Eccentricity function in distance-hereditary graphs

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Abstract

A graph $G = (V, E)$ is distance hereditary if every induced path of $G$ is a shortest path. In this paper, we show that the eccentricity function $e(v) = \max \{d(v, u) : u \in V\}$ in any distance-hereditary graph $G$ is almost unimodal, that is, every vertex $v$ with $e(v) > \text{rad}(G) + 1$ has a neighbor with smaller eccentricity. Here, $\text{rad}(G) = \min \{e(v) : v \in V\}$ is the radius of graph $G$. Moreover, we use this result to characterize the centers of distance-hereditary graphs and provide a linear time algorithm to find a large subset of central vertices, and in some cases, all central vertices. We introduce two new algorithmic techniques to approximate all eccentricities in distance-hereditary graphs, including a linear time additive 1-approximation.

1 Introduction

The eccentricity $e(v)$ of a vertex $v$ is the length of a longest shortest path from $v$ to any other vertex. In a distance-hereditary graph $G$, the length of any induced path between two vertices equals their distance in $G$ [13]. The diameter $\text{diam}(G)$ (maximum eccentricity) and radius $\text{rad}(G)$ (minimum eccentricity) of distance-hereditary graphs have been extensively studied. It was shown in [7] that with two sweeps of a Breadth-First Search (BFS) one can obtain a value that is very close to the diameter. In fact, any vertex $v$ that is furthest from an arbitrary vertex $u$ has eccentricity $e(v) \geq \text{diam}(G) - 2$. Later, Feodor Dragan and Falk Nicolai [11] showed that by using instead LexBFS (Lexicographic Breadth-First Search) one can get a vertex $v$ (last visited by a LexBFS starting at any vertex $u$) with $e(v) \geq \text{diam}(G) - 1$, and additionally if $e(v)$ is even, then $e(v)$ exactly realizes the diameter of $G$. This yielded a linear time algorithm to compute the diameter as well as a diametral pair of vertices [7,11], i.e., a pair $x,y$ such that $d(x,y) = \text{diam}(G)$. There is also a linear time algorithm to find a central vertex (a vertex with minimum eccentricity) and calculate the radius [7]. A close relationship between diameter and radius was discovered in [7,20], where it was shown that $\text{diam}(G) \geq 2\text{rad}(G) - 2$. Here, we establish further properties of the eccentricity function in distance-hereditary graphs.

We define the locality $\text{loc}(v)$ of a vertex $v$ as the minimum length of a shortest path from $v$ to a vertex with strictly smaller eccentricity. A graph’s eccentricity function is unimodal if every non-central vertex has locality 1, that is, any non-central vertex $v$ has an adjacent vertex $u$ with $e(u) < e(v)$. The unimodality of the eccentricity function has been studied in a variety of graph classes; for example, it is exactly unimodal in Helly graphs [6] and almost unimodal in $(\alpha_1, \Delta)$-metric graphs [10] (that includes all chordal graphs) and in hyperbolic graphs [1]. In particular, it was shown [10] that a vertex $v$ of a chordal graph $G$ can have $\text{loc}(v) > 1$ only under very specific conditions: that $\text{diam}(G) = 2\text{rad}(G)$, that $e(v) = \text{rad}(G) + 1$, and that $d(v,C(G)) = 2$. We show in the main theorem of Section 3 that the same conditions hold for vertices of distance-hereditary graphs with locality larger than 1. This result, which is of independent interest, is a crucial intermediate step to establish the remaining results of this paper.

The center $C(G)$ (all vertices of $G$ with minimum eccentricity and the graph induced by those vertices) of a distance-hereditary graph $G$ is also of interest. Many graph classes have a well defined center. The
center of a tree is either $K_1$ or $K_2$\(^{13}\), the center of a maximal outerplanar graph is one of seven special graphs\(^{18}\), and more generally all possible centers of 2-trees are known\(^{17}\). Graph centers have also been characterized fully for chordal graphs\(^{3}\). In distance-hereditary graphs it is known\(^{20}\) that the diameter of the center is no more than 3. This was later improved by Hong-Gwa Yeh and Gerard Chang\(^{19}\) that either $diam(C(G)) = 3$ and $C(G)$ is connected or $C(G)$ is a cograph (which may not be connected), i.e., a $P_4$-free graph. Furthermore, any cograph is the center of some distance-hereditary graph. We complete the characterization of centers of distance-hereditary graphs by investigating the instance that $G$ is not a cograph (i.e., $diam(C(G)) = 3$), in which case $C(G)$ takes the form of a graph $H$ which is further described in Section\(^6\) and moreover each such $H$ is the center of some distance-hereditary graph. We additionally describe how to find in linear time a central set $M \subseteq C(G)$ which dominates $C(G)$ (every central vertex of $G$ is in $M$ or adjacent to a vertex in $M$). We also describe how one can find all central vertices $C(G)$ in linear time when $diam(G) = 2rad(G)$.

Finally, we provide two linear time algorithms to approximate all eccentricities. The first is an additive 2-approximation of the eccentricity of any vertex $v$ based on the distances from $v$ to only two vertices $x, y$, where $x$ and $y$ is an arbitrary pair of mutually distant vertices. The second is an additive 1-approximation using the distance from $v$ to a particular central subset $M \subseteq C(G)$.

### 2 Preliminaries

Let $G = (V, E)$ be an undirected, simple (without loops or parallel edges), connected graph. Let $n = |V|$ and $m = |E|$. A path $P(v_0, v_k)$ is a sequence of vertices $v_0, ..., v_k$ such that $v_i v_{i+1} \in E$ for all $i \in [0, k - 1]$; its length is $k$. A graph $G$ is connected if there is a path between every pair of vertices. Let $d_G(x, y)$ be the distance between two vertices $x$ and $y$ in $G$, that is, the length of a shortest path from $x$ to $y$. A subgraph $H$ of a graph $G$ is called isometric if the distance in $H$ between any of its two vertices equals their distance in $G$. The eccentricity $e_G(v)$ of a vertex $v$ is the maximum distance from $v$ to any vertex. The subindex is omitted if $G$ is known by context. The diameter ($diam(G)$) and radius ($rad(G)$) of a graph $G$ is the maximum and minimum eccentricity of a vertex, respectively. The center is the set of vertices whose eccentricities are minimum: $C(G) = \{v \in V : e(v) = rad(G)\}$. It will be convenient to denote by $C(G)$ also the subgraph of $G$ induced by set $C(G)$. We define $C^k(G) = \{v \in V : e(v) \leq rad(G) + k\}$. We denote the set of furthest vertices from $v$ as $F(v) = \{u \in V : d(u, v) = e(v)\}$. Vertices $x, y$ are considered to be mutually distant if $x \in F(y)$ and $y \in F(x)$; they are called a pair of mutually distant vertices. A pair $(x, y)$ is called a diametral pair if $d(x, y) = diam(G)$. The interval $I(x, y) = \{v \in V : d(x, v) + d(v, y) = d(x, y)\}$ is the set of all vertices that are on shortest paths between $x$ and $y$. An interval slice is defined as $S_k(x, y) = \{v \in I(x, y) : d(x, v) = k\}$ for some non-negative integer $k$. We denote by $< S >$ the subgraph of $G$ induced by the vertices $S \subseteq V$. Let also $d(v, S) = \min\{d(v, u) : u \in S\}$ and $diam(S) = \max\{d_G(x, y) : x, y \in S\}$.

The neighborhood of $v$ consists of all vertices adjacent to $v$, denoted by $N(v)$, and the closed neighborhood of $v$ is defined as $N[v] = N(v) \cup \{v\}$. The $k$-th neighborhood of a vertex $v$ is the set of all vertices of distance $k$ to $v$, that is, $N^k(v) = \{u \in V : d(u, v) = k\}$. Whereas a disk of radius $k$ centered at a set $S$ (or a vertex) is the set of vertices of distance at most $k$ to some vertex of $S$, that is, $D(S, k) = \{u \in V : d(u, S) \leq k\}$. A vertex $v$ is said to be universal to a set $S$ if $N(v) \supseteq S$. Let $V = \{v_1, ..., v_n\}$. For an $n$-tuple of non-negative integers $(r(v_1), ..., r(v_n))$, a subset $M \subseteq V$ is an $r$-dominating set for a set $S \subseteq V$ in $G$ if and only if for every $v \in S$ there is a vertex $u \in M$ with $d(u, v) \leq r(v)$. We also say that $M$ $r$-dominates $S$ in $G$. If $r(v_i) = 1$ for all $i$, then we say that $M$ dominates $S$ in $G$. If $S = V$ then we say that $M$ $r$-dominates $G$. A vertex is pendant if $|N(v)| = 1$. Two vertices $v$ and $u$ are twins if they have the same neighborhood or the same closed neighborhood. True twins are adjacent; false twins are not. A graph is distance-hereditary if and only if each of its connected induced subgraphs is isometric\(^{13}\), that is, the length of any induced path between two vertices equals their distance in $G$. These graphs can also be constructed via a series of pendant or twin vertex addi-
tions \cite{2}. Two vertex sets $A$ and $B$ of $G$ said to be joined if each vertex of $A$ is adjacent to every vertex of $B$.

The following propositions provide basic information on distance-hereditary graphs necessary for the next sections.

**Proposition 1.** \cite{2,[5]} For a graph $G$, the following conditions are equivalent:

1. $G$ is distance-hereditary;
2. The house, domino, gem, and the cycles $C_k$ of length $k \geq 5$ are not induced subgraphs of $G$ (see Figure 1);
3. For an arbitrary vertex $v$ of $G$ and every pair of vertices $u, w \in N^k(v)$, that are connected in the same component of the graph $< V \setminus N^{k-1}(v) >$, we have $N(v) \cap N^k(v) = N(u) \cap N^{k-1}(v)$;
4. (4-point condition) For any four vertices $u, v, w, x$ of $G$ at least two of the following distance sums are equal: $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, and $d(u, x) + d(w, v)$. If the smaller sums are equal, then the largest one exceeds the smaller ones by at most 2.

![Figure 1: Forbidden induced subgraphs in a distance-hereditary graph.](image)

**Proposition 2.** \cite{7} Let $G$ be a distance-hereditary graph with $n$-tuple $(r(v_1), ..., r(v_n))$ of non-negative integers and $M \subseteq V$. If every vertex pair $u, v \in M$ satisfies $d(u, v) \leq r(u) + r(v) + 1$ then $M$ has an $r$-dominating clique $C$. If every vertex pair $u, v \in M$ satisfies $d(u, v) \leq r(u) + r(v)$ then there exists either a single vertex or a pair of adjacent vertices which $r$-dominates $M$.

**Proposition 3.** \cite{7} For every vertex $c$ of a distance-hereditary graph $G$, a furthest from $c$ vertex $v \in F(c)$ satisfies $e(v) \geq 2\text{rad}(G) - 3$.

**Proposition 4.** Let $G$ be a distance-hereditary graph and $x, y \in V$. Any vertex $v \in S_k(x, y)$ has $S_{k+1}(x, y) \subseteq N(v)$, i.e., neighboring interval slices are joined.

Proof. Consider a vertex $u \in S_{k+1}(x, y) \cap N(v)$ and any other vertex $w \in S_{k+1}(x, y)$. Then $u$ and $w$ are connected in $< V \setminus N^{k+1}(x) >$ via shortest paths $P(u, y)$ and $P(w, y)$. By Proposition [III], they share neighboring vertices in $S_k(x, y)$. Hence, $w \in N(v)$. □

3 Unimodality of the eccentricity function

Recall that the eccentricity function is unimodal in $G$ if every non-central vertex $v$ of $G$ has a neighbor $u$ such that $e(u) < e(v)$. We defined the locality $\text{loc}(v)$ of a vertex $v$ to be the minimum distance from $v$ to a vertex with strictly smaller eccentricity or to a central vertex. Our main result of this section is that in distance-hereditary graphs any vertex with sufficiently large eccentricity does have a neighbor with strictly smaller eccentricity. Unimodality can break only at vertices with eccentricity equal to $\text{rad}(G) + 1$, but those vertices are close (within 2) to the center $C(G)$. Moreover, this can only occur when $\text{diam}(G) = 2\text{rad}(G)$. This property of the eccentricity function of distance-hereditary graphs aligns with known results for other graph classes, such as chordal graphs and the underlying graphs of 7-systolic complexes \cite{10}.

Our proof will be based on the following two lemmas.
Lemma 1. Let $G$ be a distance-hereditary graph. If a vertex $x$ has $e(x) = \text{rad}(G) + 1$, then $d(x, C(G)) \leq 2$.

Proof. Let $c$ be a central vertex closest to $x$. Consider any vertex $v \in S_1(c, x)$ and vertex $u \in F(v)$ furthest from $v$. As $v$ is not central, $d(v, u) = \text{rad}(G) + 1$ and, by distance requirements, $d(c, u) = e(c) = \text{rad}(G)$. Hence, $u \in F(c) \cap F(v)$.

First we claim that $d(c, x) \leq 3$. Since $c \in I(v, u)$ and $v \in I(c, x)$, we have $d(u, v) + d(c, x) = d(u, c) + d(v, x) + 2$. Consider the 4-point condition on vertices $u, v, x, c$. As two distance sums must be equal, then either $d(u, x) + d(c, v) = d(u, v) + d(c, x)$ or $d(u, x) + d(v, c) = d(u, c) + d(v, x)$. We have $d(u, x) + d(c, v) \leq e(x) + 1 = \text{rad}(G) + 2$. In the first case we get $d(u, x) + d(c, v) = d(u, v) + d(c, x) = d(u, c) + d(v, x) + 2 = \text{rad}(G) + d(v, x) + 2$. Hence, $d(v, x) \leq 0$ and, by the triangle inequality, $d(x, C(G)) \leq 1$. In the second case we get $d(u, x) + d(v, c) = d(u, c) + d(v, x) = \text{rad}(G) + d(v, x)$. Hence, $d(v, x) \leq 2$ and, by the triangle inequality, $d(x, C(G)) \leq 3$, establishing the claim.

