Inertia of entanglement witness

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Entanglement witnesses (EWs) are a fundamental tool for the detection of entanglement. We study the matrix inertias of EWs with a focus on the EWs constructed by the partial transposition of states with non-positive partial transposes. We provide a method to generate more inertias from a given inertia by the relevance between inertias. Based on that we exhaust all the inertias for EWs in each qubit-qudit system. We apply our results to propose a separability criterion in terms of the rank of the partial transpose of state. We also connect our results to tripartite genuinely entangled states and the classification of states with non-positive partial transposes. Additionally, the inertias of EWs constructed by X-states are clarified.

I. INTRODUCTION

Quantum entanglement, discovered by Einstein, Podolsky, Rosen (EPR), and Schrödinger [1, 2], is a remarkable feature of quantum mechanics. It involves non-classical correlations between subsystems, and lies in the heart of quantum information theory [3, 4]. In recent decades, entanglement has been recognized as a kind of valuable resource [3, 5, 6]. It plays a central role in various quantum information processing tasks such as quantum computing [7], teleportation [8], dense coding [9], cryptography [10], and quantum key distribution [11].

Although several useful separability criteria such as positive-partial-transpose (PPT) criterion [12, 13], range criterion [13], and realignment criterion [14] were developed, all of them cannot strictly distinguish between the set of entangled states and that of separable ones. According to PPT criterion, any state with non-positive partial transpose (NPT) must be entangled. Nevertheless, the converse only holds for two-qubit and qubit-quinrit systems. There exist PPT entangled (PPTE) states in higher-dimensional Hilbert spaces [13]. It has been shown that determining whether a bipartite state is entangled or not is an NP-hard problem [15]. It is even harder to tame multipartite entanglement [16], since multipartite entangled states can be further classified as genuinely multipartite entangled states and biseparable states [17]. In 2000, Terhal first introduced the term entanglement witness (EW) by indicating that a violation of a Bell inequality can be expressed as a witness for entanglement [18]. Recently, more and more EWs have been implemented with local measurements [19–22]. Nowadays, EWs are a fundamental tool for the detection of entanglement both theoretically and experimentally [4].

EWs are observables that enable us to detect entanglement physically. It has been shown that an EW can detect PPTE states if and only if it is non-decomposable [23]. Therefore, much effort has been devoted to construct non-decomposable EWs [24–27]. Moreover, several constructions of multipartite EWs were proposed [28–31]. It is noteworthy that the partial transposition of NPT state is an easy way to construct EWs by the so-called Choi-Jamiolkowski isomorphism [32]. Furthermore, the partial transpose of an NPT state, denoted by $\rho^\Gamma$, can be used to construct optimal EWs for decomposable EWs [23]. However, it cannot be directly realized in experiments because the partial transposition is not a physical operation [33]. In a very recent paper [34] authors proposed and experimentally demonstrated conditions for mixed-state entanglement and measurement protocols based on PPT criterion. It sheds light on the experiments involving the partial transposition.

Negative eigenvalues of $\rho^\Gamma$ are a signature of entanglement. They are closely related to other problems in entanglement theory. For instance, the negativity [35], a well-known computable entanglement measure, is defined as the sum of the absolute values of negative eigenvalues. Also, by the definition of 1-distillable state [36], the more negative eigenvalues $\rho^\Gamma$ has, the more likely $\rho$ is 1-distillable. Thus, it is important to explore the negative eigenvalues of $\rho^\Gamma$. The problem of determining how many negative eigenvalues for the partial transpose of NPT state has attracted great interest [37–40]. It was first specified in [37] that $\rho^\Gamma$ has one negative eigenvalue and three positive eigenvalues for any two-qubit entangled state $\rho$. For this reason, an easier method to iden-

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tify two-qubit entangled states was proposed. That is any two-qubit state is separable if and only if det $\rho^F \geq 0$ [37]. Then some restrictions on the spectral properties of EWs were first derived in [41]. For NPT state $\rho$ supported on $\mathbb{C}^m \otimes \mathbb{C}^n$, it is known that $\rho^F$ has at most $(m-1)(n-1)$ negative eigenvalues, and all eigenvalues of $\rho^F$ lie within $[-1/2,1]$ when $\rho$ is normalized [39]. Furthermore, Nathaniel Johnston et al. discussed an interesting problem on the eigenvalues for EWs, namely the inverse eigenvalue problem [42]. This problem on EWs inspires us to investigate the matrix inertia of EW instead of considering the number of negative eigenvalues only. We will show the inertia is a finer index to characterize EWs than the number of negative eigenvalues.

In this paper, we study the inertia of EW with a focus on the partial transpose of NPT state in the qubit-qudit system. In the bipartite setting, qubit-qudit states appear in many problems, and have received a lot of attention. Several important properties of qubit-qudit states have been derived. First, all qubit-qudit NPT states are distillable [43]. However, the distillability of NPT states in the two-qutrit system still remains as a major open problem in entanglement theory. Second, a first systematic study on the separability of qubit-qudit PPT states was discussed in [44]. Moreover, the birank of qubit-qudit PPT state and the length of qubit-qudit separable state were investigated in [38]. Very recently, the absolutely separable states in qubit-qudit systems were studied in [45] for they are useful in quantum computation. Third, one of the most known analytical formulas for entanglement measures is the entanglement of formation of two-qubit states [46]. Later, a lower bound on entanglement of formation for the qubit-qudit system was derived [47]. Fourth, the optimization of decomposable EWs acting on the qubit-qudit system was studied [48, 49]. It is known that for a qubit-qudit NPT state $\rho$, $\rho^F$ is an optimal decomposable EW if and only if the range of $\rho$ contains no product vector [48].

Next, we present our main results explicitly. We first derive the lower and upper bounds on the number of negative (positive) eigenvalues for an arbitrary bipartite EW in Lemma 5. Second we completely determine the inertia of two-qubit EW in Theorem 6. It generalizes the result in [37]. Third, we show the relation between EWs and the partial transposes of NPT states in Lemma 7. Then we study the partial transpose of NPT state. In Lemma 8 we reveal the essential relevance between inertias, and propose a method to generate more inertias from a given inertia. This method is also applicable to PPT states. Thus, we can generate inertias for the partial transposes of PPT states as by-products. Moreover, the existence of product vectors in the kernel of $\rho^F$ is essential to characterize its inertia. Therefore, we discuss this problem in Lemma 9. Based on that we present a sufficient and necessary condition for a sequence to be the inertia in Theorem 10. Combining Lemma 8 and Theorem 10 we further exhaust all inertias in every qubit-qudit system in Theorem 12. Then we extend our study to general NPT states in Lemma 13. Finally, we build the connections between our results and other problems in quantum information theory. In Theorem 14 we present a separability criterion in terms of the rank of $\rho^F$. Then we propose a method to generate the inertia of $\rho^F$ for higher-dimensional state $\rho$. Using this method we can characterize the partial transposes of tripartite genuinely entangled states in a systematic way. We also indicate that the inertia of $\rho^F$ provides a tool to classify states under SLOCC equivalence. In Theorem 15 we explicitly express the eigenvalues of $\rho^F$, and quantify the number of negative ones when $\rho$ is a qubit-qudit X-state [50].

The remainder of this paper is organized as follows. In Sec. II we first clarify the notations in the whole paper. Second we introduce some necessary definitions. Finally we present useful results related to the inertia of EW. In Sec. III we study the inertia of NPT state. Second we completely determine the inertia of two-qubit EW. In Sec. IV we present a sufficient and necessary condition for a sequence to be the inertia, and exhaust all inertias in the qubit-qudit system. In Sec. V we show some applications of our results. The concluding remarks are given in Sec. VI. In the final part, we prove some of our results in the three appendices. In Appendix A we provide the proofs of results in Sec. III. In Appendix B we provide the proofs of results in Sec. IV. In Appendix C we present the proofs of results in Sec. V.

II. PRELIMINARIES

In this section we introduce the preliminaries for explaining our results on the inertia of EW. First we clarify the notations. Second we introduce some necessary definitions. Finally we present useful results related to the inertia of EW.

We use $\bigotimes_{i=1}^n \mathbb{C}^{d_i}$ to represent an n-partite Hilbert space, where $d_i$’s are local dimensions. If $\rho \in \mathcal{B}(\bigotimes_{i=1}^n \mathbb{C}^{d_i})$ is positive semidefinite, then $\rho$ is an n-partite state. Unless stated otherwise, the state in this paper is non-normalized. For a bipartite state $\rho$, we say $\rho$ is an $m \times n$ state for convenience if $\rho \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Without loss of generality, we may assume $m \leq n$. Since the two partial transposes of $\rho$ with respect to the first and second systems respectively are equivalent up to the global transposition, the two partial transposes have the same inertia. Hence, it suffices to consider the partial transpose of $\rho$ with respect to the first system, denoted by $\rho^F$. For any Hermitian operator $X$, denote by $\mathcal{R}(X), \mathcal{K}(X)$, and $r(X)$ the range, kernel and rank of $X$, respectively. Specifically, we will investigate $\mathcal{R}(\rho^F), \mathcal{K}(\rho^F)$, and $r(\rho^F)$ for a bipartite state $\rho$. We use $X \geq 0$ to represent a positive semidefinite operator $X$. Denote by $\mathcal{M}_n$ the set of $n \times n$ matrices, and by $I_n (\in \mathcal{M}_n)$ the identity matrix. In order to study the inertia of Hermitian $X$ conveniently, we shall refer to the positive (zero, negative) eigen-space of $X$ as the
subspace spanned by the eigenvectors corresponding to positive (zero, negative) eigenvalues of $X$.

In the following we introduce some necessary definitions. In Definition 1 we define EWs and decomposable EWs. In Definition 2 we introduce SLOCC equivalence. It is a useful tool to study inertia. Finally we define the matrix inertia in Definition 3.

**Definition 1**  
(i) [29] Suppose $W \in B(\bigotimes_{i=1}^n \mathcal{H}_A)$ is Hermitian. We call $W$ is an $n$-partite entanglement witness (EW) if (1) it is non-positive semidefinite, and (2) $\langle \psi | W | \psi \rangle \geq 0$ for any product vector $| \psi \rangle = \bigotimes_{i=1}^n | a_i \rangle$.

