Local Quantum Criticality in the Two-dimensional Dissipative Quantum XY Model

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Abstract

We use quantum Monte-Carlo simulations to calculate the phase diagram and the correlation functions for the quantum phase transitions in the two-dimensional dissipative XY model with and without four-fold anisotropy. Without anisotropy, the model describes the superconductor to insulator transition in two-dimensional dirty superconductors. With anisotropy, the model represents the loop-current order observed in the under-doped cuprates and its fluctuations, as well as the fluctuations near the ordering vector in simple models of itinerant antiferromagnets. These calculations test an analytic solution of the model which re-expressed it in terms of topological excitations - the vortices with interactions only in space but none in time, and warps with leading interactions only in time but none in space, as well as sub-leading interactions which are both space and time-dependent. For parameters for which the proliferation of warps dominates the phase transition, the critical fluctuations as a function of the deviation of the dissipation parameter $\alpha$ from its critical value $\alpha_c$ are scale-invariant in imaginary time $\tau$ as the correlation length in time $\xi_\tau = \tau_c e^{a_c/(\alpha_c-\alpha)^{1/2}}$ diverges. On the other hand, the spatial correlations develop with a correlation length $\xi_x \approx \xi_0 \log (\xi_\tau)$, with $\xi_0$ of the order of a lattice constant. The dynamic correlation exponent $z$ is therefore $\infty$. These results are consistent with the phenomenology proposed for the strange metal properties of the cuprates. The Monte-Carlo calculations also directly show warps and vortices. Their density and correlations across the various transitions in the model are calculated and related to those of the order parameter in the original model.

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I. INTRODUCTION

The quantum dissipative XY model was introduced \[1, 2\] to describe the observed quantum phase transition \[3\] in thin metallic films from a superconductor to insulator at a universal value of their normal state resistance. In the past few years, the model has acquired applications in other physical contexts. The new physical contexts are the quantum-critical point of the loop-current order in cuprate superconductors \[4–6\], and the itinerant antiferromagnetic (AFM) quantum-critical point \[7\], which may be of relevance to Fe-based superconductors and some heavy-fermion compounds.

Electronic-fluctuation induced superconductivity in cuprates, Fe-based superconductors, and in heavy-fermion superconductors always appears together with a normal state which does not obey the Fermi-liquid paradigm. For cuprates, the properties of the normal state, sometimes called the strange metal phase, could be phenomenologically described as a marginal Fermi liquid (MFL) \[8\], in which the coupling of electrons is to quantum-critical fluctuations which are local in space and power-law in time. Such critical fluctuations violate the paradigm of classical dynamical critical fluctuations \[9\] or their simple quantum analogs \[10–12\]. The anomalous normal-state properties have been associated with quantum criticality of an order competing with superconductivity. Thermodynamic and transport properties near the antiferromagnetic quantum-critical point of some heavy fermion compounds \[13\] and at least some of the Fe-based superconductors near their antiferromagnetic quantum-critical point are remarkably similar to these in the cuprates \[14, 15\]. The quantum-critical fluctuations of one of the heavy-Fermion compound, measured by neutron scattering \[16–18\] has also been fitted to a form of local-critical fluctuations \[19\]. All these problems share the property that they are highly anisotropic so that the fluctuation problem may be regarded as two-dimensional. The microscopic physics in these problems is of-course quite different. The universality class of local quantum-criticality appears to encompass diverse physical systems.

The classical 2D XY model can be solved by integrating over the spin-wave variables to cast the model in terms of topological excitations, the vortices, which undergo, as a function of temperature, a Kosterlitz-Thouless (KT) transition from a phase in which they occur as bound pairs of zero net-vorticity to a disordered region in which individual vortices proliferate \[20, 21\]. For the quantum dissipative XY model, the integration over the spin-wave variables
leads to a model which can be expressed in terms of two orthogonal sets of topological excitations, the vortices and the warps, which will be further described below. It is claimed that when the quantum phase transitions in the model are governed by proliferation of warps, the fluctuations at the critical point are of the form phenomenologically proposed \cite{8}. The cross-over temperature as a function of the parameters of the model, from quantum-critical fluctuations to quantum-fluctuations, was also derived. It is important to have an unambiguous check of this solution of the model and its variants by other methods. The method used here is to simulate the model with the quantum Monte-Carlo method.

In a set of Monte-Carlo calculations already done on the model \cite{22}, a rich phase diagram of the model was discovered. However, the correlations of the order parameter in some important regions were not studied. Nor were the properties of the model related to the topological excitations proposed \cite{6}. The results for all the quantities calculated here which were also calculated earlier \cite{22} are identical. In the present work, we calculate the correlation functions and relate them to those of the topological excitations which can be identified explicitly in the Monte-Carlo calculations. We show that for transitions driven by warps, the order parameter fluctuations at the critical point are scale-invariant in imaginary time $\tau$, and calculate the behavior of the correlation functions as the critical point is approached. A result beyond those derived analytically is that the spatial correlations are consistent with a length scale $\xi_x$ which grows only logarithmically with the temporal length scale $\xi_\tau$. This is consistent with a dynamical exponent $z \rightarrow \infty$, making it a model in which local quantum-criticality has been explicitly proven.

This paper is organized as follows. We introduce the model and the details of quantum Monte-Carlo method in Sec. \textbf{II}. In Sec. \textbf{II.B}, we explain how we identify the warps and vortices in the calculations. We show the obtained phase diagram and summarize the properties of three distinct phases: the Disordered, Quasi-ordered, and Ordered phases, in Sec. \textbf{III}. In Secs. \textbf{IV}, \textbf{V}, and \textbf{VI}, we show the calculated critical fluctuations of the order parameter at the transitions between them, and their relation to the change in density and correlations of warps and vortices across the transitions. We focus especially on the transition from the disordered phase to the ordered phase (Sec. \textbf{VI}), and explicitly show that the fluctuations have a temporal correlation length exponentially larger than the spatial correlation length. In Sec. \textbf{VII}, we discuss the effect of a four-fold anisotropic field, relevant to cuprates and the anti-ferromagnets. Our conclusion and the directions for future analytical calculations

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is presented in Sec. VIII.

