FRAME DRAGGING AND THE KINEMATICS OF GALACTIC-CENTER STARS

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ABSTRACT

We calculate the effects of frame dragging on the Galactic-Center stars. Assuming the stars are only slightly relativistic, we derive an approximation to the Kerr metric, which turns out to be a weak-field Schwarzschild metric plus a frame-dragging term. By numerically integrating the resulting geodesic equations, we compute the effect on Keplerian elements and the kinematics. We find that the kinematic effect at pericenter passage is proportional to \((a(1 - e^2))^{-2}\). For known Galactic-center stars it is of order \(10 \text{ m s}^{-1}\). If observed, this would provide a measurement of the spin of the black hole.

Key words: Galaxy: nucleus – relativity – stellar dynamics

1. INTRODUCTION

The center of the Milky Way is a very interesting region. It contains a massive black hole (MBH) of mass \(\sim 3 \times 10^6 M_\odot\). The central parsec contains thousands of stars. For a small but a growing number of these, due to the relatively close proximity of the MBH and short orbital periods, the orbital parameters have been accurately measured (Schödel et al. 2003; Ghez et al. 2005; Eisenhauer et al. 2005; Gillessen et al. 2008). Some of the stars have pericenter velocities as high as a few percent of \(c\). Hence, as shown in Zucker et al. (2006) the general relativistic effect of \(O(\beta^3)\) should be observable.

At \(O(\beta^3)\) general relativity predicts a new effect, which is that a spinning black hole drags the surrounding space-time along with it. Under the rotational frame-dragging effect (also known as Lense-Thirring effect), the frame of reference with minimal time dilation is one which is rotating around the object as viewed by a distant observer. If this effect could be observed for Galactic-center (GC) stars, then in principle the spin of the MBH can be measured. A method based on the orbital dynamics of the GC stars is more direct than the usual approach, which requires modeling of the effect of spin on the accretion disk (see, e.g., Narayan et al. 2008).

The effects of the \(O(\beta^3)\) terms on the Keplerian elements and on astrometry were discussed by Jaroszynski (1998) and Fragile & Mathews (2000). Will (2008) goes on to consider \(O(\beta^5)\) as well. The resulting astrometric effects are so small that they can only be observed on stars which are closer to the MBH than the observed GC stars.

In this paper we concentrate on the effects of the \(O(\beta^3)\) terms on the kinematics of the GC stars. Traditionally the effect of relativistic perturbations on orbital dynamics has been studied either using post-Newtonian celestial mechanics (Weinberg 1972) or pseudo-Newtonian equations (Semerák & Karas 1999). We adopt a different and conceptually simpler approach. We do a low-velocity perturbative expansion of the Kerr metric and then numerically integrate the resulting geodesic equations.

2. THE MODEL

Our starting point is the Kerr metric in Boyer–Lindquist coordinates (see, e.g., Misner et al. 1973):

\[
\begin{align*}
{ds}^2 &= -\frac{\Delta}{\rho^2}(dt - s \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2}((r^2 + s^2)d\phi - s\,dt)^2 \\
&\quad + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2,
\end{align*}
\]

where

\[
\Delta \equiv r^2 - 2Mr + s^2, \quad \rho^2 \equiv r^2 + s^2 \cos^2 \theta.
\]

This is equivalent to saying that the system is not very relativistic. To agree with Equation (3), we assume \(r \ll c = 1\).

The Kerr metric describes the spacetime outside a rotating black hole. The metric itself is complicated and is difficult to solve even numerically, so we make some approximations to make the solution easier. We consider the case where

\[
v^2 \sim 1/r.
\]

Here \(s\) is the spin of the black hole and \(M\) is the mass and \(G = c = 1\).

In the Kerr metric, put \(M = 1\), and Taylor expand up to \(O(\epsilon^5)\). We get the following metric:

\[
{ds}^2 = -\left(1 - \frac{2\epsilon^2}{r}\right)dt^2 + \left(1 + \frac{2\epsilon^2}{r}\right)e^{2\epsilon^2}dr^2 + e^{2\epsilon^2}d\theta^2
\]

\[+ e^{2\epsilon^2}r^2 \sin^2 \theta d\phi^2 - \frac{4\epsilon^5}{r} \sin^2 \theta dt d\phi,
\]

which is equivalent to a weak Schwarzschild field plus a frame-dragging effect. The above metric is only valid for systems with \(\beta \ll 1\). In particular, it is not valid for null geodesics, as light does not satisfy Equation (3).
Applying the Euler–Lagrange equations, we get the following geodesic equations for \( t \), \( r \), \( \theta \), and \( \phi \).

\[
\left(1 - \frac{2\epsilon^2}{r}\right) \ddot{r} + \frac{2\epsilon^2}{r^2} \dot{r} \dot{t} + O(\epsilon^5) = 0
\]

