Shifting the Phase Transition Threshold for Random Graphs and 2-SAT using Degree Constraints

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Abstract

We show that by restricting the degrees of the vertices of a graph to an arbitrary set $\Delta$, the threshold point $\alpha(\Delta)$ of the phase transition for a random graph with $n$ vertices and $m = \alpha(\Delta)n$ edges can be either accelerated (e.g., $\alpha(\Delta) \approx 0.38$ for $\Delta = \{0, 1, 4, 5\}$) or postponed (e.g., $\alpha(\Delta) \approx 0.95$ for $\Delta = \{1, 2, 50\}$) compared to a classical Erdős–Rényi random graph with $\alpha(\mathbb{Z}_{\geq 0}) = \frac{1}{2}$. In particular, we prove that the probability of graph being nonplanar and the probability of having a complex component, goes from 0 to 1 as $m$ passes $\alpha(\Delta)n$. We investigate these probabilities and also different graph statistics inside the critical window of transition (diameter, longest path and circumference of a complex component).

Finally, we introduce a 2-CNF model with restricted literal degrees, i.e. a model with a property analogous to that of graphs: the number of clauses that contain each variable, belongs to the set $\Delta$. We apply our results on random graph and prove a lower bound for the probability that a formula with $n$ variables and $m = 2\alpha(\Delta)n$ clauses is satisfiable. This probability is close to 1 for the subcritical regime $m = 2n(1 - \mu n^{-1/3}), \mu \to \infty$ and matches the lower bound of Bollobás, Borgs, Chayes, Kim, and Wilson ([BBC+01]) for the classical case. This implies that the phase transition threshold for 2-SAT can also be shifted by restricting the degrees of the literals.

1 Introduction

1.1 Shifting the Phase Transition

Consider a random Erdős–Rényi graph $G(n, m)$ [ER60], that is a graph chosen uniformly at random among all simple graphs built with $n$ vertices, conventionally labeled with distinct numbers from $\{1, 2, \ldots, n\}$, and $m$ edges. The range $m = \frac{1}{2}n(1 + \mu n^{-1/3})$ where $n \to \infty$ and $\mu$ depends on $n$, is of particular interest since there are three distinct regimes, according to how the crucial parameter $\mu$ grows as $n$ is large: as $\mu \to -\infty$, the size of the largest component is of order $\Theta(\log n)$ and the connected components are almost surely trees and unicyclic components; next, inside what is known as the critical window of transition $(\mu \in O(1))$ the largest component size is of order $\Theta((1-\mu n^{-1/3})$, $\mu \to \infty$ and matches the lower bound of Bollobás, Borgs, Chayes, Kim, and Wilson ([BBC+01]) for the classical case. This implies that the phase transition threshold for 2-SAT can also be shifted by restricting the degrees of the literals.

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As for random graphs, random 2-SAT formulae are of paramount importance in computer science, statistical physics and have been well studied (see [BBC+01] and references therein). In the 2-SAT problem, a conjunctive normal form (CNF) formula consists of \( m \) 2-clauses of the form \( x \lor y \), where \( x \) and \( y \) are literals from the set of \( n \) Boolean variables and their negations. The most accurate description of the phase transition of random 2-SAT is given by Bollobás, Borgs, Chayes, Kim, and Wilson [BBC+01] who proved that for \( m = n(1 + \mu n^{-1/3}) \) as \( \mu \to -\infty \) with \( n \), the formula is unsatisfiable with probability \( \Theta(\mu^{-3}) \); as \( \mu \) is any fixed real the probability of satisfiability is \( \Theta(1) \), and finally, as \( \mu \to +\infty \) with \( n \), the formula is satisfiable with probability \( \exp(-O(\mu^3)) \). Inside their respective windows of transition, the appearance of complex components for random graphs and the SAT/UNSAT phases for random 2-SAT, the two problems behave very similarly, see however [Kra07]. It is also relevant to indicate the link between the phase transition in random graphs and in the MAX-CUT / MAX-2-SAT problems [CGHS04, GKK16]. The analytic approach can be also successful in such problems [DMRR12].

The last decades have seen a growth of interest in delaying or advancing the phase transitions of random graphs (resp. random 2-CNF formula). Mainly, two kinds of processes have been introduced and studied:

a) the Achlioptas process where models of random graph are obtained by adding edge one by one but according to a given rule which allows to choose the next edge from a set of candidate edges [BF01, RW17],

b) the given degree sequence models where a sequence \((d_1, \ldots, d_n)\) of degrees is given and a simple graph built on \( n \) vertices is uniformly chosen from the set of all graphs whose degrees match with the sequence \( d_i \) (see [Rio12, MR95, HM12, JPRR16]).

In [BF01, RW12, RW17], the authors studied the Achlioptas process. In particular, Bohman and Frieze [BF01] were able to show that there is a random graph process such that after adding \( m = 0.535 n > 0.5 n \) edges the size of the largest component is (still) polylogarithmic in \( n \) which contrasts with the classical Erdős–Rényi random graphs. Riordan and Warnke [RW12, RW17] investigated the transition features of “bounded-size” processes. In the models of random graphs with a fixed degree sequence \( D = (d_1, \ldots, d_n) \), Joos, Perarnau, Rautenbach, and Reed [JPRR16] proved that a simple condition that a graph with degree sequence \( D \) has a connected component of linear size, is that the sum of the degrees in \( D \) which are not 2 is at least \( \lambda(n) \) for some function \( \lambda(n) \) that goes to infinity with \( n \). For the random 2-SAT formulae with prescribed literal degrees, we refer to the work of Cooper, Frieze and Sorkin [CFS07]. Several authors have also investigated 2-SAT and \( k \)-SAT transition under Achlioptas rules [SV13, Per15, DDHM13].

In the current work, our approach is rather different. We study random graphs with degree constraints that are graphs drawn uniformly at random from the set of all graphs with given number of vertices and edges with all vertices having degrees from given set \( \Delta \subseteq \mathbb{Z}_{\geq 0}, \ 1 \in \Delta \). De Panafieu and Ramos calculated asymptotic number of such graphs using methods from analytic combinatorics [dPR16]. Using their asymptotic results, we prove that random graphs with degrees from the set \( \Delta \) have their phase transition shifted from the density of edges \( \frac{m}{n} = \frac{1}{2} \) to \( \frac{m}{n} = \alpha \) for an explicit and computable constant \( \alpha = \alpha(\Delta) \) and the new critical window of transition becomes \( m = \alpha n(1 \pm \mu n^{-1/3}) \).

In addition, we also prove that the structure of such graphs inside this crucial window behaves as in the Erdős–Rényi case. For instance, we prove that extremal parameters such as the diameter, the circumference or the longest path are of order \( \Theta(n^{1/3}) \) around \( m = \alpha n \). The size of complex components of our graphs are of order \( \Theta(n^{2/3}) \) as \( \mu \) is bounded. A very similar result but about the diameter of the largest component of \( G(n, p = \frac{1}{n} + \frac{\mu}{n^{3/2}}) \) has been obtained by Nachmias and Peres [NP08] (using very different methods).

In the seminal paper of Erdős and Rényi, amongst other non-trivial properties, they discussed the planarity of random graphs with various edge densities [ER60]. The probabilities of planarity of Erdős-Rényi random graphs inside their window of transition have been since then computed by Noy, Ravelomanana, and Rué [RRN13]. In the current work, we extend this study by showing that the planarity threshold shifts from \( \frac{3}{4} \) for classical random graphs to \( \alpha n \) for graphs with degrees from \( \Delta \). More precisely, first we show that such objects are almost surely planar as \( \mu \) goes to \( -\infty \) and non-planar as \( \mu \) tends to \( +\infty \). Next, as function of \( \mu \), we compute the limiting probability that random graphs of degrees in \( \Delta \) are planar as \( \mu = O(1) \).
For the 2-sat problem, we define the degree of a literal as the number of clauses that contain this literal. We prove that if we impose degree constraints on a random 2-cnf formula, then the (possible) transition window with average edge density \( \frac{m}{n} = 1 \) is shifted to at least \( \frac{m}{n} = 2\alpha(\Delta) \).

Structure of the Article. In section 2 we state our main results and give proofs which rely on technical statements from Appendix A. Next, in section 3 we introduce our 2-cnf model and prove the lower bound for sat probability. In section 4, we give the results of simulations using the recursive method from [dPR16]. Section A contains the tools from analytic combinatorics. Then follows Appendix B with the method of moments and marking tools. Section C compares two different models: graphs with degree constraints and graphs with given degree sequence.

1.2 Preliminaries

The excess of a connected graph is the number of its edges minus the number of its vertices. For example, connected graphs with excess \(-1\) are trees, with excess \(0\) — graphs with one cycle (also known as unicycles or unicyclic graphs), connected bicycles have excess \(2\), and so on (see Figure 1). Connected graph always has excess at least \(-1\). A connected component with excess at least \(1\), is called a complex component. The complex part of a random graph is the union of its complex components.

![Figure 1: Examples of connected labeled graphs with different excess. As a whole, can be considered as a graph with total excess \(-1 + 0 + 1 + 2 = 2\)](image)

Next, we introduce the notion of a 2-core (the core) and a 3-core (the kernel) of a graph. The 2-core is obtained by repeatedly removing all vertices of degree 1 (smoothing). The 3-core is obtained from 2-core by repeatedly replacing vertices of degree two with their adjacent edges by a single edge connecting the neighbors of deleted vertices (we call this a reduction procedure). A 3-core can be a multigraph, i.e. there can be loops and multiple edges. There is only a finite number of connected 3-cores with a given excess [JKLP93]. The inverse images of vertices of 3-core under the reduction procedure, are called corner vertices (cf. Figure 3). A 2-path is an inverse image of an edge in a 3-core, i.e. a path connecting two corner vertices.

The circumference of a graph is the length of its longest cycle. A diameter of a graph is the maximal length of the shortest path taken over all distinct pairs of vertices. It is known that the problems of finding the longest path and the circumference are NP-hard.

Assume that \((x_1, \ldots, x_m, y_1, \ldots, y_m) \in \{\xi_1, \ldots, \xi_n, \overline{\xi}_1, \ldots, \overline{\xi}_n\}^{2m}\). Let \((\xi_i)_{i=1}^n\) be a collection of \(n\) Boolean variables. Given a 2-cnf formula \(\bigwedge_{i=1}^n (x_i \lor y_i)\), its digraph representation is the directed graph with the set of edges 

\[
\bigcup_{i=1}^m \left( (x_i \rightarrow y_i) \cup (y_i \rightarrow x_i) \right).
\]

If there exists an assignation of the Boolean \(\xi_i\) such that each clause of the formula is satisfiable, then we say that formula is sat, otherwise we say that it is unsat. A circuit is a directed cycle \((v_1, \ldots, v_k)\) such that \(v_1 = v_k\). We say that \(x \sim y\) if there exists a directed path from \(x\) to \(y\). See Figure 4 for an example of digraph representation.

