A polynomially bounded operator on Hilbert space which is 
not similar to a contraction

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Abstract : Let $\varepsilon > 0$. We prove that there exists an operator $T_\varepsilon : \ell_2 \to \ell_2$, such 
that for any polynomial $P$ we have $\|P(T)\| \leq (1 + \varepsilon)\|P\|_\infty$, but which is not similar to 
a contraction, i.e. there does not exist an invertible operator $S : \ell_2 \to \ell_2$ such that 
$\|S^{-1}T_\varepsilon S\| \leq 1$. This answers negatively a question attributed to Halmos after his well 
known 1970 paper ("Ten problems in Hilbert space").

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§0. Introduction

Let $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ and $T = \partial D = \{ z \mid |z| = 1 \}$.

Let $H$ be a Hilbert space.

Any operator $T : H \to H$ with $\|T\| \leq 1$ is called a contraction. By a celebrated inequality due to von Neumann [vN], we have then, for any polynomial $P$

\begin{equation}
\|P(T)\| \leq \sup_{z \in D} |P(z)|.
\end{equation}

We say that $T$ is similar to a contraction if there is an invertible operator $S : H \to H$ such that $S^{-1}TS$ is a contraction.

An operator $T : H \to H$ is called power bounded if $\sup_{n \geq 1} \|T^n\| < \infty$.

It is called polynomially bounded if there is a constant $C$ such that, for any polynomial $P$

\begin{equation}(0.1)' \quad \|P(T)\| \leq C \sup \{|P(z)| \mid z \in \mathbb{C}, \ |z| = 1 \}.
\end{equation}

Clearly, if $T$ is similar to a contraction, then it is power bounded, and actually, by von Neumann’s inequality, it is polynomially bounded.

We denote

$$\|P\|_\infty = \sup_{z \in \partial D} |P = AC(z)|.$$  

Let $A$ be the disc algebra, i.e. the closure of the space of all (analytic) polynomials in the space $C(T)$ of all continuous functions on $T$, and let $L^p(T)$ be the $L^p$-space relative to the normalized Lebesgue measure on $T$.

Let $H^p = \{ f \in L^p(T) \mid \hat{f}(n) = 0 \ \forall \ n < 0 \}$. If $X$ is a Banach space, we denote, for $p < \infty$, by $H^p(X)$ the analogous space of $X$-valued functions. In the particular case $X = B(H)$ with $H = \ell_2(I)$ ($I$ being an arbitrary set), we denote by $L^\infty(B(H))$ the space of all (classes of) bounded $B(H)$-valued functions of which all matrix coefficients are measurable. This space can be identified isometrically with the dual of the space $L^1(B(H)_*)$ of all Bochner integrable functions with values in the predual of $B(H)$, i.e. the space of all trace class operators.

If $T$ is polynomially bounded, the map $T \to P(T)$ extends to a unital homomorphism $u_T : A \to B(H)$ and $(0.1)'$ implies $\|u_T\| \leq C$. 

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In 1959, Béla Sz.-Nagy [SN] asked whether every power bounded operator $T$ is similar to a contraction. He proved that the answer is positive if $T$ is invertible and if both $T$ and its inverse are power bounded (then $T$ is actually similar to a unitary operator, see [SN]). In 1964, S. Foguel [Fo] (see also [Ha2]) gave a counterexample to Nagy’s question using some properties of Hadamard-lacunary Fourier series. Foguel’s example is not polynomially bounded (see [Le]), whence the next question: Is every polynomially bounded operator similar to a contraction?

This revised version of Sz.-Nagy’s original question was popularized by P. Halmos in [Ha1], and since then, many authors refer to it as “the Halmos problem”.

In [Pa2], Paulsen gave a useful criterion for an operator $T$ to be similar to a contraction. He proved that this holds iff the homomorphism $u_T$ is “completely bounded” (see below for more background on this notion). In that case, the operator $T$ is called “completely polynomially bounded”. In these terms, the above problem becomes: is every polynomially bounded operator completely polynomially bounded? Or equivalently, is every bounded unital homomorphism $\pi : A \to B(H)$ automatically completely bounded?

In [Pe1, Pe2] (see also [FW]), V. Peller proposes a candidate for a counterexample: let $\Gamma : H^2 \to (H^2)^*$ be a Hankel operator, i.e. such that the associated bilinear map (denoted again by $\Gamma$) on $H^2 \times H^2$ satisfies $\forall f \in A, \forall g, h \in H^2 \quad \Gamma(fg, h) = \Gamma(g, fh)$. In other words, if we denote for $f \in A$, by $M_f : H^2 \to H^2$ the operator of multiplication by $f$ and by $M_f' : (H^2)^* \to (H^2)^*$ its adjoint, we have

\begin{equation}
\label{eq:0.2}
\Gamma M_f = ^t M_f \Gamma \quad \forall f \in A.
\end{equation}

Let $H = (H^2)^* \oplus H^2$. For any polynomial $f$ consider the operator $R(f) : H \to H$ defined by the following block matrix:

\begin{equation}
\label{eq:0.3}
R(f) = \begin{pmatrix}
^t M_f & \Gamma M_f' \\
0 & M_f
\end{pmatrix}
\end{equation}

Since the coefficient $f \to \Gamma M_f'$ behaves like a derivation, we easily verify that $f \to R(f)$ is a unital homomorphism on polynomials, i.e. we have $R(1) = 1$, and, by (0.2), for all polynomials $f, g$, we have

\begin{equation}
\label{eq:0.4}
R(fg) = R(f)R(g).
\end{equation}
Hence, under the condition
\begin{equation}
\exists \ C \ \forall \ f \ \text{polynomial} \quad \|\Gamma M_f\| \leq C \|f\|_\infty,
\end{equation}
the mapping \( f \to R(f) \) defines a bounded unital homomorphism on \( A \), or equivalently the operator \( T_\Gamma = R(z) \) (i.e. \( R(\varphi_0) \) for \( \varphi_0 \) defined by \( \varphi_0(z) = z \) for all \( z \)) is polynomially bounded.

By a well known theorem of Nehari (see e.g. [Ni]), each Hankel operator \( \Gamma \) is associated to a symbol \( \varphi \in L^\infty \) so that \( \Gamma = \Gamma_\varphi \). (This corresponds to the case \( \dim(H) = 1 \) in (0.6) below.) We denote simply by \( T_\varphi \) the operator \( T_\Gamma \) with \( \Gamma = \Gamma_\varphi \). In [Pe2], Peller shows that if \( \varphi' \in BMO \) then we have (0.5) and consequently \( T_\varphi \) is polynomially bounded. The question whether this implies “similar to a contraction” was then posed by Peller [Pe2], but Bourgain [Bo2] showed that \( \varphi' \in BMO \) implies that \( T_\varphi \) is similar to a contraction, and very recently (summer 95), Aleksandrov and Peller [AP] showed that actually \( T_\varphi \) is polynomially bounded only if \( \varphi' \in BMO \). In conclusion, there is no counterexample in this class.

However, as is well known (cf. e.g. [Ni]) there is a vectorial version of Hankel operators: given a function \( \varphi \in L^\infty(B(H)) \), we can associate to it an operator (usually called a vectorial Hankel operator) \( \Gamma_\varphi : H^2(H) \to H^2(H)^* \) still satisfying the identity (0.2), with respect to the multiplication operator \( M_f \) considered as acting on \( H^2(H) \). More precisely, we can define for any \( g, h \) in \( H^2(H) \)
\begin{equation}
\Gamma_\varphi(g, h) = \int [\varphi(\xi)g(\xi), h(\xi)]dm(\xi),
\end{equation}
where \([.,.]\) is a bilinear map on \( H \) associated to a fixed isometry between \( H \) and \( H^* \). Thus, defining \( R(f) \) as in (0.3) above, we still have (0.4) and the associated operator \( T_\varphi : H^2(H)^* \oplus H^2(H) \to H^2(H)^* \oplus H^2(H) \) is polynomially bounded provided \( \Gamma = \Gamma_\varphi \) satisfies (0.5).

The main result of this paper is that, when \( H \) is infinite dimensional, there are counterexamples to the Halmos problem of the form \( T = T_\varphi \), with \( \varphi \in L^\infty(B(H)) \).

**Remark.** There is some recent related work (on operators of the form \( T_\Gamma \)) by Stafney [St] and also by S. Petrovic, V. Paulsen and Sarah Ferguson whose work I have heard of, through the seminar talks they gave in Texas at various occasions in 95.
Notation and background. In general, any unexplained notation is standard. Let \( H, K \) be two Hilbert spaces. We denote by \( B(H, K) \) (resp. \( B(H) \)) the space of all bounded linear operators from \( H \) to \( K \) (resp. from \( H \) to \( H \)) equipped with its usual norm. We denote by \( H^* \) the dual of \( H \) which, of course, is a Hilbert space canonically identifiable with the complex conjugate \( \bar{H} \) of \( H \).

We denote by \( m \) the normalized Lebesgue measure on the unit circle \( T \).

