LONG TIME BEHAVIOUR TO THE SOLUTION OF THE TWO-DIMENSIONAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATION

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Abstract. In [1], we prove the well posedness of the quasi-geostrophic equation \((QG)_\alpha\), \(1/2 < \alpha \leq 1\), in the space introduced by Z. Lei and F. Lin in [5]. In this paper we discuss the long time behaviour. Mainly, if \(2/3 < \alpha < 1\), we prove that \(\|\theta(t)\|_{X^{1-2\alpha}}\) decays to zero as time goes to infinity.

1. Introduction and statement of main results

In this paper, we study the initial value-problem for the two-dimensional quasi-geostrophic equation with sub-critical dissipation \((QG)_\alpha\),

\[
\begin{aligned}
\partial_t \theta + u \cdot \nabla \theta + k \Lambda^{2\alpha} \theta &= 0, \\
u &= R^\perp \theta = (-R_2 \theta, R_1 \theta), \\
\theta(0,.) &= \theta_0,
\end{aligned}
\]

where \(1/2 < \alpha \leq 1\) is a real number. The variable \(\theta\) represents potential temperature, \(u = (\partial_2 (-\Delta)^{-1/2} \theta, -\partial_1 (-\Delta)^{-1/2} \theta)\) is the fluid velocity. In the following.

The mathematical study of the non-dissipative case has first been proposed by Constantin, Majda and Tabak in [8] where it is shown to be an analogue to the 3D Euler equations. The dissipative case has then been studied by Constantin and Wu in [7] when \(1/2 < \alpha < 0\) and global existence in Sobolev spaces is studied by Constantin.

For \(\sigma \in \mathbb{R}\), we define the functional space

\[
(1.1) \quad X^\sigma(\mathbb{R}^2) = \left\{ f \in \mathcal{D}'(\mathbb{R}^2)/ \int_\xi |\xi|^{\sigma} |\hat{f}(\xi)| \, d\xi \right\}.
\]
The global well-posedness of \((QG)_{\alpha}\) with \(X^{1-2\alpha}(\mathbb{R}^2)\) data when \(1/2 < \alpha < 0\) is established by J. Benameur and M. Benhamed in [1]. They obtain a global existence of the solution for a small initial data in \(X^{1-2\alpha}(\mathbb{R}^2)\), more precisely

**Theorem 1.1.** Let \(\theta^0 \in X^{1-2\alpha}(\mathbb{R}^2)\). There is a time \(T > 0\) and unique solution \(\theta \in C([0,T], X^{1-2\alpha}(\mathbb{R}^2))\) of \((QG)_{\alpha}\), moreover \(\theta \in L^1([0,T], X^1(\mathbb{R}^2))\). If \(\|\theta^0\|_{X^{1-2\alpha}} < 1/4\), the solution is global and

\[
\|\theta\|_{X^{1-2\alpha}} + \frac{1 - 4\|\theta^0\|_{X^{1-2\alpha}}}{2} \int_0^t \|\theta\|_{X^1} \leq \|\theta^0\|_{X^{1-2\alpha}}, \quad \forall t \geq 0.
\]

The main purpose of this work is study the long time limit of the Fourier coefficients of the solutions of two-dimensional quasi-geostrophic equation \((QG)_{\alpha}\) when \(1/2 < \alpha \leq 1\), in this case Niche and Schonbek [4] prove that if the initial data \(\theta^0\) is in \(L^2(\mathbb{R}^2)\), then the \(L^2\) norm of the solution tends to zero but with no uniform rate, that is, there are solutions with arbitrary slow decay. If \(\theta^0 \in L^p(\mathbb{R}^2)\), with \(1 \leq p \leq 2\), they obtain a uniform decay rate in \(L^2\).

We state now our main result.

