PROOF OF DILKS’ BIJECTIVITY CONJECTURE ON BAXTER PERMUTATIONS

ZHICONG LIN AND JING LIU

Abstract. Baxter permutations originally arose in studying common fixed points of two commuting continuous functions. In 2015, Dilks proposed a conjectured bijection between Baxter permutations and non-intersecting triples of lattice paths in terms of inverse descent bottoms, descent positions and inverse descent tops. We prove this bijectivity conjecture by investigating its connection with the Françon–Viennot bijection. As a result, we obtain a permutation interpretation of the \((t, q)\)-analog of the Baxter numbers

\[
\frac{1}{\binom{n+1}{1} q^{n+1}} \sum_{k=0}^{n-1} q^{3\binom{k+1}{2}} \binom{n+1}{k} \binom{n+1}{k+1} q^{k+2} \]

where \(\binom{n}{k}_q\) denote the \(q\)-binomial coefficients.

1. Introduction

Baxter permutations originated from G. Baxter’s study \cite{Baxter} of fixed points for the composite of commuting functions. Let \(\mathcal{S}_n\) be the set of all permutations of \([n] := \{1, 2, \ldots, n\}\). A permutation \(\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n\) is a Baxter permutation if it avoids the vincular patterns \(2413\) and \(3142\), i.e., there is no indices \(1 \leq i < j < j + 1 < k \leq n\) such that

\[
\pi_{j+1} < \pi_i \leq \pi_k < \pi_j \quad \text{or} \quad \pi_j < \pi_k < \pi_i < \pi_{j+1}.
\]

Denote by \(\text{Bax}_n\) the set of all Baxter permutations in \(\mathcal{S}_n\).

By inventing a generating tree for Baxter permutations, algebraically manipulating the recurrence relation, and then magically guessing the correct enumeration formula, Chung, Graham, Hoggatt and Kleiman \cite{Chung} proved that

\[
|\text{Bax}_n| = \frac{1}{\binom{n+1}{1} \binom{n+1}{2}} \sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+2}.
\]

A bijective proof was constructed by Viennot \cite{Viennot} and a functional equation proof was provided by Bousquet-Mélou \cite{Bousquet}. The number in the right-hand side of (1.1) is denoted by \(B_n\), which is known as the \(n\)-th Baxter number. Numerous other combinatorial objects have been found to be counted by the Baxter numbers in the literature, some of which are in bijection with Baxter permutations (see \cite{Haglund, Janson, Racine, Schaeffer} and the references therein). The main

Date: December 23, 2021.

Key words and phrases. Baxter permutations; Françon–Viennot bijection; Descent bottoms; Inverse descents; Descent tops.
objective of this paper is to prove a bijectivity conjecture of Dilks \cite{5} relating three descent-based statistics on Baxter permutations to horizontal steps of non-intersecting triples of lattice paths.

**Definition 1.1** (Descent-based statistics on permutations). For any permutation \( \pi = \pi_1 \pi_2 \ldots \pi_n \in S_n \), define the following three fundamental statistics:

- \( \text{DES}(\pi) = \{ i \in [n - 1] : \pi_i > \pi_{i+1} \} \), the set of all positions of the descents of \( \pi \);
- \( \text{DT}(\pi) = \{ \pi_i : \pi_i > \pi_{i+1} \} \subseteq [2,n] \), the set of all descent tops of \( \pi \);
- \( \text{DB}(\pi) = \{ \pi_i+1 : \pi_i > \pi_{i+1} \} \), the set of all descent bottoms of \( \pi \).

For the sake of convenience, we set \( \text{IDES}(\pi) = \text{DES}(\pi-1), \text{IDT}(\pi) = \text{DT}(\pi-1) \) and \( \text{IDB}(\pi) = \text{DB}(\pi-1) \). We introduce the set of modified descent tops of \( \pi \) to be

\[ \text{\overline{DT}}(\pi) = \{ \pi_i-1 : \pi_i > \pi_{i+1} \} \subseteq [n-1] \]

and set \( \text{\overline{IDT}}(\pi) = \text{\overline{DT}}(\pi-1) \).

For a subset \( S \subseteq [n] \), a lattice path of length \( n \) (i.e., has \( n \) steps) confined to the quarter plane \( \mathbb{N}^2 \) using only vertical and horizontal steps is said to encode \( S \) if for any \( 1 \leq i \leq n \),

\[ i \in S \iff \text{the } i\text{-th step of the lattice path is horizontal} \]

Given a Baxter permutation \( \pi \in \text{Bax}_n \), define \( \Gamma(\pi) \) to be the triple of lattice paths, each consisting of \( n - 1 \) steps, where

- the bottom one starts at \((2,0)\) and encodes \( \text{IDB}(\pi) \);
- the middle one starts at \((1,1)\) and encodes \( \text{DES}(\pi) \);
- and the top one starts at \((0,2)\) and encodes \( \text{\overline{IDT}}(\pi) \).

