Locally conformal symplectic groupoids

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Abstract

Locally conformal symplectic (l.c.s.) groupoids are introduced as a generalization of symplectic groupoids. We obtain some examples and we prove that l.c.s. groupoids are examples of Jacobi groupoids in the sense of [5]. Finally, we describe the Lie algebroid of a l.c.s. groupoid.

Key words: Lie groupoids, Lie algebroids, symplectic and contact groupoids, Poisson and Jacobi groupoids, Jacobi manifolds, locally conformal symplectic manifolds.

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1 Introduction

A symplectic groupoid is a Lie groupoid $G \rightrightarrows M$ endowed with a symplectic structure $\Omega$ for which the graph of the partial multiplication is a lagrangian submanifold in the symplectic manifold $(G \times G \times G, \Omega \oplus \Omega \oplus -\Omega)$. This interesting class of groupoids, which was introduced in [1], arises in the integration of arbitrary Poisson manifolds. In fact, if $(G \rightrightarrows M, \Omega)$ is a symplectic groupoid then there exists a Poisson structure $\Lambda_0$ on $M$ and the Lie algebroid $AG$ is isomorphic to the cotangent Lie algebroid $T^*M$.

An interesting generalization of symplectic groupoids, as well as of Drinfeld’s Poisson-Lie groups, are Poisson groupoids [8].

On the other hand, a non-degenerate 2-form on a manifold $M$ of even dimension is said to be locally conformal symplectic (l.c.s.) if it is conformally related with a symplectic structure in some neighbourhood of every point of $M$ (see [3,7]). L.c.s. manifolds are interesting examples of Jacobi manifolds [3] and, in addition, they play an important role in the study of some dynamical systems, particularly conformally Hamiltonian systems (see [9]). The aim of this paper is to introduce the notion of a l.c.s. groupoid and to study some of its properties.

The paper is organized as follows. In Section 2, we recall the definition of a l.c.s. manifold and its relation with Jacobi manifolds. In Section 3, we prove that if $G$ is a contact groupoid over a manifold $M$ then it is possible
to introduce a l.c.s. structure and a Lie groupoid structure on $G \times \mathbb{R}$ (whose space of identity elements is $M \times \mathbb{R}$) and these structures are compatible in a certain way. These results motivate the definition of a l.c.s. groupoid in Section 4. Symplectic groupoids are l.c.s. groupoids and, furthermore, we prove, in Section 4, that a l.c.s. groupoid is a particular example of the so-called Jacobi groupoids, which were first introduced in [5] as a generalization of Poisson groupoids. Finally, in Section 5, we describe the Lie algebroid of a l.c.s. groupoid.

In this paper, we will use the definitions, notation and conventions introduced in our previous paper [5] (see Sections 1 and 2 in [5]).

2 Locally conformal symplectic manifolds

A manifold $M$ is said to be locally (globally) conformal symplectic (l.(g.)c.s.) manifold if it admits a non-degenerate 2-form $\Omega$ and a closed (exact) 1-form $\omega$ such that

$$\delta \Omega = \omega \wedge \Omega.$$  

The 1-form $\omega$ is called the Lee 1-form of $M$. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just the symplectic manifolds (see, for example, [3, 7]).

L.c.s manifolds are examples of Jacobi manifolds, i.e., if $(M, \Omega, \omega)$ is a l.c.s. manifold then there exists a 2-vector $\Lambda$ and a vector field $E$ on $M$ such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$. In fact, the Jacobi structure $(\Lambda, E)$ is given by

$$\Lambda(\alpha, \beta) = \Omega(b^{-1}(\alpha), b^{-1}(\beta)), \quad E = b^{-1}(\omega),$$

for all $\alpha, \beta \in \Omega^1(M)$, where $b : \mathcal{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $b(X) = i(X)\Omega$ (see [3]).

