Discrete-to-Continuous Extensions: Lovász extension and Morse theory

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Abstract

This is the first of a series of papers that develop a systematic bridge between constructions in discrete mathematics and the corresponding continuous analogs. In this paper, we establish an equivalence between Forman’s discrete Morse theory on a simplicial complex and the continuous Morse theory (in the sense of any known non-smooth Morse theory) on the associated order complex via the Lovász extension. Furthermore, we propose a new version of the Lusternik-Schnirelman category on abstract simplicial complexes to bridge the classical Lusternik-Schnirelman theorem and its discrete analog on finite complexes. More generally, we can suggest a discrete Morse theory on hypergraphs by employing piecewise-linear (PL) Morse theory and Lovász extension, hoping to provide new tools for exploring the structure of hypergraphs.

Keywords: Lovász extension; discrete Morse theory; Lusternik-Schnirelman theory; hypergraph; simplicial complex

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1 Introduction and Background

The Lovász extension is a basic tool in discrete mathematics, especially for some combinatorial optimization problems and submodular analysis [37]. It was introduced in the

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The Lovász extension appears in many areas like game theory, matroid theory, stochastic processes, electrical networks, computer vision, and machine learning [21].

In fact, a special form of the Lovász extension appeared already in the context of the Choquet integral [13] which has fruitful applications in statistical mechanics, potential theory and decision theory. Since the Lovász extension does not require the monotonicity of the set function in finite cases of the Choquet integral, it has a wider range of applications, for instance in combinatorics, for algorithms in computer science.

The discrete Morse theory on simplicial complexes was introduced by Forman [22, 23]. This theory has some deep connections with smooth Morse theory [6, 7, 26], as well as practical applications [42], and also admits several slight generalizations. It also possesses surprising applications in algebraic combinatorics and derived algebraic geometry [2]. Both this discrete Morse theory and the classical smooth one make assumptions that exclude some complicated cases such as monkey-saddle points, which makes them simpler.

Some other versions in the literatures only extend the domain to the cube [0, 1]^V or the nonnegative orthant \( \mathbb{R}^V_{\geq 0} \). In fact, many works on Boolean lattices identify \( \mathcal{P}(V) \) with the discrete cube \( \{0, 1\}^n \).
We will construct the relationship between the Morse theory of a discrete Morse function, that is, a function defined on the face set $K$ of an abstract simplicial complex, and its Lovász extension in Section 3. We define the Lovász extension on the order complex $S_K$, the simplicial complex whose vertex set is $K$ and whose faces are the inclusion chains in $K$. In this way, we can use the procedure of Lovász extension to extend a function on a set of discrete points, the vertices of $S_K$.

We restrict the Lovász extension $f^L$ to a geometric realization of the order complex $\lvert S_K \rvert$ which is a subset of the feasible domain of $f^L$. It is surprising that the Lovász extension on restricted domains leads to a fascinating connection between discrete and continuous Morse theory and Lusternik-Schnirelman theory:

**Theorem A** (Theorems 3.1, 3.2 and 3.3). For a simplicial complex with vertex set $V$ and face set $K$, let $f : K \to \mathbb{R}$ be an injective discrete Morse function. Then the following conditions are equivalent:

1. $\sigma$ is a critical point of $f$ with $\dim \sigma = i$;
2. $1_{\sigma}$ is a critical point of $f^L|_{S_K}$ with index $i$ in the sense of weak slope (metric Morse theory);
3. $1_{\sigma}$ is a critical point of $f^L|_{S_K}$ with index $i$ in the sense of Kühnel (PL Morse theory);
4. $1_{\sigma}$ is a Morse critical point of $f^L|_{S_K}$ with index $i$ in the sense of topological Morse theory.

Here the notation $\lvert S_K \rvert$ indicates a suitable restriction, more precisely the geometric realization of the order complex of $K$ whose vertices are the simplices of $K$ and whose simplices correspond to the chains in $K$, (see Section 3) for $f^{L}$ being well-defined.

Moreover, the discrete Morse vector $(n_0, n_1, \cdots, n_d)$, representing the number $n_i$ of critical points with index $i$, of $f$ coincides with the continuous Morse vector of $f^L|_{S_K}$.

Moreover, the Lusternik-Schnirelmann category theorem is preserved under Lovász extension:

$$\min_{L \in \text{LSC}_m(K)} \max_{\sigma \in L} f(\sigma) = \inf_{S \in \text{LSC}_m(\lvert S_K \rvert)} \sup_{x \in S} f^L(x),$$

where we refer to (2) and (3) for the definition of the Lusternik-Schnirelmann classes $\text{LSC}_m(K)$ and $\text{LSC}_m(\lvert S_K \rvert)$, respectively.

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2 Here and after, for convenience, we do not differentiate between a simplicial complex and its face set (both are denoted by $K$).

3 The definition of category classes for $K$ is introduced in (2) in Section 3.
In summary, Theorem A says that the Morse structures of $K$ and $|S_K|$ are coarsely equivalent, and one can translate all the results about ‘Morse data’ of a discrete Morse function $f$ on $K$ to its Lovász extension $f^L$ restricted on $|S_K|$. This also reflects the deep result from [7,26] that smooth Morse theory on a manifold is almost equivalent to the discrete Morse theory on its triangulation. The difference is that we don’t assume the complex $|K|$ to be a topological manifold, so that topological results on manifolds cannot be applied directly. Fortunately, our feasible domain $|S_K|$ is a piecewise flat geometric complex. Our proofs don’t draw heavily on the standard tools in discrete Morse theory.

The idea above allows us to establish a discrete Morse theory on hypergraphs, which helps us to understand the structure of a hypergraph from a Morse theoretical perspective. We borrow the ideas on associated simplicial complexes of hypergraphs [41] and order complexes induced by hypergraphs [45,46], as well as finite topologies on hypergraphs. With the help of associated simplicial complexes, we introduce the geometric realizations of hypergraphs, and we prove that the geometric realization of a hypergraph collapses onto the geometric realization of its order complex. As a preliminary exploration along this direction, we provide some evidence to show that we should focus on the order complex of a hypergraph which looks like a simplicial complex. Some results similar to Theorem A on such a complex-like hypergraph are presented in Section 4.

Related works: The recent paper [43] on discrete Morse theory for hypergraphs uses the so-called embedded homology on hypergraphs [11], that is, they embed the hypergraph into a simplicial complex obtained by adding some missing simplices, and define a Morse function on a hypergraph as the restriction of a Morse function on that simplicial complex. The underlying homology theory [28] takes the difference between the original hypergraph and the embedding complex into account. The theory of [43] is different from our approach. We consider both associated simplicial complexes and order complexes from the perspective of homotopy, and our definition of Morse functions on hypergraphs uses order complexes. Such an order complex is naturally defined from the hypergraph, without the need to add any further simplices. In particular, the order complex of a hypergraph in general is different from that of an embedding simplicial complex.

We provide a dictionary between different analogs of Morse theory. The relations between different versions of Morse theory have great potential to translate a problem in one context to another, thereby giving new tools for attacking problems and drawing connections. The key tool is the Lovász extension, with which we succeeded in constructing fruitful relations between certain discrete objects and their continuous analogs [31,32].

2 Preliminaries on Morse theory

Morse theory [38,39] enables us to analyze the topology of an object $M$ by studying functions $f : M \to \mathbb{R}$. In the classical case, $M$ is a manifold and $f$ is generic and differentiable. There are, however, many extensions of Morse theory in modern mathematics that do not require a smooth structure, such as the metric and topological Morse theory by the Italian school [16,17,30,33], the PS (piecewise smooth) or stratified Morse theory by Thom, Goresky and MacPherson [27], the PL Morse theory by Banchoff [4], Kühnel [5,19] and the Berlin school, as well as the discrete Morse theory by Forman [22,23].

In all such cases, a typical function $f$ on $M$ will reflect the topology quite directly,
allowing one to find CW structures on $M$ and to obtain information about their homology. The following results embody the abstract content of Morse theory, and they hold in continuous as well as in discrete cases.

