About the asymptotic formula for spectral function of the Laplace-Beltrami operator on sphere

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Abstract

In this work we established asymptotical behavior for Riesz means of the spectral function of the Laplace operator on unit sphere.

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1 Introduction

Let \( S^N \) is the \( N \)-dimensional unit sphere in \( \mathbb{R}^{N+1} \). The Laplace operator \( \Delta \):

\[
\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_N^2}, \quad u \in C_0^\infty(S^N)
\]

in spherical coordinates in \( N \) dimensions, with the parametrization \( x = r\theta \in \mathbb{R}^{N+1} \) with \( r \in [0, +\infty) \) and \( \theta \in S^N \),

\[
\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{N}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_S f
\]

where \( \Delta_S \) is the Laplace-Beltrami operator on the \( N \)-dimensional sphere, or spherical Laplacian. One can also write the term

\[
\frac{\partial^2 f}{\partial r^2} + \frac{N}{r} \frac{\partial f}{\partial r}
\]

equivalently as

\[
\frac{1}{r^N} \frac{\partial}{\partial r} \left( r^N \frac{\partial f}{\partial r} \right).
\]
As a consequence, the spherical Laplacian of a function defined on $S^N \subset R^{N+1}$ can be computed as the ordinary Laplacian of the function extended to $R^{N+1} \setminus \{0\}$ so that it is constant along rays.

Let us denote by $\lambda_0, \lambda_1, \ldots$ the distinct eigenvalues of $-\Delta_S$, arranged in increasing order. Let $H_k$ denote the eigenspace corresponding to $\lambda_k$. We call elements of $H_k$ spherical harmonics of degree $k$. It is well known (see [17]) that $\dim H_k = a_k$:

$$a_k = \begin{cases} 
1, & \text{if } k = 0, \\
N, & \text{if } k = 1, \\
\frac{(N + k)!}{N! k!} - \frac{(N + k - 2)!}{N! (k - 2)!}, & \text{if } k \geq 2
\end{cases}$$

(1)

For $a_k$ we have $a_k \approx k^{N-1}$ as $k \to \infty$.

Let $\hat{A}$ is a self-adjoint extension of the Laplace operator $\Delta_S$ in $L_2(S^N)$ and if $E_\lambda$ is the corresponding spectral resolution, then for all functions $f \in L_2(S^N)$ we have

$$\hat{A}f = \int_0^\infty \lambda dE_\lambda f.$$ 

The operator $\hat{A}$ has in$L_2(S^N)$ a complete orthonormal system of eigenfunctions

$$\{Y_1^{(k)}(x), Y_2^{(k)}(x), \ldots, Y_{a_k}^{(k)}(x)\} \subset H_k, k = 0, 1, 2, \ldots,$$

corresponding to the eigenvalues $\{l_k = k(k + N - 1)\}, k = 0, 1, 2, \ldots$.

It is easy to check that the operators $E_\lambda$ have the form

$$E_\lambda f(x) = \sum_{\lambda_n < \lambda} Y_n(f, x),$$

(2)

where

$$Y_k(f, x) = \sum_{j=1}^{a_k} Y_j^{(k)}(x) \int_{S^N} f(y) Y_j^{(k)}(y) d\sigma(y)$$

(3)

The Riesz means of the partial sums (2) is defined by

$$E_n^\alpha f(x) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha Y_k(f, x)$$

(4)

The most convenient object for a detailed investigation are the expansions of the form (4). The integral (1) may be transformed writing instead of $\hat{A}$ the integral to the right in (3) and then changing the order of integration. This yields the formula

$$E_s^\lambda f(x) = \int_{S^N} \Theta_s(x, y, \lambda) f(y) d\sigma(y)$$

(5)
with

\[ \Theta^\alpha(x, y, n) = \sum_{k=0}^{n} \left( 1 - \frac{k}{\lambda_n} \right)^\alpha Z_k(x, y). \]  

(6)

For \( \alpha = 0 \) this kernel is called the spectral function of the Laplace operator for the entire space \( S^N \).

The behavior of the spectral expansion corresponding to the the Laplace-Beltrami operator is closely connected with the asymptotic behavior of the kernel \( \Theta^\alpha(x, y, n) \).

For any two points \( x \) and \( y \) from \( S^N \) we shall denote by \( \gamma(x, y) \) spherical distance between these two points. Actually, \( \gamma(x, y) \) is a measure of angle between \( x \) and \( y \). It is obvious, that \( \gamma(x, y) \leq \pi \).

We proceed to the formulation of the fundamental results of the paper.