(ii) [23] Suppose $W \in B(\mathcal{H}_{AB})$ is a bipartite EW. If $W$ can be decomposed as $W = P + Q^T$, where $P, Q$ are both positive semidefinite, then $W$ is a decomposable EW.

Suppose $W$ is an $n$-partite EW, and $\rho$ is an $n$-partite state. If $\text{Tr}(W \rho) < 0$, we determine $\rho$ is an entangled state detected by $W$. It is conductive to understand EWs from the perspective of geometry. In convex set theory the Separation Theorem states that there is a hyperplane separating two disjoint convex sets [51]. Since the set of all separable states is convex, there is a hyperplane separating the set of all separable states and a subset of entangled ones. Here the hyperplane plays the role of EW. We illustrate how a bipartite EW detects entanglement in Fig. 1. It is known that a state is entangled if and only if it can be detected by some EW [18]. Therefore, the detection of entanglement can be transformed to construct proper EWs. The positive but not completely positive maps [32] can be used to construct EWs by the Choi-Jamiolkowski isomorphism [32]. The transpose map is a typical positive but not completely positive map. This explains why the partial transpose of NPT state is an EW.

Next, we introduce SLOCC equivalence. It is an essential concept for studying the inertia as we shall see the inertia of Hermitian matrix is invariant under SLOCC equivalence.

**Definition 2** [53] We refer to SLOCC as stochastic local operations and classical communications. Two $n$-partite pure states $| \alpha \rangle, | \beta \rangle$ are SLOCC equivalent if there exists a product invertible operation $Y = Y_1 \otimes \ldots \otimes Y_n$ such that $| \alpha \rangle = Y | \beta \rangle$.

We further extend the above definitions to spaces. Let $V = \text{span}\{| \alpha_1 \rangle, \ldots, | \alpha_m \rangle \}$ and $W = \text{span}\{| \beta_1 \rangle, \ldots, | \beta_m \rangle \}$ be two $n$-partite subspaces of $m$-dimension. $V$ and $W$ are SLOCC equivalent if there exist a product invertible operation $Y$ such that $| \alpha_i \rangle \propto Y | \beta_i \rangle$ for any $i$.

In the following we formulate the definition of inertia.

**Definition 3** Let $A \in \mathcal{M}_n$ be Hermitian. The inertia of $A$, denoted by $\text{In}(A)$, is defined as the following sequence

$$\text{In}(A) := (\nu_-, \nu_0, \nu_+),$$

where $\nu_-, \nu_0$ and $\nu_+$ are respectively the numbers of negative, zero and positive eigenvalues of $A$.

Inertia is an important concept in matrix theory. There is an essential proposition for the matrix inertia, namely Sylvester Theorem [54]. It states that Hermitian matrices $A, B \in \mathcal{M}_n$ have the same inertia if and only if there is a non-singular matrix $S$ such that $B = SAS^T$. It follows that inertas are invariant under SLOCC equivalence. It allows us to study the inertia under SLOCC equivalence.

In the last part of this section we present several useful results related to the inertia of $\rho^T$ for NPT state $\rho$.

**Lemma 4** Suppose $\rho$ is an $m \times n$ NPT state. Then

(i) [39] the number of negative eigenvalues of $\rho^T$ is in the interval $[1, (m - 1)(n - 1)]$;

(ii) [39] if $\rho$ is normalized, i.e., $\text{Tr}(\rho) = 1$, every negative eigenvalue of $\rho^T$ is not less than $-\frac{1}{2}$;

(iii) [38] if $m = 2$, for each $k \in [1, n - 1]$ there exists a state $\rho$ such that the number of negative eigenvalues of $\rho^T$ is $k$;

(iv) [42] if $\rho$ is a pure state with Schmidt rank $r$, then

$$\text{In} \rho^T = \left(\frac{r^2 - r}{2}, mn - r^2, \frac{r^2 + r}{2}\right).$$

Based on the above preliminary knowledge we are ready to study the inertia of EW.

### III. Restrictions on the Inertia of Entanglement Witness

In this section we propose restrictions on the inertia of EW. Specifically, we derive lower and upper bounds on...
the number of negative (positive) eigenvalues of an EW in Lemma 5. In virtue of these restrictions we completely determine the inertia of two-qubit EW in Theorem 6. We would like to emphasize that these restrictions will be used frequently in Sec. IV to further exhaust some inertia sets. Finally we demonstrate the relation between bipartite EWs and bipartite NPT states in Lemma 7.

First we present the lower and upper bounds on the number of negative (positive) eigenvalues of an EW.

Lemma 5 Suppose $W$ is an EW on $\mathbb{C}^m \otimes \mathbb{C}^n$.

(i) Let $E$ be the non-positive eigen-space of $W$, i.e., the sum of negative and zero eigen-spaces of $W$. Then the product vectors in $E$ all belong to the zero eigen-space of $W$. In particular, every vector in the negative eigen-space of $W$ is a pure entangled state.

(ii) The number of negative eigenvalues of $W$ is in $[1, (m-1)(n-1)]$. The decomposable EW containing exactly $(m-1)(n-1)$ negative eigenvalues exists.

(iii) The number of positive eigenvalues of $W$ is in $[2, mn - 1]$.

We give the proof of Lemma 5 in Appendix A. It is efficient to exclude several sequences to be the inertia of EW by using the restrictions in Lemma 5.

Next, we use Lemma 5 to determine the inertia of two-qubit EW. In Theorem 6 we show that every two-qubit EW has inertia $(1, 0, 3)$. This result generalizes the known conclusion that $\text{In}(\rho^T) = (1, 0, 3)$ for any two-qubit entangled state $\rho$ [37]. For this purpose we need to introduce block-positive operators [25]. Suppose $M \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is Hermitian. We call $M$ is block-positive, if

$$M := (I_m \otimes \Phi)X,$$

for some positive semidefinite operator $X \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^m)$, and some positive map $\Phi : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$. It is known that a bipartite EW is block-positive but non-positive semidefinite. In [42] there was a useful result on the eigenvalues of block-positive operators in $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$. It states that there exists a block-positive matrix $W$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ if and only if the following three inequalities hold:

$$\mu_3 \geq 0,$$

$$\mu_4 \geq -\mu_2,$$

$$\mu_4 \geq -\sqrt{\mu_1 \mu_3}.$$  

Combining Lemma 5 and the above result we can show Theorem 6 as follows.

Theorem 6 Every two-qubit EW has inertia $(1, 0, 3)$.

Proof. By Lemma 5 (ii), any two-qubit EW has exact one negative eigenvalue. Thus, there are two distributions of inertia $(1, 0, 3)$ and $(1, 1, 2)$. Here we prove that sequence $(1, 1, 2)$ is not the inertia. Assume $W$ is a two-qubit EW with inertia $(1, 1, 2)$. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ be the four eigenvalues of $W$. It follows that $\mu_4$ is negative and $\mu_3 = 0$. It contradicts with the last inequality in (4). So the assumption is not valid. Therefore, the inertia $(1, 1, 2)$ does not exist. This completes the proof. □

Theorem 6 motivates us to determine the inertias of EWs acting on higher-dimensional Hilbert spaces. As we know, the partial transpose of NPT state is an EW. Obviously, there are EWs which are not the partial transpose of NPT state. Here we construct an example to show that there exists an EW $W$ whose partial transpose $W^T$ is still an EW. Thus, $W$ cannot be the partial transpose of NPT state. Let $\alpha = (|00\rangle + |11\rangle)(|00\rangle + |11\rangle)$ and $\beta = |00\rangle|00\rangle + a|11\rangle|11\rangle + b(|01\rangle + |10\rangle)(|01\rangle + |10\rangle) + c(|01\rangle - |10\rangle)(|01\rangle - |10\rangle)$ with

$$a, b > 0, \quad c \in (0, 1/2),$$

$$2(1 + a) - (1 + b - c)^2 < 0. \quad (5)$$

One can verify that $W = \alpha^T + \beta$ is an EW, and $W^T$ is still an EW. Inspired by this example, we demonstrate the relation between bipartite EWs and bipartite NPT states in Lemma 7.

Lemma 7 Suppose $W \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is a Hermitian and non-positive semidefinite operator. Then $W$ is an EW if and only if $W^T$ is an EW or an NPT state.

Proof. Let $T$ be the set of bipartite EWs and NPT states. We first show $T$ is invariant under partial transpose. Suppose $W \in T$. If $W$ is an EW then $W^T$ is still Hermitian. Further if $W^T$ is positive semidefinite then $W^T$ is indeed an NPT state. Thus we conclude that $W^T \in T$. If $W^T$ is non-positive semidefinite one can show $W^T$ is still an EW as follows. For any product vector $|a_1, a_2\rangle$,

$$\langle a_1, a_2\rangle W^T |a_1, a_2\rangle = \langle a_1^*, a_2|W|a_1^*, a_2\rangle \geq 0.$$

Thus we conclude that $W^T \in T$. For the same reason we conclude that $W^T \in T$ if $W$ is an NPT state. Therefore, $T$ is invariant under partial transpose. This result implies that $W$ is an EW if and only if $W^T$ is an EW or an NPT state. This completes the proof. □

In experiments, an EW is usually decomposed into a sum of locally measurable observables. Then these locally measurable observables are measured individually on the constituent subsystems. Finally one obtains witness expectation value $\text{Tr}(W \rho)$ by summing the expectation values of the locally measurable observables. In [55] O. Gühne et al. introduced a general method for the experimental detection of entanglement by performing only few local measurements, assuming some prior knowledge of the density matrix. Their method is based on the minimal decomposition of witness operators into a pseudomixture of local operators. Any bipartite EW $W$ can be decomposed into a sum of projectors onto product
vectors, i.e.,
\[ W = \sum_j c_j |a_j, b_j\rangle \langle a_j, b_j| = \sum_j c_j |a_j\rangle \otimes |b_j\rangle \langle b_j|, \tag{6} \]
where the coefficients \( c_j \) are real and satisfy \( \sum_j c_j = 1 \). There is at least one coefficient has to be negative for \( W \) is an EW. This characterizes a so-called pseudomixture. Moreover, there are many different decompositions like in Eq. (6) for any EW. In [55] authors were interested in the optimal decompositions. That is the pseudomixture with minimal number of non-zero coefficients \( c_j \). Suppose \( W \) is an EW which is not the partial transpose of NPT state. It follows from Lemma 7 that \( W^T \) is also an EW. One can verify that if Eq. (6) is a decomposition of \( W \) then \( \rho \) is a decomposition of \( W^T \). It implies that the minimal number of non-zero coefficients for \( W \) is the same as that for \( W^T \).

