Ward Identities of Liouville Gravity
coupled to Minimal Conformal Matter

KEN-JI HAMADA

Institute of Physics, University of Tokyo
Komaba, Meguro-ku, Tokyo, 153, Japan

ABSTRACT

The Ward identities of the Liouville gravity coupled to the minimal conformal matter are investigated. We introduce the pseudo-null fields and the generalized equations of motion, which are classified into series of the Liouville charges. These series have something to do with the W and Virasoro constraints. The pseudo-null fields have non-trivial contributions at the boundaries of the moduli space. We explicitly evaluate the several boundary contributions. Then the structures similar to the W and the Virasoro constraints appearing in the topological and the matrix methods are realized. Although our Ward identities have some different features from the other methods, the solutions of the identities are consistent to the matrix model results.

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1. Introduction

The two dimensional quantum gravity has been studied as a toy model of the four dimensional quantum gravity. It is important to discuss what are common structures independent of the dimension. Recently there are remarkable developments in this direction [1, 2]. The problem of non linearity is the one of the most important issues of gravity. In two dimension the quantum gravity becomes exactly solvable [3, 4] and the non linear structures such as the string equations or the W and the Virasoro constraints [5, 6, 7, 8] are realized in the approaches of matrix and topological models. To make a comprehension deeper, we reexamine these structures in terms of the Liouville gravity [9, 10]. Really the non linear structures are directly related to the nature of the Hilbert space [1], the factorization of amplitudes [2, 1] and so on. Then it is important to discuss whether we have to give up the superposition principle or not. As we will see later, at least in two dimension, it appears that there is no need to abandon it.

In this talk we will discuss how the non linear structures appear in the Liouville gravity coupled to the minimal conformal field theory (CET). The quantum Liouville theory has the different features from the standard quantum field theory. The theory has two kinds of states [1]: microscopic and macroscopic ones. Microscopic states, which correspond to a branch of the local operators of Distler-Kawai [11], are dominated at small area of surface and are non-normalizable, while macroscopic states are normalizable and correspond to macroscopic loops in surface. The existence of two kinds of states is important when we discuss the non linear structures of the Liouville gravity.

In Sect.2 we summarize the results of quantum Liouville theory. Here the macroscopic states are introduced as the Hilbert space of the Liouville theory. The intermediate states of amplitudes are expanded by the macroscopic states. It is natural because these states include informations of fluctuating surface and are normalizable. The factorization of matter part is given by BPZ theory [12] in which the metric on the space of primary fields is diagonal. Then we have a question how
the non-trivial metric of scaling operators appearing in the topological and the matrix approaches, or the structures of W and Virasoro constraints, are realized from the diagonal metric of BPZ. This matter is discussed in Sect. 4 and 5.

In Sect. 3 we introduce the pseudo-null fields [9, 10], where “pseudo” means that they become exact null fields for the free theory (the cosmological constant $\mu = 0$). The pseudo-null fields can be rewritten in the form of BRST commutator. But it does not mean that they are trivial. We must take into account the measure of moduli space. Then the BRST operator picks up the non-trivial contributions from various singular boundaries of moduli space. Thus the pseudo-null fields are essentially non-zero and should satisfy the non-trivial relations, which we could see just as a generalization of the equation of motion in quantum Liouville theory coupled to the minimal conformal matter. The pseudo-null fields can be classified in the $m - 1$ series of the Liouville charge, where the central charge of minimal matter is $c_m = 1 - 6/m(m + 1)$.

In Sect. 4 and 5 we explicitly evaluate the boundary contributions and derive the various Ward identities of two dimensional gravity. Then the factorization property discussed in Sect. 2 and the fusion rule of CFT are used. The derived equations have similar structures to some of the W and Virasoro constraints $L_0$, $L_1$ and $W_{-1}$. For the Ising model we can discuss in detail and derive a closed set of the Ward identities [10]. The solutions of the identities, which are summarized in appendix, are consistent to the matrix model results.

Sect. 6 is devoted to conclusions and discussions. We consider the similarities and the differences between the Liouville gravity and the other methods. We also discuss another BRST invariant fields found by Lian-Zuckerman [13]. Since these fields have the non-standard ghost number, there are some difficulties when we consider the correlation functions of these fields.
2. Quantum Liouville Theory

The two dimensional quantum gravity is defined through the functional integrations over the metric tensor of two dimensional surface $g_{ab}$ and the matter field $m$. The partition function is

$$Z = \sum_{\chi} \kappa^{-\chi} \int [dg_{ab}][dm] \exp \left( -\frac{\mu}{2\pi} \int d^2 z \sqrt{|g_{ab}|} - S_m \right), \quad (2.1)$$

where $S_m$ is a matter action. $\chi$ is the Euler number of surfaces: $\chi = 2 - 2g$ and $\kappa$ is the string coupling constant. We use the conformal gauge $g_{ab} = e^{\gamma \phi} \hat{g}_{ab}(\hat{t})$, where $\gamma$ is a parameter given below and $\hat{g}_{ab}(\hat{t})$ is a background metric parametrized by the moduli $\hat{t}$. We choose the locally flat background metric. $\phi$ is well-known as the Liouville field. After fixing the reparametrization invariance, one can rewrite the two dimensional quantum gravity as

$$< O > = \sum_g < O >_g = \sum_g \kappa^{-\chi} \int d^2 \hat{t} \int [dm d\phi db dc] \mu(b) O e^{-S_m - S_\phi - S_{gh}}, \quad (2.2)$$

where $O$ is some operator. $S_\phi$ is the Liouville action

$$S_\phi = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} (\hat{g}^{ab} \partial_a \phi \partial_b \phi + Q \hat{R} \phi + 4\mu e^\gamma \phi) \quad (2.3)$$

and $S_{gh}$ is the ghost action. When the matter system is the minimal conformal field theory with central charge $c_m = 1 - 6/m(m+1)$, the parameters $Q$ and $\gamma$ are defined as

$$Q = \frac{4m + 2}{\sqrt{2m(m+1)}}, \quad \gamma = \frac{2m}{\sqrt{2m(m+1)}}, \quad (2.4)$$

$\mu(b)$ is the measure of the moduli space. $n$ is the number of operators.
The Liouville action has the following scaling property

\[ S_{\phi} \to S_{\phi} + \frac{Q}{2} \chi \delta \]  

(2.5)

when we change the field \( \phi \) and the cosmological constant \( \mu \) as \( \phi \to \phi + \delta, \mu \to \mu e^{-\gamma \delta} \). The constant shift of eq.(2.5) can be renormalized into the string coupling \( \kappa \).

