ON SEISMIC SIGNATURES OF RAPID VARIATION

G. Houdek\(^1\) and D. O. Gough\(^1,2\)

\(^1\)Institute of Astronomy, University of Cambridge, Cambridge CB3 0HA, UK
\(^2\)Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK

**ABSTRACT**

We present an improved model for an asteroseismic diagnostic contained in the frequency spacing of low-degree acoustic modes. By modelling in a realistic manner regions of rapid variation of dynamically relevant quantities, which we call acoustic glitches, we can derive signatures of the gross properties of those glitches. In particular, we are interested in measuring properties that are related to the helium ionization zones and to the rapid variation in the background state associated with the lower boundary of the convective envelope. The formula for the seismic diagnostic is tested against a sequence of theoretical models of the Sun, and is compared with seismic diagnostics published previously by Monteiro & Thompson (1998, 2005) and by Basu et al. (2004).

Key words: solar interior; solar oscillations; stellar oscillations.

1. INTRODUCTION

Abrupt variation in the stratification of a star (relative to the scale of the inverse radial wavenumber of a seismic mode of oscillation), such as that resulting from the (smooth, albeit acoustically relatively abrupt) depression in the first adiabatic exponent \( \gamma_1 \equiv \frac{\partial \ln p}{\partial \ln \rho} \), caused by the ionization of helium, where \( p, \rho \) and \( s \) are pressure, density and specific entropy, or from the sharp transition from radiative to convective heat transport at the base of the convection zone, induces small-amplitude oscillatory components (with respect to frequency) in the spacing of the cyclic eigenfrequencies \( \nu_{n,l} \) of seismic oscillation. One might hope that the variation of the sound speed induced by helium ionization might enable one to determine from the low-degree eigenfrequencies a measure that is directly related to, perhaps even almost proportional to, the helium abundance, with little contamination from other properties of the structure of the star.

A convenient and easily evaluated measure of the oscillatory component is the second multiplet-frequency difference with respect to order \( n \) amongst modes of like degree \( l \):

\[
\Delta_2 \nu_{n,l} \equiv \nu_{n-1,l} - 2\nu_{n,l} + \nu_{n+1,l}
\]

(Gough, 1990). Any localized region of rapid variation of either the sound speed or the density scale height, or a spatial derivative of them, which here we call an acoustic glitch, induces an oscillatory component in \( \Delta_2 \nu \) with a ‘cyclic frequency’ approximately equal to twice the acoustic depth

\[
\tau = \int_{r_{\text{glitch}}}^{R} e^{-1} \, dr
\]

dependent on the amplitude of the glitch and which decays with \( \nu \) once the inverse radial wavenumber of the mode becomes comparable with or greater than the radial extent of the glitch.

We report here on a model formula for the oscillatory contribution to the second frequency difference that is directly related to the gross properties of the glitch, and compare how well it fits actual eigenfrequency differences with corresponding fits of previously published formulae.

2. THE FITTING FUNCTION

For a nonrotating star with vanishing pressure gradient at the surface, the angular eigenfrequencies \( \omega \) of adiabatic oscillation obey a variational principle (e.g. Chandrasekhar 1963) which can be used to estimate the contribution \( \delta \omega \) to \( \omega \) from the acoustic glitch. In estimating the contribution from helium ionization we retain only the dominant glitch term (\( \delta \gamma_1 / \gamma_1 \) at fixed pressure and temperature), and we represent \( \delta \gamma_1 / \gamma_1 \) by a pair of (negative) Gaussian functions of \( \tau \) with widths \( \Delta_1 \) and \( \Delta_{11} \), whose integrals are \( \Gamma_1 \) and \( \Gamma_{11} \), and which are centred about the acoustic depths \( \tau_1 \) and \( \tau_{11} \) of the first and second ionization zones of helium beneath the seismic surface \( r = R \) of the star. After converting to cyclic frequency \( \nu = \omega / 2\pi \), the oscillatory component is given asymptotically by.
\[
\delta_{\gamma,\text{osc}\nu} \simeq -\Gamma_1 \nu_0 \left[ \nu + \frac{1}{2} (m + 1) \nu_0 \right] \\
\times \left( \mu \beta \nu_0^{-1} e^{-8 \pi^2 \nu^2 \kappa_1^2 \Delta_1 \nu^2 \cos 2\psi_1} \right) \\
+ \kappa_1^{-1} e^{-8 \pi^2 \kappa_1^2 \Delta_1 \nu^2 \cos 2\psi_1} \tag{3}
\]

