Self-Dual Effective Action of $N=4$ Super-Yang Mills

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Abstract

The full low energy effective action of $N=4$ SYM is believed to be self-dual. Starting with the first two leading terms in a momentum expansion of this effective action, we perform a duality transformation and find the conditions for self-duality. These determine some of the higher order terms. We compare the effective action of $N=4$ SYM with the probe-source description of type $II_B$ D3-branes in the $AdS_5 \times S_5$ background. We find agreement up to six derivative terms if we identify the separation of the 3-branes with a redefinition of the gauge scalar that involves the gauge field strength.

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1 Introduction

In this paper, we study the consequences of self-duality of $N = 4$ Super Yang-Mills (SYM) theory in $N = 2$ superspace. The bosonic effective action of this theory can be compared to the Dirac-Born-Infeld (DBI) action of gauge theories on branes. The DBI action is self-dual under a duality that does not act on the separation of the branes. However, the Higgs fields that parameterize this separation are in the same $N = 2$ supermultiplet as the gauge fields, and hence the Higgs fields that realize $N = 2$ supersymmetry linearly must be related by a nonlinear gauge-field dependent redefinition to the separation. This is the most striking consequence of our analysis.

Duality is a powerful tool for probing strong coupling physics: it allows us to describe strongly coupled systems using the weakly coupled Lagrangian of the dual degrees of freedom. The dual descriptions of a generic theory may in general be very different. There is however a very special theory, namely $N = 4$ Super Yang-Mills, which is believed to have isomorphic dual descriptions: the electric description has a well defined perturbative expansion when the gauge coupling $g^2/4\pi$ is weak, while the magnetic description is well defined perturbatively when a dual gauge coupling $g_D^2/4\pi = 4\pi/g^2$ is weak, i.e., the original coupling is strong. These descriptions are isomorphic in the sense that the electric effective action written in terms of the electric field strength and coupling has the same form as the magnetic effective action written in terms of the magnetic field strength and coupling. The theory is also believed to be exactly self-dual when the gauge coupling takes the value $g^2 = 4\pi$; then duality leaves the effective action unchanged. For sake of brevity, in subsequent sections we use the term self-dual in the broader sense for arbitrary $g^2$.

The isomorphy of both descriptions is well understood in the classical pure gauge action, where a first order formalism implements the change from fundamental to dual variables as a Legendre transform. Long ago it was conjectured that the isomorphy is actually a property of the full quantum theory [1]: the spectrum of BPS states remains the same, and the quantum effective action of the massless gauge sector has the same form in electric and magnetic variables. The first part of this conjecture has been tested after extending the strong-weak coupling duality to $SL(2, \mathbb{Z})$ [2].

We want to study the consequences of this conjecture by implementing duality on the $N = 4$ SYM effective action. This theory is a particular case of $N = 2$ SYM coupled to adjoint matter, and formulating it in $N = 2$ superspace is useful because the $N = 2$ superspace effective action of the massless gauge sector has a well defined expansion in the external momentum. The leading term in a momentum expansion is a $N = 2$ superpotential (a prepotential in the terminology of [3])

\[ S_{\text{eff}}^{(2)} = \Im \int d^4 x D^4 F(\tau, W) \]  

where $D^4 = D^2 Q^2$ is the chiral measure of $N = 2$ superspace, $W$ is the $N = 2$ gauge field

\[ D_{1\alpha} = D_\alpha, \quad D_{2\alpha} = Q_\alpha \] are the supercovariant derivatives associated with the Grassmann coordinates of $N = 2$ superspace. We follow the conventions of [4].
strength and \( \tau = \theta/2\pi + i4\pi/g^2 \) is the holomorphic gauge coupling. In components \( S^{(2)}_{\text{eff}} \) gives terms with at most two space-time derivatives. This \( N = 2 \) effective superpotential contains all the divergences and all the scale and \( U(1)_R \) anomalies of the theory. For \( N = 4 \) SYM the perturbative \( \left[ \frac{4}{3} \right] \) and nonperturbative \( \left[ 10 \right] \) quantum corrections to the tree level superpotential \( \mathcal{F} = \tau W^2/16\pi \) vanish.

The next term in the momentum expansion of the \( N = 2 \) effective gauge action is a \( \left[ 3 \right] \) nonholomorphic potential integrated with the full \( \left[ 3 \right] \) superspace measure. It is therefore a finite, dimensionless real function of the \( N = 2 \) gauge field strengths \( W \) and \( \bar{W} \). Since \( \mathcal{F} \) saturates all the perturbative scale and \( \left[ 1 \right] \) anomalies, the perturbative \( \left[ 3 \right] \) nonholomorphic potential must be scale and \( \left[ 1 \right] \) invariant. In the massless gauge sector this restriction completely fixes the 1-loop \( N = 2 \) nonholomorphic potential \( \left[ 5 \right] \)

\[
S^{(4)}_{\text{eff}} = \int d^4x d^8\theta \left[ \frac{c}{8\pi} \left[ \ln \frac{W}{\Lambda} + g^0(W) \right] - \ln \frac{\bar{W}}{\Lambda} + \bar{g}^0(\bar{W}) \right],
\]

where \( g^0 \) depends on gauge invariant, scale independent combinations of the \( N = 2 \) abelian field strengths (for a spontaneously broken \( SU(N_c) \) gauge theory where we keep only one unbroken \( U(1) \) background gauge multiplet, we have \( g^0(W) = 0 \)). In components \( S^{(4)}_{\text{eff}} \) contains at most four space-time derivatives. For scale invariant theories such as \( N = 4 \) SYM, the abelian nonholomorphic potential is believed to be generated only at 1-loop \( \left[ 3 \right] \), since higher loop and nonperturbative contributions would break the scale invariance of \( S^{(4)} \).

An explicit 1-loop calculation gives the value of the coefficient \( c \) for the abelian piece of \( SU(2) \) broken to \( U(1) \): \( c = 1/2\pi \left[ 4 \right] \left[ 3 \right] \left[ 1 \right] \) it was again reproduced by comparing \( N = 1 \) components. An extension of the analysis in \( \left[ 8 \right] \) to the case \( SU(N_c) \) broken to \( SU(N_c-1) \times U(1) \) is included in appendix A. Nonperturbative contributions have been studied in \( \left[ 10 \right] \) and they indeed give vanishing results.

Higher derivative contributions \( S^{(2n>4)} \) present in the effective action of \( N = 4 \) SYM must be also scale and \( \left[ 1 \right] \) invariant, dimensionless and finite.

We implement the strong-weak coupling duality on this expansion of the quantum effective action by using a first order formalism in \( N = 2 \) superspace \( \left[ 11 \right] \). In this formulation we relax the Bianchi identity constraint

\[
D^2_{ab}W = C_{ac}C_{bd}\bar{D}^{2dc}\bar{W}
\]

and we add a field strength \( W_D \) as a Lagrange multiplier that enforces the Bianchi identity\( \left[ 12 \right] \) on \( W \). Then we replace the relaxed chiral superfield \( W = D^4V \) by its field equation

\[
\frac{\partial S_{\text{eff}}}{\partial W} = 0 \Rightarrow W_D = \tau W + 2iD^4\mathcal{L}^{\text{quant}}_W
\]

\[
\frac{\partial S_{\text{eff}}}{\partial \bar{W}} = 0 \Rightarrow \bar{W}_D = \bar{\tau}\bar{W} - 2iD^4\mathcal{L}^{\text{quant}}_W.
\]

\[\text{(4)}\]

\( ^2 \)This is very clear when write the \( N = 2 \) field strength as the most general superfield obeying the Bianchi identity \( W_D = D^4D^2_{ab}V_D^{ab} \left[ 13 \right] \).
Note that the prescribed procedure guarantees that the field equations of the original action

\[ S_{\text{eff}} = \text{Im} \left( \int d^4 x \, D^4 \frac{1}{2} \tau W^2 \right) + \int d^4 x \, D^4 \bar{D}^4 \mathcal{L}^{\text{quant}} \]  

become the Bianchi identities of the dual field strength. This operation is a straightforward generalization to \( N = 2 \) superspace of the quantum duality performed in [3]. The \( N = 2 \) SYM theory studied in [3] is however very different from \( N = 4 \) SYM: Since \( S_{\text{eff}}^{(2)} \) receives quantum corrections in the \( N = 2 \) SYM case, the discussion of duality in [3] ignored higher derivative terms. We note further that in contrast to the \( N = 4 \) case, the spectrum of BPS states is not \( SL(2, Z) \) invariant.

The structure of this article is the following: in section 2 we review the duality transformation in the classical action of \( N = 2 \) SYM as a Legendre transform in the path integral of the free theory [11]. To illustrate some of the general features we encounter in these type of transformations we also study a one dimensional system where the general form of a self-dual action can be found.

In section 3 we check that the first two terms \( S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} \) in the momentum expansion of the \( N = 4 \) low energy effective action indeed have the same form in the electric and magnetic descriptions. We find that the dualization of those two leading terms produces additional six and higher space-time derivative operators in the dual variables \( S_{\text{eff}}^{(2n>4)} \), which should also be present in the original description if the theory is isomorphic under duality. Including such higher derivative terms in the electric effective action, we find that the dualization gives the same operators in magnetic variables if their coefficients obey certain relations.

