Decoding general error correcting codes and the role of complementarity

Yoshifumi Nakata,1,2,3 Takaya Matsuura,4,5 and Masato Koashi2,5

1 Yukawa Institute for Theoretical Physics, Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku, Kyoto, 606-8502, Japan.
2 Photon Science Center, Graduate School of Engineering, The University of Tokyo, Bunkyo-ku, Tokyo 113-8656, Japan
3 JST, PRESTO, 4-1-8 Honcho, Kawaguchi, Saitama, 332-0012, Japan
4 Centre for Quantum Computation & Communication Technology, School of Science, RMIT University, Melbourne, VIC 3000, Australia
5 Department of Applied Physics, Graduate School of Engineering, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan.

Among various classes of quantum error correcting codes (QECCs), non-stabilizer codes have rich properties and are of theoretical and practical interest. Decoding non-stabilizer codes is, however, a highly non-trivial task. In this paper, we show that a decoding circuit for Calderbank-Shor-Steane (CSS) codes can be straightforwardly extended to that for a general QECC. In the extension, instead of the classical decoders of the linear classical codes that define the CSS code, we use decoding measurements of a pair of classical-quantum (CQ) codes associated with the QECC to be decoded. The decoding error depends on the errors of the two decoding measurements and the degree of complementarity of the CQ codes. We then demonstrate the power of the decoding circuit in a toy model of the black hole information paradox, in which we improve decoding errors over previous approaches and further show that the black hole dynamics may be an optimal encoder for quantum information but a poor encoder for classical information.

I. INTRODUCTION

Reversing the effects of noise and recovering quantum information from a noisy quantum system are of central importance toward large-scale quantum information processing and are also offering significant insight into fundamental physics [1–17]. A common technique is quantum error correction (QEC), where quantum information is encoded into the system in the way that it is decodable even after the system experiences noises.

While the standard class of quantum error correcting codes (QECCs) is stabilizer codes, QECCs beyond stabilizer codes recently attract much attention as they have richer QEC properties. A significant advantage of non-stabilizer codes is that they have higher encoding rates than stabilizer codes and can be capacity-achieving [18–22]. As the encoding rate is an important figure of merit, it is natural to explore non-stabilizer codes so as to achieve better performance of QEC. Non-stabilizer codes are also of interest in theoretical physics since QECCs in the context of quantum many-body physics are typically non-stabilizer ones [3–6].

An important class of non-stabilizer codes is random codes. They are originally introduced as an analytical tool for investigating the achievability of the quantum capacity, but recent progress of unitary designs [23–30] and quantum technologies open the possibility of the practical use. Random codes also find many applications in theoretical physics, ranging from the foundation of statistical mechanics [1, 2], scrambling dynamics in complex many-body systems [3–6], to quantum black holes [15–17].

One of the obstacles in exploring non-stabilizer codes is to explicitly construct a decoder. In the case of stabilizer codes, a decoder can be constructed by, e.g., correcting errors in a standard manner based on the syndrome measurement and then by applying the inverse of an encoding circuit. This approach to constructing a decoder cannot be naively extended to non-stabilizer codes since measurement without destroying the logical information is not a priori given in non-stabilizer codes, and finding the syndrome measurement is far from trivial. For this reason, non-stabilizer codes have been commonly studied by the so-called decoupling approach [3, 20, 23, 31], which enables investigating the code performance without explicitly constructing a decoder.

Only a couple of explicit decoders applicable to non-stabilizer codes were proposed. One is based on the Petz map [32]. While its implementation by quantum circuits requires high computational cost [33], the Petz map can be used for decoding general QECCs [34, 35]. The other is to exploit an intrinsic relation between classical and quantum information [36–39]. In this approach, one regards a given QECC as a classical-quantum (CQ) code aiming to protect only classical information in a certain basis. That is, the input of a CQ code is restricted to one of the basis-states that represent the classical information, and the decoder is a quantum measurement to reveal the encoded classical information. A decoder of the QECC can be constructed from a pair of two decoders for CQ codes: one is for decoding general QECCs [34, 35].
that, despite a lack of a priori stabilizer structures, a decoding quantum circuit for general QECCs can be constructed similarly to stabilizer codes. More specifically, we first provide a decoding circuit for a Calderbank-Shor-Steane (CSS) code [40, 41], which is constructed simply by combining one-bit teleportation [42] with the classical decoders that the CSS code inherently possesses. We then extend the circuit by replacing the classical decoders with two decoders of CQ codes, making the circuit applicable to general QECCs. We call the extended decoding circuit a Classical-to-Quantum (C-to-Q) decoder and show that the decoding error is given by the decoding errors of the CQ codes and the degree of complementarity of the two bases that define the inputs of the CQ codes. We also show that the C-to-Q decoder with a suitable choice of decoders of the CQ codes is nearly optimal and can decode capacity-achieving QECCs.

We further demonstrate the power of C-to-Q decoder by applying it to the Hayden-Preskill protocol [3]. The protocol is a toy model of the black hole information paradox based on random coding and is of experimental interest as well [43]. Combining the C-to-Q decoder with a slight modification of the pretty-good-measurement (PGM) [44], we compute the decoding errors both for classical and quantum information. It turns out that this C-to-Q decoder not only achieves the quantum capacity of the protocol but also improves the decoding error compared to previous analyses based on decoupling, illustrating an advantage of the C-to-Q decoder. We also show that the black hole dynamics optimally encodes quantum information but poorly encodes classical information.

The C-to-Q decoder has a wide range of advantages from theoretical to practical. It is theoretically interesting not only because it advances the pursuit of explicit decoders for general QECCs but also because it quantitatively reveals the fundamental role of complementarity in decoding QECCs. Also, the C-to-Q decoder has many applications. Even for stabilizer codes, the C-to-Q decoder can apply to less restrictive situations since it does not rely on the syndrome measurement that should be all commutable. The C-to-Q decoder can also be used for experimentally benchmarking the performance of a QECC by checking that of the corresponding CQ codes. Since this is achieved merely by measuring the system, it substantially simplifies the evaluation. The C-to-Q decoder is further useful for simultaneously achieving error correction and switching codes, which has an application in quantum communication as well will be elaborated on.

This paper is organized as follows. We start with preliminaries in II. All the results are summarized in III. Proofs of the main statements about the C-to-Q decoder are given in IV. An in-depth analysis of the Hayden-Preskill protocol with the C-to-Q decoder is presented in V. Proofs of technical statements are given in Appendices after the summary and discussion in VI.

II. PRELIMINARIES

A. Notation

Throughout the paper, we write the relevant systems in the superscript, such as a Hilbert space $H^A$ of a system $A$, an operator $X^{AB}$ on $AB$, and a superoperator $E^{A\rightarrow B}$ from $A$ to $B$. A superoperator from $A$ to itself is denoted by $E^A$. The superscript will be sometimes omitted when it is clear from the context.

In this paper, we consider only orthonormal bases and do not explicitly mention that a basis is orthonormal. Let $\{ |e_j\rangle \}_j$ be a basis in $A$, and $A'$ be the system isomorphic to $A$. The state $|\Phi\rangle^{AA'} := (\dim H^A)^{-1/2} \sum_j |e_j\rangle_A \otimes |e_j\rangle^{A'}$ is called a maximally entangled state. In terms of another basis $\{ |f_j\rangle \}_j$, the state $|\Phi\rangle^{AA'}$ is given by

$$|\Phi\rangle^{AA'} = (\dim H^A)^{-1/2} \sum_j |f_j\rangle_A \otimes |f_j\rangle^{A'},$$

where $|f_j\rangle^{A'} = \sqrt{\dim H^A} |f_j\rangle |\Phi\rangle^{AA'}$. The corresponding density matrix $|\Phi\rangle^{A\otimes A'}$ is denoted by $\Phi^{AA'}$. We denote the completely mixed state on $A$ by $\pi^A := I^A / \dim H^A$, where $I^A$ is the identity operator on $A$.

The Schatten $p$-norm for a linear operator $X$ is defined by $\| X \|_p := (\text{Tr}(X^\dagger X)^{p/2})^{1/p}$ ($p \in [1, \infty]$). We particularly use the trace ($p = 1$), and operator ($p = \infty$) norms. The fidelity between quantum states $\rho$ and $\sigma$ is defined by $F(\rho, \sigma) := \| \sqrt{\rho} \sqrt{\sigma} \|_F^2$. The fidelity and the trace norm are related to each other by the Fuchs–van de Graaf inequalities as

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \| \rho - \sigma \|_1 \leq \sqrt{1 - F(\rho, \sigma)}.$$  

For any state $\rho$, which may be defined on a composite system including $A$, the collision entropy is defined by

$$H_2(\rho) = -\text{log} [\text{Tr}(\rho^A)^2].$$  

Important classes of dynamics in quantum systems are unitary, isometry, and partial isometry. An isometry $V^A\rightarrow A'$ from a system $A$ to $A'$ ($\dim H^A \leq \dim H^{A'}$) is a linear operator such that $V^A\rightarrow A' V^A\rightarrow A' = I^A$. When $\dim H^A = \dim H^{A'}$, it holds that $V^A\rightarrow A' V^A\rightarrow A' = I^A$, so the isometry is unitary. A linear operator $W^A\rightarrow A'$ is a partial isometry if both $W^A\rightarrow A' W^A\rightarrow A' = I^A$, and $W^A\rightarrow A' W^A\rightarrow A' = I^A$, are projections, the former on $A$ and the latter on $A'$. Projections, isometries, and unitaries are special classes of partial isometries.

A quantum channel $T^{A\rightarrow B}$ is a completely-positive (CP) and trace-preserving (TP) map. A map is called CP if $(\text{id}^{A'} \otimes T^{A\rightarrow B})(\rho^{A\otimes A'}) \geq 0$ for any $\rho^{A\otimes A'} \geq 0$, where id is the identity map, and is TP if $\text{Tr}(T^{A\rightarrow B}(\rho^A) = \text{Tr}(\rho^A)$.

For simplicity, we below denote $(\text{id}^{A'} \otimes T^{A\rightarrow B})(\rho^{A\otimes A'})$ by $T^{A\rightarrow B}(\rho^{A\otimes A'})$. For a given partial isometry, we sometimes denote the conjugating map by its calligraphic style, such as $V(\rho) := V\rho V^\dagger$ for an isometry $V$. A fundamental
quantum channel is the partial trace over, e.g., a system $C$, which is denoted by $\text{Tr}_C$. In this paper, a partial trace is often implicit in the sense that $\rho^A$ denotes the reduced operator on $A$ of $\rho^{AB}$, i.e., $\rho^A = \text{Tr}_B[\rho^{AB}]$.

For a superoperator $\mathcal{M}^{A \rightarrow B}$, the diamond norm is defined as

$$\|\mathcal{M}^{A \rightarrow B}\|_\diamond := \max_{\mathcal{H}_R} \max_{O^{AR}(\neq 0)} \|\mathcal{M}^{A \rightarrow B} \otimes \text{id}^R(\rho^{AR})\|_1,$$

where $\max_{\mathcal{H}_R}$ is taken over a Hilbert space $\mathcal{H}_R$ with arbitrary finite dimension, $\text{id}^R$ is the identity map on $R$, and $O^{AR}$ is any operator on $\mathcal{H}^A \otimes \mathcal{H}^R$. It suffices to take the dimension of $R$ at most the dimension of $A$ [45]. Furthermore, when the map is Hermitian-preserving, the diamond norm is given by

$$\|\mathcal{M}^{A \rightarrow B}\|_\diamond = \max_{\mathcal{H}_R} \max_{|\psi\rangle^{AR}} \|\mathcal{M}^{A \rightarrow B} \otimes \text{id}^R(|\psi\rangle\langle\psi|^{AR})\|_1,$$

where $\max_{|\psi\rangle^{AR}}$ is taken over all pure states in $\mathcal{H}^{AR}$ [45].

The Haar measure $\mathcal{H}$ on a unitary group $U(d)$ of finite degree $d$ is the unique left- and right-unitarily invariant probability measure. That is, for any measurable set $W \subset U(d)$ and $V \in U(d)$,

$$\mathcal{H}(VW) = \mathcal{H}(WV) = \mathcal{H}(V),$$

and $\mathcal{H}(\mathcal{U}(d)) = 1$. When a unitary $U$ is chosen with respect to the Haar measure $\mathcal{H}$, it is called a Haar random unitary and is denoted by $U \sim \mathcal{H}$. The average of a function $f(U)$ over a Haar random unitary is denoted by $\mathbb{E}_{U \sim \mathcal{H}}[f(U)]$.

### B. Decoding errors

A QECC is in general defined by a pair of encoding and decoding quantum channels. A QECC against a noisy quantum channel $\mathcal{N}^{B \rightarrow C}$ is the code $(\mathcal{E}^{A \rightarrow B}, \mathcal{D}^{C \rightarrow A})$ that satisfies

$$\mathcal{D}^{C \rightarrow A} \circ \mathcal{N}^{B \rightarrow C} \circ \mathcal{E}^{A \rightarrow B} \approx \text{id}^A,$$  

(7)

where $\text{id}^A$ is the identity map on $A$ that represents a logical system. As our main concern is a decoder, we use a composite channel $\mathcal{T}^{A \rightarrow C}$ of the encoding and the noisy channel. That is,

$$\mathcal{T}^{A \rightarrow C} = \mathcal{N}^{B \rightarrow C} \circ \mathcal{E}^{A \rightarrow B}.$$  

(8)

For a given QECC $(\mathcal{E}, \mathcal{D})$, we can construct a CQ code that aims to protect only classical information. Let $W = \{|j_W\rangle\}_{j=1}^d$ be a basis. If the input of the encoder $\mathcal{E}$ is restricted to a state $|j_W\rangle$ in the basis, which represents a $(\log d)$-bit classical information $j$, the code is called a CQ code in the $W$-basis. As the code is for $(\log d)$-bit classical information, a decoder is given by quantum measurement, i.e., positive-operator-valued measure (POVM), with $d$ outcomes, such as $M = \{M_j\}_{j=0}^{d-1}$. The decoding succeeds if

$$\text{Tr}[M_j \mathcal{T}^{A \rightarrow C}(|j_W\rangle\langle j_W|)] \approx 1,$$  

(9)

for all $j = 0, \ldots, d-1$. Note that the CQ code depends on the choice of the basis of the input state. Hence, various CQ codes can be defined from a single QECC.

Below, we elaborate on how the decoding errors of the QECCs and CQ codes can be quantified.

#### 1. Decoding quantum information

The decoding error of QECC $(\mathcal{E}, \mathcal{D})$ against a noisy quantum channel $\mathcal{N}$ should be defined using Eq. (7). A natural way to quantify the approximation is to use the diamond norm:

$$\Delta_\diamond(\mathcal{D}|\mathcal{T}) := \frac{1}{2} \|\text{id}^A - \mathcal{D}^{C \rightarrow A} \circ \mathcal{T}^{A \rightarrow C}\|_\diamond.$$  

(10)

Another standard way is to introduce a reference system $R$, where $\dim \mathcal{H}^A = \dim \mathcal{H}^R = d$, and to use the maximally entangled state $\Phi^{AR}$:

$$\Delta_q(\mathcal{D}|\mathcal{T}) := \frac{1}{2} \|\Phi^{AR} - \mathcal{D}^{C \rightarrow A} \circ \mathcal{T}^{A \rightarrow C}(\Phi^{AR})\|_1.$$  

(11)

These two definitions of errors are closely related since they satisfy

$$\Delta_q(\mathcal{D}|\mathcal{T}) \leq \Delta_\diamond(\mathcal{D}|\mathcal{T}) \leq d\Delta_q(\mathcal{D}|\mathcal{T}).$$  

(12)

See, e.g., [46]. Hence, when $\Delta_q(\mathcal{D}|\mathcal{T})$ is sufficiently small, so is $\Delta_\diamond(\mathcal{D}|\mathcal{T})$. From Eq. (12), we observe that $\Delta_q(\mathcal{D}|\mathcal{T})$ and $\Delta_\diamond(\mathcal{D}|\mathcal{T})$ can differ by factor $d$, which may be problematic for large $d$. This issue can be circumvented in the context of QEC. In fact, we can improve Eq. (12) by slightly modifying the encoding and decoding operations. Here, we provide two methods of doing so.

