Non-virtually abelian discontinuous group actions vs. proper $SL(2, \mathbb{R})$-actions on homogeneous spaces

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Abstract

We develop algorithms and computer programs which verify criteria of properness of discrete group actions on semisimple homogeneous spaces. We apply these algorithms to find new examples of non-virtually abelian discontinuous group actions on homogeneous spaces which do not admit proper $SL(2, \mathbb{R})$-actions.

Key words and phrases: proper action, homogeneous space, semisimple Lie algebra, GAP4, computing with simple Lie algebras.

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1 Proper actions of non-virtually abelian discrete subgroups

Assume that $G$ is a linear connected semisimple real Lie group and $H \subset G$ is a closed reductive subgroup of $G$ with finitely many connected components such that $G/H$ is non-compact. Denote by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of $G, H$ respectively. Recall that a discrete group is non-virtually abelian if it does not contain an abelian subgroup of finite index. Consider the following three conditions:

• $C1 :=$ the space $G/H$ admits a properly discontinuous action of an infinite discrete subgroup of $G$,
• \( C_2 \) := the space \( G/H \) admits a properly discontinuous action of a non-virtually abelian infinite discrete subgroup of \( G \),

• \( C_3 \) := the space \( G/H \) admits a proper action of a subgroup \( L \subset G \) locally isomorphic to \( SL(2,\mathbb{R}) \).

We will say that \( G/H \) is a \( C_i \) space, \( i = 1, 2, 3 \) if \( G/H \) fulfills the condition \( C_i \). If \( H \) is compact then every closed subgroup of \( G \) acts properly on \( G/H \) and all the three conditions above hold. However this is not the case for non-compact \( H \). For instance we have the Calabi-Markus phenomenon (see, for example [22, Corollary 4.4]):

\[ G/H \text{ is } C_1 \text{ if and only if } \text{rank}_\mathbb{R} \mathfrak{g} > \text{rank}_\mathbb{R} \mathfrak{h}. \]

One of the reasons for considering these properties comes from the idea of the "continuous analogue" due to T. Kobayashi: under the above mentioned assumptions a homogeneous space \( G/H \) admits a proper action of an infinite discontinuous group if it admits a proper action of a one-dimensional non-compact closed subgroup in \( G \). Going along this line of thinking, it was shown in [26] that a semisimple symmetric space admits a non-virtually abelian properly discontinuous action of a subgroup \( \Gamma \subset G \) if and only if it admits a proper action of a three-dimensional non-compact simple Lie subgroup of \( G \). Thus, in our terminology, for the class of symmetric spaces, \( C_2 \) and \( C_3 \) are equivalent ([26, Theorem 2.2]). Some other classes of homogeneous spaces of class \( C_3 \) were described in [10] and of class \( C_2 \) in [7]. In [1] a necessary and sufficient condition for \( G/H \) to satisfy \( C_2 \) was found. Let us explain it in some detail.

Let \( K \subset G \) be a maximal compact subgroup of \( G \) and denote by \( \mathfrak{k} \) the Lie algebra of \( K \). Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be a Cartan decomposition of \( \mathfrak{g} \) induced by a Cartan involution fixing \( \mathfrak{k} \) and choose a maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \). The little Weyl group is defined as \( W := N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \) where \( N_K(\mathfrak{a}) \) and \( Z_K(\mathfrak{a}) \) denote respectively the normalizer and the centralizer of \( \mathfrak{a} \) in \( K \). We fix a positive system of the restricted root system of \( \mathfrak{g} \) determined by \( \mathfrak{a} \) and denote by \( \mathfrak{a}^+ \) the (closed) positive Weyl chamber. The finite group \( W \) acts on \( \mathfrak{a} \) by orthogonal transformations as a finite group generated by reflections in the hyperplanes determined by simple roots of the restricted root system of \( \mathfrak{g} \). Let \( w_0 \in W \) be the longest element and put

\[
\mathfrak{b}^+ := \{ X \in \mathfrak{a}^+ | -w_0(X) = X \},
\]

\[
\mathfrak{b} := \text{Span}_\mathbb{R}(\mathfrak{b}^+).
\]

We may assume that \( \theta|_{\mathfrak{h}} \) is a Cartan involution of \( \mathfrak{h} \) so that \( \mathfrak{h} = \mathfrak{t}_h + \mathfrak{p}_h \) is a Cartan decomposition of \( \mathfrak{h} \) such that \( \mathfrak{t}_h \subset \mathfrak{t}, \mathfrak{p}_h \subset \mathfrak{p} \). We also may assume that a maximal abelian subspace \( \mathfrak{a}_h \) of \( \mathfrak{p}_h \) belongs to \( \mathfrak{a} \) (see [22]). The following theorem yields the required criterion:
Theorem 1 ([1], Theorem 1). $G/H$ is C2 if and only if for every $w$ in $W$, $w \cdot a_b$ does not contain $b$. In this case, one can choose a discrete subgroup of $G$ acting properly discontinuously on $G/H$ to be free and Zariski dense in $G$.

