REMARKS ON THE LARGE TIME BEHAVIOR OF VISCOSITY SOLUTIONS OF QUASI-MONOTONE WEAKLY COUPLED SYSTEMS OF HAMILTON–JACOBI EQUATIONS

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ABSTRACT. We investigate the large-time behavior of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton–Jacobi equations on the $n$-dimensional torus. We establish a convergence result to asymptotic solutions as time goes to infinity under rather restricted assumptions.

1. Introduction

In this paper we study the large time behavior of the viscosity solutions of the following weakly coupled systems of Hamilton–Jacobi equations

\[
\begin{align*}
(u_1)_t + H_1(x, Du_1) + c_1(u_1 - u_2) &= 0 & \text{in } \mathbb{T}^n \times (0, \infty), \\
(u_2)_t + H_2(x, Du_2) + c_2(u_2 - u_1) &= 0 & \text{in } \mathbb{T}^n \times (0, \infty), \\
u_1(x, 0) &= u_{01}(x), & u_2(x, 0) &= u_{02}(x) & \text{on } \mathbb{T}^n,
\end{align*}
\]

where the Hamiltonians $H_i \in C(\mathbb{T}^n \times \mathbb{R}^n)$ are given functions which are assumed to be coercive, i.e.,

\[
\lim_{r \to \infty} \inf \{ H_i(x, p) \mid x \in \mathbb{T}^n, |p| \geq r \} = \infty,
\]

and $u_{0i}$ are given real-valued continuous functions on $\mathbb{T}^n$, and $c_i > 0$ are given constants for $i = 1, 2$, respectively. Here $u_i$ are the real-valued unknown functions on $\mathbb{T}^n \times [0, \infty)$ and $(u_i)_t := \partial u_i / \partial t, Du_i := (\partial u_i / \partial x_1, \ldots, \partial u_i / \partial x_n)$ for $i = 1, 2$, respectively. For the sake of simplicity, we focus on the system of two equations above in Cases 1, 2 below but we can easily generalize it to general systems of $m$ equations. We are dealing only with viscosity solutions of Hamilton–Jacobi equations in this paper and thus the term “viscosity” may be omitted henceforth.

Although it is already established well that existence and uniqueness results for weakly coupled systems of Hamilton–Jacobi equations hold (see [25, 11, 23] and the references below).

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therein for instance), there are not many studies on properties of solutions of (C). Recently F. Camilli, O. Ley and P. Loreti [6] investigated homogenization problems for the system and obtained the convergence result, and the second author with F. Cagnetti and D. Gomes [5] considered new nonlinear adjoint methods for weakly coupled systems of stationary Hamilton–Jacobi equations and obtained the speed of convergence by using usual regularized equations. As far as the authors know, there are few works on the large time behavior of solutions of weakly coupled systems of Hamilton–Jacobi equations. Formally, on instance), there are not many studies on properties of solutions of (C).

1.1. **Heuristic derivations and Main goal.** First we heuristically derive the large time asymptotics for (C). For simplicity, from now on, we assume that $c_1 = c_2 = 1$. We consider the formal asymptotic expansions of the solutions $u_1, u_2$ of (C) of the form

$$u_1(x, t) = a_{01}(x)t + a_{11}(x) + a_{21}(x)t^{-1} + \ldots,$$

$$u_2(x, t) = a_{02}(x)t + a_{12}(x) + a_{22}(x)t^{-1} + \ldots \quad \text{as } t \to \infty.$$ 

Plugging these expansions into (C), we get

$$a_{01}(x) - a_{21}(x)t^{-2} + \ldots + H_1(x, Da_{01}(x)t + Da_{11}(x) + Da_{21}(x)t^{-1} + \ldots) + (a_{01}(x) - a_{02}(x))t + (a_{11}(x) - a_{12}(x)) + (a_{21}(x) - a_{22}(x))t^{-1} + \ldots = 0, \quad (1.1)$$

and

$$a_{02}(x) - a_{22}(x)t^{-2} + \ldots + H_2(x, Da_{02}(x)t + Da_{12}(x) + Da_{22}(x)t^{-1} + \ldots) + (a_{02}(x) - a_{01}(x))t + (a_{12}(x) - a_{11}(x)) + (a_{22}(x) - a_{21}(x))t^{-1} + \ldots = 0. \quad (1.2)$$

Adding up the two equations above, we have

$$H_1(x, Da_{01}t + Da_{11} + O(1/t)) + H_2(x, Da_{02}t + Da_{12} + O(1/t)) + O(1) = 0$$

as $t \to \infty$. Therefore by the coercivity of $H_1$ and $H_2$ we formally get $Da_{01} = Da_{02} \equiv 0$. Then sending $t \to \infty$ in (1.1), (1.2), we derive

$$a_{01}(x) = a_{02}(x) \equiv a_0 \quad \text{for some constant } a_0,$$

and

$$\begin{cases} H_1(x, Da_{11}(x)) + a_{11}(x) - a_{12}(x) = -a_0 & \text{in } \mathbb{T}^n, \\ H_2(x, Da_{12}(x)) + a_{12}(x) - a_{11}(x) = -a_0 & \text{in } \mathbb{T}^n. \end{cases}$$

Therefore it is natural to investigate the existence of solutions of

$$(E) \quad \begin{cases} H_1(x, Dv_1(x)) + v_1 - v_2 = c & \text{in } \mathbb{T}^n, \\ H_2(x, Dv_2(x)) + v_2 - v_1 = c & \text{in } \mathbb{T}^n. \end{cases}$$

Here one seeks for a triplet $(v_1, v_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}$ such that $(v_1, v_2)$ is a solution of (E). If $(v_1, v_2, c)$ is such a triplet, we call $(v_1, v_2)$ a pair of ergodic functions and $c$ an ergodic constant. By an analogous argument to that of the classical result of [26] we can see that there exists a solution of (E). Indeed the second author with F. Cagnetti, D. Gomes [5]
recently proved that there exists a unique constant $c$ such that the ergodic problem has continuous solutions $(v_1, v_2)$.

Hence, our goal in this paper is to prove the following large time asymptotics for (C) under appropriate assumptions on $H_i$. For any $(u_{01}, u_{02}) \in C(\mathbb{T}^n)^2$ there exists a solution $(v_1, v_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}$ of (E) such that if $(u_1, u_2) \in C(\mathbb{T}^n \times [0, \infty))^2$ is the solution of (C), then, as $t \to \infty$,

$$u_i(x, t) + ct - v_i(x) \to 0 \text{ uniformly on } \mathbb{T}^n$$

for $i = 1, 2$. We call such a pair $(v_1(x) - ct, v_2(x) - ct)$ an asymptotic solution of (C).

It is worthwhile to emphasize here that for homogenization problems, the associated cell problems do not have the coupling terms. See [6] for the detail. Therefore it is relatively easy to get the convergence result by using the classical perturbed test function method introduced by L. C. Evans [12]. But when we consider the large time behavior of solutions of (C), we need to consider ergodic problems (E) with coupling terms. This fact seems to make convergence problems for large time asymptotics rather difficult. We are not yet able to justify rigorously convergence (1.3) for general Hamiltonians $H_i$ for $i = 1, 2$ up to now. We are able to handle only three special cases which we describe below.

1.2. On the study of the large time behavior. In the last decade, a lot of works have been devoted to the study of large time behavior of solutions of Hamilton–Jacobi equations

$$u_t + H(x, Du) = 0 \text{ in } \mathbb{T}^n \times (0, \infty),$$

where $H$ is assumed to be coercive and general convergence results for solutions have been established. More precisely, the convergence

$$u(x, t) - (v(x) - ct) \to 0 \text{ uniformly on } x \in \mathbb{T}^n \text{ as } t \to \infty$$

holds, where $(v, c) \in C(\mathbb{T}^n) \times \mathbb{R}$ is a solution of the ergodic problem

$$H(x, Dv(x)) = c \text{ in } \mathbb{T}^n.$$  

(1.5)

Here the ergodic eigenvalue problem for $H$ is a problem of finding a pair of $v \in C(\mathbb{T}^n)$ and $c \in \mathbb{R}$ such that $v$ is a solution of (1.5). G. Namah and J.-M. Roquejoffre in [30] were the first to get general results on this convergence under the following additional assumptions: $H(x, p) = F(x, p) - f(x)$, where $F$ and $f$ satisfy $p \mapsto F(x, p)$ is convex for $x \in \mathcal{M}$,

$$F(x, p) > 0 \text{ for all } (x, p) \in \mathcal{M} \times (\mathbb{R}^n \setminus \{0\}), F(x, 0) = 0 \text{ for all } x \in \mathcal{M},$$

(1.6)

and

$$f(x) \geq 0 \text{ for all } x \in \mathcal{M} \text{ and } \{f = 0\} \neq \emptyset,$$

(1.7)

where $\mathcal{M}$ is a smooth compact $n$-dimensional manifold without boundary. Then A. Fathi [13] proved the same type of convergence result by using general dynamical approach and weak KAM theory. Contrary to [30], the results of [13] use strict convexity assumptions on $H(x, \cdot)$, i.e., $D_{pp}H(x, p) \geq \alpha I$ for all $(x, p) \in \mathcal{M} \times \mathbb{R}^n$ and $\alpha > 0$ (and also far more
regularity) but do not need (1.6), (1.7). Afterwards J.-M. Roquejoffre [31] and A. Davini and A. Siconolfi [10] have refined the approach of A. Fathi and they studied the asymptotic problem for (1.4) on $\mathcal{M}$ or $n$-dimensional torus. By another approach based on the theory of partial differential equations and viscosity solutions, this type of results has been obtained by G. Barles and P. E. Souganidis in [3]. Moreover, we also refer to the literatures [2, 21, 17, 18, 19] for the asymptotic problems without the periodic assumptions and the periodic boundary condition and the literatures [31, 27, 28, 29, 22, 1] for the asymptotic problems which treat Hamilton–Jacobi equations under various boundary conditions including three types of boundary conditions: state constraint boundary condition, Dirichlet boundary condition and Neumann boundary condition. We remark that results in [3, 2, 1] apply to nonconvex Hamilton–Jacobi equations. We refer to the literatures [33, 15, 16] for the asymptotic problems for noncoercive Hamilton–Jacobi equations.

