has a multiplier \( \beta \) defined so that the total vorticity \( \eta L^3 = \rho \int \omega^2 dx \), with \( \omega = \partial \wedge u \) being the vorticity, is a constant of motion; this means that:

\[
\beta(u) = \frac{\int (\partial \wedge g \cdot \omega + \omega \cdot (\omega \cdot \partial \wedge u)) \, dx}{\int (\partial \wedge \omega)^2 \, dx}
\] (1.2)

The third equation will be called the *Euler dissipative* equation, ED: it represents a non viscous ideal fluid moving in a "sticky background": \( \chi \) is a "sticky" viscosity. The model is not, as far as I know, a good model for any physical situation (i.e. for 3D–fluids), but it is interesting to consider it for comparison purposes. In 2D–fluids the sticky viscosity has interest in its own as it appears in geophysical models where the coefficient \( \chi \) is known as the Eckman viscosity. 1

The fourth equation will be called GED equation, *gaussian dissipative Euler* equation and here \( \alpha \) is a multiplier defined so that the total (kinetic) energy \( \varepsilon L^3 = \frac{\rho}{2} \int u^2 dx \) is a constant of motion in spite of the action of the force \( g \); this means that \( \alpha \) is given by:

\[
\alpha(u) = \frac{\int g \cdot u}{\int u^2} \] (1.3)

A similar equation, with the constraint that the energy contained in each “momentum shell” be a constant, was considered in [12], which is the first paper in which the idea of a reversible Navier Stokes equation is advanced and studied. The energy content of each “momentum shell” was fixed to be the value predicted by Kolmogorov theory, [14].

Note that both the GED and the GNS equations have a symmetry in \( u \), so that they are reversible in the sense that, if \( V_t \) is the flow describing the equation solution (so that \( t \to V_t u = u(t) \) is the solution with initial data \( u \)), then the transformation \( i : u \to -u \) anti-commutes with the time evolution \( V_t \):

\[
i V_t = V_{-t} i
\] (1.4)

We shall avoid (as it is, unfortunately, always the case in the current literature) considering the problem of proving the global existence and regularity of solutions to the equations (1.1) (the problem is in fact open, see [15]) and we shall consider the truncated equations with momentum cut off \( K \).

The truncation will be performed on a suitable orthonormal basis for the divergenceless fields in \( \Omega \): given the boundary conditions we consider it natural to use the basis generated by the *minimax* principle applied to the Dirichlet quadratic form \( \int_\Omega (\partial \wedge u)^2 \, dx \) defined on the

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1 I am indebted to the referee for this comment and for an appropriate reference, [13].
space of the $C^\infty(\Omega)$ divergenceless fields $u$ with $u \cdot n = 0$ on $\partial \Omega$. The basis fields $u_j$ will verify: $\Delta u_j = -E_j u_j + \partial \mu_j$, for a suitable multiplier $\mu_j$, with $u_j, \mu_j \in C^\infty$ and $E_j$ are eigenvalues).

For instance in the case of no obstacles let $u = \sum_{k \neq 0} \gamma_k e^{ik \cdot x}$ be the velocity field represented in Fourier series with $k = \frac{2\pi}{L}$ and $k \cdot \gamma_k = 0$ (incompressibility condition); here $k$ has components that are integer multiples of the "lowest momentum" $k_0 = \frac{2\pi}{L}$. Then consider the equation:

$$\dot{\gamma}_k = -\vartheta_k \gamma_k - i \sum_{k_1 + k_2 = k} (\gamma_{k_1} \cdot k_2) \Pi_{k_1} \gamma_{k_2} + g_k$$

(1.5)

where the $k$'s take only the values $0 < |k| < K$ for some momentum cut–off $K > 0$, $\vartheta_k$ is the projection on the plane orthogonal to $k$. This is an equation that defines a "truncation on the momentum sphere with radius $K$ of the equations (1.1)" if:

$$\vartheta_k = \begin{cases} -\nu k^2 & \text{NS case} \\
-\beta k^2 & \text{GNS case} \\
-\chi & \text{ED case} \\
-\alpha & \text{GED case}
\end{cases}$$

(1.6)

For simplicity we may suppose, in this no obstacles case, that the mode $k = 0$ is absent, i.e. $\gamma_0 = 0$: this can be done if, as we suppose, the external force $g$ does not have a zero mode component (i.e. it has zero average).

In order that the resulting cut–off equations be physically acceptable, and supposing that $g_k \neq 0$ only for $|k| \sim k_0$, one shall have to fix $K$ large. For instance in the NS case it should be much larger than the Kolmogorov scale $K = (\eta \nu^{-2})^{1/4}$, where $\eta \nu$ is the average dissipation rate of the solutions to (1.5) without cut–off. The scale $K$ is determined, on the basis of heuristic dimensional considerations and of the dissipation rate $\nu$–independence (as $\nu \to 0$: [16] p. 306), by setting $\nu \eta \sim L^{1/2} g^{3/2}$: see [14].

We shall use the same cut off for the other equations with $k$ replaced by the basis label $j$ and $|k|$ replaced by $\sqrt{E_j}$, which is certainly a natural choice for the GNS equation.

For the ED and GED equations the choice of $K$ should be made by developing a theory analogous to Kolmogorov’s theory. We only attempt a preliminary analysis in §6 as the latter equations are used here only for the purpose of illustrating some interesting mechanisms and theories. Below we always refer to the truncated equations, unless otherwise stated.

It is easy, in the no obstacles cases, to express the coefficients $\alpha, \beta$ for the cut off equations:

$$\alpha = \frac{\sum_{0<k<|k|<K} \overline{\gamma}_k \cdot k}{\sum_{0<k<|k|<K} \gamma_k^2} \quad \beta = \beta_i + \beta_c$$

(1.7)

$$\beta_c = \frac{\sum_{k \neq 0} k^2 \overline{g}_k \cdot k}{\sum_k k^4 \gamma_k^2}$$

$$\beta_i = -i \frac{\sum_{k_1 + k_2 + k_3 = 0} k_3^2 (\gamma_{k_1} \cdot k_2) (\gamma_{k_2} \cdot k_3)}{\sum_k k^4 \gamma_k^2}$$

where the $k$’s take only the values $0 < |k| < K$ for some momentum cut–off $K > 0$ and $\Pi_k$ is the orthogonal projection on the plane perpendicular to $k$. 

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The cases in which the region $\Omega$ contains obstacles is very similar but we cannot write simple expressions for the basis fields and therefore the equations, although formally very similar to (1.5), (1.7), cannot be written very explicitly.

The solutions of the equations (1.5), or of the corresponding ones in the obstacles cases, will be denoted $V_i^{\nu,ns} \xi, V_i^{\eta,gns} \xi, V_i^{\varepsilon,ed} \xi, V_i^{\varepsilon,ged} \xi$ when the initial datum is $\xi$. Or in general:

$$V_i^{\xi} \xi, \xi = (\nu, ns), (\eta, gns), (\chi, ed), (\varepsilon, ged)$$

(1.8)

Keeping the forcing $g$ constant we shall admit that for each equation, i.e. for each choice of $\xi$, there is a unique stationary distribution $\mu_\xi$ describing the statistics of all initial data $\xi$ that are randomly chosen with a "Liouville distribution", i.e. (in the no obstacle cases, to fix the ideas) with a distribution $\mu_0(d\gamma)$ proportional to the volume measure $\xi |K|$, where the delta function is present only in the case of the reversible equations and fixes the constants of motion to the value prescribed by the first label in $\xi$.

This means that given "any observable" $F$ on the phase space $F$ (of the velocity fields with momentum cut–off $K$) it is:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(V_i^{\xi} \gamma) dt = \int_F F(\gamma') \mu_\xi(d\gamma') \overset{\text{def}}{=} \langle F \rangle_\xi$$

(1.9)

for all choices of $\gamma$ except a set of zero Liouville measure. The distribution $\mu_\xi$ will be called the SRB distribution for the Eq. (1.5), (1.6) (see the "zero-th law" in [17], [18]).

A particular role will be played by the averages $\langle \varepsilon \rangle_\xi, \langle \eta \rangle_\xi$ as well as by the averages of $\langle \alpha \rangle_\xi, \langle \beta \rangle_\xi$ and of the entropy production rate $\sigma(\gamma)$, that is defined by the divergence of the r.h.s. of the cut off equations.

