Low dimensional cohomology of Hom-Lie algebras and $q$-deformed $W(2, 2)$ algebra

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Abstract. This paper aims to study the low dimensional cohomology of Hom-Lie algebras and $q$-deformed $W(2, 2)$ algebra. We show that the $q$-deformed $W(2, 2)$ algebra is a Hom-Lie algebra. Also, we establish a one-to-one correspondence between the equivalence classes of one dimensional central extensions of a Hom-Lie algebra and its second cohomology group, leading us to determine the second cohomology group of the $q$-deformed $W(2, 2)$ algebra. In addition, we generalize some results of derivations of finitely generated Lie algebras with values in graded modules to Hom-Lie algebras. As application we compute all $\alpha^k$-derivations and in particular the first cohomology group of the $q$-deformed $W(2, 2)$ algebra.

Key words: Hom-Lie algebras, $q$-deformed $W(2, 2)$ algebra, derivation, second cohomology group, first cohomology group.

Mathematics Subject Classification (2000): 17A30, 17A60, 17B68, 17B70.

1. Introduction

The notion of Hom-Lie algebras was initially introduced in [3] motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields. In this paper we will follow the slightly more general definition of Hom-Lie algebras given by Makhlouf and Silvestrov in [6]. Precisely, a Hom-Lie algebra is a triple $(\mathcal{L}, [\cdot, \cdot], \alpha)$ consisting of a vector space $\mathcal{L}$, a bilinear map $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and a linear map $\alpha : \mathcal{L} \to \mathcal{L}$ such that
\[
[x, y] = -[y, x], \quad \text{(skew-symmetry)}
\]
\[
\circ_{x,y,z} [[x, y], \alpha(z)] = 0, \quad \text{(Hom-Jacobi identity)}
\]
for all $x, y, z \in \mathcal{L}$, and where the symbol $\circ_{x,y,z}$ denotes summation over the cyclic permutation on $x, y, z$. One sees that the classical Lie algebras recover from Hom-Lie algebras if the twisting map $\alpha$ is the identity map. The Hom-Lie algebras were discussed intensively in [7, 8, 9, 10] while the graded cases were considered in [11, 5, 11]. But the cohomology with values in graded Hom-modules is not very clear. Therefore, one of the aims of the present

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paper is to fill this gap.

The $W(2, 2)$ Lie algebra was introduced in [13] for the study of classification of vertex operator algebras generated by vectors of weight 2. It is an extension of the Virasoro algebra. In the following we denote by $\mathcal{W}$ the centerless $W(2, 2)$ Lie algebra, which is an infinite dimensional Lie algebra generated by $L_n$ and $M_n \ (n \in \mathbb{Z})$ satisfying the following Lie brackets

$$[L_m, L_n] = (n - m)L_{m+n}, \ [L_m, M_n] = (n - m)M_{m+n}, \ [M_m, M_n] = 0, \ \text{for} \ m, n \in \mathbb{Z}.$$ 

In [12] we presented a realization of the centerless $W(2, 2)$ Lie algebra $\mathcal{W}$ by using bosonic and fermionic oscillators. The bosonic oscillator $a$ and its hermitian conjugate $a^+$ obey the commutation relations:

$$[a, a^+] = aa^+ - a^+a = 1, \ [1, a^+] = [1, a] = 0. \quad (1.1)$$

It follows by induction on $n$ that

$$[a, (a^+)^n] = n(a^+)^{n-1}, \ \text{for all} \ \ n \in \mathbb{Z}.$$ 

The fermionic oscillators $b$ and $b^+$ satisfy the anticommutators

$$\{b, b^+\} = bb^+ + b^+b = 1, \quad b^2 = (b^+)^2 = 0. \quad (1.2)$$

Moreover, we set $[a, b] = [a, b^+] = [a^+, b] = [a^+, b^+] = 0$.

**Lemma 1.1** ([12]) With notations above. The generators of the form

$$L_n \equiv (a^+)^{n+1}a, \quad M_n \equiv (a^+)^{n+1}b^+a, \ \text{for all} \ n \in \mathbb{Z}, \quad (1.3)$$

realize the centerless $W(2, 2)$ Lie algebra $\mathcal{W}$ under the commutator

$$[A, B] = AB - BA, \ \text{for all} \ A, B \in \mathcal{W}.$$ 

Now fix a nonzero $q \in \mathbb{C}$ such that $q$ is not a root of unity. We introduce the following notation

$$[A, B]_{(\alpha, \beta)} = \alpha AB - \beta BA,$$

and the $q$-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$
It is clear to see that \([-n]_q = -[n]_q\). Furthermore, one can deduce that \(q^n[m]_q - q^m[n]_q = [m - n]_q,\ q^{-n}[m]_q + q^m[n]_q = [m + n]_q\).\hspace{1cm} (1.4)\]

