On Symplectic Capacities and their Blind Spots

with Yuanpu Liang
Symplectic capacity

\[ C : S \subset \left\{ u \subset \mathbb{R}^{2n} \right\} \rightarrow [0, \infty] \]

1) \( C(U) \leq C(V) \) if \( \exists \phi \in \text{Symp} \left( \mathbb{R}^{2n}, \omega_{2n} \right) \) s.t. \( \phi(U) \leq V \).

2) \( C \left( \mathbb{R}U \right) = \chi^2 C(U) \)

3) \( C \left( B^2(1) \right) > 0 \), \( C \left( B^2(1) \times \mathbb{R}^{2n-2} \right) < \infty \)
Examples

• Gromov width = \( \sup \left\{ \| r \| \mid E \in \text{Symp}(\mathbb{R}^{2n}, \omega_{2n}) \text{ s.t.} \right\} \)

• \( \{ C_k \}_{k \in \mathbb{N}} \) Ekeland - Hofer capacities

Defined for all subsets of \( \mathbb{R}^{2n} \) in terms of periodic orbits of autonomous Hamiltonians.
Gutt-Hutchings capacities

\[ C_k, \ k \in \mathbb{N} \]

Defined for star-shaped subsets \( X \subset \mathbb{R}^{2n} \)

using \( S^1 \)-equivariant symplectic homology

\[
CH_*(X) = \begin{cases} 
\mathbb{Q}, & * \in n-1 + 2\mathbb{N} \\
0, & \text{otherwise}
\end{cases}
\]

\[ C_k(X) = \inf \left\{ L \mid \text{image } i_L: CH^L(X) \to CH(X) \text{ contains } CH_{n-1+2k}(X) \right\} \]
Conjecture (Gutt-Hutchings)

\[ C_k(X) = C_k^{\text{EH}}(X) \text{ for all } X \text{ star-shaped} \]

This holds for symplectic ellipsoids and polydisks.

\[ E(1,a) = \left\{ \frac{\pi l_1^2 + \pi l_2^2}{a} \leq 1 \right\} \subset \mathbb{R}^4 \]

\[ C_k(E(1,a)) = (\text{Sort} \left[ N \cup aN \right])[k] = C_k^{\text{EH}}(E(1,a)) \]

\[ P(1,a) = \left\{ \pi l_1^2 < 1, \pi l_2^2 < a \right\} \]

\[ C_k(P(1,a)) = k. = C_k^{\text{EH}}(P(1,a)) \]
Gutt-Hutchings formulae for convex/concave toric domains

\[ \mu : (\mathbb{C}^n = \mathbb{R}^{2n}) \rightarrow \mathbb{R}^n_{\geq 0} \]

\[ (z_1, \ldots, z_n) \mapsto (\prod |z_i|^2, \ldots, \prod |z_n|^2) \]

\[ \Omega \subset \mathbb{R}^n_{\geq 0} \leadsto X_\Omega = \mu^{-1}(\Omega) \subset \mathbb{R}^{2n} \]

\[ X_\Omega \text{ is convex if } \hat{\Omega} = \{ x \mid (1x_1, \ldots, 1x_n) \in \Omega \} \subset \mathbb{R}^n \text{ is convex.} \]

\[ X_\Omega \text{ is concave if } \Omega^C \text{ is convex in } \mathbb{R}^n_{\geq 0} \]
Theorem (Gutt-Hutchings)

1) If $X_\Omega$ is convex, then

$$c_k(X_\Omega) = \min \left\{ \| v \|_{\Omega} \mid v \in (N \cup \{0\})^n, \sum y_j = k \right\}$$

where $\| v \|_{\Omega} = \max \{ \langle v, \omega \rangle \mid \omega \in \Omega \}$

- Computing $c_k$ involves comparison of $\binom{k+n-1}{n+1}$ (similar) optimization problems
II) If $X_{\omega}$ is concave, then

$$C_{\omega}(X_{\omega}) = \max \left\{ [V]_{\omega} \mid v \in \mathbb{N}^n, \sum v_j = k+1 \right\}$$

where $[V]_{\omega} = \min \left\{ \langle v, w \rangle \mid w \in \left\{ \overline{\Omega \cap \mathbb{R}_+^n} \right\} = \partial \Omega \right\}$

i.e.

