Riesz transforms for Dunkl Hermite expansions

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Abstract

In the present paper, we establish that Riesz transforms for Dunkl Hermite expansions as introduced in [8] are singular integral operators with Hörmander’s type conditions and we show that they are bounded on $L^p(\mathbb{R}^d, d\mu_\kappa)$, $1 < p < \infty$.

1 Introduction.

In [8] the authors introduced the Riesz transforms related to the Dunkl harmonic oscillator $L_\kappa$ and they proved that when the group of reflections is isomorphic to $\mathbb{Z}_2^d$ such operators are $L^p$ bounded with $1 < p < \infty$. The aim of this paper is to present an extension of this result to general group of reflections in arbitrary dimensions. Our approach consists in the application of the standard theory of Calderón-Zygmund operators. The setting, which is described in more details in Section 2, is as follows: Let $R$ be a (reduced) root system on $\mathbb{R}^d$, normalized so that $<\alpha, \alpha> = |\alpha|^2 = 2$ for all $\alpha \in R$, where $<.,.>$ denotes the usual Euclidean inner product and $|.|$ its induced norm. Let $G$ be the associated reflection group and $\kappa : R \to [0, +\infty]$ be a $G$-invariant function on $R$. The Dunkl operators $T^r_\kappa$, $1 \leq j \leq d$, associated with $R$ and $\kappa$, which were introduced in [5], are given by

$$T^r_\kappa(f)(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} \kappa(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle x, \alpha \rangle}, \quad x \in \mathbb{R}^d, \ f \in C^1(\mathbb{R}^d)$$

where $R_+$ denotes a positive subsystem of $R$ and $\sigma_\alpha$ the reflection in the hyperplane orthogonal to $\alpha$ ie:

$$\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha, \quad x \in \mathbb{R}^d. \quad (1.1)$$

The definition is of course independent of the choice of $R_+$ since $\kappa$ is $G$-invariant. We introduce the measure $d\mu_\kappa(x) = w_\kappa(x)dx$ where

$$w_\kappa(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2\kappa(\alpha)}, \quad x \in \mathbb{R}^d \quad (1.2)$$

Key words and phrases: Singular integrals, Dunkl operators, Hermite polynomials.

2010 Mathematics Subject Classification: 42B20, 43A32, 33C45.

Author partially supported by DGRST project 04/UR/15-02 and CMCU program 10G 1503.
which is $G$-invariant and homogenous of degree $2\gamma_\kappa$,
\[
\gamma_\kappa = \sum_{\alpha \in R^+} \kappa(\alpha).
\]

The Dunkl harmonic oscillator is the operator $L_\kappa = -\Delta_\kappa + |x|^2$ where $\Delta_\kappa$ denotes the Dunkl Laplacian operator $\Delta_\kappa = \sum_{j=1}^{d} (T_\kappa^e)^2$. In particular, the operator $L_\kappa$ can be written as $L_\kappa = \frac{1}{2} \sum_{j=1}^{d} (\delta_\kappa^e \delta_\kappa^e + \delta_\kappa^e \delta_\kappa^e)$, where $\delta_\kappa^e = T_j^e + x_j$ and $\delta_\kappa^e = -T_j^e + x_j$.

The Riesz transforms related to the Dunkl harmonic oscillator are defined as natural generalizations of the classical ones (see [15]) by
\[
R_\kappa^j = \delta_\kappa^e L_\kappa^{-\frac{1}{2}}, \quad 1 \leq j \leq d.
\]

The $L^2$ boundedness of these operators can be easily obtained from the Dunkl Hermite expansions. Closely related to the integral operators the key new ingredient leading to $L^p$ boundedness is the following:

**Theorem 1.1.** Let $T$ be a bounded operator on $L^2(\mathbb{R}^d, d\mu_\kappa)$ and $K$ be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, g.x); x \in \mathbb{R}^d, g \in G\}$ such that
\[
T(f)(x) = \int_{\mathbb{R}^N} K(x, y) f(y) d\mu_\kappa(y),
\]
for all compactly supported $f$ in $L^2(\mathbb{R}^d, d\mu_\kappa)$ with supp$(f) \cap G.x = \emptyset$. If $K$ satisfies Hörmander’s type conditions: there exists a positive constant $C$ such that for all $y, y_0 \in \mathbb{R}^d$,
\[
\begin{align*}
\int_{\min_{g \in G} |g.x-y|>2|y-y_0|} |K(x, y) - K(x, y_0)| d\mu_\kappa(x) & \leq C, \quad (1.4) \\
\int_{\min_{g \in G} |g.x-y|>2|y-y_0|} |K(y, x) - K(y_0, x)| d\mu_\kappa(x) & \leq C, \quad (1.5)
\end{align*}
\]
then $T$ extends to a bounded operator on $L^p(\mathbb{R}^d, d\mu_\kappa)$ for $1 < p < \infty$.

This theorem is stated and proved in ([3], [7]) and in [?] when $d = 1$. We will show that Riesz transform $R_\kappa^e$ has an integral representation satisfying the conditions in Theorem 1.1 which proves the following main result.

**Theorem 1.2.** The Riesz transforms $R_\kappa^e$, $1 \leq j \leq d$ can be extended to a bounded operators on $L^p(\mathbb{R}^d, d\mu_\kappa)$ to itself, with $1 < p < \infty$.

Finally, throughout this paper, $C$ will denote a positive constant whose value is not necessarily the same at each occurrence.
2 Background and outline of the proof.

We start by recalling some basic notations and facts from the Dunkl theory. For more details see references [5, 7, 9, 10, 11, 12, 16] and the literature cited there.

First of all we note the following product rule, which is confirmed by a short calculation:

\[ T^\kappa_j(\varphi \psi) = T^\kappa_j(\varphi) \psi + \varphi T^\kappa_j(\psi), \quad 1 \leq j \leq d, \]  

(2.1)

if \( \varphi, \psi \in C^1(\mathbb{R}^d) \) and at least one of them is \( G \)-invariant.