(Houdek & Gough, 2007). We evaluate the phases \( \psi_1 = \psi(\hat{\tau}_1) \) and \( \psi_1 = \psi(\hat{\tau}_1) \), where \( \omega \hat{\tau} = \omega \tau + \epsilon_1 \), by representing the envelope by a plane-parallel polytrope of index \( m = 3.5 \) and adding a phase constant \( \epsilon_1 \) to \( \omega \tau \) to account for the deviation of the actual envelope from the polytrope:

\[
\psi(\tau) = \omega \tau \kappa - (m + 1) \cos^{-1} \left( \frac{m + 1}{\tau \omega} \right) + \frac{\pi}{4}. \tag{4}
\]

In equation (3), \( \epsilon_1 = \kappa(\hat{\tau}) \) etc, with \( \kappa(\tau) = [1 - (m + 1)^2 / 4 \pi^2 \nu^2 \tau^2]^{1/2} \); also \( \nu_0 \) is a characteristic value of the first frequency difference \( \Delta \nu \) whose value is half the inverse total acoustic radius \( T \) of the star: \( \nu_0 = \omega \nu_0 / 2 \pi = 1/2 T \). We have found that the ratios \( \beta = \Gamma_1 \Delta_1 / \Gamma_1 \Delta_1 \), \( \eta = \tau_1 / \tau_1 \) and \( \mu = \Delta_1 / \Delta_1 \) hardly vary amongst stellar models whose masses and radii vary by factors of at least five. We have accordingly adopted constant values, namely \( \beta = 0.45 \), \( \eta = 0.7 \) and \( \mu = 0.9 \) for solar-like stars.

To obtain a complete description of \( \Delta_2 \nu \) we must add an oscillatory contribution from the near discontinuity in the density scale height at the base \( \tau_c \) of the convection zone:

\[
\delta_{c,\text{osc}\nu} \simeq \hat{A}_c \nu_0 \nu^{-2} \left( 1 + \frac{1}{16 \pi^2 \tau_0^2 \nu^2} \right)^{-1/2} \\
\times \left[ \cos[2\psi_c + \tan^{-1}(4 \pi \tau_0 \nu)] \right], \tag{5}
\]

with

\[
\hat{A}_c = \frac{c^2}{8 \pi \nu_0} \left[ \left( \frac{d^2 \ln \rho}{dr^2} \right)_{\tau_c} + \right. \tag{6}
\]

where \( \psi_c = \psi(\tau_c + \omega^{-1} \epsilon_0) \) is the polytropic phase at the base of the convection zone, the phase constant \( \epsilon_0 \) being allowed to differ from \( \epsilon_1 \) to account for the fact that the region of the convection zone beneath the helium ionization zones is not an exact polytrope; the quantity \( \tau_0 \) is a characteristic scale over which the sound speed in the radiative interior relaxes to a smooth extrapolation from the convection zone (and which we also take to be constant, 80 s, for all the stars we consider).