**Theorem 1.1** Let \( \Theta^\alpha(x, y, n) \) is the kernel of Riesz means of the spectral expansions

1) if \( |\frac{\pi}{2} - \gamma| < \frac{n}{n+1} \frac{\pi}{2} \) then we have

\[ \Theta^\alpha(x, y, n) = O(1) \left( \frac{n^{(N-1)/2-\alpha}}{(\sin \gamma)^{(N-1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{n^{(N-3)/2-\alpha}}{(\sin \gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}} \right) + \frac{n^{-1}}{(\sin(\gamma/2))^{1+N}}; \]

2) if \( 0 \leq \gamma \leq \pi \), then we have

\[ \Theta^\alpha(x, y, n) = O(1)n^N; \]

3) if \( 0 < \gamma_0 \leq \gamma \leq \pi \), then we have

\[ \Theta^\alpha(x, y, n) = O(1)n^{N-\alpha}; \]

2 Preliminaries

In this section we give some properties of Riesz means of spectral expansions.

Let \( f \) be a function with the support in \((0, +\infty)\). If \( f \) has the local bounded variation, then the Riesz means \( f^\alpha \) for all \( \Re(\alpha) > -1 \) is defined by

\[ f^\alpha(t) = \int_0^t \left( 1 - \frac{s}{t} \right)^\alpha df(s). \]  

(7)

Using the properties of the distribution \( R_\alpha(t) = \alpha t^{\alpha-1} \), in [8] was established the generalization of M.Riesz theorem:
**Theorem 2.1** Let $\zeta$ is complex number with $\text{Re}(\zeta) > 0$, and $M_0(t)$ and $M_1(t)$ are positive increasing functions in $(0, +\infty)$. If $f(t) = 0, t < 0$, and for all $t > 0$ satisfied inequalities:

$$|f(t)| \leq M_0(t) \quad (8)$$

$$|t^\zeta f^{(\zeta)}(t)| \leq M_1(t) \quad (9)$$

then for $0 < \text{Re}(\alpha) < \text{Re}(\zeta)$ we have

$$|t^\alpha f^{(\alpha)}(t)| \leq C (1 + |\alpha|)^{\text{Re}(\zeta)+2} \left( \frac{|\alpha|}{\text{Re}(\alpha)} + \frac{|\zeta - \alpha|}{\text{Re}(\zeta) - \alpha} \right) M_0^{\text{Re}(\zeta - \alpha)}(t) M_1^{\text{Re}(\alpha)}(10)$$

Constant $C$ depends only on $\zeta$.

We are going to prove the Theorem 1.5 using this statement on Riesz means. Let us introduce the Cesaro means of spectral expansions. The Cesaro means of order $\alpha \geq 0$ is defined as

$$C^\alpha_n f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^\alpha}{A_n^\alpha} Y_k(f, x) \quad (11)$$

where $Y_k(f, x) \in H_k, k \geq 0$ and $A_m^\alpha = \frac{\Gamma(\alpha + m + 1)}{\Gamma(\alpha + 1)m!}, m = 0, 1, 2, ..., \Gamma(z)$ is Gamma function.

As a Riesz means the Cesaro means are integral operator with the kernel

$$X^\alpha(x, y, n) = \sum_{k=0}^{n} \frac{A_{n-k}^\alpha}{A_n^\alpha} Z_k(x, y), \quad (12)$$

where $Z_k$ is zonal harmonics of order $k$, which is reproducing kernel for the space $H_k, k \geq 0$. For the kernel we have

**Lemma 2.2** If $\alpha > -1$ and $|\frac{\pi}{2} - \gamma(x, y)| \leq \frac{n}{n+1} \frac{\pi}{2}$, then

$$\frac{\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} \frac{\Gamma(n + \frac{N+1}{2})}{\Gamma(n + \frac{N+1}{2})} \sin \left( (n + \frac{N+1}{2}) \gamma - \frac{N-1}{2} - \frac{\alpha}{2} \right)$$

$$+ \frac{O(n^{(N-3)/2})}{(\sin \gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{O(n^{-1})}{(\sin(\gamma/2))^{1+N}};$$

If $\alpha > -1$ and $0 < \gamma_0 \leq \gamma \leq \pi$, then for all $n > 1$

$$|X^\alpha(x, y, n)| \leq cn^{N-1-\alpha},$$

If $\alpha > -1$ and $0 < \gamma_0 \leq \gamma \leq \pi$, then for all $n > 1$

$$|X^\alpha(x, y, n)| \leq cn^N.$$
This lemma is proved in [9]. It is well known that for integer order $\alpha$ kernels $\Theta^\alpha(x, y, n)$ and $\Xi^\alpha(x, y, n)$ have same asymptotical behavior. Our purpose to extend this result for all $\alpha$.