The first method is to use a random unitary in the encoding and decoding operations. We especially use a unitary 1-design, which is a random unitary that has the same first-order moment as that of a Haar random unitary. A canonical instance of a unitary 1-design is the multi-qubit Pauli group. Using a unitary 1-design, the following statement holds. See Appendix A for the derivation.

**Proposition 1.** Let $(\mathcal{E}^{A \rightarrow B}, \mathcal{D}^{C \rightarrow A})$ be a QECC against a noisy quantum channel $\mathcal{N}^{B \rightarrow C}$. Let $U^A$ be a unitary 1-design, and define a new QECC $(\mathcal{E}_{U}^{A \rightarrow B}, \mathcal{D}_{U}^{C \rightarrow A})$ by

$$\mathcal{E}_{U}^{A \rightarrow B}(\rho^A) := \mathcal{E}^{A \rightarrow B}(U^A \rho^A U^A\dagger),$$  

(13)

$$\mathcal{D}_{U}^{C \rightarrow A}(\sigma^C) := U^\dagger \mathcal{D}^{C \rightarrow A}(\sigma^C) U^A.$$  

(14)

Then, it holds that

$$\frac{1}{2} \|\text{id}^A - \mathbb{E}_U[\mathcal{D}_{U}^{C \rightarrow A} \circ \mathcal{N}^{B \rightarrow C} \circ \mathcal{E}_{U}^{A \rightarrow B}]\|_\diamond \leq \Delta_q(\mathcal{D}|\mathcal{N} \circ \mathcal{E}),$$  

(15)

where $\mathbb{E}_U$ is the average over the unitary 1-design $U^A$. 

Proposition 1 implies that if there exists a QECC $(\mathcal{E}^{A \rightarrow B}, \mathcal{D}^{C \rightarrow A})$ that achieves a small decoding error against a noisy quantum channel $\mathcal{N}^{B \rightarrow C}$ in terms of $\Delta_q(\mathcal{D}|(\mathcal{N} \circ \mathcal{E}))$, then one can achieve the same error in terms of the diamond norm by applying a random unitary $U^A$ and $U^{A^d}$ before and after the encoding and decoding, respectively. Hence, if $\Delta_q(\mathcal{D}|(\mathcal{N} \circ \mathcal{E}))$ is small, then it is possible to correct the noise $\mathcal{N}^{B \rightarrow C}$ with the same decoding error in the diamond norm.

In the above scheme, the encoder and the decoder have to share common randomness to sample the same instance of the unitary 1-design. If one would like to avoid using common randomness, the following second method can be used. See, e.g., [45] for the proof.

**Proposition 2.** Let $(\mathcal{E}^{A \rightarrow B}, \mathcal{D}^{C \rightarrow A})$ be a QECC against a noisy quantum channel $\mathcal{N}^{B \rightarrow C}$. For any subspace $\mathcal{H}^{A_0} \subseteq \mathcal{H}^A$ with $\dim \mathcal{H}^{A_0} < d/2$, there exists a pair of quantum channels $\mathcal{F}^{A \rightarrow A_0}$ and $\mathcal{G}^{A_0 \rightarrow A}$ such that a new QECC $(\mathcal{E} \circ \mathcal{F}, \mathcal{G} \circ \mathcal{D})$ satisfies

$$\Delta_q(\mathcal{G} \circ \mathcal{D}|(\mathcal{N} \circ \mathcal{E} \circ \mathcal{F})) \leq 2\sqrt{2\Delta_q(\mathcal{D}|(\mathcal{N} \circ \mathcal{E}))}. \quad (16)$$

We observe from Proposition 2 that, if $\Delta_q(\mathcal{D}|(\mathcal{N} \circ \mathcal{E})) \ll 1$, then a small error can be achieved in terms of the diamond norm by restricting the $d$-dimensional Hilbert space $\mathcal{H}^A$ to the one with a dimension at most $d/2$. Since this corresponds to reducing one qubit of quantum information, this relation shows that one can relate the decoding error $\Delta_q$ with $\Delta_q$ at the expense of one qubit.

For these reasons, the decoding error for quantum information can be characterized well by $\Delta_q(\mathcal{D}|(\mathcal{N} \circ \mathcal{E})$ in QEC. In addition, as we will show below, the decoding error $\Delta_q(\mathcal{D}|\mathcal{T})$ is directly related to the decoding errors of the CQ codes. Thus, we use the decoding error $\Delta_q(\mathcal{D}|\mathcal{T})$ in this paper.

2. Decoding classical information

For a CQ code, the decoding error is simply given by the failure probability of the decoding measurement. Let us consider a CQ code in the $W$-basis and its decoding measurement $M = \{M_j\}_{j=0}^{d-1}$. The decoding error is defined as

$$\Delta_{cl,W}(M|\mathcal{T}) := \frac{1}{d} \sum_{i \neq j} \text{Tr}[M_j^C T^{A \rightarrow C}(|i_W\rangle\langle i_W|A)]. \quad (17)$$

This is nothing but the failure decoding probability uniformly averaged over all possible classical inputs.

This error $\Delta_{cl,W}(M|\mathcal{T})$ can be rephrased in terms of the maximally-correlated $W$-classical state $\Omega_W^{AR}$ defined by

$$\Omega_W^{AR} := \frac{1}{d} \sum_j |j_W\rangle\langle j_W|A \otimes |j_W\rangle\langle j_W|W. \quad (18)$$

To see this, we introduce a quantum channel $D_M^{C \rightarrow A}(\rho^C) = \sum_{j=1}^{d-1} \text{Tr}[M_j^C \rho^C]|j_W\rangle\langle j_W|A$. Using this, the decoding error $\Delta_{cl,W}(M|\mathcal{T})$ is rewritten as

$$\Delta_{cl,W}(D_M|\mathcal{T}) = \frac{1}{2}\|\Omega_W^{AR} - D_M^{C \rightarrow A} \otimes T^{A \rightarrow C}(\Omega_W^{AR})\|_1. \quad (19)$$

Note that optimal decoding error for classical information never exceeds that for quantum information. This is due to the facts that $\Omega_W^{AR} = C_W^{AR}$, where $C_W$ is the completely dephasing channel in the $W$-basis, and that $C_W \circ \mathcal{D}$ is equivalent to a map $D_M$ with suitably chosen POVM $M$. Using the monotonicity of the trace norm, we obtain, for any $\mathcal{T}$ and $\mathcal{D}$, that there exists a POVM $M$ such that $\Delta_{cl,W}(D_M|\mathcal{T}) \leq \Delta_q(\mathcal{D}|\mathcal{T})$.

III. MAIN RESULTS

We here summarize our results. We provide a construction of the C-to-Q decoder in III A. To complete the analysis, we comment on a possible choice of a decoder for a CQ code in III B. Finally, our results about the decoding problem in the Hayden-Preskill protocol are summarized in III C.

A. The Classical-to-Quantum decoder

To construct the C-to-Q decoder, we start with a decoding quantum circuit for a CSS code in III A 1, which is constructed by combining one-bit teleportation and the classical decoders of the CSS code. In III A 2, we extend the decoding circuit to general QECCs, namely, we provide a construction of the C-to-Q decoder. A couple of implications of the C-to-Q decoder are elaborated on in III A 3.

1. A decoding circuit for CSS codes

Let us consider an $[[N, k]]$-CSS code that encodes $k$ logical qubits into $N$ physical qubits. The code is constructed from two classical linear codes $C_1$ and $C_2$ on $N$ bits that satisfy $C_2 \subseteq C_1$. The difference between the number of logical bits for $C_1$ and that for $C_2$ should be $k$. In the CSS code, the logical Pauli-Z basis is defined by the classical code $C_1 / C_2$, and the logical Pauli-X basis is by $C_2^Z / C_1^Z$. Let $f_1$ and $f_2$ be the classical decoders for $C_1$ and $C_2$, respectively. They are both boolean functions, taking $N$ bits as an input and outputting $k$ bits. In the CSS code, bit- and phase-flip errors are independently corrected by using $f_1$ and $f_2$, respectively. Note that correcting the bit- and phase-flip errors suffice for completing error correction. See Appendix B for the details.

To decode a CSS code after error correction, one can, for instance, simply apply the inverse of the encoding
the steps of the one-bit teleportation are based only on

\[ Z^x \]

on the \( Z \)-basis, it is important to notice that the first and the sec-

dient outcome.

\[ \alpha \]

\[ R \]

fe-forward unitary \( Z^x \) to the ancillary system, which depends on the

measurement outcome \( x \in \{0, 1\}^k \).

\[ R_Z = \sum_{z \in \{0,1\}^N} \langle z | \langle z | f_1(z) \rangle \]

By this operation, a decoded bit string \( f_1(z) \in \{0, 1\}^k \)
in the Pauli-\( Z \) basis is coherently copied to the \( k \)-qubit ancillary system. If the decoding error of the classical
decoder \( f_1 \) is sufficiently small, the decoded bit string \( f_1(z) \) does not contain any bit-flip error.

The second step is also similar to one-bit teleportation, but the feed-forward unitary is based on the outcome of the
classical decoder \( f_2 \). Let \( x \in \{0,1\}^N \) be the measure-
ment outcome of the input system, when it is measured in the
Pauli-X basis. The feed-forward unitary onto the ancillary system is then given by \( Z^{f_2(x)} \), where \( f_2(x) \) is the
decoded \( k \)-bit string. As the classical decoder \( f_2 \) in the
CSS code is used for correcting the phase-flip error, if the decoding error of \( f_2 \) is sufficiently small, the \( k \) bits for
determining the feed-forward unitary are correctly cho-

We provide the whole decoding circuit in Fig. 2. By
these two steps, error correction is completed while the
quantum state is transferred to the ancillary system.
That is, error correction and decoding is simultaneously
achieved. To see how it works, it is pedagogical to con-
sider the noiseless case. Let \( |\tilde{\psi}\rangle = \sum_{j \in \{0,1\}^k} \alpha_j |j_z\rangle \) be an
encoded logical state, where \( |j_z\rangle \) is the logical Pauli-\( Z \) basis
realized in an \( N \)-qubit system. With the absence of noise, the controlled unitary with the classical decoder
\( f_1 \) simply copies \( j_z \) to the ancillary system, leading to the transformation such as

\[ |\tilde{\psi}\rangle = \sum_{j \in \{0,1\}^k} \alpha_j |j_z\rangle \mapsto \sum_{j \in \{0,1\}^k} \alpha_j |j_z\rangle \otimes |j_z\rangle. \]

If the information of \( j_z \) is erased from the input system, we
obtain \( |\tilde{\psi}\rangle \) in the ancillary system, completing the
decoding task. This can be achieved by measuring the
input system in the logical Pauli-\( X \) basis and by applying
the feed-forward unitary on the ancillary system. This is
exactly what the second step with the classical decoder
\( f_2 \) does. As a result, the state transformation

\[ \sum_{j \in \{0,1\}^k} \alpha_j |j_z\rangle \otimes |j_z\rangle \mapsto \sum_{j \in \{0,1\}^k} \alpha_j |j_z\rangle = |\psi\rangle, \]

is realized, and a decoded state is obtained in the ancil-
lary system. Note that this is nothing but one-bit tele-
portation from a logical system to an ancillary system.
The situation does not change much even in the presence of noise as far as the noise is correctable by the CSS code. This is simply because, in each step, the classical decoders $f_1$ and $f_2$ correct the bit- and phase-flip errors, respectively. In Appendix B, we provide an in-depth analysis on the decoding error $\Delta_q$ for the quantum information by this decoding circuit and show that

$$\Delta_q \leq \sqrt{\Delta_{cl,Z} + \Delta_{cl,X}},$$

where $\Delta_{cl,Z}$ and $\Delta_{cl,X}$ are the decoding errors of the classical decoders $f_1$ and $f_2$, respectively.

Note that, if one regards the CSS code as a CQ code in the $Z$-basis and that in the $X$-basis, their decoding errors are nothing but $\Delta_{cl,Z}$ and $\Delta_{cl,X}$, respectively. More specifically, denoting the encoding map of a CSS code by $\mathcal{E}$ and the noisy quantum channel by $N$, the decoding error $\Delta_{cl,W}$ for $W = X, Z$ is given by

$$\Delta_{cl,W} = \Delta_{cl,W} (M_W |N \circ \mathcal{E}),$$

where $M_W$ is the measurement followed by the classical decoding process either by $f_1$ and $f_2$.

### 2. Construction of the C-to-Q decoder

The main idea of the C-to-Q decoder is to follow the construction of the decoding circuit for CSS codes. As explained, the decoding circuit for CSS codes consists of two steps, a coherent use of one classical decoder, and the measurement and feed-forward based on the other classical decoder. Instead of the classical decoders in each step, we use decoders of two CQ codes to extend the decoding circuit.

For a given QECC, let us consider two CQ codes. One is in the $E$-basis, namely, the input is restricted to one of the pure states in the $E$-basis, and the other is in the $F$-basis. We denote the two bases by $E := \{|j_E\rangle\}_{j=0}^{d-1}$ and $F := \{|l_F\rangle\}_{l=0}^{d-1}$, respectively. We also denote the decoding measurements by $M_W := \{|M_W^c j\rangle\}_{j=0}^{d-1} (W = E, F)$, and by $\Delta_{cl,W} (M_W |N \circ \mathcal{E})$ the decoding error.

A C-to-Q decoder $D_{C\rightarrow Q}$ is based on the two decoding measurements $M_E$ and $M_F$ is constructed as in Fig. 3. Similarly to the decoding circuit for CSS codes in Fig. 2, the C-to-Q decoder consists of two steps represented by quantum channels $R_{E\rightarrow CA}$ and $Q_{F\rightarrow CA}$. The first quantum channel $R_{E\rightarrow CA}$ corresponds to the isometry for coherently recording the outcome of one decoding measurement $M_E$ into an ancillary system, and the second one $Q_{F\rightarrow CA}$ is based on the other decoding measurement $M_F$ and plays a role of the measurement and feed-forward.