(9)

\[
\left(1 + \frac{2\epsilon^2}{r}\right) \ddot{\theta} + 2\epsilon^2 \frac{\dot{r}^2}{r^2} \ddot{\phi} + \dot{t}^2 + \epsilon^2 \frac{\dot{r}^2}{r^2} + 2\epsilon^3 \sin^2 \theta \dot{t} \dot{\phi} = 0.
\]

(10)

\[
r^2 \ddot{\theta} + 2r \dot{r} \dot{\theta} = -\frac{2\epsilon^3}{r} \sin 2\theta \dot{t} + \frac{\dot{r}^2 \sin 2\phi}{2}.
\]

(11)

The leading-order terms give the Newtonian equations, terms up to \( O(\epsilon^2) \) give Schwarzschild equations, and terms up to \( O(\epsilon^3) \) give the frame-dragging effect. We remark that \( \epsilon \) as used here is just a label for keeping track of orders. For numerical work, we set \( \epsilon = 1 \).

3. RESULTS

We now proceed with the numerical integration of the geodesic Equations (9)–(12). For simplicity we choose the unit of length to be the gravitational radius \( GM/c^2 \) of the central MBH (\( \sim 5 \times 10^6 \) km) and velocity to be in units of speed of light, which makes the unit of time to be around 17 s.

We choose \( a \), \( e \), \( I \), \( \Omega \), and \( \omega \) as the free parameters, where \( a \) is the semimajor axis of the ellipse, \( e \) the eccentricity, \( I \) the inclination with respect to the spin axis of the black hole, \( \Omega \) the longitude of ascending node, and \( \omega \) is the argument of perihelion. We start the integration at apocenter and integrate from \( t = 0 \) to \( t = 2\pi a^{3/2} \), which is exactly equal to one Newtonian orbital period. We then recompute \( a \), \( e \), \( I \), \( \Omega \), and \( \omega \) at the end of one integration and hence find the effect on the Keplerian elements. Relevant formulae for the initial conditions are given in the Appendix.

We actually integrate three different sets of geodesic equations.

1. The equations for the Newtonian case (includes only zero-order \( \epsilon \) terms). The orbital elements are completely unchanged.
2. The Schwarzschild case (terms up to \( \epsilon^2 \)). There is a pericenter shift in this case which we call \( \Delta \omega_s \).
3. The frame-dragging case (terms up to \( \epsilon^3 \)). There is a shift in the node \( \Delta \Omega_{fd} \) and a further shift \( \Delta \omega_{fd} \). There is no net effect in \( a \), \( e \), and \( I \).

We have also computed the velocity differences between these cases.
We examine the numerical data to verify the known parameter dependences of the various relativistic effects or find an empirical formula. All the effects turn out to depend upon

\[ p = a(1 - e^2), \]

which is the square of the angular momentum in gravitational units.

First, the well-known expression for the pericenter shift

\[ \Delta \omega_s = \frac{6\pi}{p}, \]

has been verified numerically, as shown in Figure 1.

Next we consider the frame-dragging precession, which satisfies the relations

\[ \Delta \Omega_{fd} = \frac{4\pi s}{p^{3/2}} \]

and

\[ \Delta \omega_{fd} = -12\pi s \cos I / p^{3/2} \]

as Figure 2 verifies. For a plane perpendicular to the spin axis of the black hole (cos \( I = 1 \)), \( \Delta \Omega_{fd} + \Delta \omega_{fd} \) corresponds to the total pericenter precession which we see is equal to \(-8\pi p^{-3/2}\) (compare with Section 3 of Weinberg et al. 2005). The physical shift of the apocentre is \( a(1 + e)\Delta \Omega_{fd} \). As viewed from a distance of 8 kpc this translates into an astrometric shift of

\[ \Delta \alpha = a(1 + e)\Delta \Omega_{fd} \times 4 \mu\text{as}. \]  

It should be noted that we have calculated the change in Keplerian elements after one complete revolution around the MBH. Dividing by the orbital period \( 2\pi a^{3/2} \) gives the mean rate of change. Doing so in Equations (14), (15), and (16) gives expressions matching Equations (6), (4), and (5) of Jaroszynski (1998).

We now consider kinematic effects. Figure 3 shows the comparison between the Schwarzschild and frame-dragging effects on the three velocity components. The effect is a fraction of a kilometer per second during pericenter passage for typical GC star orbit parameters. Note that the velocity components \( v_r = \dot{r}, v_\theta = r \dot{\theta}, \) and \( v_\phi = r \sin \theta \dot{\phi} \) are actually derivatives with respect to the conformal parameter. Finally, we consider the maximum velocity difference between the frame-dragging and Schwarzschild effects (\( \Delta V_{fd} \)) and between Schwarzschild and Newtonian effects (\( \Delta V_s \)). Figure 4 shows the relations

\[ \Delta V_s \approx \frac{8e}{p^{3/2}}, \]

\[ \Delta V_{fd} \approx -8.4es \cos I / p^2. \]
All these empirical relations are fairly accurate for parameters typical of GC stars, but not exact. On the figures, the points do not exactly lie on the line which suggests that these are leading-order effects.

