Darboux coordinates for the Hamiltonian of first order

Einstein-Cartan gravity

N. Kiriushcheva and S.V. Kuzmin

Faculty of Arts and Social Science,
Huron University College and Department of Applied Mathematics,
University of Western Ontario, London, Canada

(Dated: March 26, 2010)

Abstract

Based on preliminary analysis of the Hamiltonian formulation of the first order Einstein-Cartan action (arXiv:0902.0856 [gr-qc] and arXiv:0907.1553 [gr-qc]) we derive the Darboux coordinates, which are a unique and uniform change of variables preserving equivalence with the original action in all spacetime dimensions higher than two. Considerable simplification of the Hamiltonian formulation using the Darboux coordinates, compared with direct analysis, is explicitly demonstrated. Even an incomplete Hamiltonian analysis in combination with known symmetries of the Einstein-Cartan action and the equivalence of Hamiltonian and Lagrangian formulations allows us to unambiguously conclude that the unique gauge invariances generated by the first class constraints of the Einstein-Cartan action and the corresponding Hamiltonian are translation and rotation in the tangent space. Diffeomorphism invariance, though a manifest invariance of the action, is not generated by the first class constraints of the theory.
I. INTRODUCTION

In this paper we continue our search for the gauge invariance of the Einstein-Cartan (EC) action using the Hamiltonian formulation of its first order form, which is valid in all spacetime dimensions \( D \) higher than two \((D > 2)\). This investigation was started in \([1–3]\). The complete Hamiltonian analysis of the EC action when \( D = 3 \), including the restoration of gauge invariance, was performed in \([1]\) because of the simplification in calculations which appears in \( D = 3 \). In dimensions \( D > 3 \) the calculations are much more involved and were not completed, though the preliminary results were reported in \([2]\). The main goal of the present paper is the derivation of the Darboux coordinates which, as we will show, drastically simplify the calculations at the first steps of the Dirac procedure \([4, 5]\). We hope that additional simplifications will also occur at the later steps and it will be possible to complete the Hamiltonian analysis for \( D > 3 \) and give the unique answer to the question: what is the gauge symmetry of the EC action, or the symmetry which is generated by the first class constraints.

The Hamiltonian formulation of the EC action is an old and apparently solved problem. It is claimed in many articles and included in monographs that the canonical formulation of EC theory has already been completed and its gauge symmetries are Lorentz invariance and diffeomorphism (very often only so-called “spatial” diffeomorphism). The reasons for reconsidering this claim are the following:

A) The first order form of the EC action is a uniform formulation valid in all \( D > 2 \) dimensions and it seems to us very suspicious that such a drastic change of gauge invariance in different dimensions is possible, e.g. from Poincaré in three-dimensional case (both gauge parameters with internal indices) \([1, 6]\) to Lorentz (internal) plus diffeomorphism (external) when \( D = 4 \), as stated in many papers on Hamiltonian formulations of tetrad gravity, and especially in papers and monographs on Loop Quantum Gravity (LQG) \([7, 8]\), where the spatial diffeomorphism constraint is always assumed to be present, irrespective of what variables are used. But it is clear even from the first steps of the Dirac procedure, as shown in \([1, 2]\), that diffeomorphism (neither spatial diffeomorphism, as in LQG, nor full spacetime diffeomorphism) cannot be a gauge symmetry generated by the first class constraints of the EC action, not only in \( D = 3 \), but in any dimension. The claim that spatial diffeomorphism is a gauge symmetry of tetrad gravity is the result of a non-canonical change of variables
(see Section V of [9]) that was “justified” only by such “arguments” as “convenience” and a
desire to accommodate the “expected” results. Of course, diffeomorphism is an invariance
of EC action (as it is manifestly generally covariant), as well as an invariance of EC action
under rotation and translation in internal space [10]; in fact, many other invariances can be
found in the Lagrangian formalism by constructing differential identities [3]. The statement
that the EC action is invariant under internal translation when $D = 3$ and is not invariant
in dimensions $D > 3$ is simply wrong as this contradicts known results [10, 11]. The change
of gauge symmetry from internal translation (the gauge parameter has an internal index) to
diffeomorphism (the gauge parameter is the “world” vector) does not seem to be feasible as
the first order EC action is formulated uniformly for all dimensions ($D > 2$).

Diffeomorphism is one of the invariances of the EC action but it is not a gauge symmetry
generated by the first class constraints [1, 2]. The *gauge symmetry* is a unique characteristic
of a theory and in a Hamiltonian formalism it must be uniquely derived using the Dirac pro-
cedure. According to Dirac’s conjecture [4] all the first class constraints of the Hamiltonian
formulation are responsible for the *gauge* invariance and any gauge symmetry must be deriv-
able from first class constraints [12] using, for example, the Castellani algorithm [13]. Only
after that is it possible to answer the question posed by Matschull [14] for $D = 3$ (but which
is equally well relevant in all dimensions): “what is a gauge symmetry and what is not”.
We are not aware of such a derivation for the Hamiltonian formulation of the EC action,
similar to the Einstein-Hilbert, metric, action [15, 16], where full spacetime diffeomorphism
is indeed gauge symmetry. Some arguments that so-called spatial diffeomorphism is a gauge
symmetry have been made; but this is not even a symmetry of the EC action which is
invariant under the full spacetime diffeomorphism, not under its spatial part separately.

B) It was shown some time ago that the EC action is invariant under “translations and
rotations in the tangent spaces” [10] with an algebra of generators that has “a more general
group structure than the original Poincaré group” [11]. It differs from the original Poincaré
group for $D > 3$ only by having a non-zero commutation relation between two translational
generators [10, 11]. Moreover, the explicit form of the transformations of fields were given
with two parameters that correspond to internal translation and rotation (see e.g. [11, 17]).
These results contradict the statement that the EC Lagrangian is not invariant under trans-
lation. Recently (without using the Hamiltonian formalism) translational and rotational
invariances of the first order EC action were derived in [3] by constructing the simplest
differential identities from the Euler derivatives which follow from the EC Lagrangian. To find a gauge invariance of the EC action, a Hamiltonian analysis is needed. However, the choice of what to call a gauge invariance is often based on different arguments, i.e. according to [18] “it is partly possible (and physically more plausible) to unify the two local gauge groups - Poincaré on the frames and general covariance”. Is it possible to have the gauge group that unify Poincaré and diffeomorphism? Lagrangian and Hamiltonian formalisms are equivalent; and in the Hamiltonian formalism, the gauge invariance must be derived from first class constraints [12], not to be defined arbitrarily. The number of constraints not only fixes the number of gauge parameters (which equals to the number of primary first class constraints) and their tensorial character but it also defines the number of degrees of freedom (found from the number of all constraints) [19]. Thus, internal translation and diffeomorphism cannot simultaneously be gauge group, as the number of constraints needed to accommodate both symmetries lead to a negative number of degrees of freedom, which is physically not plausible. To find a unique gauge symmetry from all possible invariances of an action cannot be done unambiguously without a Hamiltonian analysis. We do not rely on geometrical, or physical, or any other plausible argument; and our goal is to reveal the gauge symmetry of the EC theory using the Hamiltonian method.

C) The Hamiltonian analysis of the first order form of the EC action is specialized to some particular dimension, e.g. [6, 20–23]. However, such formulations might either destroy or miss some general features of the original action, which are valid in all dimensions (except the special case \(D = 2\)). For example, in constructing Darboux coordinates (the main subject of our article), it would be artificial to introduce such variables for each dimension separately - they have to be common for all dimensions as the original EC action is.

Our method of finding Darboux variables is valid in all dimensions \(D > 2\). It is not a purely mathematical interest to find the most general formulation, but it has a practical reason: the EC action is formulated in all dimensions, and so the correct methods have to produce meaningful results in all dimensions simultaneously. This will guarantee that nothing is missing or misinterpreted in the physically important four-dimensional case. Examples of formulations that were designed for only particular dimensions are: when \(D = 3\) the treatment of the EC action based on similarities (but not equivalence [14]) with the Chern-Simons action [6]; the construction of Darboux coordinates by Bañados and Contreras [23] that works only in the \(D = 4\) case and allows for neither consideration of the \(D = 3\)
limit nor dimensions higher than four. To have the correct Hamiltonian formulation of the EC action, and to find its unique gauge invariance in the physically interesting $D = 4$ case, we have to perform the analysis using an approach valid in all dimensions. An important property of using a formulation valid in all dimensions is the possibility to check the $D = 3$ limit at all stages of the calculations. The Hamiltonian formulation in the $D = 3$ case is simple, gives the consistent result and a simple Lie algebra of Poisson brackets (PBs) among the first class constraints [1]. In higher dimensions we can expect that some modifications of the Poincaré algebra will appear (we argued in [2] that the only possible modification is the non-zero PB among two translational constraints, which is exactly what happens in [10, 11]); but such modifications must disappear in the $D = 3$ limit. The calculation of constraints and their PB algebra in the Hamiltonian formulation of the EC action are still quite involved even after drastic simplification due to introduction of Darboux coordinates, and so the possibility of checking the consistency of the results by considering the $D = 3$ limit, at all stages of calculations, is extremely important. Thus, we do not specialize our analysis to a particular dimension.

The construction of Darboux coordinates which are uniform for all dimensions and simplify the Hamiltonian analysis is the main goal of this article.

The paper is organized as follows. In the next Section (II) we establish notation and provide arguments to support our expectation that Darboux coordinates exist. In Section III, based on the result of the direct Hamiltonian analysis [2], we derive Darboux coordinates. In Section IV, we show that introduction of Darboux coordinates allows one to perform the Lagrangian or Hamiltonian reduction in a much simpler manner than the Hamiltonian reduction in the direct Hamiltonian approach of [2], and to attack the most involved calculations: finding PBs among secondary first class constraints (or equivalently, to prove the closure of the Dirac procedure) which is needed to find gauge transformations of the EC action in the Hamiltonian formalism. These calculations are briefly outlined. In particular, using Dirac brackets, we demonstrate that there is a strong indication that in all dimensions the PB between translational and rotational constraints are the same and coincide with the corresponding part of the Poincaré algebra known for the $D = 3$ case (the same conclusion was made in [2] using the Castellani algorithm [13]). In the last Section (V) the results are summarized and conclusion about gauge invariance of the EC action is made. The properties of some combinations of fields that considerably simplify the calculations are collected.
in Appendix A. In Appendix B the solution of the equation that arises in the course of the Lagrangian/Hamiltonian reduction is given.

II. NOTATION AND EXPECTATIONS

In [2] we considered the Hamiltonian formulation of the Einstein-Cartan action by direct application of the Dirac procedure to its first order form [24, 25]

\[ I_{EC} = - \int d^Dx \ e \left( e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\nu(\alpha)} e^{\mu(\beta)} \right) \left( \omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\gamma)} \omega_{\nu(\gamma\beta)} \right) \]  

(1)

where the covariant N-beins \( e_{\gamma(\rho)} \) and the connections \( \omega_{\nu(\alpha\beta)} \) \( (\omega_{\nu(\alpha\beta)} = -\omega_{\nu(\beta\alpha)}) \) are treated as independent fields in all spacetime dimensions \( (D > 2) \), and \( e = \text{det} \left( e_{\gamma(\rho)} \right) \).\(^1\) Greek letters indicate covariant indices \( \alpha = 0, 1, 2, ..., (D-1) \). Indices in brackets \( (...) \) denote the internal ("Lorentz") indices, whereas indices without brackets are external or "world" indices. Internal and external indices are raised and lowered by the Minkowski tensor \( \tilde{\eta}_{\alpha\beta} = (-, +, +, ...) \) and the metric tensor \( g_{\mu\nu} = e_{\mu(\alpha)} e_{\nu(\alpha)} \), respectively (we use a tilde for any combination with only internal indices and do not use brackets in such cases, except to indicate antisymmetrization in pairs of indices). N-beins are invertible: \( e^{\mu(\alpha)} e_{\nu(\beta)} = \delta^\alpha_\beta \), \( e^{\mu(\alpha)} e^{\nu(\alpha)} = \delta^\mu_\nu \).

The Lagrangian density of (1), after an integration by parts, can be written in the following form

\[ L_{EC} \left( e_{\mu(\alpha)}, \omega_{\mu(\alpha\beta)} \right) = eB^{(\rho)(\mu(\alpha))\nu(\beta)} e_{\gamma(\rho),\mu} \omega_{\nu(\alpha\beta)} - eA^{\mu(\alpha)\nu(\beta)} \omega_{\mu(\alpha\gamma)} \omega_{\nu(\gamma\beta)} \]  

(2)

where the functions \( A^{\mu(\alpha)\nu(\beta)} \) and \( B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) are defined as

\[ A^{\mu(\alpha)\nu(\beta)} = e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\nu(\alpha)} e^{\mu(\beta)}, \quad \frac{\delta}{\delta e_{\gamma(\rho)}} \left( eA^{\mu(\alpha)\nu(\beta)} \right) = eB^{\gamma(\rho)\mu(\alpha)\nu(\beta)}. \]  

(3)

The properties of the function \( A^{\mu(\alpha)\nu(\beta)} \) and \( B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) and their further generations that considerably simplify the calculations are collected in Appendix A.

\(^1\) Usually variables \( e_{\gamma(\rho)} \) and \( \omega_{\nu(\alpha\beta)} \) are named tetrads and spin connections, but such names are specialized for \( D = 4 \). As we consider the Hamiltonian formulation in any dimension \( (D > 2) \), we will call \( e_{\gamma(\rho)} \) and \( \omega_{\nu(\alpha\beta)} \) N-beins and connections, respectively.
For the Hamiltonian formulation, where we have to separate spatial and temporal indices (not separating spacetime itself into space and time),\footnote{If one writes, for example, equations of motion of a covariant theory in components the covariance is not lost, though it is not manifest. The common statement as in \cite{26}: “Unfortunately, the canonical treatment breaks the symmetry between space and time in general relativity and the resulting algebra of constraints is not the algebra of four diffeomorphism” is groundless. In the Hamiltonian formulation of General Relativity the covariance is not manifest, but it is not broken as the gauge symmetry of the Einstein-Hilbert action, diffeomorphism, is recovered in a manifestly covariant form for the second order \cite{15,16} and the first order \cite{9} formulations.} we use 0 for an external “time” index (and (0) for an internal “time” index) and Latin letters for “spatial” external indices \(k = 1, 2, ..., (D-1)\) ((\(k\)) for “spatial” internal indices).

In Progress Report \cite{2} (the references to equations from \cite{2} are indicated as Eq. (R#)) we demonstrated that after performing the Hamiltonian reduction (i.e. elimination of part of the variables by solving the second class constraints) in all dimensions, the canonical part of the total Hamiltonian is a linear combination of secondary constraints (called “rotational” \(\chi^{0(\alpha\beta)}\) and “translational” \(\chi^{0(\sigma)}\) constraints, see Eq. (R152))

\[
H_{\text{reduced}} (e_{\mu(\rho)}, \pi^{\mu(\rho)}, \omega_{0(\alpha\beta)}, \Pi^{0(\alpha\beta)}) = \pi^{0(\rho)} \dot{e}_{0(\rho)} + \Pi^{0(\alpha\beta)} \dot{\omega}_{0(\alpha\beta)} + H_c ,
\]

where the canonical Hamiltonian (up to a total spatial derivative) is

\[
H_c = -\omega_{0(\alpha\beta)} \chi^{0(\alpha\beta)} (e_{\mu(\rho)}, \pi^{k(\rho)}) - e_{0(\sigma)} \chi^{0(\sigma)} (e_{\mu(\rho)}, \pi^{k(\rho)}) .
\]

For \(D = 4\) the reduced Hamiltonian with the same set of canonical variables \cite{4} was obtained in \cite{25}.

