Truncated Wiener-Hopf operators with Fisher-Hartwig singularities.

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Abstract

We derive the asymptotic behavior of determinants of truncated Wiener-Hopf operators generated by symbols having Fisher-Hartwig singularities. This task is achieved thanks to an asymptotic resolution of the Riemann-Hilbert problem associated to some generalized sine kernel. As a byproduct, we give yet another derivation of the asymptotic behavior of Toeplitz determinants having Fisher-Hartwig singularities. The Riemann-Hilbert problem approach to these asymptotics yields a systematic although quickly cumbersome way to compute their sub-leading asymptotics.

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1 Introduction

There is a long history of understanding the asymptotic behavior of determinants of structured matrices. It started in 1915 with the seminal work of Szegö on the asymptotic behavior of large size Toeplitz matrices. His result is known today as the strong Szegö theorem. It states that for non-vanishing and regular functions \( b \in \mathcal{C}^1 \left( \mathbb{T}, \mathbb{R}^+ \right) \)

\[
\det_m [ T [ b ]] \sim_{m \to \infty} (G [ b ])^m E [ b ] \quad \text{with} \quad T_{jk} [ b ] = c_{j-k} \quad \text{and} \quad c_k = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ik\theta} b (\theta) \quad (1.1)
\]

The constants \( G [ b ] \) and \( E [ b ] \) are expressed in terms of the Fourier coefficients of \( \log b \):

\[
G [ b ] = e^{\log b}, \quad E [ b ] = e^{\sum_{k=1}^{\infty} k[\log b]_k[\log b]_{-k}} \quad \text{with} \quad \log b_k = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ik\theta} \log b (\theta) . \quad (1.2)
\]

The result of Szegö underwent several refinements. In particular Baxter \([6]\), Hirschman \([26]\) and, finally, Ibragimov \([27]\) successively weakened Szegö’s original assumptions on \( b \). Furthermore, Widom \([39]\) considered determinants of block Toeplitz matrices and provided a clear interpretation of the constant \( E [ b ] \) in terms of an operator determinant. Despite the possibility to consider matrix valued functions \( b \) in the Szegö theorem, there are limitations of its applicability, even in the scalar case. Indeed, the theorem already breaks down in the case of symbols having zeros, power-law singularities or even jump discontinuities on the unit circle \( \mathcal{C} \). Such symbols can be represented as

\[
\sigma (\theta) = b (\theta) \prod_{p=1}^{n} \omega_{\delta_p, \gamma_p} \left( e^{i\theta} / a_p \right) \quad \text{with} \quad a_p \in \mathcal{C} , \quad (1.3)
\]

and

\[
\omega_{\delta_p, \gamma_p} (e^{i\theta}) = \frac{e^{i(\theta - \pi \text{sgn} \theta)\delta_p}}{(2 - 2 \cos \theta)^{\gamma_p}} , \quad \theta \in \left[ -\pi ; \pi \right] . \quad (1.4)
\]

In such a representation one assumes that the function \( b \) is regular enough, non vanishing on \( \mathcal{C} \) and has a vanishing winding number. The conjecture about the asymptotic behavior of Toeplitz matrices generated by such symbols goes back to Fisher and Hartwig in 1968 \([23]\). More precisely, they claimed that

\[
\det_m [ T (\sigma) ] \sim_{m \to \infty} (G [ b ])^m \sum_{l=1}^{n} \gamma_l^2 - \delta_l^2 \quad C ( [ b ] , \{ \delta \}^n_1 , \{ \gamma \}^n_1) \quad (1.5)
\]

The Fisher-Hartwig conjecture was indorsed by a couple of examples where the authors were able to compute the Toeplitz determinants explicitly. The value of the constant was later conjectured to be equal to

\[
C ( [ b ] , \{ \delta \}^n_1 , \{ \gamma \}^n_1) = E [ b ] \prod_{i=1}^{n} \frac{G (1 - \gamma_i - \delta_i) G (1 - \gamma_i + \delta_i)}{G (1 - 2\gamma_i)} \times \prod_{i=1}^{n} b_{-\gamma_i - \delta_i} (a_i) b_{+\gamma_i + \delta_i} (a_i) \prod_{p \neq q} (1 - a_p / a_q) (\delta_p + \gamma_p) (\delta_q - \gamma_q) . \quad (1.6)
\]

There \( G \) is the Barnes’ function and \( b_{\pm} \) are the Wiener-Hopf factors of \( b \), ie \( b = b_+ G [ b ] b_- \) with \( b_+ \) resp. \( b_- \) a holomorphic function on the interior, resp. exterior, of the unit disk and such that \( b_+ (z = 0) = 1 \), resp. \( b_- (z) \to 1 \) as \( z \to \infty \).
The above conjecture was first proved for some particular cases of the parameters \( \nu_k \) and \( r_k \). Basor \[2\] and, independently, Böttcher \[7\] treated the case of several jump discontinuities \( (\forall p, \gamma_p = 0) \) under the restriction \( |\Re(\delta_p)| < 1/2 \). In 1985, Böttcher and Silbermann \[10\] proved the conjecture in the case \( |\Re(\delta_p)| < 1/2 \) and \( |\Re(\gamma_p)| < 1/2 \). Finally, Ehrhardt and Silbermann \[21\] proved the conjecture in the case of a single Fisher-Hartwig type singularity for all ranges of parameters \( \delta_p \) and \( \gamma_p \) where it made sense. This allowed to prove the conjecture in most of the cases involving multiple Fisher-Hartwig singularities \[20\]. The proof was based on the so-called separation technique developed by Basor \[2\]. The Fisher–Hartwig conjecture breaks down in the case of the so-called ambiguous symbols. Basor and Tracy \[3\] raised a generalized Fisher-Hartwig conjecture for the behavior of Toeplitz determinants generated by such ambiguous symbols. This conjecture was proven recently in the framework of Riemann–Hilbert problems for orthogonal polynomials by Deift, Its and Krasovsky \[14,15\].

There exists a continuous analog of Toeplitz determinants, the Fredholm determinant of truncated Wiener-Hopf operators. The underlying operators act on functions \( g \in L^2(\mathbb{R}) \) according to the formula

\[
(1 + K) \cdot g(t) = g(t) + \int_0^x dt' K(t - t') g(t') dt' . \quad (1.7)
\]

The kernel \( K \) is traditionally defined in terms of its Fourier transform \( \sigma - 1 \), ie \( K(t) = \mathcal{F}^{-1} [\sigma - 1] (t) \). In the following, we choose the below convention for the restriction of the Fourier transform (and of its inverse) to \( L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \):

\[
\mathcal{F}^{-1} [g] (t) = \int_{\mathbb{R}} \frac{dt}{2\pi} g(\xi) e^{-it\xi} \quad \text{and} \quad \mathcal{F} [h] (\xi) = \int_{\mathbb{R}} dt h(t) e^{it\xi} . \quad (1.8)
\]

So that, for functions \( (\sigma - 1) \in L^1(\mathbb{R}) \), we have an explicit integral representation

\[
K(t) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi [\sigma(\xi) - 1] e^{-it\xi} . \quad (1.9)
\]

The question of the \( x \to +\infty \) asymptotics of \( \det [I + K] \) were first addressed by Achiezer \[1\] and Kac \[31\]. More precisely, they showed that for symbols \( \sigma \) regular enough

\[
\det [I + K] \xrightarrow{x \to +\infty} \exp \left\{ x \int_{\mathbb{R}} \log [\sigma(\xi)] d\xi + \int_0^{+\infty} \xi \log [\sigma(\xi)] \log [\sigma(-\xi)] d\xi \right\} . \quad (1.10)
\]

There exist many generalizations of this formula. These either extend the result to less regular or matrix valued symbols \( \sigma \). However, just as in the Toeplitz case, the theorem breaks down when \( \sigma \) has some jump discontinuities or power-law behavior. The continuous analogue \( \sigma \) of a symbol with Fisher-Hartwig type singularities reads

\[
\sigma(\xi) = F(\xi) \prod_{k=1}^n \sigma_{\nu_k, \nu_k} (\xi - a_k) , \quad \sigma_{\nu, \nu}(\xi) = \left( \frac{\xi + i}{\xi + i0^+} \right)^\nu \left( \frac{\xi - i}{\xi - i0^+} \right)^\nu \cdot (1.11)
\]

Note that the exponents in the definition of \( \sigma_{\nu, \nu} \) should be understood in the sense of the principal branch of the logarithm, ie \( \arg \in [\pi : \pi] \). In the above decomposition for \( \sigma \),
the function $F$ is supposed regular enough and $\sigma_{\nu_k,\nu_k}(\xi)$ has a singularity at 0:

$$\sigma_{\nu_k,\nu_k}(\xi) \sim \frac{e^{i\pi \delta_k \mathrm{sgn}(\xi)}}{|\xi|^{2\gamma_k}} \quad \text{with} \quad 2\gamma_k = \nu_k + \nu_k \quad \text{and} \quad 2\delta_k = \nu_k - \nu_k.$$  \hspace{1cm} (1.12)

There exists a continuous analogue of the Fisher-Hartwig conjecture for Toeplitz determinants and it is due to Böttcher [8]. It was inspired by the study of truncated Wiener-Hopf operators that are generated by rational symbols; indeed, in the latter case, the author was able to estimate the Fredholm determinants explicitly. This conjecture was confirmed in many particular cases: the case [12] of pure jump type singularities, the case [9] where all $\nu_k = 0$ or all $\nu_k = 0$ and finally in the case [4] of a single pure Fisher-Hartwig singularity $\sigma_{\nu,\nu}$ under the restriction $|\Re(\nu \pm \nu)| < 1$. All these results were established thanks to some identity relating determinants of truncated Wiener-Hopf operators to determinants of Toeplitz matrices, and then the use of the formulae for the asymptotic behavior of Toeplitz determinants.

Yet there exists an alternative approach to the asymptotic analysis of large size determinants of structured matrices. In true, it is well known that a Hankel matrix can be expressed as a product of the leading coefficients of the orthogonal polynomials with respect to the measure defining its entries. It was observed by Fokas, Its and Kitaev [24] that one can recast the problem of computing orthogonal polynomials into a certain Riemann-Hilbert Problem (RHP). This problem can be solved asymptotically for a large case of weights [17, 18, 36]. As noticed by Krasovsky [35], one can relate the asymptotic solution of the RHP for orthogonal polynomials with respect to a weight having a finite number of power-like singularities to the asymptotic behavior of the Hankel determinant defined in terms of this weight. Its and Krasovsky used an analogous identity to establish the asymptotic behavior of a Hankel determinant defined in terms of a gaussian weight having a jump discontinuity [29]. Moreover, still in the framework of RHP for orthogonal polynomials, Krasovsky estimated the asymptotic behavior of Toeplitz matrices on an arc and generated by symbols having jump type discontinuities [34] thanks to the relationship between polynomials orthogonal on an arc and those on a line segment. His approach presented no obstruction for a generalization to the case of root type singularities. The case of Toeplitz, Hankel and Hankel+Toeplitz determinants with Fisher-Hartwig singularities has been treated recently in the framework of the Riemann–Hilbert problem for orthogonal polynomials by Deift, Its and Krasovsky [14,15].

Truncated Wiener-Hopf operators being continuous analogs of Toeplitz matrices, there arises a natural question concerning a RHP approach to study the large $x$ behavior of Fredholm determinants for such operators. In this paper we will show how to tackle, in the framework of RHP, the $x \to +\infty$ asymptotics of determinants of truncated Wiener-Hopf operators generated by Fisher-Hartwig symbols. This treatment is based on a relationship between truncated Wiener-Hopf operators and the so-called generalized sine kernel acting on $\mathbb{R}$. The latter kernel is an integrable integral operator. Such operator can be analyzed by a RHP as observed in [30]. We asymptotically solve this RHP. The construction of its asymptotic solution is an extension of the work [33], the latter being itself a generalization of an unpublished study on the pure sine kernel by Deift, Its and Zhou. This approximate resolvent allows to compute the leading asymptotics of the Fredholm determinants of truncated Wiener-Hopf operators. The latter constitutes the main result of this article:

**Theorem 1.1** Let $I+K$ be a truncated Wiener-Hopf operator acting on the segment $[0; x]$
and generated by the symbol \((\sigma - 1) \in L^2(\mathbb{R})\) with

\[
\sigma(\zeta) = F(\zeta) \prod_{k=1}^{n} \sigma_{\nu_k \overline{\nu}_k}(\zeta - a_p), \quad a_i \in \mathbb{R}, \quad a_1 < \cdots < a_n
\]  

(1.13)

where

- \(F\) is non-vanishing and holomorphic in some open neighborhood \(U\) of the real axis;
- \(F - 1 \in L^2(\mathbb{R})\) and moreover \(F(\zeta) - 1 = O\left(|\zeta|^{-1+\epsilon}\right)\), for some \(\kappa > 0\) and \(\zeta \to +\infty\) in \(U\);
- \(\Re(\delta_k) < 1/2, \text{ and } \Re(\gamma_k) < 1/4\) for \(k \in \{1; n\}\).

Then the leading asymptotics of the 2-regularized determinant of \(I + K\) read:

\[
det_2[I + K] = G_2^*[\sigma] \cdot \left(\frac{x}{2}\right)^{\frac{n}{2} - \frac{d^2}{p}} \exp \left\{ x \int_{\mathbb{R}} \frac{d\zeta}{2\pi} \left[ \log |\sigma(\zeta)| + 1 - \sigma(\zeta) \right] \right\},
\]

(1.14)

where

\[
G_2^*[\sigma] = \exp \left\{ x \int_{\mathbb{R}} \frac{d\zeta}{2\pi} \left[ \log |\sigma(\zeta)| + 1 - \sigma(\zeta) \right] \right\},
\]

(1.15)

\[
E[F] = \exp \left\{ \int_{0}^{+\infty} \frac{d\zeta}{\xi} \left[ F^{-1}[\log F](\zeta) F^{-1}[\log F](-\zeta) \right] \right\},
\]

(1.16)

and \(F_\pm\) are the Wiener–Hopf factors of \(F\):

\[
\log F_\pm(z) = \int_{\mathbb{R}} \frac{d\zeta}{2\pi} \frac{\log F(\zeta)}{z - \zeta}.
\]

(1.17)

The estimates for the correction involve the constant \(\rho = 2\max_k |\Re(\delta_k)| < 1\). Finally \(G\) stands for the Barnes \(G\) function and we remind that \(\det_2[I + K] = \det [(I + K) e^{-K}]\).

Note that a similar result can also be established for symbols \(\sigma - 1 \in L^1(\mathbb{R})\). The above theorem reproduces all of the aforementioned results that were obtained for particular cases of singularities. However, it shows that the original conjecture doesn’t hold in its whole generality. Indeed the conjecture [8][11] predicts the presence of

\[
\prod_{k<p}^{n} \left( \frac{(a_k - a_p + i)^2}{(a_k - a_p + 2i)(a_k - a_p)} \right) \frac{\nu_k \overline{\nu}_k}{e^{\nu_k}} \prod_{k>p}^{n} \left( \frac{(a_k - a_p - i)^2}{(a_k - a_p - 2i)(a_k - a_p)} \right) \frac{\nu_p \overline{\nu}_k}{e^{\nu_k}}.
\]

(1.18)

whereas we find

\[
\prod_{k<p}^{n} \left( \frac{(a_k - a_p + i)^2}{(a_k - a_p + 2i)(a_k - a_p)} \right) \frac{\nu_k \overline{\nu}_k}{e^{\nu_k}} \prod_{k>p}^{n} \left( \frac{(a_k - a_p - i)^2}{(a_k - a_p - 2i)(a_k - a_p)} \right) \frac{\nu_p \overline{\nu}_k}{e^{\nu_k}}.
\]

(1.19)
Of course both results coincide in all the cases previously investigated \( \nu_k = \pm \nu_k \), \( \forall k \), \( \nu_k = 0 \) or \( \forall k \), \( \nu_k = 0 \). Moreover our approach opens a way, at least in principle, to asymptotically inverting truncated Wiener-Hopf operators generated by Fisher–Hartwig symbols. Such an inversion could be carried out in the spirit of the inversion for holomorphic symbols \( \sigma \) proposed in [33].

As a byproduct, using a formula [16] relating certain Fredholm determinants of a generalized sine kernel to Toeplitz matrices, we derive the asymptotic behavior of Toeplitz determinants generated by symbols having Fisher-Hartwig singularities. This computation reproduces the result of Ehrhardt [20] for the leading asymptotics and of Deift, Its and Krasovsky for the sub-leading ones [15]. Hence, we see that Toeplitz determinants and those of truncated Wiener-Hopf operators are both related to a determinant of a generalized sine kernel. The only difference being the interval on which the generalized sine kernel acts.

The article is organized as follows. In the first Section we establish the link between Fredholm determinants of truncated Wiener-Hopf operators and those of a generalized sine kernel. The second part of Section 2 is devoted to introducing some notations. In particular, we shall consider two types of symbols for \( K \). The first one belongs to \( L^2(\mathbb{R}) \) whereas the second to \( L^1(\mathbb{R}) \). Both have the same Fisher-Hartwig singularities, but in the second case, part of the function \( F(1.11) \) contains some prefactor depending on \( \delta \) in order to ensure that \( (\sigma - 1) \in L^1(\mathbb{R}) \).

In Section 3 and 4 we asymptotically solve the Riemann–Hilbert problem associated with the generalized sine kernel. In Section 5 we derive the asymptotic behavior of the resolvent of the generalized sine kernel. In Section 6 we use this asymptotic resolvent to obtain the asymptotic behavior of the Fredholm determinants of truncated Wiener-Hopf operators under investigation. We compare our results with the already existing ones. In the last Section we adapt the RHP associated to the generalized sine kernel in order to study the asymptotic behavior of Toeplitz matrices with Fisher-Hartwig symbols. We obtain the leading asymptotics of such Toeplitz determinants and also compute their first sub-leading corrections. The structure of these corrections indicates that at least part of the asymptotic series can be deduced by making shifts of certain parameters appearing in the leading asymptotics. This leads us to raise a generalization of the Basor–Tracy conjecture.

2 Some preliminary results and definitions.

In this Section we establish the link between truncated Wiener-Hopf operators and generalized sine kernels in the case of general \( L^1 \) and \( L^2 \) symbols \( \sigma - 1 \). We also present some determinant identities that will be useful in our proofs.

2.1 Truncated Wiener-Hopf and the modified sine kernel

Let \( I + K \) be a truncated Wiener-Hopf operator acting on \( L^2(\mathbb{R}) \), \text{i.e.} \( I + K \) acts on functions \( g \in L^2(\mathbb{R}) \) as follows

\[
(I + K) \cdot g(t) = g(t) + \int_0^x dt'K(t - t')g(t')dt'
\]  (2.1)

It is useful to define the kernel \( K \) in terms of its Fourier transform \( \sigma - 1 \), \text{i.e.} \( K(t) = \mathcal{F}^{-1}[\sigma - 1](t) \).
We shall focus on two cases of interest: $\sigma^{(\ell)} - 1 \in L^1(\mathbb{R})$ and $\sigma^{(\ell)} - 1 \in L^2(\mathbb{R})$. In the first case, $I + K$ is trace class and hence its determinant is well defined. In the second one, $I + K$ is Hilbert-Schmidt; hence, one ought to consider the 2-regularized determinant [37] of $I + K$ $\text{ie}$ $\det_2[I + K] = \det[(I + K)e^{-K}]$. For the purpose of this section only, we introduce the index $\ell$ in $\sigma^{(\ell)}$. This means that $(\sigma^{(\ell)} - 1) \in L^1(\mathbb{R})$, $\text{ie}$ $(\sigma^{(1)} - 1) \in L^1(\mathbb{R})$ and $(\sigma^{(2)} - 1) \in L^2(\mathbb{R})$.

**Lemma 1** Let $K^{(\ell)} = \mathcal{F}^{-1}[\sigma^{(\ell)} - 1]$, then one has the identity

$$I + K^{(1)} = \mathcal{F}^{-1} \circ M \circ D \circ (I + V^{(1)}) \circ D^{-1} \circ M^{-1} \circ \mathcal{F}.$$  \tag{2.2}

Similarly,

$$I + K^{(2)} = \mathcal{F}^{-1} \circ D \circ (I + V^{(2)}) \circ D^{-1} \circ \mathcal{F}.$$  \tag{2.3}

The operators $I + V^{(1)}$, resp. $I + V^{(2)}$, act on $L^2(\mathbb{R})$ with kernels

$$V^{(1)}(\xi, \eta) = \frac{\sqrt{\sigma^{(1)}(\xi)} - 1}{2\pi i (\xi - \eta)} \left[ e^{ix\frac{\xi - \eta}{2}} - e^{-ix\frac{\xi - \eta}{2}} \right];$$  \tag{2.4}

$$V^{(2)}(\xi, \eta) = \left[ \sigma^{(2)}(\xi) - 1 \right] \frac{e^{ix\frac{\xi - \eta}{2}} - e^{-ix\frac{\xi - \eta}{2}}}{2i\pi (\xi - \eta)}.$$  \tag{2.5}

and $D, M$ are the multiplication operator on $L^2(\mathbb{R})$

$$(Mg)(\xi) = \sqrt{\sigma^{(1)}(\xi)} - 1 \ g(\xi), \quad (Dg)(\xi) = e^{ix\xi/2} \ g(\xi).$$  \tag{2.6}

**Proof** — In any of these two cases, we have that

$$\forall g \in L^2(\mathbb{R}) \quad \mathcal{F} \left[ \int_0^x K^{(\rho)}(t - t')g(t') \ dt' \right](\xi) =$$

$$[\sigma^{(\rho)}(\xi) - 1] \int_{\mathbb{R}} d\eta \mathcal{F}[g](\eta) \ e^{ix(\xi - \eta)} \int_{\mathbb{R}} \frac{d\eta}{2i\pi (\xi - \eta)} = [O^{(\rho)} \circ V^{(\rho)} \circ O^{-1} \circ \mathcal{F}] \ [g](\xi).$$  \tag{2.7}

Where $O_1 = M \circ D$ and $O_2 = D$. Here, we precise that $V^{(1)} \circ O^{-1}$ is indeed a well defined operator on $L^2(\mathbb{R})$. \hfill \Box

These two identities relate the truncated Wiener-Hopf operator to a generalized sine kernel acting on $\mathbb{R}$. If one is able to construct the resolvent for this operator, then one is able to invert the corresponding Wiener-Hopf operator by using (2.2) or (2.3). This correspondence has already been used in [33] to build the resolvent of truncated Wiener-Hopf operators whose symbols are holomorphic, non-vanishing functions on some strip around the real axis that are decaying fast enough at infinity. In the case of symbols $\sigma^{(\ell)}$ having Fisher-Hartwig singularities as in (1.11), the expression for the leading asymptotic resolvent of the underlying generalized sine kernel is much more involved than for operators considered in [33], hence taking the Fourier transform and then obtaining some manageable result might be complicated.

The two identities given in Lemma 2 allow to establish a connection between sufficiently regularized Fredholm determinants of $I + K^{(\ell)}$ and those of the corresponding generalized sine kernel. Hence, to study the asymptotics of $\det_{\ell}[I + K^{(\ell)}]$ it is enough to focus on the ones of the associated generalized sine kernels.
Lemma 2 Let $K(\ell) = \mathcal{F}^{-1} [\sigma(\ell) - 1]$ be the kernel of the integral operator given in (2.1), then

$$\det_2 [I + K(2)] = \det_2 [I + V(2)] ,$$

and

$$\det [I + K(1)] = \det [I + V(1)] .$$

**Proof** — The second equality can be obtained thanks to the Fredholm’s series representation for the determinant of a trace class integral operator:

$$\det [I + K(1)] = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x d^n t \det_n [K(1) (t_i - t_j)]$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x \int d^n \xi \prod_{p=1}^n (\sigma(1) (\xi_p) - 1) \det_n [e^{-i \xi_p (t_p - t_j)}]$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x d^n \xi \det_n [V(1) (\xi_p, \xi_j)]$$

(2.10)

Now, let us prove the first identity. Define $R_2 (K(2)) = (I + K(2)) e^{-K(2)} - I$. Since $K_2$ is Hilbert-Schmidt, we have that $R_2 (K(2))$ is trace class [37]. Moreover, equation (2.3) implies that

$$R_2 (K(2)) = \mathcal{F}^{-1} \circ D \circ R_2 (V(2)) \circ D \circ \mathcal{F}$$

(2.11)

and $R_2 (V(2))$ is trace class as $V(2)$ is Hilbert-Schmidt. Moreover, the Fourier transform $\mathcal{F}$ its inverse $\mathcal{F}^{-1}$ as well as $D$ and $D^{-1}$ being continuous operators on $L^2 (\mathbb{R})$ we have that $R (V(2)) \circ D^{-1} \circ \mathcal{F}$ is trace class. We can thus change the order in the operator product appearing in the determinant so that

$$\det_2 [I + K(2)] = \det [I + \mathcal{F}^{-1} \circ D \circ R_2 (V(2)) \circ D^{-1} \circ \mathcal{F}]$$

$$= \det [I + R_2 (V(2)) \circ D^{-1} \circ \mathcal{F} \circ \mathcal{F}^{-1} \circ D] = \det [I + R_2 (V(2))] \; \square.$$

2.2 General assumptions

Motivated by the latter results, we consider the two Fredholm operators $I + V_\ell$ acting on $L^2 (\mathbb{R})$ and defined through equations (2.4) and (2.5). The operators are defined in terms of the symbols $\sigma_\ell$ below

$$\sigma_\ell (\xi) = F_\ell (\xi) \prod_{k=1}^n \sigma_{\nu_k, \nu_k} (\xi - a_k) .$$

(2.12)

The functions $F_\ell (\xi)$ are chosen according to

$$F_{(1)} (\xi) = b_{(1)} (\xi) \prod_{k=1}^n \left( 1 + \frac{2i \delta_k}{\xi + i} \right) ,$$

(2.13)

$$F_{(2)} (\xi) = b_{(2)} (\xi) .$$

(2.14)

and we assume that the functions $b_{(\ell)}$ are such that

- $b_{(\ell)}$ is holomorphic on some open neighborhood $U$ of $\mathbb{R}$;
\[ \bullet \ b_{(\ell)} \text{ never vanishes on } U; \]
\[ \bullet \ b_{(\ell)} (\xi) - 1 = O \left( |\xi|^{-\frac{\kappa}{2}} \right) \text{ for some } \kappa > 0, \text{ when } \xi \to \infty \text{ in } U. \]

We also make some assumptions on the exponents \( 2\delta_\ell = \nu_\ell - \nu_\ell \) and \( 2\gamma_\ell = \nu_\ell + \nu_\ell; \)
\[ \bullet \ \forall \ k, \ \Re (\gamma_\ell) < 1/2 \text{ in the } L^1 (\mathbb{R}) \text{ case; } \]
\[ \bullet \ \forall \ k, \ \Re (\gamma_\ell) < 1/4 \text{ in the } L^2 (\mathbb{R}) \text{ case; } \]
\[ \bullet \ \forall \ k, \ |\Re (\delta_\ell)| < 1/2 \text{ independently of the } L^1 (\mathbb{R}) \text{ or the } L^2 (\mathbb{R}) \text{ case.} \]

The behavior of \( \sigma_{\nu, \tau} (\xi) \) around \( \xi = 0 \) shows that the last restriction on \( \delta_\ell \) covers almost all the possible types of jump singularities \( \sigma_{(\ell)} (\xi) \) could have. However, the case \( \Re (\delta_\ell) = \pm 1/2 \) for some \( k \)'s should be treated separately. In particular, one expects additional corrections to the asymptotic formula \((6.36)\). These should have the same structure as those appearing in the Basor-Tracy conjecture for Toeplitz matrices \([3]\). We have the

**Lemma 3.** Under the above assumptions, the symbols \( \sigma_{(\ell)} \) given in \((2.12)\) are such that \( \sigma_{(1)} - 1 \in L^1 (\mathbb{R}) \) and \( \sigma_{(2)} - 1 \in L^2 (\mathbb{R}) \).

