Unipotent Jacobian Matrices and Univalent Maps

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Abstract. The Jacobian Conjecture would follow if it were known that real polynomial maps with a unipotent Jacobian matrix are injective. The conjecture that this is true even for $C^1$ maps is explored here. Some results known in the polynomial case are extended to the $C^1$ context, and some special cases are resolved.

1. Introduction

The focus of this paper is the unipotence (all eigenvalues are 1) of the Jacobian matrix of a $C^1$ (continuously differentiable) map from $\mathbb{R}^n$ to itself and whether this implies its univalence (injectivity) or invertibility (bijectivity). In the case of polynomial maps, unipotence is central to reformulations of the Jacobian Conjecture. The paper is organized as follows: a review of a number of important examples; a comparison of several established conjectures related to unipotence; a description of the goals of this paper; a number of results that parallel what is known in the polynomial case; and finally, some partial results in the general $C^1$ context.

2. Examples

Consider some simple examples of maps with unipotent Jacobian matrices and their explicit inverses.

Example 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$(u, v) = f(x, y) = (x + 5 \cos(3x + 5y), y - 3 \cos(3x + 5y))$$

Then the Jacobian matrix of $f$ is

$$J(f) = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} = \begin{bmatrix}
1 - 15 \sin(3x + 5y) & -25 \sin(3x + 5y) \\
9 \sin(3x + 5y) & 1 + 15 \sin(3x + 5y)
\end{bmatrix}$$

So $J(f)$ is unipotent; that is

$$J(f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(3x + 5y) \begin{bmatrix} -15 \\ 9 \end{bmatrix} - 25 \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

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is the sum $I + N$, where $I$ is the $(2 \times 2)$ identity matrix and $N$ is a nilpotent matrix (a power of $N$ is 0; here it is easy to verify that $N^2 = 0$). From the fact that $3u + 5v = 3x + 5y$, it follows easily that the map $f$ is invertible, with inverse

$$(x, y) = g(u, v) = f^{-1}(u, v) = (u - 5 \cos(3u + 5v), v + 3 \cos(3u + 5v))$$

**Example 2.** Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a $C^1$ upper triangular map:

$$(s, t, u, v) = f(x, y, z, w) = (x + u, y + z + w, z + c(w, w + d))$$

where $a, b, c$ are $C^1$ functions of the indicated variables and $d$ is a constant. $J(f)$ is unipotent; it is the sum of the $(4 \times 4)$ identity matrix $I$ and a matrix $N$ that is strictly upper triangular and hence nilpotent ($N^4 = 0$). The inverse of $f$ is

$$(x, y, z, w) = f^{-1}(s, t, u, v) = (s - a(t - b(u - c(v - d)), v - d), t - b(u - c(v - d), v - d), u - c(v - d), v - d)$$

**Example 3.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$(u, v, w) = f(x, y, z) = (x + z \phi(x + zy), y - \phi(x + zy), z)$$

where $\phi$ is a $C^1$ function of a single variable. Then

$$J(f) = I + \begin{bmatrix}
  z\phi'(x + zy) & z^2\phi'(x + zy) & \phi(x + zy) + zy\phi'(x + zy)
  
  -\phi'(x + zy) & 0 & -2\phi'(x + zy)
  
  0 & 0 & 0
\end{bmatrix}$$

which represents $J(f)$ as $I + N$, with $N$ nilpotent ($N$ is nilpotent since its upper left $2 \times 2$ block is). From the fact that $u + vw = x + zy$, it follows, as in the first example, that the map $f$ is invertible, with inverse

$$(x, y, z) = g(u, v, w) = f^{-1}(u, v, w) = (u - w\phi(u + vw), v + \phi(u + vw), w)$$

**Example 4.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$(u, v, w) = f(x, y, z) = (x + \phi(y - x^2), y + z + 2x\phi(y - x^2), z - (\phi(y - x^2))^2)$$

where $\phi$ is a $C^1$ function of a single variable. Then

$$J(f) = I + \begin{bmatrix}
  -2x\phi'(y - x^2) & \phi'(y - x^2) & 0
  
  2\phi(y - x^2) - 4x^2\phi'(y - x^2) & 2x\phi(y - x^2) & 1
  
  4x\phi(y - x^2)\phi'(y - x^2) & -2\phi(y - x^2)\phi'(y - x^2) & 0
\end{bmatrix}$$

which represents $J(f)$ as $I + N$, with $N$ nilpotent (it easy to verify, using a computer symbolic algebra program, that $N^3 = 0$). Note that $v - u^2 - w = y - x^2$. Put $a = \phi(y - x^2) = \phi(v - u^2 - w)$. Then $f(x, y, z) = (x + a, y + z + 2xa, z - a^2)$ and one easily obtains the inverse of $f$ by solving for $z, x$, and $y$, in that order. The result is

$$(x, y, z) = f^{-1}(u, v, w) = (u - a, v - w - 2ua + a^2, w + a^2)$$

These examples are chosen to illustrate several points.

In Example 1, it is clear that 3 and 5 could be replaced by any other pair of constants, and that the cosine could be any $C^1$ function, and the example would still have the same properties. In fact, the example is perfectly general, since any $C^1$ map of $\mathbb{R}^2$ to itself with a unipotent Jacobian matrix has, up to a constant translation, this form for some pair of constants and some $C^1$ function. A proof is supplied later.
allows them to parameterize examples with fewer variables (if \( d = 0 \), then \( w \) is a parameter in the family of 3-variable triangular maps given by the first 3 components). In addition it demonstrates the existence of an explicit, closed form inverse obtained by composition of functions. The inverse is explicit in the sense that it is represented by a (finite) formula involving algebraic operations, composition, and the functions \( a, b, c \) and the constant \( d \) that appear in the definition of \( f \).

Example 3 shows how the principles involved in the first two examples can be combined. It takes the first example, adds a coordinate \( z \) in triangular fashion, which therefore can be used as a parameter in the first example, and then it replaces \( z \) by \( 1, 5 \times z \) and \( \cos \) by \( \phi \). Engelbert Hubbers and Arno van den Essen introduced this sort of bootstrap construction; if one adds as a construction technique the replacement of a map \( f : \mathbb{R}^n \to \mathbb{R}^n \) by the map \( p \mapsto T^n f(Tp) \), where \( T \) is a matrix of constants and \( T^n \) is its classical adjoint, one is led to their New Class of Automorphisms [24]. Actually, they consider only polynomials, but over an arbitrary commutative coefficient ring with unit, which is both a more restricted and more general case than that of the \( C^1 \) maps that are the focus of this paper. All automorphisms in the New Class have unipotent Jacobian matrices and the special property that for \( f(p) = p + h(p) \), the perturbation portion, \( h : \mathbb{R}^n \to \mathbb{R}^n \), of \( f \) has a constant \( n \)-fold composition power; that is, \( h^{\circ n} = h \circ h \circ \cdots \circ h \) (\( n \) times) is a constant map. This example, with \( \phi(x + zy) = -(x + zy)^2 \), produced the first three dimension counterexample to the Markus-Yamabe conjecture [8]. In detail, the ordinary differential equation \( dp/dt = -f(p) \) in \( \mathbb{R}^3 \) has an orbit that escapes to infinity in forward time, even though the eigenvalues of \( J(-f) \), which are all \(-1\), obviously have strictly negative real parts at every point. One such orbit is \( p(t) = (x(t), y(t), z(t)) = (18e^t, -12e^{2t}, e^{-t}) \).

Example 3 is a very modest generalization of the particular case \( a = \phi(y-x^2) = y-x^2 \), which was constructed in [23], where it is shown that the perturbation part of \( f \), namely \( h(x, y, z) = (y-x^2, z+2x(y-x^2), -(y-x^2)^2) \), has nonconstant composition powers \( h^{\circ n} \) for every \( n > 0 \). Thus \( f \) is not generally an automorphism in the New Class, since that particular \( f \) is not.

3. Conjectures

The Jacobian Conjecture is a significant unsolved problem. A polynomial map with a global polynomial inverse has a Jacobian matrix whose determinant is a nonzero constant. This follows from the chain rule and the fact that the product of two polynomials is a nonzero constant if, and only if, each factor is. The Jacobian Conjecture is that the converse is true. It is sometimes known as Keller’s Jacobian Conjecture, because its first appearance in the literature appears to be [33], in which Keller proves the complex birational case. A modern formulation ([1, 16]) is

**Conjecture** 1 (The Jacobian Conjecture). Let \( k \) be a field of characteristic zero, and \( f : k^n \to k^n \) a polynomial map. Then \( f \) has a polynomial inverse if, and only if, the Jacobian matrix of \( f \) has a nonzero constant determinant.

