Halfspace Depths for Scatter, Concentration and Shape Matrices

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April 2017

ECARES working paper 2017-19
HALFSPACE DEPTHS FOR SCATTER,
CONCENTRATION AND SHAPE MATRICES

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We propose halfspace depth concepts for scatter, concentration and shape matrices. For scatter matrices, our concept extends the one from Chen, Gao and Ren (2015) to the non-centered case, and is in the same spirit as the one in Zhang (2002). Rather than focusing, as in these earlier works, on deepest scatter matrices, we thoroughly investigate the properties of the proposed depth and of the corresponding depth regions. We do so under minimal assumptions and, in particular, we do not restrict to elliptical distributions nor to absolutely continuous distributions. Interestingly, fully understanding scatter halfspace depth requires considering different geometries/topologies on the space of scatter matrices. We also discuss, in the spirit of Zuo and Serfling (2000), the structural properties a scatter depth should satisfy, and investigate whether or not these are met by the proposed depth. As mentioned above, companion concepts of depth for concentration matrices and shape matrices are also proposed and studied. We illustrate the practical relevance of the proposed concepts by considering a real-data example from finance.

1. Introduction. Statistical depth allows to measure the centrality of a given location in $\mathbb{R}^k$ with respect to a sample of $k$-variate observations, or, more generally, with respect to a probability measure $P$ over $\mathbb{R}^k$. The most famous depths include the halfspace depth (Tukey, 1975), the simplicial depth (Liu, 1990), the spatial depth (Vardi and Zhang, 2000) and the projection depth (Zuo, 2003). In the last few years, depth has also known much success in functional data analysis, where it allows to measure the centrality of a function with respect to a sample of functional data. Some instances are the band depth (López-Pintado and Romo, 2009), the functional halfspace depth (Claeskens et al., 2014) and the functional spatial depth (Chakraborty and Chaudhuri, 2014a,b). The large variety of available depths made it necessary to introduce an axiomatic approach identifying the most

*Research is supported by the IAP research network grant nr. P7/06 of the Belgian government (Belgian Science Policy), the Crédit de Recherche J.0113.16 of the FNRS (Fonds National pour la Recherche Scientifique), Communauté Française de Belgique, and a grant from the National Bank of Belgium.

†Research is supported by the FC8444 grant of the FNRS (Fonds National pour la Recherche Scientifique), Communauté Française de Belgique.

MSC 2010 subject classifications: Primary 62H20; secondary 62G35

Keywords and phrases: curved parameter spaces, elliptical distributions, robustness, scatter matrices, shape matrices, statistical depth
desirable properties of a depth function; see Zuo and Serfling (2000) in the multivariate case and Nieto-Reyes and Battey (2016) in the functional one. By using optimal transport methods, Chernozhukov et al. (2017) recently made depth suitable also for distributions with a non-convex support; we refer to Agostinelli and Romanazzi (2011) or to Paindaveine and Van Bever (2013) for an alternative solution based on the concept of local depth.

Irrespective of the depth adopted, statistical depth provides a center-outward ordering of the observations that allows to tackle in a robust and nonparametric way a broad range of inference problems; see Liu, Parelius and Singh (1999). For most depths, the deepest point is a robust location functional that extends the univariate median to the multivariate or functional setups; see, in particular, Cardot, Cénac and Zitt (2013) and Cardot, Cénac and Godichon-Baggioni (2017) for recent works on the functional spatial median. Beyond the median, depth plays a key role in the classical problem of defining a concept of multivariate quantile; see, e.g., Hallin, Paindaveine and Šiman (2010) or Serfling (2010). In line with this, the depth regions, that collect locations in \( \mathbb{R}^k \) with depth larger than or equal to a given level, are sometimes called quantile regions; see, e.g., He and Einmahl (2016) in a multivariate extreme value theory framework. In the functional case, the quantiles introduced in the seminal paper Chaudhuri (1996) may be seen as those associated with functional spatial depth; see Chakraborty and Chaudhuri (2014a,b). Both in the multivariate and functional cases, supervised classification is also a standard application of depth; see, e.g., Ghosh and Chaudhuri (2005), López-Pintado and Romo (2006), Cuesta-Albertos and Nieto-Reyes (2010), Li, Cuesta-Albertos and Liu (2012) or Paindaveine and Van Bever (2015). Finally, depth is a very natural tool for outlier detection; we refer, e.g., to Dang and Serfling (2010) in the multivariate case and to Hubert, Rousseeuw and Segaert (2015) in the functional one.

In Mizera (2002), the concept of statistical depth was extended from the above location framework to a virtually arbitrary parametric one. In a generic parametric model indexed by an \( \ell \)-variate parameter \( \theta \), the resulting tangent depth \( D(\theta_0, P_n) \) measures how appropriate a fixed parameter value \( \theta_0 \) is, with respect to the empirical measure \( P_n \) of a sample of \( k \)-variate observations \( X_1, \ldots, X_n \) at hand. This allows to order “candidate fits” \( \theta \) according to the depth function \( \theta \mapsto D(\theta, P_n) \), as one could alternatively do on the basis of the likelihood function \( \theta \mapsto L(\theta, P_n) \). While likelihoods will provide an estimator of \( \theta \) (namely, the MLE of \( \theta \)) that may suffer from a lack of robustness, the depth-based estimator maximizing \( D(\theta, P_n) \) is robust under broad conditions; see Section 4 of Mizera (2002). Moreover, likelihood-based approaches typically apply to parametric setups, whereas tangent depth can be used in a more general semiparametric framework involving infinite-dimensional nuisance parameters.

The construction proved useful in multiple linear regression, where the resulting parametric depth reduces to the regression depth concept from Rousseeuw and Hubert (1999) (\( \ell = p + 1 \), where \( p \) is the number of random covariates), in the location-scale parametric context considered in Mizera and Müller (2004) (\( \ell = 2 \)), for copulas indexed by a scalar
parameter in Denecke and Müller (2011) \((\ell = 1)\) or for correlation coefficients in Denecke and Müller (2014) \((\ell = 1)\). Tangent depth, however, cannot deal with parameters of moderate to large dimensions, as it requires evaluating the usual (that is, location) halfspace depth of a given location in \(\mathbb{R}^\ell\), which can only be achieved for rather small values of \(\ell\) (similar computational issues will affect the simplicial version of parametric depth; see Denecke and Müller (2012) and the references therein). In particular, for observations with dimensions as low as \(k = 3\) or \(4\), parametric depth cannot cope with covariance or scatter matrix parameters \((\ell = k(k + 1)/2)\), nor, a fortiori, with location-scatter parameters \((\ell = k + k(k + 1)/2)\).

The crucial role played by scatter matrices in multivariate statistics makes it highly desirable, however, to have a satisfactory depth for such parameters. The need for such a concept has actually been acknowledged in the literature. In particular, Serfling (2004) explicitly called for an extension of the Mizera and Müller (2004) location-scale depth concept into a location-scatter one. While computational issues prevent from basing this extension on tangent depth, a more ad hoc approach such as the one proposed in Zhang (2002) is suitable. Recently, another concept of scatter depth, that is very close in spirit to the one proposed in Zhang (2002), was introduced in Chen, Gao and Ren (2015). Both proposals dominate tangent depth in the sense that, for \(k\)-variate observations, they rely on projection pursuit in \(\mathbb{R}^k\) rather than in \(\mathbb{R}^{k(k+1)/2}\), which allowed Chen, Gao and Ren (2015) to consider their depth even in high dimensions, under, e.g., sparsity assumptions. Both works, however, mainly focus on asymptotic, robustness and/or minimax convergence properties of the sample deepest scatter matrix. The properties of these scatter depths thus remain largely unknown, which severely affects the interpretation of the sample concept.

In the present work, we introduce a concept of halfspace depth for scatter matrices that is close to the Zhang (2002) and Chen, Gao and Ren (2015) ones. Unlike these previous works, however, we thoroughly study the properties of the proposed depth and of the corresponding depth regions. We do so under minimal assumptions and, in particular, we do not restrict to elliptical distributions nor to absolutely continuous distributions. Interestingly, fully understanding scatter halfspace depth requires considering different geometries/topologies on the space of scatter matrices. Like Donoho and Gasko (1992) and Rousseeuw and Ruts (1999) did for location halfspace depth, we study continuity and quasi-concavity properties of scatter halfspace depth, as well as the boundedness, convexity and compacity properties of the corresponding depth regions. Existence of a deepest halfspace scatter matrix, which is not guaranteed a priori, is also investigated. We further discuss, in the spirit of Zuo and Serfling (2000), the structural properties a scatter depth should satisfy and we investigate whether the proposed depth does satisfy these. Moreover, companion concepts of depth for concentration matrices and shape matrices are proposed and studied. To the best of our knowledge, our results are the first providing structural and topological properties of depth regions outside the classical location framework. Throughout, numerical results illustrate our theoretical findings. Finally, we illustrate the practical
relevance of the proposed concepts by considering a real-data example from finance, where we show that the depth of intraday scatter and shape matrices can be used to detect days that are outlying in terms of volatility.

The outline of the paper is as follows. In Section 2, we define the proposed scatter depth concept and investigate its affine-invariance and uniform consistency properties. We also obtain explicit expressions of this depth for two particular distributions we will use as running examples in the paper. In Section 3, we derive the properties of scatter halfspace depth and scatter halfspace depth regions when considering the Frobenius topology on the space of scatter matrices, whereas we do the same for the geodesic topology in Section 4. In Section 5, we identify the desirable properties a generic scatter depth should satisfy and investigate whether or not these are met by the proposed depth concept. In Sections 6 and 7, we extend the proposed depth to concentration and shape matrices, respectively. We show the practical relevance of the new depth concepts on a real-data example in Section 8. Finally, some final comments and perspectives for future work are provided in Section 9. Several appendices collect all proofs.

Before proceeding, we list here, for the sake of convenience, some notation to be used throughout. The collection of $k \times k$ matrices, $k \times k$ invertible matrices, and $k \times k$ symmetric matrices will be denoted as $\mathcal{M}_k$, $GL_k$, and $S_k$, respectively (all matrices in this paper are real matrices). The identity matrix in $\mathcal{M}_k$ will be denoted as $I_k$. For any $A \in \mathcal{M}_k$, diag$(A)$ will stand for the $k$-vector collecting the diagonal entries of $A$, whereas, for any $k$-vector $v$, diag$(v)$ will stand for the diagonal matrix such that diag(diag$(v)$) = $v$. For $p \geq 2$ square matrices $A_1, \ldots, A_p$, diag$(A_1, \ldots, A_p)$ will stand for the block-diagonal matrix with diagonal blocks $A_1, \ldots, A_p$. Any matrix $A$ in $S_k$ can be diagonalized into $A = O \text{diag} (\lambda_1(A), \ldots, \lambda_k(A)) O'$, where $\lambda_1(A) \geq \ldots \geq \lambda_k(A)$ are the eigenvalues of $A$ and where the columns of the $k \times k$ orthogonal matrix $O = (v_1(A), \ldots, v_k(A))$ are corresponding unit eigenvectors (as usual, eigenvectors, and possibly eigenvalues, are only partly identified, but this will not play a role in the sequel). The spectral interval $\text{Sp}(A) := [\lambda_2(A), \lambda_k(A)]$ of $A$ is the interval whose endpoints are the smallest and largest eigenvalues of $A$. For any mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, we let $f(A) = O \text{diag}(f(\lambda_1(A)), \ldots, f(\lambda_k(A))) O'$. If $\Sigma$ is a scatter matrix, in the sense that $\Sigma$ belongs to the collection $\mathcal{P}_k$ of symmetric and positive definite $k \times k$ matrices, then this defines $\log(\Sigma)$ and $\Sigma^t$ for any $t \in \mathbb{R}$. In particular, $\Sigma^{1/2}$ is the unique $A \in \mathcal{P}_k$ such that $\Sigma = AA'$, and $\Sigma^{-1/2}$ is the inverse of this symmetric and positive definite square root. The unit sphere of $\mathbb{R}^k$ will be denoted as $S^{k-1} := \{x \in \mathbb{R}^k : \|x\|^2 = x'x = 1\}$. For a probability measure $P$ over $\mathbb{R}^k$, we will then let $\alpha_P := \min(\alpha_P, 1 - \alpha_P)$, where $\alpha_P := \sup_{t \in S^{k-1}} P[\{x \in \mathbb{R}^k : u'(x - \theta_P) = 0\}]$ involves the Tukey median $\theta_P$ of $P$ (we refer to the next section for a precise definition). We will say that $P$ is smooth at $\theta_P$ if the $P$-probability of any hyperplane of $\mathbb{R}^k$ containing $\theta_P$ is zero (that is, if $s_P = 0$) and that it is smooth if the $P$-probability of any hyperplane of $\mathbb{R}^k$ is zero. Finally, $\overset{D}{=}$$ will denote equality in distribution.
2. Scatter halfspace depth. We start by recalling the classical concept of location halfspace depth. To do so, let $P$ be a probability measure over $\mathbb{R}^k$ and $X$ be a random $k$-vector with distribution $P$, which allows us throughout to write $P[X \in B]$ instead of $P[B]$ for any $k$-Borel set $B$. The location halfspace depth of $\theta(\in \mathbb{R}^k)$ with respect to $P$ is then

$$HD^\text{loc}_P(\theta) = \inf_{u \in \mathbb{S}^{k-1}} P[u'(X - \theta) \geq 0].$$

The corresponding depth regions $R^\text{loc}_P(\alpha) = \{ \theta \in \mathbb{R}^k : HD^\text{loc}_P(\theta) \geq \alpha \}$ form a nested family of closed convex subsets of $\mathbb{R}^k$. The innermost depth region, namely $M^\text{loc}_P = \{ \theta \in \mathbb{R}^k : HD^\text{loc}_P(\theta) = \max_{\eta \in \mathbb{R}^k} HD^\text{loc}_P(\eta) \}$ (the maximum always exists; see, e.g., Proposition 7 in Rousseeuw and Ruts, 1999), is a set-valued location functional. For our purposes, it will be needed to identify a unique representative of $M^\text{loc}_P$. We will adopt the classical solution that consists in considering the barycentre $\theta_P$ of $M^\text{loc}_P$, which, from convexity of $M^\text{loc}_P$, has maximal depth. This particular choice of $\theta_P$, known as the Tukey median of $P$, ensures affine-equivariance, in the sense that for any $A \in GL_k$ and $b \in \mathbb{R}^k$, one has $\theta_{PA,b} = A\theta_P + b$, where the probability measure $P_{A,b}$ is such that $P_{A,b}[B] = P[A^{-1}(B - b)]$ for any $k$-Borel set $B$ (so that if $P = P_X$, then $P_{A,b} = P^{AX+b}$).

In this paper, we define the scatter halfspace depth of $\Sigma(\in \mathcal{P}_k)$ with respect to $P$ as

$$HD^\text{sc}_P(\Sigma) = \inf_{u \in \mathbb{S}^{k-1}} \min \{ P[|u'(X - \theta_P)| \leq \sqrt{u'\Sigma u}], P[|u'(X - \theta_P)| \geq \sqrt{u'\Sigma u}] \},$$

where $\theta_P$ is the Tukey median of $P$. This extends to a probability measure with arbitrary location the covariance depth concept introduced in Chen, Gao and Ren (2015). This is also close in spirit to the scatter projection depth from Zhang (2002), where, however, the centering is rather based on a univariate location functional (we do not mention an alternative solution used there to bypass centering, as the corresponding sample scatter depth is a $U$-statistic that is computationally much more expensive). While they were not considered in Zhang (2002) nor in Chen, Gao and Ren (2015), it is also of interest to introduce the corresponding depth regions

$$R^\text{sc}_P(\alpha) := \{ \Sigma \in \mathcal{P}_k : HD^\text{sc}_P(\Sigma) \geq \alpha \}, \quad \alpha \geq 0.$$ 

We will refer to $R^\text{sc}_P(\alpha)$ as the order-$\alpha$ (scatter halfspace) depth region of $P$. Obviously, $R^\text{sc}_P(0) = \mathcal{P}_k$ for any $P$.

To get a grasp of the scatter depth $HD^\text{sc}_P(\Sigma)$, it is helpful to start with the univariate case $k = 1$. There, the location halfspace deepest region is the “median interval” $M^\text{loc}_P = \arg \max_{\theta \in \mathbb{R}} \min_{\Sigma \in \mathbb{R}^d} P[X \leq \theta, P[X \geq \theta]]$ and the Tukey median $\theta_P$, that is, the midpoint of $M^\text{loc}_P$, is the usual representative of the univariate median. The scatter halfspace deepest region is then the median interval $M^\text{sc}_P := \arg \max_{\Sigma \in \mathbb{R}^d} \min_{\theta \in \mathbb{R}} P[(X - \theta_P)^2 \leq \Sigma], P[(X - \theta_P)^2 \geq \Sigma]$ of $(X - \theta_P)^2$; call it the median squared deviation interval $I_{\text{MSD}}[X]$ (or $I_{\text{MSD}}[P]$) of $X \sim P$. Below, parallel to what is done for the median, $\text{MSD}[X]$ (or $\text{MSD}[P]$) will denote the midpoint of this MSD interval. In particular, if $I_{\text{MSD}}[P]$ is a singleton, then scatter
halfspace depth is uniquely maximized at \( \Sigma = \text{MSD}[P] = (\text{MAD}[P])^2 \), where \( \text{MAD}[P] \) denotes the median absolute deviation of \( P \). Obviously, the depth regions \( R_{P}^{sc}(\alpha) \) form a family of nested intervals, \([\Sigma_{\alpha}^{-}, \Sigma_{\alpha}^{+}]\) say, included in \( P_1 = \mathbb{R}_{0}^{+} \). It is easy to check that, if \( P \) is symmetric (about \( \theta_P \)) and has an invertible cumulative distribution function \( F \), then
\[
(2.2) \quad HD_{P}^{sc}(\Sigma) = 2 \min \left( F(\theta_P + \sqrt{\Sigma}) - \frac{1}{2}, 1 - F(\theta_P + \sqrt{\Sigma}) \right)
\]
and
\[
(2.3) \quad R_{P}^{sc}(\alpha) = [(F^{-1}(\frac{1}{2} + \frac{\alpha}{2}) - \theta_P)^2, (F^{-1}(1 - \frac{\alpha}{2}) - \theta_P)^2].
\]
This is compatible with the fact that the maximal scatter depth (equal to 1/2) is achieved at \( \Sigma = \text{MSD}[P] = (\text{MAD}[P])^2 \) only.

In the multivariate case, elliptical distributions provide an important particular case. We will say that \( P = P^X \) is \( k \)-variate elliptical with location \( \theta \in \mathbb{R}^k \) and scatter \( \Sigma \in \mathcal{P}_k \) if and only if \( X \overset{D}{=} \theta + \Sigma^{1/2} Z \), where \( Z = (Z_1, \ldots, Z_k)' \) is (i) spherically symmetric about the origin of \( \mathbb{R}^k \) (in the usual sense that \( OZ \overset{D}{=} Z \) for any \( k \times k \) orthogonal matrix \( O \)) and is (ii) standardized in such a way that \( \text{MSD}[Z_1] = 1 \). With this notation, denoting by \( \Phi \) the cumulative distribution function of the standard normal, the \( k \)-variate normal distribution with location zero and scatter \( I_k \) is the distribution of \( X := W/b \), where \( b := \Phi^{-1}(\frac{3}{4}) \) and \( W \) is a standard normal random \( k \)-vector. In this Gaussian case, we obtain
\[
HD_{P}^{sc}(\Sigma) = \min \left( \inf_{u \in S_{k-1}} P[|u'X| \leq \sqrt{u'\Sigma u}], \inf_{u \in S_{k-1}} P[|u'X| \geq \sqrt{u'\Sigma u}] \right).
\]
Since the minimal and maximal values of \( u'\Sigma u \) over the unit sphere are \( \lambda_k(\Sigma) \) and \( \lambda_1(\Sigma) \), respectively, we conclude that
\[
(2.4) \quad HD_{P}^{sc}(\Sigma) = 2 \min \left( \Phi(b\lambda_k^{1/2}(\Sigma)) - \frac{1}{2}, 1 - \Phi(b\lambda_1^{1/2}(\Sigma)) \right).
\]
One can check directly that \( HD_{P}^{sc}(\Sigma) \leq HD_{P}^{sc}(I_k) = 1/2 \), with equality if and only if \( \Sigma \) coincides with the “true” scatter matrix \( I_k \) (we refer to Theorem 5.1 for a more general result). Also, a scatter matrix \( \Sigma \) belongs to the depth region \( R_{P}^{sc}(\alpha) \) if and only if
\[
\text{Sp}(\Sigma^{1/2}) \subset \left[ \frac{1}{b}\Phi^{-1}(\frac{1}{2} + \frac{\alpha}{2}), \frac{1}{b}\Phi^{-1}(1 - \frac{\alpha}{2}) \right].
\]
Extension to an arbitrary multinormal distribution is based on the following affine-invariance result, which ensures in particular that scatter halfspace depth will not be affected by possible changes in the marginal measurement units (results of this section are proved in Appendix A).

**Theorem 2.1.** (i) Scatter halfspace depth is affine-invariant in the sense that, for any probability measure \( P \) over \( \mathbb{R}^k \), \( \Sigma \in \mathcal{P}_k \), \( A \in \text{GL}_k \) and \( b \in \mathbb{R}^k \), we have \( HD_{P_{A,b}}^{sc}(A\Sigma A') = \)
$HD_P^e(\Sigma)$, where $P_{A,b}$ is as defined on page 5. Consequently, (ii) the regions $R_P^e(\alpha)$ are affine-equivariant, in the sense that, for any probability measure $P$ over $\mathbb{R}^k$, $A \in GL_k$ and $b \in \mathbb{R}^k$, we have $R_{P_{A,b}}^e(\alpha) = AR_P^e(\alpha)A'$.

This result readily entails that if $P$ is the $k$-variabe normal with location $\theta_0$ and scatter $\Sigma_0$, then

$$HD_P^e(\Sigma) = 2 \min \left( \Phi(b\lambda_k^{1/2}(\Sigma_0^{-1}\Sigma)) - \frac{1}{2}, 1 - \Phi(b\lambda_1^{1/2}(\Sigma_0^{-1}\Sigma)) \right)$$

and $R_P^e(\alpha)$ is the collection of scatter matrices $\Sigma$ for which

$$\text{Sp}(\Sigma_0^{-1/2}\Sigma^{1/2}) \subset \left[ \frac{1}{b}\Phi^{-1}(\frac{1}{2} + \frac{\alpha}{2}), \frac{1}{b}\Phi^{-1}(1 - \frac{\alpha}{2}) \right].$$

Clearly, it is straightforward to adapt the computations above to determine $HD_P^e(\Sigma)$ and $R_P^e(\alpha)$ for an arbitrary elliptical probability measure $P$ with location $\theta_0$ and scatter $\Sigma_0$. Irrespective of the elliptical probability measure considered (multinormal, $t$, power-exponential, etc.), $HD_P^e(\Sigma)$ will depend on $\Sigma$ only through $\lambda_1(\Sigma_0^{-1}\Sigma)$ and $\lambda_k(\Sigma_0^{-1}\Sigma)$.

As announced in the introduction, we also intend to consider non-elliptical probability measures. A running non-elliptical example will be the one for which the probability measure $P$ is the distribution of a random vector $X = (X_1, \ldots, X_k)'$ with independent Cauchy marginals. Using the fact that, for any non-negative real numbers $a_\ell$, $\ell = 1, \ldots, k$, the random variables $\sum_{\ell=1}^k a_\ell X_\ell$ and $(\sum_{\ell=1}^k a_\ell)X_1$ then share the same distribution, we obtain that, denoting by $\|x\|_1 = \sum_{\ell=1}^k |x_\ell|$ the $L_1$-norm of $x = (x_1, \ldots, x_k)'$,

$$P[\|u'X\| \leq \sqrt{u'\Sigma u}] = P[\|u\|_1 |X_1| \leq \sqrt{u'\Sigma u}] = 2\Psi(\sqrt{u'\Sigma u}/\|u\|_1) - 1,$$

where $\Psi$ is the Cauchy cumulative distribution function. Therefore,

$$HD_P^e(\Sigma) = \min \left( 2\Psi \left( \inf_{u \in S^{k-1}} \frac{\sqrt{u'\Sigma u}}{\|u\|_1} \right) - 1, 2 - 2\Psi \left( \sup_{u \in S^{k-1}} \frac{\sqrt{u'\Sigma u}}{\|u\|_1} \right) \right)$$

$$= 2 \min \left( \Psi \left( \min_v \sqrt{v'\Sigma v} \right) - \frac{1}{2}, 1 - \Psi \left( \max_v \sqrt{v'\Sigma v} \right) \right),$$

where the minimum and maximum in $v$ are over the unit $L_1$-sphere $\{v \in \mathbb{R}^k : \|v\|_1 = 1\}$. The technical Lemma A.1 (see Appendix A) then provides

$$HD_P^e(\Sigma) = 2 \min \left( \Psi(1/\max_s s'\Sigma^{-1}s) - \frac{1}{2}, 1 - \Psi(\sqrt{\max(\det(\Sigma))}) \right),$$

where the maximum in $s$ is over all sign vectors $s = (s_1, \ldots, s_k) \in \{-1,1\}^k$. In dimension $k = 1$, this obviously simplifies to $HD_P^e(\Sigma) = 2 \min \left( \Psi(\sqrt{\Sigma}) - \frac{1}{2}, 1 - \Psi(\sqrt{\Sigma}) \right)$, which agrees with (2.2). In dimension $k = 2$, it is easy to check that

$$HD_P^e(\Sigma) = 2 \min \left( \Psi(\sqrt{\det(\Sigma)}/s_1) - \frac{1}{2}, 1 - \Psi(\sqrt{\max(\Sigma_{11}, \Sigma_{22})}) \right),$$
where we let $s_{\Sigma} := \Sigma_{11} + \Sigma_{22} + 2|\Sigma_{12}|$. In the general $k$-dimensional case, a scatter matrix $\Sigma$ belongs to the depth region $R^c_p(\alpha)$ if and only if

$$\frac{1}{\sqrt{s'\Sigma^{-1}s}} \geq \Psi^{-1}\left(\frac{1}{2} + \frac{\alpha}{2}\right) \quad \forall s \in \{-1, 1\}^k \quad \text{and} \quad \sqrt{s'\Sigma_{\ell\ell}} \leq \Psi^{-1} \left(1 - \frac{\alpha}{2}\right) \quad \forall \ell = 1, \ldots, k.$$

The problem of identifying the scatter matrix achieving maximal depth, if any (note that existence is not guaranteed), will be considered in Section 4. Figure 1 plots scatter halfspace depth regions in the Gaussian and independent Cauchy cases above. Examples involving distributions that are not absolutely continuous with respect to the Lebesgue measure will be considered in the next sections.