II. MODEL AND METHOD

A. (2+1)D Quantum dissipative XY model

We study the (2+1)D quantum dissipative XY model on a discretized $N \times N \times N_\tau$ cubic lattice, where $N \times N$ is the spatial size in the $x$-$y$ plane and $N_\tau$ is the number of Trotter slices used to discretize the imaginary time direction. The action of this model with a four-fold anisotropic field can written as [22]:

$$S = -K \sum_{\langle x, x' \rangle, \tau} \cos(\Delta \theta_{x,x',\tau}) + \frac{K_\tau}{2} \sum_{x,\tau} (\theta_{x,\tau} - \theta_{x,\tau-1})^2$$

$$+ \frac{\alpha}{2} \sum_{\langle x, x' \rangle, \tau, \tau'} \frac{\pi^2}{N_\tau^2} \left[ \frac{\Delta \theta_{x,x',\tau} - \Delta \theta_{x,x',\tau}}{N_\tau} \right]^2 - h_4 \sum_{x,\tau} \cos(4 \theta_{x,\tau}),$$

where $x$ labels the coordinates of a lattice site in 2D spatial dimension and $\tau$ labels an imaginary time slice. $\langle x, x' \rangle$ denotes nearest neighbors. $\theta_{x,\tau}$ is the angle of the planar spins and $\Delta \theta_{x,x',\tau} \equiv \theta(x, \tau) - \theta(x', \tau)$ are bond variables connecting two spatial sites. The first term is the spatial coupling of spins as in classical XY model, while the second term is the kinetic energy. The third term describes quantum dissipations of the ohmic or Caldera-Leggett type [23]. The physical origin of such a term in the context of loop-current order in cuprates has been discussed [6]. The last term favors the four angular directions $\theta_{x,\tau} = (0, \pi/2, \pi, 3\pi/2) + 2n\pi$ when $h_4 < 0$. The model is then a soft anisotropy version of the Ashkin-Teller model which describes the symmetry of loop-current order with four discrete directions of the $\theta$-variables. We recall that the anisotropy is marginally irrelevant in the classical 2D-XY model [24] in the fluctuation regime. Strictly speaking, we should add a term with interactions $\propto \cos[2(\theta_{x,\tau} - \theta_{x',\tau})]$ to represent the Ashkin-Teller model completely. Such a term has been shown to be irrelevant in the analytic calculations [6] in the fluctuation regime. We have verified this assertion in the Monte-carlo calculations. We choose $K$, $K_\tau$, $\alpha$, and $h_4$ as independent variables and tune them separately.
B. Analytic transformation of the Model: Warps, Vortices etc.

It is useful to briefly review the analytic solution of the model in order to understand several aspects of the Monte-Carlo results including the physics of the three different phases found in Ref. [22] and the mechanism for the (almost) spatial locality of the fluctuations.

In Ref. [6], after making a Villain transformation [25] and integrating over the small oscillations or spin-waves, the action is expressed in terms of link variables which are differences of $\theta$'s at nearest neighbor sites, as shown in Fig. (1).

$$m_{x,x'}(\tau, \tau') \equiv \theta(x, \tau) - \theta(x', \tau').$$  \hfill (2)

Further

$$m = m_\ell + m_t$$ \hfill (3)

where $m_\ell$, is the longitudinal (or curl-free) part and $m_t$ is the transverse (or divergence-free) part. The appearance of $m_\ell$ is a novel feature of the quantum dissipative XY-model. Now define

$$\nabla \times m_t(x, \tau) = \rho_v(x, \tau)\hat{z},$$ \hfill (4)

so that $\rho_v(x, \tau)$ is the charge of the vortex at $(x, \tau)$, and

$$\frac{\partial \nabla \cdot m_t(x, \tau)}{\partial \tau} = \rho_w(x, \tau).$$ \hfill (5)

$\rho_w(x, \tau)$ is called the “warp” at $(x, \tau)$.

Although a continuum description is being used for simplicity of writing, it is important to do the calculation so that the discrete nature of the $\rho_v, \rho_w$ fields is always obeyed. In the numerical implementation of (2+1)D discrete lattice, given the two bonds per site $(x)$, one may construct a vector field $m_{x,\tau}$, whose components are the two directed link variables in the Cartesian directions:

$$m^x_{i,j,\tau} = \theta_{i+1,j,\tau} - \theta_{i,j,\tau},$$
$$m^y_{i,j,\tau} = \theta_{i,j+1,\tau} - \theta_{i,j,\tau},$$ \hfill (6)

as shown in Fig. (1). Here $x = (i, j)$. 

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In terms of the vortex and warp densities, the action of the model was shown to be

\[ S = \sum_{k, \omega_n} \frac{K}{k^2} |\rho_v(k\omega_n)|^2 - \frac{\alpha}{4\pi |\omega_n|} |\rho_w(k\omega_n)|^2 
- G(k, \omega_n) \left( KK_\tau - \frac{\alpha K_\tau |\omega_n|}{4\pi} - \frac{\alpha^2 k^2}{16\pi^2} \right) |\rho_w(k\omega_n)|^2, \]

where

\[ G(k, \omega_n) = \frac{1}{Kk^2 + K_\tau \omega_n^2 + \alpha |\omega_n| k^2}. \]

The first term is the action of the classical vortices interacting with each other through logarithmic interactions in space but the interactions are local in time. The second term describes the warps interacting logarithmically in time but locally in space. In the third term, the terms proportional to \( \alpha \) may be dropped in both the numerator and the denominator. Then this is just the action for a Coulomb field, which if present alone is known to not cause a transition and is therefore marginally irrelevant in the present problem. The warp and the vortex variables in the first two terms are orthogonal. With just these two terms alone, the problem is exactly soluble. If the first term dominates, one expects a transition of the class of the classical Kosterlitz-Thouless transition through binding of vortex-anti-vortex pairs in space but there is nothing to order the vortices with respect to each other in time. If the second term dominates, there is a quantum transition to a phase with binding of warp-antiwarp pairs in time but nothing to order them with respect to each other in space. Given the ordering driven by either the density of isolated vortices or of isolated warps \( \to 0 \), the flow from one to the other is determined by the third term leading to possible ordering at \( T = 0 \) both in time and space. It was derived in Ref. \[6\] that when the ordering is driven through warps the fluctuations of the order parameter at the critical point have \( 1/\tau \) correlations, in time at the critical point \[\text{[which on appropriate thermal Fourier transformation gives a spectral function } \tanh (\omega/2T)]\].

