Strongly Secure Ramp Secret Sharing Schemes for General Access Structures

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Abstract

Ramp secret sharing (SS) schemes can be classified into strong ramp SS schemes and weak ramp SS schemes. The strong ramp SS schemes do not leak out any part of a secret explicitly even in the case where some information about the secret leaks from a non-qualified set of shares, and hence, they are more desirable than weak ramp SS schemes. However, it is not known how to construct the strong ramp SS schemes in the case of general access structures. In this paper, it is shown that a strong ramp SS scheme can always be constructed from a SS scheme with plural secrets for any feasible general access structure. As a byproduct, it is pointed out that threshold ramp SS schemes based on Shamir’s polynomial interpolation method are not always strong.

1 Introduction

A secret sharing (SS) scheme [1, 13] is a method to encode a secret $S$ into $n$ shares each of which has no information of $S$, but $S$ can be decrypted by collecting several shares. For example, a $(k, n)$-threshold SS scheme means that any $k$ out of $n$ shares can decrypt secret $S$ although any $k - 1$ or less shares do not leak out any information of $S$. The $(k, n)$-threshold access structure can be generalized to so-called general access structures which consist of the families of qualified sets and forbidden sets. A qualified set is the subset of shares that can decrypt the secret, but any information does not leak out from any forbidden set. Generally, the efficiency of SS schemes is evaluated by the entropy of each share, and it must hold that $H(V_i) \geq H(S)$ where $H(S)$ and $H(V_i)$ are the entropies of secret $S$ and shares $V_i$, $i = 1, 2, \ldots, n$, respectively [5, 9].

In order to improve the efficiency of SS schemes, ramp SS schemes are proposed, which have a trade-off between security and coding efficiency [2, 10–12, 14]. For instance, in the $(k, L, n)$-threshold ramp SS scheme [2, 14], we can decrypt $S$ from arbitrary $k$ or more shares, but no information of $S$ can be obtained from any $k - L$ or less shares. Furthermore, we assume that arbitrary $k - \ell$ shares leak out about $S$ with equivocation $(\ell/L)H(S)$ for $\ell = 1, 2, \ldots, L$. In the case where $L = 1$, the $(k, L, n)$-threshold SS scheme reduces to the ordinal $(k, n)$-threshold ramp SS scheme. Hence, to distinguish ordinal SS schemes with ramp SS schemes, we call ordinal

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SS schemes perfect SS schemes. For any \((k, L, n)\)-threshold access structure, we can realize that 
\[ H(V_i) = H(S)/L \] for \(k\) and hence, ramp SS schemes are more efficient than perfect SS schemes
[2, 14]. Furthermore, ramp schemes with general access structures are studied in [10–12].

Since non-forbidden sets with \(1 \leq \ell \leq L - 1\) in ramp SS schemes are allowed to leak out
a part of a secret, it is important to analyze how the secret partially leaks out. For example,
if a secret is a personal data that consists of name, address, job, income, bank account, etc.,
any part of the secret should not leak out explicitly. However, in the case that the security is
measured by the conditional entropy, we cannot know whether or not some part of the secret
can be decrypted from a non-forbidden set. Hence, Yamamoto introduced the notion of strong
and weak ramp SS schemes [14]. A ramp SS scheme is called a strong ramp SS scheme if it does
not leak out any part of a secret explicitly from any arbitrarily \(k - \ell\) shares for \(\ell = 1, 2, \ldots, L\). A
ramp SS scheme is weak if it is not strong. But, it is not given how to construct strong ramp SS
schemes for arbitrary given general access structures although it is known for \((k, L, n)\)-threshold
ramp SS schemes in [14].

In this paper, we discuss strong ramp SS schemes with general access structures. In section
2, we define ramp SS schemes called partially decryptable (PD) ramp SS schemes, in which every
non-qualified set with \(k - \ell\) shares can decrypt explicitly \((L - \ell)/L\) parts of a secret. Then, we
clarify the relation between PD ramp SS schemes and perfect SS schemes with plural secrets.
We also point out that \((k, L, n)\)-ramp SS schemes based on Shamir’s polynomial interpolation
method are not always strong. Next, in section 3, we propose how to convert PD ramp SS
schemes into strong ramp SS schemes by using a linear transformation, and we clarify that any
access structure that can be realized as a weak ramp SS scheme can also be realized as a strong
ramp SS scheme.

