Semi-orthogonal Parseval Wavelets 
Associated with GMRAs on Local Fields 
of Positive Characteristic

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Abstract. In this article, we establish theory of semi-orthogonal Parseval wavelets associated with generalized multiresolution analysis (GMRA) for local fields of positive characteristics (LFPC) and obtain their characterization in terms of consistency equation. As a consequence, we obtain a characterization of an orthonormal (multi)wavelet associated with an MRA in terms of multiplicity function as well as dimension function. Further, we provide characterizations of Parseval scaling functions, scaling sets and bandlimited wavelets together with a Shannon-type multiwavelet. Some examples of such wavelets are also produced for LFPC.

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1. Introduction

In the recent years, wavelets on local fields of positive characteristic (LFPC) have been extensively studied by many authors including Behera and Jahan, Benedetto and Benedetto, Jiang, Li and Ji, Vyas and the first author with respect to multiresolution analysis (MRA), tight wavelet frame, low-pass filter, etc. in the references [6–10,20,22,24], but still more concepts need to be studied for its enhancement. Indeed, the development of the theory of wavelet analysis with respect to groups other than Euclidean spaces, namely, $p$-adic groups, Cantor dyadic groups, Vilenkin groups, locally compact abelian groups (LCAG), Heisenberg group, etc., has always been interesting for researchers due to its various applications [1,2,9,16,18,19,21].

Motivated by the recent work of Behera [6], our main goal is to develop the theory of semi-orthogonal Parseval (multi)wavelets associated with generalized multiresolution analysis (GMRA) in the setting of LFPC by using the concept of translation invariant spaces. A rigorous study of semi-orthogonal
Parseval (multi)wavelets and GMRAs for the Euclidean spaces has been done by many authors [3, 4, 12, 14, 15, 23]. The concept of GMRA was introduced by Baggett, Medina and Merrill in [3] for separable Hilbert spaces. They used unitary representation of the group of translations acting on the fundamental subspace $V_0$ (known as, core space) that generates the subspaces $\{V_j\}_{j \in \mathbb{Z}}$ associated with the GMRA through dilations.

Since the concept of translation invariant (TI) spaces plays an important role in the development of the theory of GMRAs in the case of Euclidean spaces [4, 12, 14], we adopt the approach of TI spaces to establish the theory of semi-orthogonal Parseval multiwavelets associated with GMRAs in the setting of LFPC and obtain necessary and sufficient conditions for the existence of a semi-orthogonal Parseval (multi)wavelet in terms of consistency equation. Further, we obtain a characterization of an orthonormal (multi)wavelet to be associated with an MRA in terms of dimension function of a (multi)wavelet in the setting of LFPC. The theory of TI spaces were developed by Currey, Mayeli and Oussa in [17] for nilpotent Lie groups and by Bownik and Ross in [13] for LCAG having a co-compact subgroup.

The existence of wavelet sets for LCAG and related groups has been discussed by Benedetto and Benedetto in [9, 10] and for Heisenberg groups by Currey and Mayeli in [16]. We also characterize bandlimited Parseval multiwavelet sets of finite order which in turn characterizes all multiwavelet sets for LFPC. Further, we obtain a necessary and sufficient condition of scaling functions associated with Parseval multiwavelets which provides a characterization of scaling sets. In this setting, we obtain various examples including a Shannon-type multiwavelet associated with MRA.

The present paper is organized as follows: In Sect. 2, we provide a brief introduction about local fields of positive characteristic [26]. Section 3 is divided into two subsections. In the first subsection, we discuss the notion of multiplicity function and spectral function associated with TI spaces for LFPC and, in the second subsection, we discuss semi-orthogonal Parseval multiwavelets, GMRAs and their main characterization theorem. We also prove that the wavelet multiplicity function satisfies a consistency equation, and the multiplicity function is equal to one in case of multiwavelets associated with MRAs. Finally, in Sect. 4, we provide a characterization of bandlimited Parseval multiwavelets for LFPC and obtain the necessary and sufficient conditions of Parseval scaling functions which generalize the characterization of orthonormal scaling functions provided by Behera and Jahan [7]. In the last section, we produce some examples of orthonormal wavelets associated with one scaling function as well as multiscale functions, and Parseval wavelets associated with scaling functions. Moreover, we discuss about scaling sets and wavelet sets in $K$.

Throughout this article, equality, in general, is almost everywhere (a.e.), that is, for any two measurable functions $f$ and $g$, $f = g$ means that $f(\cdot) = g(\cdot)$ up to a set of measure zero.
2. Basic Results and Notation

In this section, we provide a brief introduction about local fields of positive characteristic and refer to [26] for more details. Throughout the paper, $K$ denotes a local field. By a local field we mean a field which is locally compact, non-discrete and totally disconnected. The set $\mathcal{O} := \{x \in K: |x| \leq 1\}$ denotes the ring of integers which is a unique maximal compact open subring of $K$, where the absolute value $|x|$ of $x \in K$ satisfies the properties: $|x| = 0$ if and only if $x = 0; |xy| = |x||y|$, and $|x+y| \leq \max\{|x|, |y|\}$, for all $x, y \in K$. Next, we define $\mathfrak{P} := \{x \in K: |x| < 1\}$, which is called the prime ideal in $K$. In view of the total disconnectedness of $K$, there exists an element $p$ (known as prime element) of $\mathfrak{P}$ having maximum absolute value and then $\mathfrak{P} = p\mathcal{O}$. Here, it can be noted that $\mathfrak{P}$ is compact and open. Therefore, the residue space $\mathcal{Q} := \mathcal{O}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime $p$ and positive integer $c$.

For a measurable subset $E$ of $K$, let $|E| := \int_K \chi_E(x)dx$, where $\chi_E$ is the characteristic function of $E$ and $dx$ is the Haar measure for $K^+$ (locally compact additive group of $K$), so $|\mathcal{O}| = 1$. By decomposing $\mathcal{O}$ into $q$ cosets of $\mathfrak{P}$, we have $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{P}| = q^{-1}$, and hence for $x \in K \setminus \{0\} =: K^*$ (locally compact multiplicative group of $K$), we have $|x| = q^k$, for some $k \in \mathbb{Z}$. Further, notice that $\mathcal{O}^* := \mathcal{O} \setminus \mathfrak{P}$ is the group of units in $K^*$, and for $x \neq 0$, we may write $x = p^k x'$ with $x' \in \mathcal{O}^*$. In the sequel, we denote $p^k \mathcal{O}$ by $\mathfrak{P}^k$, for each $k \in \mathbb{Z}$ that is known as fractional ideal. Here, for $x \in \mathfrak{P}^k$, $x$ can be expressed uniquely as $x = \sum_{i=1}^{\infty} a_i p_i$, $a_i \in \mathcal{U}$, and $a_k \neq 0$, where $\mathcal{U} = \{c_i\}_{i=0}^{q-1}$ is a fixed full set of coset representatives of $\mathfrak{P}$ in $\mathcal{O}$.

Let $\chi$ be a fixed character on $K^+$ that is trivial on $\mathcal{O}$, but is nontrivial on $\mathfrak{P}^{-1}$, which can be found by starting with nontrivial character and rescaling. For $y \in K$, we define $\chi_y(x) = \chi(yx), x \in K$. For $f \in L^1(K)$, the Fourier transform of $f$ is the function $\hat{f}$ defined by

$$
\hat{f}(\xi) = \int_K f(x) \chi(x)dx = \int_K f(x) \chi(-\xi x)dx, \quad \xi \in K,
$$

which can be extended for $L^2(K)$.

Notation $N_0 := \mathbb{N} \cup \{0\}$. Let $\chi_u$ be any character on $K^+$. Since $\mathcal{O}$ is a subgroup of $K^+$, it follows that the restriction $\chi_u|_\mathcal{O}$ is a character on $\mathcal{O}$. Since $\chi_u|_\mathcal{O}$ is a character on $\mathcal{O}$, we have $\chi_u = \chi_v$ if and only if $u - v \in \mathcal{O}$. Hence, we have the following result [26, Proposition 6.1]:

**Theorem 2.1.** Let $\mathcal{Z} := \{u(n)\}_{n \in N_0}$ be a complete list of (distinct) coset representation of $\mathcal{O}$ in $K^+$. Then $\{\chi_u(n)|_\mathcal{O} =: \chi_u(n)\}_{n \in N_0}$ is a list of (distinct) characters on $\mathcal{O}$. Moreover, it is a complete orthonormal system on $\mathcal{O}$.

