Abstract

A coupled AKNS-Kaup-Newell hierarchy of systems of soliton equations is proposed in terms of hereditary symmetry operators resulted from Hamiltonian pairs. Zero curvature representations and tri-Hamiltonian structures are established for all coupled AKNS-Kaup-Newell systems in the hierarchy. Therefore all systems have infinitely many commuting symmetries and conservation laws. Two reductions of the systems lead to the AKNS hierarchy and the Kaup-Newell hierarchy, and thus those two soliton hierarchies also possess tri-Hamiltonian structures.

I Introduction

Systems of soliton equations come usually in hierarchies. This kind of hierarchies possesses many nice properties, for instance, Lax representations or zero curvature representations, infinitely many commuting symmetries and conservation laws, hereditary recursion structures, bi-Hamiltonian formulations and even multiple Hamiltonian formulations etc. and they are often called soliton hierarchies. Well-known examples of such soliton hierarchies (for example, see [1, 2]) contain the KdV hierarchy, the MKdV hierarchy, the AKNS hierarchy, the Kaup-Newell hierarchy, the Benjamin-Ono hierarchy [3], the Tu hierarchy [4], the Dirac hierarchy [5], the coupled KdV...
hierarchies \cite{6,7}, the coupled Harry-Dym hierarchies \cite{8}, the coupled Burgers hierarchies \cite{9} and so on. It is very interesting to search for new soliton hierarchies, even hierarchies of systems which possess only infinitely many commuting symmetries.

An idea which allows to achieve this is to construct soliton hierarchies of coupled systems of equations. It could be divided into two aspects in view of types of soliton equations. The one is to construct soliton hierarchies by coupling systems of the same type. Such examples are the coupled KdV hierarchies \cite{6,7}, the coupled Harry-Dym hierarchies \cite{8}, the coupled Burgers hierarchies \cite{9}, and the perturbation systems of the KdV hierarchy \cite{10} etc. The other is to construct soliton hierarchies by coupling systems of different types. There are few examples in this aspect. A coupled AKNS-Kaup-Newell hierarchy of complex form, recently introduced by Zhang \cite{11}, is such an example.

In this paper, motivated by Zhang’s coupled AKNS-Kaup-Newell hierarchy of complex form, we would like to propose a hierarchy of coupled AKNS-Kaup-Newell evolution equations of real form. The hierarchy will be established in the second section, in terms of hereditary symmetry operators. The required hereditary symmetry operators can be generated by observing a set of Hamiltonian operators. Zero curvature representations will be computed in the third section for all systems in the hierarchy. Interestingly the discussion of the fourth section shows that all the systems have not only bi-Hamiltonian structures but also tri-Hamiltonian structures, although Zhang didn’t present Hamiltonian structures and consequent conservation laws due to a failure in determining Hamiltonian operators \cite{11}. Therefore the resulting hierarchy has infinitely many commuting symmetries and conservation laws. Some concluding remarks are given in the last section.

II Hereditary symmetry operators

We want to establish a coupled AKNS-Kaup-Newell hierarchy in terms of hereditary symmetry operators resulted from Hamiltonian pairs. To this end, let us introduce
a set of $2 \times 2$ matrix integro-differential operators:

$$M = M(u) = \begin{pmatrix} \alpha_1 q \partial^{-1} q & \alpha_2 + \alpha_3 \partial - \alpha_1 q \partial^{-1} r \\ \alpha_2 + \alpha_3 \partial - \alpha_1 r \partial^{-1} q & \alpha_1 r \partial^{-1} r \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix},$$

where $\partial = \frac{\partial}{\partial x}$, $q = q(x,t)$, $r = r(x,t)$, and $\alpha_1, \alpha_2, \alpha_3$ are three arbitrary constants, and consider their Hamiltonian property. They are simple generalizations of the Hamiltonian operators in the AKNS case \cite{12}. The following proposition shows that they are still Hamiltonian, indeed.

**Proposition 1** The $2 \times 2$ matrix integro-differential operators defined by (3) are all Hamiltonian for any constants $\alpha_1, \alpha_2, \alpha_3$.

**Proof:** Assume that

$$X = (X_1, X_2)^T, \quad Y = (Y_1, Y_2)^T, \quad Z = (Z_1, Z_2)^T, \quad W = (W_1, W_2)^T,$$

are two-dimensional vectors of functions. Since $M$ is skew-symmetric, we only need to prove that the Jacobi identity

$$< Z, M'[MX]Y > + \text{cycle}(X, Y, Z) \equiv 0 \pmod{\partial}$$

holds for any $X, Y, Z$, where $< \cdot, \cdot >$ denotes the standard inner product of $\mathbb{R}^2$. By (3), we immediately have

$$MX = \begin{pmatrix} \alpha_2 X_2 + \alpha_3 X_{2x} + \alpha_1 q P(X) \\ -\alpha_2 X_1 + \alpha_3 X_{1x} - \alpha_1 r P(X) \end{pmatrix} := \begin{pmatrix} W_1(X) \\ W_2(X) \end{pmatrix},$$

where $P(X) = \partial^{-1}(qX_1 - rX_2)$. Following the definition of the Gateaux derivative, two objects $M'[W]$ and $M'[W]Y$ are computed as follows:

$$M'[W] = \begin{pmatrix} \alpha_1 q \partial^{-1} W_1 + \alpha_1 W_1 \partial^{-1} q & -\alpha_1 q \partial^{-1} W_2 - \alpha_1 W_1 \partial^{-1} r \\ -\alpha_1 r \partial^{-1} W_1 - \alpha_1 W_2 \partial^{-1} q & \alpha_1 r \partial^{-1} W_2 + \alpha_1 W_2 \partial^{-1} r \end{pmatrix},$$

$$M'[W]Y = \begin{pmatrix} \alpha_1 q \partial^{-1}(W_1 Y_1 - W_2 Y_2) + \alpha_1 W_1 \partial^{-1}(qY_1 - rY_2) \\ -\alpha_1 r \partial^{-1}(W_1 Y_1 - W_2 Y_2) - \alpha_1 W_2 \partial^{-1}(qY_1 - rY_2) \end{pmatrix}.$$

Now we can have

$$< Z, M'[MX]Y > = \alpha_1 (qZ_1 - rZ_2) \partial^{-1}(W_1(X)Y_1 - W_2(X)Y_2) + \alpha_1 (W_1(X)Z_1 - W_2(X)Z_2) \partial^{-1}(qY_1 - rY_2).$$

(3)
Upon observing that
\[ W_1(X)Y_1 - W_2(X)Y_2 \]
\[ = (\alpha_2 X_2 + \alpha_3 X_{2x} + \alpha_1 p(X))Y_1 - (-\alpha_2 X_1 + \alpha_3 X_{1x} - \alpha_1 p(X))Y_2 \]
\[ = \alpha_1 p(X)(qY_1 + rY_2) + \alpha_2 (X_2 Y_1 + X_1 Y_2) + \alpha_3 (X_{2x} Y_1 - X_{1x} Y_2), \]
we can make a decomposition
\[ < Z, M'[MX]Y > = R(X, Y, Z) + S(X, Y, Z) + T(X, Y, Z), \]
where \( R, S, T \) are defined by
\[ R(X, Y, Z) = \alpha_1^2 (qZ_1 - rZ_2) \partial^{-1}[P(X)(qY_1 + rY_2)] \]
\[ + \alpha_1^2 p(X)(qZ_1 + rZ_2) \partial^{-1}(qY_1 - rY_2), \]
\[ S(X, Y, Z) = \alpha_1 \alpha_2(qZ_1 - rZ_2) \partial^{-1}(X_2 Y_1 + X_1 Y_2) \]
\[ + \alpha_1 \alpha_2 (X_{2z} + X_1 Z_2) \partial^{-1}(qY_1 - rY_2), \]
\[ T(X, Y, Z) = \alpha_1 \alpha_3(qZ_1 - rZ_2) \partial^{-1}(X_{2x} Y_1 - X_{1x} Y_2) \]
\[ + \alpha_1 \alpha_3 (X_{2x} Z_1 - X_{1x} Z_2) \partial^{-1}(qY_1 - rY_2). \]

For these three functions \( R, S, T \), we can compute that
\[ R(X, Y, Z) + \text{cycle}(X, Y, Z) \]
\[ = \alpha_1^2 \partial \{ P(Z) \partial^{-1}[P(X)(qY_1 + rY_2)] \} + \text{cycle}(X, Y, Z), \]
\[ S(X, Y, Z) + \text{cycle}(X, Y, Z) \]
\[ = \alpha_1 \alpha_2 \partial [P(Z) \partial^{-1}(X_2 Y_1 + X_1 Y_2)] + \text{cycle}(X, Y, Z), \]
\[ T(X, Y, Z) + \text{cycle}(X, Y, Z) \]
\[ = \alpha_1 \alpha_3(qZ_1 - rZ_2)(X_2 Y_1 - X_1 Y_2) + \alpha_1 \alpha_3(qZ_1 - rZ_2) \partial^{-1}(X_1 Y_{2x} - X_2 Y_{1x}) \]
\[ + \alpha_1 \alpha_3 (Z_1 X_{2z} - Z_2 X_{1z}) \partial^{-1}(qY_1 - rY_2) + \text{cycle}(X, Y, Z) \]
\[ = \alpha_1 \alpha_3(qZ_1 - rZ_2)(X_1 Y_{2x} - X_2 Y_{1x}) \]
\[ + \alpha_1 \alpha_3 (Z_1 X_{2z} - Z_2 X_{1z}) \partial^{-1}(qY_1 - rY_2) + \text{cycle}(X, Y, Z) \]
\[ = \alpha_1 \alpha_3 \partial [P(Z) \partial^{-1}(X_1 Y_{2x} - X_2 Y_{1x})] + \text{cycle}(X, Y, Z). \]

They are all total derivatives and thus combining the decomposition (4) and the equalities (8), (9), (10) leads to the Jacobi identity (2). This completes the proof. ☑
Now we would like to discuss some special cases of Hamiltonian pairs starting from the above Hamiltonian operators defined by (1), which allows to generate hereditary symmetry operators and further soliton hierarchies. This idea has been successfully used to construct bi-Hamiltonian coupled KdV systems [13, 14]. An important phenomenon we want to point out is that different soliton hierarchies can be derived from Hamiltonian operators of the same type. The following discussion shows an example of such phenomenon. It is also important to realize that not all Hamiltonian pairs may generate hereditary symmetry operators. Thus care must be taken to restrict our attention to the cases where there exists at least one invertible Hamiltonian operator for each Hamiltonian pair. The required invertibility guarantees that Hamiltonian pairs can yield hereditary symmetry operators [15].