The first quantum channel $R_{E\rightarrow CA}$ is composed of an isometry $R_{E\rightarrow CC'}$ from $C$ to $CC'$. To define the isometry $R_{E\rightarrow CC'}$, let $(V_{E\rightarrow CC'}, \{P_{j}^{CC'}\}_{j=0}^{d-1})$ be a Naimark extension of the POVM $M_E$, that is, a pair of an isometry $V_{E\rightarrow CC'}$ and orthogonal projections $\{P_{j}^{CC'}\}_{j=0}^{d-1}$ such that

$$M_{E,j}^C := (V_{E\rightarrow CC'}^\dagger P_j^{CC'} V_{E\rightarrow CC'})^\dagger P_j^{CC'} V_{E\rightarrow CC'},$$

where $\sum_{j=0}^{d-1} P_j^{CC'} = I^{CC'}$. Then,

$$R_{E\rightarrow CC'A} := V_{E,inv}^{CC'} \left( \sum_{j=0}^{d-1} P_j^{CC'} \otimes |j_E\rangle\langle j_E| \right) V_{E\rightarrow CC'}^\dagger,$$

with

$$V_{E,inv}^{CC'} := V_{E\rightarrow CC'}^{-1} \otimes |e_0\rangle^{CC'} + |e_0\rangle^{CC'} \otimes (I^{CC'} - V_{E\rightarrow CC'}^{CC'} V_{E\rightarrow CC'}^{-1}),$$

where $|e_0\rangle^{CC'}$ is a unit vector in the range of $V_{E\rightarrow CC'}^{CC'}$, and $|e_0\rangle^{CC'}$ is an arbitrary unit vector in $C$. As $V_{E\rightarrow CC'}^{CC'} V_{E\rightarrow CC'}^{-1}$ is a projection, $V_{E,inv}^{CC'}$ is also an isometry. The quantum channel $R_{E\rightarrow CA}$ is then defined as

$$R_{E\rightarrow CA}(\rho^C) := \text{Tr}_C [R_{E\rightarrow CC'A}(\rho^C) R_{E\rightarrow CC'A}^\dagger].$$

To understand the action of this quantum channel $R_{E\rightarrow CA}$, we observe that the middle and the right-most terms in Eq. (26) are for coherently recording the measurement outcome by $M_E$ into an ancillary system $A$. The left-most term $V_{E,inv}^{CC'}$ is for undoing the action on $C$ as much as possible, so that the backaction to the system is suppressed. When the decoding measurement $M_E$ is projective, which is the case in CSS codes, there is no need to apply $V_{E}\ and \ V_{E,inv}$, and the quantum channel reduces to the isometry $R_E$ given by Eq. (20).

Before we proceed, we comment on the fact the quantum channel $R_{E\rightarrow CA}$ can be replaced with other channels. Another instance is to use the isometry

$$\sum_{j=0}^{d-1} M_{E,j}^C \otimes |j_E\rangle\langle j_E|^A + \sqrt{I^C - \sum_{j=0}^{d-1} (M_{E,j}^C)^2 \otimes |\text{fail}\rangle^A}$$

from $C$ to $CA$, where $|\text{fail}\rangle^A$ is the state in $A$ orthogonal to $|j_E\rangle^A$ for $j = 0, \ldots, d-1$. Accordingly, in this case, $A$ is a $(d+1)$-dimensional system. In IV A, we explain that the use of this isometry instead of $R_{E\rightarrow CA}$ leads to the same conclusion as below.

The second quantum channel $Q_{F\rightarrow CA}$ is constructed from the other decoding measurement $M_F$ as

$$Q_{F\rightarrow CA}(\rho^{CA}) = \sum_{l=0}^{d-1} \Theta_l^A \text{Tr}_C [M_{F,l}^C \rho^{CA}] \Theta_l^A.$$

This channel simply corresponds to the operations that the system $C$ is measured by $M_F$, and then, a feed-forward unitary $\Theta_l^A$ is applied to $A$ depending on the measurement outcome $l$, where

$$\Theta_l^A := \sum_{j=0}^{d-1} \delta_{\text{arg}(j_E \langle l_F | j_E)}^A |j_E\rangle^A.$$
FIG. 3. Construction of the Classical-to-Quantum decoder from POVMs $M_E$ and $M_F$, which consists of two quantum channels $R_E$ and $Q_F$. The former $R_E$ is an isometry for coherently recording the measurement outcome by the POVM $M_E$ into an ancillary system $A$. The $(V_E, \{P_j\})$ is a Naimark extension of $M_E$. The isometry $V_E,iow$ acts as if it is an inverse of the isometry $V_E$, which is for minimizing the backaction to $C$. If the decoding error $\Delta_{cl,E}(M_E|T)$ is sufficiently small, the channel $R_E$ transforms the state on $RCA$ to a noisy GHZ state. The latter quantum channel $Q_F$ plays the role of quantum erasure, which aims to delete all the $E$-classical information from $C$. The erasure succeeds if $(E,F)$ is close to MUB and $\Delta_{cl,F}(M_F|T)$ is small, transforming the noisy GHZ state in $RCA$ to a maximally entangled state between $RA$.

Definition 3 (Classical-to-Quantum decoder). Given two POVMs $M_E$ and $M_F$, a C-to-Q decoder is defined by

$$\mathcal{D}_{CtoQ}^{C \rightarrow A} = Q_{F}^{CA \rightarrow A} \circ R_{E}^{C \rightarrow CA},$$

(31)

where quantum channels $R_{E}^{C \rightarrow CA}$ and $Q_{F}^{CA \rightarrow A}$ are given in Eqs. (28) and (29), respectively.

The decoding error by the C-to-Q decoder is defined as follows (see IV A for the proof).

Theorem 4 (Decoding error by the C-to-Q decoder). Let $E^{A \rightarrow B}$ and $N^{B \rightarrow C}$ be an encoding channel of a QECC and a noisy channel, respectively. For the two bases $E = \{\langle j|E\rangle\}_{j}$ and $F = \{\langle l|F\rangle\}_{l}$ in a $d$-dimensional system $A$, let $\Delta_{cl,W} = \Delta_{cl,W}(M_{W}|N \circ E)$ be the decoding error of the corresponding CQ code in the $W$-basis by the decoder $M_{W}$ ($W = E, F$). Then, the decoding error of the C-to-Q decoder $\mathcal{D}_{CtoQ}$ constructed from $M_E$ and $M_F$ satisfies

$$\Delta_{q}(\mathcal{D}_{CtoQ}|N \circ E) \leq \sqrt{\Delta_{cl,E}(2 - \Delta_{cl,E})} + \sqrt{\Delta_{cl,F} + \Xi_{EF}}. \quad (32)$$

Here, $\Xi_{EF}$ is given by

$$\Xi_{EF} := 1 - \sum_{i=0}^{d-1} \text{Tr}[M_{F}iN \circ E(\pi^{A})]F_{BC}(\text{unif}_{d},p_{l}),$$

$$\leq 1 - \text{min}_{l=0,\ldots,d-1} F_{BC}(\text{unif}_{d},p_{l}), \quad (33)$$

$$\Xi_{EF} = 1 - \frac{1}{d} \sum_{j=0}^{d-1} \left( \frac{p_{l}(j)}{d} \right)^{2} \quad (35)$$

where $\pi^{A}$ is the completely mixed state in $A$, and $\Xi_{EF}$ is the Bhattacharyya distance between the uniform probability distribution $\text{unif}_{d}$ on $[0,d-1]$ and the probability distribution $\{p_{l}(j) = \langle j|E|l_{F}\rangle^{2}\}_{j=0}^{d-1}$ determined by the two bases $E$ and $F$.

The quantity $\Xi_{EF}$ in Theorem 4 can also be bounded from above in the form different from Eq. (34) as

$$\Xi_{EF} \leq 1 - \frac{1}{d} \sum_{i=0}^{d-1} F_{BC}(\text{unif}_{d},p_{l}) + \Delta_{cl,F}(F_{BC,\text{max}} - F_{BC,\text{min}}), \quad (36)$$

where $F_{BC,\text{max}} = \max_{l} F_{BC}(\text{unif}_{d},p_{l})$ and $F_{BC,\text{min}} = \min_{l} F_{BC}(\text{unif}_{d},p_{l})$. This may be useful especially when $\Delta_{cl,F}$ is small. See Appendix C for the derivation.

Note that the bases $E$ and $F$ are contained asymmetrically in Theorem 4, which is natural as the two decoding measurements play different roles in the C-to-Q decoding: one for the coherent isometry and the other for the measurement and feed-forward. This implies that we can construct two C-to-Q decoders from a single pair of two decoding measurements, which in general results in different decoding errors. From a practical viewpoint, using the one with a smaller error will be more useful.

Theorem 4 shows that the decoding error of the C-to-Q decoder can be split into three terms, $\Delta_{cl,E}$, $\Delta_{cl,F}$, and $\Xi_{EF}$. The first two are decoding errors of the CQ codes and analogous to the case of decoding CSS codes (see Eq. (23)). The last term $\Xi_{EF}$ characterizes how far the two bases $E$ and $F$ are from a mutually unbiased bases (MUBs). Note that the bases $E$ and $F$ define the classical inputs of the CQ codes. For instance, $\Xi_{EF} = 0$ if and only if $(E, F)$ is MUBs, and $\Xi_{EF} = 1 - 1/d$ if and only if $E = F$.

The degree of MUBs appeared in the decoding error is to some extent non-trivial and quantitatively reveals the fundamental role of complementarity in decoding QECCs. The decoding error in Theorem 4 is also of practical use for experimentally evaluating the performance of a QECC. In fact, Theorem 4 enables us to estimate the QECC performance by estimating decoding errors of two CQ codes and by checking the complementarity of the bases that define the classical information. Estimating decoding errors of CQ codes is practically more tractable than directly quantifying the performance of the QECC: while checking the QECC performance requires handling the system in a fully-quantum manner,
decoding errors of classical information can be evaluated simply by measuring the system. When we do so, it is important to understand how the complementarity of the bases affects the QEC performance since the presence of experimental imperfection, such as calibration errors in the device, may unintentionally make the encoding bases deviate from MUBs. By evaluating such errors in the device and by combining it with the decoding errors of classical information, the QEC performance can be estimated by Theorem 4.

Another possible use of the C-to-Q decoder is for switching code by preparing the ancillary system in the QEC different from the QECC against the noisy channel. This may be of particular importance in quantum communication since the QECC in local environment shall have different properties from the QEC for transmitting quantum information: fault-tolerance would be the most important in the former, while the QECC with higher encoding rate is more preferable in the latter. By using the C-to-Q decoder, one can correct errors and simultaneously switch QECCs.

Before we move on, we provide a high-level explanation about how the C-to-Q decoder works, which is basically for the same reason as the decoding circuit of the CSS code as explained in III A 1. First, because the quantum channel \( R_{C \rightarrow CA} \) is designed to coherently record the measurement outcome of \( M_E \) into the ancillary system \( A \), it generates a coherent correlation between the noisy system \( C \) and the ancillary system \( A \) in the basis \( E \). In fact, when the POVM \( M_E \) satisfies \( \Delta_{cl,E}(M_E|T) = 0 \), it holds that (see Sec. IV)

\[
R_{C \rightarrow CA}^{E} \circ T^{A \rightarrow C}(\Phi^{AR}) = \frac{1}{d} \sum_{j,i} |j_E^*\rangle\langle i_E^*| \otimes T^{A \rightarrow CA}(|i_E^*\rangle\langle j_E^*| \otimes |j_E\rangle\langle i_E|).
\]

(37)

Clearly, this state is a noisy GHZ state over \( R, C \), and \( A \), where we mean by noisy that \( T = N \circ E \) is applied. With this state, the remaining task is to transform the noisy GHZ state to the maximally entangled state between \( R \) and \( A \).

This can be accomplished by the measurement and feed-forward. For instance, a simple GHZ state \( |000\rangle_Z + |111\rangle_Z \) can be transformed to a maximally entangled state \( |00\rangle_Z + |11\rangle_Z \) by measuring one qubit in the \( X \) basis and applying a feed-forward unitary, which is simply \( Z^m \) with \( m = 0, 1 \) being the measurement outcome. This mechanism is known as the quantum eraser since the key in this process is to completely erase the \( Z \)-information from one qubit by the measurement and feed-forward. In the case of the C-to-Q decoder, the noisy GHZ state (37) can be transformed to the maximally entangled state if one could measure the noisy system \( C \) in the basis complementary to \( E \) and apply a feed-forward unitary to \( A \). However, we know only the error when \( C \) is measured either by \( M_E \) or \( M_F \), forcing us to apply \( M_F \) onto \( C \) instead of the ideal one. This results in an additional error quantified by \( \Xi(E, F) \).

An immediate corollary of Theorem 4 is the following. The corollary improves by constant factors the decoding errors of the previous decoding strategies [36–39], which are all different.

**Corollary 5.** In the same setting in Theorem 4, let \( (E, F) \) be a pair of MUBs. Then, the C-to-Q decoder \( D_{CtoQ} \) satisfies

\[
\Delta_q(D_{CtoQ}|T) \leq \sqrt{\Delta_{cl,E}(2 - \Delta_{cl,E})} + \sqrt{\Delta_{cl,F}} \leq (1 + \sqrt{2}) \max_{W=E,F} \sqrt{\Delta_{cl,W}}. 
\]

(38)

Important instances of MUBs are the (generalized) Pauli bases \( X \) and \( Z \) in multi-qubit systems and the Heisenberg-Weyl group for qudits. In these cases, the unitaries \( \Theta^A \) in Eq. (30) has a simple form. For instance, when \( E \) and \( F \) are set to be \( Z \) and \( X \), respectively, then \( \Theta_q = (Z^A)^l \), where the operator \( Z^A \) is the (generalized) Pauli-Z operator on \( A \).

The decoding circuit for CSS codes, shown in Fig. 2, is a typical instance of this kind. It may be worth noting that the upper bound on the decoding error by the decoding circuit for CSS codes (Eq. (23)) is better than Eq. (38). The difference comes from the fact that the error in the quantum channel \( Q_{E}^{CA \rightarrow C} \) for the measurement and feed-forward may in general be affected by the preceding procedure \( R_{E}^{C \rightarrow CA} \). In contrast, the two decoding procedures for CSS codes are independent and do not affect each other since the logical-Z and -X information are orthogonally encoded in the case of CSS codes. See Appendix B for the further details.

3. Implications of the C-to-Q decoder

Corollary 5 has two immediate implications. First, if decoders of the CQ codes are suitably chosen, the C-to-Q decoder is nearly optimal. Hence, the C-to-Q decoder combined with a good encoder is capable to achieve the quantum capacity, both entanglement non-assisted [18–20] and assisted ones [21, 22]. In this sense, the C-Q decoder is a capacity-achieving decoder.

To see this, let \( D_{opt} \) be an optimal decoder for \( T = N \circ E \), that is, \( \Delta_q(D_{opt}|T) \leq \Delta_q(D|T) \) for any decoder \( D \). From the optimal decoder \( D_{opt} \), we can trivially construct two decoding measurements \( M_W \) (\( W = E, F \)) for the CQ codes that satisfy \( \Delta_q(M_W|T) \leq \Delta_q(D_{opt}|T) \). Hence, the C-to-Q decoder based on \( M_E \) and \( M_F \) results in

\[
\Delta_q(D_{opt}|T) \leq \Delta_q(D_{CtoQ}|T) \leq (1 + \sqrt{2}) \sqrt{\Delta_q(D_{opt}|T)}.
\]

(40)

That is, the decoding error by the C-to-Q decoder with a proper choice of decoders of the CQ codes is at worst a square root of the optimal error.
The second implication is about encoding operations that achieve capacities of noisy quantum channels. For a given noisy quantum channel \( \mathcal{N}^{B\rightarrow C} \), the largest amount of quantum information that can be reliably transmitted by the channel is called the quantum capacity \( C_q(\mathcal{N}) \) [18–20]. To achieve this, one needs to exploit a good QECC code. Similarly, the largest amount of classical information reliably transmittable by the noisy channel is referred to as the classical capacity \( C_c(\mathcal{N}) \) [49, 50], which is achieved by a good CQ code. It trivially holds that \( C_q(\mathcal{N}) \leq C_c(\mathcal{N}) \) for any noisy channel \( \mathcal{N} \) as classical information is easier to recover than quantum one.

