Lattice QCD at finite isospin density at zero and finite temperature.

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Abstract

We simulate lattice QCD with dynamical $u$ and $d$ quarks at finite chemical potential, $\mu_I$, for the third component of isospin ($I_3$), at both zero and at finite temperature. At zero temperature there is some $\mu_I$, $\mu_c$ say, above which $I_3$ and parity are spontaneously broken by a charged pion condensate. This is in qualitative agreement with the prediction of effective (chiral) Lagrangians which also predict $\mu_c = m_\pi$. This transition appears to be second order, with scaling properties consistent with the mean-field predictions of such effective Lagrangian models. We have also studied the restoration of $I_3$ symmetry at high temperature for $\mu_I > \mu_c$. For $\mu_I$ sufficiently large, this finite temperature phase transition appears to be first order. As $\mu_I$ is decreased it becomes second order connecting continuously with the zero temperature transition.
I. INTRODUCTION

Neutron stars are made of dense cold nuclear matter – hadronic matter at high baryon-number density and low temperature. Large nuclei can be considered as droplets of nuclear matter. The relativistic heavy-ion collisions now being observed at RHIC and the CERN heavy-ion program can produce hadronic matter at high temperature and finite baryon-number density. Nuclear matter also has a finite (negative) isospin ($I_3$) density due to Coulomb interactions, and it has been suggested that at very high densities it would have a finite strangeness density. Hence it is of interest to study QCD at finite quark/baryon-number density, finite isospin density and finite strangeness density at both zero and finite temperature.

Finite density is customarily studied through the introduction of a chemical potential for the charge of interest in the action. Introducing a finite chemical potential for quark-number leads to a complex fermion determinant with a real part of indefinite sign, which precludes use of standard simulation methods which rely on importance sampling. If we include a finite chemical potential, $\mu_I$, for $I_3$ in the absence of any quark-number chemical potential, the fermion determinant remains non-negative, and simulations are possible. Such simulations can determine the QCD phase structure on one surface in the phase diagram for nuclear matter. One can hope that this will identify phases which will persist to finite baryon/quark-number density, and determine their properties.

We have performed simulations of lattice QCD with 2 flavours of light staggered quarks at finite $\mu_I$ for both zero and finite temperatures. Preliminary results of these simulations were reported at Lattice2001 [1]. We included a small explicit $I_3$-breaking interaction needed to observe spontaneous symmetry breaking on a finite lattice. In addition to allowing us to observe spontaneous breaking of $I_3$ (and parity), this term renders the fermion determinant strictly positive. Our zero temperature simulations were performed on an $8^4$ lattice at an intermediate value of the coupling constant. In the limit that our symmetry-breaking parameter vanishes, there is some $\mu_I = \mu_c$ above which $I_3$ and parity are broken spontaneously by a charged pion condensate. This is in accord with the predictions of Son and Stephanov using effective (chiral) Lagrangians [2]. We observe critical scaling consistent with the mean-field predictions of these effective (chiral) Lagrangians. In drawing these
conclusions, it is important to use equations of state of the forms predicted by such effective Lagrangians. These results are similar to those obtained in studies of the quenched version of this theory \[3\]. They are also similar to what is observed for 2-colour QCD at finite chemical potential \(\mu\) for quark-number \([4, 5, 6, 7]\). This is not surprising since the effective Lagrangian analysis of 2-colour QCD at finite \(\mu_I\) \([8]\) is similar to that for QCD at finite \(\mu_I\).

Our finite temperature simulations were performed on \(8^3 \times 4\) lattices. At sufficiently high temperature and \(\mu_I > \mu_c\), we observe the evaporation of the symmetry-breaking pion condensate. For \(\mu_I\) sufficiently large, this transition is first order. As \(\mu_I \to \mu_c\) this transition softens and appears to become second order. Such a transition from first to second order should occur at a tricritical point. Again these results are similar to what we observed for 2-colour QCD at finite quark-number chemical potential \([9]\).

Section 2 gives details of the actions and their symmetries. In section 3 we present our zero temperature results and scaling analyses. The finite temperature results are presented in section 4. Discussions, conclusions and an outline of planned extensions are given in section 5.

