ON LOCAL RESIDUAL FINITENESS OF ABSTRACT COMMENSURATORS OF FUCHSIAN GROUPS

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Abstract. Let $\Gamma$ be a finite-covolume Fuchsian group and Comm($\Gamma$) be its abstract commensurator. Then Comm($\Gamma$) contains the solvable Baumslag-Solitar groups $\langle a, b : aba^{-1} = b^n \rangle$ for any $n > 1$. Moreover, the Baumslag-Solitar group $\langle a, b : ab^2 a^{-1} = b^3 \rangle$ has an image in Comm($\Gamma$) that is not residually finite. It follows that Comm($\Gamma$) is not locally residually finite, and hence is not an inverse limit of locally residually finite groups (e.g., linear groups, mapping class groups of hyperbolic surfaces of finite type, or branch groups). This completes the picture of local residual finiteness of abstract commensurators of irreducible lattices in semisimple Lie groups, formed by Armand Borel, Gregory Margulis, G. D. Mostow, and Gopal Prasad. Our proofs are computer-assisted.

1. Introduction

Let $G$ be a group. The abstract commensurator of $G$, denoted Comm($G$), is the set of equivalence classes of isomorphisms $\phi : H_1 \to H_2$ for finite-index subgroups $H_1, H_2 \leq G$, where two isomorphisms are equivalent if they are both defined and equal on a common finite-index subgroup of $G$. Elements of Comm($G$) are called commensurators of $G$. A group is residually finite if the intersection of all finite-index subgroups is trivial. For a property $\mathcal{P}$ of groups, a group is locally $\mathcal{P}$ if every finitely generated subgroup has $\mathcal{P}$. Linear groups and mapping class groups of compact orientable surfaces are important classes of locally residually finite groups [Mal56, Gro75].

Is the abstract commensurator of the fundamental group of a hyperbolic surface of finite type (locally) residually finite? This folklore question is as old as the notion of abstract commensurators, and we answer it here. Questions about abstract commensurators of surface groups have been asked for at least 30 years [Man87].

Gilbert Baumslag showed that for residually finite $G$, the group Aut($G$) is residually finite [Bau63]. Chris Odden showed that the abstract commensurator of a closed surface of genus two is a mapping class group of the “universal hyperbolic solenoid” [Odd05]. These facts suggest that these abstract commensurators can perhaps be locally residually finite. Nevertheless, we show that the answer to the above question is “no”. To do this, we exhibit an algorithm for producing many images in Comm($G$) of a class of one-relator groups.

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1.1. The main result. The Baumslag-Solitar groups are given by the presentation
$$BS(m, n) := \langle a, b : ab^m a^{-1} = b^n \rangle.$$ 

A Fuchsian group is a discrete subgroup of the Lie group of Möbius transformations of the unit disk. Note that any fundamental group of a hyperbolic surface of finite type is a finite-covolume Fuchsian group. Moreover, any finite-covolume Fuchsian group is finitely generated.

**Theorem 1.** Let $\Gamma$ be a finite-covolume Fuchsian group, and let $n > 1$. Then the group $\text{Comm}(\Gamma)$ contains $BS(1, n)$. Moreover, $\text{Comm}(\Gamma)$ contains a non-residually finite image of $BS(2, 3)$.

Our proof of Theorem 1 appears in §2 with supplementary code appearing in the appendix. Since the abstract commensurator of a group $G$ is an invariant of the commensurability class, our theorem is a statement about the abstract commensurators of two groups: the nonabelian free group of rank two, and the fundamental group of a compact surface of genus two. Constructing commensurators of a free group is easier than constructing commensurators of a surface group. The structure of our proof reflects this: we prove the nonabelian free group case first, and then build upon it to handle the cocompact case. The first part of each case uses some of the elementary theory of hyperbolic surfaces. The second part of each case is computer-assisted, using the GAP System for Computational Discrete Algebra [GAP17].