2.1. Canonical Quantization of Liouville Theory

To discuss the Hilbert space of quantum Liouville theory we use the canonical quantization. We first change the variable from the plane coordinate \( z \) to the cylinder one \( w = \tau + i\sigma; z = e^w \). After Wick rotating \( \tau \to it \), we reach the Liouville theory in Minkowski space. Then we can set up the equal-time commutation relation

\[ [\phi(\sigma,t), \Pi(\sigma',t)] = i\delta(\sigma - \sigma') \]  

(2.6)

where \( \Pi(\sigma,t) = \frac{1}{4\pi} \partial_t \phi(\sigma,t) \) is the conjugate momentum. As a normal ordering, we adopt the free field one as used by Curtright-Thorn [14].

The energy-momentum tensor of the Liouville theory \( T_{\phi}^{\pm\pm} \equiv \frac{1}{2}(T_{\phi}^{00} \pm T_{\phi}^{11}) \) is given by

\[ T_{\phi}^{\pm\pm} = \frac{1}{8}(4\pi \Pi \pm \phi')^2 \pm \frac{Q}{4} (4\pi \Pi \pm \phi')' + \frac{\mu}{2} e^{\gamma \phi} + \frac{Q^2}{8} . \]  

(2.7)

The Hamiltonian \( H_{\phi} = L_{\phi} + \overline{L}_{\phi} \) is

\[ H_{\phi} = \frac{1}{4} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ (4\pi \Pi(\sigma))^2 + \phi'(\sigma)^2 + 4\mu e^{\gamma \phi(\sigma)} \right] + \frac{Q^2}{4} . \]  

(2.8)

The prime means the derivative with respect to \( \sigma \). \( T_{\phi}^{\pm\pm} \) satisfy the Virasoro algebra only if the parameters \( Q \) and \( \gamma \) satisfy the relation \( Q = \frac{2}{\gamma} + \gamma \). This relation is same as that derived by Distler-Kawai [11]. The central charge is found to be
\( c_\phi = 1 + 3Q^2 \). Also it can be seen that \( T^{\pm \pm}_\phi \) depends only on \( t \pm \sigma \). The conformal weight of the operator \( e^{\alpha \phi} \) is shown to be \( h_\alpha = \frac{1}{2}(\alpha Q - \alpha^2) \). Thus in spite of the interaction the values of \( c_\phi \) and \( h_\alpha \) are as if \( \phi \) is a free field. The difference is, as we will see below, that we must take the branch \( \alpha < \frac{Q}{2} \) out of two solutions of \( h_\alpha = \frac{1}{2}(\alpha Q - \alpha^2) \).

2.2. Hilbert Space of Liouville Theory

To manage the interaction term, we simplify discussions by considering the mini-superspace approximation i.e. \( e^{\gamma \phi(\sigma)} \rightarrow e^{\gamma \phi_0} \), where \( \phi_0 = \int_0^{2\pi} d\sigma \phi(\sigma)/2\pi \) is the mean value of \( \phi \). In this approximation, the Hamiltonian \( H_\phi = L_\phi + \bar{L}_\phi \) is simply

\[
H_\phi = -\frac{\partial^2}{\partial \phi_0^2} + \mu e^{\gamma \phi_0} + \frac{Q^2}{4} + N + \bar{N}
\]

(2.9)

where \( N \) and \( \bar{N} \) are the left and right-moving oscillator levels. In the case of conformal matter \( c \leq 1 \), however, the oscillator modes are canceled out by the oscillator modes of the ghost and the matter parts. In fact, when we consider the partition functions on the torus, the Dedekint \( \eta \)-functions which come from the determinants of oscillator modes are canceled out and only the zero mode contributions survive [1, 15, 16]. The derived results are exactly same as those of the matrix models. So in the following we do not attend to the oscillator modes.

The normalizable wave function for \( N = \bar{N} = 0 \) is given by using the modified Bessel function as

\[
H_\phi \Psi_p(l) = \left( p^2 + \frac{Q^2}{4} \right) \Psi_p(l),
\]

\[
\Psi_p(l) = \left( \frac{2}{\gamma} \text{psinh} \frac{2\pi}{\gamma} p \right)^{\frac{1}{2}} K_{2ip/\gamma}(2\sqrt{|l|/\gamma})
\]

(2.10)

for real \( p \), where \( l = e^{\frac{\gamma}{2} \gamma \phi_0} \). Since \( \Psi_{-p} = \Psi_p \), one can take the region \( p > 0 \). To obtain the wave function we use the boundary condition \( \Psi_p \sim \sin p \phi_0 \) (\( p > 0 \)) at
the limit $l \to 0$, which comes from the fact that the incoming wave completely reflect by the potential $e^{\gamma \phi}$. Note that there is no $p = 0$ ground state. Since the ground state is not included in the Hilbert space, we cannot define the states by acting the operators on the ground state as in the standard conformal field theory.

Now we define the state/operator identification formally by using the path integral method just like the Hartle-Hawking wave function

$$\Psi_p(l) = \int [d\phi] \psi_p(\phi) e^{-S}, \quad (2.11)$$

where $D$ is the disk with boundary $|z| = 1$ and the boundary value of $\phi$ is fixed. The operator $\psi_p(\phi)$ is located at the centre of the disk $z = 0$, which has the conformal weight $h = \bar{h} = \frac{1}{2} p^2 + \frac{Q^2}{8}$. Such a operator is given by

$$\psi_p(\phi) = \mu i p / \gamma e^{(i p + Q) \phi} + \mu -i p / \gamma e^{(-i p + Q) \phi}. \quad (2.12)$$

In general the state corresponding to the operator $e^{\alpha \phi}$ is constructed by replacing the operator $\psi_p$ into $e^{\alpha \phi}$ in (2.11). Let us consider the case that $\alpha$ is real, which corresponds to the operator of Distler-Kawai. In the mini-superspace approximation this state behaves like $\Psi_\alpha = e^{(\alpha - \frac{Q}{2}) \phi}$ at $\phi_\circ \to -\infty$. If we adopt the branch $\alpha < \frac{Q}{2}$ as a solution of $h = \frac{1}{2} (\alpha Q - \alpha^2)$, the state diverges at $\phi_\circ \to -\infty$. While for the branch $\alpha > \frac{Q}{2}$, $\Psi_\alpha$ vanishes and gives no contributions. Therefore we should take the branch $\alpha < \frac{Q}{2}$. The limit $\phi_\circ \to -\infty$ corresponds to the small area of the surface $g_{ab} = e^{\gamma \phi} g_{ab}$. So the state with $\alpha < \frac{Q}{2}$ is peaked on the small area region. We call this type of state “microscopic state”. This state is non-normalizable. On the other hand, the normalizable eigenstate of the Liouville Hamiltonian $\Psi_p$ corresponds to $\alpha = \pm i p + \frac{Q}{2}$ and oscillates at the small area region. We call it “macroscopic state”. 


