ON THE COP NUMBER OF GENERALIZED PETERSEN GRAPHS

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ABSTRACT. We show that the cop number of every generalized Petersen graph is at most 4. The strategy is to play a modified game of cops and robbers on an infinite cyclic covering space where the objective is to capture the robber or force the robber towards an end of the infinite graph. We prove that finite isometric subtrees are 1-guardable and apply this to determine the exact cop number of some families of generalized Petersen graphs. We also extend these ideas to prove that the cop number of any connected I-graph is at most 5.

1. Introduction

The game of cops and robbers on graphs was introduced by Quilliot [9] and, independently, by Nowakowski and Winkler [8]. The game is played as follows. One player, the cop player, is given a collection of \( k \) pawns called cops. She assigns each cop to a vertex of a given undirected graph \( G \). A second player, the robber player, is given a single pawn called a robber. He assigns the robber to a vertex of \( G \). The players alternate turns, with the cop player going first. On a turn, a player may move any number of her or his pawns, by moving each to an adjacent vertex or passing by remaining at the same vertex. If, after either player’s move, a cop and the robber are at the same vertex, the robber is captured and the cop player wins. The cop number, \( c(G) \), of a graph \( G \) is the least positive integer \( k \) such that \( k \) cops suffice to capture the robber in a finite number of moves. In this game, both players are assumed to have complete information about the graph and the positions of the pawns. Bonato and Nowakowski [4] have written a text which introduces and surveys many of the foundational papers on the game of cops and robbers on graphs.

The cop number of a graph is computationally expensive to compute. An algorithm described by Bonato and Chiniforooshan [3] will check whether or not \( k \) cops suffice to win on a given graph \( G \); however, the algorithm runs in \( O(n^{3k+3}) \) time, where \( n \) is the order of the graph.

There are a number of results on bounds for the cop number in terms of a graph invariant. For example, Aigner and Fromme [1] proved that if the minimum degree of a graph \( G \) is \( \delta \) and if \( G \) has girth least five, then \( c(G) \geq \delta \). Frankl [6] generalized this as follows: for each integer \( t \geq 1 \), if the minimum degree of a graph \( G \) is \( \delta \geq 2 \) and \( G \) has girth at least \( 8t - 3 \), then \( c(G) > (\delta - 1)^t \).

The present article will establish bounds for the cop number of generalized Petersen graphs. Let \( n \) and \( k \) be a positive integers such that \( n \geq 5 \) and \( 1 \leq k < n/2 \). The generalized Petersen graph, \( GP(n,k) \), is the undirected graph having vertex set \( A \cup B \), where \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \), and having the following
edges: \((a_i, a_{i+1}), (a_i, b_i),\) and \((b_i, b_{i+k})\) for each \(i = 1, \ldots, n\), where indices are to be read modulo \(n\).

In Section 3 we prove the main result of this article:

**Theorem 1.1.** The cop number of every generalized Petersen graph is less than or equal to four.

It is immediate that generalized Petersen graphs are 3-regular, and it is straightforward to check that the generalized Petersen graph \(G(n, k)\) has girth at least five if and only if \(k \neq 1\) and \(n \neq 3k, 4k\). For these graphs, the bounds of Aigner, Fromme, and Frankl establish that the cop number of a generalized Petersen graph is at least three.

By implementing the algorithm of Bonato and Chiniforooshan, we have confirmed that there are generalized Petersen graphs with cop number greater than 3; and so, by the theorem above, these graphs have cop number equal to 4.

We address the problem of determining the exact cop number of generalized Petersen graphs in Section 5. A variety of ad hoc techniques are used; however, one technique which may be of more general interest is the following (see Section 4):

**Theorem 1.2.** If \(T\) is a finite isometric subtree of a graph \(G\), then \(T\) is 1-guardable.

In Section 6 we generalize our results to \(I\)-graphs. If \(n \geq 5\) and \(0 < j, k < n/2\), the \(I\)-graph \(I(n, j, k)\) has vertex set \(A \cup B\) and has the following edges: \((a_i, a_{i+j}), (a_i, b_i), (b_i, b_{i+k})\) for each \(i = 1, \ldots, n\). Thus, \(I\)-graphs are like generalized Petersen graphs with two parameters: one for the \(A\)-vertices and one for the \(B\)-vertices; in particular, setting \(j = 1\), we see that \(I(n, 1, k) = GP(n, k)\).