2 Background and Preliminaries

Let \(V = \{V_1, V_2, \ldots, V_n\}\) be the set of all shares, and let \(2^V\) be the family of all the subsets of
\(V\). Denote a secret by an \(L\)-tuple \(S = \{S_1, S_2, \ldots, S_L\}\), and each element of \(S\) is assumed to
be a mutually independent random variable according to the uniform distribution which takes
values in a finite field \(\mathbb{F}\). We assume that \(|\mathbb{F}|\) is sufficiently large\(^1\). Then, denote by \(H(S)\) and
\(H(A)\) the entropies of the secret \(S\) and a set of shares \(A \subseteq V\), respectively.

For families \(A_\ell \subseteq 2^V\), \(\ell = 0, 1, \ldots, L\), which consist of subsets of \(V\), we define ramp SS
schemes as follows:

**Definition 1** Let \(S\) and \(\Gamma_L = \{A_0, A_1, \ldots, A_L\}\) be a given secret and a given access structure.
Then, \(\{S, V, \Gamma_L\}\) is called a ramp secret sharing (SS) scheme if every subset \(A \in A_\ell\) satisfies
the following for \(\ell = 0, 1, \ldots, L\).

\[
H(S|A) = \frac{L - \ell}{L} H(S).
\]  

\(^1\)Throughout this paper, a set of shares and a family of share sets are represented by upper case bold-face and
calligraphic font letters, respectively. For simplicity of notation, we use \(AB\) to represent \(A \cup B\) for sets \(A\) and
\(B\), and \(\{V\}\) is represented as \(V\). For example, \(AV = A \cup \{V\}\). Furthermore, let \(A - B\) be a difference set of \(A\)
and \(B\), and the cardinality of a set \(A\) is denoted by \(|A|\).
Equation (1) implies that secret $S$ leaks out from any set $A \in \mathcal{A}_\ell$ with the amount of $(\ell/L)H(S)$. Especially, $S$ can be completely decrypted from any $A \in \mathcal{A}_L$, but any $A \in \mathcal{A}_0$ leaks out no information of $S$. Hence, in the case of $L = 1$, ramp SS schemes reduce to perfect SS schemes. Without loss of generality, we can assume that $A_\ell \neq A_{\ell'}$ holds for $\ell \neq \ell'$. Furthermore, we also assume that $\bigcup_{\ell=0}^L \mathcal{A}_\ell = 2^V$.

For example, an access structure of a $(k, L, n)$-ramp SS scheme [2,14] can be defined as $\mathcal{A}_0 = \{A : 0 \leq |A| \leq k-L\}$, $\mathcal{A}_\ell = \{A : |A| = k-L+\ell\}$ for $1 \leq \ell \leq L-1$, and $\mathcal{A}_L = \{A : k \leq |A| \leq n\}$. It is shown in [10] that ramp SS schemes with general access structures can be constructed if and only if the following conditions are satisfied.

**Theorem 2 ([10])** A ramp SS scheme with access structure $\Gamma_L = \{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_L\}$ can be constructed if and only if each $\tilde{\mathcal{A}}_\ell = \bigcup_{k=\ell}^L \mathcal{A}_k, \ell = 1, 2, \ldots, L$ satisfies the monotonocity in the following sense:

$$A \in \tilde{\mathcal{A}}_\ell \Rightarrow A' \in \tilde{\mathcal{A}}_\ell \text{ for all } A' \supseteq A. \quad (2)$$

$\blacksquare$

In the case of $L = 1$, (2) in Theorem 2 coincides with the necessary and sufficient condition to realize a perfect SS scheme with an access structure $\Gamma_1 = \{\mathcal{A}_0, \mathcal{A}_1\}$, which is proved in [8].