**Theorem 1.2.** Let \(2/3 < \alpha < 1\) and \(\theta \in C(\mathbb{R}^+, X^{1-2\alpha}(\mathbb{R}^2))\) be a global solution of \((QG)_{\alpha}\) given by Theorem [4]. Then

\[
\lim_{t \to \infty} \|\theta\|_{X^{1-2\alpha}} = 0.
\]

2. THE FRAMEWORK AND PRELIMINARIES RESULTS

- For \(f\), we denote \(u_f\) the following

\[
u_f = (\partial_2(-\Delta))^{-1/2}f, \quad -\partial_1(-\Delta)^{-1/2}f).
\]

- The Fourier transform \(\mathcal{F}(f)\) of a tempered distribution \(f\) on \(\mathbb{R}^2\) is defined as

\[
\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{-ix\xi} f(x) \, dx.
\]

- The inverse Fourier formula is

\[
\mathcal{F}^{-1}(f)(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\xi} f(x) \, dx.
\]

- For any Banach space \((B, \|\cdot\|)\), any real number \(1 \leq p \leq \infty\) and any time \(T > 0\), we will denote by \(L^p_T(B)\) the space of all measurable functions \(t \in [0,T] \mapsto f(t) \in B\) such that \((t \mapsto \|f(t)\|) \in L^p([0,T])\).

- The fractional Laplacian operator \((-\Delta)^\alpha\) for a real number \(\alpha\) is defined through the Fourier transform, namely

\[
(-\Delta)^\alpha f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).
\]
3. Proof of Theorem 1.1

This proof is inspired from the work of Gallagher-Iftimie-Planchon in [3]. Let $\varepsilon > 0$, a sufficient condition on $\varepsilon$ is as follows

\begin{equation}
\varepsilon \leq 1/8.
\end{equation}

For $n \in \mathbb{N}$, put

\[ \mathcal{A}_n = \{ \xi \in \mathbb{R}^2; |\xi| \leq n \text{ and } |\theta^0(\xi)| \leq n \}. \]

$\mathcal{F}^{-1}(1_{\mathcal{A}_n} \tilde{\theta}^0)$ converge in $H^{1-2\alpha}$ to $\theta^0$. Then there exists $n_0 \in \mathbb{N}$ such that

\[ \|\theta^0 - \mathcal{F}^{-1}(1_{\mathcal{A}_n} \tilde{\theta}^0)\|_{H^{1-2\alpha}} \leq \varepsilon/4 \quad \forall n \geq n_0. \]

For $n \geq n_0$, put $\theta^0_n = \mathcal{F}^{-1}(1_{\mathcal{A}_n} \tilde{\theta}^0)$ and $w^0_n = \theta^0 - \theta^0_n$.

Then $\|w^0_n\|_{H^{1-2\alpha}} \leq \varepsilon/4$ and $\theta^0_n \in H^{1-2\alpha}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

Consider the following system

\begin{align*}
(QG_\alpha)_n \quad & \left\{ \begin{array}{l}
\partial_t w + (-\Delta)^\alpha w + w \cdot \nabla w = 0, \\
w(0) = w^0_n
\end{array} \right.
\end{align*}

For all $n \geq n_0$, $\|w^0_n\|_{H^{1-2\alpha}} \leq \varepsilon/4$. Using Theorem 1.1, we deduce that there exists a unique global solution $w_n \in C([0, T], H^{1-2\alpha}(\mathbb{R}^2)) \cap L^1([0, T], X^\alpha(\mathbb{R}^2))$.

And we have,

\begin{equation}
\|w_n\|_{H^{1-2\alpha}} + \frac{1 - 4\|w^0_n\|_{H^{1-2\alpha}}}{2} \int_0^t \|w_n\|_{H^1} \leq \|w^0_n\|_{H^{1-2\alpha}} \quad \forall t \geq 0.
\end{equation}

Also we have

\[ \left\{ \begin{array}{l}
\partial_t \theta + (-\Delta)^\alpha \theta + u_\theta \cdot \nabla \theta = 0, \\
\theta(0) = \theta^0 \in H^{1-2\alpha}.
\end{array} \right. \]

