A new cubature formula for weight functions on the disc, with error estimates

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Abstract

We introduce a new type of cubature formula for the evaluation of an integral over the disk with respect to a weight function. The method is based on an analysis of the Fourier series of the weight function and a reduction of the bivariate integral into an infinite sum of univariate integrals. Several experimental results show that the accuracy of the method is superior to standard cubature formula on the disk. Error estimates provide the theoretical basis for the good performance of the new algorithm.

1 Introduction

Recently, methods for the numerical evaluation of integrals of the form

$$I_1 (g) = \int_D g(x) \, dx = \int_0^{2\pi} \int_0^R g(r \cos \varphi, r \sin \varphi) \, r \, dr \, d\varphi$$

(1)

on the disc $D_R$ of radius $R$ in the plane $\mathbb{R}^2$ have received increased attention in the framework of the meshless local Petrov-Galerkin (MLPG) method, see [18], [19], [35], [36]. Numerical experiments in [19] have given evidence that classical rules like the piecewise midpoint quadrature rule, or the rule of Peirce (for definitions see below (39) and (40)) are superior to the Gauss-Legendre product rule which is very popular in the MLPG literature.

In the present paper we study new methods for the numerical evaluation of integrals of the type

$$I_w (f) = \int_{D_R} f(x) \, w(x) \, dx$$

(2)

where $w(x)$ is a (not necessarily non-negative) weight function on the disc $D_R$ in the plane $\mathbb{R}^2$. The introduction of a weight function is an important concept in numerical integration: the integrand $g(x)$ is decomposed into a product $f(x) \, w(x)$ where the factor $f(x)$ is a function well-approximable by polynomials (see section 3.7 in [16]) and $w(x)$ is a function of limited smoothness or with a singularity. Using the specific properties of the weight function $w(x)$ one aims to achieve a cubature formula for the integration of the function $f$ with respect to $w(x) \, dx$ which should be more accurate than using directly a cubature formulae for $g = f \cdot w$ like in (1).

The main concept underlying our method is to expand the weight function $w(x) = w(re^{i\varphi})$ into a Fourier series (4) and rely upon a similar expansion for the polynomial-like function $f(x)$, called the Almansi expansion, see (9). The reader will find an explicit description of this construction below, after all necessary notations and tools are introduced. Illustrating examples in this paper are the weight functions

$$w^{(1)} (x, y) = \frac{1 + x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad w^{(2)} (x, y) = |y|$$

(3)
where the first weight function has a singularity in 0 and the second weight function is continuous
on the closed ball but is not differentiable on the line $y = 0$ in the interior of the disk. We shall show
by numerical experiments, and by theoretical considerations as well, that our method provides in
these cases results which are better than the above-mentioned methods for the approximation of
the integral for the function $q(x) = f(x)w(x)$.

Let us now give a detailed introduction to the main topic of the present paper, the numerical
evaluation of the integral with respect to a weight function $w$. A central role in our approach plays
the Fourier series of $w$, given by

$$
w(re^{i\varphi}) = w(r \cos \varphi, r \sin \varphi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} w(k,\ell)(r) Y_{k,\ell}(\varphi). \quad (4)$$

Here we use a notation for the Fourier series which is more convenient in our context, and which is
well known from the theory of spherical harmonics (these convenient notations will be important
also for further multivariate generalizations, as in [30]): we work with the orthonormalization
of the harmonics $\cos k\varphi$ and $\sin k\varphi$, defined by

$$
Y_{0,1}(\varphi) = 1/\sqrt{2\pi} \quad (5)
$$

$$
Y_{k,1}(\varphi) = 1/\sqrt{\pi} \cos k\varphi \quad \text{and} \quad Y_{k,2}(\varphi) = 1/\sqrt{\pi} \sin k\varphi. \quad (6)
$$

for integers $k \geq 1$. Then $Y_{k,\ell}$ is an orthonormal system for $k \geq 0$, $\ell = 1, \ldots, a_k$, where $a_k = 2$
for $k \geq 1$, and $a_0 = 1$. The $(k, \ell)$-th Fourier coefficient of a complex-valued continuous function
$f(re^{i\varphi})$ is

$$
f_{k,\ell}(r) := \int_0^{2\pi} f(re^{i\varphi}) Y_{k,\ell}(\varphi) \, \text{d}\varphi \quad \text{for } k \geq 0, \ell = 1, \ldots, a_k \quad (7)
$$

and the corresponding Fourier series of $f$ is

$$
f(re^{i\varphi}) = f(r \cos \varphi, r \sin \varphi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell}(r) Y_{k,\ell}(\varphi). \quad (8)
$$

Let us recall that the Fourier series of a polynomial $p(x, y)$ is of a very special form: there exist
polynomials $\tilde{p}_{k,\ell}$ and a number $N \leq \deg p(x, y)$ such that

$$
p(x, y) = p(r \cos \varphi, r \sin \varphi) = \sum_{k=0}^{N} \sum_{\ell=1}^{a_k} \tilde{p}_{k,\ell}(r^2) r^k Y_{k,\ell}(\varphi); \quad (9)
$$

the representation [9] is called Gauss decomposition or Almansi expansion of a polynomial $p$. Hence, the Fourier coefficient $p_{k,\ell}(r)$ of a polynomial $p(x, y)$ is of the form

$$
p_{k,\ell}(r) = \tilde{p}_{k,\ell}(r^2) r^k. \quad (10)
$$

Moreover the degrees of all $\tilde{p}_{k,\ell}$ are bounded by $N - 1$ if and only if the polynomial $p(x, y)$ is
polyharmonic of order $N$, i.e. if $\Delta^N p(x, y) = 0$, where $\Delta^N$ is the $N$-th iterate of the Laplace
operator $\Delta$. These results are given a thorough treatment in see [27], [28], [44].

We consider now the integral [2], which after introducing polar coordinates, becomes

$$
I_w(f) = \int_0^{2\pi} \int_0^R f(r \cos \varphi, r \sin \varphi) \cdot w(r \cos \varphi, r \sin \varphi) \cdot \text{d}r \text{d}\varphi.
$$

We replace the weight function $w(re^{i\varphi})$ by its Fourier series, and after interchanging summation
and integration we obtain

$$
I_w(f) = \int_0^{2\pi} \int_0^R f(re^{i\varphi}) w(re^{i\varphi}) \text{d}r \text{d}\varphi = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^R f_{k,\ell}(r) w_{k,\ell}(r) \text{d}r. \quad (11)
$$
After a change of the variable quadrature (16), the integral (11) becomes

\[ I_1 (f) = \int_0^{2\pi} \int_0^R f(r \cos \varphi, r \sin \varphi) r dr d\varphi = \sqrt{2\pi} \int_0^R f_{(0,1)} (r) r dr. \tag{12} \]

Formula (11) is central to our approach since it reduces the integration of the bivariate function \( f(re^{i\varphi}) \) with respect to a weight function \( w(re^{i\varphi}) \) to the calculation of an infinite family of univariate integrals with weight functions \( w_{(k,\ell)} (r) \).

Here we come to the most crucial point of our approach: as we assume that the function \( f \) is well-approximable by a polynomial \( p \), we know that \( I_w (f) \) is close to \( I_w (p) \). By the Almansi formula (9) and (10), the one-dimensional integrals in (11) for computing \( I_{(0,1)} \) is well-approximable by a polynomial \( p \).

Hence, for a polynomial \( f \), we infer the existence of the \( \rho \)-point Gauss-Jacobi quadrature to the last integral, with measure \( \rho^{k/2} w_{(k,\ell)} (\sqrt{\rho}) \). For this reason we need the following assumption which will be made throughout the entire paper and which is called the pseudo-definiteness of the weight function:

**General Assumption**: Each Fourier coefficient \( w_{(k,\ell)} (r) \) of the weight function \( w \) does not change the sign over the interval \( (0, R) \), and it is integrable and continuous over \( (0, R) \).

Due to our assumption we infer the existence of the \( N \)-point Gauss-Jacobi quadrature with nodes and coefficients (which are either all positive or all negative)

\[ t_{1,(k,\ell)} < \ldots < t_{N,(k,\ell)} \tag{14} \]
\[ \lambda_{1,(k,\ell)}, \ldots, \lambda_{N,(k,\ell)}. \tag{15} \]

Due to the exactness of the Gauss-Jacobi quadrature for any integer \( 0 \leq s \leq 2N - 1 \) we obtain the equalities

\[ \sum_{j=1}^N \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^s = \frac{1}{2} \int_0^R \rho^s \rho^{k/2} w_{(k,\ell)} (\sqrt{\rho}) d\rho = \int_0^R \rho^{2s} r^k w_{(k,\ell)} (r) r dr. \tag{16} \]

Hence, for a polynomial \( f \), for which \( \deg f_{(k,\ell)} \leq 2N - 1 \) for all \( (k, \ell) \), by using the Gauss-Jacobi quadrature (16), the integral (11) becomes

\[ I_w (f) = \sum_{k=0}^\infty \sum_{\ell=1}^a t_{j,(k,\ell)} \cdot f_{(k,\ell)} (r) w_{(k,\ell)} (r) r dr \tag{17} \]
\[ = \frac{1}{2} \sum_{k=0}^\infty \sum_{\ell=1}^a \int_0^R \rho^s \rho^{k/2} w_{(k,\ell)} (\sqrt{\rho}) d\rho = \int_0^R \rho^{2s} r^k w_{(k,\ell)} (r) r dr, \]
\[ = I_{\text{poly}}^N (f) \]

where we have put

\[ I_{\text{poly}}^N (f) := \frac{1}{2} \sum_{k=0}^\infty \sum_{\ell=1}^a \sum_{j=1}^N \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{-s} \cdot f_{(k,\ell)} (\sqrt{\rho}) \lambda_{N,(k,\ell)}. \tag{18} \]
In [30], we defined $I^\text{poly}_N(f)$ as polyharmonic cubature of degree $N$ in arbitrary space dimension. The reason for the name is the fact that $I^\text{poly}_N$ is exact on the space of all polynomials of polyharmonic order $\leq 2N$, i.e. for each polynomial $f$ such that $\Delta^{2N} f = 0$, or more generally, on the space of smooth functions satisfying the polyharmonic equation $\Delta^{2N} f = 0$.

