LIE ELEMENTS IN THE GROUP ALGEBRA

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ABSTRACT. Given a representation $V$ of a group $G$, there are two natural ways of defining a representation of the group algebra $k[G]$ in the external power $V^\wedge m$. The set $\mathcal{L}(V)$ of elements of $k[G]$ for which these two ways give the same result is a Lie algebra and a representation of $G$. For the case when $G$ is a symmetric group and $V = \mathbb{C}^n$, a permutation representation, these spaces $\mathcal{L}(\mathbb{C}^n)$ are naturally embedded into one another. We describe $\mathcal{L}(\mathbb{C}^n)$ for small $n$ and formulate questions and conjectures for future research.

1. SETTING AND MOTIVATION

Let $V$ be a finite-dimensional representation of a group $G$ over a field $k$. For every $g \in G$ and every $m$ define linear operators $A_m(g), B_m(g) : V^\wedge m \to V^\wedge m$ as follows:

$A_m(g)(v_1 \wedge \cdots \wedge v_m) = g(v_1) \wedge \cdots \wedge g(v_m)$

$B_m(g)(v_1 \wedge \cdots \wedge v_m) = \sum_{p=1}^m v_1 \wedge \cdots \wedge g(v_p) \wedge \cdots \wedge v_m.$

(here and below $v_1, \ldots, v_m$ are arbitrary vectors in $V$). Then extend the operators $A_m, B_m : G \to \text{End}(V^\wedge m)$ to the group algebra $k[G]$ by linearity. Also take by definition $A_0(g) = 1$ (an operator $k \to k$) and $B_0(g) = 0$ for every $g \in G$.

Definition. An element $x \in k[G]$ satisfying $A_m(g) = B_m(g)$ for all $m = 0, 1, \ldots, \dim V$ is called a Lie element of $k[G]$ (with respect to the representation $V$). The set of Lie elements is denoted by $\mathcal{L}(V)$.

Besides the associative algebra structure in $k[G]$ and $\text{End}(V^\wedge m)$ consider an associated Lie algebra structure in them, taking $[p, q] = pq - qp$.

Proposition 1. Maps $A_m, B_m : k[G] \to \text{End}(V^\wedge m)$ are Lie algebra homomorphisms.

Proof. It is clear that $A_m : G \to \text{End}(V^\wedge m)$ is an associative algebra homomorphism ($A_m(xy) = A_m(x)A_m(y)$ for all $x, y \in G$), hence a Lie algebra homomorphism. For $B_m$ take $x = \sum_{g \in G} a_g g$, $y = \sum_{h \in G} b_h h$, to obtain

$B_m(x)B_m(y)v_1 \wedge \cdots \wedge v_m = \sum_{h \in G, 1 \leq p \leq m} b_h B_m(x)v_1 \wedge \cdots \wedge h(v_p) \wedge \cdots \wedge v_m$

$= \sum_{g, h \in G, 1 \leq p \leq m} a_g b_h v_1 \wedge \cdots \wedge g(h(v_p)) \wedge \cdots \wedge v_m$

$+ \sum_{g, h \in G, 1 \leq p, q \leq m, p \neq q} a_g b_h v_1 \wedge \cdots \wedge h(v_p) \wedge \cdots \wedge g(v_q) \wedge \cdots \wedge v_m,$

whence $B_m([x, y]) = [B_m(x), B_m(y)].$
Corollary. The set of Lie elements $\mathcal{L}(V) \subset k[G]$ is a Lie subalgebra.

Proposition 2. For any $x, y \in k[G]$ and any $m$ one has $yA_m(x)y^{-1} = m(yxy^{-1})$ and $yB_m(x)y^{-1} = B_m(yxy^{-1})$.

The proof is evident.

Corollary. The set $\mathcal{L}(V) \subset k[G]$ is a representation of $G$ where elements of the group act by conjugation.

This note takes its origin from the paper [1]. The paper contains a formula for the so called Hurwitz generating function which lists factorizations of a cyclic permutation $(12 \ldots n)$ to a product of transpositions. The key ingredient of the proof of the formula is the fact that $1 - (ij) \in \mathcal{L}(\mathbb{C}^n)$ where $\mathbb{C}^n$ is the permutation representation of the symmetric group (see Proposition 3 below). Any other element $x \in \mathcal{L}(\mathbb{C}^n)$ corresponds to a generalization of this result producing a formula listing factorizations of the cycle to a product of various permutations with various weights; the weights depend on $x$. Equivalently, the same formula lists graphs embedded into oriented surfaces so that their complement is homeomorphic to a disk; any $x \in \mathcal{L}(\mathbb{C}^n)$ generates a formula listing similar embeddings of multi-graphs (again, with the weights depending on $x$).

This note is a description of research in progress; see the list of questions and conjectures at the end.

2. The Symmetric Group Case

Here we take $G = S_n$, $n = 2, 3, \ldots$. Let $k = \mathbb{C}$ and $V$ be an $n$-dimensional permutation representation of $S_n$ (the group acts on elements of the basis $x_1, \ldots, x_n \in \mathbb{C}^n$ permuting their indices). We’ll be writing $\mathcal{L}_n$ for short, instead of $\mathcal{L}(\mathbb{C}^n)$.

