Generalizations of Triangle Inequalities to Spherical and Hyperbolic Geometry
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Abstract
Certain triangle inequalities involving the circumradius, inradius, and side lengths of a triangle are generalized to spherical and hyperbolic geometry. Examples include strengthenings of Euler’s inequality, \( R \geq 2r \). An extension of Euler’s inequality to a simplex in \( n \)-dimensional space is also generalized to spherical geometry.

1 Introduction
The Euclidean plane, characterized by constant Gaussian curvature \( K = 0 \), gives rise to a multitude of triangle relations. For example, Euler’s inequality relates the circumradius \( R \) and inradius \( r \) of a triangle as
\[
R \geq 2r.
\]
We define spherical geometry by a surface of constant curvature \( K = 1 \) and hyperbolic geometry by curvature \( K = -1 \). In [4], Svrtan and Veljan generalize Euler’s inequality as
\[
\begin{align*}
R &\geq 2r \\
\tan R &\geq 2 \tan r \\
\tanh R &\geq 2 \tanh r
\end{align*}
\]
maintaining the property that equality is achieved if and only if the triangle is equilateral. Our objective is to generalize other Euclidean relations in a similar manner.

Spherical geometry is modeled by the unit sphere in \( \mathbb{R}^3 \). For hyperbolic geometry we use the Klein disk model, where each point on the hyperbolic plane is identified with a point on the open unit disk. Unlike the Poincaré disk model, the Klein model is not conformal; instead, its advantage is that hyperbolic lines are straight lines. Given two points \( p \) and \( q \), we can find their hyperbolic distance \( d(p,q) \) as follows: draw the line through \( p \) and \( q \) and label its points of intersection with the boundary of the disk as \( a \) and \( b \), so that the line contains \( a, p, q, \) and \( b \) in that order. Then
\[
d(p,q) = \frac{1}{2} \ln \frac{|aq| |pb|}{|ap| |qb|}
\]
where vertical bars indicate Euclidean distances. This formula can be used to show that if a point is a Euclidean distance \( r \) from the origin, its hyperbolic distance from the origin is
\[
d(r) = \frac{1}{2} \ln \frac{1 + r}{1 - r} = \tanh^{-1} r.
\]

Our generalizations will make use of the following unifying function, also used in [1] and [2]:
\[
s(x) := \begin{cases} 
\frac{x}{2} & \text{in Euclidean geometry} \\
\sin \frac{x}{2} & \text{in spherical geometry} \\
\sinh \frac{x}{2} & \text{in hyperbolic geometry}
\end{cases}
\]
(1)
The following lemma illustrates a fundamental property of the \( s \)-function.
Lemma 1.1. In Euclidean, spherical, or hyperbolic geometry, if there exists a triangle with side lengths \( a, b, \) and \( c \), then there exists a Euclidean triangle with side lengths \( s(a), s(b), \) and \( s(c) \).

Proof. For Euclidean geometry, the statement is trivial: if \( a, b, \) and \( c \) satisfy the triangle inequality then the same is true for \( a/2, b/2, \) and \( c/2 \).

For spherical geometry, consider a triangle with side lengths \( a, b, \) and \( c \) on the unit sphere in \( \mathbb{R}^3 \). Without loss of generality we may assume all three vertices are at a spherical distance \( R \) from the north pole \((0,0,1)\). By taking the projection of the triangle onto the tangent plane \( z = 1 \) we obtain a Euclidean triangle with side lengths \( 2 \sin \frac{a}{2}/\cos R, 2 \sin \frac{b}{2}/\cos R, \) or \( k \cdot s(a), k \cdot s(b), \) and \( k \cdot s(c) \) with \( k = 2/\cos R \). Thus there exists a Euclidean triangle with side lengths \( s(a), s(b), \) and \( s(c) \).

For hyperbolic geometry, consider a triangle in the Klein disk model with side lengths \( a, b, \) and \( c \). Without loss of generality we may assume all three vertices are at a hyperbolic distance \( R \) from the origin. The Euclidean triangle determined by the same points has side lengths \( k \cdot s(a), k \cdot s(b), \) and \( k \cdot s(c) \) with \( k = 2/\cosh R \). Thus there exists a Euclidean triangle with side lengths \( s(a), s(b), \) and \( s(c) \).

