THE TRIANGLE AND THE OPEN TRIANGLE

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ABSTRACT. We show that for percolation on any transitive graph, the triangle condition implies the open triangle condition.

1. INTRODUCTION

Let $G$ be a vertex-transitive connected graph, and let $p$ be some number in $[0,1]$. We say that $p$-percolation on $G$ satisfies the triangle condition if for some $v \in G$

$$\sum_{x,y \in G} \mathbb{P}(v \leftrightarrow x)\mathbb{P}(x \leftrightarrow y)\mathbb{P}(y \leftrightarrow v) < \infty.$$ (1)

where $x \leftrightarrow y$ implies that there exists an open path between $x$ and $y$. Here and below we abuse notations by denoting “$v$ is a vertex of $G$” by $v \in G$. Of course, by transitivity, the sum is in fact independent of $v$. This note is far too short to explain the importance of the triangle condition. Suffices to say that it the triangle condition holds at the critical $p$, then many exponents take their mean-field values. See [AN84, N87, BA91, KN09] for corollaries of the triangle condition.

On the other hand, the triangle condition holds in many interesting cases, see [HS90, HHS08] for the graphs $\mathbb{Z}^d$ with $d$ sufficiently large, and [S01, S02, K] for various other transitive graphs. See [G99] or [BR06] for a general introduction to percolation.

In many applications the triangle condition (1) is not so convenient to use. One instead uses the open triangle condition, which states that

$$\lim_{R \to \infty} \max_{w \notin B(v,R)} \sum_{x,y \in G} \mathbb{P}(v \leftrightarrow x)\mathbb{P}(x \leftrightarrow y)\mathbb{P}(y \leftrightarrow w) = 0,$$

where $B(v,R)$ stands for the ball around $v$ with radius $R$ in the graph (or shortest path) distance. Clearly, the open triangle condition implies the (closed) triangle condition (recall that if $y$ and $y'$ are neighbors in the graph then $\mathbb{P}(x \leftrightarrow y) \geq c\mathbb{P}(x \leftrightarrow y')$ for some constant $c$ independent of $x$, $y$ and $y'$). The contents of lemma 2.1 of Barsky & Aizenman [BA91] is the reverse implication. The proof in [BA91] is specific to the graph $\mathbb{Z}^d$, as it uses the Fourier transform of the function $f(x) = \mathbb{P}(\vec{0} \leftrightarrow x)$. The purpose of this note is to generalize this to any transitive graph, namely

A vertex-transitive graph, and any other notion not specifically defined, may be found in Wikipedia.
Theorem. Let $G$ be a vertex-transitive graph and let $p \in [0,1]$. Assume $G$ satisfies the triangle condition at $p$. Then $G$ satisfies the open triangle condition at $p$.

This result is not particularly important. For example, in $S01, S02$ the author simply circumvents the problem by working directly with the open triangle condition. The advantage of making the triangle condition “the” marker for mean-field behavior is mostly aesthetic. The real reason for the existence of this note is to demonstrate an application of operator theory, specifically of spectral theory, to percolation. Operator theory is a fantastically powerful tool whose absence from the percolation scene is behind many of the difficulties one encounters. I aim to remedy this situation, even if by very little.

I wish to thank Asaf Nachmias for pointing out some omissions in a draft version of the paper, and Michael Aizenman for an interesting discussion of alternative proof approaches.

2. The proof

Before starting the proof proper, let us make a short heuristic argument. Define the infinite matrix

$$B(v,w) = \mathbb{P}(v \leftrightarrow w)$$

where in the notation we assume that $v \leftrightarrow v$ always so $B(v,v) = 1$. By $AN84$ $B$, considered as an (unbounded) operator on $l^2(G)$ is a positive operator. Hence the same holds for

$$Q(v,w) = \sum_{x,y} B(v,x)B(x,y)B(y,w)$$

which is just $B^3$ (as an infinite matrix or as an unbounded operator). It is possible to take the square root of any positive operator, so denote $S = \sqrt{Q}$. We get

$$Q(v,w) = \langle Q1_v, 1_w \rangle = \langle S1_v, S1_w \rangle$$

where $1_v$ is the element of $l^2(G)$ defined by

$$1_v(x) = \begin{cases} 1 & v = x \\ 0 & v \neq x. \end{cases}$$

Hence the triangle condition $Q(v,v) < \infty$ implies that $\|Sv\| < \infty$. But $S$ is invariant to the automorphisms of $G$ (as a root of $Q$ which is invariant to them) so $S1_v$ is a map of $S1_v$ under an automorphism $\varphi$ taking $v$ to $w$. But any vector in $l^2$ is almost orthogonal to sufficiently far away “translations” (namely, the automorphisms of $G$), so $\langle S1_v, S1_w \rangle \to 0$ as the graph distance of $v$ and $w$ goes to $\infty$, as required.

