Pseudoduality and Conserved Currents in Sigma Models

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Abstract

We discuss the current conservation laws in sigma models based on a compact Lie groups of the same dimensionality and connected to each other via pseudoduality transformations in two dimensions. We show that pseudoduality transformations induce an infinite number of nonlocal conserved currents on the pseudodual manifold.

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1 Introduction

In this article we investigate some properties of conserved currents on target space manifolds that are pseudodual to each other, following a method discussed in [1] we find an infinite number of conservation laws in the pseudodual manifold. We work out the currents in case of pseudodualities [2, 3] between the sigma model on an abelian group and a strict WZW sigma model [4] on a compact Lie group [5, 6] of the same dimensionality. We specialize to the case of the abelian group \( U(1) \times U(1) \times U(1) \), and of the Lie group \( SU(2) \).

The strict WZW model is the model with the Wess-Zumino term normalized so that the canonical equations of motion are given by \( \partial_- (g^{-1} \partial_+ g) = 0 \), where \( g \) is a function on spacetime taking values in some compact Lie group \( G \). Pseudoduality is defined as the on shell duality transformation mapping the solutions to the equations of motion for different sigma models. This transformation preserves the stress energy tensor though it is not a canonical transformation. We know [1, 7] that if we are given a sigma model on an abelian group, and a strict WZW sigma model on a compact Lie group, there is a duality transformation between these two manifolds that maps solutions of the equations of motion of the first manifold into the solutions of the equations of motion of the second manifold. Solutions of the equations of motions allow us to construct holomorphic [5] nonlocal conserved currents on these manifolds. Pseudoduality relations provide a way to form pseudodual currents, and we show that these currents are conserved.

Let \( \Sigma \) be two dimensional Minkowski space, and \( \sigma^\pm = \tau \pm \sigma \) be the standard lightcone coordinates. Using maps \( x : \Sigma \rightarrow M \) and \( \tilde{x} : \Sigma \rightarrow \tilde{M} \) we may write pseudoduality relations as

\[
\tilde{x}_+(\sigma) = +T(\sigma)x_+(\sigma) \quad (1.1)
\]

\[
\tilde{x}_-(\sigma) = -T(\sigma)x_-(\sigma) \quad (1.2)
\]

where \( T(\sigma) \) belongs to \( SO(n) \), and is a function of \( \sigma \).

Let \( M = G \) be a compact Lie group of dimension \( n \) with an Ad(G)-invariant metric, and \( g : \Sigma \rightarrow G \). We define the basic nonlocal conserved currents \( J^{(L)}_+ = (g^{-1} \partial_+ g) \) and \( J^{(R)}_- = (\partial_- g)g^{-1} \) on the tangent bundle of \( G \). What we demonstrate is that we can take these currents, and using the pseudoduality relations (1) we obtain currents on \( G \) (not \( \tilde{G} \)) and these currents are conserved.

We would like to search for infinitely many conservation laws [8, 9, 10] on pseudodual manifolds. We first concentrate on a simple case, where \( M = \)
$G = U(1) \times U(1) \times U(1)$ is an abelian group and $\tilde{M} = \tilde{G}$ is $SU(2)$. We show that infinite number of conservation laws of free scalar currents on $G$ enable us to construct infinite number of pseudodual current conservation on $\tilde{G}$ by means of isometry preserving orthogonal map $T$ between tangent bundles of these manifolds. We next focus our attention on a more complicated case, where $M = G$ is the Lie group $SU(2)$ and $\tilde{M} = \tilde{G}$ is $U(1) \times U(1) \times U(1)$. We find nonlocal conserved currents on $G$ and construct pseudodual free currents on $\tilde{G}$ using pseudoduality relations. We show that pseudodual free scalar currents on $\tilde{G}$ gives us infinite number of conservation laws.

2 Pseudodual Currents : Simple Case

We take $M$ as an abelian group, and the equations of motion become $\partial^2 - \phi^i = 0$, where $\phi$ is free massless scalar field. Currents on the tangent bundle of $M$ are hence given by $J^L = (\partial_+ \phi^i) X_i$ and $J^R = (\partial_- \phi^i) X_i$, where $\{X_i\}$ is a basis for the abelian Lie algebra. We notice that these currents are conserved, $\partial_- J^L = \partial_+ J^R = 0$. Now we take $\tilde{M}$ as a compact Lie group of the same dimensionality with an $\text{Ad}(G)$-invariant metric. $\{\tilde{X}_i\}$ is the orthonormal basis for the Lie algebra of $\tilde{G}$ with bracket relations $[\tilde{X}_i, \tilde{X}_j]_{\tilde{G}} = \tilde{f}^k_{ij} \tilde{X}_k$, where the structure constants $\tilde{f}^k_{ij}$ are totally antisymmetric in $ijk$. Using the map $\tilde{g} : \Sigma \rightarrow \tilde{M}$ we may write equations of motion as $\partial_- (\tilde{g}^{-1} \partial_+ \tilde{g}) = 0$. Currents on this manifold are defined by $\tilde{J}^L_+ = (\tilde{g}^{-1} \partial_+ \tilde{g})^i \tilde{X}_i$ and $\tilde{J}^R_- = [(\partial_- \tilde{g}) \tilde{g}^{-1}]^i \tilde{X}_i$. Again, by virtue of equations of motion we observe that these currents are conserved, $\partial_- \tilde{J}^L_+ = \partial_+ \tilde{J}^R_- = 0$.