**Example 1.2.** Let \( \pi = 235419786 \). Then \( \text{IDB}(\pi) = \{1,3,6,7\} \), \( \text{DES}(\pi) = \{3,4,6,8\} \) and \( \text{\overline{IDT}}(\pi) = \{3,4,7,8\} \). The triple \( \Gamma(\pi) \) of lattice paths are drawn in Fig. 1.

For \( 0 \leq k \leq n - 1 \), let \( \text{Bax}_{n,k} := \{ \pi \in \text{Bax}_n : \text{des}(\pi) = k \} \), where \( \text{des}(\pi) := |\text{DES}(\pi)| \). Denote by \( \text{Tlp}_{n,k} \) the set of all non-intersecting triples of lattice paths, each of length \( n - 1 \) using only vertical or horizontal step, from \((0,2), (1,1), (2,0) \) to \((k,n-k+1), (k+1,n-k), (k+2,n-k-1) \). In his Ph.D. thesis, Dilks \cite[Conjecture 3.5]{5} proposed the following bijectivity conjecture.

**Conjecture 1.3** (Dilks’s bijectivity conjecture). For \( 0 \leq k \leq n - 1 \), the correspondence \( \Gamma : \text{Bax}_{n,k} \to \text{Tlp}_{n,k} \) is a bijection.
We will prove Dilks’ bijectivity conjecture by investigating its connection with Viennot’s original bijection [14] between $\text{Bax}_n,k$ and $\text{Tlp}_{n,k}$ constructed by using the Françon–Viennot bijection. Dilks’ bijectivity conjecture has three interesting consequences.

For any permutation $\pi \in S_n$, define three different major indices associated with $\text{DES}(\pi)$, $\text{IDB}(\pi)$ and $\tilde{\text{IDT}}(\pi)$ as

\[
\text{maj}(\pi) = \sum_{i \in \text{DES}(\pi)} i, \quad \text{imaj}_B(\pi) = \sum_{i \in \text{IDB}(\pi)} i \quad \text{and} \quad \text{imaj}_T(\pi) = \sum_{i \in \tilde{\text{IDT}}(\pi)} i.
\]

It has already been observed by Dilks [5, Conjecture 3.4] that the following permutation interpretation of the $(t,q)$-analog of the Baxter numbers $B_n$ is a direct consequence of

(1) a natural bijection [5, Theorem 2.4] between $\text{Tlp}_{n,k}$ and plane partitions in a $k \times (n - 1 - k) \times 3$ box;

(2) and a $q$-counting formula for plane partitions in [13, Theorem 7.21.7].

**Corollary 1.4.** For any $n \geq 1$, we have

\[
\sum_{\pi \in \text{Bax}_n} t^{\text{DES}(\pi)} q^{\text{imaj}_B(\pi) + \text{maj}(\pi) + \text{imaj}_T(\pi)} = \frac{1}{\binom{n+1}{1} q \binom{n+1}{2} q} \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \left[ \begin{array}{c} n+1 \\ k \end{array} \right] \left[ \begin{array}{c} n+1 \\ k+1 \end{array} \right] \left[ \begin{array}{c} n+1 \\ k+2 \end{array} \right] t^k.
\]

A permutation $\pi \in S_n$ is alternating if

\[
\pi_1 < \pi_2 > \pi_3 < \pi_4 > \pi_5 < \cdots
\]

and is reverse alternating if all the above inequalities are reversed. In other words, if we denote $[n]_o$ (resp. $[n]_e$) the set of all odd (resp. even) integers in $[n]$, then $\pi$ is alternating if $\text{DES}(\pi) = [n - 1]_e$, while $\pi$ is reverse alternating if $\text{DES}(\pi) = [n - 1]_o$. Let

\[
C_n := \frac{1}{n + 1} \binom{2n}{n}
\]

be the $n$-th Catalan number (see [13, pp. 219-229] for many interpretations of Catalan numbers). The following result, due to Cori, Dulucq and Viennot [4], about enumeration of alternating Baxter permutations is a direct consequence of Conjecture 1.3.

**Corollary 1.5** (Cori, Dulucq and Viennot). The number of (reverse) alternating Baxter permutations of length $n$ is

(1.2) \[ C_{\lfloor n/2 \rfloor} C_{\lfloor (n+1)/2 \rfloor}. \]

The integer sequence in (1.2) appears as A005817 in the OEIS [12], where several other intriguing combinatorial interpretations are known. In particular, it has been proved in [11, Theorem 4.8] recently that this sequence also enumerates 231-avoiding ballot permutations. It would be interesting to see whether there is any bijection between these two classes of pattern avoiding permutations.

A permutation $\pi \in S_n$ is Genocchi if

\[
\pi_i > \pi_{i+1} \iff \pi_i \text{ is even}.
\]

In other words, $\pi$ is a Genocchi permutation if $\text{DT}(\pi) = [n]_o$. Genocchi permutations were introduced by Dumont [6] to interpret the Genocchi numbers. The third consequence of Conjecture 1.3 is the following new interpretation of Catalan numbers.
Corollary 1.6. The number of permutations \( \pi \in \text{Bax}_n \) such that \( \pi \) is reverse alternating and \( \pi^{-1} \) is Genocchi equals the Catalan number \( C_{\lfloor n/2 \rfloor} \).

