Hopping in the Phase Model to a Non-Commutative Verlinde Formula for Affine Fusion

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Abstract. Korff and Stroppel discovered a realization of \( su(n) \) affine fusion, the fusion of the \( su(n) \) Wess-Zumino-Novikov-Witten (WZNW) conformal field theory, in the phase model, a limit of the \( q \)-boson hopping model. This integrable-model realization provides a new perspective on affine fusion, explored in a recent paper by the author. The role of WZNW primary fields is played in it by non-commutative Schur polynomials, and fusion coefficients are thus given by a non-commutative version of the Verlinde formula. We present the extension to all Verlinde dimensions, of arbitrary genus and any number \( N \) of points. The level-dependence of affine fusion is also discussed, using the concept of threshold level, and its generalization to threshold weight.

1. Introduction

Affine fusion is found in several mathematical and physical contexts. It is a natural generalization of the tensor product of representations of simple Lie algebras; a simple truncation thereof controlled by a non-negative integer, the level.

One important physical context is conformal field theory, and more specifically, the Wess-Zumino-Novikov-Witten (WZNW) models (see [4], for example). WZNW models realize, at a fixed non-negative integer level \( k \), a non-twisted affine Kac-Moody algebra \( g^{(1)} \) based on a simple Lie algebra \( g \), or \( g_k \) for short. Their primary fields furnish representations of \( g_k \) and their operator products are governed by the corresponding affine fusion algebra.

Recently, Korff and Stroppel [8] found a much simpler physical realization of affine fusion, for the \( su(n)_k \) case, at all levels \( k \in \mathbb{N} \) together (see also [7, 13]). The phase model [2] is an integrable multi-particle model whose integrals of motion are non-commutative Schur polynomials [8]. The integrals may be diagonalized by the algebraic Bethe ansatz, and their eigenvalues are affine fusion eigenvalues [8]. The non-commutative Schur polynomials play the role in the phase model that the primary fields do in WZNW models [13]. The correlators of non-commutative Schur polynomials equal affine fusion coefficients [8] and more generally, affine Verlinde dimensions of arbitrary genus and
numbers of marked points [13]. The Korff-Stroppel formula for these correlators can be thought of as a non-commutative Verlinde formula.

This proceedings contribution is based mostly on the paper [13]. Section 2 reviews the phase model realization of affine fusion found by Korff and Stroppel [8]. Section 3 records the new results of [13]: the extension of the non-commutative Verlinde formula to higher genus and \( N \) points, and consideration of the level-dependence of fusion in the phase-model realization. In the latter, the old concept of threshold level [3, 6] plays a prominent role. Generalization to consideration of threshold weights [14] is included here as well. Section 4 is a short conclusion.

2. Phase-Model Realization of Affine \( su(n) \) Fusion

In the \( su(n) \) phase model, bosons occupy \( n \) sites on \( S^1 \), or, equivalently, the nodes of the Coxeter-Dynkin diagram of the affine Kac-Moody algebra \( \widehat{su}(n) \cong A_n^{(1)} \). For each \( j \in \{1, \ldots, n\} \), the \( j \)-th node is associated to an affine fundamental weight \( \Lambda_j \), and to boson creation and annihilation operators \( \varphi_j \) and \( \varphi_j^\dagger \), respectively. The affine dominant weight \( \hat{\nu} = \sum_{j=1}^n \nu_j \Lambda_j \) can be used to label a basis of states, where \( \nu_j \in \mathbb{N}_0 \) is the number of bosons at node \( j \), eigenvalue of the number operator \( N_j \). The complete set of possible affine dominant weights is \( \hat{P}_+ := \{ \sum_{a=1}^n \nu_a \Lambda^a | \lambda_a \in \mathbb{N}_0 \} \).