1.2. On local residual finiteness. As an immediate application of our main theorem, we get the following corollary.

**Corollary 2.** Let $\Gamma$ be a finite-covolume Fuchsian group. The group $\text{Comm}(\Gamma)$ is not locally residually finite. □

This corollary completes the picture on local residual finiteness of abstract commensurators of irreducible lattices in semisimple Lie groups. Finite-covolume Fuchsian groups are precisely lattices in the semisimple Lie group $\text{PSL}_2(\mathbb{R})$. For all semisimple Lie groups $G$ without compact factors and not isogenous to $\text{PSL}_2(\mathbb{R})$, and all irreducible lattices $L$ in $G$, strong rigidity results of G. D. Mostow, Gopal Prasad, and Gregory Margulis, combined with work of Margulis [Mar91], show that either $[\text{Comm}(L) : L] < \infty$ or $L$ is an arithmetic group. In the latter case, a theorem of Armand Borel shows $\text{Comm}(L)$ is commensurable with the $\mathbb{Q}$-points of an algebraic group. In either case, $\text{Comm}(L)$ is linear and thus locally residually finite by Malčev’s Theorem [Mal71]. Moreover, the group $\text{Comm}(N)$ is locally residually finite for $N$ any lattice in a solvable real Lie group (e.g., any finitely generated nilpotent group). Indeed, in this case $\text{Comm}(N)$ is linear [Stu15] and so is locally residually finite.

1.3. On linearity and inverse limits. We recall the definition of an inverse limit of groups. Let $(I, \leq)$ be a directed poset. Let $(A_i)_{i \in I}$ be a family of groups and $f_{ij} : A_j \to A_i$ for $i \leq j$ be homomorphisms such that

1. $f_{ii}$ is the identity on $A_i$,
2. $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

Then the **inverse limit** of the inverse system $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ is the subgroup of $\prod_{i \in I} A_i$ given by $\{ \bar{a} \in \prod_{i \in I} A_i : a_i = f_{ij}(a_j) \text{ for all } i \leq j \text{ in } I \}$. 
As a consequence of Corollary 2, these abstract commensurators are not linear. Moreover they cannot be constructed as inverse limits of linear groups, as any subgroup of a direct product of locally residually finite groups is locally residually finite.

**Corollary 3.** Let $\Gamma$ be a finite-covolume Fuchsian group. The group $\text{Comm}(\Gamma)$ is not an inverse limit of locally residually finite groups. □

This corollary is startling in the cocompact case, as Odden showed that the abstract commensurator of a cocompact Fuchsian group is a mapping class group of an inverse limit of surfaces [Odd05].

1.4. **Two non-isomorphic groups.** Let $F_2$ be the nonabelian free group of rank two. Let $\Gamma_2$ be the fundamental group of a compact surface of genus two. The groups $\text{Comm}(F_2)$ and $\text{Comm}(\Gamma_2)$ share a number of properties: L. Bartholdi and O. Bogopolski showed that neither group is finitely generated [BB10]. Moreover, finitely generated subgroups of them have solvable word problem (see §1.6). The proof of Theorem 1 shows that they both contain many infinite images of Baumslag-Solitar groups. In spite of these facts, the group $\text{Comm}(F_2)$ contains an infinite collection of groups not in $\text{Comm}(\Gamma_2)$. Indeed, $\text{Comm}(F_2)$ contains every finite group. However, Odden [Odd05 Proposition 4.5] shows that $\text{Comm}(\Gamma_2)$ acts faithfully on $S^1$ by homeomorphisms. And since $[\text{Homeo}(S^1) : \text{Homeo}_+(S^1)] = 2$ and all finite subgroups of $\text{Homeo}_+(S^1)$ are cyclic [BST18 Lemma 3.1], it follows that every finite subgroup of $\text{Comm}(\Gamma_2)$ contains a cyclic subgroup of index at most 2. In particular, $\text{Comm}(F_2)$ and $\text{Comm}(\Gamma_2)$ are not isomorphic.

1.5. **On residual finiteness of $BS(2,3)$.** Our proof explicitly identifies an element $\gamma \in BS(2,3)$ in the kernel of a surjective homomorphism $BS(2,3) \to BS(2,3)$, then concretely shows that $\gamma$ has nontrivial image under a homomorphism $BS(2,3) \to \text{Comm}(\Gamma)$. Thus, our methods prove that $BS(2,3)$ is not residually finite without using the normal forms ensured by the lemma of John L. Britton [Bri63]. Bypassing this step makes the original proof that $BS(2,3)$ is not residually finite [BS62], due to Baumslag and Donald Solitar, elementary and concrete.