From Corollary 5, we observe that an encoder of the CQ code in the X-basis that achieves the classical capacity cannot be a good encoder of the CQ code in the Z-basis unless \( C_q(\mathcal{N}) = C_c(\mathcal{N}) \). This statement follows by contradiction. Let \( \mathcal{E}_{\text{class}} \) be the encoder of the CQ code in the X-basis that achieves the classical capacity. If \( \mathcal{E}_{\text{class}} \) is also a good encoder of the CQ code in the Z-basis that achieves the classical capacity, we obtain from Corollary 5 that \( C_q(\mathcal{N}) = C_c(\mathcal{N}) \). Hence, unless \( C_q(\mathcal{N}) = C_c(\mathcal{N}) \), a good encoder for a CQ code should be specialized to a fixed basis. This is of theoretical interest since it implies that there may exist a complementary restriction on encoding operations for CQ codes. Note that this argument also applies to the entanglement-assisted capacities [21, 22].

B. Decoding classical information by projection-based PGMs

Using Theorem 4, a problem of constructing a decoder for a general QECC can be reduced to constructing decoders of the corresponding CQ codes. To complete the analysis, we consider the latter problem and provide a sufficient condition for decoding a CQ code. Since we here consider only classical information, we do not explicitly specify the basis and use a simple notation such as \( |j\rangle \).

An example of a decoder for a CQ code is a PGM, which is known to achieve the classical capacity. [49, 50]. Here, we slightly modify the PGM for the sake of simplicity of the analysis. We call the modification projection-based PGM (pPGM), which is basically the same idea as that in [50] and is sometimes referred to as pPGM as well in the literature. Let \( \Pi_j^C \) be a projection onto the support of \( T^{A\rightarrow C}(|j\rangle\langle j|) \) for a given quantum channel \( T^{A\rightarrow C} \). We define a POVM as

\[
M_{\text{pPGM}} = \left\{ \Pi_j^{1/2} \Pi_j \Pi_j^{1/2} \right\},
\]

where \( \Pi := \sum_j \Pi_j \). Note that \( \Pi \) is in general not a projection. Unlike the commonly used PGMs, we use projections \( \Pi_j \) instead of the states themselves.

Following the conventional analysis, a sufficient condition for decoding classical information by pPGM can be obtained. See IV B for the proof.

Proposition 6 (Error on decoding classical information by pPGM). Given quantum channel \( T^{A\rightarrow C} = N^{B\rightarrow C} \circ \mathcal{E}^{A\rightarrow B} \), let \( \tau_{\pi}^C \) and \( \tau_j^C \) \((j = 0, \ldots, d - 1)\) be defined as

\[
\tau_{\pi}^C := T^{A\rightarrow C}(\pi^A), \quad \text{and} \quad \tau_j^C := T^{A\rightarrow C}(|j\rangle\langle j|),
\]

where \( \pi^A \) is the completely mixed state and \( W = \{|j\rangle\rangle_{j=0}^{d-1} \) is a basis. The decoding error of the pPGM \( M_{\text{pPGM}} \) satisfies

\[
\Delta_{d,W}(M_{\text{pPGM}}|T) \leq \frac{1}{d\lambda_{\min}} \sum_{i \neq j} \text{Tr}[\tau_i \tau_j]
\]

\[
= \frac{1}{\lambda_{\min}} \left\{ \frac{d}{2H_2(C)_{\tau\pi}} - \frac{1}{d} \sum_{j=0}^{d-1} \frac{1}{2H_2(C)_{\tau_j}} \right\},
\]

where \( \lambda_{\min} := \min_{\pi \in [0,d-1]} \lambda_{\min}(\tau_{\pi}^C) \) with \( \lambda_{\min}(\sigma) \) being the minimum non-zero eigenvalue of \( \sigma \), and \( H_2(C) \) is the collision entropy of \( \rho \).

Proposition 6 implies that the decoding error of a CQ code by the pPGM is characterized by the minimum non-zero eigenvalue of \( \tau_j \), and the collision entropies of \( \tau_{\pi} \) and \( \tau_j \). Since the collision entropy of \( \tau_j \) is closely related to the minimum non-zero eigenvalues, \( \lambda_{\min} \) and \( H_2(C)_{\tau_j} \) would be the most important quantities. When the minimum non-zero eigenvalue is close to zero, we may exploit the technique of smoothing (see, e.g., [51]), but we leave the explicit use of smoothing as a future problem.

Proposition 6 provides a natural sufficient condition for a CQ code to work. It is observed from Proposition 6 that an encoder \( \mathcal{E} \) of a CQ code shall be good if \( d^{2H_2(C)_{\tau\pi}} \approx \frac{1}{3} \sum_j 2^{-H_2(C)_{\tau_j}} \). This is achieved when there is no collision by \( T = \mathcal{N} \circ \mathcal{E} \) in the sense that each pair \(|j\rangle \) and \(|i\rangle \) \((i \neq j)\) is mapped by the channel \( T \) to a pair of states with negligible overlaps in their supports. Each overlap should be negligibly small compared to \( \lambda_{\min} \). When this is the case, it is naturally expected that the pPGM works well as a decoder of classical information, as explicitly formulated in Proposition 6.

C. Decoding Hayden-Preskill by the C-to-Q decoder

To demonstrate the power of the C-to-Q decoder, we apply it to the Hayden-Preskill protocol [3]. The protocol is proposed in the context of the black hole information paradox and is closely related to random coding against the erasure noise. Hence, it is a good playground to see the advantage of the C-to-Q decoder. We start with an overview of the protocol in III C 1 and informally provide our result in III C 2.

1. Setting of the Hayden-Preskill protocol

The Hayden-Preskill protocol is a qubit-toy model of the black hole information paradox [3] and has been
This shows that the optimal decoding error on average is characterized by the collision entropy $H_2(\xi_{\text{fin}})$ of the initial state $\xi_{\text{fin}}$ of $B_{\text{in}}$. As the entropy ranges from 0 to $N$, the threshold varies from $k$ to $k + N/2$.

Recalling the fact that the error on decoding classical information does not exceed that on decoding quantum information, we also have that there exists a decoding measurement $M_W$ for classical information in the $W$-basis such that

$$\mathbb{E}_{U \sim H}[\Delta_{\text{cl,W}}(M_W|\xi,U)] \leq \mathbb{E}_{U \sim H}[\Delta_q(D|\xi,U)] \leq 2^{(\ell_{\text{th}} - \ell)/2},$$

for any basis $W$.

Although Theorem 7 provides a rigorous upper bound on the decoding error for quantum information, a decoder to achieve the bound was not explicitly given since it was based on the decoupling approach. Later, two decoders were proposed. One is based on the Petz recovery map [54] and the other is given in the form of quantum circuits and works only in a special case [55]. Both decoders turned out to be closely related to physical quantities in the protocol: namely, the former can be interpreted in terms of the spacetime geometry, and the latter in terms of the out-of-time-ordered correlators of the unitary $U$. This indicates that finding a decoder of the protocol is important to connect quantum information-theoretic analysis to the physics of quantum systems.

2. Decoding the Hayden-Preskill protocol by the C-to-Q decoder with pPGMs

We now apply the C-to-Q decoder constructed from pPGMs to the Hayden-Preskill protocol and investigate decoding errors for classical and quantum information. This can be done simply by setting the quantum channel $T$ in the C-to-Q decoder with

$$\mathcal{T}^{A \rightarrow B_{\text{rad}}S_{\text{rad}}} = \text{Tr}_{S_{\text{rad}}} [U^S(\rho_{\text{fin}} \otimes \xi_{B_{\text{in}}B_{\text{rad}}})U^†],$$

where $S = AB_{\text{in}} = S_{\text{in}}S_{\text{rad}}$ (see Fig. 4 (a) as well), and $d = \dim H_A = 2^k$. This map can be decomposed into an
“encoding” map $E^{A \to B_{rad}}$ and a “noisy” map $N^{S \to S_{rad}}$, such that $T^{S \to B_{rad}} = N^{S \to S_{rad}} \circ E^{A \to B_{rad}}$, where

$$
E^{A \to B_{rad}} : \rho^A \mapsto U^S (\rho^A \otimes \xi^{B_{rad}}) U^S, \quad (49)
$$

$$
N^{S \to S_{rad}} = \text{Tr}_{S_{rad}}. \quad (50)
$$

In the following, we consider the CQ code corresponding to the encoding map $E^{A \to B_{rad}}$. For simplicity, we consider only the codes in the Pauli-X and -Z bases, but all the results can be extended to any bases by virtue of the generality of Theorem 4. The decoding measurements and errors are denoted by $M_X$ and $M_Z$, and by $\Delta_{c.l.X}(M_X|U)$ and $\Delta_{c.l.Z}(M_Z|U)$, respectively.

In the Hayden-Preskill protocol, the dynamics $U$ is commonly assumed to be Haar random except a few cases [56]. Before we rely on this assumption, we explicitly write down the condition in Proposition 6 for the Hayden-Preskill protocol. To this end, let us fix one unitary $U^S$ acting on $S$. We use the notation that, for $W = X, Z$,

$$
\xi^{B_{rad}S_{rad}} = T^{A \to B_{rad}S_{rad}}(|jW\rangle \langle jW|)^A, \quad (51)
$$

$$
\xi^{B_{rad}S_{rad}} = T^{A \to B_{rad}S_{rad}}(\pi^A). \quad (52)
$$

See Fig. 4 (b) for the diagrams of these states. Then, a sufficient condition for decoding $W$-classical information by the corresponding pPGM $M_{pPGM,W}$ is given in terms of the minimum non-zero eigenvalue $\lambda_{\text{min}}^{B_{rad}S_{rad}}$ over $j = \{0, \ldots, 2^k - 1\}$, and the collision entropies $H_2(B_{rad}S_{rad}|\xi)$ and $H_2(B_{rad}S_{rad}|\xi_{W,j})$, which are all dependent on $U^S$:

$$
\Delta_{c.l.W}(M_{pPGM,W}|U) \leq \frac{1}{\lambda_{\text{min}}^{B_{rad}S_{rad}}} \left( 2^{k-H_2(B_{rad}S_{rad}|\xi)} - \frac{1}{2^k} \sum_j 2^{-H_2(B_{rad}S_{rad}|\xi_{W,j})} \right). \quad (53)
$$

By combining this with Theorem 4 about the C-to-Q decoder, we obtain a criteria for decoding quantum information. As the collision entropy has gravity interpretation [57], the criteria may provide a new decoding criteria in terms of gravity and space-time geometry.

We now assume that $U^S$ is a Haar random unitary. In this case, we obtain the following result. See Theorem 10 for the formal statement.

**Theorem 8** (Informal statement about decoding the Hayden-Preskill protocol). Suppose that $\text{rank}(\xi^{B_{rad}})\lambda_{\text{min}}(\xi^{B_{rad}}) < 1/2$ and that

$$
(N + k - \ell)2^{-(k+2(\ell-\ell_{th}))} \approx 0, \quad (54)
$$

for sufficiently large $N$. Then, the decoding error for the classical information by the pPGM satisfies

$$
\mathbb{E}_{U \sim \mathcal{H}}[\Delta_{c.l.W}(M_{pPGM}|U)] \lesssim 2^{2(\ell_{th}-\ell)}, \quad (55)
$$

for any basis $W$. The decoding error by the C-to-Q decoder satisfies

$$
\mathbb{E}_{U \sim \mathcal{H}}[\Delta_{c.l.W}(D_{CtoQ}|U)] \lesssim (1 + \sqrt{2})2^{2\ell_{th}-\ell}, \quad (56)
$$

where $\ell_{th} = k + \frac{N-H_2(B_{rad}|\xi)}{2}$.

This result improves the decoding errors in the Hayden-Preskill protocol compared to the previous result, Theorem 7, based on the decoupling approach. As for decoding quantum information, the improvement is by factor 2 in the error exponent, while the threshold $\ell_{th}$ is the same as previous. This should be the case since the threshold corresponds to the quantum capacity of the Hayden-Preskill protocol as already pointed out in [3] and cannot be improved.

On the other hand, as for decoding classical information, the improvement on the error exponent is by factor 4 (see Eq. (47) for the previous bound). Although this is just an upper bound, the following proposition follows due to the nature of the C-to-Q decoder.

**Proposition 9.** In the Hayden-Preskill protocol, the threshold value of $\ell$ for classical information to be decodable from the past and new radiations, $B_{rad}$ and $S_{rad}$ is $\ell_{th} = k + \frac{N-H_2(B_{rad}|\xi)}{2}$.

**Proof.** Suppose that classical information is decodable from qubits fewer than $\ell_{th}$. We can then construct a C-to-Q decoder that is able to decode quantum information from the same number of qubits, but this is not possible since $\ell_{th}$ is the threshold for decoding quantum information. Hence, $\ell_{th}$ is also the threshold for classical information.

Proposition 9 implies that the threshold value of $\ell$ for the classical information to be decodable is exactly the same as that for the quantum information. The difference between the recovery of classical and quantum information in the Hayden-Preskill protocol is only in terms of the error exponent: the recovery error of classical information decreases as $\ell$ increases quadratically faster than that of quantum information. This may be of surprise to some extent since classical information is in general easier to recover than quantum information.

In fact, these results lead to an interesting observation that a quantum black hole with Haar random dynamics is an optimal encoder of quantum information but is a poor encoder of classical information. This is best illustrated when the initial state $\xi$ of the initial black hole $B_{ini}$ is in a pure state, in which case $\ell_{th} = k + N/2$. While this number of qubits is necessary and sufficient for decoding classical information when it is encoded by the black hole dynamics, there exist better encoding schemes by which $k$-bit classical information are decodable merely from a little more than $k$ qubits. Thus, the black hole dynamics is too random to optimally encode classical information.
We finally comment on the two assumptions on Theorem 8. The first one, \( \text{rank}(\xi^{B_n}) \lambda_{\text{min}}(\xi^{B_n}) < 1/2 \), is about the property of the state \( \xi^{B_n} \). This holds especially when all the non-zero eigenvalues of \( \xi^{B_n} \) are nearly equal, which is the case for any pure state or the completely mixed state. In contrast, the second one puts a mild but additional assumption that was not taken in previous studies. For instance, the second assumption is satisfied when \( k > \log N \), or \( k = \text{constant} \) and \( \ell \geq \ell_{\text{th}} + \log N \). We strongly believe that these two assumptions are just for a technical reason and should be removable.

**IV. PROOFS OF THEOREM 4 AND PROPOSITION 6**

**A. Proof of Theorem 4**

We now prove Theorem 4.