Example 1.7. The five permutations in \( \text{Bax}_6 \) such that itself is reverse alternating and its inverse is Genocchi are

\[
214365, 215463, 324165, 325461, 435261.
\]

The above interpretation of Catalan numbers is analog to a result due to Guibert and Linusson [9], which asserts that permutations \( \pi \in \text{Bax}_n \) such that both \( \pi \) and \( \pi^{-1} \) are (reverse) alternating are counted by \( C_{\lfloor n/2 \rfloor} \). A combinatorial bijection between these two models seems not easy.

The rest of this paper is devoted to a proof of Dilks' bijectivity conjecture.

2. Proof of Dilks' bijectivity conjecture

Our starting point of the proof of Dilks' bijectivity conjecture is the following crucial observation.

Lemma 2.1. If \( \pi \in \text{Bax}_n \), then \( \pi^{-1} \in \text{Bax}_n \).

Proof. Assume that \( \pi^{-1} \) is not a Baxter permutation, then \( \pi^{-1} \) contains \( 3142 \) pattern or \( 2413 \) pattern. By the complement symmetry of these two patterns, we can assume that \( \pi^{-1} \) contains the pattern \( 2413 \), i.e., there exists indices \( 1 \leq i < j < j+1 < k \leq n \) such that \( \pi_{j+1} > \pi_i > \pi_k > \pi_j \). We aim to show that \( \pi \) contains the pattern \( 3142 \) by induction on \( l = \pi_k - \pi_i \), which will finish the proof of the lemma.

If \( l = 1 \), i.e., \( \pi_i^{-1} = \pi_k^{-1} - 1 \), then the subsequence \( (j+1) i k j \) in \( \pi \) forms an instance of \( 3142 \) pattern. If \( l > 1 \), then we need to consider two cases. If \( \pi_k^{-1} - 1 \) locates before \( \pi_j^{-1} \) in \( \pi^{-1} \), then the subsequence \( (j+1) m k j \) in \( \pi \), where the \( m \)-th (\( m < j \)) letter of \( \pi^{-1} \) is \( \pi_k^{-1} - 1 \), forms an \( 3142 \) pattern and we are done. Otherwise, \( \pi_k^{-1} - 1 \) locates after \( \pi_j^{-1} \) in \( \pi^{-1} \), then \( \pi_i^{-1} \pi_j^{-1} \pi_{j+1}^{-1} (\pi_k^{-1} - 1) \) is still a \( 2413 \) pattern in \( \pi^{-1} \), but with \( (\pi_k^{-1} - 1) - \pi_i^{-1} = l - 1 < l \). This proves that \( \pi \) contains the pattern \( 3142 \) by induction.

In view of Lemma 2.1, Conjecture 1.3 is equivalent to the assertion that the correspondence \( \pi \mapsto \Gamma'(\pi) \) is a bijection between \( \text{Bax}_{n,k} \) and \( \text{Tlp}_{n,k} \), where \( \Gamma'(\pi) := \Gamma(\pi^{-1}) \) is the triple of lattice paths defined by

- the bottom one starts at \( (2,0) \) and encodes \( \text{DB}(\pi) \);
- the middle one starts at \( (1,1) \) and encodes \( \text{IDES}(\pi) \);
- and the top one starts at \( (0,2) \) and encodes \( \text{DT}(\pi) \).

Due to the cardinality reason, it remains to show that the correspondence \( \Gamma' \) is a well-defined injection.

2.1. The correspondence \( \Gamma' \) is well defined. We will use a known recursive construction of Baxter permutations. For any \( \pi \in \mathfrak{S}_n \), a letter \( \pi_i \) is a left-to-right maxima (resp. right-to-left maxima) of \( \pi \) if \( \pi_j < \pi_i \) for all \( j < i \) (resp. \( j > i \)).

Lemma 2.2 (See [3, 2]). For any \( \pi \in \text{Bax}_n \), \( \pi \) is obtained from some \( \sigma \in \text{Bax}_{n-1} \) by inserting \( n \) into one of the following two kinds of positions:
the following result shows that $\Gamma'$ is well defined.

**Lemma 2.3.** For any $\pi \in \text{Bax}_n$, the triple of lattice paths in $\Gamma' (\pi)$ are non-intersecting.

**Proof.** By Lemma 2.2, $\pi$ is obtained from some $\sigma \in \text{Bax}_{n-1}$ by inserting $n$ in one of the following three cases:

- The letter $n$ is inserted at the end of $\sigma$. Then
  \[(DB(\pi), IDES(\pi), \tilde{DT}(\pi)) = (DB(\sigma), IDES(\sigma), \tilde{DT}(\sigma)).\]

- The letter $n$ is inserted after a right-to-left maxima $\sigma_i$, $i \neq n-1$, of $\sigma$. Then
  \[DB(\pi), IDES(\pi)) = (DB(\sigma), IDES(\sigma)) \quad \text{and} \quad \tilde{DT}(\pi) = (\tilde{DT}(\sigma) \setminus \{\sigma_i - 1\}) \cup \{n-1\}.\]