From Theorem 2, the minimal access structure $\mathcal{A}_\ell^-, \ell = 1, 2, \ldots, L$ can be defined as follows:

$$\mathcal{A}_\ell^- = \{A \in \mathcal{A}_\ell : A - \{V\} \not\in \mathcal{A}_\ell \text{ for any } V \in A\}. \quad (3)$$

**Proof of Theorem 2 ([10]):** We will prove only the sufficiency of (2) because the necessity is clear. Let $S = \{S_1, S_2, \ldots, S_L\}$ be a secret. From [8], in the case that (2) holds, we can construct a perfect SS scheme for the secret $S_\ell$ with the access structure $\tilde{\Gamma}_\ell = \{2V - \tilde{\mathcal{A}}_\ell, \tilde{\mathcal{A}}_\ell\}$ for every $\ell = 1, 2, \ldots, L$. Then, let $V_\ell = \{V_{\ell,1}, V_{\ell,2}, \ldots, V_{\ell,n}\}$ be the set of whole shares for such a perfect SS scheme with access structure $\tilde{\Gamma}_\ell$ for the secret $S_\ell$.

Now, we define $V_i = \{V_{i,1}, V_{i,2}, \ldots, V_{i,n}\}$ by collecting the $i$-th share of $V_\ell, \ell = 1, 2, \ldots, L$. Then, it is easy to check that the share set $V = \{V_1, V_2, \ldots, V_n\}$ realizes the ramp SS scheme with access structure $\Gamma_L$ for the secret $S$. In this case, we can decrypt $\{S_1, S_2, \ldots, S_L\}$ from a share set $A \in \tilde{\mathcal{A}}_\ell$, although $A$ cannot obtain any information of $\{S_\ell, S_{\ell+1}, \ldots, S_L\}$, and hence, (1) is satisfied. $\blacksquare$

In ramp SS schemes, the coding rate of the $i$-th share can be defined as $p_i = H(V_i)/H(S)$. To realize efficient ramp SS schemes, each coding rate of a ramp SS scheme should be as small as possible. Furthermore, it is known that $p_i \geq 1/L$ must hold for each $i = 1, 2, \ldots, n$ in any ramp SS scheme with $L$-level access structure $\Gamma_L$ [10,14]. From this viewpoint, the ramp SS schemes shown in the proof of Theorem 2 are not efficient. On the contrary, Okada-Kurosawa [12] presented the following example of a ramp SS scheme with a general access structure, which is more efficient than the ramp SS scheme shown in the proof of Theorem 2.

**Example 3 ([12])** Consider the following access structure $\Gamma_2^{SS}$ for a set of shares $V = \{V_1, V_2, V_3, V_4\}$.

$$\mathcal{A}_1 = \{\{V_1, V_4\}, \{V_2, V_4\}\}, \quad (4)$$

$$\mathcal{A}_2 = \{\{V_1, V_2, V_3\}\}. \quad (5)$$
Then, by letting the secret be \( S = \{ S_1, S_2 \} \), a ramp SS scheme for the access structure \( \Gamma^L_{2\times2} \) in (4) and (5) can be realized as

\[
\begin{align*}
V_1 &= \{ R_1, R_3 \}, \\
V_2 &= \{ R_2, R_4 \}, \\
V_3 &= \{ R_1 + R_4 + S_1, R_2 + R_3 + S_2 \}, \\
V_4 &= \{ R_1 + S_1, R_2 + S_1 \},
\end{align*}
\]

where \( R_1, R_2 \) and \( R_3 \) are mutually independent random numbers which take values in the same finite field \( F \).

From Example 3, it is clear that the secret \( S_2 \) can be decrypted from \( \{ V_1, V_4 \} \), but any information of \( S_1 \) cannot be obtained from the set. Hence, since \( S_1 \) and \( S_2 \) are mutually independent, it holds that \( H(S|V_1V_4) = H(S_1) = H(S)/2 \). In this way, if the partial information of the secret can be explicitly decrypted from every non-qualified set of shares, it is easy to calculate the amount of leaked information. Furthermore, we also note that such a ramp SS scheme can be considered as a special case of perfect SS schemes with \( L \) plural secrets [3, 4, 6].

In SS schemes with plural secrets, we assume that secret information is given by an \( L \)-tuple \( S^{(L)} = \{ S^{(1)}, S^{(2)}, \ldots, S^{(L)} \} \) where \( S^{(\ell)} \) are mutually independent random variables. Then, an access structure for the secret \( S^{(L)} \) is given by \( \Gamma^{(L)} = \{ A^{(1)}, A^{(2)}, \ldots, A^{(L)} \} \) where the secret \( S^{(\ell)} \) can be decrypted from any set in \( A^{(\ell)} \subseteq 2^V \) for \( \ell = 1, 2, \ldots, L \) while no information of \( S^{(\ell)} \) can be obtained from any set \( A \not\in A^{(\ell)} \).