In previous work [28], [29], [31], we have given a motivation and a detailed analysis of the polyharmonic cubature of degree $2N$ in the framework of the Polyharmonic Paradigm, explaining the natural appearance of the factor $t_{j,(k,\ell)}^{k}$ in our formulae which is related to the Gauss-Almansi decomposition of a polynomial.

As the values $f_{j,(k,\ell)}(\sqrt{t_{j,(k,\ell)}})$ are Fourier coefficients, they can be approximated by means of Discrete Cosine/Sine transform of the function $f$, by which we mean the expression

$$f_{(k,\ell)}^{(M)}(r) := \frac{2\pi}{M} \sum_{s=1}^{M} f \left( re^{\frac{2\pi i s}{M}} \right) Y_{(k,\ell)} \left( \frac{2\pi s}{M} \right)$$

The main contribution of the present paper is the Discrete Polyharmonic Cubature with parameters $(N, M, K)$, defined for integers $N \geq 1, M \geq 1$, and $K \geq 0$, by putting

$$I_{(N,M,K)}^\text{poly}(f) := \frac{1}{2} \sum_{k=0}^{K} \sum_{\ell=1}^{M} \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{k} \cdot f_{(k,\ell)}^{(M)} \left( \sqrt{t_{j,(k,\ell)}} \right)$$

$$= \frac{\pi}{M} \sum_{k=0}^{K} \sum_{\ell=1}^{M} \sum_{j=1}^{N} \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{k} \cdot Y_{(k,\ell)} \left( \frac{2\pi s}{M} \right) \cdot f \left( \sqrt{t_{j,(k,\ell)} e^{2\pi i \ell M}} \right).$$

We see that unlike [18] formula [21] for $I_{(N,M,K)}^\text{poly}(f)$ is indeed a cubature formula in the usual sense of the word. Its coefficients (weights) $\left\{ \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{k} \cdot Y_{(k,\ell)} \left( \frac{2\pi s}{M} \right) \right\}$ have varying signs but they satisfy the following remarkable inequality

$$\frac{\pi}{M} \sum_{k=0}^{K} \sum_{\ell=1}^{M} \sum_{j=1}^{N} \left| \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{k} \cdot Y_{(k,\ell)} \left( \frac{2\pi s}{M} \right) \right| \leq \sqrt{\pi} \|w\|,$$

where it is assumed that the weight function $w$ satisfies the so-called summability condition

$$\|w\| := \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_{0}^{R} |w_{(k,\ell)}(r)| r dr < \infty.$$ (23)

Hence, if the weight $w$ satisfies the summability condition, the cubature coefficients satisfy the important stability inequality [22], and experimental evidences show that all coefficients are in general very small. This inequality is proved in Theorem 2 by an application of the famous Chebyshev extremal property for the Gauss-Jacobi quadrature, see Theorem 4.1 in Chapter 4 of [33].

Let us give a short outline of the paper.

In Section 2 we provide basic properties of the discrete polyharmonic cubature formulas: the summability condition for the weight function implies that $I_{(N,M,K)}^\text{poly}$ are uniformly bounded functionals (in the parameters $N, M, K$) on the set of all polynomials. Under this assumption it follows that for $N, M, K \rightarrow \infty$, the value $I_{(N,M,K)}^\text{poly}(f)$ converges to $I_w(f)$ for any function $f$ which is continuous on the closed disk with radius $R$. Moreover we show in Theorem 3 that our cubature formula $I_{(N,M,K)}^\text{poly}$ is exact for all polynomials of the type $f(x) = r^{2s+k} Y_{(k,\ell)}(\varphi)$ where $0 \leq s \leq 2N-1$, $0 \leq k \leq M-1-K$, and $\ell = 1, 2, ..., a_k$, i.e.

$$I_{(N,M,K)}^\text{poly}(f) = \int_{D_R} f(x) w(x) dx.$$
Section 3 is devoted to error estimates for the discrete polyharmonic cubature. The error bounds are a sum of the error bounds of three successive approximations: 1. the approximation of the weight function \( w \) as a Fourier series – involving the parameter \( K \); 2. the approximation by the one-dimensional quadrature formula in radial direction – involving the parameter \( N \); 3. the approximation by the Discrete Fourier Transform – involving the parameter \( M \).

In Section 4 we provide experimental results for the discrete polyharmonic cubature with respect to the first weight function \( w^{(1)} \) in [23] for four different types of test functions, and compare the results with those obtained by the piece-wise midpoint rule ([19]) and the rule of Peirce ([41]), which are two methods used widely in practice, in particular in the Meshless Petrov-Galerkin method, see [18], [19], [35], [36]. Our methods have much higher accuracy than all other methods which might be explained by the fact that the weight function \( w^{(1)} \) has a discontinuity in 0 which has a strong negative influence on the results of the usual cubature formulas.

Section 5 contains experimental results for the second weight function \( w^{(2)} \) in [3]. Since the weight function \( w^{(2)} \) has an infinite Fourier series it has been expected that the discrete polyharmonic cubature now would perform weaker than for the first weight function. However, even in this case the accuracy has been extremely good provided that the parameter \( K \) is large enough. Finally, in Section 6 we comment on practical aspects for the numerical implementation.

2 The discrete polyharmonic cubature

At first we introduce the notion of pseudo-positive (or more general, pseudo-definite) weight functions which plays a central role in the discrete polyharmonic cubature formula.

**Definition 1** Let \( f(x, y) \) be a function on the disc with center 0 and radius \( R \) which is continuous except for 0. Then \( f \) is called pseudo-definite if every Fourier coefficient \( f_{(k, \ell)}(r) \) of \( f \) has a definite sign, thus either \( f_{(k, \ell)}(r) \geq 0 \) for all \( r \in (0, R) \) or \( f_{(k, \ell)}(r) \leq 0 \) for all \( r \in (0, R) \). The function \( f \) is called pseudo-positive if \( f_{(k, \ell)}(r) \geq 0 \) for all \( r \in (0, R) \) for all \( k \in \mathbb{N}_0, \ell = 1, \ldots, a_k \).

A simple example of a pseudo-positive function is the Poisson kernel on the unit disk given by

\[
P(x, y) = \frac{1 - x^2 - y^2}{(x - 1)^2 + y^2} = \frac{1 - r^2}{1 - 2r \cos \varphi + r^2} = 1 + \sum_{k=1}^{\infty} 2r^k \cos k \varphi.
\]

Let us recall that a cubature rule is just an expression of the type

\[
C(f) = \sum_{j=1}^{N} c_j f(x_j)
\]

where \( x_1, \ldots, x_N \) are pairwise different points in \( \mathbb{R}^n \), called nodes, and \( c_j \neq 0 \) are real constants called weights. An important property of a cubature rule, which is also a basis for widely used method for construction of cubature formulas, is the exactness for a subspace of functions: a cubature formula \( C \) is exact on a subspace of functions \( U \) with respect to a weight function \( w(x) \) if

\[
C(f) = \int f(x) w(x) \, dx \quad \text{for all } f \in U.
\]

If \( U \) is the subspace \( \mathcal{P}_m \) of all polynomials of degree \( \leq m \) then one says that a cubature formula \( C \) has degree \( m \) if \( C \) is exact on \( \mathcal{P}_m \) and if there exists a polynomial \( f \) of degree \( m + 1 \) such that \( C(f) \neq I_w(f) \). Exactness of degree \( m \) is also useful for providing error estimates using Taylor series, an approach which was emphasized already by R. von Mises, see [35], [39].

