Change of the congruence canonical form of
$2 \times 2$ and $3 \times 3$ matrices under perturbations*

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Abstract

We study how small perturbations of any given $2 \times 2$ (resp., $3 \times 3$) complex matrix may change its canonical form for congruence. We construct the Hasse diagram for the closure ordering on the set of congruence classes of $2 \times 2$ (resp., $3 \times 3$) matrices.

1 Introduction

The reduction of a complex matrix $A$ to any canonical form for congruence is an unstable operation: both the canonical form and the reduction transformation depend discontinuously on the entries of $A$. It is important to know the canonical forms of all matrices that are close to $A$; especially, if the entries of $A$ are known only approximately.

We use the congruence canonical form given by Horn and Sergeichuk [16] and study how small perturbations of a $2 \times 2$ or $3 \times 3$ complex matrix

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change its canonical form for congruence. We construct a directed graph with bijection \( v \mapsto A_v \) from the set of its vertices to the set of \( 2 \times 2 \) or, respectively, \( 3 \times 3 \) congruence canonical matrices such that there is a directed path from a vertex \( v \) to a vertex \( w \) if and only if the matrix \( A_v \) can be transformed by an arbitrarily small perturbation to a matrix whose canonical form is \( A_w \) (i.e., the congruence class of \( A_v \) is contained in the closure of the congruence class of \( A_w \)). This graph is called the closure graph for congruence classes; it shows how they relate to each other in the affine space of \( n \times n \) matrices. The closure graph is the Hasse diagram of the partially ordered set whose elements are the congruence classes and \( a \preceq b \) means that \( a \) is contained in the closure of \( b \).

We partition the set of \( 2 \times 2 \) (resp., \( 3 \times 3 \)) matrices into bundles under congruence; each bundle consists of matrices that behave alike under small perturbations. We construct the closure graph for the bundles: each vertex \( v \) represents in a one-to-one manner a bundle \( A_v \) and there is a directed path from \( v \) to \( w \) if and only if \( A_v \) is contained in the closure of \( A_w \) (i.e., each matrix in \( A_v \) can be transformed by an arbitrarily small perturbation to a matrix in \( A_w \)).

Unlike perturbations of matrices under congruence, perturbations of matrices under similarity and of matrix pencils have been much studied. A complete description of the set of Jordan canonical forms of all matrices that are sufficiently close to any given Jordan canonical matrix is presented in [21, 7]. An analogous result for Kronecker canonical forms of pencils is obtained in [23, 5] and is extended in [22] to matrix pencils with respect to the actions of \( \text{GL}_m \times \text{GL}_n \times \text{GL}_2 \), \( \text{SL}_m \times \text{SL}_n \), and \( \text{SL}_m \times \text{SL}_n \times \text{SL}_2 \). Closure relations for counter operators \( U \rightleftarrows V \) are studied in [15] and for representation of quivers in [3, 4, 24, 26].

Two square matrices \( A \) and \( B \) have the same Jordan type if there is a bijection from the set of distinct eigenvalues of \( A \) to the set of distinct eigenvalues of \( B \) that transforms the Jordan canonical form of \( A \) to the Jordan canonical form of \( B \); a set of all matrices of the same Jordan type is called a bundle, see [1]. The partition of the set of \( n \times n \) matrices into bundles is finite. The set of \( m \times n \) pencils can be also partitioned into a finite number of bundles of pencils of the same Kronecker type.

A comprehensive theory of closure relations for similarity classes of matrices, for equivalence classes of matrix pencils, and for their bundles is developed in [9]; the corresponding closure graphs can be constructed by using StratiGraph [10], which is a software tool for computing and visualizing clo-
sure hierarchies. The closure graph for $2 \times 3$ matrix pencils is constructed in [11].

2 Closure graphs

In this section we formulate theorems about closure graphs. All matrices that we consider are complex.