Random graph with degree constraints is a graph sampled uniformly at random from the set of all possible graphs \(G_{n,m,\Delta}\) having \(m\) edges and \(n\) vertices all of degrees from the set \(\Delta = \{\delta_1, \delta_2, \ldots\} \subseteq \{0,1,2,\ldots\}\), see Figure 2. The set \(\Delta\) can be finite or infinite. In this work, we require that \(1 \in \Delta\).
This technical condition allows the existence of trees and tree-like structures in the random objects under consideration. We don’t know what happens when $\frac{1}{\Delta} \notin \Delta$.

The set $\mathcal{G}_{n,m,\Delta}$ is (asymptotically) nonempty if and only if the following condition is satisfied [dPR16]:

\[ (C) \quad \text{Denote } \gcd(d_1 - d_2; d_1, d_2 \in \Delta) \text{ by periodicity } p. \text{ Assume that the number } m \text{ of edges grows linearly with the number } n \text{ of vertices, with } 2m/n \text{ staying in a fixed compact interval of } |\min(\Delta), \max(\Delta)|, \text{ and } p \text{ divides } 2m - n \cdot \min(\Delta). \]

To a given arbitrary set $\Delta \subseteq \{0,1,2,\ldots\}$, we associate the exponential generating function (EGF) $\omega(z)$:

\[
\text{SET}_\Delta(z) = \omega(z) = \sum_{d \in \Delta} \frac{z^d}{d!}. \tag{1}
\]

The domain of the argument $z$ of this function can be either considered a subset $[0,R)$ of the real axis or some subset of the complex plane, depending on the context. The function $\phi_0(z) = \frac{z\omega'(z)}{\omega(z)}$, which is called the characteristic function of $\omega(z)$, is non-decreasing along real axis [FS09, Proposition IV.5], as well as the characteristic function $\phi_1(z) = \frac{z\omega''(z)}{\omega'(z)}$ of the derivative $\omega'(z)$.

The value of the threshold $\alpha$, which is used in all our theorems, is a unique solution of the system of equations

\[
\begin{cases}
\phi_1(\hat{z}) = 1, \\
\phi_0(\hat{z}) = 2\alpha.
\end{cases} \tag{2}
\]

A unique solution $\hat{z}$ of $\phi_1(z) = 1$, $z > 0$ always exists provided that $1 \in \Delta$. This solution is computable.

## 2 Phase Transition for Random Graphs

### 2.1 Structure of Connected Components

Recall that given a set $\Delta$, its EGF is defined as $\omega(z) = \sum_{d \in \Delta} z^d/d!$, and characteristic function of $\omega(z)$ and its derivative $\omega'(z)$ are given by $\phi_0(z) = z\omega'(z)/\omega(z)$, $\phi_1(z) = z\omega''(z)/\omega'(z)$.

**Theorem 1.** Given a set $\Delta$ with $1 \in \Delta$, let $\alpha$ be a unique positive solution of (2). Assume that $m = \alpha n (1 + |\mu|^{-1/3})$. Suppose that Condition (C) is satisfied and $G_{n,m,\Delta}$ is a random graph from $\mathcal{G}_{n,m,\Delta}$.

Then, as $n \to \infty$, we have

1. if $\mu \to -\infty$, $|\mu| = O(n^{1/12})$, then

\[
P(G_{n,m,\Delta} \text{ has only trees and unicycles}) = 1 - \Theta(|\mu|^{-3}) ; \tag{3}
\]

2. if $|\mu| = O(1)$, i.e. $\mu$ is fixed, then

\[
P(G_{n,m,\Delta} \text{ has only trees and unicycles}) \to \text{constant } \in (0,1) , \tag{4}
\]

\[
P(G_{n,m,\Delta} \text{ has a complex part with total excess } \eta) \to \text{constant } \in (0,1), \tag{5}
\]

and the constants are computable functions of $\mu$.
3. if \( \mu \to +\infty, |\mu| = O(n^{1/12}) \), then

\[
\P(G_{n,m,\Delta} \text{ has only trees and unicycles}) = \Theta(e^{-\mu^3/6}\mu^{-3/4}) \quad ,
\]

\[
\P(G_{n,m,\Delta} \text{ has a complex part with excess } q) = \Theta(e^{-\mu^3/6}\mu^{3q/2-3/4}) \quad .
\]

Proof (Sketched). Consider a graph composed of trees, unicycles and a collection of complex connected components. Fix the total excess of complex components \( q \). Then, there are exactly \((n - m + q)\) trees, because each tree has an excess \(-1\).

Generating functions for each of these components are given by Lemma 5 and Lemma 7: we enumerate all possible kernels and then enumerate graphs that reduce to them under pruning and smoothing. Let \( U(z) \) be the generating function for unrooted trees, \( V(z) \) be the generating function for unicycles, \( E_j(z) \) be the generating functions for connected graphs with excess \( j \). We calculate the probability for each collection \((q_1, \ldots, q_k)\), while the total excess is \( \sum_{j=1}^k j q_j = q \). Accordingly, the probability that the process generates a graph with the described property can be expressed as the ratio

\[
\frac{n! \cdot |G_{n,m,\Delta}|^{-1}}{(n - m + q)!} [z^n U(z)^{n-m+q} e^{V(z)} \frac{E_{q_1}(z)}{q_1!} \cdots \frac{E_{q_k}(z)}{q_k!}] .
\]

Then we use an approximation of \( E_j(z) \) from Corollary 8, Lemma 7 and apply Corollary 13 with \( y = \frac{1}{2} + 3q \) in order to extract the coefficients. Note that our approach is derived from the methods from [JKLP93], and so some of our proofs are sketched.

\[ \square \]

2.2 Shifting the Planarity Threshold

Theorem 2. Under the same conditions as in Theorem 1 with a number of edges \( m = \alpha n (1 + \mu n^{-1/3}) \), let \( p(\mu) \) be the probability that \( G_{n,m,\Delta} \) is planar.

Then, as \( n \to \infty \), we have uniformly for \( |\mu| = O(n^{1/12}) \):

1. \( p(\mu) = 1 - \Theta(|\mu|^{-3}) \), as \( \mu \to -\infty \);

2. \( p(\mu) \to \text{constant} \in (0, 1) \), as \( |\mu| = O(1) \), and \( p(\mu) \) is computable;

3. \( p(\mu) \to 0 \), as \( \mu \to +\infty \).

Proof. The graph is planar if and only if all the 3-cores (multigraphs) of connected complex components are planar. As \( |\mu| = O(n^{1/12}) \), Corollary 8 implies that for asymptotic purposes it is enough to consider only cubic regular kernels among all possible planar 3-cores. Let \( G_1(z) \) be an EGF of connected planar cubic kernels. The function \( G_1(z) \) is determined by the system of equations given in [RRN13], and is computable. An EGF for sets of such components is given by \( G(z) = e^{G_1(z)} \). We give several first terms of \( G(z) \) according to [RRN13]:

\[
G(z) = \sum_{q \geq 0} \frac{g_q z^{2q}}{(2q)!^2} = 1 + \frac{5}{24} z^2 + \frac{385}{1152} z^4 + \frac{83933}{82944} z^6 + \frac{35002561}{7962624} z^8 + \ldots
\]

Thus, the number of planar cubic kernels with total excess \( q \) is given by

\[
(2q)! [z^{2q}] e^{G_1(z)} = (2q)! [z^{2q}] G(z) = \frac{g_{2q}}{(2q)!} .
\]

In order to calculate \( p(\mu) \), we sum over all possible \( q \geq 0 \) and multiply the probabilities that the 3-core is a planar cubic graph with excess \( q \) by the conditional probability that a random graph has planar cubic kernel of excess \( q \).

The probability that \( G_{n,m,\Delta} \) is planar on condition that the excess of the complex component is \( q \), is equal to

\[
\frac{n! |G_{n,m,\Delta}|^{-1}}{(n - m + q)!} [z^n U(z)^{n-m+q} e^{V(z)} \frac{g_q}{(2q)!} \frac{(T_z(z))^{2r}}{(1 - T_z(z))^{3r}}] .
\]
We can apply Corollary 13 and sum over all \( q \geq 0 \) in order to obtain the result:

\[
p(\mu) \sim \sqrt{2\pi} \sum_{q \geq 0} g_q t_3^2 A_\Delta(3q + \frac{1}{2}, \mu),
\]

where \( A_\Delta(3q + \frac{1}{2}, \mu) \) and the constant \( t_3 \) are from Corollary 13. The probabilities on the borders of the transition window, i.e. \( |\mu| \to \infty \) can be obtained from the properties of the function \( A_\Delta(y, \mu) \).

2.3 Statistics of the Complex Component Inside the Critical Window

Theorem 3. Under the same conditions as in Theorem 1, suppose that \( |\mu| = O(1), m = \alpha n(1 + \mu n^{-1/3}) \). Then, the longest path, diameter and circumference of the complex part are of order \( \Theta(n^{1/3}) \) in probability, i.e. for each mentioned random parameter there exist computable (see Lemma 16) constants \( A, B > 0 \) depending on \( \Delta \) such that the corresponding random variable \( X_n \) satisfies

\[
P \left( X_n \notin n^{1/3}(A \pm B\lambda) \right) = O(\lambda^{-2}) \quad (12)
\]

Proof. Recall that a 2-path is a path connecting two corner vertices inside a complex component, see Figure 3. In Lemma 16 we prove that the length of a randomly uniformly chosen 2-path is \( \Theta(n^{1/3}) \) in probability. This lemma also gives the explicit expressions for \( A \) and \( B \).

From Lemma 19 we obtain that the maximum height of sprouting tree over the complex part is also \( \Theta(n^{1/3}) \) in probability. Since the total excess of the complex component is bounded in probability as \( \mu \) stays bounded, and the sizes of the kernels are finite, we can combine these two results to obtain the statement of the theorem, because all the three parameters come from adding/stitching several 2-paths and tree heights.

