Photons generated by gravitational waves in the near-zone of a neutron star

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When a gravitational wave or a graviton travels through an electric or magnetic background, it could convert into a photon with some probability. In this paper, a dipole magnetic field is considered as this kind of background in both the Minkowski spacetime and the curved spacetime in the near-zone of a neutron star. In the former case, we find that the graviton traveling vertically rather than parallel to the background magnetic field could be more effectively converted into an electromagnetic radiation field. In the latter case, we focus on the situation, in which the graviton travels along the radial direction near a neutron star. The radius of a neutron star is about ten kilometers, so the gravitational wave with long wavelength or low frequency may bypass neutron stars by diffraction. For high frequency gravitational wave, the conversion probability is proportional to the distance square as that in the static electric or magnetic background case. The smaller the inclination angle between the dipole field and the neutron star north pole is, the larger magnetic amplitude will be. The term that described curved spacetime will slightly enhance this kind of probability. We estimate that this value is about the order of $\sim 10^{-14} - 10^{-10}$. Therefore, it is expectable that this kind of conversion process may have a potential to open a window for observing high frequency gravitational waves.

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I. INTRODUCTION

Field of high frequency gravitational wave (HFGW) is coming to vitality from both theoretical and experimental aspects. HFGW is theorized to be relic gravitational waves that are associated with imprints of the Big Bang like the CMB [1]. And at a very high frequency, the production of gravitational waves is possibly attributed to discrete sources, cosmological sources, brane-world Kaluza–Klein (KK) mode radiation, and plasma instabilities [2]. Ground-based GW observations are sensitive to the low frequency such as in the range from 10Hz to kHz [3]. Novel ideas have been proposed to overcome difficulties and some of them are potential approaches to probing high or even ultra high frequency gravitational waves [2–6].

When a gravitational wave propagates through a background electromagnetic field, it will slowly turn into a photon. The conversion of relic gravitational waves into photons in cosmological background magnetic fields is studied in [7], while the reverse process to probing how strong the primordial magnetic fields are generated is discussed in [8]. In this paper, we will focus on the HFGW-photon conversion in the near-zone of a typical neutron star, which has a strong surface magnetic field in range of $10^8 - 10^{15}$ G [9] and mass of 1.2 – 2.0 solar masses with radius of about 11km [10]. A neutron star can be regarded as a rotating pulsar with a magnetic dipole field, and this field has an inclination angle with the north pole of the neutron star [11].

The gravitation wave or graviton coming into the neutron star can be from distance sources and propagate like a plane wave. Dark matter can be also one of these sources [12]. One can suppose that dark matter is distributed around a neutron star and gravitational waves are generated due to quantum fluctuations [13]. These GWs may convert to photons due to the neutron star’s strong background magnetic field and then may be observed. In Section 2, the conversion process with plane gravitational waves coming from different directions under a magnetic dipole background field in the Minkowski spacetime is considered. In Section 3, the electromagnetic field in the curved space-time background around the neutron star is reviewed, and then the conversion ratio with spherical gravitational waves propagating to a typical neutron star is calculated. Section 4 is devoted to conclusions and discussions.

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II. ACTION AND EQUATIONS OF MOTION

When a gravitational wave coming from distance travels in the background of an electromagnetic field, it could slowly turn into a photon. The action describing such process is given by

\[ S = \int d^4x \sqrt{-\bar{g}} \left[ -\frac{1}{4} \bar{g}^{\mu\nu} f_{\mu\nu} + \left( \bar{g}^{\alpha\beta} \bar{F}^{\beta\nu} - \frac{1}{4} \bar{g}^{\alpha\beta} \bar{F}^{\mu\nu} \right) f_{\mu\nu} h_{\alpha\beta} \right], \]  

(1)

where \( \bar{g}_{\mu\nu} \) and \( \bar{F}_{\mu\nu} \) are the background metric and the electromagnetic field, respectively. The electromagnetic field \( f_{\mu\nu} \) is generated by the incoming gravitational wave (or the graviton) \( h_{\mu\nu} \) when it is passing through the background \( \bar{F}_{\mu\nu} \) field. Both \( f_{\mu\nu} \) and \( h_{\mu\nu} \) can be regarded as perturbations of the background fields as the following

\[ f_{\mu\nu} = \bar{F}_{\mu\nu} + \bar{g}_{\mu\nu}, \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \]  

(2)

By definition, \( f_{\mu\nu} \) is given by

\[ f_{\mu\nu} = \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  

(3)

with the electromagnetic potential \( A_\mu \). Here \( \bar{\nabla}_\mu \) stands for the covariant derivative with the background metric \( \bar{g}_{\mu\nu} \).

The second term in the action (1) could be interpreted as an effective current \( J_\mu \):

\[ S = \int d^4x \sqrt{-\bar{g}} \left[ -\frac{1}{4} \bar{g}^{\mu\nu} f_{\mu\nu} + \bar{g}_{\mu\nu} J_\mu A_\nu \right]. \]  

(4)

That is, the equation of electromagnetic field satisfies:

\[ \bar{\nabla}_\mu f^{\mu\nu} = \frac{1}{\sqrt{-\bar{g}}} \partial_\mu (\sqrt{-\bar{g}} f^{\mu\nu}) = -J_\nu. \]  

(5)

After performing partial integration and omitting total derivatives, the effective current is identified as

\[ J_\nu = -\bar{\nabla}_\lambda (C^{\lambda\alpha\beta} h_{\alpha\beta}), \]  

(6)

where we have defined

\[ C^{\lambda\alpha\beta} = \bar{g}^{\alpha\lambda} \bar{F}^{\beta\nu} - \bar{g}^{\alpha\nu} \bar{F}^{\beta\lambda} - \frac{1}{2} \bar{g}^{\alpha\beta} \bar{F}^{\lambda\nu}. \]  

(7)

It is obvious that \( C^{\lambda\alpha\beta} \) is asymmetric of the first two indexes \( \lambda\nu \), i.e. \( C^{\lambda\nu\alpha\beta} = -C^{\nu\lambda\alpha\beta} \).

III. HFGW-PHOTON CONVERSION THROUGH DIPOLE MAGNETIC FIELD BACKGROUND IN THE MINKOWSKI SPACETIME

In this section, a gravitational wave is assumed to propagate in a flat spacetime, \( g_{\mu\nu} = \eta_{\mu\nu} \), then the effective current is:

\[ J_\nu = -\bar{g}_{\alpha\nu} h_{\beta\mu} \partial^\mu \bar{F}^{\beta\alpha} + \bar{F}^{\beta\mu} \partial_\mu h_{\nu\beta} + h_{\nu\beta} \partial_\mu \bar{F}^{\beta\mu}, \]  

(8)

where the transverse-traceless (TT) gauge (\( h = h_\mu^\mu = 0, \partial_\mu h_\mu^\mu = 0 \)) is chosen. From the above equation, it is clearly that \( J^0 = -h_\mu^\mu \partial_\mu \bar{F}^{\beta\alpha} \), which means charge can be only generated by an electric background field along the \( x \) or \( y \) direction with non-zero gradient. The contribution from the graviton’s variation is the second term \( \sim \bar{F}^{\beta\mu} \partial_\mu h_{\nu\beta} \). In the TT gauge, there are two linearly independent polarization states, so the effective current density generated by the graviton has two components in a static field background.