For example the space $SL(3, \mathbb{R})/SL(2, \mathbb{R})$ fulfills the condition C1 but does not fulfill the condition C2 and therefore the condition C3 (see [1, Example 1]). For many important classes of homogeneous spaces C2 and C3 are equivalent. For example, apart from irreducible symmetric spaces, this holds for some strongly regular homogeneous spaces ([2, Theorem 2 and Corollary 1]). On the other hand, there are known exactly two examples of spaces which fulfill the condition C2 but not C3 [2, 27]. Thus for a space of reductive type we only have the following:

$$C_1 \iff C_2 \iff C_3.$$ 

The criterion for $G/H$ to be C3 is given in [22] in a general setting of the properness of the action of a Lie subgroup $L \subset G$ on $G/H$. We describe it in Subsection 2.2. Studying conditions C2, C3 and relations between them is of great importance. We refer to a recent paper [23] for a more detailed account. However, let us mention two challenging problems from [23].

**Problem 1.** Determine all pairs $(G, H)$ such that $G/H$ admits co-compact discontinuous actions of discrete subgroups $\Gamma \subset G$.

This is a long standing open problem [20, 21, 22]. There are many partial results, for example, [4, 5, 6, 9, 25], however, the general case is not settled.

**Problem 2.** Find a necessary and sufficient condition for $G/H$ to admit a discontinuous action of a group $\Gamma \subset G$ isomorphic to a surface group $\pi_1(\Sigma_g)$ with $g \geq 2$.

Note that this problem is the same as C3 for the class of symmetric spaces [26]. The general case is still open. Some partial results on homogeneous spaces which are C3 are obtained in [10]. Also, in this context an important question arises.

**Question 1.** What is the relation between classes C2 and C3?

The known two examples of homogeneous spaces $G/H$ which are C2 but not C3 are obtained in [2] and [27] by applying the criterion of the properness of the Lie subgroup action on $G/H$ (see Subsection 2.2 and Theorem 1). One shows that there is an appropriate subspace $a_m \subset a$ such that $W a_m$ does not contain $b$ and that (by Kobayashi’s criterion of proper actions and the classification of nilpotent orbits) for any $\mathfrak{sl}(2, \mathbb{R}) \hookrightarrow \mathfrak{g}$ there exists $g \in G$ such that
In this case $G/A_m$, where the subgroup $A_m \subset G$ corresponds to $a_m$, is a $C_2$ space but not a $C_3$ space. Notice that these examples are quotients obtained by explicit calculations. The calculations are fairly easy because in these cases the description of the embedding $h \hookrightarrow g$ in terms of the root systems is simple. This description enables one to write down the action of the little Weyl group on $a_m$ and use the known theoretical results. Thus, these examples are not sufficient for a qualitative understanding of the relation between $C_2$ and $C_3$. On the other hand, one can see that important Problems 1 and 2 theoretically are settled in terms of root systems and the action of the little Weyl group, by Theorems 1 and 2. The known examples of $G/H$ which are $C_2$ but not $C_3$ are obtained in the same way. Therefore, from the computational point of view these become problems which could and should be attacked in an algorithmic fashion. We pose the following.

**Problem 3.** Create computer algorithms and programs which verify $C_2$ and $C_3$ for general semisimple homogeneous spaces.

Our aim is to solve this problem. We create algorithms verifying $C_2$ versus $C_3$ and implement them in the computational algebra system GAP4 \[15\]. One can see that Theorem 1 yields a sufficient condition for the non-existence of compact quotients $\Gamma \backslash G/H$. Our approach yields methods of computer checking this as well. Note that a substantial amount of work related to the algorithms and computer programs for calculations in Lie algebras and a classification of semisimple subalgebras in semisimple Lie algebras relevant to this article was done in \[8\], \[14\], \[17\], \[18\].

The main application of our solution to Problem 3 is finding new examples of semisimple homogeneous spaces which are $C_2$ but not $C_3$. Consider the class of homogeneous spaces $G/H$, where $G$ is connected, linear and absolutely simple, and $H$ is a maximal proper semisimple subgroup of $G$ and assume that $h^c$ is maximal in $g^c$ and rank $G \leq 8$ (here $h^c$, $g^c$ denote the complexifications of $h$, $g$ respectively). The corresponding pairs $(g, h)$ are classified in \[18\]. More precisely, for each complex embedding of a maximal semisimple subalgebra $h^c \subset g^c$ and each real form $g$ of $g^c$ in \[18\] the real forms $h$ of $h^c$ are determined such that there exists an embedding $h \subset g$. If $h$ has a nontrivial centralizer in $g$ then this centralizer is also given, so that the resulting list consists of reductive subalgebras. The resulting database of embeddings of real forms of maximal semisimple subalgebras determined in \[18\] will be denoted by $\mathcal{GM}$ in this paper.
Theorem 2. For any \((g, h) \in \mathcal{G}\mathcal{M}\) such that \(\text{rank } g \leq 6\) conditions \(C2\) and \(C3\) are equivalent. There are two new examples of \((g, h) \in \mathcal{G}\mathcal{M}\) with split \(g\) and \(h\) such that \(\text{rank } g = 7, 8\) which belong to \(C2\) but not \(C3\):

\[
(e_7(7), sl(2, \mathbb{R}) \oplus f_4(4)), (e_8(8), f_4(4) \oplus g_2(2)).
\]

This theorem is proved by computation, using the implementation of the algorithms given in this paper. The arxiv version of this paper (available on https://arxiv.org/abs/2206.01069) has an ancillary file which is a short GAP program that checks that the two homogeneous spaces listed in the theorem indeed do not satisfy \(C3\).

