Temporal Aspects of Individual Fairness

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Abstract

The concept of individual fairness advocates similar treatment of similar individuals to ensure equality in treatment [DHP+12]. In this paper, we extend this notion to account for the time at which a decision is made, in settings where there exists a notion of "conduciveness" of decisions as perceived by individuals. We introduce two definitions: (i) fairness-across-time and (ii) fairness-in-hindsight. In the former, treatments of individuals are required to be individually fair relative to the past as well as future, while in the latter we only require individual fairness relative to the past. We show that these two definitions can have drastically different implications in the setting where the principal needs to learn the utility model: one can achieve a vanishing asymptotic loss in long-run average utility relative to the full-information optimum under the fairness-in-hindsight constraint, whereas this asymptotic loss can be bounded away from zero under the fairness-across-time constraint.

1 Introduction

Algorithms facilitate decisions in increasingly critical aspects of modern life – ranging from search, social media, news, e-commerce, finance, to determining credit-worthiness of consumers, estimating a felon’s risk of reoffending, determining candidacy for clinical trials, etc. Their pervasive prevalence has motivated a large body of scientific literature in the recent years that examines the effect of automated decisions on human well-being, and in particular, seeks to understand whether these effects are "fair" under various notions of fairness [DHP+12, Swe13, KMR16, ALMK16, HPS+16, Cho17, CG17, CDG18].

In this context of automated decisions, fairness is often considered in a relative sense rather than an absolute sense. In his 1979 Tanner Lectures, Amartya Sen noted that since nearly all theories of fairness are founded on an equality of some sort, the heart of the issue rests on clarifying the "equality of what?" problem [HC18 and references therein]. Equality can be desired with respect to outcomes [HPS+16], treatment [DHP+12], or even mistreatment [ZVR+17]. In this paper, we consider the equality of treatment and take the contextual (or individual) view of fairness where "similar" individuals are treated "similarly". This notion of fairness was proposed in the influential work of Dwork et al. [DHP+12] and has since been studied under several settings, e.g., see [YR18, DI18b, DI18a]. The key idea described in this work is to introduce a "Lipschitz" condition on the decisions of a classifier, such that for any two individuals \(x, y\) that are at distance \(d(x, y) \in [0, 1]\), the corresponding distributions over decisions \(M(x)\) and \(M(y)\) are also statistically close within a distance of some multiple of \(d(x, y)\).

In this work, we extend this notion of individual fairness to account for the time at which decisions are made, in settings where there exists a commonly agreed upon notion of conduciveness of decisions from the perspective of an individual; e.g., approval of a higher loan amount is more conducive to a loan applicant than the approval of a smaller amount, a shorter jail term is more conducive to a

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Thus the expected utility is $E(U) = -xp(c) + \beta x(1 - p(c)) = x(\beta - p(c)(1 + \beta)) = xf(c)$, where $f(c) = \beta - p(c)(1 + \beta)$.

Suppose that for any two contexts in $C$, we can talk about a distance between them defined by a commonly agreed-upon distance function $d_C : C \times C \to \mathbb{R}^+$. We assume that this distance function defines a metric on $C$; in particular, it satisfies the triangle inequality. Consider the following definition of an individually fair decision-rule in the spirit of [DHP+12].

**Definition 1.** [DHP+12] A decision-rule $\phi$ is $K$-Lipschitz for $K \in [0, \infty)$ if

$$|\phi(c) - \phi(c')| \leq Kd_C(c, c')$$

for all $c, c' \in C$. (1)

2 Model

**A static model.** Consider a principal responsible for mapping contexts to decisions. Contexts $c$ lie in the finite set $C \subseteq \mathbb{R}^n$ and are drawn from some distribution $D$ over $C$. Decisions $x$ are scalar lie in the set $X = [0, 1]$. For a context $c$ and decision $x$, the principal observes a random utility $U = xF$, where $F$ is a random variable drawn from some distribution $\mathcal{F}_c$ defined on a finite set $\mathcal{S} \subseteq \mathbb{R}$. For each $c \in C$, define $f(c) = \mathbb{E}_{\mathcal{F}_c}(F)$. We assume that the distribution $\mathcal{F}_c$ for each $c$, is known to the principal. A decision-rule is a function $\phi : C \to X$ that maps each context $c \in C$ to a decision in $X$.