**Theorem 1.3.** The cop number of every connected \(I\)-graph is less than or equal to five.

Acknowledgement: We are thankful for encouragement and support from the participants in the 2014 summer research experience for undergraduates at Michigan State University where this research was conducted. We are grateful for support from the National Security Agency, the National Science Foundation (NSF grant #DMS-1062817), and Michigan State University.

2. **Infinite cyclic coverings of generalized Petersen graphs**

We define an infinite analogue of a generalized Petersen graph for each positive integer \(k\). Let \(A = \{a_i \mid i \in \mathbb{Z}\}, B = \{b_i \mid i \in \mathbb{Z}\}\). The infinite graph \(GP(\infty, k)\) has vertex set \(A \cup B\) and has the following edges: \((a_i, a_{i+1}), (a_i, b_i),\) and \((b_i, b_{i+k})\) for each \(i \in \mathbb{Z}\). There is a graph homomorphism \(\pi : GP(\infty, k) \rightarrow GP(n, k)\) given by reducing the index of each vertex modulo \(n\). This map induces a regular covering map of the geometric realizations of these graphs. More precisely, there is a \(\mathbb{Z}\)-action on \(GP(\infty, k)\) defined as follows. Let \(\tau\) be a choice of generator of \(\mathbb{Z}\), let \(n \geq 5\), and define \(\tau.a_i = a_{i+n}\) and \(\tau.b_i = b_{i+n}\). Then \(\tau\) extends uniquely to an automorphism of \(GP(\infty, k)\) and the orbit space is isomorphic to \(GP(n, k)\).

The discussion below holds for any covering space \(\pi : \hat{G} \rightarrow G\) of graphs. The reader may prefer to concentrate on the special case where \(\hat{G} = GP(\infty, k)\) and \(G = GP(n, k)\) for some fixed choices of \(n\) and \(k\).

Suppose that \(X\) is a pawn assigned to a vertex \(v\) of \(G\). Let \(\pi^{-1}(X) = \{X_w \mid w \in \pi^{-1}(v)\}\) be a set of pawns in one-to-one correspondence with the set of pre-images...
of \(v\). We assign the pawn \(X_w\) to the vertex \(w\) of \(\widehat{G}\). Suppose that \(X\) moves from \(v\) to \(v' \neq v\) in \(G\) and let \(e = (v, v')\) be the corresponding edge (oriented from \(v\) to \(v'\)). There is a unique \textit{lifted move} for each \(X_w \in \pi^{-1}(X)\): the pawn \(X_w\) moves from \(w\) to \(w'\) where \(w'\) is the unique endpoint of the lift at \(w\) of the edge \(e\). For example, in \(GP(n, k)\), the edge \((a_{n-1}, a_n = a_0)\) will lift at \(a_{n-1}\) to \((a_{n-1}, a_n)\) in \(GP(\infty, k)\); and it will lift at \(a_{2n-1}\) to \((a_{2n-1}, a_{2n})\), etc. If \(X\) passes, then so does each pawn in \(\pi^{-1}(X)\).

Thus, each move of a pawn in \(G\) defines a unique move for each pawn in its pre-image. Conversely, a move of any one pawn in \(\pi^{-1}(X)\) defines a move for \(X\) by projecting this move via \(\pi\) from \(\widehat{G}\) to \(G\). We refer to this interplay informally as the \textquote{lifted game}.

Suppose that \(C\) is a cop on \(v \in V(G)\) and \(C' \in \pi^{-1}(C)\) is a cop on \(w \in \pi^{-1}(v)\). A move \(e = (w, w')\) by \(C'\) defines a move for every cop in \(\pi^{-1}(C)\) as follows: each \(C'' \in \pi^{-1}(C)\) plays the move defined by lifting the edge \(\pi(e)\) at the vertex which \(C''\) occupies. Thus, a move by one \(C' \in \pi^{-1}(C)\) defines a unique consistent move for every cop in \(\pi^{-1}(C)\), where consistency means that each of these moves projects to the same move for \(C\). When the cops in \(\pi^{-1}(C)\) move in this way, we refer to \(\pi^{-1}(C)\) as a \textit{squad} and say that these cops \textit{move as a squad} consistent with the moves of a chosen \textit{lead cop} \(C'\).

