The Apparent Fractal Conjecture: 
Scaling Features in Standard Cosmologies

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ABSTRACT

This paper presents an analysis of the smoothness problem in cosmology by focussing on 
the ambiguities originated in the simplifying hypotheses aimed at observationally verify-
ing if the large-scale distribution of galaxies is homogeneous, and conjecturing that this 
distribution should follow a fractal pattern, in the sense of having a power-law type av-
erage density profile, in perturbed standard cosmologies. This is due to a geometrical 
effect, appearing when certain types of average densities are calculated along the past 
light cone. The paper starts by reviewing the argument concerning the possibility that the 
galaxy distribution follows such a scale invariant pattern, and the premises behind the 
assumption that the spatial homogeneity of standard cosmology can be observable. Next, 
it is argued that in order to discuss observable homogeneity one needs to make a clear 
distinction between local and average relativistic densities, and showing how the different 
distance definitions strongly affect them, leading the various average densities to dis-
play asymptotically opposite behaviours. Then the paper revisits Ribeiro’s (1995) results, 
showing that in a fully relativistic treatment some observational average densities of the 
flat Friedmann model are not well defined at \( z \sim 0.1 \), implying that at this range average 
densities behave in a fundamentally different manner as compared to the linearity of the 
Hubble law, well valid for \( z < 1 \). This conclusion brings into question the widespread 
assumption that relativistic corrections can always be neglected at low \( z \). It is also shown 
how some key features of fractal cosmologies can be found in the Friedmann models. In 
view of those findings, it is suggested that the so-called contradiction between the cosmo-
logical principle, and the galaxy distribution forming an unlimited fractal structure, may 
not exist.

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1 Introduction

Cosmology, like astronomy, often needs to rely upon some transitory and simplifying assumptions in order to be able to compare theory with observations. As those assumptions, usually arising from technological constraints, are aimed at bringing a difficult problem into a workable, possibly falsifiable, level, the conclusions reached through them need, therefore, to be revised from time to time. However, it is easy to see a posteriori when a certain hypothesis has aged, but not so much so at the very moment when certain simplifying ideas need revision, or even abandonment. The reasons for that are many, but they often come about when technical means for gathering scientific data evolves, bringing new data which may be orders of magnitude more accurate than previously available, and, at the same time, the theoretical impact of such new and more precise data on those simplifying assumptions goes unnoticed. Besides, the theoretical implications of such revisions tend to be resisted, which in turn generates controversy, inasmuch as they may well lead to thorny questions which, quite often, are only reluctantly asked.

The issue of the scale where the matter distribution in the Universe would become observationally smooth involves that kind of simplifying assumptions. At the heart of this issue lies the problem of how one can observationally characterize cosmological density, and, for that purpose it is usually assumed that relativistic corrections can be neglected in cosmology at redshift ranges where distance and redshift follow a linear relation, i.e., the Hubble law.1 This can be thought of as being the cosmological Newtonian approximation, since the usual interpretation is that Newtonian cosmology represents a small and local piece of the Universe (see, e.g., Harrison 2000, p. 332), where Newtonian mechanics was long ago found to lead to the same dynamical equations as given by general relativity (Milne 1934, McCrea and Milne 1934; good reviews on these results, and their implications, can be found in Bondi 1960, Sciama 1993, and Harrison 2000). This assumption as applied to cosmology means that flat and Euclidean geometry is valid in this local observable region, with relativistically derived expressions becoming unnecessary in observational cosmology (Peebles 1980, p. 143). Then, the reasoning goes, as Newtonian and Friedmann cosmologies have homogeneous densities at the same epochs, or, stating the same in relativistic terminology, they have homogeneous spatial sections at constant time coordinates, if we take these models as our best physical representations of the Universe, their spatial homogeneity should be observed up to at least moderate redshift ranges. So, sources up to \( z \approx 1 \) are still assumed to lay within our local

1 From now on I shall call by “small redshifts” the scales where \( z < 0.1 \), by “moderate redshifts” when we have \( 0.1 \leq z < 1 \), and by “large redshifts” the scales where \( z \geq 1 \). The linearity of the redshift-distance relation is known to be valid at small and moderate redshift ranges (Sandage 1995, p. 91).
and Newtonian piece of the Universe. Those are such standard and widespread assumptions of observational cosmology that they are hardly stated openly, being almost always assumed implicitly.

Therefore, the possible observational smoothness of the Universe in fact relies on two inter-dependent and simplifying hypotheses of observational cosmology: (1) relativistic corrections may be safely disregarded in dealing with astronomical data of cosmological importance up to moderate redshift ranges, as at those limits we are supposed to be still probing within the local Newtonian piece of the Universe, and (2) an uniform distribution of mass can, in fact ought to, be inferred from astronomical data gathered at this local Newtonian universe.

However, for theoretical and practical purposes, there are at least two ways of finding the range of this Newtonian approximation. The most widely used is to take small speeds as meaning this approximation, which implies that when galaxy recession speeds are small, as compared to light speed, Newtonian mechanics is valid (Callan, Dicke and Peebles 1965). Less well-known, but equally important, is the criterion implicitly advanced by Bondi (1960), which states that it is light dynamical behaviour that determines this range. In other words, as opposed to relativity, Newtonian mechanics does not produce a dynamical theory for light and, therefore, there will be cosmological scales sufficiently large such that light can no longer be considered as being instantaneously transmitted from source to observer.

The important point here is that contrary to what may be initially thought, the practical implementation of these two criteria does not always lead to the same results. The first criterion relies upon the Hubble law being approximately written as a velocity-distance law (Harrison 1993), while Bondi’s criterion means solving the null geodesic in a fully relativistic model, obtaining expressions for the observational quantities along the observer’s past null cone, and comparing those observables with the Newtonian predictions. However, as I shall show below, in Bondi’s criterion the non-linearity of general relativity will mean that different observational quantities will have Newtonian approximations at different ranges. Therefore, we must see those two criteria as complementing each other, and this implies that the range of the Newtonian approximation, and, as a consequence, the limit up to where we can dismiss relativistic corrections in cosmology, will depend upon the observational quantities being dealt with. In other words, those limits will depend on the specific problem under analysis.

In this paper I intend to discuss the problem of the observational smoothness of the Universe, and the possibility that the large-scale mass distribution may follow a scale invariant, or fractal, pattern, in the light of Bondi’s criterion as outlined above. For this aim it

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2 Since the observed Universe is filled with stars, galaxies, etc, the second hypothesis above implies that this observed lumpiness of the Universe must originate in very local disturbances from uniformity, a phenomenon statistically similar to white noise. The concept of a correlation length was introduced with the aim of finding the maximum range of this disturbance before the uniform distribution is reached (Peebles 1993). It is therefore obvious that these two additional hypotheses, that is, lumpiness identified with white noise and the correlation statistics, are a corollary of the second simplifying hypothesis above, and, thus, they cannot survive independently from it.
is mandatory to start by discussing the theoretical problems relevant for the observational characterization of density in the cosmology, namely local versus average density, and distance definitions. For simplicity, I will use the Einstein-de Sitter model, but the analysis and most results are also valid for open and close Friedmann models. I will show that once this method is consistently and systematically applied, the two basic assumptions of observational cosmology relevant for the smoothness problem of the Universe, as described above, breakdown at small redshift ranges. Then we will be able to obtain some results long ago hypothesized for a hierarchical, or fractal, universe, without any need to drop out either the cosmological principle or the usual meaning of cosmological parameters like the Hubble constant, or even the cosmic microwave background radiation (CMBR) isotropy. I will also show that if we do not address the cosmological smoothness problem via a fully relativistic perspective of cosmological density, we may end up with some misleading conclusions drawn from the initial, but no longer valid, assumptions, which in turn will inevitably lead to false problems. Finally, concluding my analysis, I will discuss how those findings naturally lead to the conjecture that the observed fractality of the distribution of galaxies, defined here as being a system characterized by a power-law type average density profile which decays linearly at increasing distances, should be an observational effect of geometrical nature, arising in perturbed Friedmann models.

