Self-Similar Bootstrap of Divergent Series

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A method is developed for calculating effective sums of divergent series. This approach is a variant of the self-similar approximation theory. The novelty here is in using an algebraic transformation with a power providing the maximal stability of the self-similar renormalization procedure. The latter is to be repeated as many times as it is necessary in order to convert into closed self-similar expressions all sums from the series considered. This multiple and complete renormalization is called self-similar bootstrap. The method is illustrated by several examples from statistical physics.

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I. INTRODUCTION

The most powerful analytic tool for solving realistic, and therefore hard, problems in theoretical physics and applied mathematics is perturbation theory. However, perturbation series are notoriously badly behaved: in the majority of interesting cases they are divergent. A variety of mathematical techniques have been invented to assign a finite value to the sum of a divergent series. Such techniques are generally referred to as renormalization or summation methods. These methods are not only useful to theoretical and mathematical physicists but they are crucial because they provide a way to recover physical information from perturbative calculations. Probably the most common technique used to assign a meaningful value to a divergent series is the Padé summation [1]. Using the latter one converts a formal power series to a continued fraction. Truncating this fraction at successive orders one obtains the rational functions called Padé approximants. To reach a reasonable accuracy of such approximants, one usually needs to have tens of terms in a perturbative series.

Recently, a method has been suggested [2-4] permitting to ascribe meaningful values to the limits of divergent sequences by exploiting just a few terms of perturbative series. This approach, called self-similar approximation theory [2-4], is based on the following ideas.

First, one has to incorporate into the considered sequence additional functions whose role is to renormalize the sequence making it convergent. These functions, because of their role, are named governing or control functions. The latter are to be defined from fixed-point conditions [2-4]. There are several ways of introducing such control functions. One and a natural way is to include them into initial approximation [5,6]. Fixed-point equations may be written in the form of the minimal-difference condition [5-7], or the minimal-sensitivity condition [8-14]. A condition close to the latter type has also been used in the potential envelope method [15,16]. In two simple cases, of a zero-dimensional and one-dimensional anharmonic oscillator, the control functions were found analytically for an arbitrary perturbation order [17-20] by requiring the convergence of renormalized perturbation theory. Another way of introducing control functions is by using a scaled basis [21-24], in which the scaling parameters, playing the role of such functions, are given by analytical expressions with coefficients adjusted empirically from the convergence of a numerical iterative procedure. One can also define scaling parameters without using their analytical representation, directly from numerical search providing the convergence of an iterative method [25-29]. In the present paper we suggest a new way of introducing control functions, different from all variants mentioned.

Another step in the self-similar approximation theory is to construct an approximation cascade whose trajectory is bijective to the approximation sequence considered [30-32]. In this paper we construct such a cascade not for the sequence itself but for its transform. Of course, the idea of considering a transformed series instead of the initial one is not new. This is, e.g., the basis of the known Borel summation. Another example is the use of the Chebyshev transforms instead of straightforward power-series representation [33]. What is new in our approach is the use of a power-low algebraic transform with powers playing the role of control functions.

Then the constructed approximation cascade is embedded into an approximation flow. Integrating the evolution equation of the flow, we obtain a self-similar approximation [2-4, 30-32].

In this paper we define the control functions from the principle of maximal stability of the approximation cascade trajectories. This is done by minimizing the absolute values of mapping multipliers, which is equivalent to a quasifixed point condition [30-32], since multipliers tend to zero when approaching a stable fixed point.
The plan of the paper is as follows. In Sec. 2 the algebraic transforms are introduced and the main steps of the self-similar approximation theory are sketched, not going into mathematical details which can be found in Refs. 2-4, 30-32, 34. In Sec. 3 the procedure of self-similar bootstrap is defined, consisting of multiple self-similar renormalization of all sums entering into given series. In Sec. 4 a particular case of the self-similar bootstrap is considered leading to a nice representation in the form of multiple exponentials. The following Sections illustrate the approach by various examples emphasizing the generality of the method that can be applied to problems of quite different nature.

II. ALGEBRAIC TRANSFORMS

Assume that we are interested in finding a function \( f(x) \) of a real variable \( x \in (-\infty, +\infty) \). Without loss of generality, the function \( f(x) \) may be considered to be real, since the case of a complex function can be always reduced to that of two real functions. Let perturbation theory give for the function \( f(x) \) approximations \( p_k(x) \) with \( k = 0, 1, 2, \ldots \) enumerating the approximation number. The standard form of \( p_k(x) \) is a series in powers, not necessarily integer, of \( x \). The series can even include logarithms, since the latter can always be presented, using the replica trick, as an expression containing a noninteger power.

The algebraic transform is defined as

\[
P_k(x, s) = x^s p_k(x),
\]

with \( s \) real. This transform changes the powers of the series \( p_k(x) \) changing by this the convergence properties of the latter. Effectively, the approximation order increases from \( k \) to \( k + s \) as a result of Eq. (1). The transform inverse to (1) is

\[
p_k(x) = x^{-s} P_k(x, s).
\]

To construct an approximation cascade, we proceed as follows. Define the expansion function \( x = x(f, s) \) by the equation

\[
P_0(x, s) = f,
\]

where \( P_0 \) is the first available expression from Eq. (1). Substituting \( x(f, s) \) back into (1), we get

\[
y_k(f, s) \equiv P_k(x(f, s), s).
\]

The left-hand side of (4) represents a point of the approximation-cascade trajectory corresponding to approximation (1). The transformation inverse to (4) reads

\[
P_k(x, s) = y_k(P_0(x, s), s).
\]

The function (4) realizes the endomorphism

\[
y_k(f, s) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R},
\]

Consider the family \( \{y_k : k \in \mathbb{Z}_+\} \) as a dynamical system in discrete time. Since the trajectory of this dynamical system, according to (4) and (5), is bijective to the approximation sequence \( \{P_k\} \), this system was called [30-32] the approximation cascade. In order to simplify the consideration, let us pass from discrete time to continuous one. To this end, embed the approximation cascade into an approximation flow:

\[
\{y_k : k \in \mathbb{Z}_+\} \subset \{y(t, ...) : t \in \mathbb{R}_+\},
\]

which means that the trajectory \( \{y(t, f, s)\} \) of the flow has to pass through all points \( \{y_k(f, s)\} \) of the cascade trajectory. The evolution equation

\[
\frac{\partial}{\partial t} y(t, f, s) = v(y(t, f, s), s)
\]

for the approximation flow, where \( v(f, s) \) is the velocity field, can be integrated for an arbitrary time interval, say, from \( t = k - 1 \) to \( t = k^* \), which gives
\[
\int_{y_{k-1}}^{y_k} \frac{df}{v(f,s)} = k^* - k + 1;
\]

(7)

here

\[
y_k = y(k; f, s), \quad y_k^* = y(k^*, f, s).
\]

The upper limit in (7) corresponds, according to (5), to an approximation

\[
P_k^*(x, s) = y(k^*, P_0(x, s), s).
\]

(8)

The moment \( t = k^* \) is chosen so that to reach the approximation (8) by the minimal number of steps. That is, we require that the right-hand side of (7) be minimal,

\[
t_k^* \equiv \min(k^* - k + 1).
\]

(9)

Under condition (9), expression (8) is called the self-similar approximation.

To find (8) explicitly, we need to concretize in (7) the velocity field \( v(f, s) \). This can be done by the Euler discretization of (6) yielding the finite difference

\[
v_k(f, s) = y_k(f, s) - y_{k-1}(f, s).
\]

(10)

Thus, using (5), the evolution integral (7) can be written as

\[
\int_{P_{k-1}}^{P_k} \frac{df}{v_k(f, s)} = t_k^*,
\]

(11)

where

\[
P_k = P_k(x, s), \quad P_k^* = P_k^*(x, s).
\]

When no additional restrictions are imposed, the minimal number of steps for reaching a quasifixed point is, evidently, one,

\[
\text{abs} \min t_k^* = 1.
\]

(12)

Additional restrictions can be of different types. For example, if the value of the sought function at some point \( x_0 \) is known, we may require that the found approximation would coincide at this point with the given exact value. Looking for an approximation in the class of functions with a prescribed symmetry is another way of imposing restrictions. In some cases the asymptotic behavior of the sought function at \( x \to 0 \) and \( x \to \infty \) may be available. Then requiring the correct asymptotic properties also plays the role of such additional constraints. In what follows, we shall concretize these variants by illustrating them with explicit examples.