To construct pseudodual currents on the manifold $M$ we make use of the pseudoduality conditions. The pseudoduality relations between the sigma model on an abelian group and a strict WZW sigma model on a compact Lie group of the same dimension are

\begin{align}
(\tilde{g}^{-1} \partial_+ \tilde{g})^i &= +T^i_j \partial_+ \phi^j \\
(\tilde{g}^{-1} \partial_- \tilde{g})^i &= -T^i_j \partial_- \phi^j
\end{align}

where $T$ is an orthogonal matrix and $\tilde{g}^{-1} d \tilde{g} = (\tilde{g}^{-1} d \tilde{g})^i \tilde{X}_i$.

Taking $\partial_-$ of the first equation (2.1) we conclude that $T$ is a function of $\sigma^+$ only. Taking $\partial_+$ of the second equation (2.2) gives us the differential equation for $T$. 

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\[(\partial_+ T)T^{-1}]_j^i = -\tilde{f}_{kj}^i T_l^k \partial_+ \phi^l \tag{3}\]

where we used the antisymmetricity of \(f_{ikj}\) at right hand side of equation.

To get pseudodual currents on the manifold \(M\), we first solve this differential equation for \(T\), and then plug this into pseudoduality equations with an initially given \(\partial_\pm \phi^i\) and from the pseudodual currents we find that these currents are conserved.

### 2.1 An Example

We consider the sigma model based on the product group \(U(1) \times U(1) \times U(1)\) for \(M\) and a strict WZW model based on group \(SU(2)\) for \(\tilde{M}\). We may write a point on the sigma model to \(M\) as \(\phi^i X_i\), where \(i = 1, 2, 3\) and \(\{X_i\}\) are basis. Equations of motions are \(\partial_2 + \phi^i = 0\). Currents may be written as \(J_+^{(L)} = (\partial_+ \phi^i) X_i\) and \(J_+^{(R)} = (\partial_\pm \phi^i) X_i\). We learn from equations of motions that these currents are conserved.

We denote any element in \(\tilde{G}\) as \(\tilde{g} = e^{\tilde{\theta}^b \tilde{X}_b}\), where \(\{\tilde{\theta}^k\} = (\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3)\) and \(\{\tilde{X}_k\} = (-i\frac{\sigma_1}{2}, -i\frac{\sigma_2}{2}, -i\frac{\sigma_3}{2})\) is a basis for the Lie algebra of \(SU(2)\). Structure constants are \(\epsilon_{ijk}\). Equations of motion for the strict WZW model are \(\partial_-(\tilde{g}^{-1} \partial_+ \tilde{g}) = 0\), where \(\tilde{g}^{-1} d\tilde{g} = (\tilde{g}^{-1} d\tilde{g})^k \tilde{X}_k\). Currents for the Lie algebra are \(\tilde{J}_+^{(L)} = (\tilde{g}^{-1} \partial_+ \tilde{g})^k \tilde{X}_k\) and \(\tilde{J}_+^{(R)} = [(\tilde{g}^{-1} \partial_+ \tilde{g}^{-1})]_k^l \tilde{X}_k\). Again equations of motion ensure that these currents are conserved.

We first solve the ordinary differential equation for \(T\) to find the pseudodual currents. Multiplying (3) by \(T_j^k\) from right we get

\[\partial_+ T_n^i = -\tilde{f}_{kj}^i T_l^k T_n^j \partial_+ \phi^l \tag{4}\]

We put in an order parameter \(\varepsilon\) to look for a perturbation solution,

\[\partial_+ T_n^i = -\varepsilon \tilde{f}_{kj}^i T_l^k T_n^j \partial_+ \phi^l \tag{5}\]

Presumably the solution is in the form \(T = e^{\varepsilon \alpha_1} e^{\frac{1}{2}\varepsilon^2 \alpha_2} (I + O(\varepsilon^3))\), where \(\alpha_1\) and \(\alpha_2\) are antisymmetric matrices. Since we know that \(T\) is only a function of \(\sigma^+\), \(\alpha_1\) and \(\alpha_2\) are also functions of \(\sigma^+\). If we expand \(T\)

\[T = I + \varepsilon \alpha_1 + \frac{1}{2} \varepsilon^2 (\alpha_2 + \alpha_1^2) + O(\varepsilon^3) \tag{6}\]

then taking \(\partial_+\) we end up with
\[ \partial_+ T = \varepsilon \partial_+ \alpha_1 + \frac{1}{2} \varepsilon^2 [\alpha_1 (\partial_+ \alpha_1) + (\partial_+ \alpha_1) \alpha_1 + \partial_+ \alpha_2] + O(\varepsilon^3) \]  
\text{(7)}