1.6. **The word problem.** For finite-covolume Fuchsian groups, finitely generated subgroups of $\text{Comm}(\Gamma)$ have solvable word problem. Loosely speaking, words in $\text{Comm}(\Gamma)$ can be evaluated by a computer, just as words in $\text{GL}_n(\mathbb{Q})$ can be evaluated via matrix computations. This is especially useful for finitely generated subgroups of $\text{Comm}(\Gamma)$ that are not residually finite, as any such subgroup cannot be completely understood through its representations to $\text{GL}_n(\mathbb{Q})$. Please see §3 for further discussion and questions.

1.7. **The Burger-Mozes groups.** That the abstract commensurator of a free group of rank two is not locally residually finite is a well-known folklore result. The folklore proof relies on the existence of a simple, finitely presented group that is isomorphic to an amalgamated product of two finitely generated free groups over a common finite-index subgroup. The first groups of this kind were constructed by Marc Burger and Shahar Mozes [BM97]. This gives an alternative proof of Corollaries 2 and 3 in the noncompact case. It is unknown whether any of the Burger-Mozes groups embed inside the abstract commensurator of a closed oriented surface group of genus two. We conjecture that none of them embed.
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2. Proof of Theorem 1

Let $F_2$ be the free group of rank two. Let $Σ_g$ be the closed oriented surface of genus $g$. Let $Γ$ be a finite-covolume Fuchsian group. As stated in §1.1 and §1.4 the group $\text{Comm}(Γ)$ is isomorphic to either $\text{Comm}(F_2)$ or $\text{Comm}(π_1(Σ_2, *))$. We handle each case separately, as they are different enough to warrant different proofs (the first serving as an outline of the second).

It will be important to us in each case that $Γ$ satisfies the unique root property: if elements $x, y \in Γ$ satisfy $x^n = y^n$ for some $n \neq 0$, then $x = y$ [BB10, Lemma 2.2], [BH99, pages 462–463]. A consequence of the unique root property is that two isomorphisms $f : A \to B$ and $g : C \to D$ between finite-index subgroups of $Γ$ represent the same element of $\text{Comm}(Γ)$ if and only if $f|_{A∩C} = g|_{A∩C}$.

In particular, to check that $f$ represents a nontrivial element of $\text{Comm}(Γ)$, it suffices to find an element $γ ∈ Γ$ such that $f(γ) \neq γ$.

2.1. The $\text{Comm}(F_2)$ case. We first describe a method for constructing images of $\text{BS}(n, m)$ in $\text{Comm}(F_2)$. Set $F_2 = \langle A, B \rangle$. Let $π_1 : F_2 → \mathbb{Z}/m × \mathbb{Z}/n$ be the map given by $A → (1, 0)$ and $B → (0, 1)$. Let $π_2 : F_2 → \mathbb{Z}/m × \mathbb{Z}/n$ be the map given by $A → (0, 1)$ and $B → (1, 0)$. Set $Δ_1 = \ker(π_1)$ and $Δ_2 = \ker(π_2)$.

Let $φ$ be the commensurator with representative $f : F_2 → F_2$ given by $f(γ) = AγA^{-1}$. Let $ψ$ be a commensurator with representative $g : Δ_1 → Δ_2$, where $g(A^m) = A^n$; such an isomorphism $g$ exists because $Δ_1$ and $Δ_2$ are free groups of the same rank in which $A^m$ and $A^n$, respectively, are elements in free generating sets. Then the commensurator $ψ ◦ φ^m ◦ ψ^{-1}$ has a representative $h = g ◦ f^m ◦ g^{-1}$ such that for every $γ ∈ Δ_2$,

$$h(γ) = g ◦ f^m ◦ g^{-1}(γ) = g(A^m g^{-1}(γ)A^{-m}) = A^n(g ◦ g^{-1}(γ))A^{-n} = A^n γ A^{-n},$$

and hence $ψ ◦ φ^m ◦ ψ^{-1} = φ^n$. Thus, the assignment $a → ψ$ and $b → φ$ defines a homomorphism $Φ : BS(m, n) → \text{Comm}(F_2)$.

