On Clustering Incomplete Data

Eduard Eiben\textsuperscript{1}, Robert Ganian\textsuperscript{2}, Iyad Kanj\textsuperscript{3}, Sebastian Ordyniak\textsuperscript{4}, and Stefan Szeider\textsuperscript{2}

\textsuperscript{1}Royal Holloway, University of London, United Kingdom
\textsuperscript{2}TU Wien, Vienna, Austria
\textsuperscript{3}DePaul University, USA
\textsuperscript{4}University of Sheffield, United Kingdom

Abstract

We study fundamental clustering problems for incomplete data. In this setting, we are given a set of incomplete $d$-dimensional Boolean vectors (representing the rows of a matrix), and the goal is to complete the missing vector entries so that the set of complete vectors admits a partitioning into at most $k$ clusters with radius or diameter at most $r$. We develop a toolkit and use it to give tight characterizations of the parameterized complexity of these problems with respect to the parameters $k$, $r$, and the minimum number of rows and columns needed to cover all the missing entries. We show that the aforementioned problems are fixed-parameter tractable when parameterized by the three parameters combined, and that dropping any of these three parameters results in parameterized intractability. We extend this toolkit to settle the parameterized complexity of other clustering problems, answering an open question along the way. We also show how our results can be extended to data over any constant-size domain. A byproduct of our results is that, for the complete data setting, all problems under consideration are fixed-parameter tractable parameterized by $k + r$.

1 Introduction

Problem Definition and Motivation. We study fundamental clustering problems for incomplete data. In this setting, we are given a set of $d$-dimensional Boolean vectors (regarded as rows of a matrix), and some of whose entries might be missing. The objective is to complete the missing entries in order to enable a “clustering” of the $d$-dimensional vectors such that all elements in the same cluster are “similar”.

There is a wealth of research on data completion problems\cite{8, 9, 10, 20, 30, 39} due to their ubiquitous applications in recommender systems, machine learning, sensing, computer vision, data science, and predictive analytics, among others. In these areas, data completion problems naturally arise after observing a sample from the set of vectors, and attempting to recover the missing entries with the goal of optimizing certain criteria. Some of these criteria include minimizing the number of clusters into which the completed vectors can be partitioned, or forming a large cluster, where the definition of what constitutes a cluster varies from one application to another\cite{3, 19, 20, 52}.

Needless to say, the clustering problem itself (i.e., for complete data) is a fundamental problem whose applications span several areas of computing, including data mining, machine learning, pattern recognition, and recommender systems; there are several recent books\cite{2, 29, 44, 48} that provide an introduction to clustering and its applications. Clustering is the focus of extensive research in the Machine Learning and Neural Information Processing communities, with numerous papers studying
application-focused as well as purely theoretical aspects of clustering. The clustering problem is formulated by representing each element in the given set as a $d$-dimensional vector each of whose coordinates corresponds to a feature/characteristic, and the value of the vector at that coordinate reflects the score of the element with respect to that characteristic.

In many cases, the goal of clustering is to optimize the number of clusters and/or the degree of similarity within a cluster (intra-cluster similarity). To measure the intra-cluster similarity, apart from using an aggregate measure (e.g., the variance in $k$-means clustering), two measures that have been studied use the radius (maximum distance to a vector) and diameter (maximum distance between any two cluster-vectors) of the cluster. The radius is computed either with respect to a vector in the cluster itself or an arbitrary $d$-dimensional vector. Regardless of which of the above measures of intra-cluster similarity is used, the vast majority of the clustering problems that arise are NP-hard. Consequently, heuristics are often used to cope with the hardness of clustering problems, trading in a suboptimal clustering for polynomial running time. In this paper we take a different approach: we maintain the optimality of the obtained clustering by relaxing the notion of tractability from polynomial-time to fixed-parameter tractability (FPT), where the running time is polynomial in the instance size but may involve a super-polynomial factor that depends only on some problem parameter, which is assumed to be small for certain instances of interest. In the context of clustering, two natural parameters that are desirable to be small are the upper bounds on the number of clusters and on the radius/diameter. Such clusterings are suitable for many applications, as one would like the similarity level within each cluster to be high and the number of clusters not to be very large.

Contributions. Motivated by the above, we consider several fundamental clustering problems in the incomplete data setting. The first set of problems we consider consists of three problems, referred to as In-Clustering-Completion, Any-Clustering-Completion, and Diam-Clustering-Completion, that share a similar setting: In all three problems, we are given a (multi)set $M$ of $d$-dimensional vectors over the Boolean domain, some of whose entries might be missing, and two parameters $r, k \in \mathbb{N}$. For In-Clustering-Completion, the goal is to complete the missing entries so as to enable a partitioning of the set $M$ into at most $k$ clusters such that all vectors in the same cluster are within distance at most $r$ from some “center” vector that belongs to the cluster itself. The goal for Any-Clustering-Completion is the same as that for In-Clustering-Completion, except that the center vectors need not be in the set $M$ (i.e., are chosen from $\{0, 1\}^d$). For Diam-Clustering-Completion, the goal is to complete the missing entries so as to enable a partitioning of $M$ into at most $k$ clusters such that the diameter of each cluster is at most $r$. The formal problem definitions are given in Section 2.

Our first order of business is to obtain a detailed map of the parameterized complexity of the above three problems. As we show in this paper, parameterization by $k + r$ is not sufficient to achieve tractability for any of these three problems: one needs to restrict the occurrences of the unknown entries in some way as well. We do so by adopting a third parameter which captures the total number of vectors and coordinates (or, equivalently, rows and columns in a matrix representation of $M$) needed to cover all the missing entries. This parameter, which we call the covering number or simply cover, is guaranteed to be small when the unknown entries arise from the addition of a small number of new rows and columns (e.g., new users and attributes) into a known data-set. The covering number has previously been used in the context of matrix completion and is in fact the least restrictive parameter considered in that paper.

As our first technical contribution, we develop new combinatorial and algorithmic tools to settle
the parameterized complexity of In-Clustering-Completion, Any-Clustering-Completion, and Diam-Clustering-Completion w.r.t. all the considered parameterizations: We show that the three problems are fixed-parameter tractable parameterized by $k + r + \text{cover}$, and that neither of these parameters can be dropped without losing tractability.

With the initial toolkit in hand, we turn our attention to other problems in the area of clustering and data analytics. We consider the Dispersion-Completion problem, where the goal is to study the “diversity” of a given set $M$ of incomplete vectors, a notion that can be viewed as the opposite of minimizing the number of clusters in a cluster partitioning of $M$ (in the context of analysis of known data, this problem is often studied under the name \textit{diversity} \cite{11,50,32}). In Dispersion-Completion, the input is the same as that of the other problems under consideration, and the goal is to complete the missing entries of the vectors in $M$ so as to form a subset $P$ of $k$ vectors such that the distance between each pair is at least $r$.

As we show for Dispersion-Completion, parameterizing by the three parameters combined is not necessary to obtain FPT: Dispersion-Completion is FPT parameterized by $k + r$; however, losing control over the occurrence of missing data provided by \text{cover} requires a significant extension of our initial toolkit to handle this case. We complement this positive result with intractability results for the cases when any of these two parameters is dropped, and in fact obtain a full complexity picture for Dispersion-Completion w.r.t. the three considered parameters.

We then turn our attention to a fundamental clustering problem, referred to as Large Diam-Cluster-Completion, where the goal is to compute one “large-cluster” of bounded diameter. More specifically, the input for Large Diam-Cluster-Completion is the same as that for the other problems, but the goal is to complete the missing entries so as to form a subset $P$ of $k$ vectors of $M$ whose diameter is at most $r$. We establish the fixed-parameter tractability of the problem w.r.t. $r + \text{cover}$ alone. This result represents the pinnacle of our technical contribution and requires not only an extension of the toolkit used for the initial three clustering problems, but also a new technique which we dub “iterative sunflower harvesting”. We complement this result with other algorithms and lower bounds that paint a complete complexity picture for the problem.

We show an application of our toolkit and techniques to Large Diam-Cluster-Completion, where the goal is to complete the missing entries so as to form a subset $P$ of $k$ vectors of $M$ whose distance from a center vector in $\{0,1\}^d$ is at most $r$. We show that Large Any-Cluster-Completion is FPT parameterized by $k + r + \text{cover}$, answering open questions in the literature \cite{5,10} about whether the problem is fixed-parameter tractable in the easier setting where \text{cover} = 0 (i.e., when all data is known). The questions of whether the problem is FPT parameterized by $k + \text{cover}$ or by $r + \text{cover}$ remain open, even though we show membership in XP w.r.t. both parameterizations.

We also observe in this paper that, for the complete data setting, most of the clustering problems under consideration (except for Any-Clustering and Large Any-Cluster) reduce to well-known graph problems (\textit{e.g.}, Dominating Set, Clique, Independent Set) on the class of induced subgraphs of powers of the hypercube. Since our results imply that all these graph problems are FPT restricted to this graph class and the graph problems are expressible in First Order Logic (FO), a natural question to ask is whether these FPT results can be generalized to any graph problem expressible in FO logic. We answer this question negatively.

Finally, we mention that a byproduct of our results is that, for the complete data setting, all problems under consideration are fixed-parameter tractable parameterized by $k + r$. We refer to Table \ref{tab:results} for a summary of our results.

\textbf{Related Work.} The presented results are some of the first to shed light on the fine-grained complexity of the fundamental data completion problems considered in this paper. The parameterized complexity of a related problem—Matrix Completion—has recently been studied in a different
Table 1: Overview of results obtained in this paper. FPT denotes fixed-parameter tractability, paraNP-c(completeness) means that the problem remains NP-complete even after the parameters are set to a constant, W[1]-h(hardness) means that the problem is unlikely to be fixed-parameter tractable under the given parameterization, and XP means that the problem can be solved in polynomial time when the parameters are set to a constant. Definitions are provided in Section 2.

context than that of clustering [30]; the problem considered therein corresponds to the special case of In-Clustering-Completion in which the clustering radius r is 0.

On the other hand, there is an extensive bulk of research on clustering problems for complete data. Frances and Litman [27] studied Any-Clustering (the complete-data version of Any-Clustering-Completion) and showed it to be NP-hard w.r.t. the Hamming distance when k = 1; this case was referred to as the Covering Radius problem. This special case (k = 1) was also studied under the name Closest String, by Li et al. [45], who showed that the problem admits a polynomial time approximation scheme if the goal is to minimize r. Gramm et al. [37] studied Closest String from the parameterized complexity perspective and showed that it is fixed-parameter tractable parameterized by r; Theorem 13 (applied with cover = 0) can be seen as a generalization of their result to the case where k is not fixed to be 1. Gąsieniec et al. [33, 34] studied (the optimization versions of) Any-Clustering and Diam-Clustering and obtained polynomial-time algorithms as well as lower bounds for a number of cases. They also obtained 2-approximation algorithms for these problems by extending an earlier algorithm by Gonzalez [35]. Cabello et al. [7] studied the parameterized complexity of a geometric clustering problem, the k-Center Problem, which is an analogue of Any-Clustering in R^d. Gąsieniec et al. showed the parameterized intractability of this problem w.r.t. the dimension d as the parameter, even when k = 4. Finally, we mention that Boucher and Ma [5], and more recently Bulteau and Schmid [6], studied the parameterized complexity of Large Any-Cluster (albeit under a different name) with respect to several parameters, including some of the parameters under consideration in our paper.

Related results are also presented in the very recent works of Fomin, Golovach and their co-authors, which investigated the complexity of different matrix editing and clustering problems from the parameterized and approximation perspectives. These include algorithms for the problem of carefully modifying (via a small number of changes) a given matrix to one of bounded rank or a bounded number of distinct rows [25], approximation schemes for clustering the rows of a matrix subject to additional constraints [24], and a parameterized complexity study of a natural generalization of k-Means [26]. All of these results are concerned with the complete data setting.

Paper Organization. The paper is structured as follows. After introducing the relevant preliminaries in Section 2, we provide an overview of our technical contribution in Section 3. We note that this section is optional: readers interested in the full details of the results can safely skip it and proceed directly to Section 4 which introduces the tools that lie at the core of our approach. Section 5 then employs these tools to obtain FPT-algorithms for In/Any/Diam-Clustering-Completion. In Section 6 we extend these techniques to obtain algorithms for Dispersion-Completion and Large Diam-Cluster-Completion, and as an application show how the toolbox can be used for Large Any-Cluster-Completion. Section 7 is where we present all the lower bounds required to obtain the complexity classification presented in Table 1. The final two sections deal with general implications of our results: Section 8 discusses implications for graph problems, while Section 9
shows how our results generalize to matrices (vectors) over any bounded domain. We conclude the paper in Section 10 with some remarks and open questions.

2 Preliminaries

Let \( \vec{a} \) and \( \vec{b} \) be two vectors in \( \{0,1,\Box\}^d \), where \( \Box \) is used to represent coordinates whose value is unknown (i.e., missing entries). We denote by \( \Delta(\vec{a},\vec{b}) \) the set of coordinates in which \( \vec{a} \) and \( \vec{b} \) are guaranteed to differ, i.e., \( \Delta(\vec{a},\vec{b}) = \{ i \mid (\vec{a}[i] = 1 \land \vec{b}[i] = 0) \lor (\vec{a}[i] = 0 \land \vec{b}[i] = 1) \} \), and we denote by \( \delta(\vec{a},\vec{b}) \) the Hamming distance between \( \vec{a} \) and \( \vec{b} \) measured only between known entries, i.e., \( |\Delta(\vec{a},\vec{b})| \).

Moreover, for a subset \( D' \subseteq [d] \) of coordinates, we denote by \( \vec{a}[D'] \) the vector \( \vec{a} \) restricted to the coordinates in \( D' \).

There is a one-to-one correspondence between vectors in \( \{0,1\}^d \) and subsets of coordinates, i.e., for every vector, we can associate the unique subset of coordinates containing all its one-coordinates and vice-versa. We introduce the following notation for vectors to switch between their set-representation and vector-representation. We denote by \( \Delta(\vec{a},\vec{b}) \) the set of coordinates in which \( \vec{a} \) and \( \vec{b} \) measured only between known entries, and subsets of coordinates, respectively. For a set \( N \) of vectors in \( \{0,1\}^d \) and a family \( C \) of coordinate sets, we denote by \( \Delta(N) \) the set \( \{ \Delta(\vec{v}) \mid \vec{v} \in N \} \) and by \( \Delta^{-1}(C) \) the set \( \{ \Delta^{-1}(\vec{v}) \mid \vec{v} \in C \} \).

We say that a vector \( \vec{a} \in \{0,1\}^d \) is a \( t \)-vector if \( |\Delta(\vec{a})| = t \) and we say that \( \vec{a} \) contains a subset \( S \) of coordinates if \( S \subseteq \Delta(\vec{a}) \). For a subset \( S \subseteq \{0,1\}^d \) and a vector \( \vec{a} \in \{0,1\}^d \), we denote by \( \delta(S,\vec{a}) \) the minimum Hamming distance between \( \vec{a} \) and the vectors in \( S \), i.e., \( \delta(S,\vec{a}) = \min_{\vec{s}\in S} \delta(\vec{s},\vec{a}) \). We denote by \( \gamma(S) \) the diameter of \( S \), i.e., \( \gamma(S) := \max_{\vec{s}_1,\vec{s}_2\in S} \delta(\vec{s}_1,\vec{s}_2) \).

Let \( M \subseteq \{0,1\}^d \) and let \( [d] = \{1,\ldots,d\} \). For a vector \( \vec{a} \in M \), we denote by \( N_r(\vec{a}) \) the \( r \)-Hamming neighborhood of \( \vec{a} \), i.e., the set \( \{ \vec{b} \in M \mid \delta(\vec{a},\vec{b}) \leq r \} \) and by \( N_r(M) \) the set \( \bigcup_{\vec{a}\in M} N_r(\vec{a}) \). Similarly, we denote by \( N_{=r}(\vec{a}) \) the set \( \{ \vec{b} \in M \mid \delta(\vec{a},\vec{b}) = r \} \) and by \( N_{=r}(M) \) the set \( \bigcup_{\vec{a}\in M} N_{=r}(\vec{a}) \). We say that \( M^* \subseteq \{0,1\}^d \) is a completion of \( M \subseteq \{0,1,\Box\}^d \) if there is a bijection \( \alpha : M \rightarrow M^* \) such that for all \( \vec{a} \in M \) and all \( \vec{i} \subseteq [d] \) it holds that either \( \vec{a}[\vec{i}] = \Box \) or \( \alpha(\vec{a})[\vec{i}] = \vec{a}[\vec{i}] \).

We now proceed to give the formal definitions of the problems under consideration.

\begin{tabular}{|l|}
\hline
\textbf{IN-CLUSTERING-COMPLETION} \\
\hline
\textbf{Input:} & A subset \( M \) of \( \{0,1,\Box\}^d \) and \( k, r \in \mathbb{N} \). \\
\textbf{Question:} & Is there a completion \( M^* \) of \( M \) and a subset \( S \subseteq M^* \) with \( |S| \leq k \) such that \\
& \( \delta(S,\vec{a}) \leq r \) for every \( \vec{a} \in M^* \)? \\
\hline
\end{tabular}

\begin{tabular}{|l|}
\hline
\textbf{ANY-CLUSTERING-COMPLETION} \\
\hline
\textbf{Input:} & A subset \( M \) of \( \{0,1,\Box\}^d \) and \( k, r \in \mathbb{N} \). \\
\textbf{Question:} & Is there a completion \( M^* \) of \( M \) and a subset \( S \subseteq \{0,1\}^d \) with \( |S| \leq k \) such that \\
& \( \delta(S,\vec{a}) \leq r \) for every \( \vec{a} \in M^* \)? \\
\hline
\end{tabular}

\begin{tabular}{|l|}
\hline
\textbf{DIAM-CLUSTERING-COMPLETION} \\
\hline
\textbf{Input:} & A subset \( M \) of \( \{0,1,\Box\}^d \) and \( k, r \in \mathbb{N} \). \\
\textbf{Question:} & Is there a completion \( M^* \) of \( M \) and a partition \( \mathcal{P} \) of \( M^* \) with \( |\mathcal{P}| \leq k \) such that \\
& \( \gamma(\mathcal{P}) \leq r \) for every \( \mathcal{P} \in \mathcal{P} \)? \\
\hline
\end{tabular}

Observe that in a matrix representation of the above problems, we can represent the input matrix as a set of vectors where each row of the matrix corresponds to one element in our set. Of course, to precisely capture the input it seems more appropriate to consider multisets of vectors—however
tractable with an instance $I$ of FPT problems. The class $I,\kappa$ is a pair $(\text{parameters that are small in practical settings.})$ We now proceed to the formal definitions.

Parameterized Complexity. In parameterized complexity \[23, 16, 14\], the complexity of a problem is studied not only with respect to the input size, but also with respect to some problem parameter(s). The core idea behind parameterized complexity is that the combinatorial explosion resulting from the NP-hardness of a problem can sometimes be confined to certain structural parameters that are small in practical settings. We now proceed to the formal definitions.

A parameterized problem $Q$ is a subset of $\Omega^* \times \mathbb{N}$, where $\Omega$ is a fixed alphabet. Each instance of $Q$ is a pair $(I, \kappa)$, where $\kappa \in \mathbb{N}$ is called the parameter. A parameterized problem $Q$ is fixed-parameter tractable (FPT) \[23, 16, 14\], if there is an algorithm, called an FPT-algorithm, that decides whether an input $(I, \kappa)$ is a member of $Q$ in time $f(\kappa) \cdot |I|^{O(1)}$, where $f$ is a computable function and $|I|$ is the input instance size. The class FPT denotes the class of all fixed-parameter tractable parameterized problems.

A parameterized problem $Q$ is FPT-reducible to a parameterized problem $Q'$ if there is an algorithm, called an FPT-reduction, that transforms each instance $(I, \kappa)$ of $Q$ into an instance $(I', \kappa')$ of $Q'$ in time $f(\kappa) \cdot |I|^{O(1)}$, such that $\kappa' \leq g(\kappa)$ and $(I, \kappa) \in Q$ if and only if $(I', \kappa') \in Q'$, where $f$ is...
and $g$ are computable functions. By FPT-time, we denote time of the form $f(\kappa) \cdot |I|^{O(1)}$, where $f$ is a computable function. Based on the notion of FPT-reducibility, a hierarchy of parameterized complexity, the $W$-hierarchy $= \bigcup_{t \geq 0} W[t]$, where $W[t] \subseteq W[t+1]$ for all $t \geq 0$, has been introduced, in which the 0-th level $W[0]$ is the class FPT. The notions of hardness and completeness have been defined for each level $W[i]$ of the $W$-hierarchy for $i \geq 1$ \cite{16, 14}. It is commonly believed that $W[1] \not= \text{FPT}$ (see \cite{16, 14}). The $W[1]$-hardness has served as the main working hypothesis of fixed-parameter intractability. The class XP contains parameterized problems that can be solved in time $O(|I|/f(\kappa))$, where $f$ is a computable function; it contains the class $W[t]$, for $t \geq 0$, and every problem in XP is polynomial-time solvable when the parameters are bounded by a constant.

The class paraNP is the class of parameterized problems that can be solved by non-deterministic algorithms in time $f(\kappa) \cdot |I|^{O(1)}$, where $f$ is a computable function. A problem is paraNP-hard if it is NP-hard for a constant value of the parameter \cite{23}.

A parameterized problem is kernelizable if there exists a polynomial-time reduction that maps an instance $(I, \kappa)$ of the problem to another instance $(I', \kappa')$ such that (1) $|I'| \leq f(\kappa)$ and $\kappa' \leq f(\kappa)$, where $f$ is a computable function, and (2) $(I, \kappa)$ is a YES-instance of the problem if and only if $(I', \kappa')$ is. The instance $(I', \kappa')$ is called the kernel of $I$. It is well known that a decidable problem is FPT if and only if it is kernelizable \cite{16}. A polynomial kernel is a kernel whose size can be bounded by a polynomial in the parameter.

A Turing kernelization for a parameterized problem $P$ is an algorithm that, provided with access to an oracle for $P$, decides in polynomial time whether or not an input $(I, \kappa)$ is a YES-instance of $P$. During its computation, the algorithm can produce polynomially-many oracle queries on input of size $f(\kappa)$, for some computable function $f$. The function $f$ is referred to as the kernel size.

Structure of Missing Entries. As we will later show in Theorem \cite{14} and \cite{15} it is not possible to obtain fixed-parameter tractability for clustering or finding a large cluster when the occurrence of missing entries (i.e., $\square$'s) is not restricted in any way. On the other hand, when we restrict the total number of missing entries to be upper bounded by a constant or a parameter, our problems trivially reduce to the complete-data setting, since one can enumerate all values in these few missing entries by brute force. Hence, the interesting question is whether we can solve the problems when the number of $\square$'s is large but also restricted in a natural way. We do so by using the so-called covering number as a parameter, a setting which naturally captures instances where incomplete data is caused by the addition of a few new vectors (i.e., rows) and/or coordinates (i.e., columns) \cite{30}.

Formally, let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an arbitrary but fixed ordering of a subset $M$ of $\{0, 1, \square\}^d$. If $\vec{v}_i[j] = \square$, we say that $\square$ at $\vec{v}_i[j]$ is covered by row $i$ and column $j$. The covering number of $M$, denoted as $\text{cover}(M)$ or simply $\text{cover}$ where it is clear from the context, is the minimum value of $r + c$ such that there exist $r$ rows and $c$ columns in $M$ with the property that each occurrence of $\square$ is covered by one of these rows or columns. We will generally assume that for a set $M \in \{0, 1, \square\}^d$ we have computed sets $T_M$ and $R_M$ such that $\text{cover}(M) = |T_M| + |R_M|$ and each $\square$ occurring in a vector $\vec{v} \in M$ is covered by a row in $R_M$ or a column in $T_M$; we note that this computation may be done in polynomial time \cite{30}, Proposition 1], and in our algorithms parameterized by $\text{cover}(M)$ we will generally assume that $T_M$ and $R_M$ have already been pre-computed.

3 Overview of Technical Contributions

3.1 Clustering

We begin our overview by discussing the algorithmic framework we developed to deal with the In-, Any- and Diam- variants of clustering completion. It should be noted that, while the definitions of these problems differ only in the way we define clusters, one should not expect their complexity
behavior to be similar—in fact, in the case where $M$ contains no missing entries, the problems do differ in their parameterized complexity ($\text{In-CLUSTERING}$ is $\text{XP}$ parameterized by $k$ alone, while the other two problems are $\text{paraNP}$-hard) and, as we discuss in Section 8, the In- and Diam- variants are related to different graph problems. Hence it is somewhat surprising that we can provide a generic kernelization framework that can not only deal with all three clustering problems, but also provides insights that will be crucial for solving the other problems under consideration.

**Finding irrelevant vectors.** The first pillar of our algorithms for $\text{In/Any/Diam-CLUSTERING-COMPLETION}$ is the irrelevant vector technique, which is based on identifying and removing “redundant” vectors—vectors that can be omitted from the instance and later safely added back to any valid clustering of the reduced instance to yield a valid clustering of the original instance. One caveat is that, for $\text{In-CLUSTERING-COMPLETION}$, the removed vectors may serve as cluster centers, and hence, such vectors will have to be represented in the reduced instance; we will discuss how to deal with this issue later.