2.1 Preliminary results

Define the $n$-by-$n$ matrices:

$$
J_n(\lambda) := \begin{bmatrix}
\lambda & 1 & 0 \\
& \ddots & \ddots \\
0 & \ddots & 1 \\
\end{bmatrix},
\Gamma_n := \begin{bmatrix}
0 & \ddots & \ddots \\
& \ddots & 1 \\
& & -1 & 1 \\
& & 1 & 0 \\
\end{bmatrix}.
$$

Two pairs $(A, B)$ and $(C, D)$ of $m \times n$ matrices are said to be equivalent if there exist nonsingular matrices $R$ and $S$ such that $RAS = C$ and $RBS = D$. For each square matrix $A$, $A^{-T}A := (A^{-1})^T A$ is called the cosquare of $A$.

Lemma 1 ([18]). (a) Every square complex matrix is congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form

$$
H_m(\lambda) := \begin{bmatrix}
0 & I_m \\
J_m(\lambda) & 0 \\
\end{bmatrix},
\Gamma_n, \quad J_k(0),
$$

in which $\lambda \in \mathbb{C}$, $\lambda \neq (-1)^{n+1}$, each nonzero $\lambda$ is determined up to replacement by $\lambda^{-1}$, and $k$ is odd.

(b) Two $n \times n$ complex matrices $A$ and $B$ are congruent if and only if the matrix pairs $(A^T, A)$ and $(B^T, B)$ are equivalent.

(c) Two nonsingular complex matrices $A$ and $B$ are congruent if and only if their cosquares $A^{-T}A$ and $B^{-T}B$ are similar.

This canonical form for congruence was obtained in [16, 19] basing on [25, Theorem 3]; a direct proof that this form is canonical is given in [17, 18]. In all of these articles, the matrix $J_{2m}(0)$ is used instead of $H_m(0)$ (they are
congruent for each \( m \); see [25, p. 493]). We use \( H_m(0) \) to reduce the number of different types of \( 2 \times 2 \) and \( 3 \times 3 \) canonical matrices. We also use Theorem 2.1 in [13], which gives a miniversal deformation of each canonical matrix \( A \) under congruence; that is, a normal form with the minimal number of independent parameters to which all matrices \( A + E \) close to \( A \) can be reduced by transformations

\[
A + E \mapsto S(E)^T (A + E) S(E), \quad S(0) = I, \tag{2}
\]

in which \( S(E) \) smoothly depends on the entries of \( E \). Miniversal deformations of the Jordan canonical matrices and Kronecker canonical matrix pencils were obtained in [1, 2] and [8, 14].

A miniversal deformation of \( A \in \mathbb{C}^{n \times n} \) under congruence was constructed in [13] as follows. The vector space

\[
V_A := \{ C^T A + AC \mid C \in \mathbb{C}^{n \times n} \}
\]

is the tangent space to the congruence class of \( A \) at the point \( A \) since

\[
(I + \varepsilon C)^T A (I + \varepsilon C) = A + \varepsilon (C^T A + AC) + \varepsilon^2 C^T AC
\]

for all \( n \)-by-\( n \) matrices \( C \) and each small \( \varepsilon \). Let \( \mathcal{D} \) be any matrix with entries 0 and \(*\) that satisfies the condition

\[
\mathbb{C}^{n \times n} = V_A \oplus \mathcal{D}(\mathbb{C}) \tag{3}
\]

in which \( \mathcal{D}(\mathbb{C}) \) is the vector space of all matrices obtained from \( \mathcal{D} \) by replacing its stars by complex numbers. Then all matrices in \( A + \mathcal{D}(\mathbb{C}) \) that are sufficiently close to \( A \) form a miniversal deformation of \( A \). By (3), the number of stars in \( \mathcal{D} \) is equal to the codimension of the congruence class of \( A \).

Lemma 2 ([13, Example 2.1]). Let \( A \) be any \( 2 \times 2 \) or \( 3 \times 3 \) matrix. Then all matrices \( A + E \) that are sufficiently close to \( A \) simultaneously reduce by transformations (2) in which \( S(E) \) is holomorphic at zero to one of the following forms, respectively,

\[
\begin{align*}
&\text{i. } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \\
&\text{ii. } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, \\
&\text{iii. } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \\
&\text{iv. } \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}, \\
&\text{v. } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}, \\
&\text{vi. } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}, \\
\end{align*}
\]
in which \( \lambda \neq \pm 1 \), or