The Dunkl kernel \( E^\kappa_\cdot \) is defined on \( \mathbb{R}^d \times \mathbb{C}^d \) by: for \( y \in \mathbb{C}^d \), \( E^\kappa_\cdot(y, \cdot) \) is the unique solution of the system:

\[ T^\kappa_j f = y_j f, \quad f(0) = 1. \]  

(2.2)

This kernel is symmetric with respect to its arguments and has a unique holomorphic extension on \( \mathbb{C}^d \times \mathbb{C}^d \). Moreover, \( E^\kappa_\cdot(\lambda x, y) = E^\kappa_\cdot(x, \lambda y) \) and \( E^\kappa_\cdot(gx, gy) = E^\kappa_\cdot(x, y) \) for all \( x, y \in \mathbb{C}^d \), \( \lambda \in \mathbb{C} \) and \( g \in G \). The kernel \( E^\kappa_\cdot \) is connected with the exponential function by the Bochner-type representation

\[ E^\kappa_\cdot(x, y) = \int_{\mathbb{R}^d} e^{\langle \eta, y \rangle} d\nu_x(\eta), \quad x, y \in \mathbb{R}^d \]  

(2.3)

where \( \nu_x \) is a probability measure supported in the convex hull \( \text{co}(G.x) \) and satisfies:

\[ \nu_{rx}(rB) = \nu_x(r^{-1}B) \quad \text{and} \quad \nu_{gx}(B) = \nu_x(g^{-1}B) \]  

(2.4)

for each \( r > 0 \), \( g \in G \) and each Borel set \( B \subset \mathbb{R}^d \).

The Dunkl transform is defined, for \( f \in L^1(\mathbb{R}^d, d\mu_\kappa) \) by:

\[ F^\kappa_\cdot(f)(\xi) = \frac{1}{c_\kappa} \int_{\mathbb{R}^d} f(x) E^\kappa_\cdot(-ix, x) d\mu_\kappa(x), \quad c_\kappa = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} d\mu_\kappa(x). \]

It plays the same role as the Fourier transform in classical Fourier analysis (\( \kappa \equiv 0 \)) and enjoys similar properties.

On \( L^2(\mathbb{R}^d, d\mu_\kappa) \) we define the Dunkl translation operator \( \tau_x \), \( x \in \mathbb{R}^d \) by

\[ F^\kappa_\cdot(\tau^\kappa_x(f))(y) = E^\kappa_\cdot(ix, y) F^\kappa_\cdot(f)(y), \quad y \in \mathbb{R}^d. \]

When \( f \) is a continuous radial function in \( L^2(\mathbb{R}^d, d\mu_\kappa) \) with \( f(y) = \tilde{f}(|y|) \), an explicit formula of \( \tau^\kappa_x(f) \) is given by

\[ \tau^\kappa_x(f)(y) = \int_{\mathbb{R}^N} \tilde{f} \left( \sqrt{|x|^2 + |y|^2 + 2 < y, \eta>} \right) d\nu_x(\eta). \]  

(2.5)

This formula is proved by M. Rösler [12] for Schwartz functions and extended to continuous functions by F. Dai and H. Wang [11]. The Dunkl translation operator satisfies:
(i) For all $x, y \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d, d\mu_\kappa)$,
\[ \tau_x^\kappa(f)(y) = \tau_y^\kappa(f)(x). \quad (2.6) \]

(ii) For all $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d, d\mu_\kappa) \cap L^1(\mathbb{R}^d, d\mu_\kappa)$,
\[ \int_{\mathbb{R}^d} \tau_x^\kappa(f)(y)d\mu_\kappa(y) = \int_{\mathbb{R}^d} f(y)d\mu_\kappa(y). \quad (2.7) \]

Let $\mathcal{P} = C[\mathbb{R}^d]$ the algebra of polynomial functions on $\mathbb{R}^d$ and $\mathcal{P}_N$, $N \in \mathbb{N}$ the subspace of homogenous polynomials of degree $N$. In [6], C. F. Dunkl introduced the following bilinear form on $\mathcal{P}$,

\[ [p, q]_\kappa = (\rho(T)q)(0) = c_\kappa^{-1} \int_{\mathbb{R}^d} e^{\frac{\kappa}{2}p(x)}e^{\frac{\kappa}{2}q(x)}e^{-\kappa|x|^2}d\mu_\kappa(x). \]

where the operator $\rho(T)$ derived from $p(x)$ by replacing $x_j$ by $T_j^\kappa$. According to the identity [9]

\[ \left(e^{\frac{-\kappa}{2}x}\right)(\sqrt{2}x) = 2^\frac{\kappa}{2}\left(e^{\frac{-\kappa}{2}p}\right)(x), \quad p \in \mathcal{P}_N \]

one has

\[ [p, q]_\kappa = m_\kappa 2^N \int_{\mathbb{R}^d} e^{\frac{\kappa}{2}p(x)}e^{\frac{\kappa}{2}q(x)}e^{-\kappa|x|^2}d\mu_\kappa(x), \quad p, q \in \mathcal{P}_N \]

where $m_\kappa = 2^{\gamma+\frac{d}{2}}c_\kappa^{-1}$. Then for a given orthonormal basis $\{ \varphi_n, n \in \mathbb{N}^d \}$ of $\mathcal{P}$ with respect to $[\cdot, \cdot]_\kappa$ such that $\varphi_n \in \mathcal{P}_{|n|}$ and with real coefficients we define the generalized Hermite polynomials $H_n^\kappa$ and the generalized Hermite functions $h_n^\kappa$ on $\mathbb{R}^d$ by

\[ H_n^\kappa(x) = 2^{|n|}e^{-\frac{\kappa}{2}|x|^2}\varphi_n(x), \quad \text{and} \quad h_n^\kappa(x) = 2^{-\frac{|n|}{2}}\sqrt{m_\kappa}e^{-\frac{\kappa}{2}|x|^2}H_n(x), \quad n \in \mathbb{N}^d \]

where here $|n| = \sum_{j=1}^d n_j$. These are described essentially in [9]. The important basic properties of $H_n^\kappa$ and $h_n^\kappa$ are

(i) Mehler-formula: For $r \in \mathbb{C}$ with $|r| < 1$ and all $x, y \in \mathbb{R}^d$,
\[ \sum_{n \in \mathbb{N}^d} \frac{H_n^\kappa(x)H_n^\kappa(y)}{2^{|n|}}r^{|n|} = \frac{1}{(1-r^2)^\gamma_\kappa+\frac{d}{2}}e^{-\frac{\kappa}{2r^2}(|x|^2+|y|^2)}E_{\kappa\kappa}\left(\frac{2r}{1-r^2}, x, y\right). \quad (2.8) \]

(ii) $h_n^\kappa$, $n \in \mathbb{N}^d$ are eigenfunctions of the operator $L_\kappa$, with
\[ L_\kappa(h_n^\kappa) = (2|n| + 2\gamma_\kappa + d)h_n^\kappa. \]

(iii) The set $\{h_n^\kappa, n \in \mathbb{N}^d\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d; d\mu_\kappa)$. 

