The Oberth effect and relativistic rocket in the Schwarzschild background

Yu. V. Pavlov\textsuperscript{1,2} and O. B. Zaslavskii\textsuperscript{2,3}

\textsuperscript{1}Institute of Problems in Mechanical Engineering, Russian Academy of Sciences, 61 Bol’shoy pr., St. Petersburg 199178, Russia  
E-mail: yuri.pavlov@mail.ru
\textsuperscript{2}N.I. Lobachevsky Institute of Mathematics and Mechanics, Kazan Federal University, 18 Kremlyovskaya St., Kazan 420008, Russia;
\textsuperscript{3}Department of Physics and Technology, Kharkov V.N. Karazin National University, 4 Svoboda Square, Kharkov 61022, Ukraine  
E-mail: zaslav@ukr.net

Abstract

We relate the known Oberth effect and the nonrelativistic analogue of the Penrose process. When a particle decays to two fragments, we derive the conditions on the angles under which debris can come out for such a process to occur. We also consider the decay and the Oberth effect in the relativistic case, when a particle moves in the background of the Schwarzschild black hole. This models the process when a rocket ejects fuel. Different scenarios are analyzed depending on what data are fixed. The efficiency of the process is found, in particular near the horizon and for a photon rocket (when the ejected particle is massless). We prove directly that the most efficient process occurs when fuel is ejected along the rocket trajectory. When this occurs on the horizon, the efficiency reaches 100\% for a photon rocket. We also consider briefly the scenario when a rocket hangs over a black hole due to continuous ejection of fuel. Then, the fuel mass decays exponentially with the proper time.

Key words: Oberth effect, photon rocket, black hole, Penrose effect
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1 Introduction

In 1929, Oberth pointed out that using of the reactive fuel becomes more efficient for rapid motion of a rocket (see \cite{1}, pp. 200 – 201). Its engine performs more useful work than in the case of slow motion. Indeed, in the laboratory frame, fuel flowing out from a nozzle of a moving rocket can have a velocity less than that of a jet stream from a slow moving rocket. As a consequence, the kinetic energy of jet fuel can decrease, whereas the energy used for rocket acceleration can increase.

In the gravitational field, high speed is achieved in a lowest point of the trajectory. Therefore, the use of jet fuel in periastron is more efficient than in highest points. In the nonrelativistic case, this can be explained on the basis of the energy conservation law since the total energy of jet fuel is the sum of positive kinetic and negative potential energies
of fuel in the gravitational field. (In the nonrelativistic case, a reader can find simple pedagogical presentation of the Oberth effect in a methodical paper [2].)

If the sum of the potential and kinetic energies of used jet fuel is negative in the gravitational field, it was firstly noticed in [3] that the Oberth effect can be thought of as a nonrelativistic realization of the effect similar to the Penrose one in the ergosphere of a black hole [4], [5]. In this case, the total nonrelativistic energy of a rocket after using jet fuel becomes bigger than the energy of a rocket plus fuel before turning on an engine.

In the present paper, we consider the Oberth effect for a relativistic case when a rocket moves in the metric of a nonrotating black hole. But we start with the short discussion of a nonrelativistic case.

It is worth noting that general approach to the motion of a rocket in a general relativity was developed in [6], and considered for the Schwarzschild metric in [7]. There, the main accent was made on a motion of a body as such. Meanwhile, our main goal is to include the issue under consideration in a context connected with particle decay. In doing so, special attention is paid to the processes in the vicinity of a black hole horizon.

2 Nonrelativistic Case

Let a point-like particle of the mass $m$ decay to two particles at a distance $r$ from a large attracting mass $M$ ($M \gg m_0$). We denote $m_1$ and $m_2$ masses of debris. We consider process of ejection of fuel under an arbitrary angle to the direction of motion of a rocket but, with the restriction, that all particles move within the same plane.

Let us consider decay of particle 0 to 1 and 2 in point $r$. Particle 0 moves with the velocity $\vec{v}_b$, particles 1, 2 move with the velocities $\vec{v}_1, \vec{v}_2$. In the present paper, particle 2 corresponds to a rocket, particle 1 corresponds to fuel. We have for the energy

$$E_0 = \frac{m_0 v_0^2}{2} - \frac{G m_0 M}{r}.$$  \hspace{1cm} (1)

$G$ is the gravitational constant, $E_0$ is the total mechanical energy of an original body. If $E_0 \geq 0$, it can move along an unbound trajectory, then

$$E_0 = \frac{m_0 v_0^2}{2},$$ \hspace{1cm} (2)

where $v_0$ is the velocity at infinity. Thus

$$v_b^2 = v_0^2 + 2G \frac{M}{r}.$$ \hspace{1cm} (3)

The total mass

$$m_0 = m_1 + m_2.$$ \hspace{1cm} (4)

Let superscript (0) denote quantities calculated in the frame comoving with particle 0. This frame corresponds to the center of mass (CM) of particles 1 and 2. Then, the relative velocity $\vec{u}$ between particles 1 and 2 equals

$$\vec{u} = \vec{v}_2 - \vec{v}_1 = \vec{v}_2^{(0)} - \vec{v}_1^{(0)}.$$ \hspace{1cm} (5)
From (5) and the conservation law for the momentum we have

\[ \vec{v}_1 = \vec{v}_b - \frac{m_2}{m_0} \vec{u}, \]  
\[ \vec{v}_2 = \vec{v}_b + \frac{m_1}{m_0} \vec{u}, \]  

From (1), (4), (6), (7) we get

\[ E_1 + E_2 = E_0 + E_f, \]  

where

\[ E_1 = m_1 \left( \frac{v_1^2}{2} - \frac{GM}{r} \right), \quad E_2 = m_2 \left( \frac{v_2^2}{2} - \frac{GM}{r} \right), \]  

and the quantity \( E_f \) in the formula (8) is equal to

\[ E_f = \frac{\mu u^2}{2}, \]  

where \( \mu = m_1 m_2 / m_0 \) is the reduced mass. It corresponds to the energy in the center of mass frame required for ejecting fuel (fragment 1) with the velocity \( u \) with respect to the rocket (fragment 2). Indeed, one can check that

\[ T_1^{(0)} + T_2^{(0)} = E_f, \]  

where \( T_i^{(0)} \) is the kinetic energy of particle “i” in the CM frame. In the case of an ideal jet engine that converts all the energy of fuel into that of jet stream, the value of \( E_f \) would correspond just to this energy of fuel stored in some form (chemical, nuclear, etc.).

Let us denote the energy gain in the gravitational field \( E_{2\text{grav}} \), and in the absence of gravitation \( E_2^{(0)} \) (also for instantaneous ejection of fuel). Then, the difference between them is equal to

\[ E_{2\text{grav}} - E_2^{(0)} = \mu u (\vec{v}_b - \vec{v}_0). \]  

Comparison under discussion implies that in both cases all masses and velocities are the same in both cases (with and without gravitation), \( E_0 \geq 0 \) (so a rocket can start at infinity, \( \vec{v}_0 \) having the meaning of the velocity there). Also, the initial states are chosen accordingly. Thus, if we extend the tangent vector to the trajectory backward in time along the straight line, it determines the initial position of a rocket in the conditional scenario without gravitation. Then, the vector \( \vec{v}_0 \) is the characteristic of a rocket in the scenario without gravitation, it is pointed in the same direction as \( \vec{v}_b \). The case when \( \vec{u} \vec{v}_b > 0 \) corresponds to acceleration, the case when \( \vec{u} \vec{v}_b < 0 \) corresponds to deceleration.

For collinear motion (when fuel is ejected along the trajectory in the same direction),

\[ E_{2\text{grav}} - E_2^{(0)} = \mu u \left( \sqrt{\frac{v_0^2}{2} + \frac{2GM}{r}} - v_0 \right). \]  

In the particular case, for \( v_0 = 0 \), the zero velocity at infinity (13) turns into

\[ E_{2\text{grav}} - E_2^{(0)} = \mu u \sqrt{\frac{2GM}{r}} = \mu uv_p, \]  

\[ v_p \]
where \( v_p \) is the second cosmic (parabolic) velocity at distance \( r \) from the attracting center.