Then the generators \(L_n\) and \(M_n\) \((n \in \mathbb{Z})\) satisfy the following \(q\)-brackets:

\[
\begin{align*}
[L_n, L_m](q^{n-m}, q^{m-n}) &= [m - n]_q L_{m+n}, \\
[L_n, M_m](q^{n-m}, q^{m-n}) &= [m - n]_q M_{m+n}, \\
[M_n, M_m](q^{n-m}, q^{m-n}) &= 0,
\end{align*}
\]

for all \(m, n \in \mathbb{Z}\). We call this algebra the \(q\)-deformed \(W(2, 2)\) algebra, which is the second object considered in this paper. In the following we will denote \(q\)-deformed \(W(2, 2)\) algebra by \(\mathcal{W}_q\) and simply write the \(q\)-bracket as \([\cdot, \cdot]_q\). In [12] we determined quantum groups and one dimensional central extensions of \(\mathcal{W}_q\). In this paper, we will study its low dimensional cohomology theory. That is the second aim of this paper.

Throughout this paper, \(\mathbb{C}\) denotes the field of complex numbers and \(\mathbb{Z}\) denotes the set of all integers. All vector spaces and algebras are assumed to be over \(\mathbb{C}\).

2. Second cohomology group

In this section, we first recall some basic definitions and in particular central extension of Hom-Lie algebras. Then we establish a one-to-one correspondence between the equivalence classes of one dimensional central extensions of a Hom-Lie algebra and its second cohomology group with coefficients in \(\mathbb{C}\). As application we determine the second cohomology group of the \(q\)-deformed \(W(2, 2)\) algebra which is considered as a Hom-Lie algebra.

In the sequel we will often simply write a Hom-Lie algebra \((\mathcal{L}, [\cdot, \cdot], \alpha)\) as \((\mathcal{L}, \alpha)\). A Hom-Lie algebra \((\mathcal{L}, \alpha)\) is said to be \(multiplicative\) if the twisting map \(\alpha\) is an endomorphism. Let \(G\) be an abelian group. A Hom-Lie algebra \((\mathcal{L}, \alpha)\) is said to be \(G\)-\(graded\), if its underlying vector space is \(G\)-graded (i.e., \(\mathcal{L} = \oplus_{g \in G} \mathcal{L}_g\)) satisfying \([\mathcal{L}_g, \mathcal{L}_h] \subseteq \mathcal{L}_{g+h}\), and if \(\alpha\) is an even map, i.e., \(\alpha(\mathcal{L}_g) \subseteq \mathcal{L}_g\), for all \(g, h \in G\).

The theory of central extensions of Hom-Lie algebras was studied in [3, 4]. An \(extension\) of a Hom-Lie algebra \((\mathcal{L}, \zeta)\) by an abelian Hom-Lie algebra \((\mathfrak{a}, \zeta_a)\) is a commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \rightarrow & \mathfrak{a} & \xrightarrow{i} & \hat{\mathcal{L}} & \xrightarrow{\text{pr}} & \mathcal{L} & \rightarrow & 0 \\
& & \downarrow{\zeta_a} & & \downarrow{\zeta} & & \downarrow{\zeta} & & \\
0 & \rightarrow & \mathfrak{a} & \xrightarrow{i} & \hat{\mathcal{L}} & \xrightarrow{\text{pr}} & \mathcal{L} & \rightarrow & 0
\end{array}
\]
where \((\hat{L}, \hat{\zeta})\) is a Hom-Lie algebra. The extension is central if
\[
\iota(a) \subseteq Z(\hat{L}) = \{x \in \hat{L} \mid [x, \hat{L}] = 0\}.
\]

In the following we focus on the central extension of \((L, \alpha)\) by a one-dimensional center \(C\), where \(c = \iota(1)\). Note that the center \(C\) can be considered as the one-dimensional trivial Hom-Lie algebra with the identity map.

**Definition 2.1** Let \((L, \alpha)\) be a Hom-Lie algebra. A bilinear map \(\psi : L \times L \rightarrow C\) is called a 2-cocycle on \(L\) if the following conditions are satisfied
\[
\begin{align*}
\psi(x, y) &= -\psi(y, x), \\
\psi(\alpha(x), [y, z]) + \psi(\alpha(y), [z, x]) + \psi(\alpha(z), [x, y]) &= 0,
\end{align*}
\]
for all \(x, y, z \in L\).