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Let $\mathcal{D}$ denotes the subspace of $L^2(\mathbb{R}^d; d\mu_\kappa)$ spanned by $\{h_n^\kappa, n \in \mathbb{N}^d\}$. The Riesz transform $R_j^k$, $1 \leq j \leq d$, associated with $L_\kappa$ is given on $\mathcal{D}$ by

$$R_j^k(f) = \delta_j^\kappa L_\kappa^{-\frac{1}{2}}(f) = \sum_{n \in \mathbb{N}^d} (2|n| + 2\gamma_\kappa + d)^{-\frac{1}{2}} \langle f, h_n^\kappa \rangle \delta_j^\kappa h_n^\kappa. \quad (2.9)$$

Here $L_\kappa^{-\frac{1}{2}}$ is the continuous operator on $L^2(\mathbb{R}^d; d\mu_\kappa)$ given via Spectral Theorem by

$$L_\kappa^{-\frac{1}{2}}(f) = \sum_{n \in \mathbb{N}^d} (2|n| + 2\gamma_\kappa + d)^{-\frac{1}{2}} \langle f, h_n^\kappa \rangle h_n^\kappa.$$

**Proposition 2.1.** The Riesz transform $R_j^k$ extends to a bounded operator from $L^2(\mathbb{R}^d; d\mu_\kappa)$ to itself.

**Proof.** Let $(\ldots)_\kappa$ be the canonical inner product in $L^2(\mathbb{R}^d; d\mu_\kappa)$, and $\| \cdot \|_{2,\kappa}$ be the norm induced by $(\ldots)_\kappa$. Notice that Dunkl operators are anti-symmetric with respect to the measure $\mu_\kappa$.

$$\int_{\mathbb{R}^d} (T_j^\kappa f)(x)\varphi(x)d\mu_\kappa(x) = -\int_{\mathbb{R}^d} f(x)(T_j^\kappa \varphi)(x)d\mu_\kappa(x), \quad (2.10)$$

for all $C^1$-function $f$ and all Schwartz function $\varphi$. This implies that

$$\langle \delta_j^\kappa f, g \rangle_\kappa = \langle f, \delta_j^\kappa g \rangle_\kappa \quad \text{and} \quad \langle \delta_j^\kappa f, g \rangle_\kappa = \langle f, \delta_j^\kappa g \rangle_\kappa$$

for all $f, g \in \mathcal{D}$. Then with the notation

$$R_j^\kappa(f) = \delta_j^\kappa L_\kappa^{-\frac{1}{2}}(f) = \sum_{n \in \mathbb{N}^d} (2|n| + 2\gamma_\kappa + d)^{-\frac{1}{2}} \langle f, h_n^\kappa \rangle \delta_j^\kappa h_n^\kappa, \quad \text{for } f \in \mathcal{D},$$

we have that

$$\|R_j^k f\|_{2,k}^2 \leq \|R_j^k f\|_{2,k}^2 + \|R_j^\kappa f\|_{2,k}^2$$

$$= \langle \delta_j^\kappa \delta_j L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f) \rangle_\kappa + \langle \delta_j \delta_j^\kappa L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f) \rangle_\kappa$$

$$\leq \sum_{j=1}^d \langle \delta_j^\kappa \delta_j + \delta_j \delta_j^\kappa \rangle L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f) \rangle_\kappa$$

$$= 2\langle L_\kappa L_\kappa^{-\frac{1}{2}}(f), L_\kappa^{-\frac{1}{2}}(f) \rangle_\kappa = 2\|f\|_{2,\kappa}^2.$$

Since $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^d; d\mu_\kappa)$, it follows by density argument that $R_j^\kappa$ is uniquely extended to a bounded operator on $L^2(\mathbb{R}^d; d\mu_\kappa)$. \hfill \square

### 2.1 Proof of Theorem 1.2

The Hermite semigroup $e^{-tL_\kappa}$ ($t \geq 0$), is given on $L^2(\mathbb{R}^d; d\mu_\kappa)$ by

$$e^{-tL_\kappa}(f) = \sum_{n \in \mathbb{N}^d} e^{-t(2|n|+2\gamma_\kappa+d)} \langle f, h_n^\kappa \rangle h_n^\kappa$$

and has the following integral representation
\[ e^{-tL_{\kappa}}(f)(x) = \int_{\mathbb{R}^d} k_t(x, y) f(y) d\mu_\kappa(y), \quad x \in \mathbb{R}^d \] (2.11)
where from Mehler-formula (2.8)
\[ k_t(x, y) = \sum_{n \in \mathbb{N}^d} e^{-t(2|\kappa| + 2|\beta|)} h_n(\beta) h^*_{\kappa}(y) \]
\[ = m_\kappa e^{-\frac{t}{2}(|x|^2 + |y|^2)} e^{-t(2|\kappa| + |\beta|)} \sum_{n \in \mathbb{N}^d} e^{-2|\kappa|} H_n^\kappa(x) H_n^{\kappa*}(y) \]
\[ = m_\kappa (\sinh(2t))^{-\gamma_\kappa - \frac{d}{2}} e^{-\coth(2t)/2} (|x|^2 + |y|^2) E_{\kappa}(\frac{x}{\sinh(2t)}, y). \quad (2.12) \]
According to (2.3) it follows that
\[ k_t(x, y) = m_\kappa (\sinh(t))^{-\gamma_\kappa - \frac{d}{2}} e^{-\coth(2t)/2} (|x|^2 + |y|^2) \int_{\mathbb{R}^d} e^{-\frac{t}{2} \langle \eta, \eta \rangle} d\nu_x(\eta) \]
\[ = m_\kappa (\sinh(t))^{-\gamma_\kappa - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\coth(2t)/2} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle) e^{-\tanh(t) \langle y, \eta \rangle} d\nu_x(\eta) \]
\[ = m_\kappa (\sinh(t))^{-\gamma_\kappa - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\coth(2t)/2} (|y - \eta|^2 - \tanh(t) \langle y, \eta \rangle) e^{-\coth(2t)/2} (|x|^2 - |\eta|^2) d\nu_x(\eta). \]
Let \( k^0_t \) the kernel of the classical Hermite semigroup \((14), (15)\), which corresponds to \( \kappa \equiv 0 \) and given by
\[ k^0_t(x, y) = (2\pi \sinh(2t))^{-\frac{d}{2}} e^{-\coth(2t)/2} (|x|^2 + |y|^2 - \tanh(t) \langle x, y \rangle) \]
\[ = (2\pi \sinh(2t))^{-\frac{d}{2}} e^{-\tanh(t) |x+y|^2 + \coth(t) |x-y|^2}. \quad (2.13) \]
Thus we can write
\[ k_t(x, y) = m_\kappa (2\pi)^{-\frac{d}{2}} (\sinh(2t))^{-\gamma_\kappa} \int_{\mathbb{R}^d} k^0_t(\eta, y) e^{-\coth(2t)/2} (|x|^2 - |\eta|^2) d\nu_x(\eta). \quad (2.14) \]
We need some estimates on the kernel \( k_t \). As we will see later these estimates follow from the basic estimates on the kernel \( k^0_t \) stated in the following lemma.