In the following section we investigate the inertia of EW starting from the EWs constructed by the partial transpose of NPT state. Due to the relation given by Lemma 7 our results are helpful to understand the inertia of general EW.

IV. INERTIA OF THE PARTIAL TRANSPOSE OF NPT STATE

The partial transposition on an NPT state is an easy way to construct EWs and can be used to construct optimal EWs for decomposable EWs [23]. In this section we focus on the bipartite EWs constructed by the partial transpose of NPT state, and determine inertias of such EWs. In Lemma 8 we reveal the essential relevance between inertias, and propose a method to generate more inertias from a given inertia. We apply Lemma 8 to NPT states in qubit-qudit systems. The qubit-qudit states are widely investigated and have many interesting propositions. Suppose \( \rho \) is a \( 2 \times 2 \) NPT state. In Theorem 10 we show a sufficient and necessary condition for a sequence \( (a, b, c) \) to be an inertia of \( \rho^T \). Based on the above results we exhaust all inertias for \( \rho^T \) in Theorem 12. Finally, in Lemma 13 we extend our study to \( m \times n \) NPT states.

In the first part of this section we focus on the partial transposes of all states, though in this paper we are more interested in NPT states. In order to describe our results conveniently, we first denote three inertia sets:

\[ \mathcal{N}_{m,n} := \{ \text{In}(\rho^T) | \rho \text{ is an } m \times n \text{ NPT state.} \}, \]
\[ \mathcal{P}_{m,n} := \{ \text{In}(\rho^T) | \rho \text{ is an } m \times n \text{ PPTE state.} \}, \]
\[ \mathcal{S}_{m,n} := \{ \text{In}(\rho^T) | \rho \text{ is an } m \times n \text{ separable state.} \}. \]

In the following we propose an effective method to derive more inertias from a given inertia. It also reveals the relevance between inertias regarding the existence of product vectors in the kernel of \( \rho^T \).

**Lemma 8** (i) Suppose \( \rho \) is an \( m \times n \) NPT (PPTE, separable) state and \( \rho^T \) has inertia \( (a, b, c) \). Then there is a small enough \( x > 0 \) and the NPT (PPTE, separable) state
\[ \sigma := \rho + xI_{mn}, \]

such that
\[ \text{In}(\sigma^T) = (a, 0, b+c). \]

Note that if \( \rho \) is PPT, then \( a = 0 \).

(ii) Suppose \( m_1 \leq m_2 \) and \( n_1 \leq n_2 \). If
\[ (a_1, b_1, c_1) \in \mathcal{N}_{m_2,n_1}, \]
or \( (a_1, b_1, c_1) \in \mathcal{P}_{m_1,n_1}, \)
or \( (a_1, b_1, c_1) \in \mathcal{S}_{m_1,n_1}, \]
with \( a_1 + b_1 + c_1 = m_1 n_1 \), then \( \forall 0 \leq l \leq m_2 n_2 - m_1 n_1 \),
\[ (a_1 + l, b_1 + c_1 + l) \in \mathcal{N}_{m_2,n_2}, \]
or \( (a_1 + l, b_1 + c_1 + l) \in \mathcal{P}_{m_2,n_2}, \)
or \( (a_1 + l, b_1 + c_1 + l) \in \mathcal{S}_{m_2,n_2}, \]
respectively. Note that if \( (a_1, b_1, c_1) \in \mathcal{P}_{m_1,n_1} \) or \( (a_1, b_1, c_1) \in \mathcal{S}_{m_1,n_1} \), then \( a_1 = 0 \).

We show the proof of Lemma 8 in Appendix B. Using this idea one can imagine how inertias grow as local dimensions increase. We illustrate this growth process in Fig. 2. The basic idea of Lemma 8 is to add linearly independent product states into the given density matrix. We will apply this method to further characterize the inertia set \( \mathcal{N}_{2,n} \).

In the second part of this section we aim to determine the inertia set \( \mathcal{N}_{2,n} \) completely. There are two main results in this part. One is Theorem 10, where we propose a sufficient and necessary condition for a sequence in the inertia set \( \mathcal{N}_{2,n} \). The other one is Theorem 12, where we completely determine the inertia set \( \mathcal{N}_{2,n} \) for any \( n \geq 2 \).

It follows from Lemma 5 (i) that all product vectors in the non-positive eigen-space of an EW belong to the kernel of \( \mathcal{K}(\rho^T) \). The existence of product vectors in \( \mathcal{K}(\rho^T) \) is quite essential for studying \( \text{In}(\rho^T) \). For this reason we investigate how many linearly independent product vectors in \( \mathcal{K}(\rho^T) \) as follows.

**Lemma 9** Let \( \rho \) be an \( m \times n \) NPT state. Denote by \( d \) the dimension of \( \mathcal{K}(\rho^T) \). Suppose \( \rho^T \) has \( k \) negative eigenvalues, and \( d + k > (m-1)(n-1) \). Let \( l = d + k - (m-1)(n-1) \). Then there are at least \( l \) linearly independent product vectors in \( \mathcal{K}(\rho^T) \). That is
\[ \mathcal{K}(\rho^T) = \text{span}\{|a_1, b_1|, \cdots, |a_l, b_l|, |a_{l+1}|, \cdots, |a_d|\}. \]
\[(i.a) \ (a, b - 2, c) \notin N_{2,n-1}, \]
\[(i.b) \ (a, b - 1, c - 1) \notin N_{2,n-1}, \]

(ii) Suppose \( \text{In}(\rho^F) = (a, b, c) \). If \((a, b - 2, c) \in N_{2,n-1}\), and \((a, b - 1, c - 1) \in N_{2,n-1}\), then \( \rho \) can be regarded as a \( 2 \times (n-1) \) NPT state up to a local projector.

We show the proof of Theorem 10 in Appendix B. Theorem 10 demonstrates the relation between the two inertia sets \( N_{2,n-1} \) and \( N_{2,n} \) for any \( n > 2 \). Applying this result we obtain the following corollary.

**Corollary 11**  
(i) There exists a \( 2 \times n \) NPT state \( \rho \) whose partial transpose contains exact \( (n-1) \) negative eigenvalues. Further, if \( \rho^F \) has \( (n-1) \) negative eigenvalues, then \( \text{In}(\rho^F) = (n-1, 0, n+1) \).

(ii) We determine the inertia set \( N_{2,3} \) as follows.

\[
N_{2,3} = \{(1, 2, 3), (1, 1, 4), (1, 0, 5), (2, 0, 4)\}. \tag{12}
\]

(iii) Suppose \( \rho_{AB} \) is a \( 2 \times n \) NPT state. For any \( j \in [1, n-1] \), if \( \text{In}(\rho_{AB}^F) = (j, 2(n-1-j), j+2) \), then \( r(\rho_B) = j+1 \), i.e., \( \rho_{AB} \) is indeed a \( 2 \times (j+1) \) NPT state up to a local projector.

We present the proof of this corollary in Appendix B. Combining Lemma 8 and Theorem 10, and using mathematical induction we can further exhaust \( N_{2,n} \) for any \( n \geq 2 \). We will discuss the details in Theorem 12.

**Theorem 12**  
There are exact \( (n-1)^2 \) distinct iner-tias in \( N_{2,n} \), i.e.,

\[
|N_{2,n}| = (n-1)^2, \quad \forall n \geq 2. \tag{13}
\]

Furthermore, the \( (n-1)^2 \) distinct iner-tias in \( N_{2,n} \) are as follows.

\[
(1, 2(n-2) - j, j + 3), \quad \forall 0 \leq j < 2(n-2),
\]
\[
(2, 2(n-3) - j, j + 4), \quad \forall 0 \leq j < 2(n-3),
\]
\[\vdots\]
\[
(n-1, 0, n+1).
\]

We provide the proof of Theorem 12 in Appendix B. By Theorem 12 we completely determine the inertia set \( N_{2,n} \) for any \( n \geq 2 \). Using the method in Lemma 8 one can construct the example whose partial transpose has the corresponding inertia in (14). An observation from (14) is that if \( \rho \) is a \( 2 \times n \) NPT state, then \( \rho^F \) has at least one negative and three positive eigenvalues. Based on this observation we prove that any bipartite NPT state shares this property using mathematical induction.

**Lemma 13**  
If \( \rho \) is an \( m \times n \) NPT state for any \( m, n \geq 2 \), then \( \rho^F \) has at least one negative and three positive eigenvalues. Furthermore, \((1, mn - 4, 3) \in N_{m,n} \) for any \( m, n \geq 2 \).

**Proof.** It follows from Lemma 5 (ii) that \( \rho^F \) has at least one negative eigenvalue. Hence, we only need to
show \( \rho^F \) has at least three positive eigenvalues. We use mathematical induction to prove it. First, \( \rho^F \) has the property for \( m = 2 \) and any \( n \geq 2 \) from (14). Second we assume \( \rho^F \) has the property for \( m = k(\geq 2) \) and any \( n \geq 2 \). Finally we prove \( \rho^F \) has the property for \( m = k + 1 \) and any \( n \geq 2 \). We prove it by contradiction. From Lemma 5 (iii) it suffices to denote \( \ln \rho^F = (a, b, 2) \), where \( a + b = (k + 1)n - 2 \) and \( 1 \leq a \leq k(n - 1) \). It follows that

\[
    n - 2 + k \leq b \leq (k + 1)n - 3.
\]

Since \( k \geq 2 \) by assumption, we conclude that \( b \geq n \). It implies that \((a, b - n, 2) \in \mathcal{N}_{k,n}\). However, it contradicts with the induction hypothesis that \( \rho^F \) has at least three positive eigenvalues for \( m = k \) and any \( n \geq 2 \). Therefore, we conclude that \( \rho^F \) has at least three positive eigenvalues for \( m = k + 1 \) and any \( n \geq 2 \). Thus, by mathematical induction our claim holds. For the last assertion, since \( \mathcal{N}_{2,2} = \{(1, 0, 3)\} \), it follows that \((1, mn - 4, 3) \in \mathcal{N}_{m,n}\) for any \( m, n \geq 2 \). This completes the proof. \( \square \)

Lemma 13 partially improves Lemma 5 (iii). We restrict EWs here into the partial transpose of NPT state. It is interesting to ask whether all bipartite EWs share this property that the number of positive eigenvalues is at least three. It is related to the EWs with the minimal rank. A direct corollary from Lemma 13 is that if \( \rho \) is an \( m \times n \) NPT state for any \( m, n \geq 2 \), then \( \rho^F \) has rank at least four. It can be used to construct a separability criterion.