The total number of bosons \( \sum_{j=1}^n \nu_j \) equals the level of \( \hat{\nu} \), denoted \( k \). So, alternatively, the level \( k \) and an \( su(n) \) dominant weight \( \nu = \sum_{j=1}^{n-1} \nu_j \Lambda_j \) can be used to label states, since \( \nu_n = k - \sum_{j=1}^{n-1} \nu_j \). Notice that \( \nu_n \) is the affine Dynkin label. The 2 notations are useful, so we write \( |\hat{\nu}\rangle = |\nu\rangle_k \). Here \( \hat{\nu} \in \hat{P}_+ := \{ \sum_{a=1}^n \nu_a \Lambda^a | \lambda_a \in \mathbb{N}_0, \sum_{a=1}^n \nu_a = k \} \subset \hat{P}_+ \), and \( \nu \in P_+^{(k)} := \{ \sum_{a=1}^{n-1} \nu_a \Lambda^a | \lambda_a \in \mathbb{N}_0, \sum_{a=1}^{n-1} \nu_a \leq k \} \). The basis vectors are orthonormal, so

\[
\langle \lambda | \mu \rangle_{k'} = \delta_{\lambda, \mu} \delta_{k, k'} \quad \text{or} \quad \langle \hat{\lambda} | \hat{\mu} \rangle = \delta_{\hat{\lambda}, \hat{\mu}}.
\]

It is important to realize, however, that the level, the number of bosons, is not fixed in the phase model.

The algebra of \( \{ \varphi_j, \varphi_j^\dagger, N_j | j \in \{1, \ldots, n\} \} \), subject to the following relations, is known as the phase algebra:

\[
\begin{align*}
[\varphi_i, \varphi_j] &= 0, \quad [\varphi_i^\dagger, \varphi_j^\dagger] = 0, \quad [N_i, N_j] = 0, \\
[N_i, \varphi_j^\dagger] &= \delta_{i,j} \varphi_i^\dagger, \quad [N_i, \varphi_j] = -\delta_{i,j} \varphi_i, \\
N_i \left( 1 - \varphi_i^\dagger \varphi_i \right) &= 0 = \left( 1 - \varphi_j^\dagger \varphi_j \right) N_i, \\
[\varphi_i, \varphi_j^\dagger] &= 0 \quad \text{if} \; i \neq j; \quad \text{but} \quad \varphi_i^\dagger \varphi_i^\dagger = 1.
\end{align*}
\]

The last relation follows from the standard actions of \( \varphi_i \) and \( \varphi_i^\dagger \) on the basis states:

\[
\varphi_i |\hat{\nu}\rangle = \begin{cases} |\hat{\nu} - \Lambda^i\rangle, & \hat{\nu} - \Lambda^i \in \hat{P}_+; \\
0, & \text{otherwise} \end{cases}
\]
\[ \phi_i^\dagger |\hat{\nu}\rangle = |\hat{\nu} + \Lambda^i\rangle. \]

Similarly, the operator \( \phi_i^\dagger \phi_i \) can be seen to project onto states of positive \( i \)-th boson number \( \nu_i > 0 \), so that \( 1 - \phi_i^\dagger \phi_i \) projects onto \( \nu_i = 0 \) states.

The phase model is solved in the standard way; complete details are provided in [8]. The Lax matrices are

\[ L_j(u) = \begin{pmatrix} 1 & u \phi_j^\dagger \\ \phi_j & u \end{pmatrix}, \tag{3} \]

where \( u \) is a spectral parameter. The monodromy matrix is then

\[ M(u) = L_n(u) L_{n-1}(u) \cdots L_1(u) =: \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{4} \]

where the last equality is just standard notation. Integrability results because the fundamental relation

\[ R_{12}(u/v) M_1(u) M_2(v) = M_2(v) M_1(u) R_{12}(u/v), \tag{5} \]

is satisfied. The quantum Yang-Baxter equation follows, and (5) defines the so-called Yang-Baxter algebra, that includes the vanishing commutator \([B(u), B(v)] = 0\). The algebraic Bethe ansatz yields \( B(x_1) \cdots B(x_k) |\hat{0}\rangle \) for the Bethe vectors. Notice that the latter form is symmetric in the commuting creation operators \( B(x_j) \), and so completely symmetric in the variables \( x_1, \ldots, x_k \). As a result, it can be expanded in terms of the symmetric Schur polynomials:

\[ B(x_1) \cdots B(x_k) |\hat{0}\rangle = \sum_{\lambda \in P_k^+} s_{\lambda'}(x_1, \ldots, x_k) |\lambda\rangle_k. \]

