Quantum $b$-functions of prehomogeneous vector spaces of commutative parabolic type

Atsushi KAMITA

Abstract. We show that there exists a natural $q$-analogue of the $b$-function for the prehomogeneous vector space of commutative parabolic type, and calculate them explicitly in each case. Our method of calculating the $b$-functions seems to be new even for the original case $q = 1$.

0 Introduction

Among prehomogeneous vector spaces those called of commutative parabolic type have special features since they have additional information coming from their realization inside simple Lie algebras. In [7] we constructed a quantum analogue $A_q(V)$ of the coordinate algebra $A(V)$ for a prehomogeneous vector space $(L, V)$ of commutative parabolic type. If $(L, V)$ is regular, then there exists a basic relative invariant $f \in A(V)$. In this case a quantum analogue $f_q \in A_q(V)$ of $f$ is also implicitly constructed in [7]. The aim of this paper is to give a quantum analogue of the $b$-function of $f$.

Let $t^f(\partial)$ be the constant coefficient differential operator on $V$ corresponding to the relative invariant $t^f$ of the dual space $(L, V^*)$. Then the $b$-function $b(s)$ of $f$ is given by $t^f(\partial)f^{s+1} = b(s)f^s$. See [8], [13] and [3] for the explicit form of $b(s)$.

For $g \in A_q(V)$ we can also define a (sort of $q$-difference) operator $t^g(\partial)$ by

$$\langle t^g(\partial)h, h' \rangle = \langle h, gh \rangle \quad (h, h' \in A_q(V)),$$

where $\langle \ , \ \rangle$ is a natural non-degenerate symmetric bilinear form on $A_q(V)$ (see Section 3 below). We can show that there exists some $b_q(s) \in \mathbb{C}(q)[q^s]$ satisfying

$$t^f_q(\partial)f_q^{s+1} = b_q(s)f_q^s \quad (s \in \mathbb{Z}_{\geq 0}).$$

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Our main result is the following.

**Theorem 0.1.** If we have $b(s) = \prod_i(s + a_i)$, then we have

$$b_q(s) = \prod_i q_0^{s+a_i-1}[s + a_i]_{q_0} \quad (up \ to \ a \ constant \ multiple),$$

where $q_0 = q^2$ (type $B$, $C$) or $q$ (otherwise), and $[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}}$.

We shall prove this theorem using an induction on the rank of the corresponding simple Lie algebra. We remark that this result was already obtained for type $A$ in Noumi-Umeda-Wakayama [14] using a quantum analogue of the Capelli identity.

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1 Quantized enveloping algebra

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system and $W \subset \text{GL}(\mathfrak{h})$ the Weyl group. For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by $\Delta^+$ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where $I_0$ is an index set. For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$, $s_i \in W$ be the simple coroot, the fundamental weight and the simple reflection corresponding to $i$ respectively. We denote the longest element of $W$ by $w_0$. Let $(\ , \ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots $\alpha$. For $i, j \in I_0$ we set

$$d_i = \frac{\langle \alpha_i, \alpha_i \rangle}{2}, \quad a_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{(\alpha_i, \alpha_i)}.$$

We define the antiautomorphism $x \mapsto {}^tx$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ by ${}^tx_\alpha = x_{-\alpha}$ and ${}^th_i = h_i$, where $\{x_\alpha | \alpha \in \Delta\}$ is a Chevalley basis of $\mathfrak{g}$.

The quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ (Drinfel’d [4], Jimbo [3]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i^{\pm 1}\}_{i \in I_0}$.
satisfying the following relations
\[ K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \]
\[ K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \]
\[ E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \]
\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] E_i^{1-a_{ij}-k} E_j F_i^k = 0 \quad (i \neq j), \]
\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j), \]
where \( q_i = q^i \), and
\[ [m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^{m} [k]_t, \quad \binom{m}{n}_t = \frac{[m]_t!}{[n]_t! [m-n]_t!} \quad (m \geq n \geq 0). \]

For \( \mu = \sum_{i \in I_0} m_i \alpha_i \) we set \( K_\mu = \prod_i K_i^{m_i} \).

We can define an algebra antiautomorphism \( x \mapsto t^x \) of \( U_q(\mathfrak{g}) \) by
\[ t^K_i = K_i, \quad t^E_i = F_i, \quad t^F_i = E_i. \]

We define subalgebras \( U_q(\mathfrak{b}^\pm), U_q(\mathfrak{h}) \) and \( U_q(\mathfrak{n}^\pm) \) of \( U_q(\mathfrak{g}) \) by
\[ U_q(\mathfrak{b}^+) = \langle K_i^{\pm 1}, E_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{b}^-) = \langle K_i^{\pm 1}, F_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{h}) = \langle K_i^{\pm 1} \mid i \in I_0 \rangle, \]
\[ U_q(\mathfrak{n}^+) = \langle E_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle. \]

We set \( \mathfrak{h}_Z^+ = \oplus_{i \in I_0} \mathbb{Z} \varpi_i \). For a \( U_q(\mathfrak{h}) \)-module \( M \) we define the weight space \( M_\mu \) with weight \( \mu \in \mathfrak{h}_Z^+ \) by
\[ M_\mu = \{ m \in M \mid K_i m = q_i^{\mu(h_i)} m \ (i \in I_0) \}. \]

The Hopf algebra structure on \( U_q(\mathfrak{g}) \) is defined as follows. The comultiplication \( \Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \) is the algebra homomorphism satisfying
\[ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i. \]

The counit \( \epsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q) \) is the algebra homomorphism satisfying
\[ \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0. \]
The antipode $S : U_q(g) \to U_q(g)$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i.$$  

The adjoint action of $U_q(g)$ on $U_q(g)$ is defined as follows. For $x, y \in U_q(g)$ write $\Delta(x) = \sum_k x_k^{(1)} \otimes x_k^{(2)}$ and set $\text{ad}(x)(y) = \sum_k x_k^{(1)} y S(x_k^{(2)})$. Then $\text{ad} : U_q(g) \to \text{End}_{\mathbb{C}(q)}(U_q(g))$ is an algebra homomorphism.

For $i \in I_0$ we define an algebra automorphism $T_i$ of $U_q(g)$ (see Lusztig [10]) by

$$T_i(K_j) = K_j K_i^{-\alpha_{ij}},$$

$$T_i(E_j) = \begin{cases} -F_iK_i & (i = j) \\ \sum_{k=0} \left(-q_i \right)^{-k} E_i^{(-\alpha_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_i^{-1}E_i & (i = j) \\ \sum_{k=0} (-q_i)^k F_i^{(k)} F_j F_i^{(-\alpha_{ij}-k)} & (i \neq j), \end{cases}$$

where

$$E_i^{(k)} = \frac{1}{[k]_{q_i}} E_i^k, \quad F_i^{(k)} = \frac{1}{[k]_{q_i}!} F_i^k.$$  

For $w \in W$ we choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$, and set $T_w = T_{i_1} \cdots T_{i_k}$. It does not depend on the choice of the reduced expression by Lusztig [11].

It is known that there exists a unique bilinear form $(\ , \ ) : U_q(b^-) \times U_q(b^+) \to \mathbb{C}(q)$ such that for any $x, x' \in U_q(b^+), y, y' \in U_q(b^-)$, and $i, j \in I_0$

$$(y, xx') = (\Delta(y), x' \otimes x), \quad (yy', x) = (y \otimes y', \Delta(x)),$$

$$(K_i, K_j) = q^{-(\alpha_i, \alpha_j)}, \quad (F_i, E_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1},$$

$$(F_i, K_j) = 0, \quad (K_i, E_j) = 0,$$

(See Jantzen [4], Tanisaki [19]).

