Hidden Variable Quantum Mechanics from Branching from Quantum Complexity

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Beginning with the Everett-DeWitt many-worlds interpretation of quantum mechanics, there have been a series of proposals for how the state vector of a quantum system might be split at any instant into orthogonal branches, each of which exhibits approximately classical behavior. Here we propose a decomposition of a state vector into branches by finding the minimum of a measure of the net quantum complexity of the branch decomposition. We then propose a method for finding an ensemble of possible initial state vectors from which a randomly selected member, if evolved by ordinary Hamiltonian time evolution, will follow a single sequence of those branches of many-worlds quantum mechanics which persist through time. Macroscopic reality, we hypothesize, consists of an accumulating sequence of such persistent branching results. For any particular draw, the resulting deterministic system appears to exhibit random behavior as a result of the successive emergence over time of information present in the initial state but not previously observed.

I. INTRODUCTION

Microscopic particles have wave functions spread over all possible positions. Macroscopic objects simply have positions, or at least center-of-mass positions. How to apply the mathematics of quantum mechanics to extract predictions registered in the macroscopic world of positions from experiments on microscopic systems having wave functions but not definite positions is well understood for all practical purposes. But less well understood, or at least not a subject on which there is a clear consensus, is how in principle the definite positions of the macroscopic world emerge from the microscopic matter of which it is composed, which has only wave functions but not definite positions. There is a long list of proposals. In the present article we add another.

We begin in Section II with a brief reminder of “the problem of measurement” which arises for an experiment in which a microscopic system interacts with a macroscopic measuring device with both systems assumed governed by quantum mechanics. Among the proposals which address this problem are the many-worlds interpretation and environmentally-induced decoherence. Shared by these proposals is the hypothesis that the quantum state of the universe, as time goes along, naturally splits into a set of orthogonal branch states each of which displays, approximately, a distinct configuration of macroscopic reality. We will argue, however, that rules according to which these proposals are to be applied to the world are intrinsically uncertain and can be made precise only by the arbitrary choice of auxiliary parameters. The uncertainty is not simply the approximate nature of the macroscopic description of an underlying microscopic system, but rather that the branching process of the microscopic system itself, in each of these proposals, occurs according to uncertain rules. And as a consequence, it seems to me implausible that the corresponding branches are, by themselves, macroscopic reality. In addition, missing from these proposals is a mathematical structure that allows even the process of choosing the auxiliary parameters to be stated precisely. These various limitations we will try to address in a sequence of several steps.

A main feature of the proposal we present here is that branch formation does not follow from unitary time evolution by itself nor does it entail a modification of unitary time evolution. Instead, branch formation consists of an additional layer of the world that sits on top of unitary time evolution.

In Section III modifying ideas from [1], we define for a 1-dimensional lattice field theory of fermions a version of quantum complexity designed to measure the spatial structure of entanglement in a state vector. In Section IV we define an entangled state of a pair of particles each at a corresponding single point and then show in Appendix A that the state’s complexity is bounded from below and from above by quantities linear in the distance between the points. In Section V we define an entangled state of a pair of particles one of which has an extended wave function and show in Appendix B that the state’s complexity is bounded from below and above by expressions linear both in the distance between particles and in the width of the wave function. In Section VI we then propose finding a branch decomposition of any state by minimizing a measure of the decomposition’s net complexity. The net complexity minimized depends on a parameter with units of length, the branching threshold $b$, which determines the circumstances under which quantum behavior crosses over to classical. The branching threshold $b$ should in principle be measurable.

In Section VII using the decomposition of Section III we propose a branch decomposition at late time $t_{out}$, restored to some finite time $t_{in}$ by unitary time evolution, then subjected to the limit $t_{out} \to \infty$, thereby selecting only those branching results which persist through time. The set of branches found directly at a fixed finite time, we will argue, in some cases is not covariant, a problem potentially avoided by the late time limit. Macroscopic reality, we then conjecture, consists of the accumulating sequence over time of such permanent branch results. Each of the vectors in the limiting decomposition at $t_{in}$, according to this hypothesis, if evolved by the usual unitary time evolution will follow only a single sequence of
persistent branch results [8]. Macroscopic reality still emerges as an approximate description of a corresponding state vector’s time evolution. But the underlying state vector itself evolves purely deterministically through a time trajectory of branching events without additional approximations or missing parameters.

In Section VIII we consider the theory of Section VII applied to a simplified Stern-Gerlach experiment. In Section IX we apply the theory of Section VII to a model of a test of Bell’s theorem and obtain the expected quantum mechanical results. We conclude in Section X with comments about issues yet to be addressed.

Throughout this paper, we will assume flat space-time. In Section VII the limit $t_{out} \to \infty$ ignores events on a cosmological time scale.

II. PROBLEMS

Let $S$ be a microscopic system to be measured, with corresponding state space $H_S$, for which a basis is $\{|s_i>\}, i > 0$. Let $M$ be a macroscopic measuring device with corresponding state space $H_M$ containing the set of vectors $\{|m_i>\}, i > 0$. For each different value of $i > 0$ the state $|m_i>$ is a macroscopically distinct meter reading. Let $|m_0>$ be an initial state showing no reading. In the combined system-meter product state space $H_S \otimes H_M$, a measurement of $S$ by $M$ over some time interval takes each possible initial state $|s_i> |m_0>$ into the corresponding final state $|s_i> |m_i>$ with the measuring device displaying the measured value of the microscopic system’s variable

$$|s_i> |m_0> \rightarrow |s_i> |m_i>.$$  \hspace{1cm} (1)

By linearity of quantum mechanical time evolution, however, it then follows that a measurement with a linear superposition in the initial state will yield a final state also with a superposition

$$[\alpha|s_1> + \beta|s_2>] |m_0> \rightarrow$$

$$\alpha|s_1> |m_1> + \beta|s_2> |m_2>.$$  \hspace{1cm} (2)

In the measured final state, the meter no longer has a single value but a combination of two values which cannot, by itself, be connected to a recognizable configuration of a macroscopic object. The absence of a recognizable configuration for the macroscopic device is the “problem of measurement”.

The resolution of this problem proposed by the many-worlds interpretation of quantum mechanics [1, 2] is that the states $|s_1> |m_1>$ and $|s_2> |m_2>$ actually represent two different worlds. In each world the meter has a definite position but with different positions in the two different worlds. For an interaction between two systems, the splitting into separate worlds is done in the Schmidt basis, in which the density matrix of the measured system is diagonalized. Among the problems of the many-worlds interpretation, however, is that in general, for plausible models of a measurement process, the individual worlds given by the Schmidt basis do not have sufficiently narrow coordinate dispersions to count as classical reality [8]. In addition, it is unclear under what circumstances and according to what basis a system larger than just a micro system and a measuring device should be split into separate worlds.

A resolution to the first of these problems, the absence of classical behavior in the split branches, is proposed to occur through environmentally-induced decoherence [3, 4]. According to this proposal, the system-meter combination should not be considered in isolation but instead an account is required of the rest of the macroscopic environment with which the meter can interact. When the value of a macroscopic meter is changed by recording the value of a microscopic coordinate, the meter’s new state rapidly becomes entangled with a large number of other degrees of freedom in the environment

$$[\alpha|s_1> |m_1> + \beta|s_2> |m_2>]|e_0> \rightarrow$$

$$\alpha|s_1> |m_1> + \beta|s_2> |m_2> |e_2>.$$  \hspace{1cm} (3)

For a particular choice of bases for system, meter and environment, determined by the combined system’s dynamics, entanglement of the meter with the environment proceeds as quickly as possible, $|e_1>$ and $|e_2>$ almost do not mix in the course of further time development, and $|e_1>$ and $|e_2>$ include many redundant copies of the information in $|s_1> |m_1>$ and $|s_2> |m_2>$, respectively. Based on these various considerations it is argued that entangled environmental states $|e_1>$ and $|e_2>$ behave essentially as permanent classical records of the experimental results. Correspondingly, for many-worlds augmented with decoherence [5], the circumstance under which a system splits into distinct worlds is when a superposition has been produced mixing distinct values of one of these effectively classical degrees of freedom. Each distinct value of the coordinate in such a superposition goes off into a distinct world.

A step toward resolving the second problem, the absence of a criterion for branching for the universe as a whole rather than simply for some system-apparatus pair, takes the form of a theorem [6] according to which, for a system as a whole, if a particular spatial pattern of redundant records happens to occur, then there is a unique corresponding decomposition of a state vector into effectively classical branches.

A residual problem of [6, 7], however, is that the rules governing their application to the world are intrinsically uncertain. In particular, the record production needed for environmentally-induced decoherence occurs over some nonzero intervals of time and space, and perhaps is entirely completed only asymptotically in long time and large distance limits. What fraction of the initial state in Eq. (3) must become entangled with the environment for splitting into classical branches to occur? When exactly over the time interval of decoherence,
does the splitting of the world in parts occur? And since
the process extends over space, this timing will differ in
different Lorentz frames. Which is the correct choice?
These various questions may be of no practical conse-
quence in treating the meter readings as nearly classical
degrees of freedom after entanglement and using the re-
sulting values to formulate observable predictions. But
what is clear is that something is missing from the the-
ory. From outside the theory, something additional and
arbitrary needs to be supplied by hand to resolve each
of these issues. Moreover, no mathematical machinery is
present in any of these proposal which allows the process
of filling in what is missing to be stated precisely. As
a consequence of all of which it seems to me implausi-
ble that the branches provided by these accounts are by
themselves the fundamental substance of reality.
A discussion of issues concerning environmentally in-
duced decoherence and its combination with the many-
worlds interpretation of quantum mechanics appears in

[10]. The remainder of this paper consists mainly of setting
up the machinery for two conjectures intended to offer a
way to fill in the omissions which we believe prevent en-
vironmentally induced decoherence in its present formu-
lation from providing an account of macroscopic reality.
The two conjectures appear in Section [VII B]

III. COMPLEXITY

We now construct a complexity measure for a 1-
dimensional lattice field theory of fermions, at a single
instant of time, modifying ideas from [7]. No feature of
the discussion depends significantly on the lattice dimen-
sion or on the choice of fermions. Complexity measures
similar to the one we will discuss can also be constructed
in higher dimensions and for theories including bosons.

A. Lattice Fermion Hilbert Space

The state space for the system $\mathcal{H}$ we assume is a tensor
product of spaces $\mathcal{H}_x$ one for each integer valued lattice
point $x$ in some arbitrarily large but finite range $|x| \leq x_{\text{max}}$

$$\mathcal{H} = \otimes_x \mathcal{H}_x.$$  (4)

An orthogonal basis for each $\mathcal{H}_x$ consists of the vacuum
$|0\rangle_x$, 1-particle spin up and spin down states $|1\rangle_x$ and
$|−1\rangle_x$, and a 2-particle state $|2\rangle_x$, with one spin up
and the other down. The overall system vacuum $|0\rangle$
is $\otimes_x |0\rangle_x$. Let $N_x$ be the operator on $\mathcal{H}$ giving the
fermion count at site $x$ and let $N$ be the total fermion count

$$\hat{N}_x = N_x \otimes_{q \neq x} I_q,$$  (5a)

$$N = \sum_x \hat{N}_x.$$  (5b)

For convenience we now write both $\hat{N}_x$ and $N_x$ as $N_x$.

