Geometric RG Flow

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Abstract

We define geometric RG flow equations that specify the scale dependence of the renormalized effective action $\Gamma[g]$ and the geometric entanglement entropy $S[x]$ of a QFT, considered as functionals of the background metric $g$ and the shape $x$ of the entanglement surface. We show that for QFTs with AdS duals, the respective flow equations are described by Ricci flow and mean curvature flow. For holographic theories, the diffusion rate of the RG flow is much larger, by a factor $R_{AdS}^2/\ell_s^2$, than the RG resolution length scale. To derive our results, we employ the Hamilton-Jacobi equations that dictate the dependence of the total bulk action and the minimal surface area on the geometric QFT boundary data.

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1 Introduction

QFT divergences arise due to the unlimited number of UV degrees of freedom. Quantities that directly measure the number of degrees of freedom, such as the QFT effective action $\Gamma$ receive divergent contributions from the infinitude of short distance modes that fill space time. Similarly, the geometric entanglement entropy $S$ between a region of space $A$ and its complement contains a divergent term proportional to the area of the entanglement surface, due to short distance modes that straddle the boundary of $A$ [1]. These divergences are real and physical in any continuum QFT.

To produce finite quantities, one employs a standard renormalization procedure: one introduces a UV regulator, adds counterterms that cancel the divergences and then removes the cut-off. In general, this procedure requires the introduction of a renormalization group scale. The renormalized quantities depend on the RG scale $a$ via renormalization group equations. Typically, this RG evolution governs the scale dependence of space time independent quantities such as the coupling constants $\phi_I$ of the QFT lagrangian.

In this letter, we will investigate the RG evolution of the following two quantities:

1) the renormalized effective action $\Gamma_R[\phi,g]$, defined as minus the logarithm of the CFT partition function, as a function of spatially varying couplings $\phi_I$ and metric $g_{\mu\nu}$,

2) the renormalized entanglement entropy $S_R[x,a]$ of a bounded region of space, as a function of the location $x(s)$ of the entanglement surface and RG scale $a$.

We will study these quantities for $d+1$-dimensional quantum field theories that admit a holographic dual description in terms of a weakly coupled gravitational theory in $d+2$ dimensions. Using the usual AdS/CFT dictionary and the Ryu-Takayanagi formula for the geometric entanglement entropy [2,3], we will derive that these two quantities satisfy the holographic RG flow equations [4] of the form

\[
\left(a \frac{\delta}{\delta a} + \beta_n \frac{\delta}{\delta x_n}\right) S_R[x,a] = 0, \tag{1}
\]

\[
\left(a \frac{\delta}{\delta a} + \beta_I \frac{\delta}{\delta \phi_I} + \beta_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}\right) \Gamma_R[\phi,g] = 0. \tag{2}
\]

In the second equation, we identified the generators of the RG and Weyl rescalings via

\[
a \frac{\delta}{\delta a} = 2g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}. \tag{3}
\]
Figure 1: The geometric RG flow smooths out the shape of an entanglement boundary. The smoothing effect is quantified by mean curvature flow equations. A similar smoothing effect takes place for the background metric, in which case the geometric RG leads to Ricci flow.

The above RG equations take the conventional Callan-Symanzik form, except that we allow the RG scale to vary along the spatial directions. Moreover, we have included non-zero beta-functions $\beta_{\mu\nu}$ and $\beta_n$ for the space-time metric $g_{\mu\nu}$ and for the location $x_n$ of the entanglement surface! The respective flow equations

$$a \frac{\partial x_n}{\partial a} = \beta_n, \quad a \frac{\partial g_{\mu\nu}}{\partial a} = 2g_{\mu\nu} + \beta_{\mu\nu}, \quad a \frac{\partial \phi_I}{\partial a} = \beta_I.$$  

(4)

anticipate the possibility that, in addition to the couplings $\phi_I$, the entanglement surface and metric also acquire a non-trivial scale dependence. We call this geometric RG flow.

Depending on the reader’s preconceptions about the meaning of RG equations, this notion of geometric RG flow may look either unconventional and radical, or natural and self-evident. Normally one tends to think of the background metric and some given entanglement surface as fixed geometric quantities. Implicitly, one is then defining both quantities in relation to the UV fixed point QFT. Indeed, suppose that the metric and entanglement surface both have a fractal like shape, with local structure divided over all possible length scales. It is then clear that the RG flow must have the effect of smoothing out all UV features shorter than the RG scale, since the IR theory no longer notices them. This smoothing effect, depicted in fig.[1] is a first indication that there should exist a notion of geometric RG flow.
The question remains whether the geometric RG flow can be described via (4), in terms of canonically normalized geometric beta-functions $\beta_{\mu\nu}$ and $\beta_n$. Naively, one would expect that the features subject to the flow have size proportional to the RG scale, so that the associated 'beta-functions' are highly sensitive to the type of cut-off or regulator used in setting up the RG. In the following, we will show that this ambiguity can be avoided with the help of three practical restrictions. First, we will work in the regime where geometric features such as the local curvature are small, but non-negligible, compared to the RG cut-off scale. Moreover, we assume that the regulator can be arranged to preserve general covariance. Finally, we will restrict our attention to strongly coupled quantum field theories that admit a weakly coupled holographic dual description.

