Abstract

The thorough analysis of the duality properties of the Riemann curvature tensor points to possibility of extension of Einstein’s General Relativity to the nonabelian Yang-Mills theory. The motion equations of the theory are Yang-Mills’ equations for the curvature tensor. Einstein’s equations (with cosmological term to appear as an integration constant) are contained in the theory proposed. New is that now gravitational field is not exceptionally determined by matter energy-momentum but can possess its own non-Einsteinian dynamics (vacuum fluctuations, self-interaction) which is generally an attribute of nonabelian gauge field. The gravitational equations proper due to either matter energy-momentum or vacuum fluctuations are side conditions imposed on the Riemann tensor, like self-duality conditions. One of such conditions in the end results in Einstein’s equations, other ones are the gravitational instantons equations.
On General Relativity extension*

A. L. Koshkarov†

1 Introduction

There is no doubt that Einstein’s General Relativity is nonabelian gauge theory although not quite similar to conventional the Yang-Mills theory.

Nevertheless, there are rather too much arguments in favor of the theory is nonabelian. But in what way, the fact that gravitation is nonabelian does get on with widely spread and prevailing view the gravity source is energy-momentum and only energy-momentum? And how about nonabelian self-interaction? Of course, here we touch very tender spots about exclusiveness of gravity as physical field, the energy problem, etc. Still the spherically-symmetric field out of Schwarzschild’s sphere looks quite like Coulomb’s solution in Electrodynamics, the abelian theory without self-interaction. All the facts point out the General Relativity is not quite conventional nonabelian theory. In addition, Einstein’s equations are not like Yang-Mills’.

It is shown in this paper the theory can be formulated *ad exemplum* ordinary Yang-Mills’ theory with more or less standard description in the form of the Yang-Mills equation, with self-interactions and instantons. For all that, Einstein’s equations are contained in the theory rather than cancelled and do not dwindle. And their existence as themselves seems to relate to the gravity peculiarities.

For our purposes, it is essential the fact that internal (group) indices and space-time one are interchangeable, i.e. group acts in the Minkowski space-time which as a result get curved. In fact the internal space coincides with the space-time. Therefore, it is convenient to hold the viewpoint that the first two indices α, β of the curvature tensor $R_{\alpha\beta\mu\nu}$ are internal, and the second pair $\mu, \nu$ are the spacetime indices. And vice versa that’s right as well. This is the peculiar features of the gravity as the gauge theory. For this reason the gravity duality properties are even more nontrivial and interesting than those in the ordinary Yang-Mills theory.

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*University of Petrozavodsk, Russia
†email: Koshkarov@petrsu.ru
The duality properties of the Riemann tensor

The duality properties us to be interested in have been established in the article [1] which though includes some mistakes. About notations. The metric with signature $(+,−,−,−)$ in $D = 4$ pseudo-riemannian manifold is given to be metric-compatible to (Riemannian) connection in the regular way.

Let us introduce operations: 1) the left dual conjugation ($*R^\alpha_{\beta\mu\nu}$), 2) the right dual conjugation ($R^*_{\alpha\beta\mu\nu}$), and 3) twice dual conjugation ($**R^\alpha_{\beta\mu\nu}$)

\[*R^\alpha_{\beta\mu\nu} = \frac{1}{2} E^\alpha_{\beta\rho\sigma} R^\rho_{\sigma\mu\nu}, \quad R^*_{\alpha\beta\mu\nu} = \frac{1}{2} R^\alpha_{\beta\rho\sigma} E^\rho_{\sigma\mu\nu}, \quad **R^\alpha_{\beta\mu\nu} = \frac{1}{2} E^\alpha_{\beta\rho\sigma} R^\rho_{\gamma\delta} E^\gamma_{\delta\mu\nu},\]

where $E^\alpha_{\beta\rho\sigma} = \sqrt{-g} e^\alpha_{\beta\rho\sigma}$ - the Levi-Civita tensor, $g$ - the metric tensor determinant.

For example

\[**R^\alpha_{\beta\mu\nu} = R^*_{\alpha\beta\mu\nu} = -R^\alpha_{\beta\mu\nu}\]

It is usual properties of double dual conjugates in the $(+,−,−,−)$ riemannian space.