Next, we proceed to impose a natural order on $\mathcal{Z}$ which is used to develop the theory of Fourier series on $L^2(\mathcal{O})$. For this, we choose a set $\{1 = \epsilon_0, \epsilon_i\}_{i=1}^{c-1} \subset \mathcal{O}^*$ such that the vector space $\mathcal{Q}$ generated by $\{1 = \epsilon_0, \epsilon_i\}_{i=1}^{c-1}$ is isomorphic to the vector space $GF(q)$ over finite field $GF(p)$ of order $p$ as $q = p^c$. For $n \in N_0$ such that $0 \leq n < q$, we write $n = \sum_{k=0}^{c-1} a_k p^k$, where
$0 \leq a_k < p$. By noting that $\{u(n)\}_{n=0}^{q-1}$, a complete set of coset representatives of $O$ in $p^{-1}$ with $|u(n)| = q$, for $0 < n < q$ and $u(0) = 0$, we define $u(n) = (\sum_{k=0}^{c-1} a_k e_k)p^{-1}$. Now, for $n \geq 0$, we write $n = \sum_{k=0}^{s} b_k q^k$, where $0 \leq b_k < q$, and define $u(n) = \sum_{k=0}^{s} u(b_k)p^{-k}$. In general, it is not true that $u(m + n) = u(m) + u(n)$ but $u(rq^k + s) = u(r)p^{-k} + u(s)$, if $r \geq 0$, $k \geq 0$ and $0 \leq s < q^k$.

Now, we sum up the above in the following theorem (see, [26, Proposition 6.6], [7]):

**Theorem 2.2.** For $n \in \mathbb{N}_0$, let $u(n)$ be defined as above.

(a) $u(n) = 0$ if and only if $n = 0$. If $k \geq 1$, then we have $|u(n)| = q^k$ if and only if $q^{k-1} \leq n < q^k$.

(b) $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$.

(c) For a fixed $l \in \mathbb{N}_0$, we have $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$.

The following result and definition [20] will be used in the sequel:

**Theorem 2.3.** For all $l$, $k \in \mathbb{N}_0$, $\chi_{u(k)}(u(l)) = 1$.

**Definition 2.4.** A function $f$ defined on $K$ is said to be integral periodic if

$$f(x + u(l)) = f(x), \quad \text{for all } l \in \mathbb{N}_0, \ x \in K.$$  

To define the concepts of MRA and wavelets on LFPC $K$, we need notions of translation and dilation. Since $\bigcup_{j \in \mathbb{Z}} p^{-j}O = K$, we can regard $p^{-1}$ as the dilation (note that $|p^{-1}| = q$), and since $Z = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $O$ in $K$, the set $Z$ can be treated as the translation set. From Theorem 2.2, it follows that the translation set $Z$ is a subgroup of $K^+$ even though it is indexed by $\mathbb{N}_0$. We have the following definition:

**Definition 2.5.** A finite set $\Psi = \{\psi_m : m = 1, 2, ..., L\} \subset L^2(K)$ is said to be a frame wavelet (simply, framelet) in $L^2(K)$ if the affine system

$$\mathcal{A}(\Psi) := \left\{D_p^j T_k \psi_m : 1 \leq m \leq L, \ j \in \mathbb{Z}, \ k \in \mathbb{N}_0\right\}$$

forms a frame for $L^2(K)$, which means, for each $f \in L^2(K)$, there are $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{m=1}^{L} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{N}_0} |<f, D_p^j T_k \psi_m>|^2 \leq B\|f\|^2,$$

where the dilation and translation operators are defined as follows:

$$D_p^j f(x) = q^{j/2} p^{-j} f(x), \text{ and } T_k f(x) = f(x - u(k)), \quad \text{for } x \in K, j \in \mathbb{Z}, k \in \mathbb{N}_0.$$  

For $A = B$, $\Psi$ is called a tight frame wavelet (simply, tight framelet) with constant $A$, while it is said to be a Parseval multiwavelet of order $L$ for $A = B = 1$. In case of Parseval frame system $\mathcal{A}(\Psi = \{\psi\})$ for $L^2(K)$, $\psi$ is called a Parseval wavelet. If the system $\mathcal{A}(\Psi)$ is an orthonormal basis for $L^2(K)$, $\Psi$ is said to be an orthonormal multiwavelet (simply, multiwavelet)
of order $L$ of $L^2(K)$. Moreover, a framelet $\Psi$ is called a semi-orthogonal if $D_p W \perp D_p^j W$, for $j \neq j'$, where $W = \overline{\text{span}}\{T_k \psi : k \in \mathbb{N}_0, \psi \in \Psi\}$.

For $f \in L^2(K)$, $x \in K$, and $j, k \geq 0$, we note the following:

$$T_k D_p^j f(x) = D_p^j f(x - u(k)) = q^{j/2} f(p^{-j} x - p^{-j} u(k))$$

$$= q^{j/2} f(p^{-j} x - u(q^j k)) = q^{j/2} T_{q^j k} f(p^{-j} x)$$

$$= D_p^{q^j} T_{q^j k} f(x),$$

since $u(q^j k) = p^{-j} u(k)$ which shows that $T_k D_p^j = D_p^{q^j} T_{q^j k}$, for all $j, k \geq 0$.

Further, we notice that for $f \in L^2(K)$ and $\xi \in K$, we have

$$(\hat{D_p^j T_k} f)(\xi) = q^{-j/2} \chi_{u(k)}(-p^j \xi) \hat{f}(p^j \xi),$$

for $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

The following is a characterization for the system $\mathcal{A}(\Psi)$ to be a Parseval frame for $L^2(K)$ [6,8]:

**Theorem 2.6.** Suppose $\Psi = \{\psi_m : m = 1, 2, \ldots, L\} \subset L^2(K)$. Then the affine system $\mathcal{A}(\Psi)$ is a Parseval frame for $L^2(K)$ if and only if for a.e. $\xi$, the following holds:

(i) $\sum_{m=1}^L \sum_{j \in \mathbb{Z}} |\hat{\psi}_m(p^{-j} \xi)|^2 = 1$,

(ii) $\sum_{m=1}^L \sum_{j \in \mathbb{N}_0} \hat{\psi}_m(p^{-j} \xi) \hat{\psi}_m(p^{-j} (\xi + u(s))) = 0$, for $s \in \mathbb{N}_0 \setminus q \mathbb{N}_0$.

In particular, $\Psi$ is an orthonormal multiwavelet of order $L$ in $L^2(K)$ if and only if $\|\psi_m\| = 1$, for $1 \leq m \leq L$, and the above conditions (i) and (ii) hold.

**Definition 2.7.** A multiresolution analysis (MRA) with multiplicity $d$ is a sequence of closed subspaces $\{D_p^j(V)\}_{j \in \mathbb{Z}}$ of $L^2(K)$ satisfying the following properties:

(M1) $T_k V = V$, (M2) $V \subset D_p(V)$, (M3) $\bigcap_{j \in \mathbb{Z}} D_p^j(V) = \{0\}$, (M4) $\overline{\bigcup_{j \in \mathbb{Z}} D_p^j(V)} = L^2(K)$,

(M5) there are functions $\varphi_1, \ldots, \varphi_d \in V$ (say, multiscaling functions) such that

$$\{T_k \varphi_i : k \in \mathbb{N}_0, 1 \leq i \leq d\}$$

forms an orthonormal basis for $V$.

The space $V$ is said to be a core space. If we replace (M5) by (M6) as follows:

(M6) the system $\{T_k \varphi_i : k \in \mathbb{N}_0, 1 \leq i \leq d\}$ forms a Parseval frame for $V$, then the sequence $\{D_p^j(V)\}_{j \in \mathbb{Z}}$ is said to be a Parseval multiresolution analysis (PMRA), and $\varphi_1, \ldots, \varphi_d$ is called as Parseval multiscaling functions. For the case of $d = 1$, $\varphi_1$ is known as Parseval/orthonormal scaling function.

Such sequences $\{D_p^j(V)\}_{j \in \mathbb{Z}}$ satisfying conditions (M1) – (M4) are known as generalized multiresolution analyses (GMRA). This concept was introduced
by Baggett, Medina and Merill in [3] for separable Hilbert spaces. They developed GMRA structure for $L^2(\mathbb{R}^n)$. Since the core space $V$ possesses the property of TI space, it motivates us to use the theory of TI spaces for $L^2(K)$ for developing connection between the GMRA structure and (multi)wavelets or framelets for $L^2(K)$.