When each pre-image of a cop or a robber plays as a squad, there is no difference between the game played on \(\widehat{G}\) and the game played on \(G\). The purpose of playing the lifted game is to reveal strategies which may not be apparent when one studies only the structure of \(G\). The fundamental observation is that a sequence of moves in \(\widehat{G}\) which results in the capture of any lift of the robber by any lift of a cop projects to a sequence of moves in \(G\) which results in a capture of the robber.

We say that the \textit{weak cop number} of a covering of graphs \(\pi : \widehat{G} \to G\) is less than or equal to \(m\) if \(m\) squads playing on \(\widehat{G}\) can capture a single robber or force him to move arbitrarily far away from a fixed choice of a base vertex. This definition is independent of the base vertex if \(\widehat{G}\) is connected.

If \(\widehat{G}\) is a finite sheeted covering of \(G\), then the weak cop number is equal to the cop number of \(G\). But if \(G\) is an infinite sheeted cover, the weak cop number can be strictly less than the cop number. For example, for each positive integer \(k\), the infinite path having vertices \(\mathbb{Z}\) and edges \(\{(n, n + 1) \mid n \in \mathbb{Z}\}\) covers the \(k\)-cycle by reducing each vertex modulo \(k\). This covering space has weak cop number 1; but, for each \(k \geq 4\), the \(k\)-cycle has cop number 2.

It is straightforward to establish that the cop number of \(G\) is greater than or equal to the weak cop number of a covering \(\pi : \widehat{G} \to G\) and equality holds if and only if the squads have a capture strategy for any single robber in \(\widehat{G}\).

The notion of a weak cop number was introduced by Chastand, Laviolette, and Polat [3]. Lehner [7], in a recent preprint, argues in favor of the following definition which is similar to the one used here: a graph \(G\) is weakly copwin if a cop can either capture a robber or prevent him from visiting any vertex infinitely often.

**Theorem 2.1.** For each generalized Petersen graph \(GP(n, k)\), the weak cop number of \(\pi : GP(\infty, k) \to GP(n, k)\) is 2.

**Proof.** Let \(\widehat{G} = GP(\infty, k)\) and \(G = GP(n, k)\). It is clear that one squad is not sufficient since it is assumed that \(n \geq 5\). Choose \(a_0\) as the base vertex in \(\widehat{G}\). Assign two cops, \(C_1\) and \(C_2\), to \(a_0\) in \(G\). This determines an assignment of two squads
$S_1 = \pi^{-1}(C_1)$ and $S_2 = \pi^{-1}(C_2)$ to the vertices of $\hat{G}$. By abuse of notation, let $C_1$ and $C_2$ denote choices of lead cops for $S_1$ and $S_2$, respectively, with both assigned to $a_0$ in $\hat{G}$ on the first turn. Let $R$ denote the robber which is assigned to some vertex in $\hat{G}$. As in the description of the lifted game, the moves of each cop in a squad $S_i$ is determined by the moves of $C_i$.

The initial strategy of the cops is to move in such a way that, after finitely many moves, one cop, say $C_1$, occupies a vertex whose index is congruent modulo $k$ to the index of the vertex which the robber occupies. Hereafter, we refer to the index of the vertex which a pawn occupies as the *index* of the pawn. Since there are only finitely many residues modulo $k$, by moving $C_1$ from $a_0$ to $a_1$ to $a_2$, etc. and moving $C_2$ from $a_0$ to $a_{-1}$ to $a_{-2}$, etc., this is achieved in less than or equal to $k/2$ turns. The important observation is that the robber can only change the residue modulo $k$ of his index by at most one: if $R$ moves within the subgraph $A$ induced by the vertex set $A$, then his residue changes by one if and only if he does not pass; if he moves within the subgraph $B$ induced by the vertex set $B$ or if he moves from a vertex of $A$ to a vertex of $B$ or vice-versa or if he passes, then his residue does not change at all.

After possibly relabeling our cops and squads, we have that $C_1$’s index is congruent modulo $k$ to $R$’s index. Caution is needed since the other cops in the squad $S_1$ need not have indices congruent to $R$’s index; so, our choice of lead cop is important for the squad $S_1$.