II. LATTICE ACTION AND SYMMETRIES

The staggered fermion part of the action for lattice QCD with degenerate \(u\) and \(d\) quarks at a finite chemical potential \(\mu_I\) for isospin \((I_3)\) is

\[
S_f = \sum_{\text{sites}} \bar{\chi}[\not\! D(\tau_3 \mu_I) + m] \chi
\]

(1)

where \(\not\! D(\mu)\) is the standard staggered \(\not\! D\) with links in the +\(t\) direction multiplied by \(e^{\frac{i}{2} \mu}\) and those in the −\(t\) direction multiplied by \(e^{-\frac{i}{2} \mu}\) \([3]\). When \(\mu_I = m = 0\), this action has a global \(U(2) \times U(2)\) flavour symmetry under which

\[
\chi \longrightarrow \exp[i(\theta + \epsilon \phi).\tau] \chi
\]

\[
\bar{\chi} \longrightarrow \bar{\chi} \exp[-i(\theta - \epsilon \phi).\tau]
\]

(2)

where \(\tau = (1, \vec{\tau})\), \(\theta\) and \(\phi\) are site-independent 4-component “vectors”, and \(\epsilon = \epsilon(x) = (-1)^{x+y+z+t}\). Spontaneous symmetry breaking can occur in any direction in this space. If we keep \(\mu_I = 0\) and allow \(m \neq 0\), the symmetry is broken down to \(U(2)_V\). On the other hand
if we keep \( m = 0 \) and allow \( \mu_I \neq 0 \), the symmetry is broken down to \( U(1) \times U(1) \times U(1) \times U(1) \) generated by \( 1, \tau_3, \epsilon \) and \( \epsilon \tau_3 \). Finally in the general case where neither \( \mu_I \) nor \( m \) vanishes, the symmetry is reduced to \( U(1)_V \times U(1)_V \) associated with \( 1 \) and \( \tau_3 \).

In order to predict potential symmetry breaking patterns we make several simple modifications of the arguments of Son and Stephanov \[2\] to apply them to the staggered lattice action. The generic quark bilinear which creates a meson has the form

\[
M = \bar{\chi} \Gamma \chi.
\]

The propagator for such a meson obeys the inequality

\[
|\langle M(x)M^\dagger(0) \rangle| \leq \text{const} \langle S(x,0)S^\dagger(x,0) \rangle.
\]

Thus, meson operators \( M \) whose propagators are proportional to \( \langle S(x,0)S^\dagger(x,0) \rangle \), are potential Goldstone bosons. Now we note that our Dirac operator obeys

\[
\tau_{1,2}\epsilon[D(\tau_3\mu_I) + m]\epsilon \tau_{1,2} = [D(\tau_3\mu_I) + m]^\dagger,
\]

so \( i\bar{\chi}\epsilon \tau_{1,2} \chi \) are Goldstone candidates. These are linear combinations of \( \pi^+ \) and \( \pi^- \) creation operators which means that if spontaneous breaking of the remnant flavour symmetry should occur, one linear combination of \( \pi^\pm \) will become a Goldstone boson while the orthogonal linear combination will develop a vacuum expectation value – a charged pion condensate.

We note, in passing, that in the limit of massless quarks

\[
\tau_{1,2}[D(\tau_3\mu_I) + m] = -[D(\tau_3\mu_I)]^\dagger,
\]

and we have 2 additional Goldstone boson candidates, \( \bar{\chi} \tau_{1,2} \chi \), and if spontaneous symmetry breaking does occur we will have 2 Goldstone bosons rather than 1.

Since, in order to observe spontaneous symmetry breaking on a finite lattice, one needs to add a small explicit symmetry breaking term in the direction defined by the condensate, we choose to work with the fermion action

\[
S_f = \sum_{\text{sites}} [\bar{\chi}[D(\tau_3\mu_I) + m]\chi + i\lambda \epsilon \bar{\chi} \tau_2 \chi]
\]

where the term proportional to the (small) parameter \( \lambda \) serves this purpose. The Dirac operator now has the determinant

\[
\det[D(\tau_3\mu_I) + m + i\lambda \epsilon \tau_2] = \det[A^\dagger A + \lambda^2]
\]
where we have defined
\[ A \equiv \bar{\psi}(\mu_I) + m. \] (9)

(Note that this is a 1 × 1 matrix in the flavour space on which the \( \tau \)s act.) We see that adding this symmetry breaking term has the effect of rendering the determinant strictly positive, which enables us to use the hybrid molecular-dynamics (HMD) algorithm to simulate this theory. Note that this theory has 8 continuum flavours. We use the HMD method to take the required fourth root of the determinant reducing this to 2 continuum flavours. For the purpose of simulation, it is convenient to multiply the Dirac operator on the left by the matrix \( \text{diag}(1, -\epsilon) \) and on the right by the matrix \( \text{diag}(1, \epsilon) \). The transformed matrix \( \widetilde{M} \), has the same determinant as the original Dirac operator, and \( \widetilde{M}^\dagger \widetilde{M} \) is block diagonal, with the upper and lower blocks having the same determinant. This means that we use ‘noisy’ fermions and generate Gaussian noise for both upper and lower components of \( \widetilde{M} \dot{\chi} \), but only keep the upper components of \( \dot{\chi} \) after the inversion. Thus we still have only 8 flavours in the quadratic formulation. This is completely analogous to the odd-even lattice separation which prevents further species doubling in staggered lattice QCD at zero chemical potential.

