On the Energy Increase in Space-Collapse Models

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A typical feature of spontaneous collapse models which aim at localizing wavefunctions in space is the violation of the principle of energy conservation. In the models proposed in the literature the stochastic field which is responsible for the localization mechanism causes the momentum to behave like a Brownian motion, whose larger and larger fluctuations show up as a steady increase of the energy of the system. In spite of the fact that, in all situations, such an increase is small and practically undetectable, it is an undesirable feature that the energy of physical systems is not conserved but increases constantly in time, diverging for \( t \to \infty \). In this paper we show that this property of collapse models can be modified: we propose a model of spontaneous wavefunction collapse sharing all most important features of usual models but such that the energy of isolated systems reaches an asymptotic finite value instead of increasing with a steady rate.

PACS numbers: 03.65.Ta, 02.50.Ey, 05.40.–a

I. INTRODUCTION

As it is well-known, space-collapse models [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] aim at a solution of the macro-objectification or measurement problem in quantum mechanics by suitably modifying the Schrödinger equation with non-linear stochastic terms. One of the characteristic features of these models is the violation of energy-conservation for isolated systems; such a violation is determined by the stochastic process responsible for the localization mechanism, which induces larger and larger fluctuations of the wavefunction in the momentum space [4]: these increasing fluctuations, in turn, determine the increase of the energy of the system [15]. For typical values of the parameters, such an increase is very small and undetectable with present-day technology [1]; still, one would wish to restore the principle of energy conservation within space-collapse models.

In this paper we make one step towards this direction: we analyze a model of wavefunction space-collapse for which the energy of isolated systems does not increase indefinitely, but reaches an asymptotic finite value. An analogy with the quantum Brownian motion will show that the stochastic process acts like a dissipative medium which thermalizes the system to a fixed temperature (the temperature of the medium) and will suggest how to restore perfect energy conservation.

The paper is organized as follows: after a brief review of the main features of dynamical reduction models (Sec. II), we introduce the collapse-model which is the subject of the paper (Sec. III). In Sec. IV we study the master equation for the statistical operator originating from the stochastic dynamics: this will provide the rationale for the choice of the localization operator which defines the model. In Secs. V to VIII we will study in detail the most relevant properties of the model: we will analyze the time evolution of Gaussian wavefunctions (Sec. V); the collapse mechanism and collapse probability (Sec. VI); we will see that the physical predictions of the model agree with very high accuracy with standard quantum mechanical predictions and, at the same time, the model reproduces classical mechanics at the macroscopic level (Sec. VII). We will finally discuss the issue related to energy non conservation (Sec. VIII) and conclude with some final remarks (Sec. IX).
II. STRUCTURE OF DYNAMICAL REDUCTION MODELS

The typical structure of the evolution equation of collapse models is\(^{1}\):

\[
d\psi_t = \left[ -\frac{i}{\hbar} H dt + \sqrt{\lambda} (A - r_t) dW_t - \frac{\lambda}{2} \left( A^\dagger A - 2r_t A + r_t^2 \right) dt \right] \psi_t, \tag{1}\]

with:

\[
r_t = \frac{1}{2} \langle \psi_t | (A + A^\dagger) | \psi_t \rangle. \tag{2}\]

The operator \(H\) is related to the standard quantum Hamiltonian of the system, while \(A\) is the \textit{reduction operator}, i.e., the operator on whose eigenmanifolds one wants to reduce the statevector, as a consequence of the collapse mechanism; the positive constant \(\lambda\) sets the strength of the collapse mechanism. The stochastic dynamics is governed by a standard Wiener process \(W_t\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Note that the equation is non-linear but preserves the norm of the statevector.

In the literature on collapse models, the operator \(A\) is usually assumed to be self–adjoint; in such a case, and if one further assumes that it has only a discrete spectrum, it can be proven\(^{2}\) that the form of the second and third term of Eq. \((1)\), which modify the standard Schrödinger evolution, is such that:

1. The statevector collapses with respect to the “preferred basis” generated by the operator \(A\), i.e., for almost all realizations of the stochastic process there exists an eigenstate \(|a_n\rangle\) of \(A\) (depending of course on the realization of the stochastic process) such that:

\[
|\psi_t\rangle \rightarrow |a_n\rangle \quad \text{for } t \rightarrow \infty. \tag{3}\]

2. The average \(\mathbb{E}[\langle P \rangle_t] = \mathbb{E} [\langle \psi_t | P | \psi_t \rangle]\) of any operator \(P\) which commutes with \(A\) in constant in time, i.e., \(\mathbb{E} [\langle \psi_t | P | \psi_t \rangle] = \langle \psi_0 | P | \psi_0 \rangle\). In particular, if \(P\) is a projection operator relative to an eigenmanifold of \(A\), this together with property \((3)\) implies that the probability for the statevector to be reduced into the eigenmanifold associated to \(P\) is equal to \(\langle \psi_0 | P | \psi_0 \rangle\), i.e., to the probability that standard quantum mechanics associates to the collapse, as a result of a measurement of the operator \(A\). This is due to the fact that \(\langle \psi_t | P | \psi_t \rangle\) turns out to be a martingale, thanks to the particular structure of Eq. \((1)\), so that by the martingale property \(\mathbb{E}[\langle \psi_t | P | \psi_t \rangle] = \langle \psi_0 | P | \psi_0 \rangle\)\(^3\).

It is important to keep in mind that the above results are valid only when the standard Hamiltonian \(H\) either commutes with \(A\) or is equal to zero; in all other cases, such results are only approximate, the approximation depending on the value of \(\lambda\).

III. THE MODEL

In the literature, \(A\) has been mainly taken equal to the position operator \(q\), or a function of \(q\) like in the continuous version\(^2\) of the original GRW model\(^{1}\), the reason being that the operator \(q\) is the most natural candidate for localizing wavefunctions in space. As anticipated in the previous section, one consequence of such a choice is that the energy of the system increases in time, diverging for time going to infinity; it is then natural to wonder whether a different choice for \(A\) can preserve all most important features of collapse models, but at the same time cure this energy non–conservation. This problem finds a partly positive solution by making the following Ansatz\(^2\) for \(A\):

\[
A = q + i \frac{\alpha}{\hbar} p, \tag{4}\]

where \(p\) is the momentum operator. Moreover, we define the operator \(H\) as follows:

\[
H = H_0 + \frac{\lambda \alpha}{2} \{q, p\}, \tag{5}\]

where \(H_0\) is the standard quantum Hamiltonian. In the following sections we will analyze the most relevant physical properties of the model and we will focus our attention to the case of a free particle: \(H_0 = p^2 / 2m\), where \(m\) is the mass of the particle.

The model is defined in terms of the two constants \(\lambda\) and \(\alpha\); for reasons which will be clear in the following, we will assume them to vary with the mass of the particle as follows:

\[
\lambda = \frac{m}{m_0} \lambda_0, \quad \alpha = \frac{m_0}{m} \alpha_0. \tag{6}\]

\(^2\) A localization operator involving \(q\) and \(p\) has also been considered in ref.\(^{10}\) (note however that the form of the localization operator is different from ours) but with a different aim, i.e., that of studying whether the presence of a \(p\)-term instead of only a \(q\)-term can improve the localization mechanism. The authors prove that, for any physically reasonable choice of the parameters of their model, such term does not affect in an appreciable way the collapse mechanism. Here we show that a \(p\)-term is important as it can change the time evolution of the mean energy avoiding it to increase constantly in time. The authors of ref.\(^{10}\) analyze a stochastic differential equation similar to our Eq. \((4)\) where both a \(q\) and a \(p\) term are present: they mainly focus their attention on the application of the formalism to the theory of open quantum systems and decoherent histories. One of their main result is a localization theorem which we will apply to our model to prove the collapse of wavefunctions to localized states.

\(^{1}\) Of course, this form can be generalized, e.g. by adding a finite (or countable) number of operators \(A_i\), each of which is coupled to a Wiener process \(W_i\). Moreover, the Wiener processes may be complex instead of real, as assumed here.
where \( m_0 \) is a reference mass which we choose to be equal to that of a nucleon while \( \lambda_0 \) and \( \alpha_0 \) are fixed constants which we take equal to:

\[
\lambda_0 \approx 10^{-2} \text{ m}^{-2} \text{ sec}^{-1}, \quad (7)
\]

\[
\alpha_0 \approx 10^{-18} \text{ m}^2. \quad (8)
\]

As it will be shown in Sec. VII this numerical choice for the parameters guarantees that the model reproduces almost exactly the physical predictions of standard quantum mechanics at the microscopic level and reproduces classical mechanics at the macroscopic level.

