Finding uniformly most reliable graphs by counting trivial cuts

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There is a vast literature focused on network reliability evaluation. In the last decades, reliability optimization has been also addressed. Frank Boesch in 1986 introduced the concept of uniformly most reliable graph (UMRG). Later, Boesch et al. presented the first UMRGs and conjectured that some special subdivisions of the bipartite complete graph $K_{3,3}$, as well as the bipartite complete graph $K_{4,4}$, are UMRGs. Wang proved that the first conjecture is true. Wendy Myrvold confirmed that $K_{4,4}$ is also UMRG, by means of computational tests. However, thus far, there is no mathematical proof in the literature. A trivial cut is an edge-set that includes all the incident edges of a fixed node. In this article we describe a methodology to determine UMRGs based on bounding the number of trivial cuts. As a proof-of-concept it is proved that both $K_{3,3}$ and $K_{4,4}$ are UMRGs.

KEYWORDS
Graph Theory, Network Reliability, Uniformly Most Reliable Graphs.

1 | MOTIVATION

Frank Boesch, in a seminal work [2], introduced the concept of uniformly most reliable graph, or UMRG for short. He posed several conjectures, most of them are still awaiting for a resolution [14]. The interested reader can find a practical discussion in the recent survey [4]. In a foundational work, Boesch et al. provided the first nontrivial UMRGs [3]. The authors claimed without proof that the bipartite complete graph $K_{4,4}$ is UMRG. Wendy Myrvold [11] using a computational analysis found a list of all the UMRGs with 8 nodes or fewer. The list included the graph $K_{4,4}$. Nevertheless,
Throughout the document, all graphs are assumed to be simple. Consider a graph \( G = (V, E) \) whose nodes do not fail but its edges fail independently with identical probability \( \rho \). The all-terminal reliability \( R_G(\rho) \) is the probability that the resulting random graph remains connected. For convenience, we work with the unreliability \( U_G(\rho) = 1 - R_G(\rho) \).

Following the terminology from [8], we denote \( k \), the number of all the cutsets with cardinality \( k \). By the sum-rule, the unreliability polynomial can be expressed as follows:

\[
U_G(\rho) = \sum_{k=0}^{m} m_k(G)\rho^k (1 - \rho)^{m-k}. \tag{1}
\]

An \((n, m)\)-graph is a graph on \( n \) nodes and \( m \) edges. Observe that, for each pair of positive integers \( n \) and \( m \) such that \( n - 1 \leq m \leq \binom{n}{2} \), the collection of all \((n, m)\)-graphs is finite. Therefore, if we consider a fixed \( \rho \in [0, 1] \) then there exists at least one graph \( G \) that achieves the minimum unreliability, i.e., \( U_G(\rho) \leq U_H(\rho) \) for all \((n, m)\)-graphs \( H \). Further, if the previous condition holds for all \( \rho \in [0, 1] \) and all \((n, m)\)-graphs \( H \), then \( G \) is a UMRG.

The following graph-theoretic terminology will be considered. The edge connectivity \( \lambda(G) \) is the smallest \( \lambda \) such that \( m_\lambda > 0 \). A trivial cut is a cutset that includes all the incident edges of a fixed node. The degree \( d_v \) of a node \( v \) in \( V \) is the number of edges that are incident to \( v \). A graph is regular if all its nodes have identical degrees. The minimum degree of a graph \( G \) is denoted by \( \delta(G) \). A graph is super-\( \lambda \), or superconnected, if it is \( \lambda \)-regular and further, it has only trivial cutsets: \( m_\lambda = n \). In a connected graph \( G \), a bridge is a single edge \( uv \) such that \( G - uv \) is not connected. A cut-point is a node \( v \) such that \( G - \{v\} \) has more connected components than \( G \). A graph \( G \) with more than two nodes is biconnected if it is connected and it has no cut-points. A tree is an acyclic connected graph and the number of spanning trees of \( G \) is its tree-number, denoted by \( t(G) \). A matching is a set of nonadjacent edges. A perfect matching is a matching that is incident to all the nodes of a graph. The \( n \)-cycle and the \( n \)-complete graphs are denoted \( C_n \) and \( K_n \), respectively. The graphs \( C_3 \) and \( C_4 \) are called the triangle and the square respectively. The girth of a graph \( G \) is denoted by \( g(G) \), and it is the number of vertices in the smallest cycle belonging to \( G \). In the bipartite complete graph \( K_{n_1, n_2} \) the node-set \( V \) is partitioned into two parts \( A \) and \( B \) such that \( |A| = n_1 \), \( |B| = n_2 \), and the edge-set is precisely \( A \times B \). A multipartite complete graph \( K_{n_1, \ldots, n_r} \) is defined analogously, where the node-set is \( V \) is partitioned into \( r \) parts \( V_1, \ldots, V_r \) such that all the nodes belonging to \( V_i \) is joined to all the nodes belonging to \( V_j \), for all the pairs \( i \) and \( j \) such that \( j \neq i \).
3 | RELATED WORK

If $k$ is an integer such that $k \in \{0, \ldots, m\}$, then an $(n, m)$-graph $G$ is $\min_{m}$ if $m_{k}(G) \leq m_{k}(H)$ for all the $(n, m)$-graphs $H$. Furthermore, $G$ is stronger than $H$ if $m_{k}(G) \leq m_{k}(H)$ for all $k \in \{0, \ldots, m\}$. A graph $G$ is the strongest in its class if it is stronger than all the graphs in its class. From Expression 1, a strongest graph is UMRG. This sufficient criterion for a graph to become UMRG is widely adopted in the literature. Necessary conditions are available as well:

Proposition 1 (5) Consider two graphs $G$ and $H$ on $n$ vertices and $m$ edges. Then, the following assertions hold.

(i) If there exists $k \in \{0, \ldots, m\}$ such that $m_{i}(H) = m_{i}(G)$ for all $i < k$ but $m_{k}(H) < m_{k}(G)$, then there exists $\rho_{0} > 0$ such that $U_{H}(\rho) < U_{G}(\rho)$ for all $\rho \in (0, \rho_{0})$.