**Example 1.** Suppose the principal is a bank who is making loan approval decisions. The probability of loan default depends on the type $c$ of the applicant belonging to the finite set of types $C$. Suppose that for a type $c$, the probability of loan default is estimated to be $p(c)$. The decision space is $x \in [0, 1]$ representing the amount of loan sanctioned (normalized to 1). For a decision $x$, the utility of the bank is $-x$ if there is a default and it is $\beta x$ (the net present value of the interest) if there is no default, i.e., $U = xF$, where,

$$F = \begin{cases} 
-1 & \text{w.p. } p(c) \\
\beta & \text{w.p. } 1 - p(c)
\end{cases}$$

Thus the expected utility is $\mathbb{E}(U) = -xp(c) + \beta x(1 - p(c)) = x(\beta - p(c)(1 + \beta)) = xf(c)$, where $f(c) = \beta - p(c)(1 + \beta)$.
Let $\Phi_K$ be the space of $K$-Lipschitz decision-rules that map $C$ to $\mathcal{X}$. The optimization problem of the principal is to choose a $K$-Lipschitz decision-rule that maximizes the expected utility. Define,

$$U_K \triangleq \max_{\phi \in \Phi_K} \mathbb{E}_D[\phi(c)f(c)].$$

(2)

Since $C$ is finite, it is easy to see that this problem can be solved as a finite linear program.

A dynamic model. Consider now a discrete time dynamic setting where time is denoted as $t = 1, \cdots, T$ and contexts $c_t \in C$ are drawn i.i.d. from the distribution $D$ over $C$. The decisions of the principal, $x_t$ at any time $t$, lie in the set $\mathcal{X} = [0,1]$. For a context $c_t$ and corresponding decision $x_t$, the principal obtains a random utility $U_t = x_tF_t$, where $F_t$ is drawn from the distribution $F_{c_t}$ independently of the past. The average expected utility of the principal until time $T$ is given by

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[U_t] = \sum_{t=1}^{T} x_t f(c_t).$$

A policy for the principal maps the sequence of contexts seen upto time $t-1$, the corresponding decisions, and the utility outcomes, to a decision $x_t \in \mathcal{X}$ for all $t$. Note that a policy is distinct from a decision-rule: a decision-rule is a "static" object that maps every possible context to a decision, whereas a policy adaptively maps contexts to decisions as it encounters them, possibly mapping the same context to different decisions across time. We consider the following two definitions of fairness of policies.

**Definition 2. (Fairness across time)** We say that a policy is fair-across-time (FT) with respect to the function $K(s) : \mathbb{N} \to \mathbb{R}^+$ if the decisions it generates for any sequence of contexts satisfy,

$$|x_t - x_{t'}| \leq K(|t' - t|)d_C(c_t, c_{t'}) \text{ for all } t' \neq t.$$  

(3)

When $K(s) = K$ for some $K \in [0, \infty)$, we say that the policy is $K$-fair-across-time ($K$-FT).

**Definition 3. (Fairness in hindsight)** We say that a policy is fair-in-hindsight (FH) with respect to the function $K(s) : \mathbb{N} \to \mathbb{R}^+$ if the decisions it generates for any sequence of contexts satisfy,

$$x_t \geq x_{t'} - K(t' - t)d_C(c_t, c_{t'}) \text{ for all } t \geq t'.$$

(4)

When $K(s) = K$ for some $K \in [0, \infty)$, we say that the policy is $K$-fair-in-hindsight ($K$-FH).

Let $\Psi_T$ be the space of all policies for a fixed horizon $T$, and let $\Psi_{K,\text{FT}}^T$ and $\Psi_{K,\text{FH}}^T$ be the space of $T$-horizon policies that are $K$-FT and $K$-FH respectively. Consider the following two optimization problems for the principal.

$$P_{K,\text{FT}}^T : \max_{\psi \in \Psi_{K,\text{FT}}^T} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[x_t f(c_t)] \quad \text{and} \quad P_{K,\text{FH}}^T : \max_{\psi \in \Psi_{K,\text{FH}}^T} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[x_t f(c_t)].$$

(5)

