A SHORT PROOF OF THE DIMENSION CONJECTURE
FOR REAL HYPERSURFACES IN $\mathbb{C}^2$

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Abstract. Recently, I. Kossovskiy and R. Shafikov have settled the so-called
Dimension Conjecture, which characterizes spherical hypersurfaces in $\mathbb{C}^2$ via
the dimension of the algebra of infinitesimal automorphisms. In this note, we
propose a short argument for obtaining their result.

1. Introduction

Let $M$ be a 3-dimensional connected real-analytic CR-manifold of hypersurface
type. We only consider $M$ locally, and therefore one can assume that $M$ is embed-
ded in $\mathbb{C}^2$ with the CR-structure induced by the complex structure of the ambient
space. Recall that an infinitesimal CR-automorphism of $M$ is a smooth vector field
on $M$ whose flow consists of CR-transformations. For $p \in M$, denote by $\mathfrak{hol}(M, p)$
the Lie algebra of real-analytic infinitesimal CR-automorphisms of $M$ defined in a
neighborhood of $p$ on $M$, with the neighborhood $a$ priori depending on the vector
field. It is not hard to show that every element of $\mathfrak{hol}(M, p)$ is the real part of a
holomorphic vector field defined on an open subset of $\mathbb{C}^2$.

If $M$ is Levi-flat, i.e., its Levi form identically vanishes, then $M$ is locally CR-
equivalent to the direct product $\mathbb{C} \times \mathbb{R} \subset \mathbb{C}^2$, hence in this case $\dim \mathfrak{hol}(M, p) = \infty$
for all $p \in M$. On the other hand, if $M$ is Levi nondegenerate at some point,
then for every $p \in M$ one has $\dim \mathfrak{hol}(M, p) < \infty$ (see [BER, Theorem 11.5.1 and
Corollary 12.5.5]). If, furthermore, $M$ is spherical at $p$, i.e., CR-equivalent to an
open subset of the sphere $S^3 \subset \mathbb{C}^2$ in a neighborhood of $p$, then $\dim \mathfrak{hol}(M, p) = 8$.
Indeed, for every $q \in S^3$ the algebra $\mathfrak{hol}(S^3, q)$ consists of globally defined vector
fields and is isomorphic to $\mathfrak{su}_2 \mathfrak{l}$ (see [P], [C] as well as [CM], [Ta], [Sa], pp. 211–219],
[B2] for generalizations to higher CR-dimensions and CR-codimensions). Further,
the reduction of 3-dimensional Levi nondegenerate CR-structures to absolute par-
allelisms obtained by E. Cartan in [C] implies that 8 is the maximal possible value
of $\dim \mathfrak{hol}(M, p)$ provided the Levi form of $M$ at $p$ does not vanish. Moreover, as
noted in [C, part I, p. 34], results of A. Tresse in [Tr] (see also [K], [KT]) yield that
for such a point $p$ the condition $\dim \mathfrak{hol}(M, p) > 3$ forces $M$ to be spherical at $p$.

This note concerns the following conjecture, in which Levi nondegeneracy is no
longer assumed:

Conjecture 1.1. If $M$ is not Levi-flat, then for any $p \in M$ the condition
\begin{equation}
(1.1) \quad \dim \mathfrak{hol}(M, p) > 5
\end{equation}
implies that $M$ is spherical at $p$.

In the above form, the conjecture was formulated in article [KS] where the authors
called it the Dimension Conjecture and argued that it can be viewed as a variant of
Poincaré’s problème local. This statement is also a refined version, in the case $n = 2$,
of another conjecture, due to V. Beloshapka, proposed in [B3] for real hypersurfaces in \( \mathbb{C}^n \) with any \( n \geq 2 \), which so far has only been resolved for \( n \leq 3 \) (see [I2]).

In [KS], the authors proved:

**Theorem 1.2.** Conjecture 1.1 holds true.

The method of [KS] is rather involved and based on considering second-order complex ODEs with meromorphic singularity. The aim of the present paper is to provide a short proof of Theorem 1.2 by using standard facts on Lie algebras and their actions. Before proceeding, we state the following:

**Corollary 1.3.** The possible dimensions of \( \mathfrak{hol}(M, p) \) are 0, 1, 2, 3, 4, 5, 8, \( \infty \), and each of these possibilities is realizable.

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2. **Proof of Theorem 1.2 and Corollary 1.3**

Suppose that \( M \) is not Levi-flat. Then the set \( S \) of points of Levi nondegeneracy is dense in \( M \). Fix \( p \in M \) with \( \dim \mathfrak{hol}(M, p) > 5 \) and consider the algebra \( \mathfrak{hol}(M, p) \). If \( p \in S \), then, as stated in the introduction, the sphericity of \( M \) at \( p \) follows from classical results in [C], [Tr].

Assume now that \( p \notin S \). As \( \dim \mathfrak{hol}(M, p) < \infty \), there exists a neighborhood \( U \) of \( p \) in \( M \) where all vector fields in \( \mathfrak{hol}(M, p) \) are defined. Therefore, for every \( p' \in U \cap S \), the algebra \( \mathfrak{hol}(M, p) \) is a subalgebra of \( \mathfrak{hol}(M, p') \). Arguing as above, we then see that \( M \) is spherical at \( p' \). Hence, \( \mathfrak{hol}(M, p) \) can be identified with a subalgebra of \( \mathfrak{su}_2,1 \). It is not hard to show that \( \mathfrak{su}_2,1 \) has no subalgebras of dimensions 6 and 7. This is a consequence, for instance, of the proof of Lemma 2.4 in [EaI], but for the reader’s convenience we give a different argument here. Indeed, by [M], a maximal proper subalgebra of a semi-simple Lie algebra is either parabolic, or semi-simple or the stabilizer of a pseudo-torus. Therefore, all maximal subalgebras of \( \mathfrak{su}_2,1 \) up to conjugation are described as follows: (i) one parabolic subalgebra, of dimension 5; (ii) one semi-simple subalgebra, namely \( \mathfrak{so}_{2,1} \), of dimension 3; (iii) two pseudotoric subalgebras, namely \( \mathfrak{u}_2 \) and \( \mathfrak{u}_{1,1} \), both of dimension 4. In particular, \( \mathfrak{su}_2,1 \) has no subalgebras of dimension 6 and 7 as claimed.

