Statefinders and observational measurement of superenergy

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The superenergy of the universe is a tensorial quantity and it is a general relativistic analogue of the Appell’s energy of acceleration in classical mechanics. We propose the measurement of this quantity by the observational parameters such as the Hubble parameter, the deceleration parameter, the jerk and the snap (kerk) known as statefinders. We show that the superenergy of gravity requires only the Hubble and deceleration parameter to be measured, while the superenergy of matter requires also the measurement of the higher-order characteristics of expansion: the jerk and the snap. In such a way, the superenergy becomes another parameter characterizing the evolution of the universe.

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I. INTRODUCTION

It is widely known that there exists a problem of the energy-momentum of gravitational field in general relativity. Since the gravitational field may locally vanish, then one is always able to find a frame in which the energy-momentum of the gravitational field vanishes, while it does not necessarily vanish in the other frames. The physical quantities which describe gravitational field in such a coordinate-dependent way are called the gravitational field pseudotensors, or, if the matter energy-momentum tensors are added, the gravitational field complexes. The choice of a gravitational field pseudotensor is not unique so that various definitions of the pseudotensors have been proposed. However, the arbitrariness of the choice of pseudotensors inspired many authors to define quantities which describe the energy-momentum content of the gravitational field in a tensorial way. The two types of quantities have been suggested: the gravitational superenergy tensors and the gravitational superenergy tensors. In particular, the canonical superenergy tensors have successfully been calculated for the plane, the plane-fronted and cylindrical gravitational waves as well as for Friedmann, Schwarzschild, Kerr and Gödel spacetimes. It was shown, for example, that a pure gravitational wave with non-vanishing Riemann tensor components $\mathcal{R}_{iklm} \neq 0$, possesses and carries positive-definite superenergy and that there exists a relation between the positivity of superenergy and causality violation in Gödel spacetimes. This means that the gravitational superenergy tensors are very useful quantities in studying the properties of gravity.

The definition of the superenergy tensor $S_{ab}(P)$, calculated at some spacetime point $P$, which can be applied to an arbitrary gravitational as well as a matter field is

$$
S_{(a)}^{(b)}(y) = S_{a}^{b}(P) \equiv \lim_{\Omega \to P} \frac{\int_{\Omega} \left( T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P) \right) d\Omega}{\frac{1}{2} \int_{\Omega} \sigma(P; y) d\Omega},
$$

where

$$
T_{(a)}^{(b)}(y) = T_{i}^{k}(y)e_{i(a)}^{i}(y)e_{k}^{(b)}(y),
$$

$$
T_{(a)}^{(b)}(P) = T_{i}^{k}(P)e_{i(a)}^{i}(P)e_{k}^{(b)}(P) = T_{a}^{b}(P)
$$

are the tetrad components of a tensor or a pseudotensor field $T_{i}^{k}(y)$ which describe an energy-momentum, $y$ is the set of normal coordinates $\text{NC}(P)$ at a given point $P$, $\sigma(P; y)$ is the world-function, $e_{i(a)}^{i}(y), e_{k}^{(b)}(y)$ denote an
orthonormal tetrad field and its dual, respectively, $e_i^a(P) = \delta_i^a$, $e^a_i(y) e_i^b(y) = \delta^a_b$, and they are parallel propagated along geodesics through $P$, and $\Omega$ is a spacetime volume element around $P$ (usually taken as the ball of radius $r$ in the limit $r \to 0$).

Following [1], the canonical superenergy tensors of gravitation $gS_i^k$ and of matter $mS_i^k$, which where explicitly introduced in a series of papers listed in Ref. [2], have the form

$$gS_i^k(x; v^i) = \frac{2n}{9} (2v^m v^d - g^{md}) \left[ 4R_{(im)nd} R^{lm} - 2\delta^m_i R_{lmnd} + 2\delta^l_i R_{mnd} - 3R_n R_{ld} + 2R_{(i|m|n)} R_{ld} \right],$$

$$mS_i^k(x; v^i) = h^{lm} T_i^k_{;lm},$$

where $x$ are local coordinates, $v^m$ is a four-velocity vector of an observer $O$; $h^{lm} = 2v^l v^m - g^{lm}$ is an auxiliary positive-definite Riemannian metric, $T_i^k$ are the components of an energy-momentum tensor of matter, $R_{kld}$ is the Riemann curvature tensor, a semicolon denotes a covariant derivative and a comma denotes an ordinary partial derivative.