Consider explicitly only the no obstacles case: if $D_K$ is the number of modes $\\hat{k}$ with $0 < |\\hat{k}| < K$ then the number of (independent) components of $\{ \gamma_k \}$ is $2D_K$ and, see (1.5), setting $2D_K = \sum_{|\hat{k}| < K} 2 \hat{k}^2$ (which in the case with obstacles becomes $2D_K = \sum \sqrt{E_{j,K} \sqrt{E_j}}$), one finds that $\sigma$ is given by:

$$\sigma = 2D_K \nu \quad \xi = (\nu, ns)$$
$$\sigma = 2D_K \beta - \overline{\beta}_e - \overline{\beta}_i \quad \xi = (\eta, gns)$$
$$\sigma = 2D_K \chi \quad \xi = (\chi, ed)$$
$$\sigma = 2D_K \alpha - \alpha \quad \xi = (\varepsilon, ged)$$

(1.10)

where $\overline{\beta}_i, \overline{\beta}_e$ are suitably defined, e.g. in the no obstacles cases:

$$\overline{\beta}_e = \sum_{|\hat{k}|} \hat{k}^2 |\hat{k}|^2 \gamma_k^2 \hat{k}^2 - \frac{2}{\sum (\sum_{|\hat{k}|} \hat{k}^2 |\hat{k}|^2 \hat{k}^2 - \frac{2}{\sum (\sum_{|\hat{k}|} \hat{k}^2 |\hat{k}|^2 \gamma_k^2 \hat{k}^2)} (\sum_{|\hat{k}|} \hat{k}^4 |\hat{k}|^2 \gamma_k^2 \hat{k}^2)}$$

(1.11)

so that $\sigma \simeq 2D_K \beta$ for $\xi = (\eta, gns)$ and $\sigma \simeq 2D_K \alpha$ for $\xi = (\varepsilon, ged)$.

The equivalence of dynamical ensembles conjecture, [11], is the following:

Conjecture NS: The statistics $\mu_{\nu,ns}, \mu_{\eta,gns}$ of the NS equations and of the GNS equations respectively are equivalent in the limit of large Reynolds number provided the parameters $\eta$ and $\nu$ are so related that $\langle \sigma \rangle_{\nu,ns} = \langle \sigma \rangle_{\eta,gns}$.

Here equivalent means that the ratios of the averages of the same observables with respect to the two distributions approaches 1 as $R \to \infty$. The Reynolds number is defined here to be

\[ \text{Reynolds number} = \frac{u L}{v} \]

\[ \text{where } u \text{ is the velocity, } L \text{ is a characteristic length, and } v \text{ is the kinematic viscosity.} \]

\[ \text{Physica D, 105, 163–184, 1997.} \]

... è tanto nuovo e, nella prima apprezzazione, rimasto dal verisimile, che quando non si avesse modo di dilucidarla e renderla più chiara che’l Sole meglio sarebbe il tacere che’l pronunziarla; però, già che me la son lasciata scappare di bocca..., [19], p. 231.
\[ R = (\nu \eta)^{1/3} L^{4/3} \nu^{-1} \]

because the condition of equivalence can also be expressed by: \( \nu \langle \omega^2 \rangle = \nu \eta \)

and the quantity \( \nu \eta \) is the quantity that in the usual notation of Kolmogorov’s theory is called \( \varepsilon \).

We do not adhere to such notation only to avoid confusion with the quantity called (naturally) \( \varepsilon \) ("energy per unit mass") in the analysis of the ED and GED equations.

A corresponding conjecture can be formulated for the ED and GED equations:

**Conjecture ED:** The statistics \( \mu_{\chi, ed}, \mu_{\varepsilon, ged} \) of the ED equations and of the GED equations respectively are equivalent in the limit of large Reynolds number provided the parameters \( \varepsilon \) and \( \chi \) are so related that \( \langle \sigma \rangle_{\chi, ed} = \langle \sigma \rangle_{\varepsilon, ged} \).

The above stated conjectures are closely analogous to the familiar statements on the equivalence of thermodynamic ensembles, with the thermodynamic limit replaced by the limit \( R \to \infty \) of infinite Reynolds number. They can be substantially weakened for the purposes of possible applications.

It is well known that the equivalence of the ensembles in equilibrium statistical mechanics does not extend to all possible observables, but it is restricted to the local ones. The natural notion of locality is, in the cases above, locality in momentum space.

An observable \( O \) will be called "local" if its value on a particular velocity field depends only on the Fourier components of the field with wave vectors \( k \) in a range \( k_1, k_2 \) independent on the Reynolds number size.

Typical local observables are the energy content of a momentum shell, and the average (overs space) velocity near a point. The difference between the average velocity field near a point \( \bar{x} \) and an infinitesimally close field can be considered a local observable if "close" means "differing" little only near \( \bar{x} \). On the other hand the quantity \( \sigma \), total phase space contraction rate, is a non local observable.

Therefore one can expect that, in equivalent distributions, the Lyapunov exponents of the two models coincide, at least if one looks at the ones in a fixed range of values, away from the extreme values. The fluctuations of \( \sigma \) may be quite different (although the average values of this quantity will still be the same, tautologically, if the conjecture holds).

The idea of non equilibrium ensembles and their possible equivalence is not really new: the recent literature contains many hints in such direction. The clearest is perhaps [12]. See also the first of [20] (§4) and [21].

On heuristic grounds, the conjectures would be justified if one did accept that the entropy creation rate reaches its average on a time scale that is fast compared to the hydrodynamical scales. The coefficients \( \alpha \simeq (2D_K)^{-1} \sigma \) and \( \beta \simeq (2D_K)^{-1} \sigma \), see (1.10), would be confused with their time averages \( \langle \alpha \rangle_{\varepsilon, ged} \) or \( \langle \beta \rangle_{\eta, gns} \) and identified with the viscosity constants \( \nu \) or \( \chi \).

In this way the GNS and the NS equations would be equivalently good: both being the macroscopic manifestation of two equivalent microscopic dissipation mechanisms: one explicitly specified by the Gaussian constraint of constant dissipation and the other with dissipation unspecified a priori but phenomenologically modeled by a constant viscosity. Likewise one can view the GED and the ED equations as macroscopically equivalent: one with constant energy and the other with constant sticky viscosity \( \chi \).

The interest of the above conjectures is that the same physical system in which irreversible dissipation occurs (the NS or ED equations) can be described equivalently by a reversibly dissipative system (the GNS or GED equations).

For instance one can investigate the implications of the fact that for reversible systems a general principle, the chaotic hypothesis, can be reasonably assumed to hold and to imply consequences that seem to be non trivial, see [10], [18], [22], [23], [24], about fluctuations and Lyapunov spectrum.

The next section is devoted to a quick discussion of some of the established consequences of the principle and §3%§6 will deal with heuristic ideas and with describing a possible scenario for the phenomenology of the equations (1.1). The scenario will be developed without any
pretension of rigor and it will present what will appear as the simplest among many other possibilities. It leads (implicitly) to several possible experimental tests of the chaotic hypothesis and of the other ideas involved in its development: the tests can also be viewed, independently, just as interesting experiments proposals.

In the following we shall always consider the NS equations and the GNS equations with parameters fixed so that \( \mu_{\nu,\text{ns}} \) and \( \mu_{\eta,\text{gns}} \) are equivalent by the conjecture NS, and likewise we shall always consider the ED and GED equations with parameters fixed so that \( \mu_{\chi,\text{ed}} \) and \( \mu_{\varepsilon,\text{ged}} \) are equivalent by the conjecture ED.

§ 2 The fluctuation theorems.

In reference [10],[18] the chaotic hypothesis was presented as a reformulation of an older principle due to Ruelle, [1]. It gave us the possibility of some quantitative parameterless "predictions", in various cases, see also [22], [23], [24]. The hypothesis is:

Chaotic hypothesis: A chaotic many particle system or fluid in a stationary state can be regarded, for the purpose of computing macroscopic properties, as a smooth dynamical system with a transitive Axiom A global attractor. In reversible systems it can be regarded, for the same purposes, as a smooth transitive Anosov system.

The main result of [10] is the fluctuation theorem that gives a property of the variable \( p = p(\gamma) \) defined in terms of the contraction rate \( \sigma_0 \) of the attractor surface elements by:

\[
\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \sigma_0(V^t \gamma) \, dt = \langle \sigma_0 \rangle + p
\]

which can be regarded as a random variable with the distribution \( \pi_\tau(p) dp \) that it inherits from \( \mu_{\eta,\text{gns}} \) or \( \mu_{\varepsilon,\text{ged}} \); here \( \langle \sigma_0 \rangle + \) is the average of \( \sigma_0 \) with respect to the distribution \( \mu_{\eta,\text{gns}} \) or \( \mu_{\varepsilon,\text{ged}} \).

Note that \( \sigma_0 \) should not be confused with \( \sigma \), (1.10): thinking of the attractor as a smooth surface \( \sigma_0 \) is the contraction rate of its surface elements, which is different from the contraction rate \( \sigma \) of the phase space volume elements, see §4%§6.

If the conjectures of §1 are accepted \( \langle \sigma_0 \rangle + \) is also the \( \mu_{\nu,\text{ns}} \) or \( \mu_{\chi,\text{ed}} \) average of \( \sigma_0 \) or at least tends to it as \( R \to \infty \).

If \( \langle \sigma_0 \rangle + > 0 \), see [25] for a discussion of the conditions for this inequality ("Ruelle’s H-theorem"), and if \( \zeta(p) = \lim_{\tau \to \infty} \frac{1}{\tau} \log \pi_\tau(p) \) then the fluctuation theorem of [10] gives the following large deviation relation, see also [22], [26], for the equations GNS and GED:

\[
\frac{\zeta(p) - \zeta(-p)}{\langle \sigma_0 \rangle + p} = 1, \quad \text{for all } p
\]

which, in the case of nonequilibrium statistical mechanics, has been interpreted as an extension of the fluctuation dissipation theorem to large forcing fields, [24]. Here "for all" \( p \) means for all possible values of \( p \) (which is in general a bounded quantity).