![Figure 1](image.png)

Fig 1. Level sets, of order $\alpha = .2, .3$ and $A$, for the mapping $(x, y, z) \mapsto HD^c_p(V_{x,y,z})$, where $HD^c_p(V_{x,y,z})$ is the scatter halfspace depth of $V_{x,y,z} = (z, z)$ with respect to two probability measures $P$, namely the bivariate multinormal distribution with location zero and scatter $I_2$ (left) and the bivariate distribution with independent Cauchy marginals (right). The red points are those associated with $I_2$ (left) and $\sqrt{2}I_2$ (right), which are the corresponding deepest scatter matrices (see Sections 4 and 5).

We will validate through a Monte Carlo exercise the scatter halfspace depth expressions obtained in the multinormal and independent Cauchy cases. Such a numerical validation is justified by the following uniform consistency result; see (6.2) and (6.6) in Donoho and Gasko (1992) for the corresponding location halfspace depth result.

**Theorem 2.2.** Let $P$ be a smooth probability measure over $\mathbb{R}^k$ and let $P_n$ denote the empirical probability measure associated with a random sample of size $n$ from $P$. Then

$$\sup_{\Sigma \in P_k} \left| HD^c_{P_n}(\Sigma) - HD^c_P(\Sigma) \right| \to 0 \quad \text{almost surely as} \quad n \to \infty.$$

Of course, this uniform consistency result holds in particular if $P$ is absolutely continuous with respect to the Lebesgue measure. Inspection of the proof of Theorem 2.2 reveals
that the smoothness assumption is only needed to control the estimation of the Tukey
median \( \theta_P \). Hence, uniform consistency of a fixed-location version of scatter halfspace depth
would hold without the smoothness assumption. To be more precise, if one defines
\[
(2.7) \quad HD_{P, \theta_0}^{sc}(\Sigma) = \inf_{u \in S^{k-1}} \min \left( P\left[ |u'(X - \theta_0)| \leq \sqrt{u\Sigma u} \right], P\left[ |u'(X - \theta_0)| \geq \sqrt{u\Sigma u} \right] \right),
\]
where \( \theta_0 \) is fixed (to, e.g., the origin of \( \mathbb{R}^k \), like in Chen, Gao and Ren, 2015), then
\[ \sup_{\Sigma \in \mathcal{P}_k} |HD_{P, \theta_0}^{sc}(\Sigma) - HD_{P, \theta_0}^{sc}(\Sigma)| \rightarrow 0 \text{ almost surely as } n \rightarrow \infty, \]
without any assumption on \( P \).

The aforementioned Monte Carlo exercise is the following. For each possible combination
of \( n \in \{100, 500, 2000\} \) and \( k \in \{2, 3, 4\} \), we generated \( M = 1000 \) independent random
samples of size \( n \) (i) from the \( k \)-variate normal distribution with location zero and scatter \( I_k \)
and (ii) from the \( k \)-variate distribution with independent Cauchy marginals. Letting \( \Lambda_k^A := \text{diag}(I_{k-1}), \Lambda_k^B := I_k, \Lambda_k^C := \text{diag}(I_{k-1}) \), and
\[ O_k := \text{diag}\left(\begin{pmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, I_{k-2}\right), \]
we evaluated, in each sample, the depths \( HD_{P, n}^{sc}(\Sigma_{\ell}) \) of \( \Sigma_{\ell} = O_k \Lambda_k^\ell O_k^\ell \), \( \ell = A, B, C \),
where \( P_n \) denotes the empirical probability measure associated with the \( k \)-variate sample
of size \( n \) at hand (each evaluation of \( HD_{P, n}^{sc}(\cdot) \) is done by approximating the infimum
in \( u \in S^{k-1} \) by a minimum over \( N = 10000 \) directions randomly sampled from the
uniform distribution over \( S^{k-1} \)). For each \( n, k, \) and each underlying distribution (multinormal or independent Cauchy), Figure 2 reports boxplots of the corresponding \( M \) values of \( HD_{P, n}^{sc}(\Sigma_{\ell}), \ell = A, B, C \). Clearly, the results support the theoretical depth expressions
obtained in (2.4)-(2.6), as well as the consistency result in Theorem 2.2 (the bias for \( HD_{P, n}^{sc}(\Sigma_B) \) in the Gaussian case is explained by the fact that, as we have seen above, \( \Sigma_B \)
maximizes \( HD_{P}^{sc}(\Sigma) \), with a maximal depth value equal to \( 1/2 \)).

3. Frobenius topology. Our investigation of the further structural properties of the scatter halfspace depth \( HD_{P}^{sc}(\Sigma) \) and of the corresponding depth regions \( R_{n}^{sc}(P) \) depends
on the topology that is considered on \( \mathcal{P}_k \). In this section, we focus on the topology induced
by the Frobenius metric space \( (\mathcal{P}_k, d_F) \), where \( d_F(\Sigma_a, \Sigma_b) = \|\Sigma_a - \Sigma_b\|_F \) is the distance
on \( \mathcal{P}_k \) that is inherited from the Frobenius norm \( \|A\|_F = \sqrt{\text{tr}[AA^T]} \) on \( \mathcal{M}_k \). The resulting
Frobenius topology (or simply \( F \)-topology), generated by the \( F \)-balls \( B_F(\Sigma_0, r) := \{\Sigma \in \mathcal{P}_k : d_F(\Sigma, \Sigma_0) < r\} \) with center \( \Sigma_0 \) and radius \( r \), gives a precise meaning to what we call
below \( F \)-continuous functions on \( \mathcal{P}_k \), \( F \)-open/\( F \)-closed subsets of \( \mathcal{P}_k \), etc. We then have
the following result (all proofs for this section can be found in Appendix B).

**Theorem 3.1.** Let \( P \) be a probability measure over \( \mathbb{R}^k \). Then, (i) \( \Sigma \mapsto HD_{F}^{sc}(\Sigma) \)
is upper \( F \)-semicontinuous on \( \mathcal{P}_k \), so that (ii) the depth region \( R_{F}^{sc}(\alpha) \) is \( F \)-closed
for any \( \alpha \geq 0 \). (iii) If \( P \) is smooth at its Tukey median \( \theta_P \), then \( \Sigma \mapsto HD_{F}^{sc}(\Sigma) \) is \( F \)-continuous
on \( \mathcal{P}_k \).
Fig 2. Boxplots, for various values of $n$ and $k$, of $HDP^n_\mathcal{H}(\Sigma_A)$ (top row), $HDP^n_\mathcal{H}(\Sigma_B)$ (middle row) and $HDP^n_\mathcal{H}(\Sigma_C)$ (bottom row) based on $M = 1000$ independent random samples of size $n$ from the $k$-variate multivariate normal distribution with location zero and scatter $I_k$ (left) or from the $k$-variate distribution with independent Cauchy marginals (right); we refer to Section 2 for the expressions of $\Sigma_A$, $\Sigma_B$ and $\Sigma_C$.

For location halfspace depth, the corresponding continuity result was derived in Donoho and Gasko (1992) (Lemma 6.1), where the metric on $\mathbb{R}^k$ is the usual Euclidean one. The similarity between the location and scatter halfspace depths also extends to the boundedness of depth regions, in the sense that, like location halfspace depth (see Proposition 5 in Rousseeuw and Ruts, 1999), the order-$\alpha$ scatter halfspace depth region is bounded if and only if $\alpha > 0$. 
Theorem 3.2. Let \( P \) be a probability measure over \( \mathbb{R}^k \). Then, for any \( \alpha > 0 \), \( R_P^{\text{sc}}(\alpha) \) is \( F \)-bounded (that is, it is included, for some \( r > 0 \), in the \( F \)-ball \( B_F(I_k, r) \)).

This shows that, for any probability measure \( P \), \( \text{HD}_P^{\text{sc}}(\Sigma) \) goes to zero as \( \|\Sigma\|_F \to \infty \). Since \( \|\Sigma\|_F \geq \lambda_1(\Sigma) \), this means that explosion of \( \Sigma \) (that is, \( \lambda_1(\Sigma) \to \infty \)) leads to arbitrarily small depth, which is confirmed in the multinormal case in (2.4). In this Gaussian case, however, implosion of \( \Sigma \) (that is, \( \lambda_k(\Sigma) \to 0 \)) also provides arbitrarily small depth, but this is not captured by the general result in Theorem 3.2 (similar comments can be given for the independent Cauchy example in (2.6)). It is actually possible to have implosion without depth going to zero, as we show by considering the following example. Let \( P = (1 - s)P_1 + sP_2 \), where \( s \in (\frac{1}{2}, 1) \), \( P_1 \) is the bivariate standard normal, and \( P_2 \) is the distribution of \( \begin{pmatrix} 0 \\ Z \end{pmatrix} \), where \( Z \) is univariate standard normal. Then, it is possible to show that, for \( \Sigma_n := \begin{pmatrix} 1/n & 0 \\ 0 & 1 \end{pmatrix} \), we have \( \text{HD}_P^{\text{sc}}(\Sigma_n) \to 1 - s \) as \( n \to \infty \). This provides an example of a sequence of imploding scatter matrices along which scatter halfspace depth does not go to zero.

In the metric space \( (\mathcal{P}_k, d_F) \), any bounded set is also totally bounded, in the sense that, for any \( \varepsilon > 0 \), a bounded set can be covered by finitely many balls of the form \( B_F(\Sigma, \varepsilon) \). Theorems 3.1-3.2 thus show that, for any \( \alpha > 0 \), \( R_P^{\text{sc}}(\alpha) \) is both \( F \)-closed and totally \( F \)-bounded. However, since \( (\mathcal{P}_k, d_F) \) is not complete, there is no guarantee that these regions are \( F \)-compact. To show that these regions may indeed fail to be \( F \)-compact, we consider again the example from the previous paragraph. For any \( \alpha \in (0, 1 - s) \), the scatter matrix \( \Sigma_n \) belongs to \( R_P^{\text{sc}}(\alpha) \) for \( n \) large enough. However, the sequence \( (\Sigma_n) \) \( F \)-converges to \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), that does not belong to \( R_P^{\text{sc}}(\alpha) \) (since it does not even belong to \( \mathcal{P}_2 \)). Since this will also hold for any subsequence of \( (\Sigma_n) \), we conclude that, for \( \alpha \in (0, 1 - s) \), \( R_P^{\text{sc}}(\alpha) \) fails to be \( F \)-compact in this example. This provides a first discrepancy between location and scatter halfspace depths, since location halfspace depth regions associated with a positive order \( \alpha \) are always compact.

Compacity of location halfspace depth regions plays a key role in the existence of a location halfspace deepest point. In line with this, the lack of compacity of scatter halfspace depth regions may allow for probability measures for which no halfspace deepest scatter matrix exists. This is actually the case in the bivariate singular mixture example above. There, denoting as \( e_1 \) the first vector of the canonical basis of \( \mathbb{R}^2 \), any \( \Sigma \in \mathcal{P}_2 \) indeed satisfies \( \text{HD}_P^{\text{sc}}(\Sigma) \leq P[|e_1^t X| \geq e_1^t \Sigma e_1] = P[|X_1| \geq \sqrt{\Sigma_{11}}] = (1 - s)P[|Z| \geq \sqrt{\Sigma_{11}}] < 1 - s = \sup_{\Sigma \in \mathcal{P}_2} \text{HD}_P^{\text{sc}}(\Sigma) \), where the last equality follows from the fact that we identified a sequence \( \Sigma_n \) such that \( \text{HD}_P^{\text{sc}}(\Sigma_n) \to 1 - s \). This is again in sharp contrast with location halfspace depth, for which a location with maximal depth always exists; see, e.g., Propositions 5 and 7 in Rousseeuw and Ruts (1999). Identifying sufficient conditions under which a halfspace deepest scatter matrix exists requires considering another topology, namely the geodesic topology considered in Section 4 below.

The next result states that scatter halfspace depth is a quasi-concave function, which
ensures convexity of the corresponding depth regions; we refer to Proposition 1 (and to
its corollary) in Rousseeuw and Ruts (1999) for the corresponding results on location
halfspace depth.

**Theorem 3.3.** Let $P$ be a probability measure over $\mathbb{R}^k$. Then, (i) $\Sigma \mapsto HD^\text{sc}_P(\Sigma)$ is quasi-concave, in the sense that, for any $\Sigma_a, \Sigma_b \in \mathcal{P}_k$ and $t \in [0,1]$, $HD^\text{sc}_P(\Sigma_t) \geq \min(HD^\text{sc}_P(\Sigma_a), HD^\text{sc}_P(\Sigma_b))$, where we let $\Sigma_t := (1-t)\Sigma_a + t\Sigma_b$; (ii) for any $\alpha \geq 0$, $R^\text{sc}_P(\alpha)$ is convex.

Strictly speaking, Theorem 3.3 is not directly related to the $F$-topology considered
on $\mathcal{P}_k$. Yet we state the result in this section due to the link between the linear paths $t \mapsto \Sigma_t = (1-t)\Sigma_a + t\Sigma_b$ it involves and the “flat” nature of the $F$-topology (this link will become clearer below when we will compare with what occurs for the geodesic topology).

Figure 3 plots, for $k = 2,3$, the graphs of $t \mapsto HD^\text{sc}_P(\Sigma_t)$ for $\Sigma_t := (1-t)\Sigma_A + t\Sigma_C$, where $\Sigma_A$ and $\Sigma_C$ are the scatter matrices considered in the numerical exercise performed at the end of Section 2 and where $P$ is either the $k$-variate normal distribution with location zero and scatter $I_k$ or the $k$-variate distribution with independent Cauchy marginals. The same figure also provides the corresponding sample plots, based on a single random sample of size $n = 50$ drawn from each of these two distributions. All plots are compatible with the quasi-concavity result in Theorem 3.3. Figure 3 also illustrates the continuity of $t \mapsto HD^\text{sc}_P(\Sigma_t)$ for smooth probability measures $P$ (Theorem 3.1) and shows that continuity may fail to hold in the sample case.

**4. Geodesic topology.** Equipped with the inner product $<A,B> = \text{tr}[A'B]$, $\mathcal{M}_k$ is a Hilbert space. The resulting norm and distance are the Frobenius ones considered in the previous section. As an open set in $\mathcal{S}_k$, the parametric space $\mathcal{P}_k$ of interest is a differentiable manifold of dimension $k(k+1)/2$. The corresponding tangent space at $\Sigma$, which is isomorphic (via translation) to $\mathcal{S}_k$, can be equipped with the inner product $<A,B> = \text{tr}[\Sigma^{-1}A\Sigma^{-1}B]$. This leads to considering $\mathcal{P}_k$ as a Riemannian manifold, with the metric at $\Sigma$ given by the differential $ds = \|\Sigma^{-1/2}d\Sigma\Sigma^{-1/2}\|_F$; see, e.g., Bhatia (2007). The length of a path $\gamma : [0,1] \to \mathcal{P}_k$ is then given by

$$L(\gamma) = \int_0^1 \left\|\gamma^{-1/2}(t)\frac{d\gamma(t)}{dt}\gamma^{-1/2}(t)\right\|_F dt.$$ 

The resulting geodesic distance between two matrices $\Sigma_a, \Sigma_b \in \mathcal{P}_k$ is defined as

$$d_g(\Sigma_a, \Sigma_b) := \inf \{ L(\gamma) : \gamma \in \mathcal{G}(\Sigma_a, \Sigma_b) \} = \|\log(\Sigma_a^{-1/2}\Sigma_b\Sigma_a^{-1/2})\|_F,$$

where $\mathcal{G}(\Sigma_a, \Sigma_b)$ denotes the collection of paths $\gamma$ from $\gamma(0) = \Sigma_a$ to $\gamma(1) = \Sigma_b$ (the second equality in (4.1) is Theorem 6.1.6 in Bhatia, 2007). It directly follows from the definition of $d_g(\Sigma_a, \Sigma_b)$ that the geodesic distance satisfies the triangular inequality: for any $\Sigma_a, \Sigma_b, \Sigma_c \in \mathcal{P}_k$, $d_g(\Sigma_a, \Sigma_c) \leq d_g(\Sigma_a, \Sigma_b) + d_g(\Sigma_b, \Sigma_c)$. Theorem 6.1.6 in Bhatia (2007)
also states that all paths $\gamma$ achieving the infimum in (4.1) provide the same geodesic $\{\gamma(t) : t \in [0, 1]\}$ joining $\Sigma_a$ and $\Sigma_b$, and that this geodesic can be parametrized as

$$
\gamma(t) = \tilde{\Sigma}_t =: \Sigma_a^{1/2}(\Sigma_a^{-1/2}\Sigma_b\Sigma_a^{-1/2})^t\Sigma_a^{1/2}, \quad t \in [0, 1].
$$

By using the explicit formula in (4.1), it is easy to check that this particular parametrization of this unique geodesic is natural in the sense that $d_g(\Sigma_a, \tilde{\Sigma}) = td_g(\Sigma_a, \Sigma_b)$ for any $t \in [0, 1]$.

Below, we consider the natural topology associated with the metric space $(P_k, d_g)$, that is, the topology whose open sets are generated by geodesic balls of the form $B_g(\Sigma_0, r) := \{\Sigma \in P_k : d_g(\Sigma, \Sigma_0) < r\}$. This topology — call it the geodesic topology, or simply $g$-topology — defines subsets of $P_k$ that are $g$-open, $g$-closed, $g$-compact, and functions that are $g$-semicontinuous, $g$-continuous, etc. We will say that a subset $R$ of $P_k$ is $g$-bounded if and only if $R \subset B_g(I_k, r)$ for some $r > 0$ (we can safely restrict to balls centered at $I_k$ since the triangular inequality guarantees that $R$ is included in a finite-radius $g$-ball centered at $I_k$ if and only if it is included in a finite-radius $g$-ball centered at an arbitrary $\Sigma_0 \in P_k$). A $g$-bounded subset of $P$ is also totally $g$-bounded, still in the sense that, for any $\varepsilon > 0$, it can be covered by finitely many balls of the form $B_g(\Sigma, \varepsilon)$; for the sake of completeness, we prove this in Lemma C.1 (all proofs for this section are to be found in Appendix C). Since the metric space $(P_k, d_g)$ is complete (see, e.g., Proposition 10 in Bhatia and Holbrook, 2006), we conclude that any subset of $P_k$ that is both $g$-bounded and $g$-closed is also $g$-compact.
We omit the proof of the next result as it follows along the exact same lines as the proof of Theorem 3.1, once it is seen that a sequence \((\Sigma_n)\) converging to \(\Sigma_0\) in \((\mathcal{P}_k,d_g)\) also converges to \(\Sigma_0\) in \((\mathcal{P}_k,d_F)\).

**THEOREM 4.1.** Let \(P\) be a probability measure over \(\mathbb{R}^k\). Then, (i) \(\Sigma \mapsto \text{HD}_P^g(\Sigma)\) is upper \(g\)-semicontinuous on \(\mathcal{P}_k\), so that (ii) the depth region \(R_P^g(\alpha)\) is \(g\)-closed for any \(\alpha \geq 0\). (iii) If \(P\) is smooth at its Tukey median \(\theta_P\), then \(\Sigma \mapsto \text{HD}_P^g(\Sigma)\) is \(g\)-continuous on \(\mathcal{P}_k\).

The next result makes use of the notation \(s_P := \sup_{u \in S^{k-1}} P[u'(X - \theta_P) = 0]\) and \(\alpha_P := \min(s_P, 1 - s_P)\) defined in the introduction (it follows from Lemma A.4 in Appendix A that the supremum is actually always a maximum).

**THEOREM 4.2.** Let \(P\) be a probability measure over \(\mathbb{R}^k\). Then, for any \(\alpha > \alpha_P\), \(R_P^g(\alpha)\) is \(g\)-bounded, hence \(g\)-compact (if \(s_P \geq 1/2\), then this result is trivial in the sense that \(R_P^g(\alpha)\) is empty for any \(\alpha > \alpha_P\)). In particular, if \(P\) is smooth at its Tukey median \(\theta_P\), then \(R_P^g(\alpha)\) is \(g\)-compact for any \(\alpha > 0\).

This result complements Theorem 3.2 by showing that implosion always leads to a depth that is smaller than or equal to \(\alpha_P\). In particular, in the multinormal and Cauchy examples considered in Section 2, this shows that both explosion and implosion lead to arbitrarily small depth (whereas Theorem 3.2 was predicting this collapsing for explosion only).

It is not possible to improve the result in Theorem 4.2, in the sense that \(R_P^g(\alpha_P)\) may fail to be \(g\)-bounded. For instance, consider the probability measure \(P\) over \(\mathbb{R}^2\) putting probability mass 1/6 on each of the six points \((0, \pm 1/2)\) and \((\pm 2, \pm 2)\). Obviously, \(\alpha_P = s_P = 1/3\). Now, letting \(\Sigma_n = \text{diag}(1,1)\), we have \(P[|u'\Sigma_n^{-1/2}X| \leq 1] \geq 1/3\) and \(P[|u'\Sigma_n^{-1/2}X| \geq 1] \geq 1/3\) for any \(u \in S^1\) (here, \(X\) is a bivariate random vector with distribution \(P\)), which entails that

\[
\text{HD}_P^g(\Sigma_n) = \inf_{u \in S^1} \min \left( P\left[|u'X| \leq \sqrt{u'\Sigma_n u}\right], P\left[|u'X| \geq \sqrt{u'\Sigma_n u}\right] \right)
\]

\[
= \inf_{u \in S^1} \min \left( P\left[|u'\Sigma_n^{-1/2}X| \leq 1\right], P\left[|u'\Sigma_n^{-1/2}X| \geq 1\right] \right) \geq \frac{1}{3} = \alpha_P,
\]

so that \(\Sigma_n \in R_P^g(\alpha_P)\) for any \(n\). Since \(d_g(\Sigma_n, I_2) \to \infty\), \(R_P^g(\alpha_P)\) is indeed \(g\)-unbounded.

One of the benefits of working with the geodesic topology is that, unlike the Frobenius topology, it allows to show that, under mild assumptions, a halfspace deepest scatter matrix does exist. More precisely, we have the following result.

**THEOREM 4.3.** Let \(P\) be a probability measure over \(\mathbb{R}^k\) and assume that \(R_P^g(\alpha_P)\) is non-empty. Then, \(\alpha_{*P} := \sup_{\Sigma \in \mathcal{P}_k} \text{HD}_P^g(\Sigma) = \text{HD}_P^g(\Sigma_*)\) for some \(\Sigma_* \in \mathcal{P}_k\).

In particular, this result shows that for any probability measure \(P\) that is smooth at its Tukey median \(\theta_P\), there exists a halfspace deepest scatter \(\Sigma_*\). For the \(k\)-variate
multinormal distribution with location zero and scatter $I_k$, we already stated in Section 2 that $\Sigma \mapsto HD^c_P(\Sigma)$ is uniquely maximized at $\Sigma^* = I_k$, with a corresponding maximal depth equal to $1/2$, irrespective of the dimension $k$. The next result identifies the halfspace deepest scatter (and the corresponding maximal depth) for the $k$-variate distribution with independent Cauchy marginals.

**Theorem 4.4.** Let $P$ be the $k$-variate probability measure with independent Cauchy marginals. Then $\Sigma \mapsto HD^c_P(\Sigma)$ is uniquely maximized at $\Sigma^* = \sqrt{k}I_k$, and the corresponding maximal depth is $HD^c_P(\Sigma^*) = \frac{2}{\pi} \arctan \left( k^{-1/4} \right)$.