Bearing in mind the differential nature of the definition (1), it is interesting to note that the superenergy densities defined as $\rho^c = g S_i^k v^i v_k$, $\rho^m = m S_i^k v^i v_k$, calculated for an observer $O$ whose four-velocity is $v^i$, correspond exactly to the Appell’s energy of acceleration $\tilde{m} \equiv \tilde{a} \tilde{a}$ in non-relativistic mechanics. This energy of acceleration was first proposed over a century ago by Appell [2].

In Ref. [4] we studied the transformational properties of both geometrical and physical quantities under conformal transformations of the metric. We noticed that in some conformal frames the quantities under studies are of much simpler form than in original frames. Here we want to benefit from this and discuss the superenergy tensors which can be conformally transformed to simpler forms. Besides, we propose a direct observational measurement of the local superenergy tensors by the application of some standard and non-standard cosmological parameters. In order to do so, we first calculate conformal transformation rules for canonical superenergy tensors of gravity $gS_i^k$ and matter $mS_i^k$.

In general, the conformal transformation rules for the superenergy tensors are rather complicated. However, they vastly simplify when the transformed metric is conformally flat and this is for example the case of standard Friedmann universes.

Our paper is organized as follows. We give the transformation rules for superenergy tensors in Section III. In Section IV we apply these rules to conformally flat Friedmann universes. In Section IV we express the non-zero components of the superenergy tensors $gS_i^k$, $mS_i^k$ for a flat Friedmann universe by observational quantities such as the Hubble parameter $H$, the deceleration parameter $q$, the jerk $j$, and the snap (kerk) $k$. We consider this as a crucial proposal of our paper since it gives a regular measurement of superenergy of the universe. We will confine ourselves only to flat Friedmann universes and we use geometrical units ($\tilde{G} = c = 1$) and the signature $(+,-,-,-)$ throughout the paper.

II. THE TRANSFORMATION RULES FOR THE CANONICAL SUPERENERGY TENSORS

The conformal transformation defined as

$$\hat{g}_{ik} = \Omega^2(x) g_{ik}(x), \quad \Omega(x) > 0,$$

as applied to the canonical superenergy tensor of matter $mS_i^k$ written down in terms of metric $g_{ik}$, gives the superenergy tensor written down in terms of a conformal metric $\hat{g}_{ik}$ in the form

$$m\hat{S}_i^k(x; \hat{v}^i) = \hat{h}^{lm} \left( \hat{T}_i^k_{;lm} \right) \equiv \hat{h}^{lm} \left( \Omega^{-2} T_i^k \right)_{;lm} + \hat{h}^{lm} \left( \hat{c} T_i^k \right)_{;lm} \equiv mS_i^k(x; v^i) + c \hat{S}_i^k(x; \hat{v}^i),$$

where

$$\hat{T}_i^k(x) = \Omega^{-2} T_i^k(x) + c T_i^k(x) = \Omega^2 \hat{T}_i^k(x) + c \hat{T}_i^k(x), \quad \hat{T}_i^k(x) = \Omega^{-4} (x) T_i^k(x),$$

and the tensor $\hat{T}_i^k$ is the total energy-momentum tensor of matter in the new conformal frame $\hat{g}_{ik}(x)$, $T_i^k(x)$ are the components of the energy-momentum tensor of matter in the old conformal frame, $g_{ik}$, and $c T_i^k(x)$ are the components of an extra energy-momentum tensor of matter which is created by the conformal transformation [3]. It is worth mentioning that here we took the Einstein equations in the new frame $\hat{g}_{ik}$ in the form $\hat{G}_i^k = 8\pi \hat{T}_i^k$ and in
the initial gauge $g_{ik}$ in the form $G^i_k = 8\pi T^i_k$. Such a most natural procedure leads to the creation of matter by the conformal transformation $[5].$