**Proof —** We give the proof in the \( L^1 (\mathbb{R}) \) case only. The assumptions on the parameters \( \gamma_\ell \) and the local behavior of \( \sigma_{\nu_\ell, \tau_\ell} (\xi - a_\ell) \) together with \( a_1 < \cdots < a_n \) ensure that \( \sigma_{(1)} - 1 \in L^1 (\mathbb{R}) \) for any finite \( M \). It remains to check the integrability at infinity. \( \sigma_{\nu_\ell, \tau_\ell} \) decreases at infinity as:

\[
\sigma_{\nu_\ell, \tau_\ell} (\xi) = 1 - 2i\delta_k \xi^{-1} + O \left( \xi^{-2} \right) \quad (2.15)
\]

Therefore, we have

\[
\sigma_{(1)} (\xi) = b_{(1)} (\xi) \left( 1 + \frac{2i}{\xi} \sum_{k=1}^{n} \delta_k + O \left( \xi^{-2} \right) \right) \prod_{k=1}^{n} (1 - 2i\delta_k \xi^{-1} + O \left( \xi^{-2} \right)) = b_{(1)} (\xi) - 1 + O \left( \xi^{-2} \right) = O \left( |\xi|^{1+\kappa} \right) \quad , \quad (2.16)
\]

and the claim follows. \( \square \)

**2.3 The resolvent**

Let \( f^{(p)}_{\pm} \) be the solutions to the integral equations

\[
f^{(p)}_{\pm} (\xi) + \int_{\mathbb{R}} (\sigma_{(p)} (\eta) - 1) \frac{\sin x (\xi - \eta) / 2}{\pi (\xi - \eta)} f^{(p)}_{\pm} (\eta) \, d\eta = e^{\pm ix \xi / 2} . \quad (2.17)
\]

It is well known \([30]\) that the resolvent operator \( I - R_{(1)}, \) resp. \( I - R_{(2)}, \) of \( I + V_{(1)}, \) resp. \( I + V_{(2)}, \) has a simple expression in terms of \( f^{(p)}_{\pm} \). Indeed

\[
R_{(1)} (\xi, \eta) = \frac{\sqrt{\sigma_{(1)} (\xi)} - 1}{2i\pi (\xi - \eta)} \left[ f^{(1)}_{+} (\xi) f^{(1)}_{-} (\eta) - f^{(1)}_{+} (\eta) f^{(1)}_{-} (\xi) \right] . \quad (2.18)
\]
\[
R_{(2)} (\xi, \eta) = \frac{\sigma_{(2)} (\xi) - 1}{2i\pi (\xi - \eta)} \left[ f^{(2)}_{+} (\xi) f^{(2)}_{-} (\eta) - f^{(2)}_{+} (\eta) f^{(2)}_{-} (\xi) \right] . \quad (2.19)
\]

9
Lemma 4 Suppose that $\det[I + V(1)] \neq 0$, resp. $\det_2[I + V(2)] \neq 0$, then the solutions $f^{(p)}_\pm$ of (2.17) are entire functions.

Proof — Suppose that $\det[I + V(1)] \neq 0$. Then $I + V(1) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible and its inverse $I - R_{(1)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ can be constructed in terms of a Fredholm series. Thus, since $e^{\pm ix^2} \sqrt{\sigma_{(1)}(\xi)} - 1 \in L^2(\mathbb{R})$, the unique solution $\tilde{f}_\pm = \sqrt{\sigma_{(1)}(\xi)} - 1 f^{(1)}_\pm(\xi)$ of

$$\tilde{f}_\pm(\xi) + \int \limits_{\mathbb{R}} V_{(1)}(\xi,\eta) \tilde{f}_\pm(\eta) \, d\eta = e^{\pm ix^2} \sqrt{\sigma_{(1)}(\xi)} - 1$$

(2.20)

belongs to $L^2(\mathbb{R})$. We also have that

$$\forall \xi \in \mathbb{C}, \quad \eta \mapsto \sqrt{\sigma_{(1)}(\eta)} - 1 \frac{\sin x(\xi - \eta)/2}{\pi(\xi - \eta)} \in L^2(\mathbb{R}) \ .$$

(2.21)

Therefore,

- $\eta \mapsto F(\xi,\eta) \equiv f^{(1)}_\pm(\eta) \left( \sigma_{(1)}(\eta) - 1 \right) \frac{\sin x(\xi - \eta)/2}{\pi(\xi - \eta)} \in L^1(\mathbb{R})$, for all $\xi \in \mathbb{C}$;
- $\xi \mapsto F(\xi,\eta)$ is entire for almost all $\eta$.

Thus,

$$\xi \mapsto \int \limits_{\mathbb{R}} F(\xi,\eta) \, d\eta$$

(2.22)

is an entire function. By (2.17), so is $f^{(1)}_\pm$. In the case $\det_2[I + V_2] \neq 0$, we have immediately that $(1 - \sigma_{(2)}(\xi)) f^{(2)}_\pm \in L^1(\mathbb{R})$, and the rest of the proof goes the same. □

2.4 Determinant identity

We shall now derive an important determinant identity. This identity allows to obtain the leading asymptotics of truncated Wiener-Hopf operators from thoses of its resolvent.

Lemma 5 Let $V_{(1)}$ and $V_{(2)}$ be as in (2.4) and (2.5) and defined in terms of the symbol $\sigma_{(1)}$ (2.12) subject to the assumptions of subsection 2.2. We also assume that $\det[I + V_{(1)}] \neq 0$ and $\det_2[I + V_{(2)}] \neq 0$. Suppose that $\beta_p$ equals $\delta_p$ or $\gamma_p$, then the following identities hold

$$\partial_{\beta_p} \log \det \left[ I + V_{(1)} \right] = \int \limits_{\mathbb{R}} \frac{R_{(1)}(\xi,\xi)}{\sigma_{(1)}(\xi)} \, d\xi \ ,$$

(2.23)

$$\partial_{\beta_p} \log \det_2 \left[ I + V_{(2)} \right] = \int \limits_{\mathbb{R}} \left\{ \frac{R_{(2)}(\xi,\xi)}{\sigma_{(2)}(\xi)} - 1 \partial_{\beta_p} \sigma_{(2)}(\xi) - \partial_{\beta_p} V_{(2)}(\xi,\xi) \right\} \, d\xi \ .$$

(2.24)

Proof — We first treat the $L^1(\mathbb{R})$ case. Let $\{\delta_p^0,\gamma_p^0\}_{p=1}^n$ be a point in $\mathbb{C}^{2n}$ fulfilling the assumptions of subsection 2.2 for the $L^1(\mathbb{R})$ case and such that $\det[I + V_{(1)}] \neq 0$. It then follows from the Fredholm series for $\det[I + V_{(1)}]$, that the latter is a holomorphic and non-vanishing function on some open neighborhood of $\{\delta_p^0,\gamma_p^0\}_{p=1}^n$. It is in particular
differentiable and its derivatives can be expressed by using the resolvent operator \( I - R_{(1)} \). Setting \( e^{2G_1(\xi)} = \sigma_{(1)} (\xi) - 1 \), we get

\[
\partial_{\beta_p} \log \det [I + V_{(1)}] = \text{tr} \left\{ \left( I - R_{(1)} \right) \cdot \partial_{\beta_p} V_{(1)} \right\} = \int_{\mathbb{R}} d\xi \partial_{\beta_p} G_1(\xi) \left[ V_{(1)} \cdot (I - R_{(1)}) \right] (\xi, \xi) + \left[ (I - R_{(1)}) \cdot V_{(1)} \right] (\xi, \xi) \partial_{\beta_p} G_1(\xi) \]

\[
= 2 \int_{\mathbb{R}} d\xi R_{(1)} (\xi, \xi) \partial_{\beta_p} G_1(\xi) = \int_{\mathbb{R}} d\xi R_{(1)} (\xi, \xi) \frac{\partial_{\beta_p} \sigma_{(1)}(\xi)}{\sigma_{(1)}(\xi)} - 1.
\]

In the intermediary equalities we used the symmetry of the kernels as well as the cyclicity of the trace. The \( L^2 (\mathbb{R}) \) case is proved by density. Let \( \chi_{\epsilon} \) be the characteristic function of \( ]-\epsilon: \epsilon[ \). Then \( I + V_{2\epsilon} \), with \( V_{2\epsilon} (\xi, \eta) \equiv \chi_{\epsilon} (\xi) V_{(2)} (\xi, \eta) \chi_{\epsilon} (\eta) \), is trace class for all \( \epsilon > 0 \) so that

\[
\det_2 [I + V_{2\epsilon}] = \det [I + V_{2\epsilon}] e^{-\epsilon \text{tr} V_{2\epsilon}}.
\] (2.25)

As \( \det_2 [I + V_{(2)}] \neq 0 \), \( \det_2 [I + V_{2\epsilon}] \neq 0 \) for \( \epsilon \) large enough, and hence \( \det [I + V_{2\epsilon}] \neq 0 \) as well. One can then apply the results for \( L^1 (\mathbb{R}) \) kernels for the \( \beta_p \) derivative. We get,

\[
\partial_{\beta_p} \log \det_2 [I + V_{2\epsilon}] = \int_{\mathbb{R}} d\lambda \chi_{\epsilon} (\lambda) \frac{R_{2\epsilon}(\lambda, \lambda)}{\sigma_2 - 1} \partial_{\beta_p} \sigma_2 (\lambda) - \int_{\mathbb{R}} d\lambda \chi_{\epsilon} (\lambda) \partial_{\beta_p} V_{(2)} (\lambda, \lambda).
\] (2.26)

The \( \epsilon \to +\infty \) limit in the rhs becomes licit after merging the two integrals into one. \( \Box \).

The asymptotic solution of the RHP presented in the upcoming sections will allow us to construct approximate in \( x \) resolvents of \( V_1 \) and \( V_2 \) uniformly in respect to the parameters \( \delta_p \) and \( \gamma_p \). It will then remain to use these approximations to compute, in the large \( x \) limit, the integrals appearing in (2.24) or (2.23). Once this is done, it is enough to integrate the result from \( \beta_p = 0 \) to \( \beta_p \). Such an integration is analogous to the separation technique in the operator approach to asymptotics of Toeplitz determinants \( 2 \). It is then not a problem to repeatedly apply the procedure so as to obtain the asymptotics of the determinant. At this stage, it becomes clear why, in our approach, the \( L^1 (\mathbb{R}) \) case doesn’t follow from the \( L^2 (\mathbb{R}) \) one. As a matter of fact, if we want to keep jump singularities and still have an \( L^1 (\mathbb{R}) \) kernel, we ought to add an additional factor depending on the \( \delta_p \)'s as it was explicitly done for the function \( F_{(1)} \), cf (2.12). This modifies the \( \delta_p \) dependence of the integrand in (2.23) and hence the integration procedure. The result should also be, in principle modified, but eventually we see that the \( L^1 (\mathbb{R}) \) case can be obtained from the \( L^2 (\mathbb{R}) \) one by restricting correctly the parameters and replacing \( F_{(2)} \) by \( F_{(2)} \). However, since these are only minor modifications, from now on we focus on the \( L^2 (\mathbb{R}) \) case. The interested reader will find no problem in adapting the proofs to the \( L^1 (\mathbb{R}) \) case. Accordingly, from now on, we drop the \( \ell \) subscript labeling \( \sigma_{(\ell)} \), the kernels \( V_{(\ell)} \) and the resolvents \( R_{(\ell)} \). We will denote these quantities by \( \sigma, V \) and \( R \) and assume that \( \sigma = \sigma_{(2)} \) as defined in (2.12).

3 The Riemann–Hilbert Problem

We start this Section by introducing a RHP for a matrix \( \chi \). This type of RHP is adapted for constructing resolvents of integrable integral operators \( 30 \) such as the generalized sine
We then perform a few transformations of this original RHP, in order to boil it down to one where the jump matrices will be \( I_2 + o(1) \) uniformly away from the points \( a_k \) and in respect to the \( x \to +\infty \) limit. The first step will consist in finding a scalar valued function \( \alpha \) such that \( \chi \alpha^{a_k} \) has a jump matrix with 1 in its lower diagonal entry. Then we deform the original cut. The jump matrix on the new contour has the desired properties.

### 3.1 The initial Riemann-Hilbert problem

As first observed in [30], the problem of finding the resolvent of any integrable integral operator is equivalent a Riemann-Hilbert problem (RHP). Indeed, let \( f_\pm \) be the solutions of (2.17) for the corresponding \( \sigma \). Then, there exists [30] a matrix \( \chi \) allowing to reconstruct the solutions \( f_\pm \) of the integral equation (2.17), and hence the resolvent:

\[
\left( f_+ (\xi), f_- (\xi) \right) = \chi (\xi) \left( e^{ix_2 \xi}, e^{-ix_2 \xi} \right) \chi^{-1} (\xi) .
\]

This matrix \( \chi \) solves the RHP:

- \( \chi \) is analytic on \( \mathbb{C} \setminus \mathbb{R} \);
- \( \forall k \in [1; n] \), there exists \( M_k \in \text{GL}_2 (\mathbb{C}) \) such that
  \[
  \chi = M_k \left\{ I_2 + g (z) B_k + |z - a_k| (g (z) + 1) O \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \right\}, \ z \to a_k ;
  \]
- \( \chi \xrightarrow{z \to \infty} I_2 \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) ;
  \]
- \( \chi_+ (z) G (z) = \chi_- (z) ; \quad z \in \mathbb{R} .
  \]

The matrix \( B (z) \) appearing in the estimates around \( a_k \) is a rank one matrix that takes the precise form

\[
B (z) = \left( \begin{array}{cc} -1 & e^{ixz} \\ e^{-ixz} & 1 \end{array} \right) .
\]

The function \( g \) reads

\[
g (z) = \int_{\mathbb{R}} \frac{ds \sigma (s) - 1}{2i\pi z - s}
\]

and has the local behavior [25] at \( z \to a_k \):

\[
g (z) = \begin{cases} O (1) + O \left( (z - a_k)^{-2\gamma_k} \right) & \text{for } \gamma_k \neq 0 \\ O (\log (z - a_k)) & \text{for } \gamma_k = 0 \end{cases}
\]

The matrices \( M_k \) are \textit{apriori} unknown and will be determined once the solution is known. What only matters for the solvability is the invertibility of \( M_k \). Lastly, we adopt the convention that the symbol \( O (M) \) for some matrix \( M \) is to be understood entrywise \( ie \chi = O (M) \) means that \( \chi_{ij} = O (M_{ij}) \).

The jump matrix \( G \) appearing in the RHP reads

\[
G (z) = \left( \begin{array}{cc} 2 - \sigma (z) & (\sigma (z) - 1) e^{ixz} \\ (1 - \sigma (z)) e^{-ixz} & \sigma (z) \end{array} \right) .
\]

Finally, \( \chi_+ (t) \) (resp. \( \chi_- (t) \)) stands for the non-tangential limits of \( \chi (z) \) as \( z \) approaches a point \( t \) of the contour from its + (resp. -) side.
Proposition 3.1 Whenever $\Re (\gamma_k) < 1/2$, $\forall k \in [1; n]$ and $\text{det}_2 [I + K] \neq 0$, the solution to the RHP exists and is unique.

Although most of the proof is standard, see for example [36] (careful handling of the singularities) and [13] (general exposure), we include it for the sake of completeness and the reader’s convenience. The last part of the proof dealing with a cancelation of the singularities due rank $(B(z)) = 1$ is new.

Proof —

The RHP is equivalent to the singular integral equation for $\chi$:

$$\chi(z) = I_2 + \int_{\mathbb{R}} \frac{ds}{2i\pi} \left( \frac{\sigma(s) - 1}{z - s} \chi_+(s) \begin{pmatrix} -1 & e^{ixs} \\ -e^{-ixs} & 1 \end{pmatrix} \right), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.6)$$

Since we assume that $\text{det}_2 [I + K] \neq 0$, as already mentioned, the resolvent operator $I - R$ exists. Hence, one can express a solution of equation (3.6) in terms of $R$ or, equivalently, in terms of $f_\pm$ (2.17) (whose existence follows from the existence of the resolvent):

$$\chi(z) = I_2 + \int_{\mathbb{R}} \frac{ds}{2i\pi} \left( \frac{\sigma(s) - 1}{z - s} \begin{pmatrix} -f_+(s) e^{-ixs} & f_+(s) e^{ixs} \\ -f_- (s) e^{-ixs} & f_- (s) e^{ixs} \end{pmatrix} \right). \quad (3.7)$$

This proves the existence of solutions provided we show that (3.7) has the desired behavior around each point $a_k$. The solution of the jump conditions given in (3.7) can be written as

$$\chi(z) = M(z) + \left( \begin{pmatrix} -f_+(z) e^{-ixs} & f_+(z) e^{ixs} \\ -f_- (z) e^{-ixs} & f_- (z) e^{ixs} \end{pmatrix} \right) \int_{\mathbb{R}} \frac{ds}{2i\pi} \left( \frac{\sigma(s) - 1}{z - s} \right) \quad (3.8)$$

with

$$M(z) = I_2 + \int_{\mathbb{R}} \frac{ds}{2i\pi} \left( \frac{\sigma(s) - 1}{z - s} \right) \left\{ \begin{pmatrix} f_+(s) \\ f_- (s) \end{pmatrix} \begin{pmatrix} -e^{-ixs} & e^{ixs} \\ e^{-ixs} & e^{ixs} \end{pmatrix} \right\}.$$ 

The matrix $M(z)$ is holomorphic around $z = a_k$, and it is easily seen from (3.1) that

$$\begin{pmatrix} f_+(z) \\ f_- (z) \end{pmatrix} = M(z) \begin{pmatrix} e^{ixs} \\ e^{-ixs} \end{pmatrix}. \quad (3.9)$$

Therefore,

$$\chi(z) = M(z) \left\{ I_2 + g(z) \begin{pmatrix} -1 & e^{ixs} \\ e^{-ixs} & 1 \end{pmatrix} \right\} \quad \text{with} \quad g(z) = \int_{\mathbb{R}} \frac{ds}{2i\pi} \left( \frac{\sigma(s) - 1}{z - s} \right). \quad (3.10)$$
The local behavior of $g$ around $z = a_k$ can be inferred from [25]. Then, the claim for the local estimates follows by expanding the holomorphic matrices around $z = a_k$ and setting $M (a_k) = M_k$. We shall now prove that $M_k \in \text{GL}_2(\mathbb{C})$

Given any solution $\chi$ to the above RHP, $\det [\chi]$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. Since $\det [G] = 1$ we have that $\det [\chi]$ is continuous across $\mathbb{R} \setminus \cup_{p=1}^{n} \{a_p\}$ and can thus be extended to an analytic function on $\mathbb{C} \setminus \cup_{p=1}^{n} \{a_p\}$. Using the estimates for $\chi$ and the multilinearity of the determinant, we get that for $z \to a_k$

$$\det [\chi] = \det [M_k] \det [I_2 + g (z) B (z)] + O (g (z) (1 + g (z)) |z - a_k|) \ .$$

As $B (z)$ is a rank one matrix $\det [I_2 + g (z) B (z)] = g (z) \text{tr} [B (z)] = 0$, and thus the first term is a $O (1)$. The estimates for the local behavior of $g (z)$ together with the hypothesis on the parameters $\gamma_k$ ensure that $O (g (z) (1 + g (z)) |z - a_k|) = o \left (|z - a_k|^{-1} \right )$. $\det [\chi]$ has thus no pole at $z = a_k$. Its singularities at the $a_k$'s are thus of a removable type and hence $\det [\chi]$ is holomorphic around $a_k$. It follows that $\det [\chi]$ an entire function that is bounded at infinity in virtue of the normalization $\chi \xrightarrow{z \to \infty} I_2$. By Liouville’s theorem, $\det [\chi] = 1$. In particular $\det [\chi] (a_k) = 1$, which can only happen if $\det [M_k] = 1$.

We end the proof by showing the uniqueness of solutions. Let $\chi_1$ and $\chi_2$ be two solution of the RHP for $\chi$. Then as $\det [\chi_2] = 1$, $\chi_2$ is analytically invertible on $\mathbb{C} \setminus \mathbb{R}$. The matrix $\chi_1 \chi_2^{-1}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and continuous across $\mathbb{R} \setminus \bigcup_{k=1}^{n} \{a_k\}$. Moreover, using the local behavior at $z = a_k$ we get

$$\chi_1^{-1} (z) = (I_2 + t \text{Comat} (B (z)) g (z) ) M_{k,2}^{-1} + O \left ( (1 + g (z)) |z - a_k| \left ( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right ) \right ) \ .$$

Here, we used the fact that the inverse of $\chi_2$ is given by the transpose of its comatrix due to $\det [\chi_2] = 1$. We have also introduced two, a priori distinct, matrices $M_{k,\ell}$ associated with each of the solutions $\chi_\ell$. Computing the matrix products and using that $B (z) + t \text{Comat} (B (z)) = 0, B (z), t \text{Comat} (B (z)) = 0$, we get

$$\chi_1 (z) \chi_2^{-1} (z) = M_{k,1} M_{k,2}^{-1} + O \left ( (g (z) (1 + g (z)) |z - a_k| \left ( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right ) \right ) \ ,$$

The local estimates for $g$ imply that $g (z) (1 + g (z)) |z - a_k| = o \left (|z - a_k|^{-1} \right )$. Hence $\chi_1 (z) \chi_2^{-1} (z)$ has no poles at $z = a_k$. The singularities at the $a_k$’s are thus removable and, because of the asymptotic condition, we have $\chi_1 \chi_2^{-1} = I_2$. This guarantees the uniqueness of the solution to the RHP, at least for $\Re (\gamma_k) < 1/2$. \hfill \Box

Note that one doesn’t have to make such fine estimates for proving the uniqueness of solutions if one would assume that $\Re (\gamma_k) < 1/4$. However, we presented here this more complex proof as it also holds in the $L^1 (\mathbb{R})$ case.

### 3.2 A helpful scalar Riemann-Hilbert problem.

Let $\alpha$ be the solution of the following scalar Riemann-Hilbert problem

$$\alpha \text{ is analytic on } \mathbb{C} \setminus \mathbb{R} \ , \ \alpha_- (\xi) = \alpha_+ (\xi) \sigma (\xi) , \quad \xi \in \mathbb{R} \setminus \bigcup_{k=1}^{n} \{a_k\} \ , \ \alpha (\xi) \xrightarrow{\xi \to \infty} 1 \ . \quad (3.14)$$
We introduce the auxiliary functions $F$ and $\Xi$ as

\begin{equation}
\alpha_+ (z) = F_+^{-1} (z) \prod_{k=1}^{n} \left( \frac{z - a_k + i}{z - a_k + i} \right)^{\nu_k} \quad z \in H_+ ,
\end{equation}

\begin{equation}
\alpha_- (z) = F_- (z) \prod_{k=1}^{n} \left( \frac{z - a_k - i}{z - a_k - i} \right)^{\nu_k} \quad z \in H_- .
\end{equation}

$H_\pm$ is the upper/lower half-plane and $F_\pm$ are the Wiener-Hopf factors of $F$, i.e. $F = F_+ F_-$ with $F_+$ (resp. $F_-$) analytic in the upper (resp. lower) half-plane and going to 1 at $z \to \infty$ in $H_+$ (resp. $H_-$). There is no constant factor $F_0$ in the Wiener-Hopf decomposition of $F$ as $F \to 1$.

The Wiener-Hopf factors of $F$ have an integral representation either in terms of Cauchy or Fourier transforms of log $F$:

\begin{equation}
\log F_+ (z) = \int_{\mathbb{R}} \frac{d\xi}{2\pi i} \log F (\xi) \xi - z = F \left[ \Xi (\xi) F^{-1} \log F (\xi) \right] (z) \quad z \in H_+ ,
\end{equation}

\begin{equation}
\log F_- (z) = -\int_{\mathbb{R}} \frac{d\xi}{2\pi i} \log F (\xi) \xi - z = F \left[ \Xi (-\xi) F^{-1} \log F (\xi) \right] (z) \quad z \in H_- ,
\end{equation}

and $\Xi$ is Heaviside’s step function. As $F$ is analytic and non zero in $U$, it is clear from these integral representations that $F_+$ and $F_-$ have an analytic continuation to $U$. Moreover, as $F$ is non-vanishing in $U$ and the decomposition $F = F_+ F_-$ is still valid on $U$, $F_+$ and $F_-$ have no zeroes on $U$.