1.3. Main results. The first case is an analogue of the study by G. Namah, J.-M. Roquejoffre [30]. We consider Hamiltonians $H_i$ of the forms

$$H_i(x,p) = F_i(x,p) - f_i(x),$$

where the functions $F_i : \mathbb{T}^n \times \mathbb{R}^n \to [0, \infty)$ are coercive and $f_i : \mathbb{T}^n \to [0, \infty)$ are given continuous functions for $i = 1, 2$, respectively. We use the following assumptions on $F_i, f_i$.

For $i = 1, 2$

(A2) $f_i(x) \geq 0$ for all $x \in \mathbb{T}^n$;

(A3) define $\mathcal{A}_1 := \{x \in \mathbb{T}^n \mid f_1(x) = 0\}$, $\mathcal{A}_2 := \{x \in \mathbb{T}^n \mid f_2(x) = 0\}$ and then $\mathcal{A} := \mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$;

(A4) there exists $\lambda_0 \in (0,1)$ such that

$$F_i(x, \lambda p) \leq \lambda F_i(x, p)$$

for all $\lambda \in [\lambda_0, 1]$, $x \in \mathbb{T}^n \setminus \mathcal{A}$ and $p \in \mathbb{R}^n$;

(A5) $F_i(x, p) \geq 0$ on $\mathbb{T}^n \times \mathbb{R}^n$, and $F_i(x, 0) = 0$ on $\mathbb{T}^n$.

With the above special forms of the Hamiltonians, we have

**Theorem 1.1** (Convergence Result 1). Assume that the Hamiltonians $H_i$ are of the forms

$$H_i(x,p) = F_i(x,p) - f_i(x)$$

and $H_i, F_i, f_i$ satisfy assumptions (A1)–(A5), then there exists a solution $(v_1, v_2) \in C(\mathbb{T}^n)^2$ of (E) with $c = 0$ such that convergence (1.3) holds.

Notice that the *directional convexity condition* with respect to the $p$ variable on $F_i$, i.e.,

(A4′) for any $p \in \mathbb{R}^n \setminus \{0\}$ and $x \in \mathbb{T}^n$, $t \mapsto F_i(x, tp)$ is convex, together with $F_i(x, 0) = 0$ implies (A4). It is clear to see that assumption (A4) or (A4′) does not require Hamiltonians to be convex. One explicit example of Hamiltonians in
Theorem 1.1 is

\[ H_i(x,p) = F_i(x,p) - f_i(x) = \begin{cases} a_i(x)|p|^\alpha \varphi_i(p \frac{p}{|p|}) - f_i(x) & \text{for } p \neq 0, \\ -f_i(x) & \text{for } p = 0 \end{cases} \]

for some \( \alpha_i \geq 1 \), \( a_i \in C(T^n) \), \( \varphi_i \in C(S^{n-1}) \) with \( a_i, \varphi_i > 0 \) and \( f_i \) satisfy (A2)–(A3) for \( i = 1,2 \), where \( S^{n-1} \) denotes the \( (n-1) \)-dimensional unit sphere.

After this work has been completed, we learned of the interesting recent work of F. Camilli, O. Ley, P. Loreti and V. Nguyen [7], which announces results very similar to Theorem 1.1. Their result is somewhat more general along this direction. In fact they consider systems of \( m \)-equations which have coupling terms with variable coefficients instead of constant coefficients. Also, the control-theoretic interpretation of (C) is derived there.

In the second case, we consider the case where the Hamiltonians are independent of the \( x \) variable, i.e., \( H_i(x,p) = H_i(p) \) for \( i = 1,2 \). We assume that the Hamiltonians satisfy

(A6) \( H_i \) are uniformly convex, i.e.,

\[ H_i(p) \geq H_i(q) + DH_i(q) \cdot (p-q) + \alpha |p-q|^2 \]

for some \( \alpha > 0 \) and almost every \( p, q \in \mathbb{R}^n \),

(A7) \( H_i(0) = 0 \)

for \( i = 1,2 \). Our main result is

**Theorem 1.2** (Convergence Result 2). Assume that \( H_i(x,p) = H_i(p) \) for \( i = 1,2 \) and \( H_i \) satisfy assumptions (A1), (A6) and (A7), then there exists a constant \( M \) such that

\[ u_i(x,t) - M \to 0 \quad \text{uniformly on } T^n \quad \text{for } i = 1,2 \]

as \( t \to \infty \).

One explicit example of Hamiltonians in Theorem 1.2 is

\[ H_i(p) = |p - b_i|^2 - |b_i|^2 \]

for some constant vectors \( b_i \in \mathbb{R}^n \) for \( i = 1,2 \). Notice that the above Hamiltonians in general do not satisfy the conditions in the first case, particularly (A5). The idea for the proof of Theorem 1.2 can be applied to the study more general forms of Hamiltonians, e.g.,

\[ H_i(x,p) = |p - b_i(x)|^2 - |b_i(x)|^2 \]

for \( b_i \in C^1(T^n) \) with \( \text{div } b_i = 0 \) on \( T^n \) for \( i = 1,2 \) as will be noted in Remark 4.4.

In the third case, we generalize the result of G. Barles, P. E. Souganidis [3] for single equations to systems. We consider the case where the two Hamiltonians \( H_1, H_2 \) are same, i.e., \( H := H_1 = H_2 \). We normalize the ergodic constant \( c \) to be 0 by replacing \( H \) by \( H - c \) and then we assume that \( H \) satisfies

(A8) either of the following assumption \((A8)^+\) or \((A8)^-\) holds:
(A8)$^+$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p + q) \geq \eta$ and $H(x, q) \leq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \in (0, 1]$, 
\[ \mu H(x, \frac{p}{\mu} + q) \geq H(x, p + q) + \psi_\eta(1 - \mu). \]

(A8)$^-$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p + q) \leq -\eta$ and $H(x, q) \geq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \geq 1$, 
\[ \mu H(x, \frac{p}{\mu} + q) \leq H(x, p + q) - \frac{\psi_\eta(\mu - 1)}{\mu}. \]

Assumption (A8)$^+$ was first introduced in [3] to replace the convexity assumption, and it mainly concerns the set $\{H \geq 0\}$ and the behavior of $H$ in this set. Assumption (A8)$^-$ is a modified version of (A8)$^+$ which was introduced in [1] and on the contrary, it concerns the set $\{H \leq 0\}$. We can generalize them as in [3] but to simplify our arguments we only use the simplified version. See the end of Section 5.

Our third main result is

**Theorem 1.3 (Convergence Result 3).** If we assume that $H = H_1 = H_2$ and $H$ satisfies (A1), (A8) and the ergodic constant $c$ is equal to 0, then there exist a solution $(v, v) \in C(\mathbb{T}^n)^2$ of (E) with $c = 0$ such that convergence (1.3) holds.

We notice that if $H$ is smooth with respect to the $p$-variable, then (A8) is equivalent to a one-sided directionally strict convexity in a neighborhood of $\{p \in \mathbb{R}^n \mid H(x, p) = 0\}$ for all $x \in \mathbb{T}^n$, i.e.,

(A8)$'$ either of the following assumption (A8)$'^+$ or (A8)$'^-$ holds:

(A8)$'^+$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p + q) \geq \eta$ and $H(x, q) \leq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \in (0, 1]$, 
\[ D_pH(x, p + q) \cdot p - H(x, p + q) \geq \psi_\eta, \]

(A8)$'^-$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p + q) \leq -\eta$ and $H(x, q) \geq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \geq 1$, 
\[ D_pH(x, p + q) \cdot p - H(x, p + q) \geq \psi_\eta. \]

We refer the readers to [3] for interesting examples of Hamiltonians in Theorem 1.3. Our conclusions in Cases 2, 3 seem to go beyond the recent work [7].

This paper is organized as follows: in Section 2 we give some preliminary results. Section 3, Section 4, and Section 5 are respectively devoted to the proofs of Theorems 1.1–1.3. In Appendix we present the proof of the result on ergodic problems.

**Notations.** For $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $C(A)$, LSC $(A)$, USC $(A)$ and $C^k(A)$ the space of real-valued continuous, lower semicontinuous, upper semicontinuous and $k$-th
continuous differentiable functions on \( A \), respectively. We denote by \( W^{1,\infty}(A) \) the set of bounded functions whose first weak derivatives are essentially bounded. We call a function \( m : [0, \infty) \to [0, \infty) \) a modulus if it is continuous and nondecreasing on \( [0, \infty) \) and vanishes at the origin.

2. Preliminaries

In this section we assume only (A1).

Proposition 2.1 (Ergodic Problem (E) (e.g., [5, Theorem 4.2])). There exists \((v_1, v_2, \overline{H}_1, \overline{H}_2) \in W^{1,\infty}(\mathbb{T}^n)^2 \times \mathbb{R}^2\) of

\[
\begin{aligned}
H_1(x, Dv_1) + c_1(v_1 - v_2) &= \overline{H}_1 \text{ in } \mathbb{T}^n, \\
H_2(x, Dv_1) + c_2(v_2 - v_1) &= \overline{H}_2 \text{ in } \mathbb{T}^n.
\end{aligned}
\]

Furthermore, \( c_2 \overline{H}_1 + c_1 \overline{H}_2 \) is unique.

We note that solutions \( v_1, v_2 \) of (2.1) are not unique in general even up to constants. Also it is easy to see that \( \overline{H}_1, \overline{H}_2 \) are not unique as well. Take \( v'_1 = v_1 + C_1, v'_2 = v_1 + C_2 \) for some constants \( C_1, C_2 \) then

\[
\overline{H}'_1 = H_1 + c_1(C_1 - C_2), \quad \overline{H}'_2 = H_1 + c_2(C_2 - C_1),
\]

which shows that \( \overline{H}_i \) can individually take any real value. But remarkably, we have

\[
c_2 \overline{H}_1 + c_1 \overline{H}_2 = c_2 \overline{H}'_1 + c_1 \overline{H}'_2,
\]

which is a unique constant. We can get the existence result by an argument similar to a classical result in [26] (see also the proof of Proposition 3.1 below). We give the sketch of the proof for the uniqueness of \( c_2 \overline{H}_1 + c_1 \overline{H}_2 \) in Appendix for the reader’s convenience.

We assume henceforth for simplicity that \( c_1 = c_2 = 1 \). Then the ergodic constant \( c \) is unique and is given by

\[
c = \frac{\overline{H}_1 + \overline{H}_2}{2}.
\]

The comparison principle for (C) is a classical result. See [25, 11, 23], [6, Proposition 3.1] for instance.

