Choice functions based on sets of strict partial orders: an axiomatic characterisation

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Abstract Methods for choosing from a set of options are often based on a strict partial order on these options, or on a set of such partial orders. I here provide a very general axiomatic characterisation for choice functions of this form. It includes as special cases axiomatic characterisations for choice functions based on (sets of) total orders, (sets of) weak orders, (sets of) coherent lower previsions and (sets of) probability measures.

1 Introduction

A choice function $C$ is a simple mathematical model for representing choices: for any set of options $A$, it returns a subset $C(A) \subseteq A$ of options that are “chosen” from $A$, in the sense that the options in $R(A) := A \setminus C(A)$ are rejected and that the options in $C(A)$ are deemed incomparable (Seidenfeld et al., 2010; Van Camp et al., 2018; De Bock and De Cooman, 2019).

Such a choice function is often derived from a strict partial order $\succ$ on options, by choosing the options that are maximal—or undominated—with respect to this ordering, or equivalently, by rejecting the options that are dominated:

$$C_\succ(A) := \{ u \in A : (\not\exists v \in A) \ v \succ u \}$$

More generally, we can also associate a choice function with any set $O$ of such strict partial orders, by choosing the options in $A$ that are maximal with respect to at least one of the considered orderings:

$$C_O(A) := \bigcup_{\succ \in O} C_\succ(A) = \{ u \in A : (\exists \succ \in O) (\not\exists v \in A) \ v \succ u \}$$
This approach is conservative because $C_O$ will only reject an option if it is rejected—or dominated—with respect to each of the partial orders $\succ$ in $\mathcal{O}$; it can for example be used to represent conservative group decisions, by interpreting every $\succ$ in $\mathcal{O}$ as the preferences of a different group member.

Choice functions of the form $C_\succ$ and $C_O$ appear in various settings, in all sorts of forms and variations, depending on what the options are and which kinds of properties are imposed on the partial orders involved. Maximising expected utility, for example, is a well-known special case of a choice function of the type $C_\succ$, where the options are utility functions and one chooses the option(s) whose expected utility with respect to some given probability measure is highest. If the probability measure involved is only known to belong to a set—for example because different group members assign different probabilities—this naturally extends to a choice function of the type $C_O$.

The first main contribution of this paper is a set of necessary and sufficient conditions for a general choice function $C$ to be of the form $C_\succ$ or $C_O$, for the case where options are elements of a real vector space. More generally, I provide generic necessary and sufficient conditions for the representing orders to satisfy additional properties, provided these properties are expressable in an abstract rule-based form. This leads in particular to representation theorems for choice functions that are based on (sets of) total orders, (sets of) weak orders, (sets of) coherent lower previsions, (sets of) probability measures, and potentially many other types of uncertainty models.

2 Choice functions based on (sets of) proper orderings

Let $\mathcal{V}$ be a real vector space, the elements of which we call options, let $\mathcal{Q}$ be the set of all—possibly infinite—subsets of $\mathcal{V}$, excluding the empty set, and let $\mathcal{Q}_0 := \mathcal{Q} \cup \{\emptyset\}$. A choice function $C: \mathcal{Q} \rightarrow \mathcal{Q}_0$ is a function that, for any option set $A \in \mathcal{Q}$, returns a subset $C(A)$ consisting of the options in $A$ that are chosen, or rather, not rejected. The elements of $C(A)$ are deemed incomparable when it comes to choosing from $A$. We do not exclude the possibility that $C(A) = \emptyset$; this may for example be reasonable if $A$ is an infinite option set whose elements are linearly ordered. Imagine choosing the highest natural number, for example. Every natural number is rejected, yet none can reasonably be chosen.

We are particularly interested in choice functions of the form $C_\succ$ or $C_O$, corresponding to a strict partial order $\succ$ on $\mathcal{V}$ or to a set $\mathcal{O}$ of such orders, respectively. Besides being strict partial orders ($\text{PO}_1$ and $\text{PO}_2$ below), we will also require these orders to be compatible with the vector space operations of the real vector space $\mathcal{V}$ (PO$_3$ and PO$_4$). We will call such orders proper.

**Definition 1** A binary relation $\succ$ on $\mathcal{V}$ is a proper order if, for all $u, v, w \in \mathcal{V}$ and $\lambda \in \mathbb{R}_{>0} := \{\lambda \in \mathbb{R}: \lambda > 0\}$, it satisfies the following properties:
Choice functions based on sets of strict partial orders

PO\(_1\). \(u \not\succ u\) \hspace{1cm} \text{irreflexivity}

PO\(_2\). if \(u \succ v\) and \(v \succ w\), then also \(u \succ w\) \hspace{1cm} \text{transitivity}

PO\(_3\). if \(u \succ v\) then also \(\lambda u \succ \lambda v\)

PO\(_4\). if \(u \succ v\) then also \(u + w \succ v + w\).

We denote the set of all proper orders by \(\mathcal{O}\).

Since PO\(_4\) implies that \(u \succ v\) is equivalent to \(u - v \succ 0\), every proper order \(\succ\) is completely characterised by the set of options

\[
D_{\succ} := \{u \in \mathcal{V}: u \succ 0\}. \tag{3}
\]

In fact, as established in Proposition 1, proper orders are in one-to-one correspondence with convex cones \(D\) in \(\mathcal{V}\) that are blunt, meaning that they do not include 0. We call any such \(D\) a proper set of options.

**Definition 2** A set of options is a—possibly empty—subset \(D\) of \(\mathcal{V}\). It is called proper if it satisfies the following properties:

PD\(_1\). \(0 \notin D\)

PD\(_2\). if \(u \in D\) and \(v \in D\), then \(u + v \in D\)

PD\(_3\). if \(u \in D\) and \(\lambda \in \mathbb{R}_{>0}\), then \(\lambda u \in D\).

We denote the set of all proper sets of options by \(\mathcal{D}\).

**Proposition 1** A binary relation \(\succ\) on \(\mathcal{V}\) is a proper order if and only if there is a proper set of options \(D\) such that

\[
u \succ v \iff u - v \in D \text{ for all } u, v \in \mathcal{V}. \tag{4}\]

This \(D\) is then necessarily unique and equal to \(D_{\succ}\).

It follows that choice functions of the form \(C_{\succ}\) are completely determined by a single proper set of options \(D\), whereas choice functions of the form \(C_{\mathcal{O}}\) are characterised by a set \(\mathcal{D}\) of such proper sets of options.

### 3 Imposing additional properties through sets of rules

In most cases, rather than consider (sets of) arbitrary proper orders, one wishes to consider particular types of such orders, by imposing additional properties besides PO\(_1\)–PO\(_4\). Rather than treat each of these types separately, we will instead consider an abstract axiom that includes a variety of them as special cases. We will characterise this axiom by means of a set of rules. Each such rule is a pair \((A,B)\), with \(A \subseteq Q\) and \(B \in \mathcal{Q}_\emptyset\). A set of rules \(\mathcal{R}\), therefore, is a subset of \(\mathcal{P}(Q) \times \mathcal{Q}_\emptyset\), with \(\mathcal{P}(Q)\) the powerset of \(Q\).

**Definition 3** Let \(\mathcal{R}\) be a set of rules. For any proper set of options \(D\), we then say that \(D\) is \(\mathcal{R}\)-compatible if for all \((A,B)\) \(\in\) \(\mathcal{R}\):
PD_R, if \( A \cap D \neq \emptyset \) for all \( A \in \mathcal{A} \), then also \( B \cap D \neq \emptyset \).