If we compare (5) to (7), the latter may be written in tensor product form as

\[ \partial_+ T = -\varepsilon \tilde{f} \otimes (T \partial_+ \phi) \otimes T = -\varepsilon \tilde{f} \otimes [(I + \varepsilon \alpha_1) \partial_+ \phi] \otimes (I + \varepsilon \alpha_1) \]
\[ = -\varepsilon \tilde{f} \otimes \partial_+ \phi \otimes I - \varepsilon^2 \tilde{f} \otimes \alpha_1 \partial_+ \phi \otimes I - \varepsilon^2 \tilde{f} \otimes \partial_+ \phi \otimes \alpha_1 \]  
\text{(8)}

we find

\[ \partial_+ \alpha_1 = -\tilde{f} \otimes \partial_+ \phi \otimes I \]  
\text{(9)}

\[ \frac{1}{2} [\partial_+ \alpha_2 + \alpha_1 (\partial_+ \alpha_1) + (\partial_+ \alpha_1) \alpha_1] = -\tilde{f} \otimes \alpha_1 \partial_+ \phi \otimes I - \tilde{f} \otimes \partial_+ \phi \otimes \alpha_1 \]  
\text{(10)}

Solving (9) we get \( \alpha_1 \) as follows

\[ (\alpha_1)_n^i = -\int_{\sigma^+}^+ \tilde{f}_{kn} \partial_+ \phi^k d\sigma^+ = -\tilde{f}_{kn} (\phi^k + C^k) \]  
\text{(11)}

where \( C^k \) is a constant, and we choose it to be zero. Since \( \alpha_1 \) is a function of \( \sigma^+ \) only, \( \phi^k \) in the expression of \( \alpha_1 \) should involve \( \sigma^+ \), not \( \sigma^- \). From this we understand that we need to separate \( \phi \) as right moving wave \( \phi_R(\sigma^-) \) and left moving wave \( \phi_L(\sigma^+) \), i.e. \( \phi = \phi_L(\sigma^+) + \phi_R(\sigma^-) \). Hence \((\alpha_1)_n^i = -\tilde{f}_{kn} \phi_L^k\), from which we find

\[ \alpha_1 = \begin{pmatrix} 0 & \phi_L^3 & -\phi_L^2 \\ -\phi_L^3 & 0 & \phi_L^1 \\ \phi_L^2 & -\phi_L^1 & 0 \end{pmatrix} \]  
\text{(12)}

Solving (10) we obtain

\[ (\alpha_2)_n^i = -\int_{\sigma^+}^+ (\alpha_1 \partial_+ \alpha_1)_n^i d\sigma^+ - \int_{\sigma^+}^+ [(\partial_+ \alpha_1) \alpha_1]_n^i d\sigma^+ \]
\[ -2 \int_{\sigma^+}^+ \tilde{f}_{kn}^i (\alpha_1)_n^k \partial_+ \phi_L^j d\sigma^+ - 2 \int_{\sigma^+}^+ \tilde{f}_{mn}^i \partial_+ \phi_L^m (\alpha_1)_n^i d\sigma^+ \]  
\text{(13)}

which gives us the following entries of \( \alpha_2 \) with the help of (12)
\begin{align}
(\alpha_2)_1^1 &= 0 \\
(\alpha_2)_2^1 &= \int_0^{\sigma^+} [\phi_L^2(\partial_+ \phi_L^1) - \phi_L^1(\partial_+ \phi_L^2)] d\sigma^+ \\
(\alpha_2)_3^1 &= \int_0^{\sigma^+} [\phi_L^3(\partial_+ \phi_L^1) - \phi_L^1(\partial_+ \phi_L^3)] d\sigma^+ \\
(\alpha_2)_1^2 &= \int_0^{\sigma^+} [\phi_L^1(\partial_+ \phi_L^2) - \phi_L^2(\partial_+ \phi_L^1)] d\sigma^+ \\
(\alpha_2)_2^2 &= 0 \\
(\alpha_2)_3^2 &= \int_0^{\sigma^+} [\phi_L^3(\partial_+ \phi_L^2) - \phi_L^2(\partial_+ \phi_L^3)] d\sigma^+ \\
(\alpha_2)_1^3 &= \int_0^{\sigma^+} [\phi_L^1(\partial_+ \phi_L^3) - \phi_L^3(\partial_+ \phi_L^1)] d\sigma^+ \\
(\alpha_2)_2^3 &= \int_0^{\sigma^+} [\phi_L^2(\partial_+ \phi_L^3) - \phi_L^3(\partial_+ \phi_L^2)] d\sigma^+ \\
(\alpha_2)_3^3 &= 0
\end{align}

Plugging (12) and (14) into \( T \) and setting \( \varepsilon = 1 \) gives us

\begin{equation}
T_j^i = \delta_j^i + (\alpha_1)_j^i + \frac{1}{2}[(\alpha_2)_j^i + (\alpha_2)_j^i] + O(\phi^3)
\end{equation}