Proposition 2.2 (Comparison Principle for (C)). Let \((u_1, u_2) \in \text{USC}(\mathbb{T}^n \times [0, \infty))^2, (v_1, v_2) \in \text{LSC}(\mathbb{T}^n \times [0, \infty))^2\) be a subsolution and a supersolution of (C), respectively. If \( u_i(\cdot, 0) \leq v_i(\cdot, 0) \) on \( \mathbb{T}^n \), then \( u_i \leq v_i \) on \( \mathbb{T}^n \times [0, \infty) \) for \( i = 1, 2 \).

The following proposition is a straightforward application of Propositions 2.1, 2.2.

Proposition 2.3 (Boundedness of Solutions of (C)). Let \((u_1, u_2) \) be the solution of (C) and let \( c \) be the ergodic constant for (E). Then we have \(|u_i(x, t) + ct| \leq C\) on \( \mathbb{T}^n \times [0, \infty) \) for some \( C > 0 \) for \( i = 1, 2 \).
In view of the coercivity assumption on $H_i$ for $i = 1, 2$, we have the following Lipschitz regularity result.

**Proposition 2.4** (Lipschitz Regularity of Solutions of (C)). If $u_{0i} \in W^{1,\infty}(\mathbb{T}^n)$ for $i = 1, 2$, then $(u_1 + ct, u_2 + ct)$ is in $W^{1,\infty}(\mathbb{T}^n \times [0, \infty))^2$, where $(u_1, u_2)$ is the solution of (C) and $c$ is the ergodic constant.

We assume henceforth that $u_{0i} \in W^{1,\infty}(\mathbb{T}^n)$ for $i = 1, 2$ in order to avoid technicalities but they are not necessary. We can easily remove these additional requirements on $u_{0i}$. See Remark 3.5 for details.

3. First Case

In this section we consider the case where Hamiltonians have the forms $H_i(x, p) = F_i(x, p) - f_i(x)$, and $H_i, F_i, f_i$ satisfy assumptions (A1)–(A5). System (C) becomes

\[
\begin{cases}
(u_1)_t + F_1(x, Du_1) + u_1 - u_2 = f_1(x) & \text{in } \mathbb{T}^n \times (0, \infty), \\
(u_2)_t + F_2(x, Du_2) + u_2 - u_1 = f_2(x) & \text{in } \mathbb{T}^n \times (0, \infty), \\
u_1(x, 0) = u_{01}(x), & u_2(x, 0) = u_{02}(x) & \text{on } \mathbb{T}^n.
\end{cases}
\]

(C1)

In order to prove Theorem 1.1, we need several following steps.

3.1. Stationary Problems.

**Proposition 3.1.** The ergodic constant $c$ is equal to 0.

**Proof.** For $\varepsilon > 0$ let us consider a usual approximate monotone system

\[
\begin{cases}
F_1(x, Du_1^\varepsilon(x)) + (1 + \varepsilon)v_1^\varepsilon - v_2^\varepsilon = f_1(x) & \text{in } \mathbb{T}^n, \\
F_2(x, Du_2^\varepsilon(x)) + (1 + \varepsilon)v_2^\varepsilon - v_1^\varepsilon = f_2(x) & \text{in } \mathbb{T}^n.
\end{cases}
\]

(3.1)

It is easy to see that $(0, 0), (C_1/\varepsilon, C_1/\varepsilon)$ are a subsolution and a supersolution of the above for $C_1 > 0$ large enough. By Perron’s method for the monotone system, we have a unique solution $(v_1^\varepsilon, v_2^\varepsilon) \in C(\mathbb{T}^n)^2$ of (3.1). By the way of construction we have

\[
0 \leq \varepsilon v_i^\varepsilon \leq C_1 \text{ on } \mathbb{T}^n
\]

(3.2)

for $i = 1, 2$. Summing up both equations in (3.1), we have

\[
F_1(x, Du_1^\varepsilon) + F_2(x, Du_2^\varepsilon) = -\varepsilon(v_1^\varepsilon + v_2^\varepsilon) + f_1(x) + f_2(x) \leq C_2
\]

for some $C_2 > 0$. By the coercivity of $F_i$ we obtain

\[
\|Du_i^\varepsilon\|_{L^\infty(\mathbb{T}^n)} \leq C_2
\]

for $i = 1, 2$ by replacing $C_2$ by a larger constant if necessary. Therefore we see that \(\{v_i^\varepsilon\}_{\varepsilon \in (0, 1)}\) are equi-Lipschitz continuous.
We claim that there exists a constant $C_3 > 0$
\[
|v_1^\varepsilon(x) - v_2^\varepsilon(y)| \leq C_3 \text{ for all } x, y \in \mathbb{T}^n.
\]
(3.3)
Indeed setting $m_i^\varepsilon := \max_{\mathbb{T}^n} v_i^\varepsilon(z_i)$ for some $z_i \in \mathbb{T}^n$ for $i = 1, 2$. Take 0 as a test function in the first equation of (3.1) to derive
\[
F_1(z_1, 0) + (1 + \varepsilon)v_1^\varepsilon(z_1) - v_2^\varepsilon(z_1) \leq f_1(z_1),
\]
which implies
\[
v_1^\varepsilon(z_1) - v_2^\varepsilon(z_1) \leq -F_1(z_1, 0) - \varepsilon v_1^\varepsilon(z_1) + f_1(z_1) \leq C_3
\]
for some $C_3 > 0$. Thus,
\[
v_1^\varepsilon(x) - v_2^\varepsilon(y) \leq v_1^\varepsilon(z_1) - v_2^\varepsilon(y)
= v_1^\varepsilon(z_1) - v_2^\varepsilon(z_1) + v_2^\varepsilon(z_1) - v_2^\varepsilon(y) \leq C_3
\]
by replacing $C_3$ by a larger constant if necessary. This implies (3.3). In particular, $|m_1^\varepsilon - m_2^\varepsilon| \leq C_3$.

Let $w_i^\varepsilon(x) := v_i^\varepsilon(x) - m_i^\varepsilon$. Because of (3.2), $\{w_i^\varepsilon\}_{\varepsilon \in (0, 1)}$ is a sequence of equi-Lipschitz continuous and uniformly bounded functions on $\mathbb{T}^n$. Moreover they satisfy
\[
\begin{cases}
F_1(x, Dw_1^\varepsilon(x)) + (1 + \varepsilon)w_1^\varepsilon - w_2^\varepsilon = f_1(x) - (1 + \varepsilon)m_1^\varepsilon + m_2^\varepsilon \quad \text{in } \mathbb{T}^n, \\
F_2(x, Dw_2^\varepsilon(x)) + (1 + \varepsilon)w_2^\varepsilon - w_1^\varepsilon = f_2(x) - (1 + \varepsilon)m_2^\varepsilon + m_1^\varepsilon \quad \text{in } \mathbb{T}^n
\end{cases}
\]
in the viscosity solution sense. By Ascoli-Arzela’s theorem, there exists a sequence $\varepsilon_j \to 0$ so that
\[
\begin{align*}
w_i^{\varepsilon_j} & \to w_i, \\
-(1 + \varepsilon_j)m_1^{\varepsilon_j} + m_2^{\varepsilon_j} & \to \overline{H}_1 \text{ and } -(1 + \varepsilon_j)m_2^{\varepsilon_j} + m_1^{\varepsilon_j} \to \overline{H}_2
\end{align*}
\]
uniformly on $\mathbb{T}^n$ as $j \to \infty$ for some $(w_1, w_2) \in W^{1, \infty}(\mathbb{T}^n)^2$ and $(\overline{H}_1, \overline{H}_2) \in \mathbb{R}^2$. By a standard stability result of viscosity solutions we see that $(w_1, w_2, \overline{H}_1, \overline{H}_2)$ is a solution of (2.1).

We now prove that $c := (\overline{H}_1 + \overline{H}_2)/2 = 0$. Noting that $m_i^{\varepsilon_j} \geq 0$ and
\[
\frac{1}{2}\{(-(1 + \varepsilon_j)m_1^{\varepsilon_j} + m_2^{\varepsilon_j}) + (-(1 + \varepsilon_j)m_2^{\varepsilon_j} + m_1^{\varepsilon_j})\} = -\frac{1}{2}\varepsilon_j(m_1^{\varepsilon_j} + m_2^{\varepsilon_j}) \to c
\]
as $j \to \infty$, we see that $c \leq 0$. Furthermore, summing up the two equations in (2.1), we obtain
\[
2c = \overline{H}_1 + \overline{H}_2 = F_1(x, Dw_1) + F_2(x, Dw_2) - f_1(x) - f_2(x) \geq -f_1(x) - f_2(x)
\]
for almost every $x \in \mathbb{T}^n$. Since $\mathcal{A} \neq \emptyset$, we see that $c \geq 0$. Together with the above observation we get the conclusion. \qed
Theorem 3.2 (Comparison Principle for Stationary Problems). Let \((u_1, u_2) \in \text{USC} (\mathbb{T}^n)^2\), \((v_1, v_2) \in \text{LSC} (\mathbb{T}^n)^2\) be, respectively, a subsolution and a supersolution of
\[
(S1) \quad \begin{cases} 
F_1(x, Dv_1(x)) + v_1 - v_2 = f_1(x) & \text{in } \mathbb{T}^n, \\
F_2(x, Dv_2(x)) + v_2 - v_1 = f_2(x) & \text{in } \mathbb{T}^n.
\end{cases}
\]
If \(u_i \leq v_i\) on \(A\), then \(u_i \leq v_i\) on \(\mathbb{T}^n\) for \(i = 1, 2\).

The idea of the proof below basically comes from the combination of those in [20] and [25, 11, 23]. It is worthwhile to mention that the set \(A\) plays the role of the boundary as in [14, 24]. See also [8] and [7, Theorem 3.3] for weakly coupled systems of Hamilton–Jacobi equations.

