TILTING MODULES AND CELLULAR CATEGORIES

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ABSTRACT. In this paper we study categories of tilting modules. Our starting point is the tilting modules for a reductive algebraic group $G$ in positive characteristic. Here we extend the main result in [8] by proving that these tilting modules form a (strictly object-adapted) cellular category. We use this result to specify a subset of cellular basis elements, which generates all morphisms in this category. In a different direction we generalize the earlier results to the case where $G$ is replaced by the infinitesimal thickenings $G_rT$ of a maximal torus $T$ in $G$ by the Frobenius subgroup schemes $G_r$. Here our procedure leads to a special set of generators for the morphisms in the category of projective $G_rT$-modules. Our methods are rather general (applying to "quasi hereditary like" categories). In particular, there are completely analogous results for tilting modules of quantum groups at roots of unity. As examples we treat the tilting modules in the ordinary BGG category $O$, and in the modular case we examine $G = SL_2$ in some details.

1. INTRODUCTION

In [8] we proved that the category $\mathcal{T}(G)$ of tilting modules for a reductive algebraic group $G$ over a field $k$ of characteristic $p > 0$ has a natural cellular structure. The present paper explores this fact and proves further results about $\mathcal{T}(G)$, in particular about the morphisms in $\mathcal{T}(G)$. We find that the recent fundamental breakthroughs (e.g. [17], [1] and [18]) on how to describe the characters of the indecomposable tilting modules call for further understanding of the many mysteries concerning $\mathcal{T}(G)$.

Our first result is a very simple but nevertheless useful consequence of the cellularity of $\mathcal{T}(G)$. It singles out a collection of special cellular basis elements in the Hom-spaces between indecomposable tilting modules which generates all morphisms in $\mathcal{T}(G)$. Our approach to this theorem works quite generally and applies for instance to the corresponding quantum case as well as to the tilting modules in the BGG-category $O$, and in the modular case we examine $G = SL_2$ in some details.

Secondly, we extend the main result of [8] to include the case where $G$ is replaced by the subgroup scheme $G_rT$. Here $G_r$ denotes the $r$-th Frobenius kernel in $G$ and $T$ is a maximal torus. A module is tilting for $G_rT$ if and only if it is projective, so in this case we get a similar cellular structure on the category $\mathcal{P}_r$ of projective $G_rT$-modules. The periodicity in this category allows us to exhibit a finite set of special cellular basis elements, which generates all morphisms in $\mathcal{P}_r$. It is well known that the projective modules for $G_r$ (we may even take $r = 1$) contain the secrets of how to find the simple characters for $G$, see e.g. [5]. This fact together with the deep relations (cf. Section 4 below and the recent
work [18,19] between \( \mathcal{P}_r \) and \( \mathcal{T}(G) \) motivate our desire to explore further the categories \( \mathcal{P}_r \).

Once we have a set of generators for the morphisms in \( \mathcal{T}(G) \) and \( \mathcal{P}_r \) we want to find the relations among them. As we shall point out some relations are rather obvious. However, to get the full set of such relations is a much harder problem. We illustrate the complexity of this problem by giving some examples: We treat first the tilting modules in category \( \mathcal{O}(sl_3) \) and then in some details the projective \( G_rT \)-modules (especially for \( r = 1, 2 \)) when \( G = SL_2 \). In both these cases the indecomposable tilting modules are multiplicity free. This simplifies considerably the problem but as we shall see even in these categories to get the relations among our generating modules requires rather detailed analysis of the structures of indecomposable tilting modules.

As always in (modular) representation theory the group \( SL_2 \) is a natural starting point as well as a good test case, cf. [7] which describes completely the category of tilting modules for the quantum group for \( sl_2 \) at a complex root of unity as the module category for a quotient of the zigzag quiver algebra. The recent preprint [21] which contains a deep analysis of the tilting category for \( SL_2 \) going much further than we do here (exploring for instance natural transformations between modular Jones-Wenzl projectors) was an additional motivating factor for me to undertake the present work. I would like to thank D. Tubbenhauer for helpful comments and email discussions.

The paper is organized as follows. In Section 2 we deal with the category \( \mathcal{T}(G) \) of tilting modules for a reductive algebraic group \( G \). We introduce basic notations and collect (cf. [15]) the various results on representations of \( G \) that we need to establish the cellularity of \( \mathcal{T}(G) \). As already pointed out all ingredients needed for this can be found in [8]. But since we did not actually state the result in the generality we need here (in [8] we focused on proving that endomorphism rings of tilting modules are cellular algebras), we have carefully given the cellular data for \( \mathcal{T}(G) \) and proved that the list of required properties are all satisfied. We also check that with a little more care in our choice of bases we can obtain that \( \mathcal{T}(G) \) is a strictly object-adapted cellular category, a concept introduced in [12] (I’m grateful to D. Tubbenhauer for making me aware of this paper). Finally, we exhibit among our cellular basis elements a set of special ones and prove that they generate all the morphisms in \( \mathcal{T}(G) \).

Section 3 takes us on a detour to characteristic zero: We show that our procedure applies to the category of tilting modules in the BGG-category \( \mathcal{O} \). In particular this gives a set of generators for the morphisms in the principal block \( \mathcal{T}(0) \) of this category. We take the opportunity to explain how we can significantly refine this set of generators in the case where all tilting modules in \( \mathcal{T}(0) \) are multiplicity free. As an illustrative example we treat the case \( g = sl_3 \). In this example we furthermore find the relations satisfied by our (refined) set of generators.

In Sections 4 and 5 we return to the modular case but now we fix an \( r \geq 1 \) and consider the subgroup scheme \( G_rT \) of \( G \). The tilting category for \( G_rT \) coincides with the category \( \mathcal{P}_r \) of projective \( G_rT \)-modules. We check that this category is cellular. It is also a strictly object-adapted cellular category except that the poset involved do not satisfy dcc (as is
required in the definition given in [12]). Again we get a set of generators for the morphisms in $P_r$. Up to tensoring with 1-dimensional $G, T$-modules our set of generators is finite.

The final section, Section 6 is devoted to the case $G = SL_2$. Here the principal block in the category $P_1$ is a quotient of the zigzag quiver algebra with nodes parametrized by $Z$. The relations are the same as the ones describing the tilting modules for the quantum group $U_q(sl_2)$ with parameter equal to a complex toot of unity, cf. [7]. Now the complexity of $P_r$ grows rapidly with $r$. However, for $G = SL_2$ the indecomposable tilting modules for $G_r T$ are multiplicity free for all $r$. This allows us to describe a procedure for giving a nice set of generators for the morphisms in $P_r$. We treat the case $r = 2$ in some details and use this case to illustrate the sort of relations one should expect among our generators.

2. Reductive algebraic groups and their finite dimensional modules

2.1. Basic Notation. Throughout this paper $G$ will denote a connected reductive algebraic group over an algebraically closed field $k$. We assume that $p = \text{char } k$ is positive.

We choose a maximal torus $T$ in $G$ and denote by $X = X(T)$ its character group. In the root system $R \subset X$ for $(G, T)$ we fix a set of positive roots $R^+$ and denote by $X^+ \subset X$ the corresponding cone of dominant characters. Then $R^+$ defines an ordering $\leq$ on $X$ and $X^+$. It also determines uniquely a Borel subgroup $B$ whose roots are the set of negative roots $-R^+$. The opposite Borel subgroup of $B$ will be denoted $B^+$. We have $B = UT$ and $B^+ = U^+ T$, where $U$, respectively $U^+$, is the unipotent radical of $B$, respectively $B^+$.

Denote by $S$ the set of simple roots in $R^+$. If $\alpha$ is a root we set $\alpha^\vee$ equal to the corresponding coroot. Then the dominant cone $X^+$ above is given by

$$X^+ = \{ \lambda \in X | \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in S \}.$$ 

We set $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ and assume that $\rho \in X$. The bottom alcove in $X^+$ is

$$A = \{ \lambda \in X | 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+ \}.$$ 

The closure $\bar{A}$ is the corresponding set with the inequalities replaced by $\leq$.

We set $\mathcal{C}(G)$ equal to the category of finite dimensional $G$-modules. If $K$ is a closed subgroup (or subgroup scheme) of $G$ we write similarly $\mathcal{C}(K)$ for the category of finite dimensional $K$-modules. The elements of $X$ identifies with the set of 1-dimensional modules in $\mathcal{C}(T)$. These modules all extend uniquely to $B$ as well as to $B^+$. If $\lambda \in X$ we abuse notation and write also $\lambda$ for the 1-dimensional $T, B,$ or $B^+$ -module determined by $\lambda$.

The category $\mathcal{C}(T)$ is semisimple, i.e. each $M \in \mathcal{C}(T)$ splits into a direct sum of $\lambda$'s. We set

$$M_{\lambda} = \{ m \in M | tm = \lambda(t)m, t \in X \},$$

and call this the $\lambda$ weight space of $M$. We say that $\lambda$ is a weight of $M$ if $M_{\lambda} \neq 0$. We have then

$$M = \bigoplus_{\lambda} M_{\lambda},$$
with the sum performed over all weights of $M$. We denote by $\mathbb{Z}[X]$ the group algebra of the additive group $X$ and define the character of $M$ by

$$\text{ch } M = \sum_{\lambda} \dim M_{\lambda} e^\lambda \in \mathbb{Z}[X].$$

Here the sum again runs over all weights of $M$.

2.2. Standard, costandard and simple modules in $\mathcal{C}(G)$. Let $H \leq K$ be closed subgroup schemes in $G$. Then $\text{Ind}_H^K : \mathcal{C}(H) \to \mathcal{C}(K)$ denotes the induction functor. In general this functor does not preserve finite dimensionality, but we shall only consider cases where it does. This in particular includes the case where $H = B$ and $K = G$, which we now consider.

Let $\lambda \in X$. Then we set

$$\nabla(\lambda) = \text{Ind}_B^G \lambda.$$

We have $\nabla(\lambda) \neq 0$ if and only if $\lambda \in X^+$. When $\lambda \in X^+$ the socle of $\nabla(\lambda)$ is simple and we shall write

$$L(\lambda) = \text{soc}_G \nabla(\lambda).$$

Then the family $(L(\lambda))_{\lambda \in X^+}$ is up to isomorphisms the set of simple modules in $\mathcal{C}(G)$. We have for $\lambda \in X^+$

$$L(\lambda)_{\lambda} = \nabla(\lambda)_{\lambda} = k.$$

Moreover, all weights $\mu$ of $\nabla(\lambda)$ (and hence also of $L(\lambda))$ satisfy $\mu \leq \lambda$.