**Case 1:** We make a choice of an invertible Hamiltonian operator

\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

which has an inverse operator

\[
J^{-1} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

It follows from Proposition 1 that this operator \( J \) constitutes a Hamiltonian pair with \( M \) defined by (1). Therefore we can have a hereditary symmetry operator

\[
\Phi = MJ^{-1} = \begin{pmatrix}
\alpha_2 + \alpha_3 \partial - \alpha_1 q \partial^{-1} r & -\alpha_1 q \partial^{-1} q \\
\alpha_1 r \partial^{-1} r & \alpha_2 - \alpha_3 \partial + \alpha_1 r \partial^{-1} q
\end{pmatrix},
\]

where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are arbitrary. The reduction of \( \alpha_1 = -1, \alpha_2 = 0 \) and \( \alpha_3 = \frac{1}{2} \) leads to the recursion operator for the normal AKNS hierarchy [1, 2, 16].

**Case 2:** We make a choice of a Hamiltonian pair

\[
J = \begin{pmatrix}
\beta_1 q \partial^{-1} q & 1 - \beta_1 q \partial^{-1} r \\
-1 - \beta_1 r \partial^{-1} q & \beta_1 r \partial^{-1} r
\end{pmatrix}, \quad M = \begin{pmatrix}
0 & \alpha_3 \partial \\
\alpha_3 \partial & 0
\end{pmatrix}.
\]

The above proposition ensures that they constitute a Hamiltonian pair, indeed. Since the operator \( J \) has an invertible operator

\[
J^{-1} = \begin{pmatrix}
\beta_1 r \partial^{-1} r & -1 + \beta_1 r \partial^{-1} q \\
1 + \beta_1 q \partial^{-1} r & \beta_1 q \partial^{-1} q
\end{pmatrix},
\]

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we can obtain the corresponding hereditary symmetry operator
\[ \Phi = MJ^{-1} = \begin{pmatrix}
\alpha_3 \partial + \alpha_3 \beta_1 \partial q \partial^{-1} r & \alpha_3 \beta_1 \partial q \partial^{-1} q \\
\alpha_3 \beta_1 r \partial \partial^{-1} r & -\alpha_3 \partial + \alpha_3 \beta_1 r \partial \partial^{-1} r
\end{pmatrix}, \quad (14) \]
where \( \alpha_3 \) and \( \beta_1 \) are arbitrary. The reduction of \( \alpha_3 = \frac{1}{2} \) and \( \beta_1 = -1 \) leads to the recursion operator for the normal Kaup-Newell hierarchy (for example, see [17]).

More generally, we have the following case, which combines the above two cases.

**Case 3:** We make a choice of an invertible Hamiltonian operator
\[ J = \begin{pmatrix}
\beta_1 q \partial \partial^{-1} q & \beta_2 - \beta_1 q \partial \partial^{-1} r \\
-\beta_2 - \beta_1 r \partial \partial^{-1} q & \beta_1 r \partial \partial^{-1} r
\end{pmatrix}, \quad (15) \]
whose inverse operator can be given by
\[ J^{-1} = \frac{1}{\beta_2^2} \begin{pmatrix}
\beta_1 r \partial \partial^{-1} r & -\beta_2 + \beta_1 r \partial \partial^{-1} q \\
\beta_2 + \beta_1 q \partial \partial^{-1} r & \beta_1 q \partial \partial^{-1} r
\end{pmatrix}. \]
It follows from Proposition [7] that the operator \( J \) constitutes a Hamiltonian pair with the Hamiltonian operator \( M \) defined by (1). In this case, we can generate the following corresponding hereditary symmetry operator
\[ \Phi = MJ^{-1} = \frac{1}{\beta_2^2} \begin{pmatrix}
\alpha_2 \beta_2 + \alpha_3 \beta_2 \partial + (\alpha_2 \beta_1 - \alpha_1 \beta_2) q \partial^{-1} r + \alpha_3 \beta_1 \partial q \partial^{-1} r,
(\alpha_1 \beta_2 - \alpha_2 \beta_1) r \partial \partial^{-1} r + \alpha_3 \beta_1 \partial r \partial^{-1} r,
(\alpha_2 \beta_1 - \alpha_1 \beta_2) q \partial \partial^{-1} q + \alpha_3 \beta_1 \partial q \partial^{-1} q,
\alpha_2 \beta_2 - \alpha_3 \beta_2 \partial + (\alpha_1 \beta_2 - \alpha_2 \beta_1) r \partial \partial^{-1} q + \alpha_3 \beta_1 \partial r \partial^{-1} q
\end{pmatrix}, \quad (16) \]
where five constants are arbitrary but \( \beta_2 \neq 0 \).