**Morse fundamental theorem.** If $f$ has $n_i$ critical points of index $i$, $i = 0, 1, \cdots, d$, then $M$ is homotopy equivalent to a cell complex (called Morse complex) with $n_i$ cells of dimension $i$. One can write it as

\[ M \simeq \text{cell complex with } n_i \text{ cells of } \dim i \]

**Morse relation.** Denote by $P(X, A)(\cdot)$ the Poincaré polynomial\(^4\) of the pair of topological spaces $(X, A)$ over a given field $F$, where $X \supset A$. Then

\[
\sum_{a < f(x) < b} P\{f \leq f(x)\}, \{f \leq f(x)\} - \{x\})(t) = P\{f < b\}, \{f \leq a\})(t) + (1 + t)Q(t)
\]

where $a < b$, $Q(\cdot)$ is a polynomial with nonnegative coefficients.

The main aim of this section is to study the Lovász extension of a discrete Morse function on a simplicial complex, and to provide equivalences between discrete Morse theory and its Lovász extension.

For this purpose, we first clarify the notions and concepts and summarize the various Morse theories mentioned above.

- **Metric Morse theory:** Let $M$ be a metric space and $F$ a continuous function on $M$. For a point $a \in M$, there exists $\epsilon > 0$ such that there exist $\delta > 0$ and a continuous map

\[ \mathcal{H} : B_\delta(a) \times [0, \delta] \rightarrow M \]

satisfying

\[ F(\mathcal{H}(x, t)) \leq F(x) - ct, \quad \text{dist}(\mathcal{H}(x, t), x) \leq t \]

for any $x \in B_\delta(a)$ and $t \in [0, \delta]$, where $B_\delta(a)$ is an open ball in $M$. The weak slope \([17, 30, 33]\) denoted by $|dF|(a)$ is defined as the supremum of such $\epsilon$ above. A point $a$ is called a critical point of $F$ on $M$, if it has vanishing weak slope, i.e., $|dF|(a) = 0$.

The local behaviour of $F$ near $a$ is described by the so-called critical group $C_q(F, a) := H_q(\{F \leq c\} \cap U_a, \{F = c\} \cap U_a \setminus \{a\})$, $q \in \mathbb{Z}$, where $H_q(\cdot, \cdot)$ is the singular relative homology with real field coefficients, and $U_a$ is an open neighborhood of $a$. So the Morse polynomial $p(F, a)(t) := \sum_{q=0}^{d} \text{rank } C_q(F, a)t^q$ can be defined. If $C_q(F, a)$ is non-vanishing, then we say that $q$ is an index of a metric critical point $a$, and the number $p(F, a)(1)$ is called the total multiplicity of $a$. (Note that, in general, a critical point may have more than one index. The standard assumptions of Morse theory, however, exclude that possibility.)

- **Topological Morse theory:** Let $M$ be a topological space and $F$ a continuous function on $M$. A point $a \in M$ is a Morse regular point of $F$ if there exist a neighborhood $U$ of $a$ in $M$ and a continuous map

\[ \mathcal{H} : U \times [0, 1] \rightarrow M, \quad \mathcal{H}(x, 0) = x \]

\[ P(X, A)(t) := \sum_{n \geq 0} \text{rank } H^n(X, A)t^n, \quad \text{where } H^n(X, A) \text{ is the relative cohomology of the pair } (X, A). \]
satisfying 
\[ F(\mathcal{H}(x, t)) < F(x), \]
for any \( x \in U \) and \( t > 0 \). We say that \( a \) is a Morse critical point of \( F \) on \( M \) if it is not Morse regular. The index with multiplicity of a critical point is same as in the metric setting above [16].

A symmetric homological critical value [15] of \( F \) is a real number \( c \) for which there exists an integer such that for all sufficiently small \( \epsilon > 0 \), the map \( H_k((F \leq c-\epsilon)) \hookrightarrow H_k((F \leq c+\epsilon)) \) induced by inclusion is not an isomorphism [8]. Here \( H_k \) denotes the \( k \)-th singular homology (possibly with coefficients in a field).

A real number \( c \) is a homological regular value of the function \( F \) if there exists \( \epsilon > 0 \) such that for each pair of real numbers \( t_1 < t_2 \) on the interval \((c-\epsilon, c+\epsilon)\), the inclusion \( \{F \leq t_1\} \hookrightarrow \{F \leq t_2\} \) induces isomorphisms on all homology groups [8]. A real number that is not a homological regular value of \( F \) is called a homological critical value of \( F \).

- Piecewise-Linear Morse theory: Similar to the smooth setting, the PL (piecewise linear) Morse theory introduced by Banchoff requires working with a combinatorial manifold which is both a PL manifold and a simplicial complex. Here we will use the notions developed by Kühnel [5] and later by Edelsbrunner and Harer [19]. Denote by \( \text{star}_-(v) \) the subset of the star of \( v \) on which the PL function \( F \) takes values not greater than \( F(v) \). Similarly, one can define \( \text{link}_-(v) \).

Let \( M \) be a combinatorial manifold, and let \( F \) be a PL (piecewise linear) function on \( M \).

**Definition 2.1** (Kühnel [5]). A vertex \( v \) of \( M \) is said to be a PL critical point of \( F \) with index \( i \) and multiplicity \( k_i \) if \( \beta_i(\text{star}_-(v), \text{link}_-(v)) = k_i \), where \( \beta_i \) is the \( i \)-th Betti number of the relative homology group.

Equivalently, let \( \beta'_j \) be the rank of the reduced \( j \)-th homology group of \( \text{link}_-(v) \). Using this notation, we have

**Definition 2.2** (Edelsbrunner & Harer [19]). A vertex \( v \) is a PL critical point of \( F \) with index \( i \) and multiplicity \( k_i \) if \( \beta'_{i-1} = k_i \).

Clearly, a PL critical point may have many indices and multiplicities. A vertex \( v \) is called non-degenerate critical if its total multiplicity \( \sum_{i=0}^d k_i \) is equal to 1. The PL function \( F \) is called a PL Morse function if all critical vertices are non-degenerate.

- Discrete Morse theory: A discrete Morse function on an abstract simplicial complex \((V, \mathcal{K})\) is a function \( f : \mathcal{K} \rightarrow \mathbb{R} \) satisfying for any \( p \)-dimensional simplex \( \sigma \in \mathcal{K}_p \), \#U(\sigma) \leq 1 \) and \#L(\sigma) \leq 1, where

\[
U(\sigma) := \{ \tau^{p+1} \supset \sigma : f(\tau) \leq f(\sigma) \} \quad \text{and} \quad L(\sigma) := \{ \nu^{p-1} \subset \sigma : f(\nu) \geq f(\sigma) \}.
\]

**Definition 2.3** (Forman [22][23]). We say that \( \sigma \in \mathcal{K}_p \) is a critical point of \( f \) on \( \mathcal{K} \) if \#U(\sigma) = 0 and \#L(\sigma) = 0. The index of a critical point \( \sigma \) is defined to be \( \dim \sigma \).
The main results in this paper can be summarized by:

Discrete Morse data of a typical discrete Morse function \( f \) on a finite simplicial complex \( K \) equivalent to Continuous Morse data of the Lovász extension \( f^L \) restricted on a suitable domain.

While the discrete Morse data are taken here in the sense of Forman, the continuous Morse data can be in the metric, topological or PL category as described above. Precise statements are presented in the following section.

3 Relations between discrete Morse theory and its Lovász extension

We let \( K \) be an abstract simplicial complex with vertex set \( V \), and we do not distinguish between \( K \) and its face set. Let \( K_p \) be the collection of \( p \)-simplices (or \( p \)-dimensional faces) in \( K \).

**Definition 3.1.** The order complex of \( K \) is defined by

\[
S_K := \{ C \subset K : C \text{ is a chain}\},
\]

where \( C \) is a chain if for any \( \sigma_1, \sigma_2 \in C \), either \( \sigma_1 \subset \sigma_2 \) or \( \sigma_2 \subset \sigma_1 \). It is clear that the order complex \( S_K \) is itself a simplicial complex with the vertex set \( K \). Define the geometric realization of \( S_K \) by

\[
|S_K| = \bigcup_{C \in S_K} \operatorname{conv}(1_{\sigma} : \sigma \in C),
\]

where \( \operatorname{conv} \) denotes the convex hull.

**Fact:** For any function \( f : K \to \mathbb{R} \), the feasible domain \( D_K \) of its Lovász extension \( f^L \) is \( \bigcup_{t \geq 0} t|S_K| \). This means that the Lovász extension \( f^L \) is well-defined on \( |S_K| \).

