Robust and Efficient Estimation in Ordinal Response Models using the Density Power Divergence

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ABSTRACT
In real life we frequently come across data sets that involve some independent explanatory variables, generating a set of ordinal responses. These ordinal responses may be thought to depend on a continuous latent variable and a set of unknown cut-off points. The latent variable is further assumed to be linearly related to the explanatory variables which in turn drive the ordinal responses. One way of estimating the unknown parameters (i.e., the regression coefficients and the cut-offs), that comes through modelling the data, is to find its maximum likelihood estimates. Maximum likelihood estimates are noted for being fully asymptotically efficient at the true model. However, a small proportion of outlying observations, e.g., responses incoherent to the categories or unbounded covariate(s) may destabilize the likelihood function to a great extent. Therefore, the reliability of the maximum likelihood estimates is strongly affected. Minimum distance methods are widely used in the literature when robustness is a concern. In the present paper, we use the density power divergence to estimate the parameters. The roles of different link functions are analyzed through the lens of the density power divergence. Asymptotic properties of the minimum density power divergence estimator are discussed theoretically. Its robustness is investigated through the influence function analysis. Analytically, we have shown that the proposed class of estimates has a very high asymptotic breakdown point against data contamination. Numerically it is further demonstrated that the proposed method yields slope estimates that never implode towards zero. The finite-sample performance of the minimum density power divergence estimators is investigated through extensive numerical experiments, either at the model or when data contamination occurs across different link functions. It is shown to outperform the maximum likelihood estimators, producing more stable results in the face of data contamination. Moreover, our estimators are very competitive with the other robust alternatives. Finally, we wrap up this article with an application on a real data example.

KEYWORDS
Latent linear regression model, Robustness, Minimum density power divergence estimator

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1. Introduction

In recent years, the analysis of ordinal response data has become a popular topic in mainstream research. Such data arise naturally in many areas of scientific studies, for instance in psychology, sociology, economics, medicine, political science, and in several other disciplines where the final response of a subject belongs to a finite number of ordered categories based on the values of several explanatory variables in a way described later in this article. One such example may be the qualitative customer review of a particular vehicle where its price, mileage, carbon emission properties, etc., are to be taken into account to arrive at a qualitative review on an ordinal scale. While these ratings summarize many important explanatory variables and are primarily useful to a new customer, these customer feedbacks often turn out to be equally important to the manufacturer as the latter might want to learn about the statistical relationship between the ordinal response and its covariates to improve their product, or for post-manufacturing surveys to fix things.

A pioneering work in this field has been done by McCullagh [1] who advocated the use of an underlying continuous latent variable that drives ordinal responses based on some unknown cut-offs. This method has become popular as it enables us to view the ordinal response model within a unified framework of the generalized linear model (GLM); see, e.g., McCullagh [3] and Nelder et al. [2]. Moustaki [4] uses the maximum likelihood (ML) method to fit a multi-dimensional latent variable model to a set of observed ordinal variables and also discusses the related goodness-of-fit problem. Also, see Moustaki [5] for related discussions. Piccolo [6] and Iannario et al. [17] suggest a different approach which uses the response variable as a combination of a discrete mixture of uniform and a shifted binomial (CUB) random variables.

Although the area of robust statistics has a very rich and developed body of literature, applications in the direction of ordinal response data are rather limited. An early reference is Hampel [8], where, in addition to the development of a classical infinitesimal approach to robustness, some pointers about robustness in the case of binomial model fitting are discussed. Robust estimators have been developed by Victoria-Feser and Ronchetti [9] for grouped data. Ruckstuhl and Welsh [10] have considered different classes of estimators in the context of fitting a robust binomial model to a data set. Moustaki and Victoria-Feser [11, 12] have developed bounded-bias and bounded-influence robust estimators for the generalized linear latent (GLL) variable. Lack of robustness in the maximum likelihood estimates for the logistic regression model has been extensively studied in the literature (Croux et al. [13]; Müller and Neykov [14]). Croux et al. [15] have proposed a weighted maximum likelihood (WML) estimation method for the logit link function. Iannario et al. [7] dealt with robustness for the class of CUB models. More recently, Iannario et al. [18] used a weighted likelihood function where weights vary depending on the choice of link functions. Unlike the approaches of Croux et al. [15], who propose to use weights as functions of robust Mahalanobis type distances, Iannario et al. [18] considered Huber’s weight functions that combine both the generalized residuals and robust Mahalanobis distance or the normalized MAD as appropriate for different link functions. The primary objective aims at controlling the influential observations with respect to the parametric model. Recently, Scalera et al. [19] analyzed the role of different link functions towards robustness in this context.

In this article, we propose to use the density power divergence (DPD) measure, proposed by Basu et al. [23], to obtain robust and (asymptotically highly) efficient estimates in ordinal response data under the same setup as Iannario et al. [18]. The independent but non-homogeneous version of the DPD-based inference (Ghosh and Basu, [25]) is best suited for this application.
The key highlights of this paper are the following.

(a) To study the role of different link functions in estimating the slope parameter, we plot a DPD-version of the generalized residuals. As it turns out, the DPD-version of the generalized residuals stays bounded even for the commonly used links that produce unbounded generalized residuals in the MLE. This gives a clear insight into why DPD should produce robust slope estimates.

(b) Asymptotic properties of the proposed minimum DPD estimator (MDPDE) are discussed in the context of the ordinal response regression problem.

(c) Robustness of the minimum density power divergence functional is investigated through the influence function analysis. As expected, we find the effect of infinitesimal data contamination to be limited, as compared to the MLE, whenever the tuning parameter $\alpha$ is strictly positive.

(d) We have shown that under suitable assumptions, the asymptotic breakdown point of the MDPDE is $\frac{1}{2}$, the maximum possible value, compared to the very low breakdown of MLE.

(e) We have empirically shown that the implosive breakdown point of the MDPDE of the regression parameter is very high. This means that even when the sample contains a high proportion of outlying observations, we can still obtain a stable MDPDE of the slope parameter.

(f) In continuation of the earlier point, these estimates are also used to find the prediction misclassification rate. When $\alpha > 0$, the misclassification rate incurred by the MDPDE becomes much lower than the MLE.

(g) Through extensive simulation studies, we show that our proposed estimator outperforms the MLE in the presence of outlying observations at higher values of the tuning parameter $\alpha$. Also, it is almost as good as the MLE in the true model when $\alpha$ is relatively small. Our estimator also performs better than the weighted likelihood estimator proposed by Croux et al. [15], and it is very competitive to the M-estimator proposed by Iannario et al. [18]. Computationally our method is less expensive than Iannario et al. [18] (See Figure S.Fig. 4. in the supplementary material).

(h) To apply this method in real-life data examples, we make use of a tuning parameter selection algorithm, as proposed by [29]. The performance of this algorithm is validated through simulation studies.

(i) Finally, we analyze a real data example with the proposed robust estimator, where we choose the optimal robustness parameter using the above tuning parameter selection algorithm. The prediction from the resulting estimator achieves no lesser (higher in some cases) accuracy than the MLE.

The rest of this article is organized as follows. In Section 2 we state the problem under study, and briefly review the maximum likelihood estimation in the present setup. Next, we introduce the method of minimum density power divergence estimation in Section 3, and further discuss the role of generalized residuals under different error distributions in view of this divergence measure. Asymptotic properties of the minimum density power divergence estimators (i.e., consistency and asymptotic normality) are discussed in Section 5. The robustness of the proposed estimator is investigated in Section 6. In particular, we plot the gross error sensitivity in Subsection 6.1, and present an asymptotic breakdown point result in Subsection 6.2. We
also study the implosion resistance property of the slope estimates in Subsection 6.3. Finite-sample performance of the proposed estimator is compared with the MLE, and the estimators proposed by Croux et al. [15] and Iannario et al. [18] in Section 7. Next, we briefly discuss a data-driven strategy for the tuning parameter selection in Section 8. This algorithm is validated through a simulation study. Further, we apply this algorithm to a real data example in Section 9. Concluding remarks have been made in Section 10. The proofs of all the results and some additional tables and figures are provided in the Supplementary material.

2. Parametric Model and the Maximum Likelihood Estimation

Consider a random sample \( \{(x_i, Y_i) : i = 1, 2, \ldots, n\} \) of size \( n \). The \( i \)-th explanatory vector, denoted by \( x_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^T \), is assumed to be non-stochastic in \( \mathbb{R}^p \). Here \( Y_i \) is the realization of the response variable \( Y \) when conditioned on \( x_i \). Further, \( Y \) is supported on a finite set \( \chi = \{1, 2, \ldots, m\} \). Following McCullagh [1] we presume that there exists a continuous latent random variable \( Y^* \) such that it is related to \( Y \) as

\[
Y = j \iff \gamma_{j-1} < Y^* \leq \gamma_j \quad \text{for } j \in \chi.
\]

where \( -\infty = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} < \gamma_m = +\infty \) are the unknown cut-off points (thresholds) in the continuous support of the latent variable. Moreover, \( Y_i \) depends on the explanatory variable \( x_i \) as

\[
Y^*_i = x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ip}\beta_p + e_i = x_i^T\beta + e_i \quad \text{for all } i = 1, 2, \ldots, n.
\]

Here \( \beta = (\beta_1, \beta_2, \ldots, \beta_p)^T \) is a vector of regression coefficients in the latent linear (LL) regression model with \( e_i \) being a random error term. The \( e_i \)'s are assumed to be identically and independently distributed according to a known probability distribution function \( F \). The inverse of \( F \) is called the link function. We assume that \( F \) admits a probability density function \( f_i \), and further denote \( \gamma = (\gamma_1, \ldots, \gamma_{m-1})^T \). Using (2.2) we find that

\[
p_{\theta,i}(j) = Pr(Y = j|x_i) = F(\gamma_j - x_i^T\beta) - F(\gamma_{j-1} - x_i^T\beta) \quad \text{where } \theta = (\gamma, \beta),
\]

for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). It is obvious that \( p_{\theta,i}(1) = F(\gamma_1 - x_i^T\beta) \) and \( p_{\theta,i}(m) = 1 - F(\gamma_{m-1} - x_i^T\beta) \). Here the parameter space is denoted by \( \Theta \subseteq \mathbb{R}^{m+p-1} \). Later on, we shall interchangeably use the term “slope parameters” for “regression parameters”.

Now we wish to find an estimate of \( \theta \). A traditional way of doing that is to find its maximum likelihood estimate \( \hat{\theta}_{MLE} \) which maximizes the log-likelihood function

\[
\sum_{i=1}^{n} \ell(\theta; x_i, Y_i) \quad \text{where } \ell(\theta; x_i, Y_i) = \sum_{j=1}^{m} \delta_i(j) \ln p_{\theta,i}(j).
\]

Here \( \delta_i(j) = 1(Y = j|x_i) \) where \( 1(\cdot|x_i) \) symbolizes the indicator of the set \( \{Y = j\} \) given at \( x_i \), and \( \ln(\cdot) \) represents the natural logarithm. The log-likelihood function for the entire sample turns out to be the sum of individual log-likelihood functions \( \ell(\theta; x_i, Y_i) \) evaluated at each data point \( (x_i, Y_i) \), \( i = 1, 2, \ldots, n \). Under appropriate regularity conditions \( \hat{\theta}_{MLE} \) consistently estimates true \( \theta \). It is well known that the MLE is asymptotically the most efficient among
the class of consistent and uniformly asymptotically normal (CUAN) estimators. However, in practice, we hardly come across a data set that truly follows an assumed model. Often a model is deemed a good fit to a data set if the majority of the data points follow that model, leaving out only a small proportion (maybe 5% - 10%) of observations that are inconsistent with the model. Observations, that defy the assumed probability distribution, are deemed outliers with regard to that model. The reliability of the MLE becomes questionable even in the presence of a single outlier. This is a problem with the MLE whenever robustness is a concern. Often robustness comes at the cost of asymptotic efficiency, e.g., trimmed MLE, median, etc. To overcome this, it is therefore required to resort to alternative robust methodologies that would make use of the same model but yield estimators balancing between the extreme situations—asymptotic efficiency and robustness.

In the vast literature of robust statistics, methods often focus on two primary aspects to eliminate or limit the influence of outlying observations in the estimation process. The most intuitive way is to multiply the log-likelihood by a suitable weight function. Many different types of weights may be suggested along the way. Croux et al. [15] propose one such weight function that uses the robust Mahalanobis distance in the space of explanatory variables. In the second approach, a weighted average of the likelihood score is set to zero, and solved for the unknown parameter. Iannario et al. [18] take this second approach. These are expected to lead to robust estimates. Methods that deal with the parameter estimation in ordinal response models are primarily limited to these two methods. In this paper, we use the density power divergence (Basu et al. [23]) to find robust parameter estimates with high asymptotic efficiency.

3. The Density Power Divergence and Estimating Equations

The density power divergence (DPD) between two probability density functions $g$ and $q$ (with respect to a common dominating measure), indexed by a tuning parameter $\alpha$, is defined as

$$d_\alpha(g, q) = \int_S \left\{ q^{1+\alpha} - \left(1 + \frac{1}{\alpha}\right)q^\alpha g + \frac{1}{\alpha} q^{1+\alpha}g \right\} \text{ for } \alpha > 0. \quad (3.1)$$

Here $S$ denotes the support common to both $q$ and $g$. In the discrete case, this divergence may be accordingly modified by replacing the integration with the summation. Although the divergence would be undefined when we simply substitute $\alpha = 0$ in (3.1), its limit is well-defined when $\alpha \downarrow 0^+$. The latter is defined to be the value of $d_0(g, q)$. Some routine algebraic manipulation shows that

$$d_0(g, q) = \int_S g \ln \frac{g}{q}. \quad (3.2)$$

This is a version of the Kullback-Leibler divergence.

