THERE ARE NO MINIMAL EFFECTIVELY INSEPARABLE THEORIES

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Abstract. This paper belongs to the research on the limit of the first incompleteness theorem. Effectively inseparable theories (EI) can be viewed as an effective version of essentially undecidable theories (EU), and EI is stronger than EU. We examine the question: are there minimal effectively inseparable theories with respect to interpretability. We propose tEI, the theory version of EI. We first prove that there are no minimal tEI theories with respect to interpretability (i.e., for any tEI theory T, we can effectively find a theory which is tEI and strictly weaker than T with respect to interpretability). By a theorem due to Marian B. Pour-EI, we have tEI is equivalent with EI. Thus, there are no minimal EI theories with respect to interpretability. Also we prove that there are no minimal finitely axiomatizable EI theories with respect to interpretability.

1. Introduction

Since Gödel, research on incompleteness has greatly deepened our understanding of the incompleteness phenomenon. The motivation of this work is to explore the limit of the first incompleteness theorem. We can understand incompleteness in an abstract way via meta-mathematical properties of formal theories which exhibit behaviors that can be related to incompleteness/undecidability. For example, a general form of Gödel’s first incompleteness theorem says that any consistent recursively enumerable (RE) extension of Robinson Arithmetic Q (see Definition 2.7) is essentially undecidable (EU).

Meta-mathematical properties of RE theories we discuss in this paper include essentially incomplete, essentially undecidable, Creative, effectively inseparable (EI) and effectively extensible (EET) theories. For the definitions of these theories, see Definition 2.2. We give an overview of the relationships between these notions. From Lemma 2.5, essentially incomplete theories are equivalent with essentially undecidable theories. From Theorem 2.6, EET theories are equivalent with EI theories. From Lemma 2.8, an EI theory is Creative, but a Creative theory may not be EI; an EU theory may not be Creative, and a Creative theory may not be EU; the notion of EI is stronger than EU: an EI theory is EU, but an EU theory may not be EI. Typical examples of EI theories are Robinson Arithmetic Q and the theory R (see Definition 2.8).

Given a meta-mathematical property related to incompleteness/undecidability, a natural question is: are there minimal theories with this property? For essentially undecidable theories, [3] proves that there are no minimal essentially undecidable theories with respect to (w.r.t. for short) interpretability.

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Effective inseparability is an important meta-mathematical property of RE theories: (1) effective inseparability can be viewed as an effective version of essential undecidability; (2) recursion-theoretic proofs of metamathematical results tend to rely on an effectively inseparable pair of RE sets and its properties. In this work, we examine the question: are there minimal effectively inseparable theories? The answer of this question depends on how we define the notion of minimality. If we view a theory as minimal if it has a minimal number of axioms, then Robinson Arithmetic $Q$ is a minimal EI theory since $Q$ is finitely axiomatized. For a theory which is not finitely axiomatizable, if we view it as minimal if it has a minimal number of axiom schemes, then the Vaught set theory $VS$ (see Definition 2.11) is a minimal EI theory (see Fact 2.12) since $VS$ has only one axiom scheme. If we view an EI theory as minimal if, after deleting one of its axioms, the remaining theory is no longer EI, then $Q$ is a minimal EI theory since it is minimal effectively undecidable: if an axiom of $Q$ is deleted, then the remaining theory is not $EU$ (p.62). If we view a theory as minimal if it has a minimal Turing degree, then all EI theories are minimal since any EI theory has Turing degree $0'$ by Lemma 2.5. Thus, it makes no sense to ask for minimal EI theories w.r.t. Turing degree. In this paper, we examine the question whether there are minimal EI theories w.r.t. interpretability.

An effectively inseparable pair of RE sets is about sets of natural numbers. In this paper, we introduce tEI theories, the theory version of EI theories, and prove the main theorem that there are no minimal tEI theories w.r.t. interpretability: for any tEI theory $T$, we can effectively find a theory which is tEI and strictly weaker than $T$ w.r.t. interpretability. By a theorem due to Marian B. Pour-EI, we show that tEI is equivalent with EI. Thus, there are no minimal EI theories w.r.t. interpretability. Moreover, we show that there are no minimal finitely axiomatizable EI theories w.r.t. interpretability. We give two proofs of this result.

If there is a minimal EI theory, then such a theory is distinguished and could be viewed as a canonical theory of incompleteness. But we show that there are no minimal EI theories w.r.t. interpretability. The construction of weak EI theories in this paper uses purely logical methods. The research on concrete incompleteness seeks for concrete true arithmetic statements from classical mathematics which are not provable in $PA$. Both the research on meta-mathematics of arithmetic and the research on concrete incompleteness reveal that the incompleteness phenomenon is ubiquitous in both abstract formal theories and concrete mathematical theories.

This paper is structured as follows. In Section 2, we introduce basic notions and results we use in the paper. In Section 3, we prove that there are no minimal tEI theories w.r.t. interpretability: for any tEI theory $T$, we can effectively find a theory which is tEI and strictly weaker than $T$ w.r.t. interpretability. By a theorem due to Marian B. Pour-EI, we show that tEI is equivalent with EI. As a corollary, there are no minimal EI theories w.r.t. interpretability. In Section 4, we prove that there are no minimal finitely axiomatizable EI theories w.r.t. interpretability.

2. Preliminaries

In this paper, we work with first-order theories with finite signatures, and all theories are supposed to be RE. We equate a theory with the set of theorems provable in it. We always assume the arithmetization of the base theory. Given a sentence $\phi$, let $\neg \phi$ denote
the Gödel number of \( \phi \). Under arithmetization, we equate a set of sentences with the set of Gödel numbers of these sentences. Unless specifically stated, recursive functions always mean partial recursive functions in this paper.