We refer to Proposition 5.1 for the proof of the fact above and the observation below.

**Observation:**

\[
|S_K| = D_K \cap S_\infty = \bigcup_{\text{maximal chain } C \subset K} \operatorname{conv}(1_{\sigma} : \sigma \in C),
\]

where \( S_\infty = \{ x \in \mathbb{R}^n : \| x \|_\infty = 1 \} \) is the unit \( l^\infty \)-sphere, and \( \operatorname{conv} \) denotes convex hull. Maximal chains from \( K \) correspond to facets (that is, simplices not contained in the others) of \( |S_K| \).

**Lemma 3.1.** Given a discrete Morse function \( f \) on a finite simplicial complex \( K \), we have:

1. If \( \sigma \) is critical, then \( f(\tau) > f(\sigma) > f(\nu) \), whenever \( \tau \nsubseteq \sigma \nsubseteq \nu \).
2. If \( f \) is an injective Morse function and \( (\sigma, \tau) \) is a regular pair (i.e., \( \sigma \subset \tau \) with \( \dim \tau = \dim \sigma + 1 \) and \( f(\sigma) > f(\tau) \)), then
(2.1) for all \( \tau' \supseteq \tau \) with \( \tau' \not\supset \tau \setminus \sigma \), \( f(\tau') > f(\sigma) \);
(2.2) for any \( \sigma' \subsetneq \tau \) with \( \sigma' \supset \tau \setminus \sigma \), \( f(\sigma') < f(\tau) \);
(2.3) for each \( \sigma'' \subsetneq \sigma \), \( f(\sigma'') < f(\sigma) \).

Proof of Lemma 3.3. Let \( c = f(\sigma) \). Note that \( f(\nu') < c \) for all \( \nu' \subset \sigma \). If there exists \( \nu' \subset \sigma \) such that \( f(\nu') \geq c \), then \( f(\nu') \geq f(\sigma) > f(\nu') \) for all \( \nu' \supset \nu' \subset \sigma \) with \( \nu' \subset \sigma \). Since there are two \( \nu' \subset \sigma \) in \( \sigma \) containing \( \nu' \subset \sigma \), this is not compatible with the definition of a discrete Morse function. In this way, we can prove by induction on the dimension of faces of \( \sigma \) that every face \( \nu \subset \sigma \) satisfies \( f(\nu) < c \).

The proofs of the other statements are similar. \( \square \)

**Convention 1.** We use \( \cong \) and \( \simeq \) to express homeomorphism equivalence and homotopy equivalence, respectively. The link and star of some \( \sigma \in K \) will be taken on \( S_K \). The operation \( \ast \) denotes the geometric join operator \([10]\).

**Lemma 3.2.** Given an injective discrete Morse function, we have:

\[
\operatorname{link}_-(\sigma) \simeq \begin{cases} S^{\dim \sigma - 1}, & \text{if } \sigma \text{ is critical,} \\ \text{pt}, & \text{if } \sigma \text{ is regular.} \end{cases}
\]

Proof. The link of \( \sigma \) in the order complex \(|S_K|\) is the geometric join\(^5\) of

\[
\text{conv}(1_\nu : \nu \in \mathcal{C}) \cong S^{\dim \sigma - 1}
\]

and

\[
\text{conv}(1_\tau : \tau \in \mathcal{C})
\]

According to Lemma 3.1 and the definition of \( \operatorname{link}_-(\sigma) \), we obtain that if \( \sigma \) is critical, then

\[
\operatorname{link}_-(\sigma) := \operatorname{link}_-(1_\sigma) = \text{conv}(1_\nu : \nu \in \mathcal{C}) \cong S^{\dim \sigma - 1}
\]

If \((\sigma, \tau)\) is a regular pair, we note that \( \operatorname{link}_-(\sigma) \) is the join of \( \operatorname{link}_-(1_\sigma) \) and

\[
\text{conv}(1_\nu : \nu' \in \mathcal{C}) \simeq 1_\tau,
\]

where \([\sigma, \tau] := \{ \tau' \supset \tau : f(\tau') < f(\sigma) \} \). That means, \( \operatorname{link}_-(\sigma) \simeq S^{\dim \sigma - 1} \ast 1_\tau \cong B^{\dim \sigma} \simeq \text{pt} \).

Similarly, one can check that \( \operatorname{link}_-(\tau) \cong B^{\dim \tau - 1} \simeq \text{pt} \). The proof is completed. \( \square \)

**Lemma 3.3** (Kühnel [55]). Given a real-valued PL function \( f^{PL} \) on a simplicial complex \(|K|\), then the induced subcomplex of \( K \) on \( \{ v \in K_0 : f^{PL}(v) \leq t \} \) is homotopy-equivalent to the sublevel set \( \{ f^{PL} \leq t \} \).

\(^5\)The (geometric) join of subsets \( A \) and \( B \) in \( \mathbb{R}^n \) is defined as \( A \ast B := \{ t a + (1 - t) b : a \in A, b \in B, 0 \leq t \leq 1 \} \). We refer to [10] for details.
**Lemma 3.4.** Given an injective discrete Morse function, denote by $\epsilon_0 = \min\{|f(\sigma) - f(\sigma')| : \sigma \neq \sigma'\} > 0$.

If $\sigma$ is critical, then

$$\{f^L \leq t\} \cap \text{star}(1_\sigma) \simeq \begin{cases} \mathbb{S}^{\dim \sigma - 1}, & \text{if } f(\sigma) - \epsilon_0 < t < f(\sigma), \\ \mathbb{B}^{\dim \sigma}, & \text{if } f(\sigma) \leq t < f(\sigma) + \epsilon_0, \end{cases}$$

where star($1_\sigma$) is the star of 1$_\sigma$ in |$S_K$|. In consequence, 1$_\sigma$ is a topological/metric critical point of $f^L|_{S_K}$, and $f(\sigma)$ is a (symmetric) homological critical value.

**Proof.** Denote by

$$|S_\sigma| = \bigcup_{\text{maximal chain } C \subseteq P(\sigma)} \text{conv}(1_\nu : \nu \in C).$$

Then it can be checked that $|S_\sigma|$ is homeomorphic to the closed geometric simplex $\overline{\sigma}$ in $|K|$, and thus it is homeomorphic to the disc $\mathbb{B}^{\dim \sigma}$. Since $\sigma$ is critical, by Lemma 3.1 and the definition of $\epsilon_0$, for any $\nu \subsetneq \sigma$, $f(\nu) \leq f(\sigma) - \epsilon_0$. Hence, for $f(\sigma) - \epsilon_0 < t < f(\sigma)$,

$$|S_\sigma| \cap \text{star}(1_\sigma) \cap \{f^L < t\}$$

is homotopy-equivalent to

$$\bigcup_{\text{chain } C \subseteq P(\sigma) \setminus \{\sigma\}} \text{conv}(1_\nu : \nu \in C) = \partial |S_\sigma| \cong \mathbb{S}^{\dim \sigma - 1}.$$

Similarly, for any $\tau \supsetneq \sigma$, $f(\tau) \geq f(\sigma) + \epsilon_0 > t + \epsilon_0$. Therefore, together with the piecewise linearity of $f^L$, one gets that star($1_\sigma$) $\cap \{f^L < t\}$ is homotopy-equivalent to

$$|S_\sigma| \cap \text{star}(1_\sigma) \cap \{f^L < t\}$$

and thus the proof is completed.

The case of $f(\sigma) \leq t < f(\sigma) + \epsilon_0$ is similar. For more details, we may apply Lemma 3.3 to $f^L$ on $|S_K|$. Then we only need to check the homotopy type of star$_-$($\sigma$) in $|S_K|$ for $t \geq f(\sigma)$ and link$_-$($\sigma$) for $t < f(\sigma)$. According to Lemma 3.4 and similar to the proof of Lemma 3.2, we obtain that for a critical point $\sigma$, star$_-$($\sigma$) is

$$\bigcup_{\text{chain } C \subseteq P(\sigma)} \text{conv}(1_\nu : \nu \in C) \cong \mathbb{B}^{\dim \sigma},$$

and link$_-$($\sigma$) $\cong \mathbb{S}^{\dim \sigma - 1}$. The proof is completed.