We define in $\mathcal{H}$ a set $\mathcal{P}$ of product states. Define the
operators $a(x,i)$ for $i$ of -1 or 1 which annihilate the vac-
uum $|0\rangle >$, their adjoints, $a^\dagger(x,i)$, which create single
particle states and together have the anticommutation
relations

$$\{ a(x,i), a(y,j) \} = 0,$$  (6a)

$$\{ a^\dagger(x,i), a^\dagger(y,j) \} = 0,$$  (6b)

$$\{ a^\dagger(x,i), a(y,j) \} = \delta_{xy}\delta_{ij}. \quad (6c)$$

From these it is convenient to define $b^\dagger(x,i)$ to be the same
as $a^\dagger(x,i)$ for $i$ of 1 or -1 and for $i$ of 2

$$b^\dagger(x,2) = a^\dagger(x,-1)a^\dagger(x,1). \quad (7)$$

For a product in order of increasing $x_j$ with $i_j \neq 0$ we
then adopt the sign convention

$$\prod_j b^\dagger(x_j, i_j)|0\rangle > = \otimes_j |i_j > x_j \otimes q \neq x_0 ... x_n |0\rangle > . \quad (8)$$

For a normalized position wave function $p(x)$, and a
normalized spin wave function $[s(1), s(-1)]$, define creation
operators $c^\dagger(p, s)$ for extended states

$$c^\dagger(p, s) = \sum_i p(x) s(i) a^\dagger(x, i). \quad (9)$$

From a sequence of $n$ position and spin wave functions,
define an $n$-particle product state to be

$$c^\dagger(p_{n-1}, s_{n-1})...c^\dagger(p_0, s_0)|0\rangle > . \quad (10)$$

Let $\mathcal{P}$ be the set of all product states.

B. Hermitian Operator Hilbert Space

We now define a Hilbert space over the reals of Hermit-
ian operators acting on $\mathcal{H}$. For any site $x$, let $\mathcal{F}_x$ be the set
of traceless Hermitian operators acting on $\mathcal{H}_x$ which
commute with $N_x$. For any pair of nearest neighbor sites
$x < y$, let $\mathcal{F}_{xy}$ be the set of traceless Hermitian operators
acting on $\mathcal{H}_x \otimes \mathcal{H}_y$ which commute with $N_x + N_y$. $\mathcal{F}_{xy}$
can be made into a Hilbert space over the reals with the
inner product and norm

$$< f, g > = Tr( f^\dagger g),$$  (11a)

$$\| f \|^2 = < f, f > . \quad (11b)$$

Any $f \in \mathcal{F}_x$ can be viewed as $f \otimes I_y \in \mathcal{F}_{xy}$, and any
$f \in \mathcal{F}_y$ can be viewed as $I_x \otimes f \in \mathcal{F}_{xy}$. The norm in Eq.
(11b) of $f \otimes I_y \in \mathcal{F}_{xy}$ for the nearest neighbor pair $x < y$
is the same as the norm of $I_y \otimes f \in \mathcal{F}_{yx}$ for the nearest
neighbor pair $y < x$. Define $\mathcal{G}_{xy}$ to be the subspace of
$\mathcal{F}_{xy}$ orthogonal to $\mathcal{F}_x$, and $\mathcal{F}_y$.

The operators and inner product just defined can be
extended to the full space $\mathcal{H}$. For any $f_x$ in some $\mathcal{F}_x$ and
any \( g_{xy} \) in some \( G_{xy} \), define corresponding operators on \( \mathcal{H} \) by

\[
\hat{f}_{x} = f_{x} \otimes q \neq x I_{y}, \tag{12a}
\]

\[
\hat{g}_{xy} = g_{xy} \otimes q \neq x, y I_{y}. \tag{12b}
\]

Let \( K \) be a vector space over the reals of traceless, Hermitian linear operators \( k \) on \( \mathcal{H} \) commuting with \( N \) given by sums of the form

\[
k = \sum_{x} \hat{f}_{x} + \sum_{xy} \hat{g}_{xy}, \tag{13}
\]

for any collection of \( f_{x} \in F_{x} \) on a set of distinct sites \( \{x\} \), and any collection of \( g_{xy} \in G_{xy} \) on a set of distinct nearest neighbor pairs \( \{(x, y)\} \). Pairs in the set \( \{(x, y)\} \) may overlap on a single site but not on both. The inner product and norm of Eqs. (11a) and (11b) can then be extended to \( k \) of Eq. (13) and \( k' \) defined similarly according to

\[
<k, k'> = \sum_{x} <f_{x}, f_{x}'> + \sum_{xy} <g_{xy}, g_{xy}'>, \tag{14a}
\]

\[
\|k\|^2 = \sum_{x} <f_{x}, f_{x}> + \sum_{xy} <g_{xy}, g_{xy}>. \tag{14b}
\]

The inner product of Eq. (14a) is equivalent to

\[
<k, k'> = 4^{1-2x_{\max}} Tr(kk'), \tag{15}
\]

for \( k \) and \( k' \) in Eq. (13), the trace in this case taken over all of \( \mathcal{H} \). The equality of the inner products of Eqs. (14a) and (15) follows from the requirement that all \( F_{x} \) and \( G_{xy} \) are traceless, that all \( G_{xy} \) are orthogonal to the corresponding \( F_{x} \) and \( F_{y} \) and that the trace of the identity operator \( I_{x} \) on any \( \mathcal{H}_{x} \) is 4.

Although Eq. (15) could be taken as an alternative definition of \( <k, k'> \) for the space \( K \) of operators on the fermion Hilbert space of Section II A for a corresponding space of operators on a boson Hilbert space, the underlying \( \mathcal{H}_{x} \) are infinite dimensional, the trace of the identity operator \( I_{x} \) on any \( \mathcal{H}_{x} \) is not finite and there is no equivalent of Eq. (15). We thus prefer to view Eqs. (14a) and (15) as the definition of \( <k, k'> \).

C. Complexity from Unitary Trajectories

From this machinery, for any pair of states |\( \psi_{0} >\), |\( \psi_{1} >\) \( \in \mathcal{H} \) with equal particle number we define the complexity \( C(|\psi_{1} >, |\psi_{0} >) \) of |\( \psi_{1} >\) with respect to \( |\psi_{0} >\).

For \( 0 \leq t \leq 1 \), let \( k(t) \in K \) be a piecewise continuous trajectory of operators. Let the unitary operator \( U_{k}(t) \) on \( \mathcal{H} \) be the solution to the time development equation and boundary condition

\[
\dot{U}_{k}(t) = -ik(t)U_{k}(t), \tag{16a}
\]

\[
U_{k}(0) = I. \tag{16b}
\]

Among such \( k(t) \), we consider only those which in addition fulfill

\[
\xi U_{k}(1)|\psi_{0} >= |\psi_{1} >, \tag{17}
\]

for some complex number \( \xi \) with \( |\xi| = 1 \). Then the complexity \( C(|\psi_{1} >, |\psi_{0} >) \) is the minimum over all such \( k(t) \) of the integral

\[
C(|\psi_{1} >, |\psi_{0} >) = \min \int_{0}^{1} dt \| k(t) \|. \tag{18}
\]

Finally, any product state in \( P \) we assign 0 complexity. The complexity \( C(|\psi >) \) of any state |\( \psi > \) not in \( P \) is defined to be the distance to the closest product state

\[
C(|\psi >) = \min_{|\psi_{0} > \in P} C(|\psi >, |\psi_{0} >). \tag{19}
\]

For any pair of vectors \( |\psi_{1} >, |\psi_{0} > \in \mathcal{H} \) with equal particle numbers, an operator trajectory \( k(t) \) giving \( |\psi_{1} > = \xi U_{k}(1)|\psi_{0} > \) can be found even though \( K \) in Eq. (13) is restricted to nearest neighbor Hamiltonians. We show in Appendix C that the group \( G \) of all \( U_{g}(1) \) realizable as solutions to Eqs. (16a) and (16b) has as a subgroup the direct product

\[
\hat{G} = \times_{n}SU(d_{n}), \tag{20}
\]

where \( SU(d_{n}) \) acts on the subspace of \( \mathcal{H} \) with eigenvalue \( n \) of the total number operator \( N \), \( d_{n} \) is the dimension of this subspace, and the product is over the range \( 1 \leq n \leq 4 \times x_{\max} + 1 \). Missing from this range of \( n \) are only two states, the vacuum, \( n = 0 \), and the state with every site occupied by two particles, \( n = 4 \times x_{\max} + 2 \).

The restriction in Eq. (13) to nearest neighbor Hamiltonians which preserve particle number is a departure from \( K \). What is gained from the combination of these restrictions is that, as a consequence, state vectors \( |\psi > \) which carry entanglement over long distances require \( k(t) \) with many steps and thus are assigned high complexity. In Sections IV and V we consider two examples of states composed of a sum of a pair of product states, each of which individually has zero complexity, but for which the sum exhibits entanglement and as a result has complexity which is bounded both from below and from above by functions which grow linearly with the size of the region over which the entanglement extends.

IV. COMPLEXITY OF ENTANGLLED PARTICLES AT DISTANT POINTS

Let the entangled 2 particle state \( |\omega > \) be

\[
|\omega_{0} > = |1 >0 \otimes q \neq 0, n|0 > q \otimes 1 > n, \tag{21a}
\]

\[
|\omega_{1} > = |1 >0 \otimes q \neq 0, n|0 > q \otimes 1 > n, \tag{21b}
\]

\[
|\omega > = \frac{1}{\sqrt{2}} |\omega_{0} > + \frac{1}{\sqrt{2}} |\omega_{1} >. \tag{21c}
\]
for some \( n \geq 2 \). We show in Appendix A
\[
C(\ket{\omega}) \geq \frac{n\pi}{8\sqrt{2}}.
\]
(22)
The complexity of \( \ket{\omega} \) grows at least linearly with the distance between the entangled pair of particles. We obtain also in Appendix A an upper bound on \( C(\ket{\omega}) \) linear in the distance between the pairs
\[
C(\ket{\omega}) \leq (n - 1 + \frac{1}{2\sqrt{2}})\pi.
\]
(23)

V. COMPLEXITY OF AN ENTANGLED EXTENDED STATE

Although product states with extended wave functions are defined to have 0 complexity independent of the width the wave function, for entangled states with extended wave functions for one or both of the particles there is a complexity cost determined both by the distance between the states and by the width of the wave functions.

For simplicity, we present only the case with one of the two entangled particles in an extended state.