3 An example

Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid over a manifold $M$, that is, $G \rightrightarrows M$ is a Lie groupoid over $M$ with structural functions $\alpha, \beta, m$ and $\epsilon, \eta \in \Omega^1(G)$ is a contact 1-form on $G$ and $\sigma : G \to \mathbb{R}$ is an arbitrary function such that if $\oplus_{TG}$ is the partial multiplication in the tangent Lie groupoid $TG \rightrightarrows TM$, then

$$\eta_{(gh)}(X_g \oplus_{TG} Y_h) = \eta_g(X_g) + e^{\sigma(g)}\eta_h(Y_h),$$

for $(g, h) \in G^{(2)}$ and $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$ (see [2, 6]).
If \((G \rightarrow M, \eta, \sigma)\) is a contact groupoid then, using the associativity of \(\oplus_{TG}\), we deduce that \(\sigma : G \rightarrow \mathbb{R}\) is a multiplicative function, that is, \(\sigma(gh) = \sigma(g) + \sigma(h)\), for \((g, h) \in G^{(2)}\). This implies that
\[
\sigma \circ \epsilon = 0. \quad (4)
\]
Using \(\mathfrak{g}\), we also deduce that (see \([2]\))
\[
\xi(\sigma) = 0, \quad (5)
\]
\[
(\delta \eta)_{gh}(X_g \oplus_{TG} Y_h, X'_g \oplus_{TG} Y'_h) = (\delta \eta)_g(X_g, X'_g) + e^{\sigma(g)}(\delta \eta)_h(Y_h, Y'_h) + e^{\sigma(g)}(X_g(\sigma)\eta_h(Y_h) - X'_g(\sigma)\eta_h(Y_h)), \quad (6)
\]
for \((X_g, Y_h), (X'_g, Y'_h) \in T_{(g, h)}G^{(2)}\), \(\xi\) being the Reeb vector field of the contact structure \(\eta\).

Now, we will obtain a l.c.s. structure and a Lie groupoid structure on the manifold \(G \times \mathbb{R}\), both structures compatible in a certain way.

First of all, take the Lie groupoid \(G \rightarrow M\) and the multiplicative function \(\sigma : G \rightarrow \mathbb{R}\). Then, using the multiplicative character of \(\sigma\), we can define a right action of \(G \rightarrow M\) on the canonical projection \(\pi_1 : M \times \mathbb{R} \rightarrow M\) as follows
\[
(x, t) \cdot g = (\alpha(g, \sigma(g) + t)
\]
for \((x, t) \in M \times \mathbb{R}\) and \(g \in G\) such that \(\beta(g) = x\). Thus, we have the corresponding action groupoid \((M \times \mathbb{R}) \ast G \rightarrow M \times \mathbb{R}\) over \(M \times \mathbb{R}\), where
\[
(M \times \mathbb{R}) \ast G = \{(x, t), g) \in (M \times \mathbb{R}) \times G / \beta(g) = x\}
\]
(see \([3]\) for the general definition of an action Lie groupoid; see also \([4]\)). Moreover, it is not difficult to prove that \((M \times \mathbb{R}) \ast G\) may be identified with the product manifold \(G \times \mathbb{R}\) and, under this identification, the structural functions of the Lie groupoid are given by
\[
\alpha_\sigma(g, t) = (\alpha(g, \sigma(g) + t), \quad \text{for } (g, t) \in G \times \mathbb{R},
\beta_\sigma(h, s) = (\beta(h), s), \quad \text{for } (h, s) \in G \times \mathbb{R},
\]
\[
m_\sigma((g, t), (h, s)) = (gh, t), \quad \text{if } \alpha_\sigma(g, t) = \beta_\sigma(h, s),
\epsilon_\sigma(x, t) = (\epsilon(x, t), \quad \text{for } (x, t) \in M \times \mathbb{R}. \quad (8)
\]
From \(\mathfrak{g}\) and the definition of the tangent groupoid (see, for instance, \([3]\)), it follows that the projections \((\alpha_\sigma)^T, (\beta_\sigma)^T\), the inclusion \((\epsilon_\sigma)^T\) and the partial multiplication \(\oplus_{T(G \times \mathbb{R})}\) of the tangent groupoid \(T(G \times \mathbb{R}) \rightarrow T(M \times \mathbb{R})\) are given by
\[
(\alpha_\sigma)^T(X_g + \lambda \frac{\partial}{\partial t} |_t) = \alpha^T(X_g) + (\lambda + X_g(\sigma)) \frac{\partial}{\partial t} |_{t+\sigma(g)},
\]
\[
(\beta_\sigma)^T(Y_h + \mu \frac{\partial}{\partial s} |_s) = \beta^T(Y_h) + \mu \frac{\partial}{\partial s} |_s,
\]
\[
(X_g + \lambda \frac{\partial}{\partial t} |_t) \oplus_{T(G \times \mathbb{R})} (Y_h + \mu \frac{\partial}{\partial s} |_s) = X_g \oplus_{TG} Y_h + \lambda \frac{\partial}{\partial t} |_t,
\]
\[
(\epsilon_\sigma)^T(X_x + \lambda \frac{\partial}{\partial t} |_t) = \epsilon^T(X_x) + \lambda \frac{\partial}{\partial t} |_t.
\]
\[9\]
for $X_g + \lambda \frac{\partial}{\partial t}|_t \in T_{(g,t)}(G \times \mathbb{R})$, $Y_h + \mu \frac{\partial}{\partial s}|_s \in T_{(h,s)}(G \times \mathbb{R})$ and $X_x + \lambda \frac{\partial}{\partial t}|_t \in T_{(x,t)}(M \times \mathbb{R})$ (see Section 5.3.1 in [5]).

