Optimality of the recursive Neyman allocation

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Abstract

We derive a formula for the optimal sample allocation in a general stratified scheme under upper bounds on the sample strata-sizes. Such a general scheme includes SRSWOR within strata as a special case. The solution is given in terms of $\mathcal{V}$-allocation with $\mathcal{V}$ being the set of take-all strata. We use $\mathcal{V}$-allocation to give a formal proof of optimality of the popular recursive Neyman algorithm, $rNa$. This approach is convenient also for a quick proof of optimality of the algorithm of Stenger and Gabler (2005), $SGa$, as well as of its modification, $coma$, we propose here. Finally, we compare running times of $rNa$, $SGa$ and $coma$. Ready-to-use R-implementations of these algorithms are available on CRAN repository at https://cran.r-project.org/web/packages/stratallo.

1 Introduction

Optimal sample allocation in stratified sampling scheme is one of the basic problems of survey methodology. An abundant body of literature, going back to the classical optimal solution of Tchuprov (1923) and Neyman (1934) for the case of simple random sampling without replacement (SRSWOR) design in each stratum, is devoted to this issue. In recent years, there has been a growing interest in more refined allocation methods, mostly based on non-linear programming (NLP), see, e.g. Valliant, Dever and

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Kreuter (2018) and references therein. Except the NLP methods, which give only approximate solutions (typically sufficiently precise), a number of recursive allocation methods have been developed over the years. The recursive Neyman algorithm, described in Remark 12.7.1 in Särndal, Swensson, Wretman (1992), seems to be popular among practitioners. For more recent recursive methods see e.g. Kadane (2005), Stenger and Gabler (2005), Gabler, Ganninger and Münnich (2012), Friedrich, Münnich, de Vries and Wagner (2015) or Wright (2017, 2020). Another non-recursive method, based on fixed point iterations, was proposed in Münnich, Wagner and Sachs (2012). In this paper we are concerned with recursive methods.

Let $U$ be a population of size $N$. For a study variable $Y$ defined on $U$ we write $y_k = Y(k)$ to denote its value for unit $k \in U$. The parameter of interest is the total of variable $Y$ in $U$, $t_Y = \sum_{k \in U} y_k$.

To estimate $t_Y$ we consider the stratified SRSWOR design, under which the population $U$ is stratified, i.e. $U = \bigcup_{w \in W} U_w$, where the strata $U_w, w \in W$, are disjoint and non-empty and $W$ denotes the set of strata labels. We denote by $N_w$ the size of $U_w$, $w \in W$. A sample $S_w$ of size $n_w$ is drawn according to SRSWOR from $U_w, w \in W$. The draws between strata are independent. The $\pi$-estimator of $t_Y$ is given by $\hat{t}_{st} = \sum_{w \in W} \frac{N_w}{n_w} \sum_{k \in S_w} y_k$. It is design unbiased with variance

$$D_{st}^2 = \sum_{w \in W} \frac{d_w^2}{n_w} - \sum_{w \in W} \frac{d_w^2}{N_w},$$

where $d_w = N_wS_w$ and $S_w^2 = \frac{1}{N_w-1} \sum_{k \in U_w} (y_k - \bar{y}_w)^2$ with $\bar{y}_w = \frac{1}{N_w} \sum_{k \in U_w} y_k, w \in W$.

The problem of optimal sample allocation lies in the determination of the allocation vector $(n_w)_{w \in W}$ that minimizes (1) subject to

$$\sum_{w \in W} n_w = n,$$

with $n \leq N = \sum_{w \in W} N_w$.

The classical solution to this problem, called traditionally the Tchuprov-Neyman allocation (Tchuprov 1923; Neyman 1934), has the form

$$n_w^* = n \frac{d_w}{d}, \quad w \in W,$$

where $d = \sum_{v \in W} d_v$. 2
It is well-known that \((n_w^*)_{w \in \mathcal{W}}\) given by (3) may not be feasible since it may violate the natural constraints
\[
n_w \leq N_w \quad \text{for all} \quad w \in \mathcal{W}.
\] (4)

In other words, (3) may over-allocate the sample is some strata. Thus, it has to be modified. In Survey Methods and Practice (2010), the authors write "... a census should be conducted in the over-allocated strata. The overall sample size resulting from such over-allocation will then be smaller than the original sample size, so the overall precision requirements might not be met. The solution is to increase the sample in the remaining strata where \(n_w^*\) is smaller than \(N_w\) using the surplus in the sample sizes obtained from the overallocated strata." This idea is realized through the recursive Neyman algorithm (referred to by \(rNa\) in the sequel). It is a popular tool in everyday survey practice, though its optimality remains an open question.