2.3. Factorization of Amplitudes

Let us consider how the correlation function $< \prod_i O_{\alpha_i} >_{\Sigma}$ on the Riemann surface $\Sigma$ with genus $g$ factorizes into the two surfaces $\Sigma_1$ with genus $g_1$ and $\Sigma_2$ with $g_2$. Here $O_{\alpha_i}$ is the operator with the Liouville charge $\alpha_i$. The total Liouville charges of each part are $\sum_{i \in \Sigma_1} \alpha_i$ and $\sum_{i \in \Sigma_2} \alpha_i$, respectively. If they satisfy the normalizability conditions

$$\sum_{i \in \Sigma_1} \alpha_i + \frac{Q}{2}(2g_1 - 1) > 0, \quad \sum_{i \in \Sigma_2} \alpha_i + \frac{Q}{2}(2g_2 - 1) > 0, \quad (2.13)$$

the intermediate states are expanded by the normalizable macroscopic states. We normalize the macroscopic state as

$$< \bar{c}\bar{c}\psi_p \Phi_\Delta (\tilde{w} = 0) (\bar{\partial}\bar{c})(\partial c)\bar{c}\psi_q \Phi_\Delta' (w = 0) >_{g=0} = \frac{2\pi C(p^2)}{\kappa^2} \delta(p-q)\delta_{\Delta,\Delta'}, \quad (2.14)$$

where the two frames $w$ and $\tilde{w}$ are identified as $w\tilde{w} = 1$. $\Phi_\Delta$ is the primary field of minimal CFT. Then the factorization [2, 1] is

$$< O >_{\Sigma} = \sum_{\Delta} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{C(p^2)} < O_1 (\bar{\partial}c)(\partial c)\bar{c}\psi_p \Phi_\Delta (w = 0) >_{\Sigma_1} \times < \bar{c}\psi_p \Phi_\Delta (\tilde{w} = 0) O_2 >_{\Sigma_2}, \quad (2.15)$$

where we neglect the oscillator modes which do not contribute to the boundary terms.
3. Pseudo-Null Fields and Generalized Equations of Motion

Consider the Liouville system as CFT with central charge $c_\phi = 25+6/m(m+1)$. There are several null fields, for example

\begin{align*}
\chi_{\phi,1}^1 &= L_{-1}^\phi \cdot 1 , \\
\chi_{\phi,2}^1 &= \left( L_{-2}^\phi + \frac{m+1}{m} L_{-1}^\phi \right) \cdot e^{-\sqrt{\frac{m}{2(m+1)}} \phi} \equiv D_{-2}^{(1,2)} \cdot e^{-\sqrt{\frac{m}{2(m+1)}} \phi} , \\
\chi_{\phi,2}^2 &= \left( L_{-2}^\phi + \frac{m+1}{m} L_{-1}^\phi \right) \cdot e^{-\sqrt{\frac{m+1}{2m}} \phi} \equiv D_{-2}^{(2,1)} \cdot e^{-\sqrt{\frac{m+1}{2m}} \phi} .
\end{align*}

\begin{align}
(3.1)
\end{align}

In general there exist the null field $\chi_{\phi,p,q}^p$ at the level $pq$ of the primary field $e^{\beta_{p,q} \phi}$ with conformal weight $h_{p,q}$ for $1 \leq p \leq m - 1$, $1 \leq q \leq m$

\begin{align*}
&h_{p,q} = -\frac{1}{4m(m+1)} \left\{ \left[ p(m+1) + qm \right]^2 - (2m+1)^2 \right\} , \\
&\beta_{p,q} = \frac{1}{\sqrt{2m(m+1)}} \left[ 2m + 1 - p(m+1) - qm \right].
\end{align*}

\begin{align}
(3.2)
\end{align}

We write the null field as $\chi_{\phi,p,q}^p = D_{-pq}^{(p,q)} \cdot e^{\beta_{p,q} \phi}$, where $D_{-pq}^{(p,q)}$ is the proper combination of $L_{-n}^\phi$ ($n > 1$) with level $pq$. For example see eq.(3.1).

Now we construct the pseudo-null fields [9, 10]. The first non-trivial one is

\begin{align*}
N_{1,1} = L_{-1}^\phi L_{-1}^\phi \cdot \phi .
\end{align*}

\begin{align}
(3.3)
\end{align}

The dot “.” denotes that the contour surrounds the operator located on the r.h.s. of it. From the equation of motion $N_{1,1}$ is proportional to the cosmological constant operator

\begin{align*}
N_{1,1} = \frac{\gamma}{2} \mu e^{\gamma \phi} .
\end{align*}

\begin{align}
(3.4)
\end{align}

In the limit $\mu \to 0$, the r.h.s. of eq.(3.4) vanishes. So we call it a pseudo-null field.
Note that $N_{1,1}$ field is constructed by modifying the trivial null field $\chi_{1,1}^\phi$ as

$$N_{1,1} = \frac{\partial}{\partial \beta} (L_{-1}^\phi \overline{L}_{-1} \cdot e^{\beta \phi})|_{\beta=0} . \quad (3.5)$$

In the same way we can construct the pseudo-null field corresponding to the null field $\chi_{p,q}^\phi$ as

$$N_{p,q} = \frac{\partial}{\partial \beta} (D_{-pq}^{(p,q)} \overline{D}_{-pq}^{(p,q)} \cdot e^{\beta \phi} \Phi_{p,q})|_{\beta=\beta_{p,q}}$$

$$= D_{-pq}^{(p,q)} \overline{D}_{-pq}^{(p,q)} \cdot \phi e^{\beta_{p,q} \phi} \Phi_{p,q} . \quad (3.6)$$

Here, to make the physical operator, we combine the Liouville field and the matter field. $\Phi_{p,q}$ is the primary field of matter system with conformal dimension

$$\Delta_{p,q} = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)} . \quad (3.7)$$

In the analogy of the equation of motion, it is expected that $N_{p,q}$ is proportional to the dressed physical field of $\Phi_{p,q}$

$$N_{p,q} = C_{p,q} \mu^{x_{p,q}} e^{\alpha_{p,q} \phi} \Phi_{p,q} , \quad (3.8)$$

where

$$\alpha_{p,q} = \frac{2m + 1 - |p(m+1) - qm|}{\sqrt{2m(m+1)}} ,$$

$$x_{p,q} = \frac{1}{2m} [p(m+1) + qm - |p(m+1) - qm|] . \quad (3.9)$$

$C_{p,q}$ is the proportional constant, which have to be determined later. The exponent of $\mu$ is determined from the scaling property of the Liouville action (2.5).