**Theorem 4 (Restatement).** Let \( \mathcal{E}^{A \rightarrow B} \) and \( \mathcal{N}^{B \rightarrow C} \) be encoding channels of a QECC and a noisy channel, respectively. For the two bases \( E = \{ |E_i \rangle \} \) and \( F = \{ |F_i \rangle \} \) in a \( d \)-dimensional system \( A \), let \( \Delta_{\text{cl},W} = \Delta_{\text{cl},W}(M_W |N \circ \mathcal{E}) \) be the decoding error of the corresponding CQ code in the \( W \)-basis by the decoder \( M_W \) \( (W = E, F) \). Then, the decoding error of the C-to-Q decoder \( \mathcal{D}_{\text{CtoQ}} \) constructed from \( M_E \) and \( M_F \) satisfies

\[
\Delta_{\text{cl}}(\mathcal{D}_{\text{CtoQ}} |\mathcal{N} \circ \mathcal{E}) \leq \sqrt{\Delta_{\text{cl},E}(2 - \Delta_{\text{cl},E})} + \sqrt{\Delta_{\text{cl},F} + \Xi_{EF}}. \tag{57}
\]

Here, \( \Xi_{EF} \) is given by

\[
\Xi_{EF} := 1 - \sum_{l=0}^{d-1} \text{Tr}[M_{F,i}N \circ \mathcal{E}(\pi^A)] F_{BC}(\text{unif}_d, p_l), \tag{58}
\]

\[
\leq 1 - \min_{l=0, \ldots, d-1} F_{BC}(\text{unif}_d, p_l), \tag{59}
\]

where \( \pi^A \) is the completely mixed state in \( A \), and

\[
F_{BC}(\text{unif}_d, p_l) = \left( \sum_j p_l(j) \right)^2 \tag{60}
\]

is the Bhattacharyya distance between the uniform probability distribution \( \text{unif}_d \) on \([0, d - 1] \) and the probability distribution \( \{ p_l(j) = |\langle j | E_i \rangle|^2 \}_{j=0}^{d-1} \) determined by the two bases \( E \) and \( F \).

In the proof, we denote the output state of the quantum channel \( \mathcal{T}^{A \rightarrow C} \) by \( \Phi^{CR}_T \), i.e.,

\[
\Phi^{CR}_T := \mathcal{T}^{A \rightarrow C}(\Phi^{AR}), \tag{61}
\]

where \( \Phi^{AR} \) is the maximally entangled state between \( A \) and \( R \).

**Proof of Theorem 4.** As given in Eq. (31), the C-to-Q decoder consists of two quantum channels \( \mathcal{R}^{C \rightarrow CA}_E \) and \( \mathcal{Q}^{CA \rightarrow A}_F \) as

\[
\mathcal{D}_{\text{CtoQ}} = \mathcal{Q}^{CA \rightarrow A}_F \circ \mathcal{R}^{C \rightarrow CA}_E. \tag{62}
\]

We show that, when decoding errors for the CQ codes are small and the bases \( E \) and \( F \) are nearly mutually unbiased, these channels can be replaced with different ones acting on different systems, which eventually leads to Theorem 4. See also Fig. 5.

When \( \Delta_{\text{cl},E} \) is small, it holds that

\[
\mathcal{R}^{C \rightarrow CA}_E(\Phi^{CR}_T) \approx \tilde{\mathcal{R}}^{R \rightarrow RA}_E(\Phi^{CR}_T), \tag{63}
\]

where \( \tilde{\mathcal{R}}^{R \rightarrow RA}_E := \sum_j |j_E \rangle |j_E^R \rangle \otimes |j_E \rangle^A \). To see this, notice from Eqs. (26) and (27) that, since \( |e_0 \rangle^{C'} \) lies in the range of the isometry \( V^{C \rightarrow C'}_E \), we have

\[
\langle e_0 |^{C'} \mathcal{R}^{C \rightarrow CC' \cdot A}_E = \sum_{j=0}^{d-1} V^{C \rightarrow C' \cdot A}_E(j) \cdot P^{C'}_j |e_0 \rangle^{C'} \otimes |j_E \rangle^A, \tag{64}
\]

\[
= \sum_{j=0}^{d-1} M^{C,j}_E \otimes |j_E \rangle^A, \tag{65}
\]

which leads to

\[
\hat{R}^{C \rightarrow CC' \cdot A}_E \left( \tilde{\mathcal{R}}^{R \rightarrow RA}_E \otimes |e_0 \rangle^{C'} \right) \]

\[
= \left( \sum_j M^{C,j}_E \otimes \langle j_E |^A \right) \left( \sum_i |i_E \rangle^R \otimes |i_E \rangle^A \right), \tag{66}
\]

\[
= \sum_j M^{C,j}_E \otimes |j_E \rangle^R. \tag{67}
\]

Note that, if we use an isometry \( \hat{R}^{C \rightarrow CA}_E := \sum_{j=0}^{d-1} M^{C,j}_E \otimes |j_E \rangle^A + \sqrt{1 - \sum_{j=0}^{d-1} |M^{C,j}_E|^2} \otimes |\text{fail} \rangle^A \), a similar relation holds. That is,

\[
\hat{R}^{C \rightarrow CA \cdot A}_E \left( \tilde{\mathcal{R}}^{R \rightarrow RA}_E \right) \]

\[
= \sum_j M^{C,j}_E \otimes |j_E \rangle^R, \tag{68}
\]

on which the following argument is based. Hence, the isometry \( \hat{R}^{C \rightarrow CA}_E \) can be used instead of the quantum channel \( \mathcal{R}^{C \rightarrow CA}_E \) in the C-to-Q decoder.

Using a purification \( |\Phi_T \rangle^{CRS} \) of \( \Phi^{CR}_T \) by a system \( S \), we obtain
where the first line follows from the monotonicity of the trace norm, the second from the fact that both states are pure, and the third from Eq. (67).

Since $\langle j_E^r|\rho_{F|^C}|j_E^r\rangle = d^{-1}T^{A\rightarrow C}(|j_E^r\rangle\langle j_E^r|)$, it follows that

$$\text{Tr}\left[\Phi_T^{CR}\left(\sum_j M_{E,j}^C \otimes |j_E^r\rangle\langle j_E^r|\right)\right] = \frac{1}{d} \sum_j \text{Tr}\left[TA\rightarrow C(|j_E^r\rangle\langle j_E^r|)M_{E,j}^C\right],$$

(73)

which is nothing but $1 - \Delta_{cl,E}$. Thus, we arrive at

$$\left\|R_{E}^{C\rightarrow CA}(\Phi_T^{CR}) - \tilde{R}_{E}^{R\rightarrow RA}(\Phi_T^{CR})\right\|_1 \leq \sqrt{\Delta_{cl,E}(2 - \Delta_{cl,E})},$$

(74)

which implies Eq. (63) when $\Delta_{cl,E}$ is small.

We next show that, when $\Delta_{cl,E}$ is small, $Q_{F}^{CA\rightarrow A} \circ \tilde{R}_{E}^{R\rightarrow RA}(\Phi_T^{CR}) \approx \Psi^{AR}$, where

$$\Psi^{AR} := \sum_{l} q(l)|\Psi_l\rangle|\Psi_l\rangle^{AR},$$

(75)

with $q(l) = \text{Tr}[T^{A\rightarrow C}(\pi_A)M_{F,l}^C]$, and $|\Psi_l\rangle^{AR} := \sum_{j} |j_E^l\rangle\langle j_E^l| \otimes |j_E^l\rangle^{R}$. To this end, we decompose the quantum channel $Q_{F}^{CA\rightarrow A}$ as $Q_{F}^{CA\rightarrow A} = \sum_{l} Q_{F,l}^{CA\rightarrow A}$, where $Q_{F,l}^{CA\rightarrow A}$ is trace non-increasing CP map given by

$$Q_{F,l}^{CA\rightarrow A}(\rho^{CA}) := \Theta_{l}^{A} \text{Tr}_{C}[\rho^{CA}M_{F,l}^C] \Theta_{l}^{A\dagger}. $$

(76)

Note that

$$\text{Tr}[Q_{F,l}^{CA\rightarrow A} \circ \tilde{R}_{E}^{R\rightarrow RA}(\Phi_T^{CR})] = \text{Tr}[M_{F,l}^C \tilde{R}_{E}^{R\rightarrow RA}(\Phi_T^{CR})],$$

(77)

$$= \text{Tr}[M_{F,l}^C \Phi_T^{CR}],$$

(78)

$$= \text{Tr}[M_{F,l}^C T^{A\rightarrow C}(\pi_A)],$$

(79)

$$= q(l),$$

(80)

where the first line follows from the fact that $\Theta_{l}$ is unitary, the second from $\tilde{R}_{E}^{R\rightarrow RA}$ is isometry, and the third from the fact that $T^{A\rightarrow C}$ acts only onto $A$. We then have
\[
\|Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) - \Psi_{AR}\|_1 \leq 2 \sum_l q(l) \|q(l)^{-1} Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) - \Psi_{AR}^I\|_1
\]

\[
\leq 2 \sum_l q(l) \sqrt{1 - q(l)^{-1} \langle \Psi_{I}^{AR} Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) \rangle}\| \Psi_{I}^{AR}\|_1
\]

\[
\leq 2 \sum_l q(l) \left(1 - q(l)^{-1} \langle \Psi_{I}^{AR} Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) \rangle\right)
\]

\[
= 2 \sqrt{1 - \sum_l \langle \Psi_{I}^{AR} Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) \rangle\| \Psi_{I}^{AR}\|_1},
\]

where we used the definition of \(\Theta_l\), given by Eq. (30), to obtain the second equality. We hence have

\[
\langle \Psi_{I}^{AR} Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) \rangle = |l_F^* \rangle \langle l_F| \otimes M_{F,t}^C.
\]

As \(|l_F^* \rangle \langle l_F| \otimes M_{F,t}^C\rangle R = T^{A \to C}(|l_F^* \rangle \langle l_F| \otimes M_{F,t}^C\rangle, we arrive at

\[
\langle \Psi_{I}^{AR} Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) \rangle = \frac{1}{d} \text{Tr}[T^{A \to C}(|l_F^* \rangle \langle l_F| \otimes M_{F,t}^C\rangle].
\]

Substituting this into Eq. (84), we have

\[
\|Q_{F,I}^{CA} \circ \hat{R}_E^{R \to RA}(\Phi_{CR}^T) - \Psi_{AR}\|_1 \leq 2 \sqrt{1 - \sum_l \text{Tr}[T^{A \to C}(|l_F^* \rangle \langle l_F| \otimes M_{F,t}^C\rangle]}
\]

\[
\leq 2 \sqrt{\Delta_{c.l,F}^R}.
\]

Finally, we show that \(\Psi_{AR} \approx \Phi_{AR}\) if \((E,F)\) is close to MUBs. We start with

\[
\|\Psi_{AR} - \Phi_{AR}\|_1 \leq 2 \sqrt{1 - \langle \Phi|\Psi\rangle}.
\]

It is easily shown that

\[
\langle \Phi|\Psi\rangle = \sum_l q(l)|\langle \Psi_l|\Phi\rangle|^2
\]

\[
= \sum_l q(l)d^{-1/2} \sum_j |\langle j_E|l_F\rangle|^2
\]

\[
= \sum_l q(l)M_{B,C}(\text{unif}_d, p_l),
\]

where \(p_l(j) = |\langle j_E|l_F\rangle|^2\). Recalling that \(q(l) = \text{Tr}[T^{A \to C}(\pi^A)M_{F,t}^C]\), we obtain

\[
\|\Psi_{AR} - \Phi_{AR}\|_1 \leq 2 \sqrt{\Delta_{EF}}.
\]

The statement of Theorem 4 follows simply from the triangle inequality.
where the second inequality follows from the monotonicity of the trace distance and the last one from Eqs. (74), (97), and (102).

B. Proof of Proposition 6

We show Proposition 6. For clarity, we restate it here.

**Proposition 6** (Restatement: decoding error for classical information by pPGMs). Given quantum channel $T^{A\rightarrow C}$, let $\tau^C_j$ and $\tau_j$ ($j = 0, \ldots, d - 1$) be defined as

$$\tau^C_j := T^{A\rightarrow C}(\pi^A), \quad \text{and} \quad \tau_j := T^{A\rightarrow C}(|j\rangle\langle j|^A),$$

where $\pi^A$ is the completely mixed state and $W = \{|j\rangle\}_{j=0}^{d-1}$ is a basis. The decoding error of the pPGM $M_{pPGM}$ satisfies

$$\Delta_{cl, W}(M_{pPGM}|T) \leq \frac{1}{d} \sum_{i\neq j} Tr[\tau_i \tau_j]$$

where

$$\Delta_{cl, W}(M_{pPGM}|T) = \frac{1}{d} \sum_{i \neq j} Tr[\tau_i \tau_j],$$

$$\Delta_{cl, W}(M_{pPGM}|T) \leq \frac{1}{d} \sum_{i \neq j} \left( \frac{d}{2} H_2(\tau_i) + \frac{1}{d} \sum_{j=0}^{d-1} \frac{1}{2} H_2(\tau_j) \right),$$

where $\lambda_{min} := \min_{j \in [0,d-1]} \lambda_{min}(\sigma^C_j)$ with $\lambda_{min}(\sigma)$ being the minimum non-zero eigenvalue of $\sigma$, and $H_2(C)_\rho$ is the collision entropy of $\rho$.

In the proof, we do not explicitly write the superscript, such as $A$ and $C$, nor the basis $W$ in $\Delta_{cl, W}$. The pPGM is given by

$$M_{pPGM} = \{ M_j := \Pi^{-1/2} \Pi_j \Pi^{-1/2} \},$$

where $\Pi_j$ is a projection onto the support of $\tau^C_j$ and $\Pi := \sum_j \Pi_j$.

**Proof of Proposition 6.** The error in decoding classical information is rewritten as

$$\Delta_{cl}(M_{pPGM}|T) = \frac{1}{d} \sum_{j} Tr[(I - M_j)\tau_j]$$

In Appendix D, we follow the original analysis by Holevo [50] and by Schumacher and Westmoreland [49] and show that

$$\Delta_{cl}(M_{pPGM}) \leq \frac{1}{d} \sum_{i \neq j} Tr[\Pi_i \tau_j].$$

Since $\lambda_{min}(\tau_i) \Pi_i \leq \tau_i$, where $\lambda_{min}(\tau_i)$ is the minimum non-zero eigenvalue of $\tau_i$, we have

$$\Delta_{cl}(M_{pPGM}|T) \leq \frac{1}{d} \sum_{i \neq j} \frac{Tr[\tau_i \tau_j]}{\lambda_{min}(\tau_i)}$$

where

$$\lambda_{min} = \min_{j \in [0,d-1]} \lambda_{min}(\sigma^C_j)$$

and we used $\tau_d = d^{-1} \sum_j \tau_j$. This completes the proof.

V. DECODING ERRORS IN THE HAYDEN-PRESKILL PROTOCOL

We provide an in-depth analysis of decoding the Hayden-Preskill protocol by the C-to-Q decoder with pPGMs. To this end, we introduce a family of the Hayden-Preskill protocol in V A, which is to fully understand different asymptotic limits of the protocol, and provide the formal statement of Theorem 8. We then provide a single-shot analysis in Sec. VB, which is the most common form of the protocol. Based on these, we provide a proof of the formal statement in VC.