The analytic form of the tensor $cT^k_i(x)$ is

$$cT^k_i(x) = \frac{1}{8\pi} \left[ \frac{2}{\Omega} (\Omega^{-1})_{;a}g^{kd} + \frac{\delta^k_i}{\Omega^2} (3\Omega_{;de}g^{de} - \frac{(\Omega^2)_{;ad}g^{ad}}{2\Omega}) \right],$$

(8)

and this tensor $[8]$ vanishes in the old frame, $g_{ik}(x)$. After some calculations we get

$$m \hat{S}^k_i(x; \hat{v}^t) = \hat{h}^{lm}[\Omega^{-2}T^k_i]_{;lm} = \Omega^{-4}m \hat{S}^k_i(x; v^t) + \Omega^{-4}h^{lm}[\Omega^2\Omega^{-2}]_{;lm}T^k_i$$

$$+ (P^k_{tm}T^m_i)_{,l} - (P^k_{tmi}T^m_i)_{,l} - 2(\ln\Omega)_{,l}T^k_{tm}T^m_i - 4(\ln\Omega)_{,tm}T^k_i,$$

$$+ \Omega^{-5}h^{lm}\left\{2(\ln\Omega)_{,l}D^k_{tmi}T^m_i + \Gamma^k_{lp}(D^p_{tm}T^m_i - D^t_{mp}T^p_i)\right\}$$

$$- \Gamma^p_{lm}(D^k_{t}T^i_{p} - D^k_{p}T^i_{t}) - \Gamma^p_{ti}(D^k_{mt}T^m_i - D^t_{mp}T^p_k)$$

$$- D^p_{mi}[T^i_{k} - T^k_{i}] + D^k_{pi}[T^i_{p} - T^k_{i}],$$

(9)

where

$$D^a_{bc} = \Omega P^a_{bc} = \delta^a_i \Omega_{,c} + \delta^a_i \Omega_{,b} - g_{bc}g^{ad}\Omega_{,d} \right) ,$$

(10)

and

$$c\hat{S}^k_i(x; \hat{v}^t) = \hat{h}^{lm}[cT^k_i]_{;lm} = \Omega^{-2}h^{lm}[cT^k_i]_{;ml} + (P^k_{tmc}T^m_i)_{,l} - (P^k_{mci}T^m_i)_{,l}$$

$$+ \Gamma^k_{lp}(P^p_{tm}T^m_i - P^t_{mp}T^p_i) - \Gamma^p_{lm}(P^k_{pt}T^t_i - P^t_{ip}T^t_i)$$

$$- \Gamma^p_{li}(P^k_{mip}T^ip - P^t_{mp}T^t_k) + P^k_{lp}(P^p_{tp}T^t_p - P^p_{t}T^t_p)$$

$$+ P^k_{lp}(P^p_{mip}T^ip - P^t_{mp}T^t_k) - P^p_{lm}(P^k_{tp}T^t_p - P^t_{ip}T^t_k)$$

$$- P^p_{li}(P^k_{mip}T^ip - P^t_{mp}T^t_k).$$

(11)

Here $m\hat{S}^k_i$ is the canonical superenergy tensor of matter in the old frame $g_{ik}(x)$, and the tensor $c\hat{S}^k_i$ is the superenergy tensor of the matter which is created by the conformal transformation $[5]$. The latter tensor vanishes in the old frame, $g_{ik}(x)$. In order to obtain the transformation rule for the canonical superenergy of matter (under the conformal transformation $[5]$) one has to insert $[9]$ and $[11]$ into the right-hand-side of equations $[6]$. 
Now, following [4], the transformation rule for the gravitational canonical superenergy tensor $gS_{a}^{b}$ reads as