- The letter $n$ is inserted before a left-to-right maxima $\sigma_j$ of $\sigma$. Then
  \[DB(\pi) = DB(\sigma) \cup \{\sigma_j\}, \quad IDES(\pi) = IDES(\sigma) \cup \{n-1\}, \quad \tilde{DT}(\pi) = \tilde{DT}(\sigma) \cup \{n-1\}.\]

It then follows by induction on $n$ that in either case, the triple of lattice paths in $\Gamma'(\pi)$ encoding $DB(\pi)$, $IDES(\pi)$ and $\tilde{DT}(\pi))$ are non-intersecting. In fact,

- in the first case $\Gamma'(\pi)$ is obtained from $\Gamma'(\sigma)$ by adding a vertical step to each path;
- in the second case $\Gamma'(\pi)$ is obtained from $\Gamma'(\sigma)$ by adding a vertical step to the middle path and the bottom path, but changing the $(\sigma_i - 1)$-th step of the top path from horizontal to vertical and then adding a horizontal step at the end;
- in the third case $\Gamma'(\pi)$ is obtained from $\Gamma'(\sigma)$ by adding a horizontal step to the middle path and the top path, but adding a horizontal step just after the $(\sigma_j - 1)$-th step of the bottom path.

In either case, the resulting triple of lattice paths in $\Gamma'(\pi)$ are non-intersecting, which completes the proof of the lemma. \qed

2.2. Viennot’s original bijection between $\text{Bax}_{n,k}$ and $\text{Tlp}_{n,k}$. Viennot’s original bijection $\Psi$ introduced in [14] between $\text{Bax}_{n,k}$ and $\text{Tlp}_{n,k}$ consists of two main steps, the first of which is the classical Françon–Viennot bijection [8] between permutations and Laguerre histories. The main purpose here is to prove that $\Psi$ admits a direct description using the three descent-based statistics in Definition 1.1, which is key to our proof of Dilks’ bijectivity conjecture.

To begin with, let us recall the Françon–Viennot bijection. A Motzkin path of length $n$ is a lattice path confined to the quarter plane $\mathbb{N}^2$, starting from the origin, using three kinds of steps

- $U = (1, 1)$ (up step), $H = (1, 0)$ (horizontal step) and $D = (1, -1)$ (down step),
and ending at \((n, 0)\). A Motzkin path whose each horizontal step receives either blue (denoted a \(H_b\) step) or red (denoted a \(H_r\) step) is called a \(2\)-\-colored Motzkin path. Such a path is encoded as a word of length \(n\) over \(\{U, H_b, H_r, D\}\). A Laguerre history of length \(n\) is a pair \((w, \mu)\), where \(w = w_1 \cdots w_n\) is a \(2\)-\-colored Motzkin path and \(\mu = (\mu_1, \mu_2, \cdots, \mu_n)\) is a weight function satisfying \(1 \leq \mu_i \leq h_i(w)\), where

\[
h_i(w) := 1 + |\{j \mid j < i, w_j = U\}| - |\{j \mid j < i, w_j = D\}|
\]

is one plus the \textit{height} of the starting point of the \(i\)-th step of \(w\). Denote by \(\mathfrak{L}_n\) the set of all Laguerre histories of length \(n\).

For a permutation \(\pi \in \mathfrak{S}_n\), a letter \(\pi_i\) is called a \textit{valley} (resp. \textit{peak, double descent, double ascent}) of \(\pi\) if \(\pi_{i-1} > \pi_i < \pi_{i+1}\) (resp. \(\pi_{i-1} < \pi_i > \pi_{i+1}, \pi_{i-1} > \pi_i > \pi_{i+1}, \pi_{i-1} < \pi_i < \pi_{i+1}\)), where \(\pi_0 = \pi_{n+1} = 0\) by convention. The Françon–Viennot bijection \(\psi_{FV} : \mathfrak{S}_n \rightarrow \mathfrak{L}_{n-1}\) can be defined as \(\psi_{FV}(\pi) = (w, \mu)\), where for each \(i \in [n-1]\):

\[
w_i = \begin{cases} 
U & \text{if } w_i = U, \\
D & \text{if } w_i = H_r, \\
H_b & \text{if } w_i = H_b, \\
H_r & \text{if } w_i = D,
\end{cases}
\]

and \(\mu_i\) is the number of \(312\)-patterns with \(i\) representing the 2, i.e.,

\[
\mu_i = (312)_i(\pi) := 1 + \#\{j \mid j < k \text{ and } \pi_j < \pi_k = i < \pi_{j-1}\}.
\]

See Fig. 2 for an example of the bijection \(\psi_{FV}\) for \(\pi = 512439786\).