For triangle relations which involve only the side lengths, the \( s \)-function produces an immediate generalization to all three geometries.

Lemma 1.2 (proved in \cite{1}). If \( f(a,b,c) \geq 0 \) holds for all triangles with side lengths \( a, b, c \) in Euclidean geometry, then \( f(s(a),s(b),s(c)) \geq 0 \) holds for all triangles with side lengths \( a, b, c \) in all three geometries.

Proof. Suppose a triangle in Euclidean, spherical, or hyperbolic geometry has side lengths \( a, b, c \). By Lemma 1.1 there exists a triangle in Euclidean geometry with side lengths \( s(a), s(b), s(c) \). The lemma condition then implies \( f(s(a), s(b), s(c)) \geq 0 \).

The \( s \)-function also gives rise to unified formulas for the circumradius and inradius in terms of the side lengths \( a, b, c \):

\[
\frac{2s(a)s(b)s(c)}{\sqrt{s(a+b-c)s(a+c-b)s(b+c-a)s(a+b+c)}} = \begin{cases} 
R & \text{(Euclidean)} \\
\tan R & \text{(spherical)} \\
\tanh R & \text{(hyperbolic)} 
\end{cases}
\]

(2)

\[
\frac{\sqrt{s(a+b-c)s(a+c-b)s(b+c-a)}}{s(a+b+c)} = \begin{cases} 
r & \text{(Euclidean)} \\
\tan r & \text{(spherical)} \\
\tanh r & \text{(hyperbolic)} 
\end{cases}
\]

(3)
2 Strengthenings of Euler’s Inequality

In [5], Svrtan and Wu give a strengthened form of Euler’s inequality in Euclidean geometry:

\[
\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 1 \geq \frac{2}{3} \left( \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \geq 2.
\] (4)

In [1], Black and Smith prove that (4) generalizes to spherical geometry as

\[
\tan R \geq \frac{s(a)s(b)s(c) + s(a)^3 + s(b)^3 + s(c)^3}{2s(a)s(b)s(c)} \\
\geq \frac{s(a)}{s(b)} + \frac{s(b)}{s(c)} + \frac{s(c)}{s(a)} - 1 \\
\geq \frac{2}{3} \left( \frac{s(a)}{s(b)} + \frac{s(b)}{s(c)} + \frac{s(c)}{s(a)} \right) \\
\geq 2.
\]

Moreover, they demonstrate that the same generalization does not work for hyperbolic geometry. They also provide the following unified strengthening of Euler’s inequality for all three geometries.

\[
2 \leq \frac{2s(a+b+c)s(b+c-a)s(c+a-b)}{s(a)s(b)s(c)} \leq \begin{cases} 
\frac{R}{r} & \text{(Euclidean)} \\
\tan R/\tan r & \text{(spherical)} \\
\tanh R/\tanh r & \text{(hyperbolic)} 
\end{cases}
\] (5)

The strengthenings of Euler’s inequality listed below are provided in [3]. In the section to follow, these inequalities are generalized to spherical or hyperbolic geometry.

\[
\frac{R}{2r} \geq \frac{(a + b + c)(a^3 + b^3 + c^3)}{(ab + bc + ca)^2} \geq 1 \\
2R^2 + r^2 \geq \frac{1}{4}(a^2 + b^2 + c^2) \geq 3r(2R - r) \\
\frac{1}{4r^2} \geq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 \geq \frac{1}{2rR}
\] (6-8)

3 Generalization Theorem

3.1 Preliminaries

Our primary result is Theorem 3.2 which generates a simple analogue in either spherical or hyperbolic geometry for triangle inequalities relating a function of \( R \) and \( r \) to a function of \( a, b, \) and \( c \). For a triangle in Euclidean, spherical, or hyperbolic geometry with side lengths \( a, b, c \) let us define the following quantities:

\[
J := \sqrt{s(a + b - c)s(a + c - b)s(b + c - a)}, \\
\tilde{J} := \sqrt{(s(a) + s(b) - s(c))(s(a) + s(c) - s(b))(s(b) + s(c) - s(a))}.
\]