Next, we show that the map $BS(m, n) → (ψ, φ)$ is an isomorphism of groups when $m = 1$. Let $z$ be in the kernel of this map. Then $z = a^s γ$, where $γ$ is in the normal closure of $b$. Since $⟨⟨b⟩⟩ = ⟨a^t b a^{-t} : t ≤ 0⟩$, we have that $z$ is conjugate to an element of the form $a^s b^t$ for some integers $s$ and $t$. It suffices to show that $ψ^s ◦ φ^t$ is only trivial in $\text{Comm}(F_2)$ if $s = t = 0$.

If $s > 0$ then any $ψ^s$ has a representative that maps $A$ to $A^{n^s}$. By the unique root property of $F_2$, it is impossible for an automorphism of $F_2$ to send $A$ to $A^{n^s}$. Similarly, if $s < 0$ then $ψ^s$ has a representative that maps $A^{−n^s}$ to $A$, which cannot be extended to an automorphism of $F_2$. If $ψ^s ◦ φ^t$ is trivial, then $ψ^s$ has a representative that is an automorphism of $F_2$, and so $s = 0$. It is clear that $φ^t$ is trivial only if $t = 0$ because $f^t(B^ℓ) = A^t B^ℓ A^{-t}$ for all $ℓ$. This complete the proof of the $m = 1$ case.

To finish, we need to show that when $m = 2$ and $n = 3$, the image of $BS(m, n)$ as defined above is not residually finite. To do this, we need to show that element

$$b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}$$
has nontrivial image. Indeed, it is in the kernel of the surjective map $BS(2, 3) \to BS(2, 3)$ given by $a \mapsto a$ and $b \mapsto b^2$, and thus is in the residual finiteness kernel of $BS(2, 3)$.

The rest of the proof is computer-assisted. Please see the code in Appendix B and explanations there to end the proof of the Comm($\pi_1(F_2)$) case.

2.2. The Comm($\pi_1(\Sigma_2, *)$) case. As in §2.1, we begin by describing how to obtain images of $BS(n, m)$ inside Comm($\pi_1(\Sigma_2, *)$). Let $[X, Y] := XYX^{-1}Y^{-1}$ and let $\Gamma := \pi_1(\Sigma_2, *) = \langle A, B, C, D : [A, B][C, D]\rangle$.

Set $\pi_1 : \Gamma \to \mathbb{Z}/m \times \mathbb{Z}/n$ be a map satisfying $A \mapsto (1, 0)$, where in the cover corresponding to $\ker \pi_1$ the curve corresponding to $A^m$ lifts to a non-separating simple closed curve. See Figure 1 for the cover corresponding to an example of such a map. Similarly, set $\pi_2 : \Gamma \to \mathbb{Z}/m \times \mathbb{Z}/n$ be a map satisfying $A \mapsto (0, 1)$, where in the cover corresponding to $\ker \pi_1$ the curve corresponding to $A^n$ lifts to a non-separating simple closed curve. Set $\Delta_1 = \ker(\pi_1)$ and $\Delta_2 = \ker(\pi_2)$.

Let $\phi$ be the commensurator with representative $f : \Gamma \to \Gamma$ given by $f(\gamma) = A\gamma A^{-1}$. To define the commensurator $\psi$ we need some additional setup:

Let $p_1 : S_1 \to \Sigma_2$ and $p_2 : S_2 \to \Sigma_2$ be the covers corresponding to $\Delta_1$ and $\Delta_2$ respectively. Then $A^m$ and $A^n$ lift to non-separating simple-closed curves in $S_1$ and $S_2$ by construction. Any oriented surface group is uniquely determined by its Euler characteristic. Thus, by the Riemann–Hurwitz formula and the classification of surfaces $S_1$ and $S_2$ are homeomorphic. Moreover, by the Change of Coordinates Principle [FM12, p. 37] there is a homeomorphism $S_1 \to S_2$ inducing an isomorphism $g : \Delta_1 \to \Delta_2$, where $g(A^m) = A^n$. Let $\psi$ be the commensurator with representative $g$. Then the commensurator $\psi \circ \phi^n \circ \psi^{-1}$ has a representative $h = g \circ f^n \circ g^{-1}$ such that for every $\gamma \in \Delta_2$,

$$h(\gamma) = g \circ f^m \circ g^{-1}(\gamma) = g(A^m g^{-1}(\gamma) A^{-m}) = A^n (g \circ g^{-1}(\gamma)) A^{-n} = A^n \gamma A^{-n},$$

and hence $\psi \circ \phi^m \circ \psi^{-1} = \phi^n$. Thus, the assignment $a \mapsto \psi$ and $b \mapsto \phi$ defines a homomorphism $\Phi : BS(n, m) \to \text{Comm}(\Gamma)$.