In this way, the sole quantity that is not yet defined is the parameter \( s \) of the transformation (1). Recall that our aim is to find an approximate fixed point of the cascade trajectory, a quasifixed point, which, by construction, represents the sought function. Therefore, the power \( s \) of the transform in (1) is to be chosen so that to force the trajectory of the approximation dynamical system to approach an attracting fixed point. Recall that \( s \) here is nothing but a kind of the control function, so, it is to be defined by a fixed-point condition. As is discussed in the Introduction, there are several forms of equations defining fixed points. Here we opt for a condition following from the analysis of fixed-point stability properties.

Considering the mapping given by the approximation cascade, we may introduce the mapping multipliers

\[
\mu_k(f, s) = \frac{\partial}{\partial f} y_k(f, s).
\]

(13)

This is related to the local Lyapunov exponent \( \lambda_k \) through the formula \( \lambda_k = \frac{1}{k} \ln |\mu_k| \). Consequently, \( \mu_k \sim e^{\lambda_k k} \). If at increasing time, here at \( k \to \infty \), the trajectory approaches an attracting fixed point, then \( \lambda_k \to \lambda < 0 \). This implies that the multiplier \( \mu_k \to 0 \), as \( k \to \infty \). Another quantity related to the multiplier (13) is the predictability time [35] which can be defined as \( \tau_k \approx |\lambda_k|^{-1} \), or \( \tau_k \approx |k/\ln |\mu_k|| \). This is the characteristic time during which the motion along the cascade trajectory effectively approaches a fixed point. When the latter is attractive, that is, when the limit of
the local Lyapunov exponent $\lambda_k$, as $k \to \infty$, tends to a negative value $\lambda < 0$, then $\mu_k$ tends to zero, and at the same time, larger absolute values $|\lambda_k|$ lead to smaller characteristic times $\tau_k$.

These properties show that closer we are to a fixed point, smaller is the absolute value of the multiplier (13). Hence, we may define the control function $s$ as that providing the minimum of the multiplier. Instead of the multiplier (13), as a function of the variable $f$, it may be more convenient to pass to its image

$$m_k(x, s) = \mu_k(F_0(x, s), s)$$

being a function of the variable $x$. Then the control function $s = s_k(x)$ is defined by the equation

$$|m_k(x, s_k(x))| = \min_s |m_k(x, s)|.$$  

(15)

Because the minimization of the multiplier makes the trajectory more stable, we can call Eq.(15) the principle of maximal stability. And the so defined control function $s_k(x)$ can be termed the stabilizing control function, or, for brevity, the stabilizer. Notice, for comparison, that another definition of the fixed point would be to require the velocity (10) to be zero, which is exactly the minimal-difference condition [5,6]. After the stabilizer $s_k(x)$ is found from (15), we substitute it into (8) and, using the inverse transformation (2), we obtain the self-similar approximation

$$f_k^*(x) = x^{-s_k(x)}P_k^*(x, s_k(x))$$

(16)

for the sought function.

At the end of this section, let us note that the choice of control functions from fixed-point equations is rigorously justified when the fixed point is stable, that is if $|\mu_k| < 1$. When at some point $k$ the trajectory becomes unstable, we have to stop the calculational procedure at the last stable point. Another possibility is to restructure the considered perturbation series. For instance, the instability of the procedure often happens when we are trying to construct self-similar approximation for a divergent function. Assume that we are dealing with such a function $f(x)$ which diverges as $x \to \infty$. Then, the self-similar renormalization procedure may become unstable at large $x$. To avoid the instability, we can either consider the function $f^{-1}(x)$ or can rewrite the series in powers of $1/x$, treating the latter as a new expansion parameter. Usually, after this reexpansion the stability is restored. In the following, we shall illustrate this possibility by an example and will suggest a simple trick, giving the answer without the reexpansion, although being equivalent to the latter.

### III. SELF-SIMILAR BOOTSTRAP

The procedure of calculating the self-similar approximations (16), starting from a perturbative series $p_k(x)$ is now completely defined. The renormalized quantity $f^*_k(x)$ must be, by construction involving the stability properties, a much better approximation to the sought function $f(x)$ than the initial perturbative series $p_k(x)$. To improve the accuracy, we may repeat the self-similar renormalization applying it to other series that are left in (16).

For illustrating this multiple self-similar renormalization, consider explicitly a perturbative series

$$p_k(x) = \sum_{n=0}^{k} a_n x^n, \quad a_0 \neq 0,$$

(17)

containing integer powers of $x$, although, as is mentioned above, the procedure works for arbitrary noninteger powers. Following Sec.2, write the algebraic transform

$$P_k(x, s) = \sum_{n=0}^{k} a_n x^{n+s}$$

(18)

of (17). As is seen, the transform (18) corresponds to an effectively higher perturbation order, $k + s$, as compared to the initial series (17), of order $k$. Eq.(3) for the expansion function $x(f, s)$ now reads

$$P_0(x, s) = a_0 x^s = f,$$

(19)

from where

$$x(f, s) = \left(\frac{f}{a_0}\right)^{1/s}.$$  

(20)
The cascade-trajectory points in (4) become
\[ y_k(f, s) = \sum_{n=0}^{k} a_n \left( \frac{f}{a_0} \right)^{n/s+1}. \]  
(21)
The velocity field (10) writes
\[ v_k(f, s) = a_k \left( \frac{f}{a_0} \right)^{1+k/s}. \]  
(22)
Calculating the evolution integral (11), with condition (12), we get the approximation (8) in the form
\[ P_k^*(x, s) = P_{k-1}(x, s)[1 - \frac{k a_k}{s a_0^{1+k/s}} P_{k-1}^{k/s}(x, s)]^{-s/k}. \]  
(23)
The stabilizer \( s_k(x) \) is to be found from the minimization of the multiplier
\[ m_k(x, s) = \sum_{n=0}^{k} a_n (1 + \frac{n}{s}) x^n \]  
(24)
from (14). Then we obtain the self similar approximation (16),
\[ f_k^*(x) = p_{k-1}(x)[1 - \frac{k a_k}{s a_0^{1+k/s}} x^{k/s} P_{k-1}^{k/s}(x)]^{-s/k}, \]  
(25)
where \( s = s_k(x) \).
In this way, the self-similar renormalization led us from the initial perturbative series (17) to the renormalized approximation (25). The latter contains a perturbative series \( p_{k-1}(x) \) of the lower order than the initial \( p_k(x) \). This can be written as the relation
\[ f_k^*(x) = F_k(x, p_{k-1}(x)) \]  
(26)
showing that (25) depends on \( x \) and \( p_{k-1}(x) \). We may repeat the procedure renormalizing \( p_{k-1}(x) \) and getting \( f_{k-1}^*(x) \). With such a double renormalization, we come from (25) to
\[ f_k^{**}(x) = F_k(x, f_{k-1}^*(x)) = F_k(x, F_{k-1}(x, p_{k-2}(x))). \]  
(27)
Here the doubly renormalized \( f_k^{**}(x) \) is expressed through \( p_{k-2}(x) \). Repeating the self-similar renormalization \( k \) times, we obtain the \( k \)-fold self-similar approximation
\[ f_k^{**...}(x) = F_k(x, F_{k-1}(x, ... (F_1(x, a_0)...))), \]  
(28)
where we took into account that \( p_0(x) = a_0 \). It may happen that (28) contains other power series. Then we may renormalize them as well. When all power series are renormalized, so that none of them is left unrenormalized, we have
\[ f_k^{**...}(x) \to \bar{f}_k(x). \]  
(29)
This complete procedure of the self-similar renormalization of all power series, resulting in an expression \( \bar{f}_k(x) \) containing none of them, will be called the self-similar bootstrap.