Note that each quantity under the square root is positive by the triangle inequality. We can rewrite the formulas for \( R \) and \( r \) as follows.
\[
\frac{2s(a)s(b)s(c)}{J \cdot \sqrt{s(a + b + c)}} = \begin{cases} 
R \tan R & \text{(Euclidean)} \\
\tan R & \text{(spherical)} \\
\tanh R & \text{(hyperbolic)} 
\end{cases}
\]  \hspace{1cm} (9)

\[
\frac{J}{\sqrt{s(a + b + c)}} = \begin{cases} 
R \tan R & \text{(Euclidean)} \\
\tan R & \text{(spherical)} \\
\tanh R & \text{(hyperbolic)} 
\end{cases}
\]  \hspace{1cm} (10)

The following relation between \(J\) and \(\bar{J}\) was proved in [1].

**Lemma 3.1.** In spherical geometry,

\[ J \leq \bar{J} \leq \sqrt{s(a)s(b)s(c)}. \]

In hyperbolic geometry,

\[ \bar{J} \leq J \leq \sqrt{s(a)s(b)s(c)}. \]

Moreover, in both geometries, \(J = \bar{J}\) if and only if \(a = b = c\).

**Proof.** First we show \(J \leq \bar{J}\) in spherical geometry. We omit the proof of \(J \geq \bar{J}\) in hyperbolic geometry, as it can be obtained simply by reversing the inequalities and replacing sine and cosine with their hyperbolic counterparts. Note that while cosine is decreasing on the interval \([0, \pi/2]\), hyperbolic cosine is increasing on \([0, \infty)\).

To prove \(J \leq \bar{J}\) in spherical geometry, we assume \(a \geq b \geq c\) without loss of generality, then verify the following two statements separately:

\[ s(b + c - a) \leq s(b) + s(c) - s(a), \]  \hspace{1cm} (11)

\[ s(a + b - c)s(a + c - b) \leq (s(a) + s(b) - s(c))(s(a) + s(c) - s(b)). \]  \hspace{1cm} (12)

Note that \(a \geq b\) implies \(2a - b - c \geq b - c\), and since cosine is decreasing on \([0, \pi/2]\) we have

\[ \cos \frac{2a - b - c}{4} \leq \cos \frac{b - c}{4}. \]  \hspace{1cm} (13)

We also make use of the identities

\[ \sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2} \]  \hspace{1cm} (14)

and

\[ \sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y \]  \hspace{1cm} (15)

(hyperbolic sine and cosine also satisfy these identities). Now we show \([11]\):

\[ s(b + c - a) = \sin \frac{b + c - a}{2} \]

\[ = \left( \sin \frac{a}{2} + \sin \frac{b + c - a}{2} \right) - \sin \frac{a}{2} \]

\[ = 2 \sin \frac{b + c}{4} \cos \frac{2a - b - c}{4} - \sin \frac{a}{2} \]

\[ \leq 2 \sin \frac{b + c}{4} \cos \frac{b - c}{4} - \sin \frac{a}{2} \]

\[ = \sin \frac{b}{2} + \sin \frac{c}{2} - \sin \frac{a}{2} \]

\[ = s(b) + s(c) - s(a). \]
On the other hand, (12) is equivalent to
\[
\sin \frac{a + b - c}{2} \sin \frac{a + c - b}{2} \leq \left( \sin \frac{a}{2} + \sin \frac{b}{2} - \sin \frac{c}{2} \right) \left( \sin \frac{a}{2} + \sin \frac{c}{2} - \sin \frac{b}{2} \right)
\]

\[\iff \quad \sin^2 \frac{a}{2} - \sin^2 \frac{b - c}{2} \leq \sin^2 \frac{a}{2} - \left( \sin \frac{b}{2} - \sin \frac{c}{2} \right)^2
\]

\[\iff \quad \sin^2 \frac{b}{2} - \sin^2 \frac{c}{2} \leq \sin^2 \frac{b - c}{2}
\]

\[\iff \quad 2 \sin \frac{b - c}{4} \cos \frac{b + c}{4} \leq 2 \sin \frac{b - c}{4} \cos \frac{b - c}{4}
\]

which also follows from the fact that cosine is decreasing on \([0, \pi/2]\). Moreover, observe that there is equality in (11) if and only if \(a = b\) and equality in (12) if and only if \(b = c\). Thus \(J = J\) if and only if \(a = b = c\), as desired.