A. A family of the Hayden-Preskill protocol

To fully understand the protocol, we introduce a sequence of Hayden-Preskill protocols labelled by $n =$
1, 2, \ldots. This allows us to argue different asymptotic limits simultaneously. Note that the Hayden-Preskill protocol has multiple independent parameters, such as $k, N, \ell,$ and $\xi,$ and hence, various asymptotic limits can be considered. We denote the parameters for the $n$-th protocol by $(N_n, k_n, \ell_n, \xi_n)$ and introduce

$$\ell_{th.n} = k_n + \frac{N_n - H_2(B_{n})\xi_n}{2},$$

$$\Lambda_{\xi_n} = \text{rank}(\xi_n^{B_n})\lambda_{\min}(\xi_n^{B_n}) \in (0, 1].$$

From the previous result based on the decoupling approach (Theorem 7), it immediately follows that, for every $n,$ there exist a decoding map $D_n$ and decoding POVMs $M_{W_n}$ for any basis $W$ such that

$$\mathbb{E}_{U_n \sim H_n}[\Delta_{cl,W}(M_{W_n}|\xi_n, U_n)] \leq \mathbb{E}_{U_n \sim H_n}[\Delta_q(D_n|\xi_n, U_n)] \leq 2^{(\ell_{th.n} - \ell_n)/2}. \quad (118)$$

In the $n$-th protocol, we consider decoding $W$-classical information by pPGMs $M_{pPGM.W_n}$ and quantum information by the C-to-Q decoder $D_{CtoQ,n}$ constructed from the pPGM for the $Z$-classical information and that for the $X$-classical information. These decoders are all dependent on $\xi_n$ as well as $U_n.$ The decoding errors are denoted by $\Delta_{cl,W}(M_{pPGM.W_n}|\xi_n, U_n)$ for $W$-classical information and by $\Delta_q(D_{CtoQ,n}|\xi_n, U_n)$ for quantum information. For simplicity, we hereafter omit the basis $W$ in the subscript of the pPGM $M_{pPGM.W_n}$ when we refer to the decoding error $\Delta_{cl,W}.$ that is, we denote the error by $\Delta_{cl,W}(M_{pPGM.n}|\xi_n, U_n).$

Taking the average over a Haar random unitary $U_n \sim H_n,$ we define decoding errors on average:

$$\bar{\Delta}_{cl,n} = \mathbb{E}_{U_n \sim H_n}[\Delta_{cl,W}(M_{pPGM.n}|\xi_n, U_n)],$$

$$\bar{\Delta}_q,n = \mathbb{E}_{U_n \sim H_n}[\Delta_q(D_{CtoQ,n}|\xi_n, U_n)]. \quad (120)$$

Note that, due to the unitary invariance of the Haar measure, the average decoding error $\bar{\Delta}_{cl,n}$ for $W$-classical information does not depend on the choice of the basis.

The main result about the Hayden-Preskill protocol is given by the following:

**Theorem 10** (Decoding errors on the Hayden-Preskill protocol). For any Hayden-Preskill protocols that satisfies $\epsilon := \lim sup_{n \to \infty} 2(1 - \Lambda_{\xi_n}) < 1$ and

$$\lim_{n \to \infty} (N_n + k_n - \ell_n)2^{-(k_n+2(\ell_{th.n} - \ell_n))} = 0, \quad (121)$$

it holds that

$$\lim_{n \to \infty} \left(\bar{\Delta}_{cl,n} \left(\frac{2^{(\ell_{th.n} - \ell_n)}}{1 - \epsilon}\right)^{-1}\right) \leq 1, \quad (122)$$

and

$$\lim_{n \to \infty} \left(\bar{\Delta}_q,n \left(\frac{2^{(\ell_{th.n} - \ell_n)}}{1 - \epsilon}\right)^{-1}\right) \leq 1 + \sqrt{2}. \quad (123)$$

This can be obtained by the two steps. We provide a single-shot analysis in \textsc{V B} and prove Theorem 10 in \textsc{V C}.

**B. A single-shot analysis on the decoding errors**

In this subsection, we provide a proposition about the decoding errors in the Hayden-Preskill protocol for a fixed $n.$ We hence omit the labeling $n$ of the family.

**Proposition 11.** Suppose that $\Lambda_{\xi} := \text{rank}(\xi^{B_n})\lambda_{\min}(\xi^{B_n}) > 1/2$, where $\lambda_{\min}$ denotes the minimum non-zero eigenvalue. For any constants $\epsilon \in (2(1 - \Lambda_{\xi}), 1],$ it holds for any basis $W$ that

$$\mathbb{E}_{U \sim H}[\Delta_{cl,W}(M_{pPGM.W}|\xi, U)] \leq \frac{4^{\epsilon_n - \ell}}{1 - \epsilon} + \delta, \quad (124)$$

$$\mathbb{E}_{U \sim H}[\Delta_q(D_{CtoQ}|\xi, U)] \leq (1 + \sqrt{2})\frac{4^{\epsilon_n - \ell}}{1 - \epsilon} + \delta, \quad (125)$$

where $\ell_{th} = k + \frac{N - H_2(B_{n})\lambda_{\min}}{2},$ $\Lambda_{\xi}$ is given by

$$\log \delta := k + 2^{N+k-\ell+1} \left( N + k - \ell + \log \frac{5}{\epsilon} \right) - \frac{c^2 \log 2}{6} 2^{\epsilon + H_2(B_{n})\xi}, \quad (126)$$

and $c = 1 - (1 - \epsilon/2)/\Lambda_{\xi}.$

To show Proposition 11, we first show a general property about the minimum eigenvalue of a marginal state after the application of a Haar random unitary. In Proposition 12, the lageling of the systems, $A$ and $B,$ are general and not those in the Hayden-Preskill protocol.

**Proposition 12.** Let $AB$ be a composite system, and $d_A$ and $db$ be the dimensions of $A$ and $B,$ respectively. For a mixed state $\rho^{AB}$ with rank $r$ and a Haar random unitary $U^{AB},$ let $\rho^{\prime B}$ be $\text{Tr}_A[U^{AB}(\rho^{AB})].$ For any $\epsilon > 2(1 - r\lambda_{\min}(\rho^{AB})),$

$$\text{Prob}_{U \sim H} \left[ \lambda_{\min}(\rho^{\prime B}) < \frac{1 - \epsilon}{d_B} \right] \leq \frac{(5d_B)}{\epsilon} 2^{2d_B} \exp \left[ - \frac{rd_A \delta (\epsilon/2)^2}{6} \right]. \quad (127)$$

Here, $\delta (\epsilon) := 1 - \frac{1 - \epsilon}{r\lambda_{\min}(\rho^{AB})}.$

Proposition 12 is a characterization of Lemma III.4 in Ref. [58] to the case of mixed states and can be shown by the same proof technique using the $\epsilon$-net [59].

**Definition 13 ($\epsilon$-net).** For $\epsilon > 0,$ a set $N(\epsilon)$ of pure states in a Hilbert space $\mathcal{H}$ is called an $\epsilon$-net if

$$\forall \varphi \in \mathcal{H}, \; \exists \varphi' \in N(\epsilon), \; \text{such that} \; ||\varphi - \varphi'|| \leq \epsilon. \quad (128)$$

It is known that there exists an $\epsilon$-net with a sufficiently many but finite number of pure states.

**Theorem 14.** There exists an $\epsilon$-net $N(\epsilon)$ such that $|N(\epsilon)| \leq (5/\epsilon)^{2d},$ where $d := \dim \mathcal{H}.$
We also use a concentration of measure phenomena for a Haar random unitary.

**Lemma 15** (Lemma III.5 in [58]). For projections $S, Q$ on a $d$-dimensional Hilbert space $H$ and $\epsilon > 0$, it follows that

$$\text{Prob}_{U \sim H} \left[ \text{Tr}[USU^T Q] < (1 - \epsilon) \frac{sq}{d} \right] \leq \exp \left[ - \frac{s q^2}{6} \right],$$

(129)

where $s$ and $q$ are the rank of $S$ and $Q$, respectively.

Based on Theorem 14 and Lemma 15, we show Proposition 12. The proof is almost as the same in [60].

**Proof of Proposition 12.** By definition, the minimum non-zero eigenvalue is given by $\lambda_{\min}(\rho^B) = \min_{|\phi\rangle \in \mathcal{P}_B} \langle \phi | \rho^B | \phi \rangle$, where $\min_{|\phi\rangle \in \mathcal{P}_B}$ represents the minimization over all pure states $|\phi\rangle^B$ in the support of $\rho^B$. From Theorem 14, there exists an $\epsilon/d$-net $\mathcal{N}(\epsilon/d_B)$ of the Hilbert space $H^B$ with $|\mathcal{N}(\epsilon/d_B)| \leq (5d_B/\epsilon)^{2d_B}$. Hence, we have

$$\lambda_{\min}(\rho^B) \geq \min_{|\phi\rangle \in \mathcal{N}(\epsilon/d_B)} \langle \phi | \rho^B | \phi \rangle - \frac{\epsilon}{2d_B}. \tag{130}$$

Here, we used the fact that, if $||\psi\rangle \langle \psi|| - ||\psi'\rangle \langle \psi'||_1 \leq \epsilon$, then $||\langle \phi | \sigma | \psi \rangle - \langle \psi' | \sigma | \psi' \rangle|| \leq \epsilon/2$ for any state $\sigma$. This implies

$$\text{Prob}_{U \sim H} \left[ \lambda_{\min}(\rho^B) < \frac{1 - \epsilon}{d_B} \right] \leq \text{Prob}_{U \sim H} \left[ \min_{|\phi\rangle \in \mathcal{N}(\epsilon/d_B)} \langle \phi | \rho^B | \phi \rangle < \frac{1 - \epsilon/2}{d_B} \right]. \tag{131}$$

Decomposing $\rho^{AB}$ into $\sum_{j=1}^r \lambda_j |\psi_j\rangle \langle \psi_j|^{AB}$, where $|\psi_j\rangle$ are eigenstates and $\lambda_j$ are non-zero eigenvalues, and using the projection $R^{AB} = \sum_{j=1}^r |\psi_j\rangle \langle \psi_j|^{AB}$, it holds for any $|\phi\rangle \in H^B$ that

$$\langle \phi | \rho^B | \phi \rangle = \sum_{j=1}^r \lambda_j \text{Tr}[(I^A \otimes |\phi\rangle \langle \phi|)U^{AB} |\psi_j\rangle \langle \psi_j|^{AB}(U^{AB})^\dagger] \geq \lambda_{\min}(\rho^{AB}) \text{Tr}[(I^A \otimes |\phi\rangle \langle \phi|)U^{AB} R^{AB}(U^{AB})^\dagger]. \tag{132}$$

Hence, for any $|\phi\rangle^B$ and $\epsilon > 2(1 - r \lambda_{\min}(\rho^{AB}))$,

$$\text{Prob}_{U \sim H} \left[ \langle \phi | \rho^B | \phi \rangle < \frac{1 - \epsilon/2}{d_B} \right] \leq \text{Prob}_{U \sim H} \left[ \text{Tr}[(I^A \otimes |\phi\rangle \langle \phi|)U^{AB} R^{AB}(U^{AB})^\dagger] < (1 - \delta_\rho(\epsilon/2)) \frac{r}{d_B} \right] \leq \exp \left[ - \frac{dr \delta_\rho(\epsilon/2)^2}{6} \right], \tag{133}$$

(134)

where $\delta_\rho(x) := 1 - (1 - x)/(r \lambda_{\min}(\rho^{AB}))$ and the last line follows from Lemma 15.

From Eqs. (131) and (135), and using the union bound, we obtain

$$\text{Prob}_{H} \left[ \lambda_{\min}(\rho^B) < \frac{1 - \epsilon}{d_B} \right] \leq \exp \left[ - \frac{dA r \delta_\rho(\epsilon/2)^2}{6} \right], \tag{136}$$

for any $\epsilon > 2(1 - r \lambda_{\min}(\rho^{AB}))$, which concludes the proof. \hfill \square

**Proof of Theorem 11.** For simplicity, we introduce a notation for the state after the application of a unitary $U^S$, which acts on $S = AB_{in} = S_{in}S_{rad}$, when the initial state in $A$ was one of the $W$-basis $|j\rangle^A$:

$$\xi_j^{BradS}(U) = U^S(|j\rangle^A \otimes \xi_{BradS}^{in}). \tag{137}$$

Since $\xi_{BradS}^{in}$ is a pure state, so is $\xi_j^{BradS}(U)$.

Using Proposition 6, the decoding error for the W-classical information, when it is decoded by the corresponding pPGM, is given by

$$\Delta_{cl,W}(M_{pPGM,W} | \xi, U) \leq \frac{1}{2^k} \lambda_{\min} \sum_{i \neq j} \text{Tr}[(\xi_j^{BradS}(U) \xi_j^{BradS}(U))] \leq \frac{1}{2^k} \lambda_{\min} \lambda_{\min}(\xi_j^{BradS}(U)) \leq \frac{1}{2^k} \lambda_{\min}(\xi_j^{BradS}(U)). \tag{139}$$

and

$$\lambda_{\min}(\xi_j^{BradS}(U)) = \min_j \lambda_{\min}(\xi_j^{BradS}(U)). \tag{140}$$

with $\lambda_{\min}(\sigma)$ being the minimum non-zero eigenvalue of $\sigma$. The second equality holds since $\xi_j^{BradS}$ is a pure state.

To investigate $\lambda_{\min}(\xi_j^{BradS}(U))$, we apply Proposition 12 with the following identification: $A$ and $B$ in Proposition 12 corresponds to $S_{rad}$ and $S_{in}$, respectively. Then, for any $\epsilon$ such that $\epsilon \geq 2(1 - r \lambda_{\min}(\xi_{in}))$, where $r = \text{rank}(|j\rangle^j)_{j=1}^k = \text{rank}(\xi_{in})$, we have $\lambda_{\min}(\xi_j^{BradS}(U)) \leq 1 - \frac{1}{2^{k+1}}$ with probability at most

$$P_\xi(\epsilon) := \left( \frac{5 \cdot 2^{N+k-k-\epsilon}}{\epsilon} \right) \exp \left[ - \frac{2^k \epsilon c^2}{6} \right]. \tag{141}$$

For $\epsilon$ such that $\epsilon \geq 2(1 - r \lambda_{\min}(\xi_{in}))$, we define $\Upsilon_\epsilon$ as

$$\Upsilon_\epsilon := \left\{ U \in \Omega(2^N) : \lambda_{\min} \geq \frac{1 - \epsilon}{2^{N+k-\epsilon}} \right\}. \tag{142}$$

Note that $\lambda_{\min}$ depends on $U$ (see Eq. (140)). Using Eq. (141) and the union bound over the choice of $j \in \{0, \ldots, 2^k - 1\}$, the set $\Upsilon_\epsilon$ satisfies

$$\Pr(\Upsilon_\epsilon) \geq 1 - 2^k P_\xi(\epsilon). \tag{143}$$
We denote by $\mathcal{H}_e$ and $\tilde{\mathcal{H}}_e$ the probability measures induced from the Haar measure $\mathcal{H}$ by restriction to $\mathcal{U}_e$ and to the complementary set of $\mathcal{U}_e$ in $\mathcal{U}_e$, respectively. Using $E_{U \sim \mathcal{H}} = \mathcal{H}(\mathcal{U}_e)E_{U \sim \mathcal{H}_e} + (1 - \mathcal{H}(\mathcal{U}_e))E_{U \sim \tilde{\mathcal{H}}_e}$, we have

$$E_{U \sim \mathcal{H}}[\Delta_{cl,W}(M_{\text{pPGM}},W|\xi,U)] \leq \mathcal{H}(\mathcal{U}_e)E_{U \sim \mathcal{H}_e}[\Delta_{cl,W}(M_{\text{pPGM}},W|\xi,U)] + 2^k P_\xi(\epsilon),$$

where we have used Eq. (143) and that $0 \leq \Delta_{cl}(M_{\text{pPGM}}|\xi,U) \leq 1$. Further using Eqs. (138) and (142), we have

$$H(\mathcal{U}_e)E_{U \sim \mathcal{H}_e}[\Delta_{cl,W}(M_{\text{pPGM}},W|\xi,U)] \leq \frac{2N-\ell}{1-\epsilon} H(\mathcal{U}_e)E_{U \sim \mathcal{H}_e}\left[\sum_{i\neq j} \text{Tr}[\xi_i^{B_{\text{rad}}S_{\text{rad}}}(U)\xi_j^{B_{\text{rad}}S_{\text{rad}}}(U)]\right]$$

$$\leq \frac{2N-\ell}{1-\epsilon} E, \quad (145)$$

$$\leq \frac{2N-\ell}{1-\epsilon} E, \quad (146)$$

where we defined

$$E := E_{U \sim \mathcal{H}}\left[\sum_{i\neq j} \text{Tr}[\xi_i^{B_{\text{rad}}S_{\text{rad}}}(U)\xi_j^{B_{\text{rad}}S_{\text{rad}}}(U)]\right]. \quad (147)$$