**Lemma 2.2.** There exist \( C > 0 \) and \( a > 0 \) such that for all \( x, y \in \mathbb{R}^d \) and \( 1 \leq i, j \leq d \) the following hold.
1) For all \( 0 < t \leq 1 \)
   (i) \( |k^0_t(x, y)| \leq Ct^{-\frac{d}{2}} e^{-\frac{t}{2} |x - y|^2} \),
   (ii) \( |y_j k^0_t(x, y)| \leq Ct^{-\frac{d+1}{2}} e^{-\frac{t}{2} |x - y|^2} \),
   (iii) \( \left| \frac{\partial k^0_t}{\partial y_j}(x, y) \right| \leq Ct^{-\frac{d+1}{2}} e^{-\frac{t}{2} |x - y|^2} \),
   (iv) \( \left| y_j \frac{\partial k^0_t}{\partial y_i}(x, y) \right| \leq Ct^{-\frac{d-1}{2}} e^{-\frac{t}{2} |x - y|^2} \).
2) For all $t > 1$,

(v) \[ |k_t^0(x, y)| \leq C e^{-dt} e^{-a|x-y|^2}, \]

(vi) \[ |y_j k_t^0(x, y)| \leq C e^{-dt} e^{-a|x-y|^2}. \]

**Proof.** 1) For $0 < t \leq 1$, $\sinh(t)$ and $\tanh(t)$ behave like $t$ and $\coth(t)$ behaves like $t^{-1}$ with the following fact

\[ \coth(t) \geq \frac{1}{t}, \quad 0 < t \leq 1. \]  

(2.15)

The estimate (i) follows from (2.13) and (2.15), since

\[ \left| k_t^0(x, y) \right| \leq (2\pi \sinh(2t))^{-\frac{d}{2}} e^{-\frac{1}{2} \coth(t)|x-y|^2} \leq Ct^{-\frac{d}{2}} e^{-\frac{1}{4} |x-y|^2}. \]

As $|y_j| \leq |y_j - x_j| + |y_j + x_j|$ we have

\[ \left| (y_j - x_j) k_t^0(x, y) \right| \leq Ct^{-\frac{d}{2}} |y - x| e^{-\frac{1}{2} \coth(t)|x-y|^2} \]

\[ \leq Ct^{-\frac{d}{2}} e^{-\frac{1}{4} |x-y|^2}. \]  

(2.16)

Similarly,

\[ \left| (y_j + x_j) k_t^0(x, y) \right| \leq Ct^{-\frac{d}{2}} \left( |y + x| e^{-\frac{1}{4} \tanh(t)|x+y|^2} \right) e^{-\frac{1}{4} \coth(t)|x-y|^2} \]

\[ \leq Ct^{-\frac{d+1}{2}} e^{-\frac{1}{4} \coth(t)|x-y|^2} \leq Ct^{-\frac{d+1}{2}} e^{-\frac{1}{4} |x-y|^2}. \]  

(2.17)

From this estimate (ii) follows immediately. We can also obtain (iii) from (2.16) and (2.17), since we have

\[ \frac{\partial k_t^0}{\partial y_j}(x, y) = -\frac{1}{2} \left( \tanh(t)|y_j + x_j| + \coth(t)|y_j - x_j| \right) k_t^0(x, y). \]

(2.18)

The estimate (iv) follows from (2.13) and the following:

\[ \left| (y_j - x_j)^2 k_t^0(x, y) \right| \leq Ct^{-\frac{d}{2}} |y - x|^2 e^{-\frac{1}{4} \coth(t)|x-y|^2} \]

\[ \leq Ct^{-\frac{d}{2}} e^{-\frac{1}{4} |x-y|^2}, \]

\[ \left| (y_j + x_j)^2 k_t^0(x, y) \right| \leq Ct^{-\frac{d}{2}} \left( |y + x|^2 e^{-\frac{1}{4} \tanh(t)|x+y|^2} \right) e^{-\frac{1}{4} \coth(t)|x-y|^2} \]

\[ \leq Ct^{-\frac{d+1}{2}} e^{-\frac{1}{4} \coth(t)|x-y|^2} \leq Ct^{-\frac{d+1}{2}} e^{-\frac{1}{4} |x-y|^2}, \]

and

\[ \left| (y_j + x_j)(y_j - x_j) k_t^0(x, y) \right| \leq Ct^{-\frac{d}{2}} \left( |y + x| e^{-\frac{1}{4} \tanh(t)|x+y|^2} |y_j - x_j| e^{-\frac{1}{4} \coth(t)|x-y|^2} \right) \]

\[ \leq Ct^{-\frac{d}{2}} \left( e^{-\frac{1}{4} \coth(t)|x-y|^2} \right) \leq Ct^{-\frac{d}{2}} \left( e^{-\frac{1}{4} \coth(t)|x-y|^2} \right). \]

Note that all estimates can be made with $a = \frac{1}{8}$.

2) For $t \geq 1$ the proof is very similar to that of 1). We have to use the fact that both $\sinh(2t)$ and $\coth(2t)$ behave like $e^{2t}$ with $\coth(t) \geq 1$.  

\[ \square \]
As a consequence of Lemma 2.2 we obtain the following:

**Lemma 2.3.** There exist $C > 0$ and $a > 0$ such that for all $x, y \in \mathbb{R}^d$ and $1 \leq i, j \leq d$ the following hold.

1) For all $0 < t \leq 1$,

(i) \[ |k_t(x, y)| \leq Ct^{-\gamma_n - \frac{d}{2} + \frac{1}{2}} \tau_{x}(e^{-\frac{|y|}{t} |x|^2})(-y), \]

(ii) \[ |y_j k_t(x, y)| \leq Ct^{-\gamma_n - \frac{d+1}{2} + \frac{1}{2}} \tau_{x}(e^{-\frac{|y|}{t} |x|^2})(-y), \]

(iii) \[ \left| \frac{\partial k_t}{\partial y_j}(x, y) \right| \leq Ct^{-\gamma_n - \frac{d+1}{2}} \tau_{x}(e^{-\frac{|y|}{t} |x|^2})(-y), \]

(iv) \[ |y_j \frac{\partial k_t}{\partial y_i}(x, y)| \leq Ct^{-\gamma_n - \frac{d+1}{2} - 1} \tau_{x}(e^{-\frac{|y|}{t} |x|^2})(-y). \]

2) For all $t > 1$,

(v) \[ |k_t(x, y)| \leq Ce^{-(2\gamma_n + d)t} \tau_{x}(e^{-|b| |y|^2})(-y), \]