The inverse algorithm \(\psi_{FV}^{-1}\) building a permutation \(\pi\) (in \(n\) steps) from a Laguerre history \((w, \mu) \in \mathfrak{L}_{n-1}\) may be described iteratively as:

- Initialization: \(\pi = \diamond\);
- At the \(i\)-th (\(1 \leq i \leq n - 1\)) step of the algorithm, replace the \(\mu_i\)-th \(\diamond\) (from left to right) of \(\pi\) by

\[
\begin{cases} 
\diamond \diamond & \text{if } w_i = U, \\
\diamond & \text{if } w_i = H_r, \\
i & \text{if } w_i = D, \\
\diamond i & \text{if } w_i = H_b;
\end{cases}
\]

- The final permutation is obtained by replacing the last remaining \(\diamond\) by \(n\).
For example, if \((w, \mu) = (UH_rUDDDH_bUD, (1, 2, 2, 2, 1, 1, 1, 2)) \in \mathcal{L}_8\) is the Laguerre history in Fig. 2, then \(\pi = \psi_{FV}^{-1}(w, \mu)\) is built as follows:

\[
\pi = \circ \to \circ\circ \to \circ 1\circ \to \circ 12\circ \to \circ 12 \circ 3\circ \to \circ 12 43\circ \to 51243\circ \to 51243 \circ 6 \\
\to 51243 \circ 7 \circ 6 \to 51243 \circ 786 \to 512439786.
\]

A special Laguerre history \((w, \mu) \in \mathcal{L}_n\) satisfying the following two conditions is called a Baxter history:

- whenever \(w_i = U\) or \(w_i = H_b\), we have \(\mu_{i+1} = \begin{cases} 
\mu_i + 1; \\
\mu_i - 1.
\end{cases}\)

- whenever \(w_i = D\) or \(w_i = H_r\), we have \(\mu_{i+1} = \begin{cases} 
\mu_i + 1; \\
\mu_i - 1.
\end{cases}\)

Denote by \(\mathcal{B}_n\) the set of all Baxter histories of length \(n\). Viennot [14] proved that the Françon–Viennot bijection \(\psi : \mathcal{B}_n \to \mathcal{T}_{\mathcal{L}_n, \mathcal{B}}\) restricted to a bijection between \(\text{Bax}_n\) and \(\mathcal{B}_{n-1}\), which forms the first step of \(\Psi\).

Let \(\mathcal{T}_{\mathcal{L}_n, \mathcal{B}} := \bigcup_{k=0}^{n-1} \mathcal{T}_{\mathcal{L}_{n-k}}\). The second step \(\phi : \mathcal{B}_{n-1} \to \mathcal{T}_{\mathcal{L}_n, \mathcal{B}}\) of \(\Psi\) is defined as follows.

Given a Baxter history \((w, \mu) \in \mathcal{B}_{n-1}\), define \(\phi(w, \mu) \in \mathcal{T}_{\mathcal{L}_n, \mathcal{B}}\) to be the triple of lattice paths such that

- the top one and the bottom one are determined by the 2-colored Motzkin path \(w\) by requiring that the \(i\)-th step of the top one (resp. bottom one) is

  (1) vertical (resp. horizontal) if \(w_i = U\),
  (2) horizontal (resp. vertical) if \(w_i = D\),
  (3) vertical (resp. vertical) if \(w_i = H_r\),
  (4) horizontal (resp. horizontal) if \(w_i = H_b\);

- the middle one is determined by the weight function \(\mu\) and the bottom one by requiring that the coordinate of the starting point of the \(i\)-th step of the middle one is

\[
(x_i - \mu_i, y_i + \mu_i) \quad \text{for } 1 \leq i \leq n - 1
\]

if \((x_i, y_i)\) is the coordinate of the starting point of the \(i\)-th step of the bottom one.

See Fig. 2 for an example of \(\phi\) with \((w, \mu) = (UH_rUDDDH_bUD, (1, 2, 2, 2, 1, 1, 1, 2)) \in \mathcal{B}_8\).

Viennot’s original bijection \(\Psi : \text{Bax}_n \to \mathcal{T}_{\mathcal{L}_n, \mathcal{B}}\) is then set to be \(\phi \circ \psi_{FV}\), the functional composition of \(\psi_{FV}\) and \(\phi\).

For any permutation \(\pi \in \mathcal{S}_n\), introduce the variation of \(DT(\pi)\) as

\[
\widehat{DT}(\pi) := (DT(\pi) \cup \{\pi_n\}) \setminus \{n\}.
\]

Then Viennot’s original bijection \(\Psi\) admits the following direct description.

**Lemma 2.4.** For a given \(\pi \in \text{Bax}_n\), write \(\Psi(\pi) = (P_b, P_m, P_t)\) where \(P_b\) is the bottom path, \(P_m\) is the middle path and \(P_t\) is the top path in \(\Psi(\pi)\). Then

1. the \(i\)-th step of \(P_b\) is horizontal iff \(i \in \text{DB}(\pi)\), i.e., \(P_b\) encodes \(\text{DB}(\pi)\);
2. the \(i\)-th step of \(P_m\) is horizontal iff \(i \in \text{IDES}(\pi)\), i.e., \(P_m\) encodes \(\text{IDES}(\pi)\);
3. the \(i\)-th step of \(P_t\) is horizontal iff \(i \in \widehat{DT}(\pi)\), i.e., \(P_t\) encodes \(\widehat{DT}(\pi)\).