\[ \text{Diagram of } \partial \Omega \]
Capacities and the Minkowski sum

Theorem (Artstein-Avidan, Ostrover)

If $U$ and $V$ are convex bodies in $\mathbb{R}^n$, then

$$\left( c_1(U+V) \right)^{\frac{1}{2}} \geq \left( c_1(U) \right)^{\frac{1}{2}} + \left( c_1(V) \right)^{\frac{1}{2}}$$

with equality if $dU$ and $dV$ have homothetic representatives of $c_1$.

Q1. Does this inequality hold for $c_k$ with $k>1$?
Theorem (K., Liang)

For even $k$

$$c_k \left( E \left( (1+\frac{1}{k})^2, 1 \right) + E(1, (1+\frac{1}{k})^2) \right)^{1/2} < c_k \left( E \left( (1+\frac{1}{k})^2, 1 \right) \right)^{1/2} + c_k \left( E \left( (1+\frac{1}{k})^2, 1 \right) \right)^{1/2}$$

For odd $k > 1$

$$c_k \left( E \left( (1, 1) + E \left( (1-\frac{1}{k})^2, 1 \right) \right) \right)^{1/2} < c_k \left( E \left( (1, 1) \right) \right)^{1/2} + c_k \left( E \left( (1-\frac{1}{k})^2, 1 \right) \right)^{1/2}$$

* $c_k \left( E(r, b) + E(c, d) \right)$ can be made explicit.
Observation: Ostrover

Prop (A−A, 0) If a (normalized) capacity C satisfies the symplectic Brunn–Minkowski inequality then for every centrally symmetric convex body U

\[ C(U) \leq \pi \left( \frac{\text{mean-width}(U)}{2} \right)^2 \]

Applying to \( C_k \) and \( U = P(1,1) \) implies \( C_{k>1} \) do not satisfy symplectic Brunn–Minkowski for \( k \neq 3, 5, 7 \).
Steiner Formula for $U \subset \mathbb{R}^m$ convex

$$\text{Vol} \left( U + t B^m(1) \right) = \sum_{j=0}^{m} \binom{m}{j} W_j(U) t^j$$

$W_j(U) = j^{th}$ Quermassintegral of $U$

$$W_{m-1}(U) = \frac{\text{Vol}(B^m(1))}{2} \left( \text{mean-width}(U) \right)$$

Are there symplectic Steiner formulas?

$$C_k \left( U + t B^m(1) \right) = C_k(U) + a_k(U) t + c_k \left( B^m(1) \right) t^2$$
$a_k(u) = \text{"kth symplectic mean width"}$?

Artstein-Avidan, Ostrover \implies a_k(u) \geq 2 \sqrt{\pi} \sqrt{c_1(u)}

with equality if $c_1(u)$ is represented by a great circle on $SU$.

Not in the form above. For $a > \sqrt{2}$

$c_z(E(\pi, \pi t^2) + tB) = \begin{cases} 2\pi + 4\pi t + 2\pi t^2, & t \leq \frac{a - \sqrt{2}}{\sqrt{2} - 1} \\ \pi a^2 + 2\pi a + \pi t^2, & t > \frac{a - \sqrt{2}}{\sqrt{2} - 1} \end{cases}$
Relation of capacities to volume

- The $c_k(E(1,a)) = \text{Sort} \left\{ \mathbb{Z} \cup a\mathbb{Z} \right\}[k]

  'see' $\text{Vol}(E(1,a)) = \frac{a}{2}$.

- For $P(1,a)$ the $c_k(P(1,a)) = k$ are completely blind to $\text{Vol}(P(1,a)) = a$.

Q2 How do these blind spots develop?
Consider 

\[ E_p (1, a) = \left\{ \left( \frac{2}{1, \lambda^2} \right)^p + \left( \frac{1, \lambda^2}{a} \right)^p \leq 1 \right\} \]

which go from \( E(1, a) \) at \( p = 1 \) to \( P(1, a) \) as \( p \to \infty \).