A proper order \( \succ \) is called \( \mathcal{R} \)-compatible if \( D_\succ \) is \( \mathcal{R} \)-compatible. We let \( D_\mathcal{R} \) and \( \mathcal{O}_\mathcal{R} \) be the set of all \( \mathcal{R} \)-compatible proper sets of options and orders, respectively.

A first important special case are (sets of) rules of the type \( (\emptyset, B) \). Since \( \mathcal{A} = \emptyset \) makes the premise of PD_R trivially true, such a rule allows one to express that \( B \) should contain at least one option \( u \) such that \( u \succ 0 \). If \( \mathcal{V} \) is the set \( \mathcal{L}(\mathcal{X}) \) of all gambles—bounded real functions—on some state space \( \mathcal{X} \), coherence of \( D_\succ \) for example corresponds to the set of rules

\[
\mathcal{R}_C := \{ (\emptyset, \{ u \} ) : u \in \mathcal{V}_{>0} \},
\]

with \( \mathcal{V}_{>0} \) equal to \( \{ u \in \mathcal{V} : \inf u > 0 \} \) or \( \{ u \in \mathcal{V} : \inf u \geq 0, u \neq 0 \} \), depending on the specific type of coherence (Quaeghebeur, 2014). In both cases, this set of rules imposes that \( \mathcal{V}_{>0} \) should be a subset of \( D_\succ \). A second example are total orders. These correspond to the set of rules

\[
\mathcal{R}_T := \{ (\emptyset, \{ u, -u \} ) : u \in \mathcal{V} \setminus \{ 0 \} \},
\]

which imposes that \( u \succ 0 \) or \( -u \succ 0 \) for all \( u \neq 0 \). Given the properness of \( \succ \), this is equivalent to the totality of \( \succ \), meaning that \( u \succ v \) or \( v \succ u \) for all \( u, v \in \mathcal{V} \) such that \( u \neq v \).

Another special case are rules of the type \( (A, \emptyset) \). Since \( B \cap D \neq \emptyset \) cannot be true for \( B = \emptyset \), rules of this type allow one to express that there should be at least one option set \( A \in \mathcal{A} \) such that \( u \not\succ 0 \) for all \( u \in A \). If we for example want to enforce that \( u_i \not\succ v_i \) for all \( i \) in some index set \( I \), it suffices to let \( A = \{ A \} \), with \( A = \{ u_i - v_i : i \in I \} \).

Besides total orders, we can also obtain weak orders as a special case of \( \mathcal{R} \)-compatibility. This can be achieved by imposing any of the following three equivalent sets of rules:

\[
\begin{align*}
\mathcal{R}_W := & \{ (\{ u + v, -u - v \}, \{ u, -u, v, -v \} ) : u, v \in \mathcal{V} \} \\
\mathcal{R}_{W2} := & \{ (\{ u + v \}, \{ u, v \} ) : u, v \in \mathcal{V} \} \\
\mathcal{R}_M := & \{ (\text{posi}(B), B) : B \in \mathcal{Q} \text{ and } |B| < +\infty \},
\end{align*}
\]

with \( \text{posi}(B) := \{ \sum_{i=1}^n \lambda_i u_i : n \in \mathbb{N}, \lambda_i > 0, u_i \in B \} \) the set of finite positive linear combinations of options in \( B \). The first set of rules, \( \mathcal{R}_W \), is most closely related to the defining property that makes a strict partial order weak, which is that the incomparability relation—\( u \not\succ v \) and \( v \not\succ u \)—should be transitive. The second set of rules, \( \mathcal{R}_{W2} \), is the simplest yet perhaps least intuitive of the three. The third set of rules, \( \mathcal{R}_M \), corresponds to the so-called mixingness of \( D_\succ \) (De Bock and De Cooman, 2019); it also shows that a strict partial order \( \succ \) is weak if and only if \( D_\succ \) is lexicographic (Van Camp et al., 2018), in the sense that its complement \( D_\succ \) is a convex cone. The equivalence of
these three rules, as well as the fact that they indeed correspond to \( \succ \) being a weak order, follows from the properness of \( \succ \).

The notion of \( \mathcal{R} \)-compatibility can also be used to impose several properties at once; it suffices to let \( \mathcal{R} \) be the union of the sets of rules that correspond to the individual properties.

For example, if \( V = \mathcal{L}(X) \) and \( V_{\succ 0} = \{ u \in V : \inf u > 0 \} \), then partial orders that correspond to coherent lower previsions can be obtained by combining coherence with Archimedeanity (De Bock and De Cooman, 2019). Coherence, as explained above, corresponds to \( \mathcal{R}_{\text{C}} \)-compatibility. Archimedeanity of \( D_{\succ} \), on the other hand, corresponds to the set of rules

\[
\mathcal{R}_A := \left\{ \left( \{ \{ u \} \}, \{ u - \epsilon : \epsilon \in \mathbb{R}_{>0} \} \right) : u \in V \right\},
\]

(6)

where we identify \( \epsilon \in \mathbb{R}_{>0} \) with the constant gamble on \( X \) with value \( \epsilon \). To impose both properties together, it suffices to consider the single set of rules \( \mathcal{R}_{\text{CA}} := \mathcal{R}_{\text{C}} \cup \mathcal{R}_A \).

If besides coherence and Archimedeanity of \( D_{\succ} \), we additionally impose that \( \succ \) should be a weak order—or equivalently, that \( D_{\succ} \) should be mixing—the representing lower previsions become linear expectations, resulting in orders that correspond to maximising expected utility with respect to a finitely additive probability measure (De Bock and De Cooman, 2019). This can for example be achieved by imposing \( \mathcal{R}_{\text{CAM}} := \mathcal{R}_{\text{C}} \cup \mathcal{R}_A \cup \mathcal{R}_M \), where instead of \( \mathcal{R}_M \), we could have also used \( \mathcal{R}_W \) or \( \mathcal{R}_{W2} \).

As a final example, orders that are evenly continuous (Cozman, 2018) can be obtained by imposing \( \mathcal{R}_{\text{CE}} := \mathcal{R}_{\text{C}} \cup \mathcal{R}_E \), with

\[
\mathcal{R}_E := \left\{ \left( \{ \{ u_i \} : i \in \mathbb{N} \} \cup \{ \lim_{i \to \infty} (\lambda_i v - u_i) \} \right), \{ v \} \right) : u_i, v \in V, \lambda_i \in \mathbb{R}_{>0} \right\}.
\]

### 4 Proper choice functions

In order to motivate the use of choice functions of the form \( C_{\succ} \) and \( C_{\text{O}} \), without directly assuming that they must be of this type, we now proceed to provide an axiomatic characterisation for such types of choice functions. We start by introducing the notion of a proper choice function. As we will see in Theorem 4, these are exactly the choice functions of the form \( C_{\text{O}} \).