Now, let us estimate the efficiency of the gravitational Oberth effect introducing the efficiency coefficient for the jet engine. There exist different definitions of this quantity in nonrelativistic mechanics, see e.g. [8]. We define the efficiency as the ratio of two quantities. The first one is the increment of energy of a rocket without account for ejected fuel. The second one is the sum of kinetic energy \( E_{k1}^b \) of fuel before turning engine on plus the stored thermal (chemical, nuclear, etc.) energy of fuel \( E_f \):

\[
\eta = \frac{|E_2 - E_0|^b}{E_{1,k}^b + E_f}.
\]

(15)

Here, \( E_2^b \) is the mechanical energy of a rocket (without account for fuel) before turning on the engine, \( E_2 \) is that after turning it on. It is worth noting that for an observer making measurements at rest in the point where a rocket turns the engine on, the numerator of eq. (15) corresponds just to the absolute value of increment of energy, while the denominator describes the consumed energy of fuel. Such a definition corresponds just to the so-called total efficiency of jet engine [8]. The modulus in the numerator of (15) guarantees positive values of efficiency in the case of decelerating regime.

Let us find efficiency for the process under consideration. Taking into account (9), (10) and

\[
E_{1,k}^b = \frac{m_1 v_b^2}{2}, \quad E_2^b = \frac{m_2 v_b^2}{2} - \frac{Gm_2 M}{r},
\]

we obtain

\[
\eta = \frac{m_2}{m_0} |u| \left| \frac{m_1 u^2}{m_0} + \frac{2E_0}{m_0} + \frac{2GM}{r} \right| = \frac{m_2}{m_0} |u| \left[ \frac{m_1 u}{m_0} \right]^2 + v_b^2.
\]

(16)

(17)

For \( v_b = 0 \), \( \eta = m_1/m_0 \) does not depend on the velocity of speed with which fuel is being ejected.

Considering the quantity (17) as a function \( \eta(v_b) \) of a velocity \( v_b \) of a rocket with fuel in the point where engine is turned on, we find that for given \( m_1, m_2 \) the maximum is achieved for \( v_b = um_2/m_0 \). By substitution of this value into eq. (17), we find after simple transformations that

\[
\eta \left( \frac{m_2}{m_0} u \right) = \eta_{\text{max}} = 1.
\]

(18)

Then, taking into account (14) we can see that the velocity of ejected fuel is equal to zero as well as its kinetic energy. According to (15), the total chemical energy \( E_f \) stored in fuel is spent to the increment of the kinetic energy of a rocket.

3 Nonrelativistic analogue of the Penrose process

It follows from (8) that

\[
E_2 - E_0 = -E_1.
\]

(19)
From (1) – (10), one finds that

\[ E_1 = \frac{m_1}{m} E_0 + \frac{m_2 \mu}{2m} u^2 - \mu \vec{v}_b \vec{u}. \]  

(20)

In a similar way,

\[ E_2 = \frac{m_2}{m} E_0 + \frac{m_1 \mu}{2m} u^2 + \mu \vec{v}_b \vec{u}. \]  

(21)

Especially interesting case arises if \( E_1 < 0 \). Then, we gain more energy than were invested, so we deal with the nonrelativistic analogue of the Penrose process. Is it possible and under what conditions?

Writing

\[ \vec{v}_b \vec{u} = v_b |u| \cos \theta \]  

(22)

eq. (20) can be represented in the form

\[ E_1 = \frac{m_1}{m} E_0 + \frac{m_2 \mu}{2m} u^2 - \mu v_b |u| \cos \theta. \]  

(23)

Then, \( E_1 < 0 \), provided

\[ u_- < |u| < u_+, \]  

(24)

\[ u_{\pm} = \frac{m}{m_2} \left( v_b \cos \theta \pm \sqrt{\frac{2GM}{r} - v_b^2 \sin^2 \theta} \right). \]  

(25)

This can be rewritten as

\[ u_{\pm} = \frac{m}{m_2} \left( v_b \cos \theta \pm \sqrt{v_b^2 \cos^2 \theta - v_0^2} \right). \]  

(26)

The expression under the square root should be nonnegative, so

\[ \frac{2GM}{r} - v_b^2 \sin^2 \theta \geq 0, \]  

(27)

\[ \sin^2 \theta \leq \frac{v_b^2 - v_0^2}{v_b^2}, \]  

(28)

where we used (3). Eq. (24) implies that \( u_+ > 0 \), whence \( \cos \theta > 0 \). Therefore, the particle with \( E_1 < 0 \) can be ejected in the hemisphere around the direction to motion of particle 0 only. In particular, if \( \theta = 0 \),

\[ u_{\pm} = \frac{m}{m_2} \left( v_b \pm \sqrt{v_b^2 - v_0^2} \right). \]  

(29)

For the circle orbit, eq. (2) is not valid. In this case, as it is known,

\[ E_0 = -\frac{GMm}{2r}, \quad v_b^2 = \frac{GM}{r}. \]  

(30)

Then, the condition \( E_1 < 0 \) entails

\[ u_{\pm} = \frac{m}{m_2} \left( v_b \cos \theta \pm \sqrt{v_b^2 \cos^2 \theta + \frac{GM}{r}} \right) = \frac{m}{m_2} \left( v_b \cos \theta \pm \sqrt{v_b^2 \cos^2 \theta + v_b^2} \right). \]  

(31)

In this case, there is no restriction on the sign of \( \cos \theta \), so the particle with \( E_1 < 0 \) can be ejected in any direction. If we want to maximize \( E_2 \) (that is equivalent to minimizing \( E_1 \)) we must take \( \theta = 0 \). Then, all three vectors \( \vec{v}_b, \vec{u}, \vec{v}_1 \) and \( \vec{v}_2 \) are tangent to the trajectory.
4 Relativistic case. General set-up

In the nonrelativistic case, we implied the conservation of mass according to which
\[ m_0 = m_1 + m_2. \]
Now, this condition cannot be fulfilled. Indeed, if we pass to the center of mass (CM) frame, we find that for any nonzero relative velocity of fragments,
\[ E_0 = m_0 c^2, \quad E_1 > m_1 c^2, \quad E_2 > m_2 c^2, \quad E_0 = E_1 + E_2. \]
Therefore,
\[ m_0 > m_1 + m_2. \] (32)

It is worth noting that the conservation law of energy in the nonrelativistic case included also, additionally, chemical (or any other) energy of jet fuel (or energy that is consumed in the decay of a particle). In the relativistic case, this contribution enters automatically the total energy of an original body \( E_0 \). In the nonrelativistic case the kinetic energy and that of jet fuel were assumed to be small with respect to the rest energies of objects. It is this circumstance that leads to the equality \( m_0 = m_1 + m_2 \) in nonrelativistic approximation. Now, it is violated.

Now, we turn to consistent consideration of motion of a relativistic rocket in the gravitational field. Let us consider particle motion in the space-time with the metric
\[ ds^2 = -A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\omega^2, \quad d\omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \] (33)
The energy of a particle with the mass \( m \) in this metric can be found from the formula (see eq. (88.9) in [9])
\[ E = mc^2 \frac{\sqrt{A}}{\sqrt{1 - \frac{u^2}{c^2}}}, \] (34)
where \( v \) is the particle velocity measured by a static observer with fixed \( r, \theta, \varphi \). In the static metric with the interval \( ds^2 = g_{00} dx_0^2 + g_{\alpha\beta} dx^\alpha dx^\beta \), where \( \alpha, \beta \) are spatial indices, we have
\[ v = \frac{dl}{d\tau}, \quad d\tau = \frac{1}{c} \sqrt{g_{00}} dx_0, \quad dl^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \] (35)

Let a particle (rocket) moving freely in the gravitational field decay in some point with the radial coordinate \( r \) to two fragments (a rocket ejects a portion of fuel). We use the same notations \( E_i, m_i \) \((i = 0, 1, 2)\), \( u, v_b \) as in the nonrelativistic case. We assume that fuel is being ejected in the direction tangent to the trajectory (not necessarily radial). It is opposite to the direction of rocket motion in the regime of acceleration \((u > 0, \lambda > 0)\) and is in the same direction in the regime of deceleration \((u < 0, \lambda < 0)\).