Proof. Fix any \(\delta > 0\). We may choose an open neighborhood \(V\) of \(A\) and \(\lambda \in [\lambda_0, 1]\) so that \(\lambda u_i \leq v_i + \delta\) on \(V\) for \(\lambda \in [\lambda, 1]\) and \(i = 1, 2\), where \(\lambda_0\) is the constant in (A2). It is enough to show that \(\lambda u_i \leq v_i + \delta\) on \(\mathbb{T}^n \setminus V\) for \(\lambda \in [\lambda, 1]\). Fix \(\lambda \in [\lambda, 1]\) and we set \(u_i^\lambda := \lambda u_i\) and \(v_i^\delta := v_i + \delta\). We prove the above statement by a contradiction argument. Suppose that \(M := \max_{i=1,2,x \in \mathbb{T}^n \setminus V} (u_i^\lambda - v_i^\delta)(x) > 0\).

We take \(i_0 \in \{1, 2\}\), \(\xi \in \mathbb{T}^n \setminus V\) such that \(M = (u_{i_0}^\lambda - v_{i_0}^\delta)(\xi)\). We may assume that \(i_0 = 1\) by symmetry. We first consider the case where
\[
M_\lambda = (u_1^\lambda - v_1^\delta)(\xi) = (u_2^\lambda - v_2^\delta)(\xi). \tag{3.4}
\]

We define the function \(\Psi : \mathbb{T}^{2n} \to \mathbb{R}\) by
\[
\Psi(x, y) := u_1^\lambda(x) - v_1^\delta(y) - \frac{|x - y|^2}{2\varepsilon^2} - \frac{|x - \xi|^2}{2}.
\]
Let \(\Psi\) achieve its maximum at some point \((x_\varepsilon, y_\varepsilon) \in \mathbb{T}^{2n}\). By the definition of viscosity solutions we have
\[
F_1(x_\varepsilon, \frac{1}{\lambda} (\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + x_\varepsilon - \xi)) + (u_1 - u_2)(x_\varepsilon) \leq f_1(x_\varepsilon),
\]
\[
F_1(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}) + (v_1 - v_2)(y_\varepsilon) \geq f_1(y_\varepsilon).
\]
By the usual argument we may assume that
\[
x_\varepsilon, y_\varepsilon \to \xi, \quad \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} \to p \in \mathbb{R}^n
\]
as \(\varepsilon \to 0\) by taking a subsequence if necessary in view of the Lipschitz continuity of solutions. Therefore sending \(\varepsilon\) to \(0\) yields
\[
F_1(\xi, \frac{p}{\lambda}) + (u_1 - u_2)(\xi) \leq f_1(\xi), \tag{3.5}
\]
\[
F_1(\xi, p) + (v_1 - v_2)(\xi) \geq f_1(\xi). \tag{3.6}
\]
In view of (A4), (3.5) transforms to read
\[
F_1(\xi, p) + (u_1^\lambda - u_2^\lambda)(\xi) \leq \lambda f_1(\xi) \text{ for all } \lambda \in [\lambda, 1]. \tag{3.7}
\]
Note that \((v_1 - v_2)(\xi) = (v^\delta_1 - v^\delta_2)(\xi)\). By (3.4), (3.6) and (3.7) we get \(f_1(\xi) \leq \lambda f_1(\xi)\). Similarly, \(f_2(\xi) \leq \lambda f_2(\xi)\). Hence \(f_1(\xi) + f_2(\xi) \leq \lambda (f_1(\xi) + f_2(\xi))\) which is a contradiction since \(f_1(\xi) + f_2(\xi) > 0\) and \(\lambda \in (0, 1)\).

We next consider the case where
\[(u^\lambda_1 - v^\delta_1)(\xi) \neq (u^\lambda_2 - v^\delta_2)(\xi)\]

Then there exists \(a > 0\) such that \((u^\lambda_1 - v^\delta_1)(\xi) \geq (u^\lambda_2 - v^\delta_2)(\xi) + a\) and therefore by (3.6), (3.7) we obtain
\[0 > (\lambda - 1)f_1(\xi) \geq (u^\lambda_1 - v^\delta_1)(\xi) - (u^\lambda_2 - v^\delta_2)(\xi) \geq a,\]
which is a contradiction. This finishes the proof. \(\square\)

3.2. Convergence.

**Proposition 3.3** (Monotonicity Property 1). Set \(U(x, t) := u_1(x, t) + u_2(x, t)\). Then the function \(t \mapsto U(x, t)\) is nonincreasing for all \(x \in \mathcal{A}\).

*Proof.* It is easy to see that \(U\) satisfies \(U_t \leq 0\) on \(\mathcal{A}\) in the viscosity sense and we get the conclusion. \(\square\)

**Proposition 3.4** (Monotonicity Property 2). Set
\[V(x, t) := \max\{u_1(x, t), u_2(x, t)\} = \frac{1}{2}\left\{(u_1 + u_2)(x, t) + |(u_1 - u_2)(x, t)|\right\}.\]

Then the function \(t \mapsto V(x, t)\) is nonincreasing for all \(x \in \mathcal{A}\).

We notice that the result of Proposition 3.4 is included by the recent result of [7, Remark 5.7, (3)] but our proof seems to be more direct.

*Proof.* Fix \(x \in \mathcal{A}\). For \(\varepsilon, \delta > 0\) we set \(K_\varepsilon(x) := x + [-\varepsilon, \varepsilon]^n\) and
\[V_\delta(x, t) := \frac{1}{2}\left\{(u_1 + u_2)(x, t) + \langle (u_1 - u_2)(x, t)\rangle_\delta\right\},\]
where \(\langle p \rangle_\delta := \sqrt{|p|^2 + \delta^2}\). We note that \(V_\delta\) converges uniformly to \(V\) as \(\delta \to 0\).

We have for all \(t, h \geq 0\)
\[\int_{K_\varepsilon(x)} V_\delta(y, t + h) - V_\delta(y, t) dy = \int_{K_\varepsilon(x) \times [t, t+h]} (V_\delta)_t(y, s) dy ds.\]
Let \((y, s)\) be a point at which \(u_1, u_2\) are differentiable. We calculate that
\[
(V_\delta)_t(y, s) = \frac{1}{2} \{ (u_1)_t + (u_2)_t + \frac{u_1 - u_2}{(u_1 - u_2)\delta}((u_1)_t - (u_2)_t) \}
= \frac{1}{2} \{ f_1 + f_2 + \frac{u_1 - u_2}{(u_1 - u_2)\delta}(f_1 - f_2) \}
+ \frac{1}{2} \{ -F_1 - F_2 + \frac{u_1 - u_2}{(u_1 - u_2)\delta}(F_2 - F_1) \} - \frac{1}{(u_1 - u_2)\delta}(u_1 - u_2)^2
\leq \frac{1}{2} \{ f_1 + f_2 + \frac{u_1 - u_2}{(u_1 - u_2)\delta}(f_1 - f_2) \} + \frac{1}{2} \{ -F_1 - F_2 + \frac{u_1 - u_2}{(u_1 - u_2)\delta}(F_2 - F_1) \}.
\]

In view of (A5) and (A3) sending \(\delta \to 0\) yields
\[
\int_{K_{\varepsilon}(x)} V(y, t+h) - V(y, t) dy
\leq \int_{K_{\varepsilon}(x) \times [t, t+h]} \frac{1}{2} \{ f_1 + f_2 + \text{sgn}(u_1 - u_2)(f_1 - f_2) \}
+ \frac{1}{2} \{ -F_1 - F_2 + \text{sgn}(u_1 - u_2)(F_2 - F_1) \} dy ds
\leq \int_{K_{\varepsilon}(x) \times [t, t+h]} \omega_{f_1}(|x-y|) + \omega_{f_2}(|x-y|) dy ds
\leq \varepsilon^n h(\omega_{f_1}(\sqrt{n}\varepsilon) + \omega_{f_2}(\sqrt{n}\varepsilon)),
\]
where \(\omega_{f_i}\) are the moduli of continuity of \(f_i\) for \(i = 1, 2\). By dividing by \(\varepsilon^n\) and sending \(\varepsilon \to 0\) we get the conclusion. \(\square\)

**Proof of Theorem 1.1.** For any \(x \in A\) by Propositions 3.3, 3.4 we see that \((u_1 + u_2)(x, t) \to \alpha(x)\) and \(|(u_1 - u_2)(x, t)| \to \beta(x)\) as \(t \to \infty\). If \(\beta(x) > 0\), then \((u_1 - u_2)(x, t)\) converges as \(t \to \infty\) since \(t \mapsto (u_1 - u_2)(x, t)\) is continuous. The limit may be either \(\beta(x)\) or \(-\beta(x)\). Therefore \(u_1(x, t), u_2(x, t)\) converge as \(t \to \infty\). If \(\beta(x) = 0\), then we have
\[
(u_1 + u_2)(x, t) - |(u_1 - u_2)(x, t)| \leq 2u_1(x, t) \leq (u_1 + u_2)(x, t) + |(u_1 - u_2)(x, t)|,
\]
which implies \(u_1(x, t)\) and \(u_2(x, t)\) converge to \((1/2)\alpha(x)\) as \(t \to \infty\). Consequently, we see that \(u_1(x, t), u_2(x, t)\) converge for all \(x \in A\) as \(t \to \infty\).