The SS schemes for \( L \) secrets with an access structure \( \Gamma^{(L)} \) can be defined as follows:

**Definition 4 ([3])** Let \( \Gamma^{(L)} = \{ A^{(1)}, A^{(2)}, \ldots, A^{(L)} \} \) be an access structure for \( L \) secrets denoted by \( S^{(L)} = \{ S^{(1)}, S^{(2)}, \ldots, S^{(L)} \} \). Then, \( \{ S^{(L)}, V, \Gamma^{(L)} \} \) is called a SS scheme with \( L \) secrets if it satisfies for all \( \ell = 1, 2, \ldots, L \) that

\[
\begin{align*}
H(S^{(\ell)}|A) &= 0 & \text{for any } A \in A^{(\ell)}, \\
H(S^{(\ell)}|A') &= H(S^{(\ell)}) & \text{for any } A' \not\in A^{(\ell)}.
\end{align*}
\]

From [3], Definition 4 is equivalent to the following definition.

**Definition 5 ([3])** Let \( \Gamma^{(L)} = \{ A^{(1)}, A^{(2)}, \ldots, A^{(L)} \} \) be an access structure for \( L \) secrets denoted by \( S^{(L)} = \{ S^{(1)}, S^{(2)}, \ldots, S^{(L)} \} \). Let \( S^{(A)} \subseteq S \) be a subset of the secret that can be decrypted from a share set \( A \subseteq V \) according to \( \Gamma^{(L)} \), and we define that \( \overline{S^{(A)}} \) as \( S - S^{(A)} \). Then, \( \{ S^{(L)}, V, \Gamma^{(L)} \} \) is called a SS scheme with plural secrets \( S^{(L)} \) if it satisfies that

\[
\begin{align*}
H \left( S^{(A)} | A \right) &= 0, \\
H \left( \overline{S^{(A)}} | A \right) &= H \left( \overline{S^{(A)}} \right),
\end{align*}
\]

for all \( A \subseteq V \).
Based on Definition 5, we define the partially decryptable ramp SS schemes that characterize the ramp SS schemes shown in the proof of Theorem 2 and Example 3.

**Definition 6** Let \( S = \{S_1, S_2, \ldots, S_L\} \) be secrets for an access structure \( \Gamma_L = \{A_1, A_2, \ldots, A_L\} \). Then, \( \{S, V, \Gamma_L\} \) is called a partially decryptable (PD) ramp SS scheme if there exists a part of the secret information \( S_A \subseteq S \) satisfying that

\[
|S_A| = \ell 
\]

\[
H(S_A|A) = 0, \tag{15}
\]

\[
H(S_A) = H(\overline{S_A}) , \tag{16}
\]

for all \( A \in A^{(\ell)} \) where \( \overline{S_A} \overset{\text{def}}{=} S - S_A \). \( \square \)

From (15) and (16) in Definition 6, it holds that \( H(S|A) = H(S_A|A) + H(\overline{S_A}|A) = H(\overline{S_A}) \), and hence, a PD ramp SS scheme satisfies Definition 1.

Note that a PD ramp SS scheme can be regarded as a SS scheme with plural secrets. Conversely, if a SS scheme for plural secrets \( S_L \) with access structure \( \Gamma_L \) is given, we can construct a corresponding access structure of a PD ramp SS scheme for the secret \( S = \{S_1, S_2, \ldots, S_L\} = \{S^{(1)}, S^{(2)}, \ldots, S^{(L)}\} \) in the following way: Assign each share set \( A \subseteq V \) to the family \( A^{(\ell)} \) where \( \ell \) is given by

\[
\ell = \left| \{\ell' : A \in A^{(\ell')} \in \Gamma^{(L)}\} \right|. \tag{17}
\]

Then, the tuple of families \( \{A_0, A_1, \ldots, A_L\} \overset{\text{def}}{=} \Gamma_L \) can be regarded as the access structure of the PD ramp SS scheme.