(ii) If there exists $k \in \{0, \ldots, m\}$ such that $m_{i}(H) = m_{i}(G)$ for all $i > k$ but $m_{k}(H) < m_{k}(G)$, then there exists $\rho_{1} < 1$ such that $U_{H}(\rho) < U_{G}(\rho)$ for all $\rho \in (\rho_{1}, 1)$.

Harary [9] constructed graphs that achieve the maximum edge connectivity $\lambda = \lceil 2m/n \rceil$ among the class of all $(n, m)$-graphs. By definition, $m_{i}(G) = 0$ for all $i < \lambda$. Then, by Proposition 1 if $G$ is UMRG then the number of cutsets $m_{\lambda}(G)$ must be minimum among the class of all $(n, m)$-graphs. On the other hand, $m_{i}(G) = \binom{m}{i}$ for all $i > m - n + 1$, since trees are minimally connected with $n - 1$ edges. The number of spanning subgraphs with $m - n + 1$ edges is precisely the tree-number $t(G)$, so $m_{m-n+1}(G) = \binom{m}{m-n+1} - t(G)$. By Proposition 1 the maximization of the tree-number is a necessary condition for a graph to become UMRG in its class. Prior observations directly link this network design problem with distinguished graph invariants:

Corollary 1 (Necessary Criterion) If $G$ is a UMRG then it must have the maximum tree-number $t(G)$, the maximum edge connectivity $\lambda(G)$, and the minimum number of cutsets $m_{\lambda}(G)$, among all the $(n, m)$-graphs.

For convenience we say that an $(n, m)$-graph $G$ is $t$-optimal if $t(G) \geq t(H)$ for every $(n, m)$ graph $G$. Briefly, Corollary 1 asserts that any UMRG must be $t$-optimal and max-$\lambda$ min-$m_{\lambda}$.

The following theorems will be useful for our purpose:

Theorem 1 (7) All regular complete multipartite graphs are $t$-optimal.

Theorem 2 (16) If $H$ is any $(n, m)$-graph with $m \geq n$, then there exists some stronger $(n, m)$-graph $G$ that is biconnected.

A graph constellation with an updated set of the UMRGs found thus far is presented in Figure 1. The graphical representation has the corresponding graph for every pair $(n, m)$ of nodes and edges, whenever a UMRG exists. The pairs where UMRGs do not exist are marked with red circles [5, 12].

The family of sparse $(n, n-i)$ graphs are straight lines with unit slope. The reader can find trees, $n$-cycles, balanced $\theta$-graphs, and some specific subdivisions of $K_{4}$, see [3]. The green squares represent dense graphs. Observe that 3-regular graphs can be found in the straight line with slope 3/2. These graphs include $K_{4}$, Wagner [15], Petersen [13] and Yutsis [6]. Ath and Sobel [1] conjectured that special subdivisions of Wagner, Petersen, Yutsis, Heawood and Cantor-Möbius are UMRGs. It is still an open problem to determine even if Heawood and Cantor-Möbius are UMRGs. Thus far, the only 4-regular graphs include $K_{5}$, $C_{3} \cup C_{4}$, and $K_{4,4}$ (see Theorem 3).
FIGURE 1  UMRGs as a function of \((n, m)\)
4  BOUNDING TRIVIAL CUTS

We give two upper-bounds for the coefficients $m_k(G)$ using trivial cuts. These bounds combine the inclusion-exclusion principle with elementary counting. Consider the following terminology:

- For any node-set $A \subseteq V G$, we denote $|A|$ the subgraph induced by $A$ in $G$.
- The cut induced by the set $A$ is $\partial A = \{uv \in E G : u \in A, v \notin A\}$.
- For any set $A$, we denote $A^{(k)} = \{S \subseteq A : |S| = k\}$.
- The node-set composed by all nodes with degree $i$ is $V_i(G) = \{v \in V G : \deg v = i\}$.
- For any fixed node $v$, $\partial v$ denotes the set of all edges incident to $v$, that is, $\partial v = \partial \{v\} = \{uv : uv \in E(G)\}$.
- For any edge $e = uv$, $\partial e$ denotes the set of edges incident to $e$, that is, $\partial e = \partial \{u, \partial v\} = \partial u \cup \partial v$.
- Let $M^k(G)$ be the family of $k$-cutsets, $M^k(G) = \{S \in (EG)^{(k)} : G \setminus S$ disconnected$\}$.
- Clearly, $m_k(G) = |M^k(G)|$.
- When the context is clear, we write $V_i$ and $M^k$ instead of $V_i(G)$ and $M^k(G)$, respectively.
- If $H$ is a connected subgraph of $G$, $M^k_H(G)$ denotes all the $k$-cutsets containing $\partial H = \partial V H$ but no edge belonging to $H$, i.e., $M^k_H(G) = \{S \in M^k(G) : \partial H \subset S \subset E G \setminus E H\}$.
- If $H$ has only one vertex $v$ we write $M^k_v(G)$.
- Similarly, if $H$ has only two vertices $v$ and $w$ we will write $M^k_{vw}(G)$.
- For each $k \in \{1, \ldots, m\}$ and $i \leq k$ define the function $g_k$ as
  \[
g_k(i) = \binom{|G| - i}{k - i},
\]

Lemma 1  If $G$ is a graph and $A$ is a subset of of $V G$ then
  \[|\partial A| = \sum_{v \in A} \deg v - 2||A||.\]

Furthermore, if $G$ is 4-regular with girth $g$ and $S$ is a $k$-cutset such that $\partial A \subset S$ and that $|A| < g$, then $|A| \leq (k - 2)/2$. If $|A| = g$ then $k \geq 2g$.

Proof  The first part of the statement follows from the handshaking lemma and the fact that each edge in $|A|$ counts twice. Now, if $G$ is 4-regular then $|\partial A| = 4|A| - 2||A||$. If $|A| < g$ then $||A|| \leq |A| - 1$, so $|\partial A| \geq 4|A| - 2(|A| - 1)$ and $|A| \leq (|\partial A| - 2)/2$. Since $\partial A \subset S$, then $|\partial A| \leq k$ and the result follows. The proof for the case where $|A| = g$ is analogous.

