A Hermite-Minkowski type theorem of varieties over finite fields

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We show the finiteness of étale coverings of a variety over a finite field with given degree whose ramification bounded along an effective Cartier divisor. The proof is an application of P. Deligne’s theorem [4] (H. Esnault and M. Kerz in Acta Math. Vietnam. 37:531–562, 2012) on a finiteness of $l$-adic sheaves with restricted ramification. By applying our result to a smooth curve over a finite field, we obtain a function field analogue of the classical Hermite-Minkowski theorem.

1 Introduction

For a number field $F$, that is, a finite extension of $\mathbb{Q}$, the Hermite-Minkowski theorem asserts that there exist only finitely many extensions of the number field $F$ with given degree unramified outside a finite set of primes of $F$ (e.g., [15], Chap. III, Thm. 2.13; [5], Chap. V, Thm. 2.6). In [5], G. Faltings gave a higher dimensional generalization of this theorem stated as follows:

**Theorem 1.1** ([5], Chap. VI, Sect. 2.4; [9], Thm. 2.9). Let $X$ be a connected scheme of finite type and dominant (e.g., flat) over $\text{Spec} \mathbb{Z}$. Then there exist only finitely many étale coverings of $X$ with given degree.

Here, an étale covering of $X$ means a finite étale morphism $X' \to X$. The aim of this note is to give a “function field” analogue of this theorem. We begin simple observations:

- For a function field $F$ of one variable over a finite field with characteristic $p$, the Artin-Schreier equations produce infinitely many extensions of $F$ of degree $p$ which ramify only in a finite set of places (see e.g., [7], Sect. 8.23).
- For a number field $F$, (the exponents of) the discriminant of an extension of $F$ has an upper bound depending on the extension degree and the primes at which it ramifies ([16], Chap. III, Sect. 6, Prop. 13, see also remarks after the proposition). Under the conditions in the Hermite-Minkowski theorem, namely, the extension
degree and a finite set of primes are given, the discriminants of extensions of $F$ are automatically bounded.

Considering these facts together, to obtain a finiteness as above in the case of function fields we have to restrict ramification.

Now, we present the results in this note more precisely. Let $X$ be a connected and separated scheme of finite type over a finite field (we call such schemes just varieties in the following cf. Notation), and $\overline{X}$ a compactification of $X$ (cf. Sect. 2). For an effective Cartier divisor $D$ with support $|D| \subset Z = \overline{X} \setminus X$, we will introduce the notion of bounded ramification along $D$ for étale coverings of $X$ (whose ramification locus is in the boundary $Z$) in the next section (Def. 2.2). Adopting this notion, we show the following theorem.

**Theorem 1.2** (Thm. 3.4). Let $X \subset \overline{X}$ be as above. There exist only finitely many étale coverings of $X$ with bounded degree and ramification bounded by a given effective Cartier divisor $D$ with support in $Z = \overline{X} \setminus X$.

A key ingredient for the proof is (a weak form of) Deligne’s finiteness theorem on smooth Weil sheaves with bounded ramification [4] (Thm. 3.3).

For an étale covering $X' \to X$ of smooth curves over a finite field, if its degree and the discriminant are bounded, then the ramification of the covering $X' \to X$ in our sense is also bounded (Prop. 2.9):

bounded degree & discriminant $\Rightarrow$ bounded ramification.

From this, we obtain an alternative proof of the following well-known theorem:

**Corollary 1.3** ([7], Thm. 8.23.5). Let $F$ be a function field of one variable over a finite field. Then there exist only finitely many separable extensions of $F$ with bounded degree and discriminant.

On the other hand, we have another implication

bounded degree & ramification $\Rightarrow$ bounded discriminant

(see Rem. 2.12). As a result, Thm. 1.3 is equivalent to the main theorem (Thm. 1.2) for the case where $X$ is a smooth curve over a finite field.

**Contents**

The contents of this note is the following:

- Sect. 2.
We define the notion of bounded ramification along an effective Cartier divisor $D$ for étale coverings of a variety $X$ over a finite field (Def. 2.1). We also introduce the fundamental group $\pi_1(X, D)$ which classifies such étale coverings of $X$ with bounded ramification along $D$ (Def. 2.2).

We define the Swan conductor of smooth $\mathbb{Q}_l$-sheaves as in [4] (Def. 2.14). In Lem. 2.17, we give a relation between the Swan conductor of smooth $\mathbb{Q}_l$-sheaf and our notion of the bounded ramification.

• Sect. 3:
  
  We interpret the Hermite-Minkowski type finiteness as above into a property of profinite groups called smallness which is studied in [9] (Def. 3.1).
  
  The proof of the main theorem (Thm. 3.4) is given by showing the smallness of the fundamental group $\pi_1(X, D)$.
  
  We also provide some applications of our main theorem to a finiteness of representations (with finite images) of the fundamental group $\pi_1(X, D)$ (Cor. 3.6).

**Notation**

In this note, a local field is a complete discrete valuation field with perfect residue field. For such a local field $K$, we denote by

- $O_K$: the valuation ring of $K$, and
- $v_K$: the valuation of $K$ normalized as $v_K(K^\times) = \mathbb{Z}$.

For a field $F$, we denote by

- $\overline{F}$: a separable closure of $F$, and
- $G_F := \text{Gal}(\overline{F}/F)$: the Galois group of $\overline{F}$ over $F$.

We also use the following notation:

- $p$: a fixed prime number,
- $l$: a prime number $\neq p$, and
- $k$: a finite field of characteristic $p$.

Throughout this note, we assume that $l$ is invertible in all schemes we consider. For a scheme $X$, an étale covering of $X$ we mean a finite étale morphism $X' \to X$. A variety over $k$ means a separated and connected scheme of finite type over $\text{Spec}(k)$. A curve over $k$ is a variety over $k$ with dimension 1. For an integral variety $X$ over $k$, we denote by

- $k(X)$: the function field of $X$.

Following [6], Sect. 3.2, we call a pair $(X, \overline{x})$ of a scheme and a geometric point $\overline{x}$ of $X$ a **pointed scheme**. A morphism $f : (X', \overline{x}') \to (X, \overline{x})$ of pointed schemes means a morphism $f : X' \to X$ of schemes with $f(\overline{x}') = \overline{x}$.
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2 Ramification

Adding to Notation, throughout this note, we use the following notation: For a variety $X$ over a finite field $k$ (cf. Notation),

- $\overline{X}$: a compactification of $X$ (over Spec($k$)) by Nagata’s theorem (cf. [14]), that is, a proper scheme over Spec($k$) which contains $X$ as a dense open subscheme,
- $Z := \overline{X} \setminus X$: the boundary of $X$,
- $\text{Cu}(X)$: the set of the normalizations of closed integral subschemes of $X$ of dimension 1, and
- $\text{Div}_Z^+(\overline{X})$: the monoid of effective Cartier divisors $D$ on $\overline{X}$ whose support $|D|$ is in $Z$.

For each $\phi : C \rightarrow X \in \text{Cu}(X)$, we denote by

- $\overline{C}$: the smooth compactification of $C$ (which exists uniquely by a resolution of singularities),
- $\overline{\phi} : \overline{C} \rightarrow \overline{X}$: the canonical extension of $\phi$ (by the valuative criterion of properness), and
- $k(C)_x$: the completion of the function field $k(C)$ at $x \in \overline{C}$ which is a local field in our sense.

For a finite Galois extension $L/K$ of local fields, we use the following ramification filtrations (cf. [16], Chap. IV, Sect. 3):

- $(\text{Gal}(L/K)_\mu)_{\mu \in \mathbb{R}_{\geq -1}}$: the ramification filtration of $\text{Gal}(L/K)$ in the lower numbering which is given by
  $$\text{Gal}(L/K)_\mu = \{ \sigma \in \text{Gal}(L/K) \mid v_L(\sigma(\theta) - \theta) \geq \mu + 1 \},$$  
  (1)
  where $\theta \in L$ is a generator of the valuation ring $O_L$ as an $O_K$-algebra: $O_L = O_K[\theta]$ ([16], Chap. III, Sect. 6, Prop. 12). For each $\mu \in \mathbb{R}_{\geq -1}$, $\text{Gal}(L/K)_\mu$ is a normal subgroup of $\text{Gal}(L/K)$. In particular, it is known that $\text{Gal}(L/K)_1$ is the $p$-Sylow subgroup of the inertia subgroup $\text{Gal}(L/K)_0$, where $p$ is the characteristic of the residue field of $K$.
- $(\text{Gal}(L/K)^\lambda)_{\lambda \in \mathbb{R}_{\geq -1}}$: the ramification filtration of $\text{Gal}(L/K)$ in the upper numbering which is defined by the relation
  $$\text{Gal}(L/K)^{\lambda \mu} = \text{Gal}(L/K)_\mu,$$
  (2)
\begin{align}
\varphi_{L/K} (\mu) = \int_0^{\mu} \frac{dx}{(\text{Gal}(L/K)_0 : \text{Gal}(L/K)_x)} \quad (3)
\end{align}

and its inverse function is denoted by \( \psi_{L/K} \).

We note here, these ramification filtrations satisfy the following compatibility properties:

\textbf{Lemma 2.1} (\cite{16}, Chap. IV, Sect. 1, Prop. 2, and Sect. 3, Prop. 14). \textit{Let} \( L/K \) \textit{be a finite Galois extension of local fields.}

(i) \textit{For a sub extension} \( K'/K \) \textit{of} \( L \), \textit{we have}

\[ \text{Gal}(L/K')_{\mu} = \text{Gal}(L/K)_{\mu} \cap \text{Gal}(L/K'), \]

\textit{for any} \( \mu \in \mathbb{R}_{\geq 1} \).

(ii) \textit{For a sub Galois extension} \( K'/K \) \textit{of} \( L \), \textit{we have}

\[ \text{Gal}(K'/K)^{\lambda} = \text{Gal}(L/K)^{\lambda} \text{Gal}(L/K')/\text{Gal}(L/K'), \]

\textit{for any} \( \lambda \in \mathbb{R}_{\geq 1} \).

