We propose algorithms for online principal component analysis (PCA) and variance minimization for adaptive settings. Previous literature has focused on upper bounding the static adversarial regret, whose comparator is the optimal fixed action in hindsight. However, static regret is not an appropriate metric when the underlying environment is changing. Instead, we adopt the adaptive regret metric from the previous literature and propose online adaptive algorithms for PCA and variance minimization, that have sub-linear adaptive regret guarantees. We demonstrate both theoretically and experimentally that the proposed algorithms can adapt to the changing environments.

1 Introduction

In the general formulation of online learning, at each time step, the decision maker makes decision without knowing its outcome, and suffers a loss based on the decision and the observed outcome. Loss functions are chosen from a fixed class, but the sequence of losses can be generated deterministically, stochastically, or adversarially.

Online learning is a very popular framework with many variants and applications, such as online convex optimization [1, 2], online non-convex optimization [3, 4], online auctions [5], and online classification and regression [6]. What’s more, the recent advances in linear dynamical system identification [7] and reinforcement learning [8] are developed based on the ideas from online learning.

The standard performance metric for online learning measures the difference between the decision maker’s cumulative loss and the cumulative loss of the best fixed decision in hindsight [9]. We call this metric static regret, since the comparator is the best fixed optimum in hindsight. However, when the underlying environment is changing, static regret is no longer appropriate, since the comparator is a fixed point [10].

Alternatively, to capture the changes of the underlying environment, [11] introduced the metric called adaptive regret, which is defined as the maximum static regret over any contiguous time interval.

In this paper, we are mainly concerned with the problem of online Principal Component Analysis (online PCA) for adaptive settings. Previous online PCA algorithms are based on either online gradient descent or matrix exponentiated gradient algorithms [12, 13, 14, 15]. These works bound the static regret for online PCA algorithms, but do not address adaptive regret. As argued above, static regret is not appropriate under changing environments.

This paper gives an efficient algorithm for online PCA and variance minimization in changing environments. The proposed method combines the randomized algorithm from [14] with a fixed-share step [10]. This is inspired by the work of [16, 17], which shows that the hedge algorithm [18] combined with a fixed-share step provides low regret under a variety of measures, including adaptive regret.

Furthermore, we extend the idea of the additional fixed-share step to the problem of the online adaptive variance minimization under two different parameter space: unit vector space and the simplex space. We also test our algorithm’s
effectiveness in the experiment section, and show that our proposed algorithm can adapt to the changing environment faster than the previous online PCA algorithm.

While it is possible to apply the algorithm in [11] to solve the online adaptive PCA and variance minimization problems with the same order of the adaptive regret as in this paper, it requires running a pool of algorithms in parallel. Running this pool algorithms requires complex implementation that increases the running time per step by a factor of $\log T$ when compared to our algorithm.

1.1 Notation

Vectors are denoted by bold lower-case symbols. The $i$-th element of a vector $q$ is denoted by $q_i$. The $i$-th element of a sequence of vectors, $x_i$, is denoted by $x_{i,i}$.

For two probability vectors $\mathbf{q}, \mathbf{w} \in \mathbb{R}^n$, we use $d(q, w)$ to represent the relative entropy between them, which is defined as $\sum_{i=1}^n q_i \ln \left( \frac {q_i} {w_i} \right)$. The $\ell_1$-norm and $\ell_2$-norm of the vector $\mathbf{q}$ are denoted as $\|\mathbf{q}\|_1$, $\|\mathbf{q}\|_2$, respectively. $\mathbf{q}_{1:T}$ is the sequence of vectors $\mathbf{q}_1, \ldots, \mathbf{q}_T$, and $m(\mathbf{q}_{1:T})$ is defined to be equal to $\sum_{t=1}^{T-1} D_{TV}(\mathbf{q}_{t+1}, \mathbf{q}_t)$, where $D_{TV}(\mathbf{q}_t, \mathbf{q}_{t-1})$ is defined as $\sum_{i:t,q_{t,i} \geq q_{t-1,i}} (q_{t,i} - q_{t-1,i})$. The expected value operator is denoted by $\mathbb{E}$.

When we refer to a matrix, we use capital letters such as $P$ and $Q$ with $\|Q\|_2$ representing the spectral norm. For the identity matrix, we use $I$. The quantum relative entropy between two density matrices $P$ and $Q$ is defined as $\Delta(V, W_t) = \text{Tr}(V \ln V) - \text{Tr}(V \ln W_t)$, where $\ln V$ is the matrix logarithm for symmetric positive definite matrix $V$ (and $\exp(V)$ is the matrix exponential).

2 Problem Formulation

The goal of the PCA (uncentered) algorithm is to find a rank $k$ projection matrix $P$ that minimizes the compression loss: $\sum_{t=1}^T \|x_t - P^T x_t\|_2^2$. In this case, $P \in \mathbb{R}^{n \times n}$ must be a symmetric positive semi-definite matrix with only $k$ non-zero eigenvalues which are all equal to $1$.

In online PCA, the data points come in series. At each time $t$, the algorithm first chooses a projection matrix $P_t$ with rank $k$, then the data point $x_t$ is revealed, and a compression loss of $\|x_t - P_t^T x_t\|_2^2$ is incurred.

The online PCA algorithm [14] aims to minimize the static regret $\mathcal{R}_a$, which is the difference between the total expected compression loss and the loss of the best projection matrix $P^*$ chosen in hindsight:

$$\mathcal{R}_a = \sum_{t=1}^T \mathbb{E}[\text{Tr}((I - P_t)x_t x_t^T)] - \sum_{t=1}^T \text{Tr}((I - P^*)x_t x_t^T).$$

The algorithm from [14] is randomized and the expectation is taken over the distribution of $P_t$ matrices. The matrix $P^*$ is the solution to the following optimization problem with $S$ being the set of rank-$k$ projection matrices:

$$\min_{P \in S} \sum_{t=1}^T \text{Tr}((I - P)x_t x_t^T)$$

Algorithms that minimize static regret will converge to $P^*$, which is the best projection for the entire data set. However, in many scenarios the data generating process changes over time. In this case, a solution that adapts to changes in the data set may be desirable. To model environmental variation, several notions of dynamically varying regret have been proposed [10][11][6]. In this paper, we study adaptive regret $\mathcal{R}_a$ from [11], which results in the following online adaptive PCA problem:

$$\mathcal{R}_a = \max_{r,s \in [1,T]} \left\{ \sum_{t=r}^s \mathbb{E}[\text{Tr}((I - P_t)x_t x_t^T)] - \min_{U \in S} \sum_{t=r}^s \text{Tr}((I - U)x_t x_t^T) \right\}$$

In the next few sections, we will present an algorithm that achieves low adaptive regret.