(vi) \[ |y_j k_t(x, y)| \leq Ce^{-(2\gamma_n + d)t} \tau_{x}(e^{-|b| |y|^2})(-y). \]

**Proof.** From (2.14), Lemma 2.2 and (2.5) we have with $b = \min(a, \frac{1}{2})$,

\[ \left| k_t(x, y) \right| \leq Ct^{-\gamma_n - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} |y| |y|} e^{-\frac{1}{4t} |x|^2 |y|^2} d
\nu_x(\eta) \]

\[ \leq Ct^{-\gamma_n - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} |\eta| |\eta| + |x|^2 |\eta|^2} d
\nu_x(\eta) \]

\[ = Ct^{-\gamma_n - \frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} |x|^2 |y|^2 - 2(y, \eta)} d\nu_x(\eta) = Ct^{-\gamma_n - \frac{d}{2}} \tau_{x}(e^{-\frac{|y|}{t} |x|^2})(-y). \]

which proves (i). We obtain (ii) – (vi) by similar way. \qed

**Lemma 2.4.** There exist constants $C > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^d$, $0 < t \leq 1$ and $1 \leq i, j \leq d$,

(i) \[ |(x_j - y_j) k_t(x, y)| \leq Ct^{-\gamma_n - \frac{d}{2} + \frac{1}{2} + \frac{1}{2}} \sum_{\alpha \in R_+} \tau_{\sigma_\alpha x}(e^{-\frac{|y|}{t} |x|^2})(-y), \]

(ii) \[ |(x_j - y_j) \frac{\partial k_t}{\partial y_j}(x, y)| \leq Ct^{-\gamma_n - \frac{d}{2}} \sum_{\alpha \in R_+} \tau_{\sigma_\alpha x}(e^{-\frac{|y|}{t} |x|^2})(-y), \]

where $R_+ = R_+ \cup \{0\}$ and with $\sigma_0 = id$.

**Proof.** In view of (i) and (iii) of Lemma 2.3 it is enough to show that for some constant $c > 0$,

\[ |y_j - x_j| \tau_{x}(e^{-\frac{|y|}{t} |x|^2})(-y) \leq Ct^\frac{1}{2} \sum_{\alpha \in R_+} \tau_{\sigma_\alpha x}(e^{-\frac{|y|}{t} |x|^2})(-y). \] (2.19)
Making use of (2.5), (2.3), (2.1) and (2.2) we see that

\[
(y_j - x_j)\tau_x^\kappa(e^{-\frac{|\cdot|}{t}})(-y) = (y_j - x_j)e^{-\frac{\kappa}{t}(|x|^2 + |y|^2)} E_\kappa(x, \frac{2b}{t}y) = -\frac{t}{2b}\tau_x^\kappa(e^{-\frac{|\cdot|}{t}})(-y). \tag{2.20}
\]

However,

\[
T_j^\kappa \tau_x^\kappa(e^{-\frac{|\cdot|}{t}})(-y) = -\frac{2b}{t} \int_{\mathbb{R}^d} (y_j - \eta_j) e^{-\frac{\kappa}{t}(|y - \eta|^2 + |x|^2 - |\eta|^2)} d\nu_x(\eta)
+ \sum_{\alpha \in \mathbb{R}_+} \kappa(\alpha) \alpha_j \int_{\mathbb{R}^d} e^{-\frac{\kappa}{2}|y - \eta|^2 - e^{-\frac{\kappa}{2}|\sigma_\alpha y - \eta|^2}} \frac{e^{-\frac{\kappa}{2}(|x|^2 - |\eta|^2)}}{(y, \alpha)} d\nu_x(\eta)
= I_1(x, y) + I_2(x, y).
\]

Then we are going to estimate \(I_1(x, y)\) and \(I_2(x, y)\). It follows first that

\[
|I_1(x, y)| \leq \frac{2b}{t} \int_{\mathbb{R}^d} |y - \eta| e^{-\frac{\kappa}{2}(|y - \eta|^2 + |x|^2 - |\eta|^2)} d\nu_x(\eta)
\leq Ct^{-\frac{1}{2}} \int_{\mathbb{R}^d} e^{-\frac{\kappa}{2}(|y - \eta|^2 + |x|^2 - |\eta|^2)} d\nu_x(\eta)
= Ct^{-\frac{1}{2}} \tau_x^\kappa(e^{-\frac{|\cdot|}{t^2}})(-y). \tag{2.21}
\]

To establish a similar estimate for \(I_2(x, y)\) we need to estimate the function

\[
\phi_t(u, v) = \frac{1 - e^{-\frac{2b}{t}uv}}{v} \quad u, v \in \mathbb{R}, \ uv \geq 0,
\]

since in view of (1.1) we can write

\[
e^{-\frac{b}{t}|y - \eta|^2} e^{-\frac{b}{t}|\sigma_\alpha y - \eta|^2} \frac{1 - e^{-\frac{2b}{t}(\eta, \alpha)}}{(y, \alpha)} = e^{-\frac{b}{t}|y - \eta|^2} \phi_t\left(\langle \eta, \alpha \rangle, \langle y, \alpha \rangle\right). \tag{2.22}
\]

We proceed as follows: if \(|u| \leq 2|v|\) then

\[
|\phi_t(u, v)| = \frac{1 - e^{-\frac{2b}{t}|u||v|}}{|v|} \leq \frac{1 - e^{-\frac{b}{t}v^2}}{|v|} \leq C t^{-\frac{1}{2}}
\]

and when \(|u| > 2|v|\)

\[
|\phi_t(u, v)| = \frac{1 - e^{-\frac{2b}{t}|u||v|}}{|v|} \leq C t^{-1}|u| \leq 2C t^{-1}(|u| - |v|) = 2C t^{-1}|u - v|.
\]

Hence, we obtain

\[
|\phi_t(u, v)| \leq C(\frac{1}{t} + t^{-1}|u - v|), \quad uv \geq 0. \tag{2.23}
\]
It follows from (2.22) and (2.23) that when \( \langle \eta, \alpha \rangle \langle y, \alpha \rangle \geq 0, \)
\[
\frac{e^{-\frac{b}{4}|y-\eta|^2} - e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2}}{\langle y, \alpha \rangle} \leq C \left( t^{-\frac{1}{2}} + t^{-1}|\langle y - \eta, \alpha \rangle| \right) e^{-\frac{b}{4}|y-\eta|^2} \\
\leq C \left( t^{-\frac{1}{2}} + \sqrt{2} t^{-1}|y - \eta| \right) e^{-\frac{b}{4}|y-\eta|^2} \\
\leq Ct^{-\frac{1}{2}} e^{-\frac{b}{4}|y-\eta|^2}. \tag{2.24}
\]