In terms of the dual conjugates the cyclicity identity

\[R^\alpha_{\beta\mu\nu} + R^\alpha_{\nu\beta\mu} + R^\alpha_{\mu\nu\beta} = 0\]

is of the form

\[**R^\alpha_{\beta\mu} = R^*_{\alpha\beta\mu} = 0 \text{ and/or } R^*_{\alpha\beta\mu} = R^\alpha_{\beta\mu}\]

Bianchi’s identity

\[R^\mu_{\rho\sigma\delta} + R^\mu_{\rho\delta\sigma} + R^\mu_{\sigma\rho\delta} = 0\]

transforms to

\[**R^\mu_{\rho\sigma;\nu} = 0 \text{ and/or } R^*_{\mu\rho\sigma;\nu} = 0\]

Or that can be rewritten as follows

\[**R^\alpha_{\beta\mu;\nu} = 0 \text{ and/or } R^*_{\alpha\beta\mu;\nu} = 0\]

And twice dual conjugate Riemann’s tensor $**R^\alpha_{\beta\mu\nu}$ can be represented by Riemann’s and its contractions, i.e., by Ricci’s tensor and scalar, for the expression

\[E^{\alpha\beta\rho\sigma} E_{\gamma\delta\mu\nu} = \varepsilon^{\alpha\beta\rho\sigma} \varepsilon_{\gamma\delta\mu\nu}\]

can be calculated and expressed by the Kronecker $\delta$-symbols [1]:

\[**R^\alpha_{\beta\mu\nu} = -R^\mu_{\alpha\beta} + \delta^\mu_{\alpha} R^\nu_{\beta} + \delta^\nu_{\beta} R^\mu_{\alpha} - \delta^\mu_{\beta} R^\nu_{\alpha} - \delta^\nu_{\alpha} R^\mu_{\beta} - \frac{1}{2} R^{\mu\nu}_{\alpha\beta} = (1)\]

Hereafter the notations are used

\[\delta^\mu_{\alpha\beta} = \delta^\mu_{\alpha} \delta^\nu_{\beta} - \delta^\nu_{\alpha} \delta^\mu_{\beta}, \quad g_{\alpha\beta\mu\nu} = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}\]
The next important step is to expand the Riemann tensor into sum of two parts,

\[ R_{\alpha\beta\mu\nu} = \frac{1}{2} (R_{\alpha\beta\mu\nu} + R^{*}_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} - R^{*}_{\alpha\beta\mu\nu}) = R_{\alpha\beta\mu\nu} + S_{\alpha\beta\mu\nu} \quad (2) \]

where

\[ R_{\alpha\beta\mu\nu} = \frac{1}{2} (R_{\alpha\beta\mu\nu} - R^{*}_{\alpha\beta\mu\nu}), \quad S_{\alpha\beta\mu\nu} = \frac{1}{2} (R_{\alpha\beta\mu\nu} + R^{*}_{\alpha\beta\mu\nu}) \quad (3) \]

Now one can represent the tensors \( R_{\alpha\beta\mu\nu} \) and \( S_{\alpha\beta\mu\nu} \) by Riemann’s tensor and Ricci’s tensor and scalar

\[ R_{\alpha\beta\mu\nu} = \frac{1}{2} (g_{\alpha\mu} R_{\beta\nu} + g_{\beta\nu} R_{\alpha\mu} - g_{\alpha\nu} R_{\beta\mu} - g_{\beta\mu} R_{\alpha\nu} - \frac{1}{2} R g_{\alpha\beta\mu\nu} ) \quad (4) \]

\[ S_{\alpha\beta\mu\nu} = \frac{1}{2} (g_{\alpha\mu} R_{\beta\nu} + g_{\beta\nu} R_{\alpha\mu} - g_{\alpha\nu} R_{\beta\mu} - g_{\beta\mu} R_{\alpha\nu} - \frac{1}{2} R g_{\alpha\beta\mu\nu} ) \quad (5) \]

One must already say something about the tensors properties. The \( S_{\alpha\beta\mu\nu} \) is noteworthy. Note it is expressed by the Ricci tensor and scalar only, not by Riemann’s.