For $\mu \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$ let $U_q(n^-)_-\mu$ be the weight space with weight $\mu$ relative to the adjoint action of $U_q(h)$ on $U_q(n^-)$. For any $y \in U_q(n^-)_-\mu$ and $i \in I_0$ the elements $r_i(y)$ and
$r'_i(y)$ of $U_q(n^-)_{-(\mu - \alpha_i)}$ are defined by

$$
\Delta(y) \in y \otimes 1 + \sum_{i \in I_0} K_i r_i(y) \otimes F_i + \bigoplus_{0 < \nu \leq \mu \atop \nu \not\in \alpha_i} K_\nu U_q(n^-)_{-(\mu - \nu)} \otimes U_q(n^-)_{-\nu},
$$

$$
\Delta(y) \in K_\mu \otimes y + \sum_{i \in I_0} K_{\mu - \alpha_i} F_i \otimes r'_i(y) + \bigoplus_{0 < \nu \leq \mu \atop \nu \not\in \alpha_i} K_{\mu - \nu} U_q(n^-)_{-\nu} \otimes U_q(n^-)_{-(\mu - \nu)}.
$$

**Lemma 1.1.** (see Jantzen [4])

(i) We have $r_i(1) = r'_i(1) = 0$ and $r_i(F_j) = r'_i(F_j) = \delta_{ij}$ for $j \in I_0$.

(ii) We have for $y_1 \in U_q(n^-)_{-\mu_1}$ and $y_2 \in U_q(n^-)_{-\mu_2}$

$$
r_i(y_1y_2) = q_i^{\mu_i(h_i)} y_1 r_i(y_2) + r_i(y_1)y_2, \quad r'_i(y_1y_2) = y_1 r'_i(y_2) + q_i^{\mu_i(h_i)} r'_i(y_1)y_2.
$$

(iii) We have for $x \in U_q(n^\pm)$ and $y \in U_q(n^-)_{-\mu}$

$$(y, E_i x) = (F_i, E_i)(r_i(y), x), \quad (y, x E_i) = (F_i, E_i)(r'_i(y), x).$$

(iv) We have $\text{ad}(E_i)y = (q_i - q_i^{-1})^{-1}(K_i r_i(y) K_i - r'_i(y))$ for $y \in U_q(n^-)_{-\mu}$.

From Lemma 1.1 (ii) we have $r_i(F_i^n) = r'_i(F_i^n) = q_i^{-n-1}[n]_{q_i} F_i^{n-1}$.

## 2 Commutative parabolic type

For a subset $I$ of $I_0$ we set

$$
\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, \quad l_I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha, \quad n_I^\pm = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{\pm \alpha}, \quad W_I = \langle s_i \mid i \in I \rangle.
$$

Let $L_I$ be the algebraic group corresponding to $l_I$. Assume that $n_I^- \neq 0$ and $[n_I^+, n_I^-] = 0$. Then it is known that $I = I_0 \setminus \{i_0\}$ for some $i_0 \in I_0$ and $(L_I, n_I^\pm)$ is a prehomogeneous vector space. Since $n_I^-$ is identified the dual space of $n_I^\pm$ via the Killing form, we have $\mathbb{C}[n_I^\pm] \simeq S(n_I^-) = U(n_I^-)$. There exists finitely many $L_I$-orbits $C_1, C_2, \ldots, C_r, C_{r+1}$ on $n_I^\pm$ satisfying the closure relation $\{0\} = C_1 \subset C_2 \subset \cdots \subset C_r \subset \overline{C_{r+1}} = n_I^\pm$. In the remainder of this paper we denote by $r$ the number of non-open orbits on $n_I^\pm$. For $p \leq r$ we set
Fig. 1:

\[ \mathcal{I}(\mathcal{C}_p) = \{ f \in \mathbb{C}[n^+ I_p] \mid f(\mathcal{C}_p) = 0 \} \]. We denote by \( \mathcal{I}^m(\mathcal{C}_p) \) the subspace of \( \mathcal{I}(\mathcal{C}_p) \) consisting of homogeneous elements with degree \( m \). It is known that \( \mathcal{I}^p(\mathcal{C}_p) \) is an irreducible \( \mathfrak{t}_r \)-module and \( \mathcal{I}(\mathcal{C}_p) = \mathbb{C}[n^+ I_p] \mathcal{I}^p(\mathcal{C}_p) \). Let \( f_p \) be the highest weight vector of \( \mathcal{I}^p(\mathcal{C}_p) \), and let \( \lambda_p \) be the weight of \( f_p \). We have the irreducible decomposition

\[ \mathbb{C}[n^+ I_p] = \bigoplus_{\mu \in \sum_{p=1}^{r} \mathbb{Z}_{\geq 0} \lambda_p} V(\mu), \]

where \( V(\mu) \) is an irreducible highest weight module with highest weight \( \mu \) and \( V(\lambda_p) = \mathcal{I}^p(\mathcal{C}_p) \) (see Schmid \[17\] and Wachi \[21\]).

If the prehomogeneous vector space \((L_I, n^+_I)\) is regular, there exists a one-codimensional orbit \( C_r \). Then it is known that \( \mathcal{I}^r(\mathcal{C}_r) = \mathbb{C} f_r \), \( f_r \) is the basic relative invariant of \((L_I, n^+_I)\) and \( \lambda_r = -2 \omega_{i_0} \), where \( I = I_0 \setminus \{i_0\} \). The pairs \((g, i_0)\) where \((L_I, n^+_I)\) are regular are given by the Dynkin diagrams of Figure [1]. Here the white vertex corresponds to \( i_0 \).

Assume that \((L_I, n^+_I)\) is regular. For \( 1 \leq p \leq r \) we set \( \gamma_p = \lambda_{p-1} - \lambda_p \), where \( \lambda_0 = 0 \). Then we have \( \gamma_p \in \Delta^+ \setminus \Delta_I \). We denote the coroot of \( \gamma_p \) by \( h_{\gamma_p} \), and set \( \mathfrak{h}^- = \sum_{p=1}^{r} \mathbb{C} h_{\gamma_p} \).
We set

\[ \Delta^+_p = \{ \beta \in \Delta^+ \setminus \Delta_I \mid \beta|_{\mathfrak{h}^+} = (\gamma_j + \gamma_k)/2 \text{ for some } 1 \leq j \leq k \leq p \} \cup \{ \gamma_1, \ldots, \gamma_p \}, \]

\[ n^+_p = \sum_{\beta \in \Delta^+_p} g_{\pm \beta}, \]

\[ I_p = [n^+_p, n^+_p] \]

(see Wachi [21] and Wallach [22]). Note that \( \alpha_{i_0} \in \Delta^+_p \) for any \( p \) and \( \Delta^+_p = \Delta^+ \setminus \Delta_I \). Then it is known that \( (L_p, n^+_p) \) is a regular prehomogeneous vector space of commutative parabolic type, where \( L_p \) is the subgroup of \( G \) corresponding to \( I_p \). Moreover \( f_j \in \mathbb{C}[n_p] \) for \( j \leq p \), and \( f_p \) is a basic relative invariant of \( (L_p, n^+_p) \). The regular prehomogeneous vector space \( (L_{(r-1)}, n^+_{(r-1)}) \) is described by the following.

**Lemma 2.1.** (i) For \( (A_{2n-1}, n) \) we have \( r = n \) and \( (L_{(n-1)}, n^+_{(n-1)}) \cong (A_{2n-3}, n-1) \).

(ii) For \( (B_n, 1) \) we have \( r = 2 \) and \( (L_{(1)}, n^+_{(1)}) \cong (A_1, 1) \).

(iii) For \( (C_n, n) \) \((n \geq 3)\) we have \( r = n \) and \( (L_{(n-1)}, n^+_{(n-1)}) \cong (C_{n-1}, n-1) \).

(iv) For \( (D_n, 1) \) we have \( r = 2 \) and \( (L_{(1)}, n^+_{(1)}) \cong (A_1, 1) \).

(v) For \( (D_{2n}, 2n) \) \((n \geq 3)\) we have \( r = n \) and \( (L_{(n-1)}, n^+_{(n-1)}) \cong (D_{2n-2}, 2n-2) \).

(vi) For \( (E_7, 1) \) we have \( r = 3 \) and \( (L_{(2)}, n^+_{(2)}) \cong (D_6, 1) \).

3 Quantum deformations of coordinate algebras

In this section we recall basic properties of the quantum analogue of the coordinate algebra \( \mathbb{C}[n_I^+] \) of \( n_I^+ \) satisfying \( [n_I^+, n_I^+] = 0 \) (see [7]). We do not assume that \( (L_I, n_I^+) \) is regular. We take \( i_0 \in I_0 \) as in Section 2.