On the other hand, if $A(G \times \mathbb{R})$ is the Lie algebroid of the Lie groupoid $G \times \mathbb{R} \Rightarrow M \times \mathbb{R}$, then using (3)-(6) and (8)-(12), we prove the following result.

**Proposition 3.1** Let $(G \Rightarrow M, \eta, \sigma)$ be a contact groupoid. If $G \times \mathbb{R} \Rightarrow M \times \mathbb{R}$ is the Lie groupoid with structural functions given by (8), then using $\tilde{\alpha}_\sigma$, $\tilde{\beta}_\sigma$, the inclusion $c_\sigma$ and the partial multiplication $\oplus g_{\mathbb{R}}^{\mathbb{R}}$ in the cotangent groupoid $T^\ast(G \times \mathbb{R}) \Rightarrow A^\ast(G \times \mathbb{R})$ are defined by

\begin{equation}
\begin{aligned}
    \tilde{\alpha}_\sigma(g_\sigma + \gamma \delta t|_t) &= (\tilde{\alpha}(g_\sigma), \sigma(g) + t), \\
    \tilde{\beta}_\sigma(\nu_h + \zeta \delta t|_s) &= (\tilde{\beta}(\nu_h) - \zeta(\delta \sigma)_{\varepsilon(\beta(g))}, s), \\
    (\mu_g + \gamma \delta t|_t) \oplus g_{\mathbb{R}}^{\mathbb{R}}(\nu_h + \zeta \delta t|_s) &= (\mu_g + \zeta(\delta \sigma)_{\varepsilon(\beta(g))}) \oplus g_{\mathbb{R}}^{\mathbb{R}}(\nu_h + \gamma + \zeta) \delta t|_t \\
    \tilde{c}_\sigma(\mu_x, t) &= \tilde{\varepsilon}(\mu_x) + 0 \delta t|_t,
\end{aligned}
\end{equation}

for $\mu_g + \gamma \delta t|_t \in T^\ast_{(g,t)}(G \times \mathbb{R})$, $\nu_h + \zeta \delta t|_s \in T^\ast_{(h,s)}(G \times \mathbb{R})$ and $(\mu_x, t) \in A^\ast_{(x,t)}(G \times \mathbb{R}) \cong A^\ast_{\mathbb{R}}G \times \mathbb{R}$, where $\tilde{\alpha}$, $\tilde{\beta}$, $\oplus g_{\mathbb{R}}^{\mathbb{R}}$ and $\tilde{\varepsilon}$ are the structural functions of the cotangent Lie groupoid $T^\ast G \Rightarrow A^\ast G$ (see Section 5.3.1 in [5]).

Next, we define on $G \times \mathbb{R}$ the 2-form $\Omega$ and the 1-form $\omega$ given by

\begin{equation}
\Omega = -\left(\tilde{\pi}_1^\ast(\delta \eta) + \tilde{\pi}_2^\ast(\delta t) \wedge \tilde{\pi}_2^\ast(\eta)\right), \quad \omega = -\tilde{\pi}_2^\ast(\delta t),
\end{equation}

where $\tilde{\pi}_1 : G \times \mathbb{R} \to G$ and $\tilde{\pi}_2 : G \times \mathbb{R} \to \mathbb{R}$ are the canonical projections. Thus, we have that $(G \times \mathbb{R}, \Omega, \omega)$ is a locally conformal symplectic manifold of the first kind in the sense of [4]. Note that if $(\Lambda, E)$ is the Jacobi structure on $G \times \mathbb{R}$ associated with the l.c.s. structure $(\Omega, \omega)$ then

\begin{equation}
E = -\xi.
\end{equation}

Now, using (3)-(6) and (8)-(12), we prove the following result.