From these expressions it is straightforward to evaluate the second frequency difference \( \Delta_2 \nu \), to which we must add a smooth term which we represent by a third-degree polynomial in \( \nu^{-1} \):

\[
\Delta_2,\text{sm} = \sum_{i=0}^{3} \alpha_i \nu^{-i}. \tag{7}
\]

The eleven parameters \( \hat{A}_1, \hat{A}_2, \Delta_1, \tau_1, \tau_c, \epsilon_1, \epsilon_0, \) and \( \alpha_i \) are adjusted to fit by least squares the theoretical curve to the second frequency differences of the actual eigenfrequencies of the modes.

We compare the current seismic diagnostic (3)–(7) with diagnostics published previously by Monteiro & Thompson (1998, 2005) and by Basu et al. (2004).

2.1. The seismic diagnostic by Monteiro & Thompson

This diagnostic is based on a single-triangular approximation to \( \delta_{\gamma_1 / \gamma_1} \) due to He ionization. The resulting formula for the frequency perturbation is

\[
\delta_{\text{osc}\nu} \simeq b_1 \sin^2 \left( \frac{2 \pi \nu \beta}{\nu} \right) \cos (4 \pi \nu \tau_1 + 2 \epsilon_1) \\
+ \left[ \frac{c^2}{\nu^2} + \frac{c^2}{\nu^2} \right]^{1/2} \cos (4 \pi \nu \tau_c + 2 \epsilon_0), \tag{8}
\]

in which \( \beta \) is the half-width and \( b_1 \) is proportional to the height of the triangle, from which second differences can be computed. We set to zero the fitting parameter \( c_2 \), a nonzero value of which, in the original formulation, would account for an putative adiabatic extension to the convection zone produced by overshooting (which is not present in our solar models). By so doing, the seismic diagnostic (8) has the same number, 11, of fitting parameters as the seismic diagnostic (3)–(7) when adopting the third-degree polynomial (7) to represent the smooth contribution.

2.2. The seismic diagnostic by Basu et al.

In this formulation the seismic diagnostic is that used by Basu (1997), which was invented by Basu, Antia, and Narasimha (1994) for studying the base of the convection zone. Ballot et al. (2004) and Piau et al. (2005) have adopted essentially the same formula. It is not directly linked to the form of the ionization-induced acoustic glitch, but instead was motivated by the oscillatory signature of a discontinuity in sound speed, whose amplitude was modified to make it decay with frequency approximately in the manner of the second differences of the eigenfrequencies. The functional form is given by

\[
\Delta_2 \delta_{\text{osc}\nu} \simeq \left( b_1 + \frac{b_2}{\nu^2} \right) \cos (4 \pi \nu \tau_1 + 2 \epsilon_1) \\
+ \left( c_1 + \frac{c_2}{\nu^2} \right) \cos (4 \pi \nu \tau_c + 2 \epsilon_0). \tag{9}
\]

Here we set \( c_3 = 0 \), to remove the effect of the erroneous sound-speed discontinuity at the base of the convection zone and to render the functional form of the contribution at \( \tau = \tau_c \) (nearly) consistent with those in the other two signatures (5) and (8); and we adopt the third-degree polynomial (7) to represent the smooth contribution, thereby using the same number (11) of fitting parameters as the seismic diagnostics (3)–(7) and (8), (7).
at variance with the findings of Monteiro and Thompson (2005). The seismic diagnostic (3)–(7) fits $\Delta_2\nu$ of the model eigenfrequencies the best, as is evinced by its smallest values of $\chi^2$.

The quality of the fits is also illustrated in the upper panels of Figs 2–4, which show the second frequency differences over the whole frequency range considered for the central model 0. It is also clear from comparing these figures that the seismic diagnostic (3)–(7) reproduces $\Delta_2\nu$ of the model frequencies the most faithfully.