The tensor \( R_{\alpha\beta\mu\nu} \) should not be confused with the Weyl conformal tensor \( C_{\alpha\beta\mu\nu} \)

\[ R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{1}{12} R g_{\alpha\beta\mu\nu} \]

Further, when twice dual conjugating, both \( S_{\alpha\beta\mu\nu} \) and \( R_{\alpha\beta\mu\nu} \) transform simply

\[ S^{*}_{\alpha\beta\mu\nu} = +S_{\alpha\beta\mu\nu}, \quad R^{*}_{\alpha\beta\mu\nu} = -R_{\alpha\beta\mu\nu} \quad (6) \]

i.e., \( S_{\alpha\beta\mu\nu} \) and \( R_{\alpha\beta\mu\nu} \) are respectively twice selfdual and antiselfdual parts of the curvature tensor.

It makes sense to introduce a new "quantum" number — d-parity, characterizing behavior of tensors (like curvature one) under twice dual conjugation. For example, \( R_{\alpha\beta\mu\nu} \) is odd, and \( S_{\alpha\beta\mu\nu} \) - even under d-parity reflection. Two more examples of d-odd tensors are \( g_{\alpha\beta\mu\nu} \) and \( E_{\alpha\beta\mu\nu} \).

There are nontrivial equations

\[ S_{\alpha\beta\mu\nu} = 0 \quad (7) \]

and

\[ R_{\alpha\beta\mu\nu} = 0 \quad (8) \]

These equations have a direct relationship to instantons in nonabelian gauge theories. In particular in the case of \( SO(4) \) or \( SU(2) \) gauge group, they describe Belavin-Polyakov-Schwarz-Tyupkin instanton and antiinstanton [2].

Below we shall see the equation (8) describes the gravitational instantons.

Some solutions to these equations have been obtained in [1]. For example, the equation (7) has a static solution in the metric

\[ ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (9) \]
Six equations (7) with nonvanishing left member reduce to the only second order equation

\[ \lambda = -\nu, \quad \nu''(r) + \nu'(r)^2 = 2 r^2 (1 - e^{-\nu}) \]

The solution is

\[ e^\nu = 1 + C_1 r^2 + \frac{C_2}{r} \quad (10) \]

Thus central-symmetric solution to equation (7) is static and quite similar to Schwarzschild’s except for \( C_1 r^2 \). It is not without purpose and we’ll be back to this as well as to equations (7) and (8). Below we’ll see that for equation (7) with vanishing right hand side \( C_2 = 0 \).

3 From Einstein’s to gravitational Yang-Mills’ equations

Solution to equation (7) including the Schwarzschild solution suggests that it is possible to use this equation instead of Einstein’s

\[ R_{\alpha\mu} - \frac{1}{2} R g_{\alpha\mu} = \Lambda g_{\alpha\mu} + T_{\alpha\mu}, \quad R + T = -4\Lambda \quad (11) \]

although in emptiness. Really, solution to this equation in the metric (9) coincide with (10), if \( C_1 = \Lambda \). Even more so, the tensor \( S_{\alpha\beta\mu\nu} \) in the left hand side (7) is fully determined by the Ricci tensor.

At this point, we want to call attention to one little drawback to the Einstein equations. Of course, at times, there had been treating various advantages and disadvantages of these equations although the former are of the overwhelming majority. The one had most likely been discussed before. The point is that (11) are system of differential equations of the second order in metric. And the Schwarzschild solution has merely one integration constant. When more attentive treating Schwarzschild’s problem, it turns out that among the equations there are both first order equations and second order those. With all that, solutions to first order equations and second order equations are compatible provided that one of the two integration constants is strictly fixed. It is the reason that the Einstein second order equation system solution (the Schwarzschild solution) contains solely one integration constant.

This fact is of course known but completely ignored. We find on this point, Einstein’s equations are somewhat inconsistent.