Geometric RG flow is a natural notion within the context of AdS/CFT. Via holography, the operation of dividing up the CFT into an IR and UV sector, separated by a scale $a$, amounts to cutting the dual AdS space-time along a spatial slice specified by some radial location $r = r(a)$ \[\text{[7,8]}\]. Since the bulk theory contains gravity, the holographic RG equations that describe the radial evolution of this slice are defined in a diffeomorphism invariant way. We will find that, in the weakly curved regime, the geometric RG flow of the metric and entanglement surface take the following form \[\text{[4]}\]

$$\beta_n = \lambda K_n, \quad \beta_{\mu\nu} = \kappa \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right),$$

where $R_{\mu\nu}$ and $R$ denote the Ricci tensor and scalar of the boundary metric $g_{\mu\nu}$, and $K_n$ denotes the mean curvature of the entanglement surface $x_n$. Here $\lambda$ and $\kappa$ are scalar functions of the holographic couplings $\phi_I$, times appropriate powers of the RG length scale $a$. For constant $\kappa$ and $\lambda$, the geometric RG equations (4)-(5) take a very familiar form: the metric evolves via Ricci flow, and the entanglement surface evolves via mean curvature flow.

Geometric Ricci and mean curvature flow equations have been actively studied in mathematics, as an analytic tool for proving the Poincaré conjecture \[\text{[9]}\] and related geometric problems \[\text{[10]}\], and in physics, in the form of RG equations for the coupling constants of 2-dimensional sigma models in a curved target space \[\text{[11]}\], or with Dirichlet boundary conditions on a curved D-brane \[\text{[12,13,2]}\]. In both the mathematics and physics context, it proves to be helpful to formulate the geometric flow as a gradient flow of, respectively, the

\[\text{1}\]In this paper, we mostly focus on RG flow of the geometry. The spatially varying couplings $\phi_I$, however, also take active part in the geometric RG. Their $\beta$-functions take a similar form of diffusion equations $\beta_I = \nu \nabla^2 \phi_I + \bar{\beta}_I$ with $\nu$ and $\bar{\beta}_I$ scalar functions of the scalar fields $\phi_I$.

\[\text{2}\]The connection between holographic RG and Ricci flow has appeared in earlier work. The geometric beta-function $\beta_{\mu\nu}$ was first identified in \[\text{[7]}\]. In \[\text{[14]}\], the Ricci flow was studied in the supergravity duals of 6-D CFTs compactified on a Riemann surfacetimes $\mathbb{R}^4$.  

3
Einstein-Hilbert action and the area or volume functional

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{|g|} R, \quad K_n = \frac{\delta}{\delta x^n} \int \sqrt{\text{det} h}, \quad \text{ (6)} \]

with \( h_{ij} = g_{mn} \partial_i x^m \partial_j x^n \) the induced metric of the entanglement surface. These equations will be helpful in the following.

In the remainder of this letter, we will derive the above flow equations and compute the precise form of the scalar functions \( \kappa \) and \( \lambda \) for a strongly coupled QFT with a given holographic dual. Along the way, we will exhibit a direct relation between the geometric beta functions and the conformal anomaly. Finally, we present the outlines of a conceptual explanation of the geometric RG flow equations in terms the Wilsonian interpretation of the holographic renormalization group.

**Geometric RG as a Hallmark of CFTs with AdS duals**

A central mystery of AdS/CFT duality is that the bulk theory exhibits locality down to the string scale \( \ell_s \), whereas from the point of view of the boundary theory, locality would be expected to break down at a much larger length scale of order the AdS-radius \( R_{\text{AdS}} \). The latter estimate arises because boundary to bulk propagation causes a pointlike boundary perturbation to spread out into the AdS-bulk over a region of size \( R_{\text{AdS}} \). For this reason, one would have expected that in the Wilsonian RG interpretation of the radial evolution, the running RG scale, the resolution scale after coarse graining, is set by the AdS-radius. However, the bulk gravity theory tells us that this is incorrect: the resolution scale in the bulk is much smaller, of order \( \ell_s \). This discrepancy lies at the heart of the holographic duality. It also underlies the existence of geometric RG flow.

Geometric RG flow is diffusion. For example, if we linearize the Ricci tensor \( R_{\mu\nu} = \Box h_{\mu\nu} \), the Ricci flow equations literally take the form of diffusion equations. As we will show, the diffusion constant \( \kappa \) is governed by the central charge of the CFT, and proportional to \( R_{\text{AdS}}^2 \).

This interpretation of RG as diffusion sheds interesting new light on why the resolution scale in the bulk theory can be so small compared to the AdS radius. Also from the CFT perspective, there are in fact two separate RG scales in the problem: the resolution length scale \( \ell_{\text{resolution}} \), i.e. the minimal length that can be resolved in the coarse grained theory, and the diffusion length scale \( \ell_{\text{diffusion}} \), that sets the diffusion rate by which the metric and other locally varying quantities are smoothed out under the RG flow. Holographic theories

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3Here and in the following, to avoid cluttered formulas, we will often implicitly absorb a factor of \( 1/\sqrt{g} \) and \( 1/\sqrt{\text{det} h} \) in the definition of the respective variational derivatives.
are characterized by the property that there is a large hierarchy between the two scales
\[ R_{\text{AdS}} \sim L_{\text{diffusion}} \gg L_{\text{resolution}} \sim \ell_{\text{string}}. \] (7)

In other words, the mystery of why the bulk physics is local down to such small length scales is reversed. Our proposal is that the string length sets the resolution scale in the Wilson RG interpretation of the radial evolution. The classical bulk equations of motion then imply that local geometric features smooth out under the radial evolution with a diffusion rate set by the AdS-radius. So from the QFT point of view, the real unanswered question is:

*Why do localized features of a CFT with an AdS-dual diffuse so rapidly under RG flow?*

## 2 Hamilton-Jacobi

In applications of AdS/CFT duality, many computations proceed along the following lines: (i) one fixes some boundary conditions in the asymptotic AdS region, (ii) solves the bulk equations of motion and obtains the classical solution, (iii) evaluates the total classical bulk action, (iv) regulates and renormalizes the integral, by subtracting any infinities that may occur. The resulting renormalized quantity depends only on the specified boundary conditions, and thus can be interpreted as a property of CFT data.

