Existence, nonexistence, symmetry and uniqueness of ground state for critical Schrödinger system involving Hardy term

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Abstract
We study the following elliptic system with critical exponent:
\[ \begin{cases} -\Delta u_j - \frac{\lambda_j}{|x|^2} u_j = u_j^{2^*-1} + \sum_{k \neq j} \beta_{jk} \alpha_{jk} u_j^{\alpha_{jk}-1} u_k^{\alpha_{kj}}, & x \in \mathbb{R}^N, \\ u_j \in D^{1,2} (\mathbb{R}^N), & u_j > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \quad j = 1, \ldots, r. \end{cases} \]

Here \( N \geq 3, r \geq 2, 2^* = \frac{2N}{N-2}, \lambda_j \in (0, \frac{(N-2)^2}{4}) \) for all \( j = 1, \ldots, r \); \( \beta_{jk} = \beta_{kj} \); \( \alpha_{jk} > 1, \alpha_{kj} > 1 \), satisfying \( \alpha_{jk} + \alpha_{kj} = 2^* \) for all \( k \neq j \). Note that the nonlinearities \( u_j^{2^*-1} \) and the coupling terms all are critical in arbitrary dimension \( N \geq 3 \). The signs of the coupling constants \( \beta_{ij} \)'s are decisive for the existence of the ground state solutions. We show that the critical system with \( r \geq 3 \) has a positive least energy solution for all \( \beta_{jk} > 0 \). However, there is no ground state solutions if all \( \beta_{jk} \) are negative. We also prove that the positive solutions of the system are radially symmetric. Furthermore, we obtain the uniqueness theorem for the case \( r \geq 3 \) with \( N = 4 \) and the existence theorem when \( r = 2 \) with general coupling exponents.

1 Introduction
Consider the solitary wave solutions to the time-depending \( r \)-coupled nonlinear Schrödinger equations given by
\[ \begin{cases} -i \frac{\partial \Phi_j}{\partial t} = \Delta \Phi_j - \alpha_j(x) \Phi_j + \mu_j |\Phi_j|^2 \Phi_j + \sum_{i \neq j} \beta_{ij} |\Phi_i|^2 \Phi_j, \\ \Phi_j = \Phi_j(x, t) \in \mathbb{C}, \quad j = 1, 2, \ldots, r; \quad x \in \mathbb{R}^N, \quad t > 0, \\ \Phi_j(x, t) \to 0, \quad \text{as } |x| \to +\infty, \quad t > 0, \quad j = 1, 2, \ldots, r. \end{cases} \]
and may transform the system (1.1) to steady-state system:

for the self-focusing in the Subcritical case:

We briefly recall some previous works on this line. Especially in nonlinear optics, Physically, the solution of the beam in Kerr-like photorefractive media. The positive constant solitary wave solutions of the system (1.1), ones usually set ground state for the system (1.3).

We are concerned with the existence, nonexistence, symmetry and uniqueness of the 1-dimensional system (1.2) with \( N \geq 2 \), \( r \) coupled nonlinear Schrödinger system:

\[
\begin{align*}
-\Delta u_j + V_j(x)u_j &= \mu_j u_j^3 + \sum_{k \neq j} \beta_{jk} u_k^2 u_j, \quad x \in \mathbb{R}^N, \\
u_j &\geq 0, \quad x \in \mathbb{R}^N, \quad u_j &\to 0 \text{ as } |x| \to +\infty; \quad j = 1, 2, ..., r.
\end{align*}
\]

Subcritical case: When \( N \leq 3 \), then the critical Sobolev exponent \( 2^* := \frac{2N}{N-2} \in [6, +\infty] \) and hence the nonlinear terms (including the coupling terms) of (1.2) are of subcritical growth. For such cases, we call the system (1.2) subcritical which has received great interest in the last decade and large number of papers published. On this line, although we can not exhaustly enumerate all those articles, we refer the readers to [2, 5, 6, 3, 4, 8, 9, 12, 14, 18, 29, 27, 28, 30, 22, 23, 24, 31, 32, 33, 34, 35, 36, 37, 41, 43, 44, 45, 46] and the references cited therein for various existence of solutions.

Critical case: When \( N = 4 \), then the critical Sobolev exponent \( 2^* = 4 \) and thus the nonlinear terms (including the coupling terms) of (1.2) all are of critical growth. Due to the lack of compactness, this kind problems become thorny. Basically, such a system (1.2) with \( r = 2 \) and \( V_j = \text{const} \) was firstly studied in [13] (including the same system defined on a bounded domain). The positive least energy solutions and phase separation were obtained in [13]. Later, the higher dimension case (i.e., \( N \geq 5 \)) was also considered in [17] where some different phenomenon from the 3-D and 4-D cases were observed. We also note that, in [42], a partial symmetry was involved when \( N = 4 \) (and \( N = 2, 3 \)) under the premise of assuming the existence of the minimizer.

In the current paper, we are interested in the following \( r \)-coupling system:

\[
\begin{align*}
-\Delta u_j - \frac{\lambda_j}{|x|^2} u_j &= u_j^{2^*-1} + \sum_{k \neq j} \beta_{jk} \alpha_{jk} u_k^{\alpha_{jk}-1} u_j^{\alpha_{kj}}, \quad x \in \mathbb{R}^N, \\
u_j &\in D^{1,2}(\mathbb{R}^N), \quad u_j > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \quad j = 1, ..., r;
\end{align*}
\]

where \( N \geq 3, r \geq 2, \lambda_j \in (0, \frac{2}{(N-2)^2}) \) for all \( j = 1, ..., r \); and \( \beta_{jk} = \beta_{kj}, \ \alpha_{jk} > 1, \alpha_{kj} > 1 \), satisfying \( \alpha_{jk} + \alpha_{kj} = 2^* \) for all \( k \neq j \). Note that \( \alpha_{jk} \neq \alpha_{kj} \) is allowed. We are concerned with the existence, nonexistence, symmetry and uniqueness of the ground state for the system (1.3).
When \( V_j(x) = -\frac{\lambda_j}{|x|^2} \), the Hardy’s type potentials appear, then the system (1.3) arises in several physical contexts including nonrelativistic quantum mechanics, molecular physics, quantum cosmology, and linearization of combustion models. The Hardy’s type potentials do not belong to Kato’s class, so they cannot be regarded as a lower order perturbation term. In particular, any nontrivial solution is singular at \( x = 0 \). We refer to the papers [1, 19, 38, 40] for the scalar equations.

For the case of \( r = 2 \), the two-coupled system (1.3) has been studied in [16] where the positive ground state solutions are obtained and are all radially symmetric. It turns out that the least energy level depends heavily on the relations among \( \alpha_{jk} \) and \( \alpha_{kj} \). Besides, for sufficiently small coupling constants, positive solutions are also obtained via a variational perturbation approach. It is pointed out that the Palais-Smale condition cannot hold for any positive energy level, which makes the study via variational methods rather complicated, see [16]. We remark that in [15], when the coupling constant is replaced by a function decaying to zero, then the existence of ground state is obtained.

However, when \( r > 2 \), the study of system (1.3) becomes rather complicated. In particular, even for the two-coupled case of (1.3) (i.e., \( r = 2 \)), the characteristics and uniqueness of the least energy solution to (1.3) have not been solved completely in [16] (see Remarks 1.1-1.2 below). In the present paper, we give some positive answers for several standing problems related to the system (1.3). We will introduce some quite different techniques than usual. Precisely, we shall study some nonlinear constraint problems which will play an important role for exploring the multi-coupled system (1.3). We consummate the results due to [13, 15, 16, 17].

Let \( \lambda_j \in (0, \Lambda_N) \) for all \( j = 1, \ldots, r \), where \( \Lambda_N := \frac{(N-2)^2}{4} \). Set
\[
\|u\|_{\lambda_j}^2 := \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\lambda_j}{|x|^2} u^2, \quad \langle u, v \rangle_{\lambda_j} := \int_{\mathbb{R}^N} \nabla u \nabla v - \frac{\lambda_j}{|x|^2} uv, \tag{1.4}
\]
for all \( u, v \in D^{1,2} := D^{1,2}(\mathbb{R}^N) \). Denote the norm of \( L^p(\mathbb{R}^N) \) by \( |u|^p = (\int_{\mathbb{R}^N} |u|^p)^{1/p} \). Let
\[
I_{\lambda_j}(u) := \frac{1}{2} \|u\|_{\lambda_j}^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^2, \quad j = 1, \ldots, r. \tag{1.5}
\]
We call a solution \((u_1, \ldots, u_r)\) of (1.3) nontrivial if all \( u_j \neq 0, j = 1, \ldots, r \). We call that a solution \((u_1, \ldots, u_r)\) is positive if all \( u_j > 0 \) in \( \mathbb{R}^N \setminus \{0\} \) for all \( j = 1, \ldots, r \). We call a solution \((u_1, \ldots, u_r) \neq (0, \ldots, 0)\) is semi-trivial if there exists some \( i_0 \) satisfying \( u_{i_0} \equiv 0 \). Throughout this paper, we are only interested in nontrivial solutions of (1.3). Define \( \mathbb{D} := D^{1,2} \times \cdots \times D^{1,2} \) with the norm
\[
\|(u_1, \ldots, u_r)\|_{\mathbb{D}}^2 := \sum_{j=1}^r \|u_j\|_{\lambda_j}^2. \tag{1.6}
\]
Then the nontrivial solutions of (1.3) correspond to the nontrivial critical points of the \( C^1 \) functional \( J : \mathbb{D} \rightarrow \mathbb{R} \), where
\[
J(u_1, \ldots, u_r) := \sum_{j=1}^r I_{\lambda_j}(u_j) - \frac{1}{2} \sum_{1 \leq j \neq k \leq r} \beta_{jk} \int_{\mathbb{R}^N} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}. \tag{1.7}
\]
Definition 1.1. We say a solution \((u_{1,0}, ..., u_{r,0})\) of (1.3) is a ground state solution if \((u_{1,0}, ..., u_{r,0})\) is nontrivial and \(J(u_{1,0}, ..., u_{r,0}) \leq J(u_{1}, ..., u_{r})\) for any other nontrivial solution \((u_{1}, ..., u_{r})\) of (1.3).

To obtain the ground state solutions of (1.3), we define the Nehari manifold:

\[
\mathcal{N} := \{(u_{1}, ..., u_{r}) \in \mathbb{D} : \quad u_{j} \neq 0 \text{ and } \quad \|u_{j}\|_{\lambda_{j}}^{2} = \int_{\mathbb{R}^{N}} |u_{j}|^{2} + \sum_{k \neq j} \beta_{jk} \alpha_{j} \alpha_{k} |u_{j}|^{\alpha_{j}} |u_{k}|^{\alpha_{k}}, \quad j = 1, ..., r\}. \tag{1.8}
\]

Then any nontrivial solution of (1.3) belongs to \(\mathcal{N}\). Note that \(\mathcal{N} \neq \emptyset\). We set

\[
\Theta := \inf_{(u_{1}, ..., u_{r}) \in \mathcal{N}} J(u_{1}, ..., u_{r}). \tag{1.9}
\]

Hence, \(\Theta = \inf_{(u_{1}, ..., u_{r}) \in \mathcal{N}} \frac{1}{N} \sum_{j=1}^{r} \|u_{j}\|^{2}_{\lambda_{j}}\). It is easy to see that \(\Theta > 0\).

Recall the following scalar equation which has been deeply investigated in the literature (see for example [40]):

\[
\begin{cases}
-\Delta u - \frac{\lambda_{i}}{|x|^{2}} u = u^{2^{*} - 1}, & x \in \mathbb{R}^{N}, \\
u \in D^{1,2}(\mathbb{R}^{N}), & u > 0 \text{ in } \mathbb{R}^{N} \setminus \{0\},
\end{cases} \tag{1.10}
\]

which has exactly an one-dimensional \(C^{2}\)-manifold of positive solutions given by

\[
Z_{i} := \{z_{i}^{\mu}(x) = \mu \frac{A(N, \lambda_{i})}{|x|^{\alpha_{\lambda_{i}}} (1 + |x|^{2 - \alpha_{\lambda_{i}}})^{\frac{N-2}{2}}} : \mu > 0\}, \tag{1.11}
\]

where

\[
z_{i}^{1}(x) = \frac{A(N, \lambda_{i})}{|x|^{\alpha_{\lambda_{i}}} (1 + |x|^{2 - \alpha_{\lambda_{i}}})^{\frac{N-2}{2}}} \tag{1.12}
\]

and

\[
\alpha_{\lambda_{i}} = \frac{N - 2}{2} - \sqrt{\frac{(N - 2)^{2}}{4} - \lambda_{i}}, \quad A(N, \lambda_{i}) = \frac{N(N - 2 - 2\alpha_{\lambda_{i}})^{2}}{N - 2}. \tag{1.12}
\]

Moreover, all positive solutions of (1.10) satisfy

\[
I_{\lambda_{i}}(z_{i}^{\mu}) = \frac{1}{N} \|z_{i}^{\mu}\|_{\lambda_{i}}^{2} = \frac{1}{N} \frac{\|z_{i}^{\mu}\|_{\lambda_{i}}^{2}}{\|z_{i}^{\mu}\|_{2}^{2}} = \frac{1}{N} S(\lambda_{i})^{\frac{N}{2}},
\]

where

\[
S(\lambda_{i}) := \inf_{u \in D^{1,2}(\mathbb{R}^{N}) \setminus \{0\}} \frac{\|u\|_{\lambda_{i}}^{2}}{\|u\|_{2}^{2}} = \frac{\|z_{i}^{\mu}\|_{\lambda_{i}}^{2}}{\|z_{i}^{\mu}\|_{2}^{2}} = \left(1 - \frac{4\lambda_{i}}{(N - 2)^{2}}\right)^{\frac{N-4}{2}} S
\]

and \(S\) is the sharp constant of \(D^{1,2}(\mathbb{R}^{N}) \hookrightarrow L^{2^{*}}(\mathbb{R}^{N})\) (see e.g., [39]):

\[
\int_{\mathbb{R}^{N}} |\nabla u|^{2} \geq S \left(\int_{\mathbb{R}^{N}} |u|^{2^{*}}\right)^{\frac{N}{N-4}}. \tag{1.12}
\]

In the current paper, we always assume that \(\beta_{jk} = \beta_{kj}\) for \(1 \leq j, k \leq r\). Now we are ready to state the main theorems of this article.
Theorem 1.1. Assume that $N \geq 3$, $\lambda_j \in (0, \Lambda_N)$ and $\alpha_{jk} > 1$, $\alpha_{kj} > 1$, $\alpha_{jk} + \alpha_{kj} = 2^*$ for $1 \leq j, k \leq r$.

1. (Nonexistence) If $\beta_{jk} < 0$, $\forall k \neq j$, then

$$\Theta \equiv \sum_{j=1}^{N} \frac{1}{N} \left( \frac{1}{(N-2)^2} \frac{8\pi^2}{N} S \right)^{\frac{N}{2}},$$

and $\Theta$ cannot be attained, i.e., there is no ground state solution to (1.3).