Now consider the problem of minimum distance estimation. Let a data set, generated by the probability distribution function $G$, be modelled with $Q = \{Q_\theta; \theta \in \Theta\}$. $G$ and $Q_\theta$ are supposed to admit probability density functions $g$ and $q_\theta$ respectively. Then the minimum density power divergence (MDPD) functional $T_\alpha(G)$ is defined as a minimizer of $d_\alpha(g, q_\theta)$ over $\Theta$. When $\alpha = 0$ it becomes the maximum likelihood functional $T_0(G)$, and we get the minimum $L_2$ distance functional at $\alpha = 1$. Since the third term in the divergence does not involve $\theta$, the essential
objective function may be given by
\[
\int_S \left\{ q_\theta^{1+\alpha} - \left( 1 + \frac{1}{\alpha} \right) q_\theta^\alpha dG \right\} \text{ for } \alpha > 0.
\] (3.3)

In practice, the true distribution function \( G \) is unknown to us. This may be estimated using the empirical distribution function \( G_n \) based on an iid random sample. Thus the minimum density power divergence estimator is given by
\[
\hat{\theta}_\alpha := \arg \min_{\theta \in \Theta} \int_S \left\{ q_\theta^{1+\alpha} - \left( 1 + \frac{1}{\alpha} \right) q_\theta^\alpha dG_n \right\}.
\] (3.4)

Alternatively, we also use the notation \( T_\alpha(G_n) \) for \( \hat{\theta}_\alpha \). Notice that the dependency of the model on data appears linearly in (3.4). Therefore, it does not require the use of any nonparametric kernel smoothing even for the continuous models, which is unavoidably necessary for any other \( \phi \)-divergences. See Basu et al. [23] for more details. The minimum density power divergence estimator belongs to the class of M-estimators. An M-estimator \( \hat{\theta}_M \) solves
\[
\int \psi_\theta dG_n = 0
\] (3.5)

where \( \psi_\theta \) is a real-valued function. For the MDPDE, we have \( \psi_\theta = (1 + \alpha) \left\{ \int q_\theta^\alpha \nabla q_\theta - q_\theta^{\alpha - 1} \right\} \).

Here \( \nabla \) denotes the partial derivative operator with respect to \( \theta \).

When the random observations are independent but not necessarily identically distributed, the DPD may be generalized in a variety of different ways. However, we shall follow the approach of Ghosh and Basu [25]. Now, we consider our problem setup. Let \( G_i \) be the probability distribution function that generates ordinal responses \( Y_i; i = 1, 2, \ldots, n \). The sample observations \( Y_i \)s are independent but distributed according to possibly different distribution functions \( G_i \)s. It is further assumed that \( G_i \)s admit probability density functions \( g_i \)s with respect to a common dominating measure for \( i = 1, 2, \ldots, n \). For each \( i \), the true density \( g_i \) is modelled by \( p_{\theta,i} \). For the \( i \)-th data point, the DPD is given by
\[
d_\alpha(g_i, p_{\theta,i}) = \begin{cases} 
\sum_{j=1}^m p_{\theta,i}(j)^{1+\alpha} - \left( 1 + \frac{1}{\alpha} \right) p_{\theta,i}(j)^\alpha g_i(j) + \frac{1}{\alpha} g_i(j)^{1+\alpha} & \text{ when } \alpha > 0 \\
\sum_{j=1}^m g_i(j) \ln \frac{g_i(j)}{p_{\theta,i}(j)} & \text{ when } \alpha = 0.
\end{cases}
\] (3.6)

The overall divergence is defined as the arithmetic mean of individual divergence measures, i.e.,
\[
\frac{1}{n} \sum_{i=1}^n d_\alpha(g_i, p_{\theta,i}).
\] (3.7)

The minimum density power divergence functional \( \theta_\alpha \) minimizes (3.7). Notice that \( \theta_\alpha \) depends
on the true distributions $G_1, \ldots, G_n$. Essentially, $\theta_\alpha$ minimizes the following objective function

$$H(\theta) = \frac{1}{n} \sum_{i=1}^{n} H^{(i)}(\theta),$$

where $H^{(i)}(\theta)$ is obtained from (3.7) excluding its terms independent of $\theta$. To find the minimum density power divergence estimate, we substitute $\delta_i(j)$ for $g_i(j)$ in the above expressions. The empirical version of $H^{(i)}$ is given by

$$V_i(x_i, Y_i, \theta) = \begin{cases} 
\sum_{j=1}^{m} p_{\theta,i}(j)^{1+\alpha} - \left(1 + \frac{1}{n}\right) p_{\theta,i}(Y_i)\alpha & \text{when } \alpha > 0 \\
- \ln p_{\theta,i}(Y_i) & \text{when } \alpha = 0.
\end{cases}$$

Consequently, $\hat{\theta}_\alpha$ minimizes

$$H_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} V_i(x_i, Y_i, \theta)$$

over the parameter space $\Theta$. When the error distribution $F$ as in (2.3) is differentiable, the minimum density power divergence estimator $\hat{\theta}_\alpha$ can be obtained by solving the estimating equation

$$\nabla H_n(\theta) = \frac{(1 + \alpha)}{n} \left\{ \sum_{j=1}^{m} p_{\theta,i}(j)^{1+\alpha} u_{\theta,i}(j) - p_{\theta,i}(Y_i)\alpha u_{\theta,i}(Y_i) \right\} = 0,$$

where $u_{\theta,i}(j) = \nabla p_{\theta,i}(j)$ is the likelihood score function. Notice that $\nabla p_{\theta,i}(j)$ is a vector-valued function which is given by

$$\nabla p_{\theta,i}(j) = \left( \frac{\partial}{\partial \gamma^T} p_{\theta,i}(j), \frac{\partial}{\partial \beta^T} p_{\theta,i}(j) \right)^T \in \mathbb{R}^{m+p-1}.$$ 

A simple calculation shows that

$$\frac{\partial}{\partial \gamma_s} p_{\theta,i}(j) = \begin{cases} 
f(\gamma_s - x_i^T \beta) & \text{when } j = s \\
-f(\gamma_s - x_i^T \beta) & \text{when } j = s + 1 \\
0 & \text{otherwise}
\end{cases}$$

and

$$\frac{\partial}{\partial \beta_k} p_{\theta,i}(j) = \left\{ f(\gamma_{j-1} - x_i^T \beta) - f(\gamma_j - x_i^T \beta) \right\} x_{ik}.$$
for $s = 1, 2, \ldots, (m - 1)$ and $k = 1, \ldots, p$. Equations (3.13) and (3.14) would together imply

$$
\frac{1}{1 + \alpha} \cdot \frac{\partial}{\partial \gamma_s} V_i(x_i, Y_i, \theta) = \left\{ p_{\theta, s}(s)^{\alpha} f(\gamma_s - x_i^T \beta) - p_{\theta, s}(Y_i)^{\alpha - 1} f(\gamma_{Y_i} - x_i^T \beta) I(Y_i = s|x_i) \right\}
$$

$$
- \left\{ p_{\theta, s}(s + 1)^{\alpha} f(\gamma_s - x_i^T \beta) - p_{\theta, s}(Y_i)^{\alpha - 1} f(\gamma_{Y_i - 1} - x_i^T \beta) I(Y_i = s + 1|x_i) \right\},
$$

(3.15)

$$
\frac{1}{1 + \alpha} \cdot \frac{\partial}{\partial \beta_k} V_i(x_i, Y_i, \theta) = x_{ik} \left[ \sum_{j=1}^{m} p_{\theta, s}(j)^{\alpha} \left\{ f(\gamma_{j - 1} - x_i^T \beta) - f(\gamma_j - x_i^T \beta) \right\} - p_{\theta, s}(Y_i)^{\alpha - 1} \left\{ f(\gamma_{Y_i - 1} - x_i^T \beta) - f(\gamma_{Y_i} - x_i^T \beta) \right\} \right]
$$

(3.16)

for $s = 1, 2, \ldots, (m - 1)$ and $k = 1, 2, \ldots, p$. Using (3.15) and (3.16) the estimating equations in (3.11) may be further simplified. Observe that the quantity in (3.11) is unbiased when the true densities belong to the model families, i.e., $g_i(j) = p_{\theta, s}(j)$ for all $j = 1, \ldots, m$ and $i = 1, \ldots, n$. In that case, the minimum density power divergence functional is Fisher consistent, i.e., $\theta_{\alpha} = \hat{\theta}_s$ for true $\theta_s$.

Another justification for adapting the general theory of the non-homogeneous DPD over the usual one in this context is the following. Assume that $(X, Y)$ is jointly distributed according to a probability distribution that involves all the parameters of our interest. Then a single DPD can still be constructed between the data and a model using the original formulation of Basu et al. [23]. Consequently, all the parameters may be estimated by minimizing the divergence albeit with computational complexity that may arise due to modelling both the covariates and response variables in a higher dimension. This may be completely avoided or at least reduced to a great extent if we take the conditioning approach on the explanatory variables keeping the usual flavour of regression analysis as widely used in most of its applications. This, in a way, gives a reason for using the non-homogeneous version of the DPD in the context of the present situation. Not only that, the loss of efficiency incurred by the ordinary MDPDE which usually occurs in higher dimensions at a fixed $\alpha$, may be completely circumvented in this approach.

4. DPD-version of the Generalized Residuals

Next, we discuss the role of generalized residuals as introduced by Iannario et al. [18]. We know that $-\frac{1}{1 + \alpha} \sum_{i=1}^{n} V_i(x_i, Y_i, \theta)$ is akin to the log-likelihood function. It is called the $\beta$-likelihood function (cf. Fujisawa and Eguchi [24]). To find the DPD-version of the generalized residuals, we express

$$
- \frac{1}{1 + \alpha} \sum_{i=1}^{n} \frac{\partial}{\partial \beta_k} V_i(x_i, Y_i, \theta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathcal{E}_{ij}(\theta, \alpha) x_{ik} \quad \text{for all } k,
$$

(4.1)

where $\mathcal{E}_{ij}(\theta, \alpha) = \left[ p_{\theta, s}(j) - \delta_i(j) \right] e_{ij}(\theta)p_{\theta, s}(j)^{\alpha}$ with $e_{ij}(\theta)$ being the generalized residuals as in (6) of Iannario et al. [18]. Here $\mathcal{E}_{ij}(\theta, \alpha)$ plays the same role in the estimation of $\hat{\beta}_\alpha$ as
$\delta_i(j)e_{ij}(\theta)$ does for $\hat{\beta}_{ML}$, the likelihood estimates of the slope parameters. Excluding the term $[p_{\theta,i}(j) - \delta_i(j)]$, which is bounded by 2 anyway, we may consider $e_{ij}(\theta)p_{\theta,i}(j)^\alpha$ as a simple analogue for generalized residuals in the context of the DPD. To study its behaviour, denote

$$B_j(t) = A_j(j) \left[ F(\gamma_j - t) - F(\gamma_{j-1} - t) \right]^\alpha \quad \text{for } j = 1, 2, \ldots, m$$

where $t = x_i^T \beta$ and

$$A_j(t) = \frac{f(\gamma_j - t) - f(\gamma_{j-1} - t)}{F(\gamma_j - t) - F(\gamma_{j-1} - t)}.$$

Here $A_j(t)$ comes from Equation (9) in Iannario et al. [18]. As in Figure 1 - Figure 4 of Iannario et al. [18], we plot $B_j(t)$ in Figure 4.1 and Figure 4.2 in a panel for different values of the tuning parameter for the probit and complementary log-log link functions. Figures for the Cauchy and logit link are moved to the supplementary material. In all these graphs, we find that when $\alpha$ increases from 0 to 1, the magnitude of the DPD-version of the generalized residuals is significantly dampened. An outlying observation can make the generalized residuals unbounded when the probit and the complementary log-log link are used in the likelihood-based procedure. However, these outliers only have a limited impact on the DPD-based procedure. This also explains why the minimum density power divergence estimate of the slope parameter in the ordinal response models is more stable than the MLE when robustness is a concern.

**Figure 4.1.** Generalized residuals for the probit link with thresholds $\gamma = (-2.5, -1, 0, 1, 2.5)^T$.

**Figure 4.2.** Generalized residuals for the complementary log-log link with thresholds $\gamma = (-2.5, -1, 0, 1, 2.5)^T$.  

5. Asymptotic Properties

Now, we shall present the weak consistency and asymptotic normality results for \( \hat{\theta}_\alpha \). These follow from Ghosh and Basu [25] with some suitable modifications. Let us introduce the quantities \( J^{(i)}(\alpha) \) and \( \xi_i(\alpha) \) as

\[
J^{(i)}(\alpha) = \frac{1}{1 + \alpha E_g} \left[ \nabla^2 V_i(x_i, Y_i, \theta_\alpha) \right], \quad \xi_i(\alpha) = \sum_{j=1}^m u_{\theta,i}(j)p_{\theta,i}(j)g_i(j).
\]

The matrix \( J^{(i)}(\alpha) \) is assumed to be positive definite for \( i = 1, 2, \ldots, n \). Also, define

\[
\Psi_n(\alpha) = \frac{1}{n} \sum_{i=1}^n J^{(i)}(\alpha),
\]

\[
\Omega_n(\alpha) = \frac{1}{n} \sum_{i=1}^n \text{Var}_g \left[ \nabla V_i(x_i, Y_i, \theta_\alpha) \right].
\]

We now present the following assumptions that will be necessary to apply the consistency and asymptotic normality result of Ghosh and Basu [25].

(A1) The best-fitting parameter \( \theta_\alpha \) is an interior point of \( \Theta \).

(A2) The error distribution function \( F \) is thrice continuously differentiable with respect to its argument having bounded derivatives.

(A3) The matrices \( J^{(i)}(\alpha) \)s are positive definite for all \( i \), and

\[
\lambda_0 := \inf_n \left[ \min \text{ eigenvalue of } \Psi_n(\alpha) \right] > 0.
\]

(A4) The vector \( x_i = (x_{i1}, \ldots, x_{ip})^T \) is such that the following conditions are true:

\[
\frac{1}{n} \sum_{i=1}^n |x_{ij}x_{ij'}x_{ij^*}| = O(1), \quad \sup_{1 \leq i \leq n} |x_{ij}| = O(1) \quad \text{and} \quad \sup_{1 \leq i \leq n} |x_{ij}x_{ij'}| = O(1)
\]

for all \( j, j', j^* = 1, 2, \ldots, p \).

Remark 5.1. It may be easily checked that

\[
f'(x) = \begin{cases} 
\frac{e^{-x}(e^{-x}-1)}{(e^{-x}+1)^2} & \text{when } X \sim \text{Logistic}(0,1), \\
-\frac{1}{\sqrt{2\pi}} e^{-x^2/2} & \text{when } X \sim N(0,1), \\
-\frac{1}{2} \cdot \frac{e^{-x}}{(1+e^x)^2} & \text{when } X \sim \mathcal{C}(0,1), \\
e^x-e^x(1-e^x) & \text{when } F(x) = 1 - e^{-e^x}.
\end{cases}
\]

See that \( f' \)s are bounded for all these above links as long as we assume \( 0 \times \infty = 0 \). In all these cases, density functions \( f \) and \( f'' \) are also bounded. This boundedness ensures that Assumption (A2) is satisfied for these link functions. Assumption (A3) refers to the condition that the
smallest eigen root of $\Psi_n(\alpha)$ should stay positive in limit. Also, Assumption (A4) requires that the cross-products between the components of the non-stochastic covariates $x_i$s be bounded.