---

1 A density matrix is a symmetric positive semi-definite matrix with trace equal to 1. Thus, the eigenvalues of a density matrix form a probability vector.
Algorithm 1 Adaptive Best Subset of Experts
1: Input: $1 \leq k < n$ and an initial probability vector $w_1 \in B^n_{n-k}$.
2: for $t = 1$ to $T$ do
3: Use Algorithm 2 with input $d = n - k$ to decompose $w_t$ into $\sum_j p_j r_j$, which is a convex combination of at most $n$ corners of $r_j$.
4: Randomly select a corner $r = r_j$ with associated probability $p_j$.
5: Use the $k$ components with zero entries in the drawn corner $r$ as the selected subset of experts.
6: Receive loss vector $\ell_t$.
7: Update $w_{t+1}$ as:
   \begin{align}
   v_{t+1,i} &= \frac{w_{t,i} \exp(-\eta \ell_{t,i})}{\sum_j \exp(-\eta \ell_{t,j})} \tag{4a} \\
   \hat{w}_{t+1,i} &= \frac{\alpha}{n} + (1 - \alpha)v_{t+1,i} \tag{4b} \\
   w_{t+1} &= \text{cap}_{n-k}(\hat{w}_{t+1}) \tag{4c}
   \end{align}
8: end for

Algorithm 2 Mixture Decomposition [14]
1: Input: $1 \leq d < n$ and $w \in B^n_d$.
2: repeat
3: Let $r$ be a corner for a subset of $d$ non-zero components of $w$ that includes all components of $w$ equal to $\frac{|w|}{d}$.
4: Let $s$ be the smallest of the $d$ chosen components of $r$ and $l$ be the largest value of the remaining $n - d$ components.
5: update $w$ as $w - \min(ds, |w| - dl)r$ and Output $p$ and $r$.
6: until $w = 0$

3 Learning the Adaptive Best Subset of Experts

In [14] it was shown that online PCA can be viewed as an extension of a simpler problem known as the best subset of experts problem. In particular, they first propose an online algorithm to solve the best subset of experts problem, and then they show how to modify the algorithm to solve PCA problems. In this section, we show how the addition of a fixed-share step [10, 16] can lead to an algorithm for an adaptive variant of the best subset of experts problem. Then we will show how to extend the resulting algorithm to PCA problems.

The adaptive best subset of experts problem can be described as follows: we have $n$ experts making decisions at each time $t$. Before revealing the loss vector $\ell_t \in \mathbb{R}^n$ associated with the experts’ decisions at time $t$, we select a subset of
Lemma 1. With the Algorithm 1, the following lemma can be obtained:

\[ R^{\text{subexp}}_{\alpha} = \max_{[r,s] \subset [1,T]} \left\{ \sum_{t=r}^{s} \mathbb{E}[\mathbf{v}_t^T \ell_t] - \min_{\mathbf{u} \in \mathcal{S}_{\alpha}} \sum_{t=r}^{s} \mathbf{u}^T \ell_t \right\}. \]  

Here, the expectation is taken over the probability distribution of \( \mathbf{v}_t \). Both \( \mathbf{v}_t \) and \( \mathbf{u} \) are in \( \mathcal{S}_{\alpha} \) which denotes the vector set with only \( n-k \) non-zero elements equal to 1.

Similar to the static regret case from [14], the problem in Eq.(5) is equivalent to:

\[ R^{\text{subexp}}_{\alpha} = \max_{[r,s] \subset [1,T]} \left\{ \sum_{t=r}^{s} (n-k) \mathbf{w}_t^T \ell_t - \min_{\mathbf{q} \in \mathcal{B}_{n-k}^n} \sum_{t=r}^{s} (n-k) \mathbf{q}^T \ell_t \right\} \]  

where \( \mathbf{w}_t \in \mathcal{B}_{n-k}^n \), and \( \mathcal{B}_{n-k}^n \) represents the capped probability simplex defined as \( \sum_{i=1}^{n} w_{t,i} = 1 \) and \( 0 \leq w_{t,i} \leq 1/(n-k), \forall i \).

**Connection to the online adaptive PCA.** The problem from Eq.(5) can be viewed as a restricted version of the online adaptive PCA problem from Eq.(3). In particular, say that \( J - P_t = \text{diag}(\mathbf{v}_t) \). This corresponds to restricting \( P_t \) to be diagonal. If \( \ell_t \) is the diagonal of \( \mathbf{x}_t \mathbf{x}_t^T \), then the objectives of Eq.(5) and Eq.(3) are equal.

We now return to the adaptive best subset of experts problem. When \( r = 1 \) and \( s = T \), the problem reduces to the standard static regret minimization problem, which is studied in [14]. Their solution applies the basic Hedge Algorithm to obtain a probability distribution for the experts, and modifies the distribution to select a subset of the experts.

To deal with the adaptive regret considered in Eq.(6), we propose the Algorithm 1 where the corner \( r_j \) in step 3 is defined as having \( n-k \) non-zero elements equal to \( 1/(n-k) \).

The proposed Algorithm 1 is a simple modification to Algorithm 1 in [14]. More specifically, we add Eq.(4b) when updating \( \mathbf{w}_{t+1} \) in Step 7, which is called a fixed-share step. This is inspired by the analysis in [16], which shows that the online adaptive best expert problem can be solved by simply adding this fixed-share step to the standard Hedge algorithm.

With the Algorithm 1, the following lemma can be obtained:

**Lemma 1.** For all \( t \geq 1 \), all \( \ell_t \in [0,1]^n \), and for all \( \mathbf{q}_t \in \mathcal{B}_{n-k}^n \), Algorithm 1 satisfies

\[ \mathbf{w}_t^T \ell_t (1 - \exp(-\eta)) - \eta \mathbf{q}_t^T \ell_t \leq \sum_{i=1}^{n} q_{t,i} \ln(\frac{d_{t,i+1}}{d_{t,i}}) \]

**Proof.** With the update in Eq.(4), for any \( \mathbf{q}_t \in \mathcal{B}_{n-k}^n \), we have

\[ d(\mathbf{q}_t, \mathbf{w}_t) - d(\mathbf{q}_t, \mathbf{v}_{t+1}) = - \eta \mathbf{q}_t^T \ell_t - \ln(\sum_{j=1}^{n} w_{t,j} \exp(-\eta \ell_{t,j})) \]  

Also, from the proof of Theorem 1 in [14], we have \( - \ln(\sum_{j=1}^{n} w_{t,j} \exp(-\eta \ell_{t,j})) \geq (1 - \exp(-\eta)) \). Thus, we will get

\[ d(\mathbf{q}_t, \mathbf{w}_t) - d(\mathbf{q}_t, \mathbf{v}_{t+1}) \geq - \eta \mathbf{q}_t^T \ell_t + \mathbf{w}_t^T \ell_t (1 - \exp(-\eta)) \]

Moreover, Eq.(4c) is the solution to the following projection problem as shown in [14]:

\[ \mathbf{w}_t = \text{argmin}_{\mathbf{w} \in \mathcal{B}_{n-k}^n} d(\mathbf{w}, \mathbf{w}_t) \]

Since the relative entropy is one kind of Bregman divergence [19, 20], the Generalized Pythagorean Theorem holds [21]:

\[ d(\mathbf{q}_t, \mathbf{w}_t) - d(\mathbf{q}_t, \mathbf{w}_t) \geq d(\mathbf{w}_t, \hat{w}_t) \geq 0 \]

where the last inequality is due to the non-negativity of Bregman divergence.

