The interior structure of slowly rotating black holes

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Abstract
The internal structure of a slowly rotating, steadily accreting, charged black hole that is undergoing mass inflation at its inner horizon is derived. The approach is to introduce a small rotational (axial dipole) perturbation to a spherically symmetric black hole. The equations governing the angular (dipole) behaviour decouple from the radial (monopole) behaviour, so all conclusions regarding inflation in the spherical charged black hole carry through unchanged for the slowly rotating black hole. Quantities inflate only in the radial direction, not in the angular direction. Exact self-similar solutions are obtained. For sufficiently small accretion rates, the instantaneous angular motion of the accretion flow has negligible effect on the angular spacetime structure of the black hole, even if the instantaneous angular momentum of the accretion flow is large and arbitrarily oriented.

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1. Introduction
The internal structure of a realistic (astronomical) rotating black hole has remained an unsolved problem for decades (e.g. section 20.5 of [1], section 5.3 of [2]). The structure of a rotating black hole in the process of collapsing is undoubtedly thoroughly complicated [3]. However, thanks to the no-hair theorem [4–7], it is expected that a newly collapsed rotating black hole will quickly relax to the Kerr geometry. Aside from the initial collapse event, and rare occasions of high accretion such as a black hole merger, a real astronomical black hole spends the great majority of its life accurately described by the Kerr geometry down to near its inner horizon.

The reason that the Kerr geometry fails inside a black hole is that at (or rather just above) the inner horizon, the Kerr geometry is subject to the relativistic counter-streaming instability discovered by [8, 9] and called by them mass inflation. The inflationary instability is the nonlinear realization of the infinite blueshift at the inner horizon first pointed out by [10].
Mass inflation inside spherical charged black holes has been investigated extensively (see e.g. the lists of references in [11, 12]), but there have been relatively few studies of inflation inside rotating black holes [13–20] (see also the review [21]).

A primary motivation for the present paper is the question ‘What happens if I fall into a real (astronomical) rotating black hole?’, as opposed to the question that has occupied most of the literature ‘What happens as a result of gravitational collapse?’. The former problem is in ways simpler than the latter. The simplicity has the virtue of allowing exact (self-similar) solutions to be obtained, albeit in the limit of slow rotation. Towards the end of this paper, section 7, the results of the present work are compared to prior work, which is principally that of Ori [13, 16], but the potential for comparison is limited because the situations considered are different.

The present paper addresses, and solves, the problem of the internal structure of a slowly rotating, steadily accreting, charged black hole that is undergoing inflation at its inner horizon (or rather, at what would be the inner horizon if inflation were not taking place). Inflation requires the simultaneous presence of ingoing and outgoing streams near the inner horizon; but even the tiniest incident streams suffice to induce inflation. The paper follows earlier papers in this series [12, 22, 23] in assuming that ingoing and outgoing streams are being generated continuously and steadily by accretion. This assumption differs from most of the literature on inflation, including that of Ori [13, 16], which has focused on the situation where a black hole collapses and then remains isolated [24], in which case ingoing and outgoing streams are generated by a decaying Price tail [25, 26] of gravitational radiation. Real astronomical black holes are however never isolated, and it seems likely that any decaying Price tail produced by the initial collapse event will soon be overwhelmed by streams generated by accretion. This is especially true for a supermassive black hole, whose mass acquired by accretion greatly exceeds the mass of the original black hole formed by stellar collapse. Matter that falls from outside the horizon is necessarily initially ingoing, but it may become outgoing inside the horizon if it has enough charge (if the black hole is charged) or enough angular momentum (if the black hole is rotating).

The approach followed in the present paper is to introduce a small rotational (axial dipole) perturbation to a spherically symmetric, accreting black hole that is undergoing inflation. It is necessary to assume that the black hole is charged so that the unperturbed geometry has an inner horizon where inflation is taking place. In this sense the solution is not realistic, since a real black hole probably has tiny charge. However, one may hope that the solution for the internal structure of a slowly rotating, charged black hole may shed light on the real problem of a fully rotating, uncharged black hole. It should be emphasized that the problem considered in this paper is quite restricted: it considers only a single kind of perturbation, an axial dipole perturbation, not a perturbation of arbitrary type, to a spherical black hole.

As shown by [23], the radial length scale over which the interior mass and other gauge-invariant quantities (such as the Weyl curvature, and the centre-of-mass density and pressure) exponentiate during inflation is proportional to the accretion rate of the incident ingoing and outgoing streams. Thus, counter-intuitively, the smaller the accretion rate, the more rapidly inflation exponentiates as a function of radius. In the limit of a tiny accretion rate, the structure resembles a step function at the inner horizon, above which the Kerr–Newman geometry applies to an excellent approximation, and at which the interior mass and other quantities inflate to absurdly huge values while the radius scarcely changes.

Astronomically realistic black holes have small accretion rates over the great majority of their lifetimes, the accretion timescale being much longer than the black hole crossing time. For small accretion rates it might seem justifiable to assume the stationary approximation $\partial / \partial t = 0$ (which Burko [27–29] terms the homogeneous approximation, since the Killing
vector associated with time translation symmetry is spacelike inside the horizon). It is true that the spacetime is almost stationary, temporal partial derivatives being much smaller than radial partial derivatives, \( \partial/\partial t \ll \partial/\partial r \), but it is strictly stationary only in the unrealistic case of symmetrically equal ingoing and outgoing streams \[23\]. Stationarity breaks down in effect because quantities such as the interior mass \( M \) inflate so rapidly that \( \partial M/\partial t \) is never small in the inflationary zone even in the limit of infinitesimal accretion rates.

Although the slowly rotating black hole spacetimes considered in this paper are not time-translation invariant, the equations do admit solutions with conformal time-translation symmetry, or self-similarity. Self-similar solutions are considered herein, section 4.

2. Slowly rotating equations

2.1. Slowly rotating perturbation

This paper follows \[12, 22, 23\] in adopting the following line element for a general spherically symmetric background spacetime, in polar coordinates \( x^\mu \equiv \{ t, r, \theta, \phi \} \):

\[
\begin{align*}
\mathrm{d}s^2 &= -\frac{\mathrm{d}t^2}{\alpha^2} + \frac{1}{\beta_t^2} \left( \mathrm{d}r - \beta_t \frac{\mathrm{d}t}{\alpha} \right)^2 + r^2 (\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2). \\
&= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{mn} e^m_\mu e^n_\nu \, \mathrm{d}x^\mu \mathrm{d}x^\nu,
\end{align*}
\]

Through the identity

\[
\begin{align*}
\mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{mn} e^m_\mu e^n_\nu \, \mathrm{d}x^\mu \mathrm{d}x^\nu,
\end{align*}
\]

the line element (1) encodes not only a metric \( g_{\mu\nu} \), but a complete inverse vierbein \( e^m_\mu \), corresponding vierbein \( e^m_\mu \), and orthonormal tetrad \( \{ \gamma^t, \gamma^r, \gamma^\theta, \gamma^\phi \} \), satisfying \( \gamma^m \cdot \gamma^m = \eta_{mn} \) with \( \eta_{mn} \) being the Minkowski metric. The coefficients \( \beta_m \) in the line element (1) constitute the components of a covariant tetrad-frame 4-vector, the radial 4-gradient

\[
\beta_m \equiv \partial_m r,
\]

where \( \partial_m \equiv \gamma^m \cdot \partial \equiv e^m_\mu \partial/\partial x^\mu \) denotes the directed derivative along the \( \gamma^m \) tetrad axis (not to be confused with the coordinate derivative \( \partial/\partial x^\mu \)). Physically, the directed derivatives \( \partial_t \) and \( \partial_r \) are the proper time and radial derivatives measured by a person at rest in the tetrad frame. The scalar product of \( \beta_m \) with itself defines the interior, or Misner–Sharp \[30\], mass \( M \), a gauge-invariant scalar

\[
\frac{2M}{r} - 1 \equiv -\beta_m \beta^m = \beta_t^2 - \beta_r^2.
\]

Details of the vierbein, tetrad-frame connections, and tetrad-frame Riemann, Einstein, and Weyl tensors that follow from the unperturbed line element (1) are summarized in appendix A of \[23\].

Perturbations to the spherically symmetric spacetime can be characterized by the dimensionless covariant tetrad-frame vierbein perturbation \( \varphi_{mn} \) defined by

\[
e^m_\mu = \left( \delta^m_n - \varphi_{mn} \right) e^\mu_n,
\]

where the overscript \(^a\) signifies an unperturbed quantity. Because the perturbation \( \varphi_{mn} \) is of linear order, it is legitimate to raise and lower its indices with the unperturbed tetrad-frame metric (the Minkowski metric), and to transform between coordinate and tetrad frames with the unperturbed vierbein \( e^m_\mu \) and its inverse \( e^n_\mu \). Thus, the vierbein perturbation \( \varphi_{mn} \) can be regarded as a tensor field defined on the unperturbed spherically symmetric background.