\[
\begin{aligned}
1. & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} * & * & * \\ * & * \\ 0 & 0 \\ * & * \end{bmatrix}, & 2. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * \\ 0 & * \\ * & * \end{bmatrix}, \\
3. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} * & * & * \\ * & * \\ 0 & 0 \\ * & * \end{bmatrix}, & 4. & \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & * \\ * & * \end{bmatrix}, \\
5. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} * & * & * \\ * & * \\ 0 & 0 \\ * & * \end{bmatrix}, & 6. & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & * \\ * & * \end{bmatrix}, \\
7. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, & 8. & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & * \\ * & * \end{bmatrix}, \\
9. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & 0 \\ * & * \end{bmatrix}, & 10. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & * \\ * & * \end{bmatrix}, \\
11. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & 0 \\ * & * \end{bmatrix}, & 12. & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ * & * \\ 0 & * \\ * & * \end{bmatrix},
\end{aligned}
\]

in which \( \lambda, \mu \neq \pm 1 \). Each matrix in (4) and (5) has the form \( A + D \) in which \( A \) is a direct sum of blocks of the form (1) (the unspecified entries of \( A \) are zero) and the stars in \( D \) are complex numbers that tend to zero as \( E \) tends to 0. The number of stars is the smallest that can be attained by using transformations (2); it is equal to the codimension of the congruence class of \( A \).

The miniversal deformations (4) and (5) and their numbers of stars (that are equal to the codimensions of the corresponding congruence classes) will help us to construct closure graphs due to the following obvious lemma.

**Lemma 3.** The following statements hold for the arrows of the closure graph for congruence classes of \( 2 \times 2 \) or \( 3 \times 3 \) matrices:

(a) \( v \to w \) if and only if the corresponding canonical matrix \( A_v \) can be transformed by an arbitrarily small perturbation of the form (4) or (5) to a matrix whose canonical form is \( A_w \);

(b) \( v \to w \) implies that \( \text{cod}(v) > \text{cod}(w) \), where \( \text{cod}(v) \) denotes the the codimension of the congruence class of \( A_v \).
2.2 Closure graphs for congruence classes

Theorem 4. The closure graph for congruence classes of $2 \times 2$ matrices is

The vertices correspond to all the $2 \times 2$ canonical matrices for congruence:

i. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,  
ii. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,  
iii. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,  
iv. $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$,  
v. $\begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}$ ($\lambda \neq \pm 1$),  
vi. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

which are direct sums of blocks of the form (1). The graph is infinite: the point $v_\lambda$ represents the infinite family of vertices indexed by $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ with arrows $\text{iii} \rightarrow v_\lambda$ to each of these vertices. The congruence classes of canonical matrices, whose corresponding vertices are drawn on the same level, have the same codimension, which is indicated in round brackets to the right (it is equal to the number of stars in (4)).
Theorem 5. The closure graph for congruence classes of $3 \times 3$ matrices is

Its vertices correspond to all the $3 \times 3$ canonical matrices for congruence:

1. \[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]
2. \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix},
\]
3. \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\]
4. \[
\begin{bmatrix}
0 & -1 \\
1 & 1
\end{bmatrix},
\]
5. \[
\lambda.
\]
6. \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\]
7. \[
\begin{bmatrix}
0 & 1 \\
-1 & 0 \\
1
\end{bmatrix},
\]
8. \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\]
9. \[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix},
\]
10. \[
\begin{bmatrix}
0 & -1 \\
1 & 1 \\
1
\end{bmatrix},
\]
11. \[
\mu.
\]
12. \[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{bmatrix}.
\]

with $\lambda, \mu \in \mathbb{C} \setminus \{-1, 1\}$; they are direct sums of blocks of the form (1).

The graph is infinite: the points $5_\lambda$ and $11_\mu$ represent the infinite families of vertices indexed by $\lambda, \mu \in \mathbb{C} \setminus \{-1, 1\}$. The congruence classes of canonical matrices, whose corresponding vertices are drawn on the same level, have the
same codimension, which is indicated in round brackets to the right (it is equal to the number of stars in (5)).