The complex case is known to be universal [1]: that is, if the conjecture is true for \( k = \mathbb{C} \), then it is true in general. The real case \( (k = \mathbb{R}) \) implies the complex
case (consider \( \mathbb{C}^n \) as \( \mathbb{R}^{2n} \) and compare the Jacobian determinants). For any \( k \), the conjecture can be reduced to the consideration of maps of the form \( f(x) = x + (Ax)^3 \), where \( A \) is a matrix of constants and the cube \( (Ax)^3 \) is computed component-wise. If the Jacobian determinant of such a cubic-linear map is constant, then it has a unipotent Jacobian matrix. The reduction of the general case to the cubic-linear case involves introducing extra variables in general; that is, the invertibility of a polynomial map in \( n \) variables with a nonzero constant Jacobian determinant is equivalent to the invertibility of a cubic-linear map in \( m \) variables, where \( m \) is usually (significantly) larger than \( n \). So the cubic-linear variant implies the general case only if one considers all \( n \). The status of the conjecture at this moment is that the cubic-linear and related cases have been affirmatively resolved for certain low dimensions \( (n \leq 7 - \text{see } [31, 19]) \), that a few special cases work (e.g. maps with at most quadratic terms; see [1] for more), but that the general case, though true for any \( k \) and \( n = 1 \), is not known for any \( n > 1 \) for even a single field of characteristic zero.

Questions about the existence of an inverse for polynomial maps hinge on injectivity; for polynomial maps over \( \mathbb{C} \) or \( \mathbb{R} \), injectivity implies surjectivity [11, 2]. The non-vanishing of the Jacobian determinant in these two cases implies that the map is locally an analytic isomorphism. In the complex case, an injective polynomial map with a nowhere vanishing (and hence constant) Jacobian determinant is a birational map, and it has a global polynomial inverse [33, 46]. In the real case, a nowhere vanishing Jacobian determinant need not be a constant; if \( f : \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( f(x) = x + x^3 \), then \( \det J(f) = 1 + 3x^2 \) is nowhere vanishing, and \( f \) is injective, so it is a global real analytic homeomorphism, but its inverse is not polynomial. Pinchuk described [12] a class of polynomial maps \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with nowhere vanishing but non-constant Jacobian determinant that are not injective, thus refuting what was known variously as the Strong Real Jacobian Conjecture, or just the Real Jacobian Conjecture. So a nonzero constant Jacobian determinant is a necessary hypothesis in the real case of the Jacobian Conjecture (and that is what the term Real Jacobian Conjecture is now usually taken to mean).

In the real case one is naturally drawn to the question of what can be said of more general maps (real analytic, \( C^1 \), etc.). Global univalence (injectivity) of maps \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a large, well-studied topic [41, 13]. There are numerous conditions that can be imposed to obtain univalence, ranging from general topological conditions (local homeomorphism + properness) to ones more closely connected to the Jacobian matrix (positive definiteness conditions, special matrix types, Hadamard’s integral criterion). A nonzero constant Jacobian determinant, by itself, certainly does not suffice to guarantee univalence; that is, the straightforward generalization of the Jacobian Conjecture to \( C^1 \) maps is false. As a simple example of that, one can take the analytic map \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( f(x, y) = (\sqrt{2e^{x/2}} \cos (ye^{-x}), \sqrt{2e^{x/2}} \sin (ye^{-x})) \); it has Jacobian determinant 1, but is not injective (e.g. the image of \( \{x = 0\} \) is a circle). The example is taken from Brian Coomes’ paper [10]; it is also mentioned in [37, 4, 6]. Constancy of the Jacobian determinant is a global condition on the pointwise spectrum (set of eigenvalues) of the Jacobian matrix (see [47, 44] for some results in that general category). In that regard, a recent conjecture [3] is worth highlighting.

**Conjecture 2** (Chamberland). If \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is \( C^1 \) and the eigenvalues of \( J(f) \) are globally bounded away from 0, then \( f \) is injective.
This hypothesis covers the case of constant eigenvalues, and thus of a unipotent (all eigenvalues are 1) Jacobian matrix. This suggests formulating

**Conjecture 3** (*$C^1$ Unipotent Jacobian Univalence Conjecture*). If $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ and the matrix $J(f)$ is unipotent, then $f$ is injective.

This conjecture is at least known to be true for $n = 2$ (see section 3), and, by reduction to the cubic-linear case, is strong enough to imply the truth of the Jacobian Conjecture if it is established for all $n$. All the examples of $C^1$ maps with unipotent Jacobian matrices that were presented earlier are, in fact, $C^1$ automorphisms; that is, they are not just injective, but they also are surjective, and hence have a global $C^1$ inverse. Thus no counterexample has been presented yet to the somewhat stronger conjecture in which injectivity is replaced by bijectivity.

In the context of conjectures related to the eigenvalues of the Jacobian matrix, there are a few additional ones that have arisen in connection with the study of the Markus-Yamabe conjecture and its discrete (function iteration) analogue [9, 8, 40].

**Conjecture 4** (*$C^1$ Stability Conjecture*). If $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ and the eigenvalues of $J(f)$ have strictly negative real part at every point, then $f$ is injective.

Note that the 3-dimensional polynomial counterexample to the Markus-Yamabe conjecture [8] presented above (Example 3) has an orbit that escapes to infinity, but it is still injective, viewed as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$.

**Conjecture 5** (*$C^1$ Fixed Point Conjecture*). If $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ with $f(0) = 0$, and the eigenvalues of $J(f)$ have absolute value less than 1 at every point, then 0 is the unique fixed point of $f$.

The Stability Conjecture for polynomial maps implies the Jacobian Conjecture, and the Fixed Point Conjecture for polynomial maps is equivalent to the Jacobian Conjecture [5, 8, 17]. The discrete Markus-Yamabe question (DMYQ) was raised in [8]; do the hypotheses of the Fixed Point Conjecture imply global convergence of iterates (for any $x_0$, the sequence $x_{k+1} = f(x_k)$ converges to 0)? A rational counterexample to DMYQ for $n = 2$ is presented in the same paper. Polynomial counterexamples to DMYQ have been constructed for $n \geq 4$ and for $n \geq 3$ in [8], but they do have 0 as a unique fixed point.

A conjecture equivalent to the $C^1$ Unipotent Jacobian Univalence Conjecture is the following special case of the Fixed Point Conjecture. See section 3 for a proof of the equivalence of the conjectures.

**Conjecture 6** (*$C^1$ Multiple Fixed Point Conjecture*). If $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ and has two distinct fixed points, then $J(f)$ has a nonzero eigenvalue at some point.

Another way to state this is: if $J(f)$ is nilpotent, then $f$ has at most one fixed point. Viktor Kulikov poses this conjecture for polynomial maps over $\mathbb{C}$ in [33], and shows that if it is true (for all $n$), then the Jacobian Conjecture follows. The Jacobian Conjecture also follows from the real case, in view of the equivalence of this conjecture and the Unipotent Jacobian Univalence Conjecture. It should be mentioned that [9] poses a different conjecture about (complex polynomial) maps with nilpotent Jacobian matrices and refers to it as the ‘Nilpotent Conjecture’; namely, that the rows of the Jacobian matrix of such a map are linearly dependent over $\mathbb{C}$. A counterexample is given in [21].
4. Goals

The main goal of this paper is to place the conjectures relating unipotence of the Jacobian matrix and univalence of maps firmly in the $C^1$ domain. To that end, proofs of the properties of maps corresponding to the first three introductory examples are supplied in the $C^1$ case. The results are extensions of what was known in the polynomial case.

All of the conjectures of the previous section are perhaps too ambitious. Arno van den Essen, certainly an authority on the Jacobian Conjecture, has expressed, in person and in print, the opinion that it may be true for $n = 2$, but seems unlikely to be true in general. And the other conjectures all imply the Jacobian Conjecture, even if they are established only for polynomial maps (in all dimensions).

Apart from the fact that these conjectures appear to be natural and interesting in the larger domain of $C^1$ maps, there is a secondary goal in introducing them. The $C^1$ hypothesis gives them more room to be wrong, and it seems that effort should be devoted to finding specifically non-polynomial counterexamples, in the hopes that such will clarify the polynomial situation. Counterexamples may shed more light on what distinguishes the various cases in the regularity hierarchy of polynomial, rational, semialgebraic, analytic, smooth, or just $C^1$ maps. Even a failure to find counterexamples may be of some help.