Put $\theta = \theta - w_n + w_n$. Then $\theta_n$ solves the following system

\[ \left\{ \begin{array}{l}
\partial_t \theta_n + (-\Delta)^\alpha \theta_n + u_\theta \cdot \nabla \theta_n + u_{\theta_n} \cdot \nabla w_n + u_{w_n} \cdot \nabla \theta_n = 0, \\
\theta_n(0) = \theta^0_n \in H^{1-2\alpha} \cap L^2.
\end{array} \right. \]

Taking the inner product in $L^2(\mathbb{R}^2)$ with $\theta_n$, we obtain

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\theta_n\|^2_{L^2} + \|\theta_n\|_{H^\alpha} \leq |< u_\theta \cdot \nabla w_n/\theta_n >_{L^2}|.
\end{equation}

Then

\[ \frac{1}{2} \frac{d}{dt} \|\theta_n\|^2_{L^2} + \|\theta_n\|_{H^\alpha} \leq |< u_\theta \cdot \nabla w_n/\theta_n >_{L^2}| \]

\[ \leq \|u_{\theta_n} \cdot \nabla w_n\|_{L^2} + \|\theta_n\|_{L^2} \]

\[ \leq \|F(u_{\theta_n} \cdot \nabla w_n)\|_{L^2} \|\theta_n\|_{L^2} \]

\[ \leq \|F(u_{\theta_n}) \ast F(\nabla w_n)\|_{L^2}\|\theta_n\|_{L^2}. \]

By Young inequality $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, with $r = 2$, $p = 2$ and $q = 1$, we get

\[ \frac{1}{2} \frac{d}{dt} \|\theta_n\|^2_{L^2} + \|\theta_n\|_{H^\alpha} \leq C\|F(u_{\theta_n})\|_{L^2} \|F(\nabla w_n)\|_{L^1} \|\theta_n\|_{L^2} \]

\[ \leq C\|F(\nabla w_n)\|_{L^1} \|\theta_n\|^2_{L^2}. \]
Therefore,

\[ \frac{1}{2} \frac{d}{dt} \| \theta_n \|^2_{L^2} + \| \theta_n \|_{H^\alpha} \leq C \| w_n \|_{X^1} \| \theta_n \|^2_{L^2}. \]

By integrating with respect to time, we get

\[ \| \theta_n \|^2_{L^2} + 2 \int_0^t \| \theta_n \|_{H^\alpha} \leq C \| \theta^0_n \|^2_{L^2} + \int_0^t \| w_n \|_{X^1} \| \theta_n \|^2_{L^2}. \]

(3.4)

On the other hand, thanks to the Gronwall’s Lemma, we obtain

\[ \| \theta_n \|^2_{L^2} \leq \| \theta^0_n \|^2_{L^2} \exp(C \int_0^t \| w_n \|_{X^1}) \]

\[ \leq C \| \theta^0_n \|^2_{L^2}. \]

(3.5)

Combining (3.4) and (3.5), we get

\[ \| \theta_n \|^2_{L^2} + 2 \int_0^t \| \theta_n \|_{H^\alpha} \leq \| \theta^0_n \|^2_{L^2} + C_0 \| \theta^0_n \|^2_{L^2} \int_0^\infty \| w_n \|_{X^1} \leq M_n. \]

A crucial estimate towards the proof of Theorem 1.1 is the following:

**Lemma 3.1.** Let \( f \in \dot{H}^\alpha(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \). Then for all \( \frac{2}{3} < \alpha < 1 \),

\[ \| f \|_{X^{1-2\alpha}} \leq C \| f \|_{L^2}^{\frac{3\alpha-2}{2\alpha}} \| f \|_{\dot{H}^\alpha}^{\frac{2-2\alpha}{2\alpha}}. \]

(3.6)

**Proof:**

For \( \lambda > 0 \), put \( \| f \|_{X^{1-2\alpha}} = A_\lambda + B_\lambda \), were

\[ A_\lambda = \int_{|\xi| < \lambda} |\xi|^{1-2\alpha} |\hat{f}(\xi)| \, d\xi \quad \text{and} \quad B_\lambda = \int_{|\xi| > \lambda} |\xi|^{1-2\alpha} |\hat{f}(\xi)| \, d\xi \]