**V. CONNECTIONS WITH OTHER PROBLEMS**

In this section we build the connections between the inertia of EW and other aspects in quantum information theory. First we present a separability criterion based on the rank of \( \rho^F \) in Theorem 14. Second we propose a method to generate the inertia of \( \rho^F \) in higher-dimensional systems. The method is depicted in Fig. 3. Suppose \( \alpha_{AB} \) is an \( m_1 \times n_1 \) NPT state of system \( A, B \), and \( \beta_{CD} \) is an \( m_2 \times n_2 \) NPT state of system \( C, D \). Denote \( \ln(\alpha_{AB}) = (a_1, b_1, c_1) \) and \( \ln(\beta_{CD}) = (a_2, b_2, c_2) \). Let \( \rho_{(AC):(BD)} := \alpha_{AB} \otimes \beta_{CD} \) be a bipartite state of system \((AC),(BD)\). Then \( \rho_{(AC):(BD)} \) has the inertia

\[
    (a_1c_2 + a_2c_1, b_1m_2n_2 + b_2m_1n_1 - b_1b_2, a_1a_2 + c_1c_2). \tag{15}
\]

The inertia (15) can be verified directly. Since

\[
    \rho_{(AC):(BD)}^F = \alpha_{AB}^F \otimes \beta_{CD}^F,
\]

the number of negative eigenvalues is \( a_1c_2 + a_2c_1 \), and the number of positive eigenvalues is \( a_1a_2 + c_1c_2 \).

**Theorem 14** Suppose \( \rho \) is an \( n \)-partite state. Denote by \( \rho^F \) the partial transpose of \( \rho \) with respect to the subsystem \( S \subseteq \{1, \cdots, n\} \). If for any subsystem \( S \), \( \rho^F \) has rank at most three, then \( \rho \) and \( \rho^F \) are both separable.

**Proof.** Let \( S^c \) be the complement of \( S \) in \( \{1, \cdots, n\} \). First we take \( \rho \) as a bipartite state of system \( S, S^c \). It follows from Lemma 13 that if \( \rho \) is a bipartite NPT state, then \( \ln(r(\rho^F)) \geq 4 \). Thus if \( \ln(r(\rho^F)) \leq 3 \), then \( \rho \) is a bipartite PPT state in the bipartition \( S|S^c \). Thus, \( \rho^F \) is positive semidefinite, and indeed a bipartite PPT state in the bipartition \( S|S^c \). Therefore, if for any subsystem \( S \), \( \rho^F \) has rank at most three, then it implies \( \rho^F \) is PPT in any bipartition. Thus, for any subsystem \( S \), \( \rho^F \) is an \( n \)-partite PPT state. It is known that any multiparticle PPT state of rank at most three is separable [58]. Hence, for any subsystem \( S \), \( \rho^F \) is separable, and thus \( \rho \) is also separable. This completes the proof. \( \square \)

As far as we know, there are few separability criteria based on the rank of \( \rho^F \). Since \( r(\rho) \) and \( r(\rho^F) \) are different in general [38], the separability criterion in Theorem 14 sheds light on the separability problem. We may propose other useful separability criteria based on \( r(\rho^F) \) when we fully characterize the inertia of \( \rho^F \) for general state \( \rho \).

Second, we propose a method to generate the inertia of \( \rho^F \) in higher-dimensional systems. The method is depicted in Fig. 3. Suppose \( \alpha_{AB} \) is an \( m_1 \times n_1 \) NPT state of system \( A, B \), and \( \beta_{CD} \) is an \( m_2 \times n_2 \) NPT state of system \( C, D \). Denote \( \ln(\alpha_{AB}) = (a_1, b_1, c_1) \) and \( \ln(\beta_{CD}) = (a_2, b_2, c_2) \). Let \( \rho_{(AC):(BD)} := \alpha_{AB} \otimes \beta_{CD} \) be a bipartite state of system \((AC),(BD)\). Then \( \rho_{(AC):(BD)} \) has the inertia

\[
    (a_1c_2 + a_2c_1, b_1m_2n_2 + b_2m_1n_1 - b_1b_2, a_1a_2 + c_1c_2). \tag{15}
\]

The inertia (15) can be verified directly. Since

\[
    \rho_{(AC):(BD)}^F = \alpha_{AB}^F \otimes \beta_{CD}^F,
\]

FIG. 3: Here, \( \alpha_{AB} \) is an \( m_1 \times n_1 \) state with \( \ln(\alpha_{AB}) = (a_1, b_1, c_1) \), and \( \beta_{CD} \) is an \( m_2 \times n_2 \) state with \( \ln(\beta_{CD}) = (a_2, b_2, c_2) \). Then we construct an \( m_1m_2 \times n_1n_2 \) state of system \((AC),(BD)\). The inertia of \( \rho_{(AC):(BD)}^F \) is given by (15).

By splitting system \((BD)\) into two subsystems \(B, D\),
we can take $\rho_{(AC): (BD)}$ as a tripartite state of system $(AC), B, D$, i.e., $\rho_{(AC):B:D}$; and take $\rho^T_{(AC): (BD)}$ as the partial transpose of the tripartite state $\rho_{(AC):B:D}$ with respect to subsystem $(AC)$, i.e., $\rho^T_{(AC):B:D}$. In this way we can construct the tripartite genuinely entangled state $\rho_{(AC):B:D}$ using two bipartite entangled states $\alpha_{AB}$ and $\beta_{CD}$ [17]. Moreover, we conjectured in [17] that $\rho_{(AC):B:D}$ is a tripartite genuinely entangled state if both $\alpha_{AB}$ and $\beta_{CD}$ are entangled. We have shown the above conjecture holds if either $R(\alpha_{AB})$ or $R(\beta_{CD})$ is not spanned by product vectors [17]. The latest progress on this conjecture has been made in [59]. As we know, genuine multipartite entanglement is valuable resource in quantum information processing tasks [6, 60–62]. Nevertheless, it is difficult to characterize genuinely multipartite entangled (GME) states [16, 17]. Obviously, the characterization of the partial transpose of GME state is also hard. As far as we know, there are few papers discussing the inertia of the partial transpose of GME state. If the above-mentioned conjecture is true, using the method in Fig. 3 we find a systematic way to construct tripartite genuinely entangled states whose partial transposes have inertias that may be exhausted explicitly. For example, if $\alpha_{AB}$ and $\beta_{CD}$ are two $2 \times n$ NPT states, we can exhaust the inertia of $\rho^T_{(AC):B:D}$ by Theorem 12. Furthermore, we can construct a tripartite genuinely entangled state whose partial transpose has a given inertia in this way.

Third, we indicate that the inertia of $\rho^T$ can be used to classify states under SLOCC equivalence. In quantum information theory, the classification of multipartite states is one of the central problems and has received extensive attentions in the past decades [63–66]. Two main approaches of classification are the equivalence under local unitary (LU) and SLOCC operations [53]. For example, a complete classification of pure three-qubit states in terms of LU equivalence were presented in [63]. In terms of SLOCC equivalence, it has been shown that only two inequivalent classes for pure three-qubit genuinely entangled states, namely the W-state class and GHZ-state class [64]. Moreover, necessary and sufficient conditions for the equivalence of arbitrary $n$-qubit pure quantum states under LU operations were derived in [65]. A systematic classification of multipartite entanglement in terms of SLOCC equivalence were provided in [66].

In the following we introduce a classification of $m \times n$ NPT states using the inertias of their partial transposes. The inertia is invariant under SLOCC operations from Sylvester Theorem. Moreover, we conclude that if two $n$-partite mixed states of system $A_1, \ldots, A_n$ are SLOCC equivalent, then their partial transposes with respect to any $k$-partite subsystem $A_{i_1}, \ldots, A_{i_k}$ are SLOCC equivalent. (We prove this claim, i.e., Lemma 19 (i), in Appendix C.) As a result, if $\rho^T_{AB}$ and $\sigma^T_{AB}$ have different inertias, then $\rho^\otimes_N_{AB}$ and $\sigma^\otimes_N_{AB}$ are SLOCC inequivalent. Therefore, we propose a necessary condition for $\rho_{AB}$ and $\sigma_{AB}$ to be SLOCC equivalent, i.e., $\text{In}(\rho^T_{AB}) = \text{In}(\sigma^T_{AB})$.

Further, for $2 \times n$ NPT states, from Theorem 12 we conclude that there are at least $(n - 1)^2$ inequivalent families in terms of SLOCC equivalence.

Furthermore, we introduce the concept of strong SLOCC inequivalence. Suppose $\rho_{AB}$ and $\sigma_{AB}$ are both $2 \times n$ NPT states. We consider the $N$ copies of $\rho_{AB}$ and $\sigma_{AB}$, i.e., $\rho^\otimes_N_{AB}$ and $\sigma^\otimes_N_{AB}$. We find that if $\rho^T_{AB}$ and $\sigma^T_{AB}$ have different inertias, then the partial transposes of $\rho^\otimes_N_{AB}$ and $\sigma^\otimes_N_{AB}$ still have different inertias. (We prove this claim, i.e., Lemma 19 (ii), in Appendix C.) We call this relation strong SLOCC inequivalence. Physically, it implies that the collective use of many copies cannot change the inequivalence under SLOCC. The classification of states enables us to determine whether there exist SLOCC operations to transform a state to another one. The transformation between many copies of two pure multipartite states was studied in [67]. It has been shown that two transformable multipartite states under SLOCC are also transformable under multicopy SLOCC [67]. The strong SLOCC inequivalence here shows that if $\rho_{AB}$ and $\sigma_{AB}$ cannot be transformed under SLOCC, then $\rho^\otimes_N_{AB}$ and $\sigma^\otimes_N_{AB}$ cannot be transformed under multicopy SLOCC too.