Quantities we measure include the chiral condensate,
\[ \langle \bar{\psi}\psi \rangle \leftrightarrow \langle \bar{\chi}\chi \rangle, \] (10)
the charged pion condensate
\[ i\langle \bar{\psi}\gamma_5\tau_2\psi \rangle \leftrightarrow i\langle \bar{\chi}\epsilon\tau_2\chi \rangle \] (11)
and the isospin density
\[ j_0^3 = \frac{1}{V} \left\langle \frac{\partial S_f}{\partial \mu_I} \right\rangle. \] (12)

Here we have included both the lattice and continuum versions of the condensates. To get this simple continuum form for the charged pion condensate requires absorbing a factor of \( \xi_5 \) (the flavour analogue of \( \gamma_5 \)) into the definition of the \( d \)-quark field.

III. LATTICE SIMULATIONS AT ZERO TEMPERATURE

We have simulated \( N_f = 2 \) lattice QCD at finite \( \mu_I \) on an \( 8^4 \) lattice at an intermediate coupling \( \beta = 6/g^2 = 5.2 \). This \( \beta \) was chosen since it represents an approximate lower bound to estimates of the finite temperature transition value for \( N_t = 4 \) in the chiral limit. This
was used to keep finite volume effects at acceptable levels. We performed simulations at 2 different quark masses ($m = 0.025$ and $m = 0.05$) to see that varying the mass did not affect the qualitative behaviour of the theory and that we understood the effects of changing the quark mass.

At $m = 0.025$, we performed runs, each of 2000 molecular-dynamics time units in length, at 17 different $\mu_I$ values ($0 \leq \mu_I \leq 2$) for each of $\lambda = 0.0025$ and $\lambda = 0.005$. Using 2 $\lambda$ values, both chosen to be much less than $m$, enabled us to extrapolate to the $\lambda = 0$ limit, which is our ultimate interest. (We also ran at $\mu_I = 3.0$, $\lambda = 0.005$ to check saturation.)

The charged pion condensates, $i\langle \bar{\psi}\gamma_5\tau_2\psi \rangle$ from these simulations are presented in figure 1 as functions of $\mu_I$, along with a linear extrapolation to $\lambda = 0$. This extrapolation strongly suggests that the $\lambda \to 0$ condensate vanishes for $\mu_I < \mu_c \sim 0.3 - 0.45$, above which it is finite. This would indicate a phase transition to a phase in which $I_3$ symmetry is broken spontaneously by a charged pion condensate, with an associated Goldstone mode.

This behaviour is predicted by the effective (chiral) Lagrangian analyses of Son and Stephanov, which also predict that the transition should be second order with mean-field exponents [2]. We fit our extrapolated ‘data’ to the critical scaling form

$$i\langle \bar{\psi}\gamma_5\tau_2\psi \rangle = \text{const} (\mu_I - \mu_c)^{\beta_m}$$

(13)

for $\mu_I > \mu_c$ close to the transition. Fitting to this form over the range $0.4 \leq \mu_I \leq 1.0$, we find $\mu_c = 0.394(1)$, $\beta_m = 0.230(9)$ and const = 1.23(2) at a 62% confidence level. This appears inconsistent with mean field scaling for which $\beta_m = \frac{1}{2}$, and closer to the tricritical scaling for which $\beta_m = \frac{1}{4}$. Indeed, good fits to a tricritical scaling form over this range of $\mu_I$ and both $\lambda$s can be obtained (confidence level 25%).

However, as we have noted in our paper on the quenched theory [3] such fits can be deceptive and can be because the true scaling behaviour of these theories is best described by the scaling forms given by effective Lagrangian analyses. When this form of scaling is described in terms of $\mu_I - \mu_c$ rather than the natural scaling variables these forms imply, the true scaling window is very narrow. Outside this window these theories can appear to scale with $\beta_m$ which is half the true value when analyzed in terms of $\mu_I - \mu_c$. We now introduce the scaling forms (equations of state) suggested by such effective Lagrangians, both of which give mean-field scaling behaviour.
FIG. 1: Charged pion condensate as a function of $\mu_I$ for $\lambda = 0.0025$, $\lambda = 0.005$ and $\lambda \rightarrow 0$. The curves are fits of the finite $\lambda$ measurements to the scaling forms defined in the text.