Now we have the following theorem:

**Theorem 2.2** Let \((L, \alpha)\) be a Hom-Lie algebra and \(\psi : L \times L \rightarrow C\) be a bilinear map. Define on the vector space \(\hat{L} = L \oplus C\) the following bracket and linear map by
\[
\begin{align*}
[x + c, y + b]_{\hat{L}} &= [x, y]_L + \psi(x, y), \\
\hat{\alpha}(x + c) &= \alpha(x) + c,
\end{align*}
\]
for all \(x, y \in L\) and \(c, b \in C\). Then \((\hat{L}, [\cdot, \cdot]_{\hat{L}}, \hat{\alpha})\) is a Hom-Lie algebra one dimensional central extension of \((L, \alpha)\) if and only if \(\psi\) is a 2-cocycle on \((L, \alpha)\). If, in addition, \((L, \alpha)\) is multiplicative and \(\psi\) satisfies \(\psi(\alpha(x), \alpha(y)) = \psi(x, y)\), for all \(x, y \in L\), then the Hom-Lie algebra \((\hat{L}, \hat{\alpha})\) is also multiplicative.

**Proof.** Since \([\cdot, \cdot]_L\) is skew-symmetric, the new bracket \([\cdot, \cdot]_{\hat{L}}\) is skew-symmetric if and only if the map \(\psi\) is skew-symmetric. For any \(x, y, z \in L\) and \(a, b, c \in C\), we have
\[
[\hat{\alpha}(x + a), [y + b, z + c]_{\hat{L}}]_{\hat{L}} = [\alpha(x) + a, [y, z]_L + \psi(y, z)]_{\hat{L}} = [\alpha(x), [y, z]_L + \psi(\alpha(x), [y, z]_L)].
\]
Consequently,
\[
[\hat{\alpha}(x + a), [y + b, z + c]_{\hat{L}}]_{\hat{L}} + [\hat{\alpha}(y + b), [z + c, x + a]_{\hat{L}}]_{\hat{L}} + [\hat{\alpha}(z + c), [x + a, y + b]_{\hat{L}}]_{\hat{L}} = 0
\]
if and only if

\[ \psi(\alpha(x), [y, z]_\mathcal{L}) + \psi(\alpha(y), [z, x]_\mathcal{L}) + \psi(\alpha(z), [x, y]_\mathcal{L}) = 0, \]

which proves \((\hat{\mathcal{L}}, [\cdot, \cdot]_\mathcal{L}, \hat{\alpha})\) is a Hom-Lie algebra if and only if \(\psi\) is a 2-cocycle on \((\mathcal{L}, \alpha)\).

If \((\mathcal{L}, \alpha)\) is multiplicative, then we have

\[ \hat{\alpha}([x + a, y + b]_\mathcal{L}) = \hat{\alpha}([x, y]_\mathcal{L} + \psi(x, y)) \]

\[ = \alpha([x, y]_\mathcal{L}) + \psi(x, y) \]

\[ = [\alpha(x), \alpha(y)]_\mathcal{L} + \psi(x, y). \]

On the other hand, we have

\[ [\hat{\alpha}(x + a), \hat{\alpha}(y + b)]_\mathcal{L} = [\alpha(x) + a, \alpha(y) + b]_\mathcal{L} \]

\[ = [\alpha(x), \alpha(y)]_\mathcal{L} + \psi(\alpha(x), \alpha(y)). \]

According to the hypothesis that \(\psi(\alpha(x), \alpha(y)) = \psi(x, y)\) for all \(x, y \in \mathcal{L}\), we have

\[ \hat{\alpha}([x + a, y + b]_\mathcal{L}) = [\hat{\alpha}(x + a), \hat{\alpha}(y + b)]_\mathcal{L}, \text{ for all } x, y \in \mathcal{L}, a, b \in \mathbb{C}, \]

which shows that \((\hat{\mathcal{L}}, \hat{\alpha})\) is multiplicative.

Finally, we define \(\text{pr}\) and \(\iota\) as the natural projection and inclusion respectively by

\[ \text{pr} : \hat{\mathcal{L}} \rightarrow \mathcal{L}, \quad \text{pr}(x + a) = x; \]

\[ \iota : \mathbb{C} \rightarrow \hat{\mathcal{L}}, \quad \iota(a) = 0 + a. \]

Then it is easy to show that \((\hat{\mathcal{L}}, \hat{\alpha})\) is a one-dimensional central extension of \((\mathcal{L}, \alpha)\). \(\square\)

Denote by \(Z^2(\mathcal{L}, \mathbb{C})\) the vector space of all 2-cocycles on a Hom-Lie algebra \((\mathcal{L}, \alpha)\). For any linear map \(f : \mathcal{L} \rightarrow \mathbb{C}\), we can define a 2-cocycle \(\psi_f\) by

\[ \psi_f(x, y) = f([x, y]), \text{ for all } x, y \in \mathcal{L}. \quad \text{(2.5)} \]

