ON RIGHT COIDEAL SUBALGEBRAS OF QUANTUM GROUPS

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Abstract. Right coideal subalgebras are interesting substructures of Hopf algebras such as quantum groups. Examples of right coideal subalgebras are the quantum Borel part as well as quantum symmetric pairs. Classifying right coideal subalgebras is a difficult question with notable results by Schneider, Heckenberger and Kolb. After reviewing these results, as main result we prove that an arbitrary right coideal subalgebra has a particularly nice set of generators. This allows in principle to specify the set of right coideal subalgebras in a given case. As application we determine right coideal subalgebras of the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$ and discuss their representation theoretic properties.

Keywords: Quantum group, Hopf algebra, Right coideal subalgebra

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1. Introduction

The quantum group $U_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra (see e.g. [Jan96]), and it carries the structure of a Hopf algebra. A right coideal subalgebra $C \subset U_q(\mathfrak{g})$ is defined to be a subalgebra such that $\Delta(C) \subset C \otimes U_q(\mathfrak{g})$. While the Hopf subalgebras, fulfilling $\Delta(H) \subset H \otimes H$, are well understood for quantum groups and correspond to what we expect from the Lie algebra, the concept of right coideal subalgebras (RCS) appears to be much more difficult.

The study of RCS in a broader context has far-reaching applications in Skryabin's freeness results and for the structure theory of quantum groups and Nichols algebras. For example in [HS09] Heckenberger and Schneider constructed the PBW basis systematically by a flag of RCS. As a very interesting byproduct they proved that homogeneous right coideal subalgebra are in 1:1 correspondence with the Weyl group of $U_q(\mathfrak{g})$. Moreover the study of RCS yields interesting Lie-theoretic structures like the quantum Borel part $U_q(\mathfrak{g})^+$ and quantum symmetric pairs.

Recently Heckenberger and Kolb have classified homogeneous RCS (i.e. RCS $C$ with $U^0 \subset C$) in [HK11a], as well as RCS of the positive Borelpart in [HK11b]. Thus broad classes of RCS are already known:

| $C \subset U_q(\mathfrak{g})$ | $C \subset U_q(\mathfrak{g})$ |
|-----------------------------|-------------------------------|
| $U^0 \subset C$             | $C = U^+[x]U^0$               |
| $C = U^+[x]U^0U^+[y]$        | $C = U^+[x]U^0U^+[y]$         |
| $U^0 \cap C =: T_L$          | $(U^+[x]T_L)_x$               |
| General case ?               | $U^+[x]T_L$                   |

In Section 2 we review these currently known classification results on RCS.

In Section 3 we construct a set of generators for an arbitrary RCS with the property that $C^0 := C \cap U^0$ is a Hopf subalgebra. In particular we show, that one can choose a generating system, consisting of elements whose leading terms lie in $U^{\geq 0}$ resp. $U^{< 0}$. Even more we prove that we can take any generator with at most one $E$- and one $F$-leading term, which are moreover root vectors of $\psi(U^+[w^+]), U^-[w^-]$ for suitably chosen $w^+, w^- \in W$. This statement is our Main Theorem 3.11

$$\lambda_E E^{\phi_E}_\mu + \lambda_F K^{-1}_\mu \psi F^{\phi_F}_\mu + \lambda_K K^{-1}_\mu$$

In Section 4 we demonstrate our result by determining all RCS of $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$ explicitly. We find the quantum Borel part and its reflections, the quantum symmetric pairs, smaller quantum groups of smaller rank and construct many
nontrivial new RCS. With this knowledge we can also classify all so-called Borel subalgebras (i.e. RCS maximal with the property, that any finite dimensional simple representation is onedimensional) of $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$.

2. Preliminaries

Let $g$ be a finite-dimensional semisimple Lie algebra of rank $n$ over the field of complex numbers $K = \mathbb{C}$.

We denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a set of positive simple roots, by $Q$ the root lattice, and by $\Phi^+ \subset Q$ the set of all positive roots. We denote by $(\cdot, \cdot)$ the symmetric bilinear form on $\mathbb{R}^\Pi$ with the Cartan matrix $c_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$.

Our article is concerned with the quantum group $U = U_q(g)$ where $q$ is not a root of unity. There exist root vectors $E_\mu$ for all $\mu \in \Phi^+$, constructed by algebra automorphisms $T_w$ due to Lusztig for each Weyl group element $w \in W$, see [Jan96] Chapter 8.

A subalgebra $C$ of a Hopf algebra $H$ is called a right coideal subalgebra (RCS) if $\Delta(C) \subset C \otimes H$. Three essential results in the theory of coideal subalgebras of quantum groups are:

Call a right coideal subalgebra $C \subset U_q(g)$ homogeneous iff $U^0 \subset C$ (in particular $C$ is homogeneous with respect to the $Q$-grading).

**Theorem 2.1** ([HS09] Theorem 7.3). For every $w \in W$ there is an RCS $U^+[w]U^0$, where $U^+[w]$ is generated by the root vectors $E_{\beta_i}$ for all $\beta_i$ in the subset of roots

$$\Phi^+(w) = \{\alpha \in \Phi^+ \mid w^{-1}\alpha < 0\} = \{\beta_i \mid i \in \{1, \ldots, \ell(w)\}\}$$

In particular $|\Phi^+(w)| = \ell(w)$, and

$$(1) \quad v < w \text{ iff } \Phi^+(v) \subset \Phi^+(w)$$

and for the longest element $\Phi^+(w_0) = \Phi^+$.

Conversely, every homogeneous RCS $C \subset U_q^+(g)U^0$ is of this form for some $w$.

**Theorem 2.2** ([HK11a] Theorem 3.8). The homogeneous RCS $C \subset U_q(g)$ are of the form

$$C = U^+[w]U^0U^{-}[v]$$

for a certain subset of pairs $v, w \in W$.

Non-homogeneous RCS are only classified on $U^0U^{-}$ (or $U^+U^0$):

**Theorem 2.3** ([HK11b] Theorem 2.15). For $w \in W$, let $\phi : U^{-}[w] \to K$ be a character and define

$$\text{supp}(\phi) := \{\beta \in Q \mid \exists x_\beta \in U^{-}[w] \text{ with } \phi(x_\beta) \neq 0\}$$
Take any subgroup $L \subset \text{supp}(\phi)^\perp$, then there exists a character-shifted RCS

$$U^-[w]_\phi := \{ \phi(x^{(1)})x^{(2)} \mid \forall x \in U^-[w] \}$$

and an RCS $U^-[w]_\phi T_L$ with group ring $T_L = \mathbb{K}[L] \subset U^0$.

Conversely, every RCS $C \subset U^-(\mathfrak{g})U^0$ if of this form.

Orderings: Let $w \in W$ be fixed with a chosen reduced expression. We consider $\mathbb{N}^{\Phi^+(w)}$ and denote $b \in \mathbb{N}^{\Phi^+(w)}$ by $b = (b_1, \ldots, b_{|\Phi^+(w)|})$ for $b_i \in \mathbb{N}$.

There is a canonical projection $pr : \mathbb{Z}[\Phi^+(w)] \to \mathbb{Z}\Pi$ given by the addition on $\Phi^+$, i.e. $pr(b) := b_1 \beta_1 + \ldots + b_{|\Phi^+(w)|} \beta_{\Phi^+(w)}$ and we identify $\Phi^+(w)$ with the set of all unit vectors $\mathbb{N}^{\Phi^+(w)}$, i.e. let $\beta_i \in \Phi^+(w)$ be the $i$-th unit vector in $\mathbb{N}^{\Phi^+(w)}$.

- We define, respecting the choice of a reduced expression of $w$, a total ordering $<_{\text{lex}}$ on $\Phi^+(w)$ by

$$\beta_i < \beta_j \in \Phi^+(w) \iff i < j$$

This ordering is convex, i.e. $\mu < \nu \in \Phi^+(w)$ if and only if $\mu < \mu + \nu < \nu$, for $\mu + \nu \in \Phi^+(w)$ see therefore [PA] s.662.

- In dependence of this total ordering there is another partial ordering on $\mathbb{N}^{\Phi^+(w)}$, which is the lexicographical ordering $<_{\text{lex}}$ on $\Phi^+(w)$. With this ordering we can order elements $a, b \in \mathbb{N}^{\Phi^+(w)}$ with $pr(a) = pr(b)$.