For a local field \( K \), from Lem. 2.1 (ii) one can introduce

- \( (G^\lambda_K)_{\lambda \in \mathbb{R}_{\geq 1}} \) : the ramification filtration of \( G_K = \text{Gal}(\overline{K}/K) \) (in the upper numbering) which is given by

\[ G^\lambda_K = \lim_{L/K: \text{finite Galois} \subset \overline{K}} \text{Gal}(L/K)^{\lambda}, \quad (4) \]

and

- \( G^{\lambda+}_K \): the topological closure of \( \bigcup_{\lambda' > \lambda} G^{\lambda'}_K \) in \( G_K \) for \( \lambda \in \mathbb{R}_{\geq 0} \).

\textbf{Étale coverings}

\textbf{Definition 2.2} (\textit{cf.} \cite{9}, Def. 3.2). (i) \textit{Let} \( K \) \textit{be a local field. For a separable field extension} \( L/K \) \textit{(contained in} \( \overline{K} \) \textit{) and} \( \lambda \in \mathbb{R}_{\geq 0} \), \textit{we say that the ramification of} \( L/K \) \textit{is bounded by} \( \lambda \) \textit{if we have} \( G^{\lambda+}_K \subset G^\lambda_L = \text{Gal}(\overline{K}/L) \).

(ii) \textit{Let} \( X \) \textit{be a smooth curve over} \( k \), \textit{and} \( \overline{X} \) \textit{the smooth compactification of} \( X \). \textit{Let} \( D = \sum_{z \in Z} m_z [z] \in \text{Div}^+_\mathbb{Z}(\overline{X}) \) (\( m_z \in \mathbb{Z}_{\geq 0} \)). \textit{An étale covering} \( X' \to X \) \textit{is said to be of ramification bounded by} \( D \) \textit{if the extension} \( k(X')_{\mathcal{Z}'}/k(X)_{\mathcal{Z}} \) \textit{of local fields is of ramification bounded by} \( m_z \) \textit{for all} \( z \in Z \) \textit{(putting} \( m_z = 0 \) \textit{if} \( z \notin |D| \)) \textit{and for all} \( z' \in \overline{X} \) \textit{above} \( z \).
(iii) Let $X$ be a variety over $k$. For $D \in \text{Div}_k^+(\overline{X})$, an étale covering $X' \to X$ is said to be of ramification bounded by $D$ if for every $\phi : C \to X \in \text{Cu}(X)$ and for each irreducible component $C'$ of $C \times_X X'$, the ramification of the induced morphism $C' \to C$ is bounded by $\overline{\phi}^* D$, where $\overline{\phi}^* D$ is the inverse image of $D$ by $\overline{\phi} : C' \to \overline{X}$ (for the existence of $\overline{\phi}^* D$, see Lem. [2.7] below).

**Lemma 2.3.** Let $L/K$ be a finite separable extension of local fields.

(i) The extension $L/K$ is tamely ramified if and only if it is of ramification bounded by 0.

(ii) We denote by $\tilde{L}$ the Galois closure of $L/K$. For any $\lambda \in \mathbb{R}_{>0}$, the extension $L/K$ is of ramification bounded by $\lambda$ if and only if $\tilde{L}/K$ is of ramification bounded by $\lambda$.

(iii) Assume that the extension $L/K$ is Galois. For any $\lambda \in \mathbb{R}_{>0}$, the ramification of $L/K$ is bounded by $\lambda$ if and only if $\text{Gal}(L/K)^\lambda = \{1\}$ for any $\lambda' > \lambda$.

**Proof.** (ii) The “if” part follows immediately from the definition. We show the “only if” part. Assume that $L/K$ is of ramification bounded by $\lambda$ for some $\lambda \in \mathbb{R}_{>0}$. Namely, we have $G_K^{\lambda+} \subset G_L$. Since $G_L = \text{Gal}(\overline{K}/\overline{L})$ is the maximal normal subgroup of $G_K$ which is contained in $G_L$, we have $G_K^{\lambda+} \subset G_L$.

(iii) The restriction $G_K \to \text{Gal}(L/K) : \sigma \mapsto \sigma|_L$ induces $G_K^{\lambda} \to \text{Gal}(L/K)^\lambda$ from the very definition of $G_K^{\lambda}$ in (1) for any $\lambda$. For any $\lambda' > \lambda$, $G_K^{\lambda'} \subset G_L$ if and only if $\text{Gal}(L/K)^{\lambda'} = \{1\}$, and the assertion follows from it.

(i) By taking the Galois closure of the extension $L/K$ and using (ii), we may assume that $L/K$ is a Galois extension. Let $p$ be the characteristic of the residue field of $K$. As $\bigcap_{\lambda>0} \text{Gal}(L/K)^\lambda = \text{Gal}(L/K)_1$ is the $p$-Sylow subgroup of the inertia subgroup $\text{Gal}(L/K)_0$, the extension $L/K$ is tamely ramified if and only if $\text{Gal}(L/K)^\lambda = \{1\}$ for any $\lambda > 0$. The assertion (i) follows from (iii). \hfill \Box

For a pointed connected Noetherian scheme $(X, \pi)$ (cf. Notation), we define

- $\text{Cov}(X)$: the Galois category of étale coverings of $X$,
- $\pi_1(X, \pi)$: the fundamental group of $(X, \pi)$ associated to $\text{Cov}(X)$, which is defined by $\pi_1(X, \pi) = \varprojlim_{(X', \pi') \in \text{Cov}(X)} \text{Aut}_X(X')^{\text{op}}$,

where the projective limit is taken over a projective system of pointed Galois coverings $(X', \pi') \to (X, \pi)$ in $\text{Cov}(X)$ [8], Exp. V, Sect. 7), and

- $(\pi_1(X, \pi)$-sets): the category of finite sets on which $\pi_1(X, \pi)$ acts continuously on the left.
The fiber functor $Y \mapsto \text{Hom}_X(\pi, Y)$ gives an equivalence of categories

\[ \text{Cov}(X) \xrightarrow{\cong} (\pi_1(X, \pi)-\text{sets}) \]  

([6], Exp. V, Prop. 5.8; [9], Thm. 3.2.12).

Now, we assume that $X$ is a variety over $k$ (cf. Notation). For $D \in \text{Div}_{\mathbb{Z}}^+(\overline{X})$, the full subcategory $\text{Cov}(X \subset \overline{X}, D) \subset \text{Cov}(X)$ of étale coverings of $X$ with ramification bounded by $D$ also forms a Galois category (see [9], Lem. 3.3).

**Definition 2.4.** Associated to the Galois category $\text{Cov}(X \subset \overline{X}, D)$, we define the fundamental group by

\[ \pi_1(X \subset \overline{X}, D, \pi) = \varprojlim_{(X', \overline{X}) \in \text{Cov}(X \subset \overline{X}, D)} \text{Aut}_X(X')^{\text{op}}, \]

where the projective limit is taken over a projective system of pointed Galois coverings $(X', \overline{X}) \rightarrow (X, \pi)$ in $\text{Cov}(X \subset \overline{X}, D)$.

In the following, we write $\pi_1(X, D, \pi)$ and $\text{Cov}(X, D)$ when we need not specify the fixed compactification $\overline{X}$ of $X$. For the geometric point $\pi : \text{Spec}(\Omega) \rightarrow X$ with a separably closed field $\Omega$, we denote also by $\pi$ the geometric point $\text{Spec}(\Omega) \rightarrow X$ of $\overline{X}$. The functors $\text{Cov}(\overline{X}) \rightarrow \text{Cov}(X, D); \overline{Y} \mapsto \overline{Y} \times_{\overline{X}} X$, and $\text{Cov}(X, D) \rightarrow \text{Cov}(X); Y \mapsto Y$ induce homomorphisms

\[ \pi_1(X, \pi) \rightarrow \pi_1(X, D, \pi) \rightarrow \pi_1(\overline{X}, \pi) \]  

which are surjective ([8], Exp. V, Prop. 6.9).

**Lemma 2.5.** Let $f : (X', \overline{X}) \rightarrow (X, \pi)$ be a morphism of pointed varieties over $k$, and $D \in \text{Div}_{\mathbb{Z}}^+(\overline{X})$. Assume the conditions (a) and (b) below.

(a) There exists a commutative diagram:

\[
\begin{array}{ccc}
\overline{X'} & \xrightarrow{\overline{f}} & \overline{X} \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f} & X
\end{array}
\]

with a morphism $\overline{f} : \overline{X'} \rightarrow \overline{X}$ from a compactification $\overline{X'}$ of $X'$ to $\overline{X}$, where the vertical morphisms are the inclusions.

(b) The inverse image $\overline{f}^* D \in \text{Div}_{\mathbb{Z}}^+(\overline{X'})$ of $D$ exists, where $Z' = \overline{X'} \setminus X'$.

Then we have a canonical homomorphism

\[ \pi_1(X' \subset \overline{X'}, \overline{f}^* D, \overline{\pi}) \rightarrow \pi_1(X \subset \overline{X}, D, \pi). \]
Proof. We prove that the functor $\text{Cov}(X) \to \text{Cov}(X')$; $Y \mapsto Y \times_X X'$ induces a functor $\text{Cov}(X \subset \overline{X}, D) \to \text{Cov}(X' \subset \overline{X'}, f^* D)$. The latter functor gives the required homomorphism on the fundamental groups ([8], Exp. V, Sect. 6). For $h : Y \to X \in \text{Cov}(X \subset \overline{X}, D)$, it is enough to show that $Y' := Y \times_X X' \to X'$ is of ramification bounded by $f^* D$. Take any $\phi' : C' \to X' \in \text{Cu}(X')$, and we have to show that $h' : Y' \times_X C' \to C'$ gives étale coverings of ramification bounded by $\overline{\sigma}^*(f^* D)$. Here, we divide the proof into the two cases according to the image $f \circ \phi'(C')$ in $X$.

(The case where $f \circ \phi'(C')$ is a point) If the image of the composite $f \circ \phi'$ is a closed point of $X$, then $h' : Y' \times_X C' \to C'$ induces separable constant field extensions of $k(C')$, and thus unramified on the boundary of $C'$. In particular, $h'$ is of ramification bounded by $\overline{\sigma}^*(f^* D)$.