We introduce the auxiliary functions

\begin{equation}
\sigma_p (z) = F (z) \prod_{k=1}^{p} \left( \frac{z - a_k + i}{z - a_k} \right)^{\nu_k} \left( \frac{z - a_k - i}{z - a_k} \right)^{\nu_k} \prod_{k=p+1}^{n} \left\{ \left( \frac{z - a_k + i}{a_k} \right)^{\nu_k} \left( \frac{z - a_k - i}{a_k} \right)^{\nu_k} e^{2i\pi \delta_k} \right\} ,
\end{equation}

\begin{equation}
\tilde{\sigma}_p (z) = (z - a_p)^{2\nu_p} \sigma_p (z) .
\end{equation}

The function $\sigma_p$, resp. $\tilde{\sigma}_p$, is holomorphic on $\{ z : a_p < \Re z < a_{p+1} \} \cap U$, resp. $D_{a_p, \epsilon} = \{ z \in C : |z - a_p| < \epsilon \}$. Here and in the following, $\epsilon$ is such that $D_{a_p, \epsilon} \subset U$ and $D_{a_p, \epsilon} \cap D_{a_q, \epsilon} = \emptyset$ for $p \neq q$. The functions $\sigma_p$ and $\tilde{\sigma}_p$ can be thought of as the analytic parts of the local behavior of $\sigma$ on the real axis. Namely, one can continue $\sigma$ by analyticity to the domains below

\begin{equation}
\sigma (z) = \begin{cases} 
\sigma_p (z) & z \in \{ z : a_p < \Re (z) < a_{p+1} \} \cap U \\
\tilde{\sigma}_p (z) e^{2i\pi \nu_p \Xi (\Re (a_p - z))} & z \in H_+ \cap D_{a_p, \epsilon} \\
\tilde{\sigma}_p (z) e^{-2i\pi \nu_p \Xi (\Re (a_p - z))} & z \in H_- \cap D_{a_p, \epsilon}
\end{cases}
\end{equation}

\footnote{We stress that the subscript $p$ appearing in $\sigma_p$ has nothing to do with the notations of section 2. In the following, $\sigma_p$ will always refer to the definition (3.15).}
In much the same way, we split the formula for $\alpha$ into a holomorphic and a singular part:

$$
\begin{align*}
\alpha_+^2 (z) &= \hat{\sigma}_p^{-1} (z) (z - a_p)^{\nu_p} e^{-2i\varphi_p} K_p (z) , \quad z \in \mathcal{H}_+ \cap D_{a_p, \epsilon} ; \\
\alpha_-^2 (z) &= \hat{\sigma}_p (z) (z - a_p)^{-\nu_p} e^{2i\varphi_p} K_p (z) , \quad z \in \mathcal{H}_- \cap D_{a_p, \epsilon} .
\end{align*}
$$

There,

$$K_p (z) = \frac{z^{2\gamma_p} e^{i\pi \nu_p}}{F_+ (z) F_- (z)} \prod_{k=1}^{n} \frac{z - a_k - i \nu_k}{z - a_k + i \nu_k} \prod_{k=p+1}^{n} \frac{z - a_k}{z - a_k - i \nu_k} \prod_{k=p+1}^{n} \left\{ \frac{(a_k - z)^{\nu_k}}{(a_k - z)^{-\nu_k}} e^{2i\pi \gamma_k} \right\} .
$$

The analyticity of $F (z)$ on $U$ guarantees that $K_p (z)$ is holomorphic on the disk $D_{a_p, \epsilon}$. Lastly, we define $\alpha_{\alpha / \beta}^{(p)}$, the regularized version of $\alpha_{\alpha / \beta}$ around $a_p$, according to

$$
\alpha_{\alpha / \beta}^{(p)} (\xi) = \alpha_{\alpha} (z) (z - a_p)^{-\nu_p ,} , \quad \alpha_{\beta / \alpha}^{(p)} (\xi) = \alpha_{\beta} (z) (z - a_p)^{\nu_p} .
$$

So that,

$$
\frac{\alpha_{\beta / \alpha}^{(p)} (z)}{\alpha_{\alpha / \beta}^{(p)} (z)} = \hat{\sigma}_p (z) .
$$

### 3.3 The first step $\chi \to \Phi$

Let $\Phi$ be related to $\chi$ by

$$
\Phi (z) = \chi (z) [\alpha (z)]^{\sigma_3} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
$$

Then $\Phi (z)$ satisfies the following RHP:

- $\Phi$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ ;
- $\forall k \in [1 ; n]$ , there exists $M_k \in \text{GL}_2 (\mathbb{C})$ such that:

$$
\Phi = M_k \left\{ I_2 + g (z) B (z) + |z - a_k| (g (z) + 1) O \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \right\} [\alpha (z)]^{\sigma_3} ;
$$

- $\Phi \to I_2$ ;
- $\Phi_+ (z) G_\Phi (z) = \Phi_- (z) , \quad z \in \mathbb{R}$ ;

The function $g$ and the rank one matrix $B (z)$ are as given in (3.3) and (3.2). In particular $g$ has a singular behavior at $z \to a_k$ given by (3.4). $\alpha$, the solution of the scalar RHP (3.14), introduces an additional singular behavior at $z \to a_k$:

$$
\alpha (z) = \begin{cases} 
O \left( |z - a_k|^{R (\nu_k)} \right) & \text{for } z \to a_k , \ z \in \mathcal{H}_+ \\
O \left( |z - a_k|^{R (-\nu_k)} \right) & \text{for } z \to a_k , \ z \in \mathcal{H}_- 
\end{cases}
$$

Finally, the jump matrix for $\Phi$ reads

$$
G_\Phi (z) = \begin{pmatrix} 1 + P (z) Q (z) & P (z) e^{ixz} \\ Q (z) e^{-ixz} & 1 \end{pmatrix} ,
$$

16
and
\[ P(z) = \left[ 1 - \sigma^{-1}(z) \right] \alpha_+^{-2}(z), \]  
\[ Q(z) = \left[ \sigma^{-1}(z) - 1 \right] \alpha_-^2(z). \]  
(3.24)  
(3.25)

Clearly the solution of the RHP for \( \Phi \) exists as it can be built out of \( \chi \). Its uniqueness can be seen along the same lines as in proposition 3.1. Note that, because of the different possible analytic continuations of \( \sigma \) to the upper/lower half-planes \([\mathbb{R}(a_p - z)]\), \( P \) and \( Q \) will have different analytic continuations to the complex plane depending on the value of \( \Re(z) \). In particular,
\[ P(z) = \alpha_+^{-2}(z) - \frac{e^{-2i\pi p_2 \Xi(\Re(a_p - z))}}{K_p(z)} \left[ x(z - a_p) \right]^{2\delta_p} \quad z \in D_{a_p, e} \cap \mathcal{H}_+, \]  
\[ Q(z) = K_p(z) e^{-2i\pi p_2 \Xi(\Re(a_p - z))} \left[ x(z - a_p) \right]^{2\delta_p} - \alpha_+^2(z) \quad z \in D_{a_p, e} \cap \mathcal{H}_-. \]  
(3.26)

### 3.4 The second step \( \Phi \to \Upsilon \)

We now perform a transformation on \( \Phi \). The resulting matrix \( \Upsilon \) will have its jump matrices exponentially close to the identity matrix, except in the vicinities of the singularities of \( \sigma \). The jump matrix \( G_\Phi \) can be factorized into a product of an upper by a lower triangular matrix:
\[ G_\Phi = M_\up\ M_\down. \]  
(3.27)

The matrices \( M_\up \) (resp. \( M_\down \))
\[ M_\up(z) = \begin{pmatrix} 1 & P(z) e^{ixz} \\ 0 & 1 \end{pmatrix}, \]  
(3.28)
\[ M_\down(z) = \begin{pmatrix} 1 & Q(z) e^{-ixz} \\ 0 & 1 \end{pmatrix}. \]  
(3.29)

admit analytic continuations from the intervals \([ -\infty; a_1 \[ \] a_2 \[ \ldots \] a_n ; +\infty \[ \) to some interval depending domains in \( U \cap \mathcal{H}_+ \) (resp.\( U \cap \mathcal{H}_{++} \)). These analytic continuations are different if one starts from different intervals. In the following, so as to avoid any confusion, for \( \exists z > 0, M_\up(z) \) should be understood as the analytic continuation of \( M_\up(\Re(z)) \) from the interval containing \( \Re(z) \). A similar statement holds for \( M_\down(z) \).

We draw a new contour \( \Gamma = \Gamma_\up \cup \Gamma_\down \cup \bigcup_{k=1}^{n} \Gamma_\up^{(k)} \cup \Gamma_\down^{(k)} \) in \( U \). It allows to defines a piecewise holomorphic matrix function \( \Upsilon(z) \) according to Fig.[2] as readily checked, \( \Upsilon(z) \) is continuous across \( \mathbb{R} \setminus \bigcup_{i=1}^{n} \{ a_i \} \) and hence holomorphic in the interior of this new contour. By construction, \( \Upsilon \) has cuts on the exterior contour \( \Gamma_\up \cup \Gamma_\down \). The additional cuts along \( \bigcup_{k=1}^{n} \Gamma_\up^{(k)} \) and \( \Gamma_\down^{(k)} \) are due to the different analytic continuations for \( P \) and \( Q \) (and hence \( M_\up \) and \( M_\down \)) to the strips \( \{ z : a_k < \Re(z) < a_{k+1} \} \) cf \( (3.26) \).

The matrix \( \Upsilon \) solves the following RHP:
- \( \Upsilon \) is analytic in \( \mathbb{C} \setminus \Gamma \);
- \( \forall k \in \{1 \ldots n\} \), there exists \( M_k \in \text{GL}_2(\mathbb{C}) \) such that:
  \[ \Upsilon(z) = M_k \left\{ I_2 + g(z) B(z) + |z - a_k| g(z) + 1 \right\} M(z) \quad z \to a_k \]
- \( \Upsilon \to I_2 \) as \( z \to \infty \).
There, the rank one matrix $B(z)$ and the function $g(3.2)$ are as in the RHP for $\chi$. We remind that $g$ has a singular behavior at $a_k$ given by $(3.3)$. The matrix $M$ is expressed in terms of $\alpha$ and $M_{\ell/j}$ according to:

$$
M(z) = \begin{cases}
\alpha^\sigma_3 M_j(z) & z \in \left( \mathcal{H}_+ \setminus \bigcup_{k=1}^n \Gamma^{(k)}_+ \right) \cap U \\
\alpha^\sigma_3 M_l(z) & z \in \left( \mathcal{H}_- \setminus \bigcup_{k=1}^n \Gamma^{(k)}_- \right) \cap U
\end{cases}
$$

(3.30)

It is readily checked that the matrix $M(z)$ has a singular behavior at $z \to a_k$ given by

$$
M(z + a_k) = \begin{cases}
O \left( \begin{array}{cc}
|z|^{R(\nu_k)} & |z|^{\min(-R(\nu_k), R(\nu_k))} \\
0 & |z|^{-R(\nu_k)}
\end{array} \right) & \text{for } z \to 0 , z \in \mathcal{H}_+ \\
O \left( \begin{array}{cc}
|z|^{-R(\nu_k)} & 0 \\
|z|^{\min(-R(\nu_k), R(\nu_k))} & |z|^{R(\nu_k)}
\end{array} \right) & \text{for } z \to 0 , z \in \mathcal{H}_-
\end{cases}
$$

(3.31)

Finally, the jump matrices $N^{(l)}(z)$, $\overline{N}^{(l)}(z)$ are defined by

$$
N^{(l)}(z) = \begin{pmatrix} 1 & n_l(z)e^{ixz} \\ 0 & 1 \end{pmatrix} = \lim_{\epsilon \to 0} M_+^{-1}(z - \epsilon) M_+(z + \epsilon) \quad z \in \Gamma^{(l)}_+ ,
$$

$$
\overline{N}^{(l)}(z) = \begin{pmatrix} 1 & n_l(z)e^{-ixz} \\ 0 & 1 \end{pmatrix} = \lim_{\epsilon \to 0} M_-^{-1}(z - \epsilon) M_-(z + \epsilon) \quad z \in \Gamma^{(l)}_- ;
$$
and their entries read
\[ n_l(z) = \frac{[x(z-a_l)]^{2\delta}}{K_l(z)e^{ix\sigma_3}} (e^{-2ix\nu_l} - 1), \quad \text{(3.32)} \]
\[ \bar{n}_l(z) = \frac{e^{ix\sigma_3}K_l(z)}{[x(z-a_l)]^{2\delta}} (e^{2ix\nu_l} - 1). \quad \text{(3.33)} \]

The solution of the RHP for \( Y \) clearly exists and its uniqueness follows from a similar reasoning to proposition 3.1. The matrices \( Y \) and \( \chi \) are thus in a one-to-one correspondence.

Note that, apart from vicinities of the points \( a_i, i \in [1; n] \), the jump matrices for \( Y \) are exponentially close to the identity. We have almost been able to recast the original RHP into one suited for the Deift-Zhou steepest descent \[19\]; it only remains to build the parametrices around the \( a_i \)'s.

4 Construction of the Parametrices, last transformation

We first construct the parametrix for the model RHP on a small disc \( D_{0, \epsilon} \) of radius \( \epsilon > 0 \) and centered at 0. This model parametrix will be the key ingredient of the parametrices around the \( a_i \)'s.

4.1 The model parametrix.
The model parametrix \( P \) is a solution to the following RHP:

- \( P \) is analytic \( D_{0, \epsilon} \setminus \{ \Gamma_+ \cup \Gamma_- \} \);
- \( P = I_2 + O \left( \frac{1}{\epsilon x} \right), \quad z \in \partial D_{0, \epsilon} \) uniformly;
- \( \left\{ \begin{array}{l} P_+(z)N(z) = P_-(z), \quad z \in \Gamma_+ \cap D_{0, \epsilon} \\ P_+(z)\bar{N}(z) = P_-(z), \quad z \in \Gamma_- \cap D_{0, \epsilon} \end{array} \right. \)

The jump matrices of this model RHP read
\[ N(z) \begin{pmatrix} 1 & [xz]^{2\delta} e^{ixz} e^{-2ix\nu_l} \left( e^{-2ix\nu_l} - 1 \right) \\ 0 & 1 \end{pmatrix}, \]
\[ \bar{N}(z) \begin{pmatrix} K(z) & 0 \\ \frac{1}{[xz]^{2\delta}} e^{-ixz} \left( e^{2ix\nu_l} - 1 \right) & 1 \end{pmatrix}, \]
and the function \( K(z) \) is assumed to be holomorphic and non-vanishing on \( D_{0, \epsilon} \). Note that the boundary \( \partial D_{0, \epsilon} \) of the disk \( D_{0, \epsilon} \) is canonically oriented just as depicted in Fig. 3.

The RHP for \( P \) admits many solutions. For instance having one solution \( P \), one can build another one by multiplying \( P \) on the left by a holomorphic matrix on \( D_{0, \epsilon} \) that is equal to \( I_2 \) up to corrections that are uniformly an \( O(1/\epsilon x) \) on \( \partial D_{0, \epsilon} \) and hence on the whole disk \( D_{0, \epsilon} \).

A solution to the above RHP can be built thanks to the following procedure. We first assume that \( K \) is a constant. Then the Riemann-Hilbert problem for \( P \) can be solved by the standard procedure. One performs the transformation
\[ P(z) = \Theta(\zeta) \left[ \frac{\xi^2 \zeta^3}{\xi^2 \zeta^3} \right]^{-\sigma_3} \quad \text{with} \quad \zeta = xz \quad \text{(4.1)} \]
so that the jump matrices for Θ are piecewise constant, and Θ is a solution of a RHP on $D_{0,\epsilon}$. This RHP is solved explicitly in the limit $x\epsilon \to +\infty$ by the standard differential equation method [28]. It is then enough to go back to the original matrix $P$. Eventually, we get that

$$P(z) = \frac{L}{(xz)^{\delta \sigma_3}}$$

(4.2)

Here $\Psi(a,c;z)$ is Tricomi’s confluent hypergeometric function (CHF) given in (A.3). We remind that it has a cut on $\mathbb{R}^-$.

Lastly, one has $\det [P] = 1$.

Using the asymptotic behavior of Tricomi’s CHF (A.6), one readily checks that $P$ has indeed the correct asymptotic behavior. The jump conditions can be checked thanks to the monodromy properties of Tricomi’s CHF. These are (A.4) for the jump condition on $\Gamma_+$ and (A.5) in the case of the jump condition on $\Gamma_-$. Moreover, $P$ has no jump across $D_{0,\epsilon} \setminus [-\epsilon;0]$ : the discontinuity in $L$ is there to compensate the one of $(z)^{\gamma - \delta \sigma_3}$. Hence $P$ is holomorphic on $[-\epsilon;0]$. The fact that $\det [P] = 1$ can be seen as follows. We first assume that $|\Re(\gamma)| < 1/4$. Then, using the local behavior at $z = 0$ of Tricomi’s CHF (A.10), we get that $\det [P] = o(z^{-1})$. On the other hand, writing the jump conditions for $\det [P]$, one can easily convince oneself that the latter function is holomorphic on $\mathbb{C} \setminus \{0\}$. Its singularity at $z = 0$ is thus of a removable type. As $\det [P] \to 1$ when $z \to \infty$, we get that necessarily $\det [P] = 1$. To reach the case of generic parameters $\gamma$, we fix $z \neq 0$.\n
Figure 3: Set of contours in the RHP for the model parametrix.
and invoke the fact that \( \Psi (a, c; z) \) is a holomorphic function of \( a \) and \( c \). It follows that 
\[
\det [P] (z) \text{ is holomorphic in } \gamma.
\]
As it is constant in the region \( |\Re (\gamma)| < 1/4 \), we get that it is constant on \( \mathbb{C} \). Hence, \( \det [P] (z) = 1 \) for all \( z \neq 0 \). Now we get that \( \det [P] \) is bounded in every punctured neighborhood of \( z = 0 \). It thus follows that it cannot have any power-law singularity at \( z = 0 \), and \( \det [P] = 1 \).

In this way we have built a solution of the RHP for \( P \) in the case of constant functions \( K \). In order to extend this solution to functions \( K \) that are holomorphic and non-vanishing in some open neighborhood of \( \overline{D}_{0,\epsilon} \), it is enough to notice that replacing the constant \( K \) appearing in the formulae above by a holomorphic non-vanishing function \( K(z) \) on \( \overline{D}_{0,\epsilon} \) doesn’t change the analyticity of \( P \) on \( D_{0,\epsilon} \setminus (\Gamma_+ \cup \Gamma_-) \), nor its asymptotic behavior on the boundary \( \partial D_{0,\epsilon} \). As the jump conditions hold pointwise, these are also satisfied. Hence we get a solution to the general RHP for the parametrix.

### 4.2 The parametrix around \( a_k \)

Let \( P_{a_k} \) be defined as

\[
P_{a_k}(z) = \begin{pmatrix}
\Psi (\gamma_p - \delta_p; -i\xi_p) & ib_{12}^{(p)} (z) \Psi (1 + \gamma_p + \delta_p; i\xi_p) \\
-i\delta_{21}^{(p)} (z) \Psi (1 + \gamma_p - \delta_p; -i\xi_p) & \Psi (\gamma_p + \delta_p; i\xi_p)
\end{pmatrix}
\]

There we have set \( \xi_p = x (z - a_p) \) and the second argument of the CHF’s is implicitly assumed to be \( 1 + 2\gamma_p \). The piecewise constant matrix \( L_p \) reads

\[
L_p = \begin{cases}
e^{\pi i p} e^{-i\pi p \sigma_3} & -\pi/2 < \arg (z - a_p) < \pi/2 \\
e^{\pi i (\delta_p - \gamma_p)} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi \delta_p - i\pi \gamma_p} \end{pmatrix} & \pi/2 < \arg (z - a_p) < \pi \\
e^{\pi i (\delta_p + \gamma_p)} \begin{pmatrix} e^{-2i\pi \delta_p + i\pi \gamma_p} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg (z - a_p) < -\pi/2
\end{cases}
\]

and the coefficients \( b_{12}^{(p)} (z) \) and \( b_{21}^{(p)} (z) \) are

\[
b_{12}^{(p)} (z) = \frac{ie^{-i\pi \gamma_p}}{K_p(z)} \frac{\Gamma (1 - \gamma_p + \delta_p)}{\Gamma (-\gamma_p - \delta_p)} , \quad b_{21}^{(p)} (z) = -iK_p(z) e^{i\pi \gamma_p} \frac{\Gamma (1 - \gamma_p - \delta_p)}{\Gamma (\delta_p - \gamma_p)} .
\]

**Proposition 4.1** The matrix \( P_{a_k} (z) \) plays the role of a parametrix around \( a_p \) in the sense that:

- \( \forall P_{a_k}^{-1} \) is holomorphic inside of \( D_{a_p,\epsilon} \),
- \( P_{a_k}^{-1} = I_2 + O \left( 1/x^{1-2\Re (\delta_p)} \right) \) uniformly on \( \partial D_{a_p,\epsilon} \).

**Proof** — It follows from \( \det [P_{a_k}] = 1 \) that \( P_{a_k} \) is invertible and that \( P_{a_k}^{-1} = \text{Comat} (P_{a_k}) \). The fact that \( P_{a_k}^{-1} = I_2 + O \left( 1/x^{1-2\Re (\delta_p)} \right) \) uniformly on \( \partial D_{a_p,\epsilon} \) follows from the asymptotic behavior on the boundary of \( \partial D_{0,\epsilon} \) of the matrix \( P \) defined in the previous section together with the fact that the function \( K_p (z) \) appearing in the definition of \( b_{12}^{(p)} (z) \) and \( b_{21}^{(p)} (z) \) depends on \( x^{2\delta_p} \), cf. (3.18).
As $P_{ap}$ and $\Upsilon$ have the same jump matrices on $\{\Gamma^{(p)}_+ \cup \Gamma^{(p)}_\circ\} \cap D_{ap,\varepsilon}$, $\Upsilon P_{ap}^{-1}$ can be analytically continued to the punctured disk $D_{ap,\varepsilon} \setminus \{a_p\}$. It remains to see that the singularity at $z = a_p$ is removable.

The local behavior of $\Upsilon$ at $z \to a_p$ as well as the one of CHF at the origin (cf appendix (A.10)) imply that $\Upsilon P_{ap}^{-1}$ has at most a power-law singularity at $a_p$. In particular, it cannot have an essential singularity at $a_p$. The singularity can only be a pole of some finite order. To set aside this possibility, it is thus enough to check that $\Upsilon P_{ap}^{-1}$ is bounded in the quadrant $0 \leq \arg(z - a_p) \leq \pi/2$.

The addition formulae for the CHF (A.9) allow to express the product $P_{ap} M_{\tau}^{-1} \alpha^{-\sigma_3}$ as

$$
    P_{ap} (z) M_{\tau}^{-1} (z) [\alpha^\uparrow(z)]^{-\sigma_3} = \frac{(-i\zeta_p)^{\gamma_p - \delta_p}}{\alpha^\uparrow(z)} \left( -ib_{21}^{(p)} (z) \Psi (1 + \gamma_p - \delta_p; -i\zeta_p) - \Psi (\gamma_p - \delta_p; -i\zeta_p) \right) (1 - e^{i\pi z})
$$

$$
    + \frac{e^{i\zeta_p^{\gamma_p - \delta_p}}}{\Gamma (1 - 2\gamma_p) \zeta_p^{\gamma_p - \delta_p}} \left( 0 \Gamma (1 + \delta_p - \gamma_p) \Phi (1 - \gamma_p + \delta_p, 1 - 2\gamma_p; i\zeta_p) 
    - \Gamma (1 - \delta_p - \gamma_p) e^{2\pi i \gamma_p} \Phi (\delta_p - \gamma_p, 1 - 2\gamma_p; i\zeta_p) \right)
$$

(4.8)

Here, once again, the second argument of Tricomi’s CHF is assumed to be $1 + 2\gamma_p$.

Recall that the $\Phi$ functions are regular at $\zeta_p = 0$ whereas the $\Psi$ functions have a power-law singularity of the type $O(\zeta_p^{-2\gamma_p})$. Since $\alpha^\uparrow(z) / \zeta_p^{\gamma_p - \delta_p} = O(1)$ for $z \to a_p$, we get that

$$
    P_{ap} M_{\tau}^{-1} [\alpha^\uparrow]^{-\sigma_3} = \left\{ O(1) + O \left( |z - a_p|^{-2\gamma_p} \right) + O \left( \log |z - a_p| \right) \right\} O \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right)
$$

(4.9)

Moreover, using the representation (4.8), one obtains that

$$
    P_{ap} (z) M_{\tau}^{-1} (z) \alpha^{-\sigma_3} (z)^t \text{Comat} (B(z)) = O \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right).
$$

(4.10)

Hence, using the local estimates for $\Upsilon$ around $a_p$, we get that

$$
    P_{ap} (z) \Upsilon^{-1} (z) = \left\{ O(1) + O \left( |z - a_p|^{-2\gamma_p} \right) + O \left( \log |z - a_p| \right) \right\} O \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right).
$$

(4.11)

Here we have used that $\det [\Upsilon] = 1$ what implies that, for $z \to a_p$,

$$
    \Upsilon^{-1} (z) = M_{\tau}^{-1} \alpha^\uparrow M_{\tau} \left\{ I_2 + g(z)^t \text{Comat} (B(z)) + |z - a_p| (g(z) + 1) O \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \right\} M_{\tau}^{-1},
$$

(4.12)

for some $M_{\tau} \in GL(2, \mathbb{C})$. Therefore, as $\Re (2\gamma_p) < 1$, we see that $P_{ap} (z) \Upsilon^{-1} (z)$ cannot have a pole at $z = a_p$. The singularity at $z = a_p$ is hence removable and $P_{ap} (z) \Upsilon^{-1} (z)$ is analytic on $D_{ap,\varepsilon}$. As $\det \left[ P_{ap} (z) \Upsilon^{-1} (z) \right] = 1$, we have that

$$
    \Upsilon (z) P_{ap}^{-1} (z) = \text{Comat} \left( P_{ap} (z) \Upsilon^{-1} (z) \right)
$$

(4.13)

is also analytic on $D_{ap,\varepsilon}$.

\[ \square \]
4.3 The last transformation $\Upsilon \to \Omega$

The matrix

$$\Omega = \begin{cases} \Upsilon P^{-1}_{a_k} & z \in D_{a_k, \epsilon} \\ \Upsilon & z \in \mathbb{C} \setminus \bigcup_{p=1}^{n} \overline{D}_{a_k, \epsilon} \end{cases} \quad (4.14)$$

satisfies the RHP

- $\Omega$ is analytic in $\mathbb{C} \setminus \Sigma_{\Omega}$;
- $\Omega = I_2 + O \left( \frac{1}{z} \right)$, $z \to \infty$;
- \begin{align*}
\Omega_+ (z) M_r (z) &= \Omega_- (z), & z \in \Gamma_+ ; \\
\Omega_+ (z) M_+^{-1} (z) &= \Omega_- (z), & z \in \Gamma_- ; \\
\Omega_+ (z) N^{(0)} (z) &= \Omega_- (z), & z \in \Gamma_0 (1) ; \\
\Omega_+ (z) \bar{N}^{(0)} (z) &= \Omega_- (z), & z \in \bar{\Gamma}_0 (1) ; \\
\Omega_+ (z) P_{a_k} (z) &= \Omega_- (z), & z \in \partial D_{a_k, \epsilon} .
\end{align*}

The solution of this RHP for $\Omega$ exists and is unique as seen by already invoked arguments. The newly introduced contours are all depicted in Fig. 4.