Let us pick out a special sub-case of \( \alpha_2 = 0 \) and \( \beta_2 = 1 \) from the third case. If \( \alpha_3 = 0 \), we just obtain a simple hereditary symmetry operator
\[ \Phi = \begin{pmatrix}
-\alpha q \partial \partial^{-1} r & -\alpha q \partial \partial^{-1} q \\
\alpha r \partial \partial^{-1} r & \alpha r \partial \partial^{-1} q
\end{pmatrix}, \quad (17) \]
where \( \alpha = \alpha_1 \) is arbitrary. This is equivalent to the first case with \( \alpha_2 = \alpha_3 = 0 \).

If \( \alpha_3 \neq 0 \), upon resetting \( \alpha_3 = \gamma, \alpha_1 = \alpha, \alpha_3 \beta_1 = \beta \), we obtain a hereditary symmetry operator
\[ \Phi = \begin{pmatrix}
\gamma \partial - \alpha q \partial \partial^{-1} r + \beta \partial q \partial \partial^{-1} r & -\alpha q \partial \partial^{-1} q + \beta \partial q \partial \partial^{-1} q \\
\alpha r \partial \partial^{-1} r + \beta \partial r \partial \partial^{-1} r & -\gamma \partial + \alpha r \partial \partial^{-1} q + \beta \partial r \partial \partial^{-1} q
\end{pmatrix}, \quad (18) \]
where $\alpha, \beta, \gamma$ are arbitrary but $\gamma \neq 0$. Note that if we let the constant $\gamma$ go to zero, the hereditary condition for $\Phi$ with a general constant $\gamma$ becomes the one for $\Phi$ with $\gamma = 0$. Therefore the constant $\gamma$ can be chosen as zero, which doesn’t affect the hereditary property of $\Phi$. However if $\gamma = 0$, we don’t know whether the operator $\Phi$ defined by (18) is decomposable, i.e. whether there exists any Hamiltonian pair $J$ and $M$ so that $\Phi = MJ^{-1}$.

We will focus on discussing a soliton hierarchy generated by the hereditary symmetry operator in (18) because of its generality. For $\gamma \neq 0$, we can re-scale three constants to put a general case into a special case of the operator $\Phi$ defined by (18), and thus we pick out the following special case

$$\Phi = MJ^{-1} = \begin{pmatrix} \frac{1}{2} \partial - \alpha q \partial^{-1} r - \frac{1}{2} \beta \partial q \partial^{-1} r & -\alpha q \partial^{-1} q - \frac{1}{2} \beta \partial q \partial^{-1} q \\ \alpha r \partial^{-1} r - \frac{1}{2} \beta \partial r \partial^{-1} r & -\frac{1}{2} \partial + \alpha r \partial^{-1} q - \frac{1}{2} \beta \partial r \partial^{-1} q \end{pmatrix}$$

(19)

to discuss without loss of generality. To this special case, the corresponding hierarchy of evolution equations

$$(\begin{array}{c} q_t \\ r_t \end{array}) = \begin{pmatrix} \frac{1}{2} \partial - \alpha q \partial^{-1} r - \frac{1}{2} \beta \partial q \partial^{-1} r & -\alpha q \partial^{-1} q - \frac{1}{2} \beta \partial q \partial^{-1} q \\ \alpha r \partial^{-1} r - \frac{1}{2} \beta \partial r \partial^{-1} r & -\frac{1}{2} \partial + \alpha r \partial^{-1} q - \frac{1}{2} \beta \partial r \partial^{-1} q \end{pmatrix}^n \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \ n \geq 0,$$

(20)

contains two important reductions. If $\alpha \neq 0$ but $\beta = 0$, the hierarchy reduces to the AKNS hierarchy. If $\alpha = 0$ but $\beta \neq 0$, the hierarchy reduces to the Kaup-Newell hierarchy. Thus the hierarchy (20) generated by the hereditary symmetry operator (19) is called a coupled AKNS-Kaup-Newell hierarchy. All systems in the hierarchy (20) are real. Therefore the hierarchy (20) is a soliton hierarchy that we want to construct.

### III Zero curvature representations

In the previous section, we generated a coupled AKNS-Kaup-Newell hierarchy of real form by observing Hamiltonian operators. More importantly, the resulting hierarchy shares some common integrable properties. In this section, we want to show zero curvature representations for all systems in the hierarchy, and in the next section, we will establish tri-Hamiltonian structures.
To show zero curvature representations, let us impose a spectral problem

\[ \phi_x = U \phi, \quad U = U(u, \lambda) = \begin{pmatrix} \lambda & q \\ (\alpha + \beta \lambda) & -\lambda \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \]

where \( \lambda \) is a spectral parameter, and \( \alpha \) and \( \beta \) are arbitrary constants. It is customary to solve the stationery zero curvature equation \( V_x = [U, V] \) first. Suppose that

\[ V = V(u, \lambda) = \begin{pmatrix} a & b \\ (\alpha + \beta \lambda)c & -a \end{pmatrix} = \sum_{i \geq 0} \begin{pmatrix} a_i & b_i \\ (\alpha + \beta \lambda)c_i & -a_i \end{pmatrix} \lambda^{-i}, \]

and then the stationery zero curvature equation becomes

\[
\begin{cases}
  a_x = (\alpha + \beta \lambda)(qc - rb), \\
  b_x = 2\lambda b - 2qa, \\
  c_x = 2ra - 2\lambda c.
\end{cases}
\]