Assuming that the \( z \) axis is aligned with the dipole moment \( \mathbf{m} \), then the background magnetic field is given by

\[ \bar{\mathbf{B}} = B_0 \left( \frac{T_0}{T} \right)^3 \left( 2 \cos \theta \mathbf{e}_r + \sin \theta \mathbf{e}_\theta \right), \]  

(9)
FIG. 1: The vector diagram of a dipole magnetic field.

where \( \mathbf{e}_r \) and \( \mathbf{e}_\theta \) are unit coordinate vectors along the radial and the azimuthal directions in the spherical coordinates, respectively. Here \( B_0 \) is half of the magnitude in the polar direction of the dipole magnetic field, i.e. \( B_0 = B_p/2 \) where \( B_p = |\mathbf{B}(\theta = 0, r = r_0)| \). In the Cartesian coordinates this magnetic is

\[
\mathbf{B} = B_0 \left( \frac{r_0}{|\mathbf{x}|} \right)^3 \left( \frac{3zx}{|\mathbf{x}|^2} \mathbf{e}_x + \frac{3zy}{|\mathbf{x}|^2} \mathbf{e}_y + \frac{2z^2 - x^2 - y^2}{|\mathbf{x}|^2} \mathbf{e}_z \right),
\]

and its modulus is

\[
|\mathbf{B}| = B_0 \left( \frac{r_0}{|\mathbf{x}|} \right)^3 \sqrt{1 + \frac{3z^2}{|\mathbf{x}|^2}},
\]

with \( \mathbf{e}_x, \mathbf{e}_y \) and \( \mathbf{e}_z \) are the unit coordinate vectors and \( |\mathbf{x}| = r \). To see the vector diagram of a dipole magnetic field, see Fig.1. From Equ.(10), it is easy to see that \( \mathbf{B} \) is symmetric about \( x, y \) coordinates.

A. A gravitational wave propagates along the \( z \) direction

At first, we consider that only one polarization mode (e.g. \( h_{11} \)) of the gravitational wave enters along the \( z \) direction in the dipole field, see Fig.2. This mode is described by \( h_{11} = -h_{22} = e_{11} e^{ik_x x^\mu} + e_{11}^* e^{-ik_x x^\mu} \) with \( k_\mu = (w, k) = (w, k_x, k_y, k_z) = (w, 0, 0, k_z) \) and \( w^2 = k_x^2 + k_y^2 + k_z^2 = |k|^2 \). Redefine the origin of time so that \( \epsilon_{11} \) is real. In this case, the current density components are given by:

\[
J_x = -2|\epsilon_{11}|(iw\ddot{B}_y + \partial_z \dot{B}_y) e^{ik_x x^\mu},
\]

\[
J_y = -2|\epsilon_{11}|(iw\ddot{B}_x + \partial_z \dot{B}_x) e^{ik_x x^\mu},
\]

\[
J_z = 4|\epsilon_{11}|\partial_x \dot{B}_y e^{ik_x x^\mu},
\]

where the derivatives of \( B_i \) can be calculated from Equ.(10), see Sec.A 1 for details.

In the far zone \( |\mathbf{x}| \gg |\mathbf{x}'| \), the distance between the field and the source can be approximately by

\[
|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| - \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \mathbf{x}' = |\mathbf{x}| - \hat{x} \cdot \mathbf{x}'.
\]

Then we have

\[-w(t - |\mathbf{x} - \mathbf{x}'|) + \mathbf{k} \cdot \mathbf{x}' \approx -w(t - |\mathbf{x}|) - w\mathbf{u} \cdot \mathbf{x}',
\]
FIG. 2: A dipole magnetic field projects on the $y−z$ plane.

with

$$u \equiv \hat{x} - \hat{k}, \quad \hat{k} \equiv \frac{k}{w}. \quad (17)$$

The generated electric field can be calculated by using the time derivative of the retarded potential:

$$E_i(x,t) = -\partial^i A_i(x,t) = \frac{1}{4\pi} \int_{-L_x/2}^{L_x/2} \int_{-L_y/2}^{L_y/2} \int_{-L_z/2}^{L_z/2} \frac{\partial_i J_i(x',t - |x-x'|)}{|x-x'|} dx' dy' dz', \quad (18)$$

where $L_x, L_y$ and $L_z$ indicate the range where the current is not zero. After straightforward calculation, the total power radiated can be estimated by

$$P_\gamma = \frac{1}{2} \int |\mathbf{E}|^2 r^2 d\Omega \approx \frac{3}{2} \pi |e_{11}|^2 B_0^2 L^2 \left(\frac{r_0}{L}\right)^6 \left[1 + 4 \xi^2 \left(1 + \xi^2\right)^4\right],$$

with $\xi = R/L_z$ and $R = \sqrt{L_x^2 + L_y^2}/2$, see Equ.(A36) in Sec.A 2 for details. Here $L = 2L_z$ can be regarded as the distance that the graviton travels along the $z$ direction.

The stress energy tensor of the graviton is

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{8\pi G} \left(|e_{11}|^2 + |e_{12}|^2, \right), \quad (19)$$

which gives the power of the incoming graviton as the following

$$P_g = \frac{w^2 |e_{11}|^2}{8\pi G} \int_0^R \rho d\rho \int_0^{2\pi} d\theta = \frac{w^2 |e_{11}|^2}{8G} R^2. \quad (20)$$

The probability for a graviton turning into photon in background electric field is estimated by the ratio of the power radiated to that of the incoming graviton:

$$\epsilon_{g-\gamma} = \frac{P_\gamma}{P_g} = \frac{12\pi GB_0^2 L^2 \left(\frac{r_0}{L}\right)^6 \left(\frac{1}{wL}\right)^2 \left[\xi^2 (6 + 4\xi^2 + \xi^4)\left(1 + \xi^2\right)^4\right]}{12\pi GB_0^2 L^2 \left(\frac{r_0}{L}\right)^6 \left(\frac{1}{wL}\right)^2 \left[\xi^2 (6 + 4\xi^2 + \xi^4)\left(1 + \xi^2\right)^4\right]}, \quad (21)$$

which is much depressed because $wL \gg 1$ for a given source scale. For the $h_{12} = h_{21}$ mode, one can get the same graviton energy transforming rate.
B. A gravitational wave propagates along the y or x direction

Now we consider that a gravitational wave enters along the y direction with $k_\mu = (w, 0, k_y, 0)$, i.e., $k_y = w$, see Fig.2 for the background dipole magnetic field. The result will be the same for that along the x direction because the magnetic field is symmetric about x, y coordinates. For the $h_{11} = -h_{33}$ mode, the current density components are:

$$J_x = 2\epsilon_{11}(iw\bar{B}_z + \partial_y\bar{B}_z)e^{ik_\mu x^\mu},$$  \hfill (22)

$$J_y = -4\epsilon_{11}\partial_x\bar{B}_x e^{ik_\mu x^\mu},$$  \hfill (23)

$$J_z = 2\epsilon_{11}(iw\bar{B}_x + \partial_y\bar{B}_x)e^{ik_\mu x^\mu}.$$

After straightforwardly calculation, the total power radiated can be estimated by

$$P_\gamma = \frac{\pi}{2}w^2|\epsilon_{11}|^2B_0^2\epsilon_0^2L^2 \left( \frac{4L^2 + 5\epsilon^2}{\epsilon^2(L^2 + \epsilon^2)^2} - \frac{4L^2 + 5R^2}{R^2(L^2 + R^2)^2} \right) \approx 2\pi w^2|\epsilon_{11}|^2B_0^2\epsilon_0^2\frac{r_0^4}{R^2},$$

where $R = \sqrt{L_x^2 + L_y^2}/2$. Here $\epsilon$ denotes the shortest radius that the graviton can travel in the $z - x$ plane, i.e. $x^2 + y^2 \geq \epsilon^2$ in the integration (18) and $\epsilon < L, R$, see Eqn.(A47) in Sec.A2 for details. Finally, the probability for a graviton turning into photon is estimated by:

$$\epsilon_{\gamma - \gamma} = \frac{P_\gamma}{P_\gamma} = 16\pi GB_0^2\epsilon_0^2\frac{r_0^4}{\epsilon^2R^2},$$

If the order of magnitude for $(\epsilon R)^{1/2}$ is the same as $r_0$, then

$$\epsilon_{\gamma - \gamma} \approx 4\pi GB_0^2\epsilon_0^2.$$

For the $h_{13} = h_{31}$ mode, one can get the same graviton energy transforming rate.

From the results (21) and (26) one can conclude that when a gravitational wave travels perpendicular to the background magnetic field, it can be effectively converted into the electromagnetic radiation, otherwise when the propagation direction is parallel to the background magnetic field, this effect is depressed, see Fig.2 for the background magnetic field. This conclusion is consistent with that obtained from the static electromagnetic field. Therefore, in next section, we will focus on the perpendicular case, i.e. a gravitational wave is traveling along the radial direction of a neutron star.