Notice that the assumption of \(H\) being maximal in \(G\) is very interesting in the context of searching for homogeneous spaces which are \(C2\) but not \(C3\). On one hand, one should expect that homogeneous spaces which are not \(C3\) are given by “large” semisimple subgroups of \(G\) (to be more specific, by subgroups which have as large real rank as possible). On the other hand we are restricted by the condition \(C2\) (so we cannot take \(\text{rank}_\mathbb{R} g = \text{rank}_\mathbb{R} h\)).

Finally, let us stress that the main results of this article are algorithms and their implementation which check \(C2\) and \(C3\), as well as Theorem 2. These use theoretical results contained in Theorem 1 (Benoist), Theorem 3 (Kobayashi), Theorem 4, Proposition 1 (Okuda) and Theorem 5 which follows from Theorem 4 and which yields a method of computer checking whether \(G/H\) is \(C3\).

2 Lie theory approach

Let us note that our notation and terminology is close to the sources [29, 30]. Throughout this section \(G\) is a real reductive linear connected Lie group with the Lie algebra \(g\).

2.1 \(sl(2, \mathbb{R})\)-triples and antipodal hyperbolic orbits

We say that an element \(X \in g\) is hyperbolic, if \(X\) is semisimple (that is, \(ad_X\) is diagonalizable) and all eigenvalues of \(ad_X\) are real.

Definition 1. An adjoint orbit \(O_X := Ad_G X\) is said to be hyperbolic if \(X\) (and therefore every element of \(O_X\)) is hyperbolic. An orbit \(O_Y\) is antipodal if \(-Y \in O_Y\) (and therefore for every \(Z \in O_Y\), \(-Z \in O_Y\)).

Proposition 1 ([28], Proposition 4.5). Let \(X \in g\) be hyperbolic. The intersection of the complex hyperbolic orbit of \(X\) in \(g^c\) with \(g\) is a single adjoint orbit \(O_X\) in \(g\).
A triple \((h, e, f)\) of vectors in \(\mathfrak{g}\) is called an \(\mathfrak{sl}(2, \mathbb{R})\)-triple if
\[
[h, e] = 2e, \quad [h, f] = -2f, \quad \text{and} \quad [e, f] = h.
\]

In what follows we will write \(\mathfrak{sl}_2\)-triple instead of \(\mathfrak{sl}(2, \mathbb{R})\)-triple, and call \(h\) a neutral element. One can show that \(e\) is nilpotent and the adjoint orbit \(\text{Ad}_G h\) is antipodal and hyperbolic (see [11]). Furthermore there is the homomorphism of Lie groups
\[
\Phi : SL(2, \mathbb{R}) \to G
\]
defined by \(d\Phi \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = h, \quad d\Phi \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = e, \quad d\Phi \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = f.\)

2.2 Proper \(SL(2, \mathbb{R})\)-actions and a properness criterion

Let \(H\) and \(L\) be both reductive subgroups in \(G\). Let \(\theta\) be a Cartan involution of \(G\). We denote the differential of \(\theta\) by the same letter. We may assume that \(\theta\), as a Cartan involution of \(\mathfrak{g}\), has the property \(\theta|_h, \theta|_l\) are Cartan involutions of \(h, l\), respectively. We may also assume that there are maximal abelian subalgebras \(\mathfrak{a}, \mathfrak{a}_h, \mathfrak{a}_l\) in \(\mathfrak{p}, \mathfrak{p}_h\) and \(\mathfrak{p}_l\) satisfying the inclusions
\[
\mathfrak{a}_h \subset \mathfrak{a}, \mathfrak{a}_l \subset \mathfrak{a}.
\]

**Theorem 3** ([22], Theorem 4.1). In the above settings, the following conditions are equivalent:
(i) \(H\) acts on \(G/L\) properly.
(ii) \(L\) acts on \(G/H\) properly.
(iii) \(\mathfrak{a}_h \cap W\mathfrak{a}_l = \{0\}\).

We can restate these conditions in the language of hyperbolic orbits for the action of \(L = SL(2, \mathbb{R})\).