To state the defining axioms of a proper choice function, we require some additional notation. First, for any \( A \in \mathcal{Q} \) and \( w \in V \), we let \( A + w := \{ u + w : u \in A \} \), and similarly for \( A - w \). One of the properties of a proper choice function—see PC\(_1\) further on—will be that rejecting \( u \) from \( A \) is equivalent to rejecting \( 0 \) from \( A - u \). Similarly to how a proper order \( \succ \) is completely determined by \( D_{\succ} \), a proper choice function \( C \) will therefore be completely determined by the sets from which it rejects zero. In particular, as we will see in Proposition 2, the role of \( D_{\succ} \) is now taken up by
Next, for any \( A \subseteq \mathcal{Q} \), we denote by \( \Phi_A \) the collection of all maps \( \phi : A \to \mathcal{V} \) that, for all \( A \in \mathcal{A} \), select a single option \( \phi(A) \in A \). Furthermore, for any such selection map \( \phi \in \Phi_A \), we let \( \phi(A) := \{ \phi(A) : A \in \mathcal{A} \} \) be the corresponding set of selected options. If \( A \neq \emptyset \), we also let \( \Lambda_A \) be the set of all maps \( \lambda : A \to \mathbb{R}_{\geq 0} \) with finite non-empty support, so \( \lambda(A) > 0 \) for finitely many—but at least one—\( A \) in \( \mathcal{A} \) and \( \lambda(A) = 0 \) otherwise. Properness for choice functions is now defined as follows.

**Definition 4** A choice function \( C \) is proper if it satisfies the following properties:

- **PC0.** \( C(\{u\}) = \{u\} \) for all \( u \in \mathcal{V} \)
- **PC1.** if \( u \in C(A) \), then also \( u + w \in C(A + w) \), for all \( A \in \mathcal{Q} \) and \( u, w \in \mathcal{V} \)
- **PC2.** if \( \emptyset \neq A \subseteq K_C \) and if, for all \( \phi \in \Phi_A \), \( \lambda_\phi \in \Lambda_A \), then also
  \[
  \{ \sum_{A \in A} \lambda_\phi(A) \phi(A) : \phi \in \Phi_A \} \in K_C;
  \]
- **PC3.** if \( u \in A \subseteq B \) and \( u \in C(B) \), then also \( u \in C(A) \), for all \( A, B \in \mathcal{Q} \).

Each of these axioms can—but need not—be motivated by interpreting \( u \in C(A) \) as ‘there is no \( v \) in \( A \) that is better than \( u \)’, where the ‘better than’ relation satisfies the defining properties of a proper order. For PC0, PC1, and PC3, this should be intuitively clear. For PC2, the starting point is that for any \( A \in K_C \), there is at least one \( v \in A \) that is better than zero. Hence, for any \( A \subseteq K \), there is at least one \( \phi \in \Phi_A \) such that every element of \( \phi(A) \) is better than zero. Combined with PO2–PO4, or actually, PD2 and PD3, it follows that \( \sum_{A \in A} \lambda_\phi(A) \phi(A) \) should be better than zero as well.

Every proper choice function \( C \) is completely determined by a so-called proper set of option sets \( K \), which is furthermore unique and equal to \( K_C \).

**Proposition 2** A choice function \( C \) is proper if and only if there is a proper set of option sets \( K \) such that

\[
C(A) = \{ u \in A : A - u \not\in K \} \text{ for all } A \in \mathcal{Q}.
\]  

This \( K \) is then necessarily unique and equal to \( K_C \).
Hence, proper choice functions correspond to proper sets of option sets $K$. On the other hand, we know from Proposition 1 that proper orders correspond to proper sets of options $D$. Relating proper choice functions with proper orders, therefore, amounts to relating proper sets of option sets $K$ with (sets of) proper sets of options $D$. The first step consists in associating with any set of options $D$ a set of option sets

$$K_D := \{ A \in \mathcal{Q} : A \cap D \neq \emptyset \}. \quad (9)$$

It is quite straightforward to show that $D$ is proper if and only if $K_D$ is.

**Proposition 3** Consider a set of options $D$ and its corresponding set of option sets $K_D$. Then $D$ is proper if and only if $K_D$ is.

What is perhaps more surprising though is that every proper set of option sets $K$ corresponds to a set $D$ of proper sets of options $D$.

**Theorem 1** A set of option sets $K$ is proper if and only if there is a non-empty set $D \subseteq \mathcal{D}$ of proper sets of options such that $K = \bigcap \{K_D : D \in \mathcal{D}\}$. The largest such set $\mathcal{D}$ is then $\mathcal{D}(K) := \{ D \in \mathcal{D} : K \subseteq K_D \}$.

To additionally guarantee that the option sets in $\mathcal{D}$ are $\mathcal{R}$-compatible with a given set of rules $\mathcal{R}$, we introduce a notion of $\mathcal{R}$-compatibility for proper sets of option sets and choice functions.

**Definition 6** Let $\mathcal{R}$ be a set of rules. For any proper set of option sets $K$, we then say that $K$ is $\mathcal{R}$-compatible if for all $(A, B) \in \mathcal{R}$:

PK$_\mathcal{R}$, if $A \in K$ for all $A \in \mathcal{A}$, then also $B \in K$.

A proper choice function $C$ is called $\mathcal{R}$-compatible if $K_C$ is $\mathcal{R}$-compatible.

Clearly, a proper set of options $D$ is $\mathcal{R}$-compatible if and only if $K_D$ is. What is far less obvious though, is that for sets of rules $\mathcal{R}$ that are monotone a proper set of option sets $K$ is $\mathcal{R}$-compatible if and only if it corresponds to a set $\mathcal{D}$ of proper $\mathcal{R}$-compatible sets of options $D$.

**Definition 7** A set of rules $\mathcal{R}$ is monotone if for all $M \in \mathcal{Q}$:

$$(A, B) \in \mathcal{R} \text{ and } A \neq \emptyset \Rightarrow ((A \cup M : A \in A), B \cup M) \in \mathcal{R}. $$

**Theorem 2** Let $\mathcal{R}$ be a monotone set of rules. A set of option sets $K$ is then proper and $\mathcal{R}$-compatible if and only if there is a non-empty set $\mathcal{D} \subseteq \mathcal{D}_\mathcal{R}$ of proper $\mathcal{R}$-compatible sets of options such that $K = \bigcap \{K_D : D \in \mathcal{D}\}$. The largest such set $\mathcal{D}$ is then $\mathcal{D}_\mathcal{R}(K) := \{ D \in \mathcal{D}_\mathcal{R} : K \subseteq K_D \}$.

In practice, however, most sets of rules that one would like to impose on the representing sets of options are not monotone. From all the sets of rules that we considered in Section 3, for example, the only monotone ones are $\mathcal{R}_C$ and $\mathcal{R}_T$; trivially, in fact, since $\mathcal{A} = \emptyset$ for the rules in these sets.
Fortunately, a characterisation similar to Theorem 2 can also be obtained for sets of rules \( \mathcal{R} \) that are not monotone. It suffices to replace \( \mathcal{R} \) with its monotonification 
\[
\text{mon}(\mathcal{R}) := \mathcal{R} \cup \left\{ \left( \{ A \cup M : A \in \mathcal{A} \}, B \cup M \right) : (A, B) \in \mathcal{R}, A \neq \emptyset, M \in \mathcal{Q} \right\},
\]
which is the smallest monotone set of rules that includes \( \mathcal{R} \). This still leads to a representation in terms of proper \( \mathcal{R} \)-compatible sets of options because \( \mathcal{R} \)- and mon(\( \mathcal{R} \))-compatibility are equivalent for sets of options.

