Pathwise approximation of Feynman path integrals using simple random walks

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Abstract

The aim of the presented research is to give a rigorous mathematical approach to Feynman path integrals based on strong (pathwise) approximations based on simple random walks.

1 Introduction

The Schrödinger equation describing the non-relativistic motion of a single particle with mass $m = 1$ (and $\hbar = 1$) is

$$\frac{1}{i} \frac{\partial \psi}{\partial t} = \frac{1}{2} \Delta \psi - V(x)\psi, \quad \psi(0, x) = g(x),$$

(1)

where $x \in \mathbb{R}^d$, $t \geq 0$, and $\psi(t, \cdot) \in L^2(\mathbb{R}^d)$ is the complex probability amplitude of the particle. The probability density of finding the particle at the point $x$ at time $t$ is the squared modulus of the complex amplitude, $|\psi(t, x)|^2$. The potential $V$ and the initial condition $g$ should fulfill suitable assumptions for the existence and uniqueness of a solution.

Based on physics intuitions, Richard Feynman [9] suggested that the solution of this equation can be given in the form of a path integral:

$$\psi(t, x) = \frac{1}{Z} \int_{\Omega^x[0, t]} \exp \left\{ i \int_0^t \left( \frac{1}{2} \left( \frac{d\omega}{dt} \right)^2 - V(\omega(s)) \right) ds \right\} g(\omega(t)) d\omega,$$

(2)

where $\Omega^x[0, t]$ is the set of all possible trajectories $\omega$ of the particle, starting from $\omega(0) = x$, over the time interval $[0, t]$. The expression after the second integral is the Lagrangian: the kinetic energy minus the potential energy of the particle; its integral is the action integral, which should be extremal along the path of a particle in classical physics. The symbol $d\omega$ is a mathematically non-existing Lebesgue-type product measure $\prod_{0 \leq s \leq t} d\omega(s)$ over the infinite dimensional vector space of trajectories. The normalizing constant $Z$ cannot have a well-defined finite value either. However, the starting point of Feynman’s seminal paper [9]...
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is a discrete time approximation that makes perfect sense. This paper tries to follow a similar approach.

Mark Kac \[12, 13\] realized that one can give Feynman’s idea a rigorous mathematical meaning for the Schrödinger-type real-valued differential equation

\[
\frac{\partial \psi}{\partial t} = \frac{1}{2} \Delta \psi - V(x)\psi, \quad \psi(0, x) = g(x),
\]

when the unknown function \(\psi\) is real valued. From the physics point of view, here \(\psi\) can be thought of as a function of an imaginary time \(it\). In the path integral the exponential of the kinetic energy \(\exp(-\frac{1}{2}(d\omega/dt)^2)\) can be moved into the measure that would become the Wiener measure \(P^x\) over the space of continuous trajectories \(C[0, t]\) starting from the point \(x\). This way one arrives at a rigorous path integral, the celebrated Feynman–Kac formula,

\[
\psi(t, x) = \int_{C[0, t]} \exp \left( -\int_0^t V(\omega(s)) \, ds \right) g(\omega(t)) \, dP^x(\omega),
\]

which then really gives the unique solution of equation (3) when, for example, \(V\) is bounded below, piecewise continuous, and \(g\) is integrable.

There exists a huge literature that intend to give solid mathematical foundation to Feynman’s original path integral \(\int\); here is a sample of some very significant ones: \([10, 11, 2, 18, 3, 1, 16]\). A good description of different rigorous approaches can be found in \([16]\).

The aim of the present paper is to give a rigorous approach to complex-valued path integrals based on a strong (pathwise) approximation by simple, symmetric random walks. In the real-valued case, weakly convergent (that is, convergent in distribution) approximations based on simple, symmetric random walks were given, for example, in \([12, 5, 19]\). For simplicity, the paper is restricted to one spatial dimension, \(d = 1\), but could be extended to any finite dimension \(d\).

The present work was helped by many numerical experiments and computations, using Wolfram Mathematica and Maple.

## 2 Complex measure walk

Our first intention is to find a nearest neighbor, symmetric random walk approximation to the complex-valued case \([11]\) and \([2]\). Fix a positive integer \(n\) and take the measurable space \((\mathbb{R}^n, B^n)\), where \(B^n\) denotes the Borel \(\sigma\)-field in \(\mathbb{R}^n\). Take a sequence \((X_r)_{r=1}^n\), the steps of a symmetric nearest neighbor random walk; each \(X_r\) has the set of possible values \(\{-1, 0, 1\}\). Define partial sums by

\[
S_0^x = x \in \mathbb{Z}, \quad S_k^x = x + \sum_{r=1}^k X_r \quad (1 \leq k \leq n).
\]

When \(x = 0\), we simply write \(S_n\). Now the distribution of a step \(X_r\) on \(\{-1, 0, 1\}\) will be given by a complex measure \(\mu\) concentrated on \(\{-1, 0, 1\}\):

\[
\mu(X_r = 1) = \mu(X_r = -1) = p \in \mathbb{C}, \quad \mu(X_r = 0) = q \in \mathbb{C}.
\]

Presently, \(p\) and \(q\) are unknowns that we intend to determine below.
Since we want to have independent and identically distributed steps, the complex measure on \((\mathbb{R}^n, \mathcal{B}_n)\) corresponding to the sequence \((X_1, \ldots, X_n)\) will be the \(n\)th power \(\mu^n\). If it causes no ambiguity, this product measure will also be denoted by \(\mu\).

Then \((S^x_k)_{k=0}^n\) will be called a complex measure walk. Notice that the existence of an infinite product measure is not claimed; consequently, our complex measure walks will have finite lengths. If \(f : \mathbb{R}^n \to \mathbb{R}\) is an arbitrary Borel-measurable function, then the complex distribution of the random variable \(Y = f(X_1, \ldots, X_n)\) is determined by the standard rule

\[
\mu(Y \in A) := \mu^n \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : f(x_1, \ldots, x_n) \in A \} = \mu^n(f^{-1}(A)).
\]

for any Borel set \(A \in \mathcal{B}\). This rule determines the complex distribution of the random walk \(S^x_k\) as well.

Denote expectations with respect to \(\mu\) by \(E_\mu\). For example,

\[
E_\mu Y := \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \, d\mu,
\]

if \(Y = f(X_1, \ldots, X_n)\) as above.

Now take two arbitrary \(\mathbb{Z} \to \mathbb{R}\) functions \(V\) and \(g\). Then the complex difference equation corresponding to (1) is going to be

\[
\frac{1}{i} (\psi(k + 1, x) - \psi(k, x)) = \frac{1}{2} e^{-iV(x)} (\psi(k, x + 1) - 2\psi(k, x) + \psi(k, x - 1))
+ \frac{1}{i} \left( e^{-iV(x)} - 1 \right) \psi(k, x). \tag{7}
\]

The tentative solution of this difference equation is obtained by using the idea of Mark Kac to move the exponential of the kinetic energy factor \(\exp (i/2 (d\omega/dt)^2)\) (or its discrete time approximation) into the complex measure \(\mu\):

\[
\psi(k, x) = E_\mu \left\{ \exp \left( -i \sum_{r=0}^{k-1} V(S^x_r) \right) g(S^x_k) \right\} \quad (0 \leq k \leq n, x \in \mathbb{Z}). \tag{8}
\]

**Lemma 1.** Under the previous conditions, (8) is a solution of the difference equation (7) if and only if \(p = i/2\) and \(q = 1 - i\).

Then (8) is the unique solution of the above difference equation under the initial condition \(\psi(0, x) = g(x) (x \in \mathbb{Z})\).

**Proof.** Let us separate the first step of the complex measure walk:

\[
\psi(k + 1, x) = E_\mu \left\{ \exp \left( -i \sum_{r=0}^{k} V(S^x_r) \right) g(S^x_{k+1}) \right\}
= e^{-iV(x)} \sum_{j \in \{-1, 0, 1\}} E_\mu \left\{ \exp \left( -i \sum_{r=0}^{k-1} V(S^x_r + j) \right) g(S^x_{k+j}) \right\} \mu(X_1 = j)
= e^{-iV(x)} \{ \psi(k, x - 1)p + \psi(k, x)q + \psi(k, x + 1)p \}. \tag{9}
\]
That is,
\[
\frac{1}{i} (\psi(k+1, x) - \psi(k, x)) = \frac{p}{i} e^{-iV(x)} \left\{ \psi(k, x + 1) - 2\psi(k, x) + \psi(k, x - 1) \right\} \\
+ \frac{1}{i} \psi(k, x) \left( (2p + q) e^{-iV(x)} - 1 \right),
\]
which is identical with \( \Box \) if and only if \( p = i/2 \) and \( q = 1 - i \).

The uniqueness of this solution under the initial condition follows by induction over \( k \) using the recursion \( \Box \) and starting with the initial condition. \( \Box \)

While this deduction of the complex distribution of the steps of the walk has been very simple, still its result has a huge impact on everything that follows afterwards. From now on, we always assume that \( p = i/2 \) and \( q = 1 - i \). Observe the interesting fact that while the total variation of the complex measure \( \mu \) for a single step is \( 1 + \sqrt{2} > 1 \) and so the total variation for the product measure with \( n \) steps goes to \( \infty \) as \( n \to \infty \), still we have
\[
\mu(\mathbb{R}^n) = \mu((-1, 0, 1)^n) = 1 \quad \text{for any} \quad n \geq 1.
\]
The latter follows from the fact that \( \mu(\mathbb{R}) = \mu((-1, 0, 1)) = 1 \) for a single step and \( \mu \) was defined as its \( n \)th power for \( n \geq 1 \) steps.

Now let us determine the resulting complex law of the partial sums \( (S_{\ell})_{\ell=0}^n \), \( n \geq 1 \), when the initial point is \( S_0 = 0 \). This deduction is based on the standard observation that a path from the origin to a point \( j \in \mathbb{Z} \) in \( \ell \geq 0 \) steps is the result of a number (say, \( r \)) horizontal steps, and the difference of the remaining up and down steps must be \( j \) (\( |j| \leq \ell \)):
\[
\mu(S_{\ell} = j) = \sum_{r=0}^{\ell - |j|} \binom{\ell}{r} \left( \frac{\ell - r}{2} \right) \left( \frac{i}{2} \right)^{\ell-r} (1 - i)^r
\]
\[
= i^2 2^{-\ell} \ell! \sum_{r=0}^{\ell - |j|} \frac{(-1)^r 2^r (1 + i)^r}{r! (\ell - r + 1)! (\ell - r + 1)!}.
\]
(10)

Here we used the convention that a term is 0 whenever \( \frac{\ell - r + 1}{2} \) is not an integer. It follows from the above argument about the product measure that
\[
\sum_{j= -\ell}^{\ell} \mu(S_{\ell} = j) = 1.
\]
(11)

Not surprisingly, formula (10) for \( \mu(S_{\ell} = j) \) can be expressed in terms of (terminating) hypergeometric functions as well. Recall (see e.g. [6]) that a hypergeometric function is defined by a series
\[
_{2}F_{1}(a, b; c; z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{z^r}{r!},
\]
where we used the Pochhammer symbol \( (x)_r := x(x+1) \cdots (x+r-1) \) if \( r > 0 \); \( (x)_0 := 1 \) if \( r = 0 \).
**Lemma 2.** For \( n \geq 0 \) and \( |m| \leq n \) we have

\[
\mu(S_{2n} = 2m) = (-1)^n \binom{2n}{n+m} 2^{-2n} \, _2F_1(-n - m, -n + m + \frac{1}{2}; 2i),
\]

\[
\mu(S_{2n} = 2m + 1) = (-1)^n (-2 - 2i)(n - m) \binom{2n}{n+m} 2^{-2n} \, _2F_1(-n - m, -n + m + 1; \frac{3}{2}; 2i),
\]

\[
\mu(S_{2n+1} = 2m) = i(-1)^n (-2 - 2i)(n - m + 1) \binom{2n + 1}{n+m} 2^{-2n-1} \, _2F_1(-n - m, -n + m + 1; \frac{3}{2}; 2i);
\]

and for \(-n - 1 \leq m \leq n\),

\[
\mu(S_{2n+1} = 2m + 1) = i(-1)^n \binom{2n + 1}{n+m + 1} 2^{-2n-1} \, _2F_1(-n - m - 1, -n + m + 1; \frac{1}{2}; 2i).
\]

**Proof.** The proof is a simple algebraic computation with finite sums, so omitted.

We have already seen that these complex measures are related to binomial probabilities. Next we want to find recursive formulas for \( \mu(S_i = j) \). The method called Zeilberger’s algorithm, see [20, Chapter 6], can do exactly that for hypergeometric series with proper hypergeometric terms, like the one we have in formula (10). The second order linear recursions obtained with the Maple program EKHAD, included in the above book, are given next.

**Lemma 3.** For \( n \geq 0 \) with \( |j| \leq 2n \) fixed, we have the recursion

\[
\mu(S_{2n+4} = j) = (3 + 4i) \frac{(n + 1)(n + 2)(n + \frac{3}{2})(n + \frac{5}{2}) \mu(S_{2n} = j)}{(n - \frac{1}{2}) + 2)(n + \frac{3}{2}) (n - \frac{1}{2} + \frac{1}{2}) (n + \frac{3}{2})(n + \frac{5}{2})} - (2 + 4i) \frac{(n + 2)(n + \frac{3}{2})(n + \frac{5}{2}) (n^2 + \frac{3}{4}j^2 + \frac{5}{2}n + \frac{5}{2}i(n + \frac{1}{2})^2 - \frac{1}{4})}{(n - \frac{1}{2}) + 2)(n + \frac{3}{2}) (n - \frac{1}{2} + \frac{1}{2}) (n + \frac{3}{2})(n + \frac{5}{2})} \times \mu(S_{2n+2} = j).
\]

Similarly, for \( n \geq 0 \), with \( |j| \leq 2n + 1 \) fixed, we have the recursion

\[
\mu(S_{2n+5} = j) = (3 + 4i) \frac{(n + 1)(n + 2)(n + \frac{3}{2})(n + \frac{5}{2}) \mu(S_{2n+1} = j)}{(n - \frac{1}{2}) + 2)(n + \frac{3}{2}) (n - \frac{1}{2} + \frac{1}{2}) (n + \frac{3}{2})(n + \frac{5}{2})} - (2 + 4i) \frac{(n + 2)(n + \frac{3}{2})(n + \frac{5}{2}) (n^2 + \frac{3}{4}j^2 + \frac{5}{2}n + \frac{5}{2}i(n + \frac{1}{2})^2 - \frac{1}{4})}{(n - \frac{1}{2}) + 2)(n + \frac{3}{2}) (n - \frac{1}{2} + \frac{1}{2}) (n + \frac{3}{2})(n + \frac{5}{2})} \times \mu(S_{2n+3} = j).
\]
Since \( \mu(S_k = j) = \mu(S_k = -j) \), it is enough to consider the cases when \( j \geq 0 \). For any \( j \geq 0 \) fixed, the above recursion formulas give the value of \( \mu(S_k = j) \) for any \( k \) if we start, depending on parities, with \( \mu(S_1 = j) \) and \( \mu(S_{j+2} = j) \) or \( \mu(S_{j+1} = j) \) and \( \mu(S_{j+3} = j) \), respectively. In turn, the latter values can be determined by formula (10):

\[
\mu(S_j = j) = \left( \frac{j}{2} \right)^j, \quad (15)
\]

\[
\mu(S_{j+1} = j) = (1-i)(j+1)\left( \frac{j}{2} \right)^j, \quad (16)
\]

\[
\mu(S_{j+2} = j) = (j+2)\left( -\frac{1}{4} - (j+1)i \right) \left( \frac{j}{2} \right)^j, \quad (17)
\]

\[
\mu(S_{j+3} = j) = (1-i)(j+2)(j+3)\left( -\frac{1}{4} - \frac{j+1}{3}i \right) \left( \frac{j}{2} \right)^j. \quad (18)
\]

Numerical simulations showed the important conjecture that for any \( n \geq 0 \),

\[
\frac{|\mu(S_{2n} = j)|^2}{\sum_{r=-2n}^{2n}|\mu(S_{2n} = r)|^2} \approx \left( \frac{2n}{n+j} \right)^{2-2n} = \mathbb{P}(S_{2n}^* = 2j) \quad (|j| \leq n), \quad (19)
\]

where \( (S_n^*)_{n=0}^{\infty} \) is an ordinary simple, symmetric random walk. Interestingly, the approximation is good even for small values of \( n \), see Table 4.1 below. Also, the left hand side of (16) is very close to 0 when \( n < |j| \leq 2n \). There is a similar fit between the normalized \( |\mu(S_{2n+1} = j)|^2 \) and \( \mathbb{P}(S_{2n+1} = 2j - 1) \) \((-n \leq j \leq n + 1)\). Thus we define the non-negative quantities

\[
\mathbb{P}(S_\ell = j) := \frac{|\mu(S_\ell = j)|^2}{Z_\ell} \quad (|j| \leq \ell) \quad (20)
\]

as the coupled probabilities of the complex measure walk \( (S_\ell)_{\ell \geq 0} \), where

\[
Z_\ell := \sum_{j=-\ell}^{\ell} |\mu(S_\ell = j)|^2 \quad (21)
\]

is the normalizing factor. It is clear that (20) defines a probability distribution for any \( \ell \geq 0 \).