The fluctuation theorem (2.2) says that the distribution of \( p \) is multifractal, not surprisingly since \( \zeta(p) \) can be regarded as a kind of generalized sum of Lyapunov exponents in the sense of [9], [8], and the odd part of \( \zeta(p) \) is linear.

A more general fluctuation theorem concerns the joint distribution of the variable \( p \) and of any other variable \( q = q(\gamma) \) that is similarly defined in terms of an observable \( Q \) which is odd under the time reversal operation that is defined on the attractor, i.e.:

\[
\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt Q(V^t \gamma) = \langle Q \rangle + q
\]
If $\pi(\tau(p,q))$ denotes the joint probability density of the observables $p,q$ and if $\zeta(p,q) = \lim_{\tau \to \infty} \frac{1}{\tau} \log \pi(\tau(p,q))$ then it follows from the chaotic hypothesis that the distributions for $p,q$, with respect to the statistics $\mu_\xi, \xi = (\beta, gns)$ or $\xi = (\alpha, ged)$, verify:

$$\zeta(p,q) - \zeta(-p,-q) \langle \sigma_0 \rangle_{+p} = 1, \quad \text{for all } p,q \quad (2.4)$$

which, in the case of nonequilibrium statistical mechanics, has been interpreted as an *extension of the Onsager’s reciprocity* to large forcing fields, [24].

In the case of the equations (1.1), second and fourth, the above relations are applicable when the motions are chaotic, *i.e.* have at least one positive Lyapunov exponent (which should happen as soon as $R$ is large enough, excepted possibly very special cases, see [27]); and it gives an interesting parameterless prediction *if* the contraction rate $\sigma_0$ can be related to the contraction rate $\sigma$ of the GNS equations.

If the conjecture in §1 held the sense of complete asymptotic equivalence between the “ensembles” $\mu_{\eta, gns}$ and $\mu_{\nu, ns}$ (or $\mu_{\varepsilon, ged}$ and $\mu_{\chi, ed}$) then (2.2) could also hold for other models of the viscous stationary states, like the one given by the classical NS equation in particular.

It turns out, from very recent numerical and theoretical results of F. Bonetto (private communication) *in related but purely mechanical problems*, that if $\nu$ is fixed (in the NS equations) the fluctuations of $\sigma$ might be strongly affected and very different from the ones of the same $\sigma$ in the case in which (in the GNS equations) $\eta$ is fixed at the right value (as demanded by the conjecture).

Hence one has to be very cautious about extending the "equivalence" to such relation *without* further arguments to exclude that fixing $\eta$ or $\nu$ and looking at the fluctuations of $\sigma$ could be analogous to fixing the density or the chemical potential (*i.e.* considering the canonical or the grand canonical ensembles) and looking at the fluctuations of the energy (which, in the thermodynamic limit, are the same in the two ensembles).

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Hence one has to be very cautious about extending the "equivalence" to such relation *without* further arguments to exclude that fixing $\eta$ or $\nu$ and looking at the fluctuations of $\sigma$ may be more like fixing chemical potential and density (in the grand canonical and in the canonical ensembles) and looking at the density fluctuations in a box almost as large as the available volume.

In the latter case the validity of (2.2) and (2.4) could only become a way of distinguishing the two equivalent distributions $\mu_{\eta, gns}, \mu_{\nu, ns}$. Just in the same way as in equilibrium the energy fluctuations distinguish the microcanonical ensemble from the equivalent canonical ensemble (or density fluctuations distinguish the canonical and the corresponding grand canonical ensembles). The lack of equivalence only affects the fluctuations of "global" quantities like $p,q$: but this makes necessary to be more precise and to specify the class of observables for which the equivalence can be conjectured as done above in the remarks following the conjectures.

In real systems a test would require experimental ability of measuring total vorticity (or total energy) fluctuations. This is apparently *very difficult* and in any event one cannot hope to measure the total vorticity (or energy) but only the amount of vorticity (or energy) contained between the macroscopic scale of momentum $k_0$ and a certain scale $k_1$ depending on the instruments resolution power. Thus, even if one accepts GNS as a model for the motion, the fluctuation theorem relations would not apply directly to the observed data. One needs to know that the local (in momentum space) vorticity fluctuation, *i.e.* the vorticity fluctuations of the amount of vorticity below scale $k_1$, also obey the fluctuation theorem (2.2).

But this is not a consequence of the theory although one can imagine further assumptions that would imply such a "local" version of the theorem (which would apply equally well to the NS equations by the equivalence conjecture, because the statement would now involve only quantities local in the above sense). Therefore rather than entering into more conjectural territory I prefer to look at the above discussion as a suggestion of the interest of tests of the
mentioned local version of the fluctuation theorem relations, both in numerical experiments and in real fluid experiments.

A check of (2.2) or (2.4) and of the equivalence conjectures might be more accessible in the case of fluid systems (numerical or real, compared to the corresponding conjectures for interacting particles systems) because they should show chaotic motions with relatively few degrees of freedom so that the large fluctuations, that must occur in order to make possible direct testing of the fluctuation theorems, are more likely to occur and be observable.

However, the latter remark also means that the attractors have a very small size, compared to that of phase space: hence a problem in the interpretation of (2.2) is the fact that the quantity $\sigma_0$ that appears in the theorem does coincide with the easily determined (see (1.10)) contraction of volume in the phase space only if the attractor is dense in phase space.

This is a property that can be expected in the case of nonequilibrium statistical mechanics under small or moderately strong forcing, see [4] for a discussion of this point, but it cannot be expected at large forcing or in fluid systems. In the latter case one should use the contraction rate of the volume elements on the attractor, see [4], [28]. In the GNS systems it is likely that the property never really holds.

This might render (2.2) quite useless as it is usually unrealistic to hope to determine the attractor equation with accuracy sufficient to compute its area contraction per unit time (assuming, as the chaotic hypothesis implies, that the attractor can be regarded as a smooth surface).

Nevertheless in nonequilibrium statistical mechanics further analysis is possible, based on an important symmetry of the Lyapunov exponents spectrum, and one can relate easily the area contraction on the surface defined by the attractor and the total phase space volume contraction. Hence one can try to push further the analysis of [4], [28] to see if one can say something also in the case of the GNS equations.

§3 A scenario for experimental checks of the fluctuation theorem. The Lyapunov spectrum of ED and GED equations. Pairing rule and Axiom C.

What follows is a very heuristic analysis aimed at giving an argument for the explicit form that the fluctuation theorem (2.2) will take in the case of the GED equations (and perhaps by the above discussion on the equivalence conjecture also for the ED equations, see also the last comment in §7). No pretention of mathematical rigor is present and the idea is to illustrate the simplest possible scenario that I consider possible and that is compatible with the small (but not empty and quite constraining) set of exact results established elsewhere or below. The interest is (apart from the subjective feeling of a certain beauty) that the discussion suggests experiments and checks that have intrinsic interest and that do not seem to have yet been considered in the literature.

We consider first the case of (1.1) in a domain $\Omega$ with obstacles: in spite of the appearances this is an easier case because in this case we can imagine forcing the system with a locally conservative force which is not globally conservative, like a field roughly parallel to one axis and tangent to the obstacles (one can imagine a uniformly charged fluid under an electromotive constant electric field).

Note that in order to have a non trivial forcing the forcing field must be non globally conservative: otherwise its effect would be just that of altering the pressure.

The Euler equations can, in general, be regarded as hamiltonian equations for a system whose configurations are the diffeomorphisms of the box $\Omega$ (in our case a torus with, possibly, a few holes) containing the fluid: they are not directly in hamiltonian form in the same sense as the (closely analogous) Euler equations for a rigid body with a fixed point are not immediately hamiltonian (e.g. they involve half the number of actual equations of motion).

In this way the GNS or GED equations can be regarded as hamiltonian equations (approximately so, because the cut–off $K$ destroys this property) modified by the action of a non
conservative force $g$ and by the gaussian constraint that the total vorticity or the total energy are constants.

Of course we exploit the "slip" (i.e. no friction) boundary conditions in order to be able to conclude the hamiltonian nature of the Euler equations. The phase space will then consist of a space larger than the above $F$, see (1.9): its points $(\mathbf{u}, \mathbf{\delta})$ will be (cut–off) velocity fields and (cut–off) displacement fields describing the positions of the fluid particles with respect to a reference configuration. We call this the "full phase space" of the equations (1.1).

The equations for the displacements will be in all models (1.1):

$$\dot{\mathbf{\delta}} = \mathbf{u}(\mathbf{\delta}, t), \quad \mathbf{\delta}(\mathbf{x}, 0) = \mathbf{\delta}_0(\mathbf{x})$$

(3.1)

which, once $\mathbf{u}$ is known from (1.3), (1.2), permit us to compute the physical fluid flow and the positions $\mathbf{\delta}(\mathbf{x}, t)$ of the fluid particles that at time 0 were at the points $\mathbf{\delta}_0(\mathbf{x})$, away from the reference configuration position $\mathbf{x} \in \Omega$.

In the case in which the $\mathbf{u}$ verify truncated equations also (3.1) have to be truncated, for instance by replacing each $e^{i\mathbf{k} \cdot \mathbf{\delta}(\mathbf{x})}$ in $\mathbf{u}(\mathbf{\delta}, t)$ by its truncated Fourier expansion.