The Schur symmetric polynomial \( s_{\lambda'}(x_1, \ldots, x_k) \) is also an \( su(n) \) character.

The Bethe vectors diagonalize the integrals of motion, provided the Bethe ansatz equations are satisfied, and the latter imply that \( x_1, \ldots, x_k \) must equal certain \( (k+n) \)-th roots of unity. More specifically, one can put the possible sets of values of the \( x_j \) in 1-1 correspondence with dominant weights in \( P_k^+ \). Adopting the notation \( (x_1, \ldots, x_k) =: x_\sigma \) for \( \sigma \in P_k^+ \), one finds that \( s_{\lambda}(x_\sigma) \) is an affine fusion eigenvalue for \( \sigma \in P_k^+ \). The celebrated Verlinde formula

\[ s_{\lambda}(x_\sigma) s_{\mu}(x_\sigma) = \sum_{\nu \in P_k^+} (\kappa \lambda, \mu_\nu) s_{\nu}(x_\sigma), \tag{6} \]

follows. Here \( \lambda, \mu, \nu, \sigma \in P_k^+ \) and \( (\kappa \lambda, \mu_\nu) \) is an affine fusion multiplicity, and the connection with WZNW models is made.

Affine fusion multiplicities count the couplings of primary fields in a WZNW model. Affine fusion is a simple truncation of the tensor product of representations of simple Lie algebras, controlled by the level \( k \). If \( T_{\lambda,\mu}^\nu \) indicates the \( su(n) \) tensor product multiplicity, then

\[ (\kappa \lambda, \mu_\nu) \leq T_{\lambda,\mu}^\nu; \quad (\infty \lambda, \mu_\nu) = T_{\lambda,\mu}^\nu, \]

for all \( \lambda, \mu, \nu \in P_k^+ \). 

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The direct connection with affine fusion is revealed by examining the integrals of motion \([8]\). Since fusion has fixed level \(k\), we consider hopping operators \(a_j = \varphi_j^\dagger \varphi_{j-1}\), \(j \in \{1, 2, \ldots, n\}\), that do not change the number of bosons. For \(\hat{\nu} \in \hat{P}_k^+\), their action, depicted in Fig. 1, is

\[
a_i |\hat{\nu}\rangle = \begin{cases} 
|\hat{\nu} - \Lambda_{i-1} + \Lambda_i\rangle, & \hat{\nu} - \Lambda_{i-1} + \Lambda_i \in \hat{P}_k^+; \\
0, & \text{otherwise}
\end{cases}
\]

\(a_i |\hat{\nu}\rangle = \begin{cases} 
|\hat{\nu} - \Lambda_{i-1} + \Lambda_i\rangle, & \hat{\nu} - \Lambda_{i-1} + \Lambda_i \in \hat{P}_k^+; \\
0, & \text{otherwise}
\end{cases}
\]

\(\nu_1\)

\(\nu_2\)

\(\nu_3\)

\(\nu_4\)

\(\nu_5\)

\(\nu_6\)

\(\nu_7\)

\(\nu_8\)

\(a_7\)

\(\nu\)

The algebra of hopping operators \(A = \langle a_1, a_2, \ldots, a_n \rangle\) is given by:

\[
[a_i, a_j] = 0, \quad \text{if } i \neq j \pm 1 \text{ mod } n; \\
n_a a_j^2 = a_j a_i a_j, \quad a_i^2 a_j = a_j a_i a_i \quad \text{if } i = j + 1 \text{ mod } n. \quad (7)
\]