The conservation law of energy and that of the projection of momentum to the direction of motion in the point of ejection give us
\[ E_0 = E_1 + E_2, \] (36)
\[ E_0 v_b = E_2 v_2 + E_1 v_1. \] (37)
According to the relativistic law of addition of velocities,
\[ v_1 = \frac{v_2 - u}{1 - \frac{v_2 u}{c^2}}. \] (38)
5 Scenario A, general formulas

There are different scenarios depending on what data are assumed to be fixed. In this scenario, we consider $E_0$, $m_0$, $m_2$ and $u$ as known data. Then, the rest of quantities $E_1$, $E_2$, $v_1$, $v_2$ can be found from (34), (36)–(38).

For a particle with the energy $E$, mass $m$, moving in the metric (33), it follows from (34) that

$$v = c \sqrt{1 - \frac{A}{\varepsilon^2}},$$

(39)

where $\varepsilon = E/(mc^2)$ is the specific energy,

$$E v = c \sqrt{E^2 - m^2c^4A(r)}.$$  

(40)

After simple transformations, we obtain

$$\frac{v_2 - u}{c \left(1 - \frac{v_2 u}{c^2}\right)} = \frac{\sqrt{1 - A \left(1 - \frac{v^2}{c^2}\right) - A \frac{u}{c^2}}}{1 - \frac{v^2}{c^2} \left(1 - \frac{A}{\varepsilon_2} \right)}.$$  

(41)

Eq. (37) turns into

$$\frac{\varepsilon_0 m_0}{\varepsilon_2 m_2} \sqrt{1 - \frac{A}{\varepsilon_2^2}} = \left(\frac{\varepsilon_0 m_0}{\varepsilon_2 m_2} - 1\right) \frac{\sqrt{1 - A \left(1 - \frac{v^2}{c^2}\right) - A \frac{u}{c^2}}}{1 - \frac{v^2}{c^2} \left(1 - \frac{A}{\varepsilon_2} \right)} + \sqrt{1 - \frac{A}{\varepsilon_2^2}}.$$  

(42)

We remind a reader that $u > 0$ corresponds to the regime of acceleration and $u < 0$ corresponds to that of deceleration.

For a photon rocket, substituting $u = \pm c$ in eq. (42), we obtain

$$\frac{\varepsilon_0 m_0}{\varepsilon_2 m_2} \sqrt{1 - \frac{A}{\varepsilon_2^2}} = \pm \left(1 - \frac{\varepsilon_0 m_0}{\varepsilon_2 m_2}\right) + \sqrt{1 - \frac{A}{\varepsilon_2^2}}.$$  

(43)

Hereafter, the upper sign refers to the regime of acceleration, whereas the lower one corresponds to deceleration. The latter case implies that jet fuel (electromagnetic radiation) is being ejected in the direction opposite to the motion of a rocket.

6 Photon rocket and efficiency in the relativistic case

Transforming (43), one can obtain

$$\frac{\varepsilon_0 m_0}{\varepsilon_2 m_2} \left(\sqrt{1 - \frac{A}{\varepsilon_2^2}} \pm 1\right) \mp 1 = \sqrt{1 - \frac{A}{\varepsilon_2^2}}.$$  

(44)
Squaring and simplifying, we find

\[ E_2 = E_0 \left[ 1 - \frac{1}{2} \left( 1 - \frac{m_2}{m_0^2} \right) \left( 1 \mp \sqrt{1 - A \left( \frac{m_0 c^2}{E_0} \right)^2} \right) \right], \tag{45} \]

whence

\[ E_2 - E_0 = -\frac{E_0}{2} \left( 1 - \frac{m_2}{m_0^2} \right) \left( 1 \mp \sqrt{1 - A \left( \frac{m_0 c^2}{E_0} \right)^2} \right). \tag{46} \]

Now, we can estimate the efficiency of turning a photon rocket on in the strong gravitational field, i.e. the Oberth effect. If such a rocket turns the engine on near the horizon, \( A \to 0 \). Then, in the acceleration regime, it follows from \([46]\) that \( E_2 - E_0 \to 0 \). Thus the full energy of fuel transforms into the kinetic energy of a rocket. In this case, the efficiency tends to 100\%.

When a particle approaches the horizon, its energy measured by a distant observer tends to zero (infinite redshift). Thus the aforementioned equality limiting value \( E_2 \to E_0 \) complies with this property. And, it is easy to show that a body radiates photons almost radially.

The results of calculations for the case when decay of particle (turning the engine on in the acceleration regime) occurs not on the horizon, are presented on Fig. 1. In the absence of gravitational field (\( A = 1 \))

\[ E_2 - E_0 = \frac{(m_2^2 - m_0^2)c^2}{2m_0} \left( \frac{E_0}{m_0 c^2} - \sqrt{\left( \frac{E_0}{m_0 c^2} \right)^2 - 1} \right). \tag{47} \]

For the case corresponding to Fig. 1 this gives us \(-7/16 \approx -0.22\).
The role of the gravitational field is especially pronounced if $\Delta m = m_0 - m_2 \ll m_0$. Thus, for $E_0 = m_0 c^2$ we find from (47)

$$E_2 - E_0 = -\Delta mc^2 + \frac{(\Delta m)^2}{2m_0} c^2, \quad \frac{E_2 - m_2 c^2}{\Delta mc^2} = \frac{\Delta m}{2m_0},$$

(48)

so only the fraction $(\Delta m)/(2m_0)$ of the total energy of jet fuel is used for the increase of the kinetic energy of a rocket. When $\Delta m/m_0 \to 0$, it vanishes. Meanwhile, in the same case near the horizon the efficiency achieves 100%, as shown above.

## 7 Efficiency of relativistic rocket

Now, we introduce the quantitative measure of efficiency $\eta$ for the jet engine (in particular, a photon one) according to

$$\eta = \frac{|E_2 - \varepsilon_0 m_2 c^2|}{\varepsilon_0 (m_0 - m_2) c^2}. \quad (49)$$

The quantity $\varepsilon_0 m_2 c^2$ gives us an initial energy of a part of a rocket without fuel used due to turning on the engine. The factor $\varepsilon_0$ takes into account that far from an attracting body a rocket together with fuel moves with some initial velocity (if $\varepsilon_0 > 1$) or a rocket moves only within some bounded region (if $\varepsilon_0 < 1$). Then, the initial energy is equal to

$$E_2^{(0)} = \frac{m_2 c^2}{\sqrt{1 - \frac{v_0^2}{c^2}}} = \varepsilon_0 m_2 c^2. \quad (50)$$

Therefore, the numerator equals the modulus of increment of the energy without a used part of fuel. The quantity in the denominator of (49) describes the mass and the energy of fuel, where the factor $\varepsilon_0$ takes into account the energy of motion. The denominator is the total initial energy of used fuel. As in the relativistic case the energy of a body includes the rest energy, the relativistic efficiency $\eta$ does not coincide with the quantity defined in (13).

For the nonrelativistic case

$$E_2 = \frac{m_2 c^2 \sqrt{g_{00}}}{\sqrt{1 - \frac{v_0^2}{c^2}}} \approx m_2 \left( c^2 + \frac{v_0^2}{2} - \frac{GM}{r} \right), \quad \varepsilon_0 \approx 1 + \frac{v_0^2}{2c^2} - \frac{GM}{rc^2} \quad (51)$$

taking into account (7) we get from (49) that

$$\eta = \frac{m_2}{m_0 c^2} \left| \vec{v}_b \vec{u} + \frac{\Delta m u^2}{2m_0} \right| = \frac{1}{m_0 c^2} \left| m_2 \vec{v}_b \vec{u} + \frac{\mu u^2}{2} \right|. \quad (52)$$

Bearing in mind (10), we can rewrite this formulas for the case of acceleration ($\vec{v}_b \vec{u} > 0$) as

$$\eta = \frac{E_f}{m_0 c^2} + \frac{m_2 \vec{v}_b \vec{u}}{m_0 c^2}. \quad (53)$$

The first term here represents the ratio of the energy stored in fuel (for example, due to chemical or nuclear forces) to the initial energy at rest. The second terms gives us
correction just due to motion of a rocket. Thus, in the nonrelativistic case both terms are separated. If \( \vec{v}_b \) and \( \vec{u} \) are parallel, this term is maximal and is responsible for the Oberth effect which looks as an additive correction to the first contribution.