Assume now that $d(c, x) = 3$ and consider $y \in S_1(v, x)$. We next claim that $e(y) = \text{rad}(G) + 1$. By the choice of $c$, vertex $y$ is non-central and so $e(y) \geq \text{rad}(G) + 1$. Since $y \in N(v) \cap N(x)$ with $e(v) = e(x) = \text{rad}(G) + 1$, by distance requirements, $e(y) \leq \text{rad}(G) + 2$. By way of contradiction assume that $e(y) = \text{rad}(G) + 2$. Consider a furthest vertex $y^* \in F(y)$. By distance requirements, $d(v, y^*) = d(x, y^*) = \text{rad}(G) + 1$ and $d(c, y^*) = \text{rad}(G)$. Since there is a $(v, x)$-path via vertex $y$ in $< V \setminus N^{\text{rad}(G)}(y^*) >$, by Proposition 1(iii), the neighbors of $v$ and $x$ in $N^{\text{rad}(G)}(y^*)$ are shared. Therefore, $cx \in E$, contradicting with $d(c, x) = 3$. Thus, $e(y) = \text{rad}(G) + 1$ must hold.

We now obtain a general contradiction in two steps. Recall that $e(y) = e(v) = e(x) = \text{rad}(G) + 1$ and $e(c) = \text{rad}(G)$. First, consider the 4-point condition on vertices $y, y^*, c, v$. Consider three sums: $d(y, y^*) + d(c, v) = \text{rad}(G) + 2$, $d(y^*, c) + d(v, y) \leq \text{rad}(G) + 1$, and $d(y^*, v) + d(c, y) \leq \text{rad}(G) + 3$. Clearly, the first and the second sums are not equal. If the second and the third sums are equal, then $d(y^*, v) = d(y^*, c) + d(v, y) - d(c, y) \leq \text{rad}(G) - 1$, contradicting with $d(y^*, y) = \text{rad}(G) + 1$. Therefore, the first and the third sums are equal. Then $d(y^*, v) = d(y^*, y) + d(c, v) - d(c, y) = \text{rad}(G)$, let $P(y^*, v)$ be any shortest path between $y^*$ and $v$. Its length is $\text{rad}(G)$. Consider also the path $Q = P(y^*, y), y, x$ (extension of $P(y^*, v)$ that includes also $y$ and $x$). In distance-hereditary graphs, every induced path is a shortest path. As $d(x, y^*) \leq e(x) = \text{rad}(G) + 1$, $Q$ cannot be induced. As $d(y, y^*) = \text{rad}(G) + 1$, vertex $x$ must be adjacent to some vertex on $P(y^*, v)$. To avoid large induced cycles $C_k$ of length $k \geq 5$, $x$ must be adjacent to a vertex $z \in P(y^*, v)$ which is a neighbor of $v$. Thus, $d(y^*, x) = \text{rad}(G)$. Necessarily, $ze \notin E$ since $d(x, c) = 3$. We also have that $d(y^*, c) \leq e(c) = \text{rad}(G)$.

Next, consider the 4-point condition on vertices $y^*, c, x, v$. We have $d(y^*, c) + d(x, v) \leq \text{rad}(G) + 2$, $d(y^*, v) + d(c, x) = \text{rad}(G) + 3$, and $d(y^*, x) + d(v, c) = \text{rad}(G) + 1$. Since at least two sums must be equal, necessarily $d(y^*, c) = d(y^*, x) = d(v, c) - d(v, x) = \text{rad}(G) - 1$. Now all distances from $y^*$ are known. We have $d(c, y^*) = d(z, y^*) = \text{rad}(G) - 1$, $d(v, y^*) = d(x, y^*) = \text{rad}(G)$, and $d(y, y^*) = \text{rad}(G) + 1$. Since there is a $(v, x)$-path in $< V \setminus N^{\text{rad}(G)-1}(y^*) >$, by Proposition 1(iii), the neighbors of $v$ and $x$ in $N^{\text{rad}(G)-1}(y^*)$ are shared. Therefore, vertices $x$ and $c$ must be adjacent, contradicting with $d(x, c) = 3$.

Obtained contradiction proves the lemma. □

Lemma 2. Let $G$ be a distance-hereditary graph. If there is a vertex $v$ with $\text{loc}(v) > 1$, then $e(v) = \text{rad}(G) + 1$ and $\text{diam}(G) = 2\text{rad}(G)$.

Proof. Consider a vertex $u \in F(v)$, a vertex $x \in S_1(v, u)$ with minimal $|F(x)|$, and let $y \in F(x)$. By the assumption on the locality of $v$, $e(x) \geq e(v)$ and $v$ is non-central. If $d(v, y) = d(x, y) + 1$, then $e(v) \geq e(x) + d(v, y) = d(x, y) + 1 = e(x) + 1 \geq e(v) + 1$, a contradiction. Thus, $d(x, y) - 1 \leq d(v, y) \leq d(x, y)$.

Consider the 4-point condition on vertices $u, v, x, y$. We have $d(u, v) + d(x, y) = e(v) + e(x) \geq 2\text{rad}(G) + 2$, $d(u, y) + d(x, v) \leq 2\text{rad}(G) + 1$, and $d(u, x) + d(v, y) = e(v) - 1 + d(v, y) \leq e(v) + e(x) - 1$. Since two sums must be equal, necessarily $d(u, y) + d(x, v) = d(u, x) + d(v, y) \geq (e(v) - 1) + (e(x) - 1) \geq 2e(v) - 2 \geq 2\text{rad}(G)$. Therefore, $2\text{rad}(G) \geq d(u, y) \geq 2\text{rad}(G) - 1$ and $e(v) \leq \text{rad}(G) + 3/2$. Since eccentricity is an integer and $v$ is non-central, $e(v) = \text{rad}(G) + 1$ must hold.
It remains only to show that $diam(G) = 2rad(G)$. If $d(u, y) = 2rad(G)$, we are done. So, assume that $d(u, y) = 2rad(G) - 1$. We get $d(y, v) = d(u, y) + d(x, v) - d(u, x) = rad(G)$, and so $e(x) = rad(G) + 1$. Furthermore, $v \in I(x, y)$. The length of path $Q = P(u, v) \cup P(v, y)$ (the concatenation of $P(u, v)$ with $P(v, y)$ is $2rad(G) + 1$. As $d(u, y) = 2rad(G) - 1$, $Q$ is not an induced path. Hence, there are vertices $s \in P(v, u)$ and $w \in P(v, y)$ such that $sw \in E$. To avoid large induced cycles $C_k$ of length $k \geq 5$, necessarily $s \in S_1(u, x)$ and $w \in S_1(v, y)$ must hold. Then, $w$ belongs to $S_1(v, u)$ as well as $x$. Since $y \in F(x) \setminus F(w)$ (note that $e(w) = e(v) = rad(G) + 1$ as $loc(v) > 1$), by minimality of $|F(x)|$, there is a vertex $t \in F(w) \setminus F(x)$. Hence, $d(t, x) < rad(G) + 1$ and $d(t, w) = e(w) \geq rad(G) + 1$.

Now consider the 4-point condition on vertices $x, y, w, t$. We have $d(x, y) + d(w, t) = e(x) + e(w) \geq 2rad(G) + 2$, whereas $d(t, y) + d(w, x) \leq 2rad(G) + 2$ and $d(x, t) + d(w, y) \leq 2rad(G) - 1$. As $d(x, y) + d(w, t) - d(x, t) - d(w, y) \geq 3$, then only the two largest sums are equal: $d(x, y) + d(w, t) = d(t, y) + d(w, x)$. Hence, $diam(G) \geq d(t, y) = d(x, y) + d(w, t) - d(w, x) \geq 2rad(G)$. That is, $diam(G) = d(t, y) = 2rad(G)$.

We are ready to prove the main result of this section.

**Theorem 1.** Let $G$ be a distance-hereditary graph. If there is a vertex $v$ with $loc(v) > 1$, then $e(v) = rad(G) + 1$, $diam(G) = 2rad(G)$, and $d(v, C(G)) = 2$.

**Proof.** If a vertex $v$ has $loc(v) > 1$ then $d(v, C(G)) > 1$ and, by Lemma 2 $e(v) = rad(G) + 1$ and $diam(G) = 2rad(G)$. Thus, by Lemma 1 $d(v, C(G)) = 2$. □

## 4 Certificates for eccentricities

We obtain as a consequence of Theorem 1 several new results for distance-hereditary graphs on lower and upper certificates for eccentricities, which were introduced in 2 as a way to compute exactly or approximately eccentricities in a graph by maintaining upper and lower bounds. A set $L$ (set $U$) of vertices is a **lower certificate** (respectively, an **upper certificate**) for eccentricities of $G$ if it is used to obtain lower bounds (respectively, upper bounds) of eccentricities in $G$. Given all distances from a vertex $v$ to all vertices in $L \cup U$ as well as the eccentricities of vertices in $U$, we have the following lower and upper bounds for the eccentricity of any vertex $v$ 3:

$$
e_L(v) \leq e(v) \leq e_U(v),$$

where

$$e_U(v) = \min_{v \in L} d(v, x) + e(x), \quad e_L(v) = \max_{v \in L} d(v, x).$$

A lower certificate $L$ (an upper certificate $U$) is said to be **tight** if $e_L(v) = e(v)$ ($e_U(v) = e(v)$, respectively) for all $v \in V$. A **diameter certificate** is a set $U$ such that $e_U(v) \leq diam(G)$ for all $v \in V$, and therefore the diameter is realized by $\max_{v \in V} e_U(v)$. A **radius certificate** is a set $L$ such that $e_L(v) \geq rad(G)$ for all $v \in V$, and therefore the radius is realized by $\min_{v \in V} e_L(v)$. In what follows, we define the set of all diametral vertices of $G$ as $D(G) = \{v \in V : e(v) = diam(G)\}$.

In this section we show that all eccentricities can exactly be determined in distance-hereditary graphs by computing distances from vertices of $C^1(G)$ to all vertices, since $C^1(G)$ forms a tight upper certificate. We also show that in distance-hereditary graphs the set $C(G)$ is a diameter certificate and the set $D(G)$ is a radius certificate (a kind of duality between $C(G)$ and $D(G)$). This agrees with radius and diameter certificates in chordal graphs 9 but, as we show later, this does not hold for arbitrary graphs.

We use the following corollary to Theorem 1

**Corollary 1.** Let $G$ be a distance-hereditary graph.

(i) If $diam(G) < 2rad(G)$ then, for every pair of vertices $v \in V$ and $u \in F(v)$, there is a vertex $w \in I(v, u) \cap C(G)$ such that $u \in F(w)$.

(ii) For every pair of vertices $v \in V \setminus C(G)$ and $u \in F(v)$, there is a vertex $w \in I(v, u) \cap C^1(G)$ such that $u \in F(w)$. 

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Proof. Consider any vertex \( v \in V \) and \( u \in F(v) \) and proceed by induction on \( k := e(v) \). If \( k = rad(G) \), then \( w = v \) and we are done. If \( k = rad(G) + 1 \) and \( diam(G) = 2rad(G) \) then again \( u = v \) and we are done. If \( k > rad(G) + 1 \) or \( k = rad(G) + 1 \) and \( diam(G) < 2rad(G) \) then, by Theorem \( \ref{thm:radius_certificate} \), a neighbor \( z \) of \( v \) with \( e(z) = k - 1 \) satisfies \( u \in F(z) \), and we can apply the induction hypothesis.

\[ \square \]

**Lemma 3.** The set \( C^1(G) \) of a distance-hereditary graph is a tight upper certificate for all eccentricities of \( G \).

**Proof.** The statement follows from Corollary \( \ref{cor:radius_certificate} \) and the definition of a tight upper certificate.

\[ \square \]

**Lemma 4.** The center \( C(G) \) of a distance-hereditary graph \( G \) is a diameter certificate of \( G \).

**Proof.** This is clear by Corollary \( \ref{cor:radius_certificate} \) if \( diam(G) < 2rad(G) \). Additionally, in any graph \( G \) with \( diam(G) = 2rad(G) \) all central vertices \( c \in C(G) \) and every diametral pair of vertices \( x, y \) satisfy \( d(x, y) = d(x, c) + d(c, y) = d(x, c) + rad(G) = rad(G) + d(y, c) = 2rad(G) \).

\[ \square \]

**Lemma 5.** The set \( D(G) \) of a distance-hereditary graph \( G \) is a radius certificate of \( G \).

**Proof.** We first show that \( D(G) \) is a radius certificate for any graph \( G \) if \( diam(G) \geq 2rad(G) - 1 \). If \( D(G) \) is not a radius certificate, then there is a vertex \( u \in V \) such that \( max_{u \in D(G)} d(u, v) < rad(G) \). Thus, for any diametral pair \( x, y \), \( d(x, y) \leq d(x, u) + d(u, y) \leq (rad(G) - 1) + (rad(G) - 1) = 2rad(G) - 2 \), a contradiction with \( d(x, y) = diam(G) \geq 2rad(G) - 1 \).

As in a distance-hereditary, \( diam(G) \geq 2rad(G) - 2 \) holds \( \ref{thm:radius_certificate} \ref{thm:distance_inequality} \), it remains to consider only the case when \( diam(G) = 2rad(G) - 2 \). Let \( S \) be the set of vertices \( u \) such that \( d(u, t) \leq rad(G) - 1 \) for all \( t \in D(G) \). By contradiction assume \( D(G) \) is not a radius certificate and therefore \( S \) is not empty. Let \( u \in S \) be a vertex which minimizes \( |F(u)| \). Consider any diametral pair \( x, y \) and furthest from \( u \) vertex \( v \in F(u) \). Necessarily \( v \notin D(G) \) by the choice of \( u \). Since \( d(x, y) = 2rad(G) - 2 \), \( d(u, x) \leq rad(G) - 1 \) and \( d(u, y) \leq rad(G) - 1 \), clearly \( d(x, u) = d(u, y) = rad(G) - 1 \).

Consider the 4-point condition on vertices \( v, u, x, y \). We have that the largest distance sum is \( d(v, u) + d(x, y) = d(v, u) + d(u, x) + d(x, y) = 2rad(G) - 2 \), given that \( d(v, x) + d(u, y) \leq d(x, y) - 1 + rad(G) - 1 = 3rad(G) - 4 \) and that \( d(v, y) + d(u, x) \leq d(x, y) - 1 + rad(G) - 1 = 3rad(G) - 4 \). Therefore, the smaller sums are equal, establishing \( d(v, x) = d(v, y) = rad(G) - 1 \). Moreover, since the difference between the largest sum and the other sums is at most 2, we get \( d(v, u) = rad(G) \) and \( d(v, x) = d(v, y) = 2rad(G) - 3 \). So, \( u \in C(G) \).