The effective action of the \( U(1) \) gauge background in a spontaneously broken \( N = 4 \) \( SU(N_c) \) theory, is supposed to describe the dynamics of an extremal probe D3-brane in the background of \( N_c - 1 \) overlapping extremal source D3-branes [13]. This can be alternatively described by a DBI action plus WZ terms. In section 4 we compare the bosonic degrees of freedom of both actions, and we find a disagreement that can be resolved up to six-derivative terms by redefinitions of the \( N = 1 \) superfields in the \( N = 2 \) gauge multiplet: this reveals the fact that the gauge scalar of the SYM theory (in which \( N = 2 \) SUSY is linearly realized) and the separation of the 3-branes in the DBI action are not simply proportional to each other. We find that this is true only as a first order approximation: the relation also involves a nonlinear function of the YM field strengths.

Finally in section 4 we discuss open problems on this line of research, such as the perturbative/nonperturbative nature of the effective action expansion.

### 2 Two Simple Examples of self-dual actions

Let us begin by reviewing a simple example of the formalism that implements the strong-weak coupling duality transformation. We take the classical \( N = 2 \) Maxwell action in the electric description, and we relax the Bianchi identity constraint (3) on the field strength while at the same time we introduce a Lagrange multiplier as we described in the introduction.
\[ 8\pi S = \Im \int d^4x D^4 \left( \frac{1}{2} W^2 - WW_D \right). \tag{6} \]

Integrating out \( W_D \) in the path integral of the free theory imposes the Bianchi identity (3) on \( W \); alternatively, we can integrate out \( W \), which in this case is the same as replacing it by its field equation

\[ W = \frac{W_D}{\tau}. \tag{7} \]

Substituting (7) into (6) gives the dual action

\[ 8\pi S_D = \Im \int d^4x D^4 \frac{1}{2} \frac{-1}{\tau} W_D^2. \tag{8} \]

The original action is therefore equal to the dual one written with the magnetic variables \( W_D \) and \( \tau_D = -1/\tau \). The classical Maxwell action is also invariant under a real shift in \( \tau \), \( \tau \rightarrow \tau + x \), because the shifted integrand is total derivative, \( \sim \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \).

Possible self-dual functionals are actually more general than is commonly realized (see for example [14]). We find this feature when we study the duality of the quantum action. To illustrate this idea consider the following one dimensional example: given a function \( F \) and its Legendre transform

\[ \tilde{F}(y) = [F(x) - xy]_{x = F^{-1}(y)}, \tag{9} \]

self-duality implies \( \tilde{F}(y) = F(y) \) or equivalently

\[ F(F_x) = F(x) - x F_x \Rightarrow -x = F_x(F_x). \tag{10} \]

The most general solution of eqn. (10) is

\[ F(x) = \int_x^z d\xi \, g^{-1}(ig(z)) \tag{11} \]

is the most general self-dual function. In this example the function \( F(x) \) is the analogue of our quantum effective action. Since this action is constructed as a series expansion, it is useful to refine the analogy and study the Taylor series expansion of \( F(x) \). To do this, we first Taylor expand \( g \)

\[ g = g^{(1)} x + \frac{1}{3!} g^{(3)} x^3 + \frac{1}{5!} g^{(5)} x^5 + \frac{1}{7!} g^{(7)} x^7 + \cdots \tag{12} \]

where \( g^{(n)} = \frac{d^n g}{dx^n} \bigg|_{x=0} \). The inverse of \( g \) is

\[ g^{-1}(x) = \frac{1}{g^{(1)}} x - \frac{1}{3!} \frac{g^{(3)}}{(g^{(1)})^2} x^3 - \frac{1}{5!} \frac{g^{(5)} g^{(1)} - 10 (g^{(3)})^2}{(g^{(1)})^7} x^5 + \cdots \tag{13} \]

From the two equations above, one finds
\[ F_x = g^{-1}(ig(x)) = ix + \frac{i}{3}g^{(3)}x^3 + \frac{i}{6}(g^{(3)})^2x^5 + i \left( \frac{5}{36}g^{(3)} - \frac{1}{72}g^{(3)}g^{(5)} + \frac{1}{2520}g^{(7)} \right)x^7 + \left[ \frac{5}{3}g^{(3)} \left( -\frac{1}{72}g^{(3)}g^{(5)} + \frac{1}{2520}g^{(7)} \right) + \frac{85}{648}(g^{(3)})^4 \right]x^9 + \ldots \] (14)

Here the coefficient of the fifth order term, \( i\frac{5}{3}(g^{(3)})^2 \), is expressed in terms of \( g^{(3)} \), which, up to a numerical factor, is the coefficient of third order term. Similarly new parameters \( g_5 \) and \( g_7 \) appear in the seventh order coefficient, but the ninth order coefficient is a function of the parameters in the lower order coefficients. Furthermore \( g_5 \) and \( g_7 \) appear only in the combination \(-\frac{1}{72}g^{(3)}g^{(5)} + \frac{1}{2520}g^{(7)} \). In the next section, we will observe a similar behavior for the effective action of \( N = 4 \) SYM.

### 3 Dualization of the \( N = 4 \) SYM effective action

We have seen that the effective action of \( N = 4 \) SYM up to four spacetime derivatives is restricted by \( U(1)_R \) and scale invariance to be of the form

\[ 8\pi \left( S_{eff}^{(2)} + S_{eff}^{(4)} + \ldots \right) = \text{Im} \left( \int d^4x \ D^4 \frac{1}{2} \tau W^2 \right) + c \int d^4x \ D^4 \bar{D}^4 \ln W \ln \bar{W} + \ldots ; \] (15)

for explicit values of the Higgs field, the coefficient \( c \) is calculated in Appendix A. We are now ready to study if this effective action is self-dual. To dualize the action we follow the same steps as in (4): we rewrite the action as

\[ 8\pi \left( S_{eff}^{(2)} + S_{eff}^{(4)} + \ldots \right) = \text{Im} \int d^4x \ D^4 \left( \frac{1}{2} \tau W^2 - WW_D \right) + c \int d^4x \ d^8\theta \ln W \ln \bar{W} + \ldots , \] (16)

and the duality equation is given the field equation of the relaxed chiral superfield \( W \)

\[ 0 = \tau W - W_D + 2ic \frac{D^4 \ln \bar{W}}{W} + \ldots \]

\[ 0 = \bar{\tau} \bar{W} - \bar{W}_D - 2ic \frac{D^4 \ln W}{\bar{W}} + \ldots . \] (17)

This is highly nonlinear and hard to invert to find the solution \( W(W_D) \). Notice however that this solution is of the form \( W(W_D) = \frac{W_D}{\tau} + \text{higher derivatives} \). Since we are only studying self-duality up to \( S^{(4)} \) for the time being, we can solve (17) to first order in \( D^4 \) and \( \bar{D}^4 \)

\[ W = \frac{W_D}{\tau} \left( 1 - 2ic\tau \frac{D^4 \ln \bar{W}_D}{W^2} \right) + \ldots \]

\[ \bar{W} = \frac{W_D}{\bar{\tau}} \left( 1 + 2ic\bar{\tau} \frac{D^4 \ln W_D}{\bar{W}^2} \right) + \ldots \] (18)
We substitute (18) into (16) and we find

\[ 8\pi S_{\text{eff,D}} = \text{Im} \left( \int d^4x D^4 \frac{1}{2} \frac{-1}{\tau} W_D^2 \right) + c \int d^4x d^8\theta \ln W_D \ln \bar{W}_D \]

\[ + \left( ic^2 \int d^4x d^8\theta \ln W_D \frac{D^4 \ln \bar{W}_D}{\tau W_D^2} + \text{c.c.} \right) + ... \] (19)

Under the map \( W_D \to W, \tau_D = -\frac{1}{\tau} \to \tau \) this action is equivalent to (13) up to four spacetime derivatives. The dual action contains also terms with six spacetime derivatives proportional to \( 1/\tau_D \). They depend on a scale and \( U(1)_R \) invariant chiral field that contains the holomorphically normalized field strength \( \tau W^2 \)

\[ B_D = \frac{D^4 \ln \bar{W}_D}{\tau D^2} = \frac{1}{2} \frac{\bar{D}^4 \ln \left( \tau_D \bar{W}_D^2 \right)}{\tau D^2} \] (20)

The six derivative term is scale and \( U(1)_R \) invariant as a whole. It seems therefore that there must be higher order terms in the original description of the effective action, which should also be consistent with the duality conjecture. Let us test this possibility: we dualize the following effective action

\[ 8\pi \left( S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} + S_{\text{eff}}^{(6)} + ... \right) = \text{Im} \left( \int d^4x D^4 \frac{1}{2} \frac{-1}{\tau} W^2 \right) + c \int d^4x d^8\theta \ln W \ln \bar{W} \]

\[ + \left( \kappa ic^2 \int d^4x d^8\theta \ln W \frac{D^4 \ln \bar{W}}{\tau W^2} + \text{c.c.} \right) + ... \] (21)

Repeating the same steps as above and solving the field equations of \( W \) to second order in \( D^4 \) and \( \bar{D}^4 \) we find

\[ 8\pi \left( S_{\text{eff,D}}^{(2)} + S_{\text{eff,D}}^{(4)} + S_{\text{eff,D}}^{(6)} + S_{\text{eff,D}}^{(8)} + ... \right) = \text{Im} \left( \int d^4x D^4 \frac{1}{2} \frac{-1}{\tau} W_D^2 \right) \]

\[ + c \int d^4x d^8\theta \ln W_D \ln \bar{W}_D \]

\[ + c^2 \int d^4x d^8\theta \left( i(1 - \kappa) \ln \bar{W}_D \frac{D^4 \ln \bar{W}_D}{\tau \bar{W}_D^2} + \text{c.c.} \right) \]

\[ + c^3 \int d^4x d^8\theta \left( \frac{1}{2} - \kappa \right) \ln \bar{W}_D \left( \frac{D^4 \ln \bar{W}_D}{\tau \bar{W}_D^2} \right)^2 + \text{c.c.} \] (22)

\[ + (1 - \kappa - \bar{\kappa}) \frac{D^4 \ln \bar{W}_D}{\tau \bar{W}_D^2} \frac{D^4 \ln W_D}{\tau W_D^2} + ... \]

