DESCENT AND THETA FUNCTIONS FOR METAPLECTIC GROUPS

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Abstract. We establish new relations for the Fourier coefficients of the theta functions on the 4 and 6-fold covers of $GL_2$, as predicted by the conjectures of Patterson and Chinta-Friedberg-Hoffstein. These theta functions appear in the residual spectrum. To do so, we show that descent methods may be adapted to higher degree covers and show how they give relations between the Whittaker coefficients of theta functions on different groups.

1. Introduction

Let $n > 1$ and $F$ be a global field containing a full group of $n$th roots of unity, $\mu_n$. For $G$ in a broad class of algebraic groups including all Chevalley groups, there is a family of central simple extensions of the adelic points of $G$ by $\mu_n$: the metaplectic groups. If $\tilde{G}$ is a metaplectic group, the rational points $G(F)$ embed discretely and one may study automorphic forms or representations of $\tilde{G}$. The first examples of such functions, Eisenstein series, may be constructed as in the case of $G$ itself. The objects of concern in this paper are the residues of such Eisenstein series—theta functions.

Theta functions are difficult to study, even for $G = GL_2$, $n > 2$. The metaplectic Eisenstein series for $n > 2$ were introduced by Kubota [19], who showed that their Fourier coefficients are infinite sums of $n$-th order Gauss sums. Hence one can not approach the problem of determining their residues directly from this expansion. When $n = 3$, Patterson determined the cubic theta function in full [22], and showed that its Fourier coefficient at a prime $p$ was a Gauss sum attached to the cubic residue symbol $(*/p)_3$. However, for $G = GL_2$, $n > 3$, such a determination is not known, and the difficulty of this problem has a representation-theoretic origin: the Whittaker functional of the local theta representation is not unique. In spite of this non-uniqueness, there are two conjectures concerning the Fourier coefficients of the higher theta functions. In the early 1980s Patterson considered the case $n = 4$ [23]; his conjecture states that the square of the Fourier coefficient of the biquadratic theta function at a good prime $p$ is twice a normalized Gauss sum formed from a biquadratic residue symbol $(*/p)_4$. Numerical work of Eckhart and Patterson [9] has supported this conjecture, but theoretical progress has been hard to come by; a possible approach using Rankin-Selberg integrals has been offered by Suzuki [26]. More recently, Chinta, Friedberg, and Hoffstein [7] conjectured a relation (the CFH conjecture) between the Fourier coefficients at squares of the theta function on the six-fold cover of $GL_2$ and cubic Gauss sums, or equivalently the Whittaker coefficients of the cubic theta function. This conjecture has very recently been studied numerically and refined by Bröker and Hoffstein [1], working over $\mathbb{Q}(\exp(2\pi i/6))$. In

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addition, Hoffstein [17] proved Patterson’s Conjecture over the rational function field, and the CFH conjecture was established in this case as well in [7]. However, these proofs are by explicit computation of rational functions, and do not shed light on why the conjectures should be true for number fields.

The goal of this paper is to introduce a new method into the study of higher theta functions and to use it to prove representation-theoretic statements related to these two conjectures. If there were no bad places, then our work would prove the two conjectures in full (and over general base field). As we shall explain, theta representations on other, higher rank, groups intervene, and many aspects of such representations are not yet studied. A thorough analysis of these theta representations at ramified places should ultimately allow one to extend our method. Moreover, we show that the conjectures of Patterson and CFH are each the first in a series of expected relations between theta functions on different groups, and explain the general principle that gives such a relation.

The key to our work is to adapt descent constructions to covering groups. Descent is a method studied in Ginzburg, Rallis and Soudry [15] which can be described as follows. Let $G$ denote an algebraic group, and let $\Theta$ denote a small representation of $G(\mathbb{A})$, where $\mathbb{A}$ denotes the ring of adeles of $F$. In our context, a small representation may be defined as a representation which is not generic. In most examples, the representation $\Theta$ is a residue of an Eisenstein series, induced from a certain automorphic representation $\tau$ defined on a certain Levi subgroup of a parabolic subgroup of $G$. If needed, we write $\Theta_\tau$ to emphasize the dependence on $\tau$.

Let $O$ denote a unipotent orbit of the group $G$. As explained in [12], one can associate to this unipotent orbit a unipotent subgroup $U$ of $G$, and a character $\psi_U$ of $U(F)\backslash U(\mathbb{A})$. Assume that the stabilizer of this character inside a suitable parabolic subgroup of $G$ is the reductive group $H$. With this data one can construct the space of functions

$$f(h) = \int_{U(F)\backslash U(\mathbb{A})} \theta_\tau(uh) \psi_U(u) \, du$$

where $\theta_\tau \in \Theta_\tau$. Denoting by $\sigma$ the representation of $H(\mathbb{A})$ which is generated by all such $f(h)$, the main goal of the descent method is to understand the relation between the two representations $\tau$ and $\sigma$. Another class of descent integrals may also be formed incorporating Fourier-Jacobi coefficients; an example is given in (2) below.

So far all known examples of descent constructions have been in the case where $\Theta_\tau$ is a representation on a linear group or on the double cover of a reductive group. An important application has been when the correspondence between $\tau$ and $\sigma$ is functorial. This allows one to give examples of the Langlands correspondence or its inverse. Another application arises when both representations $\tau$ and $\sigma$ are generic, in which case the descent construction may be used to derive branching rules. See [13] for such examples.

In this paper we initiate the study of descent constructions using small representations defined over arbitrary covering groups. With the above application in mind, we concentrate on two specific examples. In both cases the representation $\tau$ is a theta representation constructed by Kazhdan and Patterson [18] on a 3 or 4-fold cover of the group $GL_r$ for a suitable value of $r$, and the group $H$ consists of a covering group involving a group of type $A_1$. In the first example, we use a small representation defined on the 3-fold cover of $Sp_4$. In this construction $\tau$ is the theta representation of the 3-fold cover of $GL_2$ and $\sigma$ is the theta
representation on the 6-fold cover of $SL_2$, both representations constructed in [18]. In the second case the small representation is defined on the 4-fold cover of $SO_7$, $\tau$ is defined on a 4-fold cover of $GL_3$ and $\sigma$ on a 4-fold cover of $SO_4$. The analysis of the descent for these two cases will explain the CFH and Patterson Conjectures.

In both constructions the representations $\Theta_\tau$ are constructed as residues of Eisenstein series, and the main ingredient of the construction is the smallness property of these representations. To put this in context, the unipotent orbits are a partially ordered set, and a unipotent orbit $O$ is attached to a representation if all Fourier coefficients of the form $\Pi$ for larger or incomparable orbits vanish identically and some coefficient for $O$ is nonzero. For an arbitrary metaplectic covering of a classical group, even the group $GL_r$, the determination of the unipotent orbits attached to a theta representation is not known. For the double cover of odd orthogonal groups, this determination is the main step in our paper [4] with D. Bump. In this paper, we show that in the $Sp_4$ case the representation $\Theta_\tau$ is attached to the unipotent orbit $(2^2)$, and in the $SO_7$ case it is attached to the orbit $(3^21)$.

The key is that in both cases the descent integral allows us to relate the finite part of the Whittaker coefficients of the representation $\sigma$ to those of $\tau$. Here a subtlety intervenes. There are different covers of the general linear group. Though the theta representations on some $n$-fold covers of $GL_{n-1}$ have a unique Whittaker model, in fact $\tau$ in both cases does not. However, one may use Hecke theory to simply the relations, and they then give two identities involving the Fourier coefficients of $\sigma$ and certain Gauss sums. This allows us to establish our main results, Theorems 1 and 2 which are versions of the conjectures of CFH and Patterson. We establish these results for all primes sufficiently close to 1 at the bad places. The restriction to such primes is required since the descent constructions result in local integrals at the ramified places which we do not have a precise way to compute.

The relations we establish in this paper should fit into a more general series relating certain Whittaker coefficients of higher theta functions to Gauss sums. We outline the construction in one specific case. Let $\tau$ denote the theta representation on a (certain) $(2n+1)$-fold cover of $GL_{2n}$. Using this representation construct a representation $\Theta_{\tau}^{(2n+1)}$ which is defined on the $(2n+1)$-fold cover of $Sp_{4n}$ as a residue of a certain Eisenstein series. Let $\sigma$ denote the representation of the $(4n+2)$-fold cover of $Sp_{2n}$ generated by the space of functions

$$ f(h) = \int_{U(F) \backslash U(\mathbb{A})} \tilde{\theta}(l(u)h) \, \theta_\tau(uh) \, \psi_U(u) \, du. \quad (2) $$

Here $\tilde{\theta}$ is a function in the space of the minimal representation of the double cover of $Sp_{2n}$, and $\theta_\tau$ is a function in the space of $\Theta_{\tau}^{(2n+1)}$. Also, $U$ is a certain unipotent subgroup of $Sp_{4n}$, $\psi_U$ is a character of $U(F) \backslash U(\mathbb{A})$, and $l$ is a certain homomorphism from $U$ onto the Heisenberg group with $2n+1$ variables. (In our case $\psi_U$ will be trivial but in higher rank cases it would appear.)

Then we expect that the representation $\Theta_\tau$ is attached to the unipotent orbit $((2n)^2)$. Once this is established, it should be possible to prove an identity similar to the CFH identity, relating certain values of the Whittaker coefficient of $\sigma$, a representation on the $(4n+2)$-fold cover of $Sp_{2n}$, with certain $(2n+1)$-th order Gauss sums. The case $n = 1$ is treated here. One should also be able to carry out similar constructions for orthogonal groups which generalize the construction for $SO_7$ given here, and work with even orthogonal groups as well.
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2. Notations and Preliminaries

2.1. Basic Notations. Let $F$ denote a global field and let $\mathbb{A}$ denote its ring of adeles. Since we work with covering groups, we will always require that $F$ contain certain groups of roots of unity, specified below. If $\nu$ is place of $F$, we write $F_\nu$ for the completion of $F$ at $\nu$, and if $\nu$ is a finite place we write $O_\nu$ for the ring of local integers. Let $\psi$ be a nontrivial character of $F\backslash\mathbb{A}$. Let $G$ be an algebraic group, $V$ be a unipotent subgroup of $G$, and $\psi_V$ be a given character of $V(F)\backslash V(\mathbb{A})$. Let $\pi$ denote an automorphic representation of the group $G(\mathbb{A})$ or of a cover of this group; for such covers the group $V(\mathbb{A})$ embeds canonically via the trivial section [21]. For a given vector $\varphi$ in the space of $\pi$, denote

$$\varphi^{\nu,\psi_V}(h) = \int_{V(F)\backslash V(\mathbb{A})} \varphi(vh) \psi_V(v) \, dv.$$  

We will make use of the symplectic and orthogonal groups. Let $J_n$ denote the $n \times n$ matrix with ones on the anti-diagonal and zeros elsewhere. In terms of matrices, we will represent the group $Sp_{2n}$ as the invertible linear transformations that preserve the symplectic form given by $J_n$, and the orthogonal group $SO_n$ as the invertible linear transformations that preserve the symmetric bilinear form given by $J_n$. All roots are defined with respect to the standard maximal torus of diagonal matrices and positive roots are those corresponding to the standard upper triangular unipotent subgroup. We shall denote by $w_i$ the simple reflections of the Weyl group. Order the positive roots so that in $Sp_4$, $w_1$ is the simple reflection corresponding to the short root, and in the $SO_7$, $w_1$ and $w_2$ are the simple reflections which generate the Weyl group of $GL_3$ and $w_1$ and $w_3$, the third simple reflection of $SO_7$, commute. We shall write $w[i_1i_2\ldots i_l]$ for $w_{i_1}w_{i_2}\ldots w_{i_l}$.

Let $\partial(\theta)$ denote the theta function defined on the double cover of $SL_2(\mathbb{A})$. Here $\phi$ is a Schwartz function of $\mathbb{A}$. See for example [13], pg. 8.

Let $G$ be a reductive algebraic group. We denote by $G^{(n)}(\mathbb{A})$ an $n$-fold cover of the group $G(\mathbb{A})$ as in Matsumoto [20] (if $G$ is split and connected); more generally see Brylinski-Deligne [2]. Such a cover is a group extension of $G(\mathbb{A})$ by the group of $n$-th roots of unity $\mu_n$ and is defined via a two-cocycle $\sigma$ which is constructed using local Hilbert symbols. This cover exists when $F$ contains enough roots of unity. We recall that $G(F)$ embeds discretely in $G^{(n)}(\mathbb{A})$. Similarly if $F_v$ is local we write $G^{(n)}(F_v)$ for an $n$-fold cover of $G(F_v)$.

Specifically, if $G = SL_r$, by $SL_r^{(n)}(F_v)$ we shall mean the $n$-fold cover of $SL_r$ which is described in detail in Kazhdan-Patterson [18], Section 0. This cover is determined by a two-cocycle $\sigma_n$ on $SL_r(F_v)$ which satisfies the property

$$\sigma_n(\text{diag}(h_1,\ldots, h_r), \text{diag}(k_1,\ldots, k_r)) = \prod_{i<j} (h_i, k_j)_n,$$

where $(\cdot, \cdot)_n$ is the $n$-th order local Hilbert symbol. (We will drop the subscript $n$ when it is clear from context.) This requires a choice of embedding of $\mu_n$ into $\mathbb{C}^{\times}$, which we fix but do not incorporate into the notation. We will also be concerned with the $n$-fold covers of $GL_2(\mathbb{A})$. For the general linear group, the basic cocycle $\sigma_n$ satisfying (3) may be twisted to
obtain a new cocycle $\sigma_{n,c}(g_1, g_2) = \sigma_n(g_1, g_2) (\det g_1, \det g_2)^c$ with $c \in \mathbb{Z}/n\mathbb{Z}$. See Kazhdan-Patterson [18], pg. 41. Thus there is more than one cover, and in fact the representation theory of different covers is not the same ([18], Cor. I.3.6).

In this paper we will be concerned with the 3-fold cover of $Sp_4(\mathbb{A})$ and the 4-fold covers of $SO_7(\mathbb{A})$, $SO_5(\mathbb{A})$ and $SO_4(\mathbb{A})$. These covers are obtained via restriction from covers of the special linear group. However, in forming the 3-fold cover of $Sp_4$ it is convenient to use the 2-cocycle

$$\sigma_{Sp_4}(g_1, g_2) = \sigma_3(wg_1w^{-1}, wg_2w^{-1}), \quad w = \begin{pmatrix} I_2 & \ast \\ J_2 & \ast \end{pmatrix}.$$ 

In working with this group we require that $F$ contain a full set of third roots of unity. To form the 4-fold cover of $SO_7$ we restrict the 8-fold cover of $SL_7(\mathbb{A})$ to $SO_7(\mathbb{A})$:

$$\sigma_{SO_7}(g_1, g_2) = \sigma_8(w'g_1w'^{-1}, w'g_2w'^{-1}), \quad w' = \begin{pmatrix} I_4 & \ast \\ J_3 & \ast \end{pmatrix}.$$ 

Accordingly in this case we shall assume that $F$ contains the eighth roots of unity. It is proper to regard this as a 4-fold rather than 8-fold cover as the 4-th power of the cocycle $\sigma_{SO_7}$ is almost trivial. See [4], Sect. 2. The 4-fold cover of $SO_5$ is the restriction of this cover to $SO_5$ embedded in $SO_7$ via

$$g \mapsto \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix}.$$ 

The 4-fold cover of $SO_4$ is also obtained from that for $SO_7$ via an embedding and will be described later.