Note that Eq. (145) follows from the fact that $\lambda_{\text{min}} \geq (1-\epsilon)/2^{N+k+1-\ell}$ in $\mathcal{U}_e$, on which the probability measure $\mathcal{H}_e$ is defined. We, hence, arrive at

$$E_{U \sim \mathcal{H}}[\Delta_{cl,W}(M_{\text{pPGM}},W|\xi,U)] \leq \frac{2N-\ell}{1-\epsilon} E + 2^k P_\xi(\epsilon),$$

for $\epsilon > 2(1 - r \lambda_{\text{min}}(\xi^{B_{\text{in}}}))$. In the following, we focus on $E$, which can be computed using the swap trick. We use the notation that, for a given system $X$, $X'$ denotes a copied system isomorphic to $X$. We also use the swap operator $\Xi^{X,X'} = \sum_{i,j} \langle e_i|e_j\rangle^{X}\otimes\langle e_j|e_i\rangle^{X'}$, where $\{|e_i\rangle\}$ is an orthonormal basis in $X$. A well-known property of the swap operator is that, for two operators $\omega$ and $\gamma$ on $X$, we have $\text{Tr}[\omega\gamma] = \text{Tr}[\omega^{X}\otimes\gamma^{X'}]$. Using the swap trick in $B_{\text{rad}}S_{\text{rad}}$, which together we denote by $rad$, we have the following identity relation:

$$\text{Tr}[\xi^{rad}(U)\xi^{rad}(U)] = \text{Tr}[\xi^{rad}(U)\otimes\xi^{rad}(U)]^{\text{rad},\text{rad}'}$$

$$= \text{Tr}[\xi^{SB_{\text{rad}}}(U)\otimes\xi^{SB_{\text{rad}}}(U)](\Xi^{S^{\text{rad}},S^{\text{rad}'}}), \quad (149)$$

$$= \text{Tr}[\xi^{SB_{\text{rad}}}(U)\otimes\xi^{SB_{\text{rad}}}(U)](\Xi^{S^{\text{rad}},S^{\text{rad}'}}), \quad (150)$$

where $\xi^{SB_{\text{rad}}}(U)$ and $\xi^{SB_{\text{rad}}}(U)$ are the pure states defined as $|\xi_i^{SB_{\text{rad}}}| = U^{S}(i^{A}\otimes\xi^{B_{\text{in}}})$ and so on. Using this, we have

$$E = \sum_{i\neq j} \text{Tr}[O_{ij}(\Xi^{S^{\text{rad}},S^{\text{rad}'}})], \quad (151)$$

where we defined an average operator $O_{ij}$ on $SS'B_{\text{rad}}B_{\text{rad}}'$ as

$$O_{ij} := E_{U \sim \mathcal{H}}[\xi_i^{SB_{\text{rad}}}(U)\otimes\xi_j^{SB_{\text{rad}}}(U)] \quad (152)$$

for $i \neq j$.

To compute $O_{ij}$, let

$$|\xi_i^{B_{\text{in}}B_{\text{rad}}} = \sum_m \sqrt{\xi_i^{m}}|\varphi_m^{B_{\text{in}}}\otimes|\psi_m^{B_{\text{rad}}},$$

be the Schmidt decomposition of the pure state $|\xi_i^{B_{\text{in}}}|$. We introduce

$$O_{ij}^{m'n'} := E_{U \sim \mathcal{H}}[(U^S)^{\otimes 2}(|j\rangle\langle j'|\otimes|i\rangle\langle i'|)\otimes|\varphi_m^{B_{\text{in}}}\otimes|\varphi_{n'}^{B_{\text{rad}}}\otimes|\psi_n^{B_{\text{in}}}\otimes|\psi_{m'}^{B_{\text{rad}}}, \quad (154)$$

for $i \neq j$. This is related to $O_{ij}$ as

$$O_{ij}^{m'n'} \otimes|\psi_{m'}^{B_{\text{rad}}}\otimes|\psi_{n}^{B_{\text{rad}}}.$$

The operator $O_{ij}^{m'n'}$ commutes with any unitary in the form of $V^{S} \otimes V^{S'}$ due to the unitary invariance of the Haar measure $\mathcal{H}$. From the Schur’s lemma, this implies that the operator is a linear combination of the projection onto the symmetric subspace and that onto the antisymmetric subspace. The projections to the former and the latter subspaces are given by $\frac{1}{S^S' + S^{SS'}}$ and $\frac{S^S' - S^{SS'}}{S^S' + S^{SS'}}$, respectively. Further using $\text{Tr}[O_{ij}^{m'n'}] = \delta_{m'n'}$, and $\text{Tr}[O_{ij}^{m'n'}|\varphi^S] = \delta_{m'n'} = 0$, we have

$$O_{ij}^{m'n'} = \frac{(d_S\Xi^{SS'})^2 - (\Xi^{SS'})^2}{d_S(d_S^2 - 1)} \delta_{m'n'}, \quad (156)$$

for $i \neq j$, where $d_S = 2^{N+k}$ is the dimension of the system $S$.

Substituting Eqs. (155) and (156) into Eq. (151), it follows that

$$E = 2^k(2^{k+1}) - \frac{2^{2(N+k)-\ell} - 2^\ell}{2^2(N+k) - 1} \text{Tr}[\xi^{rad}(U)^{2}], \quad (157)$$

Note that, since $\xi^{B_{\text{in}}B_{\text{rad}}}$ is a pure state, $\text{Tr}[\xi^{rad}(U)^{2}] = \text{Tr}[\xi^{rad}(U)^{2}] = 2^{-H_2(B_{\text{in}})}$, where $H_2(B_{\text{in}}) = -\log \text{Tr}[\xi^{\text{in}}]^{2}$ is the collision entropy of the state $\xi^{B_{\text{in}}}$. Hence, Eq. (148) is simplified as

$$E_{U \sim \mathcal{H}}[\Delta_{cl,W}(M_{\text{pPGM}},W|\xi,U)] \leq \frac{2^2(\ell_{th} - \ell)}{1-\epsilon}(1 - \frac{1}{2^k}) \left(1 \right. - 2^{2(N+k-\ell)} + 2^k P_\xi(\epsilon), \quad (158)$$

$$\leq \frac{2^2(\ell_{th} - \ell)}{1-\epsilon} + 2^k P_\xi(\epsilon), \quad (159)$$

where $\ell_{th} = k + \frac{N-H_2(B_{\text{in}})}{2}$.
We finally evaluate the second term $2^k P_\xi(\epsilon)$. We have
\[
\log[2^k P_\xi(\epsilon)] = k + 2^{N+k-\ell+1} \left(N + k - \ell + \log \frac{5}{\epsilon}\right) - \frac{c^2 r^2}{6} \log \epsilon.
\]
Using $\log r \geq H_2(B_m)\xi$, it holds that
\[
\log[2^k P_\xi(\epsilon)] \leq k + 2^{N+k-\ell+1} \left(N + k - \ell + \log \frac{5}{\epsilon}\right) - \frac{c^2 \log e^{\ell + H_2(B_m)\xi}}{6}.
\]
The right-hand side is exactly the form of $\log \delta$ in the statement. We hence obtain the desired conclusion:
\[
\mathbb{E}_{U \sim \mathcal{H}}[\Delta_{cl}W(M_{P_{GM}}W[\xi, U]) \leq \frac{2^{2(\ell_n - \ell)}}{1 - \epsilon} + \delta.
\]
The second statement about the average error $\mathbb{E}_{U \sim \mathcal{H}}[\Delta_{N}[D_{CtoQ}[\xi, U]]$ on decoding quantum information simply follows by combining the first statement with Corollary 5.

**C. Proof of Theorem 10**

Theorem 10 can be proved based on Proposition 11.

**Proof of Theorem 10.** Since the decoding errors are trivially zero when $N_n + k_n - \ell_n = 0$, we below consider the case when $N_n + k_n - \ell_n \geq 1$. In this case, Eq. (121) is rewritten as
\[
\lim_{n \to \infty} \left(k_n + 2(\ell_n - \ell_{th,n}) - \log(N_n + k_n - \ell_n)\right) = \infty.
\]
There exists an integer $n_0$ such that $2(1 - \Lambda_{\xi_n}) < 1$ and $k_n + 2(\ell_n - \ell_{th,n}) > 0$ for all $n \geq n_0$. For $n \geq n_0$, we take a sequence $\{\epsilon_n\}$ that satisfies $\epsilon_n \in (2(1 - \Lambda_{\xi_n}), 1]$, $\lim_{n \to \infty} \epsilon_n = \epsilon$, and
\[
\lim_{n \to \infty} \log \log \frac{1}{\epsilon_n} < 1.
\]
Note that the denominator diverges when $n \to \infty$ due to the assumption. We then apply Theorem 11 to obtain
\[
\Delta_{cl,n} \leq L_n + \delta_n
\]
for $n \geq n_0$, where $L_n = \frac{2^{2(\ell_n - \ell_{th,n})}}{1 - \epsilon}$ and $\delta_n$ is given by
\[
\log \delta_n = e_n + 2^{N_e + k_n - \ell_{th,n} + 1} \left(N_n + k_n - \ell_n + \log \frac{5}{\epsilon_n}\right) - \frac{\log 2^{2\ell_n + H_2(B_m)\xi_n}}{24}.
\]
Note that we have used an obvious bound $\epsilon_n \leq 1/2$ for all $n$.

By taking the limit of Eq. (165), it follows that
\[
\limsup_{n \to \infty} \left\{\frac{\Delta_{cl,n}}{L_n}\right\} \leq 1 + \limsup_{n \to \infty} \frac{\delta_n}{L_n}.
\]
In the following, we show that $\limsup_{n \to \infty} \delta_n/L_n = 0$. We define $p_n$, $q_n$, and $r_n$ by
\[
p_n = \ell_n + H_2(B_m)\xi_n = N_n + 2(\ell_n - \ell_{th,n}),
q_n = \log(k_n + 2(\ell_n - \ell_{th,n})),
\]
\[
r_n = N_n + k_n - \ell_n + \log(N_n + k_n - \ell_n + \log \frac{5}{\epsilon_n}).
\]
Using $p_n$, $q_n$, and $r_n$, we can write $\log \delta_n/L_n$ as
\[
\log \frac{\delta_n}{L_n} = 2^{q_n} + 2^{q_n + r_n} - \frac{\log 2^2}{6} 2^{p_n} + \log(1 - \epsilon_n).
\]
We below show that, when $n \to \infty$,
\[
p_n - q_n - r_n = p_n(1 - \frac{q_n}{p_n}) - r_n \to \infty.
\]
It follows that
\[
p_n = \ell_n + N_n + 2(\ell_n - \ell_{th,n})
q_n \leq \log(k_n + 2(\ell_n - \ell_{th,n}))
\]
\[
\leq \frac{2(\ell_n - \ell_{th,n})}{k_n + \ell_n - \ell_{th,n}}
\]
\[
\leq \frac{2(\ell_n - \ell_{th,n})}{k_n + 2(\ell_n - \ell_{th,n})},
\]
where the second line follows from $N_n + k_n - \ell_{th,n}$ since $H_2(B_m)\xi_n \geq 0$, and the last from $k_n \geq 0$. This converges to zero when $n \to \infty$ since Eq. (163) implies that $k_n + 2(\ell_n - \ell_{th,n}) \to \infty$ when $n \to \infty$. Hence, we have
\[
\limsup_{n \to \infty} \frac{q_n}{p_n} = 0.
\]
We also have
\[
p_n - r_n = k_n + 2(\ell_n - \ell_{th,n})
- \log(N_n + k_n - \ell_n + \log \frac{5}{\epsilon_n}).
\]
Since $N_n + k_n - \ell_n \geq 1$, we use $\log(x+y) \leq 1 + \log x + \log y$ for any $x, y \geq 1$ to obtain
\[
p_n - r_n \geq k_n + 2(\ell_n - \ell_{th,n}) - \log(N_n + k_n - \ell_n)
- \log \frac{5}{\epsilon_n} - 1.
\]
The right-hand side diverges when $n \to \infty$ since the first three terms diverges due to the assumption (Eq. (163)) and we have set $\epsilon_n$ such that Eq. (164) is satisfied. Hence,
\[
\lim_{n \to \infty} (p_n - r_n) = \infty.
\]
From Eqs. (176) and (179), we obtain Eq. (172), which further implies that \( \lim_{n \to \infty} \frac{\log(\delta_n/L_n)}{n} = -\infty \). Thus,

\[
\lim_{n \to \infty} \frac{\delta_n}{L_n} = 0. \tag{180}
\]

Substituting this into Eq. (167), we arrive at Eq. (122).

The statement about the error on decoding quantum information, Eq. (123), is similarly obtained. \( \square \)

VI. CONCLUSIONS AND OUTLOOKS

In this paper, we have shown that a decoding circuit of a CSS code, in which two classical decoders of the CSS code are used respectively in a controlled unitary and the measurement followed by feed-forward, can be extended to a general QECC. This is achieved by replacing the two classical decoders with the two decoding measurements of CQ codes associated with the QECC. The decoding error of the constructed decoding circuit, which we have called the C-to-Q decoder, is characterized by the two decoding errors of the CQ codes and by the degree of complementarity of the bases that define the classical inputs of the CQ codes. This is of theoretical interest since it reveals the fundamental role of the complementarity principle in decoding QECCs, and also of practical use as it has many applications, such as estimating the performance of QECC by regarding it as a CQ code and simultaneously achieving error correction and switching codes. We have also shown that the C-to-Q decoder is nearly-optimal, when the decoding measurements of the CQ codes are suitably chosen.

The power of the C-to-Q decoder has been then demonstrated in the Hayden-Preskill protocol. We have improved the decoding errors compared to the previous results and have also shown that, within the framework of the Hayden-Preskill protocol, the dynamics of black holes is an optimal encoder for quantum information but is a poor encoder for classical information.

Since QEC is a key to realizing large-scale quantum information processing, the C-to-Q decoder should have a number of applications to, i.e., as quantum communication, fault-tolerant quantum computation, quantum cryptography, and so on. This should be analogous to the Petz recovery map \cite{32}, which was originally proposed in a special context but later found many applications \cite{34, 35, 54, 61, 62} in quantum information tasks as well as in theoretical physics. It is also interesting to investigate the relations between the C-to-Q decoder and existing general constructions of decoders \cite{32–39, 63}.

Apart from applications, it is of fundamental interest to further investigate the role of complementarity in QEC. Our construction of the C-to-Q decoder quantitatively shows that, if two types of classical information defined in two complementary bases can be decoded well, so does quantum information. This observation was previously made in an implicit manner \cite{36–39}. We conjecture that this is true even if the two bases are not complementary. To be more precise, we expect that, if the two bases satisfy the condition that a strict subset of one basis does not span a subspace spanned by another strict subset of the other basis, then decoding classical information defined in such two bases is equivalent to decoding quantum information at the expense of some cost.