**Proposition 3.1** Let $(G \Rightarrow M, \eta, \sigma)$ be a contact groupoid. If $G \times \mathbb{R} \Rightarrow M \times \mathbb{R}$ is the Lie groupoid with structural functions given by (8), then $(\Omega, \omega)$ is the pull-back of the multiplicative function $\sigma$ by the canonical projection $\tilde{\pi}_1 : G \times \mathbb{R} \to G$ and $\theta$ is the 1-form on $G \times \mathbb{R}$ defined by $\theta = e^\sigma(\delta \sigma - \omega)$, then we have:

i) $m_\sigma^\ast \Omega = \tilde{\pi}_1^\ast \Omega + e^{(e_{\sigma} \tilde{\pi}_1)} \tilde{\pi}_2^\ast \Omega$;

ii) $\tilde{\alpha}_\sigma \circ \omega = 0$, $\tilde{\beta}_\sigma \circ \theta = 0$;

iii) $m_\sigma^\ast \omega = \tilde{\pi}_1^\ast \omega$, $m_\sigma^\ast \theta = e^{(e_{\sigma} \tilde{\pi}_1)} \tilde{\pi}_2^\ast \theta$;

iv) $\Lambda(\omega, \theta) = 0$, $(\theta + \omega - e_{\sigma} \circ \tilde{\beta}_\sigma \circ \omega) \circ \epsilon = 0$;

where $\tilde{\pi}_1 : G^{(i)} \to G$, $i = 1, 2$, are the canonical projections and $(\Lambda, E)$ is the Jacobi structure on $G \times \mathbb{R}$ associated with the l.c.s. structure $(\Omega, \omega)$. 
4 Locally conformal symplectic and Jacobi groupoids

Motivated by Proposition 3.1 we introduce the following definition.

**Definition 4.1** Let $G \rightrightarrows M$ be a Lie groupoid with structural functions $\alpha$, $\beta$, $m$ and $\epsilon$, $(\Omega, \omega)$ be a l.c.s. structure on $G$, $\sigma : G \to \mathbb{R}$ be a multiplicative function and $\theta$ be the 1-form on $G$ defined by

$$\theta = e^\sigma (\delta \sigma - \omega).$$

Then, $(G \rightrightarrows M, \Omega, \omega, \sigma)$ is a l.c.s. groupoid if the following properties hold:

1. $m^* \Omega = \pi_1^* \Omega + e^{(\sigma \circ \pi_1)} \pi_2^* \Omega$;
2. $\tilde{\alpha} \circ \omega = 0$, $\tilde{\beta} \circ \theta = 0$;
3. $m^* \omega = \pi_1^* \omega$, $m^* \theta = e^{(\sigma \circ \pi_1)} \pi_2^* \theta$;
4. $\Lambda(\omega, \theta) = 0$, $(\theta + \omega - \tilde{\epsilon} \circ \beta \circ \omega) \circ \epsilon = 0$;

where $\pi_i : G^{(2)} \to G$, $i = 1, 2$, are the canonical projections, $(\Lambda, E)$ is the Jacobi structure associated with the l.c.s. structure $(\Omega, \omega)$ and $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\epsilon}$ are the structural functions of the cotangent groupoid $T^*G \rightrightarrows A^*G$.

**Examples 4.2**

i) If $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid then, by Proposition 3.1.4 $(G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}, \Omega, \omega, \bar{\sigma})$ is a l.c.s. groupoid, where the structural functions of $G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}$ are defined by (3), the pair $(\Omega, \omega)$ is given by (11) and the multiplicative function $\bar{\sigma}$ is $\bar{\sigma} = \sigma \circ \tilde{\pi}_1$, $\tilde{\pi}_1 : G \times \mathbb{R} \to G$ being the canonical projection.

ii) A Lie groupoid $G \rightrightarrows M$ is said to be symplectic if $G$ admits a symplectic 2-form $\Omega$ in such a way that the graph of the partial multiplication in $G$ is a Lagrangian submanifold of the symplectic manifold $(G \times G \times G, \Omega \oplus \Omega \oplus (-\Omega))$ (see [1]). This is equivalent to say that $\Omega$ satisfies the condition $m^* \Omega = \pi_1^* \Omega + \pi_2^* \Omega$. Therefore, we conclude that $(G \rightrightarrows M, \Omega)$ is a symplectic groupoid if and only if $(G \rightrightarrows M, \Omega, 0, 0)$ is a l.c.s. groupoid.

