GLOBAL RIGIDITY OF CONJUGATIONS FOR LOCALLY NON-DISCRETE SUBGROUPS OF $\text{Diff}^\omega(S^1)$

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Abstract. We prove a global topological rigidity theorem for locally $C^2$-non-discrete subgroups of $\text{Diff}^\omega(S^1)$.

1. Introduction

In this paper we establish a topological rigidity theorem for a large class of subgroups of the group $\text{Diff}^\omega(S^1)$ consisting of (orientation-preserving) real analytic diffeomorphisms of the circle $S^1$. Indeed, the primary object studied in this paper are finitely generated, locally $C^2$-non-discrete subgroups of $\text{Diff}^\omega(S^1)$. As is often the case, our choice of restricting attention to finitely generated groups of orientation-preserving diffeomorphisms is made only to help us to focus on the main difficulties of the problem; straightforward generalizations are left to the reader. The regularity assumption ($C^\omega$) required from our diffeomorphisms, however, is a far more important point although it can substantially be weakened in several specific contexts. In this direction some possible extensions of our results to, say, smooth diffeomorphisms, are briefly discussed in the appendix of this article.

A group $G \subset \text{Diff}^\omega(S^1)$ is said to be locally $C^2$-non-discrete if there is an open, non-empty interval $I \subset S^1$ and a sequence $g_j$ of elements in $G$ satisfying the following conditions:

• We have $g_j \neq \text{id}$ for every $j \in \mathbb{N}$.
• The sequence formed by the restrictions $g_j|_I$ of the diffeomorphisms $g_j$ to the interval $I$ converges in the $C^2$-topology to the identity on $I$; see Section 2 for further details.

For the time being it suffices to know that locally $C^2$-non-discrete groups form a large class of finitely generated subgroups of $\text{Diff}^\omega(S^1)$. After stating the main results of this paper, we will further discuss these groups and provide non-trivial information on their nature.

Recall that two subgroups $G_1$ and $G_2$ of $\text{Diff}^\omega(S^1)$ are said to be topologically conjugate if there is a homeomorphism $h : S^1 \to S^1$ such that $G_2 = h^{-1} \circ G_1 \circ h$.
i.e., to every element \(g^{(1)} \in G_1\) there corresponds a unique element \(g^{(2)} \in G_2\) such that \(g^{(2)} = h^{-1} \circ g^{(1)} \circ h\) and conversely. Now we have:

**Theorem A.** Consider two finitely generated, non-abelian subgroups \(G_1\) and \(G_2\) of \(\text{Diff}^0(S^1)\). Suppose that these groups are locally \(C^2\)-non-discrete. Then every homeomorphism \(h : S^1 \to S^1\) satisfying \(G_2 = h^{-1} \circ G_1 \circ h\) coincides with an element of \(\text{Diff}^0(S^1)\).

Theorem A answers one of the questions raised in [33]. When this theorem is combined with Theorem 6.2, we also obtain:

**Theorem B.** Suppose that \(\Gamma\) is a finitely generated hyperbolic group which is neither finite nor a finite extension of \(\mathbb{Z}\), and consider two topologically conjugate faithful representations \(\rho_1 : \Gamma \to \text{Diff}^0(S^1)\) and \(\rho_2 : \Gamma \to \text{Diff}^0(S^1)\) of \(\Gamma\) in \(\text{Diff}^0(S^1)\). Assume that \(G_1 = \rho_1(\Gamma) \subset \text{Diff}^0(S^1)\) is locally \(C^2\)-non-discrete. Assume also the existence of a non-degenerate measure \(\mu\) on \(G_1\) having finite entropy and giving rise to an absolutely continuous stationary measure \(\nu\) on \(S^1\). Then every (orientation-preserving) homeomorphism \(h : S^1 \to S^1\) conjugating the representations \(\rho_1\) and \(\rho_2\) coincides with an element of \(\text{Diff}^0(S^1)\).

The main assumptions of Theorems A and B, namely the fact that our groups are locally \(C^2\)-non-discrete, cannot be dropped. Indeed, counterexamples for the previous statements in the context of discrete groups can be obtained in a variety of ways. For example, two cocompact representations in \(\text{PSL}(2, \mathbb{R})\) of the fundamental group of the genus \(g\) compact surface \((g \geq 2)\) are always topologically conjugate. However these representations are not \(C^1\)-conjugate unless they define the same point in the Teichmüller space. A wider family of counterexamples can be obtained by means of Schottky (free) groups. In fact, a Schottky group on two generators acting on \(S^1\) gives rise to an action that is structurally stable in \(\text{Diff}^0(S^1)\). Thus, by perturbing the generators inside \(\text{Diff}^0(S^1)\), we obtain numerous actions that are topologically but not \(C^1\) conjugate to the initial Schottky group (cf. [38] and references therein).

In the case of Theorem B, there is however an additional assumption regarding the existence of an absolutely continuous stationary measure \(\mu\) and this deserves a few comments (the reader is referred to Section 6 for accurate definitions). Consider then a locally \(C^2\)-non-discrete group \(G \subset \text{Diff}^0(S^1)\). To abridge the discussion assume that \(G\) leaves no probability measure on \(S^1\) invariant. Alternatively the reader may simply assume that \(G\) is isomorphic to a hyperbolic group which is neither finite nor a finite extension of \(\mathbb{Z}\). For this type of groups, the existence of absolutely continuous stationary measures is widely believed to hold in great - if not in full - generality. This belief is based on the existence of a few promising strategies to construct absolutely continuous stationary measures even though carry any of them out to full extent involves some subtle analysis. For example, it is generally believed that a Sullivan’s type construction of a discrete analogue for the Brownian motion should lead to the desired absolutely continuous stationary measure. This line of attack can further be detailed by relying on the more recent and general construction carried out by
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Connel and Muchnik in [7] which essentially reduces the problem to showing the existence of suitable spike-like diffeomorphism in the group $G$; see [7] for details. In turn a natural strategy to show that $G$ contains sufficiently many spike-like diffeomorphism consists of exploiting the denseness properties of locally $C^2$-non-discrete groups as stated in [34]. The central difficulty arising in this context stems from the fact that the mentioned “approximation” properties of $G$ are somehow local whereas the use of spikes as formulated in [7] requires a global control on the effect on certain density functions. To overcome this difficulty we need to show that “approximating sequences” as in [34] can be constructed while keeping global control on the behavior of the diffeomorphism. Since any attempt at conducting this type of analysis here would clearly take us too far from the central ideas in this work, it seems better to defer this discussion to elsewhere and simply add the corresponding assumption to the statement of Theorem B.

To complement the preceding discussion about Theorem B, we also note that topological rigidity does not hold in general when the group $\Gamma$ is $\mathbb{Z}$. A counterexample is provided by Arnold’s well-known construction of analytic diffeomorphisms of $S^1$ topologically conjugate to irrational rotations by singular homeomorphisms. Indeed, the group generated by an irrational rotation is clearly non-discrete. Concerning the possibility of generalizing Theorems A and B to higher rank abelian groups, the reader is referred to the discussions in [26] and [40]. On the other hand, by virtue of the work of Kaimanovich and his collaborators, Theorem B still holds true for other type of groups including relatively hyperbolic ones; cf. [7] and its references.

The above theorems also have consequences of considerable interest in the theory of secondary characteristic classes of (real analytic) foliated $S^1$-bundles. For example, Theorem A yields the following result.

**Corollary C.** Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two analytic foliated $S^1$-bundles. Assume that these foliated $S^1$-bundles are topologically conjugate and that the holonomy groups of $(M_1, \mathcal{F}_1)$ and of $(M_2, \mathcal{F}_2)$ are locally $C^2$-non-discrete and non-abelian. Then the Godbillon-Vey classes of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ coincide.

Concerning Corollary C, it is well known that Godbillon-Vey classes are invariant by homeomorphisms that are transversely of class $C^2$ (see [6]). By virtue of Theorem A, every topological conjugacy between $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ will necessarily be regular in the transverse direction.

The remainder of this introduction contains an overview of our approach to the proofs of Theorems A and B including the main connections with previous works as well as some interesting examples.

Very roughly speaking, the results in this paper are obtained by blending the technique of “vector fields in the closure of groups”, developed in [35] and [27] for subgroups of $\text{Diff}(\mathbb{C},0)$ and in [30] for subgroups of $\text{Diff}_\omega(S^1)$, with results related to stationary measures on $S^1$, see [9], [2], [25] and with measure-theoretic boundary theory for groups [8], [24], and [7]. We will follow a chronological order to explain the various connections between these works.
First, Shcherbakov and Nakai [35], [27] have independently studied the dynamics of non-solvable subgroups of $\text{Diff}(C,0)$ and they observed the existence of certain vector fields whose local flows were “limits” of actual elements in the pseudogroup (see Section 2 for detail). Then Ghys [17] noted that non-solvable subgroups of $\text{Diff}(C,0)$ always contain (non-trivial) sequences of elements converging to the identity. In analogy with the case of finite dimensional Lie groups, he suggested that the existence of vector fields with similar properties should be a far more general phenomenon and he went on to discuss the topological dynamics of the analogous groups of circle diffeomorphisms.

In the case of the circle, the program proposed by Ghys was fairly accomplished in [30]. In this paper, vector fields whose local flows are limits of actual elements in the initial group are said to belong to the closure of the group (see Section 2 for proper definitions). The role of “locally non-discrete subgroups of $\text{Diff}^0(S^1)$” was emphasized and it was shown that these locally non-discrete subgroups of $\text{Diff}^0(S^1)$ admit non-zero vector fields in their closure. As an application of the existence of these vector fields, the following theorem was also proved in [30]:

**Theorem ([30]).** There exists a neighborhood $\mathcal{U}$ of the identity in $\text{Diff}^0(S^1)$ with the following property. Assume that $G_1$ (resp. $G_2$) is a non-solvable subgroup of $\text{Diff}^0(S^1)$ generated by diffeomorphisms $g_{1,1}, \ldots, g_{1,N}$ (resp. $g_{2,1}, \ldots, g_{2,N}$) lying in $\mathcal{U}$. If $h : S^1 \to S^1$ is a homeomorphism satisfying $g_{2,i} = h^{-1} \circ g_{1,i} \circ h$ for every $i = 1, \ldots, N$, then $h$ coincides with an element of $\text{Diff}^0(S^1)$.

This theorem can be thought of as a local version of Theorem A. In fact, the assumption that $h$ takes a generating set formed by elements “close to the identity” to elements that are still close to the identity gives the statement in question an intrinsic local character. For example, the above theorem from [30] is satisfactory for deformations/perturbations problems but falls short of answering the same question for general groups admitting generating sets in the fixed neighborhood $\mathcal{U}$ unless the mentioned sets are, in addition, conjugated by $h$. This type of difficulty was pointed out and discussed in [33] and the method of [30] suggests that these rigidity phenomena should hold for general locally non-discrete subgroups of $\text{Diff}^0(S^1)$ (again see Section 2 for accurate definitions). The original motivation of the present work was then to shed some light on these issues.

It is mentioned in [30] that the main example of locally non-discrete subgroups of $\text{Diff}^0(S^1)$ is provided by non-solvable groups admitting a finite generating set contained in $\mathcal{U} \subset \text{Diff}^0(S^1)$, as follows from Ghys’s results in [17]. Conversely the main examples of groups that are locally discrete are provided by Fuchsian groups. The problem about understanding how the subgroups of $\text{Diff}^0(S^1)$ are split in locally discrete and locally non-discrete ones is then unavoidably raised.

Soon it became clear that locally non-discrete groups were, indeed, very common (see, for example, [32]). The problem of finding locally discrete subgroups of $\text{Diff}^0(S^1)$ beyond the context of Fuchsian groups, however, proved to be much
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harder. Nonetheless much progress has recently been made towards the understanding of the structure of locally discrete groups thanks to the works of Deroin, Kleptsyn, Navas, and their collaborators, see [10] and the survey [11] for some up-to-date information. Meanwhile, it was also observed in [34] that the Thompson-Ghys-Sergiescu subgroup of $\text{Diff}^\omega(S^1)$ is locally discrete. Whereas this example is only smooth, as opposed to real analytic, the observation in question connects with the fundamental notion of expandable point and this requires a more detailed explanation.

Fix a group $G$ of diffeomorphisms of $S^1$. A point $p \in S^1$ is said to be expandable (for the group $G$) if there is an element $g \in G$ such that $|g'(p)| > 1$. Among “large” (e.g. non-solvable) subgroups of $\text{Diff}^\omega(S^1)$ all of whose orbits are dense, $\text{PSL}(2,\mathbb{Z})$ constitutes the simplest example of group exhibiting non-expandable points. In turn, when it comes to locally non-discrete groups having all orbits dense, it is observed in [34] that all points are expandable. In particular, Thompson-Ghys-Sergiescu group must be locally discrete since it exhibits non-expandable points while having all orbits dense. Hence, a method to produce locally discrete groups consists of finding groups with non-expandable points. In a recent and interesting paper [1], V. Kleptsyn and his collaborators have made significant progress in these questions, finding in particular free subgroups of $\text{Diff}^\omega(S^1)$ which are not conjugate to Fuchsian groups and still possess non-expandable points.

Nonetheless, the full understanding of locally discrete subgroups of $\text{Diff}^\omega(S^1)$ was not yet reached (see [10] and [11] for further information). To continue our discussion, we shall then restrict ourselves to the related problem of understanding “rigidity” of topological conjugations between subgroups of $\text{Diff}^\omega(S^1)$ which, ultimately, constitutes the actual purpose of this paper. In the sequel, we then consider two topologically conjugate subgroups $G_1$ and $G_2$ of $\text{Diff}^\omega(S^1)$. Since topological rigidity is targeted, the examples provided in the beginning of the introduction indicate that one of the groups, say $G_1$, should be assumed to be $C^2$-locally non-discrete. At this level, Theorem A fully answers the question provided that $G_2$ is locally $C^2$-non-discrete as well. Thus, to make further progress, we need to investigate whether a locally $C^2$-non-discrete group $G_1$ can be topologically conjugate to a locally $C^2$-discrete subgroup $G_2$. Following our above stated results, the state-of-art of this problem can be summarized as follows.

First we assume once and for all that $G_1$ (and hence $G_2$) is minimal i.e. all of its orbits are dense in $S^1$. Moreover these groups are also assumed to be non-abelian. The material presented in Sections 2 and 3 of the present paper shows that these assumption can be made without loss of generality. Theorem B also settles the question when the groups are of hyperbolic type, up to the technical condition on the existence of absolutely continuous stationary measures. Also, if $G_2$ is conjugate to a Fuchsian group, then a conjugating homeomorphism $h$ between $G_1$ and $G_2$ cannot exist as pointed out in [33]. These general statements
apart, the existence of non-expandable points plays again a role in the problem. Thus we may consider the obvious alternative:

- All points in $S^1$ are expandable for $G_2$.
- $G_2$ has at least one non-expandable point.

In the first case, an unpublished result of Deroin asserts that the (locally $C^2$-discrete) group $G_2$ is essentially a Fuchsian group. Therefore the preceding implies that a conjugating homeomorphism between $G_1$ and $G_2$ cannot exist (cf. [33]). Alternately, the non-existence of topological conjugation between $G_1$ and $G_2$ can directly be derived from Theorem 5.1 in Section 5. In fact, the argument in Section 5 relies only on the following assumptions:

1. $G_1$ is locally $C^2$-non-discrete.
2. $G_1$ is minimal and non-abelian.
3. Every point in $S^1$ is expandable for $G_2$.

The fact that the argument of Section 5 depends only on the conditions above will also be useful in Section 6 for the proof of Theorem B.

In closing, recall that a classical problem that lends further interest to regularity properties of homeomorphisms conjugating groups actions is the possibility of having different Godbillon-Vey characteristic classes. In the case of (global) groups acting on $S^1$, our results are satisfactory for locally $C^2$-non-discrete. On the other hand, in the locally discrete case, this problem is difficult even if the groups in question arise from Fuchsian groups and we refer the reader to [18] and its references for further information.

To finish the introduction, let us provide an overview of the structure of this paper. Section 2 begins with accurate definitions for most of the notions relevant for this paper and it then goes on by reviewing some results related to Shcherbakov-Nakai theory in a form adapted to our needs. The second half of this section, namely Subsection 2.2, provides a description of the topological dynamics associated with a locally $C^2$-non-discrete subgroup of $\text{Diff}^\omega(S^1)$. This description faithfully parallels the corresponding results established in [17] for the case of groups admitting a generating set “close to the identity”.

Section 3 is devoted to proving Theorem A in different types of special situations. These include the case where the groups $G_1$ and $G_2$ have finite orbits as well as the case in which these groups are solvable but non-abelian. The results of Section 3 are implicitly used throughout the paper since they allow us to restrict our discussion to a sort of “generic case” for the group $G_1$; see Proposition 3.5. Roughly speaking, this generic situation is such that we can fix and interval $I \subset S^1$ and, for every $\epsilon > 0$, we can find a finite collection of elements in $G_1$ satisfying the following conditions:

- Diffeomorphisms in this collection are $\epsilon$-close to the identity in the $C^2$-topology on $I$.
- The collection of these diffeomorphisms generated a non-solvable subgroup of $\text{Diff}^\omega(S^1)$.

The study of this last generic case will be the object of Sections 4, 5 and 6.
In Section 4, we construct an explicit sequence of diffeomorphisms in $G_1$ converging to the identity in the $C^2$-topology on the above mentioned interval $I$. As explained in the beginning of Section 4, this construction is necessary to yield a sequence converging to the identity for which we can control the mentioned convergence rate while also estimating the growth rate of the sequence formed by the corresponding higher order derivatives. In fact, the reader will note that the very definition of a locally $C^2$-non-discrete group provides us with a sequence converging to the identity in the $C^2$-topology on some non-empty interval. This definition however does not give us any estimate on, for example, the $C^3$-norm of the diffeomorphisms in this sequence (see Section 4 for a detailed discussion). In the construction of a specific sequence converging to the identity for which estimates on the growing rate of higher derivatives are also available, we will take advantage of the fact that we can select finitely many elements of $G_1$ generating a non-solvable group and being arbitrarily close to the identity on a fixed interval $I$; cf. Proposition 3.5.

In Section 5, we shall prove Theorem A modulo Proposition 5.3 which is a by-product of the proof—if not of the statement—of Theorem F in [9]. This section is indeed devoted to the proof of Theorem 5.1 which provides a statement stronger than what is strictly needed to derive Theorem A in the “generic case” (which will be the only case under discussion after Section 3). As to Proposition 5.3, the reader will note that its use can be avoided modulo working with bounded distortion estimates for iterates of diffeomorphisms possessing parabolic fixed points, as done in [30]. The interest of Proposition 5.3 lies primarily in the fact that it makes the discussion significantly shorter by allowing us to focus exclusively on hyperbolic fixed points which, in turn, are linearizable [37].

In Section 6 we collect essentially all the results in this paper for which ergodic theory appears to be an indispensable tool. First we provide additional detail on how the proof of Proposition 5.3 can be derived from the argument given in [9] to prove their Theorem F. Then we shall state and prove Theorem 6.2 which reduces Theorem B to Theorem A. The proof of Theorem 6.2 is more involved as it combines standard facts about hyperbolic groups with results from [8] and from [24] and still depends in a crucial way on the existence of absolutely continuous stationary measures as assumed in Theorem B.

Finally the Appendix 7 contains a partial answer in the analytic category to a question raised in [8]. The argument exploits the construction carried out in Section 4. The appendix then ends with a summary of the role played by the regularity assumption ($C^\omega$) in this paper. In particular, we highlight some specific problems whose solutions would lead to non-trivial generalizations of our statements to less regular groups of diffeomorphisms.

2. Locally non-discrete groups: vector fields and topological dynamics

The definition of locally non-discrete groups is implicit in [30] and formulated in [32] and in [8]. In the analytic case, it reads as follows:
**Definition 2.1.** A subgroup $G$ of $\text{Diff}^\omega(S^1)$ is said to be locally $C^m$-non-discrete if there is a non-empty open interval $I \subseteq S^1$ and a sequence of elements $\{g_i\} \subset G$, $g_i \neq \text{id}$ for every $i \in \mathbb{N}$, whose restrictions $\{g_i|_{I}\}$ to $I$ converge to the identity in the $C^m$-topology (as maps from $I$ to $S^1$).

Concerning the above definition and the corresponding sequence $\{g_i\}$ of diffeomorphisms in $G$, it should be emphasized that that the condition $g_i \neq \text{id}$ ensures that the restriction of $g_i$ to the interval $I$ does not coincide with the identity either since our diffeomorphisms are real analytic. This is a fundamental advantage of working in the $C^\omega$-category. The analogous definition becomes therefore slightly more technical for groups of, say, smooth diffeomorphism; cf. [8].

Naturally, a group $G \subset \text{Diff}^\omega(S^1)$ is called locally $C^m$-discrete if it fails to satisfy the conditions of Definition 2.1. Unless otherwise stated, the terminology used in this paper is such that every interval is open, connected and non-empty. While the definition of locally $C^m$-non-discrete is meaningful for arbitrary values of $m \in \mathbb{N}$, in this work we shall mostly be concerned with the case $m = 2$. In a sense this comes as no surprise since Theorem 7.1 (in the Appendix) underlines the special role of the $C^2$-topology by asserting that a non-solvable, locally $C^2$-non-discrete subgroup of $\text{Diff}^\omega(S^1)$ is necessarily locally $C^\omega$-non-discrete. A more important—though also more technical and harder to immediately be appreciated—reason for the role played by locally $C^2$-non-discrete groups in our discussion stems from the following couple of remarks. First, owing to a result due to Ghys and Tsuboi [22], to prove Theorem A it suffices to prove that the conjugating homeomorphism $h$ is of class $C^1$. Secondly, the method we employ for proving rigidity theorems relies on constructing “vector fields that approximate the dynamics of the group” (see Definition 2.4) so that to conclude that $h$ is of class $C^1$ we need to find (synchronized) vector fields of class $C^1$. It turns out, however, that in the construction of vector fields “whose dynamics approximate the dynamics” there is an intrinsic loss of one derivative (see Lemma 2.7 for a prototypical situation): this loss of derivative leads us to use the $C^2$-topology to obtain the desired vector fields of class $C^1$. Finally we also point out that a simple albeit important use of the case $m = 1$ appears in connection with Lemma 6.3 and the main result in [8] which are important for Theorem B.

The preceding has the following implications in the organization of our statements: as mentioned, most of the work is conducted for the case $m = 2$ and this applies in particular to Section 4 containing several important estimates to be used in the proof of Theorem A given in Section 5. The estimates in question are accordingly related to the $C^2$-topology with exception of the very elementary Lemma 4.3 which will be needed in the proof of Theorem 7.1 as well.

It also useful to adapt Definition 2.1 to the context of pseudogroups. However, even in the analytic category, the case of pseudogroups exhibits a difficulty analogous to the one pointed out above for groups of smooth diffeomorphisms since the domain of definition of an element in a pseudogroup may be disconnected: the standard analytic continuation principle can no longer be used to
ensure that a local diffeomorphism that does not coincide with the identity in a certain region cannot coincide with the identity on another region.

**Definition 2.2.** Consider an open set $U \subset \mathbb{R}$ along with a pseudogroup $\Gamma$ of analytic diffeomorphisms from open subsets of $U$ to $\mathbb{R}$. The pseudogroup $\Gamma$ is said to be locally $C^m$-non-discrete if there is an interval (open, connected and non-empty) $I \subset U$ and a sequence of maps $\{g_i\} \subset \Gamma$ satisfying the following conditions:

1. For every $i \in \mathbb{N}$, the interval $I$ is contained in the domain of definition of $g_i$ viewed as element of the pseudogroup $\Gamma$.
2. The restriction $g_i|_I$ of $g_i$ to $I$ does not coincide with the identity map.
3. The sequence $\{g_i|_I\}$ formed by the restrictions of the $g_i$ to $I$ converge to the identity in the $C^m$-topology (as maps from $I$ to $\mathbb{R}$).