All PBs among primary \((\pi^{0(\rho)}, \Pi^{0(\alpha\beta)})\) and among primary and secondary constraints \((\chi^{0(\alpha\beta)}, \chi^{0(\sigma)})\) are zero and the PB between two rotational constraints in all dimensions is

\[
\{ \chi^{0(\alpha\beta)}, \chi^{0(\mu\nu)} \}_{D>2} = \frac{1}{2} \tilde{\eta}^{\alpha\beta\mu} \chi^{0(\mu\nu)} - \frac{1}{2} \tilde{\eta}^{\alpha\mu\nu} \chi^{0(\beta\nu)} + \frac{1}{2} \tilde{\eta}^{\beta\nu\mu} \chi^{0(\alpha\nu)} - \frac{1}{2} \tilde{\eta}^{\alpha\nu\mu} \chi^{0(\beta\mu)} ,
\]

which corresponds to Lorentz rotation in the tangent space. It is necessary to find the remaining PBs:

\[
\{ \chi^{0(\alpha\beta)}, \chi^{0(\rho)} \}_{D>3} = ? ,
\]
\[ \{ \chi^{0(\rho)}, \chi^{0(\gamma)} \}_{D>3} = \].

In the \( D = 3 \) case, the calculation of (7)-(8) is simple \[1\] and leads to:

\[ \{ \chi^{0(\alpha\beta)}, \chi^{0(\rho)} \}_{D=3} = \frac{1}{2} \tilde{n}^{\beta\rho} \chi^{0(\alpha)} - \frac{1}{2} \tilde{n}^{\alpha\rho} \chi^{0(\beta)}, \]

\[ \{ \chi^{0(\rho)}, \chi^{0(\gamma)} \}_{D=3} = 0, \]

i.e. the PB algebra of secondary constraints (6), (9) and (10) is a true Poincaré algebra; and a complete set of first class constraints when using the Castellani algorithm leads to the rotational and translational invariance in tangent space \[1\]. In higher dimensions, even knowledge of only the primary constraints is enough to conclude that it is impossible to have diffeomorphism invariance following from the first class constraints and the gauge parameters must possess internal indices, i.e. they lead to rotation and translation in the internal space. Whether it is a true Poincaré algebra or a modified Poincaré, can only be found after the PBs (7)-(8) are calculated.

In higher dimensions, calculation of the PBs of (7)-(8) is very laborious because of the complexity of the constraints \[2\]. Nevertheless, these PBs are needed to prove the closure of the Dirac procedure, and to find the transformations that are produced by the first class constraints, i.e. to answer the question (in the Hamiltonian formalism) of which symmetry (from an infinite set of symmetries of the EC action \[3\]) is the gauge symmetry of the EC action.

In the conclusion of \[2\] we discussed possible modifications of the algebra of the PBs in dimensions \( D > 3 \), based on the assumption that despite a more complicated form of constraints the algebra of secondary constraints remains Poincaré, as in the \( D = 3 \) case, or is modified Poincaré algebra. We showed that in such cases, the Lagrangian corresponding to the reduced Hamiltonian (5) remains invariant. This Lagrangian can be obtained by performing the inverse Legendre transformations and it gives us just a different first order formulation of the original EC theory (Eq. (R162))

\[ L_{\text{reduced}}(e_{\mu(\alpha)}, \pi^{k(\rho)}, \omega^{0(\alpha\beta)}) = \]
\[ \pi^k(\rho) \dot{e}_k(\rho) + \omega_0(\alpha\beta) \chi^{0(\alpha\beta)} (e_{\mu(\rho)}, \pi^k(\rho)) + \epsilon_0(\sigma) \chi^{0(\sigma)} (e_{\mu(\rho)}, \pi^k(\rho)). \] 

(11)

In the Lagrangian formalism \( \pi^k(\rho) \), as well as \( \omega_0(\alpha\beta) \), are just auxiliary variables that can be eliminated using their equations of motion (exactly as \( \omega_{\mu(\alpha\beta)} \) can be eliminated in (1)) that leads back to the second order EC action. The Lagrangian (11) gives the first order formulation of the EC action that differs in field content from (1) but they both are equivalent to second order form of the EC action (after elimination of auxiliary fields). However, the first order form (11) leads directly to the Hamiltonian.

We can say that the following operations were performed:

\[ L_{EC} \rightarrow H \overset{\text{Hamiltonian/Dirac reduction}}{\Rightarrow} H_{\text{reduced}} \overset{(\text{Eq. (4)})}{\Rightarrow} L_{\text{reduced}} \overset{(\text{Eq. (11)})}{\Rightarrow} L_{\text{EC}}. \] 

(12)

This suggests (because the Hamiltonian and Lagrangian formalisms must lead to the same result, of course, if the reductions are performed correctly [27]) that such a reduced Lagrangian should be possible to obtain directly from the EC action, so there should be some transformations which give Darboux coordinates that can simplify the calculations of \( H_{\text{reduced}} \) compared to the direct calculations [2] performed for (2). Perhaps, it can simplify calculations of the remaining PBs among secondary constraints (7)–(8) and the corresponding gauge transformations. In other words, we are looking for Darboux coordinates such that

\[ L_{EC} \overset{\text{Darboux coordinates}}{\Rightarrow} L_{EC(D)} \overset{\text{Lagrangian reduction}}{\Rightarrow} L_{\text{reduced}} \]

with

\[ L_{\text{reduced}} = L_{\text{reduced}} \overset{(\text{Eq. (11)})}{\Rightarrow} H_{\text{reduced}} \overset{(\text{Eq. (4)})}{\Rightarrow}. \] 

(13)

In the next Section, based on the result of the direct Hamiltonian analysis [2], we derive the following Darboux coordinates for spatial connections (the temporal connections \( \omega_0(\alpha\beta) \); as well as the basic variables \( e_\gamma(\rho) \), N-beins, remain unaltered)

\[ \omega_{m(\alpha\beta)} = N_{m(\alpha\beta),0n(\sigma)} F^{n(\sigma)} + e_{p(\alpha)} e_{q(\beta)} \hat{\Sigma}^{(pq)} = \omega_{m(\alpha\beta)} (F) + \omega_{m(\alpha\beta)} \left( \hat{\Sigma} \right), \] 

(14)

where \( N_{m(\alpha\beta),0n(\sigma)} \) is a non-linear, algebraic (without derivatives) combination of N-beins which is antisymmetric in \( \alpha\beta \) whose explicit form is given by (13). The field \( \hat{\Sigma}^{(pq)} \) is
antisymmetric \((\hat{\Sigma}_m^{(pq)} = -\hat{\Sigma}_m^{(qp)})\) and traceless \(\hat{\Sigma}_m^{(mq)} = \hat{\Sigma}_m^{(pm)} = 0\) with all indices being external ("world") and spatial. (Here and below we will use "hat" for combinations with only external indices and brackets used to indicate antisymmetrization in pairs of external indices.) The transformation \((14)\) is invertible, valid in all dimensions \((D > 2)\) and preserves the \(D = 3\) limit \([1]\). Note that the number of components of \(\hat{\Sigma}_m^{(pq)}\) plus \(F^{n(\sigma)}\) is the same as of \(\omega_{m(\alpha\beta)}\) in all dimensions (the number of independent components of a field, using the notation of \([28]\), is defined by \([\ldots]):\)

\[
[\omega_{m(\alpha\beta)}] = \frac{1}{2} D (D - 1) (D - 1),
\]

\[
[F^{n(\sigma)}] = D (D - 1),
\]

\[
[\hat{\Sigma}_m^{(pq)}] = \frac{1}{2} D (D - 1) (D - 3)
\]

that gives

\[
[\omega_{m(\alpha\beta)}] = [\pi^{n(\sigma)}] + [\hat{\Sigma}_m^{(pq)}].
\]

In the discussion of Darboux coordinates specialized to \(D = 4\) case appearing in \([23]\), a different field is introduced, \(\hat{\lambda}_{km} = \hat{\lambda}_{mk}\), instead of our \(\hat{\Sigma}_m^{(pq)}\). The number of components is \([\hat{\lambda}_{km}] = \frac{1}{2} D (D - 1)\) which gives the correct balance of fields (see \((18)\)) only in the \(D = 4\) case (as in this dimension \([\hat{\lambda}_{km}] = [\hat{\Sigma}_m^{(pq)}] = 6\), but supports neither a \(D = 3\) limit nor a generalization to higher dimensions. The uniform description of the EC action in all dimensions is broken by such coordinates.

In Section IV, we show that using the transformation \((14)\) not only diagonalizes the “kinetic” part of the Lagrangian (terms with temporal derivatives of N-beins), as a consequence of the following properties

\[
e B^{k(\rho)0(\alpha)m(\beta)} \omega_{m(\alpha\beta)} (F) = F^{k(\rho)}, \quad B^{k(\rho)0(\alpha)m(\beta)} \omega_{m(\alpha\beta)} (\hat{\Sigma}) = 0,
\]

but also provides a separation of variables that allows one to perform the Lagrangian or Hamiltonian reduction in a much simpler manner than in the direct Hamiltonian approach of \([2]\) (i.e. to eliminate the field \(\hat{\Sigma}_m^{(pq)}\), which corresponds to solution of the secondary second
class constraints in the Hamiltonian analysis). After elimination of $\hat{\Sigma}_{(pq)}$, the Hamiltonian (4)-(5), where only first class constraints are present, can be just read off from the reduced first order Lagrangian (11). The Hamiltonian and the first class constraints that were obtained after long and cumbersome calculations in the direct approach of [2] which started from (2) can be found almost immediately when Darboux coordinates are introduced. We show that simplifications due to introduction of the Darboux coordinates allows us to attack the most involved calculations in the direct approach: finding the remaining PBs (7)-(8) among secondary constraints (or equivalently, to prove the closure of the Dirac procedure) which is needed to find gauge transformations of the EC action in the Hamiltonian formalism. These calculations will be briefly outlined. In particular, we demonstrate that there is a strong indication that in all dimensions the PB between translational and rotational constraints are the same and coincide with the corresponding part of the Poincaré algebra (9) known for the $D = 3$ case [1] (the same conclusion was made in [2] using the Castellani algorithm):

$$\{\chi^0(\alpha\beta), \chi^0(\rho)\}_{D>2} = \frac{1}{2} \bar{\eta}^{\beta\rho} \chi^0(\alpha) - \frac{1}{2} \bar{\eta}^{\alpha\rho} \chi^0(\beta).$$

(20)

III. DERIVATION OF DARBOUX COORDINATES USING A PRELIMINARY HAMILTONIAN ANALYSIS OF THE EINSTEIN-CARTAN ACTION

Direct application of the Dirac procedure to the first order formulation of Einstein-Cartan action without specialization to a particular dimension was discussed in [2] where after performing the Hamiltonian reduction (that is, elimination of second class constraints) the total Hamiltonian (4)-(5) was obtained. However, these calculations are extremely laborious and on the last stage (closure of the Dirac procedure) become almost unmanageable with the exception of the $D = 3$ case [1]. But this preliminary Hamiltonian analysis is indispensable because it allows us to find variables at the Lagrangian level that drastically simplify the first steps of the calculations. At the Lagrangian level, only the invertability of the change of variables is usually checked; but it might happen that the change of variables even being invertible is not canonical in the Hamiltonian formulation. That is why, in finding new variables it is important to rely on the Hamiltonian analysis, especially when working with systems which have first and second class constraints. We perform a classification of fields
according to their relation to the constraints arising in the Hamiltonian formulation. This specific role of different fields can be used also at the Lagrangian level and, in particular, allows one to find Darboux coordinates that preserve equivalence with the original action and are helpful in reducing the amount of calculation that must be done. We will use the classification of fields that corresponds to the classification of constraints. Primary constraints (using a notion introduced by Bergmann), especially first class (a notion introduced by Dirac), play the most important role in the Hamiltonian formulation and define the tensorial character of the gauge parameters (see Section 5 of [9]), so we call the variables “primary variables” if in the Hamiltonian formulation the corresponding momenta enter the primary first class constraints. For the first order Einstein-Cartan action the primary variables are $e_0(\alpha)$ and $\omega_{0(\alpha\beta)}$ [2]. Second class constraints (using Dirac’s classification) irrespective of their generation (primary, secondary, etc.) can be solved for pairs of phase-space variables (this is the Hamiltonian reduction). We call such variables “second class variables”, i.e. variables that at the Hamiltonian level can be eliminated. In the case of the first order Einstein-Cartan action second class variables are $\omega_{k(\alpha\beta)}$ [2]. Primary and second class variables of the first order EC action could be already identified from the results of the first gauge-free (without a priori choice of a particular gauge) Hamiltonian formulation for $D = 4$ [25]. The importance of careful preliminary analysis before doing the Lagrange reduction was emphasized in [29]: “it seems important to develop reduction procedure within Lagrangian formulation - in a sense similar to the Dirac procedure in the Hamiltonian formulation - that may allow one to reveal the hidden structure of the Euler-Lagrange equations of motion in a constructive manner”.

The above classification is crucial because the equivalence of the Lagrangian and Hamiltonian methods dictates that if changes involving primary variables are very restrictive at the Hamiltonian level [9] then the same must be true also at the Lagrangian level. An arbitrary change of variables can lead to the loss of equivalence of two formulations even for changes which are invertible, that is the sufficient condition only for nonsingular systems. The second class fields can be eliminated and there is more freedom to redefine them; but this redefinition has to be such that their elimination does not modify the PBs for the remaining fields, if this is what happens in the Hamiltonian formalism. This imposes some restrictions; and even in this case, the invertability of transformations is only a necessary condition. In particular, in the construction of the Darboux coordinates for second class
variables (which can be a complicated expression) they have to be independent of the primary variables, i.e. their variation with respect to primary variables must be zero so as to preserve the original independence of primary and second class variables. From the above arguments it is clear that the Hamiltonian analysis is indispensable if one wants to modify the Lagrangian while keeping its equivalence with the original one (for the gauge-invariant systems). For example, the original independence of primary and second class variables,
\[
\frac{\delta \omega_k(\alpha\beta)}{\delta e_0(\rho)} = \frac{\delta \omega_k(\alpha\beta)}{\delta \omega_\mu(\rho)} = 0,
\]
should be preserved even after a change of variables in which Darboux coordinates are introduced for \( \omega_{k(\alpha\beta)} \).