**IV. MULTIPLE EXPONENTIALS**

There are particular cases of the multiple self-similar renormalization yielding to a nice exponential representation of the resulting formulas. Here we consider one such a sufficient condition when all coefficients at perturbative powers are positive, \( a_n > 0 \). Then the minimum of the multiplier (24) realizes at \( s \to \infty \). Taking this limit in (25) gives
\[ f_k^*(x) = p_{k-1}(x) \exp(\frac{a_k}{a_0} x^k). \]  
(30)
Repeating the renormalization in line with (26) and (27), we get

\[ f_{k}^{*}(x) = p_{k-2}(x) \exp\left\{ \frac{1}{a_0}(a_{k-1}x^{k-1} + a_kx^k) \right\}. \]  
(31)

Continuing so on, we obtain the \( k \)-fold approximation (28) in the form

\[ f_{k}^{*...*}(x) = a_0 \exp\left\{ \frac{1}{a_0}(a_1x + a_2x^2 + ... + a_kx^k) \right\}. \]  
(32)

As we see, the \( k \)-times repeated self-similar renormalization does not deliver us from the power series. Really, the \( k \)-star approximation (32) is expressed through a part of the initial perturbation series (17), namely, through

\[ p_k(x) - a_0 = \sum_{n=1}^{k} a_n x^n. \]

With the notation

\[ p'_k(x) = \sum_{n=0}^{k} a'_n x^n, \]
(33)
in which \( a'_n \equiv a_{n+1} \), \( n = 0, 1, 2, ..., k \), we may rewrite (32) as

\[ f_{k}^{*...*}(x) = a_0 \exp\left\{ \frac{x}{a_0} p'_k(x) \right\}. \]  
(34)

The power series \( p'_{k-1}(x) \) can be renormalized in our standard way giving the corresponding self-similar approximation

\[ f'_{k-1}(x) = a'_0 \exp\left\{ \frac{x}{a'_0} p''_k(x) \right\}, \]
(35)
in which

\[ p''_k(x) = \sum_{n=0}^{k} a''_n x^n, \quad a''_n \equiv a_{n+2}. \]
(36)

With this renormalization in mind, we transform (34) into

\[ f_{k}^{*...*}(x) = a_0 \exp\left\{ \frac{x}{a_0} f'_{k-1}(x) \right\}. \]  
(37)

Combining (35) and (37), we have

\[ f_{k}^{*...*}(x) = a_0 \exp\left\{ \frac{x}{a_0} a_1 \exp\left\{ \frac{x}{a_1} p''_{k-2}(x) \right\} \right\}. \]  
(38)

Converting \( k \) times all power series in the exponentials, with the use of the notation

\[ b_0 = a_0, \quad b_k = \frac{a_k}{a_{k-1}}, \quad k = 1, 2, ..., \]
(39)
we obtain the bootstrap self-similar approximation

\[ \tilde{f}_k(x) = b_0 \exp(b_1 x \exp(b_2 x \exp(...b_{k-1} x \exp(b_k x)))...), \]  
(40)
as is discussed in (29).

In the case of small \( x \rightarrow 0 \) the expression (40) yields

\[ \tilde{f}_k(x) \simeq c_0 + c_1 x + c_2 x^2 + c_3 x^3 \]  
(41)
with the coefficients
\[ c_0 = b_0, \quad c_1 = b_0b_1, \quad c_2 = b_0b_1(b_2 + \frac{1}{2}b_1), \quad c_3 = b_0b_1(b_2b_3 + \frac{1}{2}b_2^2 + b_1b_2 + \frac{1}{6}b_1^2). \]

Substituting here (39), we have
\[ c_0 = a_0, \quad c_1 = a_1, \quad c_2 = a_2 + \frac{a_1^2}{2a_0}, \quad c_3 = a_3 + \frac{a_2^2}{2a_1} + \frac{a_1 a_2}{a_0} + \frac{a_1^3}{6a_0^2}. \]

This shows that the asymptotic, as \( x \to 0 \), behavior of (41) and (17) coincides up to the linear terms. For the higher-order terms, \( a_n \neq b_n \) for \( n \geq 2 \). Such a renormalization of the higher order expansion coefficients is typical of the self-similar approximation theory [2-4]. This renormalization allows to extend the region of applicability of self-similar approximations, with respect to a variable \( x \), as compared to the initial perturbative series.

It is worth mentioning that the multiple, or continued, exponentials of type (40) have been studied in mathematical literature for already more than two centuries, since Euler; and a number of references can be found in Refs. [36,37]. We have derived the form (40) following the multiple self-similar renormalization for the power series (17) with positive coefficients. Of course, for other sets of coefficients in (17) the final bootstrap approximation will not necessarily take the form of a multiple exponential, as in (40), but will be a mixture of exponentials and fractions, each expression being conditioned by the principle of maximal stability (15).

Another way of getting a multiple exponential could be as follows. Consider a sequence \( \{\varphi_k(x)\} \) of the terms
\[ \varphi_1(x) = b_0 \exp(b_1x), \]
\[ \varphi_2(x) = b_0 \exp(b_1x \exp(b_2x)), \]
and so on, where the coefficients \( b_k \) are given by (39). At the \( k \)-step of the sequence \( \{\varphi_k(x)\} \) we come to (40). However, this way is justified if the approximation cascade corresponding to the sequence \( \{\varphi_k(x)\} \) possesses a stable trajectory leading to a quasifixed point representing (40). This approximation cascade can be constructed in the standard way [32,34]. From the equation \( \varphi_1(x) = f \) we find \( x = x(f) \) which is
\[ x(f) = \frac{a_0}{a_1} \ln \left( \frac{f}{a_0} \right). \]

Then we define the approximation cascade as is described in Sec.2. The corresponding trajectory \( \{z_k(f)\} \), consisting of the terms
\[ z_k(f) \equiv \varphi_k(x(f)) \]
is bijective to the approximation sequence \( \{\varphi_k(x)\} \).

The stability of the trajectory \( \{z_k(f)\} \) is checked by calculating the multipliers
\[ M_k(x) = \left| \frac{\partial z_k(f)}{\partial f} \right|_{f=\varphi_1(x)} \]
and analyzing them with respect to the stability condition \( |M_k(x)| < 1 \).

At the end of this section let us remark that we have used here the word "bootstrap" in the generally accepted meaning, as a kind of a completely self-consistent procedure permitting to construct an explicit solution to a complicated problem. This term in the close meaning was used, for example, in constructing a self-consistent distribution over particle masses in high-energy physics [38], and also in constructing the \( S \)-matrices for two-dimensional conformal field theories [39].

V. ZERO-DIMENSIONAL ANALOGS OF FIELD THEORIES

A. Non-Degenerate Vacuum

Consider the thermodynamic potential of a zero-dimensional anharmonic model (see e.g [32]) represented by the integral
\[ J(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2 - gx^4)dx, \]  

(43)

with the integrand possessing a single "vacuum" state, located at the point \( x = 0 \). The expansion of this integral in powers of the coupling parameter \( g \), around the vacuum state leads to divergent series,

\[ J(g) \sim a + bg + cg^2 + dg^3 + hg^4 + \ldots, \]  

(44)

\[ a = 1, \quad b = -\frac{3}{4}, \quad c = \frac{105}{32}, \quad d = \frac{3465}{128}, \quad h = \frac{675675}{2048}. \]

We apply here the self-similar bootstrap renormalization, guided by desire to perform as many renormalization steps, as possible. Write down the following set of approximations to the quantity \( J(g) \), analogous to the general form (17):

\[ J_0(g) = a, \]

\[ J_1(g) = a + bg, \]

\[ J_2(g) = a + bg + cg^2, \]

\[ J_3(g) = a + bg + cg^2 + dg^3, \]

\[ J_4(g) = a + bg + cg^2 + dg^3 + hg^4, \]

(45)

together with the following local multipliers, that can be found from the general representation (24):

\[ m_1(g, s) = 1 + \frac{b}{a} \frac{1 + s}{s} g, \]

\[ m_2(g, s) = m_1(g, s) + \frac{c}{a} \frac{2 + s}{s} g^2, \]

\[ m_3(g, s) = m_2(g, s) + \frac{d}{a} \frac{3 + s}{s} g^3, \]

\[ m_4(g, s) = m_3(g, s) + \frac{h}{a} \frac{4 + s}{s} g^4. \]

(46)

The analysis of (46) shows, that in the last three cases the most stable trajectories are realized at \( s \to \infty \), and that in the first case, \( s \to \infty \) also corresponds to a stable trajectory. Therefore the starting four steps of the self-similar bootstrap renormalization can be safely performed in the exponential form, leading to the intermediate renormalized quantity

\[ J^\ast\ast\ast\ast_4(g) = a \exp\left\{ \frac{1}{a} (bg + cg^2 + dg^3 + hg^4) \right\}. \]

