Brownian motions and eigenvalues on complex Grassmannian and Stiefel manifolds

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Abstract

We study Brownian motions and related random matrices diffusions on the complex Grassmannian manifolds. In particular, the distribution of eigenvalues processes related to those Brownian motions is proved to be the law of a conditioned Karlin-McGregor diffusion associated to a Jacobi process and is shown to converge in large time to the distribution of a Coulomb gas corresponding to a complex Jacobi ensemble. Along the way we obtain an algebraic form of the Berezin-Karpelević formula for the zonal spherical functions of the complex Grassmannian. We then use the Stiefel fibration to lift the Brownian motion of the complex Grassmannian to the complex Stiefel manifold and deduce a skew-product decomposition from which we prove asymptotic winding laws.

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1 Introduction

The complex Grassmannian $G_{n,k}$ is the set of $k$-dimensional complex subspaces in $\mathbb{C}^n$. Grassmannians have a very rich geometrical, combinatorial and topological structure and they naturally appear in algebraic topology, differential geometry, analysis, combinatorics and mathematical physics; see for instance [35]. In the present paper we are interested in the study of the Brownian motions and related diffusion processes on $G_{n,k}$. In that situation the interplay between stochastic differential geometry and random matrices theory appears to be particularly deep and rich. Indeed, from the Riemannian viewpoint, the Brownian motion is the diffusion associated with the Laplace-Beltrami operator of a Riemannian metric. In our case, see Section 2.1, the Riemannian metric of interest in $G_{n,k}$ is the canonical one inherited from the Stiefel fibration

$$U(k) \to V_{n,k} \to G_{n,k},$$

where $V_{n,k}$ denotes the complex Stiefel manifold that is given by the set of all unitary $k$-frames in $\mathbb{C}^n$, and where $U(k)$ is the unitary group acting on $V_{n,k}$. We prove in Theorem 2.1 and Corollary 2.2 that the Brownian motion for this Riemannian metric can be realized as a random matrix diffusion process. More precisely, we prove that if $U_t = \begin{pmatrix} X_t & Y_t \\ Z_t & W_t \end{pmatrix}$ is a Brownian motion on the unitary group $U(n)$, then the $\mathbb{C}^{(n-k)\times k}$ matrix valued process given by $W_t = X_t Z_t^{-1}$ is a Brownian motion on $G_{n,k}$. As a consequence, the Brownian motion on $G_{n,k}$ can be studied using the techniques of random matrices diffusions.

Random matrix theory is a very rich and vibrant research topic with connections to many areas of pure or applied mathematics and mathematical physics, see for instance [1] and the references therein. The fruitful idea to introduce a time-dynamic on random matrices goes back at least to Freeman Dyson [21] who quantitatively described the eigenvalues dynamics of the Hermitian Brownian motion and put forward the fundamental non-collision property exhibited by this eigenvalues process. We also refer to the early work by Eugene Dynkin [20]. Since then, non colliding processes associated with random matrices models have extensively been studied, we refer for instance to [15, 30, 24, 31, 32]. In our case we are able to show the non-colliding property and study in details the eigenvalues of the process $W_t^* W_t$. A summary of some of the main results that we prove about eigenvalues related to the Brownian motion $W_t$ is the following:

**Theorem 1.1.** Let $(W_t)_{t \geq 0}$ be a Brownian motion on $G_{n,k}$ as in Theorem 2.1. The ordered
eigenvalues process \((\rho(t))_{t \geq 0}\) of the diffusion \(((I_k - W_t^*W_t)(I_k + W_t^*W_t)^{-1})_{t \geq 0}\) has generator
\[
2 \sum_{i=1}^{k} (1 - \rho_i^2) \partial_i^2 - 2 \sum_{i=1}^{k} \left( n - 2k + (n - 2k + 2) \rho_i + 2 \sum_{\ell \neq i} \frac{1 - \rho_i^2}{\rho_i - \rho_\ell} \right) \partial_i
\]
and a density with respect to the Lebesgue measure \(dx\) given by
\[
e^{\frac{1}{4}k(k-1)(3n-4k+2)t} \prod_{i>j}(x_i - x_j) \prod_{i>j}(\rho_i(0) - \rho_j(0)) \det(p_{n-2k,0}(\rho_i(0), x_j))^{1 \leq i,j \leq k} 1_{\Delta_k}(x),
\]
where \(p_{n-2k,0}\) is the heat kernel of a one-dimensional Jacobi diffusion and
\[
\Delta_k = \{x \in [-1,1]^k, -1 \leq x_1 < \cdots < x_k \leq 1\}.
\]
Moreover, when \(t \to +\infty\), \(\rho(t)\) converges in distribution to the invariant probability measure
\[
d\nu = c_{n,k} \prod_{1 \leq i<j \leq k} (x_i - x_j)^2 \prod_{i=1}^{k} (1 - x_i)^{n-2k} 1_{\Delta_k}(x)dx.
\]
In the language of \([2]\) one can say that \((\rho(t))_{t \geq 0}\) is a Karlin-McGregor diffusion associated to a Jacobi process and conditioned by its ground state. Note that the probability measure \(\nu\) is associated to the eigenvalues of a complex Jacobi ensemble, see for instance \([28]\). Non-colliding Jacobi diffusions have also appeared in the work of V. Gorin \([23]\) as the scaling limits of some Markov chains on the Gelfand-Tsetlin graph in relation to the harmonic analysis of the infinite unitary group \(U(\infty)\) (see Remark 3.36 in \([2]\)).

Then, taking advantage of the Stiefel fibration to lift processes from \(G_{n,k}\) to \(V_{n,k}\) we obtain, in the spririt of \([34]\), a skew-product decomposition of the Brownian motion on \(V_{n,k}\).

**Theorem 1.2.** Let \((W_t)_{t \geq 0}\) be a Brownian motion on \(G_{n,k}\) started at \(W_0 \in G_{n,k}\) and let \((\Omega_t)_{t \geq 0}\) be a Brownian motion on the unitary group \(U(k)\) independent from \((W_t)_{t \geq 0}\). Let \((\Theta_t)_{t \geq 0}\) be the \(U(k)\) valued solution of the Stratonovitch stochastic differential equation
\[
\begin{cases}
d\Theta_t = \circ da_t \Theta_t \\
\Theta_0 = I_k,
\end{cases}
\]
where \(a_t = \int_{[0,t]} \eta\) and where \(\eta\) is a push-forward to \(G_{n,k}\) of the connection form of the Stiefel fibration. Then, the process
\[
\begin{pmatrix} X_t \\ Z_t \end{pmatrix} := \begin{pmatrix} W_t \\ I_k \end{pmatrix} (I_k + W_t^*W_t)^{-1/2} \Theta_t \Omega_t, \quad t \geq 0
\]
is a Brownian motion on \(V_{n,k}\).

We deduce as a byproduct, the following interesting limit result for the asymptotic windings of \(\det(Z_t)\).
Theorem 1.3. Let \( \left( \frac{X_t}{Z_t} \right)_{t \geq 0} \) be a Brownian motion on \( V_{n,k} \) with \( \det(Z_0) \neq 0 \). One has the polar decomposition
\[
\det(Z_t) = \rho_t e^{i\theta_t}
\]
where \( 0 < \rho_t \leq 1 \) and \( \theta_t \) is a real-valued continuous semimartingale such that:

- If \( k = 1 \), the following convergence holds in distribution when \( t \to +\infty \)
  \[
  \frac{\theta_t}{t} \to C_{n-1},
  \]
  where \( C_{n-1} \) is a Cauchy distribution of parameter \( n - 1 \).

- If \( k > 1 \), the following convergence holds in distribution when \( t \to +\infty \)
  \[
  \frac{\theta_t}{\sqrt{t}} \to N \left( 0, \frac{k(n-k)}{k-1} + 2n \right).
  \]

The paper is organized as follows. In Section 2, we construct the Brownian motion on \( G_{n,k} \) and show as a byproduct that the Laplacian is given in inhomogeneous coordinates by the formula
\[
\Delta_{G_{n,k}} = 4 \sum_{1 \leq i,i' \leq n-k, 1 \leq j,j' \leq k} (I_{n-k} + WW^*)_{ii'}(I_k + W^*W)_{jj'} \frac{\partial^2}{\partial W_{ij} \partial W_{i'j'}}.
\]

We then show that the invariant measure and symmetric probability measure for this Laplacian is given by \( d\mu = c_{n,k} \det(I_k + W^*W)^{-n} dW \) on \( \mathbb{C}^{(n-k) \times k} \). Since the complex Grassmannian manifold \( G_{n,k} \) is an irreducible rank \( k \) symmetric Kähler manifold, its Ricci curvature can be computed explicitly, see Calabi-Vesentini [14]. As a consequence, we obtain several quantitative functional inequalities satisfied by this invariant measure like the family of Beckner-Sobolev inequalities which include the Poincaré and log-Sobolev inequalities. In particular, one obtains an explicit rate of convergence to equilibrium for \( W_t \). Let us note that since \( G_{n,k} \) is a Kähler manifold, the constants we get from [7] for the Poincaré or log-Sobolev inequality are sharper than the ones obtained by just applying classical curvature dimension criteria.

In Section 3, we study the process \( J_t := WW^*_t, t \geq 0 \). We first show that it is a matrix diffusion process solving the stochastic differential equation
\[
dJ = \sqrt{I_k + JdB^*} \sqrt{I_k + JdB} \sqrt{I_k + JdB^*} \sqrt{I_k + JdB} dt + 2(n - k + \text{tr}(J))(I_k + J)dt
\]
where \( (B_t)_{t \geq 0} \) is a \( k \times k \)-complex-matrix-valued Brownian motion. The study of the process \( J \) is inspired by the existing body of results concerning the Wishart processes, see for instance [13] and [17] and some of the techniques we use are similar to the techniques presented by Yan Doumerc in his Phd thesis [19]. Actually the process \( J \) or rather \( (I_k - J)(I_k + J)^{-1} \) might be thought of as a complex projective analogue of the real Doumerc-Jacobi processes introduced in [19], see also [16], [25] and [26]. We then turn to the study of the eigenvalues process of \( J \). We show that it is a diffusion process with the non-colliding property and then give the
proof of Theorem 1.1. As an interesting byproduct of Theorem 1.1 we can give an algebraic formula for the zonal spherical eigenfunctions of $G_{n,k}$, see Remark 3.12. Let us also note that as a consequence of Bakry-Émery theory the functional inequalities valid for $\mu$ descend to $\nu$.

In Section 4, we study the horizontal lift to $V_{n,k}$ of the Brownian motion on $G_{n,k}$ and prove Theorem 1.2 and Theorem 1.3. Those results can be seen as an extension to the Stiefel fibration setting of results proved in [8] for the Hopf fibration. In particular, the functional $\int_{W[0,t]} \text{tr}(\eta)$ naturally plays the role of a generalized stochastic area process for the Brownian motion $W$ and Theorem 1.3 which follows from its study is a generalization of some of the results proved in that paper. To study the distribution of $\int_{W[0,t]} \text{tr}(\eta)$ and prove Theorem 1.3 the method we use, a matrix Girsanov transform, takes its root in the celebrated paper [38] by M. Yor.

In Section 5, we collect some of the most computational or routine proofs.

As a conclusion, let us point out that it has been a central problem in random matrix theory to study the limiting behavior of eigenvalues in high dimensions. Free stochastic calculus that was introduced by Kummerer-Speicher [33] and then further developed by Biane-Speicher [12] studies the limiting process of certain type of $n \times n$ random matrices diffusion when $n \to \infty$. We expect that in our situation the study of the complex Grassmannian $G_{n,\alpha n}$ and complex Stiefel spaces $V_{n,\alpha n}$ when $n \to +\infty$ and $\alpha$ is a fixed parameter will yield interesting limit results for the objects we have been studying in this paper. This will possibly be addressed in a later research project.

Notations:

- If $M \in \mathbb{C}^{n \times n}$ is a $n \times n$ matrix with complex entries, we will sometimes denote $M^* = M^T$ its adjoint.
- If $z_i = x_i + iy_i$ is a complex coordinate system
  \[ \frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right). \]
- For $k \geq 1$, $\mathfrak{S}_k$ denotes the permutation group of the set $\{1, \ldots, k\}$ and for $\sigma \in \mathfrak{S}_k$, we denote $\text{sgn}(\sigma)$ its signature.
- Throughout the paper we work on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$ that satisfies the usual conditions.
- If $X$ and $Y$ are semimartingales, we denote $\int X \circ dY$ the Itô integral, $\int X dY$ the Stratonovich integral and $\int dX dY$ or $\langle X, Y \rangle$ the quadratic covariation.
- If $X$ and $Y$ are semimartingales, we write $X \sim Y$ to indicate that $X - Y$ is a bounded variation process.
- For matrix-valued semimartingales $M$ and $N$, the quadratic variation $\int dM dN$ is a matrix such that $\left( \int dM dN \right)_{ij} = \sum_\ell \int dM_{i\ell} dN_{\ell j}$. 5
2 Brownian motion on complex Grassmannian manifolds

2.1 Geometry of the complex Grassmannian manifold and inhomogeneous coordinates

Let \( n \in \mathbb{N}, n \geq 2, \) and \( k \in \{1, \ldots, n\} \). The complex Stiefel manifold \( V_{n,k} \) is the set of unitary \( k \)-frames in \( \mathbb{C}^n \). In matrix notation we have

\[
V_{n,k} = \{ M \in \mathbb{C}^{n \times k} | M^* M = I_k \}.
\]

As such \( V_{n,k} \) is therefore an algebraic compact embedded submanifold of \( \mathbb{C}^{n \times k} \) and inherits from \( \mathbb{C}^{n \times k} \) a Riemannian structure. We note that \( V_{n,1} \) is isometric to the unit sphere \( S^{2n-1} \). There is a right isometric action of the unitary group \( U(k) \) on \( V_{n,k} \), which is simply given by the right matrix multiplication: \( M g, M \in V_{n,k}, g \in U(k) \). The quotient space by this action \( G_{n,k} := V_{n,k} / U(k) \) is the complex Grassmannian manifold. It is a compact manifold of complex dimension \( k(n-k) \). We note that \( G_{n,1} \) can be identified with the set of \( k \)-dimensional subspaces of \( \mathbb{C}^n \). In particular \( G_{n,1} \) is the complex projective space \( \mathbb{C}P^{n-1} \). Since \( G_{n,k} \) and \( G_{n,n-k} \) can be identified with each other via orthogonal complement, without loss of generality we can therefore assume throughout the paper that \( k \leq n-k \).