The final portion of the paper deals with some tractable special cases: linearizable maps, maps with bounded images, and polynomial maps with no zeros at infinity. While there are more specific individual results, what ties these all together is the fact that, in each case, if the Jacobian matrix of the map is nilpotent, then the map has a unique fixed point. The final section discusses the applicability of some of the ideas explored in this paper to broader contexts than $C^1$ maps.

5. The Planar Case

**Theorem 5.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $C^1$. Then $J(f)$ is unipotent if $f$ is of the form

$$f(x, y) = (x + b\phi(ax + by) + c, y - a\phi(ax + by) + d)$$

for some constants $a, b, c, d \in \mathbb{R}$ and some function $\phi$ of a single variable. If that is the case, then $f$ has an explicit global inverse. Conversely, if $f$ is $C^1$ and $J(f)$ is unipotent, then $f$ is of the form shown (for a $\phi$ that is $C^1$).

Some remarks are in order before the proof. The analogous result for polynomial maps over a field $k$ of characteristic zero is in [1]. The precise form described above is not spelled out, but is implicit in the proof of [1, Theorem 6.2]; [1, Corollary 6.3] explicitly states that if $f : k^n \to k^n$ is written as $f(x) = x - h(x)$ and $J(h)^2 = 0$, then for $n = 2$ or $n = 3$ it follows that $f$ is a (composition) product of elementary automorphisms, and hence invertible. A proof for polynomial maps with coefficients in a $\mathbb{Q}$-algebra that is a unique factorization domain appears in [24]; it takes the case of a field $k$ as its point of departure. The proof below is modeled on a proof by Yu Qing Chen for holomorphic maps of $\mathbb{C}^2$ to itself [7]. Marc Chamberland has also proved the same result for real analytic maps of $\mathbb{R}^2$ to itself [8], by entirely different methods.
The remainder of this section is devoted to the proof of the above theorem and some relevant observations.

By adding a constant translation, which does not affect \( J(f) \), one can assume that \( f(0) = 0 \), and then it suffices to consider the case \( c = d = \phi(0) = 0 \). If \( f \) is of the form mentioned, then it is easy to establish that \( \phi \) is \( C^1 \) and \( J(f) \) is unipotent, and to compute the inverse of \( f \) (see Example \[ \text{Example} \]). This leaves only the converse portion of the theorem to prove.

So suppose that \( J(f) \) is unipotent, and that \( f \) is \( C^1 \). Denote points in \( \mathbb{R}^2 \) by \( z = (x, y) \), and let \( h(z) = f(z) - z \). Then \( J(h) \) is nilpotent; that is, \( J(h)^2 = 0 \). If \( h \) is constant, then \( h = 0 \), and the desired representation exists. So assume that \( h \) is not identically 0. If one can show that \( f \) is of the desired form then at least one of \( a \) and \( b \) is not zero, or \( h \) would be identically zero. But then it is clear that \( f \) is \( C^1 \) if, and only if, \( \phi \) is \( C^1 \).

The main goal of the proof is to establish that

\[
(5.1) \quad h(z + (J(h)(z))\zeta) = h(z) \text{ for all } z, \zeta \in \mathbb{R}^2.
\]

For less notational clutter, write \( A = J(h)(z) \). Then the goal is to show that \( h(z + A\zeta) \) is independent of \( \zeta \).

**Remark 1.** Yu Qing Chen actually establishes this result for analytic maps \( h \) in any number of variables with \( J(h)^2 = 0 \). His proof involves computing the power series expansion for \( h(z + A\zeta) \) for fixed \( z \) at \( \zeta = 0 \) recursively, showing that all but the constant terms are zero. It has no obvious extension to \( C^m \) maps (even for \( m = \infty \)).

If \( A = 0 \) then it is clear that \( h(z + A\zeta) \) does not depend on \( \zeta \). So let \( z \in \mathbb{R}^2 \) be a point such that \( A = J(h)(z) \) is nonzero. Let \( h(z) = (h_1(z), h_2(z)) \). The Jacobian matrix \( J(h) \) is of constant rank 1 (since it is nonzero but nilpotent) in a neighborhood of \( z \), so (by a classic theorem on Jacobian matrices of constant rank) the functions \( h_1 \) and \( h_2 \) are dependent, in the precise sense that one of them can be written as a \( C^1 \) function of the other in a neighborhood of \( z \) (see, for example, [34, §98]). Assume, without loss of generality, that \( h_2(x, y) = g(h_1(x, y)) \), where \( g \) is a \( C^1 \) function of one variable. Since \( J(h)^2 = 0 \), its trace is zero, which yields \( \partial h_1 / \partial x + \partial h_2 / \partial y = \partial h_1 / \partial x + g'(h_1(x, y)) \partial h_1 / \partial y = 0 \). But then the gradient of \( h_1 \) is \( \partial h_1 / \partial y \) times the vector \((-g'(h_1(x, y)), 1)\), which is constant along level curves of \( h_1 \). This implies that the level curves of \( h_1 \) are straight line segments (locally). Furthermore, the slope of the line segment which is the level curve through a point \((x, y)\) is \( g'(h_1(x, y)) \), which is a continuous function of \((x, y)\).

Now make an affine change of coordinates, so that \( z = 0 \) and the integral curve of \( h_1 \) through \( z \) is horizontal. Then \( \partial h_1 / \partial x|_{(0,0)} = 0 \) and hence \( \partial h_1 / \partial y|_{(0,0)} \neq 0 \). Consider (in the new coordinates) a small segment of the \( y \)-axis, \( |y| \leq \eta > 0 \), along which \( \partial h_1 / \partial y \) does not vanish. Let the integral curve through \((0, y)\) have slope \( \sigma(y) \). Then \( \sigma(0) = 0 \), and \( h_1(t, \sigma(y)t + y) = h_1(0, y) \) for \( t \) small and \( |y| \) small. It follows that if \( \eta \) is small enough, one can assume that \( \sigma(y) \) is well defined and \( |\sigma(y)| \leq 1 \) for \( |y| \leq \eta \), and that there exists an \( \epsilon > 0 \), such that

\[
(5.2) \quad h_1(t, \sigma(y)t + y) = h_1(0, y) \text{ for all } |t| \leq \epsilon, |y| \leq \eta.
\]
Suppose that $\partial h_1/\partial y$ has a value of 0 at a point $(t, \sigma(y)t + y)$ with $|t| = \epsilon$ and $|y| \leq \eta$. Then there is a sequence of difference quotients $\Delta h_1/\Delta y$ with limit 0, computed using the fixed point $(t, \sigma(y)t + y)$ and a variable point $(t, \sigma(y)t + y + \Delta y)$, and one can assume that the variable point is also the endpoint of a level curve of $h_1$ through a point $(0, y')$ with $|y'| \leq \eta$ (this may involve considering one-sided differences only, if $|y| = \eta$). Connect the fixed point and the variable point back to the corresponding points $(0, y)$ and $(0, y') = (0, y + \Delta'y)$ on the $y$-axis by moving along the (straight) level curves. Compute a difference quotient $\Delta'h_1/\Delta'y$ for those points. By the invariance of $h_1$ along the level curves $\Delta'h_1 = \Delta h_1$. Since $h_1$ is monotone on the segment of the $y$-axis considered, $\Delta'y \neq 0$ and it has the same sign as $\Delta y$. By considering the quadrilateral with the 4 points in question as vertices, it is clear that the ratio between $\Delta y$ and $\Delta'y$ is bounded above and below by a function of the slopes of the level curves, and hence by absolute constants for fixed $\epsilon$ and $\eta$. This means that the difference quotients $\Delta'h_1/\Delta'y$ tend to zero, which contradicts the fact that $\partial h_1/\partial y$ does not vanish at $(0, y)$ for $|y| \leq \eta$.

The above argument establishes that the endpoints of the level curves at $|t| = \epsilon$ are nonsingular points (ones at which $J(h)$ does not vanish). The original argument then shows that both $h_1$ and $h_2$ are constant on line segments near each endpoint, which implies that the identity $h_1(t, \sigma(y)t + y) = h_1(0, y)$ can be extended past each endpoint. By compactness, $\epsilon$ can be made uniformly larger (keeping the same fixed $\eta$) in equation \ref{5.2}. Since there is no maximum value of $\epsilon$ for which equation \ref{5.2} holds, all the integral curves in question must be entire straight lines.