To sixth order, the theory is self-dual if \( \kappa = \frac{1}{2} \). For \( \kappa = \frac{1}{2} \) the eight derivative terms that we find in the dual action do not receive contributions from the dualization of \( S_{\text{eff,D}}^{(2)} + S_{\text{eff,D}}^{(4)} + S_{\text{eff,D}}^{(6)} \). In addition, since at lowest order the duality equation is still given by \( W = W_D/\tau \), the scale and \( U(1)_R \) invariant operator depending on \( \tau W^2 \)
\[ S_{\text{eff}}^{(8)}[\tau W^2] = c^3 \int d^4x d^8\theta \left( \kappa_2^{(8)} \ln \bar{W} \left( \frac{\bar{D}^4 \ln \bar{W}}{\tau W^2} \right)^2 + c.c. \right) \]
\[ + \kappa_1^{(8)} \frac{\bar{D}^4 \ln \bar{W}}{\tau W^2} \frac{D^4 \ln W}{\tau W^2} , \]
\[ (23) \]

is dualized at the lowest order into an isomorphic operator depending on \(-\tau_D W_D^2\). Hence, if we set \(\kappa = \frac{1}{2}\) in (21), we can add additional terms of the form (23) and the effective action is self-dual up to \(S_{\text{eff}}^{(8)}\) for arbitrary values of the parameters \(\kappa_1^{(8)}\) and \(\kappa_2^{(8)}\).

The appearance of \(S_{\text{eff}}^{(6)}\) and \(S_{\text{eff}}^{(8)}\) illustrates a key feature of the \(N = 4\) SYM effective action. Scale and \(U(1)_R\) imply that higher derivative operators in the effective action contain powers of the ratio \(\mathcal{D} = (1/\tau W^2)\bar{D}^4\) and its conjugate\(^3\). A term with \(2m + 4\) spacetime derivatives will have \(m\) inverse powers of \(\tau W^2, \bar{\tau} W^2\). Since the dualization of any operator \(S_{\text{eff}}^{(2m+4)}[\tau W^2, \bar{\tau} W^2]\) gives at lowest order an isomorphic expression \(S_{\text{eff}}^{(2m+4)}[-\tau_D W_D^2, -\tau_D \bar{W}_D^2]\), if \(m = 2n\) at lowest order the operator is mapped to itself under duality, while if \(m = 2n + 1\) there is an additional minus sign. \(S_{\text{eff},D}^{(2m+4)}\) receives additional contributions that we denote as \(\Delta^{(2m+4)}\). They come from the dualization of lower order operators \(S_{\text{eff}}^{(i<2m+4)}\).

\[ S_{\text{eff},D}^{(4n+4)} = S_{\text{eff}}^{(4n+4)}[\tau_D W_D^2] + \Delta^{(4n+4)}[\tau_D W_D^2] \]
\[ S_{\text{eff},D}^{(4n+6)} = -S_{\text{eff}}^{(4n+6)}[\tau_D W_D^2] + \Delta^{(4n+6)}[\tau_D W_D^2] \]
\[ (24) \]

Hence, self-duality requires \(\Delta^{(4n+4)} = 0\) and \(\Delta^{(4n+6)} = 2 \times S_{\text{eff}}^{(4n+6)}\). This means that all the coefficients of even order terms are completely arbitrary, while the coefficients of odd order operators are constrained to be some linear combination of the coefficients appearing in lower order terms. This result is analogous to that found in the one-dimensional example (14) we introduced in the previous section.

Although we can continue this procedure to construct a self-dual effective action, inverting the duality equations to obtain \(W(W_D)\) can be cumbersome at higher orders. An equivalent calculation which gives the same results, but makes it unnecessary to invert the duality equations is the following: instead of solving for \(W = W(W_D)\), we directly substitute the duality equation \(W_D = W_D(W)\) in \(S_{\text{eff}}[\tau W^2]\) and impose self-duality

\[ S_{\text{eff},D}[\tau_D W_D^2] \equiv S_{\text{eff}}[\tau W^2] - W W_D . \]
\[ (25) \]

The result of this process is an \(N = 4\) SYM effective action consistent with the duality conjecture, which depends on powers of the scale and \(U(1)_R\) invariant chiral operators \(B, D\) and of their conjugates. The details of the calculation are included in appendix B.

We seem to have found a recipe to construct a \(N = 4\) SYM effective action consistent with the duality conjecture, order by order in a momentum expansion which is at the same time

\(^3\)The holomorphically normalized chiral field strength \(\tau W^2\) is the natural object to divide the chiral operator \(\bar{D}^4\) because the dimensionless gauge coupling \(\tau\) maybe promoted to a chiral superfield with nonvanishing \(U(1)_R\) charge \(\bar{\mathcal{D}}\), but the tree level Lagrangian \(\tau W^2\) must still transform oppositely to the measure \(\bar{D}^4\).
an expansion in the gauge coupling. This is a direct consequence of the fact that the actual expansion variables are the scale and $U(1)_R$ invariant ratio $D$ and its complex conjugate.

However these terms are not the most general scale and $U(1)_R$ invariant variables we might consider. We could use for instance a “square root” $P^{ab}$ of the operator $D$ and a corresponding field $E^{ab}$:

$$P^{ab} = \frac{D^{2ab}}{\sqrt{\tau W}} , \quad E^{ab} = \frac{D^{2ab} \ln W}{\sqrt{\tau W}} = \frac{D^{2ab} \ln(\sqrt{\tau W})}{\sqrt{\tau W}}$$

and their complex conjugates. Other possible variables are complex linear superfields such as

$$D \left( \frac{D}{\sqrt{\tau W}} \ln W \right) ,$$

or fermion bilinears of the form

$$(D \ln W) \left( \frac{\bar{D}}{\sqrt{\tau W}} \ln W \right) .$$

Consequently we may form six and higher spacetime derivative terms other than the ones we have seen so far.

We want to explore if it is possible to construct an $N = 4$ SYM effective action that contains the more general variables. For simplicity we restrict our analysis to the simple variables in (26).

From our previous construction we know that any six derivative term depending on $\tau W^2$ will be dualized at lowest order to the same six derivative term up to a sign. The dualization of $S^{(2)}_{\text{eff}}$ and $S^{(4)}_{\text{eff}}$ gives contributions only to a six derivative term of the form (19) and thus $S^{(6)}_{\text{eff}}[E^{ab}, P^{ab}]$ must be dualized to an isomorphic $S^{(6)}_{\text{eff}, D}[E^{ab}, P^{ab}]$ with positive sign. This last condition restricts the form of $S^{(6)}_{\text{eff}}$ to be

$$8\pi S^{(6)}_{\text{eff}}(1/\tau, 1/|\tau|) = \int d^4 x d^8 \theta \left( i \frac{e^2}{2} \ln \bar{W} \frac{D^4 \ln \bar{W}}{\tau W^2} + \text{c.c.} \right)$$

$$+ c^2 \frac{\lambda^{(6)}}{\sqrt{\tau \bar{\tau}}} \frac{D^{2ab} \ln W}{W} \frac{D_{ab}^2 \ln W}{W} .$$

Extending our construction to even higher derivative operators we find more terms mixing the $B$ and $E^{ab}$ variables as we increase the order in the expansion. As before, the coefficients of terms which are “even” under duality remain free parameters, whereas those of “odd” terms become linear combinations of the lower order coefficients. Details of such calculation are given in the appendix B.

4 SYM effective action and 3-brane dynamics

The effective action of the abelian piece in $N = 4$ SU($N_c - 1$) $\times$ U(1) is believed to describe the dynamics of extremal 3-branes in ten dimensions [13]: the $N = 0$ massless gauge
scalar $\phi = W|$ is related to the separation of one of the D3-branes from the rest in one of the complexified transversal directions, while the $N = 0$ massless gauge field strength $\hat{D}(a\bar{Q}_\beta) W = - \hat{D}(aW_\beta) = - F_{\alpha\beta}$ describes the gauge degrees of freedom living in the D3-brane. In this section we want to establish the exact nature of this correspondence by comparing the bosonic components of the $N = 4$ SYM effective action (at fixed $<\phi> \neq 0$), with the world volume field theory that describes the dynamics of a probe D3-brane at a short distance (small $\alpha' <\phi>$) from $N_c - 1$ source D3-branes [16]. The effective action for the fields living in the 3-brane is given by a supersymmetric DBI [17] [18] action

$$S_{BI}^{bos} = \int d^4x T_3 \left( h^{-1} - \sqrt{-\det(g_{ab} + F_{ab} - h^{\frac{1}{2}} \partial_a X^i \partial_b X^i)} \right)$$ (30)

where $x^a, a = 0, \ldots, 3$ are the coordinates along the 3-brane, $X^i, i = 4, \ldots, 9$ are the transversal coordinates, $F_{ab}$ is a gauge field strength on the brane, $T_3 \sim (\alpha')^{-2}$ is the 3-brane tension and $g_{ab} = h^{-\frac{1}{2}} \eta_{ab}$ is the induced flat metric of the probe 3-brane (the normalization $h = 1 + R^4/r^4 = 1 + \frac{8N}{T_3(4\pi X^i X_i)^2}$ is induced by the extremal ten-dimensional background metric)[20].

