Homogeneous division polynomials for Weierstrass elliptic curves

Jinbi Jin
Universiteit Leiden
Mathematisch Instituut
Niels Bohrweg 1
2333 CA Leiden.
jjin@math.leidenuniv.nl

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Abstract. Starting from the classical division polynomials we construct homogeneous polynomials $\alpha_n$, $\beta_n$, $\gamma_n$ such that for $P = (x : y : z)$ on an elliptic curve in Weierstrass form over an arbitrary ring we have $nP = (\alpha_n(P) : \beta_n(P) : \gamma_n(P))$. To show that $\alpha_n$, $\beta_n$, $\gamma_n$ indeed have this property we use the a priori existence of such polynomials, which we deduce from the Theorem of the Cube.

We then use this result to show that the equations defining the modular curve $Y_1(n)_C$ computed for example by Baaziz, in fact still are correct over $\mathbb{Z}[1/n]$.

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Introduction

The main problem treated in this paper is the following.

Given an integer $n$, construct a triple of homogeneous polynomials that defines multiplication by $n$ on all Weierstrass curves (i.e. projective plane curves given by a Weierstrass polynomial) over rings.

The classical division polynomials describe multiples of affine points—these are points of the form $(a : b : 1)$—for all smooth Weierstrass curves over fields. This is a basic result that is stated in several books on elliptic curves. An example of a book that treats the theory behind this in more detail is [3]. This result is not completely satisfactory, as the division polynomials are not defined on non-affine points.

In this paper we construct for all integers $n$ three homogeneous polynomials $\alpha_n$, $\beta_n$, $\gamma_n$ (unique up to sign, and up to a multiple of the homogeneous Weierstrass polynomial) of degree $n^2$ such that for all Weierstrass curves $C$ over a ring $R$, and all “nowhere singular” points on $C$ of the form $P = (x : y : z)$, we have $nP = (\alpha_n(P) : \beta_n(P) : \gamma_n(P))$ (see the example on the next page for $\alpha_2$, $\beta_2$, $\gamma_2$). The proof that $\alpha_n$, $\beta_n$, $\gamma_n$ has this property uses the Theorem of the Cube (see e.g. [2, Thm. IV.3.3]).
As an application of this result, we then give explicit equations over \( \mathbb{Z}[1/n] \) of the modular curve \( Y_1(n) \), i.e. the moduli space of elliptic curves \( E \) together with an embedding of \( \mathbb{Z}/n\mathbb{Z} \) into \( E \). Explicit equations for its function field have been given before, for example in [1] over \( \mathbb{C} \), or in [2] over an arbitrary field. The equations for \( Y_1(n) \) we give in this paper (over \( \mathbb{Z}[1/n] \)) turn out to coincide with those given in [1, Sect. 3]. In particular, the latter define the curve \( Y_1(n) \), rather than its function field.

Overview. In Section 1.1 we define Weierstrass curves over arbitrary schemes, and give some properties of them that we will use throughout this paper. In Section 1.2 we describe the construction of \( a_n, \beta_n, \gamma_n \). In Section 1.3 we state the main theorem, Section 1.2 and 1.3 as well as the summary of Section 1.1 should be accessible for anyone with some basic knowledge about elliptic curves over fields and commutative algebra.

In Section 2.1 we consider multiplication by \( n \) on smooth Weierstrass curves, by reducing to the universal smooth Weierstrass curve and use the Theorem of the Cube to show the existence of three homogeneous polynomials of degree \( n^2 \) defining multiplication by \( n \) on smooth Weierstrass curves. We compare these polynomials with \( a_n, \beta_n, \gamma_n \), and show that they are the same. In Section 2.2 we show that \( a_n, \beta_n, \gamma_n \), in fact define multiplication by \( n \) on all Weierstrass curves, using the fact that they do so on the universal smooth Weierstrass curve.

Finally, in Section 3 we apply the main result for smooth Weierstrass curves, to obtain the aforementioned explicit description of the moduli space of elliptic curves \( E \) together with an embedding of \( \mathbb{Z}/n\mathbb{Z} \) into \( E \).

Example 1 (Doubling formula). Let \( R \) be a ring, and let \( C \) be the Weierstrass curve over \( R \) defined by the polynomial

\[
W = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3,
\]

where \( a_1, a_2, a_3, a_4, a_6 \in R \). Let \( P = (x : y : z) \) be a point on \( C \) given in homogeneous coordinates, such that \( (\partial W/\partial x)(P), (\partial W/\partial y)(P), (\partial W/\partial z)(P) \) generate the unit ideal in \( R \). Then

\[
2P = (a_2(P) : \beta_2(P) : \gamma_2(P)),
\]

where

\[
\begin{align*}
a_2 &= 2xy^3 + 3a_1x^2y^2 + (a_1^2 - 2a_2)y^2z + (a_1^3 - 3a_1a_2 + 3a_3)xyz^2 + (2a_2^2 + 2a_2^3 - 6a_4)xyz^3 + (a_1a_2^2 - 3a_2a_3 - 3a_1a_4)y^2z^2 + (a_1^2a_2 - a_1^3a_3 - 2a_1a_2a_3 - 4a_1^2a_2 - 3a_2^2 - 2a_2a_4 - 8a_6)xyz^2 + (-a_1a_2^2 + a_1a_2^2a_3 + a_2^2a_3 - 3a_1a_2^3 + 4a_1a_2a_3 - 3a_2a_3 - 9a_1a_6)xz^2 + (a_1a_2^3a_3 - a_1^2a_2a_3 - 2a_2a_3 - a_1a_3a_4 - 3a_2^2a_4 + 2a_4 - 6a_6)y^2z^3 + (-a_1a_2^3a_3 - a_1^3a_2^2a_3 + 2\alpha_2^2a_3a_4 - a_1^3a_6 - 2a_3^2 + 2a_2a_4 + 4a_1a_2^3 - 3a_1a_2a_6 - 9a_3a_6)xz^3 + (a_1a_2^2a_3 - a_1a_2^3a_6 + a_2^2a_3 - 3a_2a_3a_6 + 3a_1a_4a_6)z^4, \\
\beta_2 &= y^4 + a_1xy^3 + (a_1^2 - 2a_2)y^2z + (a_1^3 - 3a_1a_2 + 3a_3)xyz^2 + (-2a_1a_2^2 + 6a_1a_3)xz^2y + (a_1^2 - 3a_1a_2a_3 + a_1^3a_3 - 5a_2a_3 + 18a_6)y^2z^2 + (a_1^3a_2 - 2a_1a_2a_3 + a_1^3a_3 + 3a_1a_2^3 - 6a_1a_2a_4 + 3a_3a_4 + 27a_1a_6)xz^2 + (-a_1^2 + 2a_1a_2^2a_3 - a_1a_2^3a_4 + 6a_2^2a_4 - 6a_1a_3a_4 + 9a_2a_6 - 9a_1^2)xz^2 + (a_1^2a_3 - a_1a_2a_6 + 2a_3^2 - 5a_2a_3a_4 - a_1a_4 + 3a_1a_2a_6 + 18a_3a_6)y^2z^3 + (a_1^3a_2a_3 - a_1^3a_3a_4 + a_1^3a_6 + 2a_2a_3^2 - a_1^3 - a_2a_4 - 2a_1a_2^3 + 6a_2^2a_4 - 6a_1a_3a_4 + 9a_2^2a_6 - 27a_1a_6)xz^3 + (a_1a_2a_3^2 - a_1a_2^3a_6 + a_3)^2 + a_2a_3^2 - a_1a_2^3 - 2a_2a_4 - 2a_1a_2^3a_6 + 6a_1a_3a_4a_6 + a_1^3 - 9a_1a_6 + 9a_2^2a_6 + 27a_6)z^4, \\
\gamma_2 &= 8y^3z + 12a_1x^2y^2z + 6a_1^2x^2yz^2 + 12a_3y^2z^2 + (a_1^3 + 12a_1a_3)xz^2y^2 + (a_1^3a_2 + 3a_1a_3)xz^3 + (a_1^3a_6 + 3a_1^3a_3)xz^2 + (a_1^3a_6 + 3a_1a_3)a^2z^3 + (a_1^3a_6 + 3a_1^3a_3)xz^2 + (a_1^3a_6 + 3a_1^3a_3)xz^2
\end{align*}
\]
Example 2. Let \( C \) be the Weierstrass curve over \( \mathbb{Q}_2 \) defined by the polynomial \( W = y^2z - x^3 - z^3 \). Then note that \( d_2 = 2x^3 - 18xyz^2, \beta_2 = y^4 + 18x^2z^2 - 27z^4 \), and \( \gamma_2 = 8y^3z \). Let \( P \) be a point of \( C \) reducing to \( (2 : 1 : 8) \) modulo 16, which exists by Hensel’s Lemma. Note that \( (\partial W/\partial z)(P) \equiv 1 \pmod{16} \), so we can apply Example 1. Hence \( 2P \) reduces to \( (4 : 1 : 0) \) modulo 16, \( 4P \) reduces to \( (8 : 1 : 0) \) modulo 16, and \( 8P \) reduces to \( (0 : 1 : 0) \) modulo 16.