In this Section, we describe the construction of Darboux coordinates for the EC action and also illustrate the general points mentioned above. Our goal is to find the Darboux coordinates for the second class fields that simplify the Hamiltonian analysis. In [2], in the course of the Hamiltonian reduction, all \( \omega_{k(\alpha\beta)} \) were eliminated by solving the second class constraints: one part by solving the primary constraints and another part that involved primary and secondary constraints. So, it would be preferable to find such a representation of \( \omega_{k(\alpha\beta)} \) that separate its components into exactly two classes of variables, as in the Hamiltonian they were mixed leading to quite long calculations. We want to decouple them, i.e. we have to find such a transformation of an “auxiliary”, second class, field \( \omega_{k(\alpha\beta)} \) (Darboux coordinates) that diagonalizes the “kinetic part” and separate variables that can be eliminated by a Lagrangian reduction. Note that such a separation automatically appears in the \( D = 3 \) case [1].

The direct Hamiltonian analysis of the first order EC action (1) starts by introducing momenta conjugate to all independent variables. In (2) the only term that has “velocities” is
\[
L\left(e_{\gamma(\rho)}, 0\right) = eB^{\gamma(\rho)0(\alpha)\nu(\beta)}e_{\gamma(\rho), 0}\omega_{\nu(\alpha\beta)}
\]
and so the momenta corresponding to \( e_{\gamma(\rho)} \) are defined as
\[
\pi^{\gamma(\rho)} = \frac{\delta L}{\delta e_{\gamma(\rho)}, 0} = eB^{\gamma(\rho)0(\alpha)\nu(\beta)}\omega_{\nu(\alpha\beta)}.
\]
The only non-zero contributions (based on antisymmetry properties of \( B^{\gamma(\rho)\mu(\alpha)\nu(\beta)} \) (see Appendix A and also [2, 3])) are

13
or upon separating \( \pi^k(\rho) \) and \( \omega_{m(\alpha \beta)} \) into “space” and “time” components

\[
\pi^k(n) = eB^k(n)0(m(q))\omega_{m(pq)},
\]

Equations (24) and (25) lead to two primary second class constraints

\[
\phi^k(n) = \pi^k(n) - eB^k(n)0(m(q))\omega_{m(pq)} - 2eB^k(n)0(m(0))\omega_{m(q0)} \approx 0,
\]

\[
\phi^k(0) = \pi^k(0) - eB^k(0)0(m(q))\omega_{m(pq)} \approx 0.
\]

After introducing the following notation (see [2])

\[
\gamma^k(n) \equiv e^k(n) - \frac{e^k(0)}{e^0(0)}\omega^m(n),
\]

\[
E^{k(p)m(q)} \equiv \gamma^k(p)\omega^m(q) - \gamma^k(q)\omega^m(p),
\]

\[
I_{m(q)n(r)} \equiv \frac{1}{D-2}e^{m(q)}e^{n(r)} - e^{m(r)}e^{n(q)},
\]

\[
E^{k(p)m(q)}I_{m(q)n(r)} = \delta^k_n\delta^p_r,
\]

equation (26) (because it is a second class constraint in the Hamiltonian analysis) can be solved for \( \omega_{k(0)} \) (see Eq. (R46))

\[
\omega_{k(0)} = -\frac{1}{2ee^0(0)}I_{k(q)m(p)}\pi^{m(p)} - \frac{e^0(p)}{2e^0(0)}I_{k(q)m(p)}E^{m(a)n(b)}\omega_{n(ab)} + \frac{e^0(a)}{e^0(0)}\omega_{k(eq)}
\]

and (27) can be written in the following form

\[
\pi^k(0) = -ee^0(0)E^{k(p)m(q)}\omega_{m(pq)}.
\]

When \( D = 3 \) (and only when \( D = 3 \)) equation (29) can be solved for \( \omega_{m(pq)} \) and in equation (26) terms proportional to the connections \( \omega_{m(pq)} \) (with all “space” indices) cancel out, leading to separation of these two equations, (26) and (27), into equations containing
only $\omega_{m(pq)}$ and $\omega_{k(pq)}$, respectively. In addition, when $D = 3$, some terms in the Lagrangian disappear (see [1]). That is why the Hamiltonian analysis for $D = 3$ becomes so simple, as is the derivation of gauge transformations. In particular, when $D = 3$, equation (29) can be solved because $[\pi^{k(0)}] = [\omega_{m(pq)}] = 2$ (see [1]). Equation (29) for $D = 3$ is represented by two equations for two independent components of $\omega_{m(pq)}$, $\omega_{1(12)}$ and $\omega_{2(12)}$. It can be solved for $\omega_{1(12)}$ and $\omega_{2(12)}$ and the solution can be written in “covariant” form

$$\omega_{k(qp)} = \frac{1}{2ec^{0(0)}}I_{k(q)m(p)}\pi^{m(0)}.$$  

(30)

In higher dimensions, equation (29) cannot be solved in the same way. We showed in [2] that it is necessary to consider it together with the secondary constraints. In solving these constraints the combination of the form $\gamma^{m(n)}\omega_{m(pq)} = \tilde{\omega}_{n(pq)}$ was very useful (in $\tilde{\omega}_{n(pq)}$ all indices are internal), because the “trace” of this combination\(^3\) is proportional to $\pi^{k(0)}$

$$\pi^{k(0)} = 2ec^{0(0)}\gamma^{k(p)}\tilde{\omega}^{q}_{(qp)}.$$  

(31)

This suggests the introduction of variables that allow one to single out the contribution of (31), which is obviously the separation of $\tilde{\omega}_{n(pq)}$ into the trace, $\tilde{V}_q$, and the traceless, $\tilde{\Omega}_{n(pq)}$, parts

$$\tilde{\omega}_{n(pq)} = \tilde{\Omega}_{n(pq)} + \frac{1}{D-2}\left(\tilde{\eta}_{np}\tilde{V}_q - \tilde{\eta}_{nq}\tilde{V}_p\right),$$  

(32)

or equivalently

$$\omega_{m(pq)} = e^{(n)}_{m}\tilde{\Omega}_{n(pq)} + \frac{1}{D-2}\left(e_{m(p)}\tilde{V}_q - e_{m(q)}\tilde{V}_p\right)$$  

(33)

(where we have used $\tilde{\omega}_{n(pq)} = \tilde{\eta}_{hm}\tilde{\omega}^{m}_{(pq)}$.)

The variable $\tilde{\Omega}_{n(pq)}$ is an antisymmetric ($\tilde{\Omega}_{n(pq)} = -\tilde{\Omega}_{n(qp)}$) and traceless ($\tilde{\Omega}^{p}_{(pq)} = \tilde{\eta}^{pq}\tilde{\Omega}_{n(pq)} = 0$) field with all indices being internal. The necessary condition for any field re-definition (before checking the invertability) is that the number of fields is preserved, which is satisfied in our case because $[\tilde{\omega}_{n(pq)}] = [\tilde{\Omega}_{n(pq)}] + [\tilde{V}_q]$ in all dimensions. It is not difficult

\(^3\) For the original connection $\omega_{m(pq)}$, antisymmetric in internal indices, such a “trace” cannot be defined. We need combinations with all indices of the same nature and $\tilde{\omega}^{n}_{(pq)}$ (with all indices being internal) provides such a combination which naturally arises in the direct Hamiltonian analysis [2].
to demonstrate the invertability of (33). Contracting (32) with $\gamma^m(p)$ (or equally well with $\gamma^m(q)$) we obtain

$$\tilde{V}_p = \tilde{\omega}^q_{(qp)}. \quad (34)$$

Now contracting (33) with $\gamma^m(k)$ we find

$$\tilde{\Omega}^k_{(pq)} = \gamma^m(k)\omega_{m(pq)} - \frac{1}{D-2} \left( \tilde{\delta}^k_p \tilde{V}_q - \tilde{\delta}^k_q \tilde{V}_p \right) \quad (35)$$

and using (34)

$$\tilde{\Omega}^k_{(pq)} = \gamma^m(k)\omega_{m(pq)} - \frac{1}{D-2} \left( \tilde{\delta}^k_p \tilde{\omega}^n(m) - \tilde{\delta}^k_q \tilde{\omega}^n(mp) \right), \quad (36)$$

or in terms of the original connections

$$\tilde{\Omega}^k_{(pq)} = \gamma^m(k)\omega_{m(pq)} - \frac{1}{D-2} \left( \tilde{\delta}^k_p \gamma^m(n) - \tilde{\delta}^k_q \gamma^m(n) \right). \quad (37)$$

We then see that (33) is invertible. It is easy to show also that (37) is traceless.

We substitute (33) into (28) to express the connection $\omega_{k(q0)}$ in terms of new fields

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}} I_{k(q)m(p)} \pi^m(p) - \frac{e^{0(p)}}{2e^{0(0)}} I_{k(q)m(p)} E^{m(a)n(b)} \omega_{n(ab)} \left( \tilde{\Omega}, \tilde{V} \right) + \frac{e^{0(a)}}{e^{0(0)}} \omega_{k(aq)} \left( \tilde{\Omega}, \tilde{V} \right)$$

that upon substitution of $\omega_{n(ab)} \left( \tilde{\Omega}, \tilde{V} \right)$ and simple contractions gives

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}} I_{k(q)m(p)} \pi^m(p) + \frac{D-3}{D-2} e^{0(k)} \tilde{V}_q + \frac{e^{0(p)}}{e^{0(0)}} e^{(n)} \tilde{\omega}_{n(pq)}. \quad (38)$$

Using (31) and (34), we can express $\tilde{V}_q$ in terms of $\pi^k(0)$ (this is also linear in auxiliary fields redefinition with $[\pi^k(0)] = [\tilde{V}_p]$ in all dimensions)

$$\tilde{V}_q = \frac{1}{2ee^{0(0)}} e^{k(q)} \pi^k(0). \quad (39)$$

Finally, we obtain

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}} I_{k(q)m(p)} \pi^m(p) + \frac{D-3}{D-2} e^{0(k)} \frac{1}{2ee^{0(0)}} e^{m(q)} \pi^m(0) + \frac{e^{0(p)}}{e^{0(0)}} e^{(n)} \tilde{\omega}_{n(pq)}. \quad (40)$$
This is a linear transformation (in auxiliary fields) from spatial components of connections \( \omega_{m(\alpha\beta)} \) to the new set of variables \( \pi^m(\rho) \) and \( \tilde{\Omega}^n_{pq} \).

Note that this field redefinition, (40)-(41), equally well can be performed at the Lagrangian level and "momenta" \( \pi^m(\rho) \) are just new auxiliary variables that play a role of momenta conjugate to \( e^m(\rho) \) only after passing to the Hamiltonian formulation. In the Lagrangian formalism, (40)-(41) are a definition of Darboux coordinates, and from now on the auxiliary field \( \pi^m(\rho) \) will be denoted as \( F^m(\rho) \).

We can combine (40) and (41) into one "semicovariant" expression

\[
\omega_{m(\alpha\beta)} = N_{m(\alpha\beta)0n(\sigma)} F^n(\sigma) + e_{p(\alpha)} e_{q(\beta)} e_m(0) \gamma_p(\gamma) \gamma_q(\epsilon) \tilde{\Omega}_{n(bc)} \tag{42}
\]

where

\[
N_{m(\alpha\beta)0n(\sigma)} = \frac{1}{2ee^{0(0)}} \times
\]

\[
\left[ \left( \delta^0_\alpha \delta^0_\beta - \delta^0_\alpha \delta^0_\beta \right) - I_{m(\alpha)0n(\sigma)} \delta^0_\sigma + \frac{D - 3}{D - 2} e_{m(0)} e_{n(q)} \delta^0_\sigma \right] + \delta^0_\alpha \delta^0_\beta \frac{1}{D - 2} \left( e_{m(p)} e_{n(q)} - e_{m(q)} e_{n(p)} \right) \delta^0_\sigma.
\]

The advantage of going to Darboux variables

\[
L \left( e^{\mu(\alpha)}, \omega^{\mu(\alpha\beta)} \right) \to L \left( e^{\mu(\alpha)}, F^{m(\rho)}, \omega^{0(\alpha\beta)}, \tilde{\Omega}_{n(pq)} \right)
\]

is based on the following properties

\[
e^{B^0_{(\rho)}(\alpha)m(\beta)} N_{m(\alpha\beta)0n(\sigma)} = \delta^k_n \delta^p_\sigma, \quad B^{(\rho)0(\alpha)m(\beta)} \omega_{m(\alpha\beta)} \tilde{\Omega} = 0 \tag{45}
\]

that for the "kinetic part" of the original Lagrangian (2) gives a simple expression that is quadratic in fields

\[
e^{B^0_{(\rho)}(\alpha)m(\beta)} e_{k(\rho),0} \omega_{m(\alpha\beta)} = F^{k(\rho)} e_{k(\rho),0}. \tag{46}
\]

The possibility of eliminating the field \( \tilde{\Omega}_{e(pq)} \) at the Lagrangian level (Lagrangian reduction) depends on the presence of terms quadratic in this field. The semicovariant form (42)
makes the calculation quite simple as the only source of a term quadratic in $\tilde{\Omega}_{c(pq)}$ that contributes in (2) is the following (see (21))

$$-eA^{k(\alpha)m(\beta)}\omega_{k(\alpha)\gamma}\omega_{m(\beta)}^{(\gamma)}.$$  

(47)

Substitution of $\omega_{m(\alpha\beta)}(\tilde{\Omega})$ into (47) after contraction with $A^{k(\alpha)m(\beta)}$ gives

$$L\left(\tilde{\Omega}\tilde{\Omega}\right) = e\tilde{\Omega}_{b(pm)}\tilde{\Omega}^{p(nb)} - e\frac{e_{0(\alpha)}}{e_{0(0)}}\tilde{\Omega}^{q}_{\ (ap)}\frac{e_{0(\beta)}}{e_{0(0)}}\tilde{\Omega}^{p}_{\ (bq)},$$  

(48)

Upon performing variation with respect to $\tilde{\Omega}$ an equation similar to Eq. (R102) follows and, as we demonstrated in [2], it can be solved; though the second term of (48) makes calculations quite long (note that in the Darboux coordinates (42) we have the equation (48) as Eq. (R102) immediately, not after long preliminary calculations as in [2]). This suggests an additional change of Darboux coordinates separately for the part proportional to $\tilde{\Omega}_{b(pm)}$ and the first choice (as we have to keep the number of components the same) is the antisymmetric traceless field $\hat{\Sigma}^{(pq)}_{\ m}$, but with all indices being external and spatial. Such a field is defined as

$$\hat{\Sigma}^{(pq)}_{\ m} = e^{(n)}_{\ m}\gamma^{p(b)}\gamma^{q(c)}\tilde{\Omega}_{n(bc)}, \quad \hat{\Omega}^{(ab)}_{\ (k)} = \gamma^{m(k)}e_{p(\alpha)}e_{q(\beta)}\hat{\Sigma}^{(pq)}_{\ m},$$  

(49)

which is an invertible redefinition of auxiliary fields.

