Non Metric Mass.

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Abstract

Mass terms are often introduced into wave equations: for example introducing a mass term for a scalar field gives the Klein-Gordon equation \((\Box^2 - m^2)\phi = 0\). Proceeding similarly with the metric of general relativity one recovers a vanishing mass term because \(g_{abc} = 0\). For non-metric theories \(g_{abc} = -Q_{cab}\), so that the wave equation associated with the metric \((\Box^2 - M)g_{ab} = 0\) no longer entails vanishing mass. This equation can be rewritten in the form

\[ M(x) + \tilde{\nabla}_a Q^a + (\epsilon + \frac{d}{2} - 2)Q_a Q^a = 0, \]

where \(\epsilon = 0, 1, 2, or 3\), \(d\) is the dimension of the spacetime, and \(Q\) is the object of non-metricity. For any given non-metric theory it is possible to insert the metric into this wave equation and produce a non-metric mass. Alternatively one can choose this equation to be \textit{a priori}, and then try to construct theories for which it is the primary equation. This can be achieved using a simple Lagrangian theory. More ambitiously it is possible to investigate whether the introduction of non-metric mass has similar consequences to having a mass term in the Klein-Gordon and Proca equations: namely whether there are wave-like solutions, and what the rate of decay of the fields are. In order to find out a more intricate theory than the simple theory is needed. Such a theory can be found by conformally rescaling the metric and then arranging that the conformal parameter cancels out the object of non-metricity in the Schouten connection. Once this has been achieved one can conformally rescale general relativity and then compare the properties of the wave equations. On the whole its consequences are similar to \(m^2\) in the Klein-Gordon equation, the main difference being \(M\) is position dependent. The Proca \(m^2\) breaks gauge invariance, nothing similar happens for the non-metric mass \(M\) or for the Klein-Gordon \(m^2\). The dynamics of the rescaled theory are not clearly defined; the best definition criterion is the initial value problem and this is taken to signify well-defined dynamics.
1 Introduction

In field theory operators such as $(\Box^2 - m^2) = (\nabla_a \nabla^a - m^2)$ can be introduced to describe massive fields. General relativity has metric connection $\nabla_a g_{bc} = 0$, and this forces $m$ to be zero if an operator of the above type is applied. The principle question addressed here is how to change the above so that non-zero mass operators can be applied to the metric. Changing the underlying geometry to a Schouten (1954) geometry there is a non-metric connection $\nabla_a g_{bc} = -Q_{abc}$. $\Box g_{ab}$ is ambiguously defined (see 42, 43, 44, and 45), but it is not zero unless $Q_{abc} = 0$ resulting in a non-metric mass related to $Q_{abc}$. For any given non-metric theory a non-metric mass can be calculated by inserting $Q_{abc}$ into $\nabla_c g_{ab}$ and then using $M g_{ab} = \Box^2 g_{ab}$. Here however $(\Box^2 - M) g_{ab} = 0$ is thought of as the primary equation and this leads to different theories than those considered hitherto. Previously Roberts (1986) such theories have been referred to as massive theories, however in section 3 it is shown that the resulting theories have nothing in common with linearized massive theories of gravitation as described by the Fierz-Pauli equation. In section 3 a simple non-metric theory, determined by Hilbert’s Lagrangian and a Lagrangian of similar form to the Klein-Gordon scalar Lagrangian but with the object of non-metricity replacing the scalar field is presented. Physically more interesting is a theory introduced in section 4 which uses the rescaling property of non-metric spaces. This theory has problems with the consistency of the dynamics. Consistent dynamics are taken to be those for which there is a well posed initial value problem and this is discussed in section 8. Previous examples of the application of non-metricity in relativity are the original theory of Weyl (1918), scale covariant theories Canuto et al (1972), and quantum theories Smolin (1975). Non-metric theories have also been discussed in Coley (1983), Hochberg and Plunien (1991), Blanchet (1992), Poberii (1994a), Hochberg and Plumien (1991), Aldrovandi et al (1998), Socorro et al (1998), and Chen and Nester (1998). Conformal invariance suggests that non-metricity might be related to 2-dimensional dilaton theories, see for example Banks and O’Loughlin (1991). Non-metric theories are different from 2-metric theories, Aicheburg and Mansouri (1972), as the underlying geometry is different.

Having introduced theories with non-metric mass operators, the next question to be addressed is why. The main property of operators such as $(\Box^2 - m^2)$ is that they lead to differential equations which have solutions which are no longer null (or light-like). One reason is that it might be
possible to explain faster-than-light effects. Observationally apparent faster than light extra-galactic radio sources have been known for some time, Cohen et al (1977) [17]. Now there are also observations, Mirabel and Rodriguez (1994) [23], that suggest that there are super-luminal sources in the galaxy. These observations are further discussed in Pearson and Zenus (1987) [19]. The speed of propagation of the gravitational field might be connected to faster-than-light effects. It has been suggested that the speed of propagation of the gravitational field be measured directly, Roberts (1987a) [20]. If one looks at relativistic gravitational theories there are at least four ways to approach whether gravitational interaction is null or not. The first is to look at the shock waves (or characteristics) Synge (1960) [21]. This is not a good indicator as for example the shock waves of the Klein-Gordon equation are null. The second is to look at weak field approximations. It is shown in section 6 that in DeSitter (1917) spacetime these lead to the Fierz-Pauli equation with mass $m^2 = \frac{2}{3}\Lambda$, so that again these do not suggest lightlike propagation. The third is to look at exact solutions. It has been shown that in Quadratic Lagrangian theory that wave solutions are not necessarily null, Roberts (1994) [23]. Once one has a candidate exact non-null solution the speed of energy transfer can be tested. In gravitational theories the total energy is only unambiguous for asymptotically flat spacetime, for other spacetimes there are several ways of approaching this, see for example Gustavo (1994) [24], and Chen and Nester (1998) [14]. The fourth way of investigating non-null propagation is to consider non-metric operators and this is done in section 5.

The other important property of operators such as $(\Box^2 - m^2)$ is that they alter the rate of decay of the fields. The best known example of this is the Yukawa potential, which shows that the Klein-Gordon field decays quicker with a positive mass term. There are solutions to the scalar-Einstein equations, Roberts (1996) [27]; however there are none to the massive scalar-Einstein equations which suitably generalize the Yukawa potential. A different potential might have bearing on large scale dynamics: from the scale of the outer solar system Roberts (1987b) [28] to the whole Universe dynamical discrepancies grow. For example constant galactic rotation curves Rubin (1983) [24], are incompatible with dynamics from the visible mass of galactic stars, and the discrepancy is even greater for clusters of galaxies. The longer the length scale the greater the discrepancy. Potentials from Quadratic Lagrangian theory have been used to attempt to explain constant galactic rotation curves, Schmidt (1990) [30] and Roberts (1991b) [31]. How non-metric mass effects potentials is discussed in section 5. The rate of de-
cay for weak massive gravitation has been discussed in Freund et al (1969) [32], Datta and Ranna (1979) [33], and Ford and Van Dam (1980) [34].

Section 7 shows that the non-metric wave equation on its own can have an arbitrary mass parameter while being within observable bounds. In section 10 the extent to which it is possible to reinterpret the anomalously high theoretical value of the cosmological constant as simply a manifestation of non-metric mass is discussed. To do this it is necessary to generalize Freund-Rubin (1980) [35] compactification, by relaxing the usual restrictions put on the coordinates, this is similar in style to some of the 5-dimensional approaches discussed in Overduin and Wesson (1997) [36]. The cosmological implications of the rescaled theory are briefly mentioned in section 11.