Throughout this paper I shall avoid strict astronomical issues, as well as discussing the aspects of the distribution of galaxies correlation statistics, which has been at the centre of the debate, as applied to relativistic models, inasmuch as such a discussion can be found elsewhere (Ribeiro 1995). Therefore, here I shall focus on answering the questions of whether or not the standard cosmology really implies a perfectly well defined observable mean density, and also if it completely rules out an unlimited fractal pattern. Davis (1997) pointed out that these questions revolve around the concept of mean density of the Universe. However, I claim here that in fact these questions revolve around the concept of observational mean density of the Universe.

The plan of the paper is as follows. In §2, after briefly reviewing the proposal that large-scale distribution of matter follows a hierarchical, or fractal, pattern, as well as the orthodox claim of why this is untenable, I will discuss how an observed fractality needs an average density definition, and that differentiating local versus average density is essential to discuss the smoothness problem in cosmology. This point was already present in Wertz (1970, 1971) earlier contribution to the subject.

Section §3 deals with the issue of how we can define and use those two densities in a relativistic framework. I will show that Bonnor’s (1972) first attempt to discuss this problem within a relativistic context, while conceptually precise, fell short of making strict use of Bondi’s criterion due to analytical difficulties, and that led him to use an inappropriate

3 In fact, the approach presented here avoids a collision between the cosmological principle, and CMBR near isotropy, with the fractal universe model advanced by Pietronero and collaborators, as those issues become then unrelated.
distance definition. The point to be made here is that behind an observational definition of density lies an unavoidable choice to be made among the various cosmological distances. This is an essential point in order to put the discussion about the universal smoothness problem into a solid relativistic footing. §3.1 deals with this problem, proposing some criteria to be used for finding the appropriate distance definition for the problem under consideration. Then §3.2 shows unequivocally that the Friedmann cosmology does not imply a perfectly well defined apparent mean density in all scales, but also that this observational average density already becomes not well defined at small scales ($z < 0.1$), simply because when one attempts to observationally characterize the geometrical constant local density of the Friedmann models at moderate redshifts by means of Bondi's criterion, the high non-linearity of Einstein's field equations, together with the fact that volumes increase three times faster than distances, will amplify very small differences in some observables which, in turn, lead to dramatic differences in the values of the observational mean density even at small $z$. Therefore, in §3.2 it becomes clear that in order to properly characterize observational density we need to depart from the observational relations usually found in the literature, since those are only valid at very small redshifts. In other words, we need to derive new observational relations capable of taking into account the lookback time. This section also revisits some of Ribeiro's (1995) results, showing that the linearity of the distance-redshift relation does not help in the context of observational characterization of cosmological density as this relation remains well approximated by a linear relation throughout moderate redshift ranges, without any change in the value of $H_0$. Thus, it cannot be used as a test for possible dismissal of large-scale hierarchical (fractal) clustering, as has been done in the past (Sandage, Tammann and Hardy 1972; Sandage and Tammann 1975). Finally, §4 collects all those results by proposing that the observed fractal structure should arise in perturbed Friedmann cosmologies. The paper finishes with a conclusion where I argue that the near CMBR isotropy brings no difficulties to the scenario outlined here.

Some terms used in this paper are applied elsewhere with somewhat different meanings. Therefore, to avoid confusion, I shall define them immediately. Here fractality refers to the property shown by the observed large-scale distribution of galaxies of having an average density power-law type decay at increasing distances. So, in this paper, and all others where I have so far dealt with this issue, fractality means in fact observational fractality, in the astronomical sense, and only resembles non-analytical fractal sets in the sense that if we define a smooth-out average density on those sets, the properties of this average density are similar to what is found in observational cosmology data. In other words, they are both of power-law type ones. So, fractality has here an operative definition which allows us to talk about fractality, or fractal properties, in completely smooth relativistic cosmological models, where the cosmological fluid approximation is assumed.

By observational smoothness, or observational homogeneity of the Universe, I mean the

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4 In this paper the word apparent means the same as observational, in relativistic terms. In other words, all apparent results imply that they were derived along the null cone, as prescribed by Bondi's method.
possibility of inferring from observational cosmology data that the large-scale distribution of galaxies has constant average density. This is the smoothness problem in cosmology. Therefore, there must be a clear distinction between observational homogeneity and spatial homogeneity. The latter is a built-in geometrical feature defined in the standard cosmologies, while the former is the possible direct observation of the latter.

The main theses presented in this paper were briefly outlined in Ribeiro (2001). Here, however, they are introduced in a very different context, where I also provide a greatly expanded discussion, together with many additional quantitative details, new results, thoughts, and conclusions. This paper is, therefore, a follow-up to Ribeiro (2001).

2 The Hierarchical (Fractal) Clumping of Matter

The proposal that the large-scale distribution of matter in the Universe should follow a hierarchical, or fractal pattern is by no means new, dating back to almost a century ago. The first proposal was made even before relativistic cosmology itself was born in 1917. The initial suggestion that the Universe could be constructed in a hierarchical manner dates back to the very beginnings of cosmology (Fournier D’Albe 1907; Charlier 1908, 1922), with contributions made even by Einstein (Amoroso Costa 1929; Wertz 1970; Mandelbrot 1983; Ribeiro 1994; Ribeiro and Miguelote 1998). Since then it has kept reappearing in the literature, being ressurected by someone who questioned the accepted wisdom. So, the survivability of this concept is a fact which should call the attention of a future historian of science. Besides, while fractals were easily accepted in many areas of physics as bringing new and useful tools and concepts, it is amazing to witness the stiff resistance so many researches have been waging against the introduction of any kind of fractal ideas in cosmology, a fact worthy of attention (Oldershaw 1997; Ribeiro and Videira 1998; Disney 2000).

In any case, what we are witnessing now is only the latest chapter of a century old debate, which is now focused on the statistical methods used by cosmologists to study data on galaxy clustering, and whether or not the large-scale galaxy distribution follows a scaling pattern, in the sense of having a power-law average density profile. The previous chapter was between de Vaucouleurs (1970ab) and Wertz (1970, 1971; see also de Vaucouleurs and Wertz 1971) on one side, and Sandage, Tammann, and Hardy (1972) on the other side, and was mainly focused on measurements of galaxy velocity fields and deviations from uniform expansion, a topic which has also resurfaced in the recent debate (Coles 1998).

2.1 The Fractal Debate

The latest round surrounding the smoothness problem in cosmology has become known as The Fractal Debate. The controversy started with Pietronero’s (1987) claims that the usual correlation statistics employed in the characterization of the distribution of galaxies cannot be applied to this distribution, and that a novel statistical technique proposed by
him is capable of testing, rather than assuming, whether or not the galaxy distribution is uniform. The main results reached by this side of the debate are the absence of any sign of homogenization of the distribution of galaxies up to the limits of current observations, denying, thus, any usefulness to a correlation length concept (see § above), and that this distribution is well described as forming a single fractal structure, with dimension $D \approx 2$ (Coleman and Pietronero 1992; Pietronero et al. 1997; Ribeiro and Miguelote 1998; Sylos-Labini et al. 1998; Pietronero and Sylos-Labini 2000).

The other side of the debate claims, however, that the traditional statistical analysis of recent observations leads to the opposite conclusion, i.e., that the distribution of galaxies does homogenize beyond a certain small scale (Peebles 1980, 1993; Davis 1997; Wu, Lahav and Rees 1999). Besides, this group has a strong theoretical rejection to fractals under the grounds that a possible fractal pattern for the large-scale structure of the Universe cannot agree with what we know about the structure and evolution of the Universe, as this knowledge is based on the cosmological principle and the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, with both predicting spatial homogeneity for the universal distribution of matter. Therefore, this orthodox view also claims that such an observable homogenization is necessary in order to “make sense of a Friedmann-Robertson-Walker universe”, since “the FRW metric presumes large scale homogeneity and isotropy for the Universe”, and “in [this] cosmological model, the mean density of the Universe is perfectly well defined” (Davis 1997; see also Wu, Lahav and Rees 1999). Moreover, inasmuch as the cosmic microwave background radiation is isotropic, a result predicted by the FLRW cosmology, this group is, understandably, not prepared to, as it seems, give up the standard FLRW universe model and the cosmological principle, as that would mean giving up most, if not all, of what we learned about the structure and evolution of the Universe since the dawn of cosmology (Peebles 1993; Davis 1997; Wu, Lahav and Rees 1999; Martínez 1999).