This additional redefinition diagonalizes (48); and that can be checked by substitution of (49) into (48), which leads to only one term that is quadratic in $\hat{\Sigma}^{(pq)}_{\ m}$

$$L\left(\tilde{\Omega}\tilde{\Omega}\right) \Rightarrow L\left(\hat{\Sigma}\hat{\Sigma}\right) = e_{g_{qp}}\hat{\Sigma}^{(kp)}_{\ m}\hat{\Sigma}^{(mq)}_{\ k},$$  

(50)

(here $g_{qp}$ is a short-hand notation for $e_{q(\alpha)}e_{p(\alpha)}$, not an independent field).

This completes the derivation of the Darboux coordinates written down in the preceding Section in (14)

$$\omega_{m(\alpha\beta)} = N_{m(\alpha\beta)0n(\sigma)}F^{n(\sigma)} + e_{p(\alpha)}e_{q(\beta)}\hat{\Sigma}^{(pq)}_{\ m} = \omega_{m(\alpha\beta)}(F) + \omega_{m(\alpha\beta)}(\hat{\Sigma}).$$  

(51)

The second property of (45) is unaltered by the change of variables in (49) and

$$B^{k(\rho)0(\alpha)m(\beta)}\omega_{m(\alpha\beta)}(\hat{\Sigma}) = 0.$$  

(52)
Using transformation (51) we can now obtain the equivalent Lagrangian in terms of Darboux coordinates, perform a Lagrangian reduction (i.e. eliminate \( \hat{\Sigma} \)) and find the corresponding Hamiltonian as it was schematically indicated in (13); or equally well, we can start the Hamiltonian formulation using the Lagrangian in Darboux coordinates and perform the Hamiltonian reduction. Of course, using either way, we obtain the same result (4)-(5). But before we write the Lagrangian in Darboux coordinates and the corresponding Hamiltonian we would like to make a few comments.

In their discussion of Darboux coordinates specialized to the \( D = 4 \) case, the authors of [23] emphasize the non-linearity of their transformations. Non-linearity in [23] and in our (14) appears with respect to only the non-second class fields (the tetrads of \( [23] \) or N-beins in our case), and exactly opposite, linearity in the second class fields (\( \pi^{k(\rho)}, \hat{\lambda}_{km} \) in \( [23] \) and \( F^{n(\sigma)}, \hat{\Sigma}_{m}^{(pq)} \) in our case), making the transformation invertible which is a necessary condition to establish equivalence of the original formulation with the formulation in terms of Darboux coordinates.

In constructing of (43) we used the results of the Hamiltonian analysis that preserves the \( D = 3 \) limit. A different, “more covariant”, combination can be constructed that also diagonalizes the “kinetic” part of the Lagrangian (i.e. has the same properties as (43)); for example

\[
N'_{m(\alpha\beta)0n(\sigma)} = \frac{1}{e} \left[ e_{m(\sigma)} A_{n(\alpha)0(\beta)} - \frac{1}{D - 2} \left( e_{m(\alpha)} A_{n(\sigma)0(\beta)} - e_{m(\beta)} A_{n(\sigma)0(\alpha)} \right) \right] \tag{53}
\]

where

\[
A_{n(\alpha)0(\beta)} = e_{n(\alpha)} e_{0(\beta)} - e_{n(\beta)} e_{0(\alpha)}. \tag{54}
\]

Without any preliminary Hamiltonian analysis, and working only in a particular dimension (e.g. \( D = 4 \)), such a diagonalization of the “kinetic” part looks even preferable as it has a “more covariant” form that simplifies calculations and does not involve a division by \( e^{0(0)} \) as in (43). However, it does not have the correct \( D = 3 \) limit, which cannot be seen if one is working only in a particular dimension, e.g. when \( D = 4 \) (see point C in the Introduction). Using the Darboux coordinates (51) with \( N'_{m(\alpha\beta)0n(\sigma)} \), instead of \( N_{m(\alpha\beta)0n(\sigma)} \), leads to problems in the Hamiltonian analysis. The reason for this is that the transformation found using the Hamiltonian analysis, (43), preserves properties of the primary variables,
i.e. \( \frac{\delta \omega_{m(\alpha \beta)}}{\delta \epsilon_0(\rho)} = 0 \), for the Darboux coordinates with \( N_{m(\alpha \beta)0n(\sigma)} \), as

\[
\frac{\delta}{\delta \epsilon_0(\rho)} \left( N_{m(\alpha \beta)0n(\sigma)} n^{\sigma} + e_{p(\alpha)} e_{q(\beta)} \hat{\Sigma}_{m(pq)} \right) = 0,
\]

(55)

in contrast to the “more covariant” combination \( N'_{m(\alpha \beta)0n(\sigma)} \) for which

\[
\frac{\delta}{\delta \epsilon_0(\rho)} N'_{m(\alpha \beta)0n(\sigma)} \neq 0.
\]

(56)

This leads to change of the algebra of constraints, gauge invariance, etc.

This is the illustration of how a pure Lagrangian consideration of singular systems can destroy its properties if one assumes that one can always use some operations (e.g. any invertible transformation) known for non-singular Lagrangians without careful analysis and without taking into account the specifics of singular systems (see a general discussion in [29]). When constructing the Darboux coordinates, we rely on the Hamiltonian analysis and this protects us from such mistakes.\(^4\) Construction of Darboux coordinates for one particular theory, the first order EC action, is the illustration of Dirac’s general statement [4]: “I [Dirac] feel that there will always be something missing from them [non-Hamiltonian methods] which we can only get by working from a Hamiltonian”.

**IV. LAGRANGIAN AND HAMILTONIAN REDUCTIONS OF THE EC THEORY IN DARBOUX COORDINATES**

Substitution of the Darboux coordinates \( (51) \) into the original EC Lagrangian \( L_{EC} \) is a simple task, as we have only one expression for all spatial connections \( (51) \) which are the only fields that are affected by a change of variables. This gives us a different but equivalent first order formulation of the EC theory \( L_{EC(D)} \)

\[
L_{EC} \left( e_{\mu(\alpha)}, \omega_{\mu(\alpha \beta)} \right) \implies L_{EC(D)} \left( e_{\mu(\alpha)}, \omega_{0(\alpha \beta)}, F^{k(\rho)}, \hat{\Sigma}_{m(pq)} \right) =
\]

\(^4\) This example illustrates why the problems might arise in the Faddeev-Jackiw method [30]. The simplectic form in [30] was found by diagonalizing the “kinetic” part of the Lagrangian but, as we have shown, it is not enough in general, to preserve equivalence and this is the reason for “non-equivalence” of the Dirac and symplectic methods found for some models (see e.g. [31], [32] where role of second class constraints was emphasized, and [33] where the observation of non-equivalence leads the authors to the conclusion about deficiency of the Dirac method).
\[
e_{k(\rho),0} F^{k(\rho)} + \left( eB^{k(\rho)m(\alpha)0(\beta)} e_{k(\rho),m} - 2eA^{0(\alpha)k(\gamma)} \omega_k^{(\beta)} \right) F, \hat{\Sigma} \right) \omega_0^{(\alpha\beta)}
\]

\[- e_{0(\rho),k} F^{k(\rho)} + eB^{n(\rho)k(\alpha)m(\beta)} e_{n(\rho),k} \omega_m^{(\alpha\beta)} \right) F, \hat{\Sigma} \right) - eA^{k(\alpha)m(\beta)} \omega_k^{(\alpha\gamma)} \right) F, \hat{\Sigma} \right) \omega_m^{(\gamma)} \right) F, \hat{\Sigma} \right).
\]

(57)

The appearance of two terms that are quadratic in the fields (first terms in the second and third lines of (57)) is the consequence of (46). Further separating spatial connections into two parts (see (40), (11)) and performing some contractions we obtain

\[L_{EC(D)} = e_{k(\rho),0} F^{k(\rho)} + \left( \frac{1}{2} F^{k(\alpha)} e_k^{(\beta)} - \frac{1}{2} F^{k(\beta)} e_k^{(\alpha)} + eB^{k(\rho)m(\alpha)0(\beta)} e_{k(\rho),m} \right) \omega_0^{(\alpha\beta)}
\]

\[- e_{0(\rho),k} F^{k(\rho)} + eB^{n(\rho)k(\alpha)m(\beta)} e_{n(\rho),k} \omega_m^{(\alpha\beta)} \right) F, \omega_m^{(\gamma)} \right) F \]

(58)

\[+ e g_{qp} \hat{\Sigma}_m^{(kp)} \hat{\Sigma}_n^{(mq)} + 2e e^{k(\beta)} \left( e_m^{(\beta)}, q + e_q^{(\gamma)} \omega_m^{(\gamma)} \right) \hat{\Sigma}_k^{(mq)},
\]

where \( g_{qp} \) is, as before, a short-hand notation for \( e_{q(\rho)} e_{p(\rho)} \), not an independent field. Note that in the terms proportional to \( \omega_0^{(\alpha\beta)} \), there are no contributions involving \( \hat{\Sigma}_k^{(mq)} \) as \( A^{0(\alpha)k(\gamma)} \omega_k^{(\beta)} \right) \hat{\Sigma} = 0; \) and contributions with \( F^{k(\alpha)} \), instead of direct substitution, can be obtained by contracting \( B^{k(\rho)m(\alpha)0(\beta)} \) with \( e_k^{(\beta)} \) and performing an antisymmetrization that gives \( eA^{0(\alpha)k(\gamma)} \omega_k^{(\beta)} \right) F \). The last line of (58) is the result of a contraction with the explicit form of \( \omega_k^{(\alpha\gamma)} \right) \hat{\Sigma} \) (see (51)).

The last line of (58) is the algebraic expression with respect to the field \( \hat{\Sigma}_k^{(mq)} \) that can be eliminated (due to the presence of a term in the Lagrangian quadratic in this field) by using its equation of motion (Lagrangian reduction). After elimination of this field we can obtain the Lagrangian (see (11)) with \( F^{k(\rho)} \) playing the role of momenta conjugate to \( e_{k(\rho)} \) in the Hamiltonian formulation, and without the need to solve the secondary second class constraints. Thus, using the Darboux coordinates we can obtain (11), which is the same Hamiltonian derived in (2), but without having to do any long calculation. Equivalence of the Lagrangian and Hamiltonian methods allows us to interchange the order of operations, and using (58) (without the Lagrangian reduction) we can immediately write the
Hamiltonian and then perform the Hamiltonian reduction. This approach is employed in
the present paper, giving us the possibility of comparing the results obtained here with the
direct calculations of [2].

The Hamiltonian of the first order EC action written in Darboux coordi-

nates can be just read off from the corresponding Lagrangian (58), as is possible for any first order action.
The advantage of Darboux coordinates is that the primary constraints are very simple and
the second class variables $F^k(\alpha)$ and $\hat{\Sigma}_k^{(mq)}$ can be easily separated and eliminated by solving
the second class constraints (Hamiltonian reduction).

The total Hamiltonian (introducing momenta conjugate to all fields in the Lagrangian) is

$$H_T \left( e_\mu(\rho), \pi^\mu(\rho), \omega_{0(\alpha\beta)}, \Pi^{0(\alpha\beta)}, F^k(\alpha), \Pi_k(\alpha), \hat{\Sigma}_k^{(mq)}, \hat{\Pi}_k^{(mq)} \right)$$

$$= \pi^\mu(\rho) \dot{e}_\mu(\rho) + \Pi^{0(\alpha\beta)} \dot{\omega}_{0(\alpha\beta)} + \dot{F}^k(\alpha) \Pi_k(\alpha) + \dot{\hat{\Sigma}}_k^{(mq)} \dot{\hat{\Pi}}_k^{(mq)} - L,$$ (59)

where $\pi^\mu(\rho), \Pi^{0(\alpha\beta)}, \Pi_k(\alpha)$ and $\dot{\hat{\Pi}}_k^{(mq)}$ are momenta conjugate to $e_\mu(\rho), \omega_{0(\alpha\beta)}, F^k(\alpha)$ and
$\dot{\hat{\Sigma}}_k^{(mq)}$, respectively. Separating terms with “velocities” in the Lagrangian (58) we write

$$-L = -e_{k(\rho),0} F^k(\rho) + H_c$$

and then singling out terms proportional to $\omega_{0(\alpha\beta)}$ we obtain

$$H_c = -\omega_{0(\alpha\beta)} \chi^{0(\alpha\beta)} + H'_c$$ (60)

where

$$\chi^{0(\alpha\beta)} = \frac{1}{2} F^k(\alpha) e_k^{(\beta)} - \frac{1}{2} F^k(\beta) e_k^{(\alpha)} + e B^{k(\rho)0(\alpha\beta)} e_{\rho(\alpha\beta)}$$ (61)

and

$$H'_c = e_{0(\rho),k} F^k(\rho) - e B^{n(\rho)k(\alpha)m(\beta)} e_{n(\rho,k)} \omega_{m(\alpha\beta)} (F) + e A^{k(\alpha)m(\beta)} \omega_{k(\alpha\gamma)} (F) \omega_m^{(\gamma\beta)} (F)$$

$$- e g_{qp} \hat{\Sigma}_k^{(mq)} \hat{\Sigma}_m^{(kp)} - 2 e e^k(\beta) \left( e_{m(\beta),q} + e_q^{(\beta)} \omega_{m(\gamma\beta)} (F) \right) \hat{\Sigma}_k^{(mq)}.$$ (62)
The non-zero fundamental PBs are
\[
\{ e_\nu(\sigma), \pi^\mu(\rho) \} = \delta^\mu_\nu \delta^\rho_\sigma, \quad \{ \omega_0(\alpha\beta), \Pi^0(\rho\sigma) \} = \tilde{\Delta}^{(\rho\sigma)}_{(\alpha\beta)}, \quad \{ F^k(\alpha), \Pi_{m(\gamma)} \} = \delta^k_m \delta^\alpha_\gamma,
\]
(63)
\[
\left\{ \tilde{\Sigma}^{(mq)}_k, \tilde{\Pi}^{(yz)}_k \right\} = \delta^k_\nu \tilde{\Delta}^{(mq)}_{(yz)} - \frac{1}{D-2} \left( \delta^x_y \tilde{\Delta}^{(mq)}_{(kz)} - \delta^z_x \tilde{\Delta}^{(mq)}_{(ky)} \right)
\]
(64)
where
\[
\tilde{\Delta}^{(\rho\sigma)}_{(\alpha\beta)} \equiv \frac{1}{2} \left( \delta^\rho_\alpha \delta^\sigma_\beta - \delta^\rho_\beta \delta^\sigma_\alpha \right), \quad \tilde{\Delta}^{(mq)}_{(yz)} \equiv \frac{1}{2} \left( \delta^m_y \delta^q_z - \delta^m_z \delta^q_y \right).
\]
(65)

As in any first order formulation, the number of primary constraints is equal to the number of independent variables. One pair of primary constraints

\[
\phi^k(\rho) = \pi^k(\rho) - F^k(\rho) \approx 0, \quad \Pi_{m(\gamma)} \approx 0
\]
(66)
is second class. These are constraints of a special form [28] and one pair of phase-space variables can be eliminated without affecting the PBs of the remaining variables by substitution of the solution into the total Hamiltonian

\[
F^k(\rho) = \pi^k(\rho), \quad \Pi_{m(\gamma)} = 0.
\]
(67)

This is the first stage of Hamiltonian reduction and illustrates the classification (suggested in the previous Section) on primary and second class variables: \( F^k(\rho) \) is a second class variable. After this reduction the total Hamiltonian is

\[
H_T = \pi^0(\rho) \dot{\epsilon}_0(\rho) + \Pi^0(\alpha\beta) \dot{\omega}_0(\alpha\beta) + \tilde{\Sigma}^{(mq)}_k \dot{\Pi}^{(mq)}_k - \omega_0(\alpha\beta) \chi_0(\alpha\beta) (F = \pi) + H_c (F = \pi).
\]
(68)

According to the Dirac procedure, the next step is to consider the time development of the primary constraints (\( \Pi^0(\alpha\beta), \pi^0(\rho), \dot{\Pi}^{(mq)}_k \)). After the first reduction, all primary constraints obviously have zero PBs among themselves (they are momenta of canonical variables), i.e. there are no second class pairs among the primary constraints, and all of them lead to the corresponding secondary constraints, e.g.

\[
\Pi^0(\alpha\beta) = \{ \Pi^0(\alpha\beta), H_T \} = \{ \Pi^0(\alpha\beta), H_c \} = \{ \Pi^0(\alpha\beta), -\omega_0(\alpha\beta) \chi_0(\alpha\beta) \} = \chi^{0(\alpha\beta)}.
\]
(69)
The secondary rotational constraint $\chi^{0(\alpha\beta)}$ has zero PBs with all primary constraints

$$\{\chi^{0(\alpha\beta)}, \Pi^{0(\nu\rho)}\} = \{\chi^{0(\alpha\beta)}, \pi^{0(\sigma)}\} = \{\chi^{0(\alpha\beta)}, \hat{\Pi}^k_{(mq)}\} = 0. \quad (70)$$

The first and last PBs are manifestly zero and the second is just the consequence of the properties of $B^\lambda(\gamma)\mu(\alpha)\nu(\beta)$ and antisymmetry of $C^{\tau(\sigma)}\lambda(\gamma)\mu(\alpha)\nu(\beta)$ (see (A3) of Appendix A)

$$\frac{\delta}{\delta e_{0(\sigma)}} \left( eB^{k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m} \right) = eC^{0(\sigma)k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m} = 0. \quad (71)$$

The PB among two rotational constraints coincides with Lorentz algebra (6); this has been already demonstrated in [1, 2] for all dimensions.