Lemma 2  The function $g_k(i)$ is decreasing, i.e., $g_k(i) \leq g_k(i - 1)$ whenever $i \leq k$.

Proof  Recall that the identity $\binom{r}{i} = \frac{n - 1}{i - 1} \binom{n - 2}{i - 1}$ holds for any pair of positive integers $n$ and $i$. Then:
  \[g_k(i - 1) = \binom{|G| - i + 1}{k - i + 1} \leq \binom{|G| - i + 1}{k - i + 1} g_k(i).
\]

Proposition 2  Consider a graph $G = (V, E)$.
• If \( v \in VG \) and \( uv \in EG \), then \( \partial v \in M^{dv} \) and \( \partial uv \in M^{dv+dv-1} \). Furthermore,

\[
|M^k_v| = \left( \frac{\|G\| - d_v}{k - d_v} \right) = g_k(d_v), \quad \text{and} \quad |M^k_{uv}| = \left( \frac{\|G\| - d_u - d_v + 1}{k - d_u - d_v + 2} \right).
\] (2)

• Given a subset \( S \) of \( VG \),

\[
\bigcap_{v \in S} M^k_v = g_k(d_{v_1} + \cdots + d_{v_i} - \|S\|) \quad \text{where} \quad S = \{v_1, \ldots, v_i\}.
\] (3)

• In particular, if \( u, v \in VG \) then

\[
|M^k_u \cap M^k_v| = \begin{cases} 
g_k(d_u + d_v) & uv \notin E, 
g_k(d_u + d_v - 1) & uv \in E. 
\end{cases}
\] (4)

**Proof** The set \( M^k_v \) includes precisely \( d_v \) edges incident to the fixed node \( v \) and \( k - d_v \) additional edges. There are \( g_k(d_v) \) ways to choose those edges, and the first equality for \( |M^k_v| \) follows. The set \( M^k_{uv} \) includes precisely \( d_u + d_v - 1 \) edges incident to the fixed edge \( uv \) and \( k - d_u - d_v + 2 \) additional edges since \( uv \) cannot be chosen, thus proving Expression (2). Expression (3) is proved analogously, and (4) is a particular case of (3).

For each graph \( G = (V, E) \) and each subset \( A \) of \( VG \), we define the functions \( \overline{g}_k(A) \) and \( g_k(A) \) as follows,

\[
\overline{g}_k(A) = \sum_{v \in A} g_k(d_v) - \sum_{\{u,v\} \in A^{(2)}} g_k(d_u + d_v - \|\{u,v\}\|) + \cdots + (-1)^i \sum_{S \in A^{(i)}} g_k \left( \sum_{v \in S} d_v - \|S\| \right) + \cdots.
\]

\[
g_k(A) = \sum_{v \in A} g_k(d_v) - \sum_{\{u,v\} \in A^{(2)}} g_k(d_u + d_v - 1) + \cdots + (-1)^i \sum_{S \in A^{(i)}} g_k \left( \sum_{v \in S} d_v - \frac{(-1)^i + 1}{2} |S^{(2)}| \right) + \cdots.
\]

Observe that \( \|\{u,v\}\| \) equals 1 if and only if \( uv \) is an edge and 0 otherwise.

**Lemma 3** For any subset \( A \subset VG \) and any integer \( k \) such that \( k \in [0, \ldots, EG] \):

\[
\left| \bigcup_{v \in A} M^k_v \right| = \sum_{v \in A} |M^k_v| - \sum_{u, v \in A \cup \partial v} |M^k_u \cap M^k_v| + \sum_{\{u,v,w\} \in A^{(3)}} |M^k_u \cap M^k_v \cap M^k_w| - \cdots = \overline{g}_k(A).
\] (5)

and the second summation is

\[
t = \sum_{\{u,v\} \in A^{(2)}} |M^k_u \cap M^k_v| = \sum_{uv \in E[A]} g_k(d_u + d_v - 1) + \sum_{\{u,v\} \in A^{(2)} \setminus E[A]} g_k(d_u + d_v).
\] (6)

**Proof** Combine the Inclusion-Exclusion principle with (5) and the definition of \( \overline{g}_k \).

We are in position to prove some useful inequalities.

**Lemma 4** Let \( G = (V, E) \) be a graph. If \( A \subset VG \) then \( m_k(G) \geq \overline{g}_k(A) \).
Proof | It is clear that $m_k(G) = |M^k(G)|$. Additionally, by definition, $\cup_{v \in A} M^k_x(G) \subseteq M^k(G)$, and consequently, $|\cup_{v \in A} M^k_x(G)| \leq m_k(G)$. Now, the result follows from Expression \[5\].

Lemma 5 | Let $G = (V, E)$ be a graph. If $A \subseteq VG$ then $m_k(G) \geq g_k(A)$.

Proof | Combining Expression \[5\] with the inclusion $E[S] \subseteq S^{(2)}$ for any subset $S$ of $A$, we get that

\[
g_k(d_v, S) \leq \left| \bigcup_{v \in S} M^k_x \right| \leq g_k(d_v, S - |S^{(2)}|),
\]

(7)

where $d_v, S = \sum_{v \in S} d_v$. Then $g_k(A) \geq g_k(A)$, and the statement follows by Lemma 4.

The following assertion constructs sharper bounds for the number of edge-cuts $m_k(G)$ of a graph $G$.

Lemma 6 | Let $G = (V, E)$ be a graph. Consider $A \subseteq VG$ and $t = \sum_{(u,v) \in A} |M^k_x \cap M^k_y|$. Let us sort the $|A^{(2)}|$ numbers $\{g_k(d_u + d_v - 1)\}_{(u,v) \in A^{(2)}}$ increasingly and let $t'$ be its global sum. Among all those numbers consider the list of the greatest $h = |A^{(2)}| - \frac{1}{2} \sum_{v \in A} d_v$ of them, and replace those terms belonging to $t'$ by $g_k(d_u + d_v)$ to obtain a new quantity $t''$. Then $m_k(G) \geq g_k(A) + t' - t'' \geq g_k(A)$.