\text{so the entries of } T \text{ becomes}

\begin{align}
T_1^1 &= 1 - \frac{1}{2}[\phi_L^2 \phi_L^2 + \phi_L^3 \phi_L^3] \\
T_2^1 &= \phi_L^3 + \int_0^{\sigma^+} \phi_L^2(\partial_+ \phi_L^1) d\sigma^+ \\
T_3^1 &= -\phi_L^2 + \int_0^{\sigma^+} \phi_L^3(\partial_+ \phi_L^1) d\sigma^+ \\
T_1^2 &= -\phi_L^3 + \int_0^{\sigma^+} \phi_L^1(\partial_+ \phi_L^2) d\sigma^+ \\
T_2^2 &= \phi_L^1 + \int_0^{\sigma^+} \phi_L^2(\partial_+ \phi_L^1) d\sigma^+ \\
T_3^2 &= -\phi_L^2 + \int_0^{\sigma^+} \phi_L^3(\partial_+ \phi_L^1) d\sigma^+ \\
T_1^3 &= -\phi_L^3 + \int_0^{\sigma^+} \phi_L^1(\partial_+ \phi_L^2) d\sigma^+ \\
T_2^3 &= \phi_L^2 + \int_0^{\sigma^+} \phi_L^3(\partial_+ \phi_L^1) d\sigma^+ \\
T_3^3 &= -\phi_L^1 + \int_0^{\sigma^+} \phi_L^2(\partial_+ \phi_L^1) d\sigma^+
\end{align}
\[ T_2^2 = 1 - \frac{1}{2}[\phi_L^1 \phi_L^1 + \phi_L^3 \phi_L^3] \quad (16.5) \]
\[ T_3^2 = \phi_L^1 + \int_0^{\sigma^+} \phi_L^3 (\partial_+ \phi_L^2) d\sigma'^+ \quad (16.6) \]
\[ T_1^3 = \phi_L^2 + \int_0^{\sigma^+} \phi_L^1 (\partial_+ \phi_L^3) d\sigma'^+ \quad (16.7) \]
\[ T_2^3 = -\phi_L^1 + \int_0^{\sigma^+} \phi_L^2 (\partial_+ \phi_L^3) d\sigma'^+ \quad (16.8) \]
\[ T_3^3 = 1 - \frac{1}{2}[\phi_L^1 \phi_L^1 + \phi_L^2 \phi_L^2] \quad (16.9) \]

We note that \( T \) is an orthogonal matrix. The type of the field \( \phi(\sigma^+, \sigma^-) = \phi_L(\sigma^+) + \phi_R(\sigma^-) \) puts pseudoduality relations into the forms
\[ (\tilde{g}^{-1} \partial_+ \tilde{g})^i = +T_j^i \partial_+ \phi_L^j \quad (17.1) \]
\[ (\tilde{g}^{-1} \partial_- \tilde{g})^i = -T_j^i \partial_- \phi_R^j \quad (17.2) \]

We note that equation (17.1) has an invariance under \( \tilde{g}(\sigma^+, \sigma^-) \rightarrow h(\sigma^-)\tilde{g}(\sigma^+, \sigma^-) \). From this we can look for solution \( \tilde{g}(\sigma^+, \sigma^-) = \tilde{g}_R(\sigma^-)\tilde{g}_L(\sigma^+) \), so first pseudoduality relation is reduced to \( (\tilde{g}_L^{-1} \partial_+ \tilde{g}_L) = +T_j^i \partial_+ \phi_L^j \). This equation gives us the left current. Next we have to find \( \tilde{g}_R(\sigma^-) \) using second pseudoduality equation to construct right current. Plugging \( \tilde{g}(\sigma^+, \sigma^-) = \tilde{g}_R(\sigma^-)\tilde{g}_L(\sigma^+) \) into (17.2) and arranging terms we obtain

\[ \tilde{g}_R^{-1}(\sigma^-) \partial_- \tilde{g}_R(\sigma^-) = -\tilde{g}_L(\sigma^+) (X_i T_j^i \partial_- \phi_R^j) \tilde{g}_L^{-1}(\sigma^+) \quad (18) \]

where \{\tilde{X}_i\} are the Lie algebra basis of \( \tilde{g} \), and \( (\tilde{X}_i)_{jk} = \epsilon_{ijk} \). Since we want to construct pseudodual currents in the order of \( \phi^n \), we need \( T(\sigma^+) \) to the order of \( \phi^{n-1} \) to get \( \tilde{J}_+^{(L)}(\sigma^+) \) to the order of \( \phi^n \). From equation (18) we see that the knowledge of \( T \) to \( \mathcal{O}(\phi^{n-1}) \) and \( \tilde{g}_L \) to \( \mathcal{O}(\phi^{n-1}) \) allows us to construct \( \tilde{g}_R \) to \( \mathcal{O}(\phi^n) \), so we can construct \( \tilde{J}_+^{(R)}(\sigma^-) \) to \( \mathcal{O}(\phi^n) \).

First we construct \( \tilde{J}_+^{(L)}(\sigma^+) \) to the order of \( \phi^2 \), so we need \( T \) to \( \mathcal{O}(\phi^2) \)
\[ T_j^i = \delta_j^i - \tilde{f}_{kj}^i \phi_L^k + \mathcal{O}(\phi^2) \quad (19) \]