**Theorem 4** ([25], Corollary 5.5). Let \(H\) be a reductive subgroup of \(G\) and denote by \(\mathfrak{h}\) the Lie algebra of \(H\). Let \(\Phi : SL(2, \mathbb{R}) \to G\) be a Lie group homomorphism, and denote its differential by \(\phi = d\Phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}\). We put
\[
h_\phi := \phi \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \in \mathfrak{g}.
\]
Then \(SL(2, \mathbb{R})\) acts on \(G/H\) properly via \(\Phi\) if and only if the real antipodal adjoint orbit through \(h_\phi\) in \(\mathfrak{g}\) does not meet \(\mathfrak{a}_h\).

Notice that, by the classification of nilpotent orbits, the number of subspaces \(\mathbb{R}h_\phi \subset \mathfrak{a}\) is finite. Also, all \(h_\phi \in \mathfrak{a}\) are known (see [11] for details).
2.3 a-hyperbolic rank

Define the a-hyperbolic rank of a reductive Lie algebra $\mathfrak{g}$, denoted by $\text{rank}_{a-hyp}(\mathfrak{g})$, as the dimension of $\mathfrak{b}$. The a-hyperbolic rank of a reductive Lie algebra equals the sum of a-hyperbolic ranks of all its simple ideals. The a-hyperbolic rank of an absolutely simple Lie algebras can be calculated using Table 1 (see [3] for more details about the a-hyperbolic rank).

This invariant will be used in the proof of Theorem 2.

| $\mathfrak{g}$                  | $\text{rank}_{a-hyp}$ |
|-------------------------------|-----------------------|
| $\mathfrak{sl}(2k, \mathbb{R})$ | $k$                   |
| $k \geq 1$                    |                       |
| $\mathfrak{su}^*(4k)$         | $k$                   |
| $k \geq 1$                    |                       |
| $\mathfrak{so}(2k+1, \mathbb{R})$ | $k$               |
| $k \geq 1$                    |                       |
| $\mathfrak{so}(2k+1, 2k+1)$   | $2k$                  |
| $k \geq 2$                    |                       |
| $\mathfrak{e}_6(6)$           | 4                     |
| $\mathfrak{e}_6(-26)$         | 1                     |

Table 1: Real forms of simple Lie algebras $\mathfrak{g}^c$, with $\text{rank}_R(\mathfrak{g}) \neq \text{rank}_{a-hyp}(\mathfrak{g})$.

3 Checking C3 and C2

In this section we describe computational methods for checking whether a given homogeneous space $G/H$ is C3 and C2. The general theory allows us to work with their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. For the use of weighted Dynkin diagrams one can consult [3], [6].

3.1 Checking C3

The following theorem follows directly from Theorem 4. It underpins our method for checking whether $G/H$ is C3.

**Theorem 5.** Let $G \supset H$ be semisimple Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and assume (as we may) that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{t}) \oplus (\mathfrak{h} \cap \mathfrak{p})$ is a Cartan decomposition of $\mathfrak{h}$. Let $\mathfrak{a}_h \subset \mathfrak{h} \cap \mathfrak{p}$ be a maximal
abelian subspace of $h \cap p$. Let $a \subset p$ be a maximal abelian subspace of $p$ with $a_h \subset a$. Let $L \subset G$ be a subgroup isomorphic to \( \text{SL}(2, \mathbb{R}) \), with Lie algebra $l$ spanned by $h, e, f$, $h \in a$, with the usual commutation relations. Let $W$ be the little Weyl group of $g$ with respect to $a$. Then $G/H$ admits a proper action of $L$ if and only if the orbit $W \cdot h$ has no point in $a_h$.

We want to check whether there is a subgroup $L$ as in the theorem acting properly on $G/H$. We assume that we are given the Lie algebras $g, h$, along with a Cartan decomposition of $g$ as in the theorem such that $h = k_h \oplus p_h$, where $k_h = k \cap h$, $p_h = p \cap h$. We also fix a maximal abelian subspace $a \subset p$.

The first problem is to find a conjugate $a'$ of $a$ such that $a' \cap h$ is a maximal abelian subspace of $p_h$. We do this in the following way.

1. Let $a_h \subset p_h$ be a maximal abelian subspace. Set $\hat{a} = a_h$ and while $\dim \hat{a} < \dim a$ do the following
   - Compute the centralizer $\mathfrak{z} = \mathfrak{c}_g(\hat{a})$.
   - Find $x \in \mathfrak{z} \cap \mathfrak{p}$ with $x \not\in \hat{a}$ and replace $\hat{a}$ by the subalgebra spanned by $\hat{a}$ and $x$.

2. Set $a' = \hat{a}$.

We note that this works because any abelian subspace of $p$ lies in a maximally abelian subspace of $p_h$. So the spaces $\mathfrak{z}$ always contain a maximally abelian subspace.

Now we write $a$ instead of $a'$, and assume that $a_h = a \cap h$ is a maximally abelian subspace of $p_h$.