We define a subalgebra \( U_q(I_I) \) by \( U_q(I_I) = \langle K_i^{\pm 1}, E_j, F_j \mid i \in I_0, j \in I \rangle \). Let \( w_I \) be the longest element of \( W_I \), and set

\[ U_q(n_{I_I}^-) = U_q(n^-) \cap T_{w_I}^{-1} U_q(n^-). \]

We take a reduced expression \( w_I w_0 = s_{i_1} \cdots s_{i_k} \) and set

\[ \beta_t = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}), \quad Y_{\beta_t} = T_{i_1} \cdots T_{i_{t-1}}(F_{i_t}) \]

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for $t = 1, \ldots, k$. In particular $Y_{\beta_t} = F_{i_0}$. We have $\{\beta_t | 1 \leq t \leq k\} = \Delta^+ \setminus \Delta_I$. The set
\[\{Y_{\beta_1}^{n_1} \cdots Y_{\beta_k}^{n_k} | n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}\}\] is a basis of $U_q(n^-_I)$.

**Proposition 3.1.** (see [7])

(i) We have $\text{ad}(U_q(I_t)) U_q(n^-_I) \subset U_q(n^-_I)$.

(ii) The elements $Y_{\beta} \in U_q(n^-_I)$ for $\beta \in \Delta^+ \setminus \Delta_I$ do not depend on the choice of a reduced expression of $w_I w_0$, and they satisfy quadratic fundamental relations as generators of the algebra $U_q(n^-_I)$.

We regard the subalgebra $U_q(n^-_I)$ of $U_q(n^-)$ as a quantum analogue of the coordinate algebra $\mathbb{C}[n_+^I]$ of $n_+^I$.

Since $\mathbb{C}[n_+^I]$ is a multiplicity free $I$-module, for the $L_I$-orbit $C_p$ on $n_I$ there exist unique $U_q(I_t)$-submodules $\mathcal{I}_q(C_p)$ and $\mathcal{I}_p(C_p)$ of $U_q(n^-_I)$ satisfying
\[\mathcal{I}_q(C_p)_{|q=1} = \mathcal{I}(C_p), \quad \mathcal{I}_p(C_p)_{|q=1} = \mathcal{I}^p(C_p)\]
(see [7]).

**Proposition 3.2.** (see [7]) $\mathcal{I}_q(C_p) = U_q(n^-_I) \mathcal{I}_p(C_p) = \mathcal{I}_q(C_p) U_q(n^-_I)$.

Let $f_{q,p}$ be the highest weight vector of $\mathcal{I}_q(C_p)$. We have the irreducible decomposition
\[U_q(n^-_I) = \bigoplus_{\mu \in \Sigma_p} \mathcal{V}_q(\mu),\]
where $\mathcal{V}_q(\mu)$ is an irreducible highest weight module with highest weight $\mu$ and $\mathcal{V}_q(\lambda_p) = \mathcal{I}_q(C_p)$. Explicit descriptions of $U_q(n^-_I)$ and $f_{q,p}$ are given in [8] in the case where $g$ is classical, and in [12] for the exceptional cases.

Let $f$ be a weight vector of $U_q(n^-_I)$ with the weight $-\mu$. If $\mu \in m\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$, then $f$ is an element of $\sum_{\beta_1, \ldots, \beta_m \in \Delta^+ \setminus \Delta_I} C(q) Y_{\beta_1} \cdots Y_{\beta_m}$. So we can define the degree of $f$ by $\text{deg} f = m$. In particular $\text{deg} f_{q,p} = p$.

**4 Quantum deformations of relative invariants**

In the remainder of this paper we assume that $(L_I, n_+^I)$ is regular, and $\{i_0\} = I_0 \setminus I$. Then we regard the highest weight vector $f_{q,r}$ of $\mathcal{I}_q(C_r)$ as the quantum analogue of the basic relative invariant. We give some properties of $f_{q,r}$ in this section.
By $T_q\left(\mathbb{C}^r\right) = \mathbb{C}(q) f_{q,r}$ and $\lambda_r = -2w_{i_0}$, we have the following.

**Proposition 4.1.** We have

$$\text{ad}(K_i) f_{q,r} = f_{q,r}, \quad \text{ad}(E_i) f_{q,r} = 0 \quad \text{and} \quad \text{ad}(F_i) f_{q,r} = 0,$$

for any $i \in I$, and $\text{ad}(K_{i_0}) f_{q,r} = q_{i_0}^{-2} f_{q,r}$.

**Lemma 4.2.** (i) For $i \in I$ we have $r_i(U_q(n^-)) = 0$.

(ii) For $\beta \in \Delta^+ \setminus \Delta_I$ we have $r'_{i_0}(Y_{\beta}) = \delta_{\alpha_{i_0},\beta}$.

**Proof.** (i) By Jantzen [4] we have

$$\{y \in U_q(n^-)| r_i(y) = 0\} = U_q(n^-) \cap T_i^{-1} U_q(n^-).$$

On the other hand we have $U_q(n^-) \subseteq U_q(n^-) \cap T_i^{-1} U_q(n^-)$ for $i \in I$. Hence we have $r_i(U_q(n^-)) = 0$ for $i \in I$.

(ii) We show the formula by induction on $\beta$.

By the definition of $r'_{i_0}$, it is clear that $r'_{i_0}(Y_{\alpha_{i_0}}) = r'_{i_0}(F_{i_0}) = 1$.

Assume that $\beta > \alpha_{i_0}$ and the statement is proved for any root $\beta_1$ in $\Delta^+ \setminus \Delta_I$ satisfying $\beta_1 < \beta$. For some $i \in I$ we can write

$$Y_{\beta} = c \text{ ad}(F_i) Y_{\beta'} = c \left( F_i Y_{\beta'} - q^{-\langle\alpha_i,\beta'\rangle} Y_{\beta'} F_i \right),$$

where $\beta' = \beta - \alpha_i$ and $c \in \mathbb{C}(q)$. Hence we have

$$r'_{i_0}(Y_{\beta}) = c \left( F_i r'_{i_0}(Y_{\beta'}) - q^{\langle\alpha_i,\alpha_{i_0} - \beta'\rangle} r'_{i_0}(Y_{\beta'}) F_i \right).$$

If $\beta' = \alpha_{i_0}$, we have $r'_{i_0}(Y_{\beta'}) = 1$. If $\beta' \neq \alpha_{i_0}$, we have $r'_{i_0}(Y_{\beta'}) = 0$ by the inductive hypothesis. \(\square\)

**Proposition 4.3.** The quantum analogue $f_{q,r}$ is a central element of $U_q(n^-)$.

**Proof.** For $i \in I$ we have $[F_i, f_{q,r}] = \text{ad}(F_i)f_{q,r}$. By Proposition [4] we have to show $[F_{i_0}, f_{q,r}] = 0$. 

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The $q$-analogue $f_{q,r}$ is a linear combination of $Y_{\beta_1} \cdots Y_{\beta_r}$ satisfying $\sharp \{ \beta_i | \beta_i = \alpha_{i_0} \} \leq 1$ (see [3] and [12]). By using Lemma 4.2 it is easy to show that $r_{i_0}' (f_{q,r}) \neq 0$ and $r_{i_0}'^2 (f_{q,r}) = 0$. Hence we have

$$r_{i_0}'^2 (F_{i_0} f_{q,r}) = r_{i_0}'^2 (f_{q,r} F_{i_0}) = (q_{i_0}^{d_{i_0}} + 1) r_{i_0}' (f_{q,r}).$$

On the other hand there exists $c \in \mathbb{C}(q)$ such that $F_{i_0} f_{q,r} = c f_{q,r} F_{i_0}$ by Proposition 3.2, hence we have $(q_{i_0}^{d_{i_0}} + 1) r_{i_0}' (f_{q,r}) = c (q_{i_0}^{d_{i_0}} + 1) r_{i_0}' (f_{q,r})$. Therefore we obtain $c = 1$.

\[5\] b-functions and their quantum analogues

We recall the definition of the $b$-function.