Here indices are defined mod \(n\). This hopping algebra \(A\) is called the local affine plactic algebra by Korff and Stroppel [8]. The term plactic indicates a connection with Young tableaux (see [5], e.g.). \(su(n)\) tensor products can be calculated using Young tableaux in the famous Littlewood-Richardson rule. The integrals of motion of the phase model led Korff and Stroppel to a modified Littlewood-Richardson rule for affine fusion [8].

To see the relation to Young tableaux, note that the hopping operator \(a_i\) is associated with the weight \(\Lambda_i - \Lambda_{i-1}\). The set of these affine weights have horizontal parts equal to the weights of the basic \(su(n)\) irreducible representation \(L(\Lambda^1)\), that can be labelled by Young tableau \(\begin{array}{c}2 \\ 1 \\ 3\end{array}\). For example, in the \(su(3)\) case, the tableaux for \(L(\Lambda^1)\) are

\[
\begin{array}{c}2 \\ 1 \\ 3\end{array}
\]

They have been arranged here in the pattern of the weight diagram for \(L(\Lambda^1)\).

Furthermore, the action of hopping operators on basis states \(|\hat{\lambda}\rangle\), \(\hat{\lambda} \in \hat{P}_k^+\), is

\[
a_i |\hat{\lambda}\rangle = \begin{cases} 
|\hat{\lambda} - \Lambda_{i-1} + \Lambda_i\rangle, & \hat{\lambda} - \Lambda_{i-1} + \Lambda_i \in \hat{P}_k^+; \\
0, & \text{otherwise}
\end{cases}
\]
It is precisely reproduced by the product (denoted by * here) of a Young tableaux and a Young diagram that is used in the Littlewood-Richardson rule. In the su(3) example:

\[
\begin{align*}
2 \ast \begin{array}{c} \ast \ast \ast \end{array} &= \begin{array}{c} \ast \ast \ast \end{array}, \\
1 \ast \begin{array}{c} \ast \ast \ast \end{array} &= \begin{array}{c} \ast \ast \ast \end{array}, \\
3 \ast \begin{array}{c} \ast \ast \ast \end{array} &= \begin{array}{c} \ast \ast \ast \end{array} = 0.
\end{align*}
\]

Of course, more complicated Young tableaux must also be considered. But for them, the product with a Young diagram is taken column-by-column in the same way, from the right-to-left, and if no non-dominant shape is encountered, then a contribution results.

For successive tensor products, the algorithm can be repeated. Alternatively, a product \( \bullet \) of 2 Young tableaux can be defined. One finds [5] that certain “bumping” relations must be obeyed:

\[
\begin{align*}
j \ uparrow k \bullet i & = i \downarrow j \bullet k, \quad i < j \leq k; \\
i \downarrow k \bullet j & = i \downarrow j \bullet k, \quad i \leq j < k.
\end{align*}
\]

Such computations can be done in plactic algebras, however, instead of with tableaux. First, replace Young tableaux with words. The (column) word of a Young tableau is obtained by listing its entries in the order from bottom to top in the left-most column, then from bottom to top in the next-to-left-most column, continuing until the top entry of the right-most column is listed. Then replace each occurrence of digit \( j \) in the word by the corresponding generator of the plactic algebra \( \bar{a}_j \).

In this way, the bumping relations (8) are re-written as

\[
\begin{align*}
\bar{a}_{i+1}^2 \bar{a}_i & = \bar{a}_{i+1} \bar{a}_i \bar{a}_{i+1}, \quad (j = k = i + 1) ; \\
\bar{a}_i \bar{a}_{i+1} \bar{a}_i & = \bar{a}_{i+1} \bar{a}_i^2, \quad (j = k = i + 1).
\end{align*}
\]

Comparing (9) with (7), we see that bumping is compatible with hopping. That is, the relations defining the hopping algebra (7) are very similar to those resulting from the bumping process in (8). The algebra of hopping operators can therefore implement calculations similar to those involved in the Littlewood-Richardson rule for su(n) tensor products. Korff and Stroppel showed that the hopping operators of the phase model realize su(n) affine fusion, a truncation of the su(n) tensor product.