Let us quickly comment on the Riemannian structure of \( G_{n,k} \) we will be using and which is induced from the one of \( V_{n,k} \). From Example 2.3 in [5], there exists a unique Riemannian metric on \( G_{n,k} \) such that the projection map \( \pi : V_{n,k} \to G_{n,k} \) is a Riemannian submersion. From Example 2.5 in [5] and Theorem 9.80 in [11] the fibers of this submersion are totally geodesic submanifolds of \( V_{n,k} \) which are isometric to \( U(k) \). This therefore yields a fibration:

\[
U(k) \to V_{n,k} \to G_{n,k}
\]

which is often referred to as the Stiefel fibration, see also [3, 29]. We note that for \( k = 1 \) it is nothing else but the classical Hopf fibration considered from the probabilistic viewpoint in [8]:

\[
U(1) \to S^{2n-1} \to \mathbb{C}P^{n-1}.
\]

For further details on the Riemannian geometry of the complex Grassmannian manifolds we also refer to [36, 37], see in particular Theorem 4 in [36].

More concretely, the computation of the Riemannian metric (or equivalently of the Laplace-Beltrami operator) on \( G_{n,k} \) will be carried out explicitly in the next section in a convenient set of local coordinates that we now describe.

In the following, we will use the block notations as below: For any \( U \in U(n) \) and \( A \in u(n) \) we will write

\[
U = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \epsilon \end{pmatrix}
\]

where \( X \in \mathbb{C}^{(n-k)\times k}, Y \in \mathbb{C}^{(n-k)\times (n-k)}, Z \in \mathbb{C}^{k\times k}, W \in \mathbb{C}^{k\times (n-k)} \) and \( \alpha \in \mathbb{C}^{k\times k}, \beta \in \mathbb{C}^{k\times (n-k)}, \gamma \in \mathbb{C}^{(n-k)\times k}, \epsilon \in \mathbb{C}^{(n-k)\times (n-k)} \). We note that since

\[
\begin{pmatrix} X^* & Z^* \\ Y^* & W^* \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} X^* & Z^* \\ Y^* & W^* \end{pmatrix} = I_n,
\]

we have that:
\[ X^*X + Z^*Z = I_k, \quad X^*Y + Z^*W = 0, \quad Y^*Y + W^*W = I_{n-k} \]

and
\[ XX^* + YY^* = I_{n-k}, \quad ZZ^* + WW^* = I_k. \quad (2.1) \]

We consider then the open set \( O \subset V_{n,k} \) given by
\[ O = \{ (X, Z) \in V_{n,k}, \det Z \neq 0 \} \]
and the smooth map \( p : O \rightarrow \mathbb{C}^{(n-k)\times k} \) given by \( p\left( \begin{pmatrix} X \\ Z \end{pmatrix} \right) = XZ^{-1} \). It is clear that for every \( g \in U(k) \) and \( M \in V_{n,k} \), \( p(Mg) = p(M) \). Since \( p \) is a submersion from \( O \) onto its image \( p(O) \subset \mathbb{C}^{(n-k)\times k} \) we deduce that there exists a unique diffeomorphism \( \Psi \) from an open set of \( G_{n,k} \) onto \( \mathbb{C}^{(n-k)\times k} \) such that
\[ \Psi \circ \pi = p. \quad (2.2) \]

The map \( \Psi \) induces a (local) coordinate chart on \( G_{n,k} \) that we call inhomogeneous by analogy with the case \( k = 1 \) which corresponds to the complex projective space.

### 2.2 Brownian motion on \( G_{n,k} \)

In this section, we study the Brownian motion on \( G_{n,k} \) and show how it can be constructed from a Brownian motion on the unitary group \( U(n) \). First, we recall that the Lie algebra \( \mathfrak{u}(n) \) consists of all skew-Hermitian matrices
\[ \mathfrak{u}(n) = \{ X \in \mathbb{C}^{n\times n} | X = -X^* \}, \]
which we equip with the inner product \( \langle X, Y \rangle_{\mathfrak{u}(n)} = -\frac{1}{2} \text{tr}(XY) \). This induces a Riemannian metric on \( U(n) \). With respect to this inner product, an orthonormal basis of \( \mathfrak{u}(n) \) can be given by
\[ \{ E_{ij} - E_{ji}, i(E_{ij} + E_{ji}), T_{ij}, 1 \leq \ell < j \leq n \} \]
where \( E_{ij} = (\delta_{ij}(k, \ell))_{1 \leq k, \ell \leq n} \), \( T_{ij} = \sqrt{2}iE_{ij} \). A Brownian motion on \( \mathfrak{u}(n) \) is then of the form
\[ A_t = \sum_{1 \leq \ell < j \leq n} (E_{\ell j} - E_{j \ell}) B_t^{\ell j} + i(E_{\ell j} + E_{j \ell}) \tilde{B}_t^{\ell j} + \sum_{j=1}^n T_j \hat{B}_t^j, \quad t \geq 0, \]
where \( B_t^{\ell j}, \tilde{B}_t^{\ell j}, \hat{B}_t^j \) are independent standard real Brownian motions. Consider now the matrix-valued process \( (U_t)_{t \geq 0} \) that satisfies the Stratonovich stochastic differential equation:
\[
\begin{cases}
   dU_t = U_t \circ dA_t, \\
   U_0 = \begin{pmatrix} X_0 & Y_0 \\ Z_0 & W_0 \end{pmatrix}, \quad \det Z_0 \neq 0.
\end{cases}
\quad (2.3)
\]

The process \( (U_t)_{t \geq 0} \) is a Brownian motion on \( U(n) \) (which is not started from the identity). The main theorem of the section is the following:
Theorem 2.1. Let $U_t = \begin{pmatrix} X_t & Y_t \\ Z_t & W_t \end{pmatrix}$ be the solution of (2.3). Then,

$$\mathbb{P}(\inf\{t > 0, \det Z_t = 0\} < +\infty) = 0$$

and the process $(W_t)_{t \geq 0} := (X_tZ_t^{-1})_{t \geq 0}$ is a diffusion process with generator given by the diffusion operator $\frac{1}{2}\Delta_{G_{n,k}}$, where

$$\Delta_{G_{n,k}} = 4 \sum_{1 \leq i,i' \leq n-k, 1 \leq j,j' \leq k} (I_{n-k} + WW^*)_{ij}(I_k + W^*W)_{j'j} \frac{\partial^2}{\partial W_{ij} \partial W_{i'j'}}.$$

Proof. See Appendix 5.1.

The following corollary shows that $(W_t)_{t \geq 0}$ is a Brownian motion on $G_{n,k}$ which is read in inhomogeneous coordinates.

Corollary 2.2. Let $(W_t)_{t \geq 0} = (X_tZ_t^{-1})_{t \geq 0}$ be the $\mathbb{C}^{(n-k)\times k}$-valued process defined in Theorem 2.1 and $\Psi$ the map defined by (2.2) then the process $(\Psi^{-1}(W_t))_{t \geq 0}$ is a Brownian motion on $G_{n,k}$ and therefore $\Delta_{G_{n,k}}$ is the Laplace-Beltrami operator of $G_{n,k}$ in inhomogeneous coordinates.

Proof. The smooth map $p : \mathcal{O} \subset V_{n,k} \rightarrow \mathbb{C}^{(n-k)\times k}$ given by $p \left( \begin{array}{c} X \\ Z \end{array} \right) = XZ^{-1}$ is a submersion and the process $(X_tZ_t)$ is a Brownian motion on $V_{n,k}$. Let us now observe that

$$\Delta_{G_{n,k}} = 4 \sum_{1 \leq i,i' \leq n-k, 1 \leq j,j' \leq k} (I_{n-k} + WW^*)_{ij}(I_k + W^*W)_{j'j} \frac{\partial^2}{\partial W_{ij} \partial W_{i'j'}}$$

is the Laplace-Beltrami operator of a Riemannian metric on $\mathbb{C}^{(n-k)\times k}$ which is easy to compute. From Theorem 2.1 $W_t = p \left( \begin{array}{c} X_t \\ Z_t \end{array} \right)$ is a Brownian motion for this Riemannian metric. This implies that $p$ is a Riemannian submersion and thus, since $\Psi \circ \pi = p$, that $\Psi$ is an isometry. We conclude that $(\Psi^{-1}(W_t))_{t \geq 0}$ is indeed a Brownian motion on $G_{n,k}$.

Thanks to this corollary, we can refer to $W$ as a Brownian motion on $G_{n,k}$. Since $\Psi$ is an isometry, if needed, we can also identify $\mathbb{C}^{(n-k)\times k}$ with an open subset of $G_{n,k}$. Note that in this description of $G_{n,k}$ we are “missing” the boundary set $\det Z = 0$, but that this set is polar for the Brownian motion (according to Lemma 5.1). When $k = 1$ we recover the expression for the Laplacian in inhomogeneous coordinates:

$$\Delta_{CP^{n-1}} = 4(1 + |w|^2) \sum_{k=1}^{n-1} \frac{\partial^2}{\partial w_k \partial \overline{w}_k} + 4(1 + |w|^2)\mathcal{R}\overline{\mathcal{R}}$$

where

$$\mathcal{R} = \sum_{j=1}^{n-1} w_j \frac{\partial}{\partial w_j}.$$

We refer to [8] and [9] for a review of the Brownian motion on $CP^{n-1}$. 

8
2.3 Invariant probability and convergence to equilibrium

We now study the invariant probability measure on $G_{n,k}$ and the convergence to equilibrium of the Brownian motion to this measure. Let us consider on $G_{n,k}$ the probability measure given in inhomogeneous coordinates by

$$d\mu := c_{n,k} \det(I_k + W^*W)^{-n}dW$$

where $c_{n,k}$ is the normalization constant and $dW$ the Lebesgue measure on $\mathbb{C}^{(n-k)\times k}$.

Proposition 2.3. The probability measure $\mu$ is invariant and symmetric for the operator $\Delta G_{n,k}$. More precisely, for every smooth and compactly supported functions $f, g$ on $\mathbb{C}^{(n-k)\times k}$ the following integration by parts formula holds

$$\int (\Delta G_{n,k} f) g d\mu = \int f (\Delta G_{n,k} g) d\mu = -\int \Gamma(f, g) d\mu,$$

where the carré du champ operator

$$\Gamma(f, g) := \frac{1}{2} \left( \Delta G_{n,k} (fg) - (\Delta G_{n,k} f)g - (\Delta G_{n,k} g)f \right)$$

is given by

$$\Gamma(f, g) = 2 \sum_{1 \leq i, i' \leq n-k, 1 \leq j, j' \leq k} (I_{n-k} + WW^*)_{ii'}(I_k + W^*W)_{jj'} \left( \frac{\partial f}{\partial W_{ij}} \frac{\partial g}{\partial W_{i'j'}} + \frac{\partial g}{\partial W_{ij}} \frac{\partial f}{\partial W_{i'j'}} \right).$$

Proof. See Section 5.2.

We obtain an interesting corollary about the distribution of some random matrices.

Corollary 2.4. Let $U = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be a random variable on $U(n)$ be distributed according to the (normalized) Haar measure on $U(n)$. Then, the random variable

$$W = XZ^{-1} \in \mathbb{C}^{(n-k)\times k}$$

has density $c_{n,k} \det(I_k + W^*W)^{-n}$ with respect to the Lebesgue measure.

Proof. If $U = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ is distributed according to the (normalized) Haar measure on $U(n)$, then $\begin{pmatrix} X \\ Z \end{pmatrix}$ is distributed according to the (normalized) Riemannian volume measure on the Stiefel manifold $V_{n,k}$. Thus, since $p \begin{pmatrix} X \\ Z \end{pmatrix} = XZ^{-1}$ is a totally geodesic Riemannian submersion, one deduces that $XZ^{-1}$ is distributed according to the Riemannian volume of $G_{n,k}$ in inhomogeneous coordinates, which is $\mu$ thanks to Proposition 2.3.

We now discuss the convergence of the Brownian motion on $G_{n,k}$ to the invariant probability $\mu$ and related functional inequalities. The basic lemma is the following.
Lemma 2.5. The Riemannian manifold $G_{n,k}$ is an Einstein manifold with constant Ricci curvature $2n$.

