REIDEMEISTER TORSION OF A 3-MANIFOLD
OBTAINED BY A DEHN-SURGERY ALONG THE
FIGURE-EIGHT KNOT

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Abstract. Let $M$ be a 3-manifold obtained by a Dehn-surgery along the figure-eight knot. We give a formula of the Reidemeisiter torsion of $M$ for any $SL(2; \mathbb{C})$-irreducible representation. It has a rational expression of the trace of the image of the meridian.

1. Introduction

Reidemeister torsion is a piecewise linear invariant for manifolds and originally defined by Reidemeister, Franz and de Rham in 1930’s. In 1980’s Johnson developed a theory of the Reidemeister torsion from the view point of certain relation to the Casson invariant of a homology 3-sphere. He also derived an explicit formula for the Reidemeister torsion of a homology 3-sphere obtained by a $\frac{p}{q}$-Dehn surgery along any torus knot for $SL(2; \mathbb{C})$-irreducible representations. We generalized the Johnson’s formula for any Seifert fibered space [2] along his studies.

In this paper, we give a formula for 3-manifolds obtained by Dehn surgeries along the figure-eight knot. Let $K \subset S^3$ be the figure-eight knot. The knot group $\pi_1(S^3 \setminus K)$ has the following presentation.

$$\pi_1(S^3 \setminus K) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$, $l = w^{-1}\bar{w}$ and $\bar{w} = x^{-1}yxy^{-1}$. Now $x$ is a meridian and $l$ is a longitude.

Let $M$ be a 3-manifold obtained by a $\frac{p}{q}$-surgery along $K$. The fundamental group $\pi_1(M)$ admits a presentation as follows;

$$\pi_1(M) = \langle x, y \mid wx = yw, x^p l^q = 1 \rangle.$$ 

Let $\rho : \pi_1(M) \to SL(2; \mathbb{C})$ be an irreducible representation. Assume the chain complex $C_\ast(M; \mathbb{C}_\rho_p)$ is acyclic. Then Reidemeister torsion $\tau_\rho(M) = \tau(C_\ast(M; \mathbb{C}_\rho_p))$ is given by the following.

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Theorem 1.1. \[
\tau_\rho(M) = \frac{2(u - 1)}{u^2(u^2 - 5)}
\]
where \( u = \text{tr}(\rho(x)) \).

Remark 1.2. We remark the trace \( u \) cannot move freely on the complex plane in the above formula. The value \( u \) depends on the surgery coefficient \( p, q \).

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2. Definition of Reidemeister torsion
First let us describe the definition of the Reidemeister torsion for \( SL(2; \mathbb{C}) \)-representations. Since we do not give details of definitions and known results, please see Johnson [1], Milnor [5, 6, 7] and Kitano [2, 3] for details.

Let \( W \) be an \( n \)-dimensional vector space over \( \mathbb{C} \) and let \( b = (b_1, \ldots, b_n) \) and \( c = (c_1, \ldots, c_n) \) be two bases for \( W \). Setting \( b_i = \sum p_{ij}c_i \), we obtain a nonsingular matrix \( P = (p_{ij}) \) with entries in \( \mathbb{C} \). Let \([b/c]\) denote the determinant of \( P \).

Suppose \( C_* : 0 \to C_m \xrightarrow{\partial_2} C_{m-1} \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_1} C_0 \to 0 \) is an acyclic chain complex of finite dimensional vector spaces over \( \mathbb{C} \). We assume that a preferred basis \( c_i \) for \( C_i \) is given for each \( i \). Choose some basis \( b_i \) for \( B_i = \text{Im}(\partial_{i+1}) \) and take a lift of it in \( C_{q+1} \), which we denote by \( \tilde{b}_i \). Since \( B_i = Z_i = \text{Ker}\partial_i \), the basis \( b_i \) can serve as a basis for \( Z_i \). Furthermore since the sequence \( 0 \to Z_i \to C_i \to B_{i-1} \to 0 \) is exact, the vectors \((b_i, \tilde{b}_{i-1})\) form a basis for \( C_i \). Here \( \tilde{b}_{i-1} \) is a lift of \( b_{i-1} \) in \( C_i \). It is easily shown that \([b_i, \tilde{b}_{i-1}/c_i]\) does not depend on the choice of a lift \( \tilde{b}_{i-1} \). Hence we can simply denote it by \([b_i, b_{i-1}/c_i]\).