Further heuristically, there will be obtained equation to generalize Einstein’s equation (11). Then the new equation will be proclaimed as one of the basic equations describing gravity produced by matter. After that, it will be seen the new equation implies the gravitational Yang-Mills equation. At last, Einstein’s equations will be shown to follow from both the new equation and gravitational Yang-Mills’.
First expressing the Ricci tensor from (11) and substituting that in (5), we can find

\[ S_{\alpha\beta\mu\nu} = \Theta_{\alpha\beta\mu\nu} \]  

(12)

where

\[ \Theta_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\mu}T_{\beta\nu} + g_{\beta\nu}T_{\alpha\mu} - g_{\alpha\nu}T_{\beta\mu} - g_{\beta\mu}T_{\alpha\nu} - \frac{1}{2}Tg_{\alpha\beta\mu\nu}) \]  

(13)

This tensor is built from the metric tensor and the energy-momentum tensor. Further it will be used “instead” of the energy-momentum tensor. Note that the tensor \( \Theta_{\alpha\beta\mu\nu} \) is d-even like \( S_{\alpha\beta\mu\nu} \).

Now the important step follows. Let us forget the Einstein equation and instead consider the equation (12) as one of the basic equations of gravity generated by matter.

Differentiate covariantly with respect to \( x^\nu \) the equation (12)

\[ S_{\alpha\beta\mu\nu}^{\nu;\nu} = \Theta_{\alpha\beta\mu\nu}^{\nu;\nu} \]  

(14)

Having remembered what is \( S_{\alpha\beta\mu\nu} \) (5), it follows that

\[ R_{\alpha\beta\mu\nu}^{\nu;\nu} = 2\Theta_{\alpha\beta\mu\nu}^{\nu;\nu} \equiv J_{\alpha\beta\mu\nu} \]  

(15)

So, the tensor \( \Theta_{\alpha\beta\mu\nu} \) determines the matter current or the gravity matter source in the gravitational Yang-Mills equation. The current is not even conserved covariantly since the multiple covariant derivatives don’t commute.

Now making contraction over \( \alpha, \mu \)

\[ R_{\beta\nu}^{\nu;\nu} = 2\Theta_{\beta\nu}^{\nu;\nu} \]

and taking into account the Bianchi identity and explicit form of the tensor \( \Theta_{\alpha\beta\mu\nu} \) we obtain

\[ (R + T)^{\beta} = 0 \]  

(16)

After integration

\[ R + T = -4\Lambda \]  

(17)

where \(-4\Lambda\) is the integration constant.

Now we have arrived at the cross-roads. There are two alternatives. One can consider the equation (12) as a basic one. Then it implies both equations (15) and (17).

Otherwise, we can consider the gravitational Yang-Mills equation (15) as a basic one. Then we have to take (12) as a condition. This second option is preferable.

Once again let’s be back to equation (15). It is of the form of the Yang-Mills equation. The proposal is to consider it as a basic gravitational equation. And the equality (17) is the integral of motion, i. e. the conservation law.

Let’s demonstrate that the Einstein equations are implied by basic equations (15) and (12). First the conservation law (17) is obtained from (15). Then contract (12) over \((\beta, \nu)\) and eliminate \(T\) by means of (17). As a result, we have
exactly Einstein’s equations (11). Λ, the constant is obviously interpreted as a *cosmological term* appeared as an integration one!

Thus it is shown that the equations (12), (15) are equivalent to Einstein’s. It is the equations those are the basic gravitational equations. The equation (15) is the *basic dynamic* one, and another one (12) is a *side condition* which among fields singles out those generated by matter.

Coming back to equations (12), or to (7) in emptiness, we can see the equations solution in emptiness (10) includes two integration constants, one of which apparently associates with Λ. The solution describes (out of the matter distribution) the empty constant curvature space with the scale factor $1/\sqrt{\Lambda}$ and with the central-symmetry distributed about point of origin matter. Let a point mass be at origin. Then out of origin, the metric is given by (10), and $C_2$ is proportional to the mass. As for another constant, it seems to be possible only $C_1$ proportional to Λ. Thus (10) is the Schwarzschild static solution in the constant curvature space. We consider it as a manifestation of fact that gravity is non-abelian. The solution (10) describes the local geometry in the neighborhood of some spherically symmetric matter distribution. This geometry is determined by both the mass (more precisely, energy-moment) and Λ. Is that Λ the same in case of any mass or not? In other words, is cosmological constant Λ universal?