Therefore, using first pseudoduality relation (17.1)
\[ \tilde{J}_+^{(L)}(\sigma^+) = \tilde{g}_L^{-1} \partial_+ \tilde{g}_L = \tilde{X}_i \partial_+ \phi_L^i - \tilde{X}_i \tilde{f}_{kj}^i \phi_L^k \partial_+ \phi_L^j \quad (20) \]
All we need is \( \tilde{g}_L \) to the order of \( \phi \), so we need to solve \( \tilde{g}_L^{-1} \partial_+ \tilde{g}_L = \tilde{X}_i \partial_+ \phi^i_L \) for \( \tilde{g}_L(\sigma^+) \). Choosing initial condition as \( \tilde{g}_L(\sigma^+ = 0) = I \), we get
\[
\tilde{g}_L(\sigma^+) = I + \tilde{X}_i \phi^i_L + O(\phi^2)
\] (21)
Its inverse is
\[
\tilde{g}_L^{-1}(\sigma^+) = I - \tilde{X}_i \phi^i_L + O(\phi^2)
\] (22)
Plugging these into (18) we find
\[
\tilde{g}_R^{-1} \partial_- \tilde{g}_R = -(I + \tilde{X}_i \phi^i_L) \tilde{X}_i (\delta^i_j - \tilde{f}^i_{kj} \phi^k_L) \partial_- \phi^j_R (I - \tilde{X}_k \phi^k_L)
\]
\[
= -\tilde{X}_i \partial_- \phi^i_R + O(\phi^3)
\] (23)
We notice that the order of \( \phi^2 \) terms are cancelled, and \( \tilde{g}_R \) is a function of \( \sigma^- \) only. We let \( \tilde{g}_R = e^{\phi^i_R \tilde{X}_i} \xi^k \tilde{X}_k \), where \( \xi \) represents \( O(\phi^2) \). Expanding \( \tilde{g}_R \)
\[
\tilde{g}_R = (I - \phi^i_R \tilde{X}_i + \frac{1}{2} \phi^i_R \phi^j_R \tilde{X}_i \tilde{X}_j)(I + \xi^k \tilde{X}_k)
\]
\[
= I - \phi^i_R \tilde{X}_i + \frac{1}{2} \phi^i_R \phi^j_R \tilde{X}_i \tilde{X}_j + \xi^k \tilde{X}_k + O(\phi^3)
\] (24)
the inverse \( \tilde{g}_R^{-1} \) can be found from \( \tilde{g}_R = e^{-\xi^k \tilde{X}_k} \phi^i_R \tilde{X}_i \)
\[
\tilde{g}_R^{-1} = (I - \xi^k \tilde{X}_k)(I + \phi^i_R \tilde{X}_i + \frac{1}{2} \phi^i_R \phi^j_R \tilde{X}_i \tilde{X}_j)
\]
\[
= I + \phi^i_R \tilde{X}_i + \frac{1}{2} \phi^i_R \phi^j_R \tilde{X}_i \tilde{X}_j - \xi^k \tilde{X}_k + O(\phi^3)
\] (25)
It follows then that equations (24) and (25) lead to
\[
\tilde{g}_R^{-1} \partial_- \tilde{g}_R = -\partial_- \phi^i_R \tilde{X}_i + \partial_- \xi^k \tilde{X}_k + O(\phi^3)
\] (26)
and comparison with (23) evaluates \( \partial_- \xi^k = 0 \), so \( \xi^k \) is constant and we choose it to be zero. Therefore, right current can be constructed using (24) and (25) as
\[
\tilde{J}_+^{(R)}(\sigma^-) = (\partial_- \tilde{g}_R) \tilde{g}_R^{-1} = - (\partial_- \phi^i_R) \tilde{X}_i
\] (27)
we see that order of \( \phi^2 \) disappears in the expression of right current. If we explicitly write pseudodual currents on the manifold \( M \) up to the order of \( \phi^3 \) using equations (20) and (27) we get the following
\[
\tilde{J}_+^{(L)}(\sigma^+) = \tilde{X}_i [\partial_+ \phi^i_L - \tilde{f}^i_{kj} \phi^k_L \partial_+ \phi^j_L - \frac{1}{2} \int_0^{\sigma^+} (\phi^i_L \partial_+ \phi^j_L - \phi^j_L \partial_+ \phi^i_L) d\sigma^+ \\
- \tilde{f}^i_{km} \tilde{f}^m_{nj} \phi^k_L \phi^n_L \partial_+ \phi^j_L]
\] (28.1)
\[ \tilde{J}_-(\sigma^-) = -\tilde{X}_i (\partial_- \phi_R^i) \]  

(28.2)

Therefore, our currents can be written as

\[ \tilde{J}^{(\mu)} = \tilde{J}_{[0]}^{(\mu)} + \tilde{J}_{[1]}^{(\mu)} + \tilde{J}_{[2]}^{(\mu)} + \mathcal{O}(\phi^3) \]  

(29)

where \( \{\mu\} = (R, L) \). We can organize all these terms as

\[ \tilde{J}^{(\mu)}(\phi) = \sum_{n=0}^{\infty} \tilde{J}_{[n]}^{(\mu)}(\phi) \]  

(30)

It is easy to see that these currents are conserved, i.e. \( \partial_+ \tilde{J}_{[R]} = \partial_- \tilde{J}_{[L]} = 0 \), by means of the equations of motion \( \partial_+^2 \phi^i = 0 \). Since each term satisfies \( \partial_+ \tilde{J}_{[n]} = \partial_- \tilde{J}_{[n]} = 0 \) for all \( n \) separately, we have infinite number of conservation laws for each order of \( \phi \) as pointed out in [1].