Since there is the relation $h_{p,q} = h_{m+p,m+1-q} + (m+p)(m+1-q)$, the null state itself contains a null state. Therefore we can construct the family of null
physical states satisfying the physical state condition

\[ 1 = \Delta_{p,q} + h_{p,q} + pq \]

\[ = \Delta_{p,q} + h_{m+p,m+1-q} + (m+p)(m+1-q) + pq \]

\[ = \Delta_{p,q} + h_{2m+p,q} + (2m+p)q + (m+p)(m+1-q) + pq \] 

\[ \ldots \] 

The pseudo-null field \( N_{p,q} \) corresponds to the relation of the first line. From the second relation we obtain

\[ M_{p,q} = D^{(p,q)} (-pq) D^{(m+p,m+1-q)} (-m+p)(m+1-q) \cdot \phi \alpha_{m+p,m+1-q} \phi \Phi_{p,q} , \] 

which is also proportional to the dressed physical field of \( \Phi_{p,q} \). In the following we write the dressed physical field of \( \Phi_{p,q} \) as \( O_{p,q} = \bar{c}c e^{\alpha_{p,q} \phi} \Phi_{p,q} \), where we combine the ghost field. We also define \( \tilde{N}_{p,q} = \bar{c}c N_{p,q} \) and \( \tilde{M}_{p,q} = \bar{c}c M_{p,q} \). Then these fields are summarized in the table

\[
\begin{array}{cccccc}
\beta^1_n : & O_1 & - \tilde{N}_{1,1} & \ldots & \tilde{N}_{1,m} & - \tilde{M}_{1,m} & \tilde{M}_{1,m-1} \\
\beta^2_n : & O_2 & O_{2,1} & - \tilde{N}_{2,1} & \ldots & \tilde{N}_{2,m} & - \tilde{M}_{2,m} \\
\vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\beta^{m-1}_n : & O_{m-1} & \ldots & O_{m-1,1} & - \tilde{N}_{m-1,1} & \ldots & \tilde{N}_{m-1,m} \\
\end{array}
\]

Here \( O_p = O_{p,p} \) \((p = 1, \ldots, m - 1)\). The Liouville charges \( \beta^p_n \) \((p = 1, \ldots, m - 1)\) are given by

\[ \beta^p_n = -\frac{(n+p-3)m+p-1}{\sqrt{2m(m+1)}} \quad (n \neq p+1 \text{ mod } m+1) \] 

It is expected that the series \( \beta^1_n \) have something to do with the Virasoro constraints and \( \beta^i_n \) \((i = 2, \ldots, m - 1)\) with W constraints. In the following section we discuss this correspondence.
The pseudo-null field can be rewritten in the form of BRST commutator \( \tilde{N}_{p,q} = Q^B \cdot W_{p,q} \), for example \( W_{1,1} = Qs \cdot b_{-1} \cdot \bar{c}c\phi \), where \( Q^B = Q_s + \bar{Q}_s \) is the BRST charge. Thus the generalized equation of motion (3.8) can be rewritten as

\[
C_{p,q}{\mu^{x_{p,q}}}O_{p,q} = Q^B \cdot W_{p,q} ,
\]

In the following section we consider the Ward identity which is given by inserting the identity (3.14) into the correlator of scaling operators.

4. Ward Identities of 2D Quantum Gravity coupled to the Ising Model

Let us first discuss the case of the Ising model [10]. We derive various Ward identities obtained by inserting the pseudo-null field relations (3.14) into the correlation functions: \( < O >_g = \prod^n_1 O_I \prod^n_2 O_\sigma \prod^n_3 O_\varepsilon >_g \), where \( O_I = O_{1,1} \) is the cosmological constant operator. \( O_\sigma = O_{1,2} \) and \( O_\varepsilon = O_{2,1} \) are the dressed spin and energy operators.

4.1. Equation of Motion

The pseudo-null field relation of \( N_{1,1} \) is nothing but the equation of motion. In this subsection we treat this operator. As evaluating the boundary contributions, there is a problem. The operator \( W_{p,q} \) of eq.(3.14) is in general not well-defined on the moduli space, or it is not annihilated by the action of \( b_o \) and \( \bar{b}_o \). Therefore we introduce the well-defined operator \( X_{p,q} \) satisfying the condition \( b_o \cdot X_{p,q} = \bar{b}_o \cdot X_{p,q} = 0 \). The operator \( X_{1,1} \) is defined by modifying \( W_{1,1} \) slightly as \( X_{1,1} = L^{-\theta}_{-1} \cdot \bar{b}_{-1} \cdot \bar{c}c\phi \). Then the identity (3.14) for the Ising case becomes

\[
\frac{3}{2\sqrt{6}}\mu O_I = Q^B \cdot X_I + K_I .
\]

where \( X_I = X_{1,1} \) and \( K_I \) is called “ghost pieces”: \( K_I = -\frac{7}{\sqrt{6}}c_{-1} \bar{b}_{-1} \cdot \bar{c}c \), which has the non-standard asymmetric ghost number.
The Ward identity we discuss is

\[ \frac{3}{2\sqrt{6}} \mu < O_I O >_g = < Q^B \cdot X_I O >_g + < K_I O >_g. \quad (4.2) \]

The correlation function with the ghost piece \( K_I \) vanishes because it has the non-standard ghost number. The first term on r.h.s. is evaluated as follows. Taking into account the moduli and the measure for the position \( z = z_1 \) of \( Q^B \cdot X_I \), we obtain

\[ \int d^2z_1 \overline{b}_{-1} Q^B \cdot X_I (z_1) O_\alpha (z_i) = - \frac{1}{2i} \oint_{|z_1-z_i|=\epsilon} dz_1 \partial \phi (z_1) O_\alpha (z_i). \quad (4.3) \]

Here the BRST algebra \( \{ Q^B, b_{-1} \} = L_{-1} = \partial \) is used. The r.h.s. of eq.(4.3) becomes a total derivative with respect to the moduli so that finite contributions will come from the boundary of moduli space. In this case the relevant boundary is where \( Q^B \cdot X_I (z_1) \) approaches other operators \( O_\alpha (z_i) \). To evaluate the boundary term the small cut-off \( \epsilon \) is introduced and, after the calculation, we take the limit \( \epsilon \to 0 \). The contributions come from the singularity of operator product \( \partial \phi (z) O_\alpha (z_i) \). The leading singularity comes from the free field (\( \mu = 0 \)) OPE. As the next leading singularity a \( \mu \)-dependent term appear, but, in this case, does not contribute because the power of singularity is too small to give the finite value at the limit \( \epsilon \to 0 \).