In dimension $k = 1$, the Cauchy distribution in this result is symmetric about zero, hence elliptical about zero, which is compatible with the maximal depth being equal to $1/2$ there (we will see in Theorem 5.1 below that the maximal depth for elliptical distributions admitting a density with respect to the Lebesgue measure is always equal to $1/2$). For larger values of $k$, however, this provides an example where the maximal depth is strictly smaller than $1/2$. Interestingly, this maximal depth goes to zero as $k \to \infty$. Note that, for the same distribution, location halfspace depth has a maximal value equal to $1/2$, irrespective of the dimension $k$ (this follows, e.g., from Lemma 1 and Theorem 1 in Rousseeuw and Struyf, 2004).

In general, the halfspace deepest scatter $\Sigma^*$ is not unique. This will typically be the case for empirical probability measures $P_n$ (note that the existence of a halfspace deepest scatter in the empirical case readily follows from the fact that $HD^c_P(\Sigma)$ takes its values in $\{k/n : k = 0, 1, \ldots, n\}$). For several purposes, it may be useful to identify a unique representative of the halfspace deepest scatters, that would then play a similar role for scatter as the one played by the Tukey median $\theta_P$ for location. To define a unique halfspace deepest scatter, one may consider here a center of mass, that is, a scatter matrix of the form

$$\Sigma_P := \arg \min_{\Sigma \in \mathcal{P}_k} \int_{R^c_P(\alpha, \Sigma)} d_l^2(m, \Sigma) \, dm,$$

where $dm$ is a mass distribution on $R^c_P(\alpha, \Sigma)$ with total mass 1. This is a suitable solution if $R^c_P(\alpha, \Sigma)$ is $g$-bounded (hence, $g$-compact), since Cartan (1929) showed that, in a simply connected manifold with non-positive curvature (as $\mathcal{P}_k$), every compact set has a unique center of mass; see also Proposition 60 in Berger (2003). Convexity of $R^c_P(\alpha, \Sigma)$ then ensures that $\Sigma_P$ has maximal depth. Like for location, the choice of $\Sigma_P$ as a representative of the deepest scatters guarantees affine-equivariance (in the sense that $\Sigma_P A b = A \Sigma_P A'$ for any $A \in GL_k$ and any $b \in \mathbb{R}^k$).

As a final comment related to Theorem 4.3, note that if the assumption that $R^c_P(\alpha, \Sigma)$ is non-empty is violated, then it may actually be so that no halfspace deepest scatter does exist. An example is provided by the bivariate singular mixture distribution $P$ considered in Section 3. There, we have seen that no halfspace deepest scatter does exist, which is compatible with the fact that, for any $\Sigma$, $HD^c_P(\Sigma) < 1 - s = \alpha_P$, so that $R^c_P(\alpha, \Sigma)$ is empty.
5. An axiomatic approach for scatter depth. Building on the properties derived in Liu (1990) for simplicial depth, Zuo and Serfling (2000) introduced an axiomatic approach for a generic location depth $D^\text{loc}_P(\cdot) : \mathbb{R}^k \to [0, 1]$. More specifically, they argued that a location depth should satisfy the following four structural properties: (P1) affine invariance, (P2) maximality at the symmetry center (if any), (P3) monotonicity relative to any deepest point, and (P4) vanishing at infinity. Without entering into details, these properties are to be understood as follows: (P1) means that $D^\text{loc}_{P_{A,b}}(A\theta + b) = D^\text{loc}_P(\theta)$ for any $A \in \text{GL}_k$ and $b \in \mathbb{R}^k$, where $P_{A,b}$ is as defined on page 5; (P2) states that if the underlying probability measure $P$ is symmetric (in a sense to be made precise), then the symmetry center should maximize $D^\text{loc}_P(\cdot)$; according to (P3), $D^\text{loc}_P(\cdot)$ should be monotone non-increasing along any halfline originating from any $P$-deepest point; (P4) states that as $\theta$ exits any compact set in $\mathbb{R}^k$, its depth should converge to zero. There is now an almost universal agreement in the literature that (P1)-(P4) are the natural desirable properties for location depths. In the last decade, research efforts in this direction have therefore rather focused on identifying the corresponding properties depths should satisfy in other setups, particularly in functional data analysis (FDA); see, e.g., Nieto-Reyes and Battey (2016) for one of the recent references discussing depth axioms in FDA.

In view of this, it is natural to wonder what are the natural desirable properties for a scatter depth. Inspired by (P1)-(P4) above, we argue that a generic scatter depth $D^\text{sc}_P(\cdot) : \mathcal{P}_k \to [0, 1]$ should satisfy the following four properties, all involving an (unless otherwise specified) arbitrary probability measure $P$ over $\mathbb{R}^k$:

(Q1) **Affine invariance:** for any $A \in \text{GL}_k$ and $b \in \mathbb{R}^k$, $D^\text{sc}_{P_{A,b}}(A\Sigma A') = D^\text{sc}_P(\Sigma)$, where $P_{A,b}$ is still as defined on page 5;

(Q2) **Fisher-consistency under ellipticity:** if $P$ is elliptically symmetric with location $\theta_0$ and scatter $\Sigma_0$, then $D^\text{sc}_P(\Sigma_0) \geq D^\text{sc}_P(\Sigma)$ for any $\Sigma \in \mathcal{P}_k$;

(Q3) **Monotonicity relative to any deepest parameter value:** if $\Sigma_a$ maximizes $D^\text{sc}_P(\cdot)$, then, for any $\Sigma_b \in \mathcal{P}_k$, $t \mapsto D^\text{sc}_P((1-t)\Sigma_a + t\Sigma_b)$ should be monotone non-increasing over $[0, 1]$;

(Q4) **Vanishing at the boundary of the parametric space:** if $(\Sigma_n)$ $F$-converges to the boundary of $\mathcal{P}_k$ (in the sense that either $d_F(\Sigma_n, \Sigma) \to 0$ for some $\Sigma \in \mathcal{S}_k \setminus \mathcal{P}_k$ or $d_F(\Sigma_n, I_k) \to \infty$), then $D^\text{sc}_P(\Sigma_n) \to 0$.

While (Q1) and (Q3) are the natural scatter counterparts of the location properties (P1) and (P3), respectively, some comments are in order for (Q2) and (Q4).

We start with (Q2). In essence, (P2) is requiring that, whenever an indisputable location center in $\mathbb{R}^k$ exists, this location should also be flagged as most central by the location depth at hand. In the location case, a symmetry assumption is what is needed to make such a location center indisputable. A similar reasoning leads to (Q2): we argue that, for an elliptical probability measure, the “true” value of the scatter parameter is indisputable, and (Q2) then imposes that the scatter depth at hand should identify this true scatter
value as the (or at least, as a) deepest one. Of course, one may think of strengthening (Q2) by replacing the elliptical model there by a broader model in which the true value would still be clearly defined. In such a case, of course, the larger the model for which scatter depth satisfies (Q2), the better (a possibility, that we do not explore here, is to consider the union of the elliptical model and the independent component model; see Tyler et al., 2009, Nordhausen, Oja and Paindaveine, 2009, or Ilmonen and Paindaveine, 2011). This is parallel to what happens in (P2): the weaker the symmetry assumption under which (P2) is satisfied for the location depth at hand, the better (for instance, having (P2) satisfied with angular symmetry is better than having it satisfied with central symmetry only); see Zuo and Serfling (2000).

We then turn to (Q4), whose location counterpart (P4) is typically read by saying that the depth/centrality $D_{loc}^{P}(\theta_n)$ goes to zero when the point $\theta_n$ goes to infinity in the sample space. In the spirit of parametric depth (Mizera, 2002; Mizera and Müller, 2004), however, it is more appropriate to look at $\theta_n$ as a candidate location fit and to consider that (Q4) imposes that the appropriateness $D_{loc}^{P}(\theta_n)$ of this fit goes to zero as $\theta_n$ goes to infinity in the parametric space. For a location parameter, the confounding between the sample space and parametric space (both equal to $\mathbb{R}^k$) allows for both interpretations. For a scatter parameter, however, there is no such confounding (the sample space is still $\mathbb{R}^k$, but the parametric space is $\mathcal{P}_k$), and we then argue that (Q4) above is the natural scatter version of (P4): whenever $\Sigma_n$ goes to the boundary of the parametric space $\mathcal{P}_k$, scatter depth should flag it as an arbitrarily poor candidate fit. In (Q4) above, two ways are identified to reach the boundary of $\mathcal{P}_k$, namely implosion ($d_F(\Sigma_n, \Sigma) \to 0$ for some $\Sigma \in S^k \setminus \mathcal{P}_k$) or explosion ($d_F(\Sigma_n, I_k) \to \infty$). Note that this actually amounts to ask that $d_g(\Sigma_n, I_k) \to \infty$ in both cases.

Theorem 2.1 states that scatter halfspace depth satisfies (Q1). Scatter halfspace depth satisfies (Q3) as well: if $\Sigma_a$ maximizes $HD_{loc}^{P}(\cdot)$, then Theorem 3.3 indeed readily implies that $HD_{loc}^{P}(1 - t\Sigma_a + t\Sigma_b) \geq \min(\text{HD}_{loc}^{P}(\Sigma_a), \text{HD}_{loc}^{P}(\Sigma_b)) = \text{HD}_{loc}^{P}(\Sigma_b)$ for any $\Sigma_b \in \mathcal{P}_k$ and $t \in [0, 1]$. The next Fisher consistency result shows that (Q2) is also met (results of this section are proved in Appendix D).

**Theorem 5.1.** Let $P$ be an elliptical probability measure over $\mathbb{R}^k$ with location $\theta_0$ and scatter $\Sigma_0$. Then, (i) for any $\Sigma \in \mathcal{P}_k$, $\text{HD}_{loc}^{P}(\Sigma_0) \geq \text{HD}_{loc}^{P}(\Sigma)$ and the equality holds if and only if $\text{Sp}(\Sigma_0^{-1}\Sigma) \subset \mathcal{I}_{\text{MSD}}[Z_1]$, where $Z = (Z_1, \ldots, Z_k) = \Sigma_0^{-1/2}(X - \theta_0)$; (ii) in particular, if $\mathcal{I}_{\text{MSD}}[Z_1]$ is a singleton (i.e., if $\mathcal{I}_{\text{MSD}}[Z_1] = \{1\}$), then $\Sigma \mapsto \text{HD}_{loc}^{P}(\Sigma)$ is uniquely maximized at $\Sigma_0$.

While Properties (Q1)-(Q3) are satisfied by scatter halfspace depth without any assumption on $P$, Property (Q4) is not, as the bivariate mixture example considered in Section 3 shows (the sequence $(\Sigma_n)$ considered there implodes but has limiting depth $1 - s > 0$). However, Theorem 3.2 reveals that (Q4) may fail due to implosion only. More importantly, Theorem 4.2 implies that scatter halfspace depth will satisfy (Q4) at any $P$ that is smooth.
at its Tukey median \( \theta_P \).

In a generic parametric depth setup, Property (Q3) would require that the parameter space is convex, so that convex linear combinations of any two parameter values provide valid parameter values. If the parameter space rather is a non-flat Riemannian manifold, then it is natural to replace the “linear” monotonicity property (Q3) with a “geodesic” one. In the context of scatter depth, this would lead to replacing (Q3) with

\[(\widetilde{Q3}) \quad \text{Geodesic monotonicity relative to any deepest point: if } \Sigma_a \text{ maximizes } D_{sc}^P(\cdot), \text{ then, for any } \Sigma_b \in \mathcal{P}_k, \, t \mapsto D_{sc}^P(\tilde{\Sigma}_t) \text{ should be monotone non-increasing over } [0,1] \text{ along the geodesic path } \tilde{\Sigma}_t \text{ from } \Sigma_a \text{ to } \Sigma_b \text{ in (4.2)}.\]

We refer to Section 7 for an example where (Q3) cannot be considered and where \((\widetilde{Q3})\) needs to be adopted instead. In the present scatter case, however, the hybrid nature of \(\mathcal{P}_k\), which is both flat (as a convex subset of the vector space \(S_k\)) and curved (as a Riemannian manifold with non-positive curvature), allows to consider both linear and geodesic paths, hence allows to consider both (Q3) and \((\widetilde{Q3})\). Parallel to the linear case where (Q3) follows from quasi-concavity of the mapping \(\Sigma \mapsto H_{sc}^P(\Sigma)\), Property \((\widetilde{Q3})\) would follow from the same mapping being geodesic quasi-concave, in the sense that

\[H_{sc}(\tilde{\Sigma}_t, P) \geq \min(H_{sc}(\Sigma_a, P), H_{sc}(\Sigma_b, P)) \text{ along the geodesic path } \tilde{\Sigma}_t \text{ from } \Sigma_a \text{ to } \Sigma_b.\]

Geodesic quasi-concavity would actually then imply that scatter halfspace depth regions are geodesic convex, in the sense that, for any \(\Sigma_a, \Sigma_b \in R_{sc}^P(\alpha)\), the geodesic from \(\Sigma_a\) to \(\Sigma_b\) is contained in \(R_{sc}^P(\alpha)\). We refer to Dumbgen and Tyler (2016) for a recent application of geodesic convex functions and geodesic convex sets to inference on (high-dimensional) scatter matrices.

As showed in Theorem 3.3, \(\Sigma \mapsto H_{sc}^P(\Sigma)\) is quasi-concave for any probability measure \(P\). A natural question is then whether or not this extends to geodesic quasi-concavity. The answer is positive at any \(k\)-variate elliptical probability measure and at the \(k\)-variate probability measure with independent Cauchy marginals.

**Theorem 5.2.** Let \(P\) be an elliptical probability measure over \(\mathbb{R}^k\) or the \(k\)-variate probability measure with independent Cauchy marginals. Then, (i) \(\Sigma \mapsto H_{sc}^P(\Sigma)\) is geodesic quasi-concave, so that (ii) \(R_{sc}^P(\alpha)\) is geodesic convex for any \(\alpha \geq 0\).

We will show below that the mapping \(\Sigma \mapsto H_{sc}^P(\Sigma)\) may actually fail to be geodesic quasi-concave for some probability measure \(P\). Before proceeding, we comment shortly on the reason why the proof that \(\Sigma \mapsto H_{sc}^P(\Sigma)\) is quasi-concave will not apply for geodesic quasi-concavity. Writing again \(\Sigma_t = (1 - t)\Sigma_a + t\Sigma_b\), the (weighted) geometric-arithmetic matrix inequality (see, e.g., Lemma 2.1(vii) in Lawson and Lim, 2013) yields that, for any \(u \in S^{k-1}\), \(u'\Sigma_t u \leq u'\Sigma_t u \leq \max(u'\Sigma_a u, u'\Sigma_b u)\), which, in turn, provides

\[
P\left[|u'X| \leq \sqrt{u'\Sigma_t u}\right] \geq \min(P\left[|u'X| \leq \sqrt{u'\Sigma_a u}\right], P\left[|u'X| \leq \sqrt{u'\Sigma_b u}\right])
\]

\[
\geq \min(H_{sc}^P(\Sigma_a), H_{sc}^P(\Sigma_b)),
\]
hence \( \inf_{u \in S^{k-1}} P \left[ \|u'X\| \leq \sqrt{u'\Sigma_t u} \right] \geq \min(\text{HD}_P^{sc}(\Sigma_a), \text{HD}_P^{sc}(\Sigma_b)) \). Would it similarly be true that \( u'\Sigma_t u \geq \min(u'\Sigma_a u, u'\Sigma_b u) \) for any \( u \in S^{k-1} \), then we would also have

\[
P \left[ \|u'X\| \geq \sqrt{u'\Sigma_t u} \right] \geq \min(P \left[ \|u'X\| \geq \sqrt{u'\Sigma_a u} \right], P \left[ \|u'X\| \geq \sqrt{u'\Sigma_b u} \right])
\]

which would lead to \( \inf_{u \in S^{k-1}} P \left[ \|u'X\| \geq \sqrt{u'\Sigma_t u} \right] \geq \min(\text{HD}_P^{sc}(\Sigma_a), \text{HD}_P^{sc}(\Sigma_b)) \), hence would prove that \( \text{HD}_P^{sc}(\Sigma_t) \geq \min(\text{HD}_P^{sc}(\Sigma_a), \text{HD}_P^{sc}(\Sigma_b)) \). However, the inequality \( u'\Sigma_t u \geq \min(u'\Sigma_a u, u'\Sigma_b u) \) may fail for some \( u \in S^{k-1} \). For instance, with \( \Sigma_a = \text{diag}(2, 1) \), \( \Sigma_b = \text{diag}(1, 2) \), \( t = 1/2 \), and \( u = (1, 1)'/\sqrt{2} \), we have \( u'\Sigma_t u \sim \min(u'\Sigma_a u, u'\Sigma_b u) \approx -0.086 \).

We conclude this section with a numerical illustration of the geodesic quasi-concavity results in Theorems 5.2 and with an example showing that geodesic quasi-concavity may indeed fail to hold. Figure 4 provides, for three bivariate probability measures \( P \), the plots of \( t \mapsto \text{HD}_P^{sc}(\Sigma_t) \) and \( t \mapsto \text{HD}_P^{sc}(\tilde{\Sigma}_t) \), where \( \Sigma_t = (1 - t)\Sigma_a + t\Sigma_b \) is the linear path from \( \Sigma_a = I_2 \) to \( \Sigma_b = \text{diag}(0.001, 20) \) and where \( \tilde{\Sigma}_t = \Sigma_a^{1/2}(\Sigma_a^{-1/2}\Sigma_b\Sigma_a^{-1/2})^{1/2}\Sigma_a^{1/2} \) is the corresponding geodesic path. The three distributions considered are (i) the bivariate normal with location zero and scatter \( I_2 \), (ii) the bivariate distribution with independent Cauchy marginals, and (iii) the empirical distribution associated with a random sample of size \( n = 200 \) from the bivariate mixture distribution \( P = \frac{1}{4}P_1 + \frac{1}{4}P_2 + \frac{1}{4}P_3 \), where \( P_1 \) is the standard normal, \( P_2 \) is the normal with mean \( (0, 4)' \) and covariance matrix \( \frac{1}{10}I_2 \), and \( P_3 \) is the normal with mean \( (0, -4)' \) and covariance matrix \( \frac{1}{10}I_2 \). Figure 4 illustrates that (linear) quasi-concavity of scatter halfspace depth always holds, but that geodesic quasi-concavity may fail to hold. Despite this counterexample, extensive numerical experiments led us to think that geodesic quasi-concavity is the rule rather than the exception.

6. Concentration halfspace depth. In various setups, the parameter of interest is the inverse scatter matrix or concentration matrix \( \Sigma^{-1} \) rather than the scatter matrix \( \Sigma \) itself. For instance, in Gaussian graphical models, the \( (i,j) \)-entry of \( \Sigma^{-1} \) is zero if and only if the \( i \)th and \( j \)th marginals are conditionally independent given all other marginals. It may then be useful to define a depth for inverse scatter matrices. The scatter halfspace depth in (2.1) naturally leads to defining the concentration halfspace depth of \( \Sigma \) with respect to \( P \) as

\[
\text{HD}_P^{conc}(\Sigma) := \text{HD}_P^{sc}(\Sigma^{-1})
\]

and the corresponding concentration halfspace depth regions as

\[
R_P^{conc}(\alpha) := \{ \Sigma \in \mathcal{P}_k : \text{HD}_P^{conc}(\Sigma) \geq \alpha \}, \quad \alpha \geq 0.
\]

Concentration halfspace depth and concentration halfspace depth regions inherit most of the properties of their scatter antecedents, sometimes with obvious modifications. In particular, the former is affine-invariant and the latter are affine-equivariant. Concentration halfspace depth is upper \( F \)- and \( g \)-semicontinuous for any probability measure \( P \) (so that the regions \( R_P^{conc}(\alpha) \) are \( F \)- and \( g \)-closed) and \( F \)- and \( g \)-continuous if \( P \) is smooth at its
Fig 4. Plots, for various bivariate probability measures \( P \), of the scatter halfspace depth function \( \Sigma \mapsto \text{HD}_P^\Sigma(\Sigma) \) along the linear path \( \Sigma_t = (1-t)\Sigma_a + t\Sigma_b \) (red), the geodesic path \( \tilde{\Sigma}_t = \Sigma_1/2(\Sigma_a^{-1/2}\Sigma_b\Sigma_a^{-1/2})^{1/2} \) (blue), and the harmonic path \( \Sigma^*_t = ((1-t)\Sigma_a^{-1} + t\Sigma_b^{-1})^{-1} \) (orange), from \( \Sigma_a = I_2 \) to \( \Sigma_b = \text{diag}(0.001, 20) \); harmonic paths are introduced in Section 6. The probability measures considered are the bivariate normal with location zero and scatter \( I_2 \) (top left), the bivariate distribution with independent Cauchy marginals (top right), and the empirical probability measure associated with a random sample of size \( n = 200 \) from the bivariate mixture distribution described at the end of Section 5 (bottom right). The scatter plot of the sample used in the mixture example is provided in the bottom left panel.

Tukey median \( \theta_P \). While the regions \( R_P^{\text{conc}}(\alpha) \) are still \( g \)-bounded (hence also, \( F \)-bounded) for \( \alpha > \alpha_P \), the outer regions \( R_P^{\text{conc}}(\alpha), \alpha \leq \alpha_P \), here may fail to be \( F \)-bounded (this is because implosion of \( \Sigma \), under which scatter halfspace depth may fail to go below \( \alpha_P \), is associated with explosion of \( \Sigma^{-1} \)). Existence of a concentration halfspace deepest matrix is guaranteed under the same conditions on \( P \) as for scatter halfspace depth. Finally, uniform consistency still holds at any smooth \( P \).

Quasi-concavity of concentration halfspace depth and convexity of the corresponding
regions require more comments. The linear path \( t \mapsto (1-t)\Sigma_a^{-1} + t\Sigma_b^{-1} \) from a concentration matrix \( \Sigma_a^{-1} \) to another one \( \Sigma_b^{-1} \) determines a harmonic path \( t \mapsto \Sigma_t^* := ((1-t)\Sigma_a^{-1} + t\Sigma_b^{-1})^{-1} \) between the corresponding scatter matrices \( \Sigma_a \) and \( \Sigma_b \). In line with the definitions adopted in the previous sections, we will say that if \( f \) maps \( \Sigma_t^* \) onto the corresponding harmonic inequality \( \Sigma^t \). So that \( \Sigma^t \) is harmonic convex if \( \Sigma^t \) is harmonic quasi-concave if \( \Sigma^t \in R \) implies that \( \Sigma^t \in R \) for any \( t \in [0,1] \). The following result is then a direct corollary of Theorem 3.3.

**Theorem 6.1.** Let \( P \) be a probability measure over \( \mathbb{R}^k \). Then, (i) \( \Sigma \mapsto HD_P^{\text{conc}}(\Sigma) \) is harmonic quasi-concave, so that (ii) \( R_P^{\text{conc}}(\alpha) \) is harmonic convex for any \( \alpha \geq 0 \).

Actually, it is clear that \( \Sigma \mapsto HD_P^{\text{conc}}(\Sigma) \) is quasi-concave (resp., harmonic quasi-concave) if and only if \( \Sigma \mapsto HD_P^{\text{sc}}(\Sigma) \) is harmonic quasi-concave (resp., quasi-concave), so that the regions \( R_P^{\text{conc}}(\alpha) \) are convex (resp., harmonic convex) if and only if the regions \( R_P^{\text{sc}}(\alpha) \) are harmonic convex (resp., convex). In this sense, quasi-concavity and harmonic quasi-concavity are dual concepts, relative to scatter and concentration halfspace depths (which justifies the * notation in the corresponding paths \( \Sigma_t \) and \( \Sigma_t^* \)). Interestingly, \( \Sigma \mapsto HD_P^{\text{conc}}(\Sigma) \) is geodesic quasi-concave if and only if \( \Sigma \mapsto HD_P^{\text{sc}}(\Sigma) \) is geodesic quasi-concave, so that concentration halfspace depth regions are geodesic convex if and only if scatter halfspace depth regions are.

As a consequence, the quasi-concavity properties of concentration halfspace depth and convexity properties of the corresponding depth regions can be deduced from those of their scatter antecedents. Since quasi-concavity and geodesic quasi-concavity of scatter halfspace depth have been studied in the previous sections, we may focus here on harmonic quasi-concavity. The following result shows that scatter halfspace depth is harmonic quasi-concave at any \( k \)-variate elliptical probability measure and at the \( k \)-variate probability measure with independent Cauchy marginals (see Appendix D for a proof).

**Theorem 6.2.** Let \( P \) be an elliptical probability measure over \( \mathbb{R}^k \) or the \( k \)-variate probability measure with independent Cauchy marginals. Then, (i) \( \Sigma \mapsto HD_P^{\text{sc}}(\Sigma) \) is harmonic quasi-concave, so that (ii) \( R_P^{\text{sc}}(\alpha) \) is harmonic convex for any \( \alpha \geq 0 \).