Notice that a recursion relation to determine \( b \) and \( c \) may be found if we fix \( a = (\alpha + \beta \lambda)\partial^{-1}(qc - rb) \). Actually we have

\[
\begin{cases}
  b_x = 2\lambda b - 2(\alpha + \beta \lambda)q\partial^{-1}(qc - rb), \\
  c_x = 2(\alpha + \beta \lambda)r\partial^{-1}(qc - rb) - 2\lambda c,
\end{cases}
\]

which equivalently leads to

\[
\begin{pmatrix}
  -2\beta q\partial^{-1}q & 2 + 2\beta q\partial^{-1}r \\
  -2 + 2\beta r\partial^{-1}q & -2\beta r\partial^{-1}r
\end{pmatrix}
\begin{pmatrix}
  c_{i+1} \\
  b_{i+1}
\end{pmatrix}
= \begin{pmatrix}
  2\alpha q\partial^{-1}q & \partial - 2\alpha q\partial^{-1}r \\
  \partial - 2\alpha r\partial^{-1}q & 2\alpha r\partial^{-1}r
\end{pmatrix}
\begin{pmatrix}
  c_i \\
  b_i
\end{pmatrix},
\]

where \( i \geq 0 \). If we set

\[ J = \begin{pmatrix}
  -2\beta q\partial^{-1}q & 2 + 2\beta q\partial^{-1}r \\
  -2 + 2\beta r\partial^{-1}q & -2\beta r\partial^{-1}r
\end{pmatrix}, \quad M = \begin{pmatrix}
  2\alpha q\partial^{-1}q & \partial - 2\alpha q\partial^{-1}r \\
  \partial - 2\alpha r\partial^{-1}q & 2\alpha r\partial^{-1}r
\end{pmatrix}, \]

the operators \( J \) and \( M \) constitute a Hamiltonian pair, based on the result in the previous section. It is apparent that the corresponding hereditary symmetry operator \( \Phi = MJ^{-1} \) is exactly the same as the one defined by \([19]\), having noted that

\[ J^{-1} = \frac{1}{2} \begin{pmatrix}
  -\beta \partial r\partial^{-1}r & -1 - \beta \partial r\partial^{-1}q \\
  1 - \beta \partial q\partial^{-1}r & -\beta \partial q\partial^{-1}q
\end{pmatrix}. \]

The conjugate operator of \( \Phi \) reads as

\[
\Psi = \Phi^\dagger = J^{-1}M = \begin{pmatrix}
  -\frac{1}{2} \partial + \alpha r\partial^{-1}q - \frac{1}{2} \beta r\partial^{-1}q\partial & -\alpha r\partial^{-1}r - \frac{1}{2} \beta r\partial^{-1}r\partial \\
  \alpha q\partial^{-1}q - \frac{1}{2} \beta q\partial^{-1}q\partial & \frac{1}{2} \partial - \alpha q\partial^{-1}r - \frac{1}{2} \beta q\partial^{-1}r\partial
\end{pmatrix}.
\]
Therefore upon noting (23) and choosing $a_0 = 1$, we obtain a solution to the stationary zero curvature equation $V_x = [U, V]$:

$$a_0 = 1, \ b_0 = c_0 = 0; \ b_1 = q, \ c_1 = r; \ (c_{i+1} \ b_{i+1}) = \Psi (c_i \ b_i), \ i \geq 1,$$

$$a_i = \alpha \partial^{-1}(qc_i - rb_i) + \beta \partial^{-1}(qc_{i+1} - rb_{i+1}), \ i \geq 1;$$

from which we can get

$$\begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = \Psi \begin{pmatrix} r \\ q \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -r_x - \beta qr^2 \\ q_x - \beta q^2 r \end{pmatrix},$$

and

$$a_1 = \beta \partial^{-1}(qc_2 - rb_2) = -\frac{1}{2} \beta qr.$$

It should be noted that we always need to select zero constants for integration in deriving $a_i, b_i, c_i, i \geq 1$. That is, we require that $a_i|_{[u]=0} = b_i|_{[u]=0} = c_i|_{[u]=0} = 0, \ i \geq 1,$ where $[u] = (u, u_x, \cdots)$.

Now we can express the coupled AKNS-Kaup-Newell hierarchy (20) in another way. Let us define

$$u_t = K_n := J \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = \Phi^n \begin{pmatrix} 2q \\ -2r \end{pmatrix}, \ n \geq 0,$$

where $\Phi$ is defined by (19). The first three systems of the hierarchy (29) are

$$\begin{align*}
    u_t &= \begin{pmatrix} q_t \\ r_t \end{pmatrix} = K_0 = J \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 2q \\ -2r \end{pmatrix}, \\
    u_t &= \begin{pmatrix} q_t \\ r_t \end{pmatrix} = K_1 = J \begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \\
    u_t &= \begin{pmatrix} q_t \\ r_t \end{pmatrix} = K_2 = J \begin{pmatrix} c_3 \\ b_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} q_{xx} - 2\alpha \alpha^2 r - \beta q^2 r_x x \\ -r_{xx} + 2\alpha \alpha^2 r - \beta q^2 r_x x \end{pmatrix}.
\end{align*}$$

Since $K_1 = u_x$, all systems in the hierarchy (29), except the first system $u_t = K_0$, are exactly the coupled AKNS-Kaup-Newell systems in the hierarchy (20). Therefore (29) is another expression for the coupled AKNS-Kaup-Newell hierarchy (20).