Proof | Since $||A|| \leq \frac{1}{2} \sum_{v \in A} d_v$, then $|A^{(2)} \subseteq E[A]| \geq h = |A^{(2)}| - \frac{1}{2} \sum_{v \in A} d_v$. Therefore, there are at least $h$ non adjacent vertices in $A$ and $t$ has at least $h$ terms of the form $g_k(d_u + d_v)$. Keeping the $h$ greatest ones, we get that $t' \geq t'' \geq t$. Finally, since $g_k(A) - g_k(A) \geq -t + t' \geq -t'' + t'$, then by Lemma 4 we conclude that $m_k(G) \geq g_k(A) \geq g_k(A) + t' - t'' \geq g_k(A)$.

In order to sort the numbers $g_k(d_u + d_v - 1)$ we can alternatively sort the numbers $(d_u + d_v)$, since by Lemma 2 the function $g_k$ is decreasing.

5 | THE BIPARTITE GRAPHS $K_{3,3}$ AND $K_{4,4}$ ARE UMRGS

As a proof-of-concept we show that both $K_{3,3}$ and $K_{4,4}$ are UMRGs. We consider essentially the bounding methodology that combines Lemmas 4 and 6 from Section 4. The proof that $K_{3,3}$ is UMRG is elementary.

Proposition 3 | The graph $K_{3,3}$ is UMRG in the class of $(6,9)$-graphs.

Proof | Since $K_{3,3}$ has superconnectivity 3 we know that $m_k(K_{3,3}) = 0$ when $k \in \{0, 1, 2\}$. It is clear that all the $(6,9)$-graphs $G$ satisfy that $m_k(G) = \binom{9}{k}$ whenever $k \geq 5$. Furthermore, by Theorem 4, the graph $K_{3,3}$ is $t$-optimal, hence $m_4$ also attains its minimum in $K_{3,3}$.

Then, it is sufficient to prove that $m_3$ is also minimized in $K_{3,3}$. Let $G$ be any $(6,9)$-graph. If $G$ is regular, then it has six trivial 3-edge-cuts, and $m_3(G) \geq 6 = m_3(K_{3,3})$. Otherwise, $G$ is nonregular. Since $m = 9$, we get that $g_3(2) = \binom{9-2}{3-2} = 7$. By Theorem 2, we can assume, without loss of generality, that $d_v \geq 2$ for all $v \in VG$. Since $G$ is nonregular and its average degree equals 3, by the handshaking lemma $G$ must have some vertex $v$ such that $d_v = 2$. Let us apply Lemma 5 using $A = \{v\}$. Then,

\[
m_3(G) \geq g_3(A) = g_3(gr(u)) = g_3(2) = \binom{9-2}{3-2} = \binom{7}{1} = 7 > m_3(K_{3,3}).
\]

We conclude that $m_k(K_{3,3}) \leq m_k(G)$ for all $G$ in $(6,9)$ and all $k \in \{0, \ldots, 9\}$. In particular, $K_{3,3}$ is UMRG, as we wanted to prove.
We will prove that $K_{4,4}$ is the strongest in its class; in particular, $K_{4,4}$ is UMRG. Our proof strategy can be summarized in three steps. First, it is straightforward to prove that $K_{4,4}$ is min-$m_k$ for all $k$ except for $k \in \{6, 7, 8\}$; see Lemma 7. Subsequent analysis considers $k \in \{6, 7, 8\}$ separately. Then, we use the fact that $K_{4,4}$ is triangle-free to prove that it is the most reliable among all the 4-regular $(8, 16)$-graphs. Finally, the comparison with nonregular graphs considers repeated application of Lemmas 5 and 6 as well as special classifications of degree-sequences.

Lemma 7 The graph $K_{4,4}$ is min-$m_k$, for all $k \in \{0, \ldots, 5\} \cup \{9\}$

Proof Since $K_{4,4}$ is a regular complete bipartite graph, Theorem 1 asserts that $K_{4,4}$ is $t$-optimal, hence it is min-$m_9$. Furthermore, $K_{4,4}$ has superconnectivity 4, and consequently, $K_{4,4}$ is min-$m_k$ for all $k \in \{0, \ldots, 4\}$. Finally, observe that by Equation 4, for any graph $H$ it holds that

$$M^S_v(H) \cap M^S_w(H) = \emptyset \text{ if } v \neq w \text{ with } d_v + d_w \geq 7.$$ (8)

Let $H$ be a 4-regular $(8, 16)$-graph. If we set $A = V(H)$ then by Lemma 4

$$m_5(H) \geq \overline{g}_5(A) = 8g_5(4) = 96.$$

Since $K_{4,4}$ has girth 4, by Lemma 4 it has only trivial 5-cuts, i.e. $M^5 \subset \cup_r M^5_r$. Thus $m_5(K_{4,4}) = |M_5| \leq |\cup_r M^5_r| = \sum_r |M^5_r| = 8g_5(4) = 96$, and $m_5(K_{4,4}) \leq m_5(H)$, for any 4-regular $H$.

If $H$ is not 4-regular, by Theorem 2 we can assume that $\delta(H) \in \{2, 3\}$. If $\delta(H) = 2$ and $d_v = 2$ then by Lemma 5 setting $A = \{v\}$ we have $m_5 \geq g_5(A) = g_5(2) = 364 > 96$. If $\delta(H) = 3$ then we subdivide the discussion into two disjoint and exhaustive cases. If there are at least two nodes $v$ and $w$ with degree 3 then by Lemma 5 setting $A = \{v, w\}$ we have that $m_5 \geq g_5(A) = 2g_5(3) - g_5(3 + 3 - 1) = 155 > 96$. Otherwise, there is precisely one node $v$ with minimum degree 3. By the handshaking lemma, the graph $H$ must have six nodes with degree 4 and one with degree 5, hence if $d_v = d_w = 4$ and $A = \{u, v, w\}$, by Lemma 4 and 5 we get that $m_5(H) \geq \overline{g}_5(A) = g_5(3) + 2g_5(4) = 102 > 96$.