An alternative approach was introduced in Stenger and Gabler (2005), where the authors proposed another allocation algorithm (referred to by \(SGa\) in the sequel) and established its optimality. This algorithm can be described as follows: first, order strata labels in \(\mathcal{W}\) with respect to non-increasing values of \(S_w, w \in \mathcal{W}\); second, perform a sequential search for the last take-all stratum in \(\mathcal{W}\) ordered in the previous step; third, compute the Tchuprov-Neyman allocation in the remaining strata. Gabler et al. (2012) extended this approach to cover both, the upper and the lower bounds on the sample strata sizes and proposed an R-function, called \(noptcond\), as implementation of their allocation procedure. In Münnich et al. (2012), the authors used the Karush-Kuhn-Tucker (referred to by \(KKT\) in the sequel) conditions to derive optimal allocation formula expressed in terms of, so called, optimal Lagrange multiplier, being the root of a highly irregular function. They analyzed several fixed point iteration routines to speed up its calculation. Recently, integer-valued optimal allocation procedures have been developed in Friedrich et al. (2015) and Wright (2017).

In this paper, (a) we prove optimality of the recursive Neyman algorithm, (b) we introduce a modification of the Stenger-Gabler algorithm and prove its optimality, and (c) we compare computational efficiency of the recursive Neyman algorithm, the Stenger-Gabler algorithm and its proposed modifica-
Actually, we consider a more general optimization scheme described in Problem 1.

**Problem 1.** Given numbers \( n, a_w, b_w > 0, w \in \mathcal{W} \), minimize the objective function

\[
f(x) = \sum_{w \in \mathcal{W}} a_w^2 x_w^2, \quad x = (x_w)_{w \in \mathcal{W}}
\]  

subject to

\[
\sum_{w \in \mathcal{W}} x_w = n \quad \text{and} \quad 0 < x_w \leq b_w, \quad w \in \mathcal{W}.
\]

Problem 1 covers the case of stratified SRSWOR by assigning \( a_w = d_w, b_w = N_w, w \in \mathcal{W} \).

Clearly, it is feasible only if \( n \leq \sum_{w \in \mathcal{W}} b_w \). When \( n = \sum_{w \in \mathcal{W}} b_w \), the solution is trivial and is equal to \( (b_w)_{w \in \mathcal{W}} \). Therefore, we assume throughout this paper that \( n < \sum_{w \in \mathcal{W}} b_w \). Since Problem 1 is a convex optimization problem, its solution exists and is unique. It is identified in Theorem 1.1 below.

For \( \mathcal{V} \subseteq \mathcal{W} \) by \( x^\mathcal{V} = (x^\mathcal{V}_w)_{w \in \mathcal{W}} \) we denote the vector with entries

\[
x^\mathcal{V}_w = \begin{cases} 
  b_w, & \text{for } w \in \mathcal{V}, \\
  a_w s(\mathcal{V}), & \text{for } w \not\in \mathcal{V},
\end{cases}
\]  

where \( s \) is a strictly positive function defined on proper subsets of \( \mathcal{W} \) by

\[
s(\mathcal{V}) = \frac{n - \sum_{w \in \mathcal{V}} b_w}{\sum_{w \not\in \mathcal{V}} a_w}, \quad \mathcal{V} \subseteq \mathcal{W}.
\]

We refer to \( x^\mathcal{V} \) by \( \mathcal{V} \)-allocation. It turns out that the optimal allocation of the sample among strata is of the form (6) for a unique subset \( \mathcal{V} \subseteq \mathcal{W} \).