Combining Eq. (8) with Eq. (10) and expanding the left part of \( d(\mathbf{q}_t, \mathbf{w}_t) - d(\mathbf{q}_t, \mathbf{v}_{t+1}) \), we arrive at Lemma 1. \qed 

In order to upper bound the adaptive regret in Eq.(6), we also need the following lemma from [18]:

**Lemma 2.** Suppose \( 0 \leq L \leq \hat{L} \) and \( 0 < R \leq \hat{R} \). Let \( \beta = g(\hat{L}/\hat{R}) \) where \( g(z) = 1/(1 + \sqrt{2/z}) \). Then

\[ \frac{-L \ln \beta + R}{1 - \beta} \leq L + \sqrt{2LR + R} \]
Recall that the online adaptive PCA problem is below:

**Algorithm 4** Uncentered online adaptive PCA

1. **Input:** \(1 \leq k < n\) and an initial density matrix \(W_1 \in B_{n-k}^n\).
2. **for** \(t = 1\) to \(T\) **do**
3. Apply eigendecomposition to \(W_t\) as \(W_t = D \operatorname{diag}(w_t) D^T\).
4. Apply Algorithm 2 with \(d = n - k\) to the vector \(w_t\) to decompose it into a convex combination \(\sum_j p_j r_j\) of at most \(n\) corners \(r_j\).
5. Randomly select a corner \(r = r_j\) with the associated probability \(p_j\).
6. Form a density matrix \(R = (n - k) D \operatorname{diag}(r) D^T\).
7. Form a rank \(k\) projection matrix \(P_t = I - R\).
8. Obtain the data point \(x_t\), which incurs the compression loss \(\|x_t - P_t x_t\|_2^2\) and expected compression loss \((n-k) \operatorname{Tr}(W_t x_t x_t^T)\).
9. Update \(W_{t+1}\) as:

\[
V_{t+1} = \frac{\exp(W_t - \eta x_t x_t^T)}{\exp(W_t - \eta x_t x_t^T)}
\]

\[
\hat{w}_{t+1,i} = \frac{1}{n} + (1 - \alpha) \hat{w}_{t+1,i}, \quad \hat{W}_{t+1} = U \operatorname{diag}(\hat{w}_{t+1}) U^T
\]

\[
W_{t+1} = \operatorname{cap}_{n-k}(\hat{W}_{t+1})
\]

where we apply eigendecomposition to \(V_{t+1}\) as \(V_{t+1} = U \operatorname{diag}(v_{t+1}) U^T\), and \(\operatorname{cap}_{n-k}(W)\) invokes Algorithm 3 with input being the eigenvalues of \(W\).

**end for**

Now we are ready to state the following theorem to upper bound the adaptive regret \(R_{\text{subexp}}\):

**Theorem 1.** If we run the Algorithm 4 to select a subset of \(n - k\) experts, then for any sequence of loss vectors \(\ell_1, \ldots, \ell_T \in [0,1]^n\) with \(T \geq 1\), \(\min_{q \in B_{n-k}^n} \sum_{t=1}^{T} (n-k) q^T \ell_t \leq L\), \(\alpha = 1/(T(n-k)+1)\), \(D = (n-k) \ln(n+(n-k)T)+1\), and \(\eta = \ln(1+\sqrt{2D/L})\), we have

\[
R_{\text{subexp}} \leq O(\sqrt{2LD} + D)
\]

For space purposes, the proof is in the appendix.

### 4 Online Adaptive PCA

Recall that the online adaptive PCA problem is below:

\[
R_a = \max_{[r,s] \subseteq [1,n]} \left\{ \sum_{t=r}^{T} \mathbb{E}[\operatorname{Tr}((I - P_t) x_t x_t^T)] - \min_{U \in \mathcal{S}} \sum_{t=r}^{T} \operatorname{Tr}((I - U) x_t x_t^T) \right\}
\]

where \(\mathcal{S}\) is the rank \(k\) projection matrix set.

Again, inspired by [14], we first reformulate the above problem into the following 'capped probability simplex' form:

\[
R_a = \max_{[r,s] \subseteq [1,n]} \left\{ \sum_{t=r}^{T} (n-k) \operatorname{Tr}(W_t x_t x_t^T) - \min_{Q \in B_{n-k}^n} \sum_{t=r}^{T} (n-k) \operatorname{Tr}(Q x_t x_t^T) \right\}
\]

where \(W_t \in B_{n-k}^n\), and \(B_{n-k}^n\) is the set of all density matrices with eigenvalues bounded by \(1/(n-k)\). Note that \(B_{n-k}^n\) can be expressed as the convex set \(\{W : W \succeq 0, \|W\|_2 \leq 1/(n-k), \operatorname{Tr}(W) = 1\}\).

The static regret online PCA is a special case of the above problem with \(r = 1\) and \(s = T\), and is solved by Algorithm 5 in [14].

Follow the idea in the last section, we propose the Algorithm 4. Compared with the Algorithm 5 in [14], we have added the fixed-share step in the update of \(W_{t+1}\) at step 9, which will be shown to be the key in upper bounding the adaptive regret of the online PCA.

In order to analyze Algorithm 4 we need a few supporting results. The first result comes from [13]:
Theorem 2. [13] For any sequence of data points $\mathbf{x}_1, \ldots, \mathbf{x}_T$ with $\mathbf{x}_t \mathbf{x}_t^T \succeq 1$ and for any learning rate $\eta$, the following bound holds for any matrix $Q_t \in \mathbb{B}_{n-k}^n$ with the update in Eq. (11a):

$$\text{Tr}(W_t \mathbf{x}_t \mathbf{x}_t^T) \leq \frac{\Delta(Q_t, W_t) - \Delta(Q_t, V_{t+1}) + \eta \text{Tr}(Q_t \mathbf{x}_t \mathbf{x}_t^T)}{1 - \exp(-\eta)}$$

Based on the above theorem’s result, we have the following lemma:

Lemma 3. For all $t \geq 1$, all $\mathbf{x}_t$ with $\|\mathbf{x}_t\|_2 \leq 1$, and for all $Q_t \in \mathbb{B}_{n-k}^n$, Algorithm 4 satisfies:

$$\text{Tr}(W_t \mathbf{x}_t \mathbf{x}_t^T)(1 - \exp(-\eta)) - \eta \text{Tr}(Q_t \mathbf{x}_t \mathbf{x}_t^T) \leq -\text{Tr}(Q_t \ln W_t) + \text{Tr}(Q_t \ln V_{t+1})$$

Proof. First, we need to reformulate the above inequality in Theorem 2 we have:

$$\Delta(Q_t, W_t) - \Delta(Q_t, V_{t+1}) \geq -\eta \text{Tr}(Q_t \mathbf{x}_t \mathbf{x}_t^T) + \text{Tr}(W_t \mathbf{x}_t \mathbf{x}_t^T)(1 - \exp(-\eta))$$

which is very similar to the Eq. (14).

As is shown in [14], the Eq. (11c) is the solution to the following optimization problem:

$$W_t = \arg\min_{W \in \mathbb{B}_{n-k}^n} \Delta(W, \hat{W}_t)$$

(16)

As a result, the Generalized Pythagorean Theorem holds [21] for any $Q_t \in \mathbb{B}_{n-k}^n$:

$$\Delta(Q_t, \hat{W}_t) - \Delta(Q_t, W_t) \geq \Delta(W_t, \hat{W}_t) \geq 0$$

(17)

Combining the above inequality with Eq. (15) and expanding the left part, we have

$$\text{Tr}(W_t \mathbf{x}_t \mathbf{x}_t^T)(1 - \exp(-\eta)) - \eta \text{Tr}(Q_t \mathbf{x}_t \mathbf{x}_t^T) \leq -\text{Tr}(Q_t \ln W_t) + \text{Tr}(Q_t \ln V_{t+1})$$

(18)

which proves the result.

Additionally, we need the following classic bound on traces for postive semi-definite matrices. See, e.g. [12].

Lemma 4. For any positive semi-definite matrix $A$ and any symmetric matrices $B$ and $C$, $B \preceq C$ implies $\text{Tr}(AB) \leq \text{Tr}(AC)$.

In the next theorem, we show that with the addition of the fixed-share step in Eq. (11b), we can solve the online adaptive PCA problem in Eq. (12).