C. Quantum Monte-Carlo simulations

We follow the numerical procedure as in Ref. \[22\] for the Monte-Carlo simulations. We choose \( \theta_{x,\tau} \) to be a discrete variable, \( n2\pi/32 \). The system size is typically chosen as \( N = 50 \) and \( N_\tau = 200 \) unless specified. We start from a random configuration of \( \{\theta_{x,\tau}\} \). To update
the configuration, we sequentially sweep the lattice or randomly select lattice sites to change \( \theta_{x,\tau} \) to \( \theta_{x,\tau} + \theta' \), where \( \theta' \) is a random angle between \(-2\pi\) and \(2\pi\). After some warm-up sweeps, we make measurements of the interested physical quantities while updating the lattice. The number of such measurements is typically larger than \(10^6\) in our Monte-Carlo simulations. For large enough measurements, the desired thermodynamic averages and correlation functions are well approximated.

The following quantities are calculated to characterize the different phases and the transitions between them.

**Action susceptibility.** The action susceptibility is defined as

\[
\chi_S = \frac{1}{N^2 N_\tau} \left( \langle S^2 \rangle - \langle S \rangle^2 \right),
\]

where \( \langle \ldots \rangle \) denotes averaging over the \(O(10^6)\) Monte-Carlo measurements. In classical systems, as \( S = \beta H \), \( \chi_S \) is related to the specific heat, \( \chi_S = C_V/k_B \). At \( T \to 0 \), it is a measure of zero-point fluctuations which are expected to be singular at the critical point due to the degeneracy in the spectra.

**Helicity Modulus.** The helicity modulus or spatial stiffness is defined from the change of energy resulting from the slow twist of spins along the spatial direction, or

\[
\Upsilon_x = \frac{1}{N^2 N_\tau} \left\langle \sum_{(x,x')} \sum_\tau \cos(\Delta \theta_{x,x',\tau}) \right\rangle - \frac{K}{N^2 N_\tau} \left\langle \left( \sum_{(x,x')} \sum_\tau \sin(\Delta \theta_{x,x',\tau}) \right)^2 \right\rangle.
\]

In the disordered state, the two terms have comparable contributions and \( \Upsilon_x \to 0 \). In an ordered phase, the second term vanishes while \( \Upsilon_x \) becomes finite.

**Order parameter.** For XY spins, the order parameter \( M(x, \tau) = (\cos \theta_{x,\tau}, \sin \theta_{x,\tau}) \). Its modulus, the magnetization in the plane, is defined as

\[
M = \frac{1}{N^2 N_\tau} \left\langle \left| \sum_{x,\tau} e^{i\theta_{x,\tau}} \right| \right\rangle.
\]

In classical 2D XY model, the ordered phase has a quasi long-range order, where \( M \sim (1/N)^{1/(8\pi K)} \), vanishes for \( N \to \infty \). A question which we will be able to answer is whether there is a finite magnetization in the infinite size limit for the quantum dissipative XY model. We also found it illuminating to calculate \( M_{2D} \), the magnitude of magnetization in
the planes at a given time $\tau$ and then average it over the $\tau$. This is equivalent to finding the Kosterlitz-Thouless order parameter at each time slice and then averaging over the time slices.

$$M_{2D} = \frac{1}{N^2 N_\tau} \left\langle \left| \sum_x e^{i\theta_{x,\tau}} \right| \right\rangle. \quad (12)$$

By definition $M \leq M_{2D}$. Also $M = M_{2D} \neq 0$ (for $N \to \infty$) only if there is perfect long range order across time as well as space.

**Correlation Function of the Order Parameter.** The principal results for the quantum-critical fluctuations are given by the order parameter correlation functions:

$$G_\theta(x, \tau) = \frac{1}{N^2 N_\tau} \sum_{x', \tau'} \left\langle e^{i(\theta_{x+x'+\tau'\tau} - \theta_{x'+\tau'})} \right\rangle. \quad (13)$$

$G_\theta(x \to \infty, \tau \to \infty) \to M^2$ while $G_\theta(x \to \infty, \tau = 0) \to M^2_{2D}$. In Ref. [22], mean-square displacements in time $W_{2\theta}^2$ are shown, which we have reproduced. These can also be obtained from the second moment of the above correlation function at $x = 0$.

**FIG. 1:** Examples of vortex (a) and warp (b) excitations in numeric simulation. The numbers at the lattice points (in space or time) are the $\theta$’s in unit of $2\pi/32$ and are non-compact variables. The numbers in the links are the velocity fields, i.e. the difference of $\theta$’s that a link connects. (a) For the plaquette shown, $(\nabla \times \mathbf{m})_{i,j,\tau}$ is 32, or $2\pi$, showing a vortex. In (b), the change of $(\nabla \cdot \mathbf{m})_{i,j,\tau}$ for two neighboring time slices is close to $-2\pi$, showing an antiwarp.