(47)

Write down a set of approximations to the quantity \( J'(g) = bg + cg^2 + dg^3 + hg^4 \), appearing in the exponential of this expression:

\[ J'_1(g) = bg, \]

\[ J'_2(g) = bg + cg^2, \]

\[ J'_3(g) = bg + cg^2 + dg^3, \]
\[ J_4'(g) = bg + cg^2 + dg^3 + hg^4, \]

with the following local multipliers:

\[ m'_2(g, s) = 1 + \frac{c 2 + s}{b 1 + s} g, \]

\[ m'_3(g, s) = m'_2(g, s) + \frac{d 3 + s}{b 1 + s} g^2, \]

\[ m'_4(g, s) = m'_3(g, s) + \frac{h 4 + s}{b 1 + s} g^3. \]

The analysis of (48) leads us to the conclusion that the exponential renormalization is optimal at every step and, following to the standard prescriptions of Sec. 4, we transform (47) into

\[ J_4^{****}(g) = a \exp\left\{ \frac{b}{a} g \exp\left[ \frac{1}{b} (cg + dg^2 + hg^3) \right] \right\}. \]

(49)

Our routine procedure requires to renormalize now the quantity \( J''(g) = cg + dg^2 + hg^3 \), using the following approximations:

\[ J'_1(g) = cg, \]

\[ J'_2(g) = cg + dg^2, \]

\[ J'_3(g) = cg + dg^2 + hg^3, \]

and analyzing the following multipliers:

\[ m''_2(g, s) = 1 + \frac{d 2 + s}{c 1 + s} g, \]

\[ m''_3(g, s) = m''_2(g, s) + \frac{h 3 + s}{c 1 + s} g^2. \]

We conclude that the most stable trajectory corresponds, in both cases, to the exponential summation, leading to the intermediate formula

\[ J_4^{****}(g) = a \exp\left\{ \frac{b}{a} g \exp\left[ \frac{1}{b} (cg + dg^2 + hg^3) \right] \right\}. \]

(50)

The last step of the procedure, applied to the quantity \( J'''(g) = dg + hg^2 \), with the approximations set:

\[ J''_1(g) = dg, \]

\[ J''_2(g) = dg + hg^2, \]

and with the multiplier

\[ m'''_2(g, s) = 1 + \frac{h 2 + s}{d 1 + s} g, \]

again should be performed with \( s \to \infty \), and the bootstrap program is completed:

\[ \hat{J}_4(g) = a \exp\left\{ \frac{b}{a} g \exp\left[ \frac{c}{b} \exp\left[ \frac{d}{c} g \exp\left( \frac{h}{d} g \right) \right] \right] \right\}. \]

(51)
Similar expressions follow when less terms from the initial expansion are taken into account:

\[ \tilde{J}_3(g) = a \exp\left\{ \frac{b}{a} g \exp\left[ \frac{c}{b} \exp\left( \frac{d}{c} g \right) \right] \right\}, \]

\[ \tilde{J}_2(g) = a \exp\left\{ \frac{b}{a} g \exp\left[ \frac{c}{b} g \right] \right\}, \]

\[ \tilde{J}_1(g) = a \exp\left\{ \frac{b}{a} g \right\}. \]

We had pointed out already, that the last expression corresponds to the stable, but not to an optimal, i.e. the most stable, trajectory. Analyzing \( m_1(g, s) \), we obtain the optimally renormalized expression:

\[ \tilde{J}_{1o}(g) = a[1 - \frac{b}{a} \times s(g)^{-s(s)}], \quad s(g) = -\frac{bg}{a + bg}. \]  \hspace{1cm} (52)

At the point \( g = 1 \), the following numbers are generated by the sequence \( \tilde{J}_i \), \( i = 2, 3... \):

\[ \tilde{J}_1(1) = 0.472, \quad \tilde{J}_{1o}(1) = 0.512, \quad \tilde{J}_2(1) = 0.991, \quad \tilde{J}_3(1) = 0.473, \quad \tilde{J}_4(1) = 0.991. \]

We observe the two subsequences, with odd and even numbers, with values practically unchanged within each subsequence, probably embracing the correct result from below and from above, respectively. We can suspect that the corresponding sequence \( \tilde{J}_i \) possesses the two competing unstable quasifixed points, i.e behaves chaotically and, in such situation, it is appropriate to use the self-similar approximation smoothed by the Cesaro averaging procedure[32], i.e. in our case, simply to take the average over the two neighboring members of each subsequence.

This conjecture, is supported by the analysis of the corresponding sequence of the multipliers for the sequence \( \tilde{J}_i \), as discussed in Sec.4. From the initial approximation

\[ \tilde{J}_1(g) = f, \]

one can find the expansion function

\[ g = \frac{a}{b} \ln\left( \frac{f}{\sqrt{\pi} a} \right), \]

and, after the routine transformations, the following expression for the multipliers (42) can be obtained:

\[ M_1(g) \equiv 1, \]

\[ M_2(g) = \Phi_2(g) \Psi_2(g), \]

\[ M_3(g) = \Phi_3(g) \Psi_3(g), \]

\[ M_4(g) = \Phi_4(g) \Psi_4(g), \]

where

\[ \Phi_2(g) = \frac{\tilde{J}_2(g)}{a} \exp\left( \frac{c}{b} g \right), \]

\[ \Phi_3(g) = \frac{\tilde{J}_3(g)}{a} \exp\left( \frac{c}{b} g \exp\left( \frac{d}{c} g \right) \right), \]

\[ \Phi_4(g) = \frac{\tilde{J}_4(g)}{a} \exp\left( \frac{c}{b} g \exp\left( \frac{d}{c} g \exp\left( \frac{h}{d} g \right) \right) \right), \]
and

\[ \Psi_2(g) = b + cg, \]

\[ \Psi_3(g) = b + cg \exp(\frac{d}{c}g) + dg^2 \exp(\frac{d}{c}g), \]

\[ \Psi_4(g) = b + cg \exp(dg \exp(\frac{h}{d}g)) + dg^2 \exp(\frac{h}{d}g) + dh^2 \exp(\frac{h}{d}g), \]

The following values are obtained at \( g = 1 \):

\[ M_1 = 1, \quad M_2 = -0.089, \quad M_3 = 1.008, \quad M_4 = -0.089, \]

supporting our initial guess, that the approximation cascade behaves chaotically. After the Cesaro averaging, the sought value at \( g = 1 \) equals to, say, \( \tilde{J}_3(1) + \tilde{J}_4(1) = 0.731 \), deviating from the exact value 0.772 with the percentage error \(-5.228\%\), an acceptable accuracy if to remember that the initial expansion (44) gives the percentage error \(-10^4\%\). With the optimized \( \tilde{J}_{1o} \), we obtain even better estimate, 0.752, for the sought value, with the percentage error equal to \(-2.668\%\).

**B. Double-Degenerate Vacuum**

Consider the integral

\[ I(g) = \int_{-\infty}^{\infty} \exp(x^2 - gx^4)dx, \quad (53) \]

representing zero-dimensional field-theory with the integrand possessing the two maxima, located at the points

\[ \mp = \pm \frac{1}{\sqrt{2}g}, \]

where \( g \) plays the role of a coupling constant. We intend to estimate this integral in the region of intermediate couplings \( g \sim 1 \). It was pointed out in [40], that any conventional expansion, in powers of \( g \), or \( g^{-1} \), is not sufficient, since it does not take into account the existence of those degenerate maxima, corresponding to the double-degenerate "vacuum".

Within the framework of \( D \)-dimensional field theories, the existence of degenerate vacuum is taken into account, e.g., by means of the zero-energy instanton-anti-instanton solutions, and all further corrections to observables come from the excitations above the instanton-anti-instanton background and from interaction of all those quasiparticles. In our case we take into account the double-degenerate vacuum by means of the shift:

\[ x = \mp - X, \]

then expand the integral around the two saddle points and apply the self-similar renormalization to the resulting asymptotic expansion in powers of a small parameter \( g^{1/2} \), continuing it to the region of \( g \sim 1 \). Following these prescriptions, represent the integrand in the vicinity of one of the saddle points in the form:

\[ \exp(x^2 - gx^4) \approx \exp(\frac{1}{4g}) \exp(-2X^2) \exp(A(g)X^3) \approx \exp(\frac{1}{4g}) \exp(-2X^2)(1 + A(g)X^3 + ...), \]

and perform the integration, so that

\[ I(g) \approx 2 \exp(\frac{1}{4g})(a + bg^{1/2} + ...), \quad a = \sqrt{\pi}2^{-3/2}, \quad b = 2^{-3/2}. \quad (54) \]

Applying the self-similar renormalization, we readily obtain

\[ I^*(g) = 2a \exp(\frac{1}{4g}) \exp(\frac{bg^{1/2}}{a}). \quad (55) \]

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Despite the absence of dynamics in the zero-dimensional case, \( I'(g) \) consists of two factors of different nature: one of them is non-analytic in the coupling constant, resembling the well-known "instanton" term within the framework of nontrivial \( D \)-dimensional field theories, the second is analytic in \( g^{1/2} \) and resembles a contribution from the excitations above the instanton-anti-instanton background.