**Remark 2.** The argument to show that there is no maximum value for $\epsilon$ was suggested (in a somewhat different form) by Yu Qing Chen (private correspondence).

By considering $h_2$ instead of $h_1$, if necessary, it follows that there is a straight line through any point $z$ at which $A = J(h)(z) \neq 0$, such that both $h_1$ and $h_2$ are constant on that straight line. Suppose that the straight line in question is $z + tw$, where $t$ is a parameter and $w$ is a fixed nonzero vector. Differentiating with respect to $t$ yields $Aw = 0$. But $A$ is of rank 1, so its nullspace equals its image. Thus $A\zeta$ is a multiple of $w$ for any $\zeta \in \mathbb{R}^2$. This establishes equation \ref{5.1}. From here on, the argument is very similar to that of \ref{7}.

Any two such lines through different points $z$ where $J(h)(z) \neq 0$ must be parallel. For if not, they intersect, any further such line intersects at least one of the two, and $h$ is constant on the union of all those lines. But then $h$ is constant on (at least) the open set where $J(h)$ is nonzero; which is absurd. Now fix one $z$ with $A = J(h)(z) \neq 0$, and consider the line as above through it. Then $h$ is constant on every line parallel to that one (either because $J(h)$ is 0 at every point of such a line, or because it is not).

Next make an invertible linear change of coordinates (a rotation, for instance) that makes the parallel lines vertical. In the new coordinate system $J(h)$ still satisfies $J(h)^2 = 0$ (by similarity), and $h$ is a function of $x$ alone. Suppose $h(x) = (r(x), s(x))$. From $J(h)^2 = 0$, one obtains $(r')^2 = 0$, hence $r' = 0$, hence $r = 0$ (since $r(0) = 0$). So $f$ is the triangular map $f(x, y) = (x, y + s(x))$. Changing coordinates back to the original system yields the desired representation.
Example 5. The normal form given above exists, in general, only for maps that are defined (and $C^1$) on all of $\mathbb{R}^2$. The map $h(x, y) = (x/y, \ln(x/y))$ from $\{x > 0, y > 0\}$ to $\mathbb{R}^2$ has a nilpotent Jacobian matrix, but the level curves (of either component of $h$) are rays from the (excluded) origin into the first quadrant.

6. Fixed Points, Inverses, and Composition Powers

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be any map. For $a \in \mathbb{R}^n$, define $g_a(x)$ as $x - f(x) + a$.

Lemma 6.1. The family $g_a$ consists of maps with at most one fixed point if, and only if, the map $f$ is injective.

Proof. Let $p$ and $q$ be two distinct fixed points of $g_a$. Then $f(p) = a = f(q)$. And conversely.

The unipotence of $J(f)$ is equivalent to the nilpotence of any or all of the maps $g_a$. So the $C^1$ Unipotent Jacobian Univalence Conjecture and the $C^1$ Multiple Fixed Point Conjecture are equivalent, and even separately so for each dimension $n$ and particular map $f$ and family $g_a$.

Remark 3. Call a $C^1$ map $g$ non-degenerate at a point $p$ if $J(g)(p)$ does not have 1 as an eigenvalue. If $p$ is a non-degenerate fixed point of $g$, then $f(p) = 0$, where $f(x) = x - g(x)$. Since $J(f)(p)$ has no zero eigenvalue, it is invertible, and $f = 0$ has a unique solution in a neighborhood of $p$. This shows that non-degenerate fixed points are isolated. In particular, any fixed points of a map $g$ with nilpotent $J(g)$ are isolated. So the Multiple Fixed Point Conjecture holds for maps with a fixed point set that is not discrete, and also for maps with a connected fixed point set.

Given $y = f(x) = x - g(x)$, it is natural to try to solve for $x$, obtaining $x = y + g(x) = y + g(y + g(x)) = y + g(y + g(y + g(x))) = \ldots$. This procedure terminates, and yields a the inverse of $y$ under the map $f$, provided the chain of expressions eventually becomes independent of $x$. The easiest way to see this is the obvious

Lemma 6.2. $f^{-1}(y)$ consists of a single point $x$ if, and only if, $x$ is the unique fixed point of the map $w \mapsto y + g(w)$.

As a corollary, one obtains

Theorem 6.3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map, and $h : \mathbb{R}^n \to \mathbb{R}^n$ its perturbation part, defined by $h(x) = f(x) - x$. For any given point $y \in \mathbb{R}^n$, suppose that the map $\tau_y(w) = y - h(w)$ has a composition power $\tau_y^{om} = \tau_y \circ \cdots \circ \tau_y$ ($m > 0$ times) that is constant. Then $\tau_y$ has a unique fixed point $x$, and $x = f^{-1}(y)$ can be computed by starting with any initial point and applying $\tau_y$ $m$ times.

Proof. Let $p$ be the single point that is the image of $\mathbb{R}^n$ under the constant map $\tau_y^{om}$. Then $p$ is a fixed point of $\tau_y$. Furthermore, if $p$ and $q$ are fixed points of $\tau_y$, then $p = \tau_y^{om}(p) = \tau_y^{om}(q) = q$. \qed
These are the operations that provide what were called “explicit inverses” in Examples 1–3. They involve composition, but apart from that only algebraic operations and the components of the map (granted, in terms of its decomposition \( f(x) = x - g(x) \)). The simplest case is the planar case. Given the representation 
\[
f(x, y) = (x + b_0(ax + by) + c, y - a_0(ax + by) + d)
\]
the “explicit” formula
\[
\tau_y = \left( \begin{array}{cc}
1 & b_0 \\
0 & 1
\end{array} \right), \quad \tau_x = \left( \begin{array}{cc}
a_0 & 1 \\
0 & 1
\end{array} \right)
\]
where the \( \tau_y \) do not all become constant for a fixed number of composition factors. But that is akin to adding allowing an infinitary operation; in practice, fixed points of maps are often found by taking the limit of a sequence \( x, y \rightarrow (0, 0) \) is (easily shown to be) the unique fixed point of \( h \). Alternatively, \( f^{-1}(z) = z + g(z + g(w)) \) is independent of \( w \). Alternatively, \( f^{-1}(z) = z + g(z + g(-z)) = z + g(z - f(-z)) = z + g(-f(-z)) \), so \( f^{-1}(z) = z + (f(-z) - f(-f(-z))) = z - f(-z) - f(-f(-z)) \). This shows that there is nothing unique about the way of expressing the inverse in terms of the original map and composition products. Note that these expressions are universal formulas, in the sense that they are valid for every \( f \) with unipotent \( J(f) \), in exactly the same form. Both representations can legitimately be considered as explicit (even “closed form”) formulas. And they rely on a property, namely constancy of the composition power, that is a nonlinear analogue of nilpotence. The situation is reminiscent of the inversion of a unipotent matrix \( I - N \), where one has the “explicit” formula \( I + N + N^2 + \ldots + N^m \) for the inverse when \( N^{m+1} \) vanishes. Gary Meisters first drew attention to this property of composition powers in [38].

If one adds the unique fixed point operator \( \mu \) to one’s repertoire, the inverse can be expressed as \( f^{-1}(y) = \mu_x(\tau_y(x)) \), even when finite composition products of the \( \tau_y \) do not all become constant for a fixed number of composition factors. But that is akin to adding allowing an infinitary operation; in practice, fixed points of maps are often found by taking the limit of a sequence \( x_{i+1} = h(x_i) \) of iterates.

But not always. For Example 5, the composition powers \( h^i \) of the perturbation part of the map \( f(p) = p + h(p) \) are not always constant. The map \( h \) does have a unique fixed point. However, this does not imply that composition powers of such a map necessarily have a unique fixed point.

**Example 6.** let \( h(x, y, z) = (\phi(y - x^2), z + 2x\phi(y - x^2), -\phi(y - x^2)^2) \) be the perturbation part of the automorphism of Example 5. If \( \phi(0) = 0 \) and \( \phi(-1) = -1 \), then \( (0, 0, 0) \) is (easily shown to be) the unique fixed point of \( h \), but \( h^3 = h \circ h \circ h \) has a nonzero fixed point, which is therefore a periodic point of \( h \) of period 3. Specifically, \( \{(-1, 1, -1), (0, -1, 0), (-1, 0, -1)\} \) is an orbit. So \( h^3 \) has at least 4 distinct fixed points.

