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Poisson Brackets of Wilson Loops and Derivations of Free Algebras

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Abstract

We describe a finite analogue of the Poisson Algebra of Wilson Loops in Yang–Mills theory. It is shown that this algebra arises in an apparently completely different context: as a Lie algebra of vector fields on a non–commutative space. This suggests that non–commutative geometry plays a fundamental role in the manifestly gauge invariant formulation of Yang–Mills theory. We also construct the deformation of the loop algebra induced by quantization, in the large $N_c$ limit.
A central problem of particle physics is to find a formulation of Yang–Mills theory in terms of gauge invariant variables. There is a large literature on this subject, starting with pioneering work of Mandelstam. [1] Such a reformulation of Yang–Mills theory must involve as yet unknown geometrical principles, as the principle of gauge invariance would be empty. We should discover these geometrical structures by starting with the conventional formalism of gauge theory and rewriting it in terms of gauge invariant variables. A loose analogy can be made with the process by which symplectic geometry was discovered to be the foundation of classical mechanics.

In this paper we will show that the fundamental Poisson brackets of Yang–Mills theory have an interpretation as an algebra of vector fields in a non–commutative space, a sort of non–commutative generalization of the Virasoro algebra. The precise mathematical formulation is as yet possible only for a finite (regularized) version of the theory, but the ideas extend in a formal way to the continuum theory.

A natural choice of gauge invariant variable in Yang–Mills theory is the Wilson loop variable. It is just the trace of the parallel transport operator around a loop. We can describe the symplectic structure of classical Yang–Mills theory in terms of Poisson brackets of these variables. However, in the usual canonical formalism, where initial data is given on a space–like surface, this leads to either a trivial answer or to an impossibly complicated one. If the loop lies entirely on a space–like surface, the Poisson brackets will vanish since the components of the gauge field on a space–like surface commute: they are like the \( q \) variables of classical mechanics. If the loop has a finite extend into the time–like direction on the other hand, the Poisson brackets cannot be obtained without solving the equations of motion: this is like asking for \( \{q(t), q(t')\} \) at unequal times in classical mechanics. One way around this impasse is to introduce loop variables involving the electric field (which is the canonical conjugate of the Yang–Mills potential) but this does not have the elegance and simplicity of a formalism involving Wilson loop variables alone.

We showed in some recent papers [2] that the Wilson Loops in classical Yang–Mills theory which lie on a null surface satisfy simple Poisson brackets. (There is also a large literature on the null cone formalism for gauge theories. See for example, ref. [3].) In a formalism in which initial data is given on null surfaces there is thus a natural way of encoding the canonical structure of Yang–Mills theory in terms of gauge invariant variables. In this paper we will show that an analogue of this Loop algebra arises as derivations of
the free algebra on a finite number of variables. This allows us to construct a Lie group of which the finite loop algebra is the Lie algebra. Moreover, we will construct a symplectic realization analogous to the realization on the Yang–Mills phase space. We will also obtain a quantum deformation of this algebra and obtain the contraction corresponding to the large $N_c$ limit. It is conjectured that this large $N_c$-limit algebra also has a symplectic realization, but we are as yet unable to construct it. This would be of great interest in Yang–Mills theory, as it would help us discover the phase space of gauge invariant observables of that theory.

Let us now describe the situation a little more explicitly [2]. We will consider pure Yang–Mills theory on flat Minkowski space, with initial data given on a null cone. The field will then be determined at all points in the future of the cone by the Yang–Mills equations. (See Ref. [2] for details.) It will be particularly convenient to choose as initial surface the null cone at past time–like infinity, (called $\mathcal{I}^-$ in Ref. [4]) so that all points not on the cone are in its future. (This will also restore spatial translation invariance.) This can be accomplished by using a conformally equivalent metric

\[
\hat{ds}^2 = dU (dU + 2dR) - \sin^2 R \; q_{ij}(z) dz^i dz^j
\]

instead of the flat metric on Minkowski space. (Here, $q_{ij}$ is the standard metric on $S^2$.) Yang–Mills equations are conformally invariant in four dimensions, so this conformal change of the metric will not change the theory. $\frac{\partial}{\partial U}$ is a time–like vector and $\frac{\partial}{\partial R}$ is a null vector. We will regard $\frac{\partial}{\partial U}$ as defining the time direction. Also, Minkowski space corresponds to the region

\[
-\pi < U < \pi \quad -\pi < U + 2R < \pi
\]

The null cone at past time–like infinity, $\mathcal{I}^-$ will be the surface $U = -\pi$.