In the previous section, we showed that the reason why the proof of quasi-concavity of \( \Sigma \mapsto HD_P^{\text{sc}}(\Sigma) \) does not extend to geodesic quasi-concavity is that the inequality \( u'\tilde{\Sigma}_u u \geq \min(u'\Sigma_a u, u'\Sigma_b u) \) may fail to hold. Actually, as soon as this inequality fails, the corresponding harmonic inequality \( u'\tilde{\Sigma}_u u \geq \min(u'\Sigma_a u, u'\Sigma_b u) \) also fails (the weighted harmonic-geometric matrix inequality indeed implies that \( u'\tilde{\Sigma}_u u \leq u'\tilde{\Sigma}_u u \)), which explains why the proof of quasi-concavity does not extend to harmonic quasi-concavity either. As a matter of fact, scatter halfspace depth is not always harmonic quasi-concave, which we show again by providing a counterexample. Figure 4 in the previous section actually not only provided plots of scatter halfspace depth along linear and geodesic paths but also along the corresponding harmonic paths. These plots confirm that, while scatter halfspace...
depth is harmonic quasi-concave for the Gaussian and independent Cauchy examples considered there, it is not in the mixture example that already provided a counterexample for geodesic quasi-concavity. In view of the links between quasi-concavity properties of concentration and scatter halfspace depths, we can state that while concentration halfspace depth is always harmonic quasi-concave (Theorem 6.1), it is not always quasi-concave nor geodesic quasi-concave (Theorems 5.2 and 6.2, however, imply that it is both quasi-concave and geodesic quasi-concave at the Gaussian and independent Cauchy examples considered).

7. Shape halfspace depth. In many multivariate statistics problems (PCA, CCA, sphericity testing, etc.), it is sufficient to know the scatter matrix $\Sigma$ up to a positive scalar factor. In PCA, for instance, all scatter matrices of the form $c\Sigma$, $c > 0$, indeed provide the same unit eigenvectors $v_\ell(c\Sigma)$, $\ell = 1, \ldots, k$, hence the same principal components. Moreover, when it comes to deciding how many principal components to work with, a common practice is to look at the proportions of explained variance $\sum_{\ell=1}^m \lambda_\ell(c\Sigma)/\sum_{\ell=1}^k \lambda_\ell(c\Sigma)$, $m = 1, \ldots, k - 1$, which do not depend on $c$ either. In PCA, thus, the parameter of interest is a shape matrix, that is, a normalized version, $V$ say, of the scatter matrix $\Sigma$.

The generic way to normalize a scatter matrix $\Sigma$ into a shape matrix $V$ is based on a scale functional $S$, that is, on a mapping $S : \mathcal{P}_k \to \mathbb{R}_+^*$ satisfying (i) $S(I_k) = 1$ and (ii) $S(c\Sigma) = cS(\Sigma)$ for any $c > 0$ and $\Sigma \in \mathcal{P}_k$. In this paper, we will further assume that (iii) if $\Sigma_1, \Sigma_2 \in \mathcal{P}_k$ satisfy $\Sigma_2 \geq \Sigma_1$ (in the sense that $\Sigma_2 - \Sigma_1$ is positive semidefinite), then $S(\Sigma_2) \geq S(\Sigma_1)$. Such a scale functional leads to factorizing $\Sigma(\in \mathcal{P}_k)$ into

$$\Sigma = \sigma_S^2 V_S,$$

where $\sigma_S^2 := S(\Sigma)$ is the scale of $\Sigma$ and $V_S := \Sigma/S(\Sigma)$ is its shape matrix (in the sequel, we will drop the subscript $S$ in $V_S$ to avoid overloading the notation). The resulting collection of shape matrices $V$ will be denoted as $\mathcal{P}_{k,S}$. Note that the constraint $S(I_k) = 1$ ensures that, irrespective of the scale functional $S$ adopted, $I_k$ is a shape matrix. Common scale functionals satisfying (i)-(iii) are (a) $S_{tr}(\Sigma) = (\text{tr}\, \Sigma)/k$ (Dümbgen, 1998; Sirkiä et al., 2009; Taskinen, Sirkiä and Oja, 2010; Tyler, 1987), (b) $S_{det}(\Sigma) = (\text{det}\, \Sigma)^{1/k}$ (Dümbgen and Tyler, 2005, 2016; Hallin and Paindaveine, 2008; Taskinen et al., 2006; Tatsuoka and Tyler, 2000), and (c) $S_{tr}(\Sigma) = k/(\text{tr}\, \Sigma^{-1})$ (Frahm, 2009). A slightly less classical choice, that also satisfies (i)-(iii), is (d) $S_{11}(\Sigma) = \Sigma_{11}$ (Hallin, Oja and Paindaveine, 2006; Hallin and Paindaveine, 2006; Hettmansperger and Randles, 2002; Randles, 2000). The corresponding shape matrices $V$ are then normalized in such a way that (a) $\text{tr}[V] = k$, (b) $\det V = 1$, (c) $\text{tr}[V^{-1}] = k$, or (d) $V_{11} = 1$.

In this section, we propose a concept of halfspace depth for shape matrices. More precisely, for a probability measure $P$ over $\mathbb{R}^k$, we define the $(S)$-shape halfspace depth of $V(\in \mathcal{P}_{k,S})$ with respect to $P$ as

$$\text{HD}_{P,S}^{sh}(V) := \sup_{\sigma^2 > 0} \text{HD}_{P}^{sc}(\sigma^2 V),$$
where $HD_{P}^{sc}(\sigma^2V)$ is the scatter halfspace depth of $\sigma^2V$ with respect to $P$. The corresponding depth regions are defined as $R_{P,S}^{sh}(\alpha) := \{V \in P_{k,S} : HD_{P,S}^{sh}(V) \geq \alpha\}$. The resulting deepest shape matrix, if any (we study existence below), is obtained by maximizing the “profile depth” $HD_{P,S}^{sh}(V) = \sup_{\sigma^2>0} HD_{P}^{sc}(\sigma^2V)$, in the same way a profile likelihood approach would be based on the maximization of a (shape) profile likelihood of the form $L_{V}^{sh} = \sup_{\sigma^2>0} L_{\sigma^2V}$. To the best of our knowledge, such a profile depth construction has never been considered in the literature.

We start our investigation of shape halfspace depth by considering our running, Gaussian and independent Cauchy, examples. For the $k$-variate normal with location $\theta_0$ and scatter $\Sigma_0$ (hence, with $S$-shape matrix $V_0 = \Sigma_0/S(\Sigma_0)$),

$$\sigma^2 \mapsto HD_{P}^{sc}(\sigma^2V) = 2 \min \left( \Phi \left( \frac{b\sigma \lambda_k^{1/2}(V_0^{-1}V)}{\sqrt{S(\Sigma_0)}} \right) - \frac{1}{2}, 1 - \Phi \left( \frac{b\sigma \lambda_1^{1/2}(V_0^{-1}V)}{\sqrt{S(\Sigma_0)}} \right) \right)$$

(see (2.5)) will be uniquely maximized at the $\sigma^2$-value for which both arguments of the minimum are equal. Consequently, the shape halfspace depth of $V$ is given by

$$HD_{P,S}^{sh}(V) = 2 \Phi \left( c\lambda_k^{1/2}(V_0^{-1}V) \right) - 1,$$

where $c = c(V_0^{-1}V)$ is the unique solution of

$$\Phi \left( c\lambda_k^{1/2}(V_0^{-1}V) \right) - \frac{1}{2} = 1 - \Phi \left( c\lambda_1^{1/2}(V_0^{-1}V) \right).$$

At the $k$-variate distribution with independent Cauchy marginals, we have that

$$HD_{P}^{sc}(\sigma^2V) = 2 \min \left( \Psi(\sigma \max_s(sV_0^{-1}s)^{1/2}) - \frac{1}{2}, 1 - \Psi(\sigma \max_s(\text{diag}(V))^{1/2}) \right)$$

(with the same notation as in (2.6)) is still maximized at the unique $\sigma^2$-value for which both arguments of the minimum are equal, namely at $\sigma^2 = \left( \max_s(sV_0^{-1}s) / \max_s(\text{diag}(V)) \right)^{1/2}$. Therefore, the shape halfspace depth of $V$ here is given by

$$HD_{P,S}^{sh}(V) = 2 \Psi \left( \left( \max_s(sV_0^{-1}s) \max_s(\text{diag}(V)) \right)^{-1/4} \right) - 1$$

$$= \frac{2}{\pi} \arctan \left( \left( \max_s(sV_0^{-1}s) \max_s(\text{diag}(V)) \right)^{-1/4} \right).$$

Figure 5 draws, for six probability measures $P$, contour plots of $(V_{11}, V_{12}) \mapsto HD_{P,S_{tr}}^{sh}(V)$, where $HD_{S_{tr}}^{sh}(V)$ is the shape halfspace depth of $V = (V_{11}, V_{12}, V_{22})$ with respect to $P$. Letting $\Sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Sigma_B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Sigma_C = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$, the probability measures $P$ considered are those associated (i) with the bivariate normal distributions with location zero and scatter $\Sigma_A$, $\Sigma_B$ and $\Sigma_C$, and (ii) with the distributions of $\Sigma_A^{1/2}Z$, $\Sigma_B^{1/2}Z$ and $\Sigma_C^{1/2}Z$, where $Z$ has mutually independent Cauchy marginals. In each case, the “true” $S_{tr}$-shape matrix is marked in red. Note that the maximal depth is larger in the Gaussian cases than in the Cauchy ones, that depth monotonically decreases along any ray originating from the deepest shape matrix and that it goes to zero if and only if the shape matrix converges...
to the boundary of the parameter space. The shape halfspace depth contours are smooth in the Gaussian cases but not in the Cauchy ones.

In both the Gaussian and independent Cauchy examples above, the supremum in (7.2) is a maximum. In the sample case (where \( P \) is the empirical probability measure \( P_n \) associated with a sample of \( n \) observations at hand), this will always be the case since \( HD_{P}^{sc}(\sigma^2 V) \) then takes its values in \{\( k/n : k = 0, 1, \ldots, n \}\}. The following result (that, like all results of this section, is proved in Appendix E) implies in particular that, in the population case, a sufficient condition for this supremum to be a maximum is that \( P \) is smooth at its Tukey median \( \theta_P \) (which is the case for both the multinormal and independent Cauchy examples above).

**Theorem 7.1.** Let \( P \) be a probability measure over \( \mathbb{R}^k \). Fix \( V \in \mathcal{P}_{k,S} \) such that \( cV \in R_{P}^s(\alpha_P) \) for some \( c > 0 \). Then, \( HD_{P,S}^{sh}(V) = HD_{P,S}^{sc}(\sigma_V^2 V) \) for some \( \sigma_V^2 > 0 \).

The following affine-invariance/equivariance and uniform consistency results are easily obtained from their scatter antecedents.

**Theorem 7.2.** Shape halfspace depth is affine-invariant in the sense that, for any probability measure \( P \) over \( \mathbb{R}^k \), \( V \in \mathcal{P}_{k,S} \), \( A \in GL_k \) and \( b \in \mathbb{R}^k \), we have \( HD_{P,A,b,S}^{sh}(AVA'/S(AVA')) = HD_{P,S}^{sh}(V) \), where \( P_{A,b} \) is as defined on page 5. Consequently, shape halfspace depth regions are affine-equivariant, in the sense that \( R_{P,A,b,S}^{sh}(\alpha) = \{AVA'/S(AVA') : V \in R_{P,S}^{sh}(\alpha)\} \) for any probability measure \( P \) over \( \mathbb{R}^k \), \( \alpha \geq 0 \), \( A \in GL_k \) and \( b \in \mathbb{R}^k \).

**Theorem 7.3.** Let \( P \) be a smooth probability measure over \( \mathbb{R}^k \) and let \( P_n \) denote the empirical probability measure associated with a random sample of size \( n \) from \( P \). Then \( \sup_{V \in \mathcal{P}_{k,S}} |HD_{P_n,S}^{sh}(V) - HD_{P,S}^{sh}(V)| \to 0 \) almost surely as \( n \to \infty \).

Shape halfspace depth inherits the \( F^- \) and \( g^- \)-continuity properties of scatter halfspace depth (Theorems 3.1 and 4.1, respectively), at least for a smooth \( P \). More precisely, we have the following result.

**Theorem 7.4.** Let \( P \) be a probability measure over \( \mathbb{R}^k \). Then, (i) \( V \mapsto HD_{P,S}^{sh}(V) \) is upper \( F^- \) and \( g^- \)-semicontinuous on \( R_{P,S}^{sh}(\alpha_P) \), so that (ii) for any \( \alpha \geq \alpha_P \), the depth region \( R_{P,S}^{sh}(\alpha) \) is \( F^- \) and \( g^- \)-closed. (iii) If \( P \) is smooth at its Tukey median \( \theta_P \), then \( V \mapsto HD_{P,S}^{sh}(V) \) is \( F^- \) and \( g^- \)-continuous.

The \( g^- \)-boundedness part of the following result will play a key role when proving the existence of a halfspace deepest shape matrix.

**Theorem 7.5.** Let \( P \) be a probability measure over \( \mathbb{R}^k \). Then, for any \( \alpha > \alpha_P \), \( R_{P,S}^{sh}(\alpha) \) is \( F^- \) and \( g^- \)-bounded, hence \( g^- \)-compact. If \( s_P \geq 1/2 \), then this result is trivial in the sense that \( R_{P,S}^{sh}(\alpha) \) is empty for \( \alpha > \alpha_P \).
Comparing with the scatter result in Theorem 3.2, the shape result for $F$-boundedness requires the additional condition $\alpha > \alpha_P$ (for $g$-boundedness, this condition was already required in Theorem 4.2). This condition will actually be necessary for scale functionals $S$ for which implosion of a shape matrix $V$ cannot be obtained without explosion, as it is the case, e.g., for $S_{\text{det}}$ (the product of the eigenvalues of an $S_{\text{det}}$-shape matrix being equal to one, the largest eigenvalue of $V$ cannot go to infinity without the smallest one going to zero). We illustrate this on the bivariate discrete example discussed below Theorem 4.2.

The sequence of scatter matrices $\Sigma_n = \text{diag}(\frac{1}{n}, 1)$ considered there defines a sequence of $S_{\text{det}}$-shape matrices $V_n = \text{diag}(\frac{1}{\sqrt{n}}, \sqrt{n})$, that is neither $F$- nor $g$-bounded. Still, we have $HD_{P, S_{\text{det}}}^{sh}(V_n) \geq HD_{P, S_{\text{det}}}^{sh}(\Sigma_n) \geq 1/3$ for any $n$, so that $V_n \in R_{P, S_{\text{det}}}(1/3)(= R_{P, S_{\text{det}}}^{sh}(\alpha_P))$ for any $n$. In this example, thus, $R_{P, S_{\text{det}}}^{sh}(\alpha_P)$ is both $F$- and $g$-unbounded. Note also that $F$-boundedness of the regions $R_{P, S}^{sh}(\alpha)$ crucially depends on $S$. In particular, it is easy to check that the extra condition $\alpha > \alpha_P$ for $F$-boundedness is not needed for the scale functional $S_{tr}^*$ (that is, $R_{P, S_{tr}}^{sh}(\alpha)$ is $F$-bounded for any $\alpha > 0$). Finally, one trivially has that all $R_{P, S_{tr}}^{sh}(\alpha)$’s are $F$-bounded since the corresponding collection of shape matrices, $P_{k, S_{tr}}$, itself is $F$-bounded. Unlike $F$-boundedness, $g$-boundedness results are homogeneous in $S$, which further supports that the $g$-topology is the most appropriate one when investigating properties of scatter/shape depth.

As announced, the $g$-part of the previous theorem allows to show that a halfspace deepest shape matrix exists under mild conditions. More precisely, we have the following result.

**Theorem 7.6.** Let $P$ be a probability measure over $\mathbb{R}^k$ and assume that $R_{P, S}^{sh}(\alpha_P)$ is non-empty. Then, $\alpha_{P, S} := \sup_{V \in \mathcal{P}_{k, S}} HD_{P, S}^{sh}(V) = HD_{P, S}^{sh}(V_*)$ for some $V_* \in \mathcal{P}_{k, S}$.

Like for scatter, a sufficient condition for the existence of a halfspace deepest shape matrix is that the underlying probability measure $P$ is smooth at its Tukey median $\theta_P$ (since this implies that $R_{P, S}^{sh}(\alpha_P) = R_{P, S}^{sh}(0) = \mathcal{P}_{k, S}$). In particular, a halfspace deepest shape matrix exists in the Gaussian and independent Cauchy examples. For the $k$-variate probability measure with independent Cauchy marginals, it readily follows from Theorem 4.4 that $V \mapsto HD_{P, S}^{sh}(V)$ is uniquely maximized at $V_* = I_k$, with corresponding maximal depth $HD_{P, S}^{sh}(V_*) = \frac{2}{\pi} \arctan(k^{-1/4})$. The next Fisher-consistency result states that, in the elliptical case, the halfspace deepest shape matrix coincides with the “true” shape matrix.

**Theorem 7.7.** Let $P$ be an elliptical probability measure over $\mathbb{R}^k$ with location $\theta_0$ and scatter $\Sigma_0$, hence with $S$-shape matrix $V_0 = \Sigma_0/S(\Sigma_0)$. Then, (i) for any $V \in \mathcal{P}_{k, S}$, $HD_{P, S}^{sh}(V_0) \geq HD_{P, S}^{sh}(V)$; (ii) if $I_{\text{MMD}}[Z_1]$ is a singleton (i.e., if $I_{\text{MMD}}[Z_1] = \{1\}$), where $Z = (Z_1, \ldots, Z_k) \overset{D}{=} \Sigma_0^{-1/2}(X - \theta_0)$, then $V \mapsto HD_{P, S}^{sh}(V)$ is uniquely maximized at $V_0$.

We conclude this section by considering quasi-concavity properties of shape halfspace
depth and convexity of the corresponding depth regions. It should be noted that, for some scale functionals $S$, the collection $\mathcal{P}_{k,S}$ of $S$-shape matrices is not convex; for instance, neither $\mathcal{P}_{k,S_{\det}}$ nor $\mathcal{P}_{k,S_{tr}}$ is convex, so that it does not make sense to investigate whether or not $V \mapsto HD_{P,S}^{sh}(V)$ is quasi-concave for these scale functionals. It does, however, for $S_{tr}$ and $S_{11}$, and we have the following result.

**Theorem 7.8.** Let $P$ be a probability measure over $\mathbb{R}^k$ and fix $S = S_{tr}$ or $S = S_{11}$. Then, (i) $V \mapsto HD_{P,S}^{sh}(V)$ is quasi-concave, that is, for any $V_a, V_b \in \mathcal{P}_{k,S}$ and $t \in [0,1]$, $HD_{P,S}^{sh}(V_t, P) \geq \min(HD_{P,S}^{sh}(V_a), HD_{P,S}^{sh}(V_b))$, where we let $V_t := (1-t)V_a + tV_b$; (ii) for any $\alpha \geq 0$, $R_{P,S}^{sh}(\alpha)$ is convex.

As mentioned above, neither $\mathcal{P}_{k,S_{\det}}$ nor $\mathcal{P}_{k,S_{tr}}$ are convex in the usual sense. However, $\mathcal{P}_{k,S_{\det}}$ is geodesic convex, which justifies studying the possible geodesic convexity of the depth regions $R_{P,S_{\det}}^{sh}(\alpha)$ (this provides a parametric framework for which the shape version of Property (Q3) in Section 5 cannot be considered and for which it is needed to adopt Property (Q3) instead). Similarly, $\mathcal{P}_{k,S_{tr}}$ is harmonic convex, so that it makes sense to investigate the harmonic convexity of the regions $R_{P,S_{tr}}^{sh}(\alpha)$. We have the following results.

**Theorem 7.9.** Let $P$ be an arbitrary probability measure over $\mathbb{R}^k$ for which scatter halfspace depth is geodesic quasi-concave. Then, (i) $V \mapsto HD_{P,S_{\det}}^{sh}(V)$ is geodesic quasi-concave, so that (ii) $R_{P,S_{\det}}^{sh}(\alpha)$ is geodesic convex for any $\alpha \geq 0$.

**Theorem 7.10.** Let $P$ be an arbitrary probability measure over $\mathbb{R}^k$ for which scatter halfspace depth is harmonic quasi-concave. Then, (i) $V \mapsto HD_{P,S_{tr}}^{sh}(V)$ is harmonic quasi-concave, so that (ii) $R_{P,S_{tr}}^{sh}(\alpha)$ is harmonic convex for any $\alpha \geq 0$.

For the sake of illustration, Figure 6 draws contour plots of $(V_{11}, V_{12}) \mapsto HD_{P,S}^{sh}(V_S)$, for the scale functionals $S_{tr}$, $S_{\det}$ and $S_{tr}$, where $HD_{S}^{sh}(V_S)$ is the shape halfspace depth, with respect to $P$, of the $S$-shape $V_S(\in \mathcal{P}_{2,S})$ with upper-left entry $V_{11}$ and upper-right entry $V_{12}$. The probability measures $P$ considered are those associated (i) with the bivariate normal distribution with location zero and scatter $\Sigma_C = \left(\begin{array}{cc} 3 & 1 \\ 1 & 1 \end{array}\right)$, and (ii) with the distribution of $\Sigma_C^{1/2}Z$, where $Z$ has mutually independent Cauchy marginals. For $S_{tr}$, $S_{\det}$ and $S_{tr}$, the figure also shows the linear, geodesic and harmonic paths, respectively, linking the (“true”) $S$-shape associated with $\Sigma_C$ and those associated with $\Sigma_A = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$ and $\Sigma_B = \left(\begin{array}{cc} 4 & 0 \\ 0 & 0 \end{array}\right)$. The results illustrate the convexity of the regions $R_{P,S_{\det}}^{sh}(\alpha)$, along with the geodesic (resp., harmonic) convexity of the regions $R_{P,S_{tr}}^{sh}(\alpha)$ (resp., $R_{P,S_{tr}}^{sh}(\alpha)$).

**8. A real-data application.** In this section, we analyze the returns of the Nasdaq Composite and S&P500 indices from February 1st, 2015 to February 1st, 2017. During that period, for each trading day and for each index, we collected returns every 5 minutes (that is, the difference between the index at a given time and 5 minutes earlier, when available), resulting in usually 78 bivariate observations per day. Due to some missing values, the
The exact number of returns per day varies, and only days with at least 70 observations were considered. The resulting dataset comprises a total of 38489 bivariate returns distributed over \( D = 478 \) trading days.

The goal of this analysis is to determine which days, during the two-year period, exhibit an atypical behavior. In line with the fact that the main focus in finance is on volatility, atypicality here will refer to deviations from the “global” behavior either in scatter (i.e., returns do not follow the global dispersion pattern) or in scale only (i.e., returns show a usual shape but their overall size is different). Atypical days will be detected by comparing intraday estimates of scatter and shape with a global version.

Below, \( \hat{\Sigma}_{\text{full}} \) will denote the minimum covariance determinant (MCD) scatter estimate on the empirical distribution \( P_{\text{full}} \) of the returns over the two-year period, and \( \hat{V}_{\text{full}} \) will stand for the corresponding shape estimate \( \hat{V}_{\text{full}} = \hat{\Sigma}_{\text{full}} / \det(\hat{\Sigma}_{\text{full}}) \) (see, e.g., Rousseeuw and Van Driessen, 1999 or Paindaveine and Van Bever, 2014). For any \( d = 1, \ldots, D \), \( \hat{\Sigma}_d \) and \( \hat{V}_d \) will denote the corresponding estimates on the empirical distribution \( P_d \) on day \( d \).

The rationale behind the choice of MCD rather than standard covariance as an estimation method for scatter/shape is twofold. First, the former will naturally deal with outliers inherently arising in the data. Indeed, the first few returns after an overnight or weekend break are famously more volatile and their importance should be downweighted in the estimation procedure. The second reason is a partial consequence of the first. As hinted above, the global estimate will provide a baseline to measure the atypicality of any given day. This will be done, among others, using its intraday depth. It would be natural to use halfspace deepest scatter/shape matrices on \( P_{\text{full}} \) as global estimates for scatter/shape. While locating the exact maxima is a non-trivial task, the MCD shape estimator has already a high depth value \( HD_{P_{\text{full}}, \det}(\hat{V}_{\text{full}}) = 0.481 \), which makes it a very good proxy for the halfspace deepest shape matrix. For the same reason, the scaled MCD estimator \( \bar{\Sigma}_{\text{full}} = \sigma_{\text{full}}^2 V_{\text{full}} \) with \( \sigma_{\text{full}}^2 = \arg\max_{\sigma^2} \sigma^2 V_{\text{full}} \) (that, obviously, satisfies \( HD_{P_{\text{full}}}(\bar{\Sigma}_{\text{full}}) = 0.481 \)) is similarly an excellent proxy for the halfspace deepest scatter matrix. In contrast, the shape estimate associated with the standard covariance matrix (resp., the deepest scaled version of the covariance matrix) has a global shape (resp., scatter) depth of only 0.426.