Recall that the **Discrete Polyharmonic Cubature formula** with parameters \((N, M, K)\) is defined for any continuous function \( f \) on the disk of radius \( R \) by formula (21):

\[
I_{(N,M,K)}^{\text{poly}}(f) := \frac{1}{M} \sum_{k=0}^{M} \sum_{s=1}^{N} \sum_{j=1}^{N} \lambda_j(k, \ell) \int_{j(k, \ell)}^{k} Y_k(k, \ell) \left( \frac{2\pi s}{M} \right) \times f \left( \sqrt{j, k, \ell} e^{2\pi i s/M} \right).
\]

(24)
Thus $I^{\text{poly}}_{(N, M, K)} (f)$ is a cubature formula where the weights are real numbers. At first we establish the following subtle estimate (analogous to an estimate proved for the polyharmonic cubature formula in [31]):

**Theorem 2** Let $w$ be a pseudo-definite weight function with Fourier coefficients $w_{(k, \ell)}$, and let $f$ be bounded on the closed disk of radius $R$ with supremum norm

$$
\|f\|_\infty := \sup_{x^2+y^2 \leq R^2} |f(x, y)|.
$$

Then

$$
|I^{\text{poly}}_{(N, M, K)} (f)| \leq \sqrt{\pi} \|f\|_\infty \sum_{k=0}^K \sum_{\ell=1}^{\infty} \int_0^R \lambda_{j, (k, \ell)} |t_{j, (k, \ell)}| w_{(k, \ell)} (r) \, r \, dr \leq \sqrt{\pi} \|f\|_\infty \|w\|.
$$

Also, the coefficients of formula (24) satisfy inequality (22).

**Proof.** Since $|Y_{(k, \ell)} (\varphi)| \leq 1/\sqrt{\pi}$ for all $\varphi \in [0, 2\pi]$ the following estimate follows directly from [24]:

$$
|I^{\text{poly}}_{(N, M, K)} (f)| \leq \sqrt{\pi} \|f\|_\infty \sum_{k=0}^K \sum_{\ell=1}^{N} \sum_{j=1}^{N} |\lambda_{j, (k, \ell)}| t_{j, (k, \ell)}.
$$

Following (16) we define the function $G_N^{(k, \ell)} (h) := \sum_{j=1}^N |\lambda_{j, (k, \ell)}| h (t_{j, (k, \ell)})$, which due to the pseudo-definiteness of $w$, is the $N$-point Gauss-Jacobi quadrature for the integral

$$
I_{(k, \ell)} (h) := \frac{1}{2} \int_{R^2} h (\rho) \cdot \rho^{k/2} |w_{(k, \ell)} (\sqrt{\rho})| \, d\rho.
$$

Now we apply the Chebyshev extremal property of the Gauß–Jacobi quadrature (see Theorem 4.1 in Chapter 4 of [33]) which shows that

$$
G_N^{(k, \ell)} (h) \leq I_{(k, \ell)} (h)
$$

holds for any $2N$–smooth function $h (r)$ with $h^{(2N)} (\rho) \geq 0$ for all $\rho > 0$. Applying this to the function $h (\rho) = \rho^{-k/2}$, gives the inequality

$$
G_N^{(k, \ell)} (\rho^{-k/2}) = \sum_{j=1}^N |\lambda_{j, (k, \ell)}| t_{j, (k, \ell)}^{k/2} \leq \frac{1}{2} \int_{R^2} |w_{(k, \ell)} (\sqrt{\rho})| \, d\rho = \int_0^R |w_{(k, \ell)} (r)| \, r \, dr.
$$

From Theorem 2 by using standard results from functional analysis (see [15], p. 351) we obtain the following important:

**Corollary 3** Suppose that the Fourier coefficients of the pseudo-definite weight function $w$ satisfy the summability condition (see (23))

$$
\|w\| < \infty.
$$

Then the discrete polyharmonic cubature $I^{\text{poly}}_{(N, M, K)} (f)$ converges to the integral $I_w (f)$ for each continuous function on the closed disc with radius $R$ when $N, M, K$ are approaching infinity.

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1The notations in this reference differ from the present. Here we have put $d\mu_{(k, \ell)} (r) = \int_0^{2\pi} Y_{(k, \ell)} (\varphi) \, d\mu (re^{i\varphi})$ while in [33] we used to work with $d\mu_{(k, \ell)} (r) = r^k \int_0^{2\pi} Y_{(k, \ell)} (\varphi) \, d\mu (re^{i\varphi})$. 

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Let us recall that the trapezoidal sum (see [17, p. 111]), also called the composite trapezoidal rule in [23, p. 155], is defined on the set of all 2π-periodic functions \( g \) on the line, by

\[
T_M (g) := \frac{2\pi}{M} \sum_{s=1}^{M} g \left( \frac{2\pi s}{M} \right).
\]

Let now \( f(x,y) \) be defined on the disk with radius \( R \) and define \( f_r(\varphi) := f(r \cos \varphi, r \sin \varphi) \) for \( r \in [0, R] \). Then

\[
T_M (f_r \cdot Y_{(k,\ell)}) = \frac{2\pi}{M} \sum_{s=1}^{M} f \left( r \cos \left( \frac{2\pi s}{M} \right), r \sin \left( \frac{2\pi s}{M} \right) \right) Y_{(k,\ell)} \left( \frac{2\pi s}{M} \right)
\]

and the discrete polyharmonic cubature [20] can be written as

\[
I_{\text{poly}}^{(N,M,K)} (f) = \frac{1}{2} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \sum_{j=1}^{N} \lambda_{j,(k,\ell)} t_{j,(k,\ell)}^{\frac{1}{2}} T_M \left( \frac{f}{t_{j,(k,\ell)}} Y_{(k,\ell)} \right).
\]

For the following result recall that \( r^k Y_{(k,\ell)} (\varphi) \) is a polynomial of degree \( k \) in the variables \( x = r \cos \varphi \) and \( y = r \sin \varphi \), cf. [27, 44].

**Theorem 4** Let \( M \) and \( K \) be natural numbers satisfying \( M > K \). Then the discrete polyharmonic cubature \( I_{\text{poly}}^{(N,M,K)} \) with parameters \( (N,M,K) \) is exact on the linear subspace generated by the polynomials

\[
r^{2s+k} Y_{(k,\ell)} (\varphi)
\]

where \( 0 \leq s \leq 2N-1 \), \( k \leq M-1-K \), and \( \ell = 1, \ldots, a_k \).

**Proof.** We shall use that \( T_M (g) = \int_0^{2\pi} g(\varphi) \, d\varphi \) for all trigonometric polynomials \( g \) of degree \( k \leq M-1 \), see e.g. [17, p. 110]. Let \( k \leq K \) be a natural number and let \( k_1 \leq M-1-K \) be a natural number. Then it is easy to see that the product \( Y_{(k_1,\ell_1)} (\varphi) Y_{(k,\ell)} (\varphi) \) is a trigonometric polynomial of degree \( k_1 + k \leq M-1 \). It follows that

\[
T_{M,0} \left( Y_{(k_1,\ell_1)} Y_{(k,\ell)} \right) = \int_0^{2\pi} Y_{(k_1,\ell_1)} (\varphi) Y_{(k,\ell)} (\varphi) \, d\varphi = \delta_{k,k_1} \delta_{\ell,\ell_1},
\]

where \( \delta_{i,j} \) is the Kronecker symbol. Now we consider the polynomial

\[
f^{(s,k_1,\ell_1)} (r \cos \varphi, r \sin \varphi) = r^{2s+k_1} Y_{(k_1,\ell_1)} (\varphi)
\]

with \( s \leq 2N-1 \) and \( k_1 \leq M-1-K \). Then by (28) we obtain

\[
I_{\text{poly}}^{(N,M,K)} \left( f^{(s,k_1,\ell_1)} \right) = \frac{1}{2} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \sum_{j=1}^{N} \lambda_{j,(k,\ell)} t_{j,(k,\ell)}^{\frac{1}{2}} T_{M,0} \left( t_{j,(k,\ell)}^{\frac{1}{2}} Y_{(k_1,\ell_1)} Y_{(k,\ell)} \right)
\]

where

\[
\lambda_{j,(k,\ell)} := \sum_{j=1}^{N} \lambda_{j,(k,\ell)} t_{j,(k,\ell)}^{\frac{1}{2}}
\]

and

\[
\int_0^{2\pi} r^{2s+k_1} w_{(k_1,\ell_1)} (r) \, r \, dr = \int_0^{R} f^{(s,k_1,\ell_1)} (r) w_{(k_1,\ell_1)} (r) \, r \, dr.
\]
In the last row we used formula (16), due to the fact that the Gauss-Jacobi quadrature for each $(k_1, \ell_1)$ is exact for all polynomials of degree $2N-1$. In the Fourier series expansion of the polynomial $f(s,k_1,\ell_1)$ the only non-zero coefficient is $f(s,k_1,\ell_1)(r) = r^{2s+k_1}$. Hence, by formulas (17), (18) we obtain

$$I_{\text{poly}}(N,M,K)(f(s,k_1,\ell_1)) = I_w(f(s,k_1,\ell_1)).$$

**Remark 5** In [11] one can find a nice account of the early history of numerical multivariate integration until the 1950’s, commencing with the work of J.C. Maxwell [37] in 1877, of P. Appel [7], and H. Bourget [9]. A. Ahlin [1] stresses the fact that for many ad hoc formulae there are no error estimates available, and he provided error estimates for the case of product type measures.

Since the 1950’s the literature has grown considerably and very good surveys until the 1970’s can be found in the books [17], [20], [33], [43] and [47]. For a recent survey on numerical integration rules we refer to [10]. For numerical integration rules especially for the disc we refer to [11], [12], [25], [26], [48] and older work in [2], [3], [4]; see also [49], [50].