Obviously, the magnitude of the normalizing factor \( Z_\ell \) plays an important role in the following. To have an idea about it, let us consider the middle term \( \mu(S_{2n} = 0) \), or, rather, the approximate recursion for \( a_n \approx \mu(S_{2n} = 0) \) obtained from (13) when \( j = 0 \) and \( n \) is very large:

\[
a_{n+2} = (3 + 4i)a_n - (2 + 4i)a_{n+1}. \quad (22)
\]

This is a second order linear difference equation with constant coefficients. By a standard method, one can search for the solution in the form \( a_n = cq^n \); \( c, q \in \mathbb{C} \). Then one gets the characteristic equation \( q^2 + (2 + 4i)q - (3 + 4i) = 0 \) and the characteristic roots \( q_1 = 1 \) and \( q_2 = -3 - 4i \). Thus the general solution of (22) is

\[
a_n = c_1 + c_2(-3 - 4i)^n \quad (c_1, c_2 \in \mathbb{C}). \quad (23)
\]
The next section gives a more precise result.

We will need a recursion of the complex amplitudes $\mu(S_\ell = j)$ in the variable $j$ as well. For simplicity, we restrict ourselves to the case $\ell = 2n$ ($n \geq 1$) fixed and $j = 2m$ ($m \geq 0$, integer). The other cases are similar. Define $a_{n,m} := \mu(S_{2n} = 2m)$.

**Lemma 4.** If $n \geq 1$ fixed and $m$ goes from $n-1$ to 0, one gets the reverse recursion

$$a_{n,m} = \frac{\left(2 + \frac{1}{m+\frac{3}{2}}\right)\left(1 - \frac{m+\frac{3}{2}}{n}\right)\left(1 + \frac{m+\frac{1}{2}}{n}\right) - \frac{1}{2n} + i8\frac{m+1}{n}\frac{m+\frac{3}{2}}{n}}{\left(1 + \frac{1}{m+\frac{3}{2}}\right)\left(1 - \frac{m}{n}\right)\left(1 - \frac{m+\frac{1}{2}}{n}\right)} a_{n,m+1}$$

$$- \frac{\left(1 + \frac{m+\frac{3}{2}}{n}\right)\left(1 + \frac{m+2}{n}\right)}{\left(1 + \frac{1}{m+\frac{3}{2}}\right)\left(1 - \frac{m}{n}\right)\left(1 - \frac{m+\frac{1}{2}}{n}\right)} a_{n,m+2},$$

with the initial condition

$$a_{n,n} = (-1)^n 2^{-2n}, \quad a_{n,n+1} = 0.$$

**Proof.** By Lemma 2

$$a_{n,m} = (-1)^n \binom{2n}{n+m} 2^{-2n} {}_2F_1(-n - m, -n + m; \frac{1}{2}; 2i).$$

Then, first,

$$\frac{\binom{2n}{n+m+1}}{\binom{2n}{n+m}} = \frac{1 - \frac{m}{n}}{1 + \frac{m+1}{n}}.$$

Next, a second order linear recursion can be obtained for ${}_2F_1(-n - m, -n + m; \frac{1}{2}; 2i)$ in the variable $m$ by Gauss’ contiguous relations. After simplification, we obtain the second order linear reverse recursion for $a_{n,m}$ in the variable $m \geq 0$, as stated in the lemma. 

The second order recursion of Lemma 4 can be reduced to a first order recursion by introducing the ratio $\rho_{n,m} := a_{n,m+1}/a_{n,m}$ when $m$ goes from $n-1$ to 0:

$$\rho_{n,m} = \frac{\left(1 + \frac{1}{m+\frac{3}{2}}\right)\left(1 - \frac{m}{n}\right)\left(1 - \frac{m+\frac{1}{2}}{n}\right)}{\left(1 + \frac{m+\frac{3}{2}}{n}\right)\left(1 + \frac{m+\frac{1}{2}}{n}\right) \rho_{n,m+1}},$$

$$\mu_{n,m} := \left(2 + \frac{1}{m+\frac{1}{2}}\right)\left(1 - \frac{m+\frac{1}{2}}{n}\right)\left(1 + \frac{m+\frac{3}{2}}{n}\right) - \frac{1}{2n} + i8\frac{m+1}{n}\frac{m+\frac{3}{2}}{n}, \quad (20)$$

with the initial condition $\rho_{n,n} = 0$. 


3 An asymptotic solution of the recursion

Now we would like to find de Moivre–Laplace-type approximations to the “binomial formula” \( \mu(S_\ell = j) \) of \( \mu(S_\ell = j) \) when \( \ell \to \infty \). If we try to use the standard technique based on Stirling’s formula, we run into serious difficulties. The reason is that here the complex terms are wildly oscillating as \( \ell \to \infty \). Consequently, there are cancellations of bigger terms, and it is necessary to precisely handle essentially the whole range of terms \( |j| \leq \ell \) and not just the terms around the center, say \( |j| \leq \ell^{3/2} \). That is why instead of the standard approach now we want to find an asymptotic solution of the recursions obtained in the previous section.

**Theorem 1.** An asymptotic solution of the second order linear recursions \([13],[14]\) with initial conditions \([15]\) is

\[
\mu(S_\ell = j) = \sqrt{\frac{2}{\ell}} \left( \tilde{c}_{1,j} e^{\frac{2}{\ell}} + c_2 e^{i\pi j - (2 + i)\frac{2}{\ell}} (-3 - 4i)^\frac{j}{\ell} \right) + \tilde{c}_{\ell,j},
\]

(21)

where \( \tilde{c}_{1,j}, c_2 \in \mathbb{C} \) are suitable constants \( \ell \geq 1, |j| \leq \ell \). For the error term \( \tilde{c}_{\ell,j} \), we have

\[
\tilde{c}_{\ell,j} = c_2 \sqrt{\frac{2}{\ell}} e^{i\pi j - (2 + i)\frac{2}{\ell}} (-3 - 4i)^\frac{j}{\ell} \left( z_0 \frac{j^3}{2\ell^2} + z_1 \frac{2}{\ell} + \tilde{h}_{\ell,j} \right),
\]

(22)

where \( |\tilde{c}_{1,j}| \leq \tilde{k}_1, |\tilde{h}_{\ell,j}| \leq \tilde{k}_2 \ell^{-2/3} \) for any \( |j| \leq \frac{1}{2}\ell^{3/2} \) and \( \ell \geq 1 \), with suitable constants \( z_0, z_1 \in \mathbb{C} \) and \( \tilde{k}_1, \tilde{k}_2 \in (0, \infty) \).

**Proof.** The proof is long, but elementary. We concentrate on the case of \([13]\), where \( \ell = 2n \), even; but at some essential points we show to the differences that the case \( \ell = 2n + 1 \), \([13]\) causes.

Define \( a_{n,m} := \mu(S_{2n} = 2m), \quad m := \frac{j}{n} \in \frac{1}{2}\mathbb{Z}, \quad \epsilon_{n,m} := \tilde{c}_{2n,2m}, \quad c_{1,m} := \tilde{c}_{1,2m} \) and \( h_{n,m} := \tilde{h}_{2n,2m} \). Since \( a_{n,m} \) is an even function of \( m \), it is enough to consider the case \( 0 \leq m \leq \frac{1}{2}n^{3/2} \), that we assume from now on.

Dividing the numerators and denominators of \([13]\) by \( n^5 \) when \( n \geq 1 \), we get

\[
a_{n+2,m} = (3 + 4i)\beta_{n,m}a_{n,m} - (2 + 4i)\gamma_{n,m}a_{n+1,m} \]

\[
= (3 + 4i) \frac{(1 + \frac{1}{2}) (1 + \frac{\frac{2}{n}}{2}) (1 + \frac{\frac{5}{n}}{2}) (1 + \frac{\frac{7}{n}}{2})}{(1 + \frac{2 + m}{n}) (1 + \frac{2 - m}{n}) (1 + \frac{\frac{1}{n}}{2}) (1 + \frac{\frac{3}{n}}{2})} a_{n,m} \]

\[
- (2 + 4i) \frac{(1 + \frac{3}{n}) (1 + \frac{\frac{7}{n}}{2}) (1 + \frac{\frac{5}{n}}{2}) (1 + \frac{\frac{1}{n}}{2}) (1 + \frac{\frac{3}{n}}{2})}{(1 + \frac{2 + m}{n}) (1 + \frac{2 - m}{n}) (1 + \frac{\frac{1}{n}}{2}) (1 + \frac{\frac{3}{n}}{2})} \times a_{n+1,m}.
\]

A tentative asymptotic general solution of this second order linear recursion according to \([21]\) is

\[
a'_{n,m} := \frac{1}{\sqrt{\pi n}} \left( c_{1,m} e^{\frac{\pi}{n} m^2} + c_2 e^{i2\pi m - (2 + i)\frac{\pi}{n} m^2} (-3 - 4i)^n \right) \quad (c_{1,m}, c_2 \in \mathbb{C}).
\]

(24)
By (22) it is enough to show that

$$\epsilon_{n,m} = a_{n,m} - a'_{n,m} = \frac{c_2}{\sqrt{n}} e^{i2\pi m - (2+\frac{m^2}{4})(-3-4i)^n} \left( z_0 \frac{m^4}{n^3} + \frac{z_1}{n} + h_{n,m} \right),$$  \hspace{1cm} (25)$$

where \(|c_{1,m}| \leq k_1\) and \(|h_{n,m}| \leq k_2 n^{-\frac{5}{2}}\) for any \(0 \leq m \leq \frac{1}{2} n^{\frac{3}{2}}\) and \(n \geq 3\).

If we define \(\bar{a}_{n,m} := \mu(S_{2n+1} = 2m),\) \(n := \frac{1}{2} \in \mathbb{Z},\) then by (14) we similarly obtain

$$\bar{a}_{n+2,m} = (3 + 4i) \beta_{n,m} \bar{a}_{n,m} - (2 + 4i) \gamma_{n,m} a_{n+1,m}$$

$$= (3 + 4i) \frac{(1 + \frac{1}{n}) (1 + \frac{2}{n}) (1 + \frac{3}{2n}) (1 + \frac{5}{2n}) (1 + \frac{9}{4n})}{(1 + \frac{2+m}{n}) (1 + \frac{2-m}{n}) (1 + \frac{2+m}{n}) (1 + \frac{2-m}{n}) (1 + \frac{5}{4n})} \bar{a}_{n,m}$$

$$- (2 + 4i) \frac{(1 + \frac{2}{n}) (1 + \frac{5}{2n}) (1 + \frac{7}{2n}) (1 + \frac{9}{4n}) (1 + \frac{5}{4n})}{(1 + \frac{2+m}{n}) (1 + \frac{2-m}{n}) (1 + \frac{2+m}{n}) (1 + \frac{2-m}{n}) (1 + \frac{5}{4n})} \times \bar{a}_{n+1,m}. \hspace{1cm} (26)$$

In this case too, a tentative asymptotic general solution of this second order linear recursion by (21) is

$$\bar{a}_{n,m} := \frac{1}{\sqrt{n+\frac{1}{2}}} \left( c_{1,m} e^{i \frac{m^2}{n+\frac{1}{2}}} + c_2 e^{i2\pi m - (2+\frac{m^2}{4})(-3-4i)^n} \right). \hspace{1cm} (27)$$

With \(r = 2\) and \(\frac{3}{2},\) use a Taylor expansion for the first four factors in the denominator of (26) when \(n \geq 3:\)

$$\left( 1 + \frac{r \pm m}{n} \right)^{-1} = \sum_{k=0}^{\infty} (-1)^k \left( \frac{r \pm m}{n} \right)^k.$$

Then after simplification, the product of the first four factors of the denominator is

$$\left( \frac{1 + \frac{2 + m}{n}}{n} \right) \left( \frac{1 + \frac{2 - m}{n}}{n} \right) \left( \frac{1 + \frac{5}{2} + m}{n} \right) \left( \frac{1 + \frac{5}{2} - m}{n} \right)^{-1} = 1 - \frac{7}{n^2} - \frac{2m^2}{n^2} + \frac{3m^4}{n^4} - 21 \frac{m^6}{n^6} + \frac{123}{4} \frac{1}{n^2} - \frac{217}{2} \frac{1}{n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right).$$

Expanding the product of the numerator of \(\beta_{n,m}\) and \((1 + 3/(4n))^{-1}\) in (23) and then simplifying, the result is

$$1 + 6n^{-1} + 13n^{-2} + \frac{189}{16} n^{-3} + O(n^{-4}).$$

Consequently,

$$\beta_{n,m} = \frac{1 - \frac{1}{n} + \frac{2m^2}{n^2} + \frac{3m^4}{n^4} - 9 \frac{m^6}{n^6} + \frac{7}{4n^2} - \frac{51}{16n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right)}{n}, \hspace{1cm} (28)$$
Similarly, the product of the numerator of \( \gamma_{n,m} \) and \((1 + 3/(4n))^{-1}\) in (23) is

\[
1 + \frac{13}{2n} + \left( \frac{3}{5} + \frac{4}{5} \right) \frac{m^2}{n^2} + \left( \frac{12}{5} + \frac{16}{5} \right) \frac{m^2}{n^3} + \left( \frac{313}{20} - \frac{1}{20} \right) \frac{1}{n^2} + \frac{2641}{160} - \frac{1}{20} \frac{1}{n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right),
\]

so we obtain

\[
\gamma_{n,m} = 1 - \frac{1}{2n} + \left( \frac{13}{5} + \frac{4}{5} \right) \frac{m^2}{n^2} + \left( \frac{21}{5} + \frac{8}{5} \right) \frac{m^2}{n^3} + \left( \frac{49}{5} + \frac{12}{5} \right) \frac{m^2}{n^3} + \frac{9}{10} - \frac{1}{20} \frac{1}{n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right).
\]  

(29)

(Here the error terms \( O(\cdot) \) are complex valued. Then the \( O(\cdot) \) notation refers to both the real and imaginary parts.)

Combining (23), (25) and (26), the result is

\[
a_{n+2,m} = (3 + 4i) \left( \beta'_{n,m} + \Delta \beta_{n,m} \right) a_{n,m} - (2 + 4i) \left( \gamma'_{n,m} + \Delta \gamma_{n,m} \right) a_{n+1,m},
\]

where

\[
\beta'_{n,m} := 1 - \frac{1}{n} + 2 \frac{m^2}{n^2}, \quad \gamma'_{n,m} := 1 - \frac{1}{2n} + \left( \frac{13}{5} + \frac{4}{5} \right) \frac{m^2}{n^2},
\]

and for \( 0 \leq m \leq \frac{3}{n} \) and \( n \geq 3 \),

\[
\Delta \beta_{n,m} = 3 \frac{m^4}{n^2} - 9 \frac{m^2}{n^3} + \frac{7}{4n^2} - \frac{51}{16n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right)
\]

(32)

and

\[
\Delta \gamma_{n,m} = \left( \frac{21}{5} + \frac{8}{5} \right) \frac{m^4}{n^2} - \left( \frac{49}{5} + \frac{12}{5} \right) \frac{m^2}{n^3} + \left( \frac{9}{10} - \frac{1}{20} \right) \frac{1}{n^2} + \left( \frac{-107}{160} + \frac{23}{20} \right) \frac{1}{n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right).
\]

(33)

When \( \ell = 2n + 1 \), from (27), one obtains exactly the same \( \beta'_{n,m} = \beta'_{n,m} \) and \( \gamma'_{n,m} = \gamma'_{n,m} \) as above in the case of \( \ell = 2n \). Then

\[
a_{n+2,m} = (3 + 4i) \left( \beta'_{n,m} + \Delta \beta_{n,m} \right) a_{n,m} - (2 + 4i) \left( \gamma'_{n,m} + \Delta \gamma_{n,m} \right) a_{n+1,m}.
\]

(34)

On the other hand, slightly differently,

\[
\Delta \beta_{n,m} = 3 \frac{m^4}{n^2} - 11 \frac{m^2}{n^3} + \frac{9}{4n^2} - \frac{83}{16n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right)
\]

(35)

and

\[
\Delta \gamma_{n,m} = \left( \frac{21}{5} + \frac{8}{5} \right) \frac{m^4}{n^2} - \left( \frac{62}{5} + \frac{16}{5} \right) \frac{m^2}{n^3} + \left( \frac{23}{20} - \frac{1}{20} \right) \frac{1}{n^2} + \left( \frac{-431}{160} + \frac{1}{5} \right) \frac{1}{n^3} + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right).
\]

(36)
Feynman path integrals

Since these basic approximations are almost the same when \( \ell = 2n \) or \( \ell = 2n + 1 \), from now on, we deal exclusively with the even case \( \ell = 2n \).