### 7. Strong Nilpotence

Strong nilpotence is a notion that was introduced in [38] for polynomial maps \( F \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). The Jacobian matrix matrix \( J(F) \) was called strongly nilpotent if all products of \( n \) matrices of the form \( J(F)(a_i) \), for possibly distinct points \( a_i \in \mathbb{R}^n \), are zero. In [38] the Jacobian matrix of a polynomial map \( F : k^n \rightarrow k^n \) was defined to be strongly nilpotent if the matrix product \( J(F)(a_1)J(F)(a_2) \cdots J(F)(a_n) \) is zero, where the \( a_i \) are distinct sequences of independent variables of length \( n \) (that is, “generic points”). If \( k \) is infinite, this is equivalent to the obvious generalization of the first definition to \( k \). The following somewhat more general definition can be used in both these cases.
Call a family $\mathcal{F}$ of linear endomorphisms of a vector space $V$ \textit{strongly nilpotent} if there is a positive integer $r$ such that the product (i.e. composition) of any $r$ factors in the family is zero.

**Remark 4.** The use of families (indexed collections) of linear transformations, rather than sets, is sanctioned by the traditional terminology of “commuting families of endomorphisms.” It serves, perhaps, to emphasize that the factors in a product in the above definition need not be distinct.

A family $\mathcal{F}$ satisfying the above condition for a particular $r$ is said to be strongly nilpotent of index $r$. The smallest positive integer $r$ for which $\mathcal{F}$ satisfies the above condition is called the exact index of nilpotence of the family $\mathcal{F}$. The following elementary theorem can be derived from well known results on the simultaneous linear triangularizability of matrices, such as McCoy’s Theorem ([36, 30]). However, a simple proof, valid over any field $k$, is presented below. It is inspired by the proof of linear triangularizability for commuting families of endomorphisms (a theorem of Frobenius [27]) as presented in [32].

**Theorem 7.1.** Let $r$ be the exact index of nilpotence of a strongly nilpotent family of endomorphisms of a vector space $V$ of finite dimension $n$. Then $r \leq n$ and there is a basis of $V$ in which all the members of the family are represented by strictly upper triangular matrices.

**Proof.** By induction on $n$. Let $A$ be a product of $r-1$ factors from the family $\mathcal{F}$ which is non-zero (the result being trivial if $r=1$). Let $W$ be the kernel of $A$. $W$ is non-zero, of dimension less than $n$, and invariant under $\mathcal{F}$, since $AB = 0$ for any $B$ in $\mathcal{F}$. By induction, there is a non-zero $w \in W$ annihilated by $\mathcal{F}$. The family $\mathcal{F}$ acts on $V/wV$. Lift a strictly upper triangularizing basis of $V/wV$ to $v_1, \ldots, v_n \in V$ and let $v_1 = w$. Let $V_0 = \{0\}$ and $V_i = k_1 v_1 + \cdots + k_i v_i$ for $0 < i \leq n$. Then $\mathcal{F} V_i \subseteq V_{i-1}$ for $0 < i \leq n$. Thus $(v_1, \ldots, v_n)$ is a basis of $V$ in which all members of $\mathcal{F}$ are strictly upper triangular. Since the product of any $n$ strictly upper triangular matrices is zero, it follows that $r \leq n$. 

8. Linear Triangularizability

Arno van den Essen and Engelbert Hubbers showed [23] that polynomial maps over an arbitrary field which are perturbations of the identity by a map with strongly nilpotent Jacobian matrix are linearly (unit upper) triangularizable. Their proof explicitly uses the fact that polynomials are involved. The theorem below characterizes linearly triangularizable $C^1$ maps in similar terms, and also in terms of properties of composition products involving the perturbation part.

The following terminology will be used. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is (unit upper) triangular if $f = (f_1, \ldots, f_n)$ and $f_i - x_i$ depends only on those $x_j$ with $j > i$. The qualification “unit upper” will be implicitly assumed, unless otherwise stated. The map $f$ is linearly triangularizable if there is a linear mapping $S$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is invertible, and such that $S^{-1} \circ f \circ S$ is triangular. A map $\tau : \mathbb{R}^n \to \mathbb{R}^n$ will be called a translation-dilation if it has the form $\tau(x) = a + \lambda x$, where $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. All linearly triangularizable maps have unipotent Jacobian matrices and, by Theorem [23] and condition (3) below, they have explicitly computable inverses.
Theorem 8.1. Let \( f(x) = x + h(x) : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) map. Then the following conditions are equivalent:

1. \( f \) is (unit upper) linearly triangularizable;
2. The family of all matrices \( \{ J(h)(x) \} \) (for \( x \in \mathbb{R}^n \)) is strongly nilpotent;
3. Any composition product of maps from \( \mathbb{R}^n \) to itself, in which \( n \) or more of the factors equal \( h \) and the remaining factors are translation-dilations, is constant.

Proof. (1) \( \Rightarrow \) (2). Suppose \( f = S^{-1} \circ t \circ S \), with \( t \) triangular and with \( S(x) = Ax \) for an invertible matrix \( A \). Then \( J(h)(x) = A^{-1}(J(t)(S(x)))A - I \) and the matrices \( J(t)(S(x)) \) are unit upper triangular. Thus the family \( \{ J(h)(x) \} \), for \( x \in \mathbb{R}^n \), is similar to a family of strictly upper triangular matrices, and hence strongly nilpotent.

(2) \( \Rightarrow \) (3). Let \( p = p_1 \circ \cdots \circ p_s \) be a composition product of maps \( p_i : \mathbb{R}^n \to \mathbb{R}^n \). By the chain rule, \( J(p) = (J(p_1) \circ q)J(q) \), where \( q = p_2 \circ \cdots \circ p_s \). By induction, \( J(p)(x) \) is a product of \( s \) matrices that are Jacobian matrices of the factors \( p_i \) evaluated at various points. Factors that are translation-dilations yield scalar factors (multiplication by \( \lambda \)) in the matrix product, and can be moved to the front. If there are \( n \) or more factors \( p_i \) equal to \( h \), the matrix product is zero by strong nilpotence. But \( J(p) = 0 \) implies that \( p \) is constant.

(3) \( \Rightarrow \) (1). Let \( a_1, \ldots, a_n \) be \( n \) points in \( \mathbb{R}^n \). Let \( \tau_i(x) = a_i + \lambda_i x \) for non-zero \( \lambda_i \in k \). Consider the composition product \( p = h \circ \tau_1 \circ h \circ \tau_2 \circ \cdots \circ h \circ \tau_n \). By assumption it is constant. Its Jacobian matrix at a point \( J(p)(x) \) is the product of \( 2n \) matrices, \( n \) of which are scalar multiples of the identity \( (\lambda_1 I) \) and \( n \) of which have the form \( J(h)(a_i + \lambda_i(z)) \) for some point \( z \). Since \( J(p)(x) = 0 \) and the product of factors corresponding to translation-dilations is a non-zero multiple of the identity matrix, the product of the remaining factors is zero. Now vary the \( \lambda_i \) and take a limit in which all \( \lambda_i \) tend to zero to obtain \( J(h)(a_1) \cdots J(h)(a_n) = 0 \) — that is, strong nilpotence.

9. The New Class of Automorphisms

In \cite{24,25} Arno van den Essen and Engelbert Hubbers introduced a class of invertible polynomial maps more general than linearly triangularizable maps, but with some of the same properties. Maps in this New Class have been used to provide counterexamples to a number of conjectures involving spectral conditions on the Jacobian matrix and stability and linearizability of maps \cite{8,18,26}.

The invertible polynomial “maps” are defined for any coefficient ring \( A \) that is a commutative ring with 1. Because the coefficient ring can be, say, a finite field, the maps are actually polynomial morphisms of affine \( n \)-space over \( A \), many of which may correspond to the same underlying set-theoretic map \( A^n \to A^n \). A polynomial morphism of affine \( n \)-space to itself can, of course, be identified with an \( n \)-tuple \( f = (f_1, \ldots, f_n) \) of polynomials in \( n \) variables with coefficients in \( A \).

They are perturbations of the identity map; that is, each is of the form \( f(x) = x + h(x) \), where \( f \) and \( h \) are \( n \)-tuples of polynomial in \( n \) variables. Of course, that is no real restriction, since any polynomial map can be written as \( f(x) = x + h(x) \), by simplifying defining \( h(x) \) as \( f(x) - x \).
They are singled out from all polynomial maps by specifying what perturbations $h$ are permitted, and the definition of the allowed perturbations is a recursive definition in terms of the coefficient ring $A$. That is, to define the allowed perturbations $n$ variables, one starts with the allowed perturbations in $n - 1$ variables, but over the coefficient ring $A[x_n]$, and specifies what operations can be applied to these to yield allowed perturbations in $n$ variables.