Theorem 3. For any sequence of data points $\mathbf{x}_1, \ldots, \mathbf{x}_T$ with $\|\mathbf{x}_t\|_2 \leq 1$, and for $\min_{Q \in \mathbb{B}_{n-k}^n} \sum_{t=r}^s (n - k) \text{Tr}(Q \mathbf{x}_t \mathbf{x}_t^T) \leq L$, if we run Algorithm 4 with $\alpha = 1/(T(n - k) + 1)$, $D = (n - k) \ln(n(1 + (n - k)T)) + 1$, and $\eta = \ln(1 + \sqrt{2D/L})$, for any $T \geq 1$ we have:

$$R_u \leq O(\sqrt{2LD} + D)$$

For space purposes, the proof is in appendix.

5 Extension to Online Adaptive Variance Minimization

In this section, we study the closely related problem of online adaptive variance minimization. The problem is defined as follows: At each time $t$, we first select a vector $\mathbf{y}_t \in \Omega$, and then a covariance matrix $C_t \in \mathbb{R}^{n \times n}$ such that $0 \preceq C_t \preceq I$ is revealed. The goal is to minimize the adaptive regret defined as:

$$R_u^{\text{var}} = \max_{[r,s] \subseteq [1,T]} \left\{ \sum_{t=r}^s \mathbb{E}[\mathbf{y}_t^T C_t \mathbf{y}_t] - \min_{u \in \Omega} \sum_{t=r}^s \mathbf{u}^T C_t \mathbf{u} \right\}$$

(19)

where the expectation is taken over the probability distribution of $\mathbf{y}_t$. 

6
Algorithm 5 Online adaptive variance minimization over unit sphere

1: Input: an initial density matrix $Y_1 \in \mathbb{B}^n_1$.
2: for $t = 1$ to $T$ do
3: Perform eigendecomposition $Y_t = \hat{D} \text{diag}(\sigma_t) \hat{D}^T$.
4: Use the vector $y_t = \hat{D}[i, j]$ with probability $\sigma_{t, j}$.
5: Receive covariance matrix $C_t$, which incurs the loss $y_t^T C_t y_t$ and expected loss $\text{Tr}(Y_t C_t)$.
6: Update $Y_{t+1}$ as:

$$V_{t+1} = \frac{\exp(\ln Y_t - \eta C_t)}{\text{Tr}(\exp(\ln Y_t - \eta C_t))}$$

$$\sigma_{t+1, i} = \frac{\alpha}{n} + (1 - \alpha) \sigma_{t+1, i}, Y_{t+1} = \hat{U} \text{diag}(\sigma_{t+1}) \hat{U}^T$$

where we apply eigendecomposition to $V_{t+1}$ as $V_{t+1} = \hat{U} \text{diag}(V_{t+1}) \hat{U}^T$.
3: end for

This problem has two different situations corresponding to different parameter space $\Omega$ of $y_t$ and $u$.

**Situation 1:** When $\Omega$ is the set of $\{x \mid \|x\|_2 = 1\}$ (e.g., the unit vector space), the solution to $\min_{u \in \Omega} \sum_{t=1}^n u^T C_t u$ is the minimum eigenvector of the matrix $\sum_{t=1}^n C_t$.

**Situation 2:** When $\Omega$ is the probability simplex (e.g., $\Omega$ is equal to $\mathbb{B}^n_1$), it corresponds to the risk minimization in stock portfolios [22].

We will start with **Situation 1** since it is highly related to the previous section.

### 5.1 Online Adaptive Variance Minimization over the Unit vector space

We begin with the observation of the following equivalence [13]:

$$\min_{\|u\|_2 = 1} u^T C u = \min_{U \in \mathbb{B}^n_1} \text{Tr}(UC)$$

(20)

where $C$ is any covariance matrix, and $\mathbb{B}^n_1$ is the set of all density matrices.

Thus, the problem in (19) can be reformulated as:

$$R_{\text{var-unit}}^{\alpha} = \max_{[r, s] \subset [1, T]} \left\{ \sum_{t=r}^s \text{Tr}(Y_t C_t) - \min_{U \in \mathbb{B}^n_1} \sum_{t=r}^s \text{Tr}(UC_t) \right\}$$

(21)

where $Y_t \in \mathbb{B}^n_1$.

To see the equivalence between $\mathbb{E}[y_t^T C_t y_t]$ in Eq. (19) and $\text{Tr}(Y_t C_t)$, we do the eigendecomposition of $Y_t = \sum_{i=1}^n \sigma_i y_i y_i^T$. Then $\text{Tr}(Y_t C_t)$ is equal to $\sum_{i=1}^n \sigma_i \text{Tr}(y_i y_i^T C_t) = \sum_{i=1}^n \sigma_i y_i^T C_t y_i$. Since $Y_t \in \mathbb{B}^n_1$, the vector $\sigma$ is a simplex vector, and $\sum_{i=1}^n \sigma_i y_i^T C_t y_i$ is equal to $\mathbb{E}[y_1^T C_t y_1]$ with probability distribution defined by the vector $\sigma$.

If we examine Eq. (21) and (13) together, we will see that they share some similarities: First, they are almost the same if we set $n - k = 1$ in Eq. (13). Also, $x_k x_k^T$ in Eq. (13) is a special case of $C_k$ in Eq. (21).

Thus, it is possible to apply Algorithm 4 to solving the problem (21) by setting $n - k = 1$, which will not need to call the Algorithm 2 and 3 anymore. This is summarized in Algorithm 5.

Theorem 4. For any sequence of covariance matrices $C_1, \ldots, C_T$ with $0 \preceq C_t \preceq I$, and for $\min_{U \in \mathbb{B}^n_1} \sum_{t=1}^T \text{Tr}(UC_t) \leq L$, if we run Algorithm 5 with $\alpha = 1/(T + 1)$, $D = \ln(n(1 + T)) + 1$, and $\eta = \ln(1 + \sqrt{2D/L})$, for any $T \geq 1$ we have:

$$R_{\text{var-unit}}^{\alpha} \leq O(\sqrt{2LD} + D)$$

**Proof sketch.** Similar inequality can be obtained as in Lemma 5 by using the result of Theorem 2 in [13]. The rest follows the proof of Theorem 3.
In order to apply the above theorem, we need to either estimate the step size \( \eta \) heuristically or estimate the upper bound \( L \), which may not be easily done.

In the next theorem, we show that we can still upper bound the \( \mathcal{R}_n^{\text{var-unit}} \) without knowing \( L \), but the upper bound is a function of time horizon \( T \) instead of the upper bound \( L \).

Before we get to the theorem, we need the following lemma which lifts the vector case of Lemma 1 in [16] to the density matrix case:

**Lemma 5.** For any \( \eta \geq 0, t \geq 1, \) any covariance matrix \( C_t \) with \( 0 \leq C_t \leq I \), and for any \( Q_t \in \mathcal{B}_1^n \), Algorithm 5 satisfies:

\[
\text{Tr}(Y_tC_t) - \text{Tr}(Q_tC_t) \leq \frac{1}{\eta} \left( \text{Tr}(Q_t \ln V_{t+1}) - \text{Tr}(Q_t \ln Y_t) \right) + \frac{\eta}{2}
\]

For space purposes, the proof is in the appendix.

Now we are ready to present the upper bound on the regret for Algorithm 5.