The percentage error for the renormalized \( I'(1) \), calculated with respect to the exact \( I(1) = 2.762 \), is 2.462\%, and considerable improvement is achieved compared to the percentage error of the perturbative expansion (54), equal to\(-8.834\%\).

**VI. STRONG COUPLING REGIME**

**A. Zero-Dimensional Case.**

Let us apply to (43) the so-called "strong-coupling" expansion, in powers of \( 1/g \), with a quartic term of the integrand taken as a zero approximation, representing the integrand of (43) as follows

\[
\exp(-x^2 - gx^4) \approx \exp(-gx^4)(1 - x^2 + \frac{x^4}{2} + ...).
\]

After integration we obtain the expansion in inverse powers of the coupling constant

\[
J(g) \approx ag^{-1/4} + bg^{-3/4} + cg^{-5/4} + ..., \quad a = \frac{1.813}{\sqrt{\pi}}, \quad b = \frac{-0.612}{\sqrt{\pi}}, \quad c = \frac{0.227}{\sqrt{\pi}}. \tag{56}
\]

We write down the following consecutive approximations to the quantity \( J(z) \), where \( z = g^{-1/4} \),

\[
J_1(z) = az,
\]

\[
J_2(z) = az + bz^3,
\]

\[
J_3(z) = az + bz^3 + cz^5;
\]

and the multiplier \( m_2(z, s) = 1 + \frac{3z^2}{1+s} \) reaches its minimal zero value at \( s = 0.019 \), being much smaller than the minimal value of the corresponding multiplier \( m_3(z, s) = m_2(z, s) + \frac{5z^4}{1+s} \). Therefore, the renormalized quantity \( J^*_2(z) \), will correspond to the more stable trajectory, than \( J^*_3(z) \). Following the standard prescriptions of Sec.3 we obtain

\[
J^*_2(1) = ax\left[1 - \frac{2b}{a[1+s(x)]}x^2\right]^{-\left(s(x)+1\right)/2}, \quad s(x) = \frac{a + 3bx^2}{a + bx^2}.
\]

The percentage error for the renormalized quantity \( J^*_2(1) \), calculated with respect to the exact \( J(1) = 0.772 \), is 2.266\%, and a considerable improvement is reached compared to the percentage error of the perturbative expansion (53) with only starting two terms taken into account, equal to\(-12.208\%\).

Represent (43) in a little bit different form:

\[
J(g) = g^{-1/4}(a + bg^{-1/2} + cg^{-1} + ...) \equiv g^{-1/4}[a + J(g)], \tag{57}
\]

and write down the following set of approximations to \( J(g) \) , using the variable \( y = g^{-1/2} \) :

\[
\overline{J}_1(y) = by,
\]

\[
\overline{J}_2(y) = by + cy^2.
\]

The multiplier \( m_2(y, s) = 1 + \frac{cy^2}{1+s} \), is minimal at \( s = 0 \), and \( |m_2(y, 0)| < 1 \). The renormalized quantity \( \overline{J}_2(y) \) can be readily written down:

\[
\overline{J}_2 = \frac{by}{1-\frac{cy^2}{y}}
\]

Recalculating now \( J'(g) \), we obtain \( J'(1) = 0.771 \), with the percentage error \(-0.13\%\), much better than the percentage error 4.386\%, given by the initial expansion (57). Even at small \( g = 0.21 \), the percentage error given by the renormalized expression, remains less than 1\%; at the same time, the percentage error given by the initial expression reaches 43.538\%.
B. One-Dimensional Case.

Consider the dimensionless ground state energy $e(g)$ of the celebrated quantum one-dimensional quartic anharmonic oscillator, closely connected to the so-called $\varphi^4$ model in the quantum field theory (see e.g. [3]). Here $g$ stands for the dimensionless coupling constant, expressed through the parameters entering the Hamiltonian of the system by the known relation (see e.g [3,41]). The asymptotic expansion for $e(g)$ in the strong coupling limit, corresponding to $g \to \infty$, is known (see e.g.[41]) in the following form:

$$e(g) \equiv ag^{1/3} + bg^{-1/3} + cg^{-1}, \quad a = 0.667986, \quad b = 0.14367, \quad c = -0.0088. \quad (58)$$

Let us, using the experience gained while considering the strong coupling limit of the zero-dimensional field theory, renormalize the last two terms of the expansion (58). Using the notation $y = g^{-1/3}$, we write down the following set of approximations for the quantity $\tau = e - ag^{1/3}$:

$$\tau_1(y) = by,$$

$$\tau_2(y) = by + cy^3,$$

with the multiplier $m_2(y,s) = 1 + \frac{3+4}{3}y^2$, possessing minimal value at $s = 0$, when $g \geq 0.1$. The renormalized expression can be obtained following the standard prescriptions of Sec.III. Returning to the initial variable we obtain

$$e^*(g) = ag^{1/3} + \frac{b^{3/2}}{\sqrt{bys^3 - 2c}}. \quad (59)$$

An accuracy given by $e^*(g)$, can be elucidated by comparison with the "exact" numerical results (see e.g. [41]). At $g = 0.3$, the percentage error, given by (59), is equal to $-0.999\%$, at $g = 1$ it is $-0.022\%$, and at $g \to 200$ the percentage error tends to zero. To our knowledge, these results are better than those obtained by other analytical methods. On the other hand, at small $g = 0.001$, an accuracy of the formula (59) is by far inferior, compared to many other analytical methods. The reason can be understood if notice, that $e^*(0) = 0.41048$, strongly deviating from the known value $1/2$, but being much better than the infinite value predicted by the initial expansion (58). We conclude, remarking, that using the effective time $t^*$ as an optimization parameter, determined from the condition $e^*(0) = 1/2$, one can achieve better agreement of the renormalized formulae with the exact results in the region of small coupling constants. For the goal being pursued in the present paper, it is fairly enough to limit the discussion by formulae (59), designed for the intermediate and strong-coupling regimes.

VII. EQUATION OF STATE

A. System of Hard Spheres.

We demonstrate below, how the self-similar bootstrap can be applied in the theory of equations of state for simple liquids. Consider the model system of hard spheres with the diameter $d$ [42,43], where the empirical equation of state, connecting pressure $p$, temperature $T$, the number density $n$ and the reduced density $\rho = \pi nd^2/6$, is known:

$$\frac{p}{nkT} = \frac{1 + \rho + \rho^2 - \rho^3}{(1 - \rho)^3}. \quad (60)$$

The equation of state (60) agrees very well with the molecular dynamics results [42]. On the other hand, the theoretical virial formula according to Percus-Yevick [41,42], is given as follows:

$$\frac{p}{nkT} = \frac{1 + \rho + \rho^2 - 3\rho^3}{(1 - \rho)^3}. \quad (61)$$

These two expressions almost coincide at low densities, e.g at $\rho = 0.1$, the percentage error of Eq.(61), as compared with (60), equals $-0.18\%$; while for the intermediate and high densities the agreement becomes very poor, e.g. at $\rho = 0.5$, the percentage error is $-15.385\%$, and at $\rho = 0.8$ it equals $-53.112\%$.