(The case where $f \circ \phi'(C')$ is a curve) If $f \circ \phi'$ factors through $\phi : C \to X \in \text{Cu}(X)$ as in

$$
\begin{array}{ccc}
C' & \xrightarrow{\phi'} & X' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
C & \xrightarrow{\phi} & X,
\end{array}
$$

then the extension $\overline{\sigma} : \overline{C'} \to \overline{C}$ of $g$ fits into the following commutative diagram:

$$
\begin{array}{ccc}
\overline{C'} & \xrightarrow{\overline{\phi'}} & \overline{X'} \\
\downarrow{\scriptstyle \overline{\sigma}} & & \downarrow{\scriptstyle \overline{\rho}} \\
\overline{C} & \xrightarrow{\overline{\phi}} & \overline{X}.
\end{array}
$$

Since $h : Y \to X$ is of ramification bounded by $D$, the base change $Y \times_X C \to C$ produces étale coverings of $C$ which are of ramification bounded by $\overline{\sigma}^* D$. It is left to show that $h' : Y' \times_X C' = Y \times_X C' \to C'$ is of ramification bounded by $\overline{\sigma}^* (f^* D) = \overline{\sigma}^* (h^* D)$. As a result, we may assume that $f : X' \to X$ is a dominant morphism of smooth curves over $k$.

Let $D = \sum z \in \text{Div}^+(\overline{X})$, and $Y \in \text{Cov}(X \subset \overline{X}, D)$. Take an irreducible component $Y'$ of $Y \times_X X'$. We prove that the induced covering $Y' \to X'$ is of ramification bounded by $f^* D$. At $z' = \overline{Z}', \overline{X}'$ with $\overline{f}(z') = z$, the multiplicity of $f^* D$ is $e_z m_z$, where $e_z$ is the ramification index of the extension $k(X')_{z'}/k(X)_z$. The assertion carries over to the local situation as in the following lemma.

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**Lemma 2.6.** Let $K$ be a local field, and $K'/K$ a finite extension with ramification index $e$. If a finite separable extension $L/K$ is of ramification bounded by $\lambda$ for some $\lambda \in \mathbb{R}_{>0}$, then $LK'/K'$ is of ramification bounded by $e\lambda$.

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**Proof. (Reduction to $L/K$ is Galois)** We denote by $\bar{L}$ the Galois closure of $L/K$. From Lem. 2.3 (ii), $\bar{L}/K$ is of ramification bounded by $\lambda$. On the other hand, if $\bar{L}K'/K'$
is of ramification bounded by $e\lambda$, so is $LK'/K'$ by Lem. 2.3 (ii) again. We may assume that $L/K$ is a Galois extension.

(Proof of the lemma) We put $L' = LK'$ and consider the two cases below:

Case 1 ($K'/K$ is purely inseparable): Assume that $K'/K$ is purely inseparable. In this case, we show that $L'/K'$ is of ramification bounded by $\lambda$. Since we have $K' \cap L = K$, the restriction gives an isomorphism $\text{Gal}(L'/K') \cong \text{Gal}(L/K)$ and hence $[K' : K] = [L' : L] = e$. As the local field $K$ is of characteristic $p > 0$ with perfect residue field, we have $(K')^e = K$ and $(L')^e = L$. Take $\theta \in O_L$ such that $O_L = O_K[\theta]$ ([10], Chap. III, Sect. 6, Prop. 12) and this gives $O_L = O_K[\theta^e]$. For any $\sigma \in \text{Gal}(L'/K')$, we have

$$e \cdot v_L'(\sigma(\theta) - \theta) = v_L'(\sigma(\theta^e) - \theta^e) \quad (\text{since } e \text{ is a power of } p)$$

Thus, the restriction $\text{Gal}_{\mathbb{P}}$ gives an isomorphism $\text{Gal}_{\mathbb{P}}$. This implies that the extension $L'/K'$ is of ramification bounded by $\lambda$ (Lem. 2.3 (iii)).

Case 2 ($K'/K$ is separable): Assume that $K'/K$ is a separable extension. We show that $L'/K'$ is of ramification bounded by $e\lambda$. Take a finite Galois extension $M/K$ with $L' \subset M$. The Herbrand function $\psi_{K'/K}$ of the separable extension $K'/K$ is defined to be $\psi_{K'/K} = \varphi_{M/K'} \circ \psi_{M/K}$ ([10], Chap. IV, Sect. 3, Rem. 2) and it satisfies $\psi_{K'/K}(\mu/e) \leq \mu$ for any $\mu \in \mathbb{R}_{\geq -1}$. In fact,

$$\varphi_{K'/K}(\mu) = \varphi_{M/K'} \circ \psi_{M/K'}(\mu) \geq \frac{1}{e} \varphi_{M/K} \circ \psi_{M/K}(\mu) = \frac{\mu}{e},$$

where the inequality (1) follows from $\# \text{Gal}(M/K) = e \# \text{Gal}(M/K')_0$ and Lem. 2.1 (i). For any $\lambda' > e\lambda$, we have

$$\text{Gal}(M/K')^{\lambda'} \subset \text{Gal}(M/K')^{\psi_{K'/K}(\lambda'/e)} \quad (\text{by } \psi_{K'/K}(\lambda'/e) \leq \lambda')$$

$$= \text{Gal}(M/K')^{\psi_{M/K}(\lambda'/e)} \quad (\text{by Lem. 2.1 (ii)})$$

$$= \text{Gal}(M/K)^{\psi_{M/K}(\lambda'/e)} \cap \text{Gal}(M/K') \quad (\text{by Lem. 2.1 (i)})$$

$$\subset \text{Gal}(M/K)^{\lambda'/e} \quad (\text{by Lem. 2.1 (ii)})$$

$$\subset \text{Gal}(M/L) \quad (\text{by } \lambda'/e > \lambda \text{ and } L/K \text{ is of ramification bounded by } \lambda).$$

As a result, we have $\text{Gal}(M/K')^{\lambda'} \subset \text{Gal}(M/L) \cap \text{Gal}(M/K') = \text{Gal}(M/L')$ and hence

$$\text{Gal}(L'/K')^{\lambda'} \subset \text{Gal}(M/L') / \text{Gal}(M/L) = \{1\}$$

by Lem. 2.1 (ii). This implies the assertion by Lem. 2.3 (iii).

(Proof of the lemma – continued) We take the separable closure $K^s$ of $K$ within $K'$. From Case 2 above, the extension $LK^s/K^s$ is of ramification bounded by $e\lambda$. Applying Case 1 to the purely inseparable extension $K'/K^s$, the extension $L' = LK^s/K'$ is of ramification bounded by $e\lambda$ as required. □
Concerning the condition \( (b) \) in Lem. 2.5, the lemma below assures the existence of the inverse image of an effective Cartier divisor in some situations.

**Lemma 2.7** (cf. [33], Chap. IV, Sect. 21.4). Let \( f : X' \to X \) be a morphism of varieties over \( k \), and \( D \) an effective Cartier divisor on \( X \). Then the inverse image \( f^*D \) is defined in each of the following conditions:

(a) \( f \) is flat,
(b) \( X', X \) are integral, and \( f \) is a dominant morphism,
(c) \( X' \) is reduced, and for the generic point \( \xi' \in X' \) of any irreducible component of \( X' \), we have \( f(\xi') \notin |D| \), and
(d) \( f \) is a blowing up of \( X \) along a closed subscheme of \( X \).

For a normal variety \( X \) over \( k \), take a separably closed field \( \Omega \) as \( k(X) \subset \Omega \). We denote by \( \xi \) both the geometric point \( \text{Spec}(\Omega) \to \text{Spec}(k(X)) \) which is given by \( k(X) \subset \Omega \), and the geometric point \( \text{Spec}(\Omega) \to \text{Spec}(k(X)) \to X \). The map \( \text{Spec}(k(X)) \to X \) induces a canonical and surjective homomorphism

\[
G_{k(X)} = \text{Gal}(k(X)/k(X)) \simeq \pi_1(\text{Spec}(k(X)), \xi) \longrightarrow \pi_1(X, \xi),
\]

where \( k(X) \) is the separable closure of \( k(X) \) in \( \Omega \) ([8] Exp. V, Prop. 8.2; [9], Prop. 3.3.6). Its kernel is the subgroup \( \text{Gal}(k(X)/k(X)_Z) \), where \( k(X)_Z \) is the maximal extension of \( k(X) \) unramified outside \( Z \), that is, the subfield of \( k(X) \) generated by all finite separable extensions \( E \) of \( k(X) \) contained in \( k(X) \) satisfying that the normalization \( X^E \to X \) of \( X \) in \( E \) is unramified. In particular, the surjective map (7) induces an isomorphism \[
\text{Gal}(k(X)/k(X)) \simeq \pi_1(X, \xi).
\]

In the same way, the fundamental group \( \pi_1(X, D, \xi) \) also has a description

\[
\text{Gal}(k(X)_D/k(X)) \simeq \pi_1(X, D, \xi).
\]

Here, \( k(X)_D \) is the subfield of \( k(X)_Z \) generated by all finite separable extensions \( E \) of \( k(X) \) contained in \( k(X)_Z \) satisfying that the normalization \( X^E \to X \) is of ramification bounded by \( D \). When \( X \) is a smooth curve, it is easy to describe the kernel of the map \( \pi_1(X, \xi) \to \pi_1(X, D, \xi) \) given in [10] explicitly as follows:

**Lemma 2.8.** Let \( X \) be a smooth curve over \( k \), and \( \overline{X} \) the smooth compactification of \( X \). We take a geometric point \( \xi \) of \( X \) and a separable closure \( k(X) \) as above. For \( D = \sum m_z[z] \in \text{Div}_{\overline{X}}^{\text{bot}}(\overline{X}) \), we denote by \( N \) the normal closed subgroup of \( \pi_1(X, \xi) \) generated by the image of the ramification subgroup \( G_z^{n+} \) of \( G_z := \text{Gal}(k(X)_z/k(X)_z) \) by the homomorphism \( G_z \to \pi_1(X, \xi) \) which is given by choosing an embedding \( k(X) \hookrightarrow k(X)_z \) over \( k(X) \) and their conjugates in \( \pi_1(X, \xi) \), for all \( z \in Z \). Then we have

\[
\pi_1(X, \xi)/N \overset{\cong}{\longrightarrow} \pi_1(X, D, \xi).
\]
For each finite étale morphism \( f : X' \to X \) of smooth curves over \( k \), let \( \overline{f} : \overline{X'} \to \overline{X} \) be the canonical extension of \( f \) to the smooth compactifications. We denote by \( D_{X'/X} \) the discriminant for the extension of the function fields \( k(X')/k(X) \) ([16], Chap. III, Sect. 3). In the following, we write the discriminant additively as a divisor

\[
D_{X'/X} = \sum_{x \in X} v_x(D_{X'/X})[x]
\]

with the multiplicity \( v_x(D_{X'/X}) \in \mathbb{Z}_{\geq 0} \) at \( x \). The ramification locus of \( \overline{f} \) coincides with the support \( |D_{X'/X}| = \{ x \in \overline{X} \mid v_x(D_{X'/X}) > 0 \} \) ([16], Chap. III, Sect. 5, Cor. 1) so that one can consider the discriminant \( D_{X'/X} \) as a Cartier divisor with support in \( Z = \overline{X} \setminus X \).

