Progress on Holographic Three-Point Functions

Wolfgang M"uck
Dipartimento di Scienze Fisiche, Universit"a di Napoli “Federico II” and
I.N.F.N. — Sezione di Napoli
Via Cintia, 80126 Napoli, Italy

Key words Renormaliztion Group, AdS/CFT and dS/CFT Correspondence

The recently developed gauge-invariant formalism for the treatment of fluctuations in holographic renormalization group (RG) flows overcomes most of the previously encountered technical difficulties. I summarize the formalism and present its application to the GPPZ flow, where scattering amplitudes between glueball states have been calculated and a set of selection rules been found.

1 Introduction

The gauge/gravity correspondence provides an excellent tool for obtaining relevant information about supersymmetric Yang-Mills (SYM) theories from the study of their dual supergravity backgrounds, which are generated by stacks of \(D\)-branes. The most celebrated duality—known as the AdS/CFT correspondence—relates the superconformal \(\mathcal{N} = 4\) SYM theory in four dimensions (in the planar limit and at large ‘t Hooft coupling) with the type IIB supergravity on an \(AdS_5 \times S^5\) background \([1]\). More generally, it is a paradigm that the dynamics of supergravity on an (asymptotically) anti-de Sitter space encodes the correlation functions of its dual (deformed) conformal field theory. This concept has been made quantitatively precise by the AdS/CFT correspondence formula \([2, 3]\) and by holographic renormalization (see \([4, 5, 6]\) and references therein).

Holographic renormalization has taught us that the most efficient way to obtain field theory correlators is to look directly at exact one-point functions, i.e., one-point functions of gauge-invariant operators in the presence of sources, which, in principle, contain the information of all higher-point functions. Let me outline the holographic calculation taking, for simplicity, a scalar field as example. In a generic, asymptotically AdS, bulk space-time of dimension \(d + 1\), a scalar field obeying the field equations can be written as the sum of two asymptotic series,

\[
\phi(x, r) = e^{-(d-\Delta)r}(1 + \cdots)\phi(x) + e^{-\Delta r}(1 + \cdots)\tilde{\phi}(x).
\]

It is assumed that the coefficient \(\Delta\), which is the conformal dimension of the dual operator \(\mathcal{O}\) of \(\phi\), is restricted by \(d/2 < \Delta \leq d\), so that \(\mathcal{O}\) is relevant or marginal, and the first series in \([1]\) is the leading one. The variable \(r\) is the “radial” variable of the bulk, with \(r \to \infty\) representing the asymptotic region. The ellipses stand for higher order terms in the series, and the coefficients \(\phi\) and \(\tilde{\phi}\) are called the source and response functions, respectively. A priori, these functions are independent in the asymptotic analysis of the (second order) equations of motion, but imposing a regularity condition in the bulk interior functionally fixes the response to the source.

The main result of holographic renormalization is that the exact one-point function \(\langle \mathcal{O} \rangle\) is essentially given by the response function,

\[
\langle \mathcal{O}_\Delta \rangle = (2\Delta - d)\tilde{\phi} + \text{local terms.}
\]

Copyright line will be provided by the publisher
The local terms are typically scheme dependent and will not be discussed here. Hence, in order to calculate field theory three-point functions involving $O_\Delta$, one must calculate $\phi$ to quadratic order in the field fluctuations and then differentiate with respect to the sources corresponding to the other two operators.

In this talk, I will consider a generic bulk system of scalars and gravity, governed by an action of the form

$$S = \int d^{d+1}x \sqrt{g} \left[ -\frac{1}{4} \tilde{R} + \frac{1}{2} G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b + V(\phi) \right],$$

where the potential, $V(\phi)$, is given in terms of a superpotential, $W(\phi)$, by

$$V(\phi) = \frac{1}{2} G^{ab} W_a W_b - \frac{d}{d-1} W^2,$$

and the matrix $G^{ab}$ is the inverse of the $\sigma$-model metric $G_{ab}$. The notation coincides with the one used in [7], i.e., bulk quantities are adorned with a tilde, and derivatives of the potentials with respect to fields are indicated as subscripts, as in $W_a = \partial W/\partial \phi^a$.