**Theorem 1.1.** The \( \mathcal{V} \)-allocation vector \( x^\mathcal{V} \) solves Problem 1 if and only if

\[
\mathcal{V} = \{ w \in \mathcal{W} : c_w s(\mathcal{V}) \geq 1 \},
\]

where \( c_w = \frac{a_w}{b_w}, w \in \mathcal{W} \).
The proof of Theorem 1.1 based on the KKT conditions, is given in the Appendix. We use (8) to prove optimality of the recursive Neyman algorithm, rNa, in Section 2. In Section 3, following (8), we give a short proof of optimality of the Stenger-Gabler algorithm, SGa and introduce a modification, coma, for which we prove its optimality. In Section 4, we compare rNa, SGa and coma in terms of computational efficiency (also with algorithms designed for optimal allocation with double-sided constraints: noptcond of Gabler et al. (2012) and capacity scaling of Friedrich et al. (2015)). As pointed out in Männich et al. (2015), computational efficiency of allocation algorithms becomes an issue "in cases with many strata or when the optimal allocation has to be applied repeatedly, such as in iterative solutions of stratification problems". For the latter issue the reader is referred to Dalenius and Hodges (1959), Lednicki and Wieczorkowski (2003), Guning and Horgan (2004), Kozak and Verma (2006) and Baillargeon and Rivest (2011). Moreover, in Section 4 we also compare variances of estimators based on the optimal integer-valued allocation (obtained e.g. with the capacity scaling algorithm) and of those based on the integer-rounded optimal allocation (obtained e.g. with the rNa).

2 The recursive Neyman algorithm

In this section, we prove that the rNa (Remark 12.7.1 in Särndal et al., 1992), leads to the allocation that minimizes (1) under the constraints (2) and (4). For a generalized setting of this minimization problem, Problem [1], the algorithm rNa proceeds as follows:

Step 1: Let \( V_1 = \emptyset, \ r = 1 \).

Step 2: Compute \( s(V_r) \) according to (7).

Step 3: Let \( R_r = \{ w \in W \setminus V_r : c_w s(V_r) \geq 1 \} \).

Step 4: If \( R_r = \emptyset \), set \( r^* = r \) and go to Step 5;

otherwise, set \( V_{r+1} = V_r \cup R_r, \ r \leftarrow r + 1 \), and go to Step 2.

Step 5: Return \( x^{V_{r^*}} \) according to (6).

The numerical behavior of the rNa is illustrated in Table 1. Before we proceed to prove that the rNa does indeed find the optimal solution to Problem [1], we first derive in Lemma 2.1 a monotonicity property.
Table 1: For an artificial population with 20 strata, \( n = 8000 \), \( (c_w)_{w \in \mathcal{W}} \) generated as absolute values of independent Cauchy random variables and \( b_w = 1000 \), \( w \in \mathcal{W} \), the \( rNa \) assigns as take-all strata: \( R_1 = \{6, 17\} \), \( R_2 = \{15\} \), \( R_3 = \{2\} \), and it stops at iteration 4 (\( R_4 = \emptyset \)) with \( \mathcal{V}_4 = \{2, 6, 15, 17\} \). The optimal (non-integer) allocation (rounded to 0.1) is given in columns labelled \( x_{\mathcal{V}_4}^* \).

| \( w \) | \( c_w \) | \( c_w s(\mathcal{V}_1) \) | \( c_w s(\mathcal{V}_2) \) | \( c_w s(\mathcal{V}_3) \) | \( x_{\mathcal{V}_4}^* \) | \( w \) | \( c_w \) | \( c_w s(\mathcal{V}_1) \) | \( c_w s(\mathcal{V}_2) \) | \( c_w s(\mathcal{V}_3) \) | \( x_{\mathcal{V}_4}^* \) |
|-------|-------|-----------------|-----------------|-----------------|------------|-------|-------|-----------------|-----------------|-----------------|------------|
| 1     | 0.33  | 0.062           | 0.1174          | 0.1303          | 130.3      | 11    | 2.37  | 0.444           | 0.8349          | 0.927           | 927.3      |
| 2     | 2.65  | 0.480           | 0.9016          | 1.0011          | 1000       | 12    | 0.36  | 0.068           | 0.1282          | 0.1423          | 142.3      |
| 3     | 0.15  | 0.029           | 0.0543          | 0.0603          | 60.4       | 13    | 0.14  | 0.026           | 0.0493          | 0.0547          | 54.7       |
| 4     | 0.66  | 0.123           | 0.2316          | 0.2571          | 257.2      | 14    | 0.37  | 0.070           | 0.1316          | 0.1462          | 146.2      |
| 5     | 0.15  | 0.028           | 0.0519          | 0.0577          | 57.7       | 15    | 4.25  | 0.796           | 1.4967          | *               | *          |
| 6     | 15.45 | 2.895           | *               | *               | 1000       | 16    | 0.39  | 0.074           | 0.1386          | 0.1539          | 153.9      |
| 7     | 1.49  | 0.279           | 0.5239          | 0.5817          | 581.9      | 17    | 10.21 | 1.913           | *               | *               | 1000       |
| 8     | 1.74  | 0.326           | 0.612           | 0.6796          | 679.7      | 18    | 0.10  | 0.018           | 0.0339          | 0.0376          | 37.6       |
| 9     | 0.30  | 0.056           | 0.1057          | 0.1173          | 117.3      | 19    | 0.23  | 0.044           | 0.0827          | 0.0918          | 91.8       |
| 10    | 0.93  | 0.174           | 0.3278          | 0.364           | 364.1      | 20    | 0.51  | 0.095           | 0.1779          | 0.1975          | 197.6      |