Now, let us define the following half-relaxed semilimits
\[
\overline{u}_i(x) = \limsup_{t \to \infty} [u_i](x, t)\quad\text{and}\quad \underline{u}_i(x) = \liminf_{t \to \infty} [u_i](x, t).
\]
for $x \in \mathbb{T}^n$ and $i = 1, 2$. By standard stability results of the theory of viscosity solutions, $(\overline{u}_1, \overline{u}_2)$, $(\underline{u}_1, \underline{u}_2)$ are a subsolution and a supersolution of (E), respectively. Moreover, $(\overline{u}_1, \overline{u}_2) = (\underline{u}_1, \underline{u}_2)$ on $\mathcal{A}$, since $u_1, u_2$ converge on $\mathcal{A}$ as $t \to \infty$. By the comparison principle, Theorem 3.2, we obtain $(\overline{u}_1, \overline{u}_2) = (u_1, u_2)$ in $\mathbb{T}^n$ and the proof is complete. □

**Remark 3.5.** (i) The Lipschitz regularity assumption on $u_{0i}$ for $i = 1, 2$ is convenient to avoid technicalities but it is not necessary. We can remove it as follows. For each $x$ for all $x$, for all $x$, for all $x$, for all $x$, we may choose a sequence $\{u_{0i}^k\}_{k \in \mathbb{N}} \subset W^{1,\infty}(\mathbb{T}^n)$ so that $\|u_{0i}^k - u_{0i}\|_{L^{\infty}(\mathbb{T}^n)} \leq 1/k$ for all $k \in \mathbb{N}$. By the maximum principle, we have

$$\|u_i - u_i^k\|_{L^{\infty}(\mathbb{T}^n \times (0, \infty))} \leq \|u_{0i} - u_{0i}^k\|_{L^{\infty}(\mathbb{T}^n)} \leq 1/k,$$

and therefore

$$u_i^k(x, t) - 1/k \leq u_i(x, t) \leq u_i^k(x, t) + 1/k \text{ for all } (x, t) \in \mathbb{T}^n \times [0, \infty),$$

where $(u_1, u_2)$ is the solution of (C) and $(u_1^k, u_2^k)$ are the solutions of (C) with $u_{0i} = u_{0i}^k$ for $i = 1, 2$. Therefore we have

$$u_{i, \infty}(x) - 1/k \leq \liminf_{t \to \infty} u_i(x, t) \leq \limsup_{t \to \infty} u_i(x, t) \leq u_{i, \infty}(x) + 1/k$$

for all $x \in \mathbb{T}^n$, where $u_{i, \infty}(x) := \lim_{t \to \infty} u_i^k(x, t)$. This implies that

$$\liminf_{t \to \infty} u_i(x, t) = \limsup_{t \to \infty} u_i(x, t)$$

for all $x \in \mathbb{T}^n$ and $i = 1, 2$.

(ii) Notice that if $\mathcal{A} = \emptyset$ then the comparison principle for (S1) holds, i.e., for any subsolution $(v_1, v_2)$ and any supersolution $(w_1, w_2)$ we have $v_i \leq w_i$ on $\mathbb{T}^n$ for $i = 1, 2$ (e.g., [8, Theorem 3.3]). This fact implies that the ergodic constant $c$ is negative (not 0!). Indeed, by the argument same as in the proof of Proposition 3.1 we easily see that $c \leq 0$. Suppose that $c = 0$ and then the comparison principle implies that (E) has a unique solution $(v_1, v_2)$. However, that is obviously not correct since for any solution $(v_1, v_2)$ of (E), $(v_1 + C, v_2 + C)$ is also a solution for any constant $C$. In this case we do not know whether convergence (1.3) holds or not.

### 3.3. Systems of $m$-equations

This section was added after we had received the draft [7] in order for the readers to see the different ideas used in our work and [7].

In this subsection we consider weakly coupled systems of $m$-equations for $m \geq 2$

$$(u_i)_t + F_i(x, Du_i) + \sum_{j=1}^{m} c_{ij} u_j = f_i \text{ in } \mathbb{T}^n \times (0, \infty) \text{ for } i = 1, \ldots, m,$$

where $F_i$ satisfy (A1), (A5) and the convexity with respect to the $p$-variable,

$$c_{ii} \geq 0, \ c_{ij} \leq 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^{m} c_{ij} = \sum_{j=1}^{m} c_{ij} = 0 \quad (3.8)$$
for $i, j \in \{1, \ldots, m\}$ and $f_i$ satisfy (A2) and
\[ A := \bigcap_{i=1}^{m} \{ x \in \mathbb{T}^m \mid f_i(x) = 0 \} \neq \emptyset \]

then the result of Theorem 1.1 still holds. In [7] the authors first found the importance of irreducibility of coupling term. Although it is not essential in our argument, we also somehow use it below. Let us first assume for simplicity that the coefficient matrix $(c_{ij})$ is irreducible, i.e.,

(M) For any $I \subsetneq \{1, \ldots, m\}$, there exist $i \in I$ and $j \in \{1, \ldots, m\} \setminus I$ such that $c_{ij} \neq 0$.

Condition (M) will be removed in Remark 3.6 at the end of this subsection.

We just give a sketch of the formal proof for the convergence. By a standard regularization argument we can prove it rigorously in the viscosity solution sense.

We only need to prove the convergence of $u_i$ on $A$ for each $i \in \{1, \ldots, m\}$, since we have an analogous comparison principle to Theorem 3.2 when (M) holds. For $(x, t) \in \mathbb{T}^m \times [0, \infty)$, we can choose $\{i_{x,t}\}_{i=1}^m$ such that $\{1_{x,t}, \ldots, m_{x,t}\} = \{1, \ldots, m\}$ and
\[ u_{1_{x,t}}(x, t) \geq u_{2_{x,t}}(x, t) \geq \ldots \geq u_{m_{x,t}}(x, t) \]

and set $v_i(x, t) := u_{i_{x,t}}(x, t)$.

Fix $(x_0, t_0) \in A \times (0, \infty)$ and we may assume without loss of generality that
\[ 1_{x_0,t_0} = 1 \text{ and } 2_{x_0,t_0} = 2. \]

Noting that $c_{1j} \leq 0$, $u_1 \geq u_j$ for all $j = 2, \ldots, m$, and $F_1 \geq 0$, we have
\[ (v_1)_t = (u_1)_t \leq (u_1)_t + \sum_{j=1}^{m} c_{1j} u_1 \leq (u_1)_t + F_1(x_0, Du_1) + \sum_{j=1}^{m} c_{1j} u_j = 0 \]

at the point $(x_0, t_0)$, which implies that $v_1(x_0, \cdot)$ is nonincreasing for $x_0 \in A$ and therefore $v_1(x_0, \cdot)$ converges as $t \to \infty$. 
Noting that \( u_2 \geq u_j \) and \( c_{ij} \leq 0 \) for all \( i = 1, 2, j = 3, \ldots, m, \sum_{j=1}^{m} c_{2j} = 0, \) and \( F_i \geq 0, \) we have
\[
(v_1 + v_2)_t = (u_1 + u_2)_t \leq (u_1 + u_2)_t + \sum_{i=1}^{2} \sum_{j=3}^{m} c_{ij}(u_j - u_2) \\
= (u_1)_t + (u_2)_t + (c_{11} + c_{12} + c_{21} + c_{22})u_2 + \sum_{i=1}^{2} \sum_{j=3}^{m} c_{ij}u_j \\
\leq (u_1)_t + (u_2)_t + (c_{11} + c_{21})u_1 + (c_{12} + c_{22})u_2 + \sum_{i=1}^{2} \sum_{j=3}^{m} c_{ij}u_j \\
\leq (u_1)_t + (u_2)_t + F_1(x_0, Du_1) + F_2(x_0, Du_2) + \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij}u_j = 0
\]
at the point \((x_0, t_0)\). Thus,
\[
(v_1 + v_2)_t(x_0, t_0) \leq 0.
\]
Therefore \((v_1 + v_2)(x_0, \cdot)\) is nonincreasing for \( x_0 \in \mathcal{A} \). Since we have already known that \( v_1(x_0, \cdot) \) converges, we see that \( v_2(x_0, \cdot) \) converges as \( t \to \infty \).

By the induction argument, we can prove that \((v_1 + \ldots + v_k)(x_0, \cdot)\) is nonincreasing for all \( x_0 \in \mathcal{A} \) and \( k \in \{1, \ldots, m\} \), which is a generalization of Proposition 3.4. Thus, we see that
\[
v_i(x_0, t) \to w_i(x_0) \text{ as } t \to \infty \text{ for } i \in \{1, \ldots, m\},
\]
which concludes that each \( u_i(x_0, t) \) converges as \( t \to \infty \) for \( x_0 \in \mathcal{A} \).

**Remark 3.6.** (i) In general, condition (M) can be removed as follows. By possible row and column permutations, \( C := (c_{ij}) \) can be written in the block triangular form
\[
C = (C_{pq})_{p,q=1}^l
\]
where \( C_{pq} \) are \( s_p \times s_q \) matrices for \( p, q \in \{1, \ldots, l\}, \sum_{k=1}^{l} s_k = m, \) \( C_{kk} \) are irreducible for \( k \in \{1, \ldots, l\} \) and \( C_{pq} = 0 \) for \( p > q \) as in [4]. By (3.8), we can easily see that \( C_{pq} = 0 \) for \( p < q \) as well. Therefore the convergence result above can be applied to each irreducible matrix \( C_{kk} \) to yield the result.

(ii) Our approach in this general case is slightly different from the one in [7]. The convergence of each \( u_i(x, t) \) as \( t \to \infty \) for \( i \in \{1, \ldots, m\}, \) for \( x \in \mathcal{A} \) plays the key role here, while Lemma 5.6 plays the key role in [7]. See Lemma 5.6 in [7] for more details.
4. Second case

In this section we study the case where Hamiltonians are independent of the $x$-variable and then (C) reduces to

$$\begin{align*}
(u_1)_t + H_1(Du_1) + u_1 - u_2 &= 0 \quad \text{in } \mathbb{T}^n \times (0, \infty), \\
(u_2)_t + H_2(Du_2) + u_2 - u_1 &= 0 \quad \text{in } \mathbb{T}^n \times (0, \infty),
\end{align*}$$

(4.1)

$$\begin{align*}
u_1(x, 0) &= u_{01}(x), \quad u_2(x, 0) = u_{02}(x) \quad \text{on } \mathbb{T}^n.
\end{align*}$$

Proposition 4.1. The ergodic constant $c$ is equal to 0, and problem (E) has only constant Lipschitz subsolutions $(a, a)$ for $a \in \mathbb{R}$.

Proof. Since we can easily see that the ergodic constant is 0, we only prove the second statement. To simplify the presentation, we argue as if $H_i$ and $v_i$ were smooth for $i = 1, 2$ and rigorous proof can be made by a standard regularization argument. Summing up the two equations in (E) and using (A6), we obtain

$$0 \geq H_1(Dv_1) + H_2(Dv_2) \geq H_1(0) \cdot Dv_1 + \alpha|Dv_1|^2 + H_2(0) \cdot Dv_2 + \alpha|Dv_2|^2.$$ 

Integrate the above inequality over $\mathbb{T}^n$ to get

$$0 \geq \int_{\mathbb{T}^n} [DH_1(0) \cdot Dv_1 + \alpha|Dv_1|^2 + DH_2(0) \cdot Dv_2 + \alpha|Dv_2|^2] dx = \int_{\mathbb{T}^n} \alpha(|Dv_1|^2 + |Dv_2|^2) dx$$

which implies the conclusion.