If $M \in \mathcal{C}(G)$ we denote by $[M : L(\lambda)]$ the composition factor multiplicity of $L(\lambda)$ in $M$. Then the above shows that

$$[\nabla(\lambda) : L(\lambda)] = 1 \text{ and if } [\nabla(\lambda) : L(\mu)] \neq 0 \text{ then } \mu \leq \lambda.$$

Recall from [15] II.1.16 the antiautomorphism $\tau$ of $G$. Then we define $\tau M$ to be the linear dual $M^*$ of $M$ with $G$-action given by

$$gh : m \mapsto h(\tau(g)m) \text{ for all } g \in G, h \in M^*, m \in M.$$ 

The duality functor $M \mapsto \tau M$ on $\mathcal{C}(G)$ preserves weights and dimensions of weight spaces (i.e. $\text{ch } M = \text{ch } \tau M$). Therefore we see that $\tau L(\lambda) \simeq L(\lambda)$ for all $\lambda \in X^+$.

Define now

$$\Delta(\lambda) = \tau \nabla(\lambda).$$

This is often called the Weyl module (with highest weight $\lambda$) because it has character given by $\text{ch } \Delta(\lambda) = \chi(\lambda)$, where $\chi(\lambda)$ is the Weyl character.

A Weyl module is usually not simple (in contrast with what happens in characteristic zero). However, this is so for certain special weights including the following

(2.1) If $\lambda \in \tilde{A} \cap X^+$ or if $\lambda = (p^r - 1)\rho$ for some $r \geq 1$ then $\nabla(\lambda) = \Delta(\lambda) = L(\lambda)$.

This follows from the strong linkage principle [2].

We set $St_r = L((p^r - 1)\rho)$ and call this module the $r$-th Steinberg module.
2.3. Tilting modules for $G$. A crucial relation between Weyl and dual Weyl modules is

\[ \text{(2.2)} \quad \text{Let } \lambda, \mu \in X^+. \text{ Then } \text{Ext}^i_G(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k & \text{if } \mu = \lambda \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \]

A module $M \in \mathcal{C}(G)$ is said to have a \(\nabla\)-, respectively a \(\Delta\)-filtration, if there exists a sequence of submodules

\[ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \]

with $M_i/M_{i-1} \cong \nabla(\lambda_i)$, respectively $\Delta(\mu_i)$, for some $\lambda_i, \mu_i \in X^+$, $i = 1, 2, \cdots, r$.

If $M$ has a \(\nabla\)-filtration as above we set

\[ (M : \nabla(\lambda)) = |\{i| \lambda_i = \lambda\}|, \]

and in case of a \(\Delta\)-filtration we define $(M : \Delta(\mu))$ analogously. It follows from (2.2) that we have

\[ \text{(2.3)} \quad \text{If } M \text{ has a } \nabla\text{-filtration then } (M : \nabla(\lambda)) = \dim_k \text{Hom}_G(\Delta(\lambda), M) \text{ for all } \lambda \in X^+. \]

Likewise,

\[ \text{(2.4)} \quad \text{If } M \text{ has a } \Delta\text{-filtration then } (M : \Delta(\mu)) = \dim_k \text{Hom}_G(M, \nabla(\mu)) \text{ for all } \mu \in X^+. \]

We say that $Q \in \mathcal{C}(G)$ is tilting if $Q$ has both a \(\nabla\)- and a \(\Delta\)-filtration. It is easy to check that if $Q$ is tilting then

\[ (Q : \nabla(\lambda)) = (Q : \Delta(\lambda)) \text{ for all } \lambda \in X^+. \]

We have a classification of indecomposable tilting modules in $\mathcal{C}(G)$: For each $\lambda \in X^+$ there is a unique (up to isomorphisms) indecomposable tilting module $T(\lambda)$ with $T(\lambda)_\lambda = k$ and $T(\lambda)_\mu = 0$ unless $\mu \leq \lambda$, and this accounts for all indecomposable tilting modules.

Hence if $Q \in \mathcal{C}(G)$ is tilting we have unique non-negative integers $(Q : T(\lambda))$ such that

\[ Q = \bigoplus_{\lambda \in X^+} T(\lambda)^{(Q : T(\lambda))}. \]

Of course, $(Q : T(\lambda)) = 0$ for all but finitely many $\lambda \in X^+$.

**Example 1.** Note that (2.1) implies

\[ \begin{align*} 
& (1) \quad T(\lambda) = L(\lambda) \text{ for all } \lambda \in \bar{A} \cap X^+. \\
& (2) \quad T((p^r - 1)\rho) = St_r \text{ for all } r \in \mathbb{Z}_{\geq 0}. 
\end{align*} \]

Let us also state the following important property of tilting modules, see [16].

\[ \text{(2.5)} \quad \text{If } Q_1, Q_2 \in \mathcal{C}(G) \text{ are both tilting so is } Q_1 \otimes Q_2. \]
2.4. Cellular categories. Let $\mathcal{T}(G)$ be the full subcategory of $\mathcal{C}(G)$ consisting of all tilting modules for $G$. Clearly, $\mathcal{T}(G)$ is an additive (but not abelian) category and by \cite{23} $\mathcal{T}(G)$ is also a tensor category. Note that our duality functor $M \mapsto \tau M$ on $\mathcal{C}(G)$ restricts to an endofunctor on $\mathcal{T}(G)$. Moreover, since this functor preserves characters we see that $\tau T(\lambda) \cong T(\lambda)$ for all $\lambda \in X^+$. This allows us to identify any $Q \in \mathcal{T}(G)$ with its dual $\tau Q$.

In \cite{5} we proved that if $Q \in \mathcal{T}(G)$ then the endomorphism algebra $\operatorname{End}_G(Q)$ has a natural structure making it into a cellular algebra (in the sense of \cite{13}). We shall now prove more generally that $\mathcal{T}(G)$ is a cellular category (see \cite{22} for the definition of a cellular category or alternatively read it off from the proof of the following theorem). As will become clear all the necessary work for proving this result was already done in \cite{8}.

**Theorem 2.1.** $\mathcal{T}(G)$ has a natural structure as a cellular category.

In the proof of this theorem we shall need the following families of morphisms in $\mathcal{C}(G)$. Let $\lambda \in X^+$ and suppose $\varphi : M \to N$ is a morphism in $\mathcal{C}(G)$. Then $\varphi$ is in particular a $T$-homomorphism, i.e. it maps $M_\mu$ into $N_\mu$ for all $\mu \in X$. Let us denote by $\varphi_\mu : M_\mu \to N_\mu$ the restriction of $\varphi$ to $M_\mu$. We then say that $\varphi \in \mathcal{C}^{<\lambda}(G)$ if $\varphi_\mu = 0$ unless $\mu < \lambda$.

We define $\mathcal{C}^{<\lambda}(G)$ similarly. Of course, we also have the analogous families $\mathcal{T}^{<\lambda}(G)$ and $\mathcal{T}^{\leq\lambda}(G)$ of morphisms in the category $\mathcal{T}(G)$.

**Proof.** We begin by defining what will be our cell datum in $\mathcal{T}(G)$. It consists firstly of the poset $\Lambda = (X^+, \leq)$. Secondly, we need for each $Q \in \mathcal{T}(G)$ and each $\lambda \in X^+$ to give a finite set $K(Q, \lambda)$. We set

$$K(Q, \lambda) = \{(1, 2, \cdots, (Q : \Delta(\lambda))\}.$$ 

Recall that $(Q : \Delta(\lambda)) = \dim_k \operatorname{Hom}_G(\Delta(\lambda), Q)$. We choose a basis $\{g_1^\lambda(Q), g_2^\lambda(Q), \cdots, g_r^\lambda(Q)\}$ for $\operatorname{Hom}_G(\Delta(\lambda), Q)$ (so $r = (Q : \Delta(\lambda))$), and for each $i$ we lift $g_i^\lambda(Q)$ to $\bar{g}_i^\lambda(Q) \in \operatorname{Hom}_G(T(\lambda), Q)$. This is possible, since by (2.2) we have $\operatorname{Ext}^1_{\mathcal{T}}(T(\lambda)/\Delta(\lambda), Q) = 0$. Then we set $f_i^\lambda(Q) = \tau g_i^\lambda(Q) \in \operatorname{Hom}_G(Q, \nabla(\lambda))$ and $f_i^\lambda(Q) = \tau \bar{g}_i^\lambda(Q)$.

These choices allow us now to give the third ingredient of our datum for $\mathcal{T}(G)$, namely for each pair $P, Q \in \mathcal{T}(G)$ and each $\lambda$ we define the injection

$$C^\lambda(P, Q) : K(P, \lambda) \times K(Q, \lambda) \to \operatorname{Hom}_G(P, Q)$$

by the recipe $C^\lambda(P, Q)(i, j) = c_{ij}^\lambda(P, Q) := \bar{g}_i^\lambda(Q) \circ \bar{f}_j^\lambda(P)$.

We claim that our datum $(\Lambda, \{K(Q, \lambda)\} \in \mathcal{T}(G), \lambda \in \Lambda), \{C^\lambda(P, Q)|P, Q \in \mathcal{T}(G), \lambda \in \Lambda\}$ satisfies the following three properties (named C-I, C-II and C-III in \cite{22})

1. The images of the maps $C^\lambda(P, Q), \lambda \in \Lambda$ give a basis for $\operatorname{Hom}_G(P, Q)$ for all $P, Q \in \mathcal{T}(G)$. This is the content of \cite{8}, Theorem 3.1.
2. $\tau C^\lambda(P, Q) = C^\lambda(P, Q)$ for all $\lambda, i, j, P, Q$.
   
   This is clear from the definitions.
3. Let again $P, Q \in \mathcal{T}(G), \lambda \in \Lambda, i \in K(P, \lambda), j \in K(Q, \lambda)$. Take $\varphi \in \operatorname{Hom}_G(Q, R)$. Then we claim

$$\varphi \circ c_{ij}^\lambda(P, Q) = \sum_{m \in K(R, \lambda)} r_{\varphi}(i, m)c_{mj}^\lambda(P, R) \pmod{T^{<\lambda}(G)}$$
for some scalars \( r_{\varphi}(i, m) \in k \) which are independent of \( j \).