Since the Yang–Mills equations are of first order in the time variable $U$, initial data consist of prescribing the value of the gauge potential on $\mathcal{I}^-$. We can set $A_R = 0$, by a choice of gauge. Also, $A_U$ is just a Lagrange multiplier (its time derivative does not appear in the Lagrangian) so that the dynamical variables are the transverse components $A_i$. The main simplification of the null formalism is that these variables are in a sense canonically conjugate to each other with equal time Poisson brackets

\[
\{A^a_{ib}(z,R), A^c_{jd}(z',R')\} = \frac{1}{2} \delta^a_d \delta^c_b q_{ij}(z) \delta(z-z') \text{sgn}(R-R').
\]
Here $a, b = 1, 2, \cdots N_c$. We will consider the gauge group to be $U(N_c)$ so that the matrices $A_i$ are hermitian. These Poisson brackets follow from the Yang–Mills action by a straightforward application of the canonical formalism.

Define thus the inverse of the symplectic form on the phase space of Yang–Mills theory:

\[ \omega_{ijcd}^a(z, R, z', R') = \delta^a_d \delta^c_b \omega_{ij}(z, R, z', R') \]  

where,

\[ \omega_{ij}(z, R, z', R') = \frac{1}{2} \eta_{ij}(z) \delta(z - z') \operatorname{sgn}(R - R'). \]  

Now let $\xi : S^1 \to \mathcal{I}^-$ be a closed curve on the light cone. Given a Yang–Mills field $A$, we can define a complex valued function $W[\xi]$ on the space of closed curves, the trace of the parallel transport operator (holonomy) in the basic $N_c$ dimensional representation. In the case of $U(N_c)$, this loop variable is complex valued, but it satisfies the condition

\[ W[\xi] = W^*[\xi] \]  

where $\bar{\xi}$ is the curve $\xi$ with the opposite orientation.

This $W[\xi]$ is the ‘Wilson Loop’ associated to this Yang–Mills configuration. For each $\xi$, $W[\xi]$ is a function on the Yang–Mills phase space so that it is possible to compute the Poisson bracket of a pair of loop variables. We will get,

\[ \{W[\xi], W[\bar{\xi}]\} = \int ds dt \dot{\xi}^i(s) \dot{\bar{\xi}}^j(t) \omega_{ij}(\xi(s), \bar{\xi}(t)) W[\xi \circ_{st} \bar{\xi}] W. \]

Due to the delta function in the symplectic form, only points $s, t$ where $\xi^i(s) = \bar{\xi}^i(t)$ will contribute to the integral; i.e., the projections of the curves to the sphere must intersect at parameter value $s$ for the first curve and $t$ for the second. There will be a null line segment joining the points $\xi(s)$ and $\bar{\xi}(t)$ in this case. In this case, $\xi \circ_{st} \bar{\xi}$ is the product curve, defined as follows: describe the curve $\xi$ starting and ending at $s$; jump to the point $\bar{\xi}(t)$ along the null line segment; describe the curve $\bar{\xi}$ starting and ending at $\bar{\xi}(t)$; jump back to $\xi(s)$ along the null segment. Thus the product is also a closed curve. The pieces of the curve along null lines will not contribute to $W[\xi \circ_{st} \bar{\xi}]$ since we have chosen a gauge where the null component of the gauge field vanishes. Also, there will be generically only a finite number of intersection points, so that the integral on the right hand side can be actually evaluated to yield a finite sum. We will not need the explicit expression, which is given in Ref. [2]. This last property depends on the dimension of space–time being
four; in higher than four dimensions generically there are no intersections while in lower dimensions there is a continuum of such intersections.

The Poisson brackets above could have been motivated based purely on the geometry of loops on the null cone. Causality requires that loops on a null cone which have no intersections when projected to the space-like surface, must commute. This explains the delta function in $\omega(z, R, z', R')$. The factor $\text{sgn}(R - R')$ is also natural, as it simply keeps track of which event is to the future along the null direction, and makes the bracket anti-symmetric.

Indeed, geometrically, this is the most natural definitions possible for $\omega$ as well as the product of the loops. Note that the bracket is invariant under the change of parametrization of the loop. In fact the rhs will only depend on $q_{ij}$ only through the angle of intersection of the tangent vectors $\xi(s), \xi(t)$; thus the algebra is invariant under conformal transformation of the metric on the two sphere. But every metric on the two sphere is conformal to the standard one, so we see that the algebra is in fact independent of the choice of metric also.

Once the Poisson brackets are postulated, the Jacobi identity can be proved directly. The Yang–Mills phase space then arises as the solutions of some algebraic constraints satisfied by the loop variables, due to Mandelstam. We proposed that these constraints be viewed as describing the co–adjoint orbits of the above Lie algebra. In this way, Yang–Mills theories with different Unitary groups as gauge groups, would arise as different realizations of the same universal Lie algebra. For more details we refer the reader to Ref. [2]

We would like to understand the above Lie algebra of loops better. In particular we would like to have a finite analogue which can be studied by more rigorous methods; also it would be good to have a different situation in which this loop algebra arises so that we can have a point of view to Yang–Mills theory not based on the gauge group. Another natural object to study is the group associated to the above Loop algebra. Finally it is very important to understand the quantum deformations of this algebra and its large $N_c$-limit. In this paper we will in fact arrive at a finite analogue of the loop algebras and their groups, starting from considerations quite different from Yang–Mills theories; i.e., the derivations and automorphisms of Free algebras. We will also construct a quantum deformation and its large $N_c$ limit.