**Remark 2.3.** In the case of pseudogroups of maps defined on the real line, the above mentioned issue involving possibly disconnected domains of definitions can be avoided in many cases including, for example, when the pseudogroup has a finite generating set all of whose elements are defined on all of $U$. In higher dimensions however pseudogroups having elements with disconnected domains of definitions are very common and cannot easily be avoided.

For the discussion in this paper we have opted for including Condition 2 in Definition 2.2 so as to have a notion that can immediately be generalized. Yet the reader will note that for our purposes this condition is of little importance since our attention is focused on pseudogroups induced by suitable restrictions of actual groups of real analytic diffeomorphisms of the circle.

2.1. Vector fields in the closure of pseudogroups. Vector fields whose local flow can be approximated by elements in the initial group (pseudogroup) constitute a very important tool to investigate the dynamics associated with locally non-discrete groups (pseudogroups). The idea of approximating a flow by elements in a group/pseudogroup is made accurate by the following definition.

**Definition 2.4.** Consider an open set $U \subset \mathbb{R}$ along with a pseudogroup $\Gamma$ of maps from open subsets of $U$ to $\mathbb{R}$. Consider also a vector field $X$ defined on an interval $I \subset U$ and let $\Psi_X$ denote its local flow. The vector field $X$ is said to be (contained) in the $C^m$-closure of $\Gamma$ if, for every interval $I_0 \subset I$ and for every $t_0 \in \mathbb{R}_+$ such that $\Psi_X^t$ is defined on $I_0$ for $0 \leq t \leq t_0$, there exists a sequence of maps $\{g_i\} \subset \Gamma$ satisfying the conditions below:

- For every $i \in \mathbb{N}$, the interval $I_0$ is contained in the domain of definition of $g_i$ viewed as element of the pseudogroup $\Gamma$.
- The sequence $\{g_i|_{I_0}\}$ formed by the restrictions of the $g_i$ to $I_0$ converge to $\Psi_X^{t_0}: I_0 \to \mathbb{R}$ in the $C^m$-topology (where $m \in \mathbb{N} \cup \{\infty\}$).

Unless otherwise mentioned, whenever we mention a vector field $X$ belonging to the closure of a pseudogroup $\Gamma$ it is implicitly assumed that $X$ does not vanish identically. It is clear from the definitions that a pseudogroup containing some (non-identically zero) vector field in its $C^m$-closure cannot be locally $C^m$-discrete.
Before going further into the structure of the topological dynamics of locally $C^2$-non-discrete subgroups of $\text{Diff}^u(S^1)$, let us quickly revisit some results established by Shcherbakov and Nakai for pseudogroups of holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$; see [27], [35]. The discussion below is slightly simplified by the fact that only local diffeomorphisms having real coefficients will be considered. Let $\text{Diff}^u(\mathbb{R}, 0)$ denote the group of germs of orientation-preserving analytic diffeomorphisms fixing $0 \in \mathbb{R}$. Here by orientation-preserving it is meant that every $g \in \text{Diff}^u(\mathbb{R}, 0)$ satisfies $g'(0) > 0$.

First, we have:

**Lemma 2.5.** Let $\Gamma$ be a pseudogroup generated by finitely many elements of $\text{Diff}^u(\mathbb{R}, 0)$ and denote by $\Gamma_0$ the group of germs at $0 \in \mathbb{R}$ corresponding to $\Gamma$. Assume that $\Gamma_0$ is not abelian. Then $\Gamma$ contains analytic vector fields in its closure. In particular $\Gamma$ is locally $C^\infty$-non-discrete.

**Proof.** The proof is split in two cases according to whether or not $\Gamma_0$ is fully constituted by germs of diffeomorphisms tangent to the identity.

Assume first the existence of an element in $\Gamma_0$ such that $g'(0) \neq 1$. Since $g$ preserves the orientation of $S^1$, we have $g'(0) = \lambda > 0$. Thus up to replacing $g$ by its inverse $g^{-1}$, we can assume that $g'(0) = \lambda \in (0, 1)$. In this case, there are local (analytic) coordinates where $g(x) = \lambda x$; see [37]. Since $\Gamma_0$ is not abelian, there also exists another element $g_1 \neq id$ belonging to $D^1 \Gamma_0$. Though $g_1 \neq id$, the derivative of $g_1$ at $0 \in \mathbb{R}$ equals 1 since $g_1 \in D^1 \Gamma_0$ is a product of commutators. Now, by repeating the standard argument of Shcherbakov-Nakai with elements of $\Gamma_0$ having the form $\lambda^{-N(k)} \cdot g_1^k(\lambda^{N(k)} x)$, it is well known that a suitable choice of the integers $N(k)$ leads to an analytic vector field $X$ in the $C^\infty$-closure $\Gamma$, see for example [27]. The reader will note that the mentioned vector field $X$ is defined around $0 \in \mathbb{R}$ which in general does not happen for Shcherbakov-Nakai vector fields. The proof of the lemma is therefore completed provided that $\Gamma_0$ contains an element which is not tangent to the identity.

We now consider the case where every element $g$ in $\Gamma_0$ satisfies $g'(0) = 1$. Since $\Gamma_0$ is not abelian, there must exist elements $g_1, g_2 \in \Gamma_0$, $g_1, g_2 \neq id$, having different contact orders with the identity. These two elements can then be used to produce a vector field of Shcherbakov-Nakai in the $C^\infty$-closure of $\Gamma$, see for example [27]. The lemma is proved. \qed

**Lemma 2.6.** Consider a pseudogroup $\Gamma$ generated by finitely many elements of $\text{Diff}^u(\mathbb{R}, 0)$ and denote by $\Gamma_0$ the group of germs at $0 \in \mathbb{R}$ corresponding to $\Gamma$. Then the following are equivalent:

1. $\Gamma_0$ is an infinite cyclic group unless it is reduced to the identity.
2. $\Gamma$ is locally $C^m$-discrete, for every $m \in \mathbb{N} \cup \{\infty\}$.
3. $\Gamma$ does not contain vector fields in its $C^m$-closure, for every $m \in \mathbb{N} \cup \{\infty\}$.

**Proof.** Owing to Lemma 2.5 we can assume that $\Gamma_0$ is abelian otherwise none of the above statements holds. Since the elements in $\Gamma \subset \text{Diff}^u(\mathbb{R}, 0)$ are assumed to preserve the orientation of $\mathbb{R}$, it follows at once that every element different from the identity in $\Gamma_0$ has infinite order. Assuming once and for all that $\Gamma_0$ is not
reduced to the identity, consider an element \( g \neq \text{id} \) in \( \Gamma_0 \). Modulo replacing \( g \) by its inverse \( g^{-1} \), we can assume that \( g'(0) \leq 1 \). Let us then split the discussion into two cases.

**Case 1.** Suppose there is \( g \in \Gamma_0 \) such that \( g'(0) = \lambda < 1 \). Again Sternberg’s result [37] implies the existence of local analytic coordinates where \( g(x) = \lambda x \). Since \( \Gamma_0 \) is abelian, there follows that every element of \( \Gamma_0 \) coincides with a linear map of type \( x \rightarrow cx \) in the above coordinates, where \( c \in \mathbb{R}_+^* \) is a constant. In other words, \( \Gamma_0 \) is naturally identified with a multiplicative subgroup of \( \mathbb{R}_+^* \). The mutual equivalence of the above statements follows at once.

**Case 2.** Suppose now that every element in \( \Gamma_0 \) is tangent to the identity. Let then \( g \neq \text{id} \) be an element of \( \Gamma_0 \) and denote by \( Y \) the formal vector field whose time-one map coincides with \( g \).

Let \( \mathcal{T} \) be the sets of those values of \( t \in \mathbb{R} \) for which the formal flow \( \Psi_t^Y \) of \( Y \) actually defines an element of \( \Gamma_0 \). Clearly \( \mathcal{T} \) is an additive subgroup of \( \mathbb{R} \). Moreover, it is well known that the formal power series defining \( Y \) will be convergent provided that the set \( \mathcal{T} \) is not discrete in \( \mathbb{R} \), see [4], [12].

Since \( \Gamma_0 \) is abelian, it embeds in the 1-parameter group generated by the formal flow of \( Y \). In fact, the formal flow of \( Y \) is known to contain the germs of all elements in \( \text{Diff}^m(\mathbb{R},0) \) commuting with \( g \). There follows that \( \Gamma_0 \) is infinite cyclic if and only if \( \mathcal{T} \) is a discrete subgroup of \( \mathbb{R} \). In this case, there also follows that \( \Gamma \) is locally \( C^m \)-discrete and that \( \Gamma \) contains only trivial vector fields in its \( C^m \)-closure (for every \( r \in \mathbb{N} \cup \{\infty\} \)). Conversely, if \( \mathcal{T} \) is not discrete in \( \mathbb{R} \), then it must be dense. Furthermore the formal vector field \( Y \) turns out to be analytic ([4], [12]). It is now immediate to check that \( Y \) itself is contained in the \( C^\infty \)-closure of \( \Gamma \). The lemma is proved.

Shcherbakov-Nakai vector fields for non-solvable subgroups of \( \text{Diff}(\mathbb{C},0) \) were the first genuinely non-linear situation where vector fields in the closure of (countable) groups were proven to exist. Subgroups of \( \text{Diff}^m(\mathbb{R},0) \) (or even of \( \text{Diff}(\mathbb{C},0) \)) are obviously special, as opposed to groups of \( \text{Diff}^m(S^1) \), in the sense that their elements share a same fixed point, namely the origin. In addition to the existence of free discrete subgroups in \( \text{Diff}^m(S^1) \), the absence of a common fixed point for elements in free subgroups of \( \text{Diff}^m(S^1) \) is the main obstacle to extend to this context the results obtained in [27], [35]. This difficulty was overcome for the first time in [30]. The following lemma singles out the key point that is common to all constructions of vector fields having similar properties (for detailed explanations see [33]).

**Lemma 2.7.** Suppose that the pseudogroup \( \Gamma \) consisting of local diffeomorphisms from open sets of an open (non-empty) interval \( J \subset \mathbb{R} \) to \( \mathbb{R} \) contains a sequence of elements \( \{\tilde{g}_i\} \) satisfying the following conditions:

1. For every \( i \), \( \tilde{g}_i \) is defined on a fixed non-empty open interval \( I \). Moreover the restriction \( \tilde{g}_i|_I \) of \( \tilde{g}_i \) to \( I \) is different from the identity for every \( i \).
2. The sequence of local diffeomorphisms \( \tilde{g}_{i+1} \) converges to the identity in the \( C^m \)-topology.
3. There is a uniform constant $C$ such that
\[ \|\tilde{g}_i \circ \text{id}\|_{m,I} \leq C \|\tilde{g}_i \circ \text{id}\|_{m-1,I}, \]
where $\|\cdot\|_{m,I}$ (resp. $\|\cdot\|_{m-1,I}$) stands for the $C^m$-norm (resp. $C^{m-1}$-norm) of $\tilde{g}_i \circ \text{id}$ on $I$.

Then there is a (non-identically zero) vector field $X$ contained in the $C^{m-1}$-closure of $\Gamma$.

Proof. For every $i$, we consider the vector field $X_i$ defined on $I$ by the formula
\[ X_i = \frac{1}{\|\tilde{g}_i \circ \text{id}\|_{m,I}} (\tilde{g}_i(x) - x) \frac{\partial}{\partial x}. \]
It follows at once that the $C^m$-norm of $X_i$ on $I$ is bounded by 1 and, in addition, that this bound is attained in the closure of $I$. In turn, condition 3 above shows that the $C^{m-1}$-norm of $X_i$ is bounded from below by a positive constant. In fact, we have
\[ 0 < \frac{1}{C} \leq \|X_i\|_{m-1,I} \]
for every $i \in \mathbb{N}$. Owing to Ascoli-Arzela theorem, and modulo passing to a subsequence, the sequence of vector fields {$X_i$} converges in the $C^{m-1}$-topology towards a $C^{m-1}$-vector field $X$. Furthermore, $X$ is not identically zero since it must verify $\|X\|_{m-1,I} \geq 1/C > 0$. Now a standard application of Euler polygonal method shows that the vector field $X$ is contained in the $C^{m-1}$-closure of $\Gamma$ in the sense of Definition 2.4. The lemma is proved.

The method originally put forward in [30] is summarized by Proposition 2.8 below; see also [33] and [10].

**Proposition 2.8.** Consider a pseudogroup $\Gamma$ consisting of real analytic maps from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$. Assume that $\Gamma$ satisfies the two conditions below:

- There is a sequence of elements $\{g_i\} \subset \Gamma$ such that all the maps $g_i$ are defined on $I$ and none of them coincides with the identity on $I$. Moreover, this sequence converges to the identity on the $C^m$-topology on $I$.
- There is an element $f \in \Gamma$ possessing a hyperbolic fixed point $p \in I$.

Then there is an open interval $I' \subset I$ containing $p$ and a sequence of elements $\{\tilde{g}_i\}$ in $\Gamma$ satisfying the conditions of Lemma 2.7 (in particular, all the diffeomorphisms $\tilde{g}_i$ are defined on $I'$ and none of them coincides with the identity on $I$).

Proof. We shall sketch the argument since extensions of this basic idea will play an important role in Sections 4 and 5. It suffices to consider the case $m = 2$. By assumption, we have $f(p) = p$ and $f''(p) = \lambda \in (0,1)$. Since $f$ is analytic, there is a local coordinate $x$ around $p$ where $f(x) = \lambda x$ [37]. Let then $I' \subset I$ be an interval containing $p$ whose closure is contained in the domain of definition of the coordinate $x$. First, we have the following:

**Claim 1.** Without loss of generality, we can assume that $g_i(p) \neq p$ for every $i$. 


Proof of Claim 1. Suppose that \( g_i(p) = p \) for all but finitely many \( i \). If, for some large enough \( i \), we have \( g_i'(p) = 1 \) then by considering elements of the form \( \{ \lambda^{-N} g_i(\lambda^N x) \} \) (with \( i \) fixed), we can obtain a Shcherbakov-Nakai vector field defined on a neighborhood of \( p \) and contained in the \( C^\infty \)-closure of \( \Gamma \); see for example [27]. The existence of this vector field actually suffices for our purposes, yet we point out that the sequence of elements \( \{ \lambda^{-N} g_i(\lambda^N x) \} \) satisfies the conditions of Lemma 2.7.

There follows from the preceding that the proposition holds provided that there is some \( g_i \) not commuting with \( f \) and satisfying \( g_i(p) = p \). Hence it only remains to consider the possibility of having all the diffeomorphisms \( g_i \) commuting with \( f \) and satisfying \( g_i(p) = p \) (modulo dropping finitely many terms of the initial sequence). Since \( g_i \) commutes with \( f \), it must be given on \( I \) and in the coordinate \( x \) by \( g_i(x) = \lambda_i x \). However the sequence \( \{ \lambda_i \} \) converges to 1 since \( \{ g_i \} \) converges \( C^2 \) (in fact \( C^\infty \)) to the identity. In other words, the sequence \( \{ g_i \} \) satisfies all the conditions in the statement.

We may also add a further comment regarding the last possibility discussed in the proof of the above claim, the reader will note that the \( C^1 \)-closure of \( \Gamma \) contains a flow consisting of linear maps \( x \mapsto \Lambda x \) for every \( \Lambda \in \mathbb{R}^* \). Indeed, for every \( i \), \( \lambda_i \neq 1 \) since \( g_i \neq \text{id} \). There follows that the multiplicative group of \( \mathbb{R}^* \) generated by the collection of all \( \lambda_i \) is dense in \( \mathbb{R}^* \) what, in turn, ensures that the mentioned vector field lies in the \( C^1 \)-closure (indeed in the \( C^\infty \)-closure) of \( \Gamma \).

We now go back to the proof of Proposition 2.8. In what follows we assume that \( g_i(p) \neq p \) for every \( i \in \mathbb{N} \). Next, let \( \kappa_i \) be a sequence of positive integers going to infinity to be determined later. Set

\[
\tilde{g}_i = f^{-\kappa_i} \circ g_i \circ f^{\kappa_i} = \lambda^{-\kappa_i} \cdot g_i(\lambda^{\kappa_i} x).
\]

Note that the second derivative \( \tilde{g}_i'' \) of \( \tilde{g}_i \) at a point \( x \) is simply \( \tilde{g}_i''(x) = \lambda^{\kappa_i} \cdot g_i''(\lambda^{\kappa_i} x) \) provided that both sides are defined. This formula shows that \( \sup_{x \in I} \| \tilde{g}_i''(x) \| \) decreases as \( \kappa_i \) increases. On the other hand the absolute value of \( \lambda^{-n} g_i(0) \) increases monotonically with \( n \) and becomes unbounded as \( n \to \infty \) since \( g_i(0) \neq 0 \). Therefore the \( C^1 \)-norm of \( \tilde{g}_i - \text{id} \) on \( I \) also increases with \( n \). Thus, for every \( i \) fixed, we can find \( \kappa_i \in \mathbb{N}^* \) so that the following estimate holds:

\[
\sup_{x \in I} \| \tilde{g}_i''(x) \| \leq \sup_{x \in I} \| \tilde{g}_i - \text{id} \| + \| \tilde{g}_i' - 1 \|.
\]

For these choices of \( \kappa_i \) we immediately obtain

\[
\| \tilde{g}_i - \text{id} \|_{2,I} < 2 \| \tilde{g}_i - \text{id} \|_{1,I}
\]

proving the proposition.

2.2. **Topological dynamics of locally non-discrete subgroups.** The material presented in this section is very closely related to the description in [17] of the topological dynamics associated with groups generated by diffeomorphisms close to the identity. In fact, our purpose is to prove the following:
**Proposition 2.9.** Let $G \subset \text{Diff}^\omega(S^1)$ be a locally $C^2$-non-discrete group. Then either $G$ has a finite orbit or every orbit of $G$ is dense in $S^1$. Moreover, the set of points in $S^1$ having finite orbit under $G$ is itself finite. Finally, if $I$ is a connected interval in the complement of this set and $G_I$ denote the subgroup of $G$ consisting of those diffeomorphisms fixing $I$, then the action of $G_I$ on $I$ has all orbits dense in $I$.

Since our assumptions are slightly more general than those used in [17], we shall provide below a detailed proof for Proposition 2.9. We begin by recalling a well-known proposition; see for example [6], [29].

**Proposition 2.10.** Denote by $\text{Homeo}(S^1)$ the group of homeomorphisms of the circle and consider a subgroup $G \subset \text{Homeo}(S^1)$. Then one of the following holds:

1. The group $G$ possesses a finite orbit in $S^1$.
2. The $G$-orbit of every point $p \in S^1$ is dense in $S^1$.
3. There is a Cantor set $K \subset S^1$ invariant by $G$ and such that the $G$-orbit of every point $p \in K$ is dense in $K$. This set is unique and contained in the closure of the $G$-orbit of every point $p \in S^1$. \hfill \Box

Consider now a subgroup $G$ of $\text{Diff}^\omega(S^1)$. When $G$ possesses a finite orbit the statement of Proposition 2.10 can be strengthened as follows. Since $G$ has a finite orbit, rotation numbers of the elements in $G$ take values in some finite set. There follows that the subgroup $G_0$ of $G$ consisting of those diffeomorphisms fixing every point in the mentioned finite orbit has finite index in $G$. In particular $G_0$ is not reduced to the identity unless $G$ is a finite group. Assuming that $G$ is not finite and choosing $g \in G_0$, $g \neq \text{id}$, there follows that the set of all points in $S^1$ possessing finite orbit under $G$ must be finite since it is contained in the set of fixed points of $g$. Hence, we have proved:

**Corollary 2.11.** Assume that the group $G$ is infinite but has a finite orbit $O_p$. Denote by $\text{Per}(G) \subset S^1$ the set consisting of those points $q \in S^1$ whose orbit under $G$ is finite. Then $\text{Per}(G)$ is a finite set. In particular, $G$ possesses a finite index subgroup $G_0$ whose elements fix every single point in $\text{Per}(G)$. \hfill \Box

Dealing with subgroups of $\text{Diff}^\omega(S^1)$ having finite orbits will naturally involve groups of analytic diffeomorphism of the interval $[0,1]$ (i.e. the group of diffeomorphisms from $[0,1]$ to $[0,1]$ fixing the endpoints 0 and 1). In this direction, the following statement is attributed to G. Hector (see [17, page 160] for a proof).

**Proposition 2.12** (G. Hector). Let $G_I$ denote a group consisting of orientation-preserving real analytic diffeomorphisms of $[0,1]$. Suppose that the only points in $[0,1]$ that are fixed for every element in $G_I$ are precisely the endpoints 0 and 1. Suppose also that $G$ is neither trivial nor an infinite cyclic group. Then the orbit of every point $p \in (0,1)$ is dense in $(0,1)$. \hfill \Box

We are now able to prove Proposition 2.9.

**Proof of Proposition 2.9.** The core of the proof consists of showing that a subgroup $G \subset \text{Diff}^\omega(S^1)$ leaving invariant a Cantor set $K \subset S^1$ must be locally $C^2$-discrete. Equivalently a locally $C^2$-non-discrete group cannot leave a Cantor
set invariant. We begin by proving this assertion. Let then $G$ be a locally $C^2$-non-discrete subgroup of $\text{Diff}^n(S^1)$ and assume for a contradiction that $G$ leaves invariant some Cantor set $K \subset S^1$. In particular Proposition 2.10 ensures that $K$ is the unique minimal set of $G$ in $S^1$. Furthermore $K$ and the whole of $S^1$ are the only non-empty closed subsets of $S^1$ that are invariant by $G$.

Suppose for a contradiction that $G$ is locally $C^2$-non-discrete. In other words, assume the existence of an interval $I \subset S^1$ along with a sequence of elements in $\{g_i\} \subset G$ satisfying the following:

1. $g_i \neq \text{id}$ for every $i \in \mathbb{N}$ (since $G$ is constituted by analytic diffeomorphisms this condition implies that the restriction $g_i|_I$ of $g_i$ to $I$ does not coincide with the identity on $I$).

2. The sequence of restricted maps $g_i|_I : I \to S^1$ converges to the identity on the $C^2$-topology over $I$.

Now we have:

**Claim 2.** The intersection $I \cap K$ is not empty.

**Proof of Claim 2.** Suppose that $I \cap K = \emptyset$ and denote by $\tilde{I}$ the connected component of $S^1 \setminus K$ containing $I$. The endpoints of $\tilde{I}$ belong to $K$ and are automatically fixed by every element of the subgroup $G_I$ of $G$ defined by

$$G_I = \{ g \in G : g(\tilde{I}) = \tilde{I} \}.$$  

Note that the diffeomorphisms $g_i$ belong to $G_I$ provided that $i$ is large enough. Thus, modulo dropping finitely many terms of the sequence $\{g_i\}$, we can assume that every $g_i$ fixes a chosen endpoint $p$ of $\tilde{I}$. Consider a neighborhood $U$ of $p$ and the pseudogroup $\Gamma_U$ induced on $U$ by restrictions of elements in $G_I$. Since $G_I$ fixes $p$, we can also consider the group $\Gamma_p$ of germs at $p$ of elements in $\Gamma_U$. In turn, since $p \in K$ and $K$ is invariant by $G$, there follows that the $C^m$-closure of $\Gamma_U$ contains neither (standard) Shcherbakov-Nakai vector fields (asymptotically defined on an one-sided interval starting at $p$) nor vector fields defined on neighborhood of $p$. Clearly $\Gamma_p$ is not trivial since it contains the germs at $p$ of the diffeomorphisms $g_i$. Lemma 2.6 then ensures that $\Gamma_p$ must be infinite cyclic. Next, on a neighborhood of $p$ all diffeomorphisms $g_i$ are locally given as maps induced by a unique (possibly formal) local flow $\Psi$ at specific times $t_i$. The additive subgroup of $\mathbb{R}$ generated by the times $t_i$ must be discrete, otherwise the local flow $\Psi$ would actually be defined for all $t \in \mathbb{R}$ and the associated analytic vector field would be in the closure of $\Gamma_p$ which is known to be impossible. Being discrete, the subgroup of $(\mathbb{R}, +)$ generated by the times $t_i$ has a generator $t_0 > 0$. Thus, the dynamics of the group $G_I$ on $\tilde{I}$ consists of the iterations of a single diffeomorphism having the endpoints of $\tilde{I}$ fixed. In particular, the orbit of every point in $\tilde{I}$ by the diffeomorphism in question converges to a fixed point of this diffeomorphism. This contradicts the assumption that the sequence $\{g_i\}$ converges to the identity on $I \subset \tilde{I}$. $\square$

To complete the proof of the proposition, we proceed as follows. According to a classical theorem due to Sacksteder [6], [29], there is a point $p \in K$ and a diffeomorphism $f \in G$ such that $f(p) = p$ and $0 < |f'(p)| < 1$. Since $I \cap K$ is not
empty and the dynamics of $G$ on $K$ is minimal, there is no loss of generality in supposing that $p \in I \cap K$. Now, by considering the pseudogroup $\Gamma$ generated on $I$ by $f$ and by the sequence of maps $g_{r,i}$, Proposition 2.8 ensures the existence of a nowhere zero vector field $X$ defined about $p$ and contained in the $C^1$-closure of $\Gamma$. This yields a contradiction since $K$ is a Cantor set supposed to be invariant by $G$ and, hence, by $\Gamma$. The resulting contradiction then proves our claim that a locally $C^2$-non-discrete group $G \subset \text{Diff}^\omega(S^1)$ cannot leave a Cantor set $K \subset S^1$ invariant.