The two quantities of interest are precisely of this type. The QFT effective action \( \Gamma_{\alpha}[\phi, g] \) is identified with the classical bulk action of the gravity theory, with given boundary values \( \phi_I \) and \( g_{\mu\nu} \). Similarly, the Ryu-Takayanagi prescription \cite{2,3} identifies the entanglement entropy \( S[x, a] \) of a region \( A \) with boundary location specified by \( x \), with the area/volume of the minimal RT surface \( \Sigma \) in the AdS-bulk with the same boundary as the \( A \) region, \( \partial \Sigma = \partial A \). Both quantities \( \Gamma[\phi, g] \) and \( S[x, a] \) contain UV/volume divergences, and require a suitable holographic renormalization procedure.

The Hamilton-Jacobi (HJ) formalism gives direct information about the dependence of the total classical action on the initial or boundary conditions, for a brief nice review see \cite{5}. It is therefore ideally suited for this type of problem. For the effective action \( \Gamma \), this method was introduced and worked out in \cite{4,7}. Consider a \( d + 2 \) dimensional negatively curved space-time with metric
\[ ds_{d+2}^2 = G_{MN} dx^M dx^N = dr^2 + g_{\mu\nu} dx^\mu dx^\nu. \] (8)

Here \( g_{\mu\nu} \) may depend on the \( d + 1 \) space-time coordinates \( x^{\mu} \) and the radial coordinate \( r \).
We introduce the bulk gravity theory described by the \( d + 2 \)-dimensional action

\[
S_{d+2} = \int \sqrt{G} \left( R + \frac{1}{2} (\partial_M \phi)^2 + V(\phi) \right)
\]  

(9)

For definiteness, we will mostly restrict to the case that \( d = 3 \). The 5-dimensional equations of motion can be written as a Hamiltonian system with hamiltonian \( H = \int \sqrt{g} \mathcal{H} \) with

\[
\mathcal{H} = \pi^{\mu \nu} \pi_{\mu \nu} - \frac{1}{3} \pi^\mu_\mu \pi^\nu_\nu + \frac{1}{2} \pi_\mu \pi^\mu I + R + \frac{1}{2} \nabla \phi^I \nabla \phi_I + V(\phi)
\]  

(10)

the local ADM Hamiltonian. Here \( \pi_{\mu \nu} \) and \( \pi_I \) are the canonical momentum variables conjugate to \( g_{\mu \nu} \) and \( \phi_I \). Let \( \Gamma[\phi, g] \) denote the classical action [9], evaluated over a 5-d region \( r < r(a) \), with boundary values specified by \( \phi \) and \( g \). The HJ equation for \( \Gamma \) is obtained by setting the local ADM Hamiltonian equal to zero, while replacing the dual momenta by the corresponding variational derivative of the total integrated action \( \Gamma[\phi, g] \):

\[
\mathcal{H} = 0 \quad \text{with} \quad \pi_I = \frac{\delta \Gamma}{\delta \phi^I}, \quad \pi_{\mu \nu} = \frac{\delta \Gamma}{\delta g^{\mu \nu}}.
\]  

(11)

This Hamilton constraint expresses the invariance of the bulk theory under reparametrizations of the radial coordinate \( r \).

Next consider a \( d \)-dimensional Ryu-Takayanagi minimal surface \( \Sigma \), figure (2), suspended within a \( d + 1 \) dimensional constant time slice with metric

\[
ds_{d+1}^2 = g_{MN} dx^M dx^N = dv^2 + g_{mn} dx^m dx^n,
\]  

(12)

where \( x^N = (r, x^n) \). Here \( N \) runs from 1 to \( d + 1 \) and \( n \) run from 1 to \( d \). The RT formula equates the entanglement entropy of a CFT with a holographic dual to \( 1/4 \) times the \( d \)-dimensional volume of the minimal surface \( \Sigma \)

\[
S = \frac{1}{4} \int_\Sigma \sqrt{\det H}, \quad H_{AB} = g_{MN} \partial_A x^M \partial_B x^N,
\]  

(13)

where \( \Sigma \) satisfies the asymptotic condition \( \partial \Sigma = \partial A \) at the AdS boundary. Here \( H_{AB} \) denotes the induced metric on the RT minimal surface, parametrized by world-volume coordinates \( \xi_A \), with \( A = 1, \ldots, d \). Note that the formula (13) defines a reparametrization invariant

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4In addition to the Hamiltonian constraint (11), one also needs to impose the constraints \( \nabla^\mu \pi_{\mu \nu} + \pi_I \nabla^\nu \phi_I = 0 \), which upon making the same replacement as in (11) expresses the condition that \( \Gamma[\phi, g] \) is invariant underreparametrizations of the \( d + 1 \) dimensional space-time coordinates \( x^\mu \).
functional of the $\xi_A$ coordinates.

Again, we would like to write the Hamilton-Jacobi equation, that specifies the dependence of the minimal area $S[x]$ on the location of the entanglement boundary $\partial \Sigma = \partial A$. To do this, we first split the $d$ world-volume coordinates $\xi_A = (\tau, \xi_i)$ into a radial ‘time coordinate’ $\tau$ and $d-1$ ‘spatial coordinates’ $\xi_i$. We can then define the induced metric on the constant $\tau$ slice via

$$h_{ij} = g_{MN} \partial_i x^M \partial_j x^N,$$

and re-write the volume functional (13) in first order form, as follows

$$S = \int_\Sigma \left( \pi_N \dot{x}^N - N \left( g^{MN} \pi_M \pi_N + \frac{1}{4} \det h \right) - N^i \pi_N \partial_i x^N \right).$$

Here $\pi_N$ are momenta dual to $x^N$, and $N_\mu = (N, N^i)$ are Lagrange multipliers that impose