**Vortices and warps: densities and self/mutual correlations.** The curl of the vector field $\mathbf{m}$ can be calculated numerically from the four link variables of a plaquette,

$$\rho_v(x, \tau) = \frac{1}{2\pi} (\nabla \times \mathbf{m})_{i,j,\tau} = (m^x_{i,j,\tau} + m^y_{i+1,j,\tau} - m^x_{i+1,j+1,\tau} - m^y_{i,j+1,\tau})/(2\pi), \quad (14)$$
where we restrict \( m^{x,y} \) to be within \((-\pi, \pi)\) by adding or subtracting \( n2\pi \). If \((\nabla \times \mathbf{m})_{i,j,\tau} = \pm 2\pi\), we identify a vortex/antivortex, or \( \rho_v(x, \tau) = \pm 1 \). Similarly, the divergence of the vector field can be calculated from four links connected to the site

\[
(\hat{\nabla} \cdot \mathbf{m})_{i,j,\tau} = (m^x_{i,j,\tau} - m^x_{i-1,j,\tau} + m^y_{i,j,\tau} - m^y_{i,j-1,\tau})/4. \tag{15}
\]

We therefore use the following criterion to identify a warp(antiwarp) charge

\[
\rho_w(x, \tau) = 1, \quad \text{if } (\hat{\nabla} \cdot \mathbf{m})_{i,j,\tau+1} - (\hat{\nabla} \cdot \mathbf{m})_{i,j,\tau} > 2\pi - \delta\theta
\]

\[
\rho_w(x, \tau) = -1, \quad \text{if } (\hat{\nabla} \cdot \mathbf{m})_{i,j,\tau+1} - (\hat{\nabla} \cdot \mathbf{m})_{i,j,\tau} < -2\pi + \delta\theta. \tag{16}
\]

where \( \delta\theta \ll 2\pi \) to accommodate small angle changes due to spin waves. Examples of vortices and warps are also shown in Fig. (1).

After identifying the vortex and warp charges \( \rho_{v,w}(x, \tau) \) in the system, we can calculate their densities,

\[
\rho_{v,w} = \frac{1}{N^2 N_\tau} \sum_{x,\tau} \langle |\rho_{v,w}(x, \tau)| \rangle, \tag{17}
\]

as well as their correlation functions:

\[
G_{v,w}(x, \tau) = \frac{1}{N^2 N_\tau} \sum_{x',\tau'} \langle \rho_{v,w}(x + x', \tau + \tau) \rho_{v,w}(x', \tau') \rangle. \tag{18}
\]

Charge neutrality for both vortices and warps should be preserved. We verify this by calculating the net density \( \delta \rho_{v,w} = [\sum_{x,\tau} \langle \rho_{v,w}(x, \tau) \rangle]/(N^2 N_\tau) \), and find that in practice, \( |\delta \rho_v|/\rho_v < 10^{-5} \) and \( |\delta \rho_w|/\rho_w < 10^{-2} \). To capture the correlations between warps and vortices, we also calculate

\[
G_{vw} = \frac{1}{N^2 N_\tau} \sum_{x,\tau} \langle |\rho_v(x, \tau) \rho_w(x, \tau)| \rangle, \tag{19}
\]

i.e., the probability to find vortices in the vicinity of a warp and vice versa. If warps and vortices are not correlated, we expect \( G_{vw} = \rho_v \rho_w \).

### III. SUMMARY OF THE PHASE DIAGRAM

We first study the dissipative XY model [cf. Eq. (1)] without the four-fold anisotropic field \( h_4 \), whose effect is addressed in Sec. [VII]. The phase diagram in \( \alpha-K \) plane with fixed
FIG. 2: Phase diagram for the quantum dissipative XY model in $\alpha - K$ plane. Here $K_\tau = 0.01$. The transition points are determined from the non-analyticity in various static properties with a system size $N = 50$ and $N_\tau = 200$ (the transition points for infinite systems can be determined by a finite-size analysis). The area where the lines join (blue shaded area) has not been explored thoroughly enough to precisely determine how the phase boundaries meet.

$K_\tau = 0.01$ is given in Fig. (2). It is similar to the $K_\tau = 0.002$ phase diagram obtained in Ref. [22]. The “Disordered” phase (called NOR in Ref. [22]) has a transition to a “Quasi-ordered” phase (called CSC in Ref. [22]) for large enough $K$ to a phase with spatial binding of vortices as in a KT transition, but with only short-range correlations in time. The spatial correlations at $\tau = 0$ obey the KT properties. For increasing $\alpha$, this phase also orders in time to an “Ordered” phase (called FSC in Ref. [22]). For small $K$, there is a direct phase transition from the Disordered to the Ordered phase. This is in general in accord with the discussion in the previous paragraph based on the properties expected for the topological model of Eq. (7). We will show here that the transition from the Quasi-ordered to the Ordered phase as well as that from the Disordered phase to the Ordered phase occur primarily through freezing of warps. In the second transition, the vortices freeze as an accompaniment to the freezing of warps, in a manner distinct from that at the KT transition.
We summarize the properties of these three phases in Table I. The Disordered state has exponential decay in both the spatial and temporal directions. The Quasi-ordered phase is the phase with KT spatial order $M_{2D}$, which falls off slowly for large $N$. The order parameter $M$ follows $M_{2D}$ asymptotically for large spatial size, as shown in Fig. 3. The Ordered phase has long range order in both spatial and temporal directions, meaning that it goes to a finite value as $N, N_{\tau} \to \infty$.

| Quantity | Disordered | Quasi-ordered | Ordered |
|----------|------------|---------------|---------|
| $M$      | 0          | decreases $\to 0$ for $N \to \infty$ | finite  |
| $\rho_v$ | $O(1)$     | $\ll 1$       | $\ll 1$ |
| $G_v(x)$ | exponential | power law     | power law |
| $Y_x$    | 0          | finite, jump at transition | finite, no jump at transition |
| $\rho_w$ | $O(1)$     | $O(1)$        | $\ll 1$ |
| $G_w(\tau)$ | $1/\tau^2$ | $1/\tau^2$ | $1/\tau^\alpha (\alpha > 2)$ |
| $G_\theta(x,0)$ | exponential | quasi-long range | long range |
| $G_\theta(0,\tau)$ | exponential | exponential | long range |

TABLE I: Characteristic properties of three phases of the dissipative XY model. Definitions of these quantities are provided in Sec. II C.