The equations of motion stemming from the action (3) allow for a particular class of solutions with $d$-dimensional Poincaré invariance, which are called Poincaré domain walls, or holographic RG flow backgrounds. These are governed by a system of first order differential equations in terms of the radial variable $r$ (see, e.g., [8]), which implies that the background scalars, $\tilde{\phi}^a$, can have either a non-zero source, or a non-zero response, but not both. In the first case, we speak of a deformation flow generated by the insertion of the dual operator, whereas, in the second case we speak of a vev flow, since, according to (2), the dual operator acquires a non-zero vacuum expectation value. In the common nomenclature, scalars with non-zero background are called active, while those with zero background are called inert.

When M. Bianchi, M. Prisco and I started working on the calculation of three-point functions in holographic RG flows, the state-of-the-art were calculations of two-point functions [9, 10, 11, 12, 13], and some simple three-point functions [14]. Technical difficulties, which were known already from the linear analysis needed for the two-point functions, forced us to look for a new and more systematic approach to the treatment of fluctuations in holographic RG flow backgrounds. In this talk, I will introduce the gauge-invariant formalism, which we developed in [7] using reparametrization invariance as the guiding principle. This formalism elegantly overcomes the above mentioned difficulties and has been successfully applied to the calculation of scattering amplitudes in the GPPZ flow [15].

The generalization of the gauge-invariant formalism to a generic bulk system of the form (3) has been undertaken in an on-going collaboration with M. Berg and M. Haack, in which we hope to learn something about holography in non-asymptotically AdS RG flow backgrounds. In fact, supergravity-type actions of the form (3) and holographic RG flow backgrounds appear also in connection with a whole list of “famous” supergravity duals of SYM theories, e.g., the Klebanov-Strassler [16] and Maldacena-Nunez (MN) [17] solutions.

## 2 Gauge invariant formalism

### 2.1 The $\sigma$-model covariant field expansion

In the generic bulk system, which is described by the action (3), reparametrization invariance appears at two distinct places, namely, in the geometries of the bulk space and of the $\sigma$-model. As usual in gravity, invariance under space-time diffeomorphisms comes at the price of introducing redundant variables to the metric degrees of freedom. Usually, this is taken care of by gauge fixing, but our approach will be different. For now, I shall keep all metric degrees of freedom and describe later how to isolate the physical ones. The Poincaré-invariant form of the holographic RG flow backgrounds then suggests to use the ADM formalism to parametrize the metric degrees of freedom, with $g_{ij}$, $n^i$ and $n$ being the induced metric on the time-slice...
h, H, ǫ

\begin{equation}
g_{ij} = e^{2A(r)}(\eta_{ij} + h_{ij}) , \quad n_i = \nu_i , \quad n = 1 + \nu ,
\end{equation}

where \( h_{ij} , \nu_i \) and \( \nu \) denote small fluctuations. Henceforth, I shall adopt the notation that the indices of the fluctuations, as well as of the derivatives \( \partial_i \), are raised and lowered using the flat (Minkowski/Euclidean) metric. It turns out to be useful to decompose the metric fluctuations into

\begin{equation}
h^i_j = h^{TT^i}_j + \partial^i \epsilon_j + \partial_j \epsilon^i + \frac{\partial^j \partial_j}{\Box} H + \frac{1}{d-1} \delta^i_j h ,
\end{equation}

where \( h^{TT^i}_j \) denotes the traceless transversal part, and \( \epsilon^i \) is a transversal vector.

