AdS$_3$ metric from UV/IR entanglement entropies of CFT$_2$

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Abstract

How to construct the $d+1$ dimensional geometry explicitly from the dual CFT$_d$ is a widely concerned problem. Specifically, given entanglement entropies of a CFT$_2$, which is purely expressed by two dimensional parameters, can we build the dual three dimensional geometry unambiguously? To do this, one must assume nothing is known about the three dimensional geometry and starts with the most general setup. In this paper, by identifying the UV and IR entanglement entropies of a perturbed usual CFT$_2$ with the geodesic lengths, we show that, the dual geometry is uniquely determined to be asymptotic AdS$_3$. The hidden dimension is generated by the energy cut-off of the CFT$_2$, according to the holographic principle. The pure AdS$_3$ is obtained by taking the massless limit. Our derivations apply to both static and covariant scenarios. Moreover, what deserves special attention is that the ratio of the numerical factors of the UV/IR entanglement entropies are crucial to have a dual geometry. We are led to conjecture a necessary condition of holographic CFT$_2$. 
1 Introduction

The AdS/CFT correspondence plays a central role in modern theoretical physics. This conjecture states that a weakly coupled gravitational theory in a $d+1$ dimensional Anti-de Sitter (AdS) space is equivalent to a strongly coupled $d$ dimensional conformal field theory (CFT) on its conformally flat boundary $[1]$. It provides a testable realization for the holographic principle $[2, 3]$, proposed to explain the black hole entropy puzzle. Although the AdS/CFT correspondence has not yet been rigorously proved, it is justified by a lot of evidences in the last two decades.

Quantum entanglement is a manifestation of the non-local property of quantum mechanics. It is one of the most distinct features of quantum systems and measures the correlation between subsystems. For the simplest configuration, a quantum system is divided into two parts: $A$ and $B$. The Hilbert space is thus decomposed into $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The entanglement entropy of the region $A$ is defined as the von Neumann entropy, $S_A = -\text{Tr}(\rho_A \ln \rho_A)$, where $\rho_A$ is the reduced density matrix of region $A$: $\rho_A = \text{Tr}_{\mathcal{H}_B} \rho$. It is obvious that $S_A = S_B$. One of the most successful supports of the AdS/CFT correspondence is the Ryu and Takayanagi (RT) formula, which asserts the equality of the entanglement entropy (EE) of the boundary CFT and the accordingly defined minimal surface area in the bulk AdS $[4, 5, 6]$. In their work, the geodesic length (minimal surface area) in AdS$_3$ is calculated and found in agreement with the EE of CFT$_2$. This identification is then verified extensively by follow-ups, referring to a recent review $[7]$ and references therein. Since the gravity side is purely classical, RT formula is a leading order relation and receives quantum corrections in higher orders.

Motivated by the success of the RT formula, a conjecture is proposed that gravity could be interpreted as emergent structures, determined by the quantum entanglement of the dual CFT $[8, 9]$. This idea was further developed by Maldacena and Susskind. They proposed an equivalence between Einstein-Rosen bridge (ER) and Einstein-Podolsky-Rosen (EPR) experiment $[10]$. Following this logic, a crucial question has to be answered: Can we construct the dual bulk geometry, specifically the metric, from given CFT entanglement entropies? In any case, current results only (partially) verified that the EE of the CFT equals the correspondingly defined minimal surface area extending into the bulk AdS, but it is not answered if the dual geometry of the CFT is uniquely fixed to be AdS. A complete answer to this question probably needs a proof of AdS/CFT. But we may use EE = Minimal Surface Area of the dual geometry as a bridge to show the uniqueness of AdS. Of course, one must assume nothing is known about the dual geometry right from the start and make no restriction about the geometry.

This question attracts considerable attention in recent years. The major difficulty resides in how to determine the metric of the hidden dimension extending into the bulk. It is widely accepted that the hidden extra dimension is generated by the energy cut-off of the CFT and perpendicular to the boundary by the holographic principle $[11]$. Up to date, there are two major approaches on this problem. The first one is the tensor network which enables a Log-like geodesic length generated from the boundary states and gives discrete AdS space $[12]$. The
second method resorts to integral geometry. The concept of kinematic space is introduced and it is argued that the kinematic space of AdS$_3$ is dS$_2$, which can be read off from the Crofton form defined as the second derivatives with respect to two different points of the given entanglement entropy of CFT$_2$ [13]. And this method can only applies to the static scenario naturally. But both approaches are not able to construct the metric explicitly.