The difference between Definition 5 and Definition 6 is summarized as follows: In Definition 5, from a share set \( A \subseteq V \), we can decrypt a subset of secrets \( S^{(L)} \), i.e., \( S^{(A)} \), according to the access structure \( \Gamma^{(L)} \). However, in the PD ramp SS schemes defined in Definition 6, a share set \( A \in A^{(\ell)} \) decrypts some \( S_A \) which satisfies (14), i.e., \( S_A \) is not specified by the access structure \( \Gamma_L \).

We note that the amount of the leaked information about \( S \) from a share set \( A \in A^{(\ell)} \) is \( (\ell/L)H(S) \) in PD ramp SS schemes. Hence, in the sense of (1), there is no difference between Definition 1 and Definition 6. That is, both definitions guarantee the same security in the case that \( S \) is meaningless if some part of \( S \) is missing. However, if each part of \( S \) has explicit meaning, PD ramp SS schemes are not secure, and hence, not desirable.

To overcome such defects, Yamamoto defined strong ramp SS schemes as follows [14]:

**Definition 7** ([14]) Let \( S = \{S_1, S_2, \ldots, S_L\} \) and \( \Gamma_L \) be a secret and an access structure, respectively. Then, \( \{\Gamma_L, V, S\} \) is called a strong ramp SS scheme if for all \( \ell = 0, 1, \ldots, L - 1 \), \( A \in A^{(\ell)} \) satisfies (1) and

\[
H(S_{j_1, j_2, \ldots, j_{L-\ell}}) = H(S_{j_1, j_2, \ldots, j_{L-\ell}}) \tag{18}
\]

\[
\] for all \( \{S_{j_1, j_2, \ldots, j_{L-\ell}}\} \subseteq S \). \( \square \)

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3In [14], strong ramp SS schemes are defined for \( (k, L, n) \)-threshold ramp access structures.
Definition 7 implies that strong ramp SS schemes do not leak out any part of the secret explicitly from a non-qualified set $A \notin A_L$. Now, from this point of view, we review the $(k, L, n)$-threshold SS scheme based on Shamir’s interpolation method.

Remark 8 We note that the $(k, L, n)$-threshold ramp SS scheme, which is an extension of Shamir’s interpolation method [13], is not always a strong ramp SS scheme. For instance, consider a $(4, 2, n)$-threshold ramp SS scheme by using the following polynomial of degree 3 over the finite field $\mathbb{Z}_{17}$.

$$f(x) = S_1 + S_2 x + R_1 x^2 + R_2 x^3,$$

where $S = \{S_1, S_2\}$ is a secret, and $R_1$ and $R_2$ are independent random numbers. The $i$-th share is given by $V_i = f(i)$. Then, from a simple calculation of $V_3, V_6$ and $V_{15}$, we have

$$5S_2 = 7V_3 + 9V_6 + V_{15}. \quad (20)$$

This means that partial information $S_2$ can be decrypted completely from shares $V_3, V_6$ and $V_{15}$.

We also note that from share set $\{V_1, V_2, V_3\}$, we have $H(S_1|V_1V_2V_3) = H(S_1)$ for $\ell = 1, 2$, and hence, the ramp SS scheme in this example is neither PD nor strong⁴.

Remark 8 shows that it is difficult to construct strong ramp SS schemes in general. In [14], it is proposed how to construct strong $(k, L, n)$-threshold ramp SS schemes, but it is not known how to construct strong ramp SS schemes for general access structures.

Fortunately, PD ramp SS schemes with general access structure $\Gamma_L$ can easily be constructed if $\Gamma_L$ satisfies monotonicity given by (2) in Theorem 2. Furthermore, it is easy to calculate how much information leaks out from each non-qualified set in PD ramp SS schemes. Therefore, we propose a method to construct strong ramp SS schemes with general access structures based on PD ramp SS schemes.

3 Strong Ramp Secret Sharing Schemes with General Access Structures

In this section, we propose how to construct a strong ramp SS scheme with general access structure $\Gamma_L$ from a given PD ramp SS scheme with the same access structure $\Gamma_L$.