In the following sections we will compare $K_{4,4}$ versus regular and nonregular graphs in its class $(8, 16)$, respectively.

5.1 Regular Graphs

In this section we find lower bounds on the coefficient $m_k$ for any regular $(8, 16)$-graph. The bounds are precisely $m_k(K_{4,4})$. The key is Mantel’s theorem [10], which asserts that the maximum size of an $n$-vertex triangle-free graph $G$ is $[n^2/4]$, and the bound is attained if and only if $G = K_{[n/2],[n/2]}$.

Lemma 8 If $G$ is a regular $(8, 16)$-graph, then

- $m_6(G) \geq 8\binom{16-4}{2} + 16 = 544$.
- $m_7(G) \geq 8\binom{16-4}{3} - 16 + 16\binom{16-7}{1} = 1888$.
- $m_8(G) \geq 8\binom{16-4}{4} - \frac{16}{2} - 16 - 16\binom{16-7}{2} + 16\binom{16-7}{1} + 3\tau(G)\binom{16-8}{2} + s(G)/2$,

where $s(G)$ is the number of squares of $G$, $\tau(G) = 1$ if and only if $G$ has at least one triangle, or 0 otherwise. Furthermore, the equalities hold if and only if $G$ is a triangle-free graph.
Proof Any disconnected graph of order \(n\) will have a connected component of at most \(|n/2|\) vertices. If \(G\) is a \((8, 16)\)-graph and \(S \in M^k(G)\) then \(G \setminus S\) will have a connected component \(H\) of cardinality at most 4. Denote \(C'(G)\) the set of connected subgraphs of order \(i\) in \(G \setminus S\). Then,

\[
M^k(G) = \bigcup_{i=1}^{4} \bigcup_{H \in C'(G) : |\partial H| \leq k} M^k_i(G).
\]

Let \(G\) be a 4-regular \((8, 16)\)-graph. Consider some \(H \in C'(G)\) with \(i \leq 4\) and \(|\partial H| \leq 8\). There are 4 possibilities:

- \(H \in C^1\) and \(|\partial H| = 4\), for which there are precisely \(|G|\) possible \(H\)'s.
- \(H \in C^2\) and \(|\partial H| = 6\), for which there are precisely \(|G|\) possible \(H\)'s.
- \(H \in C^3\) and \(|\partial H| = 8\) or 6 depending on \(|H|\) being a 3-path or a triangle. In the former, there are precisely \(\binom{6}{3}\) \(|G| = 6|G|\) possibilities, while in the latter the number of possibilities is a function of the number of triangles, that is zero if and only if \(G = K_{4,4}\).
- \(H \in C^4\) and \(|\partial H| = 8\) or 6 or 4. If \(|\partial H| = 8\) then \(H\) is either a complete graph minus an edge or the complete graph, respectively. \(H\) is either a complete graph minus an edge or the complete graph, respectively.

Therefore, we have the following inclusions, where \(M^k_i = \bigcup_{H \in C^i} M^k_i\) and \(M^k_{i,j} = \bigcup_{H \in C^i \setminus \{H\} = j} M^k_H\).

\[
M^6 \supseteq M^6_1 \cup M^6_2. \tag{9}
\]

\[
M^7 \supseteq M^7_1 \cup M^7_2. \tag{10}
\]

\[
M^8 \supseteq M^8_1 \cup M^8_2 \cup M^8_{3,8} \cup M^8_{3,6} \cup M^8_{4,8}. \tag{11}
\]

These inclusions are equalities in triangle-free graphs. Let us consider the three cases separately:

- Note that \(M^6_1 = \{\partial v \cup \{e, f\} : v \in V, e, f \in E \setminus \partial v\}\), \(M^6_2 = \{\partial e : e \in E\}\) and \(M^6_1 \cap M^6_2 = M^6_2 \subseteq M^6_1 \cap M^6_2 = \emptyset\), for any pair of vertices \(u \neq v\) and edges \(e \neq f\). The lower bound for \(m_6\) then follows from the expression (9).
- Note that \(M^7_1 = \{\partial v \cup S : v \in V, S \in (E \setminus \partial v)^3\}\), \(M^7_2 = \{\partial e \cup \{f\} : e \in E, f \notin \partial e \cup \{e\}\}\) and both \(M^7_1 \cap M^7_2 = M^7_2 \subseteq M^7_1 \cap M^7_2 = \emptyset\) for any pair of edges \(e \neq f\), while \(|M^7_1 \cap M^7_2| = 1\) if \(uv \in E\), or 0 otherwise. The lower bound for \(m_7\) then follows from the expression (10).
- The lower bound for \(m_8\) follows from the expression (11), and it is more involved. Clearly,

\[
M^8_1 = \{\partial v \cup S : v \in V \land S \subseteq (E \setminus \partial v)^4\},
\]

\[
M^8_2 = \{\partial e \cup \{f, g\} : e \in E \land f, g \in E \setminus \partial e\},
\]

\[
M^8_{3,8} = \{\partial \{u, v, \bar{w}\} : uv, vw \in E \land \bar{w} \notin E\},
\]

\[
M^8_{3,6} = \{\partial T \cup \{e, f\} : \{T\} \simeq C_3 \land e, f \notin \partial T \land |\{e, f\} \cap E[T]| \leq 1\},
\]

\[
M^8_{4,8} = \{\partial S : |S| \simeq C_4\}.
\]

These sets are pairwise disjoint, so

\[
|M^8| \geq |M^8_1| + |M^8_2| + |M^8_{3,8}| + |M^8_{3,6}| + |M^8_{4,8}|.
\]
Let us bound each term on the right-hand side. For $|M^8_{3,6}|$, if $S$ is a square then in a regular $(8,16)$-graph the complement of the square has four edges as well. That complement is either a square or a triangle with a pendant edge. Further, if $G$ is triangle-free then $S^c$ must be a square. So, for each pair of complement squares there is only one set in $M^8_{3,6}$. If $G$ has some triangle then each set in $M^8_{3,6}$ corresponds to one or two squares; in both cases the cardinality is at least $s/2$.