In this paper, without making any pre-assumption about the geometry, we develop a method to explicitly show the dual geometry of ordinary CFT$_2$ is precisely AdS$_3$. Contrasting with efforts in literature, our strategy is based on two distinctions: the Synge’s world function [14] and IR EE of a perturbed CFT$_2$. It is obvious that constructing the geodesic length is much easier than deriving the metric directly. This was recognized in the previous work about integral geometry and kinematic space [13], but it seems the importance of Synge’s world function was not perceived. So although the authors also took derivatives with respect to two different points along the geodesics respectively, the coincidence limit was not applied to obtain the metric. In classic gravity, the Synge’s world function, defined as one half of the squared geodesic length, is the fundamental tool to study the motion of a self-driving particle in a curved background. Almost all the important (two-point) quantities in the subject are defined on the basis of the Synge’s world function. In our current problem, what is of great importance of this function is that the metric can be straightforwardly calculated from the coincidence limit of its second derivatives. Moreover, the derivatives and coincidence limit enables us to ignore all the irrelevant complexities and get exactly what we need. Nevertheless, the similarity between the kinematic space formalism and the Synge’s world function is instructive and may lead to some interactions.

The second crucial quantity we adopt is the IR EE of a perturbed CFT$_2$. Traditionally, only the UV EE is used trying to construct the dual geometry, but not successful. As one can understand, it is trivial to reduce from higher dimension to lower one, while the opposite direction is usually not easy. Superficially speaking, the UV EE is completely expressed with the flat CFT quantities and the information about the hidden dimension is lost. One speculation is that the UV EE is not expressible without a UV cut-off, though it is intrinsic to a QFT. On the other hand, we know this cut-off could be removed in the (bulk) classical gravity side. While for a massless CFT, albeit we know there is no IR cut-off, from an alternative perspective, the correlation length approaches infinity, making the IR EE not expressible. This makes us conjecture a finite IR cut-off may be of help. Eventually we can push this limit to infinity and it is possible that this procedure causes no trouble on the gravity side. Since a finite IR cut-off exists for a perturbed CFT, we anticipate the derived geometry is asymptotic AdS.

Our derivations show clearly that the frequently employed UV EE of the CFT$_2$ can only partially determine the asymptotical behavior of the coefficients of the boundary directions but no help on the energy generated direction. Remarkably, we find the EE in the IR region of a perturbed CFT$_2$ provides exactly sufficient condition to fix the energy generated direction and the residue freedom on the boundary directions. This is because the IR EE is determined by both the UV and IR energy scales, representing different values on the energy generated direction. Therefore, with both the UV and IR EE, we can fix the asymptotical form of the geodesic length in the bulk, which in turn leads to the metric of the dual geometry, the anticipated asymptotic AdS$_3$. The pure AdS$_3$, the gravity dual of the massless CFT$_2$, is accordingly obtained by taking the mass scale of the perturbed CFT$_2$ to massless limit, which is equivalent to take the correlation length of the perturbed CFT$_2$ to infinity. Moreover, the covariant case can be easily obtained with the same pattern.
Our results confirm, given EE of a CFT, one can derive the dual geometry, at least for CFT$_2$. More details of the CFT are provided, more specifically the dual geometry can be fixed. However, we find that, the ratio the numerical factors of the UV to IR EE must be 2 in order to have a gravity dual. We conjecture this might be a necessary condition for a CFT$_2$ to have a dual gravity.