IV. HFGW-PHOTON CONVERSION IN THE NEAR-ZONE OF A NEUTRON STAR

A. Briefly review of the magnetic field of a neutron star

The spacetime outside a rotating neutron star with a time-constant angular velocity can be described by the following metric:

$$ds^2 = -N^2dt^2 + \frac{1}{N^2}dr^2 - 2\Omega t^2 \sin^2\theta d\theta d\phi + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,$$

in the spherical coordinate system. Here $N^2 = 1 - \frac{\omega^2}{r^2}$ and $r_s = 2M$ is the star’s Schwarchild radius. The non-diagonal component of the metric tensor leads to dragging of the inertial frame of reference with an angular velocity $\Omega = 2J/r^3$ and here $J$ is the angular momentum.

For an observer with a 4-velocity $u^\alpha$, the covariant components of the electromagnetic field tensor are given by

$$\tilde{F}_{\alpha\beta} = 2u_{[\alpha}\tilde{E}_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}u^\gamma\tilde{B}^\delta,$$

where $\tilde{E}^\alpha$ and $\tilde{B}^\alpha$ are the electric and magnetic four-vector fields, respectively. Here $T_{[\alpha\beta]} = (T_{\alpha\beta} - T_{\beta\alpha})/2$ and $\epsilon_{\alpha\beta\gamma\delta}$ is the pseudo-tensorial expression for the Levi-Civita symbol $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$:

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\tilde{\epsilon}_{\alpha\beta\gamma\delta}, \quad \epsilon_{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}\tilde{\epsilon}_{\alpha\beta\gamma\delta}. \hfill (29)$$

The magnetic field of neutron star is caused by the perfect fluid interior region of itself rather than its rotation, while the electric field is caused by both the magnetic field and the rotation of a neutron star. It can be found that the
strength of the electric field is much smaller than that of the magnetic field, i.e. \( E \sim \Omega \times B \). Therefore, we will keep only linear terms for the angular velocity and ignore the background electric field in the following.

An observer is called a zero angular momentum observer (ZAMO) if he/she is locally stationary (at fixed values of \( r \) and \( \theta \)) but is dragged into rotation with respect to a reference that fixed with respect to a distant observer. The ZAMO has the following 4-velocity components

\[
  u^\alpha = N^{-1}(-1, 0, 0, \Omega), \quad u_\alpha = N(-1, 0, 0, 0),
\]

up to the first order of \( \Omega \). For ZAMOs, the electromagnetic nonzero components are

\[
  \bar{F}_{10} = -\bar{F}_{01} = \Omega N^{-1} r \sin \theta \bar{B}^2, \\
  \bar{F}_{20} = -\bar{F}_{02} = -r \sin \theta \bar{B}^1, \\
  \bar{F}_{12} = -\bar{F}_{21} = N^{-1} r \bar{B}^3, \\
  \bar{F}_{23} = -\bar{F}_{32} = r^2 \sin \theta \bar{B}^1, \\
  \bar{F}_{13} = -\bar{F}_{31} = -N^{-1} r \sin \theta \bar{B}^2,
\]

and

\[
  \bar{F}^{12} = -\bar{F}^{21} = \frac{N}{r} \bar{B}^3, \\
  \bar{F}^{13} = -\bar{F}^{31} = -\frac{N}{r \sin \theta} \bar{B}^2, \\
  \bar{F}^{23} = -\bar{F}^{32} = \frac{1}{r^2 \sin \theta} \bar{B}^1.
\]

where the magnetic field outside the neutron star is assumed to be a dipolar field with components observed in the zero angular momentum frame as the following [14]

\[
  \bar{B}^1 \equiv \bar{B}^\phi = \hat{f}(r) (\cos \chi \cos \theta + \sin \chi \cos \phi \sin \theta), \\
  \bar{B}^2 \equiv \bar{B}^\theta = \hat{g}(r) (\cos \chi \sin \theta - \sin \chi \cos \phi \cos \theta), \\
  \bar{B}^3 \equiv \bar{B}^\phi = \hat{g}(r) \sin \chi \sin \phi,
\]

where \( \chi \) is the angle between the magnetic moment and the polar axis of a neutron star, and

\[
  \hat{f}(r) = -\frac{3}{8} B_0 R^3 \frac{1}{M^3} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right], \\
  \hat{g}(r) = \frac{3}{8} B_0 R^3 \frac{N}{M^2 r} \left( \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right),
\]

are the relativistic corrections. Here \( R \) is the radius of the neutron star and \( B_0 \) is the value of the magnetic field in the polar direction in the Newtonian limit. Notice that the solution (39)-(41) is based on a hypothesis that there is no matter outside the star, i.e. \( \nabla_\mu F^{\mu\nu} = 0 \).

**B. Short wave approximation**

The explicit form for \( \nabla_\mu f^{\mu\nu} \) in Equ.(5) is given by

\[
  \bar{\nabla}_\mu f^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\alpha} \partial_\nu A^\alpha \right) - \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} g^{\mu\alpha} \partial_\mu A^\alpha \right).
\]

Here the gravitational field outsider a neutron star can be regarded as two parts:

\[
  \bar{g}_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu},
\]

where \( \epsilon_{\mu\nu} \sim (2M/R, J/M^2) \). Make a simple estimate, one can get \( 2M/R \approx 0.32 \sim 0.53 \) for a typical neutron star, and \( J/M^2 \leq 0.1 \) for the fastest-known millisecond pulsar PSR 1937+214[14]. The change of the background metric is assumed to be not faster than that of the generated electromagnetic radiation field. In other words, when the length
scale $L$ over which the background varies (i.e. $\partial_\mu g_{\alpha\beta}/g_{\alpha\beta} \sim O(1/L)$) is much larger than the wavelength $\lambda_{EM}$ of the generated electromagnetic radiation field $L \gg \lambda_{EM}$, the term $\partial_\mu (\sqrt{-g} \psi^\alpha \partial_\alpha A^\mu)$ in Equ.(44) can be approximated by $\tilde{g}^\alpha \partial_\alpha \partial_\mu (\sqrt{-g} A^\mu)$, which will be vanished after taking the Lorentz gauge $\nabla_\mu A^\mu = 0$. This is called the short wave approximation, which is a reasonable approximation when considering high-frequency wave. Then we can use the Green’s function in curved spacetime to obtain the retarded potential field as the following [15]:

$$A^\mu(r, t) = \frac{1}{4\pi} \int J^\mu(r', t') \psi_s(|r - r'|) dV',$$

where $\gamma$ is the determinant of the space metric, i.e. $\gamma = r^4 \sin^2 \theta/N^2$ and

$$\psi_s(r) = \int_0^\infty \sqrt{g_{rr}(R)} \frac{dR}{-g_{tt}(R) R^2} = -\ln \left(1 - \frac{2M}{r} \right),$$

$$t' = t - \int_{r'}^r \sqrt{g_{rr}(R)} \frac{dR}{-g_{tt}(R) R^2} = t - (r - r') + 2M \ln \frac{r - 2M}{r'}.$$  

At large radius, the static function $\psi_s$ will reduce to the solution in a flat space $\psi_s \sim 1/r$, and $t' \sim t - (r - r')$, see Ref.[15] for details. Furthermore, the variation of the background electromagnetic field is almost at the same order of that of the background metric, i.e. $\partial_\mu \tilde{F} / \tilde{F} \sim O(1/L)$, then the effective current (6) is approximated by

$$J^\mu(r, \theta, \phi, t) = -\nabla_\lambda (C^{\lambda\alpha\beta} h_{\alpha\beta}) = -\nabla_\lambda \left[ (\tilde{g}^{\alpha\lambda} \tilde{F}^{\beta\nu} - \tilde{g}^{\alpha\nu} \tilde{F}^{\beta\lambda} - \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{F}^{\lambda\mu}) h_{\alpha\beta} \right]$$

$$\approx -\tilde{F}^{\beta\nu} g_{\alpha\beta} \partial_\lambda h^{\lambda\alpha} + \tilde{F}^{\beta\lambda} g_{\alpha\beta} \partial_\lambda h^{\lambda\alpha},$$

where we have used the transverse trace-less gauge condition. Note that the first term of the above equation can be eliminated under the harmonic gauge condition.