Spherically symmetric spacetimes are invariant under the three-dimensional group \( O(3) \) of orthogonal transformations. It is natural to expand perturbations in eigenmodes of that
A perturbation eigenmode is characterized not only by its harmonic numbers $lm$, but also by its spin $s$, which expresses how the perturbation transforms under rotations about the radial direction. Perturbations $\psi_{mn}$ admit scalar (spin-0), vector (spin-$\pm 1$) and tensor (spin-$\pm 2$) modes. The spin of a perturbation $\psi_{mn}$ is most easily identified not in an orthonormal tetrad, as here, but rather in a spinor tetrad, where the angular tetrad axes $\gamma_\theta$ and $\gamma_\phi$ are replaced by spinor axes

$$
\gamma_+ \equiv \frac{1}{\sqrt{2}} (\gamma_\theta + i \gamma_\phi), \quad \gamma_- \equiv \frac{1}{\sqrt{2}} (\gamma_\theta - i \gamma_\phi).
$$

The general rule is that the spin of a tensor equals the sum of the +’s and −’s of its covariant indices in the spinor tetrad.

A slowly rotating spacetime is obtained from a spherically symmetric spacetime by a perturbation that is (a) cylindrically symmetric, (b) axial, meaning that the perturbation changes sign under a flip $\gamma_\phi \rightarrow -\gamma_\phi$ of the azimuthal tetrad axis, and (c) dipole, the lowest order non-spherical harmonic. The slowly rotating conditions imply that the vierbein perturbation $\psi_{mn}$ is a spin-1 dipole harmonic,

$$
\psi_{mn} \propto Y_{10}(\theta, \phi) = -\frac{\sin \theta}{\sqrt{8\pi}}.
$$

Of the 16 components of the $4 \times 4$ tensor of perturbations $\psi_{mn}$, there are four that satisfy the conditions of being spin-1 and axial, namely

$$
\psi_{t\phi}, \quad \psi_{r\phi}, \quad \psi_{\phi t}, \quad \psi_{\phi r}.
$$

The spin-1 axial character of these perturbations is manifest in a spinor tetrad, where the possible spin-1 perturbations are the four complex quantities $\psi_{t\phi}, \psi_{r\phi}, \psi_{\phi t}$ and $\psi_{\phi r}$, and the axial condition implies that each of these four perturbations is purely imaginary.

Of the four possible degrees of freedom represented by the perturbations (8), only one degree of freedom is coordinate and tetrad gauge invariant, and therefore represents a real physical perturbation. The remaining three degrees of freedom represent gauge freedoms, comprising an infinitesimal coordinate transformation of the azimuthal coordinate $\phi$ and infinitesimal tetrad (Lorentz) transformations in the $t$–$\phi$ and $r$–$\phi$ planes.

Under an infinitesimal coordinate gauge transformation of the azimuthal coordinate $\phi$,

$$
\phi \rightarrow \phi + \epsilon,
$$

the vierbein perturbations transform as

$$
\psi_{t\phi} \rightarrow \psi_{t\phi} - r \sin \theta \left( \alpha \frac{\partial \epsilon}{\partial t} + \beta_t \frac{\partial \epsilon}{\partial r} \right),
$$

$$
\psi_{r\phi} \rightarrow \psi_{r\phi} - r \sin \theta \beta_r \frac{\partial \epsilon}{\partial r},
$$

with $\psi_{t\phi}$ and $\psi_{r\phi}$ being coordinate gauge invariant. If the spacetime were either time translation invariant or conformally time translation invariant, so that $\partial \epsilon / \partial t = 0$, then the combination

$$
\beta_\gamma \psi_{t\phi} - \beta_t \psi_{r\phi}
$$

would be coordinate gauge invariant. In general, the spacetime is not time translation invariant, conformal or otherwise, but as commented in section 1 it is legitimate to think of the spacetime as being almost stationary, and therefore the combination (11) is almost coordinate gauge invariant.

The antisymmetric combinations $\psi_{t\phi} - \psi_{r\phi}$ and $\psi_{r\phi} - \psi_{t\phi}$ are not tetrad gauge invariant, the former transforming under an infinitesimal Lorentz boost in the $t$–$\phi$ plane, the latter...
transforming under an infinitesimal spatial rotation in the \( r - \phi \) plane. The symmetric combinations \( \psi_{t \phi} + \psi_{\phi t} \) and \( \psi_{r \phi} + \psi_{\phi r} \) are tetrad gauge invariant. If the spacetime were time translation invariant or conformally time translation invariant then the combination
\[
\beta_t (\psi_{t \phi} + \psi_{\phi t}) - \beta_r (\psi_{r \phi} + \psi_{\phi r})
\]
would be gauge invariant with respect to both infinitesimal coordinate and tetrad gauge transformations. Again, in general the spacetime is not time translation invariant, but it is legitimate to think of the combination (12) as being almost coordinate and tetrad gauge invariant. In terms of the almost-gauge-invariant perturbation \( \omega \) defined below, the combination (12) equals \(-r \alpha \beta_r \omega \sin \theta\). In the general case, there is no linear combination of vierbein perturbations \( \psi_{mn} \) that is fully gauge invariant, but the axial spin-0 component of the Weyl tensor provides a differential combination \( W \) of vierbein perturbations, equation (22), that is fully gauge invariant.

In view of the behaviour under gauge transformations just discussed, it is convenient to introduce four quantities \( \omega, \psi, \zeta^t \) and \( \zeta^r \) representing the different types of perturbation as follows:

\[
\begin{align*}
\omega & \text{ almost gauge invariant,} \\
\psi & \text{ coordinate transformation of } \phi, \\
\zeta^t & \text{ Lorentz boost in the } t - \phi \text{ plane,} \\
\zeta^r & \text{ spatial rotation in the } r - \phi \text{ plane,}
\end{align*}
\]

in terms of which the vierbein perturbation \( \psi_{mn} \) is
\[
\psi_{mn} = \sin \theta \begin{pmatrix}
0 & 0 & 0 & -r \alpha \omega - \beta_t \psi + \zeta^t \\
0 & 0 & 0 & -\beta_r \psi - \zeta^r \\
0 & 0 & 0 & 0 \\
-\zeta^t & \zeta^r & 0 & 0
\end{pmatrix}
\]
(14)
The perturbations \( \zeta^m \) are tetrad-frame components of the Killing vector associated with cylindrical symmetry, equation (18). The resulting perturbed vierbein is
\[
e_m^\mu = \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & \beta_t & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\zeta^t \alpha \sin \theta & 0 & 0 & \frac{1}{r \sin \theta}
\end{pmatrix}
\]
(15)
and the perturbed inverse vierbein is
\[
e_m^\mu = \begin{pmatrix}
\frac{1}{\alpha} & 0 & 0 & 0 \\
0 & \frac{1}{\beta_t} & 0 & 0 \\
0 & 0 & \frac{1}{\beta_r} & 0 \\
-\frac{1}{\Omega} \sin \theta & 0 & 0 & r \sin \theta
\end{pmatrix}
\]
(16)
where
\[
\Omega = \omega - \frac{\zeta^t \beta_r + \zeta^r \beta_t}{r \alpha \beta_r},
\]
(17a)
The quantity $\Omega_1$, equation (17a), can be interpreted as an angular velocity of the tetrad frame. The angular velocity $\Omega_1$ is not gauge invariant: it depends on the choice of the tetrad frame, that is, on the tetrad gauge perturbations $\xi^m$. The quantity $\omega$ can be interpreted as an almost gauge-invariant measure of the angular velocity.

The final column of the inverse vierbein (16) constitutes the spacelike contravariant tetrad-frame Killing vector $r \sin^2 \theta \xi_m$ associated with cylindrical symmetry $\partial / \partial \phi = e_m \phi \partial_m \equiv r \sin^2 \theta \xi_m \partial_m$.

The second column of the vierbein (15) constitutes the covariant tetrad-frame 4-vector $\beta^m$, the radial 4-gradient defined by equation (3). The azimuthal component $\beta_\phi$ of the 4-vector follows from $\partial r / \partial \phi = 0$, or equivalently $\xi^m \beta_m = 0$, which implies

$$\beta_\phi = -(\xi^t \beta_t + \xi^r \beta_r) \sin \theta.$$  (19)

The inverse vierbein (16) is equivalent to the line element

$$ds^2 = r^2 \left[\left(\frac{dt}{\alpha} + \xi^t \sin^2 \theta \, d\phi\right)^2 + \left(\frac{d\ln r}{\beta_r} - \frac{\beta_t}{\beta_r} \frac{dt}{\alpha} + \xi^r \sin^2 \theta \, d\phi\right)^2\right] + d\theta^2 + \sin^2 \theta \left(d\phi - \Psi \, d\ln r - \Omega \, dt\right)^2.$$  (20)

### 2.2. Einstein and Weyl tensors

From the line element (20), it is possible to derive the tetrad-frame connections, and tetrad-frame Riemann, Ricci, Einstein and Weyl tensors in the usual way. In the present case of a spherical spacetime subjected to an axial vector perturbation, all quantities can be classified as either scalar (spin-0) or vector (spin-1).