IV. TRANSITION FROM THE DISORDERED PHASE TO THE QUASI-ORDERED PHASE

The transition from the Disordered phase to the Quasi-ordered phase is studied by fixing $\alpha = 0.01$ and varying $K$. The static properties are show in Fig. 4. We find that above a critical value $K_c$, which weakly depends on $\alpha$, the spatial magnetization $M_{2D}$ becomes finite. As shown in Fig. 3, $M_{2D}$ decreases slowly when $N$ increases. As discussed later, this decrease is consistent with the logarithmic decrease found in earlier calculations [29]. $M \to M_{2D}$ when $N_{\tau} \ll N$ and $M \to 0$ when $N_{\tau} \gg N$. The difference between $M$ and $M_{2D}$ is also reflected in the order parameter correlations in the time direction, which shows oscillatory features at long times (not shown), which are due to the finite size of the simulations. This phase has only quasi long-range (power law) spatial order. As shown in
FIG. 3: The spatial size dependence of $M_{2D}$ and $M$ in the Quasi-ordered (top two curves) and Ordered phases (bottom 2 curves). Also shown is $\Upsilon_x$ in the Ordered phase. The parameters chosen for the Quasi-ordered phase are $K_\tau = 0.01$, $\alpha = 0.01$, $K = 1.3$ and $N_\tau = 20$, while for the Ordered phase, $K_\tau = 0.01$, $\alpha = 0.03$, $K = 0.4$ and $N_\tau = 100$. The slow decrease of the top most curve ($M_{2D}$) is just the finite size scaling to 0 at $N \to \infty$ in the Kosterlitz-Thouless phase [29], which $M$ asymptotically joins. Their behaviors are different in the Ordered Phase, where asymptotically, they are both consistent with a finite value at $N \to \infty$. The difference in $\Upsilon_x$ across the Disordered to Quasi-Ordered and across the Disordered to Ordered phases is discussed in the text.

Ref. [22], the helicity modulus $\Upsilon_x$ becomes finite in the Quasi-ordered phase. Finite size scaling of the helicity modulus $\Upsilon_x$ shows a Nelson-Kosterlitz [27] jump at $K_c$. This is related to the vortex density seen in Fig. [4], which decreases with increasing $K$, and changes slope at $K_c$. These are consistent with KT transition in classical 2D XY model. Meanwhile, we find that in the temporal direction, all quantities remain relatively unchanged from those in the disordered phase. The vortex-warp correlation $G_{vw} \approx \rho_v \rho_w$, indicating vortices and warps are not correlated, in either the Disordered phase or the Quasi-ordered phase.

We also plot the correlation functions of warps and vortices in Fig. [4]. For the equal-time vortex correlation $G_v(x,0) \equiv G_v(x,y = 0,\tau = 0)$, $G_v(0,0) = \rho_v > 0$ (not shown due to the logarithmic scale) while $G_v(x \neq 0,0) < 0$, reflecting that the vortex-antivortex correlations dominate at long distance. When $K$ increases, $-G_v(x)$ changes from an exponential decay
FIG. 4: Static properties (top panel), vortex and warp correlation functions (bottom panels) of transition from the Disordered phase to the Quasi-ordered phase. Here, $K_\tau = 0.01$, $\alpha = 0.01$, and $K$ is varied. The results shown are for $N = 50$ and $N_\tau = 200$. Note that some quantities are scaled to fit in the figure. The vortex density changes rapidly below the transition while the warp density remains smooth. Other aspects of the transition and of the Quasi-ordered phase are discussed in the text.

in the disordered phase to a power-law decay in Quasi-Ordered phase. These are consistent with the KT transition as well. The warp correlation along temporal direction at the spatial site $G_w(0, \tau)$ also satisfies $G_w(0, 0) = \rho_w > 0$ and $G_w(0, \tau \neq 0) < 0$. In this transition, it remains unchanged in asymptotic form $\propto 1/\tau^2$. 
FIG. 5: Static properties (top panel), vortex and warp correlation functions (bottom panels) of transition from the Quasi-ordered phase to the Ordered phase. Here, $K_T = 0.01$, $K = 1.5$, and $\alpha$ is varied. The results shown are for $N = 50$ and $N_T = 200$.

V. TRANSITION FROM THE QUASI-ORDERED PHASE TO THE ORDERED PHASE

We choose a suitable $K$ and tune the transition from the Quasi-Ordered phase to the Ordered phase by increasing the dissipation strength $\alpha$. Various static properties as functions of $\alpha$ and correlation functions for selected $\alpha$’s are shown in Fig. (5). The peak in the action susceptibility $\chi_S$ implies a phase transition at $\alpha_c \approx 0.02$. We find that properties characterizing spatial orders, such as $M_{2D}$, $\rho_v$ and $\Upsilon_x$, have small non-analytic changes, as already discovered in Ref. [22]. The significant changes are properties characterizing temporal order. The asymptotic behavior of the warp density is similar to that of vortex density at KT transition: it changes slope at $\alpha_c$ and decreases exponentially as $\alpha$ further increases. $M$ keeps increasing and saturates to $M_{2D}$ at $\alpha \gg \alpha_c$ (at large system sizes). The
warp correlation functions decay faster for larger \( \alpha \), changing from \( 1/\tau^2 \) in the Quasi-Ordered phase to \( 1/\tau^a \) (\( a \sim 3 \) for \( \alpha = 0.023 \) in the figure) in the Ordered phase. This indicates that warps and anti-warps, which are free in the Quasi-Ordered phase, also are bound in the ordered phase. Near \( \alpha_c \), a slower decay at large times is observed. As shown in the figure, it can be fitted as \( 1/\tau \). This is in agreement with the analytical analysis \[6\]. While the vortex-warp correlation \( G_{vw} \sim \rho_v \rho_w \) in the Quasi-ordered phase, we find \( G_{vw} > \rho_v \rho_w \) in the Ordered phase, and their difference increases when \( \alpha \) is increased from \( \alpha_c \). This implies that vortices and warps are correlated inside the Ordered phase.

VI. TRANSITION FROM THE DISORDERED PHASE TO THE ORDERED PHASE

This is the part of the problem which we shall discuss most thoroughly. We show results for a suitable value, \( K = 0.4 \) and tune \( \alpha \) across the transition at \( \alpha = \alpha_c(K) \). Similar results have been obtained for other values of these parameters across the transition, keeping \( K_\tau = 0.01 \) fixed at this low value.