The primary focus of this paper is on $C^1$ maps, and the definition in [24] is not applicable, because the relationship between functions in $n$ variables and those in $n - 1$ variables cannot be described by a simple construction that adds a variable (such as forming polynomials or power series in an additional variable), but only by fixing a variable.

An appropriate (still inductive) definition of allowed perturbations in $n$ variables is the following. The induction specifies increasingly larger subsets $H_{n,i}$ of the set of all $C^1$ maps from $\mathbb{R}^n \to \mathbb{R}^n$, all of which consist of allowed perturbations. The final, largest subset, $H_{n,n}$, is also denoted $H_n$, and a $C^1$ map $f : \mathbb{R}^n \to \mathbb{R}^n$ will be said to be in the New Class if (and only if) it has the form $f(x) = x + h(x)$, where $h \in H_n$. The definition uses the notion of a parameter of a map $f$, which refers to a coordinate variable $x_i$, for which $f_i = x_i$. Obviously, if $f(x) = x + h(x)$ and $x_i$ is a parameter of $f$, then $h_i = 0$.

**Definition 1.** Let $n$ be a positive integer, let $x = (x_1, \ldots, x_n)$ be a fixed (linear) coordinate system for $\mathbb{R}^n$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map, with components given by $h = (h_1, \ldots, h_n)$. Then define

1. $h \in H_{n,1}$ if $h_2 = h_3 = \cdots = h_n = 0$ and $h_1$ is independent of $x_1$;
2. $h \in H_{n,i}$, for $i > 1$, if $h_{i+1} = h_{i+2} = \cdots = h_n = 0$ and there exist a map $\tilde{h} \in H_{n,i-1}$, a map $T : \mathbb{R}^n \to \mathbb{R}^n$ that is $C^1$ and linear in the first $i$ coordinates with $x_{i+1}, \ldots, x_n$ as parameters, and a map $C : \mathbb{R}^n \to \mathbb{R}^n$ whose components depend only on the parameters $x_{i+1}, \ldots, x_n$, such that

\[
(*) \quad h(x) = T^a \circ \tilde{h} \circ T + C
\]

In equation (*) above, $T^a$ refers to the classical adjoint of $T$. More precisely, since $T$ has the last $n - i$ variables as parameters, and it is linear in the first $i$ variables, it can be represented as (pre-)multiplication by an $n \times n$ matrix $M$ which is block diagonal, of the special form

\[
M = \begin{bmatrix}
B & 0 \\
0 & I
\end{bmatrix}
\]

with $B$ an $i \times i$ matrix, whose coefficients will in general depend on the parameters $x_{i+1}, \ldots, x_n$. Then $T^a$ denotes the map of the same form given by replacing $M$ by its classical adjoint (matrix of cofactors). Note also that the last $n - i$ components of $C$ are necessarily 0 if equation (*) is to be satisfied.

The above definition accurately captures the spirit of the definition of the New Class of automorphisms in the polynomial case. And the New Class has the properties specified in Theorem [9.1], below, which are analogous to ones established in the polynomial case. The proofs supplied, however, are different, as [24] relies on notions of specialization and localization appropriate only to the polynomial case. The actual proof of the theorem (which is more of an elaborate verification by induction) is carried out in Appendix A.
Theorem 9.1. Let \( f(x) = x + h(x) \) be a map in the New Class; that is, \( h \in H_n \).

Then

1. If \( r \in \mathbb{R} \), then \( rh \in H_n \).
2. If \( T \) is a linear map of \( \mathbb{R}^n \) to itself, then \( T^n \circ h \circ T \in H_n \).
3. If \( C \) is a constant vector of length \( n \), then \( h + C \in H_n \).
4. \( J(h) \) is nilpotent (and thus \( J(f) \) is unipotent).
5. The composition power \( h^{\circ n} = h \circ h \circ \cdots \circ h \) (\( n \) factors) is constant.
6. The map \( f \) is invertible, with an explicit inverse.

Remark 5. If \( T \) is a change of basis map, then the expression of \( f \) in terms of the new basis is \( T \circ f \circ T^{-1} = x + T \circ h \circ T^{-1} \). From the first two properties above, \( h \in H_n \) also belongs to \( H_n \) defined in terms of the new, linearly related, coordinate system. So \( H_n \) is actually invariant under linear changes of coordinates.

Somewhat more is known in the case of polynomials with coefficients in a commutative ring \( A \) with 1. For one thing, [24] proves that if \( f \) is an automorphism in the New Class then so is its inverse, whereas the above theorem only establishes the existence of an “explicit” inverse. Also, [24] defines a subtly different class of allowable perturbations \( \overline{H_n}(A) \) that is, in general, a strict superclass of \( H_n(A) \).

Unpublished work of Arno van den Essen [22] shows that for a field \( k \) of characteristic zero one has \( \overline{H_n}(k) = H_n(k) \) for \( n < 5 \), and that the equality is false for \( n \geq 5 \). Finally, the following more refined properties of \( H_n(A) \) are proved in [25].

The relevant definitions can be found there.

Theorem 9.2. ([23]) If \( f(x) = x + h(x) \) and \( h \in H_n(A) \) then

1. \( f \) is a finite composition product of automorphisms of the form \( \exp(D) \), for locally nilpotent derivations \( D \) satisfying \( D^2(x_i) = 0, 1 \leq i \leq n \).
2. \( f \) is a stably tame automorphism.

10. Linearizability

It is instructive to consider the linear case of the Multiple Fixed Point Conjecture. A linear map \( L \) always has at least one fixed point, namely 0. If \( L \) has a second, distinct fixed point \( x \), then the equation \( x = Lx \) expresses the fact that \( x \) is an eigenvector of \( L \) with the nonzero eigenvalue 1. Roughly the same argument can be applied to a linearizable map.

Recall that \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be linearizable if there is an invertible map \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \sigma^{-1} \circ f \circ \sigma = L \) is a linear map. This is a special case of the notion of conjugate maps: \( f \) and \( g \) are conjugate maps if there is an invertible map \( \sigma \), such that \( \sigma^{-1} \circ f \circ \sigma = g \). The definition of conjugacy implies that \( f \) maps some set to itself, and that \( g \) also maps some (possibly different) set to itself. Thus one can speak of the fixed points of \( f \) and \( g \), and it is clear that they correspond to each other bijectively \((f(\sigma(q))) = \sigma(q)\) if, and only if, \( g(q) = q \). If \( q \) and \( p = \sigma(q) \) are corresponding fixed points and \( f, g \) and \( \sigma \) are all \( C^1 \) maps between open subsets of \( \mathbb{R}^n \), then the chain rule yields \( J(g)(q) = A^{-1}J(f)(p)A \), where \( A = J(\sigma)(q) \). So nilpotence at a fixed point is preserved under this type of conjugacy.

A (globally) linearizable map has at least one fixed point: the one corresponding to the fixed point 0 of the linear map. A local linearization is a conjugacy relation in
which $f$ and a linear map $L$ have corresponding fixed points $p$ and $0$. The existence of, possibly local, linearizations of various degrees of regularity (topological, $C^1$, analytic, polynomial) around a fixed point is a subject of considerable interest in a number of areas of mathematics and has a vast literature. It has captured the interest of the polynomial automorphism community recently as a possible new approach to the Jacobian Conjecture \[13, 28, 12, 18, 20, 16\].

The following theorem shows that linearizability of maps with nilpotent Jacobian matrices would imply the Multiple Fixed Point Conjecture (and hence the Jacobian Conjecture). It can also be considered as a special case in which the conjecture holds.

**Theorem 10.1.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ with nilpotent fixed points. If $f$ is globally linearizable, then $f$ has a unique fixed point.

**Proof.** Suppose that $f$ and $L$ are (globally) conjugate. Because the fixed points of $f$ are nilpotent (that is, $J(f)$ is nilpotent at each fixed point), the fixed point set of $f$ consists of isolated points (see remark in section 6), hence is countable. If $L$ has 1 as an eigenvalue, then its fixed point set is uncountable (as it contains all multiples of an appropriate eigenvector). Thus $L$ does not have 1 as an eigenvalue, hence its fixed point set consists of 0 alone, and thus $f$ has a unique fixed point. □

**Remark 6.** Note that the linearizing map $\sigma$ is not even required to be continuous for this result. The hypothesis asserts the nilpotence of the Jacobian matrix only at the fixed points of $f$, rather than generally. But that is all that is needed for this application. As far as the differentiability of $f$ is concerned, that also need be assumed only in the neighborhood of the fixed points, so that the inverse function theorem can be applied locally to show that the fixed points are isolated.