### C. A gravitational wave travels along the radial direction

A gravitational wave travels along the radial direction ($r$) with two polarization modes is described by

$$h^{22} = \frac{\tilde{g}^{33}}{\tilde{g}_{22}} h^{33} = e^{22} \frac{R}{r} e^{i k_\mu x^\mu} + e^{23} \frac{R}{r} e^{-i k_\mu x^\mu},$$

and

$$h^{23} = h^{32} = e^{23} \frac{R}{r} e^{i k_\mu x^\mu} + e^{23} \frac{R}{r} e^{-i k_\mu x^\mu},$$

with $x^\mu = (t, r, \theta, \phi)$, $k_\mu = (-w, k) = (-w, w, 0, 0)$ and $k^\mu = (w, w, 0, 0)$. Here $R$ denotes some length scales. For example, this could be the radius length scale of a neutron star. Note that $e^{22}, e^{23}$ can be real by redefining the origin of the time. Then these modes can be expressed in a complex formalism

$$h^{22} = 2e^{22} \frac{R}{r} e^{ik_\mu x^\mu}, \quad h^{33} = \frac{\tilde{g}^{22}}{\tilde{g}^{33}} h^{22}, \quad h^{23} = h^{32} = 2e^{23} \frac{R}{r} e^{ik_\mu x^\mu},$$

which satisfies $\partial_\lambda h^{\lambda\beta} \approx 0$. The index values of non-zero items in the second term of Equ.(49) are $\lambda = 1, 2$ because $\tilde{F}^{\beta0} = 0$ and $h_{\alpha\beta}$ doesn’t depend on $\phi$. Then the effect current (49) is given by

$$J^\nu(r, \theta, \phi, t) = K_1 \tilde{F}^{\beta\gamma} \tilde{g}_{\alpha\beta} h^{\nu\alpha} \approx f^\nu (r, \theta, \phi) e^{-iw(t-r)},$$

where we have defined

$$f^\nu (r, \theta, \phi) = 2K_1 \tilde{F}^{\beta\gamma} \tilde{g}_{\alpha\beta} \frac{R}{r} e^{\nu_\alpha}.$$

Here we have used

$$\partial_\lambda h^{\alpha\beta} = \left( ik_\lambda - \frac{1}{r} \delta_{\lambda,1} \right) h^{\alpha\beta} \equiv K_\lambda h^{\alpha\beta},$$

(55)
\[ K_0 = -iw, \quad K_1 = iw - \frac{1}{r} = -K_0 \left(1 - \frac{1}{iw}\right), \quad K_2 \approx 0, \quad K_3 = 0. \]  

Then the retarded potential is
\[ A^\mu(r, t) = \frac{1}{4\pi} \int J^\mu(r', t')\psi_\nu(|r - r'|)dV' \approx \frac{e^{-iw(t-r)}}{4\pi r} I^\mu, \]  

where \( r = (r, \theta, \phi) \), \( r = |r| \),
\[ t' \approx t - |r - r'| \approx t - r + \hat{r} \cdot r', \quad \hat{r} = \frac{r}{r}, \]  

and
\[ I^\mu = \int f^\mu(r')e^{-iw(F^\nu F^\nu - 1)r'}dV' = \int \sqrt{\gamma} f^\mu(r')e^{-iw(F^\nu F^\nu - 1)r'}dV'd\theta'd\phi', \]  

with \( \gamma' = r'^4 \sin^2 \theta' N^{-2} \) the determinant of the space metric. From Equ.(54), we get \( f^0 = f^1 = 0 \), which leads to \( A^0 = A^1 = 0 \) by using Equ.(57).

The components of the generated electric and magnetic radiation field are calculated by
\[ E^\nu(r, t) = F^\nu_0, \quad B_\nu(r, t) = \frac{1}{2} \epsilon_{ijk} F^{jk}, \]  

which gives \( E^1 = B^1 = 0 \) and
\[ E^2 = \partial^0 A^2(r, t) = K_0 g^{00} A^2, \]  
\[ E^3 = \partial^0 A^3(r, t) = K_0 g^{00} A^3, \]  
\[ B^2 = \partial^1 A^3(r, t) = -K_1 \sqrt{\gamma} g^{11} g^{22} A^3, \]  
\[ B^3 = \partial^1 A^2(r, t) = K_1 \sqrt{\gamma} g^{11} g^{33} A^3, \]  

where \( \epsilon_{ijk} = \sqrt{\gamma} \epsilon_{ijk} \). Here we have used the relations \( \partial_\nu A^\nu = K_\nu A^\nu \) from Equ.(57). Then we get the Poynting vector
\[ S_1 = \text{Re}(\frac{1}{4} \epsilon_{ijk} E^{jk} B^3) \]  

which has only one nonzero component:
\[ S_1 = \frac{\sqrt{\gamma}}{2} \text{Re}(E^{2*} B^3 - E^{3*} B^2) = \frac{1}{2} \text{Re} \left[ K_0 K_1 \gamma g^{00} g^{11} \left( g^{33} |A^2|^2 + g^{22} |A^3|^2 \right) \right] = -\text{Re} \left[ K_0 K_1 \frac{32\pi r^2}{\gamma} \left( g^{33} |I|^2 + g^{22} |I|^3 \right) \right]. \]  

The total power radiated is estimated by
\[ P_\gamma = \int |S_1|^2 d\Omega = \frac{1}{32\pi^2} \int \gamma |K_0 K_1| \left( g^{33} |I|^2 + g^{22} |I|^3 \right) d\Omega, \]  

where we only keep the terms up to the first order of \( 1/r \) in \( S^1 \) because terms with high orders of \( 1/r \) in the electric field can not travel to distant observers.

Note that in Equ.(59) there is an integral of \( \phi \) in range of \( [0, 2\pi] \), \( \theta \) in range of \( [0, \pi] \), so the terms that proportional to \( \sin \phi, \cos \phi \) and \( \cos \theta \) in Equ.(54) will be vanished after performing the integration. Therefore, these terms can be dropped safely to get
\[ f^2(r, \theta, \phi) = 2K_1 F^{31} \frac{R}{g^{33}} e^{23} = 2K_1 F^{31} g^{22} \frac{R}{r} e_{23} \sim G(r)e_{23}, \tag{66} \]  
\[ f^3(r, \theta, \phi) = 2K_1 F^{31} \frac{R}{g^{33}} e^{33} = -2K_1 F^{31} g^{22} \frac{R}{r} e_{22} \sim -G(r)e_{22}, \tag{67} \]
where we have defined
\[ G(r) = \frac{3}{4} K_1 B_0 R^4 \cos \chi \frac{N^2}{M^2 R^5} \left( \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right), \]  
and used Eqns.\,(37), \,(40) and \,(43). Then the power can be expressed by
\[ P_\gamma \approx \frac{1}{32\pi^2} \left( |e_{23}|^2 I^{G1} + |e_{22}|^2 I^{G2} \right), \]
where we have also defined
\[ I^{G1} = \int e^{33}(r') \left( \sqrt{\gamma} K_1 G^*(r') \right) dr' \sin \theta d\phi', \]
\[ I^{G2} = \int e^{22}(r') \left( \sqrt{\gamma} K_1 G^*(r') \right) dr' \sin \theta d\phi', \]
and
\[ I^\gamma = \int e^{-i\omega(\hat{r}' - 1) r' + i\nu(\hat{r}' - 1) r'} \sin \theta d\theta d\phi. \]

By using the short wave approximation \((wL \gg 1)\), we obtain
\[ I^{G1} \approx \frac{4\pi^2}{w^2} \int r'^4 N^{-2} K_1 G^*(r') r''^2 dr' \int N^{-1} K_0 G(r'') dr'' \sin \theta d\theta d\phi', \]
\[ I^{G2} \approx \frac{4\pi^2}{w^2} \int r'^4 N^{-2} K_1 G^*(r') r''^2 dr' \int N^{-1} K_0 G(r'') dr'' \sin^3 \theta d\theta d\phi'. \]

These two integrals can be calculated straightforwardly, see Sec.\,B for details, and they are given by
\[ I^{G1} = 16\pi w^2 \frac{\pi^2}{2} \cos^2 \chi \frac{B_0^2 L^2}{R^2} \, \left( 1 + \frac{3M}{R} \right)^2, \]
\[ I^{G2} = \frac{32\pi}{3} w^2 \frac{\pi^2}{2} \cos^2 \chi \frac{B_0^2 L^2}{R^2} \, \left( 1 + \frac{3M}{R} \right)^2, \]
so the power is
\[ P_\gamma \approx \frac{\pi w^2}{2} \left( |e_{23}|^2 + \frac{2}{3} |e_{22}|^2 \right) \cos^2 \chi \frac{B_0^2 L^2}{R^2} \, \left( 1 + \frac{3M}{R} \right)^2, \]
up to the leading order of \(M/R\). Here \(L\) denotes the distance that a gravitational wave travels along the radial direction, and we have also taken the approximation \(L \ll R\) in the above results.