### 3 Pseudodual Currents : Complicated Case

In this case we consider the pseudoduality between two strict WZW models based on compact Lie groups of dimension \( n \) with Ad-invariant metrics. If \( \{X_i\} \) are the orthonormal basis for the Lie algebra of \( G \) with commutation relations \([X_i, X_j]_G = f^k_{ij} X_k\), where \( f_{ijk} \) are totally antisymmetric in \( ijk \), and \( g : \Sigma \rightarrow M \) is the map to the target space, we may write equations of motion on \( G \) as \( \partial_- (g^{-1} \partial_+ g) = 0 \). Therefore, currents become \( J_{+}^{(L)} = (g^{-1} \partial_+ g)^j X_i \) and \( J_{-}^{(R)} = [(\partial_- g) g^{-1}]^i X_i \). These currents are conserved. We make similar assumptions for the Lie group \( \tilde{G} \). The pseudoduality equations are

\[ (\tilde{g}^{-1} \partial_+ \tilde{g})^i = +T^i_j (g^{-1} \partial_+ g)^j \]  

(31.1)

\[ (\tilde{g}^{-1} \partial_- \tilde{g})^i = -T^i_j (g^{-1} \partial_- g)^j \]  

(31.2)

where \( T \) is an orthogonal matrix. Taking \( \partial_+ \) of the first equation (31.1) we learn that \( T \) is a function of \( \sigma^+ \) only. Taking \( \partial_- \) of the second equation (31.2) we get the differential equation for \( T \)

\[ [(\partial_+ T) T^{-1}]^i = -\tilde{f}^i_{kj} T^l_k (g^{-1} \partial_+ g)^l + f^k_{ml} T^i_k T^j_l (g^{-1} \partial_+ g)^m \]  

(32)

We follow the same method as we did in the previous part to find pseudodual currents. We first solve differential equation (32) for \( T \), then replace this into the pseudoduality relations, and finally build pseudodual currents. We will see that these currents are conserved.
3.1 An Example

To illustrate all these steps in an example, we consider a strict WZW model based on Lie group \( SU(2) \) for \( G \), and a sigma model based on abelian group \( U(1) \times U(1) \times U(1) \) for \( \tilde{G} \). Using the map \( g : \Sigma \to G \), we may represent any element in \( G \) by \( g = e^{i\phi^k X_k} \), where \( \{\phi^k\} = (\phi^1, \phi^2, \phi^3) \) are commuting fields and \( \{X_k\} \) are the orthonormal basis for the Lie algebra of \( G \), and \( \{X_k\} = (-\frac{i}{2}\sigma_1, -\frac{i}{2}\sigma_2, -\frac{i}{2}\sigma_3) \) for the case of \( SU(2) \). Structure constants are \( \epsilon^{jk}_i \), and commutation relations are the familiar form of Pauli matrices, 

\[
[ -i\sigma^1, -i\sigma^2 ] = \epsilon^{k}_{ij} (-i\sigma^k).
\]

Equations of motion are

\[
\partial_{-}(g^{-1}\partial_{+}g) = 0.
\]

Nonlocal currents for the Lie algebra of \( SU(2) \) are

\[
J^{(L)}(L) + = (g^{-1}\partial_{+}g)^k X_k \quad \text{and} \quad J^{(R)}(R) - = [(\partial_{-}g)g^{-1}]^k X_k.
\]

We want to construct currents up to the order of \( \phi^2 \). If we consider infinitesimal coefficients \( \{\phi^k\} \), keeping up to second orders we may expand \( g \) as

\[
g = 1 + i\phi^k X_k - \frac{1}{2}(\phi^k \phi^l)(X_k X_l) + ...
\]

(33)

Since we are looking for \( J^{(L)}_+ \) and \( J^{(R)}_- \) up to the order of \( \phi^2 \), we need \( g \) to the order of \( \phi \), hence

\[
g = 1 + i\phi^k X_k \quad \text{(34.1)}
\]

\[
g^{-1} = 1 - i\phi^k X_k \quad \text{(34.2)}
\]

To this order the solution to equations of motion \( \partial_{-}(g^{-1}\partial_{+}g) = 0 \) is \( g = g_R(\sigma^-)g_L(\sigma^+) \), which leads to \( \phi(\sigma^+, \sigma^-) = \phi_R(\sigma^-) + \phi_L(\sigma^+) \). Thus equation (34) can be written as

\[
g_L = 1 + i\phi^k X_k \quad \text{(35.1)}
\]

\[
g_R = 1 + i\phi^k X_k \quad \text{(35.2)}
\]

and hence left and right currents can readily be obtained as

\[
J^{(L)}_+ = g_L^{-1}\partial_{+}g_L = i\partial_{+}\phi^m_L X_m + \frac{1}{2} f^m_{kl} \phi^k_L \partial_{+}\phi^l_L X_m \quad \text{(36.2)}
\]

\[
J^{(R)}_- = (\partial_{-}g_R)g_R^{-1} = i\partial_{-}\phi^m_R X_m + \frac{1}{2} f^m_{kl} \phi^k_R \partial_{-}\phi^l_R X_m \quad \text{(36.2)}
\]