Another situation where Poisson brackets of Wilson loops appear is in Chern–Simons theory. There also the spatial components of the gauge field are canonically conjugate to
each other; the Wilson loops on a space-like 2-surface satisfy the above algebra except that
\[ \omega_{i,j}(z, z') = \epsilon_{i,j}(z) \delta^2(z - z') \]  
(7)
where \( \epsilon_{i,j}(z) dz^i \wedge dz^j \) is the volume form on the space-like surface. The product of loops relative to a pair of co-incident points is defined as before as one loop followed by the other. Thus our considerations should also be of interest in the context of topological field theories.

The Free Algebra and its Automorphisms

Let \( T_M \) be the real free algebra on \( M \) variables. \([5]\) It is a graded vector space, the part of order \( m \) (for \( m = 0, 1, 2 \cdots \)) being just the set of all tensors of type \((0, m)\) on an \( M \) dimensional real vector space. Note that no symmetry of any kind is required on these tensors. The multiplication rule on the algebra is defined by the direct (or tensor) product. \( T_M \) is a non-commutative but associative algebra with identity.

More explicitly, introduce variables \( \xi^i \) for \( i = 1, \cdots M \) satisfying no relations whatever. A typical element of \( T_M \) is a polynomial in these variables,

\[ T(\xi) = \sum_{m=0}^{\infty} T_{i_1 i_2 \cdots i_m} \xi^{i_1} \xi^{i_2} \cdots \xi^{i_m} \]  
(8)

\( T_{i_1 i_2 \cdots i_m} \) are the components with respect to some basis in \( R^M \) of a tensor \( T \) of type \((0, m)\). Since \( T(\xi) \) is assumed to be a polynomial, only a finite number of terms on the right hand side of the above series are non-zero: only a finite number of the tensors \( T_{i_1 \cdots i_m} \) are non-zero.

In this language, multiplication is defined as follows

\[ (ST)(\xi) = \sum_{m,n=0} S_{i_1 \cdots i_m} T_{j_1 \cdots j_n} \xi^{i_1} \cdots \xi^{i_m} \xi^{j_1} \cdots \xi^{j_n} \]  
(9)

There is no problem with convergence of the series since only a finite number of terms are non-zero. In fact this comment applies to almost all the formally infinite series below. (The exception is where we speak of inverting a transformation of the variables.)

If these variables \( \xi \) had commuted with each other, the algebra would just have been the commutative algebra of functions (polynomials) on \( R^M \). The tensors would all have
been symmetric and multiplication would have been the symmetrized tensor product. This
algebra has as automorphisms the group of diffeomorphisms of $R^M$. (Actually a diffeo-
morphism will in general map a polynomial to an infinite series, so we will really need
$\mathcal{T}_M$ to extend to an appropriate topological vector space to make this possible.) Infinites-
imally, this would correspond to the Lie algebra of vector fields, whose components are
polynomials, which form the derivations of the commutative algebra.

Thus we can regard $\mathcal{T}_M$ as the set of ‘functions’ on a non–commutative space in the
spirit of non–commutative geometry [6]. This is perhaps the most non–commutative
case in the sense that the co–ordinates satisfy no relations at all. Now let us determine the
Lie algebra of derivations, $\mathcal{V}_M$ which will be the non–commutative analogue of the algebra
of vector fields. A derivation $v$ is determined by its effect on the generators:

$$v(\xi)^i = \sum_{m=1}^\infty v^i_{i_1\ldots i_m} \xi^{i_1} \cdots \xi^{i_m}$$

(10)

where it is assumed that only a finite number of terms in the sum are non–zero. The
effect of $v$ on an arbitrary element of $\mathcal{T}_M$ is given by the Leibnitz rule:

$$v(T)(\xi) = \sum_{m,n=1}^\infty \sum_{k=1}^m T_{i_1\ldots i_m} v^{i_k}_{j_1\ldots j_n} \xi^{i_1} \cdots \xi^{i_{k-1}} \xi^{j_1} \cdots \xi^{j_n} \xi^{i_{k+1}} \cdots \xi^{i_m}$$

(11)

A basis (analogous to the Weyl basis for $\mathfrak{gl}(M)$) for $\mathcal{V}_M$ is given by the elements $E^{i_1\ldots i_m}$
defined by

$$E^{i_1\ldots i_m}(\xi)^j = \delta^j_i \xi^{i_1} \cdots \xi^{i_m}. \quad (12)$$

In the commutative case they correspond to the vector fields $\xi^{i_1} \cdots \xi^{i_m} \frac{\partial}{\partial \xi^i}$. They satisfy
the commutation relations,

$$[E^{i_1\ldots i_m}, E^{j_1\ldots j_n}] = \sum_{l=1}^n \delta^l_i E^{j_1\ldots \hat{l}\ldots j_n} \xi^{i_1} \cdots \xi^{i_{l-1}} \xi^{i_{l+1}} \cdots \xi^{i_m}$$

$$- \sum_{k=1}^m \delta^{i_k} E^{i_1\ldots i_{k-1} j_1 \cdots j_n \hat{k} \cdots i_m}.$$ 