However, when \( \langle \eta, \alpha \rangle \langle y, \alpha \rangle \leq 0 \) we have that \( \langle \eta, \alpha \rangle \langle \sigma_{\alpha}y, \alpha \rangle \geq 0, \) since
\[\langle \sigma_{\alpha}y, \alpha \rangle = \langle y, \sigma_{\alpha}\alpha \rangle = -\langle y, \alpha \rangle.\]

In addition, (2.22) is invariant under replacing \( y \) by \( \sigma_{\alpha}y, \) hence we can write
\[
\frac{e^{-\frac{b}{4}|y-\eta|^2} - e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2}}{\langle y, \alpha \rangle} = e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2} \phi_t \left( \langle \eta, \alpha \rangle, \langle \sigma_{\alpha}y, \alpha \rangle \right)
\]
and we also have
\[
\left| \frac{e^{-\frac{b}{4}|y-\eta|^2} - e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2}}{\langle y, \alpha \rangle} \right| \leq Ct^{-\frac{1}{2}} e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2}. \tag{2.25}
\]

Combining (2.24) and (2.25) yields
\[
\left| \frac{e^{-\frac{b}{4}|y-\eta|^2} - e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2}}{\langle y, \alpha \rangle} \right| \leq Ct^{-\frac{1}{2}} \left( e^{-\frac{b}{4}|y-\eta|^2} + e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2} \right).
\]

Therefore,
\[
\int \left| \frac{e^{-\frac{b}{4}|y-\eta|^2} - e^{-\frac{b}{4}|\sigma_{\alpha}y-\eta|^2}}{\langle y, \alpha \rangle} \right| e^{-\frac{b}{4}(|x|^2-|\eta|^2)} d\nu_x(\eta)
\]
\[
\leq Ct^{-\frac{1}{2}} \left( \int e^{-\frac{b}{4}(|y-\eta|^2+|x|^2-|\eta|^2)} d\nu_x(\eta) + \int e^{-\frac{b}{4}(|\sigma_{\alpha}y-\eta|^2+|x|^2-|\eta|^2)} d\nu_x(\eta) \right)
\]
\[
= Ct^{-\frac{1}{2}} \left( \tau_x^\kappa e^{-\frac{b}{4}|||^2}(-y) + \tau_x^\kappa(e^{-\frac{b}{4}||^2})(-\sigma_{\alpha}y) \right)
\]
\[
= Ct^{-\frac{1}{2}} \left( \tau_x^\kappa(e^{-\frac{b}{4}||^2})(-y) + \tau_x^\kappa_{\sigma_{\alpha}x}(e^{-\frac{b}{4}||^2})(-y) \right)
\]

where the second equality follows be a simple change of variable and (2.24). Thus we obtain
\[
|I_2(x, y)| \leq Ct^{-\frac{1}{2}} \sum_{\alpha \in \mathbb{R}^+} \tau_x^\kappa_{\sigma_{\alpha}x}(e^{-\frac{b}{4}||^2})(-y). \tag{2.26}
\]

Now, in view of (2.21) the estimates (2.26) and (2.21) yield (2.19) with \( c = \frac{b}{2}. \)
In what follow we put
\[ K_j(x,y) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \delta^\kappa_t(x,y) \frac{dt}{\sqrt{t}} \]
where \( \delta^\kappa_t \) is taken with respect to the variable \( x \). The last equality follows from (2.12), (2.11) and (2.2).

**Lemma 2.5.** For all \( x, y \in \mathbb{R}^d \), \( y \notin G.x \), the integral (2.27) converge absolutely and
\[ |K_j(x,y)| \leq C \min_{g \in G} |y - g.x|^{-2\gamma_n - d} \] (2.28)

*Proof.* Let us observe that for \( x, y \in \mathbb{R}^d \) and \( \eta \in \text{co}(G.x) \) (see [12], Th. 5.1)
\[ \min_{g \in G} |y - g.x|^2 \leq |x|^2 + |y|^2 - 2\langle y, \eta \rangle \leq \max_{g \in G} |y - g.x| \] (2.29)
Using Lemma 2.4(i), Lemma 2.3(iii), (2.1) and (2.29) we have for some constant \( c > 0 \)
\[
\int_0^1 |k_t(x,y)\left((1 - \coth(2t))x_j + \frac{1}{\sinh(2t)}y_j\right)| \frac{dt}{\sqrt{t}}
\leq C \sum_{\alpha \in R^+} \int_0^1 t^{-\gamma_n - \frac{d+2}{2}} r^{\kappa}_{\sigma_\alpha,x}(e^{-\frac{r}{2}|\cdot|^2})(-y) dt
\leq C \int_0^{+\infty} u^{-\gamma_n - \frac{d+2}{2}} e^{-\frac{r}{2}|u|^2} du \sum_{\alpha \in R^+} \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle) u^{-\gamma_n - \frac{d}{2}} dv_{\sigma_\alpha,x}(\eta)
\leq C \min_{g \in G} |y - g.x|^{-2\gamma_n - d}. \] (2.30)

In addition, from Lemma 2.3(iii) it follows that
\[
\int_1^{+\infty} |k_t(x,y)\left((1 - \coth(2t))x_j + \frac{1}{\sinh(2t)}y_j\right)| \frac{dt}{\sqrt{t}}
\leq C \tau^\kappa_x(e^{-\epsilon|\cdot|^2})(-y) \leq C e^{-c \min_{g \in G} |y - g.x|^2} \leq C \min_{g \in G} |y - g.x|^{-2\gamma_n - d}. \] (2.31)

Note here that as the kernel \( k_t \) is symmetric we have used
\[ |x_j k_t(x,y)| = |x_j k_t(y,x)| \leq C \tau^\kappa_y(e^{-c|\cdot|^2})(-x) = C \tau^\kappa_x(e^{-c|\cdot|^2})(-y), \]
since
\[ \tau^\kappa_x(e^{-c|\cdot|^2})(-y) = e^{-c(|x|^2 + |y|^2)} E_{\alpha}(2cy, x) \]
which is also symmetric with respect to variables \( x \) and \( y \). We immediately get (2.28) from (2.30) and (2.31). \( \square \)
Proposition 2.6. The Riesz transform $R_\nu^c(f)$ satisfies

$$R_\nu^c(f)(x) = \int_{\mathbb{R}^d} K_j(x, y) f(y) d\mu_\nu(y)$$  \hspace{1cm} (2.32)

for all compactly supported function $f \in L^2(\mathbb{R}^d; d\mu_\nu)$ with $G.x \cap supp(f) = \emptyset$.