We also have to analyze curvatures carefully. One can choose a metric which is almost flat except for delta function singularities at the positions of scaling operators; \( \sqrt{g} R = 4\pi \sum_i \nu_i \delta^2 (z - z_i) \), so that \( \sum_i \nu_i = \chi \), where \( \chi \) is the Euler number of two dimensional surface with genus \( g \); \( \chi = 2 - 2g \). We do not assign the curvature to \( Q^B \cdot X_I \). To use free field operator products it is necessary to smooth out the curvature singularity in the neighborhood of the position of the operator. This is done by the coordinate transformation: \( z - z_i = (z' - z'_i)^{1-\nu_i} \), or \( dzd\bar{z} \sim \frac{dz'd\bar{z}'}{|z'-z'_i|^{\nu_i}} \). In the smooth \( z' \)-frame we can freely use the operator products.
After evaluating OPE, we finally obtain the expression

\[
\mu \frac{\partial}{\partial \mu} < \prod O_I \prod O_\sigma \prod O_\varepsilon >_g = -\frac{\mu}{2\pi} < \prod O_I \prod O_\sigma \prod O_\varepsilon >_g
\]

\[
= -\left(n_1 + \frac{5}{6}n_2 + \frac{1}{3}n_3 - \frac{7}{6}\chi\right) < \prod O_I \prod O_\sigma \prod O_\varepsilon >_g .
\]

Thus the \(\mu\)-dependence of the correlation functions† is

\[
< \prod O_I \prod O_\sigma \prod O_\varepsilon >_g = Z_{n_1,n_2,n_3}^g \mu^{\frac{7}{6} \chi - n_1 - \frac{5}{6}n_2 - \frac{1}{3}n_3} .
\]

Note that, since the path integral of Liouville field diverges, the derivation can not apply for the case \(\sum_{i=1}^{n_1+n_2+n_3} \alpha_i - \frac{Q}{2} \chi < 0\), where \(\alpha_i\) is the charge of the exponential operator. Therefore in this case we have to define the correlation function by using the differential equation such as, for example, \(-2\pi \frac{d}{d\mu} < O_\sigma O_\sigma >_g = < Q \cdot X_\sigma \prod O_\alpha >_g + \text{ghost term}\).

4.2. Ward Identities corresponding to Virasoro Constraints

In this section we consider the Ward identities given by inserting the pseudo-null field \(N_{1,2} = N_\sigma\). In the following we consider the Ward identity

\[
C_a \mu^{4/3} < O_\sigma \prod_{\alpha} O_\alpha >_g = < Q^B \cdot X_\sigma \prod_{\alpha} O_\alpha >_g + \text{ghost term},
\]

where \(O_\alpha\)'s are only the “gravitational” primary fields \(O_I\) and \(O_\sigma\). \(X_\sigma\) is the well-defined operator on moduli space: \(X_\sigma = D_2^\sigma \bar{B}_2 - \bar{c} \bar{c} \phi \bar{\phi} \). To evaluate the r.h.s. of eq.(4.6) we must take into account the 3 types of boundaries.

† \(Z_{n_1,n_2,n_3}^g\) can be directly calculated by using CFT methods [17]
As discussed in Sect.4.1 the first boundary arises from that $X_\sigma$ approaches other operators $O_\alpha$. The relevant operator product is
\[
\int_{|z_1-z_i| \geq \epsilon} d^2 z_1 b_{-1} \bar{b}_{-1} Q^B \cdot X_\sigma(z_1) O_\alpha(z_i)
\]
\[
= -\frac{1}{2i} \oint_{|z_1-z_i| = \epsilon} (dz_1 b_{-1} + d\bar{z}_1 \bar{b}_{-1}) \cdot X_\sigma(z_1) O_\alpha(z_i).
\]
(4.7)

One can see that from the power of OPE singularities between $\bar{b}_{-1} \cdot X_\sigma$ and $O_\alpha$ the integral of $d\bar{z}$ vanishes. The integral of $dz$ gives the finite contributions for $\alpha = \sigma$. Then we get
\[
-\frac{1}{2i} \oint dz b_{-1} \cdot X_\sigma(z_1) O_\sigma(z_i) = \frac{5\pi}{9\sqrt{6}} O_\varphi(z_i).
\]
(4.8)

In the interacting theory there will be the next leading OPE singularities which depend on the cosmological constant $\mu$. If one uses the free field OPE, one should add the $\mu$-dependent term, or the operators $O_\alpha(\alpha = I, \sigma)$ behave in the neighborhood of boundaries as follows
\[
O_I \rightarrow \bar{c} \bar{c}(e^{3\sqrt{6} \phi} + \eta_I \mu^{1/3} e^{\frac{1}{\sqrt{6}} \phi}),
\]
\[
O_\sigma \rightarrow \bar{c} \bar{c}(e^{2\sqrt{6} \phi} + \eta_\sigma \mu^{2/3} e^{\frac{2}{3} \sqrt{6} \phi}).
\]
(4.9)

The charge of the Liouville mode and the power of $\mu$ are determined from the restriction of conformal dimension and the scaling symmetry of the Liouville action. The phases $\eta_I$ and $\eta_\sigma$ are determined by the consistency.

