Dynamics of electromagnetic waves in Kerr geometry

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Here we are interested to study the spin-1 particle i.e., electro-magnetic wave in curved space-time, say around black hole. After separating the equations into radial and angular parts, writing them according to the black hole geometry, say, Kerr black hole we solve them analytically. Finally we produce complete solution of the spin-1 particles around a rotating black hole namely in Kerr geometry. Obviously there is coupling between spin of the electro-magnetic wave and that of black hole when particles propagate in that space-time. So the solution will be depending on that coupling strength. This solution may be useful to study different other problems where the analytical results are needed. Also the results may be useful in some astrophysical contexts.

KEY WORDS : spin-1 particle, black hole geometry, Maxwell equations

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1. INTRODUCTION

We know that Maxwell’s equations describe the dynamics of electro-magnetic wave. Four equations indicate how electric and magnetic field are dependent each other and how they propagate in space. Also electro-magnetic wave means particle of spin-1, like photon. To study the dynamics of the spin-1 particles in curved space-time one needs to look up the Maxwell’s equation in Kerr geometry (if the central gravitating object is chosen rotating and uncharged). In this context it is easier to consider the electromagnetic waves as an external perturbation in the space-time. This external perturbation can be represented as incident waves of different sorts in the space-time of the black hole. We are considering the back ground space-time as of Kerr black hole which is stationary and axisymmetric. So the perturbation can be expressed as a superposition of waves with different modes with the ‘t’ and ‘φ’ dependence given as exp[i(σt + mφ)], where σ is the frequency of the waves and m is an integer which may be positive, negative or zero. Far away from the black hole the space-time is flat where the Maxwell’s equation and its solution are well known. As the electromagnetic wave i.e., corresponding perturbation comes closer to the black hole the space-time becomes curved where usual quantum theory may not be applicable. In that region, set of Maxwell equations will be modified and corresponding global behaviour of the spin-1 particles will be deviated with respect to that of flat space.

Photon is an electro-magnetic radiation. From the study of Maxwell equations we can investigate the behaviour of the photon close to the black hole or other gravitating object which result can be applicable further for other related works and in the context of astrophysical problem where the relativistic information is important. The solutions

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of Maxwell equations in a curved space-time have a considerable importance in black hole physics as well as in the context of mathematical physics. If the spatially complete solution is possible to construct, it is very useful to study the absorption rate of the black hole, to find the corresponding Feynman Green function, for checking the stability of the black hole (say, in case of the Kerr geometry, one can study the stability of Kerr black hole), to study the black hole perturbation, to study the quasi-normal modes of the black hole, to do the second quantization of the metric (in this case Kerr metric) etc. In case of matter flows towards the compact object, out of the matter which falls towards the central body photon is there. That photon also may take part in the cooling process of the accreting matter, though the Compton wavelength of the incoming photon is very small compared to size of the black hole horizon if the black hole is chosen as formed by the supernova explosion only. On the other hand, we know all the primordial black holes of mass greater than $10^{15}$ gm are still exist in nature [1]. For this kind of black holes the interaction between the incoming photon and the black hole may be significant. Thus, naturally the dynamical behaviour of the photon, how it is coupled with space-time (how the spin of the photon couples with the rotation of compact object) is very important to study. Also, in the case of Hawking radiation, boson, fermion, graviton are emitted from the black hole and scattered in the space-time. In that context the behaviour of the electromagnetic wave is essential to look into.

In 1972, Teukolsky [2] initiated the electromagnetic perturbation in Kerr geometry and separated corresponding master equation. It was shown that, separation into radial ($r$) and angular ($\theta$) parts is possible if the particular choice of temporal ($t$) and azimuthal ($\phi$) dependency is chosen in case of stationary and axisymmetric system as given above. In next year Teukolsky [3] himself studied corresponding reflection and transmission problem of the electromagnetic waves in Kerr geometry. Then, Teukolsky & Press [4] studied the interaction of electromagnetic waves with black hole and found numerical solutions asymptotically. In 1976, Chandrasekhar [5] again looked into the separated Maxwell equations in curved space-time particularly in Kerr geometry and indicated about some features like super-radiance. Later he discussed [6] extensively about the electromagnetic, Dirac, gravitational wave equation in curved space-time and showed some asymptotic results. In 1984, Chakrabarti [7] solved the separated angular Dirac equation analytically and also listed the values of the separation constant $\lambda$ for the particle of spin $1/2$, 1 and 2. Then, various scattering phenomena from black holes have been studied by Futtermann et al. [8]. After that, recently, Mukhopadhyay & Chakrabarti [9,11], Mukhopadhyay [12,13] have started to study different behaviour of spin-$1/2$ particles in curved space-time in case of non-rotating (Schwarzschild), rotating (Kerr) and charged (Reissner-Nordström) black hole geometry. Following (almost) same approach here we would like to re-initiate the study of spin-$1$ particles in Kerr geometry. Most of the earlier studies [2-4] have given asymptotic solutions of Maxwell equations, here we are going to present spatially complete, global solution in Kerr geometry. Here one of the main difference of our approach is that after separating the equation into radial and angular part, with a further certain choice of basis, imaginary number $i$ (=$\sqrt{-1}$) is completely taken off from potential of the system. Physically, as if we consider such a frame there the potential is real which is felt by the incoming electromagnetic wave. So the final equations to be studied are explicitly real which makes easier to attack the problem (in case of [8], potential contains real and imaginary both the parts, so the potential behaviour had to be studied separately).