To reach those opposing conclusions, the validity of the methods used by both sides of this debate are, naturally, hotly disputed, and so far there has not yet been achieved a consensus on this issue. However, even if one is prone to part of the orthodox argument, i.e., that we cannot simply throw away some basic tenets of modern cosmology, like the cosmological principle and the highly successful FLRW cosmological model, when one looks in a dispassionate way at the impressive data presented by the heterodox group, one cannot dispel a certain uneasy feeling that something might be wrong in the standard observational cosmology: their results are consistent and agree with one another (Coles 1998).

Actually, one thing in common between both sides of the debate is that they seem to agree

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5 An earlier edition of Mandelbrot's book on fractals led Peebles to discuss, and dismiss, the possibility of an unlimited fractal pattern in the distribution of galaxies, as proposed by Mandelbrot (see Peebles 1980, pp. 243-249, and references therein). The reasoning behind his dismissal was, however, again based on neglecting relativistic effects at small redshifts (ibid., p. 245). Nevertheless, he kept open the possibility that other fractal sets could provide a better modelling (ibid., p. 249). This preview of The Fractal Debate seems to had attracted little attention, and interest, in the astronomical/cosmological community.
that if the distribution of galaxies does follow a scaling, or fractal, pattern up to the limits of the presently available observations, this would contradict the standard Friedmannian cosmology, which in turn would lead to the dismissal of the cosmological principle (Coleman and Pietronero 1992; Wu, Lahav and Rees 1999). This is because it would contradict the standard Friedmannian cosmology, which in turn would lead to the dismissal of the cosmological principle (Coleman and Pietronero 1992; Wu, Lahav and Rees 1999).

It must be clearly noticed that the issues behind this debate are not yet fractality in the sense of non-analytical sets, but so far it only deals with properties of a smoothed-out and averaged fractal system, whose main observational feature shows up as a decaying power-law type behaviour for the average density as plotted against increasing distances. So, when one talks about ‘observed fractality of the large-scale distribution of galaxies’, or this distribution as forming a ‘fractal system’, or, still, showing ‘fractal features’, one means this type of decaying average density power-law profile, which is consistent with scale invariant structures typically featured in fractal sets (Mandelbrot 1983; Pietronero 1987).

From this brief summary, it is clear that, to this date, the two sides of The Fractal Debate seem to be locked in antagonistic and, as it may initially appear, self-excluding viewpoints. Nevertheless, in order to link these two positions to what is discussed in §1, we need first of all to start by carefully analysing the meaning behind some terms used in this debate. The essential ones are ‘density’ and ‘observational density’.

### 2.2 Density Definitions in Cosmology

The first aspect to note is that in cosmology we may define two types of densities: a local density $\rho$ and an average density $\langle \rho \rangle$, often also called volume density and denoted by $\rho_v$. The latter is, of course, the local density averaged over larger and larger distances. If the local density is always the same, then local and average densities are equal, and, we may suppose that in the standard spatially homogeneous cosmology we will always have $\rho = \langle \rho \rangle$, as the local density is the same everywhere. However, that would be a simplistic conclusion as in standard cosmologies, both Newtonian and relativistic, the local density is only uniform at specific epochs. In other words, it is a function of time, $\rho = \rho(t)$, being the same everywhere only at fixed epochs.

Suppose now two distances $d_1$ and $d_2$ such that $d_1 < d_2$. If there exists a function $t = t(d)$ relating time and distance such that bigger distances will mean earlier times, then an object located at distance $d_1$ will be associated to time $t_1$, while another object located at distance $d_2$ is associated to an earlier epoch $t_2$ ($t_1 > t_2$). Since local density depends only on time, then $\rho(t_1) = \rho(t_2)$, which means that the average $\langle \rho \rangle = (1/2)[\rho(t_1) + \rho(t_2)] = \rho(t_1)$ or $\rho(t_2)$. This inequality between local and average density occurs provided the average is made along the curve $t = t(d)$. Therefore, even in standard cosmology local and average densities will only be equal at similar epochs.

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6 The heterodox group has recently retracted a bit from such more radical view, arguing that an open Friedmann model may be compatible with a fractal structure (Joyce et al. 2000). Some of their conclusions are similar to the ones to be shown below, although the methods used to reach them are completely different from the path taken here.
The point I wish to make here is that even in the standard cosmology we can only talk about observing its spatial homogeneity if such observational measurement is carried out at the same epoch \( t \). In other words, attempts to measure the smoothness of the Universe can only make sense if one is assured to be doing so at similar epochs, which, of course, must be within the observational errors of astronomical observations. However, when observational cosmology deals with the smoothness problem, this critical issue is usually vaguely dealt with, by simply assuming that the sources should be close enough to be guaranteed to be at the same, or very close, values of \( t \). Usually the implicit reasoning is that for “small” \( z \) we should be roughly observing at the same epochs, which are equal to our own (the observer), while for “large” \( z \) this is no longer the case. Such a reasoning already implies that in standard cosmology there must be ranges where \( \langle \rho \rangle \) will start deviating from \( \rho \), but one still lacks a method for substantially narrowing the range from what one means by “large” and “small” redshifts, allowing then a considerable grey area between them. Not surprisingly, it is exactly in this grey area that The Fractal Debate is thriving.

Anyway, the possible importance of the analysis above rests on the existence of a function \( t = t(d) \), and its possible relevance to the smoothness problem in cosmology. As I shall show below, such a function does exist when one uses Bondi’s criterion discussed in §1, and is given by the past null geodesic. Since this function is not defined in Newtonian cosmology, it can only be obtained in a fully relativistic approach to cosmology, meaning that one needs to use Bondi’s criterion from the start. Then when one finds the range where \( \rho \neq \langle \rho \rangle \), that will immediately give us the distance values where one has a breakdown of the Newtonian approach to standard cosmology, at least as far as the smoothness problem is concerned. Therefore, at the range where that happens, the two simplifying hypotheses discussed in §1 will no longer be applicable.

In hierarchical, or fractal, cosmologies, the problems above do not appear as seriously as in the standard cosmology because it has been realized long ago that in a fractal universe one needs to differentiate local from average density from the start (Wertz 1970, 1971). However, since these earlier models are only on Newtonian cosmology, it was not possible to discuss their relativistic aspects, and how and where they are related to standard cosmology. As I shall show below, this is only possible when Bondi’s criterion is applied to standard cosmology at the same time as an average density is defined in these models.

As final remarks, it is important to notice that the basic question being dealt with above is how a spatially homogeneous cosmology may appear, or be observed as, inhomogeneous, and the key to answering it lies on how one constructs an appropriate average. As shown above, this is done by averaging local densities along the past null cone. Related to this question is the opposite problem, which appears from time to time in the literature, and reads as follows. Could a spatially inhomogeneous cosmological model evolve on average like a spatially homogeneous universe model? This problem is known as “the averaging problem in cosmology”, and its main difficulty is the notion of average, whose specification and unambiguous definition is not easy to establish, mainly because it is not straightforward.
to produce an unique meaning to the averaging of tensors (Buchert 1997ab, 2000; Ellis 2000). The problem being discussed in this paper in a sense belongs to the averaging problem in cosmology, but whose motivation is opposite to the original question that established it.

3 Local Versus Average Relativistic Density

As we saw above the relationship between local and average cosmological densities in standard cosmologies is not as straightforward as it may initially appear, and implies some ambiguities and subtleties. The next question then is how those subtleties will express themselves in a relativistic setting. On this respect, Bonnor (1972) was the first to point out that in a relativistic framework the local density is the quantity entering into the energy-momentum tensor of Einstein’s equations, while the average density is obtained by averaging over a sphere of a given volume, which may be arbitrarily large. He then wrote an expression for the volume density, defined as the ratio between the integrated mass \( \int_0^r 4\pi \rho(\bar{r})\bar{r}^2d\bar{r} \), and a volume defined as \( 4\pi r^3/3 \). Here \( r \) is the radius coordinate (Bonnor 1972, eq. 3.4). Bonnor’s assumptions were a consequence of the impossibility of solving analytically the null geodesic equation of his model. Nevertheless, they bring several conceptual problems for calculating observational averages in cosmology, and unless we discuss them in detail their unchecked use may lead to an unrealistic model.