Fourth, we discuss a class of states called X-states. They are defined as states whose density matrix has nonzero elements only along its diagonal and antidiagonal in resemblance to the letter X [50]. For example, GHZ diagonal states are typical kinds of X-states [68]. X-states are important states that occur in various contexts such as entanglement [69], its decay under decoherence [70], and in describing other quantum correlations besides entanglement such as discord [71]. In Theorem 15 we study the inertia of $\rho^T$ by quantifying the number of negative eigenvalues of $\rho^T$.

Theorem 15 (i) If $\rho$ is a $2 \times n$ X-state, then $\rho^T$ has at most $\left\lfloor \frac{n}{2} \right\rfloor$ negative eigenvalues. Furthermore, there exist $2 \times n$ X-states whose partial transpose has exact $k$ negative eigenvalues, where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

One can refer to Appendix C for the proof of Theorem 15. In [72] authors considered the problems of maximizing the entanglement negativity of qubit-qutrit X-states. For this purpose they derived that there is at most one negative eigenvalue of $\rho^T$ if $\rho$ is a qubit-qutrit X-state. We generalize their result to $2 \times n$ X-states here. In the proof of Theorem 15 we formulate expressions for the eigenvalues of $\rho^T$. Therefore, using the expressions of those negative eigenvalues, one can determine the inertia of $\rho^T$, and compute the negativity of $2 \times n$ X-state.

Finally, since the transpose is a typical positive but not completely positive map, it enables PPT criterion to detect entanglement. In [42] authors considered the question of how exactly the partial transpose map can transform the eigenvalues of $\rho$. In specific, for which ordered list $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{mn} \in \mathbb{R}$ does there exist an $m \times n$ state $\rho$ such that $\rho^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{mn}$?
m = 2 in terms of how many positive and negative values among the list \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n} \).

VI. CONCLUSIONS

In this paper, we investigated the inertia of EW with a focus on the partial transpose of NPT state in the qubit-qudit system. We revealed the essential relevance between inertias, and proposed an effective method to generate more inertias from a given inertia. Using this method we exhausted all inertias in the inertia set \( \mathcal{N}_{2,n} \) for any \( n \geq 2 \). It led us to fully understand the partial transpose of NPT qubit-qudit state. We applied our results to construct a separability criterion. The connections with tripartite genuinely entangled states and the classification of NPT states were indicated. In addition, we quantified the number of negative eigenvalues of \( \rho^T \) for X-state \( \rho \).

There are some interesting problems for further study. First, it is natural to ask how many distinct inertias in the inertia set \( \mathcal{N}_{m,n} \) for \( m, n \geq 3 \) we can determine the partial transposes of (PPTE) states. Third, we may generalize the bi-rank of EWs may provide powerful separability criteria to identify PPTE states. Second, we may extend the study to the 1-distillability of NPT qubit-qudit system. We revealed the essential relevance between inertias, and proposed an effective method to quantify PPTE states. Third, we may generalize the bi-rank of EWs may provide powerful separability criteria to identify PPTE states. Second, we may extend the study to the 1-distillability of NPT qubit-qudit state. We applied our results to construct a separability criterion. The connections with tripartite genuinely entangled states and the classification of NPT states were indicated. In addition, we quantified the number of negative eigenvalues of \( \rho^T \) for X-state \( \rho \).

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Appendix A: Proofs of results in Sec. III

First of all, we prepare to show the proof of Lemma 5. For this purpose we need two essential results. The first one is a well-known conclusion on the existence of product vectors in some subspace.

Lemma 16 [73, Proposition 6.] Suppose \( \mathcal{H}^{AB} \cong \mathbb{C}^m \otimes \mathbb{C}^n \) is a bipartite Hilbert space. Any subspace of \( \mathcal{H}^{AB} \) with dimension greater than \( (m-1)(n-1) \) must contain at least one product vector. Furthermore, any subspace of \( \mathcal{H}^{AB} \) with dimension greater than \( (m-1)(n-1) + 1 \) contains infinitely many product vectors.

The second one is on the dimension of some subspace spanned by product vectors.

Lemma 17 Suppose \( \mathcal{H}^{AB} \cong \mathbb{C}^m \otimes \mathbb{C}^n \) is a bipartite Hilbert space. If \( V \) is an \( (mn-1) \)-dimensional bipartite subspace of \( \mathcal{H}^{AB} \), then \( V \) is spanned by product vectors. If \( V \) is an \( (mn-2) \)-dimensional bipartite subspace of \( \mathcal{H}^{AB} \), then \( V \) may be not spanned by product vectors.

Proof. Let \( V \) be the subspace spanned by the linearly independent vectors \( |\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_{mn-1}\rangle \) in \( \mathbb{C}^m \otimes \mathbb{C}^n \). It is known that the 3-tensor
\[
|\psi\rangle = \sum_{j=1}^{mn} |\alpha_j\rangle |j\rangle
\]
has tensor rank \( (mn-1) \) [74]. That is,
\[
|\psi\rangle = \sum_{j=1}^{mn-1} |a_j, b_j, c_j\rangle,
\]
where \( |c_j\rangle \)'s are vectors in the space
\[
\text{span}\{|1\rangle, |2\rangle, \ldots, |mn-1\rangle\}.
\]
Comparing (A1) and (A2), we obtain that \( |c_j\rangle \)'s are linearly independent. Hence,
\[
V = \text{span}\{|\alpha_1\rangle, \ldots, |\alpha_{mn-1}\rangle\}
\]
\[
= \text{span}\{|a_1, b_1\rangle, \ldots, |a_{mn-1}, b_{mn-1}\rangle\}.
\]
It follows that \( V \) is spanned by product vectors. To prove the second claim, it suffices to construct an example in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \). For example, let \( V \) be the 2-dimensional subspace spanned by \( |00\rangle \) and \( |10\rangle + |01\rangle \). One can verify that \( V \) has exact one product vector \( |00\rangle \) up to a coefficient. This completes the proof. 

Then we are ready to prove Lemma 5.

Proof of Lemma 5. First, we write \( W \) in spectral decomposition as
\[
W = \sum_{i=1}^{mn} p_i |a_i\rangle \langle a_i| \in \mathcal{B}(\mathbb{C}_m \otimes \mathbb{C}_n). \] (A6)

Suppose the inertia of \( W \) is \((q,r,mn-q-r)\). Without loss of generality, we may assume that \( p_i \leq p_{i+1} \) for any \( i \). Then the eigenvalues \( p_i < 0 \) for \( i \in [1, q] \), \( p_j = 0 \) for \( j \in [q + 1, q + r] \), and \( p_k > 0 \) for \( k \in [r + q + 1, mn] \). Since \( W \) is an EW, we have \( q \geq 1 \) and \( mn - q - r \geq 1 \).

(i) The assertion follows from the definition of EW.

(ii) If \( q \geq (m-1)(n-1) + 1 \), from Lemma 16 there is a product state \( |a, b\rangle \) in the subspace spanned by \( \{|a_1\rangle, |a_2\rangle, \ldots, |a_q\rangle\} \). So \( \langle a, b| W |a, b\rangle < 0 \). It is a contradiction with the definition of EW. Hence, \( q \leq (m-1)(n-1) + 1 \). 

\[\text{(iii) If } q = (m-1)(n-1) + 1, \text{ then } W \text{ is an EW.}\]

\[\text{(iv) If } q < (m-1)(n-1) + 1, \text{ then } W \text{ is not an EW.}\]
\((m-1)(n-1)\). The decomposable EW containing exactly \((m-1)(n-1)\) negative eigenvalues has been constructed in [40].

(iii) We prove the assertion by contradiction. Suppose \(mn - q - r = 1\). Lemma 17 implies that there is a product vector \(|a, b\rangle\) orthogonal to \(|a_{mn}\rangle\), and non-orthogonal to \(|a_1\rangle\). We have

\[
0 \leq \langle a, b | W | a, b \rangle \leq p_1 \langle a, b | \{ |a_1\rangle | a_1\rangle \rangle | a, b \rangle < 0. \quad (A7)
\]

We obtain a contradiction. Therefore, \(mn - q - r > 1\), namely that \(W\) has at least two positive eigenvalues.

This completes the proof. \(\square\)

**Appendix B: Proofs of results in Sec. IV.**

First we show the proof of Lemma 8 as follows.

**Proof of Lemma 8.** (i) Since \(\text{In}(\rho^\Gamma) = (a, b, c)\), we may assume the spectral decomposition as

\[
\rho^\Gamma = \sum_{i=1}^a \lambda_i |\alpha_i\rangle \langle \alpha_i| + 0 \cdot \sum_{j=1}^b |\beta_j\rangle \langle \beta_j| + \sum_{k=1}^c \mu_k |\gamma_k\rangle \langle \gamma_k|,
\]

where \(\lambda_i < 0\) and \(\mu_k > 0\). We choose \(x > 0\) such that

\[
x + \max_i \lambda_i < 0. \quad (B2)
\]

Therefore,

\[
\sigma^\Gamma = \rho^\Gamma + xI_{mn}
\]

\[
= \sum_{i=1}^a (x + \lambda_i) |\alpha_i\rangle \langle \alpha_i| + x \cdot \sum_{j=1}^b |\beta_j\rangle \langle \beta_j| + \sum_{k=1}^c (x + \mu_k) |\gamma_k\rangle \langle \gamma_k|.
\]

(B3)

It follows that

\[
\text{In}(\sigma^\Gamma) = (a, 0, b + c).
\]

Furthermore, if \(\rho\) is an NPT (separable) state, then \(\sigma\) is also an NPT (separable) state. If \(\rho\) is a PPTE state, there is an EW \(W\) such that \(\text{Tr}(W \rho) < 0\). So we can choose \(x > 0\) such that \(x + \max_i \lambda_i < 0\), and \(\text{Tr}(W(\rho + xI)) < 0\).

Thus \(\sigma\) is also a PPTE state.