Figure 2: The Ryu-Takayanagi minimal surface is uniquely specified by the location $x$ of the entanglement surface at the boundary, located at $r = r(\epsilon)$. The total area of the RT surface, normalized by $\frac{1}{4G_{d+2}}$, equals the holographic entanglement entropy $S[x, \epsilon]$ as a functional of the boundary location $x$ and cut-off $\epsilon$, and satisfies a Hamilton-Jacobi equation.
the Hamilton and momentum constraints, that express the reparametrization invariance of the original world-volume \[13\]. The equivalence between \[15\] and \[13\] is easily verified.\footnote{Upon eliminating the momentum variables via their equation of motion, one finds \[16\]}

The HJ equation of the minimal area functional \(S[x]\) is immediately read off from the first order action \[15\], by replacing the momentum variables in the Hamiltonian constraint with the corresponding variational derivatives of \(S[x]\)

\[
\pi_r = \frac{\partial S}{\partial r}, \quad \pi_n = \frac{\partial S}{\partial x^n}.
\]

We thus obtain the following HJ equations for \(S[x, r]\) and \(\Gamma[\phi, g]\)

\[
\left( \frac{\partial S}{\partial r} \right)^2 + g^{mn} \frac{\partial S}{\partial x^m} \frac{\partial S}{\partial x^n} = \frac{1}{4} \text{det } h, \quad (19)
\]

\[
\frac{1}{12} \left( a \frac{\delta \Gamma}{\delta a} - \left( \frac{\delta \Gamma}{\delta g^{\mu \nu}} \right)^2 - \frac{1}{2} \left( \frac{\delta \Gamma}{\delta \phi} \right)^2 \right) = V(\phi) + R + \frac{1}{2} (\nabla \phi)^2. \quad (20)
\]

In the second equation we used the definition \[3\]. These equations should be read as functional identities, and can in principle be solved by means of a derivative expansion. The solutions are not unique, but depend on integration constants, which can be thought of as initial or final conditions. A given solution uniquely specifies the radial evolution

\[
\dot{g}_{\mu \nu} = 2\pi_{\mu \nu} - \frac{3}{2} \pi^\lambda g_{\mu \nu}, \quad \dot{\phi}_I = \pi_I, \quad \dot{r} = \pi_r, \quad \dot{x}_n = \pi_n, \quad (21)
\]

via the replacement of the momenta by their ‘classical expectation values’ \[11\] and \[18\].

Equations \[19\]-\[20\] apply equally well to the IR part \(S_{ir}\) and \(\Gamma_{ir}\) of both functionals, given by the integral over the inside AdS region \(r < r(a)\), as to the UV parts, \(S_{uv}\) and \(\Gamma_{uv}\), given by the integral over outside AdS region \(r > r(a)\). A UV and IR solution combined specify a unique classical bulk trajectory. This fact was explored in \[7\], and forms the basis for the interpretation of the HJ equations as RG flow of the Wilsonian effective action.

Solving for the Lagrange multipliers \(N_i\) gives

\[
S = \frac{1}{4} \int_\Sigma \left( \frac{1}{N} (H_{00} + N^i N^j H_{ij} - 2 N^j H_{i0}) - N \text{ det } H_{ij} \right), \quad (16)
\]

Minimizing with respect to \(N\) reproduces \[13\].
3 Hamilton-Jacobi as an RG Equation

As a warm-up exercise for the next sections, let us write the Hamilton-Jacobi equation for the holographic entanglement entropy in a suggestive form, with a more direct interpretation as an RG equation. The idea is to split off the leading order divergent term from the minimal area functional $S[x, a]$. This procedure can be seen as the first step of a systematic derivative expansion. For this section, we use the pure $AdS_d$ spatial metric

$$ds^2 = R^2 \frac{da^2}{a^2} + \frac{dx^n dx_n}{a^2}, \quad (22)$$

with $R$ the AdS radius. With this metric, the HJ equation (19) takes the form

$$\left( a \frac{\partial S}{\partial a} \right)^2 + R^2 \left( a \frac{\partial S}{\partial x^i} \right)^2 = \frac{R^2}{4a^{2d-2}} \det h. \quad (23)$$

Here $h_{ij}$ has been scaled to bring out all dependence on the holographic RG scale $a$.

We now start with our derivative expansion: we write $S = S_{loc} + S_{ren}$ where the first term $S_{loc}$ is designed to cancel the r.h.s. of the HJ equation (23). We thus have

$$S = S_{loc} + S_{ren} = \frac{R}{d-1} \int \frac{1}{2a^{d-1}} \sqrt{\det h} + S_{ren}. \quad (24)$$

Since $\det h$ is independent of $a$, we immediately verify that $(a \frac{\partial S_{loc}}{\partial a})^2 = \frac{R^2}{4a^{2d-2}} \det h$. We see that the leading order term $S_{loc}$ gives the expected ‘area contribution’.

When we vary $S_{loc}[x, a]$ with respect to the location $x$, we obtain

$$\frac{\delta S_{loc}}{\delta x^n} = \frac{R}{2(d-1)a^{d-1}} K_n, \quad (25)$$

where $K_n$ is the mean curvature of the entanglement boundary $x$, measured with the $a$-independent metric $h_{\mu\nu}$. Plugging the expansion (24) into the HJ equation (23), we find after a straightforward calculation\(^6\)

$$a \frac{\delta S_{ren}}{\delta a} + \beta_n \frac{\delta S_{ren}}{\delta x_n} = b K^n K_n + \frac{a^{d-1}}{2R} \left( \left( a \frac{\delta S_{ren}}{\delta a} \right)^2 + R^2 \left( a \frac{\delta S_{ren}}{\delta x} \right)^2 \right), \quad (26)$$

\(^6\)Here we introduce the notation (c.f. footnote 3): $\frac{\delta}{\delta a} = \frac{1}{\sqrt{\det h}} \frac{\partial}{\partial a}$ and $\frac{\delta}{\delta x_n} = \frac{1}{\sqrt{\det h}} \frac{\partial}{\partial x_n}$.
with
\[
\beta_n = \frac{R^2 a^2 K_n}{(d-1)}, \quad b = \frac{R^3}{4(d-1)^2 a^{d-3}}. \tag{27}
\]

As promised, the new HJ equation (26) takes a suggestive form: it looks like an exact RG equation that prescribes the cut-off dependence of the entanglement entropy \(S[X,a]\) of a QFT. The first term on the r.h.s, with pre-coefficient \(b\) given in (27), indeed precisely matches with the expected mean curvature squared term. In \(d = 3\), it reproduces the correct logarithmic scale dependence prescribed by the conformal anomaly [15].