Since a PD ramp SS scheme with general access structure $\Gamma_L$ can always be constructed if $\Gamma_L$ satisfies (2) in Theorem 2, we assume that a PD ramp SS scheme with access structure $\Gamma_L = \{A_1, A_2, \ldots, A_L\}$ is obtained for a secret $S = \{S_1, S_2, \ldots, S_L\}$. Denote by $\phi_{\Gamma_L}(S, R)$ the encoder of such a PD ramp SS scheme with the access structure $\Gamma_L$ for the secret $S$ where $R$ represents a set of random numbers used in the encoder. Then, we choose publicly an $L \times L$ non-singular matrix $T$ and define a new encoder $\varphi_{\Gamma_L}(S', R) \overset{\text{def}}{=} \phi_{\Gamma_L}(S'T, R)$ where $S' = \{S'_1, S'_2, \ldots, S'_L\}$.

The next theorem gives the necessary and sufficient condition of $T$ that realizes a strong ramp SS scheme with the access structure $\Gamma_L$ for secret $S' = \{S'_1, S'_2, \ldots, S'_L\}$.

⁴In [7], a construction method is discussed for neither PD nor strong ramp SS schemes.

⁵Hereafter, for simplicity of notation, we identify the sets $S = \{S_1, S_2, \ldots, S_L\}$ and $S' = \{S'_1, S'_2, \ldots, S'_L\}$ with $L$-dimensional row vectors $[S_1 \ S_2 \cdots S_L]$ and $[S'_1 \ S'_2 \cdots S'_L]$, respectively.
**Theorem 9** Suppose that the encoder $\phi_T(S, R)$ of a PD ramp SS scheme with an access structure $\Gamma_L$ for a secret $S$ is given. Let $S_A$ be the partial information of the secret $S$ that can be decrypted explicitly from a share set $A$ in the PD ramp SS scheme, and denote by $I(A)$ the set of indices of $S_A$. Then, we construct a new encoder $\varphi_{T_L}(S', R) \overset{\text{def}}{=} \phi_{T_L}(S'T, R)$ for a new secret $S' = \{S'_1, S'_2, \ldots, S'_L\}$ by using a publicly opened $L \times L$ non-singular matrix $T$.

Then, the necessary and sufficient condition of $T$ to realize a strong ramp SS scheme $\{S', V, \Gamma_L\}$ is given by

$$\text{rank } [T^{-1}]_{\{1,2,\ldots,L\}-I(A)} = L - \ell,$$  \hspace{1cm} (21)

for all $A \in \mathcal{A}_\ell$, $\ell = 0, 1, \ldots, L$, where $[T^{-1}]_{\{i_1,i_2,\ldots,i_\ell\}}$ is the submatrix that consists of the $i_1$-th, $i_2$-th, $\ldots$, $i_\ell$-th rows, and the $j_1$-th, $j_2$-th, $\ldots$, $j_\ell$-th columns of $T^{-1}$. □

**Remark 10** Theorem 9 implies that any strong ramp SS schemes can be obtained from the corresponding PD ramp SS schemes without loss of coding rates. □

**Proof of Theorem 9:** Since the matrix $T$ is non-singular, $S$ has one to one correspondence with $S'$. Hence, $S'$ is also a set of $L$ mutually independent random variables according to the same uniform distribution. Therefore, it holds that $H(S) = H(S') = L \log |\mathbb{F}|$ where $\mathbb{F}$ is a finite field in which $S_\ell$, $\ell = 1, 2, \ldots, L$ take values.

Then, for any $A \in \mathcal{A}_\ell$, $\ell = 1, 2, \ldots, L$, where $\Gamma_L = \{A_0, A_1, \ldots, A_L\}$ is the access structure of the PD ramp SS scheme, we have

$$H(S'|A) = H(S|A) = \frac{L - \ell}{L} H(S) = (L - \ell) \log |\mathbb{F}| = \frac{L - \ell}{L} H(S').$$  \hspace{1cm} (22)

Therefore, (1) holds for secret $S'$. Next, from (18), we have for any $\{S'_1, S'_2, \ldots, S'_{L-\ell}\} \subseteq S'$ that

$$H(S'_1 S'_2 \cdots S'_{L-\ell}|A) = H(S [T^{-1}]_{\{1,2,\ldots,L\}-I(A)} | A)$$

$$\overset{(a)}{=} H(S_A [T^{-1}]_{\{1,2,\ldots,L\}-I(A)} | A)$$

$$\overset{(b)}{=} H(S_A | A)$$

$$\overset{(c)}{=} H(S_A) = (L - \ell) \log |\mathbb{F}| = H(S'_1 S'_2 \cdots S'_{L-\ell}),$$  \hspace{1cm} (23)

where equalities (a), (b), and (c) hold because of (15), (21) and (16), respectively.