The form for the equation of state suggested by the lowest order tree-level analysis of effective Lagrangians of the non-linear sigma model type [2] is given in terms of $\alpha$ which minimizes the effective potential

$$E = -a \mu^2 \sin^2(\alpha) - b m \cos(\alpha) - b \lambda \sin(\alpha)$$  \hspace{1cm} (14)

in terms of which

$$i\langle \bar{\psi} \gamma_5 \tau_2 \psi \rangle = b \sin(\alpha)$$  \hspace{1cm} (15)

$$\langle \bar{\psi} \psi \rangle = b \cos(\alpha)$$  \hspace{1cm} (16)
and
\[ j_0^3 = 4 \alpha \mu \sin^2(\alpha). \] (17)

\( j_0 \) is given in terms of \( \mu_c \) and \( a \), namely
\[ b = \frac{2}{m} \alpha \mu_c^2 \] (18)

The form for the equation of state which is derived from an effective Lagrangian of the linear sigma model type is obtained by extracting the values of \( R \) and \( \alpha \) which minimize the effective potential
\[ \mathcal{E} = \frac{1}{4} R^4 - \frac{1}{2} a R^2 - \frac{1}{2} b \mu^2 \sin^2(\alpha) - c m R \cos(\alpha) - c \lambda R \sin(\alpha) \] (19)
in terms of which
\[ i \langle \bar{\psi} \gamma_5 \tau_2 \psi \rangle = c R \sin(\alpha) \] (20)
\[ \langle \bar{\psi} \psi \rangle = c R \cos(\alpha) \] (21)
and
\[ j_0^3 = 2 b \mu R^2 \sin^2(\alpha). \] (22)
c is given in terms of \( \mu_c \) by
\[ c = \frac{b \mu_c^2}{m} \sqrt{a + b \mu_c^2}. \] (23)

Finally, the tricritical scaling form which we use for comparison, and which does not yield mean-field behaviour, is expressed in terms of the value of \( \phi \) which minimizes the effective potential
\[ \mathcal{E} = \frac{1}{6} \phi^6 - \frac{1}{3} c \lambda \phi^3 - \frac{1}{2} (\mu_I - \mu_c) \phi^2 - b \lambda \phi \] (24)
in terms of which,
\[ i \langle \bar{\psi} \gamma_5 \tau_2 \psi \rangle = b \phi + \frac{1}{3} c \phi^3 \] (25)
and
\[ j_0 = a \phi^2. \] (26)

We fit our measurements of the charged pion condensate for \( 0 \leq \mu_I \leq 1 \) and both \( \lambda_a \) to the linear sigma model form of equation 20. We obtained a fit with \( \mu_c = 0.4036(5) \), \( a = 0.52(1) \), \( m = 0.0253(1) \) with \( \chi^2/dof = 2.2 \). Although worse than the tricritical fit, we note that it allows a reasonable fit for \( \mu_I < \mu_c \) in addition to \( \mu_I > \mu_c \) which the tricritical
fit did not, $\mu_c = 0.426(3)$ for the tricritical fit, which is close to the value expected for the apparent tricritical scaling produced by such sigma model scaling. One of the reasons the tricritical fit gives better results is that it includes a second symmetry breaking interaction, cubic in the order parameter, which rounds off the curve as $\mu_I$ approaches 1.0 allowing a better fit to the ‘data’. Such cubic terms could be incorporated equation 19. However, now there is not one such term but several, which is why this possibility was not considered. Finally, as we shall see below, this fit makes good predictions for the chiral condensate and the isospin density.

$$\text{SU}(3) \quad N_f=2 \quad \beta=5.2 \quad m=0.025 \quad 8^4 \text{ lattice}$$

![Chiral condensate as a function of $\mu_I$](image)

**FIG. 2:** Chiral condensate as a function of $\mu_I$ for $\lambda = 0.0025$ and $\lambda = 0.005$.

In figure 2 we present the chiral condensate for the same set of simulations. The
The general characteristics of these graphs are that $\langle \bar{\psi}\psi \rangle$ remains roughly constant for $\mu_I < \mu_c$, after which it drops rapidly, approaching zero at large $\mu_I$. Although this can qualitatively be thought of as the condensate rotating from the chiral to the isospin-breaking direction, it is not a simple rotation, since the magnitude of the total condensate increases up until saturation effects take over. This contrasts with the predictions of lowest order effective Lagrangians of the non-linear sigma model type where it is a simple rotation. However, in the case of 2-colour QCD at finite quark-number chemical potential, whose effective Lagrangian is structurally very similar to that for the theory at hand, the effective Lagrangian/chiral perturbation theory calculations have been extended to next-to-leading order [11]. Here, although mean field scaling survives, the rotation of the condensate is accompanied by a rescaling. Such behaviour is reproduced at tree level by effective Lagrangians of the linear sigma model variety which we use for our fits to the pion condensate. The predictions these fits make for the behaviour of the chiral condensate (equation 21), have been superimposed on the measurements of figure 2. Except for small departures at small $\mu_I$, which we attribute to the fact that $m$ for the fit differs slightly from the true $m = 0.025$, these predictions are very good until the effects of saturation start to dominate.