**Proposition 4** Consider a set of rules \( \mathcal{R} \) and a proper set of options \( D \). Then \( D \) is \( \mathcal{R} \)-compatible if and only if it is \( \text{mon}(\mathcal{R}) \)-compatible.

**Theorem 3** Let \( \mathcal{R} \) be a set of rules. A set of option sets \( \mathcal{K} \) is then proper and \( \text{mon}(\mathcal{R}) \)-compatible if and only if there is a non-empty set \( D \subseteq \mathcal{D} \) of proper \( \mathcal{R} \)-compatible sets of options such that \( \mathcal{K} = \bigcap \{ K_D : D \in \mathcal{D} \} \). The largest such set \( \mathcal{D} \) is then \( \mathcal{D}_\mathcal{R}(\mathcal{K}) := \{ D \in \mathcal{D} : \mathcal{K} \subseteq K_D \} \).

Putting the pieces together—combining Theorem 3 with Propositions 1 and 2—we arrive at our main result: an axiomatic characterisation for choice functions of the form \( C_\mathcal{O} \), with \( \mathcal{O} \) a set of \( \mathcal{R} \)-compatible proper orders.

**Theorem 4** Let \( \mathcal{R} \) be a set of rules. Then a choice function \( C \) is proper and \( \text{mon}(\mathcal{R}) \)-compatible if and only if there is a non-empty set \( \mathcal{O} \subseteq \mathcal{O}_\mathcal{R} \) of \( \mathcal{R} \)-compatible proper orders such that \( C = C_\mathcal{O} \). The largest such set is then 
\[
\mathcal{O}_\mathcal{R}(C) := \{ \succ \in \mathcal{O}_\mathcal{R} : C_\succ(A) \subseteq C(A) \text{ for all } A \in \mathcal{Q} \}.
\]
A similar result holds without \( \mathcal{R} \)-compatibility as well. It corresponds to the special case \( \mathcal{R} = \text{mon}(\mathcal{R}) = \emptyset \), for which \( \mathcal{R} \)- and mon(\( \mathcal{R} \))-compatibility are trivially satisfied.

In order to obtain an axiomatic characterisation for choice functions of the form \( C_\succ \), with \( \succ \) a single \( \mathcal{R} \)-compatible proper order, we additionally impose that \( C \) is completely determined by its pairwise choices. Replacing \( \mathcal{R} \) by mon(\( \mathcal{R} \)) is not needed in this case.

**Definition 8** A choice function \( C \) is *binary* if for all \( A \in \mathcal{Q} \) and \( u \in A \):
\[
u \in C(A) \iff (\forall v \in A \setminus \{u\}) u \in C(\{u, v\})
\]

**Theorem 5** Let \( \mathcal{R} \) be a set of rules. Then a choice function \( C \) is proper, binary and \( \mathcal{R} \)-compatible if and only if there is an \( \mathcal{R} \)-compatible proper order \( \succ \) such that \( C = C_\succ \). This order is unique and characterised by
\[
v \succ u \iff u \notin C(\{u, v\}) \text{ for all } u, v \in \mathcal{V}.
\]
and it furthermore satisfies \( K_C = K_{D_\succ} \).

Here too, a version without \( \mathcal{R} \)-compatibility is easily obtained by setting \( \mathcal{R} = \text{mon}(\mathcal{R}) = \emptyset \).
5 Conclusions and future work

The main conclusion of this paper is that decision making based on (sets of) strict partial orders is completely characterised by specific properties of the resulting choice functions. That is, any choice function $C$ that satisfies these properties is guaranteed to correspond to a (set of) strict partial order(s). By additionally imposing a set of rules on $C$, we can furthermore guarantee that the representing orders are of a particular type. As discussed in Section 3, this includes total orders, weak orders, orders based on coherent lower previsions, orders based on probability measures (such as maximising expected utility) and evenly continuous orders, depending on the set of rules that is imposed.

In future work, I intend to study if there are other types of orders that correspond to a set of rules, as well as explain in more detail why the types of orders and sets of rules in Section 3 indeed correspond to one another. I also intend to further elaborate on the implications of Theorems 4 and 5, demonstrate how they can be applied in various contexts, and establish connections with earlier axiomatic characterisations for decision methods that are based on (sets of) orders (Savage, 1972; Nau, 1992; Seidenfeld et al., 1995; Nau, 2006; Seidenfeld et al., 2010; De Bock and De Cooman, 2019; De Bock, 2020).

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Proofs of our main results

**Proof of Proposition 1** For the ‘if’-part, we consider a binary relation $\succ$ on $\mathcal{V}$ and a proper set of options $D$ that satisfies Equation (4). That $D$ is equal to $D_\succ$ (and is hence uniquely determined by $\succ$) follows from the fact that, for any $u \in \mathcal{V}$,

$$u \in D_\succ \iff u \succ 0 \iff u - 0 \in D \iff u \in D,$$

using Equation (4) for the second equivalence. That $\succ$ satisfies PO$_4$ follows directly from Equation (4). That $\succ$ satisfies PO$_1$, PO$_2$ and PO$_3$ follows directly from Equation (4) and the fact that $D$ satisfies PD$_1$, PD$_2$ and PD$_3$, respectively.

For the ‘only if’-part, we consider a binary relation $\succ$ on $\mathcal{V}$ that is a proper order and prove that there is a proper set of options $D$ that satisfies Equation (4). In particular, we will prove that this is the case for $D = D_\succ$. That $D_\succ$ satisfies Equation (4) follows from the fact that, for any $u, v \in \mathcal{V}$:

$$u \succ v \iff u - v \succ 0 \iff u - v \in D_\succ,$$

where the first equivalence follows from PO$_4$ (with $w = -v$ and $w = v$). That $D_\succ$ satisfies PD$_1$ and PD$_3$ follows directly from Equation (3) and the fact that $\succ$ satisfies PO$_1$ and PO$_3$, respectively. It remains to show that $D_\succ$ satisfies PD$_2$. So consider any $u$ and $v$ in $D_\succ$, implying that $u \succ 0$ and $v \succ 0$. Since $u \succ 0$, PO$_4$ implies that also $u + v \succ v$. Since $u + v \succ v$ and $v \succ 0$, PO$_2$ implies that $u + v \succ 0$ and therefore, that $u + v \in D_\succ$, as desired. $\square$

**Proof of Proposition 2** For the ‘if’-part, we consider a choice function $C$ and a proper set of option sets $K$ that satisfies Equation (8). To see why $K$ is equal to $K_C$ (and is hence uniquely determined by $C$), first observe that for any $A \in \mathcal{Q}$,

$$A \in K_C \iff 0 \notin C(A \cup \{0\}) \iff (A \cup \{0\}) - 0 \in K \iff A \cup \{0\} \in K \iff A \in K,$$
Proof of Proposition 3

Consider any set of options \( D \) and let \( K_D \) be its corresponding set of option sets.