Now there only remains to discuss further the case in which $G$ has a finite orbit. The very assumption that $G$ is locally $C^2$-non-discrete implies that $G$ cannot be finite. Thus the set $\mathcal{O}_G$ of Corollary 2.11 is finite. Let $I$ be a connected component of $S^1 \sim \mathcal{O}_G$ and consider the subgroup $G_I$ of $G$ consisting of diffeomorphisms fixing $I$. To finish the proof of Proposition 2.9 it suffices to check that the action of $G_I$ on $I$ has all orbits dense. Owing to Proposition 2.12, if this does not happen then $G_I$ must be infinite cyclic. Assuming that $G_I$ is infinite cyclic, this group is also locally non-discrete. Lemma 2.6 then shows that the orbits of $G_I$ on $I$ are still dense. Proposition 2.9 is proved. □

3. RIGIDITY IN THE PRESENCE OF POINTS WITH LARGE STABILIZERS AND RELATED CASES

The purpose of this section is to prove Theorem A in some specific cases related, for example, to the existence of finite orbits for a non-solvable group (say $G_1$). We shall also settle the case in which $G_1$ happens to be a solvable group. This material will reduce the proof of Theorem A to a generic situation where, roughly speaking, the group $G_1$ is not solvable and every point in $S^1$ has cyclic (possibly trivial) stabilizer; cf. Proposition 3.5 which is the main result of this section. The generic situation described by Proposition 3.5 is, however, substantially harder and will be detailed in the subsequent sections of this paper.

In the sequel, consider a locally $C^2$-non-discrete subgroup $G_1$ of $\text{Diff}^\omega(S^1)$. Then fix an interval $I \subset S^1$ and a sequence $\{g_{1,i}\}$ of elements in $G_1$ whose restrictions $\{g_{1,i|I}\}$ to $I$ converge to the identity in the $C^2$-topology (where the analytic diffeomorphisms $g_{1,i}$ are different from the identity for every $i$). Next, let $G_2$ be another subgroup of $\text{Diff}^\omega(S^1)$ that happens to be topologically conjugate to $G_1$. The reader is reminded that the conjugating homeomorphism $h$ is assumed to preserve the orientation of the circle.

Having fixed the sequence $\{g_{1,i|I}\}$, we consider the subgroup $G_{1,r} \subset G_1$ generated by the elements $g_{1,1}, \ldots, g_{1,r}$ for every fixed value of $r \in \mathbb{N}$ (notation: $G_{1,r} = \langle g_{1,1}, \ldots, g_{1,r} \rangle$). Also given a pair of integers $r, r_0$ with $r \geq r_0$, let $G_{1,(r,r_0)}$ denote the group generated by $g_{1,r_0}, \ldots, g_{1,r}$.

Note that the group $G_1$ is assumed to be non-abelian throughout this section. At certain points, we will also need the following technical assumption:

\textit{(FOG$_t$)} For every pair of integers $r, r_0$ with $r \geq r_0$, the group $G_{1,(r,r_0)}$ possesses a finite orbit but it is not a finite group.
Whenever $FOG_r$ is assumed to hold, it can be applied in particular to the groups associated with the pairs $r = r_0$ which are nothing but the cyclic groups generated by the diffeomorphisms $g_{1,r}$. Thus assumption $FOG_r$ implies also that all the diffeomorphisms $g_{1,r}$ have finite orbits though none of them has finite order.

**Lemma 3.1.** Assume that condition $(FOG_r)$ holds. Then there is $r_0 \in \mathbb{N}$ and a finite set $P = \{p_1, ..., p_l\} \subset S^1$ all of whose points are fixed for all the groups $G_{1,(r,r_0)}$.

**Proof.** Setting $r = r_0 = 1$, let $P_1 \subset S^1$ be the set of points having finite orbit under $G_{1,1} = \langle g_{1,1} \rangle$. Owing to Corollary 2.11, the set $P_1$ consists of finitely many points. Naturally, for every $r \geq 1$, the set of points with finite orbit under the group $G_{1,r}$ is contained in $P_1$ since $G_{1,1} \subset G_{1,r}$. Denoting by $P_r \subset S^1$ the set of points having finite orbit under $G_{1,r}$, we have $P_1 \supset P_2 \supset \cdots$ so that the intersection

$$P = \bigcap_{r=1}^{\infty} P_r$$

is contained in $P_1$. Furthermore this intersection is not empty since our assumption ensures that none of the sets $P_r$ is empty. Thus, to prove the lemma, it suffices to show that the diffeomorphisms $g_{1,i}$ fix all points in $P$ provided that $i$ is sufficiently large. For this let $I_1$ denote a connected component of $S^1 \sim P$ having non-empty intersection with the open interval $I$. Since $\{g_{1,i}\}$ converges to the identity on $I$, for $i$ large enough we must have $g_{1,i}(I_1) \cap I_1 \neq \emptyset$. Since, on the other hand, the set $P$ is invariant under $g_{1,i}$, it follows at once that $g_{1,i}$ fixes every point in $P$. The lemma is proved.

Still assuming that condition $(FOG_r)$ holds and owing to Lemma 3.1, let us fix a (non-empty) finite set $P$ and an integer $r_0 \in \mathbb{N}$ such that every point in $P$ is fixed by each diffeomorphism $\{g_{1,i}\}$ with $i \geq r_0$.

Next, let us also consider the group $G_2$ along with the homeomorphism $h$. We begin by letting $g_{2,i} = h^{-1} \circ g_{1,i} \circ h$ for every $i \in \mathbb{N}$. We also pose $G_{2,r} = \langle g_{2,1}, ..., g_{2,r} \rangle$ and $G_{2,(r,r_0)} = \langle g_{2,r_0}, ..., g_{2,r} \rangle$. Next recall that $P = \{p_1, ..., p_l\}$ and let $q_j = h^{-1}(p_j)$, for $j = 1, ..., l$. It is clear that the set $Q = \{q_1, ..., q_l\}$ is constituted by fixed points of $G_{2,r}$ for every $r \geq r_0$.

Now let $p_1 \in P$ and $q_1 = h^{-1}(p_1) \in Q$ be fixed. From what precedes, the stabilizer of $p_1$ (resp. $q_1$) contains all the groups $G_{1,(r,r_0)}$ (resp. $G_{2,(r,r_0)}$) where $r \geq r_0 \in \mathbb{N}$ and $r_0$ is as above. Now we shall consider a few different possibilities involving the algebraic structure of the groups $G_{1,(r,r_0)}$.

**Proposition 3.2.** Assume condition $(FOG_r)$ holds and let $r_0$ as above be fixed. Assume also the existence of $r > r_0 \in \mathbb{N}$ such that the group $G_{1,(r,r_0)}$ is not solvable. Then the conjugating homeomorphism $h$ coincides with a real analytic diffeomorphism of $S^1$.

**Proof.** Let $\Gamma_1$ (resp. $\Gamma_2$) denote the germ of $G_{1,(r,r_0)}$ (resp. $G_{2,(r,r_0)}$) at $p_1$ (resp. $q_1$). Naturally both groups $\Gamma_1, \Gamma_2$ can be identified with non-solvable subgroups of $\text{Diff}^\omega(\mathbb{R},0)$ which are (locally) topologically conjugate by a homeomorphism induced by the restriction of $h$. A result due to Nakai [28] ensures then that $h$
is real analytic on a neighborhood of $0 \equiv p_1$. Since $p_1$ is an arbitrary point in $P$, we conclude that $h$ is analytic on a neighborhood of every point in $P$. In fact, $h$ is analytic on a neighborhood of every fixed point of $G_1, (r, r_0)$ in $S^1$ (note that the set $\text{Fix}(G_1, (r, r_0))$ of fixed points of $G_1, (r, r_0)$ may strictly contain $P$).

Now Proposition 2.12 ensures that $G_1, (r, r_0)$ has dense orbits on the connected components of $S^1 \sim \text{Fix}(G_1, (r, r_0))$. From this it promptly follows that the local analytic character of $h$ about points in $P$ extends to all of $S^1$. The proof of our proposition is over. \qed

In view of Proposition 3.2, whenever condition $(FOG_r)$ is satisfied, we can assume without loss of generality the existence of $r_0 \in \mathbb{N}$ such that all the groups $G_1, (r, r_0)$ are solvable. Note that these groups may as well be abelian since the assumption that the group $G_1$ is not abelian does not immediately imply the same holds for the groups $G_1, r$, $G_1, (r, r_0)$.

Next we shall drop condition $(FOG_r)$ and work instead with the assumption of the existence of $r_0 \in \mathbb{N}$ such that all the groups $G_1, (r, r_0)$ are solvable. To abridge notation, up to dropping finitely many terms of the initial sequence $\{g_{1, i}\}$, we can simply assume that all the groups $G_1, r$ are solvable. In the sequel the following well-known lemma on solvable subgroups of $\text{Diff}^\alpha(S^1)$ will be needed.

**Lemma 3.3.** Let $G \subset \text{Diff}^\alpha(S^1)$ be a finitely generated solvable subgroup of $\text{Diff}^\alpha(S^1)$. Then either $G$ has a finite orbit or it is topologically conjugate to a group of rotations.

**Proof.** Since $G$ is solvable, its action on $S^1$ preserves a probability measure $\mu$. Hence the support $\text{Supp}(\mu)$ of $\mu$ is a closed subset of $S^1$ invariant by $G$. Consider a minimal set $\mathcal{M}$ for $G$ contained in $\text{Supp}(\mu)$. In view of Proposition 2.10, $\mathcal{M}$ must be of one of the following types: the entire circle, a finite set or a Cantor set. Suppose first that $\mathcal{M}$ coincides with all of $S^1$. Consider then the Radon-Nikodym derivative of $\mu$ in the sense of distributions. By parameterizing the circle by the resulting cumulative distribution function a topological conjugation between $G$ and a rotation group of $S^1$ can be constructed (in particular $G$ is abelian). In turn, if the support of $\mu$ is a finite set, then this set is invariant by $G$ so that this group has finite orbits. Hence the proof of our lemma is reduced to checking that $\mathcal{M}$ cannot be a Cantor set. Since $G$ is finitely generated this last assertion follows from Sacksteder’s theorem; see [6], [29]. In other words, if $\mathcal{M}$ is a Cantor set, then there is an element $g \in G$ and a point $p \in \mathcal{M}$ such that $p$ is a hyperbolic fixed point for $g$. Now, since $g$ preserves $\mu$ and $p \in \text{Supp}(\mu) = \mathcal{M}$, there follows that the point $p$ must have strictly positive $\mu$-mass. However the measure $\mu$ is invariant by $G$ and finite which, in turn, forces the orbit of $p$ to be finite itself thus completing the proof of the lemma. \qed

Recall that all the group $G_1, r$ are assumed to be solvable.

**Proposition 3.4.** Assume that all the groups $G_1, r$ are solvable. Assume also the existence of $r_1 \in \mathbb{N}$ such that the (solvable) group $G_1, r_1$ is infinite but has a finite orbit. Then the homeomorphism $h$ conjugating $G_1$ to $G_2$ coincides with an analytic diffeomorphism of $S^1$. 

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Proof: Note that $G_{1,r_1}$ cannot be conjugate to a group of rotations since a group of rotations possessing a finite orbit must itself be finite. Thus for every $r \geq r_1$, Lemma 3.3 ensures that $G_{1,r}$ is a solvable group having a finite orbit. By intersecting these finite orbits over $r \geq r_1$, we derive the existence of a non-empty finite set all of whose points have finite orbit under $G_{1,r}$, for every $r \geq r_1$.

Let $p \in S^1$ be a point having finite orbit under $G_{1,r}$ for every $r \geq r_1$. Denote by $G_{1,r}^{(p)}$ the stabilizer of $p$ in $G_{1,r}$. Clearly $G_{1,r}^{(p)}$ induces a solvable subgroup $\Gamma_{p,r}$ of $\text{Diff}^\omega(\mathbb{R},0)$. Besides, with suitable identifications, the restriction of $h$ to a neighborhood of $p$ topologically conjugates $\Gamma_{p,r}$ to another subgroup $\Gamma_{q,r}$ of $\text{Diff}^\omega(\mathbb{R},0)$. Again the proof of the proposition becomes reduced to checking that the homeomorphism (still denoted by $h$) conjugating $\Gamma_{p,r} < \text{Diff}^\omega(\mathbb{R},0)$ to $\Gamma_{q,r} < \text{Diff}^\omega(\mathbb{R},0)$ must be analytic on a neighborhood of $0 \in \mathbb{R}$. For this, let us consider the following possibilities:

**Case 1.** Suppose that $\Gamma_{p,r}$ (an thus $\Gamma_{q,r}$) is not abelian. From the description of solvable subgroups of $\text{Diff}^\omega(\mathbb{R},0)$, there follows that solvable non-abelian subgroups of $\text{Diff}^\omega(\mathbb{R},0)$ have elements $f_1, g_1$ satisfying the following conditions (see for example [14]):

- $f_1$ has a hyperbolic fixed point at $0 \in \mathbb{R}$.
- $g_1$ is tangent to the identity at $0 \in \mathbb{R}$ (though $g_1 \neq \text{id}$).

According to Lemma 2.5, the local diffeomorphisms $f_1, g_1$ can be combined to construct a (non-identically zero) analytic vector field $X_1$ defined on a neighborhood of $0 \in \mathbb{R}$ and contained in the closure of $\Gamma_{p,r}$. A similar vector field $X_2$ can be defined by means of the elements $f_2 = h^{-1} \circ f_1 \circ h$ and $g_2 = h^{-1} \circ g_1 \circ h$ of $\Gamma_{q,r}$. Indeed the above mentioned structure of solvable subgroups of $\text{Diff}^\omega(\mathbb{R},0)$ also ensures that $f_2 = h^{-1} \circ f_1 \circ h$ is hyperbolic whereas $g_2 = h^{-1} \circ g_1 \circ h$ is tangent to the identity. By using the fact that $h$ conjugates the actions of $\Gamma_{p,r}, \Gamma_{q,r}$, there follows from the indicated constructions that $h$ conjugates $X_1$ to $X_2$ in a time-preserving manner. Thus $h$ must be analytic about $0 \in \mathbb{R}$ and this establishes the proposition in the first case.

**Case 2.** Suppose now that $\Gamma_{p,r}$ (and thus $\Gamma_{q,r}$) is an abelian subgroup for every $r \geq r_1$. Note that $\Gamma_{p,r}$ is not a finite. In fact, $G_{1,r_1} \leq G_{1,r}$ and the stabilizer of $p$ with respect to $G_{1,r_1}$ is already infinite since the orbit of $p$ under $G_{1,r_1}$ is finite whereas $G_{1,r_1}$ is an infinite group. Moreover all the groups $G_{1,r}^{(p)}$ are abelian as well ($r \geq r_1$). Now define the abelian group $G_{1,\infty}$ by letting

$$G_{1,\infty} = \bigcup_{r=r_1}^{\infty} G_{1,r}^{(p)}.$$

Next fix an element $f \in G_{1,\infty}$ different from the identity. Clearly $f(p) = p$ so that the set of fixed points of $f$ is non-empty and finite since $f$ is analytic. Because $G_{1,\infty}$ is an abelian group, the set of fixed points of $f$ is preserved by $G_{1,\infty}$. Therefore every diffeomorphism in the sequence $\{g_{1,i}\}$ must fix every point that is fixed by $f$ provided that $i$ is very large. In particular the stabilizer $G_{1,\infty}^{(p)}$ of $p$ in $G_{1,\infty}$ is an abelian group containing all the diffeomorphisms $g_{1,i}$ for large enough $i$. There follows that $G_{1,\infty}^{(p)}$ is non-discrete since $\{g_{1,i}\}$ converges to the
identity on $I$. Since $G^{(p)}_{1,\infty}$ naturally induces a subgroup of $\text{Diff}^\omega(S^1)$, we can now resort to Lemma 2.6 in Section 2 to produce vector fields $X_1, X_2$ defined around $p$ and $q$ respectively that are conjugated by $h$ in a time-preserving manner. Hence $h$ is again real analytic around $0 \in \mathbb{R}$. The proposition is proved.

To finish this section we shall establish a last reduction to the proof of Theorem A in the form of Proposition 3.5. This proposition will also summarize the preceding lemmata. To state it, recall that $I \subset S^1$ is a fixed interval for which $G_1$ contains a sequence of analytic diffeomorphisms $(g_{1,i})$, $(g_{1,i} \neq \text{id})$, whose restrictions $(g_{1,i}|_I)$ to $I$ converge to the identity in the $C^2$-topology.

**Proposition 3.5.** Let $\varepsilon > 0$ be an a priori given constant. To prove Theorem A, we can assume without loss of generality that all of the following hold:

- All the diffeomorphisms $(g_{1,i})$ are $\varepsilon$-close to the identity in the $C^2$-topology on the interval $I$.
- There is $N \in \mathbb{N}$ for which the group generated by $(g_{1,1}, \ldots, g_{1,N})$ is not solvable.
- No point $p \in S^1$ is simultaneously fixed by all diffeomorphisms $g_{1,1}, \ldots, g_{1,N}$.
- In general, every finite subset generating a non-solvable subgroup of $G_1$ cannot have a common fixed point.

To prove Proposition 3.5, let $\varepsilon > 0$ be given. Up to dropping finitely many terms from the sequence $(g_{1,i})$, it is clear that we can assume all the diffeomorphisms $(g_{1,i})$ to be $\varepsilon$-close to the identity in the $C^2$-topology on the interval $I$. Recall also that none of the analytic diffeomorphisms $g_{1,i}$ coincides with the identity (in particular their restrictions to the interval $I$ do not coincide with the identity either).

Assume there is $r_0 \in \mathbb{N}$ such that the group $G_{1,r_0}$ is not solvable. Owing to Proposition 3.2, Theorem A holds provided that the non-solvable group $G_{1,r_0}$ possesses a finite orbit. More generally, Proposition 3.2 also justifies the last assertion in the statement of Proposition 3.5. In other words, to establish Proposition 3.5 it suffices to show that Theorem A holds provided that all the groups $G_{1,r}$ are solvable ($r \in \mathbb{N}$). This will be our aim in the remainder of this section.

To begin with, recall the general fact that every finite subgroup of $\text{Diff}^\omega(S^1)$ is analytically conjugate to a rotation group. Also Lemma 3.3 informs us that every finitely generated infinite solvable group having no finite orbit is topologically conjugate to a subgroup of the rotation group. By virtue of Proposition 3.4, we can therefore assume that each individual group $G_{1,r}$ is abelian and topologically conjugate to a group of rotations. In particular each diffeomorphism $g_{1,i}$ is topologically conjugate to a rotation and hence has no fixed points since none of them coincides with the identity.

Consider again the group $G_{1,\infty} = \bigcup_{r=1}^{\infty} G_{1,r}$. In the present setting, $G_{1,\infty}$ is clearly an infinite locally non-discrete abelian group all of whose orbits are infinite. Although it is infinitely generated, the action of $G_{1,\infty}$ still preserves a probability measure $\mu_\infty$. To check this claim consider a probability measure $\mu_r$ invariant by $G_{1,r}$. Next take $\mu_\infty$ as an accumulation point of the sequence $\{\mu_r\}$. The fact that $G_{1,r} \subset G_{1,r+1}$ promptly implies that $\mu_\infty$ must be invariant by
$G_{1,r}$ for every $r \in \mathbb{N}$ and is thus invariant by $G_{1,\infty}$. Moreover $\mu_\infty$ has no atomic component since $G_{1,\infty}$ has no finite orbit. Next we have:

**Lemma 3.6.** The support $\text{supp}(\mu_\infty)$ of $\mu_\infty$ coincides with all of $S^1$.

*Proof.* The proof amounts to showing that $\text{supp}(\mu_\infty) \subset S^1$ cannot be a Cantor set. As pointed out by the referee, however, we cannot apply Sacksteder theorem ([6], [29]) to conclude the existence of an element in $G_{1,\infty}$ having a hyperbolic fixed point since $G_{1,\infty}$ is *not* finitely generated. In fact, there are examples of (infinitely generated) abelian groups of smooth diffeomorphisms with invariant Cantor sets and having no element exhibiting a hyperbolic fixed point.

In our case however we can proceed as follows. Assume aiming at a contradiction that $\text{supp}(\mu_\infty) \subset S^1$ is a Cantor set. Recall also that the sequence $\{g_{1,i}\}$, $(g_{1,i} \neq \text{id})$ converges to the identity on an interval $I$. Consider then a connected component $J$ of $S^1 \sim \text{supp}(\mu_\infty)$ intersecting $I$ non-trivially. Since $\{g_{1,i}\}$ converges to the identity on $I$, we have $J \cap g_{1,i}(J) \neq \emptyset$ for sufficiently large $i$. Owing to the invariance of $\text{supp}(\mu_\infty)$, there follows that, in fact, $g_{1,i}(J) = J$ provided that $i$ is large enough. Thus for $i$ large enough, the diffeomorphisms $g_{1,i}$ must fix the endpoints of $J$. This is however impossible since these diffeomorphisms have no fixed points, as previously pointed out. The resulting contradiction establishes the lemma.

Owing to Lemma 3.6, the distributional Radon-Nikodym derivative of $\mu_\infty$ allows us to construct a topological coordinate $H$ on $S^1$ in which $G_{1,\infty}$ becomes a group of rotations. Furthermore the following holds:

**Lemma 3.7.** In the topological coordinate $H$, the group $G_{1,\infty}$ is a dense subgroup of the group of all rotations of $S^1$.

*Proof.* Consider the map $\rho : G_{1,\infty} \to \mathbb{R}/\mathbb{Z}$ assigning to an element $g \in G_{1,\infty}$ its rotation number. Because $G_{1,\infty}$ is an abelian group, the map $\rho$ is a homomorphism so that its image $\rho(G_{1,\infty}) \subset S^1$ is a dense set of $S^1$ viewed as a multiplicative group. Moreover, the homomorphism $\rho$ is injective since, in the coordinate $H$, the rotation corresponding to an element $g \in G_{1,\infty}$ is nothing but the rotation of angle equal to the rotation number of $G$. The lemma then follows from the fact that the subgroup $\rho(G_{1,\infty})$ is clearly infinite. \hfill $\Box$

The next lemma is also elementary.

**Lemma 3.8.** Suppose that $g : S^1 \to S^1$ is a homeomorphism of the circle that commutes with a dense set $E$ of rotations. Then $g$ is itself a rotation.