The conventions used are: signature-+++, $d$ is the number of dimensions. Square, round, and Schouten [1] brackets are defined by

$$2T_{[ab]} = T_{ab} - T_{ba},$$
$$2T_{(ab)} = T_{ab} + T_{ba},$$
$$y_{\{abc\}} = y_{abc} + y_{cba} - y_{bac},$$

respectively. The Riemann tensor, Ricci tensor, Ricci scalar, and Einstein tensor are defined by

$$R_{\alpha\beta\gamma\delta} = 2\partial_{\gamma}\Gamma_{\alpha\beta}^{\delta} + 2\Gamma_{\delta}^{\alpha}\Gamma_{\beta\gamma}^{\delta},$$
$$R_{\beta\delta} = R_{\alpha\beta\alpha\delta},$$
$$R = R_{\alpha\beta}^{\alpha\beta},$$
$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R,$$

respectively. The connection use is shown by a mark above geometric objects: thus $\tilde{R}_{\alpha\beta\gamma\delta}$ is the Riemann tensor constructed with connection $\Gamma$ and $\bar{R}_{\alpha\beta\gamma\delta}$ is the Riemann tensor constructed with Christoffel connection

$$\{\alpha_{bc}\} = \frac{1}{2}g^{\alpha d}g_{\{dc,\beta\}};$$

similarly $\Box^2 = \partial_a\partial^a$ and $\Box^2 = \nabla_a\nabla^a$, where $\nabla_a$ is covariant derivative using the Christoffel connection. $\Box^2$ is ambiguous and will be defined in section 3. The commutation of covariant derivatives when the torsion vanishes is

$$X_{\alpha\beta;cd} - X_{\alpha\beta;dc} = X_{\alpha\epsilon}R_{\beta\epsilon\delta}^{\delta} + X_{\epsilon\beta}R_{\alpha\epsilon\delta}^{\delta}.$$
Stresses used include the Klein-Gordon scalar stress
\[ T_{ab} = 2\psi_a\psi_b - g_{ab}\left(\psi_c\psi^c + V(\psi^2)\right), \] (10)
where the potential \( V(\psi^2) \) usually is taken to have solely the mass term \( m^2\psi^2 \), and the fluid stress, c.f. Ellis [37] p.116
\[ T_{ab} = (\mu + p)U_a U_b + q_a U_b + U_a q_b + p g_{ab} + \Pi_{ab}, \] (11)
where \( q_a U^a = 0, \Pi_a^a = 0, \Pi_{ab} U^a = 0 \). \( U_a \) is a timelike vector field with \( U_a U^a = -1 \), \( \mu \) is the energy density of matter measured by \( U_a \), \( q_a \) is the energy flux relative to \( U_a \), \( p \) is the isotropic pressure, \( \Pi_{ab} \) is the anisotropic pressure. Einstein’s field equations are
\[ G_{ab} = \kappa T_{ab}, \] (12)
where \( \kappa = \frac{8\pi G}{c^4} \); this constant is often absorbed into the fluid or scalar field.

2 The Non-metric Mass Equation.

In field theory it is possible to produce massive fields such as the massive scalar field, which obeys the Klein-Gordon equation
\[ (\Box^2 - m^2)\phi = 0, \] (13)
and the massive vector field, which obeys Proca’s equation
\[ (\Box^2 - m^2)A_a = 0. \] (14)
For weak massive spin 2 fields there is the Fierz-Pauli (1939) equation [3]
\[ (\Box^2 - m^2)h_{ab} = 0. \] (15)
If the same type of equation is introduced for the metric
\[ (\Box^2 - M)g_{ab} = 0, \] (16)
then in general relativity \( M \) must be zero because
\[ \nabla_c g_{ab} = 0. \] (17)
To have a theory with non-vanishing mass in equation 16 it is necessary to generalize the underlying geometry so that there is a non-vanishing object of non-metricity
\[ \hat{\nabla}_a g_{bc} = Q_{a..}^b c, \quad (18) \]
Non-metric mass theory is concerned with the consequences of 16 when the covariant derivatives involve the object of non-metricity so that the mass is non-vanishing, the resulting equation is called the non-metric mass equation. For the Klein-Gordon equation 13 \( m^2 \) is identified as the mass because: i) the equation is a consequence of applying the quantization rules to a point particle, ii) the rate of decay of the field \( \phi \) is faster with a non-vanishing mass, iii) the wave solutions are slower-than-light for \( m \) positive and real; and the last two of these are investigated for non-metric mass in section 5.
Using
\[ g^{ab} g_{bc} = \delta^a_c, \quad (19) \]
and 18 gives
\[ \hat{\nabla}_a g_{bc} = -Q_{abc}. \quad (20) \]
In a Schouten (1954) geometry the connection is
\[ \Gamma^a_{bc} = \{\gamma^a_{bc}\} + \frac{1}{2} N^a_{bc}, \quad (21) \]
where \( \{\gamma^a_{bc}\} \) is the Christoffel connection and \( N^a_{bc} \) is given by
\[ N^a_{bc} = g^{cd} (-2 S_{\{bcd\}} + Q_{\{bcd\}}), \quad (22) \]
where the Schouten bracket given by equation 3 is used. Here the torsion \( S_{\{bcd\}} \) is taken to vanish and the object of non-metricity and connection are restricted to the form
\[ \hat{\nabla}_a g_{bc} = +Q_{a..}^b c = +Q_a g_{bc}, \quad (23) \]
Equations 23 reduce 22 to the connection
\[ N^a_{bc} = (Q_b \delta^a_c + Q_c \delta^a_b - g_{bd} Q^a_d), \quad (24) \]
of a semi-metric geometry (also called a Weyl geometry). Using these connections tensors can be defined as for the equations 4 5, 6, and 7; also the tensor
\[ P^a_{bcd} = R^a_{bcd} - \tilde{R}^a_{bcd}, \quad (25) \]
is of interest. Using equation 25 becomes
\[ P^a_{bcd} = \frac{1}{2} \left( (Q_{dc} - Q_{cd}) \delta^a_d + Q_{bc} \delta^a_d - Q_{bd} \delta^a_c - Q_{c,d} g_{bd} + Q_d^a g_{cb} \right) \]
\[ + \frac{1}{4} \left( -Q_b Q_c \delta^a_d + Q_b Q_d \delta^a_c - g_{cb} Q^a_d Q_d + g_{bd} Q^a_c Q_c \right) + (g_{bc} \delta^a_d - g_{bd} \delta^a_c) Q_c Q_e \) . \] (26)
Contracting
\[ P_{bd} = P^a_{bad} = \frac{1}{2} \left( Q_{db} + (1 - d) Q_{bd} \right) - \frac{1}{2} g_{bd} Q^a_a \]
\[ + \frac{d - 2}{4} \left( Q_b Q_d - g_{bd} Q^a_a \right) , \] (27)
this tensor is not symmetric unless \( Q_a \) is a gradient vector (\( Q_a = Q_a \)). \( P_{ab} \) occurs in non-metric field equations, if \( Q_a \) is not a gradient vector it results in an asymmetric stress.

Another place where a different connection is used is in the weak field perturbations off a background field. The metric is taken to be of the form
\[ g_{ab} = \bar{g}_{ab} + h_{ab} , \quad \bar{g}_{ab} = \bar{g}_{ab} - h_{ab} , \] (28)
where \( \bar{g}_{ab} \) is a given background field metric and \( h_{ab} \) is a small perturbation of the metric. Similarly to 21 decompose the connection and take \( N^a_{bc} \) to be given by \( H^a_{bc} \) where
\[ H_{abc} = h_{bac} + h_{ca;b} - h_{bc;a} . \] (29)
Let “;” be the covariant derivative using the Christoffel symbol of the background field metric. For any connection which is a sum of the Christoffel symbol and a tensor \( H^a_{bc} \), the Riemann tensor is
\[ R^a_{bcd} = \bar{R}^a_{bcd} + H^a_{(d|b|c)} + H^a_{eb} S^e_{cd} + \frac{1}{2} H^a_{[e|c]} H^e_{b} \] . (30)
In the present case cross terms in \( H_{abc} \) and the torsion \( S^a_{bc} \) are taken to vanish. Substituting equation 29 into equation 30 and using 8 for the commutation of covariant derivatives the Riemann tensor becomes
\[ R^a_{bcd} = \bar{R}^a_{bcd} - \frac{1}{2} h^a_e R^e_{bcd} - \frac{1}{2} h_{be} R^e_{c,ed} + \frac{1}{2} (h^a_{d;bc} - h^a_{db,c} - h^a_{c;bd} + h^a_{bc,d}) \] (31)
contracting and again using the commutation of covariant derivatives 8
\[ R_{bd} = \bar{R}_{bd} - \frac{1}{2} h^f_e R_{ebcd} + \frac{1}{2} h_{be} R^e_{cd} + \frac{1}{2} h_{de} R^e_{b} + \frac{1}{2} \left( h^e_{d;cb} + h^e_{b;cd} - \Box^2 h_{db} - h_{bd} \right) , \] (32)
where \( h = \dot{h}_c \). The tensor \( h_{ab} \) has ten components with four degrees of freedom. The Plebanski-Ryten (1961) \([33]\) gauge condition is

\[
[(g)_{;w}^cg_{;w}^b]_b = 0, \tag{33}
\]

with \( w \) arbitrary. When \( w = 1 \) the Ricci scalar’s second derivative terms vanish. For weak fields this coordinate condition gives the gauge condition

\[
h_{a,b}^b = w h_{a}, \tag{34}
\]

when \( w = \frac{1}{2} \) this is the harmonic gauge condition. The four degrees of freedom in \( h_{ab} \) can be removed by applying the harmonic gauge condition. When this is done equation \([32]\) becomes

\[
R_{bd} = \bar{R}_{bd} - h_{ef} \bar{R}_{ebfd} + \frac{1}{2} h_{fe} \bar{R}_{e,d} - \frac{1}{2} \Box^2 h_{bd}. \tag{35}
\]

