Componentwise perturbation bounds for the \( LU \), \( LDU \) and \( LDT^T \) decompositions

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COMPONENTWISE PERTURBATION BOUNDS FOR THE LU, LDU AND LDL\(^T\) DECOMPOSITIONS

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Abstract. We improve a componentwise perturbation bound of Sun for the LU factorization and derive a new perturbation bound for the LDU factorization. The latter bound also improves a result of Sun given for the LDL\(^T\) factorization.

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1. Introduction

Perturbation bounds for the LU, LDL\(^T\) factorizations are given by many authors (e.g., see [1], [9], [7], [8], [2]). Here we improve the componentwise LU perturbation bound of Sun [9] and derive a new perturbation bound for the LDU decomposition. These bounds are used to investigate the stability of full rank factorizations produced by Egerváry’s rank reduction procedure [4], [3]. The LDU perturbation bounds are then applied to positive definite symmetric matrices. The result is shown to be better than the LDL\(^T\) perturbation result of Sun [9].

We need the following notations. Let \(A = [a_{ij}]_{i,j=1}^n\). Then \(|A| = [|a_{ij}]_{i,j=1}^n\),

\[
diag(A) = diag(a_{11}, a_{22}, \ldots, a_{nn}),
\]

\[
tril(A,l) = [\alpha_{ij}]_{i,j=1}^n \text{ and } triu(A,l) = [\beta_{ij}]_{i,j=1}^n, \text{ where } 0 \leq |l| < n \text{ and }
\]

\[
\alpha_{ij} = \begin{cases} 
  a_{ij}, & i \geq j - l \\
  0, & i < j - l
\end{cases}, 
\]

\[
\beta_{ij} = \begin{cases} 
  a_{ij}, & i \leq j - l \\
  0, & i > j - l
\end{cases}.
\]

We also use the special notations \(tril(A) = tril(A,0), tril^*(A) = tril(A,-1), triu(A) = triu(A,0)\) and \(triu^*(A) = triu(A,1)\). The spectral radius of \(A\) will be denoted by \(\rho(A)\). For two matrices \(A, B \in \mathbb{R}^{n \times n}\) the relation \(A \leq B\) holds if and only if \(a_{ij} \leq b_{ij}\) for all \(i,j = 1, \ldots, n\). Let \(\bar{I}_k = \sum_{i=1}^k e_i e_i^T (e_i \in \mathbb{R}^n \text{ is the } i\text{th unit vector})\) for \(1 \leq k \leq n\), \(\bar{I}_k = 0\) for \(k \leq 0\) and \(\min(A,B) = [|\min(a_{ij},b_{ij}]|_{i,j=1}^n\).

In Sections 2 and 3 we derive the perturbation bound for the LU and LDU factorizations. A numerical example is shown in Section 4.
2. The LU factorization

Lemma 1 Assume that $A, B, C \in \mathbb{R}^{n \times n}$ are such that $A, B, C \geq 0$ and $\rho(B) < 1$. The maximal solution of the inequality $A \leq C + B \text{triu}(A, l)$ ($l \geq 0$) is $A^*$ ($A^* \geq C$), where $A^* e_k = \left( (I - B I_{k-1}^{l-1})^{-1} C e_k \right)$ ($k = 1, \ldots, n$). $A^*$ is the unique solution of the fixed point problem $A = f(A) = C + B \text{triu}(A, l)$. If $A_0 = (I - B)^{-1} C$, then $A_i = f(A_{i-1})$ converges to $A^*$ monotonically decreasing as $i \to +\infty$ and $0 \leq A_i - A^* \leq (I - B)^{-1} B^i (A_0 - A_1)$ ($i \geq 1$).