1 Definitions, statement of main theorem

Rings in this paper are always assumed to be commutative, associative, and unital.

1.1 Weierstrass curves over schemes

For now, we will use the language of schemes. For a summary of this section in terms of Weierstrass curves over rings, we refer to the end of this section.

We start by giving some general definitions.

Definition 3. Let \( S \) be a scheme. A genus 1 curve over \( S \) is a flat, proper \( S \)-scheme locally of finite presentation, of which the geometric fibres are integral curves of arithmetic genus 1. A pointed genus 1 curve over \( S \) is a pair \((C, s)\) of a genus 1 curve over \( S \), and a section \( s \in C(S) \). An elliptic curve over \( S \) is a pointed genus 1 curve \((C, s)\) in which \( C \) is smooth over \( S \).

We now define the main objects of study in this paper.

Definition 4. A Weierstrass curve \( C \) over \( S \) is a closed subscheme of \( \mathbb{P}^2_S \) defined by a homogeneous polynomial of the form

\[
W = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3
\]

with \( a_1, a_2, a_3, a_4, a_6 \in \mathcal{O}_S(S) \). The discriminant \( \Delta \in \mathcal{O}_S(S) \) of \( C \) is the discriminant of the equation \( W \), see e.g. \cite[Def. 2.7]{[3]}. We will show later in Proposition 6 that a Weierstrass curve over a scheme \( S \) is in fact a genus 1 curve over \( S \).

We now describe the functor of points of the Weierstrass curve \( C \) explicitly. If \( T \) is an \( S \)-scheme, \( L \) is an invertible \( \mathcal{O}_T \)-module, \( s_0, s_1, s_2 \in L(T) \), and \( f \in \mathcal{O}_S(S)[x, y, z] \) homogeneous of degree \( d \), then we have a well-defined section \( f(s_0, s_1, s_2) \in L^d(T) \). We say that \( s_0, s_1, s_2 \) generate \( L \) if \( (s_0)_x, (s_1)_x, (s_2)_x \) generate \( L_x \) as an \( \mathcal{O}_T \)-module for all \( x \in T \). The following is a special case of a well-known result, see e.g. \cite[Lem. 10103]{[3]}, as the graded \( \mathcal{O}_S \)-algebra \( \mathcal{O}_S[x, y, z]/(W) \) is generated by its degree 1 part.

Proposition 5. Let \( S \) be a scheme, and let \( C \) be a Weierstrass curve over \( S \) defined by \( W \) as in \((1)\). Then for all \( S \)-schemes \( T \), we have

\[
C(T) = \left\{ \left( L, s_0, s_1, s_2 \right) \in L \text{ invertible } \mathcal{O}_T \text{-module, } s_0, s_1, s_2 \in L(T) \text{ generating } L \mid W(s_0, s_1, s_2) = 0 \text{ in } L^3 \right\} / \
\cong.
\]

Here, \( (L, s_0, s_1, s_2) \cong (M, t_0, t_1, t_2) \) if and only if there exists an isomorphism \( L \to M \) of \( \mathcal{O}_T \)-modules mapping \( s_i \) to \( t_i \) for all \( i \in \{0, 1, 2\} \).

If \( L = \mathcal{O}_T \), then we denote the class of \( (L, s_0, s_1, s_2) \) by \( (s_0 : s_1 : s_2) \).

Proposition 6. Let \( S \) be a scheme, and let \( C \) be a Weierstrass curve over \( S \) defined by \( W \) as in \((1)\). Then \((C, (0 : 1 : 0))\) is a pointed genus 1 curve over \( S \).

Proof. Except for flatness, our claim is well-known. We check affine locally on the base that \( C \) is flat over \( S \). So assume \( S = \text{Spec } R \). Then \( C \) is covered by the two standard affine open covers in which \( y \) and \( z \), respectively, are invertible. Since \( W \) is monic with respect to \( x \), the corresponding \( R \)-algebras are free as \( R \)-modules, hence flat. We deduce that \( C \) is flat over \( S \). \( \square \)
We deduce that C is smooth (of relative dimension 1) if and only if C is smooth on all geometric fibres, i.e. $\Delta \neq 0$ in all geometric points of S (see e.g. [3, Prop. 2.25]). This holds if and only if $\Delta \in O_S(S)^\times$.

In general, let $C^{\text{sm}}$ be the smooth locus of C over S. Then we have the following description of the functor of points of $C^{\text{sm}}$.

**Proposition 7.** Let S be a scheme, and let C be a Weierstrass curve over S defined by W as in (1). Then for all S-schemes T, the set $C^{\text{sm}}(T)$ is the subset of $C(T)$ consisting of the classes of the 4-tuples $(L, s_0, s_1, s_2)$ such that the sections 

$$f \in L^2, s_0 = s_1 = s_2 = 0 \in L^3, \quad (\partial W/\partial x)(s_0, s_1, s_2), (\partial W/\partial y)(s_0, s_1, s_2), (\partial W/\partial z)(s_0, s_1, s_2) \in L^2(T)$$

generate $L^{\otimes 2}$.

**Proof.** We denote the structure morphism of C by f. Note that for all $x \in C$, we have that C is smooth in $x$ if and only if $C_{f(x)}$ is smooth in $x$, using Proposition 6. The polynomials $\partial W/\partial x, \partial W/\partial y, \partial W/\partial z$ define the singular locus on every fibre, so $C^{\text{sm}}$ is the open subscheme of C that is the complement of the common zero locus of these polynomials.

The following shows by Proposition 6 that a smooth Weierstrass curve over a scheme S admits a unique group scheme structure with zero section $(0 : 1 : 0)$, which is commutative.

**Theorem 8 ([2, Prop. II.2.7]).** Let S be a scheme, and let C be a smooth genus 1 curve over S. Then there exists a unique group scheme structure on C with zero section $(0 : 1 : 0) \in C(S)$, which is commutative.

Now we will show that any Weierstrass curve over a scheme S admits a natural group scheme structure on its smooth locus. So let C be an arbitrary Weierstrass curve over a scheme S. Note that $(0 : 1 : 0) \in C^{\text{sm}}(S)$, since $(\partial W/\partial z)(0, 1, 0) = 1$ generates $O_S$. We first consider the universal Weierstrass curve.

**Definition 9.** Let $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. Then the universal Weierstrass curve C is the Weierstrass curve over A defined by

$$W = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3 \in A[x, y, z].$$

It is universal in the following sense.