The expectations are with respect to the randomness in the sequence $(c_t)_{t \geq 1}$. Define the optimal values of these problems as $U_{K,\text{FT}}^T$ and $U_{K,\text{FH}}^T$, respectively. It is clear that both $U_{K,\text{FT}}^T \geq U_K$ and $U_{K,\text{FH}}^T \geq U_K$, since one can simply use the optimal $K$-Lipschitz decision-rule at every stage. But for small horizons, potentially, one can do better. Intuitively, this is because you may not expect to encounter all the contexts within a short horizon; hence the fairness constraints are expected to be less constraining, thus offering more flexibility in mapping contexts to decisions. But we can show that when the horizon gets longer, one can’t do any better than achieving $U_K$ on average as defined in (2).

**Proposition 2.1.** For any $K \in [0, \infty)$, $\lim_{T \to \infty} U_{K,\text{FT}}^T = \lim_{T \to \infty} U_{K,\text{FH}}^T = U_K$.

This, in particular, shows that relaxing the fairness-across-time constraint to only requiring fairness-in-hindsight doesn’t lead to any long-run gains in objective. The policy of simply choosing the optimal static $K$-Lipschitz decision-rule at every stage is approximately optimal for a large horizon $T$. In the next section, we show that the situation is drastically different when there is learning involved.

A dynamic model with learning. Now consider a situation where the distribution of the utility, given a context and a decision, is unknown to the principal and must be learned. Formally, suppose that the distribution $F_t$ equals $G_{c,w}$ for each $c \in C$, where $w \in \mathcal{W}$ for some finite set $\mathcal{W}$. We assume that for any $w$ and $w'$ in $\mathcal{W}$, there is a $c \in C$ such that $G_{c,w} \neq G_{c,w'}$. Suppose that the set $\mathcal{W}$ is known, but $w$ is unknown to the principal and must be learned by adaptively assigning decisions to contexts and observing the outcomes. Define $g(c, w)$ to be the mean of $G_{c,w}$. With some abuse of
notion, we define \( U_K(w) \) to be the maximum value of the expected utility under the optimal \( K \)-Lipschitz decision rule, given \( w \). For a large enough horizon \( T \), for any dynamic policy in \( \Psi^T_{K,\text{FT}} \) or \( \Psi^T_{K,\text{FH}} \) that doesn’t assume the knowledge of \( w \), we can compare its average expected utility against the long-run optimal benchmark \( U_K(w) \). Again, with some abuse of notation, for any \( w \)-agnostic policy \( \psi \in \Psi^T_{K,\text{FT}} \), we denote its average utility for a fixed \( w \) as \( U^T_{K,\text{FT}}(w, \psi) \). We similarly, define \( U^T_{K,\text{FH}}(w, \psi) \) to be the performance of a \( w \)-agnostic policy \( \psi \in \Psi^T_{K,\text{FH}} \).

In this case, it is easy to construct an example where for any \( \psi \in \Psi^T_{K,\text{FT}} \), \( \max_{w \in \mathcal{W}} U_K(w) - U^T_{K,\text{FT}}(w, \psi) \) is bounded away from 0 for any \( T \) large enough. A formal example is given in the appendix. Here we provide an intuition.

**Example 2.** Suppose the context for each loan applicant simply denotes whether they are aged below 45 or aged above 45, and the bank does not know whether age is positively or negatively correlated with default probability. If the first applicant is aged above 45 and is given a loan of amount \( M \), then any future applicant aged above 45 must be given \( M \) to satisfy fairness-across-time. But this decision of \( M \) loan is bound to be suboptimal when \( M \) is small but age is negatively correlated with default probability or when \( M \) is large and age is positive correlated with default probability.

This shows that the FT constraint can result in significant losses in utility relative to the static optimum when learning is involved. But the situation is not as bleak under the FH constraint as we show in following main result.