**Theorem 5.1.** Suppose the Assumptions (A1) to (A4) are true. Then the following holds:

(a) $\hat{\theta}_n \xrightarrow{P} \theta_\alpha$ as $n \to +\infty$,

(b) $\sqrt{n} \Omega_n^{-1}(\alpha)(\hat{\theta}_n - \theta_\alpha) \xrightarrow{L} \mathcal{N}(0, I_{m+p-1})$ as $n \to +\infty$.

**Remark 5.2.** Let the true distributions belong to the model families, i.e., $g_i(j) = p_{\theta_*,i}(j)$ for all $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. In this case, we get $\xi_i(0) = 0$ and $\Psi_n(0) = \Omega_n(0) = I(\theta_*)$ where $I(\theta_*)$ is the Fisher information. So the asymptotic covariance matrix of the MLE becomes $I^{-1}(\theta_*)$.

**Remark 5.3.** In simulation studies, one may compute the MSE to compare the performance of $\hat{\theta}_n$ with the MLE. In such cases, the observed efficiency (Eff) of $\hat{\theta}_n$ may be defined as $\text{Eff} = \frac{\text{MSE}_{\text{MLE}}}{\text{MSE}_{\hat{\theta}_n}}$. Because the estimators are consistent when the true distributions belong to the model family, it approximately becomes

$$\text{Eff} \approx \frac{\text{tr}(\Psi_n^{-1}(0)\Omega_n(0)\Psi_n^{-1}(0))}{\text{tr}(\Psi_n^{-1}(\alpha)\Omega_n(\alpha)\Psi_n^{-1}(\alpha))}$$

for sufficiently large $n$, where $\text{tr}(A)$ denotes the trace of a matrix $A$. When true distributions belong to the model family, smaller values of $\alpha$ should perform almost as good as the MLE. If there are some outliers in a data set, the performance of the MLE may become very unstable depending on the amount of anomaly in the data. As the MDPDE naturally downweights those outlying observations with respect to the model, we expect that the MDPDE will have superior performance over the MLE in the presence of outliers, particularly for relatively large values of $\alpha$.

**Remark 5.4.** The asymptotic covariance matrix of $\sqrt{n}\hat{\theta}_n$ is given by $(\Psi_n^{-1}(\alpha)\Omega_n(\alpha)\Psi_n^{-1}(\alpha))$. This needs to be estimated in real data analysis. A consistent estimator of $\Psi_n(\alpha)$ is obtained by plugging in $\hat{\theta}_n$ and $\delta_i$ respectively for $\theta_\alpha$ and $g_i$ in (5.2). This gives

$$\hat{\Psi}_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=1}^{m} \left\{ \nabla u_{\hat{\theta}_n,i}(j) + (1 + \alpha)u_{\hat{\theta}_n,i}(j)u_{\hat{\theta}_n,i}(j)^T \right\} p_{\hat{\theta}_n,i}(j)^{1+\alpha} \\
- \left\{ \nabla u_{\hat{\theta}_n,i}(Y_i) + \alpha u_{\hat{\theta}_n,i}(Y_i)u_{\hat{\theta}_n,i}(Y_i)^T \right\} p_{\hat{\theta}_n,i}(Y_i)^{\alpha} \right].$$

(5.8)

However $\Omega_n(\alpha)$ cannot be estimated using only a single observation $Y_i$ that comes from $g_i$, $i = 1, 2, \ldots, n$. To tackle this case, therefore, we make use of the model densities as proxies for true
densities in (5.3) substituting \( \hat{\theta}_\alpha \) for \( \theta_\alpha \). Thus we obtain

\[
\hat{\Omega}_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} u_{\hat{\theta}_\alpha, i}(j) u_{\hat{\theta}_\alpha, i}(j)^T p_{\hat{\theta}_\alpha, i}(j)^{2\alpha+1} - \hat{\xi}_i(\alpha) \hat{\xi}_i(\alpha)^T \right\},
\]

(5.9)

\[
\hat{\xi}_i(\alpha) = \sum_{j=1}^{m} u_{\hat{\theta}_\alpha, i}(j) p_{\hat{\theta}_\alpha, i}(j)^{1+\alpha}.
\]

(5.10)

These estimates will be made use of while doing the tuning parameter selection.

6. Robustness Studies

In this section, we will study the robustness of the minimum distance functional. This is, in fact, the main theme of this paper. This section contains the following three parts – influence function analysis, asymptotic breakdown point analysis, and a discussion of the implosion resistance property of the slope estimates.

6.1. Influence Function Analysis

The influence function (IF) is one of the most popular measures of robustness in studying the impact of infinitesimal data contamination on a statistical functional. Essentially, an estimate having a bounded influence function exhibits stable behaviour in the presence of very extreme outlying observations. Here, we shall present the influence function of the MDPD functional \( \theta_\alpha \) that minimizes \( H(\theta) \). Given our setup, where \( x_i \)'s are fixed carriers, an outlier may only occur in the vertical direction (i.e., in the \( Y \)-space). Therefore, having contamination at the \( i \)-direction in the vertical space only perturbs the distribution of the total mass over the set of ordinal responses given at fixed \( x_i \). This may be described as the true distribution \( G_i \) being contaminated at a point \( t_i \) resulting in the distribution \( G_i,\epsilon = (1 - \epsilon)G_i + \epsilon \Lambda_{t_i} \), where \( \Lambda_{t_i} \) is the distribution function degenerate at \( t_i = 1, 2, \ldots, m \).

Let the true distribution \( G_i \) be contaminated as \( G_i,\epsilon \) for \( j = 1, \ldots, k \) where \( k \leq n \). Through a straightforward differentiation, the influence function of \( \theta_\alpha \) when a subset of the data set is contaminated, is obtained as

\[
IF(\theta_\alpha, G_i, \ldots, G_k, t_i, \ldots, t_k) = \sum_{j=1}^{k} \frac{1}{n} \Psi_n^{-1}(\alpha) \left\{ p_{\theta_\alpha, i}(t_{ij})^\alpha u_{\theta_\alpha, i}(t_{ij}) - \xi_i(\alpha) \right\}. 
\]

(6.1)

Observe that the \( i_j \)-th summand in (6.1) is exactly the influence function when contamination is present only at the \( i_j \)-th distribution \( G_{i_j} \).

It is evident in (6.1) that the MDPD functional \( \theta_\alpha \) downweights the influence of the data points that are inconsistent with the model with weights being chosen as model densities raised to the power of \( \alpha \in [0, 1] \). In the following discussion, we consider \( k = n \). In this case, the influence function is a vector-valued function depending on \( (t_1, t_2, \ldots, t_n) \) where \( t_i \in \{1, 2, \ldots, m\} \) for all \( i \). The influence function also depends on the fixed carriers \( (x_1, x_2, \ldots, x_n) \) through the models. Since the number of levels is finite, it may be appropriate to plot the gross error sensitivity
(GES) rather than the influence function itself. The GES using (6.1) with $k = n$ is given by

$$GES(\theta_\alpha) = \max_{t_1, \ldots, t_n} ||IF(\theta_\alpha, G_1, \ldots, G_n, t_1, t_2, \ldots, t_n)||,$$  \hspace{1cm} (6.2)

where $|| \cdot ||$ is the Euclidean norm. The gross error sensitivity in (6.2) may be standardized using the asymptotic variance of $\hat{\theta}_\alpha$. See Ghosh and Basu [25] for more details. Let the components of the best-fitting parameter $\theta_\alpha$ be denoted as $\gamma_\alpha = (\gamma_{1,\alpha}, \ldots, \gamma_{(m-1),\alpha})$ and $\beta_\alpha = (\beta_{1,\alpha}, \ldots, \beta_{p,\alpha})$. The gross error sensitivity of each component may be similarly obtained using the specified component of the IF given in (6.1). These are respectively denoted by $GES(\gamma_{1,\alpha}), \ldots, GES(\gamma_{(m-1),\alpha}), GES(\beta_{1,\alpha}), \ldots, GES(\beta_{p,\alpha})$. Dependence on the true distributions is kept implicit in these notations.

Next, we plot the GES of different components of $\theta_\alpha$ related to Model 1 and Model 2 (described in Section 7) respectively in Figure 6.1 and Figure 6.2. It is clear from these graphs that $\alpha = 0$ represents the case for which GES attains its highest value. It decreases steadily as $\alpha$ increases from 0 towards 1. This gives a piece of strong evidence that when $\alpha$ is chosen close to zero the MDPD functional may tend to produce higher absolute bias at misspecified models; this indeed includes the case of maximum likelihood functional even for the logit link. As $\alpha$ increases, the MDPD functionals achieve better stability against model misspecification, particularly at larger values of $\alpha$.

6.2. Asymptotic Breakdown Point

We know that the influence function is a local measure of robustness that must be complemented with a global measure of stability. Now, the (asymptotic) breakdown point of $\theta_\alpha$ is computed for fixed $\alpha > 0$ under appropriate conditions.
Suppose that the \(i\)-th true distribution \(G_i\) is contaminated at \(\epsilon\)-proportion with a sequence of contaminating distributions \(\{K_{i,m}\}_{m=1}^{\infty}\) as \(H_{i,\epsilon,m} \equiv (1-\epsilon)G_i + \epsilon K_{i,m}\) for fixed \(n,m\). The corresponding probability density functions are denoted by \(h_{i,\epsilon,M}\), \(g_i\) and \(k_{i,M}\). We assume that the true densities \(\{g_i\}\), the models \(\{p_{\theta,i} : \theta \in \Theta\}\) and the contaminating sequence of densities \(\{k_{i,M}\}\) are all supported on \(\chi\). Let \(\theta_{\alpha}^{k_{i,M}}\) be the MDPD functional when all \(G_i\)s are \(\epsilon\)-contaminated as above, i.e.,

\[
\theta_{\alpha}^{k_{i,M}} := \arg \min_{\theta} \frac{1}{n} \sum_{i=1}^{n} d_{\alpha}(h_{i,\epsilon,M}, p_{\theta,i}). \tag{6.3}
\]

We declare that the breakdown of \(\theta_{\alpha}\) occurs at \(\epsilon\)-contamination if \(||\theta_{\alpha} - \theta_{\alpha}^{k_{i,M}}|| \rightarrow +\infty\) when \(M \rightarrow +\infty\) for fixed \(n,m\) (Simpson [32]; Park and Basu [33]). The asymptotic breakdown point of \(\theta_{\alpha}\) is defined as the least proportion of contamination \(\epsilon \in [0, \frac{1}{2}]\) such that \(||\theta_{\alpha} - \theta_{\alpha}^{k_{i,M}}|| \rightarrow +\infty\) when \(M \rightarrow +\infty\) for fixed \(n,m\). Note that, in consistent with the literature of robustness, the definition restricts the asymptotic breakdown to be at most \(\frac{1}{2}\) since we would intuitively expect the majority of the data to be uncontaminated. Define

\[
D_{\alpha}(g_i(j), p_{\theta,i}(j)) = \left\{ p_{\theta,i}^{1+\alpha}(j) - \left(1 + \frac{1}{\alpha}\right) p_{\theta,i}^\alpha(j) g_i(j) + \frac{1}{\alpha} g_i(j)^{1+\alpha} \right\} \quad \text{for } j \in \chi, \tag{6.4}
\]

and \(M_i^\alpha = \sum_j p_{\theta,i}(j)^{1+\alpha}\) for all \(i\). Let us make the following assumptions to find the asymptotic breakdown point for \(\theta_{\alpha}\). Throughout this section, \(m\) is assumed to be fixed.

(B1) \(g_i\) and \(p_{\theta,i}\) belong to a common family \(\mathcal{F}_{\theta,i}, i = 1, 2, \ldots, n\).

(B2) There exists a set \(B \subset \chi\) and a positive \(\delta_1^*(j)\) depending on \(j \in B\), such that

\[
\{k_{i,M}(j) - p_{\theta,i}(j)\} \rightarrow \delta_1^*(j) \geq 0 \quad \text{and} \quad \sum_{j \in B} k_{i,M}(j) \rightarrow 1 \quad \text{as } M \rightarrow +\infty \quad \text{for all } i \tag{6.5}
\]

uniformly for \(||\theta|| < +\infty\). This condition means that on a set \(B \subset \chi\), the contaminating sequence of densities \(k_{i,M}\) asymptotically dominate \(p_{\theta,i}\) when the parameter \(\theta\) is uniformly bounded; moreover, \(B^c\) asymptotically becomes a \(K_{i,M}\)-null set as \(M\) increases to infinity for each \(i\).

(B3) There exists a set \(C \subset \chi\) and a positive \(\delta_2^*(j)\) depending on \(j \in C\), such that

\[
\{p_{\theta,M,i}(j) - p_{\theta,i}(j)\} \rightarrow \delta_2^*(j) \geq 0 \quad \text{and} \quad \sum_{j \in C} p_{\theta,M,i}(j) \rightarrow 1 \quad \text{as } M \rightarrow +\infty \quad \text{for all } i, \tag{6.6}
\]

for any sequence \(\{\theta_M\}\) such that \(||\theta_M|| \rightarrow +\infty\) as \(M \rightarrow +\infty\). This means that as \(||\theta_M||\) diverges, the associated sequence of models \(p_{\theta_M,i}\) tend to dominate the true density \(p_{\theta,i}\) on a set \(C\), and the sequence \(p_{\theta,M,i}\) remain concentrated on \(C\) for \(i = 1, 2, \ldots, n\).

(B4)

(a) Assume that for any density \(q_i\) supported on \(\chi\), we have

\[
d_{\alpha}(c q_i, p_{\theta,i}) \geq d_{\alpha}(c p_{\theta,i}, p_{\theta,i}) \quad \text{for all } \theta, i \text{ and } 0 < \epsilon < 1. \tag{6.7}
\]
This means that $d_\alpha(eq_i, p_{\beta,i})$ will be minimized at $\theta = \theta_\alpha$ and $q_i = p_{\beta,i}$ for all $i$.