**Proof.** We will prove the three points one by one in the following. Let \(\psi_{FV}(\pi) = (w, \mu)\).
The proof of the lemma is complete. □

(1) By the construction of \( \phi \), the \( i \)-th step of \( P_b \) is horizontal iff \( w_i = U \) or \( w_i = H_b \), which in turn is equivalent to \( i \) is a valley or a double descent of \( \pi \) according to the definition of \( \psi_{FV}(\pi) \). This proves the statement in (1).

(2) The distribution of the \( i \)-th step of the middle path \( P_m \) and the bottom path \( P_b \) have the following four possibilities (see Fig. 3) that will be treated separately:

- Assume that the \( i \)-th step of \( P_m \) and \( P_b \) is in case (a), then \( i \) is a valley or a double descent in \( \pi \) and \( \mu \) satisfies \( \mu_{i+1} = \mu_i \) (according to the definitions of the two steps of \( \Psi \)). If \( i \notin IDES(\pi) \), namely, \( i+1 \) is located after \( i \) in \( \pi \), then \( \pi \) can be written as \( \pi = \pi_1 \pi_2 \cdots a i \cdots (i+1) \cdots \pi_n \) for \( a \geq i+1 \). Thus, the subsequence \( ai(i+1) \) is a 312-pattern for \( \pi \) with \( i+1 \) representing 2, which forces \( \mu_{i+1} \geq \mu_i + 1 \). A contradiction with \( \mu_{i+1} = \mu_i \), which proves that \( i \in IDES(\pi) \)

- Assume that the \( i \)-th step of \( P_m \) and \( P_b \) is in case (b), then \( i \) is a peak or a double ascent in \( \pi \) and \( \mu \) satisfies \( \mu_{i+1} = \mu_i - 1 \). In this case, we need to show that \( i \in IDES(\pi) \). If not, then \( i+1 \) is located after \( i \) and so \( \pi = \pi_1 \pi_2 \cdots c i \cdots (i+1) \cdots \pi_n \) for \( c < i \), which forces \( \mu_{i+1} \geq \mu_i \). A contradiction with \( \mu_{i+1} = \mu_i - 1 \) and thus we have \( i \in IDES(\pi) \)

- Assume that the \( i \)-th step of \( P_m \) and \( P_b \) is in case (c), then \( i \) is a valley or a double descent in \( \pi \) and \( \mu \) satisfies \( \mu_{i+1} = \mu_i + 1 \). Now we need to show that \( i \notin IDES(\pi) \). If not, \( i+1 \) is located before \( i \) in \( \pi \) and so \( \pi \) can be written as \( \pi = \pi_1 \pi_2 \cdots (i+1) \cdots i \cdots \pi_n \), which forces \( \mu_{i+1} \leq \mu_i \). This contradicts \( \mu_{i+1} = \mu_i + 1 \) and so \( i \notin IDES(\pi) \).

- Assume that the \( i \)-th step of \( P_m \) and \( P_b \) is in case (d), then \( i \) is a peak or a double ascent in \( \pi \) and \( \mu \) satisfies \( \mu_{i+1} = \mu_i \). In this case, we need to show that \( i \notin IDES(\pi) \). If not, \( i+1 \) is located before \( i \) in \( \pi \) and so \( \pi \) can be written as \( \pi = \pi_1 \pi_2 \cdots (i+1) \cdots a i \cdots \pi_n \) with \( a < i \). Then there exists at least one 312-pattern in the interval \( (i+1) \cdots a i \) with \( i \) representing 2, which forces \( \mu_{i} \geq \mu_{i+1} + 1 \). This contradicts \( \mu_{i+1} = \mu_i \) and so \( i \notin IDES(\pi) \).

This proves the statement in (2) in all cases.

(3) By the construction of \( \phi \), the \( i \)-th step (\( 1 \leq i \leq n-1 \)) of \( P_i \) is horizontal iff \( w_i = D \) or \( w_i = H_b \), which in turn is equivalent to \( i \) is a peak or a double descent of \( \pi \) according to the definition of \( \psi_{FV}(\pi) \). Since \( i \) is a peak or a double descent of \( \pi \) (notice that \( \pi_{n+1} = 0 \) by convention) iff \( i \) is in \( DT(\pi) \setminus \{n\} \) or \( \pi_n = i \), the statement in (3) follows.

The proof of the lemma is complete.
2.3. **The correspondence \( \Gamma' \) is injective.** In view of Lemma 2.4, the middle and the bottom lattice paths in \( \Psi(\pi) \) are in coincidence with those in \( \Gamma'(\pi) \) for any \( \pi \in \text{Bax}_n \). But the top lattice paths in \( \Psi(\pi) \) and \( \Gamma'(\pi) \) are different in general (compare the examples in Fig. 1 and Fig. 2). However, since \( \Psi \) is a bijection, Lemma 2.4 together with the following lemma implies that \( \Gamma' \) is injective.