**Theorem 1.** Fix a \( K \in [0, \infty) \). Then for every \( \epsilon \in (0, 1] \), there exists a sequence of \( K \)-FH policies \((\psi^T_t)_{T \in \mathbb{N}}\) such that

\[
\max_{w \in \mathcal{W}} \left( U_K(w) - \liminf_{T \to \infty} U^T_{K,\text{FT}}(w, \psi^T_t) \right) \leq \epsilon |\mathcal{C}| \max_{c, w' \in \mathcal{W}} |g(c, w')|.
\]

(6)

The idea behind this result is to choose the following \( K \)-FH policy \( \psi^T_T \). Until a random time \( T' \) (which can be thought of as a "learning" phase) defined such that at \( T' \), the parameter \( w \) is learned with a probability of error of \( 1/T \), the principal chooses \( x_t = \epsilon \) irrespective of \( c_t \). We can show that \( E(T') = O(\log T) \) and hence the expected loss in utility relative to \( U_K(w) \) in this learning phase is \( o(1) \). From that point on, the policy simply assumes the learned \( w^* \) to be the truth and chooses a \( K \)-Lipschitz decision rule defined on the decision space \( \mathcal{X}^t = [\epsilon, 1] \), assuming \( f(c) = g(c, w^*) \). The per-period expected loss in utility relative to \( U_K(w^*) \) from that point on is at most \( \epsilon |\mathcal{C}| \max_{c, w' \in \mathcal{W}} |g(c, w')| \). These facts together imply the result.

**Example 3.** Consider Example 2, where the bank is initially unaware whether age is positively or negatively correlated with default probability. In this case, the bank can approve a small amount of loan, say \( \epsilon \) to each applicant in an initial "learning" phase. Once the bank learns the correlation \( \epsilon \) from that point on is at most \( \epsilon |\mathcal{C}| \max_{c, w' \in \mathcal{W}} |g(c, w')| \). This ensures that fairness holds in the sense of FH and there is a vanishing loss in long run average utility relative to the case where the correlation structure is known.

### 3 Conclusion

Temporal aspects of fairness are highly nuanced and prominent in many areas. For example, drastically different legal decisions can be taken for similar individuals if the cases are tried at different times (think centuries). Decriminalization laws remove penalties for actions perceived as crimes in the past and such amendments are perceived as only being fair and more conducive, in line with our notion of fairness-in-hindsight. Directionality of constraints on decisions in time has also been acknowledged by the law wherein laws can be applied prospectively (affect decisions of future cases) or retrospectively (affect decisions of pending or past cases) \[\text{FT}/\text{FH}\]. In contrast, once a precedent is set by a ruling, decisions for similar contexts observed in the future must follow these precedents; whereas such rulings seldom effect past rulings \[\text{FH} \neq \text{FT}\]. In the field of revenue management and pricing, price experimentation for consumers is often perceived as unfair as similar consumers can be charged very different prices in the process of exploration to learn demand (also observed by

https://www.nytimes.com/2018/09/06/world/asia/india-gay-sex-377.html
However, if prices are slowly increased over time (for e.g. in rent control), or slowly decreased over time (for e.g. markdown sales of fashion items), this is largely deemed fair. Our work is a first step towards modeling such temporal aspects of fairness that are applicable in many such settings.

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we have an upper bound on the expected utility under any policy, equal to

Thus we have

Thus, the total side-payment over all arrivals of context \(c\), irrespective of

\(\tilde{\delta}\) estimate of the model parameter at time \(t\). Fix \(\Lambda\)

Let \(\phi\)

Proof of Proposition 2.1. Note that \(U_T^{K,FT} \leq U_T^{K,FH}\) since FT implies FH. Hence, we show the result only for \(U_T^{K,FH}\). The corresponding result for \(U_T^{K,FT}\) follows.

Fix an FH policy. At any given time \(t\), let \(u_t(c)\) be the tightest lower bound on the decision for \(c\), for each \(c \in \mathcal{C}\), based on decisions taken in the past. Note that for any decision \(x\) taken for a context \(c\) in the past, \(u_t(c) \geq x\).

First, we show that \(u_t(\cdot)\) specifies a \(K\)-Lipschitz decision-rule. To see this, consider two contexts \(c\) and \(c'\) and w.l.o.g., assume that \(u_t(c) \geq u_t(c')\). First, if \(u_t(c) = 0\), then clearly \(u_t(c) = u_t(c') = 0\).