3 Lower Bound for SAT Probability

Consider a random graph \( G_{2n,m,\Delta} \) from \( G_{2n,m,\Delta} \). Instead of labeling the vertices with natural numbers from 1 to \( 2n \), they can be labeled by \( \{1, 2, \ldots, n; \overline{1}, \overline{2}, \ldots, \overline{n}\} \). It is possible to orient its \( m \) edges in \( 2^m \) possible ways to obtain a random directed graph \( D_{n,m,\Delta} \). Then we can create a copy of this graph, replacing literals \( x_i, \overline{x}_i \) by their negations \( \overline{x}_i, x_i \) and changing the direction of each edge. Next, we combine these two digraphs into a single digraph, joining the sets of their edges.

The resulting digraph is a digraph representation of some 2-CNF formula \( F_{n,m,\Delta} \). We say that the digraph \( D_{n,m,\Delta} \) generates formula \( F_{n,m,\Delta} \), see Figure 4.

Each clause \( (x_i \vee \overline{x}_i) \) or \( (\overline{x}_i \vee \overline{x}_i) \) results in a so-called double-edge, like \( (\overline{x}_4 \vee \overline{x}_4) \) in Figure 4. If the graph \( G_{2n,m,\Delta} \) is chosen uniformly at random, then the resulting formula \( F_{n,m,\Delta} \) is not uniform. In the standard 2-SAT model all the clauses consist of strictly distinct literals, which means that in each clause \( x \vee y \) neither \( x = y \) nor \( x = \overline{y} \). However, e.g. in the case of \( \Delta = \mathbb{Z}_{\geq 0} \), it is known that the number of double edges is distributed according to Poisson distribution, therefore 2-SAT with double edges falls
Figure 4: Digraph representation and an example of digraph generating a 2-SAT formula

\((\overline{x_1} \lor x_2)(x_2 \lor \overline{x_3})(x_2 \lor \overline{x_1})(\overline{x_4} \lor x_3)(\overline{x_4} \lor x_2)(\overline{x_4} \lor x_4)\)

into the classical framework with positive probability. In our model, all the CNF-formulae with strictly distinct clauses are equiprobable.

**Theorem 4.** Let \(m = \alpha n(1 + \mu n^{-1/3})\), \(|\mu| = O(n^{1/12})\), assume that \((C)\) is satisfied. A random 2-CNF formula \(F_{n,m,\Delta}\) from the model defined above, having \(n\) literals and \(m\) clauses, and whose literal degrees belong to given set \(\Delta\), is SAT with probability

1. \(\Pr(F_{n,m,\Delta} \text{ is SAT}) \geq 1 - O(|\mu|^{-3})\) as \(\mu \to -\infty\),
2. \(\Pr(F_{n,m,\Delta} \text{ is SAT}) \geq \Theta(1)\) as \(|\mu| = O(1)\),
3. \(\Pr(F_{n,m,\Delta} \text{ is SAT}) \geq \exp(-\Theta(\mu^3))\) as \(\mu \to +\infty\).

The constant inside \(O(|\mu|^{-3})\) is the same as in Theorem 1.

**Proof.** It is well known that a formula is UNSAT if and only if there exists a contradictory circuit in its digraph representation, i.e. there exists literal \(x\) such that \(x \sim \overline{x}\) and \(\overline{x} \sim x\) (see [BBC+01]). The probability of this event can be bounded by \(n\) times the probability that \(1 \sim \overline{1}\) and \(\overline{1} \sim 1\). If there is a contradictory circuit, then there exists at least one digraph \(D_{n,m,\Delta} = D\) which generates \(F_{n,m,\Delta}\) and has a circuit inside, see Figure 5.

Figure 5: Part of a random 2-CNF containing a circuit with literals 1 and \(\overline{1}\)

Suppose that the shortest circuit connecting 1 and \(\overline{1}\) inside \(D\) has length \(\ell\). For each clause which doesn’t form a double edge there exist two possible choices of an edge which represents this clause in digraph \(D'\) which generates \(F_{n,m,\Delta}\). Thus, for each digraph \(D\), there are at most \(2^\ell\) possible digraphs \(D'\) which generate the same CNF as \(D\) and don’t have a circuit connecting 1 and \(\overline{1}\). The probability that formula \(F_{n,m,\Delta}\) is UNSAT, can be decomposed according to the total excess \(q\) of the complex part of \(D\) that generates \(F_{n,m,\Delta}\):

\[
\Pr(F_{n,m,\Delta} \text{ UNSAT}) = \sum_{q \geq 0} \Pr(F_{n,m,\Delta} \text{ is UNSAT} \mid q)\Pr(q) \\
\leq \Pr(F_{n,m,\Delta} \text{ UNSAT} \mid q = 0)\Pr(q = 0) + \sum_{q \geq 1} \Pr(q) \\
\leq n\Pr(q = 0) \sum_{\ell \geq 3} 2^\ell \Pr(D \text{ generates } F_{n,m,\Delta} \text{ and } 1,\overline{1} \in \text{ circuit of length } \ell \mid q = 0) + (1 - \Pr(q = 0)) .
\]
The sum over $\ell$ is an expectation of the random variable $E(2^L)$. $L$ is defined in subsection B.3, and can be evaluated by marking and saddle-point methods in Lemma 21. The first summand has asymptotics $\Theta(\mu^{-2}n^{-1/3})$. According to Theorem 1, the second summand is $O(|\mu|^{-3})$ which dominates the first summand.

4 Simulations

We considered random graphs with $n = 1000$ vertices, and various degree constraints. The random generation procedure of such graphs has been explained by de Panafieu and Ramos in [dPR16] and for our experiments, we implemented the recursive method. We note that this kind of sampling is not exact in the sense that the probability of obtaining a simple graph is uniform only in asymptotics.

The generator first draws a sequence of degrees and then performs a random pairing on half-edges, as in configuration model [Bol80]. We reject the pairing until the multigraph is simple, i.e. until there are no loops and multiple edges. As $|\mu| = O(1)$, expected number of rejections is asymptotically $\exp\left(-\frac{1}{2}\phi_1(\hat{z}) - \frac{1}{2}\phi_2(\hat{z})\right)$, which is $\exp(-3/4)$ in the critical window, and in the subcritical phase it is less.

Each sequence $(d_1, \ldots, d_n)$ is drawn with weight $\prod_{v=1}^n \frac{1}{d_v d_v!}$. First, we use dynamic programming to precompute the sums of the weights $(S_{i,j})$: $i \in [0, n]$, $j \in [0, 2m]$ using initial conditions and the recursive expression:

$$S_{i,j} = \sum_{d_1 + \ldots + d_i = j \in \Delta} \frac{1}{d_1 d_1!} \prod_{v=1}^i \frac{1}{d_v d_v!}, \quad S_{i,j} = \begin{cases} 1, & (i,j) = (0,0), \\ 0, & i = 0 \text{ or } j < 0, \\ \sum_{d \in \Delta} \frac{S_{i-1,j-d} d}{d!}, & \text{otherwise} \end{cases}$$

Then the sequence of degrees is generated according to the distribution

$$P(d_n = d) = \frac{S_{n-1,2m-d}}{d! S_{n,2m}}.$$

We made plots for distributions of different parameters for $\Delta = \{1, 3, 5, 7\}$, see Figure 6.

![Figure 6: Results of experiments](image)

(a) Largest excess  
(b) Largest component size  
(c) Graph diameter

Conclusion. We studied how to shift the phase transition of random graphs when the degrees of the nodes are constrained by means of analytic combinatorics [dPR16, FS09].

We have shown that the planarity threshold of those constrained graphs can be shifted generalizing the results in [RRN13]. We have also shown that when our random constrained graphs are inside their critical window of transition, the size of complex components are typically of order $n^{2/3}$ and all distances inside the complex components are of order $n^{1/3}$, thus our results about these parameters complement those of Nachmias and Peres [NP08].

In addition, we studied random 2-CNF formulae with constrained literal degrees. Our results show that if the literal degrees belong to some set $\Delta$ then with high probability a random formula with $m > n$ clauses are still SAT. In this direction, our approaches are different from the existing ones [BBC+01, CFS07] and give new insights to random Constraint Satisfaction Problems.
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References

[BBC+01] Béla Bollobás, Christian Borgs, Jennifer Chayes, Jeong Han Kim, and David Wilson. The scaling window of 2-SAT transition. *RS&AA*, 18:201 – 256, 2001.

[BF01] Tom Bohman and Alan Freize. Avoiding a giant component. *Random Structures and Algorithms*, 19(1):75–85, 2001.

[BLL98] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial Species and Tree-like Structures*. Cambridge University Press, 1998.

[Bol80] Béla Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European Journal of Combinatorics*, 1:311–316, 1980.

[Bol85] Béla Bollobás. *Random graphs*. Academic Press, Inc., London, 1985.

[CFS07] Colin Cooper, Alan Frieze, and Gregory B. Sorkin. Random 2-SAT with prescribed literal degrees. *Algorithmica*, 48:249 — 265, 2007.

[CGHS04] Don Coppersmith, David Gamarnik, Mohammad Hajiaghayi, and Gregory B. Sorkin. Random MAX SAT, random MAX CUT, and their phase transitions. *Random Structures and Algorithms*, 24(4):502–545, 2004.

[DDHM13] Varsha Dani, Josep Diaz, Thomas Hayes, and Cristopher Moore. The power of choice for random satisfiability. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 484–496. 2013.

[DMRR12] Hervé Daudé, Conrado Martínez, Vonjy Rasendrahasina, and Vlady Ravelomanana. The MAX-CUT of sparse random graphs. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pages 265–271, 2012.

[dPR16] Élie de Panafieu and Lander Ramos. Enumeration of graphs with degree constraints. *Proceedings of the Meeting on Analytic Algorithmics and Combinatorics*, 2016.

[ER60] Paul Erdős and Alfred Rényi. On the evolution of random graphs. *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei*, 5:17–61, 1960.

[FO82] Philippe Flajolet and Andrew M. Odlyzko. The average height of binary trees and other simple trees. *Journal of Computer and System Sciences*, 25:171 – 213, 1982.

[FPK89] Philippe Flajolet, Boris Pittel, and Donald E. Knuth. The first cycles in an evolving graph. *Discrete Mathematics*, 75:167 – 215, 1989.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge Press, 2009.

[GKK16] Lior Gishboliner, Michael Krivelevich, and Gal Kronenberg. On MAXCUT in strictly supercritical random graphs, and coloring of random graphs and random tournaments. *arXiv preprint arXiv:1603.04044*, 2016.