For the 'only if'-part of the statement, we assume that \( D \) is proper and prove that \( K_D \) is then proper as well. That \( K_D \) satisfies \( \text{PK}_0 \) follows from Equation (9) and the fact that \( \emptyset \notin \mathcal{Q} \). That \( K_D \) satisfies \( \text{PK}_1 \) follows from Equation (9) and \( \text{PD}_1 \). That \( K_D \) satisfies \( \text{PK}_3 \) is immediate from Equa-
tion (9). To see that $K_D$ also satisfies PK$_2$, we consider any $\emptyset \neq A \subseteq K_D$
and, for all $\phi \in \Phi_A$, some $\lambda_\phi \in \Lambda_A$. For any $A \in \mathcal{A}$, since $A \in K_D$, there
is some $u^* \in A \cap D$, which we denote by $\phi^*(A)$. For the resulting map $\phi^* \in \Phi_A$, we have that $\phi^*(A) \in D$ for all $A \in \mathcal{A}$, and therefore, because of PD$_2$ and PD$_3$, that $\sum_{A \in \mathcal{A}} \lambda_\phi (A) \phi^*(A) \in D$. Hence,
\[
\{ \sum_{A \in \mathcal{A}} \lambda_\phi (A) \phi(A) : \phi \in \Phi_A \} \in K_D.
\]

For the ‘if’-part of the statement, we assume that $K_D$ is proper and prove that $D$ is then proper as well. To prove that $D$ satisfies PD$_1$, we assume *ex absurdo* that $0 \in D$. This implies that $\{0\} \in K_D$ and therefore, because of PK$_1$, that $0 \in K_D$, contradicting PK$_0$. To prove that $D$ satisfies PD$_2$, we consider any $u, v \in D$. It then follows from Equation (9) that $\{u\} \in K_D$ and $\{v\} \in K_D$. Now let $\mathcal{A} = \{\{u\} , \{v\}\}$. Then $\emptyset \neq \mathcal{A} \subseteq K_D$ and $\Phi_A$ contains only a single function $\phi^*$, defined by $\phi^*(\{u\}) = u$ and $\phi^*(\{v\}) = v$. Let $\lambda_{\phi^*} \in \Lambda_A$ be defined by $\lambda_{\phi^*}((\{u\}) = \lambda_{\phi^*}((\{v\}) := 1$. It then follows from PK$_2$ that
\[
\{ u + v \} = \{ \lambda_{\phi^*}((\{u\}) \phi^*((\{u\}) + \lambda_{\phi^*}((\{v\}) \phi^*((\{v\}))
\] = \{ \sum_{A \in \mathcal{A}} \lambda_{\phi^*} (A) \phi(A) \} = \{ \sum_{A \in \mathcal{A}} \lambda_\phi (A) \phi(A) : \phi \in \Phi_A \} \in K_D,
\]
or equivalently, that $u + v \in D$. The proof for PD$_3$ is similar. Consider any $u \in D$ and $\lambda \in \mathbb{R}_{\geq 0}$. It then follows from Equation (9) that $\{u\} \in K_D$. Now let $\mathcal{A} = \{\{u\}\}$. Then $\emptyset \neq \mathcal{A} \subseteq K_D$ and $\Phi_A$ contains only a single function $\phi^*$, defined by $\phi^*(\{u\}) = u$. Let $\lambda_{\phi^*} \in \Lambda_A$ be defined by $\lambda_{\phi^*}((\{u\}) := \lambda$. It then follows from PK$_2$ that
\[
\{ \lambda u \} = \{ \lambda_{\phi^*}((\{u\}) \phi^*((\{u\})
\] = \{ \sum_{A \in \mathcal{A}} \lambda_{\phi^*} (A) \phi(A) \} = \{ \sum_{A \in \mathcal{A}} \lambda_\phi (A) \phi(A) : \phi \in \Phi_A \} \in K_D,
\]
or equivalently, that $\lambda u \in D$. \hfill $\Box$

**Lemma 1** Consider any non-empty set $K$ of proper sets of option sets. Then $K_K := \bigcap \{K : K \in K\}$ is a proper set of options sets as well.

**Proof** Observe that for any $A \in \mathcal{Q}_0$, since $K$ is non-empty, we have that $A \in K_K$ if and only $A \in K$ for all $K \in K$. Given this observation, the statement follows directly from Definition 5. For example, for every $K \in K$, we know from PK$_0$ that $\emptyset \notin K$. Hence, $\emptyset \notin K_K$, so $K_K$ satisfies PK$_0$. As a second example, if $A \in K_K$, then $A \in K$ for all $K \in K$. Since every $K \in K$ is proper, it follows from PK$_1$ that $A \setminus \{0\} \in K$ for all $K \in K$. Hence, $A \setminus \{0\} \in K_K$. We conclude that $K_K$ satisfies PK$_1$. The proof for PK$_2$ and PK$_3$ is completely analogous. \hfill $\Box$

**Proof of Theorem 1** For the ‘if’-part, consider any non-empty set $\mathcal{D} \subseteq \mathcal{D}$ of proper sets of options such that $K = \bigcap \{K_D : D \in \mathcal{D}\}$. Then for any $D \in \mathcal{D}$, $K_D$ is a proper set of option sets because of Proposition 3. Since $\mathcal{D}$
is non-empty, it therefore follows from Lemma 1 that $K = \bigcap \{K_D : D \in \mathcal{D}\}$ is proper set of option sets as well.

For the ‘only if’ part, we assume that $K$ is a proper set of option sets. We need to prove that there is a non-empty set $\mathcal{D} \subseteq \mathcal{D}$ of proper sets of options such that $K = \bigcap \{K_D : D \in \mathcal{D}\}$, and that $\mathcal{D}(K)$ is the largest such set. Clearly, if there is a set $\mathcal{D} \subseteq \mathcal{D}$ such that $K = \bigcap \{K_D : D \in \mathcal{D}\}$, then $\mathcal{D}$ must be a subset of $\mathcal{D}(K)$. It therefore suffices to prove the statement for the particular set $\mathcal{D} := \mathcal{D}(K)$. That is, we will prove that $\mathcal{D}(K)$ is non-empty and that $K = \bigcap \{K_D : D \in \mathcal{D}(K)\}$. To that end, we will prove that for any $A \in \mathcal{Q}_\emptyset$ such that $A \notin K$, there is a set of options $D \in \mathcal{D}(K)$ such that $A \notin K_D$. On the one hand, this implies that $\bigcap \{K_D : D \in \mathcal{D}(K)\} \subseteq K$ and therefore, since the converse set inclusion holds trivially, that $K = \bigcap \{K_D : D \in \mathcal{D}(K)\}$. On the other hand, since it follows from PK$_\emptyset$ that there is at least one option set $A \in \mathcal{Q}_\emptyset$ such that $A \notin K$—in particular, $A = \emptyset$—it also implies that $\mathcal{D}(K)$ is non-empty.

So consider any $A \in \mathcal{Q}_\emptyset$ such that $A \notin K$. We need to prove that there is some $D \in \mathcal{D}(K)$ such that $A \notin K_D$. We first consider the case $K = \emptyset$. In that case, let $D = \emptyset$. It then follows from Equation (9) that $K_D = \emptyset$, which implies that $K \subseteq K_D$ and $A \notin K_D$. Hence, since $D = \emptyset$ is trivially a proper set of options, we conclude that $D \in \mathcal{D}(K)$ and $A \notin K_D$. It remains to consider the case $K \neq \emptyset$. In that case, assume ex absurdum that $A \in K_D$ for every $D \in \mathcal{D}(K)$. We will show that this leads to a contradiction.