One begins by estimating the first term, using the Cauchy-Schwarz inequality, we get

\[ A_\lambda = \int_{|\xi| < \lambda} |\xi|^{1-2\alpha} \times |\hat{f}(\xi)| \, d\xi \]

\[ \leq \left( \int_{|\xi| < \lambda} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \left( \int_{|\xi| < \lambda} |\xi|^{2-4\alpha} \, d\xi \right)^{1/2} \]

\[ \leq \sqrt{4\pi} \| f \|_{L^2} \left( \int_0^\lambda r^{3-4\alpha} \, dr \right)^{1/2} \]

\[ \leq \sqrt{\frac{4\pi}{4-4\alpha}} \| f \|_{L^2} (\lambda^{4-4\alpha})^{1/2}; \quad \forall \alpha < 1. \]

Therefore, for all \( \alpha < 1 \),

\[ A_\lambda \leq \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{2-2\alpha} \| f \|_{L^2}. \]

(3.7)
A calculation similar to the previous yields
\[ B_\lambda = \int_{|\xi|>\lambda} |\xi|^\alpha |\hat{f}(\xi)| \times |\xi|^{1-3\alpha} \, d\xi \]
\[ \leq \left( \int_{|\xi|>\lambda} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \left( \int_{|\xi|>\lambda} |\xi|^{2-6\alpha} \, d\xi \right)^{1/2} \]
\[ \leq \sqrt{4\pi} \|f\|_{H^\alpha} \left( \int_\lambda^\infty r^{3-6\alpha} \, dr \right)^{1/2} \]
\[ \leq \sqrt{\frac{4\pi}{4-6\alpha}} \lambda^{2-3\alpha} \|f\|_{H^\alpha} \quad \forall \alpha > 2/3. \]

Then, for all \( 2/3 < \alpha < 1, \)
\[ (3.8) \quad B_\lambda \leq \sqrt{\frac{4\pi}{4-6\alpha}} \lambda^{2-3\alpha} \|f\|_{H^\alpha}. \]

Combining (3.7) and (3.8), we get
\[ \|f\|_{\chi^{1-2\alpha}} \leq \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{2-2\alpha} \|f\|_{L^2} + \sqrt{\frac{4\pi}{4-6\alpha}} \lambda^{2-3\alpha} \|f\|_{H^\alpha}. \]

We define \( \psi \), for all \( \lambda > 0 \) and \( 2/3 < \alpha < 1 \), by
\[ \psi(\lambda) = \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{2-2\alpha} \|f\|_{L^2} + \sqrt{\frac{4\pi}{4-6\alpha}} \lambda^{2-3\alpha} \|f\|_{H^\alpha}. \]

By differentiating \( \psi \), we get
\[ \psi'(\lambda) = (2 - 2\alpha) \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{1-2\alpha} \|f\|_{L^2} + (2 - 3\alpha) \sqrt{\frac{4\pi}{4-6\alpha}} \lambda^{1-3\alpha} \|f\|_{H^\alpha}. \]

In order to take the optimum in \( \lambda \), we write
\[ \psi'(\lambda) = 0 \iff (2 - 2\alpha) \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{1-2\alpha} \|f\|_{L^2} + (2 - 3\alpha) \sqrt{\frac{4\pi}{4-6\alpha}} \lambda^{1-3\alpha} \|f\|_{H^\alpha} = 0 \]
\[ \iff \lambda^{1-2\alpha} = \left( \frac{3\alpha - 2}{2 - 2\alpha} \right) \left( \frac{4-4\alpha}{4-6\alpha} \right)^{\frac{2\alpha}{2-2\alpha}} \left( \frac{\|f\|_{H^\alpha}}{\|f\|_{L^2}} \right)^{\frac{2\alpha}{2-2\alpha}}. \]