**Definition 2.1** (Basic notions).

1. We denote the RE set with index \( i \) by \( W_i \) where \( W_i = \{ x : \exists y T_1(i, x, y) \} \) and \( T_1(z, x, y) \) is the Kleene predicate (see [3]).
2. We say that a pair \((A, B)\) of disjoint RE sets is **effectively inseparable** (EI) if there is a recursive function \( f(x, y) \) such that for any \( i \) and \( j \), if \( A \subseteq W_i \) and \( B \subseteq W_j \) with \( W_i \cap W_j = \emptyset \), then \( f(i, j) \) converges and \( f(i, j) \notin W_i \cup W_j \) (see [7, p.94]).
3. We say that \( A \subseteq \mathbb{N} \) is **productive** if there exists a recursive function \( f(x) \) (called a productive function for \( A \)) such that for every number \( i \), if \( W_i \subseteq A \), then \( f(i) \in A - W_i \) (see [7, p.84]).
4. We say that \( A \subseteq \mathbb{N} \) is **creative** if \( A \) is RE and the complement of \( A \) is productive (see [7, p.84]).
5. We denote the recursive Turing degree by \( 0 \), and the jump or completion of \( 0 \) by \( 0' \) (see [7, p.256]).

Now we introduce the notions of essentially incomplete, essentially undecidable, creative, effectively inseparable and effectively extensible theories. Essentially undecidable and essentially incomplete theories are introduced in [10, p.14]. Effectively inseparable theories are introduced in [9, p.119]. Effectively extensible theories, an effective version of EU theories, are introduced in [6, Definition 9].

**Definition 2.2.** Let \( T \) be a consistent RE theory.

1. We say \( T \) is **essentially incomplete** if any consistent RE extension of \( T \) over the same language is incomplete.
2. We say \( T \) is **essentially undecidable** (EU) if any consistent RE extension of \( T \) over the same language is undecidable.
3. Let \( T_P \) be the set of Gödel numbers of sentences provable in \( T \) (i.e., \( T_P = \{ \vdash \phi : T \vdash \phi \} \)), and \( T_R \) be the set of Gödel numbers of sentences refutable in \( T \) (i.e., \( T_R = \{ \vdash \neg \phi : T \vdash \neg \phi \} \)). The pair \((T_P, T_R)\) is called the **nuclei** of the theory \( T \).
4. We say \( T \) is **creative** if \( T_P \) is creative.
5. We say \( T \) is **effectively inseparable** (EI) if \((T_P, T_R)\) is an EI pair.
6. We say \( T \) is **effectively extensible** (EET) if there exists a recursive function \( f \) such that if \( i \) is the index of a consistent RE extension \( S \) of \( T \), then \( f(i) \) outputs a sentence which is independent of \( S \) (see [6, Definition 9]).

**Fact 2.3** ([7], p.183, p.94).

1. Let \( T \) be a consistent RE theory. The theory \( T \) is **creative** iff any RE set is reducible to \( T_P \): for any RE set \( X \), there exists a recursive function \( f \) such that \( n \in X \iff f(n) \in T_P \).

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1In [6, Definition 9], effectively extensible theories are defined based on a presentation of a theory which consists of a set of axioms and a set of reference rules of the theory. But as results in [6] show, the presentation involved is not essential for the proof of the main result in [6] that EET is equivalent with EI.
There are no minimal effectively inseparable theories

Theorem 2.4 ([8], pp.172-173). There exists an essentially undecidable theory which is not Creative.

Lemma 2.5 and Theorem 2.6 establish the relationships between essentially incomplete, EU, Creative, EI and EET theories.

Lemma 2.5. Let $T$ be a consistent RE theory.

1. If $T$ is EI, then $T$ is EU.
2. If $T$ is EI, then $T$ is Creative.
3. If $T$ is Creative, then $T$ has Turing degree 0’.
4. $T$ is EU iff $T$ is essentially incomplete.
5. “$T$ is Creative” does not imply “$T$ is EU”.
6. “$T$ is EU” does not imply “$T$ is Creative”.
7. “$T$ is Creative” does not imply “$T$ is EI”.
8. Any consistent RE extension of the theory $R$ is EI.

Proof. (1) Suppose $T$ is EI, but it is not EU. Let $S$ be a consistent RE extension of $T$ such that $S$ is decidable. Suppose $(T_P, T_R)$ is EI via the recursive function $f$, and $S_P = W_i$ and $S_R$, the complement of $S_P$, is $W_j$. Since $T_P \subseteq W_i$ and $T_R \subseteq W_j$, we have $f(i, j)$ converges and $f(i, j) \notin W_i \cup W_j = \mathbb{N}$, which is a contradiction.

2. Follows from Fact 2.3(1).
3. Follows from Fact 2.3(1).
4. Follows from Theorem 2 in [10] p.15).
5. See Theorem 4.12 in [1].
6. Follows from Theorem 2.4
7. Follows from (5) since EI implies EU.
8. Follows from the definitions and the fact that $R$ is EI.

Theorem 2.6 ([6], Theorem 1). Let $T$ be a consistent RE theory. Then $T$ is EET iff $T$ is EI.

Robinson Arithmetic $Q$ and the theory $R$ were introduced by Tarski, Mostowski and R. Robinson in [10] pp.51-53, which are important base theories in the study of incompleteness and undecidability.

Definition 2.7 (Robinson Arithmetic $Q$). Robinson Arithmetic $Q$ is defined in the language $\{0, S, +, \times\}$ with the following axioms:

- $Q_1$: $\forall x \forall y (Sx = Sy \to x = y)$;
- $Q_2$: $\forall x (Sx \neq 0)$;
- $Q_3$: $\forall x (x \neq 0 \to \exists y (x = Sy))$;
- $Q_4$: $\forall x \forall y (x + 0 = x)$;
- $Q_5$: $\forall x \forall y (x + Sy = S(x + y))$;
- $Q_6$: $\forall x (x \times 0 = 0)$;
- $Q_7$: $\forall x \forall y (x \times Sy = x \times y + x)$.