Figure 3 presents the isospin ($I_3$) density as a function of $\mu_I$. While the 2 condensates are normalized to 4 flavours for comparison with previous ($\mu_I = 0$) simulations, $j_0^3$ is normalized to 8 flavours, which is the natural normalization for staggered quarks. Qualitatively, we note that $j_0^3$ is close to zero for $\mu_I < \mu_c$, rises slowly (in comparison with the pion condensate) up to $\mu_I \sim 1$, after which it starts to rise more rapidly reaching its saturation value of 3 (1/2 for each of 3 colours and 2 ‘flavours’ per site) due to fermi statistics, at $\mu_I \sim 2$. (In fact measurements made at $\mu_I = 3.0$ give a value consistent with 3). Saturation is clearly a finite lattice spacing effect and hence requires no further discussion. We also note that there is very little $\lambda$ dependence. The predictions of equation 22 using the parameters of our fit are superimposed on our ‘data’ and show good agreement out to $\mu_I \approx 0.8$. Both ‘data’ and fit are approximately linear in $\mu_I$ over this range.

As also noted by Son and Stephanov, measuring $j_0^3$ as a function of $\mu_I$ at $T = 0$ and constant volume (since $\beta$ is constant) yields the pressure $p$ and energy density $\epsilon$ as functions of $\mu_I$, since

$$ p = \int_{\mu_c}^{\mu_I} j_0^3 d\mu_I $$

(27)
FIG. 3: Isospin($I_3$) density as a function of $\mu_I$ for $\lambda = 0.0025$ and $\lambda = 0.005$.

\[
\epsilon = \int_0^{j_0^3} \mu_I d\tilde{j}_0^3 \quad (28)
\]

for $\mu_I > \mu_c$ and zero for $\mu_I < \mu_c$. In the region where $j_0^3 \approx \text{const} (\mu_I - \mu_c)$ these yield

\[
p = \frac{1}{2} \text{const} (\mu_I - \mu_c)^2 \quad (29)
\]

\[
\epsilon = \frac{1}{2} \text{const} (\mu_I^2 - \mu_c^2) \quad (30)
\]

\[
\frac{p}{\epsilon} = \frac{\mu_I - \mu_c}{\mu_I + \mu_c} \quad (31)
\]

The last of these equations is a form of the equation of state for this system in the neighbourhood of $\mu_c$. Clearly we could extend each of these expressions beyond the scaling region.
by interpolating the ‘data’ of figure 3 and performing the relevant integrals analytically or numerically.

\[ \text{SU(3) } N_f=2 \quad \beta=5.2 \quad m=0.05 \quad \text{8}^4 \text{ lattice} \]

\[
\begin{align*}
\text{x} & \quad \lambda = 0.005 \\
\text{o} & \quad \lambda = 0.01 \\
\square & \quad \lambda \to 0
\end{align*}
\]

FIG. 4: Pion condensates as functions of \( \mu_I \) for \( \lambda = 0.005 \), \( \lambda = 0.01 \) and a linear extrapolation to \( \lambda = 0.0 \). The lines are the tricritical fit described in the text.

We performed similar simulations at the same coupling \( \beta = 5.2 \) and mass \( m = 0.05 \) at \( \lambda = 0.005 \) and \( \lambda = 0.01 \). Here we concentrated on the neighbourhood of the phase transition and used more closely spaced (in \( \mu_I \)) points with somewhat lower statistics. For \( \mu_I \lesssim \mu_c \) we find acceptable fits of the pion condensate to to both the non-linear sigma model scaling form and to the tricritical scaling form (34% and 40% respectively). The fit to non-linear sigma model effective Lagrangian scaling (equation 15) enables one to extend this to low \( \mu_I \). A fit
of the ‘data’ over the complete range over which we have measurements at both $\lambda$ values — $0.4 \leq \mu_I \leq 0.7$ — yields $\mu_c = 0.569(2)$, $a = 0.0868(9)$ and $m = 0.0534(3)$, with $\chi^2/dof = 1.5$ for the non-linear sigma model form. Even though this fit is worse than those restricted to $\mu_I \gtrsim \mu_c$, we feel that the ability to fit $\mu_I < \mu_c$ in addition to $\mu_I > \mu_c$ makes the argument for this form of fit more compelling. Not only do we see qualitative consistency with the smaller mass results, but our measured values of $\mu_c$ are consistent with the expectation that $\mu_c(m = 0.05) = \sqrt{2}\mu_c(m = 0.025)$ which would be true if, indeed, $\mu_c = m_{\pi}$, from PCAC. This ‘data’ for the pion condensate with the scaling fits superimposed is plotted in figure 4. We note that the fits appear to have validity beyond the range of $\mu_I$ over which the fits were performed.