The next stage of the cops’ strategy is to move $C_1$ to match parity with the robber in the sense that both occupy vertices in the same induced subgraph, either both in $A$ or both in $B$. This can be achieved on the turn after $C_1$ has achieved a congruent index. If $R$’s next move is to a vertex in $B$, then $C_1$ moves to the unique vertex of $B$ to which the cop is adjacent. If $R$ visits a vertex of $A$, then $C_1$ plays a move (possibly passing) in $A$ which maintains the congruence of their indices. (There is only one such move if $k > 1$.) In either case, $C_1$ has maintained a congruent index and now matches parity with $R$.

On subsequent turns, $C_1$ moves so that both congruence and parity are maintained. Without loss of generality, we may assume that the index of $C_1$ is less than the index of $R$. Whenever $R$ moves within $B$, $C_1$ has a choice of two moves; we declare that $C_1$ will always move towards $R$, that is towards the vertex in $B$ with larger index. If $R$ passes in $B$ or moves towards $C_1$, then after $C_1$’s move the distance between the two pawns has decreased.

The final stage of the cops’ strategy is for $C_2$ to move in $A$ towards $R$. If $C_2$ has a higher index than $R$, then choose a new lead cop, which we will again call $C_2$, for the squad $S_2$ so that the index of $C_2$ is less than $R$’s index. On each turn, $C_2$ moves in $A$ by increasing his index.

To establish that the weak cop number is two, we prove that $R$’s index cannot remain bounded from above. Whenever $R$ moves to decrease his index or leave it unchanged, $C_2$ moves closer or $C_1$ moves closer. If $R$ succeeds in lowering his index below that of $C_2$’s index (see Figures 1 and 2), then to do so he must move in $B$ towards $C_1$. Each time this happens, $C_1$ moves closer by $2k$. Moreover, the squad $S_2$ at this point can simply choose a new lead cop, again called $C_2$, having index lower than $R$. Thus, $R$ can only evade $C_2$ in this way finitely many times. Therefore, $R$ can only decrease his index or leave it unchanged finitely many times without being captured. □
Figure 1. As $C_2$ approaches $R$, the robber may have the opportunity to lower his index below that of $C_2$ (see also Figure 2).

The following corollary is a refinement of the above proof. It says that we can, by carefully choosing a new lead cop in $GP(\infty, k)$, force the robber to move in a pre-determined direction.

**Corollary 2.2.** Two squads playing in $GP(\infty, k)$ can capture a single robber or force his index to increase without bound.

**Proof.** As in the proof of Theorem 2.1, two squads $S_1$ and $S_2$ with lead cops $C_1$ and $C_2$, respectively, play against a single robber $R$. After at most $k/2 + 1$ turns, $C_1$'s index is congruent to $R$'s index and both have the same parity, but it may be the case that the index of $C_1$ is greater than $R$'s index. Following the proof of the theorem, it is clear that the robber can be forced to decrease his index without bound. But if we want to force the robber to increase his index without bound, we must select a new lead cop for $S_1$ which has an index congruent modulo $k$ and which is less than $R$'s index. Since the indices of the cops in $S_1$ are in one-to-one correspondence with the elements of the set $\{qn + I \mid q \in \mathbb{Z}\}$, where $I$ is the index of $C_1$, we can choose a new lead cop corresponding to an index of the form $-mkn + I$.
for a sufficiently large integer $m$. Then, following the strategy in Theorem 2.1, we have the desired result.

\[\Box\]

3. **Bounding the cop number of the generalized Petersen graph**

Using the results of the previous section, we prove the main theorem stated in the introduction.

**Theorem 3.1.** The cop number of a generalized Petersen graph is less than or equal to four.

**Proof.** Fix $n$ and $k$ so that $G = GP(n, k)$ is a generalized Petersen graph. Let $\hat{G} = GP(\infty, k) \rightarrow GP(n, k)$ be the associated regular covering space. Two pairs of cops play on $G$ to capture a robber $R$. This is achieved by having each cop play the projected moves of squads consisting of their pre-images which play the lifted game in $\hat{G}$ against any single robber $\hat{R}$ in the pre-image of $R$. By Corollary 2.2, one pair of lifted cops can force $\hat{R}$ arbitrarily far to the right, i.e. force $\hat{R}$'s index to increase without bound. A second pair of lifted cops can force $\hat{R}$ arbitrarily far to
There exist generalized Petersen graphs, such as $GP(40, 7)$, which have cop number 4. That $GP(40, 7)$ does not have cop number less than 4 was verified with assistance of a computer. A summary of our findings and more precise results are given in Section 5.