**Proposition 10.** Let S be a scheme, and let C be a Weierstrass curve over S defined by W as in (1). Then there exist unique morphisms $g : C \to C$ and $h : S \to \text{Spec } A$ such that the following diagram is commutative, and such that the outer square is Cartesian.

$$
\begin{array}{ccc}
C & \longrightarrow & \mathbb{P}_A^2 \\
\uparrow{g} & & \downarrow{h} \\
C & \longrightarrow & \text{Spec } A
\end{array}
$$

**Proof.** It suffices to show this in the case that S is affine; the general case follows from it by gluing.

So suppose that $S = \text{Spec } A$, and let $\varphi$ denote the morphism $S \to \text{Spec } A$ given by the ring morphism $A \to A$ sending every $a_i$ to the corresponding coefficient of W. Then $C = C \times_A S$ (via $\varphi$), so $(\varphi_C, \varphi)$ is a pair satisfying the required properties.

Suppose that $(g, h)$ is a pair of morphisms satisfying the required properties. Then note that $C \to \mathbb{P}_A^2$ is a closed immersion, hence a monomorphism, so it follows that $g$ is the unique morphism making the diagram commute. Moreover, $h_{\mathbb{P}_A^2}$ induces a map from C to C if and only
if the induced map $\mathcal{O}_{\mathbb{P}^2(3)}(\mathbb{P}^2_A) \to \mathcal{O}_{\mathbb{P}^2(3)}(\mathbb{P}^2_A)$ (which is the map $\mathcal{A}[x,y,z]|_3 \to A[x,y,z]|_3$ induced by the ring morphism $\mathcal{A} \to A$) sends $W$ to $W$, in other words, if and only if $h = \varphi$. □

We now have the following corollary of Theorem 8.

**Corollary 11.** Let $\mathcal{S} = \text{Spec} \mathcal{A}$, and let $\mathcal{C}$ be the universal Weierstrass curve over $\mathcal{S}$. There exists a unique commutative group scheme structure on $\mathcal{C}_{\mathcal{S}}$ that has zero section $(0 : 1 : 0) \in \mathcal{C}_{\mathcal{S}}(\mathcal{S})$.

**Proof.** First note that by Proposition 6 and Theorem 8, $\mathcal{C}_{\mathcal{S}}$ admits a group scheme structure with zero section $(0 : 1 : 0) \in \mathcal{C}_{\mathcal{S}}(\mathcal{A})$. We show that it must be unique.

Note that $\mathcal{C}_{\mathcal{S}} \times_{\mathcal{A}} \mathcal{C}_{\mathcal{S}}$ is smooth over $\mathcal{A}$, with irreducible fibres, and that $\mathcal{A}$ is integral. Hence (for example by combining [8, Lem. 004Z] and [4, Prop. 17.5.7]) $\mathcal{C}_{\mathcal{S}} \times_{\mathcal{A}} \mathcal{C}_{\mathcal{S}}$ is an integral scheme. Also note that $\mathcal{C}_{\mathcal{S}}$ is separated over $\mathcal{A}$.

Now consider $\mathcal{R} = \mathbb{Z}[a_1,a_2,a_3,a_4,a_5,1/\Delta]$. Then $\text{Spec} \mathcal{R}$ is an open subscheme of $\text{Spec} \mathcal{A}$. Let $\mathcal{E} = \mathcal{C} \times_{\mathcal{A}} \text{Spec} \mathcal{R}$, and note that it is a non-empty open subscheme of $\mathcal{C}_{\mathcal{S}}$. As $\mathcal{E}$ is a smooth Weierstrass curve, $\mathcal{E}$ admits a unique group scheme structure with zero section $(0 : 1 : 0)$ in $\mathcal{E}(\mathcal{R})$. Hence the group scheme structure on $\mathcal{C}_{\mathcal{S}}$ extends that of $\mathcal{E}$, so by the above this must be the unique extension of the group scheme structure of $\mathcal{E}$ to $\mathcal{C}_{\mathcal{S}}$. □

By universality of $\mathcal{C}$, this gives a natural commutative group scheme structure on every Weierstrass curve.

We now summarise in terms of Weierstrass curves over rings.

**Summary 12.** Let $R$ be a ring. A Weierstrass curve $C$ over $R$ is a closed subscheme of $\mathbb{P}^2_R$ defined by a homogeneous polynomial of the form (1) with $a_1,a_2,a_3,a_4,a_6 \in R$. The discriminant $\Delta \in R$ of $C$ is the discriminant of (1), see e.g. [3, Def. 2.7]. The Weierstrass curve $C$ is smooth if and only if $\Delta \in R^\times$.

We have an explicit description of the set $C(R)$ of $R$-valued points of $C$ as follows. If $M$ is an invertible $R$-module, $m_0,m_1,m_2 \in M$, and $f \in R[x,y,z]$ homogeneous of degree $d$, then we have $f(s_0,s_1,s_2) \in M^{\otimes d}$. Then

$$C(R) = \left\{ (M,m_0,m_1,m_2) : \begin{array}{l}
M \text{ invertible } R\text{-module}, \quad m_0,m_1,m_2 \in M \\
Rm_0 + Rm_1 + Rm_2 = M, \quad W(m_0,m_1,m_2) = 0 \text{ in } M^{\otimes 3}
\end{array} \right\} / \sim,$$

where $(M,m_0,m_1,m_2) \cong (N,n_0,n_1,n_2)$ if and only if there exists an isomorphism $M \to N$ of $R$-modules mapping $m_i$ to $n_i$ for all $i \in \{1,2,3\}$. If $M = R$, the class of $(M,m_0,m_1,m_2)$ is denoted $(m_0 : m_1 : m_2)$.

As an example, if $R$ is a field (or more generally, any ring with trivial Picard group), then

$$C(R) = \{(x,y,z) \in \mathbb{R}^3 : W(x,y,z) = 0, \quad Rx + Ry + Rz = R\} / \mathbb{R}^\times.$$

In general, let $\mathcal{C}_{\mathcal{S}}$ be the open subscheme of $\mathcal{C}$ that is the complement of the common zero locus of $\partial W/\partial x, \partial W/\partial y, \partial W/\partial z$. Then $\mathcal{C}_{\mathcal{S}}(R)$ is the subset of $C(R)$ consisting of the classes of the 4-tuples $(M,m_0,m_1,m_2)$ such that

$$(\partial W/\partial x)(m_0,m_1,m_2), \quad (\partial W/\partial y)(m_0,m_1,m_2), \quad (\partial W/\partial z)(m_0,m_1,m_2)$$

generates $M^{\otimes 2}$.

The set $\mathcal{C}_{\mathcal{S}}(R)$ contains the point $(0 : 1 : 0) \in C(R)$, and has a natural structure of an abelian group with $(0 : 1 : 0)$ as neutral element.
1.2 Division polynomials and construction of $\delta_n$, $\beta_n$, $\gamma_n$

Let us recall the definition of division polynomials. The main reference for this is [3, Ch. 3]. For this purpose, we consider a special (but yet also generic!) smooth Weierstrass curve.

Let $K$ be the algebraic closure of the field of fractions of the ring $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. Let $E$ be the elliptic curve over $K$ defined by the homogeneous Weierstrass polynomial

$$W = y^2 z + a_1 xy z + a_3 y z^2 - x^3 - a_2 x z^2 - a_4 x^2 z - a_6 z^3.$$  

Further, let $X$, $Y$ denote the rational functions $x/z$ and $y/z$, respectively. Then the function field $K(E)$ of $E$ is the field of fractions of $K[X, Y]/(W')$, where

$$W' = Y^2 + a_1 XY + a_3 Y - X^3 - a_2 X^2 - a_4 X - a_6$$

is the affine Weierstrass polynomial. We have a leading coefficient map $\Lambda : K(E) - 0 \rightarrow K - 0$, given by $f \mapsto \{(X/Y)^{-\text{ord}_0 f}(0)\}$, where $\text{ord}_0 f$ denotes the order of $f$ at the point at infinity of $E$. This is well-defined as $X/Y$ has a simple zero at 0.