We remark here that we could use $GSpin_7$ in place of $SO_7$. The arguments below concerning the constant terms and the various Fourier coefficients go through with only minor technical changes. The application to theta functions would be the same.

When $n$ is clear, we denote by $e_{i,j}$ the $n \times n$ matrix whose $(i, j)$ entry is one and whose other entries are zero. We will also denote $e'_{i,j} = e_{i,j} - e_{n+1-j,n+1-i}$. Throughout this paper, when not specified, matrices in $G(\mathbb{A})$ or $G(F_v)$ are embedded in their covers by means of the trivial section $s(g) = (g, 1)$, with one exception. Over a local field $F_v$ such that $|n|_v = 1$ the compact subgroup $G(O_v)$ embeds canonically in $G^{(n)}$ and we shall use this embedding without comment. At the finite places where $|n|_v \neq 1$ a similar statement holds provided $G(O_v)$ is replaced by a principal congruence subgroup. Moreover, we shall always suppose that this subgroup is chosen so that the diagonal elements in it are $n$-th powers in $F_v^\times$. Lastly, we recall that the subgroup of upper triangular unipotents embeds in each cover by the trivial section $s$.

2.2. Root Exchange. In the following sections during the computations, we will carry out several Fourier expansions. One type of expansion will repeat itself several times, and therefore it is convenient to state it in generality. We shall refer to this process as root exchange. We recall this process briefly now; a detailed description is given in Ginzburg [14].

Let $G$ be the adelic points of a split algebraic group or a cover of such. Let $\alpha$ and $\beta$ be two roots (not necessarily positive) and let $x_\alpha$ and $x_\beta$ be the standard one-parameter unipotents attached to $\alpha$ and $\beta$. Let $U(\mathbb{A})$ be a unipotent subgroup that is normalized by
\(x_\alpha(t)\) and \(x_\beta(t)\) for all \(t \in \mathbb{A}\). Let \(f\) be an automorphic function. We will be concerned with the possible vanishing of the integral

\[
\int_{(F\backslash\mathbb{A})^2 U(F)\backslash U(\mathbb{A})} \int f(ux_\alpha(m)x_\beta(r)) \psi(m) \, dm \, dr.
\]

To study this, consider the following integral as a function of \(g\):

\[
L(g) = \int_{F\backslash\mathbb{A} U(F)\backslash U(\mathbb{A})} \int f(ux_\alpha(m)g) \psi(m) \, dm.
\]

We make the following hypothesis: Suppose that \(\gamma\) is any root, positive or negative, which satisfies the following commutation relations: if \(m, l, t \in \mathbb{A}\) then there exist \(u', u'' \in U(\mathbb{A})\) such that

\[
[x_\beta(l), x_\alpha(m)] = u'; \quad [x_\beta(l), x_\gamma(t)] = x_\alpha(lt)u''; \quad [x_\alpha(m), u''] = 1.
\]

Suppose that the function \(L\) is left invariant under \(x_\gamma(\delta)\) for all \(\delta \in F\), i.e. \(L(x_\gamma(\delta)g) = L(g)\) for all \(g \in G\).

In this case, we can expand the integral (5) along \(x_\gamma(t)\) where \(t \in F\backslash\mathbb{A}\). It is equal to

\[
\sum_{\delta \in F \backslash\mathbb{A}} \int \int f(ux_\alpha(m)x_\gamma(t)x_\beta(r)) \psi(m + \delta t) \, dt \, dm \, dr.
\]

Using the left-invariance properties of \(f\) under the rational elements of \(G\), changing variables, and collapsing summation over \(\delta\) with integration over \(r\), integral (5) is equal to

\[
\int_{\mathbb{A}} \int \int f(ux_\alpha(m)x_\gamma(t)x_\beta(r)) \psi(m) \, dt \, dm \, dr.
\]

Arguing as in [11], one can easily show that the above integral is zero for all choices of data if and only if the integral

\[
\int_{(F\backslash\mathbb{A})^2 U(F)\backslash U(\mathbb{A})} \int f(ux_\alpha(m)x_\gamma(t)) \psi(m) \, dm \, dt
\]

is zero for all choices of data. Hence, we deduce that the integral (5) is zero for all choices of data if and only if the integral (7) is zero for all choices of data. Referring to this process we will say that we exchange the root \(\beta\) by the root \(\gamma\). We will also call replacing the integral (5) by the equal integral (6) a root exchange.

### 3. Theta Representations

#### 3.1. Definition and Basic Properties of the Theta Representation.

In this subsection we give the definition of the theta representations for the groups \(Sp_4^{(3)}(\mathbb{A})\) and \(SO_7^{(4)}(\mathbb{A})\). These are obtained globally as residues of certain Eisenstein series. The material in this subsection is an adaptation to these groups of the construction of the theta representation for covers of the general linear group in [18]. We also refer to [4] where a similar representation for the double cover of an odd orthogonal group was constructed and studied, and [10] where further work with theta representations is carried out. We also note that a theta function on the cubic cover of \(Sp_4\) was constructed for \(F = \mathbb{Q}(\exp(2\pi i/3))\) by Proskurin [25].

Let $H$ be one of the groups $Sp_4$ and $SO_7$ and $n = 3, 4$ resp. Let $B$ denote its standard Borel subgroup and write $B = T U$ where $T$ is the maximal torus of $H$. Over a local field or over the adeles, we denote the inverse of the groups $B$ and $T$ in the covering group by $\tilde{B}$ and $\tilde{T}$. Let $Z(\tilde{T})$ denote the center of $\tilde{T}$, and fix a maximal abelian subgroup $\tilde{A}$ of $\tilde{T}$ containing $Z(\tilde{T})$. (The group $\tilde{A}$ need not be the inverse image of a subgroup of $T$.) It follows from [18] that representations of $\tilde{B}$ are determined as follows. Fix a character $\chi$ of $Z(\tilde{T})$, extend it in any way to a character of $\tilde{A}$, and then induce it to $\tilde{T}$. Then extend this representation trivially to $\tilde{B}$. Inducing the resulting representation of $\tilde{B}$ to $H^{(n)}$ we obtain a representation of this group which we denote by $Ind_{\tilde{B}}^{H^{(n)}} \chi$. The notation is justified since up to isomorphism this representation depends only on $\chi$ (and not on the choice of $\tilde{A}$, in particular). This may be carried out over a local field or over the adeles.

The global theta representation will be defined as a residue of an Eisenstein series induced from the Borel subgroup. Parameterize the maximal torus of $Sp_4$ as

$$t(a, b) = \text{diag}(a, b, b^{-1}, a^{-1})$$

and that of $SO_7$ as

$$t(a, b, c) = \text{diag}(a, b, c, 1, c^{-1}, b^{-1}, a^{-1}).$$

Unless mentioned otherwise, we will always use this parametrization.

We start with the case $H = Sp_4$. The group $Sp_4$ has two maximal parabolic subgroups. The first, which we denote by $P$, has the group $GL_2$ as its Levi subgroup. The second parabolic whose Levi subgroup is $GL_1 \times SL_2$ we shall denote by $Q$. A straightforward computation shows that when we restrict the cocycle $\sigma_{Sp_4}$ from $Sp_4$ to the Levi part of $P$, that is to $GL_2$, we obtain the complex conjugate of the $GL_2$ cocycle $\sigma_{3,1}$. This will be important later, since the Whittaker coefficient of the theta representation of the group $GL_2^{(3)}(\mathbb{A})$ is not unique when $c = 1$. For the conditions of the uniqueness, see [18] Corollary I.3.6.

Let $\chi_{s_1, s_2}$ denote the character of $T$ defined by $\chi_{s_1, s_2} = \delta_{BGL_2}^{s_1} \delta_{P}^{s_2}$. In coordinates we have $\chi_{s_1, s_2}(t(a, b)) = |a|^{s_1 + 3s_2} |b|^{-s_1 + 3s_2}$. Let $E_B^{(3)}(h, s_1, s_2)$ denote the Eisenstein series of $Sp_4^{(3)}(\mathbb{A})$ associated with the induced representation $Ind_{B}^{Sp_4^{(3)}} \chi_{s_1, s_2}$. (Here and below we often suppress the choice of section from the notation.) Following Prop. II.1.2 in [18], or Section 3 in [4] we deduce that the poles of this Eisenstein series are determined by the poles of

$$\zeta(6s_1 - 3) \zeta(18s_2 - 9) \zeta(3s_1 + 9s_2 - 6) \zeta(-3s_1 + 9s_2 - 3)$$

$$\zeta(6s_1 - 2) \zeta(18s_2 - 8) \zeta(3s_1 + 9s_2 - 5) \zeta(-3s_1 + 9s_2 - 2),$$

where $\zeta$ is the Dedekind zeta function of $F$. This ratio has a multi-residue at $s_1 = 2/3$ and $s_2 = 2/3$.\n
Definition 1. The theta representation on $Sp_4^{(3)}(\mathbb{A})$ is

$$\Theta_{Sp_4}^{(3)} = \text{res}_{s_2 = 2/3} \text{res}_{s_1 = 2/3} E_B^{(3)}(\cdot, s_1, s_2).$$

Let $\Theta_{GL_2}^{(3)}$ be the theta representation of the three-fold cover of $GL_2(\mathbb{A})$ with $c = 1$ as defined in [18]. Let $E_P^{(3)}(h, s)$ denote the space of Eisenstein series of $Sp_4^{(3)}(\mathbb{A})$ associated with the induced representation $Ind_{P(\mathbb{A})}^{Sp_4^{(3)}} \Theta_{GL_2}^{(3)} \delta_B$. Then it follows from the above, using
induction in stages, that the functions in $E^{(3)}_P(h, s_2)$ are exactly the residues at $s_1 = 2/3$ of the functions in $E^{(3)}_B(h, s_1, s_2)$. This means that we can obtain the representation $\Theta_{Sp_4}$ by the residues of a space of Eisenstein series associated with a theta representation of the Levi subgroup of a maximal parabolic. In fact the same goes for the other maximal parabolic subgroup $Q$ of $Sp_4$. Let $E^{(3)}_Q(h, s)$ denote the space of Eisenstein series of $Sp_4^3(\mathbb{A})$ associated with the induced representation $Ind_{Q(\mathbb{A})}^{Sp_4^3(\mathbb{A})} \Theta_{SL_2^3 Q}^{(3)}$. Here $\Theta_{SL_2}$ is the theta representation of the three-fold cover of $SL_2$. Then the representation $\Theta_{Sp_4}$ can be obtained as the residues at $s = 2/3$ of the Eisenstein series in $E^{(3)}_Q(h, s)$.

A local constituent of this representation is constructed in a manner similar to the two references above. Namely, let $F$, be a non-archimedean local field. Consider the induced representation $Ind_{B(F_v)}^{Sp_4^3(F_v)} \chi \Theta_2^{1/2}$. Here $\chi(t) = |a|^{2/3}b(1/3)$. Then one may define the local representation $(\Theta_{Sp_4}^{(3)}, v)$ as the image of the intertwining operator from $Ind_{B(F_v)}^{Sp_4^3(F_v)} \chi \Theta_2^{1/2}$ to $Ind_{B(F_v)}^{Sp_4^3(F_v)} \chi_2^{-1}\Theta_2^{1/2}$. Similarly for $\nu$ archimedean, so that $Sp_4^3(F_v) \cong Sp_4(\mathbb{C}) \times \mu_3$, there is a local theta representation obtained by smooth induction. More precisely, the smooth induced module is irreducible in this case, and is also the image of the intertwining operator, as in [159], Theorem I.6.4 part (c). Arguing as in the above two references one obtains that $\Theta_{Sp_4}^{(3)} = \otimes_v (\Theta_{Sp_4}^{(3)}, v).$ We omit the details.

We return to the global situation. We will need to study the constant terms of the representation $\Theta_{Sp_4}^{(3)}$ along the unipotent radicals of the two maximal parabolic subgroups. We argue in the same way as in [12], Prop. 3.4. Let $U(P)$ denote the unipotent radical of $P$. We consider the constant term

$$E^{(3), U(P)}_P(h, s) = \int_{U(P)(F) \backslash U(P)(\mathbb{A})} E^{(3)}_P(uh, s) \ du.$$ 

The space of double cosets $P(F) \backslash Sp_4^3(F)/P(F)$ contains three elements with representatives $e, w_2$ and $w[212]$. Therefore, for $m \in GL_2$ as embedded inside the Levi subgroup of $P$, we obtain

$$E^{(3), U(P)}_P(m, s, f_s) = f_s(h) + E_{GL_2}(m, M_{w_2 f_s}, s') + M_{w[212]} f_s(m).$$

Here the Eisenstein series is formed with section $f_s \in Ind_{F(\mathbb{A})}^{Sp_4^3(\mathbb{A})} \Theta_{GL_2}^{(3)} \delta_B$, the $M_w$ denote intertwining operators, and $E_{GL_2}(m, M_{w_2 f_s}, s')$ is the Eisenstein series on the 3-fold cover of $GL_2$ associated with the induction from the Borel with induction data $\delta_B \det(4/3)$ where $s' = 3s - 1$. The specific section used is

$$f'_s(m) = \int_{F \backslash A} f_s \left( \begin{pmatrix} 1 & r \\ 1 & 1 \end{pmatrix} m \right) dr$$

It is not hard to check that the first two terms are holomorphic at $s = 2/3$, the point where $E^{(3), U(P)}_P(h, s)$ has a simple pole.
A similar argument applies to the second maximal parabolic $Q$. The space of double cosets $Q(F)\backslash Sp_4(F)/Q(F)$ again consists of three elements, this time represented by $e, w_1, w[121]$. Thus, when we compute the constant term $E^{(3),U(Q)}_Q((a,h), s)$ where $a \in GL_1$ and $h \in SL_2$, we get three terms. Again, two of them will contribute zero after taking the residue at $s = 2/3$. We summarize the above discussion in the following Proposition. (We recall that by convention that elements of $H(\mathbb{A})$ are embedded in the cover via the trivial section $s$.)

**Proposition 1.** (i) Suppose that $t(a,b)$ is in $\widetilde{A}(\mathbb{A})$. Let $h = \text{diag}(a,b) \in GL_2^{(3)}(\mathbb{A})$. Then for each vector $\theta^{(3)}_{Sp_4} \in \Theta^{(3)}_{Sp_4}$ there is a vector $\theta^{(3)}_{GL_2} \in \Theta^{(3)}_{GL_2}$ such that

$$\theta^{(3),U(P)}_{Sp_4}(t(a,b)) = |\det h|\theta^{(3)}_{GL_2}(h).$$

(ii) Suppose that $t(a,1)$ is in $\widetilde{A}(\mathbb{A})$. Let $h \in SL_2^{(3)}(\mathbb{A})$. Then for all $\theta^{(3)}_{Sp_4} \in \Theta^{(3)}_{Sp_4}$ we have

$$\theta^{(3),U(Q)}_{Sp_4}(t(a,1)h) = |a|^{4/3}\theta^{(3),U(Q)}_{Sp_4}(h) = |a|^{4/3}\theta^{(3)}_{SL_2}(h).$$

**Proof.** The Proposition is a standard consequence of the constant term calculations above. For example to prove the first part we start with the identity

$$\theta^{(3),U(P)}_{Sp_4}(t(a,b)) = \text{res}_{s=2/3} M_{u[212]} f_s(t(a,b)).$$

Since the intertwining operator $M_{u[212]}$ maps $\text{Ind}^{Sp_4^{(3)}}_{P^{(3)}}(A) \Theta^{(3)}_{GL_2} \delta_B$ onto $\text{Ind}^{Sp_4^{(3)}}_{P^{(3)}}(A) \Theta^{(3)}_{GL_2} \delta_B^{-1}$ it follows that at $s = 2/3$ we obtain the factor of $\delta_B^{1/3}(h) = |\det h|$. The second part follows in a similar way.