We will use Newman-Penrose spinor formalism to write the Maxwell equations in curved space-time and then separate into the radial and angular part to solve separately. In the next section we will introduce the basic equations
of the problem. In §3, we will show analytical solutions where the behaviour of potentials for the different sets of input parameter are shown. Finally, in §4, we will make conclusions.

II. BASIC EQUATIONS OF THE PROBLEM

We know that \( F_{ij} = \partial_i A_j - \partial_j A_i \) is the electromagnetic covariant antisymmetric tensor where, \( A_i = (A_0, \vec{A}) \) \((A_0 \text{ and } \vec{A} \text{ are the scalar and vector potential respectively})\) and corresponding Maxwell equations in curved space-time is

\[
F_{ij;k} = 0, \quad g^{ik} F_{ij;k} = 0. \tag{1}
\]

Following Chandrasekhar \[6\], using Newman-Penrose formalism, \( F_{ij} \) can be written in terms of three complex scalars as

\[
\phi_0 = F_{13} = F_{ij} l^i m^j, \\
\phi_1 = \frac{1}{2} (F_{12} + F_{43}) = \frac{1}{2} F_{ij} (l^i n^j + \bar{m}^i m^j), \\
\phi_2 = F_{42} = F_{ij} \bar{m}^i m^j. \tag{2}
\]

Here, \( l^i, n^i, m^i \) and \( \bar{m}^i \) are four null basis for Newman-Penrose formalism. In terms of the tetrad components and intrinsic derivatives Maxwell equations become

\[
F_{[ab; c]} = 0, \quad \eta^{nm} F_{an|m} = 0. \tag{3}
\]

Thus, by using eqns. (2) and (3), Maxwell equations are replaced by

\[
\phi_1|1 - \phi_0|4 = 0, \quad \phi_2|1 - \phi_1|4 = 0, \\
\phi_1|3 - \phi_0|2 = 0, \quad \phi_2|3 - \phi_1|2 = 0. \tag{4}
\]

In terms of different spin coefficients, tetrads which are expressed in terms of directional derivatives, eqn. \(4\) i.e., the reduced Maxwell equations become

\[
D\phi_1 - \delta^* \phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho \phi_1 - \kappa \phi_2, \\
D\phi_2 - \delta^* \phi_1 = -\lambda \phi_0 + 2\pi \phi_1 + (\rho - 2\epsilon)\phi_2, \\
\delta \phi_1 - \Delta \phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau \phi_1 - \sigma \phi_2, \\
\delta \phi_2 - \Delta \phi_1 = -\nu \phi_0 + 2\mu \phi_1 + (\tau - 2\beta)\phi_2. \tag{5}
\]

Here, \( D, \Delta, \delta, \delta^* \) are the directional derivatives which are basically the basis vectors of the system. Different spin coefficients namely \( \pi, \alpha, \rho, \kappa, \lambda, \epsilon, \mu, \gamma, \tau, \sigma, \nu, \beta \) are given by Chandrasekhar \[6\] and which are expressed in terms of the Kerr metric coefficients. If we substitute all these in eqn. \(4\) we get

\[
\frac{1}{\rho^* \sqrt{2}} \left( L_1 - \frac{2i \sin \theta}{\rho^*} \right) \phi_0 = \left( D_1 + \frac{2}{\rho^*} \right) \phi_1, \\
\frac{1}{\rho^* \sqrt{2}} \left( L_0 + \frac{2i \sin \theta}{\rho^*} \right) \phi_1 = \left( D_0 + \frac{1}{\rho^*} \right) \phi_2. \tag{6}
\]
\[
\frac{1}{\bar{\rho} \sqrt{2}} \left( \mathcal{L}_1^1 + \frac{i \sin \theta}{\bar{\rho}^*} \right) \phi_2 = -\Delta \frac{1}{2 \rho^2} \left( \mathcal{D}_0^1 + \frac{2}{\bar{\rho}^*} \right) \phi_1,
\]
\[
\frac{1}{\bar{\rho} \sqrt{2}} \left( \mathcal{L}_0^1 + \frac{2i \sin \theta}{\bar{\rho}^*} \right) \phi_1 = -\Delta \frac{1}{2 \rho^2} \left( \mathcal{D}_0^1 - \frac{1}{\bar{\rho}^*} \right) \phi_0,
\]

where, \( \bar{\rho} = r + i a \cos \theta \), \( \bar{\rho}^* = r - i a \cos \theta \) and \( \mathcal{L}_n^1 \), \( \mathcal{D}_n^1 \), and \( \mathcal{D}_0^1 \) are defined as

