AN IMPROVEMENT TO THE NUMBER FIELD SIEVE

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Abstract

We improve the “sieve” part of the number field sieve used in factoring integer and computing discrete logarithm. The runtime of our method is shorter than that of existing methods. Under some reasonable assumptions, we prove that it is less than two-thirds of the running time of the algorithm used before asymptotically with probability greater than 0.6.

1 Introduction

General number field sieve is used in factoring integer or computing the discrete logarithm. See, for example, [1] [2] [3]. There are two time consuming parts mainly in the number field sieve. Namely, the part “sieving”, and the part “solving the linear equations”. The two parts are relatively independent and have the computational complexity in same order. In [4], the authors improved the step “solving the linear equations” for discrete logarithm problem. In this paper, we improve the step “sieve”. Our improvement work for both factoring integer and computing the discrete logarithm. The running time of our algorithm is less than the one in [5] [6] asymptotically. Under some reasonable assumptions, it is less than \( \frac{2}{3} \) of the running time of the algorithm used in [5] [6] asymptotically with probability greater than 0.6.

In section 2, we give the formulation of the problem which we want to solve, and describe the algorithm used before. In section 3, we describe our algorithm. In section 4, we prove that our algorithm is better than the algorithm used before.

2 The problem and conventional algorithm

Let us consider the following problem:

**Problem 2.1.** Let \( f(x) \) be a monic polynomial of degree \( d \) with integer coefficient that bounded by an integer \( m \) and \( K \) be an algebraic number field isomorphic to \( \mathbb{Q}[x]/(f(x)) \). Let \( \theta \) be the image of \( x \) in \( K \) and \( N m : K^\times \to \mathbb{Q}^\times \) be the norm map. Let \( u \) be a positive integer. We construct a table \( T = \{ T(b, a) \}_{0 \leq b \leq u, |a| \leq u} \) of \( u \) lines and \( 2u + 1 \) cows with

\[
T(b, a) = \begin{cases} 
0 & \text{if } (a, b) = 1; \\
(a - bm)Nm(a - b\theta) & \text{if } (a, b) \neq 1.
\end{cases}
\]
Let $y$ be a positive real number called "the smooth bound". For every element in the table, we wish to divide out all of its divisors of the form $l^e$ for all primes $l$ bounded by $y$.

The most trivial algorithm is the following:

**Algorithm 1** Sieve

1. **for** prime $0 < l \leq y$, integer $|a| \leq u$, $0 < b \leq u$, such that $T(b, a) \neq 0$ **do**
2. **while** $l | T(b, a)$ **do**
3. $T(b, a) \leftarrow T(b, a)/l$
4. **end while**
5. **end for**

The following improved algorithm is widely used in integer factoring algorithms (see [1], [6]) or algorithms of solving the discrete logarithm problem (see [2] [3] [5]).

**Algorithm 2** Sieve

1. **for** prime $l \in (0, y]$ **do**
2. $\epsilon_l \leftarrow m \mod l \in \{0, 1, \ldots, l-1\}$
3. $E_l \leftarrow \{x \in \{0, 1, \ldots, l-1\} : f(x) \equiv 0 \mod l\}$
4. **end for**
5. **for** integer $b \in (0, u]$ **do**
6. **for** prime $l \in (0, y], l \nmid b$ **do**
7. **for** $a \in [-u, u] \cap (b\epsilon_l + l\mathbb{Z})$ **do**
8. **while** $l | T(b, a)$ **do**
9. $T(b, a) \leftarrow T(b, a)/l$
10. **end while**
11. **end for**
12. **for** $a \in [-u, u] \cap (bE_l + l\mathbb{Z})$ **do**
13. **while** $l | T(b, a)$ **do**
14. $T(b, a) \leftarrow T(b, a)/l$
15. **end while**
16. **end for**
17. **end for**
18. **end for**

In Algorithm 2, we do not try to divide all the elements in the table by $l$ more, but divide those divisible by $l$ we know. Then we divide the quotient by $l$ continually as long as it is divisible by $l$. Roughly speaking, for every $b, l$, we solve the equations

$$a - bm \equiv 0 \mod l \quad \text{or} \quad Nm(a - b\theta) \equiv 0 \mod l$$

of variable $a$, and then sieve.