Why is this even a heuristic and not a full proof? Because of the benign looking expression $\langle Q1_v, 1_w \rangle$ which is in fact meaningless. $Q$ is an unbounded operator and hence it cannot be applied to any vector in $l^2(G)$, and there is nothing guaranteeing that $1_v$ will be in its domain. For example, in a sufficiently
spread-out lattice in \( \mathbb{R}^d \) one has that \( P(x \leftrightarrow y) \approx |x - y|^{2 - d} \) \cite{HHS03} which gives with a simple calculation that the triangle condition holds whenever \( d > 6 \) while \( Q1_y \in l^2 \) only when \( d > 12 \).

The proof below circumvents this problem by decomposing \( B \) into a sum of positive bounded operators using specific properties of \( B \). Somebody more versed in the theory of unbounded operators might have constructed a more direct proof.

We start the proof proper with

**Definition.** Let \( \varphi \) be an automorphism of the graph \( G \). We define the isometry \( \Phi = \Phi_\varphi \) of \( l^2(G) \) corresponding to \( \varphi \) by

\[
(\Phi(f))(v) = f(\varphi^{-1}(v)).
\]

It is easy to check that \( \Phi 1_v = 1_{\varphi(v)} \) and that the support of \( \Phi f \) is \( \varphi \) (the support of \( f \)).

**Lemma.** Let \( f \in l^2(G) \), let \( v \in G \) and let \( \delta > 0 \). Then there exists an \( R = R(f, \delta, v) \) such that for any \( w \notin B(v, R) \) and any automorphism \( \varphi \) of \( G \) taking \( v \) to \( w \) one has

\[
|\langle \Phi_\varphi f, f \rangle| < \delta
\]

**Proof.** Let \( A \subset G \) be some finite set of vertices such that

\[
\sqrt{\sum_{v \notin A} |f(v)|^2} < \frac{1}{3\|f\|} \delta.
\]

Write now

\[
f = f_{\text{loc}} + f_{\text{glob}} \text{ where } f_{\text{loc}} = f \cdot 1_A.
\]

By the definition of \( A \), \( \|f_{\text{glob}}\| < \frac{1}{3\|f\|} \delta \), and so by Cauchy-Schwarz,

\[
|\langle \Phi f, f \rangle| \leq |\langle \Phi f_{\text{loc}}, f_{\text{loc}} \rangle| + 2\|f_{\text{glob}}\| \cdot \|f_{\text{loc}}\| + \|f_{\text{glob}}\|^2 < |\langle \Phi f_{\text{loc}}, f_{\text{loc}} \rangle| + \delta.
\]

Define now

\[
R = 2 \max_{x \in A} d(v, x) + 1.
\]

To see (5), let \( w \) and \( \varphi \) be as above. We get, for any \( x \in A \),

\[
d(\varphi(x), v) \geq d(v, w) - d(\varphi(x), w).
\]

Now, \( d(\varphi(x), w) = d(\varphi(x), \varphi(v)) = d(x, v) < \frac{1}{2} R \) because \( \varphi \) is an automorphism of \( G \). Hence we get

\[
d(\varphi(x), v) > R - \frac{1}{2} R
\]

implying that \( \varphi(x) \notin A \) as it is too far. In other words, \( A \cap \varphi(A) = \emptyset \) which implies that \( \langle \Phi_\varphi f_{\text{loc}}, f_{\text{loc}} \rangle = 0 \). With (5), the lemma is proved. \[\square\]

**Proof of the theorem.** We will not keep \( p \) in the notations as it does not change throughout the proof. For every \( n \in \mathbb{N} \) and every \( v, w \in G \), let \( B_n(v, w) \) be defined by

\[
B_n(v, w) = P(v \leftrightarrow w, |\mathcal{G}(v)| = n)
\]
where $\mathcal{C}(v)$ is the cluster of $v$ i.e. the set of vertices connected to $v$ by open paths, and $|\mathcal{C}(v)|$ is the number of vertices in $\mathcal{C}(v)$. Clearly $B_n(v, w) \geq 0$ and

$$B(v, w) = \sum_{n=1}^{\infty} B_n(v, w) \quad (7)$$

where $B$ is as above $[2]$. Therefore we may write

$$Q(v, w) = \sum_{x,y} B(v, x)B(x, y)B(y, w) \leq \sum_{x,y} B(v, x) \left( \sum_{n=1}^{\infty} B_n(x, y) \right) B(y, w) = \sum_{n=1}^{\infty} \sum_{x,y} B(v, x)B_n(x, y)B(y, w) \quad (8)$$

where the change of order of summation in the last equality is justified since all terms are positive. Now, the vector

$$B1_w = (B(y, w))_{y \in G}$$

is in $l^2(G)$ because

$$\sum_{y} B(y, w)^2 \leq \sum_{y,x} B(w, y)B(y, x)B(x, w) < \infty.$$ 

Further, each $B_n$, considered as an operator on $l^2(G)$ is bounded, because the sum of the (absolute values of the) entries in each row and each column is finite. From this we conclude that $B_nB1_w \in l^2(G)$ and we may present the sum in (8) in an $l^2$ notation as