Consider the regular part of (61), defined as $r$:
\[ r = 1 + \rho + \rho^2 - 3\rho^3, \quad (62) \]

as an asymptotic, low-density expansion for the "true" regular part \( \overline{r}(\rho) \), and try to continue the expression (62) from the region of \( \rho \ll 1 \), to the region of \( \rho \leq 1 \). It seems reasonable to use for renormalization only the last three terms from (62), since the constant term describes the ideal gas behavior and we are interested in the region of high densities. Let us write down the following consecutive approximations to the quantity \( \overline{\tau} = r - 1 \):

\[
\overline{\tau}_1 = \rho, \\
\overline{\tau}_2 = \rho + \rho^2, \\
\overline{\tau}_3 = \rho + \rho^2 - 3\rho^3
\]

The multipliers are

\[
m_2(\rho, s) = 1 + \rho \frac{2 + s}{1 + s}, \\
m_3(\rho, s) = m_2(\rho, s) - 3\rho^2 \frac{3 + s}{1 + s}, \\
m'_3(\rho, s) = 1 - 3\rho^2 \frac{3 + s}{2 + s}
\]

It is admissible to apply here the self-similar bootstrap renormalization in the form of the continued exponentials, since at every step of the procedure the exponential summation is performed along the stable trajectory. Following the standard prescriptions of Sec.4, we obtain

\[
\overline{r}(\rho) = \rho \exp(\rho \exp(-3\rho)). \quad (63)
\]

The multiplier \( M(\rho) \), corresponding to (63), is given by the expression

\[
M(\rho) = \exp[\rho \exp(-3\rho)] \exp(-4\rho)(1 - 3\rho),
\]

and is very small at \( \rho > 1/3 \), e.g. \( M(0.8) = -0.061 \), signaling the robust stability of the sequence of the continued exponentials (63). Recalculating

\[
\frac{\overline{\rho}}{nkT} = \overline{r}(\rho) + 1 \quad (1 - \rho)^3 \quad (64)
\]

and comparing it to the empirical formula (60), we obtain that at \( \rho = 0.1 \), the percentage error equals \(-0.118\%\); at \( \rho = 0.5 \), the percentage error is \(-4.061\%\), and at \( \rho = 0.8 \) it equals \(-3.516\%\).

We see that the equation of state (64), obtained from the bootstrap self-similar renormalization, much better and more uniformly agrees with the computer experiment, than the initial virial expansion (61), over the entire range of densities. The agreement drastically, by 17 times, improves in the region of high densities.

**B. System of Hard Hexagons**

The model of "hard hexagons" represents a simple two-dimensional model of impenetrable molecules on the triangular lattice. The model allows an exact solution [44], and the phase transition from the liquid phase existing above the critical value of the so-called activity \( z_c = 11.0917... \), to the solid phase, existing below \( z_c \), is well studied. The equation of state, describing the dependence of the order parameter \( R \) on the activity-related parameter, can be written down in quite a complicated and not very convenient form [44]. On the other hand, the critical value of density \( \rho_c \) at the point of phase transition is known too, and equals 0.27639...[44]. Independently, the high-density expansions of the order parameter in powers of the inverse activity \( z' \equiv 1/z \), or in powers of the high-density variable \( \rho' = 1 - 3\rho \), were obtained [45]. Their quality is considered as very high, since the critical parameters could be determined from them with extremely high accuracy, using the Pade approximants in conjunction with some extrapolation methods.
We present below simple expressions for the equation of state, obtained as a continuation of the high-density expansions of the order parameter up to the point of phase transition.

The expansion of the order parameter in powers of $\rho'$ is given as follows \[45\]:

$$R = 1 - 3(\rho')^2 - 9(\rho')^3 - 36(\rho')^4 - 159(\rho')^5.\quad (65)$$

Let us apply to (65) the bootstrap self-similar renormalization based on the exponential summation at every step, and leading to the equation of state for the system of hard hexagons in the form of the continued exponentials:

$$\tilde{R}(\rho') = \exp(-\rho' \exp(3\rho' \exp(3\rho' \exp(4\rho' \exp(5\rho'))))).\quad (66)$$

The function $\tilde{R}(\rho')$ approaches zero at $\rho_c^* = 0.170005(\pm 1)$, corresponding to the $\rho_c^* = 0.276665$, deviating from the exact value by 0.1%. Thus, the renormalized equation of state (66), agrees with the initial expansion in the region of $\rho' \ll 1$ by design, and predicts the point of phase transition with very high accuracy. On the other hand, the form of continued exponential, can be justified \textit{a posteriori}, analyzing the multipliers (42), where it can be shown, after some lengthy, but routine calculations, that $M_3(\rho') \to 0$, in the region of $\rho' \approx \rho_c^*$, i.e. the stability condition is satisfied along the trajectory, described by the sequence of approximations corresponding to (66), in the vicinity of the critical point. Similarly, using the known expansion of $R$ up to the fifth order terms in $z'$, the corresponding equation of state can be obtained. In this case we found the critical $z_c^* \approx 12.1803(1)$, deviating from the exact value by 9.8%.

C. Polymer Coil

The expansion factor $\alpha$ of the polymer chain, within the framework of a standard "beads-on-string" model, is conveniently represented as a function $\alpha^2 = a^2(z)$ of the parameter $z = 2(\frac{N}{32})^{1/3}N^{1/2}B/a^2$, where $N$ is the total number of links in the chain, $a$ stands for the typical distance between the beads-monomers, and $B$ is the second virial coefficient [46,47]. We consider below only the case of a polymer coil, corresponding to $z > 0$. In the region of $z \ll 1$, the perturbation theory in powers of $z$ can be developed and for the short-range potentials one can find [46,47] that

$$\alpha^2 = a^2(z) = 1 + k_1 z + k_2 z^2 + \ldots, \quad k_1 \approx 1.28, \quad k_2 = -20.8.\quad (67)$$

One of the important problems in the physics of polymer coils, consists in the continuation of the expansion (67) to the region of $z \sim 1$ [46,47]. On the other hand, in the limit of $z \gg 1$, $\alpha$ is related to $z$ by a simple power-law

$$\alpha \sim z^{2\nu - 1},\quad (68)$$

where the critical index $\nu \geq 1/2$, can be calculated by different methods [46-48]. We propose below, using the self-similar renormalization, a simple way to continue (67) to the region of arbitrary $z$, including both known limiting cases and allowing to estimate $\nu$ from the stability condition. The problem of this type was already mentioned above, in Sec.II. From the viewpoint of the applicability of the stability conditions, it is worthwhile to study $\alpha^{-2}(z) = a(z)$, re-expanding it in powers of $z$, so that

$$a(z) \approx 1 + b_1 z + b_2 z^2 + \ldots, \quad b_1 = -1.28, \quad b_2 = 22.438.\quad (69)$$

The set of approximations to $a(z)$, including the two starting terms from (69), can be written down as follows:

$$a_0 = 1,$$

$$a_1 = 1 + b_1 z,$$

and the expression for the renormalized quantity $a_1^*$ can be readily obtained:

$$a_1^* = \left(\frac{s_1}{s_1 - b_1 z}\right)^{s_1} \Longrightarrow \left(\frac{s_1}{s_1 - b_1}\right)^{s_1} z^{-s_1} (z \to \infty),\quad (70)$$

where the stabilizer $s_1$ should be positive, if we want to reproduce in the limit of $z \to \infty$, the correct power-low behavior of the $a^2(z)$. A different set of approximations, not including into the renormalization procedure the constant term from (69), has the form:

$$\overline{a_1} = b_1 z,$$
\[ a_2^* = 1 + b_1 z [1 - \frac{b_2 z}{b_1 (1 + s_2)}]^{-1 + s_2} \Rightarrow (\frac{-b_2}{1 + s_2})^{-(1 + s_2)} b_1^{2 + s_2} z^{-s_2} (z \to \infty). \] (71)

Demanding now, that both (70) and (71) have the same power-law behavior at \( z \to \infty \), we find that

\[ s_2 = s_1 = 2(2 \nu - 1). \]

Requiring now the fulfillment of the stability criteria for the two approximations (70) and (71) in the form of the minimal-difference condition (see Introduction), we obtain the condition on the positive stabilizer \( s_1 \), i.e., \( s_1 \) should be determined from the minimum of the expression \( A \):

\[ A = \left\{ \left( \frac{-b_2}{1 + s_1} \right)^{-(1 + s_1)} b_1^{2 + s_1} - \left( \frac{s_1}{-b_1} \right)^{s_1} \right\}. \] (72)

The minimum of (72) does exist and is located at the point \( s_1 \approx 0.5 \). Correspondingly, the index \( \nu \) is equal to 0.625, in a reasonable agreement with all other theoretical and experimental estimates of this index [46-48].

