Matrix Models and One Dimensional Open String Theory

JOSEPH A. MINAHAN

Department of Physics, Jesse Beams Laboratory,
University of Virginia, Charlottesville, VA 22901 USA

ABSTRACT

We propose a random matrix model as a representation for \( D = 1 \) open strings. We show that the model is equivalent to \( N \) fermions with spin in a central potential that also includes a long-range ferromagnetic interaction between the fermions that falls off as \( 1/(r_{ij})^2 \). We find two interesting scaling limits and calculate the free energy for both situations. One limit corresponds to Dirichlet boundary conditions for the dual graphs and the other corresponds to Neumann conditions. We compute the boundary cosmological constant and show that it is of order \( 1/\log(\beta) \). We also briefly discuss a possible analog of the Das-Jevicki field for the open string tachyon.

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† Electronic mail: MINAHAN@gomez.phys.virginia.edu
Matrix models have lead to many insights in low dimensional string theory. The model that is perhaps the simplest to understand, yet ironically has the richest structure, is that which corresponds to a scalar field theory on a line coupled to gravity. This is the matrix quantum mechanics model [1] with the matrix potential tuned such that the theory is almost critical [2,3-5]. This model is equivalent to a theory of closed strings in $1 + 1$ dimensions, where time is one coordinate and the eigenvalues of the matrices $\lambda$, are related to the Liouville coordinate [6-9].

A natural extension of this work is to find the corresponding theory of open and closed strings. One suggestion is to add the term $\xi \text{Tr} \log(\phi(t) - \mu)$ to the matrix potential, and tune $\xi$ and $\mu$ to some appropriate values [10]. The trouble with this is that, unlike the zero dimensional case [11-13], this term does not arise by integrating out fields which are $1 \times N$ matrices that couple to $\phi$. These fields are fundamental representations of a global $U(N)$ group and it is precisely these fields that generate boundary terms in the Feynman diagrams. The problem with the log term is that the kinetic piece of the fundamental fields has been ignored. One might argue that it is all right to drop this term if the mass and the couplings are very large. However, it has been pointed out that the interesting critical behavior occurs when the argument of the log approaches zero [10,14]. This is precisely where the kinetic piece should become important.

In this paper we explicitly include these fundamental fields in the full lagrangian. We demonstrate that the theory is equivalent to $N$ fermions with spin which interact among each other with long-range ferromagnetic interactions. We show that there are two interesting scaling limits for the masses of the fundamental fields. One limit has the mass diverging logarithmically as $N \to \infty$, while the other case has a mass that diverges, but at a milder rate. The former case appears to correspond to an open string theory with Dirichlet boundary conditions on the dual graphs (and hence Neumann conditions on the original graphs), while the latter limit corresponds to a theory with Neumann conditions. In both cases we compute the free energy as an expansion in the open and closed string couplings. The Neumann case has almost all terms in the expansion negative definite, while
the Dirichlet case has the same form for the free energy, but with the opposite sign for the open string coupling. The Neumann case also has a critical value for the open string coupling, at which point divergences start appearing in the free energy. We also show that the boundary cosmological constant scales as \((\log \beta)^{-1}\). Finally, we discuss a possible analog to the Das-Jevicki collective coordinate field [6] for the open string tachyon.

To begin, consider a triangulated surface \(\Sigma\), with boundary \(\mathcal{B}\), where \(\mathcal{B}\) is not necessarily connected. Suppose that \(\Sigma\) is the world-sheet for a string theory which has a coordinate \(t\). Each vertex of a triangle marks a particular point in \(t\) space, \(t_v\). Then an edge, \(\ell\), connecting two vertices has associated with it a difference in \(t\), \(\Delta t_\ell = t_{v_1} - t_{v_2}\). The complete partition function should be comprised of a sum over all assignments of \(t_v\) to the vertices. This is equivalent to summing over all \(\Delta t_\ell\), so long as we impose the constraint

\[
\sum_{\ell \in \mathcal{C}} \Delta t_\ell = 0, \tag{1}
\]

where the sum over \(\ell\) is over the three edges of any triangle \(\mathcal{C}\).

The string world sheets should also have boundary conditions. For the triangulated surface \(\Sigma\), Neumann boundary conditions impose the constraint \(\Delta t_\ell = 0\) if \(\ell\) is an edge that intersects the boundary. On the other hand, Dirichlet boundary conditions impose the same constraint \(\Delta t_\ell = 0\), but now \(\ell\) is an edge that lives on the boundary.