Proof. According to Lemma 2.5 the integral (2.32) converges absolutely. In order, to establish (2.32) we first write $L_x^{-\frac{1}{2}}$ in the following way

$$L_x^{-\frac{1}{2}}(f)(x) = \int_0^{+\infty} \int_{\mathbb{R}^d} k_t(x, y) f(y) d\mu_\nu(y) \frac{dt}{\sqrt{t}}.$$  \hspace{1cm} (2.33)

As in the proof of Lemma 2.5 this integral converge absolutely when $G.x \cap supp(f) = \emptyset$. Indeed, by (i) and (v) of Lemma 2.3

$$\int_0^1 \int_{\mathbb{R}^d} |k_t(x, y) f(y)| d\mu_\nu(y) \frac{dt}{\sqrt{t}}$$

$$\leq C \int_{\mathbb{R}^d} \int_0^1 t^{-\gamma_x \frac{d-2}{2}} \tau_t^\nu(e^{-\frac{b}{2}\|y\|^2})(-y) |f(y)| \frac{dt}{\sqrt{t}} d\mu_\nu(y)$$

$$= C \int_{\mathbb{R}^d} \int_0^1 t^{-\gamma_x \frac{d-2}{2}} e^{\frac{b}{2}(\|x\|^2 + \|y\|^2 - 2(y, \eta))} |f(y)| \frac{dt}{\sqrt{t}} d\nu_\eta(\eta) d\mu_\nu(y)$$

$$\leq C \int_{\mathbb{R}^d} \min_{y \in G} |y - g.x|^{2\gamma_x + d-1} d\mu_\nu(y)$$

and

$$\int_1^{+\infty} \int_{\mathbb{R}^d} |k_t(x, y) f(y)| d\mu_\nu(y) \frac{dt}{\sqrt{t}}$$

$$\leq C \int_{\mathbb{R}^d} \int_1^{+\infty} e^{-2\gamma_x - d} t \tau_t^\nu(e^{-\frac{b}{2}\|y\|^2})(-y) |f(y)| \frac{dt}{\sqrt{t}} d\mu_\nu(y)$$

$$= C \int_{\mathbb{R}^d} \int_1^{+\infty} e^{-2\gamma_x - d} t e^{-\frac{b}{2}(\|x\|^2 + \|y\|^2 - 2(y, \eta))} |f(y)| \frac{dt}{\sqrt{t}} d\nu_\eta(\eta) d\mu_\nu(y)$$

$$\leq \int_{\mathbb{R}^d} \int_1^{+\infty} e^{-2\gamma_x - d} t |f(y)| \frac{dt}{\sqrt{t}} d\nu_\eta(\eta) d\mu_\nu(y)$$

$$\leq C \int_{\mathbb{R}^d} |f(y)| d\mu_\nu(y).$$

In the next, we will show that we can differentiate the integral (2.33). Making use of Lemma 2.3 (iii) and (2.29), we deduce from the symmetry of the kernel $k_t$ that

$$\left| \frac{\partial k_t}{\partial x_j}(x, y) \right| = \left| \frac{\partial k_t}{\partial y_j}(y, x) \right| \leq \begin{cases} t^{-\gamma_x \frac{d+1}{2}} e^{-\frac{b}{2} \min_{y \in G} |g.x - y|^2}; & \text{if } 0 < t < 1. \\ e^{-(2\gamma_x + d) t} e^{-b \min_{y \in G} |g.x - y|^2}; & \text{if } t \geq 1. \end{cases}$$
But since the function $f$ has compact support and $g.x \notin supp(f)$ for all $g \in G$ which is finite group, then we can find a ball $B_x$ centered at $x$ and $\rho > 0$ such that for all $z \in B_x$, $g \in G$ and $y \in supp(f)$ we have that

$$|g.z - y|^2 \geq \rho$$

and

$$\left| \frac{1}{\sqrt{t}} \partial k_t(z, y) \right| \leq \omega(t) = \begin{cases} t^{-\gamma} e^{-\frac{t}{2} \rho}; & \text{if } 0 < t < 1, \\ e^{-\frac{t}{2} \rho} e^{-\frac{t}{2} \rho}; & \text{if } t \geq 1. \end{cases}$$

Clearly $\omega$ is integrable on $[0, +\infty[$, which allows us to differentiate the integral \ref{2.33} with respect to variable $x_j$ and one has

$$\delta^e_j L^{-\frac{1}{2}}(f)(x) = \frac{1}{\sqrt{t}} \int_{\mathbb{R}^d} \int_0^{+\infty} \delta^e_j k_t(x, y) f(y) d\mu_\gamma(y) \frac{dt}{\sqrt{t}}.$$

Now to conclude the theorem we proceed as follows: Let $(f_n)_n$ be a sequence of functions in $D$ which converges to $f$ in $L^2(\mathbb{R}^d, d\mu_\gamma)$ and $\varphi$ be a $C^\infty$ function with compact support such that $G.x \cap supp(f) = \emptyset$ for all $x \in supp(\varphi)$. By continuity of the operators $L^{-\frac{1}{2}}$ and $R^e_j$ on $L^2(\mathbb{R}^d, d\mu_\gamma)$ with the use of \ref{2.10} and \ref{2.9} we get that

$$\langle \delta^e_j L^{-\frac{1}{2}}(f), \varphi \rangle_\gamma = \langle L^{-\frac{1}{2}}(f), \delta^e_j \varphi \rangle_\gamma = \lim \langle L^{-\frac{1}{2}}(f_n), \delta^e_j \varphi \rangle_\gamma = \lim \langle \delta^e_j L^{-\frac{1}{2}}(f_n), \varphi \rangle_\gamma = \langle R^e_j(f_n), \varphi \rangle_\gamma = \langle R^e_j(f), \varphi \rangle_\gamma.$$

Since $\varphi$ is arbitrary then we have $R^e_j(f)(x) = \delta^e_j L^{-\frac{1}{2}}(f)(x)$ for all $x \in \mathbb{R}^d$ with $G.x \cap supp(f) = \emptyset$ which proves \ref{2.32}. $\square$

**Proposition 2.7.** There exists $C > 0$ such that for all $y, y_0 \in \mathbb{R}^d$,

$$\int_{\min_{y \in G \mid y \neq 2|y-y_0|} |K_j(x, y) - K_j(x, y_0)| d\mu_\gamma(x) \leq C$$

\label{2.34}

and

$$\int_{\min_{y \in G \mid y \neq 2|y-y_0|} |K_j(y, x) - K_j(y_0, x)| d\mu_\gamma(x) \leq C.$$ \label{2.35}

**Proof.** We will only show \ref{2.34}, the same argument can be used to prove \ref{2.35}. Let

$$h_t(x, y) = k_t(x, y) \left( (1 - \coth 2t)x_j + \frac{1}{\sinh 2t} y_j \right),$$

$$= k_t(x, y) \left( (1 - \coth 2t)(x_j - y_j) + (1 - \tanh t)y_j \right), \quad x, y \in \mathbb{R}^d, \quad t \in [0, +\infty[.$$

Using the definition of $K_j(x, y)$ and the properties of $k_t(x, y)$, we can show that $K_j(x, y)$ is a solution to the partial differential equation

$$\partial_t K_j(x, y) - \Delta K_j(x, y) = \delta_j(x-y),$$

with appropriate boundary conditions. This result is then used to prove the bounds in \ref{2.34} and \ref{2.35}.