4. Guarding isometric trees

The results of this section are of independent interest. We will use these results to determine the exact value of the cop number of several families of generalized Petersen graphs in Section 5.

The distance, $d_G(u, v)$, between two vertices, $u$ and $v$, in a graph $G$ is the length of a shortest path in $G$ joining $u$ to $v$. A subgraph $H$ of $G$ is an isometric subgraph if for any two vertices, $u$ and $v$, in $H$, $d_H(u, v) = d_G(u, v) = d_G(v, u)$.

A collection of cops $\{C_i\}$ is said to guard a subgraph $H$ of a graph $G$ if these cops occupy vertices of $H$, move only in $H$, and can end each cop turn so that the following guarding condition (GC) holds:

$$(GC) \quad \forall v \in H, \exists C_i, \quad d_H(v, C_i) \leq d_G(v, R),$$

where $R$ refers to the position of the robber. If $H$ is guarded by $\{C_i\}$ and the robber were to enter $H$, then (GC) implies that the robber is immediately captured or captured on the next move by one of these cops.

A subgraph $H$ of a graph $G$ is $k$-guardable if $k$ cops, $C_1, \ldots, C_k$, can, after finitely many moves, arrange themselves so that they guard $H$. By guarding a subgraph, the cops effectively eliminate a portion of the larger graph that the robber can play on.
Lemma 4.1. Suppose that \( T \) is an isometric subtree of a graph \( G \). Suppose that a cop occupies \( c \in V(T) \) and the robber occupies \( r \in V(G) \). If \( v_1, v_2 \in V(T) \) belong to different components of \( T - \{c\} \) and \( d(c, v_1) > d(r, v_1) \), then \( d(c, v_2) < d(r, v_2) \).

Proof. If, contrary to the conclusion, \( d(c, v_2) \geq d(r, v_2) \), then
\[
d(v_1, v_2) = d(v_1, c) + d(c, v_2) > d(v_1, r) + d(r, v_2) \geq d(v_1, v_2),
\]
which is a contradiction. \(\square\)

Lemma 4.2. Suppose that \( T \) is an isometric subtree of a graph \( G \). Suppose that a cop occupies \( c_0 \in V(T) \) and the robber occupies \( r \in V(G) \). Suppose that \( v_0 \in V(T) \) and \( d(c, v_0) > d(r, v_0) \). Let \( c_1 \) be the unique vertex in \( T \) adjacent to \( c_0 \) and closer to \( v_0 \). If the cop moves to \( c_1 \), then
1. If \( v \in T \) belongs to a different component of \( T - \{c_0\} \) than \( v_0 \), then \( d(c_1, v) \leq d(r, v) \).
2. If \( v \in T \) belongs to the same component of \( T - \{c_0\} \) as \( v_0 \), then \( d(c_1, v) = d(c_0, v_0) - 1 \).

Proof. Suppose that the cop moves to \( c_1 \) as described above. If \( v \in V(T) \) is in a different component of \( T - \{c_0\} \) than \( v_0 \), then, by Lemma 4.1, \( d(c_0, v) < d(r, v) \). Therefore, \( d(c_1, v) \leq d(r, v) \). If \( v \in V(T) \) lies in the same component of \( T - \{c_0\} \) as \( v_0 \), then \( d(c_1, v_0) = d(c_0, v_0) - 1 \) because \( T \) is an isometric tree. \(\square\)

Theorem 4.3. If \( T \) is a finite isometric subtree of a graph \( G \), then \( T \) is 1-guardable.

Proof. There are two parts to the argument. First, we must show that a cop can end her turn so that the guarding condition \((GC)\) holds for \( T \). Second, we must show that after any subsequent move by the robber, she can move in \( T \) so that \((GC)\) still holds after her move.