**Lemma 2.4** ([LS91] Prop. 5.5.2). Let $w \in W$ be a Weyl group element with reduced expression $w = s_{\alpha_1} \ldots s_{\alpha_t(w)}$ and $\Phi^+(w) = \{\beta_1, \ldots, \beta_t(w)\}$. For the corresponding root vectors there is the following commutator rule. Let $\beta_i, \beta_j \in \Phi^+(w)$ with $i < j$, then:

$$[E_{\beta_i}, E_{\beta_j}] = \sum_{(a_{i+1}, \ldots, a_{j-1}) \in \mathbb{N}^{j-i-1}_0} m_{(a_{i+1}, \ldots, a_{j-1})} E_{\beta_i}^{a_{i+1}} \cdots E_{\beta_{j-1}}^{a_{j-1}}$$

For coefficients $m_{(a_{i+1}, \ldots, a_{j-1})} \in k$.

### 3. Constructing a System of Generators

**3.1. Summary.** Goal of this part is to introduce a clever choice of an algebra generating system for a right coideal subalgebra $C$ of $U = U_q(\mathfrak{g})$. That is a set $Z$ of elements in $C$, which generate as an algebra the right coideal subalgebra $C = \langle Z \rangle$.

We restrict to the case of a right coideal subalgebra $C$, for which

$$C^0 := C \cap U^0$$

is a sub Hopf algebra. In particular we show, that one can choose a generating system, consisting of elements whose leading terms lie in $U^{\geq 0}$ resp. $U^{\leq 0}$. We conjecture that one can choose a generating system which consists of elements of the
following form:

\[(2) \quad \lambda_E E^\phi_E + \lambda_F K_{\mu}^{-1} E^\phi_F + \lambda_K K_{\mu}^{-1}\]

for root vectors \(E_\mu\) resp. \(F_\nu\) in \(\psi(U^+[w_1])\) resp. \(U^-[w_2]\) respecting a special reduced expression of the elements \(w_1, w_2 \in W\) with characters \(\phi_E, \phi_F\) on the subalgebras \(\psi(U^+[w_1])T_L\) resp. \(U^-[w_2]T_L\) for some suitable \(L \subset Q\), where \(E^\phi_E := (\phi_E \otimes id)\Delta(\psi(E_\mu))\) and \(F^\phi_F := (\phi_F \otimes id)\Delta(F_\mu)\).

We denote the \(U^0\)-part of \(C\) by \(T_L := C \cap U^0\) for \(L \subset Q\) a subgroup. First we choose a set of \(\text{ad}(T_L)\)-weight vectors, generating \(C\) as algebra, and show, that we can choose elements in \(U_+^0 U_0^0 U_-^0\) with \(\nu - \gamma\) constant.

Then we prove, that we can choose elements with exactly one leading term in \(U^0\) or \(U_1^0\) respectively. It is even possible to choose root vectors as leading terms respecting a Weyl group element \(w \in W\) with a special reduced expression.

Finally, we provide a proof of the assertion that a choice of generating elements in \((U^0 + U_1^0) \cap C\) is possible and prove some further properties, which hold for \(\phi_E, \phi_F\) in [2].

First we prove statements, which hold for any choice of a reduced expression of the maximal element \(w_0 \in W\). Fix therefore in the following a reduced expression of the maximal element \(w_0\) and consider elements of \(U\) in the PBW-basis consisting of root vectors in \(U_+^0 w_0 U_0^0 U_-^0 [w_0]\).

For elements in \(\mu \in Q\) we use the partial ordering \(\prec\) on \(Q\) and on elements in \(\mathfrak{p} \in \mathbb{N}^\Phi\) the ordering \(\prec_{lex}\). We also use the projection \(\text{pr} : \mathbb{Z}\Phi^+ \to \mathbb{Z}\Pi\) given by addition in \(Q = \mathbb{Z}\Pi\), i.e. \(\text{pr}(\mathfrak{p}) := \mu_1 \beta_1 + \ldots + \mu_{|\Phi^+|} \beta_{|\Phi^+|}\). For every \(\mathfrak{p}\) there is a unique PBW-Monomial \(E_{\mathfrak{p}} \in U^+\) (the same holds in \(U^-\)) given by:

\[E_{\mathfrak{p}} := E_{\beta_{|\Phi^+|}}^\mu_{\beta_{|\Phi^+|}} \cdots E_{\beta_1}^\mu_{\beta_1}\]

The same holds in \(U^-\). We need some further technical notations:

**Definition 3.1.** For \(\mu \in Q_+, \nu \in Q, \gamma \in Q_+\) we denote the components of an element \(X \in U\) in \(Q^3\)-degree in the triangular decomposition \(U_\mu^+ U_\nu^0 U_-^-\) by \(\text{pr}_{(\mu, \nu, \gamma)}(X)\), i.e.

\[\text{pr}_{(\mu, \nu, \gamma)} : U \to U_\mu^+ K_\nu U_-^-\]

\[X_{\mu, \nu, \gamma} := \text{pr}_{(\mu, \nu, \gamma)}(X) = \sum_{\substack{\mathfrak{p} \in \mathbb{N}^\Phi \text{ with } \text{pr}(\mathfrak{p}) = \mu \\ \mathfrak{q} \in \mathbb{N}^\Phi \text{ with } \text{pr}(\mathfrak{q}) = \gamma}} c_{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}} E_{\mathfrak{p}} K_{\mathfrak{q}} F_{\mathfrak{r}}\]

This projection in the first two components \(\text{pr}_{(\nu, \gamma)}\) corresponds to \(\text{pr}_{\mu, \nu}\) which is defined in \(U^0\) as unique \(Q^2\)-graded projection \(\text{pr}_{(\alpha, \beta)} : U^0 \to U_\alpha^+ K_\beta\).
$U$ is not a $Q^3$-graded Hopf algebra, nevertheless we know:

$$\Delta(X_{\mu,\nu,\gamma}) \subset X_{\mu,\nu,\gamma} \otimes K_{\nu-\gamma} + K_{\mu+\nu} \otimes X_{\mu,\nu,\gamma} + \sum_{(\mu',\gamma') \prec (\mu,\gamma)} U_{\mu'} K_{\mu+\mu'} U_{\nu-\gamma'} \otimes U_{\mu-\mu'} U_{\nu-\gamma'} U_{\gamma+\gamma'}$$

Definition 3.2. The $U^{\geq 0}$-part resp. the $U^{\leq 0}$-part resp. the mixed part of an element $X$ are given by the following sums

$$\sum_{\mu \in Q_+, \nu \in Q} X_{\mu,\nu,0}, \quad \sum_{\nu \in Q, \gamma \in Q_+} X_{0,\nu,\gamma}, \quad \sum_{\mu \neq 0, \nu \in Q, \gamma \neq 0 \in Q} X_{\mu,\nu,\gamma}$$

Definition 3.3. For an element $X_{\mu,\nu,\gamma} = \sum_{\mu \in \mathbb{N}^+} c_{\mu,\nu,\gamma}(\phi) E_{\pi_{\mu,\nu,\gamma}} F_{\tau_{\mu,\nu,\gamma}} \in U$ and $\pi_{\mu,\nu,\gamma} \in \mathbb{N}^{Q^+}$ with $pr(\pi) = \mu$ and $\tau_{\mu,\nu,\gamma} \in \mathbb{N}^{Q^+}$ with $pr(\tau) = \gamma$ given $\pi, \tau \in \mathbb{N}^{Q^+}, \nu \in Q$ we denote the coefficient of the PBW-basis elements $E_{\pi_{\mu,\nu,\gamma}} F_{\tau_{\mu,\nu,\gamma}}$ in the PBW-Basis by

$$c_{\pi_{\mu,\nu,\gamma}}(X) \in k$$

3.2. Generating systems with $T_L$-stable root vectors.

Lemma 3.4. Let $C$ be an RCS in $U$. We can choose a generating system $Z$ of $C$, such that for all elements $X = \sum_{\nu \in Q, \mu, \gamma \in Q_+} X_{\mu,\nu,\gamma} \in Z$ the difference $\nu - \gamma$ is constant.

Proof. We use the fact, that $C$ is an RCS, the formula (3) and the PBW-basis: Let for any $\eta \in Q$ the linear form $\varphi \in U^*$ be given by $\varphi(K_{\eta}) = 1$ and $\varphi(x) = 0$ for any other PBW-Basis element. Then for any element $X \in C$ the following elements $X^{(n)}$ lie again in the right coideal subalgebra $C$:

$$X^{(n)} := (id \otimes \varphi) \Delta(X) = \sum_{\mu, \gamma \in Q_+, \nu \in Q \text{ with } \nu - \gamma = \eta} X_{\mu,\nu,\gamma}$$

On the other hand $X = \sum_{\eta} X^{(n)}$, thus we can refine a given generating system $Z$, by replacing all generators $X$ by the set of $X^{(n)}$. □

Definition 3.5. A weight vector resp. $T_L = U^0 \cap C$ is an element $x \in C$, such that there is a $\mu \in Q$, with $ad(K_{\nu})x = q^{(\mu,\nu)} x \forall \nu \in L$.