### 11. Counting Fixed Points

Let $f : X \to X$ be a continuous map of a topological space to itself and suppose that the rational homology of $X$ is finite. That is, the homology groups $H_i(X, \mathbb{Q})$ with rational coefficients are all finite dimensional rational vector spaces and all but finitely many of them are zero. The map $f$ induces an endomorphism $f_* : H_*(X, \mathbb{Q}) \to H_*(X, \mathbb{Q})$; that is, for each integer $i$ a linear map $f_{*,i} : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$. Define the Lefschetz number $L(f)$ of $f$ as the alternating sum of the traces of the induced maps: $L(f) = \sum (-1)^i \text{Tr}(f_{*,i})$. The definition appears to depend on the choice of homology theory (e.g. singular homology) and on the coefficient field $\mathbb{Q}$; as usual, it is in fact independent of those choices for “reasonable spaces” (e.g. homeomorphic to a polyhedron) and “reasonable coefficients” (a field containing the rationals). By using homology with integral coefficients (mod torsion), one can also show in that case that $L(f)$ is an integer. Two homotopic maps $X \to X$ have the same Lefschetz number.

Now let $M$ be a compact, oriented $C^1$ manifold and $f : M \to M$ a $C^1$ map. A fixed point $x$ of $f$ is said to be non-degenerate if the tangent map $df_x$ from $T_x(M)$ to itself does not have 1 as an eigenvalue. In terms of local coordinates near the fixed point, this amounts to the statement that the Jacobian matrix of $f$ in those coordinates, evaluated at $x$, does not have 1 as an eigenvalue (equivalently, it has 0 as its unique fixed point). The local Lefschetz number of $f$ at a non-degenerate
fixed point \( x \) is defined to be +1 if the determinant of \( I - df_x \) is positive, and −1 if it is negative. If all the fixed points of \( f \) are non-degenerate, then there are only finitely many of them (because they are isolated), and \( L(f) \) is the sum of the local Lefschetz numbers of \( f \) at all of its fixed points. Since the local Lefschetz numbers are all either +1 or −1, it follows that \( L(f) \) is an integer that algebraically (that is, with due attention to sign) counts the fixed points of \( f \). The non-degeneracy of all fixed points is geometrically equivalent to transversality of the intersection of the graph of \( f \) and the diagonal submanifold \( \Delta = \{(m,m)\mid m \in M\} \subset M \times M \). It can be given a purely topological definition, and the local Lefschetz number can be defined for continuous maps that satisfy the non-degeneracy condition, with the same result that \( L(f) \) is the sum of the local Lefschetz numbers. All of these facts are standard topological fare \([3, 15]\).

Obviously, a fixed point for which \( df_x \) is nilpotent will have a local Lefschetz number of +1. So if \( df_x \) is nilpotent at each fixed point of \( f \), then \( L(f) \) is the actual number of fixed points of \( f \).

Unfortunately, \( \mathbb{R}^n \) is not compact (though life would be less interesting otherwise). To apply the above results, one has to pass to a completion of \( \mathbb{R}^n \) that is a compact orientable manifold. One possibility is \( S^n \), the \( n \)-dimensional sphere, which is the 1-point compactification of \( \mathbb{R}^n \). That idea leads to the proof of the following

**Theorem 11.1.** If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \) with nilpotent fixed points and the range of \( f \) is bounded, then \( f \) has a unique fixed point.

**Proof.** Since \( f(\mathbb{R}^n) \) is bounded, so is its closure. Let \( D \) be a disc (closed ball) large enough so that the closure of \( f(\mathbb{R}^n) \) is contained in its interior. Let \( D' \) be a concentric larger disk. Let \( A \) be the closed annulus between the \( n - 1 \) spheres \( \partial D \) and \( \partial D' \). Then \( A \) is homeomorphic to \( S^{n-1} \times I \). The restriction of \( f \) to \( \partial D \) maps the \( n - 1 \) sphere \( \partial D \) into the contractible space \( D \) (because \( D \) contains the entire image of \( f \)). So it is homotopic to a constant map. Interpret the homotopy as a map of \( A \) into \( D \) which coincides with \( f \) on \( \partial D \) and is constant on \( \partial D' \). Let \( p \) be the constant terminal value of the homotopy on \( D' \). Now define a map \( F \) on \( S^n = \mathbb{R}^n \cup \{ \infty \} \) as follows: it coincides with \( f \) on \( D \), it is the selected homotopy on \( A \), and it is the constant \( p \) on \( S^n - D' \). By construction, \( F \) is continuous (and it could be made \( C^1 \) as well), and \( F(\infty) = p \). Since \( F(S^n) \subset D \), the only fixed points of \( F \) are in \( D \), hence they are those of \( f \). The homology groups of \( S^n \) with integral coefficients are \( H_0(S^n, \mathbb{Z}) = \mathbb{Z} \) and \( H_n(S^n, \mathbb{Z}) = \mathbb{Z} \), with all others 0. The induced map \( F_* \) on \( H_0 \) is the identity. On \( H_n \) it is multiplication by the degree of \( F \). Since \( F \) factors through a map to \( D \), which is contractible, \( F_* \) is 0. Therefore the Lefschetz number of \( F \) is \((-1)^0 * 1 + (-1)^n * 0 = 1 \). So \( F \) has exactly one fixed point, and thus so does \( f \).

**Remark 7.** Robert Brown notes (personal communication) that the proof can be shortened by invoking the Lefschetz-Hopf Theorem, and provides the reference \([29]\).
12. Polynomial Maps with No Zeros at Infinity

The result of the preceding section does not apply to polynomial maps in any significant way, since the only polynomial maps that have a bounded image are constant maps. In contrast, this section describes results specifically for (some) polynomial maps.

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a polynomial map. Let \( d_i \) be the total degree (degree in all the variables jointly) of the component \( f_i \) of \( f \). The algebraic degree of \( f \) is defined as \( d = \max_i d_i \). Each \( f_i \) can be written as a sum of homogeneous polynomials (forms). The form of highest degree (= \( d_i \)) is called the leading form of \( f_i \).

In [15] John Randall showed that if the Jacobian determinant of \( f \) vanishes nowhere in \( \mathbb{R}^n \) and the leading forms of the components of \( f \) have no common non-trivial zeros, then \( f \) is a proper map and hence a diffeomorphism of \( \mathbb{R}^n \) onto \( \mathbb{R}^n \).

Since each leading form is homogeneous and \( d_i > 0 \) (by the condition on the Jacobian determinant), the leading forms all vanish at 0. The condition in Randall’s Theorem is that this is the only common zero of all the leading forms.

Normally, when one speaks of zeros at infinity of a polynomial map, the reference is to common non-trivial zeros of the forms of degree \( d \) in each of the components. Then the condition that \( f \) have no zeros at infinity is equivalent to the statement that the rational map of real projective \( n \)-space to itself given by

\[
(x_0 : x_1 : \cdots : x_n) \mapsto (x_0^d : x_0^d f_1(x/x_0) : \cdots : x_0^d f_n(x/x_0))
\]

(in homogeneous coordinates) actually defines a global continuous map of real projective \( n \)-space to itself that extends the polynomial mapping \( f \). If \( d_i < d \) for some \( i \), then the form of degree \( d \) in that component is identically zero. So, if \( f \) has no zeros at infinity in the usual sense, it satisfies Randall’s condition \textit{a fortiori}.

**Example 7.** Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( f(x, y) = (x + y^3, y - x^3) \). Then \( \det J(f) = 1 + 9x^2y^2 \) and \( y^3 \) and \( x^3 \) have only \((0, 0)\) as a common zero. Thus \( f \) is a diffeomorphism.

**Remark 8.** If \( f(x) = x + (Ax)^3 \) is a map in cubic-linear form whose Jacobian determinant is constant (equivalently, \( \det J(f) = 1 \)), then \( \det A = 0 \), and \( f \) therefore does have a zero at infinity.

The following application to maps with non-degenerate Jacobian matrix is basically just Randall’s result from a different point of view.

**Theorem 12.1.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a polynomial map whose Jacobian matrix does not have 1 as an eigenvalue anywhere (e.g. \( J(f) \) is nilpotent). If the leading forms of the components of \( f \) have degrees greater than 1 and no common non-trivial zero, then \( f \) has a unique fixed point.