The system motions (describing velocity and displacement fields) can be regarded as motions with $2D_K$ degrees of freedom where, for instance in the no obstacles case, $D_K$ is the number of non zero modes $\mathbf{k}$ with $0 < |\mathbf{k}| < K$ (because each $\gamma_{-\mathbf{k}}$ has two complex components but $\gamma_{-\mathbf{k}} = \overline{\gamma_{\mathbf{k}}}$). This means that $4D_K$ coordinates are necessary to describe the motion.

Hence there are $4D_K$ Lyapunov exponents, $2D_K$ from the velocity equations (1.2) and $2D_K$ from the displacements equations (3.1).

In view of the equivalence conjectures we study the equations GNS and GED when convenient and the NS or ED when convenient.

Out of the $4D_K$ exponents one has to extract, in the GNS or GED cases, one exponent that is trivially 0 because of the conservation of the dissipation rate and one exponent that is trivially zero and corresponds to the vector field given by the r.h.s. of the GNS equation. Furthermore in the GNS or GED cases two more vanishing Lyapunov exponents are associated with other constants of motion.3

The other $2N = 4D_K - 4$ exponents, or in the ED, NS cases all the $2N = 4D_K$ exponents, can be ordered in two groups the first containing the first $N$ exponents in decreasing order and the second the remaining $N$ ones in increasing order.

The exponents of the first group are denoted $\lambda^+_j$, $j = 1, \ldots, N$ and the ones in the second group are denoted $\lambda^-_j$, $j = 1, \ldots, N$. We call the two exponents $(\lambda^+_j, \lambda^-_j)$ a pair.

We consider first in detail the ED and GED equations. In the above context it seems reasonable that in the full phase space of the GED and ED equations a pairing rule holds:

$$\frac{\lambda^+_j + \lambda^-_j}{2} = \text{const}$$

(3.2)

at least when the forcing is locally conservative as we suppose from now on unless otherwise stated. The value of the constant will be called the "pairing level" or "pairing constant", which must be $\frac{1}{2}(\sigma)_+$, see (1.10).

The pairing rule, in fact, formally holds in the present ED case. One can easily adapt the proof in [5]: this is discussed in the Appendix A1.

The rule then holds also for GED as a consequence of the conjecture ED. A direct proof can be made along the lines of the work [6]. In fact the constraint imposed by the definition of the multiplier $\alpha$, (1.7), is a constraint of the type called isokinetic in [6] and their proof seems

3 In the case of the GNS equations the helicity $\int \mathbf{\omega} \cdot \mathbf{u} \, dx$ is a constant of the motion and such is $\int \mathbf{\delta}^2 \, dx$ for the displacement equations.
to apply "without change", although I did not check the details (the appendix A1 should give the background for such an analysis).

In the cases in which (3.2) has been proved, [6], [5], it holds also in a far stronger sense: the local Lyapunov exponents, of which the Lyapunov exponents are the averages, are paired as in (3.2) to a constant that is $j$ independent but, of course, is dependent on the point in phase space. We call this the strong pairing rule. See the final comments.

Note that the Lyapunov exponents of the full system can also be easily divided into velocity exponents, i.e. the ones of the GED or ED equations, and the displacement exponents, i.e. the others (which cannot be measured from the GED or ED evolution alone but require also (3.1)). In fact if we denote symbolically by $(x, y)$ the pair $(u, \delta)$ then the jacobian matrix of the equations is described by a matrix having the form $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ where $A, B, C$ are operators.

For a further classification of the exponents we shall think that the Lyapunov exponents are divided into three classes that we call viscous, inertial and slow. The following scenario will be again summarized and enriched in the figure in §4.

1. The slow exponents ("slow pairs") consist of $M$ pairs of exponents the largest of which is $\leq 0$ and it is a velocity exponent corresponding to slow motions of the velocity field, while the other (necessarily $< 0$) exponent of the pair is a displacement exponent and corresponds to a fast approach to the stationary state of some of the displacement variables.

2. The viscous exponents ("viscous pairs") consist of $V$ negative velocity exponents describing the fast approach to the stationary state of the viscous degrees of freedom of the velocity field: their paired positive exponents are displacement exponents associated with chaotic motions of the displacement variables.

3. The remaining $2P = N - M - V$ pairs ("inertial pairs") have one $> 0$ and one $< 0$ Lyapunov exponents: $P$ of the pairs are pairs of velocity exponents and the $P$ other pairs are displacement exponents. The $P$ pairs of velocity exponents are the only pairs of exponents of the equations for the velocity field that contain one positive and one negative element: they describe the gross characteristics of the chaotic motion on the attractor. It follows that the three types of exponents can in principle be uniquely identified among the $N$ exponents of the velocity field equations, see also below.

The existence of a certain number denoted $P$ above of pairs of exponents, for the velocity field evolution, that are pairs of exponents of opposite sign does not follow simply from the fact that we are collecting together pairs containing a $> 0$ exponent. In principle the $> 0$ exponents of the velocity field could be paired with negative displacement exponents. We think that it is natural that the $> 0$ Lyapunov exponents for the velocity field are paired with $< 0$ exponents of the velocity field because we associate such pairs with the motions on the attractor. Since the GED equations are reversible it follows from [28] that if the motions are also supposed to verify a geometric property called in [28] Axiom C property (a simple extension of the paradigm of turbulent behavior, see [1], that is the Axiom A property) then there must be an equal number $P$ of positive and negative exponents for the restriction of the GED equations to their attractor. It seems therefore natural to think that they form $P$ pairs.

The equality of the number of $> 0$ and $< 0$ exponents for the motion on the attractor for the velocity fields is due to the existence, in reversible Axiom C systems, of a local time reversal map $i^*$ that transforms the attractor into itself anti-commuting with the time evolution, even when (and this is the rule in fluid dynamics) the attractor itself is not time reversal invariant: see [28]. We proceed under the assumption that the Axiom C property is verified: for a complete discussion of the property we must unfortunately refer to [28].

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4 i.e. the logarithms of the eigenvalues of $\sqrt{J_t^T J_t}$, if $J_t$ is the local jacobian matrix of the evolution operator $V_t$, other than those relative to the directions of the flow or of the imposed constraints or of other constants of motion.
In **Axiom C systems** the time reversal symmetry "cannot be lost": when it is spontaneously broken (because the attractor is not time reversal invariant) it is replaced by a "weaker" symmetry, good enough to make "effectively reversible" all the motions on the attractor, a relation similar to the one in fundamental Physics between \( T \) and \( T_{CP} \) (the latter being the "real" time reversal as the first is not a symmetry of the world we see).

(4) The other \( P \) pairs should consist of displacement exponents exhibiting a rather symmetric behavior with respect to that of the GED exponents. Below we are going to suggest that very similar properties hold for the NS and GNS equations: in that case this further appealing symmetry seems compatible with (and in fact it was suggested by) the data on the velocity Lyapunov spectrum for models ("GOY shell models") whose behavior is "believed" to be related to NS equations: see [8], figure in p. 71, taking into account that the pairing level in such data is very small because the viscosity is very small.

In the above scenario the existence of the other \( P \) pairs of displacement exponents is assumed in order to make the total count of the number of exponents correct and is not based on evidence of any other kind. The displacement exponents have been considered in the literature, [29], but no pairing rule seems to have been proposed or to have emerged yet (not surprisingly in view of the difficulty of the measurements).

Thus if we measure the Lyapunov exponents for the GED equations alone we expect to find \( P \) pairs of opposite sign exponents paired at the value \( \frac{1}{2N} \langle \sigma \rangle_+ \) for some \( P \leq N \).

It seems reasonable that the \( P \) pairs of displacement exponents coincide with the \( P \) pairs of inertial exponents for the velocity field equations: but this is not really necessary in order that an unambiguous identification of the three type of exponents be possible. They are already identified by the above properties.

However if the \( P \) pairs of velocity inertial exponents and the \( P \) pairs of displacement inertial exponents do coincide we see that, by the pairing rule, the knowledge of the Lyapunov spectrum for the velocity equations implies that all the displacement exponents are known as well: no need to compute them.

§4 **Fluctuation theorem predictions for GED and pairing rule for GED and ED.**

With the scenario developed in §3 we reconsider the fluctuation theorem and note that it is easy to check, by evaluating the divergence of the r.h.s. of equations (1.2), (3.1), that the volume in the full (2N dimensional) phase space contracts at the same rate \( \sigma \) at which the volume in velocity space does. Fluid incompressibility, and absence of displacement variables in the equations for the velocity field, imply this property.

Furthermore if the **strong pairing rule** is assumed the total volume contraction in the full phase space, including the displacement variables, will be \( \sigma(\gamma) \) and it will be related to the contraction \( \sigma_0(\gamma) \) of the area on the attractor surface by \( \sigma_0(\gamma) = \frac{2P}{2N} \sigma(\gamma) \), see [4] where the same mechanism was first exploited. This gives proportionality between the "apparent" contraction rate \( \sigma \) and the "true" contraction rate \( \sigma_0 \) on the attractor for the GNS equations.