In the special case $M = 1$, all the non–commutativity dissapears, and $\mathcal{V}_1$ is just the
algebra of polynomial vector fields on the real line. Since all the indices must take the
value one, there is just one generator with $m$ superscripts. Suppose we call it $L_{m-1}$ for
$m = 0, 1, \cdots$. Then the above commutation relation becomes

$$[L_m, L_n] = (n - m)L_{m+n} \quad \text{for } m = -1, 0, 1, 2, \cdots. \quad (13)$$
This is just the subalgebra of the Virasoro algebra on which the central term vanishes. Thus our algebras are, in a sense, generalizations of this familiar algebra.

Now let \( g_{ij} \) be a symmetric positive tensor on \( R^M \) and define \( \mathcal{V}_M^- \) to be the subalgebra of tensors that preserve the element \( g(\xi) = |\xi|^2 = g_{ij} \xi^i \xi^j \). In the commutative case these are all the vector fields tangential to the spheres centered at the origin; these preserve the distance function \( |\xi|^2 \). The simplest among these are the rotations. In the case of a Free algebra, we can see easily that the algebra \( \mathcal{V}_M^- \) consists of the set of all elements of the form

\[
v_{i_1\cdots i_m}^j = g^{ij} w_{i_0 i_1\cdots i_m}
\]

where \( w_{i_0 i_1\cdots i_m} \) is a \textit{cyclically anti-symmetric} tensor. Of course such tensors exist only when \( m \) is odd. There is a basis for \( \mathcal{V}_M^- \),

\[
G^{i_0\cdots i_m} = \sum_{k=0}^{m} (-1)^k g^{i_0 j} E_{j}^{i_{k+1}\cdots i_m i_0\cdots i_{k-1}}
\]

in which the Lie brackets become

\[
[G^{i_0\cdots i_m}, G^{j_0\cdots j_n}] = \sum_{k,l=0}^{k=m,l=n} (-1)^{k+l+1} g^{i_0 j_0} G^{i_{k+1}\cdots i_m i_1\cdots i_{k-1} j_{k+1}\cdots j_1 \cdots j_{n-1}}.
\]

In an exactly analogous fashion, let \( \omega \) be an anti-symmetric non-degenerate tensor. (Clearly this exists only if \( M \) is even, which will be assumed in the following.) This defines an element

\[
\omega(\xi) = \omega_{ij} \xi^i \xi^j.
\]

This is a symplectic analogue of the distance function. This would have vanished identically in the commutative case.

The subalgebra \( \mathcal{V}_M^+ \) which preserves \( \omega(\xi) \) is just the set of elements such that

\[
v_{i_1\cdots i_m}^j = \omega^{ii_0} w_{i_0 i_1\cdots i_m}
\]

where \( w_{i_0 i_1\cdots i_m} \) is a \textit{cyclically symmetric} tensor. There is a basis for \( \mathcal{V}_M^+ \),

\[
E^{i_1\cdots i_m} = \sum_{k=1}^{m} \omega^{i_0 j} E_{j}^{i_{k+1}\cdots i_m i_0\cdots i_{k-1}}
\]
in which the Lie brackets become

\[
[F^{i_1\cdots i_m}, F^{j_1\cdots j_n}] = \sum_{k,l=1}^{k=m,l=n} \omega^{i_k,j_l} F^{i_{k+1} \cdots i_m i_1 \cdots i_{k-1} j_{l+1} \cdots j_n j_1 \cdots j_{l-1}}.
\] (20)

Now we will show that this algebra is just a finite version of the loop algebra we found for Wilson loops. Let us think of the index \( I = i_1 \cdots i_m \) on the \( F^I \) variable as a map \( I : Z_m \to \{1, \cdots M\} \). Due to cyclic symmetry, this can be viewed as a ‘loop’ from the cyclic permutation group \( Z_m \) (which is a discrete model for the circle) to a space which contains just a finite number \( M \) of points. The product of two loops at point \( k, l \) is defined as the loop \( I \) starting at \( i_{k+1} \) and ending at \( i_{k-1} \) followed by the loop \( J \) starting at \( j_{l+1} \) and ending at \( j_{l-1} \):

\[
I \circ_{k,l} J = i_{k+1} \cdots i_m i_1 \cdots i_{k-1} j_{l+1} \cdots j_n j_1 \cdots j_{l-1}.
\] (21)

This is just the discrete analogue of the product we introduced earlier. The commutation relations of the Lie algebra \( V^+_M \) are then,

\[
[F^I, F^J] = \sum_{kl} \omega^{i_k,j_l} F^{I \circ_{k,l} J}.
\] (22)