Consider a version of Eqs. (21a) - (21c) with one of the particles in an extended state
\[
|\omega_0\rangle = \vert -1\rangle \otimes \left(\frac{1}{\sqrt{r}} \sum_{n \leq x < n + r} \vert x\rangle \otimes|0\rangle \right),
\]
(24a)
\[
|\omega_1\rangle = \vert 1\rangle \otimes \left(\frac{1}{\sqrt{r}} \sum_{n \leq x < n + r} \vert x\rangle \otimes|0\rangle \right),
\]
(24b)
\[
|\omega'\rangle = \frac{1}{\sqrt{2}}|\omega_0\rangle + \frac{1}{\sqrt{2}}|\omega_1\rangle.
\]
(24c)

In Appendix B we prove
\[
C(|\omega'\rangle) \geq \frac{n\pi}{8\sqrt{2}} + \frac{r\kappa}{2\sqrt{2}},
\]
where \( \kappa \) is
\[
\kappa = \frac{1}{r} \sum_{0 < s < r} \arcsin\left(\sqrt{\frac{s}{2r}}\right).
\]
(26)

For large \( r \), \( \kappa \) approaches
\[
\lim_{r \to \infty} \kappa = \int_{0}^{1} \arcsin\left(\sqrt{\frac{x}{2}}\right) dx.
\]
(27)

The complexity of an entangled pair with one of the particles in an extended state grows at least linearly both with the distance between the pairs and with the width of the extended state.

In Appendix B we prove also
\[
C(|\omega'\rangle) \leq (n - 1 + \frac{\pi}{2\sqrt{2}})\pi + 2\lambda r,
\]
where
\[
\lambda = \frac{1}{r} \sum_{0 < s < r} \arcsin\left(\sqrt{\frac{s}{s + 1}}\right).
\]
(29)

For large \( r \) for most of the range \( 0 < s < r \), the argument of the sum in Eq. (29) is nearly \( \frac{\pi}{2} \). Therefore
\[
\lim_{r \to \infty} \lambda = \frac{\pi}{2}.
\]
(30)

VI. BRANCHING

Using the complexity measure of Section III we now propose a decomposition of a state vector into approximately classical branches.

A. Net Complexity of a Branch Decomposition

For any \( |\psi\rangle \in \mathcal{H} \) let \( |\psi\rangle = \sum_{i} |\psi_i\rangle \) be a candidate orthogonal decomposition into branches. Let \( Q(|\psi\rangle) \) be a measure of complexity of this decomposition
\[
Q(|\psi\rangle) = \sum_{i} <\psi_i|\psi> C(|\psi_i\rangle) - b \sum_{i} <\psi_i|\psi> \ln(<\psi_i|\psi>),
\]
(31)

with branching threshold \( b \) > 0. For any choice of \( b \), the branching decomposition of \( |\psi\rangle \) is defined to be the \( \{|\psi_i\rangle\} \) which minimizes \( Q(|\psi\rangle) \). The first term in Eq. (31) is the mean complexity of the branches split from \( |\psi\rangle \). But each branch can also be thought of as specifying, approximately, some macroscopic classical configuration of the world. The second term represents the entropy of this random ensemble of classical configurations.

Since \( \mathcal{H} \) in Section IIIA is finite dimensional, the search to minimize \( Q(|\psi\rangle) \) is over a compact set of possible orthogonal decompositions. Since \( Q(|\psi\rangle) \) is nonnegative, it must have at least one minimum on this set. We will assume without proof that this minimum is unique except for a set of measure 0.

For \( b \) either extremely small or extremely large, the branches which follow from Eq. (31) will look nothing like the macro reality we see. For small enough \( b \), the minimum of \( Q(|\psi_i\rangle) \) will be dominated by the complexity term. It follows from the discussion of Section III that the minimum of the complexity term will occur for a set of branches each of which is nearly a pure, unentangled multi-particle product state. Thus bound states
will be sliced up into unrecognizable fragments. On the other hand, for very large $b$, the minimum of $Q(\{\psi_i >\})$ will be dominated by the entropy term, leading to only a single branch consisting of the entire coherent quantum state. Again, unlike the world we see.

The result of all of which is that for the branches given by minimizing $Q(\{\psi_i >\})$ of Eq. (31) to have any chance of matching the macro world, $b$ has to be some finite number.

B. Time Evolution of Optimal Branch Decomposition

Suppose $Q(\{\psi_i >\})$ is minimized at each $t$ for some evolving $|\psi(t)\rangle$. Given that time evolution is continuous, the optimal branch configuration will be a piecewise continuous function of time. For an interval the optimal configuration will move continuously, then at some isolated point in time a configuration far from the optimal configuration just preceding that point in time will acquire a lower value of $Q(\{\psi_i >\})$. The optimal configuration will then jump discontinuously to the new configuration.

Consider the circumstance under which the minimization of $Q(\{\psi_i >\})$ would cause some branch $|\psi_i >$ to split into two pieces

$$|\psi_i > = |\psi_i^0 > + |\psi_i^1 >.$$  \hspace{1cm} (32)

The terms in $Q(\{\psi_i >\})$ arising from $|\psi_i >$ before the split are

$$< \psi_i | \psi_i > [C(|\psi_i >) - b \ln(< \psi_i | \psi_i >)].$$ \hspace{1cm} (33)

The terms from $|\psi_i^0 >$, $|\psi_i^1 >$ after the split can be written in the form

$$< \psi_i | \psi_i > [\rho C(|\psi_i^0 >) + (1 - \rho) C(|\psi_i^1 >) - b \rho \ln(\rho) - b(1 - \rho) \ln(1 - \rho) - b \ln(< \psi_i | \psi_i >)].$$ \hspace{1cm} (34)

where it is convenient to express $< \psi_i^0 | \psi_i^0 >$ as $\rho < \psi_i | \psi_i >$. Thus a split will occur if

$$C(|\psi_i >) - \rho C(|\psi_i^0 >) - (1 - \rho) C(|\psi_i^1 >) > - b \rho \ln(\rho) - b(1 - \rho) \ln(1 - \rho).$$ \hspace{1cm} (35)

The condition for a split is a saving in complexity by an amount linear in $b$. Splitting occurs as soon as a certain threshold amount of complexity can be saved by the split. The largest possible value of this threshold is for $\rho$ of $\frac{1}{2}$, in which case it is given by $\ln(2)b$. Correspondingly no 2-way split will ever be accompanied by a complexity drop of more than $\ln(2)b$. Averaged over $\rho$, the complexity drop required by Eq. (35) is $\frac{b}{2}$.

A split can also reverse itself if as a result of time evolution the complexity of some branch changes sufficient to reverse the inequality in Eq. (35). It is also possible in principle for a set of some $n$ branches jointly to rearrange into a new configuration of $n+1$ branches for an accompanying saving of up to $\ln(n+1)b$. Both of these complications will be removed in Section VII as a by product of the hypothesis that branch formation minimizing $Q(\{\psi_i >\})$ occurs only at a single asymptotically late time. This hypothesis we introduce primarily to try to avoid a potential problem with Lorentz covariance of branching at any fixed finite time.

The hypothesis that branching occurs only at a single asymptotically late time, however, adds a complication to measurement of $b$ by observation at finite time. We will briefly return to this subject in Section X.

VII. DETERMINISTIC QUANTUM MECHANICS

So far, we have defined complexity and branching only for a lattice field theory. But given that the definition relies only of a field theory at a single instant of time and is thus independent of the system’s dynamics, it seems a plausible guess that the continuum limit of this construction should exit. The subject of the continuum limit of $Q(\{\psi_i >\})$ we will return to elsewhere.

A. Lorentz Covariance Problem for Branching

Assuming this goes through without a hitch, however, there is still a problem with the Lorentz transformation behavior of the branch construction rule. A discussion of this issue in a slightly different setting and a solution somewhat related to the one we will propose appear in [1].

The problem is actually a version of the original EPR paradox. In two different Lorentz frames, starting from initial states vectors related by a Lorentz transform, assume Hamiltonian time evolution and, in each frame, assume branching which minimizes the net complexity function $Q(\{\psi_i >\})$ in Eq. (31). For some period of time assume the branching in each frame is also related by a Lorentz transform. But then at a pair of points separated by a spacialike interval, suppose processes occur each of which, by itself, is sufficient to cause splitting of a branch the two events share. Assume in addition, that in one frame one of these events occurs first but, in the other frame, as a consequence of the Lorentz transform of their spacialike separation, the other event occurs first. The result will be that in the time interval between the events, the branch structure seen by the two different Lorentz frames will be different. But our goal is to be able to interpret branch state vectors as the underlying substance of reality. That interpretation clearly is not possible if branch structure is different according to different Lorentz observers.

This variant of the EPR paradox, in only slightly different clothing, we already briefly mentioned in Section XI and is a general problem for any formulation of branches.
as the substance of reality and not only a problem for branches defined by minimization of $Q(|\psi_i\rangle)$.

### B. Infinite Time Limit of Branching

What to do? A possible approach is suggested by the observation that for any distinct pair of Lorentz frames, for some $t_{out}$ sufficiently late, all interactions which cause permanent branch splitting will have finished in both frames. Thus, at least for that pair of frames, the difficulty in Section VII A might potentially be avoided.

In any frame, for $t_{out}$ sufficiently late, the corresponding state $|\psi_{out}\rangle$ will consist of individual particles and bound systems, possibly including some collection of chunks of solids, spread out over large distances, all moving away from each other. Whatever entanglement may have arisen from interactions, however, will remain recorded in the state’s structure. An example of this will be considered in Section VIII.

As a first step toward Lorentz covariance, we now add to Eq. (31) the requirement that for each $|\psi_i\rangle$, the complexity $C(|\psi_i\rangle)$ be evaluated in the rest frame of $|\psi_i\rangle$.

We now minimize $Q\{\{|\psi_{out,i}\rangle\}$ to find a corresponding branch decomposition $\{\{|\psi_{out,i}\rangle\}$ of $|\psi_{out}\rangle$, translate the result back to some some fixed $t_{in}$ to define a decomposition $\{\{|\psi_{in,i}\rangle\}$ of $|\psi_{in}\rangle$

$$|\psi_{in,i}\rangle = \exp[i(t_{out} - t_{in})H]|\psi_{out,i}\rangle, \quad (36)$$

and take the limit of $t_{out} \to \infty$. This limit is needed to insure that $t_{out}$ is after the last persistent branching event for any state in any frame. It is also required to avoid some small but nonzero dependence of the $\{|\psi_{in,i}\rangle\}$ ensemble on the choice of $t_{out}$. Any branches which may have formed but then recombined will not be included in $\{|\psi_{out,i}\rangle\}$. Only those which survived will appear.