The dual surface of \(\Sigma\) is found by bisecting all edges. This operation maps vertices to faces and vice versa, while edges are mapped to new edges at 90 degrees. The dual surface can be generated by a field theory of \(N \times N\) matrices, \(\phi_{ab}\), with cubic interactions [15,2]. The surfaces are the Feynman diagrams for this theory, with the \(\phi\) propagators forming the edges. \(\Delta t_\ell\) for an edge in \(\Sigma\) maps to \(p_\ell\), the momentum flowing through the bisecting propagator, while the constraint in (1) naturally maps to the constraint that the sum of the momenta entering any vertex is zero.
The boundaries of Σ present a slight problem when finding the dual graph. The edge dual to a boundary edge has nothing to attach to outside the boundary. We can remedy this by introducing new fields, ψₐ and χₐ which transform in the fundamental representation of a global U(N) and which couple to φₐᵦ. The propagators of these fields can then be used to tie off the ends of the φ propagators dangling over the boundary. ψₐ and χₐ are assumed to be fermionic and we will see later that two fields are necessary in order to have a nontrivial theory.

The boundary conditions in Σ lead to analogous constraints for the dual graphs [16]. Edges that are normal to the boundary are mapped to edges that are parallel. Therefore, Neumann conditions on Σ correspond to dual graphs with zero momentum in the φ propagators that run along the boundary. This means that if a large amount of momentum flows into a region of the Feynman diagram which includes a long stretch of the boundary, then the momentum should mainly flow through the propagators of the fundamental field and not the φ propagators that are next to but not actually intersecting the ψ or χ propagators. This is accomplished by making the masses of ψₐ and χₐ very large, in which case, the weights of the Feynman diagrams strongly favor a large amount of momentum flowing through the ψ and χ propagators. As the masses diverge, the correlation lengths of these fields shrink to zero and all fermion loops become localized in t. Thus, Neumann conditions on Σ lead to Dirichlet conditions for the dual theory. Likewise, Dirichlet conditions on Σ lead to dual graphs with zero momentum flowing off the boundaries. Such graphs will dominate the free energy if the masses of ψₐ and χₐ are chosen to be small. The dual graph then has Neumann boundary conditions.

Let us consider the action, previously considered by Yang [14]

\[ S = \beta \int dt \left\{ \text{Tr} \left( \frac{1}{2} \dot{\phi}^2 - V(\phi) \right) + i \dot{\psi}^+_a \psi_a + i \dot{\chi}^+_a \chi_a 
\]

\[ + \gamma \dot{\psi}^+_a \dot{\psi}_b + \gamma \dot{\chi}^+_a \dot{\chi}_b - \mu \dot{\psi}^+_a \psi_a - \mu \dot{\chi}^+_a \chi_a \right\}. \]  

We have chosen to couple the fermion fields to the square of the hermitian matrix
for later convenience, but this won’t affect the critical behavior in the double scaling limit. It is necessary to couple $\chi$ to the complex conjugate of $\phi$ in order to build open string states that are invariant under a global $SU(N)$ rotation.

In the large $N$ limit the free energy derived from this action is dominated by the planar diagrams and can be expressed as an expansion in $1/N$,

$$F = - \sum_{\text{surfaces}} N^{2(1-g)} N^{-h} (N/\beta)^{\text{area}} (\gamma^2 N/\beta)^{\text{length}} F(\Sigma, g, h, \mu),$$

where $g$ is the genus of the surface and $h$ is the number of holes. Each vertex in the dual diagram is weighted by $(N/\beta)^{1/2}$ and each vertex on the boundary is weighted by $\gamma(N/\beta)^{1/2}$. Hence in terms of the original triangulated graph, these weights lead to the terms in the free energy exponentiated by the area and length in appropriate units. $F(\Sigma, g, h, \mu)$ is a weight for the Feynman diagram corresponding to a particular surface. If $\mu$ is very large then the $\mu$ dependence of $F$ is approximately $(1/\mu)^{\text{length}}$.