To see this we express for each \( i \in K(Q, \lambda) \) the composite \( \varphi \circ g^{\lambda}_{i}(Q) \) in the basis 
\( (g^{\lambda}_{m}(R))_{m \in K(R, \lambda)} \) for \( \text{Hom}_{G}(\Delta(\lambda), R) \), i.e. we find \( r_{\varphi}(i, m) \in k \) such that \( \varphi \circ g^{\lambda}_{i}(Q) = \sum_{m} r_{\varphi}(i, m) g^{\lambda}_{m}(R) \). Then we have \( \varphi \circ \tilde{g}^{\lambda}_{i}(Q) = \sum_{m} r_{\varphi}(i, m) \tilde{g}^{\lambda}_{m}(R) \in \mathcal{T}^{\lambda}(G) \). This proves the claim. 

\[ \square \]

**Remark 1.** In [8] we work in the quantum case. However, the results in [8], Section 3 which we refer to above carry over verbatim to the modular case we are dealing with here. (Alternatively, specialize the quantum parameter in [8] to 1 which we refer to above) This works the other way as well: The same arguments as above prove that in the quantum case the category of tilting modules for the quantum group \( U_q \), \( q \) a root of unity, is also a cellular category.

If we are a little more careful with the choices made in the proof of Theorem 2.1 we can strengthen the result to get that \( \mathcal{T}(G) \) is in fact a strictly object-adapted cellular category” (or an SOACC for short). We refer to [12] for the definition of SOACC.

**Corollary 2.2.** \( \mathcal{T}(G) \) is an strictly object-adapted cellular category.

**Proof.** We consider the poset \((X^+, \leq)\) as a subset of the objects in \( \mathcal{T}(G) \) by identifying \( \lambda \in X^+ \) with \( T(\lambda) \). So now \( T(\lambda) \leq T(\mu) \) iff \( \lambda \leq \mu \). Going through the points in [12] Definition 2.4 we see that for \( \lambda \in X^+ \) and a tilting module \( P \) we may identify our set \( K(P, \lambda) \) with the set \( \text{M}(T(\lambda), X) \), respectively \( \text{E}(X, T(\lambda)) \) by associating to each \( i \in K(P, \lambda) \) the lift \( \tilde{g}^{\lambda}_{i}(P) \in \text{Hom}_{G}(T(\lambda), P) \), respectively its dual \( \tilde{f}^{\lambda}_{i}(P) \in \text{Hom}_{G}(P, T(\lambda)) \). The only thing we then need to do to make sure that our cellular datum in Theorem 2.1 satisfies the conditions in this definition is to choose as lift of the natural inclusion \( \Delta(\lambda) \hookrightarrow T(\lambda) \) the identity on \( T(\lambda) \) for any \( \lambda \in X^+ \).

\[ \square \]

**Remark 2.** It will turn out that it is convenient for the results in the next section to always work with the identity on \( T(\lambda) \) as our lift of the inclusion of \( \Delta(\lambda) \hookrightarrow T(\lambda) \). So in the rest of this paper we will fix this choice.

**2.5. Generators for tilting homomorphisms.** Set now

\[ B(G) = \{ g^{\lambda}_{i}(T_\nu(\mu)) | \lambda, \mu \in X^+, \lambda \leq \mu, i = 1, 2, \ldots, (T(\mu) : \Delta(\lambda)) \}. \]

By our choice in Remark 2 we see that \( \tilde{g}^{\lambda}_{i}(T(\mu)) = c^{\lambda}_{i1}(T(\lambda), T(\mu)) \) (note that \( K(T(\lambda), \lambda) = \{1\} \) so that the only possible second index here is 1), i.e. the elements of \( B(G) \) are certain special elements of our cellular bases.

With this notation we have

**Theorem 2.3.** The set \( B(G) \) generates via formation of direct sums, compositions and by taking duals all morphisms in \( \mathcal{T}(G) \).

**Proof.** As modules in \( \mathcal{T}(G) \) split into direct sums of indecomposable tilting modules it is enough to check that the cellular basis elements \( c^{\lambda}_{ij}(T_\nu(\mu), T_\nu(\mu)) \) belong to the set of morphisms generated by \( B(G) \) for all \( \lambda, \nu, \mu \in X^+ \). Furthermore, since (in our notation in
Section 2.4) 

\[ c^j_i(T(\nu), T(\mu)) = \bar{g}^j_i(T(\mu)) \circ \bar{f}^j_i(T(\nu)) \text{ and } \bar{f}^j_i(T(\nu)) \]

by definition is the dual of \( \bar{g}^j_i(T(\nu)) \) we see that elements in \( \text{Hom}_G(T(\mu), T(\nu)) \) are indeed linear combinations of composites of elements from \( B(G) \) with duals of such.

\[ \square \]

3. Category \( \mathcal{O} \)

In this section we consider the BGG-category \( \mathcal{O} \) for a semisimple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \). The arguments used in Section 2 applies just as well to the tilting modules in this category.

We use standard notation: \( \mathfrak{h} \) is a Cartan subalgebra in \( \mathfrak{g} \), \( \Delta(\lambda) \) is the Verma module (with respect to a Borel algebra containing \( \mathfrak{h} \)) with highest weight \( \lambda \in \mathfrak{h}^* \), and \( T(\lambda) \) the corresponding indecomposable tilting module in \( \mathcal{O} \). Our only concern will be with integral weights. We will actually only deal with the subcategory \( T(0) \) consisting of those tilting modules in \( \mathcal{O} \), which belong to the principal block (the block containing the trivial module \( \mathbb{C} \)). This means that the \( \lambda \)'s we need all belong to the orbit \( W \cdot 0 \) (\( W \) being the Weyl group for \( \mathfrak{g} \)). In particular, the indecomposable tilting modules in \( T(0) \) are \( T(x \cdot 0) \) with \( x \in W \).

Using notation similar to Section 2 and writing \( x \) instead of \( x \cdot 0 \) we define

\[ B(0) = \{ \bar{g}^x_y(T(y)) | x, y \in W, x \geq y, i = 1, 2, \cdots, (T(y) : \Delta(x)) \} \]

Here we are using the Bruhat order on \( W \). Note that \( x \geq y \) in this order iff \( x \cdot 0 \leq y \cdot 0 \) in the ordering on \( \mathfrak{h}^* \) defined by the positive roots.

Then we get exactly as in Theorem 2.3:

**Theorem 3.1.** The set \( B(0) \) generates (by forming direct sums and taking linear combinations as well as composites and duals of elements) all morphisms in \( T(0) \).

The set \( B(0) \) can often be refined. As an example let us consider the multiplicity free case (we say that \( T(0) \) is multiplicity free if \( (T(y) : \Delta(x)) \leq 1 \) for all \( x, y \in W \)):

**Theorem 3.2.** Suppose \( T(0) \) is multiplicity free case, and denote by \( \bar{g}^x_y(y) \) a lift of the unique (up to scalar) element of \( \text{Hom}(\Delta(x), T(y)) \). Then the set

\[ B'(0) = \{ \bar{g}^x_y(y) | y \leq x, y \text{ and } x \text{ are neighbors in the Bruhat graph for } W \} \]

generates the family of all morphisms in \( T(0) \).

**Proof.** It is well known that whenever \( x \geq y \) we have a unique (up to scalar) homomorphism in \( \text{Hom}_\mathcal{O}(\Delta(x), \Delta(y)) \), that this is a composite of homomorphisms between Verma modules corresponding to any sequence \( x = x_0 < x_1 < \cdots < x_r = y \) in the Bruhat graph, and that these homomorphisms are all injections. It follows that we may take \( \bar{g}^x_y(y) = \bar{g}^{x_{r-1}}_{x_r}(x_r) \circ \bar{g}^{x_{r-2}}_{x_{r-1}}(x_{r-1}) \circ \cdots \circ \bar{g}^{x_0}(x_1) \). Hence we can refine the set \( B(0) \) from Theorem 3.2 to the smaller set \( B'(0) \). \( \square \)

As an illustration of this theorem we consider the case \( \mathfrak{g} = sl_3 \). In this case we get a presentation of the tilting category \( T(0) \) in terms of a quiver algebra with relations.
Corollary 3.3. Let \( \mathfrak{g} = \mathfrak{sl}_3 \). Denote by \( s \) and \( t \) the simple reflections in \( W \) so that \( W = \{ 1, s, t, st, ts, w_0 \} \) where \( w_0 = st = ts \). If \( w \in W \) we denote by \( e_w \in \text{End}_O(T(w)) \) the identity on \( T(w) \). Moreover, we set (using the notation from Theorem 3.2)

\[
u_1 = \tilde{g}^{-w_0}(st), \nu_2 = \tilde{g}^{-w_0}(ts), \nu_3 = \tilde{g}^{-st}(s), \nu_4 = \tilde{g}^{-st}(t), \nu_5 = \tilde{g}^{-ts}(s), \nu_6 = \tilde{g}^{-ts}(t), \nu_7 = \tilde{g}^s(1), \nu_8 = \tilde{g}^t(1),
\]

and we let \( d_i \) be the dual of \( u_i \) for all \( i \). Then the set \( \{ e_w \mid w \in W \} \cup \{ u_i \mid i = 1, 2, \cdots, 8 \} \) generates all morphisms in the category \( \mathcal{T}(0) \) of principal block tilting modules for \( \mathfrak{sl}_3 \).

Moreover, the relations among these generators are the following together with their duals.

1. \( d_1u_1 = 0 = d_2u_2, u_3u_1 = u_4u_2, u_4u_1 = u_6u_2, \)
2. \( u_7u_3 = u_8u_4, u_8u_6 = u_7u_5, d_3u_4 = 0 = d_5u_5, \)
3. \( d_3u_3 = au_2d_1, d_6u_6 = au_2d_2, d_6u_4 = -au_2d_1, d_3u_5 = -au_2d_2 \) for some \( a \in \mathbb{C}^\times \),
4. \( d_7u_7 = bu_5d_5, d_8u_8 = bu_4d_4, d_8u_7 = bu_4d_3 + bu_6d_5 + ru_6u_2d_1d_3 \) for some \( b \in \mathbb{C}^\times \), \( r \in \mathbb{C} \).