The adjoint action of elements in $T_L$ on $C$ is defined, because $T_L$ is a sub Hopf algebra. In particular we can choose a generating system consisting of weight vectors of $T_L$, as $C$ is $adT_L$-stable and as $adU^0$ is diagonalizable on $U$. 
3.3. Leading terms of elements in $C$.

Definition 3.6. Given an element $X$, we define the set of all leading terms as follows:

First we define the set of all maximal $U^+$-degrees:

$$\mathcal{E}(X) := \max_{\mu, \nu, \gamma} \{0 \neq \mu \in Q_+ \mid \exists \nu \in Q, \gamma \in Q_+ \text{ with } X_{\mu, \nu, \gamma} \neq 0\}$$

For any $\mu \in \mathcal{E}(X)$ the corresponding E-leading term is

$$L_\mu(X) := \sum_{\nu \in Q, \gamma \in Q_+} X_{\mu, \nu, \gamma}$$

Similarly we can define F-leading terms.

Remark 3.7. If an element $X \in C$ has no E-leading terms, i.e. $\mathcal{E}(X) = \emptyset$, then $X \in U^{\leq 0}$.

Theorem 3.8. Let $C$ be an RCS with the property $C^0 := C \cap U^0$ is a sub Hopf-algebra. We can choose a generating system of $C$ consisting of elements whose E-leading terms lie in $U^{\geq 0}$.

Proof. For elements $X$ in $C$ let the set of PBW-Monomials of E-leading terms with non trivial F-part be defined as:

$$\mathcal{M}(X) := \left\{\overline{\nu} \in \mathbb{N}^{\Phi^+} \mid \exists \nu \in Q, 0 \neq \overline{\gamma} \in \mathbb{N}^{\Phi^+} \text{ such that } c_{\overline{\nu}, \nu, \overline{\gamma}}(X) \neq 0 \text{ und } \text{pr}(\overline{\nu}) \in \mathcal{E}(X)\right\}$$

Similarly we define for a set $Z$ of elements $X \in C$ the set $\mathcal{M}(Z) := \bigcup_{X \in Z} \mathcal{M}(X)$. For comparing elements and generating systems respecting their corresponding set $\mathcal{M}$, we define a partial ordering $\leq$ on subsets of $\mathbb{N}^{\Phi^+}$ as follows:

$\mathcal{N} \leq \mathcal{M}$ if and only if there exists for all $\overline{\nu} \in \mathcal{N}$ a $\overline{\mu} \in \mathcal{M}$, such that either $\text{pr}(\overline{\nu}) < \text{pr}(\overline{\mu})$ (i.e. the partial ordering on $Q$) or if $\text{pr}(\overline{\nu}) = \text{pr}(\overline{\mu})$ then $\overline{\nu} \leq_{\text{lex}} \overline{\mu}$ (i.e. the lexicographical ordering of PBW-Monomials). We call $\mathcal{N} < \mathcal{M}$ strictly smaller, if there exists at least one $\overline{\nu} \in \mathcal{M}$ which is not less than or equal (again respecting partial and then lexicographical ordering) to all elements $\overline{\mu} \in \mathcal{N}$.

The goal of the Theorem is to prove, that there exists a generating system $Z$ of $C$ with $\mathcal{M}(Z) = \emptyset$. In the following we give an algorithm, to construct for an element $X \in C$ with $\mathcal{M}(X) \neq \emptyset$ finitely many new elements $\mathcal{Y} = \{Y_1, \ldots, Y_n\} \subset C$ with $\mathcal{M}(\{Y_1, \ldots, Y_n\}) < \mathcal{M}(X)$ and $X \in \langle \mathcal{Y} \rangle$, i.e. $X$ lies in the subalgebra generated by the elements $Y_i$. The sets $\mathcal{M}(X)$ for single elements $X$ or finite sets of elements are finite.

By the definition of $\mathcal{N} < \mathcal{M}$, starting with a set $\mathcal{M}(X)$, in every iteration step either one of the maximal $Q$-degrees of the set decreases or (for similar $Q$-degree) in one $Q$-degree the lexicographical maximal degree decreases.

This proves, that, starting with a $\mathcal{M}(X)$ we can construct in finitely many steps a finite set $\mathcal{Y} \subset C$, for which $\mathcal{M}(\mathcal{Y}) = \emptyset$ and $X \in \langle \mathcal{Y} \rangle$. Finally we conclude, that even for an a-priori infinite generating system $Z$ of $C$ we can construct the desired
alternative generating system $Z'$ with $\mathcal{M}(Z') = \emptyset$ as union of all $\mathcal{Y}$ for all $X \in Z$; thus the claim follows.

Now we give the announced algorithm, to construct for an element $X \in C$ with $\mathcal{M}(X) \neq \emptyset$ finitely many elements $\mathcal{Y} = \{Y_1, \ldots, Y_n\} \subset C$ with $\mathcal{M}(\{Y_1, \ldots, Y_n\}) < \mathcal{M}(X)$ and $X \in \langle \mathcal{Y} \rangle$:

Let $\overline{\mu}_{\text{max}} \in \mathcal{M}(X)$ be a maximal element (maximal respecting the partial ordering $\prec$ and for all elements with the same $Q$-degree respecting the lexicographical ordering). We consider the term $X_{\mu_{\text{max}}} = E_{\mu_{\text{max}}} \sum_{\nu \in Q, \pi \in \mathbb{N}^{\Phi^+}} c_{\mu_{\text{max}}, \nu, \pi} K_\nu F_\pi$.

Due to Lemma 3.4 we can choose $X$ in a way, that $\nu - \gamma = \eta$ holds for a constant $\eta \in Q$. Thus $X_{\mu_{\text{max}}} = E_{\mu_{\text{max}}} \sum_{\nu \in \mathbb{N}^{\Phi^+}, \pi \in \mathbb{N}^{\Phi^+}} c_{\mu_{\text{max}}, \eta + \text{pr}(\gamma), \gamma} K_\eta F_\gamma$.

In the following let $\gamma = \text{pr}(\overline{\eta}), \overline{\mu} := \overline{\mu}_{\text{max}}$ and $\mu = \text{pr}(\overline{\mu})$. We can use the comultiplication to prove with the right coideal property of $C$ for all $\overline{\gamma} \in \mathbb{N}^{\Phi^+}$

\[ \sum_{\overline{\gamma}} c_{\overline{\gamma}, \eta + \gamma} K_{\eta + \gamma + \mu} F_{\overline{\gamma}} \in C \]

(4)

It is for all $\gamma$ in $Q_+$:

$\mathcal{M}(K_{\eta + \gamma + \mu}) = \emptyset = \mathcal{M}(K_{\eta + \gamma + \mu} F_{\overline{\gamma}})$

Let $\gamma_{\text{max}} = \gamma_{\text{max}}(X, \overline{\mu}_{\text{max}})$ be maximal in $Q_+$ respecting $\prec$ such that there exists a $\overline{\gamma}_{\text{max}}$ with $\text{pr}(\overline{\gamma}_{\text{max}}) = \gamma_{\text{max}}$ such that $\overline{\mu}_{\text{max}, \eta + \gamma_{\text{max}}} \neq 0$. There exists such a $\gamma_{\text{max}}$, because $\overline{\mu}_{\text{max}} \in \mathcal{M}(X)$. Consider now the coproduct of $X$:

$\Delta(X) \subset \sum_{\gamma \in Q_+} K_{\eta + \gamma_{\text{max}} + \mu} \otimes L_\mu + U \otimes U_\mu U^0 U^- + \text{terms in } U^0 (U^- \cap \text{ker} e) \otimes U_\mu U^0 U^-$

thus also $K_{\eta + \gamma_{\text{max}} + \mu}$ lies in $C$ and, as $C^0$ is a sub Hopf algebra, in particular also

\[ K_{\eta + \gamma_{\text{max}} + \mu}^{-1} \in C \]

(5)

On the other hand we get from the right coideal property for this $\gamma_{\text{max}}$

\[ E_{\overline{\mu}} K_{\eta + \gamma_{\text{max}}} + (\text{rest}) \in C \]

(6)

with $(\text{rest}) := \sum_{\mu' < \mu} (\text{rest})_{\mu', \eta + \gamma, \gamma} + \sum_{\pi, \mu' < \pi, \mu' = \mu} (\text{rest})_{\overline{\mu}, \gamma + \gamma} E_{\overline{\mu}} K_{\eta + \gamma} F_{\overline{\gamma}}$. Thus

$\mathcal{M}(E_{\overline{\mu}} K_{\eta + \gamma_{\text{max}}} + (\text{rest})) < \mathcal{M}(X)$

as in particular $\overline{\mu}_{\text{max}} \notin \mathcal{M}(E_{\overline{\mu}} K_{\eta + \gamma_{\text{max}}} + (\text{rest}))$. 