To identify redundant vectors, we show that, for each vector $\vec{v}$, we can compute a representative set (of size bounded by a function of the parameters) for all vectors at Hamming distance at most $r$ from $\vec{v}$. The properties of these representative sets ensure that all vectors outside the set are redundant. The core idea used to compute these representative sets is that if we view the $t$-neighborhood of $\vec{v}$ (for some $t \leq r$) as a set system $\mathcal{F} := \{ \Delta(\vec{v}, \vec{x}) \mid x \in N \}$, then a surprising application of the Sunflower Lemma [21] allows us to find a set $S$ of vectors (of size bounded in the parameters) such that removing any vector from $S$ preserves the maximum distance between $S$ and every possible vector of bounded distance from $\vec{v}$ (and thus, for instance, centers of “in-clusters” and “any-clusters”).

However, some obstacles need to be cleared out before the above idea can be applied to instances with incomplete data. First of all, the unrestricted occurrence of the incomplete data in the vectors in $R_M$ prevents us from handling these vectors in this way—but since there are only $\text{cover}(M)$ of them, we can omit these vectors from our consideration for now and deal with them at a later stage.

More critical is the presence of $\square$’s in the columns in $T_M$. Intuitively, what allows us to reduce the instance by removing a vector (say $\vec{f}$) from $S$ is the fact that every solution of the reduced instance maintains a large part of the sunflower together in one of the clusters—this in turn means that for every vector in the cluster there is a vector in the sunflower whose distance is at least the distance to the redundant vector $\vec{f}$. This is crucial for arguing that $\vec{f}$ can always be safely added back into that cluster. However, once we allow $\square$-entries, this can no longer be guaranteed, since whether or not we are able to add $\vec{f}$ back into the cluster depends also on how the $\square$-entries of the other vectors in the sunflower have been completed. To deal with this problem, we start by finding a significantly larger sunflower $S'$ and argue that it must contain a sub-sunflower $S$ that contains only vectors whose $\square$’s occur at precisely the same coordinates.

**Finding irrelevant coordinates.** At this point, we have all the ingredients needed to reduce the size of $M$ to a number that is upper bounded by a function of $k + r + \text{cover}(M)$. For the next step of the algorithm, we apply an irrelevant coordinate technique that will result in an equivalent instance where the number of coordinates is also bounded by a function of the parameters.

The intuition at the heart of identifying irrelevant coordinates is that, if $R_M = \emptyset$, every connected component $D$ of the compatibility graph of our instance must have bounded diameter (and, in particular, Hamming-diameter). In combination with the bounded size of this component, this allows us to obtain a set $Z$ of coordinates (whose size is bounded by a function of the parameter) such that every vector in $D$ has the same projection on all coordinates outside of $Z$. Since the number of such connected components is bounded by $k$, in the case where $R_M = \emptyset$ this would be sufficient to bound the number of coordinates by a function of the parameters.
Unfortunately, this intuition does not translate immediately to the case where \( R_M \) is non-empty: indeed, the vectors \( \{0\}^d, \{\square\}^d \) and \( \{1\}^d \) all belong to the same connected component in the compatibility graph despite having a Hamming-diameter of \( d \). A more careful analysis is needed in this case: in particular, by elaborating on the previous idea, we can find a bounded-size set \( Z' \) of coordinates such that projecting \( M \) onto \( Z' \) preserves (1) all distances between vectors inside individual components in \( M \setminus R_M \), and (2) the exact distances between \( M \setminus R_M \) and \( R_M \) up to \( r \) (or \( 2r \), for the case of \textsc{Diam-Clustering-Completion}). These properties are sufficient to ensure that coordinates outside of \( Z' \) can be safely omitted from the instance.

**Problem-specific complications.** At this point we have all that is needed to obtain a kernel for \textsc{Any-Clustering-Completion}—in particular, by finding and removing redundant vectors and coordinates we end up with an equivalent instance \((M', k, q)\) where \( M' \) has size bounded by the parameters. However, the situation is more complicated for the other two problems.

Recall that for \textsc{In-Clustering-Completion} we previously removed “redundant” vectors from \( M \) that are not needed to identify suitable centers, but these might still be required to serve as cluster centers. We now need to add a bounded number of the previously removed vectors back into the instance while ensuring that the existence of a cluster center is preserved. The main idea here can be nicely illustrated using the case of complete data: here, one can show that every vector in \( M \) that can act as a potential center for the instance on \( M' \) must be within the \( r \)-neighborhood of some vector in \( M' \), and moreover, among all (potentially many vectors within the \( r \)-neighborhood of a vector in \( M' \)), we can choose any vector which is closest w.r.t. the redundant coordinates. This way the number of potential vectors that can act as a center for a vector in \( M' \) can be bounded by the parameter. Due to the presence of incomplete data, we need to adapt this idea to deal with an additional complication: the \( \square \) entries of the vectors in \( M' \) (which can be changed without increasing the Hamming distance to the vector), can significantly increase the size of the \( r \)-Hamming neighborhood of every such vector. For instance, the potential \( r \)-Hamming neighborhood of a vector in \( M' \setminus R_M \) increases by a factor of \( 2^{|T_M|} \).

For \textsc{Diam-Clustering-Completion}, the difficulty we face now is different. In particular, recall that since we forced our sunflowers to represent vectors with \( \square \)'s in the same positions, for \textsc{In-Clustering-Completion} and \textsc{Any-Clustering-Completion} we could assume that all elements in that sunflower which end up together in a cluster will be completed the same way as the hypothetical center vector of that cluster. Not only is this assumption invalid for \textsc{Diam-Clustering-Completion}, but here in fact vectors in the same sunflower might actually be required to have different completions. Dealing with this issue requires two changes to our approach:

- we increase the size of our initial sunflowers by an additional factor of \( 2^{|T_M|} \), and
- when identifying redundant rows, we also take into account the potential distance between different vectors in the same cluster arising from the possibility of different completions of the coordinates in \( T_M \).

### 3.2 Dispersion

Our aim for \textsc{Dispersion-Completion} is to establish fixed-parameter tractability parameterized not only by \( k + r + \text{cover}(M) \), but even by \( k + r \) (i.e., regardless of the location of missing entries). As our first step, we show that all rows in an arbitrary instance \((M, k, r)\) can be, w.l.o.g., assumed to contain at most \( O(k \cdot r) \) many \( \square \)'s. It is worth noting that the arising implicit restriction to instances where the number of \( \square \)'s per row is bounded is a significantly weaker one than the one given by explicitly bounding \( \text{cover}(M) \); in particular, using this as a restriction instead of \( \text{cover}(M) \) cannot lead to fixed-parameter tractability for any of the clustering problems considered earlier.

Next, we observe that if \( M \) is sufficiently large and the \( r \)-Hamming neighborhood of each vector
is upper-bounded by a function of \( k + r \), then—since the number of \( \square \)'s is bounded—\((M,k,r)\) is a YES-instance. The argument here is analogous to the classical argument showing that INDEPENDENT SET is trivial on large bounded-degree graphs.

On a high level, we would now like to apply the irrelevant vector technique to find and remove a vector from \( M \)—since here the number of \( \square \)'s on every row is bounded, any instance reduced in this way to only contain a bounded number of vectors can be solved via a brute-force fixed-parameter procedure. However, finding an irrelevant vector is now much more difficult, primarily because the occurrence of \( \square \)'s is not restricted only to coordinates in \( T_M \); among others, this makes it impossible to use the set representation \( F = \{ \Delta(\vec{v},\vec{x}) \mid x \in \mathbb{N} \} \) for the neighborhood of \( \vec{v} \in M \) used for the previous clustering algorithms. Instead, we develop a more powerful set representation \( F' \) for vectors in the instance which also uses elements to keep track of the presence of \( \square \)'s in the neighbors of \( \vec{v} \). We can then apply the Sunflower Lemma to find a sufficiently-large sunflower in \( F' \), and in the core of the proof we argue that (1) such a sunflower consists of at most a bounded number of “important petals” (which can be identified in polynomial time), and (2) any petal that is not important represents an irrelevant vector.

### 3.3 Large Diam-Cluster-Completion

We now proceed to the pinnacle of our technical contributions: an FPT-algorithm for LARGE DIAM-CLUSTER-COMPLETION parameterized by \( r + \text{cover}(M) \). Here the unbounded size of the sought-after solution prevents us from directly applying the irrelevant vector and coordinate techniques; instead, the idea of analyzing the structure of vectors through sunflowers in their set representations will be key. This will be done via a technique which we dub: iterative sunflower harvesting.

The starting point is a brute-force branching procedure that allows us to construct a set representation \( S \) of a subset \( M' \subseteq M \) which guarantees that there is an optimal solution in \( M' \). We then prove that every hypothetical solution can be decomposed into a bounded number of sunflower cores and isolated vectors in \( S \), and moreover that we can assume w.l.o.g. that at least one such sunflower core contains a very large number of petals.

Our next task will be to find this large core via branching. The key insight here is that, while the number of “potential cores” is upper bounded by a function that depends only linearly on \( M \), we will later want to iterate this branching procedure and the number of iterations will depend on the parameters, and hence we cannot afford to branch exhaustively over all such potential cores. Instead, we use the fact that the sought-after core is large to make a pre-selection of potential cores to consider in our branching, and in this way reduce the branching factor to a function of the parameters.

Now that we have (hopefully) found the correct core, we proceed to the harvesting part. In particular, we show that all vectors left in \( M' \) whose set representation contains the core can be safely added to the solution. After removing these vectors from the instance, we then clean up the remaining instance to ensure that we can iterate the above process and branch to find another core of a large sunflower—in particular, we need to guarantee that the remaining instance still contains a large sunflower core.

### 3.4 Other Contributions

For the problems considered above, we complement almost all the FPT results with lower-bound results—interestingly, several of these are obtained via a generic transformation from graphs to matrices that could be of independent interest, which we describe in Subsection 4.4 as part of our toolkit. To complete the complexity map for these problems, we obtain several XP-algorithms.
We also use our techniques to obtain an FPT-algorithm for Large Any-Cluster-Completion; this latter result implies the FPT of Large Any-Cluster parameterized by $k + r$, which answers an open question in the literature \cite{11 6}, posed for the significantly easier complete-data setting. Finally, we prove that all our results can be extended from Boolean domain to the finite-domain setting by developing a distance-sensitive encoding.

4 The Toolkit

In this section, we present key structural results that are employed to show several algorithms and lower bounds in the paper. Surprisingly, a significant part of our toolkit and structural results for matrices are obtained by exploiting the classical sunflower lemma of Erdős and Rado. A sunflower in a set family $\mathcal{F}$ is a subset $\mathcal{F}' \subseteq \mathcal{F}$ such that all pairs of elements in $\mathcal{F}'$ have the same intersection.

**Lemma 1** \cite{21 23}. Let $\mathcal{F}$ be a family of subsets of a universe $U$, each of cardinality exactly $b$, and let $a \in \mathbb{N}$. If $|\mathcal{F}| \geq b(a - 1)^b$, then $\mathcal{F}$ contains a sunflower $\mathcal{F}'$ of cardinality at least $a$. Moreover, $\mathcal{F}'$ can be computed in time polynomial in $|\mathcal{F}|$.

4.1 Finding Redundant Vectors

The first structural lemma we introduce is Lemma 3, which plays a crucial role in showing several FPT results in this paper. Intuitively speaking, the lemma says that if the $t$-Hamming neighborhood of a vector $\vec{v}$ contains a large sunflower, then at least one of its elements can be removed without changing the maximum distance to any vector $\vec{a}$ that is of distance at most $r$ to the elements in the sunflower. The proof of Lemma 3 utilizes the straightforward Lemma 2, which captures a useful observation that is also used in other proofs.

**Lemma 2.** Let $t, r \in \mathbb{N}$. Let $N \subseteq \{0, 1\}^d$ be a set of $t$-vectors such that $\mathcal{F} := \Delta(N)$ is a sunflower with core $C$. If $|N| > r$, then for every vector $\vec{a} \in \{0, 1\}^d$ with $|\Delta(\vec{a})| \leq r$, $N$ contains a vector that has maximum distance to $\vec{a}$ among all $t$-vectors that contain $C$.

**Proof.** Let $\vec{a} \in \{0, 1\}^d$ with $|\Delta(N, \vec{a})| \leq r$ be arbitrary. Since $|N| > r$, there is a vector $\vec{n} \in N$ such that $\Delta(\vec{n}) \cap \Delta(\vec{a}) \subseteq C$, which implies that the distance of $\vec{n}$ to $\vec{a}$ is maximum among all $t$-vectors that contain $C$.

**Lemma 3.** Let $t, r \in \mathbb{N}$, $\vec{v} \in \{0, 1\}^d$, $N \subseteq N_{=t}(\vec{v})$, and $\mathcal{F} := \{ \Delta(\vec{v}, x) \mid x \in N \}$ such that $\mathcal{F}$ is a sunflower with core $C$. Then for every subset $N'$ of $N$ with $|N'| \geq r + t + 2$ and every vector $\vec{a} \in \{0, 1\}^d$ such that $\delta(N', \vec{a}) \leq r$, we have $\delta(\vec{f}, \vec{a}) \leq \max_{\vec{x} \in N' \setminus \{\vec{f}\}} \delta(\vec{x}, \vec{a})$ for every $\vec{f} \in N$. 

Figure 1: The figure shows an example for the setting in Lemma 3. Here $r = 3$ and $t = 2$ and the figure shows the vectors $\vec{v}$ and $\vec{a}$ as well as the sunflower resulting from the vectors $\vec{p}_1, \ldots, \vec{p}_7$ with $\vec{p}_i[j] = 1$ if and only if either $j = 1$ or $j = i + 1$. In this example three of the petals, i.e., the white petals $\vec{p}_5$, $\vec{p}_6$, and $\vec{p}_7$, only share the core of the sunflower with $\vec{a}$, which implies that all three of these petals are of maximum hamming distance to $\vec{a}$. 

$\vec{v} = (0, 0, 0, 0, 0, 0, 0)$

$\vec{a} = (1, 1, 1, 1, 0, 0, 0)$
Figure 4.1 illustrates an example situation for this lemma. Since \( \delta(N', a) \leq r \) and \( \delta(\bar{x}, \bar{y}) = t \) for every \( \bar{x} \in N \), we obtain that \( \delta(\bar{v}, a) \leq r + t \). Since \( |N' \setminus \{\bar{f}\}| > r + t \), we obtain from Lemma 2 that \( N' \setminus \{\bar{f}\} \) contains a vector that has maximum distance to \( a \) among all \( t \)-vectors that contain \( C \), and hence in particular to all vectors in \( N \).

The following lemma now employs Lemmas 3 and 1 to show that if the \( t \)-Hamming neighborhood of a vector \( \bar{v} \) is large enough, then at least one of its elements can be removed without changing the clustering properties of the instance.

**Lemma 4.** Let \( k, r, t \in \mathbb{N}, M \subseteq \{0, 1\}^d, \bar{v} \in M, \) and \( N := N_{\lambda}(\bar{v}) \cap M \). If \( |N| \geq t!(k(r + t + 2))^t \), then there is a vector \( \bar{f} \in N \) satisfying the following two properties:

(P1) for every set \( S \subseteq \{0, 1\}^d \) with \( |S| \leq k \) and satisfying \( \delta(S, \bar{m}) \leq r \) for every \( \bar{m} \in M \), it holds that \( \max_{\bar{y} \in M} \delta(S, \bar{y}) = \max_{\bar{y} \in M \setminus \{\bar{f}\}} \delta(S, \bar{y}); \) and

(P2) \( M \) has a partition into at most \( k \) clusters, each of diameter at most \( r \), if and only if \( M \setminus \{\bar{f}\} \) does.

Moreover, \( \bar{f} \) can be determined in time polynomial in \( M \).

**Proof.** Let \( \mathcal{F} := \{ \Delta(\bar{v}, \bar{x}) \setminus \bar{x} \in N \} \). Then \( |\mathcal{F}| = |N| \geq t!(k(r + t + 2))^t \) and \( |F| = t \) for every \( F \in \mathcal{F} \).

By Lemma 1, \( \mathcal{F} \) contains a sunflower, say \( \mathcal{F}' \), of size larger than \( k(r + t + 2) \) (and core \( C \)).

We denote by \( N(F) \) the vector in \( N \) giving rise to the element \( F \in \mathcal{F} \), i.e., \( \mathcal{F} = \Delta(\bar{v}, N(F)) \).

Moreover, for a subset \( \mathcal{F}' \subset \mathcal{F} \) we denote by \( N(\mathcal{F}') \) the set \( \{ N(F) \mid F \in \mathcal{F}' \} \). Let \( F \in \mathcal{F}' \) be arbitrarily chosen. We claim that setting \( \bar{f} \) to the vector \( N(F) \) satisfies the claim of the lemma. Since \( |\mathcal{F}' \setminus \{F\}| \geq k(r + t + 2) \), we obtain that:

(1) for every set \( S \subseteq \{0, 1\}^d \) with \( |S| \leq k \) there is a vector \( \bar{s} \in S \) such that \( |N'| \geq r + t + 2 \), where \( N' := \{ \bar{u} \in \{0, 1\}^d \mid \delta(\bar{s}, \bar{u}) = \delta(S, \bar{u}) \} \cap N(\mathcal{F}') \).

(2) For every partition \( \mathcal{P} \) of \( M \setminus \{\bar{f}\} \) into at most \( k \) sets there is a set \( P \in \mathcal{P} \) with \( |P \cap N(\mathcal{F}' \setminus \{F\})| \geq r + t + 2 \).

Towards showing (P1), let \( S \subseteq \{0, 1\}^d \) with \( |S| \leq k \). By (1), there is a vector \( \bar{s} \in S \) with \( |N'| \geq r + t + 2 \). Then \( t, r, \bar{v}, N, \) and \( N' \) satisfy the conditions of Lemma 3. By observing that \( \delta(N', \bar{s}) \leq r \), we obtain that \( \delta(\bar{f}, \bar{s}) \leq \max_{x \in N' \setminus \{\bar{f}\}} \delta(\bar{x}, \bar{s}) \). Consequently, \( \max_{\bar{y} \in M} \delta(S, \bar{y}) = \max_{\bar{y} \in M \setminus \{\bar{f}\}} \delta(S, \bar{y}) \), which shows (P1).

Towards showing (P2), first note that the forward direction holds trivially. Towards showing the other direction, let \( \mathcal{P} \) be any partition of \( M \setminus \{\bar{f}\} \) into at most \( k \) sets, each of diameter at most \( r \). By (2), there is a set \( P \in \mathcal{P} \) with \( |P \cap N(\mathcal{F}' \setminus \{F\})| \geq r + t + 2 \). Let \( N' \) be the set \( (P \cap N(\mathcal{F}' \setminus \{F\})) \cup \{\bar{f}\} \). Then \( t, r, \bar{v}, N, \) and \( N' \) satisfy the conditions of Lemma 3. By observing that \( \delta(N', \bar{p}) \leq r \) for every \( \bar{p} \in P \), we obtain that \( \delta(\bar{f}, \bar{p}) \leq \max_{x \in N' \setminus \{\bar{f}\}} \delta(\bar{x}, \bar{p}) \) for every \( \bar{p} \in P \). Hence \( P \cup \{\bar{f}\} \) has diameter at most \( r \), which implies that the partition obtained from \( \mathcal{P} \) after adding \( \bar{f} \) to \( P \) is a partition of \( M \) into at most \( k \) clusters, each of diameter at most \( r \).

### 4.2 Irrelevant Coordinates and Diameter Bound

Our clustering algorithms for In/Any/Diam-Clustering-Completion, will broadly proceed in two steps. Given an instance \( \mathcal{I} = (M, k, r) \) of In/Any/Diam-Clustering-Completion, we will first compute an equivalent instance \( (M', k, r) \) such that the size of \( M' \) can be bounded by a function of the parameter \( k + r + \text{cover}(M) \) (this is done by the irrelevant vector technique). However, since our aim is to obtain a kernel, we then still need to reduce the number of coordinates for every vector in \( M' \). That is where we use our irrelevant coordinate technique. This subsection introduces the tools and notions that are central to this technique. Throughout this section, we will assume that \( \mathcal{I} = (M, k, r) \) is the considered input instance of In/Any/Diam-Clustering-Completion.
Let $|Z(M)|$ for $M \subseteq \{0,1\}^d$ be the set of all coordinates $i$ such that at least two vectors in $M$ disagree on their $i$-th coordinate, i.e., there are two vectors $\vec{y}, \vec{y}' \in M$ such that $\{\vec{y}[i], \vec{y}'[i]\} = \{0,1\}$. Intuitively, $Z(M)$ is the set of important coordinates, since all other coordinates can be safely removed from the instance; this is because they can always be completed to the same value and hence do not influence the properties of a clustering of $M$. Note that if we could show that the number of important coordinates is bounded by a function of $M'$ and our parameter $k + r + \text{cover}(M)$, then we would obtain a kernel by simply removing all coordinates that are not important. Unfortunately, this is not the case for two reasons: First the compatibility graph $G(\mathcal{I})$ can consist of more than one component and the vectors in different components can differ in arbitrary coordinates. Furthermore, even inside a component the number of important coordinates can be arbitrary large. For instance, a component could consist of the all-zero vector, the all-one vector, and the all-zero vector. Note that the all-zero vector is crucial for this example and indeed, the next lemma shows that if we restrict ourselves to a component containing only vectors in $M \setminus R_M$, then the number of important coordinates can be bounded in terms of the diameter and the number of vectors inside the component.

**Lemma 5.** Let $M' \subseteq M \setminus R_M$ such that $G(\mathcal{I})[M']$ is connected. Then $|Z(M') \setminus T_M| \leq \gamma(M')(|M'| - 1)$.

**Proof.** Let $\vec{m} \in M'$ be arbitrary. Then for every vector $\vec{m}' \in M'$, there are at most $\gamma(M')$ coordinates in $[d] \setminus T_M$ such that $\vec{m}[i] \neq \vec{m}'[i]$. Therefore in total there are at most $\gamma(M')(|M'| - 1)$ coordinates in $[d] \setminus T_M$ for which any vector in $M'$ differs from $\vec{m}$ and hence $|Z(M') \setminus T_M| \leq \gamma(M')(|M'| - 1)$. ■

The next lemma now shows how to bound the diameter of every component in $M \setminus R_M$ in terms of our parameter $k + r + \text{cover}(M)$.

**Lemma 6.** Let $\mathcal{I} = (M, k, r)$ be an instance of In-Clustering-Completion, Any-Clustering-Completion, or Diam-Clustering-Completion and let $M' \subseteq M \setminus R_M$ be such that $G(\mathcal{I})[M']$ is connected. Then $\mathcal{I}$ is a No-instance if either:

- $\mathcal{I}$ is an instance of In-Clustering-Completion and $\gamma(M') > 3rk - r + |T_M|$;
- $\mathcal{I}$ is an instance of Any-Clustering-Completion and $\gamma(M') > 4rk - r + |T_M|$; or
- $\mathcal{I}$ is an instance of Diam-Clustering-Completion and $\gamma(M') > 2rk - r + |T_M|$.

**Proof.** We start by showing the statement for the case of In-Clustering. Assume for a contradiction that $\mathcal{I}$ is a Yes-instance of In-Clustering and let $C_1, \ldots, C_k$ be a partitioning of $M$ into $k$ clusters, each of radius at most $r$. Consider any two vectors $\vec{a}, \vec{b} \in M'$; let $v_a$ and $v_b$ be their corresponding vertices in $G(\mathcal{I})$, and let $P_{ab}$ be a shortest path between $v_a$ and $v_b$ in $G(\mathcal{I})[M']$, which exists since $G(\mathcal{I})[M']$ is connected. By minimality of $P_{ab}$, $P_{ab}$ cannot contain more than three vertices corresponding to vectors in the same cluster, as otherwise, $P_{ab}$ could be shortcut by going through the center of that cluster. It follows that the length of $P_{ab}$ is at most $3k - 1$. Since every edge in $G(\mathcal{I})$ represents a Hamming distance of at most $r$ between the two vectors to the endpoints of the edge and all $\boxempty$-entries of every vector in $M'$ is contained in $T_R$, the lemma follows. The proofs for Any-Clustering and Diam-Clustering are analogous. ■

We now already know how to bound the number of important coordinates inside a component of $M \setminus R_M$. Unfortunately, as we have illustrated previously, it is not possible to do the same for $M \setminus R_M$, let alone for the complete vector set $M$. However, the following lemma shows that there is a (small) set $D'$ of coordinates that satisfy a slightly weaker property, namely, that it preserves distances up to $r$ within every component of $M \setminus R_M$ as well as to and between the vectors in $R_M$.

**Lemma 7.** Let $M' \subseteq M$ and $r'$ be a natural number. Then there is a subset $D' \subseteq [d]$ of coordinates such that:
\( \gamma_{\text{max}}(M') \) is equal to the maximum diameter of any connected component of \( G(I)[M'] \).