For $h \in S(n_+^2) \simeq \mathbb{C}[n_+^-]$, we define the constant coefficient differential operator $h(\partial)$ by

$$h(\partial) \exp B(x,y) = h(y) \exp B(x,y) \quad x \in n_+^+, y \in n_+^-,$$

where $B$ is the Killing form on $\mathfrak{g}$. It is known that there exists a polynomial $b_r(s)$ called the $b$-function of the relative invariant $f_r$ such that for $s \in \mathbb{C}$

$$t^r f_r(\partial) f_r^{s+1} = b_r(s) f_r^s.$$

Then we have $\deg b_r = r$. The explicit description of $b_r(s)$ is given by

- $(A_{2n-1}, n)$ \quad $b_n(s) = (s + 1)(s + 2) \cdots (s + n)$
- $(B_n, 1)$ \quad $b_2(s) = (s + 1) \left( s + \frac{2n - 1}{2} \right)$
- $(C_n, n)$ \quad $b_n(s) = (s + 1) \left( s + \frac{3}{2} \right) \left( s + \frac{4}{2} \right) \cdots \left( s + \frac{n + 1}{2} \right)$
- $(D_n, 1)$ \quad $b_2(s) = (s + 1) \left( s + \frac{2n - 2}{2} \right)$
- $(D_{2n}, 2n)$ \quad $b_n(s) = (s + 1)(s + 3) \cdots (s + 2n - 1)$
- $(E_7, 1)$ \quad $b_3(s) = (s + 1)(s + 5)(s + 9)$

(see [3], [13] and [14]).

We define a symmetric non-degenerate bilinear form $\langle , \rangle$ on $S(n_+^2) \simeq \mathbb{C}[n_+^-]$ by $\langle f, g \rangle = \langle (g(\partial) f)(0).$
Lemma 5.1. (see Wachi [21]) For $f, g, h \in S(n^-_I) \simeq \mathbb{C}[n^+_I]$ we have

(i) $\langle \text{ad}(u)f, g \rangle = \langle f, \text{ad}(u)g \rangle$ for $u \in U(I)$,

(ii) $\langle f, gh \rangle = \langle t^g(\partial)f, h \rangle$.

We have for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$

$$\langle x^-_{-\beta}, x^-_{-\beta'} \rangle = \delta_{\beta, \beta'} \frac{2}{(\beta, \beta)}.$$  

The comultiplication $\Delta$ of $U(g)$ is defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in g$. We define the algebra homomorphism $\widetilde{\Delta}$ by $\widetilde{\Delta}(x) = \tau \Delta(t^x)$, where $x \in U(g)$ and $\tau(y_1 \otimes y_2) = t^{y_1} \otimes t^{y_2}$.

Since $t^x_{-\beta}(\partial)(fg) = t^x_{-\beta}(\partial)(f)g + f t^x_{-\beta}(\partial)(g)$, we have

$$\langle fg, h \rangle = \langle f \otimes g, \widetilde{\Delta}(h) \rangle.$$  

We shall define the $q$-analogue of the differential operator $t^f(\partial)$ using the $q$-analogue of $\langle , \rangle$.

We define the bilinear form $\langle , \rangle$ on $U_q(n^-_I)$ by

$$\langle f, g \rangle = (q^{-1} - q)^{\deg f} \langle f', g' \rangle,$$

for the weight vectors $f, g$ of $U_q(n^-_I)$. It is easy to show that this bilinear form $\langle , \rangle$ is symmetric. We have the following.

Proposition 5.2. Let $f, g, h \in U_q(n^-_I)$.

(i) $\langle fg, h \rangle = \langle f \otimes g, \widetilde{\Delta}(h) \rangle$, where $\widetilde{\Delta}(h) = \tau \Delta(t^h)$ and $\tau(h_1 \otimes h_2) = t^{h_1} \otimes t^{h_2}$.

(ii) For $u \in U_q(I)$ we have

$$\langle \text{ad}(u)f, g \rangle = \langle f, \text{ad}(t^u)g \rangle.$$  

(iii) The bilinear form $\langle , \rangle$ is non-degenerate.

Proof. (i) It is clear from the definition.

(ii) It is sufficient to show that the statement holds for the weight vectors $f, g$ and the canonical generator $u$ of $U_q(I)$. If $u = K_i$ for $i \in I_0$, then the assertion is obvious.
Let $u = E_i$ for $i \in I$. By Lemma 1.1 and Lemma 4.2 we have

$$(\text{ad}(E_i)f, t g) = (q_i^{-1} - q_i)^{-1}(r_i(f), t g) = (f, t g E_i).$$

On the other hand we have

$$(f, t (\text{ad}(F_i)g)) = (f, t g E_i - q_i^{-\mu(h_i)} E_i t g),$$

where $-\mu$ is the weight of $g$. Since $(U_q(n^-), E_i U_q(n^+)) = 0$ by Lemma 1.1 and Lemma 4.2 we have $(\text{ad}(E_i)f, t g) = (f, t (\text{ad}(F_i)g))$. We have $\deg f = \deg (\text{ad}(F_i)f)$, and hence the statement for $u = E_i$ holds. By the symmetry of $\langle , \rangle$ it also holds for $u = F_i$.

(iii) We take the reduced expression $w_0 = s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_m}$ such that $w_I w_0 = s_{i_1} \cdots s_{i_m}$. We define $Y_{\beta_j}$ as in Section 3. Then $\{Y_{\beta_1}^n \cdots Y_{\beta_k}^n Y_{\beta_{k+1}}^n \cdots Y_{\beta_l}^n\}$ is a basis of $U_q(n^-)$, and for $j > k$ we have $Y_{\beta_j} \in U_q(n^-) \cap U_q(I)$. Hence we have $U_q(n^-) = U_q(n^-) + \sum_{i \in I} U_q(n^-) F_i$.

Since $t U_q(n^-) = U_q(n^+)$, we have $U_q(n^+) = t U_q(n^-) + \sum_{i \in I} E_i U_q(n^+)$. Moreover, we have $(U_q(n^-), E_i U_q(n^+)) = 0$ for $i \in I$. Hence if $\langle f, g \rangle = 0$ for any $g \in U_q(n^-)$, then $\langle f, u \rangle = 0$ for any $u \in U_q(n^+)$. Thus the assertion follows from the non-degeneracy of $\langle , \rangle$. 

\[ \square \]

**Proposition 5.3.** For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ we have

$$\langle Y_\beta, Y_{\beta'} \rangle = \delta_{\beta, \beta'} \left[ \frac{(\beta, \beta)}{2} \right]^{-1}_q.$$ 

**Proof.** By the definition it is clear that $\langle Y_\beta, Y_{\beta'} \rangle = 0$ if $\beta \neq \beta'$. In the case where $\beta = \beta'$ we shall show the statement by the induction on $\beta$.

Since $Y_{\alpha_{i_0}} = F_{i_0}$, we obtain $\langle Y_{\alpha_{i_0}}, Y_{\alpha_{i_0}} \rangle = \left[ \frac{(\alpha_{i_0}, \alpha_{i_0})}{2} \right]^{-1}_q$.

Assume that $\beta > \alpha_{i_0}$ and the statement holds for any root $\beta_1$ in $\Delta^+ \setminus \Delta_I$ satisfying $\beta_1 < \beta$. Then there exists a root $\gamma$ ($< \beta$) in $\Delta^+ \setminus \Delta_I$ such that

$$Y_\beta = c_{\gamma, \beta} \text{ad}(F_i) Y_\gamma, \quad Y_\gamma = c'_{\gamma, \beta} \text{ad}(E_i) Y_\beta,$$

where $i \in I$ satisfying $\beta = \gamma + \alpha_i$ and $c_{\gamma, \beta}, c'_{\gamma, \beta} \in \mathbb{C}(q)^*$. We denote by $R$ the set of the pairs $\{\gamma, \beta\}$ as above. By Proposition 1.2 we have for $\{\gamma, \beta\} \in R$

$$\langle Y_\beta, Y_{\beta} \rangle = \langle Y_\beta, c_{\gamma, \beta} \text{ad}(F_i) Y_\gamma \rangle = c_{\gamma, \beta} \langle \text{ad}(E_i) Y_\beta, Y_\gamma \rangle = c_{\gamma, \beta} c'_{\gamma, \beta} \langle Y_\gamma, Y_\gamma \rangle = c_{\gamma, \beta} c'_{\gamma, \beta} \left[ \frac{(\gamma, \gamma)}{2} \right]^{-1}_q.$$ 

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On the other hand we have for \( \{ \gamma, \beta \} \in R \)

\[
\begin{align*}
    c_{\gamma, \beta} &= c'_{\gamma, \beta} = 1 & \text{if } (\beta, \beta) = (\gamma, \gamma),
    \\
    c_{\gamma, \beta} &= (q + q^{-1})^{-1}, c'_{\gamma, \beta} = 1 & \text{if } 4 = (\beta, \beta) > (\gamma, \gamma) = 2,
    \\
    c_{\gamma, \beta} &= 1, c'_{\gamma, \beta} = (q + q^{-1})^{-1} & \text{if } 2 = (\beta, \beta) < (\gamma, \gamma) = 4
\end{align*}
\]

(see [1] and [2]). Hence we obtain \( \langle Y_\beta, Y_\beta \rangle = \left[ \frac{(\beta, \beta)}{2} \right]_q^{-1} \).