Such a 2-cocycle is called a 2-coboundary or a trivial 2-cocycle on \(\mathcal{L}\). Let \(B^2(\mathcal{L}, \mathbb{C})\) denote the vector space of all 2-coboundaries on \(\mathcal{L}\). The quotient space

\[ H^2(\mathcal{L}, \mathbb{C}) = Z^2(\mathcal{L}, \mathbb{C})/B^2(\mathcal{L}, \mathbb{C}) \]

is called the second cohomology group of \(\mathcal{L}\) with trivial coefficients \(\mathbb{C}\). A 2-cocycle \(\psi\) is said to be equivalent to another 2-cocycle \(\phi\) if \(\psi - \phi\) is trivial. For a 2-cocycle \(\psi\), let \([\psi]\) be the equivalent class of \(\psi\). Then we have the following corollary:
**Corollary 2.3** For any Hom-Lie algebra \((L, \alpha)\), there exists a one-to-one correspondence between the equivalence classes of one dimensional central extensions of \((L, \alpha)\) and its second cohomology group \(H^2(L, \mathbb{C})\).

In the following, we consider the \(q\)-deformed \(W(2, 2)\) algebra \(W_q\). Note that \(W_q\) is not a Lie algebra, because the classical Jacobi identity does not hold (but the antisymmetry is true). By straightforward calculations, we have

\[(q^l + q^{-l})\left(\left[ L_m, L_n \right]_{q^{m-n}, q^{n-m}}, L_l \right)_{(q^{m+n-l}, q^{l-m-n})} + \text{cyclic permutations} = 0,\]  
\[(q^l + q^{-l})\left(\left[ L_m, L_n \right]_{q^{m-n}, q^{n-m}}, M_l \right)_{(q^{m+n-l}, q^{l-m-n})} + \text{cyclic permutations} = 0.\]  
(2.6)

(2.7)

Define on \(W_q\) a linear map \(\alpha\) by

\[\alpha(L_n) = (q^n + q^{-n})L_n, \quad \alpha(M_n) = (q^n + q^{-n})M_n.\]

Then, using the \(q\)-deformed Jacobi identities (2.6) and (2.7), we obtain the following result.

**Theorem 2.4** The triple \((W_q, [\cdot, \cdot]_q, \alpha)\) forms a Hom-Lie algebra.

In [12] we provided a computation of one-dimensional central extensions of \(W_q\). Hence, according to Corollary 2.3, we can determine the second cohomology group of the \(q\)-deformed \(W(2, 2)\) algebra \(W_q\) as follows:

**Proposition 2.5** \(H^2(W_q, \mathbb{C}) = \mathbb{C}\beta \oplus \mathbb{C}\gamma\), where

\[\beta(L_m, L_n) = \delta_{m,-n} \frac{[m-1]_q [m]_q [m+1]_q}{[2]_q [3]_q \langle m \rangle_q}, \quad \beta(L_m, M_n) = \beta(M_m, L_n) = 0,\]

\[\gamma(L_m, M_n) = \delta_{m,-n} \frac{[m-1]_q [m]_q [m+1]_q}{[2]_q [3]_q \langle m \rangle_q}, \quad \gamma(L_m, L_n) = \gamma(M_m, M_n) = 0,\]

and where \(\langle m \rangle_q = q^m + q^{-m}\), for all \(m, n \in \mathbb{Z}\).

### 3. Derivations of Hom-Lie algebras and \(q\)-deformed \(W(2, 2)\) Lie algebra

This section is devoted to discuss derivations of graded Hom-Lie algebras. We extend to Hom-Lie algebras some concepts and results of derivations of finitely generated Lie algebras with values in graded modules studied in [2]. As application we compute all \(\alpha^k\)-derivations and particularly the first cohomology group of the \(q\)-deformed \(W(2, 2)\) algebra.
Definition 3.1 Let $\mathcal{L}, \alpha$ be a Hom-Lie algebra. A representation of $\mathcal{L}$ is a triple $(V, \rho, \beta)$, where $V$ is a $\mathbb{C}$-vector space, $\beta \in \text{End}(V)$ and $\rho : \mathcal{L} \rightarrow \text{End}(V)$ is a $\mathbb{C}$-linear map satisfying
\[
\rho([x, y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x),
\]
for all $x, y \in \mathcal{L}$. $V$ is also called a Hom-$\mathcal{L}$-module, denoted by $(V, \beta)$ for convenience.

One recovers the definition of a representation in the case of Lie algebras by setting $\alpha = \text{id}_\mathcal{L}$ and $\beta = \text{id}_V$. For any $x \in \mathcal{L}$, define $\text{ad} : \mathcal{L} \rightarrow \text{End}(\mathcal{L})$ by $\text{ad}_x(y) = [x, y]$ for all $y \in \mathcal{L}$. Then $(\mathcal{L}, \text{ad}, \alpha)$ is a representation of $\mathcal{L}$, which is called the adjoint representation of $\mathcal{L}$.