\footnote{See Appendix C for an explicit construction of such a map in the case $(m, n) = (2, 3)$.}
Since $\Gamma$ has the unique root property, the argument for showing that the map $\text{BS}(m, n) \to \langle \psi, \phi \rangle$ is an isomorphism when $m = 1$ in §2.1 applies here verbatim. Hence, it only remains to show that when $m = 2$ and $n = 3$ the element
\[ b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1} \]
has nontrivial image, since this is in the residual finiteness kernel of $\text{BS}(m, n)$. The rest of the proof, as before, is computer-assisted. The computer computations are more difficult to implement here as surface groups do not have as much flexibility as free groups. Please see Appendix C and explanations there to end the proof. □

3. Parting thoughts and questions

3.1. If a finitely generated group $\Gamma$ has solvable word problem, then finitely generated subgroups of $\text{Aut}(\Gamma)$ have solvable word problem. A similarly straightforward proof shows that the same is true of $\text{Comm}(\Gamma)$ when $\Gamma$ is finitely presented and satisfies the unique root property. In particular, finitely generated subgroups of abstract commensurators of finite-covolume Fuchsian groups have solvable word problem.

Proposition 4. Let $\Gamma$ be a finitely presented group with solvable word problem and the unique root property. Then any finitely generated subgroup of $\text{Comm}(\Gamma)$ has solvable word problem. □

3.2. The above proposition motivates the following questions:

1. Which groups $\text{Comm}(\Gamma)$ from Proposition 4 contain a copy of a non-residually finite Baumslag-Solitar group?
2. Does there exist a torsion-free group $G$ that embeds inside $\text{Comm}(F_2)$ but not inside $\text{Comm}(\pi_1(\Sigma_2, *))$? Vice versa?
3. Let $\Gamma$ be a finite-covolume Fuchsian group. Do finitely generated subgroups of $\text{Comm}(\Gamma)$ satisfy a Tits’ alternative?

Note that a partial answer to Question 3 is given in the cocompact Fuchsian group case by [Mar00], using the fact that $\text{Comm}(\Gamma)$ embeds in $\text{Homeo}(S^1)$ [Odd05].

Appendix A. The GAP System

In the appendices that follow we use the GAP System to complete our proofs. Borrowing words from the creators [GAP17]: “GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects.” GAP is especially well-suited for our needs, with key functions that we will use to explicitly evaluate commensurators. Here is a short glossary of some of the key functions used in our code:

**GroupHomomorphismByImages** (domain, codomain, list1, list2). Inputs two groups “domain” and “codomain” with lists “list1” and “list2”. When GAP runs this command it first verifies that a well-defined homomorphism from “domain” to “codomain” sending “list1” to “list2” exists, returning “fail” otherwise. If the homomorphism requested is well-defined, this returns a homomorphism from domain to codomain where elements of “list1” consisting of generators of domain are sent to corresponding elements in “list2”.

6 KHALID BOU-RABEE AND DANIEL STUDENMUND
**Image**(map, elem). Inputs a homomorphism “map” and an element from the domain “elem”. Returns the image of element “elem” under an application of the homomorphism “map”.

**InverseGeneralMapping**(map). Inputs a homomorphism “map”. This function returns an inverse of an isomorphism of groups where the domain and codomain are not equal.

**IsomorphismSimplifiedFpGroup**(G). Inputs a finitely presented group “G”, for which GAP applies Tietze transformations to a copy in order to reduce it with respect to the number of generators, the number of relators, and the relator lengths. When we apply this to a finite-index subgroup of an oriented cocompact surface group we get a one-relator group, as expected. This function returns an isomorphism with domain G, codomain a group H isomorphic to G, so that the presentation of H has been simplified using Tietze transformations.

**IsOne**(elem). Inputs an element “elem” from a group. Returns true if “elem” is equal to the identity, and false if “elem” is not. This function is not guaranteed to terminate.