A. Static properties and correlations

The static properties shown in Fig. (6) are all non-analytic near \( \alpha_c \approx 0.0260 \). We estimate an uncertainty of \( \pm 0.0002 \) in \( \alpha_c \), due to finite size effects. The helicity modulus \( \Upsilon_x \) and magnetization \( m \) become finite for \( \alpha > \alpha_c \). We notice that both the vortex and the warp densities change slope across \( \alpha_c \). So long-range order appears to develop simultaneously along both the spatial and the temporal directions. However, the warp density has a more rapid change than the vortex density. We also observe that the mutual correlation between vortices and warps \( G_{vw} \propto \rho_w \) and \( G_{vw} > \rho_v \rho_w \) when \( \alpha > \alpha_c \), i.e., suggesting coupling of vortices to warps deep inside the Ordered phase.

The self correlation functions of vortices and warps are also shown in Fig. (6). The vortex correlation functions are relatively unchanged as \( \alpha \) changes across the transition compared to the warp correlation functions, which rather have similar changes as in the Quasi-ordered phase to the Ordered phase transition. Near \( \alpha_c \), the latter shows slower decay at long times. However, whether it has the \( 1/\tau \) behavior requires a calculation with larger time slices and
FIG. 6: Static properties (top panel), vortex and warp correlation functions (bottom panels) of transition from the Disordered phase to the Ordered phase. Here, $K = 0.01$, $K = 0.4$, and $\alpha$ is varied. The results shown are for $N = 50$ and $N = 200$. We find that when $M_{2D}$ becomes finite, $M$ varies with the size and only approaches $M_{2D}$ at large system size [see Fig. 3]. The vortex and warp correlations are discussed in the text.

more iterations to demonstrate.

B. Size dependence of $\Upsilon_x$ and $M$

Although the spatial and temporal orders appear to develop simultaneously, more significant changes in warps rather than in vortices suggest the transition is driven by the warp dynamics. We study this point by contrasting the scaling behavior of the Helicity modulus $\Upsilon_x$ at the transition from that at the KT transition. We perform a finite size scaling analysis on $\Upsilon_x$ and the order parameter $M$, and compare their behaviors with those in KT transition. The results for two set of parameters in the Quasi-ordered phase and the Ordered phase have
been shown in Fig. (3).

In the classical XY-model, the helicity modulus scales with the finite size $N$ of the system as

$$\Upsilon_x(N) = \Upsilon_x(\infty) \left(1 + \frac{1}{2 \ln N + C}\right),$$

(20)

where $C$ is an undetermined constant [28]. At the KT transition point $K = K_c$, the helicity modulus has a jump $\Upsilon_x(\infty)K_c = 2/\pi$. Both the finite size scaling and the value at the jump have been verified [22] at the Disordered to the Quasi-ordered transition. The behavior is quite different in the ordered phase. The stiffness $\Upsilon_x(N)$ in this transition already develops for $\alpha > \alpha_c$ at small sizes and remains unchanged with $N$. For $\alpha < \alpha_c$, $\Upsilon_x(N)$ decrease exponentially.

The magnetization in the Quasi-Ordered KT phase is 0 in the limit $N \rightarrow \infty$. But the passage to this limit is very slow [29]. The finite size scaling is quite different at the Ordered state as shown in Fig. (3). While $M_{2D}$ decreases with $N$ at small $N$, it is consistent with saturation at a finite value at large $N$, merging with the value of $M$. As discussed immediately after the definition of $M$ and $M_{2D}$ above, this is consistent with a truly Ordered state.

FIG. 7: The order parameter correlation functions $G_\theta(x, \tau)$ for transition from the Disordered phase to the Ordered phase. Parameters are the same as in Fig. (3). We show $G_\theta(x, \tau)$ as a function of $x$ for fixed $\tau = 2$ (left panel) and as a function of $\tau$ for fixed $x = 2$. 
C. Scaling of the order parameter and other correlation functions

The most revealing results about the critical properties are of course obtained from the order parameter correlation functions. It is seen in Fig. (7) that there exists a separatix in $G_{\theta}(x, \tau)$ for a fixed $x$ and for a fixed $\tau$ such that, for $\alpha < \alpha_c$ the asymptotic correlation $\to 0$ for large $\tau$, and for $\alpha > \alpha_c$, they tend to a constant value depending on $\alpha$. We present scaling analysis of the order parameter correlation functions on the disordered side.

![Graph](image.png)

**FIG. 8:** Scaling analysis of the order parameter correlation function $G_{\theta}(x, \tau)$ for fixed $x = x_0$ (left panel) and for fixed $\tau = \tau_0$ (right panel) from the disordered side of the transition. In the left panel, we fit each curve of $\tau G_{\theta}(x_0, \tau)$ with the form $A_\tau(x) \exp[-(\tau/\xi_\tau)^{1/2}]$, where the amplitude $A_\tau$ and the correlation length $\xi_\tau$ are fitting parameters adjusted for each $\alpha$ and $x$. In the right panel, we fit each curve of $G_{\theta}(x, \tau_0)$ with $A_x(\tau) \exp(-x/\xi_x)$ where $A_x(\tau)$ and $\xi_x$ are fitting parameters. The results of $\xi_\tau(x_0, \alpha)$ and $\xi_x(\tau_0, \alpha)$ are shown in Fig. (9). We find that $A_\tau \approx \tau_c \exp(-x/\xi_{0,x})$ with $\tau_c \approx 0.12$ and $\xi_{0,x} \approx 1.0$, and $A_x \approx (\tau_c/\tau) \exp[-(\tau/\xi_\tau(\alpha - \alpha_c))^{1/2}]$ with $\xi_\tau(\alpha - \alpha_c)$ given in Eq. (22).