Definition 3.2 Let $(V, \beta_V)$ and $(W, \beta_W)$ be two Hom-$\mathcal{L}$-modules. A linear map $f : V \rightarrow W$ is called a morphism of Hom-$\mathcal{L}$-modules if it satisfies
\[
f \circ \beta_V = \beta_W \circ f,
f(x \cdot v) = x \cdot f(v),
\]
for all $x \in \mathcal{L}$, $v \in V$.

Let $G$ be an abelian group, $(\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g, [\cdot, \cdot], \alpha)$ be a $G$-graded Hom-Lie algebra. An Hom-$\mathcal{L}$-module $V$ is said to be $G$-graded if $V = \bigoplus_{g \in G} V_g$ and $\mathcal{L}_g V_h \subseteq V_{g+h}$ for all $g, h \in G$. For any nonnegative integer $k$, denote by $\alpha^k$ the $k$-times composition of $\alpha$, i.e.,
\[
\alpha^k = \alpha \circ \alpha \circ \cdots \circ \alpha.
\]
In particular, $\alpha^0 = \text{id}$ and $\alpha^1 = \alpha$. Then we can define $\alpha^k$-derivations of $\mathcal{L}$ with values in its Hom-$\mathcal{L}$-modules.

Definition 3.3 A linear map $D : \mathcal{L} \rightarrow V$ is called an $\alpha^k$-derivation if it satisfies
\[
D \circ \alpha = \alpha \circ D,
D[x, y] = \alpha^k(x) \cdot D(y) - \alpha^k(y) \cdot D(x),
\]
for all $x, y \in \mathcal{L}$.

We recover the definition of a derivation by setting $k = 0$ in the definition above. Hence, an $\alpha^0$-derivation is often simply called a derivation in the present paper. We say that an $\alpha^k$-derivation $D$ has degree $g$ (denoted by $\text{deg}(D) = g$) if $D \neq 0$ and $D(\mathcal{L}_h) \subseteq V_{g+h}$ for any $h \in G$. Let $D$ be an $\alpha^k$-derivation. If there exists $v \in V$ such that $D(x) = \alpha^k(x) \cdot v$ for all $x \in \mathcal{L}$, then $D$ is called an inner $\alpha^k$-derivation. Denote by $\text{Der}_{\alpha^k}(\mathcal{L}, V)$ and $\text{Inn}_{\alpha^k}(\mathcal{L}, V)$ the
space of $\alpha^k$-derivations and the space of inner $\alpha^k$-derivations, respectively. In particular, let $\text{Der}(\mathcal{L}, V)$ and $\text{Inn}(\mathcal{L}, V)$ denote the space of derivations and the space of inner derivations, respectively, and write $\text{Der}(\mathcal{L}, V)_g := \{ D \in \text{Der}(\mathcal{L}, V) \mid \deg(D) = g \} \cup \{0\}$. The first cohomology group of $\mathcal{L}$ with coefficients in $V$ is defined by

$$H^1(\mathcal{L}, V) := \frac{\text{Der}(\mathcal{L}, V)}{\text{Inn}(\mathcal{L}, V)}. \quad (3.2)$$

**Remark 3.4** The set $\text{Der}_{\alpha^k}(\mathcal{L}, V)$ (resp. $\text{Inn}_{\alpha^k}(\mathcal{L}, V)$) is not close under map composition or commutator bracket. But the space of all such $\alpha^k$-derivations $\oplus_{k \geq 0} \text{Der}_{\alpha^k}(\mathcal{L}, V)$ (resp. $\oplus_{k \geq 0} \text{Inn}_{\alpha^k}(\mathcal{L}, V)$) form an Lie algebra via commutator bracket.

Now let $\mathcal{L}$ be a $G$-graded Hom-Lie algebra which is finitely generated. In the following we present two results, which can be seen as Hom versions of those obtained in [2].

**Proposition 3.5** Let $V$ be a $G$-graded Hom-$\mathcal{L}$-module. For every $D \in \text{Der}(\mathcal{L}, V)$, we have

$$D = \sum_{g \in G} D_g, \quad (3.3)$$

where $D_g \in \text{Der}(\mathcal{L}, V)_g$ and where there are only finitely many $D_g(u) \neq 0$ in the equation $D(u) = \sum_{g \in G} D_g(u)$, for any $u \in \mathcal{L}$.