Returning to non-metric spaces consider their conformal properties. Conformally rescaling the metric from \( g_{ab} \) (usually a metric of a space with Christoffel connection) to \( \hat{g}_{ab} \) a Schouten space i.e.

\[
\hat{g}_{ab} = \Omega(x)^2 g_{ab}, \tag{36}
\]

and then using \([20]\) and \([22]\) gives

\[
\nabla_c \hat{g}_{ab} = (- \Omega^2 Q_c + \partial_c (\Omega^2)) g_{ab} = (Q_c + \partial_c \ln(\Omega^2)) \hat{g}_{ab} = - \hat{Q}_{cab} = - \hat{Q} \hat{g}_{ab}, \tag{37}
\]

hence conformally rescaling the metric, as in \([27]\), and simultaneously transforming the object of non-metricity

\[
\hat{Q}_a = Q_a - \partial_a \ln(\Omega^2), \tag{38}
\]

gives the same connection, and hence the same Riemann tensor. This can also be seen directly by requiring that the connections \([8]\) and \([22]\) cancel. The equivalence of connections is frequently used to relate the solutions of general relativity to corresponding non-metric solutions. To see this note that if \( g_{ab} \) is a solution to the field equations of general relativity (as given
by equations \(12\) where \(Q_a = 0\), then dropping the hat on the object of non-metricity in the Schouten space and conformally rescaling the metric, as in \(36\), gives a solution to the field equations

\[
\begin{align*}
\dot{G}_{ab} & = \tilde{G}_{ab} + P_{ab} - \frac{1}{2}g_{ab}P \\
& = \tilde{G}_{ab} + [Q_{ba} + (1 - d)Q_{ab}] + \frac{d - 2}{4} Q_a Q_b + g_{ab} \frac{d - 2}{2} [Q_c^c + \frac{d - 3}{4} Q_a Q_c^c] \\
& = \kappa T_{ab},
\end{align*}
\]

(39)

where the object of non-metricity is given by

\[
Q_a = -\partial_a \ln(\Omega^2).
\]

(40)

This equation implies that the object of non-metricity \(Q\) is a gradient vector, therefore \(P_{ab}\) is a symmetric tensor and as \(\tilde{R}_{ab}\) is a symmetric tensor, thus \(T_{ab}\) remains a symmetric stress tensor after rescaling. Equation \(40\) can be integrated to give

\[
\Omega^2 = \exp(-Q),
\]

(41)

where the constant of integration can be absorbed into \(Q\) as it is only derivatives of \(Q\) that appear in geometrical objects. The minus sign in front of \(Q\) is unfortunately fixed as can seen from \(40\). The rescaling property is illustrated for an exact solution \(72\).

In non-metric geometry a massive wave equation of the form can be defined in four ways:

\[
\begin{align*}
(g_{ab} \nabla_a \nabla_b - M)g_{cd} & = 0, \quad \text{(for } \epsilon = 0), \\
(g_{ab} \nabla_a \nabla_b - M)g_{cd} & = 0, \quad \text{(for } \epsilon = 1), \\
(g_{ab} \nabla_a \nabla_b + M)g_{cd} & = 0, \quad \text{(for } \epsilon = 2), \\
(g_{ab} \nabla_a \nabla_b + M)g_{cd} & = 0, \quad \text{(for } \epsilon = 3).
\end{align*}
\]

(42)\(45\)

Substituting equations \(20\) and \(22\) into \(42\) and \(43\) gives

\[
M g_{cd} = -g_{ab} \nabla_a Q_{bcd} - \epsilon Q_{a:b,c} Q_{bcd},
\]

(46)

and into \(44\) and \(45\) gives

\[
M g_{cd} = -g_{ab} \nabla_a Q_{b,c:d} - (\epsilon - 2) Q_{a:b} Q_{b,c},
\]

(47)
writing out the covariant \( \Gamma \) derivatives involves unwieldy equations; however for a semi-metric space all four equations can be collected together in a straightforward equation, here called the non-metric mass equation

\[
M(x) + \tilde{\nabla}_a Q^a + (\epsilon + \frac{d}{2} - 2) Q_a Q^a = 0,
\]

where \( d \) is the dimension of the spacetime. The mass \( M(x) \) is a function of undetermined sign, this permits the definitions to have the opposite sign of \( \epsilon \) and \( d \), which in turn allows all four equations to be put in the form; this is why \( M \) is used for non-metric mass whereas \( m^2 \) occurs in the Klein-Gordon equation, the Proca equation, and the Fierz-Pauli equation. Say that there is given a non-metric spacetime (which can originate from Weyl’s or any other non-metric theory), then a non-metric mass can be calculated by substitution of \( Q_a \) into. Unlike the Barut and Haugen (1972) mass the mass \( M(x) \) is not conformally invariant. This can be seen by rescaling the metric of the non-metric spacetime and using to find the rescaled object of non-metricity; then using again gives a rescaled mass which is not the same as the original mass.

3 The Simple Theory.

Equation can generate a mass for any non-metric theory, but the theories discussed here are motivated by using this equation as a starting point. The simplest theory can be found by noting that with the choice \( \epsilon = 0, d = 4 \), \( Q_a \) is a gradient vector and

\[
M(x) = -M^* \frac{dV(Q)}{dQ}
\]

where \( M^* \) is a constant, the non-metric equation simplifies to

\[
\Box Q - M^* \frac{dV(Q)}{dQ} = 0,
\]

and this equation can be derived from the Lagrangian

\[
L_Q = -\frac{1}{2} Q_a Q^a - \frac{1}{2} M^* V(Q).
\]

A choice for a massive theory of gravity is to choose that the dynamics be given by the Lagrangian \( L_Q \) and by Hilbert’s Lagrangian \( L_H \)

\[
L_H = \sqrt{-g} \tilde{R},
\]
this choice is here called the simple non-metric massive theory of gravitation, or the simple theory for short. Such a theory is similar to Einstein-scalar theory, but based on a different geometry. It has the advantage that it has a Lagrangian formulation: resulting in its dynamics being simpler and more secure than would otherwise be the case. The disadvantage is that the field equations for it are

\[ \tilde{\Theta}_{ab} = Q_a Q_b - \frac{1}{2} g_{ab} (Q_c Q^c + M^* V(Q)) + \kappa T_{ab}, \]  

and these do not have the rescaling properties of the field equations \[39\]; similarly the rescaling properties do not hold if the left hand side of \[49\] is replaced by \( \tilde{\Theta}_{ab} \). The rescaling property is useful as it allows solutions to the field equations \[39\] to be found if solutions to the field equations of general relativity \[12\] are known, and this does not happen for the simple theory.

4 The Rescaled Theory

The theory which uses rescaling consists of the field equations \[39\] and the non-metric mass equation \[48\], with as yet no other equation. Rescaling \[48\] from a Schouten geometry (scripted S) to a metric in a space with only Christoffel connection (scripted with E) gives

\[ 0 = M(x) + \tilde{\nabla} Q^{(S)a} \left( \epsilon + \frac{d}{2} - 2 \right) Q^{(S)a} \]  

\[ = M(x) + \exp(+Q)[\tilde{\nabla}^{(E)} a \nabla \left( \epsilon + 1 \right) Q^{(E)a}]. \]  

The theory consisting of \[39\] and \[54\] in \( d = 4 \) dimensions, is called the non-metric massive theory of gravitation rescaled from general relativity, or the rescaled theory for short. The rescaled theory is physically more interesting than the simple theory, but because it is not Lagrangian formulated there are problems with the consistency of its dynamics.

A Lagrangian formulation is no guarantee of unique consistent field equations, for example just varying the metric as opposed to varying the metric and connection can give different results, see Heyl and Kerlick (1978) [40], also for Quadratic Lagrangian theory see Buchdahl (1979) [41] and Shalid-Saless (1991) [42]. Also there are dynamical theories for which no Lagrangian formulation exists. For example the theory of motion of a charged particle see Roberts (1989a) [43], where there are terms such as the Hobbs, Dewitt-Brehme, and Lorentz-Dirac terms which cannot be derived from a
Lagrangian. The possibility of the equations of motion containing more information than the Lagrangian has also been discussed in Falconi (1994) [44].

The rescaled theory with $\epsilon = 1$ has been discussed in the authors thesis, Roberts (1986) [2]. It is not unusual for theories involving the object of non-metricity to have a different number of variables than equations, for example Einstein and Bergman’s variant of Weyl’s theory has 10 equations in 13 variables. Here the rescaled theory consists of 11 geometrical objects, 10 components of the metric $g_{ab}$, and as the rescaling property requires the object of non-metricity $Q$ to be a gradient vector, only one other geometric object $Q$; it also has 11 physical quantities, rescaling forces the stress $T_{ab}$ to be symmetric and therefore it has 10 components, there is also the non-metric mass $M(x)$; the dynamical equations suggested so far are the 10 field equations 39 and the non-metric mass equation 48, thus viewed as a linear problem, and barring degeneracy of the equations, the dynamics are consistent. The dynamics are non-linear and in this section the addition of more equations is discussed; in section 8 it is found that the most elegant formulation of the initial value problem includes one of these additional equations.