In the first place, in order to relate the average density with observations, it is necessary to integrate the local density along the past light, which is the geometrical locus of astronomical observations. This means solving the null geodesic equation, a task often more difficult than solving Einstein’s field equations themselves. In addition, the best comparison with observations are obtained using number counting rather than integrated mass (see below), but, to do so, we also need the past null geodesic’s solution. Secondly, taking the radius coordinate as distance indicator is inconsistent with Bondi’s criterion outlined above. In general relativity coordinates are labels to spacetime events, and, therefore, cannot be used as distance, unless we are assured to be in a region of flat and Euclidean geometry. However, to be assured of that we need to have a prior method for determining up to where it is safe to use such approximations, which is the object of the present discussion. Since this is the aim of the method outlined here, we, therefore, cannot start with this assumption and coordinates cannot be used as distances. Finally, Bonnor’s definition of volume density is not along the null cone, but at hypersurfaces of constant time, where, astronomical observations are not located.

Clearly, what we need is some volume definition that can be compared with observational data. In other words, we need to define volume along the null cone. But, to define a volume we need to choose some distance, and since there are several ones in cosmology, we should first discuss a method for choosing the appropriate distance definition required for analysing the smoothness problem in cosmology.
3.1 Cosmological Distances

Distances in relativistic cosmology are a confusing subject, especially because there is no unifying terminology. Nevertheless, the number of distance definitions are in fact quite small. As yet, the best treatment of this issue has been given by Ellis (1971, p. 144), where one can find a rigorous relativistic treatment of observations in cosmology, leading to the most important cosmological distance definitions at section 6.4.3 of his paper. Moreover, Ellis’ treatment is general in the sense that it is valid for any cosmological model. Therefore, his analysis is not restricted to the standard cosmology, as it is the case in virtually all other treatments of this subject.

At a first thought one may think that the difficulties associated with the distance concept in cosmology can be skipped if we were to use only separations, also generically known as proper distances (Ellis and Rothman 1993), that is, line element integrals \( \int ds \). The problem with such definitions comes from the fact that separations are in general unobservable geometrical distances. One example is the absolute distance (d’Inverno 1992, p. 325), also known as interval distance (Sandage 1995, p. 13), which would require us to place a rigid rod between two astronomically separated points at the same epoch, that is, assuming \( dt = 0 \), as absolute distances are defined at constant time hypersurfaces. This is not only observationally unfeasible, but would also go against Bondi’s criterion outlined in \( \S \). Therefore, the absolute distance seems to be a device possessing little, if any, relevance in discussing the observational smoothness of the Universe. Comoving distances (also called comoving coordinate distances) do not seem to be of much use in here too, since they are in fact coordinate distances, or simply labels to spacetime events. In spherically symmetric models the one most used is the comoving radial distance coordinate, which, of course, requires the condition \( dt = 0 \) in its definition.

Thus, we must turn our attention to discussing cosmological distances along the backward null cone. However, along this hypersurface the 4-dimensional interval between two points is zero, that is, \( ds = 0 \), and, therefore, we must perform line element integrations over this specific surface, given by \( \int_C d\sigma \), where \( C \) represents the null cone hypersurface, or a line over it, and \( d\sigma \) is the line element over \( C \), and is necessarily of lower dimension. The difficulty with this procedure is that distances defined that way are not unique, a fact which leaves us no choice but, to deal only with distances which can be operationally defined by means of relationships among observational quantities calculated along the null cone. The one mostly used in astronomy is the luminosity distance \( d_L \), defined as a relationship between the observed flux \( F \) of an astronomical source and its intrinsic luminosity \( L \), in a flat and static space. One can also define the observer area distance \( d_A \), also known simply as area distance, and the galaxy area distance \( d_G \). Both \( d_A \) and \( d_G \) determine distances by comparing

\[ d_L = \frac{d_L}{d_A} = \frac{d_L}{d_G} \]

\( \footnote{It is common to call by geodesic distance the separation between two points located over some hypersurface, or lower dimensional surface. So the absolute distance is the geodesic distance defined over the surface where \( dt = d\theta = d\phi = 0 \) in spherically symmetric metrics (d’Inverno 1992, p. 325). Longair (1995, p. 362) also uses a similar terminology.} \]
a solid angle measured either at the observer or at some galaxy, and the intrinsic area of an object. Since $d_G$ implies a knowledge of this solid angle at the galaxy, it is unobservable, by definition (Ellis 1971). One can also define a \textit{parallax distance}, $d_p$, obtained by means of parallax measurements (McCrea 1935, p. 296; Ellis 1971). Inasmuch as, under our present astronomical technology, data on galaxy parallaxes are not yet available, the parallax distance will play no role in this discussion. As a final remark, although $d_\ell$, $d_A$, $d_G$, and $d_p$ are the only observationally accessible distances, it is conceivable that the observational distances could be related to distances which are not directly observational. In this case one will require more use of theory in order to provide such a link, but even so those unobservable distances will always require the use of more directly observable quantities.

The observational distances above are linked by a remarkable geometrical theorem, due to Etherington (1933), known as \textit{reciprocity theorem} (see also Ellis 1971; Schneider, Ehlers, and Falco 1992), which may be written as follows,

$$d_\ell = d_A(1 + z)^2 = d_G(1 + z).$$

In terms of bolometric (all wavelengths) flux-luminosity equations, the reciprocity theorem yields,

$$F = \frac{L}{4\pi(d_A)^2(1 + z)^4} = \frac{L}{4\pi(d_G)^2(1 + z)^2} = \frac{L}{4\pi(d_\ell)^2},$$

(2)

Since all distances above have clear definitions, we may ask, which one is correct? On this respect it is worth reproducing Allan Sandage’s wise remarks on this issue. “We cannot measure distances by placing rigid rods end to end. Rather, operational definitions of distance ‘by angular size’, ‘by apparent luminosity’, ‘by light travel time’, or ‘by redshift’ are perforce employed. Their use then requires a theory that connects the observables (luminosity, redshift, angular size) with the various notions of distances (McVittie 1974). One of the great initial surprises is that these distances differ from one another at large redshift, yet all have clear operational definitions. Which distance is ‘correct’? All are correct, of course, each consistent with their definition. Clearly, then, distance is a construct (…) operationally defined entirely by its method of measurement.” (Sandage 1988, p. 567).

In addition, he also offered a practical prescription regarding how to deal with distances. “The best that astronomers can do is to connect the observables by a theory and test predictions of that theory when the equations are written in terms of the observables alone. To this end, the concept of distance becomes of heuristic value only. It is simply an auxiliary \textit{parameter} that must drop from the final predictive equations.” (Sandage 1988, p. 567).

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8 Cosmological distances appear under different names in the literature. For instance, the observer area distance $d_A$ is called \textit{angular diameter distance} by Weinberg (1972), and \textit{corrected luminosity distance} by Kristian and Sachs (1966). The galaxy area distance $d_G$ is named \textit{effective distance} by Longair (1995, pp. 375), \textit{angular size distance} by Peebles (1993, pp. 319 and 328), and \textit{transverse comoving distance} by Hogg (1999). Such a profusion of names can only bring even more confusion to the subject, specially as some of these names are very unprecise from a geometrical viewpoint. In this paper I shall follow Ellis’ (1971) terminology as I believe his name choices are the least confusing, and, geometrically, most appropriate.
There is no question about the correctness of Sandage's remarks, and the wisdom of his prescription. However, the question remains as to what extent his prescription can be followed when dealing with the possible observational homogeneity of the Universe. The reason for that comes from the realization that the smoothness problem revolves around the concept of observational mean density, which requires some definition of volume, which, in turn, also requires some definition of distance. Thus, when we deal with an observational mean density, the distance used in its definition cannot be dropped from the final equations, and, so, it is no longer an internal parameter, but a quantity which defines the mean density itself. We therefore have to face the fact that the smoothness problem of the Universe requires us to choose a definition of distance. Sandage's prescription above is not applicable to this problem. So, there is here some subjectivity in the sense that any analysis of this issue will depend on the distance choice. If we do not make this choice explicitly, it will enter implicitly in our problem, by the back door.

At this stage one may be tempted to say that if we use the redshift instead of any distance definition, these problems are solved. However, the redshift is a distance indicator, and it will correspond to some distance in some ranges. In fact, in Einstein-de Sitter cosmology it scales with the luminosity distance at low $z$, and follows its asymptotic behaviour at the big bang (see below).