Before concluding this section, we note that, in order to find the solutions of Eq. (1) and to study their properties, it is convenient to consider also a linearized version of Eq. (1)\( ^2 \)\( ^3 \):

\[
\frac{d}{dt} \phi_t (x) = \left[ -\frac{\hbar}{i} H + \sqrt{\lambda} A \frac{d}{dt} \xi_t - \frac{\lambda}{2} A^\dagger A \right] \phi_t (x), \quad (9)
\]

where \( \xi_t \) is a standard Wiener processes defined on a new probability space \((\Omega, \mathcal{F}, \mathbb{Q})\). In ref. [4] the relation between the probability measures \( \mathbb{Q} \) and \( \mathbb{P} \) and the relation between the stochastic processes \( W_t \) and \( \xi_t \) are discussed. Here we simply recall how one can use the above linear equation to find a solution of Eq. (1):

1. Find the solution \( \phi_t \) of Eq. (9), with the initial condition \( \phi_0 = \psi_0 \).

2. Normalize the solution\( ^3 \): \( \phi_t \to \psi_t = \phi_t / \| \phi_t \| \).

3. Make the substitution: \( d\xi_t \to dW_t = d\xi_t - 2\sqrt{\lambda}r_t \); the wavefunction \( \psi_t \) thus obtained is a solution of Eq. (1).

**IV. THE MASTER-EQUATION FOR THE STATISTICAL OPERATOR**

In order to better understand the modifications to the Schrödinger dynamics induced by Eq. (1) and the motivations for the precise choice of its form, apart from the martingale structure, and in particular in order to see why the choice \( \| \phi_t \| = 0 \) for \( A \) can partially cure the problem of the energy increase, it is worthwhile considering the related equation for the statistical operator \( \rho_t \equiv \mathbb{E}[|\psi_t \rangle \langle \psi_t |] \), which is given by

\[
\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H, \rho_t] - \frac{\lambda}{2} [q, [q, \rho_t]] - \lambda \frac{\alpha^2}{2\hbar^2} [p, [p, \rho_t]] - i \frac{\lambda \alpha}{\hbar} [q, \{p, \rho_t\}], \quad (10)
\]

that is the typical structure of master-equation for the quantum description of Brownian motion, where both friction and diffusion are taken into account and positivity of the statistical operator is granted at all times. The obvious difference between Eq. (10) and the master-equation for quantum Brownian motion lies in the meaning of the coefficients, here related to the two fundamental constants of the model \( \lambda \) and \( \alpha \), rather than to the friction coefficient and the temperature of the bath. The quantum Brownian motion master-equation is in fact given by \( ^7 \)\( ^8 \)\( ^9 \):

\[
\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H_0, \rho_t] - \frac{2M}{\beta \hbar^2} [q, [q, \rho_t]] - \frac{\gamma}{8M} [p, [p, \rho_t]] - \frac{i}{\hbar} \gamma [q, \{p, \rho_t\}], \quad (11)
\]

where \( \gamma \) is the friction coefficient and \( \beta \) the inverse temperature of the background medium; the second and third term at r.h.s. account for diffusion, with coefficients proportional to the squared thermal wavelength \( \Delta x^2_\text{th} = \beta \hbar^2 / 4M \) and the squared thermal momentum spread \( \Delta p^2_\text{th} = M / \beta \) satisfying the minimum uncertainty relation \( \Delta p_\text{th} \Delta x_\text{th} = \hbar / 2 \), while the last is due to friction and ensures that the energy of the test particle asymptotically goes to the equipartition value depending only on the temperature of the bath. Note that in the quantum description friction, which accounts for the finite energy growth, is of necessity related to diffusion in order to preserve the Heisenberg uncertainty relation \( ^7 \)\( ^8 \)\( ^9 \)\( ^17 \)\( ^20 \)\( ^21 \). A fundamental result, in order to understand how Eq. (10)\( ^10 \) and therefore the striking similarity with quantum Brownian motion appears, is Holevo’s characterization of translation-covariant generators of quantum-dynamical semigroups \( ^22 \), according to which further important restrictions can be put on the operators appearing in the general so-called Lindblad structure, once symmetry under translations is taken into account. In fact, according to Holevo’s result, if the generator of the dynamics \( \mathcal{L} \) is translation-covariant, i.e., commutes with the action of the unitary representation of translations \( U(a) = \exp[-i(\hbar/a)p] \):

\[
\mathcal{L}[U(a)\rho U^\dagger(a)] = U(a)\mathcal{L}[\rho]U^\dagger(a) \quad (12)
\]

for all real \( a \), then its general structure, given that \( q \) appears at most quadratically, is the following

\[
\mathcal{L}[\rho] = -\frac{i}{\hbar} [H(p), \rho] + \mathcal{L}_G[\rho], \quad (13)
\]

with \( H(p) \) a self-adjoint operator only depending on the momentum operator \( p \) and \( \mathcal{L}_G \) (where \( G \) stands for Gaussian) is given by

\[
\mathcal{L}_G[\rho] = -\frac{i}{\hbar} [a_0 q + H_{\text{int}}(q, p), \rho] + \left[ K_{\rho} K^\dagger - \frac{1}{2} \{ K^\dagger K, \rho \} \right], \quad (14)
\]

with

\[
K = a_1 q + L_1(p), \quad a_0, a_1 \in \mathbb{R}, \quad H_{\text{int}}(q, p) = \frac{\hbar}{2l} a_1 (q L_1(p) - L_1^\dagger(p)q)
\]
and \( L_1(p) \) a generally complex function of its argument. The requirement of translational invariance is a natural and compelling one for dynamical reduction models, since the modification of quantum mechanics by a universal noise should by no way break the homogeneity of space, introducing some preferred space location. The restriction to mappings at most quadratic in the position operator \( q \) is linked to the fact that we are looking for a generalization of the most simple dynamical reduction model where \( A = q \) and the associated master-equation is given by

\[
\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H_0, \rho_t] - \frac{\lambda}{2} \{q, [q, \rho_t]\}, \tag{15}
\]

often considered in the literature (see e.g. \[4\] and references therein) even though leading to a steady energy increase. In view of Eq. (13) the most straightforward extension of Eq. (15) including a friction term proportional to velocity is obtained setting \( a_0 = 0 \), i.e., no external constant force since we are considering the modification to Schrödinger dynamics for a free particle, \( a_1 = \sqrt{\lambda} \) and \( L_1(p) = i\sqrt{\lambda}(\alpha/\hbar)p \), thus directly obtaining the Ansatz given in Eq. (4). With this choice of functions and parameters Eq. (13) gives

\[
\frac{d}{dt} \rho_t = -\frac{i}{\hbar} \left[ H_0 + \frac{\lambda \alpha}{2} \{q, p\}, \rho_t \right] + \lambda \left[ A\rho A^\dagger - \frac{1}{2} \left\{ A^\dagger A, \rho \right\} \right], \tag{16}
\]

with \( A = q + i(\alpha/\hbar)p \), as in Eq. (4), which is immediately seen to be equivalent to Eq. (10), thus giving the rationale for our choice for the operator \( A \).

Note that looking at Eq. (16) one might erroneously be led to think that the modification to Schrödinger dynamics amounts to a change in the Hamiltonian plus a free Hamiltonian of the system, giving its dynamics when it is not coupled to the environment or some noise source. On the contrary it usually happens, e.g., when the Lindblad structure appears in the reduced description of a system coupled to some reservoir, that in the commutator at r.h.s. of (17) an operator appears which is the sum of the free Hamiltonian and some other self-adjoint operator, this other contribution disappearing together with the rest of the Lindblad form when the coupling vanishes, as it correctly happens in Eq. (10) if the fundamental constant \( \lambda \) is set to zero. The general structure (17) cannot be thought of as being made up of two distinct parts, since the Lindblad characterization pertains to the structure as a whole. Note however that the free Hamiltonian can still be put into evidence in (17) according to

\[
\begin{align*}
\frac{d}{dt} \psi_t &= \left[ -\frac{i}{\hbar} H_0 dt + \sqrt{\lambda} (A - r_t) dW_t - \frac{\lambda}{2} \left( A^\dagger A - 2r_t A + r_t^2 + \frac{1}{2} (A^2 - A^\dagger 2) \right) dt \right] \psi_t, \tag{18}
\end{align*}
\]

even though in this equivalent expression the martingale structure is less evident.