IV. SIMULATIONS AT FINITE TEMPERATURE

We have performed simulations of QCD at finite $\mu_I$ and finite temperature on an $8^3 \times 4$ lattice, with $m = 0.05$. Most of these simulations were performed at $\lambda = 0.005$ and $\lambda = 0.01$, i.e. $\lambda << m$, with the objective of obtaining information about the $\lambda = 0$ limit. The goal of these simulations is to map out the region of the $(\beta, \mu_I)$ and hence the $(T, \mu_I)$ plane where $I_3$ is spontaneously broken by a charged pion condensate, and determine the nature of the phase transitions which demarcate its boundaries.

The first of these simulations was performed at a fixed, large (but well below saturation) value of $\mu_I$. The value chosen was $\mu_I = 0.8$. At $\beta$ low enough to approximate zero temperature on an $N_t = 4$ lattice, the system is in the phase where $I_3$ is spontaneously broken by a (large) pion condensate. As $\beta$ is increased we eventually reach a value $\beta = \beta_c$ at which this condensate evaporates. For $\beta > \beta_c$ we are in the phase where the pion condensate vanishes for $\lambda \to 0$ and $I_3$ is unbroken. A single $\lambda$ value, $\lambda = 0.005$ was used for these runs.

Figure 5 shows the $\beta$ dependence of the pion condensate for these simulations. We see that for $\beta \leq 5.2$ the condensate is large. Between $\beta = 5.2$ and $\beta = 5.3$ the condensate drops by an order of magnitude, and is small enough for $\beta \geq 5.3$ that we are safe to assume that it would vanish in the $\lambda \to 0$ limit. This drop is so precipitous that we suspect that it is first order, although we really need a larger lattice to confirm this.

We have also measured the Thermal Wilson Line (Polyakov Loop) during these runs.
These measurements are shown in figure 6. For $\beta \leq 5.2$, the Wilson line is small indicating confinement. For $\beta \geq 5.3$, the Wilson line becomes large indicating deconfinement. The jump between $\beta = 5.2$ and $\beta = 5.3$ is again great enough to suggest a first order transition. This behaviour of the Wilson Line indicates that this is the temperature-driven deconfinement transition.

We have also run simulations on an $8^3 \times 4$ lattice with $\beta = 4.0$ which gives us the low temperature behaviour. We chose $m = 0.05$ again and ran at $\lambda = 0.005$ and $\lambda = 0.01$ for
FIG. 6: Wilson linear a function of $\beta$ for $m = 0.05$, $\lambda = 0.005$, and $\mu_I = 0.8$ on an $8^3 \times 4$ lattice.

$0.20 \leq \mu_I \leq 2.0$ which covers both the transition from the $I_3$ symmetric phase to the phase where $I_3$ is spontaneously broken, and the approach to saturation. Our measurements of the pion condensate are shown in figure 7. Not surprisingly this graph resembles those we obtained on an $8^4$ lattice since at $\beta = 4.0$, this lattice is essentially at zero temperature.

We fit the scaling behaviour of these measurements to the non-linear sigma model scaling form of equation 15 for both $\lambda$ values and $0.2 \leq \mu_I \leq 0.8$. The fit yielded $\mu_c = 0.519(1)$, $a = 0.1400(4)$ with the quark mass fixed at $m = 0.05$ at a confidence level of 41%, which is very good. These fits are shown in figure 6. Again an acceptable tricritical fit was
possible (confidence level 48%), but only for $0.55 \leq \mu_I \leq 0.8$, yielding the expected larger estimate for $\mu_c$.

In figure 8 we present the isospin density from these runs. $j_0^3$ rises from zero at $\mu_I \sim \mu_c$. Again there is little $\lambda$ dependence. We have superimposed the form predicted from the fit to pion condensate using equation 17 on these plots. These curves are in reasonable agreement with the measured values up to $\mu_I \approx 0.7$. This indicates that the scaling window for $j_0^3$ is slightly less than that for the pion condensate. However, it is clear that the linear
or near-linear increase of this quantity with $\mu_I$ continues beyond the point where the ‘data’ and curves diverge. This is born out by the fitting $j_0^3$ to the form

$$j_0^3 = \text{const} (\mu_I - \mu_c)^{\beta_I}.$$  \hfill (32)

Fitting the $\lambda = 0.005$ ‘data’ over the range $0.5 \leq \mu_I \leq 1.1$ gives $\mu_c = 0.467(13)$, $\beta_I = 0.94(6)$ and $\text{const} = 1.22(3)$ at a confidence level of 18% while fitting the $\lambda = 0.01$ ‘data’ yields $\mu_c = 0.382(9)$, $\beta_I = 1.09(3)$, $\text{const} = 1.20(1)$ at a confidence level of 85%. These results are in excellent agreement with the effective Lagrangian prediction $\beta_I = 1$. This graph also
indicates a crossover to a more rapid increase at $\mu_I \sim 1.5$.

