ASPECTS OF THE $M_5$-BRANE \(^{a}\)

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The $\kappa$-symmetry of an open $M_2$-brane ending on an $M_5$-brane requires geometrical constraints on the embedding of the system in target superspace. These constraints lead to the $M_5$-brane equations of motion, which we review both in superspace and in component (i.e. in Green-Schwarz) formalism. We also describe the embedding of the chiral $M_5$-brane theory in a non-chiral theory where the equations of motion follow from an action that involves a non-chiral 2-form potential, upon the imposition of a non-linear self-duality condition. In this formulation, we find a simplified form of the second order field equation for the worldvolume 2-form potential, and we derive the nonlinear holomorphicity condition on the partition function of the chiral $M_5$-brane.

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1 Introduction

The $M5$-brane is an important ingredient of $M$-theory. Studies of coincident $M5$-branes, $M2$-branes stretched between $M5$-branes, and wrapping of $M5$-branes around various internal space, for example, have led to discoveries of remarkable non-perturbative phenomena.

While there are many ways to understand the existence of the $M5$-brane, it was first discovered as a classical solution of $D = 11$ supergravity. A particularly interesting way of describing the $M5$-brane is to view it as the surface on which an open $M2$-brane can end. In this picture, among other things, the full $M5$-brane equations of motion follow from the requirement of $\kappa$-symmetry of the open $M2$-brane action.

In this article, we focus on some aspects of the $M5$-brane which deal with the structure of the $M5$-brane equations of motion and the embedding of the theory into a non-chiral theory that admits an action formulation. To be more specific:

a) We review the derivation of the basic constraints on the geometry of the embedding of the $M5$-brane super worldvolume into the target superspace from the requirement of $\kappa$-symmetry of an open $M2$-brane ending on the $M5$-brane.

b) We review the key results for the $M5$-brane equations of motion following from the constraints both in superspace as well as component (i.e. Green-Schwarz) formalism.

c) We describe the embedding of the chiral $M5$-brane theory in a non-chiral theory where the equations of motion follow from an action that involves a non-chiral 2-form potential, upon the imposition of a non-linear self-duality condition.

The non-chiral formulation of the $M5$-brane does not contain any Lagrange multipliers or auxiliary fields. It is, however, equivalent to a scale invariant formulation which does contain a Lagrange multiplier scalar field and a 5-form potential. It differs, on the other hand, from the intrinsically chiral formulation which contains an auxiliary scalar field.

\[ b \] The open $M2$-brane whose boundary moves freely in target spacetime was considered briefly in [4] but it was realized that the attendant boundary condition necessarily breaks the $D = 11$ Lorentz invariance. Therefore the emphasis was put strictly on the closed supermembrane in all the early studies of the $M2$-brane.
In the non-chiral formulation, we also find a simplified form of the second order field equation for the worldvolume 2-form potential, and we derive a nonlinear holomorphicity condition on the partition function of the chiral \( M_5 \)-brane.

The results mentioned in (a) and (b) are covered in Sec. 2 and 3, and they are based on \([5]\) and \([6, 7, 8]\), respectively. The results mentioned in (c) are covered in Sec. 4, and they are essentially based on \([9]\). Sec. 4 does contain, however, some new results.

## 2 An \( M2 \)-Brane Ending on the \( M5 \)-Brane

The \( M5 \)-brane equations of motion can be derived from the considerations of an open \( M2 \)-brane ending on the \( M5 \)-brane. Consider an open \( M2 \)-brane ending on the \( M5 \)-brane whose worldvolume is a \((6|16)\)-dimensional supersubmanifold of the \((11|32)\)-dimensional target superspace. We use the notation \((D|D')\), where \(D\) is the real bosonic dimension and \(D'\) is the real fermionic dimension of a supermanifold. Thus, denoting the worldvolume of the \( M2 \)-brane by \( \Sigma \), the \( M5 \)-brane worldvolume by \( M_5 \) and the target superspace by \( \mathcal{M} \), we have the chain of embedding:

\[
\partial \Sigma \subset M_5 \subset \mathcal{M}.
\]

In this approach it is important to note the \( M2 \)-brane worldvolume \( \Sigma \) is purely bosonic, while the manifold \( M_5 \) on which it ends, which is of course the \( M5 \)-brane worldvolume, is a supermanifold. Thus the worldvolume supersymmetry of the \( M5 \)-brane is manifest, while the worldvolume supersymmetry of the \( M2 \)-brane is not manifest as it is the case in any Green-Schwarz type brane action. In this sense this formulation is a hybrid one.

The worldvolume supersymmetry of the \( M2 \)-brane can be made manifest as well, by elevating the worldvolume \( \Sigma \) into a \((3|16)\)-dimensional supermanifold \( M_2 \), thus having the superembedding chain: \( \partial M_2 \subset M_5 \subset \mathcal{M} \).

Both approaches yield the same superembedding equations for the \( M5 \)-brane. These equations, which will be derived below in the hybrid formulation, are constraints on the embedding that lead to full, covariant equations for the \( M5 \)-brane that we will spell out in Section 3.

In this section, we will consider the embedding chain \((\mathcal{M})\). For simplicity, we will take \( \partial \Sigma \) to consist of a single boundary component. We use the notations and conventions of \([5]\). In particular, we denote by \( z^\mathcal{M} = (x^\mu, \theta^I) \) the local
coordinates on \( M \) and \( A = (a, \alpha) \) is the target tangent space index. We use the ununderlined version of these indices to label the corresponding quantities on the worldsurface. The embedded submanifold \( M \), with local coordinates \( y^M \), is given as \( z^M(y) \).