The reciprocity theorem gives us a general relation among cosmological distances, but it does not tell us how to calculate them. From equations (3) it is clear that unless we have, in advance, astrophysical information about intrinsic properties of the sources, the only way we can solve equations (3) is by assuming some cosmological model, obtaining expressions for some previously chosen distance in the assumed model, and feeding those expressions, together with observational data, into equations (2). However, in discussing the possible observational smoothness of the Universe, we have an additional problem to worry about. As we saw above, testing the observable galaxy distribution homogeneity implies an implicit choice of distance. For instance, if we collect data on apparent magnitude ($F$, in fact) and does not make redshift corrections, we will end up with the luminosity distance $d_L$, as it assumes a static and non-expanding universe. On the other hand, if we do make redshift corrections, depending on the used power of $(1 + z)$ factors we may get either $d_G$ or $d_A$. In principle, $d_A$ could be determined independently from equation (2), if, by some astrophysical consideration, we are able to infer the intrinsic size of an object (Ellis 1971, p. 153). In practice, however, this is a difficult task to be performed, and can only be done to a small number of nearby objects. Due to this, such a knowledge will not affect much our discussion here as the smoothness problem requires us to know $d_A$ in large numbers, at the scale of present day redshift surveys, which count thousands of galaxies. Thus, it should be clear by now that, besides choosing a cosmological model, the way we collect and organize our astronomical data may be all that matters in our implicit choice of distance.
3.2 Distances, Volumes and Densities

In a remarkable paper, McVittie (1974) showed that all observational distances differ at large $z$, but are almost the same at moderate to small redshifts. Based on this study, and at a quick look at equations (1), one may wonder if all previous discussion is irrelevant, as all distance definitions should produce similar, if not equal, results for $z < 1$.

Here, however, another subtlety of the smoothness problem in cosmology comes into play. While all distances are similar at small $z$, the observable homogeneity of the Universe is not discussed in terms of distances, but in terms of average densities. These are theoretically constructed as being a ratio between number counting and observable volumes, where the latter are themselves formed by third power of distances, all that along the past null cone. Observable distances are, however, non-linear functions of a null geodesic (unobservable) affine parameter, meaning that average densities are highly non-linear functions along null geodesics. So, a change in distance definition can dramatically alter the behaviour of average density, no matter if those distances are similar to each other at close values of the affine parameter. We are dealing here with highly non-linear functions along the past null cone, which means that simplistic predictions about their behaviour can be very deceptive. We will return to this point below, with specific examples.

3.2.1 An Example: the Einstein-de Sitter Cosmology

Let us write the metric for the Einstein-de Sitter (EdS) model, the simplest standard cosmology, as follows ($c = G = 1$),

$$ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],$$

(3)

where $a(t)$ is the scale factor, given by

$$\left( \frac{da}{dt} \right)^2 = \frac{8\pi}{3} \rho a^2(t),$$

(4)

and the local density is

$$\rho = \frac{1}{6\pi a^3(t)}.$$  

(5)

If we label our present time hypersurface “now” as $t = 0$, then the solution of equation (4) may be written as

$$a(t) = \left( t + \frac{2}{3H_0} \right)^{2/3},$$

(6)

where $H_0$ is today’s value of the Hubble constant.

We can obtain the equation for the past light cone by integrating the past null geodesic of metric (3), $dt/dr = -a(t)$, from “here and now” ($t = r = 0$) up to $t(r)$. The solution is given by

$$3 \left( t + \frac{2}{3H_0} \right)^{1/3} = \left( \frac{18}{H_0} \right)^{1/3} - r,$$

(7)

where the radius coordinate $r$ plays the role of the parameter along the null geodesic.
As it is well known, the redshift in this cosmology is given by \(1 + z = a(0)/a(t)\), or, using equation (8),

\[
1 + z = \left(\frac{18}{H_0}\right)^{2/3} \left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^{-2},
\]

(8)
since along the null cone the scale factor becomes,

\[
a[t(r)] = \frac{1}{9} \left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^2.
\]

(9)
The distances defined by equation (1), may, in this cosmology, be obtained by means of the area distance \(d_A = r \cdot a[t(r)]\), (Ribeiro 1992b, 1995). Therefore, they may be written as,

\[
d_A = \frac{r}{9} \left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^2,
\]

(10)
\[
d_\ell = \frac{r}{9} \left(\frac{18}{H_0}\right)^{4/3} \left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^{-2},
\]

(11)
\[
d_G = \frac{r}{\left(\frac{2}{3H_0}\right)^{2/3}}.
\]

(12)
The big bang singularity hypersurface is reached when metric (3) degenerates at some early epoch. Let us call the big bang time coordinate by \(t_b\). This means that \(a(t_b) = 0\), and, with this result we can also obtain the value of the null geodesic parameter \(r\), in this case) when the past light cone reaches the big bang. Doing so, the big bag coordinates along the null geodesic may be written as,

\[
t_b = -\frac{2}{3H_0}, \quad r_b = \left(\frac{18}{H_0}\right)^{1/3}.
\]

(13)

With these coordinates we can obtain the asymptotic behaviour for the redshift and the three distances above as one approaches the big bang. Thus, the following important limits hold in EdS cosmology,

\[
\lim_{r \to r_b} z = \infty, \quad \lim_{r \to r_b} d_\ell = \infty, \quad \lim_{r \to r_b} d_A = 0, \quad \lim_{r \to r_b} d_G = 2/H_0.
\]

(14)
Notice the completely different asymptotic behaviour of the distances. The luminosity distance grows without bound, as well as the redshift, while the galaxy area distance grows up to a maximum and finite value. On the other hand, the area distance starts growing and then decreases, reaching zero at the big bang. It is not difficult to show that \(d_A\) reaches a maximum at \(z = 1.25\). These results are similar to McVittie’s (1974), although here we have reached them by means of a fully analytical study, that produced exact solutions.

If we now invert equation (11), we may write the distances in terms of the redshift, as follows,

\[
d_A = \frac{2}{H_0} \left[\frac{1 + z - \sqrt{1 + z}}{(1 + z)^2}\right] = \frac{z}{H_0} - \frac{7z^2}{4H_0} + \frac{19z^3}{8H_0} - \ldots,
\]

(15)
\[ d_\ell = \frac{2}{H_0} \left( 1 + z - \sqrt{1 + z} \right) = \frac{z}{H_0} + \frac{z^2}{4H_0} - \frac{z^3}{8H_0} + \ldots, \quad (16) \]

\[ d_G = \frac{2}{H_0} \left( 1 + z - \sqrt{1 + z} \right) = \frac{z}{H_0} - \frac{3z^2}{4H_0} + \frac{5z^3}{8H_0} - \ldots, \quad (17) \]

where the power series expansions hold for \( z < 1 \). As expected, those distances coincide on first order, but what we seek here is to determine the influence on the average densities of higher order terms at moderate redshift ranges.

Figure 1 shows a plot of distances against redshift in EdS cosmology. It is clear the different asymptotic behaviours, as well as the deviation from one another at moderate values for \( z \). Therefore, second order terms play an important role at moderate redshift ranges, and since average densities are built as third power of distances, one can expect an even more important influence of second order terms on average densities.

The next step on our analysis is to define the observational volume. It is natural to use an extension of Euclidean volume, and when doing this we end up with three different expressions for observational volume,

\[ V_A = \frac{4}{3} \pi d_A^3, \quad V_\ell = \frac{4}{3} \pi d_\ell^3, \quad V_G = \frac{4}{3} \pi d_G^3. \quad (18) \]

We may also define a volume in the so-called “redshift space”, as follows,

\[ V_z = \frac{4}{3} \pi d_z^3, \quad (19) \]
where

\[ d_z = \frac{cz}{H_0}. \]  

(20)

In the equation above the two constants are necessary for correct dimension, and light velocity was included explicitly for clarity.