**Lemma 3.5.** If $(\sigma, \tau)$ is a regular pair, then $|df^L|_{S_K}(1_\sigma) > 0$ and $|df^L|_{S_K}(1_\tau) > 0$.

**Proof.** By the definition of weak slope, we should construct a locally decreasing flow from a neighborhood of $1_\sigma$ to a neighborhood of $1_\tau$.

**Case 1.** Locally decreasing flow near $1_\sigma$: For any chain containing the pair $(\sigma, \tau)$, $f^L(1_\sigma) = f(\sigma) > f(\tau) = f^L(1_\tau)$, which means that $f^L$ is decreasing along the vector $1_\sigma \cdot 1_\tau$. Then with the help of Lemma 3.1 (2), the neighborhood of $1_\tau$ on $|S_K|$ can be decreased uniformly along the direction $1_\sigma \cdot 1_\tau$ with a small modification. Precisely, we equip $|S_K|$ with the shortest path distance ‘dist’ induced by the usual Euclidean metric on $\mathbb{R}^n$. Then, for sufficiently small $\delta < \frac{1}{9}$ and an open ball $B_\delta(1_\sigma)$ in $|S_K|$, we define the locally decreasing flow

$$\mathcal{H} : B_\delta(1_\sigma) \times [0, \delta] \to |S_K|$$
determined by $H(x, t) = (1 - t)x + t1_\tau$ if $x \in B_\delta(1_\sigma) \cap \text{star}(1_\tau)$, $t \in [0, \delta]$, and

$$H(x, t) = x + t\frac{\text{proj}(x) - x}{\|\text{proj}(x) - x\|_2}$$

when $x \in B_\delta(1_\sigma) \setminus \text{star}(1_\tau)$ and $0 \leq t \leq \min\{\delta, \|\text{proj}(x) - x\|_2\}$, where $\text{proj}(x)$ is the projection of $x$ onto $\text{star}(1_\tau) \cap \triangle(x)$, and $\triangle(x)$ is the smallest simplex of $|S_K|$ containing $x$. We refer to Fig. 1 for a sketch of the picture of the construction of the local flow $H$.

![Diagram](image)

**Figure 1:** This picture visually illustrates the construction of a locally decreasing flow near $1_\sigma$ in the proof of Lemma 3.5. In a sufficiently small neighborhood $B_\delta(1_\sigma) \cap \text{star}(1_\tau)$ of $1_\sigma$, the piecewise linear flow in $B_\delta(1_\sigma) \cap \text{star}(1_\tau)$ goes towards $1_\tau$ (see the black line in the picture), while the flow line in $B_\delta(1_\sigma) \setminus \text{star}(1_\tau)$ is orthogonal to $\text{star}(1_\tau)$ (see the red line in the picture). It follows from $f^L(1_\tau) > f^L(1_\sigma) > f^L(1_\tau)$ for any $\tau' \supsetneq \sigma$ with $\tau' \not\supset \tau$, that $f^L$ is decreasing along the local flow.

According to Lemma 3.1 (2), $f^L(1_\tau') = f(\tau') > f^L(1_\sigma) = f(\sigma) > f(\tau) = f^L(1_\tau)$ for any $\tau' \supsetneq \sigma$ with $\tau' \not\supset \tau$. Then, one can check that there exists $\epsilon > 0$ satisfying

$$f^L(H(x, t)) \leq f^L(x) - \epsilon t, \quad \text{dist}(H(x, t), x) \leq t$$

for any $x \in B_\delta(1_\sigma)$ and $t \in [0, \delta]$. We then complete the construction of a locally decreasing flow near $1_\sigma$.

**Case 2. Locally decreasing flow near $1_\sigma$:** The construction depends on Lemma 3.1 (2), as in Case 1. Slight perturbations and concrete approximations in the construction of the locally decreasing flow are necessary, but we omit the tedious and elementary process which is similar to that of Case 1.

By the deformation lemma, $1_\sigma$ is Morse regular, and by the piecewise linearity of $f^L$, $1_\sigma$ is not a critical point in the sense of weak slope. Moreover, points on $|S_K|$ other than vertices of $|S_K|$ cannot be critical points of $f^L$ if $f$ is injective.

**Theorem 3.1.** Given a finite simplicial complex with vertex set $V$ and face set $K$, let $f : K \to \mathbb{R}$ be a discrete Morse function.

If $\sigma$ is a critical point of $f$, then $1_\sigma$ is a critical point of $f^L|_{|S_K|}$ with the same index in the sense of topological/metric/PL critical point theory, and the converse holds if $f$ is further assumed to be injective.
Proof. The proof is a combination of Lemmas 3.2, 3.4 and 3.5.

**Definition 3.2.** If a generic discrete Morse function \( f : K \to \mathbb{R} \) has \( n_i \) critical points of index \( i \), we say that both \( f \) and \( K \) have discrete Morse vector \( c = (n_0, n_1, \ldots, n_d) \). Similarly, for a generic Lipschitz function on a piecewise flat metric space \( M \) having \( n_i \) critical points of index \( i \), we say that both \( f \) and \( M \) have Morse vector \( c = (n_0, n_1, \ldots, n_d) \).

Now we verify that the discrete Morse structure on a simplicial complex is equivalent to the continuous Morse structure on the restricted domain of its Lovász extension. The key idea is to translate it into PL Morse theory by barycentric subdivision. This reveals the relation between the discrete Morse vectors of \( K \) and the Morse vectors of \( |S_K| \). Such a result is relevant to the main results in [7], but we develop it here in a wider context.

**Theorem 3.2.** Given a finite simplicial complex with vertex set \( V \) and face set \( K \), let \( f : K \to \mathbb{R} \) be an injective discrete Morse function. Then the discrete Morse vector of \( f \) agrees with the continuous Morse vector of \( f^L|_{S_K} \). In particular, every discrete Morse vector on \( K \) is a Morse vector on \( |S_K| \).

Proof. The simplicial complex \((K, S_K)\) is simplicially equivalent to the simplicial complex obtained by the barycentric subdivision of \((V, K)\):

\[
(K, S_K) \quad \overset{\text{simplicially equivalent}}{\sim} \quad \text{sd}(V, K)
\]

where \( \text{sd}(V, K) \) is the barycentric subdivision of the complex \((V, K)\). Here two complexes are called simplicially equivalent (or combinatorially equivalent) if their face posets\(^6\) are isomorphic as posets. The reason for this equivalence is simply that the barycenter of a simplex in \( K \) corresponds to \( K \) as a vertex in \( S_K \), and when one simplex is a facet of another, their barycenters are connected by an edge in \( \text{sd}(V, K) \).

Thus, one may redefine a discrete function \( \hat{f} \) on the vertex set of the barycentric subdivision

\[
\hat{f} : V(\text{sd}(K)) \to \mathbb{R}
\]

via \( \hat{f}(v_\sigma) = f(\sigma) \), \( \forall \sigma \in K \), where \( V(\text{sd}(K)) \) is the vertex set of \( \text{sd}(K) \). This is essentially the viewpoint of Zaremsky when defining a Bestvina-Brady discrete Morse function in [47].

Then the Lovász extension \( f^L \) is piecewise-linearly equivalent to the piecewise linear extension \( f^{PL} \) defined by

\[
f^{PL} \left( \sum_{v \in F} t_v v \right) = \sum_{v \in F} t_v \hat{f}(v)
\]

\(^6\) Every discrete Morse function \( f \) corresponds to a unique discrete Morse vector, while there can be a variety of discrete Morse vectors on a simplicial complex \( K \) for different \( f \). Similar presentations work for a Lipschitz function on a piecewise flat metric space, that is, we refer to the Morse vector as being of both the function and the space.

\(^7\) The face poset of a complex is the set of all of its simplices, ordered by inclusion.
for any face $F$ of the simplicial complex obtained by the barycentric subdivision of $K$ and any $t_v \geq 0$ with $\sum_{v \in F} t_v = 1$. Combining the above observations, we get the following commutative diagram:

```
\begin{array}{ccc}
  f & \leftarrow & \hat{f} \\
  \downarrow & & \downarrow \\
  f^L|_{S_K} & \leftarrow & \hat{f}^{PL} \\
  \downarrow & & \downarrow \\
  PL equivalent & & PL extension
\end{array}
```

from which we derive that the Morse data of $f^L|_{S_K}$ and $\hat{f}^{PL}$ are entirely equivalent, and furthermore, the (continuous) Morse structures of $|S_K|$ and $|sd(K)|$ essentially agree with each other.