The last two non-linear terms on the r.h.s. of (26) also have the typical form of an exact RG, and are subleading for small \(a\). So in the following, we will mostly drop these terms.

The renormalized term \(S_{\text{ren}}\) is finite for \(d < 3\), but is still logarithmically divergent at \(d = 3\). In the next section we will discuss the holographic renormalization procedure, that removes this divergence. In dimension \(d\) larger than 3, the second term \(S_{\text{ren}}\) still contains divergent terms that also need to be removed by absorbing them into \(S_{\text{loc}}\). E.g. it is straightforward to extend the above analysis to \(d = 5\), and use the HJ equation to extract the form of the 6D holographic conformal anomaly.

### 4 Holographic Renormalization

Using the previous section as a guide, we now present a more systematic definition of the renormalized entanglement entropy and effective action. In the previous two sections, we have defined \(S[x,a]\) and \(\Gamma[\phi, g]\) as the bulk contribution of classical action functionals, with the asymptotic AdS region removed at some finite radial location \(r = r(a)\). This truncation can be thought of as the introduction of a covariant UV cut-off in the dual QFT.

To obtain the renormalized quantities, one needs to remove the cut-off via holographic renormalization [6].

The procedure works as follows. Pick a set of renormalized couplings \(\phi^I_R\), metric \(g^\mu_\nu_R\) and location \(x^\alpha_n\) of the entanglement boundary. These renormalized values are defined at some radial location inside the bulk of AdS space, specified by the holographic RG scale \(a\). Next,

\[\text{The quadratic terms in (26) do have physical significance: they allow for topology changing transitions along the RG flow. E.g. two disconnected components of the entanglement surface merge together. Indeed, the HJ equations for the minimal surface of a holographic Wilson line take the form of loop equations [16], of which topology changing effects are a well-known feature.} \]
evolve the holographic RG flow equations \(^{(4)}\) towards the UV region, until the scale factor \(a\) has attained the large value \(a(\epsilon) = \epsilon^{-1}a\). Evaluate the functionals \(S\) and \(\Gamma\) at this UV scale \(a(\epsilon)\). The corresponding renormalized quantities are then defined via

\[
S_R[x_R, a] = \lim_{\epsilon \to 0} S_{\text{finite}}[x(x_R, \epsilon), \epsilon^{-1}a],
\]

\[
\Gamma_R[\phi_R, g_R] = \lim_{\epsilon \to 0} \Gamma_{\text{finite}}[\phi(\phi_R, \epsilon), g(g_R, \epsilon)].
\]

Here \(x(x_R, \epsilon)\) denotes the value of \(x\) obtained by integrating the RG flow equations \(^{(4)}\) from \(a\) to \(\epsilon^{-1}a\), with initial condition \(x_R\), etc. The finite piece of the functionals is obtained by subtracting the part that diverges in the limit \(\epsilon \to 0\)

\[
S = S_{\text{div}} + S_{\text{finite}}, \quad \Gamma = \Gamma_{\text{div}} + \Gamma_{\text{finite}}.
\]

For the case \(d = 3\) of a 3+1-dimensional QFT, the divergent part takes the following form

\[
S_{\text{div}}[x, a] = \int_\Sigma f \sqrt{\det h} + S_{\text{an}}[x, a],
\]

\[
\Gamma_{\text{div}}[\phi, g] = \int \sqrt{g} (w + uR) + \Gamma_{\text{an}}[\phi, g],
\]

where \(f, w\) and \(u\) are functions of the scalar couplings \(\phi_I\), and \(S_{\text{an}}\) and \(\Gamma_{\text{an}}\) are logarithmically divergent terms, that carry the conformal anomaly of the 3+1-d QFT. To leading order in the derivative expansion, the anomaly terms satisfy the Weyl transformation property

\[
a \frac{\delta}{\delta a} S_{\text{an}}[x, a] = b K^n K_n,
\]

\[
a \frac{\delta}{\delta a} \Gamma_{\text{an}}[\phi, g] = c_1 R^\mu\nu R_{\mu\nu} + c_2 R^2,
\]

where \(b, c_1\) and \(c_2\) are suitable functions of the scalar couplings \(\phi_I\).

The precise form of the six scalar functions \(f, w, u, b, c_1\) and \(c_2\), as well as the higher curvature corrections to equations \(^{(33)}\) and \(^{(34)}\), are determined by the requirement that:

The divergent terms \(S_{\text{div}}\) and \(\Gamma_{\text{div}}\) satisfy the corresponding HJ equations \(^{(19)}\) and \(^{(20)}\).

This requirement has a clear physical motivation: \(S_{\text{div}}\) and \(\Gamma_{\text{div}}\) are designed to cancel the cut-off dependence of \(S\) and \(\Gamma\), and thus should satisfy the same radial evolution equation.