Because $\{\pi^{0(\rho)}, \chi^{0(\alpha\beta)}\} = 0$ and since the PBs among all primary constraint (69) all vanish, the time development of $\pi^{0(\sigma)}$ leads to the secondary translational constraint

$$\dot{\pi}^{0(\sigma)} = \{\pi^{0(\sigma)}, H_T\} = \{\pi^{0(\sigma)}, H_c\} = \{\pi^{0(\sigma)}, H'_c\} = -\frac{\delta H'_c}{\delta e_{0(\sigma)}} = \chi^{0(\sigma)}, \quad (72)$$

which has the explicit form

$$\chi^{0(\sigma)} = \pi^{k(\sigma)} + eC^{0(\sigma)n(\rho)k(\alpha)m(\beta)}e_{n(\rho),k}\omega_{m(\alpha\beta)}(\pi) - eB^{0(\sigma)k(\alpha)m(\beta)}\omega_{k(\alpha\gamma)}(\pi)\omega^{(\gamma)_{\alpha\beta}}(\pi)
+ eC^{0(\sigma)}g_{qq}\hat{\Sigma}^{(mq)}_k\hat{\Sigma}^{(kp)}_m + 2eA^{0(\sigma)k(\beta)}(e_{m(\beta),q} + e^{(\gamma)_{\alpha\beta}}_q\omega_{m(\gamma\beta)}(\pi))\hat{\Sigma}^{(mq)}_k. \quad (73)$$

When performing the variation in (72) we used $\frac{\delta \omega_{k(\gamma\beta)}(\pi)}{\delta e_{0(\sigma)}} = 0$ (this is easy to show using $\omega_{k(\gamma\beta)}(\pi)$ from (51) and the fact that $\frac{\delta}{\delta e_{0(\sigma)}} \left( \frac{1}{e^{0(00)}} \right) = 0$ and $\frac{\delta e_{k(\omega)}}{\delta e_{0(\sigma)}} = 0$). In all terms in $H'_c$, only $ABC$ densities are affected and their variations are simple (see Appendix A).

Contracting (73) with $e_{0(\sigma)}$ and using the $ABC$ properties (expand $A$ and $B$ in $\sigma$ and contract with $e_{0(\sigma)}$ (see (A12) of Appendix A)), we can express $H'_c$ as

$$H'_c = -e_{0(\sigma)}\chi^{0(\sigma)} + \left(e_{0(\rho)}\pi^{k(\rho)}\right)_k. \quad (74)$$

Based on the properties of the $ABC$ functions we immediately obtain that the PB of $\chi^{0(\sigma)}$ with the primary translational constraint, $\pi^{0(\mu)}$, is zero (i.e. after second variation with respect to $e_{0(\mu)}$ we will have $A, B, C$ and $D$ with two equal indices (00) which are zero because of antisymmetry of these density functions):
\( \{ \chi^0(\sigma), \pi^0(\mu) \} = 0, \) \hspace{1cm} (75)

and also

\( \{ \chi^0(\sigma), \Pi^0(\alpha\beta) \} = 0. \) \hspace{1cm} (76)

The PB of \( \chi^0(\sigma) \) with the primary constraint \( \hat{\Pi}^k_{(mq)} \) is not zero, but the time development of \( \hat{\Pi}^k_{(mq)} \) leads to the secondary constraint

\[ \hat{\Pi}^k_{(mq),0} = \left\{ \hat{\Pi}^k_{(mq)}, H'_c \right\} = -\frac{\delta H'_c}{\delta \hat{\Sigma}^k_{(mq)}(g)} = \hat{\chi}^k_{(mq)} \] \hspace{1cm} (77)

where

\[ \hat{\chi}^k_{(mq)} = e^{m(\beta)}g_{pq} - e^{n(\gamma)}g_{pm} + e^{D^k_{(mq)}} \] \hspace{1cm} (78)

with \( \hat{D}^k_{(mq)} \) being the manifestly antisymmetric and traceless combination (see Appendix B)

\[ \hat{D}^k_{(mq)} = \hat{D}^m_{mq} - \hat{D}^q_{qm} - \frac{1}{D-2} \left[ \delta^k_{m} \left( \hat{D}^n_{mq} - \hat{D}^n_{qm} \right) - \delta^k_{q} \left( \hat{D}^n_{nm} - \hat{D}^n_{mn} \right) \right], \] \hspace{1cm} (79)

built from the coefficient which appears in front of terms in (62) that are linear in \( \hat{\Sigma}^k_{(mq)} \)

\[ \hat{D}^k_{mq} = e^{k(\beta)}e^{m(\beta)},q + e^{k(\beta)}e^{(\gamma)}\omega_{m(\gamma\beta)}(\pi). \] \hspace{1cm} (80)

Note that \( \hat{D}^k_{mq} \) by itself is not antisymmetric or traceless; and (78)-(79) are the result of performing variation using the fundamental PB of (64).

The pair of constraints \( (\hat{\Pi}^k_{(mq)}, \hat{\chi}^x_{(yz)}) \) is second class because

\[ \left\{ \hat{\Pi}^k_{(mq)}, \hat{\chi}^x_{(yz)} \right\} = \hat{N}^{kx}_{(mq)(yz)}, \] \hspace{1cm} (81)

where

\[ \hat{N}^{kx}_{(mq)(yz)} = -\frac{e}{2} \left[ \delta^k_{y} \left( g_{zm}\delta^x_{q} - g_{zq}\delta^x_{m} \right) - \delta^k_{z} \left( g_{ym}\delta^x_{q} - g_{yq}\delta^x_{m} \right) \right] - \frac{1}{D-2} \left[ \delta^x_{y} \left( g_{zm}\delta^k_{q} - g_{zq}\delta^k_{m} \right) - \delta^x_{z} \left( g_{ym}\delta^k_{q} - g_{yq}\delta^k_{m} \right) \right], \] \hspace{1cm} (82)
which is manifestly antisymmetric in \((mq)\) and \((yz)\), traceless in \(km\) and \(kq\), in \(xy\) and \(xz\), non-zero and not proportional to constraints. (Here \(g_{zm}\) again denotes \(e_{z(\alpha)}e_{m(\beta)}\).) The most important property of \(\hat{N}^{kx}_{(mq)(yz)}\) is its invertability with the explicit form of inverse is given below. The pair of variables \((\hat{\Pi}^k_{(mq)}, \hat{\Sigma}^z_{(mq)})\) can be eliminated by substitution into the total Hamiltonian

\[
\hat{\Pi}^k_{(mq)} = 0, \quad \hat{\Sigma}^z_{(mk)} = \frac{1}{2} \left( \gamma^{kn} \hat{D}^m_{(nz)} - \gamma^{mn} \hat{D}^k_{(nz)} - \gamma^{ky} \gamma^{mw} g_{zz} \hat{D}^x_{(wy)} \right), \quad (83)
\]

where \(\hat{\Sigma}^z_{(mk)}\) is the solution of the constraint \((78)\) \(\hat{\chi}^k_{(mq)} = 0\) (see Appendix B). This again illustrates our classification: \(\hat{\Sigma}^z_{(mq)}\) is a second class variable as is \(F^k(\rho)\). In \((83)\) we use a short-hand notation which was originally introduced by Dirac \([34]\) for the Hamiltonian formulation of the Einstein-Hilbert action: \(\chi^{kn} \equiv g^{kn} - \frac{\hat{e}^{ak}e^{mn}}{g_{mn}}\) where \(g^{\mu\nu} = e^{\mu(\alpha)}e^{\nu(\beta)}\).

The elimination of the phase-space pair \((F^k(\rho), \Pi_k(\rho))\) by solving the corresponding second class constraint was simple as they are of a special form and it is known that in such a case the Dirac brackets (DBs) of the remaining fields coincide with their original PBs \([28]\). The pair of second class constraints \((\hat{\Pi}^k_{(mq)}, \hat{\chi}^x_{(yz)})\) is more complicated and the effect of their elimination on the PBs among the remaining canonical variables has to be checked. In \([2]\) the elimination of \(\omega_{k(\alpha\beta)}\) was performed using a different and complicated approach because of the mixture of different components \((\omega_{k(pq)}\) and \(\omega_{k(p\theta)}\)) in the equations, and it was even not clear how DBs can be calculated. After introducing Darboux coordinates and decoupling the two fields, \(F^k(\rho)\) and \(\hat{\Sigma}^z_{(mq)}\), this is possible. Let us investigate the effect of their elimination on the DBs of the remaining fields.

The Dirac bracket is defined for any pair of functions of canonical variables as \([4]\)

\[
\{\Phi, \Psi\}_{DB} = \{\Phi, \Psi\}_{PB} - \left( \left\{ \Phi, \hat{\Pi}^k_{(mq)} \right\}_{PB} \left\{ \Phi, \hat{\chi}^x_{(yz)} \right\}_{PB} \right) M^{-1} \left( \left\{ \hat{\Pi}^a_{(bc)}, \Psi \right\}_{PB} \right), \quad (84)
\]

(Here we used our set of second class constraints.) \(M^{-1}\) is the inverse of the matrix \(M\) built from the PBs of the second class constraints

\[
M = \begin{bmatrix}
\hat{\Pi}^k_{(mq)}, \hat{\Pi}^a_{(bc)} \\
\hat{\chi}^x_{(yz)}, \hat{\Pi}^a_{(bc)} \\
\hat{\chi}^x_{(yz)}, \hat{\pi}^z_{(bc)} \\
\hat{\pi}^z_{(bc)}, \hat{\pi}^x_{(bc)} \\
\hat{\pi}^z_{(bc)}, \hat{\pi}^x_{(bc)} \\
\hat{\pi}^z_{(bc)}, \hat{\pi}^x_{(bc)} \\
\end{bmatrix} = \begin{bmatrix}
0 & \hat{N}^{ka}_{(mq)(bc)} \\
-\hat{N}^{ak}_{(bc)(mq)} & \hat{\chi}^x_{(mq)(bc)} \\
\end{bmatrix}, \quad (85)
\]

Note that \(\hat{N}^{ka}_{(mq)(bc)} = \hat{N}^{ak}_{(bc)(mq)}\) (see \((82)\)).
We want to investigate the effect of eliminating the second class constraints on the properties of the remaining canonical variables and on the PBs among the functions constructed from them. \( \hat{N}^{ka}_{(mq)(bc)} \) is given by (82), and the explicit form of \( \hat{X}^{ka}_{(mq)(bc)} \) can be calculated, but the inverse of \( M \) can be defined for any \( \hat{X} \). The explicit form of \( \hat{X} \) is needed only for calculation of DB among two translational constraints, but we do not discuss this in this article. The inverse of such a matrix can be immediately found if the inverse of the off-diagonal blocks is known

\[
\hat{N}^{ka}_{(mq)(bc)} \left( \hat{N}^{-1} \right)_{ax}^{(bc)(yz)} = \hat{j}^{k(yz)}_{x(mq)}
\]

(86)

where \( \hat{j}^{k(yz)}_{x(mq)} \) is the fundamental PB for antisymmetric and traceless canonical pair defined in (64). The inverse \( \left( \hat{N}^{-1} \right)_{ax}^{(bc)(yz)} \) is

\[
\left( \hat{N}^{-1} \right)_{ax}^{(bc)(yz)} = \frac{1}{4e} \left[ g_{ax} \left( \gamma^{byz} \gamma^{cz} - \gamma^{bzz} \gamma^{cy} \right) + \delta_{x}^{b} \left( \gamma^{cy \delta_{a}^{z}} - \gamma^{cya} \delta_{a}^{z} \right) - 2D \left[ \delta_{a}^{b} \left( \gamma^{cy} \delta_{x}^{z} - \gamma^{by} \delta_{x}^{z} \right) - 2 \right] \left[ \delta_{a}^{b} \left( \gamma^{cy} \delta_{x}^{z} - \gamma^{by} \delta_{x}^{z} \right) - 2 \right] \right].
\]

(87)

\( \left( \hat{N}^{-1} \right)_{ax}^{(bc)(yz)} \) is also manifestly antisymmetric in \( (bc) \) and \( (yz) \), as well it is traceless in \( ba \) and \( ca \), in \( yx \) and \( xz \). The following properties are useful

\[
\hat{I} \cdot \hat{I} = \hat{I}, \quad \hat{I} \cdot \hat{N} = \hat{N}, \quad \hat{I} \cdot \hat{N}^{-1} = \hat{N}^{-1}.
\]

(88)

Equation (86) and properties (88) allows us to find \( M^{-1} \)

\[
M^{-1} = \left[ \begin{array}{cc} \hat{N}^{-1} \hat{X} \hat{N}^{-1} & -\hat{N}^{-1} \\ -\hat{N}^{-1} & 0 \end{array} \right], \quad M \cdot M^{-1} = \left[ \begin{array}{cc} \hat{I} & 0 \\ 0 & \hat{I} \end{array} \right].
\]

(89)

The canonical variables which remain after elimination of the pair \( (\hat{\Pi}^{z}_{(m)}), (\hat{\Sigma}^{(mk)}_{z}) \) have fundamental PBs that are not affected, as can be easily checked. Substitution of the solution of the secondary constraints (83) into (68) (Hamiltonian reduction) leads to

\[
H_{T} = \pi^{0(\rho)} e_{0(\rho)} + \Pi^{0(\alpha \beta)} \omega_{0(\alpha \beta)} - \omega_{0(\alpha \beta)} \chi^{0(\alpha \beta)} - e_{0(\sigma)} \chi^{0(\sigma)} \left( \hat{\Sigma}^{(mk)}_{z} \right) \text{ from Eq. (83)}.
\]

(90)
After substitution of $\hat{\Sigma}_2^{(mk)}$ this leads to the same result as obtained in [2]. Darboux coordinates significantly simplify the calculations.