Proof. The complex Grassmannian manifold $G_{n,k}$ is an irreducible rank $k$ symmetric Kähler manifolds and thus is an Einstein manifold, see Calabi-Vesentini [14]. The value of the Einstein constant can be seen from the expansion of the Calabi diastasis $D(0, W)$ in a neighborhood of the origin (see [14] page 502 for further details):

$$D(0, W) = \log \det(I_k + W^*W)^{-1} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} tr((W^*W)^{\ell}) = \sum_{i,j} |W_{ij}|^2 - \frac{1}{2} \sum_{i,j,p,q} W_{ij} W_{iq} W_{pq} W_{pj} + o(|W|^4).$$

From the expansion we know that $W_{ij}$ are canonical coordinates at the origin, hence we can compute the (complex) curvature tensor $R_{ijkl}$ at the origin by differentiating four times $D(0, W)$:

$$R_{ijkl}(0) = -2(\delta_{il} \delta_{su} \delta_{jv} \delta_{qt} + \delta_{iu} \delta_{ps} \delta_{jq} \delta_{tv}).$$

The (complex) Ricci tensor is then given by

$$R_{ij} = \sum_{p,q,s,t} R_{ijkl} g_{pq} g_{st} = k \delta_{iu} \delta_{jv} + (n-k) \delta_{iu} \delta_{jv} = ng_{ij}. $$

The Einstein constant of $G_{n,k}$ is thus $2n$. \hfill \square

An explicit formula for the Ricci curvature provides information about various functional inequalities satisfied by the invariant probability measure $\mu$. In particular, since $G_{n,k}$ is additionally a Kähler manifold, one deduces (see [7]) that $\mu$ satisfies the following log-Sobolev inequality

$$\int_{\mathbb{C}^{(n-k)\times k}} f^2 \ln f^2 d\mu - \left( \int_{\mathbb{C}^{(n-k)\times k}} f^2 d\mu \right) \ln \left( \int_{\mathbb{C}^{(n-k)\times k}} f^2 d\mu \right) \leq \frac{k(n-k)}{(k(n-k)+1)n} \int_{\mathbb{C}^{(n-k)\times k}} \Gamma(f,f) d\mu, \quad (2.4)$$

and the following Poincaré inequality

$$\int_{\mathbb{C}^{(n-k)\times k}} f^2 d\mu - \int_{\mathbb{C}^{(n-k)\times k}} f^2 d\mu \leq \frac{1}{4n} \int_{\mathbb{C}^{(n-k)\times k}} \Gamma(f,f) d\mu. \quad (2.5)$$

We note that the constant $\frac{1}{4n}$ is sharp for the Poincaré inequality (2.5) because the first eigenvalue of $G_{n,k}$ is indeed equal to $4n$, see Theorem 3.9. It is not known if the constant $\frac{k(n-k)}{(k(n-k)+1)n}$ is sharp or not for the log-Sobolev inequality (2.4). From this, one can easily deduces that $(W_t)_{t\geq 0}$ converges exponentially fast to equilibrium with an explicit rate that can be estimated. In particular, we obtain for instance:
Corollary 2.6. When \( t \to +\infty \), \( W_t \to \mu \) in distribution. Moreover, we have the following quantitative estimate: There exists a constant \( C > 0 \) such that for any bounded Borel function \( f \) on \( \mathbb{C}^{(n-k) \times k} \) and \( t \geq 0 \),

\[
\left| \mathbb{E}(f(W_t)) - \int_{\mathbb{C}^{(n-k) \times k}} f d\mu \right| \leq Ce^{-2nt\|f\|_\infty}.
\]

Proof. The estimate classically follows from the Poincaré inequality (2.3) by heat semigroup theory.

3 Eigenvalues process

We now turn to the study of the eigenvalues process of the random matrices \((W^*_tW_t)_{t \geq 0}\)

3.1 The \( J \) process

In this section we study the \( \mathbb{C}^{k \times k} \) valued stochastic process \( J_t := W^*_tW_t \) where, as before, \( W_t \) is a Brownian motion on \( G_{n,k} \) i.e. a diffusion with generator \( \frac{1}{2} \Delta_{G_{n,k}} \). Let \( \mathcal{H}_k \) denote the set of \( k \times k \) Hermitian matrices and let \( \hat{\mathcal{H}}_k \) be the definite positive cone in \( \mathcal{H}_k \). We assume that \( J_0 \in \hat{\mathcal{H}}_k \), and consider the stopping time

\[
T = \inf\{t > 0, J_t \notin \mathcal{H}_k \} = \inf\{t > 0, \det(J_t) = 0 \}.
\]

Theorem 3.1. Let \((J_t)_{t \geq 0}\) be given as above, then up to time \( T \), it satisfies the following stochastic differential equation

\[
dJ = \sqrt{I_k + J}d\mathbf{B} \sqrt{I_k + J} + \sqrt{J}d\mathbf{B} \sqrt{J} + J + 2(n - k + \text{tr}(J))(I_k + J)dt \tag{3.6}
\]

where \((B_t)_{t \geq 0}\) is a Brownian motion in \( \mathbb{C}^{k \times k} \).

Proof. See Section 5.3.

Our next goal is to prove that \( \mathbb{P}(T < +\infty) = 0 \) so that the stochastic differential equation (3.6) is actually defined for all \( t \geq 0 \). First we compute determinants related to \( J \) in the lemma below.

Lemma 3.2. Let \((J_t)_{t \geq 0}\) be as previously defined. We have for any \( t \geq 0 \) that

\[
d(\det(J)) = \det(J)\text{tr}\left(\frac{J^{-1/2}}{2}(I_k + J)(d\mathbf{B} + d\mathbf{B}^*)\right)
+ 2\det(J)\left(2k - 2k^2 + nk + \text{tr}(J) + (n+1-2k)\text{tr}(J^{-1})\right)dt, \tag{3.7}
\]

where \( \mathbf{B} \) is a \( k \times k \)-matrix-valued Brownian motion. As a consequence we have

\[
d(\log \det(J)) = \text{tr}\left(\frac{J^{-1/2}}{2}(I_k + J)(d\mathbf{B} + d\mathbf{B}^*)\right)
+ 2\left(k(n - 2k) + (n - 2k)\text{tr}(J^{-1})\right)dt. \tag{3.8}
\]
Proof. See Section 5.4.

**Proposition 3.3.** Assume \( J_0 \in \hat{H}_k \). Then, we have almost surely that \( T = +\infty \).

**Proof.** Consider the process \( (\Gamma_t := \log(\det(J_t)))_{t \geq 0} \). From the above lemma we have
\[
d\Gamma = tr\left(H(dB + dB^*)\right) + Vdt
\]
where \( H = J^{-1/2}(I_k + J) \) and \( V = 2\left(k(n - 2k) + (n - 2k)tr(J^{-1})\right) \). The local martingale part is a time-changed Brownian motion \( \beta_{C_t} \) and \( V \geq 2k(n - 2k) \). Hence we have
\[
\Gamma_t - \Gamma_0 - 2k(n - 2k)t \geq \beta_{C_t}.
\]
On \( \{T < +\infty\} \), we then have \( \lim_{t\to T} \beta_{C_t} = -\infty \). This implies that \( P(T < +\infty) = 0 \).

### 3.2 Eigenvalues process

In this section we study the eigenvalues of the process \((J_t)_{t \geq 0}\). We denote by \( \lambda(t) = (\lambda_i(t))_{1 \leq i \leq k} \) the eigenvalues of \( J_t \), \( t \geq 0 \). Let \( \mathcal{N} = \{\lambda \in (0, \infty)^k, \lambda_i \neq \lambda_j, \forall i \neq j\} \), and consider the stopping time
\[
\tau_{\mathcal{N}} = \inf\{t > 0, \lambda(t) \notin \mathcal{N}\}. \tag{3.9}
\]

**Theorem 3.4.** Assume \( \lambda(0) \in \mathcal{N} \). Then up to time \( \tau_{\mathcal{N}} \), the eigenvalues \( \lambda(t) = (\lambda_1, \ldots, \lambda_k)(t), t \geq 0 \) satisfy the following stochastic differential equation
\[
d\lambda_i = 2(1 + \lambda_i)\sqrt{\lambda_i}dB^i + 2(1 + \lambda_i)\left(n - 2k + 1 - (2k - 3)\lambda_i + 2\lambda_i(1 + \lambda_i)\sum_{\ell \neq i} \frac{1}{\lambda_i - \lambda_\ell}\right)dt. \tag{3.10}
\]
where \((B_t)_{t \geq 0}\) is a Brownian motion in \( \mathbb{R}^k \).

**Proof.** See Section 5.5.

The next theorem establishes the non-collision property for the eigenvalues process.

**Theorem 3.5.** Let \( \lambda \) be the eigenvalues process of \( J \) and \( \tau_{\mathcal{N}} \) the stopping time as given in (3.9). Assume that at time \( t = 0, \lambda(0) \in \mathcal{N} \). Then for all \( t \geq 0, \lambda(t) \in \mathcal{N} \) a.s. and therefore
\[
P(\tau_{\mathcal{N}} < +\infty) = 0.
\]

**Proof.** See Section 5.6.

We can simplify the stochastic differential equation (3.10) with a simple algebraic transformation.

**Corollary 3.6.** Let \( \rho_i = \frac{1 - \lambda_i}{\lambda_i} \), \( i = 1, \ldots, k \). Then, we have
\[
d\rho_i = -2\sqrt{1 - \rho_i^2}dB^i - 2\left(n - 2k + (n - 2k)\rho_i + 2\sum_{\ell \neq i} \frac{1 - \rho_\ell^2}{\rho_\ell - \rho_i}\right)dt,
\]
where \((B_t)_{t \geq 0}\) is the same Brownian motion as in (3.10).
Proof. From (3.10) we have
\[ d\rho_i = -\frac{2}{(1 + \lambda_i)^2} d\lambda_i + \frac{2}{(1 + \lambda_i)^3} d(\lambda_i) \]
\[ = -2\sqrt{1 - \rho_i^2} dB^i - 2 \left( (n - 2k + (n - 2k + 2)\rho_i) + 2 \sum_{\ell \neq i} \frac{1 - \rho_i^2}{\rho_\ell - \rho_i} \right) dt. \]

\[ \square \]

### 3.3 Distribution and limit law of the eigenvalues

In this section, we denote as before by \( \lambda \) the eigenvalues process of \( J \) and \( \rho_i = \frac{1}{1 + \lambda_i} \). Corollary 3.6 and Theorem 3.5 show that \( \rho \) is a diffusion process with generator given by
\[ \mathcal{L}_{n,k} = 2 \sum_{i=1}^{k} (1 - \rho_i^2) \partial_i^2 - 2 \sum_{i=1}^{k} \left( (n - 2k + (n - 2k + 2)\rho_i) + 2 \sum_{\ell \neq i} \frac{1 - \rho_i^2}{\rho_\ell - \rho_i} \right) \partial_i. \]

Note that we can also write
\[ \mathcal{L}_{n,k} = 2\mathcal{G}_{n-2k,0} - 4 \sum_{i=1}^{k} \sum_{\ell \neq i} \frac{1 - \rho_i^2}{\rho_\ell - \rho_i} \partial_i, \]

where \( \mathcal{G}_{\alpha,\beta} = \sum_{i=1}^{k} (1 - \rho_i^2) \partial_i^2 - (\alpha - \beta + (\alpha + \beta + 2)\rho_i) \partial_i \) is a sum of Jacobi diffusion operators on \([-1, 1]\). In fact, we will show that the semigroup generated by \( \mathcal{L}_{n,k} \) turns out to be the ground state conditioned Karlin-McGregor semigroup associated with \( 2\mathcal{G}_{n-2k,0} \). We refer to [2] for a general overview of Karlin-McGregor semigroups.