The field equations 39 and the non-metric equation 48 may not fully determine the dynamics of the rescaled theory. In this section further equations are constructed and their value briefly assessed.

An additional equation is

$$Mg_{ab} = \alpha T_{ab},$$

(55)

where $\alpha$ is a constant. This equation suffers from two drawbacks: firstly it over constrains the system by giving 10 further equations, and secondly it gives bizarre results for spherically symmetric electromagnetic fields; because of these drawbacks this equation is not used.

Another additional equation is

$$Q_a Q^a = \alpha T_{[ab]} T^{[ab]},$$

(56)

where $\alpha$ is a constant, the dependence of this equation equates the degree of asymmetry of the stress tensor to the object of non-metricity, thus it vanishes in the rescaled theory, but might be of use in other non-metric theories.

The most successful additional equation is the trace equation

$$M = \alpha T,$$

(57)
where \( \alpha \) is a constant, \( \alpha T \) is taken to be the trace of the stress before rescaling times a constant. This choice has the possibility of being derived from a quantum theory as a trace anomaly effect. Let \( W_a \) be a unit timelike vector field, so that \( W_a W^a = -1 \). The weak energy condition and the timelike convergence condition are obeyed if

\[
T_{ab} W^a W^b \geq 0, \quad (58)
\]
\[
R_{ab} W^a W^b \geq 0, \quad (59)
\]
respectively, see for example Hawking and Ellis (1973) p.89,95. Assuming the field equations

\[
\dot{G}_{ab} W^a W^b = T_{ab} W^a W^b \geq 0, \quad (60)
\]

the weak energy condition is obeyed. Thus

\[
\dot{R}_{ab} W^a W^b + \frac{1}{2} \dot{R} \geq 0. \quad (61)
\]

Assuming the time-like convergence condition is to apply to the Ricci tensor \( \dot{R} \) not \( \ddot{R} \), and noting that \( T = -\dot{R} \) gives

\[
T = \frac{M}{\alpha} \leq 0, \quad (\text{for } \alpha \neq 0). \quad (62)
\]

This implies that \( M \) and \( \alpha \) must be of opposite sign for the energy conditions to be obeyed. For a perfect fluid \( T = 3p - \mu \); therefore equation becomes

\[
\left( g_{ab} \dot{\nabla}_a \dot{\nabla}_b - (3p - \mu) \right) g_{ab} = 0, \quad (63)
\]

with similar equations for \( B, C, \) and \( D \). Equation suggests that if \( 3p > \mu \) the metric potentials \( g_{ab} \) propagate faster than the speed of light. There is empirical evidence suggesting that faster-than-light propagation might happen, see the introduction. Energy conditions are a good guide to the physical plausibility of fluids because they are macroscopic and energy conditions seem to apply in the macroscopic domain; also perfect fluids (equation with \( q_a = \Pi_{ab} = 0 \) ) have alot of arbitrariness in the pressure \( p \) and density \( \mu \) functions - energy conditions narrow the choice of these. The microscopic domain is usually described by fields not fluids. For the Klein-Gordon field the weak energy \( W \) and Ricci convergent energy \( C \) are

\[
W \equiv W^a W^b T_{ab} = 2(\psi_a W^a)^2 + \psi_a \psi^a + m^2 \psi^2, \quad (64)
\]

14
and

\[ C \equiv W^a W^b R_{ab} = 2(\psi W^a)^2 - m^2 \psi^2, \quad (65) \]

respectively, so that the weak energy \( W \) increases and the Ricci convergent energy \( C \) decreases in the presence of a mass term. The weak energy is a better measure of the energy of a field so that it appears that the introduction of mass increases a fields energy. Now the question arises: Does this happen for the introduction of non-metric mass? If the weak energy condition is taken to apply as in \((58)\) then \((58)\) and the timelike convergence condition are taken to apply to \( R \) then no information can be extracted to answer the question; furthermore if \( P_{ab} W^a W^b \) is constructed then there are several terms of undetermined sign. The initial value problem in the next section suggests that the best dynamics for the rescaled theory consist of the field equation \((58)\), the non-metric mass equation \((48)\) and the trace equation \((47)\).

5 Mass-like Properties of the Non-metric mass.

The mass-like properties of the non-metric mass are investigated by comparing both the wave-like linearized solutions of the theory and the rate of decay of the object of non-metricity to the wave solutions and rate of decay of other massive theories. The initial value problem in the previous section suggests that the best field equations for the rescaled theory are the field equations \((58)\) where the Einstein tensor is constructed with a connection which is the sum of the Christoffel connection and the semi-metric connections, the non-metric mass equation \((48)\) and the trace equation \((47)\). In the presence of a vacuum these equations become the same as the vacuum equations of general relativity.

In order to linearize these equations the metric is taken of the form of \((28)\) but with the background metric being flat spacetime,

\[ g_{ab} = \eta_{ab} + h_{ab}, \quad (66) \]

this is the same type of linearization as found in Weinberg (1972) \((47)\). Substituting \((54)\) into \((48)\) and \((41)\) gives

\[ M + \left( \frac{1}{2} h_{a} + \partial_{a} \right) Q^{a} + \epsilon Q_{a} Q^{a} = 0, \quad (67) \]

and

\[ \kappa T_{ab} = \frac{1}{2} \left( \square^2 h_{ab} - h_{b,ac} c_{a,bc} + h_{,ab} \right) - \frac{1}{2} \eta_{ab} \left( \square h - h_{cd} \right). \]
\[ + \frac{1}{2} Q_{a,b} - \frac{3}{2} Q_{b,a} + \frac{1}{2} Q_c (h_{bc,a} + h_{ac,b} - h_{ab,c}) \]
\[ + \left( \frac{1}{2} h_{c,a} \partial_c \right) Q_c^c + \frac{1}{4} Q_c Q_c^c, \quad (68) \]

where \( h = h_{c,c} \) and the trace equation \([57]\) remains unchanged. Working in the harmonic gauge (equation \([33]\) with \( \nu = \frac{1}{2} \)) and assuming that cross terms in \( Q_a \) and \( h_{ab} \), and that square terms in \( Q \) are negligible gives

\[
M = -Q_{a,a} = \alpha T, \]

\[
\Box^2 h_{ab} = -Q_{a,b} + 3Q_{b,a} + \eta_{ab}Q_{c,c} + 2\kappa(T_{ab} - \frac{1}{2}T), \quad (69)\]

To proceed further it is necessary to choose a source. Choosing a Klein-Gordon scalar field as source gives

\[
M = -Q_{a,a} = \alpha(-\psi_c \psi_c^c - 2\mu^2 \psi^2) \]

\[
\Box^2 h_{ab} = -Q_{a,b} + 3Q_{b,a} + \eta_{ab}Q_{c,c} + 2\alpha(\psi_a \psi_b + \frac{1}{2} \eta_{ab} \mu^2 \psi^2), \]

\[
(\Box^2 - \mu^2) \psi = 0. \quad (70)\]

These have solution

\[
\psi = \exp i(x + \sqrt{1 + \mu^2} t), \]
\[
Q = \frac{1}{4} \exp 2i(x + \sqrt{1 + 3k\mu^2} t), \]
\[
M = 3k \exp 2i(x + \sqrt{1 + 3k\mu^2} t). \quad (71)\]

Equations \([69]\) are not consistent for \( h_{ab} = 0 \), and the equations for \( h_{ab} \neq 0 \) are intractable. The equations \([71]\) have the properties that: the mass is position dependent, \( Q_{a,b} \) is symmetric i. e. \( Q_{a,b} = Q_{b,a} \), and \( M \) is positive (as implied by the dominant energy condition, see paragraph after \([4]\)). Also \( \psi, Q, \) and \( M \) do not propagate at the speed of light. Some alternative linearization schemes are given in the authors thesis, Roberts (1986b) \([2]\).