Then
\[ \lambda = \lambda_0 = \left( \frac{3\alpha - 2}{2 - 2\alpha} \right)^{\frac{1}{\alpha}} \left( \frac{4-4\alpha}{4-6\alpha} \right)^{\frac{\alpha}{2-2\alpha}} \left( \frac{\|f\|_{H^\alpha}}{\|f\|_{L^2}} \right)^{\frac{\alpha}{2-2\alpha}}; \quad 2/3 < \alpha < 1. \]

We have
\[ \psi(\lambda_0) = \sqrt{\frac{4\pi}{4-4\alpha}} \lambda_0^{2-2\alpha} \|f\|_{L^2} + \sqrt{\frac{4\pi}{4-6\alpha}} \lambda_0^{2-3\alpha} \|f\|_{H^\alpha} = A(\alpha) + B(\alpha). \]

Were
\[ A(\alpha) = \lambda_0^{2-2\alpha} \sqrt{\frac{4\pi}{4-4\alpha}} \|f\|_{L^2} \quad \text{and} \quad B(\alpha) = \lambda_0^{2-3\alpha} \sqrt{\frac{4\pi}{4-6\alpha}} \|f\|_{H^\alpha}. \]

We are about to estimate \( A(\alpha) \) and \( B(\alpha) \) for all \( 2/3 < \alpha < 1 \). For this purpose, we can write
\[ A(\alpha) = \left( \frac{4\pi}{3-4\alpha} \right)^{\frac{1}{2}} \left( \frac{4-4\alpha}{4-6\alpha} \right)^{\frac{1-2\alpha}{2\alpha}} \left( \frac{3\alpha - 2}{2 - 2\alpha} \right)^{\frac{2-2\alpha}{2\alpha}} \frac{\|f\|_{L^2}}{\|f\|_{H^\alpha}} \left( \frac{\|f\|_{L^2}}{\|f\|_{H^\alpha}} \right)^{\frac{2-2\alpha}{2\alpha}} \]
\[ = \left( \frac{4\pi}{4-4\alpha} \right)^{\frac{1}{2}} \left( \frac{3\alpha - 2}{2 - 2\alpha} \right)^{\frac{2-2\alpha}{2\alpha}} \left( \frac{4-4\alpha}{4-6\alpha} \right)^{\frac{1-2\alpha}{2\alpha}} \left( \frac{\|f\|_{L^2}}{\|f\|_{H^\alpha}} \right)^{\frac{2-2\alpha}{2\alpha}}. \]
And
\[ B(\alpha) = \left( \frac{4\pi}{4 - 4\alpha} \right)^{\frac{1}{2}} \left( \frac{4 - 4\alpha}{4 - 6\alpha} \right)^{\frac{2 - 3\alpha}{2\alpha}} \left( \frac{3\alpha - 2}{2 - 2\alpha} \right)^{\frac{2 - 3\alpha}{\alpha}} \| f \|_{H^\alpha} \left( \frac{\| f \|_{L^2}}{\| f \|_{H^\alpha}} \right)^{\frac{3\alpha - 2}{\alpha}}. \]

Therefore
\[
\begin{cases}
A(\alpha) \leq C_1^\alpha \| f \|_{L^2}^{\frac{3\alpha - 2}{\alpha}} \| f \|_{H^\alpha}^{\frac{2 - 2\alpha}{\alpha}} \\
B(\alpha) \leq C_2^\alpha \| f \|_{L^2}^{\frac{3\alpha - 2}{\alpha}} \| f \|_{H^\alpha}^{\frac{2 - 2\alpha}{\alpha}}.
\end{cases}
\]

From the above inequality, we get that
\[ \psi(\lambda_0) \leq C_\alpha \| f \|_{L^2}^{\frac{3\alpha - 2}{\alpha}} \| f \|_{H^\alpha}^{\frac{2 - 2\alpha}{\alpha}}. \]

Hence
\[ (3.9) \quad \| f \|_{\chi^{1-2\alpha}} \leq C_\alpha \| f \|_{L^2}^{\frac{3\alpha - 2}{\alpha}} \| f \|_{H^\alpha}^{\frac{2 - 2\alpha}{\alpha}}. \]