An interesting expression can be obtained from the equations above. If we substitute the galaxy area distance, as given by equation (17), into the volume defined by this distance in equation (18), we can easily obtain the following equation,

\[ V_G = \frac{32\pi}{3H_0^3(1+z)^{3/2}} \left[ \sqrt{1+z} - 1 \right]^{3/2}. \]  

(21)

This volume definition is exactly the same as presented by Sandage (1995, p. 19, eq. 1.42). The important point is that the expression above appears in many standard texts as if it were the only possible expression of volume as function of \( z \), ignoring the fact that there are three other volume definitions along the null cone which can be possibly used in any analysis, namely \( V_A, V_\ell, \) and \( V_z \). So, while Sandage’s (1995) presentation of cosmological observational relations is correct, it is not complete. One can define other types of observational volume and density, whose definitions are relevant to the smoothness problem in cosmology, and The Fractal Debate. In fact, it is the observational importance of Etherington’s reciprocity theorem that is currently being under appreciated in observational cosmology. We must bear this point in mind when discussing how other authors interpret cosmological observational data (see below).

To build expressions for the average density, the best method is to calculate source number counting along the past light cone. From the general expression derived by Ellis (1971, p. 159), we may obtain the bolometric number counting in the EdS model (Ribeiro 1992ab),

\[ N_c = \frac{2r^3}{9M_g}, \]  

(22)

where \( M_g \) is the average galactic rest mass (\( \sim 10^{11} M_\odot \)).

Now, average densities are easily calculated by means of the general expression \( \langle \rho \rangle = M_g N_c / V \). Since we have four types of volume, we will end up with four different average densities, with all being, in principle, obtainable from observational quantities. Considering equations (8), (15), (16), (17), (18), (19), and (22), they may be written, as follows,

\[ \langle \rho_\ell \rangle = \frac{\rho_0}{(1+z)^3}, \]  

(23)

\[ \langle \rho_z \rangle = 8\rho_0 \left[ \frac{1+z - \sqrt{1+z}}{z(1+z)} \right]^3, \]  

(24)

\[ \langle \rho_A \rangle = \rho_0 (1+z)^3, \]  

(25)

\[ \langle \rho_G \rangle = \rho_0, \]  

(26)

where,

\[ \rho_0 = \frac{3H_0^2}{8\pi} \]  

(27)
is the critical density, i.e., the EdS local density at the present time hypersurface.

The average density constructed with the galaxy area distance \(d_G\), given by equation (21), behaves as today’s local density, remaining constant along the null cone. Therefore, if one uses such a distance in the attempt to find some deviation from spatial homogeneity, even with data along the null cone, one will find none simply because choosing \(d_G\) leads to an associated constant average density. This is a feature of EdS cosmology. The other three averages are, therefore, the ones of importance for discussing observational deviations from spatial homogeneity. Their behaviour are displayed graphically in figure 2.

![Figure 2: Plot of average densities \(\langle \rho_\ell \rangle\) and \(\langle \rho_A \rangle\) respectively constructed using two different distance definitions, \(d_\ell\) and \(d_A\). The third average density \(\langle \rho_z \rangle\), is constructed in “redshift space”, that is, using \(z\) as distance in a volume definition given by equation (19). These average densities are plotted as a ratio between them and the present time local density \(\rho_0\). One can clearly see that deviations from spatial homogeneity already occur at small redshift ranges, becoming particularly large at \(z \approx 0.1\). Notice that at this same range the difference among the various distances is still small, as shown in figure 1. A 10% deviation from \(\rho_0\) occurs at \(z \approx 0.04\) (see Ribeiro 1995 for detailed calculations of relativistic corrections at low redshifts). Notice too the opposite behaviour of \(\langle \rho_A \rangle\) as compared to \(\langle \rho_\ell \rangle\) and \(\langle \rho_z \rangle\).

It is simple to see that when \(z \to 0\), \(\langle \rho_\ell \rangle = \langle \rho_z \rangle = \langle \rho_A \rangle = \rho_0\). So, these three averages tend to the present value of local density, \(\rho_0\), as they should. However, at the big bang, those averages will behave very differently. As at the big bang \(z \to \infty\) (eqs. 14), one may show that the following limits hold,

\[
\lim_{z \to \infty} \langle \rho_\ell \rangle = 0, \\
\lim_{z \to \infty} \rho = \infty,
\]
\[
\lim_{z \to \infty} \langle \rho_A \rangle = \infty, \quad (30)
\]
\[
\lim_{z \to \infty} \langle \rho_z \rangle = 0. \quad (31)
\]

These are remarkable results! Equation (28) had already been derived in Ribeiro (1992b), and equation (29) is a well known result in the literature. Equation (30) is not that surprising, since we know that the local density diverges at the big bang and we would expect that an average density would do so as well. So, the big surprise is the realization that even in standard cosmological models, some types of average densities vanish at the big bang. This is not restricted to EdS model, but occurs in all standard cosmologies (Ribeiro 1993). The fact that there are vanishing average densities even in standard cosmologies will be discussed below, in the context of The Fractal Debate.

It is worth writing power series expansions of equations (23), (24), and (25), as follows,
\[
\langle \rho_\ell \rangle = \rho_0 \left(1 - 3z + 6z^2 - 10z^3 + \ldots \right), \quad (32)
\]
\[
\langle \rho_z \rangle = \rho_0 \left(1 - \frac{9}{4} z + \frac{57}{16} z^2 - \frac{39}{8} z^3 + \ldots \right), \quad (33)
\]
\[
\langle \rho_A \rangle = \rho_0 \left(1 + 3z + 3z^2 + z^3 \right). \quad (34)
\]

Notice the existence of zeroth order terms in the expansions above, while power series expansions for the distances, as given by equations (15), (16), and (17) start on first order. As the Hubble law is a distance-redshift law, derived from the first order expansions of the distances, it is clear that due to the non-linearity of the Einstein field equations, observational relations behave differently at different redshift depths. Consequently, while the linearity of the Hubble law is well preserved in the EdS model up to \(z \approx 1\) (Ribeiro 1995), a value implicitly assumed as the lower limit up to where relativistic effects could be safely ignored, the observational average densities constructed with \(d_\ell\), \(d_A\), and \(z\) are strongly affected by relativistic effects at much lower redshift values. Then, while the zeroth order term vanishes in the distance-redshift relation, it is non-zero for the average density as plotted against redshift. This zeroth order term is the main factor for the different behaviour of these two observational quantities at small redshifts. Pietronero, Montuori and Sylos-Labini (1997) called this effect as the “Hubble-de Vaucouleurs paradox”. However, from the analysis presented here, and in Ribeiro (1995; see also Abdalla, Mohayae and Ribeiro 2000), it is clear that this is not a paradox, but just very different relativistic effects on the observables at the moderate redshift range.

This effect explains why Sandage, Tammann and Hardy (1972) failed to find deviations from uniform expansion in a hierarchical model: they were expecting that such a strong observational inhomogeneity would affect the velocity field, but it is clear now that if we take a relativistic perspective for these effects they are not correlated at the range expected by Sandage and collaborators. Notice that de Vaucouleurs and Wertz also expected that their inhomogeneous hierarchical models would also necessarily affect the velocity field, an effect also conjectured by Pietronero (1987), and change the linearity of the Hubble law at...
$z < 1$, and once such a change was not observed by Sandage, Tammann, and Hardy (1972) it was thought that this implied an immediate dismissal of the hierarchical concept. Again, this is not necessarily the case if we take a relativistic view of those observational quantities, as prescribed by Bondi’s criterion.

These findings can be summarized as follows. *The observational inhomogeneity of EdS cosmology is not related to the linearity of the Hubble law at moderate redshift ranges.* This conclusion can also be extended to open and close standard cosmologies, but with some limitations (Ribeiro 1993, 1995). Observers often use cosmological formulae which does not follow Bondi’s criterion, and so, they are often under the assumption that at the scales where observations are being made ($z < 1$) one can safely use the two simplifying assumptions discussed in §1, especially because in this range the Hubble law is observationally verified to be very linear. However, we saw above that Hubble law linearity has a range which only coincides with a constant density if we use the galaxy area distance $d_G$ as distance definition. With all other averages that does not happen. Since the observed average density is the key physical quantity for fractal characterization (Pietronero 1987; Pietronero, Montuori and Sylos-Labini 1997; Coleman and Pietronero 1992; Sylos-Labini, Montuori and Pietronero 1998; Ribeiro and Miguelote 1998), we must seek hints for fractal features in the behaviour of average densities which do not remain constant along the null cone in EdS cosmology.