Our conjecture, the first of the two mentioned in Section III, is that for a pair of frames related by some Lorentz transform $L$, branches in the set $\{|\psi_{in,i}\rangle\}$ found in one frame and in the set found in the other frame $\{|\psi_{in,i}'\rangle\}$ can be arranged in corresponding pairs related a unitary operator $U(L)$ determined by $L$

$$|\psi_{in,i}'\rangle = U(L)|\psi_{in,i}\rangle. \quad (37)$$

In any particular Lorentz frame the state of the real world at $t_{in}$, we now take as a random draw from the ensemble $\{|\psi_{in,i}\rangle\}$, each weighted with the corresponding probability $<\psi_{in,i}|\psi_{in,i}\rangle$. For any time other than $t_{in}$, the state chosen is evolved continuously in time with no further branching

$$|\psi_i(t)\rangle = \exp[-i(t - t_{in})H]|\psi_{in,i}\rangle. \quad (38)$$

The branches of a state which form at some event can be thought of as recording different values of a corresponding meter. An accumulating sequence of persistent branching results can therefore be viewed also as an accumulating sequence of persistent records. We now introduce the second of the two conjectures mentioned in Section VII which is that possible histories of macroscopic reality are accumulating sequences of those branching results which persist through time. Each final branch $|\psi_{out,i}\rangle$ will record one such macroscopic history. Each corresponding $|\psi_{in,i}\rangle$, evolved through time, will then reproduce a corresponding single history of macroscopic branch choices.

In a world which obeys purely classical mechanics, the configuration of the world on any time slice can be used to determine the configuration on all past time slices and, in effect, is a persistent record of all past configurations. The hypothesis that the macroscopic part of the quantum mechanical world is the accumulating set of persistent branches can be viewed as a kind of extension of what holds in a purely classical world.

As mentioned in passing in Section III the formation of permanent records is also an identifier of quantum degrees of freedom which become classical degrees of freedom according to environmentally induced decoherence in [4].

Each final state, at some late time after all branching is done, occurs with same probability as the corresponding initial state. But the probability of the initial state is the probability the Born rule assigns to the full set of branch results which the final state acquired over history. It follows that for any experiment which produces macroscopic results and thus, according to our hypothesis permanent records, the present theory will assign the same probability to each possible result as does the Born rule.

Some randomly chosen $|\psi_i(t)\rangle$ becomes what the real world is made out of. We obtain a kind of hidden variable theory, with the hidden variables present in the initial $|\psi_{in,i}\rangle$. They emerge in macroscopic reality only over time through their influence on the sequence of branching results unfolding as $t$ progresses.

### VIII. STERN-GERLACH EXPERIMENT

We now apply the theory of Section VII to a simplified model of a Stern-Gerlach device. In addition to a particle interacting with a magnetic field, we will need some degree of freedom to serve as the environment entangled with the experimental results and thereby causing branching. A second particle will serve that purpose.

#### A. Two Dimensional Lattice Fermions

We assume a 2-dimensional lattice in place of the 1-dimensional lattice of Section III. The state space for the system $\mathcal{H}$ becomes a tensor product of fermion spaces $\mathcal{H}_z$ one for each lattice point $(z_x, z_y)$

$$\mathcal{H} = \otimes_z \mathcal{H}_z. \quad (39)$$
As in Section III, an orthogonal basis for each $H_z$ is the vacuum $|0\rangle$ and 1-particle states $|1\rangle, |2\rangle$. The overall system vacuum $|0\rangle$ is the product $\otimes_z|0\rangle$.

To adapt the definition of complexity in Section III to hermitian operators on $H_{z_0} \otimes H_{z_1}$ for $z_0$ and $z_1$ with $z_{0x}$ and $z_{1x}$ nearest neighbors, we include also operators for which $z_{0y}$ and $z_{1y}$ are nearest neighbors. The remainder of Section III and Section VI can be adapted similarly. The linear lower bounds of Appendices A and B can be adapted by interpreting the present 2-dimensional lattice of spaces $H_{(z_x,z_y)}$ as a 1-dimensional lattice of spaces

$$H_{z_y} = \otimes_x H_{(z_x,z_y)}.$$  (40)

It is convenient to introduce a lattice spacing $d$ with respect to other length scales we will consider. Except for instants when the Stern-Gerlach device turns on a magnetic field, we assume a non-interacting Hamiltonian which, acting on any state, gives the sum of the non-relativistic kinetic energies of any particles in that state.

For momentum vector $k = (k_x, k_y)$, let $\psi(k, z_{in}, z, t)$ be a Gaussian wave function with $z_x$ and $z_y$ coordinate means

$$\bar{z}_x(t) = \int dz_x \int dz_y |z_x| \psi(k, z_{in}, z, t)|^2,$$  (41a)

$$\bar{z}_y(t) = \int dz_x \int dz_y |z_y| \psi(k, z_{in}, z, t)|^2,$$  (41b)

$$\bar{z}_x(t) = \frac{k_x t}{m} + z_{inx},$$  (41c)

$$\bar{z}_y(t) = \frac{k_y t}{m} + z_{iny},$$  (41d)

and coordinate dispersions

$$d_{z_x}(t)^2 = \int dz_x \int dz_y |z_x - \bar{z}_x(t)|^2 |\psi(k, z_{in}, z, t)|^2,$$  (42a)

$$d_{z_y}(t)^2 = \int dz_x \int dz_y |z_y - \bar{z}_y(t)|^2 |\psi(k, z_{in}, z, t)|^2,$$  (42b)

$$d_{x}(t)^2 = \frac{t^2}{4d^2m^2} + d^2,$$  (42c)

$$d_{y}(t)^2 = \frac{t^2}{4d^2m^2} + d^2.$$  (42d)

Here $m$ is the particle mass, $d$ is chosen to set the initial state dispersion, and we are using continuum expressions for the means and dispersions in place of their more complicated discrete versions.

Define the 2-particle state $|z_0, s_0, z_1, s_1\rangle$ to be

$$|z_0, s_0, z_1, s_1\rangle = |s_0\rangle \otimes |s_1\rangle \otimes z_{\neq z_0,z_1}|0\rangle.$$  (43)

and let $|k_0, z_{in0}, s_0, k_1, z_{in1}, t\rangle$ be

$$|k_0, z_{in0}, s_0, k_1, z_{in1}, t\rangle = \sum_{z_{in1}} a^t \psi(k_0, z_{in0}, 0, t) \psi(k_1, z_{in1}, z_1, t) \otimes |z_0, s_0, z_1, s_1\rangle.$$  (44)

**B. Initial State**

With these various definitions in place, assume a 2-particle initial state at $t_{in}$ of

$$|\psi, t_{in}\rangle = \frac{1}{\sqrt{2}} |k_0, z_{in0}, 1, k_1, z_{in1}, -1, t_{in}\rangle - \frac{1}{\sqrt{2}} |k_0, z_{in0}, -1, k_1, z_{in1}, 1, t_{in}\rangle,$$  (45)

with momenta and initial positions

$$k_0 = (q, 0),$$  (46a)

$$k_1 = (-q, 0),$$  (46b)

$$z_{in0} = (w, 0),$$  (46c)

$$z_{in1} = (-w, 0).$$  (46d)

This state is entangled at birth. Depending on the size of $b$ in relation to the position $w$ and the dispersion $d$, it follows from Sections III and VI that the optimal branch configuration may already split the state. Since the spin state is rotationally invariant, however, there are actually a family of branch configurations all of which would yield the same value for $Q(|\psi_i\rangle)$. By breaking rotational symmetry by some arbitrarily small amount, a particular direction of branching can be favored. However, whatever branching occurs at $t_{in}$ will recombine and be replaced by a new pattern when the magnetic field is applied, the result of which will be a single branch configuration with far lower $Q(|\psi_i\rangle)$ than any other choice.

**C. Scattering Event**

Now suppose at some later time $t_1$ a term is added to the Hamiltonian for a very short time interval causing states of particle 0 with spin 1 suddenly to acquire a momentum component $k_{0y}$ of $r$ andcausing states with spin -1 suddenly to acquire a momentum component $k_{0y}$ of $-r$. For $t > t_1$ the state becomes

$$|\psi, t\rangle = \frac{1}{\sqrt{2}} |k_{0+}, z_{in0+}, 1, k_1, z_{in1}, -1, t\rangle - \frac{1}{\sqrt{2}} |k_{0-}, z_{in0-}, -1, k_1, z_{in1}, 1, t\rangle.$$  (47)
with the new values
\[
\begin{align*}
k_{0+} &= (q, r), \\
k_{0-} &= (q, -r), \\
z_{in0+} &= (w, -\frac{rf_1}{m}), \\
z_{in0-} &= (w, \frac{rf_1}{m}).
\end{align*}
\]

This state has both entanglement between particles 0 and 1, and entanglement between the spin 1 and spin -1 states of particle 1. But it follows from Eqs. (41a-41d) that the spin 1 and spin -1 components of particle 0 will separate in the y direction with distance growing as $\frac{2rt}{m}$. The dispersion in this separation will grow as $\frac{dm}{2rt^{2}}$. If $r > \frac{1}{2\sqrt{2d}}$ the two spin components of particle 0 will completely separate as $t$ grows, and will therefore, if left entangled in a single branch, contribute complexity larger than any finite $h$. The condition on $r$ for this to happen is simply that the $y$ momentum delivered at time $t_1$ be larger than the wavefunction's $y$ momentum dispersion. This should certainly be fulfilled for any macroscopic Stern-Gerlach device.

D. Final State Ensemble, Initial State Ensemble

The end result is that at some large $t_{out}$, the ensemble of branches which minimize $Q$ will asymptotically approach
\[
|\psi, t_{out} >_{0} = |k_{0+}, z_{in0+}, 1, k_1, z_{in1}, -1, t_{out} >.
\]
\[
|\psi, t_{out} >_{1} = -|k_{0-}, z_{in0-}, -1, k_1, z_{in1}, 1, t_{out} >.
\]

Each branch will have weight $\frac{1}{2}$. In other words the Stern-Gerlach device will yield spin 1 and -1 each with probability $\frac{1}{2}$. Which is the expected result for this particular Stern-Gerlach model.

Reversing time evolution on this ensemble back to $t_{in}$ to form an initial state ensemble, then taking the limit of the branch decomposition time $t_{out} \to \infty$ gives
\[
|\psi, t_{in} >_{0} = |k_0, z_{in0}, 1, k_1, z_{in1}, -1, t_{in} >.
\]
\[
|\psi, t_{in} >_{1} = -|k_0, z_{in0}, -1, k_1, z_{in1}, 1, t_{in} >.
\]

Each of which again has weight $\frac{1}{2}$. The center of the wave packet of the first of these states, at $t_1$ will move purely in the positive $y$ direction. The center of the wave packet for the second will move purely in the negative $y$ direction.

Thus each of the initial states, from the perspective of an approximate macroscopic description, will show classical behavior. The initial state vectors, in effect, carry hidden degrees of freedom determining the result of the Stern-Gerlach experiment, but invisible to the macroscopic world until a magnetic field is applied at time $t_1$.