2. (Existence) Let $\beta_{jk} > 0$, $\forall k \neq j$, satisfy

$$\left( r + \sum_{j,k=1, j \neq k}^{r} \frac{2^* \beta_{jk}}{2} \right) / \left( \max_{j,l} B_{j,l} \right) > \left( 1 + \sum_{\alpha=1}^{r-1} \frac{\Lambda_N - \lambda_{\alpha}}{\Lambda_N - \lambda_{r}} \right) \frac{N}{N-2},$$

then (1.3) has a positive ground state solution $(u_1, ..., u_r) \in \mathbb{D}$, which is radially symmetric and whose energy satisfies

$$\Theta < \min_{l \in \{1, 2, ..., r\}} \min_{j \in A^l} B_{j,l} \frac{\max_{j,l} B_{j,l}}{N} \frac{1}{N} S(\lambda_j)^{\frac{N}{2}},$$

where $B_{j,l} = \sum_{k \neq j, k \in A^l} \beta_{jk} \alpha_{jk} + 1$, $A^l = \{1, 2, ..., r\} \setminus \{l\}$.

Theorem 1.2. Assume that $N = 3$ or $N = 4$, $\alpha_{jk} + \alpha_{kj} = 2^*$, $\alpha_{jk} \geq 2$, $\alpha_{kj} \geq 2$, $\lambda_j \in (0, \Lambda_N)$, $\beta_{jk} > 0$, $\forall k \neq j$, then any positive solution of (1.3) is radially symmetric with respect to the origin.

Next, we obtain the existence and uniqueness results about the ground state to the following critical elliptic system in $\mathbb{R}^4$ involving the Hardy’s singular term:

$$\left\{ \begin{array}{l}
-\Delta u_j - \frac{\lambda}{|x|^2} u_j = \gamma_{jj} u_j^3 + \sum_{i \neq j} \gamma_{ij} u_i^2 u_j, \\
\quad u_j(x) > 0, \quad j = 1, ..., r, \quad x \in \mathbb{R}^4 \setminus \{0\}.
\end{array} \right. \quad (1.13)$$

We have the following result.

Theorem 1.3. Considering the system (1.13). Assume that

$N = 4$, $r \geq 3$, $\lambda \in (0, \Lambda_N)$, $\gamma_{ji} = \gamma_{ij}$, $\det(\gamma_{ij}) \neq 0$, $\sum_{k} \gamma_{kj} > 0$, $i, j = 1, ..., r$;

where the matrix $(\gamma^{jk})$ represents the inverse matrix of $(\gamma_{mj})$. Then

1. (existence) $(\sqrt{\gamma_1^{11}} z_1, ..., \sqrt{\gamma_r^{11}} z_r^1)$ ($\mu > 0$) is a positive least energy solution of (1.13), where $z_1^\mu$ is a solution of (see (1.11))

$$\left\{ \begin{array}{l}
-\Delta u - \frac{\lambda}{|x|^2} u = u^3, \quad x \in \mathbb{R}^N, \\
\quad u \in D^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\},
\end{array} \right. \quad (1.14)$$
and the constant $c_j > 0$ satisfying

$$\sum_{k=1}^{\gamma_{jk}c_k = 1, \quad j = 1, ..., r.}$$

(2) (uniqueness) let $(u_1, u_2, ..., u_r)$ be any least energy solution of (1.13), then $(u_1, u_2, ..., u_r) = (\sqrt{c_1z_1}, ..., \sqrt{c_rz_r})$, where

$$\sum_{k=1}^{\gamma_{jk}c_k = 1, \quad j = 1, ..., r.}$$

Remark 1.1. When $r = 2$, $\lambda = 0$, the existence of the ground state for system (1.13) in $\mathbb{R}^4$ was firstly studied in [13].

Lastly, we consider the following two-coupled doubly critical Shr¨ odinger system:

$$\begin{cases}
-\Delta u - \lambda \frac{|x|^2}{|x|^2} u = u^{2^*-1} + \nu \alpha u^{\alpha - 1} v^\beta, \quad x \in \mathbb{R}^N, \\
-\Delta v - \lambda \frac{|x|^2}{|x|^2} v = v^{2^*-1} + \nu \beta u^{\alpha - 1} v^{\beta - 1}, \quad x \in \mathbb{R}^N.
\end{cases}
$$ (1.15)

We have the following theorem.

**Theorem 1.4.** In the system (1.15), we assume that $\lambda \in (0, \Lambda_N)$, $1 < \alpha, \beta < 2$ and $\alpha + \beta = 2^*$ (these imply $N \geq 5$).

(1) If $\nu > 0$, then $(c_1z_\mu, c_2z_\mu)$ is a positive solution of (1.15) for any $\mu > 0$, where $z_\mu$ is a solution of the following equation:

$$\begin{cases}
-\Delta u - \lambda \frac{|x|^2}{|x|^2} u = u^{2^*-1}, \quad x \in \mathbb{R}^N, \\
u \alpha x_1^{\alpha - 1} x_2^\beta = 1, \quad x \in D^{1,2}(\mathbb{R}^N), \quad u > 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\end{cases}
$$ (1.16)

(2) If

$$\nu > \left(\frac{2^*}{2} - 1\right)/\min\{d_1(\alpha, \beta), d_2(\alpha, \beta), d_3(\alpha, \beta)\},$$

then $(c_1z_\mu, c_2z_\mu)$ is a positive ground state solution of (1.15), where $c_1, c_2$ are the roots of the algebraic system about $(x_1, x_2)$:

$$\begin{cases}
x_1^{\alpha - 1} + \nu \alpha x_1^{\alpha - 1} x_2^\beta = 1, \\
x_2^{\beta - 1} + \nu \beta x_1^{\alpha - 1} x_2^\beta = 1,
\end{cases}
$$

and $d_1(\alpha, \beta), d_2(\alpha, \beta), d_3(\alpha, \beta)$ are defined as following:

$$d_1(\alpha, \beta) = 2^*(1 - \frac{\alpha}{2})^{\frac{\beta}{2}} - (1 - \frac{\alpha}{2})^{\frac{\beta}{2}} \text{ if } \alpha \neq \beta; \quad d_1(\alpha, \beta) = 2^*/2 \text{ if } \alpha = \beta;$$

$$d_2(\alpha, \beta) = \beta(1 - \frac{\beta}{2})^{1+\frac{\beta}{2}} - (1 - \frac{\alpha}{2})^{\frac{\beta}{2}} + \frac{1}{2} \alpha \beta(1 - \frac{\alpha}{2})^{1-\frac{\beta}{2}} (1 - \frac{\beta}{2})^{1-\frac{\beta}{2}};$$

$$d_3(\alpha, \beta) = \alpha(1 - \frac{\alpha}{2})^{1-\frac{\beta}{2}} (1 - \frac{\beta}{2})^{\frac{\beta}{2}} + \frac{1}{2} \alpha \beta(1 - \frac{\alpha}{2})^{1-\frac{\beta}{2}} (1 - \frac{\beta}{2})^{1-\frac{\beta}{2}}.$$
Remark 1.2. When $N \geq 5$, the existence of ground state solution is essentially proved in [16]. Here, the further characteristics is given. If $\alpha = \beta$, then
\[
\left( \frac{2^*}{2} - 1 \right) \min \{ d_1(\alpha, \beta), d_2(\alpha, \beta), d_3(\alpha, \beta) \} = \frac{2}{N}.
\]
We remark that, for the special case of $\alpha = \beta = \frac{2^*}{2}$, the uniqueness of the ground state solution of (1.15) was obtained by a different method in [16].

The paper is organized as follows. In Section 2, we develop several lemmas which will also have other applications. We give the the proof of Theorem 1.1 in Section 3, where we will use the concentration-compactness principle due to [25, 26]. In Section 4, Theorem 1.2 is proved by the moving plane method. In Section 5, we firstly construct some powerful lemmas and then obtain the existence and uniqueness results about the positive ground state. Theorems 1.3-1.4 will get proved there.

2 Preliminaries

We firstly deal with the following nonlinear algebraic equations which is important for construct the nonexistence of the ground state solution.

Lemma 2.1. Assume
\[
C_j := \| u_j \|_{2^*}^2 - \sum_{k \neq j} \frac{\alpha_{jk}^2 - \alpha_{jk} \alpha_{kj}}{2^*} \int_{\mathbb{R}^N} |\beta_{jk}| |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}} > 0, \quad j = 1, ..., r. \tag{2.1}
\]
Consider the algebraic equations about $t_j$:
\[
t_j^2 \| u_j \|_{\lambda_j}^2 = t_j^{2^*} \| u_j \|_{2^*}^2 + \sum_{k \neq j} t_j^{\alpha_{jk}} t_k^{\alpha_{kj}} \int_{\mathbb{R}^N} \beta_{jk} \alpha_{jk} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}, \quad j = 1, ..., r. \tag{2.2}
\]
where $\beta_{jk} < 0, \alpha_{jk} > 1, \alpha_{kj} > 1, \alpha_{jk} + \alpha_{kj} = 2^*, u_j \neq 0$. Then we have the following priori estimate for the positive solution of (2.2) (if any):
\[
\min_{1 \leq j \leq r} \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \leq \left( \frac{\| u_j \|_{2^*}^2}{\| u_j \|_{\lambda_j}^2} \right)^{\frac{1}{\alpha}} \leq t_j \leq \left( \sum_{j=1}^r A_j \min \{ C_1, ..., C_r \} \right)^{-\frac{1}{\alpha}}, \tag{2.3}
\]
where
\[
\alpha = 2^* - 2 = \frac{4}{N - 2}, \quad A_j = \| u_j \|_{\lambda_j}^2 > 0, \quad B_j = \| u_j \|_{2^*}^2 > 0, \quad j = 1, ..., r.
\]
In particular, the systems (2.2) has a positive solution provided that
\[
d := \frac{1}{\alpha} \max_j \left( \frac{A_j}{B_j} \right)^{-\frac{1}{\alpha}} \left( 1 + \max_j \max f_j \max_{j,m} \frac{\partial f_j}{\partial t_m} \right) < 1, \tag{2.4}
\]
where
\[
f_j(t_1, ..., t_r) := \frac{1}{A_j} \sum_{k \neq j} t_j^{\alpha_{jk} - 2} t_k^{\alpha_{kj}} \int_{\mathbb{R}^N} |\beta_{jk}| \alpha_{jk} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}. \tag{2.5}
\]
Furthermore, if
\[ \bar{\beta}_{jk} := \int_{\mathbb{R}^N} |\beta_{jk}| u_j^{\alpha_{jk}} |u_k|^{\alpha_{kj}}, \quad \forall j \neq k, \] (2.6)
all are small enough, then (2.2) admits a positive solution \((t_1, ..., t_r)\). In particular, each \(t_s\) \((s = 1, ..., r)\) satisfying
\[ t_s \to \left( \frac{\|u_s\|_{L^2}^2}{|u_s|_{L^2}^*} \right)^{\frac{1}{\alpha}} \] as all \(\bar{\beta}_{jk} \to 0, \quad \forall j \neq k. \] (2.7)

**Remark 2.1.** In Lemma 2.1 above, \(\max f_j = \max f_j(t_1,...,t_r)\) which is a finite value in view of the priori bound on \((t_1,...,t_r)\) obtained in (2.3). The same conclusion is true for \(\max |\partial f_j/\partial t_m|\).

**Proof.** Firstly, by definition we have \(\min_{1 \leq j \leq r} \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \leq \left( \frac{\|u_j\|_{L^2}^2}{|u_j|_{L^2}^*} \right)^{\frac{1}{\alpha}}\). Since \(\beta_{jk} < 0, \quad \forall j \neq k\), then we have
\[ t_j^2 |u_j|_{L^2}^* \leq t_j^2 \left( \|u_j\|_{L^2}^2 \right)^{\frac{1}{\alpha}}. \]
that is,
\[ t_j \geq \left( \frac{\|u_j\|_{L^2}^2}{|u_j|_{L^2}^*} \right)^{\frac{1}{\alpha}}. \]
Recall that \(\alpha_{jk} + \alpha_{kj} = 2^*\), by Young’s inequality, we have
\[ t_j^{\alpha_{jk}} t_k^{\alpha_{kj}} \leq \frac{\alpha_{jk}}{2^*} t_j^{2^*} + \frac{\alpha_{kj}}{2^*} t_k^{2^*}, \]
then
\[ t_j^2 A_j = t_j^2 B_j - \sum_{k \neq j} \int_{\mathbb{R}^N} |\beta_{jk}| |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}} t_j^{\alpha_{jk}} t_k^{\alpha_{kj}} \]
\[ \geq B_j t_j^{2^*} - \sum_{k \neq j} D_{jk} \frac{\alpha_{jk}}{2^*} t_j^{2^*} + D_{jk} \frac{\alpha_{kj}}{2^*} t_k^{2^*} \]
\[ = (B_j - \sum_{k \neq j} D_{jk} \frac{\alpha_{jk}}{2^*}) t_j^{2^*} - \sum_{k \neq j} D_{jk} \frac{\alpha_{kj}}{2^*} t_k^{2^*}, \]
where
\[ D_{jk} := \int_{\mathbb{R}^N} |\beta_{jk}| |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}. \]
Summing up (2.8) from $j = 1$ to $j = r$, thus

\[ \sum_{j=1}^{r} A_j t_j^2 \geq \sum_{j=1}^{r} (B_j - \sum_{k \neq j} D_{jk} \frac{\alpha_{jk}}{2^*}) t_j^2^* - \sum_{j=1}^{r} \sum_{k \neq j} D_{jk} \left( \frac{\alpha_{kj}}{2^*} \right)^* t_k^2^* \]

\[ = \sum_{j=1}^{r} (B_j - \sum_{k \neq j} D_{jk} \frac{\alpha_{jk}}{2^*}) t_j^2^* - \sum_{j=1}^{r} \sum_{k \neq j} D_{kj} \left( \frac{\alpha_{jk}}{2^*} \right)^* t_j^2^* \]

\[ = \sum_{j=1}^{r} (B_j - \sum_{k \neq j} \frac{\alpha_{jk}(D_{jk} - D_{kj})}{2^*} ) t_j^2^* \]

\[ = \sum_{j=1}^{r} C_j t_j^2^*. \]

(2.9)

Recall that $A_j > 0, C_j > 0, j = 1, \ldots, r$. For the positive solution of (2.2), without loss of generality, we assume that $t_1 = \max\{t_1, \ldots, t_r\}$, then we have

\[ C_1 t_1^2^* \leq \sum_{j=1}^{r} A_j t_j^2 \leq \sum_{j=1}^{r} A_j t_j^2, \]

that is,

\[ t_1 := \max_j t_j \leq \left( \frac{1}{C_1} \sum_{j=1}^{r} A_j \right)^{\frac{1}{\alpha}} \leq \left( \frac{1}{\min_j C_j} \sum_{j=1}^{r} A_j \right)^{\frac{1}{\alpha}}. \]

(2.10)

Hence the priori estimate is obtained. Hence, there are two positive constants $T_1 > 0, T_2 > 0$ such that for all positive solution $t_j$ of (2.2):

\[ t_j \in [T_1, T_2], \quad \forall j = 1, \ldots, r. \]

In the following, we will use Picard’s iteration to obtain the existence of positive solution of (2.2). Recall the notation of $f_j$ in (2.5), the equation (2.2) becomes

\[ t_j = \left( \frac{A_j}{B_j} + \frac{A_j}{B_j} f_j(t_1, \ldots, t_r) \right)^{\frac{1}{\alpha}} \]

\[ = \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \left( 1 + f_j(t_1, \ldots, t_r) \right)^{\frac{1}{\alpha}}, \quad j = 1, \ldots, r. \]