To proceed, let us diagonalize the matrix $\phi$, $\phi = U^\dagger \Lambda U$, where $\Lambda$ is diagonal and $U$ is an element of $SU(N)$. Moreover, let us rotate the fermion fields $\psi_a \to U^{\dagger}_{ab} \psi_b$ and $\chi_a \to \chi_b U^{\dagger}_{ba}$. Then the action (2) becomes

$$S = \beta \int dt \sum_a \left( \frac{1}{2} \dot{\lambda}_a^2 - V(\lambda_a) + i \psi_a^\dagger \dot{\psi}_a + i \chi_a^\dagger \dot{\chi}_a 
+ \gamma \psi_a^\dagger \lambda_a^2 \psi_a + \gamma \chi_a^\dagger \lambda_a^2 \chi_a - \mu \psi_a^\dagger \psi_a - \mu \chi_a^\dagger \chi_a \right)$$

$$- \sum_{a \neq b} \left( \frac{1}{2} (\dot{U}U^\dagger)_{ab} (\lambda_a - \lambda_b)^2 (\dot{U}U^\dagger)_{ba} + i \psi_a^\dagger (\dot{U}U^\dagger)_{ab} \psi_b - i (\dot{U}U^\dagger)_{ab} \chi_b \chi_a \right).$$

Letting $A = i \dot{U}U^\dagger$, we see that the action is quadratic in $A$ and hence can be integrated out after shifting variables [17,18]. To this end, consider the path integral for propagation of the vacuum state from time $t = 0$ to time $t = T$,

$$Z = \int D\phi D\psi D\psi^\dagger D\chi D\chi^\dagger e^{iS}$$

$$= \int D\lambda DU D\psi D\psi^\dagger D\chi D\chi \prod_t (\Delta(\lambda(t)))^2 e^{iS},$$
where $\Delta(\lambda(t))$ is the vandermonde determinant
\[
\Delta(\lambda(t)) = \prod_{a<b} (\lambda_a(t) - \lambda_b(t))^2.
\] (6)

Completing the square and shifting $A$, leads to the extra term in the action [18]
\[
-\beta \int dt \sum_{a \neq b} \frac{(\psi_a \psi_b - \chi_b \chi_a)(\psi_b \psi_a - \chi_a \chi_b)}{(\lambda_a - \lambda_b)^2}.
\] (7)

Noting that $U^\dagger(t + \Delta t)U(t) = 1 + i A(t + \Delta t/2) \Delta t + ..., we can replace the U integrals with A integrals, less one U integration because of the global SU(N) symmetry. Since the diagonal elements of SU(N) commute with $\Lambda$, we should restrict the integration to a subspace of $A$. If we only consider the insertions of SU(N) invariant states into the path integral, then this subspace is just the off-diagonal elements of $A$. This integration over $A_{ab}$ leads to factors that cancel off the vandermonde determinants in the Jacobian, except for a leftover contribution at each end point because of the extra integration over an SU(N) variable. Hence the partition function reduces to
\[
Z = \int D\lambda \psi^\dagger D\psi D\chi^\dagger D\chi \Delta(\lambda(0))\Delta(\lambda(T))e^{iS_{\text{eff}}},
\] (8)

where $S_{\text{eff}}$ is the effective action after integrating out $A$.

The fermion kinetic terms lead to the standard quantization for $\psi_a$ and $\chi_a$,
\[
\{\psi_a, \psi_b^\dagger\} = \delta_{ab}, \quad \{\chi_a, \chi_b^\dagger\} = \delta_{ab},
\] (9)

where a factor of $\beta^{1/2}$ has been absorbed into the fermion fields. Therefore, the hamiltonian for SU(N) invariant states is given by
\[
H = \sum_a \left(-\frac{1}{2\beta^2} \frac{\partial^2}{\partial \lambda_a^2} + V(\lambda_a) + \gamma \beta (\psi_a \psi_a^\dagger + \chi_a \chi_a^\dagger) - \frac{\mu}{\beta} (\psi_a \psi_a^\dagger + \chi_a \chi_a^\dagger) \right)
- \frac{1}{\beta^2} \sum_{a<b} \frac{(\psi_a \psi_b - \chi_b \chi_a)(\psi_b \psi_a - \chi_a \chi_b)}{(\lambda_a - \lambda_b)^2}.
\] (10)