As it was pointed out in Sec.2, the results may also be obtained, if the self-similar renormalization is applied to the initial expansion (67), for the sought function \( \alpha^*(z) \), although it is formally divergent at \( z \to \infty \) and the stabilizer should become negative to describe this divergence correctly. By simple substitution of the coefficients and changing the criteria on minimum of (72) to the maximum of the analogous expression \( K \):

\[ K = \left( \frac{-k_2}{1 + s} \right)^{-(1 + s)} k_1^{2 + s} - \left( \frac{s}{-k_1} \right)^{s} \] (73)

with respect to the now negative stabilizer \( s = 2(1 - 2\nu) \). One can see that the maximum is located at the point \( s = -0.3719 \), leading to the very reasonable estimate for the critical index \( \nu = 0.593 \). Final formulae have the following form:

\[ \alpha_1^2(z)^* = \left( \frac{s}{s - k_1 z} \right)^s, \] (74)

\[ \alpha_2^2(z)^* = 1 + k_1 z [1 - \frac{k_2 z}{k_1 (1 + s)}]^{-1 + s}. \] (75)

Both formulae, (74) and (75), with \( s \approx -0.3719 \), can be used as an approximate "equation of state" for the polymer in the whole range of the parameter \( z \), satisfying, by design, both known virial and scaling asymptotic expressions.

VIII. CRITICAL TEMPERATURE OF THE 2D ISING MODEL FROM THE EXPANSION AROUND THE DIMENSION ONE

In this Section we calculate the critical temperature \( T_c \) of the two-dimensional (2D) Ising model starting from the approximate expression obtained within the framework of the so-called quasi-chemical approximation [42]. This approximation gives \( T_c \) as a function of the coordination number \( z \):

\[ T_c(z) = \frac{-2}{\ln(1 - 2/z)}. \] (76)

The expression (76) correctly describes the limit of the one-dimensional (1D) Ising model, with \( T_c = 0 \), and at the infinite-dimensionality the limit coincides with the well known Bragg-Williams result \( T_c = z \). The expansion around the latter limit has been widely used, although its accuracy is not too good [49]. We adopt the different approach, expanding (76), in powers of the parameter \( z - 2 = \Delta \), around its correct, \( D = 1 \) (\( z = 2 \)), limit, and use the self-similar renormalization to continue the expansion valid at \( \Delta \ll 1 \) to the region of \( \Delta = 2 \), corresponding to the 2D Ising model with the quadratic lattice. The expansion of the inverse expression (76), up to the quadratic term in \( \Delta \), has the following form:
\[ T_{c}^{-1}(\Delta) \approx \frac{\ln 2}{2} \left[ 1 + \frac{1}{\ln 2} \ln \left( \frac{1}{\Delta} \right) \right] + \frac{1}{4} \Delta - \frac{1}{16} \Delta^2. \]  

We renormalize separately the logarithmic

\[ L(\Delta) = \left[ 1 + \frac{1}{\ln 2} \ln \left( \frac{1}{\Delta} \right) \right] \]  

and power-law contributions

\[ P(\Delta) = \frac{1}{4} \Delta - \frac{1}{16} \Delta^2, \]  

separating in this way, the effects of long-range and short-range contributions to \( T_c \).

The standard prescriptions of Sec.III are fully applicable to the expression (78) containing the logarithmic term. Two consecutive approximations to \( L(\Delta) \) are

\[ L_0(\Delta) = 1, \]
\[ L_1(\Delta) = 1 + \frac{1}{\ln 2} \ln \left( \frac{1}{\Delta} \right), \]

with the expansion function \( f = \Delta^s \), and the multiplier

\[ m_1(\Delta, s) = 1 - \frac{1}{\ln 2} \ln \left( \frac{1}{\Delta} \right) + \frac{1}{s}, \]

equal, at \( \Delta = 2 \), to zero as \( s \to \infty \). The velocity function has the following form:

\[ v(s, f) = -\frac{f \ln f}{\ln^2 s}. \]

Calculating the evolution integral and taking the limit of \( s \to \infty \), we obtain

\[ L^*(\Delta) = \Delta^{-1/\ln^2} \]

The expression for \( P(\Delta)^* \), can be readily written down in the case of summation along the stable trajectory, corresponding to \( s \to \infty \):

\[ P^*(\Delta) = \frac{1}{4} \Delta \exp \left( -\frac{1}{4} \Delta \right). \]

Recalculating \( T^*_c \), we obtain \( T^*_c(\Delta = 2) = 2.321 \). The percentage error equals 2.292\%, when compared to the exact Onsager result \( T_c = 2.269 \). It should be remembered, that the quasi-chemical approximation (76) works with the percentage error of 27.149\%, and that the initial expansion (77) deviates from the exact result by 76.289\%. Also, one of the best known approximate theoretical schemes, known as the Kikuchi method [42], gives the percentage error equal to 6.831%.

IX. TEMPORAL ASYMPTOTES OF THE DIFFUSION EQUATION WITH RANDOM STATIONARY NOISE

A. Poisson spectrum

Consider the diffusion of particles in a medium with randomly distributed traps, whose local density \( \alpha(\mathbf{r}) \) is described by the nonnegative Poisson random field [50-52]. The local particle density \( n(\mathbf{r}, t) \) in the presence of traps is described by the following equation:

\[ -\frac{\partial}{\partial t} n(\mathbf{r}, t) = -\nabla^2 n(\mathbf{r}, t) + \alpha(\mathbf{r}) n(\mathbf{r}, t), \]

where we put the diffusion coefficient equal to one.
This problem is formally equivalent to the Shroedinger equation with imaginary time and potential \(\alpha(r)\) [50-52]. The eigenvalues of the Shroedinger operator corresponding to (80), \(E_i\), are nonnegative. In addition, the density of states \(p(E)\), near the finite fluctuational limit of the spectrum, located at \(E = 0\), and formed due to the rare fluctuations of the potential with \(\alpha\) close to zero in large regions of space, is known explicitly:

\[
p(E) = f(E) \exp(-E^{-D/2}),
\]

where \(D\) is the dimensionality of space [53,54]. Hereafter, for the sake of simplicity, we omit the constant terms in the exponential for the density of states. Also, for \(D = 1\), the pre-exponential factor \(f(E) \sim E^{-3/2}\), is known [53] and, because of this, we consider below only the one-dimensional case. The general solution of (80) can be readily written down [52] in terms of the eigenfunctions and eigenvalues of the corresponding quantum-mechanical problem:

\[
n(r, t) = \sum_i c_i \psi_i(r) \exp(-E_i t),
\]

and, considering only contributions from the rare fluctuations of the potential, the mean density \(\langle n \rangle\) over the entire volume for \(t \to \infty\) can be represented in the form of the integral:

\[
\langle n(t) \rangle = n_0 \int_0^\infty p(E) \exp(-Et)dE,
\]

where \(n_0\) describes the initial homogeneous particle distribution. This integral can be evaluated by the method of steepest descent and the leading exponential term \(\sim \exp(-t^{1/3})\) can be written down [49-51]. Similar estimates were obtained for arbitrary \(D\) [50-52].

We will obtain below the higher-order contributions to the asymptotic expansion of (83) near the saddle point \(E = (\frac{d}{dt})^{2/3}\) and, using the self-similar renormalization, obtain the leading corrections, as \(t \to \infty\), to the pre-exponential factor \(f(E) \sim t^{-1}\).

Let us represent \(E\) in the vicinity of \(E\) as \(E = E + \epsilon\) and expand the expression

\[
\Phi(t, \epsilon) \equiv \ln[p(E) \exp(-Et)]
\]

in powers of \(\epsilon\) up to the third order terms, so that

\[
\Phi(t, \epsilon) \approx -at^{1/3} - A(t)\epsilon^2 + B(t)\epsilon^3,
\]

\[
a = 3 \times 2^{-2/3}, \quad A(t) = 3 \times 2^{-4/3}t^{5/3}, \quad B(t) = 5 \times 2^{-5/3}t^{7/3}
\]

and, expand the \(\exp\{B(t)\epsilon^3\}\) in powers of \(\epsilon\), so that

\[
\exp\{\Phi(t, \epsilon)\} \approx \exp\{-at^{1/3}\} \exp\{-A(t)\epsilon^2\} \times [1 + B(t)\epsilon^3 + ...].
\]

Now, \(\langle n(t) \rangle\) can be written down as follows:

\[
\langle n(t) \rangle \sim t^{1/6} \exp\{-at^{1/3}\}[1 + bt^{-1/6}], \quad b = 0.684,
\]

i.e, the corrections to the pre-exponential factor are obtained in the form of an expansion in inverse powers of \(t\), valid as \(t \to \infty\). Our aim is to continue this expression to the region of \(t \sim 1\).