We consider the following action for an open supermembrane ending on a superfivebrane:

\[
S = - \int_{\Sigma} d^3 \xi \left( \sqrt{-g} + \epsilon^{ijk} C_{ijk} \right) + \int_{\partial \Sigma} d^2 \sigma \epsilon_{rs} A_{rs}, \tag{2}
\]

where \( \xi^i (i = 0, 1, 2) \) are the coordinates on the membrane worldvolume \( \Sigma \), \( \sigma^r (r = 1, 2) \) are the coordinates on the boundary \( \partial \Sigma \), \( g_{ij} \) is the metric on \( \Sigma \) and \( g = \det g_{ij} \).

In addition to the usual super 3-form \( C_3 \) in (11|32) dimensional target super-space \( M \), we have introduced a super 2-form \( A_2 \) on the (6|16) dimensional superfivebrane worldvolume \( M_5 \), which is a *supersubmanifold* of \( M \). The suitable pullbacks of these superforms, and the induced metric occurring in the action are defined as:

\[
C_{ijk} := \partial_i z^M \partial_j z^N \partial_k z^P C_{PMN},
\]
\[
A_{rs} := \partial_r y^M \partial_s y^N A_{NM},
\]
\[
g_{ij} := (\partial_i z^M E_M^A) (\partial_j z^N E_N^B) \eta_{ab}, \tag{3}
\]

where \( \eta_{ab} \) is the Minkowski metric in eleven dimensions and \( E_M^A \) is the target space supervielbein. Defining the basis one-forms \( E_A^\alpha = d\xi^\alpha \partial_\xi z^M E_M^\alpha \) and \( E^A = d\sigma^r \partial_\sigma y^M E_M^A \), where \( E_M^A \) is the supervielbein on \( M_5 \), note the useful relation

\[
E_A^\alpha |_{\partial \Sigma} = E^A E_A^\alpha |_{\partial \Sigma}. \tag{4}
\]

The embedding matrix \( E_A^\alpha \) plays an important role in the description of the model, and it is defined as

\[
E_A^\alpha := E_A^M \partial_M z^M E_M^\alpha. \tag{5}
\]
The action (3) is invariant under diffeomorphisms of $\Sigma$, with suitable boundary conditions imposed on the parameter of the transformation, as well as the tensor gauge transformations $\delta C_3 = d\Lambda_2$ and $\delta A_2 = f_5^* \Lambda_2$ where $\Lambda_2$ is a super 2-form in $\mathcal{M}$ and $f_5^*$ is the pullback associated with the embedding map $f_5 : M_5 \hookrightarrow M$.

We shall now require the total action to be invariant under the $\kappa$-symmetry transformation. On the interior of $\Sigma$, they take the usual form

$$\delta_\kappa z^\alpha = 0, \quad \delta_\kappa z^\dot{\alpha} = \kappa(\xi)(1 + \Gamma_{(2)}) z^\dot{\alpha},$$

where $\delta_\kappa z^\Lambda := \delta_\kappa z^\Lambda E_M^A$ and

$$\Gamma_{(2)} := \frac{1}{3!\sqrt{-g}} \epsilon^{ijk} \Gamma_{ijk}, \quad \Gamma_i := \partial_i z^\Lambda E_M^a \Gamma_\dot{\alpha}.$$  

We also need to specify the fermionic $\kappa$-symmetry transformations of $z^\dot{\alpha}$ on the boundary $\partial \Sigma$. Without loss of generality, they take the form

$$\delta_\kappa z^\dot{\alpha} = 0, \quad \delta_\kappa z^\dot{\alpha} = \kappa(\sigma) P_\dot{\gamma}^\alpha z^\dot{\alpha} \quad \text{on } \partial \Sigma,$$

where $P_\dot{\gamma}^\alpha$ is some projector (see (11)).

We next derive the consequences of the $\kappa$-transformations specified above. To do so, we first observe that an arbitrary transformation of $y^M$ induces a transformation on $z^\Lambda$ given by

$$\delta z^\Lambda = \delta y^A E_A^\Lambda \quad \text{on } M,$$

where $\delta y^A = \delta y^M E_M^A$.

It is useful to introduce a normal basis $E_{A'} = E_A^\Lambda E_A^\Lambda$ of vectors at each point on the worldsurface. The inverse of the pair $(E_A^\Lambda, E_{A'}^\Lambda)$ is denoted by $(E_{\Lambda'}, E_{\Lambda}^A)$. It is also useful to define the projection matrices

$$E_\alpha^\beta E_{\alpha'}^{\beta'} := \frac{1}{2} (1 + \Gamma_{(5)}) z_{\alpha'}^{\beta'},$$

$$E_\alpha^\beta E_{\alpha'}^{\beta'} := \frac{1}{2} (1 - \Gamma_{(5)}) z_{\alpha'}^{\beta'},$$

where $\alpha, \alpha' = 1, \ldots, 5$.
where $\Gamma^{(5)}$, defined by these equations, satisfies $\Gamma^{(5)}_\alpha_\beta = 1$. Its explicit form is not needed at the moment, but it will be spelled out in the next section (see (33)).