Definition 2.8. Let $R$ be the theory consisting of the following axiom schemes where $L(R) = \{0, S, +, \cdot, \leq\}$ and $x \leq y := \exists z (z + x = y)$.
Ax1: \( \overline{m} + \overline{n} = \overline{m + n} \);
Ax2: \( \overline{m} \cdot \overline{n} = \overline{m \cdot n} \);
Ax3: \( \overline{m} \neq \overline{n} \), if \( m \neq n \);
Ax4: \( \forall x (x \leq \overline{n} \rightarrow x = \overline{0} \lor \cdots \lor x = \overline{n}) \);
Ax5: \( \forall x (x \leq \overline{n} \lor \overline{n} \leq x) \).

Now we introduce the notion of interpretability.

**Definition 2.9** (Translations and interpretations, [13], p.10-13).
- We use \( L(T) \) to denote the language of the theory \( T \). Let \( T \) be a theory in a language \( L(T) \), and \( S \) a theory in a language \( L(S) \). In its simplest form, a translation \( I \) of language \( L(T) \) into language \( L(S) \) is specified by the following:
  - an \( L(S) \)-formula \( \delta_I(x) \) denoting the domain of \( I \);
  - for each relation symbol \( R \) of \( L(T) \), as well as the equality relation \( = \), an \( L(S) \)-formula \( R_I \) of the same arity;
  - for each function symbol \( F \) of \( L(T) \) of arity \( k \), an \( L(S) \)-formula \( F_I \) of arity \( k + 1 \).
- If \( \phi \) is an \( L(T) \)-formula, its \( I \)-translation \( \phi_I \) is an \( L(S) \)-formula constructed as follows: we rewrite the formula in an equivalent way so that function symbols only occur in atomic subformulas of the form \( F(x) = y \), where \( x, y \) are variables; then we replace each such atomic formula with \( F_I(x, y) \), we replace each atomic formula of the form \( R(x) \) with \( R_I(x) \), and we restrict all quantifiers and free variables to objects satisfying \( \delta_I \). We take care to rename bound variables to avoid variable capture during the process.
- A translation \( I \) of \( L(T) \) into \( L(S) \) is an interpretation of \( T \) in \( S \) if \( S \) proves the following:
  - for each function symbol \( F \) of \( L(T) \) of arity \( k \), the formula expressing that \( F_I \) is total on \( \delta_I \):
    \( \forall x_0, \cdots \forall x_{k-1} (\delta_I(x_0) \land \cdots \land \delta_I(x_{k-1}) \rightarrow \exists y (\delta_I(y) \land F_I(x_0, \cdots, x_{k-1}, y))) \);
    - the \( I \)-translations of all theorems of \( T \), and axioms of equality.

The simplified picture of translations and interpretations above actually describes only one-dimensional, parameter-free, and one-piece translations. In this paper, we use this simplified notion of interpretation.

**Definition 2.10** (Interpretations II).
- A theory \( T \) is interpretable in a theory \( S \) if there exists an interpretation of \( T \) in \( S \).
- Given theories \( S \) and \( T \), let \( S \sqsubseteq T \) denote that \( S \) is interpretable in \( T \) (or \( T \) interprets \( S \)); let \( S \ll T \) denote that \( T \) interprets \( S \) but \( S \) does not interpret \( T \).
- We say that a theory \( S \) is strictly weaker than a theory \( T \) w.r.t. interpretability if \( S \ll T \).
- We say \( S \) is a minimal RE theory w.r.t. interpretability if there is no RE theory \( T \) such that \( T \ll S \).

\(^2\)For precise definitions of a multi-dimensional interpretation, an interpretability with parameters, and a piece-wise interpretation, we refer to [13, pp.10-13] for more details.
The notion of interpretability provides us a method to compare different theories in different languages. If $T$ is interpretable in $S$, then all sentences provable (refutable) in $T$ are mapped, by the interpretation function, to sentences provable (refutable) in $S$.

The theory $VS$ is introduced by Robert A. Vaught in [11] (see also [12]).

**Definition 2.11** (The Vaught set theory $VS$). The theory $VS$ is axiomatized by the schema

$$(V_n) \quad \forall x_0, \cdots, \forall x_{n-1} \exists y \forall t (t \leftrightarrow \bigvee_{i<n} t = x_i)$$

for all $n \in \omega$, asserting that $\{x_i : i < n\}$ exists.

**Fact 2.12** ([12], p.383). The theory $VS$ interprets the theory $R$ and hence is $EI$.

**Definition 2.13.** Given two RE theories $A$ and $B$, we define the theory $A \oplus B$ as follows. The signature of $A \oplus B$ is a disjoint sum of the signatures of $A$ and $B$ plus a new 0-ary predicate symbol $P$. The theory $A \oplus B$ is axiomatised by all $P \rightarrow \phi$ where $\phi$ is an axiom of $A$, plus $\neg P \rightarrow \psi$ where $\psi$ is an axiom of $B$. We call $A \oplus B$ the interpretability infimum of $A$ and $B$.

**Remark 2.14.** In this paper, we use Janiczak’s theory $J$ introduced in [3, p.136], which is a theory in the language with one binary relation symbol $E$ with the following axioms.

- $J1$: $E$ is an equivalence relation.
- $J2$: There is at most one equivalence class of size precisely $n$.
- $J3$: There are at least $n$ equivalence classes with at least $n$ elements.

We define $A_n$ to be the sentence: there exists an equivalence class of size precisely $n + 1$.

Note that the $A_n$ are mutually independent over $J$.

**Theorem 2.15.**

- $J$ is decidable (see [3, Theorem 4]).
- Over $J$, every sentence is equivalent with a Boolean combination of the $A_n$’s (see [3, Lemma 2]).

**Definition 2.16** ([5], p.7). Given $X \subseteq \mathbb{N}$, we say that $W$ is a $J, X$-theory when $W$ is axiomatised over $J$ by boolean combinations of sentences $A_s$ for $s \in X$.