We claim that there is a vertex \( w \) such that \( d(w, x) = rad(G) - 1 \) and \( d(w, y) = rad(G) - 1 \). Fix arbitrary shortest path \( P(x, u) \), \( P(y, u) \) and \( P(u, v) \). Since \( d(v, x) < d(v, u) + d(u, x) = diam(G) + 1 \), path \( Q = P(x, u) \cup P(u, v) \) is not induced. Hence, there must exist a chord between shortest path \( P(x, u) \) and shortest path \( P(u, v) \). Define vertices \( t, w \in P(u, v) \), \( s, z \in P(x, u) \), \( q, p \in P(y, u) \), as shown in Figure 2. Since \( d(v, x) = 2rad(G) - 3 \), we must have the chord \( st \in E \) or the chord \( sw \in E \). By the same argument, there must exist a chord between shortest path \( P(y, u) \) and shortest path \( P(u, v) \) which is realized by chord \( pt \in E \) or \( qw \in E \). We note that if \( st \in E \) then \( pt \notin E \) since \( d(x, y) = 2rad(G) - 2 \). Up to symmetry, we have two cases as shown in Figure 2. In case (a) we have \( zw \in E \), and in case (b) we have \( sw, qw \in E \). In either case vertex \( w \) satisfies the desired properties, establishing the claim.

We next claim that there is a vertex \( \overline{w} \) such that \( d(w, \overline{w}) \geq rad(G) \) and \( d(u, \overline{w}) \leq rad(G) - 1 \). On one hand, if \( w \notin S \) then, by definition of \( S \), there exists a vertex \( \overline{w} \in D(G) \) such that \( d(w, \overline{w}) \geq rad(G) \) and, by the choice of \( u \in S \), we have \( d(u, \overline{w}) \leq rad(G) - 1 \). On the other hand, \( w \in S \) then, by minimality of \( \{F(u)\} \) and since \( v \notin F(w) \), there exists a vertex \( \overline{w} \in F(w) \setminus F(u) \). As \( \overline{w} \notin F(u) \), \( d(u, \overline{w}) \leq rad(G) - 1 \) and, as \( \overline{w} \in F(w) \), \( d(w, \overline{w}) \geq rad(G) \), establishing the claim.

Consider now the 4-point condition on vertices \( v, u, w, \overline{w} \). Since \( v \notin D(G) \) we have \( d(v, \overline{w}) + d(w, u) \leq 2rad(G) - 3 + 2 = 2rad(G) - 1 \). We also have \( d(v, w) + d(\overline{w}, u) \leq rad(G) - 2 + rad(G) - 1 = 2rad(G) - 3 \) and \( d(v, u) + d(w, \overline{w}) \geq rad(G) + rad(G) = 2rad(G) \). Given that \( d(v, u) + d(w, \overline{w}) \) is strictly larger than the other sums, it must differ from them by at most 2. However, it differs by at least 3, giving a contradiction.

\[ \square \]
As consequences of these results, we have that if the set \( C^1(G) \) of a distance-hereditary graph \( G \) is known, then all vertex eccentricities in \( G \) can be computed in total \( O(|C^1(G)||E|) \) time by performing a BFS from each vertex of \( C^1(G) \). Similarly, if the set \( C(G) \) (\( D(G) \), respectively) of \( G \) is known, then the entire set \( D(G) \) (\( C(G) \), respectively) of \( G \) can be computed in \( O(|C(G)||E|) \) (\( O(|D(G)||E|) \), respectively) time. Although linear time algorithms for computing a central vertex, the radius, a diametral pair and the diameter of a distance-hereditary graph are known in literature \([7,11]\), no algorithms are known that compute all eccentricities (or even only the sets \( C(G) \) or \( D(G) \)) in total linear time. In Section \([7]\) we discuss how to efficiently approximate all eccentricities in a distance-hereditary graph.

We note that Lemma \([4]\) and Lemma \([5]\) do not hold for general graphs, as illustrated in Figure \([3]\) by a graph \( G \) with \( diam(G) = 6 \) and \( rad(G) = 4 \). Here \( D(G) = \{x,y\} \) and \( C(G) = \{u\} \), and all other vertices have eccentricity 5. However, \( D(G) \) is not a radius certificate since \( e_{D(G)}(u) = 3 < rad(G) \). Moreover, \( C(G) \) is not a diameter certificate since \( e_{C(G)}(v) = d(v,u) + e(u) = 8 > diam(G) \).

One aims also to minimize the size of a certificate. In trees, for example, a single diametral pair is a sufficient radius certificate rather than the full set of diametral vertices. Unfortunately this is not true for distance-hereditary graphs. The graph \( G \) in Figure \([4]\) illustrates that every diametral vertex is necessary to establish a radius certificate. Graph \( G \) consists of a clique of vertices \( \{u_1,...,u_\ell\} \) and a clique of vertices \( \{v_1,...,v_\ell\} \), where each \( u_i \) is adjacent to all vertices \( v_j \neq i \), and each \( u_i \) and \( v_i \) has a pendant vertex \( x_i \) and \( y_i \), respectively. \( G \) is distance-hereditary as it can be dismantled via a sequence of pendant and twin vertex eliminations. All vertices \( x_i \) and \( y_i \) are pendant, each \( u_i \) vertex is a false twin to \( v_i \), and the remaining \( v_i \) vertices are true twins (as they form a clique in the remaining graph). Here \( D(G) \) consists of all \( x_i \) and \( y_i \) vertices and \( C(G) \) consists of all \( u_i \) and \( v_i \) vertices, where \( diam(G) = d(x_i,y_i) = 4 \) and \( rad(G) = d(v_i,x_i) = d(u_i,y_i) = 3 \). However, any \( x_i \in D(G) \) has a vertex \( v_i \) such that \( d(v_i,t) < rad(G) \) for all \( t \in D(G) \setminus \{x_i\} \). By symmetry, the same is true for \( y_i \) and its counterpart \( u_i \). Hence, all vertices of \( D(G) \) are necessary to form a radius certificate. One can also show that all vertices of \( C(G) \) are necessary to form a diameter certificate.
Figure 4: A distance-hereditary graph $G$ for which $D(G) \setminus \{t\}$ is not a radius certificate for any $t \in D(G)$ and for which $C(G) \setminus \{c\}$ is not a diameter certificate for any $c \in C(G)$.

5 Eccentricities, mutually distant pairs and distances to the center

In this section, we show that the eccentricity of any vertex of a distance-hereditary graph is bounded by its distances to just two mutually distant vertices. Furthermore, the distance between any two mutually distant vertices is bounded by their distances to an arbitrary peripheral vertex (a vertex which is furthest for some other vertex). The unimodality behavior of the eccentricity function described in Theorem [1] gives also a relation between the eccentricity of a vertex and its distance to $C^1(G)$.

Lemma 6. Let $G$ be a distance-hereditary graph with a mutually distant pair $x, y$, and let $c \in V$ and $v \in F(c)$ be a furthest vertex from $c$. Let also $a := d(c, x)$ and $b := d(c, y)$. Then,

$$\max\{a, b\} \leq e(c) \leq \max\{\max\{a, b\}, \min\{a, b\} + 2\} \leq \max\{a, b\} + 2.$$

Moreover, if $e(c) = \max\{a, b\} + 2$, then $a = b = e(c) - 2$ and $d(v, x) = d(v, y) = d(x, y)$.

Proof. Let $v \in F(c)$. By the choice of $v$, we have $e(c) = d(c, v) \geq \max\{d(c, x), d(c, y)\}$. Consider the 4-point condition on vertices $c, v, x, y$. As $x, y$ is a mutually distant pair, we have for the three distance sums that $d(c, v) + d(x, y) = e(c) + d(x, y)$, $d(c, x) + d(v, y) \leq d(c, x) + d(x, y) \leq e(c) + d(x, y)$, and $d(c, y) + d(v, x) \leq d(c, y) + d(x, y) \leq e(c) + d(x, y)$. Clearly the first sum is largest.

We first consider the case when the first sum equals one of the latter. Suppose that $d(c, v) + d(x, y) = d(c, x) + d(v, y)$. Then, $e(c) + d(x, y) = d(c, x) + d(v, y) \leq e(c) + d(x, y)$, hence, $e(c) = d(c, x) = \max\{d(c, x), d(c, y)\}$. Suppose now that $d(c, v) + d(x, y) = d(c, y) + d(v, x)$. Then, $e(c) + d(x, y) = d(c, y) + d(v, x) \leq e(c) + d(x, y)$. Hence, $e(c) = d(c, y) = \max\{d(c, x), d(c, y)\}$. In either case, $e(c) = \max\{d(c, x), d(c, y)\}$.

We next consider the case when the two smaller sums are equal and differ from the largest one by at least 2. We have $e(c) = d(c, v) \leq d(v, y) + d(c, x) - d(x, y) + 2 = d(v, x) + d(c, y) - d(x, y) + 2$. Since $d(x, y)$ is not smaller than $d(v, y)$ and $d(v, x)$, we obtain $e(c) \leq d(c, x) + 2$ and $e(c) \leq d(c, y) + 2$, i.e., $e(c) \leq \min\{d(c, x), d(c, y)\} + 2$. Moreover, if $e(c) = \max\{d(c, x), d(c, y)\} + 2$, we must be in the latter case when the two smaller sums are equal (otherwise, $e(c) = \max\{d(c, x), d(c, y)\}$ as shown previously), and so $d(c, v) = e(c) = \min\{d(c, x), d(c, y)\} + 2$. Hence, $d(c, x) = d(c, y) = d(c, v) - 2$ and, since $d(c, x) + d(v, y) = d(c, y) + d(v, x)$, $d(v, y) = d(x, v)$ holds too. Combining this with the fact that $d(c, v) + d(x, y) - d(c, x) - d(y, v) \leq 2$, we obtain $d(x, y) \leq d(v, y)$ and, since $x, y$ are mutually distant, necessarily $d(x, y) = d(v, y) = d(x, v)$, completing the proof. \hfill \Box

Lemma 7. Let $G$ be a distance-hereditary graph with a mutually distant pair $x, y$, and let $c \in V$ and $v \in F(c)$ be a furthest vertex from $c$. Let also $a := d(v, x)$ and $b := d(v, y)$. Then,

$$\max\{a, b\} \leq d(x, y) \leq \max\{\max\{a, b\}, \min\{a, b\} + 2\} \leq \max\{a, b\} + 2.$$

Moreover, if $d(x, y) = \max\{a, b\} + 2$ then $a = b = d(x, y) - 2$ and $d(c, x) = d(c, y) = d(c, v)$. 

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Proof. The proof is analogous to that of Lemma 6 and is omitted. The only difference is that now we argue from the prospective of \(d(x, y)\) and not of \(e(c)\).

In the case when \(x\) and \(y\) form a diametral pair, Lemma 7 yields a result known from \([19]\). We have \(\text{diam}(G) = d(x, y) \leq \max\{d(v, x), d(v, y)\} + 2 \leq e(v) + 2\).

Corollary 2. \([19]\) Let \(G\) be a distance-hereditary graph. If \(v\) is a furthest vertex from any \(c \in V\), then \(e(v) \geq \text{diam}(G) - 2\).

Corollary 2 can be used to find a pair of mutually distant vertices of a distance-hereditary graph in linear time. Pick an arbitrary start vertex \(v_0\). With at most five BFSs find vertices \(v_1, \ldots, v_k\) (\(2 \leq k \leq 5\)) such that \(v_i \in F(v_{i-1})\) and \(d(v_k, v_{k-1}) = d(v_{k-1}, v_{k-2})\). Since, by Corollary 2 \(e(v_1) \geq \text{diam}(G) - 2\), there are at most two improvements on \(e(v_1)\) to get a required mutually distant pair \(v_{k-1}, v_{k-2}\). Now, by Lemma 6 the found distances from \(v_{k-1}\) and \(v_{k-2}\) to all vertices \(u \in V\) can be used to get additive 2-approximations of all eccentricities in \(G\) (see Section 7).

Figure 5(a) illustrates that the upper bounds of Lemma 6 and Lemma 7 are sharp using vertices \(u\) and \(w\) for two opposing purposes. First, \(e(u) = d(u, w) = \max\{d(x, u), d(y, w)\} + 2\), whereas \(u \in F(u)\) has \(d(x, y) = \max\{d(x, w), d(y, w)\}\). Secondly, \(e(w) = d(u, w) = \max\{d(x, w), d(y, w)\}\), whereas \(u \in F(w)\) has \(d(x, y) = \max\{d(x, u), d(y, u)\} + 2\). Recall the implications of Lemma 6 and Lemma 7 which state that for a mutually distant pair \(x, y\) and fixed vertices \(c \in V\) and \(v \in F(c)\), if either \(e(c)\) or \(d(x, y)\) is realized by its upper bound as given in the above inequalities, then the other value is realized by its lower bound. So, it is not possible to obtain for the same \(c \in V\) and \(v \in F(c)\) that both \(e(c) =\max\{d(x, c), d(y, c)\} + 2\) and \(d(x, y) =\max\{d(x, v), d(y, v)\} + 2\) are true (Figure 5(a) uses a different starting vertex \(c := u\) and then with \(c := w\)). However, we show in Figure 5(b) an example when for fixed vertices \(c\) and \(v \in F(c)\), both \(e(c) = \max\{d(x, c), d(y, c)\} + 1\) and \(d(x, y) = \max\{d(x, v), d(y, v)\} + 1\) are true.

![Figure 5: Illustration to the sharpness of Lemma 6 and Lemma 7](image)

We turn now to a relation between the eccentricity of a vertex and its distance to \(C(G)\) or \(C^1(G)\).

Lemma 8. Let \(v \in V\) be an arbitrary vertex of any graph \(G\) and let \(k\) be an integer. If \(e(v) = d(v, C^k(G)) + \text{rad}(G) + k\), then \(e(v) = d(v, C^{k+1}(G)) + \text{rad}(G) + k + 1\).