We may assume that the cop begins on a vertex \( c_0 \in V(T) \). Let \( U_0 \) be the set of vertices of \( T \) which belong to the component of \( T - \{c_0\} \) which contains all vertices of \( T \) for which \((GC)\) fails. This is well-defined by Lemma 4.1. If \( U_0 = \emptyset \), then the cop can pass and \((GC)\) holds. Otherwise, the cop moves to the unique adjacent vertex \( c_1 \in U_0 \). Let \( U_1 \) be defined analogously. Lemma 4.2 implies that \( U_1 \) is a proper subset of \( U_0 \). If \( U_1 = \emptyset \), then the first part is complete. If \( U_1 \neq \emptyset \) and the robber moves from \( r_0 \) to \( r_1 \), then Lemma 4.1 implies that the only vertices of \( T \) which are closer to \( r_1 \) than to \( c_1 \) still belong to \( U_1 \). The reason is that any vertex of \( V(T) - U_1 \) is farther from \( r_0 \) than from \( c_1 \). Therefore, this strategy can be continued. Since \( T \) is assumed to be a finite tree, there is a \( k \geq 0 \) such that \( U_k = \emptyset \).

Suppose that \((GC)\) holds and that the robber is to move. Let \( c_0 \) be the vertex occupied by the cop. If the robber moves from \( r_0 \) to \( r_1 \) so that \( d(r_1, v_1) < d(c_0, v_1) \) for some vertex \( v_1 \in V(T) \), then the cop moves to \( c_1 \), following the same strategy as above. Since \( d(c_0, v_1) \leq d(r_0, v_1) \), we have that \( d(c_1, v_1) \leq d(r_1, v_1) \). Moreover, by Lemma 4.2, \((GC)\) holds. \(\square\)
Figure 4. Above is a complete list of the generalized Petersen graphs $GP(n, k)$, with $n \leq 40$, which have cop number four.

### 5. Determining the Exact Cop Number of $GP(n, k)$

The graphs below have cop number 2.

- $GP(6, 2)$: choose two antipodal vertices on the outer rim, i.e. the induced subgraph on $A$. Up to symmetry there is only one safe starting position for $R$. Then the cops can move to block all moves of $R$ with their next move.
- $GP(8, 2)$: choose two antipodal vertices on the outer rim. Up to symmetry, there are three safe starting positions for $R$. A case-by-case analysis establishes that two cops suffice.
- $GP(n, 1)$: place on cop on a vertex of the outer rim and place the other cop on the adjacent vertex of the inner rim. Both cops move in opposite directions around their respective cycles.
- $GP(9, 3)$ and $GP(12, 3)$ have cop number 2; this has been verified with a computer.

The only other candidates for having cop number two are those of the form $GP(3k, k)$ or $GP(4k, k)$, where $k \geq 2$. It has been verified with a computer that $c(GP(12, 4)) = 3$ and $c(GP(16, 4)) = 3$.

We have verified with a computer that three cops do not suffice for each of the graphs in Figure 5; hence, by Theorem 3.1 these graphs have cop number four.

**Theorem 5.1.** The cop number of $GP(n, 3)$ is less than or equal to three.

*Proof.* Guard the isometric tree in $GP(\infty, 3)$ having vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. This disconnects $GP(\infty, 3)$. One cop in $GP(n, 3)$ plays the lifted strategy of guarding the isometric tree in $GP(\infty, 3)$. The other two cops play the lifted strategy of pushing the robber to the right (or left).
6. I-GRAPHS

In this section, we use a similar lifting strategy to bound the cop number of connected I-graphs. Given \( n \geq 5 \) and \( 0 < j, k < n/2 \), the I-graph, \( I(n, k, j) \), is the graph with vertex set \( \{ a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1} \} \) and having edges of the form \((a_i, a_{i+j}), (a_i, b_i)\) and \((b_i, b_{i+k})\) for each \( i = 1, 2, \ldots, n \) with indices read modulo \( n \). I-graphs, thus, are similar to generalized Petersen graphs except that two parameters define the adjacencies. Examples include \( I(7, 3, 2) \) and \( I(8, 2, 3) \), as shown below.

By first examining the game played on the subset of connected I-graphs for which \( k \) and \( j \) are coprime (and bounding the cop number for this family of graphs) we are able to bound from above the cop number of all connected I-graphs. To do so, we first define an infinite analogue of an I-graph for each \( k, j \in \mathbb{N} \) as follows. Given such \( k \) and \( j \), let \( A = \{ a_i \mid i \in \mathbb{Z} \} \) and \( B = \{ b_i \mid i \in \mathbb{Z} \} \). The infinite graph, \( I(\infty, k, j) \), has vertex set \( A \cup B \) and edges: \((a_i, a_{i+j}), (a_i, b_i)\), \((b_i, b_{i+k})\) for each \( i \in \mathbb{N} \). Then there is a graph homomorphism \( \pi : I(\infty, k, j) \to I(n, k, j) \) given by reducing indices modulo \( n \). In an entirely analogous manner to the lifted game for generalized Petersen graphs, we define the lifted game played on I-graphs and their corresponding cyclic coverings; each move of a pawn in the finite I-graph corresponds to a move for its associated squad in the infinite graph, and vice versa.