**Lemma 2.5.** There does not exist two Baxter permutations \( \pi, \pi' \in \text{Bax}_n \) such that
\[
\hat{\text{DT}}(\pi) = \hat{\text{DT}}(\pi'),
\]
\[
\text{IDES}(\pi) = \text{IDES}(\pi'),
\]
\[
\text{DB}(\pi) = \text{DB}(\pi'),
\]
but \( \hat{\text{DT}}(\pi) \neq \hat{\text{DT}}(\pi') \).

**Proof.** Note that \( \hat{\text{DT}}(\pi) = \hat{\text{DT}}(\pi') \) is equivalent to \( \text{DT}(\pi) = \text{DT}(\pi') \). Recall that \( \hat{\text{DT}}(\pi) = (\text{DT}(\pi) \cup \{\pi_n\}) \setminus \{n\} \). Assume to the contrary that such two Baxter permutations \( \pi \) and \( \pi' \) exist. Then \( \pi_n \neq n \) and \( \pi'_n \neq n \), for otherwise \( \text{DT}(\pi) = \text{DT}(\pi') \) would imply that \( \hat{\text{DT}}(\pi) = \hat{\text{DT}}(\pi') \). The only possibility is that
\[
(2.1) \quad \pi_n = i \neq \pi'_n = j \text{ for } 1 \leq i, j \leq n - 1 \text{ and } \hat{\text{DT}}(\pi) \setminus \{i\} = \hat{\text{DT}}(\pi') \setminus \{j\}.
\]
We aim to show that condition (2.1) could not happen when both \( \text{IDES}(\pi) = \text{IDES}(\pi') \) and \( \text{DB}(\pi) = \text{DB}(\pi') \) hold.

Without loss of generality, we can assume that (2.1) holds for \( i < j \). We write \( \Psi(\pi) = (P_b, P_m, P_t) \), where \( P_b \) is the bottom path, \( P_m \) is the middle path and \( P_t \) is the top path in \( \Psi(\pi) \). Similarly, we write \( \Psi(\pi') = (P'_b, P'_m, P'_t) \). Set \( \psi_{FV}(\pi) = (w, \mu) \) and \( \psi_{FV}(\pi') = (w', \mu') \). Since \( \text{IDES}(\pi) = \text{IDES}(\pi') \) and \( \text{DB}(\pi) = \text{DB}(\pi') \), we have \( P_m = P'_m \) and \( P_b = P'_b \) by Lemma 2.4. It then follows from the construction of \( \phi \) that the weight functions \( \mu \) and \( \mu' \) are equal. On the other hand, by Lemma 2.4 we see that condition (2.1) implies the only difference between \( P_t \) and \( P'_t \) are in \( i \)-th step and in \( j \)-th step. More precisely, the \( i \)-th step of \( P_t \) is horizontal and the \( j \)-th step of \( P_t \) is vertical, while the \( i \)-th step of \( P'_t \) is vertical and the \( j \)-th step of \( P'_t \) is horizontal. We need to distinguish two cases according to the \( i \)-th step of \( P_b = P'_b \) is vertical or horizontal. Let \( \pi^{(k)} \) (resp. \( \pi'^{(k)} \)) be the word on \([k] \cup \{\diamond\}\) after applying the \( k \)-th step in the algorithm \( \psi_{FV}^{-1} \) to retrieve \( \pi \) (resp. \( \pi' \)).

- If the \( i \)-th step of \( P_b = P'_b \) is vertical, then from the construction of \( \phi \) we have \( w_i = D \) and \( w'_i = H_r \). As \( \pi_n = i \), \( \pi^{(i)} \) is obtained from \( \pi^{(i-1)} \) by replacing the rightmost \( \diamond \) by \( i \), while \( \pi'^{(i)} \) is obtained from \( \pi'^{(i-1)} = \pi^{(i-1)} \) by replacing the rightmost \( \diamond \) (since \( \mu_i = \mu'_i \)) by \( i \diamond \). Since \( \mu_k = \mu'_k \) for \( k = i + 1, \ldots, j - 1 \), the number of \( \diamond \)'s in \( \pi'^{(j-1)} \) is one more than that in \( \pi^{(j-1)} \), which force \( \pi'_n \neq j \) (for otherwise, the rightmost \( \diamond \) of \( \pi'^{(j-1)} \) must be replaced by \( j \) or \( \diamond \!\diamond \), which is impossible because \( \mu'_j = \mu_j \), a contradiction.

- If the \( i \)-th step of \( P_b = P'_b \) is horizontal, then from the construction of \( \phi \) we have \( w_i = H_b \) and \( w'_i = U \). As \( \pi_n = i \), \( \pi^{(i)} \) is obtained from \( \pi^{(i-1)} \) by replacing the rightmost \( \diamond \) by \( i \diamond \), while \( \pi'^{(i)} \) is obtained from \( \pi'^{(i-1)} = \pi^{(i-1)} \) by replacing the rightmost \( \diamond \) (since \( \mu_i = \mu'_i \)) by \( \diamond \!\diamond \). Now the same reason as the first case leads to a contradiction with \( \pi'_n = j \).
Since both cases lead to contradictions, condition (2.1) could not happen and the proof of the lemma is complete.