As discussed at the end of §2 the **fluctuation theorem holds for the fluctuations of \( \sigma_0 \) so that the fluctuations of \( \sigma \) will verify (2.2) but with a r.h.s in which 1 is replaced by \( \frac{P}{N} \) where \( P \) is the number of pairs of Lyapunov exponents for the GED equations with one positive element.

If the number of degrees of freedom is increased by increasing \( K \) one should expect, therefore, that the constant \( \frac{P}{N} \langle \sigma \rangle_+ \) approaches \( P \langle \alpha \rangle_+ \), because the number of "true exponents" (i.e. inertial exponents) should not change as soon as \( K \) is so large that the motion is well described by the truncated equations: in fact, if there is ultraviolet stability, the attractor dimension should not depend on the truncation scale \( K \) (as long as it is large enough).
Since the conjecture of §2 implies $\langle \alpha \rangle_+ = \chi$, the constant should approach $P \chi$, at least if $R$ (i.e. the Reynolds number) is large. The fluctuation theorem will thus take the form:

$$\frac{\zeta(p) - \zeta(-p)}{p} = P \chi$$

if the variable $p$ is defined as in (2.1) but with the measurable $\sigma$ replacing the a priori difficult to measure $\sigma_0$, and $\zeta(p)$ is the limit of $\tau^{-1} \log \pi_\tau(p)$ with $\pi_\tau(p)$ being the $\mu_{\text{GED}}$ distribution of $p$. The number $P$ is accessible by measurements performed only on the GED equations and not involving the displacement variables (being the number of positive Lyapunov exponents of the GED equations).

The above analysis is somewhat conjectural but experiments, at least numerical ones, are possible to check the picture; e.g. one could attempt at:

1. checking the just derived slope $P \chi$ or
2. checking the following picture, representing the above classification of the exponents:

![Fig1](image)

The continuous line in the first graph gives the value ($\leq 0$) of the $j$–th (among $M$) slow Lyapunov exponent (as a function of $\frac{j}{M}$) of the GED equations; the dashed line is the graph of the paired exponents (of the displacement equations) and the intermediate line is the pairing constant. The exponents are defined only for $x = \frac{j}{M}$ but the graphs give, instead, continuous (or dashed) lines for visual aid.

The second graph gives the values of the $j$–th (among $P$) pair of inertial exponents of the GED equation (one positive and one negative per pair): here too we use the continuous curves even though the number of such exponents will usually be much smaller than the total number and therefore a discrete representation would be more appropriate.

The negative curve in the third graph is the graph of the $j$–th viscous exponent (out of $V$) of the GED equation; the corresponding positive curve (dashed) is the curve of the companion exponents which correspond to displacement exponents. A fourth graph giving the other $P$ displacement exponents (in pairs of one negative and one positive) would be qualitatively equal to the second graph (with the curves dashed for consistence of notation).

The graphs are not experimental data: they are just sketches illustrating the ”simplest” picture that I considered reasonably possible. They should be taken as a conjecture, and they suggest performing experimental evaluation of the exponents for a check of the ideas of the present paper: note that the pairing statement is on a much firmer footing than the others as it admits a formal proof, described in the appendix for the ED case.

What do we imagine to happen when the equations are changed by enlarging the cut off (in the velocity as well as in the displacement variables) or by changing the forcing? Suppose that
the cut off is already so large that adding one extra mode does not really affect the qualitative and quantitative features of the motion. Then adding one mode, \textit{i.e.} increasing the total dimension of the system by 2, should add one pair of viscous exponents at the end of the spectrum respectively equal to 0 and $-\chi$, as drawn in the third graph in the figure. While changing the forcing should, \textit{from time to time} as the forcing changes, change the category of some exponents. Namely the simplest picture would be that one of the vanishing slow GED exponents "becomes" positive and one of the viscous "becomes" inertial and paired with it; a symmetric evolution should take place with the displacement exponents. Or vice-versa. The attractor changes dimension by 2 units, see [28],[4], at each of such events.

In order that the latter picture be possible one needs that \textit{at a transition} the viscous spectrum bottom consists of a pair of a $>0$ displacement exponent and a negative viscous exponent coinciding with the inertial exponents top pair; and that the same should happen for the bottom pair of the inertial and the top pair of the slow spectra.

The case of periodic boundary conditions does not fit in the above analysis of the pairing rule because on the torus there is no way of forcing the system with a locally conservative but globally non conservative force field with 0 average. Nevertheless some kind of pairing might still occur under simple non conservative forcing acting only on some large scale modes, see §5.

It has been pointed out to me by F. Bonetto that consistency of the picture \textit{requires that the sum of the displacement exponents be exactly 0}: the two of us have indeed been able to verify that this is formally exactly verified in the ED equations. And this led to a correction of the graphs drawn in the figure above that I had originally drawn without taking such property into account. We shall come back on this point in a future study. Note also that the fact that the sum of the displacement exponents vanishes provides a natural test that the truncation that one is using is actually large enough for having reached cut-off independence of the asymptotic properties of the motions: this happens at the cut off value where the sum of the displacements exponents vanishes: further addition of modes only makes longer the flat part in the third graph of the figure above.

§5 \textit{The NS and GNS equations. Extension of the pairing rule.}

We turn to the NS and GNS equations, whose interest is far greater than the just studied ED or GED equations.

One is tempted to say that the scenario should be the same. However the pairing rule analysis, which is essential for the physical interpretation of the results, is no longer naively possible, not even at a heuristic level.

A pairing rule, first pointed out in special \textit{non constant} friction cases in [2], p. 281, has been proved only in the case of systems subject to special gaussian constraints, see [6], but it has apparently a much wider validity, see [2], [30], [7], [31] and it is likely to hold also in the cases GNS and NS, \textit{at least in some sense}.

But the argument in [5] implies the existence of pairing in systems that are obtained from hamiltonian systems by adding to them an irreversible constant friction term \textit{proportional to the momenta in a system of canonical coordinates}. And the argument in [6] is restricted to "isokinetic" constraints \textit{precisely} because they are reversible constraints that are obtained by adding to a hamiltonian system a suitable force proportional to the canonical momenta. Since this is an essential feature for the validity of (3.2) the latter becomes doubtful in the cases (that include NS and GNS equations) in which the friction or thermostat forces are proportional to the canonical momenta via a matrix $C$ which is not the identity (it is the laplacian in the case of the NS or GNS equations).5

5 I owe to F. Bonetto the clarification of this point.
In such cases one could envisage that (3.2) is replaced, in the GNS equations case, by a relation like:

\[(\lambda^+_j + \lambda^-_j)/2 = \langle \beta \rangle c_j \]  

(5.1)

where \( \langle \beta \rangle \) is the \( \mu_{\eta,gns} \) average (in the case of the NS equations one would write \( \nu \) instead of \( \langle \beta \rangle \)); and \( c_j \) is some suitable function of \( j \), that might be related to the spectrum of the matrix \( C \). However attempting at establishing such a connection would lead to too many too detailed assumptions at this stage and one would like not to rely on them. And from the proofs in [5], [6] it seems unlikely that a pairing rule can hold in a strong form, i.e. that (5.1) holds for the local exponents if \( \langle \beta \rangle \) is replaced by \( \beta \).

We therefore define \( c_j \) by the (5.1) without linking \( c_j \) to the matrix \( C \). However we shall suppose that (5.1) holds in a "almost local" form in the sense that on a rapid time scale (5.1) becomes true also for the local exponents. This means that, up to an error that tends to zero very quickly with the time \( \tau \), the logarithms of the eigenvalues of the matrix \((J^T D J^{-1})^{1/2}\), with \( J(x) \) being the jacobian matrix for the evolution operator \( V_x \) at \( x \), divided by \( \tau \) verify

\[\frac{1}{2}(\lambda^+_j + \lambda^-_j) = c_j \beta_j(x) \]

with \( \beta_j(x) \) denoting the average \( \frac{1}{2} \int_{-\tau/2}^{\tau/2} \beta(V_t x) dt \), still \( x \) dependent because \( \tau \) is fixed.

This property, together with the Axiom C assumption, suffices to extend, in a suitable form, the validity of the predictions (i.e. conjectures) discussed in the previous section for the ED and GED equations to the case of the NS and GNS equations as follows.

We remark that the really relevant feature of the pairing rule, as far as the applications in [4] and above are concerned, is not the constancy of the pairing but, rather, the fact that some kind of pairing takes place on a fast enough time scale. Secondly we assume that this is actually the case for the GNS equations. On this remark and on this assumption we base the analysis of the fluctuation theorem predictions for the GNS (and perhaps, as in the GED case, for the NS equations).

If a relation (5.1) holds the constants \( c_j \) will have to add up to the sum of the Lyapunov exponents. The latter can be derived as the average value of \( \sigma \): this means that \( \langle \sigma \rangle_+ = 2D_K \langle \beta \rangle_{\eta,gns} = (\sum_j c_j) \langle \beta \rangle_{\eta,gns} \), up to terms negligible as the Reynolds number tends to \( \infty \), see (1.10) and comments preceding it.