Apply now the self-similar renormalization to the quantity \(\pi(t) = 1 + bt^{-1/6}\), with the two consecutive approximations:

\[
\pi_0 = 1,
\]

\[
\pi_1 = 1 + bt^{-1/6}.
\]

Following the standard prescriptions, the renormalized quantity \(\pi(t)\) can be obtained:

\[
\pi(t) = \{t^{1/6} + \frac{b}{6s(t)}\}^{6s(t)},
\]

(85)
where the stabilizer

$$s(t) = \frac{b}{6t^{1/6} + 6b},$$

is defined as zero of the multiplier

$$m_1(s, t) = 1 + \frac{bt^{-1/6}(s - \frac{1}{6})}{s}.$$

For the intermediate region $1 \ll t < \infty$, the simple expression can be written down:

$$n^*(t) \propto t \frac{b}{6t^{1/6} + 6b},$$

which gives the correction to the pre-exponential factor in the form of continued noninteger powers. It is worth noting, that already the starting terms of the asymptotic expression (85), lead to the approximation-cascade trajectory with zero multiplier.

B. Gaussian spectrum.

Consider equation of the same type that (80), with the only difference that the Poisson potential $\alpha$ is replaced by the random potential $U(r)$ with the properties of the Gaussian "white noise", i.e. $\overline{U(r)} = 0$, $(U(r)U(r')) \propto \delta(r - r')$:

$$-\frac{\partial}{\partial t} n(r, t) = -\nabla^2 n(r, t) + U(r)n(r, t).$$ (86)

Equation of this type, but with a noise dependent both on space and time, can be easily transformed to a nonlinear Burgers equation, Kardar-Parisi-Zhang (KPZ) equation, and also it describes some other closely related physical problems [55]. Stationary random potential is not meaningful within the framework of, say, KPZ where one can think about the stationary, random in space, perturbations of a growing interface. It is also considered in biology as a model for population dynamics in the presence of a random distribution of food [55]. Equation of the type (86) with a random potential $U(r)$ can be also transformed to the corresponding Shroedinger equation with imaginary time. Therefore, one can use the known properties of the spectrum of the corresponding quantum-mechanical problem to study the $t \to \infty$ behavior of the diffusion equation with stationary, randomly distributed sources and sinks. The fluctuational limit of the spectrum is situated now at $E \to -\infty$, with the exponentially small density of states in its vicinity

$$p(E) = f(E) \exp(-|E|^{2-D/2}),$$ (87)

being formed due to the rare fluctuations of the potential with large negative values, separated from each other by the distances much larger than their own sizes [53]. Such fluctuations may be again considered separately. The eigenvalues corresponding to the eigenfunctions localized at these fluctuations are now negative, in distinction from the case considered above and the mean density evolution as $t \to \infty$ can be estimated from the following integral:

$$\langle n(t) \rangle = n_0 \int_0^{\infty} p(E) \exp(|E| t) dE.$$ (88)

We again use the method of steepest descents and follow literally the same steps as in the previous Subsection. The saddle point does exist for $1 \leq D < 4$ (except at $D = 2$, $D = 4$, where the situation becomes trivial) and is given by the expression

$$|E| = \left(\frac{2t}{4-D}\right)^{2/(2-D)}.$$

The leading exponential term has the form

$$\langle n(t) \rangle \sim \exp\{a(D)t^{\frac{4-D}{2}}\}, \quad a(D) = \left(\frac{2-D}{4-D}\right)(2-D)\frac{2}{\sigma^2},$$

with radically different behavior for $D = 1$ and $D = 3$.
corresponding to an anomalously fast growth, compared to the simple \( \exp(t) \), and, anomalously slow decay, compared to \( \exp(-t) \), respectively. We think that this difference takes its origin from the principally different properties of the corresponding Schrödinger operator, where it is known, that at \( D = 1 \) all states of the particle are localized, while at \( D = 3 \), generally speaking, both localized and delocalized states are present [53]. These basic theorems, when applied to the case of diffusion, explain why at \( 1D \), the random distribution of sources and sinks, causes an explosive instability of the density fluctuations, while in \( 3D \), the disorder can cause only longer decay times for the density fluctuations. Of course, the instability can be cured by nonlinear interactions, that should be now taken into account.

At \( D = 1 \), where \( f(E) \sim E \) [54], applying the procedure already discussed above, we obtain the expansion in the vicinity of the saddle point:

\[
\langle n(t) \rangle \sim \exp\{ |a(1)| t^3 \}, \quad D = 1,
\]

\[
\langle n(t) \rangle \sim \exp\{- |a(3)| t^{-1} \}, \quad D = 3,
\]

and the renormalized expression can be readily written down, using the same definition for \( \pi(t) \) as above:

\[
\pi(t) = \left\{ t^{3/2} + \frac{3b}{2\pi(t)^{3/2}} \right\}^{2s(t)/t} \sim t^{3b-3/2}, \quad s(t) = \frac{3b}{2(b + t^{3/2})}.
\]

At \( D = 3 \), a little bit different situation occurs, since

\[
\Phi(t, \epsilon) \approx - |a(3)| t^{-1} + A(t) \epsilon^2 - B(t) \epsilon^3,
\]

\[
A(t) = t^3, \quad B(t) = 2t^5,
\]

and, in order to guarantee the convergence of the integrals, it is reasonable to expand \( \exp(\Phi(t, \epsilon)) \) as follows:

\[
\exp\{\Phi(t, \epsilon)\} \approx \exp\{- |a(3)| t^{-1}\} \exp\{-B(t) \epsilon^3\} \times [1 + A(t) \epsilon^2 + ...],
\]

and for the \( \langle n(t) \rangle \) we obtain

\[
\langle n(t) \rangle \sim \exp\{- |a(3)| t^{-1}\} t^{-5/3}[1 + ct^{-4/3} + ...], \quad c \approx 0.296.
\]

The renormalized expression for the pre-exponential factor has the following form:

\[
\pi(t) = \left\{ t^{4/3} + \frac{4b}{3s(t)} \right\}^{4s(t)/3} \sim t^{4c+4/3}, \quad s(t) = \frac{4c}{3(c + t^{4/3})}.
\]

We have demonstrated in this Section, that the self-similar renormalization can be applied to the dynamical problems as well, generating the expressions for the pre-exponential factors in the form of continued noninteger powers.

X. CONCLUSION

We suggested here a new variant of the self-similar approximation theory permitting an easy and accurate summation of divergent series. The method is based on a power-law algebraic transformation leading to an effective increase of the order of perturbative terms. The powers of this transformation play the role of control functions governing the convergence of renormalized series. These control functions are defined by the principle of maximal stability, i.e., from the minimization of mapping multipliers. Such stabilizing control functions may be called stabilizers.

Another important novelty of the method is the multiple self-similar renormalization converting all series into closed self-similar expressions. This multiple and complete renormalization is called self-similar bootstrap. The resulting effective sum of a divergent series can be presented through analytical expressions containing exponentials and fractions, rational or irrational. In particular cases, this can be only exponentials or only fractions, depending on the behavior of control functions which dictate the resulting form. Because of much larger variety of such resulting forms, the method allows to present the answers in relatively simple analytical expressions having at the same time
quite high accuracy. The use of several types of functions, such as exponentials and various fractions, distinguishes this method from, e.g., the Padé approximants involving solely rational functions.

In order to prove that the suggested method really gives quite simple and accurate expressions for the effective sums of divergent series, we, first of all, considered several toy models cartooning the generating functionals in field theory or partition functions in statistical physics. We illustrated by these examples that the method works well in different situations, for single-well and for double-well models, for weak and strong coupling.

To stress the generality of the method, we applied it to several problems of statistical physics of quite different nature: to constructing the equation of state, to calculating the critical temperature, and to finding the time asymptotics for stochastic dynamical processes. We hope that these various and very different applications demonstrate well the validity of the method.

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