**Theorem 5.** For any sequence of covariance matrices \( C_1, \ldots, C_T \) with \( 0 \leq C_t \leq I \), if we run Algorithm 5 with \( \alpha = 1/(T + 1) \) and \( \eta = \frac{\sqrt{\ln(n(1 + T))}}{\sqrt{T}} \), for any \( T \geq 1 \) we have:

\[
\mathcal{R}_n^{\text{var-unit}} \leq O\left( \sqrt{T \ln \left( n(1 + T) \right)} \right)
\]

**Proof.** In the proof, we will use two cases of \( Q_t: Q_t \in \mathcal{B}_1^n \), and \( Q_t = 0 \).

From Lemma 5 the following inequality is valid for the case of \( Q_t \in \mathcal{B}_1^n \):

\[
\text{Tr}(Y_tC_t) - \text{Tr}(Q_tC_t) \leq \frac{1}{\eta} \left( \text{Tr}(Q_t \ln V_{t+1}) - \text{Tr}(Q_t \ln Y_t) \right) + \frac{\eta}{2}
\] (23)

Follow the same analysis as in the proof of Theorem 3, we first do the eigendecomposition to \( Q_t \) as \( Q_t = \overline{D} \text{diag}(q_t) \overline{D}^T \). Since \( \|q_t\|_1 \) is either 1 or 0, we will re-write the above inequality as:

\[
\|q_t\|_1 \text{Tr}(Y_tC_t) - \text{Tr}(Q_tC_t) \leq \frac{1}{\eta} \left( \text{Tr}(Q_t \ln V_{t+1}) - \text{Tr}(Q_t \ln Y_t) \right) + \frac{\eta}{2} \|q_t\|_1
\] (24)

We can analyze the two cases of \( Q_t \) together, because the above inequality holds under these two cases. Analyzing the term \( \text{Tr}(Q_t \ln V_{t+1}) - \text{Tr}(Q_t \ln Y_t) \) in the above inequality is the same as the analysis of the Eq. (41).

Thus, summing over \( t = 1 \) to \( T \) to the above inequality, and setting \( Q_t = Q \in \mathcal{B}_1^n \) for \( t = r, \ldots, s \) and 0 elsewhere, we will have

\[
\sum_{t=r}^s \text{Tr}(Y_tC_t) - \min_{U \in \mathcal{B}_1^n} \sum_{t=r}^s \text{Tr}(UC_t) 
\leq \frac{1}{\eta} \left( \ln \frac{n}{\alpha} + T \ln \frac{1}{1 - \alpha} \right) + \frac{\eta}{2} T,
\] (25)

since it holds for any \( Q \in \mathcal{B}_1^n \).

After plugging in the expression of \( \eta \) and \( \alpha \), we will have

\[
\sum_{t=r}^s \text{Tr}(Y_tC_t) - \min_{U \in \mathcal{B}_1^n} \sum_{t=r}^s \text{Tr}(UC_t) 
\leq O\left( \sqrt{T \ln \left( n(1 + T) \right)} \right)
\] (26)

Since the above inequality holds for any \( 1 \leq r \leq s \leq T \), we will put a \( \max_{[r,s] \subset [1,T]} \) in the left part, which proves the result.

\[\square\]
Algorithm 6 Online adaptive variance minimization over simplex

1: **Input:** an initial vector $y_1 \in B^n_1$

2: **for** $t = 1$ to $T$ **do**

3: Receive covariance matrix $C_t$.

4: Incur the loss $y_t^T C_t y_t$.

5: Update $y_{t+1}$ as:

   
   
   $$v_{t+1,i} = \frac{y_{t,i} \exp \left(-\eta(C_t y_t)_i\right)}{\sum_i y_{t,i} \exp \left(-\eta(C_t y_t)_i\right)},$$

   $$y_{t+1,i} = \frac{\alpha}{n} + (1 - \alpha)v_{t+1,i}. \tag{29a}$$

6: **end for**

5.2 Online Adaptive Variance Minimization over the Simplex space

We first re-write the problem in Eq.(19) when $\Omega$ is the simplex below:

$$R_{\text{var-sim}}^t = \max_{[r,s] \subseteq [1,T]} \left\{ \sum_{t=r}^s E[y_t^T C_t y_t] - \min_{u \in B^n_1} \sum_{t=r}^s u^T C_t u \right\} \tag{27}$$

where $y_t \in B^n_1$, and $B^n_1$ is the simplex set.

When $r = 1$ and $s = T$, the problem reduces to the static regret problem, which is solved in [13] by the exponentiated gradient algorithm as below:

$$y_{t+1,i} = \frac{y_{t,i} \exp \left(-\eta(C_t y_t)_i\right)}{\sum_i y_{t,i} \exp \left(-\eta(C_t y_t)_i\right)} \tag{28}$$

As is done in the previous sections, we add the fixed-share step after the above update, which is summarized in Algorithm 6.

With the update of $y_t$ in the Algorithm 6, we have the following theorem

**Theorem 6.** For any sequence of covariance matrices $C_1, \ldots, C_T$ with $0 \preceq C_i \preceq I$, and for $\min_{u \in B^n_1} \sum_{t=r}^s u^T C_t u \leq L$, if we run Algorithm 6 with $\alpha = 1/(T + 1)$, $c = \frac{\sqrt{2 \ln \left((1+T)n\right)+2}}{\sqrt{T}}$, $b = \frac{c^2}{2}$, $a = \frac{b}{2b+1}$, and $\eta = 2a$, for any $T \geq 1$ we have:

$$R_{\text{var-sim}}^t \leq 2 \sqrt{2L \left( \ln \left((1+T)n\right) + 1 \right) + 2 \ln \left((1+T)n\right)}$$

For space purposes, the proof is in the appendix.

6 Experiments

In this section, we use two examples to illustrate the effectiveness of our proposed online adaptive PCA algorithm:

The first example is synthetic, which shows that our proposed algorithm (denoted as Online Adaptive PCA) can adapt to the changing subspace faster than the method of [14]. The second example uses the practical dataset Yale-B to demonstrate that the proposed algorithm can have lower cumulative loss in practice when the data/face samples are coming from different persons.

The other algorithms that are used as comparators are: 1. Follow the Leader algorithm (denoted as Follow the Leader) [23], which only minimizes the loss on the past history; 2. The best fixed solution in hindsight (denoted as Best fixed Projection), which is the solution to the Problem described in Eq.(2); 3. The online static PCA (denoted as Online PCA) [14].

6.1 A Toy Example

In this toy example, we create the synthetic data samples coming from changing subspace/environment, which is a similar setup as in [14]. The data samples are divided into three equal time intervals, and each interval has 200 data
Figure 1: (a) The cumulative loss of the toy example with data samples coming from three different subspaces. (b) The detailed comparison for the two online algorithms.

Figure 2: The cumulative loss for the face example with data samples coming from 20 different persons.

samples. The 200 data samples within same interval is randomly generated by a Gaussian distribution with zero mean and data dimension equal to 20, and the covariance matrix is randomly generated with rank equal to 2. In this way, the data samples are from some unknown 2-dimensional subspace, and any data sample with $\ell_2$-norm greater than 1 is normalized to 1. Since the stepsize used in the two online algorithms is determined by the upper bound of the batch solution, we first find the upper bound and plug into the stepsize function, which gives $\eta = 0.19$. We can tune the stepsize heuristically in practice and in this example we just use $\eta = 1$ and $\alpha = 1e^{-5}$.

After all data samples are generated, we apply the previously mentioned algorithms with $k = 2$ and obtain the cumulative loss as a function of time steps, which is shown in Fig[1]. From this figure we can see that: 1. Follow the Leader algorithm is not appropriate in the setting where the sequential data is shifting over time. 2. The static regret is not a good metric under this setting, since the best fixed solution in hindsight is suboptimal. 3. Compared with Static PCA, the proposed Adaptive PCA can adapt to the changing environment faster, which results in lower cumulative loss and is more appropriate when the data is shifting over time.