The space of fields \( \phi^a \) has its own geometry, characterized by the \( \sigma \)-model metric \( G_{ab} \), which is assumed to be invertible, the inverse being called \( G^{ab} \). Hence, one can straightforwardly define the \( \sigma \)-model connection, \( G^a_{bc} \), the Riemann curvature tensor, \( R^b_{abc} \), and covariant field derivatives, denoted by \( D_a \) or by placing a bar \( \bar{\cdot} \) before the field index. Moreover, as the background is \( r \)-dependent, it is useful to introduce also a “background covariant” derivative, \( D_r \), acting on tensors in field space, such that \( G_{ab} \), evaluated on the background, is covariantly constant.

In order to exploit this notation for the fluctuation equations, it is necessary to perform the expansion of the scalar fields in a \( \sigma \)-model covariant fashion. As is well known, such an expansion is provided by the exponential map, whose generator will be called \( \varphi_i \), which, geometrically, represents the tangent vector of the geodesic connecting a background point \( \bar{\phi} \) with the point \( \phi \). In practice, calculations are simpler when carried out in Riemann normal coordinates [8].

2.2 Gauge transformations and invariants

Our main argument [7], which aims at obtaining the equations of motion in an explicitly gauge-invariant form, starts by considering the effect of diffeomorphisms on the fluctuation fields. It is well known that a diffeomorphism of the form \( x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x) \), where \( \xi^\mu \) is infinitesimal, acts as a gauge transformation on the fluctuation fields, when the background is unchanged. It turns out, though, that one can, order by order, form combinations of the fluctuation fields, which are gauge-invariant. To describe this, it is easiest to proceed in a symbolic fashion. The fluctuation fields shall be classified into two sets, \( X = \{ h, H, \epsilon^i \} \) and \( Y = \{ \phi^a, \nu , \nu^i , h^{TT^i}_j \} \), and the gauge-invariant fields shall be called \( I = \{ a^a, b, c, \delta^i , \epsilon^i \} \), where \( \delta^i \) and \( \epsilon^i \) are transversal and traceless transversal, respectively. The symbols \( X, Y, \) and \( I \) shall also be used to denote the members of the corresponding sets. Solving the definitions of the gauge-invariant variables for \( Y \) yields relations of the form

\begin{equation}
Y = I + y(X) + \alpha(X, X) + \beta(X, I) + \mathcal{O}(f^3) ,
\end{equation}

where \( y(X) \) is a linear function, quadratic terms have been included in the form of bi-linear functions \( \alpha \) and \( \beta \), and \( \mathcal{O}(f^n) \) denotes terms of at least of order \( n \) in the fluctuations. Terms of the form \( \gamma(I, I) \) do not appear, as they can be absorbed into \( I \). Hence, there is a one-to-one correspondence between the fields \( Y \) and the gauge-invariant variables \( I \). In contrast, the fields \( X \) parametrize the (unphysical) gauge degrees of freedom, as can be seen by considering the transformation law of \( h^i_j \). The explicit relations are of the form

\begin{equation}
\xi^\mu = z^\mu(\delta X) + \mathcal{O}(f^2) = \delta z^\mu(X) + \mathcal{O}(f^2) ,
\end{equation}

where \( z^\mu(X) \) are linear functionals. Eqs. (7) and (8) will play an essential role in finding the gauge-invariant form of the equations of motion.
2.3 Einstein’s equations and gauge invariance

It is our aim to re-write the equations of motion in a gauge-invariant fashion. “Gauge-invariant” here means that the final equations should contain only the fields $I$ and make no reference to $X$ and $Y$. Our guiding principle tells us that this should be possible, because the physical dynamics does not depend on the gauge.

For brevity, we shall consider Einstein’s equations, symbolically written as $E_{\mu\nu} = 0$, but it is clear that the same arguments also hold for the equations of motion for the scalar fields.

Expanding $E_{\mu\nu}$ by brute force in the fields $X$ and $Y$, and then substituting for $Y$ yields an expression in terms of $I$ and $X$ of the form

$$E_{\mu\nu} = E^{(1)1}_{\mu\nu}(X) + E^{(1)2}_{\mu\nu}(I) + E^{(2)1}_{\mu\nu}(X, X) + E^{(2)2}_{\mu\nu}(X, I) + E^{(2)3}_{\mu\nu}(I, I) + O(\delta^3).$$

The background equation is satisfied identically, and the single terms are linear or bi-linear functions of their arguments.

