KAN REPLACEMENT OF SIMPLICIAL MANIFOLDS

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Abstract. We establish a functor \( \text{Kan} \) from local Kan simplicial manifolds to weak Kan simplicial manifolds. It gives a solution to the problem of extending local Lie groupoids to Lie 2-groupoids.

1. Introduction

It is a classical topic to study the correspondence between global and infinitesimal symmetries. For us, the process from global symmetries to infinitesimal ones is called differentiation, and the inverse process is called integration. A classical example of such is in the case of Lie groups and Lie algebras,

\[
\text{Lie algebras} \xrightarrow{\text{differentiation}} \text{Lie groups} \quad \xleftarrow{\text{integration}} \quad \text{Lie algebras}
\]

However, when our symmetries become more complicated, such as \( L_\infty \)-algebras, or even \( L_\infty \)-algebroids, the integration and differentiation both become harder. The following problems have been solved for these higher symmetries: integration of nilpotent \( L_\infty \)-algebras by Getzler [5], integration of general \( L_\infty \)-algebras by Henriques [6], differentiation of \( L_\infty \)-groupoids by Ševera [13], both directions for Lie 1-algebroids by Cattaneo-Felder [2], Crainic-Fernandes [3], and from a higher viewpoint by Tseng-Zhu [11]. Here the author wants to emphasis a middle step of local symmetries missing in the above correspondence,

\[
\text{Lie algebras} \xrightarrow{\text{local integration}} \text{local Lie groups} \xleftarrow{\text{extension? restriction}} \text{Lie groups}
\]

Indeed, to do differentiation to obtain infinitesimal symmetries, we only need local symmetries. Conversely, sometimes, it is easier to obtain a local integration, avoiding some analytic issues (for example in [5] for \( L_\infty \)-algebras). In this paper, we make our first attempts towards the extension problem from local symmetries to global ones: we construct an extension from local Kan simplicial manifolds to weak Kan ones. The classical extension of local Lie group to a topological group discussed by van Est in [12] can be viewed as 1-truncation of our result. Its 2-truncation applied to local Lie groupoids provide a solution to the integration problem of Lie algebroids to Lie 2-groupoids [15]. Notice that unlike Lie algebras which one-to-one correspond to simply connected Lie groups, Lie algebroids (integrable or not) one-to-one correspond to a sort of Lie 2-groupoids with some étale property.

We use the viewpoint of Kan simplicial manifolds to describe arbitrary Lie \( n \)-groupoids.

Date: September 3, 2009.

Supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.
Recall that a simplicial manifold $X$ consists of manifolds $X_n$ and structure maps
\[ d^i_n : X_n \to X_{n-1} \] (face maps) \[ s^i_n : X_n \to X_{n+1} \] (degeneracy maps), for $i \in \{0, 1, 2, \ldots, n\}$ that satisfy suitable coherence conditions (see for example [4]). The first two examples of simplicial manifolds (actually, they are simplicial sets with discrete topology) are the simplicial $m$-simplex $\Delta[m]$ and the horn $\Lambda[m, j]$ with
\[
(\Delta[m])_n = \{ f : (0, 1, \ldots, n) \to (0, 1, \ldots, m) | f(i) \leq f(j), \forall i \leq j \},
\]
\[
(\Lambda[m, j])_n = \{ f \in (\Delta[m])_n | \{0, \ldots, j-1, j+1, \ldots, m\} \not\subseteq \{f(0), \ldots, f(n)\} \}.
\]
The horn $\Lambda[m, j]$ should be thought as a simplicial set obtained from $\Delta[m]$ by taking away its unique non-degenerate $m$-simplex as well as the $j$-th of its $m+1$ non-degenerate $(m-1)$-simplices.

\[
\Lambda[1,1] \Lambda[1,0] \Lambda[2,2] \Lambda[2,1] \Lambda[2,0] \Lambda[3,3] \Lambda[3,2] \ldots
\]

Our convention for arrows is that they are oriented from bigger numbers to smaller numbers.

Let us recall that in homotopy theory, Kan conditions say that the natural restriction map
\[ X_m = \text{hom}(\Delta[m], X) \to \text{hom}(\Lambda[m, j], X). \]
is surjective, i.e. any horn can be filled up by a simplex. They correspond to the possibility of composing and inverting various morphisms, in the language of groupoids.

With enrichment in differential geometry, Kan conditions are
\[
\text{Kan}(m, j) : \text{3} \text{ is a surjective submersion, } \quad \text{Kan!}(m, j) : \text{3} \text{ is a diffeomorphism.}
\]

But since $\text{hom}(\Lambda[m, j], X)$ is formed by taking a numerous fibre products of the $X_i$’s, it may not be a manifold. However if $\text{\text{3}}$ is a submersion for all $0 \leq j \leq m \leq m_0$, as shown in [6, Lemma 2.4], $\text{hom}(\Lambda[m_0, j], X)$ is a manifold for all $0 \leq j \leq m_0$. Hence we are allowed to define,

**Definition 1.1.** A Lie $n$-groupoid $X$ ($n \in \mathbb{N} \cup \infty$) is a simplicial manifold that satisfies $\text{Kan}(m, j), \forall m \geq 1, 0 \leq j \leq m$, and $\text{Kan!}(m, j), \forall m > n, 0 \leq j \leq m$. When $n = \infty$, a Lie $\infty$-groupoid is also called a Kan simplicial manifold.

Then Lie 1-groupoid is simply the nerve of a Lie groupoid.

To describe local Lie groupoids, we need **local Kan conditions**:
\[
\text{Kan}^l(m, j) : \text{3} \text{ is a submersion, } \quad \text{Kan!}^l(m, j) : \text{3} \text{ is injective étale.}
\]

**Definition 1.2.** A local Lie $n$-groupoid $X$ ($n \in \mathbb{N} \cup \infty$) is a simplicial manifold that satisfies $\text{Kan}^l(m, j), \forall m \geq 1, 0 \leq j \leq m$, and $\text{Kan!}^l(m, j), \forall m > n, 0 \leq j \leq m$. When $n = \infty$, a local Lie $\infty$-groupoid is also called a local Kan simplicial manifold.

Then a local Lie 1-groupoid $X$ is the nerve of a local Lie groupoid.

As soon as we have done this, it becomes clear that to associate a Kan object $\text{Kan}(X)$ to a local Kan simplicial manifold $X$, we need to do some sort of fibrant replacement in
the category of simplicial manifolds. However, simplicial manifolds do not form a model
category and we need to do it by hand. In fact, the differential category is rather special,
even the construction for simplicial presheaves cannot be used directly here. It turns out
that the object \( \text{Kan}(X) \) constructed directly by Quillen’s small object argument is not a
Kan simplicial manifold, however it is a simplicial manifold and is Kan as a simplicial set.
We also prove certain representibility conditions for \( \text{Kan}(X) \) and make it into a weak Kan
simplicial manifold (see Section 2), which is slightly weaker than a Kan one. On the other
hand, the defects of \( \text{Kan}(X) \) lie only on high levels, that is, if we perform a 2-truncation
\( \tau_2(\text{Kan}(X)) \), and the 2-truncation is still representable, then \( \tau_2(\text{Kan}(X)) \) is indeed a Lie
2-groupoid.