Consider any $\phi \in \Phi_K$ and let

$$D_\phi := \{ \sum_{B \in K} \lambda(B) \phi(B) : \lambda \in A_K \}$$

(11)

be the set of all finite positive linear combinations of options in $\phi(K)$. Then for any $B \in K$, we have that $\phi(B) \in D_\phi$—just let $\lambda(B) := 1$ and $\lambda(B') := 0$ for all $B' \in K \setminus \{B\}$—and therefore, since $\phi(B) \in B$, also that $B \in K_{D_\phi}$. Hence, $K \subseteq K_{D_\phi}$. Furthermore, by construction, $D_\phi$ clearly satisfies PD$_3$ and PD$_2$.

We now consider two cases: $0 \notin D_\phi$ and $0 \in D_\phi$. If $0 \notin D_\phi$, then $D_\phi$ satisfies PD$_1$. Since it also satisfies PD$_3$ and PD$_2$, we therefore find that $D_\phi$ is a proper set of options. Since $K \subseteq K_{D_\phi}$, this implies that $D_\phi \in \mathcal{D}(K)$. By our assumption, it then follows that $A \in K_{D_\phi}$, implying that $A \cap D_\phi \neq \emptyset$ and therefore also $(A \cup \{0\}) \cap D_\phi \neq \emptyset$. If $0 \in D_\phi$, then $0 \in (A \cup \{0\}) \cap D_\phi$, implying once more that $(A \cup \{0\}) \cap D_\phi \neq \emptyset$. Hence, in both cases, we find that $(A \cup \{0\}) \cap D_\phi \neq \emptyset$. Now let $u_\phi$ be any element of $(A \cup \{0\}) \cap D_\phi$. Since $u_\phi \in D_\phi$, it follows from Equation (11) that there is some $\lambda_\phi \in A_K$ such that $\sum_{B \in K} \lambda_\phi(B) \phi(B) = u_\phi \in A \cup \{0\}$.

In summary then, for any $\phi \in \Phi_K$, we have found some $\lambda_\phi \in A_K$ such that $\sum_{B \in K} \lambda_\phi(B) \phi(B) \in A \cup \{0\}$. Now let $A^* := \{ \sum_{B \in K} \lambda_\phi(B) \phi(B) : \phi \in \Phi_K \}$. Since $K \neq \emptyset$, it then follows from PK$_2$ that $A^* \in K$. Since $A^*$ is by construction a subset of $A \cup \{0\}$, PK$_3$ therefore implies that $A \cup \{0\} \in K$. 
This is however impossible because $A \notin K$. If $0 \in A$, this contradiction is trivial. If $0 \notin A$, this contradiction follows from $PK_1$. \hfill \Box

**Lemma 2** Consider any non-empty set $D$ of proper sets of options. Then $D_D := \bigcap \{D : D \in D\}$ is a proper set of options as well.

**Proof** Observe that for any $u \in V$, since $D$ is non-empty, we have that $u \in D_D$ if and only if $u \in D$ for all $D \in D$. Given this observation, the statement follows directly from Definition 2. \hfill \Box

**Proof of Theorem 2** For the ‘if’-part of the statement, we assume that there is a non-empty set $D \subseteq D_R$ of proper $R$-compatible sets of options such that $K = \bigcap \{K_D : D \in D\}$. It then follows from Theorem 1 that $K$ is a proper set of option sets. To show that it is also $R$-compatible, we consider any $(A, B) \in R$ such that $A \in K$ for all $A \in A$. Fix any $D \in D$. Then for all $A \in A$, since $A \in K \subseteq K_D$, we know that $A \cap D \neq \emptyset$. Since $D$ is $R$-compatible, this implies that $B \cap D \neq \emptyset$, which in turn implies that $B \in K_D$. Since this is true for every $D \in D$, we find that $B \in \bigcap \{K_D : D \in D\} = K$. Hence, $K$ is $R$-compatible.

For the ‘only if’-part, we assume that $K$ is a proper $R$-compatible set of option sets. We need to prove that there is a non-empty set $D \subseteq D_R$ of proper $R$-compatible sets of options such that $K = \bigcap \{K_D : D \in D\}$, and that $D_R(K)$ is the largest such set. Clearly, if there is a set $D \subseteq D_R$ such that $K = \bigcap \{K_D : D \in D\}$, then $D$ must be a subset of $D_R(K)$. That is, we will prove that $D_R(K)$ is non-empty and that $K = \bigcap \{K_D : D \in D_R(K)\}$. To that end, we will prove that for any $A \in Q_0$ such that $A \notin K$, there is a set of options $D_A \subseteq D_R(K)$ such that $A \notin K_{D_A}$. On the one hand, this implies that $\bigcap \{K_D : D \in D_R(K)\} \subseteq K$ and therefore, since the converse set inclusion holds trivially, that $K = \bigcap \{K_D : D \in D_R(K)\}$. On the other hand, since it follows from $PK_0$ that there is at least one option set $A \in Q_0$ such that $A \notin K$—in particular, $A = \emptyset$—it also implies that $D_R(K)$ is non-empty. So consider any $A \in Q_0$ such that $A \notin K$.

Since $K$ is proper, Theorem 1 implies that there is a non-empty set $D$ of proper sets of options such that $K = \bigcap \{K_D : D \in D\}$. Since $A \notin K$, this implies that there must be some $D \in D$ such that $K \subseteq K_D$ and $A \notin K_D$. Let $D$ be the set of all proper sets of options and consider the set $D_D := \{D \in D : K \subseteq K_D \subseteq K_D\}$, partially ordered by set inclusion. In particular, for any two $D_1, D_2 \in D_D$, we say that $D_2$ dominates $D_1$, denoted by $D_1 \triangleleft D_2$, if $D_2 \subseteq D_1$; so subsets dominate their supersets. We will use Zorn’s lemma to prove that this partially ordered set has a maximal—undominated—element. To do so, we must show that $D_D$ is non-empty and that any chain in $D_D$—any completely ordered subset of $D_D$—has an upper bound in $D_D$. That $D_D$ is non-empty follows from the fact that it contains $D$. So consider any chain $(D_i)_{i \in I}$ in $D_D$. Since $D_D$ is a non-empty set of proper sets of options, it follows from Lemma 2 that the intersection $D_I := \bigcap_{i \in I} D_i$ is
a proper set of options as well. Furthermore, since $K \subseteq K_D$, for all $i \in I$, we also have that $K \subseteq K_{D_i}$. Hence, $D_I \in \mathcal{D}_D$. Since $D_I$ is also by definition an upper bound for the chain $\{D_i\}_{i \in I}$, in the sense that $D_i \subseteq D_I$ for all $i \in I$, we conclude that $\{D_i\}_{i \in I}$ has an upper bound in $\mathcal{D}_D$. Since this is true for every chain in $\mathcal{D}_D$, and since $\mathcal{D}_D$ is non-empty, it follows from Zorn's lemma that $\mathcal{D}_D$ has a maximal element. Let $D_A$ be any such a maximal element.

Since $D_A \in \mathcal{D}_D$, we know that $D_A$ is proper and that $K \subseteq K_{D_A} \subseteq K_D$. Since $A \notin K_D$, this also implies that $A \notin K_{D_A}$. The only thing left to prove, therefore, is that $D_A$ is $\mathcal{R}$-compatible. Assume \textit{ex absurdo} that it is not. We will show that this leads to a contradiction.