For obtaining the value (15) in the nonrelativistic case, it is necessary to subtract the rest energy of jet fuel in denominator of (49).

As in the Schwarzschild metric the states with negative energy are absent, \( 0 < E_2 \leq E_0 = \varepsilon_0 m_0 c^2 \), the coefficient \( \eta \) in (49) changes in limits \( \eta \in [0, 1] \). If \( E_2 = E_0 = \varepsilon_0 m_0 c^2 \), it follows from (49) that \( \eta = 1 \) since all energy is used without loss. If the velocity of a rocket without fuel is equal to that with fuel, \( \varepsilon_0 = \varepsilon_2 \). Then, \( E_2 = m_2 \varepsilon_2 = \varepsilon_0 m_2 \), so \( \eta = 0 \) in (49).

From eq. (45) for a photon engine and arbitrary \( \varepsilon_0 \) one obtains

\[
\eta = \left| \frac{\Delta m}{2m_0} \pm \left( 1 - \frac{\Delta m}{2m_0} \right) \sqrt{1 - \frac{A}{\varepsilon_0^2}} \right|.
\]

Let a rocket move in the acceleration regime. When it approaches the attracting body, \( A \) is decreasing, \( \eta \) is increasing. In the horizon limit \( A \to 0, \eta \to 1 \).

It is worth noting that account for the rest mass of the source in the nonrelativistic case would lead for chemical sources of energy to the values of efficiency of the order \( 10^{-10} \) or less. For example, for the reaction of combustion of hydrogen, specific heat of its combustion equals 141 MJ/kg. This gives the value \( 1.6 \cdot 10^{-10} \) for the ratio of the combusted energy to the rest energy of water that forms in this reaction. If sources of energy are based on fission reaction of heavy nuclei, the ratio of the released energy to the rest energy of initial radioactive substance can reach \( 10^{-3} \). For light nuclei and the fusion reaction, the similar ratio does not exceed \( 4 \cdot 10^{-3} \).

At large distance from an attracting body, \( \eta \) takes its minimum value \( (\Delta m)/(2m_0) \) for \( E_0 = m_0 c^2 \).

For small \( \Delta m \)

\[
\Delta m \to 0 \Rightarrow \eta = \sqrt{1 - \frac{A}{\varepsilon_0^2}}.
\]

Now, we can estimate the value of the energy gain \( E_{2_{\text{grav}}} \) for the acceleration of a photon rocket as compared to the similar quantity \( E_2^{(0)} \) in the case of the absence of gravitation. As we are interested now in the acceleration regime, we take the upper sign in (46). In other words, we will estimate the value of the Oberth effect for a photon rocket.

\[
E_{2_{\text{grav}}} - E_2^{(0)} = E_0 \frac{m_0^2 - m_2^2}{2m_0^2} \left( \sqrt{1 - A \left( \frac{m_0 c^2}{E_0} \right)^2} - \sqrt{1 - \left( \frac{m_0 c^2}{E_0} \right)^2} \right) =
\]

\[
= \frac{1 - A}{2m_0} \cdot \frac{(m_0^2 - m_2^2)c^2}{\sqrt{\left( \frac{E_0}{m_0 c^2} \right)^2 - A} + \sqrt{\left( \frac{E_0}{m_0 c^2} \right)^2 - 1}}.
\]
The relative gain for the photon engine equals

$$\frac{E_{2,\text{grav}} - E_{2}^{(0)}}{E_{2}^{(0)}} = \frac{(m_0^2 - m_2^2)}{m_0^2 + m_2^2 + (m_0^2 - m_2^2)\sqrt{1 - \frac{m_0c^2}{E_0}}} \left( \sqrt{1 - A\left(\frac{m_0c^2}{E_0}\right)^2} - \sqrt{1 - \left(\frac{m_0c^2}{E_0}\right)^2} \right). \quad (57)$$

The formulas take especially simple form for the energy gain in the Schwarzschild metric ($A = 1 - r_g/r$) and $E_0 = m_0c^2$, as compared to the instant ejection of photon fuel in the absence of gravitation ($A = 1$)

$$E_{2,\text{grav}} - E_{2}^{(0)} = \frac{(m_0^2 - m_2^2)c^2}{2m_0} \sqrt{\frac{r_g}{r}}. \quad (58)$$

The relative gain, when $E_0 = m_0c^2$ is equal to

$$\frac{E_{2,\text{grav}} - E_{2}^{(0)}}{E_{2}^{(0)}} = \frac{m_0^2 - m_2^2}{m_0^2 + m_2^2} \sqrt{\frac{r_g}{r}}. \quad (59)$$

8 Rocket with nonphoton working medium

In this case, eq. (42) with respect to $E_2$ can be written as an algebraic equation of the sixth power. We will consider two particular cases admitting simple approximate solutions.

8.1 Solution in the 1st order in $u/c$

Neglecting terms of the order $u^2/c^2$, one obtains an approximate solution of (42):

$$E_2 \approx E_0 \frac{m_2}{m_0} \left[ 1 + \frac{u}{c} \left( 1 - \frac{m_2}{m_0} \right) \sqrt{1 - A\left(\frac{m_0c^2}{E_0}\right)^2} \right]. \quad (60)$$

In this case, the efficiency

$$\eta = \frac{|u| m_2}{c m_0} \sqrt{1 - \frac{A}{\varepsilon_0^2}}.$$

The energy gain in the gravitational field in the acceleration regime, as compared to that for fuel ejection in the absence of gravitation is equal to

$$E_{2,\text{grav}} - E_{2}^{(0)} \approx \frac{m_2(m_0 - m_2)}{m_0 c u} \left( \sqrt{\varepsilon_0^2 - A} - \sqrt{\varepsilon_0^2 - 1} \right). \quad (61)$$

When $E_0 = m_0c^2$, eq. (14) is reproduced.
9 Solution near horizon

For an arbitrary value of $u/c$, the solution of eq. (42) in the first approximation in $A$ takes the form

$$E_2 \approx E_0 \frac{1 + \frac{u}{c}}{\frac{u}{c} + \sqrt{\frac{u^2}{c^2} + \frac{m_2^2}{m_0^2} (1 - \frac{u^2}{c^2})}},$$

(62)

$$E_2 - E_0 \approx -E_0 \frac{1 - \frac{u}{c}}{\frac{u}{c} + \sqrt{\frac{u^2}{c^2} + \frac{m_2^2}{m_0^2} (1 - \frac{u^2}{c^2})} + \frac{m_2^2}{m_0^2} \left(1 - \frac{u^2}{c^2}\right)}.\quad (63)$$

The efficiency (49) is equal now

$$\eta = \frac{u}{c} \frac{1 + \frac{u}{c} + \frac{m_0}{m_2} \left(1 + \frac{u}{c}\right) + \sqrt{\frac{u^2}{c^2} + \frac{m_2^2}{m_0^2} \left(1 - \frac{u^2}{c^2}\right)}}{\left(1 - \frac{u^2}{c^2}\right) + \frac{m_2^2}{m_0^2} \left(1 - \frac{u^2}{c^2}\right)}.\quad (64)$$

When $u \to 0$ the efficiency also tends to zero. For a photon rocket, $u/c = 1$, the efficiency $\eta \to 1$ near the horizon that agrees with the results described above. If the quantity of ejected fuel is negligible, so $m_2 \approx m_0$, we obtain from (64) that $\eta = |u|/c$. The case when the fraction of ejected fuel is substantial, is plotted on Fig. 2.

