MINIMAL MASS BLOW-UP SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A POTENTIAL

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Abstract. We consider a mass critical nonlinear Schrödinger equation with a real-valued potential. In this work, we construct a minimal mass solution that blows up at finite time, under weaker assumptions on spatial dimensions and potentials than Banica, Carles, and Duyckaerts (2011). Moreover, we show that the blow-up solution converges to a blow-up profile. Furthermore, we improve some parts of the arguments in Raphaël and Szeftel (2011) and Le Coz, Martel, and Raphaël (2016).

1. Introduction

We consider the following nonlinear Schrödinger equation:

\[ \begin{align*}
\frac{\partial u}{\partial t} + \Delta u + g(x)|u|^N u - V(x)u &= 0, \\
(t, x) &\in \mathbb{R} \times \mathbb{R}^N,
\end{align*} \]

where \( N \in \mathbb{N} \), \( g \) is a real-valued function, and \( V \) is a real-valued potential. It is well known that if \( V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \) (see, e.g., [3]), then (1) is locally well-posed in \( H^1(\mathbb{R}^N) \). This means that for any initial value \( u_0 \in H^1(\mathbb{R}^N) \), there exists a unique maximal solution \( u \in C((0, T^*), H^1(\mathbb{R}^N)) \cap C^1((0, T^*), H^{-1}(\mathbb{R}^N)) \) for (1) with \( u(0) = u_0 \).

Moreover, the mass (i.e., \( L^2 \)-norm) and energy of the solution \( u \) are conserved by the flow, where

\[ E(u) := \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2 + \frac{4}{N}} \int_{\mathbb{R}^N} g(x)|u(x)|^{2+\frac{4}{N}} dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx. \]

Furthermore, the blow-up alternative holds:

\[ T^* < \infty \quad \text{implies} \quad \lim_{t \to T^*} \|\nabla u(t)\|_2 = \infty. \]

We define \( \Sigma^k \) by

\[ \Sigma^k := \{ u \in H^k(\mathbb{R}^N) \mid |x|^k u \in L^2(\mathbb{R}^N) \}, \quad \|u\|_{\Sigma^k}^2 := \|u\|_{H^k}^2 + \||x|^k u\|_2^2. \]

Particularly, \( \Sigma^1 \) is called the virial space. If \( u_0 \in \Sigma^1 \), then the solution \( u \) for (1) with \( u(0) = u_0 \) belongs to \( C((T_*, T^*), \Sigma^1(\mathbb{R}^N)) \).

Moreover, we consider

\[ V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \left( p \geq 2 \quad \text{and} \quad p > \frac{N}{2} \right). \]

If \( u_0 \in \Sigma^2 \), then the solution \( u \) for (1) with \( u(0) = u_0 \) belongs to \( C((T_*, T^*), \Sigma^2(\mathbb{R}^N)) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N)) \) and \( |x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N)) \).

In this paper, we investigate conditions for the potential related with the existence of a minimal mass blow-up solution for (1).

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1.1. The case \( V = 0 \) and \( g = 1 \). Firstly, we describe the results when \( V \) is a real constant and \( g = 1 \). Let \( u_V \) be a solution for (1) and define \( u(t, x) := u_V(t, x) e^{it\xi} \). Then \( u \) is a solution for (1) with \( V = 0 \) and \( g = 1 \). Therefore, we may assume that \( V = 0 \), that is, we consider

\[
i \partial u \over \partial t + \Delta u + |u|^\frac{4}{N} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
\]

It is well known ([2] [3] [4]) that there exists a unique classical solution \( Q \) for

\[-\Delta Q + \frac{4}{N} Q - |Q|^{\frac{4}{N}} Q = 0, \quad Q \in H^1(\mathbb{R}^N), \quad Q > 0, \quad Q \text{ is radial},\]

which is called the ground state. If \( \|u\|_2 = \|Q\|_2 \) \((\|u\|_2 < \|Q\|_2, \|u\|_2 > \|Q\|_2)\), we say that \( u \) has the critical mass (subcritical mass, supercritical mass, respectively).

We note that \( E_{\text{crit}}(Q) = 0 \), where \( E_{\text{crit}} \) is the energy when \( V = 0 \). Moreover, the ground state \( Q \) attains the best constant in the Gagliardo-Nirenberg inequality

\[
\|v\|_2^2 + \frac{4}{N} \leq \left( 1 + \frac{2}{N} \right) \left( \frac{\|v\|_2}{\|Q\|_2} \right)^{\frac{4}{N}} \|\nabla v\|_2^2 \quad \text{for } v \in H^1(\mathbb{R}^N).
\]

Therefore, for all \( v \in H^1(\mathbb{R}^N) \),

\[
E_{\text{crit}}(v) \geq \frac{1}{2} \|\nabla v\|_2^2 \left( 1 - \left( \frac{\|v\|_2}{\|Q\|_2} \right)^\frac{4}{N} \right)
\]

holds. This inequality and the mass and energy conservations imply that any subcritical mass solution for (5) exists globally in time and is bounded in \( H^1(\mathbb{R}^N) \).

Regarding the critical mass case, we apply the pseudo-conformal transformation

\[
u(t, x) \mapsto \frac{1}{|t|^{\frac{4}{N}}} u \left( -\frac{1}{t}, i \frac{x}{t} \right) e^{i|u|^2/2t}
\]

to the solitary wave solution \( u(t, x) := Q(x) e^{it} \). Then we obtain

\[
S(t, x) := \frac{1}{|t|^{\frac{4}{N}}} Q \left( \frac{x}{t} \right) e^{-i|u|^2/2t} e^{i|u|^2/2t},
\]

which is also a solution for (5) and satisfies

\[
\|S(t)\|_2 = \|Q\|_2, \quad \|\nabla S(t)\|_2 \sim \frac{1}{|t|} \quad (t \nearrow 0).
\]

Namely, \( S \) is a minimal mass blow-up solution for (5). Moreover, \( S \) is the only finite time blow-up solution for (5) with critical mass, up to the symmetries of the flow (see [6]).

Regarding the supercritical mass case, there exists a solution \( u \) for (5) such that

\[
\|\nabla u(t)\|_2 \sim \sqrt{\log \log |T^* - t|} \quad (t \nearrow T^*)
\]

(see [8] [9]).

1.2. Previous results. Banica, Carles, and Duyckaerts [1] present the following result for (1).

Theorem 1.1 ([4]). Let \( N = 1 \) or \( 2 \), \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \), and \( g \in C^4(\mathbb{R}^N, \mathbb{R}) \). Assume \( \left( \frac{\partial}{\partial x} \right)^\beta V \in L^\infty(\mathbb{R}^N) \) \((|\beta| \leq 2)\),

\[
\left( \frac{\partial}{\partial x} \right)^\beta g \in L^\infty(\mathbb{R}^N) \quad (|\beta| \leq 4, \beta \neq 2),
\]

and

\[
g(0) = 1, \quad \frac{\partial g}{\partial x_j}(0) = \frac{\partial^2 g}{\partial x_j \partial x_k}(0) = 0 \quad (1 \leq j, k \leq N).
\]

Then there exist \( T > 0 \) and a solution \( u \in C((0, T), \Sigma^1) \) for (1) such that

\[
\left\| u(t) - \frac{1}{\lambda(t)^{\frac{4}{N}}} Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i|u|^2/\lambda^2} e^{i\beta V(0)} \right\|_{\Sigma^1(t \searrow 0) \to 0}
\]
where \( \theta \) and \( \lambda \) are continuous real-valued functions and \( x \) is a continuous \( \mathbb{R}^N \)-valued function such that

\[
\theta(\tau) = \tau + o(\tau) \quad \text{as} \ \tau \to +\infty,
\]
\[
\lambda(t) \sim t \quad \text{and} \quad |x(t)| = o(t) \quad \text{as} \ t \searrow 0.
\]

This result means that if \( V \) and \( g \) are sufficiently smooth and bounded, then there exists a minimal mass solution that blows up at finite time with a blow-up rate \( |t|^{-1} \). The blow-up rate is identical with the blow-up rate when \( g = 1 \) and \( V = 0 \).

Le Coz, Martel, and Raphaël [5] present the following result for

\[
i \frac{\partial u}{\partial t} + \Delta u + |u|^p u + |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
\]

**Theorem 1.3** [5]. Let \( N = 1, 2, 3 \) and \( 1 < p < 1 + \frac{4}{N} \). Then for any energy level \( E_0 \in \mathbb{R} \), there exist \( t_0 < 0 \) and a radially symmetric initial value \( u_0 \in H^1(\mathbb{R}^N) \) with

\[
\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0
\]
such that the corresponding solution \( u \) for (6) with \( u(t_0) = u_0 \) blows up at \( t = 0 \) with a blow-up rate of

\[
\|\nabla u(t)\|_2 = \frac{C(p) + o_\sigma(t)}{|t|^{\sigma}},
\]

where \( \sigma = \frac{4}{4 + N(p - 1)} \) and \( C(p) > 0 \).

This result means that the attractive nonlinearity \( |u|^{p-1} u \) affects blow-up rates of blow-up solutions with critical mass. Moreover, for any energy level, there exists a blow-up solution with critical mass and the energy.

### 1.3. Main result

For the potential \( V \), we consider the following:

\begin{align}
 V & \in C^{1,1}_{\text{loc}} \mathbb{R}^N, \\
 \nabla V & = O(|x|), \\
 \nabla^2 V & = O(|x|^r) \quad \text{for some} \ r \geq 0.
\end{align}

For the function \( g \), we consider the following:

\begin{align}
 g & \in C^{3,1}_{\text{loc}} \mathbb{R}^N, \\
 g(0) & = 1, \quad \frac{\partial g}{\partial x_j}(0) = \frac{\partial^2 g}{\partial x_j \partial x_k}(0) = 0 \quad (1 \leq j, k \leq N), \\
 g, \nabla g, x \cdot \nabla g & \in L^\infty(\mathbb{R}^N), \\
 \nabla^3 g, \nabla^4 g & = O(|x|^r) \quad \text{for some} \ r_g \geq 0
\end{align}

The main result of this paper is the following, which gives an extension of Theorem 1.1

**Theorem 1.3** (Existence of a minimal mass blow-up solution). Let the potential \( V \) satisfy (7), (8), (9), and (10). Let the function \( g \) satisfy (11), (12), (13), and (14). Then there exist \( t_0 < 0 \) and a radial initial value \( u_0 \in \Sigma^1 \) with \( \|u_0\|_2 = \|Q\|_2 \) such that the corresponding solution \( u \) for (1) with \( u(t_0) = u_0 \) blows up at \( t = 0 \). Moreover,

\[
\left\| u(t, x) - \frac{1}{\lambda(t)} \sum_{\Sigma^1} Q \left( \frac{x + w(t)}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{\gamma(t)} |x + w(t)|^2 + \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
\]

holds for some \( C^1 \) functions \( \lambda : (t_0, 0) \to (0, \infty), \ b, \gamma : (t_0, 0) \to \mathbb{R}, \) and \( w : (t_0, 0) \to \mathbb{R}^N \) such that

\[
\lambda(t) = |t| (1 + o(1)), \quad b(t) = |t| (1 + o(1)), \quad \gamma(t) \sim |t|^{-1}, \quad |w(t)| = o(|t|)
\]

as \( t \nearrow 0 \).