For each day, the following measures of (a)typicality (three for scatter, three for shape) are computed:

1. The scatter depth \( HD^{\text{sc}}_{P_d}(\Sigma_{\text{full}}) \) of \( \Sigma_{\text{full}} \) in day \( d \),
2. The shape depth \( HD^{\text{sh}}_{P_d, \det}(\hat{V}_{\text{full}}) \) of \( \hat{V}_{\text{full}} \) in day \( d \),
3. The scatter Frobenius distance \( d_F(\hat{\Sigma}_d, \hat{\Sigma}_{\text{full}}) \),
4. The shape Frobenius distance \( d_F(\hat{V}_d, \hat{V}_{\text{full}}) \),
5. The scatter geodesic distance \( d_g(\hat{\Sigma}_d, \hat{\Sigma}_{\text{full}}) \), and
6. The shape geodesic distance \( d_g(\hat{V}_d, \hat{V}_{\text{full}}) \).

Of course, low depths or high distances point to atypical days. Practitioners might be
tempted to base the distances in (iii)-(vi) on standard covariance estimates, which would actually provide poorer performances in the present outlier detection exercise (due to the masking effect resulting from using a non-robust global dispersion measure as a baseline). Here, we rather use MCD-based estimates to ensure a fair comparison with the depth-based methods in (i)-(ii).

Figure 7 provides the plots of the quantities in (i)-(vi) above as a function of $d, d = 1, \ldots, D$. Major events affecting the returns during the two years are marked there with vertical lines. They are (1) the Black Monday on August 24th, 2015 (orange) when world stock markets went down substantially, (2) the crude oil crisis on January 20th, 2016 (dark blue) when oil barrel prices fell sharply, (3) the Brexit vote aftermath on June 24th, 2016 (dark green), (4) the end of the low volatility period on September 13th, 2016 (red), (5) the Donald Trump election on November 9th, 2016 (purple), and (6) the announcement and aftermath of the federal rate hikes on December 14th, 2016 (teal).

Events (1) and (2) are noticeably singled out by all outlyingness measures, displaying low depth values and high Frobenius and geodesic distances, but the four remaining events tell a very different story. In particular, event (6) exhibits a low scatter depth but a relatively high shape depth, which means that this day shows a shape pattern that is in line with the global one but is very atypical in scale (that is, in volatility size). Quite remarkably, the four distances considered fail to flag this day as an atypical one. A similar behavior appears throughout the two-month period spanning July, August and early September 2016 (between events (3) and (4)), during which the markets have seen a historical streak of small volatility. This period presents widely varying scatter depth values together with stable and high shape depth values, which is perfectly in line with what has been seen on the markets, where only the volatility of the indices was low in days that were otherwise typical. Again, the four distance plots are blind to this relative behavior of scatter and shape in the period.

Events (3) to (5) are picked up by depth measures and scatter distances, though more markedly by the former. This is particularly so for event (3), which sticks out sharply in both depths. The fact that the scatter depth is even lower than the shape depth suggests that event (3) is atypical not only in shape but also in scale. Interestingly, distance measures fully miss the shape outlyingness of this event. Actually, shape distances do not assign large values to any of the events (3) to (6) and, from March 2016 onwards, these distances stay in the same range — particularly so for the Frobenius ones in (iv). In contrast, the better ability of shape depth to spot outlyingness may be of particular importance in instances where one would like to discard the overall size of the volatility to rather focus on the shape structure of the returns.

To summarize, the detection of atypical patterns in the dispersion of intraday returns can more efficiently be performed with scatter/shape depths than on the basis of distance measures. Arguably, the fact that the proposed depths use all observations and not a sole estimate of scatter/shape allows to detect deviations from global behaviors more sharply.
As showed above, comparing scatter and shape depth values provides a tool that permits the distinction between shape and scale outliers.

9. Final comments and perspectives. In this work, we thoroughly investigated the structural properties of a concept of scatter halfspace depth linked to those proposed in Zhang (2002) and Chen, Gao and Ren (2015). While we tried doing so under minimal assumptions, alternative scatter halfspace depth concepts may actually require even weaker assumptions, but they typically would make the computational burden heavier in the empirical case. As an example, one might alternatively define the scatter halfspace depth of $\Sigma(\in \mathcal{P}_k)$ with respect to $P$ as

$$HD_{P,\text{alt}}^{sc}(\Sigma) = \sup_{\theta \in \mathbb{R}^k} HD_{P}^{sc,\theta}(\Sigma),$$

where $HD_{P}^{sc,\theta}(\Sigma)$ is the fixed-location version of scatter halfspace depth defined in (2.7). It is easy to show that this alternative scatter depth concept satisfies a uniform consistency result such as the one in Theorem 2.2 without any condition on $P$, whereas the proposed scatter halfspace depth $HD_{P}^{sc}(\Sigma)$ requires that $P$ is smooth (see Theorem 2.2). However, evaluation of $HD_{P,n}^{sc,\text{alt}}(\Sigma)$ in practice is computationally much more involved than the version $HD_{P,n}^{sc}(\Sigma)$ we focused on in the present work, which is the main reason why we favour the latter scatter depth concept. Alternative concentration and shape halfspace depth concepts may be defined along the same lines and will show the same advantages/disadvantages compared to those introduced in this paper.

Computational issues actually deserve to be explored extensively, as the concepts we proposed will make their way to applications only if efficient algorithms are available. Evaluating (good approximations of) the scatter halfspace depth $HD_{P,n}^{sc}(\Sigma)$ of a given $\Sigma$ can easily be done for very small dimensions $k = 2$ or 3 by sampling the unit sphere $S^{k-1}$, either by generating random directions (in the spirit of the “random Tukey depth”; see, e.g., Cuesta-Albertos and Nieto-Reyes, 2008) or by considering a fixed grid. Such an approach, however, is hardly applicable in higher dimensions and suitable algorithms are already needed for $k = 5$ or 6. Finally, even for small dimensions such as $k = 2$ or 3, it is a relatively challenging task to compute halfspace deepest scatter matrices: the main reason is that, while the proposed concept relies on a relatively low-dimensional (that is, $k$-dimensional) projection-pursuit approach, identifying the halfspace deepest scatter matrix requires exploring the collection of scatter matrices $\mathcal{P}_k$, that is of higher dimension, namely of dimension $k(k+1)/2$.

Beyond those related to computational aspects, perspectives for future research are rich and diverse. The proposed halfspace depth concepts for scatter, concentration and shape can be extended to other scatter functionals of interest. In particular, halfspace depths that are relevant for principal component analysis could be defined by exploiting the “profile depth” approach introduced in Section 7 (incidentally, note that the alternative definition of scatter halfspace depth in (9.1) is also based on the profile depth construction, in a global
location-scatter framework). For instance, this would lead to defining the “first principal direction” halfspace depth of $\beta (\in S^{k-1})$ with respect to the probability measure $P$ over $\mathbb{R}^k$ as

$$ HD_{P}^{1st pd} (\beta) = \sup_{\Sigma \in \mathcal{P}_{k,1,\beta}} HD_{P}^{sc} (\Sigma), $$

where $\mathcal{P}_{k,1,\beta}$ stands for the collection of scatter matrices in $\mathcal{P}_k$ for which $\Sigma \beta = \lambda_1 (\Sigma) \beta$. The halfspace deepest first principal direction is a promising robust estimator of the true underlying first principal direction, at least under elliptical assumptions. Obviously, the depth of any other principal direction, or the depth of eigenvalues, can be defined accordingly. Of course, another direction of research is to explore inferential applications of the depth concepts introduced in this work. Clearly, point estimation is to be based on halfspace deepest scatter, concentration or shape matrices; Chen, Gao and Ren (2015) partly studied this already for scatter, in a (possibly high-dimensional) fixed-location framework. Hypothesis testing is also of primary interest. In particular, a natural test for $H_0 : \Sigma = \Sigma_0$, where $\Sigma_0 \in \mathcal{P}_k$ is fixed, consists in rejecting the null for small values of $HD_{P_0}^{sc} (\Sigma_0)$. This can also be considered for shape matrices for, e.g., the problem of testing for sphericity, where the null should be rejected for small values of $HD_{P_0, S}^{sh} (I_k)$. These various topics, obviously, are beyond the scope of the present work.

**APPENDIX A: PROOFS FROM SECTION 2**

In the proofs below, we will often use the fact that

$$ HD_{P}^{sc} (\Sigma) = \inf_{u} \min \left( P \left[ |u'X| \leq \sqrt{u'\Sigma u} \right], P \left[ |u'X| \geq \sqrt{u'\Sigma u} \right] \right) = \min \left( \inf_{u} P \left[ |u'X| \leq \sqrt{u'\Sigma u} \right], \inf_{u} P \left[ |u'X| \geq \sqrt{u'\Sigma u} \right] \right), $$

where all infima are over the unit sphere $S^{k-1}$ of $\mathbb{R}^k$.

**Proof of Theorem 2.1.** Fix $A \in GL_k$ and $b \in \mathbb{R}^k$. Affine-equivariance of the Tukey median states that $\theta_{P_{A,b}} = A \theta_P + b$. Therefore, letting $u_A := A' u / \|A' u\|$, we have

$$ HD_{P_{A,b}}^{sc} (A \Sigma A') = \inf_{u \in S^{k-1}} \min \left( P \left[ |u'AX - \theta_P| \leq \sqrt{u' \Sigma A' u} \right], P \left[ |u'AX - \theta_P| \geq \sqrt{u' \Sigma A' u} \right] \right) = \inf_{u \in S^{k-1}} \min \left( P \left[ |u'_A(X - \theta_P)| \leq \sqrt{u'_A \Sigma u_A} \right], P \left[ |u'_A(X - \theta_P)| \geq \sqrt{u'_A \Sigma u_A} \right] \right) = HD_{P}^{sc} (\Sigma), $$

where the last equality follows from the fact that the mapping $u \mapsto u_A$ is a one-to-one transformation of $S^{k-1}$. \qed

We turn to the proof of the technical result that allowed to compute scatter depth for the probability measure over $\mathbb{R}^k$ with independent Cauchy marginals.
Lemma A.1. For any $\Sigma \in \mathcal{P}_k$, the maximal and minimal values of $v'\Sigma v$ when $v$ runs over the $L_1$-sphere $\{v \in \mathbb{R}^k : \|v\|_1 = \sum_{i=1}^k |v_i| = 1\}$ are $\max(\text{diag}(\Sigma))$ and $1/\max_s(s'\Sigma^{-1} s)$, respectively, where $\max_s$ is the maximum over $s = (s_1, \ldots, s_k) \in \{-1, 1\}^k$.

Proof of Lemma A.1. We start with the following considerations. In dimension $k$, the $L_1$-sphere can be parametrized as

$$v_{s,t} = (s_1 t_1, s_2 t_2, \ldots, s_{k-1} t_{k-1}, s_k (1 - t_1 - \cdots - t_{k-1}))',$$

where $s = (s_1, \ldots, s_k) \in \{-1, 1\}^k$ and $(t_1, \ldots, t_{k-1}) \in \text{Simpl}_{k-1} := \{(t_1, \ldots, t_{k-1}) : t_1 \geq 0, \ldots, t_{k-1} \geq 0, t_1 + \cdots + t_{k-1} \leq 1\}$. Clearly, symmetry of the $L_1$-sphere and of the function to be maximized/minimized allows to restrict to $s_k = 1$. Now, for any given $s$ of the form $(s_1, \ldots, s_{k-1}, 1)$, consider the function

$$f_s : \text{Simpl}_{k-1} \rightarrow \mathbb{R}$$

$$t = (t_1, \ldots, t_{k-1}) \mapsto v_{s,t}' \Sigma v_{s,t}.$$

This function is twice differentiable, with a gradient $\nabla f_s(t)$ whose $i$th component is

$$\frac{\partial}{\partial t_i} f_s(t) = (0, \ldots, 0, s_i, 0, \ldots, -1) \Sigma v_{s,t} + v_{s,t}' \Sigma (0, \ldots, 0, s_i, 0, \ldots, -1)'$$

$$= 2(0, \ldots, 0, s_i, 0, \ldots, -1) \Sigma v_{s,t}$$

and a Hessian matrix $H_s(t)$ whose $(i,j)$-entry is

$$\frac{\partial^2}{\partial t_i t_j} f_s(t) = 2(0, \ldots, 0, s_i, 0, \ldots, -1) \Sigma (0, \ldots, 0, s_j, 0, \ldots, -1)'.$$

Now, for any $z \in \mathbb{R}^{k-1}$,

$$z' H_s(t) z = 2(s_1 z_1, \ldots, s_k z_k, -z_1 - \cdots - z_k) \Sigma (s_1 z_1, \ldots, s_k z_k, -z_1 - \cdots - z_k)' \geq 0,$$

with equality if and only if $z = 0$. Therefore, $f_s$ is strictly convex over $\text{Simpl}_{k-1}$.

Let us start with the maximum. For a given $s$, strict convexity of $f_s$ implies that the maximum of $f_s$ can only be achieved at $e_{i,k-1}$, $i = 1, \ldots, k-1$, where $e_{i,\ell}$ stands for the $i$th vector of the canonical basis of $\mathbb{R}^\ell$, or at $0(\in \mathbb{R}^{k-1})$. Since $f_s(e_{i,k-1}) = \Sigma_{ii}$, $i = 1, \ldots, k-1$, and $f_s(0) = \Sigma_{kk}$, it follows that the maximal value of $f_s$ over $\text{Simpl}_{k-1}$ is $\max(\text{diag}(\Sigma))$. Since this is the case for any $s$, we conclude that the maximum of $v'\Sigma v$ over the $L_1$-sphere is itself $\max(\text{diag}(\Sigma))$.

Let us then turn to the minimum. The minimum of $f_s$, when extended into a (still strictly convex) function defined on $\mathbb{R}^{k-1}$, is the solution of the gradient conditions

$$(0, \ldots, 0, s_i, 0, \ldots, -1) \Sigma v_{s,t} = 0, \quad i = 1, \ldots, k-1.$$
Writing simply \( e_k = e_{k,k} \) for the \( k \)th vector of the canonical basis of \( \mathbb{R}^k \) and letting

\[
S_s = \begin{pmatrix}
  s_1 & -1 \\
  \vdots & \ddots \\
  s_{k-1} & -1
\end{pmatrix},
\]

these gradient conditions rewrite \( S_s \Sigma(e_k + S'_s \ell t) = 0 \) (recall that we restricted to \( s_k = 1 \)), and their unique solution in \( \mathbb{R}^{k-1} \) is \( t^\text{min}_s := -(S_s \Sigma S'_s)^{-1} S_s e_k \).

It will be useful below to have a more explicit expression of \( t^\text{min}_s \). Note that the gradient conditions above state that \( \Sigma(e_k + S'_s t^\text{min}_s) \) is in the null space of \( S_s \). Since this null space is easily checked to be \( \{ \lambda s : \lambda \in \mathbb{R} \} \), this implies that \( e_k + S'_s t^\text{min}_s = \lambda \Sigma^{-1} s \) for some \( \lambda \in \mathbb{R} \). Premultiplying both sides of this equation by \( s' \), we obtain \( 1 = \lambda s' \Sigma^{-1} s \), which yields \( \lambda = 1/(s' \Sigma^{-1} s) \). Thus,

(A.1) \[
 e_k + S'_s t^\text{min}_s = \frac{1}{s' \Sigma^{-1} s} \Sigma^{-1} s.
\]

In the first \( k - 1 \) components, this yields (after multiplication by \( s_i \))

(A.2) \[
(t^\text{min}_s)_i = \frac{s_i e_i' \Sigma^{-1} s}{s' \Sigma^{-1} s}, \quad i = 1, \ldots, k - 1,
\]

while the \( k \)th component provides (still with \( s_k = 1 \))

(A.3) \[
1 - \sum_{i=1}^{k-1} (t^\text{min}_s)_i = \frac{e_k' \Sigma^{-1} s}{s' \Sigma^{-1} s} = \frac{s_k e_k' \Sigma^{-1} s}{s' \Sigma^{-1} s}.
\]

Strict convexity of \( f_s \) implies that its minimal value over \( \mathbb{R}^{k-1} \) is \( f_s(t^\text{min}_s) \). By using (A.1), this minimal value takes the form

\[
f_s(t^\text{min}_s) = v'_{s,t^\text{min}_s} \Sigma v_{s,t^\text{min}_s} = (e_k + S'_s t^\text{min}_s)' \Sigma (e_k + S'_s t^\text{min}_s) = \frac{1}{s' \Sigma^{-1} s}.
\]

Now, consider an arbitrary sign \( k \)-vector \( s_* \) that maximizes \( s' \Sigma^{-1} s \) among the \( 2^{k-1} \) corresponding sign vectors \( s \) to be considered (the last component of \( s \) is still fixed to one). Assume for a moment that \( t^\text{min}_{s_*} \) is in the interior of \( \text{Simpl}_{k-1} \). Since it is the minimal value of \( f_s \) over \( \mathbb{R}^{k-1} \), \( f_{s_*}(t^\text{min}_{s_*}) = 1/(s_*' \Sigma^{-1} s_*) \) of course minimizes \( f_{s_*} \) over \( \text{Simpl}_{k-1} \).

Pick then another sign vector \( s \). By construction, \( 1/(s_*' \Sigma^{-1} s_*) \) is smaller than or equal to \( f_s(t^\text{min}_s) = 1/(s' \Sigma^{-1} s) \), which, as the minimal value of \( f_s \) when extended to \( \mathbb{R}^{k-1} \), can only be smaller than or equal to the minimal value of \( f_s \) over \( \text{Simpl}_{k-1} \). Therefore, \( 1/(s_*' \Sigma^{-1} s_*) \) is then the minimal value of \( v' \Sigma^{-1} v \) over the unit \( L_1 \)-sphere.

It thus remains to show that \( t^\text{min}_{s_*} \) indeed belongs to the interior of \( \text{Simpl}_{k-1} \). Equivalently (in view of (A.2)-(A.3)), it remains to show that \( s_i e_i' \Sigma^{-1} s_* > 0 \) for \( i = 1, \ldots, k \). Assume then that \( s_i e_i' \Sigma^{-1} s_* \leq 0 \) for some \( \ell \). Defining \( s_{**} \) as the vector obtained from \( s_* \) by only changing the sign of its \( \ell \)th component, that is, putting \( s_{**} := s_* - 2s_{\ell} e_\ell \), we have

\[
(s_{**}' \Sigma^{-1} s_{**} - s_*' \Sigma^{-1} s_*) = (s_* - 2s_{\ell} e_\ell)' \Sigma^{-1} (s_* - 2s_{\ell} e_\ell) - s_*' \Sigma^{-1} s_*
\]

\[
= -4s_{\ell} e_\ell' \Sigma^{-1} s_* + 4s_{\ell} e_\ell' \Sigma^{-1} e_\ell > 0.
\]
which contradicts the maximality property of $s_*$. 

The proof of Theorem 2.2 requires Lemmas A.2-A.5 below. Before proceeding, we introduce the inner and outer “slabs”

$$H_{u,c}^{\text{in}} := \{ x \in \mathbb{R}^k : |u^T x| \leq c \} \quad \text{and} \quad H_{u,c}^{\text{out}} := \{ x \in \mathbb{R}^k : |u^T x| \geq c \}.$$ 

Henceforth, the superscript “in/out” is to be read as “in (resp., out)”. We will further write $B(\theta, r)$ for the ball $\{ x \in \mathbb{R}^k : \|x - \theta\| < r \}$ and $\bar{B}(\theta, r)$ for its closure.

**Lemma A.2.** Let $P$ be a probability measure over $\mathbb{R}^k$ and $P_n$ denote the empirical measure associated with a random sample of size $n$ from $P$. Then $\theta_{P_n} \to \theta_P$ almost surely as $n \to \infty$.

**Proof of Lemma A.2.** For any $\theta \in \mathbb{R}^k$ and any probability measure $Q$ over $\mathbb{R}^k$, denote by $HD_Q^{\text{loc}}(\theta)$ the location halfspace depth of $\theta$ with respect to $Q$. Recall that we defined $\theta_Q$ as the barycentre of $M_Q^{\text{loc}} = \{ \theta \in \mathbb{R}^k : HD_Q^{\text{loc}}(\theta) = \max_{\eta \in \mathbb{R}^k} HD_Q^{\text{loc}}(\eta) \}$. It is well known that $\theta \mapsto HD_Q^{\text{loc}}(\theta)$ is upper semicontinuous; see, e.g., Lemma 6.1 in Donoho and Gasko (1992). In general, this function is not uniquely maximized at $\theta_Q$. However, it is easy to define a modified depth function $\theta \mapsto HD_Q^{\text{loc}}(\theta)$ that is still upper semicontinuous, agrees with $\theta \mapsto HD_Q^{\text{loc}}(\theta)$ on $\mathbb{R}^k / M_Q^{\text{loc}}$, and for which $\theta_Q$ is the unique maximizer. In view of the uniform consistency of location halfspace depth (see, e.g., (6.2) and (6.6) in Donoho and Gasko, 1992), the result then follows from Theorem 2.12 and Lemma 14.3 in Kosorok (2008). 

**Lemma A.3.** Let $P, Q$ be two probability measures over $\mathbb{R}^k$. Define $HD_P^{\text{in/out}}(\Sigma) := \inf_{u \in S^{k-1}} P[\theta_P + \Sigma^{1/2} H_{u,c}^{\text{in/out}}]$, with $H_{u,c}^{\text{in/out}} := H_{u,1}^{\text{in/out}}$. Then, for any $\Sigma \in \mathcal{P}_k$,

$$|HD_P^{\text{in/out}}(\Sigma) - HD_Q^{\text{in/out}}(\Sigma)| \leq \sup_{C \in C^{\text{in/out}}} |P[C] - Q[C]| + \sup_{C \in C_0^{\text{in/out}}} |P[\theta_Q + C] - P[\theta_P + C]|,$$

where we let $C^{\text{in/out}} := \{ \theta + H_{u,c}^{\text{in/out}} : (\theta, u, c) \in \mathbb{R}^k \times S^{k-1} \times \mathbb{R}^+_0 \}$ and $C_0^{\text{in/out}} := \{ H_{u,c}^{\text{in/out}} : (u, c) \in S^{k-1} \times \mathbb{R}^+_0 \}$.

**Proof of Lemma A.3.** We prove only the “in” result, since the proof of the “out” result is entirely similar. First assume that $HD_P^{\text{in}}(\Sigma) \geq HD_Q^{\text{in}}(\Sigma)$. Then, for any $\varepsilon > 0$, there exists $u_0 = u_0(\Sigma, Q, \varepsilon)$ such that $Q[\theta_Q + \Sigma^{1/2} H_{u_0}^{\text{in}}] \leq HD_Q^{\text{in}}(\Sigma) + \varepsilon$, so that

$$|HD_P^{\text{in}}(\Sigma) - HD_Q^{\text{in}}(\Sigma)| = HD_P^{\text{in}}(\Sigma) - HD_Q^{\text{in}}(\Sigma)$$

$$\leq P[\theta_P + \Sigma^{1/2} H_{u_0}^{\text{in}}] - Q[\theta_Q + \Sigma^{1/2} H_{u_0}^{\text{in}}] + \varepsilon$$

$$\leq P[\theta_Q + \Sigma^{1/2} H_{u_0}^{\text{in}}] - Q[\theta_Q + \Sigma^{1/2} H_{u_0}^{\text{in}}] + P[\theta_P + \Sigma^{1/2} H_{u_0}^{\text{in}}] - P[\theta_Q + \Sigma^{1/2} H_{u_0}^{\text{in}}] + \varepsilon$$

$$\leq \sup_{C \in C_0^{\text{in}}} |P[C] - Q[C]| + \sup_{C \in C_0^{\text{in}}} |P[\theta_Q + C] - P[\theta_P + C]| + \varepsilon.$$
Similarly, if $HD_P^\infty(\Sigma) \leq HD_Q^\infty(\Sigma)$, then, for any $\varepsilon > 0$, there exists $u_1 = u_1(\Sigma, P, \varepsilon)$ such that $P[\theta + \Sigma^{1/2}H_{u_1}^\infty] \leq HD_P^\infty(\Sigma) + \varepsilon$, so that

$$|HD_P^\infty(\Sigma) - HD_Q^\infty(\Sigma)| = HD_Q^\infty(\Sigma) - HD_P^\infty(\Sigma)$$

$$\leq Q[\theta + \Sigma^{1/2}H_{u_1}^\infty] - P[\theta + \Sigma^{1/2}H_{u_1}^\infty] + \varepsilon$$

$$\leq Q[\theta + \Sigma^{1/2}H_{u_1}^\infty] - P[\theta + \Sigma^{1/2}H_{u_1}^\infty] + P[\theta + \Sigma^{1/2}H_{u_1}^\infty] - P[\theta + \Sigma^{1/2}H_{u_1}^\infty] + \varepsilon$$

$$\leq \sup_{C \in C_0^\infty} |P[C] - Q[C]| + \sup_{C \in C_0^\infty} |P[\theta + C] - P[\theta + C]| + \varepsilon.$$

Since, in both cases, the result holds for any $\varepsilon > 0$, the result is proved. □

**Lemma A.4.** Let $P$ be a probability measure over $\mathbb{R}^k$ and $K$ be a compact subset of $\mathbb{R}^k$. For any $c > 0$, let $s_P^K(c) := \sup_{(\theta, u) \in K \times S^{k-1}} P[|u'(X - \theta)| \leq c]$ and write $s_P^K := s_P^K(0)$. Then (i) $s_P^K(c) \rightarrow s_P^K$ as $c \rightarrow 0$ and (ii) $s_P^K = P[u_0'(X - \theta_0) = 0]$ for some $(\theta_0, u_0) \in K \times S^{k-1}$.