**Proof.** First the relations involving only \( u \)’s follow from the choices we made in the proof of Theorem 3.2.

The map \( u_1 \) takes \( T(w_0) = L(w_0) \) to the socle of \( T(st) \), and \( d_1 \) maps \( T(st) \) onto its head \( L(w_0) = T(w_0) \). Hence \( d_1u_1 = 0 \). Likewise \( d_2u_2 = 0 \).

To see that \( d_4u_4 = 0 \) we observe that if \( \theta \) denotes the wallcrossing functor with respect to the wall separating the \( w_0 \)-chamber and the \( st \)-chamber then \( \theta \) takes the complex

\[
L(w_0) = T(w_0) \hookrightarrow T(ts) \to L(w_0) = T(w_0)
\]

\[
T(st) \hookrightarrow T(t) \to T(st).
\]

It is clear from the construction that the maps in this last complex are non-zero multiples of \( u_4 \) and \( d_4 \), respectively. Hence \( d_4u_4 = 0 \). An analogous argument gives \( d_5u_5 = 0 \).

The properties of the cellular bases immediately show that the relations in (3) hold up to some scalars in \( \mathbb{C} \). We now check that these scalars are non-zero. To see this choose a weight \( \mu \) on the common wall of the 1-chamber and the \( s \)-chamber. Set \( \mu = t \cdot \mu \) and \( \mu' = s \cdot \mu' \). Then the translation functor \( T_0^\mu \) takes \( u_3 \) into an inclusion \( T(\mu') \oplus T(\mu') \hookrightarrow T(\mu) \oplus T(\mu') \).

Likewise, \( T_0^\mu u_3 \) identifies \( T(\mu') \oplus T(\mu') \) with the socle of \( T(\mu) \oplus T(\mu') \) and \( T_0^\mu u_3 \) is mapping \( T(\mu) \oplus T(\mu') \) onto its head. We see that the composite is non-zero and hence so is \( d_3u_3 \). The non-vanishing of the remaining scalars is checked by similar translations onto an appropriate wall. By symmetry (coming from swapping \( s \) and \( t \)) the first two, respectively the last two scalars in (3) coincide. We shall see later that the two scalars involved sum to 0.

We now turn to the relations in (4). Here we choose a weight \( \mu \) on the wall between the \( t \)-chamber and the 1-chamber and set \( \mu' = s \cdot \mu \) and \( \mu'' = t \cdot \mu' \). Then we have an injection \( T(\mu') \hookrightarrow T(\mu) \) and a dual surjection \( T(\mu) \twoheadrightarrow T(\mu') \). The composite of these maps have image \( T(\mu'') = L(\mu'') \). Applying the translation functor \( T_0^\mu \) we obtain an injection \( T(s) \hookrightarrow T(1) \) and a surjection \( T(1) \twoheadrightarrow T(s) \). By our construction the injection is proportional to \( u_7 \) and the surjection to \( d_7 \). The image of the composite \( d_7u_7 \) is therefore...
equal to $T_0^0$ applied to $T(\mu'')$, i.e. to $T(ts)$. This means that the diagram

$$
\begin{array}{ccc}
T(s) & \xrightarrow{u_7} & T(1) \\
\downarrow d_5 & & \downarrow d_7 \\
T(ts) & \xrightarrow{u_5} & T(s)
\end{array}
$$

commutes up to a non-zero scalar in $C$, i.e. that the first identity in (4) holds. A symmetrical argument gives the second relation in (4). Again by symmetry the scalars in the two first identities in (4) coincide.

Finally, to obtain the last relation in (4) we first write $d_8 u_7$ as a linear combination of the 3 cellular basis elements in $\text{Hom}_\mathcal{O}(T(s), T(t))$

$$
d_8 u_7 = cu_4 d_3 + c'u_6 d_5 + c''u_6 u_2 d_1 d_3.
$$

Applying $d_4$, respectively $d_6$ to this equation gives via the relations (and their duals) proved so far the identities

$$
c'a' = ba', \text{ respectively } ca' + c'a = 0.
$$

Here we have named the second scalar in (3) by $a'$ (because we haven’t yet seen that it is $-a$). Likewise, precomposing with $u_3$, respectively $u_5$ leads to

$$
ca + c'a' = 0, \text{ respectively } ca' = ba'.
$$

This implies (5) and shows also that $a' = -a$.

To see that (1)-(4) are all relations among the given generators we just need to check that any path (string of $u$'s and $d$'s, say from $x$ to $y$) may be written via the above relations as a linear combination of the known cellular basis elements in $\text{Hom}_\mathcal{O}(T(x), T(y))$. This is an easy (but tedious) task which we leave to the reader. □

4. Frobenius kernels and module categories for related subgroup schemes

4.1. Subgroup schemes arising from Frobenius homomorphisms. Let $F : G \to G$ denote the Frobenius endomorphism on $G$. Then for each $r \geq 1$ we denote by $G_r$ the kernel of $F^r$ considered as a closed subgroup scheme of $G$. Likewise we have corresponding subgroup schemes $T_r, B_r, \text{ etc.}$ In addition, we shall also need the subgroup schemes

$$
G_r T = (F^r)^{-1}(T) \text{ and } G_r B = (F^r)^{-1}(B).
$$

If $M \in \mathcal{C}(G)$ we denote by $M^{(r)}$ the $r$-th Frobenius twist of $M$. This means that $M^{(r)} = M$ as $k$-vector space and that the action of $G$ on $M^{(r)}$ is given by $gm = F^r(g)m, g \in G, m \in M$. Note that $G_r$ acts trivially on $M^{(r)}$ and that $\lambda$ is a weight of $M$ iff and only if $p^r\lambda$ is a weight of $M^{(r)}$.

We define the Frobenius twists of modules in $\mathcal{C}(T), \mathcal{C}(B)$, etc. similarly.
4.2. Steinberg’s tensor product theorem and simple modules in $\mathcal{C}(G_r)$. Let $r \geq 1$ and set

$$X_r = \{ \lambda \in X^+ | \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in S \}.$$ 

The elements in $X_r$ are called the $r$-restricted weights. In particular, $X_1$ is the set of 1-restricted (or often we say just restricted) weights.

Let $\lambda \in X$. We can then write a $p$-adic expansion $\lambda = \lambda^0 + p\lambda^1 + \cdots + p^r\lambda^r$ of $\lambda$ with $\lambda^i \in X_1$ for all $i$. In this notation Steinberg’s famous tensor product theorem \cite{Steinberg} says

$$L(\lambda) = L(\lambda^0) \otimes L(\lambda^1)^{(1)} \otimes \cdots \otimes L(\lambda^r)^{(r)}.$$ 

The simple modules in $\mathcal{C}(G_r)$ are given by an equally famous theorem due to Curtis \cite{Curtis}.

$$\nabla_r(\lambda) = \operatorname{Ind}_{B^+T}^{G_rT} \lambda$$

with $\lambda^0 \in X_r, \lambda^1 \in X$. Of course $\lambda^0$ and $\lambda^1$ depend heavily on $r$, but it will always be clear from the context which $r$ we are working with.

We set $L_r(\lambda) = L(\lambda^0)_{|G_rT} \otimes p^r\lambda^1$. It follows easily from Section 4.2 that the simple modules in $\mathcal{C}_r$ are $(L_r(\lambda))_{\lambda \in X}$.

The standard and costandard objects in $\mathcal{C}_r$ may be defined rather similarly to the way we defined such objects in $\mathcal{C}(G)$ in Section 4. Namely, we set for each $\lambda \in X$

$$\nabla_r(\lambda) = \operatorname{Ind}_{B^+T}^{G_rT} \lambda$$

Alternatively, we have

$$\nabla_r(\lambda) = \operatorname{Ind}_{B^+T}^{G_rT} \lambda$$

i.e. $\nabla_r(\lambda)$ and $\Delta_r(\lambda)$ are restrictions to $G_rT$ of corresponding $G_rB$-modules.

The simple module $L_r(\lambda) \in \mathcal{C}_r$ is now realized as the socle of $\nabla_r(\lambda)$ or alternatively as the head of $\Delta_r(\lambda)$.

**Remark 3.** Note that \cite{Lusztig}, Section II.9 uses a different notation for the standard and costandard modules.

As $T$-modules (in fact as $U^+T^-$, respectively $U^-T$-modules) we have $\nabla_r(\lambda) \simeq k[U^+_r] \otimes \lambda$ and $\Delta_r(\lambda) \simeq k[U^-_r] \otimes (\lambda - 2(p^r - 1)\rho)$. It follows that in $\mathcal{C}(T)$ we have isomorphisms $\nabla_r(\lambda) \simeq \nabla_r(\mu) \otimes (\lambda - \mu)$ for all $\lambda, \mu \in X$. In the special case $\lambda = (p^r - 1)\rho$ we have isomorphisms of $G_rT$-modules

$$\nabla_r((p^r - 1)\rho) \simeq St_r \simeq \Delta_r((p^r - 1)\rho).$$

It follows that

$$\operatorname{ch}\nabla_r(\lambda) = \operatorname{ch}\Delta_r(\lambda) = \chi((p^r - 1)\rho)e^{\lambda - (p^r - 1)\rho}.$$
Note that the antiautomorphism $\tau$ we used in Section 2 to define our duality in $C(G)$ restricts to an antiautomorphism on $G_r T$. Hence it gives us a duality on $C_r$ as well. It is then a fact (which gives another explanation for the first equality in (4.3)) that in $C_r$ we have

$$\nabla_r(\lambda) \simeq \Delta_r(\lambda)$$

for all $\lambda \in X$, see [15], II.9.3(5).

4.4. Projective $G_r T$-modules. A notable difference between $C(G)$ and $C_r$ is that the first contains no projective objects whereas the second has enough projectives. Moreover, in $C_r$ all projective modules are also injective and vice versa. As we shall see shortly, to be projective in $C_r$ is the same as being tilting.

Let $\lambda \in X$ and denote by $Q_r(\lambda)$ the projective cover in $C_r$ of $L_r(\lambda)$. Then $Q_r(\lambda)$ is also the injective envelope of $L_r(\lambda)$.