**Proof.** Note that we can assume w.l.o.g. that \( G(I)[M'] \) has at most \( k \) components, since otherwise \( I \) is a trivial No-instance. But then, we obtain from Lemma 8 that the set

\[ D_0 = \bigcup_{C \text{ is a component of } G(I)[M']} Z(C) \]

has size at most \( k\gamma_{\text{max}}(M')(|M'|-1) + |T_M| \). Moreover, \( \delta(\vec{m}, \vec{m}') = \delta(\vec{m}[D_0], \vec{m}'[D_0]) \) for any two vectors \( \vec{m} \) and \( \vec{m}' \) in \( M' \setminus R_M \) that are in the same component of \( G(I)[M'] \). Hence it only remains to ensure that condition (C2) is satisfied if (at least) one of \( \vec{m} \) and \( \vec{m}' \) is in \( R_M \). To achieve this we add the following coordinates to \( D_0 \) for every two vectors \( \vec{m} \in R_M \) and \( \vec{m}' \in M' \setminus \{\vec{m}\} \):

1. if \( \delta(\vec{m}, \vec{m}') \leq r' \), then we add the (at most \( r' \)) coordinates in \( \Delta(\vec{m}, \vec{m}') \) to \( D_0 \), otherwise
2. we add an arbitrary subset of \( \Delta(\vec{m}, \vec{m}') \) of size exactly \( r' + 1 \) to \( D_0 \).

Let \( D' \) be the set obtained from \( D_0 \) in this manner. Then \( D' \) clearly satisfies (C2). Finally, \( |D'| \leq |D_0| + |R_M|(|M'|-1)(r'+1) \) since we add at most \( r'+1 \) coordinates to \( D_0 \) for every \( \vec{m} \in R_M \) and \( \vec{m}' \in M' \setminus \{\vec{m}\} \).

The following lemma now shows that keeping only the set \( D' \) of coordinates is sufficient to preserve the equivalence for our three clustering problems.

**Lemma 8.** Let \( M' \subseteq M \). Then we can compute a set \( D' \subseteq [d] \) of coordinates in polynomial-time such that:

- \( |D'| \leq (k\gamma_{\text{max}}(M') + |R_M|(|M'|-1))(2r+1) + |T_M| \) and \((M', k, r)\) is a Yes-instance of \ANY\-CLUSTERING-COMPLETION if and only if \((M'_D, k, r)\) is.
- \( |D'| \leq (k\gamma_{\text{max}}(M') + |R_M|(|M'|-1))(r+1) + |T_M| \) and \((M', k, r)\) is a Yes-instance of \IN\-CLUSTERING-COMPLETION or \DIAM\-CLUSTERING-COMPLETION if and only if \((M'_D, k, r)\) is.

Here, \( M'_D \) is the matrix obtained from \( M' \) after removing all coordinates (columns) that are not in \( D' \).

**Proof.** We start by showing the result for \ANY\-CLUSTERING-COMPLETION. Let \( D' \) be the set of coordinates obtained from Lemma 7 for \( M' \) and \( r' = 2r \) satisfying (C1) and (C2). Because of (C1), it holds that \( |D'| \leq (k\gamma(M') + |R_M|(|M'|-1))(2r+1) + |T_M| \). It remains to show that \((M', k, r)\) and \((M'_D, k, r)\) are equivalent instances of \ANY\-CLUSTERING-COMPLETION.

Clearly any solution (i.e., a completion and a clustering for that completion) for \((M', k, r)\) is also a solution for \((M'_D, k, r)\); since all we did was remove a set of coordinates, all distances in the completion can only become smaller. For the forward direction, let \( M''_D \) be a completion of \( M'_D \) leading to a solution with at most \( k \) centers \( S \subseteq \{0, 1\}^{|D'|} \) for \((M''_D, k, r)\), and assume that \( S \) is inclusion-minimal. Consider a cluster given by a center \( \vec{s} \in S \) and let \( \vec{m} \in M''_D \) be a vector in that cluster, i.e., \( \delta(\vec{s}, \vec{m}) \leq r \); note that \( \vec{m} \) exists since \( S \) is inclusion-wise minimal. Let \( \vec{m}' \in M''_D \) be any other vector in that cluster; if the cluster consists only of the vector \( \vec{m} \), we can replace it with a cluster with center \( \vec{m} \) for the instance \((M', k, r)\). Then \( \delta(\vec{m}, \vec{m}') \leq 2r \) and because \( D' \) satisfies (C2) with \( r' = 2r \) also \( \delta(\vec{m}_+, \vec{m}'_+) \leq 2r \), where \( \vec{m}_+ \) and \( \vec{m}'_+ \) are the vectors in \( M' \) corresponding to \( \vec{m} \) and \( \vec{m}' \), respectively. Therefore, all coordinates where \( \vec{m}_+ \) and \( \vec{m}'_+ \) differ are contained in \( D' \), which implies that both vectors can be completed to be equal in all other coordinates, i.e., the coordinates...
in \([d] \setminus \Delta(\bar{m}_+, \bar{m}_+');\) let \(\bar{m}_+\) be one such completion of \(\bar{m}_+.\) Since we choose \(\bar{m}_+\) arbitrarily this is true also for all other vectors in the cluster. Hence, the vector \(\bar{s}' \in \{0, 1, \square\}^d\) is equal to \(\bar{s}\) for all coordinates in \(D'\) and equal to \(\bar{m}_+\) on all other coordinates can be used as a center for \((M', k, r)\) replacing \(\bar{s}.\) Applying the same procedure for all centers in \(S,\) we obtain a solution for \((M', k, r),\) as required.

We now show the result for In-Clustering-Completion and Diam-Clustering-Completion. Let \(D'\) be the set of coordinates obtained from Lemma \(7\) for \(M'\) and \(r' = r\) satisfying (C1) and (C2). Because of (C1), it holds that \(|D'| \leq (k\gamma(M') + |R_M|(|M'| - 1))(r + 1) + |T_M|\). It remains to show that \((M', k, r)\) and \((M''_{D'}, k, r)\) are equivalent instances of In-Clustering-Completion respectively Diam-Clustering.

Note that the forward direction of the claim is again trivial for both In-Clustering-Completion and Diam-Clustering-Completion, since we are only removing coordinates and hence the distances can only become smaller. Towards showing the backward direction, we will distinguish between the case for In-Clustering-Completion and Diam-Clustering-Completion. In the former case, let \(M''_D\) be a completion of \(M'_{D'}\) leading to the (inclusion-wise minimal) solution \(S \subseteq M''_D\) for \((M''_{D'}, k, r),\) i.e., a set of at most \(k\) centers, and consider a cluster given by a center \(\bar{s} \in S.\) If the cluster does not contain any other vector apart from \(\bar{s},\) then we can replace \(\bar{s}\) with the corresponding vector in \(M'\) (and use any completion). Otherwise, let \(\bar{m} \in M''_D\) be a vector in the cluster distinct from \(\bar{s},\) i.e., \(\delta(\bar{s}, \bar{m}) \leq r.\) Because \(D'\) satisfies (C2) with \(r' = r\) also \(\delta(\bar{s}_+, \bar{m}_+) \leq r,\) where \(\bar{s}_+\) and \(\bar{m}_+\) are the vectors in \(M'\) corresponding to \(\bar{s}\) and \(\bar{m},\) respectively. Therefore, all coordinates where \(\bar{s}_+\) and \(\bar{m}_+\) differ are contained in \(D',\) which implies that both vectors can be completed to be equal in all other coordinates, i.e., the coordinates in \([d] \setminus \Delta(\bar{s}_+, \bar{m}_+');\) let \(\bar{s}_+\) be one such completion for \(\bar{s}_+.\) Since we choose \(\bar{m}_+\) arbitrarily this is true also for all other vectors in the cluster. Hence, we can use the vector \(\bar{s}_+\) as a replacement for the vector \(\bar{s}\) and complete all vectors inside the cluster for \(\bar{s}\) according to \(\bar{s}_+.\) Applying the same procedure for all centers in \(S,\) we obtain a solution for \((M', k, r),\) as required.

In the later case, i.e., the case of Diam-Clustering-Completion, let \(M''_D\) be a completion of \(M'_{D'}\) leading to the (inclusion-wise minimal) solution \(P\) for \((M''_{D'}, k, r),\) i.e., a partition of \(M'_{D'}\) into at most \(k\) clusters, and consider a cluster \(P \in P.\) Let \(\bar{m}\) be any vector in \(P;\) which exists since \(P\) is inclusion-wise minimal. Then \(\delta(\bar{m}, \bar{m}') \leq r\) for every other vector \(\bar{m}' \in P.\) Moreover, since \(D'\) satisfies (C2) also \(\delta(\bar{m}_+, \bar{m}_+') \leq r\) for the vectors \(\bar{m}_+\) and \(\bar{m}_+\) in \(M'\) corresponding to \(\bar{m}\) and \(\bar{m}',\) respectively. Hence both vectors can be completed to be equal in all other coordinates, i.e., the coordinates in \([d] \setminus \Delta(\bar{m}_+, \bar{m}_+');\) let \(\bar{m}_+\) be one such completion for \(\bar{m}_+.\) Since we choose \(\bar{m}_+\) arbitrarily this is true also for all other vectors in the cluster. Hence, completing all vectors corresponding to vectors in \(P\) according to \(\bar{m}_+\) and obtain a cluster for \((M', k, r)\) containing the same vectors. Applying the same procedure for all sets in \(P,\) we obtain a solution for \((M', k, r),\) as required.

4.3 Sunflower Fields or Representing Sets by Cores of Sunflowers

In this subsection, we provide the central component for the iterative sunflower harvesting technique. Crucial to this component is a very general structural lemma that allows us to represent a family of sets in a succinct manner in terms of sunflower cores, which we believe to be interesting in its own right. We start by stating the result in its most general form (for sets) and then later show how to adapt it to our setting.

Let \(U\) be a universe, \(B\) a system of sets over \(U\) and \(A \subseteq B.\) We say that \(A\) is an \(r\)-uniform subset of \(B\) if the following holds for every \(t \in \mathbb{N}\) and every sunflower \(S \subseteq A\) containing at least \(r + 1\) sets of cardinality \(t\) with core \(C: A\) contains every set \(B \in B\) of cardinality \(t\) such that \(C \subseteq B."
Intuitively, this property states that \( A \) contains all sets in \( B \) which are super-sets of cores of every sufficiently large sunflower in \( A \) (with sets of the same cardinality). Informally, the connection of this set property to clusters is the following: after we perform some initial pre-processing, we can show that the set representation of every maximal cluster is an \( r \)-uniform subset of the input.

The crucial insight now is that every \( r \)-uniform subset containing only subsets of bounded size admits a succinct representation, where we can completely describe the set via a collection of a bounded number of sunflower cores. This is made precise in the following lemma.

**Lemma 9.** Let \( A \) and \( B \) be two families of sets of cardinality at most \( r' \) over universe \( U \) such that \( A \subseteq B \). If \( A \) is an \( r \)-uniform subset of \( B \), then there is a set \( S \) of at most \((rr')r'\) subsets of \( U \) such that \( A \) is equal to the set of all sets in \( B \) such that \( S \subseteq B \) for some \( S \in S \). Moreover, each set \( S \in S \) is either the core of a sunflower in \( A \) with at least \( r + 1 \) petals or \(|S| = r'\).

**Proof.** We show the claim by giving a bounded depth search tree algorithm that computes a rooted (search) tree \( T \), where each node \( t \) of \( T \) is associated with a subset \( X_t \) of \( U \) and each leaf node is marked as either successful or unsuccessful. Note that the algorithm is not necessarily efficient and all the steps can be brute-forced. The set \( S \) then contains all sets associated with a successful leaf node of \( T \). Initially, \( T \) only consists of the root node for which we set \( X_r := \emptyset \). The algorithm then proceeds as follows as long as \( T \) contains a leaf that is not yet marked as either successful or unsuccessful. Let \( t \) be a leaf node of \( T \) (that has not yet been marked). We first check whether \(|X_t| = r'\) and \( X_t \in A \) or whether \( A \) contains a sunflower with at least \( r + 1 \) petals and core \( X_t \). If so we mark \( t \) as a successful leaf node. Note since \( A \) is an \( r \)-uniform subset of \( B \), \( A \) contains all sets \( B \in B \) such that \( X_t \subseteq B \). Otherwise, let \( A_X \) be the subset of \( A \) containing all set \( A \in A \) with \( X_t \subseteq A \). If \( A_X = \emptyset \), we mark \( t \) as a unsuccessful leaf node. Otherwise, let \( A'_X \) be a maximal sunflower in \( A_X \) with core \( X_t \). Note that \( A'_X \neq \emptyset \) because \( A_X \neq \emptyset \) (and we can choose the core of a sunflower containing only one set/petal arbitrarily). Moreover, \(|A'_X| \leq r\), since otherwise \( A \) contains a sunflower with at least \( r + 1 \) petals and core \( X_t \). We now claim that the set \( H := (\bigcup_{A \in A'_X} A) \setminus X_t \) is a hitting set for the sets in \( A_X \). Assume, for a contradiction, that this is not the case and let \( A \in A_X \) be such that \( H \cap A = \emptyset \). Then we can extend \( A'_X \) with \( A \) contradicting our assumption that \( A'_X \) is inclusion-wise maximal. Now, for every \( h \in H \), we add a new child node \( t_h \) to \( t \) and set \( X_{t_h} := X_t \cup \{h\} \). Note that we add at most \(|H| \leq (r'r' - |X_t|) \) children to \( t \).

Let \( S \) be the set computed by the above algorithm. Note that \( T \) has height at most \( r' \). This is because \(|X_t| = l\) for any node \( t \) of \( T \) at level \( l \) and we stop when there is at most one set in \( A \) containing \( X_t \), which implies that \( t \) is marked either as successful or unsuccessful leaf node. Moreover, since every node of \( T \) has at most \( rr' \) children, we obtain that \( T \) has at most \((rr')r'\) (leaf) nodes and hence \(|S| \leq (rr'r')r'\), as required. It remains to show that for every set \( A \) in \( A \) there is a \( S \in S \) such that \( S \subseteq A \). Let \( A \) be an arbitrary set in \( A \). We first show that there is a leaf node \( l \) such that \( X_l \) is contained in \( A \). Towards showing this, first note that \( X_r = \emptyset \) is clearly contained in \( A \) for the root node \( r \) of \( T \). Furthermore, if \( t \) is an inner node of \( T \) such that \( X_t \) is contained in \( A \), then there is a child \( t' \) of \( t \) in \( T \) with \( X_{t'} \) contained in \( A \). This follows from the construction of \( T \) since \( t \) has a child \( t'' \) for every \( h \in H \) with \( X_{t''} := X_t \cup \{h\} \), where \( H \) is a hitting set of \( A_X \) and hence also hits \( A \). Starting with the root \( r \) of \( T \) and going down the tree \( T \) by always choosing a child \( t \) such that \( X_t \) is contained in \( A \), we will eventually end up at the leaf node \( l \) such that \( X_l \) is contained in \( A \). Finally, because \( A_X \) contains \( A \), \( l \) is marked as a successful leaf node meaning that \( X_l \in S \) is contained in \( A \).

We will now show how Lemma 9 can be employed in our setting. Let \( M, M' \subseteq \{0, 1\}^d \) with \( M' \subseteq M \). Then \( M' \) is \( r \)-uniform subset of \( M \) if \( \Delta(M') \) is a \( r \)-uniform subset of \( \Delta(M) \). Here one can think of \( M \) as being (a part of) the input matrix and \( M' \) being an (inclusion-wise) maximal cluster,
which as we will show later suffices for $M'$ being a $r$-uniform subset of $M$. Using this definition, the following corollary now follows immediately from Lemma 9.

**Corollary 10.** Let $r', r \in \mathbb{N}$ and $M, M' \subseteq \{0,1\}^d$ be sets of $r'$-vectors such $M'$ is an $r$-uniform subset of $M$. There is a set $S$ of at most $(rr')^r$ subsets of $[d]$ such that $M'$ is equal to the set of all vectors $\vec{m}$ in $M$ such that $S \subseteq \Delta(\vec{m})$ for some $S \in \mathcal{S}$. Moreover, each set $S \in \mathcal{S}$ is either a core of a sunflower in $\Delta(M')$ with at least $r + 1$ petals or $|S| = r'$.

### 4.4 A Generic Reduction

Here, we present a generic construction that is used in several hardness proofs throughout the paper.

Let $G$ be a graph, where $V(G) = \{v_1, \ldots, v_n\}$ and $m = |E(G)|$, and let $\deg(v_i) \leq n - 1$ denote the degree of $v_i$ in $G$. Fix an arbitrary ordering $\mathcal{O} = (e_1, \ldots, e_m)$ of the edges in $E(G)$. For each vertex $v_i \in V(G)$, define a vector $\vec{a}_i \in \{0,1\}^m$ to be the incidence/characteristic vector of $v_i$ w.r.t. $\mathcal{O}$; that is, $\vec{a}_i[j] = 1$ if $e_j$ is incident to $v_i$ and $\vec{a}_i[j] = 0$ otherwise. Afterwards, expand the set of coordinates of these vectors by adding to each of them $n(n - 1)$ “extra” coordinates, $n - 1$ coordinates for each $v_i$, $i \in [n]$; we refer to the $n - 1$ (extra) coordinates of $v_i$ as the “private” coordinates of $v_i$. For each $v_i$, $i \in [n]$, we will choose a number $x_i \in \{0, \ldots, n - 1\}$, where the choice of the number $x_i$ will be problem dependent, and we will set $x_i$ many coordinates among the private coordinates of $v_i$ to 1, and all other extra private coordinates of $v_i$ to 0. Let $M = \{\vec{a}_i \mid i \in [n]\}$ be the set of expanded vectors, where $\vec{a}_i \in \{0,1\}^{m + n(n - 1)}$ for $i \in [n]$. We have the following straightforward observation:

**Observation 11.** For each $v_i$, where $i \in [n]$, the number of coordinates in $\vec{a}_i$ that are equal to 1 is exactly $\deg(v_i) + x_i$, and two distinct vertices $v_i, v_j$ satisfy $\delta(\vec{a}_i, \vec{a}_j) = \deg(v_i) + x_i + \deg(v_j) + x_j$ if $v_i$ and $v_j$ are nonadjacent in $G$ and $\delta(\vec{a}_i, \vec{a}_j) = \deg(v_i) + x_i + \deg(v_j) + x_j - 2$ if $v_i$ and $v_j$ are adjacent.

Throughout the paper, we denote by $\mathcal{R}$ the polynomial-time reduction that takes as input a graph $G$ and returns the set of vectors $M$ described above.

### 5 Clustering with Incomplete Data

In this section, we will show that $\text{In/Any/Diam-Clustering-Completion}$ are fixed-parameter tractable parameterized by $k + r + \text{cover}(M)$; we will then show in Section 7 that all three parameters are indeed required to obtain fixed-parameter tractability. Our algorithmic results in this section are achieved via kernelization: we will apply the irrelevant vector and irrelevant coordinate techniques to obtain an equivalent instance of size upper bounded by a function of $k + r + \text{cover}(M)$. It is well-known that the existence of a kernelization implies the fixed-parameter tractability of the problem—indeed, the resulting instance can be solved by any clustering algorithm (including, for instance, a brute force algorithm) [16, 14].

Note that this implies that also the variants $\text{In/Any/Diam-Clustering}$ for complete data are fixed-parameter tractable parameterized by only $k + r$ (and also have a polynomial kernel) and, as we will show in Subsection 7.3, both parameters are indeed required. To explain how we obtained our results, we will first by considering the general procedures for complete data first and then provide the necessary changes for the case of incomplete data. Throughout the section we will assume that $(M, k, r)$ is the given instance of $\text{In/Any/Diam-Clustering-Completion}$. Recall that, when using the parameter $\text{cover}(M)$, we will use the sets $T_M$ and $R_M$ (as defined in Section 2), where $T_M \subseteq [d]$, $R_M \subseteq M$, and $|T_M| + |R_M| = \text{cover}(M)$, and such that all \(\square\)'s in $M \setminus R_M$ occur only in coordinates in $T_M$.  

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Informal description of the algorithm for complete data: To perform kernelization, we start by identifying and removing irrelevant vectors; those are vectors that can be removed from the instance and safely added back to any valid clustering of the reduced instance to yield a valid clustering of the original instance. One caveat is that, for In-Clustering-Completion, the removed vectors may serve as cluster centers, and hence, such vectors will have to be represented in the reduced instance; we will discuss later (below) how this issue is dealt with. To identify redundant vectors, we first show that, for each vector, we can compute a “representative set” of vectors of its \((\leq r)\)-neighborhood whose size is upper bounded by a function of the parameter. The identification of representative sets is achieved via a non-trivial application of the Sunflower Lemma (and several other techniques) in Lemmas 3, 4 as well as Lemma 12 for Any-Clustering-Completion, Lemma 12 and 14 for In-Clustering-Completion, and Lemma 17 for Any-Clustering. The union of these representative sets yields a reduced instance whose number of vectors is upper bounded by a function of the parameter. For the final step of our algorithm we use the results from Subsection 4.2 to reduce the number of dimensions for every vector in the reduced instance. This will already yields the kernelization results for Any-Clustering-Completion.

As for In-Clustering-Completion, we need to ensure that the centers of the clusters in any valid solution are represented in the reduced instance. (Observe that the size of the reduced instance now is upper bounded by a function of the parameter.) To do so, we partition the set of removed vectors (i.e., not in the reduced instance) into equivalence classes based on their signature/trace on the set of important coordinates; the number of equivalence classes is upper bounded by a function of the parameter. Since each potential center must be within distance \(r\) from some vector in the reduced instance, for each (redundant) vector \(\vec{x}\) that differs in at most \(r\) important coordinates from some vector in the reduced instance, we add a vector from the equivalence class of \(\vec{x}\) (that represents \(\vec{x}\)) whose distance to the vectors in the reduced instance w.r.t. nonimportant coordinates (which all vectors in the reduced instance agree on) is minimum. Lemma 14 shows that the number of added vectors is upper bounded by a function of the parameter.

Finding Redundancy when Data is Missing. In the case of incomplete data, we will in principle employ the same general strategy that we used for clustering problems with complete data. Namely, we will again identify redundant vectors and coordinates whose removal results in an instance whose size can be bounded by our parameter. However, due to the presence of incomplete data, we need to make significant adaptations at every step of the algorithm.

Consider the first step of the algorithm, which allowed us to identify and remove redundant vectors. For this step, we can focus only on the vectors in \(M \setminus R_M\), since \(|R_M|\) is already bounded by \(\text{cover}(M)\); crucially, this allows us to assume that vectors only have \(\Box\)-entries at positions in \(T_M\).

Now consider Lemma 4 which allowed us to remove any vector, say \(\vec{f}\), in a sufficiently large sunflower occurring in the \(t\)-Hamming neighborhood of some vector \(\vec{v}\). Informally, the lemma relied on the fact that for every solution of the reduced instance, a large part of the sunflower must end up together in one of the clusters; this in turn meant that for every vector in the cluster there is a vector in the sunflower whose distance is at least the distance to the redundant vector \(\vec{f}\). This is what allowed us to argue that \(\vec{f}\) can always be safely added back into that cluster. But once we allow \(\Box\)-entries, this can no longer be guaranteed, since whether or not we are able to add \(\vec{f}\) back into the cluster depends also on how the \(\Box\)-entries of the other vectors in the sunflower have been completed.

Note that the problem above would disappear if we could ensure that a sufficiently large number of vectors from the initial sunflower that end up together in the same cluster have the \(\Box\)-entries at the exact same positions. Since we observed earlier that we can assume that all vectors
have their $\square$-entries only in $T_M$, and consequently there are at most $2^{|T_M|}$ different allocations of the $\square$-entries to these vectors, we can now enforce this by enlarging the initial sunflower by a factor of $2^{|T_M|}$. This approach allows us to obtain the following lemma, which uses Lemma 4 in a way that allows us to reduce the number of vectors for In-Clustering-Completion and Any-Clustering-Completion.

**Lemma 12.** Let $k, r \in \mathbb{N}$ and $M \subseteq \{0, 1, \square\}^d$. Then there is a subset $M'$ of $M$ with $R_M \subseteq M'$ satisfying:

1. (P1) For every $\vec{v} \in M \setminus R_M$ it holds that $|N_r(\vec{v}) \cap M' \setminus R_M| \leq 2^{|T_M|}t!(k(r + t) + 2)^t$; and
2. (P2) for every set $S \subseteq \{0, 1\}^d$ with $|S| \leq k$ and satisfying $\delta(S, \vec{m}) \leq r$ for every $\vec{m} \in M$ it holds that $\max_{\vec{y} \in M} \delta(S, \vec{y}) = \max_{\vec{y} \in M'} \delta(S, \vec{y})$.

Moreover, $M'$ can be computed in time polynomial in $M$.

**Proof.** We obtain $M'$ using the following algorithm. Initially, we set $M'$ to $M$. Then for every $\vec{v} \in M \setminus R_M$, every $Q \subseteq T_M$, and every $t$ with $1 \leq t \leq r$ we do the following.

We denote by $M_Q$ the subset of $M \setminus R_M$ with $\{i \mid \vec{a}[i] = \square\} = Q$ for every vector $\vec{a} \in M_Q$. Moreover, let $\sigma_Q : \{0, 1, \square\}^d \to \{0, 1\}^d - \{Q\}$ be the mapping that given a vector $\vec{a} \in \{0, 1, \square\}^d$ outputs the vector $\sigma_Q(\vec{a})$ which skips all the coordinates of $\vec{a}$ in $Q$, or more formally, for all $i \in [d]$ and $j = |Q \cap [i]|$, we let $\sigma_Q(\vec{a})[i - j] = \vec{a}[i]$. Note that for a $\vec{a} \in M_Q$, the mapping $\sigma_Q$ skips exactly all the coordinates with $\square$.

Therefore, for every pair of vectors $\vec{a} \in M_Q$ and $\vec{s} \in \{0, 1\}^d$, it holds $\delta(\vec{a}, \vec{s}) = \delta(\sigma_Q(\vec{a}), \sigma_Q(\vec{s}))$.