Considering explicitly the gauge transformation of $\phi$ and comparing to what one expects from the general transformation law of second-rank tensors under diffeomorphisms [using (5)], one finds

$$E^{(1)1}_{\mu\nu}(X) = 0, \quad E^{(1)2}_{\mu\nu}(X, X) = 0,$$

$$E^{(2)2}_{\mu\nu}(X, I) = [\partial_\mu z^\lambda(X)]E^{(1)2}_{\mu\nu}(I) + [\partial_\nu z^\lambda(X)]E^{(2)1}_{\mu\nu}(I) + z^\lambda(X)\partial_\lambda E^{(1)2}_{\mu\nu}(I).$$

Thus, the gauge dependent terms, which appear in the brute force expansion of $E_{\mu\nu}$ at second order, contain the first order equation, and, in an order by order analysis, can be consistently dropped. This argument generalizes recursively to higher orders. One will find that the gauge dependent terms of any given order can be consistently dropped, because they contain the equation of motion at lower orders.

Happily, we have arrived at an equation of motion, which is written in terms of the gauge-invariant variables $I$ only, and which, therefore, is explicitly gauge-invariant. Thus, we have removed the unphysical degrees of freedom without an explicit gauge fixing. The gauge-invariant equations of motion are simply found by applying the following substitution rules,

$$\phi^a \to a^a, \quad \nu \to b, \quad \nu^i \to d^i + \frac{\partial^i}{\Box} c, \quad h^i_j \to e^i_j.$$

Since $e^i_j$ is traceless and transversal, the calculational simplifications stemming from (11) are considerable.

2.4 Equations of motion

The equations of motion that follow from the action (3) are the equation for the scalar fields and Einstein’s equations, the latter being conveniently split into the normal components, $E_{rr}$, the mixed components, $E_{ir}$, and the tangential components, $E_{ij}$. After using the substitution rules (11) one finds that the scalars $a^a$ physically couple to $b$ and $c$ at the linearized level, but, in contrast to the experience in the literature, in the gauge-invariant formalism the components $E_{rr}$ and $E_{ir}$ can be solved algebraically for $b$ and $c$, so that substituting them into the scalar equation of motion yields the compact expression

$$\left[\left(\delta_0^b D_r + W^a b - \frac{W^a W_b}{W} - \frac{2d}{d-1} W \delta_0^a \right) \left(\delta_0^c D_r - W^b c + \frac{W^b W_c}{W} + \delta_0^a e^{-2A} \Box \right)\right] a^c = \tilde{J}^a,$$

where $\tilde{J}^a$ denotes quadratic interaction terms. The field $d^i$ is suppressed at linear order, which is expected from the boundary Ward identity, so that the only independent physical degrees of freedom of the metric fluctuations are $e^i_j$. Their equation of motion is found from $E_{ij}$ and reads

$$\left(\partial_r^2 - \frac{2d}{d-1} W \partial_r + e^{-2A} \Box \right) e^i_j = J^i_j.$$
The quadratic interaction terms $\tilde{J}^a$ and $J^i_j$ can be found in \cite{12,13} and depend on $a^a$ and $e^i_j$.

Equation \cite{12} is the main result of the gauge-invariant formalism. It governs the dynamics of scalar fluctuations around Poincaré domain walls in the most general case. Being a system of second order differential equations, one can use the standard Green’s function method to treat the interactions perturbatively.

## 3 Applications

### 3.1 Three-point functions in holographic RG flows

The application of \cite{12,13} to three-point functions in holographic RG flows is rather straightforward. First, since both equations of motion are second order differential equations (\Box is replaced by $-k^2$ after Fourier transforming into momentum space), it is easy to write down their formal, non-linear, solutions.