The jump matrix $v_\Omega$ for $\Omega$ is uniformly $I_2 + O \left( \frac{1}{x^{1-\rho}} \right)$ in the $L^2 (\Sigma_{\Omega})$ and $L^\infty (\Sigma_{\Omega})$ sense, ie there exists a constant $c$ such that

$$||v_\Omega - I_2||_{L^2 (\Sigma_{\Omega})} + ||v_\Omega - I_2||_{L^\infty (\Sigma_{\Omega})} \leq c x^{\rho - 1} . \quad (4.15)$$

Here, we choose the following $L^2 (\mathbb{R})$ and $L^\infty (\mathbb{R})$ matrix norms

$$||A (s)||^2_{L^2 (\Sigma_{\Omega})} = \int_{\Sigma_{\Omega}} \text{tr} \left[ A^\dagger (s) A (s) \right] |ds| , \quad ||A (s)||_{L^\infty (\Sigma_{\Omega})} = \max_{i,j} ||A_{ij} (s)||_{L^\infty (\Sigma_{\Omega})} . \quad (4.16)$$

We also remind that $\rho = 2 \max_k |\Re (\delta_k)| < 1$. By using the matrix integral equation equivalent to the RHP for $\Omega$ we see that $\Omega \xrightarrow{x \to +\infty} I_2$ uniformly. Moreover the first corrections to $\Omega$ are uniformly $O \left( \frac{1}{x^{1-\rho}} \right)$. As a consequence, $I_2$ is the unique solution of the RHP for $\Omega$, up to the uniformly $O \left( \frac{1}{x^{1-\rho}} \right)$ corrections.
5 Asymptotics of the resolvent

Recall that the resolvent $I - R$ of $I + V$ can be expressed in terms of $f_+$ and $f_-$, cf (2.19). Having solved the RHP for $\chi$ perturbatively in $z$, we use this asymptotic solution in order to obtain the asymptotics to the leading order of the functions $f_{\pm}$. The latter yield the leading asymptotic behavior of the resolvent.

5.1 The zeroth order approximants to $f_{\pm}$.

**Definition 5.1** Let $\chi^{bk}$ and $\chi^{loc}_p$ be the matrices
\[
\begin{align*}
\chi^{bk}(z) &= M_t^{-1}(z)[\alpha_t(z)]^{-\sigma_3}, \\
\chi^{loc}_p(z) &= P_{ap}(z)M_t^{-1}(z)[\alpha_t(z)]^{-\sigma_3}.
\end{align*}
\]

Let $\Omega_\epsilon$ be the solution of the RHP defined in subsection 4.3 and where all the circles in the contour $\Sigma_{\epsilon 1}$, as depicted on fig.4, have a radius $\epsilon$. Then, if $\chi$ is the solution of the RHP defined in subsection 3.1 we have
\[
\begin{align*}
\Omega_1^{-1}(z)\chi(z) &= \chi^{bk}(z), \quad z \in U \cap \{z \in \mathbb{H}_+: \Re(z) \in [a_p + \epsilon/2 : a_{p+1} - \epsilon/2]\}. \quad (5.1) \\
\Omega_2^{-1}(z)\chi(z) &= \chi^{loc}_p(z), \quad z \in D_{a_p,2\epsilon}. \quad (5.2)
\end{align*}
\]

In a sense that will become clear in the following, $\chi^{bk}$ is the leading solution of the RHP for $\chi$ when $z$ is uniformly away from the singularities at the $a_p$’s and $\chi^{loc}_p$ is the leading solution of the RHP for $\chi$ when $z$ belongs to the disk $D_{a_p,2\epsilon}$.

The advantage of using the solution of Riemann-Hilbert problems with two sizes of disks around the $a_k$’s, $\Omega_2$, and $\Omega_{3/2}$ is that the solution $\chi_+(z)$ defined by (5.1) on $\mathbb{R} \setminus \cup_{k=1}^{n} \{a_k - \epsilon; a_k + \epsilon\} + i0^+$ and by (5.2) on $\cup_{k=1}^{n} \{a_k - \epsilon; a_k + \epsilon\} + i0^+$ has a smooth correction matrix $\Omega$ around the gluing point $z = a_p \pm \epsilon$. This will simplify our forthcoming analysis when integrating the solution $\chi$ around the points $a_k = \pm \epsilon$. If we would have used a single solution $\Omega_\epsilon$, then we should have had derived additional estimates for the behavior of this matrix around $a_k = \pm \epsilon$, as a priori it could exhibit a non-smooth behavior there. Thence, we circumvent additional complications.

**Proposition 5.1** Let
\[
\begin{pmatrix}
 f_{bk}^+(z) \\
 f_{bk}^-(z)
\end{pmatrix}

\equiv \chi^{bk}(z)\begin{pmatrix}
 e_+(z) \\
 e_-(z)
\end{pmatrix}, \quad \begin{pmatrix}
 f_{loc}^{+p}(z) \\
 f_{loc}^{-p}(z)
\end{pmatrix}

\equiv \chi^{loc}_p(z)\begin{pmatrix}
 e_+(z) \\
 e_-(z)
\end{pmatrix}. \quad (5.3)
\]

Then
\[
\begin{pmatrix}
 f_{bk}^+(z) \\
 f_{bk}^-(z)
\end{pmatrix} = (\alpha_1 e_-)^{-\sigma_1} \begin{pmatrix}
 \sigma_{p+1}(z) \\
 1
\end{pmatrix} \text{ for } z \in \{a_p < \Re(z) < a_{p+1}\} \cap U, \quad (5.4)
\]

with $\sigma_p$ having already been defined in (3.15). Also
\[
\begin{pmatrix}
 f_{loc}^{+p}(z) \\
 f_{loc}^{-p}(z)
\end{pmatrix}

\equiv \frac{e^{\frac{i}{2}(\gamma_p \sigma_3 - \delta_p)x - \delta_p \sigma_3}}{\Gamma(1 - 2\gamma_p) x^{-\gamma_p}} \begin{pmatrix}
 e_+(z) \left[\hat{\alpha}_p^{(p)}(z)\right]^{-1} \\
 0
\end{pmatrix} \times \begin{pmatrix}
 \Gamma(1 + \delta_p - \gamma_p) \Phi(-\gamma_p - \delta_p, 1 - 2\gamma_p; -i\zeta_p) \\
 \Gamma(1 - \delta_p - \gamma_p) \Phi(\delta_p - \gamma_p, 1 - 2\gamma_p; i\zeta_p)
\end{pmatrix} \text{ for } z \in D_{a_p,2\epsilon}.
\]

The functions $\hat{\alpha}_p^{(p)}$ were defined in (3.19) whereas $\Phi$ is Humbert’s CHF (A.7) and $\zeta_p = x(z - a_p)$. 24
Definition 5.2

Let $\mathcal{P}_\pm$ in a vicinity of $a_p$. In the intermediary calculations we suppose that $z$ belongs to the quadrant $\{ z : a_p < R(z) < a_p + \epsilon \} \cap \mathcal{H}_+$. The final result is however valid on the whole disk as can be seen through a direct computation carried out on the other quadrants. In order to lighten the notations, the second argument of Tricomi’s CHF is undercurrented to be $1 + 2\gamma_p$ and we have set $\zeta = x (z - a_p)$.

\[
\left( \begin{array}{c}
\Psi (\gamma_p - \delta_p; -i\zeta) \\
-i b_{21}^{(p)}(z) \Psi (1 + \gamma_p - \delta_p; -i\zeta)
\end{array} \right) \\
\times \zeta e^{i \pi (\delta_p - \gamma_p) z} \left( \zeta^{\sigma_p \alpha_1} \right) e_{-}(z) \left( \begin{array}{c}
\sigma_p^{-1}(z) \\
0
\end{array} \right) \\
\left( 1 \right)
\]

With

\[
g_{21}^{(p)} = - \frac{\Gamma(1 + \delta_p - \gamma_p)}{\Gamma(-\delta_p - \gamma_p)} e^{-i\zeta} \quad g_{21}^{(p)} = - \frac{\Gamma(1 - \gamma_p - \delta_p)}{\Gamma(\delta_p - \gamma_p)} e^{i\zeta}.
\]

And finally using the recombination formulas for CHF (5.4), the claim follows. □

Remark 5.1

The function $f_{+}^{p}$, resp. $f_{-}^{p}$, are good approximates to $f_{\pm}$ in their respective domains of validity. More precisely, for $z \in \mathbb{R} \setminus \bigcup_{p=1}^{n} \{ a_p - \epsilon/2 ; a_p + \epsilon/2 \}$

\[
\left( \begin{array}{c}
f_{+}(z) - f_{+}^{p}(z) \\
f_{-}(z) - f_{-}^{p}(z)
\end{array} \right) = \left( \begin{array}{c}
\chi(z) - \chi^{p}(z) \\
e_{+}(z) - e_{-}(z)
\end{array} \right) = \left( \begin{array}{c}
\Omega^{+}(z) - I_{2} \\
f_{+}^{p}(z) - f_{+}^{p}(z)
\end{array} \right),
\]

and for $z \in D_{p,\epsilon},$

\[
\left( \begin{array}{c}
f_{+}(z) - f_{+}^{p}(z) \\
f_{-}(z) - f_{-}^{p}(z)
\end{array} \right) = \left( \begin{array}{c}
\chi(z) - \chi^{p}(z) \\
e_{+}(z) - e_{-}(z)
\end{array} \right) = \left( \begin{array}{c}
\Omega_{2p}(z) - I_{2} \\
f_{+}^{p}(z) - f_{+}^{p}(z)
\end{array} \right) .
\]

5.1.1 Uniform estimates for the resolvent

Definition 5.2

Let $R_0(\xi, \eta)$ be called the zeroth order resolvent. We define it in terms of $f_{\pm}^{p}$ as :

\[
R_{0}(\xi, \eta) = (\sigma(\xi) - 1) f_{\pm}^{p}(\xi) f_{\pm}^{p}(\eta) - f_{\pm}^{p}(\xi) f_{\pm}^{p}(\eta) ; \quad \eta, \xi \in \mathbb{R} \setminus \bigcup_{p=1}^{n} \{ a_p - \epsilon ; a_p + \epsilon \}
\]

\[
R_{0}(\xi, \eta) = (\sigma(\xi) - 1) f_{\pm}^{p}(\xi) f_{\pm}^{p}(\eta) - f_{\pm}^{p}(\xi) f_{\pm}^{p}(\eta) ; \quad \eta, \xi \in [ a_p - \epsilon ; a_p + \epsilon ] .
\]

Using the explicit expression for $f_{\pm}^{p}$ and $f_{\pm}^{p}$ we get the local expressions for the diagonal zeroth order resolvent

\[
2i\pi \sigma(\xi) R_{0}(\xi, \xi) = i\xi - \partial_{\xi} (\log \alpha_{+} \alpha_{-})(\xi) ,
\]

25
where \( \xi \in \mathbb{R} \setminus \bigcup_{p=1}^{n} [a_p - \epsilon; a_p + \epsilon] \) and we remind that \( \alpha_+ \) and \( \alpha_- \) are the boundary values of \( \alpha \) from the upper/lower half plane. Whereas for \( \xi \in ]a_p - \epsilon; a_p + \epsilon[ \)

\[
2i\pi e^{i\delta} \hat{\sigma}_p (\xi) x^{2p} R_0 (\xi, \xi) = -ix\tau (\gamma_p, \delta_p; \zeta_p) - \partial_\xi \log (\alpha^{(p)}_{i+} \alpha^{(p)}_{i-}) (\xi) \varphi (\gamma_p, \delta_p; \zeta_p). \tag{5.10}
\]

One should keep in mind that \( \zeta_p = x (\xi - a_p) \) and that \( \alpha^{(p)}_{i+} \) (resp. \( \alpha^{(p)}_{i-} \)) is the boundary value of \( \alpha^{(p)}_1 \) (resp. \( \alpha^{(p)}_{-1} \)) from the upper (resp. lower) half-plane. In the above formulæ we have introduced two functions

\[
\Gamma \left( \begin{array}{c}
1 - 2\gamma, 1 - 2\gamma \\
1 + \delta - \gamma, 1 - \delta - \gamma
\end{array} \right) \tau (\gamma, \delta; t) = -\Phi (\gamma - \delta, 1 - 2\gamma; -it) \Phi (\delta - \gamma, 1 - 2\gamma; it) + (\partial_\xi \Phi) (\gamma - \delta, 1 - 2\gamma; -it) \Phi (\delta - \gamma, 1 - 2\gamma; it)
\]

\[
+ \Phi (\gamma - \delta, 1 - 2\gamma; -it) (\partial_\xi \Phi) (\delta - \gamma, 1 - 2\gamma; it), \tag{5.11}
\]

and

\[
\Gamma \left( \begin{array}{c}
1 - 2\gamma, 1 - 2\gamma \\
1 + \delta - \gamma, 1 - \delta - \gamma
\end{array} \right) \varphi (\gamma, \delta; t) = \Phi (\gamma - \delta, 1 - 2\gamma; -it) \Phi (\delta - \gamma, 1 - 2\gamma; it) \tag{5.12}
\]

Also, we have used the standard notation

\[
\Gamma \left( \begin{array}{c}
a_1, \ldots, a_n \\
b_1, \ldots, b_m
\end{array} \right) = \frac{\prod_{k=1}^{n} \Gamma (a_k)}{\prod_{k=1}^{m} \Gamma (b_k)}. \tag{5.13}
\]

We now focus on the relationship between the exact resolvent \( R \) and the zeroth order one \( R_0 \). Observe that one can write the exact resolvent \( R \) as

\[
R (\xi, \eta) = (e_- (\xi) e_+ (\xi)) \left[ \chi^{(0)}_+ (\xi) \right]^{-1} \Omega^{-1}_+ (\xi) \Omega e^- (\eta) \left( \begin{array}{c}
é_+ (\eta) \\
é_- (\eta)
\end{array} \right), \tag{5.14}
\]

Where \( \chi^{(0)}_+ (\xi) \equiv \Omega^{-1}_+ (\xi) \chi (\xi) \) is the leading order solution of the Riemann-Hilbert problem corresponding to the choice of the radius \( \epsilon' \) for the disks \( \partial D_{a_p, \epsilon'} \) centered at the singularities of the symbol \( \sigma (\xi) \). We remind that the subscript \( \epsilon \) in \( \Omega \) indicates that the matrix \( \Omega \) is the solution of the RHP corresponding to the contour \( \Sigma_\Omega \) of section 1.3 where all the circles centered at \( a_p \) have a radius \( \epsilon \).

Note that the exact resolvent \( R (\xi, \eta) \) does not depend on the choice of \( \epsilon' \). Hence, we can chose different values for \( \epsilon' \), depending on the point \( (\xi, \eta) \) where we want to estimate the resolvent. According to the above remark, we can present the exact resolvent as

\[
R (\xi, \eta) = R_0 (\xi, \eta) + R_c (\xi, \eta). \tag{5.15}
\]

There we have introduced the correcting resolvent \( R_c \). This resolvent is defined in terms of matrices \( \Omega_{\epsilon'} \) where \( \epsilon' \) takes different values depending on the point where we are placed. More precisely,

\[
R_c (\xi, \eta; \epsilon) = \left( -f_{-}^{bk} (\xi) \right) \frac{\Omega_{-1} (\xi) \Omega_{+} (\eta) - I_2}{\xi - \eta} \left( f_+^{bk} (\eta) \right) \tag{5.16}
\]
for \( \eta, \xi \in \mathbb{R} \setminus \bigcup_{p=1}^{n} \{ a_p - \epsilon; a_p + \epsilon \} \), and

\[
R_c (\xi, \eta; 2\epsilon) = \left( -f_{\text{loc}}^{\text{loc}} (\xi) f_{\text{loc}}^{\text{loc}} (\xi) \right) \frac{\Omega^{-1}_\xi (\xi) \Omega_{2\epsilon} (\eta) - I_2}{\xi - \eta} \left( f_{\text{loc}}^{\text{loc}} (\eta) \right)
\]  
(5.17)

for \( \eta, \xi \in \{ a_p - \epsilon; a_p + \epsilon \} \). The crucial point is that the correcting resolvent is indeed small in the sense of the integration:

**Proposition 5.2** Let \( \beta_p \in \{ \delta_p, \gamma_p \} \). Then

\[
\int_{\mathbb{R}} \frac{R (\xi, \xi) - R_0 (\xi, \xi)}{\sigma (\xi) - 1} \partial_{\beta_p} \sigma (\xi) \, d\xi = O \left( x^{\rho - 1} \right)
\]  
(5.18)

**Proof** — It is a standard fact that a matrix \( \Omega_\epsilon (z) \) approaching the identity at \( \infty \) and having a jump \( \Omega_{+\epsilon} (I_2 + \Delta) = \Omega_{-\epsilon} \) on \( \Sigma_{\Omega_\epsilon} \), can be expressed in terms of solutions of a singular integral equation:

\[
\Omega_\epsilon (\xi) = I_2 + C_{\Sigma_{\Omega_\epsilon}} \left[ \Omega_{+\epsilon} \Delta \right] (\xi) \quad \text{where} \quad \Omega_{+\epsilon} (\xi) = I_2 + C_{\Sigma_{\Omega_\epsilon}} \left[ \Omega_{+\epsilon} \Delta \right] (\xi) .
\]  
(5.19)

It is readily seen from the asymptotics of CHF (A.8) that the jump matrix \( I_2 + \Delta \) for \( \Omega_\epsilon \) is such that \( ||\Delta (z)||_{L^2 (\Sigma_{\Omega_\epsilon})} = O \left( x^{\rho - 1} \right) \), but also \( ||\Delta||_{L^2 (\Sigma_{\Omega_\epsilon})} = O \left( x^{\rho - 1} \right) \). In the above equation, we have introduced the Cauchy operator

\[
C_{\Sigma_{\Omega_\epsilon}} [f] (\xi) = \int_{\Sigma_{\Omega_\epsilon}} \frac{ds}{2i\pi (\xi - s)} f (s) , \quad C_{\Sigma_{\Omega_\epsilon}}^+[f] (\eta) = \lim_{\xi \to \eta^+} C_{\Sigma_{\Omega_\epsilon}} [f] (\xi) .
\]  
(5.20)

\( \xi \to \eta^+ \) means that \( \xi \) approaches \( \eta \in \Sigma_{\Omega_\epsilon} \) non-tangentially from the left side of \( \Sigma_{\Omega_\epsilon} \). It is a classical result that \( C_{\Sigma_{\Omega_\epsilon}}^+ \) is a continuous operator on \( L^2 (\Sigma_{\Omega_\epsilon}) \). The fact that \( ||\Delta||_{L^2 (\Sigma_{\Omega_\epsilon})} = O \left( x^{\rho - 1} \right) \), leads to \( ||\Omega_{+\epsilon} - I_2||_{L^2 (\Sigma_{\Omega_\epsilon})} = O \left( x^{\rho - 1} \right) \). In its turn, this means that the entries of \( C \) such that \( \Omega_\epsilon (z) = I_2 + C/z + O \left( z^{-2} \right) \) are \( O \left( x^{\rho - 1} \right) \).

We have that

\[
\int_{\mathbb{R}} \frac{R (\xi, \xi) - R_0 (\xi, \xi)}{\sigma (\xi) - 1} \partial_{\beta_p} \sigma (\xi) \, d\xi =
\int_{\mathbb{R} \setminus \bigcup_{p=1}^{n} \{ a_p - \epsilon; a_p + \epsilon \}} R_c (\xi, \xi; \epsilon/2) \frac{\partial_{\beta_p} \sigma (\xi)}{\sigma (\xi) - 1} + \sum_{p=1}^{n} \int_{a_p - \epsilon}^{a_p + \epsilon} R_c (\xi, \xi; 2\epsilon) \frac{\partial_{\beta_p} \sigma (\xi)}{\sigma (\xi) - 1} .
\]  
(5.21)

There, we should substitute the values of the correcting resolvent \( R_c \) given in (5.17) and (5.16).

The exact formula for the correcting resolvent is

\[
R_c (\xi, \xi; \epsilon) \frac{\partial_{\beta_p} \sigma (\xi)}{\sigma (\xi) - 1} = \frac{1}{2i\pi} \sum_{i,j} f_{\mu_i}^{(0)} (\xi) f_{\mu_j}^{(0)} (\xi) \partial_{\beta_p} \sigma (\xi) \left( \Omega^{-1}_{\epsilon} (\xi) \partial_{\xi} \Omega_{\epsilon'} (\xi) \right)_{ij} .
\]  
(5.22)

There we agree upon \( \mu_1 = + \) and \( \mu_2 = - \), as well as on the fact that

\[
f_{\pm}^{(0)} (\xi) = \begin{cases} 
   f_{\pm}^{(0)} (\xi) & \xi \in \mathcal{D}_{\pm} = \mathbb{R} \setminus \bigcup_{k=1}^{n} \{ a_k - \epsilon; a_k + \epsilon \} \\
   f_{\pm}^{\text{loc}} (\xi) & \xi \in \mathcal{D}^{\text{loc}} = \bigcup_{k=1}^{n} \{ a_k - \epsilon; a_k + \epsilon \} .
\end{cases}
\]  
(5.23)
Also $\epsilon' = \epsilon/2$ for $\xi \in D_{bk}$ and $\epsilon' = 2\epsilon$ for $\xi \in D_{loc}$. The choice of two possible values for $\epsilon'$ depending whether $\xi \in D_{bk}$ or $D_{loc}$ ensures that $(\Omega_{\epsilon'}(\xi) \partial_\xi \Omega_{\epsilon'}(\xi))_{ij}$ is smooth on $\mathbb{R}$.

We first study the integral on $\mathbb{R} \setminus \bigcup_{p=1}^n [a_p - \epsilon/2; a_p + \epsilon/2]$. As already pointed out, the jump matrix $\Delta$ is uniformly $O(x^{\beta-1})$. This ensures that $\Omega(\xi) - I_2$ is also an $O(x^{\beta-1})$, for $\xi$ bounded. Moreover, by Lemma 6

$$\Omega_{\frac{1}{2}}^{-1}(\xi) \partial_\xi \Omega_{\frac{1}{2}}(\xi) = x^{\beta-1}O\left(\left|\xi\right|^\frac{1+\min(1,\kappa)}{2} \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)\right), \quad \xi \to \pm\infty. \quad (5.24)$$

As $f_{\ell k}(\xi) \sim e^{\pm i/2}\xi$ for $\xi \to \infty$, and $\partial_{\beta_p} \sigma(\xi) = O(\xi^{-1})$, we get that (5.22) is absolutely integrable on $\mathbb{R} \setminus \bigcup_{p=1}^n [a_p - \epsilon/2; a_p + \epsilon/2]$ and that the integral is an $O(x^{\beta-1})$.

It remains to study the integral of (5.22) on $[a_p - \epsilon; a_p + \epsilon]$. $\Omega_{2\epsilon}(\xi)$ is smooth on this interval and equal to $I_2 + O(x^{\beta-1})$. Moreover, for $\xi \to a_p$,

$$\partial_{\beta_p} \sigma(\xi) = O\left(\left|\xi - a_p\right|^{-2\gamma_p} \log |\xi - a_p|\right) \quad (5.25)$$

for $\beta_p \in \{\delta_p, \gamma_p\}$. The formula for $R_c$ involves four terms. They can all be treated similarly so we only explain here how to estimate the contribution of the integral around $a_p$ involving the $[f_{loc}]^2$. We denote $J_{+,+}$ this integral and have

$$\left|J_{+,+}\right| = \left|\int_{a_p - \epsilon}^{a_p + \epsilon} \partial_{\beta_p} \sigma(\xi) \left(\Omega_{2\epsilon}^{-1}(\xi) \partial_\xi \Omega_{2\epsilon}(\xi)\right)_{21} [f_{loc}]^2(\xi) \, d\xi \right|$$

$$\leq x^{\beta-2-2\Re(\delta_p)} C' \int_{-\epsilon}^{\epsilon} |z|^{-2\Re(\gamma_p)} \log |x/|z|| |\Phi(-\delta_p - \gamma_p, 1 - 2\gamma_p; -iz)|^2 \, dz. \quad (5.26)$$

We have used the bounds on all the smooth functions appearing in the first line and used the $O$ estimates for $\partial_{\beta_p} \sigma$ (5.25). $C'$ is some constant that can depend on $\epsilon$ and we have explicitly extracted the factor $x^\beta$ from the bound on the product of $\Omega$ matrices.

The integral in the last line of (5.26) is divergent for $x \to +\infty$ so that its leading $x \to +\infty$ asymptotics are obtained by substituting the $z \to +\infty$ behavior of the integrand. Since

$$\Phi(-\delta_p - \gamma_p, 1 - 2\gamma_p; -iz) = c_{\pm} z^{\delta_p + \gamma_p} (1 + o(1)) \quad \text{for} \quad z \to \pm\infty, \quad c_{\pm} \in \mathbb{C}, \quad (5.27)$$

we have

$$\int_{-\epsilon}^{\epsilon} |z|^{-2\Re(\gamma_p)} \log |x/|z|| |\Phi(-\delta_p - \gamma_p, 1 - 2\gamma_p; -iz)|^2 \, dz = O\left(\int_{-\epsilon}^{\epsilon} \log \left(\frac{x}{|t|}\right) t^{-2\Re(\delta_p)} \, dt\right)$$

$$= O\left(\frac{(\epsilon x)^{2\Re(\delta_p) + 1}}{2\Re(\delta_p) + 1} \left(-\log \epsilon + \frac{1}{2\Re(\delta_p) + 1}\right)\right). \quad (5.28)$$

Hence we have that $|J_{++,}| \leq c\epsilon x^{\beta-1}$ for some constant $c$, uniformly in $\delta_p, \gamma_p$. \hfill \Box

**Lemma 6** \hfill $\forall \epsilon' > 0$,

$$\Omega_{\epsilon'}^{-1}(\xi) \partial_\xi \Omega_{\epsilon'}(\xi) = x^{\beta-1}O\left(\left|\xi\right|^\frac{1+\min(1,\kappa)}{2} \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)\right), \quad \xi \to \pm\infty. \quad (5.29)$$

Here, we remind that $\kappa$ is such that $F(\xi) = 1 + O\left(\left|\xi\right|^{\frac{1+\min(1,\kappa)}{2}}\right)$ when $\xi \to \pm\infty$.
Proof — As already discussed, we have that \( \forall \epsilon > 0, \Omega_\epsilon (\xi) = I_2 + o(1) \) in the \( L^2 \cap L^\infty (\Sigma_{\Omega_\epsilon}) \) sense. Hence, it remains to study the asymptotic behavior of \( \partial_\epsilon \Omega_\epsilon (\xi) \).