Recall that we denote simply by \( A \) the disc algebra. Note that \( A \) is a closed subalgebra of \( H^\infty \). Often, we implicitly consider a function in \( H^\infty \) as extended analytically inside the unit disc, in such a way that we recover the original function on the circle by taking radial (or non-tangential) limits almost everywhere. When the original function on the circle is actually in \( A \), its analytic extension is continuous on \( \bar{D} \).

Consider a function \( f \) in (say) \( L^p(T, m) \) \((1 \leq p \leq \infty)\). When we sometimes abusively say that \( f \) is “analytic”, what we really mean is that \( f \in H^p \), i.e. that \( f \) extends analytically inside \( D \). In that case, throughout this paper, the derivative \( f' \) of \( f \) always means the derivative of the Taylor series of \( f \).

We denote by \( \ell^2_n \) the \( n \)-dimensional Hilbert space, and by \( \ell^2_n(H) \) the (Hilbertian) direct sum of \( n \) copies of \( H \). Moreover, we denote \( M_n = B(\ell^2_n) \).

Let \( H, K \) be Hilbert spaces. Let \( S \subset B(H) \) be a subspace. In the theory of operator algebras, the notion of complete boundedness for a linear map \( u : S \to B(K) \) has been extensively studied recently. Its origin lies in the work of Stinespring (1955) and Arveson (1969) on completely positive maps (see [Pa1, Pi4] for more details and references). Let us equip \( M_n(S) \) and \( M_n(B(K)) \) (the spaces of matrices with entries respectively in \( S \) and \( B(K) \)) with the norm induced respectively by \( B(\ell^2_n(H)) \) and \( B(\ell^2_n(K)) \).

A map \( u : S \to B(K) \) is called completely bounded (in short c.b.) if there is a constant \( C \) such that the maps \( I_{M_n} \otimes u \) are uniformly bounded by \( C \) i.e. if we have

\[
\sup_n \| I_{M_n} \otimes u \|_{M_n(S) \to M_n(B(K))} \leq C,
\]

and the c.b. norm \( \| u \|_{cb} \) is defined as the smallest constant \( C \) for which this holds.

When \( \| u \|_{cb} \leq 1 \), we say that \( u \) is completely contractive (or a complete contraction).

Let \( \mathcal{A} \subset B(H) \) be a unital subalgebra, and let \( \pi : \mathcal{A} \to B(H) \) be a unital homomorphism.
Paulsen ([Pa2]) proved that $\pi$ is completely bounded iff there is an invertible operator $S : H \to H$ such that $S^{-1}\pi(,)S$ is completely contractive.

We now return to polynomially bounded operators. An operator $T : H \to H$ will be called completely polynomially bounded if there is a constant $C$ such that for all $n$ and all $n \times n$ matrices $(P_{ij})$ with polynomial entries we have

\[(0.7) \quad \| (P_{ij}(T))\|_{B(\ell^2_n(H))} \leq C \sup_{z \in T} \| (P_{ij}(z))\|_{M_n}
\]

where $(P_{ij}(T))$ is identified with an operator on $\ell^2_n(H)$ in the natural way. Recall that $M_n$ is identified with $B(\ell^2_n)$. Note that $T$ is completely polynomially bounded iff the homomorphism $P \to P(T)$ defines a completely bounded homomorphism $u_T$ from the disc algebra $A$ into $B(H)$. Here of course we consider $A$ as a subalgebra of the $C^*$-algebra $C(T)$ which itself can be embedded e.g. in $B(L_2(T))$ by identifying a function $f$ in $C(T)$ or $L_\infty(T)$ with the operator of multiplication by $f$ on $L_2(T)$.

We can now state Paulsen’s criterion:

**Theorem 0.1.** ([Pa2]) An operator $T$ in $B(H)$ is similar to a contraction iff it is completely polynomially bounded. Moreover $T$ is completely polynomially bounded with constant $C$ (as in (0.7) above) iff there is an isomorphism $S : H \to H$ such that $\|S\|\|S^{-1}\| \leq C$ and $\|S^{-1}TS\| \leq 1$.

Actually, we only use the easy direction of this criterion, which can be derived as follows from Sz.-Nagy’s well known unitary dilation theorem. Let $T : H \to H$ be a contraction. Sz.-Nagy proved (see [SNF]) that we can find a larger Hilbert space $\hat{H}$ containing $H$ as a subspace, and a unitary operator $U$ on $\hat{H}$, such that

$$\forall n \geq 0 \quad T^n = P_H U^n_{|H}.$$  

(Here $P_H$ denotes the orthogonal projection from $\hat{H}$ to $H$.) Hence, we have $P(T) = P_H P(U)|_H$, for all polynomials $P$. In particular, this implies von Neumann’s inequality (0.1). More generally, for any matrix $[P_{ij}]$ with polynomial entries we have $[P_{ij}(T)] = j^* [P_{ij}(U)] j$ where $j^*$ is the diagonal matrix with diagonal entries all equal to $P_H$. Therefore, $\| [P_{ij}(T)] \| \leq \| [P_{ij}(U)] \| \leq \sup_{z \in T} \| (P_{ij}(z))\|_{M_n}$. 

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The last inequality is a direct consequence of the fact that \( U \) generates a commutative \( C^* \)-subalgebra of \( B(\hat{H}) \), and its spectrum lies in \( T \). Clearly, this implies

\[
\|[P_{ij}(S^{-1}TS)]\| \leq \|S\|\|S^{-1}\| \sup_{z \in T} \|(P_{ij}(z))\|_{M_n},
\]

which proves that similarity to a contraction implies complete polynomial boundedness.

For any linear map \( u : A \rightarrow B(H) \) it is easy to check that, for any \( n \) and for any finitely supported sequence \( (a_k) \) in \( M_n \), we can write:

\[
(0.8) \quad \left\| \sum a_k \otimes u(z^k) \right\|_{B(\ell_n^\infty(H))} \leq \|u\|_{cb} \sup_{z \in T} \left\| \sum a_k z^k \right\|_{M_n}.
\]

This formulation will be used below.

We refer the reader to the books [SNF, Pa1] for more background on dilation Theory, and to the lecture notes volume [Pi4] which describes the “state of the art” on similarity problems until 95.

§1. Main results

Let \( H \) be a Hilbert space. Consider an operator

\[
\Gamma: H^2(H) \rightarrow H^2(H)^*,
\]

or equivalently a bounded bilinear form \( \Gamma: H^2(H) \times H^2(H) \rightarrow \mathbb{C} \). Then \( \Gamma \) is called Hankelian (or a Hankel operator) if for any multiplication operator \( M_\varphi: H^2(H) \rightarrow H^2(H) \) by a polynomial or a function \( \varphi \) in \( A \) (or in \( H^\infty \)), we have

\[
(1.1) \quad \forall g, h \in H^2(H) \quad \Gamma(g\varphi, h) = \Gamma(g, \varphi h).
\]

Equivalently we have \( \Gamma M_\varphi = \Gamma \varphi \Gamma \). Let \( F \rightarrow \mathcal{D}(F) \) be a linear mapping from \( A \) into \( B(H^2(H), H^2(H)^*) \) which is a “derivation” in the following sense:

\[
(1.2) \quad \forall F, G \in A \quad \mathcal{D}(FG) = \Gamma(\mathcal{D}(G) + \mathcal{D}(F)M_G).
\]

Let \( \mathcal{H} = H^2(H)^* \oplus H^2(H) \). Then the mapping \( R : A \rightarrow B(\mathcal{H}) \) defined by

\[
R(F) = \begin{pmatrix}
\Gamma(M_F F) & \mathcal{D}(F) \\
0 & M_F
\end{pmatrix}
\]
clearly is a unital homomorphism.

Now assume in addition that \( \mathcal{D}(F) \) is Hankelian for any \( F \) in \( A \), i.e. that \( \mathcal{D}(F)M_\varphi = \^t M_\varphi \mathcal{D}(F) \) for all \( \varphi \) in \( A \). Then a simple computation shows by induction that

\[
\mathcal{D}(F^2) = 2\mathcal{D}(F)M_F \quad \text{and} \quad \mathcal{D}(F^n) = n\mathcal{D}(F)M_{F^{n-1}}
\]

for all \( n \geq 1 \). Therefore, for any polynomial \( P \) we must have

\[
\mathcal{D}(P(F)) = \mathcal{D}(F)M_{P'(F)},
\]

where \( P' \) is the derived polynomial. Applying this with the function \( F = \varphi_0 \) defined by \( \varphi_0(z) = z \) for all \( z \), we find

\[
\mathcal{D}(P) = \mathcal{D}(\varphi_0)M_{P'}.
\]

So if we let \( \Gamma = \mathcal{D}(\varphi_0) \) we have for all polynomials \( P \)

\[
\mathcal{D}(P) = \Gamma M_{P'}.
\]

Conversely, given any Hankel operator \( \Gamma \), if we define \( \mathcal{D}_\Gamma \) by setting \( \mathcal{D}_\Gamma(P) = \Gamma M_{P'} \), then \( \mathcal{D}_\Gamma(P) \) is Hankelian, satisfies the derivation identity (1.2) for all polynomials \( F, G \), and \( \mathcal{D}_\Gamma(\varphi_0) = \Gamma \). (Thus (1.3) defines a one to one correspondence between \( \Gamma \) and \( \mathcal{D}_\Gamma \).)