Table 1 gives the values of the various relativistic effects according to our empirical formulae for a sample of GC stars and two binary pulsars. For the GC stars we estimated $p$ using Equation (13), from the $a$ and $e$ values tabulated in Eisenhauer et al. (2005). For the binary pulsars we derived $p$ using Equation (14) and the precession rate given in Will (2006).

We see from the table that the GC stars are more relativistic than binary pulsars. The advantage is that the binary pulsars have very short orbital periods (less than a day) and hence we get many more orbits.

4. CONCLUSIONS

We see that the maximum kinematic effect in known GC stars, $\Delta V_{\text{fd}}$, is of the order of a few tens of m s$^{-1}$ during a few weeks around the pericenter passage. Although this level of accuracy is difficult to achieve for GC stars, it is not implausible. Extrasolar-planet searches regularly reach an accuracy better than 1 m s$^{-1}$ (Lovis et al. 2006) and new technologies for radial velocity measurements may be able to obtain a precision as high as 1 cm s$^{-1}$ (Li et al. 2008).

There are also two theoretical problems which remain to be solved.

1. An accurate calculation of the redshift as a function of time is required (as it is the observable quantity), rather than velocity as a function of time as calculated here. Both the kinematic and gravitational redshifts are involved. We do not know of any approximate method for calculating the redshift in this case, as our approximate metric is not valid for light. It may be necessary to calculate null geodesics in the full Kerr metric.

2. The relativistic effects have to be separated from the Newtonian effects of other masses, such as nearby stars, gas and dark-matter clouds. These could overshadow the frame-dragging contribution, especially since some Newtonian dynamical processes in the GC region can be unexpectedly strong because of resonances (Gürkan & Hopman 2007; Lückmann et al. 2008). However, Newtonian perturbations from other masses would not give the distinctive time dependence in the kinematics that frame-dragging does (Figure 3). Hence, we can be optimistic about disentangling frame-dragging from all the Newtonian effects.

APPENDIX

EVALUATING THE ORBITAL ELEMENTS

The numerical integrations in this paper are done in Boyer–Lindquist coordinates, whereas the results are presented...
in terms of Keplerian orbital elements. To convert between them, we use standard relations from celestial mechanics. In practice, we only need to use the classical formulae at or near the apocenter, so we will treat \( r, \theta, \) and \( \phi \) as ordinary spherical polar coordinates.

For the initial conditions, we need to set up a star at apocenter with given \( a, e, I, \Omega, \) and \( \omega \). Since the relativistic effects are minimal, here we set \( \dot{t} = 1 \). We start by defining a temporary Cartesian coordinate system, centered at the black hole, but oriented such that the star is on the \( x \)-axis with velocity along \(+y\). In other words, we put the star at

\[
\begin{pmatrix}
  r_{\text{apo}} \\
  0 \\
  0
\end{pmatrix}
\begin{pmatrix}
  0 \\
  v_{\text{apo}} \\
  0
\end{pmatrix}
\tag{A1}
\]

where

\[
r_{\text{apo}} = a(1 + e), \quad v_{\text{apo}} = \sqrt{\frac{1 - e}{1 + e}} \frac{1}{\sqrt{a}}. \tag{A2}
\]

Applying the rotation

\[
R_z(\Omega) R_x(I) R_z(\omega + \pi)
\tag{A3}
\]

(right operator first) gives the position and velocity in the reference Cartesian system. We then convert to spherical polar coordinates.

For the inverse process at the end of an integration, we start by computing the position \( \mathbf{r} \) and the velocity \( \mathbf{v} \) in Cartesian coordinates. We then compute the specific angular momentum \( \mathbf{h} \) and the Runge–Lenz vector \( \mathbf{e} \).

\[
\mathbf{h} = \mathbf{r} \times \mathbf{v}, \quad \mathbf{e} = \mathbf{v} \times \mathbf{h} - \frac{\mathbf{r}}{r}.
\tag{A4}
\]

The inclination \( I \) and the longitude of the ascending node \( \Omega \) are simply a way of specifying the orbital plane, and we have

\[
\Omega = \arctan(h_x, h_y) + \frac{\pi}{2}, \quad I = \arctan\left(\sqrt{h_x^2 + h_z^2}, h_z\right).
\tag{A5}
\]

Here \( \arctan \) means the two-argument form, also called \( \text{atan2} \). For the argument of the perihelion, we consider the Runge–Lenz vector (which is a vector having magnitude \( e \) and pointing toward the pericenter) in the orbital plane,

\[
\begin{pmatrix}
  e \cos \omega \\
  e \sin \omega \\
  0
\end{pmatrix}
= R_z(-I) R_x(-\Omega) \mathbf{e},
\tag{A6}
\]

and the left-hand side gives \( \omega \) and \( e \).

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