2 Spacetime metric from entanglement entropy

In classical gravity, the Synge’s world function plays a fundamental role to investigate the radiation back reaction (self-force) of a particle moving in a curved background. All the bi-tensors are defined by the Synge’s world function. A comprehensive review on this subject can be found in [15]. We only list some useful results here. Given a fixed point on a manifold $M$, and another point $x'$ which connects to $x$ through a single geodesic $x = x(\tau), \tau \in [0, t]$, such that $x(0) = x$ and $x(t) = x'$, the Synge’s world function is defined as the square of the geodesic length

$$\sigma(x, x') = \frac{1}{2} L_{\gamma_A}^2 (x, x') = \frac{1}{2} \int_0^t d\tau g_{ij} \dot{x}^i \dot{x}^j,$$

where $L_{\gamma_A}$ is the geodesic length connecting points $x$ and $x'$. This function is a bi-scalar for points $x$ and $x'$ respectively. Throughout this paper, we use $i$ and $i'$ to distinguish the two points $x$ and $x'$. The first derivative with respect to $x$ or $x'$ is the ordinary derivative, denoted as $\sigma_i = \frac{\partial \sigma}{\partial x^i}$ and $\sigma_{i'} = \frac{\partial \sigma}{\partial x^{i'}}$. It should be noted that $\sigma_i(x, x')(\sigma_{i'})$ is a vector for point $x (x')$ but a scalar for point $x' (x)$. It is not very hard to find

$$\sigma_i(x, x') = t \frac{d}{dt} x_i, \quad \sigma_{i'}(x, x') = -t \frac{d}{dt} x_{i'}.$$  

As usual, the second derivative on a single point is understood as the covariant derivative: $\sigma_{ij} \equiv \nabla_j \sigma_i$ and $\sigma_{i'j'} \equiv \nabla_{j'} \sigma_{i'}$. The quantity plays a crucial role in our derivation is the derivatives with respect to different points: $\sigma_{ij} \equiv \partial_i \sigma_j = \frac{\partial \sigma}{\partial x^i \partial x^j}$ and $\sigma_{ij} = \sigma_{j'i}$. The notation of coincidence limits for an arbitrary function is defined as

$$\lfloor f(x, x') \rfloor = \lim_{x \to x'} f(x, x').$$

It is easy to see that

$$\lfloor \sigma(x, x') \rfloor = [\sigma_i] = [\sigma_{i'}] = 0.$$  

Remarkably, the coincidence limits of the second derivatives lead to the metric

$$\lfloor \sigma_{ij'} \rfloor \equiv \lim_{x \to x'} \partial_{x_i} \partial_{x_{i'}} \left[ \frac{1}{2} L_{\gamma_A}^2 (x, x') \right] = -g_{ij} = -[\sigma_{ij}] = -[\sigma_{j'i'}].$$

The advantage of $\sigma_{ij'}$ over $\sigma_{ij}$ is that we do not need to know the connection (geometry). Thus what we need to do is to figure out the geodesic length of the yet-to-determine dual geometry.

Our strategy is to suppose we know nothing about the generated geometry. What we are going to use are exclusively restricted to the CFT EE, identifying EE with geodesic length, and holographic principle. We only
consider CFT$_2$ with infinite length in this work. We first focus on the static scenario since generalizing the results to the covariant case is easy. It is convenient to define a quantity with length dimension

$$ R \equiv \frac{2G^{(3)}c}{3}, \quad (2.6) $$

where $c$ is the central charge of the CFT$_2$ and $G^{(3)}_N$ is the Newton constant in three dimensions. The EE of CFT$_2$ in the UV region is given by

$$ S_{EE}^{UV} = \frac{c}{6} \log \left( \frac{\ell^2}{a^2} \right), \quad (2.7) $$

where $\ell = x - x'$ is the interval of the entanglement sub-region in the CFT and $a$ is the UV cut-off or lattice spacing of the CFT. Bear in mind that this UV EE is valid only in the region $\ell \gg a \to 0$. Following the proposal of RT (but without assuming the geometry), the geodesic length of the dual geometry ended on the boundary

$$ L_{\gamma \Lambda}^{UV} = 4G^{(3)}_N S_{EE} = \left( \frac{2G^{(3)}_N c}{3} \right) \log \left( \frac{\ell^2}{a^2} \right) = R \log \left( \frac{\ell^2}{a^2} \right). \quad (2.8) $$