By Proposition 5.2 and 5.3 we can regard \( \langle , \rangle \) on \( U_q(\mathfrak{n}_-^\gamma) \) as the \( q \)-analogue of \( \langle , \rangle \) on \( S(\mathfrak{n}_-^\gamma) \simeq \mathbb{C}[\mathfrak{n}_-^\gamma] \).

**Proposition 5.4.** (i) For any \( g \in U_q(\mathfrak{n}_-^\gamma) \) there exists a unique \( ^t g(\partial) \in \text{End}_{\mathbb{C}(q)}(U_q(\mathfrak{n}_-^\gamma)) \)

such that \( \langle ^t g(\partial)f, h \rangle = \langle f, gh \rangle \) for any \( f, h \in U_q(\mathfrak{n}_-^\gamma) \). In particular we have

\[
^t Y_{a_i}(\partial) = [d_i]_q^{-1}r_i',
\]

and for \( \beta > \alpha_i \)

\[
^t Y_\beta(\partial) = c_{\gamma'}\beta(\gamma')^t \text{ad}(E_i) - q_i^{-\beta'(h_i)} \text{ad}(E_i)Y_{\beta'}(\partial),
\]

where \( Y_\beta = c_{\gamma', \beta} \text{ad}(E_i)Y_{\beta'} \).

(ii) For \( f \in U_q(\mathfrak{n}_-^\gamma)_{-\mu} \) and \( g \in U_q(\mathfrak{n}_-^\gamma)_{-\nu} \) we have \( ^t g(\partial)f \in U_q(\mathfrak{n}_-^\gamma)_{-(\mu, -\nu)} \).

**Proof.** (i) The uniqueness follows from the non-degeneracy of \( \langle , \rangle \). If there exist \( ^t g(\partial) \) and \( ^t g'(\partial) \), then we have \( ^t (gg')(\partial) = ^t g'(\partial)^t g(\partial) \). Therefore we have only to show the existence of \( ^t Y_\beta(\partial) \) for any \( \beta \in \Delta^+ \setminus \Delta_L \). By Lemma 1.1 we have \( ^t Y_{a_i}(\partial) = [d_i]_q^{-1}r_i' \). Let \( \beta > \alpha_i \). Then there exists a root \( \beta'(< \beta) \) such that \( Y_\beta = c_{\gamma', \beta} \text{ad}(F_i)Y_{\beta'} \) \( (c_{\gamma', \beta} \in \mathbb{C}(q)) \). By Proposition 5.2 we can show that \( ^t Y_\beta(\partial) = c_{\gamma', \beta}(Y_{\beta'}(\partial)\text{ad}(E_i) - q_i^{-\beta'(h_i)}\text{ad}(E_i)Y_{\beta'}(\partial)) \) easily.

(ii) The assertion follows from (i).

**Lemma 5.5.** For \( i \in I \) \( \text{ad}(E_i)^t f_{q, r}(\partial) = f_{q, r}(\partial)\text{ad}(E_i) \) and \( \text{ad}(F_i)^t f_{q, r}(\partial) = f_{q, r}(\partial)\text{ad}(F_i) \).

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PROOF. Let \( y_1, y_2 \in U_q(n^-_I) \). Since \( \text{ad}(F_i) f_{q,r} = 0 \) for \( i \in I \), we have \( \text{ad}(F_i)(f_{q,r} y_2) = f_{q,r} \text{ad}(F_i) y_2 \). Hence we obtain

\[
\langle \text{ad}(E_i) f_{q,r}(\partial)(y_1), y_2 \rangle = \langle y_1, f_{q,r} \text{ad}(F_i) y_2 \rangle = \langle y_1, \text{ad}(F_i)(f_{q,r} y_2) \rangle = \langle f_{q,r}(\partial) \text{ad}(E_i)(y_1), y_2 \rangle.
\]

Similarly we obtain \( \text{ad}(F_i) f_{q,r}(\partial) = t f_{q,r}(\partial) \text{ad}(F_i) \) \( \square \)

By Proposition 5.4 and Lemma 5.5, the element \( t f_{q,r}(\partial)(f_{q,r}^{s+1}) \) \( (s \in \mathbb{Z}_{\geq 0}) \) is the highest weight vector with highest weight \( s \lambda_r = -2s \varpi_{i_0} \). Since \( U_q(n^-_I) \) is a multiplicity free \( U_q(S_I) \)-module, there exists \( \tilde{b}_{q,r,s} \in \mathbb{C}(q) \) such that

\[
t f_{q,r}(\partial)(f_{q,r}^{s+1}) = \tilde{b}_{q,r,s} f_{q,r}^s.
\]

**Proposition 5.6.** There exists a polynomial \( \tilde{b}_{q,r}(t) \in \mathbb{C}(q)[t] \) such that \( \tilde{b}_{q,r,s} = \tilde{b}_{q,r}(q_{i_0}^s) \) for any \( s \in \mathbb{Z}_{\geq 0} \).

**Proof.** Let \( \psi = \psi_1 \cdots \psi_m \), where \( \psi_j = r'_{i_0} \) or \( \text{ad}(E_i) \) for some \( i \in I \). Set \( n = n(\psi) = \sharp \{ j \mid \psi_j = r'_{i_0} \} \). For \( k \in \mathbb{Z}_{\geq 0} \) and \( y \in U_q(n^-_I)_{-\mu} \) we have

\[
r'_{i_0}(f_{q,r}^k y) = q_{i_0}^{k-1+\mu(h_{i_0})} [k]_{q_{i_0}} f_{q,r}^{k-1} r'_{i_0}(f_{q,r}) y + f_{q,r}^{k} r'_{i_0}(y)
\]

by the induction on \( k \). Note that \( q_{i_0}^{k-1+\mu(h_{i_0})} [k]_{q_{i_0}} = (q_{i_0} - q_{i_0}^{-1})^{-1} q_{i_0}^{\mu(h_{i_0})-1} ((q_{i_0}^k)^2 - 1) \). Moreover \( \text{ad}(E_i)(f_{q,r}^k y) = f_{q,r}^k \text{ad}(E_i)y \) for \( i \in I \). Hence we have

\[
\psi(f_{q,r}^{s+1}) = \sum_{p=1}^n c_p(q_{i_0}^s) f_{q,r}^{s+1-p} y_p,
\]

where \( c_p \in \mathbb{C}(q)[t] \) and \( y_p \in U_q(n^-_I) \) does not depend on \( s \).

By Proposition 5.4 \( t f_{q,r}(\partial) \) is a linear combination of such \( \psi \) satisfying \( n(\psi) = r \). The assertion is proved. \( \square \)

We set \( b_{q,r}(s) = \tilde{b}_{q,r}(q_{i_0}^s) \) for simplicity. By definition we have

\[
\langle f_{q,r}^{s+1}, f_{q,r}^{s+1} \rangle = b_{q,r}(s) b_{q,r}(s-1) \cdots b_{q,r}(0).
\]
6 Explicit forms of quantum \(b\)-functions

Our main result is the following.

**Theorem 6.1.** Let \(b_r(s) = \prod_{i=1}^{r}(s + a_i)\) be a \(b\)-function of the basic relative invariant of the regular prehomogeneous vector space \((L_I, n_I^+)\). Then the quantum analogue \(b_{q,r}(s)\) of \(b_r(s)\) is given by

\[
b_{q,r}(s) = \prod_{i=1}^{r} q_{i_0}^{s+a_i-1} [s + a_i]_{q_{i_0}} \text{ (up to a constant multiple)},
\]

where \(\{i_0\} = I_0 \setminus I\).

We prove this theorem by calculating \(b_{q,r}(s)\) in each case.