Proof. Suppose \(e(v) = d(v, C^k(G)) + \text{rad}(G) + k\) and let \(u \in F(v)\). Let \(c \in C^k(G)\) be a closest vertex to \(v\) in \(C^k(G)\). Consider an adjacent vertex \(z \in S_1(c, v)\). By the choice of \(c\), \(e(z) = \text{rad}(G) + k + 1\) and therefore \(z \in C^{k+1}(G)\). Then, \(e(v) = d(v, c) + \text{rad}(G) + k = d(v, z) + d(z, c) + \text{rad}(G) + k \geq d(v, C^{k+1}(G)) + \text{rad}(G) + k + 1\). By the triangle inequality, also \(e(c) \leq d(v, C^{k+1}(G)) + \text{rad}(G) + k + 1\).

Lemma 9. Let \(G\) be a distance-hereditary graph.

(i) If \(\text{diam}(G) < 2\text{rad}(G)\), then all vertices \(v \in V\) satisfy \(e(v) = d(v, C(G)) + \text{rad}(G)\).

(ii) All vertices \(v \in V \setminus C(G)\) satisfy \(e(v) = d(v, C^1(G)) + \text{rad}(G) + 1\).

Proof. The statements follow from Theorem 1 and Lemma 8 (see also Corollary 1).
As a consequence of Lemma \ref{lem:distance-hereditary-classes} one can compute in linear total time all eccentricities in a distance-hereditary graph \( G \) exactly if all vertices of \( C(G) \) and of \( C^1(G) \) are known. Given \( C^1(G) \), the distances \( d(v, C^1(G)) \) for all vertices \( v \notin C(G) \) can simply be computed in total linear time by a BFS(\( C^1(G) \)) starting at the set \( C^1(G) \). Additionally, the radius \( \text{rad}(G) \) of \( G \) can be computed in linear time (see \ref{lem:radius}). Thus, if \( C(G) \) and \( C^1(G) \) are known in advance, all eccentricities are computable in total linear time. We use the following two sections then to describe the structure of centers of distance-hereditary graphs and how a sufficiently large subset of \( C(G) \) can be determined so that additive 1-approximations of all eccentricities can be computed.

6 Centers of distance-hereditary graphs

In this section, we investigate the structure of centers of distance-hereditary graphs and provide their full characterization. A subset \( S \subseteq V \) is called \( m^3\)-convex if and only if \( S \) contains every induced path of length at least three between vertices of \( S \). It is known from \cite{12} that the centers of HHD-free graphs are \( m^3\)-convex. As every distance-hereditary graph is HHD-free, the centers of distance-hereditary graphs are \( m^3\)-convex, too. It is also known from \cite{19} that \( C(G) \) is either a cograph or a connected graph with \( \text{diam}(C(G)) = 3 \). As every connected subgraph of a distance-hereditary graph is isometric, when \( \text{diam}(C(G)) = 3 \), \( C(G) \) is isometric and a distance-hereditary graph. We remark that if the diameter of a set \( S \) in a distance-hereditary graph is no more than 2 then, by definition, it induces a cograph (or, equivalently, a distance-hereditary graph of diameter at most 2).

To prove our main result of this section, we will need the following auxiliary lemmas, which will be used also in Subsection \ref{subsec:centrality}.

Lemma 10. Let \( G \) be a distance-hereditary graph with \( \text{diam}(G) = 2 \text{rad}(G) \), diametral pair \( x, y \), and \( S := S_{\text{rad}(G)}(x, y) \). Then, \( S \) and \( C(G) \) are cographs with \( C(G) \subseteq S \), any vertex of \( S_{\text{rad}(G)+1}(x, y) \cup S_{\text{rad}(G)-1}(x, y) \) is universal to \( S \), and \( C^1(G) \subseteq D(S, 1) \).

Proof. Any central vertex \( c \in C(G) \) has \( d(x, c) \leq \text{rad}(G) \) and \( d(y, c) \leq \text{rad}(G) \), therefore, by distance requirements, \( C(G) \subseteq S \). By Proposition \ref{prop:shortest-path}, any vertex \( w \in S_{\text{rad}(G)+1}(x, y) \cup S_{\text{rad}(G)-1}(x, y) \) is universal to \( S \) and therefore universal to \( C(G) \). Moreover, since the diameters of \( S \) and \( C(G) \) are no more than 2, both are cographs.

Let now \( c \in C^1(G) \) and by contradiction assume \( c \notin D(S, 1) \). Necessarily \( d(x, c) \leq \text{rad}(G) + 1 \) and \( d(y, c) \leq \text{rad}(G) + 1 \). If \( d(x, c) < \text{rad}(G) \) then, by distance requirements, \( d(y, c) = \text{rad}(G) + 1 \) and \( c \in S_{\text{rad}(G)-1}(x, y) \). By Proposition \ref{prop:shortest-path}, \( c \in D(S, 1) \), a contradiction. If \( d(x, c) = d(y, c) = \text{rad}(G) \) then \( c \in S \), a contradiction. Hence, we can assume, without loss of generality, that \( d(x, c) = \text{rad}(G) + 1 \). Consider vertex \( u \neq c \) on a shortest path \( P(c, y) \) closest to \( x \), and let \( b \in S_{\text{rad}(G)+1}(x, y) \). Then, \( d(y, u) \leq \text{rad}(G) \) as \( d(y, c) \leq \text{rad}(G) + 1 \). If \( d(x, u) \geq \text{rad}(G) + 1 \), then vertices \( b \) and \( c \) are connected in \( N^\text{rad}(G)(x) \) and therefore, by Proposition \ref{prop:shortest-path} iii, share common neighbors in \( S \). So, \( c \in D(S, 1) \), a contradiction. If \( d(x, u) \leq \text{rad}(G) \), then \( d(x, u) = d(y, u) = d(y, c) = 1 = \text{rad}(G) \) as \( d(x, y) = 2 \text{rad}(G) \). Hence, \( u \in S \cap N(c) \), a contradiction.

Lemma 11. Let \( G \) be a distance-hereditary graph with \( \text{diam}(G) = 2 \text{rad}(G) - 1 \) and a diametral pair \( x, y \), and let \( A := S_{\text{rad}(G)-1}(x, y) \) and \( B := S_{\text{rad}(G)-1}(y, x) \). Then, \( C(G) \) is a cograph and any edge \( ab \in E \), where \( a \in A \) and \( b \in B \), satisfies \( C(G) \subseteq D(\{a, b\}, 1) \). Moreover, there is a vertex \( a \in A \cap C(G) \) and a vertex \( b \in B \cap C(G) \).

Proof. By Proposition \ref{prop:shortest-path} slices \( A \) and \( B \) are joined. Consider any \( c \in C(G) \) and edge \( ab \in E \) for any vertex pair \( a \in A \) and \( b \in B \). As \( d(x, y) = 2 \text{rad}(G) - 1 \) and \( e(c) = \text{rad}(G) \), \( d(x, c) < \text{rad}(G) \) implies \( d(x, c) = \text{rad}(G) - 1 \) and \( d(y, c) = \text{rad}(G) \). Hence, \( c \in A \). By symmetry, if \( d(y, c) < \text{rad}(G) \) then \( c \in B \). Assume now that \( d(x, c) = d(y, c) = \text{rad}(G) \). Then, \( b, c \in N^{\text{rad}(G)}(x) \). Consider vertex \( u \neq c \) on a shortest path \( P(c, y) \) closest to \( x \). We have \( d(y, u) \leq \text{rad}(G) - 1 \) by the choice of \( u \). If \( d(x, u) \leq \text{rad}(G) - 1 \), then
$d(x, y) \leq d(x, u) + d(u, y) \leq 2\text{rad}(G) - 2$, a contradiction. Thus, $d(x, u) \geq \text{rad}(G)$. Vertices $b$ and $c$ are connected in $V \setminus N^{\text{rad}(G)-1}(x)$ by shortest paths to $y$. By Proposition $[\text{iii}]$, $b$ and $c$ share neighbors in $A$. Therefore, $a \in N(c)$ and, by symmetry, $b \in N(c)$. Hence, any central vertex either belongs to $A \cup B$ or is universal to $A \cup B$. Thus, $C(G) \subseteq D \{a, b\}, 1$. Additionally, since any pair of vertices in $C(G)$ is at most distance 2 apart, $C(G)$ is a co-graph.

We now show the existence of vertices $a \in A$ and $b \in B$ such that $a, b \in C(G)$. Consider the family of disks $D(v, r(v))$ centered at each vertex $v$, where $r(v) = 1$ for all central vertices $v \in C(G)$ and $r(v) = \text{rad}(G) - 1$ for all others. Any two non-central vertices $u, v \in V \setminus C(G)$ have distance no more than the diameter, therefore $d(u, v) \leq 2\text{rad}(G) - 1 = r(u) + r(v) + 1$. Any two central vertices $u, v \in C(G)$ have distance no more than the diameter of the center, therefore $d(u, v) \leq 2 = r(u) + r(v)$. By definition, any central vertex $u \in C(G)$ sees any vertex $v \in V$ within $\text{rad}(G)$, and therefore $d(u, v) \leq \text{rad}(G) = r(u) + r(v)$. Hence, by Proposition $[2]$ there is an $r$-dominating clique $K$. As any non-central vertex has distance $\text{rad}(G) - 1$ to a vertex of $K$, we have $K \subseteq C(G)$. Let $a \in K$ be closest to $x$ and let $b \in K$ be closest to $y$. By distance requirements, $ab$ must be an edge with $d(x, a) = \text{rad}(G) - 1$ and $d(b, y) = \text{rad}(G) - 1$. Therefore, $a \in A$ and $b \in B$.

**Corollary 3.** Let $G$ be a distance-hereditary graph. If $\text{diam}(G) \geq 2\text{rad}(G) - 1$ then $C(G)$ is a co-graph.

It remains now to investigate the case when $\text{diam}(G) = 2\text{rad}(G) - 2$.

**Lemma 12.** Let $G$ be a distance-hereditary graph with $\text{diam}(G) = 2\text{rad}(G) - 2$ and let $M \subseteq C(G)$. If all $u, v \in M$ satisfy $d(u, v) = 2$, then there is a vertex $c \in C(G)$ that is universal to $M$.

**Proof.** Consider a disk of radius 1 centered at each $s \in M$ and a disk of radius $\text{rad}(G) - 1$ centered at each $v \in V \setminus M$. Any two vertices $u, v \in V \setminus M$ satisfy $d(u, v) \leq \text{diam}(G) = 2\text{rad}(G) - 2 = r(u) + r(v)$. Since $M \subseteq C(G)$, any $s \in M$ and $v \in V$ satisfy $d(s, v) \leq \text{rad}(G) = r(s) + r(v)$. By assumption, any two $s, t \in M$ satisfy $d(s, t) = 2 = r(s) + r(t)$. By Proposition $[2]$ there is a single vertex or a pair of adjacent vertices $r$-dominating $G$. In the former case, we are done. Thus, consider the case when there is an $r$-dominating edge $ab \in E$. We have $a, b \in C(G)$ since all vertices $v \in V$ see some end-vertex of edge $ab$ within $\text{rad}(G) - 1$. We claim that at least one end-vertex of edge $ab$ is universal to $M$. By contradiction assume there exist vertices $u, v \in M$ which are adjacent to opposite ends of edge $ab$. Without loss of generality, let $u \in N(a) \setminus N(b)$ and $v \in N(b) \setminus N(a)$. Since $d(u, v) = 2$, we get in $G$ either an induced $C_5$, or an induced house, or an induced gem. A contradiction obtained proves the lemma.

**Lemma 13.** Let $G$ be a distance-hereditary graph with $\text{diam}(G) = 2\text{rad}(G) - 2$ and a diametral pair $x, y$, and let $A := S_{\text{rad}(G)-2}(x, y)$, $S := S_{\text{rad}(G)-1}(x, y)$, and $B := S_{\text{rad}(G)-2}(y, x)$. Then $A \cup B \cup (S \cap C(G)) \subseteq C(G)$ and there is a vertex $c \in S \cap C(G)$. Moreover, $C(G) \subseteq D(S \cap C(G), 1)$.

**Proof.** Consider any $s \in A$. By Lemma $[\text{iii}]$, $c(s) \leq \max\{\max\{d(s, x), d(s, y)\}, \min\{d(s, x), d(s, y)\} + 2\} = d(s, y) = \text{rad}(G)$. Hence, $s \in C(G)$ and, by symmetry, $A \cup B \subseteq C(G)$.

By contradiction assume there is no central vertex in $S$. Let $w$ be a vertex from $S$ minimizing $|F(w)|$, and let $v \in F(w)$. Since $w \notin C(G)$, $d(w, v) \geq \text{rad}(G) + 1$. Denote by $s_1 \in S_1(w, x)$ and $s_2 \in S_1(w, y)$ two adjacent to $w$ vertices on a shortest path from $x$ to $y$. As previously established, both $s_1$ and $s_2$ are central since they belong to $A$ and $B$, respectively. Thus, $d(s_1, v) \leq \text{rad}(G)$ and $d(s_2, v) \leq \text{rad}(G)$. Therefore, $d(s_1, v) = d(s_2, v) = \text{rad}(G)$ and $d(w, v) = \text{rad}(G) + 1$. Since $s_1, s_2 \notin N^{\text{rad}(G)}(v)$ and are connected via $w$ in the graph $< V \setminus N^{\text{rad}(G)-1}(v) >$, by Proposition $[\text{iii}]$ there is a vertex $t \in N^{\text{rad}(G)-1}(v)$ adjacent to $s_1$ and $s_2$. As $t \in S_{\text{rad}(G)-1}(y, x)$ and $v \in F(w) \setminus F(t)$, by minimality of $|F(w)|$, there is a vertex $u \in F(t) \setminus F(w)$.

Our assumption, $u \notin C(G)$, i.e., $d(u, t) \geq \text{rad}(G) + 1$. Consider the 4-point condition on vertices $t, w, v, u$. We have $d(v, u) + d(w, t) = d(v, u) + 2, d(v, t) + d(u, w) \leq \text{rad}(G) - 1 + \text{rad}(G)$, and $d(v, w) + d(u, t) \geq \text{rad}(G) + 1 + \text{rad}(G) + 1 = 2\text{rad}(G) + 2$. Since the latter two sums differ by more than $2$, necessarily, $d(v, u) + d(w, t) = d(v, w) + d(u, t)$. Hence, $d(v, u) = d(v, w) + d(u, t) - d(w, t) \geq 2\text{rad}(G)$, a contradiction with $\text{diam}(G) = 2\text{rad}(G) - 2$. Thus, there is a vertex $c \in S \cap C(G)$.
Next, we establish an intermediate claim that $C(G) \subseteq D(M, 1)$, where $M := A \cup B \cup (S \cap C(G))$. By contradiction suppose there is a vertex $w \in C(G)$ with $w \notin D(M, 1)$. Consider arbitrary vertices $a \in A$ and $b \in B$. Thus, $d(a, b) = 2$, and $a, b \in M$ and, by the choice of $w$, necessarily $d(w, a) \geq 2$ and $d(w, b) \geq 2$.