We note that the subalgebras of $g$ that are isomorphic to $\text{sl}(2, \mathbb{R})$ are given by the nilpotent orbits of $g$. More precisely, we have the following. The nilpotent orbits in $g^e$ are parametrized by weighted Dynkin diagrams. Each weighted Dynkin diagram uniquely determines an element $h$ in a given Cartan subalgebra of $g^e$. This $h$ lies in a non-uniquely determined $sl_2$-triple $(h, e, f)$. Let $G^e_{\text{ad}}$ denote the adjoint group of $g^e$. Then the above construction gives a bijective correspondence between the list of weighted Dynkin diagrams and the $G^e_{\text{ad}}$-conjugacy classes of subalgebras of $g^e$ that are isomorphic to $sl_2$.

By [26, Proposition 7.8] we have that a nilpotent orbit in $g^e$ has points in $g$ if and only if its weighted Dynkin diagram matches the Satake diagram of $g$. Moreover, these are precisely the nilpotent orbits having $sl_2$-triples lying in $g$. So in the next part of our procedure we perform the following steps:

1. Obtain the list of weighted Dynkin diagrams of $g^e$.
2. For the weighted Dynkin diagrams that match the Satake diagram of $g$ compute an $sl_2$-triple $(h, e, f)$ with $h \in a$. Let $\mathcal{H}$ denote the set of all elements $h$ that are obtained in this way.

By [26, Proposition 4.5] we have that $\mathcal{H}$ is a set of representatives of the $G$-orbits of elements $h' \in g$ that lie in an $sl_2$-triple $(h', e', f')$. 

8
In order to apply Theorem 5 we need to construct the action of the little Weyl group $W$ on $\mathfrak{a}$. Let $\Sigma \subset \mathfrak{a}^{\ast}$ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ (see [30, Chapter 4, §4]). For each root $\alpha \in \Sigma$ let $P_{\alpha} = \{ a \in \mathfrak{a} \mid \alpha(a) = 0 \}$. We define $s_{\alpha} : \mathfrak{a} \to \mathfrak{a}$ to be the reflection in the hyperplane $P_{\alpha}$, where the inner product on $\mathfrak{a}$ is given by the Killing form. Then the group generated by the $s_{\alpha}$ is isomorphic to $W$ ([30, Chapter 4, Proposition 4.2]).

Now for each $h \in \mathcal{H}$ we check whether $W \cdot h \cap \mathfrak{a}_0 = \emptyset$. If this holds then the subgroup of $G$ corresponding to the subalgebra spanned by an $\mathfrak{sl}_2$-triple $(h, e, f)$ acts properly on $G/H$. If the intersection is not empty for all $h \in \mathcal{H}$ then $G/H$ does not admit a proper action of a subgroup locally isomorphic to $\text{SL}(2, \mathbb{R})$.

### 3.2 Checking C2

Our procedure for checking whether $G/H$ is a C2 space follows directly from Theorem 1. The main ingredients have already been described above. Indeed, we can compute the little Weyl group $W$ and hence its longest element. It is then a straightforward task of linear algebra to compute a basis of the space $\mathfrak{b}$. By running over $W$ we can check whether there is a $w \in W$ such that $w \cdot \mathfrak{a}_0$ contains $\mathfrak{b}$. The space $G/H$ is C2 if and only if there is no such $w$.

### 3.3 Implementation

We have implemented the two procedures of this section in the computational algebra system GAP4 [15], using the package CoReLG [12]. This package has a database $\mathcal{G}_M$ which was computed in [18]. Also we use the SLA package which has lists of weighted Dynkin diagrams for the nilpotent orbits of the semisimple complex Lie algebras.

Here we comment on two main computational problems that occur. Firstly, in order to compute the elements of $W$ we need to compute the root system of $\mathfrak{g}$ with respect to the space $\mathfrak{a}$. However, in some (but not many) cases this required eigenvalues that did not lie in the ground field that CoReLG uses. (This field is denoted $\text{SqrtField}$; here we do not go into the details.) So for a few cases our procedures could not be used.

Secondly, when $\mathfrak{g}$ is a split form of type $E_7$ or $E_8$, and the little Weyl group is equal to the usual Weyl group, the orbits $W \cdot h$ of $h \in \mathcal{H}$ can be very large. In this case we can handle large orbits by using an algorithm due to Snow ([31], see also [16, §8.6]) for running over an orbit of the Weyl group without computing all of its elements. This algorithm has been implemented in the core system of GAP4. Also we note two things:

- If we find $h$ such that $W \cdot h \cap \mathfrak{a}_0 = \emptyset$ then we can stop.
If $W \cdot h \cap a_0 \neq \emptyset$ then when enumerating the orbit $W \cdot h$ we can reasonably hope to find an element in the intersection rather quickly. So in practice it usually suffices to enumerate only part of the orbits.