[HM12] Hamed Hatami and Michael Molloy. The scaling window for a random graph with a given degree sequence. *Random Structures and Algorithms*, 41(1):99–123, 2012.

[JKLP93] Svante Janson, Donald E. Knuth, Tomasz Łuczak, and Boris Pittel. The birth of the giant component. *Random Structures and Algorithms*, 4(3):231–358, 1993.

[JPRR16] Felix Joos, Guillem Perarnau, Dieter Rautenbach, and Bruce Reed. How to determine if a random graph with a fixed degree sequence has a giant component. pages 695–703, 2016.
A Saddle-point Analysis

A.1 Symbolic Tools

For each \( r \geq 0 \), let us define \( r \)-sprouted trees: rooted trees whose vertex degrees belong to the set \( \Delta \), except the root (see Figure 7), whose degree belongs to the set \( \Delta - \ell = \{ \delta \geq 0: \delta + \ell \in \Delta \} \). Their efg \( T_\ell(z) \) can be defined recursively

\[
T_\ell(z) = z \omega^{(\ell)}(T_1(z)), \quad T_1(z) = z \omega'(T_1(z)) \quad , \quad \ell \geq 0.
\]

**Lemma 5.** Let \( U(z) \) be the efg for unrooted trees and \( V(z) \) the efg of unicycles whose vertices have degrees \( \in \Delta \). Then

\[
U(z) = T_0(z) - \frac{T_1(z)^2}{2} \quad , \quad V(z) = \frac{1}{2} \left[ \log \frac{1}{1 - T_2(z)} - T_2(z) - \frac{T_2^2(z)}{2} \right] ,
\]

where \( T_0(z), T_1(z), \) and \( T_2(z) \) are by (16).

**Remark 6.** The above statement for \( U(z) \) can be proven using the dissymmetry theorem for trees, adapted for the case with degree constraints (see [BLL98, Section 4.1], [FPK89]). In short, we can consider rooted trees \( T_0 \) and mark the vertex with label 1. Then we consider two cases, when this vertex is the root one, or not. The first case corresponds to \( U(z) \) and in the second situation we can consider two subcases, whether the label 1 belongs to the subtree induced by the first child or not (see Figure 8). Summarizing the argument, we obtain

\[
T_0(z) = U(z) + \frac{1}{2} T_1(z)^2.
\]
The expression for $V(z)$ is an application of the symbolic method of EGFs in the case of undirected cycles \((\text{CYC}_{\geq 3})\) of 2-sprouted trees, see Figure 9.

Any multigraph $M$ on $n$ labeled vertices can be defined by a symmetric $n \times n$ matrix of nonnegative integers $m_{xy}$, where $m_{xy} = m_{yx}$ is the number of edges $x - y$ in $M$. The compensation factor $\kappa(M)$ is defined by

$$\kappa(M) = 1 \left| \frac{1}{n} \prod_{x=1}^{n} \left( 2^{m_{xx}} \prod_{y=x}^{n} m_{xy}! \right) \right| .$$ (18)

A multigraph process is a sequence of $2m$ independent random vertices

$$\langle v_1, v_2, \ldots, v_{2m} \rangle , \quad v_k \in \{1, 2, \ldots, n\} ,$$

and output multigraph with the set of vertices $\{1, 2, \ldots, n\}$ and the set of edges $\{\{v_{2i-1}, v_{2i}\} : 1 \leq i \leq m\}$. The number of sequences that lead to the same multigraph $M$ is exactly $2^m m! \kappa(M)$.

**Lemma 7.** Let $\overline{M}$ be some 3-core multigraph with a vertex set $V$, $|V| = n$, having $\mu$ edges, and compensation factor $\kappa(\overline{M})$. Let $\mu_{xy}$ be the number of edges between vertices $x$ and $y$ for $1 \leq x \leq y \leq n$. 

Figure 7: Recursive construction of $T_0(z)$: the degree of the root of each subtree should belong to the set $\Delta - 1$

Figure 8: Variant of dissymetry theorem for unrooted trees with degree constraints

Figure 9: Unicycles with degree constraints
The generating function for all graphs $G$ that lead to $\overline{M}$ under reduction is
\[ \kappa(\overline{M}) \prod_{v \in \overline{V}} T_{\deg(v)}(z) \cdot \frac{P(\overline{M}, T_2(z))}{n! (1 - T_2(z))^n}. \] (19)

\[ P(\overline{M}, z) = \prod_{x=1}^{n} \left( z^{2\mu_{xx}} \prod_{y=x+1}^{n} z^{\mu_{xy} - 1}(\mu_{xy} - (\mu_{xy} - 1)z) \right). \] (20)

**Corollary 8.** Assume that $\phi_1(\hat{z}) = 1$. Near the singularity $z \sim \hat{z}$, i.e. $T_2(z) \approx 1$, some of the summands from Lemma 7 are negligible. Dominant summands correspond to graphs $\overline{M}$ with maximal number of edges, i.e. graphs with $3r$ edges and $2r$ vertices. The vertices of degree greater than 3 can be split into more vertices with additional edges. Due to [JKLP93, Section 7, Eq. (7.2)], the sum of the compensation factors is expressed as
\[ e_{r0} = \frac{(6r)!}{2^{2r}3^{2r}(3r)!(2r)!}. \] (21)

and the sum of major summands is asymptotically
\[ e_{r0} \frac{T_3(z)^{2r}}{(1 - T_2(z))^{3r}}. \] (22)

The proofs of the two previous statements are postponed until the end of Remark 9.

**Remark 9.** Let’s give an example of application of Lemma 7, first in less technical multigraph form, then for simple graphs.

Each multi-edge in the 3-core $\overline{M}$ corresponds to a sequence of trees in the initial graph $M$. Therefore, the generating function for multigraphs $M$ which reduce to one of the three depicted (see Figure 10) 3-core multigraphs consists of 3 summands:
\[ W_\Delta(z) = \frac{1}{4} \frac{T_4(z)}{(1 - T_2(z))^2} + \frac{1}{4} \frac{T_3(z)^2}{(1 - T_2(z))^3} + \frac{1}{6} \frac{T_3(z)^2}{(1 - T_2(z))^3}. \] (23)

Figure 10: All possible 3-core multigraphs of excess 1 and their compensation factors. The first one has negligible contribution because it is non-cubic

We write $T_2(z)$ because if we attach a tree on any path, the degree of the root decreases by 2. For the same reason there appear $T_3(z)$ and $T_4(z)$. If we evaluate $W_\Delta(z)$ near the pole $z = \hat{z}$, or equivalently at $T_2(z) = 1$, the first summand goes to $\infty$ slower than the second and the third. This yields asymptotic approximation
\[ W_\Delta(z) = \frac{5}{24} \frac{T_3(z)^2}{(1 - T_2(z))^3} + O\left(\frac{1}{(1 - T_2(z))^2}\right). \] (24)

The big-O notation with the generating functions means:
\[ F(z) = O(B(z)) \text{ if } [z^n]F(z) \leq c[z^n]B(z) \] (25)

for sufficiently large $n$, so from Eq. (24) we know that
\[ [z^n]W_\Delta(z) \sim [z^n] \frac{5}{24} \frac{T_3(z)^2}{(1 - T_2(z))^3}. \] (26)
With simple graphs (not multigraphs) the situation is similar. For the first core we want the path to be non-empty (because simple graphs don’t contain loops), so the generating function is \( \frac{1}{4} \left( \frac{T_1(z)T_2(z)}{1 - T_2(z)} \right)^4 \).

For the second graph we also require that both paths obtained from loops, contain at least one node inside: \( \frac{1}{4} \left( \frac{T_3(z)^2T_2(z)}{1 - T_2(z)} \right)^4 \). Then, for the third core, we need at least two of the paths contain at least one node. Collecting all the summands we obtain

\[
\tilde{W}_\Delta(z) = \frac{1}{4} \left( \frac{T_4(z)T_2(z)}{1 - T_2(z)} \right)^4 + \frac{1}{4} \left( \frac{T_3(z)^2T_2(z)}{1 - T_2(z)} \right)^4 + \frac{1}{6} \left( \frac{T_3(z)^2(3T_2(z)^2 - 2T_2(z))}{1 - T_2(z)} \right)^4.
\]

(27)

At \( z \) near \( \tilde{z} \), \( T_2(z) = 1 \), so the asymptotics of this term is again

\[
\tilde{W}_\Delta(z) = \frac{5}{24} \left( \frac{T_4(z)^2}{(1 - T_2(z))^4} \right) + O \left( \frac{1}{(1 - T_2(z))^2} \right).
\]

(28)

In the similar manner as was done in [JKLP93, Lemma 2, Eq. (9.21)], we can prove that the dominant summand in the case of simple graphs and multigraphs is the same and equals the total compensation factor of cubic kernels \( e_{r0} \) times the generating function \( \frac{T_3(z)^{\#\text{nodes}}}{(1 - T_2(z))^{\#\text{edges}}} \). We omit the factor \( T_2(z)^{\#\text{edges}} \) because it is equal to 1 (as \( z = \tilde{z} \)).

**Proof of Lemma 7.** The proof is similar to [JKLP93, Lemma 2]. We need to count the junctions of different degrees. All the paths contain vertices of degree at least 2, so we plug \( T_2(z) \) into \( P(M, z) \).

**Proof of Corollary 8.** (From [Bol85, Chapter 2]) In a cubic multigraph each vertex has 3 half-edges that need to be paired, there are 6r half-edges in total. The number of such pairings is \( (6r)! / ((3r)!2^{2r}) \). In each vertex the three half-edges can be permuted in 3! = 6 ways, so we divide by \( 6^{2r} \) to obtain finally that

\[
\text{the number of cubic multigraphs} = \frac{(6r)!}{(3r)!2^{2r}6^{2r}}.
\]

(29)

The multiple \( (2r)! \) appears because the graph has \( 2r \) vertices and we deal with exponential generating functions.

**A.2 Analytic Tools**

**Remark 10.** The crucial tool that we use in our work is the analytic lemma, Lemma 11, or equivalently, Corollary 13. Since the statements are quite cubersome, we propose an alternative way to understand the statements, by dividing the quantities involved into this theorem.

Suppose that \( |\mu| = O(1) \), and \( n \to \infty \). We treat \( A_\Delta(y, \mu) \) as a nearly constant number, while for the asymptotics the important factor is \( n^{y/3-1/2} \) with \( y = 3r + 1/2 \). The left-hand side of Eq. (33) expresses the probability of graph having complex component of excess \( r \), provided that we specify the function \( \Psi \) correctly according to our combinatorial specification.