If $d(w, a) = d(w, b) = 2$ then, by Lemma 12, applied to the set $\{w, a, b\}$, there is a central vertex $u$ adjacent to $w, a, b$. In this case $u \in S \cap C(G)$ and therefore $u \in M$, contradicting with $w \notin D(M, 1)$. Assume now, without loss of generality, that $d(w, a) \geq 3$. Consider the 4-point condition on vertices $w, x, y, a$.

We have $d(x, y) + d(w, a) \geq 2 \text{rad}(G) + 1$ is the largest sum since $d(x, w) + d(y, a) \leq 2 \text{rad}(G)$ and $d(x, a) + d(w, y) \leq 2 \text{rad}(G) - 2$. Since the smaller two sums must be equal and differ from the larger one by at most two, inequality $d(x, y) + d(w, a) \geq d(x, a) + d(y, w) + 3$ gives a contradiction which establishes the claim that $C(G) \subseteq D(M, 1)$.

Finally, we establish that $C(G) \subseteq D(S \cap C(G), 1)$. By contradiction assume there is a central vertex $w \in C(G) \setminus S$ which is not adjacent to any vertex of $S \cap C(G)$. By the previous claim, $w$ is adjacent to some vertex from $A$ or $B$. Without loss of generality, let $wa \in E$ for some vertex $a \in A$. Since $d(a, y) = \text{rad}(G)$, necessarily, $d(w, y) = \text{rad}(G) - 1$. If $d(w, y) = \text{rad}(G) - 1$ then $w \in S$, a contradiction. So, $d(w, y) = \text{rad}(G)$ must hold. Now, vertices $w$ and $a$ are connected in $V \setminus N^{\text{rad}(G)-1}(y)$. By Proposition 11(iii), $N(w) \cap N^{\text{rad}(G)-1}(y) = N(a) \cap N^{\text{rad}(G)-1}(y)$. By Proposition 11, also $S \cap C(G) \subseteq N(a) \cap N^{\text{rad}(G)-1}(y)$. Thus, $w$ is universal to $S \cap C(G)$, a contradiction.

We are ready to prove the main result of this section.

**Theorem 2.** Let $H$ be a subgraph of a distance-hereditary graph $G$ induced by $C(G)$. Either (i) $H$ is a cograph, or (ii) $H$ is a connected distance-hereditary graph with $\text{diam}(H) = 3$ and $C(H)$ is a connected cograph with $\text{rad}(C(H)) = 2$.

Furthermore, any such graph $H$ is the center of some distance-hereditary graph.

**Proof.** If $\text{diam}(G) \geq 2\text{rad}(G) - 1$ then, by Lemma 10 and Lemma 11, $H$ is a cograph. Assume now that $\text{diam}(G) = 2\text{rad}(G) - 2$ and $\text{diam}(H) = 3$ (if $\text{diam}(H) \leq 2$ then, by definition, $H$ is a cograph). Then, $H$ is a connected distance-hereditary graph [19], and so $2\text{rad}(H) - 2 \leq \text{diam}(H) \leq 2\text{rad}(H)$. On one hand, $\text{rad}(H) \geq \lceil(\text{diam}(H)/2)\rceil = 2$. On the other hand, $\text{rad}(H) \leq \lceil(\text{diam}(H) + 2)/2\rceil = 2$. Hence, $\text{rad}(H) = 2$.

So, $H$ is a connected distance-hereditary graph with $\text{diam}(H) = 3$ and $\text{rad}(H) = 2$. Consider the center $C(H)$ of $H$. First we show that $C(H)$ is connected. Let $r(u) = 1$ for each vertex $u \in H$. Then, any pair $u, v \in H$ satisfies $d_H(u, v) \leq \text{diam}(H) = r(u) + r(v) + 1$. By Proposition 2, there is a clique $K$ in $H$ dominating $H$. Since each vertex of $K$ is at most distance $2 = \text{rad}(H)$ from every vertex of $H$, $K \subseteq C(H)$ holds. Moreover, every two vertices of $C(H)$ are connected through vertices of $K \subseteq C(H)$, implying that $C(H)$ is connected in $H$. In distance-hereditary graphs every connected subgraph is isometric. Hence, $C(H)$ is an isometric subgraph of $H$. As $\text{rad}(H) = 2$, every two vertices of $C(H)$ are at distance at most 2 from each other, implying $\text{diam}(C(H)) = 2$. Thus, $C(H)$ is a connected cograph with $\text{rad}(C(H)) \leq 2$.

We will show next that $\text{rad}(C(H)) = 2$, i.e., for any $c \in C(H)$ there is a vertex $z \in C(H)$ such that $cz \notin E$. Consider a vertex $t \in F(c)$ furthest from $c$ in $G$. We have $d_G(c, t) = \text{rad}(G)$. Let $z \in H$ be a closest vertex to $t$ which is central in $G$. Since $\text{diam}(G) = 2\text{rad}(G) - 2$, by Lemma 9 we have $d_G(t, z) = d_G(t, C(G)) = e_G(t) - \text{rad}(G) \leq \text{diam}(G) - \text{rad}(G) = \text{rad}(G) - 2$. Moreover, vertices $z$ and $c$ are not adjacent since $d_G(c, t) = \text{rad}(G)$ and $d_G(t, z) \leq \text{rad}(G) - 2$. But, since $c \in C(H)$, $d_G(c, z) = d_H(c, z) \leq 2$. Therefore, $d_H(c, z) = 2$ and $d_H(t, z) = \text{rad}(G) - 2$. We next establish that $z$ belongs to $C(H)$. By contradiction, assume that there is a vertex $u \in H$ such that $d_H(z, u) > \text{rad}(H) = 2$.

Then, $d_H(z, u) = \text{diam}(H) = 3$ and, by the choice of $c (c \in C(H))$, necessarily $d_H(c, u) \leq 2$. Consider the 4-point condition on vertices $c, u, z, t$. We have that $d(c, t) + d(u, z) = \text{rad}(G) + 3$ is the largest sum since $d(c, z) + d(u, t) \leq \text{rad}(G) + 2$ and $d(c, u) + d(z, t) \leq \text{rad}(G)$. However, $d(c, t) + d(u, z) \geq d(c, u) + d(z, t) + 3$, giving a contradiction since the smaller two sums must be equal and differ from the larger one by at most two. Hence, $z$ belongs to $C(H)$ showing that every $c \in C(H)$ has a non-adjacent vertex $z \in C(H)$. 

Finally, we show that any such graph $H$ is the center of some distance-hereditary graph $G$. In what follows, we refer to Figure 6 for an illustration. If $H$ is a cograph, then one can construct a graph $G$ by simply adding to $H$ four new vertices $x, x^*, y, y^*$. Vertices $x$ and $y$ are universal to $H$, and vertices $x^*, y^*$ are pendant to $x$ and $y$, respectively. Now graph $H$ is the center of $G$ as any vertex $u$ of the cograph $H$ is at most distance 2 to any vertex of $G$, whereas $d_G(x, y^*) = 3$ and $d_G(y, x^*) = 3$. Suppose now that $H$ is a connected distance-hereditary graph with $diam(H) = 3$ and $C(H)$ is a connected cograph with $rad(C(H)) = 2$. One can construct a graph $G$ by adding to $H$ (with $C(H) = \{c_1, c_2, \ldots, c_k\}$) $\ell$ new vertices $x_1, x_2, \ldots, x_\ell$ such that each $x_i$ is pendant to $c_i \in C(H)$. Each $c_i \in C(H)$ has $d_G(c_i, u) \leq 2$ for all $u \in H$. Since $rad(C(H)) = 2$, each $c_i$ has a non-adjacent vertex $c_k \in C(H)$, and therefore $d_G(c_i, x_k) = 3$. Any vertex $u \in H \setminus C(H)$ has a vertex $v \in H \setminus C(H)$, for which $d_G(u, v) = 3$. Furthermore, any such $u$ satisfies $d_G(u, x_i) = d_G(u, c_i) + 1 \leq 3$ for each $x_i$. Since $rad(C(H)) = 2$, for any pendant $x_i$, vertex $c_i$ has a non-adjacent vertex $c_k \in C(H)$ and therefore $d_G(x_i, x_k) = 4$. Hence, $H$ is the center of $G$. 

\[ \square \]

Figure 6: Any cograph $H$ (left), and any connected distance-hereditary graph $H$ with diameter 3 where $C(H)$ is a connected cograph with radius 2 (right), is the center of some distance hereditary graph.

7 Eccentricity approximation

One common approach to approximating eccentricities in a graph $G$ is via an eccentricity $k$-approximating spanning tree $T$ [4,8,10,16], i.e., a spanning tree $T$ of $G$ such that $e_T(v) - e_G(v) \leq k$ holds for each vertex $v$ of $G$. Note that every additive tree $k$-spanner (a spanning tree $T$ of $G$ such that $d_T(x, y) \leq d_G(x, y) + k$ holds for every vertex pair $x, y$) is eccentricity $k$-approximating. However, there are graph families which do not admit any additive tree $k$-spanners and yet they have very good eccentricity approximating spanning trees. For example, for every $k$ there is a chordal graph without an additive tree $k$-spanner, though every chordal graph has an eccentricity 2-approximating spanning tree [10,16] computable in linear time [3]. Introduction of eccentricity approximating spanning trees is an attempt to weaken the restriction of additive tree spanners and instead closely approximate only distances to most distant vertices, the eccentricities. This is fruitful especially for those graphs for which additive tree $k$-spanners with small $k$ do not exist. The situation with distance-hereditary graphs is different. They have additive tree 2-spanners [15] and therefore eccentricity 2-approximating spanning trees. Furthermore, in general, the additive error 2 in an eccentricity approximating spanning tree cannot be improved. A distance-hereditary graph in Figure 7 has no eccentricity 1-approximating spanning tree. Consider any edge $uv$ of the inner $C_4$ which is not present in $T$; either $e_T(u) = e_G(u) + 2$ or $e_T(v) = e_G(v) + 2$.

\[ \square \]

Figure 7: A distance-hereditary graph in which every spanning tree is eccentricity 2-approximating.
In this section, we provide two approaches to improve eccentricity approximation in distance-hereditary graphs sans spanners.

7.1 Approximation via distances to a select few vertices

We obtain an approximation of all eccentricities in linear time using distances to a select few vertices. Our algorithm uses several consequences of results from earlier sections.

Corollary 4. Let \( x, y \) be a mutually distant pair of a distance-hereditary graph \( G \) and let \( c \in V \).

(i) If \( |d(x, c) - d(y, c)| \geq 2 \), then \( e(c) = \max\{d(x, c), d(y, c)\} \).

(ii) If \( |d(x, c) - d(y, c)| = 1 \) or \( c \in C(G) \), then \( \max\{d(x, c), d(y, c)\} \leq e(c) \leq \max\{d(x, c), d(y, c)\} + 1 \).

Proof. By Lemma 6, \( \max\{d(x, c), d(y, c)\} \leq e(c) \leq \max\{d(x, c), d(y, c)\}, \min\{d(x, c), d(y, c)\} + 2 \). If \( |d(x, c) - d(y, c)| \geq 2 \), then \( \max\{d(x, c), d(y, c)\} \geq \min\{d(x, c), d(y, c)\} + 2 \) and therefore \( e(c) = \max\{d(x, c), d(y, c)\} \). If \( |d(x, c) - d(y, c)| = 1 \), then \( e(c) \leq \max\{d(x, c), d(y, c)\} + 1 \). By contradiction assume now that \( c \in C(G) \), \( d(x, c) = d(y, c) \) and \( e(c) = d(x, c) + 2 \). By Proposition 8 we have \( d(x, y) = e(x) \geq 2 \text{rad}(G) - 3 \). By the triangle inequality, \( d(x, y) \leq d(x, c) + d(c, y) = 2(\text{rad}(G) - 2) = 2 \text{rad}(G) - 4 \), a contradiction.

In what follows, we analyze deeper the case when \( d(c, x) = d(c, y) \). As graph on Figure 5 showed, in this case, \( e(c) = \max\{d(x, c), d(y, c)\} + 2 \) may happen. However, we demonstrate that it happens not very often. First we show that if \( d(x, y) \) is odd, then still \( e(c) \leq \max\{d(x, c), d(y, c)\} + 1 \). For this we will need one auxiliary lemma.

Lemma 14. Let \( G \) be a distance-hereditary graph, and let \( x, y, c, s \in V \). If \( d(c, x) = d(c, y) \) and \( d(x, y) = 2k + 1 \) for some integer \( k \), then all vertices \( s \in S_k(x, y) \) satisfy \( d(c, x) = d(c, s) + k \).

Proof. Let \( s \in S_k(x, y) \) and \( d(c, x) = d(c, y) \). necessarily, \( d(s, y) = k + 1 \). Consider the 4-point condition on vertices \( x, y, c, s \). We have \( d(c, s) + d(x, y) = d(c, s) + 2k + 1 \), \( d(x, c) + d(s, y) = d(c, x) + k + 1 \), and \( d(c, y) + d(x, s) = d(c, x) + k \). Since at least two sums must be equal and the latter two sums are not, we consider the two remaining cases. If \( d(c, s) + d(x, y) = d(c, x) + d(s, y) \), then \( d(c, x) = d(c, s) + (2k + 1) - (k + 1) = d(c, s) + k \), and we are done. If \( d(c, s) + d(x, y) = d(c, y) + d(x, s) \), then \( d(c, x) = d(c, y) = d(c, s) + (2k + 1) - k = d(c, s) + k + 1 \). However, by the triangle inequality, \( d(c, x) \leq d(c, s) + d(s, x) = d(c, s) + k \), a contradiction.