**Theorem 2.17** ([5], Theorem 4.5). If $U$ is a consistent essentially undecidable RE theory, then we can effectively find an infinite recursive set $X$ (from an index of $U$) such that no consistent $J, X$-theory interprets $U$.

3. **There are no minimal $tEI$ theories**

In this section, we propose $tEI$ theories, the theory version of $EI$ theories, and prove that there are no minimal $tEI$ theories w.r.t. interpretability: for any $tEI$ theory $T$, we can effectively find a theory which is $tEI$ and strictly weaker than $T$ w.r.t. interpretability.

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3Our presentation of the theory $J$ follows [5, p.6]. We include the axiom $J3$ to make the proof of the following fact in Theorem 2.15 more easy: over $J$, every sentence is equivalent with a boolean combination of the $A_n$. 

Our proof uses Theorem 4.5 in [5]. Finally, based on a theorem due to Marian B. Pour-EI, we prove that tEl theories are equivalent with El theories. As a corollary, there are no minimal El theories w.r.t. interpretability.

In the definition of El theories, $W_i$ and $W_j$ are arbitrary sets of natural numbers; if we view $W_i$ as a set of sentences, it may not even be consistent. It is natural to consider the theory version of El theories in which we respectively replace $W_i$ and $W_j$ by two sets of sentences $X$ and $Y$ which respectively have the similar properties of $T_P$ and $T_R$. This is our motivation to propose tEl theories: the theory version of El theories.

Note that for any RE theory $T$, $T_P$ and $T_R$ have the following properties:

(i) If $\alpha \in T_P$ and $\alpha \vdash \beta$, then $\beta \in T_P$.

(ii) If $\alpha, \beta \in T_P$, then $\alpha \land \beta \in T_P$.

(iii) If $\alpha \in T_R$ and $\beta \vdash \alpha$, then $\beta \in T_R$.

(iv) If $\alpha, \beta \in T_R$, then $\alpha \lor \beta \in T_R$.

**Definition 3.1.**

- We say a set of sentence $X$ is a filter if the following conditions hold:
  (1) If $\alpha \in X$ and $\alpha \vdash \beta$, then $\beta \in X$.
  (2) If $\alpha, \beta \in X$, then $\alpha \land \beta \in X$.
  (3) $\perp \notin X$.

- We say a set of sentence $Y$ is an ideal if the following conditions hold:
  (A) If $\alpha \in Y$ and $\beta \vdash \alpha$, then $\beta \in Y$.
  (B) If $\alpha, \beta \in Y$, then $\alpha \lor \beta \in Y$.
  (C) $\top \notin Y$.

- We say a set $Y$ of sentences is co-consistent if $\{\neg \phi : \phi \in Y\}$ is consistent.

Note that for any consistent RE theory $T$, $T_P$ is a filter and $T_R$ is an ideal. In the theory version of El theories we will define, we respectively replace the $W_i$ and $W_j$ in the definition of El theories with a filter set of sentences and an ideal set of sentences.

**Definition 3.2.** We say an RE theory $T$ is tEl if there exists a recursive function $f$ such that if $T_P \subseteq X, T_R \subseteq Y$ and $X \cap Y = \emptyset$, where $X$ is a filter set of sentence and is RE with index $i$, and $Y$ is an ideal set of sentences and is RE with index $j$, then $f(i, j)$ converges and outputs a sentence neither in $X$ nor in $Y$ (i.e., $f(i, j) \notin X \cup Y$).

Note that in Definition 3.2, $X$ is consistent and $Y$ is co-consistent. We first show that tEl theories are closed under interpretability infimum.

**Theorem 3.3.** If $U$ and $V$ are tEl theories, then $T = U \oplus V$ is tEl.

*Proof.* Suppose $U$ is tEl with the witnessing function $f_U$ and $V$ is tEl with the witnessing function $f_V$. We want to find a recursive function $g$ such that $T = U \oplus V$ is tEl with the witnessing function $g$.

Suppose $T_P \subseteq X, T_R \subseteq Y$ and $X \cap Y = \emptyset$, where $X$ is a filter set of sentences with index $i$ and $Y$ is an ideal set of sentences with index $j$. We describe how to compute $g(i, j)$ such that $g(i, j)$ converges and $g(i, j) \notin X \cup Y$.

Define $Z_0 = \{\phi : P \to \phi \in X\}, Z_1 = \{\phi : \neg P \to \phi \in X\}, Z_2 = \{\phi : P \land \phi \in Y\}$ and $Z_3 = \{\phi : \neg P \land \phi \in Y\}$. Note that $Z_0, Z_1, Z_2$ and $Z_3$ are all RE set of sentences. Suppose $Z_0$ has index $k_0$ and $Z_1$ has index $k_1$. Note that $k_0$ and $k_1$ can be computed
effectively from $i$. Suppose $Z_3$ has index $k_2$ and $Z_3$ has index $k_3$. Note that $k_2$ and $k_3$ can be computed effectively from $j$.

Note that $U_P \subseteq Z_0, V_P \subseteq Z_1, U_R \subseteq Z_2$ and $V_P \subseteq Z_3$. Since $X$ has the closure property (1)-(2) and $Y$ has the closure property (A)-(B), $Z_0$ and $Z_1$ also have the closure property (1)-(2), and $Z_2$ and $Z_3$ also have the closure property (A)-(B).

**Claim.** Either $Z_0 \cap Z_2 = \emptyset$ or $Z_1 \cap Z_3 = \emptyset$.

**Proof.** Suppose $Z_0 \cap Z_2 \neq \emptyset$ and $Z_1 \cap Z_3 \neq \emptyset$. Take $\phi$ and $\psi$ such that $P \rightarrow \phi \in X$, $P \wedge \phi \in Y$, $\neg P \rightarrow \psi \in X$ and $\neg P \wedge \psi \in Y$.