Let us turn to construction of zero curvature representations for all coupled AKNS-Kaup-Newell systems in the soliton hierarchy (24). We need a condition of
\( \alpha^2 + \beta^2 \neq 0 \). With this condition, we have the injective property of the Gateaux derivative of \( U \) with respect to \( u \), which is required in deriving systems of evolution equations from zero curvature equations. If the condition of \( \alpha^2 + \beta^2 \neq 0 \) is not satisfied, then the systems defined by (29) are linear and separated, and thus they are all trivial.

We choose Lax operators \( V^{(n)} \) for \( n \geq 0 \) as

\[
V^{(n)} = V^{(n)}(u, \lambda) = \bar{V}^{(n)} + \Delta_n, \quad \Delta_n = \begin{pmatrix} \delta_{1n} & 0 \\ 0 & \delta_{2n} \end{pmatrix},
\]

(31)

\[
\bar{V}^{(n)} = \sum_{j=0}^{n} \begin{pmatrix} a_j & b_j \\ (\alpha + \beta \lambda)c_j & -a_j \end{pmatrix} \lambda^{n-j} = \begin{pmatrix} (\lambda^n a)_{+} & (\lambda^n b)_{+} \\ (\alpha + \beta \lambda)(\lambda^n c)_{+} & -(\lambda^n a)_{+} \end{pmatrix},
\]

(32)

where the subscript denotes to choose the polynomial part in \( \lambda \), and \( \delta_{1n} \) and \( \delta_{2n} \) are two functions to be determined. At this moment, we can compute that

\[
\bar{V}_{x}^{(n)} - [U, \bar{V}^{(n)}] = \begin{pmatrix} a_{nx} - \alpha(qc_n - rb_n) & b_{nx} + 2qa_n \\ (\alpha + \beta \lambda)(cnx - 2ra_n) & -a_{nx} + \alpha(qc_n - rb_n) \end{pmatrix},
\]

\[
\Delta_{nx} - [U, \Delta] = \begin{pmatrix} \delta_{1n,x} & q(\delta_{1n} - \delta_{2n}) \\ -(\alpha + \beta \lambda)r(\delta_{1n} - \delta_{2n}) & \delta_{2n,x} \end{pmatrix}.
\]

Therefore if we take a choice

\[
\delta_{1n} = -\delta_{2n} = -a_n + \alpha \partial^{-1}(qc_n - rb_n), \quad n \geq 1,
\]

(33)

then noting the injective property of \( U' \) under \( \alpha^2 + \beta^2 \neq 0 \), the zero curvature equation

\[
U_t - V^{(n)}_x + [U, V^{(n)}] = 0,
\]

(34)

equivalently yields the coupled AKNS-Kaup-Newell system

\[
u_t = K_n = M \begin{pmatrix} c_n \\ b_n \end{pmatrix} = J \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix}
\]

for each \( n \geq 1 \). Moreover it is easy to see that \( u_t = K_0 \) has a Lax pair \( U \) and \( \bar{V}^{(0)} \).

Therefore each coupled AKNS-Kaup-Newell system \( u_t = K_n \) has a zero curvature representation with the Lax pair \( U \) and \( V^{(n)} \) if we adopt \( \delta_{10} = \delta_{20} = 0 \). We remark that for the systems \( u_t = K_n \) with \( \alpha = \beta = 0 \), \( n \geq 0 \), the above zero curvature representations still hold, but they are not sufficient, because we lose the injective property of \( U' \) in the case of \( \alpha = \beta = 0 \).
IV Tri-Hamiltonian structures

To establish some kind of tri-Hamiltonian structures for the coupled AKNS-Kaup-Newell hierarchy, let us impose a third Hamiltonian operator

\[
N = M \Psi = \begin{pmatrix}
-\alpha q \partial^{-1} q \partial + \alpha \partial q \partial^{-1} q - \frac{1}{2} \beta \partial q \partial^{-1} q \partial, \\
-\frac{1}{2} \partial^2 + \alpha \partial r \partial^{-1} q + \alpha r^{-1} q \partial - \frac{1}{2} \beta \partial r \partial^{-1} q \partial, \\
\frac{1}{2} \partial^2 - \alpha q \partial^{-1} r \partial - \alpha \partial q \partial^{-1} r - \frac{1}{2} \beta \partial q \partial^{-1} r \partial \\
-\alpha \partial r \partial^{-1} r \partial + \alpha r^{-1} r \partial - \frac{1}{2} \beta \partial r \partial^{-1} r \partial
\end{pmatrix}.
\] (35)

It constitutes a Hamiltonian triple with \(J\) and \(M\) defined by (26), for any constants \(\alpha, \beta, \gamma\). That is, any linear combination of \(J\), \(M\) and \(N\) is still a Hamiltonian operator, which is automatically satisfied since \(J\) and \(M\) are a Hamiltonian pair.