Next lemma handles the case when \( d(x, y) \) is odd.

Lemma 15. Let \( x, y \) be a mutually distant pair of a distance-hereditary graph \( G \). If \( d(x, y) \) is odd, then any vertex \( c \in V \) has \( e(c) \leq \max\{d(x, c), d(y, c)\} + 1 \).

Proof. Let \( d(x, y) = 2k + 1 \) for some integer \( k \). By contradiction assume that \( e(c) = \max\{d(x, c), d(y, c)\} + 2 \). Consider a vertex \( v \in F(c) \). By Lemma 6 when \( e(c) = \max\{d(x, c), d(y, c)\} + 2 \), we have \( d(c, x) = d(c, y) = e(c) - 2 \), and \( d(x, y) = d(x, v) = d(y, v) \). Let \( s \in S_k(x, y) \). By Lemma 14 applied to vertex \( c \) and to vertex \( v \), we have \( d(c, x) = d(c, s) + k \) and \( d(v, x) = d(v, s) + k \). By the triangle inequality, \( e(c) = d(c, v) \leq d(c, s) + d(s, v) = (d(c, x) - k) + (d(v, x) - k) = d(c, x) + d(v, x) - 2k = d(c, x) + d(x, y) - 2k = d(c, x) + 1 \), contradicting with \( e(c) = d(c, x) + 2 \).

By Proposition 8 we have \( d(x, y) = e(x) \geq 2 \text{rad}(G) - 3 \). As Lemma 15 covers the cases \( d(x, y) = 2 \text{rad}(G) - 3 \) and \( d(x, y) = 2 \text{rad}(G) - 1 \), we consider next two remaining even values for \( d(x, y) \) (\( 2 \text{rad}(G) - 2 \) and \( 2 \text{rad}(G) \)). We will need an auxiliary lemma.

Lemma 16. Let \( G \) be a distance-hereditary graph where \( c, x, y \in V \) and \( d(x, y) = 2r - 2p \) for \( r := \text{rad}(G) \) and an arbitrary integer \( p \). If \( d(c, x) = d(c, y) \), then any \( s \in S_{r-p-1}(x, y) \) satisfies \( d(c, x) = d(c, s) + r - p - 1 \).
Proof. Let $s \in S_{r-1}(x,y)$ and so $d(s,y) = r - p + 1$. Consider the 4-point condition on vertices $x, y, c, s$. We have $d(c,s) + d(x,y) = d(c,s) + 2r - 2p$, $d(c,x) + d(s,y) = d(c,x) + r - p + 1$, and $d(c,y) + d(x,s) = d(c,x) + r - p - 1$. Since at least two sums must be equal and the latter two sums are not, we consider the two remaining cases. If $d(c,s) + d(x,y) = d(c,x) + d(s,y)$, then $d(c,x) = d(c,s) + (2r - 2p) - (r - p + 1) = d(c,s) + r - p - 1$, and we are done. If $d(c,s) + d(x,y) = d(c,y) + d(x,s)$, then $d(c,x) = d(c,s) + (2r - 2p) - (r - p - 1) = d(c,s) + r - p + 1$. However, by the triangle inequality, $d(c,x) \leq d(c,s) + d(s,x) = d(c,s) + r - p - 1$, a contradiction. \hfill \Box

Corollary 5. Let $G$ be a distance-hereditary graph with mutually distant pair $x, y$ and $d(x,y) = 2r - 2p$ for $r := \text{rad}(G)$ and $p \in \{0,1\}$. If for a vertex $c \in V$, $e(c) = \max\{d(x,c), d(y,c)\} + 2$ holds, then any $s \in S_{r-1}(x,y)$ satisfies $d(c,x) = d(c,s) + r - p - 1$.

Proof. The result follows from Lemma 6 and Lemma 16. \hfill \Box

Next lemma concerns the case when $d(x,y)$ is even.

Lemma 17. Let $x, y$ be a mutually distant pair of a distance-hereditary graph $G$ and $c \in V$ with $d(c,x) = d(c,y)$.

(i) Let $d(x,y) = 2\text{rad}(G) - 2$ and $s$ be an arbitrary vertex from $S_{\text{rad}(G)} - 2(x,y)$. Then, $e(c) = d(c,x) + 2$ if and only if $e(c) = d(c,s) + \text{rad}(G)$.

(ii) Let $d(x,y) = 2\text{rad}(G)$ and $s$ be an arbitrary vertex from $S_{\text{rad}(G)} - 1(x,y)$. Then, $e(c) = d(c,x) + 2$ if and only if $e(c) = d(c,s) + \text{rad}(G) + 1$.

Proof. Corollary 5 yields the 'only if’ direction for both statements.

By Lemma 6, we have that $e(c) \leq \max\{d(x,c), d(y,c)\} + 2 = d(x,c) + 2$. So, for the 'if' directions, it remains only to show that $e(c) \geq d(x,c) + 2$ for either value of $d(x,y)$. First, assume that $d(x,y) = 2\text{rad}(G) - 2$ and that $e(c) = d(c,s) + \text{rad}(G)$ for $s \in S_{\text{rad}(G)} - 2(x,y)$. Then, $e(c) - 2 = d(c,s) + \text{rad}(G) - 2 = d(c,s) + d(x,s) \geq d(x,c)$. Therefore, $e(c) \geq d(x,c) + 2$. Now, assume that $d(x,y) = 2\text{rad}(G)$ and that $e(c) = d(c,s) + \text{rad}(G) + 1$ for $s \in S_{\text{rad}(G)} - 1(x,y)$. Then, $e(c) - 2 = d(c,s) + \text{rad}(G) - 1 = d(c,s) + d(x,s) \geq d(x,c)$. Therefore, $e(c) \geq d(x,c) + 2$. \hfill \Box

We are ready to describe our linear time additive 2-approximation of all eccentricities in distance-hereditary graphs. Recall that finding the radius and computing a mutually distant pair of vertices can be done in linear time for distance-hereditary graphs. \cite{7}. Let $x, y$ be mutually distant vertices of $G$ and denote $r := \text{rad}(G)$. When $d(x,y)$ is even, we use an additional vertex $s \in I(x,y)$ to improve eccentricity approximations. If $d(x,y) = 2r - 2$, we pick an arbitrary vertex $s \in S_{r-2}(x,y)$. If $d(x,y) = 2r$, we pick instead an arbitrary vertex $s \in S_{r-1}(x,y)$. We assign to each vertex $c \in V$ an approximate eccentricity $\hat{e}(c)$ as follows.

$$
\hat{e}(c) = \max\{d(x,c), d(y,c)\} + \begin{cases} 
0, & \text{if } |d(x,c) - d(y,c)| \geq 2, \\
1, & \text{if } |d(x,c) - d(y,c)| = 1 \text{ or } d(x,y) \text{ is odd}, \\
1, & \text{if } d(c,x) = d(c,y), d(x,y) = 2r - 2, \text{ and } d(s,c) + r \neq d(x,c) + 2, \\
1, & \text{if } d(c,x) = d(c,y), d(x,y) = 2r, \text{ and } d(s,c) + r + 1 \neq d(x,c) + 2, \\
2, & \text{otherwise}.
\end{cases}
$$

Theorem 3. There is a linear time algorithm to approximate all eccentricities of a distance-hereditary graph within additive one-sided error 2, that is, $e(c) \leq \hat{e}(c) \leq e(c) + 2$ holds for every $c \in V$.

Proof. We use the equation described above. A constant number of breadth-first search calls obtains distances to $x, y$, and if necessary to a distinguished vertex $s$ on the interval $I(x,y)$. The run time is linear as all remaining operations are constant time comparisons for each vertex $c$ before setting an approximate
eccentricity \( \hat{e}(c) \). In all cases, by Lemma 6, \( \max\{d(x, c), d(y, c)\} \leq e(c) \leq \max\{d(x, c), d(y, c)\} + 2 \). The algorithm improves the additive error in all situations where the upper bound is known to be smaller. If \( |d(x, c) - d(y, c)| \geq 2 \) then \( \hat{e}(c) \) is the exact eccentricity, by Corollary 4. If \( |d(x, c) - d(y, c)| = 1 \) or \( d(x, y) \) is odd then \( \hat{e}(c) \) is at most one unit off from the exact eccentricity, by Corollary 4 and Lemma 15. The same error we have if \( d(x, y) = 2r - 2 \) and \( d(s, c) + r \neq d(x, c) + 2 \) or if \( d(x, y) = 2r \) and \( d(s, c) + r + 1 \neq d(x, c) + 2 \), by Lemma 17.

We have tried to get additive 1-approximations of all eccentricities via distances to a constant number of specially selected vertices, but did not succeed. The case when \( d(c, x) = d(c, y) \) and \( d(x, y) \) is even turned out to be tough. We leave this as an open question. Does there exist a small set of vertices in a distance-hereditary graph \( G \) such that the distances from an arbitrary vertex \( c \) to them define the eccentricity of \( c \) up-to an additive error of at most 1? In the next Subsection 7.2 we will succeed in getting additive 1-approximations by using distances to a specially selected central set \( S \subseteq C(G) \). The set \( S \) could be large, nevertheless, we show that it can be found in total linear time.

### 7.2 1-approximation via distances to a central set

As we have mentioned earlier, if the vertices of \( C(G) \) and \( C^1(G) \) are given, then exact eccentricities can be found in total linear time using the eccentricity equation given in Lemma 9. One can compute the radius \( \text{rad}(G) \) in linear time and then perform a breadth-first search rooted at \( C^1(G) \) to obtain the required distances. We do not know how to compute in total linear time the entire set \( C(G) \) and the entire set \( C^1(G) \). Although linear time algorithms for computing a central vertex, the radius, a diametral pair and the diameter of a distance-hereditary graph are known in literature [7, 11], there are no algorithms that compute even the set \( C(G) \) in total linear time.

In this subsection, we obtain linear time additive 1-approximations for all eccentricities in the following manner. If \( \text{diam}(G) < 2\text{rad}(G) \), we obtain a sufficient central subset \( S \subseteq C(G) \) which satisfies \( C(G) \subseteq D(S, 1) \). By the triangle inequality, one obtains \( e(v) \leq d(v, S) + \text{rad}(G) \) and \( d(v, S) \leq d(v, C(G)) + 1 \). Combined with Lemma 9 we have \( e(v) = d(v, C(G)) + \text{rad}(G) \geq d(v, S) + \text{rad}(G) - 1 \). Hence, \( \hat{e}(v) = d(v, S) + \text{rad}(G) \) yields a linear time additive 1-approximation when \( \text{diam}(G) < 2\text{rad}(G) \) since \( e(v) \leq \hat{e}(v) \leq e(v) + 1 \). If instead \( \text{diam}(G) = 2\text{rad}(G) \), we obtain all central vertices \( C(G) \) in linear time, a result also of independent interest. By the triangle inequality, \( e(v) \leq d(v, C(G)) + \text{rad}(G) \) for any vertex \( v \). Let \( w \in C^1(G) \) be a closest vertex to \( v \) in \( C^1(G) \). By Theorem 1 and the triangle inequality, \( d(v, C(G)) \leq d(v, w) + d(w, C(G)) \leq d(v, w) + 2 \). Then, combined with Lemma 9 one obtains \( e(v) = d(v, C^1(G)) + \text{rad}(G) + 1 = d(v, w) + \text{rad}(G) + 1 \geq d(v, C(G)) + \text{rad}(G) - 1 \). Hence, \( \hat{e}(v) = d(v, C(G)) + \text{rad}(G) \) yields a linear time additive 1-approximation when \( \text{diam}(G) = 2\text{rad}(G) \) since \( e(v) \leq \hat{e}(v) \leq e(v) + 1 \).

Our general approach is to consider a special set \( S \) known to contain some central vertices, then prune from \( S \) any non-central vertex. We denote by \( Pr(t, S) = \{ u \in S : d(t, S) = d(t, u) \} \) the projection of a vertex \( t \) onto the set \( S \), and by \( p(t) \) an arbitrary vertex of \( Pr(t, S) \). We define a gate of \( t \) to the projection \( Pr(t, S) \) to be a vertex \( g(t) \in V \) for which every vertex \( u \in Pr(t, S) \) satisfies \( g(t) \in N(u) \) and \( d(t, u) = d(t, g(t)) + d(g(t), u) \), that is, \( g(t) \) is on a shortest path from \( t \) to any vertex \( u \in Pr(t, S) \) and is adjacent to every vertex \( u \in Pr(t, S) \). We will prove that such gates always exist for our special set \( S \). Next, we consider the layers \( L_1, \ldots, L_p \) produced by a breadth-first search rooted at \( S \), where \( L_k = \{ v \in V : d(v, S) = k \} \). We define a set \( T \) which contains a furthest vertex from each non-central vertex of \( S \), that is, each \( w \in S \setminus C(G) \) has a vertex \( t \in F(w) \cap T \). For each \( t \in T \) we choose a gate \( g(t) \in L_1 \) and arbitrary projection vertex \( p(t) \). Clearly, if \( t \in T \cap L_k \) then \( d(t, p(t)) = k \) and \( d(t, g(t)) = k - 1 \). We characterize the non-central vertices of \( S \) as those which, for some \( t \), have no edges to \( g(t) \) or \( p(t) \), depending on the diameter of the input graph and also the layer to which \( t \) belongs.

The following intermediate lemma will be used to show the existence of \( g(t) \) for every \( t \) in each particular case of \( S \).
Lemma 18. Let $G$ be a distance-hereditary graph, $S \subseteq V$ be a vertex set, and $t$ be an arbitrary vertex from $\mathcal{L}_k$ of BFS($S$). If there is a vertex $u \in V$ with $d(t, u) \geq k$ and $Pr(t, S) \subseteq N[u]$, then any vertex of $S_1(p(t), t) \subseteq \mathcal{L}_1$ is a gate of $t$ to $Pr(t, S)$.

Proof. Let $t \in \mathcal{L}_k$ and $u \in V$ be two vertices such that $d(t, u) \geq k$ and $Pr(t, S) \subseteq N[u]$. If $|Pr(t, S)| = 1$, then any vertex $g(t) \in S_1(p(t), t)$ satisfies the desired property. Suppose now that $|Pr(t, S)| \geq 2$. Then, any two vertices $s_1, s_2 \in Pr(t, S)$ belong to $N^k(t)$ and are connected via $u$ in $< V \setminus N^{k-1}(t) >$. By Proposition 4, any vertex $g(t) \in S_1(p(t), t)$ is a gate of $t$ to $Pr(t, S)$.