![Figure 2: The plot of efficiency near event horizon in dependence of reactive fuel velocity for $m_2 \approx m_0$ (the blue dashed line) and $m_2 = 0.75m_0$ (the red line).](image)

10 General solution for small $\Delta m$

Now, let us find a general solution of eq. (42) for a nonphoton rocket for the case $\Delta m/m_0 \ll 1$. We denote

$$\frac{m_2}{m_0} = 1 - \alpha, \quad \alpha \ll 1,$$

(65)

$$\frac{\varepsilon_2}{\varepsilon_0} = 1 + \beta.$$

(66)
It is worth noting that the efficiency in this case equals

$$\eta = \frac{|\beta|}{\alpha} \quad (67)$$

We will look for $\beta$ as a function of $\alpha$. Assuming that for small $\alpha$, the quantity $|\beta| \ll 1$, we carry out calculations in the first order with respect to $\alpha$ and $\beta$:

$$\frac{\varepsilon_0 m_0}{\varepsilon_2 m_2} = 1 + \alpha - \beta, \quad (68)$$

$$\frac{A}{\varepsilon_2} = \frac{A}{\varepsilon_0} (1 - 2\beta), \quad 1 - \frac{A}{\varepsilon_2} = 1 - \frac{A}{\varepsilon_0} + 2\beta \frac{A}{\varepsilon_0}, \quad (69)$$

$$\sqrt{1 - \frac{A}{\varepsilon_0}} = \sqrt{2\beta} \, , \quad \frac{A}{\varepsilon_2} = 1, \quad \sqrt{1 - \frac{A}{\varepsilon_0} + \frac{\beta A}{\varepsilon_0} \sqrt{1 - \frac{A}{\varepsilon_0}}} \, , \quad \frac{A}{\varepsilon_2} \neq 1. \quad (70)$$

If $A/\varepsilon_0^2 = 1$, a rocket remains at rest before turning the engine on, hence the deceleration regime in this case is impossible, $\beta \geq 0$. Then, in the main approximation eq. (42) takes the form

$$0 = -(\alpha - \beta) \frac{u}{c} + \sqrt{2\beta}. \quad (71)$$

As a result, we obtain

$$\frac{A}{\varepsilon_0} = 1 \Rightarrow \beta = \alpha^2 \frac{u^2}{2c^2}, \quad \eta = \alpha^2 \frac{u^2}{2c^2}, \quad \alpha \to 0. \quad (72)$$

If $A/\varepsilon_0^2 < 1$, it follows from (42) that

$$\frac{A}{\varepsilon_0} < 1 \Rightarrow \beta = \alpha \frac{u}{c} \sqrt{1 - \frac{A}{\varepsilon_0^2}}, \quad \eta = \frac{|u|}{c} \sqrt{1 - \frac{A}{\varepsilon_0^2}}, \quad \alpha \to 0. \quad (73)$$

On the horizon, it follows from (73) that the efficiency equals

$$A = 0, \quad \frac{\Delta m}{m_0} \ll 1 \Rightarrow \eta = \frac{|u|}{c}. \quad (74)$$

### 11 Scenario B

Now, we can somewhat change the set of quantities which we assumed to be fixed. Namely, let us fix the velocity of particle 1 (fuel) $\vec{v}_1^{(0)}$ in the CM frame instead of the relative velocity $\vec{u}$. We also fix the mass $m_1$. Then, $m_2 = m - m_1$. Hereafter, we call it "scenario B". For the nonrelativistic case, the relation between scenarios A and B is quite direct. To relate them, one should express $\vec{v}_2^{(0)}$, $\vec{v}_1$ and $\vec{v}_2$ in terms of $\vec{v}_1^{(0)}$ and $\vec{u}$. As this procedure
is quite direct, we omit corresponding simple formulas. However, the situation becomes much more interesting, with relation between scenarios A and B being nontrivial. It is especially important that scenario B (in contrast to scenario A) admits simple exact algebraic solutions for all relevant quantities (see below).

Hereafter, we omit the factor $c$ if this does not lead to confusion. Let us again consider the decay of particle 0. When new particles 1 and 2 move under some angles to the original trajectory, this leads to very cumbersome formulas in relations between the center of mass (CM) frame and the static one. To simplify matter, we assume that (i) decay of particle 0 occurs just in the turning point, (ii) new particles 1 and 2 fly out along the tangent direction to the same trajectory. To avoid confusion because of using superscript “0”, hereafter we denote components of the velocity in the CM frame $\tilde{v}_1$, $\tilde{v}_2$.

Then, the standard relativistic formulas of additions of velocities give us

$$v_2 = \frac{v_b + \tilde{v}_2}{1 + v_b \tilde{v}_2},$$  \hspace{1cm} (75)$$
$$v_1 = \frac{v_b + \tilde{v}_1}{1 - v_b |\tilde{v}_1|},$$ \hspace{1cm} (76)$$

By definition, the momentum in the CM frame vanishes, the total energy is equal to the energy in the CM frame $m_0$, so we have two equations

$$\frac{m_1 \tilde{v}_1}{\sqrt{1 - \tilde{v}_1^2}} + \frac{m_2 \tilde{v}_2}{\sqrt{1 - \tilde{v}_2^2}} = 0,$$ \hspace{1cm} (77)$$
$$\frac{m_1}{\sqrt{1 - \tilde{v}_1^2}} + \frac{m_2}{\sqrt{1 - \tilde{v}_2^2}} = m_0. \hspace{1cm} (78)$$

For definiteness, we assume that $\tilde{v}_2 > 0$, $\tilde{v}_1 < 0$ that corresponds to the regime of acceleration of a rocket (which coincides with particle 0 before ejection of fuel and with particle 2 after it).

It follows from these equations that

$$\tilde{E}_2 = m_0 - \tilde{E}_1,$$ \hspace{1cm} (79)$$
$$m_2 = \sqrt{m_0^2 + m_1^2 - 2m_0 \tilde{E}_1}, \hspace{1cm} (80)$$
$$\tilde{v}_2 = -\frac{\tilde{P}_1}{m_0 - \tilde{E}_1}, \hspace{1cm} (81)$$

where

$$\tilde{P}_1 = \frac{m_1 \tilde{v}_1}{\sqrt{1 - \tilde{v}_1^2}} = -|\tilde{P}_1|, \hspace{1cm} \tilde{E}_1 = \frac{m_1}{\sqrt{1 - \tilde{v}_1^2}}. \hspace{1cm} (82)$$

Usually, in textbooks $\tilde{E}_1$ and $\tilde{E}_2$ are expressed in terms of masses (see e.g. Sec. II, 11 in [9]). But now we consider $\tilde{E}_1$ and $m_1$ as given and find characteristics of particle 2. Then, (75) gives us

$$v_2 = \frac{v_b + |\tilde{P}_1|}{m_0 - \tilde{E}_1} \frac{\tilde{P}_1}{1 + \tilde{v}_b |\tilde{P}_1|}.$$ \hspace{1cm} (83)
In the static frame, we have from (34) and (75)

\[ E_2 = \frac{\tilde{E}_2 + v_b P_2}{\sqrt{1 - v_b^2}} \sqrt{A} \]  

(84)

that generalizes the Lorentz formula typical of the flat space-time. It can be also written as

\[ E_2 = \frac{m_0 - \tilde{E}_1 + v_b P_1}{\sqrt{1 - v_b^2}} \sqrt{A}. \]  

(85)

In a similar way,

\[ E_1 = \frac{\tilde{E}_1 - v_b P_1}{\sqrt{1 - v_b^2}} \sqrt{A}. \]  

(86)

Obviously, \( E_1 + E_2 = E_0 \), as it should be. We also find

\[ \varepsilon_2 \equiv \frac{E_2}{m_2} = \frac{m_0 - \tilde{E}_1 + v_b P_1}{\sqrt{1 - v_b^2} \sqrt{m_0^2 + m_1^2 - 2m_0 \tilde{E}_1}} \sqrt{A}. \]  

(87)