The $N = 4$ SYM effective action that we have derived is completely determined up to terms with four space-time derivatives: the coefficient of $S_{eff}^{(4)}$ is given by an explicit 1-loop calculation $c = (N_c - 1)/2\pi$ (see appendix A). In addition, self-duality fixes the coefficient of some terms with six space time derivatives. We can therefore attempt to compare the bosonic components of $S_{eff}^{(2)} + S_{eff}^{(4)} + S_{eff}^{(6)}$ (setting auxiliary fields equal to zero) with the DBI action in (30).

This comparison has been performed for $S_{eff}^{(2)} + S_{eff}^{(4)}$ in a $SU(2)$ theory [7], setting $F_{\alpha\beta} = 0$ and focusing on the velocity dependent terms. Comparisons for various Dp-branes in different backgrounds have appeared in [21] [22] [23].

From reference [4] we learn that we can identify the separation of the 3-branes in two of the transversal directions with the canonically normalized gauge scalar

$$X_i X_i = \frac{1}{g_2^2 T_3} \phi \bar{\phi} , \quad i = 4, 5 , \quad N = 2\pi c .$$ (31)

Then we find an agreement between the velocity terms of the Taylor expanded BI action

$$S_{BI} = \int d^4x T_3 h^{-1} \left( 1 - \sqrt{1 - h \dot{X}^i X_i} \right) = \frac{T_3}{2} \dot{X}^i \ddot{X}^i + \frac{T_3 h}{8} \left( \dot{X}^i \ddot{X}^i \right)^2 + \ldots ,$$ (32)

and the SYM effective action

$$(S_{\phi}^{(2)} + S_{\phi}^{(4)}) = \int d^4x \left[ \frac{1}{g_2^2} \dot{\phi} \dot{\bar{\phi}} + \frac{c}{8\pi} \left( \frac{|\dot{\phi}|^4}{|\phi|^4} - \frac{\dot{\phi} \ddot{\phi}}{\phi^2} - \frac{\ddot{\phi} \dot{\bar{\phi}}}{\phi^2} + 2 \frac{|\dot{\phi}|^2}{|\phi|^2} \right) \right] .$$ (33)

---

4 We follow the conventions of [1] to define the field strength in SL(2,C) spinor index notation $F_{\alpha\beta, \dot{\alpha}\dot{\beta}} = C_{\dot{\alpha}\dot{\beta}} \dot{F}_{\alpha\beta} + C_{\alpha\beta} \bar{F}_{\dot{\alpha}\dot{\beta}}$.

5 We choose a background in which the RR scalar and 2-form are zero to simplify our comparison, and therefore the WZ term does not give a contribution to the gauge effective action. In addition the vacuum angle is set to zero, i.e., $\tau = i4\pi /g^2$.

6 For $N_c - 1$ overlapping 3-branes separated from the last one, we have $N = (N_c - 1)$ [19].

7 Here we follow [1] ignoring the spatial dependence of the gauge scalars.
In [2] it was pointed out that because of the identification [11] the SYM effective action contains acceleration terms $\dot{\phi}$, which are absent in the expansion of [12]. The explanation suggested is that we should put the action on shell and compare S-matrix elements as opposed to the effective actions of both descriptions.

We establish the correspondence of the SYM effective action and the DBI action by redefining the $N = 1$ component superfields of $S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)}$ in such a way that the undesired acceleration terms of the bosonic action are absorbed in the gauge scalar and the gauge field strength. We conjecture that the redefinition can be extended to eliminate the acceleration terms of $S_{\text{eff}}^{(6)}$ and even higher derivative terms. Then the effective action of $N = 4$ SYM written with our redefined $N = 1$ superfields becomes the $N = 1$ superspace extension of the 3-brane action. In this redefined action the second supersymmetry must be realized nonlinearly, but this is precisely what we expect [17].

Let us begin by writing the four derivative terms in $N = 1$ superspace. After integrating the Grassmann coordinates of the second supersymmetry, integration by parts and some manipulation gives

$$S_{\text{eff}}^{(4)} = \frac{c}{8\pi} \int d^4x d^2 \bar{D} \left( Q^2 \ln W \bar{Q}^2 \ln \bar{W} - Q^\alpha \ln W (i\partial_{\alpha\dot{\alpha}}) \bar{Q}^\dot{\alpha} \ln \bar{W} + \ln W \square \ln \bar{W} \right),$$

$$= \frac{c}{8\pi} \int d^4x d^4\theta \left( \left[ \frac{D^2 \phi}{\phi} - \frac{W^2}{\phi^2} \right] - D_\alpha \left[ \frac{W^\alpha}{\phi^2} \right] - \bar{D}_\dot{\alpha} \left[ \frac{\bar{W}^\dot{\alpha}}{\phi^2} \right] + D^2 \ln \phi \bar{D}^2 \ln \bar{\phi} \right),$$

$$= \frac{c}{8\pi} \int d^4x d^4\theta \left( \bar{D}^2 \phi \left[ \frac{D^2 \phi}{\phi} - \frac{W^2}{\phi^2} - (D\phi)^2 \right] - D_\alpha \left[ \frac{W^\alpha}{\phi^2} \right] + \bar{D}_\dot{\alpha} \left[ \frac{\bar{W}^\dot{\alpha}}{\phi^2} \right] + \frac{W^2 \bar{W}^2 + (D\phi)^2 (\bar{D}\bar{\phi})^2 - W^\alpha D_\alpha \phi \bar{W}^\dot{\alpha} \bar{D}_\dot{\alpha} \bar{\phi}}{\phi^2 \phi^2} \right).$$

This particular way of writing the SYM effective action in $N = 1$ superspace is very illuminating: in the last line we have terms with four fermionic superfields that we will label $S_{\text{fermi}}^{(4)}$. Their $N = 0$ bosonic components contain only powers of the velocity and gauge field strength, but not their spacetime derivatives. In addition, the remainder of $S_{\text{eff}}^{(4)}$ can be written as $\int d^4\theta \bar{D}\Delta \bar{\Psi} + \int d^2\theta W^\alpha D_\alpha i\Delta V + \text{c.c.}$ where $\Delta V$ is real. The unconstrained $N = 1$ superfields

$$\Delta \bar{\Psi} = \frac{D^2 \phi}{\phi^2} - \frac{\bar{W}^2}{\phi^2} - \frac{(D\phi)^2}{\phi^2},$$

$$i\Delta V = \frac{1}{4} \bar{D}_\dot{\alpha} \bar{W}^\dot{\alpha} - \frac{1}{4} D_\alpha W^\alpha - \frac{\bar{W}^\alpha \bar{D}_\dot{\alpha} \bar{\phi}}{\phi^2} - \frac{\bar{W}^\dot{\alpha} \bar{D}_\alpha \bar{\phi}}{\phi^2},$$

that contain the acceleration terms of $S_{\text{eff}}^{(4)}$ may therefore be absorbed in a redefinition of the superfields appearing in the kinetic term.
\[ S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} = \frac{1}{2g^2} \int d^4x d^4\theta \tilde{\phi} \tilde{\phi} - \frac{2g^2c^2}{(8\pi)^2} \int d^4x d^4\theta (D^2 \Delta \Psi)(\bar{D}^2 \Delta \Psi) \]

\[ + \frac{1}{4g^2} \int d^4xD^2 \tilde{\tilde{W}}^\alpha \tilde{W}_\alpha \frac{2}{2(8\pi)^2} \int d^4xD^2 D^2 \left( D^\alpha (i\Delta V) D^2 D_\alpha (i\Delta V) \right) \]

\[ + \frac{1}{4g^2} \int d^4xD^2 \tilde{\tilde{W}}^\alpha \tilde{W}_\alpha \frac{2}{2(8\pi)^2} \int d^4xD^2 D^2 \left( \bar{D}^\alpha (i\Delta V) D^2 \bar{D}_\alpha (i\Delta V) \right) \]

\[ + \frac{c}{8\pi} \int d^4xd^4\theta \tilde{W}^2 \tilde{\tilde{W}}^2 + \frac{(D\tilde{\phi})^2 (\bar{D}\tilde{\phi})^2 - \tilde{W}^\alpha D_\alpha \tilde{\phi} \bar{\tilde{W}}^\alpha D_\alpha \tilde{\phi}}{\phi^2 \tilde{\phi}^2}. \]

where

\[ \tilde{\phi} = \phi + \frac{2g^2c^2}{8\pi} D^2 \Delta \Psi, \quad \tilde{W}_\alpha = W_\alpha - 4g^2c^2 \frac{D^2 D_\alpha (i\Delta V)}{8\pi}. \]

The additional six derivative operators suggest that there are acceleration terms (and also part of the \(v^6, F^6\) terms in \(S_{\text{eff}}^{(4)}\)) that enter in the redefinition of the kinetic term. We therefore conjecture that all the acceleration terms \(O^{(6)}\) in \(S_{\text{eff}}^{(4)}\) can be absorbed in a redefinition of \(S_{\text{eff}}^{(2)}\) and \(S_{\text{fermi}}^{(4)}\). We also conjecture that acceleration terms and part of the velocity terms in \(S_{\text{eff}}^{(n>6)}\) can be additionally absorbed in \(S_{\text{fermi}}^{(4)}\) to make it depend in the same superfields as \(S_{\text{eff}}^{(2)}\).