When we increase the excess \( r \) by 1, the exponential index of \( n^{y/3-1/2} \) increases by 1, and the expression is multiplied by \( n \). This is exactly the combinatorial interpretation that we are looking for: the generating functions of graphs which reduce to kernels that are non-cubic, have a negligible contribution into the total probability.

In order to count the number of graphs with complex component of excess \( r \), we note that the total number of trees should compensate the total excess to \( m - n \), so the number of trees is \( n - m + r \). When we substitute this into Corollary 13, additional multiple \( n \) caused by extra excess, cancels with \( (n - m + r)! \) in the denominator. This explains why for any fixed collection of excesses of complex components \( q_1, q_2, \ldots, q_k \) the probability of having a graph with such an excess, is asymptotically a constant. A rigorous calculation of this probability involves substituting the asymptotics of \( |G_{n, m, \Delta}| \) obtained in [dPR16].
Lemma 11. Let \( m = r n = \alpha n (1 + \mu) \), where \( \nu = n^{-1/3} \), \( |\mu| = O(n^{1/12}) \), \( n \to \infty \), and \( \tilde{z} \) be a unique real positive solution of \( \phi_1(\tilde{z}) = 1 \). Let

\[
C_2 = \frac{t_3 \alpha \tilde{z}}{2(1 - \alpha)}, \quad C_3 = \frac{2 t_3 \alpha \tilde{z}}{3}, \quad t_3 = \frac{\tilde{z} \omega'''(\tilde{z})}{\omega'(\tilde{z})}.
\]

Then for any function \( \tau(z) \) analytic in \( |z| \leq \tilde{z} \) the contour integral encircling complex zero, admits asymptotic representation

\[
\frac{1}{2\pi i} \oint (1 - \phi_1(z))^{1-y} e^{nh(z; r)} \tau(z) \frac{dz}{z} \sim \nu^{2-y} (z t_3)^{1-y} \tau(z) e^{nh(z; \alpha)} \times B\Delta(y, \mu) \Big|_{z=\tilde{z}},
\]

Corollary 13. If \( \alpha n \) \( \tau \) is attained at point \( \alpha n \) the maximum is attained at point \( \alpha n \). In order to compute the probability in Corollary 13, we express the coefficient of a generating function as a contour integral. The methods for computing integrals of such kind are well-developed, for example, in [PW13]. In case of single root \( z_0 \) of the derivative \( h_z(z; r) \) we approximate the integral with Gaussian density:

\[
\frac{1}{2\pi i} \oint g(z)e^{nh(z; r)} \frac{dt}{t} \sim \left( \frac{1}{2\pi n} \right)^{1/2} \frac{g(z_0)}{z_0 \sqrt{h'(z_0)}} e^{nh(z_0)}
\]

and in case of double root we approximate the integral of exponential of \( (z - z_0)^3 \).

Therefore,

\[
\frac{1}{2\pi i} \oint g(z)e^{nh(z; r)} \frac{dt}{t} \sim \frac{1}{2\pi i} \oint g(z) \exp \left( nh(z_0; r) + nh''(z_0; r) \frac{(z - z_0)^3}{6} \right) \frac{dt}{t}.
\]

Though these techniques are quite standard in a certain community, this machinery cannot be directly applied to the case of degree constraints because we need to prove that on the circle \( z = z_0 e^{i\theta} \), \( \theta \in [0, 2\pi] \) the maximum is attained at point \( \theta = 0 \), otherwise this method is not applicable.

Remark 12. In order to compute the probability in Corollary 13, we express the coefficient of a generating function as a contour integral. The methods for computing integrals of such kind are well-developed, for example, in [PW13]. In case of single root \( z_0 \) of the derivative \( h_z(z; r) \) we approximate the integral with Gaussian density:

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\frac{1}{2\pi i} \oint g(z)e^{nh(z; r)} \frac{dt}{t} \sim \left( \frac{1}{2\pi n} \right)^{1/2} \frac{g(z_0)}{z_0 \sqrt{h'(z_0)}} e^{nh(z_0)}
\]

and in case of double root we approximate the integral of exponential of \( (z - z_0)^3 \).

Therefore,

\[
\frac{1}{2\pi i} \oint g(z)e^{nh(z; r)} \frac{dt}{t} \sim \frac{1}{2\pi i} \oint g(z) \exp \left( nh(z_0; r) + nh''(z_0; r) \frac{(z - z_0)^3}{6} \right) \frac{dt}{t}.
\]

The proof of the current lemma will be given below.

Corollary 13. If \( m = \alpha n (1 + \mu n^{-1/3}) \) and \( y \in \mathbb{R}, y \geq \frac{1}{2} \), then for any \( \Psi(t) \) analytic at \( t = 1 \) we have

\[
\frac{n!}{(n - m)!} [z^n] U(z)^{n-m} \Psi(T_2(z)) = \sqrt{2\pi} \Psi(1) A\Delta(y, \mu) n^{y/3-1/6} + O(R),
\]

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\[
\frac{n!}{(n - m)!} [z^n] U(z)^{n-m} \Psi(T_2(z)) = \sqrt{2\pi} \Psi(1) A\Delta(y, \mu) n^{y/3-1/6} + O(R),
\]
Proof of Lemma 11 and Corollary 13. Let us prove the corollary first. We start with “Stirling” approximation part. In case of classical random graphs it would be enough to apply the Stirling approximation, but in the case of degree constraints we apply the asymptotic result of [dPR16]:

\[
\frac{n!}{(n - m)! |F_{n,m,\Delta}|} = \sqrt{\frac{2\pi n}{p}} \frac{\sqrt{2\pi}}{\lambda} \exp(n \log n + (n - m) \log(n - m) - m \log 2m) \times \exp\left(\frac{1}{2} \phi_1(z_0) + \frac{1}{4} \phi_1^2(z_0) + (1 + O(n^{-1}))\right).
\]

It happens that the exponential part of Stirling and some terms that will appear in Cauchy approximation, correct error estimate,

\[
z \rightarrow \text{The statement readily follows from Lemma 11.}
\]

Let us move to the Cauchy part for obtaining formal series coefficients. After “Lagrangian” variable change \(T_1(z) \rightarrow z\) we obtain:

\[
[z^n] U(z)^{n-m} \Psi(T_2(z)) = \frac{1}{2\pi i} \oint \Psi(T_2(z)) U(z)^{n-m} \frac{dz}{(1 - T_2(z))^y} = \frac{2^{m-n}}{2\pi i} \oint \Psi(\phi_1(z)) (1 - \phi_1(z))^{1-y} e^{n h(z;r)} \frac{dz}{z}.
\]

The statement readily follows from Lemma 11.

Let us prove the lemma then. We start with specifying an integration contour, namely the circle \(z = \hat{z} e^{-\nu r}\) where \(s = \beta + it\), \(\beta > 0\), \(t \in [-\pi n^{1/3}, \pi n^{1/3}]\). We need \(\beta \rightarrow 0\) with \(n \rightarrow \infty\). Technically, for correct error estimate, \(\beta\) can be chosen from

\[
\mu = \beta^{-1} - \beta,
\]

as suggested by [JKLP93]. We need to switch to contour \(t \in (-\infty, +\infty)\) with the price of exponentially small error \(O(e^{-\max(2, |\mu|) n^{1/3}})\), we omit the details of this approximation since they are already considered in the mentioned article.

Next, there will be two variable changes. The first change of variables is \(z = \hat{z} e^{-\nu r}\). We use an approximation for \(nh(z;r)\) near the double saddle \(\hat{z}\) and critical ratio \(\alpha\). From Lemma 15 it follows that maximum value of \([e^{nh(z;r)}]\) for \(t \in [-\pi n^{1/3}, \pi n^{1/3}]\) is attained for \(t = 0\) (and also at the points \(z = \hat{z} e^{-\nu r}\) where \(d\) is a period of \(\Delta\), but we can assume without loss of generality that \(d = 1\), because otherwise, extra terms cancel out when we count the probability, since the denominator is given by expression from [dPR16]). Thus, we can choose a small \(t_0 > 0\) such that \(nh''(z)(\nu t_0)^3 \rightarrow -\infty, nh''(\hat{z})(\nu t_0)^3 \rightarrow 0\), and the absolute value of integral for \(|t| > t_0\) is negligible.

Since there is a relation \(r = \alpha(1 + \mu v)\), we can use a Taylor expansion for \(h(z, r)\) for \(z\) around \(\hat{z}\), which is uniform with respect to \((\alpha, r)\):

\[
h(z; r) = \sum_{k=0}^3 \frac{h_k^{(k)}(\hat{z}; r)(z - \hat{z})^k}{k!} + O((\nu^4)^4).\]

The first derivative turns to zero, the second and the third can be written as

\[
h''(\hat{z}, r) = \frac{(\phi_0(\hat{z}) - 2r)\phi_1'(\hat{z})}{\hat{z}(\phi_0(\hat{z}) - 2)} = \frac{t_3(\alpha - r)}{\hat{z}(\alpha - 1)},
\]

\[
h''(\hat{z}, r) = \frac{\phi_0'(\hat{z})\phi_1'(\hat{z})}{\hat{z}(\alpha - 1)} + O(\mu) \sim -\frac{4t_3(\alpha)}{\hat{z}^2},\]

hence the final approximation takes the form

\[
nh(z; r) = nh(\hat{z}; \alpha) + C_2 \mu s\hat{z} + C_3 s^2 + O((\mu^2 \hat{z}^2 + s^4)\nu),
\]

where \(C_2 = h''(\hat{z}; \alpha)\hat{z}^2/2\) and \(C_3 = -h'''(\hat{z}; \alpha)\hat{z}^3/6\) are given in the formulation. We also have

\[(1 - \phi_1(z))^{1-y} = s^{1-y} \nu^{-y}(1 + O(\nu))\]
Lemma 14. Assume that \( m = \tau n \). The coefficient at \( z^n \) of an EGF for graphs from \( F_{n, m, \Delta} \) given by an equation

\[
[z^n] U(z)^{n-m+q} \frac{e^{V(z)}}{(n - m + q)!} = e^{U(z)} E_q(z),
\]

can be expressed as

\[
\frac{2^{m-n}}{2\pi i(n - m + q)!} \oint e^{h(z;r)} g(z) \tau(z) \frac{dz}{z},
\]

where the contour contains 0, the functions \( h(z;r), g(z) \) are given by

\[
h(z;r) = r \log \omega'(z) - r \log z + (1 - r) \log(2\omega - z\omega'),
\]

\[
g(z) = (1 - \phi_1(z))^{1-y}, \quad y = 3q + \frac{1}{2}
\]

and \( \tau(z) \) doesn’t have singularities in \( |z| \leq \hat{z} \).