Proof. It follows from $A \leq C + B \text{triu}(A, l) \leq C + BA$ that $(I - B) A \leq C$. As $I - B$ is a nonsingular M-matrix by assumption we obtain the upper bound $A \leq A_0 = (I - B)^{-1} C$. As

$$\left| f(A) - f\left( \tilde{A} \right) \right| = \left| B \left( \text{triu}(A, l) - \text{triu} \left( \tilde{A}, l \right) \right) \right| \leq B \left| A - \tilde{A} \right|$$

for any two $n \times n$ matrices $A$ and $\tilde{A}$, the map $f(A)$ is a $B$-contraction [6] on $\mathbb{R}^{n \times n}$ and there is a unique fixed point $A^* = f(A^*)$. Let $X_0 \in \mathbb{R}^{n \times n}$ be arbitrary and $X_k = f(X_{k-1})$ ($k \geq 1$). Then $|A^* - X_k| \leq (I - B)^{-1} B^k |X_1 - X_0|$ ($k \geq 1$). As for any $0 \leq A \leq \tilde{A}$, $f(A) \leq f\left( \tilde{A} \right)$ holds and

$$A_1 = C + B \text{triu} \left( (I - B)^{-1} C, l \right) \leq C + B (I - B)^{-1} C = (I - B)^{-1} C = A_0,$$

the sequence $A_i = f(A_{i-1})$ tends to $A^*$ and is monotonically decreasing. We prove that $A^*$ is the maximal solution of the inequality. Assume that a solution $\tilde{A}$ exists such that $\tilde{A} \geq A^*$. Then $\tilde{A} = A^* + L + U$, where $\text{triu} (U, l) = U$ and $\text{tril} (L, l - 1) = L$. Then

$$\tilde{A} = A^* + L + U \leq C + B \text{triu} (A^* + L + U, l) \leq C + B \text{triu} (A, l) + BU$$

must hold implying that $L + U \leq BU$ and $0 \leq U \leq - (I - B)^{-1} L \leq 0$. Hence $U = L = 0$. The $k$th column of $A^*$ can be written as $A^* e_k = C e_k + B \text{triu} (A^*, l) e_k$, where $\text{triu} (A^*, l) e_k = I_{k-1} A^* e_k$. Hence we obtain $A^* e_k = \left( (I - B I_{k-1}^{l-1})^{-1} C e_k \right)$. $\blacksquare$

Remark 2 The sequence $\{A_i\}_{i \geq 0}$ gives an improving sequence of upper estimates for the maximal solution $A^*$ of the inequality.

We will use the following notations: $A^* = \phi (B, C, l)$. $A_i = \phi_i (B, C, l)$, $\phi_0 (B, C, l) = (I - B)^{-1} C$ and $\phi_i (B, C, l) = C + B \text{triu} \left( \phi_{i-1} (B, C, l), l \right)$ ($i \geq 1$). Notice that for any diagonal matrix $D$, $\phi (B, C D, l) = \phi (B, C, l) D$ and $\phi_i (B, C D, l) = \phi_i (B, C, l) D$.\[\]
Remark 3 Consider the inequality $A \leq C + \text{tril}(A, -l)B$ ($l \geq 0$) with $0 \leq A, B, C \in \mathbb{R}^{n \times n}$ and $\rho(B) < 1$. By transposition we obtain $A^T \leq C^T + B^T \text{tril}(A, -l)^T = C^T + B^T \text{tril}(A^T, l)$ the maximal solution of which is given by $\phi(B^T, C^T, l)$. The sequence $\phi_i(B^T, C^T, l)$ tends to $\phi(B^T, C^T, l)$ and is monotonically decreasing. Hence for the original inequality we have the maximal solution $\phi(B^T, C^T, l)^T$ and the monotone decreasing sequence $\phi_i(B^T, C^T, l)^T$ converging to $\phi(B^T, C^T, l)^T$.

The next theorem improves the componentwise estimate of Sun [9].