Then we can define the division polynomials as follows.

**Definition 13.** Let $n \in \mathbb{Z} - \{0\}$. The $n$-th division polynomial $\Psi_n$ is the unique rational function $f$ in $K(E)$ with divisor $\sum_{P \in E[n]} - \langle P \rangle = (n^2 - 1) \langle 0 \rangle$ and leading coefficient $n$. Moreover, we define $\Psi_0$ to be 0.

These division polynomials exist, as the degree of the divisors defining them is 0, and for all $n \in \mathbb{Z} - 0$, we have $\sum_{P \in E[n]} P = 0$.

**Remark 14.** The division polynomials satisfy the following relation for all $m, n \in \mathbb{Z}$ (see [3, Prop. 3.53])

$$\Psi_{m+n} \Psi_{m-n} = \Psi_{m+1} \Psi_{m-1} \Psi_n^2 - \Psi_{n+1} \Psi_{n-1} \Psi_m^2.$$  

Using $\Psi_1$, $\Psi_2$, $\Psi_3$, and $\Psi_4$, and the recurrence relation above, one can recursively compute the other division polynomials.

Furthermore, define the following rational functions.

$$\Phi_n = X \Psi_n^2 - \Psi_{n-1} \Psi_{n+1} \quad \Omega_n = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{1}{n^4} \big( \Psi_{2n} - \Psi_n^2 (a_1 \Phi_n + a_3 \Psi_n^2) \big) & \text{otherwise} \end{cases}$$

We list some facts in the next propositions.

**Proposition 15 ([3, Sect. 3.6]).** Let $n \in \mathbb{Z}$. Then $\Psi_n, \Phi_n, \Omega_n \in A[X, Y]/(W')$.

**Proposition 16.** Let $n \in \mathbb{Z}$. Then $\Lambda \Phi_n = n \Omega_n = 1$, and $\text{ord}_0 \Phi_n = -2n^2$, $\text{ord}_0 \Omega_n = -3n^2$.

**Proof.** This is a straightforward calculation, using that $\Lambda \Psi_n = n$ and $\text{ord}_0 \Psi_n = -(n^2 - 1)$. □

Now we can state the basic result mentioned in the introduction more precisely, in the case of our special smooth Weierstrass curve $E/K$.

**Proposition 17 ([3, Prop. 3.55]).** Let $P = (x : y : 1) \in E(K)$ be a rational point, and let $n \in \mathbb{Z}$. Then

$$nP = (\Psi_n(x, y) \Phi_n(x, y) : \Omega_n(x, y) : \Psi_n^3(x, y)).$$

We want to obtain a version of this result that works for all rational points, including the point at infinity. We will do this by choosing representatives in $A[X, Y]$ of the elements $\Psi_n \Phi_n$, $\Omega_n$, $\Psi_n^3$ of $A[X, Y]/(W')$ and then homogenising them.

However, one has to be careful with the choice of representatives. As an example of this, the usual representatives chosen for the $\Psi_n$ have $Y$-degree at most 1, which is convenient for calculations. But this also causes homogenisations to be divisible by high powers of $z$ for large
Definition 18. Let $a$ be the unique representatives of $\Psi_n\Phi_n$, $\Omega_n$, $\Psi_n^3$ with $X$-degree at most 2, and define the polynomials $a_n = A_nz^n$, $\beta_n = B_nz^n$, $\gamma_n = C_nz^n$ in $A[x, y, z, z^{-1}]$.

Uniqueness of $A_n$, $B_n$, $C_n$ is guaranteed by the fact that $W'$ is monic in $X$.

Proposition 19. Let $n \in \mathbb{Z}$. Then $a_n, \beta_n, \gamma_n \in A[x, y, z]$ and are homogeneous of degree $n^2$. Moreover, $a_n, \gamma_n \in (x, z)$ and $\beta_n \in y^n + (x, z)$.

Proof. To show that $a_n, \beta_n, \gamma_n \in A[x, y, z]$, it suffices to show that the polynomials $A_n, B_n, C_n$ have total degree at most $n^2$. Note that the monomials with $X$-degree at most 2 all have distinct orders at infinity. As $A_n, B_n, C_n$ have orders $-3n^2 + 1, -3n^2, -3n^2 + 3$ at infinity, respectively, and as they have $X$-degree at most 2, it follows that they have total degree at most $n^2$. (Here, we use that $X$ and $Y$ have poles of orders 2 and 3 at the point at infinity, respectively.)

Now $a_n, \beta_n, \gamma_n$ are by construction homogeneous of degree $n^2$, and as $Y^n$ is the unique monomial with order $-3n^2$, the result follows.

For our special smooth Weierstrass curve $E$, the problem at the point at infinity is now solved.

Proposition 20. Let $n \in \mathbb{Z}$, and let $P = (x : y : z) \in E(K)$. Then

$$nP = (a_n(P) : \beta_n(P) : \gamma_n(P)).$$

Proof. As $a_n, \beta_n, \gamma_n$ were obtained from $\Psi_n\Phi_n$, $\Omega_n$, $\Psi_n^3$ by homogenisation, the result follows from Proposition 17 for $P$ of the form $(x : y : 1)$. For $P = (0 : 1 : 0)$, the claim follows from Proposition 19.

1.3 Statement of the main theorem

Let $R$ be a ring. Given $W$ in $R[x, y, z]$ of the form [1], there is an obvious ring morphism $A \to R$ which maps $a_i \in A$ to the corresponding coefficient of $W$ in $R$. We will view $R$ as an $A$-algebra via this morphism.

Theorem 21. Let $R$ be a ring, and let $C$ be a Weierstrass curve over $R$ defined by $W$ as in [1]. Let $P = [(M, m_0, m_1, m_2)] \in \text{Cred}(R)$. Then

$$nP = \left((M^\otimes n^2, a_n(m_0, m_1, m_2), \beta_n(m_0, m_1, m_2), \gamma_n(m_0, m_1, m_2))\right)$$

In Section 2, we will state and prove the scheme-theoretic equivalent of this theorem.

If $M = R$, then $M^\otimes n^2 = R$, so if a point $P$ is of the form $(x : y : z)$, then $nP$ is also of this form, namely,

$$nP = (a_n(x, y, z) : \beta_n(x, y, z) : \gamma_n(x, y, z))$$

However, even for a smooth Weierstrass curve $C$, the subset of $R$-valued points of the form $(x : y : z)$ need not be closed under addition, as we will see in Example 23.

As $a_n, \beta_n, \gamma_n$ were obtained from $\Phi_n, \Psi_n, \Omega_n, \Psi_n^3$ by homogenisation, we then have the following immediate consequence of Theorem 21.
Corollary 22. Let \( n \in \mathbb{Z} \), and let \( C \) be a Weierstrass curve over a ring \( R \). Then for all \( P \in C^{\text{sm}}(R) \) of the form \((x : y : 1)\), we have
\[
np = (\Phi_n(x,y)\Psi_n(x,y) : \Omega_n(x,y) : \Psi^3_n(x,y)).
\]

Example 23. We give an example of a Weierstrass curve \( C \) over a ring \( R \) and two \( R \)-valued points \( P, Q \in C(R) \) of the form \((x : y : z)\) such that its sum \( P + Q \in C(R) \) is not of the form \((x : y : z)\).