It is clear that $\{f^L|_{S_K}| \leq t\}$ is homeomorphic to $\{\hat{f}^{PL} \leq t\}$. Applying Lemma 3.3, $\{\hat{f}^{PL} \leq t\}$ is homotopy-equivalent to the induced subcomplex on the sublevel set $\{\hat{f} \leq t\}$. Note that the level subcomplex induced by $\{f \leq t\}$ collapse onto the induced subcomplex on the sublevel set $\{\hat{f} \leq t\}$. So we have

$$|K(\{f \leq t\})| \simeq |Sd_K(\{\hat{f} \leq t\})| \simeq \{\hat{f}^{PL} \leq t\} \cong \{f^L|_{S_K}| \leq t\}$$

and thus the statement is proved.

Theorems 3.1 and 3.2 establish a correspondence between the geometric data of a discrete Morse function and the geometric information of its Lovász extension.

As a special homotopy invariant, the Lusternik-Schnirelmann category was created in order to provide estimates on the number of critical points for any smooth function on the manifold. While the Lusternik-Schnirelmann theory was mainly used in topology and analysis, it had far-reaching consequences in geometry as well, such as the well-known results on existence of multiple closed geodesics on manifolds.

We shall now introduce the concept of a category in the sense of critical point theory on an abstract simplicial complex $(V, K)$ at level $m$. We recall the classical Lusternik-Schnirelman category (see [12, 24, 36]) of a closed subset $S \subset |S_K|$:

$$\text{cat}(S) := \min \{k \in \mathbb{N}^+ : \exists k + 1 \text{ closed subsets } U_0, U_1, \cdots, U_k \text{ contractible in } |S_K|$$

$$\text{ and } \bigcup_{i=0}^k U_i \supseteq S\}$$

where a subset $U \subset |S_K|$ is contractible in $|S_K|$ if there exists a continuous map $\eta : [0,1] \times |S_K| \rightarrow |S_K|$ such that $\eta(0, \cdot) = \text{id}_{|S_K|}$ and $\eta(1, U) = \text{one-point set}$. With the aid of the Lusternik-Schnirelman category, we introduce a families of subsets of $K$:

$$\text{LSC}_m(K) = \{L \subset K : \text{cat}(|S_K(L)|) \geq m\}, \quad m = 0, 1, \cdots,$$

where $S_K(L)$ is the induced subcomplex of $S_K$ on $L$. Note that this is a family of subsets of $\mathcal{P}(K)$ (not $\mathcal{P}(V)$). Similarly,

$$\text{LSC}_m(|S_K|) = \{S \subset |S_K| : \text{cat}(S) \geq m, S \text{ is closed}\}$$

where references [12, 24] refer to the sets $U_0, \cdots, U_k$ as categorical sets, i.e., the inclusion map $U_i \hookrightarrow |S_K|$ is null-homotopic, $\forall i$. 

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We are now ready to establish a Lusternik-Schnirelman category theorem relating a discrete Morse function and its Lovász extension:

**Theorem 3.3 (L-S category theorem for a discrete Morse function and its Lovász extension).** Let \( f : \mathcal{K} \to \mathbb{R} \) be a discrete Morse function. Then we have a sequence of critical values:

\[
\min_{L \in \text{LSC}_m(\mathcal{K})} \max_{\sigma \in L} \sup_{S \in \text{LSC}_m(|\mathcal{S}_\mathcal{K}|)} \inf_{x \in S} f^L(x), \quad m = 0, 1, \ldots, \dim \mathcal{K}.
\]

**Proof.** Without loss of generality, we may assume that \( f : \mathcal{K} \to \mathbb{R} \) is an injective discrete Morse function. This is because any discrete Morse function can easily be replaced by a 1-1 discrete Morse function that has the same induced gradient vector field (up to a reasonable notion of equivalence). For any \( S \in \text{LSC}_m(|\mathcal{S}_\mathcal{K}|) \), \( f^L \) achieves a maximum on \( S \) at some point \( s \), that is, \( f^L(s) = \sup_{x \in S} f^L(x) \). If \( s \) does not belong to the vertex set of \(|\mathcal{S}_\mathcal{K}|\), then \( s \) is not an inner point of \( S \) according to the definition of \( f^L \). So \( s \in \partial S \cap \text{Vertex}(|\mathcal{S}_\mathcal{K}|) \), and thus we can take a small perturbation \( S' \) of \( S \) such that \( S' \in \text{LSC}_m(|\mathcal{S}_\mathcal{K}|) \) and \( \sup_{x \in S'} f^L(S') < f(s) = \sup_{x \in S} f^L(x) \). Therefore, we only need to consider such \( S \) with the property that \( \max_{x \in S} f^L(x) \) is achieved at some vertex \( v \) of \(|\mathcal{S}_\mathcal{K}|\), i.e., \( f^L(v) \geq f^L(x) \), \( \forall x \in S \). Consider the sublevel set \( \{ f^L \leq f(v) \} \). It is clear that \( \text{cat}(\{ f^L \leq f(v) \}) \geq \text{cat}(S) \geq m \) and \( \sup_{x \in S} f^L(\{ f^L \leq f(v) \}) = f(v) = \max_{x \in S} f^L(S) \).

**Claim.** \( \text{cat}(\{ f^L \leq a \}) = \text{cat}(\mathcal{S}_\mathcal{K}|_{\{ \sigma \in \mathcal{S}_\mathcal{K} : f(\sigma) \leq a \}}) \), where \( \mathcal{S}_\mathcal{K}|_{\{ \sigma \in \mathcal{S}_\mathcal{K} : f(\sigma) \leq a \}} \) is the induced closed subcomplex of \( \mathcal{S}_\mathcal{K} \) on the vertices \( \{ \sigma \in \mathcal{K} : f(\sigma) \leq a \} \) of \( \mathcal{S}_\mathcal{K} \).

**Proof.** In fact, by Lemma 3.3, there is a homotopy equivalence between \( \{ f^L \leq a \} \) and \( \mathcal{S}_\mathcal{K}|_{\{ \sigma \in \mathcal{S}_\mathcal{K} : f(\sigma) \leq a \}} \).

By the above claim, we establish the following identities

\[
\sup_{x \in S} f^L(x) = \inf_{a \in \mathbb{R} \text{ s.t. } \{ f^L \leq a \} \in \text{LSC}_m(|\mathcal{S}_\mathcal{K}|)} \sup_{x \in \{ f^L \leq a \}} f^L(x) = \min_{a \in \mathbb{R} \text{ s.t. } \mathcal{S}_\mathcal{K}|_{\{ \sigma \in \mathcal{S}_\mathcal{K} : f(\sigma) \leq a \}} \in \text{LSC}_m(\mathcal{K})} \max_{\sigma \in \mathcal{S}_\mathcal{K}|_{\{ \sigma \in \mathcal{S}_\mathcal{K} : f(\sigma) \leq a \}}} f(\sigma) = \min_{L \in \text{LSC}_m(\mathcal{K})} \max_{\sigma \in L} f(\sigma).
\]

We point out that our concept of discrete Lusternik-Schnirelman category for abstract simplicial complexes is different from that of Definition 4.3 in [25].

### 4 Discrete Morse theory on complex-like hypergraphs

In the preceding, we have established a correspondence between the discrete Morse theory on a simplicial complex \( \mathcal{K} \) with vertex set \( V \) and the continuous Morse theory on the associated order complex \( \mathcal{S}_\mathcal{K} \). Since the order complex \( \mathcal{S}_\mathcal{E} \) is still a simplicial complex when \( \mathcal{E} \) is only a hypergraph with vertex set \( V \), we can use the continuous Morse theory on that complex to define a discrete Morse theory on \( \mathcal{E} \). That is what we shall now do.

A **hypergraph** is a pair \((V, \mathcal{E})\) with \( \mathcal{E} \subset \mathcal{P}(V) \). In other words, \( \mathcal{E} \) is a general set family on \( V \). We study the combinatorial structure of a hypergraph from a topological perspective.
Since the philosophy of Morse theory is to understand the topology by functions, we should make clear what is the topology on a hypergraph. Section 4.1 indicates that it is nice to work on the order complex. Moreover, Section 4.2 reveals that we should concentrate on a hypergraph which looks like a simplicial complex if we want to establish Theorem A on hypergraphs.