Here,

\[ \tilde{E}_1 \leq \frac{m_0^2 + m_1^2}{2m_0}. \]  

(88)

### 12 Photon rocket

We can take the safe limit to the case of photon fuel, provided \( m_1 \to 0, \tilde{v}_1 \to 1 \) in such a way that \( \tilde{E}_1 \) remains finite, \( P_1 = \tilde{E}_1 \). Then,

\[ \tilde{v}_2 = \frac{\tilde{E}_1}{m_0 - \tilde{E}_1}. \]  

(89)

The condition \(|\tilde{v}_2| < 1\) gives us

\[ \tilde{E}_1 \leq \frac{m_0}{2} \]  

(90)

in agreement with (88). We also find from (80) that

\[ m_2 = \sqrt{m_0^2 - 2m_0 \tilde{E}_1}. \]  

(91)

For particle 1 we have from (86)

\[ E_1 = \tilde{E}_1 \sqrt{A} \sqrt{1 - v_b}. \]  

(92)

This is just combination of the Doppler shift and redshift — see [11] for details and references therein. It follows from (87), (91) that

\[ E_2 = E_0 - \tilde{E}_1 \sqrt{A} \sqrt{\frac{1 - v_b}{1 + v_b}}. \]  

(93)
12.1 Efficiency

Let us use the definition of the efficiency according to eq. (49). Then it follows from (34), (36) and (91) that

$$\eta = \frac{E_0 - \tilde{E}_1}{\varepsilon_0 \left( m_0 - \sqrt{m_0^2 - 2m_0 \tilde{E}_1} \right)}.$$ \hspace{1cm} (94)

In the horizon limit $A \to 0$ we have

$$\eta = 1.$$ \hspace{1cm} (95)

Let us consider the nonrelativistic case $\varepsilon_0 = 1$, $v_b \to 0$, $A = 1$ (no gravitation). Then,

$$\eta = \frac{E_0 - \tilde{E}_1 - \varepsilon_0 m_2}{(m_0 - \sqrt{m_0^2 - 2m_0 \tilde{E}_1})}.$$ \hspace{1cm} (96)

One can check that (93) agrees with (45) for a given choice of signs (acceleration regime), if one take into account eq. (80). Thus the efficiency found in Scenario B agrees with that analyzed in Sec. 7 within Scenario A.

13 Non-collinear vs collinear motion

Up to now, we considered the case of collinear motion, when both particles 1, 2 are emitted in the tangent direction of the trajectory of particle 0. Now, we consider a more general case, when all three particles have arbitrary nonzero angular momenta $L_i$ ($i = 0, 1, 2$) with the restriction $L_0 = L_1 + L_2$.

According to eqs. (19) — (30) of [10],

$$E_1 = \frac{1}{2m_0^2} \left( E_0 \Delta_+ + P_0 \delta \sqrt{D} \right),$$ \hspace{1cm} (97)

$$E_2 = \frac{1}{2m_0^2} \left( E_0 \Delta_- - P_0 \delta \sqrt{D} \right),$$ \hspace{1cm} (98)

$\delta = \pm 1$. Here, we use particle labels 1 and 2 instead of 3 and 4 respectively in [10]. In this section, we put $c = 1$ for simplicity.

$$\Delta_\pm = \tilde{m}_0^2 \pm (\tilde{m}_1^2 - \tilde{m}_2^2),$$ \hspace{1cm} (99)

$$\tilde{m}_i^2 = m_i^2 + \frac{L_i^2}{r^2},$$ \hspace{1cm} (100)

where $i = 0, 1, 2$,

$$D = \Delta_+^2 - 4\tilde{m}_0^2 m_1^2 = \Delta_-^2 - 4\tilde{m}_0^2 \tilde{m}_2^2.$$ \hspace{1cm} (101)
The direction of motion is characterized by a quantity \( \sigma \), where \( \sigma = +1 \) for motion in the outward direction and \( \sigma = -1 \) for the inward case. We are mainly interested in the situation, when particle 0 (an initial rocket) moves towards a black hole, from large radii to smaller ones. Then, at least one of particles falls in a black hole. We assume that this is particle 2, so \( \sigma_2 = -1 \).

As is explained in \cite{10}, possible scenarios can be characterized by the set \((\sigma_1, e_1, e_2, \delta)\).

\[
e_i = sgn \left( \Delta_i \sqrt{A - 2\tilde{m}_iE_0} \right),
\]

where \( i = 1, 2 \). These quantities were denoted \( \varepsilon_i \) in \cite{10}.

Then, there exist 6 scenarios. They are listed in eq. (30) of \cite{10}. We are interested in the process close to the horizon, where \( A \) is sufficiently small. Let us assume that \( \tilde{m}_i \neq 0 \) (either \( m_i \neq 0 \) or \( L_i \neq 0 \) or both). Then, \( e_1 = e_2 = -1 \) and only the following scenarios survive. If \( \delta = +1 \), only scenario \((-,-,-,+)\) is possible. If \( \delta = -1 \), we have \((-,-,-,-)\).

Now, \( \sigma_1 = \sigma_2 = -1 \), both particles fall in a black hole. Two scenarios differ only by labels 1 and 2.

### 13.1 The most efficient configuration

We are interested in the scenario that gives us the maximum possible value of \( E_2 \) for given other data. It follows from intuitively clear arguments that for the decay to be the most efficient, particle 1 must be ejected in the direction strictly opposite to the direction in which an initial particle 0 moves. For the nonrelativistic case, this can be justified very easily - see Sec. 14. However, in the relativistic case, instead of the nonrelativistic one, the situation is less obvious because of nonlinear character of relevant quantities. Below, we give an explicit proof of the aforementioned statement.

Let all masses \( m_i \), \( E_0 \) and the angular momentum \( L_0 \) be fixed. We have at our disposal one independent variable, this is the momentum \( L_2 \) (then \( L_1 = L_0 - L_2 \)). The condition of the extremum of \( E_2 \) entails

\[
\frac{\partial E_2}{\partial L_2} = 0.
\]

Then, using (97) – (101), one can obtain from (103) the equation

\[
E_0L_0\sqrt{D} = P_0\delta \left(L_0b - 2L_2m_0^2\right).
\]

From another hand, we can consider the scenario under discussion from the geometric viewpoint. If we want motion of debris to be parallel to the initial particle, the corresponding angles should coincide. Assuming all particles to move in the same plane, we see that the ratio \( v^{(3)}/v^{(1)} \) of tetrad components of the three-velocity corresponding to a static observer should have the same value for particles 0 and 2. Here, indices 1 and 3 tetrads pointed in the radial and angle directions, respectively. It is easy to find that

\[
v^{(3)} = \frac{L\sqrt{A}}{rE},
\]
\[ v^{(1)} = \sqrt{1 - \frac{A\tilde{m}^2}{E^2}}. \] (106)

This can be derived from the equations of geodesic motion or taken directly from eqs. (12), (13) of [13]. Then,

\[ \frac{L_0}{P_0} = \frac{L_2}{|P_2|}, \] (107)

\[ |P_2| = \sqrt{E_2^2 - \tilde{m}_2^2 A}. \] Using (98), one can find that

\[ |P_2| = \frac{E_0\sqrt{b} + P_0\Delta \delta}{2\tilde{m}_0^2} \] (108)

in agreement with eqs. (26), (27) of [10]. Then, after some algebra, it is easy to show that (107) is equivalent to (104). Thus the statement about the most efficient scenario is proven.

Now, we can solve eq. (104). After some algebraic manipulations, one finds

\[ L_2 = \frac{L_0}{2m_0^2} \left( b \pm \frac{E_0}{\sqrt{E_0^2 - m_0^2 A}} \sqrt{b^2 - 4m_0^2 m_2^2} \right), \quad b = m_0^2 + m_2^2 - m_1^2, \] (109)

\[ E_2 = \frac{E_0 b}{2m_0^2} \pm \frac{E_0^2 - m_0^2 A}{2m_0^2} \sqrt{b^2 - 4m_0^2 m_2^2}. \] (110)

Here, the upper sign refers to the acceleration regime, the lower one - to the deceleration regime.