Summing over the different $\mathfrak{p}$ and considering the product of these elements yields:

$$
\begin{align*}
z &:= (E_{\mathfrak{p}}K_{\eta+\gamma_{\text{max}}} + (\text{rest}) K_{\eta+\gamma_{\text{max}}+\mu}^{-1} \sum_{\mathfrak{p}} c_{\mathfrak{p},\eta+\gamma,\gamma} K_{\eta+\gamma+\mu} F_{\mathfrak{p}}) \\
&= X_{\mathfrak{p}} + \sum_{\mathfrak{p}} (\text{rest}) c_{\mathfrak{p},\eta+\gamma,\gamma} K_{\eta+\gamma_{\text{max}}+\gamma} F_{\mathfrak{p}} \in C
\end{align*}
$$

Thus for $z$ holds $\mathcal{M}(z) \leq \mathcal{M}(X)$ and $\mathcal{M}(X - z) \leq \mathcal{M}(X)$.

Obviously we can generate $X$ by the elements $z$ and $X - z$, where $z$ can be generated by elements $E_{\mathfrak{p}} K_{\eta+\gamma_{\text{max}}} + (\text{rest})$, $K_{\eta+\gamma_{\text{max}}+\mu}$ and $K_{\eta+\gamma+\mu} F_{\mathfrak{p}}$ with $\mathcal{M}(E_{\mathfrak{p}} K_{\eta+\gamma_{\text{max}}} + (\text{rest}))$, $\mathcal{M}(K_{\eta+\gamma_{\text{max}}+\mu})$, $\mathcal{M}(K_{\eta+\gamma+\mu} F_{\mathfrak{p}}) < \mathcal{M}(X)$.

Let $G$ be the finite set of all $\mathfrak{p}$ with $c_{\mathfrak{p},\eta+\gamma,\gamma} \neq 0$. Assume $\mathcal{M}(X - z) < \mathcal{M}(X)$ (in particular if $\mathfrak{p} \notin \mathcal{M}(X - z)$): Then we have constructed the desired set in $C$ as follows:

$$
\mathcal{Y} = \bigcup_{\mathfrak{p} \in G} \left\{ E_{\mathfrak{p}} K_{\eta+\gamma_{\text{max}}} + (\text{rest}), K_{\eta+\gamma_{\text{max}}+\mu}, K_{\eta+\gamma+\mu} F_{\mathfrak{p}} \right\} \\
\bigcup \left\{ X - \sum_{\mathfrak{p} \in G} (E_{\mathfrak{p}} K_{\eta+\gamma_{\text{max}}} + (\text{rest})) \cdot K_{\eta+\gamma_{\text{max}}+\mu}^{-1} \cdot K_{\eta+\gamma+\mu} F_{\mathfrak{p}} \right\}
$$

such that $\langle \mathcal{Y} \rangle \ni X$ and $\mathcal{M}(\mathcal{Y}) < \mathcal{M}(X)$. As stated above, we can thus inductively replace the generating system $Z$ by a generating system $Z'$ with $\mathcal{M}(Z') = \emptyset$, and thus, as claimed, construct a generating system consisting of elements with $E$-leading terms in $U^{\geq 0}$, so the claim is proven.

Assume $\mathcal{M}(X - z) = \mathcal{M}(X)$, then in particular $\mathfrak{p} \in \mathcal{M}(X - z)$ (maximal respecting the partial ordering $<$ and among all elements with the same $Q$-degree maximal respecting the lexicographical ordering). We construct a series of elements: $X^{(i)}$ with $X^{(1)} := X - z$ and $X^{(i)} := X^{(i-1)} - z^{(i-1)}$ for $i > 1$, by considering for any $X^{(i)}$ with $\mathcal{M}(X - z) = \mathcal{M}(X)$ (i.e. that is $\mathfrak{p} \in \mathcal{M}(X^{(i)})$) the decomposition above. Thus there exists a $j$, such that $\mathcal{M}(X^{(j)} - z^{(j)}) < \mathcal{M}(X)$ and $X$ can be generated by elements $X^{(j)} - z^{(j)}$ and $z^{(i)}$ for all $i \leq j$, where for all $i \leq j$ holds $\mathcal{M}(z^{(i)}) < \mathcal{M}(X^{(i)}) \leq \mathcal{M}(X)$. Thus, as above, we can inductively construct a generating system of the required form.

□
Lemma 3.9. Let \( C \) be an RCS as above, \( X \in C \) a weight vector of \( T_L \), then \( X \) has at most one leading term in \( U^{\geq 0} \).

Proof. Assume a weight vector \( X \) of \( T_L \) has two leading terms \( L_\mu \) and \( L_\nu \) for different elements \( \mu, \nu \in Q_+ \), then from Lemma 3.10 follows similar to the proof of Theorem 3.8 that due to \( \overline{K}_{\mu+\nu} \in C \) and \( \overline{K}_{\mu+\nu} \in C \), so in particular as \( C^0 := C \cap U^0 \) is a sub Hopfalgebra, also \( \overline{K}_{\mu-\nu} \) lies in \( C \). From the stability under \( T_L \) it follows that \( \text{ad}(K_{\mu-\nu})X = q^{|\mu-\nu|}X \). This is a contradiction to \( (\mu - \nu, \mu) \neq (\mu - \nu, \nu) \). \( \square \)

With the same argument we can normalize the leading term, such that \( L_\mu = X_\mu K_\mu^{-1} \) for a \( X_\mu \in U^*_\mu \) and \( K_\mu^{-1} \in U^0 \). The theorems above also hold for \( F \)-leading terms and we can choose a generating system of elements \( X \) which have a maximum of one \( E \)-leading term \( L_\mu \in U^{\geq 0} \) and one \( F \)-leading term \( L_\nu^- \in U^{\leq 0} \), because the two construction steps can not (or only conditionally) reverse each other since they both reduce the overall degree of the mixed terms.

Corollary 3.10. Let \( C \) be an RCS with the property \( C^0 := C \cap U^0 \) is a sub Hopfalgebra. We can choose a generating system of \( C \) consisting of elements with each at most one \( E \)- and one \( F \)-leading term \( L_\mu, L_\nu^- \), which are moreover of the form:

\[
L_\mu = X_\mu K_\mu^{-1}, \; X_\mu \in U^*_\mu \quad L_\nu^- = Y_\nu K_{\nu^-}^{-1}, \; Y_\nu \in U_{\nu^-}^{-1}
\]

For leading terms \( L_\mu \) in \( U^{\geq 0} \) holds, that the product of two such leading terms is again the leading term of an element in \( C \). Moreover, for any Linear form \( \varphi \in U^* \) also \( (id \otimes \varphi)\Delta(L_\mu) \) is a leading term of an element in \( C \). The set of all leading terms forms again an RCS \( \mathcal{L} \) in \( U^{\geq 0} \). From Theorem 3.11 there exists a \( w \in W \) such that \( \mathcal{L} = U^{+\{w\}}T_L \). Thus we can choose exactly those elements in the generating system whose leading terms are root vectors and so in particular generate all elements in \( C \) except \( C^{\leq 0} \). From these considerations, the main result of this paper follows:

Theorem 3.11. Let \( C \) be an RCS with the property \( C^0 := C \cap U^0 \) is a sub Hopf-algebra. Then we can construct a generating system of \( C \), such that any generator has at most one \( E \)- and one \( F \)-leading term, which is moreover a root vector of \( \psi(U^{+\{w^+\}}), U^{-\{w^-\}} \) with a suitable \( U^0 \)-part for suitably chosen \( w^+, w^- \in W \).

Corollary 3.12. \( C^{\geq 0} := U^{\geq 0} \cap C \) is an RCS of \( C \) so by Theorem 3.11 of the form \( \psi(U^{+\{w'\}})T_L \) for \( w' \in W \) a Weylgroup element and a character \( \phi \) on \( \psi(U^{+\{w'\}}) \). Then \( w' \leq w \) for \( w \) chosen above.
3.4. **Further conjecture of a generating system in** $U^{\geq 0} + U^{\leq 0}$. We conjecture that we can substantially intensify the previous theorem, which we neither prove here nor use in the following:

**Conjecture 3.13.** Let $w \in W$ be a Weyl group element with reduced expression $w = s_{\alpha_1} \ldots s_{\alpha_i(w)}$ and $\Phi^+(w) = \{\beta_1, \ldots, \beta_{i(w)}\}$. For the corresponding root vectors we know by [2.4] that the following commutator rule holds: Let $\beta_i, \beta_j \in \Phi^+(w)$, and $i < j$, then:

$$[E_{\beta_i}, E_{\beta_j}] = \sum_{(a_{i+1}, \ldots, a_{j-1}) \in \mathbb{N}_0^{j-1-1}} m_{(a_{i+1}, \ldots, a_{j-1})} E_{\beta_{i+1}}^{a_{i+1}} \ldots E_{\beta_{j-1}}^{a_{j-1}}$$

by [LS91] Prop. 5.5.2.