Study

\[ C_k (E_p (1, a)) \]

Lemma 1 (K.L.) For each \( k \) \( \exists \) \( p(k) < \infty \) s.t.

\[ C_k (E_p (1, a)) = k = C_k (P(1, a)) \quad \forall \ p \geq p(k) \]
Lemma 2. (K.K.) For each \( p \geq k(p) \) s.t.
\[
\frac{d}{da} \left( c_k(E_p(1,a)) \right) > 0 \quad \forall \ k > k(p)
\]
\[
\left( c_k(E_p(1,a)) = \left( \left( a \left( k-m \right) \right)^{\frac{p}{p-1}} + m^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right)
\]
for some \( m \in [1, k-1] \)

so each \( c_k(E_p(1,a)) \) with \( k > k(p) \) "sees" \( a \)

and hence \( \text{Vol}(E_p(1,a)) \)
A2 \[ \bigl\{ C_k(E_p(1,a)) \bigr\}_{k \in \mathbb{N}} \] is only blind to 
\[ \text{Vol}(E_p(1,a)) \] in the limit \( p \to \infty \).

Q3 If \( X \) is smooth does \[ \bigl\{ C_k(X) \bigr\}_{k \in \mathbb{N}} \] "see" \( \text{Vol}(X) \) ?

A3 No.
Symmetry and simplification.

**Definition.** \( \Omega \subset \mathbb{R}^n \) is symmetric if

\[
(x_1, \ldots, x_n) \in \Omega \Rightarrow (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \Omega \quad \forall \sigma \in S_n.
\]

**Theorem (K., Liang)**

1) If \( \Omega \) is symmetric and \( X := (X_1, \ldots, X_n) \) is convex, then

\[
c_k (X_\Omega) = \max_{w \in \Omega} \langle w, V(k,n) \rangle = \|V(k,n)\|_{\Omega} \quad \text{for}
\]

\[
V(k,n) = \left( \left[ \frac{k}{n} \right], \ldots, \left[ \frac{k}{n} \right], \underbrace{\left[ \frac{k}{n} \right]}_{k \text{ mod } n} \right).
\]
III) If $\Omega$ is symmetric and $X_\Omega$ is concave, then

$$c_k(X_\Omega) = \min_{w \in \overline{D}_\Omega} \langle \hat{V}(k,n), w \rangle = [\hat{V}(k,n)]_\Omega$$

where

$$\hat{V}(k,n) = \left( \left[ \frac{k+n-1}{n} \right], \ldots, \left[ \frac{k+n-1}{n} \right], \frac{k+n-1}{n}, \ldots, \frac{k+n-1}{n} \right)$$

$$\sim \frac{k+n-1}{n} \text{ mod } n \sim$$
Example

\[ y = f(x) \quad f^{-1}(y) = x \quad \Rightarrow \quad \Omega_f \text{ symmetric} \]

f2) \( f'' < 0 \)

f3) \( f'(0) \in (-\frac{1}{2}, 0) \)

\[ f\left(x(f)\right) = x(f) \quad f'(x_k) = -\frac{k-1}{k+1}, \quad k \text{ odd} \]

\[ C_k\left(\lambda_{x_2}\right) = \begin{cases} k \ x(x) \ , \ k \text{ even} \\ \frac{k-1}{2} \ x_k + \frac{k+1}{2} \ f(x_k) \ , \ k \text{ odd} \end{cases} \]
The $c_k$ are all determined by $X_k$, $X(t)$, and $f(X_k)$. 
Perturbations of $f$ away from $x'_k \neq x'(f)$ can change the volume while keeping the capacities fixed.

$$f_\delta = f + \delta \left( \frac{x_{2n-1}}{x_{2n+1}} \right) + \text{mirror bump}$$

If $\delta$ is small $\Rightarrow f_\delta$ satisfies (f1)-(f3)

$$c_k (x_{\Omega_{f_\delta}}) = c_k (x_{\Omega_f}) \quad \forall k \in \mathbb{N}$$

$$\text{Vol} (x_{\Omega_{f_\delta}}) \neq \text{Vol} (x_{\Omega_f})$$
The $\{c_k\}_{k \in \mathbb{N}}$ do not see $\text{Vol}$.