**Proposition 2.9.** Let \( f : X' \to X \) be a finite étale morphism of smooth curves over \( k \), and \( \overline{X} \) the smooth compactification. If we have \( D_{X'/X} \leq D \) for some \( D \in \text{Div}_Z^+(\overline{X}) \), then there exists \( D' \in \text{Div}_Z^+(\overline{X}) \) which depends only on \( D \) and the degree of \( f \) such that the ramification of \( f : X' \to X \) is bounded by \( D' \).

To show this proposition, we prepare the following notation and a lemma: For a finite separable extension \( K'/K \) of local fields, we denote by

- \( D_{K'/K} \subset O_K \): the discriminant of \( K'/K \), and
- \( \mathfrak{D}_{K'/K} \subset O_{K'} \): the different of \( K'/K \) (written multiplicatively as usual)

(cf. [16], Chap. III, Sect. 3).

**Lemma 2.10.** Let \( K \) be a local field. Let \( K_1 \) and \( K_2 \) be two finite separable extensions of \( K \), and \( L = K_1K_2 \) the compositum. As ideals in the valuation ring \( O_L \), we have

\[
\mathfrak{D}_{L/K} \supset \mathfrak{D}_{K_1/K} \mathfrak{D}_{K_2/K}.
\]

**Proof.** Take \( \theta \in O_{K_1} \) such that \( O_{K_1} = O_K[\theta] \) ([16], Chap. III, Sect. 6, Prop. 12). Let \( f \in O_K[T] \) be the minimal polynomial for \( \theta \) over \( K \). From the equality \( K_1 = K(\theta) \), we have \( L = K_1K_2 = K_2(\theta) \). If we denote by \( g \in O_{K_2}[T] \) the minimal polynomial for \( \theta \) over \( K_2 \), one can write \( f = gh \) in \( O_{K_2}[T] \) for some \( h \in O_{K_2}[T] \). Hence,

\[
\mathfrak{D}_{K_1/K}O_L = f'(\theta)O_L \quad \text{(by [16], Chap. III, Sect. 6, Cor. 2)}
\]

\[
= g'(\theta)h(\theta)O_L \quad \text{(by } f = gh \text{)}
\]

\[
\subset g'(\theta)O_L
\]

\[
\subset \mathfrak{D}_{L/K_2} \quad \text{(by [16], Chap. III, Sect. 6, Cor. 2 again).}
\]

From the transitivity of the differentials ([16], Chap. III, Sect. 4, Prop. 8), we obtain

\[
\mathfrak{D}_{L/K} = \mathfrak{D}_{L/K_2} \mathfrak{D}_{K_2/K} \supset \mathfrak{D}_{K_1/K} \mathfrak{D}_{K_2/K}
\]

as ideals in \( O_L \).
Hence, \( \text{Gal} \) or any \( \sigma \) Hilbert formula (\cite{16}, Chap. IV, Sect. 2, Prop. 4) gives
\[
\lambda
\]
On the other hand, for any \( \theta \), we obtain
\[
(\cite{16}, \text{Chap. III, Prop. 6}),
\]
where \( v_\lambda(D_k(\sigma)) \) is the valuation of the discriminant \( D_k(\sigma) \) at \( z \) (\cite{15}, Chap. III, Sect. 2, Cor. 2.11). It is enough to show the following lemma. \( \square \)

**Lemma 2.11.** Let \( K'/K \) be a separable extension of local fields with \( [K':K] = n \). We denote by \( v_K(D_{K'/K}) \) the valuation of the discriminant \( D_{K'/K} \). If \( v_K(D_{K'/K}) \leq m \) for some \( m \in \mathbb{Z}_{\geq 0} \), then there exists \( m' \in \mathbb{Z}_{\geq 0} \) which depends only on \( m \) and \( n \) such that the extension \( K'/K \) is of ramification bounded by \( m' \).

**Proof.** \textbf{(Reduction to Galois)} Let \( K_1, \ldots, K_n \) be the conjugate fields of \( K' = K_1 \). We denote by \( L = K_1 \cdots K_n \) the Galois closure of \( K'/K \). By using Lem. 2.10 repeatedly, we have \( \mathfrak{D}_{L/K} \supset \mathfrak{D}_{K_1/K} \cdots \mathfrak{D}_{K_n/K} \) as ideals in \( \mathcal{O}_L \). From the equality \( D_{L/K} = N_{L/K}(\mathfrak{D}_{L/K}) \) (\cite{16}, Chap. III, Prop. 6), we obtain
\[
v_K(D_{L/K}) = n!v_K(D_{K'/K}) \leq nnm.
\]
On the other hand, for any \( \lambda \in \mathbb{R}_{\geq 0} \), the ramification of \( K'/K \) is bounded by \( \lambda \) if and only if \( L/K \) is of ramification bounded by \( \lambda \) (Lem. 2.3 (ii)). We may assume that \( K'/K \) is a Galois extension.

**Proof of the lemma** It is enough to show \( \text{Gal}(K'/K)^\lambda = \{1\} \) for any \( \lambda > m \) (Lem. 2.3 (iii)). Take \( \theta \in \mathcal{O}_{K'} \) such that \( O_{K'} = O_K[\theta] \) (\cite{16}, Chap. III, Sect. 6, Prop. 12). The Hilbert formula (\cite{16}, Chap. IV, Sect. 2, Prop. 4) gives
\[
v_K(\mathfrak{D}_{K'/K}) = \sum_{\sigma \in \text{Gal}(K'/K) \setminus \{1\}} v_{K'}(\sigma(\theta) - \theta).
\]
For any \( \sigma \neq 1 \) in \( \text{Gal}(K'/K) \), we have the following inequalities
\[
m \geq v_K(D_{K'/K}) \quad \text{(from the assumption)}
\geq v_{K'}(\mathfrak{D}_{K'/K}) \quad \text{(by } D_{K'/K} = N_{K'/K}(\mathfrak{D}_{K'/K})\text{)}
\geq v_{K'}(\sigma(\theta) - \theta) \quad \text{(by } \mathcal{S}\text{)}.
\]
Hence, \( \text{Gal}(K'/K)^\lambda = \text{Gal}(K'/K)^{\lambda(1)} \subset \text{Gal}(K'/K)_\lambda = \{1\} \) for any \( \lambda > m \). \( \square \)

**Remark 2.12.** For a finite étale morphism \( f : X' \to X \) of smooth curves over \( k \), by using the Hilbert’s formula (\cite{3}) locally as in Prop. 2.4, we have the following proposition: If the ramification of \( f : X' \to X \) is bounded by \( D \) for some \( D \in \text{Div}_X^+(\mathcal{X}) \), then there exists \( D' \in \text{Div}_X^+(\mathcal{X}) \) which depends only on \( D \) and the degree of \( f \) such that \( D_{X'/X} \leq D' \).
\textbf{l-adic sheaves}

For a pointed connected Noetherian scheme \((X, \overline{x})\) (in which the fixed prime \(l\) is invertible \textit{cf.} Notation), we have an equivalence of categories (\cite{6}, Cor. 10.1.24):

\[
(\text{smooth } \mathbb{Q}_l\text{-sheaves on } X) \xrightarrow{\sim} (\mathbb{Q}_l\text{-representations } \pi_1(X, \overline{x}) \to \text{Aut}(V)). \quad (9)
\]

Here, a \(\mathbb{Q}_l\)-representation we mean a continuous homomorphism \(\rho : \pi_1(X, \overline{x}) \to \text{Aut}(V)\) with a finite dimensional \(\mathbb{Q}_l\)-vector space \(V\) (for the precise definition, see \cite{2}, 1.1.6, or \cite{6}, Sect. 10.1). The equivalence above is given by \(\mathcal{F} \mapsto \mathcal{F}_{\overline{x}} = V\), where \(\mathcal{F}_{\overline{x}}\) is the stalk of \(\mathcal{F}\) at \(\overline{x}\).

\textbf{Lemma 2.13.} Let \(f : (X', \overline{x}') \to (X, \overline{x})\) be a morphism of pointed connected Noetherian schemes, and \(\mathcal{F}\) a smooth \(\mathbb{Q}_l\)-sheaf on \(X\). Identifying the isomorphism \(\mathcal{F}_{\overline{x}} \simeq (f^* \mathcal{F})_{\overline{x}'}\) on stalks, the \(\mathbb{Q}_l\)-representation \(\rho' : \pi_1(X', \overline{x}') \to \text{Aut}((f^* \mathcal{F})_{\overline{x}'})\) corresponding to \(f^* \mathcal{F}\) makes the following diagram commutative:

\[
\begin{array}{ccc}
\pi_1(X, \overline{x}) & \xrightarrow{\rho} & \text{Aut}(\mathcal{F}_{\overline{x}}) \\
\varphi & \nearrow & \| \\
\pi_1(X', \overline{x}') & \xrightarrow{\rho'} & \text{Aut}((f^* \mathcal{F})_{\overline{x}'})
\end{array}
\]

where \(\varphi\) is the induced homomorphism of fundamental groups from \(f\), and \(\rho\) is the representation corresponding to \(\mathcal{F}\).