**Remark 1.4.** In contrast, if \( V \) satisfies (6), then any subcritical mass solution for (1) exists globally in time and is bounded in \( H^1(\mathbb{R}^N) \). This can be proved easily by the Gagliardo-Nirenberg inequality and the Sobolev embedding theorem. Therefore, the solution in Theorem 1.3 is a minimal mass blow-up solution.
1.4. **Outline of proof.** We prove Theorem 1.3 by using a simplified version with modification of the method of Le Coz, Martel, and Raphaël [5], which is based on seminal work of Raphaël and Szeftel [11]. We proceed in the following steps:

**Step 1.** For a solution $u$ for (1), we consider the following transformation:

$$u(t, x) = \frac{1}{\lambda(s)^{\frac{N}{2}}} v(s, y) e^{-i\lambda(s)|y|^2 + iy(s)}, \quad y = \frac{x + w(s)}{\lambda(s)}, \quad ds = \frac{1}{\lambda(s)^2} dt.$$

**Step 2.** Let $v = Q + \varepsilon$ for some error function $\varepsilon$. Then we obtain the equation of $\varepsilon$ (Lemmas 2.3 and 3.1):

$$0 = i \frac{\partial \varepsilon}{\partial s} + \Delta \varepsilon - \varepsilon + g(\lambda y - w)|Q + \varepsilon|^2 (Q + \varepsilon)^{1+\frac{4}{N}} - \lambda^2 V(\lambda y - w)\varepsilon + \text{modulation terms} + \text{an error term}.$$

**Step 3.** By using the modulation terms and $\varepsilon$, we estimate the parameters $\lambda$, $b$, $\gamma$, and $w$ (from Section 3 to Section 6).

**Step 4.** We construct a sequence of suitable solutions for (11) and show that the limit of the sequence is the desired minimal mass blow-up solution (Section 7).

1.5. **Comments on Theorem 1.3.** Firstly, the assumptions in Theorem 1.3 are weaker than those in Theorem 1.1. Theorem 1.3 has no restrictions on spatial dimensions. On the other hand, according to the lack of regularity of the nonlinearity $|u|^\frac{4}{N}u$, Theorem 1.1 requires the restriction $N = 1$ or 2. Although Theorem 1.3 is also affected by the lack of regularity, we overcome this difficulty by using the properties of the ground state. Regarding potentials, Theorem 1.3 requires less differentiability and integrability than Theorem 1.1. Indeed, any $V \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ satisfies (11), (7), (8), and (9). On the other hand, there exists $V$ such that it satisfies (11), (7), (8), and (9) but does not satisfy the assumption in Theorem 1.1 e.g.,

$$V(x) := \frac{1}{1 + x^2} \cos(x^4).$$

When $V$ does not satisfy (7), blow-up rates should change as the result of Le Coz, Martel, and Raphaël [5].

Regarding the function $g$, Theorem 1.3 does not require higher-order derivatives to be bounded. For example, Theorem 1.3 applies to the following function:

$$g(x) := \frac{1}{1 + x^2} \cos(x^4).$$

Secondly, we improve some parts of the arguments in Le Coz, Martel, and Raphaël [5] and Raphaël and Szeftel [10]. Although the authors of [5, 10] introduce the Morawetz functional (5, Section 5) and apply a truncation procedure to the functional, we avoid using the functional by modifying the definition of $\varepsilon$. As a result, without the truncation, we work directly in the virial space $\Sigma^1$. Moreover, the authors of [5] use the continuous dependence on the initial value for (3) in $H^s(\mathbb{R}^N)$ for some $s \in [0, 1)$. Although this continuous dependence is an important fact in the proof of the main result in [5], it is not obvious for (11). Therefore, instead of proving the continuous dependence for (11) in $H^s(\mathbb{R}^N)$ for some $s \in [0, 1)$, we use Lemma A.1 in Appendix A which gives a kind of the continuous dependence. Consequently, we provide a simpler and more general proof.

2. **Notation and preliminaries.**

We define

$$(u, v)_2 := \text{Re} \int_{\mathbb{R}^N} u(x)\overline{v(x)} dx, \quad \|u\|_p := \left( \int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}},$$

$$f(z) := |z|^\frac{4}{N}z, \quad F(z) := \frac{1}{2} + \frac{1}{N} |z|^2 + \frac{4}{N} \quad \text{for } z \in \mathbb{C}.$$
By identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \), we denote the differentials of \( f \) and \( F \) by \( df \) and \( dF \), respectively. We define
\[
\Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^\pm, \quad L_- := -\Delta + 1 - Q^\pm.
\]
Namely, \( \Lambda \) is the generator of \( L^2 \)-scaling, and \( L_+ \) and \( L_- \) come from the linearised Schrödinger operator to close \( Q \). Then
\[
L_+ - Q = 0, \quad L_+ \Lambda Q = -2Q, \quad L_- |x|^2Q = -4\Lambda Q, \quad L_+ \rho = |x|^2Q, \quad L_- Q = -\nabla Q
\]
hold, where \( \rho \in \mathcal{S}(\mathbb{R}^N) \) is the unique radial solution for \( L_+ \rho = |x|^2Q \). Furthermore, there exists \( \mu > 0 \) such that for any \( u \in H^1(\mathbb{R}^N) \),
\[
\langle L_+ \text{Re} u, \text{Re} u \rangle + \langle L_- \text{Im} u, \text{Im} u \rangle 
\]
holds (see, e.g., [7, 8, 10, 12]). Finally, we use the notation \( \lesssim \) and \( \gtrsim \) when the inequalities hold up to a positive constant. We also use the notation \( \approx \) when \( \lesssim \) and \( \gtrsim \) hold.

For the ground state \( Q \), the following property holds:

**Proposition 2.1** (E.g., [5]). For any multi-index \( \alpha \), there exist \( C_\alpha, \kappa_\alpha > 0 \) such that
\[
\left| \left( \frac{\partial}{\partial x} \right)^\alpha Q(x) \right| \leq C_\alpha Q(x), \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x).
\]
We estimate the error term \( \Psi \) that is defined by
\[
\Psi(y) := \lambda^2 V(\lambda y - w) Q(y).
\]
Without loss of generality, we may assume that \( V(0) = 0 \) (see Section 1.4).

**Proposition 2.2** (Estimate of \( \Psi \)). There exists a sufficiently small constant \( \epsilon' > 0 \) such that
\[
\left\| e^{\epsilon'|y|} \Psi \right\|_2 + \left\| e^{\epsilon'|y|} \nabla \Psi \right\|_2 \lesssim \lambda^2 (\lambda + |w|)
\]
for \( 0 < \lambda \ll 1 \) and \( w \in \mathbb{R}^N \). Moreover, for any radial function \( \varphi \in L^2(\mathbb{R}^N) \),
\[
|\langle \Psi, \varphi \rangle|_2 \lesssim \lambda^2 |w| + \lambda^4
\]
**proof.** By using Taylor’s theorem and \( V(0) = 0 \), we write
\[
\lambda^2 V(\lambda y - w) = \lambda^2 (\lambda y - w) \cdot \nabla V(0) + \sum_{|\alpha| = 2}^1 \int_0^1 \lambda^2 (\lambda y - w)^\alpha \frac{\partial^\alpha V}{\partial x^\alpha} (\tau (\lambda y - w))(1 - \tau) d\tau,
\]
\[
\lambda^2 \frac{\partial V}{\partial x_j} (\lambda y - w) = \lambda^3 \frac{\partial V}{\partial x_j} (0) + \int_0^1 \lambda^3 (\lambda y - w) \cdot \left( \nabla \frac{\partial V}{\partial x_j} \right) (\tau (\lambda y - w)) d\tau.
\]
Therefore, we have
\[
|\Psi(y)| \lesssim \lambda^2 (\lambda |y| + |w|) Q(y) + \lambda^2 (\lambda |y| + |w|)^{2+r} Q(y),
\]
\[
|\nabla \Psi(y)| \leq \lambda^3 Q(y) + \lambda^2 (\lambda |y| + |w|)^{1+r} Q + \lambda^2 (\lambda |y| + |w|) |\nabla Q(y)| + \lambda^2 (\lambda |y| + |w|)^{2+r} |\nabla Q(y)|.
\]
Therefore, according to Proposition 2.1 and the exponential decay of \( Q \) ([3, Theorem 8.1.1]), there exists a sufficiently small constant \( \epsilon' > 0 \) such that
\[
\left\| e^{\epsilon'|y|} \Psi \right\|_2 + \left\| e^{\epsilon'|y|} \nabla \Psi \right\|_2 \lesssim \lambda^2 (\lambda + |w|).
\]
Since \( \langle yQ, \varphi \rangle = 0 \) for any radial function \( \varphi \in L^2(\mathbb{R}^N) \), we obtain
\[
|\langle \Psi, \varphi \rangle|_2 = -\lambda^2 w \cdot \nabla V(0)(Q, \varphi)_2 + \sum_{|\alpha| = 2}^1 \int_0^1 \lambda^2 \left( (\lambda y - w)^\alpha \frac{\partial^\alpha V}{\partial x^\alpha} (\tau (\lambda y - w))Q, \varphi \right)_2 (1 - \tau) d\tau.
\]
Therefore, we obtain conclusion. \( \square \)

At the end of this section, we state the following standard result. For the proof, see [3].
Lemma 2.3 (Decomposition). There exists $\overline{C} > 0$ such that the following statement holds. Let $I$ be an interval and $\delta > 0$ be sufficiently small. We assume that $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$ satisfies
\[
\forall \ t \in I, \ \| \lambda(t) \frac{\partial}{\partial t} u(t, \lambda(t) y - w(t)) e^{i \gamma(t)} - Q \|_{H^1} < \delta
\]
for some functions $\lambda : I \to (0, \infty)$, $\gamma : I \to \mathbb{R}$, and $w : I \to \mathbb{R}^N$. Then there exist unique functions $\tilde{\lambda} : I \to (0, \infty)$, $\tilde{b} : I \to \mathbb{R}$, $\tilde{\gamma} : I \to \mathbb{R}/2\pi \mathbb{Z}$, and $\tilde{w} : I \to \mathbb{R}^N$ such that
\[
(15) \quad u(t, x) = \frac{1}{\lambda(t)} Q \left( t, \frac{x + \tilde{w}(t)}{\lambda(t)} \right) e^{-i \frac{b(t)}{\lambda(t)} + i \gamma(t)},
\]
\[
\left| \frac{\tilde{\lambda}(t)}{\lambda(t)} - 1 \right| + \left| \tilde{b}(t) \right| + \left| \tilde{\gamma}(t) - \gamma(t) \right|_{\mathbb{R}/2\pi \mathbb{Z}} + \left| \frac{\tilde{w}(t) - w(t)}{\lambda(t)} \right| < \overline{C}
\]
hold, where $\| \cdot \|_{\mathbb{R}/2\pi \mathbb{Z}}$ is defined by
\[
|c|_{\mathbb{R}/2\pi \mathbb{Z}} := \inf_{m \in \mathbb{Z}} |c + 2\pi m|,
\]
and that $\tilde{\varepsilon}$ satisfies the orthogonal conditions
\[
(16) \quad (\tilde{\varepsilon}, iA Q)_2 = (\tilde{\varepsilon}, |y|^2 Q)_2 = (\tilde{\varepsilon}, i\rho)_2 = 0, \quad (\tilde{\varepsilon}, y Q)_2 = 0
\]
on $I$. In particular, $\tilde{\lambda}$, $\tilde{b}$, $\tilde{\gamma}$, and $\tilde{w}$ are $C^1$ functions and independent of $\lambda$, $\gamma$, and $w$.