We conjecture additionally, that for $\beta_i, \beta_j \in \Phi^+$ with $\beta_i + \beta_j = \beta_k \in \Phi^+$ for the vector $(a_{i+1}, \ldots, a_{j-1})$ with $a_i = \delta_{ik}$ the coefficient $m_{(a_{i+1}, \ldots, a_{j-1})}$ is not zero.

**Conjecture 3.14.** We can choose a generating system of $C$ consisting of elements of the form

$$\lambda_E E_{\mu} + \lambda_F K_{\mu-\nu}^{-1} F_{\nu} + \lambda_K K_{\mu}^{-1}$$

Where $E_{\mu}$ resp. $F_{\nu}$ are root vectors respecting $\psi(U^+[w])$, resp. $U^-[w']$ for certain Weyl group elements $w, w' \in W$ and characters $\phi_E : \psi(U^+[w]) \rightarrow \mathbb{C}$, $\phi_F : U^-[w'] \rightarrow \mathbb{C}$.

**Remark 3.15.** A proof would essentially work as the proof of Theorem 3.8. But at a crucial point the special property of the ordering $\Phi^+(w)$ from conjecture 3.13 is used: Assume $X$ has not the desired form. We consider first a maximal mixed summand resp. $\prec$ and argue as in the proof above, however instead of the global maximality of the $E$-Terms we use conjecture 3.13 and argue, that this element can be replaced by other generating elements with a smaller leading term.

For all right coideal subalgebras with a generating system of the form above, we can prove some further restrictions of the $U^0$-part and the support $\text{supp}(\phi_E)$ resp. $\text{supp}(\phi_F)$.

**Lemma 3.16.** For generating elements of an RCS $C$ of the form

$$\lambda_E E_{\mu} + \lambda_F K_{\mu-\nu}^{-1} F_{\nu} + \lambda_K K_{\mu}^{-1}$$

with $\mu \neq \nu$, the following restriction holds:

- $\lambda_K = 0$ or $\lambda_F = 0$
- $\mu + \nu \perp \mu - \nu$

**Proof.** Assume $\lambda_F \neq 0$. As $L_{\mu}$ and $L_{\nu} K_{\mu-\nu}^{-1}$ are weight vectors it follows, that $K_{\mu-\nu} \in C$. The claims follow easy from the fact, that the element is an ad$(T_L)$-weight vector, i.e. in particular ad$K_{\mu-\nu}$. Thus from $\lambda_K \neq 0$ follows $(\mu - \mu, \mu) = 0$, this is a contradiction to $\mu \neq 0$ and $\lambda_F \neq 0$. The second claim, follows in the same way, as $(\mu - \nu, \mu) = -(\mu - \nu, \nu)$ due to the ad$T_L$-stability. \qed
Lemma 3.17. For the $U^0$-part $T_L$ of an RCS $C$ which is generated by elements of the form $\lambda_E F_\mu^\phi K_\mu^{-1} + \lambda_F K_\mu^{-1} F_\nu^\phi + \lambda_K K_\mu^{-1}$ with $\phi_E$, and $\phi_F$ as above $L \subset (\text{supp}(\phi_E) \cup \text{supp}(\phi_F))^\perp$ and $L \perp K_{\mu+\nu}$.

Proof. The claims follow from the ad$T_L$ stable choice of the generating system similar to Lemma 3.16

4. Examples

4.1. Right coideal subalgebras of $U_q(\mathfrak{sl}_2)$.

Data of $U_q(\mathfrak{sl}_2)$. The smallest example of a quantum group is $U_q(\mathfrak{sl}_2)$. It is the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ with root system $\Phi = \{\alpha, -\alpha\}$. It is generated as an algebra by the elements $E_a, F_a, K_a, K_a^{-1}$ with the following relations

$$[E_a, F_a] = \frac{K_a - K_a^{-1}}{q - q^{-1}}$$

where $[x, y]_q := xy - qyx$ is the q-commutator. $U_q(\mathfrak{sl}_2)$ is equipped with the following Hopf algebra structure

$$\Delta(E_a) = E_a \otimes 1 + K_a \otimes E_a \quad \Delta(F_a) = F_a \otimes K_a^{-1} + 1 \otimes F_a \quad \Delta(K_a) = K_a \otimes K_a$$

For simplicity we write in the following $E, F, K$ and $K^{-1}$ for the generators $E_a, F_a, K_a, K_a^{-1}$.

Due to chapter 3 a list of possible generating elements is up to symmetry in $E$ and $F$ given by:

1. $K^i$
2. $EK^{-1}$
3. $EK^{-1} + \lambda K^{-1}$
4. $EK^{-1} + c_F F + c_K K^{-1}$

Right coideal subalgebras of $U_q(\mathfrak{sl}_2)$. Obviously $U_q(\mathfrak{sl}_2)$ itself and the smaller version $\langle EK^{-1}, K^2, K^{-2}, F \rangle$ are RCS. The only homogeneous RCS are the Standard Borel subalgebras $U^{\geq 0}$ and $U^{\leq 0}$ and $U^0$.

In $U^{\geq 0}$ resp. $U^{\leq 0}$ there exist moreover the RCS $\langle EK^{-1}, K^j \rangle$ and $\langle F, K^j \rangle$ for some $j$ and there exist families of character shifted RCS $\langle EK^{-1} \rangle_{\phi^+}$ resp. $\langle F \rangle_{\phi^-}$ for characters on $\langle EK^{-1} \rangle$ resp. $\langle F \rangle$ given by $\phi^+(EK^{-1}) = \lambda$ and $\phi^-(F) = \lambda'$. They have the form $\langle EK^{-1} \rangle_{\phi^+} = \langle EK^{-1} + \lambda K^{-1} \rangle$ and $\langle F \rangle_{\phi^-} = \langle F + \lambda' K^{-1} \rangle$. Moreover there exists of course an RCS of the form $\langle EK^{-1} + c_F F + c_K K^{-1} \rangle$.

Next to those obvious RCS there exists also a special RCS $B = \langle EK^{-1} + \lambda K^{-1}, F + \lambda' K^{-1} \rangle$ with $\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$. The $q$-commutator of the generators
which does not lie in $U_\alpha$ leading term in the root $EZS = E\lambda K^{-1}$(7) $[EK^{-1} + \lambda K^{-1}, F + \lambda' K^{-1}]_{q^2} = \frac{q^2}{q - q^{-1}}(1 - K^{-2}) + (1 - q^2)\lambda\lambda'K^{-2} = \frac{q^2}{q - q^{-1}}1$

$B$ is closed under multiplication and thus an RCS, it is even isomorphic to the Weyl algebra and thus any finite dimensional irreducible representation is one dimensional. With the results of this paper we could prove in [Vocke16] that $B$ is even a Borel subalgebra (i.e. an RCS maximal with the property that each irreducible finite dimensional representation is one dimensional).

Note, that different right coideal subalgebras $B$ for different choices of $\lambda, \lambda'$ can be mapped into each other via the Hopf-Automorphism $E \mapsto tE, \ F \mapsto t^{-1}F$.

Classification of right coideal subalgebras. We now prove that the previous list of RCS is complete. We restrict to the case of right coideal subalgebras $C$ with the property $C^0 := C \cap U^0$ is a sub Hopf algebra, and use the knowledge obtained in the present paper on generating systems to classify all RCS of $U_q(\mathfrak{sl}_2)$: We consider the set of all possible generating elements, as in Lemma 3.12. We know there exists a generating system for each RCS whose elements lie in the set $EZS := \{K^i, c_EEK^{-1} + c_FF + c_KK^{-1}\}$ for $i \in \mathbb{Z}$ and constants $c_E, c_F, c_K \in k$. Each such generating system of an arbitrary right coideal subalgebra contains at most three elements, that is one with $E$-leading term in the root $\alpha$, one with $F$-leading term in the root $\alpha$ and one in $U^0$.

RCS $C$ with $C \cap U^0 = \emptyset$: From (7) we know, that the only RCS with $C \cap U^0 = \emptyset$ which does not lie in $U^{\geq 0}$ or in $U^{\leq 0}$, is either $\langle EK^{-1} + \lambda F + \lambda'K^{-1} \rangle$ for some $\lambda, \lambda'$ or is of the form $\langle EK^{-1} + \lambda K^{-1}, F + \lambda' K^{-1} \rangle$ with $\lambda\lambda' = \frac{q^2}{(1 - q^2)(q - q^{-1})}$.