### 3 Error Estimates for the discrete polyharmonic cubature

In this section we derive error estimates for the discrete polyharmonic cubature by considering the following chain of identities and approximations: at first the integral (2) is equal to (11), and a simple change of variables leads to (29); then the infinite sum is approximated by a finite sum (depending on $K$), and next we employ the Gauss-Jacobi quadratures (depending on $N$), and finally we use the "Discrete Fourier transform" (depending on $M$) in order to obtain the discrete polyharmonic cubature $I_{\text{poly}}(N,M,K)(f)$:

$$\int_{D_R} f(x)w(x)dx = \sum_{k_1,\ell_1=0}^{a_k} \int_0^R \int_{I_{\text{poly}}(N,M,K)} f(k_1,\ell_1)(r)w(k_1,\ell_1)(r)rdr = \frac{1}{2} \sum_{k=0}^{a_k} \sum_{\ell=1}^{\ell_1} \int_0^{R^2} f(k_1,\ell_1)\sqrt{\rho}w(k_1,\ell_1)\sqrt{\rho}\,d\rho$$

$$\approx \frac{1}{2} \sum_{k=0}^{a_k} \sum_{\ell=1}^{\ell_1} \int_0^{R^2} f(k_1,\ell_1)\sqrt{\rho}^\frac{-1}{2} \times \rho^\frac{1}{2} w(k_1,\ell_1)(\sqrt{\rho})\,d\rho$$

$$\approx \frac{1}{2} \sum_{k=0}^{a_k} \sum_{\ell=1}^{\ell_1} \int_0^{R^2} f(k_1,\ell_1)\sqrt{t_{j,(k_1,\ell_1)}} \times t_{j,(k_1,\ell_1)}^{\frac{1}{2}} w_{j,(k_1,\ell_1)}(\sqrt{\rho})\lambda_{j,(k_1,\ell_1)} =: I_{\text{poly}}(N,M,K)(f)$$

$$\approx \frac{1}{2} \sum_{k=0}^{a_k} \sum_{\ell=1}^{\ell_1} \int_0^{R^2} f(k_1,\ell_1)\sqrt{t_{j,(k_1,\ell_1)}} \times t_{j,(k_1,\ell_1)}^{\frac{1}{2}} w_{j,(k_1,\ell_1)}(\sqrt{\rho})\lambda_{j,(k_1,\ell_1)} = I_{\text{poly}}(N,M,K)(f)$$

The error between formulas (31) and (32) can be estimated as follows:

**Theorem 6** For $f \in C^{2D+1}(B_R)$ the following estimate holds

$$\left| I_{\text{poly}}(N,\infty,K)(f) - I_{\text{poly}}(N,M,K)(f) \right| \leq \frac{2\pi\zeta(2D+1)}{M^{2D+1}} \sum_{k=0}^{a_k} \sum_{\ell=1}^{\ell_1} M_{k,\ell}(f) \int_0^R w_{k,\ell}(r)\,rdr$$

where

$$M_{k,\ell}(f) := \sup_{r \in [0,R]} \sup_{\varphi \in [0,2\pi]} \left| \frac{d^{2D+1}}{d\varphi^{2D+1}} \left[ f(re^{i\varphi})Y_{k,\ell}(\varphi) \right] \right|.$$

Here $\zeta$ denotes the Riemann zeta function.
Proof. In [17, p. 110], for periodic $g \in C^{2D+1}[0, 2\pi]$ it is shown that
\[ \left| \int_0^{2\pi} g(\varphi) \, d\varphi - T_M(g) \right| \leq \frac{C_{g,D}}{M^{2D+1}}, \tag{33} \]
where the constant $C_{g,D}$ is given by
\[ C_{g,D} := 4\pi \zeta(2D+1) \sup_{\varphi \in [0,2\pi]} \left| \frac{d^{2D+1}}{d\varphi^{2D+1}} \left( g(\varphi) \right) \right|. \]
Formula (27) and the triangle inequality show that for $f$,
\[ \text{We can estimate this inequality by using the Cauchy-Schwarz inequality and the orthonormality} \]
Now we use inequality (33) for estimating the term in the middle row:
\[ |I_{(N,\infty,K)}^{poly}(f) - I_{(N,M,K)}^{poly}(f)| \leq \frac{1}{2} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \lambda_{j,(k,\ell)} \int_j^{j+\frac{\pi}{2}} f(k,\ell) \left( \sqrt{I_{j,(k,\ell)}} \right) - T_M \left( f \sqrt{I_{j,(k,\ell)}} Y_{j,(k,\ell)} \right) |. \]

Hence, by inequality (27), we obtain the final result:
\[ \left| I_{(N,\infty,K)}^{poly}(f) - I_{(N,M,K)}^{poly}(f) \right| \leq \frac{2}{M^{2D+1}} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} M_k,\ell(f) \int_j^{j+\frac{\pi}{2}} f(k,\ell) \left( \sqrt{I_{j,(k,\ell)}} \right) - T_M \left( f \sqrt{I_{j,(k,\ell)}} Y_{j,(k,\ell)} \right) |. \]

The error bound between (29) and (30) is considered in the following:

Theorem 7 Let $f \in C^{2p}(D_R)$ for some integer $p \geq 1$. Assume that the weight function $w$ satisfies
\[ 1 \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \int_0^R w(k,\ell) (\sqrt{\rho}) \, d\rho = \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \int_0^R w(k,\ell) (r) \, rdr = \|w\| < \infty \]
Then
\[ \sum_{k=K+1}^{\infty} \sum_{\ell=1}^{a_k} \int_0^R f(k,\ell) (\sqrt{\rho}) (r) \, d\rho \leq \frac{2\pi}{K^{2p}} \|w\| \|\frac{\partial^2}{\partial \varphi^2} f(re^{i\varphi})\|_{\infty}. \]

Proof. By applying a standard techniques (see e.g. Theorem 10.19 in [27]), we obtain
\[ f(k,\ell)(r) = \int_0^{2\pi} f(re^{i\varphi}) Y_{k,\ell}(\varphi) \, d\varphi = \frac{1}{k^{2p}} \int_0^{2\pi} f(re^{i\varphi}) \frac{\partial^{2p}}{\partial \varphi^{2p}} Y_{k,\ell}(\varphi) \, d\varphi \]
\[ = \frac{1}{k^{2p}} \int_0^{2\pi} \frac{\partial^{2p}}{\partial \varphi^{2p}} f(re^{i\varphi}) Y_{k,\ell}(\varphi) \, d\varphi. \]

We can estimate this inequality by using the Cauchy-Schwarz inequality and the orthonormality of $Y_{k,\ell}(\varphi)$, arriving at
\[ |f(k,\ell)(r)| \leq \frac{\sqrt{2\pi}}{k^{2p}} \|\frac{\partial^2}{\partial \varphi^2} f(re^{i\varphi})\|_{\infty}. \]
Then

\[
\left| \sum_{k=K+1}^{\infty} \sum_{\ell=1}^{a_k} \int_0^R f_{(k,\ell)} (\sqrt{\rho}) w_{(k,\ell)} (\sqrt{\rho}) \, d\rho \right| \\
\leq \sum_{k=K+1}^{\infty} \sum_{\ell=1}^{a_k} \int_0^R |f_{(k,\ell)} (\sqrt{\rho})| |w_{(k,\ell)} (\sqrt{\rho})| \, d\rho \\
\leq \sqrt{2\pi} \left\| \frac{\partial^2}{\partial \rho^2} f (re^{i\psi}) \right\|_\infty \times \sum_{k=K+1}^{\infty} \sum_{\ell=1}^{a_k} \frac{1}{k^{2\rho}} \int_0^R |w_{(k,\ell)} (\sqrt{\rho})| \, d\rho \\
\leq \sqrt{2\pi} \left\| \frac{\partial^2}{\partial \rho^2} f (re^{i\psi}) \right\|_\infty \times \frac{1}{K^{2\rho}} \sum_{k=K+1}^{\infty} \sum_{\ell=1}^{a_k} \int_0^R |w_{(k,\ell)} (\sqrt{\rho})| \, d\rho \\
\leq \sqrt{2\pi} \left\| \frac{\partial^2}{\partial \rho^2} f (re^{i\psi}) \right\|_\infty \times \frac{1}{K^{2\rho}} \|w\|.
\]

This ends the proof.

