The third symmetric potency of the circle and the Barnette sphere

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We give an elementary (not cut just paste) proof of results of Bott and Shchepin: the space of non-empty subsets of a circle of cardinality at most 3, which is called the third symmetric potency of the circle, is homeomorphic to a 3-sphere and the inclusion of the space of one element subsets is a trefoil knot. Moreover, we give an explicit simplicial decomposition of the third symmetric potency of the circle which is isomorphic to the Barnette sphere.

1 Introduction

The space of non-empty finite subsets of a topological space of cardinality at most $k$ have been studied in various areas of mathematics under various names and notations. Since our purpose of this paper is to study the case of the circle and $k = 3$, which seems to be the origin of the study of these spaces initiated by K. Borsuk [2] and R. Bott [3], we use the original name and notation.

Definition 1.1. Let $X$ be a topological space and $k$ be a positive integer. The set of non-empty subsets of $X$ of cardinality at most $k$ equipped with the quotient topology given by the canonical projection from the $k$-fold Cartesian product is called the $k$-th symmetric potency of $X$ and denoted by $(X)^{(k)}$. That is, as a set,

$$(X)^{(k)} = \{A \subset X \mid 0 < |A| \leq k\}$$

where $|A|$ denotes the cardinality of $A$, and the topology is induced by the projection

$$\pi: X^k \to (X)^{(k)}$$

given by $\pi(x_1, \ldots, x_k) = \{x_1, \ldots, x_k\}$.

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Note that we have canonical inclusions $(X)^{(1)} \subset (X)^{(2)} \subset (X)^{(3)} \ldots$, and $(X)^{(1)}$ is identified with $X$ by the projection $\pi \colon X \to (X)^{(1)}$. In [3], Bott identified the topology of $(S^1)^{(3)}$ and E. V. Schepin did the inclusion $S^1 = (S^1)^{(1)} \subset (S^1)^{(3)}$.

**Theorem 1.2** (R. Bott [3]). $(S^1)^{(3)}$ is homeomorphic to $S^3$.

**Theorem 1.3** (E. V. Schepin [8]). The inclusion $S^1 = (S^1)^{(1)} \subset (S^1)^{(3)} \cong S^3$ is a trefoil knot.

Various proofs of these Theorems are known such as given by S. Kallel and D. Sjerve [5], J. Mostovoy [7], C. Tuffley [10]. In this paper, we give an elementary (not cut just paste) proof of Theorems 1.2 and 1.3. Moreover, we give an explicit simplicial decomposition of $(S^1)^{(3)}$ which is isomorphic to the Barnette sphere [1].

## 2 Pasting

Fundamental topological properties of symmetric potencies are studied by D. Handel [4], and the following can be found as [4] Proposition 2.7. For the sake of completeness, we give a proof which is essentially the same as that of [4] Proposition 2.7.

**Proposition 2.1.** If $X$ is Hausdorff, then so is $(X)^{(k)}$.

**Proof.** Let $\pi \colon X^k \to (X)^{(k)}$ be the canonical projection.

For a subset $S \subset X$, we define subsets $E(S)$ and $I(S)$ of $(X)^{(k)}$ as follows:

$$E(S) := \left\{ A \in (X)^{(k)} \mid A \subset S \right\}$$

$$I(S) := \left\{ A \in (X)^{(k)} \mid A \cap S \neq \emptyset \right\}$$

Note that $\pi^{-1}(E(S)) = S^k$ and $I(S)^c = E(S^c)$. In particular, if $S$ is open (resp. closed), then so is $E(S)$ and hence so is $I(S)$.