**Lemma 3.7.** Consider the Vandermonde function
\[ h(\rho) = \prod_{i>j} (\rho_i - \rho_j). \]

We have for every smooth function \( f \) on \([-1, 1]^k:\)
\[ \mathcal{L}_{n,k} f = 2 \left( \frac{1}{k} \mathcal{G}_{n-2k,0}(hf) + \frac{1}{6} k(k - 1)(3n - 4k + 2)f \right). \]

**Proof.** Let
\[ \Gamma(f, g) := \frac{1}{2} (\mathcal{G}_{\alpha,\beta}(fg) - f\mathcal{G}_{\alpha,\beta}g - g\mathcal{G}_{\alpha,\beta}(f)) \]
be the carré du champ operator associated to \( \mathcal{G}_{\alpha,\beta} \). We have
\[ \Gamma(h, f) = \sum_{i=1}^{k} (1 - \rho_i^2) (\partial_i h)(\partial_i f). \]
From the definition of $h$ it is clear that
\[ \partial_i h = h \sum_{\ell \neq i} \frac{1}{\rho_i - \rho_\ell}, \]
thus, we obtain
\[ \Gamma(\log h, f) = \sum_{i=1}^{k} \sum_{j \neq i} \frac{1 - \rho_i^2}{\rho_i - \rho_\ell} \partial_i f. \]

On the other hand, thanks to a direct computation (or Proposition 12.1.1 in [19])
\[ G_{\alpha,\beta} h = \sum_{i=1}^{k} (1 - \rho_i^2) \partial_i^2 h - \sum_{i=1}^{k} (\alpha - \beta + (\alpha + \beta + 2)\rho_i) \partial_i h \]
\[ = -k(k-1) \left( \frac{k-2}{3} + \frac{\alpha + \beta + 2}{2} \right) h. \]

In particular, one has
\[ G_{n-2k,0}(h) = -\frac{1}{6} k(k-1) (3n - 4k + 2) h. \]

We conclude
\[ \frac{1}{h} G_{n-2k,0}(hf) = \frac{1}{h} (G_{n-2k,0}(h)f + G_{n-2k,0}(f)h + 2\Gamma(f, h)) \]
\[ = -\frac{1}{6} k(k-1) (3n - 4k + 2) f + G_{n-2k,0}(f) + 2\Gamma(\log h, f) \]
\[ = -\frac{1}{6} k(k-1) (3n - 4k + 2) f + \frac{1}{2} L_{n,k} f. \]

Thanks to this lemma, we can compute the density at time $t > 0$ of the random vector $\rho(t)$. To fix notations we first give some reminders about one-dimensional Jacobi diffusion operators. Let
\[ \mathcal{J}^{\alpha,\beta} = (1 - x^2) \frac{\partial^2}{\partial x^2} - ((\alpha + \beta + 2)x + \alpha - \beta) \frac{\partial}{\partial x} \]
be the one-dimensional Jacobi operator. The spectrum and eigenfunctions of $\mathcal{J}^{\alpha,\beta}$ are known and can be described in terms of the Jacobi polynomials. Let us denote by $P_m^{\alpha,\beta}(x), m \in \mathbb{Z}_{\geq 0}$ the Jacobi polynomials given by
\[ P_m^{\alpha,\beta}(x) = \frac{(-1)^m}{2^m m!(1-x)^{\alpha}(1+x)^{\beta}} \frac{d^m}{dx^m}((1-x)^{\alpha+m}(1+x)^{\beta+m}). \]
The family $\{P_m^{\alpha,\beta}(x)\}_{m \geq 0}$ is orthonormal in $L^2([-1, 1], 2^{-\alpha-\beta-1}(1+x)^{\beta}(1-x)^{\alpha}dx)$ and satisfies
\[ \mathcal{J}^{\alpha,\beta} P_m^{\alpha,\beta}(x) = -m(m + \alpha + \beta + 1) P_m^{\alpha,\beta}(x). \]
If we denote by \( p_t^{\alpha,\beta}(x,y) \) the transition density, with respect to the Lebesgue measure, of the diffusion with generator \( 2\mathcal{J}^{\alpha,\beta} \) and initiated from \( x \in (-1,1) \), then we have

\[
p_t^{\alpha,\beta}(x,y) = \frac{(1+y)^\beta(1-y)^\alpha}{2^{\alpha+\beta+1}} \sum_{m=0}^{+\infty} c_{m,\alpha,\beta} e^{-2m(m+\alpha+\beta+1)t} P_m^{\alpha,\beta}(x) P_m^{\alpha,\beta}(y),
\]

where \( c_{m,\alpha,\beta} = (2m+\alpha+\beta+1) \frac{\Gamma(m+\alpha+\beta+1)\Gamma(m+\beta+1)}{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)} \).

We can now state the main theorem of the section:

**Theorem 3.8.** Let \( \lambda \) be the eigenvalues process of \( J \) and \( \rho_i = \frac{1-\lambda_i}{1+\lambda_i} \). Let us assume that

\( \rho_1(0) < \cdots < \rho_k(0) \).

The density at time \( t > 0 \) of \( \rho(t) \) with respect to the Lebesgue measure \( dx \) on \([-1,1]^k\) is given by

\[
e^{\frac{1}{3}k(k-1)(3n-4k+2)t} \frac{1}{h(\rho(0))} e^{2G_{n-2k,2k-2}(h)}(h f)(\rho(0)) = \frac{1}{h(\rho(0))} \int_{[-1,1]^k} h(x) p_t^{n-2k,0}(\rho_1(0),x_1) \cdots p_t^{n-2k,0}(\rho_k(0),x_k)f(x) \, dx
\]

where

\( \Delta_k := \{ -1 \leq x_1 < \cdots < x_k \leq 1 \} \).

**Proof.** Let \( f \) be a smooth function defined on the simplex \( \Delta_k \). We almost everywhere extend \( f \) to \([-1,1]^k\) by symmetrization, i.e. for every permutation \( \sigma \in \mathfrak{S}_k \),

\( f(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) = f(x_1, \cdots, x_k) \).

It follows from the intertwining of generators

\[
\mathcal{L}_{n,k} = 2 \left( \frac{1}{h} G_{n-2k,2k-2}(h \cdot) + \frac{1}{6} k(k-1)(3n-4k+2) \right)
\]

that for the corresponding semigroups

\[
e^{t\mathcal{L}_{n,k}} f(\rho(0)) \]

\[
e^{\frac{1}{3}k(k-1)(3n-4k+2)t} \frac{1}{h(\rho(0))} e^{2G_{n-2k,2k-2}(h f)(\rho(0))} \]

\[
e^{\frac{1}{3}k(k-1)(3n-4k+2)t} \frac{1}{h(\rho(0))} \int_{[-1,1]^k} h(x) p_t^{n-2k,0}(\rho_1(0),x_1) \cdots p_t^{n-2k,0}(\rho_k(0),x_k)f(x) \, dx
\]

\[
e^{\frac{1}{3}k(k-1)(3n-4k+2)t} \sum_{\sigma \in \mathfrak{S}_k} \int_{\Delta_k} h(x) p_t^{n-2k,0}(\rho_{\sigma(1)}(0),x_1) \cdots p_t^{n-2k,0}(\rho_{\sigma(k)}(0),x_k)f(x) \, dx.
\]

The conclusion follows immediately. \( \square \)
We can deduce the limit law of \( \rho \).

**Theorem 3.9.** Let \( \lambda \) be the eigenvalues process of \( J \) and \( \rho_i = \frac{1-\lambda_i}{1+\lambda_i} \). Assume that
\[
\rho_1(0) < \cdots < \rho_k(0).
\]
Then, when \( t \to +\infty \), \( \rho(t) \) converges in distribution to the probability measure on \([-1,1]^k\) given by
\[
d\nu = c_{n,k} \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \prod_{i=1}^k (1 - x_i)^{n-2k} \mathbf{1}_{\Delta_k}(x)dx,
\]
where \( c_{n,k} \) is the normalization constant. Moreover, we have the following quantitative estimate: There exists a constant \( C > 0 \) such that for any bounded Borel function \( f \) on the simplex \( \Delta_k \) and \( t \geq 0 \),
\[
\left| \mathbb{E}(f(\rho(t))) - \int_{\Delta_k} f d\nu \right| \leq Ce^{-2nt\|f\|_{\infty}}.
\]

**Proof.** Using the formula (3.11) we can write
\[
p_t^{(n-2k,0)}(x,y) = (1-y)^{n-2k} \sum_{m=0}^{+\infty} C_m e^{-2m(m+n-2k+1)t} P_m^{n-2k,0}(x) P_m^{n-2k,0}(y),
\]
for some constants \( C_m \). Similarly to Section 3.9.1 in \([\text{2}]\), we now compute
\[
\det \left( p_t^{(n-2k,0)}(\rho_i(0), x_j) \right)_{1 \leq i, j \leq k}
\]
\[
= \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \prod_{i=1}^k p_t^{(n-2k,0)}(\rho_{\sigma(i)}(0), x_i)
\]
\[
= \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \prod_{i=1}^k \left( 1 - x_i \right)^{n-2k} \sum_{m=0}^{+\infty} C_m e^{-2m(m+n-2k+1)t} P_m^{n-2k,0}(\rho_{\sigma(i)}(0)) P_m^{n-2k,0}(x_i) \]
\[
= V(x) \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \sum_{m_1, \cdots, m_k = 0}^{+\infty} \prod_{i=1}^k C_m e^{-2m_i(m_i+n-2k+1)t} P_m^{n-2k,0}(\rho_{\sigma(i)}(0)) P_m^{n-2k,0}(x_i)
\]
where \( V(x) = \prod_{i=1}^k (1 - x_i)^{n-2k} \). We can now write
\[
\sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \sum_{m_1, \cdots, m_k = 0}^{+\infty} \prod_{i=1}^k C_m e^{-2m_i(m_i+n-2k+1)t} P_m^{n-2k,0}(\rho_{\sigma(i)}(0)) P_m^{n-2k,0}(x_i)
\]
\[
= \sum_{m_1, \cdots, m_k = 0}^{+\infty} \left( \prod_{i=1}^k C_m e^{-2m_i(m_i+n-2k+1)t} P_m^{n-2k,0}(x_i) \right) \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \prod_{i=1}^k P_m^{n-2k,0}(\rho_{\sigma(i)}(0))
\]
\[
= \sum_{m_1, \cdots, m_k = 0}^{+\infty} \left( \prod_{i=1}^k C_m e^{-2m_i(m_i+n-2k+1)t} P_m^{n-2k,0}(x_i) \right) \det \left( P_m^{n-2k,0}(\rho_j(0)) \right)_{1 \leq i, j \leq k}.
\]
By skew-symmetrization, we can rewrite the previous sum as

\[
\sum_{m_1<\cdots<m_k} \left( \prod_{i=1}^{k} C_{m_i} e^{-2m_i(m_i+n-2k+1)t} \right) \det \left( P_{m_i}^{n-2k,0}(x_j) \right)_{1 \leq i, j \leq k} \det \left( P_{m_i}^{n-2k,0}(\rho_j(0)) \right)_{1 \leq i, j \leq k}.
\]

When \( t \to +\infty \), the term of leading order in this sum corresponds to \((m_1, \ldots, m_k) = (0, 1, \cdots, k-1)\) and, up to a constant, is given by

\[
e^{-\frac{1}{3} k(k-1)(3n-4k+2)t} \det \left( P_{i-1}^{n-2k,0}(x_j) \right)_{1 \leq i, j \leq k} \det \left( P_{i-1}^{n-2k,0}(\rho_j(0)) \right)_{1 \leq i, j \leq k}
\]

which up to a constant is

\[
e^{-\frac{1}{3} k(k-1)(3n-4k+2)t} h(x) h(\rho(0))
\]

where, for the computation of the Vandermonde determinant, we used the fact that the Jacobi polynomial \( P_{m_i} \) is a polynomial of degree \( m \). The next order in \( t \) corresponds to \((m_1, \cdots, m_k) = (0, 1, \cdots, k-2, k)\) which yields \( e^{-2nt} \) in \([3.12]\).

\[\Box\]

**Remark 3.10.** The limit law \( \nu \) is therefore the distribution of a Coulomb gas at inverse temperature \( 2 \) with a logarithmic confinement potential

\[V(x) = -(n-2k) \ln(1-x).\]

It corresponds to a complex Jacobi ensemble in random matrix theory, see for instance \([28]\).

**Remark 3.11.** Since \( \rho_i = \frac{1-\lambda_i}{1+\lambda_i} \), we easily deduce the distribution and the limit law for the eigenvalues process \((\lambda(t))_{t \geq 0}\).

**Remark 3.12.** As a byproduct, the previous proof yields a spectral expansion for the heat kernel of \( \mathcal{L}_{n,k} \) with respect to the Lebesgue measure of the form:

\[
e^{\frac{1}{3} k(k-1)(3n-4k+2)t} \frac{h(x)}{h(\rho(0))} \sum_{m_1<\cdots<m_k} \left( \prod_{i=1}^{k} C_{m_i} e^{-2m_i(m_i+n-2k+1)t} \right) \det \left( P_{m_i}^{n-2k,0}(x_j) \right)_{1 \leq i, j \leq k} \det \left( P_{m_i}^{n-2k,0}(\rho_j(0)) \right)_{1 \leq i, j \leq k}.
\]

From spectral theory, we deduce that if \( 0 \leq m_1 < \cdots < m_k \) are integers the function

\[
\Phi_{m_1, \cdots, m_k} (\rho_1, \cdots, \rho_k) = \frac{\det \left( P_{m_i}^{n-2k,0}(\rho_j) \right)_{1 \leq i, j \leq k}}{\det \left( P_{i-1}^{n-2k,0}(\rho_j) \right)_{1 \leq i, j \leq k}}
\]

is an eigenfunction of \( \mathcal{L}_{n,k} \) associated to the eigenvalue

\[-\frac{1}{3} k(k-1)(3n-4k+2) + 2 \sum_{i=1}^{k} m_i(m_i+n-2k+1).\]

This recovers the Berezin-Karpelevič formula \([10]\) for the zonal spherical eigenfunctions on \( G_{n,k} \), see also \([27]\). In fact, our approach yields an algebraic representation of such eigenfunctions. Indeed \( \Phi_{m_1, \cdots, m_k} \) is a symmetric polynomial (a multivariate Jacobi polynomial) and if we
We refer to the book [4] for the numerous consequences of Corollary 3.14. We obtain the following result.

\[ \Phi_{m_1, \ldots, m_k}^*(M^*XM) = \Phi_{m_1, \ldots, m_k}^*(X) \]
\[ \Phi_{m_1, \ldots, m_k}^*(D) = \Phi_{m_1, \ldots, m_k}(\rho_1, \ldots, \rho_k), \]

then the function \( \Phi_{m_1, \ldots, m_k}^* \) \( (I_k - W^*W)(I_k + W^*W)^{-1} \) is an eigenfunction of \( \Delta_{G_{n,k}} \).