2. Definition

Now we try to define a functor \( \text{Kan} \) sending invertible local Kan manifold to Kan sim-
plicial manifolds by modifying directly Quillen’s small object argument. We will see that
it is not successful, however we arrive at a simplicial manifold satisfying conditions slightly
weaker than Kan. Let

\[
J := \{ \Lambda[k,j] \to \Delta[k] : 0 \leq j \leq k \geq 3, \} \cup \{ \Lambda[2,1] \to \Delta[2] \},
\]

be a subset of inclusions with respect to which Kan condition have the right lifting property.
Given a local Kan manifold \( X \), we then construct a series of simplicial manifolds

\[
X = X^0 \to X^1 \to X^2 \to \cdots \to X^\beta \to \ldots
\]

by an inductive push-out:

\[
\bigsqcup_{(\Lambda[k,j] \to \Delta[k]) \in J} \Lambda[k,j] \times \text{hom}(\Lambda[k,j], X^\beta) \to X^\beta
\]

Then we let \( \text{Kan}(X) = \text{colim}_{\beta \in \mathbb{N}} X^\beta \).

Now we make some calculation for first several steps of Kan replacement: First of all
\( X_0 = X^0_0 = X^2_0 = \cdots = \text{Kan}(X)_0 \), and

\[
X^1_1 = X_1 \cup (X_1 \times_{X_0} X_1)
\]

\[
X^2_1 = X^1_1 \cup X^1_1 \times_{X_0} X^1_1
\]

\[
= X^1_1 \cup (X_1 \times_{X_0} X_1 \cup X_1 \times_{X_0} (X_1 \times_{X_0} X_1))
\]

\[
\cup (X_1 \times_{X_0} X_1 \times_{X_0} X_1 \cup (X_1 \times_{X_0} X_1) \times_{X_0} (X_1 \times_{X_0} X_1))
\]

\[
\vdots
\]

\[
\text{Kan}(X)_1 = X_1 \cup (X_1 \times_{X_0} X_1) \cup (X^1_1 \times_{X_0} X^1_1) \cup (X^2_1 \times_{X_0} X^2_1) \cdots
\]

which we can represent them by the following picture:

\[
\text{Kan}(X)_1:
\]
A calculation shows that
\[ X_2^1 = X_2 \cup X_1 \times X_0 \cup X_1 \times X_0 \times X_1 \cup \bigcup_{j=0}^3 \mathrm{hom}(\Lambda[3, j], X) \]
\[ X_2^2 = X_2^1 \cup X_1^1 \times X_0 \cup X_1^1 \times X_0 \times X_1 \cup \bigcup_{j=0}^3 \mathrm{hom}(\Lambda[3, j], X^1) \]
\[ \vdots \]
Inside \( X_2^1 \), there are three copies of \( X_1 \times X_0 \times X_1 \). The first is an artificial filling of the horn \( X_1 \times X_0 \times X_1 \), and the second two are images of degeneracies of \( X_1 \times X_0 \times X_1 \) in \( X_1^1 \). The same for \( X_2^2 \), etc. We represent an element in \( X_2^1 \) as

\[ X_2^1 : \triangle, \quad X_1 \times X_0 \times X_1 : \cdots, \quad \mathrm{hom}(\Lambda(3, j), X_1) : \triangle \cdots 4 \text{ such} \]

plus those degenerate ones in the other two copies of \( X_1 \times X_0 \times X_1 \). Furthermore we represent an element in \( X_2^2 \) as

\[ X_2^1 \text{: described as above} \]
\[ X_1^1 \times X_0 \times X_1^1 : \cdots \]

\[ \mathrm{hom}(\Lambda[3, j], X^1) : \cdots \]

We now show that
\[ X_2^1 \to \mathrm{hom}(\Lambda[2, 0], X^1) = X_1^1 \times_{d_2, X_0, d_1} X_1^1 \]
is actually not a submersion.

We first need some technical preparation. A simplicial set \( S \) is \textit{collapsible} if it admits a filtration
\[ \text{pt} = S_0 \subset S_1 \subset \cdots \subset S_k = S \]
such that each \( S_i \) is obtained from the previous one by filling a horn, namely such that \( S_i \) can be written as \( S_i = S_{i-1} \cup \Delta[n_i] \) for some injective map \( \Lambda[n_i, l_i] \hookrightarrow S_{i-1} \). Thus we have an order for collapsible simplicial sets: we say \( S \) is not bigger than \( T \), denoted as \( S \prec T \), if \( T = S_t \) and \( S = S_s \) with \( s \leq t \) in (11). For us the notation \( S \prec T \) also indicates the inclusion map \( S \to T \). We also define the dimension of a collapsible simplicial set \( S \) as
\[ \dim S = \max_k \{ \Delta[k] \prec S \} \].

**Lemma 2.1.** Given \( S \prec T \) and a local Kan simplicial manifold \( X \), \( \mathrm{hom}(T, X) \) and \( \mathrm{hom}(S, X) \) are both manifolds, and the natural map
\[ \mathrm{hom}(T, X) \to \mathrm{hom}(S, X), \]
is always a submersion.
This is proven in [] Lemma 2.4 for Kan simplicial manifolds, but it is easy to see that it works also for local Kan ones since only the submersion condition is used.

Back to the map [], $X_1$ has several components, and the horn projection map induces on each component the following maps:

\[ X_2 \to \text{hom}(\Lambda[2,0], X), \quad \text{hom}(\Lambda[3,j], X) \to \text{hom}(\Lambda[2,0], X) \]
\[ X_1 \times_{X_0} X_1 \to X_1 \times_{X_0} (X_1 \times_{X_0} X_1), \quad X_1 \times_{X_0} X_1 \to (X_1 \times_{X_0} X_1) \times_{X_0} (X_1 \times_{X_0} X_1), \]
\[ X_1 \times_{X_0} X_1 \to X_1 \times_{X_0} X_1. \]

The morphisms in (12) are always submersions by Lemma 2.1. However the morphisms in (13) are not submersions. This implies that \text{hom}(\Delta[2], \text{Kan}(X)) \to \text{hom}(\Delta[2,0], \text{Kan}(X)) will not be a submersion. Hence \text{Kan}(X) will not be a Kan simplicial manifold, but we will prove that it satisfies

(A) \text{Kan}(X) is a simplicial manifold;
(B) moreover, \text{hom}(S, \text{Kan}(X)) is a manifold for any collapsible $S$;
(C) the natural map $\text{Kan}(X)_n \to \text{hom}(\Lambda[n,l], \text{Kan}(X))$ is surjective for all $n$ and $l$ with $0 \leq l \leq n$.

Simplicial manifolds satisfying such conditions are called weak Kan simplicial manifolds. These weak Kan simplicial manifolds as simplicial sets are indeed Kan. The submersion condition in Kan condition is replaced by condition (B). By Lemma 2.1, we can see that the submersion condition implies (B), hence weak Kan is indeed weaker than Kan. However, the usual place to use submersion condition is to guarantee some representibility for example the one in condition (B). Hence we see that in many cases we can bypass the difficulty thanks to this condition. For example, we can still talk about hypercovers of these weak Kan simplicial manifolds (even though it is not used in this paper).

Before attacking the problem, we first prepare a technical lemma:

**Lemma 2.2.** Suppose that $X^{\beta}$’s are a sequence of simplicial sets constructed by (3),

(1) if $S$ is a collapsible simplicial set, then we can decompose

\[ \text{hom}(S, X^{\beta+1}) = \bigsqcup_{a \in A} \text{hom}(S_a, X^\beta), \]

with a finite set of collapsible simplicial sets \( \{S_a : a \in A\} \ni S \) satisfying \( \dim S_a \leq \dim S \).

(2) this decomposition respects morphisms, that is if $S < T$, and both of them have a decomposition,

\[ \text{hom}(S, X^{\beta+1}) = \bigsqcup_{a \in A} \text{hom}(S_a, X^\beta), \quad \text{hom}(T, X^{\beta+1}) = \bigsqcup_{a' \in A'} \text{hom}(T_{a'}, X^\beta), \]

then there is a map $a : A' \to A$, and morphisms of simplicial sets $S_{a(a')} \to T_{a'}$ such that the natural morphism $\text{hom}(T, X^{\beta+1}) \to \text{hom}(S, X^{\beta+1})$ is induced from $\text{hom}(T_{a'}, X^\beta) \to \text{hom}(S_{a(a')}, X^\beta)$ on the level of their decompositions.