It is widely accepted that the energy scale of the CFT gives rise to a hidden holographic dimension [4, 5, 11], denoted as $y$ in this paper. So, in the UV EE (2.8), $\ell$ denotes the boundary dimension $x$ and $a$ introduces a holographic dimension $y = a$. Eqn. (2.8) only applies to the geodesics ending on the boundary. We want to make the most general extension of the expression $\frac{(x-x')^2}{y^2}$ in (2.8) to the bulk. We are going to take the generally agreed viewpoint that the energy generated holographic direction is perpendicular to the boundary surface. We first replace $a^2$ by a regular function $\hat{h}(x, x', y, y')$. Then the most general consistent extension of the proper length $\ell^2 = (x - x')^2$, including the holographic dimension, is

$$ (x - x')^2 \to (x - x')^2 k(x, x', y, y') + (y - y')^2 p(x, x', y, y'), \quad (2.9) $$

with arbitrary regular functions: $k(x, x', y, y'), p(x, x', y, y')$. Therefore, the geodesic whose endpoints locate near the boundary has the length

$$ L_{\gamma \Lambda} = R \log \left( \eta^2 \right), \quad (2.10) $$

where

$$ \eta^2 \equiv \frac{(x - x')^2 k(x, x', y, y') + (y - y')^2 p(x, x', y, y')}{\hat{h}(x, x', y, y')}. \quad (2.11) $$

Note the functions $k, p$ and $\hat{h}$ must be invariant under $(x', y') \leftrightarrow (x, y)$. In order to return back to eqn. (2.8) as $y = y' = a \to 0$, we must have $k(x, x', a, a) / \hat{h}(x, x', a, a) \sim 1/a^2$ and $(y - y')^2 p(x, x, y, y') \to 0$. These two conditions also justify the generality of $\eta^2$. The expression of $\eta$ can be simplified by dividing $p(x, x', y, y')$ on the numerator and the denominator to get

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1 This is true at least in the near boundary region as proposed by the holographic principle. If we relax this condition, there will be cross terms like $(x - x')(y - y')$ in the extension $\eta$ and in turn off-diagonal metric components. From our following derivations, it will be easy to see that in order to determine these terms, a complete EE expression covering all region of a perturbed CFT has to be supplied.

2 It is equally good to cancel $k(x, x', y, y')$. It turns out the analysis is similiar, though to some extent trickier.
\[ \eta = \sqrt{\frac{(x - x')^2 f(x, x', y, y') + (y - y')^2}{g(x, x', y, y')}}. \] (2.12)

Again, we must have \( f(x, x', a, a) / g(x, x', a, a) \sim 1/a^2 \) to recover eqn. (2.8) under the limit \( y = y' = a \to 0 \).

It is clear that eqn. (2.10) is true only for \( \eta \to \infty \). As \( (x', y') \to (x, y) \), it diverges and cannot be a valid geodesic length in the bulk. Nevertheless, as \( \eta \to 0 \), we have \( x' \to x \) and \( y' \to y \), two points approaching to have vanishing geodesic length. Thus the geodesic length can be approximated by a power series of \( \eta \) as \( \eta \to 0 \).

\[ L_{\gamma A} = R \log \chi(\eta) = \begin{cases} R \log (\eta^2), & \eta \to \infty, \\ R \log (1 + C_1 \eta^\alpha + C_2 \eta^{2\alpha} + \ldots), & \eta \to 0, \end{cases} \] (2.13)

where \( C_i \)'s are constants to be determined. So, what we need to do is to fix the functions \( f(x, x', y, y') \), \( g(x, x', y, y') \) and constant \( \alpha \). However, with the UV EE only, we cannot go further and it is clear we need more information about the EE of CFT.