RCS $C$ with $C \cap U^0 \neq \emptyset$: There is a $j$ with $K_j \in C$, thus due to Lemma 3.17 there can be no character-shifted generating element of $C$ nor a generating element with nontrivial leading terms in $E$ and $F$. Thus $C$ either lies in $U^{\geq 0}$ or in $U^{\leq 0}$ or contains as well $EK^{-1}$ as $F$ and thus also $K^2$ and $K^{-2}$, then $C$ is either $\langle EK^{-1}, K^2, K^{-2}, F \rangle$ or $U_q(\mathfrak{sl}_2)$ itself.

On representation theory of RCS of $U_q(\mathfrak{sl}_2)$. In [Vocke16] we used the results of this paper to prove some interesting results on Borel subalgebras and representation theory of RCS in $U_q(\mathfrak{sl}_2)$:

**Theorem 4.1** (Vocke16 Theorem 9.1). There are two different types of Borel subalgebras of the $U_q(\mathfrak{sl}_2)$: the Standard-Borel subalgebras $U^{\geq 0}$ and $U^{\leq 0}$ and the Borel subalgebra $B := \langle EK^{-1} + \lambda K^{-1}, F + \lambda' K^{-1} \rangle$ with $\lambda\lambda' = \frac{q^2}{(1 - q^2)(q - q^{-1})}$.

In the proof of the maximality we use the knowledge of all RCS in $U_q(\mathfrak{sl}_2)$.
Example. If we restrict the finite dimensional irreducible representations \( L(\lambda) \) on \( U \) to a Borel subalgebra \( B \) the representation is no more irreducible, but not necessary semi simple. In the case of a Borel subalgebra all the composite factors have to be one dimensional.

In \( U_q(\mathfrak{sl}_3) \): Given a highest weight vector of \( L(\lambda) \) to the only dominant weight \( \lambda = \frac{1}{2} \alpha \) and let \( x_0 := v^\lambda \), \( x_1 := v_{\lambda - \alpha} \), then the elements in \( B \) act as follows:

\[
(EK^{-1} + \lambda EK^{-1}).x_0 = \lambda q^{-1}x_0 \quad \quad \quad \quad (EK^{-1} + \lambda EK^{-1}).x_1 = \lambda qx_1 + qx_0
\]

\[
(F + \lambda' K^{-1}).x_0 = x_1 + \lambda' q^{-1}x_0 \quad \quad \quad \quad (F + \lambda' K^{-1}).x_1 = \lambda' qx_1
\]

One can easily check, that \( L(\lambda) \) has a one dimensional submodule \( \langle x_0 + \lambda (1 - q^{-2})x_1 \rangle \) to the eigenvalue \( \lambda q \) of \( EK^{-1} + \lambda EK^{-1} \) resp. \( \lambda q^{-1} \) of \( F + \lambda' K^{-1} \) and the one dimensional quotient module \( L(\lambda)/(x_0 + \lambda (1 - q^{-2})x_1) \) with Eigenvalues \( \lambda q^{-1} \) and \( \lambda' q \).

4.2. Right coideal subalgebras of \( U_q(\mathfrak{sl}_3) \).

Data of \( U_q(\mathfrak{sl}_3) \). \( U_q(\mathfrak{sl}_3) \) is the quantized universal enveloping algebra of the Lie algebra \( \mathfrak{sl}_3 \) with the root system \( \Phi = \{\alpha, \beta, \alpha + \beta, -\alpha, -\beta, -\alpha - \beta\} \) and basis \( \Pi = \{\alpha, \beta\} \). The root vectors for the simple roots are the generating elements \( E_\alpha, E_\beta \) and \( F_\alpha, F_\beta \) and those for the root \( \alpha + \beta \), depending on the underlying reduced representation, are given by:

\[
E_{\alpha\beta} := T_\alpha(E_\beta) = E_\alpha E_\beta - q^{-1} E_\beta E_\alpha \quad \quad E_{\alpha\beta} := T_\beta(E_\alpha) = -q^{-1}(E_\alpha E_\beta - q E_\beta E_\alpha)
\]

\[
F_{\alpha\beta} := T_\alpha(F_\beta) = -q(F_\alpha F_\beta - q^{-1} F_\beta F_\alpha) \quad \quad F_{\alpha\beta} := T_\beta(F_\alpha) = F_\alpha F_\beta - q F_\beta F_\alpha
\]

The quantum group is symmetric in \( \alpha \) and \( \beta \) and with the map \( \omega \) also symmetric in \( E \) and \( F \). For this reason, in the following we will only calculate the relations which do not result from symmetry from the others. The \( q \)-commutator \( \{x, y\} := xy - qyx \) relations of the root vectors \( E_{\alpha\beta} \) and \( E_{\beta\alpha} \) are:

\[
[E_{\alpha\beta}, F_{\beta\alpha}]_1 = \frac{K_{\alpha + \beta} - K_{\alpha + \beta}^{-1}}{q - q^{-1}}
\]

\[
[E_{\alpha\beta}, F_\alpha]_1 = -E_{\beta} K_\alpha^{-1} \quad \quad [E_{\beta\alpha}, F_\alpha]_1 = q^{-1} E_\beta K_\alpha
\]

\[
[E_{\alpha\beta}, E_\alpha]_q = 0 \quad \quad [E_{\beta\alpha}, E_\alpha]_q = 0
\]

The comultiplication is given by:

\[
\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha \quad \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha
\]

\[
\Delta(E_{\alpha\beta}) = E_{\alpha\beta} \otimes 1 + K_{\alpha + \beta} \otimes E_{\alpha\beta} + (1 - q^{-2}) E_\alpha K_\beta \otimes E_\beta
\]

\[
\Delta(F_{\alpha\beta}) = F_{\alpha\beta} \otimes K_{\alpha + \beta}^{-1} + 1 \otimes F_{\alpha\beta} + (q^{-1} - q) F_\beta \otimes F_\alpha K_\beta^{-1}
\]

Now we want to classify all right coideal subalgebras of \( U_q(\mathfrak{sl}_3) \). Therefore we use the knowledge of the underlying paper and first specify all possible generating
elements of any RCS $C$ in $U_q(\mathfrak{sl}_3)$ with the property $C^0 := C \cap U^0$ is a sub Hopf-algebra. In the following we will assume $C$ to be an RCS of $U_q(\mathfrak{sl}_3)$ with this property.

Possible generating elements of an RCS of $U_q(\mathfrak{sl}_3)$. From Theorem 3.11 we know, that we can choose a generating system of $C$, whose elements have a very special form. We now want to specify these possible elements explicitly. In $U_q(\mathfrak{sl}_3)$ up to symmetry in $E$ and $F$ resp. $\alpha$ and $\beta$ we can give the complete list of all possible generating elements of $C$:

**List** (possible generating elements).

1. **Homogeneous elements in $U^{\geq 0}$**
   a. Elements in $U^0$
   b. Root vectors in $U^{\geq 0}$:
      - $E_\alpha K^{-1}_\alpha$
      - $E_{\alpha\beta} K^{-1}_{\alpha+\beta}$

2. **Character-shifted root vectors in $U^{\geq 0}$**:
   a. $E_{\alpha\beta} K^{-1}_{\alpha+\beta} + \lambda_\alpha (1 - q^2) E_\beta K^{-1}_{\alpha+\beta}$
      for $\lambda_\alpha \in k^*$
      - RCS that contain this element also contain $E_\alpha K^{-1}_\alpha + \lambda_\alpha K^{-1}_\alpha$
   b. $E_{\alpha\beta} K^{-1}_{\alpha+\beta} + \lambda_{\alpha\beta} K^{-1}_{\alpha+\beta}$
      for $\lambda_{\alpha\beta} \in k^*$
      - RCS that contain this element also contain $E_\alpha K^{-1}_\alpha$
   c. $E_\alpha K^{-1}_\alpha + \lambda_\alpha K^{-1}_\alpha$
      for $\lambda_\alpha \in k^*$