In the remainder of this section, we use $\mathcal{L}_k$ to denote the $k$th layer of BFS($S$), where the set $S$ itself is defined based on the diameter of the input graph. Let $p(t) \in Pr(t, S)$ be an arbitrary projection vertex for $t$, and let $g(t)$ be a gate of $t$ to $Pr(t, S)$ (its existence we prove separately each time using Lemma 18).

Lemma 19. Let $G$ be a distance-hereditary graph with a diametral pair $x, y$ and let $d(x, y) = 2rad(G) - 2$. Set $A := S_{rad(G) - 2}(x, y)$, $S := S_{rad(G) - 1}(x, y)$, and $B := S_{rad(G) - 2}(y, x)$. Pick two vertices $a \in A$ and $b \in B$ arbitrarily and set $T := \{v \in V : d(v, a) = d(v, b) = rad(G)\}$. A vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap \mathcal{L}_{rad(G) - 1}$ such that $p(t) \notin N(w)$ and $g(t) \notin N(w)$.

Proof. We first claim that a vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap \mathcal{L}_{rad(G) - 1}$ such that $d(w, t) > rad(G)$. The “$\leftarrow$” direction is clear since $e(w) \geq d(w, t) > rad(G)$ excludes $w$ from $C(G)$. We now show “$\rightarrow$” direction. Suppose $w \in S \setminus C(G)$. By Proposition 4 any vertex of $S$, including $w$, is adjacent to $a$ and $b$. By Lemma 13, $a, b \in C(G)$ and so $e(w) = rad(G) + 1$. We now show that any $t \in F(w)$ satisfies the desired properties. Since $e(a) = e(b) = rad(G)$, $t$ is furthest from vertices $a$ and $b$. Thus, $d(a, t) = d(b, t) = rad(G)$ and so $t \in T$. As $a, b \in S_{rad(G)}(t, w)$, by Proposition 4 any vertex $u \in S_{rad(G) - 1}(t, w)$ is adjacent to both $a$ and $b$, implying $u \in S$. Hence, $d(t, S) \leq d(t, u) = rad(G) - 1$. Moreover, the existence of any vertex $s$ in $S$ with $d(t, s) < rad(G) - 1$ would contradict with $d(a, t) = rad(G)$ as $a$ and $s$ are adjacent. Thus, $t \in \mathcal{L}_{rad(G) - 1}$, establishing the claim.

We next claim that, for $w \in S$ and $t \in T \cap \mathcal{L}_{rad(G) - 1}$, $d(w, t) \leq rad(G)$ holds if and only if $p(t) \in N(w)$ or $g(t) \in N(w)$. We note that the existence of $g(t)$ is given by Lemma 18 as, by assumption, $d(t, a) = rad(G)$ and, from Proposition 4, vertex $a$ is universal to $S$. The “$\leftarrow$” direction of the claim is clear by the triangle inequality since $d(w, t) \leq d(w, p(t)) + d(p(t), t) = 1 + (rad(G) - 1)$, if $p(t) \in N(w)$, and $d(w, t) \leq d(w, g(t)) + d(g(t), t) = 1 + (rad(G) - 2)$, if $g(t) \in N(w)$. We now show the “$\rightarrow$” direction. Assume that $d(w, t) \leq rad(G)$ but still $w$ is adjacent neither to vertex $p(t)$ nor to vertex $g(t)$. Since vertex $a$ is a common neighbor of vertices $w$ and $p(t)$ and $d(t, a) = rad(G)$, either $d(w, t) = rad(G) - 1$ or $d(w, t) = rad(G)$. If $d(w, t) = rad(G) - 1$ then $w \in Pr(t, S)$ and hence $w$ must be adjacent to $g(t)$, a contradiction. If $d(w, t) = rad(G)$ then adjacent vertices $a$ and $w$ both belong to $N^{rad(G)}(t)$ and are connected in $< V \setminus N^{rad(G) - 1}(t) >$. By Proposition 4, $N(a) \cap N^{rad(G) - 1}(t) = N(w) \cap N^{rad(G) - 1}(t)$. Since $p(t) \in N(a) \cap N^{rad(G) - 1}(t)$, $w$ is also adjacent to $p(t)$, a contradiction.

Thus, for each $w \in S$, by the first claim, $w \notin C(G)$ if and only if there is a vertex $t \in T \cap \mathcal{L}_{rad(G) - 1}$ with $d(w, t) > rad(G)$. By the second claim, there is a vertex $t \in T \cap \mathcal{L}_{rad(G) - 1}$ with $d(w, t) > rad(G)$ if and only if $p(t) \notin N(w)$ and $g(t) \notin N(w)$.

Lemma 20. Let $G$ be a distance-hereditary graph with a diametral pair $x, y$ and let $d(x, y) = 2rad(G) - 1$. Set $S := S_{rad(G) - 1}(x, y)$, $B := S_{rad(G) - 1}(y, x)$. Pick an arbitrary vertex $b \in B$ and set $T := \{v \in V : d(v, b) = rad(G)\}$. A vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap \mathcal{L}_{rad(G) - 1}$ such that $p(t) \notin N(w)$ and $g(t) \notin N(w)$.

Proof. We first claim that a vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap \mathcal{L}_{rad(G) - 1}$ such that $d(w, t) > rad(G)$. The “$\leftarrow$” direction is clear since $e(w) \geq d(w, t) > rad(G)$ excludes $w$ from...
$C(G)$. We now show the “$\rightarrow$” direction. Suppose $w \in S \setminus C(G)$, and let $t \in F(w)$. By Lemma 15, $e(w) \leq \max\{d(w,x), d(w,y)\} + 1 = \text{rad}(G) + 1$. Hence $d(w, t) = \text{rad}(G) + 1$. By Lemma 11 there is an edge $v'v \in E$ where $v' \in S \cap C(G)$ and $v \in B \cap C(G)$. By Proposition 4, $v$ is adjacent to every vertex of $S$, including $w$, and so, by distance requirements, $d(v,t) = \text{rad}(G)$, that is, $v$ belongs to some shortest path $P(w,t)$ connecting $w$ with $t$. Since $d(x,w) = \text{rad}(G) - 1$ and $d(x,t) \leq 2\text{rad}(G) - 1$, the path obtained by joining a shortest path $P(x,w)$ (connecting $x$ and $w$) with $P(w,t)$ is not induced. The only possible chord connects vertices $z \in S_1(w)$ and $u \in S_1(w)$. Since $zu \in E$, $z$ belongs to the same slice of $(x,y)$ as vertex $w$, i.e., $z \in S$. Hence, $d(t,S) \leq d(t,z) = \text{rad}(G) - 1$. Moreover, the existence of any vertex $s$ in $S$ with $d(t,s) < \text{rad}(G) - 1$ would contradict with $d(v,t) = \text{rad}(G)$ as $s$ and $v$ are adjacent. Necessarily, $t \in L_{\text{rad}(G)-1}$. Since any vertex $b \in B$, by Proposition 4, $b$ is universal to $S$ and $S$ contains vertices $z$ and $w$, we get $d(t,b) = \text{rad}(G)$ (recall that $d(w,t) = \text{rad}(G) + 1$ and $d(z,t) = \text{rad}(G) - 1$). Thus, $t \in T$, establishing the first claim.

We next claim that, for $w \in S$ and $t \in T \cap L_{\text{rad}(G)-1}$, $d(w,t) \leq \text{rad}(G)$ holds if and only if $p(t) \in N(w)$ or $g(t) \in N(w)$. We note that the existence of $g(t)$ is given by Lemma 13 as, by assumption, $d(b,t) = \text{rad}(G)$ and, from Proposition 4, $b$ is universal to $S$. The “$\leftarrow$” is clear by the triangle inequality since $d(w,t) \leq d(w,p(t)) + d(p(t),t) \leq 1 + (\text{rad}(G) - 1)$, if $p(t) \in N(w)$, and $d(w,t) \leq d(w,g(t)) + d(g(t),t) \leq 1 + (\text{rad}(G) - 2)$, if $g(t) \in N(w)$. We now show the “$\rightarrow$” direction. Assume that $d(w,t) \leq \text{rad}(G)$ but still $w$ is adjacent neither to vertex $p(t)$ nor to vertex $g(t)$. Since $d(t,b) = \text{rad}(G)$ and $wb \in E$, $d(w,t) = \text{rad}(G) - 1$ or $d(w,t) = \text{rad}(G)$ must hold. If $d(w,t) = \text{rad}(G) - 1$, then $w \in Pr(t,S)$ and it must be adjacent to vertex $g(t)$, a contradiction. If $d(w,t) = \text{rad}(G)$, then adjacent vertices $b$ and $w$ both belong to $N_{\text{rad}(G)}(t)$ and are connected in $V \setminus N_{\text{rad}(G)-1}(t)$. Thus, by Proposition 1, they share common neighbors in $N_{\text{rad}(G)-1}(t)$. Since $p(t) \in N(b) \cap N_{\text{rad}(G)-1}(t)$, $w$ is also adjacent to $p(t)$, a contradiction.

Hence, for any $w \in S$, by the first claim, $w \notin C(G)$ if and only if there is a vertex $t \in T \cap L_{\text{rad}(G)-1}$ with $d(w,t) > \text{rad}(G)$. By the second claim, there is such a vertex $t$ if and only if $p(t) \notin N(w)$ and $g(t) \notin N(w)$. 

\begin{lemma}
Let $G$ be a distance-hereditary graph with a diameter pair $x,y$ and let $d(x,y) = 2\text{rad}(G)$. Set $A := S_{\text{rad}(G)-1}(x,y)$, $S := S_{\text{rad}(G)}(x,y)$, and $B := S_{\text{rad}(G)-1}(y,x)$. Pick arbitrary vertices $a \in A$ and $b \in B$ and set $T := \{v \in V : d(v,a) = d(v,b) = \text{rad}(G) \text{ or } d(v,a) = d(v,b) = \text{rad}(G)+1\}$. A vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap L_{\text{rad}(G)-1}$ such that $p(t) \notin N(w)$ and $g(t) \notin N(w)$ or there is a vertex $t \in T \cap L_{\text{rad}(G)}$ such that $g(t) \notin N(w)$.

Proof. We first claim that a vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap L_{\text{rad}(G)-1}$ such that $d(w,t) > \text{rad}(G)$ or there is a vertex $t \in T \cap L_{\text{rad}(G)}$ such that $d(w,t) > \text{rad}(G)$. The “$\leftarrow$” direction is clear since $e(w) \geq d(w,t) > \text{rad}(G)$ excludes $w$ from $C(G)$. We now show the “$\rightarrow$” direction. Suppose $w \in S \setminus C(G)$, and let $t \in F(w)$. By Lemma 13, $C(G) \subseteq S$. By Proposition 4, vertices $a$ and $b$ are adjacent to the entire slice $S$ including vertex $w$. Then, by the triangle inequality, $e(a) \leq d(a,C(G)) + \text{rad}(G) = \text{rad}(G) + 1$ and, by symmetry, $e(b) \leq \text{rad}(G) + 1$, too. Thus, $e(w) \leq d(w,a) + e(a) \leq \text{rad}(G) + 2$. As $a,b,w \notin C(G)$, necessarily, $e(a) = e(b) = \text{rad}(G) + 1 \leq e(w) \leq \text{rad}(G) + 2$. By these requirements, $\text{rad}(G) \leq d(t,a) \leq \text{rad}(G) + 1$ and $\text{rad}(G) \leq d(t,b) \leq \text{rad}(G) + 1$ must hold. Consider first, without loss of generality, that $d(t,a) = d(t,b) + 1 = \text{rad}(G) + 1$. Then, by distance requirements, $d(w,t) = \text{rad}(G) + 1$. By the 4-point condition applied to vertices $w,t,a,b$, we have $d(w,t) + d(a,b) = \text{rad}(G) + 3$, $d(w,a) + d(t,b) = \text{rad}(G) + 1$, and $d(w,b) + d(t,a) = \text{rad}(G) + 2$. Since at least two of the three sums must be equal, a contradiction arises. Therefore, $d(t,a) = d(t,b) = \text{rad}(G) + k$ for $k = 0$ or $k = 1$, that is, $t \in T$. As $a,b \in N_{\text{rad}(G)+k}(t)$ are connected by $w$ in $V \setminus N_{\text{rad}(G)+k-1}(t)$, by Proposition 1, vertices $a,b$ have a common neighbor $u \in N_{\text{rad}(G)+k-1}$. Hence, $u \in S$, too, implying $d(t,S) \leq d(t,u) = \text{rad}(G) + k - 1$. Moreover, the existence of any vertex $s$ in $S$ with $d(t,s) < \text{rad}(G) + k - 1$ would contradict with $d(a,t) = \text{rad}(G) + k$ as $s$ and $a$ are adjacent. Thus, $t \in L_{\text{rad}(G)-1}$ when $k = 0$ and $t \in L_{\text{rad}(G)}$ when $k = 1$, completing the first claim.
Note that, by Lemma 18, each \( t \in L_{\text{rad}(G)} \cup L_{\text{rad}(G)^{-1}} \) has a gate \( g(t) \in L_1 \) to \( Pr(t, S) \) since, by assumption, \( d(t, a) \geq \text{rad}(G) \) and \( a \) is universal to \( S \).

We now claim that, for vertices \( w \in S \) and \( t \in T \cap L_{\text{rad}(G)^{-1}} \), \( d(w, t) \leq \text{rad}(G) \) holds if and only if \( p(t) \in N(w) \) or \( g(t) \in N(w) \). The “\( \leftarrow \) ” direction is clear by the triangle inequality since \( d(p(t), t) = \text{rad}(G) - 1 \) and \( d(g(t), t) = \text{rad}(G) - 2 \). We now show the “\( \rightarrow \) ” direction. Assume that \( d(w, t) \leq \text{rad}(G) \) but still \( w \) is adjacent neither to vertex \( p(t) \) nor to vertex \( g(t) \). Since \( d(p(t), t) = \text{rad}(G) - 1 \), \( d(w, t) \geq \text{rad}(G) - 1 \). If \( d(t, w) = \text{rad}(G) - 1 \), then \( w \in Pr(t, S) \) and it must be adjacent to \( g(t) \), a contradiction. Hence, \( d(t, w) = \text{rad}(G) \). Since \( a \) is adjacent to \( w \) and \( d(t, a) = \text{rad}(G) \), by Proposition 18, \( N(a) \cap N^{\text{rad}(G)^{-1}}(t) = N(w) \cap N^{\text{rad}(G)^{-1}}(t) \). Since \( p(t) \in N(a) \cap N^{\text{rad}(G)^{-1}}(t) \), \( w \) is adjacent to \( p(t) \), a contradiction.