When a CFT is perturbed by a relevant perturbation, the correlation length (IR cut-off) \( \xi \) takes a finite value. We know that the UV EE (2.7) is no longer valid as \( \ell \geq \xi \). In the large \( \ell \) IR region, under the condition \( a \ll \xi \ll \ell \), the IR EE does not depend on \( \ell \) but completely determined by the ratio of the IR and UV cut-off \[ S^{IR}_{EE} = \frac{c}{6} \log \left( \frac{\xi}{a} \right), \] (2.14)

where \( \xi \equiv 1/m \) and \( m \) is the mass gap of the perturbed CFT. Note the factor 1/6, which turns out to be crucial to get the expected geometry and we will come back to this property later. It proves that the RT formula also works for the IR EE [4, 5]. Therefore, we can read off the geodesic length for the IR region from this perturbed CFT IR EE:

\[ L^{IR}_{\gamma A} = R \log \left( \frac{\xi}{a} \right). \] (2.15)

Since both \( a \) and \( \xi \) are energy scales, as suggested by the holographic principle, they are simply two points on the generated dimension \( y \), namely \( y = a \) and \( y' = \xi \). Moreover, one gets \( x - x' \to 0 \). Referring to the large \( \eta \) case in the general expression (2.13) and (2.12), we thus find

\[ L_{\gamma A} = R \log \left( \frac{(y - y')^2}{g(x, x', y, y')} \right) \approx R \log \left( \frac{(a - \xi)^2}{g(x, x, a, \xi)} \right). \] (2.16)

To be compatible with (2.15), we can fix

\[ g(x, x', y, y') = yy' \left( 1 + (x - x')^2 s(x, x', y, y') / \xi^2 \right), \] (2.17)

where for convenience we put \( 1/\xi^2 \) to make the regular function \( s(x, x', y, y') \) dimensionless. While from the discussion below eqn. (2.12), we can immediately obtain

\[ f(x, x', 0, 0) = 1 + (x - x')^2 s(x, x', 0, 0) / \xi^2. \] (2.18)
For small $y,y'$ we now can expand the function $f(x,x',y,y')$. Since the function $f(x,x',y,y')$ itself is dimensionless, we are able to rescale the dimensionful variables to dimensionless ones. We know that $L_{\gamma_A}^{UV}$ is valid when $y = y' = a \ll \ell \ll \xi$, thus we can choose the maximum value $\xi$ as the rescale parameter. Taking into account that two points are symmetric, the expansion for $f(x,x',y,y')$ is

$$f(x,x',y,y') = f(x,x',0,0) + \mu_1 (x,x') \left( \frac{y + y'}{\xi} \right) + \mu_2 (x,x') \left( \frac{y + y'}{\xi} \right)^2 + \ldots$$

(2.19)

This expansion is valid when $y,y' \ll \xi$. After obtaining the asymptotic expressions of $g(x,x',y,y')$ and $f(x,x',y,y')$, let us substitute them into the geodesic length (2.13) for the case $\eta \to 0$ ($(x - x')^2 \to 0$ and $(y - y')^2 \to 0$)

$$L_{\gamma_A} = R \log \left[ 1 + C_1 \left( \frac{(x - x')^2}{1 + (x - x')^2 s(x,x',0,0) + \mu_1 (x,x') \left( \frac{y + y'}{\xi} \right) + \ldots} \left( \frac{y y'}{1 + (x - x')^2 s(x,x',y,y')} \right)^{\alpha/2} + \ldots \right] .$$

(2.20)

As we emphasized, this expression is valid only if $y,y' \ll \xi$ in the bulk geometry. Eventually, we can use this geodesic length to calculate the metric as follows

$$[\sigma_{ij'}] = \lim_{x' \to x} \partial_{x'} \partial_{x'} \left[ \frac{1}{2} L_{\gamma_A}^2 (x,x') \right] = -g_{ij} ,$$

(2.21)

except it appears we still need to fix $\alpha$. Remarkably, with some amount of computation, one can find the following nice results

$$g_{ij} = \begin{cases} 
\text{divergent,} & \alpha < 1, \\
\text{valid,} & \alpha = 1, \\
0, & \alpha > 1,
\end{cases}$$

(2.22)

which indicates that only $\alpha = 1$ is physical. We therefore obtain the nonvanishing components of the metric

$$g_{xx} = -\frac{1}{2} \lim_{(x',y') \to (x,y)} \partial_{x'} \partial_{x'} L_{\gamma_A}^2 = C_1^2 \left( 1 + \mu_1 (x) \left( \frac{2y}{\xi} \right) + \ldots \right) \frac{R^2}{y^2} ,$$

$$g_{yy} = -\frac{1}{2} \lim_{(x',y') \to (x,y)} \partial_y \partial_{y'} L_{\gamma_A}^2 = C_1^2 \frac{R^2}{y^2} .$$