Proof. We can do a variable change \( T_1(z) = t \mapsto z \).

From the equation \( T_1(z) = z\omega'(T_1(z)) \) we obtain:

\[
z = t\omega'(t)^{-1} \mapsto z\omega'(z)^{-1},
\]

\[
dz = (\omega')^{-1}(1 - \phi_1)dt,
\]

\[
T_\ell = z\omega'(t) = t\omega(t)\omega'(t)^{-1},
\]

\[
T_\delta(z) \mapsto \phi_1(z),
\]

\[
U(z)^{n-m} \mapsto 2^{m-n}z^{n-m}(2\omega(\omega')^{-1} - z)^{n-m}
\]
Then, we separate out the singular part:

\[
\frac{U(z)^q e^{V(z)}}{z} \frac{dz}{dt} = (1 - T_2(z)) (\omega')^{-1} \omega'(t)^{-1} \frac{dz}{dt} z^{-1}
\]

\[
\times U(z)^q \sqrt{1 - T_2(z)} e^{V(z)} (1 - T_2(z)) \frac{dz}{dt} \tau(z) \phi_0(z)
\]

\[
\times \frac{1}{(1 - T_2(z))^q} \cdot \frac{1}{\sqrt{1 - T_2(z)}} d\tau(z) d\zeta(z)
\]

and the exponential one:

\[
U(z)^{n-m} z^{-m} \to 2^{m-n} \left( \frac{2 \omega'}{\omega} - z \right)^{n-m} z^{m-n} \left( \frac{\omega'}{z} \right)^n = 2^{m-n} e^{n \phi(z)}.
\]

In this section we mainly establish some asymptotic properties of \( h(z; r) \) around \( z = \tilde{z} \) and \( r = r_0 = \alpha \). Its behaviour is important for saddle-point techniques. At arbitrary point \( r = \alpha(1 + \mu n^{-1/3}) \) its derivative factors as

\[
h'_z(z; r) = \frac{\phi_0(z) - 2r}{z(\phi_0(z) - 2)}.
\]

and the dominant complex root of \( h'_z(z; r) \) (closest to zero) is a positive real number which is either the solution of \( \phi_0(z) = 2r \) or the solution of \( \phi_1(z) = 1 \). Each of the equations has unique real positive solution which we denote by \( \text{Root}_1(r) \) and \( \text{Root}_2 = \tilde{z} \), see Figure 11.

![Figure 11: Configuration of roots of \( h_z(z; r) \)](image_url)

**Lemma 15.** Let \( z_0 > 0 \), \( z_0 \leq \min(\text{Root}_1(r), \text{Root}_2) \), the periodicity of \( \Delta \) is \( p \). Then the function

\[
\Phi(\theta; r) = \Re h(z_0 e^{i\theta}; r)
\]

attains its global maximums for \( \theta \in [0, 2\pi) \) at \( p \) points \( \theta_k = \frac{2\pi k}{p} \), \( k = 0, 1, \ldots, p - 1 \).

**Proof.** Denote \( z_0 e^{i\theta} \) by \( z \). Without loss of generality we will treat the case of aperiodic \( \omega(z) \), since any \( p \)-periodic function \( \pi(z) \) can be reduced to an aperiodic one \( \varphi(z) \) by a variable change \( \pi(z) = z^p \varphi(z^p) \). \( \Phi(\theta; r) \) can be rewritten as

\[
\Phi(\theta; r) = r \log |\omega'(z)| + (1 - r) \log |2\omega - z\omega'| + C.
\]

We apply a version of Gibbs inequality for Kullback-Leibler divergence, also known as cross-entropy inequality: if \( p_1, p_2, q_1, q_2 \) are positive real numbers and \( q_1 + q_2 \leq p_1 + p_2 \) then

\[
p_1 \log \frac{p_1}{q_1} + p_2 \log \frac{p_2}{q_2} \geq 0.
\]
It is now sufficient to prove that
\[
 r \cdot \left| \frac{\omega'(z)}{\omega'(z_0)} \right| + (1-r) \cdot \left| \frac{2\omega(z) - zw'(z)}{2\omega(z_0) - z_0w'(z_0)} \right| \leq 1 .
\] (59)

Note that \( \phi_0(z_0) \leq 2r \). We first prove the inequality for \( r = \frac{1}{2} \phi_0(z_0) \). Since the function \( \omega'(z) \) has non-negative coefficients, we always have \( |\omega'(z)/\omega'(z_0)| \leq 1 \), therefore if \( r \) increases, the inequality still remains true, thus for all \( r \geq \tilde{r} \) it is also true.

Substituting \( \phi_0(z_0) = z_0\omega'(z_0)\omega(z_0)^{-1} = 2r \) we arrive to more simple inequality
\[
 |zw'(z)| + |2\omega(z) - zw'(z)| \leq 2\omega(z_0) , \quad z_0 \leq \alpha .
\] (60)

This inequality was proven by Fedor Petrov at mathoverflow [Pet16] using a beautiful geometric statement.

Let \( \gamma > \beta > 0 \) and \( 1/\beta - 1/\gamma \geq 2 \), then for any vector \( z \) with \( |z| = 1 \)
\[
|1 + \gamma z| + |1 - \beta z| \leq 2 + \gamma - \beta .
\] (61)

Let us denote \( z = e^{it} \). Differentiating the expression by \( \theta \) and finding the zeros, we obtain
\[
\frac{-2\gamma \sin \theta}{|1 + \gamma z|} + \frac{2\beta \sin \theta}{|1 + \beta z|} = 0 ,
\] (62)
which is equivalent to
\[
|z + 1/\gamma| = |z - 1/\beta| ,
\] (63)
but the middle point of the segment \([-1/\gamma, 1/\beta]\) has value greater than or equal to 1 provided that \( 1/\beta - 1/\gamma \geq 2 \), so the perpendicular bisector to this segment doesn’t contain non-real points. The geometric statement in now proven.

Let \( \omega(z) = \sum_{k=0} c_k z^k \). Since \( \phi_1(\tilde{z}) = 1 \) and \( 0 < |\tilde{z}| \leq \tilde{z} \), the inequality \( \phi_1(z) \leq 1 \) can be expanded as
\[
c_1 \geq \sum_{k \geq 2} (k^2 - 1)c_{k+1}|z|^k ,
\] (64)
and we need to prove (60), which is equivalent to
\[
\left| \sum_{k \geq 1} kc_k z^k \right| + |2c_0 + c_1z - c_3z^3 - 2c_4z^4 - \ldots | \leq 2c_0 + 2c_1|z| + 2c_2|z|^2 + \ldots
\] (65)

This is reduced by applying triangle inequality for removing terms with \( c_0 \) and \( c_2 \) and dividing by \( |z| \):
\[
|c_1 + 3c_3z^3 + \ldots | + |c_1 - c_3z^2 - 2c_4z^3 - \ldots | \leq 2c_1 + 2c_3|z|^2 + \ldots
\] (66)
Repetedly using triangle inequalities, the above can be reduced to a family of inequalities
\[
|(k^2 - 1)|z|^k + (k + 1)|z|^k + |(k^2 - 1)|z|^k - (k - 1)|z|^k | \leq (2(k^2 - 1) + 2)|z|^k ,
\] (67)
which is a partial case of the geometric statement with \( \gamma = \frac{1}{k-1} \), \( \beta = \frac{1}{k+1} \).

\[\square\]

**B Method of Moments**

In order to study the parameters of random structures, we apply the marking procedure introduced in [FS09]. We say that the variable \( u \) marks the parameter of random structure in bivariate EGF \( F(z, u) \) if \( n! [z^n u^k] F(z, u) \) is equal to number of structures of size \( n \) and parameter equal to \( k \). In this section we consider such parameters of a random graph as the length of 2-path, which corresponds to some edge of
the 3-core, and the height of random “sprouting” tree. If we treat the parameter as a random variable $X_n$ then the factorial moments can be calculated through an expression

$$
\mathbb{E} X_n (X_n - 1) \ldots (X_n - k + 1) = \left. \frac{d^k}{du^k} F(z, u) \right|_{u=1}.
$$

(68)

Recall that the number of graphs having $n$ vertices, $m$ edges, and fixed excess vector $q = (q_1, q_2, \ldots)$, can be expressed as $n$-th coefficient of the generating function

$$
\frac{U(z)^{n-m+q}}{(n-m+q)!} e^{V(z)} E_q(z),
$$

(69)

where $E_q(z) = \prod_{j=1}^{k} \frac{(E_j(z))^{q_j}}{q_j!}$, $q = \sum_{j=1}^{k} q_j$. This egf can be modified to count the moments of random variable $X_n$.

### B.1 Length of a Random 2-path

Let us fix the excess vector $q = (q_1, q_2, \ldots, q_k)$. There are in total $q = q_1 + 2q_2 + \ldots + kq_k$ connected complex components and each component has one of the finite possible number of 3-cores (see [JKLP93]). We can choose any 2-path, which is a sequence of trees, and replace it with a sequence of marked trees, see Figure 12. Let random variable $P_n$ be the length of this 2-path. Since an egf for sequence of trees is

$$
\mathbb{E}[u^{P_n}] = \frac{n! [z^n] U(z)^{n-m+q}}{(n-m+q)!} e^{V(z)} E_q(z) \frac{1 - T_2(z)}{1 - u T_2(z)},
$$

(70)

Figure 12: Marked 2-path inside complex component of some graph

#### Lemma 16

Suppose that conditions of Theorem 1 are satisfied. Suppose that there are $q_j$ connected components of excess $j$ for each $j$ from 1 to $k$. Denote by excess vector a vector $q = (q_1, q_2, \ldots, q_k)$. Inside the critical window $m = \alpha n (1 + \mu n^{-1/3})$, $|\mu| = O(1)$, the length $P_n$ of a random (uniformly chosen) 2-path is $\Theta(n^{1/3})$ in probability, i.e.

$$
P \left( P_n \notin n^{1/3} t_3 (B_1 \pm \lambda B_2) \right) \leq \frac{1}{(\lambda + o(1))^2},
$$

(71)

$$
t_3 = \frac{\omega''(z)}{\omega'(z)}, \quad B_1 = \frac{B_{\Delta}(3q + \frac{3}{2}, \mu)}{B_{\Delta}(3q + \frac{1}{2}, \mu)},
$$

$$
B_2^2 = \frac{B_{\Delta}(3q + \frac{5}{2}, \mu) B_{\Delta}(3q + \frac{1}{2}, \mu) - B_{\Delta}^2(3q + \frac{3}{2}, \mu)}{B_{\Delta}^2(3q + \frac{1}{2}, \mu)},
$$

with function $B_{\Delta}(y, \mu)$ from Lemma 11, $q = q_1 + 2q_2 + \ldots + kq_k$.