The situation with the group $SO_7$ is similar. In fact, the definition and the basic properties follow exactly as in [1]. Start with the global induced representation $\text{Ind}^{SO_7^{(4)}}_{B^{(4)}}(A) \chi_{s_1,s_2,s_3} \delta_B^{1/2}.$ Here $B$ is the Borel subgroup of $SO_7$, and $\chi_{s_1,s_2,s_3}(t(a,b,c)) = |a|^{s_1}|b|^{s_2}|c|^{s_3}$. Then form the space of Eisenstein series $E^{(4)}_{B}(h, s_1, s_2, s_3)$ associated with this induced representation. Then, as in [4], the poles of this Eisenstein series are determined by the poles of

$$\prod_{1 \leq i < j \leq 3} \zeta(4(s_i - s_j)) \zeta(4(s_i + s_j)) \prod_{1 \leq i < j \leq 3} \zeta(4(s_i - s_j + 1)) \zeta(4(s_i + s_j + 1)) \prod_{1 \leq i < j \leq 3} \zeta(4s_i + 1).$$

This quotient has a multiple residue at $s_1 = 3/4, s_2 = 2/4$ and $s_3 = 1/4$. We denote by $\Theta^{(4)}_{SO_7}$ the residue representation. Let $\chi_{\Theta}(t(a,b,c)) = |a|^{3/4}|b|^{2/4}|c|^{1/4}$. Then $\Theta^{(4)}_{SO_7}$ is a sub-quotient of $\text{Ind}^{SO_7^{(4)}}_{B^{(4)}}(A) \chi_{\Theta} \delta_B^{1/2}$. It can also be realized as a subrepresentation of $\text{Ind}^{SO_7^{(4)}}_{B^{(4)}}(A) \chi_{\Theta} \delta_B^{-1/2}$.

The group $SO_7$ has three maximal parabolic subgroups. By analogy with the $Sp_4$ case, let us now denote by $P$ the maximal parabolic subgroup whose Levi factorization is given by $GL_3(U(P))$, by $Q$ the maximal parabolic with Levi factorization $Q = (GL_2 \times SO_3)U(Q)$ and by $R$ the maximal parabolic with factorization $R = (GL_1 \times SO_3)U(R)$. The cocycle $\sigma_{SO_7}$ restricted to $GL_3 \subseteq P$ gives the standard cocycle $\sigma_{4}(g_1, g_2)$ for the 4-fold cover of $GL_3$, twisted by the eighth-order Hilbert symbol $(\det g_1, \det g_2)^{-1}$. It is not difficult to see that the construction of the theta representation in [18] goes through without essential change in this case. In particular, the Whittaker model for this representation is not unique; the argument is exactly that given in Theorem I.3.5 there using $c = -1/2$, and Corollary I.3.6 holds. We denote this representation $\Theta^{(4)}_{GL_3}$. 

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Using induction in stages we can realize the representation $\Theta_{SO_7}^{(4)}$ as a residue of a space of Eisenstein series associated with an induced representation from a maximal parabolic subgroup. Starting with the group $P$, we define the space of Eisenstein series $E^{(4)}_P(h, s)$ associated with $Ind_{P(A)}^{SO_7(A)}(\Theta_{GL_3}^{(4)})\delta_P^5$. Then, it follows similarly to the above that these Eisenstein series have simple poles at $s = 2/3$. The residue representation is $\Theta_{SO_7}^{(4)}$. For the group $Q$ we have a similar result. If we write the corresponding Eisenstein series as $\tilde{E}^{(4)}_Q(h, s)$, then we denote the residue representation by $\Theta_{SO_7}^{(6)}$. We extend it to a character $\chi = \delta_7$, and let $\delta_7 : \tilde{E}_7 \to \tilde{T}$, where $\tilde{T}$ is a maximal abelian subgroup of $A$. We then form the global induced representation $Ind_{B^{(4)}(A)}^{SL_2^{(4)}(A)}(\tilde{\chi})$. As in the previous cases it is not hard to check that this Eisenstein series has a simple pole at $s = 7/12$; we denote the residue representation by $\Theta_{SL_2}^{(6)}$. See also Prop. 5 below.

Proposition 2. (i) Suppose that $t(a, b, c)$ is in $\tilde{A}(A)$. Let $h = diag(a, b, c) \in GL_3(A)$. Then for each vector $\theta_{SO_7}^{(4)}(a, b, c) \in \Theta_{SO_7}^{(4)}$ there is a vector $\theta_{GL_3}^{(4)}(a, b, c) \in \Theta_{GL_3}^{(4)}$ such that

$$\theta_{SO_7}^{(4)}(P)(t(a, b, c)) = \det h \theta_{GL_3}^{(4)}(h).$$

(ii) Suppose that $t(a, a, 1) \in \tilde{A}(A)$. Let $h \in SO_7^{(4)}(A)$. Then, for all $\theta_{SO_7}^{(4)}(a, a, 1) \in \Theta_{SO_7}^{(4)}$ we have

$$\theta_{SO_7}^{(4)}(Q)(t(a, a, 1)) = |a|^{9/8} \theta_{SO_7}^{(4)}(Q)(h).$$

(iii) Suppose that $t(a, 1, 1) \in \tilde{A}(A)$. Let $h \in SO_7^{(4)}(A)$. Then, for all $\theta_{SO_7}^{(4)}(a, 1, 1) \in \Theta_{SO_7}^{(4)}$ we have

$$\theta_{SO_7}^{(4)}(R)(t(a, 1, 1)) = |a|^{7/4} \theta_{SO_7}^{(4)}(R)(h).$$

We remark that the power $|\det h|$ in the first part and the powers of $|a|$ in the second and third parts are just the values of $\delta^{1-s_0}$ where $s_0$ is the point where the corresponding Eisenstein series has a pole.

Let $n$ be odd. We end this section with the construction of induced representations on the even-degree cover $SL_2^{(2n)}$ of $SL_2$, following [5]. Suppose that the embeddings of $\mu_n$ and $\mu_{2n}$ into $C$ are chosen so that the Hilbert symbols satisfy $(a, b)_{2n} = (a, b)_2 (a, b)_n$. Then the cocycle $\sigma_{2n}$ on $SL_2$ satisfies $\sigma_{2n} = \sigma_2 \sigma_n$. Let $B = TU$ now denote the Borel subgroup of $SL_2$ and $A$ denote any maximal subgroup of $T$ which satisfies $\sigma_n(a_1, a_2) = 1$ for all $a_1, a_2 \in A$. Then $\tilde{A}$ is a maximal abelian subgroup of $\tilde{T}$. Let $\chi$ denote a character of the center of $\tilde{T}$. We extend it to a character $\tilde{\chi}$ of $\tilde{A}$: if $b = ((^a_a - 1), \zeta)$ is in $\tilde{A}$, define $\tilde{\chi}(b) = \chi(a) \gamma(a) \zeta$. Here $\zeta \in \mu_{2n}$, and $\gamma$ is the Weil factor attached to the nontrivial additive character $\psi$. As in [5] one may check that $\tilde{\chi}$ is indeed a genuine character of $\tilde{A}$. Inducing from $\tilde{A}$ to $\tilde{T}$ and extending in the usual way to $\tilde{B}$ we can then form the induced representation $Ind_{\tilde{B}}^{SL_2^{(2n)}}(\tilde{\chi})$.

The particular case of interest for us below is the case of the theta representation. Let $n = 3$, and let $\chi_s = \delta_9^s$. We then form the global induced representation $Ind_{B^{(6)}(A)}^{SL_2^{(6)}(A)}(\tilde{\chi}_s)$. Attached to this representation is the Eisenstein series $E^{(6)}_s(g, s)$ on $SL_2^{(6)}(A)$. As in the previous cases it is not hard to check that this Eisenstein series has a simple pole at $s = 7/12$; we denote the residue representation by $\Theta_{SL_2}^{(6)}$. See also Prop. 5 below.
3.2. Fourier Coefficients of the Theta Representation. In this subsection we study
the Fourier coefficient attached to the theta representations of the various groups under
consideration here. Our goal here is to determine the unipotent orbit attached to each such
representation in the sense of [1], Section 4, where such a determination is given for the theta
function on the double cover of an odd orthogonal group. We start with

Lemma 1. The theta representations \( \Theta^{(3)}_{Sp_4}, \Theta^{(4)}_{SO_5} \), and \( \Theta^{(4)}_{SO_7} \) are not generic.

This Lemma is a special case of a more general result of Friedberg, Goldberg and Szpruch [10]. The proof is similar to that for covers of \( GL_r \) [18]. For the convenience of the reader
we give a brief sketch here.

Proof. It suffices to show that there is no local Whittaker functional. As in subsection 3.1
let \( n = 3 \) or \( n = 4 \) for the groups under consideration, as shown. Let \( \nu \) be a place such that
\( |n_\nu| = 1 \), and let \( F_\nu \) be the corresponding local field. Fix one of the above covering groups
over \( F_\nu \). We use the same notation for the principal series as in subsection 3.1. Let \( \omega \) be a
genuine quasicharacter of \( Z(\widehat{T}) \), extend to a character \( \omega' \) of a maximal abelian subgroup
\( \widehat{A} \) containing \( Z(\widehat{T}) \) as in [18], Sect. I.1, and form the principal series, denoted in this proof
by \( V(\omega') \). The dimension of the space of Whittaker functionals on \( V(\omega) \) is easily seen to be
\( [\widehat{T} : \widehat{A}] \), and one can write down a basis by means of the regularizations \( \lambda_\eta \) of standard
Whittaker-type integrals

\[
\int_U f(\eta w_0^{-1}u) \bar{\psi}(u) \, du
\]

as \( \eta \) runs over the quotient \( \widehat{A} \backslash \widehat{T} \). Here \( \psi \) is a nondegenerate character of the maximal
unipotent subgroup \( U \). One then considers the functionals \( f \to \lambda_\eta \circ I_w(f) \) with \( \lambda_\eta \) the
the corresponding Whittaker integral for \( V(\omega'w) \). These may be expressed (on a Zariski dense
set) as linear combination \( \sum_{\eta'} \tau_w(\eta, \eta') \lambda_{\eta'} \) (compare [18], pp. 75-76). The coefficients \( \tau_w \) are seen to satisfy certain relations as in [18], Lemma I.3.3.

Suppose now that the quasicharacter \( \omega \) is exceptional. In our context, this means that
\( \omega_\alpha^{n(\alpha)} = 1 \) for all positive roots \( \alpha \), where \( \omega_\alpha \) is the composition of \( \omega \) with the canonical
embedding attached to \( \alpha \), and where in the symplectic case \( n(\alpha) = 3 \) and in the orthogonal
group case \( n(\alpha) = 4 \) if \( \alpha \) is long and \( 2 \) if \( \alpha \) is short. (Compare [18], pg. 71.) These are exactly
the characters considered in subsection 3.1 above. Then the local exceptional representation
\( V_0(\omega') \) is the image of \( V(\omega') \) under the intertwining operator \( I_{w_0} \) attached to the long Weyl
element \( w_0 \) on \( V(\omega') \). (Since \( \omega \) is regular the representations \( V(\omega') \), \( V_0(\omega') \) are independent
of the choice of the extension \( \omega' \), as in [18], Prop. I.2.2.) Now a Whittaker functional on
\( V_0(\omega') \) pulls back to one on \( V(\omega') \). But every Whittaker functional on \( V(\omega') \) may be written
as \( \sum_{\eta \in \widehat{A} \backslash \widehat{T}} c(\eta) \lambda_\eta \) where \( c : \widehat{T} \to \mathbb{C} \) satisfies \( c(at) = \delta_B(a)^{-1/2}c(t) \) for \( a \in \widehat{A} \). The Whittaker
functionals in \( V_0(\omega') \) are those on \( V(\omega') \) that are zero on the kernel of \( I_{w_0} \). This gives a system
of equations for the \( c(\eta) \), which may be expressed using the coefficients \( \tau_w(\eta, \eta') \) (see [18],
Cor. I.3.4.). This allows one to count the dimension of the space of Whittaker functionals
on \( V_0(\omega') \). As in [18], Theorem I.3.5, this dimension is seen to be at most the number of free
orbits of the Weyl group \( W \) acting on the quotient lattice \( \mathbb{Z}' / n\mathbb{Z}' \), where \( r \) is the rank of
the group. In the cases at hand it is easy to check that there are no free orbits, and so no
Whittaker functionals. \( \square \)
**Proposition 3.** The unipotent orbit attached to $\Theta_{Sp_4}^{(3)}$ is $(2^2)$ and the unipotent orbit attached to $\Theta_{SO_7}^{(4)}$ is $(3^21)$.

*Proof.* We start with the $Sp_4$ case. It follows from Lemma [1] that $\Theta_{Sp_4}^{(3)}$ is not generic. On the other hand, it follows as in [12], Theorem 3.1, that a representation on $Sp_4^{(3)}(A)$ which is not generic must have a nonzero Fourier coefficient associated with the unipotent orbit $(2^2)$.

Next consider the $SO_7$ case. We need to prove that the representation $\Theta_{SO_7}^{(4)}$ has no nonzero Fourier coefficient corresponding to the unipotent orbits $(7)$ and $(51^2)$, and has a nonzero Fourier coefficient corresponding to the orbit $(3^21)$. Lemma [1] above states that $\Theta_{SO_7}^{(4)}$ is not generic; this is equivalent to the vanishing of the Fourier coefficients for the orbit $(7)$.

Consider the Fourier coefficients corresponding to the orbit $(51^2)$. They are given as follows. Let $U'$ denote the standard unipotent radical of the parabolic subgroup of $SO_7$ whose Levi part is $GL_1^2 \times SO_3$. Then the Fourier coefficients are given by

$$\theta_{SO_7}^{U',\psi_{U',\alpha}}(h) = \int_{U'(F)\backslash U'(A)} \theta(uh) \psi_{U',\alpha}(u) \, du.$$ 

Here $\theta$ is a vector in the space $\Theta_{SO_7}^{(4)}$, and $\psi_{U',\alpha}$ is defined as follows. Let $\alpha \in F^\times$ and given $u = (u_{i,j})$ we define $\psi_{U',\alpha}(u) = \psi(u_{1,2} + u_{2,3} - \frac{1}{2} \alpha u_{2,5})$. Changing $\alpha$ by a square in $F^\times$ gives the same function for a different $h$, so we analyze $\alpha$ modulo $(F^\times)^2$. Assume first that $\alpha = 1$. Then conjugating by a suitable matrix in $SO_3(F)$, the above Fourier coefficient is zero for all choices of data if and only if the integral

$$(10) \quad \int_{U'(F)\backslash U'(A)} \theta(uh) \psi_{U'}(u) \, du$$

is zero for all choices of data. Here $\psi_{U'}(u) = \psi(u_{1,2} + u_{2,4})$.

Conjugating from left to right by $w(u) = \psi(u_{1,2} + u_{2,4})$.

Changing $\alpha$ by a square in $F^\times$ gives the same function for a different $h$, so we analyze $\alpha$ modulo $(F^\times)^2$. Assume first that $\alpha = 1$. Then conjugating by a suitable matrix in $SO_3(F)$, the above Fourier coefficient is zero for all choices of data if and only if the integral

$$(10) \quad \int_{U'(F)\backslash U'(A)} \theta(uh) \psi_{U'}(u) \, du$$

is zero for all choices of data. Here $\psi_{U'}(u) = \psi(u_{1,2} + u_{2,4})$.