Next, we will give the relation between l.c.s. groupoids and Jacobi groupoids. First, we will recall the definition of a Jacobi groupoid.

Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $TG \times \mathbb{R}$ is a Lie groupoid over $TM \times \mathbb{R}$ with structural functions given by (see Section 3 in [3])

$$\begin{align*}
(\alpha^T)_\sigma(X_g, \lambda) &= (\alpha^T(X_g), X_g(\sigma) + \lambda), \text{ for } (X_g, \lambda) \in T_gG \times \mathbb{R}, \\
(\beta^T)_\sigma(Y_h, \mu) &= (\beta^T(Y_h), \mu), \text{ for } (Y_h, \mu) \in T_hG \times \mathbb{R}, \\
(X_g, \lambda) \oplus_{TG \times \mathbb{R}} (Y_h, \mu) &= (X_g \oplus_{TG} Y_h, \lambda), \text{ if } (\alpha^T)_\sigma(X_g, \lambda) = (\beta^T)_\sigma(Y_h, \mu), \\
(\epsilon^T)_\sigma(X_x, \lambda) &= (\epsilon^T(X_x), \lambda), \text{ for } (X_x, \lambda) \in T_xM \times \mathbb{R}.
\end{align*}$$

(18)
We call this Lie groupoid the $\sigma$-inclusion $T G$

Let

**Proposition 4.5**

$G$ on a Jacobi groupoid $\mathcal{G}$ with structural functions given by (see Section 3 in [5])

\[ \sigma(E) = 0 \]

**Remark 4.4**

A Poisson groupoid is a Jacobi groupoid (the structural functions of the Lie groupoid structure on $\phi$ is a morphism of Lie groupoids over some map $\phi : T^*G \times \mathbb{R} \to T M \times \mathbb{R}$).

**Definition 4.3**

Let $G \Rightarrow M$ be a Lie groupoid, $(\Lambda, E)$ be a Jacobi structure on $G$ and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $(G \Rightarrow M, \Lambda, E, \sigma)$ is a Jacobi groupoid if the homomorphism $\#_{(\Lambda, E)} : T^*G \times \mathbb{R} \to T G \times \mathbb{R}$ given by

\[ \#_{(\Lambda, E)}(\mu_g, \gamma) = (\#_\Lambda(\mu_g) + \gamma E, -\mu_g(E)) \]

is a morphism of Lie groupoids over some map $\varphi_0 : A^*G \to T M \times \mathbb{R}$, where the structural functions of the Lie groupoid structure on $T^*G \times \mathbb{R} \Rightarrow A^*G$ (respectively, $T G \times \mathbb{R} \Rightarrow T M \times \mathbb{R}$) are given by (20) (respectively, (18)).

**Remark 4.4** A Poisson groupoid is a Jacobi groupoid $(G \Rightarrow M, \Lambda, E, \sigma)$ with $E = 0$ and $\sigma = 0$ (see [5]).

A characterization of a Jacobi groupoid is the following one. If $G \Rightarrow M$ is a Lie groupoid and $\sigma : G \to \mathbb{R}$ is a multiplicative function then $T^*G$ is a Lie groupoid over $A^*G$ with structural functions given by

\[ \tilde{\alpha}_\sigma(\mu_g, \gamma) = e^{-\sigma(\gamma)} \tilde{\alpha}(\mu_g), \quad \text{for} \quad \mu_g \in T_g^*G, \]

\[ \tilde{\beta}_\sigma(\nu_h, \zeta) = \tilde{\beta}(\nu_h) - \zeta(\delta \sigma)_\phi(\beta(\nu_h)) \quad \text{for} \quad (\nu_h, \zeta) \in T^*_hG \times \mathbb{R}, \]

\[ (\mu_g, \gamma) \oplus_{T^*G \times \mathbb{R}} (\nu_h, \zeta) = \left( (\mu_g + e^{\sigma(\gamma)} \zeta(\delta \sigma)_\phi(\nu_h)), \gamma + e^{\sigma(\gamma)} \zeta \right) \]

\[ \bar{\epsilon}_\sigma(\mu_x) = (\bar{\epsilon}(\mu_x), 0), \quad \text{for} \quad \mu_x \in A^*_G. \]

We call this Lie groupoid the $\sigma$-cotangent groupoid. Note that the canonical inclusion $T^*G \to T^*G \times \mathbb{R}$, $\mu_g \mapsto (\mu_g, 0)$, is a monomorphism of Lie groupoids.