The main goal of this paper is construction of Darboux coordinates for the EC action in an independent of a particular dimension form. The direct approach used in [2] made further calculations almost unmanageable, and the simplification due to Darboux coordinates that shortens the calculations of [90], gives us a hope of completing the Dirac procedure. We wish to demonstrate its closure, the absence of tertiary constraints, and restore gauge invariance. This result will be reported elsewhere. Here we just demonstrate that with Darboux coordinates these calculations seems to become manageable; and as an example, we consider the PB between rotational and translational constraints. We argued in [2] that the known invariance of the EC action under Lorentz rotation, Dirac’s conjecture [4] and the Castellani algorithm [13] lead to the necessity of having the PB among rotational and translational constraints being exactly the same in all dimensions and given by the corresponding part of the Poincaré algebra [9].

Let us demonstrate that indeed in all dimensions ($D > 2$) the PB among translational and rotational constraints is the same and corresponds to the Poincaré algebra [9]. This part of algebra, among the secondary constraints \( \{\chi_0^0(\sigma), \chi_0^0(\mu \nu)\} \), as well as \( \{\chi^0(\sigma), \chi^0(\rho)\} \), can be calculated using DBs, i.e. avoiding substitution of $\hat{\Sigma}$ into the translational constraint before calculating the PB (which is the longest part of such calculations) and performing this substitution only after

\[
\{\chi_0^0(\sigma), \chi_0^0(\mu \nu)\}_{DB} =
\]

\[
\{\chi^0(\sigma), \chi^0(\mu \nu)\} - \left( \begin{array}{c} \chi^0(\sigma), \tilde{\Pi}^{k}_{(mq)} \\ \chi^0(\sigma), \tilde{\chi}^{k}_{(mq)} \end{array} \right) M^{-1} \left( \begin{array}{c} \tilde{\Pi}^{a}_{(bc)}, \chi_0^0(\mu \nu) \\ \tilde{\chi}^{a}_{(bc)}, \chi_0^0(\mu \nu) \end{array} \right),
\]

(91)

where $\tilde{\Sigma}$ and $\tilde{\Pi}$ are the fundamental variables and the only non-zero PBs are given in (63-64). (Note that only the first step of reduction is performed in (67).) After calculating (91) the solution of $\hat{\Sigma}$ is substituted that gives us the final answer for $\{\chi^0(\sigma), \chi^0(\mu \nu)\}$. The advantage of this calculation is possibility to demonstrate (and also single out) contributions in the first term of (91) that gives the corresponding part of the Poincaré algebra almost manifestly

28
and in compact form. First of all, we outline the idea of such calculations. It is not difficult to demonstrate that the calculation of the first PB in (91) gives

\[ \{ \chi^{0(\sigma)}, \chi^{0(\mu\nu)} \} = \frac{1}{2} \tilde{\eta}^{\sigma\mu} \chi^{0(\nu)} (\pi, e, \Sigma) - \frac{1}{2} \tilde{\eta}^{\sigma\nu} \chi^{0(\mu)} (\pi, e, \Sigma) + R^{\sigma(\mu\nu)} (\Sigma, \pi, e). \] (92)

So, substitution of solution of \( \tilde{\Sigma} \) will not affect the first two terms in (92). What is left is to demonstrate that the remainder \( R^{\sigma(\mu\nu)} \), along with the second contribution in (91), after substitution of \( \tilde{\Sigma} \) gives zero which is a long but straightforward calculation. It is easier to prove that \( R^{\sigma(\mu\nu)} \) is zero if we consider separately terms of different nature, for example, all terms which are linear in momenta should cancel independently of the rest of contributions. This allows us to break these long and cumbersome calculations into smaller and independent pieces.

Let us outline the proof of (92). We start calculations by separating the translational constraint (73) into contributions of different order in \( \tilde{\Sigma} \)

\[ \chi^{0(\sigma)} = \chi^{0(\sigma)} (\tilde{\Sigma}^2) + \chi^{0(\sigma)} (\tilde{\Sigma}^1) + \chi^{0(\sigma)} (\tilde{\Sigma}^0). \] (93)

For the contribution quadratic \( \tilde{\Sigma} \) we almost immediately obtain the exact expression (there are no contributions into the remainder in this order, \( R^{\sigma(\mu\nu)} (\tilde{\Sigma}^2) = 0) \)

\[ \{ \chi^{0(\sigma)} (\tilde{\Sigma}^2), \chi^{0(\mu\nu)} \} = \frac{1}{2} \tilde{\eta}^{\sigma\mu} \chi^{0(\nu)} (\tilde{\Sigma}^2) - \frac{1}{2} \tilde{\eta}^{\sigma\nu} \chi^{0(\mu)} (\tilde{\Sigma}^2). \] (94)

Next contribution, linear in \( \tilde{\Sigma} \), is

\[ \{ \chi^{0(\sigma)} (\tilde{\Sigma}), \chi^{0(\mu\nu)} \} = \]

\[ \left( e_{m(\rho),q} + e_{q}^{(\gamma)} \omega_{m(\gamma\rho)} (\pi) \right) \tilde{\Sigma}^{(mq)} \left\{ 2\epsilon A^{0(\sigma)k(\rho)} \frac{1}{2} \pi^{n(\mu)} e^{(\nu)} - \frac{1}{2} \pi^{n(\nu)} e^{(\mu)} \right\} \] (95)

\[ + 2\epsilon A^{0(\sigma)k(\rho)} \tilde{\Sigma}^{(mq)} \left\{ e_{m(\rho),q} + e_{q}^{(\gamma)} \omega_{m(\gamma\rho)} (\pi) \right\} \left( \frac{1}{2} \pi^{n(\mu)} e^{(\nu)} - \frac{1}{2} \pi^{n(\nu)} e^{(\mu)} + eB^{n(\rho)m(\mu)0(\nu)} e^{(\rho, m)} \right) \].

Considering the PB in the first term of (95) and using (see (A2) of Appendix A) we obtain

\[ \left\{ 2\epsilon A^{0(\sigma)k(\rho)} \frac{1}{2} \pi^{n(\mu)} e^{(\nu)} - \frac{1}{2} \pi^{n(\nu)} e^{(\mu)} \right\} = eB^{n(\mu)0(\sigma)k(\rho)} e^{(\nu)} - (\mu \leftrightarrow \nu). \] (96)
Expanding $B$ in $n$ (see (A6) of Appendix A) and contracting it with $e_\alpha^{(\nu)}$ we have

$$
e B^{\mu(\sigma)k(\rho)} e_\alpha^{(\nu)} = e \left( \tilde{\eta}^{\mu\nu} - e^{0(\mu)} e_0^{(\nu)} \right) A^{0(\sigma)k(\rho)}$$

$$+ e \left( \tilde{\eta}^{\sigma\nu} - e^{0(\sigma)} e_0^{(\nu)} \right) A^{0(\rho)k(\mu)} + e \left( \tilde{\eta}^{\rho\nu} - e^{0(\rho)} e_0^{(\nu)} \right) A^{0(\mu)k(\sigma)}. \tag{97}$$

Three terms in (97) proportional to $e_0^{(\nu)}$ give us

$$-e e_0^{(\nu)} \left( e^{0(\mu)} A^{0(\sigma)k(\rho)} + e^{0(\sigma)} A^{0(\rho)k(\mu)} + e^{0(\rho)} A^{0(\mu)k(\sigma)} \right).$$

The expression in brackets exactly coincides with expansion of $B^{0(\mu)0(\sigma)k(\rho)}$ (see (A6) of Appendix A) which automatically equals zero because of antisymmetry of $B$ in external indices (for details see Appendix A). After antisymmetrization of the remaining terms of (96) we have

$$\left\{ 2e A^{0(\sigma)k(\rho)}, \frac{1}{2} \pi^{n(\mu)} e_\alpha^{(\nu)} - \frac{1}{2} \pi^{n(\nu)} e_\alpha^{(\mu)} \right\}$$

$$= -e \tilde{\eta}^{\mu\sigma} A^{0(\rho)k(\mu)} - e \tilde{\eta}^{\mu\rho} A^{0(\mu)k(\sigma)} + e \tilde{\eta}^{\mu\sigma} A^{0(\rho)k(\sigma)} + e \tilde{\eta}^{\mu\rho} A^{0(\nu)k(\sigma)}.$$

The first and third terms contracted with the expression in front of the PB in (95) gives exactly two rotational constraints; and the rest of terms, along with the second term in (95), contribute to the remainder. Finally,

$$\left\{ \chi^{0(\sigma)} (\hat{\Sigma}^1), \chi^{0(\mu\nu)} \right\} = \frac{1}{2} \tilde{\eta}^{\sigma\mu} \chi^{0(\nu)} (\hat{\Sigma}^1) - \frac{1}{2} \tilde{\eta}^{\sigma\nu} \chi^{0(\mu)} (\hat{\Sigma}^1) + R^{\sigma(\mu\nu)} (\hat{\Sigma}^1). \tag{98}$$

Similarly one can demonstrate (using ABC properties) that in the last order (zero order in $\hat{\Sigma}$) the same result as (98) follows (with the additional contributions into the remainder $R^{\sigma(\mu\nu)} (\hat{\Sigma}^0)$) leading to (92). This part of calculation is simple and completes the proof of (92). To complete calculations, the remainder has to be considered together with the second term of (91), and the solution of $\hat{\Sigma}$ has to be substituted at the end of calculations.

From the field content of $\chi^{0(\mu\nu)}$ (independence from $\hat{\Pi}_{(bc)}^a$) it follows that

$$\left\{ \hat{\Pi}_{(bc)}^a, \chi^{0(\mu\nu)} \right\} = 0. \tag{99}$$

Using this PB and the explicit form of $M^{-1}$ we obtain

$$\left( \left\{ \chi^{0(\sigma)}, \hat{\Pi}^k_{(mq)} \right\}, \left\{ \chi^{0(\sigma)}, \hat{\chi}^k_{(mq)} \right\} \right) M^{-1} \left( \left\{ \hat{\Pi}_{(bc)}^a, \chi^{0(\mu\nu)} \right\}, \left\{ \hat{\chi}_{(bc)}^a, \chi^{0(\mu\nu)} \right\} \right) =$$
\begin{equation}
\left\{ \chi^{0}(\sigma), \hat{\Pi}^{k}_{(mq)} \right\} \left(-\hat{N}^{-1}\right)^{(mq)(bc)}_{\ k\ a} \left\{ \check{\chi}^{a}_{(bc)}, \chi^{0(\mu\nu)} \right\}.
\end{equation}

When calculating (99), it is better to extract terms proportional to \(\check{\chi}^{a}_{(bc)}\), which after substitution of \(\hat{\Sigma}\) vanish and we are left with simple expressions

\begin{equation}
\left\{ \chi^{0}(\sigma), \hat{\Pi}^{k}_{(mq)} \right\} = -2\epsilon\epsilon^{n(\sigma)} e^{0(\beta)} \left( e_{p(\beta),\ d} + e_{d}^{(\gamma)} \omega_{p(\gamma\beta)} (\pi) \right) \left\{ \hat{\Sigma}_{n}^{(pd)}, \hat{\Pi}^{k}_{(mq)} \right\}
\end{equation}

and

\begin{equation}
\left\{ \check{\chi}^{a}_{(bc)}, \chi^{0(\mu\nu)} \right\} = -\epsilon \left\{ \check{D}^{a}_{(bc)}, \chi^{0(\mu\nu)} \right\}.
\end{equation}

Using (100), (101) together with (91), (88), antisymmetry and tracelessness of (87) and (79), we obtain

\begin{equation}
\left\{ \chi^{0(\sigma)}, \check{\chi}^{0(\mu\nu)} \right\} = \frac{1}{2} \tilde{\eta}^{\sigma\mu} \chi^{0(\nu)} - \frac{1}{2} \tilde{\eta}^{\sigma\nu} \chi^{0(\mu)} + R^{\sigma(\mu\nu)}
\end{equation}

\begin{equation}
+ 4\epsilon\epsilon^{n(\sigma)} e^{0(\beta)} \left( e_{p(\beta),\ b} + e_{b}^{(\gamma)} \omega_{p(\gamma\beta)} (\pi) \right) \left(\hat{N}^{-1}\right)^{(pd)(bc)}_{\ na} e \left\{ e_{a(\alpha)} e_{b(\alpha),\ c} + e_{a(\alpha)} e_{c}^{(\rho)} \omega_{b(\rho\alpha)} (\pi), \chi^{0(\mu\nu)} \right\}
\end{equation}

The most laborious part of calculation is a demonstration that the remainder together with the last line of (102) equals zero. We perform these calculations by separating terms of different order in \(\pi^{n(\mu)}\) that makes the analysis more manageable.

Using Darboux coordinates allows us to prove that the PB among rotational and translational constraints also supports the Poincaré algebra in all dimensions, \(D > 2\). Knowledge of this PB along with (6) is sufficient to restore rotational invariance in the Hamiltonian formulation of Einstein-Cartan action by using the Castellani procedure. This result, as well as calculation of the PB between two translational constraints and restoration of translational invariance, will be reported elsewhere.

V. DISCUSSION

Based on the results of the direct application of the Dirac procedure to the first order Einstein-Cartan action \cite{2}, we have constructed uniform Darboux coordinates valid in all
dimensions for which the first order formulation exists; i.e. when it is equivalent to the second order EC action \((D > 2)\). In particular these uniform Darboux coordinates guarantee equivalence and allow one to check the \(D = 3\) limit \([1]\) at all stages of calculations in dimensions \(D > 3\). Considerable simplification occurs when we use Darboux coordinates and it is explicitly demonstrated by obtaining the Hamiltonian formulation in a few lines compared with the direct and cumbersome calculations \([2]\) considered previously. However, we have to emphasize that the preliminary Hamiltonian analysis is indispensable for the construction of Darboux coordinates, which preserve equivalence with the original action. An arbitrary change of variables at the Lagrangian level for singular Lagrangians is an ambiguous operation because it might correspond to a non-canonical change of variables at the Hamiltonian level. For singular Lagrangians the invertability of transformations (redefinition of fields) from one set of variables to another is not a sufficient condition to preserve equivalence \([29]\); and one particular example is considered in the end of Section III (see \((53)\)). These “Darboux coordinates”, \((53)\), despite separating variables in the same way, do not preserve the \(D = 3\) limit, lead to results which are different from those found by the direct analysis and destroy equivalence. Our Darboux transformations, \((43)\), do not suffer such an ambiguity because they are based on the preliminary Hamiltonian analysis. In other words, our transformation separates variables in the same way that the Hamiltonian reduction does. This makes this transformation unique and preserves equivalence with the original action as well as the equivalence of results for the Lagrangian and Hamiltonian formulations.

