GOODWILLIE’S CALCULUS VIA RELATIVE HOMOLOGICAL ALGEBRA. THE ABELIAN CASE

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1. Introduction

We will explain how elementary concepts of relative homological algebra yield the Taylor tower for functors from pointed categories to abelian groups recovering the constructions of Johnson and McCarthy [2], [3].

Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories with enough projective objects. Let $i_* : \mathcal{C} \to \mathcal{D}$ and $i^* : \mathcal{D} \to \mathcal{C}$ be functors, such that $i^*$ is left adjoint to $i_*$. We will assume that $i_*$ is full and faithful and exact. After taking the left derived functors one obtains a pair of adjoint functors $(L(i^*), L(i_*))$ between the derived categories $D^-(\mathcal{D})$ and $D^-(\mathcal{C})$. In general, $L(i_*) : D^-(\mathcal{C}) \to D^-(\mathcal{D})$ is not a full embedding. Instead one defines a full subcategory $D_C^-(\mathcal{D})$ of $D^-(\mathcal{D})$ by

$$D_C^-(\mathcal{D}) = \{ X_n \in D^-(\mathcal{D}) | H_n(X_n) \in \mathcal{C}, n \in \mathbb{Z} \}.$$ 

Denote by $j_* : D_C^-(\mathcal{D}) \to D^-(\mathcal{D})$ the full inclusion. Then the functor $L(i_*)$ factors through $j_*$. In the favourable cases the functor $j_*$ has left adjoint $j^*$, however we do not know whether $j^*$ always exists. In the next section we will construct the functor $j^*$ under certain circumstances. Our construction is based on the elementary results of the relative homological algebra [1] and is probably well-known. In the last section we explain how the results of Section 2 imply the main results of [2], [3].

In [1] we will extend our method from abelian to nonabelian case.

2. The main construction

Let $\mathcal{A}$ be an abelian category with coproducts and let $\mathcal{P}$ be a set of objects in $\mathcal{A}$ such that each $P \in \mathcal{P}$ is projective. Define the following full subcategory

$$\mathcal{B} = \mathcal{P}^\perp = \{ A \in \mathcal{A} | \text{Hom}_\mathcal{A}(P, A) = 0, P \in \mathcal{P} \}.$$ 

It is clear that $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$. That is, $\mathcal{B}$ is closed under taking kernels, cokernels and extensions. In particular, $\mathcal{B}$ is also abelian. Denote by $i_* : \mathcal{B} \to \mathcal{A}$ the inclusion. Then $i_*$ is exact.

For any $A \in \mathcal{A}$ one puts

$$\Phi(A) = \bigoplus_{f:P \to A} P.$$ 

Here $P$ runs through all objects of $\mathcal{P}$. For a morphism $f : P \to A$ we let $in_f : P \to \Phi(A)$ be the standard inclusion. Define $i^*_A : \Phi(A) \to A$ by $\epsilon_A \circ in_f = f$ and denote $\text{Coker}(\epsilon_A)$ by $i^*_A(A)$. Since $\text{Hom}_A(P, \epsilon_A)$ is surjective one sees that $i^*_A(A) \in \mathcal{B}$. In this way one obtains a functor $i^* : \mathcal{A} \to \mathcal{B}$ which is left adjoint to $i_*$. A morphism $f : X \to Y$ in $\mathcal{A}$ is called $\mathcal{P}$-epimorphism provided $\text{Hom}_A(P, f) : \text{Hom}_A(P, X) \to \text{Hom}_A(P, Y)$ is surjective. For example, for any object $A \in \mathcal{A}$ the morphism $\epsilon_A : \Phi(A) \to A$ is a $\mathcal{P}$-epimorphism. Hence $\mathcal{P}$ is a projective class in the sense of [1] and therefore by [1] Proposition 3.1] any object $A$ has a $\mathcal{P}$-projective resolution. Thus there is a chain complex $(X_*, d)$ such that $X_n = 0$ if $n < -1$, $X_{-1} = A$, $X_n \in \mathcal{P}$ for any $n \geq 0$ and for any $P \in \mathcal{P}$ the following sequence is exact:

$$\cdots \to \text{Hom}_A(P, X_n) \to \cdots \to \text{Hom}_A(P, X_0) \to \text{Hom}_A(P, X_{-1}) \to 0.$$
It follows that \( X_s \in \text{D}^\perp_A (\mathcal{A}) \). By the standard properties of \( \mathcal{P} \)-projective resolutions the assignment \( A \mapsto X \) extends to a functor \( j^* : \text{D}^{-}(\mathcal{A}) \to \text{D}^\perp_B (\mathcal{A}) \) which turns to be left adjoint to \( j_* \).

Assume now that instead of a single set \( \mathcal{P} \), a descending sequence of sets

\[
\cdots \subseteq \mathcal{P}_n \subseteq \mathcal{P}_{n-1} \subseteq \cdots \subseteq \mathcal{P}_1
\]

is given, each of which satisfies the assumptions made in the beginning of Section 2. One obtains abelian categories \( \mathcal{B}_n = \mathcal{P}^\perp_n \) and functors \( i_n, i^*_n, j_n, j^*_n \). Clearly, \( \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \cdots \subseteq \mathcal{A} \) and for any object \( A \in \mathcal{A} \) one obtains the towers of epimorphisms

\[
A \to \cdots \to i_* i^*_n(A) \to i_* i^*_{n-1}(A) \to \cdots \to i_* i^*_1(A) \to i_* i^*_0(A)
\]

and of morphisms in \( \text{D}^{-}(\mathcal{A}) \)

\[
A \to \cdots \to j_* j^*_n(A) \to j_* j^*_{n-1}(A) \to \cdots \to j_* j^*_1(A) \to j_* j^*_0(A).
\]

3. Applications to Goodwillie’s calculus

Let \( \mathcal{M} \) be a small category with zero object \( 0 \) and finite coproduct \( \lor \). We let \( \mathcal{A} \) be the category of all functors from \( \mathcal{M} \) to the category of abelian groups. Then \( \mathcal{A} \) is an abelian category with enough projective objects. The functors \( h_a \) are small projective generators of \( \mathcal{A} \). Here \( a \) is running through all objects of the category \( \mathcal{M} \) and \( h_a \in \mathcal{A} \) is given by \( h_a = \mathbb{Z}[\text{Hom}_\mathcal{M}(a, -)] \). The obvious maps \( a \to 0 \to a \) yield a splitting \( h_a = \overline{h}_a \oplus \mathbb{Z} \), where \( \mathbb{Z} = h_0 \) is the constant functor with values equal to \( \mathbb{Z} \). Thus the collections \( \overline{h}_a, a \in \mathcal{M} \) together with \( \mathbb{Z} \) also form a family of small projective generators. Clearly \( h_{a \lor b} = h_a \otimes h_b \). It follows that the level-wise tensor product of projective objects is again a projective object. For any natural number \( n \geq 1 \) we let \( \mathcal{P}_n \) be the collection of projective objects of the form \( \overline{h}_{a_1} \otimes \cdots \otimes \overline{h}_{a_k} \), \( k > n \). One easily checks that the corresponding category \( \mathcal{B}_n = \mathcal{P}_n^\perp \) is the category of functors of degree \( \leq n \) (in the sense of Eilenberg-MacLane), while \( \text{D}^\perp_B (\mathcal{A}) \) is equivalent to the category of functors from \( \mathcal{M} \) to the category of chain complexes of abelian groups of degree \( \leq n \) (in the sense of Goodwillie). This follows from the fact that \( \text{Hom}_\mathcal{A}(\overline{h}_{a_1} \otimes \cdots \otimes \overline{h}_{a_k}, T) = cr_k T(a_1, \ldots, a_k) \), where \( cr_k \) is the \( k \)-th crossed-effect \([3]\). The last isomorphism is a trivial consequence of the Yoneda lemma and the decomposition rule: \( h_{a \lor b} = h_a \otimes h_b \). It follows that in this situation the towers constructed in Section 2 and the ones constructed in \([2]\) are equivalent.

References

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