*Proof.* Passing to the $C^0$-limit, there follows at once that $g$ commutes with all the rotations. Hence, denoting by $R_x$ the rotation of angle $x$, we have $g(x) = g(R_x(0)) = R_x(g(0)) = R_x(R_{g(0)}(x)).$ Thus $g = R_{g(0)}$ and the statement follows. \hfill $\Box$

Let us close this section with the proof of Proposition 3.5

*Proof of Proposition 3.5.* We recall that the group $G_1$ is assumed to be non-abelian. The proof amounts to showing that the initial sequence of diffeomorphisms $\{g_{1,i}\} \subset G_1$ can be chosen so as to ensure that for large enough $r \in \mathbb{N}$ the group $G_{1,r}$ cannot be topologically conjugate to a group of rotations. For this
we assume aiming at a contradiction that $G_{1,r}$ is topologically conjugate to a group of rotations for every $r \in \mathbb{N}$.

We begin by observing that the above assumption implies that the action of $G_1$ on $S^1$ has all orbits dense thanks to Lemma 3.7. On the other hand, since $G_1$ is not abelian, a classical result due to Hölder ensures the existence of an element $f \in G_1$, $f \neq \text{id}$, possessing a fixed point $p \in S^1$, see [23]. Moreover, since $G_1$ has dense orbits on $S^1$, we can assume without loss of generality that $p$ belongs to $I$.

Given the initial sequence $(g_{1,i}) \subset G_1$, we also consider the diffeomorphisms of the form $f^{-1} \circ g_{1,i} \circ f$. Clearly the sequence of diffeomorphisms $(f^{-1} \circ g_{1,i} \circ f)$ converges to the identity on $f^{-1}(I)$. However, since $f(p) = p$ and $p \in I$, up to reducing $I$, we can assume that the sequence $(f^{-1} \circ g_{1,i} \circ f)$ converges to the identity on all of $I$. Thus, up to reorganizing the indices $i$ (and dropping a finite number of them), we can assume that all the diffeomorphisms $f^{-1} \circ g_{1,i} \circ f$ are contained in the initial sequence $(g_{1,i})$. With this new definition of the sequence $(g_{1,i})$, the following holds:

**Claim 3.** The group $G_{1,\infty}$ is no longer topologically conjugate to a group of rotations.

**Proof of Claim 3.** Suppose for a contradiction that $G_{1,\infty}$ is abelian without finite orbits. Note that the new sequence $(g_{1,i})$ contains of two sets of diffeomorphisms, one denoted now by $(g_{1,k})$ and the other given by the diffeomorphisms $(f^{-1} \circ g_{1,k} \circ f)$, $k \in \mathbb{N}$.

Having fixed $k$, the diffeomorphisms $g_k$ and $f^{-1} \circ g_k \circ f$ have the same rotation number. They must therefore coincide since the “rotation number homomorphism” from $G_{1,\infty}$ to $S^1$ is one-to-one (cf. the proof of Lemma 3.7). In other words, the diffeomorphism $f$ commutes with $g_{1,k}$ for every $k \in \mathbb{N}$.

Now recall the existence of a topological coordinate $H$ where $G_{1,\infty}$ is identified with a dense group of rotations. Let $\Gamma$ be the subgroup of $G_{1,\infty}$ generated by all the elements $g_k$, $k \in \mathbb{N}$ and note that $\Gamma$ is still dense in the group of all rotations of $S^1$. Finally, always working in the coordinate $H$, the diffeomorphism $f$ commutes with all the diffeomorphisms $g_k$, and hence it commutes with a dense group of rotations. Lemma 3.8 then ensures that $f$ is itself a rotation in the coordinate $H$. Since $f$ has a fixed point, it must coincide with the identity which yields a contradiction completing the proof of the claim.

The proposition now results by repeating word-by-word the preceding discussion.

### 4. Convergence estimates for sequences of commutators

This section is devoted to providing an algorithmic way to construct diffeomorphisms converging to the identity on a suitably fixed interval. This algorithmic construction will allow for a more effective use of the assumption that our groups are locally non-discrete and it is convenient to add some explanation in this direction. Consider a locally $C^2$-non-discrete group $G \subset \text{Diff}^n(S^1)$. This means there is an interval $I \subset S^1$ and a sequence of elements $(g_i) \subset G$ satisfying
the conditions of Definition 2.1. Definition 2.1 however has the inconvenient that the sequence \( \{ g_i \} \) is \textit{a priori given} and this prevents us from having any additional control on the behavior of the diffeomorphisms \( g_i \). For example, we have no information whatsoever on the higher order derivatives of \( g_i \) and, in particular, no information on the growing rate of the sequence \( \| g_i \|_3 \), where \( \| \cdot \|_3 \) stands for the \( C^3 \)-norm. In the context of Theorem A, if \( \{ g_{1,i} \} \) is a sequence as above for the group \( G_1 \), then the corresponding sequence \( g_{2,i} = h^{-1} \circ g_{1,i} \circ h \) of elements in \( G_2 \) is known to converge to the identity only in the \( C^0 \)-topology. Nonetheless to derive non-trivial implications on the regularity of \( h \), it is natural to look for sequences as above such that \( \{ g_{2,i} \} \) forms a convergent sequence in stronger topologies as well. The main immediate virtue of the construction carried out below is to yield some estimates on the growing rate of the sequence formed by higher order derivatives of \( g_i \). These estimates will prove to be crucial for the proof of Theorem A. Finally we also point out that the mentioned construction will enable us to give a partial answer to some questions raised in [8], cf. Appendix (Section 7).

To make our discussion accurate, we place ourselves in the context of a locally \( C^2 \)-non-discrete group \( G \subset \text{Diff}^\omega(S^1) \) satisfying the conditions in Proposition 3.5. Hence we fix some interval \( I \subset S^1 \) and a collection \( S \subset G \) of elements \( g_1, \ldots, g_N \) generating a non-solvable subgroup. The diffeomorphisms \( g_i, i = 1, \ldots, N \) are also assumed to be \( \epsilon \)-close to the identity in the \( C^2 \)-topology on the interval \( I \), where the value of \( \epsilon > 0 \) will be fixed only later. Our purpose is to produce an explicit sequence of diffeomorphisms converging to the identity out of the finite set \( S = \{ g_1, \ldots, g_N \} \). In turn, the idea to obtain the desired sequence consists of iterating commutators. This will be a slight refinement of the method employed by Ghys in [17] which relies on a fast iteration technique. Indeed, the difficulty in proving convergence to the identity of sequences of iterated commutators lies in the fact that an estimate of the \( C^m \)-norm of a commutator \( [f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1} \) requires estimates on the \( C^{m+1} \)-norm of \( f_1, f_2 \). To establish the convergence of a sequence of “iterated commutators” becomes therefore tricky as at each step there is an intrinsic loss of one derivative. It is thus natural to try to overcome this difficulty by means of some suitable fast iteration scheme. The method of Ghys [17] consists then of using holomorphic extensions and the topology of uniform convergence for these extensions in order to take advantage of Cauchy formula. Owing to Cauchy formula, we can substitute the loss of one derivative by the loss of a portion of the domain of definition: hence we only need to check that the size of the region lost in the domain of definition at each step of the iteration scheme decreases fast enough to ensure that some non-empty domain is kept at the end.

Since we will work only with \( C^2 \)-convergence the same fast iteration scheme is not available, albeit some adaptations are still possible. We prefer however to introduce a slightly more elaborated iterative procedure which avoids fast convergence estimates. The idea is to add a step of \textit{renormalization} at each stage of the commutator iteration. This renormalization step has a regularizing
effect on derivatives of order two or greater. A simplified version of the same idea was already used in the proof of Proposition 2.8. One advantage of our procedure is to avoid the loss of derivatives; other advantages will become clear in the course of the discussion and these include the convergence rate to the identity of the resulting sequence; see Remark 4.7.

After this brief overview of the upcoming discussion, we begin to provide accurate definitions. We shall work with the pseudogroup generated by \( S = \langle \overline{g}_1, \ldots, \overline{g}_N \rangle \) on the interval \( I \subset \mathbb{R} \) where \( \overline{g}_1, \ldots, \overline{g}_N \) generate a non-solvable group. Also, and whereas we shall primarily think of \( \overline{g}_1, \ldots, \overline{g}_N \) as maps defined on \( I \), it is sometimes useful to keep in mind that the maps in questions are nothing but restrictions to \( I \) of global analytic diffeomorphisms of \( S^1 \) (still denoted by \( \overline{g}_1, \ldots, \overline{g}_N \), respectively).

According to Ghys [17], with the set \( S = \langle \overline{g}_1, \ldots, \overline{g}_N \rangle \) is associated a sequence of sets \( S(k), k = 1, 2, \ldots, \) inductively defined as follows:

- \( S(0) = S \),
- \( S(k) \) is the set whose elements are commutators of the form \( [\hat{f}_i^{\pm 1}, \hat{f}_j^{\pm 1}] \), where \( \hat{f}_i \in S(k - 1) \) and \( \hat{f}_j \in S(k - 1) \cup S(k - 2) \) (\( \hat{f}_j \in S(0) \) if \( k = 1 \)).

It follows from [17] that the resulting sequence of sets \( S(k) \) is never reduced to the identity since \( S = \langle \overline{g}_1, \ldots, \overline{g}_N \rangle \) generates a non-solvable group. This also yields the following:

**Lemma 4.1.** For every \( k \in \mathbb{N} \), the subgroup generated by \( S(k) \cup S(k - 1) \) is non-solvable.

**Proof.** Assume there were \( k \in \mathbb{N} \) such that \( \Gamma = \langle S(k) \cup S(k - 1) \rangle \) is solvable, where \( \langle S(k) \cup S(k - 1) \rangle \) stands for the group generated by \( S(k) \cup S(k - 1) \). Since \( \Gamma \subset \text{Diff}^p(S^1) \), there follows that \( \Gamma \) is, indeed, metabelian, i.e. its derived group \( D^1 \Gamma \) is abelian. Recalling that \( D^1 \Gamma \) is the group generated by all commutators of the form \( [\gamma_1, \gamma_2] \) where \( \gamma_1, \gamma_2 \in \Gamma \), there follows that the sets \( S(k + 1) \) and \( S(k + 2) \) are contained in \( D^1 \Gamma \). Since \( D^1 \Gamma \) is abelian, the definition of the sequence of sets \( \{S(k)\} \) promptly implies that the set \( S(k + 3) \) must coincide with \( \{\text{id}\} \). Hence the initial group generated by \( \overline{g}_1, \ldots, \overline{g}_N \) must be solvable. The resulting contradiction proves the lemma. \( \square \)

By virtue of Proposition 3.5, we obtain the following corollary:

**Corollary 4.2.** In order to prove Theorem A, we can assume that the elements in \( S(k) \cup S(k + 1) \) do not share a common fixed point (and this holds for every \( k \in \mathbb{N} \)).

From now on, we set \( I = [-a, a] \subset \mathbb{R}, a > 0 \), with the obvious identifications. Given \( \varepsilon > 0 \), we permanently fix a set of diffeomorphisms \( \overline{g}_1, \ldots, \overline{g}_N \) generating a non-solvable group and \( \varepsilon \)-close to the identity in the \( C^2 \)-topology on \( I \). The value of \( \varepsilon > 0 \) convenient for our purposes will only be fixed later. In the remainder of the section these conditions are assumed to hold without further comments.
Unless otherwise mentioned, in what follows we shall say that \( f : I' \subseteq I \subset \mathbb{R} \to \mathbb{R} \) is a diffeomorphism meaning that \( f \) is a diffeomorphism from \( I' \subset \mathbb{R} \) to \( f(I') \subset \mathbb{R} \). Let us begin our discussion by stating a simple general lemma.

**Lemma 4.3.** Given \( \varepsilon_0 > 0 \) small and \( m \geq 1 \), there is a neighborhood \( \mathcal{U}_0^m \) of the identity in the \( C^m \)-topology on \( I \) such that the commutator \( [f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1} \) between diffeomorphisms \( f_1, f_2 \in \mathcal{U}_0^m \) satisfies the two conditions below:

- Viewed as an element of the pseudogroup generated by \( f_1, f_2 \) on \( I \), the map \( [f_1, f_2] \) is well defined on \([-a + 5\varepsilon_0, a - 5\varepsilon_0]\).
- There is a constant \( C > 0 \) such that the \( C^{m-1} \)-distance

\[
\| [f_1, f_2] - \text{id} \|_{m-1, [-a+5\varepsilon_0, a-5\varepsilon_0]} \]

from \([f_1, f_2]\) to the identity on the interval \([-a + 5\varepsilon_0, a - 5\varepsilon_0]\) satisfies the estimate

\[
\| [f_1, f_2] - \text{id} \|_{m-1, [-a+5\varepsilon_0, a-5\varepsilon_0]} < C \| f_1 - \text{id} \|_{m, [-a, a]} \| f_2 - \text{id} \|_{m, [-a, a]},
\]

where \( \| f_1 - \text{id} \|_{m, [-a, a]} \) (resp. \( \| f_2 - \text{id} \|_{m, [-a, a]} \)) stands for the \( C^m \)-distance from \( f_1 \) (resp. \( f_2 \)) to the identity on the interval \( I = [-a, a] \).

The reader will note that the constant \( C \) in the above lemma depends only on the neighborhood \( \mathcal{U}_0^m \). In particular \( C \) does not increase when the neighborhood is reduced.

We now focus on the case \( m = 2 \) (see Section 7 for a more general discussion). Since we can always reduce \( \varepsilon > 0 \), the neighborhood \( \mathcal{U}_0^2 \) can be chosen as

\[
\mathcal{U}_0^2 = \{ f \in C^2([-a, a]) ; \| f - \text{id} \|_{2, [-a, a]} < \varepsilon \}
\]

where \( C^2([-a, a]) \) stands for the space of \( C^2 \)-functions defined on \([-a, a]\) and taking values in \( \mathbb{R} \). For this neighborhood \( \mathcal{U}_0^2 \), the constant provided by Lemma 4.3 will be denoted by \( C \) and the value of \( C \) does not increase when \( \varepsilon \) decreases.

Now we state a simple complement to Lemma 4.3:

**Lemma 4.4.** Up to reducing \( \varepsilon > 0 \), for every pair \( f_1, f_2 \in \mathcal{U}_0^2 \) the second derivative \( D^2 [f_1, f_2] \) of the commutator \( [f_1, f_2] \) on the interval \([-a + 5\varepsilon_0, a - 5\varepsilon_0]\) satisfies the estimate

\[
\sup_{x \in [-a+5\varepsilon_0, a-5\varepsilon_0]} |D^2 [f_1, f_2]| \leq 5 \max_{x \in (-a, a)} \left( |D^2 f_1|, |D^2 f_2|, \right),
\]

where \( D^2 f_j \) stands for the second derivative of \( f_j \), \( j = 1, 2 \).

**Proof.** The proof is elementary and we shall summarize the argument. For \( j = 1, 2 \), the very definition of \( \mathcal{U}_0^2 \) yields (see (1))

\[
1 - \varepsilon \leq |D^1 f_j| \leq 1 + \varepsilon \quad \text{and} \quad \frac{1}{1 + \varepsilon} \leq \frac{1}{|D^1 f_j|} \leq \frac{1}{1 - \varepsilon}
\]

for every \( x \in [-a, a] \). Concerning the inverses of \( f_1, f_2 \), we also have

\[
D^1 f_j^{-1} = \frac{1}{D^1 f_j^{-1}(x)} f_j \quad \text{and} \quad D^2 f_j^{-1} = -\frac{D^2 f_j^{-1}(x)}{|D^1 f_j^{-1}(x)|^3}.
\]
In particular
\[ |D_x^1 f_j^{-1}| \leq \frac{1}{1-\varepsilon} \quad \text{and} \quad |D_x^2 f_j^{-1}| \leq \frac{1}{(1-\varepsilon)^3} |D_{f_j^{-1}}(x)|. \]

Next we compute the second derivative of \([f_1, f_2]\) at a point belonging to \([-a + 5\varepsilon_0, a - 5\varepsilon_0]\). In this calculation, the points at which the several derivatives are evaluated will be omitted: since \([f_1, f_2]\) is well defined on \([-a + 5\varepsilon_0, a - 5\varepsilon_0]\), it suffices to know that all these points belong to the interval \((-a, a)\). We also observe that the very idea of providing estimates for first and second derivatives (e.g. estimating \(\varepsilon\) for the neighborhood \(\mathcal{N}_0^2\)) makes implicit use of an auxiliary metric defined on the circle which, in our case, is the flat metric. In particular the tangent maps (\(D^1 f_1, D^2 f_1, D^1 f_2, D^2 f_2\)) induced by \(f_1, f_2\) on the first and second order jet bundles on the circle can be identified with real numbers. Accordingly, when applying below the chain rule to estimate the differential of the commutator \([f_1, f_2]\), the “dot” used in our notation should properly be understood as standing for the composition of these maps on the corresponding jet bundles. All this said, we have \(D^1 [f_1, f_2] = D^1 f_1.D^1 f_2.D^1 f_1^{-1}.D^1 f_2^{-1}\) and thus
\[
D^2 [f_1, f_2] = D^2 f_1.(D^1 f_2)^2.(D^1 f_1^{-1})^2.(D^1 f_2^{-1})^2 + D^1 f_1.D^2 f_2.(D^1 f_1^{-1})^2.(D^1 f_2^{-1})^2 \\
+ D^1 f_1.D^1 f_2.D^2 f_1^{-1}.(D^1 f_2^{-1})^2 + D^1 f_1.D^1 f_2.D^1 f_1^{-1}.D^2 f_2^{-1}.
\]

Therefore, on \([-a + 5\varepsilon_0, a - 5\varepsilon_0]\), we have
\[
|D^2 [f_1, f_2]| \leq \max_{x \in [-a, a]} \left[ \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^4} + \frac{1 + \varepsilon}{(1 - \varepsilon)^4} + \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^4} + \frac{1}{(1 - \varepsilon)^4} \right] |D^2 f_1||D^2 f_2|.
\]

Up to choosing \(\varepsilon\) sufficiently small, there follows that
\[
|D^2 [f_1, f_2]| \leq 5 \max \{|D^2 f_1||D^2 f_2|\}
\]
proving the lemma.

Let us now begin the construction of a sequence of diffeomorphisms in \(G\) converging to the identity in the \(C^2\)-topology on \(I = [-a, a]\). First recall that non-solvable subgroups of \(\text{Diff}^n(S^1)\) are known to have elements with hyperbolic fixed points (see for example [13]). Let then \(F \in G\) be a diffeomorphism satisfying \(F(0) = 0\) and \(F'(0) = \lambda \in (0, 1)\). The next step is to define a new sequence \([\tilde{S}(k)]\) of subsets of \(G\). The sequence \([\tilde{S}(k)]\) will depend on a fixed integer \(n \in \mathbb{N}^*\) which will be omitted in the notation. To define the sequence \([\tilde{S}(k)]\) we proceed as follows:

- \(\tilde{S}(1)\) is the set formed by the commutators having the form \([F^{-n} \circ \tilde{f}_1 \circ F^n, F^{-n} \circ \tilde{f}_2 \circ F^n]\) where \(\tilde{f}_1, \tilde{f}_2 \in S\). Thus \(\tilde{S}(1) = F^{-n} \circ S(1) \circ F^n\).
- \(\tilde{S}(k)\) is the set formed by the commutators \([F^{-n} \circ \tilde{f}_1 \circ F^n, F^{-n} \circ \tilde{f}_2 \circ F^n]\) with \(\tilde{f}_1, \tilde{f}_2 \in \tilde{S}(k-1)\) and by the commutators \([F^{-n} \circ \tilde{f}_1 \circ F^n, F^{-2n} \circ \tilde{f}_2 \circ F^{2n}]\) with \(\tilde{f}_1 \in \tilde{S}(k-1)\) and \(\tilde{f}_2 \in \tilde{S}(k-2)\).

In other words, the sequence \([\tilde{S}(k)]\) verifies \(\tilde{S}(k) = F^{-kn} \circ S(k) \circ F^{kn}\) for every \(k \in \mathbb{N}\). Taking advantage of the fact that all our local diffeomorphisms are realized as global diffeomorphisms of the circle, we obtain the following:
Lemma 4.5. The sequence of sets $\tilde{S}(k)$ never degenerates into $\{id\}$.

Proof. When all the diffeomorphisms in question are globally viewed as diffeomorphisms of the circle, the set $\tilde{S}(k)$ is conjugate to the set $S(k)$, for every $k \in \mathbb{N}$. The statement follows then from Ghys theorem claiming that the initial sequence $S(k)$ cannot degenerate into $\{id\}$ provided that $G$ is non-solvable. □

The global realizations of our diffeomorphisms ensure that the domain of definition of elements in $\tilde{S}(k)$ are always non-empty as every diffeomorphism is clearly defined on all of $S^1$. However, going back to our local setting where the initial $C^2$-maps $\overline{g}_1, \ldots, \overline{g}_N$ are defined on $[-a,a]$ and where the domains of definition for their iterates are understood in the sense of pseudogroup, the content of the last statement becomes unclear. In other words, in the context of pseudogroups, the statement of Lemma 4.5 is only meaningful for those elements in $\tilde{S}(k)$ having non-empty domain of definition when viewed as elements of the pseudogroup in question. In any event, the estimates developed below will show that this is always the case provided that we start with a sufficiently small $\varepsilon > 0$.

The central result of this section is Proposition 4.6 below. To state it accurately and to make its content promptly available to the reader, it is convenient to explicitly summarize the conditions on which this proposition is based. In particular, since we will be dealing with a statement about convergence of sequences to the identity, there is no need to guarantee that the corresponding diffeomorphisms are different from the identity. This remark enables us to formulate the desired proposition in the context of pseudogroups. Consider a fixed interval $I = [-a,a] \subset \mathbb{R}$ ($a > 0$) along with $C^2$-diffeomorphisms $F$ and $\overline{g}_1, \ldots, \overline{g}_N$ from a neighborhood of $I = [-a,a]$ to $\mathbb{R}$. Furthermore assume that the following holds:

a) We have $F(0) = 0$ and $F'(0) = \lambda \in (0,1)$. Moreover, $\overline{g}_1, \ldots, \overline{g}_N$ are $\varepsilon$-close to the identity in the $C^2$-topology on $I = [-a,a]$ (for some $\varepsilon > 0$ to be fixed later).

b) The above defined sequence of sets $S(j)$ starts from $S(0) = \overline{g}_1, \ldots, \overline{g}_N$.

c) There are sequences $\{f_j\}$ and $\{g_j\}$ of elements in the pseudogroup generated on $I$ by $F$ and by $\overline{g}_1, \ldots, \overline{g}_N$ such that $f_j \in S(j)$ for every $j$. Moreover, for every $j$ we have $g_j = F^{-jn} \circ f_j \circ F^{jn}$ for some fixed $n \in \mathbb{N}$.

Proposition 4.6. Under the above assumptions, there are $\varepsilon > 0$ and $n \in \mathbb{N}$ such that for every sequence $\{f_j\}$ as above, the corresponding sequence $\{g_j\}$, $g_j = F^{-jn} \circ f_j \circ F^{jn}$, of elements in the pseudogroup generated on $I$ by $F$ and by $\overline{g}_1, \ldots, \overline{g}_N$ satisfies the following:

- There is $b > 0$ such that the interval $[-b,b]$ is contained in the domain of definition of every diffeomorphism $g_j$.
- The sequence of diffeomorphisms $\{g_j\}$ converges to the identity in the $C^2$-topology on the interval $[-b,b]$.