Using Taylor expansion again with \( r = 1 \) and \( 2 \), we have

\[
\delta_{n,r} := \frac{1}{\sqrt{n+r}} - \frac{1}{\sqrt{n}} \left( 1 - \frac{r}{2n} \right) = \frac{1}{\sqrt{n}} \left\{ \frac{3r^2}{8n^2} - \frac{5r^3}{16n^3} + \sum_{k=4}^{\infty} \left( \frac{1}{2} \right)^k \left( \frac{r}{n} \right)^k \right\} = \frac{1}{\sqrt{n}} \left( \frac{3r^2}{8n^2} - \frac{5r^3}{16n^3} + O(n^{-4}) \right). \tag{37}
\]

Also, with \( \alpha = 2 + i \) or \( -i \),

\[
e^{-\alpha \frac{m^2}{n^2}} = e^{-\alpha \frac{m^2}{n^2}(1+\delta)}^{-1} = e^{-\alpha \frac{m^2}{n^2}} \left( 1 + \alpha \frac{m^2}{n^2} \right) + \eta_{n,m,r}(\alpha), \tag{38}
\]

where

\[
\eta_{n,m,r}(\alpha) := e^{-\alpha \frac{m^2}{n^2}} \left\{ e^{-\alpha m^2 \left( \frac{1}{n} - \frac{1}{2} \right)} - \left( 1 + \alpha \frac{m^2}{n^2} \right) \right\} = e^{-\alpha \frac{m^2}{n^2}} \left\{ \alpha \frac{m^2}{n^2} \left( 1 + \frac{r}{n} \right)^{-1} - 1 \right\} + \frac{\alpha^2 m^4}{2 n^4} \left( 1 + \frac{r}{n} \right)^{-2} \sum_{k=3}^{\infty} \frac{1}{k!} \left( \alpha \frac{m^2}{n^2} \right)^k \left( 1 + \frac{r}{n} \right)^{-k} = e^{-\alpha \frac{m^2}{n^2}} \left\{ -\alpha^2 \frac{m^2}{n^2} + \frac{\alpha^2 m^4}{2 n^4} + O \left( \frac{m^6}{n^4} \right) \right\}. \tag{39}
\]

Using these, for any value of \( n \) and \( m \), we further approximate \( a_{n+1,m}^{(2)} \) and \( a_{n+2,m}' \) by

\[
a_{n,m}^{(1)} := \frac{1}{\sqrt{n}} \left( 1 - \frac{1}{2n} \right) \left\{ c_{1,m} e^{i \frac{m^2}{n^2}} \left( 1 - i \frac{m^2}{n^2} \right) + c_2 e^{i 2 \pi m - (2 + 1) \frac{m^2}{n^2}} \left( 1 + (2 + 1) \frac{m^2}{n^2} \right) \left( -3 - 4i \right)^{n+1} \right\}, \tag{40}
\]

\[
a_{n,m}^{(2)} := \frac{1}{\sqrt{n}} \left( 1 - \frac{1}{n} \right) \left\{ c_{1,m} e^{i \frac{m^2}{n^2}} \left( 1 - 2i \frac{m^2}{n^2} \right) + c_2 e^{i 2 \pi m - (2 + 1) \frac{m^2}{n^2}} \left( 1 + 2(2 + 1) \frac{m^2}{n^2} \right) \left( -3 - 4i \right)^{n+2} \right\}, \tag{41}
\]

respectively. Then it is easy to check that for any \( n, m, c_{1,m}, \) and \( c_2 \), we have the exact equality

\[
a_{n,m}^{(2)} = (3 + 4i) \beta_{n,m}' a_{n,m}^{(2)} - (2 + 4i) \gamma_{n,m}' a_{n,m}^{(1)}. \tag{42}
\]

This indicates that \( a_{n,m}' \), defined by \( \beta_{n,m}' \), may really be an asymptotic solution of the recursion \( \beta_{n,m}' \).

We want to show this, that is, we have to show that

\[
|h_{n,m}| \leq k_2 n^{-\frac{5}{2}} \quad (n \geq 3, |m| \leq \frac{1}{3} n^{\frac{3}{2}}) \tag{43}
\]
holds with suitable constants \( z_0, z_1 \in \mathbb{C} \) and \( k_2 \in (0, \infty) \).

Consequently, by (37)–(39) and a Taylor expansion of \((1 + 1/n)^k\) (\(k = -3, -1\)) we should have

\[
\epsilon_{n+1,m} = \frac{C_2}{\sqrt{n}} e^{i2\pi m(2 + i)\frac{m^2}{n^2}} \left( -3 - 4i \right)^{n+1} \left( 1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{5}{16n^3} + O\left( \frac{1}{n^4} \right) \right) \\
\times \left( 1 + (2 + i) \frac{m^2}{n^2} + \frac{3 + 4i m^4}{2 n^4} - (2 + i) \frac{m^2}{n^3} + O\left( \frac{m^6}{n^6} \right) \right) \\
\times \left\{ z_0 \frac{m^4}{n^5} \left( 1 - \frac{3}{n} + O\left( \frac{1}{n^2} \right) \right) + z_1 \frac{n^2}{n^5} \left( 1 - \frac{1}{n} + \frac{1}{n^2} + O\left( \frac{1}{n^3} \right) \right) + h_{n+1,m} \right\},
\]

(44)

where

\[ |h_{n+1,m}| \leq k_2 (n + 1)^{-\frac{4}{3}}, \]

(45)

with the above constants \( z_0, z_1 \in \mathbb{C} \) and \( k_2 > 0 \).

Similarly, we should have

\[
\epsilon_{n+2,m} = \frac{C_2}{\sqrt{n}} e^{i2\pi m(2 + i)\frac{m^2}{n^2}} \left( -3 - 4i \right)^{n+1} \left( 1 - \frac{1}{2n} + \frac{3}{2n^2} - \frac{5}{2n^3} + O\left( \frac{1}{n^4} \right) \right) \\
\times \left( 1 + 2(2 + i) \frac{m^2}{n^2} + 2(3 + 4i) \frac{m^4}{n^4} - 4(2 + i) \frac{m^2}{n^3} + O\left( \frac{m^6}{n^6} \right) \right) \\
\times \left\{ z_0 \frac{m^4}{n^5} \left( 1 - \frac{6}{n} + O\left( \frac{1}{n^2} \right) \right) + z_1 \frac{n^2}{n^5} \left( 1 - \frac{2}{n} + \frac{4}{n^2} + O\left( \frac{1}{n^3} \right) \right) + h_{n+2,m} \right\},
\]

(46)

where

\[ |h_{n+2,m}| \leq k_2 (n + 2)^{-\frac{4}{3}}, \]

(47)

with the above constants \( z_0, z_1 \in \mathbb{C} \) and \( k_2 > 0 \).

On the other hand, by definition,

\[
\epsilon_{n+2,m} := a_{n+2,m} - a'_{n+2,m} = (a_{n+2,m} - a^{(2)}_{n,m}) - (a'_{n+2,m} - a^{(2)}_{n,m}).
\]

(48)

Let us compute the terms here. First, by (24)–(39) and (41),

\[
da'_{n+2,m} - a^{(2)}_{n,m} = \frac{C_2}{\sqrt{n}} e^{i2\pi m(2 + i)\frac{m^2}{n^2}} \left( -3 - 4i \right)^{n+2} \\
\times \left\{ (6 + 8i) \frac{m^4}{n^5} - (8 + 4i) \frac{m^2}{n^3} + \frac{3}{2n^2} - \frac{5}{2n^3} + O\left( \frac{m^6}{n^6} \right) + O\left( \frac{1}{n^4} \right) \right\}.
\]

(49)

Similarly,

\[
a'_{n+1,m} - a^{(1)}_{n,m} = \frac{C_2}{\sqrt{n}} e^{i2\pi m(2 + i)\frac{m^2}{n^2}} \left( -3 - 4i \right)^{n+1} \\
\times \left\{ \frac{3}{2} + 2i \frac{m^4}{n^5} - (2 + i) \frac{m^2}{n^3} + \frac{3}{8n^2} - \frac{5}{16n^3} + O\left( \frac{m^6}{n^6} \right) + O\left( \frac{1}{n^4} \right) \right\}.
\]

(50)
Thus, by (30)–(33), (25), (44) and (48)–(51), after simplification we obtain that
\[ F\text{eynman path integrals} \]
Next, by (30) and (42),
\[ \text{have equality if and only if} \]
\[ + \frac{2}{5} + 4i \left( \begin{array}{c}
\epsilon_{n+1,m} (a_{n+1,m} - a_{n,m}) + \gamma_{n,m} (a_{n,m} - a_{n+1,m})
\end{array} \right) \] (51)
Thus, by (30)–(33), (25), (44) and (48)–(51), after simplification we obtain that
\[ \epsilon_{n+2,m} = \frac{c^2}{\sqrt{n}} e^{i2\pi m-2+i/n^2} (-3 - 4i)^n \]
\[ \times \left\{ z_0 \left( -7 + 24i \frac{m^4}{n^3} + (76 - 38i) \frac{m^6}{n^5} + (37 - 84i) \frac{m^4}{n^4} - (248 - 136i) \frac{m^8}{n^7} \right) \right. \]
\[ + z_1 \left( -7 + 24i \frac{1}{n} + (17 - 44i) \frac{1}{n^2} + (29 - 78i) \frac{1}{n^3} + (-76 + 38i) \frac{m^2}{n^5} \right) \]
\[ + h_{n,m} \left( 3 + 4i \left( 3 + 4i \right) \frac{1}{n} + \left( \frac{21}{4} + 7i \right) \frac{1}{n^2} + \left( \frac{153}{16} + 51i \right) \frac{1}{n^3} \right) \]
\[ + (6 + 8i) \frac{m^2}{n^5} + (9 + 12i) \frac{m^4}{n^4} - (27 + 36i) \frac{m^4}{n^3} \]
\[ + h_{n+1,m} \left( -10 + 20i \right) + (10 - 20i) \frac{1}{n} \right) \left( \frac{57}{4} - 31i \right) \frac{1}{n^2} + \left( \frac{117}{16} + 353i \right) \frac{1}{n^3} \]
\[ - (82 - 38i) \frac{m^2}{n^5} - (257 - 124i) \frac{m^4}{n^4} + (247 - 254i) \frac{m^4}{n^3} \]
\[ + (4 - 3i) \frac{1}{n^2} - \left( \frac{275}{8} + \frac{25}{8} \right) \frac{1}{n^3} + (114 + 2i) \frac{m^4}{n^4} + (7 - 74i) \frac{m^2}{n^3} \]
\[ + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right) \}
(52)
Compare the \( m^4/n^4 \) and the \( 1/n^2 \) terms, respectively, in (48) and (52). We have equality if and only if
\[ z_0 = \frac{1 + 8i}{6}, \quad z_1 = \frac{1 + i}{8}. \]
Then express \( h_{n+2,m} \) by comparing (48) and (52) again:
\[ h_{n+2,m} \]
\[ = \frac{1}{3 + 4i} \left( \frac{1}{n} + \frac{1}{n^2} + \frac{2}{5} + 4i \frac{m^2}{n^3} - \frac{51}{10}\frac{m^4}{n^4} + 3\frac{m^4}{n^3} - 9\frac{m^2}{n^5} \right) h_{n,m} \]
\[ + \frac{2}{5} + 4i \left( \frac{1}{n} + \frac{1}{n^2} + \frac{2}{5} + 4i \frac{m^2}{n^3} - \frac{51}{10}\frac{m^4}{n^4} + 3\frac{m^4}{n^3} - 9\frac{m^2}{n^5} \right) h_{n+1,m} \]
\[ + \frac{2}{5} + 4i \left( \frac{1}{n} + \frac{1}{n^2} + \frac{2}{5} + 4i \frac{m^2}{n^3} - \frac{51}{10}\frac{m^4}{n^4} + 3\frac{m^4}{n^3} - 9\frac{m^2}{n^5} \right) h_{n+1,m} \]
\[ + (4 + 2i) \frac{m^2}{n^3} - \frac{5}{2}\frac{m^4}{n^4} + (6 + 8i) \frac{m^4}{n^3} - (12 + 6i) \frac{m^4}{n^5} \]
\[ + O \left( \frac{m^6}{n^6} \right) + O \left( \frac{1}{n^4} \right) \]
Simplifying this, we obtain that
\[
\begin{align*}
&h_{n+2,m} = \frac{1}{3 + 4i} \left(1 + \frac{1}{4n^2} - (2 + 2i) \frac{m^2}{n^2} + O \left(\frac{m^4}{n^4}\right) \right) + O \left(\frac{1}{n^3}\right) \right) h_{n,m} \\
&+ \frac{2 + 4i}{3 + 4i} \left(1 + \frac{1 - 2i}{40n^2} - \frac{3 - i m^2}{5 n^2} + O \left(\frac{m^4}{n^4}\right) + O \left(\frac{1}{n^3}\right)\right) h_{n+1,m} \\
&+ \frac{-19 + 317i}{400n^3} \left[1 - 14i m^2 \frac{3}{n^3} + 1 + 8i m^8 \frac{8}{3 n^7} + O \left(\frac{m^6}{n^6}\right) + O \left(\frac{1}{n^4}\right)\right].
\end{align*}
\]

Ignoring lower order error terms, now we want to find an asymptotic solution of the inhomogeneous linear difference equation
\[
h_{n+2,m}' = \mu_{n,m} h_{n,m}' + \nu_{n,m} h_{n+1,m}' + g_{n,m},
\]
where
\[
\begin{align*}
\mu_{n,m} &:= \frac{1}{3 + 4i} \left(1 + \frac{1}{4n^2} - (2 + 2i) \frac{m^2}{n^2}\right), \\
\nu_{n,m} &:= \frac{2 + 4i}{3 + 4i} \left(1 + \frac{1 - 2i}{40n^2} - \frac{3 - i m^2}{5 n^2}\right), \\
g_{n,m} &:= \frac{-19 + 317i}{400n^3} \left[1 - 14i m^2 \frac{3}{n^3} + 1 + 8i m^8 \frac{8}{3 n^7}\right].
\end{align*}
\]

Similarly as in (24), we can conclude that an asymptotic general solution (ignoring lower order error terms again) of the corresponding homogeneous linear difference equation is
\[
h_{n,m}^{\text{hom}} := d_1,m u_{1,m}(n) + d_2,m u_{2,m}(n),
\]
where \(d_1,m, d_2,m \in \mathbb{C}\). Comparing with (59), it follows that
\[
h_{n,m}^{\text{hom}} = d_1,m e^{-\frac{\pi i}{400n^3}} + d_2,m e^{(2+2i) \frac{m^2}{n^2} + \frac{2 + 4i}{3 + 4i} (3 - 4i) - n}.
\]

Then we are going to find an (asymptotic) particular solution of the inhomogeneous difference equation (59). By variation of parameters, see e.g. [14] Section 3.2], we seek a solution as
\[
h_{n,m}^{\text{inh}} = f_{1,m}(n)u_{1,m}(n) + f_{2,m} u_{2,m}(n),
\]
where \(f_{j,m}(n)\) \((j = 1, 2)\) are unknown functions. Then one obtains a linear system of equations for the differences \(\Delta f_{j,m}(n) := f_{j,m}(n+1) - f_{j,m}(n)\):
\[
\begin{align*}
\Delta f_{1,m}(n)u_{1,m}(n+1) + \Delta f_{2,m}(n)u_{2,m}(n+1) &= 0 \\
\Delta f_{1,m}(n)u_{1,m}(n+2) + \Delta f_{2,m}(n)u_{2,m}(n+2) &= g_{n,m}.
\end{align*}
\]

Thus a particular solution is
\[
h_{n,m}^{\text{inh}} = \sum_{k=1}^{n} g_{k,m} e^{\frac{2 + 2i}{3 + 4i} (\frac{m^2}{n^2} + \frac{2 + 4i}{3 + 4i} (3 - 4i) - k) e^{-\frac{\pi i}{400(n^2 + k)}}}
\]

\[
+ \sum_{k=1}^{n} g_{k,m} e^{\frac{3 + 4i}{(2 + 2i)(n^2 + k^2))} e^{-\frac{2 + 2i}{3 + 4i} (\frac{m^2}{n^2} + \frac{2 + 4i}{3 + 4i} (3 - 4i) - k)n + 2}}.
\]

\[\text{Section 3.2,}\]
Lemma 5 below shows that $h_{n,m}^{in}$ uniformly converges to a finite complex limit for any $|m| \leq \frac{1}{3} n^\frac{2}{3}$ as $n \to \infty$:

$$h_{n,m}^{in} := \lim_{n \to \infty} h_{n,m} = \sum_{k=1}^{\infty} g_{k,m} \frac{e^{\frac{1}{4} \pi^2 i k}}{1 + (3 + 4i)^{-1} e^{\frac{2}{3} n (m + 2)}}$$  \hspace{1cm} (59)$$

For example, $h_{n,0}^{in} \approx -0.156498 + 0.849953i$.

Thus an asymptotic solution of the difference equation (62) and Lemma 5 below show that

$$h_{n,m}^{in} := h_{n,m}^{hom} + h_{n,m}^{inh},$$  \hspace{1cm} (60)$$

as defined by (57) and (58). Let us replace $h_{n,m}$ by $h_{n,m}^{inh}$ in (25), since this way we can explicitly compute approximate values of coefficients.

