Parameterization of the Equation of State and the expansion history of the very recent universe

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Motivated by a nice result shown by E. Linder, a detailed discussion on the choice of the expansion parameters of the Maclaurin series for the equation of states of a perfect fluid is presented in this paper. We show that their nice recent result is in fact a linear approximation to the full Maclaurin series as a power series of the parameter \( y = z/(1 + z) \). The power series for the energy density function, the Hubble parameter and related physical quantities of interest are also shown in this paper. The method presented here will have significant application in the precision distance-redshift observations aiming to map out the recent expansion history of the universe. In addition, a complete analysis of all known advantageous parameterizations for the equation of states to high redshift is also presented.

I. INTRODUCTION

Recently there are great advances in our abilities in cosmological observations for the quest of exploring the expansion history of the universe. It carries cosmology well beyond determining the present dimensionless density of matter \( \Omega_m \) and the present deceleration parameter \( q_0 \) of Sandage [1]. One seeks to reconstruct the entire function \( a(t) \) representing the expansion history of the universe. Earlier on, cosmologists sought only a local measurement of the first two derivatives of the scale factor \( a \), evaluated at a single time \( t_0 \). One tries instead to map out the function determining the global dynamics of the universe in the near future. While many qualitative elements of cosmology follow merely from the form of the metric, (see Weinberg [2]), deeper understanding of our universe requires knowledge of the qualitative dynamics of the scale factor \( a(t) \). This echoes the transition of energy between components signifying the epoch of radiation domination to that of matter domination. This is also a key element in the growth of density perturbations into large structure. Yet until recently the literature only considered the Hubble constant and the deceleration parameter measured today. There are now a great numbers of cosmological observational tests that will be able to probe the function \( a(t) \) more completely throughout all ages of the universe (see Sandage [3], Linder [4], [5], Tegmark [6]).

What one required is a probe capable of both precise and accurate enough observations. Indeed, a number of promising methods are being developed including the magnitude-redshift relation of Type Ia supernovae. The goal of mapping out the recent expansion history of the universe is well motivated. The thermal history of the universe, extending back through structure formation, matter-radiation decoupling, radiation thermalization, primordial nucleosynthesis, etc. is very important in the study of cosmology and particle physics, high energy physics, neutrino physics, gravitational physics, nuclear physics, and so on (see, e.g., Kolb and Turner [7]).

Recent expansion history of the universe is similarly a very promising research focus with the discovery of the current acceleration of the expansion of the universe. This includes the study of the role of high energy field theories in the form of possible quintessence, scalar-tensor gravitation, higher dimension theories, brane worlds, etc in the very recent universe. The accelerated expansion is also important to the possible fate of the universe [8], [9], [10]. It is then important to consider the use of supernova observations to obtain the magnitude-redshift law back to \( z \sim 1.7 \) and how to relate this to the scale factor-time behavior \( a(t) \) with the proposed Supernova/Acceleration Probe mission [11].

Therefore, the study of modelling different equation of state (EOS) derived from different theories plays an important role in the study of the very recent expansion history of our universe. Interestingly, there is a nice paper showing a new model that is capable of extracting important physics with the help of the linear parameterization in powers of an appropriate parameter \( z/(1 + z) \) expanding the function \( \omega \) of the equation of state [12], [13], [14], [15]. We will show in section II that one should treat the proposed new model as the leading term in the Maclaurin series associated with the EOS (cf. [14] and Fig. 1 of [15]). In section III, we will present the first four leading terms in computing the

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energy density function $\rho$, the Hubble parameter $H$, and the conformal time $\eta$ with the help of the Maclaurin series of the EOS.

II. TAYLOR EXPANSION OF THE EQUATION OF STATE

The equation of state is given by the relation $\omega = p/\rho$. In addition, the field equations of the universe can be shown to be:

\[
d\ln \rho(z) = 3(\omega + 1)d\ln(1 + z),
\]

\[
H^2 = \frac{8\pi G}{3}\rho,
\]

for a flat universe. Here $1 + z$ is the redshift defined by $a/a_0 = 1/(1 + z)$. Various linear models [12], [13] have been suggested for the EOS in the literature. We will try to point out in this paper the underlying physics behind these linear models and go further to study a more practical leading-orders expansion.