Note that $\Delta_r(\lambda)$ is projective for $B_r$. In fact, $B_r = U_r T_r$ and as a $U_r$-module we have $\Delta_r(\lambda) \simeq k[U_r]$. Likewise $\nabla_r(\lambda)_{|U_r} \simeq k[U_r^+]$ so that $\nabla_r(\lambda)$ is projective as a $B_r^+$-module.

It follows from these observations that if $M \in C_r$ has a $\Delta_r$-filtration then $M$ is projective for $B_r$, while if $M$ has a $\nabla_r$-filtration then $M$ is projective for $B_r^+$. It turns out that the converse is also true ([15], Proposition II.11.2), i.e. we have for $M \in C_r$

$$M \text{ is } B_r\text{-projective if and only if } M \text{ has a } \Delta_r\text{-filtration,}$$

and

$$M \text{ is } B_r^+\text{-projective if and only if } M \text{ has a } \nabla_r\text{-filtration.}$$

In analogy with the definition of tilting modules in $C(G)$ we say that $Q \in C_r$ is tilting if $Q$ has both a $\Delta_r$- and a $\nabla_r$-filtration. The multiplicities in such filtrations are denoted $(Q : \Delta_r(\lambda))$ and $(Q : \nabla_r(\lambda))$ in analogy with the $C(G)$-case.

By (4.4) and (4.5) we see that $Q$ is tilting if and only if $Q$ is projective for both $B_r$ and $B_r^+$. As $G_r = U_r B_r^+$ we conclude that

$$Q \in C_r \text{ is tilting if and only if } Q \text{ is projective if and only if } Q \text{ is injective.}$$

Now of course we have

$$Q \text{ is projective if and only if } Q = \bigoplus_{\lambda \in X} Q_r(\lambda)^{\oplus n_\lambda} \text{ for some } n_\lambda \in \mathbb{Z}_{\geq 0} \text{ (almost all 0).}$$

4.5. The category $\mathcal{P}_r$ of projective $G_r T$-modules. We set $\mathcal{P}_r$ equal to the subcategory of $C_r$ consisting of the projective modules. By the results in the previous subsection this is the same as the subcategory consisting of all tilting modules. We shall now deduce the exact relations between the PIM’s in $\mathcal{P}_r$ and the indecomposable tilting modules.

As in Section 4.3 we write $\lambda = \lambda^0 + p^r \lambda^1$ with $\lambda^0 \in X_r$ and $\lambda^1 \in X$. We now define

$$\tilde{\lambda} = 2(p^r - 1)\rho + w_0 \lambda^0 + p^r \lambda^1,$$

where $w_0$ denotes the longest element in the Weyl group for $G$. Then the map $\lambda \mapsto \tilde{\lambda}$ is a bijection on $X$ which fixes all elements of $-\rho + p^r X$ and carries the box $p^r \mu + X$ onto the translated box $p^r(\mu + \rho) - \rho + X$ for any $\mu \in X$. 

Weight considerations (using e.g. (4.2)) show that the highest weight of \( Q_r(\lambda) \) is \( \tilde{\lambda} \). This observation implies that the indecomposable tilting module \( T_r(\tilde{\lambda}) \) in \( C_r \) with highest weight \( \tilde{\lambda} \) is given by the following formula.

(4.8) \( T_r(\tilde{\lambda}) = Q_r(\lambda) = Q_r(\lambda^0) \otimes p^r \lambda^1 \).

Example 2. Consider the special weight \((p^r - 1)\rho \in X_r\). For this weight we have

\[ T_r((p^r - 1)\rho) = Q_r((p^r - 1)\rho) = L_r((p^r - 1)\rho) = \nabla_r((p^r - 1)\rho) = \Delta_r((p^r - 1)\rho). \]

In fact, these equations hold for all special weights, i.e. for all weights in \(-\rho + p^r X\).

Remark 4. There are some interesting partially proved conjectures which connect indecomposable objects in \( P_r \) with objects in \( C(G) \) and \( T(G) \):

(1) (The Humphreys-Verma conjecture, [14]) Let \( \lambda \in X_r \). There exists an object \( \overline{Q}_r(\lambda) \in C(G) \) such that \( Q_r(\lambda) = \overline{Q}_r(\lambda)|_{G,T} \).

This conjecture is known to hold for \( p \geq 2h - 2 \) in which case \( \overline{Q}_r(\lambda) \) is the injective envelope of \( L_r(\lambda) \) in a certain bounded subcategory of \( C(G) \), see [15] II.11.11.

(2) (Donkin’s tilting conjecture, [11] Conjecture (2.2)) Let \( \lambda \in X_r \). In the above notation \( Q_r(\lambda) = T(\tilde{\lambda})|_{G,T} \) or equivalently \( T_r(\tilde{\lambda}) = T(\tilde{\lambda})|_{G,T} \).

This conjecture was proved by Donkin for \( p \geq 2h - 2 \). It was recently shown to fail for \( p = 2 \) for \( G \) of type \( G_2 \), see [9].

4.6. Reciprocity laws. In analogy with (2.2) we have

(4.9) Let \( \lambda, \mu \in X \). Then \( \text{Ext}^i_{G,T}(\Delta_r(\lambda), \nabla_r(\mu)) = \begin{cases} k & \text{if } i = 0 \text{ and } \mu = \lambda, \\ 0 & \text{otherwise}. \end{cases} \)

We then also get analogues of (2.3) and (2.4) in \( C_r \). This implies in particular the following reciprocity law

(4.10) Let \( \lambda, \mu \in X \). Then \( (Q_r(\lambda) : \Delta_r(\mu)) = [\nabla_r(\mu) : L_r(\lambda)] \).

In fact, according to (4.9) the left hand side equals \( \dim \text{Hom}_G(Q_r(\lambda), \nabla_r(\mu)) \). This equals the right hand side, because \( Q_r(\lambda) \) is the projective cover of \( L_r(\lambda) \).

Using (4.8) we can also formulate this reciprocity in terms of indecomposable tilting modules: Let \( \lambda, \mu \in X \). Then

(4.11) \( (T_r(\tilde{\lambda}) : \Delta_r(\mu)) = [\nabla_r(\mu) : L_r(\lambda)] \).

4.7. Cellularity of \( P_r \). Arguing as in Section 2.4 we can now prove

Theorem 4.1. The category \( P_r \) has a natural cellular structure.

Proof. Set \( \Lambda = X \) and define for each \( P \in P_r \) and \( \lambda \in \Lambda \) the set \( K(P, \lambda) = \{1, 2, \cdots, (P : \Delta_r(\lambda))\} \). If also \( Q \in P_r \) define

\[ c^\lambda_{ij}(P, Q) = g^\lambda_i(Q) \circ f^\lambda_j(P), \]
where $\tilde{g}_r^\lambda(Q)$, respectively $\tilde{f}_r^\lambda(P)$, is a lift of a basis element $g_r^\lambda(Q) \in \text{Hom}_{G,T}(\Delta_r(\lambda), Q)$, respectively of a "dual basis" element $f_r^\lambda(P) \in \text{Hom}_{G,T}(P, \nabla_r(\lambda))$. This gives us exactly as in the proof of Theorem 2.1 a cellular datum for $P_r$. □

**Remark 5.** In contrast to the cellular datum for $T(G)$ presented in Theorem 2.1, the poset $(X, \leq)$ appearing in the cellular datum for $P_r$ in Theorem 3.7 does not satisfy dcc. If we ignore this then $P_r$ is also an SOACC (cf. Corollary 2.2), i.e. it satisfies all the other requirements in [12] Definition 2.4.

5. Homomorphisms in $P_r$

5.1. Weight bounds. Let $P, Q \in P_r$. Then $P$ and $Q$ are tilting modules and hence direct sums of certain $T_r(\lambda)$’s. The vector space $\text{Hom}_{G,T}(P, Q)$ is therefore a sum of certain $\text{Hom}_{G,T}(T_r(\lambda), T_r(\mu))$’s. To be precise

$$\text{Hom}_{G,T}(P, Q) = \bigoplus_{\lambda, \mu \in X} \text{Hom}_{G,T}(T_r(\lambda), T_r(\mu)) \oplus (P, T_r(\lambda); Q, T_r(\mu)).$$

We shall now prove that if $\lambda$ and $\mu$ are far apart then $\text{Hom}_{G,T}(T_r(\lambda), T_r(\mu)) = 0$. Recall the definition of $\tilde{\lambda}$ from Section 4.5.

**Proposition 5.1.** Let $\lambda, \nu \in X$. If $(T_r(\tilde{\lambda}) : \Delta_r(\nu)) \neq 0$ then $\lambda \leq \nu \leq \tilde{\lambda}$.

**Proof.** Clearly, if $(T_r(\tilde{\lambda}) : \Delta_r(\nu)) \neq 0$ then $\nu$ is a weight of $T_r(\tilde{\lambda})$. This implies that $\nu \leq \tilde{\lambda}$. Moreover, by the reciprocity (4.11) we have $(T_r(\tilde{\lambda}) : \Delta_r(\nu)) = [\nabla_r(\nu) : L_r(\lambda)]$. This gives the other inequality $\nu \geq \lambda$. □

**Remark 6.** Both inequalities in this lemma are best possible. In fact, the proof shows that $(T_r(\tilde{\lambda}) : \Delta_r(\lambda)) = 1 = (T_r(\tilde{\lambda}) : \Delta_r(\lambda))$.

**Proposition 5.2.** Let $\lambda, \mu \in X$. If $\text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\tilde{\mu})) \neq 0$ then $\mu \leq \tilde{\lambda}$ and $\lambda \leq \tilde{\mu}$.

**Proof.** We have $\text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\tilde{\mu})) \neq 0$ if and only if there exists $\nu \in X$ such that $(T_r(\tilde{\lambda}) : \Delta_r(\nu))$ and $(T_r(\tilde{\mu}) : \Delta_r(\nu))$ are both non-zero. Then by Lemma 5.1 the non-vanishing of $\text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\tilde{\mu}))$ implies that $\mu \leq \nu \leq \tilde{\lambda}$ and $\lambda \leq \nu \leq \tilde{\mu}$. □

**Remark 7.** (1) The bounds in Proposition 5.2 are always achieved: Suppose $\mu = \tilde{\lambda}$.

Then $\text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\tilde{\mu})) = \text{Hom}_{G,T}(T_r(\mu), T_r(\mu)) = k$, because $[T_r(\mu) : L_r(\mu)] = 1$. Likewise, if $\lambda = \tilde{\mu}$ then $\text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\tilde{\mu})) = \text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\lambda)) = k$.