It is expected that all curves of $\tau G_{\theta}(x_0, \tau)/A_\tau$ for difference $\alpha$ and $x_0$ collapse into a single curve $\exp(-t)$ with $t = (\tau/\xi_\tau)^{1/2}$, which are plotted (for clarity, they are rescaled by a factor $10^{(x_0/2)}$ for different $x_0$). $G_{\theta}(x, \tau_0)/A_x$ as functions of $x/\xi_x$ are plotted in the same fashion. Because of the rapid decay of the correlation function in this range of $\alpha$, it has not been numerically possible to follow its behavior for larger $x$ and $\tau$.

We find that the leading asymptotic behaviors of $G_{\theta}(x, \tau)$ can be captured in the scaling form

$$G_{\theta}(x, \tau) = \frac{1}{\tau} e^{-(\tau/\xi_\tau)^{1/2}} e^{-x/\xi_x}. \quad (21)$$
This is shown for a few fixed $x$ as a function of $\tau$ in the left panel of Fig. (8) and for a few fixed $\tau$ as a function of $x$ in the right panel of the same figure. We show the dependence of $\xi_x$ and $\xi_\tau$ on $\alpha - \alpha_c$ in Fig. (9).

In the fluctuation regime not too close to the critical point in the disordered side, for $(\alpha_c - \alpha)/\alpha_c \gtrsim 0.1$ with $\alpha_c \approx 0.0260$, we observe that in the parameter range shown, $\xi_\tau$ increases by a decade when $\alpha \to \alpha_c$ while $\xi_x$ remains relatively unchanged $\xi_x \approx \xi_{0,x} \approx 1.0$, i.e., a lattice constant. In this range of $\alpha$, the behavior of $\xi_\tau$ is consistent with

$$\xi_\tau(\alpha - \alpha_c) = \tau_c e^{a\sqrt{\alpha_c/(\alpha_c - \alpha)}},$$

(22)

where $a$ is a constant of $O(1)$. This relation, as well the leading behavior of the correlation function $G_{\theta}(x, \tau)$

$$G_{\theta}(x, \tau) \approx \frac{1}{\tau} e^{-(\tau/\xi_\tau)^{1/2}} e^{-x/\xi_{0,x}},$$

(23)

are identical to those derived analytically. $\tau_c$ is the short-time cutoff scale. It was also derived that, within factors of $O(1)$, $\tau_c = 1/\sqrt{K/K\tau}$. For the parameters chosen, $1/\sqrt{K/K\tau} = 0.16$, while the numerically obtained value is $\tau_c \approx 0.12$.

![Diagram](image.png)

FIG. 9: The left panel shows $\xi_x$ and $\xi_\tau$ as functions of $[\alpha_c/(\alpha_c - \alpha)]^{1/2}$. They have been rescaled to their respective values at $\alpha = 0.020$. For $x_0 = 0$, $\xi_\tau$ can be fitted as $\tau_c \exp[0.62 \sqrt{\alpha_c/(\alpha_c - \alpha)}]$; the numerical coefficient in the exponent changes to about 1 for $x_0 = 4$. The right panel shows the relation between $\xi_x(\alpha)$ and $\xi_\tau(\alpha)$. We find that $\xi_x/\xi_0 \sim \ln(\xi_\tau/\tau_c)$. This relation appears to become independent of $x$ and $\tau$ at large $x$ and $\tau$. Finite size effects do not permit a detailed exploration beyond $\xi_\tau/\tau_c \approx 70$. 

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However, for $\frac{\alpha - \alpha_c}{\alpha_c} \lesssim 0.1$ on the disordered side, there are deviations from Eqs. (22) and (23). For example, we notice in Fig. (7), a crossover from an exponential to a power law behavior in the spatial correlation as $\alpha \to \alpha_c$ before going to a constant value on the ordered side, consistent with true long-range order. As shown in the left panel of Fig. (9), $\xi_\tau$ also increases when $\alpha \to \alpha_c$, though much slower compared to $\xi_x$. Their monotonically growth suggests scaling one with respect to the other. In the right panel of the same figure, we show that within our numerical capabilities that

$$\xi_x \approx \xi_{0,x} \ln \left( \frac{\xi_\tau}{\tau_c} \right),$$

i.e, the spatial correlation length is consistent with growing as the logarithm of the temporal correlation length \[31\]. This means that the dynamical critical exponent is $z = \infty$. One should expect, as is consistent with Fig. (9), transients for $x \lesssim \xi_x$ and $\tau \lesssim \xi_\tau$ approaching the forms given above.

These properties, as well as what has been calculated above about the vortices, appear to be consistent with the suggestion \[6\] that when warps begin to freeze, spin-waves might develop a gap so that the vortices also order but no derivation for this was given. The approximate correlation function \[23\] is a separable function of space and time, and so is the final form of the correlation function \[21\]. However, a weak $\tau$ dependence $\propto \ln \left( \frac{\tau}{\tau_c} \right)$ cannot be excluded in $\xi_x$. This question can only be settled by further analytical calculations, possibly by a proper renormalization group calculation of the effect of the last term in Eq. (7).

From the results here as well as from Ref. \[6\], the phenemenological expression \[8\] for quantum-critical fluctuations acquires a cross-over towards purely quantum-fluctuations below a cross-over temperature $T_x \approx \xi^{-1}_x$. This presents an essential singularity at the critical point in terms of the tuning parameter of the transition, $\alpha - \alpha_c(K, K_\tau)$.

We have presented results for the correlation lengths as a function of $(\alpha - \alpha_c)$. As is evident from the phase diagram, $\alpha_c$ depends on $K$, and (not explored in this paper) on $K_\tau$, as well. Away from the meeting point of the three transitions, $\alpha_c$ depends smoothly on $K$. Therefore, we should expect that for fixed $\alpha$, the change of correlation length is the same function of $(K - K_c)$ as it is of $(\alpha - \alpha_c)$ for a fixed $K$. However, this point could benefit from further study.
VII. THE EFFECT OF FOUR-FOLD ANISOTROPY

We now turn on the four-fold anisotropic field \( h_4 \) in the Monte-Carlo simulation to study its effect. In the classical XY model, 4-fold anisotropy is marginally irrelevant \[24\]. In the quantum model, it has been argued \[6\] to be irrelevant. When \( h_4 \to \infty \), XY spins become two Ising variables, as in Ashkin-Teller model. We focus on the transition from the Disordered phase to the Ordered phase, by choosing \( K = 0.4, K_\tau = 0.01 \) and tuning \( \alpha \) for transitions for different \( h_4 \). We find that the transition persists and all quantities have similar properties across the transition as in \( h_4 = 0 \) case. In Fig. (10), we compare \( \chi_S \) and \( M \) for three different values of \( h_4 = 0, 1, 5 \). We find that up to \( h_4 = 1 \), the properties are almost the same as in \( h_4 = 0 \). In \( h_4 = 5 \), we notice that \( \alpha_c \) has been shifted to 0.0272, and the peak in \( \chi_S \) is sharper. \( M \) increases more rapidly.