\[ S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} + O^{(6)} + ... = \frac{1}{2g^2} \int d^4x d^4\theta \tilde{\phi} \tilde{\phi} + \frac{1}{4g^2} \int d^4xD^2 \tilde{\tilde{W}}^\alpha \tilde{W}_\alpha \]

\[ + \frac{1}{4g^2} \int d^4xD^2 D^2 \left( D^\alpha (i\Delta V) D^2 D_\alpha (i\Delta V) \right) \]

\[ + \frac{c}{8\pi} \int d^4xd^4\theta \tilde{W}^2 \tilde{\tilde{W}}^2 + \frac{(D\tilde{\phi})^2 (\bar{D}\tilde{\phi})^2 - \tilde{W}^\alpha D_\alpha \tilde{\phi} \bar{\tilde{W}}^\alpha D_\alpha \tilde{\phi}}{\phi^2 \tilde{\phi}^2}. \]

Note that the bosonic part of this \(N = 1\) superspace action does not contain any acceleration terms

\[ S_{\text{bos}}^{\phi,F} = \left( \frac{1}{2} \frac{\dot{\phi} \cdot \dot{\phi}}{2} + \frac{1}{4g^2} \frac{\tilde{F}^2 + \bar{\tilde{F}}^2}{2} \right) \]

\[ + \frac{c}{8\pi} \frac{1}{2} \frac{\tilde{F}^2 \tilde{\phi}^2}{2} + \frac{1}{4g^2} \frac{(\partial \tilde{\phi})^2 (\partial \tilde{\phi})^2 - \tilde{F}^3 \partial_\alpha \tilde{\phi} \tilde{F}^\alpha \partial_{\bar{\alpha}} \tilde{\phi}}{\phi^2 \tilde{\phi}^2} + \text{auxiliary} \]

Identifying

\[ F^2 = \frac{1}{4g^2 T_3} \tilde{F}^2, \quad X_i X_i = \frac{1}{g^2 T_3} \tilde{\phi} \tilde{\phi}, \quad i = 4, 5, \quad N = 2\pi c. \]
we can exactly match (41) with the Taylor expansion of (31) up to four derivatives. In fact, we expect that after absorbing $O^{(6)}$ in $S^{(2)}_{\text{eff}} + S^{(4)}_{\text{fermi}}$ the remainder of $S^{(6)}_{\text{eff}}$ contains four or more fermionic superfields (i.e. its bosonic component depends on velocities and field strengths but not their derivatives)

$$S^{(6)}_{\text{eff}} - O^{(6)} = S^{(6)}_{\text{fermi}}.$$

A fraction $\epsilon$ of the coefficient of this fermionic action is also used in the redefinition, and the rest of it can absorb acceleration terms from $S^{(2n>6)}_{\text{eff}}$ to become a functional of $\tilde{\phi}, \tilde{W}_\alpha$. Matching the bosonic components of $(1 - \epsilon)S^{(6)}_{\text{fermi}}(\tilde{\phi}, \tilde{W}_\alpha)$ with the six derivative terms of the DBI action is then a highly nontrivial test of our redefinition.

If we were to extend our redefinition of the $N = 1$ superfields in the gauge multiplet to include arbitrary higher derivative pieces, we would expect to find a redefined action containing some acceleration terms in $S^{(2n>6)}_{\text{eff}}(\tilde{\phi}, \tilde{W}_\alpha)$ [24]. Then the $N = 4$ SYM effective action and the DBI action plus its higher genus corrections would agree to all orders.

When we try to test our conjecture for $S^{(6)}_{\text{eff}}$ we find an important obstacle: we have mentioned that duality fixes the numerical coefficient of the contributions to $S^{(6)}_{\text{eff}}$ which are analytic in $1/\tau$ (see (29)). Therefore we can only test our conjecture for that particular piece $S^{(6)}_{\text{eff}}(1/\tau)$. The coefficients of contributions proportional to $1/|\tau|$ are not constrained by duality, and we can only hope that this comparison will fix them.

The detailed calculation of the higher order terms $O^{(6)}$ needed for our redefinition to work is quite tedious and we have included it in the appendix C. The main result from that calculation is the following: the only term in $S^{(6)}_{\text{eff}}(1/\tau)$ which is not absorbed in $S^{(2)}_{\text{eff}} + S^{(4)}_{\text{eff}}$ and contributes to the bosonic gauge action is given by

$$(1 - \epsilon)S^{(6)}_{\text{fermi}}(1/\tau) = -\frac{2g^2c^2}{(8\pi)^2} \int d^4x \, d^4\theta \, \frac{1}{2} \left[ (D\tilde{W})^2 + (D\tilde{\tilde{W}})^2 \right] \frac{(\tilde{W})^2(\tilde{\tilde{W}})^2}{\tilde{\phi}^4}$$

$$= -\frac{g^2c^2}{4(8\pi)^2} \int d^4x \, (\tilde{F}^2 + \tilde{\tilde{F}}^2) \frac{\tilde{F}^2 \tilde{\tilde{F}}^2}{\tilde{\phi}^4} + \text{fermions + auxiliary}. \quad (42)$$

Our identification (41) provides again an exact match to the six derivative term in the Taylor expansion of the gauge DBI action.

$$S_{\text{BI}} = \int d^4x T_3 h^{-1} \left( 1 - \sqrt{1 + h(F^2 + \tilde{F}^2) + \frac{1}{4}h^2(F^2 - \tilde{F}^2)^2} \right)$$

$$= \int d^4x \left( -T_3 \frac{F^2 + \tilde{F}^2}{2} + \frac{1}{2}T_3 h F^2 \tilde{F}^2 - \frac{1}{4}T_3 h^2 F^2 \tilde{F}^2 (F^2 + \tilde{F}^2) + \ldots \right)$$

$$= \int d^4x \left( -T_3 \frac{F^2 + \tilde{F}^2}{2} + 4N \frac{F^2 \tilde{F}^2}{\bar{r}^4} - \frac{16(N)^2 F^2 \tilde{F}^2 (F^2 + \tilde{F}^2)}{T_3 \bar{r}^8} + \ldots \right),$$

where the DBI field strength is written with $SL(2,C)$ spinor indices and we have defined $\bar{r}^2 = 4\pi X^i X^i = 4\pi \tilde{\phi} \bar{\phi}/g^2T_3$. Note that without the redefinition, setting acceleration terms
to zero in $S^{(6)}$ would not be enough to achieve the agreement between both descriptions of the 3-brane dynamics.

To try to gain some understanding about the physical meaning of our redefinition we consider again the duality transformation of the gauge scalars: in the brane picture, the DBI action in a curved background is self-dual \cite{25}, but the transversal separation $\tilde{r}^2$ remains unchanged \cite{17, 18} when we dualize the $N = 0$ gauge field strength of the DBI action

$$S_{DBI}^D = \int d^4 x \ T_3 \ h^{-1}(\tilde{r}) \left( 1 - \sqrt{1 + h(\tilde{r}) (F_D^2 + \bar{F}_D^2)} + \frac{1}{4} h^2(\tilde{r}) (F_D^2 - \bar{F}_D^2)^2 \right)$$

(44)

whereas in the SYM picture the scalar $r^2 = 4\pi \phi \bar{\phi} / g^2 T_3$ transforms in accordance with

$$r_D^2 = \frac{4\pi}{g_D^2 T_3} W_D \bar{W}_D \bigg| = \frac{g^2}{4\pi T_3} \left( \frac{4\pi}{g^2} W + 2c \tilde{D}_4 \ln \tilde{W} + \ldots \right) \left( \frac{4\pi}{g^2} \bar{W} + 2c \tilde{D}_4 \ln \bar{W} + \ldots \right) \bigg| \approx r^2 \left( 1 + \frac{4\pi c}{g^2 T_3} \frac{\bar{F}^2}{r^4} + \ldots \right) \left( 1 + \frac{4\pi c}{g^2 T_3} \frac{F^2}{r^4} + \ldots \right).$$

(45)

The self-dual scalar $\tilde{r}^2$ is therefore a nonlinear combination of $r^2$ and the gauge field strengths which remains invariant under duality (we are ignoring the velocity terms). Up to two derivatives it is given by

$$\tilde{r}^2 = \frac{4\pi}{g^2 T_3} W \left( 1 + \frac{cg^2}{4\pi} \tilde{D}_4 \ln \tilde{W} + \ldots \right) \bar{W} \left( 1 + \frac{cg^2}{4\pi} D_4 \ln W + \ldots \right) \bigg| = r^2 \left( 1 + \frac{2\pi c}{g^2 T_3} \frac{\bar{F}^2}{r^4} \right) \left( 1 + \frac{2\pi c}{g^2 T_3} \frac{F^2}{r^4} \right) + \ldots = r^2 + r_D^2 + \ldots$$

(46)

Remarkably, this is consistent with the shift \cite{35, 37} needed to remove the acceleration terms from the effective action. For a constant gauge field strength we can see that in the SYM picture the separation $r$ between the probe 3-brane and the source 3-brane is normalized by a scale that contains information about the charge density in the 3-brane (i.e. about the internal energy density)

$$\tilde{r}^4 = r^4 + \frac{4\pi c}{g^2 T_3^2} (F^2 + \bar{F}^2) + \ldots$$

(47)

Perhaps this rescaling accounts for a correction to the background metric induced by the gauge fields living in the 3-brane. In the probe-source picture this information is implicit in the separation variable.