For $1 \leq m \leq \frac{1}{3} n^\frac{2}{3}$, we choose the coefficient $c_{1,m}$ so that $a_{m,m} = (-1)^m 2^{-2m}$. (So then the error term be zero.) It implies that

$$c_{1,m} = (-1)^m 2^{-2m} \sqrt{m} e^{-im} - c_2 e^{(2+2i)m} (-3 - 4i)^m.$$  

When $m = 0$, we have to modify this, since $a'_{0,0}$ is not defined. Then we demand that $a'_{0,0} = a_{1,0} = -\frac{1}{2} - 2i$. It gives

$$c_{1,0} = -\frac{1}{2} - 2i + (3 + 4i)c_2.$$  

From these it follows that there is a positive constant $k_1$ such that $|c_{1,m}| \leq k_1$ for any $0 \leq m \leq \frac{1}{3} n^\frac{2}{3}$, as was claimed.

To find the coefficients $d_{1,m}$ and $d_{2,m}$ in (57) we demand two boundary conditions. First, we want that the (approximate) error term

$$c_{n,m} := c_2 \frac{e^{-(2+2i)\frac{n^2}{3}}}{n^\frac{2}{3}} (-3 - 4i)^n$$

$$\times \left( z_0 \frac{m^4}{n^3} + \frac{z_1}{n} + d_{1,m} e^{\frac{1}{16}} + d_{2,m} e^{(2+2i)\frac{n^2}{3}} - \frac{\sqrt{m}}{e^{\frac{1}{16}}} (-3 - 4i)^n + h_{n,m}^{inh} \right)$$  \hspace{1cm} (61)$$

be 0 when $n = m$ ($m \geq 1$) and $n = 1$ ($m = 0$). Second, we demand that the factor in parentheses above tend to 0 as $n \to \infty$. From the second condition it follows that

$$d_{1,m} = -h_{n,m}^{inh} \left( 0 \leq m \leq \frac{1}{3} n^\frac{2}{3} \right).$$  \hspace{1cm} (62)$$

From the first condition it follows that

$$d_{2,m} = \left( z_0 m + \frac{z_1}{m} + d_{1,m} e^{\frac{1}{16}} + h_{n,m}^{inh} \right) (-3 - 4i)^m e^{(2+2i)m + \frac{1}{16}}$$  \hspace{1cm} (63)$$

$(1 \leq m \leq \frac{1}{3} n^\frac{2}{3})$ and

$$d_{2,0} = \left( z_1 + d_{1,0} e^{\frac{1}{16}} + h_{n,0}^{inh} \right) (3 + 4i) e^{\frac{3}{16}} \hspace{1cm} (m = 0).$$  \hspace{1cm} (64)$$

Lemma 5 and Lemma 3 below show that

$$|d_{1,m}| \leq k_7 (1 + m^8) \hspace{1cm} (m \in \mathbb{Z}).$$
Hence (63), (64) and Lemma 5 imply that \(|d_{2,m}| \to 0\) as \(m \to \infty\). Consequently, there exists a positive constant \(k_4\) such that \(|d_{2,m}| \leq k_4\) for any \(m \in \mathbb{Z}\).

Since the error term \(h_{n,m}\) is asymptotically equal to \(h''_{n,m}\), by these and Lemma 5 we conclude that

\[
|h_{n,m}| \sim |h''_{n,m}| = \left| -h_{n,m}^{inh} e^{\frac{1}{16\pi} + d_{2,m} e^{(2+2i) \frac{n^2}{k^3}} (-3 - 4i)^{-n} + h_{n,m}^{inh}} \right| 
\]

\[
\leq \kappa_8 n^{-\frac{2}{3}} + k_4 e^{\frac{n}{n} 1/3} 5^{-n} \leq k_2 n^{-\frac{2}{3}}
\]

for any \(|m| \leq \frac{1}{3} n^{\frac{2}{3}}\), with some positive constant \(k_2\). This completes the proof of the theorem.

\[\square\]

**Lemma 5.** If \(h_{n,m}^{inh}\) is defined by (65) and \(h_{n,m}^{inh}\) is defined by (66), then

\[
|h_{n,m}^{inh}| < \kappa_7 (1 + m^8) \quad (m \in \mathbb{Z}),
\]

\[
|h_{n,m}^{inh} - h_{n,m}^{inh}| \leq \kappa_8 n^{-\frac{2}{3}} \quad (|m| \leq \frac{1}{3} n^{\frac{2}{3}}, n \geq 1),
\]

which converges to 0, uniformly in \(m\), as \(n \to \infty\). Finally,

\[
|h_{n,m}^{inh} - h_{n,m}^{inh}| \leq \kappa_9 (1 + m^2) \quad (m \in \mathbb{Z}).
\]

Here and in the proof, each \(\kappa_r\) is a finite, positive constant.

**Proof.** Let us begin with the second sum in (66). We are going to show that the limit of the second sum is 0 when \(|m| \leq \frac{1}{3} n^{\frac{2}{3}}:\)

\[
\sum_{k=1}^{n} \left| g_{k,m} e^{\left( -\frac{2+2i}{2n} \right) (\frac{1}{k^3} - \frac{1}{k^4}) (-3 - 4i)^{-n+2}} \right| 
\]

\[
\leq \kappa_1 \sum_{k=1}^{n} \frac{1 + m^2}{k^3} \frac{m^8}{k^4} e^{(-2m^2 + \frac{m^8}{n^2 k^4}) (3 - 4i)^{-n}} 
\]

\[
\leq \kappa_1 e^{\frac{m^8}{n^2 k^4}} \sum_{k=1}^{n} \frac{1 + m^2}{k^3} \frac{m^8}{k^4} (-3 - 4i)^{-n} 
\]

\[
\leq \kappa_2 (-3 - 4i)^{-\frac{2}{3}} \sum_{k=1}^{n} \frac{1 + m^2}{k^3} + \frac{m^8}{k^4} + \kappa_2 \sum_{k=[n^{\frac{2}{3}}]+1}^{n} \frac{1 + m^2}{k^3} + \frac{m^8}{k^4} 
\]

\[
\leq \kappa_3 n^{\frac{2}{3}} (-3 - 4i)^{-\frac{2}{3}} + \kappa_4 \frac{m^8}{n^2} + \frac{m^8}{n^2} \leq \kappa_5 n^{-\frac{2}{3}},
\]

which converges to 0, uniformly in \(m\), as \(n \to \infty\). In the first inequality above we used that the modulus of the denominator is clearly bigger than 5 for any \(m\) and \(k\). In the second inequality we separately considered the exponent when \(k \leq n - 2\) and when \(k = n - 1, n\). In the last inequality we used the assumption \(|m| \leq \frac{1}{3} n^{\frac{2}{3}}.\)
It is easy to see that the limit $h_{\infty,m}^{\text{inh}}$ is finite:

$$\sum_{k=1}^{n} \left| g_{k,m} \frac{e^{i\frac{\pi}{4}(\frac{k}{m} + \frac{1}{2})}}{1 + (3 + 4i)^{-1}e^{\frac{(2+2i)(m^2-\frac{1}{16})}{k+1}x}} \right| \leq \kappa_0 \sum_{k=1}^{\infty} \left( \frac{1 + m^2}{k^3} + \frac{m^8}{k^7} \right) \leq \kappa_7 (1 + m^8) < \infty$$

for any $m \in \mathbb{Z}$ and $n \geq 1$. In the first inequality above we used that the modulus of the denominator is clearly bigger than $4/5$ for any $m$ and $k$.

It follows from the above estimates that when $|m| \leq \frac{1}{3}n$, $\kappa_5 n^{-\frac{2}{3}} + \kappa_6 \sum_{k=n+1}^{\infty} \left( \frac{1 + m^2}{k^3} + \frac{m^8}{k^7} \right) \leq \kappa_8 n^{-\frac{2}{3}}$, which converges to 0, uniformly in $m$, as $n \to \infty$.

Finally, using the above estimates again, we obtain

$$|h_{\infty,m}^{\text{inh}} - h_{n,m}^{\text{inh}}| \leq \kappa_2 (3 - 4i)^{\frac{|m|}{2}} \sum_{k=\lceil \frac{m}{2} \rceil + 1}^{m} \left( \frac{1 + m^2}{k^3} + \frac{m^8}{k^7} \right) + \kappa_6 \sum_{k=m+1}^{\infty} \left( \frac{1 + m^2}{k^3} + \frac{m^8}{k^7} \right) \leq \kappa_9 (1 + m^2).$$

This completes the proof of the lemma.

Remark 1. By Theorem 1 it follows that

$$c_2 = \lim_{n \to \infty} \frac{a_{n,0}\sqrt{n}}{(-3 - 4i)^n}. $$

It is conjectured that

$$c_2 = \sqrt{(2 + i)\pi} \approx 0.40786 + i 0.0962827. $$

Taking e.g. $n = 1000$, one obtains the approximation

$$c_2 \approx \frac{a_{n,0}\sqrt{n}}{(-3 - 4i)^n} \approx 0.410514 + i 0.0969363,$$

which is relatively close to our conjecture.

Corollary 1. Theorem 7 implies that there exists a constant $C' > 0$ such that

$$\frac{\mu(S_{\ell} = j)}{\sqrt{\frac{2}{\pi} c_2 e^{i\pi j - (2 + 4i)\frac{\mu(S_{\ell} = j)}} (-3 - 4i)^{\frac{\mu(S_{\ell} = j)}}}} = 1 + \delta_{\ell,j},$$

where $|\delta_{\ell,j}| \leq C'\ell^{-\frac{3}{4}}$ for any $\ell \geq 1$ and $|j| \leq \frac{1}{4}\ell^{-\frac{3}{4}}$. Then also

$$1 - C'\ell^{-\frac{3}{4}} \leq \frac{|\mu(S_{\ell} = j)|}{\sqrt{\frac{2}{\pi} c_2 e^{i\pi j - (2 + 4i)\frac{\mu(S_{\ell} = j)}} (-3 - 4i)^{\frac{\mu(S_{\ell} = j)}}}} \leq 1 + C'\ell^{-\frac{3}{4}}.$$ (65)
The error term \((22)\) is really negligible compared to the main term for any \(\ell \geq 1\) and \(|j| \leq \frac{1}{3} \ell^2\):
\[
\frac{|\tilde{e}_{\ell,j}|}{\sqrt{\frac{1}{2} |c_2| e^{-\frac{2}{5} \ell^2}} \leq C' \ell^{-\frac{3}{2}}.}
\]

## 4 Coupled probability distributions

### 4.1 One-dimensional distributions

**Theorem 2.** If \(\ell \geq \ell_0\), with definition \((17)\) we have
\[
\frac{P(S_\ell = j + 2)}{P(S_\ell = j)} \leq e^{-2\frac{j}{\ell}} \leq 1 \quad (0 \leq j \leq \ell - 2).
\]

**Proof.** (Sketch.)

We restrict ourselves to the case where \(\ell = 2n\) and \(j = 2m\), even. The cases with other parities can be treated similarly. Define
\[
a_{n,m} := \mu(S_{2n} = 2m).
\]

The starting point is the reverse recursion \((20)\) for \(\rho_{n,m} = a_{n,m+1}/a_{n,m}\). We have to show that
\[
|\rho_{n,m}| \leq e^{-2\frac{m}{n}} \quad (n - 1 \geq m \geq 0).
\]

By Theorem \([1]\) and Corollary \([1]\) when \(n\) is large enough and \(0 \leq m \leq \frac{1}{3} n^2\), we have \(|\rho_{n,m}| \leq e^{-2\frac{m}{n}}\). So from now it is enough to consider the case \(\frac{1}{3} n^2 < m \leq n - 1\). If \(n\) is large enough, the recursion \((20)\) can be approximated by
\[
\rho_{n,m} \approx \frac{(1 - x)^2}{2(1 - x^2) + i8x^2 - (1 + x)^2\rho_{n,m+1}},
\]
where \(x := m/n \in [0,1]\). Equivalently,
\[
-\log(\rho_{n,m}) \approx 4 \tanh^{-1} x + \log \left( \frac{2(1 - x^2) + i8x^2}{(1 + x)^2} - \rho_{n,m+1} \right).
\]

An approximate solution of \((68)\) is
\[
-\log(\rho_{n,m}) \approx 4 \tanh^{-1} x + i \tan^{-1} \left( \frac{4x^2}{1 - x^2} \right).
\]

Really, substituting this for \(\rho_{n,m+1} \approx \rho_{n,m}\) in \((68)\), the error is negligible compared to \((69)\) for any \(x \in [0,1]\).

Then \((69)\) shows that \((66)\) holds true.

Simulations with Wolfram Mathematica lead to the conjecture that Theorem \([2]\) holds for any \(\ell \geq 1\), that is, \(\ell_0 = 1\).

**Theorem 3.** For the normalizing factor \((18)\) there exists a constant \(C_1 > 0\) such that for any \(\ell \geq 1\),
\[
1 - C_1 \ell^{-\frac{3}{2}} \leq \frac{Z_{\ell}}{\sqrt{\frac{1}{2} |c_2| e^{-\frac{2}{5} \ell^2}}} \leq 1 + C_1 \ell^{-\frac{3}{2}}.
\]
Feynman path integrals

For the coupled probability \([17]\) we have a constant \(C_2 > 0\) such that

\[
1 - C_2 \ell^{-\frac{1}{2}} \leq \frac{\mathbb{P}(S_\ell = j)}{\mathbb{P}(S_{2n} = m)} \leq 1 + C_2 \ell^{-\frac{1}{2}},
\]  

(71)

for any \(\ell \geq 1\) and \(|j| \leq \frac{1}{2} \ell^{\frac{3}{2}}\).

There exists a constant \(C_3 > 0\) such that

\[
1 - C_3 n^{-\frac{1}{2}} \leq \frac{\mathbb{P}(S_{2n} = m)}{\mathbb{P}^*(S_{2n}^* = 2m)} \leq 1 + C_3 n^{-\frac{1}{2}},
\]  

(72)

and

\[
1 - C_3 n^{-\frac{1}{2}} \leq \frac{\mathbb{P}(S_{2n-1} = m)}{\mathbb{P}^*(S_{2n-1}^* = 2m - 1)} \leq 1 + C_3 n^{-\frac{1}{2}},
\]  

(73)

for any \(n \geq 1\) and \(|m| \leq \frac{1}{2} n^{\frac{3}{2}}\), where \((S^*_{\ell})_{\ell \geq 0}\) is an ordinary simple, symmetric random walk w.r.t. the probability \(\mathbb{P}^*\).

\[
\mathbb{P}^*(S^*_\ell = j) = \left(\frac{\ell}{\ell+j}\right)^{2-\ell} \quad (|j| \leq \ell).
\]

(The binomial coefficient is defined to be 0 if \(\ell + j\) is odd.)

The sum of the tail terms is asymptotically negligible:

\[
\sum_{|j| > \frac{1}{2} \ell^{\frac{3}{2}}} \mathbb{P}(S_\ell = j) \leq (1 + C_4 \ell^{-\frac{1}{2}}) \sqrt{\frac{\ell}{2\pi}} e^{-\frac{\delta}{2} \ell^{\frac{3}{2}}}
\]  

(74)

for any \(\ell \geq 1\), where \(C_4 > 0\) is a constant.

Finally, we have symmetry

\[
\mathbb{P}(S_\ell = j) = \mathbb{P}(S_\ell = -j) \quad (\ell \geq 1, |j| \leq \ell).
\]  

(75)

Proof. By Corollary \([17]\) it follows that

\[
\sum_{|j| \leq \frac{1}{2} \ell^{\frac{3}{2}}} |\mu(S_\ell = j)|^2 = \frac{2|c_2|^2 \ell^3}{\ell} \sum_{|j| \leq \frac{1}{2} \ell^{\frac{3}{2}}} e^{-\frac{\delta^2}{2}} |1 + \delta_{\ell,j}|^2,
\]

where \(|\delta_{\ell,j}| \leq C' \ell^{-\frac{1}{2}}\) for any \(\ell \geq 1\). Setting \(h := 2/\sqrt{\pi}\) one gets that

\[
1 - 2 \left[ 1 - \Phi\left(h \left(\frac{1}{3} \ell^{\frac{3}{2}} - 2\right)\right) \right] \leq \frac{1}{\sqrt{2\pi}} \sum_{|j| \leq \frac{1}{2} \ell^{\frac{3}{2}}} h e^{-\frac{1}{2} (h j)^2} \leq 1 + 2 \sqrt{\frac{2}{\pi \ell}}
\]

where \(\Phi(x) := \int_{-\infty}^x \phi(t) \, dt, \phi(x) := (2\pi)^{-\frac{1}{2}} e^{-x^2/2}\). Using the well-known inequality, cf. \([17]\), Lemma 12.9),

\[
\frac{x}{x^2 + 1} \phi(x) < 1 - \Phi(x) < \frac{1}{x} \phi(x) \quad (x > 0),
\]  

(76)

one then obtains

\[
1 - C'' \ell^{-\frac{1}{2}} \leq \frac{\sum_{|j| \leq \frac{1}{2} \ell^{\frac{3}{2}}} |\mu(S_\ell = j)|^2}{\sqrt{2\pi |c_2|^2 \ell^3}} \leq 1 + C'' \ell^{-\frac{1}{2}},
\]  

(77)
for any $\ell \geq 1$ with some constant $C'' > 0$.