The variation $\delta \kappa^\alpha$ given in (8) satisfies $\bar{\kappa} P (1 - \Gamma^{(5)}) = 0$ on the boundary $\partial \Sigma$. This can be seen by multiplying the $A = \alpha$ component of (9) by $E_{\alpha \alpha}' E_{\alpha \beta}'$ and noting that $E_{\alpha \beta} E_{\beta \alpha}' = 0$. Thus the projector $P$ introduced in (8) is given by

$$P = \frac{1}{2} (1 + \Gamma^{(5)}) .$$

(11)

Next we determine $\delta \kappa^\alpha$. From (8) and (9) it follows that

$$0 = \delta \kappa^\alpha E_{\alpha \alpha} + \delta \kappa^\alpha E_{\alpha \beta} ,$$

(12)

on the boundary $\partial \Sigma$. The $a = b$ component of this equation is $0 = \delta \kappa^\alpha E_{\alpha}^a + \delta \kappa^\alpha E_{\alpha}^b$. One can check that $E_{\alpha}^b$ can be gauged away by using the bosonic diffeomorphisms of $M$, namely $\delta \kappa^\alpha E_{\alpha}^b = \eta^a$. Hence, one can set $E_{\alpha}^b = 0$, and since $E_{\alpha}^b$ is invertible, it follows that

$$\delta \kappa^\alpha = 0 ,$$

(13)

on $\partial \Sigma$, and hence on $M$. Next, using (13) in (8), we find $\delta \kappa^\alpha E_{\alpha \alpha} = \delta \kappa^\alpha E_{\alpha \alpha}$ on the boundary $\partial \Sigma$ which implies $\delta \kappa^\alpha = \delta \kappa^\alpha E_{\alpha}^\alpha$ on the boundary $\partial \Sigma$. This means that the variation $\delta \kappa^\alpha$ is an arbitrary odd-diffeomorphism, effecting the 16 fermionic coordinates of $M$, and that when restricted to $\partial \Sigma$, it agrees with the $\kappa$-symmetry transformation on $M$, which also has 16 independent fermionic parameters.

We now turn to the derivation of the constraint on the $M5$-brane embedding mentioned earlier. Using this in $a = b$ component of (12), and observing that $\delta \kappa^\alpha$ is an arbitrary odd-diffeomorphism of $M$, it follows that $E_{\alpha}^b = 0$. Recalling that $E_{\alpha}^b = 0$ as well, we get

$$E_{\alpha}^\alpha = 0 .$$

(14)

This is the superembedding condition that plays a crucial role in the description of superbrane dynamics.

Now we are ready to seek the conditions for the $\kappa$-symmetry of the action (2). Using (8) and (13) in the variation of the action, we find that the vanishing of
the terms on \( \Sigma \) imposes constraints on the torsion super 2-form \( T^A \) and the super 4-form \( H_4 = dC_3 \), such that they imply the equations of motion of the eleven dimensional supergravity. The non-vanishing parts of the target space torsion are

\[
T_{\alpha \beta}^\gamma = -i (\Gamma^\gamma)_{\alpha \beta},
\]

\[
T_{ab}^{\gamma \delta} = -\frac{1}{36} (\Gamma^{bcd})_{\gamma} H_{abcd} - \frac{1}{288} (\Gamma_{abcd})_{\gamma} H_{abcd},
\]

and \( T_{ab}^{\gamma \delta} \). The only other non-vanishing components of \( H_4 \) are

\[
H_{ab}^{\gamma \delta} = -i (\Gamma_{ab})_{\gamma \delta}.
\]

The remaining variations are on the boundary, and yield the final result

\[
\delta_\kappa S = \int_{\partial \Sigma} \epsilon^{rs} (\partial_r y^M E_M^A) (\partial_s y^N E_N^B) \delta_\kappa y^\gamma \mathcal{F}_{\gamma BA},
\]

where we have introduced the following super 3-form in \( M_5 \):

\[
\mathcal{F}_3 := dA_2 - f_5^* C_3.
\]

Since \( \delta_\kappa y^\alpha \) are arbitrary, the vanishing of (17) implies the constraint

\[
\mathcal{F}_{\gamma BA} = 0.
\]

Thus the only non-vanishing component of \( H \) is \( H_{abc} \). The constraints (14) and (15) encode elegantly all the information on the superfivebrane dynamics, as has been shown in \( [6,7,8] \).

Finally we consider the boundary conditions that arise from the variation of the action (2). The requirement of the action be stationary when the supermembrane field equations of [4] hold can readily be shown to impose the following mixed Dirichlet and Neumann boundary conditions

\[
\delta z^a|_{\partial \Sigma} = 0, \quad \left( \sqrt{-g} n^i E_i c + n_i \epsilon^{ijk} E_j E_k^a H_{abc} \right)|_{\partial \Sigma} = 0,
\]
where \( n^i \) is a unit vector normal to the boundary \( \partial \Sigma \), and \( a' \) labels the directions transverse to the fivebrane worldvolume. The reparametrization invariance of (20) imposes the boundary condition \( n^i \partial_i v^r|_{\partial \Sigma} = 0 \) and the reparametrization invariance of the leads to the further boundary condition \( n^i v^i|_{\partial \Sigma} = 0 \).