Given \( \Gamma \) (and the associated \( \mathcal{D}_\Gamma \)), let \( T_\Gamma : H^2(H)^* \oplus H^2(H) \to H^2(H)^* \oplus H^2(H) \) be the operator defined by the operator matrix

\[
T_\Gamma = \begin{pmatrix}
^t S & \Gamma \\
0 & S
\end{pmatrix}
\]

where \( S : H^2(H) \to H^2(H) \) is the shift operator i.e. \( S = M_\varphi \), and \( ^t S \) denotes its adjoint on the dual space \( H^2(H)^* \). We will show in this paper that, in the vectorial case (with \( \dim(H) = \infty \)), there are p.b. operators of this type which are not similar to a contraction.

The preceding remarks show that, for any polynomial \( P \)

\[
P(T_\Gamma) = \begin{pmatrix}
^t P(S) & \Gamma M_{P'} \\
0 & P(S)
\end{pmatrix}.
\]

Therefore if there is a constant \( C \) such that

\[
\forall P \quad \|\Gamma M_{P'}\| \leq C\|P\|_{\infty}
\]

then \( T_\Gamma \) is polynomially bounded. Our main result is the following:
Theorem 1.1. Let \((C_n)\) be any sequence in \(B(H)\) such that

\[
(1.5) \quad \forall (\alpha_n) \in \ell_2 \quad \left\| \sum \alpha_n C_n \right\|_{B(H)} \leq \left( \sum |\alpha_n|^2 \right)^{1/2},
\]

and let \((K_n)\) be an increasing sequence of positive integers such that for all \(n > 1\)

\[
(1.6) \quad 2^{n-1} < K_n \leq 2^n.
\]

Consider the operator \(u: A \rightarrow B(H)\) defined by \(u(F) = \sum_{n \geq 1} \hat{F}(K_n)C_n\).

Then there is a function \(\varphi \in L^\infty(B(H))\) such that the associated vectorial Hankel operator \(\Gamma = \Gamma_\varphi: H^2(H) \rightarrow H^2(H^*)\) satisfies (0.5) for some constant \(C\) and moreover is such that for any polynomial \(P\) (with derived polynomial denoted by \(P')\):

\[
(1.7) \quad \forall x, y \in H \quad \langle u(P)x, \overline{y} \rangle = \Gamma[P'(1 \otimes x)](1 \otimes y)
\]

where we denote by \(1 \otimes x\) the element of \(H^2(H)\) corresponding to the function taking constantly the value \(x\), and where \(y \rightarrow \overline{y}\) is an antilinear isometry on \(H\).

Corollary 1.2. There is a polynomially bounded operator (of the form \(T_\Gamma\)) which is not similar to a contraction.

Proof. Let \((C_n)\) be a sequence of operators satisfying the CAR ("canonical anticommutation relations"), i.e. such that

\[
\forall i, j \quad C_i C_j + C_j C_i = 0 \quad C_i^* C_j + C_j^* C_i^* = \delta_{ij} I.
\]

It is well known that this implies (1.5) (with equality even), see e.g. [BR, p. 11]. Moreover, if \((C'_1, ..., C'_n)\) is another \(n\)-tuple satisfying the CAR, there is a \(C^*\)-representation taking \((C_1, ..., C_n)\) to \((C'_1, ..., C'_n)\) (see [BR, Th. 5.2.5]), so that for any \(a_1, ..., a_n\) in \(M_N\) (with \(N \geq 1\)) the value of \(\| \sum_{i=1}^n a_i \otimes C_i \|_{B(\ell_2^N(H))}\) does not depend on the particular realization of \((C_1, ..., C_n)\). We denote by \(\bar{x}\) the complex conjugate of a matrix \(x\). Equivalently, if \(x \in B(H)\), we can identify \(\bar{x}\) with \(t x^* : H^* \rightarrow H^*\).

We claim that, with this choice of \((C_n)\), the mapping \(u\) is not completely bounded. To see this last fact (also well known, see e.g. [H, example 3.3]), we first fix \(n\) and recall that there
is (using the Pauli or Clifford matrices, see also [BR, p. 15]) a realization of \((C_1, \ldots, C_n)\) satisfying the CAR in the space of (say) \(2^n \times 2^n\) matrices. Then observe that we have by (0.8)
\[
\left\| \sum_{1}^{n} C_k \otimes \overline{C}_k \right\| = \left\| \sum u(z^{K_k}) \otimes \overline{C}_k \right\| \leq \|u\|_{cb} \sup_{|z|=1} \left\| \sum_{1}^{n} z^{K_k} \otimes \overline{C}_k \right\| = \|u\|_{cb} \sqrt{n}.
\]
However, since we can assume \((C_1, \ldots, C_n)\) realized in the space of \(2^n \times 2^n\) matrices, of which the unit matrix is denoted by \(I\), we have
\[
\left\| \sum_{1}^{n} C_k \otimes \overline{C}_k \right\| \geq \frac{\tr(\sum_{1}^{n} C_k C^*_k)}{\tr I} = \frac{n}{2}.
\]
(To check this, note that
\[
\left\| \sum_{1}^{n} C_k \otimes \overline{C}_k \right\| = \sup \left\| \sum_{1}^{n} \tr(C_k X C^*_k Y) \right\|
\]
where the supremum runs over all \(2^n \times 2^n\) matrices \(X, Y\) with Hilbert-Schmidt norm 1 and take \(X = Y = I(\tr I)^{-1/2}\).) Thus we conclude that \(\|u\|_{cb} \geq \sqrt{n}/2\) for all \(n\), hence \(u\) is not completely bounded. Therefore, if \(\Gamma\) satisfies (1.7), the mapping \(F \rightarrow \Gamma M_{F'}\) cannot be c.b., hence a fortiori \(F \rightarrow F(T_{\Gamma})\) is not c.b. on \(A\), which ensures by Paulsen’s criterion (easy direction) [Pa2] that \(T_{\Gamma}\) is not similar to a contraction.

Remark 1.3. Let \(T_{\Gamma} = \begin{pmatrix} tS & \Gamma \\ 0 & S \end{pmatrix}\) be an operator as in the last corollary. Fix \(\varepsilon > 0\), and consider
\[
T_{\varepsilon\Gamma} = \begin{pmatrix} tS & \varepsilon\Gamma \\ 0 & S \end{pmatrix}.
\]
Then, by (1.4), we have for all polynomials \(P\)
\[
\|P(T_{\varepsilon\Gamma})\| \leq (1 + \varepsilon C)\|P\|_{\infty},
\]
hence we can get an operator with polynomially bounded constant arbitrarily close to 1, but still not similar to a contraction.

Remark 1.4. In the preceding argument for Corollary 1.2, we are implicitly using the ideas behind a joint result of V. Paulsen and the author (included in [Pa3, Th. 4.1]). The latter result shows in particular the following: let \(m = (m(n))_{n \geq 0}\) be a scalar sequence and
let $u_m : A \to B(H)$ be the operator defined by $u_m(F) = \sum_{n \geq 0} \hat{F}(n) C_n m(n)$, with $(C_n)$ satisfying the CAR. Then $u_m$ is c.b. iff $\sum |m(n)|^2 < \infty$.

Let $(m(n))_{n \geq 0}$ be a scalar sequence such that

\begin{equation}
\sup_{n \geq 0} \sum_{2^{n-1} < k \leq 2^n} |m(k)|^2 < \infty.
\end{equation}

It is well known that this condition characterizes the Fourier multipliers from $H^1$ to $H^2$ on the circle (cf. e.g. [D, p. 103]).

With essentially the same arguments as for Theorem 1.1, we can prove more generally

**Theorem 1.5.** Let $(C_n)$ be as in Theorem 1.1. Assume that $(m(n))_{n \geq 0}$ satisfies (1.8) as above. Let $u : A \to B(H)$ be the mapping defined by

\begin{equation}
u(F) = \sum_{n \geq 0} \hat{F}(n) m(n) C_n.
\end{equation}

Then the conclusion of Theorem 1.1 still holds.

See Remark 3.3 below for an indication of proof.

**Remark 1.6.** Let $\varphi \in L^\infty(B(H))$ be the function associated to (1.9) by Theorem 1.5. By (0.6) and (1.7) the Fourier transform of $\varphi$ restricted to non positive integers is determined and we have for all $n \geq 1$

$$C_n m(n) = \int \varphi(e^{it}) ne^{i(n-1)t} dm(e^{it}).$$

Hence as a “symbol” in the sense of Nehari’s theorem as in (0.6), we can take just as well

$$\varphi(e^{it}) = \sum_{n \geq 1} n^{-1} C_n m(n) e^{-i(n-1)t}.$$  

Let $f = \sum_{k \geq 0} \hat{f}(k) z^k \in A$. Let $P_n(f) = \sum_{0 \leq k \leq n} \hat{f}(k) z^k$. Let us denote by $(e_n)$ the canonical basis in $\ell_2$. Then, assuming (1.8) the key inequality (0.5) is equivalent to the following one, perhaps of independent interest: there is a constant $C$ such that for any analytic function $f$ in $BMO$

\begin{equation}
\left\| \sum_{n \geq 1} e_n n^{-1} m(n) e^{-i(n-1)t} P_{n-1}(f') \right\|_{BMO(\ell_2)} \leq C \|f\|_{BMO},
\end{equation}
where we have denoted by $BMO$ (resp. $BMO(\ell_2)$) the classical space of $BMO$ functions on the circle with values in $\mathbb{C}$ (resp. $\ell_2$).