**Remark 3.13.** Let us observe that many functional inequalities for the invariant measure \( \nu \) and the law of \( \rho(t) \) can be obtained as a result of Bakry-Émery theory. Indeed, as before, consider the generator of the diffusion \( (\rho(t))_{t \geq 0} \):

\[ \mathcal{L}_{n,k} = 2 \sum_{i=1}^k \left( 1 - \rho_i^2 \right) \partial_i^2 - 2 \sum_{i=1}^k \left( n - 2k + (n - 2k + 2) \rho_i + 2 \sum_{\ell \neq i} \frac{1 - \rho_i^2}{\rho_\ell - \rho_i} \right) \partial_i. \]

Let now

\[ \Gamma_2(f, f) := \frac{1}{2} \left( \mathcal{L}_{n,k} \Gamma(f, f) - \Gamma(\mathcal{L}_{n,k} f, f) - \Gamma(f, \mathcal{L}_{n,k} f) \right) \]

be the Bakry’s \( \Gamma_2 \) operator where \( \Gamma \) denotes the carré du champ operator of \( \mathcal{L}_{n,k} \). By Lemma 2.5 we deduce that \( \mathcal{L}_{n,k} \) satisfies the curvature dimension inequality \( CD(2n, 2k(n - k)) \), i.e.

\[ \Gamma_2(f, f) \geq \frac{1}{2k(n - k)} \left( \mathcal{L}_{n,k} f \right)^2 + 2n \Gamma(f, f). \]

We refer to the book [4] for the numerous consequences of \( CD(2n, 2k(n - k)) \). Those applications include log-Sobolev inequality or Sobolev inequalities for \( \nu \), Gaussian concentration properties, etc...

To conclude, let us remark that by combining Theorem 3.9 with Corollary 2.6 we immediately obtain the following result.

**Corollary 3.14.** Let \( W \) be a random variable on \( \mathbb{C}^{(n-k) \times k} \) distributed according to the probability law \( \mu \). Then the ordered eigenvalues of \( (I_k - W^*W)(I_k + W^*W)^{-1} \) are distributed according to the probability measure \( \nu \).

**Proof.** Note that this last result could be proved by more direct random matrices computations. Indeed, let \( g \) be a bounded Borel function on \( \mathcal{H}_k \) the set of positive define Hermitian matrices and let \( W \) be a random variable on \( \mathbb{C}^{(n-k) \times k} \) distributed according to \( c_{n,k} \det(I_k + W^*W)^{-n} dW \). Then, from Proposition 1 in [22] one has for some normalization constant \( c'_{n,k} \)

\[ \mathbb{E}(g(W^*W)) = c_{n,k} \int_{\mathbb{C}^{(n-k) \times k}} g(W^*W) \det(I_k + W^*W)^{-n} dW \]
\[ = c'_{n,k} \int_{\mathcal{H}_k} g(S) \det(I_k + S)^{-n} \det(S)^{n-2k} dS. \]
Thus, \( S = W^* W \) is distributed as \( c'_{n,k} \det(I_k + S)^{-n} \det(S)^{n-2k} dS \). The ordered eigenvalues \( \lambda_i \)'s of \( S \) are thus distributed as

\[
c''_{n,k} \prod_{i>j} (\lambda_i - \lambda_j)^2 \left( \prod_{i=1}^k (1 + \lambda_i) \right)^{-n} \left( \prod_{i=1}^k \lambda_i \right)^{n-2k} 1_{\lambda_1 > \cdots > \lambda_n > 0} d\lambda_1 \cdots d\lambda_n,
\]

from which we deduce Corollary 3.14 after the change of variables \( \rho_i = \frac{1-\lambda_i}{1+\lambda_i} \).

\[\square\]

4 Skew-products and asymptotic windings on the complex Stiefel manifold

4.1 Connection form and horizontal Brownian motion

Let us consider the Stiefel fibration

\[ \mathbb{U}(k) \to V_{n,k} \to G_{n,k} \]

that was described in Section 2.1. According to it, one can see the complex Stiefel manifold \( V_{n,k} \) as a \( \mathbb{U}(k) \) principal bundle over \( G_{n,k} \). The next lemma gives a formula for the connection form of this bundle.

**Lemma 4.1.** Consider on \( V_{n,k} = \left\{ \begin{pmatrix} X & Z \end{pmatrix} \in \mathbb{C}^{n \times k}, X^*X + Z^*Z = I_k \right\} \) the \( \mathfrak{u}(k) \)-valued one form

\[
\omega := \frac{1}{2} \left( (X^*Z^*) d\begin{pmatrix} X \\ Z \end{pmatrix} - d(X^*Z^*) \begin{pmatrix} X \\ Z \end{pmatrix} \right) = \frac{1}{2} \left( X^*dX - dX^*X + Z^*dZ - dZ^*Z \right).
\]

Then, \( \omega \) is the connection form of the fibration \( \mathbb{U}(k) \to V_{n,k} \to G_{n,k} \).

**Proof.** We first observe that if \( p = \begin{pmatrix} X \\ Z \end{pmatrix} \in V_{n,k} \), then the tangent space to \( V_{n,k} \) at \( p \) is given by

\[ T_p V_{n,k} = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{C}^{n \times k}, A^*X + X^*A + B^*Z + Z^*B = 0 \right\}. \]

Then, if \( \theta \in \mathfrak{u}_k \), one easily computes that the generator of the one-parameter group \( \{ p \to pe^{i\theta} \}_{t \in \mathbb{R}} \) is given by the vector field on \( V_{n,k} \) whose value at \( p \) is \( \begin{pmatrix} X\theta \\ Z\theta \end{pmatrix} \). Applying \( \omega \) to this vector field yields \( \theta \). To show that \( \omega \) is the connection form it remains therefore to prove that the kernel of \( \omega \) is the horizontal space of the Riemannian submersion \( \begin{pmatrix} X \\ Z \end{pmatrix} \to XZ^{-1} \). This horizontal space at \( p \), say \( \mathcal{H}_p \), is the orthogonal complement of the vertical space at \( p \), which is the subspace \( \mathcal{V}_p \) of \( T_p V_{n,k} \) tangent to the fiber of the submersion. The previous argument shows that

\[
\mathcal{V}_p = \left\{ \begin{pmatrix} X\theta \\ Z\theta \end{pmatrix}, \theta \in \mathfrak{u}(k) \right\}.
\]
Therefore we have
\[ \mathcal{H}_p = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \in T_p V_{n,k}, \forall \theta \in u(k), \; \text{tr} \left( A^*X\theta + B^*Z\theta \right) = 0 \right\}. \]

We deduce
\[ \mathcal{H}_p = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \in T_p V_{n,k}, \; A^*X + B^*Z = X^*A + Z^*B \right\}, \]
from which it is clear that \( \omega|_{\mathcal{H}} = 0 \).

Next, our goal is to describe the horizontal lift to \( V_{n,k} \) of a Brownian motion on \( G_{n,k} \). We still denote by \( p : V_{n,k} \to G_{n,k} \) the Riemannian submersion. A continuous semimartingale \( (M_t)_{t \geq 0} \) on \( V_{n,k} \) is called horizontal if for every \( t \geq 0 \), \( \int_{M[0,t]} \omega = 0 \), where \( \int_{M[0,t]} \omega \) denotes the Stratonovich line integral of \( \omega \) along the paths of \( M \). If \( (N_t)_{t \geq 0} \) is a continuous semimartingale on \( G_{n,k} \) with \( N_0 \in G_{n,k} \), then if \( \tilde{N}_0 \in V_{n,k} \) is such that \( p(\tilde{N}_0) = N_0 \), there exists a unique horizontal continuous semimartingale \( (\tilde{N}_t)_{t \geq 0} \) on \( V_{n,k} \) such that \( p(\tilde{N}_t) = N_t \) for every \( t \geq 0 \).

The semimartingale \( (\tilde{N}_t)_{t \geq 0} \) is then called the horizontal lift at \( \tilde{N}_0 \) of \( (N_t)_{t \geq 0} \) to \( V_{n,k} \). We refer to [6] for a more general description of the horizontal lift of a semimartingale in the context of foliations. We consider on \( G_{n,k} \) the \( u(k) \) valued one-form \( \eta \) given in inhomogeneous coordinates by
\[ \eta := \frac{1}{2} \left( (I_k + W^*W)^{-1/2}(dW^*W - W^*dW)(I_k + W^*W)^{-1/2} \right. \\
\left. - (I_k + W^*W)^{-1/2}d(I_k + W^*W)^{1/2} + d(I_k + W^*W)^{1/2}(I_k + W^*W)^{-1/2} \right). \]

**Theorem 4.2.** Let \( (W_t)_{t \geq 0} \) be a Brownian motion on \( G_{n,k} \) started at \( W_0 \in G_{n,k} \). Let \( \begin{pmatrix} X_0 \\ Z_0 \end{pmatrix} \in V_{n,k} \) such that \( X_0Z_0^{-1} = W_0 \). The process
\[ \tilde{W}_t := \begin{pmatrix} W_t \\ I_k \end{pmatrix}(I_k + W_t^*W_t)^{-1/2}\Theta_t \]
is the horizontal lift at \( \begin{pmatrix} X_0 \\ Z_0 \end{pmatrix} \) of \( (W_t)_{t \geq 0} \) to \( V_{n,k} \) where \( a_t = \int_{W[0,t]} \eta \) and where \( (\Theta_t)_{t \geq 0} \) is the \( U(k) \) valued solution of the Stratonovich stochastic differential equation
\[ \begin{cases} d\Theta_t = o d a_t \Theta_t \\ \Theta_0 = (Z_0Z_0^{*})^{-1/2}Z_0. \end{cases} \]

**Proof.** As before we denote by \( p \) the submersion \( \begin{pmatrix} X \\ Z \end{pmatrix} \to XZ^{-1} \). It is easy to check that for every \( t \geq 0 \), \( p(\tilde{W}_t) = W_t \) and that \( \tilde{W}_0 = \begin{pmatrix} X_0 \\ Z_0 \end{pmatrix} \). It is therefore enough to prove that \( \tilde{W} \) is a horizontal semimartingale, i.e. that \( \int_{\tilde{W}[0,t]} \omega = 0 \). Denote
\[ X_t = W_t(I_k + W_t^*W_t)^{-1/2}\Theta_t, \; Z_t = (I_k + W_t^*W_t)^{-1/2}\Theta_t. \]
A long, but routine, computation shows that
\[
\frac{1}{2} \left( X^* \circ dX - \circ dX^* X + Z^* \circ dZ - \circ dZ^* Z \right)
\]
\[
= - \frac{1}{2} \left( \circ d\Theta^* \Theta - \Theta^* \circ d\Theta + \Theta^* \left( \circ d(I_k + J)^{-1/2} (I_k + J)^{1/2} - (I_k + J)^{1/2} \circ d(I_k + J)^{-1/2} \right) \right)
\]
\[
+ \Theta^* (I_k + J)^{-1/2} (\circ dW^* W - W^* \circ dW) (I_k + J)^{-1/2} \Theta \right).
\]
where \( J = W^* W \). Since \( \circ d\Theta^* = \circ d\Theta^{-1} = -\Theta^{-1} \circ d\Theta \Theta^{-1} \) and \( \circ d\Theta = \circ da \Theta \) with
\[
\circ da = \frac{1}{2} (I_k + J)^{-1/2} (\circ dW^* W - W^* \circ dW) (I_k + J)^{-1/2}
\]
\[
- \frac{1}{2} \left( (I_k + J)^{-1/2} \circ d(I_k + J)^{1/2} - (I_k + J)^{1/2} \circ d(I_k + J)^{-1/2} \right)
\]
we conclude that
\[
\frac{1}{2} \left( X^* \circ dX - \circ dX^* X + Z^* \circ dZ - \circ dZ^* Z \right) = 0
\]
and thus \( \int_{W[0,t]} \omega = 0 \). \( \square \)

### 4.2 Skew-product decomposition of the Stiefel Brownian motion

We now turn to the description of the Brownian motion on \( V_{n,k} \) as a skew-product.

**Theorem 4.3.** Let \((W_t)_{t \geq 0}\) be a Brownian motion on \( G_{n,k} \) started at \( W_0 \in G_{n,k} \) and let \((\Omega_t)_{t \geq 0}\) be a Brownian motion on the unitary group \( U(k) \) independent from \((W_t)_{t \geq 0}\). Let \((\Theta_t)_{t \geq 0}\) be the \( U(k) \) valued solution of the Stratonovitch stochastic differential equation

\[
\begin{cases}
\circ d\Theta_t = \circ da_t \Theta_t \\
\Theta_0 = I_k,
\end{cases}
\]

where \( a_t = \int_{W[0,t]} \eta \). The process

\[
\begin{pmatrix}
X_t \\
Z_t
\end{pmatrix} := \begin{pmatrix}
W_t \\
I_k
\end{pmatrix} (I_k + W_t^* W_t)^{-1/2} \Theta_t \Omega_t
\]

is a Brownian motion on \( V_{n,k} \).