Proposition 3 (cf. [1]). $1 - (ij) \in \mathcal{L}_n$ for all $1 \leq i < j \leq n$.

Proof. Take any $v_1, \ldots, v_m \in V$: then $A_m(1)v_1 \wedge \cdots \wedge v_m = v_1 \wedge \cdots \wedge v_m$ and

$$B_m(1)v_1 \wedge \cdots \wedge v_m = m v_1 \wedge \cdots \wedge v_m.$$ 

It follows from Proposition 2 that without loss of generality one may assume $i = 1, j = 2$. Apparently, this is enough to take for $v_s$ basic vectors: $v_s = x_i$, for all $s = 1, \ldots, m$, where $1 \leq i_1 < \cdots < i_m \leq n$ are any indices. Consider now three cases:

1. $i_1, \ldots, i_m \neq 1, 2$. Then

$$A_m((12))(x_{i_1} \wedge \cdots \wedge x_{i_m}) = x_{i_1} \wedge \cdots \wedge x_{i_m},$$

$$B_m((12))(x_{i_1} \wedge \cdots \wedge x_{i_m}) = m x_{i_1} \wedge \cdots \wedge x_{i_m},$$

so that

$$A_m(1 - (12))(x_{i_1} \wedge \cdots \wedge x_{i_m}) = 0 = B_m(1 - (12))(x_{i_1} \wedge \cdots \wedge x_{i_m}).$$

2. $i_1 = 1, i_2, \ldots, i_m \neq 1, 2$. Then

$$A_m((12))(x_1 \wedge x_{i_2} \wedge \cdots \wedge x_{i_m}) = x_2 \wedge x_{i_2} \wedge \cdots \wedge x_{i_m},$$

$$B_m((12))(x_2 + (m - 1)x_1) \wedge x_{i_2} \wedge \cdots \wedge x_{i_m}) = x_2 \wedge x_{i_2} \wedge \cdots \wedge x_{i_m},$$

so that

$$A_m(1 - (12))(x_1 \wedge x_{i_2} \wedge \cdots \wedge x_{i_m}) = (x_1 - x_2) \wedge x_{i_2} \wedge \cdots \wedge x_{i_m} = B_m(1 - (12))(x_1 \wedge x_{i_2} \wedge \cdots \wedge x_{i_m})$$
3. \(i_1 = 1, i_2 = 2\). Then

\[
A_m((12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}) = -x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m},
\]

\[
B_m((12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}) = (m - 2)x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m},
\]

so that

\[
A_m(1 - (12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}) = 2x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}
\]

\[
= B_m(1 - (12))(x_1 \wedge x_2 \wedge x_{i_3} \wedge \cdots \wedge x_{i_m}).
\]

\(\square\)

Denote by \(\iota_n : S_n \to S_{n+1}\) a standard embedding: for any permutation \(\sigma \in S_n\) take \(\iota_n(\sigma)(k) = \sigma(k)\) for any \(1 \leq k \leq n\) and \(\iota_n(\sigma)(n + 1) = n + 1\). The embedding can be extended by linearity to an algebra homomorphism \(\iota_n : \mathbb{C}[S_n] \to \mathbb{C}[S_{n+1}]\).

**Proposition 4.** \(\iota_n(L_n) \subset L_{n+1}\).

**Proof.** Let \(u = \sum_{\sigma \in S_n} a_\sigma \sigma \in L_n\). Like in Proposition 3 above, it is enough to consider the action of \(\iota_n(u)\) on \(x \overset{\text{def}}{=} x_{i_1} \wedge \cdots \wedge x_{i_m}\) where \(1 \leq i_1 < \cdots < i_m \leq n + 1\). Consider two cases.

1. \(i_m \leq n\). Then \(A_m(\iota_n(u))(x) = A_m(u)(x) = B_m(u)(x) = B_m(\iota_n(u))(x),\) so that \(\iota_n(u) \in L_{n+1}\).

2. \(i_m = n + 1\). Then \(A_m(\iota_n(u))(x) = A_{m-1}(u)(x_{i_1} \wedge \cdots \wedge x_{i_{m-1}}) \wedge x_{n+1}\). On the other hand,

\[
B_m(\iota_n(u))(x) = \left(\sum_{\sigma \in S_n} a_\sigma \sum_{p=1}^{n} x_{i_1} \wedge \cdots \wedge x_{\sigma(i_p)} \wedge \cdots \wedge x_{i_{m-1}}\right) \wedge x_{n+1} + \sum_{\sigma \in S_n} a_\sigma \cdot x.
\]

One has \(A_0(u) = \sum_{\sigma \in S_n} a_\sigma\) and \(B_0(u) = 0\). Once \(u \in L_n\), the last term in the equation above is zero, so

\[
B_m(\iota_n(u))(x) = B_{m-1}(u)(x_{i_1} \wedge \cdots \wedge x_{i_{m-1}}) \wedge x_{n+1},
\]

whence

\[
A_m(\iota_n(u))(x) = B_m(\iota_n(u))(x),\]

and again \(\iota_n(u) \in L_{n+1}\). \(\square\)

3. \(L_n\) for small \(n\)

One has \(\text{dim } L_2 = 1\). The space is spanned by \(1 - (12) \in \mathbb{C}[S_2]\), is a trivial Lie algebra and a trivial representation of \(S_2 = \mathbb{Z}/2\mathbb{Z}\).