Taylor expansion is known to be one of the best ways to extract leading contributions from a generic theory with the help of a power series expansion of some suitable field variables. The Taylor series is normally convergent due to the structure of the expansion coefficients. The power series will, however, converge quickly if the range of variable is properly chosen. Hence it is important to choose an appropriate expansion variable for the purpose of our study.

For example, one may expand the EOS, assuming to be a smooth function for all $z$, as a power series of the variable $z$ around the point $z = 0$. This will lead to the power series expansion:

\[
\omega(z) = \sum_n \frac{\omega^{(n)}(z = 0)}{n!} z^n. \tag{3}
\]

Here the summation of $n$ runs from 0 to $\infty$ and $f^{(n)}(z) = \frac{d^n f(z)}{dz^n}$. We will not specify clearly the range of summation unless it is different from the one we adopted. This convention will be applied to all the summation ranges throughout this paper for convenience.

Alternatively, one can also expand the EOS as a power series of the variable $w = 1/(1 + z)$ around the point $w(z = 0) = 1$. The result is

\[
\omega(w(z)) = \sum_n \frac{\omega^{(n)}(w = 1)}{n!} (w - 1)^n = \sum_n \frac{\omega^{(n)}(w = 1)}{n!} (-1)^n \left( \frac{z}{1 + z} \right)^n. \tag{4}
\]

Note that if we expand the EOS in power of the variable $y = z/(1 + z)$ around $y(z = 0) = 0$, we will end up with a similar power series:

\[
\omega(y(z)) = \sum_n \frac{\omega^{(n)}(y = 0)}{n!} y^n. \tag{5}
\]

The Taylor series or the Maclaurin series are normally convergent due to the structure of the expansion coefficients. Nonetheless, one would prefer to choose a more appropriate expansion parameter in order to make the series converge more rapidly. As a result, leading order terms will be enough to extract the most important physics from the underlying theory. Therefore the advantage of the $y$ expansion is that the power series converges rapidly for all range of the parameter $-1 < y = z/(1 + z) < 1$, or equivalently, $-1/2 < z < \infty$ as compared to the range $|z| < 1$ for the power series expansion of $z$ shown in Eq. (3).

One notes, however, that one should also expand all physical quantities and the field equations to the same order of precision we adopted for the EOS expansion. Higher order contributions will not be reliable unless one can show that the higher order terms do not affect the physics very much. For example, the linear order is good enough for the expansion of the EOS modelling SUGRA model [13]. This is because that the linear term fits the predicted EOS for SUGRA model to a very high precision. One readily realizes, however, that it is not easy to track the series expansion order by order due to the form of the Eq. (1) for the EOS for the $y$ expansion. This is because that after performing the integration, one needs to pay attention to the distorted integration result. Indeed, one needs to write $y = 1 - w$ in order to perform the integration involving $d\ln w$. Indeed, one will need to recombine the result back to a power series of $y$. The trouble is that the lower order terms could sometimes hide as the higher order terms in $w$. It may not be easy to track clearly the lower order $y$ expansion when we perform the expansion, for example, due to the exponential
factor in the Eq. (1). Therefore, one finds that the most natural way to expand the EOS is to expand it in terms of the variable \( x = \ln(1 + z) \) around the point \( x(z = 0) = 0 \). Indeed, this power series can be shown to be:

\[
\omega(x(z)) = \sum_{n} \frac{\omega^{(n)}}{n!} x^n. \tag{6}
\]

And this power series converge very rapidly in the range \(-1 < \ln(1 + z) < 1\), or equivalently, \(-0.63 \sim 1/e \sim z < 1/e - 1 \sim 1.72\). Here \( e \sim 2.72 \) is the natural factor. Note that this limit happens to agree with the proposed scope of the SNAP mission. We will study these different power series expansion for the EOS and its applications in the following sections.