(b) Further assume that

$$\limsup_{M \to +\infty} \left( k_{i,M}(j) \right) \leq p_{\beta,i}(j) \text{ for all } i, j, \text{ and } M^\alpha = \frac{1}{n} \sum_{i=1}^{n} M_i^\alpha < +\infty \quad (6.8)$$

for fixed $n, m$ and for all $\alpha > 0$.

**Remark 6.1.** Since we know that $p_{\beta,i}(j) = F(\gamma_j - x_i^T \beta) - F(\gamma_{j-1} - x_i^T \beta)$, Assumption (B3) can be verified in the following situations.

(S1) Any particular $\gamma_j$ decreases to $-\infty$ or increases to $+\infty$, but $\beta$'s remain bounded.

(S2) Let $\gamma_{j_1} \to -\infty$ and $\gamma_{j_2} \to +\infty$ for any $1 \leq j_1 < j_2 \leq m - 1$, but $\beta$'s remain bounded.

(S3) $\gamma_j$'s remain bounded but $\beta$'s diverge to $\pm \infty$.

Consider the scenario (S1). Suppose $\gamma_j \to -\infty$. Then $\gamma_r \to -\infty$ for $r = 1, 2, \ldots, j$ because $\gamma_1 < \gamma_2 < \cdots \gamma_{j-1} < \gamma_j$. Therefore $p_{\beta,i}(r) \to 0$ for $r = 1, 2, \ldots, j$, and $\sum_{r=j+1}^{m} p_{\beta,i}(r) \to 1$. So the probability $p_{\beta,i}(\cdot)$ will be concentrated on the set $C^j(-\infty) := \{ j+1, \ldots, m \}$. If we assume that $\gamma_j \to +\infty$. Then $\gamma_r \to +\infty$ for $r = j, \ldots, m$. This gives $p_{\beta,i}(r) \to 0$ for $r = j + 1, \ldots, m$. In this case, the probability $p_{\beta,i}(\cdot)$ gets concentrated on $C^j(+\infty) := \{ 1, 2, \ldots, j \}$.

Next we consider (S2). Now $\gamma_{j_1} \to -\infty$ and $\gamma_{j_2} \to +\infty$ such that $\gamma_{j_1} < \gamma_{j_2}$. Using the above argument we see that the probability will get concentrated on the set $C^{j_1}(-\infty) \cap C^{j_2}(+\infty) = \{ j_1 + 1, \ldots, j_2 \}$. As $\chi = C^{0}(-\infty) \cap C^{m}(+\infty)$, both the sets $C^j(-\infty), C^j(+\infty)$ are proper subsets of $\chi$ for $j = 1, \ldots, m - 1$.

In (S3), it will depend on the sign of $x_i^T \beta$. If $x_i^T \beta \to +\infty$, then all the terms $(\gamma_j - x_i^T \beta)$ goes to $-\infty$. In that case, the last term $p_{\beta,i}(m) = 1 - F(\gamma_{m-1} - x_i^T \beta) \to 1$. Thus the mass gets concentrated on the singleton set $\{m\}$. On the other hand, if $x_i^T \beta \to -\infty$, then the probability mass gets concentrated on the set $\{1\}$. In all these above cases $||\theta|| \to \infty$. These proper subsets can be taken as $B$ and $C$ as mentioned in Assumptions (B2) and (B3).

Assumption (B4)(a) ensures that the divergence in the LHS of (6.7) attains its minimum value at the models when $\theta$ being chosen as the best-fitting parameter. In (B4)(b) we state the extremity of contamination up to which the true distribution may be contaminated, but still produce reasonable MDPD estimates.

**Theorem 6.1.** Suppose $g_i, p_{\beta,i}$ and the contaminating sequence of densities $\{k_{i,M}\}_{M=1}^\infty$ are supported on $\chi$, $i = 1, \ldots, n$. Then under the Assumptions (B1) - (B4), the asymptotic breakdown point $\epsilon^*$ of the MDPD functional $\theta_\alpha$ is $\frac{1}{2}$ at the model for $\alpha > 0$.

### 6.3. Implosion Breakdown of the Slope Estimates

The notion of the breakdown discussed earlier may be called the explosive breakdown because it makes an estimator explode towards infinity. Following Croux et al. [15] we know that the lack of robustness of the MLE under the present model set-up arises also due to the implosion of the slope estimator $\hat{\beta}_{ML}$ towards zero. Let the initial sample be $Z_n = \{ z_1, \ldots, z_n \}$ where $z_i = (x_i, Y_i) : i = 1, \ldots, n$. Upon adding $m$ outliers, the initial random sample becomes $Z_{n+m} = \{z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m} \}$.
\{z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}\}. Then the implosion breakdown point of \(\hat{\beta}\) is defined as 
\[ m^- = \left\{ m \in \mathbb{N}_0 : \inf_{z_{n+1}, \ldots, z_{n+m}} \|\hat{\beta}(Z_{n+m})\| = 0 \right\}, \tag{6.9} \]

It is difficult to calculate the implosion breakdown point theoretically; we shall empirically demonstrate that the minimum density power divergence estimates of the slope parameter have a very high implosion breakdown point for \(\alpha > 0\) in situations which lead to implosion breakdown for the MLE.

For illustration, let us exhibit the plot of a random sample of size 50 as in Figure 6.3. This data set contains non-stochastic covariates \(x = (x_1, x_2)^T\) whose components are fixed at the values generated by two independent standard normal distributions. The associated error terms are distributed as the standard logistic distribution. Ordinal responses, that classify the observations into 3 categories, are generated through (2.2) using \(\beta = (-1, 1.5)^T, \gamma = (-1, 1)^T\). The norm of the true slope parameter is \(\|\beta\| = 1.803\). At the initial sample, the MLE of slope yields \(\|\beta_{ML}\| = 2.20\) with the misclassification rate equal to 36%.

However \(\hat{\beta}_{ML}\) may be completely perturbed when an outlier in the form of \((s, -s, 3)\) is added to the initial sample along the diagonal line as in Figure 6.3. Here \(x = (s, -s)\) is the projection of the outlying observation in the \(x_1, x_2\)-space. Notice that as soon as \(s\) goes outside \((-2, 2)\), the additional observation becomes outlying. However, the outlying region in the \(x_1, x_2\)-space looks more deserted in terms of the presence of fewer \(Y\)-values whenever \(s\) is positive. For each value of \(s\), the parameters of the ordinal regression model are estimated by the MDPD methods and the corresponding misclassification rate is computed. The norm of the slope estimate and the rate of misclassification are reported, respectively, in Figure 6.4 and Figure 6.5 as functions of \(s\). As expected, negative values of \(s\) do not incur much bias in the slope estimates. On the other hand, as soon as \(s\) gets positive, the impact of the added observation becomes more apparent in the ML estimation and gets even more extreme as \(s\) increases. Not only does the norm of the slope estimator go to zero, but the misclassification rate also reaches its maximum value which is about 57%. When the tuning parameter is increased by a small margin, we notice that MDPDE of \(\beta\) starts becoming resistant against implosion; and it becomes the most resistant for the minimum \(L_2\) distance estimate. Consequently, the misclassification rate decreases steadily along the way as the tuning parameter \(\alpha\) increases from 0 towards 1.

In Table 6.1 and Table 6.2 we report the minimized norm of the slope estimates when the set of outliers \(I_s = \{z_{50+i} = (s, -s, 3) : i = 1, 2, \ldots, 50\}\) are added to the initial data with \(s = 8, 10\). Also, the proportion of outliers corresponding to the lowest \(\|\hat{\beta}\|\) is reported. The third column in these tables sort of gives an idea about which value of the tuning parameter \(\alpha\) adds more resistance to the slope estimates such that it safeguards against implosion. It may be thought of as a finite sample analogue of the implosion breakdown point. Notice that \(\|\hat{\beta}\|\) is close towards zero when \(\alpha \downarrow 0^+\), resulting into the lowest tolerance of outliers, i.e., 5.7\% for \(s = 8\), and 3.8\% for \(s = 10\). Moreover, as \(s\) increases more in the positive direction, this tolerance level of MLE decreases. In this case, the MLE of \(\beta\) can accommodate a smaller proportion of outliers before it starts implosing. However, as the tuning parameter \(\alpha\) increases, the tolerance level improves significantly. Also, the decreasing trend of tolerance with the increment of \(s\) is not observed for the MDPDE. In this numerical study, we see that the implosion breakdown point of the MDPDE of \(\beta\) becomes very high for \(\alpha > 0\) when compared to the MLE. In some cases, this
Figure 6.3. Scatter plot of a simulated data set. Outliers \(((s, -s), 3)\) will be added along the diagonal line.

Figure 6.4. Norm of \(\hat{\beta}\) when a single outlier is added along the diagonal line in Figure 6.3.

Figure 6.5. Misclassification rate associated with the slope estimates.

may become as high as 50% for some positive values of \(\alpha\).
7. Numerical Studies

Samples of sizes $n = 150, 200$ are drawn from each of the following models described in Subsection 7.1, and it is repeated over 1000 (say, $B$) experiments. For any given method, let $\hat{\theta}^{(b)} = (\hat{\gamma}^{(b)}, \hat{\beta}^{(b)})$ be the estimate of $\theta$ obtained in the $b$-th experiment, $b = 1, 2, \ldots, B$. The simulated mean of the estimate is obtained as $\hat{\theta} = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}^{(b)}$. The squared biases of $\hat{\gamma}$ and $\hat{\beta}$ are respectively given by $||\hat{\gamma} - \gamma||^2$ and $||\hat{\beta} - \beta||^2$ where $|| \cdot ||$ denotes the standard Euclidean norm. The mean squared error (MSE) of $\hat{\gamma}$ is defined as $MSE(\hat{\gamma}) = \frac{1}{B} \sum_{b=1}^{B} (\hat{\gamma}^{(b)} - \gamma)^T (\hat{\gamma}^{(b)} - \gamma)$, and similarly for $\hat{\beta}$. We obtain the MSE of $\hat{\theta}$ simply by adding together the MSEs incurred in each of its components. When $\hat{\theta}$ is consistent its MSE consistently estimates the trace of the asymptotic covariance matrix.

In the above setup, we will numerically compare the performances of the minimum density power divergence estimates to those of the MLE, the robust alternatives proposed by Croux et al. [15] and those by Iannario et al. [18]. For the sake of completeness, we briefly mention here the particular choices of weight functions and tuning constants that are used in the last two methods.

In Croux et al. [15], a weighted log-likelihood function is constructed by multiplying the usual log-likelihood function with the weight function $w_i = \frac{p+3}{d_i + 3}$ at the $i$-th data point, $i = 1, \ldots, n$. Here $d_i$ denotes the robust Mahalanobis distance of the $i$-th covariate $x_i \in \mathbb{R}^p$ computed in the space of explanatory variables. The weighted maximum likelihood (WML) estimator is then obtained by maximizing the weighted log-likelihood function. Iannario et al. [18], on the other hand, suggested the multiplication of propose to multiply the usual score function by one of the following three types of weight functions

$$w_1(Y_i, x_i, \theta) = \min \left\{ 1, \frac{c}{\sum_{j=1}^{m} \delta_i(j)|e_{ij}(\theta)|} \right\}, \quad w_2(Y_i, x_i, \theta) = \min \left\{ 1, \frac{c}{\sum_{j=1}^{m} \delta_i(j)|e_{ij}(\theta)| \cdot ||x_i||} \right\},$$

$$w_3(Y_i, x_i, \theta) = \min \left\{ 1, \frac{c}{||x_i||} \right\}$$

where $c > 0$, $||x_i||$ and $e_{ij}(\theta)$ respectively denote the tuning constant, norm of $x_i$ and indicated generalized residuals. For more details about the computations of $||x_i||$ and the definition of the generalized residuals, the readers are referred to Iannario et al. [18]. Given a particular example, a weight function is chosen as the link function is chosen. Based on their numerical

| Table 6.1. Minimum of $||\hat{\beta}||$ over $I_8$ | Table 6.2. Minimum of $||\hat{\beta}||$ over $I_{10}$ |
|-----------------------------------------------|-----------------------------------------------|
| $\alpha$ | min $||\hat{\beta}||$ | Prop. of outliers | $\alpha$ | min $||\hat{\beta}||$ | Prop. of outliers |
| 0 | 0.323 | 0.0566 | 0 | 0.319 | 0.0385 |
| 0.1 | 0.374 | 0.0566 | 0.1 | 0.379 | 0.180 |
| 0.2 | 0.468 | 0.438 | 0.2 | 0.613 | 0.390 |
| 0.3 | 1.070 | 0.479 | 0.3 | 1.720 | 0.474 |
| 0.5 | 2.080 | 0.495 | 0.5 | 2.120 | 0.500 |
| 0.75 | 2.250 | 0.390 | 0.75 | 2.250 | 0.390 |
| 1 | 2.340 | 0.306 | 1 | 2.340 | 0.306 |
Model 1: The response variable $Y$, generated by (2.1), assumes the values 1, ..., 5. $Y$ depends on three dichotomous 0-1 variables $X_1, X_2, X_3$ such that at most one of them can take the value 1. The cut-off points and the regression coefficients are, respectively, given by $\gamma = (-0.7, 0, 1.5, 2.9)^T$ and $\beta = (2.5, 1.2, 0.5)^T$ for both the probit and the complementary log-log link functions.

Model 2: The response variable $Y$ is generated through the latent variable $Y^* = 1.5X + \epsilon$ where the regressor $X$ is assumed to have come from $N(0, 1)$. The categories of $Y$ are determined by the cut-off points $\gamma = (-1.7, -0.5, 0.5, 1.7)^T$ and $\gamma = (-2.1, -0.6, 0.6, 2.1)^T$, respectively, when the error component follows $N(0, 1)$ and the logistic distribution with mean 0 and variance $\frac{\pi^2}{3}$. Further, the Cauchy link is used with the same cut-offs as the logit link.

Model 3: The response variable $Y$ assumes 4 categories. It depends on two regressors $X_1 \sim N(0, 1)$ and $X_2 \sim N(0, 4)$ with $\text{Cov}(X_1, X_2) = 1.2$. The regression coefficients are given by $\beta = (1.5, 0.7)^T$. We use the cut-off points $\gamma = (-2.3, 0.2, 3)^T$ for the probit link and $\gamma = (-2.6, 0, 2.6)^T$ for the logit link.

