Thermodynamic Geometry and Deconfinement Temperature

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Abstract

The application of Riemannian geometry to the analysis of the equilibrium thermodynamics in Quantum Chromodynamics (QCD) at finite temperature and baryon density gives a new method to evaluate the transition temperature from the quark-gluon phase to the hadron phase. The results are in good agreement with the freeze-out temperature evaluated by the statistical hadronization model and with the estimates based on QCD lattice simulations.

Keywords: Information Geometry of Thermodynamics, Phase Transition, Quark Gluon Plasma

Introduction

The first application of Riemannian geometry to statistical systems dates back to 1945 \cite{1}. Thirty years later a metric for thermodynamic systems has been proposed \cite{2} and in 1979 G. Ruppeiner \cite{3}, by defining, through the Hessian of the entropy density, the metric tensor, i.e. a measure of the "distance" in the phase space, showed that the resulting line element is in inverse relation with the fluctuation probability between equilibrium states.

The main result of the approach is the "interaction hypothesis" which states that the absolute value of the scalar curvature $R$, an intensive variable (with units of a volume) evaluated by the metric, is proportional to $\xi^3$, where $\xi$ is the correlation length of the thermodynamic system. Indeed, a covariant and consistent thermodynamic fluctuation theory can be developed \cite{4}, which generalizes the classical one and offers a theoretical justification to the physical meaning of $R$. The description of the interaction in terms of $R$ is intriguing and confirms, in a statistical sense, the connection between interaction and curvature, well known from general relativity.

Following this point of view, a new method has been proposed \cite{5} to characterize a phase transition: since $|R| \propto \xi^3$, $R$ is a useful parameter to define the transition. In particular, near a first order phase transition one expects that the values of scalar curvature evaluated in the two different phases coincide. This approach, called the $R$-Crossing Method (RCM), permits to determine the phase coexistence line for Helium, Hydrogen, Neon and Argon \cite{6}, for the Lennard-Jones fluids \cite{6,7}, for ferromagnetic systems and liquid-liquid phase transitions \cite{8}. More recently, another field of application of the RCM is the study of the phase transitions of cosmological interest: the liquid-gas like first order phase transition in dyonic charged AdS black hole \cite{9} and in the Hawking-Page transitions in Gauss-Bonnet-AdS black holes \cite{10}.

A different geometrical method has been proposed to study other systems, where, in particular, there is a transition from fermionic to bosonic degrees of freedom (d.o.f.). Indeed in these cases the criterion to define the transition line is $R = 0$, i.e. the study of the sign of $R$, since it has been shown that $R > 0$ corresponds to a mostly fermionic system and $R < 0$ to a
bosonic one \[11–13\].

In this letter we apply the thermodynamic geometry to strong interaction. As well known, Quantum Chromodynamics (QCD) at finite temperature shows a transition from hadrons to a phase of deconfined quarks and gluons, a quark-gluon plasma (QGP). The initial idea of a deconfinement first order phase transition to a state of weakly interacting quarks and gluons has been now modified and there are clear indications that, near the critical temperature, \( T_c \), the system is strongly interacting and the transition is, indeed, a crossover. The estimate of the (pseudo-)critical temperature, \( T_c = 154 \pm 9 \text{ MeV} \), is based on the lattice results on the chiral susceptibility \[14\]. On the other hand, the so called freeze-out temperature, \( T_{fo} \), evaluated by the statistical hadronization model (SHM) \[15–20\], is in general larger than \( T_c \) (see however ref. \[21\]).

Concerning the application of the thermodynamic geometry to evaluate the transition temperature between the two phases, one has to recall that, at large temperature and low baryon density, the hadron formation mechanism is a transition from the QGP, which is mostly a fermionic system, to (essentially) mesons. Moreover the QCD deconfinement transition is a crossover and therefore, in this \( T - \mu_B \) regime, the criterion \( R = 0 \) is more reliable than the RCM and the results obtained by this condition are in good agreement (within 10 %) with SHM and QCD estimates.

In Section 1 the main ideas of the thermodynamic geometry and of the criterion \( R = 0 \) will be recalled; then in Sec. 2 our application to the deconfinement transition is carried out.

1. Thermodynamic Geometry and \( R=0 \) criterion

Let us consider an open thermodynamic system, of fixed volume \( V \) and \( r \) species of constituents, described in terms of the standard densities \( a^i = (a^0, a^1 \ldots a^r) \), where \( a^0 \) is the internal energy density and the other components are the number of particles of the different species.