For $M^8_{5,6}$ we just bound the cardinality by considering only one triangle $T$ and for each edge $g$ in $T$ we choose the edges $e$ and $f$ among those edges not in $\partial T \cup ET \setminus \{g\}$. Since there are three ways to choose $g$, we have \(3\left(\frac{|E| - |\partial T \cup ET \setminus \{g\}|}{2}\right) = 3\left(\frac{16 - 6(2)}{2}\right)\) possible choices for the edges $e$ and $f$.

The cardinality $|M^8_{3,8}|$ corresponds to $|G|$ choices for $v$ and \(\binom{3}{2}\) choices for $u$ and $w$ among the neighbours of $v$. Similarly, $|M^8_{2,4}|$ corresponds to $|G|$ choices for $vw$ and $\left(\frac{|G| - |\partial vw \cup \{vw\}|}{2}\right)$ choices for the edges $e$ and $f$ that are not incident to $v$ or $w$.

Finally, to bound $|M^8_4|$ note that $\partial v \cup S = \partial v' \cup S'$ if and only if either $vv' \notin E$ with $\partial v = S'$ and $\partial v' = S$ or $vv' \in E$ and $\partial v \cup S = \partial v' \cup \{e\}$ with $e \in G \setminus \partial vv'$. There are $|G| = \binom{8}{2} - 16$ possible choices for $vv' \notin E$ and $|G| \left(\frac{|G| - |\partial vv' \cup \{e\}|}{2}\right) = 16(16 - 7)$ possible choices for $vv' \in E$ and $\partial v \cup S = \partial v' \cup \{e\}$.

**Proposition 4** $K_{4,4}$ is the stronger graph among all the regular $(8,16)$-graphs. Furthermore, $m_6(K_{4,4}) = 544$, $m_7(K_{4,4}) = 1888$ and $m_8(K_{4,4}) = 4446$.

**Proof** The first part of the statement follows combining Mantel Theorem with Lemmas 7 and Lemma 8. To prove the second part of the statement use Lemma 8 with $\tau(K_{4,4}) = 0$ and $c(K_{4,4}) = \frac{4}{2}$.

### 5.2 Nonregular Graphs

We will extensively use Lemmas 4, 5 and 6. The following results consider the first $h$ nodes sorted by increasing degree. Let us call $V^hG$ to this set, i.e., if $VG = \{v_1, \ldots, v_n\}$ with $d_{v_1} \leq \cdots \leq d_{v_n}$ then $V^hG = \{v_1, v_2, \ldots, v_h\}$. This order is not unique, but the following proofs do not depend on the specific choice.

**Lemma 9** Among the class of $(8,16)$-graphs $G$ the coefficients $m_6$ and $m_7$ are minimized in $K_{4,4}$.

**Proof** Let $G$ be an $(8,16)$-graph. If $\delta(G) = 2$, by Lemma 5 with $A = V^1G = \{u\}$, i.e. $d_u = 2$, we have $m_k(G) \geq g_k(2) = m_k(K_{4,4})$ for $k \in \{6,7\}$, see Table 1. Otherwise $\delta(G) = 3$ and we discuss according to $|V_3|$, i.e., the number of nodes of degree $3$ in $G$:

- If $|V_3| \geq 3$ then by Lemma 5 with $A = V^3G$: $m_k(G) \geq g_k(3) - 3g_k(3 + 3 - 1) = 825, 1980$, for $k \in \{6,7\}$, respectively, in both cases greater than the corresponding $m_k(K_{4,4})$.
- If $|V_3| = 2$ then $|V_4| \geq 4$. Using Lemma 5 with $A = V^5G$, we have

$$m_k(G) \geq 2g_k(3) + 4g_k(4) - g_k(3 + 3 - 1) - 6g_k(3 + 4 - 1) - 3g_k(4 + 4 - 1).$$

Then $m_6(G) \geq 753 > m_6(K_{4,4})$ and $m_7(G) \geq 1972 > m_7(K_{4,4})$.

- If $|V_3| = 1$ then the degree-sequence is $(3, 4, 4, 4, 4, 4, 4, 5)$. We apply Lemma 5 with $A = V^7G$, to obtain

$$m_k(G) \geq g_k(3) + 6g_k(4) - 6g_k(3 + 4 - 1) - \frac{6}{2}g_k(4 + 4 - 1).$$
Then \( m_6(G) \geq 676 > m_6(K_{4,4}) \) and \( m_7(G) \geq 1960 > m_7(K_{4,4}) \).

### TABLE 1 Values of \( g_k(i) \) and \( m_k(K_{4,4}) \) for a \((8,16)\)-graph \( G \) and \( k = 6, 7, 8 \).

| \( k \) | \( g_k(1) \) | \( g_k(2) \) | \( g_k(3) \) | \( g_k(4) \) | \( g_k(5) \) | \( g_k(6) \) | \( m_k(K_{4,4}) \) |
|-------|---------|---------|---------|---------|---------|---------|----------------|
| 6     | 3003    | 1001    | 364     | 66      | 11      | 1       | 544           |
| 7     | 5005    | 2002    | 715     | 220     | 55      | 10      | 1888          |
| 8     | 6435    | 3003    | 1287    | 495     | 165     | 45      | 4446          |

**Lemma 10** Among the class of \((8,16)\)-graphs \( G \) the coefficient \( m_8 \) is minimized in \( K_{4,4} \), i.e. \( m_8(G) \geq \alpha = 4446 = m_8(K_{4,4}) \).

**Proof** Let \( G \) be an \((8,16)\)-graph. We split the proof into two parts according to the minimum degree of \( G \). Let us write \( g(x) \) instead of \( g_k(x) \).