On the other hand, by Theorem 2 and Corollary 1 for $\ell \geq \ell_0$,

$$
\sum_{|j| > \frac{1}{3}\ell^\frac{2}{3}} |\mu(S_\ell = j)|^2 = Z_\ell \sum_{|j| > \frac{1}{3}\ell^\frac{2}{3}} \mathbb{P}(S_\ell = j) \leq 2\ell Z_\ell \mathbb{P}\left(S_\ell = \left\lfloor \frac{1}{3}\ell^\frac{2}{3} \right\rfloor \right)
$$

$$
\leq 4|c_2|^25^\ell(1 + C'\ell^{-\frac{2}{3}})^2 \exp\left(-\frac{2}{\ell} \left(\frac{1}{3}\ell^\frac{2}{3}\right)^2\right).
$$

This together with (77) prove (70). In turn, (70) and Corollary 1 imply (71).

Using the ordinary de Moivre–Laplace theorem, cf. e.g. [24], we see that there exists a constant $C''' > 0$ such that

$$
1 - C'''n^{-\frac{2}{3}} \leq \frac{\mathbb{P}^*(S_{2n}^* = 2m)}{\sqrt{n\pi e^{-\frac{m^2}{n}}}} \leq 1 + C'''n^{-\frac{2}{3}}
$$

and

$$
1 - C'''n^{-\frac{2}{3}} \leq \frac{\mathbb{P}^*(S_{2n-1}^* = 2m - 1)}{\sqrt{(n-\frac{1}{2})\pi e^{-\frac{(m-\frac{1}{2})^2}{n-\frac{1}{2}}}}} \leq 1 + C'''n^{-\frac{2}{3}}
$$

for any $n \geq 1$ and $|m| \leq n^{\frac{1}{2}}$. These and (17) imply (72) and (73).

(78) and (70) imply that

$$
\sum_{|j| > \frac{1}{3}\ell^\frac{2}{3}} \mathbb{P}(S_\ell = j) \leq \frac{4(1 + C'\ell^{-\frac{2}{3}})^2 \exp\left(-\frac{2}{\ell} \left(\frac{1}{3}\ell^\frac{2}{3}\right)^2\right)}{(1 - C_1\ell^{-\frac{2}{3}})^{\frac{2}{\pi}}} \leq (1 + C_4\ell^{-\frac{2}{3}})4\sqrt{\frac{2\pi}{\ell}} e^{-\frac{2}{\ell} + \frac{2}{3} \frac{2}{3}}
$$

and this shows (74).

Finally, (75) follows by definition (17). This ends the proof of the theorem.

It is interesting that the random walk approximation described in Theorem 3 is rather good even for small values of $n$:

| $n$ | $m = 0$ | $m = \pm 1$ | $m = \pm 2$ | $m = \pm 3$ | $m = \pm 4$ |
|-----|---------|-------------|-------------|-------------|-------------|
| 1   | .5075/.5 | .2388/.25   | .0075/0     | 0/0         | 0/0         |
| 2   | .3478/.375 | .2584/.25  | .0642/.0625 | .0035/0     | .0000/0     |
| 3   | .3117/.3125 | .2289/.2344 | .0959/.0938 | .0012/.0156 | .0000/0     |
| 4   | .2718/.2734 | .2169/.2188 | .1085/.1094 | .0330/.0313 | .0053/.0039 |
| 5   | .2450/.2461 | .2038/.2051 | .1163/.1172 | .0446/.0439 | .0110/.0098 |
4.2 A Markov chain model and multidimensional distributions

Let us consider now the probability distributions at times $2n$ and $2n + 2$. Our goal is to find an asymptotically nearest neighbor Markov chain describing the transition from time $2n$ to time $2n + 2$, if there exists such a model at all. The assumptions we are going to use are based on the properties of the underlying complex measure walk and on the one-dimensional distributions obtained in the previous subsection.

Define
\[ p_{n,j} := \mathbb{P}(S_{2n+2} = j + 1 \mid S_{2n} = j), \quad q_{n,j} := \mathbb{P}(S_{2n+2} = j - 1 \mid S_{2n} = j). \]

We are going to use the following assumptions for $n \geq 1$:

(i) $\mathbb{P}(|S_{2n+2} - j| \geq 3 \mid S_{2n} = j) = 0$ if $|j| \leq 2n$,

(ii) $\mathbb{P}(|S_{2n+2} - j| = 2 \mid S_{2n} = j) \leq C_{5} n^{-\frac{5}{2}}$ if $|j| \leq \frac{1}{3} n^{\frac{5}{2}}$, where $C_{5} > 0$ is a constant,

(iii) the Markov chain is symmetric w.r.t. reflection about the state 0, that is, $p_{n,-j} = q_{n,j}$ if $|j| \leq 2n$.

Assumptions (i) and (ii) imply for $n \geq 1$ and $|j| \leq \frac{1}{3} n^{\frac{5}{2}}$ that
\[ |\mathbb{P}(S_{2n+2} = j \mid S_{2n} = j) - (1 - p_{n,j} - q_{n,j})| \leq C_{5} n^{-\frac{5}{2}}. \]

We want to find transition probabilities $p_{n,j} \in [0, 1]$ and $q_{n,j} \in [0, 1]$ that satisfy the above assumptions. Because of assumption (iii), it is enough to consider the cases when $j \geq 0$. Then we are led to the following Markovian inequality:
\[ |\mathbb{P}(S_{2n+2} = j) - q_{n,j+1} \mathbb{P}(S_{2n} = j + 1) - p_{n,j-1} \mathbb{P}(S_{2n} = j - 1) - (1 - p_{n,j} - q_{n,j})| \leq (\mathbb{P}(S_{2n} = j) + \mathbb{P}(|S_{2n} - j| = 2)) C_{5} n^{-\frac{5}{2}}, \quad (79) \]

if $n \geq 1$ and $0 \leq j \leq \frac{1}{3} n^{\frac{5}{2}}$.

The next theorem shows that there exists a suitable solution to this inequality.

**Theorem 4.** There exist transition probabilities $p_{n,j}$ and $q_{n,j}$ giving an asymptotically nearest neighbor Markovian solution satisfying assumptions (i) – (iii) such that
\[ |p_{n,j} - \frac{1}{4}| \leq C_{6} n^{-\frac{5}{2}}, \quad |q_{n,j} - \frac{1}{4}| \leq C_{6} n^{-\frac{5}{2}} \quad (n \geq n_{0}, |j| \leq \frac{1}{3} n^{\frac{5}{2}}), \]

with a constant $C_{6} > 0$ and a positive integer $n_{0}$.

**Proof.** Denote the left hand side and the right hand side of (79) by $L_{n,j}$ and $R_{n,j}$, respectively. By (72), for we obtain that
\[ L_{n,j} \leq \left(1 + C_{3}(n + 1)^{-\frac{5}{2}}\right) \left(\frac{2n + 2}{n + 1 + j}\right) 2^{-2n-2} - \left(1 - C_{3} n^{-\frac{5}{2}}\right) 2^{-2n} \]
\[ \times \left[q_{n,j+1} \left(\frac{2n}{n + j + 1}\right) + p_{n,j-1} \left(\frac{2n}{n + j - 1}\right) + (1 - q_{n,j} - p_{n,j}) \left(\frac{2n}{n + j}\right)\right]. \]
Then

\[ L_{n,j}^* := \frac{(n+1+j)!((n+1-j))^{2n}}{(2n)!} |U_{n,j} + C_3 n^{-\frac{1}{2}} V_{n,j}|, \quad (80) \]

where \( p_{n,j}^* := p_{n,j} - \frac{1}{4}, q_{n,j}^* := q_{n,j} - \frac{1}{4}, \)

\[ U_{n,j} := n^2 (q_{n,j}^* + p_{n,j}^* - p_{n,j-1}^* - q_{n,j+1}^*) + n \left( 2(q_{n,j}^* + p_{n,j}^*) - (2j+1) p_{n,j-1}^* - (2j-1) q_{n,j+1}^* \right) - (j^2 - 1)(q_{n,j}^* + p_{n,j}^*) - (j^2 + j)p_{n,j-1}^* - (j^2 - j)q_{n,j+1}^*, \quad (81) \]

and

\[ V_{n,j} := 2n^2 + 3n + \frac{j^2}{2} + \frac{1}{2} - U_{n,j}. \quad (82) \]

Suppose that \(|p_{n,j}^*| \leq C_6 n^{-\frac{1}{2}}\) and \(|q_{n,j}^*| \leq C_6 n^{-\frac{1}{2}}\) for all \( n \geq 1 \) and \(|j| \leq \frac{1}{3} n^{\frac{2}{3}},\)

with some constant \( C_6 > 0. \) Then it follows that

\[ L_{n,j}^* \leq C_6 n^{-\frac{1}{2}} (1 - C_3 n^{-\frac{1}{2}}) 2^{2n} \left[ \frac{2n}{n+j} + \left( \frac{2n}{n+j+2} \right) + \left( \frac{2n}{n+j-2} \right) \right], \]

where we suppose that \( n \geq n_0 \geq 1 \) and \( C_3 n_0^{-\frac{1}{2}} \leq \frac{1}{3}. \) Then

\[ R_{n,j} \geq C_5 n^{-\frac{1}{2}} (1 - C_3 n^{-\frac{1}{2}}) 2^{-2n} \left[ \frac{2n}{n+j} + \left( \frac{2n}{n+j+2} \right) + \left( \frac{2n}{n+j-2} \right) \right], \]

where we suppose that \( n \geq n_0 \geq 1 \) and \( C_3 n_0^{-\frac{1}{2}} \leq \frac{1}{3}. \) Then

\[ R_{n,j}^* := \frac{(n+1+j)!((n+1-j))^{2n}}{(2n)!} R_{n,j} \geq \frac{C_5}{2} n^{-\frac{1}{2}} \left\{ (n+1+j)(n+1-j) + \frac{(n+1-j)(n-j)(n-1-j)}{n+2+j} + \frac{(n+1+j)(n+j)(n-1+j)}{n+2-j} \right\} \]

\[ \geq \frac{C_5}{2} n^{-\frac{1}{2}} \left( \frac{11}{5} n^2 + \frac{8}{5} j n + \frac{9}{5} n + \frac{1}{5} j^2 + \frac{1}{5} j \right), \quad (84) \]

if \(|j| \leq \frac{1}{3} n^{\frac{2}{3}}.\)

We may suppose w.l.o.g. that \( C_5 \geq 6 C_3. \) Then comparing (83) and (84), we see that \( L_{n,j}^* \leq R_{n,j}^* \) for any \( n \geq n_0 \) and \(|j| \leq \frac{1}{3} n^{\frac{2}{3}} \) if \( C_6 \leq C_3/60, \) say.

Starting with \( j = 0 \) in the inequality \( L_{n,j}^* \leq R_{n,j}^* \) in the above proof and going on with values \( j = 1, 2, \ldots \) by induction, it turns out that essentially there are no other solutions satisfying assumptions (i) – (iii) beside the one claimed in Theorem 4 if \( n \) is large enough. The transition probabilities for \(|j| > \frac{1}{3} n^{\frac{2}{3}} \) are unimportant because the tail is negligible by (15) when \( n \) is large enough.

Thus Theorems 3 and 4 essentially determine the multidimensional probability distributions of the coupled process of the complex measure walk on even integer time instants for large enough \( n. \) Let us denote the set

\[ \{2n_0 \ldots 2n\} := \{2n_0, 2n_0 + 2, 2n_0 + 4, \ldots, 2n\} \]
Feynman path integrals

with some positive integers \( n > n_0 \). Take the sample space \( \mathbb{R}^{(2n_0, 2n)} \) and the corresponding Borel \( \sigma \)-field \( \mathcal{B}^{(2n_0, 2n)} \). The coupled probability distribution on the measurable space \( (\mathbb{R}^{(2n_0, 2n)}, \mathcal{B}^{(2n_0, 2n)}) \), essentially determined by Theorems 3 and 4, will be denoted by \( P^{(2n_0, 2n)} \).

The probability distribution for \( n \leq n_0 \) is not really essential for the limiting behaviour, our main interest. However, one can find a solution of assumptions (i) – (iii) for \( P^{(0, 2n_0)} \) on \( (\mathbb{R}^{(0, 2n_0)}, \mathcal{B}^{(0, 2n_0)}) \), similarly as above. Choose the solution \( p_{n,j} = \frac{1}{4} \) and \( q_{n,j} = \frac{1}{4} \) for any \( 0 \leq n \leq n_0 \). Then for \( 1 \leq n \leq n_0 \) and \( |j| \leq \frac{1}{3} n^{\frac{2}{3}} \) by (80) – (82) one obtains that

\[
L_{n,j} \leq C_3 n^{-\frac{1}{9}} \frac{2n^2 + 3n + 1}{(n + 1 + j)(n + 1 - j)} \left( \frac{2n}{n + j} \right)^{2-2n} \leq 2C_3 n^{-\frac{1}{6}}.
\]

Define

\[
P_{\min}(n_0) := \min \{ P(S_{2n} = j) : 1 \leq n \leq n_0, |j| \leq \frac{1}{3} n^{\frac{2}{3}} \}.
\]

By (79),

\[
R_{n,j} \geq 3P_{\min}(n_0) C_5 n^{-\frac{1}{5}}.
\]

Comparing these estimates for \( L_{n,j} \) and \( R_{n,j} \), it follows that the Markovian inequality (79) holds for any \( 1 \leq n \leq n_0 \) and \( |j| \leq \frac{1}{3} n^{\frac{2}{3}} \), supposing \( C_5 \geq \frac{4}{3} C_3/P_{\min}(n_0) \). The cases when \( |j| > \frac{1}{3} n^{\frac{2}{3}} \) are again unimportant: if \( n \) is large, (78) shows that the tail is negligible; when \( n \) is small, the tails can be checked by computer, see Table 4.1.

Theorems 3 and 4 also imply that asymptotically, as \( n \to \infty \), the coupled process on even integer time instants tends to a lazy random walk:

\[
p_{n,j} = q_{n,j} = \frac{1}{4}, \quad P(|S_{2n+2} - j| \geq 2 \mid S_{2n} = j) = 0,
\]

\[
P(S_{2n+2} = j \mid S_{2n} = j) = 1 - p_{n,j} - q_{n,j} = \frac{1}{2} \quad (|j| \leq 2n).
\]

On the other hand, somewhat surprisingly, if one tries to determine asymptotically nearest neighbor transition probabilities moving from a time instant \( 2n \) to \( 2n + 1 \) using the same method, then it turns out that there is no such classical Markov chain model. For example, one gets

\[
P(S_{2n+1} = 2 \mid S_{2n} = 1) = -\frac{1}{12} + O(n^{-\frac{1}{2}}).
\]

5 A strong approximation of Brownian motion based on lazy random walks

The strong approximation that will be discussed in this section is very similar to the “twist and shrink” construction of Brownian motion based on simple, symmetric random walks, see [22] and [23]. In turn, the twist and shrink method was based on Révész’s approach [21] to Knight’s construction [15].
5.1 Lazy random walks

First we define a probability space that we are going to use in this subsection. Let \( \mathbb{N} := \{0, 1, 2, \ldots \} \) be the set of natural numbers. Take the sample space \( \mathbb{R}^N \) and the countable product \( \sigma \)-field \( \mathcal{B}^N \) of the Borel sets in \( \mathbb{R} \). Let \( q^N \) be the countable power of the probability measure

\[
q\{-1\} = q\{1\} = \frac{1}{4}, \quad q\{0\} = \frac{1}{2}
\]

(85)

Let \( (Y(k))_{k=1}^{\infty} \) be the steps of a lazy random walk, each step with the probability distribution \( q \). In other words, \( Y(k) = \pi_k \), where \( \pi_k : \mathbb{R}^N \rightarrow \mathbb{R} \) is the \( k \)th coordinate projection.

The probability measure \( Q \) is defined on \( (\mathbb{R}^N, \mathcal{B}^N) \) as follows. For any \( n \geq 1 \) and Borel measurable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), the distribution of the random variable \( Z = f(Y(1), \ldots, Y(n)) \) is

\[
Q(Z \in A) := q^n(f^{-1}(A)).
\]

Using a standard diagonal procedure, over the probability space \((\Omega, \mathcal{F}, Q)\), the coordinate projections define an infinite matrix of independent and identically distributed random variables \((Y_m(k))_{m \geq 0, k \geq 1}\), each with the distribution \( q \). That is, take \( Y_0(1), Y_0(2), Y_1(1), Y_0(3), Y_1(2), Y_2(1), Y_0(4), Y_1(3), Y_2(2), Y_3(1), \ldots \) in this order. More exactly,

\[
Y_m(n) = \pi_{m+1+(m+n)(m+n-1)/2}.
\]

For each \( m \geq 0 \) define a lazy random walk by

\[
L_m(0) = 0, \quad L_m(n) = \sum_{k=1}^{n} Y_m(k). \quad \tag{86}
\]

It is clear that \( E L_m(n) = 0 \) and \( \text{Var}(L_m(n)) = n/2 \). (In this subsection \( E \) and \( \text{Var} \) are always understood w.r.t. the measure \( Q \).)