The integrals of motion can be described explicitly. The fundamental relation (5) guarantees that the transfer matrix \( T(u) := \text{tr} M(u) \) obeys \( [T(u), T(v)] = 0 \), and so is the generating function \( T(u) = \sum_{r=0}^\infty u^r e_r(A) \) of integrals of motion: \( [e_r(A), e_r(A)] = 0 \). The form of the integrals can be found from (3,4). \( e_r(A) \) indicates the \( r \)-th cyclic
elementary symmetric polynomial, the sum of all cyclically ordered products of \( r \) distinct hopping operators \( a_i \):

\[
e_r(\mathcal{A}) = \sum_{|I|=r} \prod_{i \in I} a_i.
\]

For example, with \( n = 4 \),
\[
e_2(\mathcal{A}) = a_2a_1 + a_3a_1 + a_1a_4 + a_3a_2 + a_4a_2 + a_4a_3.
\]

Because of the integrability property, one can use the Jacobi-Trudy formula to define the non-commutative Schur polynomial

\[
s_\lambda(\mathcal{A}) = \det(e_{\lambda'_{i+j}}(\mathcal{A})). \tag{10}
\]

Here \( \lambda' \) indicates the partition specified by the transpose of the Young diagram for \( \lambda \). \( \lambda'_{i+j} \) is the \( i \)-th integer in that partition.

For example, the non-commutative Schur polynomial \( s_{\lambda_1+\lambda_2}(\mathcal{A}) \) for the adjoint representation of \( su(3) \) equals the sum of:

\[
\begin{array}{ccc}
a_2a_1a_2 & a_1a_2a_1 \\
a_2a_3a_2 & (a_3a_2a_1 + a_1a_3a_2) & a_1a_3a_1 \\
a_3a_2a_3 & a_3a_1a_3
\end{array}
\tag{11}
\]

We already see a couple of important properties. First, (11) contains negative terms, whereas the \( su(n) \) analogue contains none. This is general: the vectors of any representation \( L(\lambda) \) of \( su(n) \) can be put in 1-1 correspondence with Young tableaux. But the corresponding non-commutative Schur polynomial is a sum that includes negative terms, in general. Second, there is a cyclic symmetry present in (11) that is broken in the \( su(n) \) case.

The main result of [8] is

\[
s_\lambda(\mathcal{A}) | \mu \rangle_k = \sum_{\nu \in P^+_k} (k)^{N_{\lambda,\mu}^\nu} | \nu \rangle_k. \tag{12}
\]

Inspired by the term non-commutative Schur polynomial, one can call the Korff-Stroppel equation (12) a non-commutative Verlinde formula. Using the orthonormality (1), it can be written as

\[
k \langle \nu | s_\lambda(\mathcal{A}) | \mu \rangle_k = (k)^{N_{\lambda,\mu}^\nu}.
\]

To compare with WZNW conformal field theory, consider a special case of (12),

\[
s_\lambda(\mathcal{A}) | 0 \rangle_k = | \lambda \rangle_k.
\]

This is highly reminiscent of the field-state correspondence \( \phi_\lambda(0) | 0 \rangle = | \lambda \rangle \), involving the WZNW primary field \( \phi_\lambda(z) \). Furthermore, it can be used to re-write (12) as

\[
(k)^{N_{\lambda,\mu,\nu}} = k \langle 0 | s_\lambda(\mathcal{A}) s_\mu(\mathcal{A}) s_\nu(\mathcal{A}) | 0 \rangle_k
\]
where the trivalent, directed, labeled graph that normally indicates the fusion coefficient has been drawn. Compare the last expression with the notation for the 3-point function in WZNW model: \( \langle 0 | \phi_\lambda(z_1) \phi_\mu(z_2) \phi_\nu(z_3) | 0 \rangle \). It becomes clear that the non-commutative Schur polynomial \( s_\lambda(A) \) plays the role in the phase model of the primary field \( \phi_\lambda \) in the WZNW conformal field theory.