**Proof.** We denote by \( \Delta_H \) the horizontal Laplacian and by \( \Delta_V \) the vertical Laplacian of the Stiefel fibration, see [5]. Since the submersion \( V_{n,k} \to G_{n,k} \) is totally geodesic, the operators \( \Delta_H \) and \( \Delta_V \) commute. We note that the Laplace-Beltrami operator of \( V_{n,k} \) is given by \( \Delta_{V_{n,k}} = \Delta_H + \Delta_V \) and that the horizontal lift of the Brownian motion on \( G_{n,k} \) is a diffusion with generator \( \frac{1}{2} \Delta_H \), see [6]. The fibers of the submersion \( V_{n,k} \to G_{n,k} \) are isometric to \( U(k) \), thus if \( f \) is a smooth function on \( V_{n,k} \), one has

\[
e^{\frac{1}{2} t \Delta_V} f \left( \begin{pmatrix} X \\ Z \end{pmatrix} \right) = \mathbb{E} \left( f \left( \begin{pmatrix} X_{\Omega_t} \\ Z_{\Omega_t} \end{pmatrix} \right) \right).
\]

Since \( e^{\frac{1}{2} t \Delta_V} e^{\frac{1}{2} t \Delta_H} = e^{\frac{1}{2} t \Delta_{V_{n,k}}} \), we conclude from Theorem 4.2. \( \square \)
4.3 A first limit theorem

We now give a limit theorem for the process \( \left( \int_0^t \text{tr} (W_s^* W_s) ds \right)_{t \geq 0} \) that shall be used in the next subsection. The method we use, a Girsanov transform, takes its root in the paper by M. Yor [38] and was further developed in the situation of matrix Wishart diffusions in [18] and in the situation of the Doumerc-Jacobi matrix processes in Section 9.4.2 of the thesis [19]. Our result is the following:

**Theorem 4.4.** As before, let \( J = W^* W \).

- If \( k = 1 \), the following convergence holds in distribution when \( t \to +\infty \)
  \[
  \frac{1}{t^2} \int_0^t \text{tr}(J) ds \to X,
  \]
  where \( X \) is a random variable on \([0, +\infty)\) with density \( \frac{(n-1)}{\sqrt{2\pi x^3}} e^{-\frac{(n-1)^2}{2x}} \) (i.e. \( X \) is the inverse of a gamma distributed random variable).

- If \( k > 1 \), the following convergence holds in probability when \( t \to +\infty \)
  \[
  \frac{1}{t} \int_0^t \text{tr}(J) ds \to \frac{k(n-k)}{k-1}.
  \]

The main ingredient of the proof is the following lemma from which we will be able to apply the Girsanov transform method.

**Lemma 4.5.** For every \( \alpha \geq 0 \) the process

\[
M_t^\alpha = e^{2k\alpha(n-k)t} \left( \frac{\det(I_k + J_0)}{\det(I_k + J_t)} \right)^\alpha \exp \left( -2 \int_0^t (\alpha(k-1) + \alpha^2 \text{tr}(J)) ds \right)
\]

is a martingale.

**Proof.** Consider the exponential local martingale

\[
M_t^\alpha := \exp \left( -\alpha \int_0^t \text{tr}(\sqrt{J}(d\mathbf{B} + d\mathbf{B}^*)) - 2\alpha^2 \int_0^t \text{tr}(J) ds \right),
\]

where \( \mathbf{B} \) is the Brownian motion as given in Theorem 3.1. If we denote \( V = 2k(n-k) - 2(k-1)\text{tr}(J) \), then similar computations as in Section 5.2 yield

\[
d(\log \det(I_k + J)) = \text{tr} \left( \sqrt{J}(d\mathbf{B} + d\mathbf{B}^*) \right) + V dt.
\]

Therefore

\[
\left( \frac{\det(I_k + J_t)}{\det(I_k + J_0)} \right)^\alpha = \exp \left( \alpha \left( \int_0^t \text{tr} (\sqrt{J}(d\mathbf{B} + d\mathbf{B}^*)) + V ds \right) \right),
\]

and thus

\[
M_t^\alpha = e^{2k\alpha(n-k)t} \left( \frac{\det(I_k + J_0)}{\det(I_k + J_t)} \right)^\alpha \exp \left( -2 \int_0^t (\alpha(k-1) + \alpha^2 \text{tr}(J_s) ds \right).
\]

From this expression, it is clear that there exists a constant \( C > 0 \) such that \( |M_t^\alpha| \leq Ce^{2k\alpha(n-k)t} \) and thus \( M_t^\alpha \) is a martingale. \( \square \)
We are now ready for the proof of Theorem 4.4.

Proof of Theorem 4.4. Consider the probability measure $P^\alpha$ defined by

$$P^\alpha |_{\mathcal{F}_t} = M^\alpha_t \cdot P|_{\mathcal{F}_t}.$$ 

From Girsanov theorem, the process

$$\beta_t = B_t + 2\alpha \int_0^t \sqrt{J} ds$$

is under $P^\alpha$ a $k \times k$-matrix-valued Brownian motion. Therefore, under $P^\alpha$, $J$ solves the stochastic differential equation

$$dJ = \sqrt{I_k + J} \sqrt{J} + \sqrt{J} \sqrt{I_k + J} d\beta_t + 2(n - k\alpha J + tr(J)) (I_k + J) dt.$$ 

As a consequence the distribution of the eigenvalues of $J$ under $P^\alpha$ can therefore be computed similarly as in Theorem 3.8. We now note that

$$\mathbb{E} \left( e^{-2(\alpha(k-1)+\alpha^2) \int_0^t \text{tr}(J) ds} \right) = e^{-2k(n-k)\alpha t} \mathbb{E}^\alpha \left( \frac{\det(I_k + J_t)}{\det(I_k + J_0)} \right)^\alpha$$

Therefore, denoting $\mu = \left( \frac{4(\alpha(k-1)+\alpha^2)}{k+3} \right)^\frac{1}{2}$, we obtain

$$\mathbb{E} \left( e^{-\frac{(k+3)\mu^2}{2} \int_0^t \text{tr}(J) ds} \right) = e^{-2k(n-k)\alpha t} \mathbb{E}^\alpha \left( \frac{\det(I_k + J_t)}{\det(I_k + J_0)} \right)^\alpha$$

where $\alpha = \sqrt{\frac{(k+3)\mu^2}{4} + (k-1)^2 - k-1 \frac{1}{2}}$. Hence when $k > 1$ we have

$$\lim_{t \to \infty} \mathbb{E} \left( e^{-\frac{(k+3)\mu^2}{2(t-k-1)} \int_0^t \text{tr}(J) ds} \right)$$

$$= e^{-\frac{k(n-k)(k+3)}{2(k-1)} \mu^2} \lim_{t \to \infty} \mathbb{E} \left( \frac{\det(I_k + J_t)}{\det(I_k + J_0)} \right)^{\sqrt{\frac{(k+3)\mu^2}{4t} + (k-1)^2 - k-1 \frac{1}{2}}}$$

where the last limit can be justified using the formula for the density of the eigenvalues of $J$ under a probability $P^\alpha$. When $k = 1$ we have $\mu = \alpha$ and a similar proof yields

$$\lim_{t \to \infty} \mathbb{E} \left( e^{-\frac{\mu^2}{2t} \int_0^t \text{tr}(J) ds} \right) = e^{-(n-1)\mu}.$$  

\[\square\]
4.4 Asymptotics of stochastic area and winding

By definition of $\eta$, we note that

$$
\int_{W[0,t]} \text{tr}(\eta) = \frac{1}{2} \text{tr} \left[ \int_0^t (I_k + J)^{-1/2} (\circ dW^* W - W^* \circ dW) (I_k + J)^{-1/2} \right]
$$

\begin{align*}
&= \frac{1}{2} \text{tr} \left[ \int_0^t (I_k + J)^{-1/2} (dW^* W - W^* dW) (I_k + J)^{-1/2} \right] \\
&= \frac{1}{2} \text{tr} \left[ \int_0^t (I_k + J)^{-1/2} (dW^* W - W^* dW) (I_k + J)^{-1/2} \right] 
\end{align*}

(4.13)

where as before $J = W^* W$.

From simple computations one can verify that

$$
\text{tr}(d\eta) = \frac{\partial}{\partial \log \det(I_k + W^* W)},
$$

which implies that $\text{itr}(d\eta)$ is the Kähler form on $G_{n,k}$. Therefore $\int_{W[0,t]} \text{tr}(\eta)$ can be considered as a generalized stochastic area process on $G_{n,k}$. In the theorem below we deduce large time limit distributions of such processes.

**Theorem 4.6.** Let $(W_t)_{t \geq 0}$ be a Brownian motion on $G_{n,k}$ started at $W_0 \in G_{n,k}$.

- If $k = 1$, the following convergence holds in distribution when $t \to +\infty$

$$
\frac{1}{it} \int_{W[0,t]} \text{tr}(\eta) \to C_{n-1},
$$

where $C_{n-1}$ is a Cauchy distribution of parameter $n - 1$.

- If $k > 1$, the following convergence holds in distribution when $t \to +\infty$

$$
\frac{1}{i\sqrt{t}} \int_{W[0,t]} \text{tr}(\eta) \to N \left(0, \frac{k(n-k)}{k-1}\right).
$$

**Proof.** From (5.18) we know that

$$
dW^* W - W^* dW = \sqrt{I_k + J} dB^* \sqrt{I_k + J} \sqrt{J} - \sqrt{J} \sqrt{I_k + J} dB \sqrt{I_k + J}
$$

where $(B_t)_{t \geq 0}$ is a $k \times k$-matrix-valued Brownian motion. Therefore,

$$(I_k + J)^{-1/2} (dW^* W - W^* dW) (I_k + J)^{-1/2} = dB^* \sqrt{J} - \sqrt{J} dB$$

Consider the diagonalization of $J = VAV^*$, where $V \in U(k)$ and $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_k\}$, then

$$dB^* \sqrt{J} - \sqrt{J} dB = V(V^{-1} dB^* V \sqrt{\Lambda} - \sqrt{\Lambda} V^{-1} dB V)V^{-1}.$$

Therefore from (4.13), we have in distribution that

$$
\int_{W[0,t]} \text{tr}(\eta) = iB \int_0^t \text{tr}(J) ds
$$

where $B$ is a one-dimensional Brownian motion independent from the process $\text{tr}(J)$. We conclude from Theorem 4.4. \qed
Let \( \left( X_t, Z_t \right) \) be a Brownian motion on \( V_{n,k} \). We are interested in the windings of the complex valued process \( \det(Z_t) \). From Theorem 4.3, we have the polar decomposition

\[
\det(Z_t) = \det(I_k + W_t^* W_t)^{-1/2} \det \Theta_t \det \Omega_t.
\]

**Lemma 4.7.** For every \( t \geq 0 \), \( \det \Theta_t = \exp \left( \int_{[0,t]} \text{tr}(\eta) \right) \).

**Proof.** We have

\[
\begin{cases}
    d\Theta_t = \circ d \left( \int_{[0,t]} \eta \right) \Theta_t \\
    \Theta_0 = I_k.
\end{cases}
\]

Thus from the Chen-Strichartz expansion formula \( \det \Theta_t = \exp \left( \text{tr} \left( \int_{[0,t]} \eta \right) \right) = \exp \left( \int_{[0,t]} \text{tr}(\eta) \right) \).

We immediately deduce the following corollary.

**Corollary 4.8.** Let \( \left( X_t, Z_t \right) \) be a Brownian motion on \( V_{n,k} \) with \( \det(Z_0) \neq 0 \). One has the polar decomposition

\[
\det(Z_t) = \rho_t e^{i\theta_t}
\]

where \( 0 < \rho_t \leq 1 \) and \( \theta_t \) is a continuous semimartingale such that:

- **If** \( k = 1 \), the following convergence holds in distribution when \( t \to +\infty \)

\[
\frac{\theta_t}{t} \to C_{n-1},
\]

where \( C_{n-1} \) is a Cauchy distribution of parameter \( n - 1 \).

- **If** \( k > 1 \), the following convergence holds in distribution when \( t \to +\infty \)

\[
\frac{\theta_t}{\sqrt{t}} \to N \left( 0, \frac{k(n-k)}{k-1} + 2n \right)
\]

**Proof.** From the decomposition \( \det(Z_t) = \det(I_k + W_t^* W_t)^{-1/2} \det \Theta_t \det \Omega_t \) one deduces

\[
\rho_t = \det(I_k + J_t)^{-1/2}, \quad i\theta_t = \text{tr}(\omega_t) + \int_{[0,t]} \text{tr}(\eta)
\]

where \( \omega_t \) is a Brownian motion on \( u(k) \) independent from \( W \). The conclusion follows. \( \square \)
5 Proofs

5.1 Generator of the Brownian motion on $G_{n,k}$

We divide the proof of of Theorem 2.1 in two parts, the first one proves the a.s. invertibility of $Z_t$ and the second one proves that $W_t = X_t Z_t^{-1}$ is a diffusion with generator $\frac{1}{2} \Delta_{G_{n,k}}$. Let us consider the block decomposition

$$A_t = \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \epsilon_t \end{pmatrix},$$

with $\alpha_t \in \mathbb{C}^{k \times k}$. Note that $\alpha_t$, $\beta_t = -\gamma_t^*$ and $\epsilon_t$ are independent. From (2.3) we obtain the following system of stochastic differential equations:

$$dX = X \circ d\alpha + Y \circ d\gamma = Xd\alpha + Yd\gamma + \frac{1}{2} (dXd\alpha + dYd\gamma)$$
$$dY = X \circ d\beta + Y \circ de = Xd\beta + Yde + \frac{1}{2} (dXd\beta + dYde)$$
$$dZ = Z \circ d\alpha + W \circ d\gamma = Zd\alpha + Wd\gamma + \frac{1}{2} (dZd\alpha + dWd\gamma)$$
$$dW = Z \circ d\beta + W \circ de = Zd\beta + Wde + \frac{1}{2} (dZd\beta + dWde).$$

(5.14)

Lemma 5.1. Let $Z_t$ be the bottom left corner of the $U(n)$ valued Brownian motion $U_t$ as defined above and let $\tau_Z := \inf\{t > 0, \det Z_t = 0\}$. Then $\tau_Z = +\infty$ a.s.