ACKNOWLEDGEMENTS

Y. N. is supported by JST, PRESTO Grant Number JPMJPR1865, Japan, and partially by MEXT-JSPS Grant-in-Aid for Transformative Research Areas (A) “Extreme Universe”, Grant Numbers JP21H05182 and JP21H05183, and by JSPS KAKENHI Grant Number JP22K03464. T. M. is supported by JSPS KAKENHI Grant Number 21J12744. This work is also supported by JST, Moonshot R&D Grant Number JPMJMS2061.
Appendix A: Proof of Proposition 1

We here prove Proposition 1. The statement is as follows.

**Proposition 1** (Restatement). Let $E^{A\rightarrow B}$ and $D^{C\rightarrow A}$ be encoding and decoding quantum channels against a noisy quantum channel $N^{B\rightarrow C}$. Let $U^A$ be a unitary 1-design, and define new encoding and decoding channels $E_{U}^{A\rightarrow B}$ and $D_{U}^{C\rightarrow A}$ by

$$
E_{U}^{A\rightarrow B} (\rho^A) := E^{A\rightarrow B} (U^A \rho^A U^A) \quad (A1)
$$

$$
D_{U}^{C\rightarrow A} (\sigma^C) := U^A \circ D^{A\rightarrow C} (\sigma^C) \circ U^A. \quad (A2)
$$

Then, it holds that

$$
\frac{1}{2} \| \text{id}^A - E_U [D_{U}^{C\rightarrow A} \circ N^{B\rightarrow C} \circ E_{U}^{A\rightarrow B}] \|_\diamond 
\leq \Delta_n (D | N \circ E), \quad (A3)
$$

where $E_U$ is the average over the unitary 1-design $U^A$.

**Proof.** Since $M^{twir}_{\text{id}} := E_U [D_{U}^{C\rightarrow A} \circ N^{B\rightarrow C} \circ E_{U}^{A\rightarrow B}]$ is Hermitian-preserving, it follows from Eq. (5) that

$$
\| \text{id}^A - M^{twir}_{\text{id}} \|_\diamond = \max_{|\psi\rangle^A} \| |\psi\rangle^A \langle \psi|^{AR} - M^{twir}_{\text{id}} (|\psi\rangle^A \langle \psi|^{AR}) \|_1, \quad (A4)
$$

where $R$ has the same dimension as $A$. In the following, without loss of generality, we consider a discrete unitary 1-design $\{p_j, U_j\}_j$ [64]. Using the triangle inequality and the invariance of the trace norm under unitary, we have

$$
\| \text{id}^A - M^{twir}_{\text{id}} \|_\diamond 
\leq \max_{|\psi\rangle^A} \sum_j \| p_j (U_j^A |\psi\rangle \langle \psi|^{AR} U_j^A) - M^{twir}_{\text{id}} (U_j^A |\psi\rangle \langle \psi|^{AR} U_j^A) \|_1,
$$

$$
(A5)
$$

where $M^{twir}_{\text{id}} := D^{C\rightarrow A} \circ N^{B\rightarrow C} \circ E^{A\rightarrow B}$.

We now use the fact that there exists a pure state $|\Psi\rangle^{ARC}$ on a system $ARC$ and a POVM $\{\Gamma_j\}_j$ on $C$ such that

$$
\text{Tr}_C [\Gamma_j |\Psi\rangle \langle \Psi|^{ARC}] = p_j U_j^A |\psi\rangle \langle \psi|^{AR} U_j^A. \quad (A6)
$$

By summing up over $j$, we have

$$
\Psi^{AR} = \sum_j p_j U_j^A |\psi\rangle \langle \psi|^{AR} U_j^A = \pi^A \otimes |\psi^R\rangle. \quad (A7)
$$

Here, we used the property $\sum_j \Gamma_j = I^C$ of the POVM $\{\Gamma_j\}_j$ and also the fact that $\{p_j, U_j\}_j$ is a unitary 1-design. Further tracing out $R$, we obtain $\Psi^A = \pi^A$. Since the maximally entangled state $|\Psi\rangle^{ARC}$ has the same marginal state $\Phi^A = \pi^A$, there exists an isometry $V_R^{R\rightarrow RC}$ such that

$$
|\Psi\rangle^{ARC} = V_R^{R\rightarrow RC} |\Phi\rangle^{ARC}. \quad (A8)
$$

Consider a quantum channel $\chi^{R\rightarrow RX}$ defined by

$$
\chi^{R\rightarrow RX} (\rho^R) 
= \sum_j \text{Tr}_C (\Gamma_j^{V^{R\rightarrow RC}} \rho^{RV^{R\rightarrow RC}}) \otimes |e_j\rangle \langle e_j|^X, \quad (A9)
$$

where $\{|e_j\rangle^X\}_j$ is an orthonormal basis in an ancillary system $X$. From Eqs. (A6) and (A8), we have

$$
\chi^{R\rightarrow RX} (|\Phi\rangle \langle \Phi|^A) 
= \sum_j p_j U_j^A |\psi\rangle \langle \psi|^{AR} U_j^A \otimes |e_j\rangle \langle e_j|^X. \quad (A10)
$$

This leads to
\[
\sum_j |p_j(U_j|\psi\rangle\langle\psi|^A RU_j^\dagger - \mathcal{M}A(U_j|\psi\rangle\langle\psi|^A RU_j^\dagger)|_1 = \sum_j |p_j(U_j|\psi\rangle\langle\psi|^A RU_j^\dagger - \mathcal{M}A(U_j|\psi\rangle\langle\psi|^A RU_j^\dagger)\otimes|e_j\rangle\langle e_j|^X|_1.
\]

We further assume that the decoder is symmetric over the change in the encoded value \(j\), namely,

\[
f_1(z + wG_2 + jG) = f_1(z) + j.
\]

Then, an error \(e_1\) is perfectly correctable by the decoder \(f_1(z)\) if and only if \(f_1(e_1) = 0\) since \(f_1(jG + wG_2 + e_1) = j + f_1(e_1)\).

The logical X basis \(\{|\tilde{X}\rangle\}_{i=1,...,k}\) corresponding to the F basis in Sec. III A 2 is defined as

\[
|\tilde{X}\rangle := 2^{N-k/2} \sum_{j \in \{0,1\}^k} (-1)^j |j\tilde{Z}\rangle.
\]

We similarly consider a classical decoder for identifying the X basis index \(l\) represented by \(f_2(x)\), which satisfies \(f_2(x + vH_1 + lH) = f_2(x) + l\). An X-basis error \(e_2\) is perfectly correctable by the decoder \(f_2(z)\) if and only if \(f_2(e_2) = 0\).

We now turn to check that the circuit depicted in Fig. 2 actually works as a decoder for this CSS code. Let \(|\tilde{\psi}\rangle := \sum_j \alpha_j |j\tilde{Z}\rangle\) be an arbitrary encoded state of this CSS code and assume that a Pauli error \(X^{e_1}Z^{e_2}\) has occurred on this encoded state, where \(X^y := X^{y_1} \otimes \cdots \otimes X^{y_N}\) for an \(N\)-bit string \(y = (y_1, \ldots, y_N)\). Then, by applying the isometry \(R_Z\) given in Eq. (20), we have

\[
R_ZX^{e_1}Z^{e_2}|\tilde{\psi}\rangle
\]

\[
R_ZX^{e_1}Z^{e_2}|\tilde{\psi}\rangle
\]

\[
R_ZX^{e_1}Z^{e_2}|\tilde{\psi}\rangle
\]
We then apply the X-basis measurement on the N-qubit system to obtain an outcome string \(x\) and perform a correction operation \(Z^x\) to the k-qubit system as depicted in Fig. 2. Since the outcome \(x\) satisfies \(xG_2^T = e_2G_2^T\), it is written in the form of \(x = lH + vH_1 + e_2\). The state after the error correction is then given by

\[
\sum_j \alpha_j(-1)^{jG(x^T+e_2^T)+e_1^T}Z^j f(z^j) |j + f_1(e_1)\rangle Z_j
\]

(B10)

\[
= \sum_j \alpha_j(-1)^{jG(x^T+e_1^T)+(j+f_1(e_1))(l+f_2(e_2))^T} |j + f_1(e_1)\rangle Z_j
\]

(B11)

\[
= (-1)^{e_1^T+f_1(e_1)^T} Z^f_2(e_2) X^f_1(e_1) |\psi\rangle.
\]

(B12)

Therefore, the quantum decoder succeeds if \(f_1(e_1) = 0\) and \(f_2(e_2) = 0\) are satisfied, i.e., if classical decoders for \(Z\) and \(X\) succeed, which is what we expected.

The decoding error can be easily obtained in this case. Let \(\Delta_{cl,Z}\) and \(\Delta_{cl,X}\) be the failure probability of the classical decoders \(f_1\) and \(f_2\), respectively. Then, the fidelity between the original maximally entangled state \(|\Phi\rangle\) and the output state is independent of whether or not the preceding \(X\)-error correction is independent of whether or not the preceding \(Z\)-error correction succeeds. This is not the case for general codes: decoding errors for \(Z\) and \(X\) may correlate, and thus we need to use a trick to decouple these correlations, which results in a looser bound.

\section*{Appendix C: Derivation of Eq. (36)}

In this section, we provide an upper bound of \(\Xi_{EF}\), which defined by Eq. (36) as

\[
\Xi_{EF} = 1 - \frac{1}{d} \sum_{l,m} q(l,m) F_l
\]

(C1)

For simplicity, we denote \(F_{BC}(\text{unif}, p_l)\) by \(F_l\), and \(\text{Tr}[T^{A\rightarrow C}(m_{F,l})M_{F,l}^A]\) by \(q(l,m)\). Since \(\pi\) is the completely mixed state, we have

\[
\Xi_{EF} = 1 - \frac{1}{d} \sum_{l,m} q(l,m) F_l
\]

(C2)

\[
\leq 1 - \frac{1}{d} \sum_{l,m} q(l,m) F_m - \frac{1}{d} \sum_{l,m} q(l,m) (F_l - F_m).
\]

(C3)

Using the facts that \(\sum_l q(l,m) = 1\) for any \(m\) and that

\[
\sum_{l,m} q(l,m) (F_l - F_m) = \sum_{l \neq m} q(l,m) (F_l - F_m) \geq (F_{BC,min} - F_{BC,max}) \sum_{l \neq m} q(l,m)
\]

(C4)

\[
\geq (F_{BC,min} - F_{BC,max}) d \Delta_{cl,F},
\]

(C5)

we obtain

\[
\Xi_{EF} \leq 1 - \frac{1}{d} \sum_{m} F_m + (F_{BC,max} - F_{BC,min}) \Delta_{cl,F}.
\]

(C6)

\section*{Appendix D: Derivation of Eq. (112)}

We here derive Eq. (112), namely,

\[
\Delta_{cl,Z}(M_{pPGM}|T) \leq \frac{1}{d} \sum_{l \neq j} \text{Tr}[\Pi_j^C t_{l,j}],
\]

(D1)

where \(M_{pPGM}\) is a pPGM given by

\[
M_{pPGM} = \{M_j = \Pi^{-1/2} \Pi_l \Pi^{-1/2}\},
\]

(D2)

where \(\Pi_l\) is a projection onto the support of \(t_{l,j} := T^{A\rightarrow C}(j|A)\) and \(\Pi := \sum_j \Pi_j\). In the following, we omit the superscript \(C\).

The proof is based on the same idea as [49, 50] except that we do not consider the asymptotic situation. For this reason, we need a slight modification.

Using a diagonalization of \(t_{l,j}\) such that \(t_{l,j} = \sum_{z=0}^{t_{l,j}-1} \lambda_z^j |\varphi_z^j\rangle\langle \varphi_z^j|\), where \(\lambda_z^j = \text{rank}_{t_{l,j}}\), the projection \(\Pi_j\) is given by \(\Pi_j = \sum_{z=0}^{t_{l,j}-1} |\varphi_z^j\rangle\langle \varphi_z^j|\), which leads to

\[
\text{Tr}[M_j T_j] = \sum_{z,w=0}^{t_{l,j}-1} \lambda_z^j |\langle \varphi_z^j| \Pi^{-1/2} |\varphi_w^j\rangle|^2.
\]

(D3)

Since \(\Delta_{cl,Z}(M_{pPGM}|T) = 1 - d^{-1} \sum_j \text{Tr}[M_j T_j]\), we have

\[
\Delta_{cl,Z}(M_{pPGM}|T) = \frac{1}{d} \sum_{j=0}^{d-1} \sum_{z=0}^{t_{l,j}-1} \lambda_z^j \left(1 - \sum_{w=0}^{t_{l,j}-1} |\langle \varphi_w^j| \Pi^{-1/2} |\varphi_z^j\rangle|^2\right),
\]

(D4)

\[
\leq \frac{1}{d} \sum_{j=0}^{d-1} \sum_{z=0}^{t_{l,j}-1} \lambda_z^j \left(1 - |\langle \varphi_z^j| \Pi^{-1/2} |\varphi_z^j\rangle|^2\right),
\]

(D5)

\[
\leq \frac{d}{d} \sum_{j=0}^{d-1} \sum_{z=0}^{t_{l,j}-1} \lambda_z^j \left(1 - |\langle \varphi_z^j| \Pi^{-1/2} |\varphi_z^j\rangle|^2\right),
\]

(D6)
where the first equality is due to $\sum_{z=0}^{t_j-1} \lambda_z^{(j)} = 1$ for any $j$, and the last line to $1 - x^2 \leq 2(1 - x)$.

Define two matrices, $\Gamma$ and $\Lambda$, whose matrix elements are given by

$$
\Gamma_{(j,z),(i,w)} := \langle \varphi_z^{(j)} | \varphi_w^{(i)} \rangle, \\
\Lambda_{(j,z),(i,w)} := \delta_{(j,z),(i,w)} \lambda_z^{(j)}.
$$

(D7)  

(D8)

Note that they are both positive semidefinite. By a direct calculation, it is straightforward to check that

$$
\Gamma_{1/2} := \frac{1}{d} \sum_{j,z=0}^{d-1,t_j-1} \lambda_z^{(j)} \left( 2 - 3 \Gamma_{(j,z),(j,z)} + \sum_{i=0}^{d-1} \sum_{w=0}^{t_i-1} |\Gamma_{(j,z),(i,w)}|^2 \right)
$$

(D10)

$$
= \frac{1}{d} \sum_{j \neq i} \sum_{z,w} \lambda_z^{(j)} |\Gamma_{(j,z),(i,w)}|^2
$$

(D11)

$$
= \frac{1}{d} \sum_{j \neq i} \sum_{z,w} \lambda_z^{(j)} |\langle \varphi_z^{(j)} | \varphi_w^{(i)} \rangle|^2
$$

(D12)

$$
= \frac{1}{d} \sum_{j \neq i} \text{Tr}[\Pi_i \tau_j],
$$

(D13)

where the third last line follows from the fact that $\Gamma_{(j,z),(j,w)} = \delta_{zw}$.

We finally comment on the fact that a similar statement follows from the Hayashi-Nagaoka’s lemma [65], which states that $I - M_j \leq 2(I - \Pi_j) + 4 \sum_{i \neq j} \Pi_j$. Applying this, we immediately have

$$
\Delta_{cl}(M_{pPGM}|T) \leq \frac{4}{d} \sum_{i \neq j} \text{Tr}[\Pi_i \tau_j],
$$

(D15)

where we have used $\text{Tr}[\Pi_i \tau_j] = 1$. This is slightly looser than Eq. (112) by a factor of 4.