We can then show:

**Theorem 6.1.** The cop number of a connected I-graph \( I(n, k, j) \) is less than or equal to 5.

Before proving the main result of the section involving an arbitrary connected I-graph, \( I(n, k, j) \), we reduce the problem to a simpler one, namely with \( \gcd(k, j) = 1 \). We prove the result for the special case and then extend the result to prove our general theorem.

**Theorem 6.2.** The cop number of a connected I-graph \( I(n, k, j) \) with \( k \) and \( j \) coprime is less than or equal to 5.

**Proof.** Let \( \hat{I} = I(\infty, k, j) \) and \( I = I(n, k, j) \) with \( \pi : \hat{I} \to I \) the associated projection map; let \( R \) be the robber player on \( I \). We describe a strategy in which five squads of cops capture one member of \( \pi^{-1}(R) \) playing in \( \hat{I} \); then five cops can play on \( I \), following the projected moves of their corresponding squads in \( \hat{I} \) can capture \( R \) in a finite number of moves on \( I \). That is, the strategy of the five squads in \( \hat{I} \) to capture a single member of \( \pi^{-1}(R) \) will correspond to the five cops’ capture of \( R \) playing on \( I \).

Fix one member, \( \hat{R} \) of \( \pi^{-1}(R) \) and denote five squads \( S_1, \ldots, S_5 \) of cops playing in \( \hat{I} \) with squad leaders \( C_1, \ldots, C_5 \), respectively. Firstly, \( C_1 \) and \( C_2 \) move along
π−1(A), with C1 increasing his index by j with each move, and C2 decreasing his index by j each move. Since k and j are assumed to be coprime, one of C1 and C2 will—in less than \( \max\{k, j\}/2 \) moves—obtain a congruent index, modulo k with R. Without loss of generality, suppose C1 accomplishes this first. On the next move, C1 can move onto the same subgraph as R, and will, on subsequent turns, move so as to maintain an index congruent, modulo k to that of R and to stay on the same subgraph as R. Next, C2 and C3 move along π−1(B), one increasing and the other decreasing index. By symmetric reasoning, one of C2 and C3 can match index, modulo j with that of R. Suppose, without loss of generality, that C2 does so first. Then C2 follows an analogous mirroring strategy as C1. We can repeat this process twice more, so that (up to relabeling squads), C1 and C3 are on vertices which are congruent modulo k with that of R; and C2 and C4 are on vertices congruent modulo j with that of R. Possibly reassigning squad leaders, we can also assume that the indices of C1 and C2 are strictly less than that of R, which is in turn strictly less than the indices of C3 and C4.

Once this positioning is obtained, with every move of R within π−1(A) or π−1(B), three of C1, C2, C3, and C4 can maintain distance to R while maintaining appropriate indices and remaining on the same subgraph as R. Further, one of C1, C2, C3, and C4 can reduce distance to the robber while maintaining an appropriate index and staying on the same subgraph as R. If the robber does not pass on each turn or continually switch subgraphs, he will therefore eventually be captured by one of C1, C2, C3, or C4. Since ̂I is connected, C5 can move on to ensure that the robber does not indefinitely pass or continually switch subgraphs, which forces capture on ̂I and describes a corresponding winning strategy for five cops playing on I.

Boben, Pisanski, and ˇZitnik proved that the I-graph I(n, k, j) is connected if and only if gcd(n, k, j) = 1 [2]. Thus, to obtain Theorem 6.1 we may simply reason as follows. If I(n, k, j) is a connected I-graph, then gcd(n, k, j) = 1. Therefore, in the case for which we have gcd(k, j) > 1, we may select our five lead cops to be in the same connected component of ̂I as our fixed R. Their same strategy of Theorem 2 in the lifted game follows and projects down to a capture on the finite graph I(n, k, j).

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