Note that (1.10) is actually proved below using martingale versions of $H^1$ and $BMO$, but the end result can be formulated without reference to martingales as we just did in (1.10).

**Remark 1.7.** Let us examine the converse to Theorem 1.5. Fix a sequence $(C_n)$ for which there is a constant $\delta > 0$ such that

\begin{equation}
\forall \alpha = (\alpha_n) \in \ell_2 \quad \delta (\sum |\alpha_n|^2)^{1/2} \leq \left\| \sum \alpha_n C_n \right\|.
\end{equation}

Let $(m(n))_{n \geq 0}$ be any scalar sequence. Then if the conclusion of Theorem 1.1 holds for the mapping $u$ defined in (1.9), we have necessarily (1.8). This follows from [AP]. Indeed, let $\varphi \in L^\infty(B(H))$ be a function associated to such a $u$ as in Theorem 1.1, with associated Hankel operator satisfying (0.5). For any sequence $(\beta_n)$ in the unit ball of $\ell_2$, we have a linear form $\xi$ in $B(H)^*$ with $\|\xi\|_{B(H)^*} \leq 1/\delta$ such that $\xi(C_n) = \beta_n$, for all $n$. Clearly, the function $\xi(\varphi) \in L^\infty$ defines a Hankel operator $\Gamma_{\xi(\varphi)}$ still satisfying (0.5). By [AP], this implies that $\xi(\varphi')$ is in BMO, or equivalently that $\sum_{n \geq 0} \beta_n m(n) z^n$ is in BMO. Since this holds for all $(\beta_n)$ in $\ell_2$, this means, after transposition, that $(m(n))$ is a (bounded) multiplier from $H^1$ to $\ell_2 \approx H^2$. As already mentioned, this is equivalent to (1.8) (cf. e.g. [D, p. 103]). In particular, a sequence $(K_n)$ satisfies the conclusion of Theorem 1.1 (for any $(C_n)$ satisfying (1.5) and (1.11)) if and only if it is the union of finitely many sequences which are subsequences of a sequence verifying (1.6). Equivalently, iff it is a finite union of Hadamard lacunary sequences.

Of course, our results can also be described in terms of Hankel and Toeplitz matrices. Let $G$ be the Hankel matrix (with operator entries) defined by setting for all $i, j \geq 0$

\begin{equation}
G_{ij} = m(i + j + 1)(i + j + 1)^{-1}C_{i+j+1}.
\end{equation}

For $f \in A$, let us denote by $T(f)$ the Toeplitz matrix (with scalar entries) defined by setting for all $i, j \geq 0$

$$T(f)_{ij} = \hat{f}(i - j).$$
Then Theorem 1.5 can be reformulated as follows: assuming (1.8), there is a constant $C$ such that for all $f$ in $A$ we have

\begin{equation}
\|GT(f')\| \leq C \|f\|_A.
\end{equation}

Note that, actually, the BMO norm of $f$ can be substituted to $\|f\|_A$ in (1.13), see Remark 3.2 below.

To verify (1.12), we simply apply (1.7) to $F(z) = z^n$, to show that the matrix coefficients of $\Gamma$, with respect to the decomposition $H^2(H) = \oplus_{i \geq 0} z^i H$ are given by (1.12). Then (1.13) reduces to (1.4).

§2. Martingales

We will first prove an analogue of Theorem 1.1 with $A$ replaced by its martingale version denoted by $A_m$. Then, in the next section we will deduce Theorem 1.1 from the martingale case.

Consider $\Omega = T^I$ with $I = \{1, 2, 3, \ldots\}$. Let $(Z_i)_{i \geq 1}$ denote the coordinates on $\Omega$ and let $A_n$ be the $\sigma$-algebra generated by $(Z_1, \ldots, Z_n)$ with $A_0$ the trivial $\sigma$-algebra. Every continuous function $f: \Omega \to \mathbb{C}$ defines a martingale $(f_n)_n$ by setting $f_n = E(f | A_n)$.

A martingale $(f_n)_n$, relative to the filtration $(A_n)$, is called “Hardy” if for each $n \geq 1$ the function $f_n$ depends analytically on $Z_n$ (but arbitrarily on $Z_1, \ldots, Z_{n-1}$).

Let $1 \leq p \leq \infty$. We denote by $A_m$ (resp. $H^p_m$) the subspace of $C(\Omega)$ (resp. $L^p(\Omega, P)$) formed by all $F$ which generate a Hardy martingale. As is well known $A_m$ (resp. $H_m^\infty$) is a uniform algebra in $C(\Omega)$ (resp. $L^\infty(\Omega, P)$). Moreover, for all $F, G$ in $H_m^\infty$ we have

\begin{equation}
(FG)_n = F_n G_n \quad \forall n \geq 0.
\end{equation}

In Harmonic Analysis terms, the space $A_m$ (resp. $H^p_m$) is indeed the version of the disc algebra (resp. $H^p$) associated to the ordered group $\mathbb{Z}^{(I)}$ (formed of all the finitely supported families $n = (n_i)_{i \in I}$ with $n_i \in \mathbb{Z}$), ordered by the lexicographic order, i.e. the order defined by setting $n' < n''$ iff the last differing coordinate (=“letter” with reversed alphabetical order) satisfies $n'_i < n''_i$. As explained e.g. in [Ru, Chapter 8] or in [HL], this group
has a “linear” behaviour and the associated $H^p$ spaces on it behave like the classical (unidimensional) ones. Let $P$ be the usual probability measure on $\mathbb{T}$ (= normalized Haar measure). For any Banach space $X$, we will denote by $H^p_m(X)$ ($1 \leq p \leq \infty$) the usual $H^p$-space of $X$-valued functions on the ordered group $\Omega$. Equivalently, for $p < \infty$ $H^p_m(X)$ is the closure of $A_m \otimes X$ in the space $L^p(\Omega, P; X)$ in Bochner’s sense.

Our main goal in this section is the following:

**Theorem 2.1.** There is a bounded unital homomorphism

$$\pi: A_m \longrightarrow B(H)$$

which is not completely bounded.

For each $F$ in $H^\infty_m$ we denote by $(F_n)$ the associated Hardy martingale and we set

$$dF_n = F_n - F_{n-1} \quad \forall n \geq 1, \quad dF_0 = F_0.$$

Let $\eta = (\eta_n)$ be an adapted sequence (i.e. $\eta_n$ is $A_n$-measurable for all $n \geq 0$) of bounded random variables such that there is a constant $C'$ for which:

$$\forall n \geq 0 \quad \|\eta_n\|_\infty \leq C'.$$

Consider $F \in H^\infty_m$. We introduce a mapping

$$\mathcal{D}^\eta(F): H^2_m(H) \longrightarrow H^2_m(H)^*$$

which, viewed as a bilinear map on $H^2_m(H) \times H^2_m(H)$, can be written as $\forall g, h \in H^\infty_m(H)$

$$\mathcal{D}^\eta(F)(g, h) = \mathbb{E}\left( \sum_{n \geq 1} \bar{Z}_n \eta_{n-1} dF_n[C_n g, h] \right)$$

(2.3)

where, for convenience of notation, we use a bilinear pairing on $H \times H (x, y) \rightarrow [x, y]$ on the right side of this definition, so that $Z \rightarrow [C_n g(Z), h(Z)]$ is in $H^1$ when $g, h$ are in $H^2_m(H)$. We assume that the associated map $J: H \rightarrow H^*$ is isometric.

Concerning the issue of the convergence of the series in (2.3), note that

$$\|F\|^2_2 = \sum_{n \geq 0} \mathbb{E}|dF_n|^2,$$

and we take the precaution to assume that $g, h$ belong to $H^\infty_m(H)$ (which is dense in $H^2_m(H)$). Therefore, by (1.5) the series

$$\sum_{n \geq 1} \bar{Z}_n \eta_{n-1} dF_n[C_n g, h]$$

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is convergent in $L^2(\Omega,P)$, so that (2.3) is well defined.

But actually, we will show (see below) that (2.3) defines a bounded operator from $H^2_m(H)$ to its dual, so that (2.3) eventually will make sense for all $g,h$ in $H^2_m(H)$.

We denote by $M_m(F): H^2_m(H) \rightarrow H^2_m(H)$ the operator of multiplication by $F$ when $F$ is in $H^\infty_m$.

The following identity has played a crucial role in guiding us to the present paper. We refer the reader to [LPP, Pi3] for various results related to this “derivation identity”.