The Wilson loop algebra can be understood as the limiting case where the finite set \( Z_m \) is replaced by \( S^1 \) and the set \( \{1, 2, \cdots, M\} \) is replaced by the light cone \( S^2 \times R^+ \). Thus by studying the algebra \( V^+_M \) we are studying a finite model for the algebra of Wilson loops. The algebras \( V_M, V^-_M, V^+_M \) are all graded Lie algebras. The point is that the Lie bracket of a tensor with \( m \) indices and one of with \( n \) indices has \( m + n - 2 \) indices. Thus if we assign a grade of \( m - 1 \) to the space of tensors with \( m \) indices, we have a graded Lie algebra. The range of the grading is \(-1, 0, 1, 2\); the space of grade \(-1\), if nonempty, is a subalgebra with vanishing brackets.

Although we have introduced \( V^-_M \) as a complex Lie algebra, its real form obtained by imposing

\[
F^{I^\ast} = F^{\overline{I}}
\] (23)

will be of particular interest. To save on notation we will call this real Lie algebra also \( V^+_M \). Here,

\[
\overline{I} = i_m i_{m-1} \cdots i_1
\] (24)

is the loop with the opposite orientation.
The algebra $V_M^-$ is the finite analogue of the commutation relations of the Wilson loop in super-symmetric Yang-Mills theory. In the case of supersymmetric QCD in two dimensions, for example, the bosonic components of the gauge field can be removed by gauge fixing, and the analogues of the Wilson loop variables involve only the fermionic fields in the adjoint representation. Thus it is of equal interest to study $V_M^-$; we can develop the two cases in parallel; but mostly we will speak of $V_M^+$. 

**Automorphism groups $G_M^\pm$**

It is now possible to understand the Lie groups of which $V_M^\pm$ are Lie algebras. This will solve in a finite context the problem of exponentiating the Lie algebra of Wilson loops.

First of all, let us consider a group of which $V_M$ is the Lie algebra. Consider the vector space of tensors of type $(1, m)$ for $m = 0, 1, \ldots$. In terms of the variables $\xi^i$, a typical element would be

$$\phi^i(\xi) = \sum_{m=1}^{\infty} \phi^i_{1, \ldots, i_m} \xi^1 \cdots \xi^m.$$  

(25)

Define the composition law of such functions of $\xi$ in the obvious way:

$$(\phi \circ \psi)^i(\xi) = \sum_{m=0}^{\infty} \phi^i_{1, \ldots, i_m} \phi(\xi)^1 \cdots \phi(\xi)^m.$$  

(26)

This operation is clearly associative and has identity.

If we now restrict to the subset of functions $\phi$ such that the first tensor in the series above, is invertible,

$$G_M = \{ \phi | \det \phi^i_j \neq 0 \}$$  

(27)

we have a group under the above composition law. To see this, we note that given any such $\phi$, unique inverse $\psi$ can be constructed solving the equation

$$\psi^i(\phi)(\xi) = \xi^i$$  

(28)

recursively:

$$\psi^i_j \phi^j_k = \delta^i_k,$$

$$\psi^i_j \phi^j_{k, \xi} + \psi^i_{j, \xi} \phi^j_k \phi^j_{l, \xi} = 0,$$

etc. The term of order $m$ will determine $\psi^i_{1, \ldots, i_m}$ in terms of lower order components of $\psi$ thus establishing the existence and uniqueness of an inverse. In general $\psi$ will have an
infinite number of non–zero terms even when \( \phi \) is a polynomial. \(^1\) This is an algebraic analogue of the inverse function theorem: \( \phi'_j \) is the analogue of the derivative at the origin of the function \( \phi^i(\xi) \), so that if it is invertible, we should expect \( \phi \) to be invertible at least locally.

Thus \( G_M \) is a group under the above composition law; by infinitesimalizing the composition law we see that this group has as Lie algebra \( \mathcal{V}_M \). We see that \( G_M \) is a non-commutative analogue of the diffeomorphism group of \( R^M \).

Now it is clear that groups of which the Lie algebras are \( \mathcal{V}_M^\pm \) may be defined as below:

\[
\begin{align*}
G_M^- &= \{ \phi | \det \phi'_j \neq 0; g_{ij} \phi^i(\xi) \phi^j(\xi) = g_{ij} \xi^i \xi^j \} \\
G_M^+ &= \{ \phi | \det \phi'_j \neq 0; \omega_{ij} \phi^i(\xi) \phi^j(\xi) = \omega_{ij} \xi^i \xi^j \}
\end{align*}
\]

which are just the conditions for the distance functions to be invariant.

*Symplectic Realizations*

It would obviously be interesting to look at representations of the above loop algebras. This should be interesting for example in the quantum Yang–Mills theory. However, it is quite possible that the relevant algebras are different in the quantum theory: quantization could deform the algebra itself. Therefore we study first the classical analogue of a representation, a realization of the Lie algebra \( \mathcal{V}_M^+ \) in terms of Poisson brackets of some functions on symplectic space.