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$$(10) \quad \int_{U'(F)\backslash U'(A)} \theta(uh) \psi_{U'}(u) \, du$$

is zero for all choices of data. Here $\psi_{U'}(u) = \psi(u_{1,2} + u_{2,4})$.

Conjugating from left to right by $w(u) = \psi(u_{1,2} + u_{2,4})$.

Changing $\alpha$ by a square in $F^\times$ gives the same function for a different $h$, so we analyze $\alpha$ modulo $(F^\times)^2$. Assume first that $\alpha = 1$. Then conjugating by a suitable matrix in $SO_3(F)$, the above Fourier coefficient is zero for all choices of data if and only if the integral

$$(10) \quad \int_{U'(F)\backslash U'(A)} \theta(uh) \psi_{U'}(u) \, du$$

is zero for all choices of data. Here $\psi_{U'}(u) = \psi(u_{1,2} + u_{2,4})$.
where \( \psi_U(u) = \psi(u_{2,3} + u_{3,4}) \). Arguing as in Gan-Savin \[11\], Section 7, we conclude that the integral (10) is zero for all choices of data if and only if the integral

\[
\int_{U(F) \backslash U(\mathbb{A})} \theta(u) \psi_U(u) \, du
\]
is zero for all choices of data. Write \( U = U''U(R) \) where \( U(R) \) is the unipotent radical of the maximal parabolic subgroup \( R \) of \( SO_7 \) whose Levi part is \( GL_1 \times SO_5 \), and \( U'' \) is the maximal unipotent radical of \( SO_5 \) embedded in \( SO_7 \) as in \([4]\). From the definition of \( \psi_U \) it is trivial on \( U(R) \). Writing (11) as an iterated integral and computing first the constant term along \( U(R) \), we see using Prop. 2 that we obtain the Whittaker coefficient of the theta representation of \( SO_5^{(4)} \). By Lemma [11] the representation \( \Theta^{(4)}_{SO_5} \) has no nonzero Whittaker coefficients, and hence the integral (11) is zero for all choices of data.

Next consider the case when \( \alpha \) is not a square in \( F^\times \). In this case, we choose a local unramified place \( \nu \) such that \( \alpha \) is a square in \( F_{\nu}^\times \). It is enough to prove that the Jacquet module of the local representation \( (\Theta^{(4)}_{SO_5})_{\nu} \), which corresponds to the unipotent subgroup \( U'' \) with character \( \psi_{U''} \), is zero. Following the same steps as in the global case, and using again Lemma [11] we obtain that this Jacquet module is zero.

To complete the proof of the Proposition, we need to show that \( \Theta^{(4)}_{SO_5} \) has a nonzero Fourier coefficient corresponding to the unipotent orbit \((3^21)\). Consider the Fourier coefficient

\[
\int_{U(Q)(F) \backslash U(Q)(\mathbb{A})} \theta(uh) \psi_Q(u) \, du
\]
where \( U(Q) \) is the unipotent radical of the standard maximal parabolic subgroup \( Q \) whose Levi part is \( GL_2 \times SO_3 \). The character \( \psi_Q \) is defined as \( \psi_Q(u) = \psi(u_{1,3} + u_{2,5}) \). It is not hard to show that the stabilizer of \( \psi_Q \) inside \( GL_2 \times SO_3 \) is a one-dimensional torus, and this Fourier coefficient corresponds to the orbit \((3^21)\). To show that these Fourier coefficients are not zero for all choices of data, conjugate integral (12) by the Weyl element \( w[32] \). We obtain the integral

\[
\int_{(F'\mathbb{A})^2} \int_{V(F') \backslash V(\mathbb{A})} \theta(vz(r_1, r_2)w[32]) \psi_V(v) \, dv \, dr_1 \, dr_2.
\]

Here \( V \) is the subgroup of \( U' \) such that if \( v = (v_{i,j}) \in V \), then \( v_{1,5} = v_{2,4} = v_{2,5} = 0 \). Thus its dimension is five. The character \( \psi_V \) is defined as \( \psi_V(v) = \psi(v_{1,2} + v_{2,3}) \). Finally, we define \( z(r_1, r_2) = I_7 + r_1 e_{5,2} + r_2 e_{4,3} - \frac{1}{2} r_2^2 e_{5,3} \).

Expand the integral (13) along the two-dimensional unipotent subgroup given by \( y(l_1, l_2) = I_7 + l_1 e'_{1,5} + l_2 e'_{3,4} - \frac{3}{2} l_2^2 e_{2,6} \). Performing root exchanges as explained in subsection 2.2 we replace \( z(r_1, r_2) \) by \( y(l_1, l_2) \). Thus, the integral (13) is equal to

\[
\int_{(\mathbb{A}^2)} \int_{V_1(F') \backslash V_1(\mathbb{A})} \theta(v_1 z(r_1, r_2)w[32]) \psi_{V_1}(v_1) \, dv_1 \, dr_1 \, dr_2.
\]

Here \( V_1 \) is the subgroup of \( U'' \) such that if \( v_1 = (v_1(i, j)) \in V_1 \) then \( v_1(2, 5) = 0 \).

Next, we expand the integral (14) along the subgroup \( x_1(l) = I_7 + l e'_{2,5} \). The contribution from the nontrivial orbits is zero. This follows from the fact that each such Fourier coefficient
corresponds to the unipotent orbit \((51^2)\), so was shown to vanish above. Finally, we expand along the group \(x_2(l) = I_7 + le_{3,4}^t - \frac{1}{2}P^2e_{3,5}\). As above, the contribution from the nontrivial orbit is zero, since \(\Theta^{(4)}_{SO_7}\) is not generic. Thus, the above integral is equal to

\[
\int \int_{k^2 U(F)\backslash U(\mathbb{A})} \theta(uz(r_1, r_2)w[32]) \psi_{U,1}(u) \, du \, dr_1 \, dr_2.
\]

Here \(U\) is the maximal unipotent subgroup of \(SO_7\) and \(\psi_{U,1}(u) = \psi(u_1, 2 + u_{2,3})\). Again arguing as in Gan-Savin \([11]\), we deduce that this integral is not zero for some choice of data if and only if the integral

\[
\int_{U(F)\backslash U(\mathbb{A})} \theta(u) \psi_{U,1}(u) \, du
\]

is not zero for some choice of data. Using Prop. \([2]\) part (i), we see that this integral is not zero for some choice of data if and only if the representation \(\Theta^{(4)}_{GL_3}\) is not the zero representation. Since this is so (see \([18]\)), we deduce the non-vanishing of the integral \((12)\) for some choice of data, as claimed. \(\square\)

4. The Descent Integrals

In this Section we define the descent integrals and study the constant terms and the Whittaker coefficients of the resulting functions.

4.1. The \(Sp_4\) case. Starting with the representation \(\Theta^{(3)}_{Sp_4}\), we consider the function

\[
f^{(6)}_{SL_2}(g) = \int_{(F\backslash\mathbb{A})^3} \tilde{\theta}(x, y, z) \theta^{(3)}_{Sp_4}((x, y, z)g) \, dx \, dy \, dz.
\]

Here, \((x, y, z)g\) is embedded in \(Sp_4\) as

\[
(x, y, z)g \mapsto \begin{pmatrix} 1 & x & y & z \\ 1 & 1 & y & -x \\ 1 & -x & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.
\]

The function \(\theta^{(3)}_{Sp_4}\) is a vector in the space of \(\Theta^{(3)}_{Sp_4}\), and for convenience we shall denote it by \(\theta\). Notice that \(f^{(6)}_{SL_2}(g)\) is an automorphic function defined on the group \(SL_2^{(6)}(\mathbb{A})\). Let \(\sigma^{(6)}_{SL_2}\) denote the representation of \(SL_2^{(6)}(\mathbb{A})\) generated by all the above functions.

4.1.1. The Constant Term. We start by computing the constant term of \(f^{(6)}_{SL_2}(g)\) with \(g = t(a) := \text{diag}(a, a^{-1}), a \in \mathbb{A}^\times\). Unfolding the theta function we obtain

\[
\int_{F\backslash\mathbb{A}} f^{(6)}_{SL_2} \left( \begin{pmatrix} 1 & r \\ 1 \\ a \\ a^{-1} \end{pmatrix} \right) \, dr = \int_{k^2} \int_{(F\backslash\mathbb{A})^3} \omega_{\psi}((0, y, z)(x, 0, 0)n(r)t(a)) \phi(0) \theta((0, y, z)(x, 0, 0)n(r)t(1, a)) \, dy \, dz \, dr \, dx.
\]
Here \( n(r) = (1, r) \) is embedded in \( Sp_4 \) as in \([16]\), \( t(1, a) \) is defined in \([8]\), and \( \omega_\psi \) is the Weil representation with character \( \psi \) realized in the Schrödinger model (see, for example, \([15]\) Sect. 1.2). Using the properties of the Weil representation, and conjugating by \( w_1 \) in the function \( \theta \), we deduce that integral (17) is equal to

\[
|a|^{-1/2} \gamma(a) \int \int \phi(x) \theta \left( \begin{pmatrix} 1 & y & r \\ 1 & z & y \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & 1 & 1 \\ 1 & a^{-1} & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) w_1 \left( \begin{pmatrix} 1 & x & 1 \\ 1 & 1 & 1 \\ 1 & -x & 1 \end{pmatrix} \right) \times \psi(z) \, dy \, dz \, dr \, dx.
\]

The factor \(|a|^{-1/2} \gamma(a)\) arises from the action of the Weil representation together with a change of variables in \( x \). Also, \( \gamma(a) \) is the Weil factor associated to \( \psi \).

Next we expand the above integral along the group \( I_4 + \ell_1.2 \). Since \( \Theta^{(3)}_{Sp_4} \) is not generic (see Lemma \([1]\)), the contribution from the nontrivial characters is zero. Thus integral (17) is equal to

\[
|a|^{-1/2} \gamma(a) \int \phi(x) \theta^{U, \psi_U} \left( \begin{pmatrix} a & 1 \\ 1 & a^{-1} \end{pmatrix} \right) w_1 \left( \begin{pmatrix} 1 & x & 1 \\ 1 & 1 & 1 \\ 1 & -x & 1 \end{pmatrix} \right) \, dx
\]

Here \( U \) is the standard maximal unipotent subgroup of \( Sp_4 \) and \( \psi_U \) is defined by \( \psi_U(u) = \psi(u_{2,3}) \). Note that for any nontrivial \( \psi \),

\[
\int_{F\backslash A} \theta^{(3)}_{SL_2} \left( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \right) \psi(x) \, dx
\]

is not zero for some choice of data. This may be seen using the Hecke relations and the Fourier expansion. Consequently the expression (18) is nonzero for some choice of data. Indeed, plugging in \( a = 1 \), if it were zero for all choices of data, then it follows from the second part of Prop. \([1]\) that the integral (19) is zero for all choices of data.

4.1.2. **The Whittaker Coefficient.** In this subsection we compute a Whittaker coefficient of the function \( f^{(6)}_{SL_2} \). More precisely, we compute the integral

\[
\int_{F\backslash A} f^{(6)}_{SL_2} \left( \begin{pmatrix} 1 & r \\ 1 & 1 \end{pmatrix} \right) \psi(-r) \, dr
\]

Here \( g \in SL_2^{(6)}(A) \). Beginning as in the computation of the constant term, we see that integral (20) is equal to

\[
\int \int \omega_\psi(g) \phi(x) \theta \left( \begin{pmatrix} 1 & y & z \\ 1 & r & y \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 1 \\ 1 & 1 & 1 \\ 1 & -x & 1 \end{pmatrix} g \right) \psi(z - r) \, dy \, dz \, dr \, dx.
\]
Let $\gamma_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} \gamma_0 & \gamma_0^* \end{pmatrix}$, where $\gamma_0^*$ is chosen so that the matrix is symplectic.

Since $\gamma_1 \in Sp_4(F)$, we can conjugate it from left to right. After a change of variables, the character $\psi$ is changed, and we obtain

$$\int A \int (F \setminus \mathbb{A})^3 \omega_\psi(g) \phi(x) \theta \begin{pmatrix} 1 & y & z \\ 1 & r & y \\ 1 & -y & 1 \end{pmatrix} \gamma_1 \begin{pmatrix} 1 & x \\ 1 & 1 \\ 1 & -x \end{pmatrix} g \psi(y) \, dy \, dz \, dr \, dx.$$

Next we perform a root exchange as explained in subsection 2.2. This allows us to replace the one-dimensional unipotent subgroup $z_1(r) = I_4 + re_{2,3}$ by $m(l) = I_4 + le_{1,2}$. Then we conjugate by $w_2$ from left to right, and we obtain the integral

$$\int A \int (F \setminus \mathbb{A})^3 \omega_\psi(g) \phi(x) \theta \begin{pmatrix} 1 & y & l & z \\ 1 & l & -y & 1 \end{pmatrix} w_2 z_1(r) \gamma_1 \begin{pmatrix} 1 & x \\ 1 & 1 \\ 1 & -x \end{pmatrix} g \times \psi(y) \, dy \, dz \, dr \, dx \, dl.$$

Then we expand the above integral along the unipotent group $I_4 + me_{2,3}$. The contribution from the nontrivial characters is zero since, by Lemma 1, $\Theta_{Sp_4}^{(3)}(\gamma_{\theta})$ is not generic. We thus obtain

$$\int \omega_\psi(g) \phi(x) \theta^{U,\psi_{U,1}}(w_2 z_1(r) \gamma' w_1 m(x) g) \, dr \, dx.$$

Here $\psi_{U,1}$ is defined by $\psi_{U,1}(u) = \psi(u_{1,2})$. Also, we have identified the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ with the simple reflection $w_1$ as an element of $Sp_4$, and $\gamma' = I_4 + e_{1,2} = \gamma_1 w_1^{-1}$. Finally, we conjugate $\gamma'$ to the left, and $w_2$ and $w_1$ to the right, and we deduce that integral (20) is equal to the integral

$$(21) \int \omega_\psi(g) \phi(x) \theta^{U,\psi_{U,1}}(z(r) w_2 w_1 m(x) g) \psi(r) \, dr \, dx$$

where $z(r) = I_4 + re_{3,2}$. In the above integral, $\psi(r)$ arises from the conjugation of $\gamma'$ across $z_1(r)$ and a change of variables in $U$.

As a first consequence, this computation of the Whittaker coefficient gives the following non-vanishing result.

**Lemma 2.** The representation $\sigma_{Sp_4}^{(6)}$ is not zero, and moreover for some choice of data the Whittaker coefficient (20) is not zero.

**Proof.** The representation $\sigma_{Sp_4}^{(6)}$ is not zero since we have shown that some function in this space has a nonzero constant term. To prove the nonvanishing of some Whittaker coefficient, it is enough to prove that the integral (20) is not zero for some choice of data. Let $g = e$, and assume instead that this integral is zero for all choices of data. Since $\phi$ is an arbitrary Schwartz function, this implies that the integral

$$\int \theta^{U,\psi_{U,1}}(z(r)) \psi(r) \, dr$$
is zero for all choices of data. Using the matrix $I_4 + m(e_{1,3} + e_{2,4})$, and arguing as in [11] Section 7, we deduce that $\theta^{U,U_1}$ is zero for all choices of data. Since the representation $\Theta^{(3)}_{GL_2}$ is generic, we obtain a contradiction to Prop. [11] part (i).