(2.11)
We select arbitrarily an initial value \( t_0 = (t_{1,0}, \ldots, t_{r,0}) \in [T_1, T_2]^r \), then
\[
|t_{j,n+1} - t_{j,n}| \\
= \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \left[ \left( 1 + f_j(t_{1,n}, \ldots, t_{r,n}) \right)^{\frac{1}{\alpha}} - \left( 1 + f_j(t_{1,n-1}, \ldots, t_{r,n-1}) \right)^{\frac{1}{\alpha}} \right] \\
= \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \left( 1 + f_j(\xi) \right) \left( \frac{\partial f_j}{\partial t_m}(\xi) \right)^{-1} \sum_{m=1}^{r} |t_{m,n} - t_{m,n-1}| \\
\leq \frac{1}{\alpha} \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \left( 1 + \max_j \max_m f_j \right) \max_j \left| \frac{\partial f_j}{\partial t_m} \right| \sum_m |t_{m,n} - t_{m,n-1}|,
\]
where \( \xi \) is a vector between \( t_n = (t_{1,n}, \ldots, t_{r,n}) \) and \( t_{n-1} = (t_{1,n-1}, \ldots, t_{r,n-1}) \). Add up the above inequalities from \( j = 1 \) to \( j = r \), we get that
\[
\sum_{j=1}^{r} |t_{j,n+1} - t_{j,n}| \\
\leq r \frac{1}{\alpha} \max_j \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \left( 1 + \max_j \max_m f_j \right) \max_j \left| \frac{\partial f_j}{\partial t_m} \right| \sum_m |t_{m,n} - t_{m,n-1}| \tag{2.13}
\]
\[
:= d \sum_m |t_{m,n} - t_{m,n-1}|.
\]

By the assumption \((2.4)\), \( 0 < d < 1 \), thus we may apply the classical contraction mapping principle and know that the vector sequence \( t_n = (t_{1,n}, \ldots, t_{r,n}) \) is convergent, say \( t_n = (t_{1,n}, \ldots, t_{r,n}) \to t = (t_1, \ldots, t_r) \) as \( n \to \infty \) and \( t \) is a solution of \((2.2)\). Further, by our priori estimate,
\[
\min_j \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \leq t_{j,n} \leq \left( \frac{1}{\min\{C_1, \ldots, C_r\} \sum_{j=1}^{r} A_j} \right)^{\frac{1}{\alpha}}, \quad j = 1, \ldots, r.
\]
Let \( n \to \infty \), we have
\[
\min_j \left( \frac{A_j}{B_j} \right)^{\frac{1}{\alpha}} \leq t_j \leq \left( \frac{1}{\min\{C_1, \ldots, C_r\} \sum_{j=1}^{r} A_j} \right)^{\frac{1}{\alpha}}, \quad j = 1, \ldots, r,
\]
it implies that \( t = (t_1, \ldots, t_r) \) is a positive solution of \((2.2)\). Furthermore, if \( \beta_{jk} \) (which is defined in \((2.6)\) all are small enough, then it is to see that the solvability conditions \((2.1)\) and \((2.4)\) hold. Hence, there exists a positive solution of \((2.2)\). By the priori estimate of this positive solution and in view of \((2.2)\), we get that \( t_j \to \left( \frac{\|u_j\|_2^2}{\|u_j\|_2^{2r}} \right)^{\frac{1}{\alpha}}, \quad j = 1, \ldots, r. \)

**Lemma 2.2.** If \( \Theta \) (which is defined in \((1.5)\)) is attained by \((u_1, \ldots, u_r) \in \mathcal{N} \), then it is a critical point of \( J \) (which is introduced in \((1.7)\)) provided that \( \beta_{jk} < 0, \forall k \neq j. \)
Proof. Suppose $\beta_{jk} < 0, \forall k \neq j$. Assume that $(u_1, ..., u_r) \in \mathcal{N}$ such that $\Theta = J(u_1, ..., u_r)$. Define

$$G_j(u_1, ..., u_r) = \|u_j\|_X^2 - \int_{\mathbb{R}^N} \left( |u_j|^2 + \sum_{k \neq j} \beta_{jk} \alpha_{jk} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}} \right), j = 1, ..., r.$$ 

Then there exist $L_j \in \mathbb{R}$ ($j = 1, ..., r$) such that

$$J'(u_1, ..., u_r) + \sum_{j=1}^{N} L_j G'_j(u_1, ..., u_r) = 0. \tag{2.14}$$

Testing (2.14) with $(0, ..., 0, u_i, 0, ..., 0)$ ($i = 1, ..., r$), we conclude from $(u_1, ..., u_r) \in \mathcal{N}$ that

$$\langle G'_j, (0, ..., u_j, ..., 0) \rangle = 2 \int_{\mathbb{R}^N} |\nabla u_j|^2 - 2 \int_{\mathbb{R}^N} \frac{\lambda_j}{|x|^2} u_j^2$$

$$- 2^* \int_{\mathbb{R}^N} |u_j|^{2^*} - \sum_{k \neq j} \int_{\mathbb{R}^N} \beta_{jk} \alpha_{jk}^2 |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}$$

$$= (2 - 2^*) \int_{\mathbb{R}^N} |u_j|^{2^*} - 2 \sum_{k \neq j} \int_{\mathbb{R}^N} \beta_{jk} \alpha_{jk} (2 - \alpha_{jk}) |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}.$$ 

For $k \neq j$, we have

$$\langle G'_k, (0, ..., u_k, ..., 0) \rangle = - \beta_{jk} \alpha_{jk} \alpha_{kj} \int_{\mathbb{R}^N} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}.$$ 

Then

$$\sum_{j=1}^{N} L_j \langle G'_j, (0, ..., u_i, ..., 0) \rangle = 0.$$ 

Hence

$$\left( (2^* - 2) \int_{\mathbb{R}^N} |u_j|^{2^*} + \sum_{k \neq j} |\beta_{jk}| |\alpha_{jk}| (2 - \alpha_{jk}) |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}} \right) L_j$$

$$- \sum_{i \neq j} L_i |\beta_{ij}| |\alpha_{ij}| \int_{\mathbb{R}^N} |u_j|^{\alpha_{ij}} |u_k|^{\alpha_{kj}} = 0. \tag{2.15}$$

Since $(u_1, ..., u_r) \in \mathcal{N}$, we have

$$\int_{\mathbb{R}^N} |u_j|^{2^*} > \sum_{k \neq j} |\beta_{jk}| |\alpha_{jk}| \int_{\mathbb{R}^N} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}},$$
hence

\[ \left( 2^* - 2 \right) \int_{\mathbb{R}^N} |u_j|^{2^*} + \sum_{k \neq j} |\beta_{jk}| |\alpha_{jk}(2 - \alpha_{jk})| |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}} \]

\[ > \sum_{k \neq j} |\beta_{jk}| |\alpha_{jk}(2 - \alpha_{jk})| \int_{\mathbb{R}^N} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}} \]

\[ = \sum_{k \neq j} |\beta_{jk}| |\alpha_{jk}| \int_{\mathbb{R}^N} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}, \quad j = 1, \ldots, r. \]

(2.16)

These inequalities above illustrate that the coefficient matrix of (2.15) is diagonally dominant, hence the determinant greater than 0. Combine with (2.15), we deduce that \( L_j = 0 \) \((j = 1, \ldots, r)\) and then \( J'(u_1, \ldots, u_r) = 0 \).

The next two lemmas are indispensable for the construction of the ground state solution and for the proof of its uniqueness. We also believe that they can be applied to other problems.

**Lemma 2.3.** Consider the following \( r + 1 \) inequalities,

\[
\begin{cases}
\sum_{k=1}^{r} x_k \leq 0, \\
f_j(x_1, \ldots, x_r) \geq 0, \quad j = 1, \ldots, r,
\end{cases}
\]

where \( f_j(x_1, \ldots, x_r) \) are nonnegative differentiable functions with \( f_j(0, \ldots, 0) = 0, \ j = 1, \ldots, r \). Assume that the following conditions hold:

\[
\det \left( \frac{\partial f_j}{\partial x_i} \right) \neq 0; \quad \sum_{s=1}^{r} g^{si} > 0, \quad i = 1, \ldots, r,
\]

(2.17)

where \((g_{ij}) := \left( \frac{\partial f_j}{\partial x_i} \right)\); the matrix \((g^{kl})\) represents the inverse matrix of \((g_{ij})\). Then we must have \( x_j = 0 \) for all \( j = 1, \ldots, r \).

**Proof.** Denote \( f_j(x_1, \ldots, x_r) = y_j \), then \( y_j \geq 0 \) for all \( j = 1, \ldots, r \). Note that

\[
\sum_k \frac{\partial f_j}{\partial x_k} \frac{\partial x_k}{\partial y_i} = \delta_{ij},
\]

that is,

\[
\sum_k g_{jk} \frac{\partial x_k}{\partial y_i} = \delta_{ij}.
\]

Multiply \( g^{sj} \) and sum up for \( j \) in the above equations, we get that

\[
\sum_j \sum_k g^{sj} g_{jk} \frac{\partial x_k}{\partial y_i} = \sum_j \delta_{ij} g^{sj},
\]

hence

\[
\sum_k \delta_{sk} \frac{\partial x_k}{\partial y_i} = g^{si},
\]
thus we obtain that \[
\frac{\partial x_i}{\partial y_i} = g^{si}.
\]

Then
\[
\frac{\partial}{\partial y_i} \left( \sum_{s=1}^{r} x_s(y_1, \ldots, y_r) \right) = \sum_{s=1}^{r} g^{si} > 0.
\]

This means that the function \(\sum_{s=1}^{r} x_s(y_1, \ldots, y_r)\) is strictly increasing in any direction. On the other hand, since \(y_j \geq 0\) and \(f_j(0, \ldots, 0) = 0\) for all \(j = 1, \ldots, r\), combining with \(\sum_{k=1}^{r} x_k \geq 0\), we obtain
\[
0 \geq \sum_{s=1}^{r} x_s(y_1, \ldots, y_r) \geq \sum_{s=1}^{r} x_s(0, \ldots, 0) = 0,
\]
it follows that \(y_j = 0\) and hence \(x_j = 0\) for all \(j = 1, \ldots, r\). \(\square\)

If \(f_j\) does not satisfy the initial condition \(f_j(0, \ldots, 0) = 0\), then we have the following more general version than Lemma 2.3.

**Lemma 2.4.** Consider the following nonlinear constraint problem
\[
\begin{align*}
\sum_{k=1}^{r} x_k & \leq \sum_{k=1}^{r} c_k, \\
f_j(x_1, \ldots, x_r) & \geq f_j(c_1, \ldots, c_r), j = 1, \ldots, r,
\end{align*}
\]
where \(f_j(x_1, \ldots, x_r)\) are nonnegative differentiable functions. Assume the following conditions hold:
\[
\det \left( \frac{\partial f_j}{\partial x_t} \right) \neq 0; \quad \sum_{s=1}^{r} g^{si} > 0, \quad i = 1, \ldots, r, \tag{2.18}
\]
where \((g_{ij}) = \left( \frac{\partial f_i}{\partial x_j} \right)^{-1}\) is the inverse matrix of \((g_{ij})\). Then we must have \(x_j = c_j\) for all \(j = 1, \ldots, r\).

**Proof.** Take \(h_j(x_1, \ldots, x_r) = f_j(x_1, \ldots, x_r) - f_j(c_1, \ldots, c_r)\) and make the transformation \(y_j = x_j - c_j, j = 1, \ldots, r\). Let \(l_j(y_1, \ldots, y_r) := h_j(x_1, \ldots, x_r)\). We may apply Lemma 2.3 to \(l_j(y_1, \ldots, y_r)\), then the conclusion follows. \(\square\)

**Remark 2.2.** The condition (2.18) in Lemma 2.4 may be replaced by
\[
\det \left( \frac{\partial f_j}{\partial x_t} \right) \neq 0; \quad \sum_{s=1}^{r} g^{si} \geq 0, \quad i = 1, \ldots, r; \quad \sum_{s=1}^{r} g^{s_i0} > 0 \text{ for some } i_0.
\]
Then the same conclusion as that in Lemma 2.4 holds.
3 Proof of Theorem 1.1

The proof of Theorem 1.1(1). Note the assumption $\beta_{jk} < 0, j \neq k$. Recall (1.11), it is easy to see that $z_{\mu}^i \to 0$ weakly in $D^{1,2}(\mathbb{R}^N)$ and so $(z_{\mu}^i)^{\beta} \to 0$ weakly in $L^{2/\beta}(\mathbb{R}^N)$ as $\mu \to \infty$. (here we regard $\mu^j$ as an integer in the expression $z_{\mu}^i$) hence

$$
\lim_{\mu \to +\infty} |\beta_{jk}| \int_{\mathbb{R}^N} (z_{\mu}^i)^{\alpha} (z_{\mu}^j)^{\beta} dx = \lim_{\mu \to +\infty} |\beta_{jk}| \int_{\mathbb{R}^N} (z_{\mu}^i(y))^{\alpha} (z_{\mu-i}^{j}(y))^{\beta} dy = 0, \quad 1 \leq i \neq j \leq r. 
$$

(3.1)

Then conditions (2.1), (2.4) and (2.6) in Lemma 2.1 hold for $(z_{\mu}^1, ..., z_{\mu}^r)$ when $\mu > 0$ is sufficiently large. Therefore, there exists some positive constants $\{t_j(\mu)\}_{j=1}^r$ such that $(t_1(\mu)z_{\mu}^1, ..., t_r(\mu)z_{\mu}^r) \in \mathcal{N}$. By Lemma 2.1 and in view of (1.10) (hence $\|z_{\mu}^1\|^2_{S_{j}} = |z_{\mu}^1|_{L^2}^2$), we get that $t_j(\mu) \to 1$ as $\mu \to \infty, j = 1, ..., r$. By (1.7) and (1.9), we see that

$$
\Theta \leq J(t_1(\mu)z_{\mu}^1, ..., t_r(\mu)z_{\mu}^r) = \frac{1}{N} \sum_{j=1}^r t_j^2(\mu)\|z_{\mu}^j\|^2_{S_{j}} = \frac{1}{N} \sum_{j=1}^r t_j^2(\mu)S(\lambda_j)^{N/2}.
$$

Letting $\mu \to \infty$ in the above equation, we get that

$$
\Theta \leq \frac{1}{N} \sum_{j=1}^r S(\lambda_j)^{N/2}.
$$

On the other hand, for any $(u_1, ..., u_r) \in \mathcal{N}$, we see from $\beta_{jk} < 0 (j \neq k)$ and (1.8) that

$$
\|u\|^2_{S_{j}} \leq \int_{\mathbb{R}^N} |u|^{2r} \leq S(\lambda_j)^{-2r/2}\|u\|^2_{S_{j}}, \quad j = 1, ..., r.
$$

Combining these with (1.5) and (1.7), we get that

$$
\Theta \geq \frac{1}{N} \sum_{j=1}^r S(\lambda_j)^{N/2}.
$$

Hence

$$
\Theta = \frac{1}{N} \sum_{j=1}^r S(\lambda_j)^{N/2}.
$$

(3.2)

Now we assume that $\Theta$ is attained by some $(u_1, ..., u_r) \in \mathcal{N}$, then $(|u_1|, ..., |u_r|) \in \mathcal{N}$ and $J(|u_1|, ..., |u_r|) = \Theta$. By Lemma 2.2, we know that $(|u_1|, ..., |u_r|)$ is a nontrivial solution of (1.3). By the maximum principle, we may assume that $u_j > 0$ in $\mathbb{R}^N \setminus \{0\}$ for all $j = 1, ..., r$. It follows that

$$
\int_{\mathbb{R}^N} u_i^\alpha u_j^\beta > 0 \quad \text{for any} \quad 1 \leq i, j \leq r.
$$
Then
\[ \|u\|_{\lambda_j}^2 < \int_{\mathbb{R}^N} |u|^{2^*} \, dx \leq S(\lambda_j)^{-2^*/2} \|u\|_{\lambda_j}^{2^*}, \quad j = 1, \ldots, r. \]

Therefore, it is easy to see that
\[ \Theta = J(u_1, \ldots, u_r) > \frac{1}{N} \sum_{j=1}^{r} S(\lambda_j)^{N/2}, \]
which contradicts with (3.2). This completes the proof of Theorem 1.1-(1).