Unlike the account of the experiment provided by environmentally induced decoherence by itself, however, the account provided by unitary time evolution of the initial state vectors applies through the entire process. There is no missing beat before something real emerges. The macroscopic description of the process remains approximate. But underlying the approximate macro description, the microscopic substance of reality is in place the whole time.

The split into branches of the state $|\psi, t_{out} >$ at some large $t_{out} > t_1$ in Eq. (47), done without the limit $t_{out} \to \infty$, would yield a $t_{in}$ ensemble differing from the limiting states in Eqs. (49b, 50a) by some tiny amount depending on $t_{out}$. Taking the splitting time $t_{out} \to \infty$, however, was already imposed by the goal of obtaining a branch ensemble with simple properties under Lorentz transformation.

Although in the present simplified Stern-Gerlach model complexity sufficient to cause branching of the final state arises from entanglement of the scattered wave packets across an increasing distance, this is probably not how branching would occur in a more realistic model. A more realistic model would include a large set of additional records of the scattering results copied into the environment beyond the scattered particles [3, 4, 6]. The complexity arising from this set of records would then drive branching.

IX. BELL'S THEOREM

We now consider the application of Bell’s theorem to the deterministic formulation of quantum mechanics in Section VII. Bell’s theorem asserts, essentially, that a particular class of local hidden variable theories, applied to the measurement of the correlation of a pair of spin-$\frac{1}{2}$ particles in a total angular momentum 0 state necessarily yields predictions in contradiction to the predictions of standard quantum mechanics. But the theory of Section VII is not local in the sense required for Bell’s theorem. Also, a key element of Bell’s theorem is for the spin measurements entering the correlation to be performed with spacelike separation according to spin direction choices also made at a spacelike separation. The theory of Section VII, however, allows no freedom in setting up spin measurement devices. Anything that occurs over history in this theory must be programmed into the initial state vector at time $t_{in}$. Despite these caveats, it may still be useful to show that the theory of Section VII reproduces the correlations expected of standard quantum mechanics in a Bell’s theorem experiment. Which we will now do.
A. Four Particle Initial State

We return to the 1-dimensional lattice fermion theory of Section III. We assume 4 particles in the initial state. Two to form the correlated angular momentum 0 state and two more to serve as measuring devices to record the correlated spins.

As in Section VIII we introduce a value of lattice spacing $\Delta$ small with respect to other length scales we will consider. We assume a Hamiltonian given by the sum of kinetic energies of particles in that state, along with a pair of infinite potential barriers details soon to be specified.

For momentum $k$, let $\psi(k, x, t)$ be a Gaussian wave function with coordinate means and dispersions given by the 1-dimensional version of Eqs. (4). Define the 4-particle state $|k_0, x_{i0}, s_0, ... k_3, x_{i3}, s_3, t >$ to be the 1-dimensional 4-particle analog of the state in Eqs. (43-44).

We assume a 4-particle initial state at $t_0$ of

$$|\psi, t_0 \rangle = \sum |s_0, s_1 \rangle \psi_0(s_0, s_1) \psi_2(s_2) \psi_3(s_3) \times$$

$$|k_0, x_{i0}, s_0, ... k_3, x_{i3}, s_3, t_0 \rangle \geq (51)$$

with spin wave functions

$$\psi_0(1, 1) = 0,$$  \hspace{1cm} (52a)

$$\psi_0(1, -1) = \frac{1}{\sqrt{2}},$$  \hspace{1cm} (52b)

$$\psi_0(-1, 1) = -\frac{1}{\sqrt{2}},$$  \hspace{1cm} (52c)

$$\psi_0(1, 1) = 0,$$  \hspace{1cm} (52d)

$$\psi_2(1) = 1,$$  \hspace{1cm} (52e)

$$\psi_2(-1) = 0,$$  \hspace{1cm} (52f)

$$\psi_3(1) = \cos(\theta/2),$$  \hspace{1cm} (52g)

$$\psi_3(-1) = \sin(\theta/2)$$  \hspace{1cm} (52h)

and momenta and initial positions

$$k_0 = q,$$  \hspace{1cm} (53a)

$$k_1 = -q,$$  \hspace{1cm} (53b)

$$k_2 = -q,$$  \hspace{1cm} (53c)

$$k_3 = q,$$  \hspace{1cm} (53d)

$$x_{i0} = w,$$  \hspace{1cm} (53e)

$$x_{i1} = -w,$$  \hspace{1cm} (53f)

$$x_{i2} = 3w,$$  \hspace{1cm} (53g)

$$x_{i3} = -3w.$$  \hspace{1cm} (53h)

We assume the momentum $q$ is much larger than the momentum dispersion $\frac{1}{\sqrt{4\pi}}$ in Eq. (12), and the initial positions $w, 3w$ are much larger than the position dispersion $d$ in Eq. (12).

As a consequence of the choice of initial positions and momenta, particles 0 and 2 will collide at $x = 2w$ at time $\frac{mw}{q}$, where $m$ is the particle mass. At the same time particles 1 and 3 will collide at $x = -2w$.

B. Two Scattering Events

For the collision between 0 and 2, we assume an infinite potential barrier is encountered for the components of each with spin 1 and no interaction occurs for other combinations of components. The spin 1 component of 0 is thus scattered back with momentum $-q$ and the corresponding component of particle 2 is scattered forward with momentum $q$. Other combinations of spin components pass through unaffected. Thus the final momentum of particle 2 measures the spin 1 component of particle 0.

We assume a corresponding event for collision of particles 1 and 3 at position $x$ of $-2w$ but with scattering for each particle in a state with spin wave function $[\cos(\theta/2), \sin(\theta/2)]$ and transparency for the orthogonal state with spin wave function $[-\sin(\theta/2), \cos(\theta/2)]$. The final momentum of particle 3 thus measures the component of particle 1 with spin wave function $[\cos(\theta/2), \sin(\theta/2)]$.

We assume that the potential barriers are then turned off for $t$ beyond the collision time $\frac{mw}{q}$ by a margin of, say, $\frac{mw}{q}$ to avoid additional unwanted scattering events.

As a result of the collision between particles 0 and 2 and the collision between particles 1 and 3, the states of each these pairs become entangled. And as was the case for the entangled states of the Stern-Gerlach experiment of Section VIII the consequence of these entanglements is that the total state complexity will then rise linearly with $t$. For any finite value of $b$, for some large $t_{out}$, the minimum of $Q$ will occur for a set of 4 branches $|e_0, e_1, t_{out} >$. The index $e_0$ takes values of u(p) or d(own) specifying whether the scattering of particles 0 and 2 gives a value of 1 or -1 for the spin of particle 0. The index $e_1$ takes values of u(p) or d(own) specifying whether the scattering of particles 1 and 3 gives a spin state for particle 1 with wave function $[\cos(\theta/2), \sin(\theta/2)]$ or the orthogonal state with spin wave function $[-\sin(\theta/2), \cos(\theta/2)]$. The branch states are given by

$$|e_0, e_1, t_{out} > = \eta_{e_0 e_1} \sum_{s} \psi_{e_0 e_1} (s_0, ... s_3) \times$$

$$|k_{0e_0}, x_{i0e_0}, s_0, ... k_{3e_1}, x_{i3e_1}, s_3, t_{out} > \geq (54)$$

where $|k_0, x_{i0}, s_0, ... k_3, x_{i3}, s_3, t_{out} >$ is the 1-dimensional 4-particle analog of the state in Eqs. (13-14). The spin wave function in Eq. (54) is

$$\psi_{e_0 e_1} (s_0, ... s_3) = \psi_{e_00} (s_0) \psi_{e_11} (s_1) \psi_2 (s_2) \psi_3 (s_3),$$  \hspace{1cm} (55)

where $\psi_2 (s_2)$ and $\psi_3 (s_3)$ are the same as given in Eqs. (52e, 52h), $\psi_{e_00} (s_0)$ and $\psi_{e_11} (s_1)$ are the same as $\psi_2 (s_0)$ and $\psi_3 (s_1)$, respectively, and $\psi_{e_01} (s_0)$ and $\psi_{e_10} (s_1)$ are
the vectors orthogonal to \( \psi_{d0}(s_0) \) and \( \psi_{u1}(s_1) \), respectively. The momenta \( k_{d0}, ..., k_{d3} \) in Eq. (14) are the same as \( k_0, ..., k_3 \) in Eqs. (53a - 53d) while the momenta \( k_{u0}, ..., k_{u3} \) are \( k_{d0}, ..., k_{d3} \), respectively, with signs flipped. The initial positions \( x_{i_{\text{in}0}}, ..., x_{i_{\text{in}13}} \) have values which, according Eq. (11c), put the mean positions \( \bar{x}_0(t), ..., \bar{x}_3(t) \) at the corresponding collision points at \( t \) of 4 NC. Finally, the normalizations \( \eta_{i_{\text{in}e}} \) are such that the weights of these 4 branches become

\[
< u, u, t | u, u, t > = \frac{1}{2} \sin^2 \left( \frac{\theta}{2} \right), \tag{56a}
\]
\[
< u, d, t | u, d, t > = \frac{1}{2} \cos^2 \left( \frac{\theta}{2} \right), \tag{56b}
\]
\[
< d, u, t | d, u, t > = \frac{1}{2} \cos^2 \left( \frac{\theta}{2} \right), \tag{56c}
\]
\[
< d, d, t | d, d, t > = \frac{1}{2} \sin^2 \left( \frac{\theta}{2} \right). \tag{56d}
\]

### C. Ensemble of Replicas

Now replicate this prototype experiment \( N \) times over displaced sufficiently in space and time for distinct copies not to interfere. For this combined system at large \( t_{\text{out}} \), the minimal value of \( Q \) will occur for a set of \( 4^N \) branches \( |e_0^0, e_1^0, ..., e_0^{N-1}, e_1^{N-1}, t_{\text{out}} > \) with weights

\[
< 0, 0, ..., 0, 1, ..., 0 | 0, 1, ..., 0, 0, ..., 0, t_{\text{out}} > = \frac{1}{2} \sin^2 \left( \frac{\theta}{2} \right) \sum_i a_i \left[ \frac{1}{2} \cos^2 \left( \frac{\theta}{2} \right) \right] \sum_i d_i, \tag{57}
\]

where \( a(\text{agree})_i \) is 1 for \( i \) for which \( e_0^i \) and \( e_1^i \) are \( u \) and \( u \) or \( d \) and \( d \), respectively, and 0 otherwise, and \( d(\text{disagree})_i \) is 1 for \( i \) for which \( e_0^i \) and \( e_1^i \) are \( u \) and \( d \) or \( d \) and \( u \), respectively, and 0 otherwise.