3 The Covariant M5-Brane Equations of Motion

Here, we give the nonlinear field equations of the superfivebrane equations, up to second order fermionic terms, that follow from the superembedding condition \( E_\alpha a = 0 \), which are proposed to arise equally well from the \( \mathcal{F} \)-constraint \( \mathcal{F}_{\alpha BC} = 0 \). The details of the procedures can be found in [8]. A key point is the emergence of a super 3-form \( h \) in world superspace. This form arises in the following component of the embedding matrix

\[
E_\alpha a = u_\alpha + h_\alpha a u_\alpha, \tag{21}
\]

where, upon the splitting of the indices to exhibit the \( USp(4) \) R-symmetry group indices \( i = 1, \ldots, 4 \), we have

\[
h_\alpha a b \rightarrow h_\alpha i b j = \frac{1}{6} \delta^i_j (\gamma^{abc})_{\alpha \beta} h_{abc}, \tag{22}
\]

where \( h_{abc} \) is a self-dual field defined on \( M \). The pair \((u_\alpha a, u_\alpha \alpha)\) make up an element of the group \( Spin(1,10) \).

The superembedding formalism was shown to give the following complete M5-brane equations of motion:

\[
E_\alpha a E_\alpha b (\Gamma^a)_{\beta\gamma} = 0,
\]

\[
\eta^{ab} \nabla_a E_\alpha a E_\alpha b = -\frac{1}{3} (\Gamma^{b a})_{\gamma} \beta Z_{\alpha \beta \gamma'},
\]

\[
\hat{\nabla}^c h_{abc} = -\frac{1}{32} (\Gamma^{c ab})_{\gamma} \beta Z_{\alpha \beta \gamma'}, \tag{23}
\]

where

\[
Z_{\alpha \beta \gamma'} = E_\beta \frac{\delta}{\delta} \left( E_\alpha a T_{\alpha a} \frac{\delta}{\delta} - E_\alpha \frac{\delta}{\delta} E_\alpha \gamma (\nabla_\gamma E_\delta \frac{\delta}{\delta}) E_\delta \frac{\delta}{\delta} \right) E_\frac{\delta}{\delta} \gamma'. \tag{24}
\]
Recall that the inverse of the pair \((E_A, E_{A'})\) is denoted by \((E_A^A, E_{A'}^{A'})\) and that \(A = (a, \alpha)\) label the tangential directions while \(A' = (a', \alpha')\) label the normal directions to the M5-brane worldvolume.

The target space torsion components \(T_{\underline{a} \underline{b}}\) are given in (15) and the second term involves only quantities that are bilinear in worldvolume fermions. The covariant derivative \(\hat{\nabla}\) has an additional, composite \(SO(5,1)\) connection of the form \((\nabla u)u^{-1}\) as explained in more detail in [8].

The M5-brane equations of motion (23) live in superspace. The component (i.e. Green-Schwarz) form of these equations have also been worked out. Up to fermionic bilinears, the final result is:

\[
\mathcal{E}_a (1 - \Gamma) \gamma^b m_b^a = 0 ,
\]

\[
G^{mn} \nabla_m \mathcal{F}_{npq} = Q^{-1} \left[ 4Y - 2(mY + Ym) + mYm \right]_{pq} ,
\]

\[
G^{mn} \nabla_m \mathcal{E}_{\underline{a}} = \frac{Q}{\sqrt{-g}} e^{m_1 \cdots m_6} \left( \frac{1}{6} H_{m_1 \cdots m_6} + \frac{1}{(3!)} H_{m_1 m_2 m_3} \mathcal{F}_{m_4 m_5 m_6} \right) P^{\underline{a}} .
\]

Several definitions are in order. To begin with,

\[
m_a^b := \delta_a^b - 2k_a^b , \hspace{1cm} k_a^b := h_{acd} h^{bcd} , \hspace{1cm} Q := (1 - \frac{2}{3} \text{tr} k^2) ,
\]

\[
Y_{ab} := [4 * H - 2(m * H + *Hm) + m * Hm]_{ab} ,
\]

\[
P_{\underline{a}} := \delta_{\underline{a}} - \mathcal{E}_{\underline{a}}^m \mathcal{E}_m , \hspace{1cm} *H^{ab} := \frac{1}{4! \sqrt{-g}} e^{abdef} H_{cdef} ,
\]

The fields \(\mathcal{F}_{abc}\), \(H_{\underline{a}_1 \cdots \underline{a}_4}\) and its Hodge dual \(H_{\underline{a}_1 \cdots \underline{a}_7}\) are the purely bosonic components of the superforms

\[
\mathcal{F}_3 = dA_2 - C_3 , \hspace{1cm} H_4 = dC_3 , \hspace{1cm} H_7 = dC_6 + \frac{1}{2} C_3 \wedge H_4 .
\]

The remaining nonvanishing component of \(H_7\) is

\[
H_{\alpha \beta abcde} = -i (\Gamma_{abcde})_{\underline{a} \underline{b}} .
\]
The target space indices on $H_4$ and $H_7$ have been converted to worldvolume indices with factors of $E_{m}{}^{a}$ which are the supersymmetric line elements defined as

\begin{align}
E_{m}{}^{a}(x) & := \partial_{m}{}^{a}E_{1}{}^{a} \quad \text{at } \theta = 0 , \\
E_{m}{}^{\alpha}(x) & := \partial_{m}{}^{\alpha}E_{M}{}^{\alpha} \quad \text{at } \theta = 0 .
\end{align}

The metric

\[ g_{mn}(x) := E_{m}{}^{a}E_{n}{}^{b}\eta_{ab} = e_{m}{}^{a}e_{n}{}^{b}\eta_{ab} \quad \text{is the standard GS induced metric with determinant } g, \]

and $G^{mn}$ is another metric defined as

\[ G^{mn} := (m^2)^{ab}e_{m}{}^{a}e_{n}{}^{b} . \]

Let us note that the connection in the covariant derivative $\nabla_{m}$ occurring in (25) is the Levi-Civita connection for the induced metric $g_{mn}$ up to fermionic bilinears.