The error bound between (30) and (31) is a Markov type estimate (see Theorem 14.2.2 in [15], and Theorem 44 in [31]). Using the notations (14), (15), (16) we prove:

**Theorem 8** Let $\kappa_{N,(k,\ell)}$ be the leading coefficient of the $N$-th degree orthonormal polynomial $Q_{N,(k,\ell)}$ with respect to the measure $\rho^{\frac{N}{2}} w_{(k,\ell)} (\sqrt{\rho})$ on the interval $[0, R^2]$. Then for every function $f \in C^{2N}(D_R)$ and every index $(k, \ell)$ we have

\[
I_{(k,\ell)} := \int_0^R f_{(k,\ell)} (\sqrt{\rho}) w_{(k,\ell)} (\sqrt{\rho}) \, d\rho = \sum_{j=1}^{N} f_{(j,\ell)} (\sqrt{I_{j,(k,\ell)}}) \times t_{j,(k,\ell)} \lambda_{j,(k,\ell)} \\
\leq \frac{1}{(2N)! \kappa_{N,(k,\ell)}^2} \left\| \frac{d^{2N}}{d\rho^{2N}} g_{(k,\ell)} (\rho) \right\|_\infty,
\]

where $g_{(k,\ell)} (\rho) := \rho^{-\frac{N}{2}} f_{(k,\ell)} (\sqrt{\rho})$. The difference between (30) and (31) is bounded by

\[
\sum_{k=0}^{K} \sum_{\ell=1}^{a_k} I_{(k,\ell)} \leq \frac{1}{(2N)! \kappa_{N,(k,\ell)}^2} \left\| \frac{d^{2N}}{d\rho^{2N}} g_{(k,\ell)} (\rho) \right\|_\infty.
\]

**Proof.** It is easy to see that $g_{(k,\ell)} (\rho) = \rho^{-\frac{N}{2}} f_{(k,\ell)} (\sqrt{\rho})$ is $2N$-times differentiable in the interval $(0, R^2)$. Hence, to the function $g_{(k,\ell)} (\rho) = f_{(k,\ell)} (\sqrt{\rho}) \rho^{-\frac{N}{2}} \in C^{2N} (0, R^2)$, we may apply Markov's Theorem 14.2.2 in [15], and we obtain

\[
I_{(k,\ell)} \leq \frac{1}{(2N)! \kappa_{N,(k,\ell)}^2} \left\| \frac{d^{2N}}{d\rho^{2N}} g_{(k,\ell)} (\rho) \right\|_\infty.
\]

This implies the estimate for the error between (30) and (31)

\[
\left| \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \int_0^R f_{(k,\ell)} (\rho) w_{(k,\ell)} (\sqrt{\rho}) \, d\rho - \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \int_0^R f_{(k,\ell)} (\sqrt{I_{j,(k,\ell)}}) \times t_{j,(k,\ell)} \lambda_{j,(k,\ell)} \right| \\
\leq \left| \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \int_0^R g_{(k,\ell)} (\rho) \times \rho^{\frac{N}{2}} w_{(k,\ell)} (\sqrt{\rho}) \, d\rho - \sum_{j=1}^{N} f_{(j,\ell)} (\sqrt{I_{j,(k,\ell)}}) \times t_{j,(k,\ell)} \lambda_{j,(k,\ell)} \right| \\
\leq \frac{1}{(2N)! \kappa_{N,(k,\ell)}^2} \left\| \frac{d^{2N}}{d\rho^{2N}} g_{(k,\ell)} (\rho) \right\|_\infty.
\]
Remark 9 In the case when \( \rho^k w_{(k, \ell)} (\sqrt{\rho}) \) are Jacobi weight functions explicit expressions for \( \kappa_{N,(k, \ell)} \) are well known, see e.g. [15] or Theorem 44 in [31].

Remark 10 If \( g_{(k, \ell)} (x) \) is real on the real axis and \( 2\pi \) periodic and if \( g_{(k, \ell)} (x + iy) \) is holomorphic in a strip \(|y| < \sigma\), then one may use estimates in [17, p. 110], and apply them to obtain an alternative estimate in Theorem 8.

4 Experimental results for the weight function \( w^{(1)} (x, y) \)

In this section we want to test cubature formulas for integrals of the type

\[
I_w (f) = \int_0^{2\pi} \int_0^R f (r \cos \varphi, r \sin \varphi) \cdot w^{(1)} (r \cos \varphi, r \sin \varphi) \cdot r dr d\varphi
\]

for the weight function

\[
w^{(1)} (x, y) = \frac{1 + x}{\sqrt{x^2 + y^2}} = \frac{1}{r} + \cos \varphi = \frac{\sqrt{2\pi}}{r} Y_{(0,1)} (\varphi) + \sqrt{\pi} Y_{(1,1)} (\varphi).
\]

Since \( w^{(1)} \) has only two Fourier coefficients we take \( K = 1 \) in the discrete polyharmonic cubature [20] with parameters \( (N, M, K) \). For given \( N \) one has to construct the \( N \)-point Gauss-Jacobi quadratures for the integrals

\[
\frac{1}{2} \int_0^1 P (\rho) \rho^{-1/2} d\rho \quad \text{for } k = 0, \quad \text{and} \quad \frac{1}{2} \int_0^1 P (\rho) \rho^{1/2} d\rho \quad \text{for } k = 1
\]

which are Jacobi-weight functions. Fortunately, there are excellent programmes for determining the nodes and weights of the Gauss-Jacobi quadrature providing high accuracy, see [22]. For our experiments in this section we consider four test functions:

\[
f_0 (x, y) = 1 + x^4 + y^3,
\]

\[
f_1 (x, y) = 1 + \frac{x^3}{\sqrt{x^2 + y^2}} + \frac{y^7}{x^2 + y^2} = 1 + r^2 \cos^3 \varphi + r^5 \sin^7 \varphi,
\]

\[
f_2 (x, y) = \cos (10x + 20y),
\]

\[
f_3 (x, y) = (x^2 + y^2)^{5/4} = r^{5/2}
\]

The first test function \( f_0 \) is a polynomial of degree 4, and the second \( f_1 \) is not smooth at 0. The function \( f_2 \) is of oscillatory type and an example of a test function used in the package of Genz, see [24], [43] of the form (in our case \( u = 0 \))

\[
\cos (2\pi u + ax + by) = \cos (2\pi u) \cos (ax + by) - \sin (2\pi u) \sin (ax + by).
\]

At first we present the experiments for the Discrete Polyharmonic Cubature, and then we compare it with two other standard rules which are applied to the functions

\[
g_j (re^{i\varphi}) := f_j (re^{i\varphi}) w^{(1)} (re^{i\varphi})
\]

Note that \( g_j \) is not continuous at 0 for \( j = 0, 1, 2 \), and one might argue that the standard rules do not perform too well for functions with a discontinuity. For this reason we have included the function \( f_3 (x, y) \) for which \( g_3 (re^{i\varphi}) = r^{1/2} (1 + r \cos \varphi) \) is clearly continuously differentiable, and our discrete polyharmonic cubature formula [21] performs better than the usual methods as well.

Let us note that the discrete polyharmonic cubature formula [21] needs (at most) \((2K - 1) \cdot N \cdot M\) evaluation points. Since \( w^{(1)} \) has only two non-zero Fourier coefficients we need in this case at most \(2N \cdot M\) evaluations points.
4.1 Results for the Discrete Polyharmonic Cubature Formula

In the following tables we present experimental results for the discrete polyharmonic cubature where the number $M$ of points on the circles is chosen to be equal to 9, 25, 63, 83. The reason for this choice is that we used the Fast Fourier transform for the implementation of the trapezoidal rule (depending on $M$). The number $N$ of concentric circles is chosen to be equal to 10, 15, 25, 35, 50.

Due to the exactness of the discrete polyharmonic cubature, the value $I_{\text{poly}}^{(N,M,1)}(f_0)$, according to Theorem 4, must be identical with the true value of the integral if $M$ satisfies $M \geq 6$, and $N \geq 2$. This is numerically confirmed by our experiments: for all $N$ and $M$ as above specified we obtained up to double precision that

$$I_1 \left( f_0 w^{(1)} \right) = \frac{43}{20} \pi \approx 6.754424205218060.$$

The second test function can also be integrated explicitly and the true value is

$$I_1 \left( f_1 w^{(1)} \right) = \frac{35}{16} \pi \approx 6.87223392972767.$$

Our experimental results are contained in the following table:

| N/M | 9     | 25    | 63    | 83    |
|-----|-------|-------|-------|-------|
| 10  | 6.87224296287783 | 6.87224296287783 | 6.87224296287783 | 6.87224296287783 |
| 15  | 6.87223588060173  | 6.87223588060173  | 6.87223588060173  | 6.87223588060173  |
| 25  | 6.87223420205342  | 6.87223420205342  | 6.87223420205342  | 6.87223420205342  |
| 35  | 6.87223400297000  | 6.87223400297000  | 6.87223400297000  | 6.87223400297000  |
| 50  | 6.87223394775545  | 6.87223394775545  | 6.87223394775545  | 6.87223394775545  |

| N/M | 9     | 25    | 63    | 83    |
|-----|-------|-------|-------|-------|
| 10  | 0.00000903315016 | 0.00000903315016 | 0.00000903315016 | 0.00000903315016 |
| 15  | 0.00000195087406  | 0.00000195087406  | 0.00000195087406  | 0.00000195087406  |
| 25  | 0.00000027232575  | 0.00000027232575  | 0.00000027232575  | 0.00000027232575  |
| 35  | 0.00000007324233  | 0.00000007324233  | 0.00000007324233  | 0.00000007324233  |
| 50  | 0.00000001802778  | 0.00000001802778  | 0.00000001802778  | 0.00000001802778  |

Discrete polyharmonic cubature for $f_1 w^{(1)} = (1 + r^2 \cos^3 \varphi + r^5 \sin^5 \varphi) \left( \frac{1}{r} + \cos \varphi \right)$

Obviously we obtain very good approximations: even for the case of $180 = 2 \times 9 \times 10$ evaluations the error is smaller than $10^{-5}$. Note further that in the rows we do not obtain improvements when $M$ is getting larger. This is due to the fact that $\varphi \mapsto f_1(re^{i\varphi})$ is a trigonometric polynomial of degree $\leq 7$, and the exactness of the trapezoidal rule (26) implies that there is no change when $M$ is larger than 7.