6.2 Face data Compression Example

In this example, we use the Yale-B dataset which is a collection of face images. The data is split into 20 time intervals corresponding to 20 different people. Within each interval, there are 64 face image samples. Like the previous example, we first normalize the data to ensure its $\ell_2$-norm not greater than 1. We use $k = 2$, which is the same as
the previous example. The stepsize $\eta$ is also tuned heuristically like the previous example, which is equal to 5 and $\alpha = 1e^{-4}$.

We apply the previously mentioned algorithms and again obtain the cumulative loss as the function of time steps, which is displayed in Fig.2. From this figure we can see that although there is no clear bumps indicating the shift from one subspace to another as the Fig.1 of the toy example, our proposed algorithm still has the lowest cumulative loss, which indicates that upper bounding the adaptive regret is still effective when the compressed faces are coming from different persons.

7 Conclusion

In this paper, we propose an online adaptive PCA algorithm, which augments the previous online static PCA algorithm with a fixed-share step. However, different from the previous online PCA algorithm which is designed to minimize the static regret, the proposed online adaptive PCA algorithm aims to minimize the so-called adaptive regret which is more appropriate when the underlying environment is changing or the sequential data is shifting over time. We demonstrate theoretically that our algorithm can adapt to changing environments. In experiments on changing data, our algorithm results in lower cumulative losses.

Furthermore, we extend the online adaptive PCA algorithm to online adaptive variance minimization problems over unit vectors and simplex.

For the future work, one possible direction is to extend the idea behind the proposed algorithms to other problem settings such as the online adaptive Partial Least Square problem, which can be solved with the additional ideas from [24, 25]. Another possible direction is on refining the current adaptive PCA algorithm with easily obtained hyper-parameter settings.

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Supplementary

The supplemental material contains proofs of the main results of the paper along with supporting results.

A Proof of Theorem 1

Proof. Fix 1 \leq r \leq s \leq T. We set \( q_t = q \in \mathbb{R}^n \) for \( t = r, \ldots, s \) and 0 elsewhere. Thus, we have that \( \| q_t \|_1 \) is either 0 or 1.

According to Lemma 1 for both cases of \( q_t \), we have

\[
\| q_t \|_1 w_t^T \ell_t (1 - \exp(-\eta)) - \eta q_t^T \ell_t \leq \sum_{i=1}^n q_{t,i} \ln \left( \frac{v_{t+1,i}}{w_{t,i}} \right)
\]

(30)

The analysis for \( \sum_{i=1}^n q_{t,i} \ln \left( \frac{v_{t+1,i}}{w_{t,i}} \right) \) follows the Proof of Proposition 2 in [16]. We describe the steps for completeness, since it is helpful for understanding the effect of the fixed-share step, Eq. (4b). This analysis will be crucial for understanding how the fixed-share step can be applied to PCA problems.

\[
\sum_{i=1}^n q_{t,i} \ln \left( \frac{v_{t+1,i}}{w_{t,i}} \right) = \sum_{i=1}^n \left( q_{t,i} \ln \frac{1}{v_{t,i}} - q_{t-1,i} \ln \frac{1}{v_{t,i}} \right)
\]

(31)

For the expression of \( A \), we have

\[
A = \sum_{i,q_{t,i} \geq q_{t-1,i}} (q_{t,i} - q_{t-1,i}) \ln \frac{1}{v_{t,i}} + q_{t-1,i} \ln \frac{v_{t,i}}{v_{t+1,i}}
\]

(32)

Based on the update in Eq. (4), we have \( 1/w_{t,i} \leq n/\alpha \) and \( v_{t,i}/\hat{w}_{t,i} \leq 1/(1 - \alpha) \). Plugging the bounds into the above equation, we have

\[
A \leq \sum_{i,q_{t,i} \geq q_{t-1,i}} (q_{t,i} - q_{t-1,i}) \ln \frac{n}{\alpha}
\]

\[
+ \left( \sum_{i,q_{t,i} \geq q_{t-1,i}} q_{t-1,i} + \sum_{i,q_{t,i} < q_{t-1,i}} q_{t,i} \right) \ln \frac{1}{1 - \alpha}.
\]

(33)

Telescoping the expression of \( B \), substituting the above inequality in Eq. (31), and summing over \( t = 2, \ldots, T \), we have

\[
\sum_{t=2}^T \sum_{i=1}^n q_{t,i} \ln \frac{v_{t+1,i}}{w_{t,i}} \leq m(q_1, T) \ln \frac{n}{\alpha} + \sum_{t=2}^T \left( \sum_{i=1}^n q_{t,i} \ln \frac{n}{\alpha} \right) \ln \frac{1}{1 - \alpha} + \sum_{i=1}^n q_{t,i} \ln \frac{1}{v_{t,i}}.
\]

(34)

Adding the \( t = 1 \) term to the above inequality, we have

\[
\sum_{t=1}^T \sum_{i=1}^n q_{t,i} \ln \frac{v_{t+1,i}}{w_{t,i}} \leq \| q_1 \|_1 \ln(n) + m(q_1, T) \ln \frac{n}{\alpha}
\]

(35)

\[
+ \left( \sum_{t=1}^T \| q_t \|_1 - m(q_1, T) \right) \ln \frac{1}{1 - \alpha}.
\]

Now we bound the right side, using the choices for \( q_1 \) described at the beginning of the proof. If \( r \geq 2 \), \( m(q_1, T) = 1 \), and \( \| q_1 \|_1 = 0 \). If \( r = 1 \), \( m(q_1, T) = 0 \), and \( \| q_1 \|_1 = 1 \). Thus, \( m(q_1, T) + \| q_1 \|_1 = 1 \), and the right part can be upper bounded by \( \ln \frac{n}{\alpha} + T \ln \frac{1}{1 - \alpha} \).
We will first upper bound the right part in Eq.(38). With $\eta = \frac{\ln 2}{\ln 1 + (n-k)T}$, we have

$$\sum_{t=r}^s (n-k)w_t^T x_t \leq \min_{q \in B_{n-k}^s} \sum_{t=r}^s (n-k)q^T \ell_t + \frac{D}{1 - \exp(-\eta)}$$  \hspace{1cm} (37)$$

Since the above inequality holds for arbitrary $q \in B_{n-k}^s$, we have

$$\sum_{t=r}^s (n-k)w_t^T x_t \leq \frac{\eta \min_{q \in B_{n-k}^s} \sum_{t=r}^s (n-k)q^T \ell_t + D}{1 - \exp(-\eta)}$$  \hspace{1cm} (38)$$

We will apply the inequality in Lemma 2 to upper bound the right part in Eq.(38). With $\eta = \ln(1 + \sqrt{2D/L})$, we have

$$\sum_{t=r}^s (n-k)w_t^T x_t \leq \frac{\eta}{1 - \exp(-\eta)} \sum_{t=r}^s (n-k)q^T \ell_t \leq \sqrt{2LD + D}$$  \hspace{1cm} (39)$$

Since the above inequality always holds for all intervals, $[r, s]$, the result is proved by maximizing the left side over $[r, s]$. \hfill \Box$

B Proof of Theorem 3

Proof. In the proof, we will use two cases of $Q_t$: $Q_t \in B_{n-k}^s$, and $Q_t = 0$.

We first apply the eigendecomposition to $Q_t$ as $Q_t = \tilde{D} \text{diag}(q_t) \tilde{D}^T$, where $\tilde{D} = [\tilde{d}_1, \ldots, \tilde{d}_n]$. Since in the adaptive setting, $Q_{t-1}$ is either equal to $Q_t$ or 0, they share the same eigenvectors and $Q_{t-1}$ can be expressed as $Q_{t-1} = \tilde{D} \text{diag}(q_{t-1}) \tilde{D}^T$.