Let us consider the first nonlinear system in the couple AKNS-Kaup-Newell hierarchy (20):

\[
u_t = K_2 = M \begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} q_{xx} - 2\alpha q^2 r - \beta (q^2 r)_x \\ -r_{xx} + 2\alpha qr^2 - \beta (qr^2)_x \end{pmatrix}.
\] (36)

It is apparent that this system could be written in three ways as

\[
u_t = K_2 = J \begin{pmatrix} c_3 \\ b_3 \end{pmatrix} = M \begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = N \begin{pmatrix} c_1 \\ b_1 \end{pmatrix}.
\]

Moreover a direct calculation can show three gradient vectors

\[
\begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} r \\ q \end{pmatrix} = \frac{\delta H_0}{\delta u}, \quad H_0 = qr;
\]

\[
\begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = \Psi \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -r_x - \beta qr^2 \\ q_x - \beta q^2 r \end{pmatrix} = \frac{\delta H_1}{\delta u}, \quad H_1 = \frac{1}{4} \beta q^2 r^2 - \frac{1}{4} qr_x + \frac{1}{4} qr_x;
\]

\[
\begin{pmatrix} c_3 \\ b_3 \end{pmatrix} = \Psi \begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} r_{xx} - 2\alpha qr^2 + 3\beta qr r_x + \frac{3}{2} \beta^2 q^2 r^3 \\ q_{xx} - 2\alpha q^2 r - 3\beta qq_x r + \frac{3}{2} \beta^2 q^2 r^2 \end{pmatrix} = \frac{\delta H_2}{\delta u}, \quad H_2 = \frac{1}{8} qr_{xx} + \frac{1}{8} q_{xx} r - \frac{1}{4} \alpha q^2 r^2 + \frac{3}{16} \beta q^2 rr_x - \frac{3}{16} \beta qq_x r^2 + \frac{1}{8} \beta^2 q^2 r^3.
\] (38)

Therefore a tri-Hamiltonian structure for the coupled AKNS-Kaup-Newell system (36) can be given by

\[
u_t = K_2 = J \frac{\delta H_2}{\delta u} = M \frac{\delta H_1}{\delta u} = N \frac{\delta H_0}{\delta u},
\] (40)
where three Hamiltonian functions $H_0$, $H_1$ and $H_2$ are defined by (37), (38) and (39), respectively. Based on the recursion scheme in [18, 19], this leads to a tri-Hamiltonian structure for each nonlinear system in the coupled AKNS-Kaup-Newell hierarchy

$$u_t = K_n = J \frac{\delta H_n}{\delta u} = M \frac{\delta H_{n-1}}{\delta u} = N \frac{\delta H_{n-2}}{\delta u}, \ n \geq 2. \quad (41)$$

The existence of all Hamiltonian functions $H_n$ to satisfy $\frac{\delta H_n}{\delta u} = \Psi^n \frac{\delta H_0}{\delta u}$, $n \geq 0$, is guaranteed by the hereditary property of the hereditary symmetry operator $\Phi$. They are all common conserved densities for the whole AKNS-Kaup-Newell hierarchy and commute with each other under three Poisson brackets associated with $J$, $M$ and $N$. This is because, for example, we can compute that

$$\{H_m, H_n\}_J := \int \langle \frac{\delta H_m}{\delta u}, J \frac{\delta H_n}{\delta u} \rangle \ dx = \int \langle \frac{\delta H_m}{\delta u}, J \Psi \frac{\delta H_{n-1}}{\delta u} \rangle \ dx$$

$$= \int \langle \frac{\delta H_m}{\delta u}, \Phi J \frac{\delta H_{n-1}}{\delta u} \rangle \ dx = \int \langle \Psi \frac{\delta H_m}{\delta u}, J \frac{\delta H_{n-1}}{\delta u} \rangle \ dx$$

$$= \{H_{m+1}, H_{n-1}\}_J = \cdots = \{H_n, H_m\}_J, \ m < n, \ m, n \geq 0.$$

It gives rise to the commutativity of the conserved densities $H_n$, $n \geq 0$, by combining the skew-symmetric property of the Poisson brackets. Furthermore we have

$$[K_m, K_n] = J \frac{\delta}{\delta u} \{H_m, H_n\} = 0, \ m, n \geq 0, \quad (42)$$

which implies that each coupled AKNS-Kaup-Newell system has infinitely many commuting symmetries. This may also be seen from a zero Lie derivative $L_{u_x} \Phi = 0$. The property of $L_{u_x} \Phi = 0$ also guarantees that the hereditary symmetry operator defined by (19) is a common recursion operator for all systems in the coupled AKNS-Kaup-Newell hierarchy [29].

V Concluding remarks

We have introduced a set of Hamiltonian operators and presented some corresponding hereditary symmetry operators. Therefore a coupled AKNS-Kaup-Newell hierarchy of systems of soliton equations of real form is proposed. Zero curvature representations and tri-Hamiltonian structures are established for all systems in the hierarchy.
Interestingly this coupled AKNS-Kaup-Newell hierarchy contains two different reductions of the AKNS hierarchy and the Kaup-Newell hierarchy. A natural problem we want to ask is what conditions could be found for the existence of similar coupled soliton hierarchies associated with two or more given soliton hierarchies and how one constructs such coupled soliton hierarchies if they exist.