V. SINGLE–GAUSSIAN SOLUTION

Gaussian wavefunctions are very special and they are often used to represent typical physical situations; we now show that, as for the standard Schrödinger equation, our stochastic equation preserves the form of Gaussian wavefunctions and, at the same time, we analyze their evolution in time. Let us then consider the following class of wavefunctions:

\[
\phi_t(x) = \exp \left[ -a_t(x - \overline{x}_t)^2 + i\overline{\kappa}_t x + \gamma_t \right], \tag{19}
\]

where \( a_t \) and \( \gamma_t \) are complex functions of time, while \( \overline{x}_t \) and \( \overline{\kappa}_t \) are real. By following the procedure outlined in ref. [4], one can show that the above parameters obey the following stochastic differential equations:\footnote{The superscripts “R” and “I” denote, respectively, the real and imaginary parts of the corresponding functions.}

\[
\begin{align*}
\frac{da_t}{dt} &= \left[ -\frac{2i\hbar}{m} a_t^2 - 4\lambda a_t + \lambda \right] dt, \tag{20}
\end{align*}
\]

\[
\begin{align*}
\frac{d\overline{x}_t}{dt} &= \frac{\hbar}{m} \overline{x}_t dt + \sqrt{\lambda} \left[ \frac{1}{2a_t^2} - \alpha \right] dW_t, \tag{21}
\end{align*}
\]

\[
\begin{align*}
\frac{d\overline{\kappa}_t}{dt} &= -2\lambda \overline{\kappa}_t dt - \sqrt{\lambda} \frac{a_t}{a_t^3} dW_t. \tag{22}
\end{align*}
\]

We have omitted the equation for \( \gamma_t \) since the real part can be absorbed into the normalization factor, while the imaginary part represents an irrelevant global phase.

A. The time evolution of \( a_t \)

The parameter \( a_t \) is particularly important since it is related to the spread of the wavefunction \( |\psi_t\rangle \) in position
and momentum, according to the following expressions:

\[
\sigma_q(t) = \sqrt{(q^2) - \langle q \rangle^2} = \frac{1}{2} \left( \frac{1}{a_t}\right),
\]

\[
\sigma_p(t) = \sqrt{(p^2) - \langle p \rangle^2} = \hbar \sqrt{\frac{(a_t^0)^2 + (a_t^1)^2}{a_t^0}}, \quad (23)
\]

Eq. (20) for \(a_t\) can be easily solved; one gets:

\[
a_t = -\frac{1}{2} \left[ A + i B \tanh \left( \frac{\hbar}{m} B t + k \right) \right], \quad (24)
\]

with:

\[
A = -2i \frac{\lambda \alpha m}{\hbar}, \quad B = \sqrt{\frac{4\lambda^2 \alpha^2 m^2}{\hbar^2} + \frac{2\lambda m}{\hbar}}, \quad (25)
\]

the constant \(k\) sets the initial condition \(a_0\).

After a tedious calculation, one can write explicitly the time evolution of the real and imaginary parts\(^5\) of \(a_t\):

\[
a_t^R = \frac{m \omega}{2\sqrt{2}\hbar} \left( \sin \theta \sinh(\omega_1 t + \varphi_1) + \cos \theta \sin(\omega_2 t + \varphi_2) \right) / \cosh(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi_2), \quad (26)
\]

\[
a_t^i = -\frac{m \omega}{2\sqrt{2}\hbar} \left[ \cos \theta \sinh(\omega_1 t + \varphi_1) - \sin \theta \sin(\omega_2 t + \varphi_2) \right] / \cosh(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi_2) - \frac{2\sqrt{2}\lambda \alpha}{\omega} \quad (27)
\]

where we have introduced the following two frequencies:

\[
\omega_1 = \sqrt{2} \omega \cos \theta \quad \omega_2 = \sqrt{2} \omega \sin \theta, \quad (28)
\]

the frequency \(\omega\) and the angle \(\theta\) being defined as follows:

\[
\omega = 2^{\frac{1}{4}} \sqrt{4\lambda_0^4 \alpha_0^4 + \frac{\lambda_0^2 \hbar^2}{m_0^2}} \approx 10^{-5} \text{ sec}^{-1}, \quad (29)
\]

\[
\theta = \frac{1}{2} \tan^{-1} \left[ \frac{\hbar}{2\lambda_0 \alpha_0 \omega_0} \right] \approx \frac{\pi}{4}; \quad (30)
\]

note that, due to the specific dependence of both \(\lambda\) and \(\alpha\) on \(m\) as given by Eq. (8), both \(\omega\) and \(\theta\) are independent of the mass of the particle, and thus are two constants of the model. Note also that — as it is easy to prove — if \(a_0^R > 0\), then \(a_t^R > 0\) for any subsequent time \(t\); this implies that Gaussian solutions do not diverge in time.

\[
\sigma_q(\infty) = \sqrt{\frac{\hbar}{2m\omega \sin \theta}} \approx \left( 10^{-15} \sqrt{\frac{\text{Kg} \text{ m}}{\text{m}}} \right) \text{ m}; \quad (32)
\]

The asymptotic spread decreases for increasing values of the mass of the particle, this property entailing that, as we shall discuss in more detail in Sec. VII, wavefunctions of macroscopic objects are almost always very well localized in space, so well that they practically behave like point-like particles.

The time evolution for \(\sigma_p(t)\) can also be obtained and, as it happens for the spread in position, also the spread in momentum asymptotically reaches a finite value, which is:

\[
\sigma_p(\infty) = \sqrt{\frac{\hbar m \omega \sin^2 \theta + (\cos \theta - \kappa)^2 \sin \theta}{2\sqrt{2}}} \approx \left( 10^{-19} \sqrt{\frac{\text{m}}{\text{Kg}}} \right) \text{ Kg \ sec}; \quad (33)
\]

where:

\[
\kappa = \frac{2\sqrt{2}\lambda_0 \alpha_0}{\omega} \approx 10^{-14}. \quad (34)
\]

To conclude the section, we compute the product of the two asymptotic spreads:

\[
\sigma_q \cdot \sigma_p = \frac{\hbar}{2} \sqrt{1 + \frac{(\cos \theta - \kappa)^2}{\sin^2 \theta}}, \quad (35)
\]

which is almost equal to \(\hbar / \sqrt{2}\), with our choice and for the parameters. Note however that for any choice of \(\lambda_0\) and \(\alpha_0\) Heisenberg uncertainty relations are fulfilled.

In accordance with [24], any Gaussian solution having this asymptotic values for the spread in position and

\[
^6 \text{Note that the evolution of } \sigma_q(t) \text{ (and also of } \sigma_p(t) \text{) is deterministic and depends on the noise } W_t \text{ only indirectly, through the constant } \lambda.
\]
momentum will be called a “stationary solution” of Eq. \( \text{(1)} \). Of course, the term “stationary” does not mean that such wavefunctions do not evolve in time; as a matter of fact (see the following discussion) they always undergo a random motion both in position and momentum which never stops. The term “stationary” refers only to the shape of the wavefunction: stationary solutions are special wavefunction which are Gaussian and with a fixed spread in position and momentum, given by Eqs. \( \text{(22)} \) and \( \text{(33)} \).

C. The mean in position and momentum

The mean \( \langle q \rangle_t \) in position of the wavefunction and the mean \( \langle p \rangle_t \) in momentum satisfy the following stochastic differential equations which can be derived from Eqs. \( \text{(21)} \) and \( \text{(22)} \):

\[
d\langle q \rangle_t = \frac{1}{m} \langle p \rangle_t dt + \sqrt{\lambda} \left[ \frac{1}{2a_q^2} - \alpha \right] dW_t, \tag{36}
\]

\[
d\langle p \rangle_t = -2\alpha \lambda \langle p \rangle_t dt - \sqrt{\lambda} \frac{a_t^2}{\langle p \rangle_t} dW_t. \tag{37}
\]

Their average values evolve as follows:

\[
m \frac{d}{dt} \mathbb{E} \{ \langle q \rangle_t \} = \mathbb{E} \{ \langle p \rangle_t \}, \tag{38}
\]

\[
d \frac{d}{dt} \mathbb{E} \{ \langle p \rangle_t \} = -2\alpha \lambda \mathbb{E} \{ \langle p \rangle_t \}. \tag{39}
\]

The first equation reproduces the classical relation between position and momentum of a particle while the second equation predicts that the momentum decays exponentially in time:

\[
\mathbb{E} \{ \langle p \rangle_t \} = \langle p \rangle_0 e^{-2\alpha t}, \tag{40}
\]

with

\[
2\alpha = 2\lambda \alpha_0 \approx 10^{-20} \text{ sec}^{-1}, \tag{41}
\]

which is an extremely slow decay rate, not depending on the mass of the particle.