Model 4: The response variable $Y$, that takes 3 categories, depends upon three regressors $X_1 \sim N(0, 1), X_2 \sim N(0, 4)$ and $X_3 \sim N(0, 9)$ such that $\text{Cov}(X_1, X_2) = 1.5, \text{Cov}(X_1, X_3) = 0.8$ and $\text{Cov}(X_1, X_3) = 2.5$. The regression parameters are given by $\beta = (2.5, 1.2, 0.7)^T$. The cut-off points are chosen as $\gamma = (3.8, 3.8)^T$ and $\gamma = (-4, 4)^T$, respectively, for the probit and the logit links.

Model 5: Here the response variable $Y$ is generated through $Y^* = 2.5D + 1.2X + 0.7XD + \epsilon$ where $D \sim \text{Bernoulli}(\frac{1}{2})$ and $X \sim N(0, 1)$. Here $\gamma = (-1, 1, 3)^T$ for the probit and the complementary log-log links; and $\gamma = (-1.4, 1.1, 3.4)^T$ for the logit link.

Unless otherwise stated, all the links refer to the standard ones. To be able to apply the theory developed earlier, $X$ must be non-stochastic. A justification is therefore required to bring the above models, where random covariates are involved, into the realm of our proposed theory. In all such models, we assume that the values of $X$ are fixed at the values generated by the aforesaid distributions, and the values of $Y$ are conditional on the generated (fixed) values of $X$.

We find that squared biases (with one exception at $\alpha = 0$; see Table 7.1 for $||\hat{\beta} - \beta||^2$) and MSEs decrease as the sample size increases. Now, we make the following remarks based on the simulation studies of the pure models.

- Highest efficiency is achieved at pure models when $\alpha = 0$ (MLE) for all these link functions. As $\alpha$ increases, there is a drop in the efficiencies of the MDPD estimators. However, when $\alpha = 0.1$, the loss of efficiency for using the MDPDE is roughly within 10% for Model
Table 7.1. Squared bias and MSE of the estimates at Model 1 with the probit link

| Sample size | Link     | Method | \(\|\hat{\gamma} - \gamma\|^2\) | \(\|\hat{\beta} - \beta\|^2\) | MSE(\(\hat{\gamma}\)) | MSE(\(\hat{\beta}\)) |
|-------------|----------|--------|---------------------------------|---------------------------------|------------------------|------------------------|
| 150 (200)   | probit   | MLE    | 0.00059                         | 0.00007                         | 0.01810                | 0.02113                |
|             |          |        | (0.00027)                       | (0.00008)                       | (0.01232)              | (0.01464)              |
|             |          | DPD (\(\alpha\)) | 0.1 | 0.00050                         | 0.00020                         | 0.02044                | 0.02335                |
|             |          |        | (0.00034)                       | (0.00009)                       | (0.01347)              | (0.01683)              |
|             |          |        | 0.2 | 0.00060                         | 0.00025                         | 0.02182                | 0.02456                |
|             |          |        | (0.00044)                       | (0.00012)                       | (0.01355)              | (0.01775)              |
|             |          |        | 0.3 | 0.00069                         | 0.00030                         | 0.02316                | 0.02335                |
|             |          |        | (0.00050)                       | (0.00015)                       | (0.01650)              | (0.01861)              |
|             |          |        | 0.5 | 0.00094                         | 0.00042                         | 0.02572                | 0.02894                |
|             |          |        | (0.00067)                       | (0.00023)                       | (0.01837)              | (0.02064)              |
|             |          |        | 0.8 | 0.00137                         | 0.00064                         | 0.02920                | 0.03359                |
|             |          |        | (0.00098)                       | (0.00040)                       | (0.02091)              | (0.02589)              |
|             |          |        | 1.0 | 0.00161                         | 0.00078                         | 0.03057                | 0.03642                |
|             |          |        | (0.00116)                       | (0.00053)                       | (0.02210)              | (0.02994)              |
| 1 - Model 3 and Model 5. However, it drops around 50% in Model 4. These observations are clear in Figure 7.1 to Figure 7.4.

- Generally the logit link gives better efficiency than the probit link which in turn produces more efficient estimates than the complementary log-log link at a fixed value of \(\alpha\). Also, the probit link yields better efficiency than the Cauchy link. However in Model 4, we notice that the probit link dominates (in the sense of better efficiency) the logit link up to the point around \(\alpha = 0.37\) (\(n = 150\)) and \(\alpha = 0.45\) (\(n = 200\)). Beyond that, the usual pattern follows. This is seen in Figure 7.1 and Figure 7.2. Also Figure 7.4 shows that the probit link dominates the complementary log-log link that in turn dominates the logit link up to a point close to \(\alpha = 0.45\), and thereafter the order changes when the sample size is 200. A similar thing is also noticed in Figure 7.3 when the sample size is 150 with different order of domination.

- MDPDE performs better (in terms of lower MSE) than both the Croux et al. [15] and the Iannario et al. [18] estimators in Model 1, Model 2, Model 3 (only with probit link) and Model 5. However, Iannario et al. [18] performs slightly better than the MDPDE in Model 3 (only with logit link) and Model 4 (only with probit, logit links). See Tables 7.1 - 7.5 and the additional tables in the supplementary material.
Figure 7.1. Graphs of efficiency at pure Model 1 and Model 2 for different sample sizes.
Figure 7.2. Graphs of efficiency at pure Model 3 and Model 4 for different sample sizes.
Figure 7.3. Graphs of efficiency at pure Model 5 with sample size $n = 150$.

Figure 7.4. Graphs of efficiency at pure Model 5 with sample size $n = 200$.

Table 7.2. Squared bias and MSE of the estimates at Model 2 with the Cauchy link

| Sample size | Link | Method | $||\gamma - \gamma||^2$ | $||\beta - \beta||^2$ | MSE($\gamma$) | MSE($\beta$) |
|-------------|------|--------|-------------------------|-------------------------|--------------|--------------|
| 150 (200)   | Cauchy | MLE    | 0.00025                 | 0.00044                 | 0.04725      | 0.04025      |
|             |       |        | (0.00010)               | (0.00006)               | (0.03468)    | (0.02549)    |
|             | Cauchy | DPD ($\alpha$) | 0.1 | 0.00118 | 0.00118 | 0.0698 | 0.05642 |
|             |       |        | (0.0001)               | (0.00005)               | (0.0467)     | (0.03524)    |
|             | Cauchy | DPD ($\alpha$) | 0.2 | 0.00236 | 0.00252 | 0.08108 | 0.07130  |
|             |       |        | (0.00138)               | (0.00111)               | (0.0567)     | (0.04437)    |
|             | Cauchy | DPD ($\alpha$) | 0.3 | 0.00312 | 0.00345 | 0.08731 | 0.07542  |
|             |       |        | (0.0018)               | (0.00155)               | (0.06129)    | (0.04929)    |
|             | Cauchy | DPD ($\alpha$) | 0.5 | 0.00450 | 0.00517 | 0.09456 | 0.08664  |
|             |       |        | (0.0026)               | (0.00235)               | (0.06628)    | (0.05414)    |
|             | Cauchy | DPD ($\alpha$) | 0.8 | 0.00676 | 0.00793 | 0.10515 | 0.09820  |
|             |       |        | (0.00374)               | (0.00356)               | (0.07282)    | (0.05975)    |
|             | Cauchy | DPD ($\alpha$) | 1.0 | 0.00824 | 0.00962 | 0.11155 | 0.10468  |
|             |       |        | (0.00454)               | (0.00431)               | (0.0768)     | (0.06272)    |
|             | Cauchy | Iannario ($\varepsilon$) | 0.6 | 0.00316 | 0.00350 | 0.09661 | 0.09602  |
|             |       |        | (0.00187)               | (0.00175)               | (0.06960)    | (0.06131)    |
|             | Cauchy | Iannario ($\varepsilon$) | 0.8 | 0.00311 | 0.00339 | 0.09288 | 0.08628  |
|             |       |        | (0.00183)               | (0.00166)               | (0.06642)    | (0.05806)    |
|             | Cauchy | Iannario ($\varepsilon$) | 0.9 | 0.00308 | 0.00331 | 0.09188 | 0.08290  |
|             |       |        | (0.00183)               | (0.00165)               | (0.06648)    | (0.05988)    |
|             | Cauchy | Iannario ($\varepsilon$) | 1.0 | 0.00306 | 0.00326 | 0.09114 | 0.08385  |
|             |       |        | (0.00184)               | (0.00161)               | (0.06495)    | (0.05604)    |
|             | Cauchy | Croux | 0.00289 | 0.00306 | 0.09302 | 0.08199  |
|             |       |        | (0.00185)               | (0.00164)               | (0.06625)    | (0.05269)    |
Table 7.3. Squared bias and MSE of the estimates at Model 3 with the logit link

| Sample size | Link | Method | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|------|--------|-----------------|-----------------|-----------------|-----------------|
| 150 (200)   |      | MLE    | 0.00169     | 0.00041     | 0.07806     | 0.03887       |
|             |      | DPD (α)| 0.00247     | 0.00085     | 0.08368     | 0.03928       |
|             |      | 0.1    | (0.00124)   | (0.00068)   | (0.06127)   | (0.02901)     |
|             |      | 0.2    | 0.00380    | 0.00132    | 0.09504    | 0.04365       |
|             |      | (0.00193) | (0.00096)  | (0.06069)  | (0.03227)  |
|             |      | 0.3    | 0.00596    | 0.00176    | 0.10218    | 0.04681       |
|             |      | (0.00247) | (0.00118)  | (0.07434)  | (0.03443)  |
|             |      | 0.5    | 0.00774    | 0.00281    | 0.11874    | 0.05477       |
|             |      | (0.00459) | (0.00165)  | (0.08417)  | (0.03999)  |
|             |      | 0.8    | 0.01324    | 0.00493    | 0.14840    | 0.06914       |
|             |      | (0.00671) | (0.00252)  | (0.10053)  | (0.04693)  |
|             |      | 1.0    | 0.01644    | 0.00615    | 0.16388    | 0.07690       |
|             |      | (0.00692) | (0.00300)  | (0.10902)  | (0.05129)  |

|             |      | Iannario (c) | 0.6   | 0.00003 | 0.00001 | 0.02027 | 0.01395 |
|             |      |             | 0.8   | 0.00001 | 0.00000 | 0.01760 | 0.01381 |
|             |      |             | (0.00002) | (0.00002) | (0.01596) | (0.01186) |
|             |      |             | 0.9   | 0.00002 | 0.00001 | 0.01706 | 0.01291 |
|             |      |             | (0.00002) | (0.00002) | (0.01294) | (0.01052) |
|             |      |             | 1.0   | 0.00003 | 0.00000 | 0.01763 | 0.01311 |
|             |      |             | (0.00001) | (0.00000) | (0.01210) | (0.00864) |

|             |      | Croux      | 0.00406 | 0.00138 | 0.10083 | 0.03433 |
|             |      |            | (0.00233) | (0.00113) | (0.07433) | (0.03232) |

Table 7.4. Squared bias and MSE of the estimates at Model 4 with the probit link

| Sample size | Link | Method | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|------|--------|-----------------|-----------------|-----------------|-----------------|
| 150 (200)   |      | MLE    | 0.01279     | 0.00280     | 0.19448     | 0.06218       |
|             |      | DPD (α)| 0.08153     | 0.01777     | 0.47914     | 0.12899       |
|             |      | 0.1    | (0.03230)   | (0.00709)   | (0.27740)   | (0.07516)     |
|             |      | 0.2    | 0.09886    | 0.02146    | 0.46427    | 0.12885       |
|             |      | (0.04529) | (0.00973)  | (0.30960)  | (0.08144)  |
|             |      | 0.3    | 0.11786    | 0.02552    | 0.51792    | 0.14103       |
|             |      | (0.06610) | (0.01213)  | (0.35480)  | (0.09495)  |
|             |      | 0.5    | 0.13914    | 0.03662    | 0.62212    | 0.16992       |
|             |      | (0.09103) | (0.01939)  | (0.52475)  | (0.12044)  |
|             |      | 0.8    | 0.29509    | 0.06185    | 0.98384    | 0.24715       |
|             |      | (0.15116) | (0.03113)  | (0.66670)  | (0.15294)  |
|             |      | 1.0    | 0.38689    | 0.08023    | 1.16475    | 0.28748       |
|             |      | (0.19555) | (0.04004)  | (0.76913)  | (0.17819)  |

|             |      | Iannario (c) | 1.1   | 0.00216 | 0.00190 | 0.06028 | 0.03010 |
|             |      |             | 1.4   | 0.00178 | 0.00116 | 0.05012 | 0.02313 |
|             |      |             | (0.00012) | (0.00010) | (0.03688) | (0.01029) |
|             |      |             | 1.5   | 0.00096 | 0.00104 | 0.04192 | 0.02456 |
|             |      |             | (0.00019) | (0.00011) | (0.03881) | (0.01219) |
|             |      |             | 1.7   | 0.00167 | 0.00070 | 0.05388 | 0.03880 |
|             |      |             | (0.000180) | (0.000100) | (0.04740) | (0.02285) |

|             |      | Croux      | 0.48349 | 0.09860 | 2.24010 | 0.50087 |
|             |      |            | (0.18440) | (0.03642) | (0.85851) | (0.18960) |
Table 7.5. Squared bias and MSE of the estimates at Model 5 with the complementary log-log link

| Sample size | Link   | Method         | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|--------|----------------|--------------------------------|--------------------------------|---------------------|---------------------|
| 150 (200)   | log-log| MLE            | 0.00220                       | 0.00136                       | 0.05270             | 0.05933             |
|             |        |                |                               |                               |                     |                     |
|             |        | DPD ($\alpha$) |                               |                               |                     |                     |
| 0.1         |        |                | 0.00430                       | 0.00287                       | 0.06963             | 0.07802             |
|             |        |                |                               |                               |                     |                     |
| 0.2         |        |                | 0.00611                       | 0.00428                       | 0.08108             | 0.08861             |
|             |        |                |                               |                               |                     |                     |
| 0.3         |        |                | 0.00749                       | 0.00539                       | 0.08978             | 0.09701             |
|             |        |                |                               |                               |                     |                     |
| 0.5         |        |                | 0.01250                       | 0.00930                       | 0.11549             | 0.12154             |
|             |        |                |                               |                               |                     |                     |
| 0.8         |        |                | 0.02330                       | 0.01753                       | 0.15353             | 0.16217             |
|             |        |                |                               |                               |                     |                     |
| 1.0         |        |                | 0.03010                       | 0.02260                       | 0.17581             | 0.18408             |
|             |        |                |                               |                               |                     |                     |
| 1.1         |        | Iannario ($c$) | 0.21656                       | 0.18125                       | 0.44069             | 0.42720             |
|             |        |                |                               |                               |                     |                     |
| 1.4         |        |                | 0.12067                       | 0.10590                       | 0.26810             | 0.27068             |
|             |        |                |                               |                               |                     |                     |
| 1.5         |        |                | 0.10448                       | 0.09187                       | 0.23472             | 0.25058             |
|             |        |                |                               |                               |                     |                     |
| 1.7         |        |                | 0.08478                       | 0.07408                       | 0.20196             | 0.21348             |
|             |        |                |                               |                               |                     |                     |
| Croux       |        |                | 0.00776                       | 0.00472                       | 0.09440             | 0.09516             |
|             |        |                |                               |                               |                     |                     |
|             |        |                | 0.00325                       | 0.00334                       | 0.07474             | 0.07096             |
7.2. Simulation Studies: Contaminated Models

Now the above models are contaminated in the following way.