Theorem 4 Assume that the $n \times n$ matrix $A$ has the LU decomposition $A = L_1U$, where $L_1$ is unit lower triangular and $U$ is upper triangular. Also assume that the perturbed matrix $A + \delta_A$ has the LU decomposition $A + \delta_A = (L_1 + \delta_{L_1})(U + \delta_U)$, where $L_1 + \delta_{L_1}$ is unit lower triangular and $U + \delta_U$ is upper triangular. Finally assume that $\rho(\{L_1\delta_AU^{-1}\}) < 1$. Then we have

$$|\delta_{L_1}| \leq |L_1| \text{tril}^*\left(\phi\left([L_1^{-1}\delta_AU^{-1}], [L_1^{-1}\delta_AU^{-1}], 0\right)\right), \quad (2.1)$$

$$|\delta_U| \leq \text{triu}\left(\phi\left([L_1^{-1}\delta_AU^{-1}], [L_1^{-1}\delta_AU^{-1}], 1\right)^T\right)[U]. \quad (2.2)$$

Proof. Using the relation

$$\delta_U (U + \delta_U)^{-1} + L_1^{-1}\delta_{L_1} = L_1^{-1}\delta_A (U + \delta_U)^{-1},$$

where $L_1^{-1}\delta_{L_1}$ is a strict lower triangular matrix, while $\delta_U (U + \delta_U)^{-1}$ is upper triangular, we can establish the relations

$$\text{tril}^*\left(L_1^{-1}\delta_A (U + \delta_U)^{-1}\right) = L_1^{-1}\delta_{L_1}, \quad (2.3)$$

$$\text{triu}\left(L_1^{-1}\delta_A (U + \delta_U)^{-1}\right) = \delta_U (U + \delta_U)^{-1}. \quad (2.4)$$

From relation

$$L_1^{-1}\delta_A (U + \delta_U)^{-1} = L_1^{-1}\delta_A U^{-1} - L_1^{-1}\delta_A U^{-1}\delta_U (U + \delta_U)^{-1} \quad (2.5)$$

we obtain the inequality

$$|L_1^{-1}\delta_A (U + \delta_U)^{-1}| \leq |L_1^{-1}\delta_A U^{-1}| + |L_1^{-1}\delta_A U^{-1}| \text{triu}\left([L_1^{-1}\delta_A (U + \delta_U)^{-1}]\right).$$

Applying Lemma 1 we obtain the bound

$$|L_1^{-1}\delta_A (U + \delta_U)^{-1}| \leq A^* = \phi\left([L_1^{-1}\delta_A U^{-1}], [L_1^{-1}\delta_A U^{-1}], 0\right).$$
Hence $|L_1^{-1} \delta_{L_1}| \leq \text{tril}^* (A^*)$ and $|\delta_{L_1}| \leq |L_1| \text{tril}^* (A^*)$.

Using the relation

$$\delta_U U^{-1} + (L_1 + \delta_{L_1})^{-1} \delta_{L_1} = (L_1 + \delta_{L_1})^{-1} \delta_A U^{-1},$$

where $(L_1 + \delta_{L_1})^{-1} \delta_{L_1}$ is a strict lower triangular matrix, while $\delta_U U^{-1}$ is upper triangular, we can also establish the relations

$$\text{tril}^* ((L_1 + \delta_{L_1})^{-1} \delta_A U^{-1}) = (L_1 + \delta_{L_1})^{-1} \delta_{L_1} \tag{2.6}$$

and

$$\text{triu} ((L_1 + \delta_{L_1})^{-1} \delta_A U^{-1}) = \delta_U U^{-1} \tag{2.7}.$$

> From relation

$$(L_1 + \delta_{L_1})^{-1} \delta_A U^{-1} = L_1^{-1} \delta_A U^{-1} - (L_1 + \delta_{L_1})^{-1} \delta_{L_1} L_1^{-1} \delta_A U^{-1} \tag{2.8}$$

we obtain the inequality

$$|(L_1 + \delta_{L_1})^{-1} \delta_A U^{-1}| \leq |L_1^{-1} \delta_A U^{-1}| + \text{tril}^* \left(|(L_1 + \delta_{L_1})^{-1} \delta_A U^{-1}| \right) |L_1^{-1} \delta_A U^{-1}|$$

the maximal solution of which is

$$|(L_1 + \delta_{L_1})^{-1} \delta_A U^{-1}| \leq \tilde{A}^* = \phi \left(|L_1^{-1} \delta_A U^{-1}|^T , |L_1^{-1} \delta_A U^{-1}|^T , 1 \right)^T.$$