$$gS_{a}^{b}(x; \dot{v}^t) = \Omega_{g}^{-4}S_{a}^{b}(x; u^t) + \frac{2\alpha}{9}R^{lm} \left[ (g^{ib}[\Omega^{j}]_{l} - \delta_{ij}^{[b} \Omega^{]j}k) 
+ \left( g^{ib}[\Omega^{j}] - \delta_{ij}^{[b} \Omega^{]j}k \right) \right]_{g[a][\Omega^{i}]_{m]}} 
+ \delta^{b}_{k} \left( g^{ij}[\Omega^{j}]_{l} - \delta_{ij}^{[b} \Omega^{]j}k \right) \right]_{g[i]j[\Omega^{j}]} + g_{i][j] \Omega_{k]m} \right) 
+ \frac{2\alpha}{9} \Omega^{-1}R^{lm} (\Omega^{-1})_{,mc} (g^{ic} \Omega_{al} - \delta_{i}^{[b} \Omega^{]j}k) + g_{al} \Omega^{bc} - \delta_{i}^{[b} \Omega^{]j}k) \right] 
+ \frac{4\alpha}{9} \Omega^{-2}R^{lm} \left[ 2R^{ik} \Omega_{a[i}]m] + \left( g^{ik} \Omega^{j}l - \delta_{i}^{[b} \Omega^{]j}k \right) \right]_{g[i]j[\Omega^{j}]} + g_{i][j] \Omega_{k]m} \right) 
+ \frac{4\alpha}{9} \Omega^{-1}R^{lm} \left[ 4 \delta^{i}_{a} \left( \Omega^{-1} \right)_{,al} - \left( \Omega^{-1} \right)_{,cm} R^{l}_{-} - \left( \Omega^{-1} \right)_{,an} R^{l}_{m} \right] 
+ \frac{1}{4} \Omega^{-1}R^{lm} \left[ \Omega^{ic} \left( R^{b}_{a l} + R^{b}_{l a} \right) \right] 
+ \frac{\alpha}{18} \Omega^{-4}R^{lm} (\Omega^{2})_{,rs} g^{rs} \left( \delta^{i}_{a} \left( \Omega^{2} \right)_{,cm} + g_{i[cm] \Omega^{l}_{a]} \right) 
+ \frac{1}{3} \Omega^{-3}R^{lm} (\Omega^{2})_{,rs} g^{rs} \left( \Omega^{-1} \right)_{,al} \delta^{i}_{a} + g_{al} \left( \Omega^{-1} \right)_{,an} \right] 
+ \frac{4\alpha}{3} \Omega^{-2}R^{lm} (\Omega^{2})_{,rs} g^{rs} \left( R_{a l} \delta^{i}_{a} + g_{al} \Omega^{bc} - \delta_{i}^{[b} \Omega^{]j}k \right) \right] 
+ \frac{1}{2} \Omega^{-1}R^{lm} \left( \Omega^{2} \right)_{,rs} g^{rs} \left( \delta^{i}_{a} g_{lm} - \frac{1}{2} \delta^{i}_{a} g_{al} \right). \quad (12)$$

Here $gS_{a}^{b}$ are the components of the canonical superenergy tensor of gravitation in the old frame, $g_{ik}$, and $\alpha = 1/(16\pi)$. As one can see, the transformational rules for the superenergy tensors of matter and gravitation are quite complicated. However, if the initial metric, $g_{ik}(x)$, is Minkowskian, i.e., if we confine ourselves to a conformally flat spacetime with metric $\hat{g}_{ik} = \Omega^{2}(x) \eta_{ik}$, then the above transformational formulas simplify significantly because, in the initial frame, $g_{ik}(x)$, one has

$$g_{ik} = \eta_{ik}, \quad T_{i}^{k} = 0, \quad mS_{i}^{k} = 0, \quad gS_{a}^{b} = 0, \quad R_{k}^{i} = 0, \quad R_{ik} = 0, \quad R = 0, \quad ;i = ;i. \quad (13)$$

In consequence, we have in this case that

$$mS_{a}^{b}(x; \dot{v}^t) = \hat{S}_{a}^{b}(x; \dot{v}^t), \quad (14)$$

because $mS_{i}^{k}(x; \dot{v}^t) \equiv 0$, and $\hat{S}_{i}^{k}(x; \dot{v}^t)$ is substantially simplified in comparison to [11] due to the conditions [13].

### III. THE CASE OF A FLAT FRIEDMANN UNIVERSE

Friedmann universes are conformally flat and, in a special case of its their flat geometry, we have

$$ds^{2} = a^{2}(\tau) \left( dr^{2} - dx^{2} - dy^{2} - dz^{2} \right), \quad (15)$$

which means that the conformal factor is just the scale factor $\Omega = \Omega(\tau) = a(\tau) > 0$, and $g_{ik} = \eta_{ik}$ in (5). It is important here that in (15) the conformal time $\tau$ has been introduced.