**Proof of Lemma A.4.** Clearly, $s_P^K(c)$ is increasing in $c$ over $[0, \infty)$, which guarantees that $s_P^K := \lim_{c \rightarrow 0} s_P^K(c)$ exists and satisfies $s_P^K \geq s_P^K$. Now, fix an arbitrary decreasing sequence $(c_n)$ converging to 0 and consider a sequence $((\theta_n, u_n))$ in $K \times S^{k-1}$ such that

$$P[|u_n'(X - \theta_n)| \leq c_n] \geq s_P^K(c_n) - (1/n).$$

Compactness of $K \times S^{k-1}$ guarantees the existence of a subsequence $((\theta_{n_\ell}, u_{n_\ell}))$ that converges in $K \times S^{k-1}$, to $(\theta_0, u_0)$ say. Clearly, without loss of generality, we can assume that $(u_{n_\ell}', u_0)$ is an increasing sequence and that $\|\theta_{n_\ell} - \theta_0\|$ is a decreasing sequence (if that is not the case, one can always extract a further subsequence which meets these monotonicity properties). Let then $B_\ell := B(\theta_0, \|\theta_{n_\ell} - \theta_0\|)$ and $C_\ell := \{u \in S^{k-1} : u' u_0 \geq u_{n_\ell}' u_0\}$. Clearly, $B_\ell$ and $C_\ell$ are decreasing sequences of sets, with $\cap_\ell B_\ell = \{\theta_0\}$ and $\cap_\ell C_\ell = \{u_0\}$. Therefore,

$$\lim_{\ell \rightarrow \infty} r_\ell := \lim_{\ell \rightarrow \infty} P[X \in \cup_{\ell \in \ell} \cup_{u \in C_\ell} \{x = \theta + y : |u'y| \leq c_{n_\ell}\}] = P[u_0'(X - \theta_0) = 0].$$

Now, for any $\ell$, $r_\ell \geq P[|u_{n_\ell}'(X - \theta_{n_\ell})| \leq c_{n_\ell}] \geq s_P^K(c_{n_\ell}) - (1/n_\ell)$, which implies that $s_P^K \geq P[|u_0'(X - \theta_0)| = 0] \geq s_P^K$. Therefore, $s_P^K = s_P^K = P[|u_0'(X - \theta_0)| = 0].$ □

**Lemma A.5.** Let $P$ be a smooth probability measure over $\mathbb{R}^k$ and fix $\theta_0 \in \mathbb{R}^k$. Then

$$\sup_{(u, c) \in S^{k-1} \times \mathbb{R}^+} |P[\theta + H_{u,c}^{\text{in/out}}] - P[\theta_0 + H_{u,c}^{\text{in/out}}]| \rightarrow 0$$

as $\theta \rightarrow \theta_0$.

**Proof of Lemma A.5.** We start with the “in” result. Fix $\varepsilon > 0$. Pick $c_1 > 0$ large enough to have $P[B(\theta_0, c_1/2)] \geq 1 - (\varepsilon/2)$. Pick then $c_0 \in (0, c_1)$ such that $s_P^{B(\theta_0, c_1)}(c_0) <$
According to (A.4), (A.6) and (A.7), all three quantities $Q_{\theta_0, c_0}^{\infty}$, $Q_{c_0, c_1}^{\infty}$ and $Q_{c_1, \infty}^{\infty}$ are bounded by $\varepsilon$ as soon as $\|\theta - \theta_0\| < \delta$, which concludes the proof of the “in” result.
The proof of the “out” result proceeds similarly. For the same choices of \( c_0, c_1 \) and \( \delta \), and the respective suprema \( Q_{c_0,c_1}^{\text{out}} \), \( Q_{c_0,c_1}^{\text{out}} \) and \( Q_{c_1,\infty}^{\text{out}} \), it holds

\[
Q_{c_1,\infty}^{\text{out}} \leq \sup_{u \in \mathcal{S}^{k-1}} P[\theta + H_{u,c_1}^{\text{out}}] + \sup_{u \in \mathcal{S}^{k-1}} P[\theta_0 + H_{u,c_1}^{\text{out}}]
\]

\[
\leq 2 \sup_{(\theta,u) \in B(\theta_0,c_1/2) \times \mathcal{S}^{k-1}} P[\mathbb{R}^k \setminus B(\theta_0,c_1/2)] \leq \varepsilon.
\]

Moreover, the inequality \( Q_{c_0,c_0}^{\text{out}} \leq \varepsilon \) follows from the fact that

\[
|P[\theta + H_{u,c_1}^{\text{out}}] - P[\theta_0 + H_{u,c_0}^{\text{out}}]| \leq P[\theta + H_{u,c_0}^{\text{in}}] + P[\theta_0 + H_{u,c_0}^{\text{in}}]
\]

for \( c \leq c_0 \). Finally, it can be proved that \( Q_{c_0,c_1}^{\text{out}} \leq \varepsilon \) along the exact same lines as above. The “out” result follows. \( \square \)

**Proof of Theorem 2.2.** The collection \( \mathcal{H} \) of all halfspaces in \( \mathbb{R}^k \) is a Vapnik-Chervonenkis class; see, e.g., page 152 of Van der Vaart and Wellner (1996). Hence, Lemma 2.6.17 of the same implies that \( \mathcal{H} \cap \mathcal{H} := \{ H_1 \cap H_2 : H_1, H_2 \in \mathcal{H} \} \) and \( \mathcal{H} \cup \mathcal{H} := \{ H_1 \cup H_2 : H_1, H_2 \in \mathcal{H} \} \) are also Vapnik-Chervonenkis classes. Consequently, using henceforth the notation from Lemma A.3, \( C^{\text{in}}(\mathcal{H} \cap \mathcal{H}) \) and \( C^{\text{out}}(\mathcal{H} \cup \mathcal{H}) \) are themselves Vapnik-Chervonenkis classes, which implies that

(A.8) \[ \sup_{C \in C^{\text{in}}} |P_n[C] - P[C]| \to 0 \quad \text{and} \quad \sup_{C \in C^{\text{out}}} |P_n[C] - P[C]| \to 0 \]

almost surely as \( n \to \infty \). Also, Lemmas A.2 and A.5 together entails

(A.9) \[ \sup_{C \in C^{\text{in}}_0} |P[\theta P_n + C] - P[\theta P + C]| \to 0 \quad \text{and} \quad \sup_{C \in C^{\text{out}}_0} |P[\theta P_n + C] - P[\theta P + C]| \to 0 \]

almost surely as \( n \to \infty \).

Now, by using Lemma A.3, we obtain that, for any \( \Sigma \in \mathcal{P}_k \),

\[
|HD_{P_n}(\Sigma) - HD_P(\Sigma)| = |\min(HD_{P_n}^{\text{in}}(\Sigma), HD_{P_n}^{\text{out}}(\Sigma)) - \min(HD_P^{\text{in}}(\Sigma), HD_P^{\text{out}}(\Sigma))|
\]

\[
\leq \max \left( |HD_{P_n}^{\text{in}}(\Sigma) - HD_P^{\text{in}}(\Sigma)|, |HD_{P_n}^{\text{out}}(\Sigma) - HD_P^{\text{out}}(\Sigma)| \right)
\]

\[
\leq \max \left( \sup_{C \in C^{\text{in}}} |P_n[C] - P[C]| + \sup_{C \in C^{\text{in}}_0} |P[\theta P_n + C] - P[\theta P + C]|, \right.
\]

\[
\left. \sup_{C \in C^{\text{out}}} |P_n[C] - P[C]| + \sup_{C \in C^{\text{out}}_0} |P[\theta P_n + C] - P[\theta P + C]| \right).
\]

Consequently,

\[
\sup_{\Sigma \in \mathcal{P}_k} |HD_{P_n}(\Sigma) - HD_P(\Sigma)|
\]

\[
\leq \max \left( \sup_{C \in C^{\text{in}}} |P_n[C] - P[C]| + \sup_{C \in C^{\text{in}}_0} |P[\theta P_n + C] - P[\theta P + C]|, \right.
\]

\[
\left. \sup_{C \in C^{\text{out}}} |P_n[C] - P[C]| + \sup_{C \in C^{\text{out}}_0} |P[\theta P_n + C] - P[\theta P + C]| \right),
\]
which, in view of (A.8) and (A.9), establishes the result. □

APPENDIX B: PROOFS FROM SECTION 3

Proof of Theorem 3.1. (i) Fix \( u \in S^{k-1} \). Since \( H_u^{\text{in}} := \{ x \in \mathbb{R}^k : |u'x| \leq 1 \} \) is a closed subset of \( \mathbb{R}^k \), the mapping \( P \mapsto P[H_u^{\text{in}}] \) is upper semicontinuous for weak convergence. Now, Slutsky’s lemma entails that, as \( d_F(\Sigma, \Sigma_0) \to 0 \), the measure defined by \( B \mapsto P[\theta_P + \Sigma^{1/2}B] \) converges weakly to the one defined by \( B \mapsto P[\theta_P + \Sigma_0^{1/2}B] \). Therefore, \( \Sigma \mapsto P[\theta_P + \Sigma^{1/2}H_u^{\text{in}}] \) is upper \( F \)-semicontinuous at \( \Sigma_0 \). Since \( H_u^{\text{out}} := \{ x \in \mathbb{R}^k : |u'x| \geq 1 \} \) is also a closed subset of \( \mathbb{R}^k \), the same argument shows that \( \Sigma \mapsto P[\theta_P + \Sigma^{1/2}H_u^{\text{out}}] \) is upper \( F \)-semicontinuous at \( \Sigma_0 \). Therefore,

\[
\Sigma \mapsto HD_P^{\text{sc}}(\Sigma) = \min \left( \inf_{u \in S^{k-1}} P[\theta_P + \Sigma^{1/2}H_u^{\text{in}}], \inf_{u \in S^{k-1}} P[\theta_P + \Sigma^{1/2}H_u^{\text{out}}] \right),
\]

is upper \( F \)-semicontinuous (recall that the infimum of a collection of upper semicontinuous functions is upper semicontinuous).

(ii) The result directly follows from the fact that \( R_P^{\text{sc}}(\alpha) \) is the inverse image of \( [\alpha, +\infty) \) by the upper \( F \)-semicontinuous function \( \Sigma \mapsto HD_P^{\text{sc}}(\Sigma) \).

(iii) Fix a sequence \( (\Sigma_n) \) in \( \mathcal{P}_k \) converging to \( \Sigma_0 \) with respect to the Frobenius distance. With the same notation as in the proof of (i), note that, for any \( \Sigma \),

\[
HD_P^{\text{sc}}(\Sigma) = \inf_{u \in S^{k-1}} \min \left( P[\theta_P + \Sigma^{1/2}H_u^{\text{in}}], P[\theta_P + \Sigma^{1/2}H_u^{\text{out}}] \right).
\]

For any \( n \), pick then \( u_n \in S^{k-1} \) such that

\[
\min \left( P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{in}}], P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{out}}] \right) \leq HD_P^{\text{sc}}(\Sigma_n) + \frac{1}{n}.
\]

Compactness of \( S^{k-1} \) implies that we can extract a subsequence \( (u_{n_{\ell}}) \) of \( (u_n) \) that converges to \( u_0 \in S^{k-1} \). Writing \( \mathbb{I}[C] \) for the indicator function of the set \( C \), the dominated convergence theorem then yields that

\[
P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{in}}] - P[\theta_P + \Sigma_0^{1/2}H_{u_0}^{\text{in}}] = \int_{\mathbb{R}^k} (\mathbb{I}[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{in}}] - \mathbb{I}[\theta_P + \Sigma_0^{1/2}H_{u_0}^{\text{in}}]) dP \to 0
\]

as \( \ell \to \infty \) (the smoothness assumption on \( P \) guarantees that \( \mathbb{I}[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{in}}] - \mathbb{I}[\theta_P + \Sigma_0^{1/2}H_{u_0}^{\text{in}}] \to 0 \) \( P \)-almost everywhere). Proceeding in the same way, we obtain that \( P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{out}}] - P[\theta_P + \Sigma_0^{1/2}H_{u_0}^{\text{out}}] \to 0 \) as \( \ell \to \infty \). Consequently,

\[
\lim_{\ell \to \infty} \inf_{n \to \infty} HD_P^{\text{sc}}(\Sigma_n) = \lim_{n \to \infty} \min \left( P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{in}}], P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{out}}] \right)
\]

\[
= \lim_{n \to \infty} \min \left( P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{in}}], P[\theta_P + \Sigma_n^{1/2}H_{u_{n \ell}}^{\text{out}}] \right)
\]

\[
= \min \left( P[\theta_P + \Sigma_0^{1/2}H_{u_0}^{\text{in}}], P[\theta_P + \Sigma_0^{1/2}H_{u_0}^{\text{out}}] \right) \geq HD_P^{\text{sc}}(\Sigma_0).
\]
We conclude that, if $P$ is smooth, then $\Sigma \to HD_P^\text{sc}(\Sigma)$ is also lower $F$-semicontinuous, hence $F$-continuous. \hfill \Box

**Proof of Theorem 3.2.** Fix $\alpha > 0$. Note that $\lambda_1(\Sigma) \geq ||\Sigma||_F/\sqrt{k} \geq (||\Sigma - I_k||_F - ||I_k||_F)/\sqrt{k}$ for any $\Sigma \in \mathcal{P}_k$. Therefore, denoting by $v_1(\Sigma)$ an arbitrary unit eigenvector associated with $\lambda_1(\Sigma)$, we have that, for any $\Sigma \notin B_F(I_k, r)$,

$$HD_P^\text{sc}(\Sigma) \leq \inf_{u \in \mathcal{S}_{k-1}} P\left[|u'(X - \theta_P)| \geq \sqrt{u'\Sigma u}\right]$$

$$\leq P\left[|v_1'(\Sigma)(X - \theta_P)| \geq \sqrt{\lambda_1(\Sigma)}\right] \leq P\left[\|X - \theta_P\| \geq \frac{(r-1)^{1/2}}{\lambda_1}\right],$$

which can be made strictly smaller than $\alpha$ for $r$ large enough. This confirms that, for $r$ large enough, $R_P^\text{sc}(\alpha)$ is included in the ball $B_F(I_k, r)$, hence is $F$-bounded. \hfill \Box

**Proof of Theorem 3.3.** (i) With $\Sigma_t = (1 - t)\Sigma_a + t\Sigma_b$, we clearly have that, for any $u \in \mathcal{S}_{k-1}$, $\min(u'\Sigma_a u, u'\Sigma_b u) \leq u'\Sigma u \leq \max(u'\Sigma_a u, u'\Sigma_b u)$. This entails that, for any $u \in \mathcal{S}_{k-1}$,

$$P\left[|u'(X - \theta_P)| \leq \sqrt{u'\Sigma u}\right]$$

$$\geq \min(P\left[|u'(X - \theta_P)| \leq \sqrt{u'\Sigma_a u}\right], P\left[|u'(X - \theta_P)| \leq \sqrt{u'\Sigma_b u}\right])$$

$$\geq \min(HD_P^\text{sc}(\Sigma_a), HD_P^\text{sc}(\Sigma_b))$$

and

$$P\left[|u'(X - \theta_P)| \geq \sqrt{u'\Sigma u}\right]$$

$$\geq \min(P\left[|u'(X - \theta_P)| \geq \sqrt{u'\Sigma_a u}\right], P\left[|u'(X - \theta_P)| \geq \sqrt{u'\Sigma_b u}\right])$$

$$\geq \min(HD_P^\text{sc}(\Sigma_a), HD_P^\text{sc}(\Sigma_b)).$$

The result follows. (ii) If both $\Sigma_a, \Sigma_b \in R_P^\text{sc}(\alpha)$, then Part (i) of the result entails that, for any $t \in [0, 1]$, $HD(\Sigma_t, P) \geq \min(HD_P^\text{sc}(\Sigma_a), HD_P^\text{sc}(\Sigma_b)) \geq \alpha$, so that $\Sigma_t \in R_P^\text{sc}(\alpha)$. \hfill \Box

**APPENDIX C: PROOFS FROM SECTION 4**

For the sake of completeness, we prove the following result.

**Lemma C.1.** Let $R$ be a $g$-bounded subset of $\mathcal{P}_k$. Then $R$ is totally $g$-bounded, that is, for any $\varepsilon$, there exist $\Sigma_i$, $i = 1, \ldots, m = m(\varepsilon)$ such that $R \subset \bigcup_{i=1}^m B_g(\Sigma_i, \varepsilon)$.

**Proof of Lemma C.1.** As a mapping from the metric space $(\mathcal{S}_k, d_F)$ (recall that $d_F$ denotes the Frobenius distance) to the metric space $(\mathcal{P}_k, d_g)$, $A \mapsto \exp(A)$ is continuous; see the proof of Proposition 10 in Bhatia and Holbrook (2006). Denoting, for any $A \in \mathcal{S}_k$, as $\text{vech}(A)$ the vector obtained by stacking the upper-diagonal entries of $A$ on top of each
other, the mapping \( v \mapsto \text{vech}^{-1}(v) \) from \((\mathbb{R}^{k(k+1)/2}, d_E)\) (equipped with the usual Euclidean distance \(d_E\)) to \((S_k, d_F)\) is trivially continuous, so that the mapping \( f : (\mathbb{R}^{k(k+1)/2}, d_E) \to (P_k, d_g) : v \mapsto f(v) := \exp(\text{vech}^{-1}(v))\) is also continuous.

Now, fix \( \varepsilon > 0 \), pick \( r > 0 \) such that \( R \) is included in the closed ball \( \bar{B} := \bar{B}_g(I_k, r) := \{ \Sigma \in P_k : d_g(\Sigma, I_k) \leq r \} \), and consider the resulting open covering \( \{ B_g(\Sigma, \varepsilon) : \Sigma \in \bar{B} \} \) of \( \bar{B} \).

From continuity, \( C := \{ f^{-1}(B_g(\Sigma, \varepsilon)) : \Sigma \in \bar{B} \} \) is an open covering of the closed set \( f^{-1}(\bar{B}) \) in \( \mathbb{R}^{k(k+1)/2} \). It is easy to check that, for any \( \Sigma \in \bar{B} \), \( \lambda_1(\Sigma) \leq \exp(r/\sqrt{k}) \), so that \( f^{-1}(\bar{B}) \) is bounded, hence compact. Therefore, a finite subcovering \( \{ f^{-1}(B_g(\Sigma_i, \varepsilon)) : i = 1, \ldots, m \} \) of \( f^{-1}(\bar{B}) \) can be extracted from \( C \), which provides the desired finite covering \( \{ B_g(\Sigma_i, \varepsilon) : i = 1, \ldots, m \} \) of \( \bar{B} \), hence of \( R \), with open \( g \)-balls of radius \( \varepsilon \).

**Proof of Theorem 4.2.** Assume first that \( s_P < 1/2 \) and fix \( \varepsilon > 0 \). We will then prove that \( R_F^g(s_P + \varepsilon) \) is \( g \)-bounded by showing that, for \( r > 0 \) large enough, it is included in the \( g \)-ball \( B_g(I_k, r) \). To do so, first note that (4.1) entails

\[
d_g(\Sigma, I_k) = \sqrt{\sum_{i=1}^{k} (\log \lambda_i(\Sigma))^2} \leq \sqrt{k} \max(|\log \lambda_1(\Sigma)|, |\log \lambda_k(\Sigma)|)
\]

(C.1)

\[
= \sqrt{k} \max(\log \lambda_1(\Sigma), \log \lambda_k^{-1}(\Sigma)).
\]

Therefore, \( \Sigma \notin B_g(I_k, r) \) implies that (i) \( \lambda_1(\Sigma) > \exp(r/\sqrt{k}) \) or (ii) \( \lambda_k(\Sigma) < \exp(-r/\sqrt{k}) \) (or both). In case (i),

\[
HD_F^g(\Sigma) \leq \inf_{u \in S^{k-1}} P[|u'(X - \theta_P)| \geq \sqrt{u'\Sigma u}] \leq P[|v_1'(\Sigma)(X - \theta_P)| \geq \sqrt{\lambda_1(\Sigma)}] \leq P[|v_1'(\Sigma)(X - \theta_P)| \geq \exp(r/2\sqrt{k})] \leq P[||X - \theta_P|| \geq \exp(r/2\sqrt{k})],
\]

which can be made smaller than \( \varepsilon \) (hence, smaller than \( s_P + \varepsilon \)) for \( r \) large enough. In case (ii), we have that, using the notation \( k^g_P(\cdot) \) from Lemma A.4,

\[
HD_F^{g*}(\Sigma) \leq \inf_{u \in S^{k-1}} P[|u'(X - \theta_P)| \leq \sqrt{u'\Sigma u}] \leq P[|v_1'(\Sigma)(X - \theta_P)| \leq \lambda_k^{1/2}(\Sigma)] \leq s_P^{\theta_P}(\lambda_k^{1/2}(\Sigma)) \leq s_P^{\theta_P}(\exp(-r/2\sqrt{k})),
\]

which, in view of Lemma A.4(i), can be made smaller than \( s_P + \varepsilon \) for \( r \) large enough. We conclude that, for \( \alpha > s_P \), \( R_F^g(\alpha) \) is \( g \)-bounded, hence also (Lemma C.1) totally \( g \)-bounded. Since it is also \( g \)-closed (Theorem 4.1(ii)), it is \( g \)-compact (recall from Section 4 that, in a complete metric space, any closed and totally bounded set is compact).

Finally, if \( s_P \geq 1/2 \), then, with \( u_0 \in S^{k-1} \) such that \( P[|u_0'(X - \theta_P)| = 0] = s_P \) (existence is guaranteed in Lemma A.4(ii); take \( K = \{ \theta_P \} \) there), we have \( HD_F^{g*}(\Sigma) \leq P[|u_0(X - \theta_P)| \geq \sqrt{u_0'\Sigma u_0}] \leq P[|u_0'(X - \theta_P)| > 0] = 1 - s_P \), so that \( R_F^{g*}(\alpha) \) is empty for any \( \alpha > 1 - s_P = \alpha_P \). \( \Box \)
Proof of Theorem 4.3. Let $A_P := \{ \alpha \in [0, 1] : R_P^{sc}(\alpha) \text{ is } g\text{-unbounded} \}$. Since $A_P$ is non-empty ($0 \in A_P$), $\bar{\alpha}_P := \inf A_P$ is well-defined. It follows from Theorem 4.2 that $\bar{\alpha}_P \leq \alpha_P$. By assumption, $R_P^{sc}(\bar{\alpha}_P)$ is then non-empty, and we therefore have $\alpha_{*P} = \sup_{\Sigma \in P} \text{HD}_P^{sc}(\Sigma) \geq \bar{\alpha}_P$. If $\alpha_{*P} = \bar{\alpha}_P$, then the result holds since the maximal depth $\alpha_{*P}$ is achieved at any scatter matrix in the non-empty set $R_P^{sc}(\bar{\alpha}_P)$. Assume then that $\alpha_{*P} > \bar{\alpha}_P$.

Fix $\delta > 0$ such that $\alpha_{*P} - \delta > \bar{\alpha}_P$, so that $R_P^{sc}(\alpha_{*P} - \delta)$ is $g$-compact. For any positive integer $n$, it is possible to pick a scatter matrix $\Sigma_n$ with $\text{HD}_P^{sc}(\Sigma_n) \geq \alpha_{*P} - (\delta/n)$. The $g$-compactness of $R_P^{sc}(\alpha_{*P} - \delta)$ implies that there exists a subsequence $(\Sigma_{n_k})$ that $g$-converges in $R_P^{sc}(\alpha_{*P} - \delta)$, to $\Sigma_*$ say. For any $\varepsilon \in (0, \delta)$, all terms of $(\Sigma_{n_k})$ are eventually in the $g$-closed set $R_P^{sc}(\alpha_{*P} - \varepsilon)$, so that its $g$-limit $\Sigma_*$ must also belong to $R_P^{sc}(\alpha_{*P} - \varepsilon)$. For any such $\varepsilon$, we thus have $\alpha_{*P} - \varepsilon \leq \text{HD}_P^{sc}(\Sigma_*) \leq \alpha_{*P}$, which proves that $\text{HD}_P^{sc}(\Sigma_*) = \alpha_{*P}$.  

The proof of Theorem 4.4 requires the following preliminary result.

Lemma C.2. Fix $\Sigma \in \mathcal{P}_k$ with $\max(\text{diag}(\Sigma)) \leq 1$. Then $\max_s s'\Sigma^{-1}s \geq k$ (where $\max_s$ is the maximum over $s = (s_1, \ldots, s_k) \in \{-1, 1\}^k$, with equality if and only if $\Sigma = I_k$).