We will use this property in the next subsection, to compute the six scalar functions.
5 Geometric RG Flow

The finite parts \( S_{\text{finite}} \) and \( \Gamma_{\text{finite}} \) are each given by the difference of two solutions to the HJ equations \([19]\) and \([20]\), respectively. From this it is straightforward to verify that, to leading non-trivial order in the derivative expansion, both quantities satisfy a holographic RG flow equation of the form

\[
\left( a \frac{\delta}{\delta a} + \beta_n \frac{\delta}{\delta x_n} \right) S_{\text{finite}}[x, a] = 0 \quad (35)
\]

\[
\left( a \frac{\delta}{\delta a} + \beta_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \beta_I \frac{\delta}{\delta \phi_I} \right) \Gamma_{\text{finite}}[\phi, g] = 0. \quad (36)
\]

In the second equation, the beta functions are obtained via \([4]\)

\[
\gamma = a \frac{\delta \Gamma_{\text{div}}}{\delta a}, \quad \gamma \beta_{\mu\nu} = \frac{\delta \Gamma_{\text{div}}}{\delta g_{\mu\nu}}, \quad \gamma \beta_I = \frac{\delta \Gamma_{\text{div}}}{\delta \phi_I}, \quad (37)
\]

where, in accord with the derivative expansion, one is allowed to drop the anomaly term. We thus confirm that the geometric beta function \( \beta_{\mu\nu} \) takes the form of the Ricci flow \([5]\). The coefficient \( \kappa \) in \([5]\), the scalar beta function \( \beta_I \) and the other scalar functions in \([32]\) and \([34]\) are determined by requiring that \( \Gamma_{\text{div}} \) solves the HJ equation. One finds \([4]\)

\[
\frac{1}{3} w^2 - \frac{1}{2} (\partial_I w)^2 = V, \quad \kappa = \frac{6u}{w}, \quad \beta_I = \frac{6}{w} \partial_I w, \quad (38)
\]

\[
\partial_I w \partial_I u - \frac{1}{3} wu = 1, \quad c_1 = \frac{6u^2}{w}, \quad c_2 = \frac{2}{w} \left( u^2 - \frac{3}{2} (\partial_I u)^2 \right). \quad (39)
\]

For the entanglement entropy functional \( S[x, a] \) we proceed in a similar way. The beta function \( \beta_n \) is obtained from \( S_{\text{div}} \) via

\[
\pi_r = \frac{\partial S_{\text{div}}}{\partial r} = \gamma a \frac{\partial S_{\text{div}}}{\partial a}, \quad \gamma \pi_r \beta_n = \frac{\partial S_{\text{div}}}{\partial x_n}. \quad (40)
\]

To leading order in the derivative expansion, one is again allowed to drop the anomaly term. This confirms that the RG equation for \( S[x, a] \) takes form of the the mean curvature flow eqn \([5]\). The coefficient \( \lambda \) in eqn \([5]\), and the anomaly coefficient \( b \) in eqn \([33]\) are obtained by solving the following equations

\[
\frac{1}{\gamma} = a \frac{dr}{da}, \quad a \frac{d}{da} \left( a^{d-1} f \right) = \frac{a^{d-1}}{2\gamma}, \quad \lambda = \frac{2f}{\gamma}, \quad b = \frac{f^2}{\gamma}. \quad (41)
\]
The above formulas specify all holographic beta functions, to leading non-trivial order in the derivative expansion. One can further easily verify that, for the case of pure AdS, the above result for the anomaly coefficients $c_{1,2}$ and $b$ match with the answer obtained via the standard technique, based on the Fefferman-Graham expansion $^6$.$^{15}$.

To complete the derivation of the flow equation of the renormalized quantities $S_R$ and $\Gamma_R$, we need to perform the change of variables from the ‘bare’ variables $x$ and $(\phi, g)$ to the ‘renormalized’ variables $x_R$, and $(\phi_R, g_R)$. As explained in more detail in $^4$, this step is straightforward, since the beta functions transform covariantly, as vector fields. In particular

$$
\beta_m(x) = \frac{\epsilon}{\partial \epsilon} = \beta_n(x_R) \frac{\partial x_m}{\partial x_R}. \tag{42}
$$

This equation expresses the basic principle that underlies the renormalization group, that a variation in the coordinate location $x_m$ due to a small shift in the cut-off $\epsilon$ can be compensated by an infinitesimal adjustment of the renormalized location $x_R$. Note, however, that for fixed $x_R$, the bare quantity $x_m(x_R, \epsilon)$ approaches its UV fixed point value, the location of entanglement boundary $\partial \Sigma = \partial A$ of the continuum theory. So $\beta(x_m)$ vanishes in the limit $\epsilon \to 0$. The renormalized beta function $\beta_n(x_R)$ remains finite, however, and is proportional to the extrinsic curvature $K_n$ of the entanglement boundary in units of the RG scale. An identical discussion holds for the geometric beta function $\beta_{\mu\nu}$ of the metric.

6 Wilsonian Perspective

To add a bit of insight into the physical origin of the geometric RG flow, let us recall the Wilsonian interpretation of the holographic RG $^7$.$^8$. Our discussion will be schematic.

As before, we introduce a sliding holographic RG scale by dividing the AdS spacetime into an IR region $r < r(a)$ and a UV region $r > r(a)$. Let $S_{\text{UV}}$ denote the gravity action integrated over the UV region, and $S_{\text{IR}}$ the classical action integrated over the IR region, figure (3). So both actions solve the Hamilton-Jacobi equation (20). Generalizing the standard holographic dictionary, we can identify the bulk fields with single trace couplings of the dual QFT

$$
e^{\frac{1}{\hbar}}S_{\text{IR}}[\phi, g] = \left\langle e^{\frac{i}{\hbar} \int (\dot{\phi}^I \sigma_I + h_{\mu\nu} t_{\mu\nu})} \right\rangle_{\text{IR}}, \tag{43}
$$

where the expectation value is presumed to be taken in the QFT with a covariant cut-off. Here $\hbar = 1/N$ and $h_{\mu\nu}$ denotes the metric fluctuation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. 