From $P \rightarrow \phi \in X$ and $\neg P \rightarrow \psi \in X$, by the property (1)-(2), we have $(P \wedge \phi) \vee (\neg P \wedge \psi) \in X$.

From $P \wedge \phi \in Y$ and $\neg P \wedge \psi \in Y$, by the property (B), we have $(P \wedge \phi) \vee (\neg P \wedge \psi) \in Y$.

Thus, $(P \wedge \phi) \vee (\neg P \wedge \psi) \in X \cap Y$ which contradicts that $X \cap Y = \emptyset$. \qed

Suppose $Z_0 \cap Z_2 = \emptyset$. Then $\bot \notin Z_0$ (if not, then any formula is in $Z_0$ by the property (1)), and $\top \notin Z_2$ (if not, then any formula is in $Z_2$ by the property (A)). Thus, $Z_0$ is a filter and $Z_2$ is an idea. Then $f_V(k_0, k_2)$ is defined and $f_U(k_0, k_2) \notin Z_0 \cup Z_2$. Suppose $Z_1 \cap Z_3 = \emptyset$. Then by a similar argument, $f_V(k_1, k_3)$ is defined and $f_V(k_1, k_3) \notin Z_1 \cup Z_3$.

We now check in stages simultaneously whether $f_U(k_0, k_2)$ converges, $f_V(k_1, k_3)$ converges, whether $Z_0 \cap Z_2 = \emptyset$ (i.e. whether we can find $\phi$ such that $P \rightarrow \phi \in X$ and $P \wedge \phi \in Y$) and whether $Z_1 \cap Z_3 = \emptyset$ (i.e., whether we can find $\psi$ such that $\neg P \rightarrow \psi \in X$ and $\neg P \wedge \psi \in Y$).

Suppose at stage $n$ the procedure is still running and we find at that stage:

(a) If both $f_U(k_0, k_2)$ and $f_V(k_1, k_3)$ converge, say to values $\theta$ and $\tau$, then we define $g(i, j)$ outputs $(P \wedge \theta) \vee (\neg P \wedge \tau)$.

(b) If (a) does not apply and $Z_0 \cap Z_2 \neq \emptyset$. Then $Z_0 \cap Z_2 = \emptyset$. We output $f_V(k_1, k_3)$ for $g(i, j)$.

(c) If (a) and (b) do not apply and $Z_1 \cap Z_3 \neq \emptyset$. Then $Z_0 \cap Z_2 = \emptyset$. We output $f_U(k_0, k_2)$ for $g(i, j)$.

If any of $(a, b, c)$ applied at stage $n$, we stop the procedure. If neither of $(a, b, c)$ happens at stage $n$, we proceed to stage $n+1$.

Since $X \cap Y = \emptyset$, either $Z_0 \cap Z_2 = \emptyset$ or $Z_1 \cap Z_3 = \emptyset$. If both are empty, (a) will obtain at some stage. If one is non-empty, one of the cases $(b, c)$ will obtain at some stage. So $g(i, j)$ will converge at some stage.

**Claim.** If $g(i, j)$ has its value via (b) or (c), then $g(i, j) \notin X \cup Y$.

**Proof.** Suppose $g(i, j)$ received its value via (b). Then $g(i, j) = f_V(k_1, k_3) = \tau$. We know $\neg P \rightarrow \tau \notin X$ and $\neg P \wedge \tau \notin Y$. If $\tau \in X$, then by property (1), $\neg P \rightarrow \tau \in X$ which leads to a contradiction. If $\tau \in Y$, then by property (A), $\neg P \wedge \tau \in Y$ which leads to a contradiction. So $g(i, j) \notin X \cup Y$.

By the similar argument, if $g(i, j)$ received its value via (c), then $g(i, j) \notin X \cup Y$. \qed

**Claim.** If $g(i, j)$ has its value via (a), then $g(i, j) \notin X \cup Y$.

**Proof.** If $g(i, j)$ has its value via (a), then either $Z_0 \cap Z_2 = \emptyset$ or $Z_1 \cap Z_3 = \emptyset$.

Suppose $Z_0 \cap Z_2 = \emptyset$. Then $\theta \notin Z_0 \cup Z_2$, i.e. $P \rightarrow \theta \notin X$ and $P \wedge \theta \notin Y$. We show $g(i, j) = (P \wedge \theta) \vee (\neg P \wedge \tau) \notin X \cup Y$. Suppose $(P \wedge \theta) \vee (\neg P \wedge \tau) \in X$. By the property
(1), \( P \rightarrow \theta \in X \) which leads to a contradiction. Suppose \((P \land \theta) \lor (\neg P \land \tau) \in Y \). By the property (A), \( P \land \theta \in Y \) which leads to a contradiction.

Suppose \( Z_1 \cap Z_2 = \emptyset \). Then \( \tau \notin Z_1 \cup Z_2 \), i.e. \( \neg P \rightarrow \tau \notin X \) and \( \neg P \land \tau \notin Y \). We show 
\[ g(i, j) = (P \land \theta) \lor (\neg P \land \tau) \notin X \cup Y. \]
Suppose \((P \land \theta) \lor (\neg P \land \tau) \in X \), by the property (1), \( \neg P \rightarrow \tau \in X \) which leads to a contradiction. Suppose \((P \land \theta) \lor (\neg P \land \tau) \in Y \). By the property (A), \( \neg P \land \tau \in Y \) which leads to a contradiction. \( \square \)

**Remark 3.4.** In the proof of Theorem 3.3, the argument that \( g(i, j) \) will converge at some stage is not effective. But the description of the function \( g \) is effective.

**Lemma 3.5.** If \( T \) is tEl, then \( T \) is EU.

**Proof.** Suppose \( T \) is tEl with the witnessing function \( f \). To show \( T \) is EU, by Lemma 2.5(4), it suffices to show that \( T \) is essentially incomplete. Let \( S \) be a consistent RE extension of \( T \), \( S_P \) has index \( i \) and \( S_R \) has index \( j \). Note that \( T_P \subseteq S_P \) and \( T_R \subseteq S_R \).