of function \( s \), defined in (7). This property turns out to be a convenient tool in proving optimality of the \( rNa \) as well as the algorithms we consider in Section 3.

**Lemma 2.1.** Let \( A, B \subset \mathcal{W} \) be such that \( A \cap B = \emptyset \) and \( A \cup B \subsetneq \mathcal{W} \). Then

\[
s(A \cup B) \geq s(A) \quad \text{if and only if} \quad s(A) \sum_{w \in B} a_w \geq \sum_{w \in B} b_w. \tag{9}
\]

**Proof.** Clearly, for \( \alpha, \beta, \delta \geq 0 \) and \( \gamma > 0 \), we have

\[
\frac{\alpha}{\gamma} \geq \frac{\alpha + \beta}{\gamma + \delta} \quad \text{if and only if} \quad \frac{\alpha + \beta}{\gamma + \delta} \geq \beta. \tag{10}
\]

Denote \( \mathcal{C} = A \cup B \). Take \( \alpha = n - \sum_{w \in \mathcal{C}} b_w, \gamma = \sum_{w \in \mathcal{C}} a_w, \beta = \sum_{w \in B} b_w, \delta = \sum_{w \in \mathcal{C}} a_w \). Thus, \( \frac{\alpha}{\gamma} = s(\mathcal{C}) \) and \( \frac{\alpha + \beta}{\gamma + \delta} = s(A) \). Then (9) is an immediate consequence of (10). \( \square \)

**Theorem 2.2.** The algorithm \( rNa \) solves Problem 7

**Proof.** According to (8), in order to prove that \( x_{\mathcal{V}_r}^* \) in (6) is the optimal allocation, we need to show that

\[
w \in \mathcal{V}_r^* \quad \text{if and only if} \quad c_w s(\mathcal{V}_r^*) \geq 1. \tag{11}
\]
For \( r^* = 1 \) we have \( \mathcal{V}_{r^*} = \emptyset \) and \( R_1 = \emptyset \), i.e. (11) trivially holds.

Consider \( r^* > 1 \). Since \( n < \sum_{w \in \mathcal{W}} b_w \) we have \( r^* \leq K \), where \( K \) is the number of strata.

Sufficiency: First, assume that \( c_w s(\mathcal{V}_{r^*}) \geq 1 \) and \( w \notin \mathcal{V}_{r^*} \). Then, Step 4 of \( rNa \) yields \( c_w s(\mathcal{V}_{r^*}) < 1 \) for \( w \notin \mathcal{V}_{r^*} \), which is a contradiction.

Necessity: By Step 3 of \( rNa \), we have \( s(\mathcal{V}_r) a_w \geq b_w \), \( w \in R_r \), for every \( r \in \{1, \ldots, r^* - 1\} \). Summing these inequalities over \( w \in R_r \) we get the second inequality in (9) with \( A = \mathcal{V}_r \), and \( B = R_r \). Since \( \mathcal{V}_r \cup R_r \subseteq \mathcal{W} \) and \( \mathcal{V}_r \cap R_r = \emptyset \), by Lemma 2.1 the first inequality in (9) follows. Consequently,

\[
s(\mathcal{V}_1) \leq \ldots \leq s(\mathcal{V}_{r^*}).
\] 

(12)

Now, assume that \( w \in \mathcal{V}_{r^*} \). Thus, \( w \in R_r \) for some \( r \in \{1, \ldots, r^* - 1\} \). Then, again using Step 3 of \( rNa \), we get \( c_w s(\mathcal{V}_r) \geq 1 \). Consequently, (12) yields \( c_w s(\mathcal{V}_{r^*}) \geq 1 \).