Considering equations (14) we may rewrite equation (28), and conclude that in EdS cosmology the following limit holds,

$$\lim_{d_G \to \infty} \langle \rho \rangle = 0.$$  

(35)

This result provides a remarkable link to the hierarchical (fractal) tradition. Thirty years ago James R. Wertz (1970, 1971) hypothesized that a pure hierarchical cosmology ought to obey what he called “The Zero Global Density Postulate: for a pure hierarchy the global density exists and is zero everywhere” (Wertz 1970, p. 18). Such a result was also speculated by Pietronero (1987) as a natural development of his fractal model. Therefore, what the above limit tells us is that *the Einstein-de Sitter model does obey Wertz’s zero global density postulate, a key requirement of unlimited fractal cosmologies.* This result appears naturally when one studies cosmological observational relations in a fully relativistic setting.

In addition to the conclusion above, a quick look at figure 2 shows clearly that *two types of average densities decay at increasing distances in EdS cosmology, this being another key aspect of fractal cosmologies.*

### 3.2.2 Some Common Misconceptions

In the light of the results above, we are now in position to discuss some statements that appear in the literature about what the standard and fractal cosmologies can, or cannot be. They are, in effect, misconceptions, derived from no longer valid assumptions, as discussed in §1, which lead their authors to false problems. The first important misconception is to
state that finding homogeneity in astronomical galaxy distribution data is the only way to make sense of the FLRW cosmology, and not finding it leads to its falsification.

I showed above that some key fractal features appear in the EdS cosmology. In addition, Ribeiro (1993, 1994) showed that they can also be found in all standard cosmological models (see also Humphreys, Matravers and Marteens 1998). These results were obtained without any change in the model, its metric or its basic assumptions. So, those observed fractal features appear **alongside** well-known features of the model, like obedience of the cosmological principle, linearity of Hubble law, CMBR isotropy. Moreover, cosmological parameters such as $q_0$, $\Omega_0$, $H_0$ have their usual definitions and interpretations. Therefore, recognizing observational fractality in cosmology is not necessarily incompatible with well-known tenets of modern cosmology. Nevertheless, the viewpoint currently being sustained by both sides of The Fractal Debate is opposite to this one.

What is clear from all these results is that the homogeneity of the standard cosmological models is **spatial**, that is, it is a **geometrical** feature which does not necessarily translate itself into an astronomically observable quantity (Ribeiro 1992b, 1993, 1994, 1995). That happens only on special circumstances. Although a number of authors are aware of this fact, what came as a surprise had been the calculated low redshift value where this observational inhomogeneity appears (see details in Ribeiro 1992b, 1995). Therefore, it is clear now that **relativistic effects start to play an important role in observational cosmology at much lower redshift values than previously assumed, at least as far as the smoothness problem of the Universe is concerned.**

The second common misconception is to discuss the possible evidence towards observational homogeneity/inhomogeneity in the Universe without making explicit the distance choice made in the analysis. To see how this difficulty arises, let us try to clarify some puzzles surrounding The Fractal Debate by asking the following question: which distance definition is being implicitly used by the heterodox group? A thorough discussion of this issue is beyond the scope of this paper, as it demands a detailed study of the behaviour of these distances not on bolometric measurements, but on limited frequency bandwidth, as this is how astronomical data is gathered. The problem is that limited frequency range observational relations alter the power of $(1 + z)$ factors appearing in equations (2) (Ellis 1971; see also Ribeiro 1999), and we saw above how dramatic such a change can be on the average densities. Other effects must also be considered, like the luminosity function or K-correction, which may alter even further the average densities, with unpredictable results. Nevertheless, a sketchy discussion in bolometric terms can be provided here.

If one takes redshift data and, by means of the Hubble law, transform them into distances, by using the relation $cz = H_0d_z$, making no further $(1 + z)$ factors conversion, one will be choosing as distance indicator the distance definition that scales most closely with the Hubble law linearity. Figure 1 showed that this occurs with the luminosity distance. Figure 2 showed that an average density constructed that way decreases with higher distances. Cappi et al. (1998) criticized Pietronero and collaborators handling of data by not making
the K-correction, which implies inclusion of \((1 + z)\) factors due to conversion from limited frequency bandwidth observations to bolometric ones (Ribeiro 1999). That kind of conversion can, therefore, destroy the fractal like decay of the average density, by implicitly changing the distance definition. Therefore, I suspect that Pietronero and collaborators are systematically choosing \(d_\ell\), or \(dz\), as distance in their papers, while other authors may be using other distances. That may well explain the enormous difference in behaviour that various authors, who are engaged in this debate, are finding with the same data set.

To give another example of the difficulties generated when one ignores the distance problem in cosmology, let us discuss the recent report advanced by Pan and Coles (2000) where, by using a multifractal analysis in the QDOT sample, they concluded that there is firm evidence towards its observational homogenization at larger scales. Their study starts by choosing a distance definition as given by Mattig’s formula,

\[
R = \frac{1}{H_0q_0^2(1 + z)} \left[ q_0z + (q_0 - 1) \left( \sqrt{2q_0z + 1} - 1 \right) \right],
\]

(36)

where \(R\) is their distance choice. In the context of this paper, the obvious question is, what is \(R\)? In other words, which distance definition are they implicitly choosing? They use the EdS model, and then a trivial calculation taking \(q_0 = 1/2\) reduces equation (36) to

\[
R = \frac{2}{H_0} \left( \frac{1 + z - \sqrt{1 + z}}{1 + z} \right).
\]

(37)

Comparing with equation (17) we conclude that

\[
R = d_G.
\]

(38)

So, Pan and Coles have implicitly chosen the galaxy area distance to carry out their analysis, which, then, continues by choosing cells of size \(R\) and then performing a multifractal measure. As seen above, \(d_G\) is inappropriate for such kind of data analysis as it has the in-built feature of showing no deviation from spatial homogeneity, even if the Universe is of Friedmann type (see eq. 26). The authors did not provide any justification for the use of this equation, having in fact ignored altogether the difficulties related to the distance choice problem as discussed above. Their conclusions can, therefore, be objected on the following grounds.

As they chose \(d_G\), instead of \(d_\ell\) or \(d_A\), their results cannot be related to the discussion made in here about fractal features in EdS cosmology. In fact they cannot even be related to the data analysis performed by Pietronero and collaborators as they have, most likely, been using the luminosity distance. If Pan and Coles (2000) were to change the chosen distance definition from galaxy area distance to luminosity distance, or observer area distance, that would mean multiplying their distance \(R\) by a factor of \((1 + z)\), or dividing by \((1 + z)\), respectively. Moreover, as they are using a flux limited sample, another \((1 + z)\) factor must be considered when changing from bolometric to flux limited measures (Ellis 1971, p. 161). I wonder how those changes would modify their final results. Consequently, their conclusions are of much narrower scope than stated by the authors, and their analysis is inappropriate for probing the possible observational inhomogeneity of the Universe.
4 The Apparent Fractal Conjecture

The results discussed above show that some key fractals features can already be found in the simplest possible standard cosmological model, that is, in the unperturbed Einstein-de Sitter universe. However, as the average densities constructed with \( d_\ell \) and \( z \) do not decay linearly in this model, considering all these aspects we may naturally ask whether or not a perturbed model could turn the density decay at increasing redshift depths into a power law type decay, as predicted by the fractal description of galaxy clustering. If this happens, then standard cosmology can be reconciled with a fractal galaxy distribution. Notice that there are some indications that this is a real possibility, as Amendola (2000) pointed out that locally the cold dark matter and fractal models predict the same behaviour for the power spectrum, a conclusion apparently shared by Cappi et al. (1998). In addition, confirming Ribeiro’s (1992b, 1995) conclusions, departures from the expected Euclidean results at small redshifts were also reported by Longair (1995, p. 398), and the starting point for his findings was the same as employed in here and by Ribeiro (1992b, 1995): the use of source number count expression along the null cone.

Considering all results outlined above, I feel there is enough grounds to advance the following conjecture: *the observed fractality of the large-scale distribution of galaxies should appear when observational relations necessary for fractal characterization are calculated along the past light cone in a perturbed metric of standard cosmology*. By “observational relations necessary for fractal characterization” I mean choosing \( d_\ell \) or \( d_z \) as distances, and building average densities with them, that is, deriving source number counting expressions and calculating \( \langle \rho_\ell \rangle \) and \( \langle \rho_z \rangle \) as defined above, all that along the past light cone.