For completeness we consider also the covariance matrix

\[
C(t) = \mathbb{E} \left[ \begin{pmatrix} \langle q \rangle_t - \mathbb{E} \{ \langle q \rangle_t \} \\ \langle p \rangle_t - \mathbb{E} \{ \langle p \rangle_t \} \end{pmatrix} \left( \begin{pmatrix} \langle q \rangle_t - \mathbb{E} \{ \langle q \rangle_t \} \\ \langle p \rangle_t - \mathbb{E} \{ \langle p \rangle_t \} \end{pmatrix} \right)^\dagger \right]
\]

\[
= \begin{pmatrix} C_q^2(t) & C_{qp}(t) \\ C_{qp}(t) & C_p^2(t) \end{pmatrix},
\]

whose coefficients satisfy the following equations:

\[
\frac{d}{dt} C_q^2(t) = \frac{2}{m} C_{qp}(t) + \lambda \left( \frac{1}{2a_q^2} - \alpha \right)^2, \tag{42}
\]

\[
\frac{d}{dt} C_{qp}(t) = \frac{1}{m} C_q^2(t) - 2\alpha \lambda C_{qp}(t) - \lambda \hbar \frac{a_t^2}{\langle p \rangle_t} \left( \frac{1}{2a_q^2} - \alpha \right), \tag{43}
\]

\[
\frac{d}{dt} C_p^2(t) = -4\alpha \lambda C_{p^2}(t) + \lambda \hbar^2 \left( \frac{a_t^2}{a_q^2} \right)^2. \tag{44}
\]

In Sec. VII B we will discuss the physical implications of the above equations in connection with the dynamics of macroscopic objects.

VI. ASYMPTOTIC BEHAVIOR OF THE GENERAL SOLUTION

In the previous section we have seen that any Gaussian solution converges towards a stationary solution i.e., towards a Gaussian wavefunction with a fixed finite value both for the spread in position and momentum, given by Eqs. \( \text{(32)} \) and \( \text{(33)} \). In this section we prove that the spread \( \sigma_t(t) \) of any wavefunction converges with probability one towards \( \sigma_q \): this means that any initial wavefunction converges to a localized solution; for the proof we will follow the same strategy of Ref. \( \text{(14)} \).

A. The reduction process

It is easy to see that a Gaussian stationary solution is an eigenstate of the operator:

\[
O = p - 2i\hbar a_\infty q, \tag{45}
\]

where

\[
a_\infty = \frac{m\omega}{2\sqrt{2\hbar}} \left[ \sin \theta - i (\cos \theta - \kappa) \right]; \tag{46}
\]

the proof basically consists in showing that the variance

\[
\sigma_O^2(t) \equiv \langle \psi_t | [O - \langle O \rangle] [O - \langle O \rangle] | \psi_t \rangle \tag{47}
\]

of the operator \( O \) vanishes for \( t \to \infty \).

The first step is to re–write \( \sigma_O^2(t) \) in terms of the variances associated to the operators \( q \) and \( p \):

\[
\sigma_O^2(t) = \sigma_q^2(t) + \sigma_p^2(t) - 2 \sigma_q \sigma_p \sigma_{q,p}(t) - \frac{\hbar^2}{2\sigma_q^2}, \tag{48}
\]

where we have defined:

\[
\sigma_{q,p}^2(t) = \frac{1}{2} \left[ \langle \psi_t | [q - \langle q \rangle] [p - \langle p \rangle] | \psi_t \rangle + \langle \psi_t | [p - \langle p \rangle] [q - \langle q \rangle] | \psi_t \rangle \right], \tag{49}
\]

so that for a Gaussian wavefunction such as \( \text{(19)} \)

\[
\sigma_{q,p}(t) = \sqrt{\frac{-\hbar}{2\sigma_q^2}} a_t^2, \tag{50}
\]

and \( \sigma_{q,p} \) corresponds to the value of \( \sigma_{q,p}(t) \) when the state \( | \psi_t \rangle \) is a stationary Gaussian solution.
After a rather long calculation (see Appendix A for the
details), it is possible to show that:

$$\frac{d}{dt} \mathbb{E}[\sigma_q(t)] =$$

$$= -4\lambda \mathbb{E} \left[ \sigma^2_q(t) + \sigma^4_{q,p} \left( \frac{\sigma^2_q(t) - \sigma^2_{q,p}(t)}{\sigma^2_{q,p}} \right)^2 + \frac{\hbar^2}{4\sigma_q} (\sigma^2_q(t) - \sigma^2_q)^2 \right] \leq 0. \quad (51)$$

Since \(\sigma^2_q(t)\) is by definition a positive quantity, the above
equation is consistent if and only if the r.h.s vanishes for
any \(\omega \in \Omega\), with the possible exception of a subset of
measure 0. This is particular implies both that \(\sigma^2_q(t) \to 0\)
a.s. and that \(\sigma_q(t) \to \varpi_q \) a.s., which is the desired result,
 i.e., the wavefunction converges to a localized solution.

**B. The localization probability**

Once proved that Eq. 1 with the choice 4 for the
operator \(A\) induces the localization of the wavefunction
in space, it becomes natural to analyze the probability
for a localization to occur within a given interval of the
real axis. Such an analysis can be developed along the
same line of Ref. [4].

Let us consider the probability measure:

$$\mu_t(\Delta) \equiv \mathbb{E}_\rho \left[ \| P_\Delta \psi_t \|^2 \right], \quad (52)$$

defined on the Borel sigma–algebra \(\mathcal{B}(\mathbb{R})\) of the real axis,
where \(P_\Delta\) is the projection operator associated to the
Borel subset \(\Delta\) of \(\mathbb{R}\); such a measure is identified by the
density \(p_t(x) \equiv \mathbb{E}_\rho[|\psi_t(x)|^2]\):

$$\mu_t(\Delta) = \int_\Delta p_t(x) \, dx. \quad (53)$$

The density \(p_t(x)\) corresponds to the diagonal element
\(\langle x | \rho_t | x \rangle\) of the statistical operator \(\rho_t \equiv \mathbb{E}_\rho[|\psi_t\rangle \langle \psi_t|]\), solution of the master-equation 10. In Appendix B we
show how the general solution of this master-equation
in the position representation can be obtained; the fi-
nal expression, as a function of the solution of the free
Schrödinger equation \(\rho^0\) (i.e., with \(\lambda = \alpha = 0\), is:

\[\langle q_1 | \rho_t | q_2 \rangle = \frac{1}{2\pi \hbar} \int dk \int dy \, e^{-(i/\hbar)ky} F[k, q_1 - q_2, t] \times \left( y + \frac{q_1 + q_2}{2} + \frac{q_1 - q_2}{2} e^{-2\lambda \alpha t} + \frac{kt}{2m} \left( 1 - \frac{\Gamma_t}{2\lambda \alpha t} \right) \right) \rho_t^S \left( y + \frac{q_1 + q_2}{2} - \frac{q_1 - q_2}{2} e^{-\lambda \alpha t} - \frac{kt}{2m} \left( 1 - \frac{\Gamma_t}{2\lambda \alpha t} \right) \right), \quad (54)\]

with

\[F[k, x, t] = \exp \left\{ -\frac{\lambda \alpha^2 k^2 t^2}{2\hbar^2} + \frac{1}{8\alpha \Gamma_t} \left( x e^{-\lambda \alpha t} - \frac{\Gamma_t}{2m\lambda \alpha} \right)^2 K_1(t) + 2x \left( x e^{-\lambda \alpha t} - \frac{\Gamma_t}{2m\lambda \alpha} \right) K_2(t) + x^2 K_3(t) \right\} \quad (55)\]

and

\[K_1(t) = \Gamma_t^2 + 2\Gamma_t - 4\lambda \alpha t, \quad K_2(t) = e^{-4\lambda \alpha t} + 4\lambda \alpha t e^{-2\lambda \alpha t} - 1, \quad K_3(t) = -4\lambda \alpha t e^{-4\lambda \alpha t} - \Gamma_t^2 + 2\Gamma_t e^{-2\lambda \alpha t}, \quad (56)\]

where we have defined \(\Gamma_t = 1 - e^{-2\lambda \alpha t}\). Taking the diagonal matrix elements one has:

\[p_t(x) = \frac{1}{2\pi \hbar} \int dk \int dy \, e^{-(i/\hbar)ky} F[k, 0, t] \left( y + x + \frac{kt}{2m} \left( 1 - \frac{\Gamma_t}{2\lambda \alpha t} \right) \right) \rho_t^S \left( y + x - \frac{kt}{2m} \left( 1 - \frac{\Gamma_t}{2\lambda \alpha t} \right) \right), \quad (57)\]

and

\[F[k, 0, t] = \exp \left\{ -\frac{\lambda \alpha^2 k^2 t}{2\hbar^2} + \frac{1}{32m^2 \lambda^2 \alpha^2} k^2 K_1(t) \right\} \quad (58)\]