Recall that $\lambda = F'(0)$. Owing to [37] we can find coordinates where $F(x) = \lambda x$ for every $x \in [-a,a]$. In these coordinates, $g_j$ becomes $g_j = \lambda^{-jn} f_j(\lambda^{jn} x)$. Fix
\( \epsilon_0 > 0 \) small (for example \( \epsilon_0 = a/20 \)). We choose \( \epsilon > 0 \) and \( n \in \mathbb{N} \) so that all the conditions below are fulfilled:

(A) The value of \( n \) is chosen to be the smallest positive integer for which the following conditions are satisfied:

\[
0 < \lambda^n a < a - 5\epsilon_0 \quad \text{and} \quad \lambda^n < 1/20.
\]

(B) Lemma 4.3 holds on \( \mathcal{W}_0^2 \) for some \( C > 0 \).

(C) \( -\epsilon > 0 \) is small enough to ensure that Lemma 4.4 holds and that

\[
\epsilon \max\{(\lambda^{-n} + 1)C, (\lambda^{-n} + 1)\} < 1/10.
\]

**Proof of Proposition 4.6.** Under the above conditions we are going to show that Proposition 4.6 holds with \( b = a \). The proof is by induction. First consider a diffeomorphism \( g_1 \in \tilde{S}(1) \). By assumption, \( g_1 = \lambda^{-n} f_1(\lambda^n x) \) for some \( f_1 \) given as a commutator \( [\overline{g}_{i}, \overline{g}_{j}] \) for some \( i, j \in \{1, \ldots, N\} \). Owing to Lemma 4.3, \( f_1 \) is defined on \([-a + 5\epsilon_0, a - 5\epsilon_0]\) when viewed as element of the pseudogroup generated by \( \bigcup_{i=1}^{N} \overline{g}_{i} \) on \([-a, a] \). Furthermore, the \( C^1 \)-norm of \( f_1 - \text{id} \) on \([-a + 5\epsilon_0, a - 5\epsilon_0]\) satisfies

\[
(3) \quad \|f_1 - \text{id}\|_{[-a+5\epsilon_0,a-5\epsilon_0]} < C \epsilon^2.
\]

Next observe that \( g_1 = \lambda^{-n} f_1(\lambda^n x) \) is defined on all of \([-a, a]\) since \( \lambda^n a < a - 5\epsilon_0 \). Moreover, we clearly have:

\[
\sup_{x \in [-a, a]} |g_1(x) - x| = \sup_{x \in [-a, a]} |\lambda^{-n} f_1(\lambda^n x) - x| = \lambda^{-n} \sup_{y \in [-a + 5\epsilon_0, a - 5\epsilon_0]} |f_1(y) - y|.
\]

Similarly

\[
\sup_{x \in [-a, a]} |D_x^1 g_1 - 1| = \sup_{y \in [-a + 5\epsilon_0, a - 5\epsilon_0]} |D_x^1 f_1 - 1|.
\]

In particular, we obtain

\[
(4) \quad \sup_{x \in [-a, a]} |g_1(x) - x| + \sup_{x \in [-a, a]} |D_x^1 g_1 - 1| < (\lambda^{-n} + 1)C \epsilon^2.
\]

Finally, the second derivative of \( g_1 \) at a point \( x \in [-a, a] \) is such that

\[
D_x^2 g_1 = \lambda^n D_x^2 f_1 \quad \text{so that}
\]

\[
(5) \quad \sup_{x \in [-a, a]} |D_x^2 (g_1 - \text{id})| = \sup_{x \in [-a, a]} |D_x^2 g_1| < \lambda^n 5\epsilon \max_{x \in [-a, a]} \{\|D_x^2 \overline{g}_{i}\|, \|D_x^2 \overline{g}_{j}\|\} \leq 5\lambda^n \epsilon,
\]

where we have used Lemma 4.4. Comparing estimates (4) and (5), there follows that

\[
\|g_1 - \text{id}\|_{2, [-a, a]} \leq (\lambda^{-n} + 1)C \epsilon^2 + 5\lambda^n \epsilon \leq \frac{\epsilon}{10} + \frac{\epsilon}{10} + \frac{\epsilon}{4} = \frac{\epsilon}{2},
\]

where conditions (A), (B), and (C) concerning the choices of \( \epsilon, n \), and the constant \( C \) were used. In particular, we see that \( g_1 \) belongs to \( \mathcal{W}_0^2 \). Since \( g_1 \) is an arbitrary element of \( \tilde{S}(1) \), we conclude that \( \tilde{S}(1) \subset \mathcal{W}_0^2 \) so that the procedure can be iterated. Consider then \( g_2 = \lambda^{-n} [\tilde{f}_i, \tilde{f}_j](\lambda^n x) \) where \( \tilde{f}_i, \tilde{f}_j \) belong to \( \tilde{S}(1) \cup \{\overline{g}_{i}, \ldots, \overline{g}_{N}\} \). Repeating word-by-word, the preceding argument we eventually obtain

\[
\|g_2 - \text{id}\|_{2, [-a, a]} \leq \frac{\epsilon}{2}
\]
(in particular \( g_2 \) is defined on all of \([-a, a]\)). However an element \( g_3 \in \tilde{S}(3) \) can be written as \( g_3 = \lambda^{-n}[\tilde{f}_1, \tilde{f}_2](\lambda^n x) \) where \( \tilde{f}_1, \tilde{f}_2 \) now satisfy

\[
\max\|\tilde{f}_1 - \text{id}\|_{2, [-a, a]}; \|\tilde{f}_2 - \text{id}\|_{2, [-a, a]} < \varepsilon/2.
\]

Therefore, what precedes yields

\[
\|g_3 - \text{id}\|_{2, [-a, a]} < \frac{\varepsilon}{2^j}.
\]

Now a straightforward induction shows that

\[
(6) \quad \|g_{2j} - \text{id}\|_{2, [-a, a]} < \frac{\varepsilon}{2^j}
\]

and completes the proof of Proposition 4.6. \( \Box \)

**Remark 4.7.** Consider a sequence \( g_1, g_2, \ldots \) so that \( g_j \in \tilde{S}(j) \) as above. Consider also the sequence of real numbers given by \( \|g_j - \text{id}\|_{2, [-a, a]} \). Estimate (6) shows that the subsequence of \( \|g_j - \text{id}\|_{2, [-a, a]} \) formed by those \( g_j \) with even order decays at least as \( 1/\sqrt{2^j} \). In fact, it can be shown that the entire sequence \( \|g_j - \text{id}\|_{2, [-a, a]} \) decays faster than \( \Theta^j \) for every a priori given \( \Theta > 0 \). To check this claim, we proceed as follows.

First observe that the choice of \( \varepsilon > 0 \) made in condition (C) can be modified by replacing the 1/10 on the right side of the corresponding estimate by a sufficiently small \( \delta > 0 \). Note that this change does not affect either \( n \) or the constant \( C \) whereas it allows us to obtain a finer estimate than \( \varepsilon/2 \) for \( \|g_1 - \text{id}\|_{2, [-a, a]} \). Thus the same induction argument employed above now yields a new exponential decay for the sequence \( \|g_j - \text{id}\|_{2, [-a, a]} \) where the base depends on \( \delta \) (and becomes larger when \( \delta \) becomes smaller). On the other hand, we have show that every element \( g_j \) in \( \tilde{S}(j) \) satisfies \( \|g_{2j} - \text{id}\|_{2, [-a, a]} < \varepsilon/2^j \) so that there is \( j_0 \in \mathbb{N} \) for which every element in \( \tilde{S}(j) \) satisfies the estimate in condition (C) with a fixed \( \delta > 0 \) in the place of 1/10. Thus, up to dropping finitely many terms, the sequence \( \{g_j\} \) converges to the identity faster than \( \Theta^j \). Since only finitely many terms have been dropped, there follows that the initial sequence \( \{g_j\} \) converges to the identity faster than \( \Theta^j \). This simple observation will be useful in the next section.

**Remark 4.8.** Concerning the proof of Theorem A, we can also assume without loss of generality the existence of a sequence \( \{g_j\} \) as in Proposition 4.6, and hence converging to the identity in the \( C^2 \)-topology on some interval \( I \subset S^1 \), such that \( g_j \neq \text{id} \) for every \( j \in \mathbb{N} \). Indeed, in view of Proposition 3.5 we can assume the existence of diffeomorphisms \( F \) and \( \tilde{g}_1, \ldots, \tilde{g}_N \) in \( G \) satisfying the conditions of Proposition 3.5 on a suitable interval \( I \subset S^1 \) and such that \( \tilde{g}_1, \ldots, \tilde{g}_N \) generate a non-solvable group. Since the group generated by \( \tilde{g}_1, \ldots, \tilde{g}_N \) is not solvable, there follows that none of the sets \( S(j) \) degenerate into the set containing only the identity map (see [17]). Clearly the same applies to the sets \( \tilde{S}(j) \) (Lemma 4.5).

Moreover, up to passing to a subsequence, Corollary 4.2 allows us to assume also that \( f_j(p) \neq p \) (and similarly \( g_j(p) \neq p \) ) for every a priori given point \( p \in S^1 \).
5. Expansion, bounded distortion and rigidity

In this section we shall complete the proof of Theorem A. We begin by recalling that the argument in [22] reduces the proof of Theorem A to checking that \( h \) is a diffeomorphism of class \( C^1 \). The proof of this statement under suitable conditions will be the object of the section.

To make the discussion accurate, let \( G_1 \) and \( G_2 \) be two finitely generated subgroups of \( \text{Diff}^p(S^1) \) that are conjugated by a homeomorphism \( h : S^1 \to S^1 \). By assumption, the group \( G_1 \) is locally \( C^2 \)-non-discrete. In view of the material presented in the previous sections, namely Propositions 3.5 and 4.6 as well as Remarks 4.7 and 4.8, the following conditions can be assumed to hold without loss of generality.

(1) All the orbits of \( G_1 \) are dense in \( S^1 \) (in particular \( G_1 \) has no finite orbit).

The same condition is automatically verified by \( G_2 \) since the groups are topologically conjugate.

(2) Up to using local coordinates, there is an interval \( I = [-a, a] \subset S^1 \) \((a \neq 0)\) and an element \( F_1 \) in \( G_1 \) satisfying \( F_1(0) = 0 \) and \( F_1'(0) = \lambda_1 \in (0, 1) \).

(3) For every \( \epsilon > 0 \), we can find a finite set \( \{ \tilde{g}_{1,1}, \ldots, \tilde{g}_{1,N} \} \subset G_1 \) satisfying all the conditions below:
   - \( \tilde{g}_{1,1}, \ldots, \tilde{g}_{1,N} \) are \( \epsilon \)-close to the identity in the \( C^2 \)-topology on \( I \) (where \( I = [-a, a] \) is the above chosen interval).
   - \( \tilde{g}_{1,1}, \ldots, \tilde{g}_{1,N} \) generate a non-solvable subgroup of \( \text{Diff}^p(S^1) \) having no finite orbit.
   - Consider the sequence \( \tilde{S}_1(k) \) defined in Section 4 by means of the set \( \tilde{S}_1(0) = S_1(0) = S_1 = \{ \tilde{g}_{1,1}, \ldots, \tilde{g}_{1,N} \} \) so that \( \tilde{S}_1(k) = F_1^{-kn} \circ S_1 \circ F_1^{kn} \) for every \( k \in \mathbb{N} \) and a certain fixed \( n \in \mathbb{N}^* \). Then every sequence of elements \( \{ g_{1,k} \} \) with \( g_{1,k} \in \tilde{S}(k) \) converges to the identity in the \( C^2 \)-topology on the interval \( I \).

(4) In fact, if \( \{ g_{1,k} \} \subset G_1 \) is such that \( g_{1,k} \in \tilde{S}_1(k) \), \( k \in \mathbb{N} \), then for every \( \Theta \in \mathbb{R}_+^* \), we have

\[
\lim_{k \to \infty} \left[ \frac{\| g_{1,k} - \text{id} \|_{2,[-a,a]}}{\Theta^k} \right] = 0.
\]

Next recall that a point \( p \in S^1 \) is said to be expandable for a given group \( G \subset \text{Diff}^p(S^1) \) if there is \( g \in G \) such that \( g'(p) > 1 \). Since our diffeomorphisms preserve the orientation of \( S^1 \) the conditions \( g'(p) > 1 \) and \( |g'(p)| > 1 \) are indeed equivalent. With this terminology, we state:

**Theorem 5.1.** Assume that \( G_1 \) satisfies all the conditions (1)–(4) above. Assume also that every point \( p \in S^1 \) is expandable for \( G_2 \). Then every homeomorphism \( h : S^1 \to S^1 \) conjugating \( G_1 \) to \( G_2 \) coincides with an element of \( \text{Diff}^p(S^1) \).

The following simple lemma clarifies the connection between Theorem A and Theorem 5.1.
**Lemma 5.2.** Assume that $G \subset \text{Diff}^\omega(S^1)$ is a locally $C^2$-non-discrete group satisfying conditions (1)–(4) above. Then $G$ leaves no probability measure on $S^1$ invariant. Moreover, every point $p \in S^1$ is expandable for $G$.

**Proof.** Since $G$ has all orbits dense, every probability measure invariant by $G$ must be supported on all of $S^1$. As previously seen, up to parameterizing $S^1$ by means of the cumulative effect of the corresponding (distributional) Radon-Nikodym derivative, the group $G$ becomes conjugate to a group of rotations. This is impossible since $G$ contains elements exhibiting hyperbolic fixed points.

To establish the second part of the statement, we proceed as follows. Since $G$ contains elements having hyperbolic fixed points, we can choose an interval $I = [-a, a]$ and an element $F$ in $G$ satisfying $F(0) = 0$ and $F'(0) > 1$. Furthermore, owing to the discussion in Section 2.1, we can assume without loss of generality that $I$ is equipped with a nowhere zero vector field $X$ contained in the $C^1$-closure of $G$. Consider first the case of a point $p$ lying in the interval $I$. Choose $t_0$ so that the local flow $\phi^t$ of $X$ satisfies $\phi^{t_0}(p) = 0$. The diffeomorphism $f = \phi^{-t_0} \circ F \circ \phi^{t_0}$ satisfies $f(p) = p$ and $f'(p) > 1$. Since $X$ lies in the $C^1$-closure of $G$, there follows that $\phi^{t_0}$ is the $C^1$-limit of a sequence $\tilde{f}_r$ of elements in $G$ restricted to some small neighborhood of $p$. Thus, for $r$ large enough, we conclude that $(\tilde{f}_r^{-1} \circ F \circ \tilde{f}_r)'(p) > 1$ proving the statement for points in $I$. To finish the proof of the lemma, just note that the minimal character of $G$ enables us to find a finite covering of $S^1$ by intervals satisfying the same conditions used above for the interval $I$. The lemma is proved.

**Proof of Theorem A.** It follows at once from the combination of Theorem 5.1 with Lemma 5.2.

The proof of Theorem 5.1 will occupy the remainder of this section. We begin by stating Proposition 5.3 which is a by-product of “Theorem F” in [9]. For this, first note that diffeomorphisms in $G_1$ having hyperbolic fixed points in $I$ are far from unique. We have fixed one of them, namely $F_1$. The element $F_2$ of $G_2$ verifying $F_2 = h^{-1} \circ F_1 \circ h$ has therefore a fixed point in the interval $J = h^{-1}(I)$, namely the point $q = h^{-1}(0)$. However, since $h$ is only a homeomorphism, we cannot immediately conclude that $q$ is hyperbolic for $F_2$. In fact, whereas $F_2$ certainly realizes a “topological contraction” on a neighborhood of $q$, it may happen that $F_2'(q) = 1$. The possibility of having $F_2'(q) = 1$ is a bit of an inconvenient since it would require us to work with iterations of a “parabolic map” in a context similar to the one discussed in Section 2.1. This type of difficulty, however, can be overcome with the help of Proposition 5.3 below. As mentioned the proof of Proposition 5.3 is a direct consequence of the proof of “Theorem F” in [9] and fuller details will be provided in the next section.

**Proposition 5.3.** Without loss of generality, we can assume that $F_2'(q) < 1$, where $q = h^{-1}(0)$.
Now consider a $C^1$-diffeomorphism $f : S^1 \to S^1$. Given an interval $U \subset \mathbb{R} \subset S^1$, the distortion of $f$ in $U$ is defined as

$$\omega(f, U) = \sup_{x,y \in U} \frac{|f'(x)|}{|f'(y)|} = \sup_{x \in U} \log |f'(x)| - \inf_{y \in U} \log |f'(y)|,$$

where $|.|$ stands for the absolute value. Furthermore, assuming that the map $x \mapsto \log(|D_x f|)$ has a Lipschitz constant $C_{\text{Lip}}$, the estimate

$$\omega(f, U) \leq C_{\text{Lip}} \mathcal{L}(U)$$

holds (where $\mathcal{L}(U)$ stands for the length of the interval $U$ with respect to the Euclidean metric for which the length of $S^1$ equals $2\pi$). Note also that the mentioned Lipschitz condition is satisfied provided that $f$ is of class $C^2$ on $U$. On the other, given two diffeomorphisms $f_1, f_2 : S^1 \to S^1$ as above, the estimate

$$\omega(f_1 \circ f_2, U) \leq \omega(f_1, f_2(U)) + \omega(f_2, U)$$

also holds provided that both sides are well defined.

Let us now go back to the sequence of sets $\{S_1(k)\} \subset G_1$ fixed in the beginning of the section. For every $k \in \mathbb{N}$ all the mappings in $S_1(k)$ are defined on the interval $I = [-a, a]$. Next recall that this sequence was obtained as indicated in Section 4 by means of the finite set $\{G_{1,1}, \ldots, G_{1,N}\} \subset G_1$ and of the diffeomorphism $F_1$. In particular $S_1(k) = F_1^{-kn} \circ S_1(k) \circ F_1^{kn}$. From now on, we fix a sequence $\{g_{1,k}\} \subset G_1$ of diffeomorphisms such that $g_{1,k} \neq id$ belongs to $S_1(k)$ for every $k \in \mathbb{N}$. Consider also the corresponding sequence $\{S_2(k)\} \subset G_2$ defined by means of $\{G_{2,1}, \ldots, G_{2,N}\} \subset G_2$ and of the diffeomorphism $F_2$. More precisely, we set $G_{2,j} = h^{-1} \circ G_{1,j} \circ h$ for every $j = 1, \ldots, N$ and $F_2 = h^{-1} \circ F_1 \circ h$ where $F_2$ is assumed to have a contractive hyperbolic fixed point at $h = h^{-1}(0)$ (cf. Proposition 5.3). Thus, for every $k \in \mathbb{N}$, we have $g_{2,k} = h^{-1} \circ g_{1,k} \circ h$. Finally we also pose $J = h^{-1}(I)$.

Next, for every $k \in \mathbb{N}$, let $\mathcal{P}_{l,k}$ denote the partition of the interval $I$ into $5^k$ sub-intervals having the same size and write $\mathcal{P}_{l,k} = \{I_{l,k,1}, \ldots, I_{l,k,5^k}\}$. By means of the homeomorphism $h$, the partitions $\mathcal{P}_{l,k}$ induce partitions $\mathcal{P}_{l,k} = \{J_{l,k,1}, \ldots, J_{l,k,5^k}\}$ of the interval $J$ where $J_{l,k} = h^{-1}(I_{l,k})$ for every $k \in \mathbb{N}$ and for every $j \in \{1, \ldots, 5^k\}$. Now we have:

**Lemma 5.4.** Denote by $\omega(g_{2,k}, J_{l,k})$ the distortion of $g_{2,k}$ in the interval $J_{l,k}$. Then to each $k \in \mathbb{N}$ there corresponds $l_k \in \{1, \ldots, 5^k\}$ such that the resulting sequence $k \mapsto \omega(g_{2,k}, J_{l_k,k})$ converges to zero.

**Proof.** Consider the set formed by the diffeomorphisms $\overline{G}_{2,1}, \ldots, \overline{G}_{2,N}$, $F_2$ along with their inverses. This set is therefore symmetric in the sense that whenever a diffeomorphism belongs to it so does the inverse of the diffeomorphism in question The semigroup generated by this set of diffeomorphisms coincides with the group generated by $\overline{G}_{2,1}, \ldots, \overline{G}_{2,N}$, and $F_2$. Every element in the group in question can be represented as a word in the alphabet whose letters are the diffeomorphisms in the initial symmetric set. If $\tilde{f}$ represents an element in this alphabet, i.e. a letter, the map $x \mapsto \log(|D_x f|)$ is well defined on all of $S^1$ (since...
$D_x f \neq 0$ for all $x \in S^1$). These maps are clearly Lipschitz on all of $S^1$ since $f$ is a $C^2$-diffeomorphism. Fix then a positive constant $C$ greater than the maximum among the Lipschitz constants of all the maps $x \mapsto \log(|D_x f|)$ with $f$ belonging to the alphabet in question.

The explicit construction of the sequences $\{g_{1,k}\}$ and $\{g_{2,k}\}$ makes it clear that every diffeomorphism $g_{2,k}$ can be spelled out in the above mentioned alphabet using at most $4^k + 2nk$ letters. Next let $c_1$ be a constant such that $c_1 \mathcal{L}(J) > 2\pi$ (where $\mathcal{L}(J)$ stands for the length of $J$). Note also that every diffeomorphism $f$ of the circle must satisfy $\mathcal{L}(f(J)) < 2\pi$.

Now fixed $k \in \mathbb{N}$, let $g_{2,k} = f_1 \circ \cdots \circ f_1$ denote the above mentioned spelling of $g_{2,k}$. Thus $l \leq 4^k + 2nk$ and $f_i$ belongs to \{F_2^{±1}, \bar{F}_2^{±1}, \ldots, \bar{F}_2^{±1}\} for every $i \in \{1, \ldots, l\}$. The subadditivity relation expressed by (10) combined to estimate (9) yields

$$\omega(g_{2,k}, J) \leq C \sum_{i=1}^{l-1} \mathcal{L}(f_i \circ \cdots \circ f_1(J)) + C \mathcal{L}(J) \leq c_1 C \mathcal{L}(J)(4^k + 2nk).$$

On the other hand, given an sub-interval $J_{j,k}$ in the partition $\mathcal{P}_{j,k}$ (so that $j \in \{1, \ldots, 5^k\}$). The preceding argument ensures that

$$\omega(g_{2,k}, J_{j,k}) \leq C \sum_{i=1}^{l-1} \mathcal{L}(f_i \circ \cdots \circ f_1(J_{j,k})) + C \mathcal{L}(J_{j,k}) \leq c_1 C \mathcal{L}(J_{j,k})(4^k + 2nk).$$

However, we clearly have

$$\sum_{j=1}^{5^k} \sum_{i=1}^{l-1} \mathcal{L}(f_i \circ \cdots \circ f_1(J_{j,k})) + \sum_{j=1}^{5^k} \mathcal{L}(J_{j,k}) = \sum_{i=1}^{l-1} \mathcal{L}(f_i \circ \cdots \circ f_1(J)) + \mathcal{L}(J).$$

Hence there follows that

$$\sum_{j=1}^{5^k} \omega(g_{2,k}, J_{j,k}) \leq c_1 C \mathcal{L}(J)(4^k + 2nk).$$

Finally, if $j(k)$ realizes the minimum of $j \mapsto \omega(g_{2,k}, J_{j,k})$ over the set $\{1, \ldots, 5^k\}$, we conclude that

$$\omega(g_{2,k}, J_{j(k),k}) \leq \frac{c_1 C \mathcal{L}(J)(4^k + 2nk)}{5^k}$$

which goes to zero as $k \to \infty$. The proof of the lemma is completed. \qed

As $k$ increases, we know that $g_{2,k}(y) - y$ converges uniformly to zero on all of $J$. However, when we consider the sequence of sub-intervals $J_{j(k),k}$ their diameters go to zero as well. A comparison between $\sup_{y \in J_{j(k),k}} |g_{2,k}(y) - y|$ and the length $\mathcal{L}(J_{j(k),k})$ of $J_{j(k),k}$ will however be needed. In particular, we would like to claim that the sequence of quotients $\sup_{y \in J_{j(k),k}} |g_{2,k}(y) - y|/\mathcal{L}(J_{j(k),k})$ converges to zero as $k \to \infty$. At this moment however, the above arguments do not suffice to derive this conclusion since we have no control of the ratio between the lengths of two intervals $J_{f_1(k),k}$ and $J_{f_2(k),k}$. A suitable comparison between the lengths of the mentioned intervals will however be supplied by Proposition 5.5 claiming that the conjugating homeomorphism $h$ is Hölder continuous; cf. below for details.
The next basic step in the proof of Theorem 5.1 consists of magnifying the intervals $I_{j(k),k}$ and $J_{j(k),k}$ into intervals with diameters bounded from below by some strictly positive constant. To do this, we shall resort to a slightly more straightforward version of the celebrated “Sullivan’s expansion strategy” as expounded in [29] and [36]. The main difficulty in applying Sullivan’s type of argument to our situation lies in the fact that the procedure needs to be simultaneously applied to both groups $G_1$ and $G_2$. To overcome this problem we shall first establish that the conjugating homeomorphism $h : S^1 \rightarrow S^1$ is Hölder continuous for a suitable exponent (Proposition 5.5 below). Proposition 5.5 will subsequently be combined with the several estimates involving convergence rates for the sequences $\{g_{1,k}\}_{k \in \mathbb{N}}$ and $\{g_{2,k}\}_{k \in \mathbb{N}}$ (restricted to the intervals $I_{j(k),k}$ and $J_{j(k),k}$, respectively) to yield Theorem 5.1.