Next we discuss the boundary-2 where the field $X_\sigma$ approaches the pinched point which divides the surface into two pieces. Then the factorization discussed in Sect.2 is important.

\[
< Q^B \cdot X_\sigma O >_{\Sigma} = \sum_{\Delta} \int_{-\infty}^{\infty} \frac{dp}{2\pi C(p^2)} < O_1 \int \frac{d^2 z_1}{z_1 \bar{z}_1} b_\sigma \bar{b}_\sigma Q^B \cdot X_\sigma(z_1) \int \frac{d^2 q}{q\bar{q}} b_\sigma \bar{b}_\sigma \times q^L \bar{q}^L (\partial \bar{c}) (\partial c) \bar{c} c v'_{\bar{\phi}} \Phi_\Delta (w = 0) >_{\Sigma_1} < \bar{c} c v_{-\bar{\phi}} \Phi_\Delta (\bar{w} = 0) O_2 >_{\Sigma_2},
\]
(4.10)

where $\Phi_\Delta = I, \sigma, \varepsilon$. The coordinates $w$ and $\bar{w}$ are defined in the neighborhood
of the nodes of $\Sigma_1$ and $\Sigma_2$. These are identified as $w \tilde{w} = q$, where $w = z$ and $q$ is the moduli that determines the shape of the pinch. By this identification the normalization (2.14) changes so that the operator $q^{L_0} \bar{q}^{\bar{L}_0}$ is inserted. We explicitly introduce the measure of moduli for $z_1$ and $q$. The operator $b_\sigma$ is defined by the contour integral around $w = 0$. Using the BRST algebra one can rewrite the expression into the derivatives with respect to the moduli $q$.

The boundary contributions come from the limit that $z_1$ and $q$ approach zero simultaneously. In this limit the integrand is highly peaked and we can evaluate the integral by the saddle point method. The final result becomes

$$< Q^B \cdot X_{1,2} \prod_{j \in S} O_j >_g^{b_{g2}} \simeq \sum_{s = \Sigma_1 \cup \Sigma_2} \left\{ < O_\sigma \prod_{\alpha \in X} O_\alpha >_{g_1} < O_I \prod_{\alpha \in Y} O_\alpha >_{g_2} + < O_I \prod_{\alpha \in X} O_\alpha >_{g_1} < O_\sigma \prod_{\alpha \in Y} O_\alpha >_{g_2} \right\},$$

(4.11)

where $S = X \cup Y$ means that the sum is over the possible factorizations satisfying the conditions (2.13). To derive these structures we use the fusion rule of the minimal CFT. Here we neglect the curvature contributions. Note that the metric we first introduced (4.10) is the diagonal one, but after evaluating the boundary the metric structure changes to the asymmetric form and $O_\varepsilon$ disappears such as the topological and the matrix models. This structure really corresponds to the $L_1$ Virasoro constraints.

The boundary-3 is a kind of boundary-2, where a handle is pinched. In this case the surface is not divided by the pinching. Thus we obtain

$$< Q^B \cdot X_{1,2} \prod_{j \in S} O_j >_{g}^{b_{g3}} \simeq < O_I O_\sigma \prod_{\alpha} O_\alpha >_{g-1}.$$

(4.12)

Let us consider the Ward identity with operator insertion $\prod O_\alpha = O_\sigma \prod_\alpha O_I$. 

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In general genus we get \((n \geq 3 \text{ for } g = 0)\)

\[
0 = \frac{5}{9\sqrt{6}} \pi < O_\varepsilon \prod_{I} O_{I} >_{g}
\]

\[
-2\lambda \sum_{g_{1}=0}^{g} \sum_{k=0}^{n} \left\{ \left( \begin{array}{c} n-2 \\ k \end{array} \right) \left( \frac{5}{12} \chi_{1} - \frac{1}{3} \right)^{2} + \left( \begin{array}{c} n-2 \\ k-2 \end{array} \right) \left( \frac{5}{12} \chi_{2} - \frac{1}{2} \right)^{2} \right. \\
-2 \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \left( \frac{5}{12} \chi_{1} - \frac{1}{3} \right) \left( \frac{5}{12} \chi_{2} - \frac{1}{2} \right) + \frac{25}{144} \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \sum_{i} \nu_{i}^{2} \right\} \tag{4.13}
\]

\[
\times < O_{\sigma} O_{\sigma} \prod_{I} O_{I} >_{g_{1}} < \prod_{I} O_{I} >_{g_{2}}
\]

\[
-\frac{\lambda}{72} < O_{\sigma} O_{\sigma} \prod_{I}^{n+1} O_{I} >_{g-1} \text{+ ghost term.}
\]

Here the curvature singularities are considered, which are assigned to \(n O_{I}\) operators\(^\dagger\). We determine the unknown constants except \(\lambda = \frac{8\pi^{2}}{C(p^{2}=-1/6)}\) from the consistency of the Ward identities on the sphere. In the end the next leading terms of the boundary-1 and \(C_{\alpha}\)-term are absorbed in the factorization form. In the above expression it appears as if there were no restrictions like the inequalities (2.13).

In general genus we need the contributions of the ghost term. Naively it does not contribute because of the asymmetry of the ghost number. However, for \(g \geq 1\), there will be the non-zero contribution when we evaluate the curvature singularities. To calculate the curvature contributions we used the transformation from the singular frame to the non-singular frame. Then the mapping analytic in the moduli was used. This is correct on the sphere, but for \(g \geq 1\) one can not take such a mapping globally to remove the curvature singularities. So there are possibilities that the measure makes up for the asymmetry and the ghost term contributes. Really eq.(4.13) is inconsistent if there are no contributions of the ghost term. Exceptional case is \(g = 1\), then we can choose the flat metric where all \(\nu_{i}\)'s are zero. In this case the ghost term will vanish.

\(^\dagger\) Although the expression changes by how to assign the curvatures, the final results are independent of the assignments. Furthermore as a consistency check we can see that the expression (4.13) is indeed independent of the value of \(\sum_{i} \nu_{i}^{2}\) for \(n \geq 3, g = 0\).
We also consider the Ward identity with \( n O_\sigma \) operators. The curvature singularities are assigned to \( n O_\sigma \) operators. Then we obtain

\[
0 = \frac{5}{9\sqrt{6}}\pi \left( n - \chi - \frac{5}{4} \sum_i \nu_i^2 \right) < O_\varepsilon \prod_{\sigma} O_\sigma >_g 
- 2\lambda \sum_{g_1=0}^{g} \sum_{k=0}^{n} \left\{ \left( \frac{n-2}{k} \right) \left( \frac{5}{12} \chi_1 - \frac{1}{3} \right)^2 + \left( \frac{n-2}{k-2} \right) \left( \frac{5}{12} \chi_2 - \frac{1}{2} \right)^2 \right. \\
- 2 \left( \frac{n-2}{k-1} \right) \left( \frac{5}{12} \chi_1 - \frac{1}{3} \right) \left( \frac{5}{12} \chi_2 - \frac{1}{2} \right) + \frac{25}{144} \left( \frac{n-2}{k-1} \right) \sum_i \nu_i^2 \right\} \tag{4.14}
\times < \prod_{\sigma} O_\sigma >_{g_1} < O_I \prod_{\sigma} O_\sigma >_{g_2} \\
- \frac{\lambda}{72} < O_I \prod_{\sigma} O_\sigma >_{g-1} + \text{ghost term.}
\]