Next we prove \(J^2 \leq s(a)s(b)s(c)\) for spherical geometry. Since \(s(a), s(b), s(c)\) satisfy the triangle inequality (by Lemma [1]), we can make the substitution \(s(a) = x + y, s(b) = x + z, s(c) = y + z\) which transforms the inequality into

\[(2x)(2y)(2z) \leq (x + y)(x + z)(y + z)
\]

or

\[xyz \leq \frac{2xyz + x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2}{8}
\]

which is a direct application of the arithmetic mean–geometric mean inequality.

Finally, we prove \(J^2 \leq s(a)s(b)s(c)\) for hyperbolic geometry. With the substitution \(a = x + y, b = x + z, c = y + z\) this is equivalent to

\[s(2x)s(2y)s(2z) \leq s(x + y)s(x + z)s(y + z)
\]

\[\iff \quad \sinh x \sinh y \sinh z \leq \sinh \frac{x + y}{2} \sinh \frac{x + z}{2} \sinh \frac{y + z}{2}
\]

\[\iff \quad 8 \sinh \frac{x}{2} \cosh \frac{x}{2} \sinh \frac{y}{2} \cosh \frac{y}{2} \sinh \frac{z}{2} \cosh \frac{z}{2} \leq \left( \sinh \frac{x}{2} \cosh \frac{y}{2} + \sinh \frac{y}{2} \cosh \frac{x}{2} \right) \cdots
\]

which also follows from the arithmetic mean–geometric mean inequality.

\[\square\]

3.2 Theorem and Proof

Theorem 3.2. Let \(f(x, y)\) and \(g(x, y, z)\) be homogeneous functions of degree \(n\). Suppose

\[f(R, r) \geq g(a, b, c)
\]

holds for all Euclidean triangles with side lengths \(a, b, c\) with equality if and only if \(a = b = c\). Then:

(a) if \(f \left( \frac{2M^2}{x}, x \right)\) is a decreasing function of \(x\) for \(0 < x \leq M\) then

\[f(\tan R, \tan r) \geq 2^n \cdot g(s(a), s(b), s(c)) \cdot \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{n/2}
\]

holds for all spherical triangles with equality if and only if \(a = b = c\);
Similarly, its inradius is

\[ \sqrt{f} \]

Since \( T \) is a spherical function with its hyperbolic counterparts. We only include the proof of part (a), as we can obtain the proof of part (b) by replacing all spherical functions with their hyperbolic counterparts.

Let \( T \) be a spherical triangle with side lengths \( a, b, c \). By Lemma 1.1 there exists a Euclidean triangle \( T' \) with side lengths \( a' = 2s(a), b' = 2s(b) \) and \( c' = 2s(c) \). By (2), the circumradius of \( T' \) is

\[ R' = \frac{2s(a)s(b)s(c)}{\sqrt{(s(a) + s(b) - s(c))(s(a) + s(c) - s(b))(s(b) + s(c) - s(a))(s(a) + s(b) + s(c))}} \]

= \frac{2s(a)s(b)s(c)}{J \cdot \sqrt{s(a) + s(b) + s(c)}}.

Similarly, its inradius is

\[ r' = \frac{\bar{J}}{\sqrt{s(a) + s(b) + s(c)}}. \]

Since \( T' \) is Euclidean, we know

\[ f(R', r') \geq g(a', b', c'). \] (16)

Substituting the expressions for \( a', b', c', R' \), and \( r' \), we have

\[ f \left( \frac{2s(a)s(b)s(c)}{\bar{J} \cdot \sqrt{s(a) + s(b) + s(c)}} \cdot \frac{\bar{J}}{\sqrt{s(a) + s(b) + s(c)}} \right) \geq g(2s(a), 2s(b), 2s(c)). \]