Finally, we note that the necessity of (21) is clear since equality (b) in (23) does not hold if (21) is not satisfied. □

From the proof of Theorem 9, it is sufficient to choose the matrix $T$ satisfying, instead of the condition (21), that every submatrix of $T^{-1}$ has the full rank. We note that the Hilbert matrix $T_H$ has such a property. Each element of an $L \times L$ Hilbert matrix $T_H = [t_{ij}]_{1 \leq i,j \leq L}$ is given by

$$t_{ij} = \frac{1}{x_i + y_j},$$  \hspace{1cm} (24)

where $x_i$ and $y_j$ must satisfy for all $i, j \in \{1,2,\ldots,L\}$ that

$$x_i + y_j \neq 0.$$  \hspace{1cm} (25)
Note that every submatrix of the Hilbert matrix is also a Hilbert matrix, and the determinant of the matrix $T_H$ can be calculated as follows:

$$\det T_H = \prod_{1 \leq i < j \leq L} (x_i - x_j) \prod_{1 \leq i < j \leq L} (y_i - y_j) \prod_{i=1}^{L} \prod_{j=1}^{L} (x_i + y_j).$$  \hspace{1cm} (26)

Hence, it is clear that every submatrix of $T_H$ is non-singular if and only if

$$x_i \neq x_j \quad \text{and} \quad y_i \neq y_j$$

are satisfied for $i \neq j$ in addition to (25). Since $|\mathbb{F}|$ is usually assumed to be sufficiently large in ordinal ramp SS schemes, it is easy to choose $\{x_i\}_{i=1}^{L}$ and $\{y_i\}_{i=1}^{L}$ satisfying (25) and (27).

Then, from Theorems 2 and 9, the following theorem holds.

**Theorem 11** A strong ramp SS scheme with access structure $\Gamma_L$ can be constructed if and only if each $\mathcal{A}_\ell$, $\ell = 1, 2, \ldots, L$, satisfies the monotonicity given by (2) of Theorem 2. \hspace{1cm} □

**Example 12** Note that matrices satisfying (21) may exist besides the inverse of Hilbert matrices. For example, consider the (3, 2, 3)-threshold ramp SS scheme given by (4) and (5) such that $V_1 = \{R_1, R_3\}$, $V_2 = \{R_2, R_4\}$, $V_3 = \{R_1 + R_4 + S'_1 + S'_2, R_2 + R_3 + S'_1 - S'_2\}$, and $V_4 = \{R_1 + S'_1 + S'_2, R_2 + S'_1 + S'_2\}$. It is easy to check that $V = \{V_1, V_2, V_3, V_4\}$ realizes a strong ramp SS scheme with access structure $\Gamma'_2$ for secret $S' = \{S'_1, S'_2\}$.

We note here that, in the case of the access structure $\Gamma^x_L$ in Example 3, the minimum size of $\mathbb{F}$ is 2 in order realize the PD ramp SS schemes for secret $S$ [12], although $|\mathbb{F}| \geq 3$ is required to realize a strong ramp SS schemes for $S'$ if we use the transformation $T^x$ in (28). In this way, the minimum size of $\mathbb{F}$ to realize strong ramp SS schemes generally becomes larger than that required to realize PD ramp SS schemes. □

**Remark 13** Note that the matrix $T$ described in Theorem 9 is the transformation from a PD ramp SS scheme to a corresponding strong ramp SS scheme. However, weak but not PD ramp SS schemes as shown in Remark 8 cannot always be transformed into strong ramp SS schemes by the matrix $T$ satisfying (21). For example, consider the $(3, 2, 3)$-threshold ramp SS scheme given by $V_1 = S_1 + R, V_2 = S_1 + S_2 + R$, and $V_3 = R$, where $R$ is a random number [14]. Then, these shares realize a weak but not PD ramp SS scheme. If we transform this ramp SS scheme by using $S = S'T^x$ where $T^x$ is given by (28), we have $V_1 = S'_1 + S'_2 + R, V_2 = 2S'_1 + R$, and $V_3 = R$. It is easy to check that $V_1, V_2$ and $V_3$ do not realize a strong ramp SS scheme for $S'$. □
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