For example,\footnote{Copyright line will be provided by the publisher}

\[
\alpha^a(z) = \int d^d y K^a_{\alpha}(z,y)\hat{a}^a(y) + \int d^{d+1} z' \sqrt{g(z')} \Theta^a_{\alpha}(z,z') \tilde{J}^a(z'),
\]

where $K^a_{\alpha}(z,y)$ and $\Theta^a_{\alpha}(z,z')$ denote the bulk-to-boundary propagator and the bulk Green’s function, respectively. Moreover, $\hat{a}^a$ are the prescribed sources for the scalar fields. Substituting the free solutions into $\tilde{J}^a$ in the second term yields the solution to quadratic order. By determining the asymptotic behaviour of $\hat{a}^a$ from \cite{14}, one then finds the response functions $a^a$ to quadratic order in the sources $\hat{a}^a$ and $\hat{e}^i_j$. Hence, after differentiating twice with respect to the sources and using \cite{12}, one finds the non-local terms of the three-point functions. Generically, they are of the form

\[
(\Psi_1 \Psi_2 \Psi_3) = -\delta(p_1 + p_2 + p_3) \int d^d r e^{iA} \chi_{123} K_1 K_2 K_3,
\]

where $K_i$ ($i = 1, 2, 3$) are the bulk-to-boundary propagators of the fields dual to the operators $\Psi_i$, and the operator $\chi_{123}$ is easily read off from the interaction terms $\tilde{J}^a$. To be more precise, a field re-definition to remove terms in $\tilde{J}^a$ that have two $r$-derivatives, and an integration by parts in \cite{15} might be necessary in order to achieve explicit boson symmetry of the correlators \cite{7,15}.

### 3.2 Scattering amplitudes in the GPPZ flow

The GPPZ flow \cite{19} is the gravity dual of a deformation of $\mathcal{N} = 4$ SYM theory by the insertion of the $\Delta = 3$ operator $\mathcal{O} = \delta_{AB} \text{Tr}(X^A X^B)$, where $A, B = 1, 2, 3$, and $X^A$ are three out of the six scalar fields of $\mathcal{N} = 4$ SYM theory. It is $\mathcal{N} = 1$ supersymmetric and has qualitative features in common with pure $\mathcal{N} = 1$ SYM theory, such as confinement, but not the gluino condensate. The bulk dynamics is governed by an action of the type \cite{3}, with one active scalar $\phi$ (dual to the inserted operator), one inert scalar $\sigma$, which is dual to a scalar operator $\Sigma$ in the gluino-bilinear multiplet $\text{Tr}(W_\alpha W^\alpha)$, and, as usual, the metric fluctuations are dual to the energy momentum tensor, $T^i_j$.

The holographic calculation of the two-point functions of the operators $\mathcal{O}$, $\Sigma$ and $T^i_j$ \cite{19,9,10,11,12,13} has revealed discrete, but infinite, spectra of states labelled by a positive integer $k$, which are interpreted as glueballs of spin zero ($\mathcal{O}_k$ and $\Sigma_k$) and spin two ($T_k$). The mass values of the glueballs all summarized in the second column of table 1.

The start of our collaboration was motivated mostly by the physical question to calculate three-particle scattering amplitudes between the above glueball states. In order to do this from holography, one must calculate all three-point functions of the operators $\mathcal{O}$, $\Sigma$ and $T^i_j$ from the bulk dynamics, and then amputate the external legs in a standard field theory fashion. The gauge-invariant formalism presented in this talk is, in our opinion, the most suitable and elegant approach. As published elsewhere \cite{15}, we have calculated all ten independent three-point functions of the above operators, the final results being given in terms of bulk integrals of the form \cite{15}. For the GPPZ flow, the bulk-to-boundary propagators are hypergeometric.
Table 1 Summary of mass spectra and allowed glueball decay processes. $L$ is the AdS length scale.