\textit{Proof.} For some finite extension \(E\) of \(\mathbb{Q}_l\) with valuation ring \(R = O_E\), the \(\mathbb{Q}_l\)-sheaf \(\mathcal{F}\) is represented by an \(E\)-sheaf. So we may assume that \(\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}\) is an \(\lambda\)-adic sheaf, where \(\lambda\) is a uniformizer of \(R\). Recall that the representation corresponding to \(\mathcal{F}\) is given by taking the inverse limit of the representation \(\pi_1(X, \overline{x}) \to \text{Aut}(\mathcal{F}_n)\) (cf. \cite{6}, proof of Prop. 10.1.23), and \(f^* \mathcal{F} = (f^* \mathcal{F}_n)_{n \geq 0}\) by the very definition. Without loss of generality, we may assume that \(\mathcal{F}\) is a locally constant sheaf on \(X\) with finite stalks and \(\rho : \pi_1(X, \overline{x}) \to \text{Aut}(\mathcal{F}_{\overline{x}})\) is given by the action of \(\pi_1(X, \overline{x})\) on \(\mathcal{F}_{\overline{x}}\). There exists an étale covering \(Y \to X\) such that \(Y \simeq \mathcal{F}\), where \(Y := \text{Hom}_X(-, Y)\) (\cite{6}, Prop. 5.8.1 (i)). On the other hand, we have \(f^* \mathcal{F} \simeq Y \times_X X'\) (ibid., Prop. 5.2.7). The fiber functors at \(\overline{x}\) and \(\overline{x}'\) make the following diagram commutative:

\[
\begin{array}{ccc}
\text{Cov}(X) & \xrightarrow{\simeq} & (\pi_1(X, \overline{x})\text{-sets}) \\
- \times_X X' \downarrow & & \downarrow \varphi_* \\
\text{Cov}(X') & \xrightarrow{\simeq} & (\pi_1(X', \overline{x}')\text{-sets}),
\end{array}
\]

where \(\varphi_*\) is the functor induced by the homomorphism \(\varphi\) (ibid., Prop. 3.3.1). As a result, the action of \(\pi_1(X', \overline{x}')\) on \((f^* \mathcal{F})_{\overline{x}'}\) comes from that of \(\pi_1(X, \overline{x})\) on \(\mathcal{F}_{\overline{x}}\). \qed

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**Definition 2.14** ([4], Sect. 3, see also [12], Sect. 10.1). (i) Let $K$ be a local field with residue field of characteristic $p$. For a $\mathbb{Q}_l$-representation $\rho : G_K \to \text{Aut}(V)$, the **Swan conductor** of $V$ is defined by

$$\text{Sw}(V) = \sum_{\lambda > 0} \lambda \dim(V^{G_{\lambda}^+}/V^{G_{\lambda}^-}),$$

where $V^{G_{\lambda}^+}$ and $V^{G_{\lambda}^-}$ are the fixed subspace of $V$ by $\rho(G_{\lambda}^+)$ and $\rho(G_{\lambda}^-)$ respectively.

(ii) Let $X$ be a smooth curve over $k$, and $\overline{X}$ the smooth compactification of $X$. For a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X$, the **Swan conductor** of $\mathcal{F}$ is defined to be the effective Cartier divisor

$$\text{Sw} (\mathcal{F}) = \sum_{z \in Z} \text{Sw}_z (\mathcal{F}) [z] \in \text{Div}_Z^+(\overline{X}).$$

Here, for each $z \in Z$, $\text{Sw}_z (\mathcal{F}) := \text{Sw} (\mathcal{F}|_{\text{Spec}(k(X)_z)}) \in \mathbb{Z}_{\geq 0}$ (Lem. 2.15 (i) below) is the Swan conductor of (the $\mathbb{Q}_l$-representation corresponding to) the restriction $\mathcal{F}|_{\text{Spec}(k(X)_z)}$.

(iii) Let $X$ be a variety over $k$. For $D \in \text{Div}_Z^+(\overline{X})$ and for a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X$, we say that the **ramification of** $\mathcal{F}$ is bounded by $D$ (and write as $\text{Sw}(\mathcal{F}) \leq D$ formally) if, for every $\phi : C \to X \in \text{Cu}(X)$, we have $\text{Sw}(\phi^* \mathcal{F}) \leq \phi^* D$, where $\phi^* \mathcal{F}$ is the pullback of $\mathcal{F}$ by $\phi$.

**Lemma 2.15** ([12], Thm. 4.85, [4], Sect. 3.1, (3.1)). Let $K$ be a local field with residue field of characteristic $p$.

(i) For a $\mathbb{Q}_l$-representation $V$ of $G_K$, the Swan conductor $\text{Sw}(V)$ takes a value in $\mathbb{Z}_{\geq 0}$.

(ii) For two $\mathbb{Q}_l$-representations $V$ and $V'$ of $G_K$, we have $\text{Sw}(V \oplus V') = \text{Sw}(V) + \text{Sw}(V')$.

**Remark 2.16.** For use later (in Lem. 3.7), we refer to the Artin conductor (cf. [16], Chap. IV and [19], Sect. 4).

(i) Let $K$ be a local field with residue field of characteristic $p$, and $\rho : G_K \to \text{Aut}(V)$ a $\mathbb{Q}_l$-representation of $G_K$. We assume that the restriction $\rho|_{G_0^p}$ of $\rho$ to $G_0^p$ has finite image. The **Artin conductor** of $V$ is defined by

$$\text{Ar}(V) = \int_{-1}^{\infty} \dim(V/V^{G_K^x}) dx.$$
Putting
\[ \epsilon(V) = \int_{-1}^{0} \dim(V/V^Gt) dx = \dim(V/V^Gt), \]
we have
\[ \text{Ar}(V) = \epsilon(V) + \int_{0}^{\infty} \dim(V/V^Gt) dx = \epsilon(V) + \text{Sw}(V). \]

(ii) Let \( \mathcal{F} \) be a smooth \( \mathbb{Q}_l \)-sheaf on a smooth curve \( X \) over \( k \). We assume that the corresponding \( \mathbb{Q}_l \)-representation \( \rho : \pi_1(X, \mathfrak{p}) \to \text{Aut}(\mathcal{F}_\mathfrak{p}) \) by (9) has finite image, for simplicity. The global conductors of \( \mathcal{F} \) are defined by
\[ \text{Ar}(\mathcal{F}) = \sum_{z \in Z} \text{Ar}_z(\mathcal{F})[z] \quad \text{and} \quad \epsilon(\mathcal{F}) = \sum_{z \in Z} \epsilon_z(\mathcal{F})[z] \in \text{Div}_Z(\mathfrak{X}), \]
by using the local conductors \( \text{Ar}_z(\mathcal{F}) := \text{Ar}(\mathcal{F}|_{\text{Spec}(k(x))}) \) and \( \epsilon_z(\mathcal{F}) := \epsilon(\mathcal{F}|_{\text{Spec}(k(x))}) \) respectively. From the relation of the local Artin and Swan conductors noted above, we have
\[ \text{Ar}(\mathcal{F}) = \epsilon(\mathcal{F}) + \text{Sw}(\mathcal{F}). \]  

**Lemma 2.17.** Let \((X, \mathfrak{p})\) be a pointed variety over \( k \), and \( \mathcal{F} \) a smooth \( \mathbb{Q}_l \)-sheaf on \( X \) of rank \( r \). If the corresponding \( \mathbb{Q}_l \)-representation \( \rho : \pi_1(X, \mathfrak{p}) \to \text{Aut}(\mathcal{F}_\mathfrak{p}) \) factors through \( \pi_1(X, D, \mathfrak{p}) \) for some \( D \in \text{Div}_Z(\mathfrak{X}) \), then we have \( \text{Sw}(\mathcal{F}) \leq rD \).

**Proof.** For each \( \phi : C \to X \in \text{Cu}(X) \), we show \( \text{Sw}(\phi^* \mathcal{F}) \leq \overline{\phi}^* (rD) = r(\overline{\phi}^* D) \) (the last equality follows from the additivity property of \( \overline{\phi} : \overline{\phi}^* (D + D') = \overline{\phi}^* D + \overline{\phi}^* D' \), cf. [3], Sect. 21.4.2). The assumption on \( \rho \) does not depend on the choice of the geometric point \( \mathfrak{p} \) of \( X \). By replacing the geometric point \( \mathfrak{p} \) if necessary, the morphism \( \phi : C \to X \in \text{Cu}(X) \) induces a commutative diagram
\[ \begin{array}{ccc} 
\pi_1(C, \mathfrak{p}) & \xrightarrow{\phi} & \pi_1(X, \mathfrak{p}) \\
\downarrow & & \downarrow \\
\pi_1(C, \overline{\phi}^* D, \mathfrak{p}) & \longrightarrow & \pi_1(X, D, \mathfrak{p}) 
\end{array} \]  
by Lem. 2.5, where \( \phi \) is the induced homomorphism of fundamental groups from \( \phi \), and \( \mathfrak{p} \) a geometric point of \( C \). By Lem. 2.13, the pullback \( \phi^* \mathcal{F} \) corresponds to the representation given by the composition \( \pi_1(C, \mathfrak{p}) \xrightarrow{\phi} \pi_1(X, \mathfrak{p}) \xrightarrow{\rho} \text{Aut}(\mathcal{F}_\mathfrak{p}) \). From the assumption and the diagram (11), this representation factors through \( \pi_1(C, \overline{\phi}^* D, \mathfrak{p}) \). From this, we may assume that \( X \) is a smooth curve over \( k \).
Since the Swan conductor is defined to be $Sw(\mathcal{F}) = \sum_{z \in \mathbb{Z}} Sw_z(\mathcal{F})[z]$, if we write $D = \sum_{z \in \mathbb{Z}} mz[z]$, it is enough to show $Sw_z(\mathcal{F}) \leq rm_z$ for each $z \in \mathbb{Z}$. By replacing the geometric point and using Lem. 2.13 again, the multiplicity $Sw_z(\mathcal{F})$ at $z \in \mathbb{Z}$ is given by the Swan conductor of the representation $G_z \to \pi_1(X, \mathcal{F}) \to Aut(\mathcal{F}_z)$ of $G_z := \text{Gal}(\overline{k(X)} / k(X)_z)$, where the first homomorphism is given by choosing an embedding $k(X) \hookrightarrow \overline{k(X)}_z$ over $k(X)$ as in Lem. 2.8. The assumption and Lem. 2.8 imply that this representation factors through $G_z/G_z^{m_z}$. The assertion is reduced to showing the lemma below. □

**Lemma 2.18.** Let $K$ be a local field with residue field of characteristic $p$. If a $\overline{\mathbb{Q}}_l$-representation $\rho : G_K \to \text{Aut}(V)$ annihilates $G_K^{\lambda+}$ for some $\lambda \in \mathbb{R}_{\geq 0}$, then we have $Sw(V) \leq \dim(V)\lambda$.