Hence $|\delta_U U^{-1}| \leq \text{triu} \left( \tilde{A}^* \right)$ and $|\delta_U| \leq \text{triu} \left( \tilde{A}^* \right) |U|$. This completes the proof. \smallskip

**Remark 5** If function $\phi$ is replaced by $\phi_0$ in (2.1)-(2.2) we obtain the theorem of Sun [9], Thm. 5.1). Hence, our result is sharper.

**Remark 6** Assume that the LU factorizations $A = LU_1$ and

$$A + \delta_A = (L + \delta_L) (U_1 + \delta_{U_1})$$

are such that $U_1$ and $U_1 + \delta_{U_1}$ are upper unit triangular. If $A^T = U_1^T L^T$ and $A^T + \delta_A^T = (U_1^T + \delta_{U_1}^T) (L^T + \delta_L)$ satisfy the conditions of the previous theorem we may write

$$|\delta_L| \leq |L| \text{tril} \left( \phi \left(|L^{-1} \delta_A U_1^{-1}| , |L^{-1} \delta_A U_1^{-1}| , 1 \right) \right), \tag{2.9}$$

and

$$|\delta_{U_1}| \leq \text{triu}^* \left( \phi \left(|L^{-1} \delta_A U_1^{-1}|^T , |L^{-1} \delta_A U_1^{-1}|^T , 0 \right)^T \right) |U_1| \tag{2.10}.$$

Hence, Theorem 4 is also true for the case $A = LU_1$ with unit upper triangular $U_1$. Notice, however, that we have here $\text{tril}$ and $\text{triu}^*$ instead of $\text{tril}^*$ and $\text{triu}$, respectively. This is due to the change of the unit triangular part in the LU factorization.
3. The $LDU$ factorization

Consider the $LDU$ factorization $A = L_1DU_1$ with unit lower triangular $L_1$, diagonal $D$ and unit upper triangular $U_1$. Assume that $A + \delta_A$ can be factorized so that

$$A + \delta_A = (L_1 + \delta_{L_1})(D + \delta_D)(U_1 + \delta_{U_1})$$

where $L_1 + \delta_{L_1}$ is unit lower triangular and $U_1 + \delta_{U_1}$ is unit upper triangular. For $\delta_{L_1}$ and $\delta_{U_1}$ we have the bounds (2.1) and (2.10), respectively. We now look for an estimate of $\delta_D$. We use the relation

$$L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1} = D\delta_{U_1}(U + \delta_{U_1})^{-1} + \delta_D + L_1^{-1}\delta_{L_1}(D + \delta_D),$$

where the matrix $D\delta_{U_1}(U + \delta_{U_1})^{-1}$ is strict upper triangular, $\delta_D$ is diagonal, and $L_1^{-1}\delta_{L_1}(D + \delta_D)$ is strict lower triangular. Hence

$$tril^*(L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1}) = L_1^{-1}\delta_{L_1}(D + \delta_D),$$

$$diag(L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1}) = \delta_D,$$

$$triu^*(L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1}) = D\delta_{U_1}(U_1 + \delta_{U_1})^{-1}.$$

> From relation

$$L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1} = L_1^{-1}\delta_A U_1^{-1} - L_1^{-1}\delta_A U_1^{-1} \delta_{U_1}(U_1 + \delta_{U_1})^{-1}$$

we obtain the inequality

$$|L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1}| \leq |L_1^{-1}\delta_A U_1^{-1}D^{-1}|D| + |L_1^{-1}\delta_A U_1^{-1}D^{-1}|triu^*(L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1})$$

the maximal solution of which is given by the bound

$$|L_1^{-1}\delta_A(U_1 + \delta_{U_1})^{-1}| \leq \phi(\|L_1^{-1}\delta_A U_1^{-1}D^{-1}\|, |L_1^{-1}\delta_A U_1^{-1}D^{-1}|D|, 1).$$