The only spin-0 component of the Riemann tensor that differs from its value in the unperturbed spherical background is the axial part of the spin-0 component of the Weyl tensor $C_{klmn}$. The component is proportional to the spin-0 dipole harmonic $0 Y_{10} = \cos \theta / \sqrt{4\pi}$ and is conveniently written as

$$W(t, r) \equiv \frac{1}{2} C_{t\phi r \theta} = -C_{t\phi \theta r} = C_{\phi t r \theta},$$  (21)

where $W(t, r)$ is a function only of $t$ and $r$, not of angular variables. In terms of the vierbein perturbations, $W$ is explicitly

$$W = r \alpha \beta_r \left(\frac{\partial \omega}{\partial \ln r} - \frac{\partial \Psi}{\partial t}\right).$$  (22)

The axial spin-0 component of the Weyl tensor, equation (21), hence $W$, is fully gauge invariant with respect to all relevant gauge transformations, which is to say not only infinitesimal coordinate and tetrad transformations, but also arbitrary radial Lorentz boosts.

There are four non-vanishing axial spin-1 components of the tetrad-frame Riemann tensor $R_{klmn}$, yielding the following four non-vanishing spin-1 components of the tetrad-frame Einstein tensor $G_{mn}$ and Weyl tensor $C_{klmn}$:

$$G_{\phi t} = -(R_{t\phi \theta r} + R_{r \phi t}),$$  (23a)

$$G_{\phi r} = -(R_{t \phi \theta r} + R_{r \phi t}).$$  (23b)
\[ C_{\theta\theta\phi} = -C_{\phi\phi\phi} = \frac{1}{2}(R_{\theta\theta\phi} - R_{\phi\phi\phi}), \quad (23c) \]
\[ C_{r\phi\phi} = -C_{\phi\phi r} = \frac{1}{2}(R_{r\phi\phi} - R_{\phi\phi r}). \quad (23d) \]

Being tetrad-frame quantities, all these components are automatically coordinate gauge invariant. However, none of them are tetrad gauge invariant: all spin-1 components of the Riemann tensor transform under infinitesimal spin-1 tetrad (Lorentz) transformations.

The non-vanishing spin-1 components of the tetrad-frame Einstein tensor \( G_{mn} \) are
\[ G_{t\phi} = \sin \theta \left[ \frac{1}{r^3} \partial_t (r^2 W) - \zeta (G_{tt} + G_{\phi\phi}) - \zeta (G_{tt} - G_{\phi\phi}) \right], \quad (24a) \]
\[ G_{r\phi} = \sin \theta \left[ \frac{1}{r^3} \partial_t (r^2 W) - \zeta (G_{rr} - G_{\phi\phi}) - \zeta (G_{tt} - G_{\phi\phi}) \right]. \quad (24b) \]

Note that the derivatives \( \partial_m \) in equations (24) are directed (tetrad-frame) derivatives, not partial (coordinate-frame) derivatives.

The non-vanishing spin-1 components of the tetrad-frame Weyl tensor \( C_{klmn} \) are
\[ C_{\theta\theta\phi} = \sin \theta \left( \frac{1}{2r} \partial_r W - 3\zeta C \right), \quad (25a) \]
\[ C_{r\phi\phi} = \sin \theta \left( \frac{1}{2r} \partial_t W + 3\zeta C \right), \quad (25b) \]

where \( C \) is the Weyl scalar, the polar part of the spin-0 component of the Weyl tensor
\[ C \equiv \frac{1}{2} C_{\theta\theta\phi\phi} = -C_{t\theta\phi} = -C_{t\phi\theta} = -C_{r\phi\phi} = -C_{\phi\phi r} = -\frac{1}{2} C_{\phi\phi\phi\phi} = \frac{1}{6}(G_{tt} - G_{rr} + G_{\phi\phi}) - \frac{M}{r^3}. \quad (26) \]

The Weyl scalar \( C \) is invariant with respect to arbitrary radial Lorentz boosts.

2.3. Electromagnetic field

The black hole is necessarily charged, in order that the unperturbed black hole has an inner horizon where inflation can take place.

The unperturbed spherical background admits just one component of the electromagnetic field \( F_{mn} \), the polar spin-0 component and the radial electric field \( E_r \equiv F_{tr} \). This is a monopole field and it can be written as
\[ E_r \equiv \frac{Q}{r^2}, \quad (27) \]
which defines the charge \( Q(t, r) \) interior to radius \( r \).

The rotational perturbation introduces scalar (spin-0) and vector (spin-1) axial dipole components to the electromagnetic field. The axial spin-0 part is the radial magnetic field \( B_r \equiv F_{\phi r} \). The component is proportional to the spin-0 dipole harmonic \( Y_1^{0} = \cos \theta / \sqrt{4\pi} \) and can be written as
\[ B_r \equiv \frac{B \cos \theta}{r^2}, \quad (28) \]
which defines \( B(t, r) \). Both \( Q \) and \( B \), equations (27) and (28), are fully gauge invariant with respect to infinitesimal coordinate and tetrad gauge transformations, and to arbitrary radial Lorentz boosts.
The non-vanishing axial spin-1 components of the electromagnetic field are $E_\phi \equiv F_{r\phi}$ and $B_\theta \equiv F_{r\theta}$. The source-free Maxwell’s equations $\varepsilon^{klmn}D_l F_{mn} = 0$ imply that the axial spin-1 components of the electromagnetic field must be related to $Q$ and $B$ by

\[ E_\phi = -\sin \theta \left( \frac{1}{2r} \partial_r B + \zeta^r \frac{Q}{r^2} \right), \tag{29a} \]
\[ B_\theta = -\sin \theta \left( \frac{1}{2r} \partial_r B - \zeta^r \frac{Q}{r^2} \right). \tag{29b} \]

The source Maxwell’s equations $D^m F_{mn} = 4\pi j_n$ relating the electromagnetic field $F_{mn}$ to the electric current $j_n$ are as follows:

\[ \partial_r Q = -4\pi r^2 j_t, \tag{30a} \]
\[ \partial_t Q = -4\pi r^2 j_r, \tag{30b} \]
\[ \frac{1}{2} \left( -D^r \partial_t - D^t \partial_r + \frac{2}{r^2} \right) B + \frac{2QW}{r^2} = 4\pi r \zeta^m j_m, \tag{30c} \]

where $D_m$ signifies the tetrad-frame covariant derivative.

The electromagnetic energy–momentum tensor is given by $4\pi T_{mn}^e = F_{mk} F^k_n - \frac{1}{2} \eta_{mn} F_{kl} F^{kl}$. In terms of the electric and magnetic fields (27) and (29), the axial spin-1 components of the electromagnetic energy–momentum tensor are

\[ 8\pi T_{r\phi}^e = -E_r B_\theta, \quad 4\pi T_{t\phi}^e = -E_t E_\phi. \tag{31} \]

Explicitly, from equations (29) inserted into equations (31),

\[ 8\pi T_{r\phi}^e = \sin \theta \left( \frac{Q}{r^2} \partial_r B + \zeta^r \frac{2Q^2}{r^2} \right), \tag{32a} \]
\[ 8\pi T_{t\phi}^e = \sin \theta \left( \frac{Q}{r^2} \partial_t B - \zeta^r \frac{2Q^2}{r^2} \right). \tag{32b} \]

The covariant divergence of the electromagnetic energy–momentum is $D_m T_{mn}^e = F^{mn} j_m$, and the component of this along the azimuthal Killing vector $\zeta^n$ is

\[ \zeta^n D_m T_{mn}^e = \frac{1}{2r} j^m \partial_m B. \tag{33} \]

### 2.4. Perfect fluid streams

The inflation instability requires the simultaneous presence of ingoing and outgoing streams near the inner horizon [9]. I follow [12, 23] in considering two separate streams, charged ‘baryons’ with a relativistic equation of state (a value slightly less than $1/3$ allows for the expected slow increase with temperature of the effective number of relativistic species),

\[ w \equiv p_b/\rho_b = 0.32 \quad \text{(baryons)}, \tag{34} \]

and neutral ‘dark matter’ with a pressureless equation of state,

\[ p_d/\rho_d = 0 \quad \text{(dark matter)}. \tag{35} \]

Repelled by the charge of the black hole generated self-consistently by their accretion, the baryons become outgoing inside the horizon. The dark matter falls freely into the black
hole and remains ingoing. Inflation is produced by relativistic counter-streaming between the outgoing baryons and the ingoing dark matter.

Each of the two streams constitutes a perfect fluid. The equations in this subsection are for a general perfect fluid.