| glueball | $m^2L^2$ | decay channels |
|----------|----------|----------------|
| $O_k$    | $4k(k+1)$| $\rightarrow \Sigma_k + \Sigma_1$ ($k > 1$) |
|          |         | $\rightarrow \Sigma_i + \Sigma_j$ ($i + j = k$) |
|          |         | $\rightarrow O_i + T_j$ ($i + j = k - 1$) |
| $\Sigma_k$ | $(4k-1)(k+2)$ | $\rightarrow \Sigma_i + T_j$ ($i + j = k - 1$) |
|          |         | $\rightarrow \Sigma_i + \Sigma_{k-1}$ |
|          |         | $\rightarrow \Sigma_1 + T_{k-1}$ |
| $T_k$    | $4(k+1)^2$| $\rightarrow \Sigma_i + \Sigma_j$ ($i + j = k$) |
|          |         | $\rightarrow \Sigma_k + \Sigma_1$ |
|          |         | $\rightarrow O_i + O_j$ ($i + j = k$) |

functions, and one cannot hope to perform the integrals explicitly. However, putting the external momenta on-shell, each hypergeometric function reduces to a product of the corresponding on-shell mass pole, which is amputated for the scattering amplitudes, and a Jacobi polynomial. Hence, the bulk integrals for the scattering amplitudes are elementary. Our analysis has revealed that, surprisingly, most of the kinematically allowed scattering amplitudes vanish. These selection rules have a deeper origin in certain orthogonality relations between the Jacobi polynomials. Table 1 summarizes the selection rules for the (kinematically and dynamically) allowed glueball decay processes.

Although the GPPZ flow does not correctly capture the IR dynamics of the $\mathcal{N} = 1$ supersymmetric field theory (for example, the gluino condensate is missing) we believe that the selection rules, which we have found, are insensitive to the inclusion of further non-perturbative effects, or, at least, that the effects of these corrections are very weak. Hence, a comparison with field theory data, e.g., from lattice simulations of $\mathcal{N} = 1$ SYM theory, would be very interesting.

Acknowledgements I would like to thank my collaborators Marcus Berg, Massimo Bianchi, Michael Haack and Maurizio Prisco for many inspiring discussions.

Financial support from the European Commission’s RTN Programme, contract HPRN-CT-2000-00131, from the Italian ministry of education and research (MIUR), projects 2001-1025492 and 2003-023852, and from INFN is gratefully acknowledged.

References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[2] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).
[4] M. Bianchi, D. Z. Freedman, K. Skenderis, Nucl. Phys. B 631, 159 (2002).
[5] D. Martelli, W. Mück, Nucl. Phys. B 654, 248 (2003).
[6] I. Papadimitriou, K. Skenderis, preprint [hep-th/0404176].
[7] M. Bianchi, M. Prisco, W. Mück, JHEP 11, 052 (2003).
[8] D. Z. Freedman, S. S. Gubser, K. Pilch, N. P. Warner, Adv. Theor. Math. Phys. 3, 363 (1999).
[9] O. DeWolfe, D. Z. Freedman, preprint [hep-th/0002226].
[10] G. Arutyunov, S. Frolov, S. Theisen, Phys. Lett. B 491, 295 (2000).
[11] M. Bianchi, O. DeWolfe, D. Z. Freedman, K. Pilch, JHEP 01, 021 (2001).
[12] W. Mück, Nucl. Phys. B 620, 477 (2002).
[13] M. Bianchi, D. Z. Freedman, K. Skenderis, JHEP 08, 041 (2001).
[14] M. Bianchi, A. Marchetti, Nucl. Phys. B 686, 261 (2004).
[15] W. Mück, M. Prisco, JHEP 04, 037 (2004).
[16] I. R. Klebanov, M. J. Strassler, JHEP 08, 052 (2000).
[17] J. M. Maldacena, C. Nunez, Phys. Rev. Lett. 86, 588 (2001).
[18] A. Z. Petrov, Einstein Spaces (Pergamon Press, Oxford, 1969).
[19] L. Girardello, M. Petroni, M. Porrati, A. Zaffaroni, Nucl. Phys. B 569, 451 (2000).