Lemma 4.2 (Monotonicity Property). Define

$$M(t) := \max_{i=1,2} \max_{x \in \mathbb{T}^n} u_i(x, t) \quad \text{and} \quad m(t) := \min_{i=1,2} \min_{x \in \mathbb{T}^n} u_i(x, t).$$

Then $t \mapsto M(t)$ is nonincreasing and $t \mapsto m(t)$ is nondecreasing.

Proof. Fix $s \in [0, \infty)$ and let $a = m(s)$. We have $(a, a)$ is a solution of (C2) and $a \geq u_i(x, s)$ for all $x \in \mathbb{T}^n$ and $i = 1, 2$. By the comparison principle for (C2), we have $a \geq u_i(x, t)$ for $x \in \mathbb{T}^n$, $t \geq s$ and $i = 1, 2$. Thus $t \mapsto M(t)$ is nonincreasing. Similarly, $t \mapsto m(t)$ is nondecreasing.

By Lemma 4.2, we can define

$$\overline{M} := \lim_{t \to \infty} M(t) \quad \text{and} \quad \underline{m} := \lim_{t \to \infty} m(t).$$

Proof of Theorem 1.2. If $\overline{M} = m$ then we immediately get the conclusion and therefore we suppose by contradiction that $\overline{M} > \underline{m}$ and show the contradiction.
Since \( \{u_i(\cdot, t)\}_{t>0} \) is compact in \( W^{1,\infty}(\mathbb{T}^n) \) for \( i = 1, 2 \), there exists a sequence \( T_n \to \infty \) so that \( \{u_i(\cdot, T_n)\} \) converges uniformly as \( n \to \infty \) for \( i = 1, 2 \). By the maximum principle,

\[
\|u_i(\cdot, T_n + \cdot) - u_i(\cdot, T_m + \cdot)\|_{L^\infty(\mathbb{T}^n \times (0,\infty))} \leq \|u_i(\cdot, T_n) - u_i(\cdot, T_m)\|_{L^\infty(\mathbb{T}^n)}
\]

for \( i = 1, 2 \) and \( m, n \in \mathbb{N} \). Hence \( \{u_i(\cdot, T_n + \cdot)\} \) is a Cauchy sequence in \( \text{BUC}(\mathbb{T}^n \times [0,\infty)) \) and therefore they converge to \( u_i^\infty \in \text{BUC}(\mathbb{T}^n \times [0,\infty)) \) for \( i = 1, 2 \).

By a standard stability result of the theory of viscosity solutions, \( (u_1^\infty, u_2^\infty) \) is a solution of (4.1), (4.2). Moreover for \( t > 0 \)

\[
\max_{i=1,2} \max_{x \in \mathbb{T}^n} u_i^\infty(x, t) = \lim_{n \to \infty} \max_{i=1,2} \max_{x \in \mathbb{T}^n} u_i(x, T_n + t) = \lim_{n \to \infty} M(T_n + t) = \overline{M},
\]

and similarly

\[
\min_{i=1,2} \min_{x \in \mathbb{T}^n} u_i^\infty(x, t) = \underline{m}.
\]

Let \((x_1, t_1)\) and \((x_2, t_2)\) satisfy \( \max_{i=1,2} u_i^\infty(x_1, t_1) = \overline{M} \) and \( \min_{i=1,2} u_i^\infty(x_2, t_2) = \underline{m} \). Without loss of generality, we assume that \( u_1^\infty(x_1, t_1) = \max_{i=1,2} u_i^\infty(x_1, t_1) = \overline{M} \). By taking 0 as a test function from above of \( u_1^\infty \) at \((x_1, t_1)\) we have

\[
u_1^\infty(x_1, t_1) - u_2^\infty(x_1, t_1) \leq 0
\]

and therefore we obtain \( u_1^\infty(x_1, t_1) = u_2^\infty(x_1, t_1) = \overline{M} \). Similarly we obtain \( u_1^\infty(x_2, t_2) = u_2^\infty(x_2, t_2) = \underline{m} \). In particular,

\[
\max_{x \in \mathbb{T}^n} u_i^\infty(x, t) = M, \quad \min_{x \in \mathbb{T}^n} u_i^\infty(x, t) = m
\]

for \( t > 0 \) and \( i = 1, 2 \).

On the other hand, we have

\[
(u_1^\infty + u_2^\infty)_t + H_1(Du_1^\infty) + H_2(Du_2^\infty) = 0.
\]\n
Integrate (4.4) over \( \mathbb{T}^n \), use (A6), and do the same way as in the proof of Proposition 4.1 to get

\[
0 = \frac{d}{dt} \int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t) \, dx + \int_{\mathbb{T}^n} [H_1(Du_1^\infty) + H_2(Du_2^\infty)] \, dx \\
\geq \frac{d}{dt} \int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t) \, dx + \alpha \int_{\mathbb{T}^n} (|Du_1^\infty|^2 + |Du_2^\infty|^2) \, dx \\
\geq \frac{d}{dt} \int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t) \, dx + C,
\]

where the last inequality follows from Lemma 4.3 below. Thus

\[
\frac{d}{dt} \int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t) \, dx \leq -C,
\]

which implies

\[
\lim_{t \to \infty} \int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t) \, dx = -\infty.
\]
This contradicts (4.3) and the proof is complete. □

**Lemma 4.3.** There exists a constant $\beta > 0$ depending only on $n, C$ such that

$$
\int_{T^n} |Df|^2 \, dx \geq \beta
$$

for all $f \in W^{1,\infty}(\mathbb{T}^n)$ such that $\|f\|_{W^{1,\infty}(\mathbb{T}^n)} \leq C$, $\max_{\mathbb{T}^n} f = 1$, and $\min_{\mathbb{T}^n} f = 0$.

**Proof.** We argue by contradiction. Were the stated estimate false, there would exist a sequence $\{f_m\} \subset W^{1,\infty}(\mathbb{T}^n)$ such that $\|f_m\|_{W^{1,\infty}(\mathbb{T}^n)} \leq C$, $\max_{\mathbb{T}^n} f_m = 1$, $\min_{\mathbb{T}^n} f_m = 0$, and

$$
\int_{\mathbb{T}^n} |Df_m|^2 \, dx \leq \frac{1}{m}. \tag{4.5}
$$

By Ascoli-Arzela’s theorem, we may assume there exists $f_0 \in W^{1,\infty}(\mathbb{T}^n)$ so that $f_m \to f_0$ uniformly on $\mathbb{T}^n$ by taking a subsequence if necessary. It is clear that $\max_{\mathbb{T}^n} f_0 = 1$, $\min_{\mathbb{T}^n} f_0 = 0$.

Besides, $\|f_m\|_{H^1(\mathbb{T}^n)} \leq C$ for all $m \in \mathbb{N}$. By the Rellich-Kondrachov theorem, $f_m \rightharpoonup f_0$ in $H^1(\mathbb{T}^n)$ by taking a subsequence if necessary. By (4.5), we obtain $Df_0 = 0$ a.e. Thus $f_0$ is constant, which contradicts the fact that $\max_{\mathbb{T}^n} f_0 = 1$, $\min_{\mathbb{T}^n} f_0 = 0$. □

**Remark 4.4.** (i) Assumption (A7) is just for simplicity. Indeed we can always normalize the Hamiltonians so that they satisfy (A7) by substituting $(u_1, u_2)$ with $(\overline{u}_1, \overline{u}_2)$, where

\[
\begin{cases}
\overline{u}_1(x, t) := u_1(x, t) + \frac{H_1(0) + H_2(0)}{2} t + \frac{H_1(0) - H_2(0)}{2} t \\
\overline{u}_2(x, t) := u_2(x, t) + \frac{H_1(0) + H_2(0)}{2} t
\end{cases}
\quad \text{for } (x, t) \in \mathbb{T}^n \times [0, \infty).
\]

(ii) It is clear to see that we can get a similar result for systems with $m$-equations.

(iii) The same procedure works for the following more general Hamiltonians

$$
H_i(x, p) = |p - b_i(x)|^2 - |b_i(x)|^2
$$

for $b_i \in C^1(\mathbb{T}^n)$ with $\text{div} b_i = 0$ on $\mathbb{T}^n$ for $i = 1, 2$. This type of Hamiltonians is related to the ones in some recent works on periodic homogenization of G-equation. See [9, 32] for details. The new key observation comes from the fact that

$$
\int_{\mathbb{T}^n} b_i(x) \cdot D\phi(x) \, dx = - \int_{\mathbb{T}^n} (\text{div} b_i) \phi \, dx = 0
$$

for any $\phi \in W^{1,\infty}(\mathbb{T}^n)$. This identity was also used in [32] to study the existence of approximate correctors of the cell (corrector) problem of G-equation. The divergence free requirement on the vector fields $b_i$ for $i = 1, 2$ is critical in our argument. In particular, it
forces (E) to only have constant solutions \((a, a)\) for \(a \in \mathbb{R}\). We do not know how to remove this requirement up to now.

5. Third case

In this section we consider the third case pointed out in Introduction, i.e., we assume that \(H = H_1 = H_2\) and \(H\) satisfies (A1) and (A8). Then (C) reduces to

\[
(C3) \quad \begin{cases}
(u_1)_t + H(x, Du_1) + u_1 - u_2 = 0 & \text{in } \mathbb{T}^n \times (0, \infty), \\
(u_2)_t + H(x, Du_2) + u_2 - u_1 = 0 & \text{in } \mathbb{T}^n \times (0, \infty), \\
u_1(x, 0) = u_{01}(x), \quad u_2(x, 0) = u_{02}(x) & \text{on } \mathbb{T}^n.
\end{cases}
\]

Let \((u_1, u_2)\) be the solution of (C3).

**Proposition 5.1.** The function \((u_1 - u_2)(x, t)\) converges uniformly to 0 on \(\mathbb{T}^n\) as \(t \to \infty\).