The last term in the hamiltonian leads to a repulsive force between the eigenvalues.
However, because $\psi_a$ and $\chi_a$ don’t commute with $\psi^\dagger_a$ and $\chi^\dagger_a$ respectively, it is necessary to determine the ordering of the operators in $H$ so that the theory is well defined. The ordering follows from the definition of the time derivative in the path integral. As is the case with the kinetic term for the hermitian matrix, the kinetic term for the fermions should link an otherwise uncoupled chain of fields. With this in mind, let us define the kinetic term for $\psi_a$ as

$$\sum_t i\psi^\dagger_a(t)(\psi_a(t) - \psi_a(t - \Delta t)).$$

Rotating $\psi_a$ by $U^\dagger$ and integrating out the angular variables leaves the following contribution to the path integral (ignoring the $\chi$ field):

$$Z_\psi(\lambda) = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp \left\{ i \sum_{t=0}^{T} \left[ \sum_a i\psi^\dagger_a(t)(\psi_a(t) - \psi(t - \Delta t)) \right. 
$$

$$+ \Delta t(\gamma\psi^\dagger_a(t)\lambda^2_a(t)\psi_a(t) - \mu\psi^\dagger_a(t)\psi_a(t)) \right\} 
$$

$$+ \sum_{a<b} \frac{\Delta t \psi^\dagger_a(t)\psi^\dagger_b(t)(\psi_b(t - \Delta t)\psi_a(t - \Delta t))}{\beta (\lambda_a - \lambda_b)^2} \right\}$$

where the integrations over the $\psi$ variables are at the points between $t = 0$ and $t = T$ inclusive. Then, with no $\psi$ insertions in $Z_\psi(\lambda)$, (12) reduces to

$$Z_\psi(\lambda) = C \prod_t \left( 1 - i\gamma \Delta t \sum_a (\lambda^2_a(t) - \mu) \right)$$

$$= C \exp \left( -i\beta \int_0^T dt \frac{\gamma}{\beta} \sum_a (\lambda^2_a(t) - \mu) \right),$$

where $C$ is an unimportant constant. If we now insert the $SU(N)$ invariant operator $\epsilon_{a_1a_2...a_N} \psi^\dagger_{a_1}\psi^\dagger_{a_2}...\psi^\dagger_{a_N}$ into $Z_\psi(\lambda)$ at $t = 0$, we find that the contribution to the path integral is $C$. This means that there is no contribution to the energy from the four fermi interaction if the $\psi$ states are either completely full or empty. Thus we learn that the explicit ordering given by the hamiltonian in (10) follows from our definition of the infinite matrix chain.
From the hamiltonian (10) we can determine the ground state, which should be a state that is $SU(N)$ invariant. Given an $SU(N)$ invariant state $|\Psi\rangle$, other invariant states can be constructed by acting on $|\Psi\rangle$ with not only the operator $\text{tr} \phi^n$, but also with $J^+_n$, $J^n$ and $J^z_n$, where these last three operators are defined as

$$J^+_n = \psi^\dagger_a \phi^n_{ab} \chi^\dagger_b,$$
$$J^n = \chi_a \phi^n_{ab} \psi_b,$$
$$J^z_n = \frac{1}{2} (\psi^\dagger_a \phi^n_{ab} \psi_b - \chi_a \phi^n_{ab} \chi^\dagger_b).$$

(14)

The $J^n$ operators form an algebra

$$[J^z_n, J^m_\pm] = \pm J^{n+m}_\pm \quad [J^+_n, J^m_n] = 2 J^{n+m}_z,$$

(15)

and in particular, the operators $\vec{J} \equiv \vec{J}^0$ are the generators of an $SU(2)$ lie algebra. Furthermore, after diagonalizing $\phi$ and redefining $\psi$ and $\chi$ we can construct operators $S^+_a$, $S^-_a$ and $S^z_a$, where

$$S^+_a = \psi^\dagger_a \chi^\dagger_a,$$
$$S^-_a = \chi_a \psi_a,$$
$$S^z_a = \frac{1}{2} (\psi^\dagger_a \psi_a - \chi_a \chi^\dagger_a),$$

(16)

and which satisfy the relation $\vec{J}^n = \sum_a \lambda^a_n \vec{S}_a$. Clearly, for each index $a$, $\vec{S}_a$ can be interpreted as a spin operator. Hence the hamiltonian (10) can be re-expressed as

$$H = \sum_a \left( -\frac{1}{2\beta^2} \frac{\partial^2}{\partial \lambda^2_a} + V(\lambda_a) - \frac{2}{\beta} (S^z_a - \frac{1}{2}) (\gamma \lambda^2_a - \mu) \right) + \frac{2}{\beta^2} \sum_{a<b} \frac{1/4 - \vec{S}_a \cdot \vec{S}_b}{(\lambda_a - \lambda_b)^2}. $$

(17)

The result is a system equivalent to $N$ fermions with spin in a central potential which includes a position dependent magnetic field and with Heisenberg ferromagnetic long range couplings between the spins. Although $\vec{J} \cdot \vec{J}$ does not commute with $H$, $J_z$ does, so the energy eigenstates can be classified by their $J_z$ quantum numbers.