In summary, we are able to successfully compare the effective action of $N = 4$ SYM with the DBI action of a gauge field strength living in the D3-brane up to terms with six space-time derivatives. This comparison reveals the surprising fact that the gauge scalar and the separation of the probe and source 3-branes are not simply proportional to each other.
Contrary to common lore, we find that the relation involves a nonlinear function of the field strengths living in the probe. We emphasize that our results are valid for arbitrary $g^2$ and arbitrary $N_c$. They do not contradict previous ideas [26] [27], but merely refine them: the issue of which variable $r$ transforms linearly under $N = 2$ SUSY has not been addressed in comparing the DBI action with the SYM effective action.

We have restricted our analysis to the abelian sector of $N = 4$ SYM theory in four dimensions. It seems plausible that in the unbroken gauge theory of overlapping 3-branes, an extension of the redefinition we have presented is also necessary. This would have important consequences for the identification of the fields appearing in the correlators of the superconformal four-dimensional theory that lives in the boundary of $AdS_5$ [28]. Such fields would not correspond exactly to the asymptotic states of the $N = 4$ SYM theory where the supersymmetry is linearly realized.

5 Conclusions and open problems

In this article we have succeeded in constructing the momentum expansion of a self-dual $N = 4$ SYM effective action, which by scale and $U(1)_R$ invariance turns out to be also an expansion on the inverse of the holomorphic gauge coupling $\tau = \theta_v + i \frac{4\pi}{g^2}$ (and its absolute value if we include more general contributions).

Our most striking observation is the redefinition of the Higgs field needed to reconcile $N = 2$ supersymmetry and the self-duality of the DBI action.

The expansion of the effective action that we have found is analytic in $g^2$. For $\theta_v \neq 0$, the terms that we find seem to be in conflict with the type of contributions obtained from instantons. We find the apparent contradiction that there are nonvanishing instanton contributions to $S_{eff}^{(6)}$ (and to higher order terms) [29] that introduce a dependence of the effective action on $\theta_v$, and yet are not analytic in $g^2$.

Since our analysis only specifies the coefficients of some terms but not others, it is possible that these terms can be completed to have the appropriate $\theta_v$ dependence; however, we do not see how to do this. On the other hand, our results could signal a breakdown in the instanton series. This seems unlikely in light of the results of [30].

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Appendix A

In this appendix we generalize the results of [8] to an arbitrary $SU(N_c)$ gauge group spontaneously broken to $SU(N_c - 1) \times U(1)$ or smaller subgroups. In that reference we found the nonholomorphic potential in $N = 2$ superspace of a generic $N = 4$ SYM theory as a momentum integral

$$\mathcal{H}(W, \bar{W}) = \frac{1}{2} \int \frac{dp^2}{(4\pi)^2 p^2} \text{Tr}_A \ln \left( 1 + \frac{WW + \bar{W}\bar{W}}{2p^2} \right),$$

(48)

and we were able to evaluate this integral for the abelian subgroup of $SU(2)$. The key to generalize this result is to evaluate the gauge operator $WW + \bar{W}\bar{W}$ for a particular $U(1)$ subgroup of $SU(N_c)$. This is very straightforward when we write a generator in the adjoint representation in terms of its fundamental representation

$$(T^a_A)^i_j k l = (T^a_F)^i_j k l - (T^a_F)^i_j k l.$$

(49)

We want to break $SU(N_c)$ by giving a nonzero v.e.v. to the gauge scalar aligned with a particular linear combination of Cartan generators $W^i_j k l = \langle W^a \rangle (T^a_A)^i_j k l$. The operator we want to evaluate is therefore

$$(WW + \bar{W}\bar{W})^i_j k l = (WW + \bar{W}\bar{W})^i_j k l + (WW + \bar{W}\bar{W})^i_j k l - 2W^i_j \bar{W}^k l - 2\bar{W}^i_j W^k l.$$

(50)

Once we choose a linear combination in the fundamental representation, the corresponding gauge scalars parameterize the transversal positions of the associated $N_c$ 3-branes. We are mostly interested on the configuration of $N_c - 1$ overlapping 3-branes separated from another 3-brane. The $U(1)$ gauge scalar has a v.e.v.

$$W = \langle W \rangle T^{N_c - 1}_F = \frac{1}{\sqrt{2N_c(N_c - 1)}} \text{diag}(w, w, ..., w, (1 - N_c)w).$$

(51)

Substituting this background field in (50) we find a diagonal adjoint representation matrix $(WW + \bar{W}\bar{W})_A$ with $2(N_c - 1)$ nonzero elements

$$\frac{1}{2}(WW + \bar{W}\bar{W})_A = (WW)_A = (\bar{W}\bar{W})_A = \begin{pmatrix}
0 & \dots & \dots & \dots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\frac{N_c}{2(N_c - 1)} w\bar{w} & \frac{N_c}{2(N_c - 1)} w\bar{w} & \frac{N_c}{2(N_c - 1)} w\bar{w} & 0
\end{pmatrix}.$$

(52)

This facilitates the evaluation of the trace in (48)

$$\mathcal{H}(W, \bar{W}) = \frac{N_c - 1}{(4\pi)^2} \int \frac{dp^2}{p^2} \ln \left( 1 + \frac{N_c}{2(N_c - 1)} \frac{WW}{p^2} \right).$$

(53)
Using dimensional regularization as in [8] we are able to perform the momentum integral and we obtain

\[ \int d^4 x \, d^8 \theta \, \mathcal{H}^{\text{Abel}}(W, \bar{W}) = \frac{N_c - 1}{(4\pi)^2} \int d^4 x \, D^4 \bar{D}^4 \left( \ln \frac{N_c}{2(N_c - 1)} \frac{W \bar{W}}{\mu^2} \right)^2 \]

\[ = \frac{N_c - 1}{(4\pi)^2} \int d^4 x \, D^4 \bar{D}^4 \ln W \ln \bar{W} . \tag{54} \]

If we try to perform a similar calculation for the massless nonabelian multiplets associated with the unbroken SU\((N_c - 1)\) theory we find IR divergences that need to be regulated. We therefore simplify our analysis by focusing on the U\((1)\) sector.

Notice that the numerical factor in each diagonal matrix element of (52) only changes the normalization scale in the final expression, and since we have a full N = 2 superspace measure the action is scale independent.

This observation is important when we want to consider different 3-brane configurations parameterized by other gauge scalar v.e.v.’s which are diagonal in the fundamental representation, and break SU\((N_c)\) further. We obtain again a diagonal operator \((W \bar{W})_F\) in the adjoint representation, with different coefficients in its nonzero elements. Such coefficients end up modifying the normalization scale, and they drop out of the nonholomorphic potential. The only relevant quantity is the overall number of nonzero eigenvalues in \((W \bar{W})_A\).

It is straightforward to see that these eigenvalues are given by the differences of eigenvalues of \((W \bar{W})_F\) in the fundamental representation [19]. Hence, a generically broken SU\((N_c)\) configuration\(^8\)

\[ W = \langle W_a \rangle T_F^a = \text{diag} \left( w_{11}, w_{22}, \ldots, w_{N_c N_c} \right) , \quad \sum_{i=1}^{N_c} w_{ii} = 0 , \tag{55} \]

will give a nonholomorphic potential (if \(W_{ii} = W_{jj}\) the corresponding term is absent from the sum)

\[ \sum_{i<j} S^{(4)}_{ij} = \frac{1}{(4\pi)^2} \int d^4 x \, d^8 \theta \sum_{i<j} \ln (W_{ii} - W_{jj}) \ln (\bar{W}_{ii} - \bar{W}_{jj}) . \tag{56} \]

In these more general configurations, the correspondence with the DBI action is more subtle. Since the tree level action of these abelian components can also be decomposed for an SU\((N_c)\) group

\[ S^{(2)} = \frac{R e}{2g^2} \int d^4 x D^2 Q^2 \sum_{i=1}^{N_c} W_{ii}^2 = \frac{R e}{2g^2} \int d^4 x D^2 Q^2 \sum_{i=1}^{N_c} \sum_{j=1}^{N_c} (W_{ii} - W_{jj})^2 = \sum_{i<j} S^{(2)}_{ij} \tag{57} \]

the interactions of the 3-branes seem to group pairwise [27]. We would have a probe-source DBI description for each pair \(i, j\) of 3-branes, whose Taylor expansion should be matched (up to four spacetime derivatives) with an effective action \(S^{(2)}_{ij} + S^{(4)}_{ij}\).