According to Lemma 3, the following inequality is true for both cases of $Q_t$:

$$\|q_t\| \text{Tr}(W_t x_t x_t^T)(1 - \exp(-\eta)) - \eta \text{Tr}(Q_t x_t x_t^T) \leq - \text{Tr}(Q_t \ln \tilde{W}_t) + \text{Tr}(Q_t \ln V_{t+1})$$  \hspace{1cm} (40)$$

The next steps use the same idea as in the proof of Proposition 2 in [16], but we extend it to the matrix case under adaptive setting.

Now we start analyzing the right part of the above inequality, which can be expressed as:

$$- \text{Tr}(Q_t \ln \tilde{W}_t) + \text{Tr}(Q_t \ln V_{t+1}) = \tilde{A} + \tilde{B}$$  \hspace{1cm} (41)$$

where $\tilde{A} = - \text{Tr}(Q_t \ln \tilde{W}_t) + \text{Tr}(Q_{t-1} \ln V_t)$, and $\tilde{B} = - \text{Tr}(Q_{t-1} \ln V_t) + \text{Tr}(Q_t \ln V_{t+1})$.

We will first upper bound the $\tilde{A}$ term, and then telescope the $\tilde{B}$ term.
\( \mathcal{A} \) can be expressed as:
\[
\mathcal{A} = \sum_{i \geq q_{t-1}, i} \left( -\text{Tr} \left((q_{t,i} \mathbf{d}_i \mathbf{d}_i^T - q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \ln \mathbf{\hat{W}}_t \right) \right) + \text{Tr}(q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T \ln \mathbf{V}_i) - \text{Tr}(q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T \ln \mathbf{\hat{W}}_t) \\
+ \sum_{i, q_{t,i} < q_{t-1,i}} \left( -\text{Tr} \left((q_{t,i} \mathbf{d}_i \mathbf{d}_i^T - q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \ln \mathbf{V}_i \right) \right) + \text{Tr}(q_{t,i} \mathbf{d}_i \mathbf{d}_i^T \ln \mathbf{V}_i) - \text{Tr}(q_{t,i} \mathbf{d}_i \mathbf{d}_i^T \ln \mathbf{\hat{W}}_t) \\
+ \text{Tr}(q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T \ln \mathbf{V}_i) - \text{Tr}(q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T \ln \mathbf{\hat{W}}_t) \\
\tag{42}
\]

For (1), it can be expressed as:
\[
(1) = \text{Tr} \left((q_{t,i} \mathbf{d}_i \mathbf{d}_i^T - q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \ln \mathbf{\hat{W}}_{t-1} \right) \leq \text{Tr} \left((q_{t,i} \mathbf{d}_i \mathbf{d}_i^T - q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \ln \frac{\alpha}{\alpha} \right) = (q_{t,i} - q_{t-1,i}) \ln \frac{\alpha}{\alpha} \tag{43}
\]

The inequality holds because the update in Eq. (11b) implies \( \ln \mathbf{\hat{W}}_{t-1} \leq I \ln \frac{\alpha}{\alpha} \) and furthermore, \( (q_{t,i} \mathbf{d}_i \mathbf{d}_i^T - q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \) is positive semi-definite. Thus, Lemma 4 gives the result.

The expression for (2) can be bounded as
\[
(2) = \text{Tr}(q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T \ln (\mathbf{V}_i \mathbf{\hat{W}}_{t-1}^{-1})) \leq q_{t-1,i} \ln \frac{\alpha}{1-\alpha} \tag{44}
\]

where the equality is due to the fact that \( \mathbf{V}_i \) and \( \mathbf{\hat{W}}_t \) have the same eigenvectors. The inequality follows since \( \ln (\mathbf{V}_i \mathbf{\hat{W}}_{t-1}^{-1}) \leq I \ln \frac{1}{1-\alpha} \), due to the update in Eq. (11b), while \( q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T \) is positive semi-definite. Thus Lemma 4 gives the result.

The bound (3) can be expressed as:
\[
(3) = \text{Tr} \left((-q_{t,i} \mathbf{d}_i \mathbf{d}_i^T + q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \ln \mathbf{V}_i \right) \leq 0 \tag{45}
\]

Here, the inequality follows since \( \ln \mathbf{V}_i \leq 0 \) and \( (-q_{t,i} \mathbf{d}_i \mathbf{d}_i^T + q_{t-1,i} \mathbf{d}_i \mathbf{d}_i^T) \) is positive semi-definite. Thus, Lemma 4 gives the result.

For (4), we have \( (4) \leq q_{t,i} \ln \frac{1}{1-\alpha} \), which follows the same argument as in upper bounding the term (2).

Thus, \( \mathcal{A} \) can be upper bounded as follows:
\[
\mathcal{A} \leq \sum_{i \geq q_{t-1}, i} (q_{t,i} - q_{t-1,i}) \ln \frac{\alpha}{\alpha} + \sum_{i, q_{t,i} < q_{t-1,i}} (q_{t-1,i} + \sum_{i, q_{t,i} < q_{t-1,i}} q_{t,i}) \ln \frac{1}{1-\alpha} \tag{46}
\]

Then we telescope the \( \mathcal{B} \) term, substitute the above inequality for \( \mathcal{A} \) into Eq. (41), and sum over \( t = 2, \ldots, T \) to give:
\[
\sum_{t=2}^{T} \left( -\text{Tr}(Q_t \ln \mathbf{\hat{W}}_t) + \text{Tr}(Q_t \ln \mathbf{V}_{t+1}) \right) \leq m(\mathbf{q}_1 \cdot \mathbf{T}) \ln \frac{\alpha}{\alpha} + \left( \sum_{t=2}^{T} \|\mathbf{q}_t\|_1 - m(\mathbf{q}_1 \cdot \mathbf{T}) \right) \ln \frac{1}{1-\alpha} - \text{Tr}(Q_1 \ln \mathbf{V}_2) \tag{47}
\]
Adding the $t = 1$ term to the above inequality, we have
\[
\sum_{t=1}^{T} \left( - \text{Tr}(Q_t \ln \hat{W}_t) + \text{Tr}(Q_t \ln V_{t+1}) \right)
\leq \|q_t\|_1 \ln(n) + m(q_{1:T}) \ln \frac{n}{a} + \left( \sum_{t=1}^{T} \|q_t\|_1 - m(q_{1:T}) \right) \ln \frac{1}{\alpha}.
\] (48)

For the above inequality, we set $Q_t = Q \in \mathbb{B}_n^a$ for $t = r, \ldots, s$ and 0 elsewhere, which makes $q_t = q \in \mathbb{B}_n^a$ for $t = r, \ldots, s$ and 0 elsewhere. If $r \geq 2$, $m(q_{1:T}) = 1$, and $\|q_1\|_1 = 0$. If $r = 1$, $m(q_{1:T}) = 1$, and $\|q_1\|_1 = 1$. Thus, $m(q_{1:T}) + \|q_1\|_1 = 1$, and the right part can be upper bounded by $\ln \frac{n}{a} + T \ln \frac{1}{\alpha}$.

The rest of the steps follow exactly the same as in the proof of Theorem 11.