\[
\mathcal{D}_n = \partial_r + \frac{i K}{\Delta} + 2n \frac{r - M}{\Delta}; \quad \mathcal{D}_n^1 = \partial_r - \frac{i K}{\Delta} + 2n \frac{r - M}{\Delta},
\]

\[
\mathcal{L}_n = \partial_\theta + Q + n \cot \theta; \quad \mathcal{L}_n^1 = \partial_\theta - Q + n \cot \theta,
\]

\[
K = (r^2 + a^2) \sigma + am; \quad \Delta = r^2 + a^2 - 2Mr; \quad Q = a \sigma \sin \theta + mcosec \theta.
\]

Now choosing \( \Phi_0 = \phi_0 \), \( \Phi_1 = \phi_1 \bar{\rho}^* \sqrt{2} \) and \( \Phi_2 = 2 \phi_2 (\bar{\rho}^*)^2 \) eqn. (8) reduces to

\[
\left( \mathcal{L}_1 - \frac{i \sin \theta}{\bar{\rho}^*} \right) \Phi_0 = \left( \mathcal{D}_1 + \frac{1}{\bar{\rho}^*} \right) \Phi_1,
\]
\[
\left( \mathcal{L}_0 + \frac{i \sin \theta}{\bar{\rho}^*} \right) \Phi_1 = \left( \mathcal{D}_0 - \frac{1}{\bar{\rho}^*} \right) \Phi_2,
\]
\[
\left( \mathcal{L}_0^1 - \frac{i \sin \theta}{\bar{\rho}^*} \right) \Phi_2 = -\Delta \left( \mathcal{D}_0^1 + \frac{1}{\bar{\rho}^*} \right) \Phi_1,
\]
\[
\left( \mathcal{L}_0^1 + \frac{i \sin \theta}{\bar{\rho}^*} \right) \Phi_1 = -\Delta \left( \mathcal{D}_0^1 - \frac{1}{\bar{\rho}^*} \right) \Phi_0.
\]

Now using first and last equation of (8) we get

\[
\left[ \Delta \mathcal{D}_1 \mathcal{D}_1^1 + \mathcal{L}_1^1 \mathcal{L}_1 - 2i \sigma (r + i a \cos \theta) \right] \Phi_0 = 0 \tag{9}
\]

and using second and third of (8) we get

\[
\left[ \Delta \mathcal{D}_0^1 \mathcal{D}_0 + \mathcal{L}_0^1 \mathcal{L}_1 + 2i \sigma (r + i a \cos \theta) \right] \Phi_2 = 0. \tag{10}
\]

Equations (9) and (10) are clearly separable. Now we choose, \( \Phi_0 = R_{+1}(r)S_{+1}(\theta) \), \( \Phi_2 = R_{-1}(r)S_{-1}(\theta) \) and separating the eqns. (8) and (11) into radial and angular parts we get the Teukolsky’s equation as

\[
\left( \Delta \mathcal{D}_0 \mathcal{D}_0^1 + 2i \sigma r \right) \Delta R_{+1} = \chi \Delta R_{+1},
\]
\[
\left( \mathcal{L}_0^1 \mathcal{L}_1 + 2\sigma a \cos \theta \right) S_{+1} = -\chi S_{+1},
\]
\[
\left( \Delta \mathcal{D}_0 \mathcal{D}_0 + 2i \sigma r \right) R_{-1} = \chi R_{-1},
\]
\[
\left( \mathcal{L}_0 \mathcal{L}_1 - 2\sigma a \cos \theta \right) S_{-1} = -\chi S_{-1}.
\]

Here, we use the identity \( \Delta \mathcal{D}_0 \mathcal{D}_0^1 R_{+1} = \mathcal{D}_0 \mathcal{D}_0^1 \Delta R_{+1} \) and \( \chi \) is chosen as the separation constant. Now choosing \( P_{+1} = \Delta R_{+1} \) and \( P_{-1} = R_{-1} \) and by a suitable choice of the relative normalisation of the functions \( P_{+1} \) and \( P_{-1} \) we get

\[
\Delta \mathcal{D}_0 \mathcal{D}_0 P_{+1} = \xi P_{+1}; \quad \Delta \mathcal{D}_0 \mathcal{D}_0^1 P_{+1} = \xi^* P_{-1}. \tag{12}
\]
where, $|\xi|^2 = \lambda^2 - 4\alpha^2\sigma^2$ and $\alpha^2 = a^2 + \frac{am}{\sigma}$. Thus, from the first of eqn. (12), (7a,b) and third of (11) we get

$$\xi P_{+1} = (\lambda - 2i\sigma)P_{-1} + 2iK D_0 P_{-1}. \quad (13)$$

Similarly, using second of eqn. (12), (7a,b) and first of (11) we get

$$\xi^* P_{-1} = (\lambda + 2i\sigma)P_{+1} - 2iK D_0^\dagger P_{+1}. \quad (14)$$

Then from eqns. (13) and (14) we get

$$d r P_{+1} + iK \Delta P_{+1} = \left[ (\lambda - 2i\sigma)P_{-1} - \xi P_{-1} \right], \quad (15)$$

$$d r P_{-1} - iK \Delta P_{-1} = \left[ (\lambda + 2i\sigma)P_{+1} - \xi^* P_{+1} \right]. \quad (16)$$

Now, eqns. (15) and (16) are the set of radial equations which we will solve for the radial behaviour of spin-1 particles.