Remarkably, it is the integrability of the phase model that gives rise to duality in affine fusion. The non-commutative Schur polynomials are integrals of motion, and so commute: \([s_\lambda, s_\mu] = 0\). Here we have used \( s_\lambda := s_\lambda(A) \), for short. Consequently,

\[
k\langle 0 | s_\lambda s_\mu s_\nu s_\phi | 0 \rangle_k = \sum_{\sigma \in P^k_+} k\langle 0 | s_\lambda s_\mu | \sigma \rangle_k k\langle \sigma | s_\nu s_\phi | 0 \rangle_k = \sum_{\sigma \in P^k_+} k\langle 0 | s_\lambda s_\phi | \sigma \rangle_k k\langle \sigma | s_\mu s_\nu | 0 \rangle_k.
\]

The graphical representation of this result is shown in Fig. 2.

3. Higher-Genus and Level-Dependence

Let us now turn to the new results reported in [13]. First, consider higher-genus affine fusion. The Verlinde dimension of arbitrary genus and number of points is indicated by its trivalent graph in Fig. 3. It is not difficult to show that the Korff-Stroppel non-commutative Verlinde formula extends to

\[
\langle k,g \rangle_N^{\lambda_1, \ldots, \lambda_N} = \langle \lambda_1^* \rangle \left( \sum_{a \in P^k_+} s_{a^*} s_a \right)^g s_{\lambda_2} \cdots s_{\lambda_{N-1}} | \lambda_N \rangle
\]

\[
= k\langle 0 | \left( \sum_{a \in P^k_+} s_{a^*} s_a \right)^g s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_N} | 0 \rangle_k
\]

(13)

for this case. One can then identify the handle operator as \( \sum_{a \in P^k_+} s_{a^*} s_a \).

Now consider the simple dependence of affine fusion on the level \( k \), well-described by the concept of threshold level [3, 6]. All possible fusion decompositions can be given simply by treating the level as a variable, and writing multi-sets of threshold levels as subscripts. Consider an example: the decomposition of an \( su(3) \) tensor product may be written

\[
L(\Lambda^1 + \Lambda^2)^{\otimes 2} \hookrightarrow L(0)_2 \oplus 2 L(\Lambda^1 + \Lambda^2)_{2,3} \oplus L(3\Lambda^1)_3 \\
\oplus L(3\Lambda^2)_3 \oplus L(2\Lambda^1 + 2\Lambda^2)_4.
\]

(14)
The subscripts indicate the threshold levels of the representations in the decomposition, so that
\begin{equation}
(2) \mathcal{N}_{\Lambda^1 + \Lambda^2, \Lambda^1 + \Lambda^2} = 1, \quad (k \geq 3) \mathcal{N}_{\Lambda^1 + \Lambda^2, \Lambda^1 + \Lambda^2} = 2,
\end{equation}
for example.

This threshold-level behaviour in affine fusion is reflected in the Korff-Stroppel result (12) by the striking property of non-commutative Schur polynomials: \( s_{\lambda}(\mathcal{A}) \) does not depend on the level. The phase model treats all levels \( k \in \mathbb{N}_0 \) on an equal footing, and as a consequence, the threshold behaviour of level is clear. The level-dependence can be incorporated into (12) simply by using \(|\mu\rangle_k\) with variable level.