**The proof of Theorem 1.1-(2).** Note the assumption \( \beta_{jk} > 0, \forall j \neq k \). In this part, we define
\[ \Theta' := \inf_{(u_1, \ldots, u_r) \in N'} J(u_1, \ldots, u_r), \]
where
\[ N' := \{ (u_1, \ldots, u_r) \in B \setminus \{(0, \ldots, 0)\} : \langle J'(u_1, \ldots, u_r), (u_1, \ldots, u_r) \rangle = 0 \}. \tag{3.3} \]

Note that \( N \subset N' \) and then \( \Theta' \leq \Theta \). It is easy to prove that \( \Theta' > 0 \). Moreover, it is standard to prove that
\[ \Theta' = \max_{(u_1, \ldots, u_r) \in B \setminus \{(0, \ldots, 0)\}} \frac{1}{N} \left[ \int_{\mathbb{R}^N} E(u_1, \ldots, u_r) \right]^{\frac{1}{2^*}}, \]
(3.4)

here we denote that
\[ E(u_1, \ldots, u_r) := \sum_{j=1}^{r} \left( |\nabla u_j|^2 - \frac{\lambda_j}{|x|^2} u_j^2 \right), \]
\[ F(u_1, \ldots, u_r) := \sum_{j=1}^{r} |u_j|^{2^*} + \sum_{1 \leq j < k \leq r} 2^* \beta_{jk} |u_j|^{\alpha_{jk}} |u_k|^{\alpha_{kj}}. \]

Then
\[ \int_{\mathbb{R}^N} E(u_1, \ldots, u_r) \geq (N\Theta')^{\frac{1}{2^*}} \left( \int_{\mathbb{R}^N} F(u_1, \ldots, u_r) \right)^{\frac{2}{2^*}}, \quad \forall (u_1, \ldots, u_r) \in B. \tag{3.5} \]

Before continuing to prove Theorem 1.1-(2), we have to establish three lemmas. The following lemma is the counterpart of the Brezis-Lieb Lemma (see [7] (see also [17]), here we omit the proof.

**Lemma 3.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and \((u_n, v_n)\) be a bounded sequence in \( L^{2^*}(\Omega) \times L^{2^*}(\Omega) \). If \((u_n, v_n) \to (u, v)\) almost everywhere in \( \Omega \), then
\[ \lim_{n \to \infty} \int_{\Omega} \left( |u_n|^{\alpha} |v_n|^\beta - |u_n - u|^{\alpha} |v_n - v|^\beta \right) = \int_{\Omega} |u|^{\alpha} |v|^\beta, \]
(3.6)
here \( \alpha + \beta = 2^*, \alpha > 1, \beta > 1. \)
Moreover, if
Then by letting
Lemma 3.2. Let \((u_1, \ldots, u_n) \in \mathbb{D}\) be a sequence such that
\[
\begin{aligned}
(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) &\rightharpoonup (u_1, u_2, \ldots, u_r) \text{ weakly in } \mathbb{D}, \\
(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) &\to (u_1, u_2, \ldots, u_r) \text{ almost everywhere in } \mathbb{R}^N, \\
E(u_{1,n} - u_1, u_{2,n} - u_2, \ldots, u_{r,n} - u_r) &\to \mu \text{ in the sense of measures,} \\
F(u_{1,n} - u_1, u_{2,n} - u_2, \ldots, u_{r,n} - u_r) &\to \rho \text{ in the sense of measures.}
\end{aligned}
\] (3.7)

Define
\[
\mu_\infty := \lim_{R \to \infty} \sup_{n \to \infty} \int_{|x| \geq R} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx,
\] (3.8)
\[
\rho_\infty := \lim_{R \to \infty} \sup_{n \to \infty} \int_{|x| \geq R} F(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx.
\] (3.9)

Then it follows that
\[
\|\mu\| \geq (N \Theta')^{\frac{2}{N}} \|\rho\|^{\frac{2}{N}},
\] (3.10)
\[
\mu_\infty \geq (N \Theta')^{\frac{2}{N}} \rho_\infty^{\frac{2}{N}},
\] (3.11)
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} E(u_1, \ldots, u_r) dx = \int_{\mathbb{R}^N} E(u_1, \ldots, u_r) dx + \|\mu\| + \mu_\infty,
\] (3.12)
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} F(u_1, \ldots, u_r) dx = \int_{\mathbb{R}^N} F(u_1, \ldots, u_r) dx + \|\rho\| + \rho_\infty.
\] (3.13)

Moreover, if \((u_1, u_2, \ldots, u_r) = (0, 0, \ldots, 0)\) and \(\|\mu\| = (N \Theta')^{\frac{2}{N}} \|\rho\|^{\frac{2}{N}}\), then \(\mu\) and \(\rho\) are concentrated at a single point.

Proof. In this proof we mainly follow the argument of [24]. Firstly we assume that \((u_1, u_2, \ldots, u_n) = (0, 0, \ldots, 0)\). For any \(h \in C_0^\infty(\mathbb{R}^N)\), we see from (3.5) that
\[
\int_{\mathbb{R}^N} E(hu_{1,n}, hu_{2,n}, \ldots, hu_{r,n}) dx \geq (N \Theta')^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} |h|^2 F(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \right)^{\frac{2}{N}}.
\] (3.14)

Since \(u_{j,n} \to 0, j = 1, \ldots, n\) in \(L^2_{\text{loc}}(\mathbb{R}^N)\), we have that
\[
\int_{\mathbb{R}^N} E(hu_{1,n}, hu_{2,n}, \ldots, hu_{r,n}) dx - \int_{\mathbb{R}^N} |h|^2 E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \to 0, n \to \infty.
\] (3.15)

Then by letting \(n \to \infty\) in (3.14), we obtain
\[
\int_{\mathbb{R}^N} |h|^2 d\mu \geq (N \Theta')^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} |h|^2 d\rho \right)^{\frac{2}{N}},
\] (3.16)
that is, (3.10) holds. For $R > 1$, let $\psi_R \in C^1(\mathbb{R}^N)$ be such that $0 \leq \psi_R \leq 1$, $\psi_R = 1$ for $|x| \geq R + 1$ and $\psi_R = 0$ for $|x| \leq R$. Then we see from (3.14) that
\[
\int_{\mathbb{R}^N} E(\psi_R u_{1,n}, \psi_R u_{2,n}, \ldots, \psi_R u_{r,n}) dx \\
\geq (N\Theta')^{2^*} \left( \int_{\mathbb{R}^N} |\psi_R|^{2^*} F(u_1, u_2, \ldots, u_r) dx \right)^{2^*}.
\]
Since $u_{j,n} \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ as $n \to \infty$ for all $j = 1, \ldots, r$, then
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^{2^*} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \\
\geq (N\Theta')^{2^*} \limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} |\psi_R|^{2^*} F(u_1, u_2, \ldots, u_r) dx \right)^{2^*}. \tag{3.17}
\]
Note that
\[
\int_{|x| \geq R + 1} F(u_1, u_2, \ldots, u_r) dx \leq \int_{|x| \geq R} |\psi_R|^{2^*} F(u_1, u_2, \ldots, u_r) dx \\
\leq \int_{|x| \geq R} F(u_1, u_2, \ldots, u_r) dx, \tag{3.18}
\]
so
\[
\rho_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^{2^*} F(u_1, u_2, \ldots, u_r) dx. \tag{3.19}
\]
On the other hand,
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^{2^*} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \\
= \limsup_{n \to \infty} \int_{|x| \geq R + 1} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \\
+ \limsup_{n \to \infty} \int_{R \leq |x| \leq R + 1} |\psi_R|^{2^*} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \\
= \limsup_{n \to \infty} \int_{|x| \geq R + 1} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx \\
+ \limsup_{n \to \infty} \int_{R \leq |x| \leq R + 1} |\psi_R|^{2^*} \left( \sum_{j=1}^{r} |\nabla u_{j,n}|^2 \right) dx \\
\geq \limsup_{n \to \infty} \int_{|x| \geq R + 1} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx. \tag{3.20}
\]
Letting $R \to \infty$ in the above inequality, we have that
\[
\mu_\infty \leq \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^{2^*} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) dx. \tag{3.21}
\]
Similarly,
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^2 E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx
\]
\[
= \limsup_{n \to \infty} \int_{|x| \geq R} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx
\]
\[
- \liminf_{n \to \infty} \int_{R \leq |x| \leq R+1} (1 - |\psi_R|^2) E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx
\]
\[
\leq \limsup_{n \to \infty} \int_{|x| \geq R} E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx.
\]  \hspace{1cm} (3.22)

Letting \( R \to \infty \), we see that
\[
\mu_\infty \geq \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^2 E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx.
\]  \hspace{1cm} (3.23)

Hence,
\[
\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^2 E(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx.
\]  \hspace{1cm} (3.24)

Then (3.11) follows directly from (3.17), (3.19) and (3.24). Assume moreover that
\[
\|\mu\| = (N\Theta')^{\frac{2}{p}} \|\rho\|^{\frac{2}{p'}}
\]
then by the Hölder inequality and (3.16), we have
\[
\int_{\mathbb{R}^N} |h|^{2^*} \, d\rho \leq (N\Theta')^{-\frac{2}{p-2}} \|\mu\|^{\frac{2}{p-2}} \int_{\mathbb{R}^N} |h|^{2^*} \, d\mu, \quad h \in C_0^\infty(\mathbb{R}^N).
\]  \hspace{1cm} (3.25)

From this we deduce that
\[
\rho = (N\Theta')^{-\frac{2}{p-2}} \|\mu\|^{\frac{2}{p-2}} \mu.
\]  \hspace{1cm} (3.26)

So \( \mu = (N\Theta')^{\frac{2}{p}} \|\rho\|^{-\frac{2}{p'}} \rho \), and we see from (3.16) that
\[
\|\rho\|^{\frac{2}{p'}} \left( \int_{\mathbb{R}^N} |h|^{2^*} \, d\rho \right)^{\frac{2}{p'}} \leq \int_{\mathbb{R}^N} |h|^{2^*} \, d\mu, \quad h \in C_0^\infty(\mathbb{R}^N).
\]  \hspace{1cm} (3.27)

That is, for each open set \( \Omega \), we have \( \rho(\Omega)^{\frac{2}{p'}} \rho(\mathbb{R}^N)^{\frac{2}{p'}} \leq \rho(\Omega) \). Therefore, \( \rho \) is concentrated at a single point.

For the general case, we denote that \( \omega_{j,n} = u_{j,n} - u_j, j = 1, 2, \ldots, r \). Then \( (\omega_{1,n}, \omega_{2,n}, \ldots, \omega_{r,n}) \rightharpoonup (0, 0, \ldots, 0) \) weakly in \( D \). From the Brezis-Lieb Lemma, for any nonnegative function \( h \in C_0(\mathbb{R}^N) \), we obtain that
\[
\int_{\mathbb{R}^N} hE(u_1, u_2, \ldots, u_r) \, dx = \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} hE(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx \right.
\]
\[
- \int_{\mathbb{R}^N} hE(\omega_{1,n}, \omega_{2,n}, \ldots, \omega_{r,n}) \, dx \right),
\]  \hspace{1cm} (3.28)

\[
\int_{\mathbb{R}^N} hF(u_1, u_2, \ldots, u_r) \, dx = \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} hF(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \, dx \right.
\]
\[
- \int_{\mathbb{R}^N} hF(\omega_{1,n}, \omega_{2,n}, \ldots, \omega_{r,n}) \, dx \right),
\]  \hspace{1cm} (3.29)
it follows that
\[
E(u_1, u_2, ..., u_r, n) \rightharpoonup E(u_1, u_2, ..., u_r) + \mu,
\]
\[
F(u_1, u_2, ..., u_r, n) \rightharpoonup F(u_1, u_2, ..., u_r) + \rho,
\]
in the sense of measures. Inequality (3.10) follows from the corresponding one for 
\((w_1, w_2, ..., w_r, n)\). From the Brezis-Lieb Lemma, it is easy to prove that
\[
\mu_\infty := \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| \geq R} E(w_1, w_2, ..., w_r, n) dx,
\]
\[
\rho_\infty := \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| \geq R} F(w_1, w_2, ..., w_r, n) dx.
\]
Then the inequality (3.11) can be proved in a similar way. For any \(R > 1\), we deduce
from (3.30) that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} F(u_1, u_2, ..., u_r, n)
\]
\[
= \limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} |\psi_R|^2 F(u_1, u_2, ..., u_r, n)
+ \int_{\mathbb{R}^N} (1 - |\psi_R|^2) F(u_1, u_2, ..., u_r, n) \right)
\]
\[
= \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\psi_R|^2 F(u_1, u_2, ..., u_r, n) + \int_{\mathbb{R}^N} (1 - |\psi_R|^2) F(u_1, u_2, ..., u_r)
+ \int_{\mathbb{R}^N} (1 - |\psi_R|^2) dp.
\]
Letting \(R \to \infty\), we see from (3.19) that (3.13) hold. The proof of (3.12) is similar.
This completes the proof of Lemma 3.2.