4.3. Ward Identities corresponding to \( W \) constraints

In this section we consider the case of the pseudo-null field \( N_{2,1} = N_\varepsilon \):

\[
C_1 \mu < O_\varepsilon \prod_{\alpha} O_\alpha >_g = < Q^B \cdot X_\varepsilon \prod_{\alpha} O_\alpha >_g + \text{ghost term}, \tag{4.15}
\]

where \( \alpha = I, \sigma \). The operator \( X_\varepsilon = X_{2,1} \) is defined as in the previous section by \( X_\varepsilon = D_2^\varepsilon \bar{B}_2^{\varepsilon} \cdot \bar{c}\phi \bar{e}^{-\frac{2}{\sqrt{6}} \phi \varepsilon} \). In this case we have to take into account the 4 types of boundaries.

We do not repeat the calculations in detail. The second and third boundary contributions have the following form

\[
b_{2,3} \simeq \sum_{s=S,X,\bar{X}} < O_\sigma \prod_{\alpha \in s} O_\alpha >_{g_1} < O_\sigma \prod_{\alpha \in \bar{Y}} O_\alpha >_{g_2} + < O_\sigma O_\sigma \prod_{\alpha \in S} O_\alpha >_{g-1} . \tag{4.16}
\]

The operator \( O_\varepsilon \) does not appear on the nodes. The metric structure really corresponds to the \( W_{-1} \) constraint.
Furthermore we must take into account the boundary-4 that $X_{\varepsilon}$ and two $O_{\sigma}$’s approach at a point simultaneously. In fact one can easily see that, if there is no boundary contribution of this type, the Ward identity becomes inconsistent. We do not know how to evaluate this boundary directly. Instead we assume the following form

$$Q^B \cdot X_{\varepsilon}O_{\sigma}O_{\sigma} \to O_I .$$  \hspace{1cm} (4.17)

Let us consider the Ward identity of the type: $\prod_{\alpha} O_\alpha = O_{\sigma}O_{\sigma} \prod^n O_I$. If the curvatures are assigned only to $n$ cosmological constant operators, we obtain the following Ward identity

$$0 = \frac{2\pi}{\sqrt{6}} \left( \mu \frac{\partial}{\partial \mu} + n + \frac{11}{18} \chi - \frac{55}{18} \sum_i \nu_i^2 \right) < O_{\varepsilon}O_{\sigma}O_{\sigma} \prod^{n-1}_I O_I >_g$$

$$+ \frac{1}{36} \lambda \sum_{g_1=0}^{g} \sum_{k=0}^{n} \left\{ \binom{n}{k} - 55 \left[ \binom{n-2}{k} (1-\chi_1)^2 - 2 \binom{n-2}{k-1} (1-\chi_1)(1-\chi_2) + \binom{n-2}{k-2} (1-\chi_2)^2 + \binom{n-2}{k-1} \sum_i \nu_i^2 \right] \right\}$$

$$\times < O_{\sigma}O_{\sigma} \prod^k_1 O_I >_{g_1} < O_{\sigma}O_{\sigma} \prod^{n-k}_1 O_I >_{g_2}$$

$$+ \frac{1}{72} \lambda < \prod^4_1 O_{\sigma} \prod^n_1 O_I >_{g-1} + \frac{32}{3} \pi^2 C_4 < \prod^{n+1}_1 O_I >_g + \text{ghost terms}$$

$$\text{(4.18)}$$

For the case with the operator insertions $\prod_{\alpha} O_\alpha = \prod^n O_{\sigma}$, if the curvatures are assigned only to $n$ $O_{\sigma}$, we get
\[ 0 = \frac{1}{\sqrt{6}} \mu < O_\epsilon \prod O_\sigma >_g \]
\[ - \frac{1}{72} \lambda \sum_{g_1=0}^{g} \sum_{k=0}^{n} \left\{ \binom{n}{k} - 55 \left( \binom{n-2}{k} (1 - \chi_1)^2 \right. \right. \]
\[ \left. \left. - 2 \binom{n-2}{k-1} (1 - \chi_1) (1 - \chi_2) + \binom{n-2}{k-2} (1 - \chi_2)^2 + \binom{n-2}{k-1} \sum_i \nu_i^2 \right) \right\} \]
\[ \times < \prod_{g_1} O_\sigma >_{g_1} < \prod_{n-k+1} O_\sigma >_{g_2} \]
\[ - \frac{1}{72} \lambda \left\{ \prod_{g_1} O_\sigma >_{g_1} - \frac{32}{3} \pi^2 C_4 \left\{ \frac{n(n-1)}{2} - \frac{77}{60} (n-1) \chi + \frac{11}{24} \chi^2 \right\} \right. \]
\[ \left. + \frac{11}{48} (n-3) \sum_i \nu_i^2 \right\} < O_I \prod_{g_1} O_\sigma >_{g_1} + \text{ghost terms.} \]

Unfortunately we cannot determine the constants \( \lambda \) and \( C_4 \) by the consistency. The determination of these values and the ghost terms remains as future problems. We could determine these values by using the results of ref.18, which are given by \( \lambda = \frac{2}{\sqrt{6}} \pi \) and \( C_4 = \frac{5}{6 \sqrt{6}} \pi \). Then we can derive several correlation functions on the sphere and the torus, which are consistent to the results of the two matrix model [19].

### 5. Ward Identities for Minimal CFT

In the previous section we obtain a closed set of Ward identities for the case of the Ising model. For the general minimal series it is difficult to derive a closed set of Ward identities because more complicated boundaries contribute and also the number of primary fields increases. So we only concentrate on the pseudo-null field \( N_{1,2} \). Then it is expected that the structures like \( L_1 \) equation, or metric on the space of scaling operators appearing in the matrix and the topological methods, are realized.