**Appendix B. Computer-assistance for the free group case**

The code in §B.1 concretely defines maps \( \phi \) and \( \psi \) from the proof of the free group case. Here K1 and K2 are the subgroups \( \Delta_1 \) and \( \Delta_2 \), and K1.1 corresponds to the element \( A^{-2} \) in the free group \( \langle A, B \rangle \).

Running this code outputs “false”. The GAP code verifies that each isomorphism exists and that the domains and ranges are all appropriate. In the end the GAP code computes the function

\[ w = \phi^{-1} \psi \phi^{-1} \psi^{-1} \psi \phi^{-1} \psi^{-1} \]

with input Word = \( BAB^{-1}A^{-1} \). It outputs Word10 = \( A^3BAB^{-1}A^2 \), and verifies that these are not equal (although this last part is easily done by hand).

After running the code below one can investigate properties of the map \( \psi \). For instance, here \( \psi \) is evaluated on \( A^{-2} \):

```
gap> K1.1;  
A^-2  
gap> Image(psi, K1.1);  
A^-3  
```

The variable K1.1, corresponding to the element \( A^{-2} \) in \( F_2 \), is shown here as being mapped to \( A^{-3} \).

Moreover, one can verify that \( \psi \) is an isomorphism \( K1 \rightarrow K2 \) in GAP by checking that the map is surjective and that \( K1 \) and \( K2 \) have the same index in \( f \):

```
gap> Image(psi, K1) = K2;  
true  
gap> Index(f, K1) = Index(f, K2);  
true  
```

**B.1. The code.** We ran the following code on GAP version 4.8.8. The usefulness of the code below rests on the identifications of the generator K1.1 with \( A^{-2} \) and the generator K2.2 with \( A^3 \). Be aware that other versions of GAP may have different
implementations of the function Kernel, in which \( A^2 \) and \( A^3 \) may correspond to different generators, or may not even be part of the chosen free generating set.

```gap
# Define the groups
f := FreeGroup("A", "B");;
A := DirectProduct(CyclicGroup(2), CyclicGroup(3));;

# Define conjugation map, phi:
phi := GroupHomomorphismByImages( f, f, [f.1, f.2, [f.1, f.1*f.2*f.1^(-1)]]);
phi2 := Inverse(phi);

# Define the projection maps pi1 and pi2:
pi1 := GroupHomomorphismByImages( f, A, [f.1, f.2], [A.1, A.2]);;
pi2 := GroupHomomorphismByImages( f, A, [f.1, f.2], [A.2, A.1]);;

# Running Rank ensures K1 and K2 are equipped with finite presentations
K1 := Kernel(pi1);; Rank(K1);
K2 := Kernel(pi2);; Rank(K2);

# Define the map psi, there is a great deal of flexibility here, except that
# we want K1.1 \( (A^{-2}) \) to map to K2.2^(-1) \( (A^{-3}) \).
psi := GroupHomomorphismByImages( K1, K2, [K1.1, K1.2, K1.3, K1.4, K1.5, K1.6, K1.7], [K2.2^(-1), K2.3, K2.4, K2.5, K2.6, K2.7]);;
psi2 := InverseGeneralMapping(psi);

# Evaluate the word w in the residual finiteness kernel of BS(2,3):
Word := K1.2;; Word2 := Image(phi2, Word);
Word3 := Image(psi2, Word2);; Word4 := Image(phi, Word3);
Word5 := Image(psi, Word4);; Word6 := Image(phi2, Word5);
Word7 := Image(psi2, Word6);; Word8 := Image(phi, Word7);
Word9 := Image(psi, Word8);; Word10 := Image(phi2, Word9);

# Check to see if Word10 == Word. Outputs false, as desired:
IsOne(Word10*Word^(-1));
```

**Appendix C. Computer-assistance for the cocompact case**

The code in §C.1 concretely defines maps \( \phi \) and \( \psi \) from the proof of the co-
compact Fuchsian case. Here K1 and K2 are the subgroups \( \Delta_1 \) and \( \Delta_2 \), and K1.1 corresponds to the element \( A^{-2} \) in the surface group \( g = \langle A, B, C, D : [A, B][C, D] \rangle \).