We denote by $M_Q'$ the set $M' \cap M_Q$. Now we apply Lemma 4 to $\sigma_Q(\vec{v})$ and $\sigma_Q(M_Q')$ exhaustively, i.e., as long as $|N| = |N_{\sigma}(M)| \geq t!(k(r + t) + 2)^t$, we use the lemma to find the vector $\vec{f} \in \sigma_Q(M_Q')$, we remove from $M'$ the vector $\vec{g} \in M_Q'$ such that $\sigma_Q(\vec{g}) = \vec{f}$, and apply the lemma again. Let $M'$ be the subset of $M$ obtained in this manner. Then (P1) clearly holds and (P2) follows from (P1) in Lemma 4 and the observation that, for every pair of vectors $\vec{a} \in M_Q$ and $\vec{s} \in \{0, 1\}^d$, it holds that $\delta(\vec{a}, \vec{s}) = \delta(\sigma_Q(\vec{a}), \sigma_Q(\vec{s}))$. Finally, $R_M \subseteq M'$ since we did not removed any vector in $R_M$ from $M$.

Using the above Lemma 12 together with the toolbox obtained in Subsection 4.2 (to reduce the number of relevant coordinates), we are now ready to show our first fixed-parameter algorithm for Any-Clustering-Completion.

**Theorem 13.** Any-Clustering-Completion is fixed-parameter tractable parameterized by $k + r + \text{cover}(M)$.

**Proof.** Let $(M, k, r)$ be the given instance of Any-Clustering and let $M'$ be the set obtained using Lemma 12 for $M$, $k$, and $2r$. Because $M'$ satisfies (P2), it holds that $(M, k, r)$ and $(M', k, r)$ are equivalent instances of Any-Clustering. Consider a solution $S \subseteq \{0, 1\}^d$ for $(M', k, r)$, with $|S| \leq k$, and a vector $\vec{s} \in S$. Since we can assume that $S$ is minimal, it holds that $N_r(\vec{s}) \cap M' \neq \emptyset$ for every $\vec{s} \in S$. Let $\vec{g} \in N_r(\vec{s}) \cap M'$ be arbitrarily chosen. Then $N_r(\vec{s}) \cap M' \subseteq N_{2r}(\vec{g}) \cap M'$. Moreover, since $M'$ satisfies (P1), it follows that $N_{2r}(\vec{g}) \cap M'$ and thus also $N_r(\vec{s}) \cap M'$ has size at most $\sum_{t=1}^{2r} t!(k(2r + t + 2)^t + 1$. Consequently, if $|M'| > k((\sum_{t=1}^{2r} t!(k(2r + t + 2)^t + 1)$, we can safely return that $(M, k, r)$ is a No-instance of Any-Clustering-Completion.

Thus, $|M'| \leq k((\sum_{t=1}^{2r} t!(k(2r + t + 2)^t + 1$ and it remains to reduce the number of coordinates for each vector in $M'$. Let $D'$ be the set of coordinates obtained from Lemma 8 for $M'$. Then $(M', k, r)$ and $(M'_D, k, r)$ are equivalent instances of Any-Clustering-Completion and moreover we obtain from Lemma 6 that $\gamma_{\text{max}}(M') \leq 4r k - r + |T_M|$. Therefore, we obtain:
\[ |D'| \leq (k \gamma_{\text{max}}(M') + |R_M|(|M'| - 1))(r' + 1) \]
\[ \leq (4rk - r + |T_M| + |R_M|(|M'| - 1))(2r + 1) \]

showing that the size of \( D' \) is bounded by our parameter \( k + r + \text{cover}(M) \). Hence \((M'_{\text{pr}}, k, r)\) is a kernel for \((M, k, r)\) and \textsc{Any-Clustering-Completion} is fixed-parameter tractable parameterized by \( k + r + \text{cover}(M) \).

Towards showing our kernelization result for \textsc{In-Clustering-Completion}, we need to add back some vectors that can be potential centers for the clusters containing vectors of \( M' \). The main idea for the case of complete data is the observation that every vector in \( M \) that can act as a potential center for the instance on \( M' \) must be within the \( r \)-neighborhood of some vector in \( M' \) and moreover among all (potentially many vectors within the \( r \)-neighborhood of a vector in \( M' \)), we can choose any vector, which is closest w.r.t. the unimportant coordinates, \( i.e.\), the coordinates in \( d \backslash Z(M') \). This way the number of potential vectors that can act as a center for a vector in \( M' \) can be bounded by the parameter. For the case of incomplete data we need to consider an additional complication, namely, that the \( \square \) entries of the vectors in \( M' \) (which can be changed without increasing the Hamming distance to the vector), can increase the size of the \( r \)-Hamming neighborhood of every such vector now significantly. For instance, the potential \( r \)-Hamming neighborhood of a vector in \( M' \backslash R_M \) increases by a factor of \( 2^{\gamma M'} \) and the potential \( r \)-Hamming neighborhood of a vector \( \vec{x} \) in \( R_M \) can only be bounded by \( 2^{\gamma M'} |M'| - 1) \), since potentially every important coordinate of \( \vec{x} \) could be a \( \square \).

**Lemma 14.** Let \((M, k, r)\) be an instance of \textsc{In-Clustering-Completion} and \( M' \subseteq M \) with \( R_M \subseteq M' \). Then there is a set \( M'' \) with \( M' \subseteq M'' \subseteq M \) of size at most \( |M'| + 3^{\gamma M'} |R_M| + k 3^{2|T_M|, |R_M|} \gamma_{\text{max}}(M') \gamma_{\text{max}}(|M'| - 1) \) such that there is a set \( S \subseteq M \) with \( |S| \leq k \) satisfying \( \max_{\vec{y} \in M'} \delta(S, \vec{y}) \leq r \) if and only if there is a set \( S \subseteq M'' \) with \( |S| \leq k \) satisfying \( \max_{\vec{y} \in M'} \delta(S, \vec{y}) \leq r \). Moreover, \( M'' \) can be computed in polynomial time.

**Proof.** First note that if \( |R_M| = |M| \), then \( M'' = M \) trivially satisfies the conditions of the lemma. Let \( S \subseteq M \) with \( |S| \leq k \) satisfying \( \max_{\vec{y} \in M'} \delta(S, \vec{y}) \leq r \) and let \( \vec{s} \in S \). Since we can assume that \( S \) is minimal (w.r.t. satisfying \( \max_{\vec{y} \in M'} \delta(S, \vec{y}) \leq r \)), we obtain that \( \vec{s} \) must be within the \( r \)-Hamming neighborhood of some vector \( \vec{y} \in M' \), \( i.e.\), for every \( \vec{s} \in S \) there is a vector \( \vec{y} \in M' \) such that \( \delta(\vec{s}, \vec{y}) \leq r \).

We start by defining what it means for two vectors \( \vec{m} \) and \( \vec{m}' \) to be equivalent w.r.t. the vectors in \( R_M \) in the sense that if \( \vec{m} \) can be used as a center containing a vector in \( R_M \) then so can \( \vec{m}' \) and vice versa. Namely, we say that two vectors \( \vec{m} \) and \( \vec{m}' \) are equivalent w.r.t. \( R_M \), denoted \( \equiv_{R_M} \), if and only if they agree on the coordinates in \( T_M \), \( i.e.\), \( \vec{m}[i] = \vec{m}'[i] \) for every \( i \in T_M \), and for every \( \vec{x} \) in \( R_M \) it holds that:

- \( \delta(\vec{m}, \vec{x}) = \delta(\vec{m}', \vec{x}) \) if \( \delta(\vec{m}, \vec{x}) \leq r \) and
- \( \delta(\vec{m}', \vec{x}) > r \) otherwise.

Clearly, this guarantees that if a vector \( \vec{x} \) is a part of a cluster with center \( \vec{m} \), then \( \vec{x} \) will still be contained in the cluster if \( \vec{m} \) is replaced by \( \vec{m}' \) and completed in the same manner as \( \vec{m} \). Note also that this defines an equivalence relation for the vectors in \( M \backslash R_M \) and that the number of equivalence classes is at most \( 3^{2|T_M|, |R_M|} \).

We now have to define what it means for two vectors \( \vec{m} \) and \( \vec{m}' \) to be equivalent w.r.t. the remaining vectors in \( M \), \( i.e.\), the vectors in \( M \backslash R_M \). Towards this aim let \( C \) be a component of \( G(Z)[M' \backslash R] \). Then it follows from Lemma 5 that \( |Z(C) \cup T_M| \leq \gamma(C)(|C| - 1) + |T_M| \) and hence all vectors in \( C \) agree on all coordinates outside of \( |Z(C) \cup T_M| \). We say that two vectors \( \vec{m} \) and \( \vec{m}' \) in \( M \backslash R_M \) that are in the \( r \)-Hamming neighborhood of some vector in \( C \) are equivalent w.r.t. \( C \),
denoted \( \bar{m} \equiv_C m' \), if and only if they agree on all coordinates in \( Z(C) \cup T_M \). Then \( \equiv_C \) is also an equivalence relation for all vectors that are in the \( r \)-Hamming neighborhood of some vector in \( C \) and moreover the number of equivalence classes is at most \( 2^{|Z(C)\cup T_M|} |3T_M| \). Note, however, that even if \( \bar{m} \equiv C m' \) and some vector \( \bar{c} \in C \) is contained in a cluster with center \( \bar{m} \), we cannot simple replace \( \bar{m} \) by \( \bar{m}' \) because \( \bar{m} \) and \( \bar{m}' \) might have a different Hamming distance to \( \bar{c} \) when considering the coordinates outside of \( Z(C) \cup T_M \). Nevertheless, it still suffices to keep only one vector from every equivalence class, namely, a vector that is closest to any (all) vectors in \( C \) w.r.t. the coordinates in \( [d] \setminus (Z(C) \cup T_M) \); recall that all vectors in \( C \) agree on all coordinates in \( [d] \setminus (Z(C) \cup T_M) \). Note that \( \equiv_C \) has at most \( 3^{|T_M|r}|R_M|2|Z(C)\cup T_M| = 3^{|T_M|r}|R_M|2|Z(C)\cup T_M| \) equivalence classes.

We can now combine these two equivalence relations into one. Namely, we say that two vectors \( \bar{m} \) and \( \bar{m}' \) in \( M \setminus R_M \) are equivalent w.r.t. \( C \) and \( R_M \), denoted by \( \bar{m} \equiv_{C,R_M} \bar{m}' \), if and only if \( \bar{m} \equiv_R \bar{m}' \) and \( \bar{m} \equiv_C \bar{m}' \).

Let \( M_0 \) be the set of all vectors (in \( M \setminus M' \)) defined as follows. First for every component \( C \) of \( G(\mathcal{I})[M'] \) and every equivalence class \( P \) of \( \equiv_{C,R_M} \), the set \( M_0 \) contains a vector that is closest to any/every vector in \( C \) w.r.t. the coordinates in \( [d] \setminus (Z(C) \cup T_M) \) among all vectors in \( P \). Finally, we also add to \( M_0 \) an arbitrary vector for every equivalence class of \( \equiv_{C,R_M} \). We claim that setting \( M'' \) to \( M' \cup M_0 \) satisfies the claim of the lemma. We start by showing that there is a set \( S \subseteq M \) with \( |S| \leq k \) satisfying \( \max_{\bar{m} \in M} \delta(S, \bar{m}) \leq r \) if and only if there is a set \( S \subseteq M'' \) with \( |S| \leq k \) satisfying \( \max_{\bar{m} \in M'} \delta(S, \bar{m}) \leq r \). The reverse direction is trivial since \( M'' \) is a subset of \( M \). Towards showing the forward direction let \( S \subseteq M \) with \( |S| \leq k \) satisfying \( \max_{\bar{m} \in M'} \delta(S, \bar{m}) \leq r \) and let \( \bar{s} \in S \). Then \( \bar{s} \in M \setminus R_M \) since \( R_M \subseteq M'' \). Let \( P_{\bar{s}} \) be the set of all vectors in \( M' \) that are in the cluster with center \( \bar{s} \). If \( P_{\bar{s}} \subseteq R_M \) and we replace \( \bar{s} \) with a vector \( \bar{s}' \in M'' \) such that \( \bar{s} \equiv_{C,R_M} \bar{s}' \). Since \( \bar{s} \) and \( \bar{s}' \) agree on all coordinates in \( T_M \), we can complete \( \bar{s}' \) in the same way as \( \bar{s} \). Moreover, because \( \delta(\bar{s}, \bar{m}) \leq r \) for every \( \bar{m} \in P_{\bar{s}} \) and \( \bar{m} \in R_M \), we obtain that \( \delta(\bar{s}', \bar{m}) \leq r \), as required. Otherwise, let \( \bar{m} \in P_{\bar{s}} \) be a vector with \( \bar{m} \in M' \setminus R_M \) and let \( C \) be the component of \( G(\mathcal{I})[M'] \) containing \( \bar{m} \). Then \( P_{\bar{s}} \setminus R_M \subseteq C \). We claim that we can replace \( \bar{s} \) with the vector \( \bar{s}' \) in \( \bar{m} \) such that \( \bar{s}' \equiv_{C,R_M} \bar{s} \). Since \( \bar{s}' \) agrees with \( \bar{s} \) on all coordinates in \( T_M \), we can complete \( \bar{s}' \) in the same manner as \( \bar{s} \). Let \( \bar{x} \) be a vector in \( P_{\bar{s}} \). If \( \bar{x} \notin R_M \), then \( \delta(\bar{s}, \bar{x}) \leq \delta(\bar{s}', \bar{x}) \), because \( \bar{s} \) and \( \bar{s}' \) agree on all coordinates in \( Z(C) \cup T_M \) and \( \bar{m} \) is closest to all vectors in \( C \) (and in particular to \( \bar{x} \)) w.r.t. to all other coordinates. Moreover, if \( \bar{x} \in R_M \), then \( \delta(\bar{s}, \bar{x}) \leq r \) and hence \( \delta(\bar{s}', \bar{x}) = \delta(\bar{s}, \bar{x}) \leq r \), as required.

We are now ready to bound the size of \( M_0 \) in terms of our parameter \( k + r + \text{cover}(M) \) and the size of \( M' \). Apart from the \( 3^{|T_M|r}|R_M| \) vectors (one for every equivalence class of \( \equiv_{R_M} \), \( M_0 \) contains \( 3^{|T_M|r}|R_M|2|Z(C)\cup T_M| \) vectors for every component \( C \) of \( G(\mathcal{I})[M'] \) (one for every equivalence class of \( \equiv_{R_M} \)). Since \( G(\mathcal{I})[M'] \) can have at most \( k \) components, since otherwise \( \mathcal{I} \) is a No-instance, and using the fact that \( |Z(C) \setminus T_M| \leq \gamma_{\text{max}}(M')(|M'|-1) \) for every component \( C \) of \( G(\mathcal{I})[M'] \) (Lemma 5), we obtain that:

\[
|M_0| \leq 3^{|T_M|r}|R_M| + k3^{|T_M|r}|R_M|2\gamma_{\text{max}}(M')(|M'|-1).
\]

It is straightforward to verify that \( M_0 \) can be computed in polynomial time.

With Lemma 14 in hand, we can move towards establishing the fixed-parameter tractability of In-Clustering-Completion.

**Theorem 15.** *In-Clustering-Completion* is fixed-parameter tractable parameterized by \( k + r + \text{cover}(M) \).

**Proof.** Let \( (M, k, r) \) be the given instance of In-Clustering and let \( M' \) be the set obtained using Lemma 12. Since \( M' \) satisfies (P2), it holds that \( (M, k, r) \) has a solution if and only if there is a set
$S \subseteq M$ with $|S| \leq k$ such that $\max_{\bar{y} \in M} \delta(S, \bar{y}) \leq r$. Since $M'$ satisfies (P1), we can safely return that $(M', k, r)$ is a No-instance if $|M'| > k(2^{2T_M}(\sum_{i=1}^{t!}(k(r + t) + 1)^{i}) + |R_M| + 1) = f(k, r, \text{cover}(M))$. Hence, w.l.o.g., we can assume that $|M'| \leq f(k, r, \text{cover}(M))$, and it only remains to bound the number of vectors in $M \setminus M'$ that could potentially be in a solution. Let $M''$ be the set obtained from Lemma 14 for $M$ and $M'$. Then $(M, k, r)$ and $(M'' \cup M', k, r) = (M'', k, r)$ are equivalent instances of IN-CLUSTERING-COMPLETION. Moreover, $|M''| \leq |M'| + 3^{2T_M}|R_M| + k3^22^{2T_M}|R_M|2^{\gamma_{\text{max}}(M')}(|M'| - 1)$, which together with Lemma 6 implies that $|M''|$ is also bounded by a function of $k + r + \text{cover}(M)$. Finally, it remains to reduce the number of coordinates for each vector in $M''$. Let $D'$ be the set of coordinates obtained from Lemma 16 for $M''$. Then $(M''', k, r)$ and $(M''', k, r)$ are equivalent instances of IN-CLUSTERING-COMPLETION and moreover we obtain from Lemma 6 that $\gamma_{\text{max}}(M'') \leq 3rk - r + |T_M|$. Therefore, we obtain:

$$D' \leq (k\gamma_{\text{max}}(M''') + |R_M|(|M''| - 1))(r' + 1) \leq (k(3rk - r + |T_M|) + |R_M|(|M''| - 1))(r + 1)$$

showing that the size of $D'$ is bounded by our parameter $k + r + \text{cover}(M)$. Hence, $(M''', k, r)$ is a kernel for $(M, k, r)$ and IN-CLUSTERING-COMPLETION is fixed-parameter tractable parameterized by $k + r + \text{cover}(M)$.

We now proceed to the last of the three problems considered in this section, DIAM-CLUSTERING-COMPLETION. Apart from the issue that we already had for IN-CLUSTERING-COMPLETION and ANY-CLUSTERING-COMPLETION that we require a sunflower of vectors with all vectors in the same cluster arising from the possibility of different completions of the coordinates in $M$. By Lemma 14 for the same problem, we can handle by increasing the size of the sunflower by an additional factor of $2^{2T_M}$. This leads to the following version of Lemma 12 for DIAM-CLUSTERING-COMPLETION.

**Lemma 16.** Let $k, r, t \in \mathbb{N}$, $M \subseteq \{0, 1, \square\}^d$, $\bar{v} \in M$, and let $N := N_{=t}(\bar{v}) \cap M \setminus R_M$. If $|N| \geq 2^{2T_M}t!(2^{2T_M}|k(r + t + |T_M| + 2))^{t} + 1$, then there is a vector $\bar{f} \in N$ satisfying the following property:

$M$ has a completion $M' \subseteq \{0, 1\}^d$ with a partition into at most $k$ clusters, each of diameter at most $r$, if and only if $M \setminus \{\bar{f}\}$ does.

Moreover, $\bar{f}$ can be determined in time polynomial in $M$.

**Proof.** Let us first define an equivalence relation $\sim$ over $N \setminus R_M$ depending on which subset of $T_M$ contains $\square$. That is, for two vectors $\bar{a}, \bar{b} \in N$, we say $\bar{a} \sim \bar{b}$ if and only if $\{i \mid \bar{a}[i] = \square\} = \{i \mid \bar{b}[i] = \square\}$.

Now let $N^{\square}$ be a maximum size equivalence class of $\sim$, and let $\mathcal{F} := \{\Delta(\bar{v}, \bar{x}) \mid \bar{x} \in N^{\square}\}$. Then $|\mathcal{F}| = |N^{\square}| \geq t!(2^{2T_M}|k(r + t + |T_M| + 2))^{t}$, $|\mathcal{F}| = t$ for every $F \in \mathcal{F}$, and all the vectors of $N^{\square}$ have $\square$-entries at the same indices. By Lemma 1, $\mathcal{F}$ contains a sunflower, say $\mathcal{F}'$, of size at least $2^{2T_M}|k(r + t + |T_M| + 2)$ (and core $C$).

We denote by $N^{\square}(F)$ the vector in $N^{\square}$ giving rise to the element $F \in \mathcal{F}$, i.e., $F = \Delta(\bar{v}, N^{\square}(F))$. Moreover, for a subset $\mathcal{F}'$ of $\mathcal{F}$ we denote by $N^{\square}(\mathcal{F}')$ the set $\{N^{\square}(F) \mid F \in \mathcal{F}'\}$. Let $F \in \mathcal{F}'$ be arbitrarily chosen. We claim that setting $\bar{f}$ to the vector $N^{\square}(F)$ satisfies the claim of the lemma. Since $|\mathcal{F}' \setminus \{F\}| \geq 2^{2T_M}|k(r + t + |T_M| + 2)$, we observe the following:
for every partition $\mathcal{P}$ of $M \setminus \{f\}$ into at most $k$ sets there is a set $P \in \mathcal{P}$ with $|P \cap N^\square(F \setminus \{F\})| \geq 2^{|T_M|}(r + t + |T_M| + 1) + 1$.

We are now ready to prove the lemma. First note that the forward direction of the lemma holds trivially. Towards showing the other direction, let $M_f' \subseteq \{0,1\}^d$ be a completion of $M \setminus f$ with the bijection $\alpha : M \setminus f \to M_f'$ as the completion witness, and let $\mathcal{P}$ be any partition of $M_f'$ into at most $k$ sets, each of diameter at most $r$. By the above observation, there is a set $P \in \mathcal{P}$ with $|\alpha^{-1}(P) \cap N^\square(F \setminus \{F\})| \geq 2^{|T_M|}(r + t + |T_M| + 2)$. Note that all the vectors in $N^\square(F \setminus \{F\})$ have exactly the indices in $T_M$. Hence that there is a set $P' \subseteq P$ of size $r + t + |T_M| + 2$ which was completed the same way, e.g., $\vec{a}[i] = \vec{b}[i]$ for all $\vec{a}, \vec{b} \in P'$ and all $i$ such that $\alpha^{-1}(\vec{a})[i] = \alpha^{-1}(\vec{b})[i] = \square$. Let $\alpha(f)$ be the completion of $f$ in the same way as all the other vectors in $P'$. Note that, since for all $\vec{a} \in \alpha^{-1}(P')$ it holds that $\delta(\vec{a}, \vec{v}) = t$ and all the vectors in $\alpha^{-1}(P')$ are completed the same way, it follows that there is an integer $t'$ with $t - |T_M| \leq t' \leq t + |T_M|$ such that, for every vector $\vec{a} \in \alpha^{-1}(P')$, we have $\delta(\vec{a}, \alpha(\vec{v})) = t$. Moreover, $F_{P'} := \{ \Delta(\alpha(\vec{v}), \vec{x}) \mid \vec{x} \in P' \}$ is a sunflower with $|F_{P'}| \geq r + t' + 2$.

Let $N'$ be the set $(P' \cap \alpha(N^\square(F \setminus \{F\}))) \cup \{ \alpha(f) \}$. Then $t'$, $r$, $\alpha(\vec{v})$, $N'$, $N'$ satisfy the conditions of Lemma 17. By observing that $\delta(N', \vec{p}) \leq r$ for every $\vec{p} \in P$, we obtain that $\delta(\alpha(f), \vec{p}) \leq \max_{\vec{x} \in N'} \delta(\vec{x}, \vec{p})$ for every $\vec{p} \in P$. Hence $P \cup \{ \alpha(f) \}$ has diameter at most $r$, which implies that the partition obtained from $\mathcal{P}$ after adding $\alpha(f)$ to $P$ is a partition of $M_f' \cup \alpha(f)$, which is a completion of $M$, into at most $k$ clusters, each of diameter at most $r$.

Lemma 17. Let $k, r \in \mathbb{N}$, and $M \subseteq \{0,1,\square\}^d$. Then there is a subset $M'$ of $M$ with $R_M \subseteq M'$ satisfying:

(P1) For every $\vec{v} \in M \setminus R_M$ it holds that $|N_r(\vec{v}) \cap M' \setminus R_M| \leq 2^{2|T_M|}(\sum_{t=1}^{r+|T_M|} t!(2^{T_M} k(r + t + |T_M| + 2))^t) + 1$; and

(P2) $M$ has a completion with a partition into at most $k$ clusters of diameter at most $r$ if and only if $M'$ does.

Moreover, $M'$ can be computed in time polynomial in $M$.

Proof. Initially, we set $M'$ to $M$. Then for every $\vec{v} \in M \setminus R_M$ and every $t$ with $1 \leq t \leq r + |T_M|$, we apply Lemma 16 to $\vec{v}$ and $M'$ exhaustively, i.e., as long as $|N| = |N_{|M'|}| \geq 2^{2|T_M|+1}(2^{T_M} k(r + t + |T_M| + 2)^t) + 1$, we use the lemma to find the vector $\vec{f}$, remove it from $M'$ and apply the lemma again. Let $M'$ be the subset of $M$ obtained in this manner. Then $R_M \subseteq M'$ and (P1) clearly holds. Moreover, (P2) follows from Lemma 16.

We can now prove that Diam-Clustering-Completion is fixed-parameter tractable w.r.t. the three parameters.

Theorem 18. Diam-Clustering-Completion is fixed-parameter tractable parameterized by $k + r + \text{cover}(M)$.