Let \( \eta^a_b \) be a set of complex variables satisfying the hermiticity condition

\[
\eta^a_b = \eta^{\ast b}_a. \tag{29}
\]

Here, \( i = 1, \cdots M \) and \( a, b = 1, \cdots N_c \) for some positive integer \( N_c \). We will consider only the case of even \( M \). (The indices \( a, b \) will be called color indices, since we will soon see an analogy to Yang–Mills theory.) Now impose the Poisson brackets

\[
\{ \eta^a_b, \eta^c_d \} = \omega^{ij} \delta^a_d \delta^c_b \tag{30}
\]

\(^1\) We must enlarge our space of allowed transformations to include infinite series, in order to be able to define an inverse. We dont address the issue of convergence of these series, although it should be possible to define an appropriate topology on the space of such series with respect to which \( G_M \) is a Lie group. The Lie algebra of \( G_M \) will in fact be the completion of our polynomial derivations \( \mathcal{V}_M \) in such a topology.
Thus we are just considering the real vector space $\mathbb{R}^{MN_2}$ with a symplectic form that is invariant under the adjoint action of $U(N_c)$. Now consider the space of polynomials invariant under the adjoint action of $U(N_c)$. A basis for this space is labelled by a discrete loop $I : Z_m \to \{1, 2, \cdots M\}$:

$$f^I(\eta) = \text{tr} \, \eta^{i_1} \eta^{i_2} \cdots \eta^{i_m}$$  \hspace{1cm} (31)

The cyclic symmetry of the trace assures us that $f^I$ is independent of the starting point of the loop $I$. Moreover,

$$f^{I*} = f^I$$  \hspace{1cm} (32)

Now it is a simple matter to verify that the Poisson brackets of these functions provide a realization of the Lie algebra $\mathcal{V}_M^+$:

$$\{f^I, f^J\} = \sum_{kl} \omega^{i_{1k}i_{2j}} f^{i_{k1}i_{j1}}.$$  \hspace{1cm} (33)

The analogy of this realization with the Poisson brackets of the Wilson loops is obvious.

At the level of the group, we also have an action of the group on invariant polynomials of the variables $\eta^i$ by a sort of ‘pull–back’:

$$\phi^*(h)(\eta) = h(\psi(\eta))$$  \hspace{1cm} (34)

where $\psi$ is the inverse of $\phi$ and

$$[\psi(\eta)^i]^j = \sum_{m=1}^{M} \psi^i_{i_1 \cdots i_m} \eta^{i_1} \cdots \eta^{i_m}$$  \hspace{1cm} (35)

matrix multiplication being implied on the right hand side. If we restrict to the sub–group $G^+_M$ the matrix valued function $\omega^{i_{1j}} \eta^{i_{1j}} \eta^{i_{1j}}$ is invariant under this action.

Clearly if the number of ‘colors’ $N_c$ is one, the realization described above has a large kernel. The variables $\eta^i$ then satisfy the relation

$$\eta^i \eta^j - \eta^j \eta^i = 0$$  \hspace{1cm} (36)

since they commute. The functions $f^I$ then satisfy the ‘Mandelstam identity’

$$f^{i_{k1}i_{j1}} - f^I f^J = 0$$  \hspace{1cm} (37)

relative to any way of multiplying the two loops $I$ and $J$ at points $k, l$. More generally, there will be an identity that says that the anti–symmetric part in $N_c + 1$ indices is zero;
these are the finite analogues of the Mandelstam identities. For simplicity, let us state

\[ f_{I_1 \circ I_2 \circ I_3} + f_{I_1 \circ I_2 \circ I_3} - f_{I_1 \circ I_2 \circ I_3} - f_{I_1 \circ I_2 \circ I_3} = 0. \quad (38) \]

Here, \( I_1 \) actually denotes the set \( i_1, i_2, \ldots, i_k \), \( I_2 \) denotes \( j_1, \ldots, j_k \), and \( I_3 \) refers to \( l_1, \ldots, l_k \). Circles are the products we introduced which corresponds to combining the corresponding sequences. Similarly, one can see that writing the all possible antisymmetric combinations and taking the trace, we get a relation satisfied by \( N_c + 1 \) generators of the representation. This gives us combinations of generators with all possible permutations multiplied with the appropriate sign of the permutation. If we take the cycle decomposition of a permutation \( \pi \) of \( N_c + 1 \) numbers, and denote each cycle as \( \pi_k \), we can write the result as

\[ \sum_{\pi} (-1)^\pi f^{I_{\pi_1}} f^{I_{\pi_2}} \cdots f^{I_{\pi_\pi}} = 0 \quad (39) \]

where, we used a short hand \( f^{I_{\pi_k}} \) to denote \( f^{I_{r_k-1} \circ \cdots \circ I_{r_k}} \). Here the length of the cycle \( \pi_k \) is given by \( r_k = r_{k-1} + 1 \) and circles again correspond to products.