(2.23)

It is worth noting that the functions $s(x,x',0,0)$ do not contribute to the metric. So the background metric is

$$ds^2 = C_1^2 \frac{R^2}{y^2} \left[ \left( 1 + \mu_1 (x) \left( \frac{2y}{\xi} \right) + \mu_1 (x) \left( \frac{2y}{\xi} \right)^2 + \ldots \right) dx^2 + dy^2 \right] ,$$

(2.24)

which is nothing but the asymptotic static AdS$_3$ with radius $C_1 R$. From the relation of the radius of AdS$_3$ and the central charge of CFT$_2$ [18]: $c = \frac{3R_{\text{AdS}}}{2G_N}$, we identify

$$C_1 = 1, \quad R = R_{\text{AdS}} .$$

(2.25)

Finally, we get the metric

\[7\]
ds^2 = \frac{R^2_{\text{AdS}}}{y^2} \left[ \left( 1 + \mu_1 (x) \left( \frac{2y}{\xi} \right) + \mu_1 (x) \left( \frac{2y}{\xi} \right)^2 + \ldots \right) dx^2 + dy^2 \right]. \quad (2.26)

To get the pure AdS, note the definition of the IR cut-off of the energy scale

$$\xi = \frac{1}{m}. \quad (2.27)$$

The massless CFT\textsubscript{2} is reached under $m \to 0$ ($\xi \to \infty$). Therefore, we find the metric:

$$ds^2 = \frac{R^2_{\text{AdS}}}{y^2} (dx^2 + dy^2), \quad (2.28)$$

which is pure static AdS\textsubscript{3}.

To include the time-like direction, the UV EE is

$$S_{\text{EE}}^{\text{UV}} = \frac{c}{6} \log \left( \frac{\ell^2 - (\Delta t)^2}{a^2} \right). \quad (2.29)$$

The discussion follows the same pattern as in the static case. To be most general, when writing down $\eta$ as eqn. (2.12), we also multiply an arbitrary function to $(\Delta t)^2$ and after performing the similar procedure, one is able to get

$$ds^2 = \frac{R^2_{\text{AdS}}}{y^2} \left[-G(t, x, y) \, dt^2 + F(t, x, y) \, dx^2 + dy^2 \right], \quad (2.30)$$

where

$$G(t, x, y) = 1 + \rho_1 (x, t) \left( \frac{2y}{\xi} \right) + \rho_2 (x, t) \left( \frac{2y}{\xi} \right)^2 + \ldots$$

$$F(t, x, y) = 1 + \mu_1 (x, t) \left( \frac{2y}{\xi} \right) + \mu_2 (x, t) \left( \frac{2y}{\xi} \right)^2 + \ldots$$

When $\xi \to \infty$, one obtains the pure AdS\textsubscript{3}

$$ds^2 = \frac{R^2_{\text{AdS}}}{y^2} \left(-dt^2 + dx^2 + dy^2 \right). \quad (2.31)$$

The results is in consistence with the holographic RG flow discussed in [19]. To be specific, consider a bulk field $\phi_a$ is introduced:

$$I = \frac{1}{2\kappa^2} \int d^3 x \sqrt{-g} \left[ R - \frac{1}{2} G_{ab} \partial \phi_a \partial \phi_b - V (\phi_a) \right]. \quad (2.32)$$

where $G_{ab}$ is the metric of the scalar field internal space. When the potential $V (\phi_a)$ has a critical value $V (0) = -\frac{2}{R \xi}$, the solution of the metric is

$$ds^2 = \frac{R^2_{\text{AdS}}}{y^2} \left[-dt^2 + dx^2 + F (y) \, dy^2 \right]. \quad (2.33)$$

Near the boundary $y = 0$, we have
\[ \phi_a(y) \to 0, \quad F(y) = 1 + \mu^{2\alpha}y^{2\alpha} + \cdots, \quad y \to 0, \]  
\[ (2.34) \]

where \( \mu \) is some mass scale. We easy to see this is precisely the special case of (2.30) under \( G(t, x, y) = F(t, x, y) \).