![Graph showing pion condensates as functions of $\mu_I$ for $\lambda = 0.005$, $\lambda = 0.01$ and a linear extrapolation to $\lambda = 0.0$ on an $8^3 \times 4$ lattice, at $\beta = 5.0$. The lines are the tricritical fit described in the text.](image)

**FIG. 9:** Pion condensates as functions of $\mu_I$ for $\lambda = 0.005$, $\lambda = 0.01$ and a linear extrapolation to $\lambda = 0.0$ on an $8^3 \times 4$ lattice, at $\beta = 5.0$. The lines are the tricritical fit described in the text.

Finally we have performed $m = 0.05$ simulations on an $8^3 \times 4$ lattice at $\beta = 5.0$, which lies below $\beta_c$ at high $\mu_I$, while being large enough that the effects of finite temperature should be apparent. Here again we ran with $\lambda = 0.005, 0.01$. Here we used higher statistics (2000 time-units per ‘run’) but at fewer $\mu_I$ values. Our measurements of the charged pion condensate are presented in figure 9. The transition from low to high values of this condensate shows no sign of a first order transition. The linear extrapolation to $\lambda = 0$ gives values close to zero at low $\mu_I$, rising rapidly from zero above some $\mu_c \sim 0.5$. Here we find good
fits of the ‘data’ for both $\lambda$ values and $0.2 \leq \mu_I \leq 1.0$ to the non-linear sigma model form with $\mu_c = 0.5521(5)$, $a = 0.1040(4)$, $m = 0.0523(1)$ at a confidence level of 31% and to the linear sigma model form with $\mu_c = 0.5513(5)$, $a = 2.46(1)$, $b = 0.0823(4)$, $m = 0.0518(1)$ at a confidence level of 48%. This second fit is included in figure 3. A tricritical fit is possible for $0.55 \leq \mu_I \leq 0.8$ and has a confidence level of 28%.

Thus the line of phase transitions which bound the region in the $(\mu_I, T)$ plane within which isospin ($I_3$) is broken spontaneously by a charged pion condensate, is second order for low temperatures and becomes first order at high $\mu_I$. The second order segment of this line appears to have mean-field critical exponents.

V. DISCUSSION AND CONCLUSIONS

We have simulated QCD with 2 quark flavours ($u, d$) at a finite chemical potential ($\mu_I$) for isospin, $I_3$. At zero temperature and intermediate coupling ($\beta = 6/g^2 = 5.2$) we found strong evidence for a second order transition to a phase in which $I_3$ is spontaneously broken by a charged pion condensate which also breaks parity, at $\mu_I = \mu_c$. The observed behaviour is what is predicted by effective Lagrangian methods for $\mu_I$ appreciably less than the value at which saturation, a lattice artifact, takes over. These effective Lagrangian analyses predict that $\mu_c = m_\pi$. Since we have not measured $m_\pi$ on these small lattices, all we have checked is that $\mu_c \propto \sqrt{m}$ for the 2 quark masses which we use ($m = 0.025, m = 0.05$), and found that this is true within the uncertainties of our measurements. The critical scaling appears to be well described by an equation of state suggested by these effective Lagrangian analyses, which means that the critical point has mean-field critical exponents. However, tricritical behaviour cannot be completely excluded. Such behaviour was discussed in detail in our paper on the quenched theory [3].

We also measured the isospin density ($j_0^3$) and found that it increases from zero, at or near $\mu_c$. The scaling behaviour appears to be linear close to $\mu_c$, as is predicted by effective Lagrangian analyses [2], and well described by the predictions given by the fits to the pion condensate, within the scaling window. For larger $\mu_I$ values it starts to increase considerably faster than linear. This is in qualitative agreement with the expectation that $I_3$ density should increase as $\mu_I^3$ at large $\mu_I$. We have not been able to determine if our observations
are consistent with this $\mu_I^3$ increase because of the effects of saturation which cause the isospin density to approach 3 at high $\mu_I$. Here we have indicated how the measurement of $j_0^3$ enables one to obtain the pressure ($p$) and the energy density ($\epsilon$) as functions of $\mu_I$, and given explicit expressions for these quantities in the scaling regime.

The chiral condensate remains approximately constant for $\mu_I < \mu_c$. Above $\mu_c$ it starts to fall approaching zero for large $\mu$. Again this is in agreement with expectations, and the predictions from the fits to the pion condensate. However, the expectation from lowest-order effective Lagrangian tree-level analyses, that the chiral condensate simply rotates into the direction of the charged pion condensate, is not true. However, in closely related 2-colour QCD at finite quark-number chemical potential, chiral perturbation theory calculations through next-to-leading order show that, while scaling remains mean field, the condensate does not merely rotate, but also rescales [11]. Since the structure of chiral perturbation theory (effective Lagrangians) for the two theories is so similar, we expect a similar result for QCD at finite $\mu_I$.