(2) Suppose $\lambda \in -\rho + p^rX$. Then $\lambda = \tilde{\lambda}$ so that in this case the proposition says that

$$\text{Hom}_{G,T}(T_r(\tilde{\lambda}), T_r(\tilde{\mu})) = \begin{cases} k & \text{if } \mu = \lambda, \\ 0 & \text{otherwise}. \end{cases}$$

This can also be seen via Example 2.
The proposition implies that if \( \text{Hom}_{G,T}(T_r(\nu), T_r(\eta)) \neq 0 \) then \(-2(p' - 1)\rho \leq \nu - \eta \leq 2(p' - 1)\rho \). Note that there exists weights \( \eta \) and \( \mu \) for which these bounds are realized: It is easy to check that for instance \( \text{Hom}_{G,T}(T_r(0), T_r(2(p' - 1)\rho)) = k \). Proposition 5.2 reduces the problem of finding all homomorphism spaces in \( P_r \) to a finite one:

It is enough to determine \( \text{Hom}_{G,T}(T_r(\lambda), T_r(\mu)) \) for the finite set of pairs \((\lambda, \mu)\), where \( \lambda \in X_r \) and \( \mu \) satisfies \( \mu \leq \lambda \) and \( \lambda \leq \bar{\mu} \).

### 5.2. A set of generators

The results in Section 5.1 combined with the cellularity of \( P_r \) allow us now to single out a natural finite set of generators for the family of homomorphisms in \( P_r \). Using notation as in the proof of Theorem 4.1 we set

\[
B_r^\lambda(\mu) = \{ \bar{g}^\lambda_r(T_r(\mu)) | i \in K(T_r(\mu), \lambda) \}
\]

for all \( \lambda, \mu \in X \). Note that the lift \( \bar{f}^\lambda_r(T_r(\lambda)) \) of the basis element in \( \text{Hom}_{G,T}(T_r(\lambda), \nabla_r(\lambda)) = k \) is always chosen to be the identity by our convention from Remark 2 so that in fact \( B_r^\lambda(\mu) \) consists of cellular basis elements.

**Theorem 5.3.** The set \( B_r = \bigcup_{\lambda \in X, \lambda \leq \mu \leq \bar{\lambda}} B_r^\lambda(\mu) \) generates via formation of direct sums, compositions, dualizing, and tensoring with elements from \( p^*X \) all morphisms in \( P_r \).

**Proof.** As modules in \( P_r \) split into direct sums of indecomposable tilting modules it is enough to check that the cellular basis elements \( c_{ij}^\lambda(T_r(\nu), T_r(\mu)) \) are generated by \( B_r \) for all \( \lambda, \nu, \mu \in X \). Furthermore, since \( c_{ij}^\lambda(T_r(\nu), T_r(\mu)) = \bar{g}^\lambda_r(T_r(\mu)) \circ \bar{f}^\lambda_r(T_r(\nu)) \) we are done if we check that all \( \bar{g}^\lambda_r(T_r(\mu)) \) belong to the set generated by \( B_r \). Note that here \( \mu \geq \lambda \). By tensoring with an appropriate element of \( p^*X \) we may assume \( \lambda \in X_r \). Finally, Proposition 5.2 ensures that \( K(T_r(\mu), \lambda) \) is empty unless \( \mu \leq \bar{\lambda} \). \( \square \)

### 5.3. The strong linkage principle in \( C \) and \( T \)

The strong linkage principle in \( C \) may be formulated as follows, see [15], II.9.15.

(5.1) Let \( \lambda, \mu \in X \). If \([\Delta_r(\lambda) : L_r(\mu)] \neq 0\) then \( \mu \) is strongly linked to \( \lambda \).

Here we may of course replace \( \Delta_r \) by \( \nabla_r \).

Strongly linked weights are in particular linked in the sense that they belong to the same orbit of \( W_\rho \) under the dot action. Recall that if \( \alpha \in R \) then the reflection \( s_\alpha \) acts on \( X \) by \( s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \lambda \in X \). For each \( m \in Z \) we have a corresponding affine reflection \( s_{\alpha,m} \) given by \( s_{\alpha,m} \lambda = s_\alpha \lambda + m\rho \). The affine Weyl group \( W_\rho \) is the group generated by these affine reflections for \( \alpha \in R, m \in Z \). The dot action of \( W_\rho \) on \( X \) is then the above action shifted with \(-\rho\), i.e. \( w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W_\rho, \lambda \in X \).

When we combine (5.1) with (4.11) we get (using notation as in Section 4.5)

(5.2) Let \( \lambda, \nu \in X \). If \([\Delta_r(\lambda) : \Delta_r(\nu)] \neq 0\) then \( \lambda \) is strongly linked to \( \nu \).

We immediately get

**Proposition 5.4.** Let \( \lambda, \mu \in X \). If \( \text{Hom}_{G,T}(T_r(\lambda), T_r(\mu)) \neq 0 \) then \( \mu \in W_\rho \cdot \lambda \).
Remark 8. Note that $\tilde{\lambda} = w_0 \cdot \lambda + p^r(\lambda^1 - w_0 \cdot \lambda^1)$ so that since $\lambda^1 - w_0 \cdot \lambda^1 \in \mathbb{Z}R$ we have $\tilde{\lambda} \in W_p \cdot \lambda$. Hence we can replace $\tilde{\lambda}$ and $\tilde{\mu}$ by $\lambda$ and $\mu$ in this proposition.

5.4. The translation principle in $C_r$ and $\mathcal{P}_r$. It follows from the results in Section 5.3 that all composition factors of an indecomposable tilting module $T_r(\lambda)$ have highest weights in $W_p \cdot \lambda$. This is more generally true (e.g. because tilting modules are the same as projective modules) for the composition factors of any indecomposable module in $C_r$. In other words, since $\tilde{A}$ is a fundamental domain for the action by $W_p$ on $X$ we can split any module $M \in C_r$ as

$$M = \bigoplus_{\lambda \in \tilde{A}} \text{pr}_{\lambda}(M).$$

Here $\text{pr}_{\lambda}(M)$ is the largest submodule of $M$ such that its composition factors all have highest weights in $W_p \cdot \lambda$. This allows us for each $\lambda, \mu \in \tilde{A}$ to define translation functors $T_{\mu}^\lambda$ on $C_r$ and $\mathcal{P}_r$, see [15], II.9.22. We set $C_{\mu}^\lambda = \text{pr}_{\lambda}(C_r)$. If $\lambda, \mu \in A$ then $T_{\mu}^\lambda$ is an equivalence between $C_{\mu}^\lambda$ and $C_{\mu}^\lambda$. A special consequence of this is

Proposition 5.5. Let $\lambda, \mu \in A$. Then $\text{Hom}_{G_r,T}(T_r(x \cdot \lambda), T_r(y \cdot \lambda)) \simeq \text{Hom}_{G_r,T}(T_r(x \cdot \mu), T_r(y \cdot \mu))$ for all $x, y \in W_p$.

Note that $A = \emptyset$ if $p$ is less than the Coxeter number for $R$. Hence this proposition is empty for small primes.

Another well known result (see [15] II.11.10(1)) is

Proposition 5.6. Let $\lambda \in \tilde{A}$ and $\mu \in X$. Then $T_{\rho}^\lambda T_r(-\rho + p^r \mu) \simeq T_r(\lambda + p^r \mu)$.

Note that by (4.2) we have that $T_r(-\rho + p^r \mu) = \Delta_r(-\rho + p^r \mu)$. From the well known behavior of translation functors on standard modules in $C_r$ we get

Corollary 5.7. Let $\lambda \in \tilde{A}$ and $\mu \in X$. Then $(T_r(\lambda + p^r \mu) : \Delta_r(\nu)) = \begin{cases} 1 & \text{if } \nu \in -\rho + p^r \mu + W\lambda, \\ 0 & \text{otherwise.} \end{cases}$

In fact, Proposition 5.6 and Corollary 5.7 are special cases of the following more general result

Proposition 5.8. Let $\lambda \in A$ and $\mu \in \tilde{A}$. Suppose $w$ is an element of $W_p$ for which $w \cdot \lambda$ is maximal in the set $\{wx \cdot \lambda | x \cdot \mu = \mu\}$. Then

(1) $T_{\mu}^\lambda T_r(w \cdot \mu) \simeq T_r(\lambda)$.
(2) $(T_r(w \cdot \lambda) : \Delta_r(y \cdot \lambda)) = (T_r(w \cdot \mu) : \Delta_r(y \cdot \mu))$ for $y \in W_p$.

Proof. (1) is the infinitesimal analogue of [3], Proposition 5.2. Then we get (2) from the adjointness of $T_{\mu}^\lambda$ and $T_{\lambda}^\mu$ as follows: $(T_r(w \cdot \lambda) : \Delta_r(y \cdot \lambda)) = \dim_k \text{Hom}_{G_r,T}(\Delta_r(y \cdot \lambda), T_r(w \cdot \lambda)) = (T_r(w \cdot \mu) : T_{\lambda}^\mu \Delta_r(y \cdot \lambda)) = (T_r(w \cdot \mu) : \Delta_r(y \cdot \mu))$ because $T_{\lambda}^\mu \Delta_r(y \cdot \lambda) \simeq \Delta_r(y \cdot \mu)$. \qed

Using Proposition 5.8 we can now strengthen the conditions in (6.2) Let $\lambda \in X$ be $p$-regular, i.e. $\lambda \in W_p \cdot A$. Then we have
Corollary 5.9. If \((T_r(\tilde{\lambda}) : \Delta_r(\nu)) \neq 0\) for some \(\nu \in X\) then \(\lambda\) is strongly linked to \(\nu\) and \(\nu\) is strongly linked to \(\tilde{\lambda}\).