Given $U \subset \mathbb{R}^{2n}$ convex with all smooth,

$$\sup \frac{\text{Vol}(V)}{\text{Vol}(W)} < \infty \quad \text{where}$$

$$c_k(V) = c_k(W) = c_k(U) \quad \forall \quad k \in \mathbb{N}.$$

Without convexity, the answer is No!
In the concave toric setting, consider

\[ y = h(x) \]

\[ \Omega_h \]

\[ h_1) \quad h^\perp = h \]

\[ h_2) \quad h'' > 0 \]

\[ h_3) \quad h'(0) \in (-\infty, -2) \]

Thm (K, L.) implies

\[ c_k (X_{\Omega_h}) = \begin{cases} 
  (k+1) x(h) & \text{for } k \text{ odd} \\
  \frac{k+1}{2} x_k + \frac{k}{2} h(x_k) & \text{for } k \text{ even}
\end{cases} \]

where \[ h'(x_k) = -\frac{k+2}{k} \]
$$C_k \left( X \omega_{h \nu} \right) = C_k \left( X \omega_h \right)$$

$$\text{Vol} \left( X \omega_{h \nu} \right) \rightarrow \infty \quad \text{as} \quad \sigma \rightarrow \infty$$
Q4 Are the $c_k$ independent?

A4 Yes

For $j = 2n+1$ consider $f$ as above and form

$$f_{\delta} = f + \delta \left( f_{x_j} \right) + \text{mirror bump}$$

$$C_k \left( X_{\Omega_{f_i}} \right) = C_k \left( X_{\Omega_{f}} \right) \quad \text{for } k \neq j$$

$$C_j \left( X_{\Omega_{f_i}} \right) > C_j \left( X_{\Omega_{f}} \right) \quad \left( = C_j \left( X_{\Omega_l} \right) + \left( \frac{n+1}{2} \right) \delta \right)$$

(\textit{Rmk: Volume is independent of } \delta.\)
For \( j = 2n \) use a similar trick in concave setting

\[
h_{\delta} = h + \delta \left( \frac{x_{j-2}}{x_j} \right) + \text{mirror bump}
\]

\( \exists \exists ? X \) such that for all \( j \in \mathbb{N} \)

there is a \( Y_j \) such that

\[
C_k (Y_j) = C_k (X) \quad \forall \ k \neq j \quad \text{and} \quad C_j (Y_j) \neq C_j (X).
\]
Q5: If $U \subset \mathbb{R}^4$ has smooth boundary and is strictly convex, do the $c_k(U)$ and $\text{Vol}(U)$ determine $U$ up to symplectomorphism?

A5: No.

Tools: ECH capacities of Hutchings, $c_k^{ECH}$, and algorithm for $c_k^{ECH}(X_{2n})$ developed by Choi, Christofaro-Gardiner, Frenkel, Hutchings, Ramos.
Strategy

- Consider $U = X_{2h}$ as above.
- Deform $h$ away from $x_k \cdot x(h)$ as

$$h_\delta = h + \delta \left( \frac{X_{2h}}{2h} \right) + \text{mirror bump}$$

Note the $c_k$ and $\text{Vol}$ are unchanged.

- Show $\Rightarrow C_{X}^{\text{ECH}} (X_{2h_\delta}) \neq C_{X}^{\text{ECH}} (X_{2h})$

for some $l$. 
Example from K. L

$$\alpha(t) = \left( 2 \sin \left( \frac{t}{2} \right) - t \cos \left( \frac{t}{2} \right), \ 2 \sin \left( \frac{t}{2} \right) + (2\pi + t) \cos \left( \frac{t}{2} \right) \right)$$

( Ramos ) \quad \in [0, 2\pi]

$$\alpha(t) - (x, y) = \text{graph of } h$$

- The $c_k^{\text{EIT}}(X, 2h)$ depend on $h$ at points $X_1, X_2, X_{11}, X_{12}, X_{21}, X_{22}$. 