**Proof.** Since the procedure to form $X^{\beta+1}$ by $X^\beta$ is the same as the one to form $X^1$ by $X$, we only have to prove the two statements for $\beta = 0$. Since $X$ is arbitrary, the decomposition in (1) is clearly unique. We use an induction on the size of $S$ and $T$. The initial assumption is verified in the calculation we did earlier in this section. It is clear that (1) holds for $\Delta[m]$ for $m \in \mathbb{N}$. Now with a fixed $n$, we consider a horn filling diagram as we mentioned in the...
process of (11)  

\[
\begin{array}{ccc}
S & \longrightarrow & T \\
\Lambda[k,j] & \longrightarrow & \Delta[k]
\end{array}
\]

with \( \dim S \leq \dim T \leq n - 1 \), and \( k \leq n - 1 \).

We suppose that

(i) statement (1) is true for all \( S' \) with \( S' \preceq S \);

(ii) statement (2) is true for \( S' \preceq T' \) and \( S' \preceq \Delta[n] \) when \( \dim S' \leq \dim T' \leq n - 1 \) and when (11) verifies for \( S' \) and \( T' \).

To finish the induction, we will prove that

- statement (1) holds for \( T \);
- statement (2) holds for \( S \preceq T \), and \( T \preceq \Delta[n] \) if such a map \( T \rightarrow \Delta[n] \) exists.

First of all, we apply \( \text{hom}(\cdot, X^1) \) to (15) and apply the induction hypothesis to \( \text{hom}(S, X^1) \), \( \text{hom}(\Lambda[k,j], X^1) \) and \( \text{hom}(\Delta[k], X^1) \), then we have

\[
\text{hom}(T, X^1) = \text{hom}(S, X^1) \times_{\text{hom}(\Lambda[k,j], X^1)} \text{hom}(\Delta[k], X^1)
\]

\[
= \bigsqcup_{a' \in A'} \text{hom}(S'_{a'}, X) \times_{\bigsqcup_{a \in A} \text{hom}(S_a, X)} \bigsqcup_{a'' \in A''} \text{hom}(S''_{a''}, X)
\]

\[
= \bigsqcup_{b \in B} \text{hom}(T_b, X).
\]

Here \( T_b \) is formed when \( a(a') = a(a'') \) by

\[
\begin{array}{ccc}
S'_a & \longrightarrow & T_b \\
S_{a(a')} & \longrightarrow & S''_{a''}
\end{array}
\]

We obtain a map \( B \rightarrow A' \) defined by \( b \mapsto a' \) and morphisms \( S'_a \rightarrow T_b \). They induce the morphisms \( \text{hom}(T_b, X) \rightarrow \text{hom}(S'_{a'}, X) \), hence the morphism \( \text{hom}(T, X^1) \rightarrow \text{hom}(S, X^1) \).

It’s not hard to see that \( T \in \{ T_b \} \) by induction hypothesis and (15).

Suppose \( \text{hom}(\Delta[n], X^1) = \bigsqcup_{c \in C} (D_c, X) \). If there is a map \( T \preceq \Delta[n] \), by restriction, we obtain maps \( S \preceq \Delta[n] \), \( \Lambda[k,j] \preceq \Delta[n] \), and \( \Delta[k] \preceq \Delta[n] \) which fit in the following commutative diagram:

\[
\begin{array}{ccc}
S & \longrightarrow & \Delta[n] \\
\Lambda[k,j] & \longrightarrow & \Delta[k]
\end{array}
\]

By induction hypothesis, we have

- the morphism \( \text{hom}(\Delta[n], X^1) \rightarrow \text{hom}(S, X^1) \) is induced by a map \( a' : C \rightarrow A' \) and morphisms \( S'_{a'(c)} \rightarrow D_c \);

- the morphism \( \text{hom}(\Lambda[k,j], X^1) \rightarrow \text{hom}(\Lambda[k,j], X^1) \) is induced by a map \( a : C \rightarrow A \) and morphisms \( S_{a(c)} \rightarrow D_c \);
• the morphism \( \text{hom}(\Delta[n], X^1) \to \text{hom}(\Delta[k], X^1) \) is induced by a map \( a'' : C \to A'' \) and morphisms \( S''_{a''(c)} \to D_c \).

We see that \( \text{hom}(D_c, X) \to \text{hom}(S_{a(c)}, X) \) induces \( \text{hom}(\Delta[n], X^1) \to \text{hom}(\Lambda[k,j], X^1) \), and the composed morphism \( \text{hom}(D_c, X) \to \text{hom}(S'_{a'(c)}, X) \) induces \( \text{hom}(\Delta[n], X^1) \to \text{hom}(S, X^1) \to \text{hom}(\Lambda[k,j], X^1) \), which is the same morphism as \( \text{hom}(\Delta[n], X^1) \to \text{hom}(\Lambda[k,j], X^1) \). Hence by uniqueness of the decomposition, we have \( a(a'(c)) = a(c) \) and similarly \( a(c) = a(a''(c)) \), and a commutative diagram

\[
\begin{array}{ccc}
S'_{a'(c)} & \longrightarrow & D_c \\
\downarrow & & \downarrow \\
S_{a(a')} & \longrightarrow & S''_{a''(c)}
\end{array}
\]

Then \( T_{b(c)} \) defined by the pushout diagram

\[
\begin{array}{ccc}
S''_{a''(c)} & \longrightarrow & T_{b(c)} \\
\downarrow & & \downarrow \\
S_{a(c)} & \longrightarrow & S'_{a(a'(c))}
\end{array}
\]

has a canonical map \( T_{b(c)} \to D_c \). By the property of \( \text{hom}(T, X^1) \) being the fibre product, these canonical maps induce the map \( \text{hom}(\Delta[n], X^1) \to \text{hom}(T, X^1) \) via the maps \( \text{hom}(D_c, X) \to \text{hom}(T_{b(c)}, X) \).

**Proposition-Definition 2.3.** The operation \( \text{Kan} \) constructed in (6) is a functor from the category of local Kan manifolds \( X \) to the one of weak Kan simplicial manifolds.

**Proof.** The construction of \( \text{Kan} \) makes it clear that it is functorial. Since \( \text{Kan}(X) = \text{colim}_\beta X^\beta \), given any finite simplicial set \( A \) (a collapsible simplicial set \( S \) is such), the natural map of sets is an isomorphism,

\[
\text{colim}_\beta \text{hom}(A, X^\beta) \cong \text{hom}(A, \text{Kan}(X)).
\]

Moreover by Lemma 2.2,

\[
\text{hom}(S, X^{\beta+1}) = \text{hom}(S, X^\beta) \bigsqcup (\sqcup \text{hom}(S_a, X^\beta)),
\]

We then use Lemma 2.2 recursively, and obtain that for any collapsible simplicial set \( T \),

\[
\text{hom}(T, X^\beta) = \sqcup \text{hom}(T_p, X),
\]

for a finite set of collapsible simplicial sets \( T_p \). Hence \( \text{hom}(S, X^\beta) \) and \( \text{hom}(S_a, X^\beta) \) are manifolds because \( X \) is local Kan. By (17) and (16), \( \text{hom}(S, \text{Kan}(X)) \) is a disjoint union of manifolds.
So it remains to show that \(Kan(X)\) is Kan as a simplicial set. We take an element \(A \to B\) of \(J\) and a solid arrow diagram,

\[
\begin{array}{ccc}
A & \to & Kan(X) \\
\downarrow & & \downarrow \\
B & \to & pt
\end{array}
\]

then we must show that the dotted arrow exists. By the isomorphism (16), the map \(A \to Kan(X)\) factors through \(X^\beta \to Kan(X)\) for some \(\beta\) and we have the solid arrow diagram

\[
\begin{array}{ccc}
A & \to & X^\beta & \to & X^{\beta+1} & \to & Kan(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \to & pt & \to & pt & \to & pt
\end{array}
\]

Since \(X^{\beta+1}\) is constructed as the push-out in (6), the dotted arrow naturally exists, and this dotted arrow defines the one in (18).