Now we want to test the function $f_2(x, y) = \cos (ax + by)$.

It is not difficult to see that the first Fourier coefficient of $f_2$ satisfies

$$f_{2,(0,1)}(r) = \sqrt{2 \pi} J_0 \left( \sqrt{a^2 + b^2} r \right), \quad \text{and}$$

$$f_{2,(1,1)}(r) = f_{2,(1,2)}(r) = 0,$$

where $J_n$ is the Bessel function of the first kind of order $n$ defined by

$$J_n(x) = \left( \frac{x}{2} \right)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s! (n+s)!} \left( \frac{x}{2} \right)^{2s};$$

(38)
Then the first Fourier coefficient \( g_{(0,1)} (r) \) of the function \( g := f_2 \cdot w^{(1)} \) can be computed (using the orthogonality relations of spherical harmonics):

\[
g_{(0,1)} (r) = \frac{1}{\sqrt{2\pi}} f_{2,(0,1)} (r) w^{(1)}_{(0,1)} = J_0 \left( \sqrt{a^2 + b^2}r \right) \frac{\sqrt{2\pi}}{r}.
\]

It follows that

\[
I_1 \left( f_2 w^{(1)} \right) = \sqrt{2\pi} \int_0^R g_{(0,1)} (r) r dr = 2\pi \int_0^R J_0 \left( \sqrt{a^2 + b^2}r \right) dr.
\]

Using the power series expansion of the Bessel function in (38) the integral can be evaluated up to any accuracy. For \( a = 10 \) and \( b = 20 \) we see that up to double precision we have

\[
I_{w^{(1)}} (f_2) = I_1 \left( f_2 w^{(1)} \right) = 0.301310995335215
\]

Our experiments for the discrete polyharmonic cubature provide the following table:

| \( N \) | \( M \) | \( 9 \) | \( 25 \) | \( 63 \) | \( 83 \) |
|-------|-------|-------|-------|-------|-------|
| 10    | -0.08102057453745 | 0.31409913156633 | 0.30131093100867 | 0.30131093100867 |
| 15    | -0.08102397430499 | 0.31409915892923 | 0.3013109533522 | 0.3013109533522 |
| 25    | -0.08102401217317 | 0.31409919589293 | 0.3013109533522 | 0.3013109533522 |
| 35    | -0.08102401237119 | 0.31409919589293 | 0.3013109533522 | 0.3013109533522 |
| 50    | -0.08102401237809 | 0.31409919589293 | 0.3013109533522 | 0.3013109533522 |

| \( \text{Error} \) | \( N \) | \( M \) | \( 9 \) | \( 25 \) | \( 63 \) | \( 83 \) |
|-----------------|-------|-------|-------|-------|-------|-------|
| 10              | 0.38233156987266 | 0.01278813623111 | 0.00000006432655 | 0.00000006432655 |
| 15              | 0.38233496964021 | 0.01278820055772 | 0.00000000000000 | 0.00000000000000 |
| 25              | 0.38233500750838 | 0.01278820055772 | 0.00000000000000 | 0.00000000000000 |
| 35              | 0.38233500770641 | 0.01278820055772 | 0.00000000000000 | 0.00000000000000 |
| 50              | 0.38233500771330 | 0.01278820055772 | 0.00000000000000 | 0.00000000000000 |

**Discrete polyharmonic cubature** for \( f_2 w^{(1)} = (\cos (10x + 20y)) \left( \frac{1}{r} + \cos \varphi \right) \).

Note that the formula is sensitive with respect to the values of \( M \) : if \( M \) is 9 then large deviations occur (even if \( N \) is large) , for \( M = 25 \) the approximation error is 0.01 and the number \( N \) of circles does not influence much the results. For \( M = 63 \) and \( N = 10 \) the approximation error is very small:

\[
0.301310995335215 - 0.30131093100867 = 0.00000064326545
\]

Next we consider the test function \( f_3 (x, y) = r^{3/2} \), for which the integrand \( f_3 w^{(1)} = r^{3/2} (1 + r \cos \varphi) \) is smooth. Then the explicit computation gives

\[
\int_0^{2\pi} \int_0^1 r^{3/2} (1 + r \cos \varphi) r dr d\varphi = 1.79519580205131
\]
with the following table:

| N/M | 9     | 25    | 63    | 83    |
|-----|-------|-------|-------|-------|
| 10  | 1.79513323182095 | 1.79513323182095 | 1.79513323182095 | 1.79513323182095 |
| 15  | 1.79518029482336  | 1.79518029482336  | 1.79518029482336  | 1.79518029482336  |
| 25  | 1.79519315318245   | 1.79519315318245   | 1.79519315318245   | 1.79519315318245   |
| 35  | 1.795197859942     | 1.795197859942     | 1.795197859942     | 1.795197859942     |
| 50  | 1.7951956405565    | 1.7951956405565    | 1.7951956405565    | 1.7951956405565    |

**Discrete polyharmonic cubature** for \( f_3 w^{(1)} = r^{5/2} \left( \frac{1}{r} + \cos \varphi \right) \)

### 4.2 Comparison with the piece-wise midpoint rule

The **piecewise midpoint quadrature rule** (see e.g. [19]) is rather geometric: subdivide the disk of radius \( R \) by concentric circles of radius

\[
r_j = \frac{j^2 - j + 1}{3} R \approx \frac{j}{N} R \text{ for } j = 1, \ldots, N
\]

and radial half-lines with angle \( 2\pi i/M \) for \( i = 1, \ldots, M \). Then the integral over each subdomain is approximated by the evaluation of the integrand at the centroid of the sector multiplied by the area of the sector, given by \( (j - \frac{1}{2}) \left( \frac{R}{N} \right)^2 \frac{2\pi}{M} \). Formally, we define:

**Definition 11** The piecewise midpoint quadrature rule, is given by:

\[
I_{mid}^{N,M}(f) := \frac{2\pi R^2}{M \cdot N^2} \sum_{j=1}^{N} \sum_{s=1}^{M} \left( j - \frac{1}{2} \right) f \left( r_j \cos \frac{2\pi (s - \frac{1}{2})}{M}, r_j \sin \frac{2\pi (s - \frac{1}{2})}{M} \right).
\]

(39)

The results for the first test function \( f_0 \) are contained in the following table.

| N=M | \( I_{N,M}^{mid}(f_0 w^{(1)}) \) | Error |
|-----|---------------------------------|-------|
| 5   | 6.29394814952597                | 0.46047665569209 |
| 10  | 6.55266428574299                | 0.2017591947507 |
| 20  | 6.65272561900472                | 0.10169858621334 |
| 100 | 6.73395357471790                | 0.02047063505016 |
| 200 | 6.74418070869065                | 0.01024349652741 |

**Midpoint cubature** for \( f_0 w^{(1)} = (1 + x^4 + y^3) \left( \frac{1}{r} + \cos \varphi \right) \)

**True Value** is: \( \frac{43}{20} \pi \approx 6.754424205218060 \)

Note that the case \( N = 200 \), i.e. 40000 evaluation points, still leads to an error of 0.01. The discrete polyharmonic cubature was exact for this case.

For the second test function \( f_1 \) we have a similar pattern:
Midpoint cubature for \( f_1w^{(1)} = (1 + r^2 \cos^3 \varphi + r^3 \sin^7 \varphi) \left( \frac{1}{r} + \cos \varphi \right) \)

True value is \( \approx 6.872 233 929 727 67 \)

and we see that the error is quite big; for \( 200 \times 200 \) evaluation points it is bigger than 0.01.

For the third test function \( f_2(x, y) = \cos(10x + 20y) \) we obtain the following table:

| \( N = M \) | \( 9 \) | \( 25 \) | \( 63 \) | \( 83 \) |
|---|---|---|---|---|
| 10 | -0.1904404544101284 | 0.0936727156130806 | 0.105393884431863 | 0.105393884431863 |
| 15 | -0.120671303989885 | 0.207320729809817 | 0.219935669864855 | 0.219935669864855 |
| 25 | -0.0641280979279670 | 0.207320729809817 | 0.219935669864855 | 0.219935669864855 |
| 35 | -0.0400423613447862 | 0.207320729809817 | 0.219935669864855 | 0.219935669864855 |
| 50 | -0.0220967384451548 | 0.207320729809817 | 0.219935669864855 | 0.219935669864855 |

Midpoint cubature for \( f_2w^{(1)} = \cos(10x + 20y) \left( \frac{1}{r} + \cos \varphi \right) \).

Finally we consider the differentiable test function \( f_3(x, y) = r^{5/2} \). We obtain the following results:

| \( N = M \) | \( 9 \) | \( 25 \) | \( 63 \) | \( 83 \) |
|---|---|---|---|---|
| 10 | 0.49175144943649 | 0.207638279722134 | 0.195917110903352 | 0.195917110903352 |
| 15 | 0.421822993525100 | 0.146454786534294 | 0.134145613265731 | 0.134145613265731 |
| 25 | 0.365439093263182 | 0.093990265525398 | 0.081375325470359 | 0.081375325470359 |
| 35 | 0.3413553560000011 | 0.0709936571240705 | 0.058293964325133 | 0.058293964325133 |
| 50 | 0.323407733780370 | 0.0536207774061552 | 0.040859738062723 | 0.040859738062723 |

Midpoint cubature for \( f_3w^{(1)} = r^{5/2} \left( \frac{1}{r} + \cos \varphi \right) \).