3. **Elements in $U$ with non trivial $F$- and $E$-leading term**
   a. $E_{\alpha\beta} K^{-1}_{\alpha+\beta} + \lambda^+_\alpha (1 - q^2) E_\beta K^{-1}_{\alpha+\beta} + c_F F_{\alpha\beta} + c_F (q^{-1} - q) \lambda^-_\alpha F_{\beta} K^{-1}_\alpha$
      for $c_F \in k^*$ and $\lambda^+_\alpha, \lambda^-_\alpha \in k$ with $\lambda^+_\alpha \lambda^-_\alpha = \frac{q^2}{(1 - q^2)(q^{-1} - q)}$
      - RCS that contain this element also contain $E_\alpha K^{-1}_\alpha + \lambda^+_\alpha K^{-1}_\alpha$ and $F_{\alpha} + \lambda^-_\alpha K^{-1}_\alpha$
   b. $E_{\alpha\beta} K^{-1}_{\alpha+\beta} + \lambda_\alpha (1 - q^2) E_\beta K^{-1}_{\alpha+\beta} + c_F F_{\alpha\beta} + c_F \lambda_\beta (q^{-1} - q) F_{\alpha} K^{-1}_{\alpha+\beta} + c_K K^{-1}_{\alpha+\beta}$
      for $c_F \in k^*$ and $\lambda^+_\alpha, \lambda^-_\beta, c_K \in k$ with $\lambda^+_\alpha \lambda^-_\beta c_K = \lambda^+_\alpha c_K = 0$
      - RCS that contain this element also contain $E_\alpha K^{-1}_\alpha + \lambda^+_\alpha K^{-1}_\alpha$ and $F_{\beta} + \lambda^-_\alpha K^{-1}_\beta$
   c. $E_{\alpha\beta} K^{-1}_{\alpha+\beta} + c_\alpha (1 - q^2) q^{-1} F_{\alpha} E_\beta K^{-1}_{\alpha+\beta} - c_\alpha c_\beta q^{-2} F_{\alpha\beta} + (1 - q^2) q^{-1} c'_\beta c_\alpha F_{\alpha} K^{-1}_{\alpha+\beta} + c'_\alpha (1 - q^2) E_\beta K^{-1}_{\alpha+\beta} + c'_\alpha c'_\beta (1 - q^{-1}) q^{-1} K^{-1}_{\alpha+\beta}$
      for $c_\alpha, c_\beta \in k^*$ and $c'_\alpha, c'_\beta \in k$ are either 0 or determined by $c_\alpha, c_\beta$
      - RCS that contain this element also contain $E_\alpha K^{-1}_\alpha + c_\alpha F_{\alpha} + c'_\alpha K^{-1}_\alpha$ and $E_\beta K^{-1}_\beta + c_\beta F_{\beta} + c'_\beta K^{-1}_\beta$
(d) $E_{\alpha} - (1 - q^2)c_\alpha E_\alpha F_\alpha K_\beta^{-1} - q c_\alpha c_\beta F_\alpha + \frac{q c_\alpha - c_\beta}{1 - q} K_\alpha^{-1}$

for $c_\alpha, c_\beta \in k^*$

- RCS that contain this element also contain $E_\beta K_\alpha^{-1} + c_\alpha F_\alpha$ and $E_\alpha K_\beta^{-1} + c_\beta F_\beta$ and $K_\alpha K_\beta^{-1}$

(e) $E_\alpha K_\beta^{-1} + c_F F_\beta$

for $c_F \in k^*$

- RCS that contain this element also contain $K_\beta^{-1} K_\alpha$

(f) $E_\alpha K_{\alpha+\beta}^{-1} + c_F F_\alpha$

for $c_F \in k^*$

- RCS that contain this element also contain $K_{\beta}^{-1}, F_\beta$

(g) $E_\alpha K_{\alpha}^{-1} + c_F F_\alpha + c_K K_{\alpha}^{-1}$

for $c_F \in k^*$ and $c_K \in k$

**Right coideal subalgebras of $U_q(\mathfrak{sl}_3)$**. We now want to give a complete list of all RCS of $U_q(\mathfrak{sl}_3)$ with the property $C^0 := C \cap U^0$ is a sub Hopfalgebra. Therefore we will first specify the homogeneous RCS ($U^0 \subset C$) in the positive Borel Part and give a classification of homogeneous RCS, as done in [HK11a]. This gives us easily a list of all homogeneous generated RCS of $U_q(\mathfrak{sl}_3)$. Then we will use [HK11b] for explicitly specifying all RCS in $U_q(\mathfrak{sl}_3^{\geq 0})$. We then give a complete list of all RCS which are generated by these so called character-shifted elements (i.e. generated by elements of type [4.2.3]).

Lastly, we use the results of this paper to consider general RCS by giving a list of all RCS which are generated by at least one element of type [4.2.3].

Homogeneous RCS of the positive Borel part of $U_q(\mathfrak{sl}_3)$. In the positive Borelpark of the $U_q(\mathfrak{sl}_3)$ lie up to isomorphism exactly six types of homogeneous RCS, i.e. one per Weyl group element:

- $U^0$
- $U^0(s_{\alpha}) U^0 = \langle E_\alpha, U^0 \rangle$
- $U^0(s_{\beta}) U^0 = \langle E_\beta, U^0 \rangle$
- $U^0(s_{\alpha} s_{\beta}) U^0 = \langle E_\alpha, E_{\alpha \beta}, U^0 \rangle$
- $U^0(s_{\beta} s_{\alpha}) U^0 = \langle E_\beta, E_{\beta \alpha}, U^0 \rangle$
- $U^0_{\geq 0}$

The RCS in $U^{\leq 0}$ are constructed analogously. If we want to construct homogeneous generated RCS which are not necessary homogeneous (in the sense $U^0 \subset C$), we can exchange $U^0[s]$ by $\psi(U^0[s])$ and $U^0$ by $T_L$ for some subgroup $L \subset \Phi$.

Homogeneous RCS of $U_q(\mathfrak{sl}_3)$. From [HK11b] we know all homogeneous RCS in $U$. If we consider RCS generated by homogeneous elements we get up to isomorphism and symmetry the following list of remaining homogeneous RCS:

- $\psi(U^0[s_{\alpha}]) T_L U^{-[s_{\alpha}]}$ for $K_\alpha^2 \subset L$
- $\psi(U^0[s_{\alpha} s_{\beta}]) T_L U^{-[s_{\alpha}]}$ for $K_\alpha^2 \subset L$
- $\psi(U^0[s_{\beta}]) T_L U^{-[s_{\alpha}]}$
- $\psi(U^0[s_{\beta} s_{\alpha}]) T_L U^{-[s_{\alpha}]}$ for $K_\alpha^2 \subset L$
RCS in the positive Borel part of $U_q(\mathfrak{sl}_3)$. The RCS $C$ in $U^{\geq 0}$ are by [HS09] generated via character shifting. We give here explicitly up to isomorphism all types of non homogeneous RCS in the positive Borel part of $U_q(\mathfrak{sl}_3)$. Consider first the connected RCS in $C$ (i.e. those for which $U^0 \cap C = k \cdot 1$).

There is the RCS $\langle E_a K_{\alpha}^{-1} \rangle_{\phi_a}$ with character $\phi_a$ defied by $\phi_a(E_a K_{\alpha}^{-1}) = \lambda_a$. Moreover there is the RCS $\langle E_a K_{\alpha}^{-1}, E_{a\beta} K_{\alpha+\beta}^{-1} \rangle_{\phi_a}$ with character defined by $\phi_a(E_a K_{\alpha}^{-1}) = \lambda_a$ and $\phi_a(E_{a\beta} K_{\alpha+\beta}^{-1}) = 0$, moreover there is the RCS $\langle E_a K_{\alpha}^{-1}, E_{a\beta} K_{\alpha+\beta}^{-1} \rangle_{\phi_{a\beta}}$ for a character $\phi_{a\beta}$ on $\langle E_a K_{\alpha}^{-1}, E_{a\beta} K_{\alpha+\beta}^{-1} \rangle$ with $\phi_{a\beta}(E_a K_{\alpha}^{-1}) = 0$ and $\phi_{a\beta}(E_{a\beta} K_{\alpha+\beta}^{-1}) = \lambda_{a\beta}$.

All RCS $C$ in $U_q(\mathfrak{sl}_3)^{\geq 0}$ have the form $C = C^+ T_L$ for a $C^+$ as above. Due to [HK11b] the following restriction holds for $T_L$: $L \subset \text{supp}(\phi)^{\perp}$. Thus in the cases above $T_L$ can have the following form: In the case of $\phi_a$ $T_L$ lies in $\langle K_{2\beta+a}, K_{2\beta+a} \rangle$. In the case of $\phi_{a+\beta}$ $T_L$ lies in $\langle K_{\alpha-\beta}, K_{\alpha-\beta}^{-1} \rangle$. The character-shifted RCS of the $U_q(\mathfrak{sl}_3)^{\leq 0}$ are constructed similarly.

To give now a complete list of RCS in $U_q(\mathfrak{sl}_3)$ we consider step by step RCS with generating elements in the list 12. In particular we list to each generating element all RCS up to symmetry in E and F resp. in $\alpha$ and $\beta$ which contain this element in their generating set, such that it cannot be written as a sum of other generating elements.