4.1 Topologies on hypergraphs

There are several natural approaches available to define a topology on a hypergraph. The concept of finite topology is introduced and studied by Alexandrov [1], and later by Stong from the perspective of algebraic topology [44].

There are several ways to endow a finite hypergraph \((V, \mathcal{E})\) with a topology on its hyperedge set \(\mathcal{E}\).

**Definition 4.1.** The (down) finite topology \(\mathcal{T}\) on \(\mathcal{E}\) is generated by the base \(\{U_e\}_{e \in \mathcal{E}}\), where \(U_e = \{e' \in \mathcal{E} : e' \subseteq e\}\).

**Definition 4.2.** The (up) finite topology \(\mathcal{T}'\) on \(\mathcal{E}\) is generated by the base \(\{U_{e'}^\text{up}\}_{e \in \mathcal{E}}\), where \(U_{e'}^\text{up} = \{e' \in \mathcal{E} : e' \supseteq e\}\).

**Definition 4.3.** The order complex of \(\mathcal{E}\) denoted by
\[
\mathcal{S}_\mathcal{E} := \{\mathcal{C} \subset \mathcal{E} : \mathcal{C} \text{ is a chain}\}
\]
collects all inclusion chains in \(\mathcal{E}\). It is clear that \(\mathcal{S}_\mathcal{E}\) is a simplicial complex with the vertex set \(\mathcal{E}\). Define the geometric realization of \(\mathcal{S}_\mathcal{E}\) by
\[
|\mathcal{S}_\mathcal{E}| = \bigcup_{\mathcal{C} \in \mathcal{S}_\mathcal{E}} \text{conv}(\mathbf{1}_e : e \in \mathcal{C}).
\]
associated simplicial complex [41].

**Definition 4.4.** The associated simplicial complex \((V, \mathcal{K}_\mathcal{E})\) is the smallest simplicial complex \(\mathcal{K}_\mathcal{E}\) containing \(\mathcal{E}\). Each hyperedge \(e\) corresponds to an open simplex \(|e|\) in the geometric realization \(|\mathcal{K}_\mathcal{E}|\). The geometric realization \(|\mathcal{E}|\) is then defined as \(\bigcup_{e \in \mathcal{E}} |e|\) in the geometric simplicial complex \(|\mathcal{K}_\mathcal{E}|\).

As a subset of \(|\mathcal{K}_\mathcal{E}|\), the geometric realization (or underlying space) \(|\mathcal{E}|\) may be neither closed nor open (see Examples 4.1 and 4.2). In contrast, the order complex \(|\mathcal{S}_\mathcal{E}|\) is closed.
In this section, we provide some detailed descriptions on the relations among these objects.

**Definition 4.5.** (Section 1.4 in [9]) A weak homotopy equivalence between two topological spaces $X$ and $Y$ is a continuous map $X \to Y$ (or $Y \to X$) which induces isomorphisms in all homotopy groups.

Two topological spaces $X$ and $Y$ are weakly homotopy equivalent (denoted by $\text{weak} \simeq$) if there exists a finite sequence of topological spaces $X = X_0, X_1, \ldots, X_n = Y$ such that there are weak homotopy equivalences $X_i \to X_{i+1}$ (or $X_{i+1} \to X_i$) for every $0 \leq i \leq n - 1$.

It is interesting that the above four topologies are pairwise distinct, but they are weakly homotopy equivalent. In detail, we have:

**Proposition 4.1.** The topologies as stated above are (weakly) homotopy equivalent, i.e.,

$$(\mathcal{E}, \mathcal{T}) \text{ weak} \simeq (\mathcal{E}, \mathcal{T'}) \text{ weak} \simeq |\mathcal{E}| \simeq |\mathcal{S}_\mathcal{E}|.$$

**Proof.** The proof of $(\mathcal{E}, \mathcal{T}) \text{ weak} \simeq |\mathcal{S}_\mathcal{E}|$ is essentially established in [10] for finite topologies (see also [9]). Consider a new hypergraph $(V, \mathcal{E}')$ defined by $\mathcal{E}' = \{e' : e \in \mathcal{E}\}$, where $e' = V \setminus e$ is the complement of $e$ in $V$. Then $(\mathcal{E}, \mathcal{T}) \text{ weak} \simeq |\mathcal{S}_\mathcal{E}|$. Since the order complex associated with an order is homeomorphic to the order complex associated with its inverse order (see [9]), we have $|\mathcal{S}_{\mathcal{E}'}| = |\mathcal{S}_\mathcal{E}|$.

To prove $|\mathcal{E}| \simeq |\mathcal{S}_\mathcal{E}|$, we only need to show that $|\mathcal{S}_\mathcal{E}|$ is a strong deformation retract of $|\mathcal{E}|$.

Let $K$ be the smallest simplicial complex containing $\mathcal{E}$, and note the following facts:

1. $|\mathcal{E}| = |K| \setminus \bigcup_{e \in K \setminus \mathcal{E}} |e| = \bigcap_{e \in K \setminus \mathcal{E}} (|K| \setminus |e|)$ and $|\mathcal{S}_\mathcal{E}| = |K| \setminus \bigcup_{e \in K \setminus \mathcal{E}} \text{star}_{\mathcal{S}_K}(e) = \bigcap_{e \in K \setminus \mathcal{E}} (|K| \setminus \text{star}_{\mathcal{S}_K}(e))$, where $\text{star}_{\mathcal{S}_K}(e)$ is the relatively open geometric realization of the star neighborhood of $e$ in $|\mathcal{S}_K|$.
2. \(|\text{star}_{SE}(e)| \simeq |e| \simeq \text{pt}\) for any \(e \in \mathcal{E}\); \(|\text{star}_{SK}(e)| \simeq |\text{star}_K(e)| \simeq |e| \simeq \text{pt}\) for any \(e \in \mathcal{K}\).

3. There is a deformation \(\eta_e\) from \(|\text{star}_{SE}(e)| \setminus |e|\) to \(|\text{link}_{SE}(e)|\), for any \(e\).

4. There is a natural deformation \(\eta'_e : (|\mathcal{K}| \setminus |e|) \times [0, 1] \rightarrow |\mathcal{K}| \setminus |e|\) with \(\eta'_e(|\mathcal{K}| \setminus |e|, 1) = |\mathcal{K}| \setminus |\text{star}_{SE}(e)|\), for any \(e\).

5. Let \(\mathcal{K} \setminus \mathcal{E} = \{e^1, \ldots, e^k\}\). Then the composition of the deformations \(\eta'_{e^1}, \ldots, \eta'_{e^k}\), gives a deformation from \(|\mathcal{E}|\) to \(|SE|\), according to

\[
\eta'_{e^1}(\eta'_{e^2}(\cdots (\eta'_{e^k}(|\mathcal{E}|, 1), \cdots), 1), 1) = \eta'_{e^1}(\eta'_{e^2}(\cdots (\eta'_{e^k}(\bigcap_{i=1}^k(|\mathcal{K}| \setminus |e^i|), 1), \cdots), 1), 1)
\]

\[= \bigcap_{i=1}^k(|\mathcal{K}| \setminus |\text{star}_{SE}(e^i)|) = |SE|.
\]

The proof is completed.

**Example 4.1.** Let \(V = \{1, 2, 3, 4\}\) and \(\mathcal{E} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{1, 3, 4\}\}\). We show the geometric realization \(|\mathcal{E}|\) and the order complex \(|SE|\) in the left picture in Remark 1 below.

**Example 4.2.** Let \(V = \{1, 2, 3, 4\}\) and \(\mathcal{E} = \{\{1\}, \{2\}, \{4\}, \{1, 2, 3\}, \{1, 3, 4\}\}\). The geometric realization \(|\mathcal{E}|\) and the order complex \(|SE|\) are displayed on the right picture in Remark 1.

**Remark 1.** The conclusion \(|\mathcal{E}| \simeq |SE|\) in Proposition 4.1 can be regarded as a special Nerve Theorem for Čech-type complexes. Indeed, we can see \(|SE|\) as the nerve of \(|\mathcal{E}|\) in a certain sense. To visually understand this point, we draw the nerve complex of a special cover of the underlying space \(|\mathcal{E}|\) for Examples 4.1 and 4.2 respectively:

As in the case of simplicial complexes, the vertices of \(SE\) correspond to the barycenters of the hyperedges in \(\mathcal{E}\). And the deformation process from \(|\mathcal{E}|\) to \(|SE|\) can be visualized as the following picture (for Examples 4.1 and 4.2 respectively):
are more geometrically intuitive.

we would like to work on the geometric realization

The embedded homology of a hypergraph

Remark 4.

|E| ≃ |N|

Remark 3.

|N| set

Remark 2.