According to the values 7 and 8 for \(\lambda_0\) and \(\alpha_0\), the above expressions can be expanded for small7 \(\lambda \alpha\), leading

---

7 This means that we are considering only times \(t \ll (\lambda \alpha)^{-1} \approx 10^{20} \text{ s.}\)
to:

\[ p_t(x) \simeq \frac{1}{2\pi \hbar} \int dk \int dy e^{-(i/k)ky} \exp \left\{ - \frac{\lambda k^2 y^2 + \lambda \alpha^2 k^2 / 2m t^3}{6m^2 t^3} \right\} \langle y + x + \lambda \alpha k t^2 / 2m | \rho_t | y + x - \lambda \alpha k t^2 / 2m \rangle. \]  

(59)

We now focus on the case of a macroscopic object (let us say \( m \geq 1 \) g); one can further approximate the above expression by noting that for such values of \( m \) the exponential factor appearing in Eq. (59) damps all matrix elements such that the term \( \lambda \alpha k t^2 / 2m \) is not vanishingly small; e.g. when \( \lambda \alpha k t^2 / 2m \geq 10^{-15} \) m, then the second exponential in the above equation is much smaller than \( e^{-10^{15} (m/\sec)} \). We can then neglect the two terms in the matrix elements and perform the integration over \( k \), and we get:

\[ p_t(x) \simeq \sqrt{\frac{2t}{\pi}} \int dy e^{-\beta_t y^2} p_t^S(x + y) \]  

(60)

with

\[ \beta_t = \frac{3}{2\hbar^2} \lambda \left[ 1 + 3 \left( \frac{m \alpha}{\hbar} \right)^2 \right] \frac{1}{t^3} \left\{ \begin{array}{ll} \simeq 10^{43} \left( \frac{m}{\text{kg}} \right) \left( \frac{\text{sec}}{t} \right)^3 & \text{for: } t \geq 10^{-11} \text{ s}, \\
\geq 10^{65} \left( \frac{m}{\text{kg}} \right) \left( \frac{\text{sec}}{t} \right)^3 & \text{for: } t \leq 10^{-11} \text{ s} \end{array} \right. \]  

(61)

The exponent in (60) is extremely peaked with respect to the typical values the probability density \( p_t^S(x) \) associated to the wavefunction of a macroscopic object takes, so that with very high accuracy we have:

\[ p_t(x) \simeq p_t^S(x). \]  

(62)

As discussed in ref. 4, the probability measure \( \mu_t(\Delta) \) which we have shown to be extremely close to the quantum probability obtained from the free Schrödinger equation can be interpreted as a probability measure for the collapse of the wavefunction of the macroscopic object within \( \Delta \), when \( \Delta \) corresponds to an interval of the real axis of width greater or equal e.g. to \( 10^{-5} \) cm 4.

To conclude, the previous analysis shows that under the above listed conditions the probability for the wavefunction of a macro-object to be localized within an interval of the real axis is almost equal to the corresponding quantum probability as given by the Born rule.

VII. DYNAMICS OF MICROSCOPIC AND MACROSCOPIC SYSTEMS

In this section we discuss how our reduction model is related both to quantum and to classical mechanics. Our aim is to show that at the microscopic level the physical predictions of the model are almost identical to standard quantum predictions and that, at the same time, the model with high accuracy reproduces classical mechanics at the macro–level.

A. Micro–systems: comparison with standard quantum mechanics

Microscopic system cannot be directly observed, and their properties can be analyzed only by resorting to suitable measurement procedures. All physical predictions of our model, concerning the outcome of measurements, have the form \( \mathbb{E}_\psi [\langle \psi_\psi | S | \psi_\psi \rangle] \) where \( S \) is a suitable self–adjoint operator, typically a projection operator and it is easy to show that \( \mathbb{E}_\psi [\langle \psi_\psi | S | \psi_\psi \rangle] \equiv \text{Tr}[S \rho_t] \), where the statistical operator \( \rho_t \equiv \mathbb{E}_\psi [\langle \psi_\psi \rangle] \) satisfies Eq. (10). Accordingly, as already discussed in Sec. IV the testable effects of the stochastic process on the wavefunction are similar to the effect induced by a quantum environment on the particle, when both friction and diffusion are taken into account.

With our choice 1 and 2 for \( \lambda_0 \) and \( \alpha_0 \), the testable effects of the stochastic process are of the same order of magnitude of those induced by the interaction of the system with particles and radiation of the intergalactic space 25: such effects are very small and masked by most other sources of decoherence, so that they can be tested only by resorting to sophisticated experiments 26, 27. This implies that the physical predictions of our model are very close to standard quantum mechanical predictions.

B. Macro–objects: comparison with classical mechanics

A macroscopic object is made of elementary constituents strongly interacting among each other and, according to our model, its dynamics is governed by the following stochastic differential equation, which is the straightforward generalization of Eq. 1 to a system.
of \( N \) particles:

\[
d\psi_t(\{x\}) = \left[ -\frac{i}{\hbar} H_{\text{TOT}} dt + \sum_{n=1}^{N} \sqrt{\lambda_n} (A_n - r_{nt}) dW^r_t \right. \\
\left. - \sum_{n=1}^{N} \frac{\lambda_n}{2} \left( A_n^\dagger A_n - 2r_{nt} A_n + r_{nt}^2 \right) \right] \psi_t(\{x\});
\]

\hspace{1cm} (63)

the symbol \( \{x\} \equiv x_1, x_2, \ldots, x_N \) represents the \( N \) spatial coordinates of the configuration space of the composite system; \( W^1_t, W^2_t, \ldots, W^N_t \) are \( N \) independent Wiener processes; \( r_{nt} = \langle \psi_t|[A_n + A_n^\dagger]|\psi_t\rangle / 2 \) and the localization operators \( A_n \) are given by expressions (4), with \( q \) replaced by \( q_n \), the position operator of the \( n \)-th particle, and \( p \) replaced by \( p_n \), the corresponding momentum operator. Furthermore:

\[
\lambda_n = \frac{m_n}{m_0} \lambda_0 \quad \alpha_n = \frac{m_0}{m_n} \alpha_0
\]

and

\[
H_{\text{TOT}} = H_{\text{TOT}}^0 + \sum_{n=1}^{N} \frac{\lambda_n \alpha_n}{2} \{q_n, p_n\}, \quad (65)
\]

\( H_0 \) being the standard quantum Hamiltonian for the composite system.