...
\bullet \quad X_{22} \text{ lies away from } x_k \cap x(h)

\bullet \quad \text{For } h_\delta = h + \delta \left( \frac{1}{x_{22}} \right) + \text{mirror bump}

\mathcal{E}_{\text{ch}} \left( X_{\Omega h_\delta} \right) = \mathcal{E}_{\text{ch}} \left( X_{\Omega h} \right) + \delta

\text{for all } h \text{ sufficiently small } \delta > 0.
Yuanpu’s Proof of the simplified formulas

Given $\Omega$ symmetric s.t. $X_\Omega$ is convex. Need

$$c_k(X_\Omega) = \min \left\{ \| v \|_{\Omega} \left| v \in (\mathbb{N} \cup \{0\})^n, \sum v_j = k \right\}$$

$$\leq \| V(k,n) \|_{\Omega}^{k \mod n}$$

For $V(k,n) = \left( \left\lfloor \frac{k}{n} \right\rfloor, \ldots, \left\lfloor \frac{k}{n} \right\rfloor, \left\lceil \frac{k}{n} \right\rceil, \ldots, \left\lceil \frac{k}{n} \right\rceil \right)$

and $\| v \|_{\Omega} = \max_{w \in \Omega} \langle v, w \rangle$. 
Def: $v \in \mathbb{R}^n$ is ordered if $v_1 \leq v_2 \leq \ldots \leq v_n$.

Eg: $V(k, n)$ is ordered.

Symmetry of $\Omega \Rightarrow \| (v_1, \ldots, v_n) \|_2 = \| (v_{\sigma_1(n)}, \ldots, v_{\sigma_n(n)}) \|_2$, $\forall \sigma \in S_n$.

$\tilde{S}(k, n) = \{ v \in (\mathbb{N} \cup \{0\})^n \mid \sum v_j = k \}, v \text{ ordered} \}$

$C_k(X_\Omega) = \min \left\{ \| v \|_2 \mid v \in \tilde{S}(k, n) \right\}$. 
Consider the map $T: \tilde{S}(k,n) \to \tilde{S}(k,n)$

$V = (V_1, V_2, \ldots, V_i, \ldots, V_n)$

\[ V \mapsto \begin{cases} 
  (V_1, \ldots, V_i, V_{i+1}, \ldots, V_{n-1}, V_n \ldots V_n) & \text{if } V_n > V_{i+1} \\
  V & \text{otherwise}
\end{cases} \]

- $\text{Fix}(T) = \{V(k,n)\}$ and $T^j(V) = V(k,n)$ for $j \gg 1$
Prop \[ \| T(v) \|_{\Omega} \leq \| v \|_{\Omega} \]

this settles things
Lemma \( \| v \|_2 = \langle v, w \rangle \) for an ordered \( w \in \Omega \)

Let assume \( \| v \|_2 = \langle v, w \rangle \) and \( W_j > W_{j+1} \)

Set \( \tilde{w} = (w_1, \ldots, w_{j-1}, w_j, \ldots, w_n) \)

\[
\langle v, \tilde{w} \rangle - \langle v, w \rangle = v_j w_{j+1} + v_{j+1} w_j - v_j w_j - v_{j+1} w_{j+1}
\]

\[
= (v_j - v_{j+1})(w_{j+1} - w_j) - v_j w_{j+1} - v_{j+1} w_j
\]

\( \geq 0 \)

Since \( \langle v, w \rangle = \max_{w \in \Omega} \langle v, w \rangle \) we have \( \langle v, \tilde{w} \rangle = \langle v, w \rangle \)

Proof of Prop: \( \| D(v) \|_2 \leq \| v \|_2 \)
lemma \Rightarrow \| T(v) \|_{\mathcal{S}_2} = \langle T(v), w \rangle \text{ for } w \text{ ordered}

\| v \|_{\mathcal{S}_2} - \| T(v) \|_{\mathcal{S}_2} \geq \langle v, w \rangle - \langle T(v), w \rangle

= \left( v_1 w_t + v_n w_{n-T} \right) - (v_1 + 1) w_t + (v_n - 1) w_{n-t}

= w_{n-t} - w_t

\geq 0 \quad \text{since } w \text{ is ordered}