(2.4) $\forall F,G \in H^\infty_m$ \quad $\mathcal{D}^\eta(FG) = \mathcal{D}^\eta(F)M_m(G) + tM_m(F)\mathcal{D}^\eta(G)$.

Let $u^\eta: H^\infty_m \rightarrow B(H)$ be the mapping defined by

$$u^\eta(F) = \sum_{n \geq 1} E(dF_n \bar{Z}_n \eta_{n-1}) C_n$$

where $(C_n)$ is a CAR sequence as before. The same argument as in the preceding section shows that, if $\eta$ is suitably chosen, for instance if $\eta_n = 1$ identically for all $n$, $u^\eta$ is not c.b., even when restricted to $A_m$. Therefore, in this case, any homomorphism $\pi$ admitting $u^\eta$ as a “coefficient” a fortiori is not c.b. either.

Moreover we will see that if $V: H \rightarrow H^2_m(H)$ is the mapping defined by $V(x) = 1 \otimes x$, then we have (note that $tV: H^2_m(H)^* \rightarrow H^*$)

(2.5) $\forall F \in H^\infty_m$ \quad $Ju^\eta(F) = tV\mathcal{D}^\eta(F)V$

where $J: H \rightarrow H^*$ is the isometric isometry associated to the bilinear pairing $(x,y) \rightarrow [x,y]$ on $H$. From (2.4) and (2.5), Theorem 2.1 is immediate. Indeed, (2.4) shows that the mapping $\pi^\eta(F) = \begin{pmatrix} tM_m(F) & \mathcal{D}^\eta(F) \\ 0 & M_m(F) \end{pmatrix}$

is a unital homomorphism. By (2.5), $\mathcal{D}^\eta$ is not c.b. if $\eta$ is suitably chosen, hence a fortiori $\pi^\eta$ is not c.b. and we are done.

The proof of (2.5) is immediate, so it remains, to complete the proof, to check (2.4) and prove that $F \rightarrow \mathcal{D}^\eta(F)$ is bounded. The key for (2.4) is the observation that for any $A_{n-1}$-measurable function $\varphi_{n-1}$ and for any pair $F,G$ say in $H^\infty_m$ we have $\forall n \geq 1$

(2.6) $E(\bar{Z}_n dF_n dG_n \varphi_{n-1}) = 0$. 

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Indeed, integrating first in $Z_n$ alone we find 0 since $F_n, G_n$ are analytic in $Z_n$, and $F_{n-1}$ can be obtained from $F_n$ by setting $Z_n = 0$ in $F_n$ (Jensen’s formula). Now, using (2.6) repeatedly it is easy to verify that for all $F$ in $H_m^\infty$ we have
\begin{equation}
D^n(F)(g, h) = E\left(\sum_{n \geq 1} \bar{Z}_n \eta_{n-1} dF_n [C_n g_{n-1}, h_{n-1}]\right).
\end{equation}
Hence, using
\[
d(FG)_n = F_n G_n - F_{n-1} G_{n-1} = dF_n G_{n-1} + F_{n-1} dG_n + dF_n dG_n
\]
and using (2.6) and (2.7) again, we find
\[
D^n(FG)(g, h) = E\left(\sum_{n \geq 1} Z_n \eta_{n-1} dF_n [C_n G_{n-1} g_{n-1}, h_{n-1}]\right)
+ E\left(\sum_{n \geq 1} \bar{Z}_n \eta_{n-1} dG_n [C_n g_{n-1}, F_{n-1} h_{n-1}]\right)
= D^n(F)(Gg, h) + D^n(G)(g, Fh).
\]
This establishes (2.4). We now turn to the boundedness of $F \rightarrow D^n(F)$ on $H_m^\infty$.

Recall that the predual of $B(H)$ can be naturally identified with the projective tensor product $H \bar{\otimes} H$. Therefore, if $g, h$ are in the unit ball of $H_m^2(H)$ then $g \otimes h$ can be viewed as an element of the unit ball of $H_m^1(H \bar{\otimes} H)$. Thus, we can use the easy direction of the “vectorial Nehari Theorem” to argue that
\begin{equation}
\|D^n(F)\| \leq \left\|\sum \bar{Z}_n \eta_{n-1} dF_n C_n\right\|_{H_m^1(H \bar{\otimes} H)^*}.
\end{equation}
Now let $v: \ell_2 \rightarrow B(H)$ be the mapping defined by $v(e_n) = C_n$ which by (1.5) satisfies $\|v\| \leq 1$. Then (2.8) can be estimated through $\ell_2$:
\begin{equation}
\|D^n(F)\| \leq \left\|\sum \bar{Z}_n \eta_{n-1} dF_n e_n\right\|_{H_m^1(\ell_2)^*}.
\end{equation}
On the other hand, let $H_M^1(\ell_2)$ be the space formed of all $(\mathcal{A}_n)$-adapted $\ell_2$-valued martingales $f = (f_n)_{n \geq 0}$ such that $f^* = \sup \|f_n\|$ is integrable. We equip $H_M^1(\ell_2)$ with the norm $f \rightarrow \mathbf{E}(f^*)$. We claim that the natural inclusion
\[
H_m^1(\ell_2) \rightarrow H_M^1(\ell_2)
\]
taking \( f \) to the associated martingale \((f_n)\) is bounded. Indeed, this is entirely classical: let \( f \in H^1_m(\ell_2) \). Since \((f_n)\) is “analytic” (= a Hardy martingale), by Jensen’s inequality the sequence \( (\|f_n\|^p) \) is a submartingale for any \( p > 0 \), then choosing \( p = 1/2 \), the desired result follows from Doob’s maximal inequality in \( L_2 \) (cf. e.g. [Du, p. 307]). This proves the claim. Thus, \( f \to E(f^*) \) is an equivalent norm on \( H^1_m(\ell_2) \). As is well known ([Bu, Du, Ga]), the latter norm is equivalent to the norm \( f \to E \left( \sum_{n \geq 0} \|df_n\|^2 \right)^{1/2} \).

Moreover, the dual of \( H^1_M(\ell_2) \) can be identified with the space \( \text{BMO}_M(\ell_2) \) defined as the space of all martingales \( y = (y_n)_{n \geq 0} \) such that for all \( n \geq 1 \)

\[
\sup_{n \geq 1} \left\| E(\|y - y_{n-1}\|^2 \mid A_n) \right\| \leq \infty
\]

equipped with the “norm” (up to an additive constant)

\[
\|\|y\|| = \left( \sup_{n \geq 1} \left\| E(\|y - y_{n-1}\|^2 \mid A_n) \right\| \right)^{1/2}
\]

It follows from these remarks that there are numerical constants \( K' \) and \( K \) such that if we denote by \((e_n)\) the canonical basis of \( \ell_2 \) and if we set \( y = \sum \tilde{Z}_n \eta_{n-1} dF_n e_n \) then we have

\[
\|y\|_{H^1_m(\ell_2)^*} \leq K' \|y\|_{H^1_m(\ell_2)^*} \leq K [\|y\| + (E\|y\|^2)^{1/2}]
\]

But by a simple computation we find (recalling (2.2) and denoting \( E_n \) for \( E( \cdot \mid A_n) \))

\[
E_n \|y - y_{n-1}\|^2 = E_n \left\| \sum_{k \geq n} e_k (\tilde{Z}_k \eta_{k-1} dF_k - E_{n-1} (\tilde{Z}_k \eta_{k-1} dF_k)) \right\|^2 \leq 2C' \sum_{k \geq n} |dF_k|^2 + E_{n-1} \|dF_k\|^2 \leq 2C' (E_n |F - F_{n-1}|^2 + E_{n-1} |F - F_{n-1}|^2) \leq 4C' \|F\| \leq 16C' \|F\|_\infty.
\]

Thus we obtain

\[
\|\|y\|| \leq 2C' \|\|F\|| \leq 4C' \|F\|_\infty = 4C' \|F\|_{H^1_m}.
\]
similarly we have \((\mathbb{E}\|y\|^2)^{1/2} \leq C\|F\|_2\) and by (2.9) and (2.10) we conclude

\[
(2.12) \quad \|\mathcal{D}^n(F)\| \leq 5C'K\|F\|_{H^\infty_m}.
\]

This ends the proof of Theorem 2.1.

Recapitulating, we have actually proved the following statement.

**Theorem 2.2.** Let \((\eta_n)\) be an adapted sequence satisfying (2.2). Let

\[
\mathcal{D}^n : H^\infty_m \to B(H^2_m(H), H^2_m(H)^*)
\]

be the mapping defined by (2.3). Then \(\mathcal{D}^n\) is a bounded linear operator satisfying the identities (2.4) and (2.5). Moreover, for each \(F\) in \(H^\infty_m\), \(\mathcal{D}^n(F)\) is “Hankelian”, meaning it satisfies the following identity

\[
(2.13) \quad \forall G \in H^\infty_m \quad \mathcal{D}^n(F) M_m(G) = t M_m(G) \mathcal{D}^n(F).
\]

**Proof.** Everything has already been explicitly proved, except (2.13) which follows immediately from the definition (2.3) of \(\mathcal{D}^n(F)\).