The tetrad-frame energy–momentum $T^{mn}$ of a perfect fluid of proper density and pressure $\rho$ and $p$ moving at tetrad-frame 4-velocity $u^m$ is

$$T^{mn} = (\rho + p)u^m u^n + p\eta^{mn}. \tag{36}$$

Since the spacetime is cylindrically symmetric, any geodesic is characterized by a conserved azimuthal angular momentum, a gauge-invariant scalar. The azimuthal angular momentum (per unit mass) of a stream is given by the covariant azimuthal component $\nu_\phi$ of the coordinate-frame 4-velocity $\nu^\mu$ (the coordinate-frame 4-velocity $\nu^\mu$ is written with a Greek upsilon to avoid any possible confusion with the tetrad-frame 4-velocity $u^m$, written with a Latin $u$):

$$\nu_\phi = e^m_\phi \eta^a_{am} u^a = r \sin^2 \theta \zeta_m u^m. \tag{37}$$

It is convenient to rewrite the azimuthal angular momentum $\nu_\phi$ of a stream in terms of an angular momentum parameter $a$ defined by

$$\nu_\phi \equiv a \sin^2 \theta. \tag{38}$$

If the stream is slowly rotating, and if it has no $\theta$ motion ($\nu^\theta = 0$), so that its rest frame belongs to the system of slowly rotating tetrads considered in section 2.1, then $a = r \zeta_t$ in the tetrad rest-frame of the stream. However, equations (37) and (38) remain valid in general, regardless of the angular motion of the stream. Equations (37) and (38) imply that the azimuthal component $u_\phi$ of the tetrad-frame 4-velocity of a stream is related to the time and radial components $u_t$ and $u_r$, and to its angular momentum parameter $a$, by

$$u_\phi = \sin \theta \left( \frac{a}{r} - \zeta_t u_t - \zeta_r u_r \right). \tag{39}$$

With $u^\phi$ related to $u^t$ and $u^r$ by equation (39), it follows from equation (36) that the axial spin-1 components of the tetrad-frame energy–momentum tensor of a perfect fluid are

$$T^{t\phi} = \sin \theta \left[ \frac{a(\rho + p)u^t}{r} - \zeta_t (T^{tt} + T^{t\phi}) - \zeta_r T^{tr} \right], \tag{40a}$$

$$T^{r\phi} = \sin \theta \left[ \frac{a(\rho + p)u^r}{r} - \zeta_r (T^{rr} - T^{t\phi}) - \zeta_t T^{tr} \right]. \tag{40b}$$

The component of the covariant divergence of the perfect-fluid energy–momentum along the azimuthal Killing vector $\zeta^a$ is

$$\zeta_a D_m T^{mn} = \frac{1}{r^3} \{ (\partial_\phi + h_\phi) [r^2 a(\rho + p)u^t] + (\partial_r + h_r) [r^2 a(\rho + p)u^r] \}, \tag{41}$$

where $h_m \equiv \Gamma^a_{trm}$ are tetrad-frame connection coefficients, equations (64) of [23].

### 2.5. Einstein equations

The Einstein equations are

$$G_{mn} = 8\pi T_{mn}. \tag{42}$$

With the Einstein tensor from equation (24) and the electromagnetic and perfect fluid energy–momentum tensors from equations (31) and (40), the axial spin-1 Einstein equations (42) reduce to
\[ \partial_t (r^2 W) = Q \partial_t B - 8\pi r^2 \sum_{\text{streams}} a(\rho + p)u', \]
\[ \partial_r (r^2 W) = Q \partial_r B - 8\pi r^2 \sum_{\text{streams}} a(\rho + p)u', \]

where the sum is over the accreting baryonic and dark-matter streams. These equations (43) are gauge invariant, the parts depending on the gauge perturbations \( \zeta' \) and \( \zeta'' \) in equations (24), (31) and (40) having canceled between the left- and right-hand sides.

If the underlying sources of energy–momentum \( T_{m\nu} \) are arranged to satisfy conservation of energy–momentum, as they should, then one of the two Einstein equations (43) can be treated as redundant, since it serves simply to enforce overall covariant energy–momentum conservation in the azimuthal direction
\[ \zeta^m D^v G_{mn} = 0. \]

That there is effectively only one independent axial spin-1 Einstein equation is consistent with the fact that there is only one physical degree of freedom in the axial spin-1 vierbein perturbations.

### 2.6. Energy exchange

To complete the equations, it is necessary to specify the exchange of energy between each pair of species.

I assume for simplicity that the dark matter and the baryons stream freely through each other without interacting.

For the interaction between charged baryons and the electromagnetic field, I assume Ohm’s law in the rest frame of the baryons,
\[ j_m = \sigma E_m \quad (m = r, \phi), \]
where \( \sigma \) is an electrical conductivity. The azimuthal current \( r \zeta^m j_m \) that appears on the right-hand side of the Maxwell equation (30c) is then, in the rest frame of the baryons,
\[ r \zeta^m j_m = - \left( a_jt + \frac{\sigma}{2} \partial_t B \right). \]

### 3. Slowly rotating Kerr–Newman

As long as the accretion rate is small, as is true for astronomically realistic black holes, the spacetime down to near the inner horizon is well approximated by the slowly rotating Kerr–Newman [31, 32] geometry. The Kerr–Newman geometry is characterized by the mass \( M_* \), charge \( Q_* \) and angular momentum parameter \( a_* \) of the black hole, all constants.

The monopole part of the slowly rotating Kerr–Newman solution coincides with Reissner–Nordström. The interior mass \( M(r) \) and charge \( Q(r) \) are
\[ M = M_* - \frac{Q_*^2}{2r}, \quad Q = Q_* \]

If the time coordinate \( t \) is chosen to reflect the stationarity of the spacetime, \( \partial / \partial t = 0 \), and scaled globally to synchronize with proper time at rest at infinity, then the vierbein coefficient \( \alpha \) is related to the vierbein coefficients \( \beta_m \) by
\[ \alpha = \frac{1}{\beta_r}. \]
The radial 4-gradient $\beta^m$, equation (3), is a tetrad-frame 4-vector, and different choices of $\beta^t$ or $\beta^r$ correspond to different gauge choices of radial Lorentz boost of the tetrad frame. For example, $\beta^t = 0$ recovers the standard diagonal Reissner–Nordström line element, while $\beta^r = 1$ yields the Gullstrand–Painlevé [33] line element.

The dipole part of the slowly rotating Kerr–Newman geometry is described by the gauge-invariant vierbein perturbation $\omega(r)$, and by the gauge-invariant quantities $W(r)$ and $B(r)$ characterizing the axial spin-0 Weyl and electromagnetic scalars, equations (22) and (28):

$$\omega = \frac{2a_* M^3}{r^5}, \quad (49a)$$
$$W = \frac{a_*}{r^2} \left( -3M_* + \frac{2Q_*^2}{r} \right), \quad (49b)$$
$$B = \frac{2a_* Q_*}{r}. \quad (49c)$$

The Kerr–Newman geometry is most commonly expressed in the Boyer–Linquist [34] tetrad, where the spin-1 components of the Einstein, Weyl, and electromagnetic tensors all vanish identically. The Boyer–Linquist tetrad is attained with the gauge choices

$$r \xi^t = -a_* \beta^r, \quad r \xi^r = a_* \beta^t. \quad (50)$$

The coordinate gauge perturbation $\psi$, or equivalently $\Psi$, equation (17b), plays no role, but it simplifies the line element (20) to set

$$\Psi = 0. \quad (51)$$

4. Self-similar solutions

The slowly rotating, accreting black hole equations admit self-similar solutions. Though idealized as a model of reality, the similarity solutions are exact solutions that provide insight into the general case. The similarity solutions in this section are extensions of those discussed by [12, 22, 23].

As has been seen in section 2, the equations governing the slowly rotating perturbation are completely decoupled from the spin-0 monopole equations governing the spherically symmetric background. It follows that the behaviour of inflation in a slowly rotating black hole is identical to that in a spherical black hole. This conclusion is true subject to the consistency condition that the slowly rotating perturbation remains small through inflation. As seen in section 4.5, the slowly rotating perturbation remains small during and well after inflation, so the solutions are indeed self-consistent.

Eventually, the rotational perturbation does become large, section 4.6, signalling a breakdown in the perturbation assumption, but this occurs only at tiny scales, deep into the post-inflationary collapse regime. For small accretion rates, the Weyl curvature scalar $C$ already greatly exceeds the Planck scale before the rotational perturbation becomes large, so the breakdown of perturbation theory is irrelevant, since presumably quantum gravity or other physics (such as particle collisions between ingoing and outgoing streams at super-Planck energies [23]) intervenes. However, if the accretion rate is sufficiently large, such as may occur during the initial collapse of a black hole, then the rotational perturbation could become large before the curvature hits the Planck scale. The calculations of the present paper are insufficient to predict the kind of singularity that might ensue at small radius.
4.1. Similarity assumption

Self-similarity is the assumption that the system possesses conformal time translation invariance, that is, there exists a conformal time \( T \) such that the system at any one conformal time is a scaled copy of the system at any other conformal time.

The line element (20) can be cast in a form that has explicit conformal symmetry by transforming the time and radial coordinates \( t \) and \( r \) to conformal time and radial coordinates \( T \) and \( R \) defined by

\[
t = T, \quad \ln r = R + T. \tag{52}
\]

The fact that it is possible to choose the time coordinate to be conformal time, \( t = T \), reflects the gauge freedom in the choice of time coordinate. In terms of the conformal coordinates \( T \) and \( R \), the line element (20) is

\[
ds^2 = r^2 \left\{ - (\xi^t \, dT + \zeta^t \sin^2 \theta \, d\phi)^2 + \left( \frac{dR}{Br} + \xi^r \, dT + \zeta^r \sin^2 \theta \, d\phi \right)^2 \right. \\
+ \left. d\theta^2 + [\sin \theta (d\phi - \Psi \, dR) + \xi^\theta \, dT] \right\}. \tag{53}
\]

The circumferential radius \( r \) (squared) appears as an overall conformal factor. Conformal time translation symmetry requires that all conformal inverse-vierbein coefficients; the coefficients of differentials inside the braces in equation (53) are functions only of conformal radius \( R \), not of conformal time \( T \).