Since $D_A$ is proper but not $\mathcal{R}$-compatible, it follows from Definition 3 that there is some $(A^*, B^*) \in \mathcal{R}$ such that $A^* \cap D_A \neq \emptyset$ for all $A^* \in A$ but $B^* \cap D_A = \emptyset$. There are now two possibilities: $A^* = \emptyset$ and $A^* \neq \emptyset$. As we will see, they both lead to a contradiction. If $A^* = \emptyset$, it follows from the $\mathcal{R}$-compatibility of $K$ that $B^* \subseteq K \subseteq K_{D_A}$, contradicting the fact that $B^* \cap D_A = \emptyset$. It remains to consider the case $A^* \neq \emptyset$. In that case, let $M := \mathcal{V} \setminus D_A$. Then $M \in Q$ because $PD_1$ implies that $0 \notin D_A$, so $0 \in M$ and hence $M \neq \emptyset$. Since $(A^*, B^*) \in \mathcal{R}$ and $A^* \neq \emptyset$, the monotonicity of $\mathcal{R}$ implies that $\{A^* \cup M : A^* \in A^*\}, B^* \cup M) \in \mathcal{R}$. Consider now any $D \in \mathcal{D}$ and any $A^* \in A^*$. If $D \subseteq D_A$, then $K \subseteq K_D \subseteq K_{D_A} \subseteq K_D$, so $D \in \mathcal{D}_D$. This is impossible because $D_A$ is a maximal element of $\mathcal{D}_D$ with respect to $\subseteq$ and therefore a minimal element with respect to $\subset$. Hence, we find that either $D = D_A$ or $D \cap (\mathcal{V} \setminus D_A) \neq \emptyset$. If $D = D_A$, then since $A^* \cap D_A \neq \emptyset$ and therefore $(A^* \cup M) \cap D_A \neq \emptyset$, we find that $A^* \cap M \subseteq K_{D_A} = K_D$. If $D \cap (\mathcal{V} \setminus D_A) \neq \emptyset$, then $D \cap M \neq \emptyset$ and therefore also $D \cap (A^* \cup M) \neq \emptyset$, which implies that $A^* \cup M \in K_D$. Hence, in both cases, we find that $A^* \cap M \subseteq K_D$. Since this is true for every $D \in \mathcal{D}$ and $A^* \in A^*$, and since $K = \bigcap\{K_D : D \in \mathcal{D}\}$, it follows that $A^* \cup M \subseteq K$ for all $A^* \in A^*$. Since $\{A^* \cup M : A^* \in A^*\}, B^* \cup M) \in \mathcal{R}$, the $\mathcal{R}$-compatibility of $K$ therefore implies that $B^* \cup M \subseteq K \subseteq K_{D_A}$, so $(B^* \cup M) \cap D_A \neq \emptyset$. Since $B^* \cap D_A = \emptyset$, this implies that $M \cap D_A \neq \emptyset$, which is clearly impossible because $M = \mathcal{V} \setminus D_A$. Hence, we find that $D_A$ must indeed be $\mathcal{R}$-compatible.

\textbf{Proof of Proposition 4} Since $\mathcal{R}$ is a subset of $\text{mon}(\mathcal{R})$, we trivially have that $\text{mon}(\mathcal{R})$-compatibility implies $\mathcal{R}$-compatibility. To prove the converse, assume that $D$ is $\mathcal{R}$-compatible and consider any $(A, B) \in \text{mon}(\mathcal{R})$ such that $A \cap D \neq \emptyset$ for all $A \in A$. We need to prove that $B \cap D \neq \emptyset$. If $(A, B) \in \mathcal{R}$, this follows directly from the $\mathcal{R}$-compatibility of $D$. Otherwise, since $(A, B) \in \text{mon}(\mathcal{R})$, there are $(A^*, B^*) \in \mathcal{R}$ and $M \in Q$ such that $A = \{A^* \cup M : A^* \in A^*\}, B = B^* \cup M$ and $A^* \neq \emptyset$. Since $A \cap D \neq \emptyset$ for all $A \in A$, this implies that $(A^* \cup M) \cap D \neq \emptyset$ for all $A^* \in A^*$. We now consider two cases: $M \cap D \neq \emptyset$ and $M \cap D = \emptyset$. If $M \cap D \neq \emptyset$, then also $B \cap D = (B^* \cup M) \cap D \neq \emptyset$, as desired. If $M \cap D = \emptyset$, then for all $A^* \in A^*$, we infer from $(A^* \cup M) \cap D \neq \emptyset$ that $A^* \cap D \neq \emptyset$. Since $D$ is $\mathcal{R}$-compatible and $(A^*, B^*) \in \mathcal{R}$, this implies that $B^* \cap D \neq \emptyset$. Hence, also in this second case, $B \cap D = (B^* \cup M) \cap D \neq \emptyset$. \hfill \Box
Proof of Theorem 3 Since mon($\mathcal{R}$) is clearly a monotone set of rules, we know from Theorem 2 that $K$ is proper and mon($\mathcal{R}$)-compatible if and only if there is a non-empty set $D \subseteq D_{\text{mon}(\mathcal{R})}$ such that $K = \bigcap\{K_D : D \in D\}$, and that the largest such set $D$ is $D_{\text{mon}(\mathcal{R})}(K) := \{D \in D_{\text{mon}(\mathcal{R})} : K \subseteq K_D\}$. Since we know from Proposition 4 that $D_{\text{mon}(\mathcal{R})} = D_\mathcal{R}$ and hence also $D_{\text{mon}(\mathcal{R})}(K) = D_\mathcal{R}(K)$, this concludes the proof. □

Proof of Theorem 4 For the ‘if’-part of the statement, we assume that there is a non-empty set $O \subseteq O_\mathcal{R}$ of $\mathcal{R}$-compatible proper orders such that $C = C_O$. Let $D := \{D_\succ : \succ \in O\}$. Then $D$ is non-empty because $O$ is, and the sets of options in $D$ are proper and $\mathcal{R}$-compatible because of Proposition 1 and Definition 3. Hence, $D$ is a non-empty set of proper $\mathcal{R}$-compatible sets of options. It therefore follows from Theorem 3 that $K := \bigcap\{K_D : D \in D\}$ is a proper mon($\mathcal{R}$)-compatible set of option sets. Now observe that for any $A \in \mathcal{Q}$:

$$C_O(A) = \{u \in A : (\exists \succ \in O)(\exists v \in A) v \succ u\} = \{u \in A : (\exists D \in D)(\exists v \in A) v - u \in D_\succ\} = \{u \in A : (\exists D \in D)(A - u) \cap D = \emptyset\} = \{u \in A : (\exists D \in D)A - u \notin K_D\} = \{u \in A : A - u \notin \bigcap\{K_D : D \in D\}\} = \{u \in A : A - u \notin K\},$$

where the first equality follows from Equation (2), the second from Proposition 1, the fifth from Equation (9) and the last from our choice of $K$. Since $K$ is proper, it therefore follows from Proposition 2 that $C$ is proper and that $K = K_C$. Since $K_C = K$ is mon($\mathcal{R}$)-compatible, it furthermore follows from Definition 4 that $C$ is mon($\mathcal{R}$)-compatible.