### II.A. Reduction of Angular equations

Now, in angular set of equation of (11), if we choose the integrating factor $\sin\theta$ and consider the independent variable $\theta$ transforms as $\frac{d}{dr} = \sin\theta \frac{d}{d\theta}$ such that

$$U = \log \left| \tan \frac{\theta}{2} \right|, \quad (17)$$

we get the transformed set of angular Maxwell equations as

$$\frac{d^2 S_{\pm 1}}{dU^2} + J_{\pm 1}^2 S_{\pm 1} = 0, \quad (18)$$

where, $J_{\pm 1}^2 = \sin^2 \theta (\lambda \pm Q' - \csc^2 \theta \mp Q \cot \theta - Q^2 \pm 2a\sigma \cos \theta)$, prime denotes the derivative with respect to $\theta$.

According to the transformation of variable $\theta$ (which runs from 0 to $\pi$) to $U$ (which runs from $-\infty$ to $+\infty$) the reduced eqn. (18) is now wave equation in cartesian like coordinate system with wave vector $J_{\pm 1}$ which contains the information of angular potential felt by the spin-up and spin-down incoming particles.

### II.B. Reduction of Radial equations

We define the new set of basis for spin-1 particles as $P_{\pm} = P_{+1} \pm iP_{-1}$. So from eqns. (13) and (16) we get

$$\left[ \frac{d}{dr} - \left( \frac{\sigma r}{K} + \frac{\xi}{2K} \right) \right] P_+ = \left[ \frac{K}{\Delta} - \frac{\lambda}{2K} \right] P_- e^{i\pi/2}, \quad (19)$$

where $\xi$ is real $|\xi|$ and given as $\xi = \xi^* = \sqrt{\lambda^2 - 4\alpha^2\sigma^2}$. Further, defining $\psi_\pm = P_\pm e^{\mp i\pi/4}$ eqn. (19) reduces to

$$\left[ \frac{d}{dr} - P_1(r) \right] \psi_+ = -W \psi_. \quad (20)$$

Similarly, we can get another one as,

$$\left[ \frac{d}{dr} - P_2(r) \right] \psi_- = W \psi_+. \quad (21)$$
Here, \( P_1(r) = \frac{\sigma r}{W} + \frac{\lambda}{2r} \), \( P_2(r) = \frac{\sigma r}{W} - \frac{\lambda}{2r} \) and \( W = \frac{\lambda}{2r} - \frac{K}{r} \). Now, eqns. (20) and (21) are the coupled radial equations for spin-1 particles in the particular choice of basis system as given above. To get appropriate solution we will decouple those and change the independent variable \( r \) to \( V \) as is given below.

Decoupling eqns. (20) and (21) for \( \psi_+ \), we get

\[
\frac{d^2 \psi_+}{dr^2} - \left( P_1(r) + P_2(r) + \frac{W'}{W} \right) \frac{d \psi_+}{dr} + \left( W^2 - P_1' + \frac{W'}{W} P_1 + P_1 P_2 \right) \psi_+ = 0, \tag{22}
\]

where, prime denotes the derivative with respect to \( r \). Now, we consider the integrating factor \( |W(r^2 + \alpha^2)|^{-1} \) and variable transformation as \( \frac{d}{dV} = |W(r^2 + \alpha^2)|^{-1} \frac{d}{dr} \) such that

\[
V = \left( \frac{\lambda}{2\sigma} \right) r - \sigma \left[ \frac{r^3}{3} + Mr^2 + r(2\alpha^2 + 2(2M^2 - a^2) + a^2) - \frac{(\alpha^2 + r_{+}^2)^2}{2\sqrt{M^2 - a^2}} \log(r - r_{-}) + \frac{(\alpha^2 + r_{+}^2)^2}{2\sqrt{M^2 - a^2}} \log(r - r_{+}) \right], \tag{23}
\]

where, \( r_{\pm} = M \pm \sqrt{M^2 - a^2} \) and \( \alpha^2 = a^2 + am/\sigma \). Thus from eqn. (22) we get

\[
\frac{d^2 \psi_+}{dV^2} + \mathcal{K}_+^2 \psi_+ = 0, \tag{24}
\]

where, \( \mathcal{K}_+^2 = \frac{(W^2 - P_1' + \frac{W'}{W} P_1 + P_1 P_2)}{W^2(r^2 + \alpha^2)^2} \). Similarly, decoupling eqns. (20) and (21) for \( \psi_- \) we get