4.2. The $SO_7$ Case. We start by describing the descent integral. Recall that $U(R)$ denotes the standard unipotent radical of the maximal parabolic subgroup of $SO_7$ whose Levi part is $GL_1 \times SO_5$. We define a character of this group by $\psi_{U(R)}(u) = \psi(u_1,a)$. It is not hard to check that the stabilizer of $\psi_{U(R)}$ inside $SO_5$ contains the split orthogonal group $SO_4$. We define the descent integral by

$$f^{(4)}_{SO_4}(g) = \int_{U(R)(F) \backslash U(R)(A)} \theta^{(4)}_{SO_7}(ug) \psi_{U(R)}(u) \, du.$$ 

Here $\theta^{(4)}_{SO_7}$ is a vector in the space of $\Theta^{(4)}_{SO_7}$, and we write $\theta$ if there is no confusion. As in the previous case, we denote by $\theta^{(4)}_{SO_4}$ the automorphic representation of $SO_4^{(4)}(A)$ generated by the space of functions given by the integral (22).

4.2.1. The Constant Term. We start with the constant term of the descent. In other words, we first compute

$$\int_{(F \backslash A)^2} f^{(4)}_{SO_4} \left( \begin{pmatrix} 1 & r_1 & r_2 & -r_1r_2 & 1 & 1 \\ 1 & -r_2 & -r_1 & a & b & b^{-1} \\ 1 & 1 & a^{-1} & 1 \end{pmatrix} \right) \, dr_1 \, dr_2,$$

where $a, b \in A^\times$. Here the embedding of the matrices in the argument of $f^{(4)}_{SO_4}$ into $SO_7$ is given by

$$(I_7 + r_1 e'_{2,3})(I_7 + r_2 e'_{2,5}) \text{ and } \text{diag}(1, a, b, 1, b^{-1}, a^{-1}, 1).$$

To compute the constant term we first start by exchanging the unipotent subgroup $y_1(x) = I_7 + x_1 e'_{1,2}$ with $I_7 + le'_{2,3} - \frac{l^2}{2} e_{2,5}$. Then conjugating from the left by $w_1$, the integral (23) is equal to

$$\int_{A \backslash U(Q)(F) \backslash U(Q)(A)} \theta(vw_1 y_1(x_1) t(1, a, b)) \psi_{U(Q)}(v) \, dv \, dx_1.$$ 

Here $t(1, a, b)$ is the torus element in (24) (cf. [9]), and we recall that $U(Q)$ is the standard unipotent radical of the maximal parabolic subgroup of $SO_7$ whose Levi part is $GL_2 \times SO_3$. The character $\psi_{U(Q)}$ is defined by $\psi_{U(Q)}(v) = \psi(v_{2,4})$.

Expand this integral along the unipotent group $I_7 + re'_{1,2}$. The nontrivial characters contribute zero to the expansion, since each of them is a Fourier coefficient associated with the unipotent orbit $(51^2)$. By Prop. [3] this is zero. Thus we are left with the contribution from the constant term. Next we exchange the unipotent subgroup $y_2(x_2) = I_7 + x_2 e'_{2,3}$ with $I_7 + le'_{3,4} - \frac{l^2}{2} e_{3,5}$. Conjugating by $w_2$ from left to right the above integral is equal to

$$\int_{A^2 \backslash V_2(F) \backslash V_2(A)} \theta(v_2w_2 y_2(x_2) w_1 y_1(x_1) t(1, a, b)) \psi_{V_2}(v_2) \, dv_2 \, dx_1 \, dx_2.$$
where \( \psi \) of unipotent subgroups as in the computation of the constant term. We obtain the integral exactly the integral (12). Therefore we may perform the same Fourier expansion that leads from Prop. 3 that the nontrivial characters contribute zero to the expansion. Thus we deduce that integral (23) is equal to

\[
(25) \quad \int_{\mathcal{A}^2} \theta^{U,\psi_{U,2}}(w_2y_2(x_2)y_1(x_1)) t(1,a,b) \, dx_1 \, dx_2.
\]

Here \( \psi_{U,2} \) is defined by \( \psi_{U,2}(u) = \psi(u_{3,4}). \)

4.2.2. The Whittaker Coefficient. In this subsection we compute a certain Whittaker coefficient of the function \( f^{(4)}_{SO_4}(g) \). More precisely, we compute the integral

\[
(26) \quad \int_{(F \backslash A)^2} f^{(4)}_{SO_4}(\left(\begin{array}{ccc} 1 & r_1 & r_2 \\ 1 & 1 & -r_1 \\ 1 & 1 & 1 \end{array}\right) \left(\begin{array}{c} a \\ 1 \\ a^{-1} \end{array}\right)) \psi(r_1 - \frac{1}{2}r_2) \, dr_1 \, dr_2.
\]

The embedding of these matrices in \( SO_7 \) is given in (24), and we start with the same exchange of unipotent subgroups as in the computation of the constant term. We obtain the integral

\[
\int_{\mathcal{A}} \left( \int_{U(Q)(F)/(U(Q)(\mathcal{A}))} \theta(u y_1(x_1)t(1,a,1)) \psi'_{U(Q)}(u) \, du \right) \, dx_1
\]

where \( \psi'_{U(Q)} \) is defined as \( \psi'_{U(Q)}(u) = \psi(u_{1,4} + u_{2,3} - \frac{1}{2}u_{2,5}) \). Let

\[
\gamma_1 = \left(\begin{array}{cc} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{array}\right), \quad \gamma_2 = k(1) \left(\begin{array}{cc} r_1 & -1 \\ -1 & 1 \end{array}\right) k(1) \text{ with } k(1) = \left(\begin{array}{ccc} 1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 1 & 1 & 1 \end{array}\right)
\]

and denote \( \gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_1^t) \in SO_7(F) \). Then \( \gamma \) normalizes the subgroup \( U(Q) \). Conjugating this element from left to right changes the character \( \psi'_{U(Q)} \) to a new character whose value on \( u \in U(Q) \) is \( \psi(u_{1,3} + u_{2,5}) \). Thus the integral over \( U(Q) \) with this character is exactly the integral (12). Therefore we may perform the same Fourier expansion that leads from integral (12) to integral (15). We conclude that the above integral is equal to

\[
\int_{\mathcal{A}^3} \theta^{U,\psi_{U,1}}(z(r_1, r_2)w[32]y_1(x_1)t(1,a,1)) \, dr_1 \, dr_2 \, dx_1.
\]

The \( \gamma_2 \) part of \( \gamma \) commutes with \( y_1(x_1)t(1,a,1) \), hence we can conjugate it to the right, and ignore it by means of a change of the vector \( \theta \). Performing the Bruhat decomposition of \( \gamma_1 \) and changing variables in \( x_1 \), the above integral becomes

\[
\int_{\mathcal{A}^3} \theta^{U,\psi_{U,1}}(z(r_1, r_2)w[32]y_1(1)w_1y_1(x_1)t(1,a,1)) \, dr_1 \, dr_2 \, dx_1.
\]

Conjugating the matrix \( y_1(1) \) to the left and the matrix \( w[321] \) to the right and changing vectors, we obtain the integral

\[
(27) \quad |a|^{-1} \int_{\mathcal{A}^3} \theta^{U,\psi_{U,1}}(z(r_1, r_2) t(a,1,1) y_2(x_1)) \psi(r_1) \, dr_1 \, dr_2 \, dx_1.
\]
Here \(y_2(x_1) = I_\gamma + x_1 e_{5,1}'\), the factor of \(|a|^{-1}\) is obtained from the change of variables in \(x_1\), and \(\psi(r_1)\) is obtained from the conjugation of \(y_1(1)\) to the left and a change of variables.

Similarly to Lemma 2 we have

**Lemma 3.** The representation \(\sigma_{SO_4}^{(4)}\) is not zero, and moreover for some choice of data the Whittaker coefficient \((26)\) is not zero.

5. **Identification of the Descent Integrals With Theta Representations**

In this section we let \(H\) be one of the groups \(Sp_4, SO_7\). Let \(n = 3\) if \(H = Sp_4\), and \(n = 4\) if \(H = SO_7\). We let \(H^{(n)}\) denote the \(n\)-fold cover of \(H\). In the first case we let \(G = SL_2\), and in the second case we let \(G = SO_4\). Let \(m = 6\) in the first case and \(m = 4\) in the second.

We also recall that we have fixed a maximal abelian subgroup \(\tilde{A}(\mathbb{A})\) of \(\tilde{A}(\mathbb{A}) \subseteq H^{(n)}(\mathbb{A})\). We shall specify a specific choice, written \(\tilde{T}_0\), later in this section. In the previous sections we constructed nonzero representations \(\sigma_{G}^{(m)}\) of the group \(G^{(m)}(\mathbb{A})\), and computed their constant term and a certain Whittaker coefficient. We have

**Proposition 4.** The representations \(\sigma_{G}^{(m)}\) are in \(L^2(G(F) \backslash G^{(m)}(\mathbb{A}))\). Moreover they lie in the discrete spectrum.

**Proof.** To prove this when \(H = Sp_4\) we use the identity of integral \((17)\) with integral \((18)\), and the identity in Prop. 1, part (ii). It follows from these results that if \(\text{diag}(a, 1, 1, a^{-1})\) lies in \(\tilde{A}(\mathbb{A})\), then the integral \((18)\) is equal to

\[
|a|^{5/6} \gamma(a) \int_{\tilde{A}} \phi(x) \phi_U \psi_U \left( \begin{array}{cc} 1 & x \\ 1 & 1 - x \end{array} \right) dx.
\]

From this we deduce the identity

\[
\int_{F \backslash A} f_{SL_2}^{(6)} \left( \begin{array}{cc} 1 & r \\ 1 & a \end{array} \right) dr = |a|^{5/6} \gamma(a) \int_{F \backslash A} f_{SL_2}^{(6)} \left( \begin{array}{cc} 1 & r \\ 1 & 1 \end{array} \right) dr
\]

for all diagonal matrices which lie in the corresponding maximal abelian subgroup of the diagonal elements of \(SL_2\). This means that the constant term defines an \(SL_2^{(6)}(\mathbb{A})\) mapping from the representation \(\sigma_{SL_2}^{(6)}\) into the representation \(\text{Ind}_{\tilde{B}(\mathbb{A})}^{SL_2^{(6)}(\mathbb{A})} \delta_B^{-1/12} \delta_B^{1/2}\). Indeed, it follows from the discussion at the beginning of Section 3 that the induced representation is determined uniquely by the values of the character on a maximal abelian subgroup of the diagonal elements.

It follows from Jacquet’s criterion, see [21], the Lemma in I.4.11 (pg. 74), that \(\sigma_{SL_2}^{(6)}\) is in \(L^2(G(F) \backslash G^{(m)}(\mathbb{A}))\). Also, since this induced representation is not unitary we deduce that \(\sigma_{SL_2}^{(6)}\) lies in the discrete spectrum.
The case when \( H = SO_7 \) is similar. Starting with the fact that integral (23) is equal to integral (25), we then use the two last parts of Prop. 2. We deduce the equality

\[
\int_{(F\backslash A)^2} f_{SO_4}^{(4)} \left( l(r_1, r_2) \begin{pmatrix} ab & b^{-1} \\ b & a^{-1}b^{-1} \end{pmatrix} \right) dr_1 \, dr_2 = \left| ab \right|^{3/4} \int_{(F\backslash A)^2} f_{SO_4}^{(4)}(l(r_1, r_2)) \, dr_1 \, dr_2
\]

where

\[
l(r_1, r_2) = \begin{pmatrix} 1 & r_1 & r_2 & -r_1 r_2 \\ 1 & -r_2 & 1 & -r_1 \\ 1 & 1 & 1 \end{pmatrix}
\]

and \( \text{diag}(ab, b, b^{-1}, a^{-1}b^{-1}) \), embedded via (24), lies in \( \tilde{A}(A) \). Hence, the constant term maps \( \sigma_{SO_4}^{(4)} \) into the principal series \( \text{Ind}_{B(A)}^{SO_4(A)} \delta_B^{-1/8} \delta_B^{3/2} \). As in the previous case, we deduce that \( \sigma_{SO_4}^{(4)} \) is in \( L^2(G(F)\backslash G^{(m)}(A)) \) and lies in the discrete spectrum. \( \square \)

From Prop. 4, we deduce that \( \sigma_G^{(m)} \) is a direct sum of irreducible representations. Since the constant term is nonzero, there is at least one irreducible summand of \( \sigma_G^{(m)} \) which lies in the residual part of the spectrum. Denote such a summand by \( \Pi_G^{(m)} \). Notice that from the computations of the exponents, we know that \( \Pi_G^{(m)} \) and \( \Theta_G^{(m)} \) are nearly equivalent.

**Proposition 5.** We have \( \Pi_G^{(m)} = \Theta_G^{(m)} \).

**Proof.** Consider the case of \( G = SL_2 \). Let \( E_{SL_2}^{(6)}(g, s) \) denote the Eisenstein series defined on \( SL_2(A) \) and associated with the induced representation \( \text{Ind}_{B(A)}^{SL_2(A)} \delta_B^{1/2} \), as in subsection 3.1. The poles of this Eisenstein series are determined by the ratio \( \zeta(12s - 6)/\zeta(12s - 5) \). Hence the Eisenstein series has a residue representation which is obtained at \( s = 7/12 \), and this residue is the representation \( \Theta_{SL_2}^{(6)} \). Arguing similarly to [18] or to [4] Theorem 3.2, one shows that the residue representation is irreducible. On the other hand, it follows from Prop. 4 that \( \Pi_{SL_2}^{(6)} \) is a sub-representation of \( \text{Ind}_{B(A)}^{SL_2(A)} \delta_B^{-1/12} \delta_B^{3/2} \) or a quotient of \( \text{Ind}_{B(A)}^{SL_2(A)} \delta_B^{3/12} \delta_B^{5/12} = \text{Ind}_{B(A)}^{SL_2(A)} \delta_B^{7/12} \). Thus \( \Pi_{SL_2}^{(6)} = \Theta_{SL_2}^{(6)} \).

The argument when \( G = SO_4 \) is similar. \( \square \)

Let \( M \) denote an algebraic group and consider a covering group \( M^{(n)}(A) \) for some number \( n \). Let \( \pi \) denote an irreducible representation of \( M^{(n)}(A) \), and write \( \pi = \pi_{\inf} \otimes \pi_{\fin} \). Here \( \pi_{\inf} \) is the product of all local representation \( \pi_{\nu} \) such that \( \nu \) is an infinite place, and \( \pi_{\fin} \) is the product over all finite places in \( F \).

Assume that \( \pi \) is generic, i.e. that the integral

\[
W_{\pi}(m) = \int_{U(M)(F)\backslash U(M)(A)} \varphi_{\pi}(um) \psi_{U(M)}(u) \, du
\]
is not zero for some choice of data. Here $U(M)$ is the maximal unipotent radical of $M$, $\psi_{U(M)}$ is a generic character of $U(M)(F)/U(M)(A)$, and the function $\varphi_\pi$ is a vector in the space of $\pi$. Choose a function $\varphi_\pi$ which corresponds to a factorizable vector in $\pi$. Then

**Lemma 4.** With the above notations we have $W_\pi(m) = W_{\pi_{\text{of}}}(m_{\text{fin}})W_{\pi_{\text{fin}}}(m_{\text{fin}})$. Here $W_{\pi_{\text{of}}}$ is a Whittaker functional defined over all infinite places. Also, $W_{\pi_{\text{fin}}}$ is a Whittaker functional on the representation $\otimes_{\nu \in \Phi} \pi_\nu$, where $\Phi$ is the set of all finite places in $F$.

**Proof.** Since all infinite places are complex, the cover is trivial in those places. Therefore, the space of Whittaker functionals defined on these local representations is one-dimensional, and hence can be separated from the other places. (For a similar argument see the Basic Lemma in [24], pg. 117.)