Let \( R = \mathbb{Z}[\sqrt{-5}] \), let \( K = Q(\sqrt{-5}) \), and consider the smooth Weierstrass curve \( C \) over \( R \) given by
\[
W = y^2z + xyz + yz^2 - x^3 - 4xz^2 + 6z^3,
\]
and the two points
\[
P = (9 : 23 : 1) \quad Q = (3411\sqrt{-5} : 26488 + 117\sqrt{-5} : -3645\sqrt{-5})
\]
in \( C(R) \). Note that \( P \) and \( Q \) define points of the form \((x : y : z)\) in \( C(R) \) as well, as the \( R \)-submodule of \( K \) generated by their coordinates are trivial, but that the \( R \)-submodule of \( K \) generated by the coordinates of \( P + Q \) is invertible, but not trivial, so \( P + Q \) is not of the form \((x : y : z)\) in \( C(R) \).

2 Proof of the main theorem

Here is the scheme-theoretic equivalent of Theorem 21.

Theorem 24. Let \( S \) be a scheme, and let \( C \) be a Weierstrass curve over \( S \) defined by \( W \) as in (1). Let \( T \) be an \( S \)-scheme, and let \( P = ([\mathcal{L}, s_0, s_1, s_2]) \in C^{\text{sm}}(T) \). Then
\[
np = \left[ (\mathcal{L} \otimes \mathcal{O}_S^n, \alpha_n(s_0, s_1, s_2), \beta_n(s_0, s_1, s_2), \gamma_n(s_0, s_1, s_2)) \right].
\]

We will first prove this for smooth Weierstrass curves, and we will do this by first using the Theorem of the Cube, by considering a universal point on a universal smooth Weierstrass curve.

2.1 The universal smooth Weierstrass curve and the proof in the smooth case

Definition 25. The universal smooth Weierstrass curve \( E \) is the Weierstrass curve over \( \text{Spec} \, \mathbb{R} \) (where \( \mathbb{R} = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta] \)) defined by the homogeneous polynomial
\[
W = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3 \in \mathbb{R}[x, y, z].
\]
The universal point \( P \) on \( E \) is the point in \( E(\mathbb{R}) \) corresponding to the identity map on \( \mathbb{R} \).

Note here that \( P = \left[ (\mathcal{O}_E(1), x, y, z) \right] \), and that it is universal in the sense that any morphism \( S \to E \) of \( \mathbb{R} \)-schemes factors through the identity on \( E \). Moreover, \( E \) is universal in the following sense.

Proposition 26. Let \( E \) be a smooth Weierstrass curve over \( S \). Then there exist unique morphisms \( g : E \to E \) and \( h : S \to \text{Spec} \, \mathbb{R} \) such the following diagram is commutative, and such that the outer square is Cartesian.

\[
\begin{array}{ccc}
E & \longrightarrow & \mathbb{P}^2_E \\
\downarrow g & & \downarrow h \\
E & \longrightarrow & \mathbb{P}^2_{\text{Spec} \, \mathbb{R}} \\
\end{array}
\]

Proof. Mimic the proof of Proposition 10. \( \square \)
We want to consider the multiples of $\mathcal{P}$. Note that for $n \in \mathbb{Z}$, the point $n\mathcal{P}$ is the point corresponding to the multiplication-by-$n$ map $\mu_n$ on $\mathcal{E}$. The following is a special case of the Theorem of the Cube.

**Theorem 27** ([6], Thm. IV.3.3). Let $n_1, n_2, n_3 \in \mathbb{Z}$. Then there is a canonical isomorphism of $\mathcal{O}_{\mathcal{E}}$-modules

$$\bigotimes_{i \in \{1,2,3\}} \mu_{n_i}^* \mathcal{O}_{\mathcal{E}}(1)^{(-1)^{n_i}} = \mathcal{O}_{\mathcal{E}}.$$

**Proposition 28.** Let $n \in \mathbb{Z}$. Then $\mu_n^* \mathcal{O}_{\mathcal{E}}(1) \cong \mathcal{O}_{\mathcal{E}}(n^2)$.

**Proof.** As $\mathcal{E}$ is a smooth Weierstrass curve, we have $\mathcal{O}_{\mathcal{E}}(1) = \mathbb{I}^{-3}(0)$; the identity on $\mathcal{E}$ can be given by $[(\mathcal{O}_{\mathcal{E}}(1), x, y, z)]$, as well as $[(\mathbb{I}^{-3}(0), X, Y, 1)]$, where the latter is in the notation of [6, Sect. 2.2]. As $\mu^{-1}$ fixes the zero section, we deduce that $\mu_{n}^* \mathcal{O}_{\mathcal{E}}(1) \cong \mathcal{O}_{\mathcal{E}}(1)$. Hence it suffices to show that for all positive $n \in \mathbb{Z}$, we have $\mu_n^* \mathcal{O}_{\mathcal{E}}(1) \cong \mathcal{O}_{\mathcal{E}}(n^2)$.

For $n = 2$, we apply [Theorem 27] with $n_1 = n_2 = 1$, $n_3 = -1$ to see that $\mu_2^* \mathcal{O}_{\mathcal{E}}(1) \cong \mathcal{O}_{\mathcal{E}}(4)$. For $n > 2$, our claim follows by induction, by applying [Theorem 27] with $n_1 = n - 2$, and $n_2 = n_3 = 1$.

As a consequence, we have the following important observation.

**Corollary 29.** Let $n \in \mathbb{Z}$. Then there exist homogeneous elements $a'_n$, $b'_n$, $\gamma'_n$ in $\mathcal{R}[x, y, z]/(W)$ of degree $n^2$ such that

$$[(\mathcal{O}_{\mathcal{E}}(n^2), a'_n, b'_n, \gamma'_n)] \in \mathcal{E}(\mathcal{E})$$

is the point corresponding to $\mu_n$. These elements are unique up to a common unit in $\mathcal{R}$.

**Proof.** We simply note that the global sections of $\mathcal{O}_{\mathcal{E}}(n^2)$ are the homogeneous elements of degree $n^2$ in $\mathcal{R}[x, y, z]/(W)$, and that $\text{Aut}_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}}(n^2)) = \mathcal{O}_{\mathcal{E}}(n^2)^* = \mathcal{R}^\times$.

Translating this observation in terms of the functor of points of $\mathcal{E}$ using the universality of $\mathcal{P}$ then gives the following.

**Corollary 30.** Let $n \in \mathbb{Z}$. Let $T$ be a scheme over $\mathcal{R}$, and let $P = [(\mathcal{L}, s_0, s_1, s_2)] \in \mathcal{E}(T)$. Then we have

$$nP = [(\mathcal{L} \otimes n^2, a'_n(s_0, s_1, s_2), b'_n(s_0, s_1, s_2), \gamma'_n(s_0, s_1, s_2))].$$

Finally, using the universality of $\mathcal{E}$, we obtain the following.

**Corollary 31.** Let $n \in \mathbb{Z}$. Let $S$ be a scheme, and let $E$ be a smooth Weierstrass curve over $S$ defined by $W$ as in [11]. Let $T$ be an $S$-scheme, and let $P = [(\mathcal{L}, s_0, s_1, s_2)] \in \mathcal{E}(T)$. Then we have

$$nP = [(\mathcal{L} \otimes n^2, a'_n(s_0, s_1, s_2), b'_n(s_0, s_1, s_2), \gamma'_n(s_0, s_1, s_2))].$$

We now finish the proof of [Theorem 24] in the smooth case.

**Proof of Theorem 24 (smooth case).** We will now show that up to a common unit in $\mathcal{R}$, the triples $(\alpha_n, \beta_n, \gamma_n)$ and $(\alpha'_n, \beta'_n, \gamma'_n)$ are equal. First observe that we have $\beta_0, \beta'_0 \in \mathcal{R}^\times, \alpha'_0 = 0 = \alpha_0$, and $\gamma'_0 = 0 = \gamma_0$. Hence we may assume that $n \neq 0$.