We have performed simulations at finite temperature ($T$) in addition to finite $\mu_I$. In particular we have heated the system at fixed $\mu_I = 0.8 > \mu_c(T = 0)$ (in lattice units) by increasing $\beta$. On our $N_t = 4$ lattice there is some $\beta = \beta_c (5.2 \leq \beta_c \leq 5.3)$ at which the charged pion condensate evaporates. This transition appears to be first order. Such first order behaviour was predicted for $\mu_I$ large enough by Son and Stephanov [2] who argued that at high $\mu_I$ the fermions would effectively decouple, and the phase transition would be that for pure glue, which is known to be first order. We note from our observations that the situation is not quite this simple. The pure glue transition on an $N_t = 4$ lattice occurs at $\beta \approx 5.7$ [12, 13]. Since our observations place $\beta_c$ somewhat lower than this, and close to the value of the $\mu_I = 0$ finite temperature transition, the quarks are having an effect. We also note that Son and Stephanov suggest that the first order deconfinement transition at high temperature is distinct from the $I_3$ symmetry restoring transition. Our evidence is that these 2 transitions are coincident. This means that the second order segment of the phase boundary can be considered as driven purely by the chemical potential, which makes their argument for $O(2)$ universality less compelling. We note that, in our paper on 2-colour QCD at finite $\mu$ and $T$ [3], we present an alternative argument for the first-order finite-temperature transition which that theory exhibits for large $\mu$. 
In addition, we have simulated our $N_t = 4$ system at fixed $\beta$, varying $\mu_I$. In particular we have performed simulations at $\beta = 4.0$ which is at near-zero temperature, and $\beta = 5.0$ where the system is clearly at a finite temperature. Both of these simulations showed a second order transition. Again the scaling was well described by the scaling forms suggested by effective Lagrangian analyses which indicates that they have mean-field critical exponents.

We have noted throughout this paper the similarity between 3-colour QCD at finite $\mu_I$ and 2-colour QCD at finite quark-number chemical potential, $\mu$. The correspondence is seen by identifying $\frac{1}{2} \mu_I$ with $\mu$, the charged pion condensate with the diquark condensate and the isospin density with the quark number density. This similarity is seen both in simulations and in effective Lagrangian analyses. The expected position of the zero temperature transition is the same $\mu_I = m_\pi (\mu = \frac{1}{2} m_\pi)$. Its nature — second order with mean-field exponents — is the same as predicted [11] and observed [3, 7] for 2-colour QCD. In both systems the spontaneous symmetry breaking is in the Goldstone mode (superfluid). At finite temperature the condensate evaporates at a transition which is first order for $\mu_I (\mu)$ large enough. This line of first order transitions softens to second order and the line of second order transitions connects to the zero temperature transition. Thus, we can use 2-colour QCD results at finite $\mu$ as a guide as to QCD at finite $\mu_I$.

We are extending these simulations to a larger lattice ($12^3 \times 24$) and weaker coupling where we hope to observe the expected mean-field transition more clearly distinguished from the tricritical alternative and rule out the $O(2)$ alternative. This lattice will also enable us to measure the spectrum of Goldstone and pseudo-Goldstone excitations as functions of $\mu_I$. More extensive spectrum analyses at $\mu_I = 0$ will give us a more definitive scale for these phenomena, in addition to a value for $m_\pi$ with which to compare $\mu_c$. Configurations will be stored so that we can make other spectroscopy measurements at finite $\mu_I$. We will also extend the finite temperature simulations to a $12^3 \times 6$ lattice since it is difficult to determine the order of the continuum transitions from $8^3 \times 4$ lattices. In addition we will study the instantons at large $\mu_I$ since it is believed that instantons and their interactions have a relatively simple structure at large isospin density, analogous to what has been predicted for large quark-number density [14].

We are also starting lattice QCD simulations including both a chemical potential $\mu_I$ for isospin and a chemical potential $\mu_s$ for strangeness. Here effective Lagrangian analyses
have indicated that there is a competition between the formation of pion and kaon condensates as $\mu_\pi$ and $\mu_s$ are varied independently and that the boundary between the region with a pion condensate and that with a kaon condensate is a line of first order transitions \[13\].

Although our simulations have been limited to zero baryon number density, it is interesting to know how much of this analysis is relevant to systems with finite baryon number density and isospin density. If it does have relevance, charged pion condensates could contribute to the equation-of-state of nuclear matter and thus be important in understanding the physics of neutron stars and perhaps large nuclei.

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