Proof. We have left to check that if \((T_r(\lambda) : \Delta_r(\nu)) \neq 0\) then \(\nu\) is strongly linked to \(\lambda\). By tensoring with an appropriate element of \(p^a X\) we may assume \(\lambda \in X_r\). Denote by \(A'\) the alcove containing \(\lambda\). If \(A' = A\) we are done by Corollary 5.7. So suppose \(A' > A\) and assume inductively that the statement holds for weights in all alcoves in \(A\) the alcove containing a weight in the interior of this common wall. Then by Proposition 5.8 we see that \(\nu\) is a summand of \(T_r(\lambda)\). We have left to check that if \((\nu - \rho) \leq \lambda\) and let \(A''\) an alcove in \(X_r\) with \(A'' < A'\) which share a wall with \(A'\). Let \(\mu\) be a weight in the interior of this common wall. Then by Proposition 5.8 we see that \(T_r(\lambda)\) is a summand of \(T_r^\mu T_r(\lambda'')\) where \(\lambda''\) is the element of \(A''\) strongly linked to \(\lambda\). Then \((T_r(\lambda) : \Delta_r(\nu)) \leq (T_r^\mu T_r(\lambda'') : \Delta_r(\nu)) = (T_r(\lambda'') : \Delta_r(\nu)) + (T_r(\lambda'') : \Delta_r(\nu'))\) where \(\nu'\) is the weight different from \(\nu\) for which \(\Delta_r(\nu')\) is a \(\Delta_r\)-factor of \(T_r^\mu T_r(\lambda'').\) The statement now follows from the induction hypothesis. \(\square\)

Let \(P_r(0)\) denote the principal block in \(P_r\) corresponding to \(0 \in \Lambda\), i.e \(P_r(0)\) is the subcategory of \(P_r\) whose indecomposable modules are \((T_r(w \cdot 0))_{w \in W_r}\). Set \(\Lambda(0) = W_p \cdot 0\) and let \(\leq_{SL}\) denote the ordering on \(\Lambda(0)\) given by \(\lambda \leq_{SL} \mu\) iff \(\lambda\) is strongly linked to \(\mu\). Then we get the from the above

Corollary 5.10. (1) \(P_r(0)\) is a cellular category with weight poset \((\Lambda(0), \leq_{SL})\).

(2) \(B_r(0) = \{\mu^i(0)(T_r(y \cdot 0)|x, y \in W_p, x \cdot 0 \leq_{SL} y \cdot 0, i = 1, 2 \ldots, (T_r(y \cdot 0) : \Delta_r(x \cdot 0))\}\) generates the morphisms in \(P_r(0)\).

5.5. The Steinberg linkage class in \(C_r\) and \(P_r\). Consider the special weight \((p - 1)\rho \in X\). As \(s_\alpha \cdot (p - 1)\rho = (p - 1)\rho - p\alpha\) for all simple roots \(\alpha\) we see that the linkage component in \(C_r\) corresponding to \((p - 1)\rho\) consists of all \(M \in C_r\) whose composition factors have highest weights in \((p - 1)\rho + p\mathbb{Z}R\). For the purposes in this paper it will be convenient to consider the (possibly bigger) class consisting of all \(M \in C_r\) with composition factors belonging to \(\{L_r(\lambda)|\lambda + p \rho \in pX\}\). We shall denote this component of \(C_r\) by \(St_r\).

If \(r = 1\) then \(St_1\) is a semisimple category with simple modules \(L_1(-\rho + p\lambda), \lambda \in X\).

Note that in fact \(L_1(-\rho + p\lambda) = St_1 \otimes (p(\lambda - \rho)) = \Delta_r(-\rho + p\lambda) = T_1(-\rho + p\lambda),\) so that \(St_1\) is contained in \(P_1\).

Suppose \(r > 1\). The Frobenius homomorphism on \(G\) restricts to a homomorphism \(G_r \to G_{r-1}\). Via this homomorphism we can make any \(M \in C_{r-1}\) into a \(G_r\)-module. The resulting module in \(C_r\) is denoted \(M^{(1)}\). We then define a functor \(\Phi_{r-1} : C_{r-1} \to C_r\)

by \(\Phi_{r-1}M = St_1 \otimes M^{(1)}\).

We observe that \(\Phi_{r-1}\) takes values in \(St_r\). In fact, it gives an equivalence of categories \(\Phi_{r-1} : C_{r-1} \cong St_r\) with inverse functor \(\text{Hom}_{G_1}(St_1, -)\), cf. \([H]\). This equivalence of categories carries simple modules to simple modules, (co)standard modules to (co)standard modules, and tilting
modules to tilting modules, see [4], Theorem 3.1. In particular, we see that the restriction of \( \Phi \) to \( \mathcal{P}_{r-1} \) gives an equivalence
\[
(5.3) \quad \mathcal{P}_{r-1} \cong \mathcal{S}_r \cap \mathcal{P}_r.
\]

Let us also record the following consequences of the above.

**Proposition 5.11.** Let \( r > 1 \) and take \( \lambda \in X \). Then the functor \( \Phi_{r-1} \) carries the tilting module \( T_{r-1}(\lambda) \in \mathcal{P}_{r-1} \) to \( T_r((p-1)\rho + p\lambda) \in \mathcal{P}_r \), and if also \( \mu \in X \) we get an isomorphism of \( k \)-spaces
\[
\Hom_{G_{r-1}}(T_{r-1}(\lambda), T_{r-1}(\mu)) \cong \Hom_{G_r}(T_r((p-1)\rho + p\lambda), T_r((p-1)\rho + p\mu)).
\]

Finally, we observe that for \( r > 1 \) we have a sequence of subcomponents of \( \mathcal{S}_r \)
\[
\mathcal{S}_{r_1} \subset \mathcal{S}_{r_1}^{-1} \subset \cdots \subset \mathcal{S}_{r_2} \subset \mathcal{S}_r
\]
defined inductively by setting \( \mathcal{S}_{r_1} = \Phi_{r-1}(\mathcal{S}_{r-1}^{-1}) \). The smallest of these, \( \mathcal{S}_{r_2} \), is a semisimple category with simple objects \( L_r(-\rho + p^r\lambda) \), \( \lambda \in X \), compare Remark 5(2).

6. The \( SL_2 \)-case

In this section we take \( G = SL_2 \). Then \( X = Z \), \( R = \{\pm 2\} \) and we choose \( R^+ = \{2\} \) so that \( X^+ = Z_{\geq 0} \). We shall describe the corresponding \( \mathcal{P}_r \)'s and in particular the homomorphisms in these categories by using the cellular structures discussed in the previous sections.

Note that in this case the affine Weyl group \( W_p \) is the infinite dihedral group with generators \( s \) and \( t \), the reflections (with respect to the dot actions) in \( -1 \) and \( p-1 \), respectively. The alcove \( A \) is the interval \( \{0,1,\cdots,p-2\} \).

We shall generally ignore \( p = 2 \) leaving to the reader to slightly modify the following in order to cover this case.

6.1. \( r=1 \). When \( r = 1 \) we have essentially only two components of \( \mathcal{P}_1 \) to consider, namely the one corresponding to the orbit \( W_p \cdot 0 \) and the one corresponding to \( W_p \cdot (-1) \cup W_p \cdot (p-1) \), respectively. The first orbit equals \( 2pZ \cup (-2 + 2pZ) \) and the corresponding block in \( \mathcal{C}_1 \) is equivalent to any other regular block (obtained by taking instead of 0 another point in \( A \)). The second gives the component \( \mathcal{S} t_1 \), i.e the semisimple subcategory of \( \mathcal{P}_1 \) with simple modules \( (L_1(-1 + mp))_{m \in Z} \).

There is nothing more to say about \( \mathcal{S} t_1 \), so let’s turn to the regular block \( \mathcal{P}_1(0) \) associated to \( 0 \in A \). By Proposition 5.8 we have \( T_1(0) = T_{-1}^{0} T_{1}(-1) = T_{-1}^{0} \Delta_{1}(-1) \). Likewise \( T_{1}(2p-2) = T_{-1}^{0} T_{1}(p-1) = T_{-1}^{0} \Delta_{1}(2) \). So we see that both \( T_{1}(0) \) and \( T_{1}(2p-2) \) are two \( \Delta_{1} \)-factors, namely \( \Delta_{1}(0) \) and \( \Delta_{1}(2) \), respectively. This allows us to determine all morphisms in \( \mathcal{P}_1(0) \) by using the general results from Section 5.

Set \( P_m = T_1(0) \otimes 2mp \) and \( Q_m = T_1(2p-2) \otimes 2pm \). Write \( P = P_0 \) and \( Q = Q_0 \). Then any indecomposable tilting module in \( \mathcal{P}_1(0) \) is either isomorphic to \( P_m \) or to \( Q_m \) for some \( m \in Z \). Using Proposition 5.2 we get
\[
(6.1) \quad \Hom_{G_{1}}(P_m, P_{m'}) = \Hom_{G_{1}}(Q_m, Q_{m'}) = 0 \text{ if } m \neq m'.
\]
and

\[ (6.2) \quad \text{Hom}_{G, T}(P_m, Q_{m'}) = \text{Hom}_{G, T}(Q_{m'}, P_m) = 0 \text{ if } m - m' \not\in \{0, 1\}. \]

Let now \( u_0 \), respectively \( u_1 \) be a basis element in \( \text{Hom}_{G, T}(P, Q) \), respectively \( \text{Hom}_{G, T}(Q, P) \) (in our standard notation \( u_0 = \bar{g}^0(T_{2p} - 2) \)) and \( u_1 = \bar{g}^{2p-2}(T(2p)) \). We denote their dual elements in \( \text{Hom}_{G, T}(Q, P) \), respectively \( \text{Hom}_{G, T}(P, Q) \) by \( d_0 \) and \( d_1 \). Then for any \( m \in Z \) we set \( u_{2m} = u \otimes 2pm \in \text{Hom}_{G, T}(P_m, Q_m) \) and \( u_{2m+1} = u_1 \otimes 2pm \in \text{Hom}_{G, T}(Q_m, P_{m+1}) \). We let again \( d_{2m} \) and \( d_{2m+1} \) denote their duals.

It is easy to see that \( u_n \circ d_n = d_{n+1} \circ u_{n+1} \) for all \( n \). Setting \( \epsilon_n = u_n \circ d_n \) we see that \( \text{End}_{G_1}(P_n) = \text{span}_k \{id_{P_n}, \epsilon_n\} \) and \( \text{End}_{G_1}(Q_n) = \text{span}_k \{id_{P_n}, \epsilon_n\} \).