\section{Proof of Lemma 5}

\textbf{Proof.} We first deal with the term $\text{Tr}(Q_t \ln V_{t+1})$. According to the update in Eq. (22a), we have
\[
\text{Tr}(Q_t \ln V_{t+1}) = \text{Tr} \left( Q_t \ln \left( \frac{\exp(\ln Y_t - \eta C_t)}{\text{Tr}(\exp(\ln Y_t - \eta C_t))} \right) \right)
= \text{Tr} \left( Q_t(\ln Y_t - \eta C_t) \right) - \ln \left( \text{Tr}\left( \exp(\ln Y_t - \eta C_t) \right) \right),
\] (49)
since $Q_t \in \mathbb{B}_n^1$ and $\text{Tr}(Q_t) = 1$.

As a result, we have $\text{Tr}(Q_t \ln V_{t+1}) - \text{Tr}(Q_t \ln Y_t) = -\eta \text{Tr}(Q_t C_t) - \ln \left( \text{Tr}\left( \exp(\ln Y_t - \eta C_t) \right) \right)$.

Thus, to prove the inequality in Lemma 5 it is enough to prove the following inequality
\[
\eta \text{Tr}(Y_t C_t) - \frac{\eta^2}{2} + \ln \left( \text{Tr}\left( \exp(\ln Y_t - \eta C_t) \right) \right) \leq 0
\] (50)

Before we proceed, we need the following lemmas:

\textbf{Lemma 6 (Golden-Thompson inequality).} For any symmetric matrices $A$ and $B$, the following inequality holds:
\[
\text{Tr}\left( \exp(A + B) \right) \leq \text{Tr}\left( \exp(A) \exp(B) \right)
\]

\textbf{Lemma 7 (Lemma 2.1 in \cite{12}).} For any symmetric matrix $A$ such that $0 \leq A \preceq I$ and any $\rho_1, \rho_2 \in \mathbb{R}$, the following holds:
\[
\exp(A \rho_1 + (I-A) \rho_2) \preceq A \exp(\rho_1) + (I-A) \exp(\rho_2)
\]

Then we apply the Golden-Thompson inequality to the term $\text{Tr}\left( \exp(\ln Y_t - \eta C_t) \right)$, which gives us the inequality below:
\[
\text{Tr}\left( \exp(\ln Y_t - \eta C_t) \right) \leq \text{Tr}(Y_t \exp(-\eta C_t)).
\] (51)

For the term $\exp(-\eta C_t)$, by applying the Lemma 7 with $\rho_1 = -\eta$ and $\rho_2 = 0$, we will have the following inequality:
\[
\exp(-\eta C_t) \preceq I - C_t(1 - \exp(-\eta)).
\] (52)

Thus, we will have
\[
\text{Tr}(Y_t \exp(-\eta C_t)) \leq 1 - \text{Tr}(Y_t C_t)(1 - \exp(-\eta)),
\] (53)
and
\[
\text{Tr}\left( \exp(\ln Y_t - \eta C_t) \right) \leq 1 - \text{Tr}(Y_t C_t)(1 - \exp(-\eta)),
\] (54)
since $Y_t \in \mathbb{B}_n^1$ and $\text{Tr}(Y_t) = 1$.

Thus, it is enough to prove the following inequality
\[
\eta \text{Tr}(Y_t C_t) - \frac{\eta^2}{2} + \ln \left( 1 - \text{Tr}(Y_t C_t)(1 - \exp(-\eta)) \right) \leq 0
\] (55)

Since $\ln(1 - x) \leq -x$, we have
\[
\ln \left( 1 - \text{Tr}(Y_t C_t)(1 - \exp(-\eta)) \right) \leq - \text{Tr}(Y_t C_t)(1 - \exp(-\eta)).
\] (56)
Thus, it suffices to prove the following inequality:

\[(\eta - 1 + \exp(-\eta)) \text{Tr}(Y_t C_t) - \frac{\eta^2}{2} \leq 0 \tag{57}\]

Note that by using convexity of \(\exp(-\eta)\), \(\eta - 1 + \exp(-\eta) \geq 0\).

By applying Lemma 4 with \(A = Y_t, B = C_t,\) and \(C = I,\) we have \(\text{Tr}(Y_t C_t) \leq \text{Tr}(Y_t) = 1.\) Thus, when \(\eta \geq 0,\) it is enough to prove the following inequality

\[\eta - 1 + \exp(-\eta) - \frac{\eta^2}{2} \leq 0. \tag{58}\]

This inequality follows from convexity of \(\frac{\eta^2}{2} - \exp(-\eta)\) over \(\eta \geq 0\). \(\Box\)

**D Proof of Theorem 6**

*Proof.* First, since \(0 \leq C_t \leq I,\) we have \(\max_{i,j} |C_t(i,j)| \leq 1.\)

Before we proceed, we need the following lemma from [13].

**Lemma 8** (Lemma 1 in [13]). Let \(\max_{i,j} |C_t(i,j)| \leq \frac{r}{2},\) then for any \(c_t \in B^s,\) any constants \(a\) and \(b\) such that \(0 \leq a \leq \frac{b}{1+a},\) and \(\eta = \frac{2b}{1+a},\) we have

\[ay_t^T C_t y_t - bu_t^T C_t u_t \leq d(u_t, y_t) - d(u_t, v_{t+1})\]

Now we apply Lemma 8 under the conditions \(r = 2, a = \frac{b}{20+1}, \eta = 2a,\) and \(b = \frac{c}{2}.\)

Recall that \(d(u_t, y_t) - d(u_t, v_{t+1}) = \sum_i u_{t,i} \ln \left(\frac{u_{t+1,i}}{y_{t,i}}\right).\) Combining this with the inequality in Lemma 8 and the fact that \(\|u_t\|_1 = 1,\) we have

\[a \|u_t\|_1 y_t^T C_t y_t - b u_t^T C_t u_t \leq \sum_i u_{t,i} \ln \left(\frac{v_{t+1,i}}{y_{t,i}}\right) \tag{59}\]

Note that the above inequality is also true when \(u_t = 0.\)

Note that the right side of the above inequality is the same as the right part of the Eq.(30) in the proof of Theorem 1. Then we will set \(u_t = u = \arg\min_{q \in B^s} \sum_{r=1}^s q^T C_t q\) for \(t = r, \ldots, s,\) and 0 elsewhere. Summing from \(t = 1\) up to \(T,\) gives the following inequality:

\[a \left(\sum_{t=r}^s y_t^T C_t y_t\right) - b \left[\min_{u \in B^s} \sum_{t=r}^s u^T C_t u\right] \leq \ln \frac{n}{\alpha} + T \ln \frac{1}{1-\alpha} \tag{60}\]

Since \(\alpha = 1/(1+T), T \ln \frac{1}{1-\alpha} \leq 1.\) Then the above inequality becomes

\[a \left(\sum_{t=r}^s y_t^T C_t y_t\right) - b \left[\min_{u \in B^s} \sum_{t=r}^s u^T C_t u\right] \leq \ln ((1+T)n) + 1 \tag{61}\]

Plugging in the expressions of \(a = c/(2c+2), b = c/2,\) and \(c = \sqrt{\frac{2 \ln ((1+T)n) + 2}{2L}}\) we will have

\[
\sum_{t=r}^s y_t^T C_t y_t - \min_{u \in B^s} \sum_{t=r}^s u^T C_t u \\
\leq c \left[\min_{u \in B^s} u^T C_t u\right] + \frac{2c+1}{c} \left(\ln ((1+T)n) + 1\right) \\
\leq cL + 2\frac{c+1}{c} \left(\ln ((1+T)n) + 1\right) \\
= 2L \left(\ln ((1+T)n) + 1\right) + 2\ln ((1+T)n) \tag{62}\]

Since the inequality holds for any \(1 \leq r \leq s \leq T,\) the proof is concluded by maximizing over \([r, s]\) on the left. \(\Box\)