**Proposition 4.5**

Let $G \Rightarrow M$ be a Lie groupoid, $(\Lambda, E)$ be a Jacobi structure on $G$ and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $(G \Rightarrow M, \Lambda, E, \sigma)$ is a Jacobi groupoid if and only if the following conditions hold:

i) $\#_\Lambda : T^*G \to T G$ is a Lie groupoid morphism over some map $\varphi_0 : A^*G \to T M$ from the $\sigma$-cotangent groupoid $T^*G \Rightarrow A^*G$ to the tangent Lie groupoid $T G \Rightarrow T M$.
ii) $E$ is a right-invariant vector field on $G$ and $E(\sigma) = 0$.

iii) If $X_0 \in \Gamma(AG)$ is the section of the Lie algebroid $AG$ satisfying $E = \dot{X}_0$, we have that

$$\#_\Lambda(\delta\sigma) = \dot{X}_0 - e^{-\sigma}\dot{X}_0.$$

Proof: Suppose that $(G \supseteq M, \Lambda, E, \sigma)$ is a Jacobi groupoid. Then, proceeding as in the proof of Proposition 4.4 in \cite{5}, we deduce that i), ii) and iii) hold.

A similar computation proves the converse.

Now, we will show that a l.c.s. symplectic groupoid is a particular example of a Jacobi groupoid.

**Theorem 4.6** Let $G \supseteq M$ be a Lie groupoid, $(\Omega, \omega)$ be a l.c.s. structure on $G$ and $\sigma : G \to \mathbb{R}$ be a multiplicative function. If $(\Lambda, E)$ is the Jacobi structure associated with the l.c.s. structure $(\Omega, \omega)$ then $(G \supseteq M, \Omega, \omega, \sigma)$ is a l.c.s. groupoid if and only if $(G \supseteq M, \Lambda, E, \sigma)$ is a Jacobi groupoid.

Proof: Assume that $(G \supseteq M, \Omega, \omega, \sigma)$ is a l.c.s. groupoid. If $\mu_g \in T_g^*G$ and $\nu_h \in T_h^*G$ satisfy the relation $\tilde{\alpha}_\sigma^*(\mu_g) = \tilde{\beta}_\sigma^*(\nu_h)$, then using \cite{2}, \cite{14} and the definition of the partial multiplication $\oplus_{TG}$ in the tangent Lie groupoid $TG \supseteq TM$, we obtain that $\alpha^T(\#_\Lambda(\mu_g)) = \beta^T(\#_\Lambda(\nu_h))$ and, in addition,

$$(i(\#_\Lambda(\mu_g) \oplus_{TG} \nu_h))\Omega_{(gh)}(X_g \oplus_{TG} Y_h) = (i(\#_\Lambda(\mu_g) \oplus_{TG} \#_\Lambda(\nu_h)))\Omega_{(gh)}(X_g \oplus_{TG} Y_h),$$

for $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$. Thus (see \cite{2}), it follows that $\#_\Lambda(\mu_g) \oplus_{TG} \nu_h = \#_\Lambda(\mu_g) \oplus_{TG} \#_\Lambda(\nu_h)$ and, therefore, the map $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism over some map $\tilde{\varphi}_0 : A^*G \to TM$, between the $\sigma$-cotangent groupoid $T^*G \supseteq A^*G$ and the tangent groupoid $TG \supseteq TM$. In particular, this implies that

$$\alpha^T \circ \#_\Lambda = \tilde{\varphi}_0 \circ \tilde{\alpha}_\sigma^*, \quad \beta^T \circ \#_\Lambda = \tilde{\varphi}_0 \circ \tilde{\beta}_\sigma^*.$$  \hspace{1cm} (21)

Now, from \cite{2}, we deduce that $E = -\#_\Lambda(\omega)$. Using this relation, \cite{15} and \cite{21}, we have that the vector field $E$ is $\sigma$-vertical.