Vertical outliers: In this paper, a data point is said to be a vertical outlier when \( Y \) takes the highest categorical value in a manner such that it is inconsistent with the covariates. We add 5\% and 10\% vertical outliers to the data sets generated by Model 1 to Model 5 across all the aforesaid link functions. Numerical values are reported in Tables 7.6 - 7.15.

Horizontal outliers: We have developed the theory where the covariates are assumed to be non-stochastic. However, we might still come across a situation where a small proportion of \( x \)-values with high magnitudes may destabilise the MLE. We refer to these \( x \)-values as horizontal outliers because they do not naturally correspond to the ordinal responses. Let the covariate in the data sets, simulated through Model 2 with the probit link, be contaminated with 5\% – 10\% horizontal outliers, where the outlying values are chosen to be 5.

Misspecification of links: In this case, data sets are generated through Model 1 using the probit link. However, the complementary log-log is used in estimation.

For convenience, we only present the graphs when the sample size is fixed at \( n = 200 \). This is followed throughout the simulation studies only for the contaminated models. However, results corresponding to both these sample sizes are reported in the respective tables.

Now we present the following general remarks:

- In contaminated models MDPDEs with \( \alpha > 0 \) clearly perform better than the MLE. As \( \alpha \) increases its efficiency increases up to a point, then sometimes it decreases slowly. But all the way, it performs much better than the MLE. In Figures 7.5 - 7.8 we notice that the complementary log-log link performs better than the probit link in Model 1. Also in Model 2, we find that the probit link yields better efficiency than the logit which in turn holds an edge over the Cauchy link. But, at lower values of the tuning parameter, the Cauchy link dominates the logit link. In both Model 3 and Model 4 the probit link performs better than the logit link. In Model 5 the complementary log-log link performs better than the probit link which performs better than the logit link.

- Let the data sets generated through Model 2 be contaminated by horizontal outliers as described before. The graphs of the efficiency are plotted for both the 5\% and 10\% level of contamination in Figure 7.9 and Figure 7.10. Also, we report the squared bias and squared MSE in Table 7.14 only for the 5\% data contamination. We find that the estimated efficiency improves with the increment of \( \alpha \); after some point, it sharply increases and reaches its maximum at some point between \( \alpha = 0.25 \) and \( \alpha = 0.4 \). After that, the estimated efficiency slowly drops, but still stays higher than the MLE. Also, the MDPDE performs better than the M-estimates of Iannario et al. [18].

- When comparing the MSE, we see that MDPDE performs better than both the Croux et al. [15] and the Iannario et al. [18] methods in Model 1 with the probit link, Model 2 with the Cauchy and Model 5 with the complementary log-log link, and also in the case when the probit link is misspecified in Model 1. Although the Iannario et al. [18] beats the MDPDE in Model 3 with the logit link, our method performs better than the Croux et al. [15] method.
Figure 7.5. Graphs of efficiency when data generated by Model 1 and Model 2 are vertically contaminated at 5% and 10% levels of contamination.
Figure 7.6. Graphs of efficiency when data generated by Model 3 and Model 4 are vertically contaminated at 5% and 10% levels of contamination.
Table 7.6. Squared bias and MSE when 5% vertical outliers are added to data generated by Model 1 with the probit link

| Sample size | Link   | Method | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|--------|--------|--------------------------------|--------------------------------|---------------------|---------------------|
| 150 (200)   | probit | MLE    | 0.01836                        | 0.01357                        | 0.03743             | 0.03508             |
|             |        |        | (0.02104)                      | (0.01564)                      | (0.03319)           | (0.03140)           |
| DPD (α)     |        |        |                                |                                |                     |                     |
| 0.1         |        |        | 0.01550                        | 0.01123                        | 0.03274             | 0.03271             |
|             |        |        | (0.01749)                      | (0.01306)                      | (0.02982)           | (0.02909)           |
| 0.2         |        |        | 0.01533                        | 0.00971                        | 0.03193             | 0.03210             |
|             |        |        | (0.01550)                      | (0.01150)                      | (0.02854)           | (0.02798)           |
| 0.3         |        |        | 0.01118                        | 0.00792                        | 0.03095             | 0.03188             |
|             |        |        | (0.01301)                      | (0.00961)                      | (0.02706)           | (0.02708)           |
| 0.5         |        |        | 0.00691                        | 0.00454                        | 0.02969             | 0.03237             |
|             |        |        | (0.00833)                      | (0.00659)                      | (0.02474)           | (0.02590)           |
| 0.8         |        |        | 0.00312                        | 0.00159                        | 0.02941             | 0.03549             |
|             |        |        | (0.00411)                      | (0.00252)                      | (0.02334)           | (0.02650)           |
| 1.0         |        |        | 0.00201                        | 0.00076                        | 0.02912             | 0.03770             |
|             |        |        | (0.00283)                      | (0.00140)                      | (0.02283)           | (0.02749)           |
| Iannario (c)|        |        |                                |                                |                     |                     |
| 1.1         |        |        | 0.65004                        | 0.33193                        | 0.70072             | 0.39141             |
|             |        |        | (0.62722)                      | (0.32905)                      | (0.66663)           | (0.36653)           |
| 1.4         |        |        | 0.38608                        | 0.22975                        | 0.38034             | 0.26487             |
|             |        |        | (0.34148)                      | (0.22876)                      | (0.36280)           | (0.25362)           |
| 1.5         |        |        | 0.23591                        | 0.11497                        | 0.29638             | 0.25526             |
|             |        |        | (0.22664)                      | (0.11298)                      | (0.29271)           | (0.26179)           |
| 1.7         |        |        | 0.06634                        | 0.02597                        | 0.17265             | 0.23807             |
|             |        |        | (0.06672)                      | (0.02308)                      | (0.10344)           | (0.07125)           |
| Croux       |        |        | 0.01827                        | 0.01364                        | 0.03867             | 0.03327             |
|             |        |        | (0.02186)                      | (0.01566)                      | (0.03327)           | (0.03142)           |
### Table 7.7. Squared bias and MSE when 10% vertical outliers are added to data generated by Model 1 with the probit link

| Sample size | Link     | Method   | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|----------|----------|-------------------------------|-------------------------------|---------------------|---------------------|
| 150 (200)   | probit   | MLE      | 0.06437                       | 0.04571                       | 0.08052             | 0.06505             |
|             |          | DPD (α)  |                               |                               |                     |                     |
|             |          | 0.1      | 0.05960                       | 0.04291                       | 0.07531             | 0.06284             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.2      | 0.05647                       | 0.04049                       | 0.07276             | 0.06100             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.3      | 0.05235                       | 0.03734                       | 0.06950             | 0.05920             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.5      | 0.04207                       | 0.02859                       | 0.06174             | 0.05434             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.8      | 0.02725                       | 0.01523                       | 0.05101             | 0.04892             |
|             |          |          |                               |                               |                     |                     |
|             |          | 1.0      | 0.02073                       | 0.00892                       | 0.04599             | 0.04124             |
|             |          |          |                               |                               |                     |                     |
|             |          | Iannario (c) |                      |                               |                     |                     |
|             |          | 1.1      | 0.50295                       | 0.28278                       | 0.54448             | 0.32853             |
|             |          |          |                               |                               |                     |                     |
|             |          | 1.4      | 0.25635                       | 0.17064                       | 0.28689             | 0.21103             |
|             |          |          |                               |                               |                     |                     |
|             |          | 1.5      | 0.15558                       | 0.08586                       | 0.19719             | 0.14505             |
|             |          |          |                               |                               |                     |                     |
|             |          | 1.7      | 0.04151                       | 0.02526                       | 0.09104             | 0.07660             |
|             |          |          |                               |                               |                     |                     |
|             |          | Croux    | 0.06531                       | 0.04584                       | 0.08197             | 0.06534             |
|             |          |          |                               |                               |                     |                     |

### Table 7.8. Squared bias and MSE when 5% vertical outliers are added to data generated by Model 2 with the Cauchy link

| Sample size | Link     | Method   | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|----------|----------|-------------------------------|-------------------------------|---------------------|---------------------|
| 150 (200)   | Cauchy   | MLE      | 0.01566                       | 0.00444                       | 0.08885             | 0.07220             |
|             |          | DPD (α)  |                               |                               |                     |                     |
|             |          | 0.1      | 0.01319                       | 0.00327                       | 0.07317             | 0.05511             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.2      | 0.01347                       | 0.00375                       | 0.08207             | 0.06229             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.3      | 0.01262                       | 0.00349                       | 0.08502             | 0.06942             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.5      | 0.01056                       | 0.00251                       | 0.08754             | 0.07431             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.8      | 0.00788                       | 0.00125                       | 0.09167             | 0.07902             |
|             |          |          |                               |                               |                     |                     |
|             |          | 1.0      | 0.00852                       | 0.00170                       | 0.09462             | 0.08237             |
|             |          |          |                               |                               |                     |                     |
|             |          | Iannario (c) |                      |                               |                     |                     |
|             |          | 0.6      | 0.01963                       | 0.00362                       | 0.09957             | 0.08104             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.8      | 0.01177                       | 0.00373                       | 0.09455             | 0.07745             |
|             |          |          |                               |                               |                     |                     |
|             |          | 0.9      | 0.01715                       | 0.00386                       | 0.09505             | 0.07938             |
|             |          |          |                               |                               |                     |                     |
|             |          | 1.0      | 0.01668                       | 0.00397                       | 0.09193             | 0.07512             |
|             |          |          |                               |                               |                     |                     |
|             |          | Croux    | 0.01604                       | 0.00448                       | 0.09273             | 0.07449             |
Table 7.9. Squared bias and MSE when 10% vertical outliers are added to data generated by Model 2 with the Cauchy link

| Sample size | Link | Method    | $||\gamma - \gamma||^2$ | $||\beta - \beta||^2$ | $MSE(\gamma)$ | $MSE(\beta)$ |
|-------------|------|-----------|--------------------------|------------------------|----------------|----------------|
| 150 (200)   | Cauchy MLE | 0.08868   | 0.03983                  | 0.14961                | 0.09796        |
| DPD (α)     | 0.6   | 0.09577   | 0.03679                  | 0.16811                | 0.10414        |
|             | 0.8   | 0.09401   | 0.03748                  | 0.15930                | 0.10102        |
|             | 0.9   | 0.09227   | 0.03805                  | 0.15655                | 0.10015        |
|             | 1.0   | 0.09099   | 0.03853                  | 0.15444                | 0.09946        |
| Iannario (c)|       | 0.08982   | 0.04025                  | 0.15346                | 0.09912        |
| Croux       |       | 0.08921   | 0.04424                  | 0.13672                | 0.08328        |

Table 7.10. Squared bias and MSE when 5% vertical outliers are added to data generated by Model 3 with the logit link

| Sample size | Link | Method    | $||\gamma - \gamma||^2$ | $||\beta - \beta||^2$ | $MSE(\gamma)$ | $MSE(\beta)$ |
|-------------|------|-----------|--------------------------|------------------------|----------------|----------------|
| 150 (200)   | logit MLE | 0.09434   | 0.01487                  | 0.15705                | 0.05238        |
| DPD (α)     | 0.1   | 0.06391   | 0.00938                  | 0.12838                | 0.04431        |
|             | 0.2   | 0.04892   | 0.00627                  | 0.12044                | 0.04389        |
|             | 0.3   | 0.03573   | 0.00369                  | 0.11897                | 0.04411        |
|             | 0.5   | 0.01934   | 0.00095                  | 0.11330                | 0.04819        |
|             | 0.8   | 0.00908   | 0.00001                  | 0.12563                | 0.05788        |
|             | 1.0   | 0.00656   | 0.00005                  | 0.13555                | 0.06321        |
| Iannario (c)|       | 0.00927   | 0.00004                  | 0.12278                | 0.04410        |
| Croux       |       | 0.00997   | 0.00014                  | 0.02866                | 0.01818        |
|             | 0.8   | 0.00679   | 0.00009                  | 0.02593                | 0.01637        |
|             | 0.9   | 0.00886   | 0.00012                  | 0.02399                | 0.01663        |
|             | 1.0   | 0.00878   | 0.00013                  | 0.02415                | 0.01577        |
|             |       | 0.00990   | 0.00019                  | 0.02246                | 0.01476        |
| Croux       |       | 0.08967   | 0.01427                  | 0.15754                | 0.05292        |
Table 7.11. Squared bias and MSE when 10% vertical outliers are added to data generated by Model 3 with the logit link

| Sample size | Link        | Method | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|-------------|--------|-------------------------------|-------------------------------|---------------------|---------------------|
| 150 (200)   | logit       | MLE    | 0.36860                       | 0.07404                       | 0.41935             | 0.10829             |
|             |             |        | (0.37264)                     | (0.07911)                     | (0.42266)           | (0.11010)           |
| DPD (a)     |             |        |                               |                               |                     |                     |
| 0.1         |             |        | 0.30176                       | 0.05413                       | 0.35734             | 0.08980             |
|             |             |        | (0.36911)                     | (0.06532)                     | (0.34978)           | (0.08219)           |
| 0.2         |             |        | 0.25484                       | 0.04331                       | 0.31373             | 0.08001             |
|             |             |        | (0.26981)                     | (0.04507)                     | (0.36684)           | (0.07238)           |
| 0.3         |             |        | 0.21064                       | 0.03276                       | 0.27549             | 0.07158             |
|             |             |        | (0.21814)                     | (0.03498)                     | (0.26821)           | (0.06551)           |
| 0.5         |             |        | 0.14538                       | 0.01830                       | 0.22511             | 0.06313             |
|             |             |        | (0.15677)                     | (0.02121)                     | (0.21604)           | (0.05351)           |
| 0.8         |             |        | 0.09205                       | 0.00860                       | 0.18919             | 0.06802             |
|             |             |        | (0.10601)                     | (0.01207)                     | (0.18052)           | (0.05421)           |
| 1.0         |             |        | 0.07560                       | 0.00618                       | 0.18670             | 0.06528             |
|             |             |        | (0.08997)                     | (0.00980)                     | (0.17075)           | (0.05884)           |
| Iannario (c)|             |        |                               |                               |                     |                     |
| 0.6         |             |        | 0.00551                       | 0.00069                       | 0.05755             | 0.03125             |
|             |             |        | (0.00700)                     | (0.00063)                     | (0.05788)           | (0.02746)           |
| 0.8         |             |        | 0.00668                       | 0.00059                       | 0.05680             | 0.02908             |
|             |             |        | (0.00731)                     | (0.00078)                     | (0.05857)           | (0.02949)           |
| 0.9         |             |        | 0.00719                       | 0.00123                       | 0.06402             | 0.03791             |
|             |             |        | (0.00639)                     | (0.00077)                     | (0.05680)           | (0.02869)           |
| 1.0         |             |        | 0.00673                       | 0.00073                       | 0.05609             | 0.02974             |
|             |             |        | (0.00645)                     | (0.00059)                     | (0.05413)           | (0.02740)           |
| Croux       |             |        | 0.35907                       | 0.07206                       | 0.41123             | 0.10671             |
|             |             |        | (0.37283)                     | (0.07520)                     | (0.41637)           | (0.10117)           |