Clearly, \( S_P \) is a filter and \( S_R \) is an ideal. Then \( f(i, j) \) converges and \( f(i, j) \notin S_P \cup S_R \). That is, \( f(i, j) \) outputs a sentence independent of \( S \). Thus, \( S \) is incomplete. \( \square \)

**Theorem 3.6** (The \( s-m-n \) theorem, [7], p.23). For any \( m, n \geq 1 \), there exists a recursive function \( s_n^m \) of \( m + 1 \) variables such that for all \( x, y_1, \ldots, y_m, z_1, \ldots, z_n \), we have
\[ \phi(s_n^m(x, y_1, \ldots, y_m))(z_1, \ldots, z_n) = \phi_x^{m+n}(y_1, \ldots, y_m, z_1, \ldots, z_n). \]

**Lemma 3.7.** Suppose \((Y, Z)\) is an effectively inseparable pair of RE sets. Define the theory \( V = J + \{A_n : n \in Y\} + \{\neg A_n : n \in Z\} \), where \( A_n \) is the sentence defined in Remark 2.1. Then \( V \) is El.

**Proof.** We want to find a recursive function \( h(i, j) \) such that if \( W_P \subseteq W_i, V_R \subseteq W_j \) and \( W_i \cap W_j = \emptyset \), then \( h(i, j) \) converges and \( h(i, j) \notin W_i \cup W_j \).

Define the function \( f : n \rightarrow A_n \). Clearly, \( f \) is a total recursive function. By s-m-n theorem, there is a recursive function \( g \) such that \( f^{-1}[W_i] = W_{g(i)} \). Suppose \((Y, Z)\) is effectively inseparable via the recursive function \( t(i, j) \). Define \( h(i, j) = f(t(g(i), g(j))). \)

Clearly, \( h \) is recursive.

Suppose \( V_P \subseteq W_i, V_R \subseteq W_j \) and \( W_i \cap W_j = \emptyset \). Note that \( Y \subseteq f^{-1}[V_P] \subseteq f^{-1}[W_i] = W_{g(i)} \), and \( Z \subseteq f^{-1}[V_R] \subseteq f^{-1}[W_j] = W_{g(j)} \). Note that \( W_{g(i)} \cap W_{g(j)} = \emptyset \) since \( W_i \cap W_j = \emptyset \). Since \((Y, Z)\) is El via the function \( t \), we have \( t(g(i), g(j)) \) converges and \( t(g(i), g(j)) \notin W_{g(i)} \cup W_{g(j)} \). Then, \( h(i, j) \) converges and \( h(i, j) \notin W_i \cup W_j \). Thus, \( V \) is El. \( \square \)

**Theorem 3.8.** There are no minimal tEl theories w.r.t. interpretation: for any tEl theory \( U \), we can effectively find a theory which is tEl and strictly weaker than \( U \) w.r.t. interpretation.

**Proof.** Let \( U \) be an tEl theory. By Lemma 3.5, tEl theories are essentially undecidable. Theorem 2.17 applies to tEl theories. Thus, we can effectively find an infinite recursive set \( X \) (from an index of \( U \)) such that no consistent \( J, X \)-theory interprets \( U \). Let \((Y, Z)\) be an El pair of RE sets which are subsets of \( X \). Define the theory \( V \) as \( V = J + \{A_n : n \in Y\} + \{\neg A_n : n \in Z\} \). By Lemma 3.7, \( V \) is El and hence is tEl. Let \( T = U \oplus V \). Note that from a given tEl theory \( U \), we can effectively find such a theory \( T \). By Theorem
Theorem 3.11. For any consistent RE theory $T$, the following statements are equivalent:

1. $T$ is EI;
2. $T$ is tEI;
3. $T$ is EET.

Proof. Clearly, EI implies tEI. We show that tEI implies EET. Suppose $T$ is tEI with the witnessing recursive function $f$. There exists a recursive function $h$ such that $\{\phi : \neg \phi \in W_i\} = W_{h(i)}$. Define $g(i) = f(i, h(i))$. Clearly, $g$ is recursive. For any $i$, if $S$ is a consistent RE extension of $T$ with index $i$, then $h(i)$ is the index of $S_R$. Note that $T_P \subseteq S_P, T_R \subseteq S_R, S_P$ is a filter and $S_R$ is an ideal. Then $f(i, h(i))$ converges and $f(i, h(i)) \notin S_P \cup S_R$. Thus, $g(i)$ outputs a sentence independent of $S$. Hence, $T$ is EET with the witnessing function $g$. Finally, by Theorem 2.6 EET implies EI.  

As a corollary of Theorem 3.8 and Theorem 3.11, we have:

Theorem 3.12. There are no minimal EI theories w.r.t. interpretability: for any EI theory $T$, we can effectively find a theory which is EI and strictly weaker than $T$ w.r.t. interpretability.

4. THERE ARE NO MINIMAL FINITELY AXIOMATIZABLE EI THEORIES

In this section, we prove that there are no minimal finitely axiomatizable EI theories w.r.t. interpretability. We give two proofs of this result in Theorem 4.9 and Theorem 4.11. Both proofs use the properties of the theory $T_N$ (see Definition 4.1) and Theorem 4.6 whose proof uses Smullyan’s Theorem 4.5. Theorem 4.9 assumes piecewise interpretations and we can show that given a finitely axiomatizable EI theory $T$, we can effectively find a finitely axiomatizable EI theory, which is strictly weaker than $T$ w.r.t. interpretability. One main tool of Theorem 4.9 is Harvey Friedman’s Theorem 4.8. The proof of Theorem 4.11 does not assume piecewise interpretations, and uses an argument in the proof of Theorem 3.1 in [5].