For a branch randomly chosen from the ensemble in Eq. (57), consider the average of \( a_i - d_i \) over its copies of the experiment. The result will be that branch’s estimate for the average of the product of observed values of the two spins. It follows from Eq. (57) that the result may be viewed as the average of \( N \) independent choices of a random variable that takes the value 1 with probability \( \sin^2 \left( \frac{\theta}{2} \right) \) and takes the value -1 with probability \( \cos^2 \left( \frac{\theta}{2} \right) \).

By the central limit theorem, for large \( N \) we expect any randomly chosen branch to give nearly the result

\[
\frac{1}{N} \sum_i (a_i - d_i) = \sin^2 \left( \frac{\theta}{2} \right) - \cos^2 \left( \frac{\theta}{2} \right) = -\cos(\theta), \tag{58a}
\]

\[
= -\cos(\theta), \tag{58b}
\]

which is the expected result from standard quantum mechanics. This is the result which Bell’s theorem rules out for its class of local hidden variable theories.

### X. Conclusion

In Section [11] we argued that the branching which follows from environmentally induced decoherence by itself looks like its missing something. The conjectures in Section [VII B] propose to fill in what’s missing.

Multiple questions follow. What more can be said to support or refute the hypothesis that macroscopic reality consists of the accumulating set of persistent branching results? What are the consequences, or possible ways to handle, the limitation that \( t_{\text{out}} \rightarrow \infty \) in Section [VII] ignores a cosmological time scale? Does the lattice definition of net complexity in Section [V IA] or something close to it, have a continuum limit, and if it does, is the branch structure which it gives rise to Lorentz covariant in the sense proposed in Section [VII B]? Would the continuum limit differ for Hermitian operators entering Eq. (13) allowed to span more than 2 lattice sites? Are there other constants which might show up in addition to \( b \) if more complicated choices of operators are allowed in Eq. (13)?

At least for free lattice field theories and for simple interacting theories, it seems some of these questions might be answerable. There is now a large amount of mathematically rigorous work on the continuum limits of lattice field theories which might be drawn on. Generally, what has emerged is that an infinite range of possible lattice theories all reduce to some much smaller set of continuum theories often specified by only a finite dimensional space of parameters.

Finally, how could \( b \) be measured? From multiple non-interacting copies of some single system, as in Section [IX] the mean value of an observed complexity, via Eq. (11), might be used to find \( b \). Among the problems with this proposal, however, are that it needs to be done in the limit \( t_{\text{out}} \rightarrow \infty \).

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Appendix A: Complexity of Entangled Particles at Distant Points

The proof of Eq. (22) proceeds as follows. The trajectories \( k(t) \in K \) and \( U_k(t) \) which determine any \( C(\omega, |\psi\rangle) \), according to Eqs. (16a) - (18), we characterize by a corresponding set of trajectories of Schmidt spectrum vectors. We then find the rotation matrices which govern the motion of these vectors as \( t \) changes. A bound on the time integral of the angles which occur in these matrices by a time integral of \( \| k(t) \| \) yields Eq. (22).

1. Schmidt Spectra

For a trajectory \( k(t) \in K \), let \( U_k(t) \) be the solution to Eqs. (16a) and (16b). Define \( |\psi(t)\rangle > \)

\[
|\psi(t)\rangle = U_k(t)|\psi\rangle >, \quad (A1)
\]

for a 2 particle product state \( |\psi\rangle > \), and assume that \( k(t) \) has been chosen to give

\[
\xi|\psi(1)\rangle := |\omega\rangle >, \quad (A2)
\]

for some complex \( \xi \) with \( |\xi| = 1 \).

For any \( p \) in the range \( 0 \leq p < n \), split the state space \( \mathcal{H} \) into the product

\[
Q_p = \otimes_{q \leq p} \mathcal{H}_q, \quad (A3a)
\]

\[
R_p = \otimes_{q > p} \mathcal{H}_q, \quad (A3b)
\]

\[
\mathcal{H} = Q_p \otimes R_p, \quad (A3c)
\]

Since the number operators \( N_{Q_p} \), on \( Q_p \) and \( N_{R_p} \) on \( R_p \) commute, \( |\psi(t)\rangle > \) can be decomposed into a sum of terms each of which is an eigenvector of both operators. The \( N_{Q_p}, N_{R_p} \) eigenvalue pairs which occur must sum to 2 since \( |\psi(t)\rangle > \) is an eigenvector of the total particle number operator \( N \) with eigenvalue 2. The sum of the \( Q_p \otimes R_p \) Schmidt decompositions of \( N_{Q_p}, N_{R_p} \) eigenvectors then gives a Schmidt decomposition of \( |\psi(t)\rangle > \), which can be arranged in the form

\[
|\psi(t)\rangle = \sum_i s_{ip}(t)|\phi_{ip}(t)\rangle > |\chi_{ip}(t)\rangle >, \quad (A4a)
\]

\[
N_{Q_p}|\phi_{ip}(t)\rangle > = 0, \quad (A4b)
\]

\[
N_{R_p}|\chi_{ip}(t)\rangle > = 2|\chi_{ip}(t)\rangle >, \quad (A4c)
\]

\[
N_{Q_p}|\phi_{ip}(t)\rangle > = 2|\phi_{ip}(t)\rangle >, \quad (A4d)
\]

\[
N_{R_p}|\chi_{ip}(t)\rangle > = 0, \quad (A4e)
\]

\[
N_{Q_p}|\phi_{ip}(t)\rangle > = |\phi_{ip}(t)\rangle >, i \geq 2, \quad (A4f)
\]

\[
N_{R_p}|\chi_{ip}(t)\rangle > = |\chi_{ip}(t)\rangle >, i \geq 2, \quad (A4g)
\]

with \( |\phi_{ip} \rangle < Q_p, |\chi_{ip} \rangle < R_p \), all Schmidt spectrum values \( s_{ip}(t) \) real nonnegative, and the \( s_{ip}(t) \) in decreasing order for \( i \geq 2 \). The normalization of \( |\psi(t)\rangle > \) implies

\[
\sum_i s_{ip}(t)^2 = 1. \quad (A5)
\]

As \( t \) goes from 0 to 1, the Schmidt spectrum vector \( [s_{0p}(t), s_{1p}(t), ...] \) follows a smooth trajectory across a (high dimensional) unit sphere. A lower bound can be placed on the total angular rotation of \( [s_{0p}(t), s_{1p}(t), ...] \).

The initial Schmidt vector \( [s_{00}(0), s_{10}(0), ...] \), since it is from a 2 particle product state \( |\psi(0)\rangle = |\psi\rangle > \), has the form \( [s_{00}(0), s_{10}(0), s_{20}(0), 0, ...] \) with at most 3 nonzero entries. The final Schmidt vector \( [s_{00}(1), s_{10}(1), ...] \), according to Eqs. (21a) - (21d), has the form \( [0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ...] \) with exactly 2 nonzero entries. Thus over the interval \( t \) from 0 to 1, \( s_{00}(t), s_{10}(t), ... \) must go through a sequence of rotations with a total angle of at least \( \frac{\pi}{4} \). For the small time interval from \( t \) to \( t + \delta t \), let \( u_{ip}(t) \) and \( \theta_p(t) \) be

\[
s_{ip}(t + \delta t) = s_{ip}(t) + \delta t u_{ip}(t), \quad (A6a)
\]

\[
|\theta_p(t)|^2 = \sum_i |u_{ip}(t)|^2. \quad (A6b)
\]

We then have

\[
\sum_{0 \leq p < n} \int_0^1 |\theta_p(t)| dt \geq \frac{n\pi}{4}. \quad (A7)
\]

2. Schmidt Rotation Matrix

The rotation of \( [s_{00}(t), s_{10}(t), ...] \) during the interval \( t \) to \( t + \delta t \) will be determined by \( k(t) \). In Eq. (13), the only term which will give rise to a nonzero value for \( \theta_p(t) \) is \( \dot{g}_{p}(p+1)(t) \). All other terms in Eq. (13) will either contribute a unitary transformation on \( Q_p \) and identity on \( R_p \) or an identity on \( Q_p \) and unitary on \( R_p \). Either of these two alternatives will leave \( s_{00}(t), s_{10}(t), ... \) unchanged. The effect of \( \dot{g}_{p}(p+1)(t) \) on \( s_{00}(t), s_{10}(t), ... \) over the interval \( t \) to \( t + \delta t \) can therefore be determined from the simplification

\[
|\psi(t + \delta t)\rangle = \exp[i\delta t \dot{g}_{p}(p+1)(t)]|\psi(t)\rangle >. \quad (A8)
\]
From $|\psi(t + \delta t)\rangle$ of Eq. (A8), construct the density operator $\rho_p(t + \delta t)$ by a partial trace over $R_p$, for which it is convenient to use the basis for $R_p$ taken from the Schmidt decomposition of $|\psi(t)\rangle$ at the beginning of the time interval

$$\rho_p(t + \delta t) = \sum_i <\chi_ip|\psi(t + \delta t)> <\psi(t + \delta t)|\chi_ip>.$$ (A9)

An eigenvector decomposition of $\rho_p(t + \delta t)$ exposes the vector $[s_{ip}(t + \delta t), s_{ip}(t + \delta t), ...]$

$$\rho_p(t + \delta t) = \sum_i [s_{ip}(t + \delta t)]^2 |\phi_{ip}(t + \delta t)\rangle <\phi_{ip}(t + \delta t)|.$$ (A10)

A bound state perturbation expansion to leading order in small $\delta t$ applied to Eqs. (A8), (A9) and (A10) then gives for $u_{ip}(t)$ of Eq. (A6a)

$$u_{ip}(t) = \sum_j r_{ijp}(t)s_{jp}(t),$$ (A11)

for the rotation matrix $r_{ijp}(t)$

$$r_{ijp}(t) = -\text{Im}(\phi_{ip}(t) <\chi_{jp}(t)|\chi_{jp}(t)>).$$ (A12)

3. Rotation Angle Bounds

Since $\hat{g}_{p(p+1)}(t)$ preserves $N_p + N_{p+1}$, it is useful to decompose it into a linear combination of an operator $\hat{z}_{p(p+1)}^0(t)$ which, acting on $N_p + N_{p+1}$ eigenvectors, gives nonzero results only for eigenvalue 0, and an operator $\hat{z}_{p(p+1)}^1(t)$ which, acting on $N_p + N_{p+1}$ eigenvectors, gives nonzero results only for $N_p + N_{p+1}$ eigenvalues other than 0

$$\hat{g}_{p(p+1)}(t) = a_{p(p+1)}^0(t)\hat{z}_{p(p+1)}^0(t) + a_{p(p+1)}^1(t)\hat{z}_{p(p+1)}^1(t).$$ (A13)

Both operators we assume normalized to 1

$$\|\hat{z}_{p(p+1)}^i(t)\| = 1.$$ (A14)

It follows that $\hat{z}_{p(p+1)}^0(t)$ is constant over $t$ and given by the projection operator

$$\hat{z}_{p(p+1)}^0(t) = |0 >_p |0 >_{p+1} <0|_p <0|_{p+1} \otimes q\neq p,p+1 I_q.$$ (A15)