Next, if for some time \(t' < t\), the context \(c\) was mapped to decision \(u_t(c)\), then from the FH constraint, it follows that \(u_t(c') \geq u_t(c) - Kd_C(c, c')\). Thus \(|u_t(c) - u_t(c')| \leq Kd_C(c, c')\). Finally, suppose that either the context \(c\) had never appeared before time \(t\), or it had appeared and the highest decision taken for this context so far is some \(x < u_t(c)\) (note again that the highest decision in the past for context \(c\) cannot be larger than \(u_t(c)\)). In this case, there is some other context \(c^*\) that was not mapped to some decision \(x^*\) at some time in the past and \(u_t(c) = x^* - Kd_C(c^*, c)\) (since \(u_t(c)\) is the tightest lower bound). But this also means that \(u_t(c') \geq x^* - Kd_C(c^*, c')\). Thus \(u_t(c') - u_t(c) \geq K(d_C(c^*, c) - d_C(c^*, c'))\).

But by the triangle inequality, we have \(d_C(c^*, c') \leq d_C(c^*, c) + d_C(c, c')\). Thus we have \(u_t(c') \geq u_t(c) - Kd_C(c, c')\). Thus again, \(|u_t(c) - u_t(c')| \leq Kd_C(c, c')\). This shows that \(u_t(\cdot)\) is \(K\)-Lipschitz.

Now consider the decision rule \(\phi_t\) chosen by the policy at time \(t\). Our overall proof strategy is as follows. We will bound from above the expected utility under \(\phi_t\) at time \(t\) by the expected utility of the decision rule \(u_t(\cdot)\) plus a side-payment. Since \(u_t(\cdot)\) is a \(K\)-Lipschitz decision-rule, the expected utility under this decision-rule is at most \(U_T^{K}\). Additionally, we will show that over the total side-payments are bounded by a constant independent of \(T\).

To see this, suppose that \(\phi_t\) is replaced by \(u_t(\cdot)\). Now any loss in expected utility due to this switch can be compensated by a side-payment of \((\phi_t(c) - u_t(c))|f(c)|\) to the principal in the event that \(c\) arrives at time \(t\), for each \(c \in \mathcal{C}\). Moreover, if \(c\) arrives at time \(t\), then at time \(t+1\), \(u_{t+1}(c) = \phi_t(c)\).

Thus, the total side-payment over all arrivals of context \(c\), irrespective of \(T\), is at most \(|f(c)|\). Thus we have an upper bound on the expected utility under any policy, equal to \(TU_K + \sum_{c \in \mathcal{C}} |f(c)|\). This implies the result.

\[\square\]

Proof of Theorem 1. Let \(L_t(w)\) be the likelihood of \(w \in \mathcal{W}\) based on observations until time \(t\). Define \(\Lambda_t(w, w') = \log L_t(w)/L_t(w')\). Let \(w^* = \arg \max_{w \in \mathcal{W}} L_t(w)\) be the maximum likelihood estimate of the model parameter at time \(t\). Fix \(\epsilon \in (0, 1]\). The \(K\)-FH policy \(\psi_T^\epsilon\) is defined as follows.

1. **Learning phase:** While \(\min_{w \in \mathcal{W}; w \neq w^*} \Lambda_t(w^*, w) \leq T\), assign \(x_t = \epsilon\).

2. **Exit from learning phase:** If \(\min_{w \in \mathcal{W}; w \neq w^*} \Lambda_t(w^*, w) > T\) define \(w^* = w^*_t\) and permanently enter the exploitation phase.

3. **Exploitation phase:** Use the static optimal decision rule in \(\mathcal{X} = [\epsilon, 1]\) assuming the model parameter is \(w^*\), i.e., use the decision rule that solves:

   \[
   \max_{\phi \in \mathcal{X}} \mathbb{E}[\phi(c)g(c, w^*)]
   \]

   \[
   \text{s.t. } |\phi(c) - \phi(c')| \leq Kd_C(c, c') \text{ for all } c, c' \in \mathcal{C}.
   \]

First, observe that this policy is FH. To see this, note that the policy is fixed irrespective of the context in the learning phase and hence FH. In the exploitation phase it is FH with respect to any time in the exploitation phase since the exploitation phase uses a \(K\)-Lipschitz decision rule. Finally, it is also FH with respect to the learning phase in the exploitation phase since decisions for each context only increase in going from learning to exploitation.
Next, for a fixed $T$, if we define $T' \leq T$ be the random time at which learning phase ends, then we can show that $\mathbb{E}(T') = O(\log T)$ (this follows from Lemma 4.3 in [RTA89]). Moreover, if we denote $P_w(w^* \neq w)$ to be the probability of learning an incorrect model parameter $w^*$ when the true parameter is $w$, then we can show that $P_w(w^* \neq w) \leq 1/T$. This follows from the fact that under the true $w$, the sequence of likelihood ratios $\Lambda_t(w, w')$ is a martingale and hence by Doob’s martingale inequality [Ros96], $P(\max_{t \leq T} \Lambda_t(w, w') > T) \leq 1/T$ for any $w' \neq w$.