### 3. Multiplicity Function Associated with Semi-orthogonal Parseval Wavelets for LFPC

Given a finite family $\Psi \subset L^2(K)$, we define its space of negative dilates $V$ as follows:

$$V = \text{span}\{q^{j/2} \psi(p^{-j} \cdot -u(k)) : j < 0, k \in \mathbb{N}_0, \psi \in \Psi\}.$$  

We say that a framelet $\Psi$ comes from a GMRA if its space of negative dilates $V$ satisfies $(M1) - (M4)$. In addition, if $V$ satisfies $(M5)$, then $V$ is said to be associated with an MRA.

Since for a semi-orthogonal framelet $\Psi$, its space of negative dilates $V$ and the space $W = \text{span}\{T_k\psi : k \in \mathbb{N}_0, \psi \in \Psi\}$ satisfy

$$\bigoplus_{j \in \mathbb{Z}} D_j^p W = L^2(K), \quad V = \bigoplus_{j \leq -1} D_j^p W = \left(\bigoplus_{j \geq 0} D_j^p W\right) \perp,$$

it can be easily seen that every semi-orthogonal framelet $\Psi$ comes from a GMRA. A study of translation invariant spaces will be useful to obtain (multi)wavelet, or a semi-orthogonal framelet from a GMRA. We refer to [5,6,13,14,23] and references therein for the theory of translation invariant spaces in different scenarios.

The following is our main theorem for this section:

**Theorem 3.1.** Suppose that $\Psi$ is a semi-orthogonal Parseval multiwavelet with $L$ generators and $V$ is the space of negative dilates of $\Psi$. Then, $\{D_j^p(V)\}_{j \in \mathbb{Z}}$ is a GMRA such that the multiplicity function $m_V(\xi) < \infty$ for a.e. $\xi$, and

$$\sum_{d=0}^{q-1} m_V(p(\xi + u(d))) \leq L + m_V(\xi), \text{ for a.e. } \xi.$$  

Conversely, if $\{D_j^p(V)\}_{j \in \mathbb{Z}}$ is a GMRA satisfying the above conditions, then there exists a semi-orthogonal Parseval (multi)wavelet $\Psi$ (with at most $L$ generators) associated with this GMRA.

### 3.1. Multiplicity Function for TI Spaces

Suppose $Z = \{u(k) : k \in \mathbb{N}_0\}$. A closed subspace $V$ of $L^2(K)$ is said to be translation invariant under $Z$, in short, $Z$-TI if $f \in V$ implies $T_k f \in V$ for all $k \in \mathbb{N}_0$. The following results play an important role in finding building blocks of all $Z$-TI spaces [6].
Theorem 3.2. (A) Let $V_\varphi := \text{span} \{T_k \varphi : k \in \mathbb{N}_0\}$, for $\varphi \in L^2(K)$. Then $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, for some integral periodic function $r \in L^2(\mathcal{O}, w)$, where
\[
w(\xi) = \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2.
\]

The support of $w$ is known as the spectrum of $V_\varphi$ which is denoted by $\Omega$.

(B) Let $\varphi \in L^2(K)$. Then a necessary and sufficient condition for the system $\{T_k \varphi : k \in \mathbb{N}_0\}$ to be a Parseval frame for the $V_\varphi$ is as follows:
\[
\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = \chi_{\Omega}(\xi), \text{ a.e.}
\]

The following is an immediate application of the fiberization map as a consequence of Plancherel theorem that provides a notion of multiplicity function:

Theorem 3.3. Let $V$ be a $\mathbb{Z}$-TI subspace of $L^2(K)$. Then the image of $V$ under the fiberization map $T$ is given as follows:
\[
T(V) = \{F \in L^2(\mathcal{O}, l^2(\mathbb{Z})) : F(\xi) \in J(\xi)\}, \text{ where } J(\xi) = \{Tf(\xi) : f \in V\}, \text{ for } \xi \in \mathcal{O},
\]
where the fiberization map $T : L^2(K) \rightarrow L^2(\mathcal{O}, l^2(\mathbb{Z}))$ is defined by $Tf(\xi) = \left(\hat{f}(\xi + u(p))\right)_{p \in \mathbb{N}_0}$. Moreover, the multiplicity function satisfies the following condition:
\[
m_{T(V)}(\xi) = \sum_{n=1}^{M} \sum_{k \in \mathbb{N}_0} |\hat{\varphi}_n(\xi + u(k))|^2,
\]
where $V = \bigoplus_{i=1}^{M} V_{\varphi_i}$, and $\varphi_i$ is a Parseval frame generator for the $\mathbb{Z}$-TI space $V_{\varphi_i}$.

(The mapping $J : \mathcal{O} \rightarrow \{\text{closed subspaces of } l^2(\mathbb{Z})\}$ is known as range function, and the dimension of $J(\xi)$ is the multiplicity function of $V$ which is denoted by $m_{T(V)}(\xi)(= \dim J(\xi))$, for $\xi \in \mathcal{O}$. Throughout, we assume that the range function $J$ is measurable.)

Proof. Let $V$ be a $\mathbb{Z}$-TI subspace of $L^2(K)$. Then, there exists a countable set of functions $\{\varphi_i\}_{i=1}^{M}$ belonging to $V$ such that $V = \bigoplus_{i=1}^{M} V_{\varphi_i}$, where $M$ is a natural number or infinity. Let us consider a $\mathbb{Z}$-TI space $V_{\varphi_i}$ with a Parseval frame generator $\varphi_i$ and a spectrum $\Omega_i$, for $1 \leq i \leq M$. Then it is enough to find the image of $V_{\varphi_i}$ under the transformation $T$. For this, as every function $f \in V_{\varphi_i}$ satisfies $\hat{f}(\xi) = r(\xi)\hat{\varphi}_i(\xi)$, for some integral periodic function $r \in L^2(\mathcal{O}, \Omega_i)$, we have
\[
T(f)(\xi) = \left(\hat{f}(\xi + u(p))\right)_{p \in \mathbb{N}_0} = r(\xi)\left(\hat{\varphi}_i(\xi + u(p))\right)_{p \in \mathbb{N}_0} = r(\xi)T(\varphi_i)(\xi),
\]
and hence we obtain
\[
T(V_\varphi) = \{ F \in L^2(\mathcal{O}, l^2(\mathcal{Z})) : F(\xi) = r(\xi)T(\varphi_i)(\xi), r \in L^2(\mathcal{O}, \Omega_i) \}
\]
\[
= \{ F \in L^2(\mathcal{O}, l^2(\mathcal{Z})) : F(\xi) \in J_i(\xi) \},
\]
where \( J_i(\xi) = \text{span}\{ T(\varphi_i)(\xi) \}, \) for \( \xi \in \mathcal{O} \). Therefore, the result follows for \( \mathcal{Z} \)-TI space \( V \) by noting that \( J(\xi) = \text{span}\{ T(\varphi_i)(\xi) : i = 1, 2, \ldots, M \} = \bigoplus_{i=1}^{M} J_i(\xi) \).

Moreover, the multiplicity function satisfies
\[
m_V(\xi) = \dim J(\xi) = \sum_{i=1}^{M} \dim J_i(\xi) = \sum_{i=1}^{M} \sum_{k \in \mathbb{N}_0} \left| \hat{\varphi}_i(\xi + u(k)) \right|^2,
\]
for a.e. \( \xi \in K \). This is followed by the fact that the dimension of subspace \( J_i(\xi) \) of \( l^2(\mathcal{Z}) \) is one for a.e. \( \xi \in \mathcal{O} \cap \Omega_i \) and zero for a.e. \( \xi \in \mathcal{O} \) outside of \( \Omega_i \).