Now we only have to verify that the dotted arrow in (18) exists for \(\Lambda[1,j] \to \Delta[1]\) for \(j = 0, 1\) and \(\Lambda[2,j] \to \Delta[2]\) for \(j = 0, 2\). We have \(X_0^\beta = X_0\), and

\[
\text{hom}(\Lambda[1,j], X^{\beta+1}) = X_0^{\beta+1} = X_0, \quad \text{hom}(\Delta[1], X^{\beta+1}) = X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta,
\]

thus the map \(\text{hom}(\Delta[1], X^{\beta+1}) \to \text{hom}(\Lambda[1,j], X^{\beta+1})\) being the pull-back of \(d_1\) or \(d_0\), has to be a surjective submersion. Now we prove that if \(X^\beta\) is invertible, then \(X^{\beta+1}\) is also invertible.

\[
\text{hom}(\Lambda[2,2], X^{\beta+1})^\beta = X_1^{\beta+1} \times_{d_1,X_0,d_1} X_1^{\beta+1}
\]

\[
= (X_1^\beta \sqcup X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta) \times_{d_1,X_0,d_1} (X_1^\beta \sqcup X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta)
\]

\[
= X_1^\beta \times_{d_1,X_0,d_1} X_1^\beta \times_{d_1,X_0,d_1} (X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta) \sqcup (X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta) \times_{d_1,X_0,d_1} X_1^\beta \sqcup \ldots
\]

Since \(X^\beta\) is invertible, \(X_1^\beta \times_{d_1,X_0,d_1} X_1^\beta \cong X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta\). Hence

\[
X_1^\beta \times_{d_1,X_0,d_1} (X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta) \cong X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta,
\]

\[
(X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta) \times_{d_1,X_0,d_1} X_1^\beta \cong X_1^\beta \times_{d_1,X_0,d_1} X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta \cong (X_1^\beta \times_{d_0,X_0,d_1} X_1^\beta) \times_{d_0,X_0,d_1} X_1^\beta,
\]

\[
\ldots
\]

It is easy to continue to verify that \(X^{\beta+1}\) is invertible. Then the final result follows from (16).

Given an invertible local Kan manifold \(X\), we call \(Kan(X)\) the **Kan replacement** of \(X\).

Even through \(Kan(X)\) is not Kan, its 2-truncation \(\tau_2(Kan(X))\) behaves well. We define \(n\)-truncation \(\tau_n\) (it is called \(\tau_{\leq n}\) in [6 Section 3]), of a simplicial manifold \(X\) as,

\[
\tau_n(X)_k = X_k, \forall k \leq n - 1, \quad \tau_n(X)_k = X_k / \sim_k, \forall k \geq n,
\]
where two elements $x \sim_k y$ in $X_k$ if they are homotopic and have the same $n$-skeleton. Since in the procedure, taking a quotient is involved, the result $\tau_n(X)$ might not be a simplicial manifold anymore. We view it as a simplicial stack. When $X$ is Kan, $\tau_n(X)$ viewed as a simplicial set is always a discrete $n$-groupoid. It is representable, namely it is indeed a simplicial manifold, if and only if the quotient $X_n/\sim_n$ is representable because the higher levels are decided by $X_n/\sim_n$. Even though $\text{Kan}(X)$ is not a Kan manifold, we still have

**Proposition 2.4.** When $\text{Kan}(X)_2/\sim_2$ is representable, $\tau_2(\text{Kan}(X))$ is a Lie 2-groupoid.

**Proof.** As a simplicial set, $\text{Kan}(X)$ is Kan. Hence $\text{hom}(\Delta[n], \tau_2(\text{Kan}(X))) \cong \text{hom}(\Lambda[n, j], \tau_2(\text{Kan}(X)))$, for $n \geq 3$. Especially, $\tau_2(\text{Kan}(X))_3 \cong \text{hom}(\Lambda[3, 0], \tau_2(\text{Kan}(X)))$. Since the higher layers are determined by the first four layers,

$$\tau_2(\text{Kan}(X)) = \text{Cosk}^3 \circ \text{Sk}^3(\tau_2(\text{Kan}(X))),$$

by the same argument in [10] Section 2.3, to show $\tau_2(\text{Kan}(X))$ is a Lie 2-groupoid, we only need to show that $\text{hom}(\Lambda[3, 0], \tau_2(\text{Kan}(X)))$ is representable and $\text{Kan}(m \leq 2, j)$ for $\tau_2(\text{Kan}(X))$. In fact the induction argument there already shows that the representability of $\text{hom}(\Lambda[3, 0], \tau_2(\text{Kan}(X)))$ is implied by $\text{Kan}(m \leq 2, j)$ for $\tau_2(\text{Kan}(X))$ given $\tau_2(\text{Kan}(X))_2 = \text{Kan}(X)_2/\sim_2$ is representable. Hence we only need to show $\text{Kan}(m \leq 2, j)$.

As shown in Def.-Prop. 2.3, $\text{hom}(\Delta[1], X^{\beta+1}) \to \text{hom}(\Lambda[1, j], X^{\beta+1})$ being the pull-back of $d_1$ or $d_0$, is a surjective submersion, hence $\text{hom}(\Delta[1], \text{Kan}(X)) \to \text{hom}(\Lambda[1, j], \text{Kan}(X))$ is a surjective submersion. This is $\text{Kan}(1, j)$ for $\text{Kan}(X)$, hence for $\tau_2(\text{Kan}(X))$.

The surjective part in $\text{Kan}(2, j)$ is automatically satisfied: since $\text{Kan}(X)$ is Kan as a simplicial set, the composed map

$$\text{Kan}(X)_2 \to \tau_2(\text{Kan}(X)) \xrightarrow{p} \text{hom}(\Lambda[2, j], \text{Kan}(X)) = \text{hom}(\Lambda[2, j], \tau_2(\text{Kan}(X))),$$

is surjective, hence the desired map $p$ is also surjective. We only need to show the submersion part. Then what happened to the degenerate faces where the horn projection map is not a submersion for $\text{Kan}(X)_2$? An element $\eta \in \text{Kan}(X)_2$ can be described as a tree as stated in Lemma 3.3 If all the vertices of the tree are triangles in $X_2$, then the horn projection map is a submersion for $\text{Kan}(X)_2$, hence $p$ is a submersion. The problem happens exactly when the tree contains at least one vertex coming from one of the three copies of $X_1 \times X_0 X_1$. But these bad pieces as in [13] and [14] are all homotopic via elements in $\text{Kan}(X)_3$ to the boundary of good pieces as in [12], where the submersion holds. Hence the submersion part is also true for the 2-truncation. 

$$\square$$

3. Universal Properties

Given a local Lie 1-groupoid $W$ (or the nerve of a local Lie groupoid), then it extends to a Lie 2-groupoid $\tau_2(\text{Kan}(W))$. In [15], we verified that $\tau_2(\text{Kan}(W))$ is always a Lie 2-groupoid (even though $\tau_1(\text{Kan}(W))$ might not be Lie) with universal property.