**Remarks.** The fact that the duality \(H^1, BMO\) for martingales (see the classical reference [FS]) extends to the Hilbert space valued case is a well known fact. The proof given in the first pages of Garsia’s book [Ga] extends almost verbatim. See also [Du, Pet].

Incidentally, the fact that the vectorial Nehari Theorem is valid in this setting is also known (perhaps not so “well”). All the ingredients for this are available. Indeed, it suffices to know that any positive matrix valued function \(W: \Omega \to M_n\) such that \(W \geq \varepsilon I\) for some \(\varepsilon > 0\) admits a factorization of the form \(W = F^*F\) with \(F: \Omega \to M_n\) analytic and “outer”. The latter can be proved as in the case of the disc, see [HL].

In the disc case, the classical references for this are [Sa] and [Pag]. Concerning vectorial Hankel operators, see also [Pe3], and consult [Tr] for more recent refinements. Finally, we also refer to [Pi2] for Banach space versions of Nehari’s theorem.

**Remark 2.3.** Although it is not needed to understand the sequel, we should point out that the arguments in the next section are based on properties of Brownian motion, following
a well known avenue in Probability Theory. Let $0 \leq r_0 \leq \ldots \leq r_{n-1} < r_n < 1$ be an increasing sequence of radii with $r_n \to 1$ when $n \to \infty$.

Let $(B_t)_{t>0}$ be the standard complex Brownian motion starting from the origin and let 

$$T_n = \inf \{ t > 0 \mid |B_t| = r_n \}$$

be the usual stopping time. We denote by $T = T_\infty$ the exit time from the unit disc and by $(A_t)_{t>0}$ the Brownian filtration. Let $F_n = A_{T_n}$ for all $n \geq 0$. Consider a function $F$ in $H^\infty$ and extend it analytically inside $D$. Then the martingale $(\psi_n)$ defined below is identical in distribution with $B_{T_n}$ and $F(\psi_n)$ coincides in distribution with $F(B_{T_n})$. Note that $B_{T_{n-1}}$ is uniformly distributed on $\{ z \mid |z| = r_{n-1} \}$. Moreover for any fixed $z$ with $|z| = r_{n-1}$ the distribution of $B_{T_n}$ conditional to $B_{T_{n-1}} = z$, is given by a homothetic of the Poisson kernel. Thus, the formulae that we are using below are entirely classical, they relate the Poisson kernels (and the M"obius transformations of the unit disc) with the distributions (or certain conditional distributions) of the random variables $B_{T_n}$. We refer the reader e.g. to [Du, Pet] or to the article [Bu] for more information.

§3. Proof of Theorem 1.1

In this section, we deduce Theorem 1.1 from its martingale counterpart Theorem 2.2. the idea for this deduction is quite simple: we introduce a specific element $\psi$ in $H^\infty_m$ with associated Hardy martingale $(\psi_n)_{n \geq 0}$ such that for each $n \geq 1$

$$\forall Z \in T \quad |\psi_n(Z)| = 1 - 2^{-n}.$$ 

Then we consider the unital homomorphism

$$\sigma : A \to H^\infty_m$$ 

defined, for $F$ in $A$, by the composition

$$\sigma(F) = F(\psi).$$ 

Actually, we can also give a meaning to this definition for $F$ in $H^\infty$: in that case we first extend $F$ inside the disc, then we consider the uniformly bounded martingale $(F(\psi_n))$ and
we define $\sigma(F)$ as the weak-$\ast$ limit in $L^\infty(\Omega, P)$ of the sequence $(F(\psi_n))$. When $F$ is in $A$, we recover $F(\psi)$ since (by, say, the martingale convergence theorem) $\psi_n$ tends to $\psi$ almost surely.

In addition, $\psi$ will be chosen uniformly distributed over the circle, so that $\sigma$ extends by density to an isometric embedding

$$
\sigma_2 : H^2(H) \to H^2_m(H).
$$

Moreover, we will find an adapted sequence $(\eta_n)_{n \geq 1}$ satisfying (2.2) and such that the mappings $u$ and $u^\eta, D^\eta$ defined respectively in sections 1 and 2 satisfy for all polynomial $F$ in $A$

(3.1) $D^\eta(\sigma(F)) = D^\eta(\psi)M_m((\sigma(F'))),$

(3.2) $u(F) = u^\eta(\sigma(F)).$

Then, by (2.5) and (2.12), the mapping $\Gamma : H^2(H) \to H^2(H)^*$ defined by

$$
\Gamma = \sigma_2 D^\eta(\psi)\sigma_2
$$

satisfies all the properties listed in Theorem 1.1.

Consider first the disc at the origin. For any analytic function $F$ in $A$ (considered as extended analytically inside $D$), we can write

$$
F(0) = \int_{\partial D} F(\xi)dm(\xi) \quad \text{and} \quad F'(0) = \int_{\partial D} \xi[F(\xi) - F(0)]dm(\xi)
$$

where $dm(\xi)$ is normalized Lebesgue measure on $\partial D$. The Möbius transformations are usually denoted for any $z$ in $D$ by $\zeta \to \varphi_z(\zeta)$. For simplicity, we set

$$
\forall z \in D, \forall \zeta \in T \quad \Phi(z, \zeta) = \varphi_z(\zeta) = \frac{\zeta + z}{1 + \overline{z}\zeta}.
$$

Now consider $0 \leq r < s < 1$ and $z$ with $|z| = r$. After a suitable change of variables, we can rewrite the preceding formulae as follows:

(3.3) $F(z) = \int F(s\Phi(z/s, \xi))dm(\xi)$
and

\[ F'(z)s(1 - \frac{|z|^2}{s^2}) = \int \bar{\xi}[F(s\Phi(z/s, \xi)) - F(z)]dm(\xi). \]  

For all \( n \geq 0 \), let

\[ r_n = 1 - 2^{-n}, \]

so that \( r_n < 1 \) and \( r_n \to 1 \). We now define by induction a sequence of functions \( (\psi_n) \) on \( \Omega = T' \). We set first \( \psi_0 = 0 \) and, for all \( n \geq 1 \) and all \( Z = (Z_n) \in \Omega \)

\[ \psi_n(Z) = r_n \Phi(r_n^{-1}\psi_{n-1}(Z), Z_n). \]

Note that \( \psi_n \) depends only on \( Z_1, ..., Z_n \), so that we can write (slightly abusively) \( \psi_n(Z) = \psi_n(Z_1, ..., Z_n) \). Note that for any \( Z \) in \( \Omega \), the function \( z \to \psi_n(Z_1, ..., Z_n, z) \) extends to an analytic function in a neighborhood of \( \bar{D} \) [namely \( z \to r_n \Phi(r_n^{-1}\psi_{n-1}(z), z) \)] such that \( \psi_n(Z_1, ..., Z_n, 0) = \psi_{n-1}(Z_1, ..., Z_{n-1}) \). Therefore \( (\psi_n)_{n \geq 0} \) is a Hardy martingale defining an element \( \psi \) in \( H^\infty_m \). Moreover since \( |\psi_n(Z)| \equiv r_n \), we have \( \|\psi\|_{H^\infty_m} = 1 \). More

generally, for any \( F \) in \( A \), we have by (3.3) (=by Jensen’s formula)

\[ \int F(\psi_n(Z))dm(Z_n) = F(\psi_n(Z_1, ..., Z_{n-1}, 0)) = F(\psi_{n-1}(Z)). \]

Therefore, \( (F \circ \psi_n)_{n \geq 0} \) is a bounded Hardy martingale so that \( F \to \sigma(F) = F \circ \psi \) is a
unital homomorphism from \( A \) to \( H^\infty_m \). Consider \( F \) as extended to an analytic function on \( D \). Integrating (3.5) we obtain

\[ \int F(\psi_n)dP = \int F(\psi_{n-1})dP = ... = F(0) \]

hence

\[ \int F(\psi_n)dP = \int F(r_n\xi)dm(\xi). \]

The latter identity ensures (since it remains true for all harmonic functions in a neighborhood of \( \bar{D} \)) that the random variable \( \psi_n \) is uniformly distributed over the circle \( \{z \mid |z| = r_n\} \). Since \( \psi_n \) tends a.s. to \( \psi \), \( \psi \) is uniformly distributed over the unit circle.
Then (3.4) yields for all $n \geq 1$

$$
F'(\psi_{n-1}(Z)) \frac{r_n^2 - r_{n-1}^2}{r_n} = \int \bar{Z}_n[F(\psi_n(Z)) - F(\psi_{n-1}(Z))]dm(Z_n).
$$

We have clearly for all $r < 1$ and for all $m \geq 0$

$$
r^{m-1}m\hat{F}(m) = \int e^{-i(m-1)t}F'(re^{it})dm(e^{it})
$$

where $F = \sum_{n \geq 0} \hat{F}(n)z^n$ is an arbitrary function in $A$. Taking $m = K_n$ and $r = r_{n-1}$ we obtain

$$(r_{n-1})^{K_n-1}K_n\hat{F}(K_n) = \int \xi^{K_n-1}F'(r_{n-1}\xi)dm(\xi)
$$

hence by (3.6)

$$
= \int [r_{n-1}\bar{\psi}_{n-1}]^{K_n-1}F'(\psi_{n-1})dP.
$$

Let $\xi_{n-1} = [r_{n-1}\bar{\psi}_{n-1}]$. Note that $|\xi_{n-1}| = 1$. After multiplication of both sides of (3.7) by $\xi_{n-1}^{K_n-1}$ and integration with respect to $dm(Z_1)\ldots dm(Z_{n-1})$, we obtain for all $n > 1$

$$
\hat{F}(K_n) = \int \eta_{n-1}\bar{Z}_n[F(\psi_n(Z)) - F(\psi_{n-1}(Z))]dP(Z)
$$

with

$$
\eta_{n-1} = \xi_{n-1}^{K_n-1} \frac{r_n}{r_n^2 - r_{n-1}^2} \frac{1}{K_n(r_{n-1})^{K_n-1}}.
$$

The first terms being irrelevant, we may as well assume that $K_1 = 1$, so that taking $\eta_0 = 1$ identically, we still have (3.8) for $n = 1$. Since we assume $2^{n-1} < K_n \leq 2^n$ for $n > 1$, we clearly have (2.2) for some numerical constant $C''$.