Let $x = \{x_1, \ldots, x_l\}, y = \{y_1, \ldots, y_m\} \in (X)^{(k)}$ be two distinct points. We may assume that $x_1 \not\in y$. Since $X$ is Hausdorff, we can find open subsets $U, V \subset X$ satisfying $x_1 \in U, y = \{y_1, \ldots, y_m\} \subset V$ and $U \cap V = \emptyset$. Then $I(U)$ and $E(V)$ are open and $x \in I(U), y \in E(V)$. Since $V \subset U^c, E(V) \subset E(U^c) = I(U)^c$ whence $I(U) \cup E(V) = \emptyset$. \[\square\]

Our starting point is Morton’s prism. Consider the following subspaces of $\mathbb{R}^3$ (see Figure 1):

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid x \leq y \leq z \leq x + 1 \text{ and } 0 \leq x + y + z \leq 1\}$$

$$S = \{(x, y, z) \in P \mid x = y \text{ or } y = z \text{ or } z = x + 1\}$$

$$D = \{(x, y, z) \in P \mid x = y = z \text{ or } x = y = z - 1 \text{ or } x + 1 = y = z\}$$

The vertices of $P$ are the following:

- $0 : (0, 0, 0)$
- $1 : (-2/3, 1/3, 1/3)$
- $2 : (-1/3, -1/3, 2/3)$
- $4 : (0, 0, 1)$
- $5 : (1/3, 1/3, 1/3)$
- $6 : (-1/3, 2/3, 2/3)$
H. R. Morton [6] used the prism $P$ to describe symmetric products of the circle: the symmetric product $(S^1)^3/\Sigma_3$ is obtained from $P$ by identifying the bottom face $[0,1,2]$ and the top face $[4,5,6]$, hence homeomorphic to a solid torus.

Let $p: P \to (S^1)^3$ be the map defined by $p(x, y, z) = \{e^{2\pi ix}, e^{2\pi iy}, e^{2\pi iz}\}$. Define an equivalence relation on $P$ by $(x, y, z) \sim (x', y', z') \iff p(x, y, z) = p(x', y', z')$ and give $P/\sim$ the quotient topology. Let $\bar{p}$ be the induced map:

$$\bar{p}: P/\sim \to (S^1)^3$$

**Proposition 2.2.** The map $\bar{p}: P/\sim \to (S^1)^3$ is a homeomorphism. Moreover, restrictions of $\bar{p}$ give homeomorphisms $S/\sim \to (S^1)^2$ and $D/\sim \to (S^1)^1$:

$$D/\sim \subset S/\sim \subset P/\sim$$

$$\cong \bar{p} \cong \bar{p} \cong \bar{p}$$

$$S^1 \subset (S^1)^1 \subset (S^1)^2 \subset (S^1)^3$$

**Proof.** It is easy to see that the map $p: P \to (S^1)^3$ is surjective. Since $P$ is compact and $(S^1)^3$ is Hausdorff, the map $\bar{p}$ is a homeomorphism. It is straightforward to see that $p^{-1}((S^1)^2) = S$, $p^{-1}((S^1)^1) = D$, and the rest of the assertions follow.

It is straightforward to see the following.

**Lemma 2.3.** The equivalence relation $\sim$ is given by affinely identifying the following triangles (2-simplices) of the boundary of the prism $P$:

- $[0,1,2]$ and $[4,5,6]$
- $[0,1,6]$ and $[4,1,6]$
- $[1,2,4]$ and $[5,2,4]$
- $[2,0,5]$ and $[6,0,5]$
These identifications can be visually seen as follows.

We first identify the sides. By a half twist clockwise of the top of the prism, we get an octahedron of Figure 3 and expand the top and the bottom of the octahedron to get a parallelepiped of Figure 4. Further twisting and pushing down the top to identify the sides, we get the triangular bipyramid of Figure 5. Figure 6 shows $S/\sim$ in the triangular bipyramid. We see that $S/\sim$ is a full and a half twisted Möbius band and $D/\sim$ is its boundary. In particular, $D/\sim$ is a trefoil knot in the triangular bipyramid.

Now, $P/\sim$ is obtained by identifying the surface of the upper and the lower pyramids of the triangular bipyramid, we see that $P/\sim$ is homeomorphic to $S^3$.

You may find a JavaScript animation of these identifications at [9].