Thus, we have \( \mathfrak{hol}(M, p) = \mathfrak{su}_2,1 \). Consider the isotropy subalgebra \( \mathfrak{hol}_0(M, p) \subset \mathfrak{hol}(M, p) \), which consists of all vector fields in \( \mathfrak{hol}(M, p) \) vanishing at \( p \). Clearly, \( \dim \mathfrak{hol}_0(M, p) \geq 5 \), and we obtain, again by the nonexistence of codimension one and two subalgebras in \( \mathfrak{su}(2,1) \), that either \( \dim \mathfrak{hol}_0(M, p) = 5 \) or \( \dim \mathfrak{hol}_0(M, p) = 8 \). In the former case, it follows that the orbit of \( p \) under the corresponding local action of \( SU(2,1) \) is open. Since \( M \) is spherical at every point \( p' \in U \cap S \), we then see that \( M \) is spherical at \( p \) as required.

Suppose now that \( \dim \mathfrak{hol}_0(M, p) = 8 \), i.e., \( \mathfrak{hol}_0(M, p) = \mathfrak{su}_2,1 \). As shown in [GS] (see pp. 113–115 therein), an action of a semisimple Lie algebra \( \mathfrak{g} \) by real-analytic vector fields on a real-analytic manifold \( X \) can be linearized near a fixed point \( x \), i.e., there exist local coordinates in a neighborhood of \( x \) on \( X \) in which all vector fields arising from \( \mathfrak{g} \) are linear. It then follows that \( \mathfrak{su}_2,1 \) has a nontrivial real 3-dimensional representation. On the other hand, it is easy to see that no such representation exists. Indeed, assuming the contrary and complexifying, we obtain a complex 3-dimensional representation of \( \mathfrak{sl}_3(\mathbb{C}) \). Up to isomorphism, this is the standard (defining) representation, hence the standard action of \( \mathfrak{su}_2,1 \) on \( \mathbb{C}^3 \) must have an invariant totally real 3-dimensional subspace, and it is straightforward to
verify that no such subspace in fact exists. This contradiction completes the proof of the theorem.

\[ \square \]

Remark 2.1. The argument contained in the last paragraph of the above proof provides a short way of answering the question asked in the title of article [B4].

Next, to prove Corollary 1.3, we only need to observe that each of the integers 0, 1, 2, 3, 4, 5 is realizable as \( \text{dim} \; \mathfrak{hol}(M, p) \). The realizability of 0, 2, 3, 4, 5 follows from the examples given in [B4, p. 143], [KL, Table 1], [St], so it only remains to find an example with \( \text{dim} \; \mathfrak{hol}(M, p) = 1 \). Consider the hypersurface \( \Gamma_1 \) given in coordinates \( z, w \) in \( \mathbb{C}^2 \) by the equation

\[ \text{Im} \; w = |z|^2 + (\text{Re} \; z^2)|z|^2. \]

By Theorem 3 of [B1], the stability group of \( \Gamma_1 \) at the origin consists only of the transformations \( z \mapsto \pm z, w \mapsto w \), hence \( \mathfrak{hol}_0(\Gamma_1, 0) = 0 \). One can further show (e.g., by Maple-assisted computations) that \( \mathfrak{hol}(\Gamma_1, 0) \) is spanned by the vector field \( \partial/\partial w + \partial/\partial \overline{w} \). Another example is given by the hypersurface \( \Gamma_2 \) defined as

\[ \text{Im} \; w = |z|^2 + (\text{Re} \; w)|z|^2. \]

In this case, the stability group at the origin consists of all rotations in \( z \) (see, e.g., [EzI, p. 1159]), and one can further Show that every element of \( \mathfrak{hol}(\Gamma_2, 0) \) vanishes at the origin. Hence, \( \mathfrak{hol}(\Gamma_2, 0) \) is spanned by \( iz\partial/\partial z - i\overline{z}\partial/\partial \overline{z} \). One can produce many more examples of this kind by considering hypersurfaces of the form \( \text{Im} \; w = f(|z|^2, \text{Re} \; w) \), where \( f \) is real-analytic and in general position.

\[ \square \]

Remark 2.2. As we noted in the proof of Theorem 1.2, \( \mathfrak{su}_{2,1} \) has only one, up to conjugation, 5-dimensional subalgebra (which is parabolic), and this is exactly the subalgebra that occurs in the examples with \( \text{dim} \; \mathfrak{hol}(M, p) = 5 \) given in [B4], [KL]. In all these cases, one has \( \mathfrak{hol}(M, p) = \mathfrak{hol}_0(M, p) \). Explicitly classifying the manifolds with \( \text{dim} \; \mathfrak{hol}(M, p) = 5 \) requires a much greater effort, and article [KS] makes progress in this direction by showing that every such manifold has to be a “sphere blowup” (as defined above the statement of Theorem 3.10 therein).

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