Now, since the distribution of $\psi_n$ is uniform over the circle of radius $r_n$, we clearly have, for all $F$ in $A$

$$
\|F\|_{H^2} = \lim_{r \to 1} \uparrow (\int |F(r\xi)|^2 dm(\xi))^{1/2} = \sup_n \|F(\psi_n)\|_2 = \|F \circ \psi\|_{H^2},
$$

so that $\sigma_2$ as defined above is an isometry. Moreover, (3.8) immediately implies (3.2).

Finally, we verify (3.1). This is now easy. Indeed, by (2.13) we know that, for each $F$ in $A$, $D(F)$ as defined above satisfies (1.1) (i.e. it is Hankelian), therefore by the remarks
preceding Theorem 1.1, we have necessarily (3.1) for any polynomial $F$. This completes the proof of Theorem 1.1.

**Remark 3.1.** Let $F$ be an analytic function on $D$ with boundary values in the classical space of BMO over the circle. Then it is well known that the martingale $F(\psi_n)$ is in the space $BMO_m$ considered in the preceding section and (with suitable choice of norms) we have $\|F(\psi)\|_{BMO_m} \leq \|F\|_{BMO}$. Therefore, we actually have proved Theorem 1.1 with (0.5) replaced by the following stronger inequality

$$(0.5)' \quad \exists C \quad \forall f \text{ polynomial} \quad \|\Gamma f\| \leq C |||f|||$$

where (say)

$$|||f||| = \sup_{z \in D} (\int |f(\zeta) - f(z)|^2 P^z(d\zeta))^{1/2}.$$

**Remark 3.2.** To prove Theorem 1.5, let us define

$$\eta_{n-1,k} = \zeta_{n-1} \cdot \frac{r_n}{r_n^2 - r_{n-1}^2} \cdot \frac{1}{k(r_{n-1})^{k-1}}.$$

Note that there is a constant $C'$ such that $\|\eta_{n-1,k}\|_\infty \leq C'$ for all $n$ and all $k$ with $2^{n-1} < k \leq 2^n$. Moreover, we have for all $F$ in $A$

$$\hat{F}(k) = \int \eta_{n-1,k} \tilde{Z}_n [F \psi_n(Z) - F \psi_{n-1}(Z)]dP(Z).$$

We can now introduce the following modified version of $D^n$: for all $g, h$ in $H^2_m(H)$

$$D(F)(g, h) = E \left( \sum_{n \geq 1} \tilde{Z}_n dF_n \sum_{2^{n-1} < k \leq 2^n} m(k) \eta_{n-1,k} [C_k g, h] \right).$$

Then, the same argument as above yields Theorem 1.5.

**Remark 3.3.** There is a well known “dictionary” between the classical theory of $H^1$ and BMO for the disc and the corresponding theory for Brownian martingales. Although I did not see it, it seemed very likely that there should be a way to prove that the operator $\Gamma$ appearing in Theorem 1.1 satisfies (0.5) using only the classical theory, and effectively this has recently been done by S. Kisliakov (personal communication).
In the opposite direction, it is easy to modify the approach of §2 to exhibit an “Ito-Clifford” integral (cf. [BSW]) with the properties of Theorem 1.1.

§4. Operator space interpretations

In this section, we give various consequences of the previous results for operator spaces or (non self-adjoint) operator algebras. In the theory of operator spaces, the relevant morphisms are completely bounded maps and the corresponding isomorphisms are called complete isomorphisms. See e.g. [BP, ER, Pa1, Pi4].

We start by an operator space theoretic reformulation of Theorem 1.1. Fix a number $c > 1$. Let $[A]_c$ be the disc algebra equipped with the operator space (actually operator algebra) structure induced by the embedding $A \subset \bigoplus_{\pi \in I(c)} B(H_\pi)$ taking $a$ to $\oplus_{\pi \in I(c)} \pi(a)$, where $I(c)$ is the class of all unital homomorphisms $\pi: A \rightarrow B(H_\pi)$ with $\|\pi\| \leq c$. Note that, as Banach algebras $[A]_c$ and $A$ are the same with equivalent norms, namely we have $\|a\|_A \leq \|a\|_{[A]_c} \leq c\|a\|_A$ for all $a$ in $A$. However, as operator spaces, they are quite different. As we will see this is a consequence of Theorem 1.1 (and, by Theorem 0.1, it is is equivalent to Corollary 1.2).

Indeed, consider the operator space max($\ell_2$) in the sense of [BP]. The latter can be defined as follows. Let $I$ be the class of all linear maps $v: \ell_2 \rightarrow B(H_v)$ with $\|v\| \leq 1$. Let $J: \ell_2 \rightarrow \bigoplus_{v \in I} B(H_v)$ be the isometric embedding defined by $J(x) = \oplus_{v \in I} v(x)$. Then the operator space max($\ell_2$) can be defined as the range of $J$. Clearly, by Theorem 1.1, the mapping $P: [A]_c \rightarrow \text{max}(\ell_2)$ defined by $PF = (\hat{F}(K_n))_{n \geq 1}$ is completely bounded (see the proof of Theorem 4.1 below for details). Moreover, cf. e.g. [Pi1, p. 69] $P$ is surjective.

Similarly, replacing $A$ by $A_m$, we can define the operator algebra $[A_m]_c$. Let us denote by $P_m: A_m \rightarrow \ell_2$ the mapping defined by

$$\forall F \in A_m \quad P_m(F) = \sum_{n \geq 1} E(dF_n Z_n) e_n.$$
In other words $P_m$ is essentially the same as $u^0$ when all $\eta_n$’s are identically equal to 1. Then, it is easy to check (by the same argument as above) that $P_m$ is a bounded surjection of $A_m$ onto $\ell_2$. As observed in §2, it is not completely bounded from $A_m$ to $\max(\ell_2)$, but Theorem 2.2 implies that it is completely bounded as a map from $[A_m]_c$ to $\max(\ell_2)$. More generally, as an immediate consequence of Theorem 1.5, we have

**Theorem 4.1.** Let $c > 1$. Let $(m(n))_{n \geq 0}$ be any scalar sequence. If $(m(n))_{n \geq 0}$ satisfies (1.8), then the mapping $M : [A]_c \to \max(\ell_2)$ defined (for $F \in A$) by $M(F) = (m(n)\hat{F}(n))$ is completely bounded. However, if we view $M$ as acting from $A$ to $\max(\ell_2)$, then it is completely bounded iff $\sum |m(n)|^2 < \infty$.

**Proof.** It suffices to show that for some constant $C$, we have for any $v \in I$, $\|vM\|_{cb([A]_c,B(H_v))} \leq C$. Let $C_n = v(e_n)$. Note that (1.5) holds, so that Theorem 1.1 and Remark 1.3 imply that there is a homomorphism $\pi \in I(c)$ and operators $V_1, V_2$ so that we can write, for all $F$ in $A$

$$vM(F) = V_1 \pi(F)V_2.$$  

Moreover, we can achieve this with $\|V_1\|\|V_2\| \leq K_c$ for some constant $K_c$ depending only on $c$. This implies $\|vM\|_{cb([A]_c,B(H_v))} \leq \|V_1\|\|V_2\| \leq K_c$, hence $\|M\|_{cb([A]_c,\max(\ell_2))} \leq K_c$. This proves the first part. The second one follows from a joint result of V. Paulsen and the author (cf. [Pa3, Th. 4.1]).

**Corollary 4.2.** We have for any $c > 1$ complete isomorphisms

$$\frac{[A]}{\ker P} \simeq \max(\ell_2) \quad \text{and} \quad \frac{[A_m]}{\ker P_m} \simeq \max(\ell_2).$$

**Proof.** Indeed, $[A]_c/\ker (P) \approx \ell_2$ as Banach spaces, so that the complete boundedness of the mapping $\max(\ell_2) \to [A]_c/\ker(P)$ is obvious by maximality. For the inverse mapping, the complete boundedness follows immediately from the preceding statement. The case of $A_m$ is analogous, we skip the details.