We would like to emphasize that one can obtain a flat, and also curved Friedman universes with all their energetic and superenergetic content from the flat Minkowski spacetime by a suitable conformal transformation (see e.g. [4] for details). An analogous statement is true for any other conformally flat spacetime so that we do not need any quantum fluctuations of the Minkowski vacuum to create a Friedmann or any other conformally flat spacetime - the idea which is commonly used in quantum cosmology (see e.g. [8]). Amazingly, a classical conformal transformation of the initial Minkowski metric is sufficient to do the job.
The components of the superenergy tensors $g\hat{S}_k^i(x;v^i)$ and $m\hat{S}_k^i(x;v^i)$ for the Friedmann metric (15) can be calculated by using the formulas (6), (9), (11), (12), simplified by (13), and give
\begin{align}
g\hat{S}_0^0 &= \frac{\alpha}{9\sigma^3}(204a^2 - 396aa'^2a'' + 188a^2a'^{n^2}), \\
g\hat{S}_1^1 &= g\hat{S}_2^2 = g\hat{S}_3^3 = \frac{\alpha}{3\sigma^3}(10a^4 - 22aa'^2a'' + \frac{20}{3}a^2a'^{n^2}), \\
m\hat{S}_0^0 &= \frac{12\alpha}{a^3}(a'^{n^2} + a''a''') - \frac{16a^2a'''}{a} + 22a^4, \\
m\hat{S}_1^1 &= m\hat{S}_2^2 = m\hat{S}_3^3 = 4\alpha(\frac{a^m + 11a'aa''}{a} - 4a''a'''} + 40a^2a'' - 22a^4),
\end{align}
where $a' \equiv \frac{da}{d\tau}$, $a'' \equiv \frac{d^2a}{d\tau^2}$, $a''' \equiv \frac{d^3a}{d\tau^3}$, $a'''' \equiv \frac{d^4a}{d\tau^4}$. For a zero pressure dust-filled flat Friedmann universe one gets
\begin{equation}
\Omega(\tau) = a(\tau) = \frac{A^3}{9\tau^2}, \quad 0 < \tau < \infty,
\end{equation}
where
\begin{equation}
A \equiv (6\pi\rho a^3)^{1/3} = \text{const} > 0,
\end{equation}
and we have from (16)-(19)
\begin{align}
g\hat{S}_0^0 &= \frac{618192\alpha}{A^{12}\tau^{12}} > 0, \quad g\hat{S}_1^1 = g\hat{S}_2^2 = g\hat{S}_3^3 = \frac{23328\alpha}{A^{12}\tau^{12}} > 0, \\
g\hat{S} &= g\hat{S}_0^0 + g\hat{S}_1^1 + g\hat{S}_2^2 + g\hat{S}_3^3 = \frac{688176\alpha}{A^{12}\tau^{12}} > 0, \\
m\hat{S}_0^0 &= \frac{897544\alpha}{A^{12}\tau^{12}} > 0, \quad m\hat{S}_1^1 = m\hat{S}_2^2 = m\hat{S}_3^3 = -\frac{1259712\alpha}{\kappa A^{12}\tau^{12}} < 0, \\
m\hat{S} &= m\hat{S}_0^0 + m\hat{S}_1^1 + m\hat{S}_2^2 + m\hat{S}_3^3 = \frac{1417176\alpha}{\kappa A^{12}\tau^{12}} > 0.
\end{align}

IV. OBSERVATIONAL MEASUREMENT OF SUPERENERGY

Bearing in mind the final output for superenergy tensors given by (16)-(19), we notice that they can be expressed in terms of some standard and non-standard cosmological parameters. The well-known characteristics of the universe expansion are the Hubble parameter $H$, and the deceleration parameter $q$:
\begin{equation}
H = \frac{\dot{a}}{a}, \quad q = -\frac{1}{H^2} = -\frac{\ddot{a}a}{\dot{a}^2},
\end{equation}
while the new characteristics are the jerk $j$ (10), and the snap (kerk) [11]
\begin{equation}
j = \frac{1}{H^3} \frac{\dddot{a}}{a} = -\frac{\dddot{a}a^2}{\dot{a}^3}, \quad k = -\frac{1}{H^4} \frac{\ddot{a}a^3}{a^4} = -\frac{\ddot{a}a^3}{a^4}.
\end{equation}
A general form of these parameters can be expressed as [4]
\begin{equation}
x^{(i)} = (-1)^{i+1} \frac{1}{H^i} \frac{a^{(i)}}{a} = (-1)^{i+1} \frac{a^{(i)}a^{i-1}}{a^{i+1}},
\end{equation}
with $i$ - the natural number, ‘$(i)$’ means the $i$th derivative while ‘$i$’ is just a power.