\[
\frac{d^2 \psi_-}{dV^2} + \mathcal{K}_-^2 \psi_- = 0, \tag{25}
\]

where, \( \mathcal{K}_-^2 = \frac{(W^2 - P_1' + \frac{W'}{W} P_1 + P_1 P_2)}{W^2(r^2 + \alpha^2)^2} \). Finally, we have set of second order differential wave equations (24) and (25) which are easier to attack to study the radial behaviour of spin-1 particles. Usually new transformed variable \( V \) varies from \( -\infty \) to \( +\infty \). But, if the frequency \( \sigma \) of the incident wave is such that

\[
W^2(r^2 + \alpha^2)^2 = 0 \tag{26}
\]

at \( r = r_c > r_+ \), \( V - r \) relation becomes multivalued. For that frequency range wave vector \( \mathcal{K}_\pm \) diverges at \( r = r_c \) outside the horizon and the energy extraction from the space-time may be possible in the range \( r_+ \) to \( r_c \). Here \( r_c \) may play the same role as the radius of ergosphere. We know that, in case of a rotating black hole energy extraction may be possible from the ergo-region. Chandrasekhar [6] conjectured from the asymptotic solution of the electromagnetic wave that the super-radiance is exist. He showed that, if the frequency of the incident wave is less than of certain cut-off value (i.e., \( \lesssim \frac{am}{2Mr_+} \)), potential barrier of the system diverges and it varies as \( \frac{1}{(r - |\alpha|)} \) close to the singular point. Here, our calculations tally well with that conjecture. Equations (24) and (25) are the wave equations in cartesian like coordinate system where \( \mathcal{K}_\pm \) contains the information of the radial potential felt by the particles in the new basis system.

Now, from eqn. (26) we see the singular points are, where potential diverges for

\[
\sigma = \sigma_c = \frac{\pm \Delta^{1/2} \sqrt{\lambda} - 2am}{2(a^2 + r^2)}, \tag{27a}
\]

\[
\sigma = \sigma_c = \frac{-am}{r^2 + a^2}. \tag{27b}
\]

Thus, there are three possible frequencies where the potential may diverge and the super-radiance occurs. But not necessarily for all three \( \sigma_c \)'s potential will diverge at \( r \geq r_+ \) for particular \( a, m, M \). If the frequencies of the incident
electromagnetic wave are such that, at a radius \( r \geq r_+ \) at least one of the eqns. (27a,b) satisfy then for that frequency range super-radiance is expected. For the other values of \( \sigma \) the potential as well as \( K^2_\pm \) behaviour does not have any singular point as is shown in Figs. 1 and 2a,b below. Here, the frequencies for which super-radiance is expected to occur are looking different from that given by Chandrasekhar [1]. Actually, as we change the dependent variable \((R_{\pm 1} \to \psi_{\pm})\) and independent variable \((r \to V)\) here in a different manner with respect to that of Chandrasekhar, frequency expressions are looking different apparently but the physical meanings are same for both the cases. It is interesting to note that for super-radiance to occur either eqn. (27a) or (27b) or both have to be satisfied at a radius \( r \leq r_c \) whatever be the \( \sigma \). Thus, it is very obvious that for all \( \sigma \) eqns. (27a,b) do not satisfy, which is again in agreement with Chandrasekhar’s calculation.

### III. SOLUTION

Here we will discuss about the solution of the reduced Maxwell equations described in previous section. We mainly will concentrate on the solution of eqns. (18), (24), (25) and will understand about the dynamical behaviour of the spin-1 particles in Kerr geometry.