Let us denote by

$$M_\alpha(t) = \frac{t^{(N-1)/2}}{(\sin \gamma)(N-1)/2(\sin(\gamma/2))^{1+\alpha} + \frac{t^{(N-3)/2}}{(\sin \gamma)(N+1)/2(\sin(\gamma/2))^{1+\alpha} + \frac{t^{-1}}{(\sin(\gamma/2))^{1+N}};$$

If $\zeta$ is integer, then for the kernel $\Theta^\zeta(x, y, n)$ we have

$$|\Theta^0(x, y, n)| \leq M_0(t);$$

and

$$t^\zeta|\Theta^\zeta(x, y, n)| \leq M_\zeta(t).$$

So using the statement of Theorem 2.1 we obtain

$$|t^\alpha\Theta^\alpha(t)| \leq C(1 + |\alpha|)^{Re(\zeta)+2} \left( \frac{|\alpha|}{Re(\alpha)} + \frac{\zeta - \alpha}{Re(\zeta - \alpha)} \right) M_0^{Re(\zeta)} M_1^{Re(\zeta)}.$$ (13)

So finally by simplifying we get

$$t^\alpha \Theta^\alpha(x, y, n) \leq \frac{t^{(N-1)/2}}{(\sin \gamma)(N-1)/2(\sin(\gamma/2))^{1+\alpha} + \frac{t^{(N-3)/2}}{(\sin \gamma)(N+1)/2(\sin(\gamma/2))^{1+\alpha} + \frac{t^{-1}}{(\sin(\gamma/2))^{1+N}};$$

3 The estimates for maximal operators

In this section we are going to prove the estimation for maximal operators of the Riesz means by using the results of previous section. Let us recall some standard definitions from harmonic analysis on unit sphere.

Spherical ball $B(x, r)$ of radius $r$ and with the center at point $x$ defined by $B(x, r) = \{y \in S^N : \gamma(x, y) < r\}$. For integrable function $f(x)$ the maximal function of Hardy-Littelwood

$$f^*(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{S^N} |f(y)|d\sigma(y)$$ (14)

is finite almost everywhere on sphere. The maximal function $f^*$ plays a major role in analysis and has been much studied (see [17]). In particular, for any $p > 1$ and if $f \in L^p$, then there exists constant $c_p$, such that

$$\|f^*\|_{L^p} \leq \frac{c_p(N)}{p-1}\|f\|_{L^p},$$

where $c_p$ has no singularities at point $p = 1$. 
Theorem 3.1 Let $\alpha > \frac{N-1}{2}$ then for all $f \in L_1(S^N)$ we have

$$E^\alpha_n f(x) \leq \frac{c_\alpha(N)}{\alpha - \frac{N-1}{2}} \left( f^*(x) + f^*(\bar{x}) \right)$$

(15)

where $\gamma(x, \bar{x}) = \pi$.

Proof. Since Riesz means of the Fourier-Laplace series are integral operator with the kernel $\Theta^\alpha(x, y, n)$, we use the asymptotic behavior of this kernel in Theorem 15.

Using this estimates for the kernel $\Theta^\alpha(x, y, n)$ we can estimate the Riesz means of the spectral expansions:

$$E^\alpha_n f(x) = \int_{S^N} \Theta^\alpha(x, y, n) f(y) d\sigma(y)$$

separate into four parts as follow

$$E^\alpha_n f(x) = \int_{S^N} \Theta^\alpha(x, y, n) f(y) d\sigma(y) =$$

$$\int_{\gamma(x, y) < \frac{1}{n}} \Theta^\alpha(x, y, n) f(y) d\sigma(y) + \int_{\frac{1}{n} < \gamma(x, y) \leq \frac{\pi}{2}} \Theta^\alpha(x, y, n) f(y) d\sigma(y) +$$

$$\int_{\frac{\pi}{2} < \gamma(x, y) \leq \pi - \frac{1}{n}} \Theta^\alpha(x, y, n) f(y) d\sigma(y) + \int_{\pi - \frac{1}{n} < \gamma(x, y) \leq \pi} \Theta^\alpha(x, y, n) f(y) d\sigma(y).$$

and for estimate each part let us apply the Theorem 15

$$|E^\alpha_n f(x)| \leq C \left( n^N \int_{\gamma(x, y) < \frac{1}{n}} |f(y)| d\sigma(y) + n^{\frac{N+1}{2}} \int_{\frac{1}{n} < \gamma(x, y) \leq \frac{\pi}{2}} (\sin \gamma)^{-\frac{N+1}{2}} |f(y)| d\sigma(y) +$$