For \(p = 1, \ldots, r\) we define \(\Delta_{(p)}^+, L_{(p)}\) and \(n_{(p)}^\pm\) as in Section 2. We define the subalgebra \(U_q(n_{(p)}^-)\) of \(U_q(n_I^-)\) by

\[U_q(n_{(p)}^-) = \langle Y_{\beta} | \beta \in \Delta_{(p)}^+ \rangle.\]

Then \(U_q(n_{(p)}^-)\) is a \(q\)-analogue of \(\mathbb{C}[n_{(p)}^+]\), and \(f_{q,p} \in U_q(n_{(p)}^-)\) is a \(q\)-analogue of basic relative invariant \(f_p\) of the regular prehomogeneous vector space \((L_{(p)}, n_{(p)}^+)\). We denote by \(b_{q,p}(s)\) the \(q\)-analogue of the \(b\)-function of \(f_p\).

The regular prehomogeneous vector space \((L_{(1)}, n_{(1)}^+)\) is of type \((A_1, 1)\), and we have

\[U_q(n_{(1)}^-) = \langle F_{i_0} | c \in \mathbb{C}(q)^* \rangle.\]

Since \(r'_{i_0}(F_{i_0}^{s+1}) = q_{i_0}^{s+1} [s + 1]_{q_{i_0}} F_{i_0}^s\), we obtain

\[b_{q,1}(s) = c^2 [d_{i_0}]_q^{-1} q_{i_0}^s [s + 1]_{q_{i_0}}.\]

If we determine \(a_p(s) \in \mathbb{C}(q)\) by

\[
\langle f_{q,p}^s, f_{q,p}^s \rangle = a_p(s) \langle f_{q,p-1}^s, f_{q,p-1}^s \rangle,
\]

then we have \(b_{q,p}(s) = \frac{a_p(s + 1)}{a_p(s)} b_{q,p-1}(s)\). Therefore we can inductively obtain the explicit form of \(b_{q,r}\).

The next lemma is useful for the calculation of \(a_p(s)\).

**Lemma 6.2.** (i) For \(\beta \in \Delta^+ \setminus \Delta_I\) we have

\[t^i Y_{\beta}(\partial)(f_{q,r}^n y) = t^i Y_{\beta}(\partial)(f_{q,r}^m y) \text{ad}(K_{\beta}^{-1}) y + f_{q,r}^n Y_{\beta}(\partial) y \quad (y \in U_q(n_I^-)).\]
Proof. (i) This is proved easily by the induction on $\beta$. Note that $\operatorname{ad}(E_i)(f_{q,r}) = 0$ for $i \in I$.

(ii) Since $f_{q,r}$ is a central element of $U_q(\mathfrak{n}_r)$, this follows from (i).

(iii) Let $\beta \in \Delta^+_p \setminus \Delta^+_{p-1}$. Then there exists some $j \in I$ such that $\beta = \mathbb{Z}_{\geq 0} \alpha_j + \sum_{i \neq j} \mathbb{Z}_{\geq 0} \alpha_i$ and $\gamma \in \sum_{i \neq j} \mathbb{Z}_{\geq 0} \alpha_i$ for any $\gamma \in \Delta^+_{p-1}$. Hence we have $U_q(\mathfrak{n}^-_{(p-1)})(\lambda_{p-1}-\beta) = \{0\}$, and the statement follows.

Let us give $a_p(s)$ in each case.

Let $(L_I, \mathfrak{n}_r^+)$ be the regular prehomogeneous vector space of type $(A_{2n-1}, n)$. Then the number of non-open orbits $r$ is equal to $n$, and $d_i = d_n = 1$. Here we label the vertices of the Dynkin diagram as in Figure 1.

Let $1 \leq i, j \leq n$. We set $\beta_{ij} = \alpha_{n-i+1} + \alpha_{n-i+2} + \cdots + \alpha_{n+j-1}$, and $Y_{ij} = Y_{\beta_{ij}}$. For two sequences $1 \leq i_1 < \cdots < i_p \leq n$, $1 \leq j_1 < \cdots < j_p \leq n$ we set

$$(i_1, \ldots, i_p|j_1, \ldots, j_p) = \sum_{\sigma \in S_p} (-q)^{\ell(\sigma)} Y_{i_1,j_{\sigma(1)}} \cdots Y_{i_p,j_{\sigma(p)}}.$$ 

Then we have $f_{q,p} = (1, \ldots, p|1, \ldots, p)$ (see (6)). It is easy to show the following formula.

$$(6.1) f_{q,p} = \sum_{k=1}^{p} (-q)^{p-k} Y_{p,k}(1, \ldots, p-1|1, \ldots, k, \ldots, p).$$

Note that $\beta_{p,k} \in \Delta^+_p \setminus \Delta^+_{p-1}$ and $(1, \ldots, p-1|1, \ldots, k, \ldots, p) = \operatorname{ad}(F_{n+k} \cdots F_{n+p-1}) f_{q,p-1}$ in (6.1).

Since $\beta_{p,i} \in \Delta^+_p \setminus \Delta^+_{p-1}$ for $1 \leq i \leq p$, by Lemma 5.2 we have

$$tY_{p,i}(\theta)(f_{q,p} f_{q,p-1}^{s_2}) = q^{s_2-1}[s_1]_q f_{q,p-1} f_{q,p}^{s_2} tY_{p,i}(\theta)(f_{q,p}) \operatorname{ad}(K_{\beta_{p,i}}^{-1})(f_{q,p-1}).$$

On the other hand we have the following.

Lemma 6.3.

$$(i,j)(\theta) f_{q,p} = (-q)^{i+j-2}(1, \ldots, \tilde{i}, \ldots, p|1, \ldots, \tilde{j}, \ldots, p) \quad (1 \leq i, j \leq p).$$

Proof. By Proposition 5.4, the statement is proved by the induction on $i, j$ easily. □
Since $\text{ad}(K_{p,i}^{-1})(f_{q,p-1}) = qf_{q,p-1}$ if $i \leq p - 1$ and $f_{q,p-1}$ if $i = p$, we have

$$
\langle f_{q,p}^{s_1}f_{q,p}^{s_2}, f_{q,p}^{s_1}f_{q,p}^{s_2} \rangle = \sum_{i=1}^{p}(-q^{-1})^{p-i}\langle Y_{p,i}(\partial)(f_{q,p}^{s_1}f_{q,p}^{s_2}), g_i f_{q,p}^{s_1-1}f_{q,p}^{s_2} \rangle
$$

$$
= \sum_{i=1}^{p-1}(-q)^{2i-2}q^{s_1+s_2-1}\langle s_1 | q (f_{q,p}^{s_1-1}g_i f_{q,p}^{s_2-1}, f_{q,p}^{s_1-1}g_i f_{q,p}^{s_2-1})
+ (-q)^{2p-2}q^{s_1-1}\langle s_1 | q (f_{q,p}^{s_1-1}f_{q,p}^{s_2+1}, f_{q,p}^{s_1-1}f_{q,p}^{s_2+1}),
$$

where $g_i = (1, \ldots, p - 1|1, \ldots, \tilde{i}, \ldots, p)$.

Now we have for $1 \leq i \leq p - 1$

$$
g_i = \text{ad}(F_{n+i})g_{i+1}, \quad g_{i+1} = \text{ad}(E_{n+i})g_i,
$$

$$
\text{ad}(E_{n+i})f_{q,p} = f_{q,p-1} = 0, \quad \text{ad}(F_{n+i})f_{q,p} = 0, \quad \text{ad}(F_{n+i})f_{q,p-1} = \delta_{i,p-1}g_{p-1}
$$

(see [1]). Therefore we have

$$
\text{ad}(E_{n+p-1} \ldots E_{n+i+1}E_{n+i})(f_{q,p}^{s_1-1}g_i f_{q,p}^{s_2}) = \text{ad}(E_{n+p-1} \ldots E_{n+i+1})(f_{q,p}^{s_1-1}g_i f_{q,p}^{s_2})
= \ldots = \text{ad}(E_{n+p-1})(f_{q,p}^{s_1-1}g_{p-1}f_{q,p}^{s_2}) = q^{-s_2}f_{q,p}^{s_1-1}f_{q,p}^{s_2+1},
$$

and

$$
f_{q,p}^{s_1-1}g_i f_{q,p}^{s_2-1} = \text{ad}(F_{n+i})(f_{q,p}^{s_1-1}g_{i+1} f_{q,p}^{s_2}) = \ldots = \text{ad}(F_{n+i} \ldots F_{n+p-2})(f_{q,p}^{s_1-1}g_{p-1}f_{q,p}^{s_2}).
$$