A characteristic of massive field theories is that they decay at a faster rate than the corresponding massless ones. In the simple theory the rate of decay of the fields is difficult to ascertain because there is no known spherically symmetric solution for the stress of the massive Klein-Gordon
equation 13 coupled to Einstein’s equations 12. For the rescaled theory, it will be shown in section 7 that just the assumption of the field equations and the non-metric equation implies the Eddington-Robertson parameters \( \Omega^2 = \exp(-Q) \), now requiring the metric to be asymptotically flat both before and after rescaling implies that \( Q \to \pm 0 \) as \( r \to \infty \); however there is no general way of telling if \( Q \) decays as \( \frac{1}{r} \) or \( \frac{\exp(-r)}{r} \). To investigate this it is necessary to resort to exact solutions. The solution must be spherically symmetric with a non-zero trace. The non-zero trace is required so that the trace equation 57 gives a non-metric mass. The interior Schwarzschild solution has trace \( 3p - \mu \) but it is not dependent on the radial coordinate and so might give misleading results, in any case the resulting equations are intractable; therefore the most general static spherically symmetric solution to the massless scalar field equations (the Klein-Gordon equation 13 with \( V = 0 \)) is used.

The static spherically symmetric scalar-Einstein solution is (see Roberts (1985) [48] and references therein)

\[
\begin{align*}
ds^2 &= - \exp\left(-\frac{2\mu}{r}\right)dt^2 + \exp\left(\frac{2\mu}{r}\right)(\frac{\eta}{r})^4 \text{cosech}^4\left(\frac{\eta}{r}\right) dr^2 \\
&\quad + \exp\left(\frac{2\mu}{r}\right)\text{cosech}^2\left(\frac{\eta}{r}\right)(d\theta^2 + \sin^2 d\phi), \\
\phi &= - \frac{\sigma}{r}.
\end{align*}
\]

(72)

where \( \eta^2 = \sigma^2 + \mu^2 \) with \( \mu \) is the ADM mass and \( \sigma \) the scalar charge. The trace equation 57 gives

\[
M = \alpha T = -2\frac{\alpha \sigma^2}{\eta^4} \exp\left(-\frac{2\mu}{r}\right) \text{sinh}^4\left(\frac{\eta}{r}\right)
\]

(73)

For 72, the \( \epsilon = 0 \) equation of 48 becomes

\[
0 = M(x) + \frac{r^2}{\eta^4} \exp\left(-\frac{2\mu}{r}\right) \text{sinh}^4\left(\frac{\eta}{r}\right) \exp(Q) \left[(r^2Q')' + (\eta - 1)Q'\right].
\]

(74)

Eliminating \( M(x) \) from \( 73 \) and \( 74 \) gives

\[
2\alpha \sigma^2 \frac{\exp(-Q)}{r} = \frac{2Q'}{r} + (\epsilon - 1)Q' + Q''.
\]

(75)
which can be put in the form

\[ 2\alpha\sigma^2 \frac{\exp(-Q)}{r} = (rQ' \exp((\epsilon - 1)Q))', \tag{76} \]

this has the general solution

\[ Q = -\frac{k}{r}, \quad (\epsilon = 1), \tag{77} \]

\[ \exp((\epsilon - 1)Q) = l + (1 - \epsilon)\frac{k}{r}, \quad (\epsilon \neq 1), \tag{78} \]

where \( l \) and \( k \) are constants. This solution can give asymptotically flat rescaled spacetimes. For \( \epsilon \neq \frac{2}{3} \) this differential equation has solution

\[ \exp\left(\frac{Q}{2}\right) = \sqrt{\frac{\alpha - \sigma}{2\epsilon - 3}} r, \tag{79} \]

which is not asymptotically flat as \( \exp(-Q) \) diverges as \( r \to \infty \). For \( \epsilon = 1 \) there are also the solutions

\[ \exp(Q) = 3\alpha \cosh(c - \frac{\sigma}{\sqrt{3}r}), \quad (\alpha > 0), \tag{80} \]

\[ \exp(Q) = -3\alpha \sinh(c - \frac{\sigma}{\sqrt{3}r}), \quad (\alpha < 0), \tag{81} \]

where \( c \) is a constant. These solutions are asymptotically flat when

\[ c = \text{arccosh}\left(\frac{1}{\sqrt{3\alpha}}\right), \quad (\alpha > 0), \tag{82} \]

\[ c = \text{arcsinh}\left(\frac{1}{\sqrt{-3\alpha}}\right), \quad (\alpha < 0), \tag{83} \]

a particular case of the first of these is when \( c = 0 \) and \( 3\alpha = 1 \). There are similar solutions with the hyperbolic functions replaced by trigonometrical ones.

The trigonometrical functions would appear to be unphysical because of the periodic properties they give to \( Q \). If the dominant energy condition is satisfied and the mass is taken to be positive then \( Q \) must negative which does not happen for \( \text{80} \). For \( \text{79} \) and \( \text{80} \) with the conformal factor \( \exp(-Q) \) the solution is not asymptotically flat and \( Q \) diverges as \( r \to \infty \). This could be overcome by choosing \( c \) in \( \text{81} \) to be larger than any length scale under consideration. Perhaps a better alternative is to relax the requirement that
the dominant energy condition is satisfied and choose solution 70, doing this $Q$ is well-behaved. After some calculation the Eddington-Robertson parameters for the spacetime with conformal factor 80 and with constant $c = 0$ are found to be $\alpha = \gamma = 1$, and $\beta = 1 - \frac{a^2}{6}$. $\beta$ is unity in both Schwarzschild and for the solution 72, the smaller $\beta$ has the interpretation that the centralfugal force is larger, thus the solution acts as though it has more mass than it does. This observation suggests that non-metric theories and conformal factors might have application to the "missing mass" problem. In passing it is worth noting that Bekenstein's (1974) trick for producing massless conformal invariant scalar field solutions from massless scalar solutions, introduces hyperbolic conformal factors similar to the above. Using this the whole of the above discussion could be carried out for conformal scalar fields in a similar manner.

From Roberts (1991c) equation A3.5 the ADM mass is

$$A.\, t^{(S)} = \lim_{r \to \infty} \exp(-Q) [A.\, t^{(E)} - \frac{\partial Q}{\partial R^2} \delta^E_{\theta \theta}],$$

$$= \mu - \lim_{r \to \infty} Q'.$$  \hspace{1cm} (84)

From 77, 78 and 79

$$Q' = \frac{k}{r}, \hspace{1cm} (\epsilon = 1),$$

$$Q' = \frac{k}{r^2} \frac{1}{1 + \left(1 - \epsilon\right) \frac{k}{r}}, \hspace{1cm} (\epsilon \neq 1),$$

$$Q' \approx \frac{2\sigma}{\sqrt{3r}} (c - \frac{\sigma}{\sqrt{3r}}),$$  \hspace{1cm} (85)

respectively and it is apparent that in the limit $r \to \infty$, $Q'$ vanishes. Therefore the ADM mass of solutions 77 and 78 is still $\mu$. For all the rescaled asymptotically flat solutions found the ADM mass remains unaltered after rescaling.

6 Linearized Waves in DeSitter Spacetime.

In this section it is shown that the traceless part of the linearized metric perturbations of DeSitter spacetime obey the Fierz-Pauli equation, with
mass $m^2 = \frac{2}{3}\Lambda$. This implies that weak gravitational waves travel slower-than-light in DeSitter spacetime, and are super luminal in Anti-DeSitter spacetime.

Start with a weak gravitational field in the form 66. If the cross terms in $h_{ab}$ are taken to be negligible so that the vacuum Einstein equations reduce to

$$\Box^2 h_{ab} = 0,$$  \hspace{1cm} (86)

as there is no mass term in this equation weak gravitational waves in Minkowski spacetime travel at the speed of light. Now a similar analysis in DeSitter spacetime leads to the Fierz-Pauli (1939) \[3\] equation 15 with $m$ real and weak gravitational waves traveling slower-than-light in DeSitter (1917) \[22\] spacetime, and with $m$ imaginary and weak gravitational waves travelling faster-than-light in Anti-DeSitter spacetime.