Consider the smooth Weierstrass curve $E$ over $K$ as defined in [Section 1.2]. Note that it now suffices to show that $\alpha_n/\alpha'_n = \beta_n/\beta'_n = \gamma_n/\gamma'_n$ as rational functions in $K(E)$, and that this is a unit in $\mathcal{R}$, i.e. of the form $\pm \alpha_i$ for some integer $i$.

Observe that for all $P = (x : y : 1) \in E(K) - 0$, we have, by [Proposition 17] and [Corollary 31]

$$(\alpha'_n(P) : \beta'_n(P) : \gamma'_n(P)) = nP = (\Phi_n(P) \Psi_n(P) : \Omega_n(P) : \Psi_n^3(P))$$

for $n > 0$ and

$$(\alpha_n(P) : \beta_n(P) : \gamma_n(P)) = nP = (\Phi'_n(P) \Psi'_n(P) : \Omega'_n(P) : \Psi'_n^3(P))$$

for $n < 0$.
First note that by definition, \( \alpha_n / \gamma_n = \Phi_n / \Psi_n^2 \) and \( \beta_n / \gamma_n = \Omega_n / \Psi_n^3 \). Hence \( \alpha_n / \gamma_n = \alpha'_n / \gamma'_n \) and \( \beta_n / \gamma_n = \beta'_n / \gamma'_n \), as rational functions in \( K(E) \). As \( \mu_n \) is surjective, it follows that \( \text{im} \mu_n \) is infinite, hence none of \( \alpha'_n, \beta'_n, \gamma'_n \) are identically zero. So \( \alpha_n / \alpha'_n = \beta_n / \beta'_n = \gamma_n / \gamma'_n \) in \( K(E) \). (These are actually rational functions, as both the numerators and denominators are homogeneous of degree \( n^2 \).)

Let \( \theta_n = \beta_n / \beta'_n \). Then we need to prove that \( \theta_n \) is in fact a unit in \( \mathcal{R} \).

First note that for all \( P \in E(K) \), we have \((\alpha'_n(P), \beta'_n(P), \gamma'_n(P)) \neq (0, 0, 0) \). Hence \( \theta_n \) has no poles on \( E \). It follows that \( \theta_n \) has no zeroes on \( E \) either, hence \( \theta_n \) is constant, i.e. \( \theta_n \in K^\times \).

As the homogeneous Weierstrass polynomial \( W \) is monic in \( x \), it follows that \( \mathcal{R}[x, y, z] / (W)n^2 \) is a free \( \mathcal{R} \)-module, with a basis consisting of monomials. Since \( \theta_n \) is by definition a quotient of two elements of \( \mathcal{R}[x, y, z] / (W)n^2 \), which is moreover a constant, it follows that \( \theta_n \) is in the field of fractions of \( \mathcal{R} \).

Now observe that \( \mathcal{R} \) is a unique factorisation domain. Hence we can write \( \theta_n = f / g \) with \( f, g \in \mathcal{R} \) in such a way that \( f \) and \( g \) have no common factors. Then \( f / g = g / \beta_n \) in \( \mathcal{R}[x, y, z] / (W) \).

Thus \( g \) divides all coefficients of \( \beta_n \). Now note that for the point \( P = (0 : 1 : 0) \), we have \( (\alpha'_n(P) : \beta'_n(P) : \gamma'_n(P)) = (0 : 1 : 0) \), hence the \( y^n \)-coefficient of \( \beta'_n \) is a unit. Therefore, \( g \) is a unit as well, which implies that \( \theta_n \in \mathcal{R} \).

Finally, note that \( \theta_n \) divides all coefficients of \( \beta_n \). But because \( \Delta \Omega_n \neq 1 \) and \( \text{ord}_\mathfrak{p} \Omega_n = -3n^2 \), the coefficient of \( y^n \) in \( \beta_n \) must also be 1, as \( \beta_n \) is the homogenisation of a representative of \( \Omega_n \).

Hence \( \theta_n \in \mathcal{R}^\times \), which finishes the proof.

2.2 The universal Weierstrass curve and the proof in the general case

As in the case of smooth Weierstrass curves, there exists a universal Weierstrass curve \( \mathcal{C} \) (as defined in \[Section 1\]) and a universal point \( \mathcal{P} = [(O_{\mathcal{C}sm}(1), x, y, z)] \in \mathcal{C}^\text{sm}(\mathcal{C}^\text{sm}) \), corresponding to the identity map on \( \mathcal{C}^\text{sm} \). Moreover, the universal smooth Weierstrass curve \( \mathcal{E} \) is the complement of the closed subscheme of \( \mathcal{C}^\text{sm} \) defined by the elliptic discriminant \( \Delta \).

**Proposition 32.** Let \( n \in \mathbb{Z} \). Then we have, in \( \mathcal{C}^\text{sm}(\mathcal{C}^\text{sm}) \),

\[ n\mathcal{P} = [(O_{\mathcal{C}sm}(n^2), \alpha_n, \beta_n, \gamma_n)]. \]

**Proof.** Write \( n\mathcal{P} = [(\mathcal{L}, s_0, s_1, s_2)] \in \mathcal{C}^\text{sm}(\mathcal{C}^\text{sm}) \). By \[Proposition 28\] and \[Corollary 1\], we have \( \mathcal{L}|_\mathfrak{e} \cong O_{\mathcal{C}sm}(n^2)|_\mathfrak{e} \). Note that \( \mathcal{C}^\text{sm} \) is noetherian integral separated regular, therefore the Weil divisor class group is naturally isomorphic to the Picard group.

Note moreover that \( \Delta \) is irreducible and that \( \text{div} \mathcal{E} = V(\Delta) \), so \( V(\Delta) \) is a principal prime divisor. As \( \mathcal{E} = \mathcal{C}^\text{sm} - V(\Delta) \), it follows that the natural map \( \text{Pic} \mathcal{C}^\text{sm} \to \text{Pic} \mathcal{E} \) is an isomorphism. Hence \( \mathcal{L} \cong O_{\mathcal{C}sm}(n^2) \), so we assume without loss of generality that \( \mathcal{L} = O_{\mathcal{C}sm}(n^2) \).

Next, we note that by the smooth case of \[Theorem 24\] we have

\[ u \cdot (s_0|_\mathfrak{e}, s_1|_\mathfrak{e}, s_2|_\mathfrak{e}) = (\alpha_n|_\mathfrak{e}, \beta_n|_\mathfrak{e}, \gamma_n|_\mathfrak{e}) \]

for a unique unit \( u \) of \( \mathcal{R} = \mathcal{A}_\Delta \). As \( \mathcal{E} \) is dense in \( \mathcal{C}^\text{sm} \), we deduce that \( u \cdot (s_0, s_1, s_2) = (\alpha_n, \beta_n, \gamma_n) \).