#### III.A. Angular Solution

From eqn. (11), we see that for Schwarzschild black hole, solution of angular set of equations is nothing but the standard spherical harmonics \( Y_{lm}(\theta, \phi) \) with integral azimuthal quantum number \( m \) [14,15]. In that occasion \( \mathcal{L}^\dagger \) and \( \mathcal{L} \) simply act as raising and lowering operator respectively of spin-s for \( Y_{lm}(\theta, \phi) \). Corresponding eigenvalue which is the separation constant \( \lambda \) for spin-1 particles is appeared as \( \lambda^2 = l(l + 1) \) [14], which is well known as the total angular momentum of the system where \( l \) is the angular momentum quantum number. For the case of spin-\( 1/2 \) particles, Chakrabarti [7] found the angular solution in Kerr geometry in terms of the combination of different spherical harmonics and Clebsch-Gordan coefficients in perturbative method with \( \sigma \) as the perturbative parameter. He also found the separation constant which was modified from that of Schwarzschild case in terms of Clebsch-Gordan coefficients. Here, our approach to find the angular solution is different from that of Chakrabarti. Actually our method is easily possible to apply here even for angular equation as particles are massless. Thus to get angular solution we will solve eqn. (18) by IWKB method which was applied earlier for the solution of radial Dirac equation only. By this IWKB (instantaneous WKB) method we can get the analytical solution of the second order wave equation which is described in detail in earlier papers [8,11,12]. By this method the transmission and reflection coefficients are calculated at the each location. Here, the particle under consideration is moving in a potential field varying with the location. As the potential barrier felt by the particle is changing with space (see figures for potential plot), the transmission and reflection amplitudes should be space-dependent. Thus from the IWKB solution we can get the local values of the reflection and transmission coefficient and the spatially complete solution is possible to construct. One can have more clear feeling about the local transmission and reflection coefficients if the potential barrier of the system is thought to compose of a large number of square steps. The size and number of the different steps are chosen in such a way that the overall combination of all the steps follows the pattern of the actual potential barrier [14]. Now, one can solve this
as a barrier problem where the barrier under consideration contains a large number of square steps. The transmission and reflection coefficients are calculated at each junction out of the transmission in the previous junction, from what the meaning of local values of the transmission and reflection coefficient is well-understood. In the earlier papers [9,10], in the context of the solution of Dirac equation it was seen that in both the ways (by IWKB method and the way when barrier is considered to compose by a large number of square steps) the solutions are matching perfectly. Here, throughout the study we will concentrate on IWKB solution (not the barrier method as [11]) to obtain an analytical result. The solution of IWKB method is valid if the wave number $k$ of the incoming photon at a location $x$ satisfies the condition $\frac{1}{k} \frac{dk}{dx} \ll k$. Thus, the applicability of IWKB method depends on the validity of the WKB approximation method.

In the eqn. (18), $\lambda$ is already known following the calculation of Chakrabarti [7]. Thus all the parameters are known in (18). As $J_{\pm1}$ varies in entire region and we are interested about the dynamics of the waves in that region itself, from eqn. (18) we get the solution

$$S_{\pm1} = A_{a\pm} e^{+iu_{a\pm}} + B_{a\pm} e^{-iu_{a\pm}}$$

by IWKB method, where the coefficients $A_{a\pm}$ and $B_{a\pm}$ are space-dependent related as $A_{a\pm}^2 + B_{a\pm}^2 = J_{\pm1}$ because sum of the transmission and reflection coefficients are always unity. The ikonals are defined as $\int J_{\pm1} dU = u_{a\pm}$. 


Fig. 1: Variation of $J^2_+$ and $J^2_-$ as functions of $U$ for frequency $\sigma = 0.5$ and Kerr parameter $a = 0$ (solid curve), 0.2 (dotted curve), 0.4 (dashed curve), 0.6 (long-dashed curve), 0.8 (dot-dashed curve) and 1 (dot-long-dashed curve). $a = 0$ corresponds to Schwarzschild solution for which $\lambda = 2$. Following [7] we get other $\lambda$s as $\lambda^2 = 1.704382, 1.417009, 1.136979, 0.863197, 0.594325$ for $a\sigma = 0.1, 0.2, 0.3, 0.4, 0.5$ respectively; $l = 1, m = -1, M = 1$.

Now we will display the solutions for a few sets of physical parameter. The physical parameters are chosen in such a manner that the interaction of photon with black hole should be significant. Therefore, corresponding Compton wavelength should be of same order to the black hole radius, i.e., $\frac{1}{\sigma} \sim M + \sqrt{M^2 - a^2}$ (as $\hbar = G = c = 1$), otherwise the result is insignificant. Also we choose $M = 1$ for all the figures. Figure 1 shows the variation of $J^2_+$ and $J^2_-$ with respect to $U$ for different rotation parameters ($a = 0 \rightarrow 1$) of the black hole for a particular frequency of the incident wave ($\sigma = 0.5$). It is seen that in a certain range both $J^2_+$ and $J^2_-$ attain negative values and corresponding ikonals become imaginary. Recalling Schrödinger wave equation, we can say for that range of $U$ total energy of the incoming particle is less than the potential energy of the system and particle is inside the barrier. In the other range, total energy of the spin-1 particle (in that coordinate system) is greater than or equal to the potential energy of the system as $J^2_+$ attains greater than or equal to zero respectively. In the case of $J^2_+$, at around $\theta = \frac{\pi}{2}$ particle enters into the
barrier and remains inside upto $\theta = \pi$. On the other hand, behaviour of $J_2$ is just opposite. In that case particle energy becomes greater than or equal to the potential energy of the system at around $\theta = \pi/2$ but for $\theta \leq \pi/2$ the particle remains inside the barrier. In this way, with the increment of $\theta$ values, particle goes inside the barrier and then comes out from it alternatively and corresponding $U$ runs from $-\infty$ to $\infty$ and $\infty$ to $-\infty$ alternatively. As the rotation of the black hole decreases curvature of the space-time reduces, which reflects in the behaviour of $J_2$ whose height is lowered down. The results with particular $a$ parameter but different $\sigma$s will be same as Fig. 1 as it depends only on $a\sigma$ not on the individual values of $a$ and $\sigma$.