Here we have $g_{p-1}f_{q,p-1} = q^{-1}f_{q,p-1}g_{p-1}$, and hence

$$
f_{q,p}^{s_1-1}g_i f_{q,p}^{s_2-1} = q^{-s_2}[s_2 + 1]_q^{-1} \text{ad}(F_{n+i} \ldots F_{n+p-2})f_{q,p}^{s_1-1}f_{q,p}^{s_2+1}.
$$

By Proposition 5.2 we obtain

$$
\langle f_{q,p}^{s_1}, f_{q,p}^{s_2}, f_{q,p}^{s_1}, f_{q,p}^{s_2} \rangle = q^{s_1+1}\langle s_1 | q (q^{-s_2}[s_2 + 1]_q^{-1} \sum_{i=1}^{p-1}q^{2i-2} + q^{2p-2})(f_{q,p}^{s_1-1}f_{q,p}^{s_2+1}, f_{q,p}^{s_1-1}f_{q,p}^{s_2+1})
$$

$$
= q^{p+s_2-2}\langle s_1 | q (p + s_2) (q^{-s_2}[s_2 + 1]_q^{-1} f_{q,p}^{s_1-1}f_{q,p}^{s_2+1}, f_{q,p}^{s_1-1}f_{q,p}^{s_2+1}).
$$

From this formula we have the following.

**Proposition 6.4.** Let $(L, n^-)$ be a regular prehomogeneous vector space of type $(A_{2n-1}, n)$.

We have

$$
a_p(s) = q^{\frac{s(s + 2p - 3)}{2}} \prod_{i=1}^{s}[i + p - 1].
$$
In particular $b_{q,p}(s) = q^{s+p-1}[s + p]_q b_{q,p-1}(s)$, and we have the quantum $b$-function

$$b_{q,n}(s) = \prod_{p=1}^{n} q^{s+p-1}[s + p]_q.$$ 

Next, we assume that $(L_I, n^+_I)$ is regular of type $(D_{2n}, 2n)$. We label the vertices of the Dynkin diagram as in Figure [1], then $d_{i_0} = d_{2n} = 1$. There exist $n$ non-open orbits on $n^+_I$. Let $1 \leq i < j \leq 2n$. Set

$$\beta_{ij} = \{ \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} \ (j < 2n),$$

$$\alpha_i + \cdots + \alpha_{2n-2} + \alpha_{2n} \ (j = 2n),$$

and $Y_{ij} = Y_{\beta_{ij}}$. For a sequence $1 \leq i_1 < i_2 < \cdots < i_{2p} \leq 2n$, we set

$$(i_1, i_2, \ldots, i_{2p}) = \sum_{\sigma \in \hat{S}_{2p}} (-q^{-1})^{l(\sigma)} Y_{i_{\sigma(1)}, i_{\sigma(2)}} \cdots Y_{i_{\sigma(2p-1)}, i_{\sigma(2p)}}$$

where $\hat{S}_m = \{ \sigma \in S_m \mid \sigma(2k-1) < \sigma(2k+1), \sigma(2k-1) < \sigma(2k) \text{ for all } k \}$. Then we have $f_{q,p} = (j^p_1, j^p_2, \ldots, j^p_{2n})$, where $j^p_i = 2n - 2p + k$ (see [4]). We can easily show the following description of $f_{q,p}$ similar to (6.1).

$$f_{q,p} = \sum_{k=2}^{2p} (-q)^{2-k} Y_{j^p_1, j^p_k} (j^p_2, \ldots, j^p_k, \ldots, j^p_{2p}) = \sum_{k=2}^{2p} (-q)^{2-k} Y_{j^p_1, j^p_k} \text{ad}(F_{j^p_{k-1}} \cdots F_{j^p_{k-2}}) f_{q,p-1}.$$ 

Note that $\beta_{j^p_1, j^p_k} \notin \Delta^+_{(p-1)}$. Hence we can use Lemma 6.2.

By using the induction on $i, j$, we can show the following lemma.

**Lemma 6.5.** We have

$$t_{Y_{j^p_1, j^p_k}}(\partial) f_{q,p} = (-q)^{4n-1-k-k'} (j^p_1, \ldots, j^p_k, \ldots, j^p_{2p})$$

for $1 \leq k < k' \leq 2p$.

Similarly to the case of type $A$, we obtain the following.

**Proposition 6.6.** Let $(L_I, n^-_I)$ be a regular prehomogeneous vector space of type $(D_{2n}, 2n)$. We have

$$a_p(s) = q^{\frac{(4p+k-5)}{2}} \prod_{j=1}^{s} [j + 2p - 2]_q.$$ 

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In particular \( b_{q,p}(s) = q^{s+2p-2}[s + 2p - 1]q b_{q,p-1}(s) \) we have the quantum \( b \)-function

\[
b_{q,n}(s) = \prod_{p=1}^{n} q^{s+2p-2}[s + 2p - 1]q.
\]

Let \( (L_I, n^+_I) \) be the regular prehomogeneous vector space of type \( (B_n, 1) \). We label the vertices of the Dynkin diagram as in Figure 1, then \( d_{i_0} = d_1 = 2 \). There exist two non-open orbits on \( n^+_I \). Let \( 1 \leq i \leq 2n - 1 \). We set \( Y_i = Y_{\beta_i} \), where

\[
\beta_i = \begin{cases} 
\alpha_1 + \cdots + \alpha_i & (1 \leq i \leq n) \\
\alpha_1 + \cdots + \alpha_{2n-i} + 2\alpha_{2n-i+1} + \cdots + 2\alpha_n & (n + 1 \leq i \leq 2n - 1).
\end{cases}
\]

We have

\[
f_{q,1} = Y_1 = F_1,
\]

\[
f_{q,2} = \sum_{i=1}^{n-1} (-q_{i_0})^{i+1-n} Y_{n+i} Y_{n-i} + (q + q^{-1})^{-2} q^{-1} (-q_{i_0})^{1-n} Y^2_n
\]

(see 4). Note that \( \beta_i \notin \Delta^+_1 \) if \( i \neq 1 \). On the other hand we have the following.

**Lemma 6.7.**

\[
i Y_i(\partial)f_{q,2} = \begin{cases} 
(q + q^{-1})^{-1}(-q_{i_0})^{i-1} Y_{2n-i} & (1 \leq i \leq n) \\
-(q + q^{-1})^{-1}(-q_{i_0})^{i-2} Y_{2n-i} & (n + 1 \leq i \leq 2n - 1).
\end{cases}
\]

Similarly to the case of type \( A \), we obtain the following.

**Proposition 6.8.** Let \( (L_I, n^{-}_I) \) be a regular prehomogeneous vector space of type \( (B_n, 1) \). We have

\[
a_2(s) = (q + q^{-1})^{-s} q_{i_0}^{-\frac{s(s+2n-4)}{2}} \prod_{i=1}^{s} \left[ i + \frac{2n - 3}{2} \right]_{q_{i_0}}.
\]

In particular we have the quantum \( b \)-function

\[
b_{q,2}(s) = (q + q^{-1})^{-2} q_{i_0}^s [s + 1]_{q_{i_0}} q_{i_0}^{s + \frac{2n-3}{2}} \left[ s + \frac{2n - 1}{2} \right]_{q_{i_0}}.
\]

Let \( (L_I, n^+_I) \) be the regular prehomogeneous vector space of type \( (D_n, 1) \). We label the vertices of the Dynkin diagram as in Figure 1, then \( d_{i_0} = d_1 = 1 \). There exist two non-open
orbits on \( \mathfrak{n}_1^+ \). Let \( 1 \leq i \leq 2n - 2 \). We set \( Y_i = Y_{\beta_i} \), where
\[
\beta_i = \begin{cases} 
\alpha_1 + \cdots + \alpha_i & \text{for } 1 \leq i \leq n - 1 \\
\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n & \text{for } i = n \\
\alpha_1 + \cdots + \alpha_{2n-i} + 2\alpha_{2n-i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & \text{for } n + 1 \leq i \leq 2n - 2.
\end{cases}
\]

Then we have \( f_{q,1} = Y_1 = F_1 \), and \( f_{q,2} = \sum_{i=1}^{n-1} (-q)^{i+1-n}Y_{n+i-1}Y_{n-i} \) (see [3]). We have the following results similar to those of type \((B_n, 1)\).