It is not known how to calculate the exact spectrum of (17). However, for some choices of $\gamma$ and $\mu$ we can at least find the ground state. First consider the
case \( \gamma < 0 \). If \( \mu \geq 0 \), then the magnetic field, while position dependent, points in the same direction for all values of \( \lambda \). Hence, the ground state \( | \downarrow \rangle \), has all spins pointing down in the direction of the field and \( J_z = -N/2 \). The reduced hamiltonian for spins with this configuration is [14]

\[
H_\downarrow = \sum_a \left( -\frac{1}{2\beta^2} \frac{\partial^2}{\partial \lambda_a^2} + V(\lambda_a) + \frac{2}{\beta} (\gamma \lambda_a^2 - \mu) \right).
\]

(18)

Since all spins are down, the fermion interaction term has dropped out. If we now let \( \mu < 0 \), then \( | \downarrow \rangle \) is not necessarily the ground state anymore, since the field is no longer pointing in the same direction for all \( \lambda \). For some negative value of \( \mu \), the lowest energy state with \( J_z = -N/2 \) becomes degenerate with the lowest energy state with \( J_z = 1 - N/2 \). Hence, this value of \( \mu \) is a critical point for the fermion mass.

Next consider the case \( \gamma > 0 \). If \( \mu < 0 \), then the field is pointing in the same direction for all \( \lambda \), but now the ground state has all spins up. Examining the hamiltonian (17), we find that there is no longer any \( \gamma \) dependence in the reduced hamiltonian. However, if we assume that \( V(\lambda) \) is infinite in the range \( |\lambda| > a \), and if \( \mu > \gamma a^2 \), then the field points in the same direction for all \( |\lambda| < a \) and the ground state is spin down. In this case, the reduced hamiltonian (18) is valid, and there exists a critical value of \( \mu \), \( \mu_0 \approx \gamma a^2 \).

The net effect of the \( \psi \) and \( \chi \) fields on the ground state is to shift the potential for \( \lambda \), and with proper scaling of \( \gamma \), to shift the fermi energy. Let the potential \( V(\lambda) \) be given by

\[
V(\lambda) = \begin{cases} 
(\lambda^2 - \lambda^4/(2a^2))/\alpha' & -a \leq \lambda \leq a \\
\infty & |\lambda| > a
\end{cases}
\]

\( (19) \)

where \( \alpha' \) is the Regge slope. This potential has maxima at \( \lambda = \pm a \), and at these points \( V''(\pm a) = -1/\alpha' \). We have chosen the infinite wall to lie at the local maxima so that we can have a small magnetic field near these points and
still be able to compute the ground state. Adding the term in (18) shifts the maxima to $\pm a(1 + 2\gamma\alpha'/\beta)^{1/2}$ and changes the second derivatives at these points to $-(1 + 2\gamma\alpha'/\beta)/\alpha'$. Furthermore, if $\mu$ is tuned close to $\mu_0$ then the magnetic field is small near the local maxima.

The quartic potential is rather unwieldy to work with, so instead let us use the potential

$$V(\lambda) = -\frac{\lambda^2}{\alpha'} \quad 0 \leq \lambda \leq a$$

$$= \infty \quad \lambda < 0 \quad \text{or} \quad \lambda > a.$$  \quad (20)

Now the effect of the boundary fields is to multiply $V(\lambda)$ by $(1 + 2\gamma\alpha'/\beta)$. The fermi energy for the potential in (20) is easily found using a semiclassical approximation for the density of states [2]. Inserting this approximation into an integral over phase space gives an expression for $N$,

$$N = \beta \int_0^a d\lambda \int \frac{dp}{2\pi} \theta(\varepsilon_F - p^2/2 + \lambda^2/\alpha').$$

The integral is straightforward, giving the relation

$$N = \sqrt{2\beta(-\alpha'\varepsilon_F)} \left( \frac{a}{\sqrt{-\alpha'\varepsilon_F}} \frac{a^2}{\sqrt{-\alpha'\varepsilon_F}} - 1 - \cosh^{-1}(a/\sqrt{-\alpha'\varepsilon_F}) \right).$$  \quad (22)