Proof of Lemma C.2. We prove the result by induction. Clearly, the result holds for $k = 1$. Assume then that the result holds for $k$. Writing

$$\Sigma = \begin{pmatrix} \Sigma_- & v \\ v' & \Sigma_{k+1,k+1} \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} s_- \\ s_{k+1} \end{pmatrix},$$

the classical formula for the inverse of a block partitioned matrix yields

$$s'\Sigma^{-1}s = s'_-\Sigma_-^{-1}s_- + \frac{(s'_-\Sigma_-^{-1}v - s_{k+1})^2}{\Sigma_{k+1,k+1} - v'\Sigma_-^{-1}v}.$$  

By induction assumption, there exists $s_-$ such that

$$s'_-\Sigma_-^{-1}s_- \geq k + 1 + \frac{(s'_-\Sigma_-^{-1}v - s_{k+1})^2 - (\Sigma_{k+1,k+1} - v'\Sigma_-^{-1}v)}{\Sigma_{k+1,k+1} - v'\Sigma_-^{-1}v}.$$  

By induction assumption, there exists $s_-$ such that

Now, irrespective of $s_-$, choosing $s_{k+1} = -\text{sign}(s'_-\Sigma_-^{-1}v)$ yields

$$(s'_-\Sigma_-^{-1}v - s_{k+1})^2 - (\Sigma_{k+1,k+1} - v'\Sigma_-^{-1}v)^2.$$

since $\Sigma_{k+1,k+1} \leq 1$. Jointly with (C.3), this provides $\max_s s'\Sigma^{-1}s \geq k + 1$.

Now, assume that $\max_s s'\Sigma^{-1}s = k + 1$. We consider two cases. (a) $\max_{s_-} s'_-\Sigma_-^{-1}s_- > k$. Pick an arbitrary $s_{*s}$ such that $s'_{*s}\Sigma^{-1}s_{*s} > k$. Then, with $s_s = (s'_{*s}, s_{*s})'$, we have

$$s'_{*s}\Sigma^{-1}s_s > k + 1 + \frac{(s'_{*s}\Sigma^{-1}v - s_{*s,k+1})^2 - (\Sigma_{k+1,k+1} - v'\Sigma^{-1}v)}{\Sigma_{k+1,k+1} - v'\Sigma^{-1}v}.$$
Choosing again $s_{*}^{*} = -\text{sign}(s_{*}^{*} \Sigma^{-1} v)$ makes the third term of the righthand side non-negative, which implies that $\max_{s} s^{'} \Sigma^{-1} s > k+1$, a contradiction. (b) $\max_{s} s^{*} \Sigma^{-1} s_{*} = k$. By induction assumption, we must then have $\Sigma_{*} = I_{k}$. For any $s = (s^{*}, s_{k+1})^{T}$, (C.2) thus yields

$$s^{'} \Sigma^{-1} s = s_{*}^{*} s_{*} + \frac{(s_{*}^{*} v - s_{k+1})^{2}}{\Sigma_{k+1,k+1} - v^{T} v} = k + \frac{(s_{*}^{*} v - s_{k+1})^{2}}{\Sigma_{k+1,k+1} - v^{T} v}. $$

Since $\max_{s} s^{*} \Sigma^{-1} s = k+1$, we must have that

$$1 = \max_{s} \frac{(s_{*}^{*} v - s_{k+1})^{2}}{\Sigma_{k+1,k+1} - v^{T} v} = \frac{(1 + \sum_{l=1}^{k} |v_{l}|)^{2}}{\Sigma_{k+1,k+1} - v^{T} v} = \frac{c}{d}. $$

Since $c \geq 1$ and $d \leq 1$ (recall that $\Sigma_{k+1,k+1} \leq 1$), this imposes that $c = d = 1$, which leads to $v = 0$ and $\Sigma_{k+1,k+1} = 1$. Jointly with $\Sigma_{*} = I_{k}$, this shows that we must have $\Sigma = I_{k+1}$, which establishes the result. □

**Proof of Theorem 4.4.** First note that, with $\Sigma_{*} = \sqrt{k} I_{k}$, (2.6) yields

$$HD_{P}^{sc}(\Sigma_{*}) = 2 \min \left( \Psi \left( k^{-1/4} \right) - \frac{1}{2}, 1 - \Psi \left( k^{1/4} \right) \right) = \frac{2}{\pi} \arctan \left( k^{-1/4} \right) $$

and fix an arbitrary $\Sigma \in \mathcal{P}_{k}$. If $\max(\text{diag}(\Sigma)) > \sqrt{k}$, then

$$HD_{P}^{sc}(\Sigma) \leq 2 \left( 1 - \Psi \left( \sqrt{\max(\text{diag}(\Sigma))} \right) \right) < 2 \left( 1 - \Psi \left( k^{1/4} \right) \right) = HD_{P}^{sc}(\Sigma_{*}). $$

If $\max(\text{diag}(\Sigma)) \leq \sqrt{k}$, then Lemma C.2 yields $\max_{s} s^{*} \Sigma^{-1} s = k^{-1/2} \max_{s} s^{*} (k^{-1/2} \Sigma)^{-1} s \geq k^{-1/2}$, so that

$$HD_{P}^{sc}(\Sigma) \leq 2 \left( \Psi \left( 1 / \max_{s} \sqrt{s^{*} \Sigma^{-1} s} \right) - \frac{1}{2} \right) \leq 2 \left( \Psi \left( k^{-1/4} \right) - \frac{1}{2} \right) = HD_{P}^{sc}(\Sigma_{*}). $$

We conclude that $HD_{P}^{sc}(\Sigma_{*}) \geq HD_{P}^{sc}(\Sigma)$ for any $\Sigma \in \mathcal{P}_{k}$. Now, assume that $HD_{P}^{sc}(\Sigma) = HD_{P}^{sc}(\Sigma_{*})$ for some $\Sigma \in \mathcal{P}_{k}$. If $\max(\text{diag}(\Sigma)) > \sqrt{k}$, we can only have $HD_{P}^{sc}(\Sigma) < HD_{P}^{sc}(\Sigma_{*})$, as showed in (C.4). Thus we must have $\max(\text{diag}(\Sigma)) \leq \sqrt{k}$, and by assumption, all inequalities in (C.5) should be equalities. In view of Lemma C.2, this implies that $k^{-1/2} \Sigma = I_{k}$, which establishes the result. □

**APPENDIX D: PROOFS FROM SECTIONS 5 AND 6**

**Proof of Theorem 5.1.** We start with the case $\theta_{0} = 0$ and $\Sigma_{0} = I_{k}$, for which

$$\min(P[|u^{'} X| \leq \sqrt{u^{'} \Sigma u}], P[|u^{'} X| \geq \sqrt{u^{'} \Sigma u}]) = \min(P[|X_{1}| \leq \sqrt{u^{'} \Sigma u}], P[|X_{1}| \geq \sqrt{u^{'} \Sigma u}]) $$

for any $u \in S^{k-1}$, so that

$$HD_{P}^{sc}(\Sigma) = \inf_{z \in \mathcal{S}^{p}(\Sigma)} \min(P[X_{1}^{2} \leq z], P[X_{1}^{2} \geq z]) \leq \min(P[X_{1}^{2} \leq 1], P[X_{1}^{2} \geq 1]) = HD_{P}^{sc}(I_{k}), $$

where $HD_{P}^{sc}(I_{k})$ is the halfspace depth of the standard normal distribution.
where the equality holds if and only if \( \text{Sp}(\Sigma) \subset \mathcal{I}_{\text{MSD}}[X_1] \). The result for a general location \( \theta_0 \) and scatter \( \Sigma_0 \) readily follows from affine invariance and the identity \( \text{Sp}(AB) = \text{Sp}(BA) \). (ii) By definition, if \( \mathcal{I}_{\text{MSD}}[Z_1] \) is a singleton, then this singleton must be \( \{1\} \). Consequently, if \( HD^p_\Sigma(\Sigma) = HD^q_\Sigma(\sigma_0) \), then Part (i) of the result entails that \( \lambda_k(\Sigma_0^{-1}\Sigma) = \lambda_1(\Sigma_0^{-1}\Sigma) = 1 \). This implies that \( \Sigma_0^{-1}\Sigma = I_k \), which establishes the result. \( \square \)

We turn to the proofs of Theorems 5.2 and 6.2, that require the following preliminary results for the elliptical case (Lemma D.1) and for the independent Cauchy case (Lemma D.2).

**Lemma D.1.** For any \( \Sigma_a, \Sigma_b, \in \mathcal{P}_k \) and \( t \in [0, 1] \), let \( \tilde{\Sigma}_t := \Sigma^{1/2}_a (\Sigma^{-1/2}_a \Sigma_b \Sigma^{-1/2}_a)^t \Sigma^{1/2}_a \) and \( \Sigma^*_t := ((1 - t)\Sigma^{-1}_a + t\Sigma^{-1}_b)^{-1} \). Then (i) \( \lambda_1(\Sigma^*_t) \leq \lambda_1(\tilde{\Sigma}_t) \leq \max(\lambda_1(\Sigma_a), \lambda_1(\Sigma_b)) \) and (ii) \( \lambda_k(\tilde{\Sigma}_t) \geq \lambda_k(\Sigma^*_t) \geq \min(\lambda_k(\Sigma_a), \lambda_k(\Sigma_b)) \).

**Proof of Lemma D.1.** (i) With the usual order on positive semidefinite matrices (\( A \leq B \) iff \( B - A \) is positive semidefinite), the (weighted) harmonic-geometric-arithmetic inequality (see, e.g., Lemma 2.1(vii) in Lawson and Lim, 2013)

\[
(\Sigma^*_t) \leq \tilde{\Sigma}_t \leq \Sigma_t := (1 - t)\Sigma_a + t\Sigma_b
\]

holds for any \( t \in [0, 1] \). This implies that

\[
\lambda_1(\Sigma^*_t) \leq \lambda_1(\tilde{\Sigma}_t) \leq \lambda_1(\Sigma_t).
\]

Indeed, if, e.g., the second inequality in (D.2) does not hold (the argument for the first inequality is strictly the same), then, denoting as \( \bar{v}_{1t} \) an arbitrary eigenvector associated with \( \lambda_1(\tilde{\Sigma}_t) \), we have \( \bar{v}_{1t}^t \tilde{\Sigma}_t \bar{v}_{1t} = \lambda_1(\tilde{\Sigma}_t) > \lambda_1(\Sigma_t) \geq \bar{v}_{1t}^t \Sigma_t \bar{v}_{1t} \), which contradicts (D.1).

Hence, (D.2) holds and provides

\[
\lambda_1(\Sigma^*_t) \leq \lambda_1(\tilde{\Sigma}_t) \leq \max_{u \in \mathcal{S}^{d-1}} u^t((1 - t)\Sigma_a + t\Sigma_b)u \leq (1 - t) \max_{u \in \mathcal{S}^{d-1}} u^t\Sigma_a u + t \max_{u \in \mathcal{S}^{d-1}} u^t\Sigma_b u
\]

\[
= (1 - t)\lambda_1(\Sigma_a) + t\lambda_1(\Sigma_b) \leq \max(\lambda_1(\Sigma_a), \lambda_1(\Sigma_b)),
\]

as was to be showed. (ii) Proceeding in a similar way as above, it is readily showed that (D.1) implies that \( \lambda_k(\tilde{\Sigma}_t) \geq \lambda_k(\Sigma^*_t) \). Using this, we obtain

\[
\lambda_k(\tilde{\Sigma}_t) \geq \lambda_k(\Sigma^*_t) = \lambda_k^{-1}((\Sigma^*_t)^{-1})
\]

\[
= \left( \max_{u \in \mathcal{S}^{d-1}} u^t((1 - t)\Sigma^{-1}_a + t\Sigma^{-1}_b)u \right)^{-1} \geq \left( (1 - t)\lambda_1(\Sigma^{-1}_a) + t\lambda_1(\Sigma^{-1}_b) \right)^{-1}
\]

\[
= \left( (1 - t)\lambda_k^{-1}(\Sigma_a) + t\lambda_k^{-1}(\Sigma_b) \right)^{-1} \geq \min(\lambda_k(\Sigma_a), \lambda_k(\Sigma_b)),
\]

since any weighted harmonic mean of two real numbers is a convex linear combination of these. \( \square \)
Lemma D.2. For any $\Sigma_a, \Sigma_b \in \mathcal{P}_k$ and $t \in [0, 1]$, let $\tilde{\Sigma}_t := (1 - t)\Sigma_a + t\Sigma_b$ and $\Sigma^*_t := ((1 - t)\Sigma_a^{-1} + t\Sigma_b^{-1})^{-1}$. Then,

$$\max(\text{diag}(\Sigma^*_t)) \leq \max(\text{diag}(\tilde{\Sigma}_t)) \leq \max(\text{diag}(\Sigma_a)), \max(\text{diag}(\Sigma_b))$$

and

$$\max_s s'(\Sigma^*_t)^{-1} \leq \max_s \left(\max s'\Sigma_a^{-1}s, \max s'\Sigma_b^{-1}s\right)$$

where $\max_s$ is the maximum over $s = (s_1, \ldots, s_k) \in \{-1, 1\}^k$, so that both the mappings $\Sigma \mapsto \max(\text{diag}(\Sigma))$ and $\Sigma \mapsto \max_s s'(\Sigma^*_t)^{-1}s$ are geodesic and harmonic quasi-convex.

**Proof of Lemma D.2.** The result in (D.3) readily follows from the fact that the weighted harmonic-geometric-arithmetic inequality $\Sigma^*_t \leq \tilde{\Sigma}_t \leq (1 - t)\Sigma_a + t\Sigma_b$ yields $(\Sigma^*_t)^{\ell\ell} \leq (\tilde{\Sigma}_t)^{\ell\ell} \leq (1 - t)(\Sigma_a)^{\ell\ell} + t(\Sigma_b)^{\ell\ell} \leq \max((\Sigma_a)^{\ell\ell}, (\Sigma_b)^{\ell\ell})$ for any $\ell = 1, \ldots, k$. Turning to (D.4), the harmonic-geometric inequality implies that $\tilde{\Sigma}_t^{-1} \leq (\Sigma^*_t)^{-1}$, which readily yields $\max_s s'\tilde{\Sigma}_t^{-1}s \leq \max_s s'(\Sigma^*_t)^{-1}s$. Consequently, it only remains to prove the second inequality in (D.4). To do so, choose an arbitrary $s_s$ such that $\max_s s'(\Sigma^*_t)^{-1}s = s'_s((1 - t)\Sigma_a^{-1} + t\Sigma_b^{-1})s_s$. Then

$$\max_s s'((1 - t)\Sigma_a^{-1} + t\Sigma_b^{-1})s_s \leq (1 - t)\max_s s'\Sigma_a^{-1}s + t\max_s s'\Sigma_b^{-1}s = \max(s's'\Sigma_a^{-1}s, \max s's'\Sigma_b^{-1}s),$$

which establishes the result. \( \Box \)

We can now prove Theorems 5.2 and 6.2.

**Proof of Theorem 5.2.** (i) We start by considering the case where $P$ is an elliptical probability measure over $\mathbb{R}^k$ with location $\theta_0$ and scatter $\Sigma_0$, where we first prove the result for $\theta_0 = 0$ and $\Sigma_0 = I_k$. Then we have

$$\text{HD}^{sc}_P(\tilde{\Sigma}_t) = \inf_{u \in S^{k-1}} \min\left(P[|u'X| \leq \sqrt{u'\Sigma_t u}], P[|u'X| \geq \sqrt{u'\Sigma_t u}]\right)$$

$$= \min\left(P[|X_1| \leq \sqrt{u'\Sigma_t u}], \inf_{u \in S^{k-1}} P[|X_1| \geq \sqrt{u'\Sigma_t u}]\right)$$

$$= \min(P[|X_1| \leq \lambda^{1/2}_k(\Sigma_t)], P[|X_1| \geq \lambda^{1/2}_k(\Sigma_t)]).$$

Since Lemma D.1 entails that

$$P[|X_1| \leq \lambda^{1/2}_k(\Sigma_t)] \geq P[|X_1| \leq \min(\lambda^{1/2}_k(\Sigma_a), \lambda^{1/2}_k(\Sigma_b))]$$

$$= \min(P[|X_1| \leq \lambda^{1/2}_k(\Sigma_a)], P[|X_1| \leq \lambda^{1/2}_k(\Sigma_b)]) \geq \min(\text{HD}^{sc}_P(\Sigma_a), \text{HD}^{sc}_P(\Sigma_b))$$

and

$$P[|X_1| \geq \lambda^{1/2}_k(\Sigma_t)] \geq P[|X_1| \geq \max(\lambda^{1/2}_k(\Sigma_a), \lambda^{1/2}_k(\Sigma_b))]$$

$$= \min(P[|X_1| \geq \lambda^{1/2}_k(\Sigma_a)], P[|X_1| \geq \lambda^{1/2}_k(\Sigma_b)]) \geq \min(\text{HD}^{sc}_P(\Sigma_a), \text{HD}^{sc}_P(\Sigma_b)),$$
the result for \( \theta_0 = 0 \) and \( \Sigma_0 = I_k \) follows from (D.6).

We now prove the result in the elliptical case with arbitrary values of \( \theta_0 \) and \( \Sigma_0 \). To this end, let \( A = \Sigma_0^{-1/2} \) and note that the square roots (in \( \mathcal{P}_k \)) of \( \Gamma_a := A \Sigma_a A' \) and \( \Gamma_b := A \Sigma_b A' \) are of the form \( \Gamma_a^{1/2} = A \Sigma_a^{1/2} O_a \) and \( \Gamma_b^{1/2} = A \Sigma_b^{1/2} O_b \), for some \( k \times k \) orthogonal matrices \( O_a, O_b \). Consequently,

\[
\tilde{\Gamma}_t := A \tilde{\Sigma}_t A' = A \Sigma_a^{1/2} (\Sigma_a^{-1/2} \Sigma_b \Sigma_a^{-1/2}) t \Sigma_a^{1/2} A' = \Gamma_a^{1/2} O_a \Gamma_a^{-1/2} \Gamma_b \Gamma_a^{-1/2} O_a \Gamma_a^{1/2} = \Gamma_a^{1/2} (\Gamma_a^{-1/2} \Gamma_b \Gamma_a^{-1/2}) t \Gamma_a^{1/2}
\]
describes a geodesic path from \( \Gamma_a \) to \( \Gamma_b \). Since the result holds at \( P_0 = P_{A_1, -A \theta_0} \) (where the notation \( P_{A, b} \) was defined on page 5), affine invariance then entails that

\[
\text{HD}_{P}^{\text{sc}}(\tilde{\Sigma}_t) = \text{HD}_{P_b}^{\text{sc}}(\tilde{\Gamma}_t) \geq \min(\text{HD}_{P_b}^{\text{sc}}(\Gamma_a), \text{HD}_{P_b}^{\text{sc}}(\Gamma_b)) = \min(\text{HD}_{P}^{\text{sc}}(\Sigma_a), \text{HD}_{P}^{\text{sc}}(\Sigma_b)),
\]
as was to be showed.

We now turn to the case where the probability measure \( P \) over \( \mathbb{R}^k \) has independent Cauchy marginals. Fix \( \Sigma_a, \Sigma_b \in \mathcal{P}_k \) and consider the geodesic path \( \tilde{\Sigma}_t, t \in [0, 1], \) from \( \tilde{\Sigma}_a \) to \( \tilde{\Sigma}_b \). Recall that \( \text{HD}_{P}^{\text{sc}}(\Sigma) = 2 \min \left( \Psi(1/\max_{s} \sqrt{\Sigma_{s}^{-1}s}) - \frac{1}{2}, 1 - \Psi(\sqrt{\max_{s} \text{diag}(\Sigma)}) \right) \), where \( \Psi \) stands for the Cauchy cumulative distribution function; see (2.6). Lemma D.2 readily entails that

\[
2 - 2 \Psi\left(\sqrt{\max_{s} \text{diag}(\tilde{\Sigma}_t)}\right) \geq \min\left(2 - 2 \Psi\left(\sqrt{\max_{s} \text{diag}(\Sigma_a)}\right), 2 - 2 \Psi\left(\sqrt{\max_{s} \text{diag}(\Sigma_b)}\right)\right)
\]

(D.7)

\[
\geq \min(\text{HD}_{P}^{\text{sc}}(\Sigma_a), \text{HD}_{P}^{\text{sc}}(\Sigma_b)).
\]

Lemma D.2 also provides \( \max_{s} s \Sigma_{s}^{-1}s \leq \max_{s} s \Sigma_{a}^{-1}s, \max_{s} s \Sigma_{b}^{-1}s \), which rewrites \( 1/\max_{s} (s \Sigma_{s}^{-1}s)^{1/2} \geq \min(1/\max_{s} (s \Sigma_{a}^{-1}s)^{1/2}, 1/\max_{s} (s \Sigma_{b}^{-1}s)^{1/2}) \). This implies that

\[
2 \Psi\left(1/\max_{s} (s \Sigma_{s}^{-1}s)^{1/2}\right) - 1 \geq \min\left(2 \Psi\left(1/\max_{s} (s \Sigma_{a}^{-1}s)^{1/2}\right) - 1, 2 \Psi\left(1/\max_{s} (s \Sigma_{b}^{-1}s)^{1/2}\right) - 1\right)
\]

(D.8)

\[
\geq \min(\text{HD}_{P}^{\text{sc}}(\Sigma_a), \text{HD}_{P}^{\text{sc}}(\Sigma_b)).
\]

From (D.7)-(D.8), it readily follows that \( \text{HD}_{P}^{\text{sc}}(\tilde{\Sigma}_t) \geq \min(\text{HD}_{P}^{\text{sc}}(\Sigma_a), \text{HD}_{P}^{\text{sc}}(\Sigma_b)) \), which concludes the proof of Part (i).

(ii) Let then \( P \) be an arbitrary probability measure over \( \mathbb{R}^k \) satisfying Part (i) of the result. For any \( \Sigma_a, \Sigma_b \in R_{P}^{\text{sc}}(\alpha) \), we then have \( \text{HD}_{P}^{\text{sc}}(\tilde{\Sigma}_t) \geq \min(\text{HD}_{P}^{\text{sc}}(\Sigma_a), \text{HD}_{P}^{\text{sc}}(\Sigma_b)) \geq \alpha \), so that \( \tilde{\Sigma}_t \in R_{P}^{\text{sc}}(\alpha) \).

\[
\text{Proof of Theorem 6.2.} \text{ The proof is entirely similar to the proof of Theorem 5.2. In the elliptical case, the affine-invariance argument is based on the identity } \Gamma_t^* := A \Sigma_t^* A' = ((1 - t)\Gamma_a^{-1} + t \Gamma_b^{-1})^{-1}, \text{ with } \Gamma_a := A \Sigma_a A' \text{ and } \Gamma_b := A \Sigma_b A'. \]

APPENDIX E: PROOFS FROM SECTION 7

Proof of Theorem 7.1. We may restrict to the case where $\text{HD}^{sh}_{P,S}(V) > \alpha_P$ (indeed, the assumptions ensure that $\text{HD}^{sh}_{P,S}(V) \geq \alpha_P$ and that the result holds if $\text{HD}^{sh}_{P,S}(V) = \alpha_P$).

For any $\alpha > \alpha_P$, consider then $I_\alpha := I_\alpha(V,P) := \{\sigma^2 \in \mathbb{R}_{+}^2 : \sigma^2 V \in R_P^{sc}(\alpha)\} = \{\sigma^2 \in \mathbb{R}_{+}^2 : \text{HD}^{sc}_P(\sigma^2 V) \geq \alpha\}$. The convexity of $R_P^{sc}(\alpha)$ (Theorem 3.3(ii)) implies that $I_\alpha$ is an interval. Since $\alpha > \alpha_P$, Theorem 4.2 shows that $R_P^{sc}(\alpha)$ is $g$-bounded, which implies there exist $\eta_\alpha > 0$ and $M_\alpha > \eta_\alpha$ such that $I_\alpha \subset [\eta_\alpha, M_\alpha]$. Since, moreover, Theorem 3.1 implies that $\sigma^2 \mapsto \text{HD}^{sc}_P(\sigma^2 V)$ is upper semicontinuous, $I_\alpha$ is also closed, hence (still for $\alpha > \alpha_P$) compact.

Now, fix $\delta > 0$ such that $\text{HD}^{sh}_{P,S}(V) - \delta > \alpha_P$. For any $n$, pick then $\sigma^2_n$ in the (non-empty) interval $I_{\text{HD}^{sh}_{P,S}(V) - (\delta/n)}$. The resulting sequence $(\sigma^2_n)$ is in the compact set $I_{\text{HD}^{sh}_{P,S}(V) - \delta}$, hence admits a subsequence $(\sigma^2_{n_k})$ converging in $\mathbb{R}_{+}^2$, to $\sigma^2$, say. Fix then an arbitrary $\varepsilon \in (0, \delta)$. For $\ell$ large enough, all $\sigma^2_{n_k}$ belong to the closed set $I_{\text{HD}^{sh}_{P,S}(V) - \varepsilon}$, so that $\sigma^2$ also belongs to $I_{\text{HD}^{sh}_{P,S}(V) - \varepsilon}$. This shows that $\text{HD}^{sh}_{P,S}(V) - \varepsilon \leq \text{HD}^{sc}_P(\sigma^2 V) \leq \text{HD}^{sh}_{P,S}(V)$. Since $\varepsilon$ can be taken arbitrarily small, the result is proved.