\textbf{Lemma 3.3.} Let \(\beta_{jk} > 0\), for any \(1 \leq j \neq k \leq r\), then (1.3) has a solution
\((u_1, u_2, ..., u_r) \in \mathcal{D}' \setminus \{(0, 0, ..., 0)\}\) (possibly semi-trivial) such that \(J(u_1, u_2, ..., u_r) = \Theta'\) and that \(u_j \geq 0 (j = 1, ..., r)\) are radially symmetric with respect to the origin.
Moreover, if further
\[
\Theta' < \min_{l \in \{1, 2, ..., r\}} \min_{j \in A^l} B_{j,l} \lambda_j^{-\frac{N+2}{2}} \frac{1}{N} S(\lambda_j)^{\frac{N}{2}},
\]
then \((u_1, u_2, ..., u_r) \in \mathcal{D}\) is a positive ground state solution of (1.3) and
\[
\Theta' = J(u_1, u_2, ..., u_r), \text{ where } B_{j,l} = \sum_{k \neq j, k \in A^l} \beta_{jk} \alpha_{jk} + 1, A^l = \{1, 2, ..., r\} \setminus \{l\}.
\]

\textbf{Proof.} For \((u_1, u_2, ..., u_r) \in \mathcal{N}'\) with \(u_j \geq 0, j = 1, ..., r\), we denote by \((u_1^*, u_2^*, ..., u_r^*)\)
for its Schwartz symmetrization. Then by the properties of Schwartz symmetrization,
we see from $\lambda_j > 0, \beta_{jk} > 0, j \neq k$ that

$$\sum_{j=1}^r \int_{\mathbb{R}^N} |\nabla u_j^*|^2 - \frac{\lambda_j}{|x|^2} |u_j^*|^2 \leq \sum_{j=1}^r \int_{\mathbb{R}^N} |u_j^*|^2 + \sum_{1 \leq j < k \leq r} 2^* \beta_{jk} |u_j^*|^\alpha_{jk} |u_k^*|^{\alpha_{jk}}.$$

Therefore, there exists $0 < t^* \leq 1$ such that $(t^* u_1^*, t^* u_2^*, ..., t^* u_r^*) \in \mathcal{N}'$, and that

$$J(t^* u_1^*, t^* u_2^*, ..., t^* u_r^*) = \frac{1}{N} (t^*)^2 \left( \sum_{j=1}^r |u_j^*|^2 \right) \leq \frac{1}{N} \left( \sum_{j=1}^r |u_j^*|_{\lambda_j}^2 \right) = J(u_1, u_2, ..., u_r). \quad (3.34)$$

We can take a minimizing sequence $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n}) \in \mathcal{N}'$ such that

$$(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n}) = (\tilde{u}_{1,n}^*, \tilde{u}_{2,n}^*, ..., \tilde{u}_{r,n}^*), \quad (3.35)$$

and $J(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n}) \to \Theta'$ as $n \to \infty$. Define the Levy concentration function

$$Q_n(R) := \sup_{y \in \mathbb{R}^N} \int_{B(y,R)} F(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n})dx.$$

Since $\tilde{u}_{j,n} \geq 0$ ($j = 1, 2, ..., r$) are radially nonincreasing, we have that

$$Q_n(R_n) = \int_{B(0,R_n)} F(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n})dx.$$

Then there exists $R_n > 0$ such that

$$Q_n(R) = \int_{B(0,R_n)} F(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n})dx = \frac{1}{2} \int_{\mathbb{R}^N} F(\tilde{u}_{1,n}, \tilde{u}_{2,n}, ..., \tilde{u}_{r,n})dx.$$

Define

$$(u_{1,n}(x), u_{2,n}(x), ..., u_{r,n}(x)) = (R_n^{2-n} u_{1,n}(R_n x), R_n^{2-n} u_{2,n}(R_n x), ..., R_n^{2-n} u_{r,n}(R_n x)).$$

By a direct computation, we know that $(u_{1,n}, u_{2,n}, ..., u_{r,n}) \in \mathcal{N}'$, $J(u_{1,n}, u_{2,n}, ..., u_{r,n}) \to \Theta'$ and that $u_j \geq 0$ are radially nonincreasing. Moreover,

$$\int_{B(0,1)} F(u_{1,n}, u_{2,n}, ..., u_{r,n})dx = \frac{1}{2} \int_{\mathbb{R}^N} F(u_{1,n}, u_{2,n}, ..., u_{r,n})dx \quad (3.36)$$

From (3.34), we know that $(u_{1,n}, u_{2,n}, ..., u_{r,n})$ are uniformly bounded in $\mathbb{D}$. Then passing to a subsequence, there exist $(u_1, u_2, ..., u_r) \in \mathbb{D}$ and finite measures $\mu, \rho$ such that (3.7) holds. Then by Lemma 3.2 we see that (3.10), (3.13) hold. Note that

$$\sum_{j=1}^r |u_{j,n}|_{\lambda_j}^2 = \int_{\mathbb{R}^N} F(u_{1,n}, u_{2,n}, ..., u_{r,n})dx \to N\Theta', \quad \text{as } n \to \infty.$$
From (3.7)-(3.13), we have that
\[ N\Theta' = \int_{\mathbb{R}^N} F(u_1, u_2, \ldots, u_r) \, dx + \|\rho\| + \rho_\infty, \]  
(3.37)
\[ N\Theta' \geq (N\Theta')^* \left[ \int_{\mathbb{R}^N} F(u_1, u_2, \ldots, u_r) \, dx + \|\rho\| \left\| \frac{\rho}{\rho_\infty} \right\|_{L^\infty} \right]. \]  
(3.38)
Therefore, \( \int_{\mathbb{R}^N} F(u_1, u_2, \ldots, u_r) \, dx, \rho \) and \( \rho_\infty \) are equal to either 0 or \( N\Theta' \). By (3.36), (3.37), (3.38), we have \( \rho_\infty \leq \frac{1}{2} N\Theta' \), hence \( \rho_\infty = 0 \). If \( \|\rho\| = N\Theta' \), then
\[ \int_{\mathbb{R}^N} F(u_1, u_2, \ldots, u_r) \, dx = 0 \]
and so \((u_1, u_2, \ldots, u_r) = (0, 0, \ldots, 0)\). On the other hand, since \( \|\mu\| \leq N\Theta' \), we deduce from (3.10) that \( \|\mu\| = (N\Theta')^* \left\| \frac{\rho}{\rho_\infty} \right\|_* \). Then Lemma 3.2 implies that \( \rho \) is concentrated at a single point \( y_0 \), and we see from (3.36), (3.37) and (3.38), that
\[ \frac{1}{2} N\Theta' = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} F(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \]
\[ \geq \lim_{n \to \infty} \int_{B(y_0, 1)} F(u_{1,n}, u_{2,n}, \ldots, u_{r,n}) = \|\rho\|, \]
a contradiction. Therefore, \( \int_{\mathbb{R}^N} F(u_1, u_2, \ldots, u_r) \, dx = N\Theta' \). Since \( \sum_{j=1}^r \|u_j\|^2_\lambda_j \leq N\Theta' \), we deduce from (3.36), (3.37) and (3.38) that
\[ N\Theta' = \sum_{j=1}^r \|u_j\|^2_\lambda_j = \int_{\mathbb{R}^N} F(u_1, u_2, \ldots, u_r) \, dx = \lim_{n \to \infty} \sum_{j=1}^r \|u_{j,n}\|^2_\lambda_j, \]
then
\[ \sum_{j=1}^r \|u_{j,n} - u_j\|^2_\lambda_j = \sum_{j=1}^r \|u_{j,n}\|^2_\lambda_j + \|u_j\|^2_\lambda_j - 2(u_{j,n}, u_j)_{\lambda_j} \to 0, \text{ as } n \to \infty, \]
that is, \((u_{1,n}, u_{2,n}, \ldots, u_{r,n}) \to (u_1, u_2, \ldots, u_r)\) strongly in \( \mathbb{D} \), \((u_1, u_2, \ldots, u_r) \in \mathcal{N}'\) and \( J(u_1, u_2, \ldots, u_r) = \Theta' \). Recall that \( \Theta' > 0 \), hence \((u_1, u_2, \ldots, u_r) \neq (0, 0, \ldots, 0)\).

By the definition of \( \mathcal{N}' \) and using the Lagrange multiplier method, it is standard to prove that \( J(u_1, u_2, \ldots, u_r) = 0 \). Therefore, \((u_1, u_2, \ldots, u_r)\) is a solution of (1.1).

Now assume that \( \Theta' < \min_{i \in \{1, 2, \ldots, r\}} \min_{j \in A^i} B_{j,l} \frac{\lambda_j^2}{2} \frac{1}{N} S(\lambda_j) \frac{1}{N} \). Then it is easy to prove that
\[ u_j \neq 0, \quad j = 1, 2, \ldots, r. \]

In fact, note that \( \alpha_{jk} + \alpha_{kj} = 2^* \),
\[ \int_{\mathbb{R}^N} |u_j|^{2^*} |u_k|^{\alpha_{kj}} \leq \frac{\alpha_{jk}}{2^{*}} \int_{\mathbb{R}^N} |u_j|^{2^*} + \frac{\alpha_{kj}}{2^{*}} \int_{\mathbb{R}^N} |u_k|^{2^*}, \]

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Lemma 3.3. By the maximum principle, a contradiction. Hence, $u_j \equiv 0$, then
\[
\sum_{j=1}^r \int_{\mathbb{R}^N} |u_j|^2 + \sum_{1 \leq j < k \leq r} 2^*-\beta_{jk}|u_j|^{\alpha_j}|u_k|^{\alpha_k} \\
\leq \sum_{j \in A^l} \left( \sum_{k \neq j, k \in A^l} \beta_{jk} \alpha_j + 1 \right) \int_{\mathbb{R}^N} |u_j|^2 \\
\leq \sum_{j \in A^l} B_{j,l} \int_{\mathbb{R}^N} |u_j|^2,
\]
where $A^l = \{1, 2, \ldots, r\} \setminus \{l\}$, then
\[
\Theta' = \inf_{(u_1, u_2, \ldots, u_r) \in \mathbb{D} \setminus \{(0,0, \ldots, 0)\}} \frac{1}{N} \left[ \frac{1}{N} \left( \sum_{j=1}^r \int_{\mathbb{R}^N} |u_j|^2 \right) \right]^{\frac{N}{2}} \\
\geq \inf_{(u_1, u_2, \ldots, u_r) \in \mathbb{D} \setminus \{(0,0, \ldots, 0)\}} \frac{1}{N} \left[ \frac{1}{N} \left( \sum_{j \in A^l} B_{j,l} \int_{\mathbb{R}^N} |u_j|^2 \right) \right]^{\frac{N}{2}} \\
\geq \min_{j \in A^l} B_{j,l}^{-\frac{N-2}{2}} \frac{1}{N} S(\lambda_j)^{\frac{N}{2}},
\]
a contradiction. Hence, $u_j \neq 0$ for all $j = 1, 2, \ldots, r$. That is, $(u_1, u_2, \ldots, u_r) \in \mathbb{N}'$ and so $J(u_1, u_2, \ldots, u_r) = \Theta' = \Theta$. Hence, $(u_1, u_2, \ldots, u_r)$ is a ground state solution of (1.3). By the maximum principle, $u_j > 0$ in $\mathbb{R}^N \setminus \{0\}$. This completes the proof of Lemma 3.3.

Finishing the proof of Theorem 1.1-(2). We apply Lemma 3.3. It suffices to prove (3.39). Let $\beta_{jk} > 0, j \neq k$. Without loss of generality, we assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$, then $S(\lambda_r) \leq \cdots \leq S(\lambda_2) \leq S(\lambda_1)$. Denote that $d_\alpha := \frac{\lambda_1 - \lambda_2}{\alpha^2}$, for $1 \leq \alpha \leq r$. By Hardy's inequality, we know that $\|u\|_{\lambda_\alpha} \leq d_\alpha \|u\|_{\lambda_1}$ for all $u \in D^{1,2}(\mathbb{R}^N), \alpha = 1, 2, \ldots, r$. Then we deduce from (3.40) that
\[
\Theta' \leq \inf_{(u_1, u_2, \ldots, u_r) \in \mathbb{D} \setminus \{(0,0, \ldots, 0)\}} \frac{1}{N} \left[ \frac{1}{N} \left( \sum_{j=1}^r |z_j|^2 \right) \right]^{\frac{N}{2}} \\
\leq \inf_{(u_1, u_2, \ldots, u_r) \in \mathbb{D} \setminus \{(0,0, \ldots, 0)\}} \frac{1}{N} \left[ \frac{1}{N} \left( \sum_{j=1}^r |z_j|^2 \right) \right]^{\frac{N}{2}} \\
= \left[ \frac{1}{N} \left( \sum_{j=1}^r |z_j|^2 \right) \right]^{\frac{N}{2}} \frac{1}{N} S(\lambda_r)^{\frac{N}{2}} \\
< \min_{l \in \{1,2,\ldots,r\}} \min_{j \in A^l} B_{j,l}^{-\frac{N-2}{2}} \frac{1}{N} S(\lambda_j)^{\frac{N}{2}}.
\]
provided that

\[
(r + \sum_{j,k=1,j\neq k}^{r} \frac{2^*}{2} \beta_{jk})/(\max B_{j,l}) > (1 + \sum_{\alpha=1}^{r-1} d_{\alpha})^{\frac{N}{N-2}},
\]

where, \(B_{j,l} = \sum_{k \neq j,k \in A^l} \beta_{jk} \alpha_{jk} + 1\), \(A^l = \{1, 2, ..., r\} \setminus \{l\}\). Hence, the conclusion follows from Lemma 3.3.

\[
\Box
\]

4 Proof of Theorem 1.2

In this section, we will use the moving plane method to prove Theorem 1.2. In the sequel, we assume that \(N = 3\) or \(N = 4\), \(r \geq 3\), \(\alpha_{jk} + \alpha_{kj} = 2^*, \alpha_{jk} \geq 2, \alpha_{kj} \geq 2, \beta_{jk} > 0\) for \(k \neq j\) and \(\lambda_j \in (0, \Lambda_N)\) for all \(j = 1, ..., r\). Let \((u_1, ..., u_r)\) be any positive solution of (4.1). For \(\lambda < 0\), we consider the following reflection:

\[
x = (x_1, x_2, ..., x_N) \mapsto x^\lambda = (2\lambda - x_1, x_2, ..., x_N),
\]

where \(x \in \Sigma^\lambda := \{x \in \mathbb{R}^N : x_1 < \lambda\}\). Define \(u_j^\lambda(x) := u_j(x^\lambda)\), then

\[
u_j(x) = u_j^\lambda(x) \quad \text{for} \quad x \in \partial \Sigma^\lambda, \quad \text{where} \quad \Sigma^\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}.
\]

Define \(w_j^\lambda(x) := u_j^\lambda(x) - u_j(x)\) for \(x \in \Sigma^\lambda\), then

\[
w_j^\lambda = 0, \quad x \in \partial \Sigma^\lambda.
\]

Recall that \(u_j(x)\) and \(u_j^\lambda(x)\) satisfy the following equations

\[
-\Delta u_j - \frac{\lambda_j}{|x|^2} u_j = u_j^{2^*-1} + \sum_{k \neq j} \alpha_{jk} \beta_{jk} u_j^{\alpha_{jk}-1} u_k^{\alpha_{kj}},
\]

\[
-\Delta u_j^\lambda - \frac{\lambda_j}{|x|^2} u_j^\lambda = (u_j^\lambda)^{2^*-1} + \sum_{k \neq j} \alpha_{jk} \beta_{jk} (u_j^\lambda)^{\alpha_{jk}-1} (u_k^\lambda)^{\alpha_{kj}}, \tag{4.1}
\]

thus we have

\[
-\Delta w_j^\lambda \geq \frac{\lambda_j}{|x|^2} w_j^\lambda + b_j^\lambda + \sum_{k \neq j} b_{jk} w_k^\lambda,
\]

here

\[
b_{jj}^\lambda = \frac{(u_j^\lambda)^{2^*-1} - u_j^{2^*-1}}{u_j^\lambda - u_j} + \sum_{k \neq j} \alpha_{jk} \beta_{jk} (u_j^\lambda)^{\alpha_{jk}-1} \frac{(u_j^\lambda)^{2^*-1} - u_j^{\alpha_{jk}-1}}{u_j^\lambda - u_j} \geq 0,
\]

\[
b_{jk}^\lambda = \alpha_{jk} \beta_{jk} (u_j^\lambda)^{2^*-1} (u_j^\lambda)^{\alpha_{jk}-1} \frac{u_j^{\alpha_{jk}-1}}{u_j^\lambda - u_j} \geq 0.
\]