3. Uniformity estimates for modulation terms

From this section to Section 6, we prepare lemmas for the proof of Theorem 1.3.

Given $t_1 < 0$ which is sufficiently close to 0, we define $s_1 := -t_1^{-1}$ and $\lambda_1 = b_1 = s_1^{-1}$. Let $u(t)$ be the solution for (10) with an initial value
\[
(17) \quad u(t_1, x) := \frac{1}{\lambda_1} Q \left( x \frac{1}{\lambda_1} \right) e^{-i \frac{b_1}{\lambda_1} + i \frac{|x|^2}{\lambda_1}}.
\]

Note that $u \in C(((T_*, T^*), \Sigma^2(\mathbb{R}^N))$ and $|x| \nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$. Moreover,
\[
\text{Im} \int_{\mathbb{R}^N} u(t_1, x) \nabla u(t_1, x) dx = 0
\]
holds.

Since $u$ satisfies the assumption in Lemma 2.3 in a neighbourhood of $t_1$, there exist decomposition parameters $\tilde{\lambda}_{t_1}$, $\tilde{b}_{t_1}$, $\tilde{\gamma}_{t_1}$, $\tilde{w}_{t_1}$, and $\tilde{\varepsilon}_{t_1}$ such that (15) and (16) hold in the neighbourhood. We define the rescaled time $s_{t_1}$ by
\[
s_{t_1}(t) := s_1 - \int_t^{t_1} \frac{1}{\lambda_{t_1}(\tau)^2} d\tau.
\]

Moreover, we define
\[
t_{t_1} := s_{t_1}^{-1}, \quad \lambda_{t_1}(s) := \tilde{\lambda}_{t_1}(t_{t_1}(s)), \quad b_{t_1}(s) := \tilde{b}_{t_1}(t_{t_1}(s)), \quad \gamma_{t_1}(s) := \tilde{\gamma}_{t_1}(t_{t_1}(s)), \quad w_{t_1}(s) := \tilde{w}_{t_1}(t_{t_1}(s)), \quad \varepsilon_{t_1}(s, y) := \tilde{\varepsilon}_{t_1}(t_{t_1}(s), y).
\]
For the sake of clarity in notation, we often omit the subscript $t_1$. Furthermore, let $I_{t_1}$ be the maximal interval of the existence of the decomposition such that (15) and (16) hold and we define
\[
J_{s_1} := s_{t_1}(I_{t_1}).
\]

Additionally, let $s_0 (\leq s_1)$ be sufficiently large and
\[
s' := \max \{ s_0, \inf J_{s_1} \}.
\]

Let $K$ be sufficiently large and $L$ and $M$ satisfy
\[
L = \frac{3}{2} + \frac{1}{K}, \quad 1 < M < 2(L - 1).
\]

Moreover, we define $s_*$ by
\[
s_* := \inf \{ \sigma \in (s', s_1) \mid (15) \text{ holds on } [\sigma, s_1] \},
\]
Secondly, we obtain
\[ \|u\|_{H^1}^2 + b(s)^2 \|y(u)\|_2^2 < s^{-2L}, \]
\[ s\lambda(s) - 1 < s^{-M}, \quad |sb(s) - 1| < s^{-M}, \quad |w(s)| < s^{-\frac{1}{2}}. \]

Note that for all \( s \in (s_*, s_1] \), we have
\[ s^{-1}(1 - s^{-M}) < \lambda(s), b(s) < s^{-1}(1 + s^{-M}). \]

Finally, we define
\[ \text{Mod}(s) := \left( \frac{1}{2} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2, 1 - \frac{\partial \gamma}{\partial s}, \frac{\partial w}{\partial s} \right). \]

The goal of this section is to estimate of \( \text{Mod}(s) \).

In the following, positive constants \( C \) and \( \epsilon \) are sufficiently large and small, respectively. If necessary, we retake \( s_0 \) and \( s_1 \) sufficiently large in response to \( \epsilon \).

**Lemma 3.1** (The equation for \( \epsilon \)). On \( J_{s_1} \),
\[ \Psi = i \frac{\partial \epsilon}{\partial s} + \Delta \epsilon - \epsilon + g(\lambda y - w)f(Q + \epsilon) - f(Q) - \lambda^2 V(\lambda y - w) \epsilon \]
\[ - i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \Lambda(Q + \epsilon) + \left( 1 - \frac{\partial \gamma}{\partial s} \right)(Q + \epsilon) + \left( \frac{\partial b}{\partial s} + b^2 \right) \frac{|w|^2}{4}(Q + \epsilon) \]
\[ - \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \frac{|w|^2}{2}(Q + \epsilon) + i \frac{\partial w}{\lambda} \cdot \nabla(Q + \epsilon) + \frac{1}{2} \frac{\partial w}{\lambda} \cdot y(Q + \epsilon). \]

**proof.** This result is proven via direct calculation. \( \Box \)

**Lemma 3.2.** For \( g \),
\[ g(\lambda y - w) = 1 + \frac{1}{6} \sum_{|\alpha|=3} (\lambda y - w)^\alpha \frac{\partial^\alpha g}{\partial x^\alpha}(0) + \frac{1}{6} \sum_{|\alpha|=4} \int_0^1 (\lambda y - w)^\alpha \frac{\partial^\alpha g}{\partial x^\alpha}(\tau(\lambda y - w))(1 - \tau)^3 d\tau \]
\[ \frac{\partial g}{\partial x_j}(\lambda y - w) = \frac{1}{2} \sum_{|\alpha|=2} (\lambda y - w)^\alpha \frac{\partial^\alpha g}{\partial x^\alpha}(0) + \frac{1}{2} \sum_{|\alpha|=3} \int_0^1 (\lambda y - w)^\alpha \frac{\partial^\alpha g}{\partial x^\alpha \partial x_j}(\tau(\lambda y - w))(1 - \tau)^2 d\tau \]
hold.

**proof.** This result is proven via direct calculation by Taylor’s theorem. \( \Box \)

**Lemma 3.3.** For all \( s \in (s_*, s_1] \),
\[ |(\text{Im} \epsilon(s), \nabla Q)| \lesssim s^{-2}. \]

**proof.** According to a direct calculation, we have
\[ \frac{d}{dt} \text{Im} \int_{\mathbb{R}^N} u(t, x) \nabla \psi(t, x) dx = 2 \left( g|u|^{\dot{\Psi}} u - V u(t), \nabla u(t) \right)_2 \left( - \frac{1}{\lambda + \frac{\lambda}{2}} \nabla g|u|^{\dot{\Psi}} u + \nabla V u(t), u(t) \right)_2. \]

Firstly, according to \( \mathcal{S} \), we obtain
\[ |(\nabla V u(t), u(t))_2| = \left| \left( \langle \nabla V (\lambda(t)y - \tilde{w}(t))(Q + \tilde{\epsilon}(t)), Q + \tilde{\epsilon}(t) \rangle \right) \left( Q + \tilde{\epsilon}(t) \right) \right|_2 \lesssim \|Q + \epsilon\|_2^2 + \|Q + \tilde{\epsilon}(t)(Q + \epsilon)\|_2^2 \lesssim 1. \]

Secondly, we obtain
\[ \frac{1}{2} + \frac{1}{\lambda} \left( \nabla g|u|^{\dot{\Psi}} u + \nabla V u(t), u(t) \right)_2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} (\nabla g)(\lambda y - w) F(Q + \epsilon) dy \]
\[ = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} (\nabla g)(\lambda y - w) (F(Q + \epsilon) - F(Q) - dF(Q)(\epsilon)) dy + \frac{1}{\lambda^2} \int_{\mathbb{R}^N} (\nabla g)(\lambda y - w) (F(Q) + dF(Q)(\epsilon)) dy. \]
Therefore, since \((\nabla g)(\lambda y - w)Q = O(\lambda^2 + |w|^2)\), we obtain

\[
\left| \left( \nabla g|u\right|^2 u + \nabla V u(t), u(t) \right|_2 \lesssim \frac{1}{\lambda^2} \|\varepsilon\|^2_{H^1} + 1 + \frac{|w|^2}{\lambda^2} \lesssim 1
\]

Accordingly, we obtain

\[
\left| \text{Im} \int_{\mathbb{R}^N} u(t(s), x) \nabla \mathcal{R}(t(s), x) dx \right| \lesssim \int_{s}^{s_1} \lambda(\sigma)^2 \|\left( \nabla V u(t(\sigma)), u(t(\sigma)) \right)\|_2 d\sigma \\
\lesssim \int_{s}^{s_1} \sigma^{-2} d\sigma \lesssim s^{-1}.
\]

Therefore, we obtain

\[
2(\text{Im} \varepsilon(s), \nabla Q)_2 + (\varepsilon(s), i \nabla \varepsilon(s))_2 + \frac{b}{2} \int_{\mathbb{R}^N} y |Q(y) + \varepsilon(s, y)|^2 dy \\
= \lambda \text{Im} \int_{\mathbb{R}^N} u(t(s), x) \nabla \mathcal{R}(t(s), x) dx \\
= O(s^{-2})
\]

Moreover, from (18) and the orthogonal conditions (10), we obtain

\[
2(\varepsilon(s), i \nabla \varepsilon(s))_2 + b \int_{\mathbb{R}^N} y |Q(y) + \varepsilon(s, y)|^2 dy = 2(\varepsilon(s), i \nabla \varepsilon(s))_2 + b \int_{\mathbb{R}^N} y |\varepsilon(s, y)|^2 dy \\
= O(s^{-2L}).
\]

Consequently, we obtain (20). \(\square\)

**Lemma 3.4** (Estimation of modulation terms). For all \(s \in (s_*, s_1]\),

(21) \(2(\varepsilon(s), Q)_2 = -\|\varepsilon(s)\|^2_2\),

(22) \(|\text{Mod}(s)| \lesssim s^{-3}\),

(23) \(\left| \frac{\partial b}{\partial s} + b^2 \right| \lesssim s^{-2L}\).

**proof.** According to the mass conservation, we have

\[
2(\varepsilon, Q)_2 = \|u\|^2_2 - \|Q\|^2_2 - \|\varepsilon\|^2_2 = -\|\varepsilon\|^2_2,
\]

meaning (21) holds.

From Lemma 3.1

\[
i \frac{\partial \varepsilon}{\partial s} = L_+ \text{Re} \varepsilon + i L_- \text{Im} \varepsilon - (g(\lambda y - w) - 1) f(Q + \varepsilon) - f(Q + \varepsilon) - f(Q) - df(Q)(\varepsilon) + \lambda^2 V(\lambda y - w)\varepsilon \\
+ i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \Lambda(Q + \varepsilon) - \left( 1 - \frac{\partial \lambda}{\partial s} \right) (Q + \varepsilon) - \left( \frac{\partial b}{\partial s} + b^2 \right) \frac{|y|^2}{4} (Q + \varepsilon) \\
+ \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) b \frac{|y|^2}{2} (Q + \varepsilon) - i \frac{1}{\lambda} \frac{\partial w}{\partial s} \cdot \nabla (Q + \varepsilon) - \frac{1}{\lambda} \frac{\partial w}{\partial s} \cdot y(Q + \varepsilon) + \Psi
\]

holds.