Definition 2.1. The torsion \( \tau(C_*) \) is given by the alternating product
\[
\prod_{i=0}^{m} [b_i, b_{i-1}/c_i](-1)^{i+1}.
\]

Remark 2.2. It is easy to see that \( \tau(C_*) \) does not depend on the choices of the bases \( \{b_0, \ldots, b_m\} \).
Now we apply this torsion invariant of chain complexes to the following geometric situations. Let $M$ be a finite CW-complex and $\tilde{M}$ a universal covering of $M$. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C_*(\tilde{M}; \mathbb{Z})$ has the structure of a chain complex of free $\mathbb{Z}[\pi_1(M)]$-modules. We denote the 2-dimensional vector space $C_2$ by $V$. Using a representation $\rho: \pi_1(M) \to SL(2; \mathbb{C})$, $V$ has the structure of a $\mathbb{Z}[\pi_1(M)]$-module. Then we denote it by $V_\rho$ and define the chain complex $C_*(M; V_\rho)$ by $C_*(\tilde{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} V_\rho$. Here we choose a preferred basis 
\[
\{ \tilde{u}_1 \otimes e_1, \tilde{u}_1 \otimes e_2, \cdots, \tilde{u}_k \otimes e_1, \tilde{u}_k \otimes e_2 \}
\] of $C_2(M; V_\rho)$ where $\{e_1, e_2\}$ is a canonical basis of $V = \mathbb{C}^2$ and $u_1, \cdots, u_k$ are the $q$-cells giving the preferred basis of $C_2(M; \mathbb{Z})$. We suppose that all homology groups $H_*(M; V_\rho)$ are vanishing. In this case we call $\rho$ an acyclic representation.

**Definition 2.3.** Let $\rho: \pi_1(M) \to SL(2; \mathbb{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(M)$ is defined to be the torsion $\tau(C_*(M; V_\rho))$.

**Remark 2.4.**

1. We define the $\tau_\rho(M) = 0$ for a non-acyclic representation $\rho$.
2. The Reidemeister torsion $\tau_\rho(M)$ depends on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant. See Johnson [1] and Milnor [5, 6, 7].

Here we recall the Reidemeister torsion of the torus and the solid torus.

**Proposition 2.5.** Let $\rho: \pi_1(T^2) \to SL(2; \mathbb{C})$ be a representation.

1. This representation $\rho$ is an acyclic representation if and only if there exists an element $z \in \pi_1(T^2)$ such that $tr(\rho(z)) \neq 2$.
2. If $\rho$ is acyclic, then it holds $\tau_\rho(T^2) = 1$.

Next we consider the solid torus $S^1 \times D^2$ with $\pi_1(S^1 \times D^2) \cong \mathbb{Z}$ generated by $x$.

**Proposition 2.6.** Let $\pi_1(S^1 \times D^2) \to SL(2; \mathbb{C})$ be a representation. Then it holds

\[
\tau(S^1 \times D^2; V_\rho) = \frac{1}{\det(\rho(l) - E)} = \frac{1}{2 - \text{tr}(\rho(l))}
\]

for a generator $l \in \pi_1(S^1 \times D^2) \cong \mathbb{Z}$. Here $E$ is the identity matrix in $SL(2; \mathbb{C})$. 
From here we assume $M$ is a compact 3-manifold with an acyclic representation $\rho : \pi_1(M) \to SL(2; \mathbb{C})$. Here we take a torus decomposition of $M = A \cup \mathbb{T}^2 B$. For simplicity, we write the same symbol $\rho$ for a restricted representation to subgroups $\pi_1(A)$, $\pi_1(B)$ and $\pi_1(T^2)$ of $\pi_1(M)$.

By this torus decomposition, we have the following exact sequence:

$$0 \to C_*(T^2; V_\rho) \to C_*(A; V_\rho) \oplus C_*(B; V_\rho) \to C_*(M; V_\rho) \to 0.$$

**Proposition 2.7.** Let $\rho : \pi_1(M) \to SL(2; \mathbb{C})$ be a representation which restricted to $\pi_1(T^2)$ is acyclic. Then $H_*(M; V_\rho) = 0$ if and only if $H_*(A; V_\rho) = H_*(B; V_\rho) = 0$. In this case it holds

$$\tau_\rho(M) = \tau_\rho(A)\tau_\rho(B).$$

We apply this proposition to any 3-manifold obtained by Dehn-surgery along a knot. Now let $M$ be a closed 3-manifold obtained by a $p/q$-surgery along the figure eight knot $K$. Under the presentation

$$\pi_1(E(K)) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$, $l = w^{-1}\bar{w}$ and $\bar{w} = x^{-1}yx^{-1}$, $x$ is a meridian and $l = w^{-1}\bar{w}$ is a longitude.