Table 5

| \( N = M \) | \( F_{N,N}^{mid} (f_1w^{(1)}) \) | Error |
|---|---|---|
| 5 | 6.479 185 720 369 13 | 0.393 048 209 358 541 |
| 10 | 6.671 455 800 859 80 | 0.290 778 128 867 871 |
| 20 | 6.770 780 856 013 81 | 0.101 450 737 138 588 |
| 100 | 6.851 773 117 091 46 | 0.020 468 126 319 494 |
| 200 | 6.861 992 887 600 82 | 0.010 241 042 126 847 |

Table 6

If we compare the \( 2 \times 9 \times 35 = 630 \) valuations with precision \( 10^{-6} \) in Table 3 with the present \( 10000 \) evaluations with precision \( 10^{-4} \), we see how much better is our cubature formula, even if the integrand is a \( C^1 \) function.
4.3 Comparison with the generalized Peirce Rule

S. De and K.J. Bathe discussed in [18] (see also see [36, p. 65]) the following rule: For given \(N\), let \(\rho_j, j = 1, \ldots, N\) be the nodes of the Gauss quadrature \(G_N\) on \([0, R^2]\) with corresponding weights \(w_j\), further \(\alpha\) be a real parameter and \(M\) a natural number: then

\[
I^{\text{Peirce}, \alpha}_{N,M}(f) := \frac{\pi}{M} \sum_{j=1}^{N} w_j \sum_{s=1}^{M} f\left(\sqrt{\rho_j} \cos \frac{2\pi (s + \alpha)}{M}, \sqrt{\rho_j} \sin \frac{2\pi (s + \alpha)}{M}\right)
\]

(40)

is called the generalized Peirce rule. The rule of Peirce [41] is obtained by setting \(N = m + 1\), \(M = 4m + 4\), and \(\alpha = 0\).

Remark 12 Let us note that the Discrete Polyharmonic Cubature for the constant weight function \(w \equiv 1\) (hence \(K = 0\) in (20)) is identical with formula (40) with \(\alpha = 0\). Thus the numerical evaluation of this formula can be carried out with our programme.

The next table gives the results for the rule of Peirce \(I^{\text{Peirce},0}_{N,M}(f_0w^{(1)})\) for the first test function:

| N/M | 9   | 25   | 63   | 83   |
|-----|-----|------|------|------|
| 10  | 6.49387212 | 6.49387212 | 6.49387212 | 6.49387212 |
| 15  | 6.577936813 | 6.577936813 | 6.577936813 | 6.577936813 |
| 25  | 6.64152541  | 6.64152541  | 6.64152541  | 6.64152541  |
| 35  | 6.67370918  | 6.67370918  | 6.67370918  | 6.67370918  |
| 50  | 6.700258414 | 6.700258414 | 6.700258414 | 6.700258414 |

Peirce Cubature rule for \(f_0w^{(1)} = (1 + x^4 + y^3) \left(\frac{1}{\sqrt{r}} + \cos \varphi\right)\)

True value is \(\frac{43}{20}\pi \approx 6.754424205218060\)

Even in the case \(N = 25, M = 25\), with the number of evaluation points equal to \(NM = 25 \cdot 25 = 625\), we have an error of

\[6.754424205218060 - 6.64152541 = 0.10727166421806\]

In case of \(NM = 50 \cdot 83 = 4150\) evaluation points (which is about 6 times bigger than 625) the error is only about twice smaller:

\[6.754424205218060 - 6.700258414 = 0.05416579121806\]

The experiments with the test functions \(f_1, f_2\) and \(f_3\) have shown a similar behaviour.

5 Experimental results for the weight function \(w^{(2)}(x, y)\)

In this section we discuss a weight function which is of quite different nature compared to the first weight function. The function

\[w^{(2)}(re^{i\varphi}) := |y| = |r \sin \varphi|\]

(41)

is simple in the sense that it is homogeneous in the variable \(r\). However it has an infinite Fourier series since

\[|\sin \varphi| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos (2k\varphi)}{4k^2 - 1}\]

The weight function \(w^{(2)}\) is pseudo-definite since its orthonormalized Fourier coefficients have a definite sign:

\[w_{(0,1)}(r) = \frac{2\sqrt{2}}{\sqrt{\pi}} r\] and \[w_{(2k,1)}(r) = -\frac{4}{\sqrt{\pi} 4k^2 - 1} r \text{ for } k \geq 1.\]
We recall our main integration formula (2)

$$I_w(f) = \int_0^{2\pi} \int_0^R f(re^{i\varphi}) w(re^{i\varphi}) r dr d\varphi = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^R f_{(k,\ell)}(r) w_{(k,\ell)}(r) r dr.$$  \hspace{1cm} (42)

5.1 Results for the discrete polyharmonic cubature

Further we consider the test function

$$f_4(x, y) = 30x^{12}. \hspace{1cm} (43)$$

Note that

$$I_{w(2)}(f_4) = I_1(30x^{12}|y|) = \int_0^1 \int_0^{2\pi} 30r^{12}\cos^{12}(\varphi) |\sin \varphi| r^2 dr d\varphi = \frac{8}{13} \approx 0.6153846153846160. $$

Since $f_4(x, y)$ is a polynomial of degree 12 the Fourier coefficients satisfy $f_{4,(k,\ell)}(r) = 0$ for $k > 13$, hence, we may take $K = 12$. For $M > 24$ and $N > 12$ the polynomial $f(x, y)$ will be reproduced, i.e. holds

$$I_{poly(N,M,12)}(f) = I_{w(2)}(f) = I_1(f_{w(2)}).$$

This can be seen in the following table:

| N \ M | 9     | 25     | 63     | 83  |
|------|-------|--------|--------|-----|
| 10   | 0.5609353695139790 0.6153846153846160 0.6153846153846160 0.6153846153846160 |
| 15   | 0.5609353656165750 0.6153846153846150 0.6153846153846150 0.6153846153846150 |
| 25   | 0.5609353655516600 0.6153846153846170 0.6153846153846170 0.6153846153846170 |
| 35   | 0.5609353655539850 0.6153846153846140 0.6153846153846140 0.6153846153846140 |
| 50   | 0.5609353655539860 0.6153846153846170 0.6153846153846170 0.6153846153846170 |

| Error | 9     | 25     | 63     | 83  |
|-------|-------|--------|--------|-----|
| 10    | 0.0544492458706369 0.0000000000000000 0.0000000000000000 0.0000000000000000 |
| 15    | 0.0544492497680410 0.0000000000000000 0.0000000000000000 0.0000000000000000 |
| 25    | 0.0544492498304560 0.0000000000000000 0.0000000000000000 0.0000000000000000 |
| 35    | 0.0544492498306309 0.0000000000000000 0.0000000000000000 0.0000000000000000 |
| 50    | 0.0544492498306299 0.0000000000000000 0.0000000000000000 0.0000000000000000 |

Discrete Polyharmonic cubature for $K = 12$ and $f_{w(2)} = 30x^{12}|y|$.

True value is $\approx 0.6153846153846160$.

Next we consider the test function

$$f_5(x, y) := |y|. \hspace{1cm} (44)$$

Since it is not smooth we should not expect to obtain too good results for the discrete polyharmonic cubature. The exact value of the integral can be computed:

$$I_1(|y| |y|) = \int_0^1 \int_0^{2\pi} r^2 |\sin \varphi|^2 d\varphi dr = \int_0^1 r^3 dr \cdot \int_0^{2\pi} |\sin \varphi|^2 d\varphi $$

$$= \frac{1}{4} \pi \approx 0.785398163397448$$

We took $K = 12$, as in the last experiment, and we obtained a table of results where the error is almost the same for all $N$ and $M$, and is around $10^{-3}$. If we take $K = 22$ the results are much
Table 10

| N\M | 9             | 25             | 63             | 83             |
|------|---------------|----------------|----------------|----------------|
| 10   | 0.785206660  | 0.785352337    | 0.785367124    | 0.785369362    |
| 15   | 0.785208297  | 0.785358970    | 0.785373081    | 0.785375274    |
| 25   | 0.785208235  | 0.785361119    | 0.785374994    | 0.785377171    |
| 35   | 0.785208149  | 0.785361440    | 0.785375276    | 0.785377452    |
| 50   | 0.785208109  | 0.785361541    | 0.785375364    | 0.785377539    |
| Error| 0.0001915037 | 0.0000458267    | 0.0000310393    | 0.0000288009    |
| 10   | 0.0001915037 | 0.0000458267    | 0.0000310393    | 0.0000288009    |
| 15   | 0.0001898666 | 0.0000391939    | 0.0000250824    | 0.0000228890    |
| 25   | 0.0001899281 | 0.0000370443    | 0.0000231697    | 0.0000209919    |
| 35   | 0.0001900142 | 0.0000367231    | 0.0000228870    | 0.0000207116    |
| 50   | 0.0001900544 | 0.0000366227    | 0.0000227993    | 0.0000206248    |

Discrete Polyharmonic cubature for $K = 22$ and $f_5^w(2) = |y|^2$

True value is $\approx 0.785398163397448$

Finally, we look again at the oscillating test function (36), namely, $f_2(x, y) = \cos(10x + 20y)$. Using the expansion

$$C_{a,b}(x, y) := \cos(ax + by) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (ax + by)^{2n}$$

one can prove that the Fourier coefficients $C_{a,b}^{(2k+1,\ell)}(r)$ are zero and

$$C_{a,b}^{(2k,\ell)}(r) = 2\pi Y_{2k,\ell}(\varphi_{a,b}) (-1)^k J_{2k} \left( r \sqrt{(a^2 + b^2)} \right)$$

where $J_{2k}$ are the Bessel functions, see formula (38), and $\varphi_{a,b}$ is the angle given by $\tan \varphi_{a,b} = \frac{b}{a}$.