In this case, more care must be taken in defining the map \( \psi : K1 \rightarrow K2 \) because
K1 and K2 are not free groups. Explicitly defining maps from K1 to K2 using the
GAP function GroupHomomorphismByImages usually results in maps that GAP
cannot verify are well-defined isomorphisms (moreover, finding such maps from
scratch is prohibitively difficult). To get around this obstruction, we first simplify
the presentations of $K_1$ and $K_2$ before finding an isomorphism (akin to diagonalizing a matrix before doing computations).

We briefly explain the construction of $\psi$ in the code now. First, the code defines group homomorphisms $\text{Iso}_1 : K_1 \to \text{fp}_1$, $\text{Hom}_1 : \text{fp}_1 \to \text{Image}_1$, $\text{Iso}_2 : K_1 \to \text{fp}_2$, and $\text{Hom}_2 : \text{fp}_2 \to \text{Image}_2$. GAP does not view $K_1$ and $K_2$ as finitely presented groups, and so the maps $\text{Iso}_1$ and $\text{Iso}_2$ are simply isomorphisms to their finite presentations (computed from the fact that $K_1$ and $K_2$ are finite-index normal subgroups). The maps $\text{Hom}_1$ and $\text{Hom}_2$ are maps to simplified finite presentations. The resulting maps from this construction are version dependent, so running our code in a different version of GAP may result in lastMap not being well-defined (in which case GAP will throw a fault). The images of the maps $\text{Hom}_1$ and $\text{Hom}_2$ have the following presentations. The red color indicates a part of the relation that is significantly different from the rest.

\begin{align*}
\text{Image}_1 &= \langle F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14} : \\
& \quad [F_{11}, F_{12}][F_4, F_5][F_1^{-1}F_6F_1][F_{13}, F_{14}][F_9, F_{10}][F_6^{-1}F_7, F_8][F_2, F_3] = 1 \rangle, \\
\text{Image}_2 &= \langle F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14} : \\
& \quad [F_5, F_6][F_4][F_{13}, F_{14}][F_7, F_8][F_1^{-1}F_4^{-1}F_1][F_3, F_{10}][F_2, F_3][F_{11}, F_{12}] = 1 \rangle.
\end{align*}

The image of $A^{-2}$ under $\text{Hom}_1 \circ \text{Iso}_1$ is $F_1$ in Image1. Moreover, $F_4$ maps to $A^3$ under $(\text{Hom}_2 \circ \text{Iso}_2)^{-1}$. Then lastMap : Image1 $\to$ Image2 is defined to be an isomorphism taking $F_1$ to $F_4$. The resulting map $\psi : K_1 \to K_2$ is then defined by the composition

$$
\psi := \text{flipHom} \circ \text{Iso}_2^{-1} \circ \text{lastMap} \circ \text{Hom}_1 \circ \text{Iso}_1,
$$

where flipHom sends $A$ to $A^{-1}$, correcting that lastMap sends $A^{-2}$ to $A^3$.

After verifying that the map $\psi$ is well-defined, the computer uses the finite presentation to verify that the input and output of the function

$$
W = \phi^{-1} \psi \phi^{-1} \psi^{-1} \phi^{-1}
$$

are not identical when evaluated at $K_1.2 = C$ in $g$ (one can also do this last check by hand using Dehn’s algorithm). The output to this evaluation, Word10, is:

$$
A^{-1}B^{-2}A^{-1}B^{-2}(A^{-1}C^2D^2C^{-1}D^{-1}A^2B^2A^{-2}A^{-2}B^2A^{-3}B^2A^{-1}D^2C^2D^{-2}C^{-2}A^2B^{-2}A^{-2}B^{-2}A^{-2}B^2A^{-2}A^{-1}D^{-3}A^{-1}B^{-2}A^{-2}B^{-2}A^{-2}B^{-2}A^{-2}D^2C^2D^{-2}C^{-2}A^2B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}A^{-1}B^{-2}A^{-2}D^{-3}
$$

As before, $K_1.1$ corresponds to $A^{-2}$ and applying $\psi$ to it yields $A^{-3}$, as desired:

```
gap> K1.1;  
A^&-2  
gap> Image(\psi, K1.1);  
A^&-3
```