Fortunately, each \( Y_m(n) \) has the same distribution as \( S_n^2/2 \), where \((S_n^*)_{n \geq 0}\) is a simple, symmetric random walk started from the origin. \((S_n^*)_{n \geq 1}\) can be obtained as partial sums of the coordinate projections on \((\mathbb{R}^N, \mathcal{B}^N, Q^*)\), where \( Q^* \) is the probability measure obtained from \( q^N \), where \( q^*(1) = q^*(-1) = \frac{1}{2} \). Consequently,

\[
Q(L_m(n) = j) = \sum_{r=0}^{n-|j|} \binom{n}{r} \binom{n-r}{n-r+j} \left( \frac{1}{2} \right)^{n-r} \left( \frac{1}{2} \right)^{r}
\]

\[= \binom{2n}{n+j} 2^{-2n} = Q^*(S_{2n}^* = 2j), \quad \tag{87}\]

for \( n \geq 0 \) and \( |j| \leq n \). Clearly, \( L_m \) is strong Markovian.

For \( m \geq 0 \) we define stopping times \( T_m^*(0) := 0, \)

\[
T_m^*(k+1) := \inf\{n : n > T_m^*(k), |L_m(n) - L_m(T_m^*(k))| = 1\} \quad (k \geq 0). \quad \tag{88}\]

Then the random variables \((T_m^*(k+1) - T_m^*(k))_{k \geq 0}\) are independent and identically distributed,

\[
Q(T_m^*(1) = j) = 2^{-j} \quad (j \geq 1),
\]
a geometric distribution with parameter $\frac{1}{2}$. Then it follows that
\[ E(T_m^*(1)) = 2, \quad \text{Var}(T_m^*(1)) = 2. \] (89)
The moment generating function of $T_m^*(1)$ is
\[ E(e^{uT_m^*(1)}) = \frac{e^u}{2 - e^u}, \]
and of its standardized version is
\[ E(e^{u(T_m^*(1) - 2)/\sqrt{2}}) = \frac{e^{-u/\sqrt{2}}}{2 - e^{u/\sqrt{2}}}. \] (90)
finite if $u < \sqrt{2} \log 2 \approx 0.980258$.

It also follows that $T_m^*(k)$ has negative binomial distribution:
\[ Q(T_m^*(k) = j) = \binom{j - 1}{k - 1} 2^{-j} \quad (j \geq k \geq 1) \]
and
\[ E(T_m^*(k)) = 2k, \quad \text{Var}(T_m^*(k)) = 2k. \]

5.2 A construction of Brownian motion

A key tool in the construction is a large deviation lemma, cf. Lemma 2 and the remarks after it in [22].

**Lemma 6.** Suppose that $(Y_j)_{j \geq 1}$ is a sequence of independent and identically distributed random variables on a probability space $(\Omega, F, \mathbb{P})$: $E(Y_j) = 0$, $E(Y_j^2) = 1$, and the moment generating function $E(e^{uY_j}) < \infty$ in an interval $|u| \leq u_0$, where $u_0 > 0$. Let $Z_0 = 0$ and $Z_n = \sum_{j=1}^{n} Y_j$ (n \geq 1) be the partial sums. Then for any $C > 1$ there exists a positive $N_0(C)$ (possibly depending on the distribution of $Y_j$ as well) such that for any $N \geq N_0(C)$ we have
\[ Q \left( \max_{0 \leq n \leq N} |Z_n| \geq (2CN \log N)^{\frac{1}{2}} \right) \leq N^{1-C}. \]

**Proof.**
\[
Q \left( \max_{0 \leq n \leq N} |Z_n| \geq (2CN \log N)^{\frac{1}{2}} \right) \leq \sum_{n=0}^{N} Q \left( |Z_n| \geq (2CN \log N)^{\frac{1}{2}} \right) \\
\leq \sum_{0 \leq n < (\log N)^{\delta}} Q \left( |Z_n| \geq (2CN \log N)^{\frac{1}{2}} \right) + \sum_{(\log N)^{\delta} \leq n \leq N} Q \left( \frac{|Z_n|}{\sqrt{n}} \geq (2C \log N)^{\frac{1}{2}} \right) \] (91)

Let us estimate the first sum in (91) using an exponential Chebyshev’s inequality:
\[
\sum_{0 \leq n < (\log N)^{\delta}} Q \left( |Z_n| \geq (2CN \log N)^{\frac{1}{2}} \right) \\
\leq \sum_{1 \leq n < (\log N)^{\delta}} \left\{ \left( E(e^{u_0 Y_j}) \right)^n + \left( E(e^{-u_0 Y_j}) \right)^n \right\} e^{-u_0 (2CN \log N)^{\frac{1}{2}}}.\]
We want to show that this is not larger than $\frac{1}{2} N^{1-C}$ if $N$ is large enough. Define
$c_Y := \max (\mathbb{E}(e^{u_0 Y}), \mathbb{E}(e^{-u_0 Y}))$ and take logarithm; then we have to show

$$4 \log \log N + \log 2 + (\log N)^4 \log c_Y - u_0 (2CN \log N)^{1/2} + \log 2 + (C - 1) \log N \leq 0.$$ 

Upon dividing by $N^{1/2}$, we see that it really holds if $N$ is large enough.

To estimate the second sum of (91) under the assumptions of the lemma, we may use the large deviation theorem in \cite{8, Section XVI.6}:

$$\lim_{n \to \infty} \frac{Q(Z_n/\sqrt{N} \geq x_n)}{1 - \Phi(x_n)} = 1, \quad \Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

supposing $x_n \to \infty$ and $x_n = o(n^{1/2})$. Since now $x_n = (2C \log N)^{1/2}$, where $(\log N)^4 \leq n \leq N$, these assumptions hold. Using inequality (76),

$$\sum_{(\log N)^4 \leq n \leq N} Q \left( \left| \frac{Z_n}{\sqrt{N}} \right| \geq (2C \log N)^{1/2} \right) \leq N 2(1 + \epsilon)(1 - \Phi((2C \log N)^{1/2})) \leq \frac{1}{2} Ne^{-C \log N} = \frac{1}{2} N^{1-C},$$

if $N$ is large enough. This completes the proof of the lemma. \hfill \square

Note that in the above lemma $N$ and $N_0(C)$ can be positive real numbers.

**Hoeffding’s lemma** is a useful addition to the previous lemma in the special case of bounded random variables $(Y_j)_{j \geq 1}$. Suppose that $(Y_j)_{j \geq 1}$ is a sequence of independent and identically distributed random variables and $b_j \leq Y_j \leq a_j$. Let $Z_0 = 0$ and $Z_n = \sum_{j=1}^{n} Y_j \ (n \geq 1)$ be the partial sums. Then for any $x > 0$,

$$Q \left\{ \left| Z_n - \mathbb{E}(Z_n) \right| \geq x \left( \frac{1}{n} \sum_{j=1}^{n} (a_j - b_j)^2 \right)^{1/2} \right\} \leq 2 e^{-x^2/4}. \quad (92)$$

Here we are going to modify the “twist and shrink” construction of Brownian motion (Wiener process) discussed in \cite{22}. Here we use a sequence of lazy random walks $(L_m(n))_{n \geq 0}, \ (m \geq 0)$, instead of simple, symmetric random walks applied in \cite{22}. The sample functions of the approximations will be continuous broken lines over the half line $\mathbb{R}_+ = [0, \infty)$ (the time axis). So we take the sample space $\Omega := C(\mathbb{R}_+)$ with the topology of uniform convergence on compact sets, and the corresponding Borel $\sigma$-field $\mathcal{F}$, the smallest $\sigma$-field containing all open sets in $\Omega$. Since the continuous sample functions will be functions of the elements of infinite matrix of lazy random walk steps $(Y_m(k))_{m \geq 0, k \geq 1}$ defined in Subsection 5.1, the probability measure can be extended to $(\Omega, \mathcal{F})$ and will also be denoted by $Q$.

The first approximation $(B_0(t))_{t \geq 0}$ is based on $(L_0(n))_{n \geq 0}$. At the integer time instants $n \in \mathbb{Z}_+$, $B_0(n) := L_0(n)$. For any real $t \in (n, n+1)$ we interpolate:

$$B_0(t) := B_0(n) + \{B_0(n+1) - B_0(n)\}(t-n).$$

Before we define the second approximation based on $(L_1(n))_{n \geq 0}$ we need an important procedure of the construction called “twisting”. Its aim is to make
For twisting we use the stopping times given by (88). A part of \( L \) between two consecutive stopping times \( T_1^*(k) \) and \( T_1^*(k+1) \) will be called a bridge of \( L_1 \). Remember that \( L_1(T_1^*(k+1)) - L_1(T_1^*(k)) = \pm 1 \). Two bridges of \( L_1 \) is going to correspond to one step of \( L_0 \). A horizontal step of \( L_0 \) is mimicked by a combination of a consecutive up and down (or down and up) bridges of \( L_1 \). An up (respectively, a down) step of \( L_0 \) will correspond to two consecutive up (respectively, two consecutive down) bridges of \( L_1 \). If a bridge of \( L_1 \) does not fulfills these rules, then we reflect the bridge.

More exactly, let us see the details.

1. **Twisting.** \( L_0 \) is not twisted, \( \tilde{L}_0(n) = L_0(n) \) for each \( n \geq 0 \). By induction, suppose that \( \tilde{L}_j \) is already defined. The next twisted walk will be defined in terms of \( L_{j+1} \) and \( L_j \).

   - For \( k \geq 1 \) take \( \tilde{Y}_j(k) := \tilde{L}_j(k) - \tilde{L}_j(k-1) \).
   - (a) If \( \tilde{Y}_j(k) = 0 \) and
     \[
     \epsilon_k := \left\{ \begin{array}{ll}
     Y_{j+1}(n) & \text{if } T_{j+1}^*(2k-2) < n \leq T_{j+1}^*(2k-1), \\
     -\epsilon Y_{j+1}(n) & \text{if } T_{j+1}^*(2k-1) < n \leq T_{j+1}^*(2k). 
     \end{array} \right.
     \]
   - (b) if \( \tilde{Y}_j(k) = \pm 1 \), then
     \[
     \epsilon_{k,1} := \tilde{Y}_j(k) \{ L_{j+1}(T_{j+1}^*(2k-1)) - L_{j+1}(T_{j+1}^*(2k-2)) \} = \pm 1, \\
     \tilde{Y}_{j+1}(n) := \epsilon_{k,1} Y_{j+1}(n) & \text{if } T_{j+1}^*(2k-2) < n \leq T_{j+1}^*(2k-1); \\
     \epsilon_{k,2} := \tilde{Y}_j(k) \{ L_{j+1}(T_{j+1}^*(2k)) - L_{j+1}(T_{j+1}^*(2k-1)) \} = \pm 1, \\
     \tilde{Y}_{j+1}(n) := \epsilon_{k,2} Y_{j+1}(n) & \text{if } T_{j+1}^*(2k-1) < n \leq T_{j+1}^*(2k). 
     \]

Finally, we put
\[
\tilde{L}_{j+1}(0) := 0, \quad \tilde{L}_{j+1}(n) := \sum_{r=1}^{n} \tilde{Y}_{j+1}(r) \quad (n \geq 1).
\]

It is important that a twisted lazy random walk \((\tilde{L}_j(n))_{n \geq 0}\) is still a lazy random walk, because as a stochastic process it is not affected by the applied reflections, compare with [22, Lemma 1].

2. **Shrinking.** Now the sample functions of the \( m \)th approximation make steps of length \( 2^{-m} \) in times \( 2^{-2m} \). Thus
\[
B_m(n2^{-2m}) := 2^{-m} \tilde{L}_m(n) \quad (n \in \mathbb{Z}_+) \tag{93}
\]
and for any real \( t \in (n2^{-2m}, (n + 1)2^{-2m}) \),
\[
B_m(t) := B_m(n2^{-2m}) + 2^{2m} \{ B_m((n + 1)2^{-2m}) - B_m(n2^{-2m}) \} (t - n2^{-2m}).
\]

The refinement property of the approximation is
\[
B_{m+1}(T_{m+1}^*(2k)2^{-2(m+1)}) = B_m(k2^{-2m}) \quad (k \geq 0, m \geq 0). \tag{94}
\]
Its meaning is that \( B_{m+1} \) takes the same values as \( B_m \), in the same order, at stopping times \( T_{m+1}^*(2k)2^{-2(m+1)} \) that correspond to the times \( k2^{-2m} \). There is a time lag in general, but these lags a.s. uniformly converge to 0 on any finite time interval as \( m \to \infty \), as easily follows from the next lemma by the Borel–Cantelli lemma.

**Lemma 7.** Let the stopping times \( T_m^*(k) \) be defined by (59). For any \( C > 1 \) there exists a positive \( N_0(C) \) such that for any \( T > 0, \ m \geq 1, \ T2^{2m} \geq N_0(C) \) we have

\[
Q \left\{ \sup_{k2^{-2m} \in [0,T]} |T_{m+1}^*(2k)2^{-2(m+1)} - k2^{-2m}| \geq (\alpha C m2^{-2m}T \log_4 T)^{\frac{1}{2}} \right\} \leq (T2^{2m})^{1-C},
\]

where \( \alpha = \frac{1}{2} + \log 2 \) and \( \log_4 T := \max(1, \log T) \).

**Proof.** We are going to use Lemma 5 for the i.i.d. terms

\[
Y_j = (T_{m+1}^*(2j) - T_{m+1}^*(2j-2) - 4)/2 \quad (j \geq 1).
\]

By (59) and (60), these terms are standardized and have a finite moment generating function in a neighborhood of the origin. Take \( N = T2^{2m} \), with \( m \geq 1 \) fixed. Then

\[
2CN \log N \leq 2(1 + \log 4)Cm2^{2m}T \log_4 T,
\]

and

\[
Q \left\{ \sup_{k2^{-2m} \in [0,T]} |T_{m+1}^*(2k) - 4k|/2 \geq (2(1 + \log 4)Cm2^{2m}T \log_4 T)^{\frac{1}{2}} \right\} \leq (T2^{2m})^{1-C},
\]

which is equivalent to the statement of the lemma. \( \square \)

A consequence of the next lemma is that the sequence \( \{B_m\}_{m \geq 0} \) a.s. uniformly converges on any finite interval as \( m \to \infty \).

**Lemma 8.** Let the sequence of approximations \( \{B_m\}_{m \geq 0} \) be defined by (63). For any \( C > 1 \) there exists a positive \( N_1(C) \) such that for any \( T > 0, \ m \geq 1, \ T2^{2m} \geq N_1(C) \) we have

\[
Q \left\{ \sup_{j \geq 1} \sup_{t \in [0,T]} |B_{m+j}(t) - B_m(t)| \geq 39 \cdot 2^{-\frac{1}{4}} \log_4 T \right\} \leq \frac{5}{1 - 4^{1-C}} (T2^{2m})^{1-C},
\]

where \( T := \max(1, T) \) and \( \log_4 T := \max(1, \log T) \).

**Proof.** Step 1. Fix \( m \) and consider the difference of the \( m \)th and \( (m+1) \)th approximations at the time instants \( t_k := k2^{-2m} \in [0,T] \):

\[
B_{m+1}(t_k) - B_m(t_k) = B_{m+1}(4k2^{-2(m+1)}) - B_{m+1}(T_{m+1}^*(2k)2^{-2(m+1)}) = 2^{-m-1}(\tilde{L}_{m+1}(4k) - \tilde{L}_{m+1}(T_{m+1}^*(2k))).
\]
Let $D_{T,m} := C m^{\frac{1}{2}} 2^{-\frac{3}{2} M} T^\frac{1}{2} (\log_* T)^{\frac{3}{2}}$. Then

$$Q \left\{ \sup_{t_k \in [0,T]} |B_{m+1}(t_k) - B_m(t_k)| \geq \beta D_{T,m} \right\} \leq Q(A_{T,m}) + \sum_{1 \leq k \leq T^{2m}} Q \left\{ \sup_{\{j|j-4k| \leq N'\}} |\tilde{L}_{m+1}(j) - \tilde{L}_{m+1}(4k)| \geq \beta 2^{m+1} D_{T,m} \right\},$$

where

$$A_{T,m} := \left\{ \sup_{t_k \in [0,T]} |T_{m+1}^*(2k) - 4k| \geq N' \right\}, \quad N' := 4(\alpha m 2^{2m} T \log_* T)^{\frac{1}{2}},$$

$\alpha = \frac{1}{2} + \log 2$, and $\beta$ will be chosen below. Now we can apply Lemma 7 to the first term on the right hand side of (96) and Hoeffding’s inequality (92) to the second. We apply Hoeffding’s inequality to the partial sums $Z_n = L_m(n)$, so $E(Z_n) = 0$, $a_j = -b_j = 1$, and for $|j-4k| \leq N'$ we have

$$Q \left\{ |\tilde{L}_{m+1}(j) - \tilde{L}_{m+1}(4k)| \geq x \sqrt{N'} \right\} \leq Q \left\{ |\tilde{L}_{m+1}(j) - \tilde{L}_{m+1}(4k)| \geq x \sqrt{|j-4k|} \right\} \leq 2 e^{-\frac{x^2}{2}}. $$

Suppose $T > 0$, $m \geq 1$, $T^{2m} \geq N_0(C)$. Then Lemma 7 and (92) imply that

$$Q \left\{ \sup_{t_k \in [0,T]} |B_{m+1}(t_k) - B_m(t_k)| \geq \beta D_{T,m} \right\} \leq (T^{2m})^{1-C} + T^{2m} 2^{N'} e^{-\frac{x^2}{2}}, $$

where $x$ is chosen so that $x \sqrt{N'} = \beta 2^{m+1} D_{T,m}$. It follows that

$$x = \frac{\beta 2^{m+1} C m^{\frac{1}{2}} 2^{-\frac{3}{2} M} T^\frac{1}{2} (\log_* T)^{\frac{3}{2}}}{2(\alpha m 2^{2m} T \log_* T)^{\frac{1}{2}}} = \frac{\beta}{\alpha} C^{\frac{1}{2}} m^{\frac{1}{2}} (\log_* T)^{\frac{3}{2}}. $$

On the other hand, we demand that $e^{-\frac{x^2}{2}} \leq (N')^{-C}$, and $C' > 1$ is chosen so that $(N')^{1-C'} \leq (T^{2m})^{(1-C')/2} \leq (T^{2m})^{-C}$. This implies $C' \geq 2C + 1$, so $C' = 3C$ is a suitable choice. There exists a positive $N_1(C) \geq N_0(C)$ such that

$$(2C' \log N')^{\frac{1}{2}} \leq (2C' \log(T^{2m}))^{\frac{1}{2}} \leq (6(1 + \log 4) C m \log_* T)^{\frac{1}{2}}$$

if $T^{2m} \geq N_1(C)$. Thus

$$x \geq (12 \alpha C m \log_* T)^{\frac{1}{2}} \quad (98)$$

satisfies our demand. Comparing (97) and (98) implies $\beta \geq 2 \sqrt{3} \alpha^{\frac{3}{2}}$, e.g. $\beta = 4$ is good.