The averaged energy momentum tensor of the gravitational wave in a curved spacetime can be estimated by
\[ t_{\mu\nu} = \frac{1}{32\pi G} \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta}, \]
in which the energy flux along the \(r\) direction is \(t_{10}\). Then the power of the traveling gravitational wave is
\[ P_g = \int |K_1 K_0^*| \left( \left( \hat{g}^{22} \right)^2 |e_{22}|^2 + \hat{g}^{22} \hat{g}^{23} |e_{23}|^2 \right) \frac{R^2}{r^2} r^2 \sin^2 \theta d\Omega \approx \frac{w^2}{2GR^2} \left( \frac{2}{3} |e_{22}|^2 + |e_{23}|^2 \right). \]

Finally, the probability for a graviton turning into a photon in the near zone of a neutron star is estimated by the ratio of the power radiated to that of the traveling graviton:
\[ \epsilon_{\gamma - \gamma} = \frac{P_\gamma}{P_g} \approx \pi GB_0^2 L^2 \cos^2 \chi \left( 1 + \frac{3M}{R} \right)^2, \]
in the leading order of $M/R$ and $L \ll R$. From Equ.(80), the probability is proportional to the distance square, which is the same as that in the static electric or magnetic background case. The smaller of the inclination angle $\chi$, the larger of the magnetic amplitude, then this probability will be larger. The term caused by the curved spacetime is proportional to $(1 + 3M/R)^2$, which will slightly enhance this kind of probability.

The measured surface magnetic field of $\sim 1.6 \times 10^{13}$G for Swift J0243.6+6124 is the strongest for all known neutron starts with detected electron cyclotron resonance scattering features[16]. By taking the values of $M/R \sim 0.21$, $\chi = 60^\circ$ from [17] and $L \sim 1$km, one can estimate the probability as follows

$$\epsilon_{g-\gamma} \approx 2.1 \times 10^{-38} \left( \frac{B_0L}{T \cdot m} \right)^2 \cos^2 \chi \left( 1 + \frac{3M}{R} \right)^2 \approx 3.57 \times 10^{-14}. \quad (81)$$

For a magnetar with a higher magnetic field, such as $\sim 10^{15}$G, the order of magnitude of this conversion rate will reach $\sim 10^{-10}$. Compare another situation, in which gravitons travel in the universe between galaxies. There are more gravitons in the universe than we believed. A typical galaxy has a size of 100000 light years ($10^{21}$m) and an average magnetic field of $10^{-9}$ T, and most of the fields are not turbulent, the conversion probability is on the order of $10^{-14}$ (versus $\sim 10^{40}$ for thermal radiation). Therefore, it has a greater opportunity to detect HFGWs through this conversion phenomenon in the near-zone of a neutron star or even a magnetar.

V. CONCLUSION

In this paper, we calculate the ratio of graviton-photon conversion under the backgrounds of a dipole magnetic field in a Minkowski spacetime and a typical neutron star magnetic field in a slightly rotational curved spacetime. In the former case, we find that when a GW travels perpendicular to the background magnetic field, it can be effectively converted into an electromagnetic radiation field, but this conversion will be depressed if the GW travels parallel to the background magnetic field; so in the latter case, we focus on the case of that a GW travels along the radial direction near a neutron star. The conversion probability is proportional to the distance square, which is the same as that in the static electric or magnetic background case. The smaller the inclination angle $\chi$ is and the larger the magnetic amplitude is, the higher this probability will be. The term caused by the curved spacetime is proportional to $(1 + 3M/R)^2$, which will slightly enhance this kind of probability. This conversion probability is on the order of $\sim 10^{-14} - 10^{-10}$ for a neutron star or a magnetar.

In recent years, some new ideas for searching for dark matters through GWs have been raised[18]. A graviton that could convert into a photon in the near-zone of a typical neutron star will have a frequency $\nu \gtrsim c/R \sim 2.7 \times 10^7$Hz. Therefore, this conversion process may open a window to observe HFGW, and we believe that to further study on a high-frequency GW and its conversion process is worthwhile. On the other hand, with the assistance of gravitational waves, it is also possible to enlighten a new effective theorem and a new observation method for dark matters.

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Appendix A: Detail calculations in the Minkowski spacetime

1. The background dipole magnetic field and the effective current

From Equ.(10), the derivatives of background magnetic field with respect to coordinates are given by

\[ \partial_x \bar{B}_x = \partial_y \bar{B}_z = 3B_0 r_0^3 \frac{x}{|x|^5} \left( 1 - 5 \frac{z^2}{|x|^2} \right), \]  
(A1)

\[ \partial_x \bar{B}_y = \partial_y \bar{B}_z = 3B_0 r_0^3 \frac{y}{|x|^5} \left( 1 - 5 \frac{z^2}{|x|^2} \right), \]  
(A2)

\[ \partial_z \bar{B}_y = \partial_y \bar{B}_x = -15B_0 r_0^3 \frac{xy}{|x|^7}, \]  
(A3)

\[ \partial_x \bar{B}_x = 3B_0 r_0^3 \frac{z}{|x|^5} \left( 1 - 5 \frac{x^2}{|x|^2} \right), \]  
(A4)

\[ \partial_y \bar{B}_y = 3B_0 r_0^3 \frac{z}{|x|^5} \left( 1 - 5 \frac{y^2}{|x|^2} \right), \]  
(A5)

\[ \partial_z \bar{B}_z = -3B_0 r_0^3 \frac{z}{|x|^5} \left( 1 + 5 \frac{z^2}{|x|^2} \right). \]  
(A6)

By using the definition of \( \tilde{F}^{\mu \nu} \):

\[ \tilde{F}^{12} = \bar{B}_z, \quad \tilde{F}^{13} = -\bar{B}_y, \quad \tilde{F}^{23} = \bar{B}_x, \]  
(A7)

the current components can be obtained from Equ.(8) for the \( h_{11} = -h_{22} \) mode

\[ J_x = 2|e_{11}| \left[ ik_y \bar{B}_z - k_z \bar{B}_y \right] e^{ik_\mu x^\mu}, \]  
(A8)

\[ J_y = 2|e_{11}| \left[ ik_x \bar{B}_z - k_z \bar{B}_x \right] e^{ik_\mu x^\mu}, \]  
(A9)

\[ J_z = 2|e_{11}| \left( \partial_x \bar{B}_y + \partial_y \bar{B}_x \right) e^{ik_\mu x^\mu} = 4|e_{11}| \partial_y \bar{B}_x e^{ik_\mu x^\mu}, \]  
(A10)

where we have used \( h_{11} = 2|e_{11}|e^{ik_\mu x^\mu} \) in the complex formalism and \( \partial_i h_{11} = ik_i h_{11} \), for \( i = x, y, z \).

The current components for the \( h_{12} = h_{21} \) mode are

\[ J_x = 2|e_{11}| \left[ ik_x \bar{B}_z - k_z \bar{B}_x \right] e^{ik_\mu x^\mu}, \]  
(A11)

\[ J_y = 2|e_{11}| \left[ ik_y \bar{B}_z - k_z \bar{B}_y \right] e^{ik_\mu x^\mu}, \]  
(A12)

\[ J_z = 2|e_{11}| \left( \partial_x \bar{B}_y - \partial_y \bar{B}_x \right) e^{ik_\mu x^\mu}, \]  
(A13)

where we have used \( h_{12} = 2|e_{11}|e^{ik_\mu x^\mu} \) and \( \partial_i h_{12} = ik_i h_{12} \), for \( i = x, y, z \).