\(^8\)Note added in proof: a derivation of the same N = 2 nonholomorphic potential corresponding to this configuration appeared in [31, 32] shortly after this manuscript was completed.
Appendix B

In this appendix, we will simply quote the higher order result. We give the self-dual gauge effective action that depends on scale and \(U(1)_{\pi} \) invariant chiral variables up to terms with sixteen space-time derivatives. Then we present the gauge effective action that depends on \(D \) and \(D_{ab} \) up to eight spacetime derivatives. Since the expression is rather long, the following notation is convenient

\[
\mathcal{D} \equiv \frac{D^4}{\tau \sqrt{2}}, \quad B \equiv \frac{D^4 \ln \bar{W}}{\tau \sqrt{2}}, \quad \int \equiv \int d^4 x \, D^4 \bar{D}^4.
\] (58)

Using this definition, the effective action is

\[
8\pi S_{\text{eff}} = 8\pi \left( S^{(2)} + S^{(4)} + S^{(6)} + S^{(8)} + S^{(10)} + S^{(12)} + S^{(14)} + S^{(16)} + \cdots \right)
\]

\[
= \text{Im} \int d^4 x \, D^4 \frac{1}{2} \tau W^2 + c \int \ln W \, \ln \bar{W} \left[ + i c^2 \int \frac{1}{2} \ln \bar{W} B \\
+ c^3 \int \left( \frac{1}{2} \kappa_1^{(8)} B \bar{B} + \kappa_2^{(8)} \ln \bar{W} B^2 \right) \right.
\]

\[
+ i c^4 \int \left( \kappa_1^{(10)} \ln W \, B^3 + \kappa_2^{(10)} B^2 \bar{B} + \kappa_3^{(10)} B \mathcal{D}(B) \right)
\]

\[
+ c^5 \int \left( \kappa_1^{(12)} \ln \bar{W} \, B^4 + \kappa_2^{(12)} B^3 \bar{B} + \frac{1}{2} \kappa_3^{(12)} B^2 \bar{B}^2 + \frac{1}{2} \kappa_4^{(12)} \mathcal{D}(B) \mathcal{D}(\bar{B}) \\
+ \kappa_5^{(12)} B^2 \mathcal{D}(B) + \kappa_6^{(12)} B \bar{B} \mathcal{D}(B) \right)
\]

\[
+ i c^6 \int \left( \kappa_1^{(14)} \ln \bar{W} \, B^5 + \kappa_2^{(14)} B^4 \bar{B} + \kappa_3^{(14)} B^3 \bar{B}^2 + \kappa_4^{(14)} B^2 \bar{B}^3 \bar{D}(B) \\
+ \kappa_5^{(14)} B^2 \mathcal{D}(B^2) + \kappa_6^{(14)} B(\mathcal{D}(B))^2 + \kappa_7^{(14)} B^2 \bar{B} \mathcal{D}(B) \right)
\]

\[
+ \kappa_8^{(14)} B \bar{B}^2 \mathcal{D}(B) + \kappa_9^{(14)} B \mathcal{D}(B) \mathcal{D}(\bar{B}) + \kappa_{10}^{(14)} \mathcal{D}(B) \mathcal{D}(\bar{B}) \mathcal{D}(B^2) \bar{D}(B) \bar{D}(\bar{B}) \\
+ \kappa_{11}^{(14)} \mathcal{D}(B) \mathcal{D}(\mathcal{D}(B)) \right)
\]

\[
+ c^7 \int \left( \kappa_1^{(16)} \ln \bar{W} B^6 + \kappa_2^{(16)} B^5 \bar{B} + \kappa_3^{(16)} B^4 \bar{B}^2 + \frac{1}{2} \kappa_4^{(16)} B^3 \bar{B}^3 + \kappa_5^{(16)} B^4 \mathcal{D}(B) \right)
\]

\[
+ \kappa_6^{(16)} B^3 \bar{B} \mathcal{D}(B) + \kappa_7^{(16)} B^3 \bar{B} \bar{D}(B) + \kappa_8^{(16)} B^3 \frac{\mathcal{D}(B^2) + \kappa_9^{(16)} B^2 \mathcal{D}(B) \bar{D}(B) \bar{D}(\bar{B})} \\
+ \kappa_{10}^{(16)} B^2 \bar{B} \mathcal{D}(B) \mathcal{D}(\bar{B}) \mathcal{D}(B) \bar{D}(B) \bar{D}(\bar{B}) \bar{D}(B) \bar{D}(\bar{B}) \bar{D}(B) \bar{D}(\bar{B}) \bar{D}(B) \bar{D}(\bar{B}) \bar{D}(B) \bar{D}(\bar{B}) \bar{D}(B) \bar{D}(\bar{B}) \bar{D}(B) \bar{D}(\bar{B}) \\
+ \kappa_{19}^{(16)} \mathcal{D}(B)^2 \mathcal{D}(B) + \kappa_{20}^{(16)} \mathcal{D}(B^3) \mathcal{D}(B) \mathcal{D}(\Delta(B)) + \kappa_{21}^{(16)} \mathcal{D}(B^2) \mathcal{D}(\Delta(B)) \right) \\
+ \frac{1}{2} \kappa_{22}^{(16)} \mathcal{D}(B^2) \mathcal{D}(\bar{B}^2) \right] + \ldots + c.c.
\] (60)

where \( \kappa_1^{(8)} \), \( \kappa_3^{(10)} \), \( \kappa_4^{(12)} \), \( \kappa_4^{(16)} \), \( \kappa_{15}^{(16)} \) and \( \kappa_{22}^{(16)} \) are real and

\[
\kappa_1^{(10)} \equiv -4\kappa_2^{(8)} + \frac{7}{6}.
\]
\begin{align}
\kappa_2^{(10)} & \equiv -2 \kappa_1^{(8)} - 3 \kappa_2^{(8)} + 2 \\
\kappa_3^{(10)} & \equiv 2 \kappa_1^{(8)} - 1 \\
\kappa_1^{(14)} & \equiv -8 \kappa_1^{(12)} - 72 \kappa_2^{(8)} + 8 \kappa_2^{(8)} + \frac{121}{5} \\
\kappa_2^{(14)} & \equiv -5 \kappa_1^{(12)} - 20 \kappa_1^{(8)} - 6 \kappa_2^{(12)} - 104 \kappa_2^{(8)} + 8 \kappa_1^{(8)} \kappa_2^{(8)} + 48 \\
\kappa_3^{(14)} & \equiv 2 \kappa_2^{(12)} + 16 \kappa_1^{(8)} + 32 \kappa_2^{(8)} + 14 \kappa_3^{(8)} - 4 \kappa_3^{(12)} - 12 \kappa_2^{(8)} \kappa_2^{(8)} - \frac{76}{3} \\
\kappa_4^{(14)} & \equiv \kappa_2^{(12)} + 16 \kappa_2^{(8)} + \frac{38}{3} \kappa_1^{(8)} - 4 \kappa_3^{(12)} - 4 \kappa_1^{(8)} \kappa_2^{(8)} - \frac{38}{3} \\
\kappa_5^{(14)} & \equiv 4 \kappa_1^{(8)} + 6 \kappa_2^{(8)} - 2 \kappa_5^{(12)} - \frac{9}{2} \kappa_2^{(8)} - 4 \\
\kappa_6^{(14)} & \equiv 2 \kappa_1^{(8)} + \kappa_6^{(12)} + 2 \kappa_2^{(8)} - 2 \\
\kappa_7^{(14)} & \equiv -4 \kappa_6^{(12)} - 20 \kappa_1^{(8)} + 2 \kappa_5^{(12)} - 12 \kappa_2^{(8)} - 12 \kappa_2^{(8)} + 6 \kappa_1^{(8)} \kappa_2^{(8)} + 2 \kappa_3^{(12)} + 20 \\
\kappa_8^{(14)} & \equiv 3 \kappa_2^{(12)} + 20 \kappa_1^{(8)} + 4 \kappa_6^{(12)} - 2 \kappa_1^{(8)} + 36 \kappa_2^{(8)} - 25 \\
\kappa_9^{(14)} & \equiv 14 \kappa_1^{(8)} + 6 \kappa_2^{(8)} + 2 \kappa_6^{(12)} - 2 \kappa_5^{(12)} - 2 \kappa_4^{(12)} - 2 \kappa_1^{(8)} - 2 - 10 \\
\kappa_{10}^{(14)} & \equiv 6 \kappa_1^{(8)} + 3 \kappa_2^{(8)} - 3 \kappa_1^{(8)} \kappa_2^{(8)} - 2 \kappa_5^{(12)} - 2 \kappa_4^{(12)} - 4 \\
\kappa_{11}^{(14)} & \equiv -2 \kappa_1^{(8)} + \kappa_4^{(12)} + \frac{1}{2} \kappa_1^{(8)} + 1
\end{align}

In section 3, we pointed out that self-duality does not exclude operators such as (26). We also introduced a simplified notation for them

\[ \mathcal{P}^{ab} = \frac{D^{2ab}}{\sqrt{\tau W}} \quad , \quad E^{ab} = \frac{D^{2ab} \ln W}{\sqrt{\tau W}} = \frac{D^{2ab} \ln(\sqrt{\tau W})}{\sqrt{\tau W}} . \]

It is possible to construct a self-dual effective action that includes these operators

\begin{align}
8\pi S_{\text{eff}} &= 8\pi \left( S^{(2)} + S^{(4)} + S^{(6)} + S^{(8)} + S^{(10)} + \ldots \right) \\
&= \mathcal{I} \mathcal{M} \left( \int d^4x \ D^4 \frac{1}{2} \tau W^2 \right) + c \int d^4x \ d^8\theta \ \ln W \ \ln \bar{W} \\
& \quad \left[ + c^2 \int \left( i \frac{\tau}{2} B \ln \bar{W} + \frac{1}{2} \lambda^{(6)} E \cdot E \right) \\
& \quad + c^3 \int \left( \frac{1}{2} \kappa^{(8)} B \bar{B} + i \lambda^{(6)} B(\mathcal{P} \cdot E - \bar{E} \cdot E) + \kappa^{(8)} B^2 \ln W \right) \\
& \quad + c^4 \int \left( \left( -8 i \kappa^{(8)} + i \frac{7}{3} \right) B^2 \ln \bar{W} + \left( 2 i \kappa^{(8)} - i \right) B \mathcal{D}(B) \\
& \quad + \left( -6 i \kappa^{(8)} - 4 i \kappa^{(8)} + 4 i \right) \bar{B}^2 \bar{B} \\
& \quad + \lambda^{(10)} B^2 (\bar{E} \cdot E - \mathcal{P} \cdot E) + \frac{1}{4} \mathcal{L}^{(10)} B \bar{B} \left( \bar{E} \cdot E - \mathcal{P} \cdot E - \bar{E} \cdot (\bar{E}) \right) \\
& \quad + \lambda^{(10)} B \mathcal{E}(B) + \lambda^{(10)} \bar{E} \cdot E \mathcal{D}(B) + \lambda^{(10)} \mathcal{P} \cdot (\bar{E}) \mathcal{D}(B) \\
& \quad + \frac{1}{4} \lambda^{(10)} \mathcal{P} \cdot \mathcal{B} \cdot \mathcal{P}(B) \\
& \quad - \frac{i}{2} \left( \lambda^{(6)} \right)^2 \left( \bar{E} \cdot E - \mathcal{P} \cdot E \right) \mathcal{D} \left( \bar{E} \cdot E - \mathcal{P} \cdot E + \ldots \right) + c.c. \right] \end{align}
Note that we could have also added a term $c^3 \lambda^{(8)} (\vec{E} \cdot E)^2$ to $S_{\text{eff}}^{(8)}$ which would be mapped to itself at lowest order under duality, preserving the self-duality at this order.