In the next section, we give a more explicit homeomorphism from $P/\sim$ to $S^3$ as a simplicial map.
3 The Barnette sphere

Note that, by adding a vertex in the interior of the upper tetrahedron of the triangular bipyramid of Figure 5, we obtain a simplicial decomposition of the bipyramid as in Figure 6. By identifying the vertices 3 and 7 (more precisely, by identifying faces [0, 1, 3] and [0, 1, 7], [1, 2, 3] and [1, 2, 7], [2, 0, 3] and [2, 0, 7]), we obtain a simplicial decomposition of the 3-sphere which is known as the Barnette sphere. We denote the Barnette sphere by $S^3_B$.

We give a simplicial decomposition of Morton’s prism $P$ as follows. Add the barycenters of the bottom and top faces, the mid points of vertical edges and a point in the interior of $P$ as vertices. As in the previous section, we slightly twist the top of the prism and stretch the bottom to get a polyhedron in Figure 7. We triangulate it as in the figure. We denote this simplicial complex by the same symbol $P$.

The vertices and facets of $P$ and $S^3_B$ are listed in the following table.

| vertices       | $P$            | $S^3_B$        |
|----------------|----------------|----------------|
| facets         | $[4, 5, 7, 9], [5, 6, 7, 10], [4, 6, 7, 8], [4, 7, 8, 9], [5, 7, 9, 10], [6, 7, 8, 10], [4, 1, 8, 9], [5, 2, 9, 10], [0, 6, 8, 10], [0, 8, 10, 11], [1, 8, 9, 11], [2, 9, 10, 11], [0, 1, 8, 11], [1, 2, 9, 11], [0, 2, 10, 11], [7, 8, 9, 10], [8, 9, 10, 11], [0, 1, 2, 11], [0, 1, 2, 3] | $[0, 1, 3, 9], [1, 2, 3, 10], [0, 2, 3, 8], [0, 3, 8, 9], [1, 3, 9, 10], [2, 3, 8, 10], [0, 1, 8, 9], [1, 2, 9, 10], [0, 2, 8, 10], [0, 8, 10, 11], [1, 8, 9, 11], [2, 9, 10, 11], [0, 1, 8, 11], [1, 2, 9, 11], [0, 2, 10, 11], [3, 8, 9, 10], [8, 9, 10, 11], [0, 1, 2, 11], [0, 1, 2, 3] |

Clearly, this simplicial decomposition of $P$ induces a simplicial decomposition of $P/\sim$. We define a map

$$q: \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \rightarrow \{0, 1, 2, 3, 8, 9, 10, 11\}$$
by

\[ q(i) = \begin{cases} 
  i - 4, & 4 \leq i \leq 7 \\
  i, & \text{otherwise.}
\end{cases} \]

The map \( q \) defines a simplicial map \( q : P \rightarrow S^3_B \), and it induces an isomorphism of simplicial complexes:

\[ \bar{q} : P/\sim \xrightarrow{\cong} S^3_B \]

Moreover, under this simplicial decomposition, \( S/\sim \) is the 2-dimensional subcomplex in Figure 9 which is a full and a half twisted Möbius band, and \( D/\sim \) is its boundary.

\[
\begin{array}{c|c|c}
S/\sim & D/\sim \\
\hline
\text{vertices} & \{0, 1, 2, 8, 9, 10\} & \{0, 1, 2, 8, 9, 10\} \\
\text{facets} & [0, 1, 8], [0, 2, 8], [1, 8], [2, 8] & [0, 9], [2, 9], [0, 10], [1, 10] \\
\end{array}
\]

Thus, we obtained the following:

**Theorem 3.1.** \( (S^1)^{(3)} \) is homeomorphic to \( S^3 \).

\( (S^1)^{(2)} \) is homeomorphic to a Möbius band, and \( (S^1)^{(1)} \) is its boundary. \( (S^1)^{(2)} \) is included in \( (S^1)^{(3)} \cong S^3 \) as a full and a half twisted Möbius band.

In particular, the inclusion \( S^1 = (S^1)^{(1)} \subset (S^1)^{(3)} \cong S^3 \) is a trefoil knot.
Figure 9: $S/\sim$ and $D/\sim$

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