Define

\[
\Omega_j^\lambda := \{x \in \Sigma^\lambda : w_j^\lambda(x) < 0\}, \quad j = 1, ..., r. \tag{4.2}
\]
Since \( u_j \in L^2(\mathbb{R}^N) \) and \( \Omega_j^\lambda \subset \Sigma^\lambda \), there exists \( \lambda_0 \to -\infty \) such that

\[
\| b_{jj}^\lambda \|_{L^\infty(\Omega_j^\lambda)} \to 0, \\
\| b_{jk}^\lambda \|_{L^\infty(\Omega_j^\lambda)} \to 0, \\
\| u_j^{\alpha_k} b_k^{\alpha_k} \|_{L^\infty(\Omega_j^\lambda \cap \Omega_k^\lambda)} \to 0.
\]

(4.3)

**Step 1:** We claim that for any \( \lambda \leq \lambda_0 \), all \( w_j^\lambda > 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \). For this aim, we define

\[
w_j^\lambda := -\max\{-w_j^\lambda, 0\}, \quad j = 1, \ldots, r,
\]

then \( w_j^\lambda \in D^{1,2}(\mathbb{R}^N) \) and

\[
\int_{\Omega_j^\lambda} |\nabla w_j^\lambda|^2 \leq \int_{\Omega_j^\lambda} \frac{\lambda_j}{|x|^2} |w_j^\lambda|^2 + \int_{\Omega_j^\lambda} b_{jj}^\lambda |w_j^\lambda|^2 + \sum_{k \neq j} \int_{\Omega_j^\lambda \cap \Omega_k^\lambda} b_{jk}^\lambda w_k^{\alpha_k} w_j^\lambda,
\]

\[
\leq \frac{\lambda_j}{\Lambda_N} \int_{\Omega_j^\lambda} |\nabla w_j^\lambda|^2 + \| b_{jj}^\lambda \|_{L^\infty(\Omega_j^\lambda)} \| w_j^\lambda \|_{L^2(\Omega_j^\lambda)}^2
\]

\[
+ \sum_{k \neq j} \| b_{jk}^\lambda \|_{L^\infty(\Omega_j^\lambda \cap \Omega_k^\lambda)} \| w_k^{\alpha_k} \|_{L^2(\Omega_k^\lambda)} \| w_j^\lambda \|_{L^2(\Omega_j^\lambda)} \| w_j^\lambda \|_{L^2(\Omega_j^\lambda)}^2,
\]

then we have

\[
(1 - \frac{\lambda_j}{\Lambda_N}) \int_{\Omega_j^\lambda} |\nabla w_j^\lambda|^2
\]

\[
\leq \| b_{jj}^\lambda \|_{L^\infty(\Omega_j^\lambda)} \| w_j^\lambda \|_{L^2(\Omega_j^\lambda)}^2
\]

\[
+ \sum_{k \neq j} \| b_{jk}^\lambda \|_{L^\infty(\Omega_j^\lambda \cap \Omega_k^\lambda)} \| w_k^{\alpha_k} \|_{L^2(\Omega_k^\lambda)} \| w_j^\lambda \|_{L^2(\Omega_j^\lambda)} \| w_j^\lambda \|_{L^2(\Omega_j^\lambda)}^2.
\]

Denote that

\[
a_{jj}^\lambda := 1 - \frac{\lambda_j}{\Lambda_N} - \left( \| b_{jj}^\lambda \|_{L^\infty(\Omega_j^\lambda)} + \sum_{k \neq j} \frac{1}{2} \| b_{jk}^\lambda \|_{L^\infty(\Omega_j^\lambda \cap \Omega_k^\lambda)} \right),
\]

then by (4.3), we see that by letting \( |\lambda_0| \to \infty \), \( \lambda_0 < 0 \), we can make sure that

\[
a_{jj}^\lambda \geq \frac{1}{2} \left( 1 - \frac{\lambda_j}{\Lambda_N} \right) \geq \min_j \frac{1}{2} \left( 1 - \frac{\lambda_j}{\Lambda_N} \right) := a.
\]

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By this notation

\[ a \int_{\Omega_j^\lambda} |\nabla w_{j,-}^\lambda|^2 \leq \sum_{k \neq j} \frac{1}{2} \| b_{jk}^\lambda \|_{L^2(\Omega_j^\lambda \cap \Omega_k^\lambda)} \| w_{k,-}^\lambda \|_{L^2(\Omega_j^\lambda)} \]

\[ \leq \sum_{k \neq j} \frac{1}{2} \| b_{jk}^\lambda \|_{L^2(\Omega_j^\lambda \cap \Omega_k^\lambda)} \frac{1}{S} \int_{\Omega_k^\lambda} |\nabla w_{j,-}^\lambda|^2. \]

Summing up the above inequalities, we have

\[ a \sum_{j=1}^r \int_{\Omega_j^\lambda} |\nabla w_{j,-}^\lambda|^2 \leq \sum_{j=1}^r \sum_{k \neq j} \frac{1}{2} \| b_{jk}^\lambda \|_{L^2(\Omega_j^\lambda \cap \Omega_k^\lambda)} \frac{1}{S} \int_{\Omega_k^\lambda} |\nabla w_{j,-}^\lambda|^2 \]

\[ = \sum_{k=1}^r \sum_{j \neq k} \frac{1}{2} \| b_{kj}^\lambda \|_{L^2(\Omega_j^\lambda \cap \Omega_k^\lambda)} \frac{1}{S} \int_{\Omega_k^\lambda} |\nabla w_{j,-}^\lambda|^2 \]

\[ = \sum_{j=1}^r \sum_{k \neq j} \frac{1}{2} \| b_{kj}^\lambda \|_{L^2(\Omega_j^\lambda \cap \Omega_k^\lambda)} \frac{1}{S} \int_{\Omega_k^\lambda} |\nabla w_{j,-}^\lambda|^2. \]

Recall (4.3), we can let \( \lambda_0 \) go to \( -\infty \) such that

\[ \sum_{k \neq j} \frac{1}{2} \| b_{kj}^\lambda \|_{L^2(\Omega_j^\lambda \cap \Omega_k^\lambda)} \frac{1}{S} \leq 1. \]

It follows that

\[ a \sum_{j=1}^r \int_{\Omega_j^\lambda} |\nabla w_{j,-}^\lambda|^2 \leq \frac{1}{2} \int_{\Omega_j^\lambda} |\nabla w_{j,-}^\lambda|^2, \]

hence we have the following inequality

\[ \int_{\Omega_j^\lambda} |\nabla w_{j,-}^\lambda|^2 \leq 0. \]

Since \( a > 0 \), we have

\[ \int_{\Omega_j^\lambda} |\nabla w_{j,-}^\lambda|^2 = 0, \quad j = 1, \ldots, r. \]

Therefore, we have the following alternative conclusion for any \( j \): either

\[ w_{j,-}^\lambda(x) \equiv \text{const} \quad \text{in} \quad \Omega_j^\lambda, \quad m(\Omega_j^\lambda) > 0, \quad (4.4) \]

or

\[ m(\Omega_j^\lambda) = 0, \quad (4.5) \]

where \( m \) represents the Lebesgue measure. Notice that \( w_{j,-}^\lambda = 0 \) on \( \partial \Omega_j^\lambda \), hence, by (4.4) we have that \( w_{j,-}^\lambda(x) \equiv 0 \) in \( \Omega_j^\lambda \). This is, \( w_j^\lambda \geq 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \). Look (4.5) now,
it just says that \( w_j^\lambda \geq 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \) in another way. In summary, we have that \( w_j^\lambda \geq 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \). Recall (4.1), we see that

\[
- \Delta w_j^\lambda \geq \lambda_j \left( \frac{1}{|x|^2} - \frac{1}{|x|^2} \right) u_j^\lambda(x) > 0,
\]

holds in \( \Sigma^\lambda \setminus \{0^\lambda\} \). Then by the strong maximum principle, we have \( w_j^\lambda > 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \).

**Step 2:** Define \( \lambda^* = \sup \{ \lambda < 0 : w_j^\lambda > 0 \text{ in } \Sigma^\lambda \setminus \{0^\lambda\}, \forall \lambda < \lambda \} \). Then we claim that \( \lambda^* = 0 \).

Assume by contradiction that \( \lambda^* < 0 \). By the continuity we have \( w_j^{\lambda^*} \geq 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \). By a similar argument as in Step 1, we have \( w_j^{\lambda^*} > 0, j = 1, ..., r \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \). By the absolutely continuity of the integral, there exists a \( \lambda \) with \( 0 < \lambda > \lambda^* \) such that

\[
\begin{align*}
|b_{j1}^{\lambda}|_{L^p(\Omega_1^\lambda)} & \to 0, \\
|b_{jk}^{\lambda}|_{L^p(\Omega_1^\lambda)} & \to 0, \\
||u^{\alpha\beta}_{jk}|_{L^p(\Omega_1^\lambda \cap \Omega_2^\lambda)} & \to 0, \quad \lambda \to \lambda^*.
\end{align*}
\]

(4.6)

Then we can follow the same proof as in Step 1, we can find a \( \lambda \), which satisfies \( 0 > \lambda > \lambda^* \) and \( w_j^{\lambda^*} > 0 \) in \( \Sigma^\lambda \setminus \{0^\lambda\} \) as \( \lambda \) closing to \( \lambda^* \), which contradicts to the definition of \( \lambda^* \). Therefore, \( \lambda^* = 0 \).

**Step 3.** We show that \( u_j (j = 1, ..., r) \) are radially symmetric with respect to the origin. Since \( \lambda^* = 0 \), then we can carry out the above procedure in the opposite direction, namely we can take the transform \( y = (y_1, y_2, ..., y_r) = (-x_1, x_2, ..., x_r) \), then moving plane by Step 1 and Step 2 about \( y_1 \), we can derive that \( u_j (j = 1, ..., r) \) are symmetric with respect to 0 in the \( x_1 \) direction. Since we take the orthogonal transform \( y = (y_1, y_2, ..., y_r) = A(x_1, x_2, ..., x_r) \) arbitrarily, where \( A \) is a \( r \) order orthogonal matrix, we can derive that \( u_j (j = 1, ..., r) \) are symmetric with respect to 0 in any direction. It follows that \( u_j (j = 1, ..., r) \) are radially symmetric with respect to the origin. This completes the proof of Theorem 1.2.

### 5 Proofs of Theorems 1.3, 1.4

Firstly, we will prepare several lemmas which are essential to the proof of Theorems 1.3, 1.4. We remark that these lemmas are also interesting from its own perspective.

**Lemma 5.1.** Consider the following nonlinear constraint problem:

\[
\begin{aligned}
x_1 + x_2 & \leq c_1 + c_2, \\
f_1(x_1, x_2) \coloneqq x_1^{p - 1} + \nu \alpha x_1^{q - 1} x_2^\delta - f_1(c_1, c_2), \\
f_2(x_1, x_2) \coloneqq x_2^{p - 1} + \nu \beta x_1^\delta x_2^{q - 1} - f_2(c_1, c_2).
\end{aligned}
\]

(5.1)
If $\nu > (p - 1)/\min\{d_1(\alpha, \beta), d_2(\alpha, \beta), d_3(\alpha, \beta)\}$, then
\[ x_1 = c_1, \quad x_2 = c_2, \]
where $N \geq 5, p = \frac{2p}{\nu}, \alpha + \beta = 2p^* = 2p$ and $\alpha > 0, \beta > 0, c_i > 0, x_i > 0; i = 1, 2$;
\[ d_1(\alpha, \beta) = 2p(1 - \frac{\alpha}{2})\beta \left(1 - \frac{\beta}{2}\right) \] if $\alpha \neq \beta; d_1(\alpha, \beta) = p$ if $\alpha = \beta,$
\[ d_2(\alpha, \beta) = \beta(1 - \frac{\beta}{2})^{1-\frac{\alpha}{2}}(1 - \frac{\alpha}{2})^{\frac{\beta}{2}} + \frac{1}{2}\alpha\beta(1 - \frac{\alpha}{2})^{1-\frac{\alpha}{2}}(1 - \frac{\beta}{2})^{\frac{\beta}{2}-1}, \]
\[ d_3(\alpha, \beta) = \alpha(1 - \frac{\alpha}{2})^{1-\frac{\alpha}{2}}(1 - \frac{\beta}{2})^{\frac{\beta}{2}} + \frac{1}{2}\alpha\beta(1 - \frac{\alpha}{2})^{1-\frac{\alpha}{2}}(1 - \frac{\beta}{2})^{\frac{\beta}{2}-1}. \]
In particular if $\alpha = \beta = p$ in (5.3), we have the concise form, that is,
\[ d_1(\alpha, \beta) = d_2(\alpha, \beta) = d_3(\alpha, \beta) = p, \]
hence, under this case, $x_1 = c_1$ and $x_2 = c_2$ if $\nu > \frac{2p}{N}$.