III. POWER SERIES OF \( Z/(1 + Z) \)

We will show in details how to extract the leading terms in the \( y \)-expansion with \( y = \left(\frac{z}{1+z}\right) \) around the point \( y(z = 0) = 0 \). One can write the expansion coefficient as \( \omega_n = \frac{(1+\omega)^{(n)}(y=0)}{n!} \) such that the power series for the expansion of the EOS becomes

\[
1 + \omega(y(z)) = \sum_{n} \omega_n y^n. \tag{7}
\]

Hence the Eq. (1) can be shown to be

\[
d\ln \rho(y) = 3(1 + \omega) \frac{dy}{1-y} = d \left[ \sum_{n} \sum_{k} 3\omega_n \frac{y^{n+k+1}}{n+k+1} \right]. \tag{8}
\]

Note that we are now expanding with respect to the smooth function \((1 + \omega)\), instead of \(\omega\), as a Maclaurin series for convenience. Therefore, one can integrate above equation to obtain

\[
\rho(y) = \rho_0 \exp\left[ 3 \sum_{n} \sum_{k} \omega_n \frac{y^{n+k+1}}{n+k+1} \right] \equiv \rho_0 X(y) = \rho_0 \sum_{n} X_n y^n. \tag{9}
\]

Note that one needs to expand the function \( \rho \) as a power series of \( y \) too in order to extract the approximated solution with appropriate order. The expansion coefficient \( X_n \) is defined as \( X_n = X^{(n)}(y = 0)/n! \). Here, the superscript in \( X' \) denotes differentiation with respect to the argument \( y \) of the function \( X(y) \). One can show that \( X' = X Y' \) with \( Y = 3 \sum_{n} \sum_{k} \omega_n y^{n+k} \). In addition, one can show that

\[
Y^{(l)}(y) = 3 \sum_{n} \sum_{k} \frac{(n + k)!}{(n + k - l)!} \omega_n y^{n+k-l}. \tag{10}
\]

Hence one has

\[
Y^{(l)}(y = 0) = 3(l!) \sum_{n=0}^{l} \omega_n. \tag{11}
\]

One can also show, for example, that

\[
X'' = X(Y^2 + Y') \tag{12}
\]

\[
X''' = X(Y^3 + 3YY' + Y'') \tag{13}
\]

\[
X^{(4)} = X(Y^4 + 6Y^2Y' + 3(Y')^2 + 4YY'' + Y'''). \tag{14}
\]

This series does not appear to have a more compact close form for the multiple differentiation with respect to \( y \). One can, however, put the equations as a more compact format:

\[
X^{(l+1)} = X[Y + \frac{d}{dy}]^l Y. \tag{15}
\]
It appears, however, that one needs to do it manually even it is straightforward. We will only list the leading terms as this is already suitable expansion for our purpose at this moment.

Hence one has

\begin{align*}
X_0 &= 1, \\
X_1 &= 3\omega_0, \\
X_2 &= \frac{1}{2}[9\omega_0^2 + 3\omega_0 + 3\omega_1], \\
X_3 &= \frac{1}{6}[27\omega_0^3 + 27\omega_0(\omega_0 + \omega_1) + 6(\omega_0 + \omega_1 + \omega_2)], \\
X_4 &= \frac{1}{24}[81\omega_0^4 + 162\omega_0^2(\omega_0 + \omega_1) + 27(\omega_0 + \omega_1)^2 + 72\omega_0(\omega_0 + \omega_1 + \omega_2) + 18(\omega_0 + \omega_1 + \omega_2 + \omega_3)].
\end{align*}

Therefore, one can expand the final expression for the energy density \(\rho\) accordingly. Indeed, the result is

\[
\rho = \rho_0\{1 + 3\omega_0 y + \frac{1}{2}[9\omega_0^2 + 3\omega_0 + 3\omega_1]y^2 + \frac{1}{6}[27\omega_0^3 + 27\omega_0(\omega_0 + \omega_1) + 6(\omega_0 + \omega_1 + \omega_2)]y^3 \\
+ \frac{1}{24}[81\omega_0^4 + 162\omega_0^2(\omega_0 + \omega_1) + 27(\omega_0 + \omega_1)^2 + 72\omega_0(\omega_0 + \omega_1 + \omega_2) + 18(\omega_0 + \omega_1 + \omega_2 + \omega_3)]y^4\}
\]