Because of Lemma 2.5, the algorithm for the inverse \((\Gamma')^{-1}\) can be constructed as follows. **The algorithm for the inverse \((\Gamma')^{-1}\).** Given a triple of lattice path \((P_b, P_m, P_t)\) \(\in \mathcal{T}_{lp_{n,k}}\), we can retrieve the permutation \(\pi = (\Gamma')^{-1}(P_b, P_m, P_t)\) according to the following two cases.

- If the last step of \(P_t\) is vertical, then form the new top path \(P'_t\) starting at \((0, 2)\) of the same length that encodes \(\{i + 1 : \text{the } i\text{-th step of } P_t \text{ is horizontal}\}\). Define \(\pi\) to be the permutation \(\Psi^{-1}(P_b, P_m, P'_t)\).
- If the last step of \(P_t\) is horizontal, then let \(S := \{i + 1 : \text{the } i\text{-th step of } P_t \text{ is horizontal}\} \setminus \{n\}\).

Find the unique integer \(j \in [n - 1] \setminus S\) such that

1. the new top path \(P'_t\) starting at \((0, 2)\) of the same length that encodes \(S \cup \{j\}\) does not intersect \(P_m\);
2. and the distance between the starting point of the \(j\)-th step of \(P_m\) and that of \(P'_t\) is \(\sqrt{2}\) (i.e., a diagonal unit).

Define \(\pi\) to be the permutation \(\Psi^{-1}(P_b, P_m, P'_t)\).

By the construction of the algorithm \(\psi_{FV}^{-1}\) for building a permutation \(\pi\) from a Laguerre history \((w, \mu) \in \mathcal{L}_{n-1}\), in order to guarantee that \(\pi_n = j\), we must require that \(j\) is the smallest index with \(w_j = D\) or \(H_b\) and \(\mu_j = h_j(w)\), which under \(\phi\) is equivalent to the distance requirement in (2) above. The fact that \(\Gamma'\) is a bijection guarantees the existence and uniqueness of such a \(j\).

**Acknowledgement**

The authors are very grateful to professor Viennot for explaining his bijective proof of (1.1) in [14]. This work was supported by the National Science Foundation of China grants 11871247 and the project of Qilu Young Scholars of Shandong University.

**References**

[1] G. Baxter, On fixed points of the composite of commuting functions, Proc. Amer. Math. Soc., 15 (1964), 851–855.
[2] M. Bousquet-Mélou, Four classes of pattern-avoiding permutations under one roof: generating trees with two labels, Electron. J. Combin., 9 (2003), #R19.
[3] F.R.K. Chung, R.L. Graham, V.E. Hoggatt, Jr. and M. Kleiman, The number of Baxter permutations, J. Combin. Theory Ser. A, 24 (1978), 382–394.
[4] R. Cori, S. Dulucq and G. Viennot, Shuffle of parenthesis systems and Baxter permutations, J. Combin. Theory Ser. A, 43 (1986), 1–22.
[5] K. Dilks, Involutions on Baxter Objects, and \(q\)-Gamma Nonnegativity, Ph.D. thesis, University of Minnesota, 2015.
[6] D. Dumont, Interprétations combinatoires des nombres de Genocchi (in French), Duke Math. J., 41 (1974), 305–318.
[7] S. Felsner, É. Fusy, M. Noy and D. Orden, Bijections for Baxter families and related objects, J. Combin. Theory Ser. A, 118 (2011), 993–1020.
[8] J. Franel and G. Viennot, Permutations selon leurs pics, creux, doubles montées et double descentes, nombres d’Euler et nombres de Genocchi (in French), Discrete Math., 28 (1979), 21–35.
[9] O. Guibert and S. Linusson, Doubly alternating Baxter permutations are Catalan (FPSAC 1997), Discrete Math., 217 (2000), 157–166.
[10] Z. Lin and D. Kim, Refined restricted inversion sequences, Ann. Comb., 25 (2021), 849–875.
[11] Z. Lin, D.G.L. Wang and T. Zhao, A decomposition of ballot permutations, pattern avoidance and Gessel walks, arXiv:2103.04599.
[12] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, http://oeis.org, 2021.
[13] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Stud. Adv. Math., vol. 62, Cambridge University Press, Cambridge, 1999.
[14] G. Viennot, A bijective proof for the number of Baxter permutations, 3rd Séminaire Lotharingien de Combinatoire, Le Klebach, 1981.
[15] S.H.F. Yan and Y. Yu, Pattern-avoiding inversion sequences and open partition diagrams, Theoret. Comput. Sci., 841 (2020), 186–197.

(Zhicong Lin) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao 266237, P.R. China
Email address: linz@sdu.edu.cn

(Jing Liu) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao 266237, P.R. China
Email address: 202012008@mail.sdu.edu.cn