Let \(v\) be any vertex of (6) or (8), and let \(A_v\) be the corresponding canonical matrix for congruence. The set of canonical forms of all matrices in a sufficiently small neighborhood of \(A_v\) consists of the matrices \(A_w\) such that there is a directed path from \(v\) to \(w\) (including the “lazy” path of length 0). The closure of the congruence class of \(A_v\) is equal to the union of the congruence classes of all matrices \(A_w\) such that there is a directed path from \(w\) to \(v\) (including the “lazy” path).

### 2.3 Closure graphs for bundles under congruence

The definitions of bundles of matrices under similarity and of bundles of matrix pencils are given in terms of Jordan and Kronecker canonical forms [1, 9].

Analogously, bundles of matrices under congruence could be defined in terms of canonical matrices from Lemma [1] or in terms of tridiagonal canonical matrices [12, Theorem 1.1], or in terms of other canonical matrices for congruence. But we define them in a form that does not depend on canonical matrices, using the closure ordering on the set of congruence classes.

We say that two vertices \(v\) and \(w\) of the closure graph for congruence classes are equivalent if there are no arrows \(v \rightarrow w\) and \(v \leftarrow w\), and for each vertex \(x\)

- \(v \rightarrow x\) if and only if \(w \rightarrow x\),
- \(v \leftarrow x\) if and only if \(w \leftarrow x\)

(i.e., the closure graph has an automorphism that interchanges \(v\) and \(w\) and preserves the other vertices).

This is an equivalence relation. The partition of the set of vertices into equivalence classes induces the partition of \(\mathbb{C}^{n \times n}\) into bundles under congruence: by a bundle under congruence we mean the union of all congruence classes of matrices that correspond to equivalent vertices.
Theorem 6. (a) The closure graph for bundles of $2 \times 2$ matrices under congruence is

\[
\begin{array}{c}
\text{v & vi} \\
\downarrow \\
\text{iv} \\
\downarrow \\
\text{iii} \\
\downarrow \\
\text{ii} \\
\downarrow \\
\text{i}
\end{array}
\]

Its vertices represent the bundles that contain the canonical matrices (7) (all the matrices $v_\lambda$ and $v_i$ belong to the same bundle represented by the vertex $v & vi$).

(b) The closure graph for bundles of $3 \times 3$ matrices under congruence is

\[
\begin{array}{c}
\text{11} \\
\downarrow \\
\text{10} \\
\downarrow \\
\text{9} \\
\downarrow \\
\text{8} \\
\downarrow \\
\text{7} \\
\downarrow \\
\text{6} \\
\downarrow \\
\text{5} \\
\downarrow \\
\text{4} \\
\downarrow \\
\text{3} \\
\downarrow \\
\text{2} \\
\downarrow \\
\text{1}
\end{array}
\]

Its vertices represent the bundles that contain the canonical matrices (9); the
bundles that are represented by the vertices 5 and 11 contain all the matrices $5_\lambda$ and $11_\mu$, respectively.

3 Proof of Theorem 4

For each vertex $v$ of (6), we denote by $A_v$ the canonical matrix that corresponds to $v$.

Step 1: Let us verify that all the arrows in (6) are correct; that is, for each arrow $v \rightarrow w$ the matrix $A_v$ can be transformed by an arbitrarily small perturbation to a matrix that is congruent to $A_w$. We write $A \sim B$ if $A$ is congruent to $B$; that is, $S^T A S = B$ for some nonsingular $S$. Expressing $S$ as a product of elementary matrices, we obtain that

$$\begin{align*}
A \sim B \text{ if and only if } A \text{ can be reduced to } B \text{ by simultaneous elementary transformations of rows and columns.} \quad (12)
\end{align*}$$

Let $\varepsilon$, $\delta$, and $\zeta$ be arbitrarily small complex numbers.