Moreover, one can verify that $\psi$ is an isomorphism $K_1 \to K_2$ in GAP by checking that the map is surjective and that $K_1$ and $K_2$ have the same index in $g$:

```
gap> Image(\psi, K1) = K2;  
true  
gap> Index(g, K1) = Index(g, K2);  
true
```
C.1. **The code.** We ran the following code on GAP version 4.8.8. Note again that different versions of GAP may have different implementations of key functions, such as IsomorphismFpGroup and IsomorphismSimplifiedFpGroup, which are used below to find explicit presentations of $\Delta_1$ and $\Delta_2$. Because the definition of $\psi$ uses the presentations output by these functions, the definition below may not determine an isomorphism in other versions of GAP.

```gap
# Define the groups:
f := FreeGroup( "A", "B", "C", "D" );;
comm := function (a,b) return a*b*a^(-1)*b^(-1);; end;;
g := f / [ comm(f.1,f.2)*comm(f.3,f.4) ];;
A := DirectProduct(CyclicGroup(2), CyclicGroup(3));;

# Setup for defining the map psi.
# Note that g.1, g.2, g.3, g.4 correspond to the images of A, B, C, D in f.
P1 := GroupHomomorphismByImages ( g, A, [g.1,g.2,g.3,g.4], [A.1, A.2, A.1^0, A.1^0] );;
P2 := GroupHomomorphismByImages ( g, A, [g.1,g.2,g.3,g.4], [A.2, A.1, A.1^0, A.1^0] );;
K1:= Kernel(P1);
K2:= Kernel(P2);

# Simplify the presentations of K1 and K2, while keeping track of
# the maps from K1 and K2 to their reduced presentations
Iso1 := IsomorphismFpGroup( K1 );;
fp1 := Image(Iso1);
Hom1 := IsomorphismSimplifiedFpGroup(fp1);
Image1 := Image(Hom1);

Iso2 := IsomorphismFpGroup( K2 );;
fp2 := Image(Iso2);
Hom2 := IsomorphismSimplifiedFpGroup(fp2);
Image2 := Image(Hom2);

# To define the map psi from K1 to K2, first define a map, lastMap,
# between the reduced presentations Image1 and Image2 of K1 and K2,
# respectively.

# Conjugation map for defining lastMap
conj := ConjugatorAutomorphism(Image1, Image1.1^(-1)*Image1.6);;
apc := function (a) return Image(conj, a);; end;;

# A map to swap g.1 with g.1^(-1) so we get an image of BS(2,3)
# and not BS(-2,3)
flipHom := GroupHomomorphismByImages( g, g, [g.1, g.2, g.3, g.4],
```
[g.1^(-1), g.2^(-1), g.2^(-1)*g.1^(-1)*g.3*g.1*g.2, 
g.2^(-1)*g.1^(-1)*g.4*g.1*g.2];;

# This defines an isomorphism between reduced presentations of 
# K1 and K2.
lastMap := GroupHomomorphismByImages ( Image1, Image2, 
[Image1.1,apc(Image1.2),apc(Image1.3),apc(Image1.4),apc(Image1.5), 
Image1.6,apc(Image1.7),apc(Image1.8),Image1.9,Image1.10, 
apc(Image1.11), apc(Image1.12),Image1.13,Image1.14, 
Image2.4,Image2.2,Image2.3,Image2.5,Image2.6,Image2.1, 
Image2.9,Image2.10,Image2.7,Image2.8,Image2.11,Image2.12, 
Image2.13,Image2.14]);;

# Define the map psi:
psi := CompositionMapping(flipHom, InverseGeneralMapping(Iso2), 
InverseGeneralMapping(Hom2), lastMap, Hom1, Iso1);
psi2 := Inverse(psi);

# Define conjugation map:
phi := GroupHomomorphismByImages ( g, g, [g.1, g.2, g.3, g.4], 
[g.1, g.1*g.2*g.1^(-1), g.1*g.3*g.1^(-1), g.1*g.4*g.1^(-1)]);
phi2 := Inverse(phi);

# Evaluate the word w in the residual finiteness kernel of BS(2,3):
Word := K1.2;; Word2 := Image(phi2, Word);
Word3 := Image(psi2, Word2);; Word4 := Image(phi, Word3);
Word5 := Image(psi, Word4);; Word6 := Image(phi2, Word5);
Word7 := Image(psi2, Word6);; Word8 := Image(phi, Word7);
Word9 := Image(psi, Word8);; Word10 := Image(phi2, Word9);

# Check to see if Word10 == Word. Outputs false, as desired:
IsOne(Word10*Word^(-1));

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