Thus (96) gives the result of Step 1:

$$Q \left\{ \sup_{t_k \in [0,T]} |B_{m+1}(t_k) - B_m(t_k)| \geq \beta D_{T,m} \right\} \leq 5(T^{2m})^{1-C}. $$

(99)
Step 2. Let $D_{T,m}^* = Cm \hat{2}^{-\frac{1}{2}} T^\frac{3}{2} (\log s, T) \hat{2} \geq D_{T,m}$. By (29), $|B_{m+1}(t_k) - B_m(t_k)| < 4 D_{T,m}^*$ on an event $A_{T,m}$ such that $\mathbb{Q}(A_{T,m}) \geq 1 - 5(T2^{2m})^{1-C}$. With $m$ fixed, consider an interval $[t_k, t_{k+1}]$. First, $|B_m(t_{k+1}) - B_m(t_k)| \leq 2^{-m} \leq 2^{\frac{1}{2}} D_{T,m}^*$. Second, $B_{m+1}$ makes 4 steps of magnitude $\leq 2^{\frac{1}{2}} D_{T,m}^*$. Similarly, at time $t_k + \frac{1}{2}$ the deviation cannot be larger than $4 D_{T,m}^* + 2 \cdot 2^{\frac{1}{2}} \leq (4 + 2^{\frac{1}{2}}) D_{T,m}^*$. Hence

$$
\mathbb{Q} \left\{ \sup_{t \in [0, T]} |B_{m+1}(t) - B_m(t)| \geq \left( 4 + 2^{\frac{1}{2}} \right) D_{T,m}^* \right\} \leq 5(T2^{2m})^{1-C}.
$$

Let us use the fact (obtained by Wolfram Mathematica) that for any $m \geq 1$,

$$
\sum_{j=0}^{\infty} (m+j)^{\frac{1}{2}} 2^{-\frac{m+j}{2}} \leq \frac{2^{\frac{1}{2}}}{2} \sum_{j=0}^{\infty} (1+j)^{\frac{1}{2}} 2^{\frac{j}{2}} < \frac{65}{8} m^{\frac{3}{2}} 2^{\frac{3}{4}}.
$$

Then

$$
\mathbb{Q} \left\{ \sup_{j \geq 1} \sup_{t \in [0, T]} |B_{m+j}(t) - B_m(t)| \geq \frac{65}{8} \left( 4 + 2^{\frac{1}{2}} \right) D_{T,m}^* \right\} \leq \sum_{j=0}^{\infty} \mathbb{Q} \left\{ \sup_{t \in [0, T]} |B_{m+j+1}(t) - B_{m+j}(t)| \geq \left( 4 + 2^{\frac{1}{2}} \right) D_{T,m+j}^* \right\} \leq \sum_{j=0}^{\infty} 5(T2^{2(m+j)})^{1-C} = \frac{5}{1 - 4^{1-C}} (T2^{2m})^{1-C}.
$$

This proves the lemma.

We only sketch the proof of the next theorem, because it is pretty standard; moreover, it is essentially the same as the proof of [23] Theorem 3.

**Theorem 5.** As $m \to \infty$, the approximations $(B_m(t))_{t \geq 0}$ a.s. converge to Brownian motion $(B(t))_{t \geq 0}$. (The variance of an increment $B(t) - B(s)$ is $(t-s)/2$.) Moreover,

$$
\mathbb{Q} \left\{ \sup_{t \in [0, T]} |B(t) - B_m(t)| \geq 39 \cdot Cm^{\frac{1}{2}} 2^{-\frac{1}{2}} T^\frac{3}{2} (\log s, T) \right\} \leq \frac{5}{1 - 4^{1-C}} (T2^{2m})^{1-C}, \quad (100)
$$

**Proof.** (Sketch.)

The a.s. convergence follows from Lemma 8 by the Borel–Cantelli lemma. (100) also follows from Lemma 8.

For any $0 \leq s < t$, as $m \to \infty$,

$$
B(t) - B(s) \sim B_m(t) - B_m(s) \sim 2^{-m} \left( \tilde{L}_m(|t2^{2m}|) - \tilde{L}_m(|s2^{2m}|) \right) = \left( \frac{t-s}{2} \right)^{\frac{1}{2}} \frac{\tilde{L}_m(|t2^{2m}|) - \tilde{L}_m(|s2^{2m}|)}{(t-s)/2} \sim \mathcal{N} \left( 0, \frac{t-s}{2} \right).
$$
where \( N(\mu, \sigma^2) \) denotes a Gaussian random variable with expectation \( \mu \) and variance \( \sigma^2 \).

Finally, for any \( 0 \leq s < t \leq u < v \), the increments \( B(v) - B(u) \) and \( B(t) - B(s) \) are independent, because for any \( m \geq 1 \), the approximating increments

\[
2^{-m} \left( \tilde{L}_m(\lfloor v 2^m \rfloor) - \tilde{L}_m(\lfloor u 2^m \rfloor) \right) \quad \text{and} \quad 2^{-m} \left( \tilde{L}_m(\lfloor t 2^m \rfloor) - \tilde{L}_m(\lfloor s 2^m \rfloor) \right)
\]

are independent. \( \square \)

6 A strong approximation of Brownian motion based on coupled random walks

Take the measurable space \((\Omega, \mathcal{F})\) as in Subsection 5.2. Fix a positive integer \( T \). The value of the positive integer \( m \) should be large enough, as specified later. We also need an infinite matrix of independent, simple, symmetric random walk steps \((Z_j(k))_{j \geq 0, k \geq 1}\), similarly defined as the matrix of lazy random walk steps in Subsection 5.1:

\[
\mathbb{Q}^* \left( Z_j(k) = \frac{1}{2} \right) = \mathbb{Q}^* \left( Z_j(k) = -\frac{1}{2} \right) = \frac{1}{2}.
\]

First, take complex measure walk steps \( X_r \) \((1 \leq r \leq 2T)\) on the set \( \{1, 2, \ldots, 2T\} \), see Section 2. Augment this with simple, symmetric random walk steps \((Z_0(k))_{k \geq 0}\) on the set \( \{2T + 1, 2T + 2, 2T + 3, \ldots\} \) and denote the so obtained infinite walk by \((S_0(n))_{n \geq 0}\): \( S_0(0) = 0 \),

\[
S_0(n) = \sum_{r=1}^{n} X_r \quad (1 \leq n \leq 2T), \quad S_0(2T + k) = S_0(2T) + \sum_{r=1}^{k} Z_0(r) \quad (k \geq 1).
\]

Second, take complex measure walk steps \( X_r \) \((2T + 1 \leq r \leq 2T + 2^4 T)\) on \( \{1, 2, 3, \ldots, 2^4 T\} \). Augment this with simple, symmetric random walk steps \((Z_1(k))_{k \geq 0}\) on the set \( \{2^4 T + 1, 2^4 T + 2, 2^4 T + 3, \ldots\} \), and denote the so obtained infinite walk by \((S_1(n))_{n \geq 0}\): \( S_1(0) = 0 \),

\[
S_1(2^4 T + k) = S_1(2^4 T) + \sum_{r=1}^{k} Z_1(r) \quad (k \geq 1).
\]

And so on, at the \( m \)th walk, take complex measure walk steps \( X_r \) \((1 + 2T + 2^4 T + 2^7 T + \cdots + 2^{3m-2} T \leq r \leq 2^T + 2^4 T + 2^7 T + \cdots + 2^{3m+1} T)\) on the set \( \{1, 2, \ldots, 2^{3m+1} T\} \). Augment this with simple, symmetric random walk steps \((Z_m(k))_{k \geq 0}\) on the set \( \{2^{3m+1} T + 1, 2^{3m+1} T + 2, 2^{3m+1} T + 3, \ldots\} \), and denote the so obtained infinite walk by \((S_m(n))_{n \geq 0}\): \( S_m(0) = 0 \),

\[
S_m(2^{3m+1} T + k) = S_m(2^{3m+1} T) + \sum_{r=1}^{k} Z_m(r) \quad (k \geq 1).
\]
When they are large enough, consider an integer 1. In sum, we needed a corresponding coupled probability measure \( P \) on \( (\Omega^{(0...2N)}, B^{(0...2N)}) \), on the even non-negative integers, for the finite triangular matrix \((S_j(2k)) (0 \leq j \leq m, 0 \leq k \leq 2^{(j+1)}T)\) of random walks defined above. Each row of this matrix was augmented by infinitely many simple symmetric random walk steps \( Z_m(r) \). This way, we are given the probability measure \( P \times Q^* \) on the set of even positive integers, that is, on \((\mathbb{R}^{2N}, B^{2N})\). Since the continuous sample functions will be functions of the elements of infinite matrix of the steps defined above, the probability measure can be extended to \((\Omega, B)\) and will be denoted by \( P \).

Because of the symmetry (75) of \( S_j(2k) \) about 0, \( \mathbb{E} S_j(2k) = 0 \) for any \( j \geq 0 \) and \( k \geq 0 \). By (72) and (74), there exists a constant \( C_7 > 0 \) such that

\[
(1 - C_7 k^{-\frac{1}{2}}) \frac{k}{2} \leq \text{Var}(S_j(2k)) \leq (1 + C_7 k^{-\frac{1}{2}}) \frac{k}{2}
\]

for any \( j \geq 0 \) and \( k \geq 1 \). By assumption (i) in Subsection 1.2, \( S_j \) has bounded increments, \( |S_j(2k+2) - S_j(2k)| \leq 2 \).

Now we are going to define stopping times, similarly to (88). Take \( j \geq 1 \). Define \( T_j(0) := 0 \), and for any \( k \geq 0 \),

\[
T_j(k + 1) := \inf\{2n : 2n > T_j(k), |S_j(2n) - S_j(T_j(k))| \geq 1\}. \tag{101}
\]

To determine the distribution and moments of these stopping times, at least when they are large enough, consider an integer \( n \geq n_0 \), where \( n_0 \) is defined in Theorem 1.1. Put

\[
\tau_n := \inf\{\ell \geq 1 : |S_1(2n + 2\ell) - S_1(2n)| \geq 1\}.
\]

Below it is supposed that \( n \) is sufficiently large. Then

\[
\mathbb{P}(\tau_n = 1) = \mathbb{P}\left(|S_1(2n)| \leq \frac{1}{3} n^{\frac{1}{2}} \right) \mathbb{P}(|S_1(2n + 2) - S_1(2n)| \geq 1) + \mathbb{P}\left(|S_1(2n)| > \frac{1}{3} n^{\frac{1}{2}} \right) \mathbb{P}(|S_1(2n + 2) - S_1(2n)| \geq 1)
\]

So by Theorems 3, 4 and assumption (ii) in Subsection 1.2, we obtain that

\[
\mathbb{P}(\tau_n = 1) \leq \frac{1}{2} + 2C_9 n^{-\frac{1}{12}} + C_9 n^{-\frac{1}{12}} + (1 + C_4 (2n)^{-\frac{1}{2}}) 4 \sqrt{\frac{n}{\pi}} e^{-\frac{\pi}{8}(2n)^{\frac{1}{2}}} \leq \frac{1}{2} + C_9 n^{-\frac{1}{12}},
\]

where \( C_9 > 0 \) is a suitable constant. Similarly,

\[
\mathbb{P}(\tau_n = 1) \geq \left(\frac{1}{2} - 2C_9 n^{-\frac{1}{12}}\right) \left(1 - (1 + C_4 (2n)^{-\frac{1}{2}}) 4 \sqrt{\frac{n}{\pi}} e^{-\frac{\pi}{8}(2n)^{\frac{1}{2}}}\right) \geq \frac{1}{2} - C_9 n^{-\frac{1}{12}}.
\]
These imply that for any \( k \geq 1 \),
\[
\left( \frac{1}{2} - C_8 n^{-\frac{4}{3}} \right)^k \leq \mathbb{P}(\tau_n = k) \leq \left( \frac{1}{2} + C_8 n^{-\frac{4}{3}} \right)^k.
\]
(Compare with Subsection 5.1.) So for the first two moments we get
\[
\mathbb{E}(\tau_n) \leq 6 \left( \frac{1 - 2 C_8 n^{-\frac{4}{3}}}{(1 + 2 C_8 n^{-\frac{4}{3}})^3} \right) \leq \mathbb{E}(\tau_n) \leq 6 \left( \frac{2 C_8 n^{-\frac{4}{3}} + 1}{(1 - 2 C_8 n^{-\frac{4}{3}})^3} \right)
\]
That is, there exists a constant \( C_9 > 0 \) such that
\[
2(1 - C_9 n^{-\frac{4}{3}}) \leq \mathbb{E}(\tau_n) \leq 2(1 + C_9 n^{-\frac{4}{3}}),
\]
\[
2(1 - C_9 n^{-\frac{4}{3}}) \leq \text{Var}(\tau_n) \leq 2(1 + C_9 n^{-\frac{4}{3}}).
\]
Also,
\[
\frac{e^n(1 - 2 C_9 n^{-\frac{4}{3}})}{2 - e^n(1 - 2 C_9 n^{-\frac{4}{3}})} \leq \mathbb{E} e^{u \tau_n} \leq \frac{e^n(1 + 2 C_9 n^{-\frac{4}{3}})}{2 - e^n(1 + 2 C_9 n^{-\frac{4}{3}})},
\]
where \( \tau_n^* := (\tau_n - \mathbb{E}(\tau_n))(\text{Var}(\tau_n))^{-\frac{1}{2}} \) and \( C_{10} > 0 \) is a suitable constant. It follows that the moment generating function \( \mathbb{E} e^{u \tau_n^*} \) is finite in a neighborhood \(|u| \leq u_0, u_0 > 0\), if \( n \geq n_1 = n_2(C_{10}) \geq n_0 \).

At this point it is natural trying to generalize Lemma 6 to the current situation. The random variables we have to consider are \( (\tau_j^*(k))_{k \geq k_1} \) (\( j \geq j_1 \)), where \( \tau_j(k) := T_j(k) - T_j(k - 1) \), \( \tau_j^*(k) := (\tau_j(k) - \mathbb{E}\tau_j(k))(\text{Var}(\tau_j(k)))^{-\frac{1}{2}} \), and the positive integers \( k_1 \) and \( j_1 \) are chosen so that
\[
T_j(k - 1) \geq n_1 \quad \mathbb{P} - \text{a.s. for any } k \geq k_1 \text{ and } j \geq j_1.
\]
We suppose that \( m \geq j_1 \). By the previous computations, the random variables \( \tau_j^*(k) \) are asymptotically independent and identically distributed w.r.t. \( \mathbb{P} \), and otherwise satisfy all the other conditions of Lemma 6. We suppose now that the statement of the lemma is valid to this case as well.

Now we are ready to begin a "twist and shrink" construction of Brownian motion, based on the the random walks \( (S_j(2k))_{k \geq 0} \) (\( j \geq 0 \)). It is going to be a slight modification of the one discussed in Subsection 5.2.

(1) \( \text{T} \text{W} \text{i} \text{s} \text{t} \text{i} \text{n} \text{g} \). The first approximation is based on \( (S_0(n))_{n \geq 0} \). It is not twisted: \( \tilde{S}_0(n) = S_0(n) \) for all \( n \geq 0 \). By induction, suppose that the \( j \)th twisted random walk \( \tilde{S}_j(n)_{n \geq 0} \) is already defined where \( j \geq 0 \). The next twisted walk will be based on \( S_{j+1} \) and \( \tilde{S}_j \). Note that reflections will be based only on even
time instants, although a bridge reflected will contain the included odd time instants as well.

For $k \geq 1$ define $\tilde{Y}_j(k) := \tilde{S}_j(2k) - \tilde{S}_j(2k - 2)$.