Next, suppose that $(g, h) \in G^{(2)}$ and denote by $R_h : G_{\beta(h)} \to G_{\alpha(h)}$ the right-translation by $h$. Then, \cite{2} and \cite{16} imply that

$$(i(E_{(gh)})\Omega_{(gh)})(X_g \oplus_{TG} Y_h) = (i((R_h)^\#(E_{(g)}))\Omega_{(gh)})(X_g \oplus_{TG} Y_h),$$

for $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$. Consequently, $E$ is a right-invariant vector field and there exists $X_0 \in \Gamma(AG)$ such that $E = -\dot{X}_0$.

On the other hand, if $X_{e^\sigma}$ is the hamiltonian vector field of the function $e^\sigma$, $X_{e^\sigma} = e^\sigma\#_\Lambda(\delta\sigma) + e^\sigma E$, it is clear that $X_{e^\sigma} = \#_\Lambda(\theta)$. Using this equality,
and proceeding as in the proof of the fact that $E$ is right-invariant, we conclude that $X_{e^\sigma}$ is a left-invariant vector field. Moreover, if $x$ is a point of $M$, then relation (17) implies that $X_{e^\sigma}(e(x)) = -X_0(\epsilon(x))$. Thus, $\#_\Lambda(\delta \sigma) = X_0 - e^{-\sigma}X_0$.

Finally, since $\Lambda(\omega, \theta) = 0$ and $E = -\#_\Lambda(\omega)$, we obtain that $E(\sigma) = 0$. Therefore, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.

In a similar way, we prove the converse.

**Remark 4.7** Using Theorem 4.6 we directly deduce that a symplectic groupoid is a Poisson groupoid. This result was proved in [8].

## 5 The Lie algebroid of a l.c.s. groupoid

Let $(G \rightrightarrows M, \Omega, \omega, \sigma)$ be a l.c.s. groupoid and $\theta$ the 1-form on $G$ given by (13). Then, the 1-form $e^{-\sigma}\theta$ is closed and since $\beta \circ \theta = 0$, it follows that $\theta$ is basic with respect to the projection $\alpha$. Thus, there exists a unique 1-form $\theta_0$ on $M$ such that $\alpha^* \theta_0 = e^{-\sigma}\theta$. It is clear that $\theta_0$ is closed.

Now, denote by $(\Lambda, E)$ the Jacobi structure on $G$ associated with the l.c.s. structure $(\Omega, \omega)$. Then, $\#_\Lambda(\theta)$ is the hamiltonian vector field $X_{e^\sigma}$ of the function $e^\sigma$. Moreover, from Theorem 4.6 and using the results in [5] (see Proposition 5.6 in [5]), we deduce that there exists a Jacobi structure $(\Lambda_0, E_0)$ on $M$ in such a way that the couple $(\alpha, e^\sigma)$ is a conformal Jacobi morphism between the Jacobi manifolds $(G, \Lambda, E)$ and $(M, \Lambda_0, E_0)$. This implies that

$$\#_{\Lambda_0(\alpha(g))}(\theta_0(\alpha(g))) = e^{\sigma(\alpha(g))}(\alpha^T_g \circ \#_{\Lambda_0}(\alpha^T_g) \circ (\alpha^T_g)^*)(\theta_0(\alpha(g)))$$

$$= \alpha^T_g(X_{e^\sigma}(g)) = E_0(\alpha(g)),$$

for $g \in G$, where $(\alpha^T_g)^* : T^*_\alpha(g)M \to T^*_gG$ is the adjoint map of the tangent map $\alpha^T_g : T_gG \to T_{\alpha(g)}M$. Therefore, we have proved the following result.

**Proposition 5.1** Let $(G \rightrightarrows M, \Omega, \omega, \sigma)$ be a l.c.s. groupoid and $\theta$ the 1-form on $G$ given by (13). Then, there exists a unique 1-form $\theta_0$ on $M$ such that $\alpha^* \theta_0 = e^{-\sigma}\theta$. Furthermore, $\theta_0$ is closed and $\#_{\Lambda_0}(\theta_0) = E_0$.

Next, we will describe the Lie algebroid associated with a l.c.s. groupoid.