Table 7.12. Squared bias and MSE when 5% vertical outliers are added to data generated by Model 5 with the complementary log-log link

| Sample size | Link        | Method | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|-------------|--------|-------------------------------|-------------------------------|---------------------|---------------------|
| 150 (200)   | log-log     | MLE    | 0.67107                       | 0.34065                       | 0.75620             | 0.46745             |
|             |             |        | (0.79469)                     | (0.41913)                     | (0.84674)           | (0.50749)           |
| DPD (a)     |             |        |                               |                               |                     |                     |
| 0.1         |             |        | 0.33965                       | 0.16504                       | 0.45120             | 0.27767             |
|             |             |        | (0.41723)                     | (0.21468)                     | (0.51548)           | (0.29913)           |
| 0.2         |             |        | 0.08889                       | 0.03262                       | 0.17620             | 0.12563             |
|             |             |        | (0.11621)                     | (0.04806)                     | (0.18499)           | (0.11780)           |
| 0.3         |             |        | 0.02325                       | 0.00645                       | 0.11282             | 0.10273             |
|             |             |        | (0.03436)                     | (0.01254)                     | (0.10290)           | (0.08164)           |
| 0.5         |             |        | 0.00233                       | 0.00113                       | 0.10606             | 0.11394             |
|             |             |        | (0.00784)                     | (0.00131)                     | (0.07781)           | (0.07851)           |
| 0.8         |             |        | 0.00112                       | 0.00454                       | 0.12874             | 0.14493             |
|             |             |        | (0.00167)                     | (0.00099)                     | (0.08337)           | (0.09388)           |
| 1.0         |             |        | 0.00022                       | 0.00019                       | 0.13590             | 0.15810             |
|             |             |        | (0.00091)                     | (0.00059)                     | (0.08772)           | (0.11153)           |
| Iannario (c)|             |        |                               |                               |                     |                     |
| 1.1         |             |        | 0.03569                       | 0.07658                       | 0.28077             | 0.31508             |
|             |             |        | (0.02280)                     | (0.05765)                     | (0.21503)           | (0.23128)           |
| 1.4         |             |        | 0.00845                       | 0.03774                       | 0.20197             | 0.22277             |
|             |             |        | (0.00888)                     | (0.02608)                     | (0.15595)           | (0.16131)           |
| 1.5         |             |        | 0.00433                       | 0.02996                       | 0.18551             | 0.20015             |
|             |             |        | (0.00420)                     | (0.01810)                     | (0.14522)           | (0.14065)           |
| 1.7         |             |        | 0.00154                       | 0.01992                       | 0.17534             | 0.18171             |
|             |             |        | (0.00175)                     | (0.01341)                     | (0.15017)           | (0.14222)           |
| Croux       |             |        | 0.74334                       | 0.39614                       | 0.81978             | 0.50701             |
|             |             |        | (0.85310)                     | (0.40098)                     | (0.89923)           | (0.53671)           |
Table 7.13. Squared bias and MSE when 10% vertical outliers are added to data generated by Model 5 with the complementary log-log link

| Sample size | Link   | Method | $||\gamma - \hat{\gamma}||^2$ | $||\beta - \hat{\beta}||^2$ | $\text{MSE}(\hat{\gamma})$ | $\text{MSE}(\hat{\beta})$ |
|-------------|--------|--------|-------------------------------|-------------------------------|-----------------------------|-----------------------------|
| 150 (200)   | log-log| MLE    | 1.35901                       | 0.67394                       | 1.39573                     | 0.75649                     |
|             |        |        | (1.39844)                     | (0.70918)                     | (1.42347)                  | (0.68956)                  |
|             |        | DPD (α)|                               |                               |                             |                             |
|             |        | 0.1    | 1.12067                       | 0.53078                       | 1.17625                     | 0.63136                     |
|             |        |        | (1.17012)                     | (0.56991)                     | (1.20153)                  | (0.65803)                  |
|             |        | 0.2    | 0.64292                       | 0.27279                       | 0.75860                     | 0.39454                     |
|             |        |        | (0.67148)                     | (0.29143)                     | (0.74108)                  | (0.38141)                  |
|             |        | 0.3    | 0.21846                       | 0.08975                       | 0.35056                     | 0.26630                     |
|             |        |        | (0.26717)                     | (0.10096)                     | (0.34617)                  | (0.17944)                  |
|             |        | 0.5    | 0.06191                       | 0.01273                       | 0.16822                     | 0.12938                     |
|             |        |        | (0.07849)                     | (0.01925)                     | (0.15566)                  | (0.10010)                  |
|             |        | 0.8    | 0.02076                       | 0.00066                       | 0.13952                     | 0.13650                     |
|             |        |        | (0.03506)                     | (0.00658)                     | (0.11800)                  | (0.09665)                  |
|             |        | 1.0    | 0.01543                       | 0.00005                       | 0.13761                     | 0.14375                     |
|             |        |        | (0.02927)                     | (0.00230)                     | (0.11410)                  | (0.10652)                  |
|             |        | Iannario (c)|                          |                               |                             |                             |
|             |        | 1.1    | 0.00817                       | 0.03981                       | 0.28032                     | 0.27322                     |
|             |        |        | (0.00939)                     | (0.03204)                     | (0.23622)                  | (0.21724)                  |
|             |        | 1.4    | 0.02240                       | 0.02028                       | 0.26673                     | 0.22526                     |
|             |        |        | (0.02133)                     | (0.01424)                     | (0.22134)                  | (0.16984)                  |
|             |        | 1.5    | 0.02654                       | 0.01753                       | 0.28295                     | 0.23761                     |
|             |        |        | (0.02542)                     | (0.01157)                     | (0.24144)                  | (0.19900)                  |
|             |        | 1.7    | 0.03521                       | 0.03100                       | 0.30673                     | 0.22814                     |
|             |        |        | (0.04570)                     | (0.01024)                     | (0.25802)                  | (0.16984)                  |
|             |        | Croux | 1.35182                       | 0.67836                       | 1.39702                     | 0.75624                     |
|             |        |        | (1.40714)                     | (0.71510)                     | (1.44104)                  | (0.77266)                  |

Figure 7.9. Graphs of efficiency when data generated by Model 2 is horizontally contaminated at 5% level of contamination.

Figure 7.10. Graphs of efficiency when data generated by Model 2 is horizontally contaminated at 10% level of contamination.
### Table 7.14. Squared bias and MSE when 5% horizontal outliers are added to data generated by Model 2 with the probit link

| Sample size | Link     | Method | $||\hat{\gamma} - \gamma||^2$ | $||\hat{\beta} - \beta||^2$ | $MSE(\hat{\gamma})$ | $MSE(\hat{\beta})$ |
|-------------|----------|--------|-------------------------------|-------------------------------|---------------------|---------------------|
| 150 (200)   | probit   | MLE    | 0.17980                       | 1.08089                       | 0.19615             | 1.08790             |
|             |          | DPD (α)|                               |                               |                     |                     |
| 0.1         |          | 0.10358 | 0.62587                       | 0.13361                       | 0.71104             |                     |
|             |          |        | (0.12947)                     | (0.76388)                     | (0.14917)           | (0.81316)           |
| 0.2         |          | 0.08011 | 0.00175                       | 0.03179                       | 0.05376             |                     |
|             |          |        | (0.00639)                     | (0.00290)                     | (0.02494)           | (0.04390)           |
| 0.3         |          | 0.06028 | 0.000419                      | 0.03098                       | 0.02279             |                     |
|             |          |        | (0.00010)                     | (0.00015)                     | (0.02335)           | (0.02075)           |
| 0.5         |          | 0.00063 | 0.001411                      | 0.03461                       | 0.03072             |                     |
|             |          |        | (0.00033)                     | (0.00074)                     | (0.02026)           | (0.02579)           |
| 0.8         |          | 0.00122 | 0.00266                       | 0.04123                       | 0.03939             |                     |
|             |          |        | (0.00092)                     | (0.00152)                     | (0.02122)           | (0.02651)           |
| 1.0         |          | 0.00168 | 0.00358                       | 0.04369                       | 0.04522             |                     |
|             |          |        | (0.00101)                     | (0.00206)                     | (0.03434)           | (0.03336)           |
|             |          | Iannario (c) | 0.01437 | 0.17591 | 0.05819 | 0.21230 |                     |
|             |          |        | (0.01114)                     | (0.16005)                     | (0.04167)           | (0.18579)           |
| 1.4         |          | 0.00419 | 0.02228                       | 0.05082                       | 0.08047             |                     |
|             |          |        | (0.00290)                     | (0.04204)                     | (0.02071)           | (0.06532)           |
| 1.5         |          | 0.00267 | 0.03273                       | 0.05350                       | 0.05880             |                     |
|             |          |        | (0.00179)                     | (0.02567)                     | (0.02463)           | (0.03524)           |
| 1.7         |          | 0.00094 | 0.01042                       | 0.02978                       | 0.03424             |                     |
|             |          |        | (0.00059)                     | (0.00654)                     | (0.02201)           | (0.02088)           |
|             |          | Croux  | 0.17578                       | 0.10969                       | 0.10868             |                     |
|             |          |        | (0.18979)                     | (1.12763)                     | (0.20143)           | (1.13247)           |

### Table 7.15. Squared bias and MSE when the probit link is misspecified with the complementary log-log link in Model 1

| Sample size | Link     | Method | $||\gamma - \gamma||^2$ | $||\beta - \beta||^2$ | $MSE(\gamma)$ | $MSE(\beta)$ |
|-------------|----------|--------|----------------------------|----------------------------|----------------|----------------|
| 150 (200)   | log-log  | MLE    | 0.22799                     | 0.10405                     | 0.27993        | 0.20401        |
|             |          | DPD (α)|                               |                               |                     |                     |
| 0.1         |          | 0.19688 | 0.14477 | 0.24136 | 0.18171 |                     |
|             |          |        | (0.18326)                     | (0.14289)                     | (0.21630)        | (0.16870)        |
| 0.2         |          | 0.20842 | 0.14284 | 0.25887 | 0.18398 |                     |
|             |          |        | (0.19078)                     | (0.14061)                     | (0.22980)        | (0.16835)        |
| 0.3         |          | 0.21571 | 0.13956 | 0.27373 | 0.18054 |                     |
|             |          |        | (0.20093)                     | (0.13634)                     | (0.23818)        | (0.16785)        |
| 0.5         |          | 0.23037 | 0.14024 | 0.29848 | 0.20461 |                     |
|             |          |        | (0.21439)                     | (0.13696)                     | (0.26034)        | (0.17922)        |
| 0.8         |          | 0.23812 | 0.13576 | 0.30620 | 0.21919 |                     |
|             |          |        | (0.22256)                     | (0.14158)                     | (0.26900)        | (0.19087)        |
| 1.0         |          | 0.23509 | 0.14644 | 0.30301 | 0.22220 |                     |
|             |          |        | (0.22104)                     | (0.14322)                     | (0.26707)        | (0.19585)        |
|             |          | Iannario (c) | 4.07929 | 1.13771 | 4.27116 | 1.30679 |                     |
|             |          |        | (4.05829)                     | (1.05697)                     | (4.00773)        | (1.19122)        |
| 1.4         |          | 2.22459 | 0.29767 | 2.45129 | 0.46903 |                     |
|             |          |        | (2.09108)                     | (0.24344)                     | (2.24104)        | (0.41603)        |
| 1.5         |          | 0.86343 | 0.50844 | 1.20427 | 0.60684 |                     |
|             |          |        | (0.79926)                     | (0.54682)                     | (0.80930)        | (0.62439)        |
| 1.7         |          | 0.80507 | 0.57209 | 0.92810 | 0.66357 |                     |
|             |          |        | (0.74604)                     | (0.55233)                     | (0.81903)        | (0.60622)        |
| Croux       |          | 0.22830 | 0.14335 | 0.28051 | 0.20474 |                     |
|             |          |        | (0.21362)                     | (0.16199)                     | (0.24918)        | (0.19011)        |
8. Data Driven Selection of Tuning Parameter

We have seen in simulation studies that small values of $\alpha$ increase the efficiency of an MDPD estimator under the model whereas its robustness increases with $\alpha$. However, an experimenter will not know, a priori, the amount of contamination in a data set. So a data-driven strategy for tuning parameter selection is required to apply this method in real-life data sets. Among different existing methods, we will follow the approach of Warwick and Jones [29]. They suggest constructing an empirical version of the mean square error, and further minimizing it over the tuning parameter. This method has been generalized by Ghosh and Basu [26], and further extended by Basak et al. [27]. Minimizing the empirical MSE has been shown to provide satisfactory performance in selecting an appropriate tuning parameter based on a data set. The empirical version of the asymptotic mean square error, expressed as a function of the tuning parameter and a pilot $\theta_P$, is given by

$$\text{MSE}_\alpha(\theta_P) = (\hat{\theta}_\alpha - \theta_P)^T \left( \hat{\Omega}_n^{-1}(\alpha) \hat{\Omega}_n(\alpha) \hat{\Omega}_n^{-1}(\alpha) \right).$$  \tag{8.1}$$

The optimal $\alpha$ that minimizes (8.1) may depend on the choice of the pilot $\theta_P$ as well. In Table 8.1, we compare the replication-MSE and the squared bias of the fitted (in the sense of minimized (8.1)) MDPDEs with that of the MLEs for data sets generated through Model 3. In doing so, we have considered both $\hat{\theta}_{0.5}$ and $\hat{\theta}_1$ as pilot values. We see that the replication-MSE of the fitted MDPDEs corresponding to the pilot $\hat{\theta}_{0.5}$ gives a much better approximation of the replication-MSE of the MLE (smallest in this case) in pure data sets across two different links. Also, note that the squared biases due to the pilot $\hat{\theta}_1$ become almost twice the values generated by the MLE in pure data for both the probit and the logit links.