\( \xi^m \) in the line element (53) are the tetrad-frame components of the Killing vector associated with conformal time translation invariance,

\[
\frac{\partial}{\partial T} = e^m_T \partial_m \equiv r \xi^m \partial_m. \tag{54}
\]

In terms of the vierbein coefficients of the line element (20), the conformal Killing vector \( \xi^m \) is

\[
\xi^m = \left\{ \frac{1}{r\alpha}, \frac{1}{\beta_r} \left( - \frac{\beta_t}{r\alpha} + 1 \right), 0, - (\Omega + \Psi) \sin \theta \right\}. \tag{55}
\]

The scalar product of the conformal Killing vector \( \xi^m \) which itself defines the horizon function \( \Delta \), a coordinate and tetrad gauge-invariant scalar,

\[
\Delta \equiv - \xi_m \xi^m = \xi^t \xi^t - \xi^r \xi^r. \tag{56}
\]

The horizon function is positive outside the horizon, zero at the horizon and negative inside the horizon, reflecting the fact that the Killing vector \( \xi^m \) is respectively timelike, null, and spacelike outside, at and inside the horizon. The scalar product of the conformal Killing vector \( \xi^m \) with the radial 4-gradient \( \beta_m \) follows from \( \partial \ln r/\partial T = 1 \), which implies

\[
\xi^m \beta_m = 1. \tag{57}
\]

In self-similar solutions, all quantities are proportional to some power of circumferential radius \( r \) and that power can be determined by dimensional analysis. Dimensional analysis shows that the conformal coordinates \( \{ T, R, \theta, \phi \} \), the coordinate metric \( g_{\mu\nu} \) and the vierbein perturbation \( \phi_{mn} \) are all dimensionless. In particular,

\[
\beta_m, \quad \xi^m, \quad \omega, \quad \Psi, \quad \zeta^m \text{ are dimensionless.} \tag{58}
\]

The vierbein \( e_m^\mu \), inverse vierbein \( e^m_\mu \), tetrad-frame connections \( \Gamma_{kmn} \) and tetrad-frame Riemann tensor \( R_{klmn} \), energy–momenta \( T_{mn} \) and electromagnetic field \( F_{mn} \), scale as

\[
e^m_\mu \propto r^{-1}, \quad e^\mu_m \propto r, \quad \Gamma_{kmn} \propto r^{-1}, \quad R_{klmn} \propto r^{-2}, \quad T_{mn} \propto r^{-2}, \quad F_{mn} \propto r^{-1}. \tag{59}
\]
Various specific quantities that appear in section 4.2 scale as
\[ W \propto r^0, \quad Q \propto r, \quad B \propto r, \quad a \propto r. \]  
\[ (60) \]

### 4.2. Similarity equations

Four equations (63)–(66) govern the angular structure of the slowly rotating black hole, one for each of the baryons, dark matter, gravity and electromagnetic field. For simplicity, the equations in this subsection are for zero conductivity, \( \sigma = 0 \), as assumed in the model of section 4.5. Equations for finite conductivity are given for reference in the appendix.

I follow [12, 22] in choosing the integration variable \( x \) to be the dimensionless baryonic time defined by \( dx \equiv r^{-1}d\tau \), where \( \tau \) is the proper time along the worldline of the baryons. Equivalently
\[ \frac{d}{dx} = r \partial_t \]  
\[ (61) \]
where \( \partial_t \) is the directed time derivative in the rest frame of the baryons. The dimensionless baryonic time \( x \) increases monotonically, since proper time does. The dark matter falls into the black along a different trajectory than the baryons, and it is therefore necessary to distinguish the dark matter radius \( r_d \) and dark matter tetrad frame from the baryonic radius and tetrad frame, as detailed in [12]. I treat the baryonic frame as the default frame, and for brevity drop the baryonic subscript \( b \) from \( r \) and \( \xi_m \) evaluated in the baryon frame.

In self-similar solutions, the volume of a Lagrangian volume element is proportional to \( \frac{4}{3} \pi r^3 \xi^r \). Angular momenta \( L \) of Lagrangian volume elements can be defined to be \( L \equiv \frac{4}{3} \pi r^3 \xi^r a(\rho + p) \). Explicitly, for each of the baryon (subscripted \( b \)) and dark matter (subscripted \( d \)) streams, whose equations of state are given by equations (34) and (35),
\[ L_b \equiv \frac{4}{3} \pi r^3 \xi^r_a (1 + w) \rho_b, \]  
\[ (62a) \]
\[ L_d \equiv \frac{4}{3} \pi r^3 d \xi^r_a \rho_d. \]  
\[ (62b) \]
Covariant conservation \( \xi_n D_m (T^{mn}_{b} + T^{mn}_{d}) = 0 \) of electromagnetic plus baryon energy along the azimuthal Killing vector \( \xi^r \), equations (33) and (41), yields the following equation of conservation of angular momentum of baryons in their rest frame (note that for self-similar solutions the tetrad-frame connection \( h_r \) in equation (41) is \( h_r = \partial_t \ln (r \xi^r) \)):
\[ \frac{QB}{6} + L_b = \text{constant.} \]  
\[ (63) \]
Similarly, covariant conservation \( \xi_n D_m T^m_{d} = 0 \) of dark matter energy implies conservation of angular momentum of dark matter in its frame,
\[ L_d = \text{constant.} \]  
\[ (64) \]
Equation (64) should be interpreted as meaning that the angular momentum \( L_d \) is constant along the path of the dark matter, not along the path of the baryons, which is different. The dimensionless combination \( L_d / r_d^2 \) is however the same in any frame.

The operator \( \xi^m \partial_m \), equation (54), vanishes when acting on any dimensionless quantity. Taking \( \xi^t \) times equation (43a) plus \( \xi^t \) times equation (43b) yields an integral of motion for the axial Weyl scalar \( W \),
\[ W = \frac{QB}{2r^2} - \frac{3L_b}{r^2} - \frac{3L_d}{r_d^2}. \]  
\[ (65) \]
Equation (30c) governing the radial magnetic field scalar $B$, coupled with equation (46) and the assumption of vanishing conductivity, yields a second-order ordinary differential equation for $B$,

$$
- \frac{1}{2\Delta} \left[ -1 + \left( -\frac{\Delta}{\xi'} \frac{d}{dx} + \frac{\xi'}{\xi} \right)^2 \right] B + B + 2QW = \frac{a_b Q}{r\xi'}.
$$

(66)

For self-similar solutions, the dimensionless vierbein coefficient $\omega$, the conformal angular velocity, is (coordinate and tetrad) gauge-invariant and is related to the axial Weyl scalar $W$ by, from equation (22),

$$
- \frac{1}{2\xi'} \frac{d\omega}{dx} = W.
$$

(67)

The vierbein coefficients $\xi^m$, the tetrad-frame components of the azimuthal Killing vector, are not tetrad gauge invariant. A traditional choice of angular gauge is the principal, or Weyl, gauge, defined to be the frame in which the spin-1 components (25) of the Weyl tensor vanish. In the principal tetrad,

$$
\xi^m = -\frac{a M_* \xi^m}{M_*},
$$

(68)

where $M_*$ and $M_*$ are the mass and mass accretion rate of the black hole as measured by a distant observer (see section 4.3), and the angular momentum parameter $a$ is

$$
a \equiv \frac{M_*}{6M_*r^2C\xi'} \frac{dW}{dx}.
$$

(69)

The factor of $M_*/M_*$ in equation (68) brings $a$ to the same scale as the angular momentum parameters $a_b$ and $a_d$ of the baryons and dark matter, equation (38). The factor follows from scaling from conformal time $T$ to Kerr–Newman time $t_{KN}$ at the sonic radius, where $\partial \ln r/\partial T = 1$, whereas $\partial \ln r/\partial t_{KN} = M_*/M_*$, so that

$$
\frac{dT}{dt_{KN}} = \frac{M_*}{M_*} \text{ at the sonic radius.}
$$

(70)

The angular momentum parameter $a_*$ of the black hole itself can be defined to be the value that would be measured by a distant observer if there were no mass or charge outside the sonic radius. Matching the Kerr–Newman (49b) and self-similar (65) values of the axial Weyl scalar $W$ at the sonic radius (section 4.3) yields

$$
a_* M_* = -\frac{r^2W + QB}{3} = \frac{QB}{6} + L_b + L_d \text{ at the sonic radius.}
$$

(71)

### 4.3. Spherical boundary conditions

The boundary conditions of the unperturbed spherically symmetric self-similar solutions are set at a sonic point outside the outer horizon, where the infalling baryonic fluid accelerates from subsonic to supersonic. The mass $M_*$ and charge $Q_*$ of the black hole can be defined to be the mass and charge that a distant observer would see if there were no matter or charge outside the sonic radius:

$$
M_* = M + \frac{Q^2}{2r}, \quad Q_* = Q \text{ at the sonic radius.}
$$

(72)

The extra mass $Q^2/2r$ added to the interior mass $M$ is the mass–energy in the electric field outside the sonic radius, given no charge outside the sonic radius.
For the spherically symmetric solutions, there are three dimensionless boundary conditions at the sonic radius: (1) the mass accretion rate $\dot{M}_*$; (2) the charge-to-mass ratio $Q_*/M_*$; (3) the dark-matter-to-baryon density ratio $\rho_d/\rho_b$. The solutions scale with the mass $M_*$ of the black hole.