The current components for the \( h_{11} = -h_{33} \) mode are

\[ J_x = 2|e_{11}| \left[ ik_y \bar{B}_z - k_z \bar{B}_y \right] e^{ik_\mu x^\mu}, \]  
(A14)

\[ J_y = -2|e_{11}| \left( \partial_x \bar{B}_z + \partial_z \bar{B}_x \right) e^{ik_\mu x^\mu} = -4|e_{11}| \partial_x \bar{B}_z e^{ik_\mu x^\mu}, \]  
(A15)

\[ J_z = 2|e_{11}| \left( ik_x \bar{B}_y - k_y \bar{B}_x \right) e^{ik_\mu x^\mu}, \]  
(A16)

where we have used \( h_{11} = 2|e_{11}|e^{ik_\mu x^\mu} \) and \( \partial_i h_{11} = ik_i h_{11} \), for \( i = x, y, z \).

The current components for the \( h_{13} = h_{31} \) mode are

\[ J_x = 2|e_{11}| \left[ ik_x \bar{B}_y - k_y \bar{B}_x \right] e^{ik_\mu x^\mu}, \]  
(A17)

\[ J_y = 2|e_{11}| \left( \partial_x \bar{B}_z - \partial_z \bar{B}_x \right) e^{ik_\mu x^\mu}, \]  
(A18)

\[ J_z = 2|e_{11}| \left[ ik_y \bar{B}_z - k_z \bar{B}_y \right] e^{ik_\mu x^\mu}, \]  
(A19)
where we have used \( h_{13} = 2|e_{11}|e^{ik_{x}x_{n}} \) and \( \partial_{i}h_{13} = ik_{i}h_{13} \), for \( i = x, y, z \).

2. The generated electric field \( E \) and its total power

The component of current density can be expressed as

\[
J_{i}(t, x, y, z) = f_{i}(x, y, z)e^{ik_{x}x_{n}} = f_{i}(x, y, z)e^{i(k \cdot x - ut)},
\]

with some function \( f_{i}(x, y, z) \). By using Eqs.(16) and (18), we get

\[
E_{i}(x, t) = -\frac{1}{4\pi|x|} \int \partial_{i}J_{i}(x', t - |x - x'|)dV',
\]

\[
= \frac{iwe^{-iw(t - |x|)}}{4\pi|x|} \int f_{i}(x', y', z')e^{-iwu \cdot x'}dV',
\]

where \( dV' = dx'dy'dz' \). The total power radiated can be calculated by:

\[
P = \frac{1}{2} \int \int |E|^{2}r^{2}d\Omega = \frac{1}{2} \sum_{i=1}^{3} \int |E_{i}|^{2}r^{2}d\Omega
\]

\[
= \frac{w^{2}}{32\pi^{2}} \sum_{i=1}^{3} \int d\Omega \int f_{i}(x', y', z')e^{-iwu \cdot x'}dV' \int f_{i}^{*}(x'', y'', z'')e^{iwu \cdot x''}dV''
\]

\[
= \frac{w^{2}}{32\pi^{2}} \sum_{i=1}^{3} \int d\Omega f_{i}(x', y', z')f_{i}^{*}(x'', y'', z'')e^{-i\omega(\hat{k} \cdot (x' - x'')}}dV'dV''. \quad (A23)
\]

a. A gravitational wave propagates along the z direction

In this case, \( k_{z} = w \) and \( \hat{k}_{z} = 1 \), then the total power radiated can be estimated from Equ.(A23) as the following

\[
P = \frac{w^{2}}{32\pi^{2}} \sum_{i=1}^{3} \int f_{i}(x', y', z')f_{i}^{*}(x'', y'', z'')I(x', x'')dV'dV'', \quad (A24)
\]

where

\[
I(x', x'') = \int e^{-i\omega(\hat{k} \cdot (x' - x''))} \frac{d\hat{x}d\hat{y}}{\hat{z}}
\]

\[
= \int_{\hat{x}=-1}^{\hat{x}=1} \int_{\hat{y}=-1}^{\hat{y}=1} e^{-i\omega(\hat{x}' - \hat{x}')(\hat{y}' - \hat{y}')}e^{-i\omega(\hat{z}' - \hat{z})} \frac{d\hat{x}d\hat{y}}{\hat{z}}. \quad (A25)
\]

When the wavelength of the photon (\( \sim 1/w \)) is much smaller than the scale of the source (\( L \)), i.e. \( wL \gg 1 \), there is a resonance in the region of

\[
wL(1 - \hat{z}) = wL(1 - \cos \theta) \ll 2\pi,
\]

which means

\[
1 - \cos \theta \approx 1 - 1 + \frac{\theta^{2}}{2} \ll \frac{2\pi}{wL} \quad (A27)
\]

, i.e.

\[
\theta \ll \sqrt{\frac{4\pi}{wL}}, \quad \text{or} \quad \hat{z} \approx 1, \quad (A28)
\]
which is very similar to the main lobe of the antenna. Then the integral in Equ.(A25) can be approximated by
\[
I(x',x'') \approx \frac{1}{w^2} \int_{\hat{w}=-\infty}^{\hat{w}=\infty} e^{-i\hat{w}(x'-x'')} d\hat{w} \int_{\hat{w}=-\infty}^{\hat{w}=\infty} e^{-i\hat{w}(y'-y'')} d\hat{w} = \frac{4\pi^2}{w^2} \delta(x' - x'') \delta(y' - y'').
\]

The total power from Equ.(A24) is
\[
P = \frac{1}{8} \sum_{i=1}^{3} \int_{-L}^{L} f_i(x',y',z') f_i^*(x'',y'',z'') \delta(x' - x'') \delta(y' - y'') dx' dy' dz' dx'' dy'' dz''
= \frac{1}{8} \sum_{i=1}^{3} \int_{-L}^{L} f_i(x',y',z') f_i^*(x',y',z'') dx' dy' dz' dx'' dy'' dz''
= \frac{1}{8} \sum_{i=1}^{3} \int \left[ F_i(x',y') F_i^*(x',y') \right] dx' dy',
\]
where we have defined
\[
F(x',y') = \int_{-L}^{L} f_i(x',y',z') dz'.
\]

In the case of a static electric background, by taking \( f_x = 2wE|e_{11}| \), we recover the result with \( L = L_z/2 \):
\[
P = \frac{1}{2} w^2 E^2 |e_{11}|^2 L_z^2 L_x L_y.
\]

For the \( h_{11} = -h_{22} \) mode, by using the current components (A8-A10), we get
\[
\int_{-L}^{L} f_{x'}(x',y',z') dz' = -2|e_{11}| \int_{-L}^{L} (i\omega B_y' + \partial_{x'} B_y') dz' = -12|e_{11}| B_0 r_0^3 L \left( \frac{y'}{x'^2 + y'^2 + L^2} \right)^{5/2},
\]
\[
\int_{-L/2}^{L/2} f_{y'}(x',y',z') dz' = -12|e_{11}| B_0 r_0^3 L \left( \frac{x'}{x'^2 + y'^2 + L^2} \right)^{5/2},
\]
\[
\int_{-L}^{L} f_{z'}(x',y',z') dz' = 4|e_{11}| \int_{-L}^{L} \partial_{x'} B_y' dz' = -60|e_{11}| B_0 r_0^3 \int_{-L}^{L} \frac{x'y'z'}{|x'|^3} dz' = 0.
\]

And then
\[
P = 18|e_{11}|^2 B_0^2 r_0^6 L^2 \int \frac{x^2 + y^2}{(x'^2 + y'^2 + L^2)^{5/2}} dx' dy',
= 18|e_{11}|^2 B_0^2 r_0^6 L^2 \int_{0}^{R} \frac{\rho^2}{(\rho^2 + L^2)^{5/2}} \rho d\rho \int_{0}^{2\pi} d\psi,
= 24\pi|e_{11}|^2 B_0^2 L^2 \left( \frac{r_0}{L_z} \right)^6 \left[ 1 - \frac{1 + 4\xi^2}{(1 + \xi^2)^4} \right],
\]
where we have replaced \( L = L_z/2 \) and defined \( \xi = R/L_z, R = \sqrt{L_z^2 + L_y^2}/2 \).