The space \(L_3\) contains elements \(1 - (12), 1 - (23)\) and \(1 - (13)\) by Proposition 3. By the corollary of Proposition 1 it also contains \([1 - (12), 1 - (23)] = (123) - (132)\) (by \((i_1 \ldots i_k) \in S_n\) we mean a cyclic permutation sending every \(i_a\) to \(i_{a+1} \text{ mod } k\)). Easy calculations show that these elements form a basis in \(L_3\), so that \(\text{dim } L_3 = 4\). The space \(L_3\) splits, as a representation of \(S_3\), to the trivial representation \(V_0\) (spanned by \(1 - (12)/3 - (13)/3 - (23)/3\), sign representation \(V_1\) (spanned by \(123 - (132)\)) and a two-dimensional representation \(V_2\) (spanned by \((12) - (13), (13) - (23)\) and \((23) - (12)\); the elements sum up to zero, and any two of them form a basis). As a Lie algebra \(L_3\) is a direct sum of the center \(V_0\) and a three-dimensional subalgebra spanned by \(V_1 \cup V_2\). (This statement is partly true for any \(n\): \(L_n\) contains a trivial representation, which lies in its center as a Lie algebra.)

The space \(L_4\) contains, by Proposition 3, the 6 elements \(1 - (ij), 1 \leq i < j \leq 4\). By Propositions 1 and 2 it also contains all the elements \((ijk) - (ikj) = [1 - (ij), (1 -
\[ (jk), 1 \leq i < j < k \leq 4 \text{ (totally 4)}, \text{ and the elements } \gamma_1 = [1 - (14), (123) - (132)] = (1234) + (1432) - (1243) - (1342) \text{ and } \gamma_2 = [1 - (24), (123) - (132)] = (1243) + (1342) - (1234) - (1423). \]

Easy computer-assisted computations show that these 12 elements form a basis in \( L_4 \).

As a representation of \( S_4 \), \( L_4 \) contains a 6-dimensional representation spanned by \( 1 - (ij), 1 \leq i < j \leq 4 \); it splits into a trivial representation spanned by \( \sum_{1 \leq i<j \leq 4} (ij) \), a 3-dimensional representation of the type \((3,1)\) and a 2-dimensional representation of the type \((2,2)\). Another 4-dimensional subrepresentation of \( L_4 \) is spanned by \( (ijk) - (ikj), 1 \leq i < j < k \leq 4 \); it splits into a sign representation (spanned by \( \sum_{1 \leq i<j<k \leq 4} (ijk) - (ikj) \)) and a 3-dimensional representation of the type \((2,1,1)\). The elements \( \gamma_1 \) and \( \gamma_2 \) span a 2-dimensional subrepresentation. Totally, \( L_4 \) contains a trivial representation, a sign representation, two copies of a 2-dimensional representation and two nonisomorphic 3-dimensional representations.

4. Questions and conjectures

4.1. Dimension and representations. For an arbitrary \( n \), what is the dimension of \( L_n \)? A refinement of the question: find the Frobenius character \( R_n = \sum_{|\lambda|=n} a_{\lambda} \chi_{\lambda} \) of the representation \( L_n \); here the sum runs over all partitions of \( n \), \( a_{\lambda} \) is the multiplicity in \( L_n \) of the irreducible representation of \( S_n \) of the type \( \lambda \), and \( \chi_{\lambda} \) is the Schur polynomial corresponding to \( \lambda \).

4.2. Generators.

Conjecture. The Lie algebra \( L_n \) is generated by the elements \( \nu_{ij} = 1 - (ij), 1 \leq i < j \leq n \).

Computations confirm the conjecture for \( n \leq 5 \).

4.3. Action on the original representation. The elements of \( L(V) \subset k[G] \) act in the original representation \( V \) of the group \( G \). This action may have a kernel. These kernels and quotients of \( L(V) \) by them sometimes exhibit interesting properties:

Conjecture. Let \( K_n \) be a kernel of the action of \( L_n \) in the permutation representation \( \mathbb{C}^n \). Then \( \dim L_n/K_n = (n - 1)! \). The repeated commutators

\[ [\ldots [\nu_{i_1}, \nu_{i_2}], \nu_{i_3}], \ldots, \nu_{n-1,i_{n-1}}] \]

for all \( i_1, \ldots, i_{n-1} \) such that \( s + 1 \leq i_s \leq n \) for all \( s = 1, \ldots, n - 1 \) form a basis in \( \dim L_n/K_n \).

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References

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