A key relation between $h_{abc}$ and $F_{abc}$ follows from the dimension-0 components of the Bianchi identity $dF_{3} = -H_{4}$, and is given by

\[ h_{abc} = \frac{1}{4}m_{d}F_{bcd} . \]

The matrix $\Gamma$ is the $\theta = 0$ component of the matrix $\Gamma_{(5)}$ introduced above in (11) and it is given by

\[ \Gamma = -\bar{\Gamma} + \frac{1}{4}h^{mnp}\Gamma_{mnp} = -[\exp (-\frac{1}{4}\Gamma^{mnp}h_{mnp})] \bar{\Gamma} , \]

where

\[ \bar{\Gamma} := \frac{1}{6!\sqrt{-g}}e_{m_{1} \cdots m_{6}}\Gamma_{m_{1} \cdots m_{6}} , \]

\[ \Gamma_{m} := E_{m}{}^{a}\Gamma_{a} , \quad \Gamma^{b} := \Gamma^{m}e_{m}{}^{b} , \quad e_{m}{}^{b} := E_{m}{}^{a}m_{a}{}^{b} . \]

\(^{c}\text{We have rescaled the } F_{3} \text{ of } [8] \text{ by a factor of } 4.\)
The $\kappa$-symmetry transformation rules are

\begin{align*}
\delta_\kappa \bar{z}^a &= 0 , \\
\delta_\kappa \bar{\omega}^a &= \kappa^\frac{1}{2} (1 + \Gamma) \bar{\omega}^a , \\
\delta_\kappa h_{abc} &= - \frac{1}{16} m_{d[a} c^{d}(1 - \Gamma) \Gamma_{bc]} \kappa , \quad (35)
\end{align*}

where $\Gamma$ is given by (33). The $\kappa$-symmetry transformations are the fermionic diffeomorphisms of the $M5$-brane worldvolume with parameter $\kappa^\alpha = \kappa \bar{\omega}_\alpha \bar{\omega}^\alpha$. Thus, using (19), it follows immediately that

\begin{equation}
\delta_\kappa F_3 = \{ d, i_\kappa \} F_3 = - i_\kappa \bar{H}_3 , \quad (36)
\end{equation}

which can also be verified by direct computation by combining (32) and (37). The equations of motion (25) have been shown to be equivalent to those which follow from an action with auxiliary scalar field (32).

We conclude this section by elucidating the consequences of the central equation (32). To this end, we first note the useful identities

\begin{align*}
h_{abc} h^{cde} &= \delta^{[c}_{[a} k^d_{b]} , \\
k^{ac} k^b_c &= \frac{1}{6} \delta^{ac}_{[a} h_{[b]d} k^2 , \\
k_{ad} h_{bcd} &= k_{[a}^d h_{bc]d} , \quad (37)
\end{align*}

which are consequences of the linear self-duality of $h_{abc}$. Taking the Hodge dual of (32) one finds $\ast \mathcal{F}_{abc} = - \mathcal{F}_{abc} + 2 Q^{-1} m_a d \mathcal{F}_{bcd}$. Using the identity $m^2 = 2m - Q$, we readily find the nonlinear self-duality equation

\begin{equation}
\ast \mathcal{F}_{mnp} = Q^{-1} G_{m}^q q \mathcal{F}_{npq} . \quad (38)
\end{equation}

This equation can be expressed solely in terms of $\mathcal{F}_3$. To do this, we first insert (32) into (37), which yields the identities

\begin{align*}
\mathcal{F}_{abc} \mathcal{F}^{cde} &= 2 \delta^{[c}_{[a} X^d_{b]} + \frac{1}{2} k^{ac} X^b_{c} X^d_{b} + 2 (K^2 - 1) \delta^{[c}_{[a} \delta^d_{b]} ,
\end{align*}

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\[ X_{ac}X_b^c = 4K^2(K^2 - 1)\eta_{ab}, \]
\[ X_a^dF_{bcd} = X_{[a^d}F_{bc]d}. \]  \hspace{1cm} (39)

where we have defined

\[ K := \sqrt{1 + \frac{1}{24}F_{abc}F_{abc}}, \]  \hspace{1cm} (40)
\[ X_{ab} := \frac{1}{2}K \ast F_{a^cd}F_{bcd}. \]  \hspace{1cm} (41)

Next we derive the identities

\[ Q(K + 1) = 2, \]
\[ X_{ab} = \frac{1}{2}F_{acd}F_{b^{cd}} - \frac{1}{12}\eta_{ab}F_{cde}F_{cde} = 4K(1 + K)k_{ab}. \]  \hspace{1cm} (42)

We can now express (38) entirely in terms of \( F_3 \) by deriving the identity

\[ Q^{-1}G_{mn} = K\eta_{mn} - \frac{1}{2}K^{-1}X_{mn}. \]  \hspace{1cm} (43)

Another way of writing (38) is

\[ \mathcal{F}_{abc} = \frac{1}{2}(1 + K)^{-2}F_{ade}\mathcal{F}_{+dfc}\mathcal{F}_{fbc}, \]  \hspace{1cm} (44)

where \( K \) is a root of the quartic equation

\[ (K + 1)^3(K - 1) = \frac{1}{24}\mathcal{F}_{+abc}\mathcal{F}_{ade}\mathcal{F}_{+def}\mathcal{F}_{fbc}. \]  \hspace{1cm} (45)