**Proof.** Let $L = \overline{K}^{\text{Ker}(\rho)}$ be the extension of $K$ corresponding to Ker$(\rho)$. From the assumption, we have $G_K^{\lambda+} \subset \text{Ker}(\rho) = G_L$. The ramification of $L/K$ is bounded by $\lambda$. In particular, we have $V^{G_K'} = V$ for any $\lambda' > \lambda$. We obtain

$$Sw(V) = \sum_{0 < \lambda \leq \lambda'} \lambda' \dim(V^{G_K^{\lambda+}}/V^{G_K^{\lambda'}}) \leq \lambda \dim(V).$$

□

3 Finiteness

**Smallness**

We recall a property of profinite groups called *smallness*. This notion is also refereed as “type (F)” in [17], Chap. III, Sect. 4.1.

**Definition 3.1** ([9], Def. 2.1). A profinite group $G$ is said to be *small* if there exist only finitely many open subgroups $H$ with $(G : H) \leq n$ for any $n \in \mathbb{Z}_{\geq 1}$.

For example, a topologically finitely generated profinite group is small ([9], Prop. 2.4). Using this notion, one can interpret the Hermite-Minkowski theorem as follows: For a number field $F$ and a finite set $S$ of primes of $F$, the Galois group $G_S$ of the maximal Galois extension of $F$ unramified outside $S$ is small.

**Proposition 3.2** ([9], Sect. 2). Let $G$ and $G'$ be profinite groups.

(i) $G$ is small if and only if there exist only finitely many open normal subgroups $N$ with $(G : N) \leq n$ for any $n \in \mathbb{Z}_{\geq 1}$.

(ii) If $G$ is small and $N$ is a closed normal subgroup of $G$, then the quotient group $G/N$ is small.
(iii) If $G$ and $G'$ are small, then their free product $G * G'$ is also small.

**Main theorem**

Recall that a smooth Weil sheaf $\mathcal{F}$ on a variety $X$ over $k$ consists of a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X \times \text{Spec}(k)$ and an action of the Weil group $\mathbb{W}_k = \mathbb{W}(\mathbb{F}_k) \subset G_k$ on $\mathcal{F}$ ([2], Def. 1.1.10 (i)). For a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X$, the pullback of $\mathcal{F}$ on $X \times \text{Spec}(k)$ by the projection $X \times \text{Spec}(k) \to X$ and the action of $\mathbb{W}_k$ which is given by the restriction of that of $G_k$ produce a smooth Weil sheaf on $X$. This construction gives a fully faithful functor

$$(\text{smooth } \mathbb{Q}_l\text{-sheaves on } X) \to (\text{smooth Weil sheaves on } X).$$

(12)

The Weil sheaves in the essential image of the above functor are said to be \textit{étale} ([2], 1.3.2). In fact, for a general Weil sheaf $\mathcal{F}$ on $X$ and $D \in \text{Div}^+_p(X)$, the condition $\text{Sw}(\mathcal{F}) \leq D$ is defined in the same manner as Def. 2.14 using the Weil group $\mathbb{W}(X, \mathfrak{p}) \subset \pi_1(X, \mathfrak{p})$ ([1], Sect. 3). For a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$, we have

$$\text{Sw}(\mathcal{F}) \leq D \text{ in the sense of Def. 2.14} \iff \text{Sw(\mathcal{F} as an étale Weil sheaf)} \leq D.$$

(13)

From this, we often identify smooth $\mathbb{Q}_l$-sheaves with the corresponding étale Weil sheaves. Adding to [2], for more details on Weil sheaves, see also [11], Chap. I, and [12], Sect. 10.

Following [11], for $r \in \mathbb{Z}_{\geq 1}$, we denote by

- $\mathcal{R}_r(X)$: the set of smooth Weil sheaves on $X$ of rank $r$ up to isomorphism and up to semi-simplification.

**Theorem 3.3** ([3], Thm. 2.1). Let $X$ be a smooth variety over $k$, and $\overline{X}$ a normal compactification of $X$ such that $Z = \overline{X} \setminus X$ is the support of an effective Cartier divisor on $\overline{X}$. Then for any $r, n \in \mathbb{Z}_{\geq 1}$ and $D \in \text{Div}^+_p(X)$, the set of irreducible sheaves $\mathcal{F} \in \mathcal{R}_r(X)$ with

- $\text{Sw}(\mathcal{F}) \leq D$, and
- $\det(\mathcal{F}) \otimes n = 1$

is finite, where $\det(\mathcal{F}) := \bigwedge^r \mathcal{F}$ is the determinant of $\mathcal{F}$.

From the Galois correspondence within the Galois category $\text{Cov}(X, D)$ for a variety $X$ over $k$, Thm. 1.2 is equivalent to the smallness of the fundamental group $\pi_1(X, D, \mathfrak{p})$ which is stated in Thm. 3.4 below. For another geometric point $\overline{\mathfrak{p}}$ of $X$, we have an isomorphism $\pi_1(X, D, \overline{\mathfrak{p}}) \cong \pi_1(X, D, \mathfrak{p})$ and two such isomorphisms differ by an inner automorphism of $\pi_1(X, D, \mathfrak{p})$ ([6], Prop. 3.2.13). Since we are interested in the smallness, we omit the base point $\mathfrak{p}$ of $X$ from the fundamental groups and write $\pi_1(X, D)$ as well as $\pi_1(X)$. Under this convention, it must be noted that a canonical homomorphism or a commutative diagram concerning these groups makes sense viewing it up to conjugates.
Theorem 3.4. Let $X$ be a variety over a finite field $k$ of characteristic $p$, $\overline{X}$ a compactification of $X$, and $Z = \overline{X} \setminus X$. Then $\pi_1(X, D)$ is small for any $D \in \text{Div}^+_X(\overline{X})$.

Proof. First, we reduce the assertion to the situation in which Thm. 3.3 holds, that is, $X$ is smooth over $\text{Spec}(k)$ and the compactification $\overline{X}$ is normal with the boundary $Z = \overline{X} \setminus X$ which is the support of an effective Cartier divisor on $\overline{X}$.

(Reduction to reduced) Let $\overline{X}_{\text{red}}$ and $X_{\text{red}}$ be the reduced closed subschemes associated to $\overline{X}$ and $X$ respectively. The natural morphisms $f : X_{\text{red}} \hookrightarrow X$ and $\overline{f} : \overline{X}_{\text{red}} \hookrightarrow \overline{X}$ which are closed immersions give the following commutative diagram

$$
\begin{array}{ccc}
\overline{X}_{\text{red}} & \xrightarrow{j} & \overline{X} \\
\downarrow j_{\text{red}} & & \downarrow j \\
X_{\text{red}} & \xleftarrow{f} & X,
\end{array}
$$

where $j_{\text{red}}$ is the morphism associated to the inclusion map $j$ ([3], Chap. I, 5.1.5). Since we have $|\overline{X}_{\text{red}}| = |\overline{X}|$ and $|X_{\text{red}}| = |X|$ as the underlying topological spaces, the induced morphism $j_{\text{red}}$ is a dominant open immersion ([3], Chap. I, Prop. 5.1.6). Here, $\overline{X}_{\text{red}}$ is reduced, and for the generic point $\xi$ of an irreducible component of $\overline{X}_{\text{red}}$, we have $\overline{f}(\xi) \notin Z = \overline{X} \setminus X$. In particular, $\overline{f}(\xi) \notin |D|$. The inverse image $\overline{f}^{-1}D \in \text{Div}^+_X(\overline{X}_{\text{red}})$ is defined (the case (c) in Lem. 2.7). By Lem. 2.3, there is a commutative diagram

$$
\begin{array}{ccc}
\pi_1(X_{\text{red}}) & \xrightarrow{\pi_1(f)} & \pi_1(X) \\
\downarrow & & \downarrow \\
\pi_1(X_{\text{red}}, \overline{f}^{-1}D) & \xrightarrow{\pi_1(\overline{f})} & \pi_1(X, D),
\end{array}
$$

(14)

where the vertical homomorphisms are surjective as we noted in (6). Since the top horizontal homomorphism in (14) is known to be bijective ([8], Exp. IX, Prop. 1.7), the bottom is surjective. If we assume that $\pi_1(X_{\text{red}}, \overline{f}^{-1}D)$ is small, then so is $\pi_1(X, D)$ by Prop. 3.2 (ii). Therefore, we may assume that $X$ and $\overline{X}$ are reduced.

(Reduction to integral) Since $\overline{X}$ is Noetherian, it has only a finite number of irreducible components $\overline{X}_1, \ldots, \overline{X}_n$. The irreducible components of $X$ are given by $X_i := \overline{X}_i \cap X$ for $i = 1, \ldots, n$ ([13], Prop. 2.4.5(b)). We endow these components with the reduced closed subscheme structure. For each $i$, we obtain the following commutative diagram

$$
\begin{array}{ccc}
\overline{X}_i & \xleftarrow{\overline{f}_i} & \overline{X} \\
\downarrow & & \downarrow \\
X_i & \xleftarrow{f_i} & X,
\end{array}
$$
where \( f_i \) is the natural morphism and \( f_i \) is induced by \( f_i \). The inverse image \( D_i := f_i^*D \in \text{Div}_{\overline{X_i}}(\overline{X_i}) \) exists. In fact, for the generic point \( \xi_i \in \overline{X_i} \), we have \( f_i(\xi_i) \notin \overline{D} \) (the case (c) in Lem. 2.7). The canonical homomorphism \( \pi_1(X_i, D_i) \to \pi_1(X, D) \) exists for each \( i \) by Lem. 2.5. The collection of morphisms \( \{ f_i \}_{1 \leq i \leq n} \) induces \( n \pi_1(X) \to \pi_1(X) \), which is an effective descent morphism (\[8\], Exp. IX, see also [18], Thm. 5.2). The descent theory (\[8\], Exp. IX, Thm. 5.1, see also [18], Cor. 5.3) says that there exist finitely many generators \( \gamma_1, \ldots, \gamma_m \) such that we have a surjective homomorphism

\[
\bigotimes_{i=1}^n \pi_1(X_i) \ast \langle \gamma_1, \ldots, \gamma_m \rangle \twoheadrightarrow \pi_1(X),
\]

where \( \langle \gamma_1, \ldots, \gamma_m \rangle \) is the profinite completion of the free group on the set \( \{ \gamma_1, \ldots, \gamma_m \} \).