For this purpose, we need to show some universal properties of our Kan replacement. It should be stable under Morita equivalence of simplicial manifolds (whatever that is), and if some simplicial manifold $X$ is already Kan, $\text{Kan}(X)$ should be Morita equivalent to $X$. Hence let’s first begin with an introduction of these concepts such as Morita equivalence.

1This means that $d_i x = d_i y$, 0 ≤ $i$ ≤ $k$, and there exists $z \in X_{k+1}$ such that $d_k(z) = x$, $d_{k+1}(z) = y$, and $d_i z = s_{k-i}d_i x = s_{k-i}d_i y$, 0 ≤ $i$ < $k$. 

---

\[ \text{Kan Replacement of Simplicial Manifolds} \] 9
3.1. Morita equivalence of local Kan manifolds. The reader’s first guess is probably that a morphism \( f : X \to Y \) of simplicial manifolds ought to be a simplicial smooth map i.e. a collection of smooth maps \( f_n : X_n \to Y_n \) that commute with faces and degeneracies. We shall call such a morphism a strict map from \( X \) to \( Y \). Unfortunately, it is known that, already in the case of usual Lie groupoids, such strict notions are not good enough. Indeed there are strict maps that are not invertible even though they ought to be isomorphisms. That’s why people introduced the notion of Hilsum-Skandalis bimodules \([9]\). Here is an example of such a situation: consider a manifold \( M \) with an open cover \( \{ U_{\alpha} \} \). The simplicial manifold \( X \) with \( X_n = \bigsqcup_{\alpha_1, \ldots, \alpha_n} U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \) maps naturally to the constant simplicial manifold \( M \). All the fibers of that map are simplices, in particular they are contractible simplicial sets. Nevertheless, that map has no inverse.

The second guess is then to define a special class of strict maps which we shall call hypercovers. A map from \( X \) to \( Y \) would then be a zig-zag of strict maps \( X \sim Z \to Y \), where the map \( Z \to X \) is one of these hypercovers.

Another alternative however equivalent way to define a generalized morphism of simplicial manifolds follows from \([7, \text{Section 2.4}]\)’s Cartesian fibrations. In this paper, we use the zig-zag method with the notion of hypercover.

Our hypercover is very much inspired from the notion of hyper cover of étale simplicial objects \([1, 4]\) and of trivial fibration of Quillen for simplicial sets \([10]\).

Recall \([8, \text{Section I.3}]\), given a pointed Kan simplicial set \( X \), i.e. \( X_0 = \text{pt} \), its homotopy groups are given by
\[
\pi_n(X) := \{ x \in X_n | d_i(x) = \text{pt} \text{ for all } i \}/ \sim
\]
where \( x \sim x' \) if there exists an element \( y \in X_{n+1} \) such that \( d_0(y) = x \), \( d_1(y) = x' \), and \( d_i(y) = \text{pt} \) for all \( i > 1 \). When \( X_0 \) is not necessarily a point, \( \pi_n \) is a sheaf over \( X_0 \) in general.

**Lemma 3.1.** Given a map \( S \to T \) of pointed Kan simplicial sets, if for any \( n \geq 0 \) and any commutative solid arrow diagram
\[
\begin{array}{ccc}
\partial \Delta[n] & \longrightarrow & S \\
\downarrow & & \downarrow \\
\Delta[n] & \longrightarrow & T
\end{array}
\]
there exists a dotted arrow that makes both triangles commute, then this map is a homotopy equivalence, i.e. \( \pi_n(S) = \pi_n(T) \). Here \( \partial \Delta[n] \) stands for the boundary of the n-simplex.

The proof is standard.

Translating the condition of Lemma 3.1 into hom spaces gives:

**Definition 3.2.** A strict map \( f : Z \to X \) of local Kan simplicial manifolds is a hypercover if the natural map
\[
Z_m = \text{hom}(\Delta[m], Z) \to \text{hom}(\partial \Delta[m] \to \Delta[m], Z \to X)
\]
is a surjective submersion for all \( 0 \leq m \).

Here \( \text{hom}(A \to B, Z \to X) \) denotes the pull-back spaces of the form \( \text{hom}(A, Z) \times_{\text{hom}(A, X)} \text{hom}(B, X) \), where the maps are induced by some fixed maps \( A \to B \) and \( Z \to X \). This notation indicates that the space parameterizes all commuting diagrams of the form
\[
\begin{array}{ccc}
A & \longrightarrow & Z \\
\downarrow & & \downarrow \\
B & \longrightarrow & X
\end{array}
\]
where we allow the horizontal arrows to vary but we fix the vertical ones.

Similarly, we can define hypercover for Lie $n$-groupoids:

**Definition 3.3.** A strict map $f : Z \to X$ of Lie $n$-groupoids is a hypercover if the natural map $(20)$ is a surjective submersion for all $0 \leq m < n$ and is an isomorphism when $m = n$.

**Remark 3.4.** As proved in [14], if $f : Z \to X$ is a hypercover of Lie $n$-groupoids, then $(20)$ is automatically an isomorphism for all $m > n$.

As in the case of Definition 1.1, we need to justify that the pull-back $\text{hom}(\partial \Delta[m] \to \Delta[m], Z \to X)$ is a manifold. This is rather surprising since the spaces $\text{hom}(\partial \Delta[m], Z)$ need not be manifolds (for example take $m = 2$ and $Z$ the cross product groupoid associated to the action of $S^1$ on $\mathbb{R}^2$ by rotation around the origin). We justified this in [14] for Kan simplicial manifolds, but it is clear that only the submersion property is needed, hence the same proof works for local Kan manifolds.

**Definition 3.5.** Two local Kan simplicial manifolds $X$ and $Y$ are Morita equivalent if there is another local Kan simplicial manifold $Z$ such that both of the maps $X \sim Z \sim Y$ are hypercovers. In [14, Section 2], we show that this definition does give an equivalence relation. We call it Morita equivalence of local Kan simplicial manifolds.

We also define Morita equivalence of Lie $n$-groupoids exactly in the same fashion using hypercover of Lie $n$-groupoids.

Hypercover of Lie $n$-groupoids may also be understood as a higher analogue of pull-back of Lie groupoids. Let $X$ be a 2-groupoid and $Z_1 \Rightarrow Z_0$ be two manifolds with structure maps as in (11) up to the level $n \leq 1$, and $f_n : Z_n \to X_n$ preserving the structure maps $d^0_n$'s and $s^{n-1}_k$'s for $n \leq 1$. Then $\text{hom}(\partial \Delta[n], Z)$ still makes sense for $n \leq 1$. We further suppose that $f_0 : Z_0 \to X_0$ (hence $Z_0 \times Z_0 \times X_0 \times X_0 X_1$ is a manifold) and $Z_1 \to Z_0 \times Z_0 \times X_0 \times X_0 X_1$ are surjective submersions. That is to say that the induced map from $Z_1$ to the pull-back $\text{hom}(\partial \Delta[k], Z) \times \text{hom}(\partial \Delta[k], X) X_k$ are surjective submersions for $k = 0, 1$. Then we form

$$Z_2 = \text{hom}(\partial \Delta[2], Z) \times \text{hom}(\partial \Delta[2], X) X_2,$$

which is a manifold (see [14, Lemma 2.4]).

Moreover there are $d^2_2 : Z_2 \to Z_1$ induced by the natural projections $\text{hom}(\partial \Delta[2], Z) \to Z_1$;

$s^1_1 : Z_1 \to Z_2$ by

$$s^0_1(h) = (h, h, s^0_0(d^1_1(h)), s^1_0(f_1(h)));
\quad s^1_1(h) = (s^0_0(d^1_1(h)), h, h, s^1_1(f_1(h)));$$

$m_i : \text{hom}(\Lambda[3, i], Z) \to Z_2$ by for example

$$m_0((h_2, h_5, h_2, \bar{\eta}_1), (h_4, h_5, h_0, \bar{\eta}_2), (h_1, h_3, h_0, \bar{\eta}_3)) = (h_2, h_4, h_1, m_0(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)),$$

and similarly for other $m'$s.