Furthermore let \( I \) be the set of \( P \) inertial pairs (i.e. of pairs of Lyapunov exponents \( \lambda^+_j, \lambda^-_j \) with one positive element) and suppose that the (5.1) becomes valid on a sufficiently fast time scale, then the values of \( \langle \sigma \rangle_+ \) and \( \langle \sigma_0 \rangle_+ \) would have ratio (see [4]) \((\sum_{j \in I} c_j)/(\sum_j c_j)\) so that:

\[\langle \sigma_0 \rangle = \frac{\sum_{j \in I} c_j}{\sum_j c_j} \langle \sigma \rangle_+ = (\sum_{j \in I} c_j) \langle \beta \rangle_{\eta,gns} = (\sum_{j \in I} c_j) \nu \]  

(5.2)

having used the conjecture NS of §1 equating \( \langle \beta \rangle_{\eta,gns} \) to \( \nu \).

Then if a local time reversal exists on the attractor (i.e. if the geometric Axiom C is assumed as well, [28], for the dynamics generated by the GNS equations) the fluctuations of the observable \( \sigma \) will have a "free energy" (or a "generalized sum of Lyapunov exponents" to adhere to the terminology in [9], [8]) \( \zeta(p) \), in the sense of (4.1), with an odd part \( p \mathcal{F} \nu \), with \( \mathcal{F} \) defined in (5.2). This is a property whose validity can be conceivably tested in moderately turbulent GNS systems. At least the linearity in \( p \) of \( \zeta(p) - \zeta(-p) \) should be observable. For the NS equations the same comments in §2 and at the end of §7 for the ED equations has to be made: the fluctuation theorem might not hold but a local version of it may hold (see §2 and §7).

Note also that, in all cases, the pairing rule is trivially valid in the case of no forcing: in fact the equivalence criterion in the conjecture in §2 requires that in absence of forcing one has to take \( \eta = 0 \) or \( \varepsilon = 0 \): i.e. the stationary state is, in that case, the trivial (non chaotic) flow \( \mathbf{u} = \mathbf{0}, \mathbf{\beta} = \text{const.} \).
The assumption that the forcing be locally conservative has not been used and disappears together with the constancy of the pairing: the above more general pairing hypothesis (see (5.1) and the comment following it) is more "flexible" and does not require the special hypothesis of local conservativity of the forcing.

§6 Relation between the NS and ED equations. The barometric formula.

Finally we discuss another main point of our analysis.

In reference [11] the argument leading to the conjecture NS above can be interpreted as saying that NS and ED are also in some sense equivalent.

The argument is based on the constancy of the dissipation rate in a stationary flow at high Reynolds number and on the microscopic reversibility. In some sense the GNS equations emerge as even more natural than the NS equations.

A criticism can be raised, however. In fact one can argue that the energy is also constant in a stationary state and one could develop the argument in [11] to imply that the GED equations are also a good model for a fluid motion.

Since clearly one should not expect NS and ED to be equivalent this looks at first as an unsolvable logical contradiction. Which can furthermore be conceivably easily checked to occur.

However on further thought the contradiction can be resolved and one should think that all what has been deduced is that there should be a relation between the stationary states of ED (or GED equivalently) and of NS (or GNS). The relation to which I think is the kind of relation that one also finds in equilibrium statistical mechanics in gases in a strongly varying external field of intensity $g$, like the gravity field.

Locally a gas in a field looks just like a homogeneous gas in equilibrium, but globally over a length scale $H$ over which the external potential really changes ($\beta m g H \sim 1$, if $\beta$ is the inverse temperature and $m$ the particles mass) one will see that pressure and the density are not constants and one gets the barometric formula, see [32].

Likewise we can expect that the stationary states of ED (or equivalently of GED) are also "locally" equivalent to stationary states for NS (or GNS): in the sense that if we only look at observables depending on field components $\frac{\partial}{\partial x_k}$ with modes $k$ on a certain scale $|k| \sim \kappa$ whose size depends on the dissipation then we should see essentially no difference. The precise relation that determines $\kappa$ will be called barometric formula: it should be easy to determine the formula on the basis of dimensional considerations. Locality is here to be interpreted in momentum space rather than in coordinate space.

The determination of the barometric formula amounts essentially at a development of the analogous of the Kolmogorov theory for the ED equations.

We now attempt at a partial development of such theory, in the no obstacles case for simplicity, on the basis of a few assumptions that deserve further attention and perhaps criticism. We follow closely the ideas (and imitate the assumptions) of the exposition of Kolmogorov’s theory in [14]. We set $\rho = 1$.

It seems reasonable to suppose that in the ED case the stationary distribution equipartitions the energy among the modes, i.e. $\langle |\gamma_{\frac{1}{2}} k| \rangle^2 = \gamma^2$ for all $k$ in the “inertial range” $L^{-1} \ll |k| \ll k_\chi$ where $k_\chi$ is the "Kolmogorov scale", to be determined below. Hence $\gamma^2 (k_\chi L)^3 = \varepsilon$.

Then a velocity variation characteristic of the momentum scale $\kappa$ is given by the expression $v_\kappa^2 = \langle (\sum_{k \in [\kappa/2, \kappa]} \gamma_{\frac{1}{2}} k)^2 \rangle$ and, assuming statistical independence of the distribution of the various $\gamma_{\frac{1}{2}} k$, we get $v_\kappa^2 = (k_\chi L)^3 \gamma^2$ up to a constant factor.

The cut-off scale $k_\chi$ has to be, on dimensional basis, a momentum scale formed with the quantities $k_0$, $\sqrt{\chi^2/\varepsilon}$ and $\chi/\sqrt{gL}$ or, in case of ultraviolet instability of the equations solutions, it might even depend on the cut off $K$ necessary to make the equations well defined. Hence it cannot be determined without a more detailed theory of the equations.
For purposes of comparison we note that the quantity called \( \varepsilon \) in the Kolmogorov’s theory ("K41-theory"), see [14], corresponds to \( \eta \nu \) of the present paper.

In this case the energy distribution (i.e. the amount \( K(k)dk \) of energy per unit volume and between \( k \) and \( k + dk \) is \( K(k) = \frac{3\varepsilon}{4\pi} \frac{k^2}{k'}, \) for \( k < k' \): very different from the Kolmogorov’s \( k^{-5/3} \) law.

In the K-41 theory a key role is played by the quantity \( v_{3}\kappa \) which is identical to \( \eta \nu \) for all \( k_0 \ll \kappa \ll k_\nu \). Therefore we compute the value of \( v_{3}\kappa \) in our case and we find:

\[
\frac{v_{3}\kappa}{\varepsilon \chi} = \frac{((kL)^3\gamma^2)^{3/2} \kappa}{\varepsilon \chi} = \frac{(k\kappa L)^3\gamma^2}{\varepsilon \chi} \frac{\kappa}{k'} \left( \frac{\kappa}{k'} \right)^{11/2} = \varepsilon^{3/2} k_0 \left( \frac{\kappa}{k'} \right)^{11/2} \tag{6.1}
\]

and we see that the quantity \( v_{3}\kappa \) does depend on \( \kappa \) in the ED case.

Given \( \kappa \) the SRB statistics for the ED equations driven with a total energy \( \varepsilon \) gives to this quantity the same value that it has in the SRB statistics for the NS equation driven with a total vorticity \( \eta \) if:

\[
\frac{\varepsilon \chi}{\eta \nu} = \text{const} \kappa^{-\frac{1}{4}} \tag{6.2}
\]

provided (of course) \( \kappa \) is greater than the Kolmogorov scales \( k_\nu, k'_\chi \); since the constant depends on \( k'_\chi \), the above discussion leads only to the determination of the exponent 11/2.

The “barometric formula” is then the statement of equivalence between NS and ED on scale \( \kappa \), i.e. if one only looks at field properties depending on \( \gamma_k \) for \( \frac{1}{2} \kappa < |k| < \kappa \), if (6.2) holds and \( \kappa \gg k_\nu, k'_\chi \).

If we look at a different scale \( \kappa' = 2^n \kappa \) for some (large) \( n \) then we can expect equivalence between ED (or GED) and NS (or GNS) but the pairs \( \varepsilon, \eta \) should now be such that (6.2) holds on the new scale: the analogy with the usual barometric formula for the Boltzmann Gibbs distribution in the gravity field justifies the name given to (6.2). We see that \( \eta \nu \) plays the role of the gravity, \( \varepsilon \chi \) plays the role of the chemical potential and \( \kappa/k'_\chi \) plays the role of the height.

The above analysis seems to be fully consistent with the numerical results in [12] who first proposed, in a different context, a picture very close to the one developed here.

It is clear that this point of view has several consequences: for instance in particular it tells us that that the shape \( j \rightarrow c_j \) of the pairing curve in (5.1) cannot be arbitrary (i.e. \( \langle \beta \rangle c_j \sim \nu \chi^2 \) if the modes are ordered in increasing order). This is a point on which I hope to return in a later analysis.

Also: the equivalence between NS and ED on a given momentum scale makes more interesting the ideas in [23] and a test of the Onsager reciprocity derived in the latter paper seems now quite feasible and seems also to have consequences for the real NS equations.

A further remark is that although (6.2) depends on the validity of the K-41 theory and of the corresponding theory for ED equations the barometric formula can be developed independently of such theories: hence any modification of the K-41 theory (and of the corresponding theory for ED) will lead to a barometric formula, with a relation between \( \varepsilon, \eta, \kappa \), possibly more complicated than (6.2).