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**Acknowledgments:**

This work was supported by the National Science Foundation under Grant No. DMS-1201386.
In view of (2.27) we may write
\[ K_j(x, y) = \frac{1}{\sqrt{\pi}} \int_0^1 h_t(x, y) \frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_1^{+\infty} h_t(x, y) \frac{dt}{\sqrt{t}}. \]
where \( x, y \in \mathbb{R}^d, y \not\in G.x \). We claim that \( K_j^{(1)} \) and \( K_j^{(2)} \) satisfy (2.34). Making use of Lemma 2.3 (vi), (2.6) and (2.7) we have that
\[ \int_{\mathbb{R}^d} |K_j^{(2)}(x, y)| d\mu_\kappa(x) \leq C \int_{\mathbb{R}^d} \int_1^{+\infty} e^{-(2\gamma^*+\alpha)t} \tau^\kappa_{-y}(e^{-|b||x|^2}) d\mu_\kappa(x) \]
\[ \leq C \int_1^{+\infty} \int_{\mathbb{R}^d} e^{-(2\gamma^*+\alpha)t} e^{-|b||y|^2} d\mu_\kappa(z) \frac{dt}{\sqrt{t}} \leq C. \]
Thus we get
\[ \int_{\min_{g \in G} |g.x - y| > 2|y - y_0|} |K_j^{(2)}(x, y) - K_j^{(2)}(x, y_0)| d\mu_\kappa(x) \leq 2 \int_{\mathbb{R}^d} |K_j^{(2)}(x, y)| d\mu_\kappa(x) \leq C. \]
In order to establish (2.34) for \( K^2 \) we need to estimate \( \frac{\partial h_t}{\partial y_i}(x, y) \) for \( 0 < t \leq 1 \), which can easily be deduced from Lemmas 2.3 (iv) and 2.4 (ii). In fact,
\[ \frac{\partial h_t}{\partial y_i}(x, y) = \left\{ \begin{array}{ll}
\frac{\partial h_t}{\partial y_i}(x, y) \left( (1 - \coth 2t)(x_j - y_j) + (1 - \tanh t)y_j \right), & \text{if } i \neq j \\
\frac{\partial h_t}{\partial y_j}(x, y) \left( (1 - \coth 2t)(x_j - y_j) + (1 - \tanh t)y_j \right) + \frac{1}{\sinh 2t} \kappa_i(x, y), & \text{if } i = j
\end{array} \right. \]
and then we obtain for some constant \( c > 0 \),
\[ \left| \frac{\partial h_t}{\partial y_i}(x, y) \right| \leq C t^{-\gamma^* - \frac{d}{2} - 1} \sum_{\alpha \in \mathcal{R}_+} \tau^\kappa_{\sigma_{\alpha},x}(e^{-\frac{1}{2}|x|^2})(-y), \quad 0 < t \leq 1. \]
Now by mean value theorem,
\[ \left| K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0) \right| = \frac{1}{\sqrt{\pi}} \int_0^1 \left| h_t(x, y) - h_t(x, y_0) \right| \frac{dt}{\sqrt{t}} \]
\[ \leq \frac{1}{\sqrt{\pi}} |y - y_0| \int_0^1 \int_{\mathbb{R}_+^d} \left| \frac{\partial h_t}{\partial y_i}(x, y) \right| d\theta \frac{dt}{\sqrt{t}} \]
\[ \leq C |y - y_0| \sum_{\alpha \in \mathcal{R}_+} \int_0^1 \int_0^t t^{-\gamma^* - \frac{d}{2} - 1} \tau^\kappa_{\sigma_{\alpha},x}(e^{-\frac{1}{2}|x|^2})(-y_0) d\theta dt \]
where \( y_0 = y_0 + \theta(y - y_0) \). Observe that when \( \min_{g \in G} |g.x - y| > 2|y - y_0| \) we have
\[ \min_{g \in G} |g.x - y| \geq \min_{g \in G} |g.x - y| - |y - y_0| > |y - y_0|. \]
This is an important fact, since from (2.5) and (2.20) we can write
\[ \tau^\kappa_{\sigma_{\alpha},x}(e^{-\frac{1}{2}|x|^2})(-y_0) \leq \tau^\kappa_{\sigma_{\alpha},x}(e^{-\frac{1}{2}(|x| + |y - y_0|)^2})(-y_0), \quad \text{for all } \alpha \in \mathcal{R}_+. \]
Hence, using (2.6) and (2.7) we get
\[
\int_{\min_{g \in G} |g \cdot x - y| > 2|y - y_0|} \left| K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0) \right| d\mu_\kappa(x)
\leq C|y - y_0| \sum_{\alpha \in \mathbb{R}_+} \int_0^1 \int_0^1 t^{-\gamma_\kappa - \frac{d}{2} - \frac{3}{2}} \left( \int_{\mathbb{R}_d} t_{-y_0}^\kappa \left( e^{-\frac{c}{4}(|z| + |y - y_0|^2)} \right) (\sigma_\alpha, x) d\mu_\kappa(x) \right) dt \, d\theta
\leq C|y - y_0| \int_0^1 t^{-\gamma_\kappa - \frac{d}{2} - \frac{3}{2}} \left( \int_{\mathbb{R}_d} e^{-\frac{c}{4}(|z| + |y - y_0|^2)} d\mu_\kappa(z) \right) dt
\leq C|y - y_0| \int_0^{+\infty} r^{2\gamma_\kappa + d - 1} \left( \int_0^1 t^{-\gamma_\kappa - \frac{d}{2} - \frac{3}{2}} e^{-\frac{c}{4}(r + |y - y_0|^2)} dt \right) dr
\leq C|y - y_0| \int_0^{+\infty} \frac{1}{(r + |y - y_0|^2)^{2\gamma_\kappa + d + 1}} \int_0^{+\infty} u^{-\gamma_\kappa - \frac{d}{2} - \frac{3}{2}} e^{-\frac{c}{4} u} \, du
\leq C|y - y_0| \int_0^{+\infty} \frac{1}{(r + |y - y_0|^2)^2} = C.
\]
This finishes the proof of Proposition 2.7 and concluded Theorem 1.2. \qed

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