It is interesting to note that in determining the parameters in the formulae the fitting procedure gives greater weight to the low-frequency end of the ionization-induced component $\Delta_{2,\gamma}$, where the amplitude is the greatest, and greater weight to the high-frequency end of the contribution $\Delta_{2,\epsilon}$ from the base of the convection zone, where misfits cannot be offset by adjustments to $\Delta_{2,\epsilon}$; therefore, the diagnostics (8), (7) and (9), (7) fail principally in the mid-range. Some large-scale discrepancy can be seen also at high frequency for formula (9), (7) because $\Delta_{2,\gamma}$ decays with frequency too slowly. We acknowledge that these discrepancies might be either smaller or hardly greater than the random errors in imminent asteroseismic data; nevertheless, fitting an erroneous formula can lead to the inferences from that fitting being biased, and should therefore be avoided.

ACKNOWLEDGEMENTS

GH is grateful for support from the Particle Physics and Astronomy Research Council.

REFERENCES

[1] Ballot J., Turck-Chièze, S., García R.A., 2004, A&A, 423, 1051
[2] Basu S., 1997, MNRAS, 288, 527
[3] Basu S., Antía H., Narasimha D., 1994, MNRAS, 267, 209
[4] Basu S., Mazumdar A., Antía H.M., Demarque P., 2004, MNRAS 350, 277
[5] Chandrasekhar S., 1963, ApJ 138, 896
[6] Gough D.O., 1990, in Progress of Seismology of the Sun and Stars, Lecture Notes in Physics, Vol. 367, Y. Osaki, H. Shibahashi, eds. Springer Verlag, p. 283
[7] Gough D.O. Novotny, E., 1990, Solar Phys., 128, 267, 209
[8] Houday G., Gough D.O. 2007, MNRAS, in press
[9] Monteiro M.J.P.F.G., Thompson M., 1998, in Deubner F.-L., Christensen-Dalsgaard J., Kurtz D., eds. Proc. IAU Symp.185, New Eyes to See Inside the Sun and Stars. Kluwer, Dordrecht, p. 317
[10] Monteiro M.J.P.F.G., Thompson M., 2005, MNRAS, 361, 1187
[11] Piau L. Ballot J. Turck-Chièze S., 2005, A&A, 430, 571
Figure 2. Top: The symbols are second differences $\Delta^2 \nu$, defined by equation (1), of low-degree frequencies obtained from adiabatic pulsation calculations of the central model (0). The curve is the seismic diagnostic (3)–(7) whose eleven parameters have been adjusted to fit the data by least squares. Bottom: Model properties determined from the seismic diagnostic (3)–(7) for the nine test models. Models with varying age are connected by solid lines, those with varying $Z$ by dashed lines. Some model identifiers are written in the upper left panel. The smallest (central) frequency value used in the least-square fitting is $\nu \approx 1322 \mu$Hz, the largest is $\nu \approx 4058 \mu$Hz. The lower right panel is the standard measure $\chi^2$ for the goodness of the least-squares fit.
Figure 3. Top: The symbols are second differences $\Delta_2\nu$, defined by equation (1), of low-degree frequencies obtained from adiabatic pulsation calculations of the central model 0. The curve is the seismic diagnostic (8),(7) whose eleven parameters have been adjusted to fit the data by least squares. Bottom: Model properties determined from the seismic diagnostic (8),(7) for the nine test models. The smallest (central) frequency value used in the least-square fitting is $\nu \approx 1322 \mu$Hz, the largest is $\nu \approx 4058 \mu$Hz. The lower right panel is the standard measure $\chi^2$ for the goodness of the least-squares fit.
Figure 4. Top: The symbols are second differences $\Delta_2 \nu$, defined by equation (1), of low-degree frequencies obtained from adiabatic pulsation calculations of the central model 0. The curve is the seismic diagnostic (9),(7) whose eleven parameters have been adjusted to fit the data by least squares. Bottom: Model properties determined from the seismic diagnostic (9),(7) for the nine test models. The smallest (central) frequency value used in the least-square fitting is $\nu \approx 1322 \mu$Hz, the largest is $\nu \approx 4058 \mu$Hz. The lower right panel is the standard measure $\chi^2$ for the goodness of the least-squares fit.