From the orthogonal properties (10) we have

\[
0 = \frac{d}{ds}(i \varepsilon, \Lambda Q)_2 = \frac{d}{ds}(i \varepsilon, i|Q|^2) = \frac{d}{ds}(i \varepsilon, \rho)_2, \quad 0 = \frac{d}{ds}(i \varepsilon, iyQ)_2
\]

For \(v = \Lambda Q, i|y|^2 Q, \rho, \text{ or } iy_j Q\), the following estimates hold:

\[
|f(Q + \varepsilon) - f(Q) - df(Q)(\varepsilon)||v| \lesssim |\varepsilon|^2, \quad |(\lambda^2 V(\lambda y - w)\varepsilon, v)|_2 \lesssim s^{-3} \|\varepsilon\|_2.
\]
Firstly, we obtain
\[
0 = \left( \frac{\partial \epsilon}{\partial s}, \Lambda Q \right)_2 \\
= -2 \left( \text{Re} \epsilon, Q \right)_2 - (f(Q + \epsilon) - f(Q), (g(\lambda y - w) - 1)\Lambda Q)_2 - (f(Q), (g(\lambda y - w) - 1)\Lambda Q)_2 \\
+ O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) + \frac{1}{4}\|yQ\|^2 \left( \frac{\partial b}{\partial s} + b^2 \right) + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right) + O \left( \lambda^2|w| + \lambda^4 \right) \\
= O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) + O \left( \lambda^2|w| \right) + \frac{1}{4}\|yQ\|^2 \left( \frac{\partial b}{\partial s} + b^2 \right) + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right).
\]
Therefore, we obtain
\[
\left| \frac{\partial b}{\partial s} + b^2 \right| \lesssim s^{-2L} + s^{-\left(\frac{\lambda}{2} + \frac{\lambda}{2} \right)} |\text{Mod}(s)|.
\]
Secondly, we obtain
\[
0 = \left( i \frac{\partial \epsilon}{\partial s}, i |Q|^2 \right)_2 \\
= -4 \left( \text{Im} \epsilon, \Lambda Q \right)_2 - (f(Q + \epsilon) - f(Q), (g(\lambda y - w) - 1)i|Q|^2)_2 - (f(Q), (g(\lambda y - w) - 1)i|Q|^2)_2 \\
+ O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) - \|yQ\|^2 \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right) \\
= O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) - \|yQ\|^2 \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right).
\]
Therefore, we obtain
\[
\left| 1 - \frac{\partial \lambda}{\lambda} \right| \lesssim s^{-2L} + s^{-\left(\frac{\lambda}{2} + \frac{\lambda}{2} \right)} |\text{Mod}(s)|.
\]
Thirdly, we obtain
\[
0 = \left( \frac{\partial \epsilon}{\partial s}, \rho \right)_2 \\
= \left( \text{Re} \epsilon, |Q|^2 \right)_2 - (f(Q + \epsilon) - f(Q), (g(\lambda y - w) - 1)\rho)_2 - (f(Q), (g(\lambda y - w) - 1)\rho)_2 \\
+ O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) - (Q, \rho)_2 \left( 1 - \frac{\partial \gamma}{\partial s} \right) + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right) + \left| \frac{\partial b}{\partial s} + b^2 \right| + b \left| \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right| + O \left( \lambda^2|w| + \lambda^4 \right) \\
= O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) + O \left( \lambda^2|w| \right) - (Q, \rho)_2 \left( 1 - \frac{\partial \gamma}{\partial s} \right) + \left| \frac{\partial b}{\partial s} + b^2 \right| + b \left| \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right| + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right).
\]
Therefore, we obtain
\[
\left| 1 - \frac{\partial \gamma}{\partial s} \right| \lesssim s^{-2L} + s^{-\left(\frac{\lambda}{2} + \frac{\lambda}{2} \right)} |\text{Mod}(s)|.
\]
Fourthly, we obtain
\[
0 = \left( \frac{\partial \epsilon}{\partial s}, iy_j Q \right)_2 \\
= \left( \text{Im} \epsilon, \frac{\partial Q}{\partial y_j} \right)_2 - (f(Q + \epsilon) - f(Q), (g(\lambda y - w) - 1)iy_j Q)_2 - (f(Q), (g(\lambda y - w) - 1)iy_j Q)_2 \\
+ O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) - \frac{1}{\lambda} \frac{\partial w_i}{\partial s} \left( \frac{\partial Q}{\partial y_j}, y_j Q \right) - \frac{1}{2 \lambda} \frac{\partial w_i}{\partial s} \|y_j Q\|^2 + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right) \\
= O(s^{-2}) + O \left( \|\epsilon\|_{H^1}^2 \right) + O \left( s^{-3}\|\epsilon\|_{H^1} \right) - \frac{1}{\lambda} \frac{\partial w_i}{\partial s} \left( \frac{\partial Q}{\partial y_j}, y_j Q \right) - \frac{1}{2 \lambda} \frac{\partial w_i}{\partial s} \|y_j Q\|^2 + O \left( s\|\epsilon\|_{H^1} |\text{Mod}| \right).
\]
Therefore, we obtain
\[
\left| \frac{\partial w}{\partial s} \right| \lesssim s^{-3} + s^{-1} |\text{Mod}(s)|.
\]

Accordingly, we obtain
\[
|\text{Mod}(s)| \lesssim s^{-3} + s^{-1} |\text{Mod}(s)|,
\]
so that (22) holds. Moreover, from (24), we obtain (23). □

4. Modified energy function

In this section, we proceed with a modified version of the technique presented in Le Coz, Martel, and Raphaël [5] and Raphaël and Szeftel [10]. Let \( m \), \( \epsilon_1 \), and \( \epsilon_2 \) satisfy
\[
1 < 1 + \epsilon_1 < \frac{m}{2} < L, \quad 0 < \epsilon_2 < \frac{m \epsilon_1}{16},
\]
where \( \mu \) is from the coercivity (14) of \( L_+ \) and \( L_- \). Moreover, we define
\[
H(s, \varepsilon) := \frac{1}{2} \|\varepsilon\|^2_{H^1} + \epsilon_2 b^2 \|y\|_{L^2}^2 - \int_{\mathbb{R}^N} g(\lambda y - w) (F(Q(y) + \varepsilon(y)) - F(Q(y)) - dF(Q(y))(\varepsilon(y))) \, dy
\]
\[
+ \frac{1}{2} \lambda^2 \int_{\mathbb{R}^N} V(\lambda y - w) |\varepsilon(y)|^2 \, dy,
\]
\[
S(s, \varepsilon) := \frac{1}{\lambda^m} H(s, \varepsilon).
\]

Lemma 4.1 (Coercivity of \( H \)). For all \( s \in (s_*, s_1] \),
\[
H(s, \varepsilon) \geq \frac{\mu}{4} \|\varepsilon\|^2_{H^1} + \epsilon_2 b^2 \|y\|_{L^2}^2.
\]

**proof.** Firstly, we have
\[
\int_{\mathbb{R}^N} \left( F(Q(y) + \varepsilon(y)) - F(Q(y)) - dF(Q(y))(\varepsilon(y)) - \frac{1}{2} d^2 F(Q(y))(\varepsilon(y), \varepsilon(y)) \right) \, dy = O \left( \|\varepsilon\|^2_{H^1} + \|\varepsilon\|^2_{H^1} \right).
\]

Secondly, according to (14), we have
\[
\left| \lambda^2 \int_{\mathbb{R}^N} V(\lambda y - w) |\varepsilon(y)|^2 \, dy \right| = o \left( \|\varepsilon\|^2_{H^1} \right).
\]

Thirdly, we have
\[
|\{(g(\lambda y - w) - 1) (F(Q(y) + \varepsilon(y)) - F(Q(y)) - dF(Q(y))(\varepsilon(y)))| \leq |(g(\lambda y - w) - 1)| (Q^+ + |\varepsilon|^2) |\varepsilon|^2 \leq (\lambda^3 + |w|^3) |\varepsilon|^2 + |\varepsilon|^2 + |\varepsilon|^2 + \Phi.
\]
Therefore, we obtain
\[
\int_{\mathbb{R}^N} g(\lambda y - w) - 1) (F(Q(y) + \varepsilon(y)) - F(Q(y)) - dF(Q(y))(\varepsilon(y))) \, dy = o(\|\varepsilon\|^2_{H^1})
\]
Finally, since
\[
\|\varepsilon\|^2_{H^1} - \int_{\mathbb{R}^N} d^2 F(Q(y))(\varepsilon(y), \varepsilon(y)) \, dy = (L_+ \text{Re} \varepsilon, \text{Re} \varepsilon)_2 + (L_- \text{Im} \varepsilon, \text{Im} \varepsilon)_2,
\]
we obtain Lemma 4.1. □

From Lemma 4.1 and the definition of \( S \), we obtain the following:

Corollary 4.2 (Estimation of \( S \)). For all \( s \in (s_*, s_1] \),
\[
\frac{1}{\lambda^m} \left( \frac{\mu}{4} \|\varepsilon\|^2_{H^1} + \epsilon_2 b^2 \|y\|_{L^2}^2 \right) \leq S(s, \varepsilon) \lesssim \frac{1}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + b^2 \|y\|_{L^2}^2 \right).
\]
Lemma 4.3. For all \( s \in (s_*, s_1] \),

\begin{align*}
(25) \quad |((g(\lambda y - w) - 1)f(Q + \varepsilon) - f(Q), \Delta \varepsilon)_2| & \lesssim \|\varepsilon\|^2_{H^1}, \\
(26) \quad |((g(\lambda y - w) - 1)f(Q + \varepsilon) - f(Q), \nabla \varepsilon)_2| & \lesssim \|\varepsilon\|^2_{H^1}, \\
(27) \quad |\lambda^2 (V(\lambda y - w)\varepsilon, \Delta \varepsilon)_2| & \lesssim s^{-2} (\|\varepsilon\|^2_{H^1} + b^2 \|y\|\varepsilon\|^2_2), \\
(28) \quad |\lambda^2 (V(\lambda y - w)\varepsilon, \nabla \varepsilon)_2| & \lesssim s^{-3} (\|\varepsilon\|^2_{H^1} + b^2 \|y\|\varepsilon\|^2_2).
\end{align*}

\textit{proof.} Firstly,

\[ \nabla (g(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon))) = \lambda(\nabla g)(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) + g(\lambda y - w) \nabla (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) \]

\[ = \lambda(\nabla g)(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) + g(\lambda y - w) \nabla (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) \]

Therefore, we obtain

\[ g(\lambda y - w) \nabla (F(Q + \varepsilon) - F(Q)) \]

\[ = - \lambda(\varepsilon) (\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) - \lambda(\nabla g)(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) \]

\[ g(\lambda y - w) \nabla (F(Q + \varepsilon) - F(Q)) \]

and

\[ g(\lambda y - w) \nabla (F(Q + \varepsilon) - F(Q)) \]

so that \( (25) \) and \( (26) \) hold.

For \( (27) \), from \( (8) \), a direct calculation shows

\[ (V(\lambda y - w)\varepsilon, \Delta \varepsilon)_2 = -\frac{1}{2} (\lambda y \cdot (\nabla V)(\lambda y - w)\varepsilon, \varepsilon)_2 = O(\lambda \|y\|\varepsilon\|2(1 + |\lambda y - w|)\varepsilon\|2). \]

Therefore, we obtain \( (27) \).