- If \( \delta(G) = 2 \), we will consider three cases as a function of \( |V_2(G)| \) and \( |V_3(G)| \).
  - If \( |V_2(G)| \geq 2 \), by Lemma 5 with \( A = V^2G \), i.e. \( A = \{u, v\} \subset V_2(G) \), we have,
    \[
    m_8(G) \geq 2g(2) - g(2 + 2 - 1) = 2 \times 3003 - 1287 > \alpha.
    \]
  - If \( |V_2(G)| = 1 \) and \( |V_3| \geq 2 \), thus by Lemma 4 with \( A = V^3G \), we discuss according to the set \( E' = E[A] \) of edges of the subgraph induced by \( A = \{u, v, w\} \) with \( d_u = 2 \) and \( d_v = d_w = 3 \):
    \[
    m_8(G) \geq g(2) + 2g(3) + \begin{cases} -2g(2 + 3) - g(3 + 3) + g(2 + 3 + 3) & E' = \emptyset, \\
                                   -2g(2 + 3) - g(3 + 3 - 1) + g(2 + 3 + 3 - 1) & E' = \{vw\}, \\
                                   -g(2 + 3 - 1) - g(2 + 3) - g(3 + 3) + g(2 + 3 + 3 - 1) & E' = \{uv\} \text{ or } \{uw\}, \\
                                   -g(2 + 3 - 1) - g(2 + 3) - g(3 + 3 - 1) - g(2 + 3 + 3 - 2) & E' = \{uv, vw\} \text{ or } \{uw, vw\}, \\
                                   -2g(2 + 3 - 1) - g(3 + 3) + g(2 + 3 + 3 - 2) & E' = \{uv, uw\}, \\
                                   -2g(4) - g(3 + 3 - 1) + g(2 + 3 + 3 - 3) & E' = \{uv, uw, vw\}. \end{cases}
    \]

    which is greater than 4587 > \( \alpha \) in the six cases.

- If \( |V_2(G)| = 1 \) and \( |V_3| \leq 1 \) we consider all possible degree sequences, which are: \((2, 3, 4, 4, 4, 4, 4, 7), (2, 3, 4, 4, 4, 4, 4, 6), (2, 3, 4, 4, 4, 5, 5, 5), (2, 4, 4, 4, 4, 4, 6), (2, 4, 4, 4, 4, 4, 5, 5), \) and apply Lemma 5 with \( A = V^2G \). For the first three sequences with the prefix \((2, 3, 4, 4, 4, \ldots)\), we have
    \[
    m_8(G) \geq g(2) + g(3) + 3g(4) - g(2 + 3 - 1) - 3g(2 + 4 - 1) - 3g(3 + 4 - 1) \\
    - 3g(4 + 4 - 1) + 3g(2 + 3 + 4) + 3g(3 + 4 + 4) + g(4 + 4 + 4) \\
    - 3g(2 + 3 + 4 + 4 - C_2^4) - g(3 + 4 + 4 + 4 - C_2^4) \\
    - g(2 + 4 + 4 + 4 - C_2^4) + g(2 + 3 + 4 + 4 + 4) \geq 4595 > \alpha.
    \]
The terms in the second and last line are null, because the arguments of \( g \) are all greater than 8. Finally, for the last two sequences with the prefix \((2,4,4,4,\ldots)\), we apply Lemma 6 to the same \( A \) to obtain

\[
m_{8}(G) \geq g(2) + 4g(4) - 2g(2 + 4 - 1) - 2g(2 + 4) - 10g(4 + 4 - 1) \\
- 4g(2 + 3 \times 4 - \binom{4}{2}) = 4505 > \alpha, 
\]

where the null terms such as \( g(2 + 3 + 4) \) were not written.

- If \( \delta(G) = 3 \) we discuss according to \( |V_{3}| \) and \( |V_{4}| \):
  - If \( |V_{3}| \geq 5 \) then, using Lemma 5 with \( A = V^{5}G \),
    \[
m_{8}(G) \geq 5g(3) - 10g(3 + 3 - 1) - 5g(3 + 3 + 3 - 6) = 4560 > \alpha. 
    \]
  - If \( |V_{3}| = 4 \) and \( |V_{4}| \geq 1 \), note that \( \|V_{3}\| \leq 4 \) since \( G \) is biconnected. Using Lemma 4 with \( A = V^{5}G \),
    \[
m_{8}(G) \geq 4g(3) + g(4) - 4g(3 + 3 - 1) - 2g(3 + 3) - 4g(3 + 4 - 1) \\
- g(3 + 3 + 3 - 4) - 4g(3 + 3 + 4 - 6) = 4676 > \alpha. 
\]
  - If \( |V_{3}| = 4 \) and \( |V_{4}| = 0 \) then the degree sequence is \((3,3,3,3,5,5,5,5)\). By Lemma 4 with \( A = V^{5}G \) and the previous observation about the biconnectivity of \( G \),
    \[
m_{8}(G) \geq 4g(3) + g(5) - 4g(3 + 3 - 1) - 2g(3 + 3) - 4g(3 + 5 - 1) \\
- g(3 + 3 + 3 + 4 - 6) - 4g(3 + 3 + 5 + 4 - 8) \geq 4599 > \alpha. 
\]
  - If \( |V_{3}| = 3 \) and \( |V_{4}| \geq 3 \), if \( A = V^{6}G \) and \( h = \|V_{3}\| \), then \( h \in \{0,1,2,3\} \) and by Lemma 4 we have
    \[
m_{8}(G) \geq 3g(3) + 3g(4) - hg(3 + 3 - 1) - (3 - h)g(3 + 3) - (9 - 2h)g(3 + 4 - 1) \\
- 2hg(3 + 4) - 3g(4 + 4 - 1) - 3g(3 + 3 + 3 + 4 - 6) \\
- 9g(3 + 3 + 4 + 4 - 6) - g(3 + 3 + 4 + 4 + 4 - 8) \geq 4599 > \alpha. 
\]

The remaining possible degree-sequences are:

\((3,3,3,4,4,5,5,5), (3,3,4,4,4,4,4,6), (3,3,4,4,4,4,5,5), (3,4,4,4,4,4,4,5)\).