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Figure 3: The operation of dividing up the CFT into an IR and UV sector, separated by an RG scale $a$, amounts to cutting the dual AdS space-time along a spatial slice specified by some radial location $r = r(a)$. The figure illustrates the application of this procedure to the holographic entanglement surface.

The partition function of the continuum QFT is obtained by gluing the UV and IR regions together via \[ Z_{\text{QFT}} = \int [d\phi \, dg] \langle e^{\frac{i}{\hbar} \int (\phi^l O_l + h^{\mu\nu} T_{\mu\nu})} \rangle_{\text{IR}} e^{\frac{i}{\hbar} S_{\text{UV}}[\phi,g]} \]

\[ = \int [d\phi \, dg] e^{\frac{i}{\hbar} S_{\text{IR}}[\phi,g] + \frac{i}{\hbar} S_{\text{UV}}[\phi,g]} \]

In the large $N$ limit, the integral can be replaced by its saddle point approximation. The saddle point values of the metric and couplings are thus determined via \[ \delta \delta S_{\text{UV}} + \delta S_{\text{IR}} = 0, \]

\[ \delta S_{\text{UV}} = \delta S_{\text{IR}} + \langle O \rangle_{\text{IR}} = 0, \]

\[ \delta g^{\mu\nu} (S_{\text{UV}} + S_{\text{IR}}) = \delta S_{\text{UV}} + \langle T_{\mu\nu} \rangle_{\text{IR}} = 0. \]

From the point of view of the classical bulk gravity theory, these are the continuity equations.
that ensure that the UV and IR solutions are smoothly joined together into a single global
classical background. From the boundary QFT point of view, they are saddle point equations
that determine the value of single trace couplings in terms of the expectation value of the
dual operators. The saddle point value of the total classical action $S_{UV} + S_{IR}$ can be identified
with the quantum 1PI effective action of the QFT

$$\Gamma_R[\phi_R, g_R] = \min_{\phi, g} \left( S_{UV}[\phi, g] + S_{IR}[\phi, g] \right),$$

(48)

where the minimum is taken over the field values at the holographic RG scale $r = r(a)$,
while keeping the asymptotic UV boundary conditions fixed. As explained in more detail
in [7], the fact that the UV and IR actions both solve the HJ equation implies that $\Gamma_R$
satisfies the geometric flow equations (2) with beta-functions determined via

$$\gamma(2g_{\mu\nu} + \beta_{\mu\nu}) = -\frac{\delta S_{UV}}{\delta g_{\mu\nu}} = \langle T_{\mu\nu} \rangle_{IR},$$

(49)

$$\gamma\beta_I = -\frac{\delta S_{UV}}{\delta \phi_I} = \langle O_I \rangle_{UV}.$$  \hspace{.5cm} (50)

The solution to this system of equations, up to the first few orders in the derivative expan-
sion, is given in the previous section.

The Wilsonian interpretation of the holographic RG is now immediate. The Wilson
effective action of the QFT, with the UV cut-off in place, is defined such that after performing
the functional integral over the IR degrees of freedom, one obtains the continuum QFT
partition function $Z_{QFT}$. The appropriate Wilsonian effective action is thus obtained simply
by removing the IR expectation values in the formula (44)

$$e^{\frac{i}{\hbar} S_{\text{eff}}(O, T)} = \int [d\phi \, dg] \, e^{\frac{i}{\hbar} \int (\phi^I + h^{\mu\nu} T_{\mu\nu}) - h^{\mu\nu} S_{UV}[\phi, g]].$$

(51)

The key point emphasized in [8] is that the integral over the single trace couplings on the
right-hand side implies that the effective action $S_{\text{eff}}(O, T)$ contains multi-trace couplings.
In particular, it contains terms that are quadratic and higher order in the stress tensor $T_{\mu\nu}$.

We thus learn from the Wilsonian perspective that geometric RG flow arises because
the effective space-time metric, defined as the local coupling constant dual to $T_{\mu\nu}$, contains
operator valued terms proportional to $T_{\mu\nu}$ itself. On a curved background, the stress tensor
acquires a non-trivial expectation value, which by general covariance and to leading order is
bound to be proportional to the Ricci tensor, with pre-coefficient proportional to the central
charge of the CFT. This explains why the effective background metric of the CFT evolves via Ricci flow.

Next let us look at the geometric entanglement entropy $S[x,a]$ from this perspective. As we will argue, the geometric RG flow of $S[x,a]$ can be derived from the geometric flow of the effective action $\Gamma[\phi,g]$. This is not entirely surprising, since entanglement entropy can be represented as the limit of a QFT partition sum on a space-time with a conical singularity $[3,17]$. Our discussion will continue to be schematic.

Vacuum entanglement is a property of the QFT vacuum state $|0\rangle$. Via the decomposition (44) of the QFT into an UV and IR sector, the state $|0\rangle$ factorizes into an entangled sum of UV and IR vacuum states

$$
|0\rangle = \sum_{\phi,g} |0_{\text{IR}}\rangle_{\phi,g} |0_{\text{UV}}\rangle_{\phi,g},
$$

(52)

where $|0_{\text{IR}}\rangle_{\phi,g}$ denotes the vacuum of the IR QFT with single trace couplings $(\phi,g)$, while $|0_{\text{UV}}\rangle_{\phi,g}$ denotes the vacuum of the bulk theory in the UV region $r > r(a)$, with IR boundary conditions $(\phi,g)$. The vacuum state $|0\rangle$ thus contains non-trivial UV/IR entanglement.