4 The \( M5 \)-brane Action for an Unconstrained 2-Form Potential

While the superembedding constraints yield the covariant equations of motion it is desirable to have an action from which these equations of motion can be derived. There exists a universal action formula which emerges naturally in the superembedding approach\[^{14}\]. However, as we shall see in Section 5, this action formalism is not directly applicable to branes with self-dual field strengths in the worldvolume such as the \( M5 \)-brane.
A manifestly target space supercovariant and $\kappa$-invariant action, which contains an auxiliary scalar and from which the self-duality condition can be derived as an equation of motion has been constructed. However, as has been argued by Witten, any attempt to even define a proper partition function for the $M$-brane using this action requires a choice of the auxiliary scalar whose topological class in general breaks some of the symmetries of $M$-theory. The root of this problem lies in the fact that the theory, in effect, describes a chiral 2-form.

One resolution of this problem involves the embedding of the chiral theory into non-chiral one. At the classical level, this amounts to finding an action involving an unconstrained 2-form potential $A_2$ such that its field equation is equivalent to the Bianchi identity $dF_3 = -H_4$ and the action is $\kappa$-invariant upon the imposition of the self-duality condition.

At the quantum level, the decoupling of the unwanted chirality components amounts to imposing a constraint on the partition function $Z[C_3]$, which, at the linearized level, reads

$$D_{abc}Z[C_3] = -i\sqrt{-g} \left\langle F^+_{abc} \right\rangle,$$

(46)

where the functional derivative $D_{abc}$ is defined as

$$D_{abc} := \frac{\delta}{\delta C_{abc}} + \frac{1}{\sqrt{-g}} \star C_{abc}.$$

(47)

Thus $D_{abc}Z = 0$, and only $F^+_{abc}$ couples to $C_{abc}$.

Motivated by these considerations, a non-chiral extension of the $M$-brane was constructed in, where it was found that the connection between $\kappa$-symmetry and self-duality is sufficient to determine the action and the non-linear self-duality condition.

The key to the construction of the action is the fact that the non-linear self-duality condition can be written in the following form

$$\star F_3 = \frac{\partial K}{\partial F_3}.$$

(48)

where $K$ can be computed with the help of the identities presented in the previous section. The result is
\[ K = 2 \sqrt{1 + \frac{1}{4} F^2 + \frac{1}{88} (F^2)^2 - \frac{1}{12} F_{abc} F^{bcd} F_{def} F^{efa} }, \] (49)

modulo terms whose \( F \)-derivatives vanish when (38) holds. One also finds the useful relation:

\[ K = 2K \quad \text{for} \quad \star F = \frac{\partial K}{\partial F} . \] (50)

In view of (48), a suitable action is given by

\[ S = \int \left( \frac{1}{2} K - Z_6 \right), \] (51)

where the Wess-Zumino term

\[ Z_6 = C_6 - \frac{1}{2} C_3 \wedge F_3 . \] (52)

The rational behind this action is as follows: treating \( F_3 \) as subject only to the Bianchi identity \( dF_3 = -H_4 \), and varying (51) with respect to the unconstrained two-form potential \( A_2 \) one finds that

\[ d \left( \star \frac{\partial K}{\partial F} \right) = -H_4 . \] (53)

Combining this second order equation for \( A_2 \) with the Bianchi \( dF_3 = -H_4 \), we find that the only possible self-duality condition that can be imposed is precisely (48). Moreover it was shown in [9] that the action (51) has the \( \kappa \)-symmetry characterized by

\[ \delta_\kappa Z_6 = \kappa \delta Z_6 , \quad \delta_\kappa A_2 = i_\kappa C_3 , \] (54)

provided that the self-duality condition (48) is satisfied and the \( \kappa \)-parameter is projected as

\[ \kappa_M = \kappa \frac{1}{2} (1 + \Gamma') \mu^M E_2 \mu^M , \] (55)

\[ ^{d}\text{Notice that the } C_6 \text{ term in the action does not affect the } A_2 \text{ field equations. Nonetheless it is needed for } \kappa \text{-symmetry of the action.} \]
where

\[
\Gamma' = K^{-1} \left( -\Gamma + \frac{1}{12} \ast F^{mnp} \Gamma_{mnp} \right).
\]

(56)

The equivalence between the \(\kappa\)-symmetry transformations (54)-(55) and those which arise from the superembedding formalism as given in (35)-(36), follows from the identities

\[
(1 + \Gamma')(1 - \Gamma) = 0 = (1 + \Gamma)(1 - \Gamma'),
\]

(57)

which can be shown with the help of (32) and (39).

We emphasize that the self-duality condition (38) does not follow directly as an equation of motion from the action (51). Instead, as we saw above, it is recovered as the only self-dual truncation of the theory that is consistent in the sense that it interchanges the Bianchi identity \(dF_3 = -H_4\) with the tensor field equation (53). Actually, the form of \(K\) and the self-duality condition (38) can also be understood by starting from an action of the form \(S = \int (\frac{1}{2} \ast K - Z_6)\) and demanding invariance under the \(\kappa\)-symmetry transformations of the form (54)-(55).