**Proof.** The statement of the lemma is just an application of Chebyshev’s inequality to the first and the second moment. Essentially, we need to prove that

$$
\mathbb{E} P_n \sim n^{1/3} t_3 \frac{B_{\Delta}(3q + \frac{3}{2}, \mu)}{B_{\Delta}(3q + \frac{1}{2}, \mu)}, \quad \mathbb{E} P_n (P_n - 1) \sim n^{2/3} 2 t_3^2 \frac{B_{\Delta}(3q + \frac{5}{2}, \mu)}{B_{\Delta}(3q + \frac{1}{2}, \mu)},
$$

(72)

which is just a consequence of Lemma 11 and Eq. (68). □
B.2 Height of a Random Sprouting Tree

Let $x(z) = \omega'(z)$. Consider recursive definition for the generating function of simple trees whose height doesn’t exceed $h$:

$$T^{[h+1]}(z) = z x(T^{[h]}(z)) , \quad T^{[0]}(z) = 0 .$$  \hfill (73)

The framework of multivariate generating functions allows to mark height with a separate variable $u$ so that the function

$$F(z, u) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{h=0}^{n} A_n^{[h]} u^h$$  \hfill (74)

is the BGF for trees, where $A_n^{[h]}$ stands for the number of simple labelled rooted trees with $n$ vertices, whose height equals $h$.

Flajolet and Odlyzko [FO82] consider the following expressions:

$$H(z) = \left. \frac{d}{du} F(z, u) \right|_{u=1} , \quad D_s(z) = \left. \frac{d^s}{du^s} F(z, u) \right|_{u=1} .$$  \hfill (75)

Generally speaking, $H(z) = D_1(z)$ is a particular case of $D_s(z)$, but their analytic behaviour is different for $s = 1$ and $s \geq 2$.

**Lemma 17** ([FO82, pp. 42–50]). The functions $H(z)$ and $D_s(z)$, $s \geq 2$ satisfy

$$H(z) \sim \alpha \log \varepsilon(z) , \quad D_s(z) \sim (\frac{\varepsilon}{z})^{-s} s! \Gamma(s) \xi(s) z^{-s+1} ,$$  \hfill (76)

$$\alpha = 2 \frac{x'(\bar{z})}{x''(\bar{z})} , \quad \varepsilon(z) = \bar{z} \left( 1 - \frac{z}{\rho} \right)^{1/2} \left( 2 x''(\bar{z}) x'(\bar{z}) \right)^{1/2} , \quad \rho = \bar{z} x^{-1}(\bar{z}) = (x'(\bar{z}))^{-1} .$$

Here, $\Gamma(s)$ is a gamma-function, and $\xi(s)$ is Riemann zeta-function.

We don’t represent their proof here, but would like to remark that it has great methodological impact. For our purposes we need the asymptotic equivalence $\sim$ only in the circle of analyticity $|z| < \rho$.

Recall that

$$T_1(z) = z \omega'(T_1(z)) , \quad T_\ell(z) = z \omega^{(\ell)}(T_1(z)) , \quad \ell \geq 0$$  \hfill (77)

From local expansion at $z = \rho$ of $z = z(T_1)$ it is easy to show that

$$z \sim \rho - (T_1(z) - \bar{z})^2 \left( \frac{x''(\bar{z}) \bar{z}}{2 x'^2(\bar{z})} \right)$$  \hfill (78)

and consequently, since $T_\ell(z) = z x^{(\ell)}(T_1(z))$,

$$T_1(z) = \bar{z} - \sqrt{\frac{2 \xi}{x'}} \sqrt{1 - \frac{z}{\rho}} + O(1 - z \rho^{-1}) ,$$  \hfill (79)

$$T_2(z) = 1 - \sqrt{\frac{2 \xi x''}{x'^3}} \sqrt{1 - \frac{z}{\rho}} + O(1 - z \rho^{-1}) .$$  \hfill (80)

So we have $\varepsilon(z) \sim \bar{z}^{1/2} (1 - T_2(z))$.

Actually, there are two kinds of sprouting trees that we have to distinguish: the first ones are attached to the vertices with degree from $\Delta - 2$, and the second — to the vertices with degree from $\Delta - 3$, we will treat these cases separately.

Now we can introduce random variables $H_n^{(2)}$, $H_n^{(3)}$ equal to the height of a randomly uniformly chosen sprouting tree (of the first and second type respectively), conditioned on excess number $q = (q_1, q_2, \ldots, q_k)$, and their moment generating functions:

$$\mathbb{E}[H_{n(1)}] = \frac{[z^n] U(z)^{n-m+q} e^{V(z)} E_q(z) F_2(z, u)}{[z^n] U(z)^{n-m+q} e^{V(z)} E_q(z) T_2(z)} ,$$  \hfill (81)

$$\mathbb{E}[H_{n(2)}] = \frac{[z^n] U(z)^{n-m+q} e^{V(z)} E_q(z) F_3(z, u)}{[z^n] U(z)^{n-m+q} e^{V(z)} E_q(z) T_2(z)} ,$$  \hfill (82)

where $F_2(z, u)$ and $F_3(z, u)$ are the corresponding BGF for 2- and 3-sprouted trees.
Lemma 18. Around \( z = \rho \) the derivatives of \( F_2(z) \) and \( F_3(z) \) with respect to \( u \) at \( u = 1 \) can be expressed as

\[
\frac{d^n}{du^n} F_2(z, u) \bigg|_{u=1} \sim \frac{z^n}{z^m} \frac{d^n}{du^n} F(z, u) \bigg|_{u=1}, \quad \frac{d^n}{du^n} F_3(z, u) \bigg|_{u=1} \sim \frac{z^n}{z^m} \frac{d^n}{du^n} F(z, u) \bigg|_{u=1}.
\]

Proof. We only present the main idea of the proof, omitting the technical details of how the error term is treated — we refer to [FO82] for the details of transfer theorems and sum approximations.

Consider more general specification, where root degree can belong to the set \( \Phi \) whose EGF is given by \( \varphi(z) = \sum_{d \in \Phi} (d!)^{-1} \). As said before, let \( T^{[h]}(z) \) be an EGF for trees of height \( \leq h \) given by Eq. (73). Then the EGF \( T^{[h]}(z) \) for rooted trees, whose root belongs to \( \Phi \) with height bounded by \( h \), can be written as

\[
T^{[h+1]}_\Phi(z) = z \varphi(T^{[h]}_1(z)), \quad T^{[h]}_\Phi(z) = 0.
\]

Then, there is a second-order Taylor expansion

\[
T_\Phi(z) - T^{[h+1]}_\Phi(z) = z(T_1(z) - T^{[h]}_1(z)) \varphi'(T_1(z)) \times \\
\left[ 1 - (T_1 - T^{[h]}_1) \frac{\varphi''(T_1)}{2\varphi'(T_1)} + O \left( (T_1 - T^{[h]}_1)^2 \right) \right] .
\]

Denoting \( T_1 - T^{[h]}_1 = e_h(z), T_\Phi - T^{[h]}_\Phi = \tilde{e}_h(z) \), we get approximate expansions

\[
F(z, u) \sim u T_1(z) + (u - 1)z \sum_{h \geq 1} u^h e_h(z) \kappa'(T_1),
\]

\[
F_\Phi(z, u) \sim u \varphi(T_1(z)) + (u - 1)z \sum_{h \geq 1} u^h e_h(z) \varphi'(T_1),
\]

so in order to calculate the ratio of derivatives with respect to \( u \) at the vicinity of \( z = \rho \) we note that the terms \( \kappa'(\tilde{z}) \) and \( \varphi'(\tilde{z}) \) provide the ratio of the coefficients of main asymptotics. \( \square \)

Lemma 19. Inside the critical window \( m = \alpha n(1 + \mu n^{-1/3}), |\mu| = O(1) \), the maximal height \( H_n \) of a sprouting tree, is of \( O(n^{1/3}) \) in probability, i.e.

\[
\mathbb{P} \left( \max H_n > \lambda n^{1/3} \right) = O(\lambda^{-2}) .
\]

Actually, the average height of a sprouting tree (if the tree is taken uniformly at random) appears to be \( \Theta(\log n) \) (which seems to be a new result), but when we take the maximum over all possible \( \Theta(n^{1/3}) \) trees, and apply Chebyshev inequality, this factor disappears.

Proof of Lemma 19. We prove the statement for 2-sprouting trees (with root degree from \( \Delta - 2 \)), and for 3-sprouting trees the proof is the same up to a constant term.

The ratio of the expressions in the numerator and denominator can be treated in terms of Lemma 11. After “Lagrangian” variable change \( T_1(z) = t \to z \) the ratio in \( E H_{n(1)} \) becomes proportional to

\[
\frac{C_1 \int (1 - \phi_1(z))^{1-y} e^{nh(z;r)} \log(1 - \phi_1(z)) dz / z}{C_2 \int (1 - \phi_1(z))^{1-y} e^{nh(z;r)} dz / z}
\]

with \( y = 3q + \frac{1}{2} \), and after the second variable change \( z = \tilde{z} e^{-s\nu}, s = a + it \) the main asymptotics term will become

\[
\left( C_1 \int \cdots \right) / \left( C_2 \int \cdots \right) \sim \tilde{C}_1(\mu) \log n ,
\]

(88)
For the second factorial moment we obtain
\[
\tilde{C}_2(\mu)n^{1/3} + O(1 + |\mu|^4),
\] (89)
so from Chebyshev inequality:
\[
P\left(|H_n(1) - \tilde{C}_1 \log n| \geq \lambda C_2 n^{1/6}\right) \leq \frac{1}{(\lambda + o(1))^2}. \tag{90}
\]
Since 2-path length is \(\Theta(n^{1/3})\) in probability, we can control the maximal tree height:
\[
P(H_n \geq \lambda C_2 n^{1/3}) = O(\lambda^{-2} n^{-1/3}),
\]
\[
P(\max H_n \geq \lambda C_2 n^{1/3}) = O(\lambda^{-2}). \tag{91}
\]

### B.3 Circuits in Directed Graphs

Let’s introduce egfs for directed rooted trees and directed unrooted trees:
\[
\tilde{T}_r(z) = \frac{1}{2} T_r(2z), \tag{92}
\]
\[
\tilde{U}(z) = \frac{1}{2} T_0(2z) - \frac{1}{4} T_2(2z). \tag{93}
\]
In the first case, each edge can be oriented in two ways, but the number of edges is equal to the size of the tree (the number of nodes) minus one, hence the multiple \(1/2\). In the second case, it can be shown that \(\tilde{U}(z) = \frac{1}{4} U(2z)\) using the same argument: each edge can be oriented in two possible ways, and the number of edges is equal to the tree size minus one.