Then $Z_2 \Rightarrow Z_1 \Rightarrow Z_0$ is a Lie 2-groupoid and we call it the pull-back 2-groupoid by $f$. Moreover $f : Z \to X$ is an equivalence with the natural projection $f_2 : Z_2 \to X_2$. 

![Diagram](https://via.placeholder.com/150)
3.2. **Lemmas.** What we wish to prove is: If \( X \) is already a Kan simplicial manifold, then \( X \xrightarrow{\sim} \text{Kan}(X) \) are Morita equivalent. It is very easy to prove for simplicial set. Since the procedure of Kan replacement is basically to fill out horns, the geometric realization of \( \text{Kan}(X) \) and \( X \) are homotopic to each other. Since \( X \) is Kan, this is equivalent to \([19]\). However, the missing tool of homotopy theory of simplicial manifolds (which do not form a model category, but building a certain machinery as a suitable replacement of model category should be the eventual correct method to prove these lemmas) prevents us to apply this proof directly. In fact, in the case of simplicial sets, one can easily obtain a morphism \( \pi : \text{Kan}(X) \to X \) such that the composition \( X \to \text{Kan}(X) \to X \) is the identity. Then it is straightforward to check that \( \text{Kan}(X) \to X \) has the correct lifting property. However, \( \pi \) is not unique (basically it depends on the choice of fillings in the Kan condition). Hence when generalized to a differential category, \( \pi \) is in general not a continuous morphism. This forces us to use another proof. Here we provide a proof for Lie 2-groupoids \( W \).

**Lemma 3.6.** If \( W \) is a Lie 2-groupoid, then \( \tau_2(\text{Kan}(W)) \) is a Lie 2-groupoid which is Morita equivalent to \( W \).

**Proof.** Usually, we do not have a direct map from \( \tau_2(\text{Kan}(W)) \) to \( W \) because there is no (unique) multiplication map \( W_1 \times_{W_0} W_1 \to W_1 \) (even when there exists such a multiplication, we will encounter the issue of surjective submersions). Hence we must construct a middle step.

A more natural way to describe this is to use the corresponding stacky groupoid \( \mathcal{G} \Rightarrow W_0 \), where \( \mathcal{G} \) is presented by the Lie groupoid \( G_1 \Rightarrow G_0 \), with \( G_0 = W_1 \) and \( G_1 \) the set of bigons in \( W_2 \), and the multiplication \( \mathcal{G} \times_{W_0} \mathcal{G} \to \mathcal{G} \) is presented by bimodule \( E_m = W_2 \). The bimodules of various compositions of multiplication from various copies of \( \mathcal{G} \) to \( \mathcal{G} \) are presented by various fibre product of \( W_2 \)'s. For example, the bimodule \( W_2 \times_{d_1, W_1, d_2} W_2 \) with the moment map \( J_l \) to \( W_1 \times_{W_0} W_1 \times_{W_0} W_1 \) and \( J_r \) to \( W_1 \), presents the multiplication

\[
m \circ (m \times \text{id}) : (\mathcal{G} \times_{W_0} \mathcal{G}) \times_{W_0} \mathcal{G} \to \mathcal{G}.
\]

To simplify the notation, we denote a \( k \)-times fibre product as \( \square \times^k \) when it’s clear from the context. We construct \( Z_0 = W_0 = W_0 \) and, \( Z_1 \) is the disjoint union of these bimodules \( W_2 \times^k \) presenting different compositions of multiplication,

\[
Z_1 = W_1 \sqcup W_2 \sqcup (W_2 \sqcup W_2 \times^2 \sqcup W_2 \times^2 \sqcup W_2 \times^3) \sqcup \ldots.
\]

It is best to be understood as the following picture:

![Diagram](image)

That is, we fill out horns in \( \text{Kan}(W)_1 \) by replacing \( W_1 \times^n \) with \( W_2 \times^{(n-1)} \). The projections of \( Z_1 \to \text{Kan}(W)_1 \) and \( Z_1 \to W_1 \) are simply the disjoint union of the left and right moment maps respectively. These projections are both surjective submersions.

To show that \( \tau_2(\text{Kan}(W)) \) is Morita equivalent to \( W \), we only have to show that the pullback 2-groupoids on \( Z \) are the same, that is

\[
(21) \quad \text{Kan}(W)_2/_{\sim} \times_{\text{hom}(\partial \Delta[2], \text{Kan}(W))} \text{hom}(\partial \Delta[2], Z) \cong W_2 \times_{\text{hom}(\partial \Delta[2], W)} \text{hom}(\partial \Delta[2], Z).
\]
If the map \( p : M \to N \) is surjective and admit local section at any point in \( N \), then the pull-back groupoid \( G_1 \times_M N \Rightarrow G_0 \times_M N \) is free and proper if and only the original groupoid \( G_1 \Rightarrow G_0 \) is so. Since this is our case, the isomorphism (21) automatically implies that \( Kan(W)/\sim_2 \) is representable. By Prop. 2.4, \( \tau_2(Kan(W)) \) is a Lie 2-groupoid.

We denote the two pullbacks by the map \( Z_1 \to W_1 \) and \( Z_1 \to Kan(W)_1 \) to \( Z_1 \) by \( W|_Z \) and \( Kan(W)|_Z \) respectively, and we construct morphisms

\[
\pi : (Kan(W)|_Z)_2 \to (W|_Z)_2, \quad \iota : (W|_Z)_2 \to (Kan(W)|_Z)_2,
\]

and prove \( \pi \circ \iota = id \) and \( \iota \circ \pi \sim id \) up to something in \( (Kan(W)|_Z)_3 \). Then the above isomorphism follows naturally. Notice that \( Kan(W) \) is not a Lie 2-groupoid usually, but pull-back described in Section 3.1 works also when \( X \) is a local Kan manifold. We form \( (X|_Z)_n = hom(\partial \Delta[n], X \to Z) \), where \( \partial \Delta[n] \) denotes of taking the 1-dimensional skeleton. By [14, Lemma 2.4], \( (X|_Z)_n \) are manifolds. Then it’s easy to check that \( \tau_2(Kan(W)|_Z) = \tau_2(Kan(W))|_Z \).

We first construct \( \iota \). Let \( S \) be a simplicial polygon with three marked points, namely a simplicial set constructed inductively

\[
\Delta[2] = S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \ldots \hookrightarrow S_i \ldots,
\]

by push-out \( S_{i+1} = S_i \cup_{\Delta[i]} \Delta[2] \) and the three marked points are the vertices of \( S_0 \). With these three marked points, the \( S_i \)'s can be viewed as generalized triangles with their three sides a concatenation of line segments. In this sense, we also have the three facial maps \( d^k \).