The Ward identity is given by substituting the relation (3.14) with \((p,q) = (1,2)\) into correlation functions. For simplicity we consider the correlation func-
tion with the operators $O_j \equiv O_{j,j}$ $(j = 1, \cdots, m - 1)$, which corresponds to the gravitational primary fields. Then

$$C_{1,2} \mu^{m+1 \over m} < O_{1,2} \prod_{j \in S} O_j >_g = < Q^B \cdot W_{1,2} \prod_{j \in S} O_j >_g . \quad (5.1)$$

The operator $O_{1,2} = O_{m-1,m-1}$ corresponds to the first gravitational primary $O_{m-1}$. We evaluate the boundary contributions of the r.h.s. of eq.(5.1). The contributions of the first boundaries are given by

$$< Q^B \cdot W_{1,2} \prod_{j \in S} O_j >_g^{b_1} \simeq \sum_{k \in S \setminus \{k \neq 1\}} < O_{k,k-1} \prod_{j \in S \setminus \{j \neq k\}} O_j >_g + \text{next leading terms}, \quad (5.2)$$

where $O_{k,k-1}$ corresponds to the gravitational descendant $\sigma_1(O_k)$ $(k = 2, \cdots, m - 1)$. Here we neglect the curvature contributions and the normalization of scaling operators. The contributions from the boundaries-2 and -3 are given by

$$< Q^B \cdot W_{1,2} \prod_{j \in S} O_j >_g^{b_2,3} \simeq \sum_{k=1}^{m-1} \left\{ < O_k O_{m-k} \prod_{j \in S} O_j >_{g-1} \right. \left. + \sum_{S = \mathbf{X} \cup \mathbf{Y}} \sum_{g = g_1 + g_2} \left( \sum_{k \in S \setminus \{X \cup Y\}} < O_k \prod_{j \in X} O_j >_{g_1} < O_{m-k} \prod_{j \in Y} O_j >_{g_2} \right) \right\}, \quad (5.3)$$

where $S = \mathbf{X} \cup \mathbf{Y}$ means that the sum is over the possible factorizations satisfying the conditions (2.13). To derive these structures we use the fusion rule of the minimal CFT. As discussed in the case of the Ising model, the l.h.s. of eq.(5.1) and the next leading terms of eq.(5.2) are used to complete the factorization form of eq.(5.3). Then we finally obtain the following structure

$$0 \simeq \sum_{k \in S \setminus \{k \neq 1\}} < O_{k,k-1} \prod_{j \in S \setminus \{j \neq k\}} O_j >_g \left. \right. \left. \right. \left. + \sum_{k=1}^{m-1} \sum_{S = \mathbf{X} \cup \mathbf{Y}} \sum_{g = g_1 + g_2} \left( \sum_{k \in S \setminus \{X \cup Y\}} < O_k \prod_{j \in X} O_j >_{g_1} < O_{m-k} \prod_{j \in Y} O_j >_{g_2} \right) \right\}, \quad (5.4)$$
The equation really has the similar structures to the $L_1$ Virasoro constraint. The metric on the space of scaling operators appearing in the other methods are realized explicitly. It is essentially determined by the fusion rule of CFT and the conservation of the total Liouville charge.

It probably needs to discuss all $N_{p,q}$ fields to determine all correlation functions of $O_{p,q}$. It appears, however, that $M_{p,q}$ and the others do not give the essentially new informations as far as one considers only the correlation functions of the operators in the Kac table $O_{p,q}$.

The difference from the other methods is that the l.h.s. of eq.(5.4) vanishes and the identity is closed only by the operators $O_{p,q}$ in the Kac table, while $L_1$ equation appearing in the other methods has the structure that the l.h.s. of eq.(5.4) is the correlation function with the gravitational descendant $\sigma_3(O_1)$:

$$<\sigma_3(O_1) \prod_{j \in S} O_j >_g.$$

6. Conclusions and Discussions

We have discussed the Ward identities of the Liouville gravity coupled to the minimal CFT. We found the series of the pseudo-null fields and the generalized equations of motion. The various Ward identities given by inserting these equations into correlation functions are derived. Especially for the Ising model we give a closed set of the identities. Then the several interesting structures similar to the matrix and the topological methods appeared. The identities we discussed have the similar structures to the W and Virasoro constraints; $L_0$, $L_1$ and $W_{-1}$ for the Ising model and $L_1$ for the general case. Really the boundary-2 structure has the same metric on the space of scaling operators as that in the other two methods. Also the Ward identities corresponding to $W_{-1}$ constraints require the new boundary different from the Virasoro constraints as in the other methods. It should be stressed that these non-linear structures are derived from the factorization (2.15), where the intermediate states are expanded by the normalizable Hilbert states of the Liouville and CFT. Since the states are defined by using the path integral just
like the Hartle-Hawking states, it might be expected that we also need not abandon
the superposition principle in higher dimensional quantum gravity.

The differences from the other methods are related to the problem of the grav-
itational descendants in the Liouville gravity. Our Ward identities are closed only
by the dressed operators corresponding to the Kac table of CFT and just have the
same form as the W and Virasoro constraints given by setting the gravitational
descendants outside of the Kac table to zero. The pseudo-null fields have indeed
the similar properties to the gravitational descendants outside of the Kac table.
These properties are ruled by the Liouville charges and the fusion rule of CFT.
Therefore the gravitational descendants should have the same Liouville charge and
matter field as that listed in table (3.12).

Lian and Zuckerman [13] found a series of BRST invariant states with these
properties as a candidate of gravitational descendants. For example one can easily
construct the states with the same Liouville charge and matter field as that of $N_{1,2}$
and $N_{2,1}$

$$R_{1,2} = \left\{ b_{-2} c_1 + \frac{m+1}{m} \left( L_{-1}^{\phi} - L_{-1}^{m} \right) \right\} \cdot e^{\beta_{1,2} \Phi_{1,2}} ;$$

$$R_{2,1} = \left\{ b_{-2} c_1 + \frac{m}{m+1} \left( L_{-1}^{\phi} - L_{-1}^{m} \right) \right\} \cdot e^{\beta_{2,1} \Phi_{2,1}} .$$

(6.1)

The BRST invariance of these states are proved by using the null states of the
Liouville and the matter sectors. Note that these states have zero ghost number.
If the measure of moduli space is taken into account, the correlation function with
these fields vanishes by the ghost number conservation. This situation might have
something to do with that the Ward identities of the Liouville gravity have the
form mentioned above. If one wants the non-vanishing correlation functions, it is
necessary to change the measure of moduli.

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