### 3 Our algorithm

There is unnecessary computing still in algorithm 2. In fact, we can almost know which $T(b, a)$ can be divided by $l$ again, after it divided by $l$ first. Roughly speaking, for every $b, l$, we can almost can solve the equations

$$a - bm \equiv 0 \mod l^k \quad \text{or} \quad Nm(a - b\theta) \equiv 0 \mod l^k$$
of variable $a$, for any $k$, and then sieve. Our new algorithm consists of 3 parts.

**First**, we divide out all $l$-power divisors caused by the term $(a - bm)$: Let $\epsilon_i^{(1)}$ be the residue for $m$ module $l$. We divide $T(b, a)$ by $l$ and write the quotient in $T(b, a)$ for all $a \in [-u, u] \cap (b\epsilon_i^{(1)} + l\mathbb{Z})$. Let $\epsilon_i^{(2)}$ be the residue for $m$ module $l^2$. We divide $T(b, a)$ by $l$ and write the quotient in $T(b, a)$ for all $a \in [-u, u] \cap (b\epsilon_i^{(2)} + l^2\mathbb{Z})$. 

**Second**, we divide out all the $l$-power divisors caused by the term $Nm(a - b\theta)$, for all $a, b$ such that $b \in [0, u]$ is coprime to $l$, and $a \mod l$ is a single root of the equation $Nm(a - b\theta) \equiv 0 \mod l$. By lemma 3.1 below we can do this as follows: Let $E_i^{(1)} \subset \{0, 1, \cdots l - 1\}$ be the set of single roots of the equation $f(x)$ module $l$. We can directly compute $E_i^{(1)}$ by solve equation. We divide $T(b, a)$ by $l$ and write the quotient in $T(b, a)$ for all $a \in [-u, u] \cap (bE_i^{(1)} + l\mathbb{Z})$. Let $E_i^{(2)} \subset \{0, 1, \cdots l^2 - 1\}$ be the set of single roots of the equation $f(x)$ module $l^2$. We can directly compute $E_i^{(2)}$ from $E_i^{(1)}$ by Newton’s method. We divide $T(b, a)$ by $l$ and write the quotient in $T(b, a)$ for all $a \in [-u, u] \cap (bE_i^{(2)} + l^2\mathbb{Z})$. 

**Finally**, we divide out all the $l$-power divisors caused by the term $Nm(a - b\theta)$, for all $a, b$ such that $b \in [0, u]$ is coprime to $l$, and $a \mod l$ is a multiple root of the equation $Nm(a - b\theta) \equiv 0 \mod l$. By lemma 3.1 and lemma 3.2 below we can do this as follows: Let $E_i^{(1)} \subset \{0, 1, \cdots l - 1\}$ be the set of multiple roots of the equation $f(x)$ module $l$, we can directly compute $E_i^{(1)}$ by solving the equation. We divide $T(b, a)$ by $l$ and write the quotient in $T(b, a)$ for all $a \in [-u, u] \cap (bE_i^{(1)} + l\mathbb{Z})$. Lemma 3.2 below tells us that whether a root of $f(x) \equiv 0 \mod l$ can be lifted to a root of $f(x) \equiv 0 \mod l^2$ is only dependent on its residue class module $l$. Let $E_i^{(2)}$ be the subset of $E_i^{(1)}$ whose elements can be lifted to solutions of $f(x) \equiv 0 \mod l^2$. We can compute $E_i^{(2)}$ by $\mathcal{E}_i^{(1)}$’s tests. Then we divide $T(b, a)$ by $l$ and write the quotient in $T(b, a)$ for all $a \in [-u, u] \cap (bE_i^{(2)} + l\mathbb{Z})$ one after another until $l \mid T(b, a)$. 

Now we give statements and proofs of lemma 3.1 and lemma 3.2 mentioned above.

**Lemma 3.1.** If $l \nmid b$, there is a bijective

$\{x \in \mathbb{Z}/l^e\mathbb{Z}; f(x) \equiv 0 \mod l^e\} \rightarrow \{a \in \mathbb{Z}/l^e\mathbb{Z}; Nm(a - b\theta) \equiv 0 \mod l^e\}$

for all $e > 0$. Moreover, in the situation $e = 1$, the images of simple roots are simple, and the image of multiple roots are multiple.