Although the solution (28) is looking like a wave solution containing incident and reflected parts but there is nothing like that physically. This appears like that because of the method we choose and corresponding transformation of variable $\theta$ to cartesian like variable $U$. Following [9], [10] and [12] we can find out the $\theta$-dependent expressions of $A_{a\pm}$ and $B_{a\pm}$ but there is no relation to transmission and reflection wave amplitudes. To calculate the $\theta$-dependent expressions of $A_{a\pm}$ and $B_{a\pm}$ we need to impose the boundary conditions properly. In case of $J_2^+$, for $U \rightarrow -\infty$ (actually for $U \lesssim -5$, see Fig. 1) both $A_{a+}$ and $B_{a+}$ should be constant which are same as for pure WKB solution. But in the region $U \sim 0 \rightarrow \infty$, as because the wave is inside the potential barrier, $B_{a+}$ must be zero otherwise the solution will diverge at $U \sim \infty$. In case of $J_2^-$, the boundary conditions for $A_{a-}$ and $B_{a-}$ should be appeared just in an opposite way as is depicted from the behaviour of $J_2^-$ too. But it can be reminded that this solution is only valid for $\frac{1}{J_2^+} \frac{dJ_2^+}{dr} \ll J_2^-$.  

### III.B. Radial Solution

To find the solution of eqns. (24) and (25) we again follow IWKB method where $K_{\pm}$ a is function of $r$ and we are interested about the solution at the varying $K_{\pm}$ region. The solution is

$$\psi_{\pm} = \frac{A_{r\pm}}{\sqrt{K_{\pm}}} e^{+iu_{r\pm}} + \frac{B_{r\pm}}{\sqrt{K_{\pm}}} e^{-iu_{r\pm}},$$

where, the space dependent transmitted and reflected amplitudes are related as $A_{r\pm}^2 + B_{r\pm}^2 = K_{\pm}$ and the ikonals are defined as $\int K_{\pm} dV = u_{r\pm}$. In a similar way as the case for angular solution sum of the space-dependent transmission and reflection coefficients are unity always. Close to the black hole horizon $B_{r\pm}$ should be $\sim 0$; because of high gravitative power there is virtually no outwards reflection. Also, far away from the black hole, both $A_{r\pm}$ and $B_{r\pm}$ are constant as potential does not vary there (see Fig. 2a,b). These are the essential boundary conditions to obtain space-dependent solutions. Far away from the black hole, solutions will merge to that of WKB solution. As matter comes closer, spatial variance comes into the picture and IWKB method is needed to obtain an analytical space-dependent solution. This solution is valid for the entire range of $r$ and all values of $a$ (even for Schwarzschild $a = 0$ case). As for different physical parameter set, variations of $K_{\pm}$ are different, corresponding space-dependent behaviour of transmission and reflection coefficients will be different.

Here we display the solutions for a few physical parameter set for which the interaction between incoming photon and black hole is significant, as explained in §III.A. Figure 2a shows the behaviour of the square of wave numbers $K_{\pm}$ with respect to new variable $V$ for a particular frequency $(\sigma)$ and different angular momentum of the black
hole \((a)\). It is seen that as \(a\) value increases, curvature effect on the space-time increases as a result corresponding strength of the wave number increases. Similar features are seen in Fig. 2b where \(\mathcal{K}^2_+\) and \(\mathcal{K}^2_-\) are shown for a particular Kerr parameter \((a = 0.5)\) and different \(\sigma\)s. As the frequency of the incident wave becomes higher, its energy becomes higher too, as a result it needs to overcome the less strength of barrier. On the other hand with the decrement of frequency its energy decreases as well as it feels higher potential barrier in the motion. For both the cases, in entire region of \(V\), \(\mathcal{K}^2_{\pm}\) never be negative, which indicates total energy of the incoming wave is always greater than or equal to the potential energy of the system. It can be checked that for some parameter set, \(\mathcal{K}^2_{\pm}\) becomes negative, which indicates the existence of the potential barrier whose peak is higher than the energy of the incoming particle as was seen in the case of Dirac particles in Kerr geometry [10].

**Fig. 2a:** Variation of \(\mathcal{K}^2_+\) (solid curve) and \(\mathcal{K}^2_-\) (dotted curve) as functions of \(V\) for a particular frequency \(\sigma = 0.5\) and different Kerr parameter \(a\) as marked on each set of curve. Corresponding values of \(\lambda\) for different plots are given in Fig. 1; \(l = 1, m = -1, M = 1\).
Fig. 2b: Variation of $\mathcal{K}_+^2$ (solid curve) and $\mathcal{K}_-^2$ (dotted curve) as functions of $V$ for a particular Kerr parameter $a = 0.5$ and different frequency of incident wave as marked on each set of curve. Corresponding values of $\lambda$ for different plots are given in Fig. 1; $l = 1$, $m = -1$, $M = 1$. 
Figure 2c shows the space-dependent reflection and transmission coefficients for the frequency of particle $\sigma = 0.5$ and angular momentum of the compact object $a = 0.4$. Following $[7]$ we get corresponding $\chi = 1.190382$. In Fig. 2a, the corresponding behavior of wave number is shown. Far away from the black hole ($V \to -\infty$) small $K_+$ indicates higher barrier height and high rate of reflection. As the wave comes closer to the black hole ($V \to \infty$) potential barrier decreases as well as wave number $K_+$ increases which results particle becomes free or almost free. So in that region transmission is 100% or almost 100%. At the other locations, these transmission and reflection coefficients are varying according to the behaviour of potential as well as $K_+$. Again, this solution is only valid as the condition $\frac{1}{K_+} \frac{dK_+}{dV} << K_+$ satisfies.