We focus on the case \( \xi \to +\infty \) as the \( \xi \to -\infty \) case can be treated similarly. We decompose the contour \( \Sigma_{\Omega_\epsilon} = \Sigma_{\Omega_\epsilon}^L \cup \Sigma_{\Omega_\epsilon}^R \), with

\[
\Sigma_{\Omega_\epsilon}^L = \Sigma_{\Omega_\epsilon} \cap \left\{ z : \Re (z) < \frac{\xi}{2} \right\}, \quad \Sigma_{\Omega_\epsilon}^R = \Sigma_{\Omega_\epsilon} \cap \left\{ z : \Re (z) \geq \frac{\xi}{2} \right\}.
\] (5.30)

Then, using the Cauchy integral representation for \( \Omega_\epsilon \) (5.19), we get

\[
\left| \partial_\epsilon \Omega_\epsilon (\xi) \right|_{ij} \leq \frac{2}{\pi \xi^2} \left| \partial_\epsilon \Omega_\epsilon \right|_{L^2 (\Sigma_{\Omega_\epsilon})} \| \Delta \|_{L^2 (\Sigma_{\Omega_\epsilon})} + || I^R_{ij} ||. \] (5.31)

Where \( I + \Delta \) stands for the jump matrix for \( \Omega_\epsilon \) and

\[
I^R = \int_{\Sigma_{\Omega_\epsilon}^R} \frac{ds}{2\pi (\xi - s)^2} \Omega_{+\epsilon} \Delta (s).
\] (5.32)

Reminding that \( \| A \|^2_{L^2 (\Sigma_{\Omega_\epsilon})} = \int_{\Sigma_{\Omega_\epsilon}} |ds| \operatorname{tr} \left[ A^\dagger (s) A (s) \right] \), we get

\[
| I^R_{ij} |^2 \leq \frac{1}{4\pi^2} \left| \partial_\epsilon \Omega_\epsilon \right|^2_{L^2 (\Sigma_{\Omega_\epsilon})} \int_{\Sigma_{\Omega_\epsilon}^R} \frac{|ds|}{|s - \xi|^4} \operatorname{tr} \left\{ \Delta^\dagger (s) \Delta (s) \right\}.
\] (5.33)

But taking the explicit formula for \( \Delta (s) \) and using the estimates for the asymptotic behavior of \( \sigma_{\nu_k, \nu_k} (\xi) \), as well as the assumptions on the asymptotic behavior of the function \( F (\xi) = 1 + O \left( |\xi|^{-\frac{1 + \kappa}{2}} \right) \), for some \( \kappa > 0 \), we get that

\[
\operatorname{tr} \left\{ \Delta^\dagger (s) \Delta (s) \right\} = \frac{e^{-2x\delta}}{s^{1+\min(1,\kappa)}} \left( 1 + o(1) \right) \text{ at } s \to \infty.
\] (5.34)

Where \( \delta = \text{dist} \left( \Sigma_{\Omega_\epsilon}^R, \Re \right) > 0 \) uniformly in \( |\xi| \) large enough. Therefore, for such \( \xi \)

\[
\int_{\Sigma_{\Omega_\epsilon}^R} \frac{|ds|}{|s - \xi|^4} \operatorname{tr} \left\{ \Delta^\dagger (s) \Delta (s) \right\} \leq 2 \int_\frac{\xi}{2}^{+\infty} \frac{e^{-2x\delta}}{|s - \xi - i\delta|^4 \left( |s + i\delta| \right)^{1+\min(1,\kappa)}}
= e^{-2x\delta} O \left( \frac{1}{\xi^{1+\min(1,\kappa)}} \right).
\] (5.35)

It follows that

\[
| I^R_{ij} | = e^{-2x\delta} O \left( \frac{1}{\xi^{1+\min(1,\kappa)}} \right),
\] (5.36)

for some constant \( c \). The Lemma follows once upon replacing \( || \Delta \||_{L^2 (\Sigma_{\Omega_\epsilon})} = O \left( x^{\rho - 1} \right) \) in (5.31). \( \Box \)

29
6 Asymptotics of the truncated Wiener-Hopf determinant

In this Section we shall compute the leading asymptotics of determinants of truncated Wiener-Hopf operators that are Hilbert-Schmidt. Note that, for \( \xi \in \mathbb{R} \), we can decompose \( \sigma_{\nu_k, \mathcal{P}} (\xi - a_k) \) into

\[
\sigma_{\nu_k, \mathcal{P}} (\xi - a_k) = e^{g_k(\xi)} e^{h_k(\xi)}.
\]

Here,

\[
g_k(\xi) = \delta_k \left[ \log (\xi - a_k - i) - \log (\xi - a_k + i) + 2i\pi \mathbb{E}(a_k - \xi) \right],
\]

\[
h_k(\xi) = \gamma_k \left[ \log (\xi - a_k + i) + \log (\xi - a_k - i) - 2\log |\xi - a_k| \right],
\]

and we remind that \( \mathbb{E} \) is Heaviside’s step function. It follows,

\[
\partial \beta_k \sigma (z) = \sigma (z) \partial \beta_k (g_k (z) + h_k (z)) \quad \text{for} \quad \beta_k \in \{ \delta_k, \gamma_k \}.
\]

6.1 Integration

Before carrying out the integrals appearing in the formulas for \( \partial \delta_1 \log \det \left[ I + V \right] \) and \( \partial \gamma_1 \log \det \left[ I + V \right] \), we first establish a useful integration Lemma.

**Lemma 7** Let \( R \) be a Riemann integrable function on \( \mathbb{R} \), \( g \in C^1 (I, \mathbb{C}) \), and \( I \) an interval such that \( \circ I \supset [-\epsilon_1; \epsilon_2] \), where \( \epsilon_1 \geq 0, \epsilon_2 \geq 0 \). Then

\[
\epsilon_2 \int_{-\epsilon_1}^{\epsilon_2} x \left( g(t) - g(0) \right) R(xt) \, dt = \lim_{\epsilon \to +\infty} \epsilon \int_{-\epsilon}^{\epsilon} R(t) + o(1) .
\]

The \( o(1) \) corresponds to terms vanishing in the ordered limit \( x \to +\infty \) and then \( \epsilon_i \to 0 \).

**Proof —**

One has

\[
\int_{-\epsilon_1}^{\epsilon_2} (g(t) - g(0)) R(xt) \, dt = \int_0^{\epsilon_2} \int_{-\epsilon_1}^{\epsilon_2} \int_{-\epsilon_1}^{\epsilon_2} d'y \, dy \, \left( \int_{-\epsilon_1}^{\epsilon_2} d't \, R(t) \right) - \int_{-\epsilon_1}^{\epsilon_2} \int_{-\epsilon_1}^{\epsilon_2} d'y \, \left( \int_{-\epsilon_1}^{\epsilon_2} d't \, R(t) \right).
\]

Since \( R \) is Riemann integrable, we have that \( (a, b) \mapsto \int_a^b R(t) \, dt \) is continuous and has a finite limit at \( \infty \). Hence it is bounded, say by \( M > 0 \). It follows from (6.6) that

\[
\left| \int_{-\epsilon_1}^{\epsilon_2} x \, dt \left( g(t) - g(0) \right) R(t) \right| \leq (\epsilon_1 + \epsilon_2) \, M \, \max_{[-\epsilon_1; \epsilon_2]} |g'| \quad \square.
\]

We now establish the first separation identity.
Proposition 6.1 Let $I + V$ be the generalized sine kernel defined by a symbol $\sigma - 1$ satisfying the $L^2(\mathbb{R})$ assumptions. Then the below identity holds

$$\log \left( \frac{\det_2 [I + V]}{\det_2 [I + V]|_{\delta_1 = 0}} \right) = x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left\{ \frac{\sigma}{\sigma|_{\delta_1 = 0}} - \sigma + \sigma|_{\delta_1 = 0} \right\} + \delta_1^2 \log \left( \frac{2}{x} \right)$$

$$+ \delta_1 \log \left( \frac{\alpha_+^{(2...n)}(a_1 + i) \alpha_-^{(2...n)}(a_1 - i)}{\alpha_+^{(2...n)}(a_1 + i0^+) \alpha_-^{(2...n)}(a_1 - i0^+)} \right)$$

$$+ \log \left( \frac{G(1 - \gamma_1 + \delta_1) G(1 - \gamma_1 - \delta_1)}{G(1 - \gamma_1) G(1 - \gamma_1)} \right) + o(1). \quad (6.8)$$

There, $G$ stands for the Barnes’ function and the $o(1)$ vanishes in the $x \to +\infty$ limit. The $o(1)$ is uniform in $\delta_p$ and $\gamma_p$.

The functions $\alpha_+^{(2...n)}$ refer to the values in the upper/lower half planes of the solution to the scalar RHP for $\alpha$ with vanishing exponents $\nu_1$ and $\nu_1$, ie

$$\alpha_+^{(2...n)}(z) = F_+^{-1}(z) \prod_{k=2}^{n} \left( \frac{z - a_k}{z - a_k + i} \right)^{\nu_k},$$

$$\alpha_-^{(2...n)}(z) = F_- (z) \prod_{k=2}^{n} \left( \frac{z - a_k - i}{z - a_k} \right)^{\nu_k}. \quad (6.9)$$

Finally, the notation $|\delta_1 = 0$ means that we ought to set $\delta_1 = 0$ without altering all the other parameters.

Proof — The asymptotic formulae for the resolvent (5.10), (5.9) combined with (2.24) lead to

$$\partial_{\delta_1} \log \det_2 [I + V] = \int_{\mathbb{R}} \left\{ \frac{R_0(\xi, \xi)}{\sigma(\xi) - 1} \partial_{\delta_1} \sigma(\xi) - \partial_{\delta_1} V(\xi, \xi) \right\} d\xi$$

$$+ \int_{\mathbb{R}} \frac{R(\xi, \xi) - R_0(\xi, \xi)}{\sigma(\xi) - 1} \partial_{\delta_1} \sigma(\xi) d\xi.$$
to infinity when $\epsilon \to 0$. Writing the result of integration explicitly yields

$$\partial_{\delta_1} \log \det_2 [I + V] = \int_{J_e} \frac{d\xi}{2\pi} \left\{ \partial_{\delta_1} \log \sigma - \partial_{\delta_1} \sigma \right\} - \int_{J_e} \frac{d\xi}{2\pi} \left( \partial_\xi \log \alpha_+ + \alpha_- \right) (\xi) \partial_{\delta_1} g_1 (\xi)$$

$$\sum_{p=1}^n \int_{\delta_{\delta_1} \sigma (t) + \partial_{\delta_1} g_1 (t)}^{a_p + \epsilon} \left[ x - \frac{2i\delta_p}{x (t - a_p) + \text{sgn} (t - a_p)} + i\partial_\xi \log \hat{\alpha}_1 \hat{\alpha}_2 (t) \right] \frac{dt}{2\pi}$$

$$+ \sum_{p=1}^n \int_{\delta_{\delta_1} \sigma (t) + \partial_{\delta_1} g_1 (t + a_p)}^{a_p + \epsilon} \left[ -ix\tilde{\tau} (\gamma_p, \delta_p; xt) - \left( \partial_\xi \log \hat{\alpha}_1 \hat{\alpha}_2 (t) + a_p \right) \tilde{\varphi} (\gamma_p, \delta_p; xt) \right]$$

$$+ \int_{-\epsilon}^{0} \left[ -ix\tilde{\tau} (\gamma_1, \delta_1; xt) - \left( \partial_\xi \log \hat{\alpha}_1 \hat{\alpha}_2 (t) + a_1 \right) \tilde{\varphi} (\gamma_1, \delta_1; xt) \right] + O (x^{\alpha-1}) \quad (6.10)$$

There $J_e = \mathbb{R} \setminus \bigcup_{p=1}^n [a_p - \epsilon; a_p + \epsilon]$. Also, we have introduced $\tilde{g}_1$, the smooth part of $g_1$, as well as $\tilde{\tau}$ and $\tilde{\varphi}$, the Riemann integrable regularizations of the functions $\tau$ and $\varphi$ introduced in (6.12), (6.11):

$$\tilde{g}_1 (\xi) = \delta_1 (\log (\xi - a_1 - i) - \log (\xi - a_1 + i))$$

$$\tilde{\tau} (\gamma_p, \delta_p; xt) = e^{-i\delta_p \text{sgn} (t)} |t|^{-2\gamma_p} \tau (\gamma_p, \delta_p; xt) + 1 - \frac{2i\delta_p}{xt + \text{sgn} (t)}$$

$$\tilde{\varphi} (\gamma_1, \delta_1; xt) = e^{-i\delta_1 \text{sgn} (t)} |t|^{-2\gamma_1} \varphi (\gamma_1, \delta_1; xt) + 1$$

The Riemann integrability of $\tilde{\tau}$ and $\tilde{\varphi}$ is part of the conclusions of Corollary B.1 given in Appendix B.

The integrals in the third and fourth line of (6.13) can be estimated using Lemma 7. One gets that

$$\int_{-\epsilon}^{0} \left[ -ix\tilde{\tau} (\gamma_1, \delta_1; xt) - \left( \partial_\xi \log \hat{\alpha}_1 \hat{\alpha}_2 (t) \right) (t + a_1) \tilde{\varphi} (\gamma_1, \delta_1; xt) \right]$$

$$= -i \int_{-\infty}^{0} \tilde{\tau} (\gamma_1, \delta_1; t) \left( \partial_\xi \log \hat{\alpha}_1 \hat{\alpha}_2 (t) \right) (a_1) \int_{-\infty}^{0} \tilde{\varphi} (\gamma_1, \delta_1; t) dt + o (1)$$

$$= -2\delta_1 + (\gamma_1 + \delta_1) \psi (\gamma_1 - \delta_1 - (\delta_1 - \gamma_1) \psi (\delta_1 - \gamma_1) + i\pi \gamma_1 + o (1) \quad (6.14)$$

There we have used the value of the integrals of $\tilde{\tau} (\gamma, \delta; t)$ and $\tilde{\varphi} (\gamma, \delta; t)$ given in Corollary B.1. The $o (1)$ stands for terms that vanish in the ordered limit $x \to +\infty$ and then $\epsilon \to 0$. Very similarly

$$\sum_{p=1}^n \int_{\delta_{\delta_1} \sigma (t) + \partial_{\delta_1} g_1 (t + a_p)}^{a_p + \epsilon} \left[ -ix\tilde{\tau} (\gamma_p, \delta_p; xt) - \left( \partial_\xi \log \hat{\alpha}_1 \hat{\alpha}_2 (t) \right) (t + a_p) \tilde{\varphi} (\gamma_p, \delta_p; xt) \right]$$

$$= \sum_{p=1}^n -\frac{i}{2\pi} \partial_{\delta_1} \tilde{g} (a_p) \times -2\pi \gamma_p = \sum_{p=1}^n \gamma_p \partial_{\delta_1} \tilde{g} (a_p) \quad. \quad (6.15)$$

The equality

$$\sum_{p=1}^n \int_{a_p - \epsilon}^{a_p + \epsilon} i\partial_\xi g_1 (t) \partial_\xi \log \left( \hat{\alpha}_1 \hat{\alpha}_2 (t) \right) \frac{dt}{2\pi} = O (\epsilon) \quad (6.16)$$

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holds as we deal with a finite sum of integrals of piecewise smooth functions over intervals of length $2\epsilon$. In order to estimate the integral appearing in the second line of (6.13), we need to change its expression a little

$$
\sum_{p=1}^{n} \int_{a_p - \epsilon}^{a_p + \epsilon} \partial_{\delta_1} g_1(t) \frac{2i\epsilon \delta_p}{x(t-a_p) + \text{sgn}(t-a_p)2\pi} dt
$$

$$
= 2i\pi \int_{-\epsilon}^{0} \frac{2\delta_1 x dt}{xt-1} + \int_{0}^{\epsilon} \sum_{p=1}^{n} \partial_{\delta_1} \left( \tilde{g}_1(t+a_p) - \tilde{g}_1(a_p-t) \right) \frac{2i\epsilon \delta_p xt dt}{xt+1} 2\pi
$$

$$
= -2\delta_1 \log(x+1) + O(\epsilon) = -2\delta_1 \log x + o(1) \quad (6.17)
$$

There we have used that $t^{-1} (\tilde{g}_1(t+a_p) - \tilde{g}_1(a_p-t))$ is smooth and $xt/(xt+1) \leq 1$. This ensures that the corresponding integrals are a $O(\epsilon)$. The $o(1)$ term stands, once again, for terms vanishing in the ordered limit $x \to +\infty$ and then $\epsilon \to 0$.

We now explain how to treat the integral containing $\partial_{\delta} \log(a_{+}a_{-})$. We start by integrating by parts:

$$
- \int J_{\infty} \frac{d\xi}{2i\pi} (\partial_{\delta} \log(a_{+}a_{-}))(\xi) \partial_{\delta_1} g_1(\xi) = \int J_{\infty} \frac{d\xi}{2i\pi} \log(a_{+}a_{-})(\xi) \partial_{\delta_1 \xi} \hat{g}_1(\xi)
$$

$$
- \frac{1}{2i\pi} \sum_{p=1}^{n} \partial_{\delta_1 \hat{g}_1}(a_p - \epsilon) \log[a_{+}a_{-}](a_p - \epsilon) - \partial_{\delta_1 \hat{g}_1}(a_p + \epsilon) \log[a_{+}a_{-}](a_p + \epsilon)
$$

$$
- \log(a_{+}a_{-})(a_1 - \epsilon) \quad (6.18)
$$

There, we have explicitly separated the regular part $\partial_{\delta_1 \hat{g}}$ of $\partial_{\delta_1} g$ from the one containing a jump. The sum appearing in the second line can be computed up to $O(\epsilon \log \epsilon)$ terms by using the local behavior of $a_{+}a_{-}$ around $a_p$:

$$
a_{+}a_{-}(a_p \mp \epsilon) = \hat{\alpha}_1^{(p)}(a_p \mp \epsilon + i0^+) \hat{\alpha}_1^{(p)}(a_p \mp \epsilon - i0^+) e^{2i\pi \gamma_1 \Xi(\pm 1)} |\epsilon|^{-2\delta_p} \quad (6.19)
$$

Hence, up to $O(\epsilon \log \epsilon)$ terms, only the discontinuous part of $a_{+}a_{-}$ contributes to the sum:

$$
- \frac{1}{2i\pi} \sum_{p=1}^{n} \left\{ \partial_{\delta_1 \hat{g}_1} \log[a_{+}a_{-}](a_p - \epsilon) - \partial_{\delta_1 \hat{g}_1} \log[a_{+}a_{-}](a_p + \epsilon) \right\}
$$

$$
= - \sum_{p=1}^{n} \gamma_p \partial_{\delta_1 \hat{g}_1}(a_p) + O(\epsilon \log \epsilon) \quad (6.20)
$$

The singular behavior of $a_{+}a_{-}$ around $a_1$ leads to

$$
- \log[a_{+}a_{-}](a_1 - \epsilon) = 2\delta_1 \log \epsilon - i\pi \gamma_1 - \log \left[ \alpha_{1}^{(2,...,n)}(a_1 + i0^+) \alpha_{1}^{(2,...,n)}(a_1 - i0^+) \right] \quad (6.21)
$$

$\alpha_{1}^{(2,...,n)}$ have been defined in (6.9).

Up to $o(1)$ corrections, it is now possible to replace the integral over $J_{\epsilon}$ appearing in the rhs of (6.18) by one over $\mathbb{R}$ as the integrand has integrable singularities at the points $a_k$. The resulting integral over $\mathbb{R}$ can then be evaluated by computing the residues at
\( \xi = a_1 \pm i \) thanks to the fact that \( \alpha_{\pm} \) is analytic in the upper/lower half-plane and goes to 1 when \( z \to \infty \) in \( \mathcal{H}_\pm \). One gets:

\[
\int_{\mathbb{R}} \frac{d\xi}{2i\pi} \log (\alpha_+ \alpha_-) (\xi) \partial_{\xi}^2 g_1 (\xi) = \log \left( \frac{\alpha_+^{(2-n)} (a_1 + i) \alpha_-^{(2-n)} (a_1 - i)}{\alpha_+^{(2-n)} (a_1 + i) \alpha_-^{(2-n)} (a_1 - i)} \right) + 2\delta_1 \log 2 \ . \tag{6.22}
\]

At the end of the day we get,

\[
- \int_{J_\epsilon} \frac{d\xi}{2i\pi} (\partial_{\xi} \log \alpha_+ \alpha_-) (\xi) \partial_{\delta_1} g_1 (\xi) = - \log \left( \frac{\alpha_+^{(2-n)} (a_1) \alpha_-^{(2-n)} (a_1)}{\alpha_+^{(2-n)} (a_1 + i) \alpha_-^{(2-n)} (a_1 - i)} \right) + (\gamma_1 + \delta_1) \psi (-\gamma_1 - \delta_1) + \delta_1 \log 2e - i\pi \gamma_1 - \sum_{p=1}^{n} \partial_{\delta_1} g_1 (a_p) \gamma_p + o(1) \ . \tag{6.23}
\]

Putting all the different results together we get

\[
\partial_{\delta_1} \log \det_2 [I + V_2] = x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \{ \partial_{\delta_1} \log \sigma - \partial_{\delta_1} \sigma \} + 2\delta_1 \log \left( \frac{2}{x} \right) - 2\delta_1
\]

\[
- \log \left( \frac{\alpha_+^{(2-n)} (a_1) \alpha_-^{(2-n)} (a_1)}{\alpha_+^{(2-n)} (a_1 + i) \alpha_-^{(2-n)} (a_1 - i)} \right) + (\gamma_1 + \delta_1) \psi (-\gamma_1 - \delta_1) + \delta_1 \log 2e - i\pi \gamma_1 - \sum_{p=1}^{n} \partial_{\delta_1} g_1 (a_p) \gamma_p + o(1) \ . \tag{6.24}
\]

In particular, the two log \( \epsilon \) contributions from (6.23) and (6.17) cancel each other. We stress that \( o(1) \) stands for vanishing terms in the ordered limit \( x \to +\infty \) and \( \epsilon \to 0 \). It remains to integrate this result from 0 up to \( \delta_1 \). This is licit as the remainders \( o(1) \) are uniform in \( |\Re(\delta_1)| < 1/2 \). The integral of the \( \psi \) functions yields Barnes’ functions due to the formula (A.14). Finally, we get that

\[
\log \left( \frac{\det_2 [I + V]}{\det_2 [I + V]_{|\delta_1 = 0}} \right) x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left\{ \log \left( \frac{\sigma}{\sigma_{|\delta_1 = 0}} \right) - \sigma + \sigma_{|\delta_1 = 0} \right\} + \delta_1^2 \log \left( \frac{2}{x} \right)
\]

\[
= \delta_1 \log \left( \frac{\alpha_+^{(2-n)} (a_1 + i) \alpha_-^{(2-n)} (a_1 - i)}{\alpha_+^{(2-n)} (a_1 + 0^+) \alpha_-^{(2-n)} (a_1 - 0^+)} \right) + \log \left( \frac{G (1 - \gamma_1 + \delta_1) G (1 - \gamma_1 - \delta_1)}{G (1 - \gamma_1) G (1 - \gamma_1)} \right) + o(1) \ . \tag{6.25}
\]

This means that, for all \( \epsilon > 0 \), the limit

\[
\lim_{x \to +\infty} \left\{ \log \left( \frac{\det_2 [I + V]}{\det_2 [I + V]_{|\delta_1 = 0}} \right) - x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left\{ \log \left( \frac{\sigma}{\sigma_{|\delta_1 = 0}} \right) - \sigma + \sigma_{|\delta_1 = 0} \right\} + \delta_1^2 \log x \right\} \tag{6.26}
\]

exists. It is given by the rhs of (6.25), where \( o(1) \) are \( \epsilon \) dependent terms that go to 0 when \( \epsilon \to 0 \). As the lhs of (6.25) is \( \epsilon \)-independent the \( x \to +\infty \) limit cannot depend on \( \epsilon \), therefore the value of the constant can be computed by sending \( \epsilon \to 0 \). The claim then follows. \( \Box \)
Proposition 6.2 Under the assumptions of the previous proposition, the following identity holds

\[
\log \left( \frac{\det_2 [I + V]|_{s_1 = 0}}{\det_2 [I + V]|_{s_1 = \gamma_1 = 0}} \right) = x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left\{ \log \left( \frac{\sigma|_{s_1 = 0}}{\sigma|_{s_1 = \gamma_1 = 0}} \right) - \sigma|_{s_1 = 0} + \sigma|_{s_1 = \gamma_1 = 0} \right\} + \gamma_1^2 \log \left( \frac{x}{2} \right) + \gamma_1 \log \left( \frac{\alpha^{(2,n)} (a_1 + i) \alpha^{(2,n)}_+ (a_1 - i) + \log \left( \frac{G (1 - \gamma_1)}{G (1 - 2\gamma_1)} \right) + o(1)}{\frac{1}{2} \log det [I + V]} \right). 
\]

Proof — Following very analogous steps to the \( \delta_1 \)-derivative, one shows that

\[
\partial_{\gamma_1} \log \det_2 [I + V]|_{s_1 = 0} = x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left\{ \partial_{\gamma_1} \log \sigma|_{s_1 = 0} - \partial_{\gamma_1} \sigma|_{s_1 = 0} \right\} 
\]

\[
- \int_{\epsilon}^{\epsilon} \frac{d\xi}{2\pi} \left[ \partial_{\gamma_1} \log (\alpha_+ + \sigma_-) \right] \partial_{\gamma_1} h_1(\xi) + \sum_{p=2}^{n} \gamma_p \partial_{\gamma_1} h_1(a_p) 
\]

\[
+ \int_{-\epsilon}^{\epsilon} \frac{dt}{2\pi} \partial_{\gamma_1} h_1(z + a_1) \left[ -i \chi \tilde{\tau}(\gamma_1, 0; xt) - \left( \partial_{\xi} \log \tilde{\alpha}_{+}^{(1)} + \tilde{\alpha}_{+}^{(1)} \right) (t + a_1) \tilde{\varphi}(\gamma_1, 0; xt) \right] + o(1). 
\]

More precisely, we have replaced the integration of the exact resolvent by one involving the approximate one \( R_0 \) for the price of \( O(x^{p-1}) \) corrections. Then we have applied the integration Lemma 7 to estimate asymptotically the integrals around \( \{a_k\}_{k=2}^{n} \) that involve the CHF. These estimates produced the sum \( \sum_{p=2}^{n} \gamma_p \partial_{\gamma_1} h_1(a_p) \) and some \( o(1) \) corrections.

These, as before, are vanishing in the ordered limit \( x \to +\infty \) and then \( \epsilon \to 0 \).