3 Revisiting the Stenger and Gabler methodology

Stenger and Gabler (2005, Lemma 1) proposed another allocation algorithm and proved its optimality. In contrast to the \( rNa \), their algorithm is based on ordering the set of strata labels \( \mathcal{W} \) with respect to \( (c_w)_{w \in \mathcal{W}} \). In this section, we adjust Stenger and Gabler’s algorithm to the general scheme of Problem 1 (such adjusted algorithm is also referred to by \( SGa \)) and give a short proof of its optimality based on (8).

We also propose a modification called \( coma \). To describe both algorithms it is convenient to introduce the notation: \( \mathcal{V}_1 = \emptyset \) and \( \mathcal{V}_r = \{1, \ldots, r - 1\} \) for \( r > 1 \).

The algorithm \( SGa \) proceeds as follows:

Step 1: Reorder \( \mathcal{W} = \{1, \ldots, K\} \) according to \( c_1 \geq \ldots \geq c_K \).

Step 2: Let \( r = 1 \).

Step 3: Compute \( s(\mathcal{V}_r) \) according to (7).

Step 4: If \( c_r s(\mathcal{V}_r) < 1 \), set \( r = r^* \) and go to Step 5;
otherwise, let \( r \leftarrow r + 1 \), and go to Step 3.

Step 5: Return \( x^{\mathcal{V}_{r^*}} \) according to (6).
Proposition 3.1. The algorithm SGa solves Problem 1.

Proof. According to (8), in order to prove that \( x^{V_{r^*}} \) in (6) is the optimal allocation, we need to show that

\[
 r \in V_{r^*} \text{ if and only if } c_r s(V_{r^*}) \geq 1. \tag{13}
\]

For \( r^* = 1 \) we have \( V_{r^*} = \emptyset \) and \( c_r s(V_{r^*}) < 1 \) for all \( r \in W \), i.e. (13) trivially holds.

Consider \( r^* > 1 \). As in the previous proof, we have \( r^* \leq K \).

Sufficiency: Assume that \( c_r s(V_{r^*}) \geq 1 \) and \( r \notin V_{r^*} \), i.e. \( r \geq r^* \). By Step 4 of SGa, we have \( c_{r^*} s(V_{r^*}) < 1 \). Since \( (c_r)_{r=1,\ldots,K} \) is non-increasing, it follows that \( c_r s(V_{r^*}) < 1 \) for \( r \geq r^* \), which is a contradiction. Thus, \( r \in V_{r^*} \).

Necessity: For \( r \in V_{r^*} \), in view of Step 4 of SGa, we have \( c_r s(V_{r}) \geq 1 \). Thus, the second inequality in (9) is satisfied with \( A = V_{r} \) and \( B = \{r\} \). Since \( V_{r} \cup \{r\} \subseteq W \) and \( V_{r} \cap \{r\} = \emptyset \), by Lemma 2.1 the first inequality in (9) follows. Consequently,

\[
 s(V_1) \leq \ldots \leq s(V_{r^*}).
\]

Hence \( c_r s(V_{r^*}) \geq 1 \) for \( r \in V_{r^*} \). \( \square \)

Finally, we propose a new algorithm, coma (named after change of monotonicity algorithm), which is a modification of the approach from Stenger and Gabler (2005). To describe this algorithm, it is convenient to denote \( s(W) = 0 \). The set of strata labels \( W \) needs ordering as in SGa. The algorithm coma proceeds as follows:

Step 1: Reorder \( W = \{1, \ldots, K\} \) according to \( c_1 \geq \ldots \geq c_K \).

Step 2: Let \( r = 1 \).

Step 3: Compute \( s(V_r) \) and \( s(V_{r+1}) \) according to (7).

Step 4: If \( s(V_r) > s(V_{r+1}) \), set \( r^* = r \) and go to Step 5;

otherwise, let \( r \leftarrow r + 1 \), and go to Step 3.

Step 5: Return \( x^{V_{r^*}} \) according to (6).

Proposition 3.2. The algorithm coma solves Problem 1.
Table 2: For the artificial population considered in Table 1 we apply SGa and coma (\(w\) refers to strata labels as in Table 1, i.e. before ordering performed in Step 1 of both algorithms). In every iteration 1–4, one stratum is assigned to the set of take-all strata: 6, 17, 15 and 2. Both procedures stop at iteration 5.