**Proof.** Consider the map \( g(x) = x - f(x) \). Then \( g \) has no zero eigenvalues anywhere, and so the Jacobian determinant of \( g \) vanishes nowhere. The leading forms of the components of \( f \) and of \( g \) are negatives of each other. By Randall’s Theorem, \( g \) is a diffeomorphism, so the equation \( g(x) = 0 \) has a unique solution, which is also the unique solution to \( x = f(x) \).

For maps with no zeros at infinity, the statement is not as complicated.
Theorem 12.2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with no zeros at infinity. If $J(f)$ does not have 1 as an eigenvalue anywhere (e.g. $J(f)$ is nilpotent), then $f$ has a unique fixed point.

Proof. If the algebraic degree, $d$, of $f$ is zero, then $f$ is constant, and hence it has a unique fixed point. If $d = 1$, then the Jacobian matrix of $f$ is constant, with no eigenvalue of 1. Thus $g(x) = x - f(x)$ is an affine map with a nonsingular linear part, so it is bijective, and hence $f$ has a unique fixed point. If $d > 1$, the fact that $f$ has no zeros at infinity implies that the same is true for $g$. Though $g$ may have some components $g_i$ of degree $d_i < d$, Randall’s condition must be satisfied, since the leading forms of the components of degree $d$ have no common non-trivial zero. So $g$ is a diffeomorphism, and $f$ has a unique fixed point. \qed

13. A Note on Scope

A uniform context of $C^1$ maps of $\mathbb{R}^n$ to itself was adopted throughout for clarity. Clearly, differentiability assumptions can be dropped in some results, where only the group structure of $\mathbb{R}^n$ plays a role (e.g. Theorem 12.3). At the other extreme, a lot of the results can be extended to situations in which differentiability is only assumed to exist in an abstract sense (as a derivation), provided that it has adequate formal properties. Such properties include being defined on all polynomial maps (and perhaps others), being defined on compositions, and satisfying the chain rule. However, closure (the derivative is differentiable) and an inverse function theorem are not always needed. As a substitute for continuity one can use an axiom to the effect that constant is equivalent to finite-valued and to a vanishing derivative. The sort of situations to which one can then apply the results include, for example, holomorphic maps over $\mathbb{C}$, rigid analytic maps over a $p$-adic field, $C^1$ semialgebraic maps over a real closed ground field, and other classes of maps of $k^n$ to itself for $k$ a field of characteristic zero.

Appendix A. Proof of Theorem 9.1

The proof of Theorem 9.1 consists of a sequence of lemmas and propositions. The following notation will be used. For a given $1 \leq i \leq n$, let $v$ denote the “live” variables and $z$ the parameters; $v = (x_1, \ldots, x_i)$ and $z$ is either the empty sequence (if $i = n$) or $z = (x_{i+1}, \ldots, x_n)$ (if $i < n$). All functions, matrix entries, and components will be $C^1$, and dependence on the variables and parameters will be denoted in the usual fashion (for example, $f(x_i, z)$ denotes a function or map independent of $x_1, \ldots, x_{i-1}$). Juxtaposition is used for matrix and matrix-vector products. The notation $M^a$ will be used for the classical adjoint of the matrix $M$. The inductive definition of $h \in \mathcal{H}_{n,i}$ can then be stated as follows: $h$ has the form $S \circ \tilde{h} \circ T$ with $\tilde{h} \in \mathcal{H}_{n,i-1}$, $T(v, z) = (M(z)v, z)$, and $S(v, z) = (M^a(z)v + \eta(z), z)$ for an $i \times i$ matrix $M$ and an $i$-vector $\eta$ of functions that depend only on $z$ and not on $v$. In the inductive proofs that follow the base case $(i = 1)$ is always obvious and will not be explicitly checked.

Lemma A.1. Let $\delta(z), \eta(z)$ and $N(z)$ be, respectively, a scalar, a vector of length $i$, and an $i \times i$ matrix, consisting of functions that depend only on $z$. If $h$ belongs to $\mathcal{H}_{n,i}$, then so do $\delta(z)h, h + (\eta(z), 0)$ and $(N^a(z)v, z) \circ h \circ (N(z)v, z)$. \hfill \qed
Proof. Obvious induction. The last claim uses $N^nM^n = (MN)^n$.

This establishes the first three claims in the theorem when $i = n$.

Lemma A.2. Suppose $h \in \mathcal{H}_{n,i}$. Then the $i$-th leading principal submatrix of $J(h)$ is nilpotent.

Proof. The matrix in question consists of the first $i$ rows and columns of $J(h)$. Temporarily fix the values of the parameters. With $z$ fixed, apply the chain rule to see that the matrix is $M^aNM$, where the leading principal submatrix of rank $i - 1$ of $N$ is nilpotent of index $i - 1$ (by induction) and the last row of $N$ is 0. It follows that $N^i = 0$ and hence the same is true for $(M^aNM)^i = M^aNM^aN \ldots M = det(M)^{i-1}M^a(N^i)M = 0$. Since that is true for every fixed value of $z$, the desired result holds.

This establishes the fourth claim in the theorem when $i = n$.

Lemma A.3. If $\delta(z)$ and $\epsilon(z)$ are a scalar function and an $i$-vector of functions that depend only on $z$, and $h \in \mathcal{H}_{n,i}$, then $h \circ (\delta(z)v + \epsilon(z), z) \in \mathcal{H}_{n,i}$.

Proof. $h(v, z) \circ (\delta(z)v + \epsilon(z), z) = (M^a(z)v + \eta(z), z) \circ \tilde{h}(v, z) \circ (M(z)v, z) \circ (\delta(z)v + \epsilon(z), z) = (M^a(z)v + \eta(z), z) \circ \tilde{h}(M(z)\delta(z)v + \tilde{M}(z)\epsilon(z), z) = (M^a(z)v + \eta(z), z) \circ \tilde{h}(v, z) \circ (M(z)v, z)$, where $\tilde{h}(v, z) = \tilde{h}(\delta(z)v + \gamma(z), z)$ and $\gamma(z) = M(z)\epsilon(z)$. By induction, the fact that all the parameters are also parameters for $i - 1$, it follows $\tilde{h} \in \mathcal{H}_{n,i-1}$ and hence that $h$ has the desired form.

Lemma A.4. If $h \in \mathcal{H}_{n,i}$ then the composition power $h^{\circ i} = h \circ h \circ \cdots \circ h$ ($i$ times) depends only on the parameters $z$.

Proof. Let $j = i - 1$. $h^{\circ i} = (S(v, z) \circ \tilde{h}(v, z) \circ T(v, z) \circ S(v, z))^{\circ j} \circ S(v, z) \circ \tilde{h}(v, z) \circ T(v, z) \circ S(v, z)$, where $T(v, z) = (M(z)v, z)$ and $S(v, z) = (M^a(z)v + \eta(z), z)$. $T(v, z) \circ S(v, z) = (M(z)M^a(z)v + \tilde{M}(z)\eta(z), z) = (\delta(z)v + \epsilon(z), z)$ with $\delta(z) = det(M(z))$ and $\epsilon(z) = M(z)\eta(z)$. By the previous lemma, and the fact that parameters for $i$ are parameters for $j = i - 1$, it follows that $\tilde{h}(v, z) \circ T(v, z) \circ S(v, z) \in \mathcal{H}_{n,j}$. By induction, the result of raising this transformation to the composition power $j$ depends only on the parameters for $j$ – that is, on $x_i$ and $z$. Denote the power by $g(x_i, z)$. Then $h^{\circ i} = S(v, z) \circ g(x_i, z) \circ \tilde{h}(v, z) \circ T(v, z)$. But $g(x_i, z) \circ \tilde{h}(v, z) = g(0, 0)$ since all components of $\tilde{h}$ are zero from the $i$-th on. So $h^{\circ i}$ depends only on $z$.

This establishes the fifth claim in the theorem when $i = n$.

Proposition A.5. For $f$ as in the theorem, $f$ has an explicit inverse. Specifically, for each fixed $y$, the value of $f^{-1}(y)$ is the unique value of the composition power $(y - h(x))^{\circ n}$ (which is a constant map; that is, independent of $x$).

Proof. Let $y \in \mathbb{R}^n$. By claims (1)–(3), the map $y - h(x)$ is in $\mathcal{H}_n$. By claim (5), its $n$-th composition power is constant. Since this is true for every $y$, it follows from Theorem 3.2 that $f^{-1}(y) = (y - h(x))^{\circ n}$ (composition power).

This establishes the sixth claim in the theorem.
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