Now we turn to the proof of the theorem

Applying the last lemma to \( \theta_n \), we get
\[ (3.10) \quad \| \theta_n \|_{\chi^{1-2\alpha}} \leq C \| \theta_n \|_{L^2}^{\frac{3\alpha - 2}{\alpha}} \| \theta_n \|_{H^\alpha}^{\frac{2 - 2\alpha}{\alpha}}. \]

Then
\[ \| \theta_n \|_{\chi^{1-2\alpha}}^{\frac{\alpha}{\alpha}} \leq C \| \theta_n \|_{L^2}^{\frac{3\alpha - 2}{\alpha}} \| \theta_n \|_{H^\alpha}^{\frac{2 - 2\alpha}{\alpha}}. \]

Using (3.5), we deduce that
\[ (3.11) \quad \| \theta_n \|_{\chi^{1-2\alpha}}^{\frac{\alpha}{\alpha}} \leq C \| \theta_n \|_{H^\alpha}^{\frac{2}{\alpha}} \| \theta_n \|_{H^\alpha} \]

hence, after integration in time between 0 and \( \infty \), we obtain
\[ (3.12) \quad \int_0^\infty \| \theta_n \|_{\chi^{1-2\alpha}}^{\frac{\alpha}{\alpha}} \leq C \int_0^\infty \| \theta_n \|_{H^\alpha}^{\frac{2}{\alpha}}. \]

Using the continuity and the integrability of the function \( t \mapsto \| \theta_n(t) \|_{\chi^{1-2\alpha}}^{\frac{\alpha}{\alpha}} \) on \([0, \infty)\), we infer that
\[ (3.13) \quad \exists t_0 \geq 0 \text{ such that } \| \theta_n(t_0) \|_{\chi^{1-2\alpha}}^{\frac{\alpha}{\alpha}} \leq \left( \frac{\varepsilon}{4} \right)^{\frac{\alpha}{\alpha}}. \]

Now, put
\[ \theta(t_0) = (\theta(t_0) - w_n(t_0) + w_n(t_0)) = \theta_n(t_0) + w_n(t_0). \]

Thus, by (3.12), we get
\[ \| \theta(t_0) \|_{\chi^{1-2\alpha}} \leq \| \theta_n(t_0) \|_{\chi^{1-2\alpha}} + \| w_n(t_0) \|_{\chi^{1-2\alpha}} \leq \frac{\varepsilon}{4} + \| w_n(t_0) \|_{\chi^{1-2\alpha}} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \]
Therefore
\begin{equation}
\|\theta(t_0)\|_{X^{1-2\alpha}} \leq \frac{\varepsilon}{2} \leq \varepsilon.
\end{equation}
Using (3.14) and the theorem 1.1, we deduce that there exists a unique \( \gamma \in C([0, \infty), X^{1-2\alpha}(\mathbb{R}^2)) \cap L^1([0, \infty), X^1(\mathbb{R}^2)) \), solution of
\[
\begin{cases}
\partial_t \Gamma + u \cdot \nabla \Gamma + k \Lambda^2 \alpha \Gamma = 0, \\
\Gamma(0) = \theta(t_0),
\end{cases}
\]
such that
\[
\|\gamma\|_{X^{1-2\gamma}} + \frac{1 - 4\|\gamma^0\|_{X^{1-2\alpha}}}{2} \int_0^t \|\gamma\|_{X^1} \leq \|\gamma(0)\|_{X^{1-2\alpha}}, \quad \forall t \geq 0.
\]
The uniqueness gives
\[\forall t \geq 0 \quad \gamma(t) = \theta(t_0 + t).\]
Then
\[\|\theta(t_0 + t)\|_{X^{1-2\alpha}} = \|\gamma(t)\|_{X^{1-2\alpha}} \leq \|\gamma(0)\|_{X^{1-2\alpha}} \leq \varepsilon.\]
Thus Theorem 1.2 is now proved.

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