The fundamental groups \( \pi_1(X_i, D_i) \) and \( \pi_1(X, D) \) are quotients of \( \pi_1(X_i) \) and \( \pi_1(X) \) respectively as in [14]. The canonical homomorphisms \( \pi_1(X_i, D_i) \to \pi_1(X, D) \) and the composition \( \langle \gamma_1, \ldots, \gamma_m \rangle \to \pi_1(X) \to \pi_1(X, D) \) give a commutative diagram

\[
\begin{array}{c}
\bigotimes_{i=1}^n \pi_1(X_i) \ast \langle \gamma_1, \ldots, \gamma_m \rangle \\
\downarrow \\
\bigotimes_{i=1}^n \pi_1(X_i, D_i) \ast \langle \gamma_1, \ldots, \gamma_m \rangle \\
\to \pi_1(X).
\end{array}
\]

From the diagram (15), the bottom horizontal map is surjective. As the free profinite group \( \langle \gamma_1, \ldots, \gamma_m \rangle \) is topologically finitely generated, it is small ([9], Prop. 2.4). If we assume \( \pi_1(X_i, D_i) \) is small for all \( i \), the free product \( \bigotimes_{i=1}^n \pi_1(X_i, D_i) \ast \langle \gamma_1, \ldots, \gamma_m \rangle \) is small by Prop. 3.2 (iii) and the same holds for \( \pi_1(X, D) \) by Prop. 3.2 (ii). Thus, we may assume that \( X \) and \( \overline{X} \) are integral.

(Reduction to normal) Let \( \overline{f} : \overline{X} \to \overline{X} \) be the normalization of \( \overline{X} \). The morphism \( \overline{f} \) is finite ([13], Cor. 4.1.30) and hence proper. The restriction \( f : X' := \overline{f}^{-1}(X) \to X \) of \( \overline{f} \) gives

\[
\begin{array}{ccc}
\overline{f} & \longrightarrow & \overline{f}^{-1} \longrightarrow \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X
\end{array}
\]

which is commutative. The morphism \( f \) is also the normalization of \( X \) (from the very definition of the normalization, [13], Cor. 4.1.19). By the descent theory again ([8], Exp. IX, Thm. 5.1, see also [18], Cor. 5.3), \( \pi_1(X) \) is a quotient of the free product of \( \pi_1(X') \) and a free profinite group \( \langle \gamma_1, \ldots, \gamma_m \rangle \) as

\[
\pi_1(X') \ast \langle \gamma_1, \ldots, \gamma_m \rangle \twoheadrightarrow \pi_1(X).
\]
The inverse image $D' := \overline{f}^*(D)$ on $\overline{Y}$ exists by Lem. 2.7 (the case (b)). From Lem. 2.5 we have the following commutative diagram:

\[
\begin{array}{cccccc}
\pi_1(X') \ast \langle \gamma_1, \ldots, \gamma_m \rangle & \longrightarrow & \pi_1(X) \\
\downarrow & & \downarrow \\
\pi_1(X', D') \ast \langle \gamma_1, \ldots, \gamma_m \rangle & \longrightarrow & \pi_1(X, D).
\end{array}
\]

The bottom homomorphism in the diagram (16) becomes surjective. By Prop. 3.2 (ii) and (iii) as above (the reduction to integral), the assertion is reduced to the situation where $X$ and $\overline{X}$ are normal.

**(Reduction to smooth)** Take the smooth locus $U$ of $X$. This $U$ forms an open subscheme of $X$ ([13], Prop. 8.2.40). Since $X$ and hence $U$ are normal, $\pi_1(X)$ and $\pi_1(U)$ are quotient of the absolute Galois group $G_{k(X)}$ and $G_{k(U)}$ respectively ([7]). As these quotient maps are canonical (up to inner automorphisms), we have the following commutative diagram:

\[
\begin{array}{ccc}
G_{k(U)} & \longrightarrow & G_{k(X)} \\
\downarrow & & \downarrow \\
\pi_1(U) & \longrightarrow & \pi_1(X).
\end{array}
\]

By the commutativity, the bottom horizontal map is surjective. From Lem. 2.10 there is a commutative diagram

\[
\begin{array}{ccc}
\pi_1(U) & \longrightarrow & \pi_1(X) \\
\downarrow & & \downarrow \\
\pi_1(U \subset \overline{X}, D) & \longrightarrow & \pi_1(X \subset \overline{X}, D).
\end{array}
\]

Since the top horizontal homomorphism $\pi_1(U) \rightarrow \pi_1(X)$ in the diagram above is surjective, so is the bottom. From Lem. 3.2 (ii), we may assume that $X$ is smooth over $k$.

**(Reduction to $Z = |E|$ for some $E \in \text{Div}^+(X)$)** Let $\overline{\mathcal{f}} : \overline{X}' \rightarrow \overline{X}$ be the blowing up of $\overline{X}$ along the reduced closed subscheme $Z = \overline{X} \setminus X \hookrightarrow \overline{X}$. Note that $\overline{X}'$ is integral ([13], Chap. II, Prop. 8.1.4) (but may not be normal), $\overline{\mathcal{f}}$ is proper and induces an isomorphism $X' := \overline{\mathcal{f}}^{-1}(X) \rightarrow X'$ ([13], Prop. 8.1.12 (b) and (d)) which makes the following diagram commutative:

\[
\begin{array}{ccc}
\overline{X}' & \rightarrow & \overline{X} \\
\downarrow & & \downarrow \\
X' & \rightarrow & X.
\end{array}
\]

Hence, we have a dominant open immersion $X' \hookrightarrow \overline{X}'$ whose complement $\overline{X}' \setminus X'$ is the support of an effective Cartier divisor on $X'$ ([13], Prop. 8.1.12 (e)). The inverse image
\(D' := \mathfrak{p}^* D\) exists (the case (d) in Lem. 2.7). Lem. 2.5 gives a commutative diagram

\[
\begin{array}{ccc}
\pi_1(X') & \xrightarrow{\sim} & \pi_1(X) \\
\downarrow & & \downarrow \\
\pi_1(X' \subset X', D') & \rightarrow & \pi_1(X \subset X, D).
\end{array}
\]

(17)

From this diagram, the bottom horizontal map \(\pi_1(X' \subset X', D') \rightarrow \pi_1(X \subset X, D)\) is surjective. Using Prop. 3.2 (ii), we may assume that \(Z = \mathfrak{p} \setminus X \) is the support of an effective Cartier divisor \(E\) on \(X\).

(Reduction to a normal compactification) Let \(\overline{f} : \overline{X}' \rightarrow \overline{X}\) be the normalization. This morphism \(\overline{f}\) is finite and induces an isomorphism \(X' := \overline{f}^{-1}(X) \rightarrow X\) as \(X\) is normal. From Lem. 2.5 the diagram

\[
\begin{array}{ccc}
\overline{X}' & \xrightarrow{\overline{f}} & \overline{X} \\
\uparrow & & \uparrow \\
X' & \xrightarrow{\sim} & X
\end{array}
\]

induces a commutative diagram

\[
\begin{array}{ccc}
\pi_1(X') & \xrightarrow{\sim} & \pi_1(X) \\
\downarrow & & \downarrow \\
\pi_1(X' \subset \overline{X}', D') & \rightarrow & \pi_1(X \subset \overline{X}, D),
\end{array}
\]

(18)

where \(D' = \overline{f}^*(D)\). In particular, we obtain a surjective homomorphism \(\pi_1(X' \subset \overline{X}', D') \rightarrow \pi_1(X \subset \overline{X}, D)\). Accordingly, there exists an open immersion \(X' \hookrightarrow \overline{X}\) with \([\overline{X} \setminus X'] = [\overline{f}^* E]\). Therefore, without loss of generality, we may assume that \(X\) is smooth and \(\overline{X}\) is a normal compactification of \(X\) whose boundary \(Z = \overline{X} \setminus X\) is the support of an effective Cartier divisor on \(\overline{X}\).

Next, we show the smallness of the fundamental group \(\pi_1(X, D)\) under these assumptions.

(Proof of the smallness) For \(r \in \mathbb{Z}_{\geq 1}\), we denote by \(\mathcal{S}_r\) the set of open normal subgroups \(N\) of \(\pi_1(X, D)\) with \((\pi_1(X, D) : N) = r\). By Prop. 3.2 (i), it is enough to show \(\# \mathcal{S}_r < \infty\). For each \(N \in \mathcal{S}_r\), consider the regular (and semi-simple) representation \(\overline{\pi}_N : \pi_1(X, D)/N \rightarrow GL_r(\overline{Q}_l)\) of the finite group \(\pi_1(X, D)/N\) (with \(l \neq p\)). By composing the natural homomorphisms (cf. 39), \(\overline{\pi}_N\) induces

\[
\rho_N : \pi_1(X) \rightarrow \pi_1(X, D) \rightarrow \pi_1(X, D)/N \rightarrow GL_r(\overline{Q}_l).
\]
We denote by $F_N \in \mathcal{R}_r(X)$ the étale Weil sheaf on $X$ associated with $\rho_N$ (cf. [9] and (12)). By considering the irreducible decomposition, $F_N$ consists of irreducible components $F$ of rank $\leq r$ satisfying

- $\text{Sw}(F) \leq \text{Sw}(F_N) \leq rD$ (by Lem. 2.15 (ii), Lem. 2.17 and (13)), and

- $\det(F)^{\text{or}} = \det(F_N)^{\text{or}} = 1$ (for $\pi_1(X, D) : N) = r$, and hence $\det(\rho_N)^{\text{or}} = 1$).

Thm. 3.3 implies that there exist only finitely many such irreducible sheaves. For we have $\#S_r = \# \{ F_N \}_{N \in S_r}$, the assertion $\#S_r < \infty$ follows.