If the degree-sequence is \((3,3,3,4,4,5,5,5)\), by Lemma 5 with \( A = V \),

\[
m_{8}(G) \geq 3g(3) + 2g(4) + 3g(5) - 3g(3 + 3 - 1) - g(4 + 4 - 1) \\
- 3g(5 + 5 - 1) - 6g(3 + 4 - 1) - 9g(3 + 5 - 1) - 6g(4 + 5 - 1) \\
- 2g(3 + 3 + 3 + 4 - 6) - 3g(3 + 3 + 5 + 5 - 6) - 3g(3 + 3 + 4 + 4 - 6) \\
= 4461 > \alpha. 
\]

If the degree-sequences is \((3,3,4,4,4,4,4,6)\) with \( V_{3} = \{u, v\} \), then the maximum number of edges between
The last sum is 4663 if $h = 0$ and 4615 if $h = 1$, both greater than $\alpha$.

Finally, if $|V_3| \leq 2$ then, the only possible degree-sequences, are $(3, 3, 4, 4, 4, 5, 5)$ or $(3, 4, 4, 4, 4, 4, 4, 5)$, the study is more involved. We will lower bound the cardinality of $M' = \bigcup_{v \in V_G} M^f_v \cup \bigcup_{e \in E_G} M^e_v$, applying Lemma 5 to find a bound on $|\bigcup_{v \in V_G} M^f_v|$, and detailed analysis for the remaining terms. By the inclusion-exclusion principle:

$$|M'| = \left| \bigcup_{v \in V_G} M^f_v \right| + \left| \bigcup_{e \in E_G} M^e_v \right| - \left| \bigcup_{v \in V_G} M^f_v \cap \bigcup_{e \in E_G} M^e_v \right|$$

$$\geq \left| \bigcup_{v \in V_G} M^f_v \right| + \left| \bigcup_{e \in E_G} M^e_v \right| - 18,$$

where the second equality holds since $M^f_v \cap M^e_w = \emptyset$ for all $e, f \in E$, while the last inequality holds since $M^f_v \cap M^e_w$ is the empty sets unless $d_v = 3$, $e = uw$ with $d_u = d_w = 4$ and $v$ is adjacent to $u$ or $w$, which are only 9 cases for $|V_3| = 1$ and at most 18 cases for $|V_3| = 2$. Now, we will first bound the first term on the right-hand side of the last inequality, and then, we will bound the second term. For the first term, let us consider the Lemma 5 with $A = V$. For the first sequence we have

$$\left| \bigcup_{v \in V_G} M^f_v \right| \geq 2g(3) + 4g(4) + 2g(5) - f(3 + 3 - 1) - 6g(4 + 4 - 1) - 8g(3 + 4 - 1)$$

$$- 4g(3 + 5 - 1) - 8g(4 + 5 - 1) - \left( \begin{array}{c} 4 \\ 2 \end{array} \right) g(3 + 3 + 4 + 4 - 6) = 4255,$$

and for the second sequence we have

$$\left| \bigcup_{v \in V_G} M^e_v \right| \geq g(3) + 6g(4) + g(5) - 6g(3 + 4 - 1) - g(3 + 5 - 1) - 6g(4 + 5 - 1)$$

$$= 4137.$$

Now, let us consider the sum $S = \sum_{e \in E_G} |M^f_e|$. If $e = uv$, then $|M^f_e| = \binom{16 - d_v - d_u + 1}{8 - d_u - d_v}$, which has the four possible values shown in Table 2 since there are three possible values of the degrees. Thus, a possible lower bound for that sum $S$ is the minimum of the function $f(a, b, c, d) = 16a + 36b + 36c + 8d$ subject to the constraints $a + b = 5, 6, d = 9, 10$ and $a + b + c + d = 16$ for the first sequence and subject to the constraints $a + b = 3, d \leq 5$ and $a + b + c + d = 16$ for the second sequence.
TABLE 2 Values of $|M_{dv}|$ according with $d_u$ and $d_v$

| $d_u$ | $d_v$ | $|M_{dv}|$ |
|-------|-------|----------|
| 3     | 4     | 168      |
| 3     | 5     | 36       |
| 4     | 4     | 36       |
| 4     | 5     | 8        |

These minimum values are attained at $(a, b, c, d) = (0, 5, 0, 11)$ and $(a, b, c, d) = (0, 3, 5, 8)$, respectively. Therefore, $S \geq 268$ and $436$ respectively. All the bounds together give us: $m_8(G) \geq 4255 + 268 - 18 = 4505$ and, $m_8(G) \geq 4162 + 436 - 18 = 4580$, both greater than $\alpha$.

Proposition 5 Among the class of nonregular $(8, 16)$-graphs, $K_{4,4}$ is $\text{min-}m_k$ for all $k \in \{0, \ldots, 16\}$.

Proof By Lemmas 7, 9 and 10 we know that $K_{4,4}$ is $\text{min-}m_k$ for each $k \in \{0, \ldots, 9\}$. Now, set $k \in \{10, \ldots, 16\}$. Observe that if we remove $k$ edges to any $(8, 16)$-graph then the resulting subgraph is not connected. Therefore, $m_k(K_{4,4}) = m_k(G) = \binom{16}{k}$ for each $k \in \{10, \ldots, 16\}$.

Theorem 3 The complete bipartite graph $K_{4,4}$ is the strongest in its class. In particular, $K_{4,4}$ is uniformly most reliable in the class of $(8, 16)$-graphs.

Proof Combining Propositions 4 and 5 we conclude that the complete bipartite graph $K_{4,4}$ is the strongest in its class. In particular, any strongest graph in its class is uniformly most reliable, and the result follows.

6 CONCLUSIONS AND TRENDS FOR FUTURE WORK

Uniformly most reliable graphs (UMRGs) represent a synthesis in network reliability analysis. Finding them is a hard task not well understood. An exhaustive comparison is computationally prohibitive for most cases. Prior works in the field try to globally minimize the cutsets. This minimization could be enriched with our bounding methodology. As a proof-of-concept, here we show that both $K_{3,3}$ and $K_{4,4}$ are UMRGs. There are several trends for future work. Even though all UMRGs share special properties such as the greatest girth, there are no proofs available for these conjectures. A powerful methodology to find UMRGs is not known. In this paper we propose a novel methodology to count trivial cutsets, which could be used for the discovery of UMRGs. As future work, we want to study if the complete bipartite graphs are always UMRGs.

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