If \( m_1 = 0 \) (a photon rocket), eq. (110) coincides with eq. (45).

### 13.2 Near-horizon expansion

Near the horizon when \( A \to 0 \),

\[ P_0 = E_0 - \frac{A\tilde{m}_0^2}{2E_0} + O(A^2), \] (111)

\[ L_2 \approx \frac{L_0}{2m_0^2} \left( b \pm \sqrt{b^2 - 4m_0^2 m_2^2} \right), \] (112)

\[ E_2 \approx \frac{E_0 b}{2m_0^2} \pm \sqrt{b^2 - 4m_0^2 m_2^2}. \] (113)

According to (110), in this limit the efficiency in the acceleration regime

\[ \eta = \frac{m_0}{m_0 - m_2} \left( b \pm \sqrt{b^2 - 4m_0^2 m_2^2} \right) - \frac{m_2}{2m_0^2} \] (114)

For a photon rocket \( m_1 = 0 \) and \( \eta \to 1 \) in agreement with (54).
14 Decay in the turning point

Let us consider the decay of particle 0. When new particles 1 and 2 move under some angles to the original trajectory, this leads to very cumbersome formulas. To simplify matter and concentrate on the physically relevant situation, we discuss now the following scenario. (i) Decay of particle 0 occurs just in the turning point, (ii) new particles 1 and 2 fly out along the tangent direction to the same trajectory. This means that the radial velocity of each particle vanishes, so \(P_0 = P_1 = P_2 = 0\). Then, according to eqs. (97)–(101) in the point of decay \(r_d\),

\[
E_i = \tilde{m}_i \sqrt{A_d} = m_i \sqrt{A_d} \sqrt{1 + \frac{x_i^2}{m_i^2}},
\]

where \(x_i \equiv L_i/r_d, i = 0, 1, 2\), \(A_d = A(r_d)\), where we used the definition (100). From the conservation of energy, we obtain

\[
\tilde{m}_0 = \tilde{m}_1 + \tilde{m}_2.
\]

It follows from (116)

\[
x_1 = \frac{d_+}{2m_0^2} x_0 \pm \frac{\tilde{m}_0}{2m_0^2} \sqrt{D_0},
\]

\[
x_2 = \frac{d_-}{2m_0^2} x_0 \pm \frac{\tilde{m}_0}{2m_0^2} \sqrt{D_0},
\]

\[
D_0 = d_+^2 - 4m_1^2 m_0^2 = d_-^2 - 4m_2^2 m_0^2,
\]

\[
d_+ = m_0^2 + m_1^2 - m_2^2, \quad d_- = m_0^2 + m_2^2 - m_1^2.
\]

These formulas resemble those for the energy in Sec. IV of [10], with \(\tilde{m}_i\) replaced with \(m_i\) in proper places.

A velocity of particle \(i\) is directed along the \(\phi\) axis. Using the tetrads attached to a static observer, one easily finds that in the turning point. We obtain that (e.g., see eq. (44) in Ref. [12]) its value

\[
\varepsilon_i^{(3)} = \frac{\sqrt{AL_i}}{r \varepsilon_i} = \frac{\sqrt{E_i^2 - m_i^2 A}}{E_i} = \sqrt{1 - \frac{m_i^2}{E_i^2}}.
\]

In the acceleration regime we have for particle 2 (rocket) we should take the upper sign in (118). Below, we assume in this section that a rocket is photonic, so \(m_1 = 0\). Then, we have from (118) in the acceleration regime (assuming \(x_0 > 0\))

\[
x_1 = \frac{x_0}{2}(1 - \alpha^2) - \frac{m_0}{2} \sqrt{1 + \frac{x_0^2}{m_0^2} (1 - \alpha^2)} \equiv m_0 g_-(\frac{x_0}{m_0}, \alpha).
\]

\[
\frac{x_2}{m_2} = \frac{x_0}{2m_0 \alpha}(1 + \alpha^2) + \frac{1}{2\alpha} \sqrt{1 + \frac{x_0^2}{m_0^2} (1 - \alpha^2)} \equiv f_+(\frac{x_0}{m_0}, \alpha),
\]
\( \alpha \equiv m_2/m_0 \). By substitution into (115), we obtain

\[
\varepsilon_2 = \sqrt{A} \sqrt{1 + f^2_\pm \left( \frac{x_0}{m_0}, \alpha \right)}, \quad E_2 = m_0 \alpha \varepsilon_2,
\]

(124)

\[
E_1 = E_0 - E_2 = x_1 \sqrt{A} = m_0 g \left( \frac{x_0}{m_0}, \alpha \right) \sqrt{A},
\]

(125)

where \( A \) is taken in the point of decay.

### 15 Example: escape from the ISCO

Let particle 0 rotate around a black hole on a circle orbit. Then, for a given \( L = L/m \), there are two radii, \( r_A \) (stable) and \( r_P \leq r_A \) (unstable) \[9\],

\[
\frac{r_A}{r_g} = \frac{L^2}{r_g^2} \left[ 1 + \sqrt{1 - \frac{3r_g^2}{L^2}} \right],
\]

(126)

\[
\frac{r_P}{r_g} = \frac{L^2}{r_g^2} \left[ 1 - \sqrt{1 - \frac{3r_g^2}{L^2}} \right],
\]

(127)

\[
\varepsilon^2 = U_{ef}(r_c) = \frac{2A^2L^2}{r_g^2},
\]

(128)

\[
L^2 \geq 3r_g^2.
\]

(129)

In the case of equality, both roots coincide, so

\[
\frac{r_A}{r_g} = 3 = \frac{r_P}{r_g}, \quad \varepsilon^2 = \frac{8}{9}, \quad \frac{L}{r_g} = \sqrt{3}, \quad x_A \equiv \frac{mL}{r_A} = x_P = \frac{m}{\sqrt{3}}, \quad \tilde{m} = \frac{2m}{\sqrt{3}}.
\]

(130)

\[
\frac{x_A}{m} = \frac{1}{\sqrt{3}}, \quad \frac{\tilde{m}}{m} = \frac{2}{\sqrt{3}}.
\]

(131)

\[
A = \frac{2}{3}, \quad \sqrt{A} = \frac{2}{\sqrt{3}}.
\]

(132)

\[
\varepsilon = \frac{2\sqrt{2}}{3} = \sqrt{\frac{8}{9}}.
\]

(133)

This corresponds to the innermost stable circular orbit (ISCO).

In this case, eqs. (123), (124) give us

\[
\varepsilon_2 = \frac{3 + \alpha^2}{3\sqrt{2} \alpha}.
\]

(134)

Escape can occur if \( \varepsilon \geq 1 \). Then, \( \alpha \leq \alpha_- \), where

\[
\alpha_- = \frac{3 - \sqrt{3}}{\sqrt{2}} < 1.
\]

(135)
In doing so, the velocity at infinity

\[ v_\infty = \sqrt{1 - \frac{1}{\varepsilon_2^2}}. \]  

(136)

If \( \alpha \to \alpha_- \), \( \varepsilon_2 \to 1 \),

\[ \frac{E_2}{E_0} = \frac{3}{4} (3 - \sqrt{3}) \approx 0.951, \]

(137)

\[ \frac{E_1}{E_0} = \frac{3\sqrt{3} - 5}{4} \approx 0.049. \]

(138)

### 16 Continuous ejection and hovering over the horizon

Let us consider continuous process of fuel ejection. We assume that a body of mass \( m \) ejects portion of fuel with the small mass \( dm' \). Then, in the absence of external forces, the conservation law reads

\[ D(mu^\mu) + dm'w^\mu = 0, \]

(139)

where \( u^\mu \) corresponds to a rocket, \( w^\mu \) corresponds to fuel. For the flat space-time see discussion, e.g. in [14], pages 284–285. In (139) \( D \) denotes covariant differential, so

\[ a^\mu = \frac{Du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}, \]

(140)

where \( a^\mu \) is the four-acceleration, \( \Gamma^\mu_{\alpha\beta} \) are Christoffel symbols. Eq. (139) can be rewritten in the form

\[ dmu^\mu + ma^\mu d\tau + dm'w^\mu = 0, \]

(141)

\( \tau \) is the proper time.