For the ‘only if’-part of the statement, we consider any proper mon($\mathcal{R}$)-compatible choice function $C$. It then follows from Proposition 2 and Definition 4 that $K_C$ is a proper mon($\mathcal{R}$)-compatible set of option sets. Because of Theorem 3, this implies that there is a non-empty set $D$ of proper $\mathcal{R}$-compatible sets of options such that $K_C = \bigcap\{K_D : D \in D\}$. With any $D$ in $D$, we now associate a binary relation $\succ_D$ on $\mathcal{V}$, defined by

$$u \succ_D v \iff u - v \in D \text{ for all } u, v \in \mathcal{V}. \quad (12)$$

Since $D$ is proper, it follows from Proposition 1 that $\succ_D$ is a proper order and that $D_{\succ_D} = D$. Since $D$ is proper and $\mathcal{R}$-compatible, it therefore follows from Definition 3 that $\succ_D$ is $\mathcal{R}$-compatible. Hence, if we let $O := \{\succ_D : D \in D\}$, then $O$ is a set of $\mathcal{R}$-compatible proper orders—$O \subseteq O_\mathcal{R}$—and, since $D$ is non-empty, $O$ is non-empty as well. That $C = C_O$ follows from the fact that, for all $A \in \mathcal{Q}$:
Choice functions based on sets of strict partial orders

\[ C(A) = \{ u \in A : A - u \notin K_C \} \]
\[ = \{ u \in A : A - u \notin \bigcap \{ K_D : D \in \mathcal{D} \} \} \]
\[ = \{ u \in A : (\exists D \in \mathcal{D}) (A - u) \cap D = \emptyset \} \]
\[ = \{ u \in A : (\exists D \in \mathcal{D}) (\exists v \in A) v - u \in D \} \]
\[ = \{ u \in A : (\exists D \in \mathcal{D}) (\exists v \in A) v \succ_D u \} \]
\[ = \{ u \in A : (\exists \succ \in \mathcal{O}) (\exists v \in A) v \succ u \} = C_{\mathcal{O}}(A), \]

where the first equality follows from Equation (9), the sixth from Equation (12) and the last from Equation (2).

For the final part of the statement, we consider any non-empty set \( \mathcal{O} \subseteq \mathcal{O}_R \) such that \( C = C_{\mathcal{O}} \). Then for any \( \succ \in \mathcal{O} \) and \( A \in \mathcal{V} \), it follows from Equation (2) that \( C_{\succ}(A) \subseteq C_{\mathcal{O}}(A) = C(A) \). Since \( \mathcal{O} \subseteq \mathcal{O}_R \), this implies that \( \mathcal{O} \subseteq \mathcal{O}_R(C) \) and therefore, since \( \mathcal{O} \) is non-empty, also that \( \mathcal{O}_R(C) \) is non-empty. Furthermore, for any \( A \in \mathcal{V} \), we find that

\[ C(A) = C_{\mathcal{O}}(A) \subseteq C_{\mathcal{O}_R(C)}(A) = \bigcup_{\succ \in \mathcal{O}_R(C)} C_{\succ}(A) \subseteq C(A), \]

where the first set-inclusion follows from the fact that \( \mathcal{O} \subseteq \mathcal{O}_R(C) \), the second equality follows from Equation (2) and the last set-inclusion follows from the definition of \( \mathcal{O}_R(C) \). Hence \( C = C_{\mathcal{O}_R(C)} \). It follows that \( \mathcal{O}_R(C) \) is indeed the largest non-empty set \( \mathcal{O} \) of \( R \)-compatible proper orders such that \( C = C_{\mathcal{O}} \).

\[ \square \]

Proof of Theorem 5 For the ‘if’-part of the statement, we consider a choice function \( C \) and an \( R \)-compatible proper order \( \succ \) such that \( C = C_{\succ} \). Then for any \( u, v \in \mathcal{V} \):

\[ u \notin C(\{u, v\}) \iff u \notin C_{\succ}(\{u, v\}) \iff (\exists w \in \{u, v\}) w \succ u \]
\[ \iff u \succ u \text{ or } v \succ u \iff v \succ u, \]

using Equation (1) for the second equivalence and PO\(_1\) for the last one. The order \( \succ \) is therefore characterised by Equation (10) and hence uniquely determined by \( C \). Since it follows from Equation (1) and (2) that \( C_{\succ} = C_{\{\succ\}} \), we furthermore know from Theorem 4 that \( C = C_{\succ} = C_{\{\succ\}} \) is proper and mon(\( R \))-compatible. Since \( R \subseteq \text{mon}(\mathcal{R}) \), \( C \) is therefore also \( R \)-compatible. That \( C \) is binary follows from the fact that, for all \( A \in \mathcal{V} \) and \( u \in A \):

\[ u \in C(A) \iff u \in C_{\succ}(A) \iff (\exists v \in A) v \succ u \]
\[ \iff (\exists v \in A) u \notin C(\{u, v\}) \]
\[ \iff (\forall v \in A) u \in C(\{u, v\}) \]
\[ \iff (\forall v \in A \setminus \{u\}) u \in C(\{u, v\}), \]
For the ‘only if’-part of the statement, we consider a choice function $C$ that is proper, binary and $R$-compatible. Let $\succ$ be the binary relation that is defined by Equation (10). For all $A \in \mathcal{Q}$ and $u \in A$, we then have that

\[
    u \in C(A) \iff (\forall v \in A \setminus \{u\}) u \in C(\{u, v\})
\]

\[
    \iff (\forall v \in A) u \in C(\{u, v\})
\]

\[
    \iff (\exists v \in A) u \notin C(\{u, v\}) \iff (\exists v \in A) v \succ u \iff u \in C_\succ(A),
\]

where the first equivalence follows from the binarity of $C$, the second follows from $PC_0$, the fourth follows from Equation (10) and the last from Equation (1). Hence, $C = C_\succ$. It remains to show that $K_C = K_{D_\succ}$ and that $\succ$ is an $R$-compatible proper order. For any $A \in \mathcal{Q}$, we have that

\[
    A \in K_C \iff 0 \notin C(A \cup \{0\}) \iff (\exists v \in (A \cup \{0\}) \setminus \{0\}) 0 \notin C(\{0, v\})
\]

\[
    \iff (\exists v \in A \setminus \{0\}) 0 \notin C(\{0, v\})
\]

\[
    \iff (\exists v \in A) 0 \notin C(\{0, v\})
\]

\[
    \iff (\exists v \in A) v \succ 0
\]

\[
    \iff (\exists v \in A) v \in D_\succ
\]

\[
    \iff A \cap D_\succ \neq \emptyset \iff A \in K_{D_\succ},
\]

where the first equivalence follows from Equation (7), the second from the binarity of $C$, the fourth from $PC_0$, the fifth from Equation (10), the sixth from Equation (3) and the last from Equation (9). Hence, $K_C = K_{D_\succ}$. Since $C$ is proper and $R$-compatible, it follows from Proposition 2 and Definition 6 that $K_C$ is a proper set of option sets that is $R$-compatible. Since $K_C = K_{D_\succ}$, it therefore follows from Proposition 3 that $D_\succ$ is proper and from Definitions 3 and 6 and Equation (9) that $D_\succ$ is $R$-compatible. Consider now any $u, v \in V$. Then

\[
    u \succ v \iff v \notin C(\{v, u\}) \iff 0 \notin C(\{0, u - v\}) \iff u - v \succ 0 \iff u - v \in D_\succ,
\]

where the first and third equivalence follow from Equation (10), the second follows from $PC_1$ and the fourth follows from Equation (3). Since $D_\succ$ is a proper set of options, it therefore follows from Proposition 1 that $\succ$ is a proper order. That $\succ$ is $R$-compatible, finally, follows from Definition 3 and the properness and $R$-compatibility of $D_\succ$. \hfill $\square$