For the \( h_{12} = h_{21} \) mode, one can see that Equ.(A11) has the same formalism as (A9) up to a minus sign, and Equ.(A12) is the same as Equ.(A8) since \( k_x = k_y = 0 \). The only difference is the integral of function \( f_z \):
\[
\int_{-L}^{L} f_z(x',y',z') dz' = 2|e_{11}| \int_{-L}^{L} (\partial_{y'} B_x' - \partial_{x'} B_y') dz' = 30|e_{11}| B_0 r_0^3 \int_{-L}^{L} \frac{z}{|x'|^5} \left( \frac{x'^2 - y'^2}{|x'|^2} \right) dz' = 0.
\]

So the total power is the same as that for the \( h_{11} \) mode.
b. A gravitational wave propagates along the y (or x) direction

In this case, \( k_y = w \) and \( \hat{k}_y = 1 \). Take the same approach as that in Sec. A2a to get the total power radiated approximately

\[
P = \frac{1}{8} \sum_{i=1}^{3} \int \left[ F_i(x', z') F_i^*(x', z') \right] dx' dz', \tag{A38}
\]

where

\[
F(x', z') = \int_{-L}^{L} f_i(x', y', z') dy'. \tag{A39}
\]

The minus sign in front of Eqn. (A38) coming from the transformation \( d\bar{y}/\bar{z} = -d\bar{z}/\bar{y} \). For the \( h_{11} = -h_{33} \) mode, by using the current components (A8-A10), we get

\[
\int_{-L}^{L} f_{x'}(x', y', z') dy' = 2|e_{11}| \int_{-L}^{L} (i w \bar{B}_{z'} + \partial_y \bar{B}_{z'}) dy'
\]

\[
= \frac{4iw|e_{11}| B_0 r_0^3 L}{(x'^2 + z'^2)^2(x'^2 + z'^2 + L^2)^{1/2}} \left[ 3 \left( \frac{L^2}{x'^2 + z'^2 + L^2} \right) z'^2 - (x'^2 + z'^2) \right], \tag{A40}
\]

\[
\int_{-L}^{L} f_{y'}(x', y', z') dy' = -4|e_{11}| \int_{-L}^{L} \partial_x \bar{B}_{z'} dy'
\]

\[
= \frac{-8|e_{11}| B_0 r_0^3 L x'}{(x'^2 + z'^2)^2(x'^2 + z'^2 + L^2)^{3/2}} \left[ 3(x'^2 + z'^2) + 2L^2 - z'^2 \left( 15 + \frac{5L^2(x'^2 + z'^2) + 8L^4}{(x'^2 + z'^2)(x'^2 + z'^2 + L^2)} \right) \right], \tag{A41}
\]

\[
\int_{-L}^{L} f_{z'}(x', y', z') dy' = 2|e_{11}| \int_{-L}^{L} (i w \bar{B}_{x'} + \partial_y \bar{B}_{x'}) dy'
\]

\[
= \frac{4iw|e_{11}| B_0 r_0^3 L x'}{(x'^2 + z'^2)^2(x'^2 + z'^2 + L^2)^{3/2}} \left[ 3(x'^2 + z'^2) + 2L^2 \right], \tag{A42}
\]

for \( x'^2 + z'^2 > 0 \). For the line \( x'^2 + z'^2 = 0 \), we have

\[
\int_{-L}^{L} f_{x'}(x', y', z') dy' = 2|e_{11}| \bar{B}_{z'} \bigg|_{y'=-L}^{y'=-L} = -4|e_{11}| B_0 r_0^3 \frac{L}{L^2}, \tag{A43}
\]

and

\[
\int_{-L}^{L} f_{y'}(x', y', z') dy' = \int_{-L}^{L} f_{z'}(x', y', z') dy' = 0. \tag{A44}
\]

Thus, the contribution from line \( x'^2 + z'^2 = 0 \) can be neglected. In the following we assume that the graviton can travel in the \( z - x \) plane with a shortest radius \( \epsilon \), i.e., \( x'^2 + y'^2 \geq \epsilon^2 \).

After transforming \( z' = \rho \cos \psi, x' = \rho \sin \psi \), we get \( dx'dz' = -\rho d\rho d\theta \). Then it leads to

\[
\int_{\epsilon}^{R} \rho d\rho \int_{0}^{2\pi} F_2^2 d\psi = \int_{\epsilon}^{R} \rho d\rho \frac{16w^2|e_{11}|^2 B_0^2 r_0^6 L^2}{\rho^3(\rho^2 + L^2)^2} \int_{0}^{2\pi} \left[ \left( 3 - \frac{L^2}{\rho^2 + L^2} \right) \cos^2 \psi - 1 \right] d\psi
\]

\[
= 4\pi w^2 |e_{11}|^2 B_0^2 r_0^6 L^2 \int_{\epsilon}^{R} \frac{4L^4 + 12L^2 \rho^2 + 11\rho^4}{\rho^4(\rho^2 + L^2)^3} d\rho = \frac{4\pi w^2 |e_{11}|^2 B_0^2 r_0^6 L^2}{4\rho^2(\rho^2 + L^2)^2} \left[ \frac{8L^2 + 11\rho^2}{\rho^2(\rho^2 + L^2)^2} \right] \bigg|_{\rho=\epsilon} \tag{A45}
\]

\[
\int_{\epsilon}^{R} \rho d\rho \int_{0}^{2\pi} F_2^2 d\psi = 64|e_{11}|^2 B_0^2 r_0^6 L^2 \int_{\epsilon}^{R} \frac{dp}{\rho^3(\rho^2 + L^2)^3} \int_{0}^{2\pi} \sin^2 \psi \left[ 3\rho^2 + 2L^2 - \rho^2 \cos^2 \psi \left( 15 + \frac{5L^2 \rho^2 + 8L^4}{\rho^2(\rho^2 + L^2)} \right) \right] d\psi
\]

\[
= 8\pi |e_{11}|^2 B_0^2 r_0^6 L^2 \int_{\epsilon}^{R} 32L^8 + 160L^6 \rho^2 + 320L^4 \rho^4 + 300L^2 \rho^6 + 117\rho^8 d\rho
\]

\[
= \frac{8\pi |e_{11}|^2 B_0^2 r_0^6 L^2}{8\rho^4(\rho^2 + L^2)^4} \left[ \frac{64L^6 + 256L^4 \rho^2 + 339L^2 \rho^4 + 156\rho^6}{\rho^2(\rho^2 + L^2)^4} \right] \bigg|_{\rho=\epsilon} \tag{A46}
\]
and
\[ \int_0^r \rho d\rho \int_0^{2\pi} F_{x}^2 d\psi = 16w^2|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^2 \int_0^R \rho d\rho \int_0^{2\pi} \frac{(3\rho^2 + 2L^2)^2}{\rho^3(\rho^2 + L^2)^3} \sin^2 \psi \cos^2 \psi d\psi \]
\[ = 4\pi w^2|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^2 \int_\epsilon^R \frac{(3\rho^2 + 2L^2)^2}{\rho^3(\rho^2 + L^2)^3} d\rho = -4\pi w^2|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^2 \frac{8L^2 + 9\rho^2}{4\rho^2 (L^2 + \rho^2)^3} \bigg|_{\rho=\epsilon} . \]

Then the total power is
\[ P = -\frac{\pi}{2} w^2|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^2 \left( \frac{4L^2 + 5\rho^2}{\rho^2 (L^2 + \rho^2)^2} \right) \bigg|_{\rho=\epsilon} \]
\[ - \pi|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^2 \left( \frac{64L^6 + 256L^4\rho^2 + 339L^2\rho^4 + 156\rho^6}{8\rho^4 (L^2 + \rho^2)^4} \right) \bigg|_{\rho=\epsilon} , \]

where the second term could be neglected when \( w \epsilon \gg 1 \).