(ii) If \((a_1, b_1, c_1) \in N_{m_1, n_1}(P_{m_1, n_1}, S_{m_1, n_1})\), it follows from (i) that

\[
(a_1, 0, b_1 + c_1) \in N_{m_1, n_1}(P_{m_2, n_2}, S_{m_2, n_2}).
\]

Suppose \(\rho\) is an \(m_1 \times n_1\) NPT (PPTE, separable) state with \(\text{In}(\rho^\Gamma) = (a_1, 0, b_1 + c_1)\). Using the spectral decomposition we write \(\rho^\Gamma\) as

\[
\rho^\Gamma = \sum_{j=1}^{b_1} \lambda_j |\psi_j\rangle \langle \psi_j| + \sum_{j=b_1+1}^{b_1+c_1} \mu_j |\psi_j\rangle \langle \psi_j|, \quad (B4)
\]

where \(\lambda_j < 0, \mu_j > 0\), and \(\{|\psi_j\rangle\}_{j=1}^{b_1+c_1}\) is an orthonormal basis of \(\mathbb{C}^{m_1} \otimes \mathbb{C}^{n_1}\). By adding proper zero rows and columns in the original density matrix of \(\rho\) we construct an \(m_2 \times n_2\) NPT (PPTE, separable) state \(\tilde{\rho}\), and

\[
\text{In}(\tilde{\rho}^\Gamma) = (a_1, m_2 n_2 - m_1 n_1, b_1 + c_1).
\]

We again write \(\tilde{\rho}^\Gamma\) in spectral decomposition as

\[
\tilde{\rho}^\Gamma = \sum_{j=1}^{m_2 n_2 - m_1 n_1} \lambda_j |\tilde{\psi}_j\rangle \langle \tilde{\psi}_j| + \sum_{j=m_2 n_2 - m_1 n_1 + 1}^{b_1+c_1} \mu_j |\tilde{\psi}_j\rangle \langle \tilde{\psi}_j| + 0 \cdot \sum_{j=1}^{m_2 n_2 - m_1 n_1} |\phi_j\rangle \langle \phi_j|,
\]

where \(\{|\phi_j\rangle\}_{j=1}^{m_2 n_2 - m_1 n_1}\) is the set of product vectors \(|p, q\rangle\) with either \(m_1 < p \leq m_2\) or \(n_1 < q \leq n_2\). Let

\[
\sigma^\Gamma := \tilde{\rho}^\Gamma + \sum_{x_1}^l |j_x, k_x\rangle \langle j_x, k_x|,
\]

where \(\{|j_x, k_x\rangle\}_{x_1=1}^l \subseteq \{|\phi_j\rangle\}_{j=1}^{m_2 n_2 - m_1 n_1}\) is a subset of any \(l\) orthonormal product vectors. Thus,

\[
\text{In}(\sigma^\Gamma) = (a_1, m_2 n_2 - m_1 n_1 - l, b_1 + c_1 + l)
\]

for any \(0 \leq l \leq m_2 n_2 - m_1 n_1\). Since the state \(\sigma\) is obtained by adding product states into the density matrix of \(\tilde{\rho}\), it follows that \(\sigma\) is also an \(m_2 \times n_2\) NPT (PPTE, separable) state.

This completes the proof. \(\square\)

The basic idea of Lemma 8 is to add a convex combination of linearly independent product states into the original density matrix. Using the similar idea we can also determine the inertia sets \(S_{m,n}\) and \(P_{m,n}\) as by-products.

**Corollary 18** Suppose \(\rho\) is an \(m \times n\) PPT state. Then

\[
\text{In}(\rho^\Gamma) = (0, mn - r(\rho^\Gamma), r(\rho^\Gamma)). \quad (B7)
\]

(i) If \(\rho\) is separable then any given integer \(r(\rho^\Gamma) \in [1, mn]\) exists.

(ii) If \(\rho\) is a PPTE state, and

\[
k := \max\{\nu_0(0, \nu_0, \nu_+), \nu_+ \in P_{m,n}\},
\]

then \((0, mn - r(\rho^\Gamma), r(\rho^\Gamma)) \in P_{m,n}\) for any given integer \(r(\rho^\Gamma) \in [mn - k, mn]\).

**Proof.** Since \(\rho\) is a PPT state, \(\rho^\Gamma\) has no negative eigenvalue. Hence, Eq. (B7) holds.

(i) It suffices to construct specific examples to prove
this assertion. Let
\[ \rho = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} |i\rangle\langle j|, \]  
(B8)
where \( c_{ij} \)'s are non-negative real numbers, and exact \( p \in [1, mn] \) elements of \( \{c_{ij}\} \) are positive. It follows that \( r(\rho) = r(\rho^F) = p \) for any \( p \in [1, mn] \).

(ii) Suppose \( \rho \) is a PPTE state which satisfies \( r(\rho^F) = mn - k \). We write \( \rho^F \) in spectral decomposition as
\[ \rho^F = \sum_{j=1}^{mn-k} \lambda_{ij} |\psi_j\rangle\langle \psi_j|, \]  
(B9)
where \( \lambda_{ij} \)'s are positive. It follows that \( \mathcal{R}(\rho^F) = \operatorname{span}\{|\psi_j\rangle\}_{j=1}^{mn-k} \). Thus we can assume the kernel of \( \rho^F \) is spanned by \( k \) linearly independent product vectors, i.e.,
\[ \mathcal{K}(\rho^F) = \operatorname{span}\{|a_j, b_j\rangle\}_{j=1}^{k}. \]

Hence, for any \( 1 \leq p \leq k \) we define
\[ \sigma_p^F := \sum_{j=1}^{mn-k} \lambda_{ij} |\psi_j\rangle\langle \psi_j| + \sum_{j=k+1}^{p} |a_j, b_j\rangle\langle a_j, b_j|. \]  
(B10)
One can verify \( \sigma_p \) is a PPTE state which satisfies \( r(\sigma_p^F) = mn - k + p, \forall 1 \leq p \leq k \).

This completes the proof.

Second we provide the proof of Lemma 9 as follows.

**Proof of Lemma 9.** Using the spectral decomposition we can write \( \rho^F \) as
\[ \rho^F = -\sum_{j=1}^{k} |v_j\rangle\langle v_j| + \sum_{j=k+1}^{mn-d} |v_j\rangle\langle v_j|, \]  
(B11)
where \( \{|v_j\rangle\}_{j=1}^{mn-d} \) are pairwisely orthogonal. Assume \( \mathcal{K}(\rho^F) = \operatorname{span}\{|u_1\rangle, \ldots, |u_d\rangle\} \). It follows from Lemma 16 that any subspace of \( \mathbb{C}^m \otimes \mathbb{C}^n \) whose dimension is \( (m-1)(n-1) + 1 \) contains at least one product vector. Thus, there exist proper coefficients such that
\[ |a_1, b_1\rangle = \sum_{i=1}^{k} x_i |v_i\rangle + \sum_{j=k+1}^{mn-d} y_j |u_j\rangle. \]  
(B12)
Since \( \rho^F \) is an EW, it follows that \( \langle a_1, b_1| \rho^F |a_1, b_1\rangle \geq 0 \). Thus we conclude that \( x_i = 0, \forall 1 \leq i \leq k \) in Eq. (B12).

That is
\[ |a_1, b_1\rangle = \sum_{j=1}^{(m-1)(n-1) + 1 - k} y_j |u_j\rangle \in \mathcal{K}(\rho^F). \]  
(B13)
Up to a permutation of \( \{|u_j\rangle\}_{j=1}^{(m-1)(n-1) + 1 - k} \), we can assume \( y_1 \neq 0 \). In the same way, there exist proper coefficients such that
\[ |a_2, b_2\rangle = \sum_{j=2}^{(m-1)(n-1) + 2 - k} y_j |u_j\rangle \in \mathcal{K}(\rho^F). \]  
(B14)
Similarly we assume that \( y_2 \neq 0 \). Repeating this process we obtain that there are at least \( l \) linearly independent product vectors in \( \mathcal{K}(\rho^F) \). Moreover, we conclude that
\[ \mathcal{K}(\rho) = \operatorname{span}\{|u_1\rangle, \ldots, |u_d\rangle\} = \operatorname{span}\{|a_1, b_1\rangle, \ldots, |a_1, b_1\rangle, |u_{l+1}\rangle, \ldots, |u_d\rangle\}. \]  
(B15)
This completes the proof.

Third we show the proof of Theorem 10 as follows.

**Proof of Theorem 10.** (i) We first prove the "Only if" part. It is equivalent to prove the claim that if the sequence \( (a, b, c) \) satisfies that either \( (a, b-2, c) \in \mathcal{N}_{2,n} \) or \( (a, b-1, c-1) \in \mathcal{N}_{2,n} \), then \( (a, b, c) \in \mathcal{N}_{2,n} \). If \( (a, b-2, c) \in \mathcal{N}_{2,n} \), then \( (a, b, c) \in \mathcal{N}_{2,n} \) naturally. If \( (a, b-1, c-1) \in \mathcal{N}_{2,n} \), then \( (a, b+1, c-1) \in \mathcal{N}_{2,n} \) naturally. Suppose \( \sigma \) is a \( 2 \times n \) state, and \( \text{In}(\sigma^F) = (a, b+1, c-1) \). Since \( a + b + 1 > n \), from Lemma 9 there is a product vector in \( \mathcal{K}(\sigma^F) \), namely \( |e, f\rangle \). Let
\[ \tilde{\sigma} := \sigma + |e^*, f\rangle\langle e^*, f| \]
It follows that \( \text{In}(\tilde{\sigma}^F) = (a, b, c) \), and thus \( (a, b, c) \in \mathcal{N}_{2,n} \).
So the "Only if" part holds.