**Lemma 6.9.**
\[
\text{Y}^i(\partial) f_{q,2} = \begin{cases} 
(-q)^i Y_{2n-1-i} & \text{for } 1 \leq i \leq n - 1 \\
(-q)^i Y_{2n-1-i} & \text{for } n \leq i \leq 2n - 2.
\end{cases}
\]

**Proposition 6.10.** Let \((L_I, \mathfrak{n}_I^-)\) be a regular prehomogeneous vector space of type \((D_n, 1)\).
We have
\[
a_2(s) = q^{s(s+2n-5)/2} \prod_{i=1}^{s} [i + n - 2]_q.
\]
In particular we have the quantum \(b\)-function
\[
b_{q,2}(s) = q^s [s + 1]_q q^{s + n - 2} [s + n - 1]_q.
\]

Let \((L_I, \mathfrak{n}_I^+)\) be the regular prehomogeneous vector space of type \((E_7, 1)\). We label the vertices of the Dynkin diagram as in Figure [4], then \(d_{10} = d_1 = 1\). There exist three non-open orbits on \( \mathfrak{n}_1^+ \).

For \( 1 \leq j \leq 27 \), we denote by \( Y_j \) and \( \psi_j \) the generators of irreducible \( U_q(I_I) \)-modules \( V_q(\lambda_1) \) and \( V_q(\lambda_2) \) respectively (see [12] for the explicit descriptions of \( Y_j \) and \( \psi_j \)). Note that \( Y_j = Y_{\beta_j} \) for some \( \beta_j \in \Delta^+ \setminus \Delta_I \), \( U_q(\mathfrak{n}_2^-) = \langle Y_1, \ldots, Y_{10} \rangle \), and \( \psi_{27} = f_{q,2} \). Now \((L_{(2)}, \mathfrak{n}_{(2)}^+)\) is of type \((D_6, 1)\), hence we have \( b_{q,2}(s) = q^s [s + 1]_q q^{s + 4} [s + 5]_q \).

The \(q\)-analogue \( f_{q,3} \) of the basic relative invariant is given by
\[
f_{q,3} = \sum_{j=1}^{27} (-q)^{|\beta_j|} - 1 Y_j \psi_j
\]
\[
= (1 + q^8 + q^{16})Y_{27} \psi_{27} + \frac{q^{-10} + q^{-8} - q^{-4} + 1 + q^4}{1 + q^2} \sum_{j=11}^{26} (-q)^{|\beta_j|} - 1 Y_j \psi_j,
\]
where \(|\beta| = \sum_{i=1}^{7} m_i \) for \( \beta = \sum_{i=1}^{7} m_i \alpha_i \).
Lemma 6.11. For $1 \leq j \leq 27$ we have $tY_j(\partial) f_{q,3} = \left(1 + q^8 + q^{16}\right)(-q)^{|\beta_j|} \psi_j$.

Then we have the following.

Proposition 6.12. Let $(L_I, n_I^{-})$ be a regular prehomogeneous vector space of type $(E_7, 1)$. We have

$$a_3(s) = \left(1 + q^8 + q^{16}\right)^2 q^{8[s+15]} \prod_{i=1}^{8} [i+8] q.$$ 

Therefore we have the quantum $b$-function

$$b_{q,3}(s) = \left(1 + q^8 + q^{16}\right)^2 q^{s+8} [s + 9] q b_{q,2}(s)$$

$$= \left(1 + q^8 + q^{16}\right)^2 q^s [s + 1] q q^{s+4} [s + 5] q q^{s+8} [s + 9] q.$$ 

Finally, we assume that $(L_I, n_I^{+})$ is the regular prehomogeneous vector space of type $(C_n, n)$. We label the vertices of the Dynkin diagram as in Figure 1, and then $d_{i_0} = d_n = 2$. There exist $n$ non-open orbits on $n_I^{+}$. Let $1 \leq i \leq j \leq n$. We set $\beta_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n$ and $Y_{ij} = c_{ij} Y_{\beta_{ij}}$, where $c_{ij} = q + q^{-1}$ if $i = j$ and 1 if $i \neq j$. For $i < j$ we define $Y_{ji}$ by $Y_{ji} = q^{-2} Y_{ij}$. Then we can write for $1 \leq p \leq n$

$$f_{q,p} = \sum_{\sigma \in S_p} (-q)^{-l(\sigma)} Y_{i_{p(1)}, i_{p(1)}}^{p} \cdots Y_{i_{p(n), i_{p(n)}}}^{p},$$

where $i_k^p = n + k - p$ (see [5]).

Lemma 6.13.

$$f_{q,p} = Y_{i_{1,1}}^{p} f_{q,p-1} + \sum_{k=2}^{p} \frac{(-q)^{1-k}}{q + q^{-1}} Y_{i_{k-1,1}}^{p} \text{ad}(F_{i_{k-1}} \cdots F_{i_{2}} F_{i_{1}}) f_{q,p-1}.$$ 

Proof. We denote the right handed side of the statement by $g_p$. It is easy to show that the coefficient of $Y_{i_{1,1}}^{p} \cdots Y_{i_{p-1,1}}^{p}$ in $f_{q,p}$ is equal to that in $g_p$. Moreover the weight of $f_{q,p}$ is equal to that of $g_p$. Hence it is sufficient to show that $g_p$ is the highest weight vector. Since $(L_{(p)}, n_{(p)}^+ \simeq (C_p, p)$, we have only to show the statement in the case where $p = n$. We can easily show that $\text{ad}(E_j) g_n = 0$ for $2 \leq j \leq n - 1$. 

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Let us show $\text{ad}(E_1)g_n = 0$. For $2 \leq j \leq n$ we define $\varphi_j$ by $\varphi_2 = \text{ad}(E_1)g_n$ and $\varphi_{j+1} = \text{ad}(E_j)\varphi_j$. We denote the weight of $\varphi_j$ by $\mu_j$. Then we have $\mu_j \in -\alpha_1 + \sum_{i \neq 1} \mathbb{Z} \alpha_i$. It is easy to show that $\text{ad}(E_k)\varphi_j = 0$ for any $k \neq j$. In particular $\text{ad}(E_k)\varphi_n = 0$ for any $k \in I$.

On the other hand we have the irreducible decomposition

$$U_q(n^-) = \bigoplus_{\mu \in \sum_{j=1}^n \mathbb{Z} \lambda_j} V_q(\mu),$$

and if $\mu \in \sum_{j=1}^n \mathbb{Z} \lambda_j$, then $\mu \in 2\mathbb{Z} \alpha_1 + \sum_{i \neq 1} \mathbb{Z} \alpha_i$. Hence $\mu_n \notin \sum_{j=1}^n \mathbb{Z} \lambda_j$, and we have $\varphi_n = 0$. We obtain $\varphi_j = 0$ for any $j$ by the induction. \[\square\]

Note that $Y_{i_1i_p}^{p-1} \notin U_q(n_{(p-1)}^-)$ for $1 \leq k \leq p$. Hence we can use Lemma 5.2.

We can prove the following lemma.

**Lemma 6.14.**

$$i Y_{i_1i_k}^{p-1}(\partial) f_{q,p} = \begin{cases} (-q)^{2p-2}(q + q^{-1}) f_{q,p-1} & (k = 1) \\ (-q)^{2p-k} \text{ad}(F_{i_1}^{p-1} \cdots F_{i_k}^{p-1})(f_{q,p-1}) & (k \leq 2). \end{cases}$$

Similarly to the case of type $A$, we obtain the following.

**Proposition 6.15.** Let $(L_I, n^-)$ be a regular prehomogeneous vector space of type $(C_n, n)$. We have

$$a_p(s) = (q + q^{-1})^s q_0^{\frac{s(s+p-2)}{2}} \prod_{i=1}^s \left[ i + \frac{p-1}{2} \right]_q.$$

In particular $b_{q,p}(s) = (q + q^{-1}) q_0^{\frac{s+p+1}{2}} \left[ s + \frac{p+1}{2} \right]_q b_{q,p-1}(s)$, and we have the quantum $b$-function

$$b_{q,n}(s) = (q + q^{-1})^n q_0^{\frac{n+p+1}{2}} \left[ s + \frac{p+1}{2} \right]_q.$$

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