Then we introduce an egf \(\tilde{V}(z)\) for so-called unicycuits, which are directed graphs obtained from undirected unicycle graphs by directing edges, and having all edge directions of the cycle in the same direction (clockwise or counterclockwise). We start with a more simple egf for circuits, i.e. directed graphs consisting of one circuit. Their egf is the same as egf for usual cycle operator:
\[
\text{circuit}_{>2}(z) = \log \frac{1}{1-z} - z - \frac{z^2}{2}. \tag{94}
\]
Then,
\[
\tilde{V}(z) = \text{circuit}_{>2}\left(\tilde{T}(z)\right). \tag{95}
\]

Let us define \(\tilde{V}^\bullet(z, u)\) such that \(n! |z^n u^k| \tilde{V}^\bullet(z, u)\) is equal to the number of directed unicycle graphs (i.e. connected digraphs of excess 0 having one circuit inside) weighted by \(2^\ell\) where \(\ell\) is the length of the circuit, and the graphs are equipped with two marked distinct vertices on the circuit.

For distinct vertices using the double-marking operator \(z^2 \frac{d^2}{dz^2}\), we obtain
\[
\tilde{V}^\bullet(z, u) = z^2 \frac{d^2}{dz^2} \text{circuit}_{>2}(uz) \bigg|_{z = \tilde{T}(z)}, \tag{96}
\]
\[
\tilde{V}^\bullet(z, u) = \frac{\left(u \tilde{T}(z)\right)^2}{\left(1 - u \tilde{T}(z)\right)^2} - \left(u \tilde{T}(z)\right)^2. \tag{97}
\]
This can be explained combinatorially in a different manner: the circuit with two marked vertices on a cycle is just a pair of sequences of length at least one (minus the case of two sequences of length 1 each because it doesn’t correspond to a simple graph).

Given \(n, m = 2rn, \Delta\), consider random graph \(D = D_{n,m,\Delta}\) constructed by the rules described in section 3 such that the total excess of the complex part is equal to 0, i.e. there are only trees and unicycles. Let us define random variable \(L\) as follows:
\[
L = \begin{cases} -\infty, & \text{if } 1, \mathcal{T} \notin \text{ strongly connected component in } D, \\ \ell, & \text{if } 1, \mathcal{T} \in \text{ circuit of length } \ell \text{ in } D. \end{cases} \tag{98}
\]
Lemma 20. The expectation $E(2^L)$ is equal to
\[
\frac{[z^2n]U^2(z)2n-me^\Delta V(z)\tilde{V}^\ast(z)}{2n(2n-1)[z^2n]U^2(z)2n-me^\Delta V(z)} = \left(\frac{z}{\omega'(\hat{z})}\right)^2 \oint \frac{e^{2nh(z;r)}g(z)\tau(z)\frac{dz}{z}}{e^{2nh(z;r)}g(z)\frac{dz}{z}},
\]
where
\begin{itemize}
  \item $\tilde{V}^\ast(z) = V^\ast(z, 2)$,
  \item $h(z; r)$ is from Lemma 11,
  \item $g(z) \sim C_1\sqrt{1-z}$, $\tau(z) \sim \frac{C_2}{(1-z)^2}$ for some constants $C_1, C_2$.
\end{itemize}

Proof. Note that weighted number of graphs that have 1 and $\Delta$ differs from \(O\) times the total number of weighted graphs with marked nodes, because at the marked places there can be any labels 1, 2, $\bar{2}$, ..., $n, \bar{n}$, and there are 2n(2n - 1) ways to choose two labels.

Then, if a digraph has 2n vertices, $m$ edges, and doesn’t have a complex part, then the number of trees is equal to 2n - $m$. The expected value of $u^L$ is equal to the ratio of 2n-th coefficient in marked generating function (with trees, and a set of unicycles with one marked unicircuit graph) and 2n-th coefficient in EGF for all digraphs without complex part.

The integral expression can be obtained by Lagrangian variable substitution $z \mapsto t = T_y(2z)$ in Cauchy integral formula after cancelling constant multiples (with respect to $z$) in the numerator and denominator. The coincidence with $h(z; r)$ from Lemma 11 is not surprising, because putting the directions on the edges doesn’t change the point of phase transition. Additional multiple $z^k$ that we gain from the method of moments, is transformed into $\left(\frac{z}{\omega'(\hat{z})}\right)^2$.

Lemma 21. Let $m = 2\alpha n(1 + \mu n^{-1/3})$, $L$ — random variable described above. Then $nE(2^L) = O(n^{-1/3}\mu^{-2})$.

Proof. We apply Lemma 11 to an expression from Lemma 20 to obtain the final answer. The fact that instead of $nh(z; r)$ we have $2nh(z; r)$ in the exponent doesn’t change the asymptotic rate, it only changes the constants.

An expression $2n(2n - 1)$ in the denominator contributes $O(n^{-2})$. Then, the ratio of the integrals is
\[
\oint \frac{e^{2nh(z;r)}g(z)\tau(z)\frac{dz}{z}}{\oint e^{2nh(z;r)}g(z)\frac{dz}{z}} = \Theta \left(\frac{A_\Delta(3q + 2 + \frac{1}{2}, \mu)}{A_\Delta(3q + 1, \mu)}\right),
\]
where $A_\Delta(y, \mu)$ is from Corollary 13.

In the subcritical phase, i.e. when $\mu \rightarrow -\infty$ it holds $A(y, \mu) \sim \frac{1}{\sqrt{2\pi|\mu|^{y-1/2}}}$. The function $A_\Delta(y, \mu)$ differs from $A(y, \mu)$ by a constant multiple (for fixed $y$) depending on $\Delta$. Therefore, the final asymptotic bound is
\[
nE(2^L) \leq \frac{n \cdot n^2\mu^{-2}}{n(n-1)} = O(n^{-1/3}\mu^{-2}).
\]

C Degree Sequence and Degree Constraints Models

In this section, we give some insights about the two different models. Furthermore, we emphasize that the equation (2) for threshold point is expressed through exponential generating functions, and it’s unlikely to obtain it using some other method. The analysis using generating function is usually more precise, and using our machinery we were able to track the structure of connected components, which has not been done before.
C.1 Analyzing Hatami–Molloy Framework

In [HM12], the authors consider two parameters depending on sequence of degrees \( \mathcal{D} = (d_v)_{v \in G} \):

\[
Q := Q(\mathcal{D}) := \frac{\sum_{v \in G} d_v^2}{2|E|} - 2, \tag{100}
\]

\[
R := R(\mathcal{D}) := \frac{\sum_{v \in G} d_v(d_v - 2)^2}{2|E|}. \tag{101}
\]

They prove that if \(|Q| = O(n^{-1/3} R^{2/3})\) then, with high probability, the size of the largest component will be of order \(\Theta(n^{2/3} R^{2/3})\), i.e. the density of edges corresponds to the so-called critical phase. The second parameter \(R\) is just bounded by a constant in probability.

We can try to show that (a) \(\mathbb{E}Q = O(n^{-1/3})\) and (b) \(\text{Var } Q = O(n^{-1})\).

The second statement is more simple if we assume that the sequence \(d_v\) consists of almost independent entries and \(\text{Var } d_v^2\) is asymptotically constant. In this case,

\[
\text{Var } Q(\mathcal{D}) \approx n^4 m^2 \text{Var } d_1^2 = O(n^{-1}). \tag{102}
\]

To track the expectation \(\mathbb{E}Q(\mathcal{D})\) inside our critical phase \(m = \alpha n(1 + \mu n^{-1/3})\), we use method of moments and marking procedure from analytic combinatorics [FS09] to study \(d_v\).

We choose the vertex with label 1, which can be inside a tree, unicycle or a complex component. To track the degree of corresponding vertex, we introduce additional variable \(u\), so that

\[
G(z, u) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{k=0}^n a_{n,k} u^k \tag{103}
\]

is an egf for graphs with \(m\) edges, \(n\) vertices with vertex labelled 1 having \(k\) neighbors. After two applications of operator \(\frac{d}{du}\), we obtain the expectation of square \(\mathbb{E}d_1^2\) by dividing corresponding quantities:

\[
[z^n] \left[ \left( \frac{d}{du} \right)^2 G(z, u) \right]_{u=1} \tag{104}
\]

In either of three cases we are able to apply the analytic lemma (Lemma 11 / Corollary 13) to obtain the final expressions. We don’t want to complexify this answer by direct computation: we expect the answer to be \(\mathbb{E}d_1^2 = 4\alpha + O(n^{-1/3})\). In this case, after substitution, we obtain:

\[
\mathbb{E}Q(\mathcal{D}) = \frac{4n\alpha + O(n^{2/3})}{2m} - 2 = O(n^{-1/3}) \tag{105}
\]

We combine the estimates for expectation and variance with Chebyshev inequality to bound the parameter \(Q\) in probability.

C.2 Analyzing Proportions of Vertices of Fixed Degree

Another option is to study the distribution of vertices with some given degrees. This is possible by means of marking the corresponding variables.

In construction of a rooted tree, we used \(\Delta\)-SET operator, so given \(d \in \Delta\), in order to mark vertices of degree \(d\), we mark corresponding vertices inside the tree, whose number of descendants is equal to \((d - 1)\), so finally instead

\[
T(z) = z \omega(T(z)) \tag{106}
\]

we have

\[
T(z, u) = z \cdot \left( \omega + (u - 1) \frac{z^d}{d!} \right) \circ T(z) \tag{107}
\]

We don’t provide direct calculations either, but we predict that using this method we can prove that the proportion of vertices of each degree is asymptotically constant, so the number of such vertices is linear. However, we admit that our method restricts to the case when \(|\mu| = O(n^{1/12})\). Thanks for reading this article.