**Proof.** For any $g \in G$, define a homogeneous linear map $D_g : \mathcal{L} \to V$ as follows: for any $u \in L_h$ with $h \in G$, write $D(u) = \sum_{p \in G} u_p$ with $u_p \in V$, then set $D_g(u) = u_{g+h}$. Clearly, $D_g$ is well defined and $D_g \in \text{Der}(\mathcal{L}, V)_g$. Also, (3.3) is true. $\square$

**Proposition 3.6** Let $V$ be a $G$-graded Hom-$\mathcal{L}$-module such that

(a) $H^1(\mathcal{L}_0, V_g) = 0$, for $g \in G \setminus \{0\}$.

(b) $\text{Hom}_{\mathcal{L}_0}(\mathcal{L}_g, V_h) = 0$, for $g \neq h$.

Then

$$\text{Der}(\mathcal{L}, V) = \text{Der}(\mathcal{L}, V)_0 + \text{Inn}(\mathcal{L}, V).$$

**Proof.** Let $D$ be a derivation from $\mathcal{L}$ into its Hom-$\mathcal{L}$-module $V$. According to (3.3) we can decompose $D$ into its homogeneous components $D = \sum_{g \in G} D_g$ with $D_g \in \text{Der}(\mathcal{L}, V)_g$. Suppose that $g \neq 0$. Then $D_g|_{\mathcal{L}_0}$ is a derivation from $\mathcal{L}_0$ into the Hom-$\mathcal{L}_0$-module $V_g$. By virtue of (a), $D_g|_{\mathcal{L}_0}$ is inner, i.e., there exists $v_g \in V_g$ such that $D_g(u) = u \cdot v_g$ for all $u \in \mathcal{L}_0$. Consider $\psi_g : \mathcal{L} \to V$ defined by $\psi_g(x) := D_g(x) - x \cdot v_g$ for all $x \in \mathcal{L}$. Then $\psi_g$ is a derivation.
of degree $g$ which vanishes on $L_0$. Hence $\psi_q$ is a morphism of Hom-$L_0$-modules and condition (b) entails the vanishing of $\psi_q$ on $L_h$ for every $h \in G$. Consequently, $D_q \in \text{Inn}(L, V)$, which completes the proof.

In the following we focus on the $q$-deformed $W(2,2)$ algebra $W_q$ as a Hom-Lie algebra $(W_q, [,], \alpha)$ defined in Theorem 2.4. Obviously, $W_q$ is $\mathbb{Z}$-graded by

$$W_q = \bigoplus_{n \in \mathbb{Z}} W_q^n,$$

where $W_q^n = \text{span}_\mathbb{C}\{L_n, M_n\}$.

Note that $\mathcal{M} := \text{span}_\mathbb{C}\{M_n\}$ is an ideal of $(W_q, \alpha)$, or in other words, $\mathcal{M}$ is an adjoint Hom-$W_q$-module. In addition, $W_q$ is finitely generated by $\{L_1, L_{-1}, M_1\}$. Let $D$ be an $\alpha^k$-derivation of $W_q$. For all $m, n \in \mathbb{Z}$, we have

$$(q^m + q^{-m})^k[D(L_n), L_m] + (q^n + q^{-n})^k[L_n, D(L_m)]_q = [m - n]D(L_{m+n}), \quad (3.4)$$

$$(q^m + q^{-m})^k[D(L_n), M_m] + (q^n + q^{-n})^k[L_n, D(M_m)]_q = [m - n]D(M_{m+n}). \quad (3.5)$$

Now we aim to determine all $\alpha^k$-derivation of $W_q$. First, we compute the $(\alpha^0)$-derivations of $W_q$. Denote by $\text{Der}(W_q)$ and $\text{Inn}(W_q)$ the set of all derivations and the set of all inner derivations, respectively. Let $\text{Der}(W_q)_m$ be the set of derivations of degree $m$.

**Lemma 3.7** $H^1(W_q^0, W_q^m) = 0$ for any nonzero integer $n$.

**Proof.** Note that $[L_0, X_0]_q = 0$, for any $X_0 \in W_q^0 = \text{span}\{L_0, M_0\}$. Let $D$ be any element in $\text{Der}(W_q^0, W_q^m)$. Then it follows $D(L_0) \in W_q^m$. Applying $D$ to $[L_0, X_0]_q = 0$, we have $[n]D(X_0) = [L_0, D(X_0)] = [X_0, D(L_0)]$. Consequently, $D(X_0) = [X_0, v]$ with $v = \frac{1}{m}D(L_0)$ in $W_q^n$. In other words, $D$ is an inner derivation from $W_q^0$ into its adjoint module $W_q^n$. \hfill $\square$