As we already know all homogeneous RCS, we can restrict to the case of an RCS $C$ which is not generated by only homogeneous elements, thus it contains at least one generating element from point 2) or 3) in [12]. First we consider those RCS which are generated only by generating elements from point 1) or 2), thus every generating element lies either in $U^{\geq 0}$ or in $U^{\leq 0}$. We list all possible RCS containing an element of 2) step by step up to symmetry in E and F resp. $\alpha$ and $\beta$.

2a) For an arbitrary $0 \neq \lambda_a, \lambda_{a}^{-1} \in k^*$ with $\lambda_a \lambda_{a}^{-1} = \frac{q^2}{(1-q^2)(q^2-q^{-2})}$ and $i \in \mathbb{N}$ there exist the following RCS:

- $\langle E_{a\beta} K_{\alpha+\beta}^{-1} + \lambda_a (1 - q^{-2}) E_{a} K_{\alpha}^{-1}, E_a K_{\alpha}^{-1} + \lambda_a K_{\alpha}^{-1}, (K_a K_{\beta}^{-1}) \rangle$
- $\langle E_{a\beta} K_{\alpha+\beta}^{-1} + \lambda_a (1 - q^{-2}) E_{a} K_{\alpha}^{-1}, E_a K_{\alpha}^{-1} + \lambda_a K_{\alpha}^{-1}, K_a K_{\beta}^{-1}, F_{\beta} \rangle$
- $\langle E_{a\beta} K_{\alpha+\beta}^{-1} + \lambda_a (1 - q^{-2}) E_{a} K_{\alpha}^{-1}, E_a K_{\alpha}^{-1} + \lambda_a K_{\alpha}^{-1}, K_a K_{\beta}^{-1}, F_{\beta}, F_{a\beta} \rangle$
- $\langle E_{a\beta} K_{\alpha+\beta}^{-1} + \lambda_a (1 - q^{-2}) E_{a} K_{\alpha}^{-1}, E_a K_{\alpha}^{-1} + \lambda_a K_{\alpha}^{-1}, (K_a K_{\beta}^{-1})^i, F_{a} + \lambda_a K_{\alpha}^{-1} \rangle$
4.2 3) in their generating set, such that it cannot be written as a sum of other
\[ c_2b) \text{ For an arbitrary } 0 \]
\[ 3a) \text{ For an arbitrary } 0 \]
\[ 3c) \text{ Of course there exist the RCS in 2a). For an arbitrary } 0 \]
\[ 2b) \text{ For an arbitrary } 0 \neq \lambda_\alpha \lambda_\beta \in k^* \text{ with } \lambda_\alpha \lambda_\beta = = \frac{q^2}{(1-q^2)(q-q^{-1})} \text{ and } i \in \mathbb{N} \text{ there exist the following RCS:} \]
\[ 2c) \text{ Of course there exist the RCS in 2a). For an arbitrary } 0 \neq \lambda_\alpha, \lambda_\beta \in k^* \text{ with } \lambda_\alpha \lambda_\beta = = \frac{q^2}{(1-q^2)(q-q^{-1})} \text{ and } i \in \mathbb{N} \text{ there exist moreover the following RCS:} \]
\[ \text{ Now we give a complete list of all RCS which contain at least one element from } \]
\[ 3a) \text{ For an arbitrary } 0 \neq \lambda_\alpha, \lambda_\beta \in k^* \text{ with } \lambda_\alpha \lambda_\beta = = \frac{q^2}{(1-q^2)(q-q^{-1})} \text{ there exist the following RCS:} \]
\[ \text{ 3b) For an arbitrary } c_F \neq 0 \text{ and } i \in \mathbb{N} \text{ there are two options:} \]
\[ \text{ 3c)} \text{ For } 0 \neq c_\alpha, c_\beta \in k^* \text{ and suitably chosen } c_\alpha', c_\beta' \text{ there is only one possible RCS given by} \]
\[ \text{ 3d) For an arbitrary } 0 \neq c_\alpha, c_\beta \in k^* \text{ there is only one possible RCS given by} \]
\[ \text{ 3e) Of course there is the RCS of (3d)). For an arbitrary } c_F \neq 0 \text{ there are three more types of RCS:} \]
Angular borel subalgebras in $U$ with characters as above and $\lambda$,

- Type 3: The RCS
  - $\mathcal{C}_3$: Of course there exist the RCS from 3c). For an arbitrary $0 \neq c_F \in k^*$ there are exactly two such Borel subalgebras. These are algebraic to $\psi$. $U$.
  - $\mathcal{C}_4$: For an arbitrary $0 \neq c_F \in k^*$ and $i \in \mathbb{N}$ there are moreover the following RCS:
    - $\langle E_\alpha K_{\alpha + \beta}^{-1} + c_F F_{\beta}, K_{\beta}^{-1}, F_{\beta}, E_{\beta}, E_{\beta} K_{\alpha}^{-1} - c_F F_{\alpha} \rangle$
  - $\mathcal{C}_5$: Of course there exist the RCS from 3c). For an arbitrary $0 \neq c_F \in k^*$ and $i \in \mathbb{N}$ there are moreover the following RCS:
    - $\langle E_\alpha K_{\alpha + \beta}^{-1} + c_F F_{\beta}, K_{\beta}^{-1}, F_{\beta}, E_{\beta}, E_{\beta} K_{\alpha}^{-1} - c_F F_{\alpha} \rangle$

As an application of the underlying paper we could classify in all Borel subalgebras (maximal RCS with the property that each simple finite dimensional representation is one dimensional) of $U_q(\mathfrak{sl}_3)$ with the knowledge of all RCS in $U_q(\mathfrak{sl}_3)$. Next we give the construction of the three types of Borel subalgebras which appear to be all possible Borel subalgebras due to the classification result from $\mathcal{V}$ocke(16).

Type 1: Standard Borel subalgebras. $U^{\geq 0}$ and $U^{\leq 0}$ are the so-called standard Borel subalgebras.

Type 2: RCS with a non degeneracy property. The Borel subalgebras $\psi(U^+[w^+])_{\phi^+} TLTU^-[w^-]_{\phi^-}$ with $\Phi^+(w^+) \cap \Phi^+(w^-) = supp(\phi^+) \cap supp(\phi^-)$ are up to symmetry isomorphic to algebra to $\psi(U^+[w^+])_{\phi^+} \langle K_{2\beta + \alpha}, K_{2\beta + \alpha}^{-1} U^-[s_\alpha]_{\phi^-}$ with $\phi^+(E_\alpha K_{\alpha}^{-1}) = \lambda_{\alpha}^+$ and 0 otherwise, as well $\phi^-(F_{\alpha}) = \lambda_{\alpha}^-$, such that $\lambda_{\alpha}^+ \lambda_{\alpha}^- = \frac{q^2}{(1-q)(q^{-1})}$. More precisely there are up to symmetry exactly two such Borel subalgebras. These are

$$\psi(U^+[w^+])_{\phi^+} \langle K_{2\beta + \alpha}, K_{2\beta + \alpha}^{-1} U^-[s_\alpha]_{\phi^-}$$

with characters as above and

$$\psi(U^+[s_\alpha s_\beta])_{\phi^+} \langle K_{\alpha - \beta}, K_{\alpha - \beta}^{-1} U^-[s_\beta s_\alpha]_{\phi^-}$$

with $\phi^+(E_\alpha K_{\alpha + \beta}^{-1}) = \lambda_{\alpha \beta}^+$ also $\phi^-(F_{\alpha}) = \lambda_{\alpha \beta}^-$, such that $\lambda_{\alpha \beta}^+ \lambda_{\alpha \beta}^- = \frac{q^2}{(1-q)(q^{-1})}$

Type 3: The RCS $\psi(U^+[s_\alpha s_\beta])_{\phi^+} \langle K_{2\beta + \alpha}, K_{2\beta + \alpha}^{-1} U^-[s_\alpha s_\beta]_{\phi^-}$. The third type of triangular Borel subalgebras in $U_q(\mathfrak{sl}_3)$ is of the form

$$\psi(U^+[s_\alpha s_\beta])_{\phi^+} \langle K_{2\beta + \alpha}, K_{2\beta + \alpha}^{-1} U^-[s_\alpha s_\beta]_{\phi^-}$$

with $\phi^+(E_\alpha K_{\alpha}^{-1}) = \lambda_{\alpha}^+$ and 0 otherwise, and $\phi^-(F_{\alpha}) = \lambda_{\alpha}^-$ and 0 otherwise, such that $\lambda_{\alpha}^+ \lambda_{\alpha}^- = \frac{q^2}{(1-q)(q^{-1})}$ as above.
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