For our purposes we can approximate this equation as

$$N = \frac{\sqrt{2\beta}}{2\pi\sqrt{\alpha'}} \left( a^2 + \frac{1}{4}(-\alpha'\varepsilon_F) \log(-\varepsilon_F) + O(\varepsilon_F) \right).$$  \quad (23)

Shifting the potential leads to a new fermi energy $\tilde{\varepsilon}_F$, which satisfies the equation

$$N = \frac{\sqrt{2\beta}}{2\pi\sqrt{\alpha'}} \left( a^2 \sqrt{1 + 2\gamma\alpha'/\beta} + \frac{1}{4}(-\alpha'\tilde{\varepsilon}_F) \log(-\tilde{\varepsilon}_F) + O(\tilde{\varepsilon}_F) \right).$$  \quad (24)

The value of $\mu$ does not effect this expression so long as the ground state remains

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\* We should note that for $\gamma > 0$, the points $\pm a(1 + 2\gamma\alpha'/\beta)^{1/2}$ are in the region where the potential is infinite. But this won’t matter because, as we will see later, $a\gamma\alpha'/\beta$ is much smaller than $((V(a) - \varepsilon_F)/\alpha')^{1/2}$. This is the distance from the wall where the potential equals the fermi energy.
spin down. Equating the expressions for $N$ in (23) and (24) leads to the relation

$$\gamma = \frac{\beta}{4a^2}(\bar{\varepsilon}_F - \varepsilon_F) \log(-\varepsilon_F).$$ (25)

It has been argued [4,5,19] that the proper double scaling limit is found by fixing $\beta\sqrt{\alpha'}\varepsilon_F$ to be constant while taking the limit $N \to \infty$ and $\beta/N$ to its critical value. Thus, we discover that $\gamma$ must have a logarithmic divergence in the scaling limit in order to shift $\beta\varepsilon_F$ by a finite amount.

In order to understand the significance of this shift, let us consider the free energy in the scaling limit. Many authors have argued that this free energy is given by [3,4,5,19]

$$F = \frac{1}{4\pi\beta\sqrt{\alpha'}} \left( -\frac{1}{g_{cs}^2} \left( 1 - \alpha' \kappa g_{cs} \right)^2 \log(-\varepsilon_F) - \frac{1}{3} \log(-\varepsilon_F) \right. \right.$$

$$\left. - \sum_{m=1}^{\infty} (2^{2m+1} - 1) \frac{|B_{2m+2}|}{m(m+1)} (\beta\sqrt{\alpha'}\varepsilon_F)^{-2m} \right),$$ (26)

where $B_{2m}$ are the Bernoulli numbers. Based on this expansion, it seems clear that $(-\beta\sqrt{\alpha'}\varepsilon_F)^{-1}$ should be interpreted as the closed string coupling, $g_{cs}$. If we shift the fermi energy to $\bar{\varepsilon}_F$ and define $\kappa = \beta(\bar{\varepsilon}_F - \varepsilon_F)/\sqrt{\alpha'}$, then the relevant part of the free energy becomes

$$F = -\frac{1}{4\pi\beta\sqrt{\alpha'}} \left( \frac{1}{g_{cs}^2} \left( 1 - \alpha' \kappa g_{cs} \right)^2 \log(-\varepsilon_F) - \alpha' \kappa g_{cs} \right.$$

$$\left. + \frac{3}{2}(\alpha' \kappa g_{cs})^2 - 2 \sum_{n=2}^{\infty} \frac{(\alpha' \kappa g_{cs})^n}{n(n-1)(n-2)} \right)$$

$$+ \frac{1}{3} \log(-\varepsilon_F) - \frac{1}{3} \sum_{n=1}^{\infty} (\alpha' \kappa g_{cs})^n$$

$$\left. - \sum_{m=1}^{\infty} (2^{2m+1} - 1) \frac{|B_{2m+2}|}{m(m+1)} g_{cs}^{2m} \frac{(\alpha' \kappa g_{cs})^n(2m + n)!}{(2m)!} \right),$$ (27)

Shifting the bulk cosmological constant shifts $\varepsilon_F$ and $\bar{\varepsilon}_F$ by an equal amount and thus keeps $\kappa$ fixed while scaling $g_{cs}$. A surface of genus $g$ with $h$ holes should scale
as \((g_{cs})^2(1-g)+h\). Hence the terms in the expansion of the free energy have the proper scaling if \(\kappa\) is interpreted as an open string coupling constant.