Hence $|\delta_D| \leq |D| \text{diag} (\phi(\|L_1^{-1}\delta_A U_1^{-1}D^{-1}\|, |L_1^{-1}\delta_A U_1^{-1}D^{-1}|, 1))).$

We may get another estimate by using the expression

$$(L_1 + \delta_{L_1})^{-1}\delta_A U_1^{-1} = (D + \delta_D)\delta_{U_1} U_1^{-1} + \delta_D + (L_1 + \delta_{L_1})^{-1}\delta_{L_1} D,$$
where the matrix \((D + \delta_D) \delta_U U_1^{-1}\) is strict upper triangular, \(\delta_D\) is diagonal, and 
\((L_1 + \delta_{L_1})^{-1} L_1, D\) is strict lower triangular. Hence

\[
\text{tril}^* \left( (L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} \right) = (L_1 + \delta_{L_1})^{-1} \delta_L D, \tag{3.5}
\]

\[
\text{diag} \left( (L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} \right) = \delta_D, \tag{3.6}
\]

\[
\text{triu}^* \left( (L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} \right) = (D + \delta_D) \delta_U U_1^{-1}. \tag{3.7}
\]

> From relation

\[
(L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} = L_1^{-1} \delta_A U_1^{-1} - (L_1 + \delta_{L_1})^{-1} \delta_L, L_1^{-1} \delta_A U_1^{-1}
\]

we obtain the inequality

\[
\left| (L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} \right| \leq |D| |D^{-1} \delta_A U_1^{-1}| + + \text{tril}^* \left( (L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} \right) |D^{-1} \delta_A U_1^{-1}|.
\]

It has the maximal solution

\[
\left| (L_1 + \delta_{L_1})^{-1} \delta_A U_1^{-1} \right| \leq \phi \left( |D^{-1} \delta_A U_1^{-1}|^T, |D^{-1} \delta_A U_1^{-1}|^T |D|, 1 \right)^T.
\]

Hence \(|\delta_D| \leq |D| \text{diag} \left( \phi \left( |D^{-1} \delta_A U_1^{-1}|^T, |D^{-1} \delta_A U_1^{-1}|^T, 1 \right) \right).\) We now have two estimates for \(|\delta_D|\). As in general \(|AD| \neq |DA|\) these two estimates are different. We can establish

**Theorem 7** Assume that the \(n \times n\) matrix \(A\) has the LDU decomposition \(A = L_1 D U_1\), where \(L_1\) is unit lower triangular, \(D\) is diagonal and \(U_1\) is unit upper triangular. Also assume that the perturbed matrix \(A + \delta_A\) has the LDU decomposition \(A + \delta_A = (L_1 + \delta_{L_1}) (D + \delta_D) (U_1 + \delta_{U_1})\), where \(L_1 + \delta_{L_1}\) is unit lower triangular and \(U_1 + \delta_{U_1}\) is unit upper triangular. Finally assume that \(\max \left( \rho \left( \Gamma_{L_1} \right), \rho \left( \Gamma_{U_1} \right) \right) < 1\) holds with \(\Gamma_{L_1} = L_1^{-1} \delta_A U_1^{-1} D^{-1}\) and \(\Gamma_{U_1} = D^{-1} L_1^{-1} \delta_A U_1^{-1}\). Then the following inequalities are satisfied:

\[
|\delta_{L_1}| \leq |L_1| \text{tril}^* \left( \phi \left( \Gamma_{L_1}, \Gamma_{L_1}, 0 \right) \right), \tag{3.9}
\]