Proof. Let $(J_t) := (Z_t Z_t^*)_{t \geq 0}$. From (5.14) and (5.15) we have $dZ = Zd\alpha + Wd\gamma - nZdt$, hence

$$dZZ^* = Zd\alpha Z^* + Wd\gamma Z^* - nJdt.$$

Note that $I_k - ZZ^* = WW^*$, we have

$$dZZ^* = (Zd\alpha + Wd\gamma)(\alpha^* Z^* + d\gamma^* W^*) = Zd\alpha Z^* + Wd\gamma^* W^*$$
$$= 2kdZ^* + 2kdt(I_k - ZZ^*) = 2kI_k dt.$$

Therefore we obtain

$$dJ = dZZ^* + ZdZ^* + dZZ^*$$
$$= Zd\gamma^* W^* + Wd\gamma Z^* + \left(2kI_k - 2nJ\right)dt.$$

Let $B_t = \int_0^t (J)^{-1/2} Zd\gamma^* W^* (I_k - J)^{-1/2}$. We can easily check that $B_t$, $t \geq 0$ is a Brownian motion on $H_k$—the collection of $k \times k$ Hermitian matrices. The martingale part of $dJ$ is then given by

$$Zd\gamma^* W^* + Wd\gamma Z^* = \sqrt{J} dB \sqrt{I_k - J} + \sqrt{I_k - J} dB^* \sqrt{J}.$$

Next we prove $\tau_Z = \infty$ a.s. Apply similar calculation as for (3.8) (see Section 5.4) to $J$ we have that

$$d(\log \det(J)) = tr \left(J^{-1/2} \sqrt{I_k - J}(dB + dB^*) \right) - 2k(n - k)dt.$$
Since the local martingale part of the above SDE is a time-changed Brownian motion $\beta_C t$, we have

$$\log \det(J_t) - \log \det(J_0) - 2k(n-k)t = \beta_C t.$$  

On $\{\tau_Z < \infty\}$, we have $\lim_{t \to \tau_Z} \log \det(J_t) = -\infty$. This implies that $\lim_{t \to \tau_Z} \beta_C t = \infty$, Since Brownian motion never goes to infinity without oscillating, we conclude that $\mathbb{P}(\tau_Z < \infty) = 0$. \[
\]

**Proof of Theorem 2.1.** Since

$$d\alpha = \sum_{1 \leq i < j \leq k} (E_{ij} - E_{ji}) dB^{ij}_t + i(E_{ij} + E_{ji}) dB^{ij}_t + \sum_{j=1}^k T_j dB^j_t$$

we easily compute that

$$d\alpha d\alpha = \left( \sum_{1 \leq i < j \leq k} (E_{ij} - E_{ji})(E_{ij} - E_{ji}) - (E_{ij} + E_{ji})(E_{ij} + E_{ji}) + \sum_{j=1}^k T_j T_j \right) dt$$

$$= -2(k-1)I_k dt - 2I_k dt = -2kI_k dt$$

and similarly we can prove that

$$d\beta d\gamma = -d\beta d\beta^* = -2(n-k)I_k dt.$$  

Hence we get

$$dX d\alpha = X d\alpha d\alpha = -2kX dt, \quad dY d\gamma = X d\beta d\gamma = -2(n-k)X dt$$

$$dZ d\alpha = Z d\alpha d\alpha = -2kZ dt, \quad dW d\gamma = Z d\beta d\gamma = -2(n-k)Z dt.$$  

Therefore we obtain that

$$dX = X \circ d\alpha + Y \circ d\gamma = X d\alpha + Y d\gamma - nX dt$$

$$dZ = Z \circ d\alpha + W \circ d\gamma = Z d\alpha + W d\gamma - nZ dt.$$  

Now consider the process $W_t := X_t Z_t^{-1}$, which satisfies that

$$dW = dX Z^{-1} + X dZ^{-1} + dX dZ^{-1}.$$  

Note $ZdZ^{-1} = -dZZ^{-1} - dZdZ^{-1}$, hence we have

$$dW = dX Z^{-1} - WdZ Z^{-1} - WdZdZ^{-1} + dX dZ^{-1}$$

$$= (Xd\alpha + Yd\gamma - nXdt)Z^{-1} - W(Zd\alpha + Wd\gamma - nZdt)Z^{-1} - WdZdZ^{-1} + dX dZ^{-1}$$

$$= Yd\gamma Z^{-1} - WWd\gamma Z^{-1} - WdZdZ^{-1} + dX dZ^{-1}.$$  

Note that for the finite variation part of $dW$ we have

$$-WdZdZ^{-1} + dX dZ^{-1} = WdZZ^{-1}dZ^{-1} - dX Z^{-1}dZ Z^{-1}$$

$$= W(Zd\alpha + Wd\gamma)Z^{-1}(Zd\alpha + Wd\gamma)Z^{-1} - (Xd\alpha + Yd\gamma)Z^{-1}(Zd\alpha + Wd\gamma)Z^{-1}$$

$$= WZZd\alpha Z^{-1} - X(d\alpha d\alpha)Z^{-1} = 0.$$  

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Hence we have
\[ dW = Y d\gamma Z^{-1} - W W d\gamma Z^{-1}. \]

We are now in position to prove that \( W \) is a matrix diffusion process using the above formula. Since for \( 1 \leq i \leq n - k, 1 \leq j \leq k \),

\[ dW_{ij} = \sum_{\ell=1}^{k} (Y - WW)_{i\ell} (d\gamma Z^{-1})_{\ell j}, \]

we have
\[ dW_{ij} dW_{i'j'} = \sum_{\ell,m=1}^{k} (Y - WW)_{i\ell} (Y - WW)_{i'm} (d\gamma Z^{-1})_{\ell j} (d\gamma Z^{-1})_{mj'}. \]

Moreover, since
\[ (d\gamma Z^{-1})_{\ell j} (d\gamma Z^{-1})_{mj'} = \sum_{p,q=1}^{k} (d\gamma)_{\ell p} (Z^{-1})_{p j} (d\gamma)_{mq} (Z^{-1})_{q j'} = 2 \delta_{ml} dt \sum_{p=1}^{k} (Z^{-1})_{pj} (Z^{-1})_{pj'} \]

we have
\[ dW_{ij} dW_{i'j'} = 2 ((Y - WW)(Y - WW)^T)_{ij} ((ZZ^*)^{-1})_{jj'} dt. \] (5.16)

From (2.1) we know that
\[-XZ^{-1} W Y^* = XX^*,\]

plug into (5.16) we then obtain
\[ dW_{ij} dW_{i'j'} = 2 (I_{n-k} + WW^*)_{ij} ((ZZ^*)^{-1})_{jj'} dt \]
\[ = 2 (I_{n-k} + WW^*)_{ij} (I_k + W^* W)_{jj'} dt. \] (5.17)

Therefore, we conclude that \( (W_t)_{t \geq 0} \) is a diffusion whose generator is given by \( \frac{1}{2} \Delta_{G_{n,k}} \). \( \Box \)

### 5.2 Computation of the volume measure on \( G_{n,k} \)

**Proof.** We denote \( \partial_{ij} = \frac{\partial}{\partial W_{ij}}, \partial_{ij} = \frac{\partial}{\partial W_{ij}}, A_{ii'jj'} = (\delta_{ii'} + (WW^*)_{ii'}) (\delta_{jj'} + (W^* W)_{jj'}), \) and \( \rho = c_{n,k} \det(I_k + W^* W)^{-n}. \) Let us denote

\[ \mathcal{T}(f, g) = 2 \sum_{1 \leq i, i' \leq n-k, 1 \leq j, j' \leq k} (I_{n-k} + WW^*)_{ii'} (I_k + W^* W)_{jj'} \left( \frac{\partial f}{\partial W_{ij}} \frac{\partial g}{\partial W_{i'j'}} + \frac{\partial g}{\partial W_{ij}} \frac{\partial f}{\partial W_{i'j'}} \right) \]
By integration by parts we have
\[- \frac{1}{2} \int (\Delta G_{n,k} f) g \, d\mu \]
\[= \sum_{1 \leq i,i' \leq n-k, 1 \leq j,j' \leq k} \int (\partial_{ij} f) \overline{\partial}_{i'j'} (A_{ii'jj'}) g \rho \, d\mu + \int (\overline{\partial}_{i'j'} f) \partial_{ij} (A_{ii'jj'}) g \rho \, d\mu \]
\[= \frac{1}{2} T(f,g) + \sum_{1 \leq i,i' \leq n-k, 1 \leq j,j' \leq k} \left( \int [(\partial_{ij} f) (\overline{\partial}_{i'j'} A_{ii'jj'}) + (\overline{\partial}_{i'j'} f) (\partial_{ij} A_{ii'jj'})] g \rho \, d\mu \right) \]
\[+ \int [(\partial_{ij} f) (\partial_{i'j'} \rho) + (\overline{\partial}_{i'j'} f) (\partial_{ij} \rho)] g A_{ii'jj'} \, d\mu \]
\[= \frac{1}{2} T(f,g) + R. \]

Since
\[\overline{\partial}_{i'j'} A_{ii'jj'} = W_{ij}(\delta_{i'j'} + (W^*W)_{i'j'}) + (\delta_{ii'} + (WW^*)_{ii'}) W_{i'j'},\]
and
\[\partial_{ij} A_{ii'jj'} = W_{i'j}(\delta_{i'j'} + (W^*W)_{i'j'}) + (\delta_{ii'} + (WW^*)_{ii'}) W_{ij},\]
we have
\[\sum_{1 \leq i' \leq n-k, 1 \leq j' \leq k} \overline{\partial}_{i'j'} A_{ii'jj'} = n (W(I_k + J))_{ij} \]
and
\[\sum_{1 \leq i \leq n-k, 1 \leq j \leq k} \partial_{ij} A_{ii'jj'} = n (W(I_k + J))_{i'j'}.\]

Moreover, since
\[\overline{\partial}_{i'j'} \det(I_k + J) = \det(I_k + J) \sum_{1 \leq p,q \leq k} ((I_k + J)^{-1})_{pq} \overline{\partial}_{i'j'} (I_k + J)_{pq} \]
\[= \det(I_k + J) \left( W(I_k + J)^{-1} \right)_{i'j'},\]
and
\[\partial_{ij} \det(I_k + J) = \det(I_k + J) \left( W(I_k + J)^{-1} \right)_{ij},\]
we have
\[\overline{\partial}_{i'j'} \rho = -n \rho \left( W(I_k + J)^{-1} \right)_{i'j'}, \quad \partial_{ij} \rho = -n \rho \left( W(I_k + J)^{-1} \right)_{ij}.\]

We then have
\[\sum_{i'j'} (\overline{\partial}_{i'j'} A_{ii'jj'}) g \rho + (\overline{\partial}_{i'j'} \rho) g A_{ii'jj'} = 0\]
and
\[\sum_{i,j} (\partial_{ij} A_{ii'jj'}) g \rho + (\partial_{ij} \rho) g A_{ii'jj'} = 0.\]

This gives \( R = 0. \)
5.3 Stochastic differential equation for $J$

Proof of Theorem 3.1. We use the notations of the proof of Theorem 2.1. Recall that

$$dW = (Y - WW)d\gamma Z^{-1}.$$  

We first compute the martingale part of $dJ$:

$$dJ \sim dW^*W + W^*dW$$

$$\sim (Z^{-1})^*d\gamma^*(Y - WW)^*W + W^*(Y - WW)d\gamma Z^{-1}$$

$$\sim (\sqrt{I_k + J})^*(\sqrt{I_k + J})^*(\sqrt{J})^* + \sqrt{J}\sqrt{I_k + J}dB\sqrt{I_k + J}$$  (5.18)

where $B_t$ is a $k \times k$-matrix-valued stochastic process that satisfies

$$dB = \sqrt{I_k + J}^{-1}d\gamma Z^{-1}W^*(Y - WW)d\gamma Z^{-1}\sqrt{I_k + J}^{-1}.$$  

Since for any $1 \leq i, j \leq m$,

$$dB_{ij} = \sum_{k,\ell, s, t, q} (\sqrt{I_k + J}^{-1})_{ik}(\sqrt{J}^{-1})_{k\ell}(W^*)_{\ell s}(Y - WW)_{st}(d\gamma)_{tp}(Z^{-1})_{pq}(\sqrt{I_k + J}^{-1})_{qj},$$

we obtain that

$$\frac{dB_{ij}dB_{i'j'}}{2dt} = \sum_{k,\ell, s, t, q} (\sqrt{I_k + J}^{-1})_{ik}(\sqrt{J}^{-1})_{k\ell}(W^*)_{\ell s}(W)_{s't'}(Y - WW)(Y - WW)^*_{ss'}(Z^{-1})_{pq}(\sqrt{I_k + J}^{-1})_{qj}.$$  

Here we use the fact that $(d\gamma)_{tp}(d\gamma)_{t'q'} = 2dt\delta_{tt'}\delta_{pp'}$. Moreover, since

$$(Y - WW)(Y - WW)^* = I_{n-k} + WW^*,$$

we have

$$\sum_{s, s'} (W)_{ss'}((Y - WW)(Y - WW)^*)_{ss'}W_{s't'} = (J + J^2)_{tt'}.$$  

Also since $(Z^{-1})^*Z^{-1} = I_k + J$, we obtain

$$\frac{dB_{ij}dB_{i'j'}}{2dt} = \delta_{ii'}\delta_{jj'}.$$  

Hence $B$ is a $k \times k$-matrix-valued Brownian motion. It remains to compute the bounded variation part of $J$. We see that the bounded variation part in $dJ$ is given by $dW^*dW$. From (5.17) we have for any $1 \leq i, j \leq k$,

$$(dW^*dW)_{ij} = \sum_{t=1}^{n-k} (d(W^*)_{t'i}(dW)_{tj} = 2\sum_{t=1}^{n-k} (I_{n-k} + WW^*)_{tt}(I_k + W^*W)_{ij}dt$$

Therefore we have

$$dW^*dW = 2(n - k + tr(WW^*))(I_k + J)dt$$

and the proof is complete.  