Second we prove the "If" part by contradiction. Assume that there is a \( 2 \times n \) NPT state \( \rho \) such that \( \text{In}(\rho^F) = (a, b, c) \). Since \( a + b > n - 1 \), it follows from Lemma 9 that there is a product vector in \( \mathcal{K}(\rho^F) \). Thus we can assume \( |0, 0\rangle \in \mathcal{K}(\rho^F) \) up to SLOCC equivalence. Also, we obtain that \( |0, 0\rangle \notin \mathcal{K}(\rho) \). Hence, the matrix of \( \rho^F \) is as follows.
\[ \rho^F = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \]  
(B16)
where

\[
M_{11} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & \rho_{22} & \rho_{23} & \cdots & \rho_{2n} \\
0 & \rho_{32} & \rho_{33} & \cdots & \rho_{3n} \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & \rho_{n2} & \rho_{n3} & \cdots & \rho_{nn}
\end{bmatrix},
\]

\[
M_{12} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & \rho_{2(n+2)} & \rho_{2(n+3)} & \cdots & \rho_{2(2n)} \\
0 & \rho_{3(n+2)} & \rho_{3(n+3)} & \cdots & \rho_{3(2n)} \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & \rho_{n(n+2)} & \rho_{n(n+3)} & \cdots & \rho_{n(2n)}
\end{bmatrix},
\]

\[
M_{21} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & \rho_{(n+2)2} & \rho_{(n+2)3} & \cdots & \rho_{(n+2)n} \\
0 & \rho_{(n+3)2} & \rho_{(n+3)3} & \cdots & \rho_{(n+3)n} \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & \rho_{(2n)2} & \rho_{(2n)3} & \cdots & \rho_{(2n)n}
\end{bmatrix},
\]

\[
M_{22} = \begin{bmatrix}
\rho_{(n+1)(n+1)} & \rho_{(n+1)(n+2)} & \cdots & \rho_{(n+1)(2n)} \\
\rho_{(n+2)(n+1)} & \rho_{(n+2)(n+2)} & \cdots & \rho_{(n+2)(2n)} \\
\rho_{(n+3)(n+1)} & \rho_{(n+3)(n+2)} & \cdots & \rho_{(n+3)(2n)} \\
0 & \vdots & \vdots & \vdots \\
\rho_{(2n)(n+1)} & \rho_{(2n)(n+2)} & \cdots & \rho_{(2n)(2n)}
\end{bmatrix}
\]

\[(B17)\]

If \(\rho_{(n+1)(n+1)} = 0\), since \(\rho\) is positive semidefinite, we conclude that

\[\rho_{(n+1)j} = \rho_{j(n+1)} = 0, \quad \forall n + 2 \leq j \leq 2n.\]

It implies that \([1, 0] \in \mathcal{K}(\rho^F)\), and \([1, 0] \notin \mathcal{N}_2\). Thus, \(\rho\) is indeed a \(2 \times (n - 1)\) state up to a local projector. It implies that \((a, b - 2, c) \notin \mathcal{N}_2\). Next, we consider \(\rho_{(n+1)(n+1)} > 0\). Using a locally invertible operator \(I_2 \otimes V\), where \(V\) is an \(n \times n\) invertible matrix, we obtain that

\[
(I \otimes V)\rho^F (I \otimes V) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},
\]

\[(B18)\]

where

\[
M_{22}' = \begin{bmatrix}
\rho_{(n+1)(n+1)} & 0 & \cdots & 0 \\
0 & \rho_{(n+2)(n+2)} & \cdots & \rho_{(n+2)(2n)} \\
0 & \rho_{(n+3)(n+2)} & \cdots & \rho_{(n+3)(2n)} \\
0 & \vdots & \vdots & \vdots \\
0 & \rho_{(2n)(n+2)} & \cdots & \rho_{(2n)(2n)}
\end{bmatrix}
\]

\[(B19)\]

Since the inertia is invariant under invertible operations by Sylvester Theorem, we conclude that the inertia of

\[
\sigma^F = \begin{bmatrix}
\rho_{22} & \cdots & \rho_{2n} & \rho_{2(n+2)} & \cdots & \rho_{2(2n)} \\
\rho_{32} & \cdots & \rho_{3n} & \rho_{3(n+2)} & \cdots & \rho_{3(2n)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{n2} & \cdots & \rho_{nn} & \rho_{n(n+2)} & \cdots & \rho_{n(2n)} \\
\rho_{(n+2)2} & \cdots & \rho_{(n+2)n} & \rho_{(n+2)(n+2)} & \cdots & \rho_{(n+2)(2n)} \\
\rho_{(n+3)2} & \cdots & \rho_{(n+3)n} & \rho_{(n+3)(n+2)} & \cdots & \rho_{(n+3)(2n)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{(2n)2} & \cdots & \rho_{(2n)n} & \rho_{(2n)(n+2)} & \cdots & \rho_{(2n)(2n)}
\end{bmatrix}
\]

\[(B20)\]

is \((a, b - 1, c - 1)\). Since \(\sigma\) is a \(2 \times (n - 1)\) state, it follows that \((a, b - 1, c - 1) \notin \mathcal{N}_2\). It contradicts with the condition (i.b) that \((a, b - 1, c - 1) \notin \mathcal{N}_2\). Therefore, we conclude that \((a, b, c) \notin \mathcal{N}_2\). So the “If” part holds.

(ii) Since \(\text{In}(\rho^F) = (a, b, c)\) with \(a + b > n - 1\), it follows from Lemma 9 that there is a product vector in \(\mathcal{K}(\rho^F)\). Up to SLOCC equivalence, we may assume \([0, 0] \in \mathcal{K}(\rho^F)\). Thus, one can similarly write \(\rho^F\) in the form as Eq. (B16). For the entry \(\rho_{(n+1)(n+1)}\) in \(M_{32}\) given by (B17), if it is positive, then we can transform \(\rho\) into a \(2 \times (n - 1)\) state \(\sigma\) by using a locally invertible operation, namely Eq. (B18). Similarly, \(\sigma^F\) expressed by Eq. (B20) has inertia \((a, b - 1, c - 1)\). It contradicts with \((a, b - 1, c - 1) \notin \mathcal{N}_2\). Therefore, we conclude that \(\rho_{(n+1)(n+1)} = 0\). It implies \([1, 0] \notin \mathcal{K}(\rho^F)\). Thus, \(\rho\) is indeed a \(2 \times (n - 1)\) state up to a local projector, and \((a, b - 2, c) \notin \mathcal{N}_2\). So assertion (ii) holds.

This completes the proof.

Fourth, we show the proof of Corollary 11 as follows.

**Proof of Corollary 11.** (i) It follows from Lemma 4 (iii) that such a state \(\rho\) whose partial transpose contains exact \((n - 1)\) negative eigenvalues exists. Then we show \(\text{In}(\rho^F)\) can only be \((n - 1, 0, n + 1)\). First for any \(k > 0\) the sequence \((n - 1, k, n + 1 - k)\) satisfies the condition \(n - 1 + k > n - 1\). Second it follows from Lemma 5 (ii) that the partial transpose of any \(2 \times (n - 1)\) NPT state has at most \((n - 2)\) negative eigenvalues. Hence, we conclude that for any \(k > 0\),

\[
(n - 1, k, n + 1 - k) \notin \mathcal{N}_2, n - 1.
\]

Note that if \(k < 2 < 0\) or \(n + 1 - k < 0\), then such inertia \((n - 1, k, n + 1 - k)\) naturally does not exist. Similarly if \(k - 1 < 0\) or \(n - k < 0\), then such inertia \((n - 1, k - 1, n - k)\) does not exist. Therefore, it follows from Theorem 10 (i) that \((n - 1, k, n + 1 - k) \notin \mathcal{N}_2, n\) for any \(k > 0\).

(ii) It follows from Lemma 5 that the number of negative eigenvalues of \(\rho^F\) is either one or two, and the number of positive eigenvalues of \(\rho^F\) lies in \([2, 5]\). Thus, \(\text{In}(\rho^F)\) can only be the following seven sequences:

\[
(1, 3, 2), (1, 2, 3), (1, 1, 4), (1, 0, 5), (2, 2, 2), (2, 1, 3), (2, 0, 4).
\]

\[(B21)\]
First we construct concrete examples to show the four inertiases in Eq. (12) exist.

\[
\begin{align*}
\rho_1 &= (|00\rangle + |11\rangle)(|00\rangle + |11\rangle), \\
\rho_2 &= (|00\rangle + |11\rangle)(|00\rangle + |11\rangle) + |02\rangle\langle 02|, \\
\rho_3 &= (|00\rangle + |11\rangle)(|00\rangle + |11\rangle) + \frac{1}{10}I_6, \\
\rho_4 &= (|00\rangle + |11\rangle)(|00\rangle + |11\rangle) + (|01\rangle + |12\rangle)(|01\rangle + |12\rangle).
\end{align*}
\] (B23)

One can verify

\[
\begin{align*}
\text{In}(\rho_1) &= (1, 2, 3), \quad \text{In}(\rho_2) = (1, 1, 4), \\
\text{In}(\rho_3) &= (1, 0, 5), \quad \text{In}(\rho_4) = (2, 0, 4).
\end{align*}
\] (B24)

Second we exclude other three sequences in (B22). It follows from Theorem 6 that \( N_{2,2} = \{(1, 0, 3)\}. \) Then from Theorem 10 (i) the three sequences \((1,3,2), (2,2,2), (2,1,3)\) do not belong to \( N_{2,3}. \)

(iii) We prove it by contradiction. Assume \( r(\rho_B) = k(\neq j + 1). \) If \( k < j + 1, \) then \( r(\rho_{AB}) < 2k < 2(j + 1). \) It contradicts with \( \text{In}(\rho_{AB}) = (j, 2(n - 1 - j), j + 2). \) If \( k > j + 1 \geq 2, \) then we can assume \( r(\rho_{AB}) = (j, 2(k - 1 - j), j + 2) \) by taking \( \rho_{AB} \) as a \( 2 \times 2 \) state. Since \( j < k - 1, \) from (14) we obtain \( (j, 2(k - 2 - j), j + 2) \in N_{2,k-1}. \) Moreover, for any inertia \( (a,b,c) \in N_{2,n}, \forall n \geq 2, \) an observation from (14) is that \( c - a \geq 2. \) Hence, \( (j, 2(k - 2 - j) + 1, j + 1) \notin N_{2,k-1}. \) Straightforward calculation yields that \( j + 2(k - 1 - j) > k - 1 \) from \( k > j + 1. \) It follows from Theorem 10 (ii) that \( \rho_{AB} \) can be regarded as \( 2 \times (k - 1) \) state. It implies \( r(\rho_{AB}) \leq k - 1. \) It contradicts with the assumption \( r(\rho_{AB}) = k. \) Therefore, we conclude that \( r(\rho_{AB}) = j + 1. \)

This completes the proof. \( \Box \)

Finally we provide the proof of Theorem 12 as follows.