We can now derive the non-linear version of the constraint (46) by starting from the action (51) and the formal definition

\[
Z[C_3] = \int DA_2 \ e^{iS}.
\]

(58)

Using the functional derivative (47) we get

\[
D_{abc} e^{iS} = \frac{i}{2} \sqrt{-g} \left( \frac{\partial K}{\partial C_{abc}} - \ast (dA_2)_{abc} + \ast C_{abc} \right) e^{iS}
\]

\[
= -\frac{i}{2} \sqrt{-g} \left( \frac{\partial K}{\partial F_{abc}} + \ast F_{abc} \right) e^{iS}.
\]

(59)

which means that

\[
D_{abc} Z[C_3] = -\frac{i}{2} \sqrt{-g} \left( \frac{\partial K}{\partial F_{abc}} + \ast F_{abc} \right).
\]

(60)
Since the right side is a projection onto the nonlinearly self-dual part of \( F_3 \), this is a proper generalization of the constraint (46) to the nonlinear case. The full consequences of this constraint remain to be investigated.

The M5-Brane Equations of Motion

The \( \kappa \)-symmetry transformations (54)-(55) map the non-linear self-duality condition (48) into the \( z^\Delta \) equations of motion, which therefore must agree with the corresponding results (25) obtained from the superembedding. It is nonetheless instructive to demonstrate the equivalence of the field equations by direct computation. The equations of motion following from the action (51) followed by the use of the self-duality equation (38) are:

\[
E_m J^m = 0 \tag{61}
\]

\[
G^{pq} \nabla_p F_{qmn} = -Q \left( * H_{mn} + \frac{1}{2} F_{mnp} F^{pqr} * H_{qr} \right),
\]

\[
G^{mn} \nabla_m E_n = \frac{Q}{\sqrt{-g}} \epsilon^{m_1 \cdots m_6} \left( \frac{1}{6} H^{a}_{m_1 \cdots m_6} + \frac{1}{(3!)} H^a_{m_1 m_2 m_3} F_{m_4 m_5 m_6} \right) P_a \psi,
\]

where the symmetric bispinor \( J^m \) is given by

\[
J^m = \Gamma^m \bar{\Gamma} + \frac{1}{2} * F^{mpn} \Gamma_{np} - Q^{-1} G^{mn} \Gamma_n. \tag{62}
\]

The \( z^\Delta \) field equation arises as an admixture of the variation with respect to \( z^\Delta \) and the tensor field equation obtained by varying the action with respect to \( \delta z^\Delta := V^\Delta \) and \( \delta A_2 = i_{V^C} \bar{C}_3 \), such that \( \delta F_3 = -i_{\bar{H}_4} \).

In obtaining the field equations (61), it is useful to realize that \( Q^{-1} G_{mn} \) actually is the energy-momentum tensor associated with the composite metric \( g_{mn} \):

\[
\frac{\delta S_{kin}}{\delta g_{mn}} = \frac{1}{2} \sqrt{-g} Q^{-1} G^{mn}, \tag{63}
\]

where \( S_{kin} = \int \frac{1}{2} * \kappa \). The invariance of \( S_{kin} \) under the bosonic worldvolume diffeomorphism \( \delta g_{mn} = 2 \nabla_{(m} \xi_{n)} \) and \( \delta F_3 = d \xi A_2 - i \xi \bar{H}_4 \) then implies

\[
\nabla_n (Q^{-1} G^{mn}) = -\frac{1}{6} H^{mpn} * F_{npq}, \tag{64}
\]
provided that (53) holds.

Next, we compare the equations of motion (61) with (25) obtained from superembedding. The scalar field equations are already in the same form. To show equivalence of the Dirac and tensor equations to those obtained in superembedding formalism requires some work. Let us begin with the Dirac equation. The $\kappa$-symmetry of this equation implies that

$$
(1 + \Gamma) J^m = 0 = J^m(1 - \Gamma^T).
$$

Eq. (65) then implies

$$
(1 + \Gamma) J^m = 0 = J^m(1 - \Gamma^T).
$$

From (33) it follows that $\Gamma = -X^{-1} \Gamma X$ and $\Gamma^T = -X \Gamma X^{-1}$ where we have introduced $X := \exp(\frac{1}{6} h^{abc} \Gamma_{abc})$, which implies that $(1 - \Gamma) X J^m = 0 = J^m X (1 + \Gamma)$. On the other hand, from $m^{ab} \Gamma_b = -\Gamma m^{ab} \Gamma_b$ and $(1 - \Gamma) \Gamma = 1 - \Gamma$ it follows that the Dirac equation in (25) is annihilated from the right by $1 + \Gamma$. Hence, upon multiplication from right by $X$ the Dirac equation (61) will be proportional to (25). Indeed one can verify that $(1 - \Gamma) m^{ab} \Gamma_b = -Q J^a X$, which shows the equivalence between the two Dirac equations.

As for the tensor field equation, we have verified that it turns into the one obtained in the superembedding formalism (see (25)) by using the identity $m^2 = 2m - Q$. It is worth noting that the field equation for $A_2$ given in (53) is considerably simpler than the form it takes in (25).