Therefore, we conclude that \( \partial_{-}J^{(L)}_+ = \partial_{+}J^{(R)}_- = 0 \), i.e., currents are conserved on \( G \). We first solve equation (32) to figure out the pseudodual currents. Since \( f^{k}_{ij} = 0 \), we have
\[ [(\partial_+ T) T^{-1}]^i_j = f^k_m T^i_k T^j_l (g^{-1}_L \partial_+ g_L)^m \]  (37)

this may be reduced to

\[ [(\partial_+ T)]^i_n = f^k_m T^i_k (g^{-1}_L \partial_+ g_L)^m \]  (38)

and putting in an order parameter \( \epsilon \) we get

\[ [(\partial_+ T)]^i_n = \epsilon f^k_m T^i_k (g^{-1}_L \partial_+ g_L)^m \]  (39)

We adapt to an exponential solution \( T = e^{\epsilon \alpha_1} e^{\frac{1}{2} \epsilon^2 \alpha_2} (I + \mathcal{O}(\epsilon^3)) \), where \( \alpha_1 \) and \( \alpha_2 \) are antisymmetric matrices, and expanding this solution we get

\[ T = I + \epsilon \alpha_1 + \frac{1}{2} \epsilon^2 (\alpha_2 + \alpha_1^2) + \mathcal{O}(\epsilon^3) \]  (40)

taking \( \partial_+ \) of \( T \) leads to

\[ \partial_+ T = \epsilon \partial_+ \alpha_1 + \frac{1}{2} \epsilon^2 [\partial_+ (\alpha_1 \alpha_1) + (\partial_+ \alpha_1) \alpha_1 + \partial_+ \alpha_2] + \mathcal{O}(\epsilon^3) \]  (41)

expressing (39) in tensor product form

\[ \partial_+ T = \epsilon f (g^{-1}_L \partial_+ g_L) \otimes T = \epsilon f (g^{-1}_L \partial_+ g_L) \otimes (I + \epsilon \alpha_1) \]

\[ = \epsilon f (g^{-1}_L \partial_+ g_L) + \epsilon^2 f (g^{-1}_L \partial_+ g_L) \otimes \alpha_1 \]  (42)

and comparing this with (41) we obtain \( \alpha_1 \)

\[ (\alpha_1)^i_n = \int_0^{\sigma^+} f^i_m (g^{-1}_L \partial_+ g_L)^m d\sigma^+ \]  (43)

\[ = i f^i_m \phi^m_L + \frac{1}{2} f^i_m f^m_{kl} \int_0^{\sigma^+} \phi^k_L \partial_+ \phi^l_L d\sigma^+ \]

\[ = i \epsilon f^i_m \phi^m_L + \frac{1}{2} \int_0^{\sigma^+} (\phi^m_L \partial_+ \phi^i_L - \phi^i_L \partial_+ \phi^m_L) d\sigma^+ \]

this expression leads to the following entries

\[ (\alpha_1)_1^1 = 0 \]  (44.1)

\[ (\alpha_1)_2^1 = -i \phi^2_L - \frac{1}{2} \int_0^{\sigma^+} [\phi^1_L (\partial_+ \phi^2_L) - \phi^2_L (\partial_+ \phi^1_L)] d\sigma^+ \]  (44.2)

\[ (\alpha_1)_3^1 = i \phi^3_L + \frac{1}{2} \int_0^{\sigma^+} [\phi^3_L (\partial_+ \phi^1_L) - \phi^1_L (\partial_+ \phi^3_L)] d\sigma^+ \]  (44.3)
$$(\alpha_1)_1^2 = i\phi^3_L + \frac{1}{2} \int_0^{\sigma^+} [\phi^i_L (\partial_+ \phi^2_L) - \phi^2_L (\partial_+ \phi^1_L)] d\sigma^+$$  \hspace{1cm} (44.4)

$$(\alpha_1)_2^2 = 0$$  \hspace{1cm} (44.5)

$$(\alpha_1)_3^2 = -i\phi^1_L - \frac{1}{2} \int_0^{\sigma^+} [\phi^2_L (\partial_+ \phi^3_L) - \phi^3_L (\partial_+ \phi^2_L)] d\sigma^+$$  \hspace{1cm} (44.6)

$$(\alpha_2)_3^3 = -i\phi^1_L - \frac{1}{2} \int_0^{\sigma^+} [\phi^2_L (\partial_+ \phi^3_L) - \phi^3_L (\partial_+ \phi^2_L)] d\sigma^+$$  \hspace{1cm} (44.7)

$$(\alpha_2)_3^3 = 0$$  \hspace{1cm} (44.8)

and

$$\partial_+ \alpha_2 = 2f(g_L^{-1} \partial_+ g_L) \otimes \alpha_1 - [\alpha_1 (\partial_+ \alpha_1) + (\partial_+ \alpha_1) \alpha_1]$$

Hence, $\alpha_2$ is obtained as

$$\alpha_2^n = \int_0^{\sigma^+} (\phi^n_L \partial_+ \phi^n_L - \phi^n_L \partial_+ \phi^n_L) d\sigma^+$$ \hspace{1cm} (45)

We see that this is equivalent to (13), and entries are the same as the negative of (14). Therefore, we can find $T$ by means of (43) and (45), and setting $\varepsilon = 1$

$$T^n_l = \delta^l_n + i\varepsilon_{mn} \phi^m_L + \frac{1}{2}(\delta^n_m \phi^m_L \phi^m_L - \phi^n_L \phi^m_L)$$ \hspace{1cm} (46)