Appendix C

In this appendix we present the detailed calculation of the $N = 1$ components with six space-time derivatives which are needed in the SYM effective action for our redefinition \(^{[31]}\) to work. Let us begin our analysis identifying the six derivative operators that are implied by the redefinition of two and four derivative operators (for brevity we write $V$ and $\Psi$ instead of $\Delta V$ and $\Delta \Psi$)

\[
O^{(6)} = \int d^4x \, d^4\theta \left( \frac{2g^2c^2}{(8\pi)^2} \left( (D^2\Psi)(\bar{D}^2\Psi) - [D^\alpha(iV)\bar{D}^2D_\alpha(iV)] - [\bar{D}^\alpha(iV)D^2\bar{D}_\alpha(iV)] \right) \right.
\]
\[
+ \frac{c}{8\pi} \left[ \frac{4g^2c^2}{8\pi} D^2D_\alpha(iV) \right] W_\alpha(\bar{W})^2 + (W)^2 \left[ \frac{4g^2c^2}{8\pi} D^2\bar{D}^\alpha(iV) \right] \bar{W}_\alpha
\]
\[
\left. + \frac{c}{8\pi} D^\alpha \phi \left[ \frac{4g^2c^2}{8\pi} D_\alpha\bar{D}^2\bar{\Psi} \right] (\bar{D}\bar{\phi})^2 + (D\phi)^2 \bar{D}^\alpha\bar{\phi} \left[ \frac{4g^2c^2}{8\pi} \bar{D}_\alpha D^2\Psi \right] \bar{W}_\alpha \bar{D}_\alpha\bar{\phi} \right).
\]

(63)

To simplify our analysis, we ignore all terms containing the auxiliary superfields $D^2\phi$, $D_\alpha W^\alpha$ and their conjugates. The first type can be absorbed as a higher order correction in the redefinition of the scalar kinetic term $\bar{\phi}\phi$ and the second can be absorbed in a redefinition of the gauge spinor kinetic term $W^2 + c.c.$ Such higher order corrections will in turn require acceleration terms from $S_{\text{eff}}^{(8)}$ and $S_{\text{eff}}^{(10)}$. We know the general form of these higher corrections but not their precise numerical coefficients, and therefore we can only establish that the required acceleration terms are generically present in the effective action\(^{[31]}\).

After integrating by parts and some algebra we find that it is useful to classify the terms without auxiliary superfields by counting the number of fermionic superfields

\[
O_0^{(6)} = \frac{g^2c^2}{(8\pi)^2} \int d^4x \, d^4\theta \left( 2 \frac{1}{2}(D\bar{W})^2 \frac{1}{2}(\bar{D}\bar{W})^2 \right) \phi^4\phi^3 + 2 \frac{1}{2}(\bar{D}\bar{D}\phi)^2 \frac{1}{2}(D\bar{D}\bar{\phi})^2 \phi^3\phi^3 + 3 \frac{1}{2}(D\bar{W})^2 \frac{1}{2}(\bar{D}\bar{\phi})^2 \phi^2\phi^4
\]
\[
+ 3 \frac{1}{2}(\bar{D}^\alpha W^\beta)(\bar{D}_\alpha\bar{D}_\beta\bar{\phi})(\bar{D}^\alpha\bar{\phi})(\bar{D}_\alpha D_\beta\bar{\phi}) \phi^3\phi^3 \right).
\]

(64)

\(^{9}\text{It is worth mentioning that the elimination of auxiliary superfields through redefinition of the physical ones is a feature we have encountered in the study of the supersymmetric formulation of the BI action}^{[33]}\) and the 3-brane living in six dimensions \(^{[34]}\).
\[ O_2^{(6)} = \frac{g^2 c^2}{(8\pi)^2} \int d^4x \, d^4\theta \left\{ \frac{1}{2} \left( \bar{D} \bar{D} \bar{\phi} \right)^2 \left( \bar{D} \bar{W} \beta \bar{\phi} \right) \bar{D} \bar{\phi} + \frac{1}{2} \left( \bar{D} \bar{W} \beta \bar{\phi} \right)^2 \bar{D} \bar{\phi} \right\} + \frac{1}{2} \left( \bar{D} \bar{W} \beta \bar{\phi} \right)^2 \bar{D} \bar{\phi} \bar{D} \bar{\phi} \\
+ \frac{1}{2} \left( \bar{D} \bar{W} \beta \bar{\phi} \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} + \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \\
+ \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \\
+ \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \\
+ \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \right\} + \text{c.c.} \]  

(65)

\[ O_4^{(6)} = \frac{g^2 c^2}{(8\pi)^2} \int d^4x \, d^4\theta \left\{ \frac{1}{2} \left( \bar{D} \bar{D} \bar{\phi} \right)^2 \left( \bar{D} \bar{W} \beta \bar{\phi} \right) \bar{D} \bar{\phi} + \frac{1}{2} \left( \bar{D} \bar{W} \beta \bar{\phi} \right)^2 \bar{D} \bar{\phi} \right\} + \frac{1}{2} \left( \bar{D} \bar{W} \beta \bar{\phi} \right)^2 \bar{D} \bar{\phi} \bar{D} \bar{\phi} \\
+ \frac{1}{2} \left( \bar{D} \bar{W} \beta \bar{\phi} \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} + \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \\
+ \frac{1}{2} \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \\
+ \frac{1}{2} \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \left( \bar{D} \bar{D} \phi \right) \bar{D} \bar{\phi} \right\} + \text{c.c.} \]  

(66)
We can evaluate now the $N = 1$ components of $S_{\text{eff}}^{(6)}(1/\tau)$ in (21). Dropping again any term depending on auxiliary superfields we obtain

\[
S_{\text{eff}}^{(6)}(1/\tau) = \frac{g^2 c^2}{(8\pi)^2} \int d^4x \, d^4\theta \left[ \frac{1}{2} (D\bar{\psi})^2 - \frac{1}{2} (D\Phi)^2 - \frac{1}{2} (D\bar{W})^2 + \frac{1}{2} (D\bar{\theta})^2 \right]
\]

\[
+ 6 \frac{1}{2} (D\bar{\theta})^2 (D\bar{W})^2 (D\phi)^2 + 4 \frac{1}{2} (D\bar{W})^2 (D\bar{\Phi})^2 + 8 \frac{1}{2} (D\bar{W})^2 (D\bar{\phi})^2 (W)^2
\]

\[
+ 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 - 2 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + \frac{1}{2} (D\bar{\Phi})^2 (W)^2
\]

\[
+ 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2
\]

\[
+ c.c. \quad \text{(67)}
\]

We can see that all the terms in this action except the second and the last three have precisely the correct numerical coefficients for them to be absorbed in the redefinition of the superfields appearing in $S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)}$. Using part of the last term in the redefinition all we are left with is

\[
S_{\text{eff}}^{(6)} - \epsilon S_{\text{fermi}}^{(6)} = \frac{g^2 c^2}{(8\pi)^2} \int d^4x \, d^4\theta \left[ - 2 \frac{1}{2} (D\bar{\psi})^2 (W)^2 (W)^2 + \frac{1}{2} (D\bar{\theta})^2 (W)^2 \right]
\]

\[
+ 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + 1 \frac{1}{2} (D\bar{\Phi})^2 (W)^2 + c.c. \quad \text{(68)}
\]

The first term is precisely the six derivative contribution we are looking for in the redefined $N = 1$ action. The last three we expect to cancel against similar terms in $S_{\text{eff}}^{(6)}(1/|\tau|)$. In
any case such contributions contain products of velocities and gauge field strengths, and therefore do not affect our comparison with the gauge part of the DBI action.

Since we do not know the exact form $S_{eff}^{(6)}(1/|\tau|)$ we can only try to guess the operators present in this piece of the effective action and fix the coefficients by matching their $N = 1$ components with $O^{(6)}(1/|\tau|)$. This is a difficult task, further complicated by the ambiguity of integration by parts. It is worth mentioning that the $N = 1$ components of the operators in (26) are different from those in (67) and therefore our partial result is unchanged by the $\mathcal{P}^{ab}$ terms. The analysis of these contributions will be presented in a future publication.

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