The Čech complex for a hypergraph.

|E| homotopic to

|N|

Proposition 4.2

According to the inclusion relation

Propositions 4.1 and 4.2 suggest to consider the order complex instead of other complexes like the Čech and the Vietoris-Rips complex with respect to $\mathcal{E}$, which do not include the precise topological data of $\mathcal{E}$ (see Remarks 2 and 3 for some comments).

Proposition 4.2 (Homology properties). For $A \subset B \subset \mathcal{E}$ and $A' \subset B' \subset \mathcal{E}$ with $B \setminus A = B' \setminus A'$, we can regard $(V, A), (V, B), (V, A')$ and $(V, B')$ as the sub-hypergraphs of $(V, \mathcal{E})$.

Then we have

\[
H(B, A) := H(|B|, |A|) \cong H(B_{\mathcal{E}}, A_{\mathcal{E}}) \cong H(B'_{\mathcal{E}}, A'_{\mathcal{E}}) \cong H(|S_B|, |S_A|)
\]

\[
\cong H(|S_{B'}|, |S_{A'}|) \cong H(B'_{\mathcal{E}}, A'_{\mathcal{E}}) \cong H(|S_{B'}|, |A'|) := H(B', A').
\]

Propositions 4.1 and 4.2 suggest to consider the order complex instead of other complexes like the Čech and the Vietoris-Rips complex with respect to $\mathcal{E}$, which do not include the precise topological data of $\mathcal{E}$ (see Remarks 2 and 3 for some comments).

Remark 2. The Čech complex for $\mathcal{E}$ is the simplicial complex with vertex set $\mathcal{E}$ and face set $\mathcal{N}_\mathcal{E} := \{ E' \subset \mathcal{E} : \bigcap_{e \in E'} e \neq \emptyset \}$.

However, it can be verified that $|\mathcal{N}_\mathcal{E}| \simeq |\mathcal{K}_\mathcal{E}|$. If $(V, \mathcal{E})$ is a simplicial complex, then $|\mathcal{N}_\mathcal{E}| \simeq |\mathcal{K}_\mathcal{E}| = |\mathcal{E}| \simeq |\mathcal{S}_\mathcal{E}|$. But if $(V, \mathcal{E})$ is not a simplicial complex, $|\mathcal{N}_\mathcal{E}|$ may not be homotopic to $|\mathcal{E}| \simeq |\mathcal{S}_\mathcal{E}|$. Due to this reason, we shall work on the order complex instead of the Čech complex for a hypergraph.

Remark 3. The Vietoris-Rips complex for $\mathcal{E}$ is the simplicial complex with vertex set $\mathcal{E}$ and face set $\mathcal{VR}_\mathcal{E} := \{ E' \subset \mathcal{E} : e' \cap e \neq \emptyset, \forall e', e \in E' \}$. Also, $|\mathcal{VR}_\mathcal{E}| \not\simeq |\mathcal{E}|$ in general.

According to the inclusion relation $S_\mathcal{E} \subset \mathcal{N}_\mathcal{E} \subset \mathcal{VR}_\mathcal{E}$ and Remark 2, it is better to work with the order complex $|S_\mathcal{E}|$.

Remark 4. The embedded homology of a hypergraph $\mathcal{E}$ introduced in [11] is different from $H(\mathcal{E})$. As an embedded homology may not be the homology of any simplicial complex [28], we would like to work on the geomtric realization $|\mathcal{E}|$ and the order complex $|S_\mathcal{E}|$, which are more geometrically intuitive.
4.2 Discrete Morse function on special hypergraphs

To establish a Morse theory on hypergraphs, we can first work on a complex-like hypergraph, whose combinatorial structure and topological structure are similar to a simplicial complex. The next definition for Morse functions on hypergraphs follows verbatim from the original definition of Morse functions on simplicial complexes by Forman.

Definition 4.6. An edge pair \((e', e)\) is called sequential if \(e' \subseteq e\) and there is no other \(e'' \supsetneq e\). A function \(f : \mathcal{E} \to \mathbb{R}\) is a simple discrete Morse function if it has the property that for any \(e \in \mathcal{E}\), \#\{sequential pair \((e', e) : f(e') \geq f(e)\}\) \(\leq 1\) and \#\{sequential pair \((e, e') : f(e) \geq f(e')\}\) \(\leq 1\). An edge \(e\) is called a critical point of a simple discrete Morse function \(f\) if \{sequential pair \((e', e) : f(e') \geq f(e)\}\) \(= \emptyset = \{sequential pair \((e, e') : f(e) \geq f(e')\}\). We say that \(e\) has height \(k\) if there are at most \(k\) edges, \(e^1, \ldots, e^k\), in a chain of the form \(e^1 \supsetneq e^2 \supsetneq \cdots \supsetneq e^k \supsetneq e\). A critical point \(e\) of \(f\) has index \(k\) if the height of \(e\) is \(k\).

We have a preliminary result for special hypergraphs and the corresponding typical functions, which is a slight generalization of Forman’s discrete Morse theory.

Theorem 4.1. For a finite hypergraph \((V, \mathcal{E})\), assume that \(\mathcal{E}\) has the properties that the geometric realization \(|\{e' \in \mathcal{E} : e' \supsetneq e\}|\) is homotopic to a sphere for any \(e\), and the geometric realization \(|\{e'' \in \mathcal{E} : e'' \subset e, e'' \notin \{e', e\}\}|\) is contractible for any sequential edge pair \((e', e)\). Let \(f : \mathcal{E} \to \mathbb{R}\) be a simple discrete Morse function with exactly one critical point of index \(k\). Then the geometric realization \(|\mathcal{E}|\) is homotopy equivalent to a CW-complex with one \(k\)-cell.

Definition 4.7 (complex-like hypergraph). A finite hypergraph satisfying the conditions in Theorem 4.1 is called a complex-like hypergraph.

It is clear that every simplicial complex is a complex-like hypergraph. Example 4.2 shows a complex-like hypergraph which is not a simplicial complex, while Example 4.1 shows a hypergraph which is not complex-like. One may find that the hypergraph in Example 4.2 and the simplicial complex \(K = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}\) have the same order complex. However, there exists a complex-like hypergraph whose order complex doesn’t agree with any order complex on simplicial complexes (see Example 4.3). So, the phrase ‘complex-like’ essentially refers to ‘like a simplicial complex in topology’.

The complex-like condition is technical but natural, ensuring the validity of Lemmas 3.2, 3.4 and 3.5. In the sequel, all the results regarding Morse theory on a hypergraph require the complex-like condition. Actually, Example 4.4 indicates that the main theorems in this paper do not hold for general hypergraphs.

We then define the Lovász extension restricted on \(|\mathcal{S}_\mathcal{E}|\) as follows.

According to Proposition 5.1, the Lovász extension \(f^L\) is well-defined on \(|\mathcal{S}_\mathcal{E}|\) for any \(f : \mathcal{E} \to \mathbb{R}\). Indeed, the feasible domain of \(f^L\) is

\[
D_\mathcal{E} = \begin{cases} 
\bigcup_{t \geq 0} t|\mathcal{S}_\mathcal{E}| = \{tx : t \geq 0, x \in |\mathcal{S}_\mathcal{E}|\} \subset \mathbb{R}_\geq 0, & \text{if } V \notin \mathcal{E}, \\
\bigcup_{t \geq 0} t|\mathcal{S}_\mathcal{E}| + \mathbb{R}1_V = \{tx : t \geq 0, x \in |\mathcal{S}_\mathcal{E}|\} + \text{span}(1_V), & \text{if } V \in \mathcal{E}.
\end{cases}
\]
In any case, \( f^L \) is well-defined on the polyhedral cone \( \bigcup_{t \geq 0} t|S| \). We shall restrict the Lovász extension \( f^L \) on \( |S| \).