To answer the question about possible modifications of the Poincaré algebra of PBs among the secondary constraints of the EC Hamiltonian in dimensions \(D > 3\), we need to complete the calculations of PBs. In particular, if \(\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\} = \frac{1}{2} \eta^{\sigma\mu} \chi^{0(\nu)} - \frac{1}{2} \eta^{\sigma\nu} \chi^{0(\mu)}\) (and there is a strong indication that this is the case) and \(\{\chi_{0(\alpha)}, \chi^{0(\beta)}\} = 0\), then the N-bein gravity is the Poincaré gauge theory in all dimensions and the \(D = 3\) case is not special at all. Note (we discussed this in \([2]\)) that in higher dimensions, the constraints are much more complicated and having the same algebra does not mean that the gauge transformations must be exactly the same as for \(D = 3\). If \(\{\chi_{0(\alpha)}, \chi^{0(\beta)}\} \neq 0\), but proportional to secondary first class constraints, we still have closure of the Dirac procedure, all constraints are first class, the gauge generators can be found and the gauge transformations can be restored. In this case, N-bein gravity for \(D > 3\) is the gauge theory, but with the modified Poincaré
algebra. For example, if (the most general case)
\[
\{ \chi^0(\alpha), \chi^0(\beta) \} = \tilde{F}^{(\alpha\beta)}_{(\mu\nu)} \chi^0(\mu\nu) + \tilde{M}^{(\alpha\beta)}_{\nu} \chi^0(\nu)
\]
with structure functions $\tilde{F}^{(\alpha\beta)}_{(\mu\nu)}$ and $\tilde{M}^{(\alpha\beta)}_{\nu}$, which are $\tilde{F}^{(\alpha\beta)}_{(\mu\nu)} = 0$ and $\tilde{M}^{(\alpha\beta)}_{\nu} = 0$ when $D = 3$, then one can say that in all dimensions the EC theory is a gauge theory with a generalized Poincaré algebra among secondary first class constraints that degenerates into the true Poincaré algebra when $D = 3$. A similar result as (103) has been known for a long time and was presented in [10, 11, 17]; but it was not obtained using the Hamiltonian procedure and it was written for generators, not for the PBs of constraints. The complete Hamiltonian analysis will show whether the algebra among constraints is Poincaré or modified Poincaré.

The exact form of algebra of secondary constraints is important; but we already have enough evidence to make a conclusion about the gauge invariance of the EC action. It is certain that gauge invariance is the translation and Lorentz rotation in the internal space and that diffeomorphism (either spatial or full) is not a gauge invariance of N-bein gravity generated by first class constraints. This conclusion is based on the following arguments.

The parameters characterizing the gauge transformations are defined by the tensorial nature of primary first class constraints. Both of them, $\Pi^{0(\alpha\beta)}$ and $\pi^{0(\sigma)}$, have internal indices, so do the corresponding gauge parameters $r^{(\alpha\beta)}$ and $t^{(\sigma)}$ (see Section 5 of [9] for more details). This gauge symmetry corresponds to rotation and translation in the tangent space. The gauge parameter of diffeomorphism, $\xi^\mu$, has an external index, which can be accommodated only if the corresponding primary first class constraint has also an external free index. This does not happen in the case of N-bein gravity if the Dirac procedure is performed correctly and a non-canonical change of variable is not made (see [9]). Formulations that claim to have the “spatial diffeomorphism constraint” or any further consideration based on such a constraint for tetrad gravity is the product of non-canonical change of variables, which has the same origin as in metric gravity.\(^5\) Gauge invariance is a unique characteristic of a

\(^5\) The loss of full diffeomorphism invariance due to a non-canonical change of variables in metric gravity was discussed in [15, 16, 35]. A lapse with canonicity leads to big (or rather a devastating) shift from covariant General Relativity (Einstein-Hilbert and Einstein-Cartan actions) to some non-covariant models like “geometrodynamics” for metric gravity and inspired by it non-covariant models for tetrads (see Section V of [9] for discussion on this topic). This, of course, propagates into further analysis (i.e. quantization) of these models.
singular system and follows from its unique constraint structure, i.e. it is derivable from
the first class constraints in accordance with Dirac’s conjecture [4] and using the Castellani
algorithm [13] (the only true algorithm to restore gauge invariance [9]). Gauge invariance
is unique, but it does not presume the absence of additional symmetries in the action. In
[12] such symmetries, which are not derivable from constraints, are called “trivial” but
this name seems to us to be a little bit confusing as e.g. the non-gauge symmetry of the
EC action, diffeomorphism, can hardly be called “trivial”. It is more preferable, without
introducing any new terminology, to just have “symmetries of the action” (there could be
many) and “a gauge symmetry” (a unique one) that follows from the Hamiltonian analysis
or from basic differential identities at the Lagrangian level [3].

The Hamiltonian analysis allows us to single out “what is a gauge symmetry and what
is not” [14]. In [3] we showed, using differential identities, that the translation in the
internal space is an invariance of the EC action. This fact has long been known; such
transformations were written in [11, 17], and are exactly the same as we obtained. This
makes the common statement that “translation is not invariance” absolutely groundless and
somewhat mysterious. In [3] we also argue that two invariances, translation in the internal
space and diffeomorphism, cannot coexist (simultaneously present) as gauge invariances in
Hamiltonian formulations as the number of first class constraints needed to generate both of
them would lead to a negative number of degrees of freedom (for relation between the number
of constraints and degrees of freedom see [19]). The only possible way to reconcile these two
symmetries, as we stated in [3], is that there exists a canonical transformation that converts
our constraints (90) into a different set of constraints which support diffeomorphism. In this
article we argue that this is impossible and such a canonical transformation does not exist.
Let us look at this from the Hamiltonian and Lagrangian points of view.

From the Hamiltonian point of view, the known canonical transformations for the first
and second order Einstein-Hilbert actions [3, 35, 36] always preserve the form-invariance
of the constraint algebra that would be destroyed by any transformation that changes the
tensorial character of the primary constraints needed to have diffeomorphism as a gauge
invariance of the EC action (as the gauge parameter of diffeomorphism $\xi^\mu$ is a true vector;

---

6 In Section 3.1.5. of [12] “trivial gauge transformations” are defined and all transformations are classified
by using the Hamiltonian method and “the transformations are of no physical significance because in the
Hamiltonian formalism they are not generated by a constraint”.

34
for more details see \[9\]). In addition, in formulations which are related by a canonical transformation, constraints are different, but the gauge transformations are the same and for new and old fields can be obtained one from another without any need for a field dependent redefinition of gauge parameters (such a field dependent redefinition is an indication of having a non-canonical change of variables). The gauge parameters which are responsible for diffeomorphism, $\xi^\mu$, and translation, $t_\sigma$, cannot be related without involving fields as they have a different tensorial dimension. Therefore, the field dependent redefinition of parameters is needed. Consequently, there is no canonical transformation between such formulations which give diffeomorphism and translation as gauge symmetries. And so, the Hamiltonian formulation with the constraints that would produce diffeomorphism invariance is not equivalent to the Hamiltonian formulation with translation in the tangent space as a gauge symmetry.

From the Lagrangian point of view, using the 16 components of the tetrads (in the $D = 4$ case) we can restore the 10 components of the metric tensor but “not vice versa” \[37\]. Tetrads are “world” vectors and are invariant under diffeomorphism, as any vector or tensor does in a generally covariant theory. From the diffeomorphism invariance of tetrads we can derive invariance under diffeomorphism for any combination of tetrads, in particular, for $e_\mu^{(\alpha)}e_\nu^{(\alpha)} = g_{\mu\nu}$ we obtain

$$
\delta_{\text{diff}}\left(e_\mu^{(\alpha)}e_\nu^{(\alpha)}\right) = e_\mu^{(\alpha)}\left(-e_\rho^{(\alpha)}\xi_\rho^\mu - e_\nu^{(\alpha)}\xi_\rho^\rho\right) + \left(-e_\rho^{(\alpha)}\xi_\rho^\nu - e_\mu^{(\alpha)}\xi_\rho^\rho\right)e_\nu^{(\alpha)}
$$

$$
= -\xi_{\mu,\nu} + g_\rho\mu\nu\xi^\rho - \xi_{\nu,\mu} + g_\rho\nu\mu\xi^\rho - g_\mu\nu\rho\xi^\rho = \delta_{\text{diff}} g_{\mu\nu}. \tag{104}
$$

Note that in this case there is no need for a field dependent redefinition of gauge parameters.

We can also perform the inverse operation: from the diffeomorphism of the metric tensor (which is the gauge invariance of Einstein-Hilbert (EH) action \[9, 15, 16, 35\]) we can derive the diffeomorphism of tetrads (more details of this derivation and discussion about gauge symmetries of the metric tensor and tetrads are given in \[1\]). But we cannot obtain Lorentz or translational invariances in the tangent space of the EC action from the diffeomorphism of the metric tensor. The reason for this is simply that the EC and EH actions are not equivalent and neither are the corresponding Hamiltonians: they have a different number of phase-space variables, different constraints, PB algebras, tensorial dimension of primary
constraints and different gauge invariances.

In the Lagrangian formalism, if we perform a change of variables that keep the equivalence of two formulations, build differential identities and restore invariance of each formulation, then the invariance of one formulation must be derivable from the invariance of another, using the same original redefinition of fields and without redefinition of parameters. If a field dependent redefinition of parameters is needed, then the change of variables that was performed is not canonical at the Hamiltonian level. Hence, such a change of variables destroy equivalence also at the Lagrangian level. As an illustration of this, we again compare metric General Relativity (GR) and ADM gravity. For metric GR the Hamiltonian \cite{15, 16} and Lagrangian \cite{38} methods give the same gauge transformation, diffeomorphism. For the ADM gravity the Hamiltonian and Lagrangian methods also produce the same invariance, which is different from diffeomorphism (compare \cite{39} and \cite{40}). This is consistent with the fact that the Hamiltonian and Lagrangian approaches give equivalent descriptions of the same system. However, it is clear from \cite{39} and \cite{40}, that the gauge transformations of ADM gravity do not coincide with diffeomorphism. It is not a surprise as ADM gravity is not equivalent to GR (see \cite{9, 16}). Only after a field dependent redefinition of parameters is performed, is it possible to find “equivalence between diffeomorphism and gauge transformations” of ADM gravity \cite{39}. The same redefinition is also needed at the Lagrangian level \cite{40} which, according to the authors, demonstrates “the equivalence between the gauge and diff parameters by devising of the one to one mapping”. The same is true for the EC gravity: the field dependent redefinition of parameters is needed to relate translation and diffeomorphism, so only non-canonical transformations can relate two such Hamiltonian formulations.

An additional argument to support our point of view is related to differential identities from which the invariances of a Lagrangian can be found. As we showed in \cite{3}, considering the EC Lagrangian as an example, all differential identities can be constructed from a few basic differential identities. One out of many identities leading to invariances of the EC action a gauge identity can be singled out using the following arguments. Differential identities leading to gauge invariances for known theories are always the simplest: they are built starting by contracting derivatives ($\partial_\mu$) with the Euler derivatives ($E$). For example: for Maxwell theory it is $\partial_\mu E^\mu$, for Yang-Mills - $\partial_\mu E^{\mu(a)}$, for the second order metric GR-$\partial_\mu E^{\mu\nu}$. Variation of the action with respect to the fundamental (basic) fields of a theory
defines a tensorial character of a differential identity. For the first order EC action the Euler derivatives are $E^\mu(\alpha) = \frac{\delta L_{EC}}{\delta \dot{e}^\mu(\alpha)}$ and $E^\mu(\alpha\beta) = \frac{\delta L_{EC}}{\delta \omega^\mu(\alpha\beta)}$ [3]. Thus, the basic (the most fundamental) differential identities can be constructed starting from $\partial_\mu E^\mu(\alpha)$ and $\partial_\mu E^\mu(\alpha\beta)$ that lead to the following identities $I^{(\alpha)} = \partial_\mu E^\mu(\alpha) + \ldots$ and $I^{(\alpha\beta)} = \partial_\mu E^\mu(\alpha\beta) + \ldots$(see [3]) which give rise to the translational and rotational invariances in the tangent space [10].

The Hamiltonian method applied to singular systems (the Dirac procedure) always leads to first class constraints that allow the restoration of the gauge invariance. The Hamiltonian formulation of the EC action leads to the first class constraints with the PB algebra that describe internal translation and rotation. This is clear from the first steps of the Dirac procedure and the tensorial character of the primary first class constraints [1, 2]. The same result, translational and rotational invariances, also follows from the analysis of basic differential identities at the Lagrangian level [3].

From the equivalence of the descriptions given by the Hamiltonian and Lagrangian formulations, we can conclude that diffeomorphism is not a gauge symmetry derivable from the first class constraints of the EC action and its Hamiltonian. Gauge symmetries are translation and rotation in the tangent space, which makes the EC theory very similar to the Yang-Mills theory. This is a classical result which, in particular, is related to the classical background of Loop Quantum Gravity (LQG). One can say that this is a classical question and it has nothing to do with quantum issues of LQG, e.g. according to [41] “at the current stage, with the author’s [ours] focus on classical questions while the debated issues of loop quantum gravity arise after quantization, their [our] criticism is rather empty”. The logic of this statement is simple 7: it does not matter what a model is quantized, it does not matter whether the result is correct or not, because one can always debate the result obtained after quantization. In our opinion, if on the classical level the true gauge invariance was missing because the Hamiltonian formulation produces something different (e.g. “spatial diffeomorphism” of LQG) then the formulation, which is used for quantization, is classically not equivalent to the original Einstein-Cartan action. With such a discrepancy at a very basic level, to debate the results that arise after quantization of a classically different formulation is rather an empty exercise (or a waste of time) for anyone who is interested in quantizing

---

7 This logic cannot be debated in a scientific journal (debates cannot be published) perhaps because of authorships of the statement [41].
Einstein gravity. Any quantization starts from a classical action. What classical action is quantized in LQG? Contrary to the undebatable view expressed in [41], there is a different opinion that coincides with our point of view. In the conclusion to the review article on LQG of Nicolai et al. [42], one can read “...despite the optimism prevalent in many other reviews, more attention should be paid to basic aspects and unresolved problems of the theory” and a few lines later the authors repeat again ”...there are still too many problems at a basic level that need to be addressed and resolved”. We hope that our results provide enough arguments to re-focus attention to the basic, classical, issues from “the debated issues of loop quantum gravity” that “arise after quantization”.