Classically, the solution (10) if you wish could be interpreted as exhibition of asymptotic freedom in gravity.

It specially should be noted that the vacuum (in emptiness) solution to the equation (12), i.e. (10) is nontrivial, as distinct from the Einstein theory. That is, this solution does not just reduce to the Minkowski spacetime. There are both static solutions (10) and nonstatic ones with the de Sitter asymptotic. For example, for (closed) Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left( d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$

(18)

the equation (12) in emptiness for the scale factor $a(t)$ is of the form

$$a\ddot{a} - \dot{a}^2 - 1 = 0.$$

The vacuum solution is

$$a(t) = a_0 \cosh \frac{t - t_0}{a_0}.$$

Similarly, for the open metric

$$ds^2 = dt^2 - a^2(t) \left( d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$

(19)

the equation for $a(t)$

$$a\ddot{a} - \dot{a}^2 + 1 = 0,$$

has a solution

$$a(t) = a_0 \sinh \frac{t - t_0}{a_0}$$

or

$$a(t) = a_0 \sin \frac{t - t_0}{a_0}.$$
For the latter case $a(t)$ is alternating in sign that seems not to be of physical meaning.

One can use the model equations e. g. to construct cosmological models. That’s done. On cursory examination, the Einstein-Friedmann cosmology remains intact. However now the cosmological term seems to be the necessary element of the theory. It should be experimentally measured in the observation cosmology. In the sense, “the dark matter problem” might seem otherwise. Universality of $\Lambda$-term in this approach is open to question.

4 Non-Einsteinian gravity

It is quite clear that the equations (15) are wider than the Einstein General Relativity. Namely, any real gravitational fields is considered to obey these equations. Of all fields, the Einstein theory extracts the ones to be generated by the matter energy-momentum. Within the theory proposed, the extraction happens by imposing the side condition (12). This condition is analogous to self-duality conditions for instantons in the Yang-Mills theory. However it is not vacuum one. It is possible to treat some other conditions which might extract non-Einsteinian solutions for gravitational fields.

Let us try to discuss possible conditions for gravity. Quite general side condition for equation (15) is of the form

$$R_{\alpha\beta\mu\nu} = \kappa R^{\ast}_{\alpha\beta\mu\nu} + \epsilon^\ast R^{\ast}_{\alpha\beta\mu\nu} + \lambda g_{\alpha\beta\mu\nu} + \zeta E_{\alpha\beta\mu\nu} + 2 \Theta_{\alpha\beta\mu\nu}$$

This is more general than (12). This implies the basic equation (15) fulfilled. In the matter presence $\Theta_{\alpha\beta\mu\nu} \neq 0$, and taking $\epsilon = -1, \kappa, \lambda, \zeta = 0$, we obtain the equation (12). Then the Einstein equation holds and the source conservation takes place. Consequently, the gravitational field equations imply motion equations of matter in the gravitation field generated by the matter.

All that will not occur with alternative set of constants $\epsilon, \kappa, \lambda, \zeta$. Still admissibility of such a condition is open to question.

In the case of $\Theta_{\alpha\beta\mu\nu} = 0$ we deal with the vacuum side conditions. Vacuum solutions to equation (15) could be called gravitational instantons. General representation for instanton is

$$R_{\alpha\beta\mu\nu} = \kappa R^{\ast}_{\alpha\beta\mu\nu} + \epsilon^\ast R^{\ast}_{\alpha\beta\mu\nu} + \lambda g_{\alpha\beta\mu\nu} + \zeta E_{\alpha\beta\mu\nu}$$

If the condition holds then gravitational Yang-Mills equation (13) will be obeyed. The constants $\kappa, \epsilon, \lambda, \zeta$ are not quite arbitrary and should be determined.

Alternatively instead of (20), one can consider the equation

$$R_{\alpha\beta\mu\nu} = \kappa R^{\ast}_{\alpha\beta\mu\nu} + \epsilon R^{\ast}_{\alpha\beta\mu\nu} + \lambda g_{\alpha\beta\mu\nu} + \zeta E_{\alpha\beta\mu\nu}.$$
to be fulfilled.