### 3 A necessary condition for a holographic CFT

In the general expression of (2.12), simply for convenience, we let the power of \((y - y')\) be two to exactly match the IR EE. Now, let us suppose the IR EE is multiplied by a number \( A \) and identify it as the geodesic length

\[ S_{EE}^{IR} = A \frac{c}{6} \log \left( \frac{\xi}{a} \right) \implies L_{\gamma A}^{IR} = R \log \left( \frac{\xi}{a} \right)^A. \]

\[ (3.35) \]

It is not very hard to check that in order to be compatible with this IR EE and the UV EE in (2.3), \( \eta \) must take the form

\[ \eta = \sqrt{\frac{(x - x')^2 f(x, x', y, y') + (y - y')^4}{g(x, x', y, y')}}, \]

\[ (3.36) \]

and \( g(x, x', y, y') = (yy')^A(1 + \Delta x^2 s(x, x', y, y')/\xi^2 + \cdots), f(x, x', 0, 0) = (yy')^{A-1}(1 + \Delta x^2 s(x, x', 0, 0)/\xi^2 + \cdots) \).

However, when we substitute this \( \eta \) to calculate the metric, \( g_{yy} \) is found to be

\[ g_{yy} = \begin{cases} 
0, & A = 0, 1/2; A > 1, \\
\text{valid,} & A = 1, \\
\text{divergence,} & \text{otherwise.} 
\end{cases} \]

\[ (3.37) \]

Therefore, we conclude that there is no dual gravity for \( A \neq 1 \), which provides a necessary condition to determine if a CFT\(_2\) is holographic. This is consistent with that for a single interval, though there are two boundary points, the holographic dual IR EE only counts one point.

### 4 Discussion and conclusion

In this work, merely use the (perturbed) CFT\(_2\) entanglement entropy, \( EE = \text{geodesic length} \), and general guidance from the holographic principle, we explicitly show that the dual geometry of ordinary CFT\(_2\) is exactly the asymptotic AdS\(_3\), for both static and covariant cases. The pure AdS\(_3\) metric is obtained by taking the relevant perturbation of the CFT to vanishing limit. To have a dual (AdS) gravity, the ratio of the numeric factors of IR/UV EE must be exactly tuned.

In our calculation, we had assumed that the function \( \eta \) has a general expression (2.11) without cross terms such as \((x - x')(y - y')\). It is reasonable from the holographic principle, at least in the near boundary region. From our derivations, it is clear that more complete expressions of EE has to be supplied to fix this kind of terms. This is understandable since we have EE for two extreme regions only and these two regions are separated far away without any interface. Moreover, we can only precisely fix the leading terms of the metric. However, it is convincing from our derivations that, once detailed EE in finite region calculated from a specific CFT is supplied, the subleading terms of the bulk metric can be derived, as expected from AdS/CFT.
In a previous work [20], we observed that the Riemann Normal Coordinates (RNC) metric of AdS$_3$ can be straightforwardly read off from the entanglement entropy of CFT$_2$. Since RNC takes the geodesics as the basis, from the relationship of $\eta$ in (2.11) and the metric, the reason becomes clear now.

We only considered the infinite system for CFT$_2$ in this paper. Generalizations to the finite length or finite temperature systems are certainly important and the periodic condition may cause some subtleties. But it is quite interesting if the BTZ black hole property can be exhibited somehow. Another highly nontrivial and important extension is to discuss the higher dimensional theories. Our results does imply that it is possible to determine the dual geometry once giving the EE of a CFT, provided this CFT has a holographic dual.

Acknowledgements  We are deeply indebted to Bo Ning for many illuminating discussions and suggestions. This work cannot be done without her help. We are also very grateful to S. Kim, H. Nakajima and Song He for very helpful discussions and suggestions. This work is supported by the NSFC (Grant No. 11005016, 11175039 and 11375121).

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