Proof. Let $(M, k, r)$ be the given instance of Diam-Clustering-Completion and let $M'$ be the set obtained using Lemma 17. Because $M'$ satisfies (P2), it holds that $(M, k, r)$ and $(M', k, r)$ are equivalent instances of Diam-Clustering-Completion. Moreover, because $M'$ satisfies (P1), we obtain that every cluster of diameter at most $r$ can contain at most $(2^{T_M}(\sum_{t=1}^{r+|T_M|} t!(2^{T_M} k(r + t + |T_M| + 2)^t) + |R_M| + 1))$ vectors, which implies that if $|M'| > k(2^{T_M}(\sum_{t=1}^{r+|T_M|} t!(2^{T_M} k(r + t + |T_M| + 2)^t) + |R_M| + 1))$, we can safely return that $(M, k, r)$ is a No-instance. Consequently, $|M'| \leq k(\sum_{t=1}^{r+|T_M|} t!(k(r + t + 2)^t) + 1)$ and it remains to reduce the number of coordinates for each vector in $M'$. Let $D'$ be the set of coordinates obtained from Lemma 8 for $M'$. Then $(M', k, r)$ and
(M_D, k, r) are equivalent instances of DIAM-CLUSTERING-COMPLETION and moreover we obtain from Lemma 6 that \( \gamma_{\text{max}}(M') \leq 2r - r + |T_M| \). Therefore, we obtain:

\[
|D'| \leq (k\gamma_{\text{max}}(M') + |R_M|(\|M'| - 1))(r' + 1)
\]

\[
\leq (k(2r - r + |T_M|) + |R_M|(\|M'| - 1))(r + 1)
\]

showing that the size of \( D' \) is bounded by our parameter \( k + r + \text{cover}(M) \). Hence \((M_D', k, r)\) is a kernel for \((M, k, r)\) and IN-CLUSTERING-COMPLETION is fixed-parameter tractable parameterized by \( k + r + \text{cover}(M) \).

\[
\]

\section{Extensions and Applications of the Toolkit}

\subsection{Dealing with Unstructured Missing Data: Dispersion-Completion}

In this subsection, we design an algorithm for Dispersion-Completion which, unlike the other algorithms presented in this paper, remains efficient even when the number and placement of unknown entries is not explicitly restricted on the input.

We begin with a simple lemma that allows us to deal with vectors \((i.e., rows)\) with a large number of missing entries. Note that even after exhaustively applying this lemma to remove such vectors, we will still not have as much control over the occurrence of \( \square \)'s as when parameterizing by \( \text{cover}(M) \). For brevity, let a \( k \)-dispersion set be a set containing \( k \) vectors which have pairwise Hamming distance at least \( r + 1 \).

\begin{lemma}
Let \( \mathcal{I} = (M, k, r) \) be an instance of Dispersion-Completion where \( k \geq 1 \) and let \( \vec{v} \in M \) be a vector containing more than \( (k - 1) \cdot (r + 1) \)-many \( \square \)'s. Then \( \mathcal{I} \) is a YES-instance if and only if \( \mathcal{I}' = (M \setminus \{\vec{v}\}, k - 1, r) \) is a YES-instance. Moreover, a completion and \( k \)-dispersion set for \( \mathcal{I} \) can be computed from a completion and \( (k - 1) \)-dispersion set for \( \mathcal{I}' \) in linear time.
\end{lemma}

\begin{proof}
The forward direction is trivial: for any completion \( M^* \) of \( M \) and \( k \)-dispersion set \( S \) in \( M^* \), we can obtain a \((k - 1)\)-dispersion set and completion for \( \mathcal{I}' \) by simply removing \( \vec{v} \) from \( M^* \) and \( S \).

For the backward direction, consider a completion \( M'^* \) of \( M' = M \setminus \vec{v} \) and a \((k - 1)\)-dispersion set \( S = \{s_1, \ldots, s_{k - 1}\} \) in \( M'^* \). Let us choose an arbitrary set \( C \) of \( (k - 1) \cdot (r + 1) \) coordinates in \( \vec{v} \) that all contain \( \square \), and let us then partition \( C \) into \( k \)-many subsets \( \alpha_1, \ldots, \alpha_k \) each containing precisely \( r + 1 \) coordinates. Now consider the vector \( \vec{v}^\alpha \) obtained from \( \vec{v} \) as follows:

- for each \( i \in [k - 1] \) and every coordinate \( j \in \alpha_i \), set \( \vec{v}^\alpha[j] \) to the opposite value of \( \vec{s}_i[j] \) \( (i.e., \vec{v}^\alpha[j] = 1 \text{ if and only if } \vec{s}_i[j] = 0) \);
- for every other coordinate \( j \) of \( \vec{v}^\alpha \), we set \( \vec{v}^\alpha[j] = \vec{v}[j] \) if \( \vec{v}[j] \neq \square \) and \( \vec{v}^\alpha[j] = 0 \) otherwise.

Clearly, \( M^* = M'^* \cup \{\vec{v}^\alpha\} \) is a completion of \( M \). Moreover, since \( \vec{v}^\alpha \) differs from each vector in \( S \) in at least \( r + 1 \) coordinates, \( S \cup \{\vec{v}^\alpha\} \) is a \( k \)-dispersion set in \( M^* \).

Next, we show that instances which are sufficiently large and where each vector only “interferes with” a bounded number of other vectors are easy to solve. For ease of presentation, let \( \zeta(k, r, t) = 3^{(k - 1)(r + 1)} \cdot 4 \cdot \left( (k - 1) \cdot 3^{(k - 1)(r + 1) + r + t} \right)^t \) be the exact meaning of “sufficiently large” in this case.

\begin{lemma}
Let \( \mathcal{I} = (M, k, r) \) be an instance of Dispersion-Completion. If \( |M| \geq k \cdot r \cdot \zeta(k, r, r) \) and \( |N_i(\vec{v})| < \zeta(k, r, t) \) for every \( \vec{v} \in M \) and \( t \leq r \), then a \( k \)-dispersion set in \( \mathcal{I} \) can be found in polynomial time.
\end{lemma}
Proof. One can find a solution to \( \mathcal{I} \) by iterating the following greedy procedure \( k \) times: choose an arbitrary vector \( \vec{v} \), add it into a solution, and delete all other vectors with Hamming distance at most \( r \) from \( \vec{v} \). By the bound on \( |N_i(\vec{v})| \), each choice of \( \vec{v} \) will only lead to the deletion of at most \( r \cdot \zeta(k, r, r) \) vectors from \( M \). Moreover, since \( \delta \) measures the Hamming distance only between known entries, any completion of the missing entries can only increase (and never decrease) the Hamming distance between vectors. Hence, the size of \( M \) together with the bounded size of the Hamming neighborhood of \( \vec{v} \) guarantee that this procedure will find a solution of cardinality \( k \) in \( \mathcal{I} \) which will remain valid for every completion of \( M \).

We can now move on to the main part of the proof: a procedure which either outputs a solution outright or finds an irrelevant vector.

Lemma 21. Let \( \mathcal{I} = (M, k, r) \) be an instance of Dispersion-Completion such that \( |N_i(\vec{v})| \geq \zeta(k, r, t) \) for some vector \( \vec{v} \in M \) and \( t \leq r \) and such that each vector in \( M \) contains at most \((k - 1) \cdot (r + 1)\) \( \Box \)'s. There is a polynomial-time procedure that finds a vector \( \vec{f} \in M \) satisfying the following properties:

- \((M, k, r)\) is a YES-instance if and only if \((M \setminus \{\vec{f}\}, k, r)\) is a YES-instance, and
- A completion and dispersion set for \( \mathcal{I} \) can be computed from a solution and dispersion set for \( \mathcal{I}' \) in linear time.

Proof. We will begin by constructing a set system over the neighborhood of \( \vec{v} \), albeit with a bit more detail than the set systems used previously (which used \( \Delta(\vec{a}, \vec{b}) \) as the set of elements). Let \( Z = \{ z \in [d] \mid \vec{v}[z] = \Box \} \) be the set of coordinates where \( \vec{v} \) is incomplete. Clearly, since \( |N_i(\vec{v})| \geq 3^{(k-1)} \cdot (r+1) \cdot t! \cdot (k-1) \cdot (3(k-1) \cdot (r+1) + r + t) \) and \( |Z| \leq (k-1) \cdot (r+1) \), we can find a subset \( N \subseteq N_i(\vec{v}) \) of vectors whose cardinality is at least \( t! \cdot (k-1) \cdot (3(k-1) \cdot (r+1) + r + t) \) such that all vectors in \( N \) are the same on the coordinates in \( Z \), i.e., \( \forall \vec{x}, \vec{y} \in N : \forall z \in Z : \vec{x}[z] = \vec{y}[z] \).

Now, let \( F \) be a set containing 2 elements for each coordinate \( j \in [d] \setminus Z \) of vectors in \( M \): the element \( \Box_j \) and the element \( D_j \). We construct a set system \( \mathcal{F} \) over \( F \) as follows: for each vector \( \vec{x} \in N \), we add a set \( \vec{x} \) to \( \mathcal{F} \) that contains:

- \( \Box_j \) if and only if \( \vec{x}[j] = \Box \), and
- \( D_j \) if and only if \( \vec{x}[j] \neq \vec{v}[j] \).

Observe that, since \( \vec{x} \) contains at most \((k - 1) \cdot (r + 1)\) \( \Box \)'s by assumption and since \( \vec{x} \) differs from \( \vec{v} \) in at most \( t \) many completed coordinates, every set in \( \mathcal{F} \) has cardinality at most \((k - 1) \cdot (r + 1) + t \). This means we can apply Lemma 1 to find a sunflower \( \mathcal{F}' \) in \( \mathcal{F} \) of cardinality at least \((k - 1) \cdot (3(k-1) \cdot (r+1) + r + t) + 1 \); for ease of presentation, we will identify the elements of \( \mathcal{F}' \) with the vectors they represent. Let \( \vec{f} \) be an arbitrarily chosen vector from \( \mathcal{F}' \); we claim that \( \vec{f} \) satisfies the properties claimed in the lemma, and to complete the proof it suffices to establish this claim.

The backward direction is trivial: if \( \mathcal{I}' \) is a YES-instance then clearly \( \mathcal{I} \) is a YES-instance as well. It is also easy to observe that a completion and dispersion set for \( \mathcal{I} \) can be computed from a solution and dispersion set for \( \mathcal{I}' \) in linear time (adding a vector does not change the validity of a solution). What we need to show is that if \( \mathcal{I} \) is a YES-instance, then so is \( \mathcal{I}' \) (i.e., \( (M \setminus \vec{f}, k, r) \)); moreover, this final claim clearly holds if \( \mathcal{I} \) admits a solution that does not contain \( \vec{f} \).

So, assume that \( M \) admits a completion \( M^* \) which contains a \( k \)-dispersion set \( S = \{ \vec{f}, \vec{s}_1, \ldots, \vec{s}_{k-1} \} \).

Let \( C \) be the core of the sunflower \( \mathcal{F}' \), and note that all vectors in \( \mathcal{F}' \) have precisely the same content in the coordinates in \( C \).

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Finding a replacement for $\vec{f}$. We would now like to argue that, for some completion which we will define later, $\mathcal{F}'$ contains a vector that can be used to replace $\vec{f}$ in the solution. To do so we will use some ideas from Lemmas 3 and 1 in our toolbox, but in a more careful way due to the lack of control over $\square$’s and the more expressive set representation.

Let $\vec{s}_i \in S$ be an arbitrary vectors. First, let us consider the case that, in $M$, $\vec{s}_i$ differs from $\vec{v}$ in more than $3(k-1) \cdot (r+1) + r + t$ coordinates (i.e., $\vec{v}[j] \neq \vec{s}_i[j]$ in $M$ for at least $3(k-1) \cdot (r+1) + r + t$ choices of $j$). Then every vector in $\mathcal{F'}$ will have Hamming distance greater than $r$ from $\vec{s}_i$ regardless of the completion.

Indeed, for every vector $\vec{f}' \in \mathcal{F}'$ there are at most $3(k-1) \cdot (r+1)$ coordinates $j$ such that at least one of $\vec{v}[j], \vec{s}_i[j], \vec{f}'$, meaning that there are at least $t + r$ other coordinates where $\vec{v}$ differs from $\vec{s}_i$ and which are guaranteed to be complete—and since $\delta(\vec{f}', \vec{v}) = t,$ $\vec{f}$ must hold that $\delta(\vec{f}', \vec{s}_i) > r$ (by the triangle inequality). Hence indeed every vector in $\mathcal{F}'$ must have distance at least $r + 1$ from $\vec{s}_i$, and in this case we will create a set $S_i = \emptyset$ (the meaning of this will become clear later).

Now, consider the converse case, i.e., that $\vec{s}_i$ differs from $\vec{v}$ in at most $3(k-1) \cdot (r+1) + r + t$ coordinates. We may now extend the sunflower $\mathcal{F}'$ by adding a set representation of $\vec{s}_i$, i.e., a set $Q_i$ which contains $\square$, if and only if $\vec{s}_i[j] = \square$ and $D_j$ if and only if $\vec{s}_i[j] \neq \vec{v}[i]$ (for all $j \in [d] \setminus Z$).

Observe that $|Q_i| \leq 3(k-1) \cdot (r+1) + r + t$, and in particular $Q_i \setminus C$ intersects with at most $3(k-1) \cdot (r+1) + r + t$ elements of $\mathcal{F}'$. Let $S_i$ be the set of all such elements, i.e., elements of $\mathcal{F}'$ which have a non-empty intersection with $Q_i$ outside of the core (formally, with $Q_i \setminus C$).

To conclude the proof, we will show that there is a completion $M''$ of $M'$ such that any arbitrarily chosen vector $\vec{f}'$ in the non-empty set $\mathcal{F}' \setminus (\{\vec{f}'\} \cup \bigcup_{i \in [k-1]} S_i)$ can replace $\vec{f}$ in the $k$-dispersion set $S$.

Arguing Replaceability. Consider a new completion $M''$ of $M \setminus \vec{f}$ obtained as follows:

- For each vector $\vec{v} \in \mathcal{F}' \setminus S$, we complete
  1. the $\square$’s in $C \cup Z$ precisely in the same way as $\vec{f}$, and
  2. for every other $\square$ at coordinate $j$, we set $\vec{v}[j] = -(\vec{v}[j] - 1)$ (i.e., to the opposite of $\vec{v}$; recall that $\vec{v}[j] \neq \square$ since $j \notin Z$);
- all other $\square$’s in all other vectors in $M \setminus \vec{f}$ are completed in precisely the same way as in $M'$.

Since $M''$ precisely matches $M'$ on all vectors in $S \setminus \vec{f}$, it follows that $S_i \setminus \vec{f}$ is a $(k-1)$-dispersion set in $M''$. Moreover, consider for a contradiction that $\delta(\vec{f}', \vec{s}_i) \leq r$ for some $\vec{s}_i \in S$ after completion, i.e., in $M''$. Then clearly $\vec{s}_i$ could not differ from $\vec{v}$ in more than $3(k-1) \cdot (r+1) + r + t$ coordinates in $M'$, since—as we already argued—in this case every vector in $\mathcal{F}'$ will have Hamming distance greater than $r$ from $\vec{s}_i$ regardless of the completion.

Hence, we must be in the case where $\vec{s}_i$ differed from $\vec{v}$ in at most $3(k-1) \cdot (r+1) + r + t$ coordinates in $M$. Now consider how $\delta(\vec{f}', \vec{s}_i)$ differs from $\delta(\vec{f}, \vec{s}_i)$. First of all, there is no difference between these two distances on the coordinates in $Z \cup C$ due to our construction of $M''$ and choice of $N$. For the remaining coordinates, we will consider separately the set $X$ of coordinates in the petals of $\vec{f}$ and $\vec{f}'$ (i.e., the set $\{j \in [d] \setminus (Z \cup C) \mid \vec{f}'[j] \neq \vec{f}[j] \lor \vec{f}'[j] \neq \vec{v}[j])$), and the set $X = [d] \setminus (C \cup Z \cup X)$ of all remaining coordinates. It follows that $\vec{v}[j] = \vec{f}[j] = \vec{f}'[j]$ for all coordinates $j \in Y$, and hence there is no difference between the two distances on these coordinates either.

So, all that is left is to consider the difference between $\delta(\vec{f}', \vec{s}_i)$ and $\delta(\vec{f}, \vec{s}_i)$ on the coordinates in $X$. Among these coordinates, $\vec{f}$ can only differ from $\vec{s}_i$ in at most $t - |C|$ many coordinates—notably in the coordinates of its own petal—because the coordinates in the petal of $\vec{f}$ do not intersect with $Q_i$. On the other hand, our construction guarantees that $\vec{f}'$ differs from $\vec{s}_i$ in at least $t - |C|$ coordinates in $X$; more precisely, on all coordinates in the petal of $\vec{f}'$, since on these coordinates (1)
\( \bar{s}_i \) is equal to \( \vec{v} \) and (2) \( \vec{f}' \) differs from \( \vec{v} \).

In summary, we conclude that \( \delta(\vec{f}', \bar{s}_i) \geq \delta(\vec{f}, \bar{s}_i) \) and hence \( (S \setminus \{\vec{f}\}) \cup \{\vec{f}'\} \) is a \( k \)-dispersion set in \( M'^* \), as claimed.

We can now establish our main result for Dispersion-Completion.

**Theorem 22.** Dispersion-Completion is fixed-parameter tractable parameterized by \( k + r \).

**Proof.** The algorithm proceeds as follows. Given an instance \( I = (M, k, r) \) of Dispersion-Completion, it first checks whether \( M \) contains a vector with more than \( (k - 1) \cdot (r + 1) \) squares; if yes, it applies Lemma \([19]\) and restarts on the reduced instance. Second, it checks whether \( |M| \geq k \cdot r \cdot \zeta(k, r, r) \); if not, it uses the fact that the number of \( \square \)'s and the number of rows is bounded by a function of the parameter to find a completion and a \( k \)-dispersion set in \( I \) (or determine that one does not exist) by brute force.

Third, it checks whether each vector \( \vec{v} \) satisfies \( |N_t(\vec{v})| < \zeta(k, r, t) \) for every \( t \in [r] \); if yes, then it solves \( I \) by invoking Lemma \([20]\). Otherwise, it invokes Lemma \([21]\) to reduce the cardinality of \( M \) by \( 1 \) and restarts. If the algorithm eventually terminates with a \( "NO" \), then we know that the initial input was a NO-instance; otherwise, it will output a solution which can be transformed into a solution for the original input by the used lemmas.

\[ \blacksquare \]

### 6.2 Finding a Single Large Cluster: Large Diam-Cluster-Completion

In this subsection we show how our techniques developed for In/Any/Diam-Clustering-Completion can be employed and extended for Large Diam-Cluster-Completion. Our main algorithmic results are that Large Diam-Cluster-Completion is fixed-parameter tractable parameterized by \( r + \text{cover}(M) \) and in XP parameterized by \( k \) only. Together with Theorem \([13]\) (showing \( W[1] \)-hardness for Large Diam-Cluster w.r.t. \( k \)) and Theorem \([45]\) (showing \( W[1] \)-hardness for Large Diam-Cluster-Completion w.r.t. \( k \) even for \( r = 0 \)) this gives us a complete complexity landscape for Large Diam-Cluster-Completion parameterized by any combination of the parameters \( k, r, \) and \( \text{cover}(M) \). We also show that Large Diam-Cluster-Completion has a Turing kernel parameterized by \( k + r + \text{cover}(M) \).

#### 6.2.1 Large Diam-Cluster-Completion Parameterized by \( k + r + \text{cover}(M) \)

We start by showing that Large Diam-Cluster-Completion parameterized by \( k + r + \text{cover}(M) \) is fixed-parameter tractable. We will later show an even stronger result, namely, that the same result already holds if we only parameterize by \( r + \text{cover}(M) \). However, showing the result here is important for the following reasons (1) we obtain a Turing kernel whose size is polynomial in \( k \), (2) we use the algorithm here as a subprocedure for our result for \( r + \text{cover}(M) \), and (3) the techniques developed here (namely the two lemmas below) can also be employed for Large Any-Cluster-Completion.

The main approach behind our Turing kernel is to guess to vectors in the cluster of maximum distance. The following lemma shows how this can be used to pre-process the instance (by identifying vectors that cannot be in a cluster with the two vectors of maximum distance). Note that this and the following lemma only deal with the case that the solution cluster uses at least two vectors from \( M \setminus R_M \). In order to make our exposition more concise, we say that a set \( P \) is a Diam-cluster (or \( |P| \)-Diam-cluster) if \( \forall \vec{p}, \vec{q} \in P : \delta(\vec{p}, \vec{q}) \leq r \), and similarly \( P \) is an Any-cluster (or \( |P| \)-Any-cluster) if there exists a vector \( \vec{v} \in \{0, 1\}^d \) such that \( \delta(P, \vec{v}) \leq r \).
Lemma 23. Let \( \vec{v} \) and \( \vec{u} \) be two vectors in \( M \setminus R_M \) and let \( t \) be an integer with \( \delta(\vec{v}, \vec{u}) \leq t \leq \delta(\vec{v}, \vec{u}) + |T_R| \). Then any vector \( \vec{m} \) for which either \( |\Delta(\vec{v}, \vec{m})| > t \), \( |\Delta(\vec{v}, \vec{m})| > t \), or \( |\Delta(\vec{v}, \vec{m}) \cap \Delta(\vec{u}, \vec{m})| > t/2 \) cannot be contained in an Any/Diam-cluster containing \( \vec{v} \) and \( \vec{u} \), in which \( \vec{v} \) and \( \vec{u} \) are two vectors of maximum distance.

Proof. Clearly the lemma holds for any vector \( \vec{m} \) satisfying either \( |\Delta(\vec{v}, \vec{m})| > t \) or \( |\Delta(\vec{v}, \vec{m})| > t \), since we are only looking for clusters where \( \vec{v} \) and \( \vec{u} \) are of distance \( t \) to one another and have maximum distance within the cluster. Hence let \( \vec{m} \) be a vector satisfying \( |\Delta(\vec{v}, \vec{m}) \cap \Delta(\vec{u}, \vec{m})| > t/2 \). Since \( \vec{v} \) and \( \vec{u} \) differ in exactly \( t \) coordinates in \( \Delta(\vec{v}, \vec{u}) \cup T_R \), it follows that \( \vec{m} \) differs in at least \( t/2 \) of those coordinates with either \( \vec{v} \) or \( \vec{u} \). But then \( \vec{m} \) differs in more than \( t \) coordinates from either \( \vec{u} \) or \( \vec{v} \) and hence cannot be part of a cluster with \( \vec{v} \) and \( \vec{u} \) that has a maximum distance of \( t \) between any two vectors.

The following lemma now shows that the instance obtained after identifying two vectors of maximum distance and removing the vectors identified by the previous lemma, cannot have two many vectors; since otherwise it is a Yes-instance.

Lemma 24. Let \( \vec{v} \) and \( \vec{u} \) be two vectors in \( M \setminus R_M \) and let \( t \) be an integer with \( \delta(\vec{v}, \vec{u}) \leq t \leq \delta(\vec{v}, \vec{u}) + |T_R| \). Let \( M' \) be the set of all vectors in \( M \setminus (R_M \cup \{\vec{v}, \vec{u}\}) \) obtained after removing all vectors \( \vec{m} \) for which either \( |\Delta(\vec{v}, \vec{m})| > t \), \( |\Delta(\vec{v}, \vec{m})| > t \), or \( |\Delta(\vec{v}, \vec{m}) \cap \Delta(\vec{u}, \vec{m})| > t/2 \). If \( |M'| > k3^{(T_M + t)} + |R_M| \), then \( M' \) contains a Diam-cluster of size at least \( k \) and diameter \( t \), which is also an Any-cluster of radius \( t/2 \).

Proof. Assume that there are at least \( k3^{(T_M + t)} \) vectors \( \vec{m} \) in \( M' \). Then there is a set \( S \subseteq M' \) of size at least \( k \) that agrees on all the coordinates in \( \Delta(\vec{v}, \vec{u}) \cup T_M \); since \( |\Delta(\vec{v}, \vec{u}) \cup T_M| \leq t + |T_M| \) and there are at most \( 3^{(T_M + t)} \) possible assignments to these coordinates. Since \( \vec{v} \) and \( \vec{u} \) are equal on all other coordinates, i.e., the coordinates in \( [d] \setminus (\Delta(\vec{v}, \vec{u}) \cup T_M) \), we obtain that any such vector \( \vec{m} \) disagrees with \( \vec{v} \) (or \( \vec{u} \)) on at most \( t/2 \) of these coordinates; since otherwise \( |\Delta(\vec{v}, \vec{m}) \cap \Delta(\vec{u}, \vec{m})| > t/2 \) and we would have removed \( \vec{m} \) from \( M \setminus (R_M \cup \{\vec{v}, \vec{u}\}) \). Hence, after completing all \( k \) vectors in \( S \) in the same way, which is possible since they agree on the coordinates in \( T_M \), we obtain that the pairwise distance between any two of these vectors is at most \( t \). But then these vectors form a Diam-cluster. Moreover, the vectors also form an Any-Cluster as witnessed by the center that is 0 at all coordinates in \( [d] \setminus (\Delta(\vec{v}, \vec{u}) \cup T_M) \) and equal to any (every) vector in \( S \) at the coordinates in \( (\Delta(\vec{v}, \vec{u}) \cup T_M) \).

We are now ready to show that Large Diam-Cluster-Completion has a Turing kernel parameterized by \( k + r + \text{cover}(M) \).