Next, let \( V \) be a \( \mathcal{Z} \)-TI space with the range function \( J \). Suppose \( P_{J(\xi)} \) is the orthogonal projection on \( J(\xi) \) for a.e. \( \xi \in \mathcal{O} \). Then the spectral function of \( V \) is the mapping \( \sigma_V : K \to [0, 1] \) given by
\[
\sigma_V(\xi + u(k)) = \| P_{J(\xi)} e_{u(k)} \|_{l^2(\mathcal{Z})}^2, \quad \text{for } \xi \in \mathcal{O} \text{ and } k \in \mathbb{N}_0,
\]
where \( \{ e_{u(k)} \}_{k \in \mathbb{N}_0} \) denotes the standard orthonormal basis of \( l^2(\mathcal{Z}) \). Some properties of spectral function \( \sigma_V \) have been discussed by Behera in [6]. The following are important results that will be used in the sequel:

**Theorem 3.4.** The spectral function satisfies the following properties:

- **(A)** Let \( V \) be a \( \mathcal{Z} \)-TI space of \( L^2(K) \) with a set of generators \( \{ \varphi_n \}_{n=1}^{N} \) (\( N \) is natural number or infinite). If the system \( \{ T_k \varphi_n : k \in \mathbb{N}_0, 1 \leq n \leq N \} \) forms a Parseval frame for the space \( V \), then the function \( \sigma_{V, \varphi}(\xi) = \sum_{n=1}^{N} | \hat{\varphi}_n(\xi) |^2, \) defined for a.e. \( \xi \in K \), does not depend on the choice of generators.

- **(B)** Let \( \nu \) denote the set of all \( \mathcal{Z} \)-TI spaces of \( L^2(K) \). Then there exists a unique mapping \( \sigma : \nu \to L^\infty(K) \) such that \( \sigma_{V, \varphi}(\xi) = | \hat{\varphi}(\xi) |^2 \)
\[
(\sum_{k \in \mathbb{N}_0} | \hat{\varphi}(\xi + u(k)) |^2)^{-1} \in \text{supp } \hat{\varphi}, \text{ otherwise zero, which is } V = \bigoplus_{i=1}^{\infty} V_{\varphi, i}, \text{ that implies } \sigma_V = \sum_{i=1}^{\infty} \sigma_{V, \varphi}.
\]

- **(C)** If \( V \) is a \( \mathcal{Z} \)-TI space with a decomposition \( V = \bigoplus_{n=1}^{N} V_{\varphi_n} \), where \( \varphi_n \) is a Parseval frame generator for \( V_{\varphi_n} \) and \( N \) is a natural number or infinity, then \( \sigma_V(\xi) = \sum_{n=1}^{N} | \hat{\varphi}_n(\xi) |^2, \) a.e.

- **(D)** If \( V \) is a \( \mathcal{Z} \)-TI space, then \( D_{p}(V) \) is \( p\mathcal{Z} \)-TI space, and \( \sigma_{D_{p}(V)}(\xi) = \sigma_{V}(p\xi), \) a.e.

- **(E)** If \( V \) is a \( \mathcal{Z} \)-TI space, then \( m_V(\xi) = \sum_{k \in \mathbb{N}_0} \sigma_V(\xi + u(k)), \) a.e.

**Proof.** (A) The result follows by noting that
\[
\sigma_V(\xi + u(k)) = \| P_{J(\xi)} e_{u(k)} \|_{l^2(\mathcal{Z})}^2 = \sum_{n=1}^{N} | \langle P_{J(\xi)} e_{u(k)}, T(\varphi_n)(\xi) \rangle_{l^2(\mathcal{Z})} |^2
\]
\[
= \sum_{n=1}^{N} | \langle e_{u(k)}, T(\varphi_n)(\xi) \rangle_{l^2(\mathcal{Z})} |^2 = \sum_{n=1}^{N} | \hat{\varphi}_n(\xi + u(k)) |^2.
\]
(B) If \( V_\varphi \) is a \( \mathcal{Z} \)-TI space, then the system \( \{ T_k \psi \}_{k \in \mathbb{N}_0} \) is a Parseval frame for the \( \mathcal{Z} \)-TI space \( V_\varphi \), where the Fourier transform of \( \psi \) is given by

\[
\hat{\psi}(\xi) = \begin{cases} 
\hat{\varphi}(\xi) \left( \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 \right)^{-1/2} & \text{for } \xi \in \text{supp} \hat{\varphi} \\
0 & \text{otherwise},
\end{cases}
\]

and hence \( \{ \mathcal{T}(\psi)(\xi) \} \) is a Parseval frame for the range function \( J_{V_\varphi}(\xi) \). So the value of \( \sigma_{V_\varphi} \) follows by noting that for all \( \xi \in \mathcal{O} \) and \( k \in \mathbb{N}_0 \), we have

\[
\sigma_{V_\varphi}(\xi + u(k)) = \| P_{J_{V_\varphi}(\xi)} e_{u(k)} \|^2 = | \langle e_{u(k)}, \mathcal{T}(\psi)(\xi) \rangle |^2 \\
= |\hat{\psi}(\xi + u(k))|^2.
\]

The remaining portion of the result follows by considering \( V = \bigoplus_{n=1}^\infty V_{\varphi_n} \), with the corresponding range functions \( J_V \) and \( J_{V_{\varphi_n}} \), and noting that

\[
\sigma_V(\xi + u(k)) = \| P_{J_V(\xi)} e_{u(k)} \|_{l^2(\mathcal{Z})}^2 = \sum_{n=1}^\infty \| P_{J_{V_{\varphi_n}}(\xi)} e_{u(k)} \|^2 \\
= \sum_{n=1}^\infty \sigma_{V_{\varphi_n}}(\xi + u(k)).
\]

(D) Consider for any \( f \in D_p(V) \) and \( \gamma \in \mathfrak{p} \mathcal{Z} \), i.e., \( f = D_p g \) for some \( g \in V \) and \( \mathfrak{p}^{-1} \gamma = u(l) \), for \( u(l) \in \mathcal{Z} \). Then for \( \xi \in K \), we have

\[
f(x - \gamma) = f(x - \mathfrak{p} u(l)) = D_p g(x - \mathfrak{p} u(l)) = q^{1/2} g(\mathfrak{p}^{-1} x - u(l)) = D_p (T_l g)(x).
\]

Since \( T_l g \in V \), this shows that \( D_p(V) \) is \( \mathfrak{p} \mathcal{Z} \)-TI. Next, it suffices to show the result for \( \mathcal{Z} \)-TI space \( V_\varphi \) which has a Parseval frame generator \( \varphi \). For this, let \( f \in D_p(V_\varphi) \). Then, we have

\[
\| f \|^2_2 = \| (D_p)^{-1} f \|^2 = \sum_{k \in \mathbb{N}_0} | \langle f, D_p T_k \varphi \rangle |^2 \\
= \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{q-1} | \langle f, D_p T_{qk+i} \varphi \rangle |^2 \\
= \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{q-1} | \langle f, D_p T_{qk}(T_i \varphi) \rangle |^2 \\
= \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{q-1} | \langle f, T_k(D_p(T_i \varphi)) \rangle |^2,
\]

in view of the following facts: the map \( D_p \) is unitary on \( L^2(K) \); for \( k \geq 0 \) and \( 0 \leq i \leq q - 1 \), \( u(qk + i) = \mathfrak{p}^{-1} u(k) + u(i) \); and \( D_p T_{qk} = T_k D_p \), for \( k \geq 0 \). Since \( T_i \varphi \in V_\varphi \), this shows that the system \( \{ D_p(T_i \varphi) \}_{i=0}^{q-1} \) is a Parseval frame for the space \( D_p(V_\varphi) \).

Note that (C) and (E) follow immediately. \( \square \)
In the upcoming subsection, we discuss semi-orthogonal Parseval multiwavelets, GMRAs and their main characterization theorem that provide a connection between the dimension function and the multiplicity function of a (multi)wavelet.

3.2. Semi-orthogonal Parseval Wavelets and GMRAs

Recall that every semi-orthogonal framelet $\Psi$ comes from a GMRA. Now we look into its converse, that is, when a GMRA gives rise to a wavelet, or a semi-orthogonal framelet with the help of knowledge of translation invariant spaces for LFPC obtained from previous subsection. Before providing proof of our main theorem of this section, we have to note the following result which is analogous to a result of Bownik [11]:

If $V$ is a $Z$-TI space and $N = \|m_V\|_\infty$ ($N$ is a natural number or infinity), then there exist a set of functions $\{\varphi_n\}_{n=1}^N$ such that $V = \bigoplus_{n=1}^N V_{\varphi_n}$, where $\varphi_n$ is a Parseval frame generator of $V_{\varphi_n}$.