It should be remarked that the above analysis is based on the conjectures NS and ED in §2 (i.e. it is independent on the chaotic hypothesis) and the observables involved are observables relative to a fixed shell, so that they are "local" in the sense discussed in §2, after conjecture ED. This is necessary as the conjectures (may) fail for (some) non local obserbables, see §2.

§7 Overview and concluding remarks.

1) The analysis is based on Ruelle’s proposal of considering the SRB distributions as the "physical distributions" describing stationary states of generic mechanical systems, i.e systems
with at least one positive Lyapunov exponent: technically this is formulated by assuming the validity of the "chaotic hypothesis" of §2, [10]. Even in equilibrium this hypothesis is stronger than the ergodic hypothesis: therefore there is no hope to prove it in "any" system. But one can analyze its consequences, just as in the case of the ergodic hypothesis.

The consequences should be something like non equilibrium thermodynamics and turbulence theory: they can be expected to be of general nature (as Boltzmann’s heat theorem, i.e. the exactness of \((dU + pdV)/T\)). Noting that the same system can be described equivalently by several different equations (e.g. a fluid can be described by microscopic equations in terms of its molecules or by equations for a macroscopic continuum) one may expect that the stationary states can be described by several probability distributions. In [11] the analogy between this obvious remark and the possibility of many equivalent ensembles in equilibrium statistical mechanics was pointed out (with reference to fluids but the same ideas apply to molecular models). Here I tried to give an example of how one can pursue the analogy to build the notion of "equivalent dynamical ensembles": the idea seems to be slowly emerging, independently, in various areas and the first attempt can be identified in [12].

2) The mechanism for the generation of equivalent ensembles (i.e. SRB distributions if one follows Ruelle) is illustrated by considering the two pairs of equations in (1.1). In [11] there is a heuristic argument of why one can consider the first two equations (NS and GNS) equivalent: conjecture NS above. The second conjecture is based on the same type of reasoning.

Basically I state that the time scale over which the variable coefficient \(\beta\) or \(\alpha\) in (1.1) reaches its average value (respectively equal to the viscosities \(\nu\) and \(\chi\)) should be much faster than the time scales for the hydrodynamic evolution of the NS or ED equations.

3) The equivalence conjecture leads to studying the stationary ensembles in the most convenient case, which may be NS (respectively ED) or GNS (respectively ED) depending on which property one studies. Just as in the case of statistical mechanics it is sometimes more convenient to study the canonical rather than the microcanonical ensemble (e.g. Boltzmann’s heat theorem is very conveniently studied in the canonical ensemble while entropy theory is more natural in the microcanonical ensemble).

The main application of the chaotic hypothesis is the "fluctuation theorem" ((2.4)) which is a rigorous consequence of the chaotic hypothesis (a mathematically minded treatment of the original proof given in [10] can be found in the third of [20] for the case of maps or in [26] for flows).

One does not know whether the GNS or GED equations verify the hypothesis: therefore it is interesting to test the consequence (2.4) of the fluctuation theorem. Its verification would provide stronger grounds for the hypothesis because it is an exact parameterless prediction (to use an ambitious comparison one can say that it is as an exact consequence of the chaotic hypothesis as Boltzmann’s heat theorem is of the ergodic hypothesis: both hypotheses are clearly violated in interesting case, but at least the heat theorem is always valid).

4) Checking the fluctuation theorem result for the GED or GNS equations meets a major obstacle: namely (by the equivalence conjecture) the GED or GNS systems are "far from equilibrium". This means that the stationary distributions will live on small attracting and basically unknown sets (i.e. with closure smaller than the whole phase space and possibly fractal). This difficulty was already met in experimental attempts at checking the chaotic hypothesis in the case of simple gas conduction models in [4].

In that case the difficulty was not really severe as the attractor dimension was almost maximal, but it stimulated a proposal for a general solution based on a very remarkable property of the Lyapunov exponents in certain dissipative systems. This property is rigorously established for systems which are locally hamiltonian (e.g. forced by an electromotive force) and subject to a viscous force proportional to the momenta, or alternatively subject to a constraint of constant kinetic energy ("isokinetic" systems) imposed via Gauss’ minimal constraint principle. The result is that the Lyapunov exponents can be arranged in pairs, even the local ones, with constant average [5], [6].
5) In this paper I have shown that the GED equations and the ED equations fall, respectively, under the assumptions of the mentioned papers except that the systems here have infinite dimension: this is achieved by considering the full lagrangian equations for the fluid motion (the theorem does not hold if one only looks at the velocities of the fluid, i.e. unless the phase space is enlarged to take into account the displacements of the fluid elements). The average of the pairs is equal to the viscosity $\chi$ in the ED case and, in the GED case, is equal to the average viscosity $\langle \alpha \rangle$ or (for the local exponents) to the viscosity $\alpha$, see (1.1).

The infinite dimensionality makes the result formal and one can think that it could be made rigorous if one could establish ultraviolet stability for the ED or GED equations (i.e. that the equations can be cut off in momentum space at a large enough, Reynolds number dependent, cut-off): but this is out of question in 3D, as one does not even know the corresponding property for the NS equation, [15]. In the 2D case (not discussed here) there is more hope: even the Euler equations in this case do not have ultraviolet problems (i.e. smooth data evolve into smooth solutions). But unfortunately the existence and regularity theorems for the Euler equations are not constructive (at least I do not know of any proof that does not use a compactness argument or a monotonicity argument) and I see no hope to tackle the question until a constructive theory of 2D Euler equation is available (I find it surprising that this seems hardly considered a problem and the 2D Euler existence and uniqueness theory is considered "completely understood").

6) Once pairing is (formally) established one can have recourse to the ideas in [4] which provide a simple relation between the phase space contraction in the full phase space and the one on the (unknown) attractor: a property that permits us to put the fluctuation theorem in a simple and usable form, (4.1), provided one can show that the motion on the attractor is reversible.

Of course even in the case of the GED equations that are reversible the motion on the attractor will not be reversible (because time reversal transforms the attractor into a repeller). The problem has been studied in [28]: where we investigated under which general geometric conditions one could establish that time reversal is an undestructible symmetry, i.e. the conditions under which even when time reversal symmetry $i$ is "spontaneously broken" (as one can interpret the formation of an attractor smaller than phase space) still one can define a new transformation $i^*$ acting on the attractor only and reversing the time (i.e. $i$ and $i^*$ are related in the same way as $T$ and $TCP$ in fundamental Physics). In [28] it is shown that a very natural and simple geometric property exists which has precisely the above feature of rendering time reversal undestructible.

The property is a global form of the Axiom A property: the latter is a property that seems to be rather widely accepted as closely related to chaotic systems, [33]. A related property is the Axiom B: which is a global version of the Axiom A (which should be regarded as a property only of the attractor and not of the whole phase space). The Axiom B, particularly if one interprets literally the original definition (by Smale, [34]), is not exactly what is needed (and it seems to be not even structurally stable, unlike the Axiom A property). In [28] we show that a "minor modification" of Axiom B, that we call Axiom C, makes (as a mathematical theorem), time reversal undestructible in the above sense; furthermore as conjectured in [28], and as checked recently, Axiom C systems are structurally stable.

Hence the fluctuation theorem applies to GED if: i) the chaotic hypothesis is strengthened (and simplified) by saying that "chaotic reversible systems are Axiom C systems for the purposes ..." (see §2), ii) the pairing rule, (3.2) proved here formally, is assumed valid and iii) the heuristic argument, given in [4] (and necessary to interpret the experimental results of that work), connecting the total phase space contraction to the one on the attractor is assumed. It takes the form of (4.1). Note that the three properties above are independent and, unless one...
Physica D, 105, 163–184, 1997.

dismisses formal proofs (the formality being the application of a finite dimensional argument to an infinite dimensional case), i) and ii) are based on mathematical theorems.

7) Independently on the applications of the chaotic hypothesis to GED, the pairing rule for the GED and ED equations establishes a very strong and remarkable connection between the Lyapunov exponents of the velocity field evolution and the ones of the lagrangian description of the same fluid (twice as many). In §4 we try to spell out this relation: the picture that emerges is very appealing, but the reader is warned that it is only the "simplest" possible consistent with the pairing rule. It is not easy to think of others but I am afraid that there might be others. This analysis is independent of the chaotic hypothesis.

8) To apply the above discussion to the GNS (reversible) equations a new difficulty arises: namely I feel that it is unreasonable to even think that the pairing rule holds for the NS or GNS equations (i.e. if one has a friction proportional to the laplacian of the field or a gaussian "isovorticity" constraint). But I propose that a more general pairing rule holds, (5.1), essentially on the basis that the time scale for the stabilization of the average of paired Lyapunov exponents is shorter than the other time scales involved. This is my suggestion for the extension of the pairing rule to non isokinetic systems: it is proposed as the "minimal" assumption necessary to deduce a testable form of the fluctuation theorem. It would be interesting to try to check its validity (or simply the validity of the fluctuation theorem for GNS).

9) Finally partly inspired by the ideas in [12] I try to establish a relation between NS and ED equations: this requires a formulation of the Kolmogorov theory for the ED equations. It is easy to see that such a theory risks to be controversial. Hence I refrain from formulating it in complete form and I only assume that there is an "inertial range" where the energy is equipartitioned among the modes. This Leads to a scaling law, that I have called barometric formula in §6, wich is relevant in the comparison between the NS or GNS equations and the ED and GED equations, as a consequence of the conjectures NS and ED. This is independent on the Chaotic hypothesis.