If \(\gamma < 0\), then \(\tilde{\varepsilon}_F > \varepsilon_F\) and \(\kappa > 0\). In this case all terms in the expansion of the free energy are negative definite, except for those terms inside the curly brackets in (27). From (27) we also see that \(\kappa = 1/(g_{cs}\sqrt{\alpha'})\) is a critical value for the coupling constant since \(\tilde{\varepsilon}_F = 0\) for this value, and thus the free energy for surfaces of genus \(g \geq 1\) diverges when summing over the number of holes. The significance of this is not presently clear.

Letting \(\gamma > 0\) means that \(\kappa < 0\) and therefore the terms in (27) alternate sign. Furthermore, \(\mu\) must satisfy

\[
\mu > \mu_0 \approx \gamma a^2 = -(\kappa\sqrt{\alpha'}/4) \log(-\varepsilon_F)
\]

in order that the ground state is spin down. Therefore, in the scaling limit, the mass of the boundary generating fields diverges and hence, the dual graphs have Dirichlet boundary conditions.

We can give a rough estimate of the correction to \(\mu_0\) coming from the ferromagnetic interaction. If we flip one spin up, then the ferromagnetic term will try to symmetrize the up spin among all the fermions. This is counteracted by the position dependent field which tries to push the spin to a point where the field is small. We will approximate these effects by assuming that the up spin is symmetrized among the fermions less than a distance \(2y\) from the top of the potential at \(\lambda = 0\). Hence, we will say that the average position of the up spin is at \(\lambda = y\) and that it feels the \(1/r^2\) potential from the down spins that are greater than a distance \(y\) away from it. Thus, we estimate the shift in energy for turning up one spin as

\[
\Delta E \sim \frac{1}{\beta^2} \int \frac{d\lambda}{y} \frac{\rho(\lambda)}{\lambda^2} - \frac{2\Delta \mu_0}{\beta} + \frac{2}{\beta} \gamma y^2
\]

where \(\rho(\lambda)\) is the density of states, \(\frac{dn}{d\lambda}\). Plugging in the expression for \(\gamma\) in (25)
and using the semiclassical estimate

\[ \rho(\lambda) = \frac{\sqrt{2\beta}}{\pi\sqrt{\alpha'}} \sqrt{\lambda^2 + \tilde{\varepsilon}_F \alpha' \lambda} \]  

(30)

gives

\[ \Delta E \sim \frac{\sqrt{2}}{\pi\beta \sqrt{\alpha'}} \log y + \frac{\sqrt{\alpha' \kappa \log(-\varepsilon_F)}}{\beta a^2} y^2 - \frac{2\Delta \mu_0}{\beta}. \]  

(31)

Minimizing \( \Delta E \) leads to the estimates

\[ y \sim \frac{a}{(\alpha' \kappa \log(-\varepsilon_F))^{1/2}}, \quad \Delta E \sim (\sqrt{\alpha'/\beta})^{-1}(\log(\alpha' \kappa \log(-\varepsilon_F)) - \Delta \mu_0). \]  

(32)

Therefore the correction to \( \mu_0 \) is of order

\[ \Delta \mu_0 \sim \log(\alpha' \kappa \log(-\varepsilon_F)). \]  

(33)

Actually, given the roughness of our estimate, it is not clear that we can distinguish between a double log and a constant. In particular, flipping the spin will certainly alter the density of states, and if the spin pushes the eigenvalues out far enough, then the double log should be replaced with a constant. In any case the correction \( \mu_0 \) is much smaller than \( \mu_0 \)

Now suppose that \( \gamma < 0 \) and \( \mu \approx 0 \). Let us calculate the energy to flip one spin at the bottom of the quartic potential. We can play the same game to estimate the critical value of \( \mu \). In this case turning up one spin shifts the energy by

\[ \Delta E \sim \frac{2\sqrt{2}}{\beta \pi \sqrt{\alpha'}} \int_{y} dx \frac{\sqrt{x^2 + a^2}}{x^2} + \frac{2\mu_0}{\beta} - \frac{\kappa \sqrt{\alpha' \log(-\varepsilon_F)}}{2a^2} y^2. \]  

(34)

This then leads to the estimate

\[ \mu_0 \sim -(a\beta)^{-1}(-\alpha' \kappa \log(-\varepsilon_F))^{1/3} \]  

(35)

in order to keep \( \Delta E \) small. Hence in order to have some relevance in the double scaling limit, the Neumann case also requires a divergent mass for the fundamental
fields, although the divergence is milder than in the Dirichlet case. In fact, the mass is small in the sense that \( \mu_0 / \gamma \to 0 \) in the double scaling limit.