Now, we are interested in the situation when a particle halts and hovers over the horizon. Then, \( u^r = 0 \). From \( a^\mu u_\mu = 0 \) it follows that \( a^0 = 0 \) as well.

As a result, we have from the \( t \)-component of (141) that

\[ dm u^0 + dm' w^0 = 0. \]

(142)

For particle at rest,

\[ u^0 = \frac{1}{\sqrt{A}}, \]

(143)

whence

\[ dm' = -\frac{dm}{w^0 \sqrt{A}}. \]

(144)

From the radial component of (141) we have

\[ ma^r d\tau + dm' w^r = 0. \]

(145)
Here,
\[ a^r = \Gamma^r_{00} (u^0)^2 = \frac{A'}{2}, \]  
\[ a^2 = g_{\mu\nu} a^\mu a^\nu = \frac{A'^2}{4A}. \]  

Combining these formulas, one obtains
\[ \frac{dm}{d\tau} = m \frac{a^r \sqrt{A}}{w^r} w^0. \]  

Let we use tetrads \( e_{(a)\mu} \), attached to a static observer, so in the coordinates \((t, r, \theta, \phi)\)
\[ e_{(0)\mu} = \left( -\sqrt{A}, 0, 0, 0 \right), \]  
\[ e_{(1)\mu} = \left( 0, \frac{1}{\sqrt{A}}, 0, 0 \right), \]  
\[ e_{(2)\mu} = r(0, 0, 1, 0), \]  
\[ e_{(2)\mu} = r \sin \theta(0, 0, 0, 1). \]

Then,
\[ w^{(1)} = \frac{1}{A} \frac{w^r}{w^0}, \] 
whence in our case
\[ \frac{w^r}{w^0} = -A |u| \] 
since eq. [38] \( v_2 = 0 \) and \( w^r < 0 \) in the deceleration regime. Using also the normalization condition, one finds easily
\[ w^0 = \sqrt{\frac{(w^0)^2 + A}{A}}, \]  
\[ w^r = -\frac{|u| \sqrt{A}}{\sqrt{1 - u^2}}, \]  
\[ w^0 = \frac{1}{\sqrt{A} \sqrt{1 - u^2}}. \]  

Then, we have from [148] for a given fixed \( r \)
\[ m = m_0 \exp(-B\tau), \]  
where
\[ B = \frac{A'}{2\sqrt{A}|u|}, \] 
and we assumed that at \( \tau = 0, m = m_0, m' = 0 \). This agrees with general formula (5) of Ref. [22].

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Then, taking into account (144), we infer
\[ m' = m_0 \sqrt{1 - u^2} [1 - \exp(-B\tau)], \]  
(160)

\[ E_{\text{loc}} = m_0 [1 - \exp(-B\tau)], \]  
(161)

where \( E_{\text{loc}} = m'/\sqrt{1 - u^2} \) is a local energy measured by a stationary observer and expended by a rocket because of fuel ejection. This quantity admits a safe limit to the case of a photon rocket, when \( u \to 1, m' \to 0 \) simultaneously, \( E_{\text{loc}} \) being finite.

Meanwhile, there are serious restrictions on the possibility of the process because of necessity to have a big initial mass. It is seen from eq. (158) that fuel supply necessary for hovering should be exponentially large as compared to the rocket mass. From (158), we have the hovering proper time
\[ \Delta \tau = \frac{1}{B} \log \frac{m_0}{m} \]  
(162)

and the hovering time for a distant observer
\[ \Delta t = \frac{1}{\sqrt{AB}} \log \frac{m_0}{m} \]  
(163)

where \( m_0 \) is initial mass the rocket with fuel and \( m \) is final mass.

In the Schwarzschild case we see that the rate with which mass decreases
\[ -\frac{dm}{mdt} = \sqrt{AB} = \frac{r_+ c^2}{2r^2 |u|} \]  
(164)

remains finite even near the horizon and changes smoothly when a point of hovering becomes closer and closer to the horizon. Meanwhile, for a local observer, the corresponding rate is
\[ -\frac{dm}{md\tau} = B = \frac{r_+ c^2}{2r^2 |u| \sqrt{1 - \frac{r_+}{r}}} \]  
(165)

In the horizon limit, because of the redshift factor \( \sqrt{A} \) in (159), this rate changes crucially depending on a point and diverges when the horizon is approached.

To illustrate the situation by concrete examples, we assume that a rocket with a photon engine hovers over the horizon of a black hole that has the Sun mass and the initial mass of a rocket is of the order \( 10^{12} \) kg. For substance with usual density the size of such a spacecraft has the order 1 km. Then, after the time of the order \( 10^{-3} \) s the remaining mass of a rocket with fuel cannot exceed the mass of the atom of hydrogen \( 1.67 \cdot 10^{-27} \) kg.

In another example, a photon super-rocket with the mass equal to that of Earth \( (5.97 \cdot 10^{24} \) kg) hovers over a black hole in the center of Milky Way. We assume that a black hole has the mass equal to \( 4.3 \cdot 10^6 \) masses of Sun. Then, after one and a half hours of hovering the mass of a rocket with remnants of fuel cannot exceed 100 kg.
17 Conclusion

We studied the Oberth effect in the relativistic case. In particular, we showed that this effect enables us to convert the total energy \(E = mc^2\) of jet fuel into the kinetic energy of a photon rocket, provided the process occurs near the black hole horizon of the Schwarzschild case. In this sense, the relativistic Oberth effect for nonrotating black holes is very close to the Penrose process, when decay of a particle inside the ergosphere of a rotating black hole produces debris whose energy measured at infinity exceeds the initial energy. It is possible when one of new particles remains on an orbit with a negative energy. As for a nonrotating neutral black hole the negative energy states are absent, the Penrose effect is absent as well. However, near the horizon there are states with the energy whose value is as small as one likes. As a result, after decay the energy of one of fragments can be close to the energy of an initial particle as close as one likes. For the evaluation of role of the Oberth effect we introduced the efficiency in the relativistic case and discussed its behavior in some typical cases. We directly proved that the ejection of fuel along the trajectory corresponds to the most efficient scenario for given initial energy, angular momentum and fixed masses of particles participating in decay. We also considered the process in which a rocket continuously ejects fuel in such a way that this enables it to hover over a horizon. Then, the consumption of fuel should change exponentially with respect to the proper time.

It is worth noting that sometimes an oversimplified interpretation of the Oberth effect is used that does not show the essence of matter properly. For example, in the paper “Oberth effect” from Wikipedia it is said that “the Oberth effect, wherein the use of a reaction engine at higher speeds generates a greater change in mechanical energy than its use at lower speeds”. This does not into account a simple circumstance: we must at first drive a rocket at high speed starting from low velocities. Then, on the first stage of the process an ineffective expenses of fuel are inevitable. The idea suggested by Oberth consisted in the gain of high velocities due to the action of the gravitational field. Then, the use of an engine in the periastron of a trajectory becomes effective [15]. In modern astronautics, the Oberth effect is used for gravitational maneuvers [16]. In corresponding situations, the relativistic effects are small or even negligible. Meanwhile, in our paper, we showed that the efficiency of the acceleration of the jet engine can reach 100% on the event horizon. Then, the whole energy (including the rest one) is spent to the growth of the rocket energy.

We would also like to stress the following difference between the nonrelativistic and relativistic versions of the Oberth effect. In the first case, we saw that, according to (53), the efficiency admits splitting to the contribution due to the energy stored in fuel and kinematic correction (the Oberth effect is described just the second term giving an additive correction to the first term). However, in the relativistic case (when velocities are high and the gravitational field is strong) such decomposition is not valid and the manifestation of the Oberth effect is essentially nonlinear.

We hope that consideration of the simplest case of the static spherically symmetric black hole will be useful for further generalization to rocket movement in the vicinity of rotating black holes.
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