**Proof.** By Lemma 2.4 we only need to check that the matrix $F = (\frac{\partial f}{\partial x}) := (F_{ij})$ satisfies $\det(F) < 0, F_{22} - F_{12} < 0, F_{11} - F_{21} < 0$. By a direct computation, we have
\[
F = \begin{pmatrix}
(p-1)x_1^{p-2} + \nu\alpha(x_1\frac{\alpha}{2} - 1)x_1^{\frac{\beta}{2}} - x_2^{\frac{\beta}{2}} & \frac{1}{2}\nu\alpha\beta x_1^{\frac{\beta}{2}} - x_2^{\frac{\beta}{2}} - 1 \\
\frac{1}{2}\nu\alpha\beta x_1^{\frac{\beta}{2}} - x_2^{\frac{\beta}{2}} - 1 & (p-1)x_2^{p-2} + \nu\beta(x_2\frac{\beta}{2} - 1)x_2^{\frac{\alpha}{2}} - x_1^{\frac{\alpha}{2}}
\end{pmatrix},
\]
\[
F^{-1} = \frac{1}{\det(F)} \begin{pmatrix}
A_0 & -\frac{1}{2}\nu\alpha\beta x_1^{\frac{\beta}{2}} - x_2^{\frac{\beta}{2}} - 1 \\
\frac{1}{2}\nu\alpha\beta x_1^{\frac{\beta}{2}} - x_2^{\frac{\beta}{2}} - 1 & (p-1)x_1^{p-2} + \nu\alpha(x_1\frac{\alpha}{2} - 1)x_1^{\frac{\beta}{2}} - x_2^{\frac{\beta}{2}} - 2
\end{pmatrix},
\]
where $A_0 := (p-1)x_1^{p-2} + \nu\beta(x_2\frac{\beta}{2} - 1)x_2^{\frac{\alpha}{2}} - 2$ and
\[
\det(F) = \left\{ (p-1)^2 + (\nu^2\alpha\beta(x_1\frac{\alpha}{2} - 1)(x_2\frac{\beta}{2} - 1) - \frac{1}{4}\nu^2\alpha^2\beta^2)((x_1\frac{\alpha}{2})^{\alpha-p} + (p-1)\nu\beta((x_1\frac{\alpha}{2} - 1)(x_1\frac{\alpha}{2} - 1)(x_1\frac{\beta}{2} - 1)(x_x\frac{\beta}{2} - 1)) \right\} x_1^{p-2}x_2^{p-2}.
\]
When $\alpha \neq \beta$, since $\frac{\alpha}{2} - 1 - \frac{1}{4}\alpha\beta = 1 - p < 0$, we have
\[
\det(F) \leq \left\{ (p-1)^2 + (p-1)\nu\beta((x_1\frac{\beta}{2} - 1)(x_1\frac{\beta}{2})^{\beta} + (p-1)\nu\alpha((x_1\frac{\alpha}{2} - 1)(x_1\frac{\beta}{2})^{\beta}) \right\} x_1^{p-2}x_2^{p-2}.
\]
When $\alpha = \beta$,

\[
\det(F) \leq \left\{ (p-1)^2 + \nu^2 \alpha^2 \beta^2 \left( \frac{\alpha}{2} - 1 \right) \right\}^2 - \\
\frac{1}{4} \nu^2 \alpha^2 \beta^2 + (p-1) \nu \beta \left( \frac{\beta}{2} - 1 \right) \left( \frac{x_1}{x_2} \right)^{\frac{p}{2}} + (p-1) \nu \alpha \left( \frac{\alpha}{2} - 1 \right) \left( \frac{x_2}{x_1} \right)^{p-\frac{p}{2}} x_2^{p-2} x_1^{2-2} \\
= (p-1)^2 + \nu^2 p^2 (1-p) + \nu (p-1) p \left( \frac{p}{2} - 1 \right) \left( \frac{x_1}{x_2} \right)^{\frac{p}{2}} + \\
\nu (p-1) p \left( \frac{p}{2} - 1 \right) \left( \frac{p}{2} - 1 \right) \left( \frac{x_1}{x_2} \right)^{\frac{p}{2}}.
\]

Let

\[
h_1(x) := (p-1) \nu \beta \left( \frac{\beta}{2} - 1 \right) x_2^{\frac{p}{2}} + (p-1) \nu \alpha \left( \frac{\alpha}{2} - 1 \right) \left( \frac{1}{x} \right)^{p-\frac{p}{2}}, 0 < x := \frac{x_1}{x_2} < \infty,
\]

hence,

\[
h_1'(x_0) = 0 \Rightarrow x_0 = \left[ \frac{1 - \frac{\beta}{2}}{1 - \frac{\alpha}{2}} \right].
\]

It is easy to see that $x_0$ is the maximum point of $h_1(x)$ in the interval $(0, \infty)$, so

\[
\det(F) < 0 \iff h_1(x) < 0 \iff \nu > (p-1)/d_1(\alpha, \beta).
\]

Next we estimate $F_{22} - F_{12}$:

\[
F_{22} - F_{12} = x_2^{p-2} \left\{ (p-1) - \nu \beta (1 - \frac{\beta}{2}) \left( \frac{x_1}{x_2} \right)^{\frac{p}{2}} - \frac{1}{2} \alpha \beta \nu \left( \frac{x_1}{x_2} \right)^{\frac{p}{2} - 1} \right\},
\]

\[
h_2(x) := (p-1) - \nu \beta (1 - \frac{\beta}{2}) x_2^{\frac{p}{2}} - \frac{1}{2} \alpha \beta \nu x_2^{\frac{p}{2} - 1}, 0 < x := \frac{x_1}{x_2} < \infty,
\]

\[
h_2'(x_0) = 0 \iff x_0 = \frac{1 - \frac{\beta}{2}}{1 - \frac{\alpha}{2}},
\]

it is easy to see that $x_0$ is the maximum point of $h_2(x)$ in the interval $(0, \infty)$, so

\[
F_{22}(x) - F_{12}(x) < 0 \iff h_2(x) < 0 \iff \nu > (p-1)/d_2(\alpha, \beta).
\]

Note that

\[
F_{11} - F_{21} = x_1^{p-2} \left\{ (p-1) - \nu \alpha (1 - \frac{\alpha}{2}) \left( \frac{x_2}{x_1} \right)^{\frac{p}{2}} - \frac{1}{2} \alpha \beta \nu \left( \frac{x_2}{x_1} \right)^{\frac{p}{2} - 1} \right\}.
\]

Let

\[
h_3(x) := (p-1) - \nu \alpha (1 - \frac{\alpha}{2}) x_1^{\frac{p}{2}} - \frac{1}{2} \alpha \beta \nu x_1^{\frac{p}{2} - 1}, 0 < x := \frac{x_2}{x_1} < \infty,
\]

then

\[
h_3'(x_0) = 0 \iff x_0 = \frac{1 - \frac{\alpha}{2}}{1 - \frac{\beta}{2}}.
\]
and it is easy to see that \(x_0\) is the maximum point of \(h_3(x)\) in the interval \((0, \infty)\). Therefore,

\[
F_{11}(x) - F_{21}(x) < 0 \iff h_2(x) < 0 \iff \nu > (p - 1)/d_3(\alpha, \beta). \tag{5.9}
\]

Combine with (5.7), (5.8), (5.9), we see that

\[
\sum_{j=1}^{2} F^{ij} > 0, \quad i = 1, 2.
\]

Then by Lemma 2.4 the conclusion follows. \(\Box\)

**Lemma 5.2.** Consider the symmetric matrix \(\gamma = (\gamma_{ij})\). Assume \(\det(\gamma) \neq 0\) and

\[
\sum_{k=1}^{r} \gamma_{jk} c_k = 1, \quad j = 1, \ldots, r. \tag{5.10}
\]

View \((\sum_{j=1}^{r} c_j)\) as a function of \(\gamma_{ml}\). Then

\[
\frac{\partial}{\partial \gamma_{ml}} \left( \sum_{j=1}^{r} c_j \right) = -c_m c_l.
\]

**Proof.** Let \((\gamma^{sj})\) represent the inverse matrix of \((\gamma_{jk})\). Derivative on both side of (5.10) with respect to \(\gamma_{ml}\) for any fixed \(m, l\), we have

\[
\sum_{k} \left( \gamma_{jk} c'_k + \delta_{mj} \delta_{kl} c_k \right) = 0,
\]

that is,

\[
\sum_{k} \gamma_{jk} c'_k = -\delta_{ml} c_l,
\]

\[
\sum_{j} \sum_{k} \gamma^{sj} \gamma_{jk} c'_k = -\sum_{j} \delta_{mj} c_l \gamma^{sj},
\]

\[
\sum_{k} \delta_{sk} c'_k = -\sum_{j} \delta_{mj} c_l \gamma^{sj},
\]

\[
c'_s = c_l \gamma^{sm}.
\]

Hence,

\[
\sum_{s=1}^{r} c'_s = -c_l \sum_{s=1}^{r} \gamma^{sm}. \tag{5.11}
\]

On the other hand, we see from (5.10) that

\[
\sum_{j} \sum_{k} \gamma_{jk} c_k \gamma^{sj} = \sum_{j} \gamma^{sj},
\]

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\[ \sum_{k} \delta_{ks} c_k = \sum_{s} \gamma^{s} c_s, \]
hence,
\[ c_s = \sum_{s} \gamma^{s} c_s. \]
Since the matrix \( \gamma = (\gamma_{ij}) \) is symmetric, combine the equality above with (5.11), we have
\[ \sum_{s=1}^{r} c_s' = -c_m c_t. \]

**The proof of Theorem 1.3** Consider the matrix defined by
\[
\gamma := \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1r} \\
\gamma_{21} & \gamma_{22} & \cdots & \gamma_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{r1} & \gamma_{r2} & \cdots & \gamma_{rr}
\end{pmatrix},
\]
where \( \gamma_{jk} = \gamma_{kj} \). We consider the following critical elliptic system involving Hardy singular terms
\[
\begin{cases}
-\Delta u_j - \frac{\lambda}{|x|^2} u_j = \gamma_{jj} u_j^3 + \sum_{k \neq j} \gamma_{ij} u_i^2 u_j, & x \in \mathbb{R}^4, \\
u_j(x) > 0, & x \in \mathbb{R}^4 \setminus \{0\}, \quad j = 1, ..., r.
\end{cases}
\]

**Remark 5.1.** As previous definitions in (1.7), (1.8) and (1.9), we may introduce the corresponding functional, Nehari manifold and the least energy for the Eq. (5.13). We adopt the same notations by \( J, \mathcal{N}, \theta \) respectively as defined in (1.7), (1.8) and (1.9) though the constants (coefficients) are replaced by those corresponding to Eq. (5.13).

Recall that the matrix \( \gamma \) is invertible and the sum of each row of the inverse matrix \( \gamma^{-1} \) is greater than 0, it follows that the equation
\[ \sum_{k=1}^{r} \gamma_{jk} c_k = 1, \quad j = 1, ..., r, \]
has a solution \( (c_1, ..., c_r) \) satisfying \( c_j > 0 \) \( (j = 1, ..., r) \) and so \( (\sqrt{c_1} z, ..., \sqrt{c_r} z) \) is a nontrivial solution of (5.13) (where \( z \) is a solution of (1.14)) and
\[ \Theta = J(\sqrt{c_1} z, ..., \sqrt{c_r} z) = \sum_{j=1}^{r} c_j \Theta_1, \quad \text{where} \quad \Theta_1 = I_{\lambda}(z) \quad \text{(see (1.3) with } \lambda_j = \lambda). \]

**The proof of Theorem 1.3-(1)**. Let \( \{(u_{1,n}, ..., u_{r,n})\} \subset \mathcal{N} \) be a minimizing sequence for \( \Theta \), that is, \( J(u_{1,n}, ..., u_{r,n}) \to \Theta \). Define
\[ d_{i,n} = \left( \int_{\Omega} u_{i,n}^4 dx \right)^{1/2}, \quad i = 1, ..., r. \]
Then by (5.14), we have
\[
2 \sqrt{\Theta_1} d_{j,n} \leq \int_{\mathbb{R}^4} |\nabla u_{j,n}|^2 - \frac{\lambda}{|x|^2} u_{j,n}^2 \\
= \int_{\mathbb{R}^4} \gamma_{jj} u_{j,n}^4 + \sum_{k \neq j} \int_{\mathbb{R}^4} \gamma_{kj} u_{k,n}^2 u_{j,n}^2 \\
\leq \gamma_{jj} d_{j,n}^2 + \sum_{k \neq j} \gamma_{kj} d_{j,n} d_{k,n}.
\]

On the other hand
\[
2 \sqrt{\Theta_1} \sum_{i=1}^r d_{i,n} \leq 4 J(u_{1,n}, \ldots, u_{r,n}) \leq 4 \sum_{j=1}^r c_j \Theta_1 + o(1),
\]
thus we have
\[
\left\{ \begin{array}{l}
\sum_{i=1}^r d_{i,n} \leq \sum_{i=1}^r c_i 2 \sqrt{\Theta_1} + o(1), \\
\gamma_{ii} d_{i,n} + \sum_{k \neq j} \gamma_{ki} d_{k,n} \geq 2 \sqrt{\Theta_1}.
\end{array} \right.
\]
Recall (5.14), then the inequalities above are equivalent to
\[
\left\{ \begin{array}{l}
\sum_{i=1}^r (d_{i,n} - c_i 2 \sqrt{\Theta_1}) \leq o(1), \\
\gamma_{ii} (d_{i,n} - c_i 2 \sqrt{\Theta_1}) + \sum_{k \neq i} \gamma_{ki} (d_{k,n} - c_k 2 \sqrt{\Theta_1}) \geq 0,
\end{array} \right. \\
i = 1, \ldots, r.
\]
By Lemma 2.3, we have \( d_{i,n} \to c_i 2 \sqrt{\Theta_1} \) as \( n \to \infty \), and
\[
4 \Theta = \lim_{n \to \infty} 4 J(u_{1,n}, \ldots, u_{r,n}) \geq \lim_{n \to \infty} 2 \sqrt{\Theta_1} \sum_{i=1}^r d_{i,n} = 4 \sum_{i=1}^r c_i \Theta_1.
\]
Combining this with (5.15), one has that
\[
\Theta = \sum_{j=1}^r c_j \Theta_1 = J(\sqrt{c_1} z, \ldots, \sqrt{c_r} z),
\]
and so \((\sqrt{c_1} z, \ldots, \sqrt{c_r} z)\) is a positive least energy solution of (5.13). \( \square \)

**The proof of Theorem 1.3-(2).** Namely, we need to prove the uniqueness of the ground state of (5.13). Let \((u_{1,0}, \ldots, u_{r,0})\) be any least energy solution of (5.13). Firstly we define the real functions with variables \((t_1, \ldots, t_r) \in \mathbb{R}^r:\)
\[
f_j(t_1, \ldots, t_r) := \int_{\mathbb{R}^4} t_j \gamma_{jj} u_{j,0}^4 + \sum_{k \neq j} \int_{\mathbb{R}^4} t_k \gamma_{kj} u_{k,0}^2 u_{j,0}^2 - \int_{\mathbb{R}^4} |\nabla u_{j,0}|^2 - \frac{\lambda}{|x|^2} u_{j,0}^2.
\]
Here we regard \( \gamma_{ml} \) (for any fixed \((m, l)\) satisfying \( 1 \leq m, l \leq r \)) as the variable. Recalling the definitions of \( J, N \) and \( \Theta \), they all depend on \( \gamma_{ml} \). Hence, we now adopt
the notations \( J(\gamma_{ml}) \), \( N(\gamma_{ml}) \) and \( \Theta(\gamma_{ml}) \) in this proof. With the definitions above, we have \( f_j(1,\ldots,1) = 0 \) and

\[
\frac{\partial f_j}{\partial t_i} = \gamma_{ij} \int_{\mathbb{R}^4} u_{i,0}^2 u_{j,0}^2.
\]

Define the matrix:

\[
F := \left( \frac{\partial f_j}{\partial t_i} \right)_{(1,\ldots,1)}.
\]

Since the matrix \( \gamma \) defined in (5.12) is positively definite, so is the following matrix \( (\gamma_{ij} \int_{\mathbb{R}^4} u_{i,0}^2 u_{j,0}^2) \). Hence, \( \det(F) > 0 \). Therefore, by the Implicit Function Theorem, the functions \( t_j(\beta_{ml}) \) are well defined and of class \( C^1 \) on \( (\gamma_{ml} - \delta_1, \gamma_{ml} + \delta_1) \) for some \( 0 < \delta_1 \leq \delta \). Moreover, \( t_j(\gamma_{ml}) = 1, j = 1,\ldots,r \), and so we may assume that \( t_j(\gamma_{ml}) > 0 \) for all \( \gamma_{ml} \in (\gamma_{ml} - \delta_1, \gamma_{ml} + \delta_1) \) by choosing a small \( \delta_1 \). From \( f_k(t_1(\gamma_{ml}),\ldots,t_r(\gamma_{ml})) \equiv 0 \), it is easy to prove that:

\[
\sum_{j=1}^N \frac{\partial f_k}{\partial t_j} t_j'(\gamma_{ml}) = -\frac{\partial f_k}{\partial \gamma_{ml}}.
\]