We take an open tubular neighborhood $N(K)$ of $K$ and its knot exterior $E(K) = S^3 \setminus N(K)$. We denote its closure of $N(K)$ by $\bar{N}$ which is homeomorphic to $S^1 \times D^2$. Since this 3-manifold $M$ is obtained by Dehn-surgery along $K$, we have a torus decomposition

$$M = E(K) \cup \bar{N}.$$

Let $\rho : \pi_1(E(K)) = \pi_1(S^3 \setminus K) \to SL(2; \mathbb{C})$ be a representation which extends to $\pi_1(M)$. In this case it holds the following.

**Proposition 2.8.** If $\rho$ is acyclic on $\pi_1(T^2)$, then

$$\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\bar{N}).$$

Further if all chain complexes are acyclic, then

$$\tau_\rho(M) = \frac{\tau_\rho(E(K))}{2 - \text{tr}(\rho(l))}.$$
Lemma 3.1. Let $X, Y \in SL(2, \mathbb{C})$. If $X$ and $Y$ are conjugate and $XY \neq YX$, then there exists $P \in SL(2, \mathbb{C})$ s.t.

$$PXP^{-1} = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \quad PYP^{-1} = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}. \tag{1}$$

We apply this lemma to irreducible representations of $\pi_1(E(K))$. For any irreducible representation $\rho$, we may assume that its representative of this conjugacy class is given by

$$\rho_{s,t} : \pi_1(E(K)) \to SL(2, \mathbb{C}) \quad (s, t \in \mathbb{C} \setminus \{0\})$$

where

$$\rho_{s,t}(x) = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \quad \rho_{s,t}(y) = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix} \tag{2}$$

Simply we write $\rho$ to $\rho_{s,t}$ for some $s, t$. We compute the matrix

$$R = \rho(w)\rho(x) - \rho(y)\rho(w) = (R_{ij})$$

to get the defining equations of the space of the conjugacy classes of the irreducible representations.

- $R_{11} = 0$,
- $R_{12} = 3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2t + t^2$,
- $R_{21} = 3t - \frac{t}{s^2} - s^2t + 3t^2 - \frac{t^2}{s^2} - s^2t^2 - t^3 = tR_{12}$,
- $R_{22} = 0$.

Hence $R_{12} = 0$ is the equation of the space of the conjugacy classes of the irreducible representations.

This equation

$$3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2t + t^2 = 0 \tag{3}$$

can be solved in $t$ as

$$t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2} \tag{4}$$

Here it can be seen that $L = \rho(l) = (l_{ij})$ is given by the followings:

**Lemma 3.2.**

- $l_{11} = 1 - \frac{t}{s^2} + s^2t - t^2 - \frac{t^2}{s^4} + \frac{t^2}{s^2} + s^2t^2 - t^3 - \frac{t^3}{s^2}$
- $l_{12} = \frac{t^2}{s^3} + \frac{t^2}{s^2} - \frac{t}{s} - st^2$
- $l_{21} = \frac{t^2}{s^3} - \frac{2t^2}{s} - 2st^2 + s^3t^2 + \frac{t^3}{s^3} - \frac{2t^3}{s} - 2st^3 + s^3t^3 - \frac{t^4}{s} - st^4$
- $l_{22} = 1 + \frac{t}{s^2} - s^2t - t^2 + \frac{t^2}{s^2} - s^2t^2 + s^4t^2 - t^3 - s^2t^3$
Here we get the trace of direct computation.

\[
\text{tr}(\rho(l)) = 2 - 2t^2 + \frac{t^2}{s^4} + st^2 - 2t^3 - \frac{t^3}{s^2} - s^2t^3
\]

It is easy to see that \(\text{tr}(\rho(l)) \neq 2\) if \(u = s + \frac{1}{s} = 2\). Hence there exists an element \(z \in \pi_1(T^2)\) s.t. \(\text{tr}(\rho(z)) \neq 2\). This means \(\rho\) is always acyclic on \(T^2\). Now we have

\[
\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\bar{N})
\]

Here we obtain the Reidemeister torsion of \(E(K)\) as follows. See [3] for precise computation.

**Proposition 3.3.**

\[
\tau_\rho(E(K)) = -2(u - 1)
\]

where \(u = s + \frac{1}{s}\).

By substituting

\[
t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}
\]

in \(\text{tr}(\rho(l))\), we get the following proposition.

**Proposition 3.4.**

\[
\tau_\rho(\bar{N}) = -\frac{1}{u^2(u^2 - 5)}.
\]

Therefore we obtain the following formula:

\[
\tau_\rho(M) = \tau_\rho(E(K))\tau_\rho(\bar{N})
\]

\[
= (-2(u - 1)) \left( -\frac{1}{u^2(u^2 - 5)} \right)
\]

\[
= \frac{2(u - 1)}{u^2(u^2 - 5)}.
\]

**Remark 3.5.** The representations for \(u^2 - 5 = 0\) are degenerate into reducible representation from irreducible representations.

**References**

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