Now, we add 10% and 15% vertical outliers in the data sets. As we found, the fitted MDPDEs perform better than the MLE, and the pilot $\hat{\theta}_1$ gives the lowest replication-MSE and the squared bias. This gives a clear indication that the more robust we choose a pilot value, the fitted MDPDEs become more resistant to the outliers, but also they lose their efficiency along the way. Since $\hat{\theta}_{0.5}$ is highly robust (though less robust than $\hat{\theta}_1$) and they yield better-fitted MDPDEs in pure data, $\hat{\theta}_{0.5}$ chosen as a pilot would provide a nice trade-off between the efficiency and robustness. This conclusion also validates the empirical evidence of Ghosh and Basu [26].

| Model | Link | Pilot | $\epsilon = 0$ | $\epsilon = 0.10$ | $\epsilon = 0.15$ |
|-------|------|-------|----------------|------------------|------------------|
| Model 3 | probit | $\hat{\theta}_{0.5}$ | 0.18278 (0.01067) | 2.87686 (2.79551) | 4.00026 (3.94187) |
|       |      | $\hat{\theta}_1$   | 0.19797 (0.01283) | 0.54000 (0.35334) | 1.07291 (0.89671) |
| logit |      | $\hat{\theta}_{0.5}$ | 0.24429 (0.02210) | 0.46164 (0.25809) | 0.82525 (0.63395) |
|       |      | $\hat{\theta}_1$   | 0.29862 (0.00528) | 0.91008 (0.67627) | 1.70114 (1.50296) |
|       |      | $\hat{\theta}_{0.5}$ | 0.34358 (0.01027) | 0.77661 (0.50389) | 1.39069 (1.14921) |

This demonstrates the effectiveness of the algorithm in picking out a suitable tuning parameter based on a data set. We apply this strategy in the next section to analyze a wine quality data set. Going forward, we will use the $\hat{\theta}_{0.5}$ as our pilot.
9. Real Data Analysis

We analyze a white wine quality data set that is available in the UCI Machine Learning Repository. This data set contains 11 independent variables and an ordinal response variable. These continuous variables \( x_i \)s determine the wine’s quality which is rated on a scale of 3 – 9. As we see, the values of the covariates vary a lot. They are standardized using the median (\( med \)) and the mean absolute deviation (MAD) as

\[
x_{ij}^* = \frac{x_{ij} - med(x_i)}{1.4828 \times MAD(x_i)} \text{ where } x_i = (x_{i1}, \ldots, x_{ip})^T \text{ and } j = 1, 2, \ldots, p = 11
\]  

(9.1)

for each \( i \)-th data point, to aid better convergence for the optimization algorithms. Having done that, we find the parameter estimates using both the probit and the logit links. In Table 9.1 and Table 9.2 the minimum density power divergence estimates are reported along with the 95%-trimmed MLE (trim. MLE). The 95%-trimmed MLE refers to the computation of the MLE after deletion of the statistical units whose covariates satisfying

\[
(x_i^* - \hat{\mu})^T S^{-1}(x_i^* - \hat{\mu}) \geq \chi^2_{0.95, 4897} \text{ for all } i.
\]  

(9.2)

Here \( \hat{\mu} \) and \( S \) respectively denote the robust location and scale estimates of the data cloud \( \{x_i^*; i = 1, \ldots, 4898\} \), and \( \chi^2_{0.95, 4897} \) being the upper 5% point of central \( \chi^2 \)-distribution with 4897 degrees of freedom. More precisely, \( \hat{\mu} \) is obtained by taking component-wise medians, whereas \( S \) is constructed from the median absolute deviation for each row of the data matrix. As we find in Table 9.1 that when \( \alpha = 0, 0.1, 0.3 \), the MDPDE estimates of \( \beta_1, \beta_2, \beta_3 \) differ with the trimmed MLE. This might give an idea that the optimum tuning parameter might occur somewhere above 0.3 because \( \alpha \) close to zero gives unstable estimates. This speculation is fairly supported in Table 9.3 which gives the optimum tuning parameter as \( \alpha = 0.39 \) with the lowest value of the MSE for the probit link. When the logit link is used, the optimum tuning parameter turns out \( \alpha = 0.68 \). Here MSE is computed using (8.1) with the pilot being chosen as \( \hat{\theta}_{0.5} \). Also, we notice that, for the logit link, the matrix \( \hat{\Psi}_n(\alpha) \) is singular at \( \alpha = 0 \). Therefore, we report \( \alpha = 0.01 \) in Table 9.2 in place of \( \alpha = 0 \). We also report the total standard error (SE) in the second-last row of these two tables. The SE is computed as the sum of asymptotic standard deviations divided by the squared root of the sample size. For these links, SE generally increases with \( \alpha \), therefore it is the squared bias term that drives the MSE to have a parabolic shape along with \( \alpha \).

Next, to measure the performance of these estimates we split the entire data set into two parts, namely the training and test data sets. The training data set consisting of 75% data points is used to estimate the parameters that are further used to predict the wine quality levels in the test data set. The proportion of these cases, where the true levels match the predicted values in the test data set, measures the prediction accuracy of a particular method. In Table 9.1 we notice that the MDPDE with \( \alpha > 0 \) produces better accuracy than both the MLE (53.6%) and the trimmed MLE (54%) for the probit link. However in Table 9.2 we find that both the trimmed MLE produces slightly better accuracy (55.4% respectively) than the MDPDE with \( \alpha > 0 \).

Now we proceed to find the optimum value of the tuning parameter using the strategy of Warwick and Jones [29]. In Figure 9.1 we see that MSE decreases with \( \alpha \) roughly up to the point \( \alpha = 0.39 \), after that the curve moves slightly upwards. This implies that MDPDE with \( \alpha > 0 \)
Table 9.1. Parameters estimates in the wine quality data set with the probit link

| Parameters | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ | 95%-trim. MLE |
|------------|--------------|----------------|----------------|----------------|----------------|---------------|---------------|
| $\beta_1$  | 0.06268      | 0.07435        | 0.11204        | 0.16994        | 0.11992        | 0.11688       | 0.13085       |
| $\beta_2$  | -0.2509      | -0.25329       | -0.25605       | -0.25542       | -0.25602       | -0.25796      | -0.2695       |
| $\beta_3$  | 0.00053      | -0.00015       | 0.00791        | 0.01135        | 0.01241        | 0.01230       | -0.01282      |
| $\beta_4$  | 0.61871      | 0.63508        | 0.72930        | 0.73082        | 0.72955        | 0.70955       | 0.6382        |
| $\beta_5$  | -0.00358     | -0.00514       | -0.00374       | -0.0037        | -0.00344       | -0.00282      | -0.07922      |
| $\beta_6$  | 0.08815      | 0.11128        | 0.12612        | 0.1368         | 0.14413        | 0.14893       | 0.12994       |
| $\beta_7$  | -0.01651     | -0.02224       | -0.03029       | -0.04818       | -0.06126       | -0.07476      | -0.01060      |
| $\beta_8$  | -0.66650     | -0.71373       | -0.90893       | -0.92605       | -0.93696       | -0.91311      | -0.68881      |
| $\beta_9$  | 0.13937      | 0.15234        | 0.18989        | 0.20338        | 0.21168        | 0.21468       | 0.15736       |
| $\beta_{10}$ | 0.09548 | 0.1016 | 0.11539 | 0.12403 | 0.13168 | 0.13694 | 0.09799 |
| $\gamma_1$ | -2.99276 | -3.13133 | -3.36179 | -3.43993 | -3.49516 | -3.47491 | -3.10773 |
| $\gamma_2$ | -2.05813 | -2.10963 | -2.18638 | -2.23159 | -2.26262 | -2.27492 | -2.19545 |
| $\gamma_3$ | -0.43326 | -0.43169 | -0.43081 | -0.42982 | -0.42819 | -0.42475 | -0.46651 |
| $\gamma_4$ | 1.06414 | 1.07661 | 1.10986 | 1.13198 | 1.14808 | 1.15783 | 1.02281 |
| $\gamma_5$ | 2.25861 | 2.29456 | 2.39137 | 2.45673 | 2.50497 | 2.53885 | 2.26259 |
| $\gamma_6$ | 24.6606 | 24.6605 | 24.6605 | 24.6605 | 24.6605 | 24.6605 | 24.6605 |

Table 9.2. Parameters estimates in the wine quality data set with the logit link

| Parameters | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ | 95%-trim. MLE |
|------------|----------------|----------------|----------------|----------------|----------------|---------------|---------------|
| $\beta_1$  | 0.29494 | 0.21962 | 0.23269 | 0.22743 | 0.21692 | 0.20681 | 0.29497 |
| $\beta_2$  | -0.45306 | -0.47338 | -0.44141 | -0.43249 | -0.42688 | -0.42918 | -0.45306 |
| $\beta_3$  | 0.00639 | 0.02054 | 0.01721 | 0.01900 | 0.01979 | 0.01745 | 0.00638 |
| $\beta_4$  | 1.40803 | 1.39715 | 1.35650 | 1.32555 | 1.27118 | 1.22729 | 1.40802 |
| $\beta_5$  | -0.13639 | -0.00420 | -0.00147 | -0.0008 | -0.00151 | 0.00141 | -0.13660 |
| $\beta_6$  | 0.15613 | 0.20760 | 0.21563 | 0.22599 | 0.23991 | 0.23934 | 0.15611 |
| $\beta_7$  | 0.01588 | -0.01629 | -0.05074 | -0.07677 | -0.10496 | -0.11556 | 0.01587 |
| $\beta_8$  | -1.76746 | -1.79154 | -1.76990 | -1.73325 | -1.66398 | -1.61093 | -1.76747 |
| $\beta_9$  | 0.38776 | 0.35698 | 0.36654 | 0.37110 | 0.36943 | 0.36313 | 0.38776 |
| $\beta_{10}$ | 0.19478 | 0.20459 | 0.21147 | 0.22260 | 0.23045 | 0.23322 | 0.19478 |
| $\beta_{11}$ | 0.4817 | 0.49613 | 0.48572 | 0.48776 | 0.51220 | 0.52506 | 0.48172 |
| $\gamma_1$ | -19.18072 | -19.17419 | -19.17860 | -19.18054 | -19.18059 | -19.18051 | -19.18072 |
| $\gamma_2$ | -4.07523 | -4.04373 | -3.93328 | -3.91250 | -3.92988 | -3.92934 | -4.07523 |
| $\gamma_3$ | -0.74738 | -0.74259 | -0.72432 | -0.70962 | -0.71042 | -0.70239 | -0.74737 |
| $\gamma_4$ | 1.84985 | 1.8959 | 1.89781 | 1.91055 | 1.91435 | 1.91503 | 1.84983 |
| $\gamma_5$ | 4.29262 | 4.26090 | 4.24203 | 4.26516 | 4.28086 | 4.30877 | 4.29263 |
| $\gamma_6$ | 20.87745 | 20.87584 | 20.87700 | 20.87743 | 20.87745 | 20.87745 | 20.87745 |
| $\gamma_{10}$ | 1.33471 | 1.08470 | 1.09194 | 1.1541 | 1.24054 | 1.3822 | NA |

Accuracy 0.53551 0.54367 0.54449 0.54367 0.54286 0.54041 0.54017
Table 9.3. Optimum tuning parameter along with estimated MSE, Accuracy and SE.

| Method | Link functions | $\alpha$ | MSE  | SE   | Accuracy |
|--------|----------------|---------|------|------|----------|
| MDPDE  | probit         | 0.39    | 0.07358 | 0.78485 | 0.54286 |
|        | logit          | 0.68    | 0.17656 | 1.23131 | 0.53959 |

Figure 9.1. Graphs of MSE for the probit link

Figure 9.2. Graphs of MSE for the logit link

performs better than the MLE. This fact is fairly corroborated by the trend of the accuracy values as well. From the discussions of the simulation results, we can fairly conclude that this data set contains outlying observations with respect to the probit link function. Similarly, for the logit link, we observe a similar trend of the MSE values in Figure 9.2. We notice that MSE is minimized when the tuning parameter $\alpha = 0.39, 0.68$ respectively for the probit and the logit links. These values are reported in Table 9.3. Given this data set, we obtain different optimized MSE values for different link functions. We may take up this as a future research problem in choosing an appropriate link function in this setup.

10. Conclusions

The lack of robustness in the likelihood-based inferential procedures poses a major challenge in modelling ordinal response data. Here we explore an alternative robust methodology to estimate the parameters in these statistical models through minimization of the density power divergence. It is based on the theory of an independent but non-homogeneous version of the DPD. We see how the choice of tuning parameters enables the MDPDE to achieve a higher degree of stability against different types of outliers that are inconsistent with a probabilistic model. The robustness of these estimators is discussed through the influence function and breakdown point analysis. Numerically, we have also shown that MDPD estimates also have a high implosive breakdown point at model misspecification. Robustness and asymptotic optimality are generally two competing concepts. The balance between these two is hard to achieve. We have demonstrated through the simulation studies how it is possible to find a suitable trade-off between these two extremities through the proper choice of a tuning parameter. Moreover, our proposed estimates perform better than Croux et al. [15]. Also, they are very competitive with Iannario et al. [18]. Factoring in all such possible challenges, we believe that the use of the MDPDE in the ordinal response models provides a useful tool for the applied scientist.
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Disclosure Statement

No potential conflict of interest is reported by the authors.

Data Availability Statement

The data set that supports the findings of this study is openly available in the UCI Machine Learning Repository at https://archive.ics.uci.edu/ml/datasets/wine+quality. See [34] to know more about this data set.

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