For \( (28) \), from \( (8) \), a direct calculation shows

\[ (V(\lambda y - w)\varepsilon, \nabla \varepsilon)_2 = -\frac{1}{2} (\lambda(\nabla V)(\lambda y - w)\varepsilon, \varepsilon)_2 = O(\lambda \|\varepsilon\|\varepsilon\|2(1 + |\lambda y - w|)\varepsilon\|2). \]

Therefore, we obtain \( (28) \). \( \square \)

We define \( \kappa \) by

\[ \kappa := \frac{1}{4} - \frac{1}{2K} = \frac{2 - L}{2}. \]

Lemma 4.4 (Derivative of \( H \) in time). For all \( s \in (s_*, s_1] \),

\[ \frac{d}{ds} H(s, \varepsilon(s)) \geq -b \left( \frac{4\varepsilon_2}{\varepsilon_1} \|\varepsilon\|^2_{H^1} + \left( \frac{m}{2} + 1 + \varepsilon_1 \right) \varepsilon_2 b^2 \|y\|\varepsilon\|^2_2 + C' s^{-4} \right). \]
**Proof.** Firstly, we have

\[
\frac{d}{ds}H(s, \varepsilon(s)) = \frac{\partial H}{\partial s}(s, \varepsilon(s)) + \left\langle \frac{\partial H}{\partial \varepsilon}, \frac{\partial \varepsilon}{\partial s} \right\rangle.
\]

Secondly, we have

\[
\frac{\partial H}{\partial \varepsilon} = -\Delta \varepsilon + \varepsilon + 2\varepsilon b^2|y|^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon
\]

\[
= L_+ \Re \varepsilon + iL_- \Im \varepsilon + 2\varepsilon b^2|y|^2 \varepsilon - (g(\lambda y - w) - 1) df(Q)(\varepsilon)
\]

\[
- g(\lambda y - w)(f(Q + \varepsilon) - f(Q) - df(Q)(\varepsilon)) + \lambda^2 V(\lambda y - w) \varepsilon,
\]

\[
i \frac{\partial \varepsilon}{\partial s} = \frac{\partial H}{\partial \varepsilon} - 2\varepsilon b^2|y|^2 \varepsilon - (g(\lambda y - w) - 1)f(Q) + \text{Mod}_{op}(Q + \varepsilon) + \Psi,
\]

where

\[
\text{Mod}_{op} v := i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \Delta v - \left( 1 - \frac{\partial \gamma}{\partial s} \right) v - \left( \frac{\partial b}{\partial \varepsilon} + b^2 \right) \frac{|y|^2}{4} v + \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) b |y|^2 \varepsilon - i \frac{\partial w}{\partial \varepsilon} \cdot \nabla v - \frac{1}{2\lambda} \frac{\partial w}{\partial \varepsilon} \cdot y v.
\]

For \( \frac{\partial H}{\partial \varepsilon} \), we have

\[
\frac{\partial H}{\partial \varepsilon} = 2\varepsilon b\frac{\partial b}{\partial \varepsilon} ||y||^2 \varepsilon - \int_{\mathbb{R}^N} \left( \frac{\partial \lambda}{\partial \varepsilon} y - \frac{\partial w}{\partial \varepsilon} \right) \cdot (\nabla g)(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) dy
\]

\[
+ \lambda^2 \frac{1}{\lambda} \frac{\partial \lambda}{\partial \varepsilon} \int_{\mathbb{R}^N} V(\lambda y - w) |\varepsilon|^2 dy + \frac{1}{2} \lambda^2 \int_{\mathbb{R}^N} \left( \frac{\partial \lambda}{\partial \varepsilon} y - \frac{\partial w}{\partial \varepsilon} \right) \cdot (\nabla V)(\lambda y - w) |\varepsilon|^2 dy.
\]

Moreover, since \( \frac{1}{\lambda} \frac{\partial \lambda}{\partial \varepsilon} \approx -b, \frac{\partial \lambda}{\partial \varepsilon} \approx -b^2 \), and \( \lambda \approx b \), we have

\[
2\varepsilon b\frac{\partial b}{\partial \varepsilon} ||y||^2 \varepsilon \geq -2(1 + \varepsilon)\varepsilon b^3 ||y||^2 \varepsilon.
\]

\[
\left| \lambda^2 \int_{\mathbb{R}^N} V(\lambda y - w) |\varepsilon|^2 dy \right| \lesssim \lambda^2 \frac{\varepsilon^2}{\varepsilon^2} ||\varepsilon||^2_{H^1},
\]

\[
\left| \int_{\mathbb{R}^N} \lambda y \cdot (\nabla V)(\lambda y - w) |\varepsilon|^2 dy \right| \lesssim b ||y||_2 ||(1 + \lambda |y| + |w|)\varepsilon||_2 \lesssim ||\varepsilon||^2_{H^1} + b^2 ||y||^2_2,
\]

\[
\left| \int_{\mathbb{R}^N} (\nabla V)(\lambda y - w) |\varepsilon|^2 dy \right| \lesssim ||\varepsilon||_2 ||(1 + \lambda |y| + |w|)\varepsilon||_2 \lesssim ||\varepsilon||^2_{H^1} + b^2 ||y||^2_2,
\]

\[
\left| \int_{\mathbb{R}^N} (\nabla g)(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) dy \right| \lesssim ||\varepsilon||^2_{H^1},
\]

\[
\left| \int_{\mathbb{R}^N} (\lambda y - w) \cdot (\nabla g)(\lambda y - w) (F(Q + \varepsilon) - F(Q) - dF(Q)(\varepsilon)) dy \right| \lesssim \int_{\mathbb{R}^N} (\lambda y - w) \cdot (\nabla g)(\lambda y - w) (Q \frac{\varepsilon}{\varepsilon^2} + |\varepsilon| \frac{\varepsilon^2}{\varepsilon^2}) |\varepsilon|^2 dy
\]

\[
\lesssim (\lambda + |w|) ||\varepsilon||^2_2 + ||\varepsilon||^2_{H^1} + \frac{b^4}{\varepsilon^2}.
\]

Therefore,

\[
(29) \quad \frac{\partial H}{\partial \varepsilon} \geq -2(1 + \varepsilon)\varepsilon b^3 ||y||^2 \varepsilon + o \left( ||\varepsilon||^2_{H^1} + b^2 ||y||^2_2 \right).
\]

For \( \left\langle \frac{\partial H}{\partial \varepsilon}, 2\varepsilon b^2 |y|^2 \varepsilon \right\rangle \), we have

\[
\left\langle \frac{\partial H}{\partial \varepsilon}, 2i\varepsilon b^2 |y|^2 \varepsilon \right\rangle
\]

\[
= (-\Delta \varepsilon + \varepsilon + 2\varepsilon b^2 |y|^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon, 2i\varepsilon b^2 |y|^2 \varepsilon)
\]

\[
= 4\varepsilon b^2 (\nabla \varepsilon, iy \varepsilon)_2 - 2\varepsilon b^2 \left( g(\lambda y - w) \left( Q + \varepsilon \frac{\varepsilon^2}{\varepsilon^2} - Q \frac{\varepsilon^2}{\varepsilon^2} \right) Q, |y|^2 \varepsilon \right)_2.
\]

Therefore, we obtain

\[
(30) \quad \left| \left\langle \frac{\partial H}{\partial \varepsilon}, 2\varepsilon b^2 |y|^2 \varepsilon \right\rangle \right| \leq 4\varepsilon b^2 ||\varepsilon||_{H^1} ||y||^2_2 + o(||\varepsilon||^2_{H^1}).
\]
For $\langle \frac{dH}{d\varepsilon}, (g(\lambda y - w) - 1)f(Q) \rangle$, we have

$$\langle \frac{dH}{d\varepsilon}, i(g(\lambda y - w) - 1)f(Q) \rangle$$

$$= (-\Delta \varepsilon + \varepsilon + 2\varepsilon^2 b^2 b^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w)\varepsilon, i(g(\lambda y - w) - 1)f(Q))$$

$$= (\nabla \varepsilon, i\lambda g(\lambda y - w)f(Q) + i(g(\lambda y - w) - 1)df(Q)\nabla Q)_2 + O(s^{-3}\|\varepsilon\|_{H^1})$$

$$= O(s^{-3}\|\varepsilon\|_{H^1}).$$

Therefore, we obtain

$$\left| \langle \frac{dH}{d\varepsilon}, (g(\lambda y - w) - 1)f(Q) \rangle \right| \leq \varepsilon_3 b\|\varepsilon\|^2_{H^1} + \frac{C'}{\varepsilon_3} s^{-5}. \quad (31)$$

For $\langle i\frac{\partial H}{\partial \varepsilon}, \Psi \rangle$, we have

$$\left| \langle i\frac{\partial H}{\partial \varepsilon}, \Psi \rangle \right| \leq \varepsilon_3 b\|\varepsilon\|^2_{H^1} + \frac{C'}{\varepsilon_3} s^{-5}. \quad (32)$$

Next, we consider $\langle i\frac{\partial H}{\partial \varepsilon}, \text{Mod}_{op} Q \rangle$. Firstly, for $\langle i\frac{\partial H}{\partial \varepsilon}, i\Lambda Q \rangle$, we have

$$\langle \frac{\partial H}{\partial \varepsilon}, \Lambda Q \rangle$$

$$= \langle L_+ \text{Re} \varepsilon + iL_- \text{Im} \varepsilon + 2\varepsilon^2 b^2 |y|^2 \varepsilon - g(\lambda y - w) - 1) df(Q)\varepsilon$$

$$- g(\lambda y - w)(f(Q + \varepsilon) - f(Q) - df(Q)\varepsilon) + \lambda^2 V(\lambda y - w)\varepsilon, i\Lambda Q \rangle$$

$$= -2(\text{Re} \varepsilon, Q)_2 + O(s^{-2}\|\varepsilon\|_{H^1}) + O(s^{-3}\|\varepsilon\|_{H^1}) + O(\|\varepsilon\|^2_{H^1}) + O(s^{-3}\|\varepsilon\|_{H^1}).$$

Therefore, we obtain

$$\left| \langle i\frac{\partial H}{\partial \varepsilon}, i\Lambda Q \rangle \right| \lesssim \|\varepsilon\|^2_{H^1} + s^{-4}. \quad (33)$$

Secondly, for $\langle i\frac{\partial H}{\partial \varepsilon}, Q \rangle$, we have

$$\langle \frac{\partial H}{\partial \varepsilon}, iQ \rangle$$

$$= \langle L_+ \text{Re} \varepsilon + iL_- \text{Im} \varepsilon + 2\varepsilon^2 b^2 |y|^2 \varepsilon - g(\lambda y - w) - 1) df(Q)\varepsilon$$

$$- g(\lambda y - w)(f(Q + \varepsilon) - f(Q) - df(Q)\varepsilon) + \lambda^2 V(\lambda y - w)\varepsilon, iQ \rangle$$

$$= O(s^{-2}\|\varepsilon\|_{H^1}) + O(s^{-3}\|\varepsilon\|_{H^1}) + O(\|\varepsilon\|^2_{H^1}) + O(s^{-3}\|\varepsilon\|_{H^1}).$$

Therefore, we obtain

$$\left| \langle i\frac{\partial H}{\partial \varepsilon}, iQ \rangle \right| \lesssim \|\varepsilon\|^2_{H^1} + s^{-4}. \quad (34)$$
Therefore, we obtain
\[
\left\langle i \frac{\partial H}{\partial \epsilon}, \frac{1}{2} |y|^{2Q} \right\rangle \lesssim \|\epsilon\|_{H^1}^2 + s^{-4}.
\]