### 3.4 Computational results

The package CoReLG contains the database $\mathcal{GM}$. We have used our implementation to check whether $G/H$ is C2 and C3 where the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ are taken from this database (if the subalgebra $\mathfrak{h}$ is reductive rather than semisimple we have taken its derived algebra instead). We have done this for non-compact $\mathfrak{g}$ of rank up to 6, and for $\mathfrak{g}$ equal to the split forms of type $E_7$ and $E_8$. We have obtained the following results

- For the simple $\mathfrak{g}$ up to rank 6 there are 842 maximal semisimple subalgebras in total. We found no examples of $G/H$ that were C2 but not C3. Of these subalgebras 13 could not be dealt with by our programs for the reasons in Section 3.3. We list them in Table 2. Analyzing this table, we see that 9 of the spaces from this list do not yield $G/H$ which is C2 but not C3 by theoretical considerations. This is because for such spaces either $\text{rank}_{\mathbb{R}} \mathfrak{g} = \text{rank}_{\mathbb{R}} \mathfrak{h}$, or $\text{rank}_{\mathbb{R}} \mathfrak{h} = 1 < \text{rank}_{\mathfrak{a} \text{-hyp}} \mathfrak{g}$ and the result follows from the classification of nilpotent orbits, according to Section 4.1. The following homogeneous spaces from Table 2 are not covered by these arguments:

$$(\mathfrak{sp}(3,3), \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(1,3)), (\mathfrak{so}(4,6), \mathfrak{so}(2,3)), (\mathfrak{e}_6(2), \mathfrak{sl}(3, \mathbb{R})), (\mathfrak{e}_6(2), \mathfrak{g}_2(2)).$$

Here we apply one more procedure to verify that these spaces are C3. We will show the following.

**Proposition 2.** Let $G/H$ be a homogeneous space of absolutely simple non-compact Lie group $G$ and a reductive subgroup $H$. Assume that $\text{rank}_{\mathfrak{a} \text{-hyp}} \mathfrak{g} > 1$ and that $\text{rank}_{\mathbb{R}} \mathfrak{h} = 2$. Let $P$ denote the set of 2-planes in $\mathfrak{a}$ with the following property: $P \in P$ if and only if it is generated by a pair of vectors $w_1 h_1, w_2 h_2, w_i \in W$ and $h_i$ are neutral elements of some $\mathfrak{sl}_2$-triple, $i = 1, 2$. Then $G/H$ is C3 or $a_0 \in P$.

**Proof.** Assume that $G/H$ is not C3 and $a_0 \not\in P$. If an orbit of a neutral element of some $\mathfrak{sl}_2$-triple does not meet $a_0$ then $G/H$ is C3 (by Theorem 1). Therefore, we may assume that there is one $w_1 h_1 \in a_0$. Each $W$-orbit of any neutral element must intersect $a_0$. It follows that this orbit must intersect the line $\langle w_1 h_1 \rangle$, because otherwise one would get $a_0 = \langle w_1 h_1, w_2 h_2 \rangle$ which is not the case. Hence, each $W$-orbit belongs to $\langle w_1 h_1 \rangle$. Since we can assume that $\mathbb{R}^+ (w_1 h_1) \subset a^+$, where $a^+$ is the pos-
itive Weyl chamber, we get \( \text{rank}_{a}\text{-hyp} \, \mathfrak{g} = 1 \) by \([28, \text{Theorem 1.1}]\). Thus, \( a_h \in \mathcal{P} \) or \( G/H \) is C3. A contradiction.

This result implies the following.

**Proposition 3.** Let \( G/H \) be a homogeneous space such that \( \mathfrak{g} = \mathfrak{sp}(3,3) \), \( \mathfrak{g} = \mathfrak{so}(4,6) \) or \( \mathfrak{g} = \mathfrak{e}_6(2) \) and \( \text{rank}_R \, \mathfrak{h} = 2 \). Then \( G/H \) is C3.

**Proof.** Note that for the Lie algebras \( \mathfrak{g} = \mathfrak{sp}(3,3) \), \( \mathfrak{g} = \mathfrak{so}(4,6) \) and \( \mathfrak{g} = \mathfrak{e}_6(2) \) the a-hyperbolic rank is greater than 1. By Proposition \([2]\) it is sufficient to show that for any reductive subgroup \( H \) of \( G \) with \( a_h = P \in \mathcal{P} \), the homogeneous space \( G/H \) is C3. This is verified by the procedure described below. It is implemented and executed with the required result.

**Remark 1.** Here we do not assume that \( (\mathfrak{g}, \mathfrak{h}) \in \mathcal{G}\mathcal{M} \), since the procedure which checks C3 is general and can be applied to all \( P \in \mathcal{P} \).

Before writing down the procedure for verifying C3 in Proposition 3, let us mention the following. Condition C3 is expressed in terms of the action of the Weyl group acting on vectors which can be read off the weights of the weighted Dynkin diagram. However, the straightforward procedure does not work, because the number of required operations is too big. It is worth to consider the two procedures below.

**Straightforward procedure**

1. Write down all weighted Dynkin diagrams for \( \mathfrak{g}^c \) which match the Satake diagram for \( \mathfrak{g} \).

2. Write down vectors \( \mathcal{H} = \{A_1, \ldots, A_t\} \subset \mathfrak{a} \) whose coordinates are read off the weights of the weighted Dynkin diagram (here \( t \) is the number of the non-trivial weighted Dynkin diagrams of \( \mathfrak{g}^c \) which match the Satake diagram).