Since \( f \) and \( g \) are homogeneous of degree \( n \) we can write this as

\[ f \left( \frac{2s(a)s(b)s(c)}{J}, J \right) \cdot \left( \frac{1}{s(a) + s(b) + s(c)} \right)^{n/2} \geq 2^n \cdot g(s(a), s(b), s(c)). \] (17)

The condition that \( f \left( \frac{2M^2}{x}, x \right) \) is a decreasing function of \( x \) on \((0, M]\) implies \( f \left( \frac{2M^2}{x_1}, x_1 \right) \geq f \left( \frac{2M^2}{x_2}, x_2 \right) \) for all \( x_1, x_2 \) satisfying \( 0 < x_1 \leq x_2 \leq M \). By Lemma 3.1, we have \( 0 < J \leq \bar{J} \leq \sqrt{s(a)s(b)s(c)} \) for spherical triangles. Taking \( x_1 = J, x_2 = \bar{J} \) and \( M = \sqrt{s(a)s(b)s(c)} \) yields

\[ f \left( \frac{2s(a)s(b)s(c)}{J}, J \right) \geq f \left( \frac{2s(a)s(b)s(c)}{J}, J \right). \] (18)

Combining (17) and (18), we have

\[ f \left( \frac{2s(a)s(b)s(c)}{J}, J \right) \cdot \left( \frac{1}{s(a) + s(b) + s(c)} \right)^{n/2} \geq 2^n \cdot g(s(a), s(b), s(c)). \]

Now we multiply both sides by the quantity \( \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{n/2} \) to get the desired right hand side of

\[ 2^n \cdot g(s(a), s(b), s(c)) \cdot \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{n/2} \]
while the left hand side becomes

\[ f \left( \frac{2s(a)s(b)s(c)}{J}, \frac{1}{s(a+b+c)} \right) \cdot \left( \frac{1}{s(a+b+c)} \right)^{n/2}. \]

Since \( f \) is homogeneous we can rewrite the left hand side:

\[ f \left( \frac{2s(a)s(b)s(c)}{J}, \frac{1}{s(a+b+c)} \right) \cdot \left( \frac{1}{s(a+b+c)} \right)^{n/2} = f \left( \frac{2s(a)s(b)s(c)}{J \cdot \sqrt{s(a+b+c)}}, \frac{J}{\sqrt{s(a+b+c)}} \right) \]

by formulas (9) and (10). So, we have

\[ f(\tan R, \tan r) \geq 2^n \cdot g(s(a), s(b), s(c)) \cdot \left( \frac{s(a) + s(b) + s(c)}{s(a+b+c)} \right)^{n/2} \]

as desired.

We only need to show that equality is achieved if and only if \( a = b = c \). Note that equality is achieved if and only if there is equality in both (17) and (18). Since (17) is equivalent to (16), there is equality if and only if \( a' = b' = c' \) which is true if and only if \( a = b = c \). On the other hand, there is equality in (18) if and only if \( J = \bar{J} \), which occurs if and only if \( a = b = c \) by Lemma 3.1.

The following corollary immediately generalizes (4) and (6) to spherical geometry.

**Corollary 3.3.** Let \( \frac{R}{r} \geq g(a, b, c) \) be an inequality which holds for all Euclidean triangles with circumradius \( R \), inradius \( r \), and side lengths \( a, b, c \). Then for all spherical triangles,

\[ \frac{\tan R}{\tan r} \geq g(s(a), s(b), s(c)). \]

**Proof.** Let \( f(x, y) = x/y \). Then we have

\[ f(R, r) \geq g(a, b, c) \]

with \( f \) and \( g \) homogeneous functions of degree \( n = 0 \). Now since

\[ f \left( \frac{2M^2}{x}, x \right) = \frac{2M^2}{x^2} \]

is a decreasing function of \( x \) for \( x > 0 \), part (a) of Theorem 3.2 yields

\[ f(\tan R, \tan r) \geq g(s(a), s(b), s(c)) \]

for all spherical triangles, as desired.

**3.3 Examples**

To further illustrate the use of Theorem 3.2 we apply it to (7) and (8). These two statements give rise to four separate inequalities relating \( R \) and \( r \) to the side lengths \( a, b, c \) for which we can find an analogue in either spherical or hyperbolic geometry.