It follows that

$$I_{w^2}(f) = 4 \int_0^1 J_0 \left( r \sqrt{a^2 + b^2} \right) r^2 dr + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2k\varphi_{a,b})}{4k^2 - 1} \int_0^1 J_{2k} \left( r \sqrt{a^2 + b^2} \right) r^2 dr. \ (45)$$

We may estimate the error using this expression.

In formula (21), for the Discrete Polyharmonic Cubature, we have taken $K = 22$. We
obtain the following results:

| N | M | 9 | 25 | 63 | 83 |
|---|---|---|----|----|----|
| 10 | | 0.096718846427391 | 0.01447243304185 | 0.014477271351135 | 0.014477271351135 |
| 15 | | 0.096642162991824 | 0.014472441635349 | 0.014477279682299 | 0.014477279682299 |
| 25 | | 0.096593708566055 | 0.014472441635349 | 0.014477279682299 | 0.014477279682299 |
| 35 | | 0.096592741482745 | 0.014472441635350 | 0.014477279682299 | 0.014477279682299 |
| 50 | | 0.096597855764522 | 0.014472441635349 | 0.014477279682299 | 0.014477279682299 |

| Error | N | M | 9 | 25 | 63 | 83 |
|---|---|---|---|----|----|----|
| 10 | | 0.082241566745092 | 0.000004846378115 | 0.000000008331165 | 0.000000008331165 |
| 15 | | 0.082164883309524 | 0.000004838046951 | 0.000000000000001 | 0.000000000000001 |
| 25 | | 0.082116428883756 | 0.000004838046950 | 0.000000000000000 | 0.000000000000000 |
| 35 | | 0.082115461800446 | 0.000004838046950 | 0.000000000000000 | 0.000000000000000 |
| 50 | | 0.082120576082222 | 0.000004838046950 | 0.000000000000000 | 0.000000000000000 |

Discrete Polyharmonic cubature for $K = 22$ and $f_2 w^{(2)} = \cos (10r \cos \varphi + 20r \sin \varphi) |r \sin \varphi|$

True value is $\approx 0.0144772796822995$

5.2 Comparison with piece-wise midpoint rule

We want to give briefly the results for the piece-wise midpoint rule for the test function $f_4 (x, y) = 30x^{12}$ with the weight $w^{(2)} (x, y) = |y|$.

We see that even with $500 \times 500$ evaluation points the error is around 0.0001. On the other hand, for the test function $f_5 (x, y) = |y|$ the piecewise midpoint rule is very good, since the integrand is just $|y|^2$ but we do not have exactness.

We omit a discussion of the Peirce rules since the results are very similar to the case of the midpoint cubature rule.

6 Aspects of the Numerical Implementation

The discrete polyharmonic cubature (20) is defined for pseudo-definite weight functions $w (re^{i\varphi})$ on the disk. It is important to determine numerically the nodes and weights for the quadrature of degree $N$ for the univariate integrals

$$\frac{1}{2} \int_0^{R^2} P (\rho) \rho^{k/2} w^{(k,\ell)} (\sqrt{\rho}) d\rho$$

with high accuracy. The nodes are the zeros of the orthogonal polynomial $P_n (\rho)$ of degree $N$ with respect to the measure $\rho^{k/2} w^{(k,\ell)} (\sqrt{\rho})$. Thus any implementation of our algorithm depends on reliable and stable software for finding the nodes and weights of the corresponding quadrature. We
have judiciously chosen the weights \( w^{(1)} \) and \( w^{(2)} \) such that
\[
w_{(k,\ell)} (\sqrt{p}) = C_{(k,\ell)} r^{\alpha(k,\ell)} (1 - r)^{\beta(k,\ell)}
\]
are weight functions of Jacobi type, i.e. of the form \( x^\alpha (1 - x)^\beta \) with \( \alpha, \beta \geq -1 \) in the interval \([0,1]\).
For these weight functions fast and highly accurate programmes for finding the nodes and weights are available (see e.g. [22] or the website: https://www.cs.purdue.edu/archives/2002/wxg/codes) which we have used in our Matlab programs. A more general situation, namely when the Fourier coefficients \( w_{(k,\ell)} (\sqrt{p}) \) are linear combinations of Jacobi type weight functions, is easy to handle.

In our Matlab program we used the readily implemented Matlab function for the Fast Fourier Transform: then the Fourier series is in the form
\[
\sum_{d=0}^{\infty} d \omega_d (r) e^{ik\rho}.
\]
with the usual relations \( w_0 (r) = \frac{1}{2} w_{(0,1)} (r) \) and
\[
w_{(k,1)} (r) = \frac{1}{2} \left( w_k (r) + w_k (r) \right) \quad \text{and} \quad w_{(k,2)} (r) = \frac{1}{2i} \left( w_k (r) - w_k (r) \right).
\]

Here \( w_k (r) := \frac{1}{2\pi} \int_0^{2\pi} w (re^{i\rho}) e^{-ik\rho} d\rho \). For the computation of the approximation \( f_{(k,\ell)} \) to the Fourier coefficients \( f_{(k,\ell)} (r) \) of the integration function \( f (x, y) \) we use the Fourier series representation
\[
\int (re^{i\rho}) = \sum_{k=-\infty}^{\infty} f_k (r) e^{ik\rho}, \quad \text{where} \quad f_k (r) := \frac{1}{2\pi} \int_0^{2\pi} f (re^{i\rho}) e^{-ik\rho} d\rho.
\]
Hence, the integral (11) can be written in the form
\[
I (f) = \sum_{k=-\infty}^{\infty} \int_0^{R} 2\pi f_k (r) w_k (r) r dr = \sum_{k=-\infty}^{\infty} \int_0^{R} \pi f_k (\sqrt{p}) w_k (\sqrt{p}) dp.
\]

In our program we choose always odd \( M \), and the discrete Fourier approximation to \( f_k (r) \) given by
\[
f_k^{(M)} (r) := f^{(M)}_{(k,1)} (r) - i f^{(M)}_{(k,2)} (r) = \frac{1}{M} \sum_{s=1}^{M} \int (re^{2\pi i s/M}) e^{-2\pi i ks/M}
\]
which is the link to the Fast Fourier transform. By subdividing the interval \([0,2\pi]\) into \( M \) subintervals we obtain an approximation
\[
f_k^{M} (r) := \frac{1}{M} \sum_{s=1}^{M} \int (re^{2\pi is/M}) e^{-2\pi is/M}
\]
of \( f_k (r) \) which is just the Discrete Fourier transform (DFT) for the data points \( f (re^{2\pi is/M}) \) for \( s = 1, ..., M \).
7 Concluding Remarks

The discrete polyharmonic cubature formula $I_{(N,M,K)}^{\text{poly}}(f)$ provides excellent numerical results for integrating functions on the disk in the plane with respect to a weight function $w$. A possible drawback for applications might be the high number of evaluations points needed in the formula given by $(2K-1) \cdot N \cdot M$ evaluation points. Since the evaluation of a function value $f(x)$ might be very costly, it is a natural question whether one could modify the formula so that we would need less function evaluations.

In a forthcoming paper [32] we introduce a new cubature formula related to the above formula (21) with a high degree of "approximative exactness", which uses a spline approximation in direction $r$ of the function $f_M^{(k,\ell)}(r)$. The point evaluations for this formula are on a regular grid in both directions $\varphi$ and $r$, and the number of knots is $N_1 \times M$, where $N_1$ is the number of spline knots in direction $r$. We call this a hybrid cubature since we combine spline methods with a cubature formula. In mathematical terms the hybrid formula is of the form:

$$I_{(N,M,K)}^{\text{poly}}(f) \approx \frac{1}{2} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \sum_{j=1}^{N} SPL \left[ \left\{ f_M^{(M)}(R_m) \right\}_{m=0}^{N_1} \right] \left( \sqrt{t_j^{(k,\ell)}} \right) \times t_j^{(k,\ell)} \lambda_j^{(k,\ell)}, \quad (47)$$

where $SPL \left[ \left\{ f_M^{(M)}(R_m) \right\}_{m=0}^{N_1} \right] (t)$ is a univariate spline interpolation function with nodes $\{R_m\}_{m=0}^{N_1}$ for the data $\left\{ f_M^{(M)}(R_m) \right\}_{m=0}^{N_1}$.

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