**Lemma 5.2.** Set \(\gamma(t) := \max_{x \in \mathbb{T}^n}(u_1 - u_2)(x, t)\). Then \(\gamma\) is a subsolution of

\[
\dot{\gamma}(t) + 2\gamma(t) = 0 \text{ in } (0, \infty). \tag{5.1}
\]

**Proof of Lemma 5.2.** Let \(\phi \in C^1((0, \infty))\) and \(\tau > 0\) be a maximum of \(\gamma - \phi\). Choose \(\xi \in \mathbb{T}^n\) such that \(\gamma(\tau) = u_1(\xi, \tau) - u_2(\xi, \tau)\). We define the function \(\Psi\) by

\[
\Psi(x, y, t, s) := u_1(x, t) - u_2(y, s) - \frac{1}{2\varepsilon^2}(|x - y|^2 + (t - s)^2) - |x - \xi|^2 - (t - \tau)^2 - \phi(t).
\]

Let \(\Psi\) achieve its maximum at some \((\bar{x}, \bar{y}, \bar{t}, \bar{s})\). By the definition of viscosity solutions we have

\[
\frac{\bar{t} - \bar{s}}{\varepsilon^2} + 2(\bar{t} - \tau) + \dot{\phi}(\bar{t}) + H(\bar{x}, \frac{\bar{x} - \bar{y}}{\varepsilon^2} + 2(\bar{t} - \xi)) + u_1(\bar{x}, \bar{t}) - u_2(\bar{x}, \bar{t}) \leq 0,
\]

\[
\frac{\bar{t} - \bar{s}}{\varepsilon^2} + H(\bar{y}, \frac{\bar{x} - \bar{y}}{\varepsilon^2}) + u_2(\bar{y}, \bar{s}) - u_1(\bar{y}, \bar{s}) \geq 0.
\]

Subtracting the two inequalities above, we obtain

\[
2(\bar{t} - \tau) + \dot{\phi}(\bar{t}) + H(\bar{x}, \frac{\bar{x} - \bar{y}}{\varepsilon^2} + 2(\bar{t} - \xi)) - H(\bar{y}, \frac{\bar{x} - \bar{y}}{\varepsilon^2})
\]

\[
+ u_1(\bar{x}, \bar{t}) - u_2(\bar{x}, \bar{t}) - (u_2(\bar{y}, \bar{s}) - u_1(\bar{y}, \bar{s})) \leq 0. \tag{5.2}
\]

By the usual argument we may assume that

\[
\bar{x}, \bar{y} \to \xi, \quad \bar{t}, \bar{s} \to \tau, \quad \frac{\bar{x} - \bar{y}}{\varepsilon^2} \to p \tag{5.3}
\]

as \(\varepsilon \to 0\) by taking a subsequence if necessary. Sending \(\varepsilon \to 0\) in (5.2), we get

\[
\dot{\phi}(\tau) + 2\gamma(\tau) \leq 0,
\]

which is the conclusion. \qed
Lemma 5.3 (Key Lemma). Let $C$ be the constant fixed in (5.5). 

(i) Assume that (A8)$^+$ holds. For any $\eta \in (0, \eta_0]$ there exists $s_\eta > 0$ such that the pair of
the functions \((\alpha^+_\eta, \beta^+_\eta)\) is a supersolution of
\[
\begin{align*}
\max \{ (\alpha^+_\eta)'(s) + \frac{\psi_n}{C}(\alpha^+_\eta(s) - 1) + F(\alpha^+_\eta(s) - \beta^+_\eta(s)), \alpha^+_\eta(s) - 1 \} &= 0 \quad \text{in} \ (s, \infty), \quad (5.8) \\
\max \{ (\beta^+_\eta)'(s) + \frac{\psi_n}{C}(\beta^+_\eta(s) - 1) + F(\beta^+_\eta(s) - \alpha^+_\eta(s)), \beta^+_\eta(s) - 1 \} &= 0 \quad \text{in} \ (s, \infty), \quad (5.9)
\end{align*}
\]
where
\[
F(r) := \begin{cases} 
Cr & \text{if } r \geq 0, \\
\frac{r}{C} & \text{if } r < 0.
\end{cases}
\]

(ii) Assume that \((A8)^-\) holds. For any \(\eta \in (0, \eta_0]\) there exists \(s_\eta > 0\) such that the pair of the functions \((\alpha^-_\eta, \beta^-_\eta)\) is a subsolution of
\[
\begin{align*}
\min \{ (\alpha^-_\eta)'(s) + \frac{\psi_n}{C} \cdot \frac{\alpha^-_\eta(s) - 1}{\alpha^-_\eta(s)} + F(\alpha^-_\eta(s) - \beta^-_\eta(s)), \alpha^-_\eta(s) - 1 \} &= 0 \quad \text{in} \ (s, \infty), \quad (5.10) \\
\min \{ (\beta^-_\eta)'(s) + \frac{\psi_n}{C} \cdot \frac{\beta^-_\eta(s) - 1}{\beta^-_\eta(s)} + F(\beta^-_\eta(s) - \alpha^-_\eta(s)), \beta^-_\eta(s) - 1 \} &= 0 \quad \text{in} \ (s, \infty). \quad (5.11)
\end{align*}
\]

Proof. We only prove (i), since we can prove (ii) similarly. Fix \(\mu \in (0, \eta_0]\). By abuse of notations we write \(\alpha, \beta\) for \(\alpha^+_\eta, \beta^+_\eta\). Recall that \(\alpha(s), \beta(s) \leq 1\) for any \(s \geq 0\). By Lemma 5.2, there exists \(s_\eta > 0\) such that \(|u_1(x, t) - u_2(x, t)| \leq \eta/2\) for all \(x \in \mathbb{T}^n\) and \(t \geq s_\eta\).

We only consider the case where \((\alpha - \phi)(s) > (\alpha - \phi)(\sigma)\) for some \(\phi \in C^1((0, \infty))\), \(\sigma > s_\eta, \delta > 0\) and all \(s \in [\sigma - \delta, \sigma + \delta] \setminus \{\sigma\}\), since a similar argument holds for \(\beta\). Since there is nothing to check in the case where \(\alpha(\sigma) = 1\), we assume that \(\alpha(\sigma) < 1\). We choose \(\xi \in \mathbb{T}^n\) and \(\tau \geq \sigma\) such that
\[
\alpha(\sigma) = \frac{u_1(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)}{u_1(\xi, \sigma) - v(\xi)} =: \frac{\alpha_2}{\alpha_1}.
\]
We write \(\alpha\) for \(\alpha(\sigma)\) henceforth.

Set \(K := \mathbb{T}^n \times \{ (t, s) \mid t \geq s, s \in [\sigma - \delta, \sigma + \delta] \}\). For \(\varepsilon \in (0, 1)\), we define the function \(\Psi : K \to \mathbb{R}\) by
\[
\Psi(x, y, z, t, s) := \frac{u_1(x, t) - v(z) + \eta(t - s)}{u_1(y, s) - v(z)} - \phi(s) + \frac{1}{2\varepsilon^2}(|x - y|^2 + |x - z|^2 + |x - \xi|^2 + (t - \tau)^2).
\]
Let $\Psi$ achieve its minimum over $K$ at some $(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{s})$. Set
\[
\bar{\alpha} := \frac{\alpha}{\alpha_1},
\bar{p} := \frac{\bar{y} - \bar{x}}{\varepsilon^2}, \text{ and } \bar{q} := \frac{\bar{z} - \bar{x}}{\varepsilon^2}.
\]
We have by the definition of viscosity solutions
\[
\begin{cases}
-\eta - 2\bar{\alpha} \bar{t} + H(\bar{x}, D_x u_1(\bar{x}, \bar{t})) + (u_1 - u_2)(\bar{x}, \bar{t}) \geq 0, \\
-\frac{1}{\alpha}(\eta + \bar{\alpha}_1 \phi'(\bar{s})) + H(\bar{y}, D_y u_1(\bar{y}, \bar{s})) + (u_1 - u_2)(\bar{y}, \bar{s}) \leq 0, \\
H(\bar{z}, D_z v(\bar{z})) \leq 0,
\end{cases}
\tag{5.12}
\]
where
\[
D_x u_1(\bar{x}, \bar{t}) = \bar{\alpha}_1 \{\bar{p} + \bar{q} + 2(\xi - \bar{x})\},
\]
\[
D_y u_1(\bar{y}, \bar{s}) = \frac{\bar{\alpha}_1}{\alpha} \bar{p},
\]
\[
D_z v(\bar{z}) = \frac{\bar{\alpha}_1}{1 - \alpha} \bar{q}.
\]
By taking a subsequence if necessary, we may assume that
\[
\bar{x}, \bar{y}, \bar{z} \to \xi \text{ and } \bar{t} \to \tau, \bar{s} \to \sigma \text{ as } \varepsilon \to 0.
\]
Since $u_i, v$ are Lipschitz continuous, we have
\[
\frac{|\bar{x} - \bar{y}|}{\varepsilon^2} + \frac{|\bar{x} - \bar{z}|}{\varepsilon^2} \leq M
\]
for some $M > 0$ and all $\varepsilon \in (0, 1)$. We may assume that
\[
\bar{p} := \frac{\bar{y} - \bar{x}}{\varepsilon^2} \to p, \bar{q} := \frac{\bar{z} - \bar{x}}{\varepsilon^2} \to q
\]
as $\varepsilon \to 0$ for some $p, q \in B(0, M)$.

Sending $\varepsilon \to 0$ in (5.12) yields
\[
-\eta + H(\xi, \tilde{P} + Q) + (u_1 - u_2)(\xi, \tau) \geq 0,
\]
\[
-\frac{1}{\alpha(\sigma)}(\eta + \alpha_1 \phi'(\sigma)) + H(\xi, P) + (u_1 - u_2)(\xi, \sigma) \leq 0,
\]
\[
H(\xi, Q) \leq 0,
\tag{5.13}
\]
where
\[
P := \frac{\alpha_1}{\alpha} p, \quad Q := \frac{\alpha_1}{1 - \alpha} q, \quad \tilde{P} = \alpha (P - Q).
\]
Recalling that $(u_1 - u_2)(\xi, \tau) \leq \eta/2$, we have
\[
H(\xi, \tilde{P} + Q) \geq \eta/2.
\]
Therefore, by using \((A8)^+\), we obtain
\[
H(\xi, \tilde{P} + Q) \leq \alpha H(\xi, P) - \psi_\eta (1 - \alpha)
\] (5.14)
for some \(\psi_\eta > 0\).