This conjecture has theoretical and observational implications. On the theoretical side, one can no longer ignore the distance choice, and all calculations must clearly start with one. On the observational side, a careful analysis is necessary about the way data is collected, reduced and organized, as an implicit distance choice may occur during this process.

If this conjecture proves, even partially, correct, fractals in cosmology would no longer be necessarily seen as opposed to the cosmological principle. Notice that this can only happen in circumstances where fractality is characterized by an *observed*, smoothed out, and averaged fractal system, as opposed to building a fractal structure in the very spacetime geometrical structure, as initially thought necessary to do for having fractals in cosmology (Mandelbrot 1983; Ribeiro 1992a). Thus, the usual tools used in relativistic cosmology, like the fluid approximation, will remain valid. As a possible consequence of this conjecture, a detailed characterization of the observed fractal structure could provide direct clues for the kind of cosmological perturbation necessary in our cosmological models, and this could shed more light in issues like galaxy formation.

A recent attempt to check the validity of this conjecture showed, although in a restricted perturbative sense, that this conjecture is sound as an apparent fractal pattern did emerge from the model (Abdalla, Mohayaee, and Ribeiro 2000).
5 Conclusions

In this paper I have presented an analysis of the smoothness problem of the Universe by focussing on the ambiguities arising from the simplifying hypotheses aimed at observationally verifying whether or not the large-scale distribution of galaxies is homogeneous. After briefly reviewing The Fractal Debate, it is pointed out that in order to analyse the possible observational homogeneity of the Universe one requires to make a clear distinction between local and average density in a relativistic framework. Then I showed that the different cosmological distance definitions strongly affect the average density. An example, using the Einstein-de Sitter cosmological model, is worked out, where I showed that various observational average densities can be defined in this cosmology, with the majority of them not leading to well defined values at $z \approx 0.1$. I also revisited the discussion made in Ribeiro (1995), which showed that the linearity of the Hubble law does not imply in an observationally homogeneous density distribution of dust at the moderate redshift ranges ($0.1 \leq z < 1$). Finally, I propose a conjecture stating that the large-scale galaxy distribution should follow a fractal pattern if observational relations necessary for fractal characterization are evaluated along the past light cone. All these results were obtained without any change in the standard cosmological models metric, meaning that its observational fractality, as described in here, appears in cosmologies which obey the cosmological principle, and have a near isotropy of the cosmic microwave background radiation.

As discussed above, the divide caused by The Fractal Debate may not be as radical as presented by both sides, and that it is possible to build a bridge between both opinions, reconciling them by means of a change in perspective regarding how we deal with observations in cosmology. What was seen above is that there is already enough theoretical evidence to suggest that the observed fractality can be accommodated within the standard cosmology, where it would stem from the special way we are forced to collect, organize and display our observational data on galaxy distribution. And this special observational data collection and organization are, in turn, a consequence of the underlining geometrical structure of Friedmannian spacetime. Under this theoretical perspective, the cosmological principle, uniform Hubble expansion, CMBR isotropy, and well defined meanings for the cosmological parameters, such as $\Omega_0$, can survive, together with the observational fractality obtained by the heterodox group mentioned above. This perspective has the advantage of preserving most of what we have learned with the standard FLRW cosmology, and, at the same time, making sense of Pietronero and collaborators’ data, which, as seen above, can no longer be easily dismissed.

At this point a relevant question arises immediately, and requires an answer. If a FLRW cosmology may be observed to look like an universe with properties of fractals, could this effect be nullified by using relevant distances to match the way observations have actually been carried out? In order to answer this question, it is important to point out first of all that actual observations on galaxy distribution basically consist of tables listing integrated...
$F$, $z$, and projected angle positions on the sky. These are the real observations, and all else is theory and interpretation. To state that there could be a manner such that one can mimic the ways in which observations have “actually”, or “really” been carried out is the same as to say that there is an actual, or real, cosmological distance. As we have seen above, searching for a “true” distance is a futile exercise, and the same is also true regarding attempts to say how observations are actually made. It is the handling of data that matters here, and not the actual observations.

Therefore, at this stage it must be very clear that the essential issues dealt with in this paper can be summarized as follows. While observers have been handling astronomical data for quite a while, and claiming that they are consistent with a spatially homogeneous FLRW cosmology, the main point raised in here is that, from a theoretical viewpoint, the same FLRW cosmology may be consistent with an universe with observed fractal properties, as supported by data collected by Pietronero and collaborators. In fact, claims that the only way to make sense of FLRW model is by observing homogeneity, ignores the richness of the standard model by sticking to a somewhat narrow interpretation of its observables features. The various ways that data is handled (e.g., K-correction) and effects are considered (e.g., galaxy evolution) will affect interpretation of data, and what I have discussed above is that fractal properties might also be part of those interpretations, and should not be considered as extraneous, irrelevant, or wrong cosmological data handling. Indeed, if the observational fractality is one of the many possible interpretations of galaxy distribution data, one may speculate that its fractal dimension may become an important cosmological parameter, perhaps to be taken into consideration in any model of galaxy formation.

Another way of summarizing the results of this paper is by noticing that while observational cosmologists were aware of of the possible apparent inhomogeneity of the standard model, which may also be called densities changing with time, or, still, lookback effects, it is clear that this phenomenon occurs at close ranges in EdS cosmology. The link with fractal properties, that is, a smoothed-out and averaged fractal system possessing properties of power-law average density decay, as originally proposed by Pietronero (1987), but whose roots can be found in Wertz’s (1970) work, occurs because some of these observational densities decay, rather than increase, at deeper distances, or, which is the same, earlier times. Cosmologists have been working with the hypotheses that, (1) this effect should not be important for $z < 1$, i.e., the effect of densities changing with time is not relevant for the observational determination of whether or not the Universe is observationally homogeneous, and (2), because the local density diverges at the big bang, the same should happen to all density definitions. It should be clear by now that these two hypotheses have difficulties when we consider the full consequences of the reciprocity theorem in observational quantities. Therefore, the analysis presented in standard texts is not complete. There is more to be said on those issues than can be found in standard texts, and this paper attempts at adding some ideas and results in the context of the possible observational homogeneity of the Universe.
Reconciling two seemingly disparate universe models through the recognition of the importance of a previously dismissed physical effect would not be new in observational cosmology. During the 1910’s there was observational evidence supporting two opposing models about the size and structure of the Milky Way, the Kapteyn Universe and Shapley’s Model. The former sustained that the Sun was located near the centre of an approximately oblate spheroidal distribution of stars, whose dimension was estimated to be about 8.5kpc, while the latter located the Sun at the edge of the stellar system and estimated the size of the Milky Way as being 100kpc. As at that time the nature of the nebulae, that is, the question of whether or not they were structures belonging to the Milky Way or external objects placed at distances much greater than the size of the Milky Way, was still an unresolved issue (solved later by Hubble), these two models were effectively dealing with the observable Universe of the time, and, therefore, such a discrepancy was, perhaps, the main puzzle in observational cosmology of that epoch, and which effectively led to a split in the astronomical community. The public confrontation of these two views took place in April 1920, and this event is now known as “The Great Debate”, although the final reconciliation between them was only reached in the 1930’s, when astronomers generally recognized that the apparent stellar distribution is dominated by the effects of absorption. Kapteyn himself allowed this possibility, but as he only considered Rayleigh scattering as the possible source of obscuration, he dismissed this effect once he found it to be small, while we now know that the dominant obscuration source is dust absorption. On perspective, it is clear now that both sides of the debate had elements of the truth, as we perceive it today, and even the heliocentric Kapteyn Universe is not that absurd since the Sun does lie close to the centre of a local loose cluster of stars.¹

The historical lesson to be learned from this episode is that controversial issues in cosmology are not necessarily solved with the simple dismissal of one side of the debate, as happened to be the case in the 1920’s controversy of static versus expanding universe or, later, the steady state versus evolving universe, in the 1960’s. Based on the theses exposed above, it is the opinion of this author that The Fractal Debate may well be overcome in a similar manner as the issues surrounding The Great Debate, but in this case only when one recognizes that relativistic effects and their consequences must be fully considered in observational cosmology.

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¹ See Binney and Merrifield (1998, pp. 5-15), for a historical overview of this earlier split of the astronomical community.
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