Now, we proceed as follows to find a proof of our main result:

Theorem 3.5. If $\Psi = \{\psi_l\}_{l=1}^L \subset L^2(K)$ is a semi-orthogonal Parseval multiwavelet with $L$ generators and $V = \bigoplus_{j<0} D^j_p(W)$, where $W = \text{span}\{\psi_l(\cdot-u(k)) : k \in \mathbb{N}_0, l=1,2,\ldots,L\}$, then

$$m_V(\xi) = \sum_{l=1}^L \sum_{j=1}^\infty \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(p^{-j}(\xi+u(k)))|^2, \: a.e.,$$

and hence $\int_O m_V(\xi) d\xi \leq \frac{L}{q-1}$. Moreover, it satisfies the consistency equation

$$m_V(\xi) + L \geq \sum_{d=0}^{q-1} m_V(p(\xi+u(d))), \: \text{for} \: \xi \in K.$$

Proof. Since $L^2(K) = V \oplus \bigoplus_{j\geq0} D^j_p(W)$, we have $\sigma_V + \sum_{j\geq0} \sigma_{D^j_p(W)} = 1$ in view of Theorem 3.4, and hence, $\sigma_V(\xi) = \sum_{l=1}^L \sum_{j=1}^\infty |\hat{\psi}_l(p^{-j}\xi)|^2$. This follows by noting that

$$\sigma_V(\xi) = 1 - \sum_{j\geq0} \sigma_{D^j_p(W)}(\xi) = \sum_{l=1}^L \sum_{j \in \mathbb{Z}} |\hat{\psi}_l(p^j\xi)|^2 - \sum_{l=1}^L \sum_{j\geq0} |\hat{\psi}_l(p^j\xi)|^2 = \sum_{l=1}^L \sum_{j=1}^\infty |\hat{\psi}_l(p^{-j}\xi)|^2,$$

because $\sigma_{D^j_p(W)}(\xi) = \sum_{l=1}^L |\hat{\psi}_l(p^j\xi)|^2$, for $j \geq 0$. Therefore, the first result follows by writing $m_V(\xi) = \sum_{k \in \mathbb{N}_0} \sigma_V(\xi+u(k))$ from Theorem 3.4. Now, we have
\[
\int_{\mathcal{O}} m_V(\xi) d\xi = \int_{\mathcal{O}} \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(p^{-j}(\xi + u(k)))|^2 d\xi \\
= \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} \int_{\mathcal{O}+u(k)} |\hat{\psi}_l(p^{-j}\xi)|^2 d\xi \\
= \sum_{l=1}^{L} \sum_{j=1}^{\infty} \frac{1}{q_j} \int_{\mathcal{K}} |\hat{\psi}_l(\xi)|^2 d\xi \leq \frac{L}{q-1}.
\]

For the consistency equation, we have
\[
\sum_{d=0}^{q-1} m_V(p(\xi + u(d))) = \sum_{d=0}^{q-1} \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(p^{-j}(p\xi + pu(d) + u(k)))|^2 \\
= \sum_{d=0}^{q-1} \sum_{l=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(p^{-j}(\xi + u(d) + p^{-1}u(k)))|^2 \\
= \sum_{l=1}^{L} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(\xi + u(k))|^2 \\
+ \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{\psi}_l(p^{-j}(\xi + u(k)))|^2 \\
\leq L + m_V(\xi). 
\]

**Corollary 3.6.** If \( f \in L^2(K) \), then the collection \( \{f(\cdot - u(k)) : k \in \mathbb{N}_0\} \) is a Parseval frame for \( \text{span}\{f(\cdot - u(k)) : k \in \mathbb{N}_0\} \) sequence if and only if \( m_f \) satisfies the following consistency equation
\[
\sum_{d=0}^{q-1} m_f(p(\xi + u(d))) \leq 1 + m_f(\xi),
\]
where \( m_f(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} |\hat{f}(p^{-j}(\xi + u(k)))|^2 \). Equality holds if and only if the system \( \{f(\cdot - u(k)) : k \in \mathbb{N}_0\} \) is orthonormal in \( L^2(K) \).

Next, we define multiplicity function associated with a wavelet as follows:

**Definition 3.7.** If \( \Psi \) is a multiwavelet, then there exists a multiplicity function associated with it. This function is called the wavelet multiplicity function.

**Corollary 3.8.** Let \( \Psi \) be a multiwavelet, and let \( m : \mathcal{O} \to \mathbb{N}_0 \) be its associated multiplicity function. Then \( m(\xi) = D_\Psi(\xi) \), where the wavelet dimension function \( D_\Psi(\xi) \) is defined by
\[
D_\Psi(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{N}_0} \left|\hat{\psi}_l(p^{-j}(\xi + u(k)))\right|^2.
\]
The following result gives a characterization of a multiwavelet associated with an MRA:

**Theorem 3.9.** If \( \Psi = \{\psi_l\}_{l=1}^{q-1} \subset L^2(\mathbb{R}) \) is an orthonormal multiwavelet, and \( m \) is its associated multiplicity function, then \( \Psi \) is associated with an MRA if and only if \( m \equiv 1 \), a.e.

**Proof.** Suppose \( \Psi \) is an MRA multiwavelet. Then there exists \( \varphi \in V \) such that the system \( \{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\} \) is an orthonormal basis for \( V \), and hence \( \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \), a.e. Therefore, we have \( m(\xi) = 1 \) since \( V = \text{span}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\} \) is a \( 2 \)-TI space. Conversely, assume that \( \Psi \in L^2(\mathbb{R}) \) is a multiwavelet, and \( m \) is its associated multiplicity function such that \( m(\xi) = 1 \). Then, we have to show that \( \Psi \) is an MRA multiwavelet. For this, consider \( V = \bigoplus_{j<0} D_p^j(W) \), where \( W = \text{span}\{\psi_l(\cdot - u(k)) : k \in \mathbb{N}_0, l = 1,2,\ldots, q-1\} \). Then, the multiplicity function \( m(\xi) = 1 \), and hence, \( V = V_\varphi \), where \( \varphi \) is a Parseval frame generator for \( V_\varphi \) and spectrum of \( V_\varphi \) is equal to \( K \). Therefore, we have \( \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \), a.e. and hence, \( \{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\} \) is an orthonormal basis for \( V \). \( \square \)

**Proof of Theorem 3.6.** Suppose that \( \Psi \) is a semi-orthogonal Parseval multiwavelet with \( L \) generators and the spaces \( V \) and \( W \) are defined as follows:

\[
W = \text{span}\{T_k \psi : k \in \mathbb{N}_0, \psi \in \Psi\}, \quad V = \text{span}\{\psi_{j,k} : j < 0, k \in \mathbb{N}_0, \psi \in \Psi\}.
\]

Then, we have

\[
\int_{\mathbb{O}} m_V(\xi) d\xi = \int_{K} \sigma_V(\xi) d\xi = \sum_{\psi \in \Psi} \sum_{j=1}^{\infty} \int_{K} |\tilde{\psi}(p^{-j}\xi)|^2 d\xi = \sum_{\psi \in \Psi} ||\psi||^2/(q-1) \leq L/(q-1) < \infty.
\]

Hence, \( m_V(\xi) < \infty \). Since \( W \oplus V = D_p(V) \), we have

\[
\sigma_W(\xi) + \sigma_V(\xi) = \sigma_{D_p(V)}(\xi) = \sigma_V(p\xi).
\]

This implies that

\[
m_W(\xi) + m_V(\xi) = \sum_{d=0}^{q-1} m_{V}(p(\xi + u(d))).
\]

Since \( m_W(\xi) \leq L \), we get the result.

Conversely, from

\[
\sum_{d=0}^{q-1} m_{V}(p(\xi + u(d))) - m_V(\xi) \leq L
\]

and

\[
m_W(\xi) + m_V(\xi) = \sum_{d=0}^{q-1} m_{V}(p(\xi + u(d))),
\]
we have $m_W(\xi) \leq L$ which implies that $W$ has a set $\Psi$ having generators less than or equal to $L$. Since $V = \bigoplus_{j \leq -1} D^0_j(W)$, we infer that $\Psi$ is a semi-orthogonal Parseval multiwavelet associated with the GMRA $\{D^0_j(V)\}_{j \in \mathbb{Z}}$.