We first introduce the theory $T_N$ of numbers.

Definition 4.1 (The theory $T_N$, [13], p.4). The theory $T_N$ consists of the following axioms:
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For more details about the theory \( \text{TN} \), see [13, p.4]. A \( \Delta^0_0 \)-formula is pure if all bounding terms are variables and all occurrences of terms are in sub-formulas of the form \( \text{S}x = y \), \( x + y = z \) and \( x \times y = z \). A \( \Sigma^0_1 \)-sentence is pure if it is equivalent with the form \( \exists x \phi(x) \) where \( \phi \) is a pure \( \Delta^0_0 \)-formula. We can transform any \( \Sigma^0_1 \)-sentence into a pure \( \Sigma^0_1 \)-sentence. In this section, we assume that all \( \Sigma^0_1 \)-sentences are rewritten in pure form.

Let \( \psi = \exists x \phi(x) \), where \( \phi \) is a pure \( \Delta^0_0 \)-formula. Define the finitely axiomatized theory \([\psi]\) as follows:

\[
[\psi] = \text{TN} + \exists x \exists y < x \phi(y).
\]

Definition 4.2 ([13], p.4). Suppose \( \varphi = \exists x A(x) \) and \( \psi = \exists x B(x) \) are two \( \Sigma^0_1 \)-sentences.

We Define:

1. \( \varphi \preceq \psi \triangleq \exists x (A(x) \land \forall y < x \neg B(y)) \);
2. \( \varphi < \psi \triangleq \exists x (A(x) \land \forall y \leq x \neg B(y)) \);
3. If \( \theta \) is \( \varphi \leq \psi \), then \( \theta^\perp = \psi < \varphi \);
4. If \( \theta \) is \( \varphi < \psi \), then \( \theta^\perp = \psi \leq \varphi \).

Fact 4.3 ([13], Theorem 1; [3], p.5). Suppose \( \varphi, \psi \) are \( \Sigma^0_1 \)-sentences.

1. If \( \psi \) does not hold, then \([\psi]\) \( \supset \) \( \text{R} \).
2. If \( \psi \) holds and we allow piecewise interpretations, then \([\psi]\) \( \preceq \) \( \text{TN} \).
3. If \( \varphi \preceq \psi \), then \([\psi]\) \( \vdash \varphi \).
4. Let \( A = \varphi \preceq \psi \). If \( \varphi \) (or \( \psi \)) holds, then either \( A \) holds or \( A^\perp \) holds.

Now we prove that \( \text{El} \) theories are closed under interpretability infimum. An important tool we use is Smullyan’s Theorem 4.5.

Definition 4.4. Let \( (A, B) \) and \( (C, D) \) be disjoint pairs of RE sets. We say \( (A, B) \) is \textit{semi-reducible} to \( (C, D) \) if there is a recursive function \( f(x) \) such that \( f(x) \in C \) if \( x \in A \), and \( f(x) \in D \) if \( x \in B \).

Theorem 4.5 ([9], pp.70-126). For any RE theory \( T \), \( T \) is \( \text{El} \) iff any disjoint pair \( (A, B) \) of RE sets is semi-reducible to \( (T_P, T_R) \).

Theorem 4.6. If \( U \) and \( V \) are \( \text{El} \) theories, then \( T = U \oplus V \) is \( \text{El} \).

\(^4\)For the notion of piecewise interpretability, we refer to [13, p.13].
Proof. Suppose $U$ and $V$ are $\mathcal{E}l$ theories, and $T = U \oplus V$. By Theorem 3.10, it suffices to show that any disjoint pair of RE sets is semi-reducible to $(T_P, T_R)$.

Let $(A, B)$ be any disjoint pair of RE sets. Since $U$ is $\mathcal{E}l$, there is a recursive function $f_1$ such that:

(i) if $n \in A$, then $f_1(n) \in U_P$;
(ii) if $n \in B$, then $f_1(n) \in U_R$.

Since $V$ is $\mathcal{E}l$, there is a recursive function $f_2$ such that:

(i) if $n \in A$, then $f_2(n) \in V_P$;
(ii) if $n \in B$, then $f_2(n) \in V_R$.

Note that $U_P \subseteq \{\phi : P \rightarrow \phi \in T_P\}$, $V_P \subseteq \{\phi : \neg P \rightarrow \phi \in T_P\}$, $U_R \subseteq \{\phi : P \land \phi \in T_R\}$, $V_R \subseteq \{\phi : \neg P \land \phi \in T_R\}$. Define $g(n) = (P \rightarrow f_1(n)) \lor (\neg P \rightarrow f_2(n))$. Since $f_1$ and $f_2$ are recursive, $g$ is also recursive.

Suppose $n \in A$. Then $P \rightarrow f_1(n) \in T_P$ and $\neg P \rightarrow f_2(n) \in T_P$. Thus, if $n \in A$, then $g(n) \in T_P$.

Suppose $n \in B$. Since $P \land f_1(n) \in T_R$, we have:

(1) \[ T \vdash P \rightarrow \neg f_1(n). \]

Since $\neg P \land f_2(n) \in T_R$, we have:

(2) \[ T \vdash \neg P \rightarrow \neg f_2(n). \]

From (1) and (2), we have: $T \vdash (P \land \neg f_1(n)) \lor (\neg P \land \neg f_2(n))$.

Thus, if $n \in B$, then $g(n) \in T_R$. Hence, $(A, B)$ is semi-reducible to $(T_P, T_R)$. \hfill \square

Remark 4.7. By the similar argument as in Theorem 3.10, Theorem 4.4 can also be used to give a second proof of Theorem 3.12.

Theorem 4.8 (Harvey Friedman). Assuming that we allow piecewise interpretations, for any finitely axiomatizable theory $A$, if $\top \prec A$, then there exists a finitely axiomatizable theory $B$ such that $\top \prec B \prec A$.