Note that the units of \( \mathcal{R} \) are \( \pm \Delta^i \) with \( i \in \mathbb{Z} \). If we pull back both triples via the zero section (i.e. we evaluate all of the sections at \( (0 : 1 : 0) \)), we see that both \( u s_0(0, 1, 0) \) and \( u s_2(0, 1, 0) \) are zero, and \( u s_1(0, 1, 0) = 1 \), as global sections of \( O_{\mathcal{A}_\Delta} \). i.e. elements of \( \mathcal{A} \). Hence we have \( s_0(0, 1, 0) = s_2(0, 1, 0) = 0 \), so as \( s_0, s_1, s_2 \) generate \( \mathcal{L} \), we deduce that \( s_1(0, 1, 0) \) is a unit in \( \mathcal{A} \). So \( u \) is a unit in \( \mathcal{A} \) as well, i.e. equal to 1 or \(-1\). \[\square\]

\[Theorem 24\] follows from this by the universality of \( \mathcal{C} \) and \( \mathcal{P} \). Moreover, as the polynomials \( \alpha_n, \beta_n, \gamma_n \) were defined as homogenisations of \( \Phi_n, \Psi_n, \Omega_n, \Psi_n^3 \), it immediately follows that we have the following scheme-theoretic equivalent of \[Corollary 22\].
Corollary 33. Let \( n \in \mathbb{Z} \). Let \( S \) be a scheme, and let \( C \) be a Weierstrass curve over \( S \) defined by \( W \) as in \( \text{[1]} \). Let \( T \) be an \( S \)-scheme, and let \( P \in C(T) \) be of the form \((x : y : 1)\), such that \( a_1y - 3x^2 - 2a_2x - a_4, 2y + a_1x + a_3 \) generate \( \mathcal{O}_T \). Then we have
\[
nP = (\Phi_n(x, y) \Psi_n(x, y) : \Omega_n(x, y) : \Psi_n^3(x, y)).
\]

3 Application to a moduli problem
In this section, we use Corollary 33 to give an explicit description of \( Y_1(n) \) over \( \mathbb{Z}[1/n] \). These turn out to coincide with the descriptions of \( Y_1(n)_C \) given for example by \([1]\).

3.1 Definitions
Let \( S \) be a scheme. By Proposition 36 all smooth Weierstrass curves over a scheme \( S \) (together with the section \((0 : 1 : 0)\)) are also elliptic curves over \( S \). Moreover, we have the following partial converse, treated for example in \([5, \text{Sect. 2.2}]\).

Theorem 34. Let \( E \) be an elliptic curve over \( S \). Then Zariski locally on the base, \( E \) is isomorphic to a smooth Weierstrass curve.

By Theorem 8 any elliptic curve admits a unique commutative group scheme structure that has the given zero section as neutral element.

We now make the following definition.

Definition 35. Let \( E \) be an elliptic curve over \( S \), and let \( n \) be a positive integer. A \( \mathbb{Z}/n\mathbb{Z} \)-embedding into \( E \) is a section \( P \in E(S) \) such that \(nP = 0 \), and such that \( P \) has order \( n \) in every geometric fibre of \( E \) over \( S \).

In \([5, \text{Sect. 1.4}]\), this is called an “étalement of exact order \( n \)” (which is the same as a “point of exact order \( n \)” if \( n \) is invertible in \( S \)). This condition on the point \( P \) is equivalent to the resulting homomorphism \((\mathbb{Z}/n\mathbb{Z})_S \to E\) being a closed immersion, and if \( S \) is the spectrum of a field in which \( n \) is invertible, then it is even equivalent to being a point of order \( n \) in \( E(S) \).

Following \([5]\), we define the category \( \text{Ell} \) of elliptic curves as the category in which the objects are pairs \((E, S)\) with \( E \) an elliptic curve over \( S \), and in which \( \text{Hom}_{\text{Ell}}((E, S), (E', S')) \) is the set of pairs \((a, s)\) of morphisms \( a: E \to E' \) and \( s: S \to S' \) making the diagram
\[
\begin{array}{ccc}
E & \xrightarrow{a} & E' \\
\downarrow & & \downarrow \\
S & \xrightarrow{s} & S'
\end{array}
\]
Cartesian, which thereby induce an isomorphism \( E \to E' \times_{S'} S \).

Let \( n \) be a positive integer. Then let \( \mathcal{P}(n) \) be the naive \( \Gamma_1(n) \) moduli problem \( \text{Ell} \to \) Set attaching to a pair \((E, S)\), the set of \( \mathbb{Z}/n\mathbb{Z} \)-embeddings into the elliptic curve \( E \) over \( S \). (We call it naive since we consider, in the terminology of \([5, \text{Sect. 1.4}]\), “étale points of exact order \( n \)” instead of “points of exact order \( n \)”.) In the remainder of this section, we will study this moduli problem for \( n \geq 4 \).

3.2 \( \mathbb{Z}/n\mathbb{Z} \)-embeddings and classical division polynomials
In this section, we state some divisibility properties of division polynomials, and their implications for \( \mathbb{Z}/n\mathbb{Z} \)-embeddings. The notation is as in Section 1.2. Moreover, recall that \( \mathcal{R} = \mathcal{A}_\Delta \).

Proposition 36. Let \( n \in \mathbb{Z} - \{0\} \) be a non-zero integer. Then \( \Phi_n \) and \( \Psi_n \) generate \( \mathcal{R}[X, Y]/(\mathcal{W}') \) as an ideal.
Proof. Let \( R = \mathcal{R}[X,Y]/(W', \Phi_n) \). We show that \( \Psi_n \) is invertible in this ring, or equivalently, that \( \Psi_n \) does not lie in any maximal ideal of \( R \). So let \( m \) be a maximal ideal of \( R \), and let \( k \) be an algebraic closure of \( R/m \). Let \( E \) be the smooth Weierstrass curve corresponding to the canonical morphism \( \mathcal{R} \to k \), and let \( P = (X : Y : 1) \in E(k) \). Note that the image of any \( f \in \mathcal{R}[X,Y]/(W') \) in \( k \) is \( f(P) \).

Suppose for a contradiction that \( \Psi_n(P) = 0 \). As \( X\Psi_n(P)^2 - \Psi_{n+1}(P)\Psi_n(P) = \Phi_n(P) = 0 \), we see that either \( \Psi_{n-1}(P) = 0 \) or \( \Psi_{n+1}(P) = 0 \). Hence by Corollary 33, we have \( nP = 0 \) and either \( (n-1)P = 0 \) or \( (n+1)P = 0 \), which implies that \( P = 0 \). This is a contradiction. We deduce that \( \Psi_n \) does not lie in any maximal ideal of \( R \), as desired. \( \square \)

Using the formula in Corollary 33, we immediately deduce the following.

**Corollary 37.** Let \( S \) be a scheme, let \( E \) be a smooth Weierstrass curve over \( S \). Moreover, let \( P \in E(S) \) be of the form \( (a : b : 1) \) and let \( n \) be an integer. Then \( nP \) is non-zero in every geometric fibre of \( E \) over \( S \), then \( nP \) is non-zero in every geometric fibre of \( E \) over \( S \), and let \( n \) be an integer. Then \( nP \) is non-zero in every geometric fibre if and only if \( P \) is of the form \( (a : b : 1) \) with \( a, b \in O_S(S) \), and \( \Psi_n(P) \in O_S(S)^{\times} \).

This gives us an explicit way of expressing when a section \( P \) of a smooth Weierstrass curve \( E \) defines a \( \mathbb{Z}/n\mathbb{Z} \)-embedding.

**Corollary 38.** Let \( S \) be a scheme, let \( E \) be a smooth Weierstrass curve over \( S \). Moreover, let \( P \in E(S) \) and let \( n \) be an integer. Then \( nP \) is non-zero in every geometric fibre if and only if \( P \) is of the form \( (a : b : 1) \) with \( a, b \in O_S(S) \), and \( \Psi_n(P) \in O_S(S)^\times \).

If we invert \( n \) in \( R \), we can show more. First, we show the following.

**Lemma 40.** Let \( n \in \mathbb{Z} - \{0\} \), and let \( d \) be a divisor of \( n \). Then \( \Psi_d | \Psi_n \) in \( \mathcal{R}[X,Y]/(W') \).

Proof. Let \( R = \mathcal{R}[X,Y]/(W', \Psi_d) \), and let \( E \) be the smooth Weierstrass curve corresponding to the canonical morphism \( \mathcal{R} \to R \). Let \( P = (X : Y : 1) \in E(R) \). Again, note that the image of any \( f \in \mathcal{R}[X,Y]/(W') \) in \( k \) is \( f(P) \). By Corollary 37, we now see that \( dP = 0 \) in \( E(R) \), hence \( nP = 0 \), so \( \Psi_n = 0 \) in \( R \). We deduce that \( \Psi_d \mid \Psi_n \) in \( \mathcal{R}[X,Y]/(W') \). \( \square \)

In the same way as in Proposition 36, we now prove the following proposition.