**Lemma 3.8** $\text{Hom}_{W_q^0}(W_q^m, W_q^n) = 0$ for $m \neq n$.

**Proof.** Let $f \in \text{Hom}_{W_q^0}(W_q^m, W_q^n)$ with $m \neq n$. For any $X_m \in W_q^m$, we have

$$(q^n + q^{-n})(f(X_m)) = \alpha(f(X_m)) = f(\alpha(X_m)) = (q^m + q^{-m})f(X_m),$$

leading to $f(X_m) = 0$, since $m \neq n$. Hence, $f = 0$. \hfill $\square$

Now according to Proposition 3.6, we have the following result:

**Proposition 3.9** $\text{Der}(W_q) = \text{Der}(W_q)_0 + \text{Inn}(W_q)$. 

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Thanks to Proposition 3.9, the study of $\text{Der}(\mathcal{W}_q)$ reduces to that of its constituent of degree zero. Let $D$ be an element of $\text{Der}(\mathcal{W}_q)_0$. For any integer $n$, assume that

$$D(L_n) = a_n L_n + b_n M_n, \quad D(M_n) = c_n L_n + d_n M_n, \quad (3.6)$$

where the coefficients are complex numbers. Applying $D$ to $[L_0, L_n]_q = [n]_q L_n$, one can obtain $D(L_0) = 0$, i.e., $a_0 = b_0 = 0$. Using (3.4), we have

$$a_{m+n} = a_m + a_n, \quad b_{m+n} = b_m + b_n, \quad \text{for } m \neq n.$$ 

Let $m = -n$. Then we have

$$a_{-m} = -a_m, \quad b_{-m} = -b_m, \quad \text{for } m \neq 0.$$ 

Furthermore, we have

$$a_m = ma_1, \quad b_m = mb_1, \quad \text{for all } m \in \mathbb{Z}.$$ 

Similarly, using (3.5) we have

$$c_{m+n} = c_n, \quad d_{m+n} = a_m + d_n, \quad \text{for } m \neq n,$$

from which it follows

$$c_m = c_0, \quad d_m = ma_1 + d_0, \quad \text{for all } m \in \mathbb{Z}.$$ 

Applying $D$ to $[M_1, M_0] = 0$, we have $c_0 = 0$. It follows that $c_m = 0$ for all $m \in \mathbb{Z}$. Hence, there exist $a, b, d \in \mathbb{C}$ such that

$$D(L_n) = n(aL_n + bM_n), \quad D(M_n) = (na + d)M_n, \quad \text{for all } n \in \mathbb{Z}. \quad (3.7)$$

From the discussions above we obtain the following result:

**Proposition 3.10** All the derivations of $\mathcal{W}_q$ is

$$\text{Der}(\mathcal{W}_q) = \text{span}_\mathbb{C}\{D\} \oplus \text{Inn}(\mathcal{W}_q),$$

where $D$ is defined by (3.7).

**Corollary 3.11** The first cohomology group of $\mathcal{W}_q$ with values in its adjoint module is one-dimensional.
Next we compute the $\alpha^1$-derivations of $\mathcal{W}_q$. Let $D$ be an $\alpha^1$-derivation of degree $s$. Assume that
\[ D(L_n) = a_{s,n}L_{n+s} + b_{s,n}M_{n+s}, \quad D(M_n) = c_{s,n}L_{n+s} + d_{s,n}M_{n+s}, \]
where the coefficients are complex numbers. Then from equation (3.4) we obtain
\[ [m - n]_q a_{s,mn} = (q^m + q^{-m})[m - s - n]_q a_{s,n} + (q^n + q^{-n})[m + s - n]_q a_{s,m}, \tag{3.8} \]
\[ [m - n]_q b_{s,mn} = (q^m + q^{-m})[m - s - n]_q b_{s,n} + (q^n + q^{-n})[m + s - n]_q b_{s,m}. \tag{3.9} \]
We first consider the case of $s \neq 0$. Taking $m = 0$ in (3.8), we have
\[ (2[s + n]_q - [n]_q) a_{s,n} = (q^n + q^{-n})[s - n]_q a_{s,0}. \]
Furthermore,
\[ a_{s,n} = \frac{(q^n + q^{-n})[s - n]_q}{(2[s + n]_q - [n]_q)} a_{s,0}. \tag{3.10} \]
Plugging (3.10) into (3.8), we have
\[ \frac{(q^{m+n} + q^{-m-n})[m - n]_q [s - m - n]_q}{2[s + m + n]_q - [m + n]_q} a_{s,0} = \frac{(q^m + q^{-m})(q^n + q^{-n})[m - s - n]_q [s - n]_q a_{s,0}}{2[s + n]_q - [n]_q} + \frac{(q^m + q^{-m})(q^n + q^{-n})[m + s - n]_q [s - m]_q}{2[s + m]_q - [m]_q} a_{s,0}. \]
Let $m = s$ in the equation above, we have
\[ \frac{(q^{s+n} + q^{-s-n})[s - n]_q [-n]_q}{2[2s + n]_q - [s + n]_q} a_{s,0} = \frac{(q^s + q^{-s})(q^n + q^{-n})[s - n]_q [-n]_q}{2[s + n]_q - [n]_q} a_{s,0}. \tag{3.11} \]
Then taking $n = -s$ in (3.11), we get
\[ \frac{[2s]_q [s]_q}{[s]_q} a_{s,0} = \frac{(q^s + q^{-s})^2 [2s]_q [s]_q}{[s]_q} a_{s,0}. \]
It follows $a_{s,0} = 0$ since $s \neq 0$. Then we have $a_{s,n} = 0$ for $n \in \mathbb{Z}$ and $s \neq 0$ by (3.10).
Similarly, from (3.9) we can deduce that $b_{s,n} = 0$ for $s \neq 0$ and $n \in \mathbb{Z}$.