As a concluding remark I point out that while the fluctuation theorem cannot be applied to the NS equations but only to the GNS equations (because it involves "nonlocal" observables, so that the equivalence conjecture might be stretched too much) one can still think of applying it in a local version: this is in analogy with the inapplicability of the gaussian fluctuation theory to the energy distribution, in equilibrium statistical mechanics, in the microcanonical ensemble. The energy distribution is gaussian in the canonical ensemble ("central limit theorem") and a delta function in the microcanonical ensemble. Nevertheless the fluctuations of energy in both ensembles are gaussian (and equal) if one looks at the energy in a local region (i.e. small compared to the total volume). I think that a similar picture holds for the fluctuation theorem but this requires a longer discussion on which I hope to come back in the future, because it would open the possibility of checking the fluctuation theorem in many more experiments.

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6 For the sake of clarity about what I have in mind by a possible "local version" of the fluctuation relations for NS or GNS equations I give an example. Let $k_0$ be the momentum scale of the container and $k_1$ be a fixed higher momentum scale ($\nu$–independent). Then one can define the entropy production in the given scale range to be $\sigma_{k_1} = 2\pi k_1 \beta_{k_1}$, where $\pi k_1$ is twice the number of modes below $k_1$ (see (1.10)) and $\beta_{k_1}$ is defined by (1.2) with the fields $u$, $\omega$ truncated at momentum $k_1$. The distribution $\pi(p)$ of the quantity $p$ (defined as in (2.1) but with $\sigma_{k_1}$ replacing $\sigma$) generates a function $\zeta(p)$. A "local fluctuation relation" could then be (4.1) with $P_\chi$ replaced by $P_\nu$, see (5.2).
Appendix A1: The Hamiltonian formalism for Euler equations and Dressler’s theorem for ED.

To check the applicability of the results on pairing of [5] to the ED equations we must check that the equations can be written, in canonical coordinates for some Hamiltonian function $H$, in the form:

\[
\dot{q} = \frac{\partial}{\partial p} H \\
\dot{p} = -\frac{\partial}{\partial q} H + F - \chi p
\]  

where $F$ is such that $\frac{\partial}{\partial q} F_i = \partial_i F_j$ without being necessarily $F = -\frac{\partial}{\partial q} V$ for some globally defined $V$ (the latter would be a trivial case). The labels for the components of $q$ are $x, i$ with $x \in \Omega$ and $i = 1, 2, 3$. The partial derivatives are, correspondingly, functional derivatives; we shall ignore this because a “formally proper” analysis is easy and leads to the same results. By “formal” we do not mean rigorous, but only rigorous if the functions we consider have suitably strong smoothness properties: a fully rigorous treatment is of course impossible for want of reasonable existence, uniqueness and regularity theorems for the Euler equations or the Navier-Stokes equations in 3 dimensions.

Consider first the Euler equations. They can be derived from the Lagrangian:

\[
\mathcal{L}_0(\dot{\delta}, \delta) = \frac{\rho}{2} \int \dot{\delta}^2 d\varphi
\]

(\(\rho = \text{density}\) defined on the space $\mathcal{D}$ of the diffeomorphisms $\varphi \to \delta(\varphi)$ of the box $\Omega$, by imposing the ideal constraint:

\[
\det J = \det \frac{\partial \delta}{\partial \varphi}(\varphi) = \frac{\partial}{\partial \varphi} \delta_1 \wedge \frac{\partial}{\partial \varphi} \delta_2 \cdot \frac{\partial}{\partial \varphi} \delta_3 = 1
\]

In fact, if $Q(\varphi)$ is a Lagrange multiplier, the stationarity condition for:

\[
\mathcal{L}(\dot{\delta}, \delta) = \frac{\rho}{2} \int \dot{\delta}^2 d\varphi + \int Q(\varphi(\delta)) \det J(\delta(\varphi)) d\varphi
\]

leads to, using $J(\delta(\varphi)) d\varphi = d\delta$:

\[
\rho \delta = -\frac{\partial}{\partial \varphi} Q
\]

so that setting $u(\delta(\varphi)) = \dot{\delta}(\varphi)$, $p(\delta) = Q(\varphi(\delta))$ if $\delta = \delta(\varphi)$, we see that:

\[
\frac{d u}{dt} = -\frac{1}{\rho} \frac{\partial}{\partial \varphi} p
\]

which are the Euler equations. And the multiplier $Q(\varphi)$ can be computed as:

\[
Q(\varphi(\delta)) = p(\delta) = -[\Delta^{-1}(\frac{\partial}{\partial \varphi} \cdot \frac{\partial}{\partial \varphi} u)] \delta
\]

where the functions in square brackets are regarded as functions of the variable $\delta$ and the differential operators also differentiate over such variable. After the computation the variable $\delta$ has to be set equal to $\delta(\varphi)$.

Therefore by using the Lagrangian:
we generate Lagrangian equations for which the "surface" $\Sigma$ of the incompressible diffeomorphisms in the space $D$ is invariant: these are the diffeomorphisms $x \rightarrow \delta(x)$ such that $J(\delta) = \partial_1 \delta_1 \wedge \partial_2 \delta_2 \cdot \partial_3 \delta_3 = 1$ at every point $x \in \Omega$.

Then $\Sigma$ is invariant in the sense that the solution to the Lagrangian equations with initial data "on $\Sigma$", i.e. such that $\delta \in \Sigma$ and $\partial \cdot \dot{\delta} = 0$, evolve remaining "on $\Sigma$".

The Hamiltonian for the Lagrangian (A1.8) is obtained by computing the canonical momentum $\pi(x)$ and the Hamiltonian as:

$$H(\pi, q) = \frac{1}{2}(G(q)\pi, \pi)$$

where $G(q)$ is a suitable quadratic form that can be read directly from (A1.8) (but it has a somewhat involved expression of no interest for us), and the ... (that can also be read from (A1.8)) are terms that vanish if $\delta \in \Sigma$ and $\partial \cdot \dot{\delta} = 0$, i.e. they vanish on the incompressible motions.

The above is well known and shows that the Euler flow can be interpreted as a geodesic flow on the surface $\Sigma$ of the incompressible diffeomorphisms of the box $\Omega$ enclosing the fluid, see appendix 2 in [35].

Modifying the Euler equations by the addition of a force $f(x)$ such that locally $f(x) = -\partial \Phi(x)$ means modifying the equations into:

$$\frac{du}{dt} = -\partial p - \partial_x \Phi$$

which can be derived from a lagrangian:

$$\mathcal{L}_i(\dot{\delta}, \delta) = \int \left( \frac{\rho \dot{\delta}(x)^2}{2} - \left[ \Delta^{-1}(\partial_u \cdot \partial \omega) \right] \dot{\delta}(x) (\det J(\delta)|_\Sigma - 1) \right) dx$$

Hence the ED equations have the form:

$$\dot{q} = \partial_p H$$
$$\dot{p} = -\partial_q H - E - \chi p$$

at least as far as the motions which have an incompressible initial datum are concerned. This is true because the ED equations which have an incompressible initial datum evolve it by keeping it incompressible.

The Lyapunov exponents of the equation (A1.13) verify the pairing rule by the analysis in [5]. However the pairing takes place in the full phase space of the diffeomorphisms of $\Omega$, including the incompressible ones.

It is not difficult to see, by using that the constraint to stay on the surface $\Sigma$ is holonomic, that one can find canonical coordinates $\pi, \kappa, \pi^\perp, \kappa^\perp$ describing the motions on $\Sigma$ or, respectively, transversally to it. And the equations for $\pi^\perp, \kappa^\perp$ are, near $\Sigma$ and for $\pi^\perp$ small,
\dot{\pi}^\perp = -\chi \pi^\perp \quad \text{and} \quad \dot{\kappa}^\perp = \pi^\perp \quad \text{so that the corresponding Lyapunov exponents are trivially paired in pairs } 0, -\chi \text{ with pairing sum } -\frac{\chi}{2}.

Since we have seen above that all the exponents are paired at the level \( \frac{\chi}{2} \), this means that all the physically interesting exponents (relative to the incompressible motions, \textit{i.e.} relative to the \( \pi, \kappa \) coordinates) are also paired at the same level, as claimed in \( \S 4 \).

\textbf{Erratum}

In Sec. 3 the statement

"The pairing rule, in fact, formally holds in the present ED case at least when the forcing is locally conservative ..."

is an error as a "locally conservative forcing" cannot admit a periodic potential, hence any supposed pairing rule cannot be based on such assumption. Therefore the analysis of Sec. 3 can only be regarded as speculative assuming an approximate pairing (nevertheless evidence for the existence of some kind of pairing is present in the literature).

Sec. 1,2,6 present the main idea of equivalent evolutions and do not need any pairing property; Sec. 4,5 remain valid if an approximate pairing rule is supposed; but the claims on its formal validity for ED and GED (stated in Sec.3 and appendix A1 and mentioned in items 5,7 in Sec.7 ), have to be retracted as a consequence of the error in Sec.3.

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