In the \( su(3) \) example just mentioned, the weight-0 part of the non-commutative Schur polynomial \( s_{\Lambda^1 + \Lambda^2} \) is \( a_3 a_2 a_1 + a_1 a_3 a_2 + a_2 a_1 a_3 - 1 \). But
\begin{align*}
(a_3 a_2 a_1 + a_1 a_3 a_2 + a_2 a_1 a_3 - 1) |\Lambda^1 + \Lambda^2 + (k-2)\Lambda^3\rangle &= \{ \theta(k-3) + \theta(k-2) \} |\Lambda^1 + \Lambda^2 + (k-2)\Lambda^3\rangle,
\end{align*}
confirming the presence of \( L(\Lambda^1 + \Lambda^2)_{2,3} \) in (14). Here we have used \( |\Lambda^1 + \Lambda^2 + (k-2)\Lambda^3\rangle = \theta(k-2)|\Lambda^1 + \Lambda^2 + (k-2)\Lambda^3\rangle \) and
\begin{equation}
\theta(x) := \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}
\end{equation}

As already emphasized, an advantage of the phase-model realization of affine fusion is that, unlike in the WZNW model, the level is not fixed—it is just the total particle number. Changes in level can be described in a simple, algebraic way by the operators \( \varphi_i^\dagger, \varphi_i \) of the phase algebra (7).

Let us consider threshold multiplicities [13] in this spirit. The threshold multiplicity \( (t) n_{\lambda,\mu}^{\nu} \) is the contribution to a fusion multiplicity \( (k) N_{\lambda,\mu}^{\nu} \) at a fixed threshold level \( t \), so that
\begin{equation}
(k) N_{\lambda,\mu}^{\nu} = \sum_t (t) n_{\lambda,\mu}^{\nu}.
\end{equation}
We also find
\begin{equation}
(k) n_{\lambda,\mu}^{\nu} = (k) N_{\lambda,\mu}^{\nu} - (k-1) N_{\lambda,\mu}^{\nu},
\end{equation}
where we have put \( (k-1) N_{\lambda,\mu}^{\nu} = 0 \) if any of \( \lambda, \mu, \nu \) are not in \( P_{k-1}^+ \).

Notice that \( \varphi_n^\dagger |\mu\rangle_k = |\mu\rangle_k \). So we calculate
\begin{equation}
[s_{\lambda}(\mathcal{A}), \varphi_n^\dagger] |\mu\rangle_{k-1} = \sum_{\nu \in P_k^+} (k) N_{\lambda,\mu}^{\nu} |\nu\rangle_k - \sum_{\nu \in P_{k-1}^+} (k-1) N_{\lambda,\mu}^{\nu} \varphi_n |\nu\rangle_{k-1}.
\end{equation}
So the phase-model version of (19) is
\[ k \langle \nu | s_\lambda(A), \varphi_n^\dagger \rangle_{k-1} = \langle (k) n_{\lambda,\mu} \rangle. \] (21)

Once a particular non-commutative Schur polynomial \( s_\lambda(A) \) is calculated, the interesting operator \([s_\lambda(A), \varphi_n^\dagger]\) is easy to write down, since
\[ [a_i, \varphi_n^\dagger] = \delta_{i,1} \varphi_n^\dagger (1 - \varphi_n^\dagger \varphi_n) . \] (22)

Recall that \( 1 - \varphi_n^\dagger \varphi_n \) projects onto states with 0 particles at the \( n \)-th node.

To conclude this section, let us consider the threshold weight \([14]\), a concept that generalizes threshold level, in the phase model. Instead of treating the level and the \( n \)-th Dynkin label as special, as in (12), now let the weight \( \hat{\mu} \) be variable in
\[ s_\lambda(A) | \hat{\mu} \rangle = \sum_{\nu \in \hat{P}_+^n} \langle (k) N_{\lambda,\mu}^\nu | \hat{\mu} \rangle, \quad k = \sum_{c=1}^n \mu_c, \quad k = \sum_{c=1}^n \nu_c . \] (23)

The existence of a threshold level \( t \) implies a threshold value for the Dynkin label \( \mu_n \), and threshold weight \( \hat{\theta} \) will take account of all the \( n \) Dynkin labels of \( \hat{\mu} \). A threshold weight \( \hat{\theta} = \sum_{c=1}^n \theta_c \Lambda^c \) may be defined for each term \( a_{j_1} \cdots a_{j_k} a_{j_1} \) in \( s_\lambda(A) \).