Proof. We employ the Gödel fixed point construction to find a sentence $\theta$ such that $PA \vdash \theta \leftrightarrow (A \leq A \oplus [\theta]) \leq (A \oplus [\theta] \leq T)$.

Claim. $A \not\leq A \oplus [\theta]$.

Proof. Suppose $A \leq A \oplus [\theta]$ holds. By Fact 3.3(4), either $\theta$ holds or $\theta^\perp$ holds.

Case one: Suppose $\theta$ holds. By Fact 3.3(2), $[\theta] \leq T$. Since $A \leq A \oplus [\theta] \leq [\theta]$, we have $A \leq T$, which is a contradiction.

Case two: Suppose $\theta$ does not hold. Then $\theta^\perp$ holds. By definitions, it is easy to check that $\theta^\perp \leq \theta$ holds. By Fact 3.3(3), $[\theta] \vdash \theta^\perp$. Since $[\theta] \vdash \theta^\perp \land \theta$ and $\theta^\perp \land \theta \vdash \perp$, we have $[\theta] \vdash \perp$. Since $(A \leq A \oplus [\theta]) \leq (A \oplus [\theta] \leq T)$ does not hold and $A \leq A \oplus [\theta]$ holds, we have $A \oplus [\theta] \not\leq T$ holds. Since $[\theta] \vdash \perp$, we have $A \oplus [\theta] = A$. Thus $A \not\leq T$, which is a contradiction. \hfill \square

By the similar argument, we can show that $A \oplus [\theta] \not\leq \top$. Thus, $\top \prec A \oplus [\theta] \prec A$. \hfill \square

This proof is simple than Friedman’s proof in [2], and the idea of this proof is from [13].
Theorem 4.9. Assuming that we allow piecewise interpretations, if $T$ is a finitely axiomatizable $EI$ theory, then we can effectively find a finitely axiomatized $EI$ theory which is strictly weaker than $T$ w.r.t. interpretability.

Proof. This follows from Theorem 4.8. Let $\theta$ be the sentence defined in Theorem 4.8. Note that since $T$ and $[\theta]$ are finitely axiomatizable, the theory $S = T \oplus [\theta]$ as in Theorem 4.8 is also finitely axiomatizable. By Theorem 4.8, $\top \vdash S \vdash T$. Recall that the fixed point construction of the sentence $\theta$ in the proof of Theorem 4.8 is effective. Thus, we can effectively find such a theory $S$. Suppose $\theta$ holds. Since we allow piecewise interpretations, by Fact 4.3(2), $[\theta] \vdash \top$. Thus, $S \vdash \top$, which contradicts that $\top \vdash S$. Hence, $\theta$ does not hold. By Fact 4.3(1), $[\theta] \supseteq R$. By Lemma 2.5(8), $[\theta]$ is $EI$. By Theorem 4.6, $S$ is $EI$. □

Now, we give another proof which does not assume piecewise interpretability, but its proof is non-constructive in some sense.

Lemma 4.10. If there is a minimal finitely axiomatizable $EI$ theory w.r.t. interpretability, then it is a minimum finitely axiomatizable $EI$ theory w.r.t. interpretability.

Proof. Suppose $T$ is a minimal finitely axiomatizable $EI$ theory w.r.t. interpretability. Suppose $S$ is any finitely axiomatizable $EI$ theory. Consider the theory $T \oplus S$. By Theorem 4.6, $T \oplus S$ is a finitely axiomatizable $EI$ theory. By the minimality of $T$, $T \leq T \oplus S$ and thus $T \leq S$. Hence, $T$ is the minimum finitely axiomatizable $EI$ theory w.r.t. interpretability. □

Theorem 4.11. There are no minimal finitely axiomatizable $EI$ theories w.r.t. interpretability.

Proof. By Theorem 4.6, if $S$ and $T$ are finitely axiomatizable $EI$ theories, then $S \oplus T$ is also a finitely axiomatizable $EI$ theory. By Lemma 4.10, it suffices to show that there is no minimum finitely axiomatizable $EI$ theory w.r.t. interpretability. The following argument comes from Theorem 3.1 in [5]. The key observation comes from Lemma 2.5(8). Suppose $T$ is the minimum finitely axiomatizable $EI$ theory w.r.t. interpretability. Consider any $\Sigma^0_2$-sentence $\sigma$. If $\sigma$ holds, then $[\sigma]$ has a finite model and thus $T \not\vdash [\sigma]$. If $\sigma$ does not hold and $[\sigma]$ is consistent, since $[\sigma] \supseteq R$ by Fact 4.3(1), $[\sigma]$ is $EI$ by Lemma 2.5(8), and thus $T \leq [\sigma]$ since $T$ is the minimum finitely axiomatizable $EI$ theory. If $[\sigma]$ does not hold and is inconsistent, then $T \vdash [\sigma]$. Thus, $\sigma$ does not hold if and only if $T \leq [\sigma]$, which contradicts the fact that the set of $\sigma$ such that $T \leq [\sigma]$ is an $RE$ set. □

Remark 4.12. The proof of Theorem 4.11 does not tell us, given a finitely axiomatizable $EI$ theory $T$, how to construct another finitely axiomatizable $EI$ theory which is strictly weaker than $T$ w.r.t. interpretability. It only tells us that if there is a minimal finitely axiomatizable $EI$ theory w.r.t. interpretability, this leads to a contradiction. In this sense, the proof of Theorem 4.11 is non-constructive.

We conclude the paper with some perspectives for future work. The work in this paper can be extended from two perspectives. We could examine the existence of minimal RE theories with meta-mathematical properties with respect to other notions of reducibility except for Turing degree and interpretability that we have considered in this paper. On the other hand, we could examine the existence of minimal RE theories w.r.t.
interpretability for other meta-mathematical properties that we have not examined. For example, a natural question is: are there minimal Creative theories w.r.t. interpretability. We did not explore it here.

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