Fourthly, for \( \left\langle i \frac{\partial H}{\partial \epsilon}, i \nabla Q \right\rangle \), we have
\[
\left\langle \frac{\partial H}{\partial \epsilon}, \frac{\partial Q}{\partial y_j} \right\rangle = \left\langle L^+ \Re \epsilon + i L^- \Im \epsilon + 2 \epsilon_2 b^2 |y|^2 \epsilon - (g(\lambda y - w) - 1) \partial f(Q)(\epsilon) - g(\lambda y - w)(f(Q + \epsilon) - f(Q) - \partial f(Q)(\epsilon)) + \lambda^2 V(\lambda y - w)\epsilon, i \frac{\partial Q}{\partial y_j} \right\rangle
\]
\[
= \partial(\|\epsilon\|_{H^1}) + O(s^{-3}\|\epsilon\|_{H^1}) + O(\|\epsilon\|_{H^1}^2) + O(s^{-3}\|\epsilon\|_{H^1}).
\]
Therefore, we obtain
\[
\left\langle i \frac{\partial H}{\partial \epsilon}, i \nabla Q \right\rangle \lesssim \|\epsilon\|_{H^1}^2 + s^{-4}.
\]

Fifthly, for \( \left\langle i \frac{\partial H}{\partial \epsilon}, yQ \right\rangle \), we have
\[
\left\langle \frac{\partial H}{\partial \epsilon}, iy_j Q \right\rangle = \left\langle L^+ \Re \epsilon + i L^- \Im \epsilon + 2 \epsilon_2 b^2 |y|^2 \epsilon - (g(\lambda y - w) - 1) \partial f(Q)(\epsilon) - g(\lambda y - w)(f(Q + \epsilon) - f(Q) - \partial f(Q)(\epsilon)) + \lambda^2 V(\lambda y - w)\epsilon, iy_j Q \right\rangle
\]
\[
= \left( \Im \epsilon, \frac{\partial Q}{\partial y_j} \right) + O(s^{-3}\|\epsilon\|_{H^1}) + O(\|\epsilon\|_{H^1}^2) + O(s^{-3}\|\epsilon\|_{H^1}).
\]
Therefore, we obtain
\[
\left\langle i \frac{\partial H}{\partial \epsilon}, yQ \right\rangle \lesssim \|\epsilon\|_{H^1}^2 + s^{-2}.
\]

Accordingly, we obtain
\[
\left( i \frac{\partial H}{\partial \epsilon}, \text{Mod}_{\text{op}} Q \right) \lesssim s^{-3}(\|\epsilon\|_{H^1}^2 + s^{-4}) + s^{-2}(\|\epsilon\|_{H^1}^2 + s^{-4}) + s^{-3}(\|\epsilon\|_{H^1}^2 + s^{-2}).
\]

Finally, we consider \( \left\langle i \frac{\partial H}{\partial \epsilon}, \text{Mod}_{\text{op}} \epsilon \right\rangle \). Firstly, for \( \left\langle i \frac{\partial H}{\partial \epsilon}, i \Lambda \epsilon \right\rangle \), we have
\[
\left\langle \frac{\partial H}{\partial \epsilon}, \Lambda \epsilon \right\rangle = \left\langle - \Delta \epsilon + \epsilon + 2 \epsilon_2 b^2 |y|^2 \epsilon - g(\lambda y - w)(f(Q + \epsilon) - f(Q)) + \lambda^2 V(\lambda y - w)\epsilon, \Lambda \epsilon \right\rangle
\]
\[
= O(\|\epsilon\|_{H^1}^2) + O(b^2\|y\|_{L^2}^2) + O(\|\epsilon\|_{H^1}^2).
\]
Therefore, we obtain
\[
\left\langle i \frac{\partial H}{\partial \epsilon}, i \Lambda \epsilon \right\rangle \lesssim \|\epsilon\|_{H^1}^2.
\]
Secondly, for \( \left\langle i \frac{\partial H}{\partial \varepsilon}, \varepsilon \right\rangle \), we have

\[
\left\langle -\Delta \varepsilon + \varepsilon + 2 \varepsilon b^2 |y|^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon, i \varepsilon \right\rangle = - \left( g(\lambda y - w) \left( |Q + \varepsilon|^{\frac{4}{N}} - Q^{\frac{4}{N}} \right) Q, i \varepsilon \right)_2 = O(\|\varepsilon\|_{H^1}^2).
\]

Therefore, we obtain

\[
\left| \left\langle i \frac{\partial H}{\partial \varepsilon}, \varepsilon \right\rangle \right| \lesssim \|\varepsilon\|_{H^1}^2.
\]

Thirdly, for \( \left\langle i \frac{\partial H}{\partial \varepsilon}, |y|^2 \varepsilon \right\rangle \), we have

\[
\left\langle -\Delta \varepsilon + \varepsilon + 2 \varepsilon b^2 |y|^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon, i |y|^2 \varepsilon \right\rangle = \left( \nabla \varepsilon, i y \varepsilon \right)_2 - \left( g(\lambda y - w) \left( |Q + \varepsilon|^{\frac{4}{N}} - Q^{\frac{4}{N}} \right) Q, i |y|^2 \varepsilon \right)_2 = O(\|\varepsilon\|_{H^1} \|y \varepsilon\|_2).
\]

Therefore, we obtain

\[
\left| \left\langle i \frac{\partial H}{\partial \varepsilon}, |y|^2 \varepsilon \right\rangle \right| \lesssim s \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y \varepsilon\|_2^2 \right).
\]

Fourthly, for \( \left\langle i \frac{\partial H}{\partial \varepsilon}, \nabla \varepsilon \right\rangle \), we have

\[
\left\langle -\Delta \varepsilon + \varepsilon + 2 \varepsilon b^2 |y|^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon, \nabla \varepsilon \right\rangle = -4 \varepsilon b^2 (\varepsilon \varepsilon)_2 + \left( g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon, \nabla \varepsilon \right)_2 = O(\|\varepsilon\|_{H^1}^2 + b^2 \|y \varepsilon\|_2^2).
\]

Therefore, we obtain

\[
\left| \left\langle i \frac{\partial H}{\partial \varepsilon}, \nabla \varepsilon \right\rangle \right| \lesssim \|\varepsilon\|_{H^1}^2 + b^2 \|y \varepsilon\|_2^2.
\]

Fifthly, for \( \left\langle i \frac{\partial H}{\partial \varepsilon}, y \varepsilon \right\rangle \), we have

\[
\left\langle -\Delta \varepsilon + \varepsilon + 2 \varepsilon b^2 |y|^2 \varepsilon - g(\lambda y - w)(f(Q + \varepsilon) - f(Q)) + \lambda^2 V(\lambda y - w) \varepsilon, iy \varepsilon \right\rangle = \left( \frac{\partial \varepsilon}{\partial y_j}, i \varepsilon \right)_2 - \left( g(\lambda y - w) \left( |Q + \varepsilon|^{\frac{4}{N}} - Q^{\frac{4}{N}} \right) Q, iy \varepsilon \right)_2 = O(\|\varepsilon\|_{H^1}^2).
\]

Therefore, we obtain

\[
\left| \left\langle i \frac{\partial H}{\partial \varepsilon}, y \varepsilon \right\rangle \right| \lesssim \|\varepsilon\|_{H^1}^2.
\]

Accordingly, we obtain

\[
\left| \left\langle i \frac{\partial H}{\partial \varepsilon}, \text{Mod}_\varepsilon \varepsilon \right\rangle \right| \lesssim s^{-3} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y \varepsilon\|_2^2 \right) + s^{-2} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y \varepsilon\|_2^2 \right).
\]
Combining inequalities (29), (30), (31), (32), (33), and (34), we obtain
\[
\frac{d}{ds} H(s, \varepsilon(s)) = \frac{\partial H}{\partial s}(s, \varepsilon(s)) + \left\{ \frac{\partial H}{\partial \varepsilon}(s, \varepsilon(s)) \right\} + \frac{\partial \varepsilon}{\partial s}(s, \varepsilon(s)) \geq -2(1 + \varepsilon) \varepsilon_2 b^3 \| y \varepsilon \|_2^2 + o \left( b \left( \| \varepsilon \|_{H^1}^2 + b^2 \| y \varepsilon \|_2^2 \right) \right) - 4 \varepsilon_2 b^2 \| \varepsilon \|_{H^1} \| y \varepsilon \|_2 + o(b \| \varepsilon \|_{H^1}^2)
\]
\[
- 2 \varepsilon_3 b \| \varepsilon \|_{H^1}^2 - \frac{C'}{\varepsilon_3} s^{-5} + o \left( b \left( \| \varepsilon \|_{H^1}^2 + b^2 \| y \varepsilon \|_2^2 \right) \right) - C'' s^{-5}
\]
\[
\geq - b \left( 2 \left( \frac{\varepsilon_2}{\varepsilon_1} + \varepsilon_3 + \varepsilon \right) \| \varepsilon \|_{H^1}^2 + 2 \left( 1 + \varepsilon_1 + \varepsilon \right) \varepsilon_2 b^2 \| y \varepsilon \|_2^2 - \frac{C'}{\varepsilon_3} s^{-4} \right)
\]
\[
\geq - b \left( \frac{4 \varepsilon_2}{\varepsilon_1} \| \varepsilon \|_{H^1}^2 + \left( \frac{m}{2} + 1 + \varepsilon_1 \right) \varepsilon_2 b^2 \| y \varepsilon \|_2^2 - C'' s^{-4} \right),
\]
so that we obtain Lemma 4.4. □

**Lemma 4.5** (Derivative of S in time). For all \( s \in (s_*, s_1] \),
\[
\frac{d}{ds} S(s, \varepsilon(s)) \geq \frac{b}{\lambda^m} \left( \| \varepsilon \|_{H^1}^2 + b^2 \| y \varepsilon \|_2^2 - s^{-(2L+\kappa)} \right).
\]

**proof.** According to (22), Lemma 4.3, and Lemma 4.4 we have
\[
\frac{d}{ds} S(s, \varepsilon(s)) = m \frac{b}{\lambda^m} H(s, \varepsilon(s)) - m \frac{1}{\lambda^m} \left( \frac{1}{\lambda} \frac{\partial H}{\partial s} + b \right) H(s, \varepsilon(s)) + \frac{1}{\lambda^m} \frac{d}{ds} H(s, \varepsilon(s)) \geq \frac{b}{\lambda^m} \left( \frac{m \mu}{4} - \frac{4 \varepsilon_2}{\varepsilon_1} \right) \| \varepsilon \|_{H^1}^2 + \left( \frac{m}{2} - (1 + \varepsilon_1) \right) \varepsilon_2 b^2 \| y \varepsilon \|_2^2 - C'' s^{-4} \right).
\]
Therefore, we obtain Lemma 4.5. □

5. **Bootstrap**

In this section, we establish the estimates of the decomposition parameters by using a bootstrap argument and the estimates obtained in Section 4.