**Proof.** It is because

$$Nm(a - b\theta) = (-b)^d Nm(\frac{a}{b} - \theta) = (-b)^d f(\frac{a}{b})$$

\[\blacksquare\]

**Lemma 3.2.** Let $x, y$ be two integers and $f$ be a polynomial over $\mathbb{Z}$. Assume $x \equiv y \mod l$ is a multiple root of $f(x) \equiv 0 \mod l$. Then $x \mod l^2$ is a root of $f(x) \equiv 0 \mod l^2$ if and only if $y \mod l^2$ is a root of $f(x) \equiv 0 \mod l^2$.

**Proof.** Let $y = x + kl$ where $k \in \mathbb{Z}$. If $x \mod l^2$ is a root of $f(x) \equiv 0 \mod l^2$, we have $f(y) \in f(x) + f'(x)kl + l^2\mathbb{Z}$

3
by Taylor expansion. On the other hand, we know \( f'(x) \equiv 0 \mod l \). Therefore
\[
f(y) \in f(x) + l^2 \mathbb{Z}
\]

\[\square\]

4 Complexity analysis

We will compare the computational complexity of Algorithm 2 and Algorithm 3. Considering the practical situation, we make the assumption that \( y \leq K u \) for some constant \( K \).

For \((l, b) = 1, c > 0\), Let
\[
A_{l}^{b, s} := \{a | a \leq u; (a, b) = 1, a \text{ is a single root of } Nm(a - b\theta) \equiv 0 \mod l\}
\]
\[
A_{l}^{b, m} := \{a | a \leq u; (a, b) = 1, a \text{ is a multiple root of } Nm(a - b\theta) \equiv 0 \mod l\}
\]
\[
A_{l}^{b, s} := \{a \in A_{l}^{b, s}; Nm(a - b\theta) \equiv 0 \mod l^c\}
\]
\[
A_{l}^{b, m} := \{a \in A_{l}^{b, m}; Nm(a - b\theta) \equiv 0 \mod l^c\}
\]
\[
A_{l}^{b} := \{a | a \leq u; (a, b) = 1, a - bm \equiv 0 \mod l^c\}
\]

In Algorithm 2, the complexity of line 1–line 4 is an infinitesimal of the complexity of line 5–line 18 as \( u \to \infty \). From line 5, the complexity of sieving the elements in the \( b \)-th line of the table by prime \( l \) is

\[
C_l^b = \begin{cases} \#B_l^b & (T(b, a) \leftarrow T(b, a)/l \text{ for } a \in B_l^b) \\ +\#B_l^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in B_l^b \text{ again}) \\ +\#B_{l}^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in B_{l}^b) \\ +\#B_{l}^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in B_{l}^b) \\ \vdots & \vdots \\ +\#A_l^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in A_l^b) \\ +\#A_l^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in A_l^b) \\ +\#A_l^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in A_l^b) \\ +\#A_l^b & (\text{try to divide } T(b, a) \text{ by } l \text{ for } a \in A_l^b) \\ +\cdots \\ = \#B_l^b + \#B_{l}^b + \#B_{l}^b + \#B_{l}^b + \cdots \\ +\#A_l^b + \#A_l^b + \#A_l^b + \#A_l^b + \cdots \\ +\#A_l^b + \#A_l^b + \#A_l^b + \#A_l^b + \cdots \\ = \#B_l^b(1 + 1 + 1 + \cdots) \\ +\#A_l^b(1 + 1 + 1 + \cdots) \\ +\#A_l^b + \#A_l^b + \#A_l^b + \#A_l^b + \cdots \\ = \#B_l^b - 1 + \#A_l^b + \#A_l^b + \#A_l^b + \cdots \\ +\#A_l^b + \#A_l^b + \#A_l^b + \#A_l^b + \cdots \\ +\cdots \end{cases}
\]

Therefore the total complexity of Algorithm 2 is
\[
(1 + o(1)) \sum_{b \in [1, a]} \sum_{\text{prime } l \in [2, b], l \mid b} C_l^b \quad \text{as } u \to \infty
\]

In the first part of Algorithm 3, the complexity of line 2–line 6 is an infinitesimal of the
Algorithm 3 sieve