We sum up these findings as follows (using the above notations)

**Proposition 6.1.** The two homomorphisms \( u_0 \) and \( u_1 \) generate via formation of direct sums, compositions, dualization, and tensoring by \( \pm 2p \) all morphisms in the regular block \( \mathcal{P}_1(0) \) of \( \mathcal{P}_1 \). They satisfy the relations

\[ u_1 \circ u_0 = 0 = (u_0 \otimes 2p) \circ u_1 \text{ and } u_0 \circ d_0 = d_1 \circ u_1, \quad u_1 \circ d_1 = (d_0 \circ u_0) \otimes 2p. \]

**Remark 9.** This proposition should be compared with [7], Proposition 2.30.

6.2. \( r \geq 2 \). First we shall prove that for \( SL_2 \) we always have the following multiplicity freeness.

**Proposition 6.2.** Let \( r \geq 1 \) and let \( m \in Z \). Then \( (T_r(m) : \Delta_r(n)) \in \{0, 1\} \) for all \( n \in Z \).

**Proof.** We shall use induction on \( r \). The case \( r = 1 \) is taken care of in the previous section. So assume \( r > 1 \). As \( St_r \cap \mathcal{P}_r \) is equivalent to \( \mathcal{P}_{r-1} \) we conclude that the proposition holds for \( m \in -1 + p\mathbb{Z} \). So we may assume \( m = m_1p + m_0 \) with \( 0 \leq m_0 \leq p - 2 \). Then Proposition 5.8 shows that we may obtain the \( \Delta_r \)-factors of \( T_r(m) \) from the \( \Delta_r \)-factors of \( T_r(m_1p - 1) \) as follows. For each \( \Delta_r(np - 1) \) occurring in \( T_r(m_1p - 1) \) (this will be with multiplicity 1 as we have just seen) we get two \( \Delta_r \)-factors of \( T_r(m) \), namely those with highest weights in the two alcoves \( np + A \) and \( (n - 1)p + A \). By the linkage principle we know that when \( \Delta_r(np - 1) \) occurs in \( T_r(m_1p - 1) \) then \( \Delta_r((n \pm 1)p - 1) \) do not and hence \( \Delta_r(m') \) cannot occur twice in \( T_r(m) \) for any \( m' \in \mathbb{Z} \).

**Remark 10.** (1) The composition factor multiplicities \([\Delta_r(m) : L_r(n)]\) are all 0 or 1. This follows from the proposition by the reciprocity law (4.11). Alternatively, it follows from the observation that all weights of \( \Delta_r(m) \) have multiplicity 1.

(2) We could turn the arguments around and use (1) (and its alternative proof) to prove the proposition. The proof we have given has the advantage that it gives a recipe for finding the \( \Delta_r \)-factors of all \( T_r(m) \).

As a consequence of Proposition 6.2 we get that if \( (T_r(n) : \Delta_r(m)) \neq 0 \) then there is exactly one cellular basis element in \( \text{Hom}_{G, T}(T_r(m), T_r(n)) \) of weight \( m \), namely (in our standard notation) \( \bar{g}^m(T_r(n)) \). We shall write

\[ u_r(m, n) = \bar{g}^m(T_r(n)). \]
Recall that by our convention from Remark 2 we have \( u_r(m, m) = \text{id}_{T_r(m)} \).

With this notation the result in Theorem 5.3 reads as follows for \( G = SL_2 \).

**Theorem 6.3.** The set
\[
B_r = \{ u_r(m, n) \mid 0 \leq m < p^r, \ m \leq n \leq 2p^r - 2 - m \}
\]
generates the morphisms in \( P_r \).

If we restricts ourselves to the principal block \( P_r(0) \) in \( P_r \) associated with \( 0 \in A \) then this theorem combined with the strong linkage principle give

**Corollary 6.4.** The set
\[
B_r(0) = \{ u_r(m, n) \mid 0 \leq m < p^r, \ m \leq n \leq 2p^r - 2 - m \text{ and } n, m \equiv (2p) 0, -2 \}
\]
generates the morphisms in \( P_r(0) \).

6.3. \( r = 2 \). Up to equivalence we have the following two components of \( P_2 \).

1. The Steinberg component \( SL_2 \cap P_2 \).
2. The \( p \)-regular block \( P_2(0) \) associated to \( 0 \in A \).

As the Steinberg component is equivalent to \( P_1 \) this case is taken care of in Section 6.1.
So let’s consider (2). We set
\[
P_0 = T_2(0), P_1 = T_2(2p - 2), P_2 = T_2(2p), \cdots, P_{2p-1} = T_2(2p^2 - 2),
\]
and if \( j = i + 2ap \) with \( 0 \leq i \leq 2p - 1, \ a \in \mathbb{Z} \)
\[
P_j = P_i \otimes 2ap^2.
\]
Here \( P_0 \) and \( P_p \) have two \( \Delta_2 \)-factors. In fact, \( (P_0(0) : \Delta_2(0)) = (P_2 : \Delta_2(-2)) = 1 = (P_p : \Delta_2(p^2 + p - 2)) = (P_p : \Delta_2(p^2 - p)) \) and these are the only \( \Delta_2 \)-factors of \( P_0 \) and \( P_p \). The other \( P_i \)’s all have four \( \Delta_2 \)-factors, namely \( (P_1 : \Delta_2(2p - 2)) = (P_1 : \Delta_2(0)) = (P_1 : \Delta_2(-2p)) = (P_2 : \Delta_2(2p)) = (P_2 : \Delta_2(2p - 2)) = (P_2 : \Delta_2(-2p)) = (P_2 : \Delta_2(-2p - 2)) = 1, \) etc, and finally, \( (P_{2p-1} : \Delta_2(2p^2 - 2)) = (P_{2p-1} : \Delta_2(2p^2 - 2p)) = (P_{2p-1} : \Delta_2(2p^2 - 2p)) = (P_{2p-1} : \Delta_2(2p^2 - 2)) = (P_{2p-1} : \Delta_2(0)) = 1, \) and these are the only \( \Delta_2 \)-factors of \( P_1, P_2, \cdots, P_{2p-1} \).

We now define some homomorphisms between these \( P_j \)’s as follows. For each \( i \in \mathbb{Z} \) we let \( \lambda_i \) denote the highest weight of \( P_i \), i.e.
\[
\lambda_i = \begin{cases} 
  ip & \text{if } i \text{ is even}, \\
  -2 + (i+1)p & \text{if } i \text{ is odd}.
\end{cases}
\]
Define then the following maps
\[
u_i = \tilde{g}_i^{\lambda_i}(P_{i+1}), \ i = -p, -p+1, \cdots, -2, 0, 1, \cdots, p-2,
\]
\[
u'_i = \tilde{g}_i^{\lambda_i}(P_{p-i}), \ i = 1, 2, \cdots, p-1,
\]
\[
u'_{-i} = \tilde{g}_{-i}^{\lambda_{-i}}(P_{-i}, P_i), \ i = 1, 2, \cdots, p-1.
\]
We denote by \( d_i \) and \( d'_i \) the corresponding dual elements. Then we have
Proposition 6.5. The above homomorphisms \( \{ u_i \mid -p \leq i \leq p-2, i \neq -1 \} \) and \( \{ u'_i \mid -p+1 \leq i \leq p-1, i \neq 0 \} \) generate via formation of direct sums, composites, dualisation, and tensoring with \( \pm 2p^2 \) all morphisms in \( \mathcal{P}_2(0) \).

Proof. We shall use the general results from Theorem 5.3 combined with the above detailed knowledge of which \( \Delta_2 \)-factors occur in \( P_i = T_2(\lambda_i) \). If we fix \( i \) with \( 0 \leq i < p \) it follows from the above list of \( \Delta_2 \)-factors of \( P_i \) that \( B_{\lambda_i}^2(\lambda_j) \) is empty except for \( j = i, i+1, 2p-i, 2p-i-1 \) (if \( i = 0 \) or \( p-1 \) there are only three such \( j \)). We analyse each of these possibilities and get

For \( j = i \) we have \( B_{\lambda_i}^2(\lambda_i) = \{ id_{P_i} \} \),

for \( j = i+1 \) we have \( B_{\lambda_i}^2(\lambda_{i+1}) = \{ u_i \} \),

for \( j = 2p-i \) we have \( B_{\lambda_i}^2(\lambda_{2p-i}) = \{ u'_i \} \),

and for \( j = 2p-i-1 \) we have \( B_{\lambda_i}^2(\lambda_{2p-i-1}) = \{ u'_{i-1} \circ u_i \} \).

A similar analysis for \( -p \leq i < 0 \) finishes the proof. \( \square \)

Proposition 6.6. The generators \( u_i \) and \( u'_i \) in Proposition 6.5 satisfy the following relations

1. \( u_{i+1}u_i = 0, i = 0, 1, \ldots, p-3, p, p+1, \ldots, 2p-3 \),
2. \( u'_{2p-i}u'_i = 0 \) and \( (u'_i \otimes 2p^2)u'_{2p-i} = 0, i = 1, 2, \ldots, p-1 \),
3. The following diagrams (where we have only drawn the up-arrows” - for each arrow there is a dual reverse arrow)

\[
\begin{array}{ccc}
P_i & \xrightarrow{u'_i} & P_{2p-i} \\
\downarrow u_i & & \downarrow u'_{2p-i-1}
\end{array}
\begin{array}{ccc}
P_{i+1} & \xrightarrow{u'_{i+1}} & P_{2p-i-1}
\end{array}
\]

commute, i.e. \( d_{2p-i-1} \circ u'_i = u'_{i+1} \circ u_i \) and \( u_{2p-i-1} \circ u'_{i+1} = u'_i \circ d_i \).
4. \( u_{i-1} \circ d_{i-1} = d_i \circ u_i \) and \( u'_{i-1} \circ d'_{i-1} = d'_i \circ u'_i \).
5. \( d'_i \circ d_{2p-i} \circ u'_{i-1} \circ d_{i-1} = d'_i \circ u_{2p-i-1} \circ u_{i+1} \circ u_i \).

Proof. The relations (1) and (2) follow from the fact that the corresponding Hom-spaces are zero, cf. Proposition 5.2 and the above determinations of \( \Delta_2 \)-factors of the \( P_i \)’s. (3) - (5) are proved by brute force using the structure of the \( Q_2(\lambda) \)’s from \( \square \) Section 5.

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