It can be mentioned that this method of solution is also applicable to a range of parameters which demonstrates super-radiance where potential diverges at a certain location $r > r_+$. In early, this divergent nature of potential was depicted in case of fermions $[10]$, though for the case of fermion super-radiance does not occur. In the present case, if we choose for example, $a = 0.5$, $\sigma = 0.2$ and corresponding $\chi = 1.305519$ $[7]$, $m = -1$, it is found that potential
as well as $K^2_{±}$ diverge at a certain location $r > r_+$ and $V − r$ relation becomes multivalued. For this parameter set super-radiance is expected to occur. Similarly, different other parameter set can be chosen where potentials as well as wave numbers diverge.

VI. CONCLUSIONS

Here we have studied the solution of Maxwell’s equation in curved space-time particularly in Kerr geometry analytically. By the solution we can get the dynamical behaviour of spin-1 particles around black holes. We started with Maxwell equations in curved space-time especially in Kerr geometry then separated it into radial and angular parts. Furthermore, the radial and angular variables are given transformation in such a way that the equations are reduced to one dimensional wave equations in cartesian like coordinate system which is easier to attack. Following the earlier works by the author radial and angular solutions of Maxwell equations are obtained which we had only asymptotic knowledge \[6\] so far. Here we give the spatially complete analytical solution which is helpful for further studies those are already mentioned in the Introduction. Here the incident and reflected amplitudes are not constant, which vary according to the variation of the potential. Thus the reflection and transmission rate of the incident photon are space-dependent. On the occasion for the solution of Dirac equation in a black space-time author initiated a technique namely IWKB method to obtain the space-dependent behaviour of the transmission and reflection coefficient and corresponding spatially complete solution, then it was verified with the numerical results \[11\] \[12\]. Here we have used same IWKB method to obtain solutions of Maxwell equations. The detail technique, i.e., how to obtain the space-dependent analytical expressions for transmission and reflection amplitudes are not repeated here. Here, one of the main emphasis is to construct an analytical complete solution of the electromagnetic wave in a curved space-time, according to my knowledge which was unavailable so far in the literature although the asymptotic as well as numerical solutions were available. Once we have the analytical solution, we can apply it obtain the Feynman Green function, second quantization in a specific metric, here for Kerr metric. The technique applied here is quantum mechanical where $S_{±1}$ and $ψ_{±}$ are wave functions and the corresponding amplitudes of the wave are number. Once we have this knowledge of solution we can make second quantization in curved space-time where $S_{±1}$ and $ψ_{±}$ are no longer wavefunctions but fields and the corresponding amplitudes become operator. Also in the context of Hawking radiation, to check black hole’s stability and its perturbation, to find the quasi-normal modes of the black hole this solution is useful, as this is valid at close to the black hole as well as away from it.

Also we have studied here how the electromagnetic wave face different potential barrier for different physical parameter in a black hole space-time. As the radial functions $R_{±1}$ are given a certain choice of transformation, the potential in the final second order wave equation becomes real (unlike the earlier works like \[6\] etc.) which is now easier to study. According to that we concentrated on to study the transmission and reflection coefficients of the incoming wave. It is seen that depending on the frequency of the incoming electromagnetic wave and the angular momentum of the central object potential as well as wave number of the system change. Thus, it is concluded that the black hole can distinguish the spin-1 particles with different physical parameters. Earlier it was shown \[11\] by the author that black hole can distinguish Dirac particles of different frequency and mass and act as mass-spectrograph. Now globally
we can say that black hole can act as mass-spectrograph irrespective of the spin, frequency and mass of the particle (although in case of electromagnetic wave there is no question of rest mass). Furthermore, the coupling between the spin of black hole and that of particle parts important role, as a result, particle of the particular physical parameter will face different potential barrier for different black holes. Also the same particle with spin-up and spin-down in case of a particular black hole will face totally different potential. Thus we can conclude that the gravitational field is very sensitive on the spin of the particle.

Another important issue is the super-radiance. It was seen [10] that for Dirac particle there is no existence of super-radiance. In case of spin-1 particle super-radiance is very obvious feature as was conjectured by Chandrasekhar earlier [6]. But the existence of super-radiance disappears as particle flips in its spin. For the same sign of spin of the black hole and particle (here $a$ and $m$ respectively) with positive energy of the particle super-radiance does not occur, but, if any one of them flips in spin super-radiance may appear for a particular range of frequency.

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