We have a natural embedding \( W \hookrightarrow Kan(W) \), but this embedding does not give \( W|_Z \to Kan(W)|_Z \). In fact, take an element \( (w, \partial z) \in (W|_Z)_2 = \cup_i hom(S_i, W) \) for a certain set of \( S_i \)'s, then \( (w, \partial z) \notin (Kan(W)|_Z)_2 \) since \( \partial w \) the boundary of \( w \), is not \( \partial z \) under the map \( \partial_2 Z \to \partial_2 Kan(W) \). Here \( \partial_k \square = hom(\partial \Delta[k], \square) \). To construct \( \iota \) we need to construct a morphism \( \mu_i : hom(S_i, W) \to Kan(W)_2 \) inductively, such that it commutes with the facial map \( d_k \) for \( k = 0, 1, 2 \),

\[
\begin{array}{c}
\text{hom}(S_i, W) \xrightarrow{d_k} \text{hom}(d^k S_i, W) \\
\downarrow \mu_i \\
Kan(W)_2
\end{array}
\]

Then \( \iota(w, \partial z) := (\mu_i(w), \partial z) \) where \( w \in hom(S_i, W) \).
Step 1: We first prove the case $i = 1$. We simplify the notation by $K := Kan(W)$.

\[(23) \quad \text{hom}(S_1, W) = W_2 \times_{d_k, W_1, d_1} W_2 \]

\[\text{by } W_1 \times W_0 W_1 \hookrightarrow K_2 \]

\[W_2 \times_{d_k, W_1, d_1} W_2 \times W_1 \times W_0, K_2 \]

\[\text{by } W \hookrightarrow K \]

\[\text{hom}(\Lambda[3, k'], K) \]

\[\text{Kan}(3, k') \]

\[\text{hom}(\Delta[3], K) \]

\[\xrightarrow{d_k'} K_2 \]

But this map does not commute with the facial map (see (22)). For this purpose, we only need to compose with the following one,

\[(24) \quad K_2 \to (K_2 \times_{W_1} (W_1 \times W_0 W_1)) \times_{K_1 \times W_0 W_1} W_1 \times W_0 W_1 \]

\[\text{W}_1 \times W_0 W_1 \hookrightarrow K_2 \]

\[\text{hom}(\Lambda[3, k''], K) \to K_2 \]

\[\text{good boundary} \]

\[\text{Step 2: Now suppose we have such a map } \text{hom}(S_i, W) \to K, \text{ then we can construct a map } \text{hom}(S_{i+1}, W) \to K \text{ as below,} \]

\[\text{hom}(S_{i+1}, W) = \text{hom}(S_i, W) \times_{W_i} W_2 \xrightarrow{\partial} \text{hom}(\partial S_{i+1}, W) = \text{hom}(\partial S_i, W) \times_{W_1} \partial_2 W \]

\[\text{By Lemma 3.7} \]

\[\text{hom}(S_i, W) \times_{d_k, K_1, d_1} K_2 \xrightarrow{\partial} \text{hom}(\partial S_i, W) \times_{K_1} \partial_2 K \]

\[\text{By hom}(S_i, W) \to K_2 \]

\[K_2 \times_{d_k, K_1, d_1} K_2 = \text{hom}(S_1, K) \xrightarrow{\partial} \text{hom}(\partial S_1, K) \]

\[\text{Similarly as Step 1, replace } W \text{ by } K \]

\[K_2 \]
Lemma 3.7. There is a natural morphism $K_1 \times_{W_1} W_2 \to K_2$.

We leave this to the readers as an exercise involving the Kan condition of $K$.

To construct $\pi$, we first construct a local morphism $f : Kan(W) \to W$ inductively. The first step is to construct $f_1^n$ by

$$
W_1^n = \text{hom} \left( \Delta[n], W \coprod_{\prod \Lambda[k,j] \times \text{hom}(\Lambda[k,j], W)} \Delta[k] \times \text{hom}(\Lambda[k,j], W) \right)
$$

$$
\rightarrow W_n \coprod \text{hom}(\Delta[n], \Delta[k]) \times \text{hom}(\Delta[k], W)
$$

$$
\rightarrow W_n.
$$

In the second last step we use the strict Kan condition $\text{hom}(\Lambda[k,j], W) \cong W_k$ when $k \geq 2$ and we choose a local section $\text{hom}(\Lambda[2,1], W) \to W_2$ when $k = 2$. The last step follows from the composition $\text{hom}(\Delta[n], \Delta[k]) \times \text{hom}(\Delta[k], W) \to W_n$ and thus both spaces in the coproduct have a natural map to $W_n$.

Suppose that $f_\beta : W^\beta \to W$ is constructed. Then $f^{\beta+1}$ is the composition of the following natural morphisms

$$
W_{n}^{\beta+1} = \text{hom} \left( \Delta[n], W^\beta \coprod_{\prod \Lambda[k,j] \times \text{hom}(\Lambda[k,j], W^\beta)} \Delta[k] \times \text{hom}(\Lambda[k,j], W^\beta) \right)
$$

$$
\rightarrow \text{hom} \left( \Delta[n], W \coprod_{\prod \Lambda[k,j] \times \text{hom}(\Lambda[k,j], W)} \Delta[k] \times \text{hom}(\Lambda[k,j], W) \right)
$$

$$
= W_1^n \xrightarrow{f_1^n} W_n.
$$

Then $f$ is the colimit of $f^\beta$.

More geometrically, if we view an element in $Kan(W)_2$ as a set of small triangles of $W_2$ touching together, $f_2$ is basically to compose these small triangles into a big one in $W_2$ with a choice of filling for each $W_1 \times_{W_0} W_1$, which is given by $f_1$.

Now when we make a choice of fillings for a $W_1 \times_{W_0} W_1$ on the boundary, instead of choosing some filling given by $f_1$, we choose the element $\partial z \in \text{hom}(\partial \Delta[2], Z)$, then this element in $W_2$ is denoted by $f_2(x) \circ \partial z$. 
Thus $f_2$ induces a map
\begin{equation}
Kan(W)_2 \times \hom(\partial \Delta[2], Kan(W)) \hom(\partial \Delta[2], Z) \overset{\pi}{\to} W_2 \times \hom(\partial \Delta[2], W) \hom(\partial \Delta[2], Z).
\end{equation}
as $(x, \partial z) \mapsto (f_2(x) \circ \partial z, \partial z)$. In Lemma 3.8 we give a combinatorial proof that this map does not depend on the choice of fillings. Hence we obtain a well-defined global map $\pi$.

Then it is not hard to see that $\pi \circ \iota = id$ since $\pi$ is exactly the opposite procedure of $\iota$.

The procedure to form $\pi$ and $\iota$ is basically to use $Kan! (3, j)$ to compose (for example (23)), hence $\iota \circ \pi$ and $id$ differ by something in $Kan(W)_3$. \hfill $\Box$

**Lemma 3.8.** The map $\pi$ does not depend on the choice of sections in the construction of $f$.

**Proof.** We denote an element in $\eta \in Kan(W)_2$ by a bicolored tree

A point is black if it represents a weird triangle, i.e. a triangle comes from the first copy of $X_1 \times X_0 X_1$ which serves as an artificial filling; otherwise, it is white. For the other two copies of $X_1 \times X_0 X_1$, they are degenerate ones and can not glue directly with a normal triangle in $W_2$. Since degenerate elements play the role of identities in composition $\pi$, we here ignore them. We prove the result by induction on the number of generations and the number of points in the youngest generation. It is obvious for the initial case.

Now take three siblings points in the youngest generation, if all of them are white, then we use $Kan (3, j)$ without a choice and we end up with an element $\eta' \in Kan(W)_2$ which has a fewer number of generations or a fewer number of points in the youngest generation. Done!

If one of the three siblings is black, then there is precisely one black one in these three siblings, which we denote by $x$. Since $\hom(\partial \Delta[2], Z)$ will give the fillings for the weird triangles on the border of $\eta$, to show the independence, we only have to deal with the inner triangles. Then some ancestor of $x$ must have a black descendant $y$, because a weird triangle must lie on the side of another triangle (which is the parent of $y$).