The analysis of the equation (20) is rather complicated so we shall restrict our consideration to particular cases.

We have already discussed the vacuum solutions to the equation (12). Here is another possibility: \( \epsilon = 1; \kappa, \zeta = 0, \lambda \) is arbitrary. Then we have the non-Einsteinian equation

\[
\mathcal{R}_{\alpha\beta\mu\nu} = \lambda g_{\alpha\beta\mu\nu} \tag{21}
\]

Both left and right members of the equation are d-even. Contracting over \( \alpha, \mu \) and \( \beta, \nu \), we obtain

\[ R = 12\lambda \]

Let’s remind we have the matter vacuum, hence \( T = 0 \). It seems just in cosmology \( \lambda \) to relate to the cosmology term. Generally it is arbitrary and possibly relates to the local vacuum fluctuations of the gravitational fields in Universe.

The equation (21) can be solved in the (closed) Robertson-Walker metric. The equation for \( a(t) \)

\[ a\ddot{a} + \dot{a}^2 + 1 = -2\lambda a^2 \tag{22} \]

has a solution

\[ a(t) = \sqrt{C_1 \exp(2\sqrt{-\lambda}t) + C_2 \exp(-2\sqrt{-\lambda}t) - \frac{1}{2\lambda}} \tag{23} \]

It’s not analytic in \( \lambda \). Note a simple constant solution

\[ a(t) = a_0 = -\frac{1}{2\lambda} \]

This solution describes the empty constant positive curvature space. Could not it be called a gravipole? Really it is the same solution as the static solution to equation (10) in emptiness, i.e. with \( C_2 = 0 \). It is impossible to pass directly on to \( \lambda = 0 \) in this metric. However that corresponds to solution to equation (21) as the empty Minkowski space.

In the case \( \lambda = 0 \), there is timeldependent solution in the metric (13)

\[ a(t) = a_0 \sqrt{1 - \left(\frac{t - t_0}{a_0}\right)^2} \]

There are similar solutions in the open Robertson-Walker metric as well. Interestingly, the matter motion (e. g. the test point mass) in the vacuum gravitational fields is already not determined by the field equations but obeyed the geodesic equation.

New approach allows more directly than before to discuss topological effects in gravitation. Really, the conditions (21) are ”topological”. Projecting (21) onto \( R^{\alpha\beta\mu\nu} \) (i.e. multiplying and contracting) results in

\[ R^{\alpha\beta\mu\nu} \mathcal{R}_{\alpha\beta\mu\nu} - 2\lambda R = \kappa R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} + \epsilon R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} \]

After integration, we can see that (in case of convergence) topological numbers can be expressed in terms of invariants quadratic and linear in curvature. Non-trivial topological solutions seem to exist in manifolds with Euclidean signature.
5 Conclusion

So, new version of gravity is proposed. It is in form and fact the nonabelian Yang-Mills theory of gravitational field with own rich dynamics and nontrivial topology. The theory contains the Einstein General Relativity.

It is not impossible to avoid a question: what are the Einstein equations? Do they express any conservation law? Or are they any compatibility conditions due to the gauge group peculiarity? It should be specially noted that in the new theory the dynamics is described by the equation (15) in presence of the side condition (12) and/or others. And the Einstein equations themselves are sequel of this condition and the conservation law (17). We have to consider various conditions as well as Einstein’s equations as constraints. Perhaps that might change the situation with quantizing gravity.

As for application the theory to astrophysics and cosmology, it is the next job ahead. At once one can say that standard Einstein-Friedmann cosmological model seems not to change. Cosmological term situation may get more definitive. It is not matter of a taste: to work or not to work with that. One must to measure that. Once again one have to say that it is not clear whether Λ is universal.

Nothing to say as yet about black hole physics in new approach. Task in hand is to search for the nontrivial topology solutions. It may be the time to pass over from gossip about spacetime foam and quantizing gravity to practice.

The theory proposed is natural from viewpoint of interactions unity. Gauge invariance and duality are ideas underlying. Some of these ideas are not yet exhausted in gravity and of interest to apply in the Yang-Mills theory. But it is another topic.

References

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