**Proof of Theorem 12.** First we show the \((n-1)^2\) sequences in (14) belong to \( N_{2,n}. \) It follows from Corollary 11 (i) that

\[(j - 1, 0, j + 1) \in N_{2,j}, \; \forall 2 \leq j \leq n.\]

Then from Lemma 8 (ii) we conclude that \( \forall 2 \leq j \leq n, \)

\[(j - 1, 2(n - j) - l, j + 1 + l) \in N_{2,n}, \; \forall 0 \leq l \leq 2(n - j).\] (B25)

Thus the number of distinct inertiases in \( N_{2,n} \) is at least

\[\sum_{j=2}^{n} (2(n - j) + 1) = (n - 1)^2.\]

Second we show except the \((n-1)^2\) sequences in (14) there is no other inertia in \( N_{2,n}. \) We prove this claim using mathematical induction. First, it follows from Theorem 6 and Corollary 11 (ii) that this claim holds for \( n = 2, 3. \) Assume \( |N_{2,n}| = (n - 1)^2 \) holds for \( n = k. \) Next, we need to show \( |N_{2,n}| = (n - 1)^2 \) holds for \( n = k + 1. \)

From (B25), it is equivalent to prove that

\[(j - 1, b, c) \notin N_{2,k+1}, \] (B26)

for any \( 2 \leq j \leq k + 1, \) where \( b > 2(k + 1 - j) \) and \( b + c = 2(k + 1) + 1 - j. \) Straightforward computation yields that \( j - 1 + b > 2k + 1 - j \geq k. \) Thus we can apply Theorem 10 (i) to prove (B26). This is equivalent to prove that

\[(j - 1, b - 2, c), \quad (j - 1, b - 1, c - 1) \notin N_{2,k}, \] (B27)

for any \( 2 \leq j \leq k + 1, \) where \( b > 2(k + 1 - j) \) and \( b + c = 2(k + 1) + 1 - j. \)

According to the induction hypothesis we obtain

\[N_{2,k} = \left\{ (j - 1, 2(k - j) - l, j + 1 + l) \mid 0 \leq l \leq 2(k - j), \forall 2 \leq j \leq k \right\}. \] (B28)

Thus for any \( 2 \leq j \leq k, \) \( (j - 1, b - 2, c) \notin N_{2,k} \) and only if \( 0 \leq b - 2 \leq 2(k - j). \) This is a contradiction with the condition \( b > 2(k + 1) \) below (B27). Similarly for any \( 2 \leq j \leq k, \) using Eq. (B28) we obtain that \( (j - 1, b - 1, c - 1) \notin N_{2,k} \) if and only if \( 0 \leq b - 1 \leq 2(k - j). \) So we obtain the same contradiction. We have proven (B27) for \( 2 \leq j \leq k. \)

It remains to prove (B27) for \( j = k + 1. \) It follows from Lemma 5 (ii) that the partial transpose of any \( 2 \times k \) NPT state has at most \( k - 1 \) negative eigenvalues. Thus, both \( (j - 1, b - 2, c) \) and \( (j - 1, b - 1, c - 1) \) do not belong to \( N_{2,k} \) if \( j = k + 1. \) We have proven (B27) for \( j = k + 1. \) Combining with the last paragraph, we have proven (B27). The equivalence of (B26) and (B27) implies that \( |N_{2,n}| = (n - 1)^2 \) holds for \( n = k + 1. \)

To sum up, according to mathematical induction we conclude that \( |N_{2,n}| = (n - 1)^2 \) for any \( n \geq 2. \) This completes the proof. \( \Box \)

**Appendix C: Proofs of results in Sec. V.**

First we show the following results.

Lemma 19 (i) If two \( n \)-partite mixed states of system \( A_1, ..., A_n \) are SLOCC equivalent, then their partial transposes with respect to any \( k \)-partite subsystem \( A_{j_1}, ..., A_{j_k} \) are SLOCC equivalent. (ii) Suppose \( \rho_{AB} \) and \( \sigma_{AB} \) are both \( 2 \times n \) NPT states of system \( A, B. \) If \( \rho_{AB} \) and \( \sigma_{AB} \) have different inertiases, then the partial transposes of \( \rho_{AB} \) and \( \sigma_{AB} \) still have different inertiaces for any \( N \) copies.

**Proof.** (i) Suppose \( \rho \) and \( \sigma \) are two \( n \)-partite mixed states of system \( A_1, ..., A_n, \) and they are SLOCC equivalent. Let \( \rho^\Lambda \) and \( \sigma^\Lambda \) be the partial transposes of \( \rho \) and \( \sigma \) respectively, with respect to first \( k \)-partite subsystem
\( A_1, \cdots, A_k \). Up to a permutation of subsystems, it suffices to show that \( \rho^X \) and \( \sigma^X \) are SLOCC equivalent. By Definition 2 there is a locally invertible operator
\[
X = V_1 \otimes V_2 \otimes \cdots \otimes V_n
\]
such that \( X \rho X^\dagger = \sigma \).

Let
\[
X^\Gamma := V_1^T \otimes \cdots \otimes V_k^T \otimes V_{k+1} \otimes \cdots \otimes V_n.
\]
One can verify \((X^\Gamma)^\dagger \rho^X X^\Gamma = \sigma^X\). Therefore, \( \rho^X \) and \( \sigma^X \) are SLOCC equivalent.

(ii) Denote \( \text{In}(\rho^X_{AB}) = (a_1, b_1, c_1) \) and \( \text{In}(\sigma^X_{AB}) = (a_2, b_2, c_2) \). It follows that
\[
(\rho^X_{AB})^\Gamma \otimes N = (\rho^X_{AB})^\otimes N, \quad (\sigma^X_{AB})^\Gamma \otimes N = (\sigma^X_{AB})^\otimes N.
\]

Straightforward calculation yields
\[
\nu_-( (\rho^X_{AB})^\Gamma ) = \sum_{k=\text{odd}} \frac{N!}{k!} a_1^{N-k} c_1^{-k} = \frac{(a_1 + c_1)^N - (a_1 - c_1)^N}{2},
\]
\[
\nu_+ ( (\rho^X_{AB})^\Gamma ) = \sum_{k=\text{even}} \frac{N!}{k!} a_1^{N-k} c_1^{-k} = \frac{(a_1 + c_1)^N + (a_1 - c_1)^N}{2}.
\]

Similarly we obtain
\[
\nu_-( (\sigma^X_{AB})^\Gamma ) = \frac{(a_2 + c_2)^N - (a_2 - c_2)^N}{2},
\]
\[
\nu_+ ( (\sigma^X_{AB})^\Gamma ) = \frac{(a_2 + c_2)^N + (a_2 - c_2)^N}{2}.
\]

If \( \text{In}((\rho^X_{AB})^\Gamma) = \text{In}((\sigma^X_{AB})^\Gamma) \), then
\[
\nu_-( (\rho^X_{AB})^\Gamma ) = \nu_-( (\sigma^X_{AB})^\Gamma ) ,
\]
\[
\nu_+ ( (\rho^X_{AB})^\Gamma ) = \nu_+ ( (\sigma^X_{AB})^\Gamma ) .
\]

From (14) we have \( a_1 < c_1 \) and \( a_2 < c_2 \). Hence, Eq. (C3) is equivalent to \( a_1 = a_2 \) and \( c_1 = c_2 \). It implies \( \text{In}(\rho^X_{AB}) = \text{In}(\sigma^X_{AB}) \). We obtain a contradiction. Therefore, assertion (ii) holds.

This completes the proof. \( \square \)

Second we present the proof of Theorem 15 as follows.

**Proof of Theorem 15.** Denote by \( \rho_X \) the \( 2 \times n \) X-state. The density matrix of an arbitrary \( 2 \times n \) X-state can be parametrized as
\[
\rho_X = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix},
\]
where
\[
M_{11} = \text{diag}(a_1, a_2, \cdots, a_n),
\]
\[
M_{22} = \text{diag}(b_n, b_{n-1}, \cdots, b_1),
\]
\[
M_{12} = \begin{bmatrix} 0 & \cdots & 0 & r_1 e^{i\theta_1} \\ \vdots & \ddots & \vdots & \vdots \\ r_n e^{i\theta_n} & \cdots & 0 & 0 \end{bmatrix},
\]
and for all \( j \), \( a_j, b_j, r_j \) are non-negative real numbers. With a proper permutation \( P \), we have \( P \rho_X P^\dagger = \otimes_{j=1}^n B_j \), where \( B_j = \begin{bmatrix} a_j & r_j e^{i\theta_j} \\ r_j e^{-i\theta_j} & b_j \end{bmatrix} \). Thus the eigenvalues of \( \rho_X \) can be formulated as
\[
\lambda_j^+ = \frac{a_j + b_j}{2} + \sqrt{r_j^2 + d_j^2},
\]
\[
\lambda_j^- = \frac{a_j + b_j}{2} - \sqrt{r_j^2 + d_j^2},
\]
where \( d_j = \frac{a_j - b_j}{2} \) for any \( j \). Since \( \rho^X \) is positive semidefinite, it follows that \( \forall j, \lambda_j^+ \) and \( \lambda_j^- \) are non-negative. This is equivalent to
\[
r_j \leq \sqrt{a_j b_j}, \forall j.
\]

Since \( \rho^X_{AB} \) is still an X-type matrix, one can similarly formulate the eigenvalues of \( \rho_X \) as
\[
\mu_j^+ = \frac{a_j + b_j}{2} + \sqrt{r_{n+1-j}^2 + d_j^2},
\]
\[
\mu_j^- = \frac{a_j + b_j}{2} - \sqrt{r_{n+1-j}^2 + d_j^2}.
\]

It follows from Eq. (C8) that \( \mu_j^+ \geq 0, \forall j \), and \( \mu_j^- \) is negative if \( r_{n+1-j} > \sqrt{a_j b_j} \). Using this inequality and (C7), the number of negative eigenvalues of \( \rho_X \) is that of \( r_{n+1-j} \) satisfying
\[
\sqrt{a_n b_n} \cdots \sqrt{a_{n+1-j} b_{n+1-j}} \geq r_{n+1-j} > \sqrt{a_j b_j}.
\]

To satisfy the inequality, we can exclude the case that \( n \) is odd and \( j = \lceil \frac{n}{2} \rceil \). Next, we obtain two inequalities by setting \( j = k \) and \( j = n + 1 - k \) for every \( k \leq \lfloor \frac{n}{2} \rfloor \) in (C9). One can verify that at most one of the two inequalities holds. So the number of negative eigenvalues of \( \rho_X \) is at most \( \lfloor \frac{n}{2} \rfloor \). For a fixed \( k \leq \lfloor \frac{n}{2} \rfloor \), by choosing proper parameters we can make (C9) hold if and only if \( 1 \leq j \leq k \). Thus the corresponding X-state given by Eq. (C4) is one whose partial transpose has \( k \) negative eigenvalues. This completes the proof. \( \square \)

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