(a) If $\tilde{Y}_j(k) = 0$ and

$$\epsilon_k := \{S_j(T_{j+1}(2k - 1)) - S_{j+1}(T_{j+1}(2k - 2))\} \times \{S_{j+1}(T_{j+1}(2k)) - S_{j+1}(T_{j+1}(2k - 1))\} = \pm 1,$$

then we set

$$\tilde{X}_{j+1}(n) := \begin{cases} S_{j+1}(n) - S_{j+1}(n - 1) & \text{if } T_{j+1}(2k - 2) < n \leq T_{j+1}(2k - 1), \\ -\epsilon_k(S_{j+1}(n) - S_{j+1}(n - 1)) & \text{if } T_{j+1}(2k - 1) < n \leq T_{j+1}(2k). \end{cases}$$

(b) If $\tilde{Y}_j(k) = \pm 1$, let

$$\epsilon_{k,1} := \tilde{Y}_j(k)\{S_{j+1}(T_{j+1}(2k - 1)) - S_{j+1}(T_{j+1}(2k - 2))\} = \pm 1,$$

$$\tilde{X}_{j+1}(n) := \epsilon_{k,1} \tilde{Y}_{j+1}(n) & \text{if } T_{j+1}(2k - 2) < n \leq T_{j+1}(2k - 1);$$

$$\epsilon_{k,2} := \tilde{Y}_j(k)\{S_{j+1}(T_{j+1}(2k)) - S_{j+1}(T_{j+1}(2k - 1))\} = \pm 1,$$

$$\tilde{X}_{j+1}(n) := \epsilon_{k,2} \tilde{Y}_{j+1}(n) & \text{if } T_{j+1}(2k - 1) < n \leq T_{j+1}(2k).$$

Finally,

$$\tilde{S}_{j+1}(0) := 0, \quad \tilde{S}_{j+1}(n) := \sum_{r=1}^{n} \tilde{X}_{j+1}(r) \quad (n \geq 1).$$

It is important that a complex measure walk and also, a simple, symmetric random walk preserve their basic properties, because as stochastic processes they are not affected by the applied reflections.

(2) Shrinking. The sample functions of the $j$th approximation make steps of length $2^{-j}$ in time-steps $2^{-2j}$. Thus

$$B_j(n2^{-2j}) := 2^{-j} \tilde{S}_j(n) \quad (n \geq 0) \quad (105)$$

and for any real $t \in (n2^{-2j}, (n + 1)2^{-2j})$,

$$B_j(t) := B_j(n2^{-2j}) + 2^{2j} \{B_j((n + 1)2^{-2j}) - B_j(n2^{-2j})\}(t - n2^{-2j}).$$

The refinement property of the approximation is

$$B_{j+1}(T_{j+1}(2k)2^{-2(j+1)}) = B_j(2k2^{-2j}) \quad (k, j \geq 0). \quad (106)$$

Its meaning is that $B_{j+1}$ takes the same values as $B_j$, in the same order, at stopping times $T_{j+1}(2k)2^{-2(j+1)}$ that correspond to the times $2k2^{-2j}$. There is a time lag in general, but these lags a.s. uniformly converge to 0 on any finite time interval as $j \to \infty$, as easily follows from the next lemma by the Borel–Cantelli lemma. The next lemma is a slight modification of Lemma 7 thus its proof is omitted.
Lemma 9. Let the stopping times $T_j(k)$ be defined by (101). For any $C > 1$ there exists a positive $N_0(C)$ such that for any $T > 0$, $j \geq j_1$, $T_{2^j} \geq N_0(C)$ we have

$$\mathbb{P}\left\{ \sup_{2k2^{-2j} \in [t, T]} |T_{j+1}(2k)2^{-2(j+1)} - 2k2^{-2j}| \geq (\alpha_0 Cj2^{-2j} T \log_* T)^\frac{1}{2} \right\} \leq (T2^j)^{1-C},$$

where $t_1 := k_1 2^{-2j}$, $\alpha_0 > 0$ is a suitable constant, and $\log_* T := \max(1, \log T)$.

By (72) and the first part of the present section, the increments $S_j(2k) - S_j(2k - 2)$ are asymptotically independent and identically distributed random variables w.r.t. $\mathbb{P}$, with zero expectation, and are bounded by 2, at least when $k$ is large enough, $k \geq k_1$. (Suppose that $k_1$ was chosen so.) Assuming that Hoeffding’s inequality (92) can be generalized to this case as well, we can state the following slight modification of Lemma 8 whose proof is therefore omitted.

Lemma 10. Let the sequence of approximations $(B_j)_{j \geq 0}$ be defined by (105). For any $C > 1$ there exists a positive $N_1(C)$ such that for any $T > 0$, $j \geq j_1$, $T_{2^j} \geq N_1(C)$ we have

$$\mathbb{P}\left\{ \sup_{k \geq 1} \sup_{t \in [t_1, T]} |B_{j+k}(t) - B_j(t)| \geq \beta_0 Cj2^{j+2} T \left( \log_* T \right)^\frac{1}{2} \right\} \leq \frac{5}{1 - 4^{1-C}} (T2^j)^{1-C},$$

where $t_1 = k_1 2^{-2j}$, $\beta_0 > 0$ is a suitable constant, $T_* := \max(1, T)$ and $\log_* T := \max(1, \log T)$.

As in the case of Theorem 5, this lemma implies the following theorem, whose proof is again omitted.

Theorem 6. As $j \to \infty$, the approximations $(B_j(t))_{t \geq 0}$ $\mathbb{P}$-a.s. converge to Brownian motion $(B(t))_{t \geq 0}$. (The variance of an increment $B(t) - B(s)$ is $(t - s)/4$.) Moreover,

$$\mathbb{P}\left\{ \sup_{t \in [k_1 2^{-2m}, T]} |B(t) - B_m(t)| \geq \beta_0 Cm2^{-m} T_m \left( \log_* T_m \right)^\frac{1}{2} \right\} \leq \frac{5}{1 - 4^{1-C}} (T2^m)^{1-C}, \quad (107)$$

where $k_1$ is defined by (104) and the other constants are the same as in Lemma 10.

The really important consequence of this theorem is that the random walk $(B_m(t))_{t \leq T}$ coupled to a complex measure walk is arbitrarily close to a Brownian motion a.s. if $m$ is large enough.
7 Convergence and the Schrödinger equation

Temporarily fix an integer $m \geq j_1$, where $j_1$ is defined by (103). Introduce the notations $\Delta t := 2^{-m}$, $\Delta := 2^{-2m}$, and $t_{r} := r2^{-2m}$, where $r \geq 0$ is an integer. Now we extend the discrete path integral to the twisted and shrunk random walks $B_{m}^{x}(t_{r}) := x + B_{m}(t_{r})$ ($x \in \mathbb{R}$). It means that if $x_{j} := 2^{-m}$ ($j \in \mathbb{Z}$) and $B_{m}(t_{r}) = x_{j}$, then $B_{m}(t_{r+1})$ is concentrated on $\{x_{j-1}, x_{j}, x_{j+1}\}$ and

$$\mu(B_{m}(t_{r+1}) = x_{j+1}) = \mu(B_{m}(t_{r+1}) = x_{j-1}) = \frac{i}{2}, \quad \mu(B_{m}(t_{r+1}) = x_{j}) = 1 - i.$$ 

Take functions $V, g : \mathbb{R} \to \mathbb{R}$ and define

$$\psi_{m}(t_{k}, x) := \mathbb{E}_{\mu} \left\{ \exp \left( -i \sum_{r=0}^{k-1} V(B_{m}^{x}(t_{r})) \Delta t \right) g(B_{m}^{x}(t_{k})) \right\}.$$  
(108)

By the argument of Lemma 11 this discrete path integral is the unique solution of the discrete Schrödinger equation

$$\frac{1}{i} \psi_{m}(t_{k+1}, x) - \psi_{m}(t_{k}, x) = \frac{1}{2} e^{-iV(x) \Delta t} \psi_{m}(t_{k}, x + \Delta x) - 2 \psi_{m}(t_{k}, x) + \psi_{m}(t_{k}, x - \Delta x)$$

$$\frac{1}{(\Delta x)^{2}} + \frac{1}{i} e^{-iV(x) \Delta t} - \frac{1}{\Delta t} \psi_{m}(t_{k}, x), \quad \psi_{m}(0, x) = g(x).$$  
(109)

**Theorem 7.** Suppose that $V, g \in C^{2}$ and $V, V', V'', g, g', g''$ are bounded. Then there exists a function $\psi \in C^{1,2}(\mathbb{R}^{+} \times \mathbb{R} \to \mathbb{C})$ that solves Schrödinger equation (7) for $(t, x) \in [0, T] \times \mathbb{R}$ and for any $\epsilon > 0$ there exists an $m_{0}$ such that

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} |\psi_{m}(t^{(m)}, x) - \psi(t, x)| < \epsilon \quad \text{if} \quad m \geq m_{0},$$

where $t^{(m)} := [2^{2m}]2^{-2m}$.

**Proof.** Define the complex valued random variables

$$Y_{m}(t_{k}, x) := \exp \left( -i \sum_{r=0}^{k-1} V(B_{m}^{x}(t_{r})) \Delta t \right) g(B_{m}^{x}(t_{k})),\quad Y(t, x) := \exp \left( -i \int_{0}^{t} V(B^{x}(s)) ds \right) g(B^{x}(t)) \quad ((t, x) \in \mathbb{R}^{+} \times \mathbb{R}),$$

where $B^{x}(t) := x + B(t)$ and $B$ is defined in Theorem 6. Then by our assumptions it follows that for any $\epsilon > 0$ there exists an $m_{0}$ such that

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} |Y_{m}(t^{(m)}, x) - Y(t, x)| < \epsilon \quad \text{P-a.s.} \quad \text{if} \quad m \geq m_{0};$$

$$\sup_{(t, x) \in \mathbb{R}^{+} \times \mathbb{R}} |Y_{m_{1}}(t^{(m_{1})}, x) - Y_{m_{2}}(t^{(m_{2})}, x)| < \epsilon \quad \text{P-a.s.} \quad \text{if} \quad m_{1}, m_{2} \geq m_{0}.$$ 

Hence

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} |\psi_{m_{1}}(t^{(m_{1})}, x) - \psi_{m_{2}}(t^{(m_{2})}, x)| < \epsilon \quad \text{if} \quad m_{1}, m_{2} \geq m_{0}. \quad (110)$$
by our assumptions we similarly obtain that

\[
\langle \partial_{xx} \psi_m \rangle (t_k, x) = \mathbb{E}_\mu \left\{ \exp \left( -i \sum_{r=0}^{k-1} V(B_{m}^r(t_r)) \Delta t \right) \left[ \left( -i \sum_{r=0}^{k-1} V'(B_{m}^r(t_r)) \Delta t \left( g(B_{m}^r(t_k)) - g(B_{m}^r(t_k)) \Delta t \right) \right) - i \sum_{r=0}^{k-1} V''(B_{m}^r(t_r)) \Delta t \right] \right\},
\]

by our assumptions we similarly obtain that

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \left| \langle \partial_{xx} \psi_{m_1} \rangle (t^{(m_1)}), x) - \langle \partial_{xx} \psi_{m_2} \rangle (t^{(m_2)}, x) \right| < \epsilon \quad \text{if} \quad m_1, m_2 \geq m_0.
\]

\[(110)\] and \[(111)\] imply that there exists a function \( \psi \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{C}) \) such that

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \left| \psi_m(t^{(m)}, x) - \psi(t, x) \right| < \epsilon,
\]

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \left| \langle \partial_{xx} \psi_m \rangle (t^{(m)}, x) - \langle \partial_{xx} \psi \rangle (t, x) \right| < \epsilon, \quad \text{if} \quad m \geq m_0.
\]

Thus

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \left| \frac{1}{2} e^{-iV(x) \Delta t} \psi_m(t_k, x + \Delta x) - 2 \psi_m(t_k, x) + \psi_m(t_k, x - \Delta x) \right| \left( \frac{\Delta x}{2} \right)^2
\]

\[
+ \frac{1}{\Delta t} \left. \langle \partial_{xx} \psi \rangle (t, x) \right| < \epsilon
\]

if \( m \geq m_0 \). By \[(109)\] this implies that \( \psi \) is differentiable w.r.t. \( t \) and \( \psi \) solves Schrödinger equation \[(1)\] for \((t, x) \in [0, T] \times \mathbb{R}\).

\( \square \)

Finally, let us see the convergence of a normalized complex measure describing the motion of the twisted and shrunken random walk \( B_m \). By Corollary \[(1)\]

\[
\mu(S_\ell = j) \sim \sqrt{\frac{\mathcal{C}_2}{\ell}} e^{i \pi j - (2 + i) \frac{\ell^2}{2} (-3 - 4i) \frac{\ell}{2}},
\]

where \( \sim \) denotes asymptotic equality as \( \ell \to \infty \). It means that

\[
\mu(B_m(t_k) = x_j) \sim \frac{\mathcal{C}_2}{\ell_k} \exp \left( i \pi x_j 2^m - (2 + i) \frac{x_j^2}{2t_k} \right) (-3 - 4i)^{1/2} 2^{2m-1} 2^{-m}.
\]

Using the independence of increments of \( B_m \) with the complex measure, it
follows that for \(0 = t_{k_0} < t_{k_1} < t_{k_2} < \cdots < t_{k_d}\),

\[
\mu(B_m(t_{k_1}) = x_{j_1}, B_m(t_{k_2}) = x_{j_2}, \ldots, B_m(t_{k_d}) = x_{j_d}) \sim \frac{2^d c_d^2}{\prod_{r=1}^d (t_{kr} - t_{k(r-1)})} \exp \left( i \pi x_d 2^m - (2 + i) \sum_{r=1}^d \frac{(x_{jr} - x_{j(r-1)})^2}{2(t_{kr} - t_{k(r-1)})} \right) \\
\times (-3 - 4i) t_{kd} 2^{m - 1} 2^{-md}.
\]

Here we introduce a complex normalizing factor, which is somewhat similar to the square root of \(Z_\ell\) in (70) and which will divide the measure \(\mu\):

\[
Z_{td}^* := \sqrt{\frac{4\pi}{2 + i} 2^{d} t_{d} 2^{m - 1}}.
\]

Moreover, we consider only those time instants \(t_d\) for which

\[
\left(1 + \frac{1}{\pi} \arctan \frac{4}{3}\right) t_{d} 2^{m - 1}
\]

is an even positive integer, and only those points \(x_d\) for which \(x_d 2^m\) is an even integer. These points are dense on the \(t\) and \(x\) axis, respectively, as \(m \to \infty\). This way we arrive at the time homogeneous and spatially homogeneous complex transition function

\[
\nu_t(x, dy) := \sqrt{\frac{2 + i}{2\pi t}} e^{-\left(2 + i\right) \frac{x^2}{2t}} dy.
\]

The following argument is taken from [16]. This \(\nu\) satisfies the Chapman-Kolmogorov equation and

\[
\int_{\mathbb{R}} \nu_t(x, dy) = 1, \quad \|\nu_t(x, \cdot)\| = \frac{5^4}{2^2}.
\]

Thus \(\nu\) generates a strongly continuous, regular semigroup over \(C_0(\mathbb{R})\) (continuous functions vanishing at infinity) by the formula

\[
(T_t f)(x) := \int_{\mathbb{R}} f(y) \nu_t(x, dy).
\]

Then we introduce the one-point compactification \(\hat{\mathbb{R}}\) and for \(0 \leq s < t\) the path space \(\hat{\mathbb{R}}^{[s,t]}\) (infinite product of \([s,t]\) copies). Then \(\nu\) defines a family of linear functionals on cylindric functions \(C_y^{[s,t]}\) on the path space by the formula

\[
\nu_{s,t}^* (\phi_{t_0, \ldots, t_{k+1}}) := \int_{\hat{\mathbb{R}}^{k}} f(x, y_1, \ldots, y_{k+1}) \nu_{t_0} - t_0 (x, dy_1) \nu_{t_1} - t_1 (y_1, dy_2) \ldots \nu_{t_{k+1}} - t_k (y_k, dy_{k+1}),
\]

where \(k \geq 0, s = t_0 < t_1 < \cdots < t_k < t_{k+1} = t, x \in \mathbb{R}, f : \hat{\mathbb{R}}^{k+2} \to \mathbb{C}\) is a bounded Borel function, and

\[
\phi_{t_0, \ldots, t_{k+1}}^y(\cdot) := f(y(t_0), \ldots, y(t_{k+1})).
\]
Then by continuity, \( \nu_{s,t}^x \) can be extended to a unique bounded linear functional on \( C(\mathbb{R}^{[s,t]}) \), and consequently there exists a Borel measure \( \mu_{s,t}^x \) on the path space \( \mathbb{R}^{[s,t]} \) such that

\[
\nu_{s,t}^x(F) = \int F(y(\cdot)) \mu_{s,t}^x(dy(\cdot)) \quad (F \in C(\mathbb{R}^{[s,t]})),
\]

\[
(T_t f)(x) = \int f(y(t)) \mu_{s,t}^x(dy(\cdot)).
\]

It is an open question whether the function \( \psi(t,x) \) of Theorem 7 can be obtained by this Borel measure \( \mu_{s,t}^x \) from the complex random variables \( Y(t,x) \) or not.

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