Theorem 25. Large Diam-Cluster-Completion parameterized by \( k + r + \text{cover}(M) \) has a Turing-kernel containing at most \( m = k3^{(T_M + r)} + |R_M| + 2 \) vectors each having at most \( \max\{rm - 1, |T_M|, \binom{|R_M|}{2}(r + 1)\} \) coordinates.

Proof. We distinguish three cases: (1) the solution cluster contains at least two vectors in \( M \setminus R_M \), (2) the solution cluster contains exactly one vector in \( M \setminus R_M \), and (3) the solution cluster does not contain any vector in \( M \setminus R_M \).

We start by showing the result for case (1). Our first aim is to reduce the number of vectors in \( M \setminus R_M \). As a first step, we guess two vectors \( \vec{v} \) and \( \vec{u} \) of \( M \setminus R_M \), that are farthest apart in the cluster w.r.t. to all vectors in \( M \setminus R_M \); this is possible since we assume that the cluster contains at least two vectors in \( M \setminus R_M \). We then guess the exact distance, say \( t \), between \( \vec{v} \) and \( \vec{u} \) in a completion leading to the cluster. Note \( t \) can be anywhere between \( \delta(\vec{v}, \vec{u}) \) and \( \min\{\delta(\vec{v}, \vec{u}) + |T_R|\} \),
we guessed that \( P \) contains \( |\mathcal{P}| \) elements.

The completion of at least two vectors in \( \mathcal{M} \) algorithm for every sub-instance of the Turing kernel. Hence, we can assume that

\[
\text{Algorithm in Theorem 25 to obtain a Turing kernel, where we can run any exponential time size \( \text{Diam} \) by just} \tag{6.2.2}
\]

\[
\text{cover} \]

\[
\text{Large Diam-Cluster-Completion parameterized by} \ r + \text{cover}(M) \]

In this subsection, we show that \( \text{Large Diam-Cluster-Completion} \) is actually FPT parameterized by just \( r + \text{cover}(M) \). We will give an algorithm that finds a completion \( M^* \) of \( M \) and a maximum size \( \text{Diam-Cluster} P^* \) in \( M^* \). We first discuss some initial branching and pre-processing that is exhaustive and introduces some structure into the instance.

Let \( (M,k,r) \) be an instance of \( \text{Large Diam-Cluster-Completion} \). If \( k \leq |R_M| + 2 \), we use the algorithm in Theorem 25 to obtain a Turing kernel, where we can run any exponential time algorithm for every sub-instance of the Turing kernel. Hence, we can assume that \( P^* \) contains a completion of at least two vectors in \( M \setminus R_M \). We can guess the subset \( P_R \) of \( R_M \) that will be completed to a vector in \( P^* \) and restrict our attention to finding a \( \text{Diam-Cluster} \) in \( M \setminus R_M \) of size \( |P^* \setminus R_M| \) that is compatible with \( P_R \). Towards finding a cluster in \( M \setminus R_M \), we first guess two elements \( \vec{v} \) and \( \vec{u} \), together with their completions \( \vec{v}^* \) and \( \vec{u}^* \), of \( M \setminus R_M \), such that \( \vec{v}^* \) and \( \vec{u}^* \) are both in \( P^* \) and are the farthest vectors apart in \( P^* \setminus R_M \); fix \( r_{\max} = \delta(\vec{v}^*, \vec{u}^*) \). We also remove all other vectors that can be completed into \( \vec{v}^* \) or \( \vec{u}^* \), and reduce \( k \) accordingly, and hence, we do not
keep duplicates of the two vectors that we already know to be in $P^*$. We then normalize all vectors in $M$ so that $\vec{v}$ becomes the all-zero vector, i.e., we replace $\vec{v}$ by the all-zero vector, and for every other vector $\vec{w} \neq \vec{v}$, we replace it with the vector $\vec{w}'$ such that $\vec{w}'[i] = 0$ if $\vec{v}[i] = \vec{w}[i]$, $\vec{w}'[i] = \Box$ if $\vec{w}'[i] = \Box$, and $\vec{w}'[i] = 1$, otherwise. Finally, for each vector $\vec{w} \in M \setminus R_M$, we compute the set $\Lambda(\vec{w})$ of all completions of $\vec{w}$ at distance at most $r_{\text{max}}$ from both $\vec{v}$ and $\vec{v}$. Note that $\Lambda(\vec{w})$ can be computed in $O(2^{\ell_M} \cdot d)$ time for each vector in $M \setminus R_M$, where $d$ is the dimension of the vectors in $M$. We then remove all vectors $\vec{w}$ with $\Lambda(\vec{w}) = \emptyset$ from $M \setminus R_M$.

We will extend the notation of $\Lambda(\vec{w})$ to $\Lambda_\ell(\vec{w})$, for a multiset $C$ of vectors from $\{0, 1\}^d$, such that $\Lambda_C(\vec{w})$ is the set of all completions of vector $\vec{w}$ at distance at most $r_{\text{max}}$ to all vectors in $C$. We are now ready to show that after normalizing vectors in $P^*$, the multiset $P^* \setminus R_M$ is an $r$-uniform subset of $M^*$.

**Lemma 26.** Let $(M, k, r)$ be an instance of **Large Diam-Cluster-Completion**, let $M^*$ be a completion of $M$ and let $P^*$ be a Diam-cluster in $M^*$ of maximum size such that $\vec{0} \in P^*$. Then for every $N \subseteq M^*$, $P^* \setminus N$ is an $r$-uniform subset of $M^* \setminus N$.

**Proof.** Let $t \in \mathbb{N}$ and $S \subseteq (P^* \setminus N)$ such that $|S| > r$, for all $\vec{w} \in S$ it holds that $|\Delta(\vec{w})| = t$, and $\mathcal{F} = \Delta(S)$ is a sunflower with core Let $\vec{m}$ be an arbitrary $t$-vector in $M^* \setminus (N \cup P^*)$ containing $X$ and let $\vec{c}$ be an arbitrary vector in $P^*$. Then $|\Delta(\vec{c})| = \delta(\vec{0}, \vec{c}) \leq r$ and it follows from Lemma 2 that $S$ contains a vector $\vec{n}$ that has maximum distance to $\vec{c}$ among all $t$-vectors in $M^*$ containing $X$. Hence, the distance between $\vec{m}$ and $\vec{c}$ is at most that between $\vec{n}$ and $\vec{c}$, which, since both $\vec{n}$ and $\vec{c}$ are in $P^*$, is at most $r$. Consequently, the distance between $\vec{m}$ and any vector in $P^*$ is at most $r$, contradicting the maximality of $P^*$. 

Since $P^* \setminus N$ is an $r$-uniform subset of $M^* \setminus N$, by Corollary 10 applied separately for each $r' \in [r]$, there exists a set $S = \{(S_1, r_1), \ldots, (S_{\ell}, r_{\ell})\}$, with $\ell \leq \sum_{r' \in [r]}(rr')^{r'} \leq r^{2r+1}$, such that $P^* \setminus N$ contains precisely all the vectors $\vec{w}$ in $M^*$, such that for some $(S_i, r_i)$, $i \in [\ell]$, $S_i \in \Delta(\vec{w})$ and $|\Delta(\vec{w})| = r_i$.

We now introduce some notations. We call the pair $(S_i, r_i)$ an $r_i$-center (of $P^* \setminus N$ in $M^* \setminus N$). We say that a vector $\vec{w} \in \{0, 1\}^d$ is compatible with $r_i$-center $(S_i, r_i)$ if $\vec{w} \in S_i$ and $S_i \subseteq \Delta(\vec{w})$ and $\vec{w} \in \{0, 1\}^d$ is compatible with $S$ if it is compatible with some $(S_i, r_i) \in S$. Moreover, for a set $S = \{(S_1, r_1), \ldots, (S_{\ell}, r_{\ell})\}$ and a multiset $C$ of vectors from $\{0, 1\}^d$, we say that $S$ defines $C$, if every vector $\vec{c} \in C$ is compatible with $S$ and for every $(S_i, r_i) \in S$ there is a vector in $C$ compatible with $(S_i, r_i)$. We say $S$ properly defines $C$, if $|S| \leq r^{2r+1}, S$ defines $C$, and for every $(S_i, r_i) \in S$ either $|S_i| = r_i$ and the unique vector that is compatible with $(S_i, r_i)$ is in $C$ or $|S_i| < r_i$ and $C$ contains a set $N$ of $r + 1$ $r_i$-vectors such that $\Delta(N)$ forms a sunflower. Note that if every vector in $C$ has at most $r_{\text{max}}$ 1’s, then since $C$ is an $r$-uniform subset of $C$, it follows from Corollary 10 that there always exists a set $S$ that properly defines $C$.

**Observation 27.** Let $C$ be a multiset of vectors from $\{0, 1\}^d$, with $\max_{\vec{v} \in C} |\Delta(\vec{v})| \leq r$. Then there exists a set $S$ of at most $r^{2r+1}$ $r_i$-centers that properly defines $C$.

Before we continue with our FPT-algorithm, we first note that the above lemma together with Corollary 10 already give an XP-algorithm due to the following argument. All vectors in $P^*$ have at most $r$ ones. Hence there are at most $q^{O(r^{2r+2})}$ possibilities for the set $S$ that properly defines $P^*$ and we can enumerate all of them. Let $S$ be the correct guess. For each $(S_i, r_i) \in S$ such that $|S_i| = r_i$, there is only one possible $r_i$-vector that contains $S_i$. If our guess is correct, then $P^*$ contains at least one, and by maximality of $P^*$, $P^*$ has to contain all vectors in $M$ that can be completed to this particular vector. If $(S_i, r_i) \in S$ such that $|S_i| < r_i$, then $P^*$ contains a sunflower containing $r_i$-vectors of size at least $r + 1$. Clearly, $P^*$ contains all the vectors that can be completed
to an \( r_i \)-vector containing \( S_i \), because Lemma 26 holds for any completion \( M^* \) of \( M \) that contains \( P^* \) as a subset. We claim that we can complete such vectors arbitrarily. The proof of the claim is given in the lemma below, which will also be useful for our \( \text{FPT} \)-algorithm. This then concludes \( \text{XP} \)-algorithm, as we can complete the vectors not selected to the \( \text{Diam} \)-cluster arbitrarily.

**Lemma 28.** Let \((M, k, r)\) be an instance of \( \text{Large Diam-Cluster-Completion}, M^* \) a completion of \( M \), \( P^* \) a \( \text{Diam} \)-cluster in \( M^* \) of maximum size with \( \vec{0} \in P^* \), and \( N \subseteq M^* \). If \( S \) properly defines \( P^* \setminus N \), then for every pair of vectors \( \vec{w}_1, \vec{w}_2 \in \{0,1\}^d \) compatible with \( S \) it holds that \( \delta(\vec{w}_1, \vec{w}_2) \leq r \).

**Proof.** Let \( \vec{w}_i, i \in \{1, 2\}, \) be compatible with the \( r_i \)-center \((S_i, r_i)\). If \( |S_i| < r_i, i \in \{1, 2\}, \) then let \( i \subseteq P^* \setminus N \) be set of \( r_i \)-vectors such that \( F_i = \Delta(N_i) \) is a sunflower with core \( S_i \). Note that such a set \( N_i \) always exists by the definition of \( S \). Moreover, if \( |S_i| = r_i \), then \( \vec{w}_i \) is uniquely defined and already in \( P^* \setminus N \). Hence, if both \( |S_1| = r_1 \) and \( |S_2| = r_2 \), then the lemma holds.

If \( |S_1| = r_1 \) and \( |S_2| \leq r_2 \), then \( P^* \setminus N \) contains a vector identical to \( \vec{w}_1 \), \( \Delta(\vec{w}_1) = r_1 \leq r \) and by Lemma 2 there is a vector \( \vec{v}_2 \) in \( N_2 \) that has the maximum distance to \( \vec{w}_1 \) among all \( r_2 \)-vectors that contain \( C_2 \) and \( \delta(\vec{w}_1, \vec{v}_2) \leq r \). The case where \( |S_1| < r_1 \) and \( |S_2| = r_2 \) is symmetric.

Finally, if both \( |S_1| < r_1 \) and \( |S_2| < r_2 \), then for every \( \vec{v}_1 \in N_1 \) is \( \Delta(\vec{v}_1) = r_1 \leq r \) and by Lemma 2 there is a vector \( \vec{v}_2 \) in \( N_2 \) that has the maximum distance to \( \vec{v}_1 \) among all \( r_2 \)-vectors that contain \( C_2 \). Hence, \( \vec{w}_2 \) has distance at most \( r \) to all vectors in \( N_1 \). Now \( \Delta(\vec{w}_2) = r_2 \leq r \) and by Lemma 2 there is a vector \( \vec{v}_1 \) in \( N_1 \) that has the maximum distance to \( \vec{w}_2 \) among all \( r_1 \)-vectors that contain \( C_1 \). Since \( \delta(\vec{w}_2, \vec{v}_1) \leq r \), it follows that \( \delta(\vec{w}_1, \vec{w}_2) \leq r \). □

We are now ready to give our iterative sunflower harvesting procedure, which allows us to obtain the \( \text{FPT} \)-algorithm. Namely, we show that instead of enumerating all \( d^{2r+2} \) possible sets \( S \) of \( r_i \)-centers to find the one that properly defines \( P^* \setminus R_M \), it suffices to enumerate only \( f(r, |T_M|) \) “important” \( r_i \)-centers for each choice of \( \vec{u}^* \) and \( \vec{w}^* \), where \( f \) is some function that depends only on \( r \) and \( |T_M| \). Moreover, we can enumerate these possibilities in \( \text{FPT} \)-time. The idea is to enumerate possible sets \( S \) such that we compute pairs \((S_i, r_i)\) in \( S \) ordered by size from largest to smallest, give a lower bound on the size of the next largest \( r_i \)-center, and show that only \( g(r, |T_M|) \) such large \( r_i \)-center exist, for some function \( g \). Note that if we guess an \( r_i \)-center \((S_i, r_i)\) with \( |S_i| < r_i \), then if our guess is correct, we can, by Lemma 28, already complete the vectors that can be completed to \( r_i \)-vector containing \( S_i \) and include all of them in the \( \text{Diam} \)-cluster. The following lemma shows that if we have some partial \( \text{Diam} \)-cluster \( C \) computed, then we can pre-process the remaining instance such that it contains a \( \text{Diam} \)-cluster compatible with \( C \) of size at least \( 2^r \) fraction of the remaining instance. This gives us that the intersection of \( P^* \) and the remaining instance has to be large, and in turn, implies that one of the remaining \( r_i \)-centers or \( P^* \) has size roughly at least \( 2^r 2^{r+1} - |R_M| \) fraction of the remaining instance. Recall that for a vector \( \vec{w} \in \{0,1,\Xi\}^d \) and multiset \( C \) of vectors from \( \{0,1\}^d \), \( \Lambda_C(\vec{w}) \) denotes the set of all completions of vector \( \vec{w} \) at distance at most \( r_{\text{max}} \) to all vectors in \( C \), i.e., \( \max_{\vec{c} \in C} \{\delta(\vec{c}, \vec{w})\} \leq r_{\text{max}} \).

**Lemma 29.** Let \( C \) be a \( \text{Diam} \)-cluster of vectors in \( \{0,1\}^d \) such that \( \{\vec{0}, \vec{u}\} \subseteq P \) for some \( r_{\text{max}} \)-vector \( \vec{u} \in \{0,1\}^d \), \( r_{\text{max}} \in [r] \), such that \( \delta(\vec{a}, \vec{b}) \leq r_{\text{max}} \) for every distinct \( a, b \in C \), and let \( M' \) be a multiset of vectors \( \vec{w} \) with \( \Lambda_C(\vec{w}) \neq \emptyset \). Then there exists \( P \subseteq M' \) and a completion \( P^* \) of \( P \) such that \( |P| \geq |M'|/2^{r_{\text{max}}} \) and \( P \cup C \) is a \( \text{Diam} \)-cluster of diameter \( r_{\text{max}} \).

**Proof.** Fix a completion \( M^* \) of \( M' \) such that each vector \( \vec{w} \in M' \) is completed to \( \vec{c}_{\vec{w}} \in \Lambda_C(\vec{w}) \). Now let \( S \subseteq \Delta(\vec{u}) \) and let \( M^*_S \) be the subset of \( M^* \) containing precisely all the vectors \( \vec{w} \) such that \( \Delta(\vec{w}) \cap \Delta(\vec{u}) = S \). We claim that \( C \cup M^*_S \) is a \( \text{Diam} \)-cluster. First note that \( \delta(\vec{a}, \vec{b}) \leq r_{\text{max}} \) for every distinct \( a, b \in C \) and also is true if \( \vec{a} \in C \) and \( \vec{b} \in M^*_S \). It remains to show that \( \delta(\vec{a}, \vec{b}) \leq r_{\text{max}} \).
for every distinct \( \vec{a}, \vec{b} \in M'_S \). Let \( |\Delta(\vec{a})| = a \) and \( |\Delta(\vec{a})| = b \). Then \( \delta(\vec{a}, \vec{b}) \leq a + b - 2|S| \). Now \( a \leq r_{max} \) and \( b \leq r_{max} \) because \( \vec{0} \in C \) and hence \( a - |S| \leq r_{max} - |S| \) and \( b - |S| \leq r_{max} - |S| \). Moreover \( \delta(\vec{a}, \vec{a}) = r_{max} - |S| + a - |S| \leq r_{max} \). Therefore, \( a - |S| \leq |S| \) and similarly \( b - |S| \leq |S| \). Hence \( \delta(\vec{a}, \vec{b}) \leq a + b - 2|S| \leq \min(2|S|, 2r_{max} - 2|S|) \). It follows that either \( |S| \leq r_{max}/2 \) and \( \delta(\vec{a}, \vec{b}) \leq r_{max} \) or \( |S| \geq r_{max}/2 \) and \( \delta(\vec{a}, \vec{b}) \leq 2r_{max} - 2|S| \leq r \). Because there are \( 2^{r_{max}} \) possibilities for the set \( S \) and each vector in \( M' \) is in one of sets \( M'_S \) and lemma follows.

As a corollary, we obtain the following lemma. Before we state the lemma, we introduce some additional notation that will be useful. Let \( S = \{(S_1, r_1), \ldots, (S_t, r_t)\} \) and let \( \vec{w} \in \{0, 1, \square\} \). We say that \( \vec{w} \) is compatible with \( S \) if there exists a completion \( \vec{w}^* \in \{0, 1\}^d \) of \( \vec{w} \), called witness of compatibility, and a pair \( (S_i, r_i) \in S \) such that \( S_i \subseteq \Delta(\vec{w}^*) \) and \( |\Delta(\vec{w}^*)| = r_i \). If \( \vec{w} \) is compatible with \( S \), we will denote by \( \zeta^S(\vec{w}) \) the set of witnesses of compatibility for \( \vec{w} \) and \( S \). If \( \vec{w} \) is compatible with \( \{(S', r')\} \), we simply say \( \vec{w} \) is compatible with \( (S', r') \).

**Lemma 30.** Let \( P^* \) be a maximum \( \text{DIAM-cluster} \) in \((M, k, r)\), \( S \) the set of \( r_i \)-centers that properly define \( P^* \setminus R_M, S' \subseteq S \), let \( C' \) be the multiset of vectors \( \vec{w} \in M \setminus R_M \) with \( \Delta(\vec{w}) \neq \emptyset \) and \( C \) the multiset containing a vector \( \vec{w} \in \zeta^S(\vec{w}) \) for every \( \vec{w} \in C' \). Let \( M' \) be the multiset consisting of all the vectors \( \vec{w} \in M \setminus (C \cup R_M) \) with \( \Delta(\vec{w}) \neq \emptyset \). Then there exists \( (S_i, r_i) \subseteq S \setminus S' \) such that at least \(|M'|/2^{r_{max}} - |R_M|/|r_{max} + 1| \) vectors in \( M' \) are compatible with \( (S_i, r_i) \).

**Proof.** By Lemma 29, \( M' \) contains a \( \text{DIAM-cluster} \) \( P' \) of size at least \(|M'|/2^{r_{max}} \) such that \( C \cup P' \) can be completed to a \( \text{DIAM-cluster} \). From Lemma 28 it follows that every vector in \( P^* \setminus (R_M \cup C') \) is at distance at most \( r_{max} \) to every vector in \( C \) and hence it is in \( M' \). Hence \( C \cup (P^* \setminus (R_M \cup C')) \) is a \( \text{DIAM-cluster} \) of size \(|P^* \setminus R_M| \). Because \( C \cup P' \) can be completed to a \( \text{DIAM-cluster} \), it follows from maximality of \( P^* \) that \(|P^* \setminus (R_M \cup C')| \geq |P' \setminus R_M| \geq |M'|/2^{r_{max}} - |R_M| \). But every vector in \( P^* \setminus (R_M \cup C') \) is compatible with some \( (S_i, r_i) \in S \setminus S' \) and \(|S \setminus S'| \leq |S| \leq 2^{r_{max} + 1} \) and the lemma follows.

Now we are ready to describe the algorithm that enumerates all “important” sets of \( r_i \)-centers. The algorithm is given by calling procedure \( \text{FindClusters}(\{\xi^S(\vec{w})\}, M, \emptyset) \) described in Algorithm 1.

**Lemma 31.** Let \((M, k, r)\) be an instance of \( \text{LARGE DIAM-CLUSTER-COMPLETION} \). Then the procedure \( \text{FindClusters}(\{\xi^S(\vec{w})\}, M \setminus R_M, \emptyset) \) runs in \( \text{FPT-time parameterized by } r \) plus \( \text{cover}(M) \).

**Proof.** We will prove by induction on \( r_{2^{r_{max} + 1} - |S|} \) that the algorithm \( \text{FindClusters}(C \cup \{\xi^S(\vec{w})\}, M \setminus R_M, S) \) runs in time \( (r_{2^{r_{max} + 1} - 2^{2^{r_{max}} + |T_M|} \cdot |R_M|})^{2^{r_{max} + 1} - |S|} \cdot |M|^3 \cdot |d| \).

Clearly, if \( r_{2^{r_{max} + 1} - 1 - |S|} \leq 0 \) the algorithm in constant time outputs \( \emptyset \). Otherwise, it proceeds to compute the set \( M' \). Note that \( \vec{w} \in M \setminus R_M \) has at most \( |T_M| \) \( S_i \)'s and at most \( 2^{2^{r_{max} + 1} - |T_M|} \cdot |M'| \) \( S_i \)'s pairs \( (S_i, r_i) \). For each we can compute sets \( V \) and \( V' \) in time \( O(2^{2^{r_{max} + 1} - |C|} \cdot |M'| \cdot |C|) \). If \( |V'| < (|M'|/2^{r_{max}} - |R_M|)/2^{r_{max} + 1} \), the algorithm continues to different different choice of \((S, r_i')\) else it calls subroutine \( \text{FindClusters}(C \cup \{\xi^S(\vec{w})\} \cup V, M \setminus R_M, S \cup \{(S, r_i')\}) \). It remains to show that the subroutine \( \text{FindClusters}(C \cup \{\xi^S(\vec{w})\} \cup V, M \setminus R_M, S \cup \{(S, r_i')\}) \) is called in at most \( r_{2^{r_{max} + 1} \cdot 2^{2^{r_{max} + 1} - |T_M|} \cdot |R_M| \cdot r_{2^{r_{max} + 1} - |S|} \) branches, because the running time we get from our inductive hypothesis for \( \text{FindClusters}(C' \cup \{\xi^S(\vec{w})\}, M \setminus R_M, S', \emptyset) \), with \( |S'| = |S| + 1 \) dominates the running time so far.

\footnote{With a slight abuse of notation we will allow set operations between sets of completed and uncompleted vectors assuming the natural bijection between completed and uncompleted vectors.}

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\textbf{ALGORITHM 1:} The procedure \texttt{FindClusters}(C, M, S).

\textbf{Data:} a multiset $C$ of vectors from $\{0,1\}^d$, a multiset $M$ of vectors from $\{0,1,\Box\}^d$ with at most $r_{\text{max}}$ ones, and a set $S = \{(S_1,r_1), (S_2,r_2), \ldots, (S_q,r_q)\}$. Moreover, each vector $\vec{c} \in C$ is associated with a distinct vector $\vec{w}_c \in M$ such that $c$ is a completion of $\vec{w}_c$.