The answer to Matshull’s question [14] “what is a gauge symmetry and what is not” is given for the EC theory in any dimension. This is enough for covariant methods of quantization (based on the Lagrangian formalism). However, for canonical quantization, the Hamiltonian formulation of the EC action has to be completed. The unique Darboux coordinates, the main subject of this article, provide a great simplification of the Hamiltonian analysis. The modification of the algebra of PBs among first class secondary constraints (see (103)) compared to the $D = 3$ case and the restoration of gauge invariance (as consistency check) from the constraint structure of the EC Hamiltonian using the Castellani algorithm [13] is under our current investigation and will be reported elsewhere.

ACKNOWLEDGMENTS

The authors are grateful to A.M. Frolov, P.G. Komorowski, D.G.C. McKeon, and A.V. Zvelindovsky for numerous discussions and suggestions. The partial support of The Huron University College Faculty of Arts and Social Science Research Grant Fund is greatly acknowledged.

**Appendix A: ABC properties**

Here we collect properties of the $ABC$ functions that were introduced in considering the Hamiltonian formulation of N-bein gravity [1, 2]. They also turn out to be very useful in the Lagrangian formalism [3].

These functions are generated by consecutive variation of the N-bein density.
The first important property of these density functions is their total antisymmetry: interchange of two indices of the same nature (internal or external), e.g.

\[ A^{\nu(\beta)\mu(\alpha)} = -A^{\nu(\alpha)\mu(\beta)} = -A^{\mu(\beta)\nu(\alpha)} \]  

with the same being valid for \( B, C, D, \) etc. In particular, the presence of two equal indices of the same nature (both internal or both external) makes the functions \( A, B, \) etc. equal zero.

The second important property is their expansion using an external index

\[ B^{\tau(\rho)\mu(\alpha)\nu(\beta)} = e^{\tau(\rho)} A^{\mu(\alpha)\nu(\beta)} + e^{\tau(\alpha)} A^{\mu(\beta)\nu(\rho)} + e^{\tau(\beta)} A^{\mu(\rho)\nu(\alpha)}, \]  

or an internal index

\[ B^{\tau(\rho)\mu(\alpha)\nu(\beta)} = e^{\tau(\rho)} A^{\mu(\alpha)\nu(\beta)} + e^{\mu(\rho)} A^{\nu(\alpha)\tau(\beta)} + e^{\nu(\rho)} A^{\tau(\alpha)\mu(\beta)}, \]  

\[ C^{\tau(\rho)\lambda(\sigma)\mu(\alpha)\nu(\beta)} = e^{\tau(\rho)} B^{\lambda(\sigma)\mu(\alpha)\nu(\beta)} - e^{\tau(\sigma)} B^{\lambda(\alpha)\mu(\beta)\nu(\rho)} + e^{\tau(\alpha)} B^{\lambda(\beta)\mu(\rho)\nu(\sigma)} - e^{\tau(\beta)} B^{\lambda(\rho)\mu(\sigma)\nu(\alpha)} \]  

The third property involves their derivatives
\[ (eA^{\nu(\beta)\mu(\alpha)})_{,\sigma} = \frac{\delta}{\delta e_{\lambda(\gamma)}} (eA^{\nu(\beta)\mu(\alpha)}) e_{\lambda(\gamma),\sigma} = eB^{\lambda(\gamma)\nu(\beta)\mu(\alpha)} e_{\lambda(\gamma),\sigma}, \quad (A10) \]

\[ (eB^{\tau(\rho)\nu(\beta)\mu(\alpha)})_{,\sigma} = \frac{\delta}{\delta e_{\lambda(\gamma)}} (eB^{\tau(\rho)\nu(\beta)\mu(\alpha)}) e_{\lambda(\gamma),\sigma} = eC^{\tau(\rho)\lambda(\gamma)\nu(\beta)\mu(\alpha)} e_{\tau(\rho),\sigma}. \quad (A11) \]

We also use the contraction of \( B^{\tau(\rho)\mu(\alpha)\nu(\beta)} \) with a covariant \( e_{\tau(\lambda)} \):

\[ e_{\tau(\lambda)} B^{\tau(\rho)\mu(\alpha)\nu(\beta)} = \tilde{\delta}^{\rho}_{\lambda} A^{\mu(\alpha)\nu(\beta)} + \tilde{\delta}^{\alpha}_{\lambda} A^{\mu(\beta)\nu(\rho)} + \tilde{\delta}^{\beta}_{\lambda} A^{\mu(\rho)\nu(\alpha)}. \quad (A12) \]

The above properties considerably simplify the calculations.

**Appendix B: Solution of the equation of motion for \( \hat{\Sigma} \)**

To eliminate the \( \hat{\Sigma} \) field in the course of the Lagrangian or Hamiltonian reduction, we perform variation of (58) or (62) with respect to \( \hat{\Sigma} \) and solve this equation for \( \hat{\Sigma} \). The corresponding part of the Lagrangian (Hamiltonian), quadratic and linear in \( \hat{\Sigma} \), after changing dummy indices and performing some contractions is

\[ L \left( \hat{\Sigma} \right) = e g_{pq} \hat{\Sigma}_{k}^{(mq)} \hat{\Sigma}_{\lambda}^{(kp)} + 2 e \hat{\Sigma}_{m}^{(pq)} \hat{D}_{pq}^{m}, \quad (B1) \]

where \( g_{qp} = e_{q(\alpha)} e_{p(\alpha)} \) and

\[ \hat{D}_{pq}^{m} = e^{m(\beta)} e_{p(\beta),q} + e^{m(\beta)} e_{p(\gamma)} N_{p(\gamma)0n(\sigma)} \pi^{n(\sigma)}. \quad (B2) \]

Note that there are no symmetries in this expression for \( pq \) indices (e.g., \( \hat{D}_{pq}^{m} \not= -\hat{D}_{qp}^{m}, \hat{D}_{pq}^{m} \not= \hat{D}_{qp}^{m} \)), which is clear from its explicit form. Of course, we can do further contraction in the second term of (B2); but to find the solution it is not necessary as it can be expressed in terms of the whole \( \hat{D}_{pq}^{m} \) and the separation of it into contributions with momenta and spatial derivatives of covariant N-bein is sufficient on this stage and keep the expressions in compact form.

Variations of a traceless antisymmetric field (see the fundamental PB (64)) is

\[ \frac{\delta \hat{\Sigma}^{(mq)}_{k}}{\delta \hat{\Sigma}^{(yz)}_{\lambda}} = \delta^{\lambda}_{k} \hat{\Delta}^{(mq)}_{(yz)} - \frac{1}{D-2} \left[ \delta^{\lambda}_{y} \hat{\Delta}^{(mq)}_{(kz)} - \delta^{\lambda}_{z} \hat{\Delta}^{(mq)}_{(ky)} \right]. \quad (B3) \]
It is clear from (B3) that this expression is antisymmetric in $mq$ and $yz$ and equals zero if the traces of $\hat{\Sigma}_k^{(mq)}$ or $\hat{\Sigma}_x^{(yz)}$ are taken. Variation of (B1) gives

$$\hat{\Sigma}_y^{(px)} g_{zp} - \hat{\Sigma}_z^{(px)} g_{yp} = \hat{D}^x_{(yz)}$$  \hspace{1cm}  (B4)$$

where

$$\hat{D}^x_{(yz)} = \hat{D}^x_{yz} - \frac{1}{D - 2} \left[ \delta^x_y \left( \hat{D}^m_{mz} - \hat{D}^m_{zm} \right) - \delta^x_z \left( \hat{D}^m_{my} - \hat{D}^m_{ym} \right) \right],$$  \hspace{1cm}  (B5)$$

which is manifestly antisymmetric and traceless as it should be after variation with respect to the field with such properties.

To solve equation (B4) we use Einstein’s permutation \[43\]. To do this we must have three indices of the same nature, either all external or all internal, and in the same position, covariant or contravariant. We can achieve this by contracting (B4) with $g_{wx}$

$$g_{wx} \hat{\Sigma}_y^{(px)} g_{zp} - g_{wx} \hat{\Sigma}_z^{(px)} g_{yp} = g_{wx} \hat{D}^x_{(yz)}.$$  \hspace{1cm}  (B6)$$

Now we have combinations with three free external indices in covariant position and can use the permutation $(wyz) + (yzw) - (zwy)$ that gives us

$$2 g_{yx} \hat{\Sigma}_z^{(px)} g_{wp} = g_{wx} \hat{D}^x_{(yz)} + g_{yx} \hat{D}^x_{(zw)} - g_{zx} \hat{D}^x_{(wy)}.$$  \hspace{1cm}  (B7)$$

To find explicitly $\hat{\Sigma}_z^{(px)}$, we have to use the Dirac inverse $\gamma^{km} = g^{km} - g^{0m} g^{0k}$. (We repeat, it is not a new variable, but a short-hand notation for a particular combination of N-bein fields). After contracting (B7) with $\gamma^{ky} \gamma^{mw}$ we obtain the solution

$$\hat{\Sigma}_z^{(mk)} = \frac{1}{2} \left( \gamma^{kb} \hat{D}^m_{(bz)} - \gamma^{mb} \hat{D}^k_{(bz)} - \gamma^{ky} \gamma^{mw} g_{ex} \hat{D}^z_{(wy)} \right).$$  \hspace{1cm}  (B8)$$

Of course, solution for antisymmetric and traceless field is antisymmetric (RHS of (B8) is manifestly antisymmetric) and traceless (contracting (B8) with $\delta^z_m$ or $\delta^z_k$).

At this stage, we can check the $D = 3$ limit. The solution for $\hat{\Sigma}_z^{(mk)}$ (B8) was obtained for all dimensions $D > 2$ and it has to vanish when $D = 3$. It is not difficult to check, taking $\hat{\Sigma}_z^{(mk)}$ from (B8) and using the exact expressions of $\hat{D}^m_{(bz)}$ (B5) and $\hat{D}^m_{pq}$ (B2), that such a limit is preserved
\[ \lim_{D=3} \hat{\Sigma}_z^{(mk)}(\pi) = \lim_{D=3} \hat{\Sigma}_z^{(mk)}(e,s) = 0. \]  \hspace{1cm} (B9)

Actually, to demonstrate (B9), it is not necessary to substitute the explicit form of \( \hat{D}_{(bz)}^m \). When \( D = 3 \), there are only two independent components of \( \hat{\Sigma}_z^{(mk)} \) and they are \( \hat{\Sigma}_1^{(12)} = \hat{\Sigma}_2^{(12)} = 0 \) because \( \hat{\Sigma}_z^{(mk)} \) is antisymmetric and traceless with only spatial indices.

Substitution of (B8) back into (58) or (74) gives the reduced Lagrangian or Hamiltonian, respectively.

[1] A.M. Frolov, N. Kiriushcheva and S.V. Kuzmin, arXiv:0902.0856 [gr-qc].
[2] N. Kiriushcheva and S.V. Kuzmin, arXiv:0907.1553 [gr-qc].
[3] N. Kiriushcheva and S.V. Kuzmin, arXiv:0907.1999 [gr-qc].
[4] P.A.M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Sciences, Yeshiva University, New York, 1964.
[5] P.A.M. Dirac, Can. J. Math. 2 (1950) 129; ibid. 3 (1951) 1.
[6] E. Witten, Nucl. Phys. B 311 (1988) 46.
[7] R. Gambini, J. Pullin, Loops, Knots, Gauge Theories and Quantum Gravity, Cambridge University Press, Cambridge, 1996.
[8] T. Thiemann, Modern canonical quantum general relativity, Cambridge University Press, Cambridge, 2007.
[9] N. Kiriushcheva and S.V. Kuzmin, arXiv:0912.3396 [gr-qc].
[10] A. Trautman, Ann. N.Y. Acad. Sci. 262 (1975) 241.
[11] F.W. Hehl, P. von der Heyde, G.D. Kerlick, J.M. Nester, Rev. Mod. Phys. 48 (1976) 393.
[12] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, Princeton, New Jersey, 1992.
[13] L. Castellani, Ann. Phys. 143 (1982) 357.
[14] H-J. Matschull, Class. Quantum Grav. 16 (1999) 2599.
[15] N. Kiriushcheva, S.V. Kuzmin, C. Racknor and S.R. Valluri, Phys. Lett. A 372 (2008) 5101.
[16] N. Kiriushcheva and S.V. Kuzmin, arXiv:0809.0097 [gr-qc].
[17] M. Leclerc, Int. J. Mod. Phys. D 16 (2007) 655.
[18] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne’eman, Physics Report 258 (1995) 1.
M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B 332 (1990) 169.

J. F. Plebanski, J. Math. Phys. 18 (1977) 2511-2520.

A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244.

P. Peldan, Class. Quantum Grav. 11 (1994) 1087.

M. Bañados and M. Contreras, Class. Quantum Grav. 15 (1998) 1527.

J. Schwinger, Phys. Rev. 130 (1963) 1253.

L. Castellani, P. van Nieuwenhuizen and M. Pilati, Phys. Rev. D 26 (1982) 352.

J. Pullin, in: Cosmology and Gravitation: Xth Brazilian School of Cosmology and Gravitation; 25th Anniversary (1977-2002), AP conference proceedings, vol. 668 (2003) p. 141; arXiv:gr-qc/0209008.

N. Kiriushcheva and S.V. Kuzmin, Ann. Phys. 321 (2006) 958.

D.M. Gitman and I.V. Tyutin, Quantization of Fields with Constraints, Springer, Berlin, 1990.

B. Geyer, D.M. Gitman and I.V. Tyutin, J. Phys. A: Math. Gen. 36 (2003) 6587.

L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692.

H. J. Rothe and K. D. Rothe, J. Phys. A: Math. Gen. 36 (2003) 1671.

A. Shirzad, M. Mojiri, J. Math. Phys. 46 (2005) 012702.

J.A. García and J.M. Pons, Int. J. Mod. Phys. A 13 (1998) 3691.

P.A.M. Dirac, Proc. Roy. Soc. A 246 (1958) 333.

K.R. Green, N. Kiriushcheva and S.V. Kuzmin, arXiv:0809.1198 [gr-qc].

A. Einstein, Sitzungsber. preuss. Akad. Wiss., Phys.-Math. K1 (1928) 217; The Complete Collection of Scientific Papers (Nauka, Moskva, 1966) vol. 2, p. 223; English translation of this and a few other articles is available from: <http://www.lrz-muenchen.de/~aunzicker/ae1930.html> and A. Unzicker, T. Case, arXiv:physics/0503046.

S. Samanta, Int. J. Theor. Phys. 48 (2009) 1436.

P. Mukherjee, A. Saha, Int. J. Mod. Phys. A 24 (2009) 4305.

R. Banerjee, S. Gangopadhyay, P. Mukherjee, D. Roy, arXiv:0912.1472 [gr-qc].

Report of an anonimous referee on [35], Gen. Rel. Grav., December 22, 2009.

H. Nicolai, K. Peeters and M. Zamaklar, Class. Quantum Grav. 22 (2005) R193.

A. Einstein, Sitzungsber. preuss. Akad. Wiss., Phys.-Math. K1 (1925) 414; The Complete Collection of Scientific Papers (Nauka, Moskva, 1966) vol. 2, p. 171; English translation of this and
a few other articles is available from: <http://www.lrz-muenchen.de/~aunzicker/ae1930.html>

and A. Unzicker, T. Case, arXiv:physics/0503046