**Proof.** The proof follows from the equivalence (9) after referring to Proposition 3.1.

Table 2 shows how SGa and coma work for the artificial population with 20 strata considered in Table 1.

### 4 Numerical experiments

In the simulations, using the R software (2019), we compared the computational efficiency of rNa, SGa and coma as well as some known algorithms for optimal allocation. Since the efficiency of SGa and coma turned out to be quite similar, we present results only for the coma. The R code used in our experiments is available at [https://github.com/rwieczor/recursive_Neyman](https://github.com/rwieczor/recursive_Neyman).

Two artificial populations with several strata were constructed by iteratively (100 and 200 times) binding collections of observations (each collection of 10000 elements) generated independently from lognormal distributions with varying parameters. The logarithms of generated random variables have mean equal to 0 and standard deviations equal to \(\log(1+i)\), \(i = 1, \ldots, N_{\text{max}}\), where \(N_{\text{max}}\) is equal 100 and 200, respectively. In each iteration, the strata were created using the geometric stratification method of Gunning and Horgan (2004) (implemented in the R package *stratification*) with parameter 10 being the number of strata and targeted coefficient of variation equal to 0.05.

For these populations, we calculated the following vectors of parameters: populations sizes, \((N_w)_{w \in W}\).
and population standard deviations in strata, \( (S_w)_{w \in W} \), both needed for the allocation algorithms. Finally, the original order of strata was rearranged by a random permutation. In this way two populations with 507 and 969 strata were created (for details of the implementation see the R code available on GitHub repository).

We used the `microbenchmark` R package for numerical comparisons of computational efficiency of the algorithms. The results obtained are presented in Fig. 1. The simulations suggest that the \( rNa \) is typically more efficient than \( coma \) (and \( SGa \)). However, this is not always the case as Fig. 2 shows. For the experiment referred to in Fig. 2, we created an artificial data set with significant differences in standard deviations between the strata.

We also compared computational efficiency of \( rNa \) and \( coma \) with the algorithms: \( \text{noptcond} \) of Gabler et al. (2012) and \( \text{capacity scaling} \) of Friedrich et al. (2015), designed for, respectively, non-integer and integer allocation under both lower and upper bounds for the sample strata sizes. Numerical experiments show that these two algorithms (with lower bounds set to zero) were considerably slower than \( rNa \) and \( coma \). Moreover, the design variances obtained for the optimal non-integer allocation before and after rounding (we used optimal rounding of Cont and Heidari (2014)) and for the optimal integer allocation, were practically indistinguishable, see Table 3.

One may argue that in real life applications the number of strata may not be large and so differences in computational efficiency are of marginal importance. Nevertheless, in some applications, like census-related surveys, the number of strata can be counted even in tens of thousands (there were more than 20 000 strata in German Census 2011, see Burgard and Münnich (2012)). The issue of computational complexity of optimal allocation algorithms has been addressed e.g. in Münnich et al. (2012). Computational efficiency of optimal allocation algorithms becomes a crucial issue in iterative solutions of stratification problems, see e.g. Lednicki and Wieczorkowski (2003), Baillargeon and Rivest (2011) and Barcaroli (2014). In such procedures the allocation routine may be typically repeated a very large number of times (e.g. millions or more, depending on desired accuracy of approximations).
Figure 1: Comparison of running times of R-implementations coma and rNa for two lognormal populations. Top graphs show the empirical median of performance times calculated from 100 repetitions. Numbers of iterations of the rNa follow its graph. Counts of take-all strata are inside bars of bottom graphs.
Figure 2: Comparison of running times of R-implementations of coma and rNa. Top graphs show the empirical median of performance times calculated from 100 repetitions. Computations were done for an artificial population with $S_w = 10^w$, and $N_w = 1000$, $w = 1, \ldots, 20$. Numbers of iterations of the rNa follow its graph. Counts of take-all strata are inside bars of bottom graphs: the first two are 0, 1.
Table 3: Variances $D^2$ and $D_0^2$ are based on optimal non-integer and optimal integer allocations respectively. For variances $\tilde{D}^2$, based on rounded optimal non-integer allocation, we systematically get $\tilde{D}^2/D_0^2 = 1$ (up to five decimal digits).