Combining Eqs. (A6b), (A11) - (A13) gives

$$|\theta_p^0(t)|^2 \leq |\theta_p^0(t)| + |\theta_p^1(t)|,$$ (A16a)

$$|\theta_p^0(t)|^2 = \sum_j |u_{jp}^1(t)|^2,$$ (A16b)

with the definition

$$u_{jp}^1(t) = -a_{p(p+1)}^1(t)\sum_k \text{Im}(\phi_{jp}(t)|<\chi_{jp}(t)| \hat{z}_{p(p+1)}^1(t)|\phi_{kp}(t) > |\chi_{kp}(t) > s_{kp}(t)).$$ (A17)

For $u_{jp}^0(t)$ we then have

$$u_{jp}^0(t) = a_{p(p+1)}^i(t)\sum_k \text{Im}(\phi_{jp}(t)|<\chi_{jp}(t)| \hat{z}_{p(p+1)}^0(t)|\phi_{kp}(t) > |\chi_{kp}(t) > s_{kp}(t)).$$ (A18)

since the set of $|\phi_{ip}(t) > |\chi_{jp}(t) >$ is an orthonormal basis for $H$ and therefore

$$\text{Im}(\phi_{jp}(t)|<\chi_{jp}(t)| I|\phi_{kp}(t) > |\chi_{kp}(t) > = 0.$$ (A19)

But in addition

$$|\psi(t) > = \sum_k |\phi_{kp}(t) > |\chi_{kp}(t) > s_{kp}(t).$$ (A20)

Eqs. (A15), (A16a) - (A20), then give

$$[\theta_p^0(t)]^2 \leq [a_{p(p+1)}^0(t)]^2 <\psi(t)|I - \hat{z}_{p(p+1)}^0(t)|<\psi(t)|.$$ (A21)

For $u_{jp}^1(t)$, since $\hat{z}_{p(p+1)}^1(t)$ is nonzero only on the subspace with $N_p + N_{p+1}$ nonzero, we have

$$u_{jp}^1(t) = -a_{p(p+1)}^1(t)\sum_k \text{Im}(\phi_{jp}(t)|<\chi_{jp}(t)| \hat{z}_{p(p+1)}^1(t)|I - \hat{z}_{p(p+1)}^0(t)|<\psi(t)|.$$ (A22)

Eqs. (A16b) and (A22) give

$$[\theta_p^1(t)]^2 \leq [a_{p(p+1)}^1(t)]^2 <\psi(t)|I - \hat{z}_{p(p+1)}^0(t)|<\psi(t)|.$$ (A23)

But by Eq. (A14), the underlying operator $[\hat{z}_{p(p+1)}^1(t)]^2$ on $H_p \otimes H_{p+1}$ has trace 1 and therefore all eigenvalues bounded by 1. Thus Eq. (A23) implies

$$[\theta_p^1(t)]^2 \leq [a_{p(p+1)}^1(t)]^2 <\psi(t)|I - \hat{z}_{p(p+1)}^0(t)|<\psi(t)|.$$ (A24)

Eqs. (A16a), (A21) and (A24) give

$$\sum_p |\theta_p(t)|^2 \leq \sum_p [[a_{p(p+1)}^0(t)]^2 + [a_{p(p+1)}^1(t)]^2] \times \sqrt{<\psi(t)|I - \hat{z}_{p(p+1)}^0(t)|<\psi(t)|}.$$ (A25)
Thus for at most 4 values of \( p \), namely \( x_0, x_0 - 1, x_1, x_1 - 1 \). Thus

\[
\sum_p < \psi (t) | I - z^0_p (p_{p+1}) | \psi (t) > \leq 4. \tag{A27}
\]

In addition, by Eqs. (13), (14), (A13) and (A14),

\[
\sum_p | a^0_p (p_{p+1}) (t) | + | a^1_p (p_{p+1}) (t) |^2 \leq 2 \sum_p | a^0_p (p_{p+1}) (t) |^2 + | a^1_p (p_{p+1}) (t) |^2 \leq 2 \| k(t) \|^2. \tag{A28}
\]

Assembling Eqs. (A26), (A27) and (A28) gives

\[
\sum_p | \theta_p (t) | \leq 2 \sqrt{2} \| k(t) \|. \tag{A29}
\]

Eqs. (18), (A7) and (A29) then yield

\[
C (| \omega >, | \psi >) \geq \frac{n \pi}{8 \sqrt{2}}. \tag{A30}
\]

and therefore

\[
C (| \omega >) \geq \frac{n \pi}{8 \sqrt{2}}. \tag{A31}
\]

4. Complexity Upper Bound

An upper bound on \( C (| \omega >) \) is given by \( C (| \omega >, | \psi >) \) for any product state \( | \psi > \), which in turn is given by \( \int_0^1 \| k(t) \| \) for any trajectory \( k(t) \) connecting \( | \psi > \) to \( | \omega > \), one of which we now construct for

\[
| \psi > = | 1 > > 1 > > 0 > > 0 > > 0 > q. \tag{A32}
\]

Define \( k_0 \) to be

\[
k_0 = - i (| a_0 > > b_0 | - | b_0 > > a_0 |) \otimes q \neq 0, 1 I_q. \tag{A33}
\]

where

\[
| a_0 > = | 1 > > 1 > > 0 > > 0, \tag{A34a}
\]

\[
| b_0 > = | 1 > > 1 > > 1 > > 1. \tag{A34b}
\]

In addition, for \( 1 \leq i < n \), define \( k_i \) to be

\[
k_i = - i (| a_i > > b_i | - | b_i > > a_i | + | c_i > < d_i | - | d_i > < c_i |) \otimes q \neq i, i+1 I_q. \tag{A35}
\]

where

\[
| a_i > > | 0 > > 1 > > i > > 1, \tag{A36a}
\]

\[
| b_i > = | 1 > > i > > 1, \tag{A36b}
\]

\[
| c_i > = | 0 > > i > > 1 > > i+1, \tag{A36c}
\]

\[
| d_i > = | - 1 > > i > > 1 > > i+1. \tag{A36d}
\]

From these states, define the projection operator

\[
P_0 = (| a_0 > > b_0 | + | b_0 > > a_0 |) \otimes q \neq 0, 1 I_q. \tag{A37}
\]

and for \( 1 \leq i < n \) the projections

\[
P_i = (| a_i > > b_i | + | b_i > > a_i | + | c_i > < d_i | - | d_i > < c_i |) \otimes q \neq i, i+1 I_q. \tag{A38}
\]

We then have for any \( i \geq 0 \)

\[
\exp(i \theta k_i) = (I - P_i) + \cos(\theta) P_i + i \sin(\theta) k_i. \tag{A39}
\]

Define iteratively

\[
| \psi_0 > = \exp(\frac{i \pi}{4} k_0) | \psi >, \tag{A40a}
\]

\[
| \psi_i > = \exp(\frac{i \pi}{2} k_i) | \psi_{i-1} >, i \geq 1. \tag{A40b}
\]

Eqs. (A37) - (A39) then imply

\[
| \psi_{n-1} > = | \omega >. \tag{A41}
\]

The norms of these operators are

\[
\| k_0 \| = \sqrt{2}, \tag{A42a}
\]

\[
\| k_i \| = 2, i \geq 1. \tag{A42b}
\]

We therefore have

\[
C (| \omega >, | \psi >) \leq (n - 1 + \frac{\pi}{2 \sqrt{2}}) \pi. \tag{A43}
\]

Eq. (A43) then implies

\[
C (| \omega >) \leq (n - 1 + \frac{1}{2 \sqrt{2}}) \pi. \tag{A44}
\]

Appendix B: Entangled Extended State

1. Lower Bound on the Complexity of an Entangled Extended State

Repeating Section (A1) assume \( k(t) \) and \( U_k(t) \) solve Eqs. (10a) and (10b) and connect some 2 particle product state \( | \psi > \) to \( | \omega > \) of Eqs. (24a) - (24c) according to Eqs. (A1) and (A2).
For any \( p \) in the range \( 0 \leq p < n + r \), split \( \mathcal{H} \) into \( \mathcal{Q}_p \) and \( \mathcal{R}_p \) according to Eqs. (A3a) - (A3c). The Schmidt decomposition of \( |\psi(t)\rangle \) will again be given by Eqs. (A4a) - (A4g) and the initial Schmidt vector \( |s_0p(0), s_1p(0), \ldots\rangle \), since it is from a 2 particle product state \( |\psi(0)\rangle := |\psi\rangle \), has, as before, the form \( |s_0p(0), s_1p(0), s_2p(0), 0, \ldots\rangle \) with at most 3 nonzero entries. For \( 0 \leq p < n \), the final Schmidt vector \( |s_0p(1), s_1p(1), \ldots\rangle \) remains \([0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots]\) with exactly 2 nonzero entries, requiring again a total rotation angle of at least \( \pi \). For \( \theta_p(t) \) of Eqs. (A4a) and (A6b), Eq. (A7) is then repeated.

For \( n \leq p < n + r \), the initial Schmidt vector \( |s_0p(0), s_1p(0), \ldots\rangle \) still has at most 3 nonzero entries \( |s_0p(0), s_1p(0), s_2p(0), 0, \ldots\rangle \). But as a result of the extended wave functions in Eqs. (B2) and (A29) then give

\[
|\psi_{n-1}\rangle = \exp(i\theta_1k_1)|\psi_{n-1}\rangle,
\]

\[
\sin(\theta_1) = \sqrt{\frac{n + r - i - 1}{n + r - i}}.
\]

Eqs. (A3b) and (A3c) imply

\[
|\psi_i\rangle := \sqrt{\frac{i + 1 - n}{2r}}|\psi_{0i}\rangle + \sqrt{\frac{i + 1 - n}{2r}}|\psi_{1i}\rangle + \sqrt{\frac{n + r - i - 1}{2r}}|\psi_{2i}\rangle + \sqrt{\frac{n + r - i - 1}{2r}}|\psi_{3i}\rangle
\]

where

\[
|\psi_{0i}\rangle := |1 - 1 > 0 \otimes (\frac{1}{\sqrt{i + 1 - n}} \sum_{n \leq x \leq i} |1 - 1 > x \otimes q \not= 0, x > 0 > q),
\]

\[
|\psi_{1i}\rangle := |1 > 0 \otimes (\frac{1}{\sqrt{i + 1 - n}} \sum_{n \leq x \leq i} |1 - 1 > x \otimes q \not= 0, x > 0 > q),
\]

\[
|\psi_{2i}\rangle := |1 - 1 > 0 \otimes |1 - 1 > i + 1 \otimes q \not= 0, i + 1 > 0 > q,
\]

\[
|\psi_{3i}\rangle := |1 > 0 \otimes |1 - 1 > i + 1 \otimes q \not= 0, i + 1 > 0 > q.
\]