Because our coupled AKNS-Kaup-Newell hierarchy includes the AKNS hierarchy and the Kaup-Newell hierarchy as two simple reductions, tri-Hamiltonian structures can be constructed for the AKNS hierarchy and the Kaup-Newell hierarchy, based on the obtained tri-Hamiltonian structures of the coupled AKNS-Kaup-Newell hierarchy. The corresponding tri-Hamiltonian structure for the Kaup-Newell system of nonlinear derivative Schrödinger equations has been raised recently in [20] and a nonlinearization problem has been manipulated for the associated spectral problem [21]. We believe that some other nice properties may also be achieved for the the coupled AKNS-Kaup-Newell hierarchy.

It is worthy pointing out that by using a similar deduction to one in Section III, a general hereditary symmetry operator defined by (18) can be constructed from the following spectral problem

\[ \phi_x = U\phi, \quad U = U(u, \lambda) = \begin{pmatrix} \frac{1}{2\gamma} \lambda & q \\ \frac{1}{2\gamma}(\alpha - \frac{\beta}{\gamma} \lambda) r & -\frac{1}{2\gamma} \lambda \end{pmatrix} \]  

with the same constants \( \alpha, \beta, \gamma \) as ones in (18). It is apparent that the condition of \( \gamma \neq 0 \) is required, but \( \alpha \) and \( \beta \) may be equal to zero. Only a condition of \( \alpha^2 + \beta^2 \neq 0 \) is needed for \( \alpha \) and \( \beta \), in order to guarantee the injective property of the Gateaux derivative \( U' \). It also deserves to mention that the Hamiltonian operators defined by (1) can lead to other hierarchies of systems of evolution equations. For example, a hierarchy of bi-Hamiltonian systems \( u_t = \Phi^n u_x, \ n \geq 0 \), can be generated from a hereditary symmetry operator \( \Phi \) defined by (17). What is more, we can make another choice of an invertible Hamiltonian operator

\[ J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \]  

13
which has an inverse operator

\[ J^{-1} = \begin{pmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{pmatrix}. \]

It constitutes a Hamiltonian pair together with \( M \) defined by (1). Thus we can have the corresponding hereditary symmetry operator

\[ \Phi = MJ^{-1} = \begin{pmatrix} \alpha_2 \partial^{-1} + \alpha_3 - \alpha_1 q \partial^{-1} r \partial^{-1} & \alpha_1 q \partial^{-1} q \partial^{-1} \\ \alpha_1 r \partial^{-1} r \partial^{-1} & -\alpha_2 \partial^{-1} + \alpha_3 - \alpha_1 r \partial^{-1} q \partial^{-1} \end{pmatrix}. \quad (45) \]

This generates a new hierarchy \( u_t = \Phi^n u_x, n \geq 0 \), which is an inverse hierarchy of the Kaup-Newell hierarchy. In conclusion, Hamiltonian operators of the same type may lead to soliton hierarchies of different types.

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References

[1] A. C. Newell, *Solitons in mathematics and Physics* (SIAM, Philadelphia, 1985).

[2] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[3] A. S. Fokas and B. Fuchssteiner, Phys. Lett. A 86, 341–345 (1981).

[4] G. Z. Tu, Sci. in China A 32, 142–153 (1989).

[5] W. X. Ma, Chin. Ann. Math. B 18, 79–88 (1997).

[6] M. Boiti, P. J. Caudrey and F. Pempinelli, Nuovo Cimento B 83, 71–87 (1984).

[7] M. Antonowicz and A. P. Fordy, Physica D 28, 345–357 (1987).

[8] M. Antonowicz and A. P. Fordy, J. Phys. A: Math. Gen. 21, L269-L275 (1988).

[9] W. X. Ma, J. Phys. A: Math. Gen. 26 L1169–L1174; W. X. Ma and Z. X. Zhou, Prog. Theoret. Phys. 96, 449–457 (1996).

[10] W. X. Ma and B. Fuchssteiner, Phys. Lett. A 213, 49–55 (1996).

[11] B. C. Zhang, Ann. of Diff. Eqs. 13, 408–418 (1997).

[12] F. Magri, in: *Nonlinear Evolution Equations and Dynamical Systems*, Lectures Notes in Physics Vol. 120, eds. M. Boiti, F. Pempinelli and G. Soliani (Springer-Verlag, Berlin), pp233–263(1980).

[13] W. X. Ma, J. Phys. A: Math. Gen. 31, 7585–7591 (1998).

[14] W. X. Ma and M. Pavlov, Phys. Lett. A 246, 511–522 (1998).

[15] B. Fuchssteiner and A. S. Fokas, Physica D 4, 47–66 (1981).

[16] W. X. Ma and W. Strampp, Phys. Lett. A 185, 277-286 (1994).

[17] W. X. Ma, Q. Ding, W. G. Zhang and B. Q. Lu, Il Nuovo Cimento B 111, 1135–1149 (1996).

[18] F. Magri, J. Math. Phys. 19, 1156–1162 (1978).

[19] I. M. Gel’fand and I. Y. Dorfman, Funct. Anal. Appl. 13, 248–262 (1979).

[20] W. X. Ma and R. G. Zhou, preprint (1998).

[21] R. G. Zhou and W. X. Ma, preprint (1998).