Calling \( s(a) \) the entropy density of the system, the fluctuation probability density, \( P_f \), in Gaussian approximation is given by \[4\]

\[
P_f \propto \exp \left\{ -\frac{V}{2} g_{ij} \Delta a^i \Delta a^j \right\},
\]

where

\[
g_{ij} = -\frac{1}{k_B} \frac{\partial^2 s(a)}{\partial a^i \partial a^j}
\]

is the metric tensor and the line element \((\Delta l)^2 = g_{ij} \Delta a^i \Delta a^j\) is an invariant, positive definite quadratic form (as a consequence of the maximum entropy principle, see \[4\] for detailed explanations). Starting from Eq. \(2\), the calculation of the scalar curvature \( R \) is straightforward; we will use the standard intensive quantities in the entropy representation:

\[ F^\mu \equiv \frac{\partial s(a)}{\partial a^\mu} = \left( \frac{1}{T}, -\mu^1 T \ldots -\mu^r T \right), \]

where \( \mu^i \) are the chemical potentials of the different species and \( T \) is the overall temperature. In this “frame”, the metric depends on the derivatives of the thermodynamic potential \( \phi = P/T \), where \( P \) is the total pressure of the system \[4\].

In two dimensions, the scalar curvature \( R \) contains all the informations about the geometry of the Riemannian manifold and its expression is considerably simplified:

\[
R = \frac{k_B}{2g^2} \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{122} \\ \phi_{111} & \phi_{112} & \phi_{122} \\ \phi_{112} & \phi_{122} & \phi_{222} \end{vmatrix},
\]

where \( k_B \) is the Boltzmann’s constant, \( g \) is the determinant of the metric and the usual comma notation for derivatives has been used. For example, \( \phi_{12} \) indicates the derivative of \( \phi \) with respect to the first coordinate \( \beta = 1/T \) and the second coordinate \( (\gamma = -\mu/T) \). The metric elements are

\[
g_{ij} = \frac{1}{k_B} \phi_{,ij}
\]
As discussed in the introduction, the interaction hypothesis has been confirmed for a relevant number of different systems. In particular, the most interesting developments are in the field of real fluids where the absolute value of $R$ is a direct measure of the size of organized mesoscopic fluctuating structures in thermodynamic systems [22].

Within the thermodynamic geometry approach, the physical meaning of the sign of $R$ is still under debate. However, the picture is clear enough in the case of pure fluids [6, 7, 23] and of quantum gases, where $R$ turns out to be positive for repulsive fermi interactions and negative in the bosonic case [11]. Particularly, in [13], the authors study a two dimensional ideal anyon gas of particles obeying fractional statistics finding that the sign of the scalar curvature of this system is a function of the parameter $\alpha$ that specifies the particle content ($\alpha = 0$ corresponds to bosons, $\alpha = 1$ to fermions, and $0 < \alpha < 1$ to intermediate statistics). The sign changes at $\alpha = 1/2$ in the classical limit (dotted-dashed line in Fig. 1) and the $R = 0$ condition is satisfied by slightly lower values of $\alpha$ (continuous line) by deviations from the classical behavior (see ref. [13] for details). Moreover, the sign of $R$ can hide information on the underlying interactions for black holes. For example, in [24] one shows that the scalar curvature remains negative for the metastable phase of the black hole, but changes sign at the Hawking-Page transition temperature that, therefore, can be associated with the condition $R = 0$.

In section 2 we implement the constraints $R = 0$ for QCD by considering the QGP phase. We have preliminary verified [25], by the Hadron Resonance Gas model in the considered $T - \mu_B$ range, that in the hadronic sector $R < 0$ and, from this point of view, the sign of $R$ for QCD signals a crossing between a fermion dominated system (the QGP) and a hadronic (mesonic) phase.

2. Transition line and $R=0$ in QCD

The criterion $R = 0$ will be applied to QCD by considering two thermodynamic variables, $\beta = 1/T$ and $\gamma^2$, i.e. a 2-dimensional thermodynamic metric (see Eq. (4)). The potential $\phi = P/T$ is evaluated by lattice QCD data in the quark-gluon phase, available at small chemical potential, and then our calculations are reliable in the range $\gamma^2 < 1$, where a series expansion is feasible. However the criterion $R = 0$ is completely general.