(21)

to the order of \(y^4\). Note that one keep the order of precision to \(y^4\) in computing the energy density \(\rho\) even we are expanding the EOS only to the order of \(y^3\). This is due to the special structure in the energy momentum conservation law 1. In addition, one can show that the Hubble parameter \(H = H_0 X^{1/2}\) with \(H_0 = \sqrt{8\pi G\rho_0/3}\). And the expansion for \(X^{1/2}\) can be obtained by replacing all \(\omega_n\) with \(\omega_n/2\) in writing the expansion for \(X\). Therefore one has

\[
H = H_0\{1 + \frac{3}{2}\omega_0 y + \frac{1}{2}[9\omega_0^2 + 6(\omega_0 + \omega_1)]y^2 + \frac{1}{48}[27\omega_0^3 + 54\omega_0(\omega_0 + \omega_1) + 24(\omega_0 + \omega_1 + \omega_2)]y^3 \\
+ \frac{1}{192}[162\omega_0^4 + 162\omega_0^2(\omega_0 + \omega_1) + 54(\omega_0 + \omega_1)^2 + 144\omega_0(\omega_0 + \omega_1 + \omega_2) + 72(\omega_0 + \omega_1 + \omega_2 + \omega_3)]y^4\}
\]

(22)

Note also that one can also compute the conformal time according to the expression:

\[
H_0\eta = \int_0^z dz' X' \frac{1}{2} = \int dy \frac{X^{-\frac{3}{2}}}{(1 - y)^2}.
\]

(23)

Knowing that \(1/(1 - y)^2 = \sum_n (n + 1)y^n\), one can show that

\[
H_0\eta = \int dy X^{-\frac{3}{2}} \sum_n (n + 1)y^n.
\]

(24)

Therefore, one can easily compute the expansion of \(\eta\) in a straightforward manner.

**IV. POWER SERIES OF \(\ln(1 + Z)\)**

We will show in details how to extract the leading terms in the \(x\)-expansion with \(x = \ln(1 + z)\) around the point \(x(z = 0) = 0\). One can write the expansion coefficient as \(\omega_n = \frac{(1 + \omega)^{(n)}(x=0)}{n!}\) such that the power series for the expansion of the EOS becomes

\[
1 + \omega(x) = \sum_n \omega_n x^n.
\]

(25)

Note that we use the same notation for \(\omega_n\) in different parameterizations for convenience. Hence the Eq. (1) can be shown to be

\[
d\ln \rho(x) = 3(1 + \omega)dx = d \left[\sum_n 3\omega_n \frac{x^{n+1}}{n + 1}\right].
\]

(26)
Note that we are now expanding the physical quantities with respect to the function \((1+\omega)\) instead of \(\omega\) for convenience. Therefore, one can integrate above equation to obtain

\[
\rho(x) = \rho_0 \exp\left[ 3 \sum_n \frac{\omega_n x^{n+1}}{n+1} \right] \equiv \rho_0 X(x) = \rho_0 \sum_n X_n x^n. \tag{27}
\]

Note that one needs to expand the function \(\rho\) as a power series of \(x\) too in order to extract the approximated solution with appropriate order. The expansion coefficient \(X_n\) is defined as \(X_n = X^{(n)}(x = 0)/n!\). One can show that \(X' = XY\) with \(Y = 3 \sum_n \omega_n x^n = 3(1 + \omega)\). Therefore, one has

\[
Y^{(l)}(y = 0) = 3(l!) \omega_l. \tag{28}
\]

One can also show, for example, that

\[
X'' = X(Y^2 + Y') \tag{29}
\]
\[
X''' = X(Y^3 + 3YY' + Y'') \tag{30}
\]
\[
X^{(4)} = X(Y^4 + 6Y^2Y' + 3(Y')^2 + 4YY'' + Y'''). \tag{31}
\]

In addition, one can show that

\[
X^{(l)} = \sum_n \sum_k 3X_n \omega_k \frac{(n+k)!}{(n+k-l+1)!} x^{n+k-l+1}. \tag{32}
\]

Therefore, one has

\[
X_0^{(l)} = \sum_{n+k=l-1, k>0} 3X_n \omega_k (l-1)!. \tag{33}
\]

Hence one obtains the recurrence relation for the expansion coefficients of \(X_n\):

\[
X_l = \frac{3}{l} \sum_{n=0}^{l-1} \sum_k X_n \omega_k. \tag{34}
\]

As a result, one has, for example,

\[
X_0 = 1, \quad X_1 = 3\omega_0, \tag{35}
\]
\[
X_2 = \frac{1}{2}[9\omega_0^2 + 3\omega_1], \quad X_3 = \frac{1}{6}[27\omega_0^3 + 27\omega_0\omega_1 + 6\omega_2], \tag{36}
\]
\[
X_4 = \frac{1}{24}[81\omega_0^4 + 162\omega_0^2\omega_1 + 27\omega_1^2 + 72\omega_0\omega_2 + 18\omega_3]. \tag{37}
\]