$$Q(v, w) = \sum_{n=1}^{\infty} \langle B_nB1_v, B1_w \rangle. \quad (9)$$

Next we employ the argument of Aizenman & Newman [AN84] to show that $B_n$ is a positive operator. This means that $B_n(v, w) = B_n(w, v)$ (which is obvious) and that $\langle B_n f, f \rangle \geq 0$ for any (real-valued) $f \in l^2$. It is enough to verify this for $f$ with finite support. But in this case we can write

$$\langle B_n f, f \rangle = \sum_{v,w} f(v)f(w) P(v \leftrightarrow w, |\mathcal{C}(v)| = n) = \mathbb{E}\left( \sum_{v,w} f(v)f(w) 1_{|v \leftrightarrow w, |\mathcal{C}(v)| = n} \right) = \mathbb{E}\left( \sum_{\mathcal{C} \text{ s.t.} |\mathcal{C}| = n} \sum_{v,w \in \mathcal{C}} f(v)f(w) \right) = \mathbb{E}\left( \sum_{\mathcal{C} \text{ s.t.} |\mathcal{C}| = n} \left( \sum_{v \in \mathcal{C}} f(v) \right)^2 \right) \geq 0,$$

where $(\ast)$ is where we used the fact that $f$ has finite support to justify taking the expectation out of the sum. The notation $1_E$ here is for the indicator of the event $E$. Thus $B_n$ is positive.

We now apply the spectral theorem for bounded positive operators to take the square root of $B_n$. See [EMT04], lemma 6.3.5 for the specific case of taking the root of a positive operator and chapter 7 for general spectral theory. Denote
\[ S_n = \sqrt{B_n}. \] This implies, of course, that \( S_n^2 = B_n \) but also that \( S_n \) is positive and that it commutes with any operator \( \Phi \) that commutes with \( B_n \).

Returning to (9) we now write
\[
Q(v, w) = \sum_{n=1}^{\infty} \langle S_n^2 B_1 v, B_1 w \rangle = \sum_{n=1}^{\infty} \langle S_n B_1 v, S_n B_1 w \rangle. \tag{10}
\]

The fact that \( Q(v, v) < \infty \) therefore implies that
\[
\sum_{n=1}^{\infty} ||S_n B_1 v||^2 < \infty. \tag{11}
\]

Our only use of the triangle condition.

Fix now some \( \epsilon > 0 \). By (11) we can find some \( N \) such that
\[
\sum_{n=N+1}^{\infty} ||S_n B_1 v||^2 < \frac{1}{2} \epsilon. \tag{12}
\]

Since \( S_n B_1 v \in l^2(G) \), we can use the lemma, and we use it with
\[
f_{\text{lemma}} = S_n B_1 v \quad v_{\text{lemma}} = v \quad \delta_{\text{lemma}} = \frac{\epsilon}{2N}.
\]

We get some \( R_n \) such that for any \( \varphi \) taking \( v \) outside of \( B(v, R_n) \),
\[
|\langle \Phi \varphi S_n B_1 v, S_n B_1 v \rangle| \leq \frac{\epsilon}{2N}.
\]

Some standard abstract nonsense shows that the invariance of \( B_n \) i.e. the fact that \( B_n(x, y) = B_n(\varphi(x), \varphi(y)) \) implies that \( B_n \Phi = \Phi B_n \). Hence also \( S_n \Phi = \Phi S_n \) so
\[
\langle \Phi S_n B_1 v, S_n B_1 v \rangle = \langle S_n B \Phi B_1 v, S_n B_1 v \rangle = \langle S_n B_1 \varphi(v), S_n B_1 v \rangle.
\]

Define \( R = \max\{R_1, \ldots, R_N\} \). We get, for every \( w \not\in B(v, R) \),
\[
\sum_{n=1}^{N} \langle S_n B_1 v, S_n B_1 w \rangle \leq N \delta = \frac{1}{2} \epsilon. \tag{13}
\]

(12) takes care of the other sum,
\[
\sum_{n=N+1}^{\infty} \langle S_n B_1 v, S_n B_1 w \rangle \leq \sum_{n=N+1}^{\infty} ||S_n B_1 v|| \cdot ||S_n B_1 w|| = \\
\quad \quad \quad = \sum_{n=N+1}^{\infty} ||S_n B_1 v||^2 < \frac{1}{2} \epsilon. \tag{14}
\]

We are done. We get that for any \( w \not\in B(v, R) \),
\[
Q(v, w) = \sum_{n=1}^{\infty} \langle S_n B_1 v, S_n B_1 w \rangle \leq \epsilon
\]
as required. \(\square\)
Closing remark. Comparing the proof here to that of Barsky & Aizenman [BA91], it seems as if there is something missing in their argument. This is not true. Justifying the change of order of summation in [BA91] is completely standard — for example, by examining Cesàro sums — and does not deserve any special remark.

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