Noting that
\[
\beta(\sigma) \leq \frac{u_2(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)}{u_2(\xi, \sigma) - v(\xi)} = \frac{\beta_2}{\beta_1},
\]
we calculate that
\[
(u_1 - u_2)(\xi, \tau) - \alpha(u_1 - u_2)(\xi, \sigma)
= - (u_2(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)) + \alpha(u_2(\xi, \sigma) - v(\xi))
= - \beta_1 \left( \frac{\beta_2}{\beta_1} - \alpha \right)
\leq - \beta_1 (\beta(\sigma) - \alpha(\sigma)).
\]

Therefore by (5.14) and (5.13),
\[
\eta \leq H(\xi, \tilde{P} + Q) + (u_1 - u_2)(\xi, \tau)
\leq \alpha \left( \frac{1}{\alpha} (\eta + \alpha_1 \phi'(\sigma)) - (u_1 - u_2)(\xi, \sigma) \right) - \psi_\eta (1 - \alpha) + (u_1 - u_2)(\xi, \tau)
\leq \eta + \alpha_1 \phi'(\sigma) - \psi_\eta (1 - \alpha) + \beta_1 (\alpha(\sigma) - \beta(\sigma)),
\]
which implies
\[
\phi'(\sigma) + \frac{\psi_\eta}{C}(\alpha(\sigma) - 1) + \frac{\beta_1}{\alpha_1} (\alpha(\sigma) - \beta(\sigma)) \geq 0.
\]
Combining the above inequality with the fact that \(1/C \leq \beta_1/\alpha_1 \leq C\), we have
\[
\phi'(\sigma) + \frac{\psi_\eta}{C}(\alpha(\sigma) - 1) + F(\alpha(\sigma) - \beta(\sigma)) \geq 0.
\]

\(\square\)

**Lemma 5.4.**
(i) Assume that \((A8)^+\) holds. The functions \(\alpha_\eta^+(s)\) and \(\beta_\eta^+(s)\) converge to 1 as \(s \to \infty\) for each \(\eta \in (0, \eta_0]\).
(ii) Assume that \((A8)^-\) holds. The functions \(\alpha_\eta^-(s)\) and \(\beta_\eta^-(s)\) converge to 1 as \(s \to \infty\) for each \(\eta \in (0, \eta_0]\).

**Proof.** Fix \(\eta \in (0, \eta_0]\). We first recall that, by definition,
\[
\alpha_\eta^+(s) \leq 1 \leq \alpha_\eta^-(s), \quad \beta_\eta^+(s) \leq 1 \leq \beta_\eta^-(s)
\]
for any \(s \geq 0\). On the other hand, one checks easily that the pairs
\[
(1 + (\gamma_1 - 1) \exp(-\frac{\psi_\eta}{C} t), 1 + (\gamma_1 - 1) \exp(-\frac{\psi_\eta}{C} t))
\]
and
\[(1 + (\gamma_2 - 1) \exp(-\frac{\psi_\eta t}{C\gamma_2}), 1 + (\gamma_2 - 1) \exp(-\frac{\psi_\eta t}{C\gamma_2}))\]
are, respectively, a subsolution and a supersolution of (5.8)-(5.9) and (5.10)-(5.11) for \(\gamma_1 = \min\{\alpha_\eta^+(0), \beta_\eta^+(0)\}\), and \(\gamma_2 = \max\{\alpha_\eta^-(0), \beta_\eta^-(0)\}\). Therefore, by the comparison principle in [25, 11, 23], we get
\[\alpha_\eta^+(s), \beta_\eta^+(s) \geq 1 + (\gamma_1 - 1) \exp(-\frac{\psi_\eta t}{C})\]
and
\[\alpha_\eta^-(s), \beta_\eta^-(s) \leq 1 + (\gamma_2 - 1) \exp(-\frac{\psi_\eta t}{C\gamma_2})\]
which give us the conclusion. \(\Box\)

By Lemma 5.4, we immediately get the following proposition.

**Proposition 5.5** (Asymptotically Monotone Property).

(i) **(Asymptotically Increasing Property)**

Assume that (A8)\(^+\) holds. For \(\eta \in (0, \eta_0]\), there exists a function \(\delta_\eta : [0, \infty) \to [0, 1]\) such that
\[\lim_{s \to \infty} \delta_\eta(s) = 0\]
and
\[u_i(x, s) - u_i(x, t) - \eta(t - s) \leq \delta_\eta(s)\]
for all \(x \in \mathbb{T}^n, t \geq s \geq 0\) and \(i = 1, 2\).

(ii) **(Asymptotically Decreasing Property)**

Assume that (A8)\(^-\) holds. For \(\eta \in (0, \eta_0]\), there exists a function \(\delta_\eta : [0, \infty) \to [0, 1]\) such that
\[\lim_{s \to \infty} \delta_\eta(s) = 0\]
and
\[u_i(x, t) - u_i(x, s) - \eta(t - s) \leq \delta_\eta(s),\]
for all \(x \in \mathbb{T}^n, t \geq s \geq 0\) and \(i = 1, 2\).

Theorem 1.3 is a straightforward result of the above proposition. See [3, Section 4] or [1, Section 5] for the detail.

Finally we remark that if we want to deal, at the same time, with the Hamiltonians of the form
\[H(x, p) := |p| - f(x),\]
we can generalize Theorem 1.3 as in [3]. We replace (A8) by

(A9) Either of the following assumption (A9)\(^+\) or (A9)\(^-\) holds:

(A9)\(^+\) There exists a closed set \(K \subset \mathbb{T}^n (K\) is possibly empty) having the properties
(i) min_{p \in \mathbb{R}^n} H(x, p) = 0 for all x \in K,

(ii) for each \( \varepsilon > 0 \) there exists a modulus \( \psi_\varepsilon(r) > 0 \) for all \( r > 0 \) and \( \eta_0^\varepsilon > 0 \) such that for all \( \eta \in (0, \eta_0^\varepsilon) \) if dist \((x, K) \geq \varepsilon, H(x, p + q) \geq \eta \) and \( H(x, q) \leq 0 \) for some \( x \in \mathbb{T}^n \) and \( p, q \in \mathbb{R} \), then for any \( \mu \in (0, 1] \),

\[
\mu H(x, \frac{p}{\mu} + q) \geq H(x, p + q) + \psi_\varepsilon(\eta)(1 - \mu).
\]

(A9) There exists a closed set \( K \subset \mathbb{T}^n \) (\( K \) is possibly empty) having the properties

(i) min_{p \in \mathbb{R}^n} H(x, p) = 0 for all x \in K,

(ii) for each \( \varepsilon > 0 \) there exists a modulus \( \psi_\varepsilon(r) > 0 \) for all \( r > 0 \) and \( \eta_0^\varepsilon > 0 \) such that for all \( \eta \in (0, \eta_0^\varepsilon) \) if dist \((x, K) \geq \varepsilon, H(x, p + q) \geq -\eta \) and \( H(x, q) \geq 0 \) for some \( x \in \mathbb{T}^n \) and \( p, q \in \mathbb{R} \), then for any \( \mu \in (0, 1] \),

\[
\mu H(x, \frac{p}{\mu} + q) \leq H(x, p + q) - \psi_\varepsilon(\eta)(\mu - 1)\mu.
\]

Theorem 5.6. The result of Theorem 1.3 still holds if we replace (A8) by (A9).

Sketch of Proof. By the argument same as in the proof of Propositions 3.3, 3.4 we can see \((u_1 + u_2)|_K\) and \(\max\{u_1, u_2\}|_K\) are nonincreasing and therefore we see that \(u_i\) converge uniformly on \(K\) as \(t \to \infty\) for \(i = 1, 2\).

Setting \(K_\varepsilon := \{x \in \mathbb{T}^n \mid d(x, K) \geq \varepsilon\}\), we see that \(u_i\) are asymptotically monotone on \(\mathbb{R}^n \setminus K_\varepsilon\) for every \(\varepsilon > 0\), which implies that \(u_i\) converges uniformly on \(\mathbb{R}^n \setminus K\) as \(t \to \infty\) for \(i = 1, 2\) as in [3].

6. Appendix

We present a sketch of the proof based on Proposition 2.1 from [5] for the reader’s convenience.

Sketch of the proof of Proposition 2.1. Without loss of generality, we may assume \(c_1 = c_2 = 1\). The existence of \((v_1, v_2, H_1, H_2)\) can be proved by repeating the argument same as in the first part of Proposition 3.1. We here only prove that \(H_1 + H_2\) is unique.

Suppose by contradiction that there exist two pairs \((\lambda_1, \lambda_2) \in \mathbb{R}^2\) and \((\mu_1, \mu_2) \in \mathbb{R}^2\) such that \(\lambda_1 + \lambda_2 < \mu_1 + \mu_2\) and two pair of continuous functions \((v_1, v_2), (\overline{v}_1, \overline{v}_2)\) such that \((v_1, v_2), (\overline{v}_1, \overline{v}_2)\) are viscosity solutions of the following systems

\[
\begin{cases}
H_1(x, Dv_1) + v_1 - v_2 = \lambda_1 \\
H_2(x, Dv_2) + v_2 - v_1 = \lambda_2
\end{cases}
\text{in } \mathbb{T}^n,
\]
and
\[
\begin{cases}
H_1(x, D\overline{v}_1) + \overline{v}_1 - \overline{v}_2 = \mu_1 \\
H_2(x, D\overline{v}_2) + \overline{v}_2 - \overline{v}_1 = \mu_2
\end{cases}
\text{ in } \mathbb{T}^n,
\]
respectively.

For a suitably large constant $C > 0$, \((v_1 + \frac{\lambda_2 - \lambda_1}{2} - \frac{\lambda_1 + \lambda_2}{2} t - C, v_2 - \frac{\lambda_1 + \lambda_2}{2} t - C)\) and \((\overline{v}_1 + \frac{\mu_2 - \mu_1}{2} - \frac{\mu_1 + \mu_2}{2} t + C, \overline{v}_2 - \frac{\mu_1 + \mu_2}{2} t + C)\) are respectively a subsolution and a supersolution of (C). By the comparison principle for (C), Proposition 2.2, we obtain particularly
\[
v_1 + \frac{\lambda_2 - \lambda_1}{2} - \frac{\lambda_1 + \lambda_2}{2} t - C \leq \overline{v}_1 + \frac{\mu_2 - \mu_1}{2} - \frac{\mu_1 + \mu_2}{2} t + C,
\text{ in } \mathbb{T}^n \times [0, \infty)
\]
which contradicts the fact that $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$. \hfill \Box

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