Rather than trying to compute the geometric entanglement entropy of the factorized vacuum state (52), it is more practical to look for a way to split the entanglement entropy of the full vacuum state into an UV and IR contribution. Let $\rho_A[x,a]$ denote the density matrix of region $A$, bounded by the entanglement surface parametrized by $x$, obtained by tracing over all states associated with the geometric complement of $A$. The entanglement entropy of region $A$ is then obtained via

$$
S[x,a] = -\text{tr}(\rho_A \log \rho_A) = -\frac{\partial}{\partial n} \log \text{tr}(\rho^n_A) \big|_{n=1},
$$

(53)

where the trace is over the Hilbert space of $A$. Here we used the standard replica trick to represent the von Neumann entropy of $\rho_A$ as the $n \to 1$ limit of the Renyi entropy $S_n[x,a] = \frac{1}{1-n} \log \text{tr}(\rho^n_A)$. The Renyi entropy can in turn be represented as the partition function of the QFT on a Euclidean space-time with conical singularity, a negative deficit angle $2\pi(1-n)$, along the entanglement surface. Since this space-time with the conical singularity is specified by some specific background geometry $g_{\mu\nu}$, albeit a slightly singular one, this places us in the same setting as above. In other words, the Renyi entropy is a quantum effective action $\Gamma[\phi,g]$ for a particular conical background metric $g_{\mu\nu} [3,17,18]$.

We can thus write the Renyi entropy as a sum of a UV and an IR contribution, exactly as in eqn (48). Since the background metric data include the location of the entanglement
boundary $x$, the geometric entanglement entropy involves minimization with respect to $x$

$$S_R[x_R, a] = \min_x (S_{\text{IR}}[x, a] + S_{\text{UV}}[x, a]),$$

(54)

where $S_{\text{IR}}$ and $S_{\text{UV}}$ both satisfy the HJ equation. The fact that, in accord with the Wilsonian philosophy, this entanglement entropy does not depend on the RG scale $a$ is expressed in terms of an RG flow equation of the form $[1]$, with exact geometric beta function $\beta_n$ given by eqns $[40]$ with $S_{\text{div}}$ replaced by $S_{\text{UV}}$. The Wilsonian RG evolution of the entanglement entropy and the effective action thus follow the same general pattern.

7 Discussion

In this paper, we derived Hamilton-Jacobi equation for the holographic entanglement entropy formula of Ryu and Takayanagi, and proposed an interpretation in terms of a geometric RG flow in the dual QFT. We gave a unified treatment of the entanglement entropy and the quantum effective action, and emphasized that in both cases, the RG equations include generalized beta functions that prescribe how the geometry smooths out under flow. Here we briefly highlight some properties of the geometric RG flow and list some open ends.

**Universality, Diffusion and Irreversibility**

Geometric RG is a universal property of QFTs with weakly curved holographic duals. This class of theories have a large central charge and a gap in the spectrum of conformal dimensions. Combined, these two properties guarantee that there is a regime in which the RG scale dependence of the partition function and entanglement entropy can be characterized by low energy gravity and minimal surface equations.

We have argued that geometric RG flow is a dynamical diffusion effect that is intimately linked to but distinct from the coarse graining that drives the RG evolution. The diffusion constant is set by the AdS curvature radius $R$, which can be made arbitrarily large compared to the resolution scale, set by the string length $\ell_s$. This indicates that geometric flow is a dynamical effect. This intuition is supported by the QFT interpretation of the HJ equations in terms of the appearance of multi-trace couplings in the Wilsonian effective action.

When evolved from UV to IR, geometric RG flow rapidly smooths out all local features smaller than the AdS radius. This flow is irreversible. Suppose we can specify the geometry of space-time or of the entanglement surface at some intermediate RG scale, with local
features that are smaller than the AdS radius, but still large compared to the string scale. Running the RG backwards, by integrating the bulk gravity equations of motion towards the UV, is then like trying to run a diffusion equation backwards in time. The geometry will develop a singularity at some finite RG time, much before reaching the UV boundary.

**RG Fixed Points and Topology**

With conventional RG, one often considers flows from a UV fixed point to some IR fixed point. The fixed points of the curvature flow (5) are Ricci flat space-times and minimal surfaces with zero extrinsic curvature. To get a non-trivial flow between different fixed points, one would need to consider situations with non-trivial topology. One could also consider modifying the equations by absorbing a non-trivial scaling factor into the RG step, so that the fixed geometries are allowed to have non-zero constant curvature.

When the entanglement region consists of two or more disconnected but sufficiently proximate regions in the UV, it is possible that while evolving the RG towards the IR, the different components of the entanglement boundary merge together. If in addition, one also considers space-time geometries with non-trivial topology, one can easily imagine setting up RG flows between different topologically distinct fixed points geometries.

**QFT Derivation?**

An outstanding challenge is to find a pure QFT derivation of the geometric RG flow equations. This may seem impossible, because geometric flow is a property of strongly coupled quantum field theories with holographic duals. In the case of AdS$_3$/CFT$_2$, however, it is known how to derive the 2+1-D bulk gravity equations of motion (in Hamilton-Jacobi form) from the 2-D conformal anomaly and Virasoro Ward-identities of the boundary CFT. In that case, it is possible to give a CFT proof of the geometric RG flow equations. However, due to the low dimension and absence of intrinsic geometry of the entanglement boundaries, the Ricci and mean curvature flow degenerate into a rather trivial form. It may still be worth trying, via clever use of various techniques (such as Ward identities of the stress tensor, conformal anomaly, cone geometries, conformal perturbation theory, etc) to find a derivation of the geometric flow from a CFT in $d \geq 2$. A somewhat tantalizing hint is that the beta functions of the geometric RG look identical to those of an (open) string world sheet sigma model. This suggests that an open string representation of the QFT may be a helpful tool [20].
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