**Proposition 3.1** (Generalization of (7) (left inequality) to spherical geometry). All spherical triangles satisfy

\[ 2\tan^2 R + \tan^2 r \geq \left( s(a)^2 + s(b)^2 + s(c)^2 \right) \left( \frac{s(a) + s(b) + s(c)}{s(a+b+c)} \right) \]  

with equality if and only if \( a = b = c \).
Proof. The leftmost inequality in (7) states that all Euclidean triangles satisfy
\[ 2R^2 + r^2 \geq \frac{1}{4}(a^2 + b^2 + c^2). \]  

Let \( f(x, y) = 2x^2 + y^2 \) and \( g(x, y, z) = \frac{1}{4}(x^2 + y^2 + z^2) \). Note that \( f \) and \( g \) are homogeneous of degree 2, and (20) can be stated as
\[ f(R, r) \geq g(a, b, c). \]

To prove the desired spherical analogue, we only need to confirm that
\[ f \left( \frac{2M^2}{x}, x \right) = \frac{8M^4}{x^2} + x^2 \]

is a decreasing function of \( x \) for \( x \in (0, M] \). We take its derivative with respect to \( x \):
\[
\frac{d}{dx} \left( \frac{8M^4}{x^2} + x^2 \right) = -\frac{16M^4}{x^3} + 2x \]
\[
= \frac{2x^4 - 16M^4}{x^3}
\]

which is indeed negative for \( 0 < x \leq M \). Theorem 3.2 tells us that for spherical triangles,
\[ f(\tan R, \tan r) \geq 2^2 \cdot g(s(a), s(b), s(c)) \cdot \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{2/2} \]
or
\[ 2 \tan^2 R + \tan^2 r \geq (s(a)^2 + s(b)^2 + s(c)^2) \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right). \]

Proposition 3.2 (Generalization of (7) (right inequality) to hyperbolic geometry.) All hyperbolic triangles satisfy
\[ (s(a)^2 + s(b)^2 + s(c)^2) \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right) \geq 3 \tanh r(2 \tanh R - \tanh r) \]  
with equality if and only if \( a = b = c \).

Proof. The rightmost inequality in (7) states that all Euclidean triangles satisfy
\[ \frac{1}{4}(a^2 + b^2 + c^2) \geq 3r(2R - r). \]

Let \( f(x, y) = -3y(2x - y) \) and \( g(x, y, z) = -\frac{1}{4}(x^2 + y^2 + z^2) \). Then \( f \) and \( g \) are homogeneous of degree 2 and
\[ f(R, r) \geq g(a, b, c) \]
holds for all Euclidean triangles. By part (b) of Theorem 3.2 since
\[ f \left( \frac{2M^2}{x}, x \right) = -3x \left( \frac{4M^2}{x} - x \right) = 3x^2 - 12M^2 \]
is an increasing function of \( x \) for all \( x > 0 \), the inequality
\[ f(\tanh R, \tanh r) \geq 2^2 \cdot g(s(a), s(b), s(c)) \cdot \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{2/2} \]

is a decreasing function of \( x \) for \( x \in (0, M] \). We take its derivative with respect to \( x \):
\[
\frac{d}{dx} \left( \frac{8M^4}{x^2} + x^2 \right) = -\frac{16M^4}{x^3} + 2x \]
\[
= \frac{2x^4 - 16M^4}{x^3}
\]

which is indeed negative for \( 0 < x \leq M \). Theorem 3.2 tells us that for spherical triangles,
which is equivalent to
\[
(s(a)^2 + s(b)^2 + s(c)^2) \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right) \geq 3 \tanh r (2 \tanh R - \tanh r)
\]
holds for all hyperbolic triangles.

Theorem 3.2 also proves the following two propositions, generalizing (8).

**Proposition 3.3** (Generalization of (8) (left inequality) to spherical geometry). All spherical triangles satisfy
\[
\frac{1}{\tan^2 r} \geq \left( \frac{1}{s(a)^2} + \frac{1}{s(b)^2} + \frac{1}{s(c)^2} \right) \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{-1}
\]
with equality if and only if \(a = b = c\).