### 4.4 Angular boundary conditions

What angular boundary conditions must be adjoined to the boundary conditions of the background spherically symmetric solution?

First, the baryonic and dark matter angular momentum parameters $a_b$ and $a_d$, equation (38), must be constants at the sonic radius, independent of polar angle $\theta$. This means that the angular flow patterns of the infalling baryons and dark matter are dipoles, with angular velocity proportional to $\sin \theta$.

Equation (66) governing the radial magnetic field scalar $B$ is a second-order differential equation, so requires two boundary conditions. However, one boundary condition is fixed by the requirement that the solution passes smoothly through the horizon, which is a singular point of the differential equation. I use an interactive shooting method to locate the desired solution. The remaining boundary condition on $B$ is in principle set by the requirement that $B$, which is proportional to the radial component of a dipole magnetic field, should go to zero at infinity (absent distant current sources). However, the self-similar solutions generally do not continue consistently to infinity. Instead, I simply set $B$ equal to the Kerr–Newman value (49c) at the sonic radius.

The angular momentum parameters $a_b$ and $a_d$ of the baryons and dark matter, together with the value of $B$ at the sonic radius, determine the total angular momentum parameter $a_*$ of the black hole through equation (71). Since all angular quantities scale in proportion to $a_*$, the angular behaviour of the solutions is determined in effect by a single dimensionless boundary condition, the ratio $a_d/a_b$ of the dark matter to baryonic angular momentum parameters ($B$ being fixed to its Kerr–Newman value at the sonic radius).

### 4.5 Similarity model

Figure 1 shows a self-similar solution with accretion rate $\dot{M}_* = 0.01$, charge-to-mass $Q_*/M_* = 0.8$ and dark-matter-to-baryon density ratio $\rho_d/\rho_b = 0.1$ at the sonic radius. This is the same model shown in figures 4, 6 and 7 of [23]. The ratio $a_d/a_b$ of dark-matter-to-baryon angular momentum parameters has been set to 1 far from the black hole

$$\frac{a_d}{a_b} = 1.$$  \hspace{1cm} (73)

The accretion rate $\dot{M}_* = 0.01$ (which essentially means that the sonic radius is expanding at 0.01 of the speed of light, as measured by a distant observer) is much larger than any typical astronomical accretion rate. The advantage of a large accretion rate is that it is possible to follow the complete evolution of the spacetime through inflation to collapse without quantities overflowing numerically. The smaller the accretion rate, the more rapidly inflation exponentiates [23], triggering numerical overflow.

The top panel of figure 1 shows the various components of the tetrad-frame energy–momentum tensor measured in a frame that is centre-of-mass in the radial direction, $T^{rr} = 0$, and principal in the angular direction, where the spin-1 components of the Weyl tensor, equation (25), vanish. The centre-of-mass energy density and pressure $T^{tt}$ and $T^{rr}$ inflate rapidly at a radius $r \approx 0.4$ close to where the inner horizon would have been in the absence of inflation. The centre-of-mass energy and pressure arise almost entirely from the streaming
energy and pressure produced by the relativistic counter-streaming between the outgoing baryons and the ingoing dark matter. The proper energy and pressure of either of the baryon or dark matter streams in their own frames change only modestly.

The angular components $T^{\phi}$ and $T^{\phi}$ in the top panel of figure 1 are plotted in units such that the black hole angular parameter is unity, $a_\bullet = 1$, and should be multiplied by the true $a_\bullet$. 

Figure 1. Self-similar model with accretion rate $M_\bullet = 0.01$, charge-to-mass $Q_\bullet/M_\bullet = 0.8$ and dark-matter-to-baryon density ratio $\rho_d/\rho_b = 0.1$, at the sonic radius. Lines are dashed where a quantity is negative. All quantities are in geometric units, $c = G = M = 1$. Perturbed quantities (everything except $T^{\phi}$ and $T^{\phi}$) are in units where $a_\bullet = 1$ and should be multiplied by the actual $a_\bullet$, a small number. (Top) Components of the tetrad-frame energy–momentum tensor $T^{\mu\nu}$ in a frame that is centre-of-mass in the radial direction and principal in the angular direction. (Upper middle) Axial spin-0 Weyl and electromagnetic fields $W$ and $B$, equations (22) and (28). (Lower middle) Vierbein coefficient $\omega_\mu$ (the scaling adjusts from conformal time to Kerr–Newman time at the sonic radius, equation (70)). (Bottom) Angular momentum parameters $a_b$ and $a_d$ of the baryons and dark matter, equation (38), and $a$ of the black hole in the principal frame, equation (69).
a small number. Thus although the figure appears to show that, in the Kerr–Newman regime \( r \gtrsim 0.4 \), the angular components \( T^{\phi \phi} \) and \( T^{r \phi} \) are comparable to the radial components \( T^{\phi \phi} \) and \( T^{r r} \), in fact the angular components are much smaller. In the inflationary regime \( r \lesssim 0.4 \), the angular components \( T^{\phi \phi} \) and \( T^{r \phi} \) inflate, but they inflate more slowly than the radial components \( T^{\phi \phi} \) and \( T^{r r} \). Thus the angular components, already a small perturbation in the Kerr–Newman regime, become an even smaller perturbation in the inflationary regime.

The upper middle panel of figure 1 shows the gauge-invariant axial Weyl and electromagnetic scalars \( \mathcal{W} \) and \( \mathcal{B} \), equations (22) and (28), while the lower middle panel shows the dimensionless gauge-invariant vierbein perturbation \( \omega \), equation (67). All three quantities remain well behaved during inflation, showing no sign of any inflationary behaviour. This demonstrates that the assumption of a slowly rotating perturbation is consistent: the perturbation remains small throughout inflation.

Note that \( \omega \) obtained from equation (67) is a dimensionless conformal angular velocity.

Figure 1 shows \( \omega \) multiplied by \( \dot{M}_e/M_* \), equation (70), which converts \( \omega \) to a true angular velocity that can be compared directly to the Kerr–Newman angular velocity (49a).

The lower panel of figure 1 shows various angular momentum parameters \( a \). The dark matter angular momentum parameter \( a_d \) is constant, reflecting the fact that the dark matter is neutral and pressureless, and therefore falls freely with constant angular momentum. The baryonic angular momentum parameter \( a_b \) decreases inwards, as the charged baryons lose angular momentum into the magnetic field, in accordance with equation (63). The two angular momentum parameters \( a_b \) and \( a_d \) remain well behaved in the inflationary regime: they do not inflate. The parameter \( a \) (with no subscript), equation (69), is defined to be the angular momentum parameter in the principal frame. In the Kerr–Newman geometry, this angular momentum parameter would be constant and equal to 1 in the units \( a_* = 1 \) used in the figure. Figure 1 shows that \( a \) in the self-similar solution is indeed close to 1 in the Kerr–Newman regime \( r \gtrsim 0.4 \). What appear to be glitches in \( a \) at \( r \approx 0.63 \) and again at \( r \approx 0.39 \) arise because \( dW/dx \) and \( C \), whose ratio yields \( a \), go through zero at very slightly different radii (in Kerr–Newman, the zeros in \( dW/dx \) and \( C \) at \( r \approx 0.63 \) occur at exactly the same radius). No physical divergence is associated with the glitches, since \( a \) is a derived quantity, not a physical perturbation.

The angular structure of the self-similar solution is determined by four equations, of which three are algebraic, equations (63)–(65), and the fourth is a second-order differential equation for the radial magnetic field scalar \( B \), equation (66). Only the fourth equation poses a challenge to understanding its behaviour. Figure 2 shows that in the Kerr–Newman regime \( r \gtrsim 0.4 \), the magnetic field is closely approximated by the Kerr–Newman form (49c). Inflation takes place just above the inner horizon, and in the inflationary regime \( r \lesssim 0.4 \), the horizon function \( \Delta \) is tiny and \( \xi^l/\xi^r \approx -1 \). In this regime, the differential equation (66) for \( B \) simplifies to

\[
\left( 1 + \left( \frac{\Delta}{\xi^r} \frac{d}{dx} - 1 \right) \right) B = 0.
\]

Introduce the Regge–Wheeler coordinate \( r^* \) defined by

\[
\frac{d}{dr^*} \equiv -\frac{\Delta}{\xi^r} \frac{d}{dx}.
\]

Then equation (74) simplifies to

\[
\frac{d}{dr^*} \left( \frac{d}{dr^*} - 2 \right) B = 0,
\]
Figure 2. Radial magnetic field $B$ in units $a_* = 1$ (left scale), and Regge–Wheeler coordinate $r^*$ (right scale) for the model shown in figure 1. The dashed lines superposed on the magnetic field $B$ show approximations valid, respectively in the Kerr–Newman (red) and inflationary (blue) regimes, equations (49c) and (77).

whose general solution is

$$B = B_0 + B_1 e^{2r^*}$$  \hspace{1cm} (77)

where $B_0$ and $B_1$ are constants. Figure 2 shows a fit to approximation (77) with the constants $B_0$ and $B_1$ treated as free parameters. The fit is excellent. Analytic and numerical investigation indicates that the constant $B_1$ depends on boundary conditions (such as $\rho_d/\rho_b$ or $a_d/a_b$) and does not tend to some universal limit in the limit of small accretion rates (for example, $B_1$ does not vanish).