**Corollary 3.10.** Let $V$ be a $\mathbb{Z}$-TI space such that the multiplicity function of $V$ is integrable and satisfies the following consistency equation $m_V(\xi) + L = \sum_{d=0}^{q-2} m_V(p(\xi + u(d)))$. Then there exists a set of functions $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ in $D_p(V) \cap V \equiv W$ such that the system $\{\psi_1(\cdot - u(k)) : k \in \mathbb{N}_0, 1 \leq l \leq L\}$ is an orthonormal basis for $W$.

## 4. Bandlimited Wavelets for LFPC

The present section is devoted to the study of characterizations of bandlimited Parseval multiwavelets as well as Parseval scaling functions for LFPC. In the recent work of the first two authors [25], they have established the following characterization:

**Theorem 4.1.** Let $\Psi = \{\psi_m\}_{m=1}^L \subset L^2(K)$ be such that for each $m \in \{1, 2, \ldots, L\}$, $|\hat{\psi}_m| = \chi_{W_m}$, and $W = \bigcup_{m=1}^L W_m$ is a union of measurable subsets of $K$. Then $\Psi$ is a semi-orthogonal Parseval multiwavelet in $L^2(K)$ if and only if the following hold:

(i) $\{p^j W : j \in \mathbb{Z}\}$ is a measurable partition of $K$, and

(ii) for each $m \in \{1, 2, \ldots, L\}$, the set $\{W_m + u(k) : k \in \mathbb{N}_0\}$ is a measurable partition of a subset of $K$.

Such a set $W$ is said to be a Parseval multiwavelet set of order $L$.

As a consequence, we have the following:

**Corollary 4.2.** Let $\Psi = \{\psi_m\}_{m=1}^L \subset L^2(K)$ be such that for each $m \in \{1, 2, \ldots, L\}$, $|\hat{\psi}_m| = \chi_{W_m}$, and $W = \bigcup_{m=1}^L W_m$ is a union of measurable subsets of $K$. Then $\Psi$ is a multiwavelet in $L^2(K)$ if and only if the following hold:

(i) $\{p^j W : j \in \mathbb{Z}\}$ is a measurable partition of $K$, and

(ii) for each $m \in \{1, 2, \ldots, L\}$, the set $\{W_m + u(k) : k \in \mathbb{N}_0\}$ is a measurable partition of $K$.

Such a set $W$ is said to be a multiwavelet set of order $L$.

The result given below is a necessary and sufficient condition of Parseval scaling functions for LFPC which generalizes the characterization of orthonormal scaling functions given in [7]:

**Theorem 4.3.** Let $\varphi \in L^2(K)$. Then, $\varphi$ is a Parseval scaling function associated with a PMRA $\{D^0_j(V)\}_{j \in \mathbb{Z}}$ if and only if the following conditions hold:

(i) the system $\{\varphi(\cdot - u(k))\}_{k \in \mathbb{N}_0}$ is a Parseval frame for $V = \overline{\text{span}}\{\varphi(\cdot - u(k))\}_{k \in \mathbb{N}_0}$ in $L^2(K)$,

(ii) $\lim_{j \to \infty} |\hat{\varphi}(p^j \xi)| = 1$, a.e. $\xi \in K$, and
(iii) there exists an integral periodic function $m_0$ in $L^2(O)$ such $\hat{\varphi}(\xi) = m_0(\xi)\hat{\varphi}(p\xi)$, a.e.

Proof. (ii) and (iii) are straightforward in view of Theorem 5.1 of [7], and (i) follows by noting Theorem 3.2.

Next, we illustrate a characterization of a Parseval scaling function $\varphi$ such that $|\hat{\varphi}| = \chi_S$, for some measurable set $S$ of $K$. Such set $S$ is known as Parseval scaling set.

**Theorem 4.4.** A function $\varphi$ such that $|\hat{\varphi}| = \chi_S$, for some measurable set $S$ of $K$, is a Parseval scaling function of a PMRA if and only if

1. $\{S + u(k) : k \in \mathbb{N}_0\}$ is a measurable partition of a subset of $K$,
2. $\bigcup_{j \in \mathbb{N}} p^{-j}S = K$, and
3. $S \subset p^{-1}S$.

Moreover, Parseval multiwavelet set(s) $W$ associated with PMRA can be obtained by $W = p^{-1}S\setminus S$, and hence $S = \bigcup_{j \in \mathbb{N}} p^jW$.

Proof. Suppose $\varphi$ is a Parseval scaling function such that $|\hat{\varphi}| = \chi_S$. Then from (i) of Theorem 4.3 and Theorem 3.2, we have

$$\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = \sum_{k \in \mathbb{N}_0} \chi_S(\xi + u(k)) = \chi_\Omega(\xi),$$

which implies that the set $\{S + u(k) : k \in \mathbb{N}_0\}$ is a measurable partition of a subset of $K$. From (iii) of Theorem 4.3,

$$\hat{\varphi}(\xi) = m(\xi)\hat{\varphi}(p\xi) \Rightarrow |\hat{\varphi}(\xi)| = |m(\xi)||\hat{\varphi}(p\xi)| \Rightarrow \chi_S(\xi) \leq \chi_{p^{-1}S}(\xi),$$

since $|m(\xi)| \leq 1$, which gives $S \subset p^{-1}S$. Now, from (ii) of Theorem 4.3,

$$\lim_{j \to +\infty} |\hat{\varphi}(p^j\xi)|^2 = 1 \Rightarrow \lim_{j \to +\infty} \chi_S(p^j\xi) = 1 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}_0 \text{ such that}$$

$$|\chi_S(p^j\xi) - 1| = |\chi_S(p^j\xi) - \chi_K(\xi)| = |\chi_K(p^{-j}S)| < \epsilon,$$

whenever $j > N$ that implies $\bigcup_{j \in \mathbb{N}} p^{-j}S = K$, since $S \subset p^{-1}S$.

Conversely, suppose $\{S + u(k) : k \in \mathbb{N}_0\}$ is a measurable partition of a subset of $K$, $\bigcup_{j \in \mathbb{Z}} p^{-j}S = K$, and $S \subset p^{-1}S$. Then, it suffices to see the conditions (i), (ii) and (iii) of Theorem 4.3 for the proof. Now, for this, the measurable partition $\{S + u(k) : k \in \mathbb{N}_0\}$, of a subset of $K$, implies

$$\sum_{k \in \mathbb{N}_0} \chi_S(\xi + u(k)) = \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = \chi_\Omega(\xi);$$

$$S \subset p^{-1}S \Rightarrow \chi_S \leq \chi_{p^{-1}S} \Rightarrow |\hat{\varphi}(\xi)| = |m(\xi)||\hat{\varphi}(p\xi)|,$$

where $\theta$ is a unimodular and integral periodic function. Next, from $\bigcup_{j \in \mathbb{N}} p^{-j}S = K$, we have $\bigcup_{j > N} p^{-j}S = K$, for $j > N$. This follows from the fact that
\[
S \subset p^{-1}S \Rightarrow \lim_{j \to +\infty} \chi_S(p^j \xi) = 1 \Rightarrow \lim_{j \to +\infty} |\hat{\varphi}(p^j \xi)|^2 = 1.
\]

\[\square\]

**Corollary 4.5.** A function \( \varphi \) such that \( |\hat{\varphi}| = \chi_S \), for some measurable set \( S \) of \( K \) is an orthonormal scaling function of an MRA if and only if

1. \( \{S + u(k) : k \in \mathbb{N}_0\} \) is a measurable partition of \( K \),
2. \( \bigcup_{j \in \mathbb{Z}} p^{-j}S = K \), and
3. \( S \subset p^{-1}S \).

Moreover, multwavelet set(s) \( W \) of order \( q-1 \) associated with MRA can be obtained by \( W = p^{-1}S \setminus S \), and hence \( S = \bigcup_{j \in \mathbb{N}} p^jW \), called as orthonormal scaling set.