3. If some of \( A_1, \ldots, A_t \) are on the same line, then choose only one of these, getting \( A_1, \ldots, A_k \).

4. Check, if there exist \( w_1, \ldots, w_k \in W \) such that \( w_1 A_1, \ldots, w_k A_k \) lie in a 2-plane.

5. If \( w_1, \ldots, w_k \in W \) do not exist, then \( G/H \) cannot be "not C3".

The above procedure does not require the knowledge of the embedding \( a_h \subset \mathfrak{a} \). By Theorem \([4]\) \( G/H \) is of class C3, if and only if at least one \( W \)-orbit of the element \( h_{a_h} \) does not meet \( a_h \). In our case \( \dim a_h = 2 \), therefore, Step 4 of the algorithm is a necessary condition to get \( G/H \) not belonging to C3. This procedure cannot be implemented directly,
because the number of operations is too big (it exceeds $10^{60}$). However, we find another procedure to settle the cases of our 4 spaces.

Now we will describe a procedure which verifies $C_3$ for homogeneous spaces in Proposition 3. By some abuse of terminology we will say that a subspace $P \in \mathcal{P}$ is $C_3$, if for some $h \in \mathcal{H}$, $W \cdot h \cap P = \emptyset$.

**Procedure for verifying $C_3$ in Proposition 3**

1. Compute the following sets of vectors in $\mathfrak{a}$:
   
   $$WA_j = \{w_s A_j \mid w_s \in W\}, \quad X = \bigcup_{j=1}^k WA_j$$

2. Create the set $\mathcal{P}'$ of 2-planes in $\mathfrak{a}$ spanned by $A_1$ and a vector from $X$, so
   
   $$P \in \mathcal{P}' \iff P = \langle A_1, x \rangle, \quad x \in X \setminus \{\pm A_1\}.$$  

3. For each $P \in \mathcal{P}'$ verify $C_3$ for $P$.

4. If a given $P \in \mathcal{P}'$ is not $C_3$, then stop. In such case there exists $\mathfrak{h}$ reductive in $\mathfrak{g}$ such that $\text{rank}_\mathbb{R} \mathfrak{h} = 2$ and $G/H$ is not $C_3$ (one can take $\mathfrak{h} := P$).

5. If every $P \in \mathcal{P}'$ is $C_3$, then $G/H$ is $C_3$, for any reductive $H$ of real rank 2.

**Justification of the procedure**

The justification is given by the following.

**Proposition 4.** The procedure described above verifies $C_3$ for any pair $(\mathfrak{g}, \mathfrak{h})$ as in Proposition 3.

**Proof.** By Proposition 2 we need to verify $C_3$ only for $\mathfrak{a}_h \in \mathcal{P}$. Thus, we need to show that we can restrict ourselves to checking $C_3$ for any $P \in \mathcal{P}'$, that is, to fix $A_1$. If there exists a 2-plane $V \subset \mathfrak{a}$ which meets every orbit $WA_j$, then $V$ is spanned by some $w_i A_1$ and some other $x \in X$. This is because $\text{rank}_{\mathfrak{a}_{\text{hyp}}} \mathfrak{g} > 1$ and because of [28, Theorem 1.1]. After conjugating $V$ by an element of the Weyl group $W$ we may assume that $V$ is spanned by $A_1$ and some $x' \in X$. Since our criteria of properness from Subsection 2.2 are invariant with respect to conjugation in $G$, the result follows.

One can roughly estimate the number of operations considering the case of the biggest little Weyl group of $E_{6(2)}$. The number of $A_j$ is 14 (see [11, Section 8.4]), the number of elements in the little Weyl group is
Take into account, that $A_1$ is fixed, so $|\mathcal{P}| \leq |X| - 1$ (in our case, $\leq 15 \cdot 1151$).

- Let $\mathfrak{g}$ be split of type $E_7$ or $E_8$. We also checked the maximal semisimple subalgebras of these Lie algebras. Again a few subalgebras could not be dealt with by our programs. However, we did find two examples of homogeneous spaces that are C2 but not C3:
  
  $$(\mathfrak{e}_7(7), \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{f}_4(4)), (\mathfrak{e}_8(8), \mathfrak{f}_4(4) \oplus \mathfrak{g}_2(2)).$$