We further show the scaling results of the spin correlation functions for \( h_4 = 5 \) in Fig. (11). We find similar behaviors as in \( h_4 = 0 \) case, which indicates that the transition is also of the local critical type.

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FIG. 10: \( \chi_x \) and \( M \) for different \( h_4 \). Here, \( K = 0.4, K_\tau = 0.01 \) and \( \alpha \) is varied. The system size is kept the same, \( N = 50 \) and \( N_\tau = 200 \).
FIG. 11: Order parameter correlation functions for $h_4 = 5$. The left panel shows scaling analysis of the $x = 0$ spin correlation function $G_\theta(0, \tau)$ from the disordered side of the transition. The fitting curve is similar to that in $h_4 = 0$ case, $G_\theta(0, \tau) = (\tau_0/\tau) \exp\left[-\sqrt{\tau/\xi_\tau}\right]$ with $\tau_0 \approx 0.08$. The inset shows $\xi_\tau/\tau_0$ as a function of $\sqrt{\alpha_c/(\alpha_c - \alpha)}$. One finds that $\xi_\tau(\alpha)/\tau_0 \approx 0.5 \exp\left[\sqrt{\alpha_c/(\alpha_c - \alpha)}\right]$, with $\alpha_c = 0.0272$. The right panel shows equal-time spin correlation functions $G_\theta(x, 0)$ as functions of $x$. For $x \lesssim 10$, they have also the same form as in $h_4 = 0$ case $G_\theta(x, 0) = \exp(-x/\xi_x)$ with $\xi_x \approx 0.8$.

VIII. DISCUSSION

In this paper, the properties of the dissipative quantum XY model have been investigated by Monte-Carlo simulations to verify and extend the analytical calculations in Ref. [6] and the previous Monte-carlo simulations in Ref. [22]. While nothing from calculations at finite $N$ and $N_\tau$ is writ in stone, we have found properties consistent with local quantum-criticality of the form proposed in Ref. [8] and derived in Ref. [6] with a crossover in time/temperature to the disordered quantum state. Very importantly, we have also found a new result: a spatial correlation length which however varies very slowly, consistent with logarithmically, with the temporal correlation length. It is hoped that this result can also be derived analytically. This requires a proper treatment of the third term in the action of Eq. (7).

It is not the purpose of this paper to discuss the experiments which may be related to the findings here. But a few comments about future directions in relation to both theory and experiments may be worth-while.

The dissipative quantum XY model was first proposed [1, 2] in connection with the superconductor to insulator transition in thin superconducting films [3]. Quite correctly,
the transition as a function of dissipation was proven. But the fluctuation spectra in various calculations [30] in two dimensions were not obtained in a controlled manner and do not agree with the results presented here and in Ref. [6]. (However, the results for the one-dimensional array of Josephson junctions in a dissipative environment [32] are closely related to the results here and in Ref. [6].) Nor do the results of these calculations give the rich phase diagram found in [22] and here, which is suggested by re-expression of the model in terms of warps and vortices. It would be interesting to think of how experiments might discover the different phases in a superconducting thin film. We are also not aware of experiments to probe the fluctuation spectra at the superconductor to insulator transitions. This would also be very interesting to pursue, possibly by studying fluctuations across a Josephson junction to a three-dimensional superconductor below its transition temperature. To fully understand such possible experiments, the present work should be extended to include (the equivalent of) a magnetic field. In the work described here, the kinetic energy parameter $K_\tau$ was kept fixed at a low value. It is worth investigating if the phase diagram changes as $K_\tau$ is increased, in particular whether there is a phase transition as well as a phase of the purely 3D classical XY model.

The dissipative quantum XY model (with four-fold anisotropy) has also been proposed [4] as a model for the observed order [5] in the under-doped region of the cuprates. It is very satisfying that the phenomenological quantum-critical fluctuations, which alone have been successful in explaining the diverse anomalies in the strange metal region of these compounds, have now been proven to be the property of the fluctuations of the observed order. It is remarkable that some of the same anomalies observed in the cuprates in this region also occur in the AFM quantum-critical region of some of the heavy-fermions and in the Fe-based superconductors. This has led to the inquiry and proof [7] that a simple model of itinerant AFM can be transformed canonically to the model treated here.

An important point in relation to consistency of the theory of actual phenomena in itinerant fermion systems, as opposed to the quantum XY model by itself should be mentioned. The point of view adopted [4, 6] has been that collective modes in some itinerant fermion systems with interactions among them can be found whose properties can be described by an XY model. The resulting effective model then is that of non-interacting fermions and the XY model for the collective variables together with an interaction between the fermions and the collective variables. These interactions introduce the dissipation we have included in the
XY model. They also renormalize the fermions. In relation to systematically understanding the experimental properties of the interacting fermions, (at least) two important questions must be addressed. The conservation of the total number of degrees of freedom in the effective model and whether the fluctuation spectra of the XY model is modified by including the renormalization of the fermions in the dissipation. The first question may be important only for quantitative considerations, but the second is fundamental to the stability of the theory. In connection with the local critical fluctuation spectra and the resulting marginal Fermi-liquid renormalization, this question was addressed through showing that conservation laws are obeyed and the singularities of the fluctuation spectra (at least in the long wave-length limit) are unaltered by re-calculating the complete vertex for fluctuations in the presence of exact insertions of the renormalized fermions.

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