By the expansion of the pressure $P$ as a power series in the parameter $\gamma^2$ around the point $\mu_B = 0$, one can write the thermodynamical potential $\phi$ as

$$\phi(\beta, \gamma) = A(\beta) + B(\beta)\gamma^2 + C(\beta)\gamma^4 + D(\beta)\gamma^6 + \cdots .$$

One can show [25] that the scalar curvature, $R$, can be also expressed by an analogous power series, i.e.

$$R(\beta, \gamma) = \sum_{n=0} R_{O(2n)} \gamma^{2n} .$$
where the $2n$-th coefficient, $R_{O(2n)}$, is a function of the first $2(n+1)$ coefficients of the expansion for the potential $\phi$ given by Eq. (6) [26].

For example, the first coefficient, $R_{O(0)}$, depends on the first and second coefficient of the $\phi$ series expansion and it is given by

$$R_{O(0)} = \frac{1}{2} \frac{B'}{A'' B} \left( \frac{A''}{A'} \frac{B'}{B} \right) = \frac{1}{2} \frac{\dot{P}_0}{P_0} \left[ 3 + T \frac{\chi_2}{\chi_2} \right] \left[ \frac{P_0}{P_0} - \frac{\chi_2}{\chi_2} \right],$$

(8)

where $\cdot'$ and $\cdot''$ denote, respectively, the derivative with respect to $\beta$ and $T$; $P_0(\beta)$ is the pressure and $\chi_2(\beta) = \partial^2 (P/T^4)/\partial \gamma^2$, both at $\mu_B = 0$.

By the results of the HotQCD collaboration [26] for the pressure, $P^L(\beta, \gamma)$, in the high temperature and low baryon-chemical potential regime, the coefficients of the series expansion of the corresponding thermodynamical potential in the quark-gluon phase, $\phi^L(\beta, \gamma)$, turn out to be [23]

$$A^L(\beta) = P^L_0(\beta) \beta,$$

$$B^L(\beta) = \frac{\chi^L_1(\beta)}{2! \beta^3},$$

$$C^L(\beta) = \frac{\chi^L_2(\beta)}{4! \beta^3},$$

$$D^L(\beta) = \frac{\chi^L_3(\beta)}{6! \beta^3},$$

(9)

where $P^L_0(\beta) = P^L(\beta, 0)$ and $\chi^L_{2n} = \frac{\partial^{2n} (p^L)}{\partial (\beta T)^n}|_{T=0}$.

Following the previous discussion, the crossing line from QGP to a mostly bosonic system can be evaluated by implementing the condition $R = 0$.

Figure 2 shows the scalar curvature $R$ from Eq. (7). The black lines are from lattice data obtained by the condition $n_S = n_Q = 0$, while the red ones are for $n_S = 0$ and $n_Q/n_B = 0.4$ (n$S$, n$Q$, n$B$ being the strangeness, charge and baryon number densities respectively), following the procedure of Ref. [26, 27] where n$Q$ is considered as a function of $\mu_B$. The continuous lines are for $\mu_B = 0$ MeV, the dashed ones for $\mu_B = 80$ MeV and the dotted lines for $\mu_B = 135$ MeV.

In Figure 3 is plotted the critical temperature $T_c(\mu_B)$ at which the scalar curvature of the QGP phase crosses the $R = 0$ line, both for $n_Q = n_S = 0$ (continuous black line) and for $n_S = 0$ and $n_Q/n_B = 0.4$ (black dotted line), compared with lattice results [26, 27] and the freeze-out temperature obtained by ALICE [28] and STAR [29, 30] collaborations. The yellow curve gives the crossover temperature, the blue and black grid bands are obtained by considering fixed values of the energy density (blue) or of the entropy density (grid) (see Ref. [21, 27] for details).

**Comments and Conclusions**

The introduction of a thermodynamic metric, the calculation of corresponding scalar curvature, $R$, and the $R = 0$ method applied to the quark-gluon phase give a estimate of the transition temperature in good agreement with QCD lattice simulations and the SHM freeze-out temperature. This new method to determine the critical line has been applied for small baryon density since it requires a reliable evaluation of the thermodynamic potential $\phi$. However the geometrical approach is quite general.
and the calculations at large baryon density can be analytically done once \( \phi \) is known.

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