DeSitter and Anti-DeSitter spacetime have Riemann tensor

$$R_{abcd} = \frac{\Lambda}{3}(g_{ad}g_{bc} - g_{ac}g_{bd}).$$ \hspace{1cm} (87)

where $\Lambda > 0$ for DeSitter spacetime and $\Lambda < 0$ for Anti-DeSitter spacetime. Taking DeSitter and Anti-DeSitter spacetimes to be the background so that equation 28 applies becomes

$$R_{ab} = \bar{R}_{ab} + \frac{\Lambda}{3}(4h_{ab} - \bar{g}_{ab}h) - \frac{1}{2}\Box^2 h_{ab}. \hspace{1cm} (88)$$

Taking

$$R_{ab} = \Lambda g_{ab}, \hspace{1cm} \bar{R}_{ab} = \bar{\Lambda}\bar{g}_{ab}, \hspace{1cm} (89)$$

equation 88 becomes

$$(\Lambda - \bar{\Lambda})g_{ab} = \frac{\Lambda}{3}(h_{ab} - \bar{g}_{ab}h) - \frac{1}{2}\Box^2 h_{ab}. \hspace{1cm} (90)$$

Assuming that differences in the perturbed value of the cosmological constant to that of the background value is sufficiently weak so that $\bar{\Lambda} = \Lambda$ equation 90 becomes

$$0 = \frac{\Lambda}{3}(h_{ab} - \bar{g}_{ab}h) - \frac{1}{2}\Box^2 h_{ab}. \hspace{1cm} (91)$$

The trace of this equation is

$$(\Box^2 + 2\Lambda)h = 0. \hspace{1cm} (92)$$
Defining $\psi_{ab}$ as the traceless part of $h_{ab}$:

$$\psi_{ab} \equiv h_{ab} - \frac{1}{4}\bar{g}_{ab}h,$$  \hfill (93)

from 92 and 93 $\psi$ obeys the Fierz-Pauli equation 15

$$(\Box^2 - \frac{2}{3}\Lambda)\psi_{ab} = 0,$$  \hfill (94)

with mass

$$m^2 = \frac{2}{3}\Lambda.$$  \hfill (95)

### 7 Solar System Bounds on Non-metric Mass.

In this section the Eddington-Robertson parameters of theories involving the non-metric mass are discussed. The Eddington-Robertson parameters are what are measured in solar system tests of gravitational theories, see for example Weinberg (1972) [47], in general relativity they are unity. In order to calculate the Eddington-Robertson parameters for the simple theory it is necessary to know how the spherically symmetric solution for a massive Klein-Gordon scalar field 13; such a solution is not known, therefore no information on the Eddington-Robertson parameters of the simple theory can be given. For the rescaled theory the trace equation 57 gives the non-metric mass dependent on the trace of the stress; because the trace of the stress vanishes, the solar system exterior to the sun will be modeled by the Schwarzschild solution as in general relativity. In this section it is shown, using only the field equations 39 and the non-metric mass equation 48 (and not the trace equation 57), that the non-metric mass can have any value while the Eddington-Robertson parameters remain unity. The Schwarzschild solution in isotropic coordinates is

\[
\begin{align*}
ds^2 &= - \left(1 - \frac{M}{2r}\right)^2 dt^2 \\
&\quad + \left(1 + \frac{M}{2r}\right) d\sigma^2, \hfill (96)
\end{align*}
\]

where

\[
\begin{align*}
\quad d\sigma^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \hfill (97)
\end{align*}
\]
To introduce the effects of non-metric mass choose a conformal factor

\[ \Omega^2 = \exp(\alpha \bar{r}^n), \] (98)

expanding the exponential gives

\[
\begin{align*}
    ds^2 &= -(1 + \alpha \bar{r}^n + \frac{1}{2} \alpha^2 \bar{r}^{2n} - 2 \frac{M}{\bar{r}} - 2Ma\bar{r}^{n-1} + 2\frac{M^2}{\bar{r}})dt^2 \\
    &\quad + (1 - 2\frac{M}{\bar{r}} + a\bar{r} + aM\bar{r}^{n-1})d\sigma^2.
\end{align*}
\] (99)

Now if \( n \geq -3 \) the Eddington-Robertson parameters are all unity and the solution is indistinguishable from the Schwarzschild solution. From (48) and (98) the non-metric mass is

\[
\begin{align*}
    M &= -\bar{\nabla}_a Q^a - \epsilon Q_a Q^a \\
    &= -\alpha \bar{r}^{n-2}(1 + \frac{M}{2\bar{r}})^{-5}(1 - \frac{M}{2\bar{r}})^{-1}(1 - n + \frac{M(n - 3)}{4\bar{r}}) \\
    &\quad - \epsilon a^2 n^2 \bar{r}^{2n-1}(1 + \frac{M}{2\bar{r}}).
\end{align*}
\] (100)

For \( \bar{r} \geq M \)

\[
    M \approx -\alpha n \bar{r}^{n-2}(1 - n + \epsilon an \bar{r}^n).
\] (101)

Now for \( n = -3 \) and at \( r \) a fixed value of \( r \) the non-metric mass is

\[
    M = 3\alpha \left( 4 - \frac{3\alpha \epsilon}{\bar{r}^2} \right),
\] (102)

the constant \( \alpha \) is arbitrary and hence the non-metric mass can be made as large as required by choosing a suitable value of \( \alpha \).

8 The Initial Value Problem for the Rescaled Theory

In this section the initial value problem is discussed following the treatment in Alder et al (1965) for general relativity. The conventions used here are Latin indices \( a, b, \ldots \) take the values 0 to 3 and Greek indices \( \alpha, \beta, \ldots \)
take values 1 to 3, which is the opposite convention from Alder et al (1965) [55].

Considerations are restricted to the rescaled theory with a massive scalar field as source. The field equations [39], the non-metric equation [48], and the trace equation [57] become

\[ R_{ab} = 2\psi_a \psi_b + g_{ab} \mu^2 \psi^2, \]  
\[ M = -\tilde{\nabla}_a Q^a - \epsilon Q_a Q^a, \] \[ M = -\alpha (\psi_a \psi^a + 2\mu^2 \psi^2), \]

respectively. Initial data is prescribed on a surface \( S \) described by the equation \( x^0 = 0 \). For the metric initial data

\[ g_{ab}, \quad g_{ab,0}, \]  
are given; the derivatives of the metric interior to the surface \( g_{ab,i} \) can be calculated by differentiation in the surface, but \( g_{ab,00} \) has to be determined.

A first choice of additional initial data is

\[ Q, \quad M, \quad \psi, \]  
the derivatives of the object of non-metricity interior to the surface \( Q \) are given by differentiation in the surface. The value of \( M \) is given by the initial data, then \( \psi_i \) can be calculated by differentiating in the surface, and \( \psi_0 \) determined by equation [105]. From [1] and [3] \( \bar{R}_{ab} = \bar{R}_{ab} + P_{ab} \) where \( P_{ab} \) is given by [27]. \( P_{ab} \) is given in terms of \( Q \) and its derivatives which can be determined by the procedure above. Thus equation [103] becomes

\[ \bar{R}_{ab} = \frac{1}{2} g^{cd} (\epsilon g_{ad,bc} - \epsilon g_{bc,ad} + g_{ab,cd} + g_{cd,ab}) + A_{ab}. \]

Now \( A_{ab} \) consists of known data of the metric \( g_{ab} \) and its first derivatives, the object of non-metricity \( Q \) and its derivatives, the scalar field and its derivatives. Hence the initial value problem has been reduced to the same form as that for general relativity as discussed in Alder et al (1965) [55], and thus equations [103], [104], and [105] give a fully determined initial value problem.

A second choice of additional data

\[ Q, \quad Q_0, \quad M, \quad \psi. \]
in the rescaled theory the object of non-metricity is a gradient vector, because the components of the object of non-metricity $Q_0$ can be calculated by differentiating $Q$ in the surface the initial data $Q$ is equivalent to the initial data $Q$ and $Q_0$. Thus this \[109\] set of additional data is equivalent to the first set \[107\].

A third choice of initial data is

$$Q_a, \psi, \psi_0. \quad (110)$$

Then equation \[105\] gives $M$ immediately, next equation \[104\] gives $Q_{00}$ and then the problem reduces to that of general relativity as above.

A fourth choice of initial data is

$$Q_a, M, \psi, \psi_0. \quad (111)$$

and with this choice it is not necessary to introduce the trace equation in the form \[105\].

The shock waves or characteristics of the metric $g_{ab}$ and the scalar field are null, this follows in a straightforward manner using the standard approach Synge (1960) \[21\] V.7. In the simple and rescaled theory $Q_a$ is a gradient vector and the standard approach works; if the object of non-metricity $Q_a$ is not a gradient vector, but simply a vector obeying a first order equation \[24\] then a shock solution has to be redefined as a solution that does not give a unique value for $Q_{a,0}$, again however the shock solutions are null.

9 The Non-recoverability of the Fierz-Pauli Equation.

The dynamics of both the simple and the rescaled theories have the unusual feature of position dependent mass. This feature is not unique to the theories discussed here, it also occurs in conformally invariant mass theories of Page (1936) \[50\], Fulton et al (1962) \[51\], and Barut and Haugen (1972) \[38\], and in the fluid symmetry breaking theories of the author, Roberts (1989) \[53\]. Except for spins $0, \frac{1}{2},$ and $1$ fields the presence of a constant mass term and covariance seem to be exclusive, thus to have higher spin fields with constant mass covariance must be broken in some way.