**Applications**

Let $F$ be an algebraically closed field, $X$ a variety over $k$, and $D \in \text{Div}_{\geq 1}(\overline{X})$. As in Sect. 4 of [9], in the following, we derive some finiteness results of representations $\pi_1(X, D) \to \text{GL}_r(F)$ from our main theorem (Thm. 3.4) (so that we still omit the base point). Here, we endow $\text{GL}_r(F)$ with the discrete topology. We define

$$\pi_1(X, D)^0 := \text{Ker}(\varphi : \pi_1(X, D) \to \pi_1(\text{Spec}(k)) = G_k),$$

where the homomorphism $\varphi$ is induced from the structure morphism $X \to \text{Spec}(k)$. Since $G_k(\simeq \hat{\mathbb{Z}})$ is abelian, $\pi_1(X, D)^0$ does not depend on the choice of the (omitted) geometric point.

**Definition 3.5.** A representation $\rho : \pi_1(X, D) \to \text{GL}_r(F)$ is said to be **geometric** if it satisfies $\rho(\pi_1(X, D)) = \rho(\pi_1(X, D)^0)$.

When $X$ is normal, this is equivalent to that the corresponding extension of the function field $k(X)$ contains no constant field extension.

**Corollary 3.6.** Let $X$ be a normal variety over $k$, $\overline{X}$ a compactification of $X$, and $Z = \overline{X} \setminus X$. For $r \in \mathbb{Z}_{\geq 1}$ and $D \in \text{Div}_{\geq 1}(\overline{X})$, we have the following:

1. *There exist only finitely many isomorphism classes of semi-simple geometric representations $\pi_1(X, D) \to \text{GL}_r(F)$ with solvable image.*

2. *If the characteristic of $F$ is 0, then there exist only finitely many isomorphism classes of semi-simple geometric representations $\pi_1(X, D) \to \text{GL}_r(F)$.*

To show this corollary, we prepare some notation and a lemma on a finiteness of the abelian fundamental group. For a topological group $G$, we denote by

- $G^\vee = \{ \chi \in \text{Hom}(G, Q/\mathbb{Z}) \mid \text{continuous} \}$: the Pontryagin dual group of $G$,

- $G^{\text{ab}} = G/[G, G]$ : the abelianization of $G$, where $[G, G]$ is the topological closure of the commutator subgroup of $G$. 

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The structure map $X \to \text{Spec}(k)$ as in (19) induces

$$\pi_1(X, D)^{ab,0} := \text{Ker}(\varphi^{ab} : \pi_1(X, D)^{ab} \to \pi_1(\text{Spec}(k))^{ab} = G_k).$$

**Lemma 3.7.** Let $X$ be a normal variety over $k$, and $\overline{X}$ a compactification of $X$. For any $D \in \text{Div}_Z^+(\overline{X})$, we have $\#\pi_1(X, D)^{ab,0} < \infty$.

**Proof.** As in the proof of the main theorem (Thm. 3.4), we reduce the lemma to the case where $\overline{X}$ is normal and the boundary $Z = \overline{X} \setminus X$ is the support of an effective Cartier divisor on $\overline{X}$.

(Reduction to $Z = [E]$ for some $E \in \text{Div}_Z^+(\overline{X})$) Let $\overline{p} : \overline{X}' \to \overline{X}$ be the blowing up along the reduced closed subscheme $Z = \overline{X} \setminus X$. As in (17), there exists a commutative diagram

$$
\begin{array}{ccc}
\pi_1(X')^{ab} & \xrightarrow{\simeq} & \pi_1(X)^{ab} \\
\downarrow & & \downarrow \\
\pi_1(X' \subset \overline{X}', \overline{p}^*D)^{ab} & \xrightarrow{\simeq} & \pi_1(X \subset \overline{X}, D)^{ab}
\end{array}
$$

with $X' = (\overline{p})^{-1}(X)$, and the bottom horizontal map becomes surjective. We may assume that $Z = \overline{X} \setminus X$ is the support of an effective Cartier divisor on $\overline{X}$.

(Reduction to a normal compactification) Let $\overline{f} : \overline{X}' \to \overline{X}$ be the normalization. As in (18), the isomorphism $X' := \overline{f}^{-1}(X) \cong X$ induces a commutative diagram

$$
\begin{array}{ccc}
\pi_1(X')^{ab} & \xrightarrow{\simeq} & \pi_1(X)^{ab} \\
\downarrow & & \downarrow \\
\pi_1(X' \subset \overline{X}', \overline{f}^*D)^{ab} & \xrightarrow{\simeq} & \pi_1(X \subset \overline{X}, D)^{ab}
\end{array}
$$

The bottom map is surjective so that we may assume that $\overline{X}$ is normal and there exists $E \in \text{Div}_Z^+(\overline{X})$ such that $Z = [E]$.

(Proof of the lemma) Recall that there exists a canonical isomorphism $(\pi_1(X)^{ab})^\vee \cong H^1(X, \mathbb{Q}/\mathbb{Z})$, where $H^1(X, \mathbb{Q}/\mathbb{Z})$ denotes the étale cohomology group ([1] Exp. 1, Sect. 2.2.1). By composing this with the dual of $\pi_1(X)^{ab} \to \pi_1(X, D)^{ab}$ induced from (9), we have an injective homomorphism

$$\psi : (\pi_1(X, D)^{ab})^\vee \hookrightarrow H^1(X, \mathbb{Q}/\mathbb{Z}).$$

(20)

Following [10], Def. 2.4, we define

$$\text{fil}_D H^1(X, \mathbb{Q}/\mathbb{Z}) := \{ \chi \in H^1(X, \mathbb{Q}/\mathbb{Z}) \mid \text{Ar}(\chi) \leq D \}.$$
Here, the condition “\(\text{Ar}(\chi) \leq D\)” is defined by \(\text{Ar}(\phi^*\chi) \leq \overline{\phi}^* D\) for every \(\phi : C \to X \in \text{Cu}(X)\) using the Artin conductor \(\text{Ar}(\phi^*\chi)\) of the character \(\phi^*\chi : \pi_1(C)^{ab} \to \mathbb{Q}/\mathbb{Z}\) (which is corresponding to \(\phi^*\chi \in H^1(C,\mathbb{Q}/\mathbb{Z})\)) \((10)\), Chap. IV, Sect. 3). We fix \(\mathbb{C}^\times \simeq (\mathbb{Q}_\ell)^\times\). One can consider \(\chi \in (\pi_1(X,D)^{ab})^\vee\) as a \(\mathbb{Q}_\ell\)-representation (with finite image)

\[
\pi_1(X) \xrightarrow{\phi} \pi_1(X,D) \xrightarrow{\psi} \pi_1(X,D)^{ab} \xrightarrow{\chi} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{GL}_1(\mathbb{Q}_\ell)}
\]

of \(\pi_1(X)\), where \(\pi_1(X,D) \to \pi_1(X,D)^{ab}\) is the quotient map. We denote by \(\mathcal{F}_\chi\) the corresponding \(\mathbb{Q}_\ell\)-sheaf on \(X\) (by \((9)\)). It satisfies \(\text{Sw}(\mathcal{F}_\chi) \leq D\) by Lem. \(2.17\). For every \(\phi : C \to X \in \text{Cu}(X)\), we have

\[
\text{Ar}(\phi^*\chi) = \text{Ar}(\phi^*\mathcal{F}_\chi) \quad \text{as \text{Rem. 2.16}}.
\]

(21)

Take \(E \in \text{Div}^+_Z(X)\) such that \(Z = |E|\) and we denote by \(E_{\text{red}}\) the reduced divisor associated to \(E\). Putting \(D' = E_{\text{red}} + D\), we obtain

\[
\text{Ar}(\phi^*\chi) = \varepsilon(\phi^*\mathcal{F}_\chi) \quad \text{as \text{Rem. 2.17}}.
\]

This implies that the image of \(\psi\) defined in \((20)\) is contained in \(\text{fil}_{D'} H^1(X,\mathbb{Q}/\mathbb{Z})\). By taking the dual of \(\psi : (\pi_1(X,D)^{ab})^\vee \to \text{fil}_{D'} H^1(X,\mathbb{Q}/\mathbb{Z})\), we have a surjective homomorphism \(\psi^\vee : (\text{fil}_{D'} H^1(X,\mathbb{Q}/\mathbb{Z}))^\vee \to \pi_1(X,D)^{ab}\) using the Pontryagin duality theorem. (In fact, the group \((\text{fil}_{D'} H^1(X,\mathbb{Q}/\mathbb{Z}))^\vee\) is denoted by \(\pi_1^{ab}(X,D')\) in \((10)\).) The structure map \(X \to \text{Spec}(k)\) induces a commutative diagram

\[
\begin{array}{ccc}
(\text{fil}_{D'} H^1(X,\mathbb{Q}/\mathbb{Z}))^\vee & \xrightarrow{\psi^\vee} & \pi_1(X,D)^{ab} \\
\downarrow & & \downarrow \varepsilon^{ab} \\
H^1(\text{Spec}(k),\mathbb{Q}/\mathbb{Z})^\vee & \xrightarrow{\sim} & \pi_1(\text{Spec}(k))^{ab}.
\end{array}
\]

Here, the left vertical map is given by \(H^1(\text{Spec}(k),\mathbb{Q}/\mathbb{Z}) \to H^1(X,\mathbb{Q}/\mathbb{Z})\). As \(\overline{X}\) is normal, the kernel of the left vertical map (which is \(\pi_1^{ab}(X,D'^0)\) in the sense of \((10)\)) is finite by Cor. 1.2 in \((10)\). Our claim \(\#\pi_1(X,D)^{ab,0} < \infty\) follows from this and the commutative diagram above. \(\square\)

**Proof of Cor. 3.6.** As in the proof of Lem. 4.4 in \((9)\), by induction on \(n\), we also obtain the finiteness of the quotient \(\pi_1(X,D)^0/\pi_1(X,D)^0(n)\) for any \(n \in \mathbb{Z}_{>1}\), where \((\pi_1(X,D)^0(n))\) is the \(n\)-th commutator subgroup \((9), \text{Def. 4.2}\). Starting from this, the same proofs of Thm. 4.5 (ii) and Thm. 4.6 (ii) in \((9)\) work and the assertions follow. \(\square\)
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