\textbf{Result:} A set $C = \{(C_1,S_1), \ldots, (C_t,S_t)\}$

1. if $|S| > r^{2r+1}$ then  
   2. return $\emptyset$;
3. end
4. $C = \emptyset$;
5. Let $M'$ be the set of vectors $\vec{w} \in M \setminus C$ with $\Lambda_C(\vec{w}) \neq \emptyset$;
6. if $|M'| \leq 2^r \cdot |R_M| + 1$ then
   7. end
   8. $C = \emptyset$;
9. foreach $S$ such that there exists $\vec{w} \in M'$ with $S \subseteq \{i \mid \vec{w}[i] = 1 \lor \vec{w}[i] = \Box\}$ do
10.   \hspace{1em} foreach $r' \in \{S, \ldots, r\}$ do
11.     \hspace{2em} Let $V$ be the multiset of vectors $\vec{a} \in M'$ with $\zeta^*(S,r')(\vec{a}) \cap \Lambda_C(\vec{a}) \neq \emptyset$;
12.     \hspace{2em} Let $V'$ be the multiset that contains for each $\vec{a} \in V$ arbitrary one vector $\vec{a}' \in \zeta^*(S,r')(\vec{a}) \cap \Lambda_C(\vec{a})$;
13.     \hspace{2em} if $|V'| \geq (|M'|/2^r - |R_M|)/r^{2r+1}$ then
14.         \hspace{3em} Let $C = C \cup \text{FindClusters}(C \cup V', M, S \cup (S,r'))$;
15.     \hspace{2em} end
16.   \hspace{1em} end
17. end
18. return $C$;

To do so, let us fix a vector $\vec{w} \in M'$ and let us compute in how many sets $V$ it can appear. Vector $\vec{w}$ can appear in $V$ for some pair $(S,r')$ only if $\zeta^*(S,r')(\vec{w}) \neq \emptyset$. That can only happen if $S \subseteq \{i \mid \vec{w}[i] = 1 \lor \vec{w}[i] = \Box\}$. As $|\{i \mid \vec{w}[i] = 1 \lor \vec{w}[i] = \Box\}| \leq r_{\text{max}} + |T_M|$ it follows that $\vec{w}$ can appear in at most $2^{r+|T_M|} |M'|$ multisets. Hence, the sum of sizes of multisets $V$ for all pairs $(S,r')$ is at most $2^r |T_M| |M'|$ and the lemma follows by distinguishing between two cases depending on whether $|M'| \leq 2^r |R_M|$ or $|M'| > 2^r |R_M|$.

\textbf{Lemma 32.} Let $(M,k,r)$ be an instance of \textsc{Large Diam-Cluster-Completion} and let $P^*$ be a \textsc{Diam-Cluster} of maximum size with $\{\vec{0}, \vec{a}\} \subseteq P^*$ for some $r$-vector $\vec{a} \in \{0,1\}^d$. Then $\text{FindClusters}(\{\vec{0}, \vec{a}\}, M \setminus R_M, \emptyset)$ contains a pair $(C \cup \{\vec{0}, \vec{a}\}, S)$ such that $P^* \setminus (R_M \cup \{\vec{0}, \vec{a}\})$ is properly defined by some superset of $S$ and $C$ is a \textsc{Diam-Cluster} defined by $S$ of size $|P^* \setminus (R_M \cup \{\vec{0}, \vec{a}\})|$.

\textbf{Proof.} Let $P \subseteq M \setminus (R_M \cup \{\vec{u}, \vec{v}\})$ be the multiset of vectors that are completed to vectors in $P^*$, and let $S$ properly define $P^* \setminus (R_M \cup \{\vec{u}, \vec{v}\})$, and $S' \subseteq S$. Let $C'$ be the set of vectors $\vec{w} \in P$ with $\zeta^{S'}(\vec{w}) \neq \emptyset$. We will show the following claim that holds whenever $P \setminus C' \neq \emptyset$.

\textbf{Claim 1.} $\text{FindClusters}(C^* \cup \{\vec{u}, \vec{v}\}, M \setminus R_M, S')$ for some multiset $C^*$ containing a vector $\vec{w}_c \in \zeta^S(\vec{w})$ for every $\vec{w} \in C'$ calls as a subroutine $\text{FindClusters}(C^{**} \cup \{\vec{u}, \vec{v}\}, M \setminus R_M, S' \cup (S_i,r_i))$, where $(S_i,r_i) \in S \setminus S'$ and $C^{**}$ contains a vector $\vec{w}_{c'} \in \zeta^{S' \cup (S_i,r_i)}(\vec{w})$ for every vector $\vec{w} \in P$ with $\zeta^{S' \cup (S_i,r_i)}(\vec{w}) \neq \emptyset$.

\textbf{Proof of Claim.} Let $M'$ be the set of vectors computed on line 5. Note that every vector in $P' \setminus C'$ is compatible with $S \setminus S'$ and hence by Lemma 28 it is in $M'$. Therefore, we will enumerate over all $r_i$-centers in $S \setminus S'$ on lines 9 and 10. Let $V$ and $V'$ be the multisets computed on lines 11 and 12 respectively. Then $C' \cup V$ contains all vectors $\vec{w}$ in $P$ with $\zeta^{S' \cup (S_i,r_i)}(\vec{w}) \neq \emptyset$. Moreover, $C^* \cup V'$
contains a vector \( \vec{w}_c \in \zeta(S \cup \{(S, r_i)\})(\vec{w}) \) for every vector \( \vec{w} \in P \) with \( \zeta(S \cup \{(S, r_i)\})(\vec{w}) \neq \emptyset \). By Lemma 30 for some \((S_i, r_i) \in S' \) it follows from inductive application of Claim 1 that \( \text{FindClusters}(\{\vec{0}, \vec{u}\}, M \setminus R_M, \emptyset) \) calls as a subroutine \( \text{FindClusters}(C \cup \{\vec{0}, \vec{u}\}, M \setminus R_M, S') \), where \( C \) contains a vector \( \vec{w}_C \in \zeta(S)(\vec{w}) \) for every vector \( \vec{w} \in P \). It follows that \(|C| = |P| = |P^* \setminus (R_M \cup \{\vec{0}, \vec{u}\})| \) and \( C \) is by its definition defined by \( S' \subseteq S \). Since \( P^* \setminus (R_M \cup \{\vec{0}, \vec{u}\}) \) is properly defined by \( S' \), it follows from Lemma 28 that \( C \) is a \text{Diam}-cluster. It remains to show that \( \text{FindClusters}(C \cup \{\vec{0}, \vec{u}\}, M \setminus R_M, S') \) actually adds the pair \((C \cup \{\vec{0}, \vec{u}\}, S')\) to the output \( C \) on line 7.

Let \( M' \) be the set of vectors computed on line 5. If \(|M'| \geq 2r \cdot |R_M| + 1\), then by Lemma 29 there is a \text{Diam}-cluster of size \(|C \cup \{\vec{0}, \vec{u}\}| + |R_M| + 1 > |P^*|\), which contradicts the choice of \( P^* \). Hence the condition on line 6 is satisfied and the pair \((C, S)\) is added to \( C \).

**Theorem 33.** \text{Large Diam-Cluster-Completion} is fixed-parameter tractable parameterized by \( r + \text{cover}(M) \).

**Proof.** Let \((M, k, r)\) be an instance of \text{Large Diam-Cluster-Completion}. If \( k \leq \text{cover}(M) + 2 \), we run the algorithm from Theorem 25. Otherwise, every maximum size \text{Diam}-cluster contains at least two vectors in \( M \setminus R_M \). Let us fix some maximum cluster \( P \) in \((M, k, r)\) together with its completion \( P^* \). We branch over all pairs of vectors in \( \vec{v}, \vec{u} \in M \setminus R_M \) as the two vectors completed to two vectors that are farthest apart in \( P^* \). We branch over all completions \( \vec{v}^* \) and \( \vec{u}^* \) of vectors \( \vec{v} \) and \( \vec{u} \) and normalize the instance so that \( \vec{v}^* = \vec{0} \). In the correct branch it holds after normalization that \( \vec{v}^*, \vec{0} \in P^* \). We fix \( r_{\text{max}} = |\Delta(\vec{v}^*)| \). Let \( S \) be the set that properly defines \( P^* \setminus (R_M \cup \{\vec{0}, \vec{u}^*\}) \).

By Lemmas 31 and 32 the procedure \( \text{FindClusters}(\{\vec{0}, \vec{u}\}, M \setminus R_M, \emptyset) \) computes in \( \text{FPT} \)-time a set \( C \) of pairs of form \((C_i, S_i)\) such that there exists \((C_i, S_i) \in C \) such that \(|C_i| = |P^* \setminus R_M| \) and some subset \( S' \subseteq S \) defines \( C_i \setminus \{\vec{0}, \vec{u}^*\} \). It follows from Lemma 28 that \( C_i \cup (P^* \cap R_M) \) is \text{Diam}-cluster. For each \((C_i, S_i) \in C \), we enumerate all \( 2^{|R_M|} \) subsets of \( R_M \). It remains to show that if \( P^* \cap R_M \) is a completion of a multiset \( P_R \subseteq R_M \), we can find a completion \( P_R^* \) or \( P_R \) such that \( C_i \cup P_R^* \) is a \text{Diam}-cluster in \( \text{FPT} \)-time. Since all sets \( S' \) with \((S', r') \in S_i \) have size at most \( r \) and \(|S_i| \leq r^{2r+1} \), it suffices to show the following claim:

**Claim 2.** For every \( \vec{w} \in P_R \), if coordinate \( j \in [d] \) is not in \( \delta(\vec{u}) \cup \bigcup_{(S', r') \in S_i} S' \), then we can safely set \( \vec{w}[j] = 0 \).

**Proof of Claim.** Let \( \vec{v}^* \) be the completion of \( \vec{u} \) in \( P^* \) and let \( \vec{w}^* \) be the completion of \( \vec{w} \) where we set all \( \Box \)'s in \( \vec{w} \) at coordinates not in \( \Delta(\vec{u}) \) or any set \( S' \) such that \((S', r') \in S_i \) to zero and set all remaining \( \Box \)'s as in \( \vec{w}^* \).

Let \( \vec{c} \in C_i \) be compatible with \((S', r') \in S_i \). If \( \vec{c} = \vec{u} \) or \( r = |S'| \), then \( \vec{w}^* \) and \( \vec{w}^* \) are the same on coordinates in \( \Delta(\vec{c}) \) and \( \Delta(\vec{w}^*) \leq |\Delta(\vec{w}^*)| \). Moreover, \( \vec{c} \) is also in \( P^* \) (or some copy of \( \vec{c} \)) as it is the unique vector that is compatible with \((S', r') \in S_i \) and \( S_i \subseteq S \). Hence \( \delta(\vec{c}, \vec{w}) \leq r \). If \(|S'| < r'\), then \( P^* \) contains a set \( N \) of \( r' \)-vectors of size \( r + 1 \) such that \( F = \Delta(N) \) is a sunflower with core \( S' \). Hence, by Lemma 2, there is a vector \( \vec{n} \in N \) with maximum distance to \( \vec{w}^* \) among all vectors in \([0, 1]^d \) that contain \( S' \). It is rather straightforward to see that \( \delta(\vec{c}, \vec{w}^*) \leq \delta(\vec{n}, \vec{w}^*) \leq r \). 

It follows from the above claim that it suffices to branch over all completions of \( P_R \) that set all \( \Box \)'s not in \( \Delta(\vec{u}) \cup \bigcup_{(S', r') \in S_i} S' \) to 0. There are at most \( r^{O(r^2|P_R|)} \) such completions and the theorem follows.
6.2.3 LARGE DIAM-CLUSTER-COMPLETION Parameterized by $k$

We show next that LARGE DIAM-CLUSTER-COMPLETION parameterized by $k$ is in XP. Note that Theorem 13 shows that this cannot be improved to an FPT-algorithm even for the case of LARGE DIAM-CLUSTER.

**Theorem 34.** LARGE DIAM-CLUSTER-COMPLETION is in XP parameterized by $k$.

**Proof.** Let $(M, k, r)$ be an instance of LARGE DIAM-CLUSTER-COMPLETION. The algorithm works by enumerating all potential clusters $C$ of size exactly $k$, and then uses a reduction to an ILP instance with $f(k)$ variables to check whether $C$ can be completed into a cluster. Since there are at most $|M|^k$ many potential clusters of size exactly $k$, it only remains to show how to decide whether a given set $C$ of exactly $k$ vectors in $M$ can be completed into a DIAM-cluster. Let $M_C$ be the submatrix of $M$ containing only the vectors in $C$. Then $M_C$ has at most $3^k$ distinct columns, and moreover, each of those columns can be completed in at most $2^k$ possible ways. Let $T$ be the set of all columns occurring in $M_C$ and for a column $\vec{t} \in T$, let $F(\vec{t})$ be the set of all possible completions of $\vec{t}$, and let $\#(t)$ denote the number of columns in $M_C$ equal to $\vec{t}$. For a vector $\vec{f} \in \{0, 1\}^k$ (representing the completion of a column), let $T(\vec{f})$ denote the subset of $T$ containing all columns $\vec{t}$ with $\vec{f} \in F(\vec{t})$. Moreover, for every $i$ and $j$ with $1 \leq i < j \leq k$ (representing the $i$-th and the $j$-th vector in $C$), we denote by $FD(i, j)$ the set of all vectors (completions of columns) $\vec{f} \in \{0, 1\}^k$ such that $\vec{f}[i] \neq \vec{f}[j]$.

We are now ready to construct an ILP instance $\mathcal{I}$ with at most $3^k2^k$ variables that is feasible if and only if $C$ can be completed into a DIAM-cluster. $\mathcal{I}$ has one variable $x_{\vec{t}, \vec{f}}$ for every $\vec{t} \in T$ and every $\vec{f} \in F(\vec{t})$ whose value (in a feasible assignment) represents how many columns of type $\vec{t}$ in $M_C$ will be completed to $\vec{f}$. Moreover, $\mathcal{I}$ has the following constraints:

- One constraint for every $\vec{t} \in T$ stipulating that every column of type $\vec{t}$ in $M_C$ is completed in some manner:

$$\sum_{\vec{f} \in F(\vec{t})} x_{\vec{t}, \vec{f}} = \#(t).$$

- For every every $i$ and $j$ with $1 \leq i < j \leq k$ (representing the $i$-th and the $j$-th vector in $C$), one constraint stipulating that the Hamming distance between the $i$-th and the $j$-th vector in $C$ does not exceed $r$:

$$\sum_{\vec{f} \in FD(i, j)} x_{\vec{t}, \vec{f}} \leq r.$$

This completes the construction of $\mathcal{I}$ and it is straightforward to verify that $\mathcal{I}$ has a feasible assignment if and only if $C$ can be completed to a DIAM-cluster. Since $\mathcal{I}$ has at most $3^k2^k$ variables, and since it is well known that ILP can be solved in FPT-time w.r.t. the number of variables [43], $\mathcal{I}$ can be solved in FPT-time w.r.t. $k$. \hfill $\blacksquare$

6.3 Application: LARGE ANY-CLUSTER-COMPLETION

In this subsection we show how our techniques developed for IN/ANY/DIAM-CLUSTERING-COMPLETION can be employed and extended for LARGE ANY-CLUSTER-COMPLETION. Our main algorithmic results are that LARGE ANY-CLUSTER-COMPLETION is fixed-parameter tractable parameterized by $k + r + \text{cover}(M)$ and in XP parameterized by either $k$ or $r + \text{cover}$ alone. Together Theorem 45 (showing $W[1]$-hardness for LARGE DIAM-CLUSTER-COMPLETION w.r.t. $k$ even for $r = 0$) this
gives us an almost complete picture for the parameterized complexity of LARGE ANY-CLUSTER-COMPLETION for any combination of the parameters \( k \), \( r \), \( \text{cover}(M) \). The only two remaining questions are whether the XP result for \( r + \text{cover}(M) \) can be improved to an fpt-result (as this has been the case for LARGE DIAM-CLUSTER-COMPLETION) and whether it is possible to obtain an FPT-algorithm either for \( k \) or \( k + \text{cover}(M) \).

### 6.3.1 LARGE ANY-CLUSTER-COMPLETION Parameterized by \( k + r + \text{cover}(M) \).

We start by showing that like LARGE DIAM-CLUSTER-COMPLETION also LARGE ANY-CLUSTER-COMPLETION has a Turing kernel parameterized by \( k + r + \text{cover}(M) \). The approach is similar to the approach used in Subsection 6.2.1 and will, in particular, make use of Lemma 23 and Lemma 24.

**Theorem 35.** LARGE ANY-CLUSTER-COMPLETION parameterized by \( k + r + \text{cover}(M) \) has a Turing-kernel containing at most \( m = k3^{T_M |+2r} + |R_M| + 2 \) vectors each having at most \( \max\{2r(m − 1) + |T_M|, (|R_M|/2)(2r + 1)\} \) coordinates.

**Proof.** We distinguish three cases: (1) the solution cluster contains at least two vectors in \( M \setminus R_M \), (2) the solution cluster contains exactly one vector in \( M \setminus R_M \), and (3) the solution cluster does not contain any vector in \( M \setminus R_M \).

We start by showing the result for case (1). Our first aim is to reduce the number of vectors in \( M \setminus R_M \). As a first step, we guess two vectors \( \vec{v} \) and \( \vec{u} \) of \( M \setminus R_M \), that are farthest apart in the cluster w.r.t. to all vectors in \( M \setminus R_M \); this is possible since we assume that the cluster contains at least two vectors in \( M \setminus R_M \). We then guess the exact distance, say \( t \), between \( \vec{v} \) and \( \vec{u} \) in a completion leading to the cluster. Note that \( t \) can be anywhere between \( \delta(\vec{v}, \vec{u}) \) and \( \min\{\delta(\vec{v}, \vec{u}) + |T_R|\} \), but also at most \( 2r \). We now remove all vectors \( \vec{m} \) from \( M \) for which either \( \delta(\vec{v}, \vec{m}) > t \), \( \delta(\vec{u}, \vec{m}) > t \), or \( |Δ(\vec{v}, \vec{m}) \cap Δ(\vec{u}, \vec{m})| > t/2 \). Note that this is safe because of Lemma 23.

It now follows from Lemma 24 that if \( M \setminus (R_M \cup \{\vec{v}, \vec{u}\}) \) contains more than \( k3^{T_M |+t} + |R_M| \) vectors, then we can return a trivial YES-instance of LARGE ANY-CLUSTER-COMPLETION. Otherwise, we obtain that \( |M \setminus R_M| ≤ k3^{T_M |+t} + 2 \). We now also need to add back the vectors in \( R_M \). Clearly, if a vector in \( R_M \) differs in more than \( t \) coordinates from \( \vec{v} \), it cannot be part of an ANY-cluster containing \( \vec{v} \), and we can safely remove it from \( R_M \). Hence every vector in \( M \) now differs from \( \vec{v} \) in at most \( t \) coordinates, which implies that \( Z(M) ≤ t(|M| − 1) + |T_M| \). We now remove all coordinates outside of \( Z(M) \) from \( M \), since these can always be completed in the same manner for all the vectors. Now the remaining instance is a kernel containing at most \( m = k3^{T_M |+t} + 2 + |R_M| \) vectors each having at most \( t(m − 1) + |T_M| \) coordinates. This completes the proof for the case that the solution cluster contains at least two vectors in \( M \setminus R_M \).

In the following, let \( t = 2r \).

For the second case, i.e., the case that the solution cluster contains exactly one vector in \( M \setminus R_M \), we first guess the vector say \( \vec{m} \) in \( M \setminus R_M \) that will be included in the solution cluster and then remove all other vectors from \( M \setminus R_M \). This already leaves us with at most \( |R_M| + 1 \) vectors and it only remains to reduce the number of relevant coordinates. Because we guessed that \( \vec{m} \) will be in the solution cluster, we can now safely remove all vectors \( \vec{m}' \in M \), with \( δ(\vec{m}, \vec{m}') > t \). Now every remaining vector differs from \( \vec{m} \) in at most \( t \) coordinates and hence \( Z(M) ≤ t|R_M| + |T_M| \), which gives us the desired kernel.

For the third case, i.e., the case that the solution cluster contains no vectors in \( M \setminus R_M \), we first remove all vectors in \( M \setminus R_M \). This leaves us with only \( |R_M| \) vectors and it only remains to reduce the number of coordinates. To achieve this, we employ an idea similar to the idea used in Lemma 7, i.e., we compute a set of coordinates that preserves the distance up to \( t \) between any pair.
of vectors. Namely, we compute a set $D$ of relevant coordinates starting from $D = \emptyset$ by adding the following coordinates to $D$ for every two distinct vectors $\vec{m}$ and $\vec{m}'$ in $M$:

- if $|\Delta(\vec{m}, \vec{m}')| \leq t$, we add $\Delta(\vec{m}, \vec{m}')$ to $D$ and otherwise
- we add an arbitrary subset of at most $t+1$ coordinates in $\Delta(\vec{m}, \vec{m}')$ to $D$.

Let $M_D$ be the matrix obtained from $D$ after removing all coordinates/columns in $D$. Using ideas similar to the ideas employed in Lemma 8, it is now straightforward to show that $(M, k, r)$ and $(M_D, k, r)$ are equivalent instances of Large Any-Cluster-Completion. Since $|D| \leq \binom{|R_M|}{2}(t+1)$, the remaining instance has at most $|R_M|$ vectors each having at most $\binom{|R_M|}{2}(t+1)$ coordinates. □

6.3.2 Large Any-Cluster-Completion Parameterized by $r + \text{cover}(M)$

Similarly as for Large Diam-Cluster-Completion, we can guess two vectors from $M \setminus R_M$ in a maximum ANY-Cluster $P^*$ and complete these two vectors. After normalizing w.r.t. one of these vectors, the solution vector has at most $r$ ones and hence it is not difficult to see that an equivalent $P^*$ is $r$-uniform subset of $M^*$ for some completion of $M$. Moreover, we can keep in $M$ only vectors with at most $2r$ ones and hence $P^*$ is also properly defined by a set $S$ of size at most $4r^{d+1}$. We note that this already leads to an XP-algorithm. Unfortunately, it is not clear how to obtain an equivalent of Lemma 29 and Lemma 30 in this setting: unlike in the case where we were searching for a Diam-cluster, here we cannot choose which vectors are compatible with a hypothetical center of a sought-after ANY-cluster independently (indeed, which vectors become “too far” to be considered would depend on parameters of the hypothetical center that are outside of our immediate control). That being said, our techniques for Large Diam-Cluster-Completion leads to an XP-algorithm parameterized by $r + \text{cover}(M)$ for Large Any-Cluster-Completion. However, we will give an alternative simpler proof.

Theorem 36. Large Any-Cluster-Completion is in XP parameterized $r + \text{cover}(M)$.

Proof. Let $(M, k, r)$ be an instance of Large Any-Cluster-Completion. If $k \leq \text{cover}(M)$, we can use the FPT-algorithm parameterized by $k + r + \text{cover}(M)$ given by Theorem 35. Otherwise, a maximum size ANY-cluster contains at least one vector in $M \setminus R_M$. We guess one such vector and its completion $\vec{v}$. This is at most $2|T_M| \cdot |M \setminus R_M|$ branches. We then normalize all the vectors in $M$ such that $\vec{v} = \vec{0}$. Hence the center vector for an ANY-cluster containing $\vec{0}$ has at most $r$ ones and there are at most $O(d^{r+1})$ vectors with at most $r$ ones. We enumerate all of them. Once we have the correct center vector $\vec{c} \in \{0, 1\}^d$, it is simple to check whether vector $\vec{w} \in \{0, 1, \square\}^d$ can be completed to a vector at distance at most $r$ to $\vec{c}$ by simply completing $\vec{w}$ to match $\vec{c}$ on all $\square$.

6.3.3 Large Any-Cluster-Completion Parameterized by $k$

We start by showing that Large Any-Cluster-Completion is in XP parameterized by $k$ via an algorithm that works quite similar to the XP-algorithm for Large Diam-Cluster-Completion given in Theorem 34.

Theorem 37. Large Any-Cluster-Completion is in XP parameterized by $k$.

Proof. Let $(M, k, r)$ be an instance of Large Any-Cluster-Completion. The algorithm works by enumerating all potential clusters $C$ of size exactly $k$ and it then uses a reduction to an ILP instance with $f(k)$ variables to check whether $C$ can be completed into a cluster. Since there are at most $|M|^k$ many potential clusters of size exactly $k$, it only remains to show how to decide whether a given set $C$ of exactly $k$ vectors in $M$ can be completed into an ANY-cluster. Let $M_C$ be the submatrix of $M$ containing only the vectors in $C$. Then $M_C$ has at most $3^k$ distinct columns and
moreover each of those columns can be completed in at most \(2^k\) possible ways. Let \(T\) be the set of all columns occurring in \(M_C\) and for a column \(\bar{t} \in T\), let \(F(\bar{t})\) be the set of all possible completions of \(\bar{t}\), let \(F(T) = \bigcup_{\bar{t} \in T} F(\bar{t})\), and let \(#(t)\) denote the number of columns in \(M_c\) equal to \(\bar{t}\). Moreover, for a vector \(\bar{f} \in \{0,1\}^k\) (representing the completion of a column), let \(T(\bar{f})\) denote the subset of \(T\) containing all columns \(\bar{t}\) with \(\bar{f} \in F(\bar{t})\).

We are now ready to construct an ILP instance \(I\) with at most \(3^{2k}k^2\) variables that is feasible if and only if \(C\) can be completed into an \(\text{Any}\)-cluster. \(I\) has one variable \(x_{\bar{t},\bar{f},b}\) for every \(\bar{t} \in T\), every \(\bar{f} \in F(\bar{t})\), and every \(b \in \{0,1\}\), whose value (in a feasible assignment) represents for how many columns of type \(\bar{t}\) in \(M_C\) that will be completed to \(\bar{f}\) the center of the cluster is set to \(b\). Moreover, \(I\) has the following constraints:

- One constraint for every \(\bar{t} \in T\) that ensures that every column of type \(\bar{t}\) in \(M_C\) is completed in some manner:
  \[ \sum_{\bar{f} \in F(\bar{t})} x_{\bar{t},\bar{f},b} = #(t). \]
- For every \(i\) with \(1 \leq i \leq k\) (representing the \(i\)-th vector in \(C\)), one constraint that ensures that the Hamming distance between the \(i\)-th vector in \(C\) and the center is at most \(r\):
  \[ \sum_{\bar{f} \in F(T) \bar{f}[i]=0 \land \bar{t} \in T(\bar{f})} x_{\bar{t},\bar{f},1} + \sum_{\bar{f} \in F(T) \bar{f}[i]=1 \land \bar{t} \in T(\bar{f})} x_{\bar{t},\bar{f},0} \leq r. \]

This completes the construction of \(I\) and it is straightforward to verify that \(I\) has a feasible assignment if and only if \(C\) can be completed to an \(\text{Any}\)-cluster. Because \(I\) has at most \(3^{2k}k^2\) variables and it is known that ILP can be solved in \(\text{FPT}\)-time w.r.t. the number of variables \(\|43\|\), we obtain that \(I\) can be solved in \(\text{FPT}\)-time w.r.t. \(k\).