Finally, we claim that, for vertices \( w \in S \) and \( t \in T \cap L_{\text{rad}(G)} \), \( d(w, t) \leq \text{rad}(G) \) if and only if \( N_w \neq \{ \} \). For the other direction, assuming that \( d(w, t) \leq \text{rad}(G) \), we have that \( w \) belongs to the projection \( Pr(t, S) \) and therefore it must be adjacent to gate vertex \( g(t) \).

Hence, for any \( w \in S \), by the first claim, \( w \notin C(G) \) if and only if there is a vertex \( t \in T \cap (L_{\text{rad}(G)^{-1}} \cup L_{\text{rad}(G)}) \) with \( d(w, t) > \text{rad}(G) \). By the second claim, there is a vertex \( t \in T \cap L_{\text{rad}(G)^{-1}} \) with \( d(w, t) > \text{rad}(G) \) if and only if \( p(t) \notin N(w) \) and \( g(t) \notin N(w) \). By the final claim, there is a vertex \( t \in T \cap L_{\text{rad}(G)} \) with \( d(w, t) > \text{rad}(G) \) if and only if \( g(t) \notin N(w) \).

We can now describe our algorithms that find a sufficient central subset \( M \subseteq C(G) \), when \( \text{diam}(G) \leq 2\text{rad}(G) - 1 \), and the entire center \( C(G) \), when \( \text{diam}(G) = 2\text{rad}(G) \), in total linear time.

**Lemma 22.** Let \( G \) be a distance-hereditary graph with \( \text{diam}(G) = 2\text{rad}(G) - 2 \). There is a linear time algorithm that finds a set \( M \subseteq C(G) \) such that \( C(G) \subseteq D(M, 1) \).

**Proof.** Let \( x, y \) be a diametral pair of \( G \) and set \( r := \text{rad}(G) \). The radius \( \text{rad}(G) \) of \( G \) and a diametral pair \( x, y \) can be found in linear time by known algorithms from [7][11]. Let \( A := S_{r-2}(x, y) \), \( B := S_{r-2}(y, x) \), and \( S := S_{r-1}(x, y) \). By Lemma 13, \( A \cup B \subseteq C(G) \) and there is a central vertex in \( S \). Also, by Lemma 13, \( C(G) \subseteq D(S \cap C(G), 1) \). In what remains we describe how to find \( S \cap C(G) \) in linear time.

With \( \text{BFS}(x) \) and \( \text{BFS}(y) \) we find the sets \( A, B, S \). Pick an arbitrary \( a \in A \), \( b \in B \) and perform \( \text{BFS}(a) \) and \( \text{BFS}(b) \), producing the set \( T = \{ v \in V : d(v, a) = d(v, b) = \text{rad}(G) \} \). Perform \( \text{BFS}(S) \) to get layers \( L_1, L_2, \ldots, L_{r-1} \). As a result we get also a BFS\((S)\)-tree with big supernode \( S \) as the root; we will need the ancestor information from that tree. As a vertex \( v \) is added to \( L_i \), record with it the ancestor vertex \( p(v) \in S \) and the ancestor vertex \( g(v) \in L_i \). Clearly, \( v \) is a vertex from \( Pr(v, S) \) and, by Lemma 18, \( g(v) \) is a gate of \( v \) to \( Pr(v, S) \). By Lemma 19, a vertex \( v \in S \) is not central if and only if there is a vertex \( t \in T \cap L_{\text{rad}(G)} \) such that \( p(t) \notin N(w) \) and \( g(t) \notin N(w) \). Hence, the neighborhood of any central vertex in \( S \) must contain \( p(t) \) or \( g(t) \) for every \( t \in T \cap L_{\text{rad}(G)} \). Thus, \( S \cap C(G) = S \cap \bigcap_{t \in T \cap L_{r-1}} [N(p(t)) \cup N(g(t)) \] To complete the proof we show that the necessary set operations can be efficiently computed.

It is clear that, if any two different vertices of \( T \) have the same gate vertex, then their projections are also the same. Let \( G := \{ g_1, g_2, \ldots, g_t \} \) denote the set of unique gate vertices, that is, each \( g_i \) is equal to \( g(t) \) for one or more vertices \( t \in T \cap L_{r-1} \). Let \( P := \{ p_1, \ldots, p_k \} \) denote the set of unique projection vertices, where \( k \leq t \) and each \( p_i \) is equal to \( p(t) \) for one or more vertices \( t \in T \cap L_{r-1} \). In the BFS\((S)\)-tree, each \( g(t) \) has one parent vertex \( p(t) \), however, there may be multiple gate vertices having the same projection vertex as the parent. For each \( p_i \) let \( G(p_i) \subseteq G \) denote the set of gate vertices for which \( p_i \) is the parent. Hence, by the distributive property for sets, one obtains

\[
S \cap C(G) = S \cap \bigcap_{t \in T \cap L_{r-1}} [N(p(t)) \cup N(g(t))] = S \cap \bigcap_{p \in P} [N(p) \cup \bigcap_{g \in G(p)} N(g)].
\]

For a given \( p \in P \), the set \( [N(p) \cup \bigcap_{g \in G(p)} N(g)] \) can be found in time \( O(\text{deg}(p) + \sum_{g \in G(p)} \text{deg}(g)) \).
and its size is at most $\deg(p) + \min_{g \in G(p)} \deg(g)$. Thus, the entire complexity of finding $S \cap C(G)$ is
\[
|S| + O\left(\sum_{p \in \mathcal{P}} [\deg(p) + \sum_{g \in G(p)} \deg(g)]\right) + O\left(\sum_{p \in \mathcal{P}} [\deg(p) + \min_{g \in G(p)} \deg(g)]\right) \leq
\]
\[
|S| + O(|E|) + O(|E|) = O(|S| + |E|).
\]

**Lemma 23.** Let $G$ be a distance-hereditary graph with $\text{diam}(G) = 2\text{rad}(G) - 1$. There is a linear time algorithm that finds a set $M \subseteq C(G)$ such that $C(G) \subseteq D(M, 1)$.

**Proof.** Let $x, y$ be a diametral pair of $G$ and set $r := \text{rad}(G)$. Let $S := S_{\text{rad}(G)-1}(x, y)$, $B := S_{\text{rad}(G)-1}(y, x)$. By Lemma 11 there are vertices $a \in S$ and $b \in B$ such that $a, b \in C(G)$ and $C(G) \subseteq D\{a, b\}, 1$. It is enough to show how to find $S \cap C(G)$ in linear time. By symmetry, one may apply the same method to find $B \cap C(G)$.

With $\text{BFS}(x)$ and $\text{BFS}(y)$ find the sets $S$ and $B$. Pick an arbitrary $b \in B$ and perform $\text{BFS}(b)$ to produce the set $T = \{v \in V : d(v, b) = \text{rad}(G)\}$. The remaining part of the proof is exactly the same as in the proof of Lemma 22 and is omitted. One needs only to replace the reference to Lemma 19 which is for the case when $\text{diam}(G) = 2\text{rad}(G) - 2$, with the reference to Lemma 20 which is for the case when $\text{diam}(G) = 2\text{rad}(G) - 1$.

The case when $\text{diam}(G) = 2\text{rad}(G)$, we formulate as a theorem, since in this case we compute the entire center $C(G)$ in linear time.

**Theorem 4.** Let $G$ be a distance-hereditary graph with $\text{diam}(G) = 2\text{rad}(G)$. There is a linear time algorithm that finds the entire center $C(G)$ of $G$.

**Proof.** The proof is analogous to that of Lemma 22 and Lemma 23. Let $x, y$ be a diametral pair of $G$ and set $r := \text{rad}(G)$. Let $A := S_{\text{rad}(G)-1}(x, y)$, $S := S_{\text{rad}(G)}(x, y)$, and $B := S_{\text{rad}(G)-1}(y, x)$. By Lemma 10 $C(G) \subseteq S$. In what remains we describe how to find $S \cap C(G)$ in linear time.

With $\text{BFS}(x)$ and $\text{BFS}(y)$ we find the sets $A, B, S$. Pick arbitrary vertices $a \in A$, $b \in B$ and perform $\text{BFS}(a)$ and $\text{BFS}(b)$, producing the set $T = \{v \in V : d(v, a) = d(v, b) = \text{rad}(G)\}$ or $d(v, a) = d(v, b) = \text{rad}(G) + 1$. Perform $\text{BFS}(S)$ to get layers $\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_r$ and a $\text{BFS}(S)$-tree for ancestry relations. As a vertex $v$ is added to $\mathcal{L}_i$, record with it the ancestor vertex $p(v) \in S$ and the ancestor vertex $g(v) \in \mathcal{L}_i$. We have that $p(v)$ is a vertex from $Pr(v, S)$ and $g(v)$ is a gate of $v$ to $Pr(v, S)$. By Lemma 21 a vertex $w \in S$ is not central if and only if there is a vertex $t \in T \cap \mathcal{L}_{\text{rad}(G)-1}$ such that $p(t) \notin N(w)$ and $g(t) \notin N(w)$ or there is a vertex $t \in T \cap \mathcal{L}_{\text{rad}(G)}$ such that $g(t) \notin N(w)$. Hence, the neighborhood of any central vertex in $S$ must contain $p(t)$ or $g(t)$, for every $t \in T \cap \mathcal{L}_{\text{rad}(G)-1}$, and it must contain $g(t)$, for every $t \in T \cap \mathcal{L}_{\text{rad}(G)}$.

Let $\mathcal{G} := \{g_1, g_2, ..., g_t\}$ denote the set of unique gate vertices, that is, each $g_i$ is equal to $g(t)$ for one or more vertices $t \in T \cap \mathcal{L}_{r-1}$. Let $\mathcal{P} := \{p_1, ..., p_k\}$ denote the set of unique projection vertices, where $k \leq \ell$ and each $p_i$ is equal to $p(t)$ for one or more vertices $t \in T \cap \mathcal{L}_{r-1}$. In the $\text{BFS}(S)$-tree, each $g(t)$ has one parent $p(t)$, however, there may be multiple gate vertices having the same projection vertex as the parent. For each $p_i$ let $\mathcal{G}(p_i) \subseteq \mathcal{G}$ denote the set of gate vertices for which $p_i$ is the parent. Hence, by the distributive property for sets, one obtains
\[
S \cap C(G) = S \cap \bigcap_{t \in T \cap \mathcal{L}_{r-1}} [N(p(t)) \cup N(g(t))] \cap \bigcap_{t \in T \cap \mathcal{L}_r} N(g(t)) = S \cap \bigcap_{p \in \mathcal{P}} [N(p) \cup \bigcup_{g \in \mathcal{G}(p)} N(g)] \cap \bigcap_{t \in T \cap \mathcal{L}_r} N(g(t)).
\]
As before (see the proof of Lemma 22) $C(G) = S \cap C(G)$ can be found in $O(|E|)$ time. 

We summarize the results in the following theorem.
Theorem 5. There is a linear time algorithm to approximate all eccentricities of a distance-hereditary graph within additive one-sided error 1, that is, for every \( v \in V \), a value \( \hat{e}(v) \) is returned such that \( e(v) \leq \hat{e}(v) \leq e(v) + 1 \).

Proof. Let \( G \) be a distance-hereditary graph. By Corollary 2, \( \text{diam}(G) \geq 2\text{rad}(G) - 2 \).

If \( \text{diam}(G) = 2\text{rad}(G) - 2 \) or \( \text{diam}(G) = 2\text{rad}(G) - 1 \), apply the algorithm described in Lemma 22 or Lemma 23, respectively, to obtain a set \( M \subseteq C(G) \subseteq D(M,1) \). For any \( v \in V \), by the triangle inequality, \( e(v) \leq d(v,M) + \text{rad}(G) \) and \( d(v,M) \leq d(v,C(G)) + 1 \). Combined with Lemma 9, we get \( e(v) = d(v,C(G)) + \text{rad}(G) \geq d(v,M) + \text{rad}(G) - 1 \). Hence, setting \( \hat{e}(v) = d(v,M) + \text{rad}(G) \), we have \( e(v) \leq \hat{e}(v) \leq e(v) + 1 \).

If \( \text{diam}(G) = 2\text{rad}(G) \), apply the algorithm described in Theorem 4 to obtain all central vertices \( C(G) \). By the triangle inequality, \( e(v) \leq d(v,C(G)) + \text{rad}(G) \) for any vertex \( v \). Let \( w \in C^1(G) \) be a closest vertex to \( v \) in \( C^1(G) \). By Theorem 1 and the triangle inequality, \( d(v,C(G)) \leq d(v,w) + d(w,C(G)) \leq d(v,w) + 2 \). Then, combined with Lemma 9 one obtains \( e(v) = d(v,C^1(G)) + \text{rad}(G) + 1 = d(v,w) + \text{rad}(G) + 1 \geq d(v,C(G)) + \text{rad}(G) - 1 \). Hence, setting \( \hat{e}(v) = d(v,C(G)) + \text{rad}(G) \), we get \( e(v) \leq \hat{e}(v) \leq e(v) + 1 \).

8 Concluding remarks

We have shown that the eccentricity function in distance-hereditary graphs is almost unimodal. We used this result to fully characterize centers of distance-hereditary graphs and to provide a simple additive 2-approximation of all eccentricities and finally an additive 1-approximation. Both approximation algorithms work in linear time. Two interesting questions left open. First, can all eccentricities (or at least the sets \( C(G) \) and/or \( D(G) \)) in distance-hereditary graphs be computed exactly in total linear time? Secondly, does there exist a small constant size set of vertices in a distance-hereditary graph \( G \) such that the distances from an arbitrary vertex \( v \) to them define the eccentricity of \( v \) up-to an additive error of at most 1?

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