Recall that a map $f : U \subset S^1 \rightarrow S^1$ is said to be $\alpha$-Hölder continuous on the interval $U$ if the supremum

\begin{equation}
\sup_{x, y \in U, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}
\end{equation}

is finite (where the bars $|.|$ stand for the fixed Euclidean metric on $S^1$). The above definition is local in the sense that the length of $U$ and of $f(U)$ are assumed to be smaller than $\pi$ so that the above indicated distances are well defined. We shall say that $f$ is $\alpha$-Hölder continuous if its restriction to every interval $U$ satisfying $\max\{\mathcal{L}(U), \mathcal{L}(f(U))\} < \pi$ is $\alpha$-Hölder continuous on $U$. With this terminology, we have:

**Proposition 5.5.** There is $\alpha > 0$ such that the homeomorphisms $h : S^1 \rightarrow S^1$ and $h^{-1} : S^1 \rightarrow S^1$ are both $\alpha$-Hölder continuous.

The idea to prove Proposition 5.5 is very simple and relies on the fact that none of the groups $G_1$ and $G_2$ has non-expandable points. We first provide an outline of the argument which might well be sufficient to convince readers familiar with Sullivan’s expansion strategy and related results in one-dimensional dynamics. To prove that $h$ is Hölder continuous we will rely on the fact that all points in $S^1$ are expandable for the group $G_2$. By permuting the roles of $G_1$ and $G_2$ in the subsequent discussion we would also conclude the Hölder continuity of $h^{-1}$.

To begin with note that the definition of $\alpha$-Hölder continuity is only non-trivial when $|x - y|$ becomes arbitrarily small: indeed, if $|x - y|$ is bounded from below by some positive constant then the supremum appearing in equation (11) is clearly finite. Hence to check that $h$ is $\alpha$-Hölder continuous we only need to check that

$$\limsup_{i \to \infty} \frac{|h(x_i) - h(y_i)|}{|x_i - y_i|^\alpha}$$

is finite for every sequence $\{(x_i, y_i) \subset S^1 \times S^1, x_i \neq y_i\}$ for every $i \in \mathbb{N}$, satisfying $\lim |y_i - x_i| = 0$. Fix then some small $\delta > 0$ and consider those diffeomorphisms in $G_2$ expanding the interval $[h(x_i), h(y_i)]$ to a size greater than $\delta$. The existence of these diffeomorphisms follows from the fact that $G_2$ does not have
non-expandable points. Among the mentioned diffeomorphisms in $G_2$, we fix one $F_{2,i} \in G_2$ which, in addition, has a minimal spelling in the letters of the alphabet given by $\mathcal{G}_{2,1}, \ldots, \mathcal{G}_{2,N}$, $F_2$ and their inverses. The fact that all points in $S^1$ can be expanded by $G_2$ ensures that the number of letters involved in the spelling of $F_{2,i}$ is comparable to $-\ln(|h(y_i) - h(x_i)|)$. More precisely, there follows from Sullivan’s expansion strategy that the mentioned number is bounded by some constant times $-\ln(|h(y_i) - h(x_i)|)$. On the other hand, this number is clearly bounded from below by some constant times $-\ln(|h(y_i) - h(x_i)|)$ since the derivatives of all the corresponding “letters” are uniformly bounded on $S^1$.

Since $G_1$ is conjugate to $G_2$, the interval $[x_i, y_i]$ can also be expanded to some uniform size (depending only on $\delta$ and on $h$) by using diffeomorphisms $F_{1,i}$ in $G_1$ whose spelling in the alphabet formed by $\mathcal{G}_{1,1}, \ldots, \mathcal{G}_{1,N}$, $F_1$ and their inverses uses the same number of letters as $F_{2,i}$. In particular this number must be bounded from below by some constant times $-\ln(|y_i - x_i|)$ (whether or not $G_1$ has non-expandable points since this is the easy direction of the above mentioned estimate). Thus the quotient

$$\frac{\ln(|h(y_i) - h(x_i)|)}{\ln(|y_i - x_i|)}$$

is bounded from below by a strictly positive constant which yields the Hölder continuity of $h$.

In the sequel we provide full detail on the above argument for readers less familiar with the corresponding techniques. In particular Sullivan’s expansion strategy is summarized by Lemma 5.6 below.

We begin by recalling that to each point $p \in S^1$ there corresponds a diffeomorphism $f_{1,p} \in G_1$ with $f'_{1,p}(p) > 1$ (Lemma 5.2). Owing to the compactness of $S^1$, there is a finite covering $U_1 = \{U_{1,1}, \ldots, U_{1,s}\}$ of $S^1$ by open connected intervals $U_{1,i}$, $i = 1, \ldots, s$, satisfying the following conditions:

- For $2 \leq i \leq s - 1$, the interval $U_{1,i}$ intersects only the intervals $U_{1,i-1}$ and $U_{1,i+1}$. The interval $U_{1,1}$ (resp. $U_{1,s}$) intersects only the intervals $U_{1,2}$ and $U_{1,s}$ (resp. $U_{1,s-1}$ and $U_{1,1}$).
- To each interval $U_{1,i}$ there corresponds a diffeomorphism $f_{1,i} \in G_1$ such that $f'_{1,i}(x) > 1$ for every $x \in U_{1,i}$ (recall that $G_1$ and $G_2$ preserve the orientation of $S^1$).

Let $m_1 > 1$ be given as

$$m_1 = \min_{i \in \{1, \ldots, s\}} \{\inf_{U_{1,i}} f'_{1,i}\}.$$  

Similarly let $M_1 = \max_{i \in \{1, \ldots, s\}} \{\sup_{U_{1,i}} f'_{1,i}\}$. Next let $L > 0$ denote the minimum of the lengths of the sets $U_{1,1} \cap U_{1,s}$ and $U_{1,i} \cap U_{1,i+1}$ (for $i = 1, \ldots, s - 1$) so that every interval $[a, b] \subset S^1$ of length less than $L$ is contained in some interval $U_{1,i}$ ($i \in \{1, \ldots, s\}$). For $[a, b]$ as indicated, the derivative of $f_{1,i}$ is not less than $m_1 > 1$ at every point in $[a, b]$ and the length $\mathcal{L}(f_{1,i}([a, b]))$ of $f_{1,i}([a, b])$ is at least $m_1 \mathcal{L}([a, b]) \geq \mathcal{L}([a, b])$. When $\mathcal{L}(f_{1,i}([a, b]))$ is still less than $L$, $f_{1,i}([a, b])$ is again contained in some interval $U_{1,i}$. Thus $f_{1,i} f_{1,i} f_{1,i}(\{a, b\})$ has length greater...
than or equal to \( m^2 \mathcal{L} ([a, b]) \) and the procedure can be continued provided that \( \mathcal{L} (f_1, i_1 \circ f_1, i_1, ([a, b])) < L \). Thus we have proved the following:

**Lemma 5.6.** To every interval \([a, b] \subset S^1\) of length less than \( L\), we can assign an element \( F_1,[a,b] \in G_1\) satisfying the following conditions:

1. \( F_1,[a,b] = f_1, i_1 \circ \cdots \circ f_1, i_1 \) where each \( i_1 \) belongs to \([1, \ldots, s]\).
2. For every \( l \in \{1, \ldots, r\} \), \( f_1, i_1 \circ \cdots \circ f_1, i_1, ([a, b]) \) is contained in \( U_{1, i_1} \) (where \( f_1, i_1 \circ \cdots \circ f_1, i_1, ([a, b]) = [a, b] \) if \( l = 1 \)).
3. We have
   \[
   L \leq \mathcal{L} (F_1,[a,b]([a, b])) \leq L M_1.
   \]

Recalling that \( \mathcal{U}_1 = \{U_{1,1}, \ldots, U_{1,s}\} \), we define a new covering \( \mathcal{U}_2 \) of \( S^1 \) by letting \( \mathcal{U}_2 = \{U_{2,1}, \ldots, U_{2,s}\} \) where \( U_{2,i} = h^{-1}(U_{1,i}) \) for every \( i = 1, \ldots, s \). To every diffeomorphism \( F_1,[a,b] = f_1, i_1 \circ \cdots \circ f_1, i_1 \in G_1\) as above, we assign the corresponding diffeomorphism

\[
F_2,[a,b] = f_2, i_1 \circ \cdots \circ f_2, i_1 = h^{-1} \circ F_1,[a,b] \circ h
\]

where \( f_2, i_1 = h^{-1} \circ f_1, i_1 \circ h \) for every \( l \in \{1, \ldots, r\} \). Clearly the diffeomorphism \( F_2,[a,b] \) takes the (small) interval \( h^{-1}([a, b]) \) to the interval \( h^{-1}(F_1,[a,b]([a, b])) \) whose diameter is bounded from below by a positive constant since \( h \) is uniformly continuous \((S^1\) is compact). Moreover we can still define

\[
M_2 = \max_{l \in \{1, \ldots, s\}} \{\sup_{U_{2,i}} f^l_2 \}
\]

so that \( M_2 > 1 \). However at this point we cannot ensure that \( \inf_{l \in \{1, \ldots, s\}} f^l_2 > 1 \) for a given \( l \in \{1, \ldots, s\} \).

We are now able to complete the proof of Proposition 5.5.

**Proof of Proposition 5.5.** By using the above introduced coverings \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) of \( S^1 \), we are going to show the existence of \( \alpha > 0 \) so that \( h \) is \( \alpha \)-Hölder continuous. By reversing the roles of \( G_1 \) and \( G_2 \) as indicated above, the same argument implies the \( \alpha \)-Hölder continuity of \( h^{-1} \) as well (up to reducing \( \alpha > 0 \)). In fact, all points in \( S^1 \) are known to be expandable for \( G_1 \) (Lemma 5.2) while, in the case of \( G_2 \), this condition is satisfied by assumption.

As already pointed out, the claim that \( h \) is \( \alpha \)-Hölder continuous has a local character. More precisely, considering points \( c \neq d \) in \( S^1 \), we need to find constants \( C \in \mathbb{R}^+ \) and \( \alpha > 0 \) such that

\[
|h(d) - h(c)| \leq C |d - c|^\alpha
\]

provided that \( |d - c| \) is small. Here the vertical bars \( |\cdot| \) stand for the distance between the corresponding points for the fixed Euclidean metric (i.e. \( |d - c| = \mathcal{L} ([c, d]) \)). Owing to the previous discussion and to the fact that both \( h \) and \( h^{-1} \) are uniformly continuous since \( S^1 \) is compact, there easily follows the existence of a uniform \( \tau > 0 \) so that all the considerations made in the course of the proof are valid provided that \( |d - c| < \tau \). We shall then proceed to prove that \( h \) is \( \alpha \)-Hölder continuous on intervals whose length does not exceed \( \tau \) which clearly implies the proposition.
We therefore consider \(c, d\) as before and let \([a, b] = h([c, d])\). Without loss of generality, \(h\) preserves the orientation so that we set \(a = h(c)\) and \(b = h(d)\). The next step consists of expanding the interval \([a, b]\) by means of the procedure summarized by Lemma 5.6. With the notation used in this lemma, we find \(F_{1,[a,b]} = f_{1,i_1} \circ \cdots \circ f_{i_1,1} \in G_1\) such that
\[
L \leq \mathcal{L}(F_{1,[a,b]}([a, b])) \leq M_1 L.
\]

Consider now the corresponding element \(F_{2,[a,b]} = h^{-1} \circ F_{1,[a,b]} \circ h\) in \(G_2\). We also set \(F_{2,[a,b]} = f_{2,i_2} \circ \cdots \circ f_{2,1}\), as previously indicated. There exists a uniform \(\delta > 0\) so that
\[
\mathcal{L}(F_{2,[a,b]}([c, d])) \geq \delta > 0.
\]
Indeed, just note that \(F_{2,[a,b]}([c, d]) = h^{-1} \circ F_{1,[a,b]}([a, b])\) so that the claim follows from the uniform continuity of \(h^{-1}\) since \(\mathcal{L}(F_{1,[a,b]}([a, b])) \geq L > 0\).

Consider now the number \(r\) of diffeomorphisms \(f_{1,i} (i \in \{1, \ldots, s\})\) appearing in the above indicated spelling of \(F_{1,[a,b]}\). By construction, at each iteration of \(f_{1,i}\) the corresponding interval is expanded by a factor bounded from below by \(m_1 > 1\). Hence we obtain
\[
\frac{|b - a|}{LM_1} \leq 1.
\]
In particular, there follows that
\[
r \leq \frac{1}{\ln m_1} (\ln(LM_1) - \ln|b - a|).
\]

On the other hand, considering \(F_{2,[a,b]} = f_{2,i_2} \circ \cdots \circ f_{2,1}\), there also follows that at each iteration of \(f_{2,i}\) the interval in question cannot be expanded by a factor exceeding \(M_2\). Hence, we similarly obtain \(|d - c|/M_2^r \geq \delta\) so that
\[
1 \leq \frac{|d - c| M_2^r}{\delta}.
\]
Putting together estimates (13) and (15), we conclude that
\[
|b - a| \leq \frac{LM_1}{\delta} |d - c| \left(\frac{M_2}{m_1}\right)^r.
\]
Without loss of generality, we can assume \(M_2 > m_1\) for otherwise the preceding estimate implies at once that \(h\) is Lipschitz. Set \(\overline{c} = \ln(M_2/m_1)/\ln m_1\) so that \(\overline{c} > 0\) since \(M_2 > m_1\) and \(m_1 > 1\). Moreover estimate (16) becomes
\[
|b - a| \leq C_1 |d - c| \exp(\ln|b - a|^{\overline{c}})
\leq C_1 |d - c||b - a|^{-\overline{c}}
\]
for some suitable constant \(C_1\). Hence \(h\) is \(\alpha\)-Hölder continuous for \(\alpha = 1/(1 + \overline{c})\). The proof of the proposition is completed.

We are almost ready to prove Theorem 5.1. The last ingredient needed in our proof consists of a simple estimate for the second derivatives of \(F_{1,[a,b]}\) and of \(F_{2,[a,b]}\). This is as follows. Keep the preceding notation and fix again intervals \([a, b]\) and \([c, d]\) such that \(h([c, d]) = [a, b]\). Then we have:
LEMMA 5.7. There are constants $\overline{C} > 0$ and $\beta \in (0, 1)$ such that
\[
\max \left\{ \sup_{x \in [a, b]} |D^2 F_{1, i[a, b]}(x)|; \sup_{y \in [c, d]} |D^2 F_{2, i[a, b]}(y)| \right\} \leq \overline{C}|b - a|^{\ln \beta}.
\]
Proof. Let $\overline{M}$ be a constant satisfying
\[
\max_{i=1, \ldots, s} \left\{ \sup_{U_{i,j}} |D^2 f_{1,i}|; \sup_{U_{i,j}} |D^2 f_{2,i}| \right\} < \overline{M}.
\]
First we will show the existence of constants $\overline{C}_1$ and $\beta_1$ for which
\[
\sup_{x \in [a, b]} |D^2 F_{1, i[a, b]}(x)| \leq \overline{C}_1|b - a|^{\ln \beta_1}.
\]
We begin by recalling that $F_{1, i[a, b]} = f_{1, i} \circ \cdots \circ f_{1, i}$. For $x_0 \in [a, b]$ and $l \in \{1, \ldots, r - 1\}$, let $x_l = f_{1, i} \circ \cdots \circ f_{1, i}(x_0)$. Thus we have $F_{1, i[a, b]}(x_0) = f_{1, i}^l(x_{r-1}) \circ \cdots \circ f_{1, i}(x_0)$ and
\[
D^2 F_{1, i[a, b]}(x_0) = \prod_{l=1}^{r} f'_{1, i}(x_{l-1}) \left[ \sum_{j=1}^{r} \left( \frac{D^2 f_{1, i}(x_{j-1})}{f'_{1, i}(x_{j-1})} f'_{1, i}(x_{j-2}) \cdots f'_{1, i}(x_0) \right) \right].
\]
Hence
\[
|D^2 F_{1, i[a, b]}(x_0)| \leq \overline{M} M_1^{2r}.
\]
On the other hand, recall that $r \leq (\ln LM_1 - \ln |b - a|) / \ln m_1$ (estimate (14)). Setting $\overline{C}_1 = \overline{M} M_1^{2\ln LM_1 / \ln m_1}$, there follows that
\[
|D^2 F_{1, i[a, b]}(x_0)| \leq \overline{C}_1|b - a|^{-2\ln M_1 / \ln m_1}.
\]
Since $M_1 \geq m_1 > 1$, the exponent $-2\ln M_1 / \ln m_1$ is negative and hence has the form $\ln \beta_1$ for some $\beta_1 \in (0, 1)$. This proves the first assertion. To complete the proof of the lemma it only remains to show that a similar estimate holds for $|D^2 F_{2, i[a, b]}|$ on $[c, d]$. However, a repetition word-by-word of the above argument yields constants $\overline{C}_2$ and $\beta_2 \in (0, 1)$ such that
\[
|D^2 F_{2, i[a, b]}(y_0)| \leq \overline{C}_2 |d - c|^{\ln \beta_2}
\]
for every $y_0 \in [c, d]$. The desired estimate is then an immediate consequence of Proposition 5.5. The lemma is proved. \qed

Proof of Theorem 5.1. In what follows we keep all the notation introduced in the course of this section. Consider the interval $I \subset S^1$ (resp. $J = h^{-1}(I) \subset S^1$) and the sequence of partitions $\mathcal{P}_{I,k}$ on $I$ (resp. $\mathcal{P}_{J,k}$ on $J$). More precisely, consider the sequences of intervals $k \rightarrow I_{l,k}$ and $k \rightarrow J_{l,k}$ as in Lemma 5.4.\footnote{Note that $I_{l,k} = [a_k, b_k]$ and $J_{l,k} = [c_k, d_k]$ so that $a_k = h(c_k)$ and $b_k = h(d_k)$.}

Next set $I_{l,k} = [a_k, b_k]$ and $J_{l,k} = [c_k, d_k]$ so that $a_k = h(c_k)$ and $b_k = h(d_k)$. Also $a > 0$ is fixed so that both homeomorphisms $h$ and $h^{-1}$ are $\alpha$-Hölder continuous (Proposition 5.5). Now, for each $k \in \mathbb{N}$ fixed, let $F_{1, [a_k, b_k]}$ be the element of $G_1$ obtained by means of Lemma 5.6. Thus we have $F_{1, [a_k, b_k]} = f_{1, i_k} \circ \cdots \circ f_{1, i_1}$ where each $i_j$ belongs to $\{1, \ldots, s\}$. Analogously we define $F_{2, [a_k, b_k]} \in G_2$ so that $F_{2, [a_k, b_k]} = h^{-1} \circ F_{1, [a_k, b_k]} \circ h$. In particular
\[
F_{2, [a_k, b_k]} = f_{2, i_k} \circ \cdots \circ f_{2, i_1}
\]
with \( i_l \in \{1, \ldots, s\} \).

By construction, all the intervals of the form \( \{F_{1,1(a_k,b_k)}([a_k,b_k])\} \subset S^1 \) have length comprised between \( L \) and \( LM_1 \). Hence, up to passing to a subsequence, we can assume that \( F_{1,1(a_k,b_k)}(a_k) \to \bar{a} \) and \( F_{1,1(a_k,b_k)}(b_k) \to \bar{b} \) where \( \bar{a} \neq \bar{b} \). To abridge notation, we refer to this by saying that the mentioned intervals converge towards the open interval \( \bar{I} = (\bar{a},\bar{b}) \subset S^1 \). Finally set also \( \tilde{I} = h^{-1}(\bar{I}) = (\tilde{a},\tilde{b}) \subset S^1 \) so that \( F_{2,1(a_k,b_k)}(c_k) \to \tilde{c} \) and \( F_{2,1(a_k,b_k)}(d_k) \to \tilde{d} \).

Consider now the sequences of diffeomorphisms \( \{\tilde{f}_{1,k}\} \subset G_1 \) and \( \{\tilde{f}_{2,k}\} \subset G_2 \) obtained by setting
\[
\tilde{f}_{1,k} = F_{1,1(a_k,b_k)} \circ g_{1,k} \circ F_{1,1(a_k,b_k)}^{-1} \quad \text{and} \quad \tilde{f}_{2,k} = F_{2,1(a_k,b_k)} \circ g_{2,k} \circ F_{2,1(a_k,b_k)}^{-1}.
\]

Claim 4. The sequence \( \{\tilde{f}_{1,k}\} \subset G_1 \) (resp. \( \{\tilde{f}_{2,k}\} \subset G_2 \)) converges to the identity in the \( C^0 \)-topology on compact parts of \( \bar{I} \) (resp. \( \tilde{I} \)).

Proof of Claim 4. Consider first the sequence \( \{\tilde{f}_{1,k}\} \) and a point \( x \in \bar{I} \). By construction the point \( y = F_{1,1(a_k,b_k)}^{-1}(x) \) lies in \( I_{k,k} = [a_k,b_k] \) provided that \( k \) is large enough. Therefore
\[
|\tilde{f}_{1,k}(x) - x| = |F_{1,1(a_k,b_k)} \circ g_{1,k} \circ F_{1,1(a_k,b_k)}^{-1}(x) - x| \\
\leq \sup_{[a_k,b_k]} |D^1 F_{1,1(a_k,b_k)}| |g_{1,k}(y) - y| \\
\leq M_1^{r_k} |g_{1,k}(y) - y|.
\]
However \( r_k \) is bounded by estimate (14) which yields
\[
M_1^{r_k} \leq \text{const} |b_k - a_k|^{-\ln M_1/\ln m_1}
\]
for some constant const. Since in addition \( |b_k - a_k| \) equals \( 5^{-k} \) up to a multiplicative constant, we obtain
\[
|\tilde{f}_{1,k}(x) - x| \leq \text{Const} 5^{k \ln M_1/\ln m_1} |g_{1,k}(y) - y|
\]
which converges to zero as \( k \to \infty \) by virtue of condition (4) in the beginning of the section (here Const stands for some suitable new constant).

It remains to show the same holds for the sequence \( \{\tilde{f}_{2,k}\} \). Setting \( z = h^{-1}(x) \) and \( w = h^{-1}(y) \), the same argument used above yields
\[
|\tilde{f}_{2,k}(z) - z| \leq \text{Const} \cdot 5^{k \ln M_1/\ln m_1} |g_{2,k}(w) - w|
\]
for a new constant \( \text{Const}' \). However the \( \alpha \)-Hölder continuity of \( h^{-1} \) ensures that
\[
|g_{2,k}(w) - w| \leq |g_{1,k}(y) - y|^{\alpha}
\]
so that the claim follows again from condition (4). \( \square \)

We now consider the problem of \( C^1 \)-convergence for the sequences \( \{\tilde{f}_{1,k}\} \) and \( \{\tilde{f}_{2,k}\} \). We begin by recalling that the restriction of \( g_{1,k} \) to \( I_{k,k} \) converges \( C^2 \) (in particular \( C^1 \)) to the identity. On the other hand, the restriction of \( g_{2,k} \) to \( I_{k,k} \) is known to satisfy the following conditions:
\[
\sup_{w \in I_{k,k}} |g_{2,k}(w) - w| \to 0.
\]

(B) The sequence \( \{\bar{\omega}(g_{2,k}, J_{l_2,k})\} \) formed by the distortion of \( g_{2,k} \) on \( J_{l_2,k} \) converges to zero.