We apply Lemma 4 to our situation. Consider first the case $H = S^4, G = SL_2$. There is a choice of data such that integral (21) defines the Whittaker coefficient of the representation $\Pi_\nu^{(\ell)}$. In view of the factorization in Lemma 4 we may consider the contribution to this coefficient from the finite places. In other words, choosing factorizable vectors in the corresponding representations, plugging $g = \text{diag}(a, a^{-1})$, we obtain the integral

$$|a|_\Phi^{-1/2} \gamma(a) \int_{A_\Phi} \phi_\Phi(x) L_\Phi(z(r)t(a, 1)w_2w_1m(x)) \psi(r) \, dr \, dx.$$  

Here $A_\Phi = \prod_{\nu \in \Phi} F_\nu$, $|a|_\Phi = \prod_{\nu \in \Phi} |a_\nu|_\nu$, and $\phi_\Phi = \prod_{\nu \in \Phi} \phi_\nu$ where $\phi_\nu$ is a Schwartz function of $F_\nu$ at the place $\nu$ such that for almost all $\nu$ it is the unramified function. Also $L_\Phi$ is the function defined as follows.

Let $U$ continue to denote the standard maximal unipotent subgroup of $S^4$. Denote by $\psi_{U,1}$ the character of $U$ defined by $\psi_{U,1}(u) = \psi(u_{1,2})$, as in (21). For $\theta \in \Theta_{S^4}^{(3)}$ let $\theta^{U,\psi}$ be the corresponding coefficient; it defines a functional $\ell$ on the space of $\Theta_{S^4}^{(3)}$. This functional may be viewed as the convolution of the intertwining operator with the Whittaker coefficient on the $GL(2)$ part. Note that because of the lack of uniqueness of the Whittaker model, the functional $\ell$ is not factorizable. However, the intertwining operator is factorizable and because of the uniqueness of the Whittaker functional at the infinite places, we can separate out the functional at the infinite places. Let $\ell_\Phi$ denote the corresponding functional defined on $\otimes_{\nu \in \Phi} (\Theta_{S^4}^{(3)})_\nu$. Then $L_\Phi$ is defined by $L_\Phi(g) = \ell_\Phi(\rho(g)\theta)$ where $\rho$ is right translation.

Thus we have $L_\Phi(uh) = \psi_{U,1}(u)L_\Phi(h)$ for all $u \in U(A_\Phi)$. Also, since we assume that $L_\Phi$ is obtained from a factorizable vector $\theta$, there is a compact open group $K_\theta := \prod_{\nu \in \Phi} K_\nu$ such that $L_\Phi(hk) = L_\Phi(h)$, where for almost all places $K_\nu$ is the maximal compact subgroup $S^4_\nu(O_\nu)$ of $S^4(F_\nu)$, and at the remaining places $K_\nu$ is a principal congruence subgroup. Let $T$ denote the maximal torus of $S^4$ which consists of diagonal elements. We shall now choose a maximal abelian subgroup of $T(A_\Phi)$. Let $T_\nu$ denote the inverse image of $T_\nu$ in $S^4_\nu(F_\nu)$. For each place, we choose a maximal abelian subgroup of $\tilde{T}_\nu$, denoted $\tilde{T}_\nu^0$, as follows. Let $p$ denote a generator of the prime ideal in the ring of integers of $F_\nu$. Then, from the properties of the local Hilbert symbol $(\cdot, \cdot)_3$ of $F_\nu$, we have $(p, -p)_3 = 1$. Since $-1$ is a cube, it follows that $(p^{k_1}, p^{k_2})_3 = 1$ for all integers $k_1$ and $k_2$. Let $\tilde{T}_\nu^0$ denote the group generated by the center of $T_\nu$ and the group of all matrices of the form $\text{diag}(p^{k_1}, p^{k_2}, p^{-k_2}, p^{-k_1})$. Finally, let $\tilde{T}^0 = \prod_{\nu \in \Phi} \tilde{T}_\nu^0$. We fix this as the maximal abelian subgroup in the $S^4$ case from now on.
Given \( t = \text{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \), denote \( t' = \text{diag}(t_1, t_2) \in GL_2 \). Then it follows from Prop. 1 that for \( t \in \widetilde{T}^0 \) we have

\[
L_\Phi(t) = |t_1 t_2| W_{\Theta^{(3)}_{GL_2,\text{fin}}} (t') L_\Phi(e).
\]

Here the function \( W_{\Theta^{(3)}_{GL_2,\text{fin}}} \) is the finite part of the Whittaker function on a suitable vector in the representation \( \otimes_{\nu \in \Phi} (\Theta^{(n)}_{GL_2})_\nu \). Also, the factor \( |t_1 t_2| \) is equal to \( \delta_P^{-s_0} \) where \( s_0 = 2/3 \) is the point where \( \Theta^{(3)}_P \) is defined as a residue of an Eisenstein series. See Section 3.

The situation in the case when \( H = SO_7 \) is similar. We recall that \(-1\) is a fourth power in \( F^\times \). Applying the same reasoning we are led to consider the global integral

\[
|a|_\Phi^{-1} \int_{A^3_{\Phi}} L_\Phi(z(r_1, r_2) t(a, 1, 1) y_2(x)) \psi(r_1) dr_1 dr_2 dx.
\]

Similarly to the previous case, the function \( L_\Phi \) here is obtained from a functional on \( \Theta^{(4)}_{SO_7} \) applied to a vector in this space; the function \( L_\Phi \) satisfies the property \( L_\Phi(uh) = \psi_{U,1}(u) L_\Phi(h) \) for all \( u \in U(A_{\Phi}) \), where now \( U \) is the maximal unipotent subgroup of \( SO_7 \) and \( \psi_{U,1}(u) = \psi(u_1, u_2, u_3) \), as in (15) (recall that \( \Theta^{(4)}_{SO} \) is not generic; a factorization similar to Lemma [4] holds for similar reasons). Analogously to property (29), one has

\[
L_\Phi(t) = |t_1 t_2 t_3| W_{\Theta^{(4)}_{GL_3,\text{fin}}} (t') L_\Phi(e)
\]

where the group \( \widetilde{T}^0 \) is defined similarly to the prior case, and for \( t = t(t_1, t_2, t_3) \in \widetilde{T}^0 \), we denote \( t' = \text{diag}(t_1, t_2, t_3) \in GL_3 \).

6. Some Local Computations

Let us introduce the following notation for the normalized \( n \)-th order Gauss sums. Suppose that \( \nu \) is a finite place with local uniformizer \( p \), \( |p^{-1}|_{\nu} = q \), \( \psi_{\nu} \) is an additive character of conductor \( O_{\nu} \), \( |b|_{\nu} \leq 1 \), and \( j \in \mathbb{Z} \), \( (j, n) = 1 \). Then we write

\[
\int_{|\epsilon|_{\nu} = 1} (\epsilon, p)^j_{\nu} \psi_{\nu}(b p^{-1} \epsilon) d\epsilon = q^{-1/2} G_j^{(n)}(b, p).
\]

Here \( G_j \) is a normalized \( n \)-th order Gauss sum modulo \( p \). We extend the notation to composite moduli as in [7], pg. 151; thus if \( p, q \) are local uniformizers at places \( \nu_1, \nu_2 \) resp., then

\[
G_j^{(n)}(b, pq) = (p, q)^j_{\nu_2} (q, p)^j_{\nu_1} G_j^{(n)}(b, p) G_j^{(n)}(b, q),
\]

where we have indicated the fields for the two local residue symbols.

Fix \( \nu \) to be a finite place such that all data is unramified at \( \nu \). In this section we will compute the integrals (28) and (30) at \( a = (a_{\nu})_{\Phi} \in A^X_{\Phi} \) defined as follows. For all \( \nu' \neq \nu \) we assume that \( a_{\nu'} \) is a unit. At the place \( \nu \) we let \( a_{\nu} \) be a positive power of a local uniformizer \( p_{\nu} \). Let \( \theta = \otimes \theta_{\nu'} \) be a factorizable vector. Notice that in both cases, we have \( t(a, 1) \) and \( t(a, 1, 1) \) are in the corresponding groups \( \widetilde{T}^0 \), and that outside of \( \nu \) the element \( p_{\nu} \) is a unit.

Consider first the \( Sp_4 \) case. We can ignore the integration over \( x \) in the integral (28) in the following sense. In the previous Section we defined the compact group \( K_\theta = \prod_{\nu' \in \Phi} K_{\nu'} \).
such that \( L_\Phi(hk) = L_\Phi(h) \). Choose a Schwartz function \( \phi \) such that \( \phi_{\nu'} \) is one if \( |x_{\nu'}| \) is so small that \( m(x_{\nu'}) \in K_{\nu'} \), and zero otherwise. Then the integral (28) is equal to

\[
|a|^{-1/2} \gamma(a) \int_{A_\Phi} \Phi(z(r) t(a, 1) w_2 w_1) \psi(r) dr.
\]

In the \( SO_7 \) case, we argue as follows. Given a place \( \nu' \) which is unramified, let \( y_3(m) = I_7 + me_{1,6}' \) such that \( y_3(m) \in K_{\nu'} \). Consider the integral

\[
\int_{A_\Phi} \Phi(z(r_1, r_2) t(a, 1, 1) y_2(x_{\nu'}) \psi(r_1) dr_1 dr_2.
\]

Using the right invariant property of \( L_\Phi \) at the place \( \nu' \), we have

\[
\int_{A_\Phi} \Phi(z(r_1, r_2) t(a, 1, 1) y_2(x_{\nu'}) y_3(m) \psi(r_1) dr_1 dr_2.
\]

Conjugating \( y_3(m) \) to the left and using the left-invariance properties of \( L_\Phi \) under the group \( V \), we obtain \( Z(x_{\nu'}) = \psi(ma_{\nu'} x_{\nu'}) Z(x_{\nu'}) \). From the definition of \( a \), we know that \( a_{\nu'} \) is a unit. Thus \( Z(x_{\nu'}) \) is zero unless \( |x_{\nu'}| \leq 1 \). In the bad places we may argue similarly to see that the integral (29) is zero if \( |x_{\nu'}| \) is large. Since the function \( Z(x_{\nu'}) \) is locally constant, this means that the integration over \( x_{\nu'} \in F_{\nu'} \) can be replaced by a finite sum of right translations, leading to an adjustment of the function \( L_\Phi \).

We now consider the \( Sp_4 \) case and the integral (33). We claim that by a suitable change involving \( L_\Phi \) we can replace the integration over \( A_\Phi \) with an integration over \( F_{\nu'} \). This follows from an argument similar to the one above. More precisely, for an unramified place \( \nu' \neq \nu \), define the function \( Z_1(x_{\nu'}) = L_\Phi(z(x_{\nu'}) t(a, 1, 1)) \). Let \( z_1(m) = I_4 + me_{1,3}' \) with \( m \in F_{\nu'} \), \( |m_{\nu'}| \leq 1 \). Then \( z_1(m) \in K_{\nu'} \). Arguing as above we obtain \( Z_1(x_{\nu'}) = \psi(ma_{\nu'} x_{\nu'}) Z_1(x_{\nu'}) \).

Again, from the definition of \( a \), we know that \( a_{\nu'} \) is a unit. Hence if \( Z_1(x_{\nu'}) \neq 0 \) then \( z(x_{\nu'}) \in K_{\nu'} \). Similarly, in a ramified place \( \nu' \), the integral is zero if \( |x_{\nu'}| \) is sufficiently large. Recall that the function \( L_\Phi \) is obtained by applying the functional \( \ell_\Phi \) to a suitable vector \( \theta \in \Theta_{Sp_4} \) (see the discussion following (28)). Replacing \( \theta \) by a suitable finite sum of translations which involve the bad places only and then applying this functional, we thus obtain a new function which we denote by \( \tilde{L}_\Phi \), such that the integral (33) is equal to

\[
|a|^{-1/2} \gamma(a) \int_{F_{\nu'}} \tilde{L}_\Phi(z(r_{\nu'}) t(a, 1)) \psi_{\nu'}(r_{\nu'}) dr_{\nu'}.
\]

Here \( \psi_{\nu'} \) is the local constituent of \( \psi \).

To simplify notation we drop the subscript \( \nu \). To compute the integral (34), we first conjugate \( t(a, 1) \) to the right. Then we write this integral as a sum of two integrals, integrating separately over the regions \( |r| \leq 1 \) and \( |r| > 1 \). Since \( \nu \) is an unramified place, we may dispose of the \( r \) variable in the first integral. Using the property (29), the contribution of \( |r| \leq 1 \) to integral (34) is

\[
|a|^{-1/2} \gamma(a) \Phi_{1/2}^{G(3)}_{23} \left( \begin{array}{c} a \\ 1 \end{array} \right) \tilde{L}_\Phi(a).
\]
In the summand that corresponds to the integration over $|r| > 1$, we perform the Iwasawa decomposition of

$$z(r) = \begin{pmatrix} 1 & 1 & 1 \\ r & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r^{-1} & 1 \\ 1 & 1 & r \end{pmatrix} k$$

where $k$ is in the maximal compact subgroup. This implies that the second integral is equal to

$$|a|^{-1/2} \gamma(a) \int_{|r|>1} \tilde{L}_\Phi \begin{pmatrix} a & 1 & 1 \\ 1 & a^{-1} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & r^{-1} & 1 \\ 1 & 1 & r \end{pmatrix} \psi(r) \, dr.$$ 

Write $r = p^{-m} \epsilon$ where $m \geq 1$ and $|\epsilon| = 1$. Then the factorization

$$\begin{pmatrix} a & 1 & 1 \\ 1 & p^m \epsilon^{-1} & p^{-m} \epsilon \\ 1 & a^{-1} & 1 \end{pmatrix} = \begin{pmatrix} a & p^m & a^{-1} \\ p^{-m} & a^{-1} & 1 \\ 1 & \epsilon^{-1} & \epsilon \end{pmatrix}$$

contributes a factor of $(\epsilon, p^m)_3$ due to the 2-cocycle. (Since $(p, p)_3 = 1$ we do not record an additional factor of $(a, p^m)_3$.) Thus we find that the second integral is equal to

$$|a|^{-1/2} \gamma(a) \sum_{m=1}^\infty q^m \tilde{L}_\Phi \begin{pmatrix} a & p^m & 1 \\ p^{-m} & p^{-m} & 1 \\ 1 & a^{-1} & 1 \end{pmatrix} \int_{|\epsilon|=1} (\epsilon, p^m)_3 \psi(p^{-m} \epsilon) \, d\epsilon.$$ 

If $m > 1$, the inner integral in (36) is zero, and if $m = 1$ it gives

$$\int_{|\epsilon|=1} (\epsilon, p)_3 \psi(p^{-1} \epsilon) \, d\epsilon = q^{-1/2} G_1^{(3)}(1, p).$$

Let $\iota(p)$ be the finite idele which is 1 outside of $\nu$ and $p$ at $\nu$. Then we have shown that the second integral gives a contribution of

$$|a|^{1/2} \gamma(a) q^{-1/2} G_1^{(3)}(1, p) W_{\Theta_{GL_2}^{(3)}}(\iota(p)) \tilde{L}_\Phi(\epsilon).$$

Given a Whittaker function on a group $M$ we define the normalized Whittaker coefficient $\tau_M(t) = \delta_B^{-1/2}(t) W_M(t)$ where $B$ is the Borel subgroup of $M$. With this normalization, the coefficients $\tau_{GL_2}^{(4)}$ and $\tau_{SL_2}^{(6)}$ attached to the Whittaker functions of the respective theta representations are precisely the theta coefficients described in [7].