| $(\mathfrak{g}, \mathfrak{h})$ | $(\mathfrak{g}, \mathfrak{h})$ |
|-----------------|-----------------|
| $(\mathfrak{su}(3), \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(3))$ | $(\mathfrak{su}(2,5), \mathfrak{so}(2,5))$ |
| $(\mathfrak{so}(6), \mathfrak{sl}(2, \mathbb{R}))$ | $(\mathfrak{so}(6,7), \mathfrak{sl}(2, \mathbb{R}))$ |
| $(\mathfrak{sp}(4, \mathbb{R}), \mathfrak{sl}(2, \mathbb{R}))$ | $(\mathfrak{sp}(5, \mathbb{R}), \mathfrak{sl}(2, \mathbb{R}))$ |
| $(\mathfrak{sp}(3), \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(1,3))$ | $(\mathfrak{sp}(6, \mathbb{R}), \mathfrak{sl}(2, \mathbb{R}))$ |
| $(\mathfrak{so}(4,6), \mathfrak{so}(2,3))$ | $(\mathfrak{so}(6,2), \mathfrak{su}(1,2))$ |
| $(\mathfrak{e}_6(2), \mathfrak{su}(3, \mathbb{R}))$ | $(\mathfrak{e}_6(2), \mathfrak{g}_2(2))$ |
| $(\mathfrak{f}_4(4), \mathfrak{sl}(2, \mathbb{R}))$ | $(\mathfrak{f}_4(4), \mathfrak{su}(2, \mathbb{R}))$ |

Table 2: Spaces with rank $\mathfrak{g} \leq 6$ described in Section 3.3.

## 4 Proof of Theorem 2

The algorithms checking C2 and C3 for the spaces in the database are applied after filtering it with respect to certain conditions (see Subsection 4.1). Also, the database \cite{18} which we use yields $\mathfrak{h} \subset \mathfrak{g}$ up to an abstract isomorphism.

To prove Theorem 2 it is sufficient to show that for every pair $(\mathfrak{g}, \mathfrak{h})$ in Table 2 the condition C2 and C3 are equivalent for any embedding $\mathfrak{h}$ into $\mathfrak{g}$. But this can be done using Corollary 1, Proposition 3 and Theorem 6.

### 4.1 Filtering of the database

We exclude some cases when considering the homogeneous spaces from the database $\mathcal{G}\mathcal{M}$. Our filtering is based on the following theoretical results.

**Theorem 6** (\cite{3}, Theorem 8). Let $G$ be a connected semisimple linear Lie group and $H$ a reductive subgroup with a finite number of components. The following holds:

1. if rank$_{\text{a-hyp}} \mathfrak{h} = \text{rank}_{\text{a-hyp}} \mathfrak{g}$, then $G/H$ does not admit discontinuous actions of non-virtually abelian discrete subgroups,
2. If \( \text{rank}_{a\text{-hyp}} \mathfrak{g} > \text{rank}_R \mathfrak{h} \) then \( G/H \) admits a discontinuous action of non-virtually abelian discrete subgroup.

**Proposition 5.** Assume that \( \text{rank}_{a\text{-hyp}} \mathfrak{g} > 1 \). Then there exist two copies of \( \mathfrak{sl}(2, \mathbb{R}) \) in \( \mathfrak{g} \), with neutral elements \( h_1, h_2 \in \mathfrak{a} \) such that \( R h_1 \) does not meet the orbit \( W(h_2) \).

**Proof.** It is clear that the Weyl group cannot send a half-line lying in the positive Weyl chamber to other half-line in the positive Weyl chamber. By [28, Theorem 1.1], \( \mathfrak{b} \) is spanned by neutral elements (contained in \( \mathfrak{b}^+ \)) of some copies of \( \mathfrak{sl}(2, \mathbb{R}) \) in \( \mathfrak{g} \).

**Corollary 1.** If \( \text{rank}_R \mathfrak{h} = 1 < \text{rank}_{a\text{-hyp}} \mathfrak{g} \) then \( (\mathfrak{g}, \mathfrak{h}) \) is C3.

**Proof.** Assume that \( (\mathfrak{g}, \mathfrak{h}) \) is not C3. Take \( h_1, h_2 \in \mathfrak{a} \) given in Proposition 5. By Theorem 4 there exists \( w_1 \in W \) such that \( R w_1 h_1 = a_6 \) (as \( a_6 \) is 1-dimensional) and \( w_2 \in W \) such that \( w_2 h_2 \in a_6 \). But this contradicts Proposition 5.

The Calabi-Markus phenomenon and Theorem 6 eliminate the necessity to consider the cases
1. \( \text{rank}_R \mathfrak{g} = \text{rank}_R \mathfrak{h} \),
2. \( \text{rank}_{a\text{-hyp}} \mathfrak{g} = \text{rank}_{a\text{-hyp}} \mathfrak{h} \)

since they are not C2. Finally, we eliminate the case \( \text{rank}_R \mathfrak{h} = 1 < \text{rank}_{a\text{-hyp}} \mathfrak{g} \). This follows from Corollary 1.

**Remark 2.** Looking for \( G/H \) which are C2 but not C3, when \( \text{rank} \mathfrak{g} > 6 \) we consider the cases

\[ \text{rank}_{a\text{-hyp}} \mathfrak{g} > \text{rank}_R \mathfrak{h} \]

since C2 is satisfied by Theorem 6 and C3 is checked by the general procedure.

**4.2 Completion of the proof of Theorem 2**

We complete the proof of Theorem 2 running the computer program which checks the remaining cases in the database.

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