4.6. Behaviour at small radii

In the self-similar model of the previous subsection, section 4.5, the dimensionless conformal angular velocity $\omega$, equation (67), was found to be approximately constant during and after inflation, as illustrated in figure 1. The dimensionless conformal angular velocity $\omega$ is the only angular vierbein perturbation that is (coordinate and tetrad) gauge invariant, so a sufficient condition for perturbation theory to hold is that $\omega$ remains small.

Figure 3 shows the behaviour of the dimensionless conformal angular velocity $\omega$, in units $a_* = 1$, down to tiny radii, deep into the post-inflationary collapse regime (unlike in figure 1, $\omega$ is plotted in dimensionless form in figure 3, without the dimensional scaling factor $\dot{M}_*/M_*$. Figure 3 shows that $\omega$ remains constant down to $r \sim 10^{-40}$, at which point $\omega$ starts increasing, signalling a breakdown in perturbation theory. Figure 3 also shows the gauge-invariant dimensionless horizon function $\Delta$, equation (56). Inflation drives the (negative) horizon function to a tiny value, $|\Delta| \sim 10^{-110}$, but the horizon function starts increasing again in absolute value during the collapse phase following the end of inflation. Eventually the horizon function $\Delta$ exceeds unity in absolute value, and it is at this point that the angular velocity $\omega$ starts increasing, and perturbation theory begins to break down. The axial spin-0 fields $B$ and $r^2 W$, which are not plotted in figure 3, remain sensibly constant to tiny scales, where $\omega$ is diverging.
Figure 3. Gauge-invariant dimensionless conformal angular velocity $\omega$, dimensionless horizon function $\Delta$, interior mass $M$ and Weyl curvature scalar $C$, as a function of baryonic radius $r$, for the model shown in figure 1. The conformal angular velocity $\omega$ is in units where $a_0 = 1$; the radius $r$, mass $M$ and Weyl scalar $C$ are in geometric units $c = G = M_\bullet = 1$. Lines are dashed where quantities are negative. The radial (horizontal) scale is broken into two regimes in order to bring out the behaviour both near the horizon (right) and at small radii (left). Inflation over the radial range $r \sim 0.4$–$0.1$ is followed by collapse at $r \lesssim 0.1$.

Figure 3 also shows the interior mass $M$ and the Weyl curvature scalar $C$, equation (26). Inflation exponentiates the mass $M$ and the (negative) Weyl scalar $C$ to a huge value, $|C| \sim M \sim 10^{115}$ in geometric units. During the subsequent collapse phase, the mass $M$ is approximately constant, while the Weyl scalar, which is dominated by the mass term in equation (26), $C \propto -M/r^3$, diverges as $r^{-3}$. The Weyl curvature at the end of inflation is $|C| \sim 10^{19}(M_\bullet/M_\odot)^{-2}$ in Planck units, which exceeds Planck for any astronomically reasonable black hole mass $M_\bullet$. Thus, it might be expected that quantum gravitational effects would become important well before collapse has reached the radius where perturbation theory breaks down.

The accretion rate in the model of figure 3 is $M_\bullet = 0.01$, which from an astronomical perspective is large (a sonic radius expanding at 0.01 of the speed of light). As shown by [23], the radial length scale over which inflation exponentiates is proportional to the accretion rate, so that, counter-intuitively, smaller accretion rates lead to more rapid and larger inflation. For the curvature to remain sub-Planckian down to a sufficiently small radial scale that perturbation theory breaks down, the accretion rate must be larger than that $M_\bullet = 0.01$ in figure 3. Such a large accretion rate is likely to occur only during the initial collapse of a black hole or during a black hole merger.

The calculations of the present paper are inadequate to predict the character of the general relativistic singularity that might form at small radius if the accretion rate were large enough. The fact that the angular perturbation eventually becomes large at tiny radii accords with the expectation that, as long as the ratio of angular-momentum-to-mass is finite, however small, the singularity should not be the same as Schwarzschild. It is possible that the singularity would be BKL-like [35–37]. It is also possible that there could be a renewed period of inflation.
5. Small accretion rate

The self-similar model of the previous section, section 4, has the virtue of yielding exact solutions, but it is unrealistic in the sense that it requires a finely tuned accretion flow, in which the accreting streams have a small dipole angular velocity that remains steady over the age of the black hole.

In fact however, as will now be shown, for sufficiently small accretion rates, condition (81), the instantaneous angular motion of the accretion flow has a negligible effect on the spacetime geometry of the slowly rotating black hole. For small enough accretion rates, the equations describing the angular spacetime structure of a slowly rotating black hole simplify to equations (82) and (83), independent of the instantaneous angular motion of the accreting streams.

For small accretion rates, the interior charge $Q$ can be treated as effectively constant as a function of radius,

$$Q \approx Q_* = \text{constant}. \quad (78)$$

Initially, before inflation, the charge is constant simply because the small accretion rate implies a small charge density. During inflation, as first pointed out by [38], Lagrangian volume elements remain little distorted even though the tidal force (Weyl curvature) exponentiates hugely. The physical reason for the mild distortion is that a volume element experiences only a tiny proper time during inflation, so that the tidal force, though huge, has too little time to do damage. Because volume elements change little during inflation, and because charge density is conserved and the initial charge density is small, therefore the charge density remains always small, and the interior charge $Q$ is effectively constant.

For constant interior charge $Q$, the Einstein equation (43b) integrates along the worldline of any tetrad frame to

$$r^2 W = Q_* B - 3a_* M_* + 8\pi \sum_{\text{streams}} \int_{\text{sonic}} a(\rho + p)u r^2 \frac{dr}{\beta_t}. \quad (79)$$

Here the integral with respect to proper time $\tau$ along the worldline of the tetrad has been converted to an integral with respect to radius $r$ using $dr/d\tau \equiv \partial_r r \approx \beta_t$. The constant $-3a_* M_*$ of integration in the integral (79) is established at the sonic radius, where equation (71) holds.

For small accretion rates, the integral over either of the baryonic or dark matter streams in equation (79) is small

$$\int a(\rho + p)u r^2 \frac{dr}{\beta_t} \approx 0. \quad (80)$$

Before inflation, the integral is small because the small accretion rate implies a small proper density $\rho$ and pressure $p$. During inflation, the proper density and pressure $\rho$ and $p$ of a stream change only mildly (because Lagrangian volume elements distort mildly), and, thanks to conservation of angular momentum, the angular momentum parameter $a$ of the stream likewise changes little. The only factor that could potentially become large in the integrand of the integral (80) is the ratio $u'/\beta_t$. The integrals in equation (79) can be evaluated in any tetrad frame (the same frame for both streams); one option is to choose the no-going frame, defined to be the frame where $\beta_t = 0$. Relative to the no-going frame, the streaming velocity $u'$ and $\beta_t$ both increase exponentially, but they increase approximately in proportion to each other [23]. The result is that the integral (80) remains small because $\rho + p$ is small, and the other factors in the integrand do not spoil that conclusion.
The integral over either stream in equation (79) involves the product $a(\rho + p)$ of the angular momentum parameter and the proper density plus pressure. The integrals are small compared to the other terms in equation (79) provided that $a(\rho + p) \ll a_*$ in geometric units ($c = G = M_* = 1$). If the streams are slowly rotating, $a \sim a_*$, then the integrals are small provided that $\rho + p \ll 1$. But if the accretion rate is sufficiently small that it satisfies the stronger condition that

$$\rho + p \ll a_* \quad \text{(geometric units)},$$

then the integrals remain small even if the angular momentum parameter $a$ is large, of order unity in geometric units. The fact that the integrals remain small, equation (80), even for large angular momenta means that the instantaneous angular motion of the accreting streams has effectively negligible effect on the spacetime structure of the slowly rotating black hole, even in the case that the accreting streams have large angular momentum and any angular motion, not just cylindrically symmetric rotation about the rotation axis. Of course, ultimately the angular momentum of the black hole is equal to the cumulative angular momentum of the accreted streams, but the instantaneous angular motion is essentially irrelevant.

In approximations (78) and (80), valid for small enough accretion rates, equation (79) simplifies to

$$r^2 W \approx Q_* B - \frac{3}{a} M_*,$$

which relates the axial Weyl scalar $W$ to the axial magnetic field scalar $B$ without any dependence on the baryon or dark matter streams.