As custom, we separate the motion of the center of mass from the internal motion. To this end let:

\[
Q = \frac{1}{M} \sum_{n=1}^{N} m_n q_n \quad \left( M = \sum_{n=1}^{N} m_n \right)
\]

(66)

be the position operator associated to the center-of-mass coordinate \( R \), and \( \hat{q}_n \) the position operators associated to the internal coordinates \( x_n = x_n - R \) \((n = 1, \ldots, N)\); let also \( P \) and \( \hat{p}_n \) be the corresponding momentum operators. Then, if \( H_{\text{TOT}}^0 \) can be written as the sum of a term \( H_{\text{CM}}^0 \) associated to the center of mass and a term \( H_{\text{REL}}^0 \) associated to the internal motion, it is easy to prove that the dynamics of the two types of degrees of freedom decouple; in particular the equation for the center of mass — the only one we consider here — becomes:

\[
d\psi_t(R) = \left[ -\frac{i}{\hbar} H_{\text{CM}} dt + \sqrt{\lambda_{\text{CM}}} (A_{\text{CM}} - r_{\text{CM},t}) dW^r_t \\
- \frac{\lambda_{\text{CM}}}{2} \left( A_{\text{CM}}^\dagger A_{\text{CM}} - 2r_{\text{CM},t} A_{\text{CM}} + r_{\text{CM},t}^2 \right) \right] \psi_t(R),
\]

\hspace{1cm} (67)

with:

\[
H_{\text{CM}} = H_{\text{CM}}^0 + \frac{\lambda_{\text{CM}} \alpha_{\text{CM}}}{2} \{Q, P\}, \quad (68)
\]

\[
r_{\text{CM},t} = \frac{1}{2} \langle \psi_t|[A_{\text{CM}}^\dagger + A_{\text{CM}}]|\psi_t\rangle, \quad (69)
\]

\[
A_{\text{CM}} = Q + \frac{\alpha_{\text{CM}}}{\hbar} P, \quad (70)
\]

and \( \lambda_{\text{CM}} \) and \( \alpha_{\text{CM}} \) defined by Eqs. (10), with \( m \) equal to the total mass \( M \) of the composite system. Note that the separation of the center-of-mass motion from the relative motion, for the non-Schrödinger terms of Eq. (63), is possible because of the specific dependence of the parameters \( \lambda_n \) and \( \alpha_n \) on the masses \( m_n \) of the particles as given by Eq. (65).

According to Eq. (64), the center of mass behaves like a particle whose dynamics, in the free case, has been discussed in detail in Secs. IV and VI. We now show how the large numerical value for \( M \), typical of macroscopic objects, affects the time evolution of the center-of-mass wavefunction.

1. **Collapse rate.** According to Eq. (64), and assuming that the wavefunction is not already localized in space, i.e., that \( \sigma_q(t) \gg \sigma_r \), one has:

\[
\left| \frac{d}{dt} E[\sigma_r^2(t)] \right| \geq \frac{\hbar^2}{\sigma_r^4} \quad (71)
\]

with \( 2\lambda_0 \omega^2 \sin^2 \theta / m_0 \approx 10^{15} \text{Kg}^{-1} \text{m}^{-2} \text{sec}^{-3} \). In the microscopic case the r.h.s. of Eq. (71) is in general small and negligible as we expect it to be, since the reduction mechanism must be ineffective on microscopic systems; in the macroscopic case instead it rapidly becomes big, due to the large value of the mass \( m \), ensuring that any initial wavefunction rapidly converges towards a localized solution. We can then assume that, possibly after a very short transient period, the wavefunction describing the motion of the center of mass of any macro-object is practically localized in space.

2. **Behavior of the stationary solution for macroscopic objects.** We now discuss the time evolution of a typical localized wavefunction, i.e., a Gaussian stationary solution. Eqs. (63) and (64) imply that the two maxima in position and momentum of such wavefunctions fluctuate around their mean values; we now show that in the macroscopic regime these fluctuations are extremely small.

As a matter of fact Eqs. (12) to (14) imply for a stationary solution:

\[
C_{\text{q}^2}(t) = \lambda_0^2 t - \frac{\hbar^2}{2\lambda_0^2 m} \left( \cos \theta - \kappa \right) \frac{\sin \theta}{\sin \theta} \left( 1 - e^{-2\lambda_0 t} \right)
\]

\[
+ \frac{\hbar^2}{16\lambda_0^2 \alpha^2 m^2} \left( \cos \theta - \kappa \right)^2 \frac{\sin \theta}{\sin \theta} \left( 1 - e^{-4\lambda_0 t} \right),
\]

\hspace{1cm} (72)

\[
C_{\text{p}^2}(t) = \frac{\hbar^2}{4\alpha} \left( \frac{\cos \theta - \kappa}{\sin \theta} \right)^2 \left( 1 - e^{-4\lambda_0 t} \right),
\]

\hspace{1cm} (73)

where \( \ell \) is defined as follows:

\[
\ell = -\frac{\hbar}{2\lambda_0 m} \left( \frac{\cos \theta - \kappa}{\sin \theta} \right) - \left( \sqrt{2\hbar} / m \omega \sin \theta - \alpha \right).
\]

\hspace{1cm} (74)
Since the exponential factors decay very slowly (we remember that $\lambda \alpha \approx 10^{-20}$ sec$^{-1}$) it is physically significant to consider only the linear term of their Taylor expansion; one then gets:

$$C_{q^2}(t) \approx \frac{\hbar^2 \lambda_0}{m_0 \omega^2} \frac{t}{m} \sim 10^{-33} \left( \frac{\text{Kg}}{m} \right) \left( \frac{t}{\text{sec}} \right) \text{m}^2$$

$$C_{p^2}(t) \approx \frac{\hbar^2 \lambda_0}{m_0 \alpha t} \sim 10^{-43} \left( \frac{m}{\text{Kg}} \right) \left( \frac{t}{\text{sec}} \right) \text{Kg} \cdot \text{m}^2 \cdot \text{sec}^{-2}$$

which are very small quantities, when $m$ is the mass of a macro-object. Accordingly, in the macroscopic case the actual value of the two peaks in position and momentum of a stationary Gaussian solution are very close to their average values which, as we have seen, evolve in time according to Newton’s laws for a free particle moving in a (very weakly) dissipative medium. This proves that with high accuracy a stationary solutions for the center of mass of a macro-system practically behave like a point moving in space according to the laws of classical mechanics.

VIII. TIME EVOLUTION OF THE MEAN ENERGY

We now discuss one of the main purposes of our work, i.e., we show that within our collapse model the energy of an isolated system does not increase with a constant (even if negligible) rate, but reaches an asymptotic finite value. Indeed, this result is entailed by Eq. (10) for the localization operator, which motivated our choice for the statistical operator, that with high accuracy a stationary solutions for the energy of the system, a criticized feature [15] of the standard Schrödinger equation with a stochastic behavior to which a temperature $T$ can be associated and such that, with good accuracy, can be treated like a Wiener process. This would imply not only that the medium acts on the wavefunction by changing its state according to Eq. (1), but also that the wavefunction acts back on the medium according to equations which still have to be studied. The above suggestion opens the way to the possibility that by taking into account the energy of both the quantum system and the stochastic medium one can restore perfect energy conservation not only on the average but also for single realizations of the stochastic process. A similar proposal has been considered by P. Pearle [22] and by S. Adler [23].

These ideas will be subject of future research; we conclude by noting that, whatever its nature can be, the stochastic medium cannot be quantum in the usual sense since its coupling to the particle is not a standard coupling between two quantum systems: Eq. (1), in fact, is not the standard Schrödinger equation with a stochastic potential.

IX. CONCLUSIONS

We have presented and analyzed a collapse–model which preserves the standard quantum mechanical predictions and reproduces classical mechanics at the macroscopic level, at the same time avoiding the infinite growth of energy of the system, a criticized feature [17] of the space collapse models appeared in the literature. This has been obtained by drawing on an analogy with quantum Brownian motion, where friction effects, directly related to diffusion for the preservation of Heisenberg un-
certainty relation, guarantee that the energy of the test particle reaches a finite value depending on the parameters of the model. The model is also characterized by the fact that the related master-equation complies with the general structure of translation-covariant generators of quantum-dynamical semigroups obtained by Holevo [22], so that symmetry under translation is correctly kept into account, as compulsory for dynamical reduction models, which should not introduce any preferred space location.

Nevertheless the exploited analogy with the quantum Brownian motion master-equation, typically used for the description of dissipation and decoherence, should by no way induce confusion on the different nature of the two issues of decoherence and of the measurement or macro–objectification problem in quantum mechanics; as it has been stressed also in recent publications [25, 29] decoherence does not provide a solution to the measurement problem. This important conceptual difference notwithstanding, dynamical reduction models and models of environmental decoherence both impinging on the same mathematical inventory, typically used in the theory of open quantum system [31], so that results obtained in the one field can often be fruitfully exploited in the other.

An extension of this approach might be pursued in order to cope with the infinite energy growth also in the other.