Put \( \hat{\text{wt}}(a_j) := \Lambda^j - \Lambda^{j-1}, \text{wt}(a_{j_1} \cdots a_{j_k}) := \text{wt}(a_{j_1}) + \text{wt}(a_{j_k}), \) etc. Then the \( c \)-th Dynkin label of the threshold weight is
\[ \theta_c(a_{j_1} \cdots a_{j_k} a_{j_1}) = -\min \{ 0, \text{wt}(a_{j_1}), \ldots, \text{wt}(a_{j_k}) \} . \] (24)

That is, \( -\theta_c(a_{j_1} \cdots a_{j_k} a_{j_1}) \) is the minimum value of the \( c \)-th Dynkin label of the weights of the sequence \( a_{j_1}, a_{j_2}, \ldots, a_{j_k} \).

For an \( su(3) \) example, take \( \lambda = \Lambda^1 + \Lambda^2 \), and consider \( a_3 a_2 a_1 \) in \( s_\lambda(A) \). The weight sequence associated with \( a_3 a_2 a_1 \) is \( \{ \text{wt}(a_1), \text{wt}(a_2 a_1) \} = \{ \Lambda^1 - \Lambda^3, \Lambda^2 - \Lambda^3, 0 \} \).

Therefore \( \hat{\theta}(a_3 a_2 a_1) = \Lambda^3 \). This threshold weight tells us that \( a_3 a_2 a_1 \) contributes to \( s_{\Lambda^1 + \Lambda^2}(A) | \hat{\mu} \rangle \) iff \( \hat{\mu} - \hat{\theta}(a_3 a_2 a_1) = \hat{\mu} - \Lambda^3 \in \hat{P}_+ \).

4. Conclusion

The Korff-Stroppel integrable realization of \( su(n) \) affine fusion \([8]\) leads to the non-commutative Verlinde formula (12) for affine fusion coefficients, that can be extended to a non-commutative formula (13) for arbitrary Verlinde dimensions \([13]\). The realization offers a new perspective on affine fusion, that should deepen our understanding. It already makes clear certain features, like the existence of threshold levels (and weights) \([13]\).

Let us conclude with a brief discussion of possible future work.

Perhaps the most important further development would be to construct the phase-model realizations of affine fusion for all complex, simple Lie algebras. The next-to-simplest case is likely the Lie algebra \( sp(2n) \cong C_n \), since tableaux work almost as well in this case as for \( su(n) \cong A_{n-1} \). Some progress is reported in \([1]\), where a quantum group approach is used.
From the physical point of view, it would be useful to derive the phase-model realization from affine fusion in another physical context. For example, it is well known that the $G/G$ gauged WZNW model is a topological field theory, with correlation functions equalling the affine Verlinde dimensions [10]. Okuda and Yoshida [9] have already found the Bethe ansatz equations of the phase model in the path-integral formulation of the $U(n)/U(n)$ model, and other indications of the connection. A more direct relation would be helpful, as well as a manifestation of the phase model in the $su(n)$ Chern-Simons theory, which also has Verlinde dimensions as some of its expectation values [12].

More technically, formulas for $s_\lambda(A)$, besides the Jacobi-Trudy formula (10), would likely prove useful. The properties of non-commutative Kostka polynomials should be explored. Of course, similar formulas for non-commutative characters for other Lie algebras are desirable, too.

We are hopeful that progress along these lines can be made in the near future.

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