**Proof.** With reference to Theorem 5.1 of [7] and in view of Theorem 4.4, the conditions (ii) and (iii) are straightforward. The condition (i) follows from the fact that \( \{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\} \) is an orthonormal system in \( L^2(K) \) characterized by

\[
\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1, \text{ a.e. } \xi,
\]

(follows by substituting \( \Omega = K \) in Theorem 3.2), which on substitution \( |\hat{\varphi}| = \chi_S \) gives rise to

\[
\chi_K(\xi) = 1 = \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = \sum_{k \in \mathbb{N}_0} \chi_S(\xi + u(k)) = \sum_{k \in \mathbb{N}_0} \chi_S(\xi - u(k))
\]

\[
= \sum_{k \in \mathbb{N}_0} \chi_{S+u(k)}(\xi) = \chi_{\bigcup_{k \in \mathbb{N}_0}(S+u(k))}(\xi),
\]

for a.e. \( \xi \in K \) in view of Theorem 2.2. Therefore, the set \( \{S + u(k) : k \in \mathbb{N}_0\} \) is a partition of \( K \) up to null set. \( \square \)

### 5. Example

In the upcoming section, we provide some examples of orthonormal wavelets associated with one scaling function as well as multiscaling function and Parseval wavelets associated with scaling functions. The construction of such wavelets depends upon \( \mathfrak{P}^k = p^k \mathcal{O} \) for \( k \in \mathbb{Z} \) in view of the fact that \( K = \bigcup_{j \in \mathbb{Z}} p^{-j}(p^{-1} \mathcal{O}\setminus \mathcal{O}) \) (disjoint unions having role of dilation) and \( K = \bigcup_{k \in \mathbb{N}_0}(\mathcal{O} + u(k)) \) (disjoint unions having role of translation).

Now, we start with an example of orthonormal multiwavelet of order \( q-1 \) which is of Shannon type:

**Example 5.1 (Shannon-type multiwavelet with one scaling function).** Let us consider the ring of integers \( \mathcal{O} \) in \( K \), and let the set \( \{u(n)\}_{n=0}^{q-1} \) be a complete set of distinct coset representatives of \( \mathcal{O} \) in \( \mathfrak{P}^{-1} \) with \( u(0) = 0 \), and \( |u(n)| = q \), for \( 0 < n < q \). Suppose \( \Psi = \{\psi_1, \psi_2, \ldots, \psi_{q-1}\} \) is a collection of functions in \( L^2(K) \) such that the Fourier transform

\[
\hat{\psi}_i(\xi) = \chi_{\mathcal{O}+u(i)}(\xi) = \begin{cases} 
1 & \text{if } \xi \in \mathcal{O} + u(i) \\
0 & \text{otherwise},
\end{cases}
\]

for a.e. \( \xi \in K \) in view of Theorem 2.2. Therefore, the set \( \{S + u(k) : k \in \mathbb{N}_0\} \) is a partition of \( K \) up to null set. \( \square \)
for all $i, 1 \leq i \leq q - 1$. Then, $\Psi$ is an orthonormal multiwavelet of order $q - 1$ (for more details, see [25]).

Next, let us write $W = \bigcup_{j=1}^{q-1}(\mathcal{O} + u(j)) = \mathcal{P}^{-1}\setminus\mathcal{O}$. Then, it can be noticed that $W$ is a multiwavelet set of order $q - 1$. Further, if we consider the set $S = \bigcup_{j \in \mathbb{N}} p^jW = \bigcup_{j \in \mathbb{N}} p^j(\mathcal{P}^{-1}\setminus\mathcal{O})$, then $S = \mathcal{O}$ since $p\mathcal{O} \subset \mathcal{O}$ and $\mathcal{P}^{-1} = p^{-1}\mathcal{O}$, and the set $S = \mathcal{O}$ satisfies conditions (i), (ii) and (iii) of Corollary 4.5. Therefore $S$ is an orthonormal scaling set in the local field $K$ of positive characteristic, and hence $\Psi$ is associated with an MRA whose scaling function $\varphi$ is defined by $\widehat{\varphi} = \chi_{\mathcal{O}}$.

In particular, for $c = 1$ and $p = 2$, the additive group $K^+$ of the local field $K = F^{(c)}$ is the binary group Dirichlet. In this group for the scaling function $\varphi$ with $\widehat{\varphi} = \chi_{\mathcal{O}}$, we have wavelet $\psi$ with $\widehat{\psi} = \chi_{p^{-1}\mathcal{O}\setminus\mathcal{O}}$. If we restore the wavelet, then we get

$$\psi(x) = r_0(x) = \begin{cases} 
-1 & \text{for } x \in \mathcal{O}\setminus p\mathcal{O} \\
1 & \text{for } x \in p\mathcal{O}.
\end{cases}$$

It is binary Haar function and $r_0$ is Rademacher function. When $c = 1$ and $p \geq 2$, we have

$$\psi(x) = r_0(x) = e^{2\pi i x/p}$$

for $x \in \mathcal{O} + u(j)$, $j = 0, 1, 2, \ldots, p - 1$ and $\varphi_l(x) = r_0(x, l = 1, 2, \ldots, p - 1$, and hence we have $p$-ary Haar system. Further for the case of $c > 1$ and $p \geq 2$, we obtain $c$-dimensional $p$-ary Haar system. This system is usually called the generalized Haar system.

The following is an example of multiwavelet associated with multiscaling functions, but is not generated by unique scaling function:

**Example 5.2 (Multiwavelets with multiscaling functions).** Let us consider a set of functions $\Phi = \{\varphi_j\}_{j=0}^{q^m-1} \subset L^2(K)$, where $m \in \mathbb{N}$, and the Fourier transform $\widehat{\varphi}_j$ of $\varphi_j$ is defined as follows:

$$\widehat{\varphi}_j(\xi) = \chi_{\mathcal{O}+u(j)}(\xi), \ \ a.e. \ \xi \in K,$$

for each $j \in \{0, 1, 2, \ldots, q^m - 1\}$. Next, we suppose that $S_j = \mathcal{O} + u(j)$, for $j \in \{0, 1, 2, \ldots, q^m - 1\}$, then we have $S := \bigcup_{j=0}^{q^m-1}(\mathcal{O} + u(j)) = \mathcal{P}^{-m}$ in view of the fact that $\{u(j)\}_{j=0}^{q^m-1}$ is a complete set of distinct coset representatives of $\mathcal{O}$ in $\mathcal{P}^{-m}$ with $u(0) = 0$, and for $k \in \mathbb{N}$, $|u(n)| = q^k$, for $q^{k-1} \leq n < q^k$. Hence from the expression $W := p^{-1}S\setminus S$, we have

$$W = \bigcup_{j=q^m}^{q^{(m+1)-1}}(\mathcal{O} + u(j)) = \bigcup_{j=q^m}^{q^{(m+1)-1}} W_j,$$

where $W_j := \mathcal{O} + u(j)$, and $\{u(j)\}_{j=0}^{q^{(m+1)-1}}$ is a complete set of distinct coset representatives of $\mathcal{O}$ in $\mathcal{P}^{-(m+1)}$. Here, notice that $S \subset p^{-1}S$ since $\mathcal{P}^{-m} \subset \mathcal{P}^{-(m+1)}$; $W = p^{-m}(p^{-1}\mathcal{O}\setminus\mathcal{O}) = \mathcal{P}^{-(m+1)}\setminus\mathcal{P}^{-m}$; and
Further observe that for each \( j \in \{0,1,\ldots,q^m-1\} \), the system
\[
\{S_j + u(k) : k \in \mathbb{N}_0\} = \{\mathcal{O} + u(j) + u(k) : k \in \mathbb{N}_0\} = \{\mathcal{O} + u(l) : l \in \mathbb{N}_0\}
\]
is a measurable partition of \( K \), i.e.,
\[
\bigcup_{k \in \mathbb{N}_0} (S_j + u(k)) = K,
\]
in view of Theorem 2.2 and the fact that the system \( \{u(l)\}_{l \in \mathbb{N}_0} \) is a complete set of distinct coset representatives of \( \mathcal{O} \) in \( K \). Now, \( W \) is a multiwavelet set of order \( q^m+1 - q^m = q^m(q-1) \) which is associated with MRA having multiplicity \( q^m \) with scaling functions \( \varphi_j \), \( j \in \{0,1,\ldots,q^m-1\} \), following from Corollary 4.5.