**Proposition 3.4** (Generalization of (8) (right inequality) to spherical and hyperbolic geometry).

(a) All spherical triangles satisfy
\[
\frac{1}{3} \left( \frac{1}{s(a)} + \frac{1}{s(b)} + \frac{1}{s(c)} \right)^2 \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{-1} \geq \frac{2}{\tan r \tan R}
\]
with equality if and only if \(a = b = c\).

(b) All hyperbolic triangles satisfy
\[
\frac{1}{3} \left( \frac{1}{s(a)} + \frac{1}{s(b)} + \frac{1}{s(c)} \right)^2 \left( \frac{s(a) + s(b) + s(c)}{s(a + b + c)} \right)^{-1} \geq \frac{2}{\tanh r \tanh R}
\]
with equality if and only if \(a = b = c\).

### 4 Euler’s Inequality in \(n\)-Dimensional Space

We now turn to higher-dimensional space, namely \(n\)-dimensional spherical space, modeled by the unit sphere in \(\mathbb{R}^{n+1}\). We focus on the generalization of the known inequality
\[
R \geq nr
\]
for an \(n\)-dimensional Euclidean simplex with circumradius \(R\) and inradius \(r\).

**Theorem 4.1** (Extension of Corollary 3.3). Let \(\frac{R}{r} \geq f(\{d_{ij}\})\) be an inequality which holds for an \(n\)-dimensional Euclidean simplex (where \(R\) is the circumradius, \(r\) is the inradius, and \(\{d_{ij}\}\) is the set of edge lengths). Then \(\frac{\tan R}{\tan r} \geq f(\{s(d_{ij})\})\) holds for an \(n\)-dimensional spherical simplex.

**Proof.** Assume without loss of generality that a spherical \(n\)-simplex, lying on the unit sphere in \(\mathbb{R}^{n+1}\), has its circumcenter at the north pole \(C = (0, \ldots, 0, 1)\). We may also assume the incenter lies on the two-dimensional \(x_1 x_{n+1}\) plane. By projecting the simplex onto the \(n\)-dimensional hyperplane tangent to the unit sphere at \(C\) (with equation \(x_{n+1} = 1\)) we obtain an \(n\)-dimensional Euclidean simplex with circumradius \(\tan R\) and some inradius \(r'\). On the other hand, the image of its insphere under the same projection is an inscribed ellipsoid of the Euclidean simplex. By symmetry, this ellipsoid has the same radius \(b\) in all directions except along the \(x_1\)-axis where it is longer. If \(d_{ij}\) denotes the length of the spherical edge between vertices \(P_i\) and \(P_j\) then the corresponding Euclidean edge length is \(k \cdot \sin \frac{d_{ij}}{2} = k \cdot s(d_{ij})\) where \(k = 2 / \cos R\). By similarity, there exists a Euclidean simplex...
Figure 2: The projection of the simplex, circumsphere, and insphere onto the tangent plane $x_{n+1} = 1$.

with side lengths $\{s(d_{ij})\}$ and the same circumradius-to-inradius ratio. Since the inequality is true for a Euclidean simplex we have

$$\frac{\tan R}{r'} \geq f(\{s(d_{ij})\})$$

and so it is sufficient to show

$$\frac{\tan R}{\tan r} \geq \frac{\tan R}{r'}.$$

or

$$r' \geq \tan r.$$

Let $O$ be the origin, $I$ the image of the spherical incenter and $Z$ the point where the ray starting at $I$ and heading in the $x_2$-direction intersects the inscribed ellipse (see Fig. 2). Observe that $OZI$ is a right triangle yielding $\tan r = \frac{IZ}{IO}$. It is clear that $IO \geq 1$ (as the $x_{n+1}$-coordinate of $I$ is equal to 1) so we have $\tan r \leq IZ$. But $IZ \leq b$, and $b \leq r'$ as a sphere of radius $b$ with the same center as the inscribed ellipse would be contained in the ellipse and also the simplex. Thus $\tan r \leq r'$. \qed

Theorem 4.1 immediately generalizes (25) to spherical geometry.

**Theorem 4.2.** An $n$-dimensional simplex in spherical geometry with circumradius $R$ and inradius $r$ satisfies

$$\tan R \geq n \tan r.$$

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