Proof of Theorem 7.2. From Theorem 2.1, we readily obtain

$$\text{HD}^{sh}_{P_A,b,S}(AVA'/S(AVA')) = \sup_{\sigma^2 > 0} \text{HD}^{sc}_{P_A,b}(\sigma^2 AVA'/S(AVA'))$$

$$= \sup_{\sigma^2 > 0} \text{HD}^{sc}_{P_A,b}(\sigma^2 AVA') = \sup_{\sigma^2 > 0} \text{HD}^{sc}_P(\sigma^2 V) = \text{HD}^{sh}_{P,S}(V),$$

which establishes the result.

Proof of Theorem 7.3. Consider two arbitrary probability measures $P, Q$ on $\mathbb{R}^k$. Fix $V \in \mathcal{P}_{k,S}$ and assume (without loss of generality) that $\text{HD}^{sh}_{P,S}(V) \leq \text{HD}^{sh}_{Q,S}(V)$. Then, for any $\varepsilon > 0$, there exists $\sigma^2_\varepsilon > 0$ such that $\text{HD}^{sh}_{Q,S}(V) \leq \text{HD}^{sc}_Q(\sigma^2_\varepsilon V) + \varepsilon$, so that

$$|\text{HD}^{sh}_{P,S}(V) - \text{HD}^{sh}_{Q,S}(V)| = \text{HD}^{sh}_{Q,S}(V) - \text{HD}^{sh}_{P,S}(V)$$

$$\leq \text{HD}^{sc}_Q(\sigma^2_\varepsilon V) + \varepsilon - \text{HD}^{sc}_P(\sigma^2_\varepsilon V) \leq \sup_{\Sigma \in \mathcal{P}_k} |\text{HD}^{sc}_P(\Sigma) - \text{HD}^{sc}_Q(\Sigma)| + \varepsilon.$$

Since this holds for any $\varepsilon > 0$ and since $V$ is arbitrary, we have that

$$\sup_{V \in \mathcal{P}_{k,S}} |\text{HD}^{sh}_{P,S}(V) - \text{HD}^{sh}_{Q,S}(V)| \leq \sup_{\Sigma \in \mathcal{P}_k} |\text{HD}^{sc}_P(\Sigma) - \text{HD}^{sc}_Q(\Sigma)|.$$

The result then follows from Theorem 2.2.

Proof of Theorem 7.4. (i) Fix a shape matrix $V_0 \in R^{sh}_{P,S}(\alpha_P)$ and assume, ad absurdum, that there exists a sequence $(V_n)$ in $\mathcal{P}_{k,S}$ that $g$-converges (resp., $F$-converges) to $V_0$ and such that $\limsup_{n \to \infty} \text{HD}^{sh}_P(V_n) > \text{HD}^{sh}_{P,S}(V_0)$. Extracting a subsequence if necessary, we can fix $\varepsilon > 0$ small enough to have $\text{HD}^{sh}_{P,S}(V_0) + \varepsilon < \text{HD}^{sh}_{P,S}(V_n)$ for any $n$. 

Fix then, for any $n$, $\sigma_n^2 > 0$ such that $\text{HD}^{sc}_P(\sigma_n^2 V_n) > \text{HD}^{sh}_{P,S}(V_n) - \varepsilon/2$, which yields

\[(E.1) \quad \text{HD}^{sc}_P(\sigma_n^2 V_n) > \text{HD}^{sh}_{P,S}(V_0) + \varepsilon/2 \geq \alpha_P + \varepsilon/2.\]

Now, we can assume without loss of generality that $V_n$ belongs to a neighborhood of $V_0$ that is $g$-compact in $P_{k,S}(P)$. Since (E.1) implies that $\sigma_n^2 V_n$ belongs, for any $n$, to the $g$-bounded (Theorem 4.2) scatter depth region $R^sc_P(\alpha_P + \varepsilon/2)$, the sequence $(\sigma_n^2)$ then stays away from 0 and $\infty$ (that is, the $\sigma_n^2$’s belong to a common compact set of $\mathbb{R}^+_0$).

Consequently, there exists a subsequence $(\sigma_{n\ell}^2)$ such that $(\sigma_{n\ell}^2 V_{n\ell})$ $g$-converges (resp., $F$-converges) to $\sigma^2_0 V_0$, say. In view of (E.1), we therefore found $\varepsilon > 0$ such that, for any $\ell$,

$$\text{HD}^{sc}_P(\sigma_{n\ell}^2 V_{n\ell}) > \text{HD}^{sh}_{P,S}(\sigma^2_0 V_0) + \varepsilon/2,$$

where $(\sigma_{n\ell}^2 V_{n\ell})$ $g$-converges (resp., $F$-converges) to $\sigma^2_0 V_0$, which contradicts the scatter depth upper semicontinuity result in Theorem 4.1 (resp., in Theorem 3.1). (ii) The result follows from the fact that $R^sh_P(\alpha)$ is the inverse image of $[\alpha, +\infty)$ by the upper $F$- and $g$-semicontinuous function $V \mapsto \text{HD}^{sh}_{P,S}(V)$. (iii) Since the supremum of lower semicontinuous functions is a lower semicontinuous function, Theorems 3.1 and 4.1(iii) yield that $V \mapsto \text{HD}^{sh}_{P,S}(V)$ is lower-semi continuous. The result then follows from Part (i) and the fact that the smoothness of $P$ at its Tukey median implies that $R^sh_{P,S}(\alpha_P) = R^sh_{P,S}(0) = P_{k,S}$. \hfill $\square$

The proof of Theorem 7.5 requires the following lemma.

**Lemma E.1.** Let $S$ be a scale functional, that is a mapping from $P_k$ to $\mathbb{R}^+_0$ that satisfies the properties (i)-(iii) in page 22. Then, $\lambda_k(V) \leq 1 \leq \lambda_1(V)$ for any $V \in P_{k,S}$.

**Proof of Lemma E.1.** (a) Factorize $V$ into $V = O \text{diag}(\lambda_1(V), \ldots, \lambda_k(V))O'$, where $O$ is a $k \times k$ orthogonal matrix. Then, since $\lambda_k(V)I_k = O \text{diag}(\lambda_k(V), \ldots, \lambda_k(V))O' \leq V \leq O \text{diag}(\lambda_1(V), \ldots, \lambda_1(V))O' = \lambda_1(V)I_k$ (where $A \leq B$ still means that $B - A$ is positive semidefinite), the properties of a scale functional yield $\lambda_k(V) = S(\lambda_k(V)I_k) \leq S(V) \leq S(\lambda_1(V)I_k) = \lambda_1(V)$. \hfill $\square$

**Proof of Theorem 7.5.** We start with the proof of the result for $s_P < 1/2$ and $g$-boundedness. We fix $\varepsilon > 0$ and intend to prove that $R^sh_{P,S}(s_P + \varepsilon)$ is $g$-bounded by showing that, for $r > 0$ large enough, it is included in the $g$-ball $B_g(I_k, r)$. To do so, fix a shape matrix $V(\in P_{k,S})$ that does not belong to $B_g(I_k, r)$ ($r$ is to be chosen later). In view of (C.1), we then have (i) $\lambda_1(V) > \exp(r/\sqrt{k})$ or (ii) $\lambda_k(V) < \exp(-r/\sqrt{k})$ (or both).

We start with case (i). Fix (so far, arbitrarily) $\sigma_0^2 > 0$. Then for any $\sigma^2 \in (0, \sigma_0^2]$, Lemma E.1 entails that (denoting by $v_k(V)$ an arbitrary unit vector associated with $\lambda_k(V)$)

$$\text{HD}^{sc}_P(\sigma^2 V) \leq \inf_{u \in S^{k-1}} P\{|u'(X - \theta_P)| \leq \sigma \sqrt{u'Vu} \} \leq P\{|v_k(V)(X - \theta_P)| \leq \sigma \lambda_k^{1/2}(V) \} \leq P\{|v_k(V)(X - \theta_P)| \leq \sigma_0 \} \leq s^\theta_P(\sigma_0),$$

(E.2)

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where we used the notation $s_P^K(\cdot)$ introduced in Lemma A.4. By using this lemma, pick then $\sigma_0^2 > 0$ such that $s_P^{(\theta_p)}(\sigma_0) < s_P^{(\theta_p)} + (\varepsilon/2) = s_P + (\varepsilon/2)$. Denoting by $v_1(V)$ an arbitrary unit vector associated with $\lambda_1(V)$, we then have that, for any $\sigma^2 \in [\sigma_0^2, \infty)$,

$$
HD_P^{sc}(\sigma^2 V) \leq \inf_{u \in \mathcal{S}^{k-1}} P\left[\left|u'(X - \theta_P)\right| \geq \sigma \sqrt{u'V u}\right] \leq P\left[\left|v_1'(V)(X - \theta_P)\right| \geq \sigma \lambda_1^{1/2}(V)\right]
$$

(E.3) \quad \leq P\left[\left|v_1'(V)(X - \theta_P)\right| \geq \sigma_0 \exp\left(r/2\sqrt{k}\right)\right] \leq P\left[\left|X - \theta_P\right| \geq \sigma_0 \exp\left(r/2\sqrt{k}\right)\right],

which, for $r$ large enough, can be made smaller than $\varepsilon/2$ (hence, smaller than $s_P + (\varepsilon/2)$). For $r$ large enough, thus, (E.2)-(E.3) guarantee that $HD_{P,S}^{sh}(V) = \sup_{\sigma^2 > 0} HD_P^{sc}(\sigma^2 V) < s_P + \varepsilon$, as was to be showed.

We then turn to case (ii). By picking $\sigma_0$ large enough, we have that, for any $\sigma^2 \in [\sigma_0^2, \infty)$,

$$
HD_P^{sc}(\sigma^2 V) \leq \inf_{u \in \mathcal{S}^{k-1}} P\left[\left|u'(X - \theta_P)\right| \geq \sigma \sqrt{u'V u}\right] \leq P\left[\left|v_1'(V)(X - \theta_P)\right| \geq \sigma \lambda_1^{1/2}(V)\right]
$$

(E.4) \quad \leq P\left[\left|v_1'(V)(X - \theta_P)\right| \geq \sigma_0\right] \leq P\left[\left|X - \theta_P\right| \geq \sigma_0\right] < \varepsilon/2,

where we used Lemma E.1. For any $\sigma^2 \in (0, \sigma_0^2]$, we then have

$$
HD_P^{sc}(\sigma^2 V) \leq \inf_{u \in \mathcal{S}^{k-1}} P\left[\left|u'(X - \theta_P)\right| \leq \sigma \sqrt{u'V u}\right] \leq P\left[\left|v_1'(V)(X - \theta_P)\right| \leq \sigma \lambda_1^{1/2}(V)\right]
$$

(E.5) \quad \leq P\left[\left|v_1'(V)(X - \theta_P)\right| \leq \sigma_0 \exp\left(-r/2\sqrt{k}\right)\right] \leq s_P^{(\theta_p)}(\sigma_0 \exp\left(-r/2\sqrt{k}\right) < s_P + (\varepsilon/2),

for $r$ large enough. Thus, for $r$ large enough, (E.4)-(E.5) still yield that $HD_{P,S}^{sh}(V) = \sup_{\sigma^2 > 0} HD_P^{sc}(\sigma^2 V) < s_P + \varepsilon$, as was to be showed. We thus conclude that, for $\alpha > s_P$, $R_{P,S}^{sh}(\alpha)$ is $g$-bounded (its $g$-compacity then follows from the same argument as in the proof of Theorem 4.2).

The proof for $F$-boundedness (still for $s_P < 1/2$) follows along the same lines and is actually simpler since only one of both cases (i)-(ii) above is to be considered. Recall indeed that, as seen in the proof of Theorem 3.2, $V \notin B_F(I_k, r)$ implies that $\lambda_1(V) > (r - 1)/k^{1/2}$, so that the same reasoning as in case (i) above allows to show that for any $\varepsilon > 0$, there exists $r = r(\varepsilon)$ such that $V \notin B(I_k, r)$ implies $HD_{P,S}^{sh}(V) < s_P + \varepsilon$. This establishes that $R_{P,S}^{sh}(\alpha)$ is $F$-bounded for $\alpha > s_P$.

Finally, if $s_P \geq 1/2$, then, with $u_0 \in \mathcal{S}^{k-1}$ such that $P[|u_0'(X - \theta_P)| = 0] = s_P$ (existence is guaranteed in Lemma A.4(ii), with $K = \{\theta_p\}$), we have

$$
HD_P^{sc}(\sigma^2 V) \leq P[|u_0'(X - \theta_P)| \geq \sigma \sqrt{u_0'V u_0}] \leq P[|u_0'(X - \theta_P)| > 0] = 1 - s_P
$$

for any $\sigma^2 > 0$, so that $HD_{P,S}^{sh}(V) \leq 1 - s_P$. Therefore, $R_{P,S}^{sh}(\alpha)$ is empty for any $\alpha > 1 - s_P = \alpha_p$. \hfill \Box

**Proof of Theorem 7.6.** The proof follows along the exact same lines as that of Theorem 4.3, hence is not reported here. \hfill \Box

**Proof of Theorem 7.7.** (i) For any $V \in \mathcal{P}_{k,S}$, Theorem 5.1(i) readily implies that $HD_{P,S}^{sh}(V) = \sup_{\sigma^2 > 0} HD_P^{sc}(\sigma^2 V) \leq HD_P^{sc}(\Sigma_0)$. Since $HD_P^{sc}(\Sigma_0) = HD_P^{sc}(S(\Sigma_0)V_0) \leq \sigma_0^2$ for $\sigma_0 > 0$, we then have $HD_{P,S}^{sh}(V) < \sup_{\mathcal{P}_{k,S}} \inf_{\sigma^2 > 0} \sup_{\mathcal{P}_{k,S}} \inf_{\sigma^2 > 0} \sigma_0^2$. \hfill \Box
$HD_{P,S}^{sh}(V_0)$, the result follows. (ii) Before proceeding, note that since $P$ is an elliptical probability measure, we have $s_P = P[\{\theta_0\}]$. Moreover, we must have $s_P < 1/2$ (otherwise, $P[|Z_1| = 0] \geq P[\{\theta_0\}] = s_P \geq 1/2$, so that $0 \in I_{MSD}[Z_1]$, a contradiction). Now, assume, ad absurdum, that there exists $V \in P_{k,S} \setminus \{V_0\}$ with $HD_{P,S}^{sh}(V) = HD_{P,S}^{sh}(V_0)$. Since $HD_{P,S}^{sh}(V_0) = HD_P^{sc}(\Sigma_0) \geq 1/2$, we must have $HD_{P,S}^{sh}(V) \geq 1/2$. Since $\alpha_P = s_P < 1/2$, there exists $\sigma^2 > 0$ such that $HD_P^{sc}(\sigma^2 V) \geq s_P$. Therefore, Theorem 7.1 ensures that $HD_{P,S}^{sh}(V) = HD_P^{sc}(\sigma^2 V)$ for some $\sigma^2 > 0$. We therefore have $HD_P^{sc}(\sigma^2 V) = HD_{P,S}^{sh}(V) = HD_{P,S}^{sh}(V_0) = HD_P^{sc}(\Sigma_0) = HD_P^{sc}(S(\Sigma_0) V_0)$. Then, since $\det \tilde{\Sigma}$ is proportional to the identity matrix, which implies that $V = V_0$. □

Proof of Theorem 7.8. (i) Consider first the scale functional $S_{tr}$ and fix $\varepsilon > 0$. By definition, there exist positive real numbers $\sigma^2_a$ and $\sigma^2_b$ such that $HD_P^{sc}(\sigma^2_a V_a) \geq HD_{P,S_{tr}}^{sh}(V_a) - \varepsilon$ and $HD_P^{sc}(\sigma^2_b V_b) \geq HD_{P,S_{tr}}^{sh}(V_b) - \varepsilon$. Consider then the linear path $\Sigma_t = (1 - t)\Sigma_a + t\Sigma_b$ from $\Sigma_a = \sigma^2_a V_a$ to $\Sigma_b = \sigma^2_b V_b$. Letting $h(t) = t\sigma^2_b / ((1 - t)\sigma^2_a + t\sigma^2_b)$, the $S_{tr}$-shape matrix associated with $\Sigma_t$ is

\[
\frac{k}{\text{tr}[\Sigma_t]} \Sigma_t = \frac{k}{\text{tr}[(1 - t)\sigma^2_a V_a + t\sigma^2_b V_b]} ((1 - t)\sigma^2_a V_a + t\sigma^2_b V_b)
\]

\[
= \frac{k}{\text{tr}[(1 - h(t)) V_a + h(t) V_b]} ((1 - h(t)) V_a + h(t) V_b) = V_{h(t)}.
\]

Since $h : [0, 1] \to [0, 1]$ is a one-to-one mapping, Theorem 3.3 yields that

$$HD_{P,S_{tr}}^{sh}(V_t, P) = HD_{P,S_{tr}}^{sh}\left(\frac{k}{\text{tr}[\Sigma_{h^{-1}(t)}]} \Sigma_{h^{-1}(t)}^{-1}, P\right) \geq HD_P^{sc}(\Sigma_{h^{-1}(t)})$$

\[
\geq \min(\text{HD}_P^{sc}(\Sigma_a), \text{HD}_P^{sc}(\Sigma_b)) \geq \min(\text{HD}_{P,S_{tr}}^{sh}(V_a), \text{HD}_{P,S_{tr}}^{sh}(V_b)) - \varepsilon,
\]

for any $t \in [0, 1]$. Since this holds for any $\varepsilon > 0$, Part (i) of the result is proved for $S = S_{tr}$. The proof for $S = S_{11}$ is along the exact same lines, hence is omitted. As for Part (ii), it strictly follows like Part (ii) of Theorem 5.2. □

Proof of Theorem 7.9. (i) Fix $\varepsilon > 0$. By definition, there exist $\sigma^2_a > 0$ and $\sigma^2_b > 0$ such that $HD_P^{sc}(\sigma^2_a V_a) \geq HD_{P,S_{det}}^{sh}(V_a) - \varepsilon$ and $HD_P^{sc}(\sigma^2_b V_b) \geq HD_{P,S_{det}}^{sh}(V_b) - \varepsilon$. Consider then the geodesic path $\tilde{\Sigma}_t = \Sigma_a^{1/2} (\Sigma_a^{-1/2} \Sigma_b \Sigma_a^{-1/2}) t \Sigma_a^{1/2}$ from $\Sigma_a = \sigma^2_a V_a$ to $\Sigma_b = \sigma^2_b V_b$. Then, since $\det \tilde{\Sigma}_t = (\det \Sigma_a)^{1-t} (\det \Sigma_b)^{t}$, it is easy to check that the $S_{det}$-shape matrix associated with $\tilde{\Sigma}_t$ is $(\det \tilde{\Sigma}_t)^{-1/k} \tilde{\Sigma}_t = V_a^{1/2} (V_a^{-1/2} V_b V_a^{-1/2})^{t} V_a^{1/2} =: \tilde{V}_t$. Therefore, using Theorem 5.2, we obtain

$$HD_{P,S_{det}}^{sh}(\tilde{V}_t) \geq HD_P^{sc}(\det \tilde{\Sigma}_t^{1/k} \tilde{V}_t) = HD_P^{sc}(\tilde{\Sigma}_t)$$

\[
\geq \min(\text{HD}_P^{sc}(\Sigma_a), \text{HD}_P^{sc}(\Sigma_b)) \geq \min(\text{HD}_{P,S_{det}}^{sh}(V_a), \text{HD}_{P,S_{det}}^{sh}(V_b)) - \varepsilon.
\]

Part (i) of the result follows since $\varepsilon > 0$ is arbitrary above. As for Part (ii), it is obtained again as in Part (ii) of Theorem 5.2. □
Proof of Theorem 7.10. (i) Fix $\varepsilon > 0$. By definition, there exist positive real numbers $\sigma_a^2$ and $\sigma_b^2$ such that $\text{HD}_{P,S_{tr}}^\text{sc}(V_a) \geq \text{HD}_{P,S_{tr}}^\text{sh}(V_a) - \varepsilon$ and $\text{HD}_{P,S_{tr}}^\text{sc}(\sigma_b^2 V_b) \geq \text{HD}_{P,S_{tr}}^\text{sh}(V_b) - \varepsilon$.

Consider then the harmonic path $\Sigma^* = ((1 - t)\Sigma_a^{-1} + t\Sigma_b^{-1})^{-1}$ from $\Sigma_a = \sigma_a^2 V_a$ to $\Sigma_b = \sigma_b^{-2} V_b$. Then, letting $h(t) = t\sigma_a^{-2}/((1 - t)\sigma_a^{-2} + t\sigma_b^{-2})$, the $S_{tr}^*$-shape matrix associated with $\Sigma^*$ is

$$
\frac{\text{tr}[(\Sigma^*)^{-1}]}{k} \Sigma^* = ((1 - t)\sigma_a^{-2} + t\sigma_b^{-2})((1 - t)\sigma_a^{-2}V_a^{-1} + t\sigma_b^{-2}V_b^{-1})^{-1}
$$

$$
= ((1 - h(t))V_a^{-1} + h(t)V_b^{-1})^{-1} =: V_{h(t)}^*
$$

Since $h : [0,1] \to [0,1]$ is a one-to-one mapping, we obtain that, for any $t \in [0,1]$,

$$
\text{HD}_{P,S_{tr}}^\text{sh}(V_{h(t)}^*) = \text{HD}_{P,S_{tr}}^\text{sh} \left( \frac{\text{tr}[(\Sigma_{h^{-1}(t)})^{-1}]}{k} \Sigma_{h^{-1}(t)}^* \right) \geq \text{HD}_{P}^\text{sc}(\Sigma_{h^{-1}(t)}^*)
$$

$$
\geq \min(\text{HD}_{P}^\text{sc}(\Sigma_a), \text{HD}_{P}^\text{sc}(\Sigma_b)) \geq \min(\text{HD}_{P,S_{tr}}^\text{sh}(V_a), \text{HD}_{P,S_{tr}}^\text{sh}(V_b)) - \varepsilon.
$$

Since this holds for any $\varepsilon > 0$, Part (i) of the result is proved. Part (ii) strictly follows like Part (ii) of Theorem 5.2. \qed

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Fig 5. Contour plots of \((V_{11}, V_{12}) \mapsto HD_{P, \text{Str}}(V)\), for several bivariate probability measures \(P\), where \(HD_{P, \text{Str}}^S(V)\) is the shape halfspace depth, with respect to \(P\), of \(V = (V_{11}, V_{12}, -V_{11})\). Letting \(\Sigma_A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\), \(\Sigma_B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\) and \(\Sigma_C = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\), the probability measures \(P\) considered are those associated (i) with the bivariate normal distributions with location zero and scatter \(\Sigma_A\), \(\Sigma_B\) and \(\Sigma_C\) (top, middle and bottom left), and (ii) with the distributions of \(\Sigma_A^{1/2} Z\), \(\Sigma_B^{1/2} Z\) and \(\Sigma_C^{1/2} Z\), where \(Z\) has mutually independent Cauchy marginals (top, middle and bottom right). In each case, the “true” \(\text{Str}\)-shape matrix is marked in red.
Fig 6. Contour plots of $(V_{11}, V_{12}) \mapsto HD_{P,S}^{sh}(V_S)$, for $S_{tr}$ (top), $S_{det}$ (middle) and $S^{*}_{tr}$ (bottom), where $HD_{P,S}^{sh}(V_S)$ is the shape halfspace depth, with respect to $P$, of the $S$-shape $V_S(\in P_{2,S})$ with upper-left entry $V_{11}$ and upper-right entry $V_{12}$. The probability measures $P$ considered are those associated (i) with the bivariate normal distribution with location zero and scatter $\Sigma = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ (left) and (ii) with the distribution of $\Sigma^{1/2} Z$, where $Z$ has mutually independent Cauchy marginals (right). In each case, the (“true”) $S$-shape associated with $\Sigma_C$ is marked in red and those associated with $\Sigma_A$ and $\Sigma_B$ from Figure 5 are marked in black. Linear paths (top), geodesic paths (middle) and harmonic paths (bottom) between these three shapes are drawn.
Fig 7. Plots of (i) $H_{SD}^P(\bar{\Sigma}_{full})$, (ii) $H_{SD}^P(\hat{V}_{full})$, (iii) $d_F(\hat{\Sigma}_d, \hat{\Sigma}_{full})$, (iv) $d_F(\hat{V}_d, \hat{V}_{full})$, (v) $d_g(\hat{\Sigma}_{det}, \hat{\Sigma}_{null})$ and (vi) $d_g(\hat{V}_{det}, \hat{V}_{full})$, as a function of $d$, for the MCD scatter and shape estimates described in Section 8. The horizontal dotted lines in (i)-(ii) correspond to the global depths $H_{SD}^P(\bar{\Sigma}_{null})$ and $H_{SD}^P(\hat{V}_{null})$, respectively. Vertical lines mark the six events listed in Section 8.