Again we note that $T$ is an orthogonal matrix. Now using pseudoduality equations (31)

$$\partial_+ \phi^i_L = +T^i_j (g^{-1} \partial_+ g)^j = +T^i_j (g_L^{-1} \partial_+ g_L)^j$$ \hspace{1cm} (47.1)

$$\partial_+ \phi^i_R = -T^i_j (g^{-1} \partial_+ g)^j = -T^i_j (g_L^{-1} (\partial_+ g_R) g_L)^j$$ \hspace{1cm} (47.2)

since we are trying to find $\partial_+ \phi^i_L$ and $\partial_+ \phi^i_R$ up to the order of $\phi^2$, we need $T$ to the order of $\phi$, hence using

$$T^n_l = \delta^l_n + i\varepsilon_{mn} \phi^m_L + \mathcal{O}(\phi^2)$$ \hspace{1cm} (48)
\[ \partial_+ \tilde{\phi}_L^i = T_j^i (g_L^{-1} \partial_+ g_L)^j \]
\[ = [\delta_j^i + i e^i_{mj} \phi_L^m] [i \partial_+ \phi_L^j + \frac{1}{2} e_{kl}^i \phi_L^k \partial_+ \phi_L^l] \]
\[ = i \partial_+ \phi_L^i - \frac{1}{2} e^i_{kl} \phi_L^k \partial_+ \phi_L^l + O(\phi^3) \] (49.1)

\[ \partial_- \tilde{\phi}_R^i = -T_j^i (g^{-1} \partial_- g)^j = -T_j^i (g_L^{-1} (g_R^{-1} \partial_- g_R) g_L)^j \]
\[ = -[\delta_j^i + i e^i_{mj} \phi_L^m] [(1 - i \phi^k_R X_k)(1 - \phi^k_R X_k)(i \partial_- \phi_R^k X_k) - \frac{1}{2} \phi_R^m \partial_- \phi_R^l X_l X_m] - \frac{1}{2} \phi_R^i \partial_- \phi_R^m X_l X_m (1 + i \phi^k_L X_k) \]
\[ = -[\delta_j^i + i e^i_{mj} \phi_L^m] [i \partial_- \phi_R^j + e^j_{mn} \phi_L^m \partial_- \phi_R^n + \frac{1}{2} e^j_{lm} \phi_R^l \partial_- \phi_R^m] \]
\[ = -i \partial_- \phi_R^i - \frac{1}{2} e^i_{lm} \phi_R^l \partial_- \phi_R^m \] (49.2)

We see that these are free scalar currents on the tangent bundle to pseudodual manifold \( \tilde{\mathcal{G}} \). Since \( \partial_+ \tilde{\phi}_L^i \) depends only on \( \sigma^+ \), and \( \partial_- \tilde{\phi}_R^i \) only on \( \sigma^- \), these pseudodual free scalar currents are conserved provided that equations of motion for free scalar fields hold. We go back to equations of motion to see that these pseudodual tangent bundle components take us to pseudodual conserved currents. Equations of motion, \( \partial_- (g^{-1} \partial_+ g)^i = 0 \), imply that \( \partial_+ \phi^i = 0 \). Obviously we find out the pseudodual conservation laws \( \partial_{\pm}^2 \phi^i = 0 \) in all \( \phi \)-orders using these conditions.

4 Discussion

We observed that nonlinear character of WZW models results in an infinite number of terms in transformation matrix \( T \), which in turn leads to construct infinite number of nonlocal currents in pseudodual manifold. Calculations here were motivated by the fact that sigma models have Lie group structures, and \( T \in SO(n) \). Hence structure of Lie groups together with perturbation calculations reflects the nonlinear characteristic of sigma models. It is obvious that pseudoduality transformation preserved the pseudodual currents in our cases where one model based on an abelian group \( U(1) \times U(1) \times U(1) \) in two cases we discussed. However, One can consider general Lie group valued
fields for both models, and see that this would also yield conserved currents on pseudodual model. We considered three dimensional models for simplicity but this can be extended to any dimension. Calculation of these currents gives us curvatures by means of Cartan structural equations

\[ dw^i + w^i_j \wedge w^j = 0 \quad (50.1) \]
\[ dw^i_j + w^i_k \wedge w^k_j = \frac{1}{2} R^i_{jkl} w^k \wedge w^l \quad (50.2) \]

where \( w^i = J^i \) and \( w^i_j = \frac{1}{2} f^i_{kj} J^k \) is the antisymmetric connection, and \( J \) stands for both \( J^{(L)} \) and \( J^{(R)} \). These currents form an orthonormal frame on pullback bundle \( g^*(TG) \). Since we considered abelian models, and hence obtained scalar currents, it is easily noted that curvatures are zero. In general case where sigma models based on general Lie groups, curvatures will be constant and opposite. This shows that sigma models are based on symmetric spaces as pointed out in [3]. The calculations and results of this paper can be applied to pseudoduality relations between symmetric space sigma models to construct nonlocal currents, and as a result curvatures. In more extreme cases it would be interesting to discuss the pseudoduality in supersymmetric sigma models, and get constraints for pseudoduality transformation.

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