To complete the equations governing the spacetime structure for small accretion rates, equation (82) must be supplemented by an equation for the magnetic field $B$, specifically equation (30c) with, since the accretion rate is small, vanishing right-hand side. Self-similar solutions provide a guide to what happens. As illustrated in figure 2, $B$ is well approximated by the slowly rotating Kerr–Newman solution (49c) before inflation sets in, and by approximation (77) in the inflationary regime. Approximation (77) depends on the Regge–Wheeler coordinate $r^*$, which has the property that, as illustrated in figure 2, the derivative $d r^*/d \ln r$ goes through a maximum near the onset of inflation, and then becomes progressively smaller during and after inflation. The net effect is that $B$ flattens out and becomes constant during and after inflation, which is true even though the constant $B_1$ in approximation (77), which is effectively determined by matching to the Kerr–Newman approximation (49c) at the onset of inflation, is non-zero. For small accretion rates, the transition from the Kerr–Newman regime to the regime where $B$ is constant becomes increasingly abrupt and $B$ simplifies to

$$B \approx \frac{2 a_* Q_*}{\max(r, r_-)},$$

where $r_- = M_* - \sqrt{M_*^2 - Q_*^2}$ is the radius of the erstwhile inner horizon, which is destroyed by inflation. Approximation (83) has been derived on the assumption of self-similarity, which is premised on the assumption of constant accretion rate $M_*$. However, since approximation (83) is independent of the value of the accretion rate, it may be guessed that approximation (83) remains valid even for a variable accretion rate, just so long as the accretion rate remains small.

The axial spin-0 Weyl scalar $W$ is in a fundamental sense the (coordinate and tetrad) gauge-invariant quantity that describes the angular part of the gravitational field. The Weyl scalar $W$ is related to the angular vierbein coefficients $\omega$ and $\psi$ by equation (22), which in the self-similar case becomes equation (67), which depends only on $\omega$. Converted to a derivative with respect to $\ln r$ instead of $x$, the self-similar equation (67) for $\omega$ becomes $-(\beta t/2 \xi') \omega d x/\ln r = W$. In the inflationary regime, the coefficient $-\beta_t/2 \xi'$ is so large, for
small accretion rates, that $\frac{d\omega}{d\ln r}$ is tiny, so that $\omega$ is essentially constant as a function of radius $r$ (beware this statement fails deep into the post-inflationary collapse regime, section 4.6). The result is that, for small accretion rates, the vierbein coefficient $\omega$ is approximately equal to the Kerr–Newman form (49a) prior to inflation and to a constant during and after inflation:

$$\omega \approx a \left( \frac{2M}{\max(r, r_-)} - \frac{Q}{\max(r, r_-)} \right)^3. \quad \text{(84)}$$

As with equation (83), equation (84) has been derived on the assumption of self-similarity, but in fact the equation is independent of the value of the accretion rate, so it may be guessed that approximation (84) remains valid even for an accretion rate that varies with time, just so long as the accretion rate remains small.

Equations (82)–(84) look enticingly simple, but one should be wary of interpreting them too literally, especially equation (84). For example, if the approximation $\omega \approx \text{constant}$ in the inflationary regime were taken literally, then it would imply, from equation (67), that $\mathcal{W} \approx 0$, which is false. Thus, it would be incorrect to insert approximation (84) into the line element (20) and imagine that the consequent Riemann tensor would be valid. The problem is that inflation generates exponentially huge numbers, and something that by itself appears sensibly equal to zero may become significant when multiplied by a huge number.

6. Limitation

The slowly rotating models considered in this paper have the limitation that the rates of accretion of ingoing and outgoing streams are assumed to be constant as a function of angular position about the black hole. This limitation is inherited from the assumption that the slow rotation is a small perturbation of a background spacetime that is spherically symmetric. The previous section showed that the angular motion of the accreting streams has negligible effect, for small accretion rates. But the angular motion is distinct from the angular dependence of the accretion rate.

It would be possible to consider, in addition to the slowly rotating perturbation considered in this paper, perturbations to the accretion rate as a function of angular position over the black hole. This could be the subject of future investigation.

7. Comparison to prior work

Prior work [13–20] on the interior structure of astronomically realistic rotating black holes has focused on the situation of a black hole in the late stages of gravitational collapse. In this case, mass inflation is produced by counter-streaming between ingoing and outgoing gravitational waves produced by a decaying Price [25] tail of radiation.

The work was initiated by Ori [13] and concluded by him [16]. Based on an asymptotic analysis of gravitational waves near the inner horizon generated by the Price tail, Ori argues, first, that wave amplitudes, though diverging, are nevertheless dominated by linear as opposed to higher order perturbations of the Kerr metric, and second, that the perturbations can be separated in plain spin-weighted spherical harmonics as opposed to the more complicated frequency-dependent spheroidal harmonics of the Teukolsky [39, 40] equation. Ori arrived at two principal conclusions, first, that the perturbations are dominated by quadrupole ($l = 2$) modes, the lowest order of gravitational waves, and second, that the inflationary instability as seen by a freely falling observer is oscillatory. The oscillations result from the rotational motion of the observer relative to the gravitational waves and occur only for waves that are...
non-axisymmetric ($m \neq 0$). Although self-consistent, Ori’s arguments lack full mathematical rigor and it would be desirable to check them independently.

Sadly, the present work has little to say about Ori’s conclusions. In the present paper, inflation is produced by monopole ($l = 0$) streams of ingoing and outgoing matter (dark matter and baryons), as opposed to by gravitational waves. The only perturbation considered here is an axisymmetric dipole ($l = 1, m = 0$), which excludes the non-axisymmetric ($m \neq 0$) modes responsible for Ori’s oscillations.

8. Summary

The internal structure of a slowly rotating, steadily accreting black hole that is undergoing inflation at (just above) its inner horizon has been derived. The approach has been to introduce a slowly rotating (axial dipole) perturbation to a spherically symmetric, accreting black hole undergoing inflation. It has been shown that the rotational perturbation remains small throughout inflation, so the perturbation assumption is self-consistent.

The equations governing the angular (dipole) behaviour, section 2, decouple from the radial (monopole) behaviour. Consequently all of the conclusions concerning inflation in a spherically symmetric, accreting black hole carry through unchanged for a slowly rotating black hole. This conclusion supports [9] conjecture that inflation in rotating black holes may be similar to inflation in spherical black holes.

Exact self-similar solutions for slowly rotating, steadily accreting black holes have been obtained, section 4. The self-similar solutions require a special accretion flow, in which the accreting streams have a small dipole angular velocity that remains steady over the age of the black hole.

The rotational perturbation does eventually become large, section 4.6, signalling a breakdown in perturbation theory, but this occurs only at tiny scales, deep into the post-inflationary collapse regime. For realistically small accretion rates, by this radius the Weyl curvature has already inflated well above the Planck curvature, so presumably quantum gravity in some form intervenes and the breakdown of perturbation theory is irrelevant. Regardless of accretion rate, in no case does a classical general relativistic null singularity form. A crucial assumption of the theorems [24, 26] that prove that a null singularity form is that the black hole remains isolated forever following collapse. However, a real black hole is never isolated and forever is a long time. As shown by [23], at least in spherical black holes, if quantum gravity is set aside, then inflation is generically followed by collapse, not by a null singularity.

It has been shown, section 5, that for sufficiently small accretion rates the instantaneous angular motion of the accretion flow has a negligible effect on the spacetime geometry of the slowly rotating black hole. The angular structure of the black hole is described by the axial spin-0 Weyl and magnetic field scalars $W$ and $B$, which for small accretion rates are given approximately by equations (82) and (83), independent of the instantaneous angular motion of the accretion flow.

This paper has confined itself to the problem of a rotational dipole perturbation on a spherically symmetric background. This is perhaps the simplest problem that one could possibly think of in addressing the problem of the interior structure of a rotating black hole. The simplicity has the virtue that it enables exact self-similar solutions. But the results obtained represent only a small step towards the ultimate goal of solving the interior structure of an astronomically realistic, fully rotating black hole.

The prior results of Ori [13, 16] are considered briefly in section 7. Unfortunately the present paper has little to say about Ori’s conclusions, because the situations are different.
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Appendix. Finite conductivity

This appendix gives, for reference, the self-similar equations for the case of non-vanishing baryonic electrical conductivity $\sigma$. Equation (A.3) replaces equation (63), while equation (A.4) replaces equation (66).

To admit self-similar behaviour, the conductivity must have dimension $\sigma \propto r^{-1}$. If the conductivity is assumed to depend only on the baryon proper density $\rho_b$, then, since $\rho_b \propto r^{-2}$, it follows that (the following repeats equation (48) of [22])

$$\sigma = \frac{\kappa \rho_b^{1/2}}{(4\pi)^{1/2}},$$

(A.1)

where $\kappa$ is a phenomenological dimensionless conductivity coefficient. A dimensionless conductivity $s$ can be defined as (the following repeats equation (51) of [22])

$$s \equiv 4\pi \sigma r = \kappa (4\pi r^2 \rho_b)^{1/2}.$$

(A.2)

In terms of the dimensionless electrical conductivity $s$, the equation expressing conservation of the angular momentum of baryons is

$$\frac{d}{dx} \left( \frac{Q B_6 + L}{6} \right) = -s Q B,$$

(A.3)

and the equation governing the radial magnetic field scalar $B$ is

$$-\frac{1}{2\Delta} \left[ -1 + \left( \frac{\Delta}{\xi r} \frac{d}{dx} + \frac{\xi t}{\xi r} \right)^2 \right] B + B + 2 Q W = \frac{1}{r \xi r} \left( 1 + s \xi t \right) - s \frac{d B}{2 \, dx}.$$

(A.4)

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