**Lemma 5.1.** There exists a sufficiently small \( \varepsilon_3 > 0 \) such that for all \( s \in (s_*, s_1] \),
\[
\| \varepsilon(s) \|_{H^1}^2 + b(s) \| y \varepsilon(s) \|_2^2 \lesssim s^{-(2L+\kappa)},
\]
\[
| s \lambda(s) - 1 | < (1 - \varepsilon_3) s^{-M},
\]
\[
| s b(s) - 1 | < (1 - \varepsilon_3) s^{-M},
\]
\[
| w(s) | \lesssim s^{-2}.
\]

**proof.** Let \( C_1 \) be sufficiently large and \( s_1 \) defined by
\[
s_1 := \inf \left\{ \sigma \in [s_*, s_1] \mid \| \varepsilon(\tau) \|_{H^1}^2 + b(\tau) \| y \varepsilon(\tau) \|_2^2 \leq C_1 \tau^{-(2L+\kappa)} \quad (\tau \in [\sigma, s_1]) \right\}.
\]
Assume \( s_1 > s_* \). Then
\[
\| \varepsilon(s_1) \|_{H^1}^2 + b(s_1) \| y \varepsilon(s_1) \|_2^2 = C_1 s_1^{-(2L+\kappa)}
\]
holds. Moreover, let \( s_\dagger \) defined by
\[
s_\dagger := \sup \left\{ \sigma \in [s_1, s_1] \mid \| \varepsilon(\tau) \|_{H^1}^2 + b(\tau) \| y \varepsilon(\tau) \|_2^2 \geq \tau^{-(2L+\kappa)} \quad (\tau \in [s_1, \sigma]) \right\}.
\]
Then, since \( s_\dagger < s_1 \),
\[
\| \varepsilon(s_\dagger) \|_{H^1}^2 + b(s_\dagger) \| y \varepsilon(s_\dagger) \|_2^2 = s_\dagger^{-(2L+\kappa)}
\]
holds.
From Corollary 4.2 and Lemma 5.3,
\[ \frac{C_1}{\lambda^m} \left( \frac{H}{4} \left\| \varepsilon \right\|_{H^1}^2 + b^2 \left\| y \varepsilon \right\|_2^2 \right) \leq S(s, \varepsilon) \leq \frac{C_2}{\lambda^m} \left( \left\| \varepsilon \right\|_{H^1}^2 + b^2 \left\| y \varepsilon \right\|_2^2 - s^{-2(L + \kappa)} \right) \]
\[ \frac{C_3 b}{\lambda^m} \left( \left\| \varepsilon \right\|_{H^1}^2 + b^2 \left\| y \varepsilon \right\|_2^2 - s^{-2(L + \kappa)} \right) \leq \frac{d}{ds} S(s, \varepsilon(s)) \]
hold. Then \( s \mapsto S(s, \varepsilon(s)) \) is monotonically increasing on \( [s_1, s_1] \). Therefore, we obtain
\[ C_1 C_1 s_1^{-2(L + \kappa)} = C_1 \left( \left\| \varepsilon(s_1) \right\|_{H^1}^2 + b(s_1)^2 \left\| y \varepsilon(s_1) \right\|_2^2 \right) \leq \lambda(s_1)^m S(s_1, \varepsilon(s_1)) \leq \lambda(s_1)^m S(s_1, \varepsilon(s_1)) \leq C_2 \frac{\lambda(s_1)^m}{\lambda(s_1)^m} \left( \left\| \varepsilon(s_1) \right\|_{H^1}^2 + b(s_1)^2 \left\| y \varepsilon(s_1) \right\|_2^2 \right) \leq C_2 \frac{\lambda(s_1)^m}{\lambda(s_1)^m} s_1^{-2(L + \kappa)} \leq \frac{2C_2}{s_1^{-2(L + \kappa)-1}} s_1^{-2(L + \kappa)}. \]
Accordingly, we obtain
\[ C_1 C_1 \leq 2C_2. \]
It is a contradiction since \( C_1 \) is sufficiently large.
We prove \( \frac{d}{ds} (s \lambda) \leq s^{-1} (1 + \epsilon) \left( s^{-M} + C s^{-2(L - 1)} \right) \leq (1 + \epsilon) s^{-M+1} \)
and \( \lambda(s_1) = s_1^{-1} \), we have
\[ |s \lambda - 1| \leq \int_{s_1}^{s_1} (1 + \epsilon) \sigma^{-M+1} d\sigma \leq \frac{1 + \epsilon}{M} s^{-M}. \]
Therefore, \( 36 \) holds since \( M > 1 \). Next, we prove \( 37 \). Since
\[ \left| \frac{b}{\lambda} - 1 \right| \leq \int_{s_1}^{s_1} \sigma^{-2(L - 1)} d\sigma \lesssim s^{-2(L - 1)}, \]
we have
\[ |s b - s \lambda| \lesssim s^{-2(L - 1)}. \]
Consequently, we have
\[ |s b - 1| \leq |s b - s \lambda| + |s \lambda - 1| \leq s^{-2(L - 1)} + \frac{1 + \epsilon}{M} s^{-M}. \]
Therefore, \( 37 \) holds. Finally, since
\[ |w(s)| \leq \int_{s_1}^{s_1} |\text{Mod}(\sigma)| d\sigma \lesssim \int_{s_1}^{s_1} \sigma^{-3} d\sigma \lesssim s^{-2}, \]
we obtain \( 38 \). \( \square \)

From Lemma 5.1 and the definition of \( s_\ast \), we obtain the following:

**Corollary 5.2.** If \( s_0 \) is sufficiently large, then \( s_\ast = s' \).

**Lemma 5.3.** If \( s_0 \) is sufficiently large, then \( s' = s_0 \).

**Proof.** We prove \( s' \leq s_0 \) by contradiction. Assume that for any \( s_0 \gg 1 \), there exists \( s_1 > s_0 \) such that \( s' > s_0 \). In the following, we consider the initial value \( 17 \) in response to such \( s_1 \) and the corresponding solution \( u \) for \( 1 \).

Let \( t' := \inf I_{s_1} \). Then \( s' = \inf J_{s_1} > s_0 \) holds. Furthermore, we have
\[ \left\| \left( s \lambda \right) \frac{\partial}{\partial s} u(s, \lambda(s) y - w(s) e^{-i\gamma(s)} - Q(y) \right\|_{H^1} = \left\| \varepsilon(s) \right\|_{H^1} \leq \frac{\delta}{4}. \]
for all \( s \in (s', s_1) \). Since \( t_{t_1}((s', s_1)) = (t', t_1) \), we have
\[
\left\| \lambda(t) \nabla u(t, \tilde{\lambda}(t)y - \tilde{w}(t)) e^{-i\gamma(t)} - Q(y) \right\|_{H^1} \leq \frac{\delta}{4}
\]
for all \( t \in (t', t_1) \). We consider three cases \( t' > T_\ast \), \( t' = T_\ast > -\infty \), and \( t' = -\infty \).

Firstly, assume \( t' > T_\ast \). Then \( \lambda \) and \( \tilde{\lambda} \) are bounded on \((s', s_1)\) and \((t', t_1)\), respectively, according to [18] and Corollary 5.2. Then, by setting \( t \) sufficiently close to \( t' \), we have
\[
\left\| \lambda(t) \nabla u(t', \tilde{\lambda}(t)y - \tilde{w}(t)) e^{-i\gamma(t)} - Q(y) \right\|_{H^1} < \delta.
\]
Therefore, there exists the decomposition of \( u \) in a neighbourhood of \( t' \) according to Lemma 2.3. Its existence contradicts the maximality of \( I_{t_1} \).

Next, assume \( t' = T_\ast > -\infty \). Then \( \|\nabla u(t)\|_2 \to \infty \) \((t \searrow t')\) holds according to the blow-up alternative. Also, \( \|\nabla u(s)\|_2 \to \infty \) \((s \searrow s')\) holds. Then since
\[
\|u(s)\|_2 + \lambda(s) \|\nabla u(s)\|_2 \lesssim 1,
\]
we have \( \lambda(s) \to 0 \) \((s \searrow s')\). Therefore, we obtain
\[
|s\lambda(s) - 1| \to 1, \quad s^{-M} \to s'^{-M} < \frac{1}{2} \quad (s \searrow s'),
\]
which contradicts [30].

Finally, assume \( t' = -\infty \). Then there exists a sequence \((s_n)_{n \in \mathbb{N}}\) that converges to \( s' \) such that \( \lim_{n \to \infty} \lambda(s_n) = \infty \) holds. Therefore, we obtain
\[
|s_n\lambda(s_n) - 1| \to \infty, \quad s_n^{-M} \to s'^{-M} < 1 \quad (n \to \infty),
\]
which contradicts [30].

Consequently, we obtain \( s' \leq s_0 \). \( \square \)

6. Conversion of estimates

In this section, we rewrite the estimates for \( s \) in Lemma 5.1 into estimates for \( t \).

**Lemma 6.1 (Interval).** Let \( s_0 \) be sufficiently large. Then there exists \( t_0 < 0 \) such that
\[
[t_0, t_1] \subset s_{t_1}^{-1}([s_0, s_1]), \quad |s_{t_1}(t) - |t|| \lesssim |t|^{M+1} \quad (t \in [t_0, t_1])
\]
hold for all \( t_1 \in (t_0, 0) \).

**Proof.** Firstly, \([t_{t_1}(s_0), t_1] = s_{t_1}^{-1}([s_0, s_1])\) holds. For all \( s \in [s_0, s_1] \), we have
\[
t_1 - t_{t_1}(s) = s^{-1} - s_1^{-1} + \int_{s_1}^{s} \sigma^{-2} (\sigma \lambda_{t_1}(\sigma) + 1)(\sigma \lambda_{t_1}(\sigma) - 1) \, d\sigma
\]
since \(-s_1^{-1} = t_1 = t_{t_1}(s_0)\). Therefore, we have
\[
\frac{1}{2} s^{-1} \leq s^{-1} (1 - 3s^{-M}) \leq |t_{t_1}(s)| \leq s^{-1} (1 + 3s^{-M}) \leq 2s^{-1}.
\]
Accordingly, we obtain \( \frac{1}{2} |t_{t_1}(s)| \leq s^{-1} \leq 2 |t_{t_1}(s)| \). According to \( s_{t_1}^{-1} = t_{t_1} \), we obtain
\[
\frac{1}{2} |t| \leq s_{t_1}(t)^{-1} \leq 2 |t|.
\]
Consequently, according to (39), we obtain
\[
||t| - s_{t_1}(t)^{-1}| \leq 3s_{t_1}(t)^{-1}|t|^{M+1} \leq 3 \cdot 2^{M+1} |t|^{M+1}.
\]
Furthermore, since
\[
t_{t_1}(s_0) = -|t_{t_1}(s_0)| \leq -\frac{1}{2} s_{t_1}(t_{t_1}(s_0))^{-1} = -\frac{1}{2} s_0^{-1}
\]
and \( s_0 \) is independent of \( t_1 \) according to Lemma 5.3, we obtain Lemma 6.1. \( \square \)
Lemma 6.2 (Conversion of estimates). For all \( t \in [t_0, t_1] \),
\[
\tilde{\lambda}_t(t) = |t| \left( 1 + \epsilon_{\tilde{\lambda}_t(t)}(t) \right), \quad \tilde{b}_t(t) = |t| \left( 1 + \epsilon_{\tilde{b}_t(t)}(t) \right), \quad |\tilde{w}_t(t)| \leq |t|^{2L^{-1}},
\]
\[
\|\tilde{v}_t(t)\|_{H^1} \leq |t|^{L + \frac{4}{N}}, \quad \|\tilde{y}|\tilde{v}_t(t)\|_2 \leq |t|^{L + \frac{4}{N} - 1}
\]
holds for some functions \( \epsilon_{\tilde{\lambda}_t(t)} \) and \( \epsilon_{\tilde{b}_t(t)} \). Furthermore,
\[
\sup_{t \in [t_0, 0]} \left| \epsilon_{\tilde{\lambda}_t(t)}(t) \right| \leq |t|^M, \quad \sup_{t \in [t_0, 0]} \left| \epsilon_{\tilde{b}_t(t)}(t) \right| \leq |t|^M.
\]