These are precisely the analogues of the identities satisfied by the Wilson loop for finite \( N_c \) (See Ref. [2]). They simply describe the fact that \( f^I \) is the trace of an \( N_c \times N_c \) matrix. As \( N_c \to \infty \) these identities should disappear which must be a reason for the simplicity of the large \( N_c \) limit.

We remark that if we introduce Grassmann variables \( \psi_b^{ia} \) which anti-commute,

\[ \psi_b^{ia} \psi_d^{jc} + \psi_d^{jc} \psi_b^{ia} = 0 \quad (40) \]

and satisfy the super-Poisson bracket

\[ \{ \psi_b^{ia}, \psi_d^{jc} \} = g^{ij} \delta_d^a \delta_b^c \quad (41) \]

we also have a super-symplectic realization of \( V_M^- \):

\[ G^I \mapsto \text{tr}\psi^{i_1} \cdots \psi^{i_m}. \quad (42) \]

Clearly these \( G^I \) are cyclically anti-symmetric and a short computation will show that their super-Poisson brackets form a realization of \( V_M^+ \). By replacing the above Grassmann algebra by a Clifford algebra

\[ \psi_b^{ia} \star \psi_d^{jc} + \psi_d^{jc} \star \psi_b^{ia} = \hbar g^{ij} \delta_d^a \delta_b^c \quad (43) \]
we also have a quantum deformation for $V^+_M$, analogous to the one in the next section.

Quantum deformation

It is interesting to see what happens to the above realization upon quantization. One approach to quantization is the deformation of the commutative product of the functions of $\eta^a_b$ by the so-called Moyal product:

$$f \ast g(\eta) = \left[ e^{-i \frac{1}{\hbar} \sum_{i,j} \tilde{\omega}_{ij} \tilde{n}^a_i \tilde{n}^b_j} f(\eta) g(\eta^I) \right]_{\eta=\eta^I}. \tag{44}$$

(This particular definition of the product corresponds to Weyl ordering.) If we apply this multiplication rule to the $U(N_c)$--invariant polynomials $f^I$, we will get a noncommutative associative algebra. The commutator of this multiplication defines a Lie algebra, which is a quantum deformation of our loop algebra $V^{-}_M$. To first order in $\hbar$ this commutator is just the Poisson bracket, so that in this limit we recover the previous algebra as a contraction of the quantum algebra. But the general answer is quite formidable, at each order $r$ in $\hbar$, there will be terms involving up to $r$ products of loops.

On the other hand, it is to be expected that some simplifications will occur in the limit as $N_c \to \infty$. The point is that the leading contribution will come from terms where there are the largest number of possible independent traces, so that we must keep the terms with the largest number of loops. All the other terms are subleading order. Nevertheless, it turns out that there is such a term of leading order in $\frac{1}{N_c}$ at each order in $\hbar$; the limit $N_c \to \infty$ is quite different from the limit $\hbar \to 0$. But this is also a ‘classical’ limit in that the commutators of color invariant observables is of order $\frac{1}{N_c}$, so that they become simultaneously measurable in the limit $N_c \to \infty$. It is of utmost importance to understand the large $N_c$ limit of gauge theories; our discussion identifies the canonical structure (Poisson brackets of loop variables) of color singlet observables in the large $N_c$ limit.

Let us now calculate the deformed brackets more explicitly. First of all note that

$$\frac{\partial f^I}{\partial \eta^a}_b = 0 \tag{45}$$

unless $k$ is equal to one of the elements of the loop $\{i_1,i_2,\cdots i_m\}$. In the case $k = i_\mu$ for some $\mu = 1,2,\cdots m$,

$$\frac{\partial f^I}{\partial \eta^a}_b = \left[ \eta^{i_{\mu+1}} \eta^{i_{\mu+2}} \cdots \eta^{i_m} \eta^{i_{\mu+1}} \cdots \eta^{i_{\mu-1}} \right]_a^b. \tag{46}$$
Thus differentiation with respect to cuts the loop at the point with parameter value $\mu$.

More generally,

$$\frac{\partial^r f^I}{\partial \eta_{b_1}^{k_1 a_1} \cdots \partial \eta_{b_r}^{k_r a_r}} = 0$$