5.4 Determinant of the $J$ process

Proof of lemma 3.2. By Itô’s formula we know that

$$d(\det(J)) = \sum_{i,j=1}^{k} \frac{\partial \det(J)}{\partial J_{ij}} dJ_{ij} + \frac{1}{2} \sum_{i,j,j'=1}^{k} \frac{\partial^2 \det(J)}{\partial J_{ij} \partial J_{ij'}} dJ_{ij} dJ_{ij'}.$$ 

First we know that

$$\frac{\partial \det(J)}{\partial J_{ij}} = \frac{\partial}{\partial J_{ij}} \sum_{i=1}^{k} J_{ii} J_{ij} = J_{ji},$$

where $J = \det(J) J^{-1}$ is the cofactor of $J$. Hence the first order term in the above SDE is $\det(J) tr(J^{-1} dJ)$. Next, for any $1 \leq i, j, i', j' \leq k$ we have

$$\frac{\partial^2 \det(J)}{\partial J_{ij} \partial J_{i'j'}} = \frac{\partial J_{ji}}{\partial J_{i'j'}} = \frac{\partial \det(J)}{\partial J_{ii'}} (J^{-1})_{ji} + \det(J) \frac{\partial (J^{-1})_{ji}}{\partial J_{i'j'}}.$$

The first term on the right hand side is obviously $\det(J) (J^{-1})_{j'i'} (J^{-1})_{ji}$. To compute the second term, note that

$$\frac{\partial J}{\partial J_{i'j'}} J^{-1} + J \frac{\partial J^{-1}}{\partial J_{i'j'}} = 0,$$

which gives that

$$\frac{\partial (J^{-1})_{ji}}{\partial J_{i'j'}} = -(J^{-1})_{j'i'} (J^{-1})_{ji}. $$

Hence

$$\frac{\partial^2 \det(J)}{\partial J_{ij} \partial J_{i'j'}} = (\det(J)) \left( (J^{-1})_{ji} (J^{-1})_{j'i'} - (J^{-1})_{j'i} (J^{-1})_{ji} \right).$$

Moreover, from (3.6) we know that

$$dJ_{ij} dJ_{i'j'} = 2 dt \left( (J + J^2)_{ij} I_k + J \right)_{ij} + (J + J^2)_{ij} (I_k + J)_{ij},$$

thus

$$d(\det(J)) = \det(J) tr(J^{-1} dJ) + \sum_{i,j,j'=1}^{k} \det(J) \left( (J^{-1})_{ji} (J^{-1})_{j'i'} - (J^{-1})_{j'i} (J^{-1})_{ji} \right) \left( (J + J^2)_{ij} I_k + J \right)_{ij} + (J + J^2)_{ij} (I_k + J)_{ij} \right) dt$$

$$= \det(J) tr(J^{-1} dJ) + 2 \det(J) \left( tr(2 I_k + J + J^{-1}) - tr(I_k + J) tr(I_k + J^{-1}) \right) dt$$

$$= \det(J) tr(J^{-1} dJ) + 2 \det(J) \left( 2k - k^2 - (k - 1)(tr(J) + tr(J^{-1})) - tr(J) tr(J^{-1}) \right) dt.$$ 

From (3.6) we know

$$tr(J^{-1} dJ) = tr \left( J^{-1/2} (I_k + J) (dB + dB^*) \right) + 2(n - k + tr(J)) tr(I_k + J^{-1}) dt.$$ 

Hence we obtain (3.7). As a direct consequence of $d(\det(J), \det(J)) = 4 tr(2 I_k + J + J^{-1}) dt$ we then have (3.8).
5.5 Stochastic differential equations for the eigenvalues

Proof of Theorem 3.4. We label the eigenvalues by \( \lambda_1 \geq \cdots \geq \lambda_k \). Note \( J \) is Hermitian, hence it can be diagonalized by \( J = V \Lambda V^* \) where \( V \in \mathbf{U}(k) \) and \( \Lambda = \text{diag}\{ \lambda_1, \ldots, \lambda_k \} \). Let \( dU = dV^* \circ V \) and \( dN = V^* \circ dJ \). Then

\[
d\Lambda = dU \circ \Lambda - \Lambda \circ dU + dN
\]

hence

\[
d\lambda_i = dN_{ii}, \quad dU_{ij} = \frac{1}{\lambda_i - \lambda_j} \circ dN_{ij} \quad \text{for} \ i \neq j.
\]

From (3.6) we know that

\[
\frac{(dJ)_{ij}(dJ)_{i'j'}}{2dt} = (J + J^2)_{ij}(I_k + J)_{ij} + (J + J^2)_{ij}(I_k + J)_{i'j'}.
\]

We can then compute that

\[
\frac{(dN)_{ij}(dN)_{i'j'}}{2dt} = \sum_{p,p',\ell,\ell'} V_{ip}^* V_{i'p'}^* V_{\ell j} V_{\ell'j'} ((J + J^2)_{p'\ell'}(I_k + J)_{p'\ell} + (J + J^2)_{p'\ell'}(I_k + J)_{\ell'j'})
\]

\[
= (V^*(J + J^2)V)_{ij}(V^*(I_k + J)V)_{ij} + (V^*(J + J^2)V)_{ij'}(V^*(I_k + J)V)_{i'j}
\]

\[
= (\Lambda + \Lambda^2)_{ij}(I_k + \Lambda)_{ij} + (\Lambda + \Lambda^2)_{ij'}(I_k + \Lambda)_{i'j}.
\]

If we denote by \( dM \) the local martingale part of \( dN \) and \( dF \) the finite variation part, then from (3.6) we know that

\[
\frac{dF}{2dt} = V^*(n - k + tr(J))(I_k + J)V + \frac{1}{2} \frac{(dV^*dJV + V^*dJdV)}{2dt}
\]

\[
= (n - k + tr(J))(I_k + \Lambda) + \frac{1}{2} \frac{dUdN + dN^*dU^*}{2dt}.
\]

Since

\[
\frac{(dUdN)_{ij}}{2dt} = \sum_{\ell \neq i} \frac{1}{\lambda_i - \lambda_\ell} \frac{dN_{ii}dN_{ij}}{2dt}
\]

\[
= \delta_{ij} \sum_{\ell \neq i} \frac{(1 + \lambda_i)(1 + \lambda_\ell)(\lambda_i + \lambda_\ell)}{\lambda_i - \lambda_\ell}
\]

we obtain that \((dUdN)^* = dUdN\). Hence

\[
dF_{ij} = 2dt\delta_{ij} \left( (n - k + \sum_{\ell=1}^k \lambda_\ell)(1 + \lambda_i) + \sum_{\ell \neq i} \frac{(1 + \lambda_i)(1 + \lambda_\ell)(\lambda_i + \lambda_\ell)}{\lambda_i - \lambda_\ell} \right)
\]

\[
= 2dt\delta_{ij}(1 + \lambda_i) \left( n - 2k + 1 - (2k - 3)\lambda_i + 2\lambda_i(1 + \lambda_i) \sum_{\ell \neq i} \frac{1}{\lambda_i - \lambda_\ell} \right).
\]

At last, we have that

\[
dM_{ii}dM_{jj} = dN_{ii}dN_{jj} = 2dt((\Lambda + \Lambda^2)_{ij}(I_k + \Lambda)_{ij} + (\Lambda + \Lambda^2)_{ij}(I_k + \Lambda)_{ji})
\]

\[
= 4dt\delta_{ij}\lambda_i(1 + \lambda_i)^2.
\]
Hence
\[ dM_{ii} = 2(1 + \lambda_i) \sqrt{\lambda_i} dB^i \]
where the \( B^i \)'s are independent standard real Brownian motions. We conclude
\[ d\lambda_i = dM_{ii} + dF_{ii} = 2(1 + \lambda_i) \sqrt{\lambda_i} dB^i + 2(1 + \lambda_i) \left( n - 2k + 1 - (2k - 3) \lambda_i + 2\lambda_i(1 + \lambda_i) \sum_{\ell \neq i} \frac{1}{\lambda_i - \lambda_{\ell}} \right) dt. \]

\[ \square \]

5.6 Non-collision property for the eigenvalues

Proof of Theorem 3.5. Let \( \rho_i = \frac{1 - \lambda_i}{1 + \lambda_i}, i = 1, \ldots, k, \) and let \( \tau = \inf\{t > 0 \mid \exists i < j, \rho_i(t) = \rho_j(t)\} \) be the first colliding time. We then want to show that \( P(\tau < +\infty) = 0 \), namely for every \( t \geq 0 \) we almost surely have
\[ \rho_1(t) < \cdots < \rho_k(t). \]
Let \( h = \Pi_{i>j} (\rho_i - \rho_j) \). Using similar idea as previously, let us consider the process
\[ \Omega_t := V(\rho_1(t), \ldots, \rho_k(t)), \]
where \( V(\rho_1, \ldots, \rho_k) = \frac{1}{2} \log h = \frac{1}{2} \sum_{i>j} \log(\rho_i - \rho_j) \). We can compute that
\[ d\Omega_t = \sum_{i=1}^{k} \left( \partial_i V d\rho_i + \frac{1}{2} \partial_i^2 V d\langle \rho_i \rangle \right) = \mathcal{L}_{n,k} V dt + dM_t \quad (5.19) \]
where \( M_t \) is a local martingale satisfying \( dM_t = -2 \sum_{i=1}^{k} \sqrt{1 - \rho_i^2} (\partial_i V) dB^i \) and
\[ \mathcal{L}_{n,k} = 2 \sum_{i=1}^{k} (1 - \rho_i^2) \partial_i^2 - 2 \sum_{i=1}^{k} \left( n - 2k + (n - 2k + 2) \rho_i + 2 \sum_{\ell \neq i} \frac{1 - \rho_i^2}{\rho_i - \rho_\ell} \right) \partial_i. \]
For any \( 1 \leq i \leq k, \)
\[ \partial_i V = \frac{\partial_i h}{2h}, \quad \partial_i^2 V = \frac{\partial_i^2 h}{2h} - \frac{(\partial_i h)^2}{2h^2}. \]
Since \( \sum_{i=1}^{k} \partial_i h = 0 \) and \( \sum_{i=1}^{k} \rho_i \partial_i h = \frac{k(k-1)}{2} h \), we that
\[ \sum_{i=1}^{k} \partial_i V = 0, \quad \sum_{i=1}^{k} \rho_i \partial_i V = \frac{k(k-1)}{4}. \]
Hence
\[ \mathcal{L}_{n,k} V = \sum_{i=1}^{k} (1 - \rho_i^2) \left( \frac{\partial_i^2 h}{h} + \frac{(\partial_i h)^2}{h^2} \right) - \frac{(n - 2k + 2)k(k-1)}{2}. \]
At last by a direct computation (for instance see Proposition 12.1.1 in [19]) we know that
\[ \sum_{i=1}^{k} \rho_i^2 \frac{\partial_i^2 h}{h} = \frac{1}{3} k(k-1)(k-2), \quad \sum_{i=1}^{k} \frac{\partial_i^2 h}{h} = 0, \]

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therefore we obtain that
\[
L_{n,k}V = -\frac{3n - 4k + 2}{6}k(k-1) + \sum_{i=1}^{k} (1 - \rho_i^2) \frac{(\partial_i h)^2}{h^2} \geq -\frac{3n - 4k + 2}{6}k(k-1).
\]
Plug back into (5.19) we have for any \( t \geq 0 \),
\[
M_t \leq \Omega_t - \Omega_0 + \frac{3n - 4k + 2}{6}k(k-1)t.
\]
On \( \{ \tau < +\infty \} \), by letting \( t \to \tau \) we have the right hand side of the above inequality goes to \(-\infty\). This implies that \( M_\tau = -\infty \). However, since \( M_t \) is a time changed Brownian motion, we then obtain that \( \{ \tau < +\infty \} \) is a null set.

**Remark 5.2.** As a corollary, we deduce that if the rank of \( J_0 \) is \( k \) then it also \( k \) for every \( J_t, t \geq 0 \).

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