\textbf{proof.} Firstly, we define \( \epsilon_{\tilde{\lambda}_t(t)}(t) := \frac{\tilde{\lambda}_t(t)}{|t|} - 1 \). According to (69) and Lemma 6.1, we have
\[
\left| \epsilon_{\tilde{\lambda}_t(t)}(t) \right| = \left| \left( s_{t_1}(t) \tilde{\lambda}_t(t) - 1 \right) \frac{1}{s_{t_1}(t)|t|} + \frac{1}{s_{t_1}(t)|t|} - 1 \right| \leq |t|^M.
\]
Similarly, we define \( \epsilon_{\tilde{b}_t(t)}(t) := \frac{\tilde{b}_t(t)}{|t|} - 1 \) and obtain estimates of \( \tilde{b}_t(t) \) and \( \tilde{w}_t(t) \).\(\square\)

7. PROOF OF THEOREM 1.3

\textbf{Proof of Theorem 1.3.} Let \( (t_n)_{n \in \mathbb{N}} \subset (t_0, 0) \) be an increasing sequence such that \( \lim_{n \to \infty} t_n = 0 \). For each \( n \in \mathbb{N} \), let \( u_n \) be the solution for (1) with the initial value
\[
u_n(t_n, x) := \frac{1}{\lambda_{1,n}^{\frac{N}{4}}} Q \left( \frac{x}{\lambda_{1,n}^{\frac{1}{2}}} \right) e^{-\frac{|b_{1,n}|}{2} \left| x \right|^2 - \frac{|w_{1,n}|^2}{|\lambda_{1,n}|^2} + i\gamma_{1,n}(t)}
\]
at \( t_n \), where \( b_{1,n} = \lambda_{1,n} = s_{n}^{-1} = -t_n \).

According to Lemma 2.2 there exists the decomposition
\[
u_n(t, x) = \frac{1}{\lambda_n(t)^{\frac{N}{4}}} \left( Q + \tilde{\varepsilon}_n \right) \left( t, \frac{x + \tilde{w}_n(t)}{\lambda_n(t)} \right) e^{-\frac{|b_n(t)|}{2} \left| x + \tilde{w}_n(t) \right|^2 - \frac{|w_n(t)|^2}{|\lambda_n(t)|^2} + i\gamma_n(t)}
\]
on \([t_0, T_n]\). Then \( (u_n(t_0))_{n \in \mathbb{N}} \) is bounded in \( \Sigma^1 \). Therefore, up to a subsequence, there exists \( u_{\infty}(t_0) \in \Sigma^1 \) such that
\[
u_n(t_0) \to u_{\infty}(t_0) \quad \text{weakly in } \Sigma^1.
\]
Moreover, as in Section 3.2 in [5], we see that
\[
u_n(t_0) \to u_{\infty}(t_0) \in L^2(\mathbb{R}^N) \quad (n \to \infty).
\]

Let \( u_{\infty} \) be the solution for (1) with the initial value \( u_{\infty}(t_0) \) and \( T^* \) be the supremum of the maximal existence interval of \( u_{\infty} \). Moreover, we define \( T := \min\{0, T^*\} \). For any \( T' \in [t_0, T] \), we have \([t_0, T'] \subset [t_0, t_n]\) if \( n \) is sufficiently large. Then there exists \( n_0 \) such that
\[
\sup_{n \geq n_0} \|u_n\|_{L^\infty([t_0, T'], \Sigma^1)} \lesssim (1 + |T'|^{-1}) (1 + |t_0|^4)
\]
holds. According to Lemma 6.1
\[
u \to u_{\infty} \quad \text{in } C([t_0, T'], L^2(\mathbb{R}^N)) \quad (n \to \infty)
\]
holds. In particular, \( u_n(t) \to u_{\infty}(t) \) in \( \Sigma^1 \) for any \( t \in [t_0, T] \). Furthermore, we have
\[
\|u_{\infty}(t_0)\|_2 = \lim_{n \to \infty} \|u_n(t_0)\|_2 = \lim_{n \to \infty} \|u_n(t_n)\|_2 = \|Q\|_2.
\]

According to weak convergence in \( H^1(\mathbb{R}^N) \) and Lemma 6.2 we decompose \( u_{\infty} \) to
\[
u_{\infty}(t, x) = \frac{1}{\lambda_{\infty}(t)^{\frac{N}{4}}} (Q + \tilde{\varepsilon}_{\infty}) \left( t, \frac{x + \tilde{w}_{\infty}(t)}{\lambda_{\infty}(t)} \right) e^{-\frac{|b_{\infty}(t)|}{2} \left| x + \tilde{w}_{\infty}(t) \right|^2 - \frac{|w_{\infty}(t)|^2}{|\lambda_{\infty}(t)|^2} + i\gamma_{\infty}(t)}
\]
on \([t_0, T]\). Furthermore, as \( n \to \infty,
\lambda_n(t) \to \lambda_{\infty}(t), \quad b_n(t) \to b_{\infty}(t), \quad w_n(t) \to w_{\infty}(t), \quad e^{i\gamma_n(t)} \to e^{i\gamma_{\infty}(t)}, \quad \\
\tilde{\varepsilon}_n(t) \to \tilde{\varepsilon}_{\infty}(t)
\]
weakly in \( \Sigma^1\).
hold for any $t \in [t_0, T)$. Therefore, we obtain
\[
\lambda_\infty(t) = |t| (1 + \epsilon_{\lambda_\infty}(t)), \quad \hat{b}_\infty(t) = |t| (1 + \epsilon_{\hat{b}_\infty}(t)), \quad |\hat{w}_\infty(t)| \lesssim |t|^{2L-1},
\]
\[
|\tilde{\varepsilon}_\infty(t)|_{H^1} \lesssim |t|^{L + \frac{N}{2}}, \quad \|y|\tilde{\varepsilon}_\infty(t)\|_{L^2} \lesssim |t|^{L + \frac{N}{2} - 1}, \quad \epsilon_{\lambda_\infty}(t) \lesssim |t|^M, \quad \epsilon_{\hat{b}_\infty}(t) \lesssim |t|^{M'}
\]
from the uniform estimates in Lemma 6.2. Consequently, we obtain Theorem 1.3.

\[\Box\]

**Appendix A. A fact regarding the Schrödinger equation**

In this section, we describe a certain continuous dependence on the initial values used in the proof of Theorem 1.3. For notation, see [3]. We consider a more general Schrödinger equation

\[(40) \quad i \frac{\partial u}{\partial t} + \Delta u + g(u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.\]

For $g = g_1 + \cdots + g_k$, we consider the following assumptions:

(a) There exists $G_j \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ such that $G_j' = g_j$.

(b) There exist $r_j, \rho_j \in [2, 2^*)$ such that for any $M < \infty$, there exists $L(M) < \infty$ such that
\[
\|g_j(u) - g_j(v)\|_{\rho_j} \leq L(M)\|u - v\|_{r_j}
\]
for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1} + \|v\|_{H^1} \leq M$.

(c) For any $u \in H^1(\mathbb{R}^N)$,
\[
\text{Im} g_j(u)\overline{\varphi} = 0 \quad \text{a.e. in } \mathbb{R}^N.
\]

Here, $\rho'$ is the Hölder conjugate and $2^*$ is the Sobolev conjugate, i.e., $2^* := \frac{2N}{N-2} \quad (N \geq 3), \quad 2^* := \infty \quad (N = 1, 2)$.

**Lemma A.1.** Let $g = g_1 + \cdots + g_k$ satisfy (a), (b), and (c). For $\varphi$ and $\varphi \in H^1(\mathbb{R}^N)$, let $u_n$ and $u$ be solutions for $[10]$ with $u_n(0) = \varphi_n$ and $u(0) = \varphi$, respectively. Moreover, we assume that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$ and that for any bounded closed interval $J \subset (-T_{\min}(\varphi), T_{\max}(\varphi))$, there exists $m \in \mathbb{N}$ such that $\sup_{n \geq m} \|u_n\|_{L^\infty(J, H^1)} < \infty$. Then
\[
\lim_{n \to \infty} u_n(t) \to u(t) \quad \text{weakly in } H^1(\mathbb{R}^N) \quad \text{for any } t \in J.
\]

**proof.** We may assume that $T_1, T_2 > 0$ and $J = [-T_1, T_2]$. Then we define
\[
M := \|u\|_{L^\infty(J, H^1)} + \sup_{n \geq m} \|u_n\|_{L^\infty(J, H^1)}.
\]

Furthermore, we define
\[
\mathcal{G}_j(u)(t) := i \int_0^t T(t - s)g_j(u(s))ds, \quad \mathcal{H}(u)(t) := T(t)\varphi + \mathcal{G}_1(u)(t) + \cdots + \mathcal{G}_k(u)(t).
\]

Similarly, we define $\mathcal{G}_j(u_n)$ and $\mathcal{H}(u_n)$. According to Duhamel’s principle, we have $u = \mathcal{H}(u)$ and $u_n = \mathcal{H}(u_n)$.

Let $n \geq m$ and $0 < T \leq \min\{T_1, T_2\}$. Moreover, let $(q, r), (q_j, r_j)$, and $(\gamma_j, \rho_j)$ be admissible pairs. Then, according to the Strichartz estimate and [10], we have
\[
\|T(t)\varphi_n - T(t)\varphi\|_{L^q(\mathbb{R}, L^r)} \leq C\|\varphi_n - \varphi\|_{L^2},
\]
\[
\|\mathcal{G}_j(u_n) - \mathcal{G}_j(u)\|_{L^q((-T, T), L^r)} \leq C(M)T_j^{\frac{1}{q} - \frac{1}{\gamma_j}} \|u_n - u\|_{L^\gamma((-T, T), L^{r_j})}.
\]

For $v, w \in C([-T, T], H^1(\mathbb{R}^N))$, we define
\[
d(v, w) := \|v - w\|_{L^\infty((-T, T), L^2)} + \sum_{j=1}^k \|v - w\|_{L^\gamma((-T, T), L^{r_j})}.
\]

Then we have
\[
d(u_n, u) = d(\mathcal{H}(u_n), \mathcal{H}(u)) \leq C\|\varphi_n - \varphi\|_{L^2} + d(u_n, u)C(M)\sum_{j=1}^k T_j^{\frac{1}{q} - \frac{1}{\gamma_j}}.
\]
Since there exists $T(M) > 0$ such that $C(M) \sum_{j=1}^{k} T(M) \gamma_j \leq \frac{1}{2}$, we obtain
\[
\|u_n - u\|_{L^\infty((-T(M), T(M)), L^2)} \leq d(u_n, u) \leq C \|\varphi_n - \varphi\|_{L^2} \to 0 \quad (n \to \infty),
\]
which yields the conclusion.

Finally, $(u_n(t))_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and converges to $u(t)$ in $L^2(\mathbb{R}^N)$ for any $t \in I$. Therefore, $(u_n(t))_{n \in \mathbb{N}}$ weakly converges to $u(t)$ in $H^1(\mathbb{R}^N)$.

\[\square\]

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