The integral around \( a_1 \) should be considered separately as it is a little different in respect to the already studied integrals. We obtain

\[
\int_{-\epsilon}^{\epsilon} \frac{dt}{2\pi} \partial_{\gamma_1} h_1(t + a_1) \left[ -i \chi \tilde{\tau}(\gamma_1, 0; xt) - \left( \partial_{\xi} \log \tilde{\alpha}_{+}^{(1)} + \tilde{\alpha}_{+}^{(1)} \right) (t + a_1) \tilde{\varphi}(\gamma_1, 0; xt) \right] 
\]

\[
= o(1) + o \left( \frac{\log x}{x\epsilon} \right) + \int_{-\epsilon x}^{\epsilon x} \frac{dt}{2\pi} \log \left( \frac{|t|}{x} \right) \tilde{\tau}(\gamma_1, 0; t) 
\]

\[
= 2\gamma_1 \left( \psi(1 - \gamma_1) - 2\psi(1 - 2\gamma_1) + 1 \right) - 2\gamma_1 \log x + o(1). 
\]

During the estimation of the above integral, one finds that the contributions stemming from the regular part of \( \partial_{\gamma_1} h_1 \) only produce \( o(1) \) corrections as it vanishes at \( t = a_1 \). Also one gets that the integral of the irregular part (equal to \( \log |\xi| \)) versus \( \tilde{\varphi} \) produces at most \( O(\log x/x) \) corrections. The remaining integral in the second line can be estimated thanks to corollary [B.1] One should however use that fact that as \( \tilde{\tau}(\gamma_1, 0; t) \) decreases at infinity as an oscillating factor dumped by \( 1/t \),

\[
\log x \int_{-\epsilon x}^{\epsilon x} \frac{dt}{2\pi} \tilde{\tau}(\gamma_1, 0; t) = \log x \int_{\mathbb{R}} \frac{dt}{2\pi} \tilde{\tau}(\gamma_1, 0; t) + o \left( \frac{\log x}{x\epsilon} \right). 
\]
It now remains to study the limiting value of the integral in the second line of (6.28). In the case of the $\gamma_1$-derivative, this integral should be handled with greater care. Indeed, 

\[- \int_{J_\varepsilon} \frac{d\xi}{2i\pi} \left( \partial_\xi \log \alpha_+\alpha_- \right) (\xi) \partial_{\gamma_1} h_1 (\xi) =
\]

\[- \gamma_1 \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{d\xi}{2i\pi} \left( \frac{2i}{\xi^2 + 1} - \frac{1}{\xi - i0^+} + \frac{1}{\xi + i0^+} \right) \left( \log (\xi^2 + 1) - 2 \log |\xi| \right)
\]

\[= - \frac{1}{2i\pi} \sum_{p=2}^{n} \left[ \log (\alpha_{+,\alpha_{-,p}}^{(2\ldots n)}) \partial_{\gamma_1} h_1 \right] (a_p - \epsilon) - \left[ \log (\alpha_{+,\alpha_{-,p}}^{(2\ldots n)}) \partial_{\gamma_1} h_1 \right] (a_p + \epsilon)
\]

\[+ \int_{\mathbb{R}} \log [\alpha_+\alpha_-]^{(2\ldots n)} (\xi + a_1) \left\{ \frac{i}{(\xi - i)(\xi - i0^+)} - \frac{i}{(\xi + i)(\xi + i0^+)} \right\} \frac{d\xi}{2i\pi} \quad (6.31)
\]

We have integrated by parts and decomposed the result into the integration of the singular part (line 2), and the regular part involving $\alpha_{+,\alpha_{-,p}}^{(2\ldots n)} \alpha_{+}^{(2\ldots n)}$ (lines 3 and 4). The latter functions have already been defined in (6.9). We have

\[\frac{1}{\xi + i0^+} - \frac{1}{\xi - i0^+} = 2i\pi \delta (\xi) \quad (6.32)
\]

in the sense of distributions ($\delta (\xi)$ stands for the Dirac mass at zero). We can thus drop this functions from the first integral appearing on the rhs of the equation above. In particular, this integral is finite even if, a priori, it involves terms $\log |\xi| / (\xi \pm i0^+)$ that are integrated at distance $\epsilon$ from zero. The sum appearing in the second line of (6.31) is handled similarly to the case of the $\delta_1$-derivative, ie by separating the smooth/singular parts of $\alpha_{+,\alpha_{-,p}}^{(2\ldots n)}$ around $a_p$, and then neglecting all the $O (\epsilon \log \epsilon)$ contributions. Finally, one can send $\epsilon$ to zero in the integral appearing in the last line of (6.31). This produces some corrections that go to zero with $\epsilon$ due to the integrability of the integrand. The resulting integral over $\mathbb{R}$ can then be computed by the residues at $\xi = \pm i0^+$ and $\xi = \pm i$, exactly as it was done in the proof of proposition [6.1]. At the end of the day,

\[- \int_{J_\varepsilon} \frac{d\xi}{2i\pi} \left( \partial_\xi \log \alpha_+\alpha_- \right) (\xi) \partial_{\gamma_1} h_1 (\xi) = -2\gamma_1 \log 2 - \sum_{p=2}^{n} \gamma_p \partial_{\gamma_1} h_1 (a_p)
\]

\[+ \log \left( \frac{\alpha_+^{(2\ldots n)}(a_1 + i)}{\alpha_+^{(2\ldots n)}(a_1)\alpha_+^{(2\ldots n)}(a_1 - i)} \right) + o (1) \quad (6.33)
\]

Hence,

\[\partial_{\gamma_1} \log \det_2 [I + V]_{\delta_1 = 0} = x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left\{ \partial_{\gamma_1} \log \sigma_{|\delta_1 = 0} - \partial_{\gamma_1} \sigma_{|\delta_1 = 0} \right\} + 2\gamma_1 \log \left( \frac{x}{2} \right)
\]

\[+ \log \left( \frac{\alpha_+^{(2\ldots n)}(a_1 + i)}{\alpha_+^{(2\ldots n)}(a_1)\alpha_+^{(2\ldots n)}(a_1 - i)} \right) + 2\gamma_1 \left( \psi (1 - \gamma_1) - 2\psi (1 - 2\gamma_1) + 1 \right) + o (1) \quad (6.34)
\]

We now integrate (6.34) with respect to $\gamma_1$. The operation preserves the $o (1)$ symbols as they are uniform in $\gamma_1$. The $\psi$ functions are integrated thanks to (A.14). Once upon
integration, sending first $x$ to infinity and then $\epsilon$ to zero settles the value of the constant term. □

6.2 Asymptotics of the Fredholm determinant

**Theorem 6.1** Let $I + K$ be a truncated Wiener-Hopf operator acting on the segment $[0; x]$ and generated by the symbol $\sigma - 1$ with

$$\sigma(\xi) = F(\xi) \prod_{k=1}^{n} \sigma_{\nu_k, \varphi_k}(\xi - a_k) \quad , \quad a_i \in \mathbb{R} \quad a_1 < \cdots < a_n \ ,$$

(6.35)

where

- $F$ is holomorphic and non-vanishing in some open neighborhood of the real axis such that $F - 1 \in L^2(\mathbb{R})$, and even $F(\xi) - 1 = O\left(\xi^{-\frac{1+\kappa}{2}}\right)$, for some $\kappa > 0$;
- $\Re(\gamma_k) < 1/4$ and $|\Re(\delta_k)| < 1/2$, $\forall k \in [1; n]$.

Then the leading asymptotics of $\det_2[I + K]$ read:

$$\det_2[I + K] = G_2^*[\sigma] \left(\frac{1}{2} \sum_{p=1}^{n} \gamma_p^2 - \delta_p^2\right) E[F] \prod_{k=1}^{n} \frac{G(1 + \delta_k - \gamma_k) G(1 - \delta_k - \gamma_k)}{G(1 - 2\gamma_k)}$$

$$\prod_{k=1}^{n} \frac{F^\nu_k(a_k)}{F_{\nu_k}(a_k)} \prod_{k \neq p} \left(\frac{(a_k - a_p + i)^2}{(a_k - a_p + 2i)(a_k - a_p)}\right)^{\gamma_k \nu_p} \left(1 + O(x^{\rho-1})\right) .$$

(6.36)

We have defined

$$G_2^*[\sigma] = \exp \left\{ x \int_{\mathbb{R}} \frac{d\xi}{2\pi} \left[ \log(\sigma)(\xi) + 1 - \sigma(\xi) \right] \right\} ,$$

(6.37)

$$E[F] = \exp \left\{ \int_{0}^{+\infty} d\xi \xi F^{-1}[\log F](\xi) F^{-1}[\log F](-\xi) \right\} .$$

(6.38)

We also remind that $\rho = 2 \max |\Re(\delta_k)|$.

**Proof** — The result follows by a recursive applications of propositions 6.1 and 6.2. At the end of the recursion, one also needs to invoke the Aheizer-Kac formula for the 2-determinant of the truncated Wiener-Hopf operator generated by the $L^2(\mathbb{R})$ symbol $F - 1$ so as to fix the constant $E[F]$ and the $F$ dependent part of $G_2^*[\sigma]$. □

The leading asymptotics of Wiener-Hopf operators generated by general Fisher-Hartwig symbols (6.36) reproduces all the previously know results: $\forall k \nu_k = 0$, or $\forall k \varphi_k = 0$ proven in [9], $\forall k \gamma_k = 0$ proven in [12] and also the case of a pure Fisher-Hartwig singularity $n = 1$ and $F = 1$ [4]. We refer the reader to [11] for a restatement of all the know results in a language very close to the one used in this article. One should only pay attention to the different definition of $\sigma_{\nu_k, \varphi_k}$ between this article and the book [11]. Indeed $\nu_k$, resp. $\varphi_k$, differ by an overall minus sign with respect to the conventions of the latter book.

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However, our result disproves the continuous analog of the Fisher-Hartwig conjecture \[8\] in its broad generality. Although most of the factors between the formula and the conjecture coincide, the latter predicts the presence of
\[
\prod_{k<p} \left( \frac{(a_k - a_p + 1)^2}{(a_k - a_p)^2 + 4} \frac{\tau_k \nu_p}{(a_k - a_p)^2} \right)^{\nu_k \nu_p}
\]
whereas we find
\[
\prod_{k<p} \left( \frac{(a_k - a_p + i)^2}{(a_k - a_p + 2i) (a_k - a_p)} \right)^{\tau_k \nu_p} \frac{(a_k - a_p - i)^2}{(a_k - a_p - 2i) (a_k - a_p)} \frac{\tau_p \nu_k}{(a_k - a_p)^2} \]
\[6.39\]

The difference comes from the presence of \(\tau_p \nu_k\) in the second exponent instead of \(\tau_k \nu_p\).

Of course in all the cases previously investigated, the difference between the conjecture and the present result was not appearing as either the factor was not present or \(\nu_p\) and \(\nu_k\) were related by a sign.

To end this Section we would like to stress that it is not a problem to obtain the sub-leading asymptotics of truncated Wiener-Hopf with Fisher-Hartwig symbols by the so-called \(x\)-derivative method:
\[
\partial_x \log \det \left[ I + V \right] = -\frac{i}{2} \text{tr} \{ \chi_1 \sigma_3 \} \quad \text{with} \quad \chi(z) = I_2 + \frac{\chi_1}{z} + o\left(z^{-1}\right) \quad \text{for} \quad z \to \infty.
\]
The above is a straightforward generalization of the identity for the pure sine kernel given in [16]. The \(x \to +\infty\) asymptotics for \(\chi_1\) can be obtained by solving perturbatively the singular integral equation satisfied by \(\Omega\). We do not present the calculations here as we are going to derive the sub-leading asymptotics for the Toeplitz determinant case investigated in the next Section.

7 Toeplitz matrices with Fisher-Hartwig type symbols

In this Section, we adapt the previous analysis of the generalized sine kernel. In this way we obtain, in the framework of Riemann-Hilbert problems, the asymptotic behavior of Toeplitz determinants generated by symbols \(\sigma\) having Fisher-Hartwig singularities. This approach is based on an observation made by Deift, Its and Zhou in [16] concerning the relationship between the Fredholm determinant of a sine kernel on a circle and a Toeplitz determinant. Our results reproduce those obtained by T.Ehrhardt in his thesis [20]. Moreover, the Riemann-Hilbert approach allows to compute sub-leading asymptotics to any order in a quite systematic, although quickly cumbersome way. At the end of this Section we shall establish the first sub-leading asymptotics of Toeplitz matrices with Fisher-Hartwig singularities. We observe that these sub-leading asymptotics of Toeplitz determinants partly restore the independence on the choice of a Fisher-Hartwig type representant for the symbol \(\sigma\). Indeed the jumps of \(\sigma\) are characterized by parameters \(\delta_k\). A shift of any \(\delta_k\) by an integer describes the same jump. The freedom of choice of a Fisher-Hartwig representant for \(\sigma\) is broken if one considers the leading asymptotics only. These correspond to the choice \(|\Re(\delta)| < 1/2\). However, part of the sub-leading asymptotics (the so-called oscillating ones) we obtain correspond to shifts \(\delta_i \mapsto \delta_i + 1\), \(\delta_j \mapsto \delta_j - 1\) in the parameters appearing in the leading asymptotics. These sub-leading asymptotics shed a light on the mechanism appearing in the asymptotics for ambiguous case type symbols that has been conjectured by Basor and Tracy [3] and proven recently by Deift, Its and Krasovsky [14].
Indeed, terms that were subdominant in the asymptotic series for a generic set of parameters become of the same order of magnitude as the leading asymptotics when some of the parameters $\delta_p$ and $\gamma_p$ are set to these specific ambiguous values. The global structure of the sub-leading asymptotics seems to follow the scheme already pointed out in [33]. We formulate a conjecture on this global structure at the end of this Section. Our conjecture can be seen as a generalization of the Basor-Tracy conjecture: we believe that the full asymptotic series for $\det_m [T [\sigma]]$ results of a $1 - \delta$ periodization of only a small part of the asymptotic series.

7.1 The Riemann-Hilbert Problem

Let us consider an integral operator acting on the unit circle $\mathcal{C}$ with the kernel

$$V (z, z') = \sqrt{\sigma (z)} - 1 \sqrt{\sigma (z')} - 1 \frac{z^m z'^m - z - z^m z'^m}{2i\pi (z - z')} ,$$

(7.1)

where

$$\sigma (z) = b (z) \prod_{k=1}^{n} \left( 1 - \frac{z}{a_k} \right)^{-\nu_k} \left( 1 - \frac{a_k}{z} \right)^{-\nu_k} , \quad a_k \in \mathcal{C} .$$

(7.2)

There we assume that $b$ is holomorphic and non-vanishing in a vicinity of $\mathcal{C}$ and has zero winding number. One can actually characterize the singular behavior of $\sigma$ on the contour of integration more explicitly. Namely,

$$\left( 1 - \frac{z}{a_k} \right)^{-\nu_k} \left( 1 - \frac{a_k}{z} \right)^{-\nu_k} = e^{i\delta_k (\theta - \theta_k - \pi \text{sgn} (\theta - \theta_k))} (2 - 2 \cos (\theta - \theta_k))^{\nu_k} , \quad \frac{z}{a_k} = e^{i(\theta - \theta_k)} , \theta - \theta_k \in ] - \pi ; \pi [ .$$

(7.3)

We have set, just as in the preceding sections,

$$2\delta_k = \nu_k \quad 2\gamma_k = \nu_k .$$

(7.4)

Here, we assume that $|\Re (\delta_k)| < 1/2$ and $\Re (\gamma_k) < 1/2$.

The authors of [16] observed that for $m \in \mathbb{N}$ the pure sine kernel on the unit circle is of finite rank. This property persists in the case of the integral operator under investigation as:

$$V (z, z') = \sqrt{\sigma (z)} - 1 \sqrt{\sigma (z')} - \sum_{p=1}^{m} \frac{z^{p-1} z'^{m-p}}{2i\pi} .$$

(7.5)

Hence we have that

$$\det_{\mathcal{C}} [I + V] = \det_m \left[ \delta_{jk} + \int_{0}^{2\pi} \frac{d\theta}{2\pi} \left( \sigma (e^{i\theta}) - 1 \right) e^{i\theta (k-j)} \right]$$

(7.6)

$$= \det_m \left[ \int_{0}^{2\pi} \frac{d\theta}{2\pi} \sigma (e^{i\theta}) e^{i\theta (k-j)} \right] .$$

(7.7)

We used the subscript $\mathcal{C}$ in order to insist that the $lhs$ is the Fredholm determinant of an integral operator acting on the unit circle $\mathcal{C}$ whereas the $rhs$ is the determinant of an $m \times m$ matrix.

Hence, the asymptotics of Toeplitz matrices with symbols having jump and power-law singularities will follow from those of the Fredholm determinant of the integral operator.
defined in (7.1). The only significant difference between the kernel (7.1) and the one considered in the preceding Sections is the interval on which they act. Most of the steps in the derivation of the asymptotics are very similar. We only insist on the most striking differences.

7.2 Asymptoic solution of the Riemann-Hilbert Problem

We consider the RHP for a piecewise analytic matrix $\chi$ having a jump on the unit circle $C$:

- $\chi$ is analytic on $C \setminus C$;
- $\forall k \in [1 ; n ]$, there exists $M_k \in \text{GL}_2 (\mathbb{C})$ such that
  \[ \chi = M_k \left\{ I_2 + g (z) B (z) + |z - a_k| (g (z) + 1) O \left( \frac{1}{|z|} \right) \right\} , \ z \to a_k ; \]
- $\chi \to I_2$;
- $\chi_+ (z) G (z) = \chi_- (z) ; \ z \in C$.

There, just as for the Wiener-Hopf case, the rank one matrices $B_k$ read

\[ B_k = \begin{pmatrix} -1 & z^m \\ -z^{-m} & 1 \end{pmatrix} . \quad (7.8) \]

The function $g$ is also defined similarly

\[ g (z) = \int_{\mathcal{C}} \frac{ds}{2i\pi} \frac{\sigma (s) - 1}{z - s} . \quad (7.9) \]

It has a singular behavior at $z = a_k$ of the type

\[ g (z) = \begin{cases} O (1) + O \left( (z - a_k)^{-2\gamma_k} \right) & \text{for } \gamma_k \neq 0 \\ O (\log (z - a_k)) & \text{for } \gamma_k = 0 \end{cases} \text{ when } z \to a_k . \quad (7.10) \]

We finally precise that the unit circle $\mathcal{C}$ is oriented canonically (ie the + side of the contour corresponds to the interior of the circle) and that the jump matrix $G$ reads

\[ G (z) = \begin{pmatrix} 2 - \sigma (z) (\sigma (z) - 1) \sigma (z) \\ (1 - \sigma (z)) z^{-m} \end{pmatrix} . \quad (7.11) \]

We now define a new matrix $\Upsilon$ according to

- $\Upsilon = \chi \alpha^{\sigma_3} , \text{ for } z \text{ being in the exterior of } \Gamma_- \text{ and the interior of } \Gamma_+$;
- $\Upsilon = \chi \alpha^{\sigma_3} M_1^{-1} , \text{ for } z \text{ between } \Gamma_- \text{ and } \mathcal{C}$;
- $\Upsilon = \chi \alpha^{\sigma_3} M_1^{\dagger} , \text{ for } z \text{ between } \Gamma_+ \text{ and } \mathcal{C}$.

Here $\alpha$ is the solution of the scalar RHP

\[ \alpha \text{ analytic on } \mathbb{C} \setminus \mathcal{C} \quad \alpha_- = \sigma \alpha_+ , \quad z \in \mathcal{C} \setminus \cup_{k=1}^{n} \{ a_k \} \quad \alpha \to 1 \text{ when } z \to \infty . \quad (7.12) \]
This scalar RHP can be solved explicitly in terms of the canonical Wiener-Hopf factors of $b$: $b = b_+ G [b] b_-$. One has $\alpha = \alpha_\uparrow$ on $D_{0,1}$ and $\alpha = \alpha_\downarrow$ on $\mathbb{C} \setminus \overline{D}_{0,1}$, with
\[ \alpha_\uparrow (z) = b_+^{-1}(z) G [b]^{-1} \prod_{k=1}^{n} \left( 1 - \frac{z}{a_k} \right)^{\nu_k} \]  \hspace{1cm} (7.13)
\[ \alpha_\downarrow (z) = b_- (z) \prod_{k=1}^{n} \left( 1 - \frac{a_k}{z} \right)^{-\nu_k} \]  \hspace{1cm} (7.14)

The matrices $M_\uparrow/\downarrow$ defining $\Upsilon$ read
\[ M_\uparrow (z) = \begin{pmatrix} 1 & (1 - \sigma^{-1}) \alpha_\uparrow^{-2}(z) z^m \\ 0 & 1 \end{pmatrix}, \quad M_\downarrow (z) = \begin{pmatrix} 0 & 1 \\ (\sigma^{-1} - 1) \alpha_\downarrow^2(z) z^{-m} & 1 \end{pmatrix}. \]  \hspace{1cm} (7.15)

We stress that, just as for the Wiener-Hopf case, the matrix $M_\uparrow/\downarrow (r e^{i\theta})$ should be understood as the analytic continuation of $M_\uparrow/\downarrow (e^{i\theta})$ from a small neighborhood of $e^{i\theta}$ to the ray $[e^{i\theta}; r e^{i\theta}]$. One readily sees that $\Upsilon$ satisfies the RHP
\[ \Upsilon = \chi \sigma_3 \Upsilon_\uparrow \]  \hspace{1cm} $z \to a_k$
\[ \Upsilon \to I_2 \]  \hspace{1cm} $z \to \infty$
\[ \{ \begin{array}{ll}
\Upsilon_+(z) M_\uparrow (z) = \Upsilon_- (z); & z \in \Gamma_+ \\
\Upsilon_+(z) M_\downarrow^{-1} (z) = \Upsilon_- (z); & z \in \Gamma_-
\end{array} \]

Figure 5: Contour for the RHP $\Upsilon$ and the associated contour $\Gamma$.

- $\Upsilon$ is analytic in $\mathbb{C} \setminus \Gamma$;
- $\forall k \in [1; n]$, there exists $M_k \in \text{GL}_2 (\mathbb{C})$, such that:
\[ \Upsilon (z) = M_k \left\{ I_2 + g (z) B (z) + |z - a_k| (g (z) + 1) O \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \right\} M (z) \quad z \to a_k \]
- $\Upsilon \to I_2$;
\[ \begin{aligned}
\{ & \mathbb{Y}_+ (z) N^{(l)} (z) = \mathbb{Y}_- (z) ; \quad z \in \Gamma_+^{(l)} \\
& \mathbb{Y}_+ (z) \bar{N}^{(l)} (z) = \mathbb{Y}_- (z) ; \quad z \in \Gamma_-^{(l)} \}, \quad l \in \left[ 1 ; n \right]. 
\end{aligned} \]

The function \( g \) and the rank one matrices \( B (z) \) are as defined above. Moreover, the matrix \( M \) reads

\[ M (z) = \begin{cases}
\alpha^{\sigma_3} M_+ & z \in \left\{ D_{0,1} \setminus \cup_{k=1}^{n} \Gamma_+^{(k)} \right\} \cap U \\
\alpha^{\sigma_3} M_+^{-1} & z \in \left\{ U \setminus \left( D_{0,1} \cup_{k=1}^{n} \Gamma_+^{(k)} \right) \right\} \cap U.
\end{cases} \tag{7.16} \]

The local behavior at \( z \to a_k \) of \( M \) can be inferred from the one of \( \alpha_{+}^{\pm} \) and the explicit formulae for \( M_{+/-} \). The jump matrices \( N^{(l)} (z) \), \( \bar{N}^{(l)} (z) \) are defined as

\[ \begin{aligned}
N^{(l)} (z) &= \begin{pmatrix}
1 & n_l (z) \left( \frac{z}{a_l} \right)^m \\
0 & 1
\end{pmatrix} \quad z \in \Gamma_+^{(l)} , \\
\bar{N}^{(l)} (z) &= \begin{pmatrix}
1 & \pi_l (z) \left( \frac{a_l}{z} \right)^m \\
0 & 1
\end{pmatrix} \quad z \in \Gamma_-^{(l)} ,
\end{aligned} \]

and their entries read

\[ \begin{aligned}
n_l (z) &= \frac{[-im \log (z/a_l)]^{2\delta_l}}{K_l (z)} \left( e^{-2i\pi \tau_l} - 1 \right) , \quad (7.17) \\
\pi_l (z) &= \frac{K_l (z)}{[-im \log (z/a_l)]^{2\delta_l}} \left( e^{2i\pi \tau_l} - 1 \right) . \quad (7.18)
\end{aligned} \]

We have defined, analogously to the Wiener-Hopf case,

\[ K_l (z) = \frac{b_- (z) e^{-i\pi \gamma_l}}{b_+ (z) \left( 1 - z/a_l \right) \log (z/a_l)} \frac{1}{[-im \log (z/a_l)]^{2\delta_l}} \left( \frac{z}{a_l} \right)^{\pi_l} \prod_{r \neq 1} \left( 1 - \frac{z}{a_r} \right)^{-\nu_r} \left( 1 - \frac{a_r}{z} \right)^{-\tau_r} . \tag{7.19} \]

It is not a problem to see that the parametrix around \( a_p \) can be chosen as

\[ P_{a_p} (z) = \begin{pmatrix}
\Psi (\gamma_p - \delta_p; -i\gamma_p) & \imath \delta_{12}^{(p)} (z) \Psi (1 + \gamma_p + \delta_p; i\gamma_p) \\
-\imath \delta_{21}^{(p)} (z) \Psi (1 + \gamma_p - \delta_p; -i\gamma_p) & \Psi (\gamma_p + \delta_p; i\gamma_p)
\end{pmatrix} \frac{L_p}{(\zeta_p)^{d_p^3 - \gamma_p}} . \tag{7.20} \]

Where we have set \( \zeta_p = -im \log (z/a_p) \) and the second parameter of the CHF’s is implicitly assumed to be \( 1 + 2\gamma_p \). The piecewise constant matrix \( L_p \) reads

\[ L_p = \begin{pmatrix}
e^{i\frac{\pi \delta_p}{2}} e^{-i\frac{\pi \gamma_p}{2} \sigma_3} & -\pi/2 < \arg (\zeta_p) < \pi/2 \\
e^{\frac{\pi (\delta_p - \gamma_p)}{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi \delta_p - i\pi \gamma_p} \end{pmatrix} & \pi/2 < \arg (\zeta_p) < \pi, \\
e^{\frac{\pi (\delta_p + \gamma_p)}{2}} \begin{pmatrix} e^{-2\pi \delta_p + i\pi \gamma_p} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg (\zeta_p) < -\pi/2
\end{pmatrix} \tag{7.21} \]

and the coefficients \( \delta_{12}^{(p)} (z) \) and \( \delta_{21}^{(p)} (z) \) are given by

\[ \begin{aligned}
\delta_{12}^{(p)} (z) &= \frac{i e^{-i\pi \gamma_p}}{K_p (z)} \Gamma (1 - \gamma_p + \delta_p) , \quad \delta_{21}^{(p)} (z) = -i K_p (z) e^{i\pi \gamma_p} \Gamma (1 - \gamma_p - \delta_p) \frac{\Gamma (\delta_p - \gamma_p)}{\Gamma (\delta_p - \gamma_p)} . \tag{7.22}
\end{aligned} \]
The solution of the RHP for \( \Omega \), clearly exists and is unique. Moreover it is uniformly independent, so that we can also use them for a generalized sine kernel acting on equation (5.11) and (5.12).