Therefore, one can expand the final expression for the energy density \(\rho\) accordingly. Indeed, one has

\[
\rho = \rho_0 \{1 + 3\omega_0 y + \frac{1}{2}[9\omega_0^2 + 3\omega_1]y^2 + \frac{1}{6}[27\omega_0^3 + 27\omega_0\omega_1 + 6\omega_2]y^3 \\
+ \frac{1}{24}[81\omega_0^4 + 162\omega_0^2\omega_1 + 27\omega_1^2 + 72\omega_0\omega_2 + 18\omega_3]y^4 \}. \tag{38}
\]

In addition, one can show that the Hubble parameter \(H = H_0 X^{1/2}\) with \(H_0 = \sqrt{8\pi G\rho_0}/3\). And the expansion for \(X^{1/2}\) can be obtained by replacing all \(\omega_n\) with \(\omega_n/2\) in writing the expansion for \(X\). The result is

\[
H = H_0 \{1 + \frac{3}{2}\omega_0 y + \frac{1}{8}[9\omega_0^2 + 6\omega_1]y^2 + \frac{1}{48}[27\omega_0^3 + 54\omega_0\omega_1 + 24\omega_2]y^3 \\
+ \frac{1}{384}[81\omega_0^4 + 324\omega_0^2\omega_1 + 108\omega_1^2 + 288\omega_0\omega_2 + 144\omega_3]y^4 \}. \tag{39}
\]
Note also that one can also compute the conformal time according to the expression:

$$H_0 \eta = \int_0^z dz' X^{-\frac{1}{2}} = \int dy X^{-\frac{1}{2}} \frac{X}{(1 - y)^2}. \quad (42)$$

Knowing that $1/(1 - y)^2 = \sum_n (n + 1) y^n$, one can show that

$$H_0 \eta = \int dy X^{-\frac{1}{2}} \sum_n (n + 1) y^n. \quad (43)$$

Therefore, one can easily compute the expansion of $\eta$ in a straightforward manner.

V. CONCLUSION

The proposed Supernova/Acceleration Probe (SNAP) will carry out observations aiming to determine the components and equations of state of the energy density, providing insights into the cosmological model, the nature of the accelerating dark energy, and potentially clues to fundamental high energy physics theories and gravitation. As a result, we are motivated by a nice result shown in Ref. [13] to study the physics underlying the model presented there.

A detailed discussion on the choices of the expansion parameters of the Maclaurin series for the equation of states of a perfect fluid is presented in this paper accordingly. For example, the Maclaurin series of the EOS is expanded as power series of the variables $y = z/(1 + z)$ and $x = \ln(1 + z)$ respectively. We also show how to obtain the power series for the energy density function, the Hubble parameter and related physical quantities of interest. The method presented here will have significant application in the precision distance-redshift observations end to map out the recent expansion history of the universe, including the present acceleration and the transition to matter dominated deceleration. We also show that the nice recent result in Ref. [13] is in fact a linear approximation to the full Maclaurin series as a power series of the parameter $y = z/(1 + z)$.

Since we can power expand all smooth EOS into a convergent power series, it is more practical for the future probe to measure local derivatives of the EOS. Note that $\omega_0$ represents the value of $1 + \omega$ at time $t_0$ or $z = 0$. In addition, $\omega_1$ is the slope of the curve $\omega$ vs $y$ or $\omega$ vs $x$ at $z = 0$. Therefore, knowing the values of the expansion coefficients will enable us to plot the entire EOS throughout the range of convergence. One may need to use different expansion series depending on the convergent speed of the power series. For example, it appears that the leading order term in the $y$ expansion is good enough to obtain a nice result for the SUGRA prediction. This is because the leading term is close enough to the theoretical prediction [13]. Nonetheless, one expects that a few leading terms in the Maclaurin series will be able to offer close enough result for us in the future.

In addition, the result shown here is independent of the choice of the time $t_0$. The local measurement of the expansion coefficients can be extended to compare the expansion coefficient at any time. One probably should trust less on the results other than the local predictions.

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