APPENDIX A: THE MEAN RATE OF CHANGE OF THE VARIANCE \( \sigma_0 \)

In order to derive Eq. (61), we first compute the average value of the time derivative of the second moments of the operators \( q \) and \( p \) and of their symmetrized correlation. Using the evolution equation (11), we obtain through Itô calculus:

\[
\frac{d}{dt} \mathbb{E}[\sigma_q^2(t)] = \frac{2 \mathbb{E}[\sigma_{q,p}^2(t)]}{m} - 4 \lambda \mathbb{E}[\sigma_q^4(t)] + 4 \alpha \lambda \mathbb{E}[\sigma_q^2(t)],
\]

(A1)

\[
\frac{d}{dt} \mathbb{E}[\sigma_p^2(t)] = -4 \lambda \mathbb{E}[\sigma_{q,p}^4(t)] - 4 \alpha \lambda \mathbb{E}[\sigma_p^2(t)] + \lambda \hbar^2,
\]

(A2)

\[
\frac{d}{dt} \mathbb{E}[\sigma_{q,p}^2(t)] = \frac{\mathbb{E}[\sigma_{q,p}^2(t)]}{m} - 4 \alpha \lambda \mathbb{E}[\sigma_{q,p}^2(t)] \sigma_q^2(t).
\]

(A3)

Since \( \alpha \) is constant for a stationary solution, the stationary values of variances and correlation are such that the right-hand sides of (A1) to (A3) vanish:

\[
\frac{\sigma_q^2}{m} - 2 \alpha \lambda \sigma_q^4 + 2 \alpha \lambda \sigma_q^2 = 0,
\]

\[
\sigma_{q,p}^4 + \alpha \lambda \sigma_{q,p}^2 \sigma_q^2 = 0,
\]

\[
\frac{\sigma_p^2}{m} - 4 \lambda \sigma_{q,p}^2 \sigma_q^2 = 0.
\]

Moreover Eq. (38) can be rewritten as:

\[
\sigma_{q,p}^2 \sigma_q^2 - \sigma_{q,p}^4 = \frac{\hbar^2}{4}.
\]

(A7)

For convenience we define the quantity \( X, Y \) and \( Z \) by the equations:

\[
\sigma_q^2(t) = \sigma_q^2(1 + X),
\]

\[
\sigma_p^2(t) = \sigma_p^2(1 + Y),
\]

\[
\sigma_{q,p}^2(t) = \sigma_{q,p}^2(1 + Z),
\]

so that Eq. (A8) becomes

\[
\sigma_0^2(t) = \sigma_p^2(X + Y) - 2 \alpha \lambda \sigma_{q,p}^2 Z,
\]

(A8)

where we also made use of the Eq. (A4).

Using Eqs. (A1) to (A3) we obtain for the average value of the time derivative of \( \sigma_0^2 \):

\[
\frac{d}{dt} \mathbb{E}[\sigma_0^2(t)] = w_1 \mathbb{E}[X] + w_2 \mathbb{E}[Y] + w_3 \mathbb{E}[Z] - 4 \lambda \left( \sigma_q^4 \mathbb{E}[X]^2 + \sigma_{q,p}^4 \mathbb{E}[Z]^2 - 2 \sigma_{q,p}^4 \mathbb{E}[XZ] \right)
\]

(A9)

with

\[
w_1 = -8 \lambda \left( \sigma_q^2 \sigma_{q,p}^2 - \frac{1}{2} \alpha \sigma_q^2 \sigma_{q,p}^4 \right),
\]

(A10)

\[
w_2 = -2 \left( 2 \alpha \lambda \sigma_q^2 + \frac{\sigma_{q,p}^2}{m} \sigma_q^2 \right),
\]

(A11)

\[
w_3 = 2 \frac{\sigma_{q,p}^2 \sigma_q^2}{m} \sigma_q^2.
\]

(A12)

From Eqs. (A7) and (A5) we get for \( w_1 \)

\[
w_1 = -4 \lambda \sigma_q^4 \sigma_{q,p}^2,
\]

(A13)

while using Eq. (A4) one can prove that

\[
w_2 = w_1.
\]

(A14)

Finally, from Eq. (A6), we find that \( w_3 \) and \( w_1 \) are related as:

\[
w_3 = -\frac{\sigma_{q,p}^4}{\sigma_{q,p}^2} w_1
\]

(A15)

Acknowledgements

This work was partially supported by INFN and by MIUR under Cofinanziamento and FIRB. The work of A.B. was supported by the EU grant No. 507 547 93 - 1, AOST. Nr. 851601 - 5. We thank S. L. Adler for useful suggestions on a draft of the manuscript.
so that we have

\[ w_1X + w_2Y + w_3Z = w_1 \left( X + Y - \frac{2\mathbf{r} q^4}{\mathbf{q}^2 q^4} Z \right) = -4\lambda q^4 \sigma^2(t), \]

where the last equality is obtained through Eqs. (A16) and (A13). Finally, the above result together with Eq. (A7) allow us to write

\[ \frac{d}{dt} E[\sigma^2(t)] = -4\lambda E \left[ \sigma^2 X - 2Z^2 \right], \]

(A17)

which is just Eq. (51).

**APPENDIX B: GENERAL SOLUTION OF THE MASTER-EQUATION**

We now show how to obtain the general solution of the master-equation

\[ \frac{d}{dt} \rho_t = \frac{i}{\hbar} [H_0, \rho_t] - \frac{\lambda}{2} \{ q, [q, \rho_t] \} - \frac{\lambda \alpha^2}{2\hbar^2} [ p, \{ p, \rho_t \} ] - i \frac{\lambda \alpha}{\hbar} \{ q, \{ p, \rho_t \} \}, \]

(B1)
given in (10), which has the same form of the quantum Brownian motion master-equation, partially following an appendix of [22]. In particular we want to express the general solution in the position representation \( \langle q_1 | \rho_t | q_2 \rangle \) as a function of the solution of the pure Schrödinger equation \( \langle q_1 | \rho^S_t | q_2 \rangle \), in order to point out the corrections to the position probability density due to the non-linear stochastic modification of the Schrödinger dynamics.

As a first step we want to express the solution of (B1) as a function of the generic initial condition \( \rho_0 \) according to

\[ \langle q_1 | \rho_t | q_2 \rangle = \int dq_{10}dq_{20} G(q_{10}, q_{20}; t; q_1, q_2, 0) \langle q_1 | \rho_{0} | q_{20} \rangle, \]

(B2)

where \( G \) is the Green function solution of the partial differential equation associated to (B1) in the position representation satisfying the following initial condition

\[ G(q_{10}, q_{20}; t; q_{10}, q_{20}, 0) \xrightarrow{t \to 0} \delta(q_{1} - q_{10}) \delta(q_{2} - q_{20}). \]

(B3)

Once \( G \) is known one may immediately express the general solution as a function of the solution of the Schrödinger equation by means of the free propagator \( G_0 \)

\[ \langle q_1 | \rho_t | q_2 \rangle = \int dq_{10}dq_{20} \int du dv G(q_{10}, q_{20}; t; q_1, q_2, 0) \times G_0(q_{10}, q_{20}, 0; u, v, t) \langle u | \rho^S_t | v \rangle. \]

(B4)

For calculational purposes it is however convenient to consider the quantity

\[ \tilde{\rho}_t(k, x) = \text{Tr} \left( \rho_t e^{-i(qk + xp)} \right), \]

(B5)
corresponding to the characteristic function associated to the Wigner function. The master-equation (B1) thus becomes

\[ \frac{\partial}{\partial t} \tilde{\rho}_t(k, x) = \frac{k}{m} \frac{\partial}{\partial x} \tilde{\rho}_t(k, x) - \frac{\lambda}{2} x^2 \tilde{\rho}_t(k, x) \]

\[ - \frac{\lambda \alpha^2}{2\hbar^2} k^2 \tilde{\rho}_t(k, x) - 2\lambda \alpha x \frac{\partial}{\partial x} \tilde{\rho}_t(k, x), \]

(B6)

while \( B2 \) and \( B4 \) take the form

\[ \tilde{\rho}_t(k, x) = \int dk_0 dx_0 \tilde{G}(k, x; t; k_0, x_0, 0) \tilde{\rho}_0(k_0, x_0) \]

(B7)

and

\[ \tilde{\rho}_t(k, x) = \int dk_0 dx_0 \int dr ds \tilde{G}(k, x; t; k_0, x_0, 0) \times \tilde{G}(k_0, x_0; 0; r, s, t) \tilde{\rho}^S_t(r, s) \]