**Remark.** The preceding two statements remain valid with $H^\infty$ and $H^\infty_m$ in the place of $A$ and $A_m$. 

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Corollary 4.3. When $c > 1$, the operator spaces $[A]_c$ and $[A_m]_c$ are not exact (in the sense of e.g. [JP]).

Proof. Indeed, by [BP] we know that $\max(\ell_2)^* \cong \text{an operator subspace of a commutative } C^*\text{-algebra (namely with min}(\ell_2))$. Hence, by [JP, Corollary 1.7], if $[A]_c$ was exact, the mapping $tP : \max(\ell_2)^* \to ([A]_c)^*$ would be 2-summing, which is absurd since it is an isomorphism on $\ell_2$.

Remark. Let $P^d_c$ be the subspace of $[A]_c$ spanned by the polynomials of degree at most $d$. With the notation of [JP], the preceding argument shows that for some absolute constant $\delta > 0$ we have for all $c > 1$ and all $d$

$$\delta(c - 1)\sqrt{\ln(d)} \leq d_{SK}(P^d_c).$$

In [Bo2], Bourgain proves an upper estimate for polynomially bounded $n \times n$ matrices, which we will now be able to bound from below. But first we consider another parameter (related to Theorem 0.1) which we will estimate rather precisely. For any $n$, we will denote by $f(c,n)$ the norm of the identity mapping from $M_n(A)$ to $M_n([A]_c)$. Equivalently, we have

$$f(c,n) = \sup \|(P_{ij}(T))\|_{M_n(B(H))}$$

where the supremum runs over all polynomially bounded operators $T$ with $\|u_T\| \leq c$ and over all $n \times n$ matrices $(P_{ij})$ with polynomial entries such that $\sup_{z \in T} \|(P_{ij}(z))\|_{M_n} \leq 1$.

Theorem 4.4. There is an absolute constant $K > 0$ such that for all $c > 1$ and all integers $n$ we have

$$K^{-1}(c - 1)\sqrt{n} \leq f(c,n) \leq Kc\sqrt{n}.$$ 

Proof. Let $(a_{ij})$ be an $n \times n$ matrix with entries in $B(H)$. It is easy to check that

$$\|(a_{ij})\|_{B(\ell_2^2(H))} \leq \sqrt{n} \sup_i \|(\sum_j a_{ij}a^*_{ij})^{1/2}\|_{B(H)}.$$

Let $(P_{ij})$ be an $n \times n$ matrix of polynomials as above such that $\sup_{z \in T} \|(P_{ij}(z))\|_{M_n} \leq 1$. Then, for each $i$ and any $z$ in $T$, we have $\sum_j |P_{ij}(z)|^2 \leq 1$. By a result due to Bourgain
(see (21) in [Bo 2], this result uses [Bo1, Th. 2.2]) there is a numerical constant $K_1$ such
that for each $i$

$$\|(\sum_j P_{ij}(T)P_{ij}(T)^*)^{1/2}\| \leq K_1 c.$$ 

Hence by (4.2) we conclude that

$$f(c, n) \leq K_1 c \sqrt{n}.$$ 

To prove the converse direction in (4.1), we will use the following known fact on random
matrices (the idea to use this in this context comes from M. Junge’s unpublished indepen-
dent proof of the already mentioned result from [Pa3]): there is a numerical constant $K_2$
such that for each $n$ there is an $n$-tuple of unitary matrices $U_1, ..., U_n$ in $M_n$ such that

$$∀ (\alpha_i) \in \ell_2^n \quad \|\sum_{i=1}^n \alpha_i U_i\|_{M_n} \leq K_2 \left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2}. \quad (4.3)$$

For a proof see e.g. [TJ, p. 323]. Let $C_i = (K_2)^{-1} U_i$ for $i = 1, 2, ..., n$ and $C_i = 0$ for
$i > n$. Then (1.5) is satisfied, but on the other hand since the $U_i$’s are (finite dimensional)
unitaries, we have $\|\sum_1^n U_i \otimes \overline{U_i}\| = n$ hence $\|\sum_1^n C_i \otimes \overline{C_i}\| = (K_2)^{-2} n$.
Let $u : A \to M_n$ be the mapping associated to $(C_i)$ as in Theorem 1.1. Then by Theorem
1.1 and Remark 1.3, there is, for some numerical constant $K_3$, a polynomially bounded
operator $T$ with $\|u_T\| \leq c$ and operators $V_1, V_2$ with $\|V_1\| \|V_2\| \leq K_3 (c - 1)^{-1}$ such that,
for any polynomial $P$, we have

$$u(P) = V_1 P(T) V_2.$$ 

On the other hand, arguing as in the proof of Corollary 1.2 we find that there is a matrix
$(P_{ij})$ with $\sup_{z \in T} \|(P_{ij}(z))\|_{M_n} \leq 1$, such that

$$\|(I \otimes u)(P_{ij})\|_{M_n(M_n)} \geq K_2^{-2} \sqrt{n}.$$ 

Hence this implies

$$\|(P_{ij}(T))\|_{B(\ell_2^2(H))} \geq K_3^{-1} K_2^{-2} \sqrt{n}(c - 1),$$

which yields the left side of (4.1) for a suitable constant $K$. ■
In [B2], Bourgain proves that if \( T \in M_N \) satisfies \( \|u_T\| \leq c \) there is an invertible \( S \in M_N \) for which \( \|S^{-1}TS\| \leq 1 \) and satisfying

\[
\|S^{-1}\|\|S\| \leq Kc^4 \ln(N + 1)
\]

for some absolute constant \( K \) (independent of \( N \) or \( c \)). By Theorem 0.1, this is equivalent to

\[
\|u_T\|_{cb} \leq Kc^4 \ln(N + 1).
\]

It is unclear how sharp this estimate is asymptotically. However, as a simple direct consequence of Theorem 1.1, we have

**Theorem 4.5.** There is a constant \( \delta > 0 \) with the following property: for any \( N \) and \( c > 1 \), there is \( T \in M_N \) polynomially bounded with \( \|u_T\| \leq c \) such that any \( S \) invertible in \( M_N \) with \( \|S^{-1}TS\| \leq 1 \) must satisfy

\[
\delta(c - 1)\sqrt{\ln(N + 1)} \leq \|S^{-1}\|\|S\|.
\]

**Proof.** Fix an integer \( n \) and let \( C_i \) be as in the preceding proof with \( \dim(H) = n \) and \( C_i = 0 \) for all \( i > n \). Let \( \Gamma = \Gamma_\varphi \) be as in Theorem 1.1. For any polynomial \( F \), we again let \( \mathcal{D}(F) = \Gamma_\varphi M_{F''} \). Note that by Remark 1.6, we have \( \hat{\varphi}(k) = 0 \) for all \( k \leq -2^n \).

Observe that \( \mathcal{D}(F) = \Gamma_\varphi M_{F''} = \Gamma_\varphi F'' \), so that \( \mathcal{D}(F) = 0 \) if \( F \in z^{2^n+1}A \).

Let us denote \( K = H^2(H)/z^{2^n+1}H^2(H) \) (or equivalently \( K = H^2(H) \oplus z^{2^n+1}H^2(H) \)) and for any \( F \) in \( A \) let \( Q_F : K \to K \) be the compression of \( M_F \) to \( K \). The observation immediately preceding implies further that, for any polynomial \( F \), \( \mathcal{D}(F) \) vanishes on both \( H^2(H) \times z^{2^n+1}H^2(H) \) and \( z^{2^n+1}H^2(H) \times H^2(H) \). Therefore, \( \mathcal{D}(F) \) defines unambiguously a linear map \( \Delta(F) : K \to K^* \) satisfying \( \|\Delta(F)\| \leq \|\mathcal{D}(F)\| \) hence (by Theorem 1.1) we have for any polynomial \( F \)

\[
\|\Delta(F)\| \leq C\|F\|_{\infty}.
\]

Finally, let \( \mathcal{H} = K^* \oplus K \), let \( \varepsilon = C^{-1}(c - 1) \) and let \( r(F) : \mathcal{H} \to \mathcal{H} \) be defined by

\[
r(F) = \begin{pmatrix}
tQ_F & \varepsilon\Delta(F) \\
0 & Q_F
\end{pmatrix}
\]

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Then, it is easy to check that $F \to r(F)$ is a bounded homomorphism with $\|r\| \leq c$ for which there are contractions $V_1$ and $V_2$ such that $V_1 r(z^{2k}) V_2 = \varepsilon C_k$ for all $k = 1, 2, \ldots, n$. By the same argument as in the preceding proof, we know that this implies

\begin{align}
(4.4) \quad \|r\|_{cb} \geq \varepsilon (K_2)^{-2} \sqrt{n}.
\end{align}

But on the other hand we have $\dim(H) = 2 \dim(K) = 2(2^n + 1) \dim(H) = 2(2^n + 1)n$ so that $n \approx \ln(\dim(H))$, which shows that (4.4) yields the announced result, modulo Paulsen’s criterion (cf. Theorem 0.1). □

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