The reader will note that condition (B) above is nothing but Lemma 5.4 whereas condition (A) follows from Proposition 5.5. In fact, the \( \alpha \)-Hölder continuity of \( h \) ensures that \( \mathbb{L}(J_{l_2,k}) \geq \mathbb{L}(I_{l_2,k})^{1/\alpha} \) while the \( \alpha \)-Hölder continuity of \( h^{-1} \) yields \( \sup_{w \in J_{l_2,k}} |g_{2,k}(w) - w| \leq \sup_{x \in I_{l_2,k}} |g_{1,k}(y) - y|^{\alpha} \). Thus the limit in question results from the fact that

\[
\lim_{k \to \infty} \left| \frac{\|g_{1,k} - \text{id}\|_{[2,\lbrack-a,a\rbrack]}}{\Theta^k} \right| = 0
\]

for every \( \Theta \in \mathbb{R}_+^* \) (condition (4) as stated in the beginning of this section).

Owing to Proposition 5.3 and to the fact \( G_1 \) acts minimally on \( S^1 \), we choose a point \( p \in \bar{I} \) such that the following condition holds: there are conjugate elements \( \bar{F}_1 \in G_1 \) and \( \bar{F}_2 \in G_2 \) (\( \bar{F}_2 = h^{-1} \circ \bar{F}_1 \circ h \)) such that \( \bar{F}_1 \) has a hyperbolic fixed point in \( p \) whereas \( \bar{F}_2 \) has a hyperbolic fixed point in \( q = h^{-1}(p) \). Owing to Corollary 4.2 we can assume without loss of generality that \( \bar{f}_{1,k}(p) \neq p \) for every \( k \in \mathbb{N} \) (which also implies that \( \bar{f}_{2,k}(q) \neq q \)).

The next step consists of estimating the derivative of \( \bar{f}_{1,k} \) at a point \( x \in \bar{I} \). For \( y = F_{1,\lbrack a_k,b_k\rbrack}(x) \), we clearly have \( \bar{f}_{1,k}^2(x) = D_{g_{1,k}(y)} F_{1,\lbrack a_k,b_k\rbrack} g_{1,k}(y) D_{F_{1,\lbrack a_k,b_k\rbrack}^{-1} g_{1,k}(y)} \). Thus,

\[
|\bar{f}_{1,k}(x)| \leq |D_{g_{1,k}(y)} F_{1,\lbrack a_k,b_k\rbrack} - D_{g_{1,k}(y)} F_{1,\lbrack a_k,b_k\rbrack}| |g_{1,k}(y)| + |g_{1,k}(y)|
\]

\[
\leq \sup_{\lbrack a_k,b_k\rbrack} |D^2 F_{1,\lbrack a_k,b_k\rbrack}||g_{1,k}(y) - y||g_{1,k}(y) + g_{1,k}'(y)|.
\]

On the other hand, \( |b_k - a_k| \) is bounded by a uniform constant times \( 5^{-k} \). Thus Lemma 5.7 yields

\[
\sup_{\lbrack a_k,b_k\rbrack} |D^2 F_{1,\lbrack a_k,b_k\rbrack}| \leq \text{const} 5^{-k \ln \beta}.
\]

Therefore condition (4) ensures that \( \sup_{\lbrack a_k,b_k\rbrack} |D^2 F_{1,\lbrack a_k,b_k\rbrack}||g_{1,k}(y) - y| \) converges to zero as \( k \) goes to infinity. Since \( \{g_{1,k}\} \) converges \( C^1 \) to the identity, there follows that the restriction of \( \bar{f}_{1,k} \) to every compact part of \( \bar{I} \) converges \( C^1 \) to the identity as well. The claim below shows that a similar phenomenon holds for the sequence \( \{\bar{f}_{2,k}\} \) as well.

**Claim 5.** The sequence \( \{\bar{f}_{2,k}\} \) converges \( C^1 \) to the identity on \( \bar{I} \).

**Proof of Claim 5.** The argument is more subtle and builds on the previous discussion. Recalling that \( q = h^{-1}(p) \), we set \( q_k = F_{2,\lbrack a_k,b_k\rbrack}^{-1}(q) \). Let also \( \lambda_k = g_{2,k}'(q_k) \).

We also immediately note that Lemma 5.7 still yields \( \sup_{\lbrack c_k,d_k\rbrack} |D^2 F_{2,\lbrack a_k,b_k\rbrack}| \leq \text{const} 5^{-k \ln \beta} \) for a suitable constant const. For \( z \in \bar{I} \) and \( w = F_{2,\lbrack a_k,b_k\rbrack}^{-1}(z) \), the argument used above now provides

\[
|\bar{f}_{2,k}(z)| \leq \sup_{\lbrack c_k,d_k\rbrack} |D^2 F_{2,\lbrack a_k,b_k\rbrack}||g_{2,k}(w) - w||g_{2,k}'(w) + g_{2,k}'(w)|.
\]

Again

\[
\sup_{\lbrack c_k,d_k\rbrack} |D^2 F_{2,\lbrack a_k,b_k\rbrack}||g_{2,k}(w) - w||g_{2,k}'(w)|
\]

\[
\leq \text{const} 5^{-k \ln \beta}.
\]

The argument above ensures that \( \sup_{z \in \bar{I}} |\bar{f}_{2,k}(z)| \) converges to zero as \( k \) goes to infinity.
converges to zero so that $|\tilde{f}_{1,k}'(z)|$ becomes arbitrarily close to $g_{2,k}'(w)$. In turn, owing to Lemma 5.4, the derivative $g_{2,k}'(w)$ becomes arbitrarily close to $\lambda_k$.

Finally we can assume that $\lambda_k$ converges to some $\tilde{\lambda} \in \mathbb{R}$ for $\lambda_k$ is uniformly bounded since the lengths of the intervals $\tilde{f}_{2,k}(\tilde{j})$ are clearly so. Summarizing what precedes, the sequence of maps $\{\tilde{f}_{2,k}'\}$ converges uniformly on $\tilde{j}$ to the constant $\tilde{\lambda}$. To conclude that $\tilde{\lambda} = 1$, just note that the sequence of primitives $\{\tilde{f}_{2,k}\}$ converges uniformly to the identity on $\tilde{j}$ (Claim 4). This ends the proof of Claim 5.

To finish the proof of Theorem 5.1 we proceed as follows. Consider again the sequences of maps $\{\tilde{f}_{1,k}\} \subset G_1$ and $\{\tilde{f}_{2,k}\} \subset G_2$. By construction, we have $\tilde{f}_{2,k} = h^{-1} \circ \tilde{f}_{1,k} \circ h$ for every $k \in \mathbb{N}$. Furthermore $\{\tilde{f}_{1,k}\}$ (resp. $\{\tilde{f}_{2,k}\}$) converges $C^1$ to the identity on $\tilde{I}$ (resp. $\tilde{J}$). From this point, the standard argument relies on synchronized vector fields (see [30]). This is as follows.

Recall that $\tilde{f}_{1,k}(p) \neq p$ (resp. $\tilde{f}_{2,k}(q) \neq q$) for every $k \in \mathbb{N}$. Moreover there are conjugate elements $\tilde{F}_1 \in G_1$ and $\tilde{F}_2 \in G_2$ which have hyperbolic fixed points in $p$ and $q$, respectively. In suitable local coordinates around $p \approx 0$ (resp. $q \approx 0$), $\tilde{F}_1$ becomes a homothety $x \rightarrow \Lambda_1 x$ (resp. $\tilde{F}_1, z \rightarrow \Lambda_2 z$. Here both $\Lambda_1$ and $\Lambda_2$ belong to $(0,1)$. Consider the effect of the conjugations $\tilde{F}_1^{-j} \circ \tilde{f}_{1,k} \circ \tilde{F}_1^j$ on $\tilde{f}_{1,k}$ for $k$ fixed and $j \in \mathbb{N}$. As there follows from [30] (cf. also Section 2.1) if $\tilde{j}(k)$ is a suitably chosen sequence with $\tilde{j}(k) \rightarrow \infty$, the conjugate diffeomorphisms $\tilde{F}_1^{-j(k)} \circ \tilde{f}_{1,k} \circ \tilde{F}_1^{j(k)}$ and $\tilde{F}_2^{-j(k)} \circ \tilde{f}_{2,k} \circ \tilde{F}_2^{j(k)}$ converge in the $C^1$-topology, respectively on $\tilde{I}$ and $\tilde{J}$, to non-trivial translations. Thus, we actually obtain non-zero constant vector fields $\tilde{X}_1$ and $\tilde{X}_2$ contained in the $C^1$-closures of $G_1$ and $G_2$, respectively, and whose flows $\phi_1^t$ and $\phi_2^t$ satisfy the equation

$$h \circ \phi_2^t(z) = \phi_1^t \circ h(z)$$

whenever both sides are well defined. By fixing $z$ and letting $t$ takes values around $0 \in \mathbb{R}$, we conclude that $h$ is of class $C^1$ on a neighborhood of $z \in \tilde{J}$.

The fact that the dynamics of $G_1$ and $G_2$ are minimal then implies that $h$ is of class $C^1$ on the entire circle. The proof of Theorem 5.1 is completed.

6. Ergodic Theory and Conjugate Groups

Here we shall apply some probabilistic methods to the study of topologically conjugate groups of circle diffeomorphisms. In the course of the discussion we will also indicate how Proposition 5.3 follows from the methods of [9]. The main result of this section is Theorem 6.2 which can be combined to Theorem A to immediately yield our Theorem B in the Introduction.

Throughout the section, we fix two topologically conjugate subgroups $G_1$ and $G_2$ of $\text{Diff}^\omega(S^1)$. The group $G_1$ is assumed to be locally $C^2$-non-discrete and, in fact, it is assumed to satisfy all the conditions (1)–(4) in the beginning of Section 5.

**Lemma 6.1.** None of the groups $G_1$ and $G_2$ leaves a probability measure on $S^1$ invariant.
Proof: The statement holds for $G_1$ thanks to Lemma 5.2. The conclusion concerning $G_2$ then arises from the fact that these two groups are topologically conjugate.

The above lemma is all that is needed to derive Proposition 5.3 from the proof of “Theorem F” in [9].

Proof of Proposition 5.3. Since the group $G_1$ acts minimally on $S^1$, we only need to show the existence of a diffeomorphism $F_1 \in G_1$ having a hyperbolic fixed point at some point $p \in S^1$ and such that the corresponding diffeomorphism $F_2 = h^{-1} \circ F_1 \circ h$ in $G_2$ has a hyperbolic fixed point at $q = h^{-1}(p)$.

The essential point here is that we can provide a coherent notion of “random iteration” simultaneously valid for both groups $G_1$ and $G_2$. In fact, since they are topologically conjugate, we can think that $G_1$ and $G_2$ are isomorphic to an abstract group $G$. Fix then a finite generating set $\Sigma$ for $G$ which contains elements and their inverses (i.e. $\Sigma$ generates $G$ as semigroup). Up to fixing an auxiliary measure $\mu$ on $\Sigma$ which is symmetric (i.e. gives the same mass to an element and to its inverse) and non-degenerate (i.e. every element in $\Sigma$ has strictly positive $\mu$-mass), the shift space $\Omega = \Sigma^\mathbb{N}$ equipped with the probability product measure $\mu^\mathbb{N}$ yields a natural notion of random iteration for the group $G$. By means of the above mentioned isomorphisms, we obtain natural shift spaces $\Omega_1$ and $\Omega_2$ along with a natural bijection (induced by $h$) between these two sets. The bijection in question preserves the corresponding product measures on $\Omega_1$ and $\Omega_2$. In particular, $h$ takes a full-measure subset of $\Omega_1$ to a full-measure subset of $\Omega_2$.

Now, since $G_1$ (resp. $G_2$) does not preserve a probability measure on $S^1$, the proof of Theorem F in [9] actually shows that a “long random iteration” of elements in $G_1$ (resp. $G_2$) yields a diffeomorphism all of whose fixed points are hyperbolic (and there exists at least one such fixed point). If none of these “long random iteration” were taken by $h$ to an element of $G_2$ having only hyperbolic fixed points, then $h$ would take a full-measure subset of $\Omega_1$ to a null-measure subset of $\Omega_2$ which is impossible in view of the above remark. The proof of the proposition is completed.

The rest of this section is devoted to the proof of Theorem 6.2 stated below. This theorem concerns the potential existence of topologically conjugate groups $G_1$ and $G_2$ acting on $S^1$ with $G_1$ being locally $C^2$-non-discrete whereas $G_2$ is locally $C^2$-discrete. This discussion will lead to the proof of Theorem B in the introduction. We begin by stating Theorem 6.2. For the rest of this section, $\Gamma$ will always denote an abstract hyperbolic group which is neither finite nor a finite extension of $\mathbb{Z}$. The notion of entropy for measures as those considered in the statement of Theorem 6.2 is also recalled below.

**Theorem 6.2.** For $\Gamma$ as above, let $\rho_1 : \Gamma \to \text{Diff}^0(S^1)$ be a faithful representation of $\Gamma$ in $\text{Diff}^0(S^1)$ and set $G_1 = \rho_1(\Gamma)$. Assume that $G_1$ is locally $C^2$-non-discrete and that there is a non-degenerate measure with finite entropy on $\Gamma = G_1$ giving
rise to an absolutely continuous stationary measure \( \nu \) on \( S^1 \). Then every subgroup \( G_2 \subset \text{Diff}^{\omega}(S^1) \) topologically conjugate to \( G_1 \) is locally \( C^2 \)-non-discrete as well.

**Proof of Theorem B.** Just note that the statement follows immediately from the combination of Theorem A and Theorem 6.2.

In turn, the proof of Theorem 6.2 relies on the combination of a few deep results including Theorem 1.1 of [8] and Kaimanovich’s theorem in [24]. For suitable background on hyperbolic groups and on measure theoretic methods in group theory, the reader is referred to [24], [39], and [20]. Yet, to make the statement of Theorem 6.2 intelligible, we shall briefly revisit the notion of stationary measures. Consider a generating set \( \Sigma \) for a group \( G \) containing elements and their inverses so that \( \Sigma \) generates \( G \) as a semigroup (the groups considered in the course of the discussion will be \( G_1 \), \( G_2 \), and \( \Gamma \)). Recall that a measure \( \mu \) on \( G \) is said to be non-degenerate if there is \( \Sigma \) as above such that \( \mu \) gives strictly positive mass to every element of \( \Sigma \). Similarly if \( \mu \) gives the same measure to an element and to its inverse, then \( \mu \) is said to be symmetric.

For a non-degenerate measure \( \mu \) as above, the entropy of \( \mu \) is defined by

\[
H(\mu) = -\sum_{\gamma \in \Gamma} \mu(\gamma) \ln \mu(\gamma).
\]

Assume now that \( G \) acts on a compact metric space \( M \) (in our case we will have \( G_1 \), \( G_2 \) acting on \( S^1 \), and \( \Gamma \) acting on its geometric boundary \( \partial \Gamma \)). In this case the choice of \( \mu \) on \( G \) gives rise to the notion of stationary measures on \( M \). Namely, a probability measure \( \nu \) on \( M \) is called stationary if for every Borel set \( \mathcal{B} \subset M \), we have

\[
\nu(\mathcal{B}) = \sum_{g \in G} \mu(g) \nu(g^{-1}(\mathcal{B})).
\]

With these definitions in place, consider the above groups \( \Gamma \) and \( G_1 = \rho_1(\Gamma) \) so that \( G_1 \) is locally \( C^2 \)-non-discrete. By assumption, there is a non-degenerate measure of finite entropy on \( \Gamma \cong G_1 \) leading to an absolutely continuous stationary measure for the action of \( G_1 \) on \( S^1 \). The existence of absolutely continuous stationary measures however is not needed until we effectively start the proof of Theorem 6.2 and for this reason we shall conduct a slightly more general discussion for the time being.

By way of contradiction we assume in the sequel that the statement of Theorem 6.2 is false. Thus we can assume the existence of a locally \( C^2 \)-discrete group \( G_2 \subset \text{Diff}^{\omega}(S^1) \) which is topologically conjugate to \( G_1 \). The proof of Theorem 6.2 relies heavily on properties of stationary measures and the structure of our argument can be described as follows. First we can assume without loss of generality that \( G_2 \) is indeed locally \( C^1 \)-discrete; cf. Lemma 6.3. We consider then a suitable measure \( \mu \) on \( \Gamma = G_1 \cong G_2 \) (non-degenerate and with finite entropy) and denote by \( \nu_1 \) (resp. \( \nu_2 \)) the corresponding stationary measure for the action of \( G_1 \) (resp. \( G_2 \)) on \( S^1 \). Owing to a result due to Deroin [8], we know that the Furstenberg boundary of \( G_2 \) can essentially be identified with \( (S^1, \nu_2) \). On the other hand, Kaimanovich [24] shows that this Furstenberg boundary can also be
modeled by the geometric boundary \( \partial \Gamma \) of the hyperbolic group \( \Gamma \). Putting all these identifications together, we obtain a measurable isomorphism between the action of \( G_1 \) on \( (S^1, v_1) \) and the action of \( \Gamma \) on \( \partial \Gamma \). The action of \( \Gamma \) in \( \partial \Gamma \) is however “locally discrete” in a \( C^0 \)-sense (Lemma 6.5) whereas the action of \( G_1 \) on \( S^1 \) is locally non-discrete. A priori this is not a contradiction since the equivariant map between \( S^1 \) and \( \partial \Gamma \) is only measurable. However, if \( v_1 \) is absolutely continuous, the classical Lusin theorem yields topological constraints on the measurable map in question and these constraints are sufficient to derive the desired contradiction.

The preceding discussion will be made accurate in what follows. Let then \( \Gamma, G_1 \), and \( G_2 \) be as above and recall that \( \rho_1 \) stands for the representation \( \rho_1 : \Gamma \rightarrow G_1 \subset \text{Diff}^0(S^1) \). By post-composing \( \rho_1 \) with a conjugating homeomorphism \( h \), we obtain another faithful representation \( \rho_2 : \Gamma \rightarrow \text{Diff}^0(S^1) \) satisfying

\[
\rho_2(\gamma) = h^{-1} \circ \rho_1(\gamma) \circ h
\]

for every \( \gamma \in \Gamma \) and where \( G_2 = \rho_2(\Gamma) \). In other words, the representations \( \rho_1 \) and \( \rho_2 \) are topologically conjugated by \( h \). We begin with the following lemma.

**Lemma 6.3.** Without loss of generality we can assume that the group \( G_2 \) is locally \( C^1 \)-discrete.

**Proof.** The proof relies on Theorem 5.1. In fact, according to this theorem, \( G_2 \) must have a non-expandable point provided that it is locally \( C^2 \)-discrete. Hence to prove the lemma it suffices to check that a locally \( C^1 \)-non-discrete subgroup of \( \text{Diff}^0(S^1) \) (having all orbits dense and leaving no probability measure invariant) expands every point \( p \in S^1 \).

Consider then a diffeomorphism \( F_2 \in G_2 \) having a hyperbolic fixed point \( q \in S^1 \). In local coordinates around \( q \approx 0 \), we then have \( F_2(x) = \lambda x \) for some \( \lambda \in (0, 1) \). Next, suppose that \( G_2 \) is locally \( C^1 \)-non-discrete. By using the minimal character of \( G_2 \), we then obtain a sequence \( g_{2,j} \) of diffeomorphisms in \( G_2 \) \((g_{2,j} \neq \text{id} \) for all \( j \in \mathbb{N} \)) which converges to the identity on a small interval \((-\varepsilon, \varepsilon)\) around \( q \approx 0 \) (for some \( \varepsilon > 0 \)). Again the discussion in Section 2.1 allows us to assume that \( g_{2,j}(0) \neq 0 \) for every \( j \in \mathbb{N} \). Thus, as shown in the end of the proof of Theorem 5.1, there is a sequence of positive integers \( m(j) \rightarrow \infty \) such that the corresponding diffeomorphisms \( F_2^{-m(j)} \circ g_{2,j} \circ F_2^{m(j)} \) converge in the \( C^1 \)-topology on \((-\varepsilon, \varepsilon)\) to a non-trivial translation. There also follows that the local flow of the vector field \( \partial / \partial x \) on \((-\varepsilon, \varepsilon)\) is contained in the \( C^1 \)-closure of \( G_2 \). Since \( q \approx 0 \in (-\varepsilon, \varepsilon) \) is clearly expandable for \( G_2 \), we conclude that every point in \((-\varepsilon, \varepsilon)\) must be expandable for \( G_2 \) as well. The lemma follows since \( G_2 \) acts minimally on \( S^1 \).

Now denote by \( \partial \Gamma \) the geometric boundary of the hyperbolic group \( \Gamma \), see [20]. The boundary \( \partial \Gamma \) is a compact metric space which is effectively acted upon by the group \( \Gamma \) itself. Thus we often identify an element \( \gamma \in \Gamma \) with the corresponding automorphism of \( \partial \Gamma \) (still denoted by \( \gamma \)).

Next consider a finite subset \( \Sigma = \{\overline{\gamma_1}, \ldots, \overline{\gamma_r}, \overline{\gamma_1^{-1}}, \ldots, \overline{\gamma_r^{-1}}\} \) of \( \Gamma \) generating \( \Gamma \) as a semigroup. Consider a corresponding non-degenerate measure \( \mu \) on \( \Gamma \) (note
that the support of \( \mu \) does not need to be contained in \( \Sigma \). Once \( \mu \) is fixed, there is a unique stationary measure \( \nu_T \) on \( \partial \Gamma \) associated with the action of \( \Gamma \) on \( \partial \Gamma \) (cf. [24]). In other words, for every Borel set \( \mathcal{B} \in \partial \Gamma \), we have

\[
\nu_T(\mathcal{B}) = \sum_{\gamma \in \Gamma} \mu(\gamma) \nu_T(\gamma^{-1}(\mathcal{B}))
\]

where \( \gamma_i(\mathcal{B}) \) refers to the identification of \( \gamma \in \Gamma \) with the corresponding automorphism of \( \partial \Gamma \).

Now let \( g_{1,i} \in G_1 \) (resp. \( g_{2,i} \in G_2 \)) be defined as \( g_{1,i} = \rho_1(\bar{g}_i) \) (resp. \( g_{2,i} = \rho_2(\bar{g}_i) \)), \( i = 1, \ldots, r \). We also pose

\[
\Sigma_1 = \{g_{1,1}, \ldots, g_{1,r}; g_{1,1}^{-1}, \ldots, g_{1,r}^{-1}\} \quad \text{and} \quad \Sigma_2 = \{g_{2,1}, \ldots, g_{2,r}; g_{2,1}^{-1}, \ldots, g_{2,r}^{-1}\}.
\]

Since both representations \( \rho_1 \) and \( \rho_2 \) from \( \Gamma \) to \( \text{Diff}^\omega(S^1) \) are one-to-one, the groups \( G_1 \) and \( G_2 \) become equipped with the probability measure \( \mu \) up to the evident identifications.

Going back to the action of \( G_1 \) on \( S^1 \), Lemma 5.2 allows us to apply the main theorem of [9] to ensure the existence of a unique stationary measure \( \nu_1 \) for \( G_1 \) (with respect to \( \mu \)). The support of \( \nu_1 \) is all of \( S^1 \) since \( G_1 \) is minimal. It is also well known that \( G_1 \) gives no mass to points. Analogous conclusions hold for the stationary measure \( \nu_2 \) on \( S^1 \) arising from \( G_2 \) and \( \mu \). Now the combination of [8] with [24] yields the following.