We summarize the results obtained here in the following Proposition. Recall that we have fixed a local uniformizer $p_\nu$ at $\nu$.

**Proposition 6.** Suppose that $\nu$ is an unramified place. For each vector in $\Theta_{SL_2}^{(6)}$ there is a vector in $\Theta_{GL_2}^{(3)}$ such that for all finite ideles $a$ which are units outside of $\nu$ and a positive
power of the local uniformizer $p_\nu$ at $\nu$, the following equality holds:

\begin{equation}
\tau_{\Theta_{SL_2}^{(6)}; fin}(a \quad a^{-1}) = \gamma(a) \left( \tau_{\Theta_{GL_2}^{(3)}; fin}(a \quad 1) + \gamma_1(1, p_\nu) \tau_{\Theta_{GL_2}^{(3)}; fin}(a \quad t(p_\nu)) \right) \tau_{\Theta_{SL_2}^{(6)}; fin}(e).
\end{equation}

Here the coefficients $\tau_{\Theta_{SL_2}^{(6)}; fin}$ and $\tau_{\Theta_{GL_2}^{(3)}; fin}$ depend on choices of vector but we have suppressed this from the notation. The factor $\tau_{\Theta_{SL_2}^{(6)}; fin}(e)$ on the right hand side of (37) comes from $\widetilde{L}_\Phi(e)$.

Note also that if we choose the vector in $\Theta_{SL_2}^{(6)}$ so that $\tau_{\Theta_{SL_2}^{(6)}; fin}(e) = 1$ then it follows from (37) that the corresponding vector in $\Theta_{GL_2}^{(3)}$ satisfies $\tau_{\Theta_{GL_2}^{(3)}; fin}(e) = 1$.

Next we analyze the $SO_7$ integral (30). After omitting the integration over $x$, we consider the integral

$$|a|_\Phi^{-1} \int_{K_\phi^2} L_\Phi(z(r_1, r_2)t(a, 1, 1)) \psi(r_1) \, dr_1 \, dr_2.$$ 

Arguing as in the $Sp_4$ case, immediately after Eqn. (33), we may reduce to the computation of the local integral

$$|a|_\nu^{-1} \int_{F_2} \tilde{L}_\Phi(z(r_1, r_2)t(a, 1, 1)) \psi_\nu(r_1) \, dr_1 \, dr_2,$$

where $\tilde{L}_\Phi$ is obtained from a suitable sum of right translates, similarly to the $Sp_4$ case above. We omit the place $\nu$ from the notation. Consider first the integration over $|r_1| \leq 1$. From the right-invariance property of $L_\Phi$ we obtain that $\tilde{L}_\Phi(z(r_1, r_2)t(a, 1, 1)) = \tilde{L}_\Phi(z(0, r_2)t(a, 1, 1))$. In this case we claim that we may also limit the domain of integration to $|r_2| \leq 1$. Indeed, if $|r_2| > 1$, then we use the right-invariance of $L_\Phi$ by the maximal compact subgroup (at $\nu$) to deduce that $\tilde{L}_\Phi(h) = \tilde{L}_\Phi(hy(m))$ for $y(m) = I_7 + m e_{2,4} - \frac{m^2}{2} e_{2,6}$ with $|m| \leq 1$. Conjugating this matrix from right to left, we deduce that $\tilde{L}_\Phi(z(0, r_2)t(a, 1, 1)) = 0$ in this case. Hence using (31) we obtain the contribution of

$$W_{\Theta_{GL_2}^{(4)}; fin}(a \quad 1) \quad \tilde{L}_\Phi(e) = |a| \tau_{\Theta_{GL_2}^{(4)}; fin}(a \quad 1) \tilde{L}_\Phi(e)$$

where $\tau$ is the normalized Whittaker coefficient of the theta function.

Next consider the domain $|r_1| > 1$. Performing an Iwasawa decomposition similarly to (35), we obtain the contribution of

$$|a|^{-1} \int_{F \setminus |r_1| > 1} \tilde{L}_\Phi(t(a, 1, 1)t_1(r_1^{-1})z(0, r_2)) \psi(r_1) \psi(r_2^2 r_1) \, dr_1 \, dr_2.$$ 

Here $t_1(r_1^{-1}) = \text{diag}(1, r_1^{-1}, r_1^{-1}, 1, r_1, r_1)$ and the factor of $\psi(r_2^2 r_1)$ is obtained from the conjugation of the unipotent matrix obtained from the Iwasawa decomposition to the left and a change of variables.
As above, using the matrix \( y(m) \) we deduce that the integral vanishes unless \(|r_2| \leq 1\). Thus we obtain

\[
|a|^{-1} \int_{|r_1|>1} \tilde{L}_\Phi(t(a,1,1)t_1(r_1^{-1})) \psi(r_1) |r_1| \int_{|r_2| \leq 1} \psi(r_2 r_1) dr_2 dr_1.
\]

Write \( r_1 = p^{-m} \epsilon \) with \( m \geq 1 \) and \(|\epsilon| = 1\). The factorization \( t_1(p^{-m} \epsilon) = t_1(p^{-m}) t_1(\epsilon) \) contributes the factor \((\epsilon, p)^{2m}_4 (\epsilon, p)^{m}_8 = (\epsilon, p)^m_4 \) from the 2-cocycle. We thus obtain

\[
(38) \quad |a|^{-1} \sum_{m=1}^\infty q^{2m} \tilde{L}_\Phi(t(a,1,1)t_1(p^m)) \int_{|\epsilon|=1} \int_{|r_2| \leq 1} (\epsilon, p)^m_4 \psi(p^{-m} \epsilon) \psi(r_2^2 p^{-m} \epsilon) dr_2 d\epsilon.
\]

Writing \( r_2 = p^n \eta \) with \(|\eta| = 1\), the inner integration is equal to

\[
(39) \quad \sum_{n=0}^\infty q^{-n} \int_{|\epsilon|=1} \int_{|\eta|=1} (\epsilon, p)^m_4 \psi(p^{-m} \epsilon) \psi(p^{2n-m} \eta^2 \epsilon) d\eta d\epsilon.
\]

If \( n \geq 1 \), we notice that \( \psi(p^{-m} \epsilon) \psi(p^{2n-m} \eta^2 \epsilon) = \psi(p^{-m} \epsilon (1 + p^{2n} \eta^2)) \). Since, for \( n \geq 1 \), \( 1 + p^{2n} \eta^2 \) is a unit which is a fourth power, we can make a change of variables in \( \epsilon \) to remove this factor. Therefore, summing over \( n \geq 1 \) we obtain the contribution

\[
\sum_{n=1}^\infty q^{-n} (1 - q^{-1}) \int_{|\epsilon|=1} (\epsilon, p)^m_4 \psi(p^{-m} \epsilon) d\epsilon = \begin{cases} q^{-3/2} G_1^{(4)}(1, p) & \text{if } m = 1 \\ 0 & \text{if } m > 1. \end{cases}
\]

Here the inner integral vanishes if \( m > 1 \) due to the oscillation in \( \psi \). Thus the contribution to the integral (38) from the summands with \( n \geq 1 \) is

\[
|a| q^{-1/2} G_1^{(4)}(1, p) \tau_{\text{G0,lin}}(a) \begin{pmatrix} a \\ \iota(p) \\ \iota(p) \end{pmatrix} \tilde{L}_\Phi(\epsilon).
\]

Finally, we consider the contribution to the sum (39) from the term \( n = 0 \). It is equal to

\[
\int_{|\epsilon|=1} \int_{|\eta|=1} (\epsilon, p)^m_4 \psi(p^{-m} \epsilon) \psi(p^{-m} \eta^2 \epsilon) d\eta d\epsilon.
\]

We use the identity

\[
\int_{|\eta|=1} \psi(p^{-m} \epsilon \eta^2) d\eta = \int_{|\eta|=1} \psi(p^{-m} \epsilon \eta) (1 + (\eta, p)^2_4) d\eta.
\]

Plugging this into the above integral and changing variables \( \eta \mapsto \eta e^{-1} \), we obtain

\[
\int_{|\epsilon|=1} (\epsilon, p)^m_4 \psi(p^{-m} \epsilon) d\epsilon \int_{|\eta|=1} \psi(p^{-m} \epsilon \eta) d\eta + \int_{|\epsilon|=1} (\epsilon, p)^{3m}_4 \psi(p^{-m} \epsilon) d\epsilon \int_{|\eta|=1} (\eta, p)^2_4 \psi(p^{-m} \epsilon) d\eta.
\]
If $m > 1$ this expression is zero, and for $m = 1$ we obtain $-q^{-3/2} G_1^4(1,p) + q^{-1} G_3^4(1,p)$. Therefore the contribution to (38) is

$$\left( G_3^4(1,p) \tau_{\Theta_{GL_3}^4,\mathrm{fin}} \begin{pmatrix} a & \tau(p) \\ \tau(p) & \tau(p) \end{pmatrix} - q^{-1/2} G_1^4(1,p) \tau_{\Theta_{GL_3}^4,\mathrm{fin}} \begin{pmatrix} a & \tau(p) \\ \tau(p) & \tau(p) \end{pmatrix} \right) \tilde{L}_\Phi(e).$$

Combining all terms we obtain the following Proposition.

**Proposition 7.** Suppose that $\nu$ is an unramified place. For each vector in $\Theta_{SO_4}^4$ there is a vector in $\Theta_{GL_3}^4$ such that for all finite ideles $a$ which are units outside of $\nu$ and a positive power of the local uniformizer $p_\nu$ at $\nu$, the following equality holds:

$$\tau_{\Theta_{SO_4}^4,\mathrm{fin}} \begin{pmatrix} a & 1 \\ 1 & a^{-1} \end{pmatrix} = \left( \tau_{\Theta_{GL_3}^4,\mathrm{fin}} \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix} + G_3^4(1,p_\nu) \tau_{\Theta_{GL_3}^4,\mathrm{fin}} \begin{pmatrix} a & \tau(p_\nu) \\ \tau(p_\nu) & \tau(p_\nu) \end{pmatrix} \right) \tau_{\Theta_{SO_4}^4,\mathrm{fin}}(e).$$

Here, as in Prop. 6, the coefficients $\tau_{\Theta_{SO_4}^4,\mathrm{fin}}$ and $\tau_{\Theta_{GL_3}^4,\mathrm{fin}}$ depend on choices of vector which we have suppressed from the notation, and if we choose the vector in $\Theta_{SO_4}^4$ so that $\tau_{\Theta_{SO_4}^4,\mathrm{fin}}(e) = 1$ then the corresponding vector in $\Theta_{GL_3}^4$ satisfies $\tau_{\Theta_{GL_3}^4,\mathrm{fin}}(e) = 1$.

Props. 6 and 7 easily extend to $a$ which are units outside of $\nu$ and powers of $p_\nu$ times units at $\nu$ by $K_\nu$-invariance, taking into account the relevant cocycles. We shall extend them further below.

### 7. The CFH Conjecture

The goal of this section is to address Conjecture 1 of [7], which describes the square coefficients of the theta function on the 6-fold cover of $GL_2$. As we shall see, we can indeed establish this result, but because we do not have good information about the representation $\Theta_{Sp_4}^{(3)}$ at its ramified places, we must limit ourselves to coefficients which are products of sufficiently nice primes.

In this section we let $S$ denote the set of all bad places for the representation $\Theta_{Sp_4}^{(3)}$, and all places for which $\psi$ is not of full conductor. The main goal is to study the integral

$$W_{\Theta_{SL_2}^6,\mathrm{fin}} \begin{pmatrix} a & a^{-1} \\ a^{-1} & a \end{pmatrix} = |a|^{-1/2} \gamma(a) \int_{\mathbb{A}_\Phi} L_\Phi(t(a,1)z(r)) \psi(r) dr$$

for certain values of $a \in \mathbb{A}_\Phi^\times$. Indeed, starting with the integral (28), we have chosen the Schwartz function $\phi$ as was specified at the beginning of Section 6 and also absorbed the Weyl element $w_2 w_1$ into the choice of vector. As explained in Section 5 the function $L_\Phi$ is defined via a functional applied to a vector in the space $\Theta_{Sp_4}^{(3)}$. 


First, we choose data such that the integral (10) is not zero for $a = 1$; for every place outside of $S$ we choose the unramified vector. It follows from the proof of Prop. 6 that

\begin{equation}
W_{\Theta_{SL_2}^f}(e) = \int_{F_S} L_{\Phi}(z(r)) \psi_S(r) \, dr.
\end{equation}

Here $F_S$ is the product of $F_\nu$ over all $\nu \in S$, and $\psi_S = \prod_{\nu \in S} \psi_\nu$. Since we do not know how to compute this integral, we can only choose vectors at the bad places such that this integral is not zero. We shall denote such a vector by $\theta$.

As explained above, there is a compact open subgroup $K_\theta = \prod_{\nu \in \Phi} K_\nu$ which fixes the vector $\theta$, with $K_\nu = S\mathfrak{p}_\nu(O_\nu)$ for $\nu \in \Phi$, $\nu \notin S$. Next we specify a set of elements $R \subset \mathbb{A}_\Phi^\times$ defined as follows. Let $\nu$ denote a place outside of $S$, and let $p \in F$ (or $p_\nu$ if it is necessary to show the place $\nu$ explicitly) denote a generator of the maximal ideal in $O_\nu$ such that the diagonal matrices $t(p^k, p^l) \in K_\nu$ for all $k, l \geq 0, \nu \in \Phi$, $\nu' \neq \nu$. This will be true provided $|p|_{\nu'} = 1$ for all $\nu' \neq \nu$ and $p$ is sufficiently close to 1 at the places $\nu' \in S$. By Dirichlet’s theorem, there are infinitely many such $p$. Given such a $p$, let $a_\nu = (p, p, \ldots) \in \mathbb{A}_\Phi^\times$. We then define $R$ to be the multiplicative set generated by the elements $a_\nu$ as $\nu$ varies over places outside of $S$. Notice that if $a, b \in R$ then also $a^k b^l \in R$ for all natural numbers $k$ and $l$. It is convenient to denote such an element $a_\nu$ by $p$ when the place $\nu$ is clear. Recall also that $\iota(p)$ denotes the element in $\mathbb{A}_\Phi^\times$ which is $p$ at the $\nu$ component and 1 elsewhere. With these choices, we have $L_{\Phi}(t(p^k, p^l)) = L_{\Phi}(t(\iota(p^k), \iota(p^l)))$ for all $k, l \geq 0$. We emphasize that the set $R$ depends on the stabilizer of the vector $\theta$ that is chosen above.

The main result in this section is the following.

**Theorem 1.** Let $m_1, m_2 \in R$ be square-free and coprime. Suppose $\tau_{SL_2^{(6)}}(e) = 1$. Then

\begin{equation}
\tau_{SL_2^{(6)}}(m_1^2 m_2^2, m_1^{-1} m_2^{-2}) = \gamma(m_1) G_1^{(3)}(1, m_1) G_1^{(3)}(1, m_2)^2 \left( m_2^2 m_1^2 \right)^2 \sum_{m_1 = d_1 d_2} \left( \frac{d_2}{d_1} \right) \frac{1}{3}.
\end{equation}

This theorem is essentially Conjecture 1 of [7]. However, note that it depends on a choice of a specific vector. The quantities also depend on the choice of the additive character $\psi$.