1: (First)
2: for prime $l \in (0, y]$ do
3:   for $e = 1, 2, \cdots \log_l [u(m + 1)]$ do
4:     $\epsilon_l^{(e)} \leftarrow m \mod l^e \in \{0, 1, \cdots, l^e - 1\}$
5:   end for
6: end for
7: for integer $b \in (0, u]$ do
8:   for prime $l \in (0, y], l \nmid b$ do
9:     for $e = 1, 2, \cdots \log_l [u(m + 1)]$ do
10:    for $a \in [-u, u] \cap (be_l^{(e)} + l^e \mathbb{Z})$ do
11:     $T(b, a) \rightarrow T(b, a)/l$
12:   end for
13: end for
14: end for
15: end for
16: (Second)
17: for prime $l \in (0, y]$ do
18:   for $e = 1, 2, \cdots \log_l [m(d + 1)u^d]$ do
19:    $E_l^{(e)} \leftarrow \{x = 0, 1, \cdots l^e - 1 : x$ is a single root of $f(x) \equiv 0 \mod l^e\}$
20: end for
21: end for
22: for integer $b \in (0, u]$ do
23:   for prime $l \in (0, y], l \nmid b$ do
24:     for $e = 1, 2, \cdots \log_l [m(d + 1)u^d]$ do
25:      for $a \in [-u, u] \cap (bE_l^{(e)} + l^e \mathbb{Z})$ do
26:       $T(b, a) \leftarrow T(b, a)/l$
27:     end for
28:   end for
29: end for
30: end for
31: (Finally)
32: for prime $l \in (0, y]$ do
33:    $\tilde{E}_l^{(1)} \leftarrow \{x = 0, 1, \cdots l - 1 : x$ is a multiple root of $f(x) \equiv 0 \mod l\}$
34:    $\tilde{E}_l^{(2)} \leftarrow \{x \in \tilde{E}_l^{(1)} : f(x) \equiv 0 \mod l^2\}$
35: end for
36: for integer $b \in (0, u]$ do
37:   for prime $l \in (0, y], l \nmid b$ do
38:     for $a \in [-u, u] \cap (b\tilde{E}_l^{(1)} + l\mathbb{Z})$ do
39:      $T(b, a) \leftarrow T(b, a)/l$
40:   end for
41: for $a \in [-u, u] \cap (b\tilde{E}_l^{(2)} + l\mathbb{Z})$ do
42:     while $l \mid T(b, a)$ do
43:      $T(b, a) \leftarrow T(b, a)/l$
44:   end while
45: end for
46: end for
47: end for
Proposition 4.2. The complexity of line 17–line 21 is an infinitesimal of the complexity of line 22–line 30 as \( u \to \infty \). The latter is

\[
\frac{\frac{1}{l-1} B_b^h}{l-1} + \frac{1}{l-1} A_{i_2}^{b,s} + \frac{1}{l-1} A_{i_3}^{b,s} + \cdots = \frac{1}{l-1} A_{i_1}^{b,s} + \frac{1}{l-1} A_{i_2}^{b,m} + \frac{1}{l-1} A_{i_3}^{b,m} + \cdots
\]

The complexity of line 32–line 35 is an infinitesimal of the complexity of line 36–line 47 as \( u \to \infty \). The latter is

\[
\frac{1}{l-1} A_{i_1}^{b,m} + \frac{1}{l-1} A_{i_2}^{b,m} + \frac{1}{l-1} A_{i_3}^{b,m} + \cdots
\]

Therefore the total complexity of Algorithm 3 is

\[
(1 + o(1)) \sum_{b \in [1, u]} \sum_{i \in [2,y], l} D_i^b \quad \text{as } u \to \infty,
\]

where

\[
D_i^b = \frac{1}{l-1} B_b^h + \frac{1}{l-1} A_{i_2}^{b,s} + \frac{1}{l-1} A_{i_3}^{b,s} + \frac{1}{l-1} A_{i_4}^{b,m} + \frac{1}{l-1} A_{i_5}^{b,m} + \cdots
\]

It is easy to see that the complexity of Algorithm 3 is less than the complexity of Algorithm 2 asymptotically. Moreover, if for any \( (l, b) = 1 \) we have \( A_{i_2}^{b,m} = 0 \) and then we have \( D_i^b < \frac{2}{3} C_i^b \), because

\[
\frac{1}{l-1} B_b^h \leq \frac{2}{3} A_{i_2}^{b,m} \quad \text{for any } (l, b) = 1
\]

\[
\frac{1}{l-1} A_{i_2}^{b,s} \leq \frac{2}{3} A_{i_2}^{b,m} \quad \text{for any } (l, b) = 1
\]

\[
A_{i_4}^{b,m} < \frac{2}{3} A_{i_4}^{b,m} \quad \text{for any } (l, b) = 1
\]

\[
A_{i_5}^{b,m} = 0 \quad \text{for any } (l, b) = 1 \text{ and all } e > 1
\]