(47)

unless the set \{k_1, k_2 \cdots k_r\} is a subset of the set \{i_1, i_2 \cdots i_m\}. Suppose \{\mu_1, \mu_2 \cdots \mu_r\} \subset \{1, 2, \cdots m\} and moreover that $\mu_1 < \mu_2 \cdots < \mu_r$. Then we can see that

$$\frac{\partial^r f^I}{\partial \eta_{b_1}^{i_{\nu_1} a_1} \cdots \partial \eta_{b_r}^{i_{\nu_r} a_r}} = \left[ \eta^{i_{\nu_1}+1} \eta^{i_{\nu_1}+2} \cdots \eta^{i_{\nu_r}-1} \right]_{a_1}^{b_1}$$

$$\left[ \eta^{i_{\mu_2}+1} \eta^{i_{\mu_2}+2} \cdots \eta^{i_{\mu_r}-1} \right]_{a_2}^{b_2}$$

$$\cdots \left[ \eta^{i_{\mu_r}+1} \eta^{i_{\mu_r}+2} \cdots \eta^{i_{\mu_1}-1} \right]_{a_r}^{b_r}$$

which corresponds to cutting the loop at points $\mu_1, \mu_2, \cdots \mu_r$. It is clearly convenient to introduce the matrix, for $\mu_1 < \mu_2 \in \{1, 2, \cdots m\}$

$$P_a^b(I(\mu_1, \mu_2)) = \left[ \eta^{i_{\mu_1}+1} \eta^{i_{\mu_1}+2} \cdots \eta^{i_{\mu_2}-1} \right]_{a}^{b}$$

(48)

which represents the parallel transport operator for the piece of the loop $I$ from $\mu_1$ to $\mu_2$.

Then, for $\mu_1 < \mu_2 \cdots < \mu_r$,

$$\frac{\partial^r f^I}{\partial \eta_{b_1}^{i_{\nu_1} a_1} \cdots \partial \eta_{b_r}^{i_{\nu_r} a_r}} = P_a^{b_1}(I(\mu_1, \mu_2))P_a^{b_2}(I(\mu_2, \mu_3)) \cdots P_a^{b_r}(I(\mu_r, \mu_1)).$$

Now let us consider the general term in the definition of the deformed product of color invariant functions of the $\eta$:

$$f^I \ast f^J = f^I f^J + \sum_{r=1}^{\infty} \frac{1}{r!} \left( -\frac{i \hat{h}}{2} \right)^r \omega^{i_{\nu_1} j_{\nu_1}} \cdots \omega^{i_{\nu_r} j_{\nu_r}}$$

$$\frac{\partial^r f^I}{\partial \eta_{b_1}^{i_{\nu_1} a_1} \cdots \partial \eta_{b_r}^{i_{\nu_r} a_r}} \frac{\partial^r f^J}{\partial \eta_{a_1}^{j_{\nu_1} b_1} \cdots \partial \eta_{a_r}^{j_{\nu_r} b_r}}.$$

We can, using the symmetry of the derivatives, and relabelling of indices, always bring the indices in the first derivative factor to the order $\mu_1 < \mu_2 \cdots < \mu_r$. However, once this is done, there is no reason that the indices $\nu_1, \nu_2 \cdots \nu_r$ are in any particular order. This is
because the contraction of the color indices 

\[ \mu_k \] to \( \nu_k \). Thus, the general term in the 
series will involve quite complicated ways of contracting the color indices.

In the large \( N_c \) limit, however, the leading term will have the largest number of traces. This will happen when the \( \nu \) indices are in decreasing order: \( \nu_1 > \nu_2 \cdots > \nu_r \). This is the term that involves a product of \( r \) Wilson loops, so that the algebra reduces in the large \( N_c \) limit to

\[
 f^I \ast f^J = f^I f^J + \sum_{r=1}^{\infty} \sum_{\mu_1 < \mu_2 \cdots < \mu_r} \left( -\frac{i \hbar}{2} \right)^r \omega^{i_{\mu_1},j_{\nu_1}} \cdots \omega^{i_{\mu_r},j_{\nu_r}} \\
 f^{I(\mu_1,\mu_2)} \circ (\nu_2,\nu_1) f^{R(\mu_2,\mu_3)} \circ (\nu_3,\nu_2) \\
 \cdots \cdots f^{R(\mu_r,\mu_1)} \circ (\nu_1,\nu_r).
\]

Here \( I(\mu_1,\mu_2)J(\nu_2,\nu_1) \) for example is the loop \( i_{\mu_1+1} i_{\mu_1+2} \cdots i_{\mu_2-1} j_{\nu_2+1} \cdots j_{\nu_1-1} \). This is still somewhat complicated, but not as bad as it is in the general case. There would be additional terms involving fewer than \( r \) Wilson loops on the right hand side. Of course, the series on the right hand side has only a finite number of terms, since \( f^I \) are polynomials. We would like to have an alternative interpretation of this algebra.

In any case we can define a vector space with basis \( f^I \), and the above multiplication law turns it into an associative algebra with identity. The invertible elements of this algebra will form a group, whose Lie algebra will be given by the commutator. This Lie algebra and group will generalize \( G^+_M \) and \( \mathcal{V}^+_M \) to the quantum theory in the large \( N_c \) limit.

This algebra also should have an interpretation in terms of non-commutative geometry. In fact it is already clear that it is a generalization of the algebra of pseudo-differential operators to non-commutative geometry. We hope to return to this issue in a future publication.

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