The simplest situation in this case is when the other black descendant is a (true) cousin (namely their direct ancestors are siblings)

\begin{equation}
\text{We choose a triangle $\eta_{034}$, and we are given triangle $\eta_{014}$, $\eta_{134}$, $\eta_{123}$, $\eta_{234}$, $\eta_{024}$. The procedure is to compose $\eta_{034}$, $\eta_{014}$, $\eta_{134}$ first to obtain $\eta_{013}$ by $Kan(3, 3)$; then to compose}
\end{equation}
Hence by the induction hypothesis, we will choice-independently end up with an element $f$ more remote cousin, but we can reduce them to the simple situation above: $\eta_{Kan}$ is a surjective submersion. This implies that $X$ to hom$(\Lambda[n+1, j], X) \cong X_{n+1}$ $\overset{d_j}{\rightarrow} X_n \rightarrow Y_n$ and hom$(\Delta[n+1, j], X)$ $\cong X_{n+1}$ $\overset{d_j}{\rightarrow} X_n \overset{\partial}{\rightarrow} \text{hom}(\partial \Delta[n], X)$. This gives us a map hom$(\Delta[n+1, j], X) \overset{\partial}{\rightarrow} \text{hom}(\partial \Delta[n], X) \rightarrow \Delta[n], X \rightarrow Y)$. With this map, we rewrite hom$(\partial \Delta[n+1] \rightarrow \Delta[n+1], X \rightarrow Y) \cong \text{hom}(\partial \Delta[n+1, j], X) \times_{\text{hom}(\partial \Delta[n] \rightarrow \Delta[n], X \rightarrow Y)} X_n$.

Since $X \sim Y$ as local Kan simplicial manifolds, the following map

$$X_{n+1} \cong \text{hom}(\Lambda[n+1, j], X) \rightarrow \text{hom}(\Lambda[n+1, j], X) \times_{\text{hom}(\partial \Delta[n] \rightarrow \Delta[n], X \rightarrow Y)} X_n$$

is a surjective submersion. This implies that $X_n \rightarrow \text{hom}(\partial \Delta[n] \rightarrow \Delta[n], X \rightarrow Y)$ is injective. However, $X_n \rightarrow \text{hom}(\partial \Delta[n] \rightarrow \Delta[n], X \rightarrow Y)$ is a surjective submersion by the condition of hypercovers. Hence $X_n \cong \text{hom}(\partial \Delta[n] \rightarrow \Delta[n], X \rightarrow Y)$, which shows $X \sim Y$ as Lie $n$-groupoids.

This implies
Corollary 3.10. Two Lie n-groupoids $X \rightarrow Y$ are Morita equivalent as local Kan simplicial manifolds if and only if they are Morita equivalent as Lie n-groupoids.

Lemma 3.11. If $\phi : X \rightarrow Y$ is a hypercover of local Kan simplicial manifolds, and if $\text{Kan}(X)_2/_{\sim}2$ is representable, then both $\tau_2(\text{Kan}(X))$ and $\tau_2(\text{Kan}(Y))$ are Lie 2-groupoids and the induced map $\tau_2(\text{Kan}(X)) \rightarrow \tau_2(\text{Kan}(Y))$ is a hypercover of Lie 2-groupoids.

Proof. We first show that if $\phi : K \rightarrow K'$ is a hypercover of Kan simplicial sets (i.e. (20) is surjective instead of a surjective submersion), then the natural map

$$\tau_n(K)_n \xrightarrow{f} \text{hom}(\partial \Delta[n] \rightarrow \Delta[n], \tau_n(K) \rightarrow \tau_n(K')),$$

is an isomorphism. Notice that the right hand side is simply $\text{hom}(\partial \Delta[n], K) \times_{\text{hom}(\partial \Delta[n], K')} \text{hom}(\Delta[n], \tau_n(K'))$. Thus we have a commutative diagram

$$\begin{array}{ccc}
K_n & \xrightarrow{g} & \text{hom}(\partial \Delta[n] \rightarrow \Delta[n], K \rightarrow K') \\
\tau_n(K)_n & \xrightarrow{f} & \text{hom}(\partial \Delta[n] \rightarrow \Delta[n], \tau_n(K) \rightarrow \tau_n(K')),
\end{array}$$

where $\rightarrow$ denotes surjective maps. Then $f$ must be surjective because $f \circ g$ being surjective implies $f$ being surjective.

Take $(\delta x_n, [y_n]) \in \text{hom}(\partial \Delta[n], K) \times_{\text{hom}(\partial \Delta[n], K')} \text{hom}(\Delta[n], \tau_n(K'))$. If both $[x_n], [x'_n] \in \tau_n(K)_n$ map to $(\delta x_n, [y_n])$, that is the boundary $\partial x_n = \partial x'_n = \delta x_n \in \text{hom}(\partial \Delta[n], K)$ and $\phi_n(x'_n) = y_n' \sim y_n \sim y_n'' = \phi_n(x_n)$, then $y_n''$ and $y_n'$ differ by a certain element $y_{n+1} \in K'_{n+1}$. Since $K_{n+1} \rightarrow \text{hom}(\partial \Delta[n+1] \rightarrow \Delta[n+1], K \rightarrow K')$, is surjective, there exists $x_{n+1}$ such that $\phi_{n+1}(x_{n+1}) = y_{n+1}$ and $x_n, x'_n$ differ by $x_{n+1}$. This proves that $[x_n] = [x'_n] \in \tau_n(K)_n$. Hence $f$ is also injective.

If the map $p : M \rightarrow N$ is surjective and admits local section at any point in $N$, then the pull-back groupoid $G_1 \times_M N \Rightarrow G_0 \times_M N$ is free and proper if and only the original groupoid $G_1 \Rightarrow G_0$ is so. Since this is our case, the isomorphism (27), when applied to $n = 2$ and $K = \text{Kan}(X) K' = \text{Kan}(Y)$, implies that $\text{Kan}(Y)/_{\sim}2$ is representable. Hence $\tau_2(\text{Kan}(X))$ and $\tau_2(\text{Kan}(Y))$ are Lie 2-groupoids by Prop. 2.4.

Now we only need to verify that the morphism

$$\tau_2(\text{Kan}(X))_m \rightarrow \text{hom}(\partial \Delta[m] \rightarrow \Delta[m], \tau_2(\text{Kan}(X)) \rightarrow \tau_2(\text{Kan}(Y)))$$

is a surjective submersion for $m = 0, 1$. For $m = 0$ it is implied by $X_0 \rightarrow Y_0$ being a surjective submersion. For $m = 1$, by induction, we need to show that the natural map

$$X_1^{\beta+1} \rightarrow \text{hom}(\partial \Delta[1] \rightarrow \Delta[1], X^{\beta+1} \rightarrow Y^{\beta+1}),$$

is a surjective submersion supposing the same is true for $\beta$. We have

$$X_1^{\beta+1} = X_1^{\beta} \sqcup \text{hom}(\Lambda[2, 1], X^{\beta}), \quad Y_1^{\beta+1} = Y_1^{\beta} \sqcup \text{hom}(\Lambda[2, 1], Y^{\beta}).$$

The right hand side of (28) decomposes into two terms $I, II$ according to the decomposition of $Y_1^{\beta+1}$,

$$I = \text{hom}(\partial \Delta[1], X^{\beta}) \times_{\text{hom}(\partial \Delta[1], Y^{\beta})} \text{hom}(\Delta[1], Y^{\beta}),$$

$$II = \text{hom}(\partial \Delta[1], X^{\beta}) \times_{\text{hom}(\partial \Delta[1], Y^{\beta})} \text{hom}(\Lambda[2, 1], Y^{\beta}).$$
By the induction hypothesis, $X^β_1 \to I$ is a surjective submersion. Further by Lemma 2.5 (take $S = T = \Lambda[2,1]$, and $T' = \partial \Delta[1]$),

$$\text{hom}(\Lambda[2,1], X^β) \to II$$

is a surjective submersion. Thus (28) is a surjective submersion. \qed

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