It is obvious that nowadays we can measure $H$ and $q$ [12], however, the observational determination of the value of jerk by using type Ia supernovae sample is more difficult. Despite that it has been performed [13] and the claim is that $j_0 > 0$. Similar investigations may perhaps also be possible for higher-order characteristics such as kerk/snap, kerk etc. Some hints about that are given in [11].

Now, our objective is to express the non-vanishing components of the canonical superenergy tensors (16)-(19) in terms of the above observational parameters (20) and (27). Since the standard definition of these parameters uses
the derivatives of the scale factor with respect to the cosmic time $t$ instead of the derivatives with respect to the conformal time $\tau$, we will use the conformal time definition formula

$$\dot{\tau} \equiv \frac{dr}{dt} = \frac{1}{a(t)}, \quad (29)$$

to express all the derivatives with respect to the cosmic time $t$ by the derivatives with respect to the conformal time $\tau$ and rewrite the observational parameters $H, q, j, k$ in terms of the derivatives with respect to the conformal time $\tau$. By doing so, one can express the non-zero components of the superenergy tensors (16)-(19) in terms of the observational parameters $H, q, j, k$ as

$$g \hat{S}^0_0 = \frac{4\alpha H^4}{9} (47q^2 + 5q - 1), \quad (30)$$

$$g \hat{S}^1_1 = g \hat{S}^2_2 = g \hat{S}^3_3 = \frac{2\alpha H^4}{9} (10q^2 + 13q - 8), \quad (31)$$

$$m \hat{S}^0_0 = 12\alpha H^4 (q^2 + 10q + j + 8), \quad (32)$$

$$m \hat{S}^1_1 = m \hat{S}^2_2 = m \hat{S}^3_3 = 4\alpha H^4 (q - 4j - k + 4). \quad (33)$$

Theoretically, for a dust-filled flat Friedman universe one has $q = 1/2, \ j = 1, \ k = 7/2$. So, in this case both superenergy densities $g \hat{S}^0_0$ and $m \hat{S}^0_0$ for a comoving observer $O$ are positive-definite. One can see from the formulas (30)-(33) that the measurement of the superenergy of gravity requires only the measurement of the Hubble parameter $H$ and the deceleration parameter $q$, while the measurement of the superenergy of matter requires also the measurement of the jerk, $j$, and the snap (kerk), $k$.

We claim that our formulas (30)-(33) are of great physical importance because they allow a direct measurement of the components of the canonical superenergy tensors for Friedman universes from large-scale cosmological observations.

Using (29) and taking into account the observational values of the $H_0 = 71\ km/s/Mpc, q_0 = -0.55, j_0 > 0$, we then obtain the values of the superenergy near the point of measurement on Earth at the present moment of the evolution of the universe. Also, by using the rules of the time-evolution of these parameters we may express the superenergy throughout the whole evolution of the universe. This means that both superenergy tensors are observational quantities and can be added to a set of the standard parameters characterizing the evolution of the universe.

V. CONCLUSION

We have calculated the conformal transformation rules for the canonical superenergy tensors, gravitation and matter. The general rules we obtained are pretty complicated, but they vastly simplify when the spacetime under study is conformally flat. This happens, for example, for the Friedman universes.

We have applied our general formulas to the flat Friedman universes and obtained relatively simple expressions for the components of the canonical superenergy tensors of matter and gravitation. Finally, we have expressed these components in terms of the observational parameters: the Hubble parameter, the deceleration parameter, the jerk and the snap (kerk).

According to current observations, our Universe is likely to be the flat Friedman universe. Then the formulas (30)-(33) we obtained, allow a direct measurement of the superenergy densities of the Universe from large-scale cosmological observations.

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