• $i \rightarrow ii$ and $i \rightarrow iii$ since

$$\begin{bmatrix}
0 & \varepsilon \\
-\varepsilon & 0
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
\varepsilon & 0 \\
0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.$$ 

• $ii \rightarrow iv$ since

$$\begin{bmatrix}
0 & 1 \\
-1 & \varepsilon
\end{bmatrix} \sim \begin{bmatrix}
0 & -1 \\
1 & 1
\end{bmatrix}.$$ 

• $iii \rightarrow iv$ since

$$\begin{bmatrix}
1 & \varepsilon \\
-\varepsilon & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 \\
-1 & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & -1 \\
1 & 1
\end{bmatrix}.$$ 

• $iii \rightarrow v_0$ and $iii \rightarrow vi$ since

$$\begin{bmatrix}
1 & \varepsilon \\
0 & 0
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & 0 \\
0 & \varepsilon
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.$$ 

• $iii \rightarrow v_\lambda$ for each $\lambda \notin \{-1, 0, 1\}$; it suffices to find arbitrarily small $\varepsilon$ and $\delta$ such that

$$A := \begin{bmatrix}
1 & 0 \\
\varepsilon & \delta
\end{bmatrix} \quad \text{is congruent to} \quad B := \begin{bmatrix}
0 & 1 \\
\lambda & 0
\end{bmatrix}; \quad (13)$$
i.e., by Lemma 1(c), that $A^{-T}A$ is similar to $B^{-T}B = \text{diag}(\lambda, \lambda^{-1})$; i.e., that the eigenvalues of $A^{-T}A$ are $\lambda$ and $\lambda^{-1}$; i.e., that the roots of the polynomial

$$\det(A^T x - A) = \begin{vmatrix} x - 1 & \varepsilon x \\ -\varepsilon & \delta x - \delta \end{vmatrix} = \delta[x^2 + (\varepsilon^2/\delta - 2)x + 1]$$

are $\lambda$ and $\lambda^{-1}$. This polynomial is equal to $\delta(x - \lambda)(x - \lambda^{-1})$ if $\varepsilon^2/\delta - 2 = -\lambda - \lambda^{-1}$, which can be satisfied by arbitrarily small $\varepsilon$ and $\delta$.

**Step 2:** Let us verify that all arrows of the closure graph are drawn in (6). We write $u \not\rightarrow v$ if the closure graph does not have the arrow $u \rightarrow v$; that is, all matrices obtained from $A_u$ by sufficiently small perturbations are not congruent to $A_v$. Due to Lemma 3(a), we can consider only perturbations of $A_u$ from the list (4).

The evident statement “if $v \leftarrow u \not\rightarrow w$ then $v \not\rightarrow w$” and Lemma 3(b) ensure that we only need to prove the absence of the arrows $\text{ii} \rightarrow v_\lambda$ and $\text{ii} \rightarrow \text{vi}$:

- $\text{ii} \not\rightarrow v_\lambda$ since

  $$A := \begin{pmatrix} \varepsilon & 1 \\ -1 + \delta & \zeta \end{pmatrix} \quad \text{is not congruent to} \quad B := \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

  for each fixed $\lambda$ and arbitrarily small $\varepsilon, \delta, \zeta$. Otherwise by Lemma 1(c) their cosquares must be similar, but $A^{-T}A \rightarrow I_2$ as $\varepsilon, \delta, \zeta \rightarrow 0$.

- $\text{ii} \not\rightarrow \text{vi}$ since

  $$\begin{pmatrix} \varepsilon & 1 \\ -1 + \delta & \zeta \end{pmatrix} \sim I_2,$$

  otherwise the skew-symmetric parts of these matrices must be congruent.

### 4 Proof of Theorem 5

We follow the notation of Section 3.

**Step 1:** Let us verify that all the arrows in (8) are correct. For every $A, B \in \mathbb{C}^{2 \times 2}$ and $C \in \mathbb{C}^{1 \times 1}$, if a matrix that is congruent to $B$ can be obtained by an arbitrarily small perturbation of $A$, then a matrix that is congruent to $B \oplus C$ can be obtained by an arbitrarily small perturbation of $A \oplus C$. Thus, arrows of (6) ensure the correctness of the following arrows of
$$\text{\square}:$$

$$1 \to 2 \text{ by } i \to \text{ii}, \quad 3 \to 6 \text{ by } iii \to \text{vi},$$

$$1 \to 3 \text{ by } i \to \text{iii}, \quad 3 \to 7 \text{ by } i \to \text{ii},$$

$$2 \to 4 \text{ by } \text{ii} \to \text{iv}, \quad 6 \to 8 \text{ by } iii \to \text{vi},$$

$$3 \to 4 \text{ by } \text{iii} \to \text{iv}, \quad 7 \to 10 \text{ by } \text{ii} \to \text{iv}.$$  

$$3 \to 5\lambda \text{ by } iii \to v\lambda,$$

Analogously, $$2 \to 7$$ since an arbitrarily small perturbation of $$[0]$$ gives a matrix $$[\varepsilon]$$ that is congruent to $$[1].$$

The correctness of the remaining arrows of (8) is proved as follows.