In the case of $s = 0$, we simply write $a_{0,n}$ as $a_n$. Then it can be deduced from (3.8) that
\[ a_{m+n} = (q^m + q^{-m})a_n + (q^n + q^{-n})a_m, \text{ for } m \neq n. \tag{3.12} \]
Let $m = 0$ in (3.12), we have

$$a_n = -(q^n + q^{-n})a_0,$$  \hspace{1cm} (3.13)

which implies $a_n = a_{-n}$ for $n > 0$. Taking $m = -n$ in (3.12), we have

$$a_0 = (q^n + q^{-n})(a_n + a_{-n}).$$  \hspace{1cm} (3.14)

Substituting (3.13) into (3.14), we have $a_0 = 0$ and $a_n = 0$ for all $n \in \mathbb{Z}$. Similarly, we can deduce that $b_{0,m} = 0$ for all $m \in \mathbb{Z}$ by using (3.9) where $s = 0$. Hence, we have proved that

$$a_{s,m} = b_{s,m} = 0,$$ \hspace{1cm} for all $m, s \in \mathbb{Z},$

or, in other words, we get $D(L_m) = 0$ for $m \in \mathbb{Z}$.

It remains to determine $D(M_n)$ for all $n \in \mathbb{Z}$. Using $D(L_n) = 0$, we can deduce from (3.5) that

$$[m - n]_q c_{s,m+n} = (q^n + q^{-n})[m + s - n]_q c_{s,m},$$  \hspace{1cm} (3.15)

$$[m - n]_q d_{s,m+n} = (q^n + q^{-n})[m + s - n]_q d_{s,m}.$$  \hspace{1cm} (3.16)

Let $m = 0$ in (3.15) and (3.16), respectively. We have

$$-[n]_q c_{s,n} = (q^n + q^{-n})[s - n]_q c_{s,0},$$  \hspace{1cm} (3.17)

$$-[n]_q d_{s,n} = (q^n + q^{-n})[s - n]_q d_{s,0}.$$  \hspace{1cm} (3.18)

Taking $n = 0$ in (3.15) and (3.16), respectively, one has

$$[m]_q c_{s,m} = 2[m + s]_q c_{s,m},$$  \hspace{1cm} (3.19)

$$[m]_q d_{s,m} = 2[m + s]_q d_{s,m}.$$  \hspace{1cm} (3.20)

Taking $m = 0$ in (3.19) and (3.20), respectively, we have $c_{s,0} = d_{s,0} = 0$ for $s \neq 0$. Then it follows from (3.17) (resp. (3.18)) that $c_{s,n} = 0$ (resp. $d_{s,n} = 0$) for $n \in \mathbb{Z}$ and $s \neq 0$.

If $s = 0$, then it follows from (3.15) that

$$[m - n]_q c_{0,m+n} = (q^n + q^{-n})[m - n]_q c_{0,m}.$$  \hspace{1cm} (3.21)

Let $n = 0$ in (3.21), we have $[m]_q c_{0,m} = 2[m]_q c_{0,m}$. It follows that $c_{0,m} = 0$ for $m \neq 0$. Taking $n = -m$ in (3.21), we have $[2m]_q c_{0,0} = (q^m + q^{-m})[2m]_q c_{0,m}$, leading us to $c_{0,0} = 0$. Similarly, we can deduce from (3.16) that $d_{0,m} = 0$ for all $m \in \mathbb{Z}$. Thereby, the following proposition is proved.
Proposition 3.12 If $D$ is an $\alpha^1$-derivation of $W_q$, then $D = 0$.

With the similar discussions as above, we can compute $\alpha^k$-derivations of $W_q$ for $k > 1$ and thus we have all the $\alpha^k$-derivations of $W_q$ for $k > 0$ determined.

Proposition 3.13 For $k > 0$, all the $\alpha^k$-derivations of $W_q$ are zero.

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