**Proof.** The Theorem is a direct consequence of identity (37) in Prop. 6. To explain this succinctly we carry out the details for $m_1, m_2$ products of small numbers of primes; the general case follows by exactly the same arguments. Let $p$ and $q$ be two elements in $R$ corresponding to the distinct places $\nu_1$ and $\nu_2$, resp., and write $\iota(p), \iota(q)$ for the finite ideles which are 1 outside of places $\nu_1, \nu_2$ resp. where they are $p, q$ resp. Suppose $\tau_{SL_2^{(6)}}(e) = 1$.

Arguing as in the proof of Prop. 6 we obtain the identities

\begin{equation}
\tau_{SL_2^{(6)}}(p^i) = \gamma(p^i) \left( \tau_{\Theta_{GL_2}^{(3)}}(p^i) \right) + G_1^{(3)}(1, p) \tau_{\Theta_{GL_2}^{(3)}}(p^i) \iota(p).
\end{equation}

and

28
Indeed, to prove (43) one argues as in the proof of Prop. 6 starting from Eqn. (34). The integral in that equation, adapted to our case, is given by

\[
\int \int L_\phi(t(p^jq^j, 1)z(r_{\nu_1})z(r_{\nu_2})) \psi_{\nu_1}(r_{\nu_1}) \psi_{\nu_2}(r_{\nu_2}) \, dr_{\nu_1} \, dr_{\nu_2},
\]

where \( p \) (resp. \( q \)) corresponds to place \( \nu_1 \) (resp. \( \nu_2 \)). To obtain (43) one breaks the domain of integration in (44) into four pieces corresponding to \(|r_{\nu_k}| \leq 1, |r_{\nu_k}| > 1\) for \( k = 1, 2 \). Each of the four summands in equation (43) is obtained from one of the four pieces. Eqn. (42) is similar.

To simplify (42), we may use \( K_\theta \)-invariance to write

\[
\tau_{\Theta_{GL_2}^{(3)}}(p^jq^j, 1) = \tau_{\Theta_{GL_2}^{(3)}}(p^j, 1).
\]

Then since the central character is trivial at all archimedean places and functions in \( \Theta_{GL_2}^{(3)} \) are automorphic, in particular left-invariant under \( pI_2 \), this equals

\[
\tau_{\Theta_{GL_2}^{(3)}}(p^{j-1}, 1).
\]

Thus

\[
\tau_{SL_2^{(6)}, fin}(p^j, p^{-i}) = \tau_{\Theta_{GL_2}^{(3)}, fin}(p^j, 1) + G_1^{(3)}(1, p) \tau_{\Theta_{GL_2}^{(3)}, fin}(p^{j-1}, 1).
\]

To simplify (43) we recall that the cocycle in the group \( GL_2^{(3)} \) as induced from the cocycle on \( Sp_4^{(3)} \) is given by the conjugate of

\[
\sigma_{3,1} \left( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = (ab, cd)_3(a, d)_3.
\]

Thus \( \tau_{\Theta_{GL_2}^{(3)}, fin} \) transforms by the adelic version of \( \sigma_{3,1} \). All Hilbert symbols in this Section are cubic; since we will be concerned with elements of \( F \) embedded in different completions \( F_\nu \), we shall sometimes write the Hilbert symbol in \( F_\nu \) as \( (\ , \ )_\nu \).
Using the right invariant property of the function $\tau_{GL_2, fin}$ under $K_\theta$, we obtain

$$
\tau_{GL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ \tau(p) \end{pmatrix} \right) = \tau_{GL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ \tau(p) \end{pmatrix} \right).
$$

Since we have $(p, p) = (q, p) = 1$ by Hilbert reciprocity, we may factor

$$
\left( \begin{pmatrix} p^i q^j & 1 \\ \tau(p) \end{pmatrix} \right) = \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right).
$$

in the adelic metaplectic group (recall that the matrices are embedded via the trivial section $s$). Then we see that

$$
\tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right) = \tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right),
$$

again using the left-invariance of functions in $\Theta_{GL_2}^{(3)}$ under $pI_2$.

Similarly

$$
\tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right) = \tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right),
$$

Finally, to analyze the last term in (43) (where it turns out some care is needed due to the cocycle), we begin by observing

$$
\tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right) = \tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right),
$$

by the right invariance under $K_\theta$. Carrying out a cocycle computation we obtain that the rightmost term in the above equality is

$$(p, q) \nu_1 \tau_{GL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ \tau(p) \end{pmatrix} \right).$$

In a similar way, using the right invariance by $\left( \begin{pmatrix} 1 & \tau(p) \end{pmatrix} \right) \in K_\theta$ we find that the above expression is equal to

$$(p, q) \nu_1 (q, p) \nu_2 \tau_{GL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ \tau(p) \end{pmatrix} \right).$$

Arguing as above, we conclude that

$$
\tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right) = \tau_{GL_2, fin} \left( \begin{pmatrix} p & 1 \\ p \end{pmatrix} \right).
$$

Combining all this, equation (43) can be written as

(45)

$$
\tau_{SL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ p^{-1} q^{-j} \end{pmatrix} \right) = \gamma(p, q) \tau_{GL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ p^{-1} q^{-j} \end{pmatrix} \right) + \gamma(p, q) \tau_{GL_2, fin} \left( \begin{pmatrix} p^i q^j & 1 \\ p^{-1} q^{-j} \end{pmatrix} \right).
$$

In the last term we used (32).
With this preparation, we may prove the Theorem. Consider first the case of a single prime $p \in R$. The identity

$$\tau_{SL_2^{(6)\, fin}} \left( \frac{p}{p^{-1}} \right) = 2 \gamma(p) G_1^{(3)}(1, p)$$

follows directly from (42) with $i = 1$, since

$$\tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{p}{1} \right) = \tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{\nu(p)}{1} \right) = G_1^{(3)}(1, p).$$

This evaluation of the Whittaker coefficient of the cubic theta function here follows from a well-known Hecke operator argument (Kazhdan-Patterson, [18], Sect. 1.4; Hoffstein [16], Prop. 5.3).

Similarly, (42) with $i = 2$ gives

$$\tau_{SL_2^{(6)\, fin}} \left( \frac{p^2}{p^{-2}} \right) = G_1^{(3)}(1, p)^2$$

since $\gamma(p^2) = 1$ and

$$\tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{p^2}{1} \right) = 0,$$

again via Hecke operators.

The case of more than one prime is similar. To prove

$$\tau_{SL_2^{(6)\, fin}} \left( \frac{pq}{pq^{-1}q^{-1}} \right) = 2 \gamma(pq) \left( G_1^{(3)}(1, p)G_1^{(3)}(1, q) + G_1^{(3)}(1, pq) \right),$$

we use (45) with $i = j = 1$. All terms have already been evaluated except for

$$\tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{pq}{1} \right) = \tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{\nu(p)\nu(q)}{1} \right) \tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{\nu(q)^{-1}q^{-1}}{1} \right).$$

Using invariance under $K_\theta$ and computing the cocycles, one sees that

$$\tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{pq}{1} \right) = (p, q)_{\nu_2} \tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{\nu(p)\nu(q)}{1} \right).$$

This equals $G_1^{(3)}(1, pq)$, again via Hecke operators.

And to prove

$$\tau_{SL_2^{(6)\, fin}} \left( \frac{pq^2}{pq^{-1}q^{-2}} \right) = 2 \gamma(p) G_1^{(3)}(1, p) G_1^{(3)}(1, q)^2 \left( \frac{q}{p} \right)^2,$$

(recall $\gamma(p) = \gamma(p^{-2})$), we use $i = 1, j = 2$, recalling that

$$\tau_{\Theta_{GL_2^{(3)}\, fin}} \left( \frac{pq^2}{1} \right) = 0,$$

and employing (32), the link between local Hilbert symbols and global power residue symbols, and reciprocity. (Note $(p, q)_v = 1$ for all $v \in S$ since $p, q \in R$.)

This completes the proof of the theorem. □
Remarks. 1. Let $Z$ denote the center of $GL_2$. Using the Kubota cocycle for $GL_2$ as in \[18\], pg. 41, it is easy to see if $F$ is a local field containing enough roots of unity then for a metaplectic cover of any degree, $\tilde{SL}_2(F)$ and $\tilde{Z}(F)$ commute in $\tilde{GL}_2(F)$. Moreover for the double cover $\tilde{Z}(2)$ is split with section $\gamma$. Using this, the descent integral and Theorem 1 may easily be extended to $\tilde{Z} \cdot \tilde{SL}_2 \subseteq GL_2(6)$, giving exactly the formula predicted in \[7\], pg. 155 for $m_1, m_2 \in R$.

2. The numerical work of \[1\] detects a sixth root of unity which does not appear here; however here $\tau_{\text{fin}}$ includes a contribution from the places in $S$ while the coefficients computed in \[1\] do not. Also, \[1\] consider specific vectors and the authors there note that some choices give cleaner results than others. By contrast, we are imposing the condition that the data is normalized so that $(41)$ is 1, but are not able to specify the exact vector satisfying this condition.

3. If $F = \mathbb{Q}(e^{2\pi i/3})$ then the cubic Kubota map is known to have level 3, and by the work of Bass-Milnor-Serre it extends to a homomorphism of the same level for $Sp_4$. Thus in this case it is reasonable to hope that there is a vector $\theta$ as above which is fixed by the adelic group $K_0$ which is the full subgroup $Sp_4(\mathcal{O}_v)$ away from 3 and is the principal congruence subgroup of level 3 at the prime dividing 3. Since any prime $p$ with $(p, 3) = 1$ may be adjusted by a unit to be $\equiv 1 \mod 3$, in that case Theorem 1 would hold for a set of integers generating all ideals prime to 3.

8. Patterson’s Conjecture

The situation in the $SO_7$ case is similar. Recall that for the quartic theta function, the coefficients at cubes of primes are zero and those at squares are known. Patterson’s conjecture concerns the value at square-free arguments. We have constructed a theta function on the 4-fold cover of $SO_4$. Via the identification of $D_2$ with $A_1 \times A_1$, the coefficients of the theta function on the 4-fold cover of $SO_4$ are squares of the quartic theta function for $GL_2$.

More precisely, let

$$ (GL_2 \times GL_2)^0 = \{(g_1, g_2) \in GL_2 \times GL_2 : \det g_1 = \det g_2\}. $$

Then there is a natural map from $(GL_2 \times GL_2)^0$ to $SO(4)$ as in \[3\], Ch. 30, with kernel $Z$ consisting of scalar matrices in $GL_2$ embedded diagonally. Under this map, the torus $\text{diag}(a, 1, 1, a^{-1}) \in SO_4$ may be identified with the torus consisting of pairs $((a^i_1), (a^i_1))$ in $Z \setminus (GL_2 \times GL_2)^0$. This identification identifies the coefficients of the theta function on the 4-fold cover of $SO_4$ with squares of the corresponding coefficients of the quartic theta function for $GL_2$.

We define the set $R$ as in the $Sp_4$ case. Let $m(a) = \text{diag}(a, 1, 1, a^{-1}) \in SO_4$. We have

**Theorem 2.** Let $a \in R$ be square-free. Suppose that $\tau_{SO_4^{(4)}}(e) = 1$. Then

$$ \tau_{SO_4^{(4)}, \text{fin}}(m(a)) = C_3^{(4)}(1, a) \sum_{d_1d_2 = a} \left( \frac{d_1}{d_2} \right)_2. $$

**Proof.** Once again for notational convenience we consider prime and coprime elements $p, q$ in $R$, corresponding to distinct places $\nu_1, \nu_2$ and consider $a = p^i q^j$, $i, j = 0, 1$. These arguments extend to more than two primes without difficulty. Suppose that $\tau_{SO_4^{(4)}}(e) = 1$. 

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The starting point is the proof of Prop. 7. Indeed, arguing as in the $Sp_4$ case we obtain the following two identities. First, for integers $i \geq 1$ we have

$$
\tau_{SO_4^{(i)}, \text{fin}}(m(p^i)) = \\
\tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i & 1 \\ 1 & 1 \end{pmatrix} + G_3^{(i)}(1, p) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(p) \\ \iota(p) & 1 \end{pmatrix}.
$$

Here $m(a) = \text{diag}(a, 1, 1, a^{-1}) \in SO_4$. The second identity, valid for all integers $i, j \geq 1$, is

$$
\tau_{SO_4^{(i)}, \text{fin}}(m(p^i q^j)) = \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix} + G_3^{(i)}(1, p) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(p) \\ \iota(p) & 1 \end{pmatrix} \\
+ G_3^{(i)}(1, q) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} q^j & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(q) \\ \iota(q) & 1 \end{pmatrix} \\
+ G_3^{(i)}(1, p) G_3^{(i)}(1, q) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(p) \iota(q) \\ \iota(p) \iota(q) & 1 \end{pmatrix}.
$$

Using the fact that $(p, p) = (q, q) = 1$, we argue as in the proof of Thm. 1 to obtain the identities

$$
\tau_{SO_4^{(i)}, \text{fin}}(m(p^i)) = \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i & 1 \\ 1 & 1 \end{pmatrix} + G_3^{(i)}(1, p) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i & 1 \\ 1 & 1 \end{pmatrix}
$$

and

$$
\tau_{SO_4^{(i)}, \text{fin}}(m(p^i q^j)) = \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix} + G_3^{(i)}(1, p) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix} + G_3^{(i)}(1, q) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix} + G_3^{(i)}(1, p) G_3^{(i)}(1, q) \tau_{GL_3^{(i)}, \text{fin}} \begin{pmatrix} p^i q^j & 1 \\ 1 & 1 \end{pmatrix}.
$$

In deriving (48) from (46) the only difficulty is to check that we get the right cocycle so as to obtain the last summand of (48). To explain this point, starting with the last summand
of (46), by $K_\theta$-invariance we have

$$
\tau_{GL_3^{(4)}, fin} \left( \begin{pmatrix} \frac{p^j q^j}{q} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(p)\iota(q) \\ \iota(p)\iota(q) & 1 \end{pmatrix} \right)
= \tau_{GL_3^{(4)}, fin} \left( \begin{pmatrix} \frac{p^j q^j}{q} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(p)\iota(q) \\ \iota(p)\iota(q) & 1 \end{pmatrix} \right).$$

Multiplying the last two matrices, the cocycle contributes a factor of $(p,q)^3_{\nu_1}$, where $(\ , \ )_{\nu_1}$ is the local 4-th order residue symbol in $F_{\nu_1}$. Adjusting on the right by a similar factor involving $p$, we obtain

$$
\tau_{GL_3^{(4)}, fin} \left( \begin{pmatrix} \frac{p^j q^j}{q} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \iota(p)\iota(q) \\ \iota(p)\iota(q) & 1 \end{pmatrix} \right) = (p,q)_{\nu_1}^3 (q,p)_{\nu_2}^3 \tau_{GL_3^{(4)}, fin} \left( \begin{pmatrix} \frac{p^j q^j}{q} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{pq}{q} \\ \frac{pq}{q} & 1 \end{pmatrix} \right).
$$

Thus by (32) we obtain the last term in (48). The desired conclusion now follows by putting $i, j = 0, 1$ and using the values of $\tau_{GL_3^{(4)}, fin}$. Note that we have

$$
G_3^{(4)}(1,p)G_3^{(4)}(1,q) = \left( \frac{q}{p} \right)_2 G_3^{(4)}(1,pq),
$$

since under our assumptions

$$
\left( \frac{q}{p} \right)_4 = \left( \frac{p}{q} \right)_4.
$$

\[\square\]

Remark. Eqn. (47) with $i = 2$ gives

$$
\tau_{SO_4^{(4)}, fin} (m(p^2)) = G_3^{(4)}(1,p)^2.
$$

In fact, this is also known via Hecke theory, cf. [7], pg. 153.

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