One can prove that, just as for the Wiener-Hopf kernel, if \( R \) denotes the exact resolvent of the kernel (7.1),

\[
\int_{\mathcal{C}} dz \frac{R(z, z) - R_0(z, z)}{\sigma - 1} \partial_{\beta_p} \sigma = o(1)
\]

whenever \( z \in \bigcup_{k=1}^n \mathcal{C} \cap D_{ak, \epsilon} \). The functions \( \tau(\gamma, \delta; z) \) and \( \varphi(\gamma, \delta; z) \) have been defined in (5.11) and (5.12).

7.3 The resolvent and asymptotics of \( \log \det [I + V] \)

Just as in the case of kernels acting on \( \mathbb{R} \), one can reconstruct the approximate resolvent \( R_0 \) in terms of Humbert’s CHFs:

\[
R_0(z, z) = \frac{1 - \sigma^{-1}(z)}{2i\pi} \left\{ \frac{m}{z} - \partial_z \log (\alpha_+ \alpha_-)(z) \right\},
\]

for \( z \in \mathcal{C} \setminus \left( \bigcup_{k=1}^n \mathcal{C} \cap D_{ak, \epsilon} \right) \) and

\[
R_0(z, z) = e^{-i\pi \delta_p} \frac{\sigma(z) - 1}{2i\pi m^2 \varphi(z)} \left\{ -\frac{m}{z} \tau(\gamma_p, \delta_p; -im \log z/a_p) \right\}
\]

whenever \( z \in \bigcup_{k=1}^n \mathcal{C} \cap D_{ak, \epsilon} \). The functions \( \tau(\gamma, \delta; z) \) and \( \varphi(\gamma, \delta; z) \) have been defined in (5.11) and (5.12).

One should observe that the proof of the differential identities (2.23) is contour independent, so that we can also use them for a generalized sine kernel acting on \( \mathcal{C} \). Moreover one can prove that, just as for the Wiener-Hopf kernel, if \( R \) denotes the exact resolvent of the kernel (7.1),

\[
\int_{\mathcal{C}} dz \frac{R(z, z) - R_0(z, z)}{\sigma - 1} \partial_{\beta_p} \sigma = o(1)
\]

The \( o(1) \) terms vanish in the ordered limit \( x \to +\infty \) and then \( \epsilon \to 0^+ \). Harping on the steps for the integration in the case of a kernel acting on \( \mathbb{R} \) we get.
Proposition 7.1 Up to o (1) terms in the \( x \to +\infty \) limit, one has

\[
\log \left( \frac{\det [1 + V]}{\det [I + V]|_{t_1 = 0}} \right) = m \int_{\mathcal{C}} \frac{dz}{2i\pi z} \log \left( \frac{\sigma}{\sigma|_{t_1 = 0}} \right) - \delta_1 \log m + \delta_1 \log \left( \frac{\alpha^{(2 \ldots n)}_{r+}}{(a_1) \alpha^{(2 \ldots n)}_{r-}} (a_1) \right) + \log \left( G (1 - \gamma_1 + \delta_1) G (1 - \gamma_1 - \delta_1) \right) + o (1) .
\]

(7.26)

Also, \( \alpha_{r/+,/-} \) stand for their boundary values from the \( \pm \) sides of \( \mathcal{C} \).

Proof — The proof goes exactly the same as in the case of Wiener-Hopf operators. The only notable difference is that one needs an additional version of the integration lemma.

Let \( \epsilon > 0 \), and \( I \) be a sub-interval of \( \mathbb{R} \) such that \( \forall \mathcal{C} \supset [-\epsilon, \epsilon] \). Then \( \forall h \in \mathcal{C}^1 (I, \mathbb{C}) \), and for any Riemann integrable on \( \mathbb{R} \) function \( R (t) \) that it is decreasing as a power-law when \( |t| \to \infty \), one has

\[
\int_{-\epsilon}^{\epsilon} mdt R (mt) h (t) \log [2 (1 - \cos t)] = h (0) \int_{\mathbb{R}} \log [t^2] R (t) dt + o (1) .
\]

(7.29)

The \( o (1) \) term is vanishing in the ordered limit \( m \to +\infty \) and then \( \epsilon \to 0 \). The proof is straightforward by applying the original integration Lemma \[7\] to the function \( h (t) \log [2 (1 - \cos t)] - h (0) \log [t^2] \) that is \( \mathcal{C}^1 (I, \mathbb{C}) \). Moreover the power-law decrease of \( R (t) \) at infinity ensures that the contributions of the boundary which are of the type \( \log m \int_{-\epsilon}^{\infty} dt R (t) \) will indeed be subdominant with respect to \( o (1) \). \( \Box \)

We point out that

\[
\int_{\mathcal{C}} \frac{dz}{2i\pi z} \log \left( \frac{\sigma|_{t_1 = 0}}{\sigma|_{t_1 = \gamma_1 = 0}} \right) = 0 \quad \int_{\mathcal{C}} \frac{dz}{2i\pi z} \log \left( \frac{\sigma|_{t_1 = 0}}{\sigma|_{t_1 = \gamma_1 = 0}} \right) .
\]

(7.30)

We chose to include these factors in the proposition above so as to make the parallel with the Wiener-Hopf case more obvious. By repeatedly applying Proposition \[7.1\] and then invoking the strong Szeg#o limit theorem we get
Theorem 7.1 Let $T[\sigma]$ be an $m \times m$ Toeplitz matrix generated by the symbol
\[ \sigma (\theta) = b(z) \prod_{k=1}^{n} \left( 1 - \frac{z}{a_k} \right)^{-\nu_k} \left( 1 - \frac{a_k}{z} \right)^{-\nu_k} \quad a_k \in \mathcal{C}, \ z = e^{i\theta} \] (7.31)

where $b$ is analytic and non-vanishing in some open neighborhood of the unit circle $\mathcal{C}$ and $2\delta_k = \nu_k - \nu_k$, $2\gamma_k = \nu_k + \nu_k$ are such that $|\Re(\delta_k)| < 1/2$, $\Re(\gamma_k) < 1/2$.

Then the leading asymptotics of $\det_m [T[\sigma]]$ are given by
\[ \det_m [T[\sigma]] = (G[b])^m E[b] \prod_{p=1}^{n} G(1 - \gamma_p + \delta_p) G(1 - \gamma_p - \delta_p) \]
\[ \prod_{p=1}^{n} b_{\gamma_p}^{-\gamma_p} (a_p) b_{\gamma_p}^{\gamma_p} (a_p) \prod_{p \neq q}^{n} \left( 1 - \frac{a_p}{a_q} \right)^{(\gamma_p + \delta_p)(\gamma_q - \gamma_p)} \]
\[ (1 + o(1)) \] (7.32)

where,
\[ G[b] = e^{[\log b]_0} \quad E[b] = e^{\sum_{k=1}^{\infty} \frac{k}{k!} \log b}_{[\log b]_{-k}} \quad [\log b]_k = \int_{0}^{2\pi} \frac{d\theta}{2\pi} e^{ik\theta} \log b(\theta). \] (7.33)

This reproduces the result of T. Ehrhardt [20]. One should pay attention that the exponents $\delta_i$ and $\gamma_i$ correspond to the exponents $\beta_i$ and $-\alpha_i$ in Ehrhardt’s notations.

7.4 The sub-leading asymptotics

The authors of [16] found a way to express the discrete derivative of $\log T_m [\sigma]$ in terms of the RHP data. Their result reads
\[ \frac{T_{m+1}[\sigma]}{T_m[\sigma]} = \chi_{11}(z = 0). \] (7.34)

Where $\chi_{11}(0)$ stands for the upper diagonal entry of the solution to the RHP for $\chi$ given at the beginning of this Section. Their proof also works, word for word, in the case of the more complicated kernel we consider, so we omit it here. It is now enough to determine the sub-leading asymptotics of the matrix $\Omega$ defined by (7.23) thanks to the singular integral equation:
\[ \Omega(z) = I_2 + \int_{\Sigma_\Omega} \frac{ds}{2\pi (z - s)} \Omega_+(s) \Delta(s) \] (7.35)

with $I_2 + \Delta$ being the jump matrix for $\Omega$. The method for computing the corrections is standard. We send the reader to [33] for more details. We stress that we did not chose this reference because of its originality with respect to the perturbation theory of such integral equations. We rather chose it as there, the perturbation theory is applied in notation quite close to the ones we use.

It is easy to see that the jump matrix $\Delta$ is exponentially vanishing with respect to $x$ away of the disks $\partial D_{a_p \epsilon}$. However for $s \in \partial D_{a_p \epsilon}$ it admits the asymptotic expansion
\[ \Delta(s) = \sum_{\ell=1}^{M} \frac{1}{\ell! m^{[\log (s/a_p)]}} \Delta^{(p)}_{\ell}(s) + O(m^{-M+\rho-1}). \] (7.36)
We have set, using \((a)_n = \Gamma (a+n)/\Gamma (a)\),

\[
\Delta_{\ell}^{(p)} (s) = \begin{pmatrix}
\frac{1}{\pi^2 - \gamma_p^2} & \frac{i \Delta_{12}^{(p)} (s)}{\gamma_p^2 - \gamma_p^2} \\
-\frac{i \Delta_{21}^{(p)} (s)}{\gamma_p^2 - \gamma_p^2} & 1
\end{pmatrix} \times \begin{pmatrix}
\gamma_p - \delta_p & \ell (-\gamma_p - \delta_p) \\
0 & 0
\end{pmatrix} \frac{\ell^\ell}{(\gamma_p + \delta_p)_{\ell} (\delta_p - \gamma_p)_{\ell}}.
\]

(7.37)

and the functions \(\Delta_{12}^{(p)} (s)\) and \(\Delta_{21}^{(p)} (s)\) defined in \((7.22)\) exhibit a slight dependence on \(m\) that is a \(O(m^\epsilon)\). The standard considerations of a perturbative approach to \((7.35)\) lead to

\[
\Omega (0) = I_2 + \frac{\Omega_1 (0)}{m} + \frac{\Omega_2 (0)}{m^2} + O \left( \frac{1}{m^3}, \frac{(a_i/a_j)^m m^{3\rho}}{m^4} \right).
\]

(7.38)

Where, for \(z \in \mathbb{C} \setminus \bigcup_{p=1}^n \partial D_{a_{p,\epsilon}}\),

\[
\Omega_1 (z) = - \sum_{p=1}^n \frac{1}{1 - z/a_p} \left\{ \Delta_{1}^{(p)} (a_p) + \frac{1 - z/a_p}{\log (z/a_p)} \mathbf{1}_{D_{a_{p,\epsilon}}} (z) \right\},
\]

(7.39)

with \(\mathbf{1}_A\) being the indicator function of \(A\), and the expression for \(\Omega_2 (0)\) is already more involved:

\[
\Omega_2 (0) = - \sum_{p=1}^n \frac{d}{ds} \left\{ \frac{\Delta_{2}^{(p)} (s)}{2s} \left[ \frac{s - a_p}{\log (s/a_p)} \right]^2 \right\}_{s=a_p}\bigg|_{s=a_p}
+ \sum_{p \neq \ell} \sum_{p=1}^n \frac{\Delta_{2}^{(p)} (a_p) \Delta_{1}^{(p)} (a_p)}{1 - a_p/a_p}
+ \sum_{p=1}^n \frac{d}{ds} \left\{ \frac{\Delta_{2}^{(p)} (s)}{\log (s/a_p)} \right\}_{s=a_p} \cdot \frac{\Delta_{1}^{(p)} (a_p)}{a_p}.
\]

(7.40)

We get

\[
\frac{T_m [\sigma]}{T_{m-1} [\sigma]} = G [\sigma] \left\{ 1 + \frac{[\Omega_1 (0)]_{11}}{m} + \frac{[\Omega_2 (0)]_{11}}{m^2} + O \left( \frac{1}{m^3}, \frac{(a_i/a_j)^m m^{3\rho}}{m^4} \right) \right\}.
\]

(7.41)

This leads to

\[
\log T_m [\sigma] = m \log G [\sigma] + \log m \sum_{p=1}^n \left( \gamma_p - \delta_p \right)^2 + K + \frac{O_{osc}}{m^2} (1 + o (1)) + \frac{N_{osc}}{m} (1 + o (1)).
\]

(7.42)

We recover the first two terms of the asymptotics of \(\log T_m [\sigma]\) given in \((7.32)\). The discrete \(m\) derivative method does not allow to determine the constant \(K\), but this is irrelevant in what concerns the structure of corrections. \(Osc\) stands for what we call the oscillating corrections, whereas \(Nosc\) for the non-oscillating ones. More explicitly,

\[
Osc = \sum_{p \neq \ell} \Gamma \left( 1 - \gamma_p - \delta_p, 1 - \gamma_p + \delta_p \right) b_- (a_{\ell}) b_+ (a_p) m^{2\ell - 2\delta_p} (a_p/a_{\ell}) \left( a_p/a_{\ell} - 1 \right) (a_p/a_{\ell} - 1)
\]

\[
\prod_{r=1}^n \frac{(1 - a_r/a_{\ell})^{-\gamma_p}}{(1 - a_r/a_{\ell})^{-\gamma_p}}.
\]

(7.43)
whereas

\[ N_{\text{osc}} = \sum_{p=1}^{n} a_p \left( \delta_p^2 - \gamma_p^2 \right) \left[ \partial_z \log K_p(z) \right]_{z=a_p} + \frac{1}{2} \sum_{p \neq \ell}^{n} \left( \delta_\ell^2 - \gamma_\ell^2 \right) \left( \delta_p^2 - \gamma_p^2 \right) \frac{a_\ell/a_p + 1}{a_\ell/a_p - 1} \]  

(7.44)

The oscillating corrections have a very nice relationship with the leading term \( T^{(0)} (\{\gamma_i\}, \{\delta_i\}; m) |b| \) of the asymptotics given in (7.32). Indeed, one readily checks that,

\[ \text{Osc} = \sum_{p \neq q}^{n} T^{(0)} \left( \{\gamma_i\}, \left\{ \delta_i \right\}_{i \neq p,q}, \delta_p + 1, \delta_q - 1 \right) ; m \left[ b'_{pq} \right] \]  

(7.45)

with \( b'_{pq} (z) = a_p b(z) / a_q \). Such simultaneous changes \((\delta_p, \delta_q, b) \mapsto (\delta_p + 1, \delta_q - 1, b'_{pq})\) leave the value of the symbol \( \sigma \) given in (7.2) unchanged. Hence, this gives strong arguments supporting the Basor-Tracy conjecture as already a small part of the different Fisher-Hartwig representations for \( \sigma \) is present in the sub-leading asymptotics for \( T_m [\sigma] \).

However, our computations do not allow to give the proof of the Basor-Tracy conjecture in some particular cases where some terms in \( \text{Osc} \) become of the same order than \( T^{(0)} (\{\gamma_i\}, \{\delta_i\}; m) |b| \). Although we formally reproduce some particular results of the conjecture (for instance \( \Re(\delta_j) = 1/2 = \Re(\delta_k) \) for some \( j \) and \( k \)) we cannot consider this limiting case as we do not have a control of the remainder. However, we raise the following generalization of the Basor-Tracy conjecture:

**Conjecture 7.1** The sub-leading asymptotics of a Toeplitz matrix with Fisher-Hartwig symbols slowly restore the broken by \( T^0 \) independence with respect to the choice of a Fisher-Hartwig representation for \( \sigma \). More precisely the asymptotics have the structure

\[ T_m [\sigma] \sim \sum_{n \in \mathbb{Z}, \sum n_i = 0} \tilde{T} (\{\gamma_i\}, \{\delta_i + n_i\}; m) [b_{n_i}] , \quad b_{n_i} (z) = \prod_{p=1}^{n} a_p^{n_p} b(z) . \]  

(7.46)

With

\[ \tilde{T} (\{\gamma_i\}, \{\delta_i\}; m) |b| \sim T^{(0)} (\{\gamma_i\}, \{\delta_i\}; m) |b| \left( 1 + \sum_{k=1}^{+\infty} \frac{C_k (\{\gamma_i\}, \{\delta_i\}, b)}{m^k} \right) \]  

(7.47)

having no oscillating terms with \( m \). Note that the notation \( \sim \) stands for the equality in the sense of asymptotic series.

This conjecture is a natural extension of the \( \nu \) periodicity conjecture raised in [33], and, of course, of the Basor-Tracy conjecture. We stress that one could raise a similar type of conjecture in what concerns the Wiener-Hopf case.

### 8 Conclusion

In this article we have proven the formula for the leading asymptotic of Fredholm determinants of truncated Wiener-Hopf operators generated by symbols having Fisher-Hartwig singularities. As a byproduct we reproduced, in the framework of Riemann-Hilbert problems, the leading asymptotics of Toeplitz matrices having Fisher-Hartwig singularities. We were also able to compute the first sub-leading asymptotics of Toeplitz matrices having Fisher-Hartwig singularities. These give support to the Basor-Tracy conjecture. We
proposed an extension of the latter conjecture. Our results were based on a connection between Toeplitz determinants and those of Wiener-Hopf operators: both are related to the so-called generalized sine kernel acting either on the unit circle \( \mathbb{C} \) or the real axis \( \mathbb{R} \). In the case of Fisher-Hartwig singularities, this generalized kernel has some jump discontinuities and power-law singularities on the contour.

An open question is the construction of an explicit asymptotic resolvent for truncated Wiener-Hopf operators \( I + K \) generated by symbols having Fisher-Hartwig type singularities. Indeed the resolvent is known in the Fourier space, so it would be enough to take the inverse Fourier transform so as to have the resolvent of \( I + K \). Also, it would be interesting to find a proof for the conjecture we have raised above.

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A Some properties of confluent hypergeometric function

Tricomi’s confluent hypergeometric function \( \Psi (a, c; z) \) is one of the solutions to the differential equation [5]:

\[
zy'' + (c - z) y' - ay = 0 \quad \text{(A.1)}
\]

For generic \( a \) and \( c \), \( \Psi (a, c; z) \) has a power-law singularity at the origin, and a cut on \( \mathbb{R}^- \). It can be defined, for instance, by its Mellin-Barnes type integral representation in terms of Euler’s \( \Gamma \) function

\[
\Psi (a, c; z) = \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma \left( a + s, -s, 1 - c - s \right) \frac{z^s ds}{2\pi i}, \quad \text{(A.2)}
\]

that is valid for \(-\Re (a) < \gamma < \min (0, 1 - \Re (c))\) and \(-3\pi/2 < \arg (z) < 3\pi/2\). The latter integral representation is then supplemented by an analytic continuation. In the above formula we have used the standard hypergeometric type notation

\[
\Gamma \left( \frac{a_1, \ldots, a_n}{b_1, \ldots, b_m} \right) = \prod_{k=1}^{n} \Gamma (a_k) \prod_{k=1}^{m} \Gamma (b_k). \quad \text{(A.3)}
\]

Tricomi’s CHF satisfies the monodromy properties

\[
\Psi \left( a, c; z e^{2i\pi} \right) = e^{-2i\pi a} \Psi (a, c; z) + \frac{2i\pi e^{-i\pi a + z}}{\Gamma (a, 1 + a - c)} \Psi (c - a, c; -z) \quad \text{(A.4)}
\]

for \( \Im z < 0 \) and

\[
\Psi \left( a, c; z e^{-2i\pi} \right) = e^{2i\pi a} \Psi (a, c; z) - \frac{2i\pi e^{i\pi a + z}}{\Gamma (a, 1 + a - c)} \Psi (c - a, c; -z) \quad \text{(A.5)}
\]
for $\Im z > 0$. $\Psi (a, c; z)$ has an asymptotic expansion at $z \to \infty$ given by

$$\Psi (a, c; z) = \sum_{n=0}^{M} \frac{(-1)^n (a)_n (a-c+1)_n}{n!} z^{-a-n} + O \left( z^{-M-a} \right), \quad -\frac{3\pi}{2} < \arg(z) < \frac{3\pi}{2}. \quad (A.6)$$

Humbert’s CHF $\Phi (a, b; z)$ is another solution of $\left( A.1 \right)$. $\Phi (a, c; z)$ is an entire function that is defined in terms of its series expansion around $z = 0$

$$\Phi (a, c; z) = \sum_{n=0}^{+\infty} \frac{(a)_n}{n! (c)_n} z^n, \quad c \notin \mathbb{Z}^-. \quad (A.7)$$

It has the asymptotic expansion around $\infty$

$$\Phi (a, c; z) = \frac{\Gamma (c)}{\Gamma (c-a)} \left( \frac{e^{i\pi \epsilon}}{z} \right)^a \sum_{n=0}^{M} \frac{(a)_n (a-c+1)_n}{n! (-z)^n} + O \left( \left| z \right|^{-a-M-1} \right)$$

$$+ \frac{\Gamma (c)}{\Gamma (a)} e^{\epsilon z a-c} \sum_{n=0}^{N} \frac{(c-a)_n (1-a)_n}{n! z^n} + O \left( \left| e^{\epsilon z} z^{-1-c-N} \right| \right). \quad (A.8)$$

There are many relations between these two different CHF. In particular

$$e^{(c-a+1)i\pi} z^{1-c} \Phi (a-c+1, 2-c; z) = \left\{ \Gamma \left( \frac{2-c}{1-a} \right) \Phi (a, c; z) - \Gamma \left( \frac{2-c}{1+a-c} \right) e^{\epsilon z} \Phi (c-a, c; -z) \right\} \quad (A.9)$$

where $\epsilon = \text{sgn} (\Im z)$ and it is assumed that $\arg(z) \in ]-\pi/2; \pi/2[$. One can also express $\Psi (a, c; z)$ in terms of $\Phi (a, c; z)$ thus allowing to access to the singularity structure of Humbert’s CHF at the origin:

$$\Psi (a, c; z) = \Gamma \left( \frac{1-c}{a-c+1} \right) \Phi (a, c; z) + \Gamma \left( \frac{c-1}{a} \right) z^{-c} \Phi (a-c+1, 2-c; z). \quad (A.10)$$

Actually a CHF is some limiting case of the Gauss hypergeometric function. This function is one of the solutions of the hypergeometric equation. We recall its series expansion around $z = 0$:

$$\begin{aligned}
\mathbf{2F}_1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) &= \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n, \quad c \notin \mathbb{Z}^-.
\end{aligned} \quad (A.11)$$

The above solution is regular around $z = 0$ and can be continued to large value of $z$ thanks to the identity

$$\begin{aligned}
\mathbf{2F}_1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) &= \Gamma \left( \begin{array}{c} c, b-a \\ b, c-a \end{array} \right) (-z)^{-a} \mathbf{2F}_1 \left( \begin{array}{c} a, 1+a-c \\ 1+a-b \end{array} ; z^{-1} \right)
\end{aligned}$$

$$+ \Gamma \left( \begin{array}{c} c, a-b \\ a, c-b \end{array} \right) (-z)^{-b} \mathbf{2F}_1 \left( \begin{array}{c} b, 1+b-c \\ 1+b-a \end{array} ; z^{-1} \right). \quad (A.12)$$

One can also consider multi-variable generalizations of hypergeometric functions. We give here the definition of the Appell function of the second kind in terms of a double series that is convergent provided that $|y| + |z| < 1$:

$$\mathbf{F}_2 \left( \begin{array}{c} a, b, c \\ d, e \end{array} ; y, z \right) = \sum_{n,m \geq 0} \frac{(a)_{m+n} (b)_n (c)_m}{n! m! (d)_n (e)_m} y^n z^m. \quad (A.13)$$
We finally point out that the Barnes G-function admits an integral representation in terms of $\psi$, the logarithmic derivative of Euler’s Gamma function.

$$G(z + 1) = (2\pi)^{\frac{z}{2}} \exp \left\{ -\frac{z(z-1)}{2} + \int_0^z t\psi(t) \, dt \right\} \Re(z) > -1 . \quad (A.14)$$

B Integrals of CHF

B.1 Series Expansion of Appell function of large arguments.

Using the Mellin-Barnes type integral representation for CHFs [5], Erdelyi was able to evaluate the Laplace transform of products of CHFs in terms of Lauricella’s function [22]. The Lauricella function associated to an integral involving a product of two Humbert’s CHF is better known as the Appell function of the second kind (A.13). In terms of this function, Erdelyi’s result reads:

$$f(p, \alpha_1, \alpha_2; s) \equiv \int_0^{+\infty} \frac{dt \, e^{-st}}{t^{2\gamma+1-\epsilon-p}} \Phi(-\delta - \gamma, 1 - 2\gamma; -i\alpha_1 t) \Phi(\delta - \gamma, 1 - 2\gamma; i\alpha_2 t)$$

$$= s^{2\gamma-p-\epsilon} \Gamma(\epsilon + p - 2\gamma) F_2 \left( p + \epsilon - 2\gamma; -\gamma - \delta, \delta - \gamma \over 1 - 2\gamma, 1 - 2\gamma ; -i\frac{\alpha_1}{s}, i\frac{\alpha_2}{s} \right) . \quad (B.1)$$

Such integrals have been considered in [32], in the case where $\epsilon$ and $p$ are integers and the integral is absolutely convergent. Here, we study the behavior of such integrals when $\epsilon$ is close to zero. In that case, one cannot apply the integration procedure presented in [32] as it only applies to integer $\epsilon$. Moreover, we consider products of CHF that aren’t decaying sufficiently fast at infinity. Hence one should regularize the integrals before taking the $s \to 0^+$ limit. Once a regularization is performed, this limit can be computed thanks to a series expansion of the second Appell function around $\infty$.

We study $f(p, \alpha_1, \alpha_2; s)$ as it is the generating function for all the integrals that appear in the evaluation of the trace of the resolvent. The precise procedure for computing these integrals will be explained in this Appendix. The idea is to derive a series expansion for $F_2$ at $\infty$ and then use it to compute, after a proper regularization, the $s \to 0^+$ limit of $f(p, \alpha_1, \alpha_2; s)$.