The idea of covariance breaking is best illustrated by considering weak field gravity. Consider the weak field equations of general relativity in the
harmonic gauge

$$\Box^2 h_{ab} = 2\kappa(T_{ab} - \frac{1}{2}\eta_{ab} T),$$  \hspace{1cm} (112)

and the Fierz-Pauli (1939) \cite{3} equation \cite{15} for a massive spin 2 field. Now the weak field equations \cite{112} are covariant, the Fierz-Pauli equation \cite{15} is not, and furthermore it cannot be derived from a covariant Lagrangian, see for example Freund et al (1969) \cite{32} or Datta and Rana (1979) \cite{33}. To produce the Fierz-Pauli equation from the weak field equations it is necessary to equate $2\kappa(T_{ab} - \frac{1}{2}\eta_{ab} T)$ to give $m^2h_{ab}$. This is here called “breaking covariance”. There appears to be no neat way of doing this for stresses such as those of a perfect fluid or a scalar field; in this case covariance breaking appears to be unnatural. Observations show that the Fierz-Pauli equation is incorrect no matter how small $m^2$ in \cite{15} is, see for example Datta and Rana (1979) \cite{33}. The solar system, exterior to the sun, is not a vacuum and there must be a small stress present (see for example Roberts (1998) \cite{52}): thus in this case the observational evidence shows that this small stress does not reduce to (or covariantly break into) $m^2h_{ab}$ term of the Fierz-Pauli equation \cite{15}.

In the fluid symmetry breaking theory of Roberts (1989) \cite{53} the mass of a vector field $X$ is given by

$$X = -a \left(-1 + \sqrt{1 - \frac{1}{3b\mu_1}}\right),$$  \hspace{1cm} (113)

where $a$ and $b$ are positive real constants, and $\mu_1$ is a fluid density. Now the fluid density is position dependent, hence the vector fields mass is also position dependent. This is simply overcome by assuming that there is no appreciable density fluctuation in the region under consideration. In this case there is no covariance breaking because the stress with a constant fluid density remains covariant.

The above examples motivate the question: Is it possible to recover the Fierz-Pauli equation from the non-metric theories discussed here? Consider just the $\epsilon = 0$ rescaled theory, the other theories being similar, re-arranging the dynamical equations gives

$$2\Box^2 h_{ab} \approx \tilde{R}_{ab} = \tilde{\nabla}_a Q_b + Q_a Q_b + \kappa T_{ab} + \frac{1}{2}g_{ab} \left(M(1 - 2\frac{\kappa}{\alpha}) - Q_a Q^a\right).$$  \hspace{1cm} (114)

To reduce this equation to the Fierz-Pauli equation \cite{15} it is necessary to equate the right hand side to $m^2h_{ab}$ and the same problems that occur doing
this for weak field general relativity re-occur. Thus it must be concluded that there is no natural way of breaking general covariance in order to arrive at the Fierz-Pauli equation \[^{[3]}\]. In other words non-metric mass theory and massive spin-2 linearized theory are entirely distinct. The Fierz-Pauli equation \[^{[15]}\] might be derivable from the infrared properties of other theories, see Frondsal and Heidenreich (1992) \[^{[56]}\].

### 10 The Cosmological Constant.

In this section the extent to which it is possible to re-interpreted the cosmological constant as due to non-metric mass is investigated. The cosmological constant is predicted to be many orders of magnitude larger than the upper bound on its size from cosmological considerations Zel’dovich (1968) \[^{[57]}\]. The predictions of an anomalous cosmological constant comes from symmetry breaking in grand unified theories. The presence of a cosmological constant cannot explain orbital irregularities in the outer solar system, Roberts (1987) \[^{[28]}\]. Here first attempts are made to absorb the cosmological constant in four dimensions and re-interpreted it as non-metric mass. Secondly attention is turned to higher dimensional theories (see for example Overduin and Wesson (1997) \[^{[36]}\]), where the standard Freund-Rubin \[^{[35]}\] compactification is generalized.

DeSitter spacetime can be expressed in conformally flat form \[^{[36]}\]. The idea is to absorb the cosmological constant $\Lambda$ by choosing a conformal factor $\Omega^2 = \exp(-Q)$ so that the rescaled space has a flat metric and non-vanishing object of non-metricity $Q$; then $Q$ is used to generate non-metric mass via equation \[^{[48]}\].

DeSitter spacetime can be put in the explicitly static form and rescaled by $\exp(-Q)$ to give

$$ds^2 = -\exp(-Q) \left(1 - \frac{\Lambda r}{3}\right) dt^2 + \exp(-Q) \left(1 - \frac{\Lambda r}{3}\right)^{-1} dr^2 + \exp(-Q) r^2 d\Sigma^2. \tag{115}$$

Defining the luminosity coordinate

$$R^2 = \exp(-Q) r^2, \tag{116}$$

gives

$$ds^2 = -\exp(-Q) \left(1 - \frac{\Lambda R}{3}\right) dt^2 + \left(1 + \frac{1}{2} R Q' \right) (1 - \Lambda R \exp(Q)/3)^{-1} dR^2 + R^2 d\Sigma^2, \tag{117}$$
where \( ' \) is \( \frac{\partial}{\partial r} \). To produce the null form define

\[
R^* = \int (1 + \frac{1}{2} R Q'(1 - \frac{1}{3} \Lambda R \exp(Q))) \exp(\frac{1}{2} Q) dR^2
\]

\[
= \int (R \exp(\frac{1}{2} Q') (1 - \frac{1}{3} \Lambda R \exp(Q))) dR,
\]

(118)

note that \( R^* \) is given by simple arc hyperbolic and trigonometrical functions depending on the sign of the cosmological constant and the value of \( R \). Now define \( v \equiv t + R^*, w \equiv t - R^* \) to give

\[
ds^2 = - \left( \exp(\frac{-Q}{3}) - \frac{1}{3} \Lambda R \right) dv dw + R^2 (d\theta^2 + \sin(\theta) d\phi^2).
\]

(119)

Choosing \( \exp(-Q) = 1 + \frac{1}{3} \Lambda R \) gives a spacetime in which the metric is flat; however there is curvature present, the Riemann curvature invariant \( \hat{R}_{abcd} \hat{R}^{abcd} \) is unaltered by the coordinate transformations. With the choice \( \exp(-Q) = 1 \) the non-zero curvature is in \( \hat{R}_{abcd} \hat{R}^{abcd} \), the different choice of \( \exp(-Q) \) essentially "shifts" it to \( P_{abcd} P^{abcd} \). The non-metric mass equation [18] becomes

\[
0 = M + \frac{2}{r} Q' \exp(Q) \left( 1 - \frac{2}{3} \Lambda r \right) + Q'' \exp(Q) \left( 1 - \frac{1}{3} \Lambda r \right),
\]

(120)

where \( ' \) is now \( \frac{\partial}{\partial r} \). What we would like to happen is that substituting \( \exp(-Q) = 1 + \frac{1}{3} \Lambda R \) into [120] we would get an equation for \( M \) in terms of \( \Lambda \). If this happened we would be able to say that theories with large cosmological constant are entirely equivalent to massive theories of gravitation with mass of the order of the Planck mass. Unfortunately this does not happen. Substituting gives

\[
0 = M - 2\Lambda + \frac{2}{3} \Lambda^2 r^2,
\]

(121)

and we have an \( r \) dependent mass. The requirements of both a flat metric and a space dependent quantity are hard to reconcile. Ignoring the \( r \) dependent terms suggests that non-metric mass is of the same order as the cosmological constant.

An alternative approach is to first find solutions of the wave equation [120] with constant \( M \) and then ask what is the resulting structure of spacetime; however the non-metric mass equation [120] has so far proved intractable. We could ask whether we get solutions if we require the spacetime to be Schwarzschild spacetime. This would have the interpretation
that the cosmological constant is equivalent to massive gravitation with the
Schwarzschild solution around each particle; however requiring \(\exp(-Q) - \frac{1}{3} \Lambda R = 1 - \frac{2M}{R}\) produces complex equations which again appear to be intractable.