**Lemma 6.4.** There is a measurable isomorphism \( \theta_2 \) from \( (\partial \Gamma, \nu_T) \) to \( (S^1, \nu_2) \).

**Proof.** Whereas \( G_2 \) was initially assumed to be locally \( C^2 \)-discrete, Lemma 6.3 shows that \( G_2 \) is, in fact, locally \( C^1 \)-discrete. Recalling that the measure \( \mu \) is assumed to have finite entropy, we apply Theorem 1.1 of [8] to the action of \( G_2 \) on \( S^1 \). Since \( G_2 \) is locally \( C^1 \)-discrete and \( \mu \) has finite entropy, all the conditions required by the theorem in question are satisfied so that the Poisson boundary of \( G_2 \) coincides with its \( (G_2, \mu) \)-boundary (see [8], [7] for terminology).

In turn, Kaimanovich theorem in [24] ensures that the Poisson boundary of \( G_2 \) can be identified with \( (\partial \mathcal{F}, \nu_T) \) (recall that \( G_2 \) is isomorphic to the fixed hyperbolic group \( \Gamma \)). Thus, to complete the proof of the lemma, it suffices to show that \( (G_2, \mu) \)-boundary of \( G_2 \) can be identified with \( (S^1, \nu_2) \). When \( G_2 \) is proximal, i.e. when every closed interval can be mapped to intervals of arbitrarily small length by means of elements of \( G \), this is exactly the contents of [2] and [25] since \( G_2 \) leaves no probability measure invariant (see Lemma 5.2).

In the general case, there is a finite topological quotient of \( S^1 \) where \( G_2 \) induces a proximal action. The construction of this quotient begins with the observation that arbitrarily small intervals of \( S^1 \) can always be expanded by the dynamics of \( G_2 \) beyond some uniform positive length. In fact, \( G_2 \) is topologically conjugate to \( G_1 \) which, in turn, clearly satisfies this condition since every point in \( S^1 \) is expandable for \( G_1 \) (Lemma 5.2). The existence of the desired quotient then follows from an argument due to Ghys, see [19], page 362.

To complete the proof of the lemma, just note that the quotient in question is endowed with a unique stationary measure \( \nu'_2 \). The pair \( (S^1, \nu'_2) \) is the \( (G, \mu) \)-boundary of the quotient owing to the result of Antonov and Kleptsyn-Nal’ski.
Finally, the \((G_2, \mu)\)-boundary of \(G_2\) can then be identified with \(S^1\) equipped with the pull-back (still denoted by \(v_2\)) of \(v'_2\) by the projection map. The lemma is proved. 

It is implicitly understood in the statement of Lemma 6.4 that \(\theta_2\) is \(\Gamma\)-equivariant in the sense that \(\theta_2^* v_2 = v_\Gamma\) and

\[
\theta_2 \circ \gamma(x) = \rho_2(\gamma) \circ \theta_2(x)
\]

for every \(\gamma \in \Gamma\) and \(v_\Gamma\)-almost all point \(x \in \partial \Gamma\). We are now ready to prove Theorem 6.2.

**Proof of Theorem 6.2.** Let \(h : S^1 \to S^1\) be a homeomorphism conjugating \(G_1\) to \(G_2\). By way of contradiction, we have assumed that \(G_1\) is locally \(C^2\)-non-discrete whereas \(G_2\) is locally \(C^2\)-discrete. From now on we fix \(\mu\) on \(\Gamma \approx G_1 \approx G_2\) satisfying the previous conditions and such that, in addition, the corresponding stationary measure \(v_1\) for \(G_1\) is absolutely continuous.

Recall also that \(v_1\) (resp. \(v_2\)) is the unique stationary measure for \(G_1\) (resp. \(G_2\)) with respect to \(\mu\) (see [9]). From the uniqueness of the stationary measure there follows again that \(h^* v_1 = v_2\).

Consider the measurable isomorphism \(\theta_2 : (\partial \Gamma, v_\Gamma) \to (S^1, v_2)\) of Lemma 6.4 and define a new measurable isomorphism \(\theta_1 : (\partial \Gamma, v_\Gamma) \to (S^1, v_1)\) by letting \(\theta_1 = h \circ \theta_2\). The equivariant nature of \(\theta_2\) expressed by Equation (19) combines with the fact that \(h^* v_1 = v_2\) to yield

\[
\theta_1 \circ \gamma(x) = \rho_1(\gamma) \circ \theta_1(x)
\]

for every \(\gamma \in \Gamma\) and \(v_\Gamma\)-almost all point \(x \in \partial \Gamma\). Furthermore \(\theta_2^* \nu_1 = \nu_\Gamma\). Up to eliminating null measure sets, we fix once and for all a Borel set \(\mathcal{B} \subset \partial \Gamma\) having full \(v_\Gamma\)-measure and such that Equation (20) holds for every \(x \in \mathcal{B}\) and every \(\gamma \in \Gamma\) (in particular both sides of this equation are well defined). To complete the proof of the proposition, we are going to show that the existence of \(\theta_1\) is not compatible with the fact that \(G_1\) is locally \(C^2\)-non-discrete. To do this, we proceed as follows.

Fix an interval \(I \subset S^1\) along with a sequence of elements \(\{g_j\} \subset G_1, g_j \neq \text{id}\) for every \(j \in \mathbb{N}\), whose restrictions to \(I\) converge to the identity in the \(C^2\)-topology. The existence of \(I\) and of \(\{g_j\}\) clearly follows from the assumption that \(G_1\) is locally \(C^2\)-non-discrete. Now Lusin approximation theorem [5] ensures the existence of a Cantor set \(K\) satisfying the following conditions:

1. \(K \subset I \cap \theta_1(\mathcal{B})\), i.e. \(K\) is contained in the domain of definition of \(\theta_1^{-1}\).
2. The restriction of \(\theta_1^{-1}\) to \(K\) is continuous from \(K\) to \(\partial \Gamma\) (where the reader is reminded that \(\partial \Gamma\) is a compact metric space).
3. \(v_1(K) \geq 9v_1(I)/10\).

Next, for each \(j\), let \(\gamma_j \in \Gamma\) be such that \(\rho_1(\gamma_j) = g_j\). **Claim 6.** There is a Cantor set \(K_{\Gamma} \subset \partial \Gamma\) such that the restrictions of the elements \(\gamma_j\) to \(K_{\Gamma}\) converge uniformly to the identity.
Proof of Claim 6. Since \( \{g_j\} \) converges to the identity in the \( C^1 \)-topology and \( \nu_1 \) is absolutely continuous, there follows that \( \nu_1(K \cap g_j^{-1}(K)) \) converges to \( \nu_1(K) \) as \( j \to \infty \). Therefore, up to passing to a subsequence, we can assume that

\[
K_{\infty} = K \cap \bigcap_{j=1}^{\infty} g_j^{-1}(K)
\]

is an actual (non-empty) Cantor set. Furthermore, by construction, \( K_{\infty} \subset K \) and \( g_j(K_{\infty}) \subset K \) for every \( j \in \mathbb{N}^+ \). Finally let \( K_{\Gamma} = \theta_1^{-1}(K_{\infty}) \).

To complete the proof of the claim, note that the restriction of \( \theta_1 \) to \( K_{\Gamma} \) is continuous since \( \theta_1^{-1} \) is continuous and one-to-one on the Cantor set \( K \) (and \( K_{\infty} \subset K \)). On the other hand, on \( K_{\Gamma} \) we have

\[
\gamma_j = \theta_1^{-1} \circ g_j \circ \theta_1
\]

i.e. the left hand side is well defined on \( K_{\Gamma} \). Since \( \theta_1 \) is continuous on \( K_{\Gamma} \) and \( \theta_1^{-1} \) is continuous on \( g_j \circ \theta_1(K_{\Gamma}) \subset K \), the fact that \( g_j \) converges uniformly (and actually \( C^1 \)) to the identity implies the claim. \( \square \)

We have just found a sequence \( \{\gamma_j\} \) of elements in \( \Gamma \), \( \gamma_j \neq \text{id} \) for every \( j \), whose restrictions to a (non-empty) Cantor set \( K_{\Gamma} \subset \partial \Gamma \) converge uniformly to the identity. The theorem now follows from Lemma 6.5 below claiming that such a sequence cannot exist in a finitely generated hyperbolic group. \( \square \)

To state Lemma 6.5 recall that every element \( \gamma \in \Gamma \) can be identified with the corresponding automorphism of \( \partial \Gamma \). Naturally \( \gamma \) can equally well be identified with its translation action on \( \Gamma \) which happens to be an isometry for the natural left-invariant metric on \( \Gamma \) (see [20]).

**Lemma 6.5.** Let \( \Gamma \) be a hyperbolic group which is neither finite nor a finite extension of \( \mathbb{Z} \). Let \( K_{\Gamma} \) be a Cantor set contained in the boundary \( \partial \Gamma \) of \( \Gamma \) and let \( \{\gamma_j\} \) be a sequence of elements in \( \Gamma \) thought of as automorphisms of \( \partial \Gamma \). Assume that the sequence \( \{\gamma_j|_{K_{\Gamma}}\} \) obtained by restricting \( \gamma_j \) to \( K_{\Gamma} \) converges uniformly to the identity. Then we have \( \gamma_j = \text{id} \) for large enough \( j \in \mathbb{N} \).

**Proof.** The lemma is certainly well known to the specialists albeit we have not been able to find it explicitly stated in the literature. In the sequel, the reader is referred to the chapters 7 and 8 of [20] for background material.

Assume for a contradiction that \( \gamma_j \neq \text{id} \) for every \( j \in \mathbb{N} \). Consider also a base point \( w \in \Gamma \) along with the sequence \( \gamma_j(w) \). Since \( \gamma_j \) acts as an isometry of \( \Gamma \), there follows that the sequence \( \{\gamma_j(w)\} \) leaves every compact part of \( \Gamma \). Thus, up to a passing to a subsequence, we assume that \( \gamma_j(w) \to b \in \partial \Gamma \).

Next fix another point \( a \in \partial \Gamma \sim K_{\Gamma} \), \( a \neq b \), and consider the family of metrics \( d_{c,a,w'} \) on \( \partial \Gamma \sim \{a\} \) for a fixed (small) \( c > 0 \) and where \( w' \in \Gamma \) (see [20, page 141]). Let \( \beta_a \) denote the Busemann function relative to the point \( a \in \partial \Gamma \). Since \( \gamma_j(w) \to b \), with \( b \neq a \), it follows from the general properties of Busemann functions that

\[
\beta_a(w, \gamma_j(w)) \to -\infty
\]
(cf. [20, page 136]). In particular, there is some uniform constant \( C \) such that
\[
\frac{1}{C} \exp(-\varepsilon \beta_a(w, \gamma_j(w))) \leq \frac{d_{\varepsilon, a, \gamma_j(w)}(x, y)}{d_{\varepsilon, a, w}(x, y)} \leq C \exp(-\varepsilon \beta_a(w, \gamma_j(w)))
\]
see [20, page 141]. In other words, the metric \( d_{\varepsilon, a, \gamma_j(w)} \) is bounded from below and by above by the metric \( d_{\varepsilon, a, w} \) multiplied by suitable constants going to infinity as \( j \to \infty \). However, by construction, these metrics also satisfy
\[
d_{\varepsilon, a, \gamma_j(w)}(\gamma_j(x), \gamma_j(y)) = d_{\varepsilon, a, w}(x, y).
\]
Therefore
\[
\frac{d_{\varepsilon, a, w}(x, y)}{d_{\varepsilon, a, w}(\gamma_j(x), \gamma_j(y))} \to \infty
\]
uniformly for every pair \( x \neq y \) in \( \partial \Gamma \sim \{a\} \). The desired contradiction now arises by choosing \( x \neq y \in K \) so that \( \gamma_j(x) \to x \) and \( \gamma_j(y) \to y \). The proof of the lemma is completed.

7. Appendix: On Locally \( C^r \)-Non-Discrete Groups

For \( r \geq 2 \), every subgroup \( G \) of \( \text{Diff}^r(S^1) \) that is locally \( C^r \)-non-discrete is clearly locally \( C^l \)-non-discrete for every \( l \leq r \). A sort of converse for the above claim also holds in most cases. This is the content of the theorem below.

**Theorem 7.1.** Let \( G \subset \text{Diff}^r(S^1) \) be a non-solvable group and assume that \( G \) is locally \( C^2 \)-non-discrete. Then \( G \) is locally \( C^\infty \)-non-discrete.

To prove Theorem 7.1 we shall use the same technique of regularization (or renormalization) employed in Section 4. By assumption there is an open (non-empty) interval \( I \subset S^1 \) and a sequence \( \{f_j\} \), \( f_j \neq \text{id} \) for every \( j \in \mathbb{N} \), of elements in \( G \) whose restrictions to \( I \) converge to the identity in the \( C^2 \)-topology. In fact, arguing as in Section 3, we can assume without loss of generality that the following holds: for every given \( \varepsilon > 0 \), there is a finite set \( \overline{f}_1, \ldots, \overline{f}_N \) of elements in \( G \) satisfying the two conditions below.

- The group \( G_{(\varepsilon, N)} \subset G \) generated by \( \overline{f}_1, \ldots, \overline{f}_N \) is not solvable.
- For every \( i \in \{1, \ldots, N\} \), the restrictions of \( \overline{f}_i \) and of \( \overline{f}_i^{-1} \) to the interval \( I \) are \( \varepsilon \)-close to the identity in the \( C^2 \)-topology on \( I \).

First we state:

**Proposition 7.2.** If \( \varepsilon > 0 \) is small enough, then the group \( G_{(\varepsilon, N)} \) is locally \( C^r \)-non-discrete for every \( r \in \mathbb{N} \).

The proof of Theorem 7.1 can be derived from Proposition 7.2 as follows.

**Proof of Theorem 7.1.** We can assume once and for all that \( G_{(\varepsilon, N)} \) has no finite orbits, otherwise Theorem 7.1 follows at once from the discussion in Section 2.1. In turn, it is clearly sufficient to prove that the subgroup \( G_{(\varepsilon, N)} \) is locally \( C^\infty \)-non-discrete provided that \( \varepsilon > 0 \) is small enough. This is equivalent to finding an open, non-empty interval \( I_{\infty} \subset S^1 \) on which “\( G_{(\varepsilon, N)} \) is locally \( C^r \)-non-discrete for every \( r \in \mathbb{N} \)”.

Almost every $f_j$ is not the identity for every $j \in \mathbb{N}$, of elements in $G_{(\epsilon,N)}$ whose restrictions to $I_{\infty}$ converge to the identity in the $C'$-topology.

On the other hand, by assumption, to every $r \in \mathbb{N}$ there corresponds a non-trivial sequence $\{f_j \cdot C_r\}_{j \in \mathbb{N}}$ of elements in $G_{(\epsilon,N)}$ whose restriction to some open, non-empty interval $I_r$ converges to the identity in the $C'$-topology on $I_r$. Thus the only difficulty to derive Theorem A lies in the fact that the intervals $I_r$ depend on $r$. To show that these intervals can be chosen in a uniform way, we proceed as follows.

First recall that $G_{(\epsilon,N)}$ contains an element $F$ exhibiting a hyperbolic fixed point. Furthermore $S^1$ can be covered by finitely many intervals $J_1, \ldots, J_j$ such that each interval $J_i$ is equipped with a constant (non-zero) vector field $X_i$ in the $C^1$-closure of $G_{(\epsilon,N)}$; cf. Theorem 3.4 of [34] (which, in particular, recovers the fact that all orbits of $G$ are dense in $S^1$). By using these constant vector fields and the diffeomorphism $F$, we obtain a sequence $F_r$ of elements in $G_{(\epsilon,N)}$ satisfying the following conditions:

- The diffeomorphism $F_r$ has an attracting hyperbolic fixed point $p_r$ lying in $I_r$.
- The basin of attraction of $p_r$ with respect to $F_r$ has length greater than a certain $\delta > 0$ (in other words, there is $\delta > 0$ such that $F_r$ has no other fixed point on a $\delta$-neighborhood of $p_r$).

Now each interval $I_r$ can be “re-scaled” by means of $F_r$ so as to have length bounded from below by $\delta$. More precisely, fixed $r$ and $n_r \in \mathbb{N}$, the sequence of elements of $G_{(\epsilon,N)}$ given by $j \mapsto F^{-n_r} \circ f_j \cdot C_r \circ F^{n_r}$ clearly converges to the identity in the $C'$-topology on the interval $I_r = F^{-n_r}(I_r)$. The above stated conditions on the diffeomorphisms $F_r$ then ensure that $n_r$ can be chosen so that $I_r = F^{-n_r}(I_r)$ has length bounded from below by $\delta > 0$. Up to passing to a subsequence, the sequence of intervals $\{I_r\}$ must converge to a uniform interval $I_\infty$ satisfying the desired conditions. The proof of Theorem 7.1 is completed.

As in Section 4, we consider the sequence of sets $S(k)$ defined by means of the initial set $S = S(0) = \{f_1, \ldots, f_N\}$. Since the group generated by $f_1, \ldots, f_N$ is not solvable, none of the sets $S(k)$ is reduced to the identity diffeomorphism.

We can now prove Proposition 7.2.

Proof of Proposition 7.2. The proof is essentially by induction. First we are going to prove that $G_{(\epsilon,N)}$ is locally $C^3$-non-discrete. To do this, we proceed as follows. Consider a fixed set $\{f_1, \ldots, f_N\}$ generating a non-solvable group $G_{(\epsilon,N)}$ as before. Assume moreover that for every $i = 1, \ldots, N$, both diffeomorphisms $f_i$ and $f_i^{-1}$ are $\epsilon$-close to the identity in the $C^2$-topology on $I$ where the value of $\epsilon > 0$ will be fixed later on.

As already seen, the group $G_{(\epsilon,N)}$ contains an element $F$ exhibiting a hyperbolic fixed point in $I$. Without loss of generality, we can assume that this fixed point coincides with $0 \in I \subset \mathbb{R}$. Furthermore in suitable coordinates, $F$ becomes a homothety $x \mapsto \lambda x$ on all of the interval $I$. Still keeping the notation of Section 4, consider the sequence of sets $S(k)$ given by $S(k) = F^{-kn} \circ S(k) \circ F^{kn}$ for
some \(n \in \mathbb{N}^*\) fixed. We will show that the diffeomorphisms in \(\tilde{S}(k)\) converge to the identity in the \(C^3\)-topology on \(I\) provided that \(n\) is suitably chosen.

Claim 7. There is \(n \in \mathbb{N}\) such that every non-trivial sequence \(\{\tilde{f}_k\}\), with \(\tilde{f}_k \in \tilde{S}(k)\), converges to the identity in the \(C^3\)-topology on \(I\).

Proof of Claim 7. Fix a sequence \(\{\tilde{f}_k\}\) as in the statement. In Section 4 it was seen that these elements converge to the identity in the \(C^2\)-topology. More precisely, we have

\[
\|\tilde{f}_k - \text{id}\|_{2,1} < \frac{\epsilon}{\sqrt{2^k}}
\]

for every diffeomorphism \(\tilde{f}_k \in \tilde{S}(k)\) and for a suitable fixed \(n\). To show that convergence takes place in the \(C^3\)-topology as well, we first estimate the third derivative \(D^3[\tilde{f}_1, \tilde{f}_2]\) of a commutator \([\tilde{f}_1, \tilde{f}_2] = \tilde{f}_1 \circ \tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \tilde{f}_2^{-1}\). For this we shall use the fact that \(\tilde{f}_1, \tilde{f}_2\) and their inverses \(\tilde{f}_1^{-1}, \tilde{f}_2^{-1}\) are \(C^2\)-close to the identity. Recall then that higher order derivatives of a composed function are given by Faà di Bruno formula which, in the present case, simply means

\[
D^3(f_1 \circ f_2) = D^3_{f_2(x)} f_1(D_x f_2)^3 + 3D^2_{f_2(x)} f_1 D_x^2 f_2 + D^1 f_1 D_x f_2 + D^3 f_1 D_x^3 f_2.
\]

Thus, if \(\epsilon > 0\) is sufficiently small, we have

\[
|D^1(f_1 \circ f_2)| \leq 3 \max \sup_I |D^1(f_1 - \text{id})|, \sup_I |D^1(f_2 - \text{id})|);
\]

\[
D^2(f_1 \circ f_2) \leq 3 \max \sup_I |D^2 f_1|, \sup_I |D^2 f_2|);
\]

\[
D^3(f_1 \circ f_2) \leq 3 \max \sup_I |D^3 f_1|, \sup_I |D^3 f_2|).
\]

Similar estimates also hold for \(D^1(\tilde{f}_1^{-1} \circ \tilde{f}_2^{-1})\), \(D^2(\tilde{f}_1^{-1} \circ \tilde{f}_2^{-1})\), and \(D^3(\tilde{f}_1^{-1} \circ \tilde{f}_2^{-1})\).

If \(\epsilon > 0\) is small enough, then the preceding estimates can also be applied to \((f_1 \circ f_2) \circ (f_1^{-1} \circ f_2^{-1})\) so as to yield

\[
D^3(f_1, f_2) \leq 10 \max \sup_I |D^3 f_1|, \sup_I |D^3 f_2|, \sup_I |D^3 f_1^{-1}|, \sup_I |D^3 f_2^{-1}|
\]

provided that \(f_1, f_2, f_1^{-1}, f_2^{-1}\) are \(\epsilon\)-close to the identity in the \(C^2\)-topology. From estimate (23), there follows that

\[
D^3(F^{-n} \circ [f_1, f_2] \circ F^n) = D^3(\lambda^{-n} \cdot [f_1, f_2](\lambda^n x)) \leq 10\lambda^{2n} \max \sup_I |D^3 f_1|, \sup_I |D^3 f_2|, \sup_I |D^3 f_1^{-1}|, \sup_I |D^3 f_2^{-1}|).
\]

If \(n\) is chosen so that \(\lambda^{2n} < 1/10\), there follows that the third order derivatives of elements in \(\tilde{S}(1)\) are smaller than the maximum of the third order derivatives of elements in \(\tilde{S}(0)\). This procedure can be iterated to higher order commutators by virtue of Estimate (21) so that third order derivatives of elements in \(\tilde{S}(k)\) actually decay geometrically with \(k\). The claim results at once.
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claim (the general Faà di Bruno formulas can be used in the context). The detail is left to the reader.

**Final comments.** We close this paper by pointing out a couple of specific issues involved in our regularization scheme for iterated commutators, as explained above and in Section 4. First, the reader will note that the analytic assumption is not needed in order to ensure the corresponding diffeomorphisms converge to the identity. The importance of the analytic assumption lies in the fact that the sequence of sets $S(k)$ (and hence $\tilde{S}(k)$) does not degenerate into $\{\text{id}\}$. As mentioned this result is due to Ghys [17] and has a formal algebraic nature: it depends on ensuring that a $C^\infty$-diffeomorphism $f$ of $S^1$ coincides with the identity so long there is a point in $S^1$ at which $f$ is $C^\infty$-tangent to the identity. It would be nice to know whether or not there are finitely generated pseudo-solvable, yet non-solvable, groups in $\text{Diff}^\infty(S^1)$.

Finally note also that our regularization technique falls short of working in the $C^1$-case. Therefore, even in the analytic category, we have not proved that a locally $C^1$-non-discrete subgroup of $\text{Diff}^\omega(S^1)$ is also locally $C^\infty$-non-discrete. Although this statement is very likely to hold, the renormalization procedure $x \mapsto \lambda x$ used here does not decrease the first order derivative of the diffeomorphism and this accounts for the special nature of locally $C^1$-non-discrete groups. To overcome this difficulty, our iteration scheme must be further elaborated. This can probably be done by suitably adding further “take the commutator” steps so as to keep control on the growing rate of first order derivatives.

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