(B8)

respectively, where \( \tilde{\rho}^S_t(r, s) \) is again the solution of the free Schrödinger equation, \( G_0 \) is simply given by

\[ \tilde{G}(k, x; t; k_0, x_0, 0) = \delta(k - k_0) \delta \left( x - x_0 - \frac{k_0}{m} t \right), \]

(B9)

and \( \tilde{G} \) satisfies the following initial condition:

\[ \tilde{G}(k, x; t; k_0, x_0, 0) \xrightarrow{t \to 0} \delta(k - k_0) \delta(x - x_0). \]

(B10)

Eq. (B8) can be brought back to (B4) by exploiting the inverse relation of Eq. (B5), i.e.,

\[ \langle q_1 | \rho_t | q_2 \rangle = \frac{1}{2\pi \hbar} \int dk e^{-i(k(q_{1} - q_{2}))} \tilde{\rho}_t(k, q_1 - q_2). \]

(B11)

The key observation in order to determine \( \tilde{G} \) is the fact that Eq. (B1) preserves Gaussian states, so that given the Gaussian Ansatz:

\[ \tilde{\rho}_t(k, x) = \exp \left\{ -c_1 k^2 - c_2 k x - c_3 x^2 - ic_4 k - ic_5 x - c_6 \right\} \]

(B12)

one easily obtains the evolved state by expressing the coefficients at time \( t \), \( c_i(t) \), as a function of the initial state characterized by the values of the coefficient at time zero. Considering an initial Gaussian state \( \tilde{\rho}_0^{k_0, \epsilon, \eta} (k, x) \) which in the limit of small \( \epsilon \) and \( \eta \) approximates the Dirac delta, according to

\[ \tilde{\rho}_0^{k_0, \epsilon, \eta} (k, x) \xrightarrow{\epsilon, \eta \to 0} \delta(k - k_0) \delta(x - x_0) \]

(B13)

one has

\[ \tilde{G}(k, x; t; k_0, x_0, 0) = \lim_{\epsilon, \eta \to 0} \tilde{\rho}_t^{k_0, \epsilon, \eta} (k, x). \]

(B14)

Coming back to the Ansatz (B12), the coefficients satisfy the equations
\[ \dot{c}_1(t) = \frac{c_2(t)}{m} + \frac{\lambda\alpha^2}{2\hbar^2} \]
\[ \dot{c}_2(t) = \frac{2c_3(t)}{m} - 2\lambda\alpha c_2(t) \]
\[ \dot{c}_3(t) = \frac{\lambda}{2} - 4\lambda\alpha c_3(t) \]
\[ \dot{c}_4(t) = \frac{c_5(t)}{m} \]
\[ \dot{c}_5(t) = -2\lambda\alpha c_5(t) \]

with solutions
\[ c_1(t) = c_1(0) + c_2(0) \frac{\Gamma_t}{2m\lambda\alpha} + c_3(0) \left( \frac{\Gamma_t^2}{4m^2\lambda^2\alpha^2} - \frac{1}{32m^2\lambda^2\alpha^2} \right) \]
\[ c_2(t) = c_2(0) e^{-2\lambda\alpha t} + c_3(0) \frac{\Gamma_t e^{-2\lambda\alpha t}}{m\lambda\alpha} + \frac{\Gamma_t^2}{8m\lambda^2} \]
\[ c_3(t) = \frac{1}{8\alpha} + \left( c_3(0) - \frac{1}{8\alpha} \right) e^{-4\lambda\alpha t} \]
\[ c_4(t) = c_4(0) + c_5(0) \frac{\Gamma_t}{2m\lambda\alpha} \]
\[ c_5(t) = c_5(0) e^{-2\lambda\alpha t} \]

with \( \Gamma_t = 1 - e^{-2\lambda\alpha t} \) and \( c_6 \) simply a constant. For the choice
\[ \hat{\rho}_0^{k_0x_0,\eta}(k, x) = \frac{1}{\pi \sqrt{\eta}} \exp \left\{ -\frac{1}{\epsilon} (k - k_0)^2 \right\} \exp \left\{ -\frac{1}{\eta} (x - x_0)^2 \right\} \]

one has, exploiting (B16)
\[ \hat{\rho}_t^{k_0x_0,\eta}(k, x) = \frac{1}{\pi \sqrt{\eta}} \exp \left\{ -\frac{1}{\epsilon} (k - k_0)^2 \right\} \exp \left\{ -\frac{1}{\eta} \left( \frac{\Gamma_t k}{2m\lambda\alpha} - (x_0 - xe^{-2\lambda\alpha t}) \right)^2 \right\} \]
\[ \times \exp \left\{ -\frac{\lambda\alpha^2}{2\hbar^2} k^2 t \right\} \exp \left\{ \frac{1}{8\alpha} \left[ \frac{k^2}{4m^2\lambda^2\alpha^2} - kx \frac{\Gamma_t}{m\lambda\alpha} - x^2 (1 - e^{-4\lambda\alpha t}) \right] \right\} \]

so that taking the limit one has
\[ \hat{G}(k, x, t; k_0, x_0, 0) = \delta(k-k_0) \delta \left( \frac{\Gamma_t k}{2m\lambda\alpha} - (x_0 - xe^{-2\lambda\alpha t}) \right) \exp \left\{ -\frac{\lambda\alpha^2}{2\hbar^2} k^2 t \right\} \exp \left\{ \frac{1}{8\alpha\Gamma_t} \left[ x_0^2 K_1(t) + 2xx_0 K_2(t) + x^2 K_3(t) \right] \right\} \]

with
\[ K_1(t) = \Gamma_t^2 + 2\Gamma_t - 4\lambda\alpha t \]
\[ K_2(t) = e^{-4\lambda\alpha t} + 4\lambda\alpha te^{-2\lambda\alpha t} - 1 \]
\[ K_3(t) = -4\lambda\alpha te^{-4\lambda\alpha t} - \Gamma_t^2 + 2\Gamma_t e^{-2\lambda\alpha t} \]

and therefore (B16) now explicitly becomes
\[ \hat{\rho}_t(k, x) = \exp \left\{ -\frac{\lambda\alpha^2}{2\hbar^2} k^2 t \right\} \exp \left\{ \frac{1}{8\alpha\Gamma_t} \left[ x^2 \left( e^{-2\lambda\alpha t} + \frac{\Gamma_t k}{2m\lambda\alpha} \right)^2 K_1(t) + 2x \left( xe^{-2\lambda\alpha t} + \frac{\Gamma_t k}{2m\lambda\alpha} \right) K_2(t) + x^2 K_3(t) \right] \right\} \]
\[ \times \hat{\rho}_t^S \left( k, xe^{-2\lambda\alpha t} + \frac{\Gamma_t k}{2m\lambda\alpha} \left( 1 - \frac{2\lambda\alpha t}{\Gamma_t} \right) \right) . \]

Exploiting the inversion formula (B11) together with the expression
\[ \hat{\rho}_t(k, x) = \int dy \, e^{+ky} \langle y + \frac{r}{2} | \rho_t | y - \frac{r}{2} \rangle \]
equivalent to (B5) one finally obtains the desired explicit expression for (B4):

\[
\langle q_1 | \rho_t | q_2 \rangle = \frac{1}{2 \pi \hbar} \int dk \int dy e^{-\left(\frac{\Delta}{\hbar}\right)ky} \exp \left\{ -\frac{\lambda^2}{2 \hbar^2} k^2 t \right\} \times \exp \left\{ \frac{1}{8 \alpha \Gamma_t} \left[ \left( q_1 - q_2 \right) e^{-2\lambda t} - \frac{\Gamma_t}{\lambda \alpha k} \right] ^2 K_1(t) + 2\left( q_1 - q_2 \right) \left( q_1 - q_2 \right) e^{-2\lambda t} - \frac{\Gamma_t}{\lambda \alpha k} \right\} \times \left\langle y + \frac{q_1 + q_2}{2} + \frac{q_1 - q_2}{2} e^{-2\lambda t} + \frac{kt}{2m} \left( 1 - \frac{\Gamma_t}{2\lambda \alpha} \right) \right| \rho_t^S \left| y + \frac{q_1 + q_2}{2} - \frac{q_1 - q_2}{2} e^{-2\lambda t} - \frac{kt}{2m} \left( 1 - \frac{\Gamma_t}{2\lambda \alpha} \right) \right\rangle. \quad (B3)
\]

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