We then have

\[
|\psi_{n+r-2}\rangle := \omega'.
\]

Combining Eqs. (A42a), (A42b), (A40a), (A40b), (B6a) and (B6b) gives

\[
C(|\omega'\rangle, |\psi\rangle) \leq (n - 1 + \frac{\pi}{2\sqrt{2}})\pi + 2\lambda r,
\]

where

\[
\lambda = \frac{1}{r} \sum_{0 < s < r} \arcsin(\sqrt{\frac{s}{s + 1}}).
\]

For large \( r \) for most of the range \( 0 < s < r \), the argument of the sum in Eq. (B11) is nearly \( \frac{\pi}{2} \). Therefore

\[
\lim_{r \to \infty} \lambda = \frac{\pi}{2}
\]

Eq. (B10) implies

\[
C(|\omega'\rangle) \leq (n - 1 + \frac{\pi}{2\sqrt{2}})\pi + 2\lambda r.
\]
Appendix C: Complexity Group

We now show that the group of $G$ all $U_k(1)$ realizable as solutions to Eqs. (16a) and (16b) has as a subgroup the direct product

$$\hat{G} = \times_n SU(d_n),$$

where $SU(d_n)$ acts on the subspace of $H$ with eigenvalue $n$ of the total number operator $N,$ $d_n$ is the dimension of this subspace, and the product is over the range $1 \leq n \leq 4x_{max} + 2.$ Missing from this range of $n$ are only two states, the vacuum, $n = 0,$ and the state with every site occupied by two particles, $n = 4x_{max} + 2.$

For any integer $-x_{max} + 1 \leq p \leq x_{max},$ divide $H$ into the product

$$Q_p = \otimes_{z \leq p} H_z,$$

$$R_p = \otimes_{z > p} H_z,$$

$$\mathcal{H} = Q_p \otimes R_p.$$  \hfill (C2a) \hfill (C2b) \hfill (C2c)

Let $K_p$ be the vector space of operators given by Eq. (13) constructed from the set $\mathcal{F}_x$ of traceless particle number preserving Hermitian operators on $H_x$ with $x \leq p$ and the set $\mathcal{G}_{xy}$ of traceless particle number preserving Hermitian operators on nearest neighbor $H_x \otimes H_y$ orthogonal to $\mathcal{F}_x$ and $\mathcal{F}_y$ with $x, y \leq p.$ Let $G_p$ be the group on $H$ of all $U_k(1)$ realizable as solutions to Eq. (16a) for $k(t) \in K_p.$

The group $G_p$ consists of all operators of the form $\exp(i\hbar \mathbf{P})$ for $\mathbf{P} \in L_p,$ where $L_p$ is the Lie algebra generated by $K_p.$ Said differently, $L_p$ is the smallest vector space of operators such that $K_p \subseteq L_p$ and, in addition, for any $h_0, h_1 \in L_p,$ and any real $r_0, r_1,$ there are $h_2, h_3 \in L_p$ given by

$$h_2 = r_0 h_0 + r_1 h_1,$$

$$h_3 = i[h_0, h_1].$$  \hfill (C3a) \hfill (C3b)

The requirement that $L_p$ be closed under sums in Eq. (C3a) follows from the Trotter product formula applied to the large $t$ limit of

$$[\exp(it^{-1}r_0 h_0) \exp(it^{-1}r_1 h_1)]^t.$$ \hfill (C4)

The requirement that $L_p$ be closed under commutation in Eq. (C3b) follows from the Baker-Campbell-Hausdorff formula applied to the large $t$ limit of

$$[\exp(it^{-1/2}h_0) \exp(it^{-1/2}h_1) \exp(-it^{-1/2}h_0) \exp(-it^{-1/2}h_1)]^t.$$ \hfill (C5)

By induction on $p,$ we will show that $G_p$ includes the subgroup $\hat{G}_p$

$$\hat{G}_p = \times_n \hat{G}_{pn},$$

$$\hat{G}_{pn} = SU(d_{pn}) \otimes \otimes_{z \geq p} I_z,$$ \hfill (C6a) \hfill (C6b)

where $SU(d_{pn})$ acts on the subspace $Q_{pn}$ of $Q_p$ with eigenvalue $n$ of the total number operator $N,$ and $d_{pn}$ is the dimension of $Q_{pn}.$ The product in Eq. (C6a) is over $1 \leq n \leq 2x_{max} + 2.$ Eqs. (C6a) (C6b) for the case $p = x_{max}$ become Eq. (C1).

For $p = -x_{max} + 1,$ Eqs. (C6a) and Eqs. (C6b) follow immediately from the definition of $K_p.$ Assuming Eqs. (C6a) and Eqs. (C6b) for some $p - 1,$ we will prove them for $p.$

Let $S_{pn}$ be an orthonormal basis for $Q_{pn}$ consisting of all vectors of the form

$$|\psi\rangle = \otimes_{x \leq p} |\psi(x)\rangle_x,$$ \hfill (C7a)

$$|\psi(x)\rangle \in \{-1, 0, 1, 2\},$$ \hfill (C7b)

$$\sum_{x \leq p} |\psi(x)\rangle_n = n.$$ \hfill (C7c)

For any pair of distinct $|\psi\rangle >, |\psi_1\rangle > \in S_{pn},$ and $2 \times 2$ traceless Hermitian $h,$ define

$$H(|\psi\rangle >, |\psi_1\rangle >, h) =$$

$$\sum_{ij} |\psi_1\rangle < \left< \psi_j \right| h_{ij} \otimes \otimes_{z > p} I_z.$$ \hfill (C8)

The set of such $H(|\psi\rangle >, |\psi_1\rangle >, h)$ is a linear basis for the Lie algebra $L_{pn}$ of the group $\hat{G}_{pn}$ of Eq. (C6b).

Thus to prove Eqs. (C6a) and (C6b) for $p$ it is sufficient to show that any $H(|\psi\rangle >, |\psi_1\rangle >, h)$ for $|\psi\rangle >, |\psi_1\rangle > \in S_{pn}$ and $2 \times 2$ traceless Hermitian $h,$ given the induction hypothesis, is contained in the Lie algebra generated by $L_{p-1}m$ for some $m$ and all

$$\hat{k} = k \otimes \otimes_{z \neq p-1, p} I_z,$$ \hfill (C9)

for traceless, Hermitian number preserving $k$ on $H_{p-1} \otimes H_p.$

We will work backwards starting from some $H(|\psi\rangle >, |\psi_1\rangle >, h)$ for $|\psi\rangle >, |\psi_1\rangle > \in S_{pn}.$ Since $|\psi\rangle >$ and $|\psi_1\rangle >$ have the same value of total $N$ on the region $x \leq p,$ it follows that a $U_0$ can be found in $\hat{G}_{p-1}$ such that

$$|\psi_2\rangle > = U_0 |\psi\rangle >,$$ \hfill (C10a)

$$|\psi_3\rangle > = U_0 |\psi_1\rangle >,$$ \hfill (C10b)

are orthogonal vectors in $S_{pn}$ with equal total particle counts on the region $p - 1 \leq x \leq p.$ The particle count difference between $|\psi\rangle >$ and $|\psi_1\rangle >$ at point $p$ is at most 2, and equal and opposite to the difference between the corresponding totals on the region $x \leq p - 1.$ This compensating difference can be moved by $U_0$ to the point $p - 1.$ A $k$ in Eq. (C9) can then be found such that

$$U_1 = \exp(ih)$$ \hfill (C11a)

$$|\psi_4\rangle > = U_1 |\psi_2\rangle >,$$ \hfill (C11b)

$$|\psi_5\rangle > = U_1 |\psi_3\rangle >,$$ \hfill (C11c)

$$|\psi_4\rangle > = |\phi_4\rangle > \otimes |r\rangle >,$$ \hfill (C11d)

$$|\psi_5\rangle > = |\phi_5\rangle > \otimes |r\rangle >,$$ \hfill (C11e)

for some $r \in \{-1, 0, 1, 2\}$ and $|\phi_4\rangle >$ and $|\phi_5\rangle >$ orthogonal vectors in $S_{p-1}m$ with $m = n - |r|.$
It is then possible to find a $U_2$ in $\hat{G}_{p-1}$ such that

\begin{align}
|\psi_6 > &= U_2 |\psi_4 >, \\
|\psi_7 > &= U_2 |\psi_5 >, \\
|\psi_6 > &= |\chi > \otimes 1 >_{p-1} \otimes |r >_p, \quad (C12a) \\
|\psi_7 > &= |\chi > \otimes 1 >_{p-1} \otimes |r >_p, \quad (C12b) \\
|\psi_6 > &= |\chi > \otimes | -1 >_{p-1} \otimes |r >_p, \quad (C12c) \\
|\psi_7 > &= |\chi > \otimes |1 >_{p-1} \otimes |r >_p, \quad (C12d)
\end{align}

for a some $|\chi >$ in $S_{p-2m-1}$.

Combining Eqs. (C10a) - (C12d), the induction hypothesis implies unit determinant unitary operators $U_0, U_1, U_2$ such that

$$
U_2 U_1 U_0 H(|\psi_0 >, |\psi_1 >, h) U_0^\dagger U_1^\dagger U_2^\dagger = \\
|\chi > < \chi | \otimes \sum_{ij} |i >_{p-1} < j |_{p-1} h_{ij} \\
\otimes |r >_{p} < r |_{p} \otimes z >_{p-1} I_p, \quad (C13)
$$

for $i$ and $j$ now summed over $-1$ and 1 with the original index value 0 of $h_{ij}$ replaced by $-1$.

The expression on the right-hand side of Eq. (C13) can then be obtained from a commutator between an operator $\hat{k}$ from Eq. (C9) and an operator $\hat{g} \in L_{p-1m}$ for $m = n - |r|$. For 2 x 2 traceless Hermitian $k_{ij}$, define

$$
\hat{k} = \sum_{ij} |i >_{p-1} < j |_{p-1} k_{ij} \otimes |r >_{p} < r |_{p} \otimes z >_{p-1} I_p, \quad (C14)
$$

and for a 2 x 2 traceless Hermitian $g_{ij}$, define

$$
\hat{g} = |\chi > < \chi | \otimes \sum_{ij} |i >_{p-1} < j |_{p-1} g_{ij} \otimes z >_{p-1} I_p. \quad (C15)
$$

Choose $k_{ij}$ and $g_{ij}$ so that their commutator as 2 x 2 matrices fulfills

$$
h = i [\hat{k}, \hat{g}]. \quad (C16)
$$

Combining Eqs. (C13), (C14), (C15) and (C16) then gives

$$
H(|\psi_0 >, |\psi_1 >, h) = U_0^\dagger U_1^\dagger U_2^\dagger i [\hat{k}, \hat{g}] U_2 U_1 U_0, \quad (C17)
$$

which completes the induction step and therefore the proof of Eq. (C1).