5 Final remarks and conclusions

We proved that the recursive Neyman algorithm, $rNa$, is optimal under upper bounds on sample strata sizes. The approach is based on optimality of the $V$-allocation (6) and (8), derived from the KKT conditions. We also proposed a modification, $coma$, of the algorithm, $SGa$, of Stenger and Gabler (2005) and, using the $V$-allocation, we gave short proofs of optimality for both algorithms.

Simulation comparisons of computational efficiency showed that $rNa$ is much faster than $coma$ and $SGa$ (the latter two being of similar efficiency) and its relative efficiency increased with the sample fraction. Nevertheless, there may exist situations when $coma$ (or $SGa$) happen to be more efficient than $rNa$. Ready-to-use R-implementations of the algorithms are available on CRAN repository in the new package stratallo:

https://cran.r-project.org/web/packages/stratallo

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A Appendix: Convex optimization scheme and the KKT conditions

*Proof of Theorem 1.1* Problem 1 belongs to a class of optimization problems of the form: minimize a strictly convex function \( f : (0, \infty)^m \to \mathbb{R} \), under constraints

\[
g_i(x) \leq 0, \quad i = 1, \ldots, r, \quad \text{and} \quad h_j(x) = 0, \quad j = 1, \ldots, s,
\]
satisfied for all \( x \in (0, \infty)^m \), where \( g_i, i = 1, \ldots, r \), are convex and \( h_j, j = 1, \ldots, s \), are affine. It is well known, see e.g. Boyd and Vanderberghe (2004), that in such case there exists a unique \( x^* \in (0, \infty)^m \), such that \( f \) attains its global minimum at \( x = x^* \). The minimizer, \( x^* \), can be identified through the set of equations/inequalities, known as the KKT conditions:

There exist \( \lambda_i \in \mathbb{R}, i = 1, \ldots, r \), and \( \mu_j \in \mathbb{R}, j = 1, \ldots, s \), such that

\[
\nabla f(x^*) + \sum_{i=1}^r \lambda_i \nabla g_i(x^*) + \sum_{j=1}^s \mu_j \nabla h_j(x^*) = 0, \quad (14)
\]
and

\[
h_j(x^*) = 0, \quad \lambda_i \geq 0, \quad \lambda_i g_i(x^*) = 0, \quad g_i(x^*) \leq 0 \quad (15)
\]
for \( i = 1, \ldots, r, j = 1, \ldots, s \).

We consider the KKT scheme with the objective function \( f \) defined in (5) and with

\[
h(x) = \sum_{w \in \mathcal{W}} x_w - n \quad \text{and} \quad g_w(x) = x_w - b_w, \quad w \in \mathcal{W}.
\]
Thus
\[ \nabla f(x) = -\left(\frac{a_w^2}{x_w^2}\right)_{w \in W}, \quad \nabla h(x) = (1, \ldots, 1), \quad \nabla g_w(x) = 1_w, \quad w \in W, \]
where $1_w$ denotes the vector with all entries 0, except the entry with label $w$ which is 1.

Then, (14) and (15) read: there exist $\lambda_w \geq 0$, $w \in W$, and $\mu \in \mathbb{R}$ such that
\[ -\frac{a_w^2}{x_w^2} + \lambda_w + \mu = 0, \quad w \in W, \tag{16} \]
\[ \sum_{w \in W} x_w^* = n, \quad x_w^* \leq b_w, \quad w \in W, \quad w \in W, \tag{17} \]
and
\[ \lambda_w (x_w^* - b_w) = 0, \quad w \in W. \tag{18} \]

Since Problem 1 is a convex optimization problem, its solution exists and is unique. Therefore, to prove Theorem 1.1 it suffices to show that conditions (16)–(18) are satisfied for $x^* = x^V$ with $V$ defined in (8).

Let $\mu = s^2(V) > 0$, with $s(V)$ defined in (7), $\lambda_w = c_w^2 - \mu$, $w \in V$, and $\lambda_w = 0$ for $w \not\in V$. Note that (8) yields $\lambda_w \geq 0$ for $w \in V$. Then (16), as well as the equality in (17), i.e.
\[ \sum_{w \in V} x_w^V = \sum_{w \in V} b_w + s(V) \sum_{w \not\in V} a_w = n, \]
are satisfied. Inequalities in (17) are trivial for $w \in V$, and for $w \not\in V$ they follow from (8). Finally, (18) holds true, since $\lambda_w = 0$ for $w \not\in V$ and $x_w^V - b_w = 0$ for $w \in V$. \qed