Let \[ \text{LSC}_m(E) = \{ E' \subset E : \text{cat}(|S|(E')) \geq m \} \]
where \( S(E') \) is the induced subcomplex of \( S \) on \( E' \). Then, Theorems 3.1, 3.2 and 3.3 can also be generalized to this setting of hypergraphs:

**Theorem 4.2.** For a hypergraph \((V, E)\) under the assumptions of Theorem 4.1 let \( f : E \to \mathbb{R} \) be an injective discrete Morse function. Then the following conditions are equivalent:

1. \( e \) is a critical point of \( f \) with index \( i \);
2. \( 1_e \) is a critical point of \( f^L|_{|S|} \) with index \( i \) in the sense of weak slope (metric Morse theory);
3. \( 1_e \) is a critical point of \( f^L|_{|S|} \) with index \( i \) in the sense of Kühnel (PL Morse theory);
4. \( 1_e \) is a Morse critical point of \( f^L|_{|S|} \) with index \( i \) in the sense of topological Morse theory.

Moreover, the discrete Morse vector \((n_0, n_1, \ldots, n_d)\), representing the number \( n_i \) of critical points with index \( i \), of \( f \) coincides with the continuous Morse vector of \( f^L|_{|S|} \).

Moreover, the Lusternik-Schnirelmann category theorem is preserved under Lovász extension:

\[ \min_{E' \in \text{LSC}_m(E)} \max_{c \in E'} f(c) = \inf_{S \in \text{LSC}_m(|S|)} \sup_{x \in S} f^L(x), \]

The proof of Theorem 4.2 follows verbatim the proof of Theorem A in Section 3 and thus we omit it.

**Example 4.3.** The setting of Theorem 4.2 is a little wider than Theorem A. For example, taking \( V = \{1, 2, 3, 4\} \) and \( E = \{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\} \). It is clear that \( E \) is not a simplicial complex. Moreover, its order complex \( S_E \) cannot be an order complex of any simplicial complex. However, one can check that \((V, E)\) is complex-like, and both Theorem 4.2 and Theorem 4.3 are valid for \((V, E)\).

Theorem 4.2 doesn’t hold for a general hypergraph. In fact, we need the complex-like setting to obtain appropriate analogs of Lemmas 3.2, 3.4 and 3.5.

**Example 4.4.** Let \( V = \{1, \ldots, n\} \) and \( E = \{\{1\}, \{1, 2\}, \ldots, \{1, \ldots, k\}, \ldots, \{1, \ldots, n\}\} \).

Then the function \( f : E \to \mathbb{R} \) defined by \( f(\{1, \ldots, k\}) = k \) is a discrete Morse function, and every \( e \in E \) is a critical point of \( f \), but every \( 1_e \) with \#e \notin \{1, n\} \) is a regular point of \( f^L|_{|S|} \). Therefore, Theorem 4.2 fails for such a hypergraph.

It is expected that a more general Morse theory of hypergraphs and more applications will be developed in the future. The key idea is that the definition of critical points of a general function \( f \) on \( E \) is translated into the PL critical point theory of its restricted Lovász extension \( f^L|_{|S|} \).

**Definition 4.8.** Given a finite hypergraph \((V, E)\) and a function \( f : E \to \mathbb{R} \), we say that \( e \in E \) is a critical point of \( f \) if \( 1_e \) is a critical point of \( f^L|_{|S|} \) in the sense of PL Morse theory.
5 Conclusions and Discussions

In this paper, we present a systematic approach on constructing Lovász extensions of discrete Morse functions, with which we build the foundational theory to translate one Morse theory to another. Our dictionary on translating different Morse theories is useful and has the potential to solve some problems in discrete Morse theory. We can work on the barycentric subdivision (or the order complex) of a simplicial complex to explore the critical simplexes. Based on our approach, we provide a new definition of a discrete Lusternik-Schnirelmann category.

We also propose a general definition of critical points of a function on the edge set of a hypergraph, and this allows us to study the Morse theory on hypergraphs by employing the PL Morse theory on polyhedral structures. This idea works well for certain hypergraphs, e.g., the complex-like hypergraphs in this paper, and it is possible to get further results along this direction.

We leave the following two open problems for future research:

**Question 5.1.** Can we modify a discrete Morse function $f$ on a simplicial complex to a function with fewer critical simplices by exploiting the structure of $f^L$ to perform handle-cancellations?

**Question 5.2.** Can we get a concise formulation of the discrete Morse theory on hypergraphs regarding the definition of critical points introduced in Definition 4.8?

Finally, we should point out that there are many other ways to introduce topological structures on hypergraphs, for example following ideas related to independence complexes [20] and Hom complexes [18][34]. Analogously to various techniques developed for simplicial complexes, for special hypergraphs or posets having good patterns, one could apply methods from analysis and homotopy theory to study these discrete structures. To better understand these structures, further exploration of the relationship between different topological structures on families of hypergraphs is needed. The frontier of research in topological, geometrical, and dynamical combinatorics has the potential to provide new mathematical tools in data science.

Acknowledgement

Dong Zhang would like to thank Professor Kung-Ching Chang for his long-term guidance, encouragement and support in mathematics.

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Appendix

Proposition 5.1. Given a set family \( \mathcal{E} \subset \mathcal{P}(V) \) on a finite set \( V \), let
\[
\mathcal{S}_E := \{ C \subset \mathcal{E} : C \text{ is a chain under the inclusion relation} \}.
\]
Then \( (\mathcal{E}, \mathcal{S}_E) \) forms an abstract simplicial complex with vertex set \( \mathcal{E} \) and face set \( \mathcal{S}_E \).

Let \( |\mathcal{S}_E| \) be a geometric realization determined by the simplicial map \( e \mapsto 1_e, \forall e \in \mathcal{E} \). Then
\[
\mathcal{D}_E = \begin{cases} 
\{ tx : t \geq 0, x \in |\mathcal{S}_E| \} \subset [0, \infty)^n, & \text{if } V \notin \mathcal{E}, \\
\{ tx : t \geq 0, x \in |\mathcal{S}_E| \} + \text{span}(1_V), & \text{if } V \in \mathcal{E}.
\end{cases}
\]
Proof. By the definition, $(\mathcal{E}, S_\mathcal{E})$ is a simplicial complex with the facet set consisting of maximal chains under the inclusion relation.

We determine the feasible domain $\mathcal{D}_\mathcal{E}$ of the Lovász extension of any function $f : \mathcal{E} \to \mathbb{R}$ as follows.

1. If $V \in \mathcal{E}$, then according to the definition of $\mathcal{D}_\mathcal{E}$, there holds

$$\mathcal{D}_\mathcal{E} = \{ x \in \mathbb{R}^n : V^t(x) \in \mathcal{E}, \forall t \in (-\infty, \max x) \}$$

$$= \left\{ x \in \mathbb{R}^n \mid x = \sum_{i \geq 0} t_i 1_{e_i} \text{ with } V = e_0 \supset e_1 \supset e_2 \supset \cdots , t_0 \in \mathbb{R}, e_i \in \mathcal{E}, t_i \geq 0, i \geq 1 \right\}$$

$$= \bigcup_{\text{maximal chain } C \subseteq \mathcal{E}} \text{cone}(1_e : e \in C) + \text{span}(1_V)$$

$$= \{ tx : t \geq 0, x \in |S_\mathcal{E}| \} + \text{span}(1_V).$$

2. If $V \not\in \mathcal{E}$, we first show that one must assume $\min_i x_i = 0$. Otherwise, $\min_i x_i \neq 0$, then the Lovász extension has a term $\min_i x_i f(V)$ which needs the data of $f(V)$, but $V \not\in \mathcal{E}$ and thus it is impossible to get the value of $f(V)$. Therefore, $\min_i x_i$ should be set as 0. Similar to the above case, for $x \in \mathbb{R}^n$ with $\min_i x_i = 0$,

$$\mathcal{D}_\mathcal{E} = \{ x \in \mathbb{R}^n : V^t(x) \in \mathcal{E}, \forall t \in [0, \max x) \}$$

$$= \left\{ x \in \mathbb{R}^n \mid x = \sum_{i \geq 1} t_i 1_{e_i} \text{ with } V \neq e_1 \supset e_2 \supset \cdots , e_i \in \mathcal{E}, t_i \geq 0, i \geq 1 \right\}$$

$$= \bigcup_{\text{maximal chain } C \subseteq \mathcal{E}} \text{cone}(1_e : e \in C)$$

$$= \{ tx : t \geq 0, x \in |S_\mathcal{E}| \}.$$

The proof is completed.