$$+ n^{\frac{N-3}{2} - \alpha} \int_{\frac{\pi}{2} < \gamma(x, y) \leq \pi} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |f(y)| d\sigma(y) + n^{-1} \int_{\pi - \frac{1}{n} < \gamma(x, y) \leq \pi} (\sin \gamma)^{-1-N} |f(y)| d\sigma(y) +$$

$$+ n^{N} \int_{\pi - \frac{1}{n} < \gamma(x, y) \leq \pi} |f(y)| d\sigma(y) + n^{\frac{N+1}{2} - \alpha} \int_{\pi/2 < \gamma(x, y) \leq \pi - \frac{1}{n}} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |f(y)| d\sigma(y) +$$

$$+ n^{\frac{N-3}{2} - \alpha} \int_{\pi/2 < \gamma(x, y) \leq \pi} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |f(y)| d\sigma(y) + n^{-1} \int_{\pi/2 < \gamma(x, y) \leq \pi - \frac{1}{n}} (\sin \gamma)^{-1-N} |f(y)| d\sigma(y) \right).$$
Let us denote by $U_n(x)$ and $V_n(x)$ the first four member and the last four member, respectively:

$$U_n(x) = n^N \int_{\gamma(x,y) < \frac{1}{n}} |f(y)|d\sigma(y) + n^{\frac{N-1}{2} - \alpha} \int_{\frac{1}{n} < \gamma(x,y) \leq \frac{\pi}{2}} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |f(y)|d\sigma(y) +$$

$$+ n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n} < \gamma(x,y) \leq \frac{\pi}{2}} (\sin \gamma)^{-\frac{N+3}{2} - \alpha} |f(y)|d\sigma(y) + n^{-1} \int_{\frac{1}{n} < \gamma(x,y) \leq \frac{\pi}{2}} (\sin \gamma)^{-1-N} |f(y)|d\sigma(y)$$

and

$$V_n(x) = n^N \int_{\pi/2 < \gamma(x,y) \leq \pi} |f(y)|d\sigma(y) + n^{\frac{N-1}{2} - \alpha} \int_{\frac{\pi}{2} < \gamma(x,y) \leq \pi - \frac{1}{n}} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |f(y)|d\sigma(y) +$$

$$+ n^{\frac{N-3}{2} - \alpha} \int_{\frac{\pi}{2} < \gamma(x,y) \leq \pi - \frac{1}{n}} (\sin \gamma)^{-\frac{N+3}{2} - \alpha} |f(y)|d\sigma(y) + n^{-1} \int_{\frac{\pi}{2} < \gamma(x,y) \leq \pi - \frac{1}{n}} (\sin \gamma)^{-1-N} |f(y)|d\sigma(y)$$

It is not hard to see that $U_n(\overline{x}) = V_n(x)$, where $\overline{x}$ os opposite point to $x \in S^N$, i.e. $\gamma(x, \overline{x}) = \pi$.

If we define function $F(t)$ by

$$F(t) = \int_{\gamma(x,y) < t} |f(y)|d\sigma(y)$$

then it is easy to show that

$$F(t) \leq C \ t^N \ f^*(x)$$

Due to definition of $F(t)$ we can rewrite the expression of $U_n(x)$ as follow:

$$U_n(x) = n^N F(1/n) + n^{\frac{N-1}{2} - \alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} (\sin t)^{-\frac{N+1}{2} - \alpha} F'(t)dt +$$

$$+ n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} (\sin t)^{-\frac{N+3}{2} - \alpha} F'(t)dt + n^{-1} \int_{\frac{1}{n}}^{\frac{\pi}{2}} (\sin t)^{-1-N} F'(t)dt.$$
\[ U_n(x) \leq C f^*(x) \left( 1 + n^{\frac{N-1}{2} - \alpha} \int_{1/n}^{\pi/2} t^{-\frac{N-1}{2} - \alpha} dt + n^{-\alpha} \int_{1/n}^{\pi/2} t^{-\frac{N-1}{2} - \alpha} dt + n^{-1} \int_{1/n}^{\pi/2} \frac{dt}{t^2} \right) \]

The interior of right part of the last inequality may be computed exactly and

\[ U_n(x) \leq \frac{C_1}{\alpha - \frac{N-1}{2}} f^*(x) \]

Analogously for \( V_n(x) \) we have

\[ V_n(x) \leq \frac{C_2}{\alpha - \frac{N-1}{2}} f^*(x) \]

So finally for Riesz means we have

So providing the inequality \( \alpha > \frac{N-1}{2} \) we have

\[ E_\alpha^* f(x) \leq \frac{C\alpha(N)}{\alpha - \frac{N-1}{2}} (f^*(x) + f^*(x)) \quad (16) \]

Theorem 3.1 is proved.

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