**Lemma 8** Let $f(p, \alpha_1, \alpha_2; s)$ be defined in terms of the Appell function of the second kind as in (B.1). Then $f$ admits a series expansion around $s = 0$:

$$\Gamma \left( \frac{-\gamma - \delta - \gamma}{1 - 2\gamma, 1 - 2\gamma} \right) f(p, \alpha_1, \alpha_2; s) = S_1(p, \alpha_1, \alpha_2; s) + S_2(p, \alpha_1, \alpha_2; s) + S_3(p, \alpha_1, \alpha_2; s) \quad (B.2)$$

Where the $S_i$ are series involving Gauss’ functions

$$S_1(p, \alpha_1, \alpha_2; s) = \frac{\alpha_1^2 \alpha_2^{p+\epsilon-2\gamma}}{\gamma + p + \epsilon - 2\gamma} \sum_{n \geq 0} \left( \frac{\alpha_1}{\alpha_2} \right)^n n! \binom{p+n-2\gamma+\epsilon, p+n+\epsilon}{1-\gamma-\delta+p+n+\epsilon} (-i\frac{s}{\alpha_2})$$

$$\times \Gamma \left( \frac{n+p+\epsilon-2\gamma, \delta+\gamma-p-\epsilon-n, -\delta-\gamma+n}{1+n-2\gamma, 1-p-n-\epsilon, 1+n} \right) \quad (B.3)$$
We remind the asymptotics behavior of a hypergeometric function of a large argument \cite{5}:

\[ S_2 (p, \alpha_1, \alpha_2, \epsilon; s) = \frac{e^{i \pi \delta}}{\Gamma^2 (p+\epsilon)} \sum_{n \geq 1} 2 F_1 \left( \begin{array}{c} n + p + \delta - \gamma, \gamma + \delta + p + n \\ 1 + p + n \end{array} ; - \frac{s}{\alpha_2} \right) \times \frac{\alpha_1^{n+p+\gamma-\epsilon}}{\alpha_2^{n+p+\delta-\gamma}} \Gamma \left( \begin{array}{c} n + p + \delta - \gamma, n - \epsilon, \epsilon - \delta - \gamma - n \\ 1 + n + \delta - \gamma - \epsilon, 1 - \gamma - \delta - p - n, 1 + p + n \end{array} \right) \] (B.4)

\[ S_3 (p, \alpha_1, \alpha_2, \epsilon; s) = e^{i \pi (\delta + p)} e^{-i \pi \epsilon} \Gamma \left( \begin{array}{c} -\gamma + \delta \\ 1 - \gamma - \delta \end{array} \right) \frac{\pi a_1^{\gamma+\delta-p} a_2^{\gamma-\delta}}{\sin (\pi \epsilon) s^p} \times \sum_{n \geq 0} \left( \frac{s}{i \alpha_1} \right)^n \left[ \left( \frac{i}{s} \right)^{\epsilon} \Gamma \left( \begin{array}{c} n - \gamma - \delta \\ 1 + \delta - \gamma - n, 1 + n \end{array} \right) 2 \Phi_1 \left( \begin{array}{c} \delta - \gamma, \gamma + \delta \\ 1 + n - p - \epsilon \end{array} ; -i \frac{s}{\alpha_2} \right) - \alpha_1^{-\epsilon} \Gamma \left( \begin{array}{c} n - \gamma - \delta + \epsilon \\ 1 + \delta - \gamma - n - \epsilon, 1 + n + \epsilon \end{array} \right) 2 \Phi_1 \left( \begin{array}{c} \delta - \gamma, \gamma + \delta \\ 1 + n - p \end{array} ; -i \frac{s}{\alpha_2} \right) \right] \] (B.5)

**Proof** — We consider the Appell function of the second kind as in \cite{A13} and assume the following dependence between the parameters

\[ a = p + \epsilon + b + c \quad \text{and} \quad f = d = 1 + b + c = 1 + a - p - \epsilon \] (B.6)

with \( \epsilon \) small and complex. The series expansion of the second Appell function can be re-summed into a Mellin-Barnes type integral representation \cite{38}:

\[ \Gamma \left( \begin{array}{c} a, b \\ d \end{array} \right) F_2 \left( \begin{array}{c} a, b \\ c \end{array} ; x, y \right) = \int_{\mathcal{C}} \frac{ds}{2i\pi} \Gamma \left( \begin{array}{c} a + s, b + s, -s \\ f + s \end{array} \right) \times 2 F_1 \left( \begin{array}{c} a + s, c \\ f \end{array} ; y \right) (-x)^s \] (B.7)

One possible choice of the contour \( \mathcal{C} \) is depicted in fig. \[6\] and the integral is convergent provided \( |\arg (-x)| + |\Im (y)| < \pi \). This can be seen using the asymptotics of \( \Gamma (z) \) in the region \( |\arg (s)| < \pi \). One has

\[ \Gamma \left( \begin{array}{c} a + s, b + s, -s \\ f + s \end{array} \right) = 2i \pi s \text{sgn} (\Im (s)) s^{a+b-f-1} e^{-\pi |\Im (s)|} (1 + O (s^{-1})) \quad s \to +i\infty \] (B.8)

We remind the asymptotics behavior of a hypergeometric function of a large argument \cite{5}:

\[ 2 F_1 \left( \begin{array}{c} a + s, c \\ f \end{array} ; y \right) = \left\{ e^{i \pi \Gamma} \left( \begin{array}{c} f \\ f - c \end{array} \right) (sy)^{-c} + e^{sy+ay} (sy)^{c-f} \right\} \left[ 1 + O (s^{-1}) \right] \] (B.9)

So that, all together

\[ \left| \Gamma \left( \begin{array}{c} a + s, b + s, -s \\ f + s \end{array} \right) 2 F_1 \left( \begin{array}{c} a + s, c \\ f \end{array} ; y \right) (-x)^{-s} \right| \leq C \left| s^{a+b+2f-1+c} \exp \left\{ -\Im (s) \arg (-x) + |\Im (y) \Im (s)| - \pi |\Im (s)| \right\} \right| \] (B.10)

where \( C \) is some computable constant. The integrand of \cite{B.7} is thus absolutely integrable provided \( |\arg (-x)| + |\Im (y)| < \pi \).
We now split \( (B.7) \) in two by using the analytic continuation of Gauss function for for large \( y \) \( (A.12) \):

\[
\Gamma \left( \frac{a}{d}, \frac{b}{f}, \frac{c}{f} \right) F_2 \left( \frac{a}{d}; \frac{b}{d}, \frac{c}{f} : x, y \right) = L_1 + L_2
\]

\[
L_1 = \int_{\mathcal{C}} \frac{ds}{2i\pi} \Gamma \left( \frac{a + s, b + s, -s, c - a - s}{d + s, f - a - s} \right) 2F_1 \left( \frac{a + s, a + s + 1 - f}{a + 1 + s - c}, \frac{1}{y} \right) \frac{(-x)^s}{(-y)^{a+s}}
\]

\[
L_2 = \int_{\mathcal{C}} \frac{ds}{2i\pi} \Gamma \left( \frac{a - c + s, b + s, -s, c}{d + s, f - c} \right) 2F_1 \left( \frac{c + 1 - f, c}{c + 1 - a - s}, \frac{y^{-1}}{y} \right) \frac{(-x)^s}{(-y)^{a+s}} \quad (B.11)
\]

We were able to split the integral into two parts as each of them converges separately. The separate convergence of the integrals is readily seen from the asymptotic of the Gauss hypergeometric function [5]:

\[
2F_1 \left( \frac{\alpha + s, \beta + s}{\gamma + s} ; z \right) = (1 - z)^{\gamma - \alpha - \beta - s} 2F_1 \left( \frac{\gamma - \alpha, \gamma - \beta}{\gamma + s} ; z \right) \sim (1 - z)^{\gamma - \alpha - \beta - s} \left( 1 + O \left( s^{-1} \right) \right) \quad , \quad |z| < 1. \quad (B.12)
\]

Hence putting

\[
g (x, y; s) = \Gamma \left( \frac{a + s, b + s, c - a - s, -s}{d + s, f - a - s} \right) 2F_1 \left( \frac{a + s, a - f + s + 1}{a - c + s + 1}, \frac{1}{y} \right) \frac{(-x)^s}{(-y)^{a+s}}, \quad (B.13)
\]

and using the asymptotic behavior of the Gamma function we get

\[
|g (x, y; s)| \leq C |s|^{\Re(a+b+c-d-f-1)} e^{-\pi |\Im(s)| - \Im(s)(\arg(-x) - \arg(1-y^{-1}) - \arg(-y))}. \quad (B.14)
\]

\( L_1 \) is thus convergent in the region defined by the equation

\[
|\arg(-x) - \arg(1-y^{-1}) - \arg(-y)| < \pi \quad \text{and} \quad |y| > 1. \quad (B.15)
\]
Similar calculation lead to the conclusion that \( L_2 \) is convergent in the region \( |\arg(-x)| < \pi \).

\( L_1 \) can be computed as a sum over the poles located at the right of \( \mathcal{C} \). These are
\[
s = n, \ n \in \mathbb{N} \quad \text{and} \quad s = -b - \epsilon + n, \ n \in \mathbb{N}^*.
\]

One eventually arrives to
\[
L_1 = -\frac{\sin \pi \epsilon}{\sin \pi (b + \epsilon)} ( -y )^{-a} \sum_{n \geq 0} \left( \frac{-x}{y} \right)^n \Gamma \left( \frac{a + n, b + n, p + \epsilon + n}{d + n, 1 + n + \epsilon + p + b, 1 + n} \right)
\]
\[
\times \ 2F_1 \left( \frac{a + n, p + n + \epsilon}{1 + b + \epsilon + n + p} ; y^{-1} \right) - \frac{\sin \pi b}{\sin \pi (b + \epsilon)} (-x)^{-b - \epsilon} ( -y )^{-p - c} \sum_{n \geq 1} \left( \frac{-x}{y} \right)^n
\]
\[
\times \Gamma \left( \frac{p + c + n, n - \epsilon, p + n - b}{1 + c - \epsilon + n, 1 + n - \epsilon - b, 1 + p + n} \right) 2F_1 \left( \frac{p + c + n, p + n - b}{1 + n + p} ; y^{-1} \right)
\]
\[
\text{(B.16)}
\]

Similarly \( L_2 \) can be computed as a sum over the poles located to the left of \( \mathcal{C} \); these are \( \{ -b - n, -b - n - \epsilon \} \) where \( n \in \mathbb{N} \). This becomes apparent when one normalizes the Gauss function:
\[
2F_1 \left( \frac{a, b}{c} ; z \right) = \Gamma ( c ) \ 2F_1 \left( \frac{a, b}{c} ; z \right).
\]

so that \( \Phi \) is an entire function of the parameters \( a, b \) and \( c \). The result reads:
\[
\frac{(-1)^p ( -y )^{-\epsilon} \sin \pi (d - b)}{(-x)^{b+\epsilon}} \sum_{n \geq 0} \left( \frac{-x}{x} \right)^n \Gamma \left( \frac{b + n, n - c}{1 + n} \right) \ 2F_1 \left( \frac{-b, c}{1 + n - \epsilon} ; y^{-1} \right)
\]
\[
- \frac{\sin \pi (d - b - \epsilon)}{x^n \sin \pi (d - b)} \Gamma \left( \frac{b + n + \epsilon, \epsilon + n - c}{1 + \epsilon + n} \right) \ 2F_1 \left( \frac{-b, c}{1 + n - \epsilon} ; y^{-1} \right)
\]
\[
\text{(B.18)}
\]

The joint condition for the convergence of \( L_1 \) and \( L_2 \) defines an open subset \( O \) of \( \mathbb{C}^2 \).

Hence we can continue the series representation for the second Appell function to the largest open subset in \( \mathbb{C}^2 \) containing \( O \) where the series is convergent. In particular, the series representation is well defined for the range of parameters that we use. Specifying the values of \( a, b, c, d, f \) to the ones of the Lemma we obtain the claimed result. \( \Box \)

### B.2 Useful integrals

We will use the series expansion for \( F_2 \) in order to compute some integrals of products of \( CHF \). We remind the definitions of the functions \( \tau ( \gamma, \delta; t ) \) and \( \varphi ( \gamma, \delta; t ) \)

\[
\Gamma \left( \frac{1 - 2\gamma, 1 - 2\gamma}{1 + \delta - \gamma, 1 - \delta - \gamma} \right) \tau ( \gamma, \delta; t ) = - \Phi ( -\gamma - \delta, 1 - 2\gamma, -it ) \Phi ( \delta - \gamma, 1 - 2\gamma; it )
\]
\[
+ ( \partial_t \Phi ) ( -\gamma - \delta, 1 - 2\gamma, -it ) \Phi ( \delta - \gamma, 1 - 2\gamma; it )
\]
\[
+ \Phi ( -\gamma - \delta, 1 - 2\gamma; -it ) ( \partial_t \Phi ) ( \delta - \gamma, 1 - 2\gamma; it ) \]

and

\[
\Gamma \left( \frac{1 - 2\gamma, 1 - 2\gamma}{1 + \delta - \gamma, 1 - \delta - \gamma} \right) \varphi ( \gamma, \delta; t ) = \Phi ( -\gamma - \delta, 1 - 2\gamma, -it ) \Phi ( \delta - \gamma, 1 - 2\gamma; it )
\]

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Corollary B.1 Let $|\Re(\delta)| < 1/2$, $\Re(\gamma) < 1/2$ and $\tau(\gamma, \delta; t)$, $\varphi(\gamma, \delta; t)$ be as above, then

$$
\int_{-\infty}^{0} dt \left( \frac{e^{i\pi t}}{|t|^{2\gamma}} \varphi(\gamma, \delta; t) - 1 \right) = 2i\delta \quad , \quad \int_{\mathbb{R}} dt \left( \frac{e^{-i\pi \delta \text{sgn}(t)}}{|t|^{2\gamma}} \varphi(\gamma, \delta; t) - 1 \right) = 0
$$

$$
\int_{-\infty}^{0} dt \left( \frac{e^{i\pi t}}{|t|^{2\gamma}} \tau(\gamma, \delta; t) + 1 - \frac{2i\delta}{t - 1} \right) = -2i\delta - \pi \gamma + i(\gamma + \delta) \psi(-\gamma - \delta) + i(\delta - \gamma) \psi(\delta - \gamma)
$$

$$
\int_{\mathbb{R}} dt \left( \frac{e^{-i\pi \text{sgn}(t)} \delta}{|t|^{2\gamma}} \tau(\gamma, \delta; t) + 1 - \frac{2i\delta}{t + \text{sgn}(t)} \right) = -2\pi \gamma
$$

$$
\int_{\mathbb{R}} dt \log(|t|) \left( |t|^{-2\gamma} \varphi(\gamma, 0, t) - 1 \right) = -2\pi \gamma
$$

$$
\int_{\mathbb{R}} dt \log(|t|) \left( |t|^{-2\gamma} \tau(\gamma, 0, t) + 1 \right) = 2\pi \gamma(\psi(1 - \gamma) - 2\psi(1 - 2\gamma) + 1)
$$

The Riemann integrability of the integrands is part of the conclusion.

Proof — Using the asymptotic behavior of Trico’s CHF (A.6) as well as (A.9) one can readily convince oneself that for $|\Re(\delta)| < 1/2$

$$
\Phi(-\delta - \gamma, 1 - 2\gamma; -it) \Phi(\delta - \gamma, 1 - 2\gamma; it) \sim_{t \rightarrow +\infty} e^{i\pi \delta t^{2\gamma} \Gamma} \left( \begin{array}{c} 1 - 2\gamma, 1 - 2\gamma \\ 1 + \delta - \gamma, 1 - \delta - \gamma \end{array} \right) \left\{ 1 + O\left( \frac{e^{\pm it}}{t^{1+2\delta}} \right) \right\}
$$

(B.19)

$$
(\partial_z \Phi)(-\delta - \gamma, 1 - 2\gamma; -it) \Phi(\delta - \gamma, 1 - 2\gamma; it) + \Phi(-\delta - \gamma, 1 - 2\gamma; -it)(\partial_z \Phi)(\delta - \gamma, 1 - 2\gamma; it) \sim_{t \rightarrow +\infty} e^{i\pi \delta t^{2\gamma} \Gamma} \left( \begin{array}{c} 1 - 2\gamma, 1 - 2\gamma \\ 1 + \delta - \gamma, 1 - \delta - \gamma \end{array} \right) \left\{ \frac{2i\delta}{t} + O\left( \frac{e^{\pm it}}{t^{1+2\delta}} \right) \right\}.
$$

(B.20)

Where the terms that are sub-leading to $e^{\pm it t^{\pm 2\delta - 1}}$ are already absolutely integrable. For $|\Re(\delta)| < 1/2$, $e^{\pm it t^{\pm 2\delta - 1}}$ is only Riemann integrable, thus all integrals should be understood in this sense. Moreover the asymptotics in the region $t \rightarrow -\infty$ can be inferred from those at $t \rightarrow +\infty$ if one starts with the complex conjugated parameters $\delta^*$ and $\gamma^*$ and then takes the complex conjugate of the asymptotic series. This settles the question about the Riemann integrability of the different integrands.

The proof of this Corollary is straightforward although quite long. The principle of the proof is to extract the divergent and constant terms in the $s \rightarrow 0^+$ limit from the expansion of $f(p, \alpha_1, \alpha_2, c; s)$ around $s \rightarrow 0^+$.

We shall explain in detail how to obtain the first four integrals. The remaining two are obtained in a similar fashion, although computations become more and more involved. At the end of the proof we list all the summation identities that are necessary to compute the
Now, the first integral in the list of (B.19) is obtained by considering (B.24) in the case of $s \to 0^+$ limit in the other cases. Recall the notation introduced in (B.1). Then one has

\[
\int_0^{+\infty} \frac{dt}{t^{2\gamma}} \left\{ \varphi(\gamma, \delta, t) - t^{2\gamma} e^{i\pi \delta} \right\} = \lim_{s \to 0^+} \int_0^{+\infty} \frac{dt}{t^{2\gamma}} e^{-st} \left\{ \varphi(\gamma, \delta, t) - t^{2\gamma} e^{i\pi \delta} \right\} = \Gamma \left( 1 + \delta - \gamma, 1 - \delta - \gamma \right) \left( f(1, 1, 1, 0; s) - \frac{e^{i\pi \delta}}{s} \right). \tag{B.21}
\]

Since we compute the $s \to 0^+$ limit, it is enough to determine $f(1, 1, 1, 0; s)$ up to $o(1)$ with respect to $s \to 0^+$. One easily sees that $S_1(1, 1, 1, 0; s) = 0$, cf. (B.3). Since we compute $S_2(1, 1, 1, 0; s)$ up to $O(s)$ terms, we can already replace Gauss’ function by 1; the latter only contributes to higher orders terms in $s$. Then, for this particular choice of the constants $p, \alpha_1, \alpha_2$ and $\epsilon$, the second sum boils down to

\[
S_2(1, 1, 1, 0; s) = -i e^{i\pi \delta} \sum_{n \geq 1} \Gamma \left( \frac{n}{n + 2} \right) + O(s) = -i e^{i\pi \delta} + O(s). \tag{B.22}
\]

Finally, we estimate $S_3(1, 1, 1, 0; s)$ \[B.5\]. This term is the most complicated one. Indeed, due to the presence of the factor $1/(s \sin \pi \epsilon)$ in front of the sum, one must compute the linear in $\epsilon$ terms of the sum (the zeroth order vanishes as it should be). Also, it is enough to expand the sum up to $O(s \log s)$ as such terms won’t contribute to the result after the $s \to 0^+$ limit is performed. After some computations one gets

\[
S_3(1, 1, 1, 0; s) = \Gamma \left( \frac{\delta - \gamma, -\delta - \gamma}{1 + \delta - \gamma, 1 - \delta - \gamma} \right) e^{i\pi \delta} (s^{-1} - 2i\delta) + i e^{i\pi \delta} + O(s \log s). \tag{B.23}
\]

Adding up all the three contributions, we see that the $s^{-1}$ part cancels with the one coming from the regularization term in (B.21). The remaining terms combine to give

\[
\int_0^{+\infty} \frac{dt}{t^{2\gamma}} \left\{ \varphi(\gamma, \delta, t) - t^{2\gamma} e^{i\pi \delta} \right\} = -2i\delta e^{i\pi \delta}. \tag{B.24}
\]

Now, the first integral in the list of (B.19) is obtained by considering (B.24) in the case of parameters $\delta^*$ and $\gamma^*$, changing variables $t \to -t$, and then taking the complex conjugate of the whole expression.

We now explain how to evaluate the integrals involving $\tau(\delta, \gamma; t)$. Since we have already established the value of integrals involving $\varphi(\delta, \gamma; t)$, we only need to compute

\[
e^{-ix\delta} \int_0^{+\infty} \frac{dt}{t^{2\gamma}} \left\{ (\partial_2 \Phi)(-\gamma - \delta, 1 - 2\gamma; -it) \Phi(\delta - \gamma, 1 - 2\gamma; it) + \Phi(-\gamma - \delta, 1 - 2\gamma; -it) (\partial_2 \Phi)(\delta - \gamma, 1 - 2\gamma; it) - \frac{2i\delta t^{2\gamma} e^{i\pi \delta}}{t + 1} \Gamma \left( \frac{1 - 2\gamma, 1 - 2\gamma}{1 + \delta - \gamma, 1 - \delta - \gamma} \right) \right\} = -i \lim_{s \to 0^+} \left\{ [\partial_{\alpha_2} f(0, 1, 1, 0; s) - \partial_{\alpha_1} f(0, 1, 1, 0; s)] e^{-i\pi \delta} + 2i\delta \Gamma \left( \frac{1 - 2\gamma, 1 - 2\gamma}{1 + \delta - \gamma, 1 - \delta - \gamma} \right) \Psi(1, 1; s) \right\}. \tag{B.25}
\]
In this case, only the first term appearing in \( S_1 \) contributes:

\[
[(\partial_{\alpha_2} - \partial_{\alpha_1}) \cdot S_1](0, 1, 1, 0; s) = ie^{-i\pi \gamma} \left( \frac{1}{\delta - \gamma} \right) + O(s) \quad \text{(B.26)}
\]

Already for \( S_2 \) one has to compute some less trivial sums

\[
[(\partial_{\alpha_2} - \partial_{\alpha_1}) \cdot S_2](0, 1, 1, 0; s) = ie^{i\pi \delta} \left( \frac{1}{1 - 2\gamma, 1 - 2\gamma} \right) \sum_{n \geq 1} \frac{1}{n (n + \delta - \gamma)} + \frac{1}{n (n + \delta + \gamma)} \left( 1 + O(s) \right)
\]

\[
= ie^{i\pi \delta} \left( \frac{1}{1 - 2\gamma, 1 - 2\gamma} \right) \left( \frac{\psi(1) - \psi(1 + \delta - \gamma)}{\gamma - \delta} + \frac{\psi(1 + \delta + \gamma) - \psi(1)}{\gamma + \delta} \right) \left( 1 + O(s) \right) \quad \text{(B.27)}
\]

And we have used \( \sum_{n \geq 1} \frac{1}{n (n - a)} = \frac{\psi(1) - \psi(1 - a)}{a} \). Finally,

\[
[(\partial_{\alpha_2} - \partial_{\alpha_1}) \cdot S_3](0, 1, 1, 0; s) = -e^{i\pi \delta} \left( \frac{\delta - \gamma, -\delta - \gamma}{1 + \delta - \gamma, 1 - \delta - \gamma} \right)
\]

\[
\{2\delta [\log (i/s) + 2\psi(1) - \psi(-\delta - \gamma) - \psi(1 + \delta - \gamma)] + 1 \} (1 + O(s \log s)) \quad \text{(B.28)}
\]

Adding together the three contributions we get

\[
e^{-i\pi \delta} \left( \frac{1 + \delta - \gamma, 1 - \delta - \gamma}{1 - 2\gamma, 1 - 2\gamma} \right) [(\partial_{\alpha_2} - \partial_{\alpha_1}) \cdot f](0, 1, 1, 0; s) =
\]

\[
- 2\delta [\log (i/s) + \psi(1) - \psi(-\delta - \gamma) - \psi(1 + \delta - \gamma)] - (\gamma + \delta) \psi(\delta - \gamma) + \frac{2\delta}{\gamma - \delta} + (\gamma - \delta) \psi(1 + \delta + \gamma) + \frac{\pi (\delta - \gamma) e^{-i\pi (\gamma + \delta)}}{\sin \pi (\gamma + \delta)} + O(s \log s)
\]

\[
= 2\delta (\log s - \psi(1)) - i\pi \gamma + (\gamma + \delta) \psi(-\delta - \gamma) + (\delta - \gamma) \psi(\delta - \gamma) + O(s \log s) \quad \text{(B.29)}
\]

We used the addition formulae for the \( \psi \) function

\[
\psi(1 + z) - \psi(z) = \frac{1}{z} \quad , \quad \psi(1 + z) - \psi(-z) = -\pi \cot \pi z ,
\]

in order to obtain the last line. The leading asymptotics of Tricomi’s CHF around zero \( \Psi(1, 1; s) = -\log s + \psi(1) + O(s \log s) \) allows to take the \( s \to 0^+ \) limit in (B.25). We get

\[
e^{-i\pi \delta} \int_0^\infty \frac{dt}{t^{2\gamma}} \left\{ \varphi(\gamma, \delta; t) + \tau(\gamma, \delta; t) - \frac{2i\delta e^{i\pi \delta}}{t + 1} \right\} = -\pi \gamma - i (\gamma + \delta) \psi(-\delta - \gamma) - i (\delta - \gamma) \psi(\delta - \gamma) \quad \text{(B.31)}
\]

The other integrals involving \( \tau \) are then obtained from the latter results by the standard manipulations that we have already described.

The value of the last two integrals appearing in the Corollary is obtained by a similar procedure. Namely,

\[
\int_0^\infty \frac{dt \log t}{t^{2\gamma}} \{ \varphi(\gamma, 0; t) - t^{2\gamma} \}
\]

\[
= \lim_{s \to 0^+} \left\{ \Gamma \left( \frac{1 - \gamma, 1 - \gamma}{1 - 2\gamma, 1 - 2\gamma} \right) (\partial_e f)(1, 1, 1, 0; s) - \frac{\psi(1) - \log s}{s} \right\} \quad \text{(B.32)}
\]

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Finally, using standard properties of the \( \psi \) function we get that

\[
-\imath \Gamma \left( \frac{1}{1 - 2\gamma}, 1 - 2\gamma \right) \left( (\partial_{\alpha_{2\epsilon}} \epsilon - \partial_{\alpha_{1\epsilon}} \epsilon) f \right) (0, 1, 1, 0; s) = \gamma \pi \{ \psi (1 - \gamma) - 2\psi (1 - 2\gamma) \} + O (s \log s) .
\]
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