Hence

\[
t_j'(\gamma_{ml}) = -\sum_{k=1}^N \frac{\partial f_k}{\partial \gamma_{ml}} F_{k}^* f_j,
\]

here \( F^* := (F_{kj}^*) \) denotes the adjoint matrix of \( F \). From (5.16), we have

\[
\frac{\partial f_k}{\partial \gamma_{ml}} = \delta_{km} \int_{\mathbb{R}^4} u_{m,0}^2 u_{i,0}^2 dx, \quad \text{where } \delta_{km} \text{ is the Kronecker notation},
\]

hence

\[
t_j'(\gamma_{ml}) = -\sum_k \delta_{km} \int_{\mathbb{R}^4} u_{m,0}^2 u_{i,0}^2 dx \frac{F_{kj}^*}{\det(F)},
\]

that is,

\[
t_j'(\gamma_{ml}) = -\int_{\mathbb{R}^4} u_{m,0}^2 u_{i,0}^2 dx \frac{F_{kj}^*}{\det(F)}.
\]

By the Taylor’s expansion, we see that

\[
t_j'(\gamma_{ml}) = 1 + t_j'(\gamma_{ml})(\gamma_{ml} - \gamma_{ml}) + \mathcal{O}((\gamma_{ml} - \gamma_{ml})^2).
\]

Note that \( t_j(t_1(\gamma_{ml}),\ldots,t_r(\gamma_{ml})) \equiv 0 \) implies that

\[
(\sqrt{t_1(\gamma_{ml})} u_{1,0}, \ldots, \sqrt{t_1(\gamma_{ml})} u_{r,0}) \in N(\gamma_{ml}),
\]

therefore

\[
J(\gamma_{ml}) \leq E_{\gamma_{ml}}(\sqrt{t_1(\gamma_{ml})} u_{1,0}, \ldots, \sqrt{t_1(\gamma_{ml})} u_{r,0})
\]

\[
= \frac{1}{4} \sum_{j=1}^r t_j(\gamma_{ml}) \int_{\mathbb{R}^4} |\nabla u_{j,0}|^2 - \frac{\lambda}{|x|^2} u_{j,0}^2 dx
\]

\[
= J(\gamma_{ml}) + \frac{1}{4} D(\gamma_{ml} - \gamma_{ml}) + \mathcal{O}((\gamma_{ml} - \gamma_{ml})^2),
\]

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where

\[ D := \sum_{j=1}^{r} t_j'(\gamma_{ml}) \int_{\mathbb{R}^4} |\nabla u_{j,0}|^2 - \frac{\lambda}{|x|^2} u_{j,0}^2 \, dx \]

\[ = \sum_{j=1}^{r} t_j'(\gamma_{ml}) \left( \int_{\mathbb{R}^4} \gamma_{jj} u_{j,0}^4 \, dx + \sum_{k \neq j} \gamma_{kj} \int_{\mathbb{R}^4} u_{k,0}^2 u_{j,0}^2 \, dx \right) \]

\[ = -\int_{\mathbb{R}^4} u_{m,0}^2 u_{l,0}^2 \sum_{j=1}^{r} \frac{F_{mj}^*}{\det(F)} (F_{jj} + \sum_{k \neq j} F_{kj}^* \gamma_{jj}) \]

\[ = -\int_{\mathbb{R}^4} u_{m,0}^2 u_{l,0}^2 \sum_{j=1}^{r} \frac{1}{\det(F)} \sum_{h=1}^{r} \delta_{km} \det(F) \]

Here we have used (5.17). It follows that

\[ J'(\gamma_{ml}) - J'(\gamma_{ml}) \geq \frac{D}{4} + O(\gamma_{ml} - \gamma_{ml}) \]

as $\gamma_{ml} \not\to \gamma_{ml}$ and so $J'(\gamma_{ml}) \geq \frac{D}{4}$. Similarly, we have $J'(\gamma_{ml}) - J'(\gamma_{ml}) \leq \frac{D}{4} + O(\gamma_{ml} - \gamma_{ml})$ as $\gamma_{ml} \not\to \gamma_{ml}$, that is, $J'(\gamma_{ml}) \leq \frac{D}{4}$ in this case. Hence,

\[ J'(\gamma_{ml}) = \frac{D}{4} = -\frac{1}{4} \int_{\mathbb{R}^4} u_{m,0}^2 u_{l,0}^2. \]

On the other hand, by Lemma 5.2 we have

\[ J'(\gamma_{ml}) = -c_m c_l \Theta_1 = -\frac{1}{4} c_m c_l \int_{\mathbb{R}^4} z^4. \]

Hence,

\[ \int_{\mathbb{R}^4} u_{m,0}^2 u_{l,0}^2 = c_m c_l \int_{\mathbb{R}^4} z^4. \]

Define

\[ (\bar{u}_1, ..., \bar{u}_r) := \left( \frac{1}{\sqrt{c_1}} u_{1,0}, ..., \frac{1}{\sqrt{c_r}} u_{r,0} \right). \]

Combine this with the following identity:

\[ \gamma_{jj} c_j + \sum_{k \neq j} \beta_{kj} c_k = 1. \]
and \((u_{1,0}, \ldots, u_{r,0}) \in \mathcal{N}\), we get that
\[
\int_{\mathbb{R}^4} (|\nabla \tilde{u}_j|^2 - \frac{\lambda}{|x|^2} \tilde{u}_j^2) \, dx = \frac{1}{c_j} \int_{\mathbb{R}^4} (|\nabla u_{j,0}|^2 - \frac{\lambda}{|x|^2} u_{j,0}^2) \, dx
\]
\[
= \frac{1}{c_j} \left( \int_{\mathbb{R}^4} \gamma_{jj} u_{j,0}^4 \, dx + \sum_{k \neq j} \gamma_{kj} \int_{\mathbb{R}^4} u_{k,0}^2 u_{j,0}^2 \, dx \right)
\]
\[
= \frac{1}{c_j} \left( \mu_j c_j^2 + \sum_{k \neq j} \gamma_{kj} c_k c_j \right) \int_{\mathbb{R}^4} \omega^4 \, dx
\]
\[
= \left( \gamma_{jj} c_j + \sum_{k \neq j} \gamma_{kj} c_k \right) \int_{\mathbb{R}^4} \omega^4 \, dx
\]
\[
= \int_{\mathbb{R}^4} \tilde{u}_j^4 \, dx.
\]
Then by (5.14), we have
\[
\frac{1}{4} \int_{\mathbb{R}^4} \left( |\nabla \tilde{u}_j|^2 - \frac{\lambda}{|x|^2} \tilde{u}_j^2 \right) \, dx \geq \Theta_1, \quad j = 1, \ldots, r.
\]
(5.18)
Hence,
\[
\Theta = \sum_{j=1}^r c_j \Theta_1 = \frac{1}{4} \sum_{j=1}^r \int_{\mathbb{R}^4} (|\nabla u_{j,0}|^2 - \frac{\lambda}{|x|^2} u_{j,0}^2) \, dx
\]
\[
= \frac{1}{4} \sum_{j=1}^r \int_{\mathbb{R}^4} (|\nabla \tilde{u}_j|^2 - \frac{\lambda}{|x|^2} \tilde{u}_j^2) \, dx
\]
\[
\geq \sum_{j=1}^r c_j \Theta_1.
\]
This implies that
\[
\frac{1}{4} \int_{\mathbb{R}^4} (|\nabla \tilde{u}_j|^2 - \frac{\lambda}{|x|^2} \tilde{u}_j^2) \, dx = \Theta_1, \quad j = 1, \ldots, r.
\]
Then we see that \(\tilde{u}_j(j = 1, \ldots, r)\) are the positive least energy solutions of (1.10). We see from the fact that
\[
-\Delta \tilde{u}_j - \frac{\lambda}{|x|^2} \tilde{u}_j = \gamma_{jj} c_j \tilde{u}_j^3 + \sum_{k \neq j} \gamma_{kj} c_k \tilde{u}_k^2 \tilde{u}_j = \tilde{u}_j^3
\]
and
\[
\gamma_{jj} c_j \tilde{u}_j^2 + \sum_{k \neq j} \gamma_{kj} c_k \tilde{u}_k^2 = \tilde{u}_j^2,
\]
and hence
\[
\gamma_{jj} c_j + \sum_{k \neq j} \gamma_{kj} c_k \left( \frac{\tilde{u}_k}{\tilde{u}_j} \right)^2 = 1.
\]
Since the matrix $\gamma$ is invertible, we get that $\frac{\hat{\varphi}}{m_j} = 1, k \neq j$. That is, $\tilde{u}_k = \tilde{u}_j, k \neq j$. Denote that $U = \tilde{u}_1$, then $(u_1,0,...,u_r,0) = (\sqrt{c_1}U',...,\sqrt{c_r}U)$, where $U$ is a positive least energy solution of (1.10).

The proof of Theorem 1.4 We consider the following doubly critical Shrödinger system (i.e., (1.15)) on $\mathbb{R}^N$:

$$
\begin{cases}
-\Delta u - \frac{\lambda}{|x|^2} u = u^{2^*-1} + \nu\alpha u^{\alpha-1} v^\beta, \\
-\Delta v - \frac{\lambda}{|x|^2} v = v^{2^*-1} + \nu\alpha u^{\alpha} v^{\beta-1}.
\end{cases}
$$

(5.19)

Let $p = 2^*/2$. It is easy to see that the following system

$$
\begin{cases}
f_1(x_1, x_2) : = x_1^{p-1} + \nu\alpha x_1^{\frac{\alpha}{2}-1} x_2^{\frac{\beta}{2}} = 1, \\
f_2(x_1, x_2) : = x_2^{p-1} + \nu\beta x_1^{\frac{\alpha}{2}} x_2^{\frac{\beta}{2}-1} = 1,
\end{cases}
$$

(5.20)

admits a positive solution $(c_1, c_2)$ for any $\nu > 0$. In fact, from the first equality of (5.20), we know that $x_2 = (2\nu\alpha)^{-\frac{1}{p}} (1 - x_1^{p-1})^{\frac{2}{p}} x_1^{\frac{\alpha}{2}}$. The system (5.20) admitting a positive solution is equivalent to the equation

$$
f(x_1) := (\nu\alpha)^{-\frac{1}{p}(p-1)} x_1^{\frac{\alpha}{2}+\frac{\beta}{2}(p-1)} (1 - x_1^{p-1})^{\frac{2}{p}} - \nu \beta (\nu\alpha)^{-\frac{\alpha}{p}} x_1^{\frac{\alpha}{2}} (1 - x_1^{p-1})^{-\frac{2}{p}} - 1 = 0
$$

has a root in the interval $(0, 1)$. Since $f(0) = -1$ and $\lim_{x_1 \to 1^-} f(x_1) = +\infty$, the conclusion follows from the Mean Value Theorem. Hence, $(\sqrt{c_1}z, \sqrt{c_2}z)$ is a nontrivial solution of (5.19) and

$$
0 < \Theta \leq J(\sqrt{c_1}z, \sqrt{c_2}z) = (c_1 + c_2)\Theta_1.
$$

(5.21)

Now we assume that $\nu > (p-1)/\min\{d_1(\alpha, \beta), d_2(\alpha, \beta), d_3(\alpha, \beta)\}$, and we shall prove that $\Theta = J(\sqrt{c_1}z, \sqrt{c_2}z)$. Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a minimizing sequence for $\Theta$, that is, $J(u_n, v_n) \to \Theta$. Define

$$
d_{1,n} = \left( \int_{\mathbb{R}^N} |u_n|^{2p} \, dx \right)^{\frac{1}{p}}, \quad d_{2,n} = \left( \int_{\mathbb{R}^N} |v_n|^{2p} \, dx \right)^{\frac{1}{p}}.
$$

By (1.5) and (1.7), we have

$$
(N\Theta_1)^{2/N} d_{1,n} \leq \int_{\mathbb{R}^N} (|\nabla u_n|^2 - \frac{\lambda}{|x|^2} u_n^2) = \int_{\mathbb{R}^N} (|u_n|^{2p} + \nu\alpha |u_n|^{\alpha} v_n^\beta) \\
\leq d_{1,n}^p + \nu\alpha d_{1,n}^{\alpha/p} d_{2,n}^{\beta/p},
$$

$$
(N\Theta_1)^{2/N} d_{2,n} \leq \int_{\mathbb{R}^N} (|\nabla v_n|^2 - \frac{\lambda}{|x|^2} v_n^2) = \int_{\mathbb{R}^N} (|v_n|^{2p} + \nu\beta |u_n|^\alpha v_n^\beta) \\
\leq d_{2,n}^p + \nu\beta d_{1,n}^{\alpha/p} d_{2,n}^{\beta/p}.
$$
Since $J(u_n, v_n) = \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 - \frac{1}{|x|^2} u_n^2 - \frac{1}{|x|^2} v_n^2)$, by (5.21), we have

$$
\begin{cases}
(N\Theta_1)^{2/N} (d_{1,n} + d_{2,n}) \leq NJ(u_n, v_n) \leq N(c_1 + c_2)\Theta_1 + o(1),
\end{cases}
$$

$$
\begin{align*}
d_{1,n}^{p-1} + \nu\alpha x_1^{\alpha/2-1} d_{2,n}^{\beta/2} &\geq (N\Theta_1)^{2/N}, \\
d_{2,n}^{p-1} + \nu\beta x_1^{\alpha/2-1} d_{2,n}^{\beta/2} &\geq (N\Theta_1)^{2/N},
\end{align*}
$$

(5.22)

First, this means that $d_{1,n}, d_{2,n}$ are uniformly bounded. Passing to a subsequence we may assume that $d_{1,n} \to d_1, d_{2,n} \to d_2$. It is easy to check that $d_1 > 0, d_2 > 0$.

Denote

$$x_1 = \frac{d_1}{(N\Theta_1)^{1-\frac{\alpha}{2}}}, \quad x_2 = \frac{d_2}{(N\Theta_1)^{1-\frac{\alpha}{2}}}.$$ 

By a simple scaling we can transform (5.22) to

$$
\begin{align*}
x_1 + x_2 &\leq c_1 + c_2, \\
x_1^{x_1^{\alpha/2-1}} x_2^{\beta/2} &\geq 1, \\
x_2^{x_1^{\alpha/2-1}} x_2^{\beta/2} &\geq 1.
\end{align*}
$$

By Lemma 5.1 we see that $x_1 = c_1, x_2 = c_2$. It follows that

$$d_{1,n} \to c_1 (N\Theta_1)^{1-\frac{\alpha}{2}}, \quad d_{2,n} \to c_2 (N\Theta_1)^{1-\frac{\alpha}{2}}, \quad \text{as} \ n \to \infty$$

and

$$N\Theta = \lim_{n \to \infty} NJ(u_n, v_n) \geq (N\Theta_1)^{2/N} (d_{1,n} + d_{2,n}) = N(c_1 + c_2)\Theta_1.$$

Combing this with (5.21), we have

$$\Theta = (c_1 + c_2)\Theta_1 = J(\sqrt{c_1}z, \sqrt{c_2}z),$$

and therefore, $(\sqrt{c_1}z, \sqrt{c_2}z)$ is a positive least energy solution of (5.19). 

\begin{flushright}
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\end{flushright}

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