Finally, if we denote $U^c_K(w)$ to be the optimal value of the optimization problem (7) when $w^* = w$, then we can show that $U^c_K(w) \geq U_K(w) - \epsilon |C| \max_{c \in C, w' \in W} |g(c, w')|$. This is because we can take the optimal $K$-Lipschitz decision rule $\phi$ in $X = [0, 1]$ that attains utility $U_K(w)$, and we can define a new decision rule $\phi'$ such that $\phi'(c) \triangleq \phi(c)$ if $\phi(c) \geq \epsilon$ and $\phi'(c) \triangleq \epsilon$ otherwise. It is easy to verify that this decision rule is $K$-Lipschitz and all the decisions are in $X^*$; hence it is feasible in problem (7). Clearly, the expected utility of this decision rule is at least $U_K(w) - \epsilon |C| \max_{c \in C, w' \in W} |g(c, w')|$. This implies the claim.

Thus, we finally have that for a fixed model parameter $w$, the total expected utility $U^T_{Q-FH}(w, \psi^T)$ under the $Q$-FH policy is at least

$$\left(1 - \frac{1}{T}\right) \left(U_K(w) - \epsilon |C| \max_{c \in C, w' \in W} |g(c, w')|\right) \left(T - E(T')\right).$$

Dividing by $T$ and taking the limit as $T \to \infty$ implies the result.

\[\square\]

B  Formal version of Example 2

Suppose that $C = W = \{0, 1\}$. $g(c, w)$ is expressed in the matrix below, where the first row (column) corresponds to $c = 0$ ($w = 0$).

$$[g(c, w)]_{C \times W} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Suppose that $c_t = 0$ with probability 0.9 and $c = 1$ with probability 0.1 independently for all $t$. Suppose that $K = 1/2$. In this case, it is easy to see that the $K$-Lipschitz constraint is superfluous if $w$ is known: if $w = 0$, then the optimal $K$-Lipschitz decision-rule chooses $x_t = 1$ when $c_t = 0$ and $x_t = 0$ when $c_t = 1$; if $w = 1$, then it chooses $x_t = 0$ when $c_t = 0$ and $x_t = 1$ when $c_t = 1$.

Now consider a $w$-agnostic policy $\psi \in \Psi^T_{FT}$ for some large $T$. The average utility under this policy cannot be worse than in the setting where $w$ is revealed to the policy by an oracle in time period 2. In this setting, suppose that if $c_1 = 0$ then the policy chooses some $x_1 \in \{0, 1\}$. We can show that whatever $x_1$ may be, it forces a long-run loss that is bounded away from 0 relative to $U^*_K$ for at least one of the two possible $w$, even in the case where $w$ is revealed by an oracle in time period 2.

To see this, note that for a fixed $x_1$, if $w$ is revealed to be 0 at time 2, a near-optimal policy (for a large $T$) for any $t \geq 2$ under the constraint that $x_t = x_1$ if $c_t = 1$ is to map $c_t = 1$ to $x_t = 0$. If instead $w$ is revealed to be 1 at time 2, a near-optimal policy (for a large $T$) under the constraint that $x_t = x_1$ if $c_t = 0$ is to map $c_t = 1$ to $x_t = 1$. The per period loss relative to $U^*_K(0)$ from time 2 onwards in the first case is $0.9 \times 1 - x_1$ and the per period loss relative to $U^*_K(1)$ from time 2 onwards in the second case is $0.9 \times x_1$. Thus in expectation over the randomness in $c_1$, $\max_{w \in W} U_K(w) - U^T_{K-FT}(w, \psi) \geq (T - 1)/T \times 0.9 \times 0.9 \times \max(x_1, 1 - x_1) = 0.9^2 \times 0.5 \times (1 - 1/T)$.