7 Lower-Bound Results

We dedicate this section to showing that the parameterizations used in the presented \(\text{FPT}\) algorithms presented in Sections 5 and 6 are necessary to achieve tractability. Obviously, lower-bound results for clustering problems for complete data carry over to their counterparts for incomplete data. Therefore, we will omit restating these results for the incomplete data case.

7.1 Lower-Bound Results for Complete Data

It is known that \(\text{Any-Clustering}\) is \(\text{NP}\)-complete for \(r = 2\) (see Section 3 of previous work by Jiao et al. \(\|42\|\)). Our first hardness results show that the other two clustering problems are also \(\text{NP}\)-complete for constant values of \(r\).

**Theorem 38.** \(\text{In-Clustering}\) is \(\text{NP}\)-complete for \(r = 4\), and \(\text{Diam-Clustering}\) is \(\text{NP}\)-complete for \(r = 6\).

**Proof.** For \(\text{In-Clustering}\), we give a polynomial-time reduction from the \(\text{Dominating Set}\) problem on 3-regular graphs (3-DS), which is \(\text{NP}\)-complete \(\|31\|\), to the restriction of \(\text{In-Clustering}\) to instances where \(r \leq 4\). Given an instance \((G,k)\) of 3-DS, where \(V(G) = \{v_1,\ldots,v_n\}\), set \(x_i = 0\) for \(i \in [n]\), and apply the reduction \(R\) (described in Subsection 4.4) to \(G\) to obtain the set of vectors \(M\). By Observation \(\|11\|\) for any two vertices \(v_i,v_j \in V(G)\), where \(i \neq j\), we have \(\delta(\bar{a}_i,\bar{a}_j) = 6\) if
Theorem 39. Diam-Clustering is NP-complete for $k = 3$.

Proof. Consider the problem of deciding whether a graph on $n$ vertices can be partitioned into three cliques, referred to as 3-Clique Partitioning henceforth. This problem is NP-hard via a trivial reduction (that complements the edges of the graph) from the NP-hard problem [15] 3-Coloring.
We can now reduce 3-Clique Partitioning to the restriction of Diam-Clustering to instances in which the number of desired clusters, $k$, is 3, using the generic construction given in Subsection 4.4.

Given an instance $G$ of 3-Clique Partitioning, where $V(G) = \{v_1, \ldots, v_n\}$, we set $x_i = n - 1 - \deg(v_i)$ for $i \in [n]$, and apply the polynomial-time reduction $R$ to $G$ to produce the set of vectors $M$. The polynomial-time reduction from 3-Clique Partitioning produces the instance $(M, 3, 2n - 4)$ of Diam-Clustering. By Observation [11] for any two distinct vertices $v_i, v_j \in V(G)$, $\delta(\vec{a}_i, \vec{a}_j) = 2n - 2$ if $v_i$ and $v_j$ are nonadjacent and $\delta(\vec{a}_i, \vec{a}_j) = 2n - 4$ if $v_i$ and $v_j$ are adjacent.

If $G$ can be partitioned into 3 cliques, then the vectors in $M$ corresponding to the vertices in each clique form a cluster of diameter $2n - 4$, and hence $(M, 3, 2n - 4)$ is a Yes-instance of Diam-Clustering. Conversely, if $(M, 3, 2n - 4)$ is a Yes-instance of Diam-Clustering, $M$ can be partitioned into 3 clusters, each of diameter at most $2n - 4$. The vertices in $G$ corresponding to the vectors in each of the 3 clusters are pairwise adjacent, and hence form a clique in $G$. It follows that $G$ can be partitioned into 3 cliques, and $G$ is a Yes-instance of 3-Clique Partitioning.

Finally, we note that, unlike the previous two problems, In-Clustering admits a simple polynomial-time brute-force algorithm for every fixed value of $k$ where the order of the polynomial depends on $k$. However, one can still exclude fixed-parameter tractability via a reduction from Dominating Set.

Observation 40. In-Clustering can be solved in time $O(|M|^k|M|kd)$.

Proof. The result follows using a brute-force algorithm that enumerates each subset of $k$ vectors in $M$ as the potential centers of the $k$ clusters sought. For each such subset $S \subseteq M$ of $k$ vectors, the algorithm iterates through the vectors in $M$, placing each vector $\vec{a} \in M$ into the cluster containing the vector in $S$ whose distance to $\vec{a}$ is minimum and is at most $r$; if no vector in $S$ has distance at most $r$ to $\vec{a}$, the enumeration is discarded, as it does not lead to a solution. If the algorithm manages to place each vector $\vec{a} \in M$ into a cluster containing a vector in $S$ whose distance to $\vec{a}$ is at most $r$, the algorithm accepts. Enumerating all subsets of $k$ vectors in $M$ takes time $O(|M|^k)$. Iterating through each vector in $M$, and finding its closest vector in the enumerated $k$-subset of $M$, takes time $O(|M|kd)$. The theorem follows.

Theorem 41. In-Clustering is W[2]-complete parameterized by $k$.

Proof. We prove W[2]-hardness by giving a reduction from Dominating Set (DS), which is W[2]-hard [16], to In-Clustering parameterized by $k$. The reduction is very similar to that in the proof of Theorem [38] albeit that its starting point is Dominating Set (on general graphs) rather than 3-DS. Given an instance $(G, k)$ of DS, where $V(G) = \{v_1, \ldots, v_n\}$, we set $x_i = n - 1 - \deg(v_i)$ for $i \in [n]$, and apply the polynomial-time reduction $R$ to $G$ to produce the set of vectors $M$. The reduction from DS to In-Clustering produces the instance $I = (M, k, 2n - 4)$ of In-Clustering.

By Observation [11] for any two distinct vertices $v_i, v_j \in V(G)$, $\delta(\vec{a}_i, \vec{a}_j) = 2n - 2$ if $v_i$ and $v_j$ are nonadjacent and $\delta(\vec{a}_i, \vec{a}_j) = 2n - 4$ if $v_i$ and $v_j$ are adjacent. The proof that $(G, k)$ is a Yes-instance of DS iff $(M, k, 2n - 4)$ is a Yes-instance of In-Clustering now follows by similar arguments to those in the proof of the same statement in Theorem [38].

Finally, membership in W[2] can be shown via a reduction from In-Clustering to DS that constructs the compatibility graph $G(I)$ of the given instance $I$ in polynomial time, and uses the observation that there is a direct correspondence between a dominating set in $G$ of size $k$ and a solution for $I$.

Theorem 42. Dispersion is NP-complete and W[1]-complete parameterized by $k$. 
Theorem 43. LARGE DIAM-Cluster is W[1]-complete parameterized by $k$.

Proof. The proof is the same as that of Theorem 42, albeit that the reduction is from the W[1]-complete problem CLIQUE (on general graphs) [16] instead of INDEPENDENT SET.

7.2 Lower-Bound Results for Incomplete Data

The earlier results in this section already show that out of the three considered parameters, $k$ and $r$ must both be used if one wishes to obtain fixed-parameter algorithms for the clustering problems under consideration. In the case of clustering of incomplete data, the only two questions that remain are whether one also needs to use the covering number $\text{cover}(M)$, and whether it is possible to extend the polynomial-time algorithm for IN-CLUSTERING to IN-CLUSTERING-COMPLETION. We resolve these questions below.

Theorem 44. IN-CLUSTERING-COMPLETION, ANY-CLUSTERING-COMPLETION, DIAM-CLUSTERING-COMPLETION are NP-complete even if $k = 3$ and $r = 0$.

Proof. We give a polynomial-time reduction from 3-COLORING as follows. Let $G$ be the given instance of 3-COLORING with edges $e_1, \ldots, e_{|E(G)|}$ and let $M$ be the set of vectors containing a vector $\overrightarrow{v} \in \{0, 1, \square\}^{|E(G)|}$ for every $v \in V(G)$ such that $\overrightarrow{v}[i] = \square$ if $e_i$ is not incident with $v$, $\overrightarrow{v}[i] = 0$ if $e_i = \{u, v\}$ is incident with $v$ and $v < u$, and $\overrightarrow{v}[i] = 1$ otherwise; here we assume an arbitrary but fixed ordering $<$ of the vertices of $G$. It is now straightforward to verify that $G$ has a 3-coloring if and only if the vectors in $M$ can be partitioned into three sets such that $\delta(x, y) = 0$ for every $x, y \in M$ contained in the same set, which in turn is true if and only if $(M, 3, 0)$ is a Yes-instance of IN-CLUSTERING-COMPLETION, ANY-CLUSTERING-COMPLETION, or DIAM-CLUSTERING-COMPLETION.

Theorem 45. LARGE IN/ANY/DIAM-CLUSTER-COMPLETION are W[1]-complete parameterized by $k$ even if $r = 0$.

Proof. The proof of the theorem is similar to that of Theorem 44, albeit that the starting point of the reduction is the W[1]-hard problem INDEPENDENT SET [16]. The proof is the same for each of three problems in the theorem, and hence, we only present it for LARGE IN-CLUSTER-COMPLETION. (Note that for $r = 0$, the two problems LARGE IN-CLUSTER-COMPLETION and LARGE ANY-CLUSTER-COMPLETION are identical.)
Given an instance \((G, k)\) of \textsc{Independent Set}, where \(G\) has edges \(e_1, \ldots, e_{|E(G)|}\), fix an arbitrary ordering on the vertices in \(G\). Let \(M\) be the set of vectors containing a vector \(\vec{v} \in \{0, 1, \Box\}^{|E(G)|}\) for every \(v \in V(G)\) such that \(\vec{v}[i] = \Box\) if \(e_i\) is not incident with \(v\), \(\vec{v}[i] = 0\) if \(e_i = \{u, v\}\) is incident with \(v\) and \(v < u\), and \(\vec{v}[i] = 1\) otherwise. It is now straightforward to check that \(G\) has an independent set of size at least \(k\) if and only if there is a completion \(M^*\) of \(M\) and a subset \(S \subseteq M^*\) of cardinality at least \(k\), such that the distance between every pair of vectors in \(S\) is 0 (and hence the distance between any fixed vector in \(S\) and any other vector in \(S\) is 0).

Completeness follows after observing that \textsc{Large In-Cluster-Completion} is polynomial-time \textsf{FPT}-reducible to the \textsc{Independent Set} problem on the compatibility graphs corresponding to the instances of \textsc{Large In-Cluster-Completion}.

\textbf{Theorem 46.} \textsc{In-Clustering-Completion} is \textsf{NP}-complete even if \(k = 1\) and there is only one row containing \(\Box\)-entries.

\textit{Proof.} We give a polynomial-time reduction from \textsc{Closest String}, which is well-known to be \textsf{NP}-hard even for binary alphabets \cite{27}. Let \((S, r)\) with \(S = (s_1, \ldots, s_n)\) and \(s_i \in \{0, 1\}^L\) for every \(i \in [n]\) be the given instance of \textsc{Closest String}. Then the set \(M\) of vectors contains one vector \(s_i\) for every \(i \in [n]\) and additionally the vector \(\vec{q} = \{\Box\}^L\). It is easy to observe that for every \textsc{Yes}-instance of \((M, 1, r)\) there exists a solution which completes \(\vec{q}\) to the closest string of \((S, r)\), and hence \((S, r)\) is a \textsc{Yes}-instance of \textsc{Closest String} if and only if \((M, 1, r)\) is a \textsc{Yes}-instance of \textsc{In-Clustering-Completion}. \hfill \square

8 From Clustering to Graph Problems on Hypercubes

In this section, we discuss the implications of the results in Sections 5 and 6 and Subsection 7.1 for fundamental problems defined on induced subgraphs of powers of the hypercube graph.

In particular, the \(d\)-dimensional hypercube graph is the graph \(Q_d\) whose vertex set is the set of all Boolean \(d\)-dimensional vectors, and two vertices are adjacent if and only if their two vectors differ in precisely 1 coordinate. We can then define the class \(Q^r_d\) of all induced subgraphs of powers of the hypercube graphs as the class of all graphs that are induced subgraphs of the \(r\)-th power of \(Q_d\). We note that, in line with the commonly used definition of hypercube graphs \cite{17, 28}, we consider the vertices in \(Q^r_d\) to be vectors and hence every graph \(G \in Q^r_d\) contains an explicit characterization of its vertices as vectors.

In this setting, it is straightforward to observe that \textsc{In-Clustering}, \textsc{Diam-Clustering}, \textsc{Dispersion} and \textsc{Large Diam-Cluster} are precisely the \textsc{Dominating Set}, \textsc{Partition Into Cliques}, \textsc{Independent Set} and \textsc{Clique} problems, respectively, on \(Q^r_d\). Therefore, all the upper and lower bound results derived in this paper pertaining to these clustering problems hold true for their corresponding graph problems on \(Q^r_d\).

\textbf{Corollary 47.} Given \(r, d, k \in \mathbb{N}\) and a graph \(G \in Q^r_d\), determining whether \(G\) has a:

- dominating set of size \(k\) is \textsf{FPT} parameterized by \(k + r\);
- partition into \(k\) cliques is \textsf{FPT} parameterized by \(k + r\);
- independent set of size \(k\) is \textsf{FPT} parameterized by \(k + r\);
- clique of size \(k\) is \textsf{FPT} parameterized by \(r\).

We note that all the tractability results outlined in Corollary 47 are tight, which follows from the lower-bound results in Section 7 in the sense that dropping any parameter from our parameterizations leads to an intractable problem.
Observing that three of the graph properties in the problems discussed above are expressible in First Order Logic (FO) and result in FO formulas whose length is a function of the parameter $k$, an interesting question that ensues from the above discussion is whether these positive results can be extended to the generic problem of First-Order Model Checking \cite{46,38}, formalized below. We will show next that the answer to this question is negative—and, in fact, remains negative even when we restrict ourselves to induced subgraphs of hypercubes (i.e., for $r = 1$).

| Q-FO-MODEL-CHECKING |
|-----------------------|
| **Input:** A first-order (FO) formula $\phi$, integers $d, r$, and a graph $G \in Q^r_d$. |
| **Parameter:** $|\Phi|$ |
| **Question:** Does $G \models \Phi$? |

We denote by FO-MODEL-CHECKING the general FO Model Checking problem on graphs, i.e., C-FO-MODEL-CHECKING with C being the class of all graphs.

**Lemma 48.** Let $H$ be an arbitrary graph. There is a graph $G \in Q^1_{|V(H)|+|E(H)|}$ such that $G$ is isomorphic to the graph $H'$ obtained from $H$ after subdividing every edge of $H$ exactly once and attaching a leaf to every vertex resulting from a subdivision. Moreover, $G$ can be computed from $H$ in polynomial time.

**Proof.** Let $n = |V(H)|$ and $m = |E(H)|$. To prove the lemma, we construct a matrix representation $M \in \{0,1\}^{n+m}$ of $H'$ which has one row (vector) for every vertex in $H$ and where two vertices in $H'$ are adjacent if and only if their corresponding rows in $M$ have Hamming distance at most 1. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of $H$, and $e_1, \ldots, e_m$ be an arbitrary ordering of its edges. Then, $M$ contains one row $r_i$ for every $i \in [n]$ that is 1 at its $i$-th entry and 0 at all other entries. Moreover, for every edge $e_\ell = \{v_i, v_j\} \in E(H)$, $M$ contains the following two rows:
- the row $r_e$ (corresponding to the degree-3 vertex in $H'$ obtained from $e$) that is 1 at the $i$-th and $j$-th entries, and 0 at all other entries; and
- the row $r'_e$ (corresponding to the leaf in $H'$ obtained from $e$) that is 1 at the $i$-th, $j$-th, and $(n+\ell)$-th entries, and 0 at all other entries.

This completes the construction of $M$. Clearly, two rows in $M$ have Hamming distance at most one if and only if their corresponding vertices in $H'$ are adjacent, as required. \hfill $\blacksquare$

**Theorem 49.** Q-FO-MODEL-CHECKING is $\mathcal{W}[t]$-hard for every $t \in \mathbb{N}^*$.

**Proof.** We give a parameterized reduction from FO MODEL CHECKING, which is $\mathcal{W}[t]$-hard for every $t \in \mathbb{N}^*$. Let $I := (\Phi, H)$ be an instance of FO MODEL CHECKING. We will show the theorem by constructing the equivalent instance $I' := (\Phi', G)$ such that $G \in Q^1_d$ and $|\Phi| \leq f(|\Phi'|)$ for some computable function $f$ and value $d$ that is polynomially bounded in the input size. $G$ is obtained from $H$ in the same manner as in Lemma \cite{48}. Moreover, $\Phi'$ is obtained from $\Phi$ as follows:
- Let $\phi_V(x)$ be the formula that holds for a variable $x$ if and only if $x$ corresponds to one of the original vertices in $G$, i.e., $\phi_V(x) := \forall y E(x,y) \exists z \neq x \land E(y,z)$;
- replace every subformula of the form $\exists x \phi$ (for some variable $x$ and some subformula $\phi$ of $\Phi$) with the formula $\exists x \phi_V(x) \land \phi$;
- replace every subformula of the form $\forall x \phi$ (for some variable $x$ and some subformula $\phi$ of $\Phi$) with the formula $\forall x \phi_V(x) \rightarrow \phi$; and
- replace every atom $E(x,y)$, where $E$ is the adjacency predicate and $x$ and $y$ are variables, with the formula $\exists s E(x,s) \land E(s,y) \land x \neq y$.

It is straightforward now to show that $H \models \Phi$ if and only if $G \models \Phi'$, and that $|\Phi'| \leq 20|\Phi|$. Moreover, because of Lemma \cite{48}, $G' \in Q^1_d$, as required. \hfill $\blacksquare$
9 Going Beyond Boolean Domain

In this section, we will expand our scope to also consider two generalizations of the clustering problems under consideration that allow for larger domain sizes. To this end, we will consider two ways for measuring distance, notably the Hamming distance and the Manhattan distance, in matrices over $Q = \{0, 1, \ldots, q - 1, \square\}$ for some $q \geq 2$.

| HAM-IN-CLUSTERING$_q$ |
|-------------------------|
| **Input:** A subset $M$ of $\{0, 1, \ldots, q - 1, \square\}^d$ and $k, r \in \mathbb{N}$. |
| **Question:** Is there a completion $M^*$ of $M$ and subset $S \subseteq M$ with $|S| \leq k$ such that $\delta (S, \vec{a}) \leq r$ for every $\vec{a} \in M$? |

| MAN-IN-CLUSTERING$_q$ |
|------------------------|
| **Input:** A subset $M$ of $\{0, 1, \ldots, q - 1, \square\}^d$ and $k, r \in \mathbb{N}$. |
| **Question:** Is there a subset $S \subseteq M$ with $|S| \leq k$ such that for every $\vec{a} \in M$ there exists $\vec{s} \in S$ such that $\sum_{t=1}^{d} |a[t] - s[t]|$ is at most $r$? |

The generalizations of the other problems to higher domains w.r.t. the Hamming and Manhattan distance, respectively, are defined analogously. Observe that for $q = 2$, the problems we obtain are precisely those we introduced in Section 2. Our aim in this section is to extend our results from matrices (and vectors) over the Boolean domain to the above generalizations, and the main tool we will use are two encodings of domain values. In particular, we define the two encodings $\alpha : [q] \cup \{\square\} \rightarrow \{0, 1, \square\}^q$ and $\beta : [q] \cup \{\square\} \rightarrow \{0, 1, \square\}^q$, where $\alpha (i)$ is the binary encoding of $2^i$ and $\beta (i)$ is the unary encoding of $i$ if $i \neq \square$ and $\alpha (i) = \beta (i) = \square^q$, otherwise. Moreover, for $\vec{v} \in \{0, 1\}^d$, we let $\alpha (\vec{v})$ and $\beta (\vec{v})$ be the vectors in $\{0, 1\}^{qd}$ obtained from $\vec{v}$ by replacing every coordinate $i \in [d]$ with a block of $q$ coordinates equal to $\alpha (i)$ and $\beta (i)$, respectively.

**Example:** Assume $Q = \{0, 1, 2, \square\}$ and $d = 2$. Then $\alpha ((0, 2)) = (0, 0, 1, 1, 0, 0)$ and $\beta ((0, 2)) = (0, 0, 0, 0, 1, 1)$.

It is easy to verify that there is a direct correspondence between the distances in a matrix $M$ over $Q^d$ and the Hamming distances in the matrix over $\{0, 1, \square\}^{qd}$ obtained by applying the respective encoding function on $M$.

**Observation 50.** For each $\vec{a}, \vec{b} \in Q^d$ it holds that $\delta (\vec{a}, \vec{b}) = \delta (\alpha (\vec{a}), \alpha (\vec{b}))$ and that $\sum_{t=1}^{d} |a[t] - b[t]| = \delta (\beta (\vec{a}), \beta (\vec{b}))$.

For each $i \in [d]$, we will call the set of coordinates $\{(i - 1) \cdot q + 1, (i - 1) \cdot q + 2, \ldots, (i - 1) \cdot q + q\}$ a block. Consider a matrix $M$ obtained by applying $\alpha$ (or $\beta$) on a matrix $M'$. A completion $M^*$ of $M$ is block-preserving w.r.t. $\alpha$ (respectively $\beta$) if for each vector $\vec{v} \in M^*$ the $i$-th block of $\vec{v}$ is equal to $\alpha (i)$ (respectively $\beta (i)$) for some $i \in Q$. Equivalently, $M^*$ is block-preserving w.r.t. $\alpha$ (or $\beta$) if it is can be obtained by applying $\alpha$ (or $\beta$, respectively) on the elements of some completion of the matrix $M'$.

For a problem $\text{PROB}$ (one of the problems considered in this paper), let $\text{PROB}_\alpha$ and $\text{PROB}_\beta$ be the adaptation of $\text{PROB}$ to the case where we additionally require the completion $M^*$ of $M$ to be block-preserving (w.r.t. $\alpha$ or $\beta$). Since both encodings only increase the dimension of the vectors by a constant factor, Observation 50 allows us to reduce the completion problems over $Q$ to the question of finding block-preserving completions of Boolean matrices. In particular, it is relatively easy to show that all the developed algorithmic techniques can be extended to the block-preserving variants of the problems. For instance, finding and removing irrelevant vectors is not affected by
blocks. Additionally, finding and removing irrelevant coordinates is safe as long as one always treats all coordinates of a block in the same manner. Moreover, when we need to consider a completion of certain \(\Box\)-entries, we will only consider the completions that are block-preserving. For the algorithm for **Large Any-Cluster-Completion** (Theorem 35), where we construct a cluster center out of the core of a sunflower, we note that this is block-preserving since the blocks of the center will consist of the intersection (logical AND) of two (or more) valid blocks, and it is easy to verify that such an intersection again gives a valid block for both \(\alpha\) and \(\beta\). Finally, for our algorithms based on a reduction to ILP (Theorems 34 and 37), we note that instead of having ILP variables for single columns one needs to consider ILP variables for blocks.

**Corollary 51.** Let \(\text{Prob}\) be one of the problems considered in this paper, i.e., \(\text{In/Any/Diam-Clustering-Completion}, \text{Dispersion-Completion}, \text{Large Diam-Cluster-Completion},\) and **Large Any-Cluster-Completion** as well as their complete variants, and parameterization \(\iota \subseteq \{k, r, \text{cover}\}\):

- If \(\text{Prob}_\alpha\) is \(\text{FPT}\) (or \(\text{XP}\)) parameterized by \(\iota\), then so is \(\text{HAM-Prob}_\alpha\);
- If \(\text{Prob}_\beta\) is \(\text{FPT}\) (or \(\text{XP}\)) parameterized by \(\iota\), then so is \(\text{MAN-Prob}_\beta\).

Note that the corollary implies that all our \(\text{FPT}\)-results and \(\text{XP}\)-results also apply in the finite domain case.

10 Conclusion

We provided a systematic study of the parameterized complexity of fundamental clustering problems for incomplete data. Our results draw a detailed map of the complexity landscape for the studied problems and showcase a sharp contrast between the settings that are fixed-parameter tractable and those which are not, where for the latter case we obtain \(\text{paraNP}\)-completeness w.r.t. many parameterizations.

Our \(\text{FPT}\) result for **Large Any-Cluster-Completion** answers an open question about the parameterized complexity of the complete-data version of the problem w.r.t. the parameterization by \(k + r\), studied in the context of finding a closest string, with possible outliers, to a given set of strings. Two open questions ensue from our results for **Large Any-Cluster-Completion**, which are determining its parameterized complexity w.r.t. the parameterizations by \(r + \text{cover}\) and by \(k + \text{cover}\).

Last but certainly not least, we believe that the insights and techniques showcased in this paper are of general interest. Indeed, in essence they show that vectors over a bounded domain which are packed in dense clusters have non-trivial combinatorial properties that only become accessible through a suitable set representation. We hope that these insights and techniques turn out to be useful as well in other settings.

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