\[
|\delta_D| \leq |D| \min \left\{ \text{diag} \left( \phi \left( \Gamma_{L_1}, \Gamma_{L_1}, 1 \right) \right), \text{diag} \left( \phi \left( \Gamma_{U_1}^T, \Gamma_{U_1}^T, 1 \right) \right) \right\}, \tag{3.10}
\]

\[
|\delta_{U_1}| \leq \text{triu}^* \left( \phi \left( \Gamma_{U_1}^T, \Gamma_{U_1}^T, 0 \right) \right) |U_1|. \tag{3.11}
\]
Remark 8 If \( \phi \) is replaced by \( \phi_0 \), we obtain the following weaker estimates:

\[
|\delta_{L_1}| \leq |L_1| \text{tril}^* \left( (I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right),
\]

(3.12)

\[
|\delta_{U_1}| \leq \text{triu}^* \left( \Gamma_{U_1} (I - \Gamma_{U_1})^{-1} \right) |U_1|,
\]

(3.13)

\[
|\delta_D| \leq |D| \min \left( \text{diag} \left( (I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right), \text{diag} \left( \Gamma_{U_1} (I - \Gamma_{U_1})^{-1} \right) \right).
\]

(3.14)

Next we specialize the above result for symmetric and positive definite matrices. In such a case \( \Gamma_{L_1} = \Gamma_{U_1}^T \) (\( \Gamma_{L_1} = \bar{L}_1^{-1} \delta_A \bar{L}_1^{-T} \)) and we have the following

Corollary 9 Assume that \( A \) is symmetric and positive definite and its perturbation \( \delta_A \) is such that \( A + \delta_A \) remains symmetric and positive definite. If \( A \) and \( A + \delta_A \) are written in the forms \( A = L_1 D L_1^T \) (\( D \geq 0 \)) and

\[
A + \delta_A = (L_1 + \delta_{L_1}) (D + \delta_D) \left( L_1^T + \delta_{L_1}^T \right),
\]

respectively, then

\[
|\delta_{L_1}| \leq |L_1| \text{tril}^* (\phi (\Gamma_{L_1}, \Gamma_{L_1}, 0))
\]

(3.15)

and

\[
|\delta_D| \leq D \text{diag} (\phi (\Gamma_{L_1}, \Gamma_{L_1}, 1)).
\]

(3.16)

Replacing \( \phi \) by the weaker estimate \( \phi_0 \), we obtain the following bounds:

\[
|\delta_{L_1}| \leq |L_1| \text{tril}^* \left( (I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right)
\]

(3.17)

and

\[
|\delta_D| \leq D \text{diag} \left( (I - \Gamma_{L_1})^{-1} \Gamma_{L_1} \right).
\]

(3.18)

We recall that Sun ([9], Thm. 3.1) for symmetric positive definite matrices proved that

\[
|\delta_{L_1}| \leq |L_1| \text{tril}^* \left( E_{ld} \left( I - \text{diag} (D^{-1} E_{ld}) \right)^{-1} D^{-1} \right),
\]

(3.19)

\[
|\delta_D| \leq \text{diag} (E_{ld})
\]

(3.20)

with

\[
E_{ld} = \left( I - L_1^{-1} \delta_A L_1^{-T} \right) D^{-1} \left( L_1^{-1} \delta_A L_1^{-T} \right).
\]

(3.21)
We compare now estimates (3.17)-(3.18) and (3.19)-(3.20), respectively. We exploit the fact that for any diagonal matrix \( D \), \(|AD| = |A||D|\) and \( \text{diag}(AD) = \text{diag}(A)D \) hold. We can write

\[
(I - \Gamma L_1)^{-1} \Gamma L_1 = (I - |L_1^{-1}\delta AL_1^{-T}|D^{-1})^{-1} |L_1^{-1}\delta AL_1^{-T}|D^{-1} = E_{ld}D^{-1}
\]

and then estimate (3.18) yield

\[
|\delta D| \leq \text{diag} \left( (I - |L_1^{-1}\delta AL_1^{-T}|D^{-1})^{-1} |L_1^{-1}\delta AL_1^{-T}| \right) = \text{diag}(E_{ld}).
\]