Now the role of non-metric gravitation in higher dimensional theories
is considered, in particular its is investigated by generalizing the Freund-
Rubin (1980) [35] compactification c.f. Roberts (1991a) [58]. The bosonic
part of the Cremmer-Julia-Scherk equations for \(d = 11\) supergravity are

\[
R_{MN} = (F_{MPQR}F_{N...}^{PQR} - \frac{1}{12} g_{MN} F^2),
\]

(122)

\[
\partial_M(\sqrt{g^{(11)}} F^{MN_1N_2N_3}) = \kappa \epsilon^{P_1P_2P_3P_4Q_1Q_2Q_3} N_1N_2N_3 F_{P_1P_2P_3P_4FQ_1Q_2Q_3},
\]

(123)

where

\[
\kappa_1 = \frac{1}{48} \pi G,
\]

\[
a, b, \ldots = 1, 2, 3, 4,
\]

\[
M, N, \ldots = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,
\]

\[
i, j, \ldots = 5, 6, 7, 8, 9, 10, 11,
\]

\[
g^{(11)} = \det g_{MN},
\]

\[
g^7 = \det g_{ij}.
\]

(124)

Compactification is considered onto

\[
d\bar{s}^2 = d\Sigma^2_{4-} + Z(x^a) d\Sigma^2_7,
\]

(125)

where \(d\Sigma^2_{4-}\) is the Anti-DeSitter metric, \(d\Sigma^2_7\) is the maximally symmetric seven dimensional metric and \(Z(x^a)\) is a function of \(x^a\), it is the addition of \(Z(x^a)\) that generalizes the Freund-Rubin compactification, \(Z = 1\) gives the
Freund-Rubin solution. As in the Freund-Rubin compactification only one component of \(F^{MNPO}\) is considered, namely \(F_{abcd}\). Integrating the second equation of (123) gives

\[
F_{abcd} = f(\neq x^a) \epsilon_{abcd}(g^{(11)})^{-\frac{1}{2}},
\]

(126)

absorbing \(g^{(4)}\) into \(f(\neq x^a)\) gives

\[
F_{abcd} F^{abcd} = f^2(\neq x^a) \epsilon_{abcd} \epsilon_{abcd} |g^{(11)}|^{-1}
\]
The equation of (122) now becomes
\[ R_{ab} = -\frac{1}{6}\kappa_1 f^2 g_{ab}, \quad R_{ij} = +\frac{1}{12}\kappa_1 f^2 g_{ij}. \] (128)

The first equation of (128) has as a solution Anti-DeSitter spacetime with cosmological constant \( \Lambda = -\frac{1}{2}\kappa_1 f^2 \) dependent on the internal space.

For the internal space to be small it is necessary to have \( f^2 \) large and thus a large cosmological constant. There are two methods of reducing the size of the cosmological constant. The first is to rescale the equation (126) by a factor of \( \exp(-Q) \) and assume that the resulting small cosmological constant \( \delta^2 \) is independent of the internal indices and defined by
\[ \delta^2 \equiv f^2 \exp(-Q) \] (129)

Substituting this equation into the 11 dimensional non-metric massive wave equation (48) gives
\[ M(x^i) = -\frac{2}{f} f^i, -\frac{4}{f} (\epsilon + 3) f^i f_i, \] (130)

and thus the large cosmological constant \( -\frac{1}{6}\kappa_1 f^2 \) has been replaced by a small cosmological constant \( -\frac{1}{6}\kappa_1 \delta^2 \) with the introduction of the mass (131). The alternative approach is to note that Köttler’s solution (the Schwarzschild solution with cosmological constant) is also a solution of the equation (126) and then again attempt to absorb the cosmological constant by rescaling the line element.

11 Cosmology and Massive Gravitation.

The simple theory equations are related to Hoyles’ (1948) \( C \) field equations which are used in steady state cosmology. Defining
\[ \frac{1}{2} p = -Q c Q^c c - V - 3H^2, \] (131)
and
\[ \frac{1}{2} \mu = -Q c Q^c c + V + 3H^2, \] (132)
the simple field theory field equations (53) become

$$G_{ab} + C_{ab} = \kappa T,$$

(133)

where

$$C_{ab} = -(p + \mu)U_a U_b - (p + 3H^2)g_{ab},$$

(134)

is Hoyles’ \( C \) field, \( U_a \) is a time-like vector, \( H \) is the Hubble constant, \( p \) is a pressure, and \( \mu \) is a density. Hoyle requires that

$$C_{ab} = \nabla_a \nabla_b C,$$

(135)

and this is an unusual constraint if applied directly to the equations (134).

For the rescaled theory note that from the form of the Roberts on-Walker line element

$$ds^2 = -dt^2 + R^2(t) \left( (1 - kr^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right),$$

(136)

is unaltered if the metric is conformally rescaled. For example if the line element is multiplied by a conformal factor \( \exp(-Q) \) and then \( dT = \exp(-\frac{1}{2}Q) \) and \( \bar{R} = \exp(-Q)R \) are defined, the metric remains of the same form. Thus massive gravitation does not alter cosmology as the Hubble constant becomes defined in terms of \( \bar{R} \) instead of \( R \) etc... Non-metricity has been related to inflation in the work of Stelmach (1991) \[59\] and Poberii (1994b) \[60\].

12 Conclusion.

In section 2 the approach to massive gravitation advocated here is motivated by analogy with the massive wave equations of other field theories. The resulting theory appears to have nothing in common with the Fierz-Pauli equation because: firstly it is fully covariant, secondly it is based on the Weyl geometry, and thirdly it does not conflict with observations: the Fierz-Pauli equation \[53\] has none of these properties. It was shown in section 9 that there appears to be no natural way of breaking the covariance of the non-metric massive theory to produce the Fierz-Pauli equation.

A brief description of semi-metric or Weyl geometries was given in section 2. The properties of conformal rescaling in this geometry was explained. Four different equations can be produced by analogy with other massive field theories, and it was shown that all four can be expressed by equation \[33\].

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The mass in this equation is position dependent, as is the mass in the theories of Page (1939) [50], Fulton et al (1962) [51], Barut and Haugen (1972) [38], and Roberts (1989) [53]; in the first three of these the mass is conformally invariant. Two theories which involve the non-metric mass equation were suggested. The first is the simple theory, which is given by Hilbert’s Lagrangian $L_H$ and by the Lagrangian $L_Q$ [51] which has similar form to the Klein-Gordon scalar Lagrangian; this theory does not have the rescaling property. The second is the non-metric massive theory of gravitation rescaled from general relativity. It was shown in section 7 that with the dynamics given by the field equations [51] and the non-metric equation [48] while the non-metric mass can have any value. However it is argued in sections 4 and 8 that the dynamics are best defined when there is an additional equation, called the trace equation [57], which equates the non-metric mass to the trace of the stress tensor. When the trace of the stress tensor vanishes the non-metric mass vanishes, thus in a vacuum the field equations are the same as for general relativity. A consequence of this is that if the spacetime exterior to the sun is taken to be a vacuum, then it is still modeled by the Schwarzschild solution.

The major problem with the rescaled theory is to produce a consistent set of equations to govern the dynamics. This problem arises because the non-metric equation [48] can only be derived from a Lagrangian when $\epsilon = 0$ and the mass term takes the form [49] - these restrictions are incompatible with the rescaling property. The field equations [33] and the non-metric equation [48] consist of 11 geometrical objects, 11 physical quantities, and 11 equations and thus viewed as a linear problem these equations are sufficient, however the problem is not linear and in section 4 additional equations are discussed. In most field theories, for example Maxwell and Proca theories, the massive theory produces a trace for the stress tensor and the massless does not: because of this one of the additional equations [57] called the trace equation, which relates the mass of gravitation to the trace of the stress, appears to be physically well motivated. The initial value problem set up in section 8 is most elegantly formulated with the addition of the trace equation.

A difficulty (with the initial assumption that the non-metric equation should be of the same form as wave equations in other massive theories) is that in other massive theories this is in part motivated by both wave-like solutions to the wave equation, and rates of decay which are faster than in the corresponding massless case: it is not immediately clear that the non-metric equation [48] has the same consequences. In section 4 it was remarked that the trace equation would seem to imply that for a perfect fluid with
$3p > \mu$ the metric potentials propagate faster-than-light. In section 8 it was mentioned that the shock wave solutions for all the fields are null, but even the Klein-Gordon equation in flat space has null shock solutions. A linearization scheme for the rescaled theory with a massive scalar field, in which the object of non-metricity, the mass, and the massive scalar field all behave as non-null waves was given in section 5, but to produce an explicit form of the weak field metric proved impossible as the equations involved are intractable. In section 5 the rates of decay of the fields were investigated and the rescaled scalar-Einstein spacetime was found to provide a completely tractable model, which illustrated that the fields have the desired properties. Intuitively it would be expected that the massive theory would produce a greater inward force than the massless theory of general relativity, and this might have bearing on the problem of ”missing mass” in galactic dynamics. The scalar-Einstein spacetime has all the Eddington-Robertson parameters equal to unity; however after rescaling the spacetime has the $\beta$ Eddington-Robertson parameter less than unity. Using the weak field limit it can be shown that this implies that the inward force is greater in the massive case. Thus non-metric massive gravitation allows a qualitative explanation of the missing mass in galactic dynamics.

In section 10 it was shown that it is possible to interpret the theoretically very high cosmological constant produced in spontaneous compactification as due to gravitational mass. To do this it was necessary to generalize Freund-Rubin [33] compactification. In section 11 the cosmological consequences of the theory were discussed. For the simple theory, cosmology can be made in a form similar to Hoyle’s $C$ field cosmology Hoyle (1948) [54] and Weinberg (1972) [47]. For the rescaled theory, because of the conformal invariance of the Robertson-Walker line element, the cosmological theory is the same as that for general relativity.

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