For the \( h_{13} = h_{31} \) mode, one can see that Equ.(A17) has the same formalism as (A16) up to a minus sign, and Equ.(A19) is the same as Equ.(A14). The only difference is the integral of the function \( f_y \):
\[ \int_{-L}^L f_y'(x', y', z') dy' = 2|\epsilon_{11}| \int_{-L}^L (\partial_x \hat{B}_{z'} - \partial_z \hat{B}_{x'}) dy' \]
\[ = \frac{4|\epsilon_{11}| B_0 r_0^3 L z'}{(x'^2 + z'^2)(x'^2 + z'^2 + L^2)^{3/2}} \left[ 6(x'^2 + z'^2) + 4L^2 + \frac{(z'^2 - x'^2)(15(x'^2 + z'^2)^4 + 20L^2(x'^2 + z'^2) + 8L^4)}{(x'^2 + z'^2)(x'^2 + z'^2 + L^2)} \right] , \]

and then
\[ \int_0^R \rho d\rho \int_0^{2\pi} F_{y}^2 d\psi = \int_\epsilon^R \rho d\rho \int_0^{2\pi} \cos^2 \psi \left( 6\rho^2 + 4L^2 + \frac{(\cos^2 \psi - \sin^2 \psi)(15\rho^4 + 20L^2\rho^2 + 8L^4)}{\rho^2 (L^2 + \rho^2)^2} \right)^2 d\psi \]
\[ = 8\pi|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^6 \int_\epsilon^R \frac{(160L^8 + 800L^6\rho^2 + 1552L^4\rho^4 + 1380L^2\rho^6 + 477\rho^8)}{\rho^5 (\rho^2 + L^2)^5} d\rho \]
\[ = -8\pi|\epsilon_{11}|^2 B_{0}^2 \rho^6 R^6 \left( \frac{320L^{10} + 1280L^8\rho^2 + 1939L^6\rho^4 + 1468L^4\rho^6 + 672L^2\rho^8 + 192\rho^{10}}{8L^4\rho^4 (L^2 + \rho^2)^4} \right) + \frac{24}{L^6} \log \frac{\rho^2 + L^2}{\rho^2} \bigg|_{\rho=\epsilon} . \]

Therefore, the total power is the same as that with \( h_{11} \) mode since the above contribution can be neglected when \( w \epsilon \gg 1 \).

Appendix B: Detail calculations in the near-zone of a neutron star

To calculate Equ.(72), one needs to perform the following integration
\[ \int e^{-iw\theta(r'\theta' - r''\theta'')} e^{-iw \sin \theta (\sin \theta' \phi' - \sin \theta'' \phi'')} dr' dr'' r'^{2r''} \sin \theta' d\theta' d\phi' \sin \theta'' d\theta'' d\phi'' d\Omega . \]

(B1)

By using the short wave approximation and the following transformations
\[ x = r \theta , \quad y = r \sin \theta \phi , \quad dx \approx r d\theta , \quad dy \approx r \sin \theta d\phi , \]

(B2)

and
\[ \hat{x} = \theta , \quad \hat{y} = \sin \theta \phi , \quad d\hat{x} \approx d\theta , \quad d\hat{y} \approx d\theta \phi , \]

(B3)

we get
\[ 4\pi^2 \int \frac{dx dy}{w^2} \int \delta(x' - x'') \delta(y' - y'') dx' dy' dx'' dy'' \]
\[ \approx 4\pi^2 \int \delta(x' - x'') \delta(y' - y'') dx' dy' dx'' dy'' = 4\pi^2 \int dx' dy' = 4\pi^2 \int r'^{2} \sin \theta' d\theta' d\phi' . \]

(B4)
To calculate the radial part in the integrals Equs.(73) and (74), the integral variable is changed to $\eta = 2M/r$ in the following:

\[
\int r^4 N^{-2} K_1 G^* (r') dr' = \frac{3}{4} B_0 R^4 \cos \chi \int \frac{|K_1|^2}{M^2 r'^3} \left( \frac{r'}{M} \ln N^2 + \frac{1}{N^2} + 1 \right) dr'
\]

\[
= \frac{3}{4 M^3} B_0 R^4 \cos \chi \int \left( \frac{w^2 + 1}{r^2} \right) \left[ \ln N^2 + \frac{M}{r'} \left( \frac{1}{N^2} + 1 \right) \right] dr'
\]

\[
= \frac{3}{4 M^3} B_0 R^4 \cos \chi \int \left( \frac{w^2 + 1}{4 M^3} \eta^2 \right) \left[ \ln(1 - \eta) + \frac{1}{2} \eta \left( \frac{1}{1 - \eta} + 1 \right) \right] \frac{2M}{\eta^2} d\eta
\]

\[
= \frac{3}{4 M^4} B_0 R^4 \cos \chi \left[ \frac{3M}{2r} - \frac{M^2}{2r^2} + \left( \frac{3}{4} - w^2 M^2 + \frac{r M^2 w^2}{M^3} \right) \ln \left( 1 - \frac{2M}{r} \right) \right] \bigg|_{R}^{R+L}
\]

\[
\approx \frac{3}{4 M^4} B_0 R^4 \cos \chi \left[ -2 - \frac{1}{4 M^2} - \frac{1}{6} \frac{8M^3}{r^3} \right] \left( \frac{M^2 w^2}{R^5} \right) \bigg|_{R}^{R+L}
\]

\[
\approx \frac{w^2}{8 M^2} B_0 R^4 \cos \chi \left[ \frac{4M^2}{R^5} (1 - (1 + L/R)^{-2}) + \frac{8M^3}{R^5} (1 - (1 + L/R)^{-3}) \right]
\]

\[
= w^2 \cos \chi B_0 LR \left( 1 + \frac{3M}{R} \right)
\]

and

\[
\int N^{-1} K_0^* G'' (r'') dr'' = \frac{3}{4} B_0 R^4 \cos \chi \int \frac{|K_0|^2}{M^2 r'^5} \left( \frac{r''}{M} \ln N^2 + \frac{1}{N^2} + 1 \right) dr''
\]

\[
= \frac{3}{4 M^3} B_0 R^4 \cos \chi \int \frac{w^2}{r'^4} \left[ \ln N^2 + \frac{M}{r''} \left( \frac{1}{N^2} + 1 \right) \right] dr''
\]

\[
= \frac{3}{4 M^3} B_0 R^4 \cos \chi \int \frac{w^2}{(2M)^3} \eta^4 \left[ \ln(1 - \eta) + \frac{1}{2} \eta \left( \frac{1}{1 - \eta} + 1 \right) \right] \frac{2M}{\eta^2} d\eta
\]

\[
= \frac{3w^2}{4 M^3} B_0 R^4 \cos \chi \int \eta^2 \left[ \ln(1 - \eta) + \frac{1}{2} \eta \left( \frac{1}{1 - \eta} + 1 \right) \right] d\eta
\]

\[
= \frac{3w^2}{(2M)^6} B_0 R^4 \cos \chi \left[ \frac{10M}{3r} + \frac{5(2M)^2}{6r^2} + \frac{5(2M)^3}{9r^3} - \frac{(2M)^4}{4r^4} + \frac{5}{3} - \frac{2(2M)^3}{3r^3} \right] \ln \left( 1 - \frac{2M}{r} \right) \bigg|_{R}^{R+L}
\]

\[
\approx \frac{3w^2}{(2M)^6} B_0 R^4 \cos \chi \left[ \frac{(2M)^6}{18r^6} + \frac{(2M)^7}{14r^7} \right] \bigg|_{R}^{R+L}
\]

\[
\approx \frac{3w^2}{(2M)^6} B_0 R^4 \cos \chi \left( \frac{1}{18R^6} (1 - (1 + L/R)^{-6}) + \frac{M}{7R^7} (1 - (1 + L/R)^{-7}) \right)
\]

\[
\approx \frac{w^2 B_0 L \cos \chi}{R^3} \left( 1 + \frac{3M}{R} \right),
\]

where we only keep the leading order of $M/R$, and take the approximation $L \ll R$ in the final results of the above integrals.

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