- $$4 \to 9$$ since due to (12)

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix},$$

the first matrix is reduced to the second by adding the last column multiplied by $$-\varepsilon^{-1}$$ to the first and second columns; the same transformations of rows does not change the matrix.

- $$\{5\lambda, 6\} \to 9$$ (i.e., $$5\lambda \to 9$$ and $$6 \to 9$$) since

$$\begin{bmatrix} 0 & 1 & 0 \\ \lambda & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -i\varepsilon \\ i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & i & 0 \\ i & 1 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}.$$  

- $$8 \to 12$$ since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(by Lemma 1(c))}$$

and

$$\begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^{-1} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -\varepsilon \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$
• 9 → \{10, 11_\mu, 12\}. For the deformation

\[
A := \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\varepsilon & 0 & \delta \\
\end{bmatrix},
\]

of the matrix 9, we have

\[
|A^T x - A| = \begin{vmatrix}
0 & -1 & \varepsilon x \\
x & 0 & -1 \\
-\varepsilon & x & \delta x - \delta \\
\end{vmatrix} = \varepsilon x^3 - \varepsilon + x(\delta x - \delta)
\]

(14)

\[
= \varepsilon(x^3 + \delta \varepsilon^{-1} x^2 - \delta \varepsilon^{-1} x - 1)
\]

\[
= \varepsilon(x - 1)(x^2 + (\delta \varepsilon^{-1} + 1)x + 1).
\]

To verify 9 → 10, we set \(\delta = \varepsilon\) and obtain

\[
|A^T x - A| = \varepsilon(x - 1)(x + 1)^2.
\]

Thus, the eigenvalues of \(A^T A\) are 1, -1, -1. Among the cosquares of the nonsingular matrices in (9) only the cosquares of matrices 7 and 10 have these eigenvalues. Lemma II(c) ensures that \(A\) is congruent to one of these matrices; that is, 9 → 7 or 9 → 11. But 9 → 7 by Lemma III(b).

Let us verify 9 → 11_\mu. Any complex number is represented in the form \(\delta \varepsilon^{-1}\) with arbitrarily small \(\delta\) and \(\varepsilon\); hence for every nonzero \(\mu\) there exist arbitrarily small \(\delta\) and \(\varepsilon\) such that

\[
x^2 + (\delta \varepsilon^{-1} + 1)x + 1 = (x - \mu)(x - \mu^{-1}).
\]

By (14), the eigenvalues of \(A^{-T} A\) are 1, \(\mu, \mu^{-1}\). Only the cosquare of the matrix 11_\mu in (9) has these eigenvalues. Lemma II(c) ensures 9 → 11_\mu if \(\mu \neq 0\). The arrow 9 → 11_0 exists because

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & \delta \\
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & \delta \\
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & \delta \\
\end{bmatrix}.
\]

To verify 9 → 12, we set \(\delta = -3\varepsilon\). Then \(|A^T x - A| = (x - 1)^3\). By Lemma III(c), \(A\) is congruent to

\[
8. \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \text{ or } 12. \begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & -1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

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from the list (9). But the matrix 8 cannot be congruent to \( A \) by Lemma 3(b).

**Step 2:** Using Lemma 3(a), we verify that none of the arrows of the closure graph was omitted in (8). Due to the evident statements

\[
\begin{align*}
\text{if } v &\leftarrow u \rightarrow w \text{ then } v \not\rightarrow w, \\
\text{if } u \not\rightarrow v \leftarrow w \text{ then } u \not\rightarrow w,
\end{align*}
\]

and Lemma 3(b), we need to verify the absence of the arrows

\[
\begin{align*}
2 &\not\rightarrow 5_\lambda \text{ for each } \lambda \neq \pm 1, \\
\{4, 5_\lambda, 6\} &\rightarrow 7, \quad \{4, 5_\lambda\} \rightarrow 8, \quad \{7, 8\} \rightarrow 9.
\end{align*}
\]