Dualization of the Non-Chiral Action

As a check of the formalism, let us briefly discuss the dualization of the non-chiral theory. To this end we introduce a dual 2-form potential $\tilde{A}_2$ as a Lagrange multiplier for the Bianchi identity $dF_3 = -H_4$ and integrate out $A_2$ via its field strength $F_3$, which yields the dual partition function

$$
\tilde{Z}_{[C_3]} = \int D F_3 \, D \tilde{A}_2 \, e^{iS - \frac{1}{2} \int \tilde{A}_2 \wedge d(F_3 + C_3)} := \int D \tilde{A}_2 \, e^{i\tilde{S}}. \tag{67}
$$

The dual action $\tilde{S}$ can easily be computed in the saddle point approximation which yields

17
\[ \tilde{S} = \int \left( \frac{1}{2} \ast \tilde{K} - \tilde{Z}_6 \right), \]  

(68)

where we have used the notations \( \ast \tilde{K} := \ast K - \tilde{F}_3 \wedge F_3 \) and \( \tilde{Z}_6 := C_6 - \frac{1}{2} C_4 \wedge \tilde{F}_3 \) where \( F_3 \) is supposed to be expressed in terms of the dual field strength \( \tilde{F}_3 \) via

\[ \tilde{F}_3 := d\tilde{A}_2 - C_4 = \ast \frac{\partial K}{\partial F_3}. \]  

(69)

Varying (68) with respect to \( \tilde{A}_2 \) and using (69), one finds the dual second order tensor field equation \( dF_3 = -H_4 \), where, as mentioned above, \( F_3 \) is expressed in terms of \( \tilde{F}_3 \) via (69). Thus, in the saddle point approximation, the chiral truncation of the dual theory is given by \( \tilde{F}_3 = F_3 \), which from (69) is seen to be equivalent to the condition (48) for chiral truncation of the original non-chiral theory. In other words, the non-chiral theory and its dual have equivalent chiral truncations.

A Scale Invariant Formulation of the Non-Chiral Action

The action discussed at length in the previous section is equivalent to the scale invariant form of the action constructed in [9]. In the latter formulation, a worldvolume Lagrange multiplier scalar field \( \lambda \), and a worldvolume 5-form potential \( A_5 \), with field strength \( F_6 = dA_5 + Z_6 \), are introduced. This construction is parallel to the scale invariant formulations of super p-brane actions that has been known for sometime [17, 18].

The \( M5 \)-brane action constructed in [9] is given by

\[ S' = \frac{1}{2} \int \ast \lambda \left( \frac{1}{2} K^2 - (\ast F_6)^2 \right). \]  

(70)

To verify that the equations of motion following from this action are equivalent to those which follow from (51), we begin by varying (70) with respect to \( \tilde{A}_2 \) and the 5-form potential \( A_5 \). This yields two first order field equations, namely \( \ast F_6 = \pm \frac{1}{2} K \) (where the sign reflects the duality of the 2-form potential), which determines \( A_5 \) up to a gauge, and \( d(\lambda \ast F_6) = 0 \), which we solve by taking \( \lambda = 2T_5 \ast K^{-1} \), where the integration constant \( T_5 \) is the \( M5 \)-brane tension. The
remaining equations of motion from $S'$ follow by varying $z^M$ and the 2-form potential $A_2$. Denoting a general variation of this kind by $\delta$, we find that

$$
\delta S' = \frac{1}{2} \int \lambda \left[ \star \left( \left( \frac{1}{4} K^2 + (\star F_6)^2 \right) \frac{1}{\sqrt{-g}} \delta \sqrt{-g} + \frac{1}{2} K \delta K \right) - 2(\star F_6) \delta Z_6 \right].
$$

(71)

Using the relations $\star F_6 = \frac{1}{2} K$ and $\lambda = 2T_5 K^{-1}$ in this formula, one then immediately finds $\delta S' = T_5 \delta S$.

**Relation to an M5-Brane Action in Superembedding Approach**

It has been shown how Green-Schwarz type actions can be systematically constructed for most branes starting from the superembedding approach. The construction yields a general action formula. The only branes for which this formula runs into an obstacle are those which contain worldvolume chiral $p$-form potentials, such as the M5-brane. It is, nonetheless, interesting to see the result one obtains by a naive application of this action formula to this case. In doing so, we will find an action which is closely related to the one discussed above.

The application of the action formula to the M5-brane proceeds as follows. Defining

$$
W_7 := dZ_6 = H_7 + \frac{1}{2} H_4 \wedge F_3
$$

(72)

and using the fact that de Rham cohomology of the supermanifold $M$ coincides with that of its body $M$ one can always write

$$
W_7 = dK_6
$$

(73)

for some *globally* defined 6-form $K_6$ on $M$. Furthermore, since none of the target space fields or the worldsurface fields has negative dimension, it follows that the only non-vanishing component of $K_6$ is the purely bosonic one. In components this means

$$
K_{\alpha A_1 \cdots A_5} = 0.
$$

(74)
The application of the general action formula of [4] to the present case gives the functional

$$S'' := \int_{M_0} i^*(K_6 - Z_6) ,$$  \hspace{1cm} (75)$$

where $i : M_0 \hookrightarrow M$ is the embedding of the body $M_0$ of $M$ into $M$. $S''$ is by construction only defined for self-dual $h_{abc}$ or, equivalently, for the 3-form field strength $F_{abc}$ obeying the non-linear self-duality condition (48). $S''$ is manifestly invariant under reparametrizations of $M_0$ and the $\kappa$-transformations (35) and (36) as these transformations are generated by $i^*v$, where $v$ is a supervectorfield on $M$, and $\delta_{i^*v}i^*L_6 = i^*[d, i_v]L_6 = di^*i_vL_6$ where $L_6 = K_6 - Z_6$.

To find $K_6$ we insert the constraint (74) into (73). After some algebra one finds that the dimension 0 component yields

$$\star K_6 = K ,$$  \hspace{1cm} (76)$$

where $K$ is given by (40). Therefore, in view of (50), $S''$ is simply the restriction of the action (51) to the constraint surface defined by the non-linear self-duality condition (48).

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