Therefore we get

**Proposition 4.1.** Let \( K > 0 \) be a constant. Let \( u \to \infty \) and \( y < Ku \). Then the complexity of Algorithm 3 is less than the complexity of Algorithm 2 asymptotically. Moreover, if for any \( (l, b) = 1 \) we have \( A_{i_2}^{b,m} = 0 \), then the complexity of Algorithm 3 is less than \( \frac{2}{3} \) of the complexity of Algorithm 2 asymptotically. 

The following proposition tells us that the condition "for any \( (b, l) = 1, A_{i_2}^{b,m} = 0 \)" has much chance to be realized.

**Proposition 4.2.** Suppose \( f(x) \) is a random polynomial of degree \( d \) over \( \mathbb{Z} \) such that \( f(x) \mod t^2 \) is uniform distribution on \( \{ h(x) \in \mathbb{Z}/t^2\mathbb{Z}[x] ; \deg h \leq d \} \) for all prime \( l \leq y \), and \( \{ R_l = 0 \}_{\text{prime } l \leq y} \) are independent random events, where \( R_l := \{ x \in \mathbb{Z}/t^2\mathbb{Z} ; f(x) \equiv 0 \mod t^2, x \text{ is a multiple root of } f(x) \equiv 0 \mod l \} \) for any prime \( l \leq y \). Then the probability of event \( A_{i_2}^{b,m} = 0 \), for any \( b \in (0, u) \), prime \( l \leq y \), s.t. \( (l, b) = 1 \) is greater than 0.6.

**Proof.** For any prime \( l \), we have

\[
P(\exists R_l \neq 0) = P(\exists i = 0, 1, \cdots, l - 1, \text{ s.t. } i \in R_l) \quad (\text{lemma 3.2})
\]

\[
\leq \sum_{i=0}^{l-1} P(i \in R_l)
\]
For any \( i = 0, 1, \ldots, l - 1 \), the assumption that \( f(x) \mod l^2 \) is uniform distribution on \( \{ h(x) \in \mathbb{Z}/l^2\mathbb{Z}[x]; \deg h \leq d \} \) and lemma 4.3 below show

\[
P(f(i) \equiv 0 \mod l^2, \ f'(i) \equiv 0 \mod l) = \frac{1}{l^3}
\]

i.e. \( P(i \in R_i) = \frac{1}{l} \). Hence \( P(\notin R_i) \leq \frac{1}{l^2} \). The assumption that \( \{ R_i = \phi \} \) prime \( i \leq y \) are independent random events implies

\[
P(\sum_{\text{prime } l \leq y} \notin R_i = 0) = \prod_{\text{prime } l \leq y} P(\notin R_i = 0) = \prod_{\text{prime } l \leq y} (1 - \frac{1}{l})
\]

\[
> \prod_{l: \text{prime}} (1 - \frac{1}{l^2}) = \frac{1}{\zeta(2)} > 0.6,
\]

where \( \zeta(s) \) is the Riemann’s Zeta function.

Let \( R_i^b : = \{ a \in \mathbb{Z}/l^2\mathbb{Z}; Nm(a - b\theta) \equiv 0 \mod l^2, a \text{ is a multiple root of } Nm(a - b\theta) \equiv 0 \mod l \} \). From lemma 3.1, we know

\[
\notin R_i^b = \notin R_i \quad \text{for all } (b, l) = 1
\]

Therefore

\[
P(\notin A_{l}^{b, m} = 0, \text{ for any } b \in (0, u], \text{ prime } l \leq y, \text{ s.t } (l, b) = 1)
\geq P(\notin R_i^b = 0, \text{ for any } b \in (0, u], \text{ prime } l \leq y, \text{ s.t } (l, b) = 1)
= P(\notin R_i = 0, \text{ for any prime } l \leq y) > 0.6
\]

Now we give the statement and proof of lemma 4.3 mentioned above.