We would like to calculate the change in energy to turn up one spin if there are already \( s \) spins pointing up. This then leads to a density of up spins \( \frac{ds}{dE} \). It is not clear how to do this, but a reasonable guess for this behavior is that the density scales as \( \beta \log(-\varepsilon_F) \) for the Dirichlet case and \( \beta \) for the Neumann case. We postulate that the extra log term in the Dirichlet case is a result of the magnetic field being small at the top of the potential. Near this point, while there are fewer fermions around whose spin can flip, there are also fewer fermions to oppose such a flip. The guess is that the latter effect dominates the former.

Turning to the one point functions of \( \vec{J}^n \), it is clear that only \( \langle J^z_n \rangle \) is nontrivial. Actually, based on the structure of the hamiltonian in (17), the relevant operator is \( (1/N)(\text{tr} \phi^n - 2J^z_n) \), which couples to the fermion loops. Because of the structure of the vacuum, the one-point function for this operator is the same as the one-point function for \( (2/N) \text{tr} \phi^n \). Note that if \( \mu >> \mu_0 \) then those states that are constructed by acting on the vacuum with \( J^z_n \) decouple from the theory. In this case, the theory is essentially equivalent to a theory containing only closed strings, since now all \( J^z_n \) operators can be replaced by \( \text{tr} \phi^n \) operators. The only effect of \( \psi \) and \( \chi \) in this case is to renormalize the closed string coupling constant.

One interesting operator is \( \mathcal{P} = 1 - 2J^0_z/N \), where \( \mathcal{P} \) generates linear transformations of \( \mu \). If \( \mu \geq \mu_0 \), then the one point function for \( \mathcal{P} \) is

\[
\langle \mathcal{P} \rangle = 2. \quad (36)
\]

In the case of dual graphs with Dirichlet boundary conditions, the length dependence of the free energy for a particular surface is \( (\gamma/\mu)^{\text{length}} \). Therefore, \( \mathcal{P} \) generates a shift of the boundary cosmological constant. For \( \mu \gtrsim \mu_0 \), the boundary cosmological constant is, in some appropriate units,

\[
\mu_B = (\mu - \mu_0)/\mu_0 = \frac{4(\mu - \mu_0)}{\kappa \sqrt{\alpha'} \log(-\varepsilon_F)}. \quad (37)
\]
As in the case for closed strings, the cosmological constant should approach zero as the system nears the critical point. Since $P$ generates transformations of order unity, the value of $\mu - \mu_0$ should also be of order unity in order that it have some relevance in the scaling limit. In other words, if $\mu$ is shifted by order unity, then the energies of states with one spin pointing up are shifted by order $1/\beta$, which is enough to have some effect on the physics. Thus, we find that $\mu_B$ is of order $(-\log(-\varepsilon_F))^{-1}$.

It is instructive to compare this with the behavior of the boundary cosmological constant for the one-matrix model. For the $k^{th}$ multicritical point, the bare boundary cosmological constant scales as $\mu_B = \beta^{-2/(2k+1)}\mu_R$, where $\mu_R$ is the renormalized value [12,13]. As $k \to \infty$, $c$ approaches 1 and the $\beta$ dependence of $\mu_B$ is $\beta^\epsilon$ with $\epsilon \to 0$. This result compares favorably with our result for the one-dimensional chain.

The open string vertex operators should be constructed out of some combination of the $\vec{J}^n$ operators, which raise and lower the spins of the fermions. This suggests a possible connection to the Das-Jevicki collective field formulation of the tachyon field [6]. Recall that these authors showed there was a massless field associated with the fluctuations of the eigenvalue density, which they identified with the closed string tachyon. Analogously, it seems that the open string tachyon should be associated with the density of up pointing spins. This means that in the classical picture, there should be a connection between propagating tachyons and spin waves in the matrix model.

In order to compute nontrivial open string scattering amplitudes, it is necessary to compute the spectra of states that have some spins up. It is not clear if it will be possible to do this exactly, but it is worthwhile to point out that the interaction term between the fermions in (17) is somewhat similar to the interaction studied by Calogero [20]. Calogero had the $1/r^2$ potentials but not the spin-spin interaction. In any case, he showed that with a harmonic central potential, his theory was exactly solvable. It would thus seem not totally pointless to pursue exact solutions
of (17). Work on this issue and others is in progress.

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