- Prove that \( 2 \not\rightarrow 5_\lambda \) for each \( \lambda \neq \pm 1 \). To the contrary, suppose that an arbitrarily small deformation \( A \) of the form 2 in (13) is congruent to the matrix \( 5_\lambda \):

\[
A := \begin{pmatrix}
\alpha & 1 & 0 \\
-1 + \beta & \gamma & 0 \\
\varepsilon & \zeta & \eta
\end{pmatrix} \sim \begin{pmatrix}
0 & 1 & 0 \\
\lambda & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  

(15)

Then rank \( A = 2 \), and so \( \eta = 0 \).

If \( \zeta \neq 0 \), then

\[
A \sim \begin{pmatrix}
\alpha & 0 & 0 \\
-1 + \beta & 0 & 0 \\
\varepsilon & \zeta & 0
\end{pmatrix} \sim \begin{pmatrix}
* & 0 & 0 \\
-1 + \beta & 0 & 0 \\
0 & \zeta & 0
\end{pmatrix} \sim \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

since \(-1 + \beta \neq 0\) for a small \( \beta \). The last matrix is congruent to the matrix 9:

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \sim \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]

which is not congruent to \( 5_\lambda \).

If \( \varepsilon \neq 0 = \zeta \), then we interchange the first two rows and the first two columns in \( A \) and reason as in the previous case.

If \( \varepsilon = \zeta = 0 \), then the congruence (15) with \( \eta = 0 \) implies

\[
\begin{pmatrix}
\alpha & 1 \\
-1 + \beta & \gamma
\end{pmatrix} \sim \begin{pmatrix}
0 & 1 \\
\lambda & 0
\end{pmatrix}
\]

because of the uniqueness in Lemma 1(a), which is impossible since \( ii \not\rightarrow v_\lambda \) in (5).
4 \rightarrow 7. Let a deformation of the form 4 in (5) be congruent to the matrix 7:
\[
\begin{bmatrix}
\alpha & -1 & 0 \\
1 & 1 & 0 \\
\beta & \gamma & \delta
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
for some arbitrarily small \(\alpha, \beta, \gamma, \delta\). Then \(\delta \neq 0\) and the symmetric parts of these matrices have distinct ranks; a contradiction.

5 Proof of Theorem 6
(a) The union of the congruence classes corresponding to the vertices \(v_\lambda\) and \(v_i\) in (6) is a bundle, which is represented by the vertex \(v & v_i\) in (10). The congruence classes that correspond to the other vertices of (6) are bundles. Thus, the closure graph for congruence classes (6) without the
vertices $v_\lambda$ and $v_i$, and without the arrows $ii \rightarrow v_\lambda$ and $ii \rightarrow v_i$ is a subgraph of (10).

The arrow $iv \rightarrow v$ is correct since

$$\begin{bmatrix} 0 & -1 + \varepsilon \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 + \varepsilon \\ 1 & 0 \end{bmatrix}.$$  

(b) The union of the congruence classes that correspond to the vertices $5_\lambda$ (respectively, $11_\mu$) in (8) is a bundle, which is represented by the vertex 5 (respectively, 11) in (11). The congruence classes that correspond to the other vertices of (8) are bundles. In view of the graph (8), it remains to verify that each arrow with the vertex 5 or 11 in (11) is correct and that we did not omit any arrow with these vertices.

The arrows $4 \rightarrow 5$, $6 \rightarrow 5$, $10 \rightarrow 11$, and $5 \rightarrow 9$ in (11) follow from the arrows $4 \rightarrow 5_\lambda$, $6 \rightarrow 5_\lambda$, $10 \rightarrow 11_\mu$, and $5_\lambda \rightarrow 9$ in (8). There are no arrows $5 \rightarrow 7$ and $5 \rightarrow 8$ since there are no oriented paths from $5_\lambda$ to 7 and 8 in (8). There are no arrows $7 \rightarrow 5$ and $8 \rightarrow 5$ since the canonical matrices that correspond to 7 and 8 have rank 3 and the canonical matrices that correspond to 5 have rank 2.

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