**Lemma 4.3.** Let \( l \) be a prime and \( d \) be a positive integer. For any \( i \in \mathbb{Z}/l^2\mathbb{Z} \), we have

\[
P(h(i) \equiv 0 \mod l^2, h'(i) \equiv 0 \mod l \mid h(x) \in \mathbb{Z}/l^2\mathbb{Z}[x], \deg h = d, \text{monic}) = \frac{1}{l^3}
\]

**Proof.** Consider the surjective homomorphism of abelian group

\[
\{ h(x) \in \mathbb{Z}/l^2\mathbb{Z}[x]; \deg h \leq d \} \rightarrow \mathbb{Z}/l^2\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}, \ h(x) \mapsto (h(i) \mod l^2, h'(i) \mod l).
\]

We have

\[
P(h(i) \equiv 0 \mod l^2, h'(i) \equiv 0 \mod l \mid h(x) \in \mathbb{Z}/l^2\mathbb{Z}[x], \deg h \leq d) = \frac{1}{l^3}.
\]

Similarly, we have

\[
P(h(i) \equiv 0 \mod l^2, h'(i) \equiv 0 \mod l \mid h(x) \in \mathbb{Z}/l^2\mathbb{Z}[x], \deg h \leq d - 1) = \frac{1}{l^3}.
\]

Let

\[
H := \{ h \in \mathbb{Z}/l^2\mathbb{Z}[x]; \deg f \leq d \}
\]

\[
H_0 := \{ h \in H; h(i) \equiv 0 \mod l^2, h'(i) \equiv 0 \mod l \}
\]

\[
H_{l\mathbb{Z}} := \{ h \in H; \text{the leading coefficient of h is in } l\mathbb{Z} \}
\]

\[
H_{l\mathbb{Z}}^0 := H_0 \cap H_{l\mathbb{Z}}
\]

\[
H_c := \{ h \in H; \text{the leading coefficient of h is } c \} \quad \text{for any } c \in \mathbb{Z}/l^2\mathbb{Z}
\]

\[
H_c^0 := H_c \cap H^0 \quad \text{for any } c \in \mathbb{Z}/l^2\mathbb{Z}
\]
It is easy to see that
\[ H_0 = \{ h \in \mathbb{Z}/l^2\mathbb{Z}[x]; \deg g \leq d - 1 \} \]
\[ H_0' = \{ h \in H_0; h(i) \equiv 0 \mod l^2; h'(i) \equiv 0 \mod l \} \]

Hence, we have
\[ P(h \in H_0' | h \in H_0) = P(h \in H^0 | h \in H) = \frac{1}{l^3} \]

Let us consider the commutative diagram of abelian groups
\[
\begin{array}{cccccc}
0 & \longrightarrow & H_1^{l\mathbb{Z}} & \longrightarrow & H_0^{\mathbb{Z}/l^2\mathbb{Z}} & \longrightarrow & 0 \\
0 & \longrightarrow & H_0^{l\mathbb{Z}} & \longrightarrow & H & \longrightarrow & \mathbb{Z}/l^2\mathbb{Z} & \longrightarrow & 0 \\
\end{array}
\]

where the map from $H$ to $\mathbb{Z}/l^2\mathbb{Z}$ is defined by $h \mapsto (\text{the leading coefficient of } h)$. The vertical map in the right side is an identity, hence we have
\[ P(h \in H_0' | h \in H_0) = P(h \in H^0 | h \in H) = \frac{1}{l^3} \]

On the other hand,
\[
P(h \in H_0' | h \in H_0) = \sum_{c=0}^{l-1} P(h \in H_0' | h \in H_c)P(h \in H_c | h \in H_0) = \sum_{c=0}^{l-1} P(h \in H_0' | h \in H_c)P(h \in H_c | h \in H_0) = \sum_{c=0}^{l-1} P(h \in H_0' | h \in H_c)P(h \in H_c | h \in H_0)
\]
\[ = \sum_{c=0}^{l-1} P(h \in H_0' | h \in H_c) + \sum_{c=1}^{l-1} P(h \in H_0' | h \in H_c)
\]
\[ = \frac{1}{l} + \sum_{c=1}^{l-1} P(h \in H_0' | h \in H_c)
\]

Hence we have
\[ \sum_{c=1}^{l-1} P(h \in H_0' | h \in H_c) = \frac{l-1}{l^3} \]