The following are examples of Parseval multiwavelets of order 1 as well as \( q-1 \) in \( L^2(K) \):

**Example 5.3 (Parseval wavelets).** Let \( m \in \mathbb{N}_0 \). Suppose \( \psi \) is a function in \( L^2(K) \) whose Fourier transform is defined as follows:
\[
\hat{\psi}(\xi) = \chi_{\mathbb{P}^m \mathcal{O}^*}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{P}^m \mathcal{O}^* = \mathbb{P}^m \backslash \mathbb{P}^{m+1} \\
0 & \text{otherwise.} \end{cases}
\]
Then, \( \psi \) is a Parseval multiwavelet of order 1 in view of the following:

(i) The system \( \{p^j(\mathbb{P}^m \mathcal{O}^*) : j \in \mathbb{Z}\} \) is a measurable partition of \( K \), since \( \bigcup_{j \in \mathbb{Z}} p^{-j} \mathcal{O} = K \), \( \mathcal{O} \subset \mathbb{P}^{-1} \), and \( p^{-1} \mathcal{O}^* = \mathbb{P}^{-1} \backslash \mathcal{O} \).
(ii) The system \( \{\mathbb{P}^m \mathcal{O}^* + u(k) : k \in \mathbb{N}_0\} \) is a measurable partition of a measurable subset of \( K \) since \( \{\mathcal{O} + u(k) : k \in \mathbb{N}_0\} \) is a measurable partition of \( K \) and \( \mathbb{P}^m \subset \mathcal{O} \).

Now, for each \( m \in \mathbb{N} \), let us consider the set \( W = \mathbb{P}^m \mathcal{O} \backslash \mathbb{P}^{m+1} \mathcal{O} \), which is a Parseval multiwavelet set \( W \) of order 1. Further, we consider the set \( S = \bigcup_{j \in \mathbb{N}} p^j W = \mathbb{P}^{m+1} \mathcal{O} \). Then, \( S \) is a Parseval scaling set in view of conditions (i), (ii) and (iii) of Theorem 4.4, and hence \( \psi \) is associated with a PMRA whose Parseval scaling function \( \varphi \) is defined by \( \hat{\varphi} = \chi_{\mathbb{P}^{m+1} \mathcal{O}} \).

**Example 5.4 (Parseval multiwavelets).** Let \( m \in \mathbb{N} \). Suppose \( \Psi = \{\psi_1, \psi_2, \ldots, \psi_{q-1}\} \) is a collection of functions in \( L^2(K) \) whose Fourier transforms are defined as follows for each \( i, 1 \leq i \leq q-1 \):
\[
\hat{\psi}_i(\xi) = \chi_{\mathbb{P}^m + \mathbb{P}^m u(i)}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{P}^m + \mathbb{P}^m u(i) \\
0 & \text{otherwise.} \end{cases}
\]

Then, in view of Example 5.1, with \( \mathbb{P}^m = \mathbb{P}^m \mathcal{O} \), \( \mathbb{P}^m \subset \mathcal{O} \), and the system \( \{\mathbb{P}^m W_i : 1 \leq i \leq q-1\} \) is a measurable partition of \( \mathbb{P}^{m-1} \mathcal{O}^* \), where \( W_i = \mathcal{O} + u(i) \), \( \Psi \) is a Parseval multiwavelet of order \( q-1 \). Here, note that \( |\mathbb{P}^m| = \)}
\( \frac{1}{q^m} < 1 \), as \( q \geq 2 \), and for \( k, k' \in \mathbb{N}_0, (k \neq k') \), we have
\[
|(p^m W_i + u(k)) \cap (p^m W_i + u(k'))| = q^m |(W_i + p^{-m} u(k)) \cap (W_i + p^{-m} u(k'))|
= q^m |(W_i + u(q^m k)) \cap (W_i + u(q^m k'))|
= 0,
\]
as the system \( \{W_i + u(k) : k \in \mathbb{N}_0\} \) is a measurable partition of \( K \).

Now, consider the Parseval multiwavelet set \( W \) of order \( q - 1 \) given by
\[
W = \bigcup_{j=1}^{q-1} p^m (O + u(j)) = p^m (\mathcal{P}^{-1} \setminus O),
\]
where \( m \in \mathbb{N} \). Then, the set \( S = \bigcup_{j \in \mathbb{N}} p^j W = p^m \mathcal{O} \) is a Parseval scaling set in view of conditions (i), (ii) and (iii) of Theorem 4.4, and hence \( \Psi \) is associated with a PMRA whose Parseval scaling function \( \varphi \) is defined by \( \hat{\varphi} = \chi_{p^m \mathcal{O}} \).

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**References**

[1] Albeverio, S., Evdokimov, S., Skopina, M.: \( p \)-adic nonorthogonal wavelet bases. Proc. Steklov Inst. Math. **265**(1), 135–146 (2009)

[2] Albeverio, S., Evdokimov, S., Skopina, M.: \( p \)-adic multiresolution analysis and wavelet frames. J. Fourier Anal. Appl. **16**, 693–714 (2010)

[3] Baggett, L.W., Medina, H.A., Merrill, K.D.: Generalized multi-resolution analyses and a construction procedure for all wavelet sets in \( \mathbb{R}^n \). J. Fourier Anal. Appl. **5**(6), 563–573 (1999)

[4] Bakić, D.: Semi-orthogonal Parseval frame wavelets and generalized multiresolution analyses. Appl. Comput. Harmon. Anal. **21**(3), 281–304 (2006)

[5] Barbieri, D., Hernández, E., Mayeli, A.: Bracket map for the Heisenberg group and the characterization fo cyclic subspaces. Appl. Comput. Harmon. Anal. **37**, 218–234 (2014)

[6] Behera, B.: Shift-invariant subspaces and wavelets on local fields. Acta Math. Hungar. **148**(1), 157–173 (2016)

[7] Behera, B., Jahan, Q.: Multiresolution analysis on local fields and characterization of scaling functions. Adv. Pure Appl. Math. **3**(2), 181–202 (2012)

[8] Behera, B., Jahan, Q.: Characterization of wavelets and MRA wavelets on local fields of positive characteristic. Collect. Math. **66**(1), 33–53 (2015)

[9] Benedetto, J.J., Benedetto, R.L.: A wavelet theory for local fields and related groups. J. Geom. Anal. **14**(3), 423–456 (2004)
[10] Benedetto, R.L.: Examples of wavelets for local fields, Wavelets, frames and operator theory, 27–47, Contemp. Math., 345, Amer. Math. Soc., Providence, RI (2004)

[11] Bownik, M.: The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$. J. Funct. Anal. 177(2), 282–309 (2000)

[12] Bownik, M.: Baggett’s problem for frame wavelets, Representations, wavelets, and frames, 153–173, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA (2008)

[13] Bownik, M., Ross, K.A.: The structure of translation-invariant spaces on locally compact abelian groups. J. Fourier Anal. Appl. 21(4), 849–884 (2015)

[14] Bownik, M., Rzeszotnik, Z.: On the existence of multiresolution analysis of framelets. Math. Ann. 332(4), 705–720 (2005)

[15] Bownik, M., Rzeszotnik, Z., Speegle, D.: A characterization of dimension functions of wavelets. Appl. Comput. Harmon. Anal. 10(1), 71–92 (2001)

[16] Currey, B., Mayeli, A.: Gabor fields and wavelet sets for the Heisenberg group. Monatsh. Math. 162(2), 119–142 (2011)

[17] Currey, B., Mayeli, A., Oussa, V.: Characterization of shift-invariant spaces on a class of nilpotent Lie groups with applications. J. Fourier Anal. Appl. 20(2), 384–400 (2014)

[18] Farkov, Yu A.: Orthogonal wavelets on locally compact abelian groups. Funct. Anal. Appl. 31(4), 294–296 (1997)

[19] Farkov, Yu A.: Multiresolution analysis and wavelets on Vilenkin groups, Facta Universitatis (NIS) Ser. Electron. Energy 21, 309–325 (2008)

[20] Jiang, H.K., Li, D.F., Jin, N.: Multiresolution analysis on local fields. J. Math. Anal. Appl. 294(2), 523–532 (2004)

[21] Lang, W.C.: Wavelet analysis on the Cantor dyadic group. Houst. J. Math. 24(3), 533–544 (1998)

[22] Li, D.F., Jiang, H.K.: The necessary condition and sufficient condition for wavelet frame on local fields. J. Math. Anal. Appl. 345(1), 500–510 (2008)

[23] Rzeszotnik, Z.: Characterization theorems in the theory of wavelets, Ph.D. thesis, Washington University (2000)

[24] Shukla, N.K., Vyas, A.: Multiresolution analysis through low-pass filter on local fields of positive characteristic. Complex Anal. Oper. Theory 9(3), 631–652 (2015)

[25] Shukla, N.K., Maury, S.C.: Super-wavelets on local fields of positive characteristic. Math. Nachr. 291(4), 704–719 (2018)

[26] Taibleson, M.H.: Fourier analysis on local fields. Princeton Univ. Press, Princeton (1975)

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