As \((I - \text{diag} (D^{-1}E_{ld}))^{-1} \geq I\) and \(E_{ld} (I - \text{diag} (D^{-1}E_{ld}))^{-1} D^{-1} \geq E_{ld}D^{-1}\), the bound (3.19) satisfies

\[
|L_1| \text{tril}^* \left( E_{ld} (I - \text{diag} (D^{-1}E_{ld}))^{-1} D^{-1} \right) \geq |L_1| \text{tril}^* \left( (I - \Gamma L_1)^{-1} \Gamma L_1 \right).
\]

Thus it follows that Theorem 7 improves the special \( LDL^T\) perturbation result of Sun ([9], Thm. 3.1).

4. Final remarks

Computer experiments on symmetric positive definite MATLAB test matrices indicate that estimate \( \phi_1 \) is often so good as \( \phi \) itself. We could observe significant difference between the estimates if \( \Gamma L_1 \) was relatively large. A typical result is shown in Figure 4.1.

Here we display the maximum difference between the components of the bound and the true error matrix for Example 6.1 of [9] to which we added 20 random symmetric matrices with elements of the magnitude \( 5 \times 10^{-3} \). Hence, the line marked with + denotes estimate (3.19) of Sun, the line with triangles denotes the estimate (3.17), the solid line denotes estimate \( \phi_1 \), while the line with circles denotes the best estimate.

The estimates of Theorems 4 and 7 are optimal, if one accepts inequalities of the form \( A \leq C + B \text{triu}(A, l) \) \((A, B, C \geq 0)\) in the estimation process. We can solve, however, the equation \( A = C + B \text{triu}(A, l) \) without any nonnegativity condition. Hence we can give exact expressions for the perturbation errors. For example, in case of Theorem 4 we can prove the following result.

**Theorem 10** For \( k = 1, \ldots, n \) we have

\[
\delta L_k e_k = \begin{bmatrix}
0 \\
- \left( L_2^{(k)} \left( L_1^{(k)} \right)^{-1} \delta_1^{(k)} - \delta_2^{(k)} \right) \left( L_1^{(k)} U_1^{(k)} + \delta_1^{(k)} \right)^{-1} \bar{e}_k
\end{bmatrix}
\]

and

\[
e_k^T \delta U_1 = \begin{bmatrix}
0, \bar{e}_k^T \left( L_1^{(k)} U_1^{(k)} + \delta_1^{(k)} \right)^{-1} \left( \delta_1^{(k)} \left( U_1^{(k)} \right)^{-1} U_2^{(k)} - \delta_4^{(k)} \right)
\end{bmatrix}.
\]
where \( \hat{e}_k \in \mathbb{R}^k \) is the \( k \)th unit vector,

\[
L_1 = \begin{bmatrix}
L_1^{(k)} & 0 \\
L_2^{(k)} & L_3^{(k)}
\end{bmatrix}, \quad U = \begin{bmatrix}
U_1^{(k)} & U_2^{(k)} \\
0 & U_3^{(k)}
\end{bmatrix}, \quad \delta_A = \begin{bmatrix}
\delta_1^{(k)} \\
\delta_2^{(k)} \\
\delta_3^{(k)}
\end{bmatrix}
\]

and \( L_1^{(k)}, U_1^{(k)}, \delta_1^{(k)} \in \mathbb{R}^{k \times k} \).

It does not seem easy to find componentwise estimates better than those of Theorem 4. We can obtain, however, better result than those of Chang and Paige [2].

Finally we remark that either from Theorem 4 or Theorem 7 we can easily obtain normwise perturbation estimates slightly weaker than those of Barrlund [1] by simply using the relation \( \|A\|_F = \|A\|_F \) and \( \phi_0 \) instead of \( \phi \).

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