For any $c \in (\mathbb{Z}/l^2\mathbb{Z})^\times$, we have a commutative diagram of sets
\[
\begin{array}{cccc}
H_0' & \sim & H_0' \\
\downarrow & & \downarrow \\
H_c & \sim & H_c
\end{array}
\]

where the horizontal map is defined by $h \mapsto ch$. Hence we have
\[ P(h \in H_0' | h \in H_c) = P(h \in H_0' | h \in H_c) \text{ for any } c \in (\mathbb{Z}/l^2\mathbb{Z})^\times \]

Therefore
\[
\begin{align*}
\frac{1}{l^3} = & \quad P(h \in H^0 | h \in H) \\
= & \quad \sum_{c \in \mathbb{Z}/l^2\mathbb{Z}} P(h \in H^0 | h \in H_c)P(h \in H_c | H) \\
= & \quad \sum_{c \in \mathbb{Z}/l^2\mathbb{Z}} P(h \in H_0^c | h \in H_c)P(H_c | H) \\
= & \quad \sum_{c \in \mathbb{Z}/l^2\mathbb{Z}} P(h \in H_0^c | h \in H_c) + \sum_{c=1}^{l-1} P(h \in H_0^c | h \in H_c) + \sum_{c \in (\mathbb{Z}/l^2\mathbb{Z})^\times} P(h \in H_0^c | h \in H_c) 	imes \frac{1}{l^3}
\end{align*}
\]
\[ = \left( \frac{1}{l^3} + \frac{l-1}{l^3} + (l^2 - l) P(h \in H_0^0 | h \in H_1) \right) \times \frac{1}{l^3} \]
Hence we have
\[ P(h \in H_0^1 \mid h \in H_1) = \frac{1}{l^3} \]

Finally, from proposition 4.1 and proposition 4.2, we get the main conclusion of this paper:

**Proposition 4.4.** Let \( K > 0 \) be a constant. Let \( u \to \infty \) and \( y < Ku \), then the complexity of Algorithm 3 is less than the complexity of Algorithm 2 asymptotically. Moreover, suppose \( f(x) \) is a random polynomial of degree \( d \) over \( \mathbb{Z} \) such that \( f(x) \mod l^2 \) is uniform distribution on \( \{h(x) \in \mathbb{Z}/l^2\mathbb{Z}; \deg h \leq d \} \) for all prime \( l \leq y \), and \( \{R_l = \phi \}_{\text{prime } l \leq y} \) are independent random events, where \( R_l := \{x \in \mathbb{Z}/l^2\mathbb{Z}; f(x) \equiv 0 \mod l^2, x \text{ is a multiple root of } f(x) \equiv 0 \mod l \} \) for any prime \( l \leq y \), then the complexity of Algorithm 3 is less than \( \frac{2}{3} \) of the complexity of Algorithm 2 asymptotically with probability greater than 0.6. ■

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**References**

[1] J.P. Buhler, H.W. Lenstra, Jr., C. Pomerance, Factoring Integers with the Number Field Sieve, in A.K. Lenstra and H.W. Lenstra, Jr. (eds), The Development of the Number Field Sieve, Lecture Notes in Mathematics 1554, Springer-Verlag, New York, 1993, pp. 50-94

[2] D. M. Gordon, "Discrete logarithms in GF(p) using the number field sieve", SIAM J. Discrete Math. 6:1 (1993), 124-138

[3] O. Schirokauer, Discrete logarithms and local units. Philos. Trans. Roy. Soc. London Ser. A, vol. 345, The Royal Society, London, 1993, pp. 409-423.

[4] A. Joux, R. Lercier, Improvements to the general number field sieve for discrete logarithms in prime fields. A comparison with the Gaussian integer method. Math. Comp. 72 (2003), no. 242, 953967 (electronic).

[5] O. Schirokauer, The impact of the number field sieve on the discrete logarithm problem in finite fields. Algorithmic number theory: lattices, number fields, curves and cryptography, 397420, Math. Sci. Res. Inst. Publ., 44, Cambridge Univ. Press, Cambridge, 2008.

[6] P. Stevenhagen, The number field sieve. Algorithmic number theory: lattices, number fields, curves and cryptography, 83100, Math. Sci. Res. Inst. Publ., 44, Cambridge Univ. Press, Cambridge, 2008.