**Proposition 41.** Let \( n \in \mathbb{Z} - \{0\} \), and let \( d \) be a divisor of \( n \). Then \( \Psi_d \) and \( \Psi_n/\Psi_d \) generate \( \mathcal{R}[1/n, X, Y]/(W') \) as an ideal.

Proof. Let \( R = \mathcal{R}[1/n, X, Y]/(W', \Psi_d) \). We show that \( \Psi_n/\Psi_d \) is invertible in \( R \), or equivalently, that \( \Psi_n/\Psi_d \) does not lie in any maximal ideal of \( R \). So let \( m \) be a maximal ideal of \( R \), and let \( k \) be an algebraic closure of \( R/m \). Let \( E \) be the smooth Weierstrass curve over \( k \) corresponding to the canonical morphism \( \mathcal{R} \to k \), and let \( P = (X : Y : 1) \in E(k) \).

Note that \( \text{div} \, \Psi_i = \sum_{P \in E}|i| - (i^2 - 1)(0) \) for all non-zero integers \( i \) invertible in \( k \), so \( \text{div} \, \Psi_n/\Psi_d = \sum_{P \in E}|n| - (n^2 - d^2)(0) \). As \( \Psi_d(P) = 0 \), it now follows that \( (\Psi_n/\Psi_d)(P) \neq 0 \) in \( k \). We deduce that \( \Psi_n/\Psi_d \) is not in any maximal ideal of \( R \), as desired. \( \square \)

If we then define the elements \( F_n \in \mathcal{R}[X,Y]/(W') \) recursively by

\[
F_n = \begin{cases} 
1 & \text{if } n = 1 \\
\Psi_n \prod_{d|n, d < n} F_d^{-1} & \text{if } n \geq 2
\end{cases}
\]

then we have the following consequence of Corollary 39.
Corollary 42. Let \( S \) be a scheme, let \( E \) be a Weierstrass curve over \( S \). Moreover, let \( P \in E(S) \) and let \( n \) be a non-zero integer that is invertible in \( S \). Then \( P \) is a \( \mathbb{Z}/n\mathbb{Z} \)-embedding if and only if \( P \) is of the form \( (a : b : 1) \) with \( a, b \in \mathcal{O}_S(S) \) and \( F_n(P) = 0 \).

3.3 The Tate normal form and representability of \( \mathcal{P}(n) \)

We recall the following well-known theorem, and its proof.

Theorem 43 (Tate normal form). Let \( S \) be a scheme, let \( E \) be an elliptic curve over \( S \), and let \( P \in E(S) \) be such that \( 2P \neq 0 \) and \( 3P \neq 0 \) in every geometric fibre of \( E \) over \( S \). (For example, if \( P \) is a \( \mathbb{Z}/n\mathbb{Z} \)-embedding with \( n \geq 4 \).) Then there exist unique \( s \in \mathcal{O}_S(S), t \in \mathcal{O}_S(S)^\times \) such that there exists a unique \( S \)-isomorphism from \( E \) over \( S \) to the smooth Weierstrass curve \( C \) given by the Weierstrass polynomial

\[
y^2z + (1 + s)xyz + tyz^2 - x^3 - tx^2z,
\]

identifying \( P \in E(S) \) and \( (0 : 0 : 1) \in C(S) \).

Proof. We work affine locally on \( S \). Since we require uniqueness of \( s \) and \( t \) and of the isomorphism, we can glue the obtained local isomorphisms afterwards to get the desired isomorphism. Hence we may assume that \( E/S \) is in fact a smooth Weierstrass curve, and that \( S = \text{Spec} R \) for some ring \( R \).

For the existence, suppose that \( E \) is defined by the Weierstrass equation

\[
y^2z + a_1xyz + a_3y^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3.
\]

Then note that \( P \neq 0 \) on all geometric fibres, so \( P \) is of the form \( (x : y : 1) \). So after a suitable translation, we may assume that \( P = (0 : 0 : 1) \) and hence \( a_6 = 0 \). As \( 2P \neq 0 \) on all geometric fibres, it then follows that \( a_3 \in R^\times \). Hence after composing with a suitable shearing parallel to the line \( y = 0 \), we may assume that \( a_4 \) is zero. Finally, as \( 3P \neq 0 \) on all geometric fibres, we have \( a_2 \in R^\times \), hence after a suitable scaling, we find an isomorphism between \( E \) and a Weierstrass curve \( C \) over \( S \) of the desired form.

For uniqueness, note that by [5, Sect. 2.2], any isomorphism between Weierstrass curves over \( S \) corresponds to

\[
(O(1), a^2x + az, a^3y + bx + cz, z) \in \mathbb{P}_S(\mathbb{P}_S^2)
\]

on the ambient \( \mathbb{P}_S^2 \) with \( a \in R^\times \), \( b, c \in R \). Let \( s, s', t, t' \in R^\times \), and let \( C, C' \) be the Weierstrass curves over \( S \) defined by the polynomials

\[
y^2z + (1 + s)xyz + tyz^2 - x^3 - tx^2z \quad \text{and} \quad y^2z + (1 + s')xyz + t'yz^2 - x^3 - t'x^2z,
\]

respectively. Suppose that \( i: C \to C' \) is an isomorphism sending \( (0 : 0 : 1) \) to \( (0 : 0 : 1) \) and that it is given on the ambient \( \mathbb{P}_S^2 \) by

\[
(O(1), a^2x + az, a^3y + bx + cz, z) \in \mathbb{P}_S(\mathbb{P}_S^2)
\]

for \( a \in \mathcal{O}_S(S)^\times \), \( b, c \in \mathcal{O}_S(S) \). Then \( a = c = 0 \) as \( i(0 : 0 : 1) = (0 : 0 : 1) \), \( a = 1 \) as the \( yz^2 \) - and \( x^2z \) -coefficients sum to 0, and \( b = 0 \) as the \( xz^2 \) -coefficient is zero. Hence \( s = s', t = t' \), and \( i \) is the identity, which shows the uniqueness. \( \square \)

In order to state the corollaries, we introduce some more notation. Let \( \delta = \Delta(1+s,t,t,0,0) \in S \). Moreover, for any integer \( n \), let \( \psi_n = \Psi_n(0,0) \in \mathbb{Z}[s,t] \) and \( f_n = F_n(0,0) \in \mathbb{Z}[s,t] \). Now we have the following consequences of Corollary 39 and Corollary 42.

Corollary 44. Let \( n \) be a positive integer. The moduli problem \( \mathcal{P}(n) \) is representable (over \( \mathbb{Z} \)) by

\[
\text{Spec } \mathbb{Z}[s,t,\delta^{-1},p_n^{-1}]/(\psi_n).
\]
Here, \( p_n = \prod_{d|n, 0 < d < n} \psi_d. \) The universal elliptic curve over \( Y_1(n) \) is the Weierstrass curve given by the Weierstrass polynomial

\[
y^2z + sxyz + tyz^2 - x^3 - tx^2z,
\]
together with the point \( P = (0 : 0 : 1) \).

**Corollary 45.** Let \( n \) be a positive integer. The moduli problem \( \mathcal{P}(n) \) is representable over \( \mathbb{Z}[1/n] \) by

\[
Y_1(n)_{\mathbb{Z}[1/n]} = \text{Spec} \mathbb{Z}[s, t, \delta^{-1}, 1_n] / (f_n).
\]

The universal elliptic curve over \( Y_1(n) \) is the Weierstrass curve given by the Weierstrass polynomial

\[
y^2z + sxyz + tyz^2 - x^3 - tx^2z,
\]
together with the point \( P = (0 : 0 : 1) \).

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