**Theorem 5.2** Let $(G \rightrightarrows M, \Omega, \omega, \sigma)$ be a l.c.s. groupoid, $AG$ be the Lie algebroid of $G$, $(\Lambda, E)$ be the Jacobi structure on $G$ associated with the l.c.s. structure $(\Omega, \omega)$ and $(\Lambda_0, E_0)$ be the corresponding Jacobi structure on $M$. Then, the map $\Psi : \Omega^1(M) \to \mathfrak{T}_L(G)$ between $\Omega^1(M)$ and the space of left-invariant
vector fields on $G$ defined by $\Psi(\mu) = e^\sigma \#_A(\alpha^* \mu)$ induces an isomorphism between the vector bundles $T^*M$ and $AG$. Under this isomorphism, the Lie bracket on $\Gamma(AG) \cong \mathfrak{X}_L(G)$ and the anchor map of $AG$ are given by

$$
\begin{align*}
[\mu, \nu]_{(\Lambda_0, E_0, \theta_0)} &= L_{\#\Lambda_0(\mu)} \nu - L_{\#\Lambda_0(\nu)} \mu - \delta(\Lambda_0(\mu, \nu)) \\
#_{(\Lambda_0, E_0, \theta_0)}(\mu) &= \#_{\Lambda_0}(\mu),
\end{align*}
$$

for $\mu, \nu \in \Omega^1(M)$, where $\theta_0$ is the 1-form on $M$ considered in Proposition 5.4.

Proof: Let $\mu$ be a 1-form on $M$. Since the map $\#_A : T^*G \to TG$ is a morphism between the $\sigma$-cotangent groupoid and the tangent groupoid $TG \rightrightarrows TM$, we obtain that vector field $\tilde{X} = \Psi(\mu)$ is $\beta$-vertical. Moreover, if $(g, h) \in G^{(2)}$ and $L_g : G^{\alpha(g)} \to G^{\beta(g)}$ is the left-translation by $g$ then, using (14), we deduce that

$$(i(\tilde{X}(gh)) \Omega_{(gh)})(Y_g \oplus TG Z_h) = (i((L_g)_h^*(\tilde{X}(h))) \Omega_{(gh)})(Y_g \oplus TG Z_h).$$

for $(Y_g, Z_h) \in T_{(g, h)}G^{(2)}$. This proves that $\tilde{X} \in \mathfrak{X}_L(G)$.

Conversely, if $\tilde{X} \in \mathfrak{X}_L(G)$ and $\tilde{\mu}$ is the 1-form on $G$ defined by $\tilde{\mu} = -i(\tilde{X})\Omega$ then, from (14), it follows that $e^\sigma \alpha^* \mu = \tilde{\mu}$, $\mu$ being the 1-form on $M$ given by $\mu = e^* \tilde{\mu}$. This implies that $\Psi(\mu) = \tilde{X}$.

On the other hand, using that the map $\#_A : \Omega^1(G) \to \mathfrak{X}(G)$ is an isomorphism of $C^\infty(G, \mathbb{R})$-modules, we conclude that $\Psi$ is an isomorphism of $C^\infty(M, \mathbb{R})$-modules.

Now, suppose that $X, Y \in \Gamma(AG)$. We have that the left-invariant vector field $\Xi$ is $\alpha$-projectable to a vector field $a(X)$ on $M$. In addition, if $\mu$ and $\nu$ are the 1-forms on $M$ satisfying $\Psi(\mu) = \Xi$ and $\Psi(\nu) = Y$, then a long computation, using (1) and Definition 4.1, shows that

$$
a(X) = \#_{(\Lambda_0, E_0, \theta_0)}(\mu), \quad i([\Xi, Y]) \Omega = -e^\sigma \alpha^* [\mu, \nu]_{(\Lambda_0, E_0, \theta_0)}.\$$

This ends the proof of our result. $\blacksquare$

Remark 5.3 Let $(G \rightrightarrows M, \Omega)$ be a symplectic groupoid. Then, the Jacobi structure on $M$ is Poisson, that is, $E_0 = 0$ (see Section 5.2 in [5]) and the 1-form $\theta_0$ on $M$ identically vanishes. Thus (see Theorem 5.2), $AG$ is isomorphic to the cotangent Lie algebroid $T^*M$. This result was proved in [1].

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