Halos of Modified Gravity*

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We describe how a certain simple modification of general relativity, in which the local cosmological constant is allowed to depend on the space-time curvature, predicts the existence of halos of modified gravity surrounding spherically-symmetric objects. We show that the gravitational mass of an object weighed together with its halo can be much larger than its gravitational mass as seen from inside the halo. This effect could provide an alternative explanation of the dark-matter phenomenon in galaxies. In this case, the local cosmological constant in the solar system must be some six orders of magnitude larger than its cosmic value obtained in the supernovae type Ia experiments. This is well within the current experimental bounds, but may be directly observable in the future high-precision experiments.

The long-standing problem of gravitationally detectable but otherwise unobserved dark matter can, in fact, suggest that general relativity theory needs to be modified under certain physical conditions which are not encountered in the solar system. This case has become particularly strong after the discovery — made by Milgrom in 1983 and confirmed afterwards — that the need for dark matter in galaxies arises as soon as the Newtonian acceleration of test bodies reaches a tiny universal value $a_0 \simeq 2 \times 10^{-10}$ m/s$^2$, which happens to be of the order of $cH_0$, where $H_0$ is the current Hubble parameter, and $c$ is the speed of light [1]. Universality of this kind can hardly be explained in the dark-matter paradigm, in which the relative amount and distribution of dark and visible matter can vary significantly from object to object, reflecting the haphazard history of formation of individual gravitationally bound systems [2].

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In general relativity, Newtonian acceleration itself is not a locally observable quantity, in view of the strong equivalence principle inherent in this theory. Therefore, in order to substantiate the principles of modified gravity and to construct a self-consistent relativistic theory exhibiting effects of dark matter, one has either to introduce additional special vector fields, with respect to which acceleration can be defined, or to choose another invariant indicating the conditions under which gravity is to be modified. The first approach is realized in the tensor–vector–scalar theories of modified gravity \[3\]; the second approach is implemented, in particular, in the modified theory of gravity which is the subject of this essay.

A general-relativistic quantity most closely connected with Newtonian relative acceleration is, of course, tidal acceleration. For two test bodies at a small relative proper distance $d$ in an external gravitational field, the relative acceleration is proportional to $d$:

$$\ddot{d} \equiv a_{\text{tidal}} = \beta d.$$  \hspace{1cm} (1)

This is what is called tidal acceleration, and the constant of proportionality $\beta$ is part of the curvature tensor, characterizing the external gravitational field. It looks reasonable to ask whether one can construct a mathematically consistent theory in which curvature invariants play the role of indicators signalling the breakdown of general relativity. At first sight, such a modification would require terms in the action involving higher powers of the curvature tensor, resulting in a higher-order metric theory of gravity with its extra degrees of freedom and severe problems of instability. It is surprising, therefore, that a formulation of general relativity exists in which the suggested modifications can easily be implemented without introducing any new degrees of freedom (in particular, without increasing the order of differential equations).

The theory which is the subject of this essay is a slight modification of the less known, albeit rather old, formulation of general relativity in the language of self-dual two-forms due to Plebański \[4\]. The Plebański action can be written as

$$S = \frac{1}{8\pi G} \int \text{Tr} \left[ BF + B \left( \Psi - \frac{1}{3} \Lambda \right) B \right],$$  \hspace{1cm} (2)

where the $\mathfrak{su}(2)$-Lie-algebra valued two-form $B$ carries information about the space-time metric, and $F$ is the curvature of an $\mathfrak{su}(2)$ connection $A$. The quantity $\Psi$ is a symmetric traceless $3 \times 3$ matrix, which is the Weyl curvature in this formulation. Wedge products of
forms is assumed in (2), and the symbol $\Lambda$ stands for the usual cosmological constant. A remarkable feature of formulation (2) of general relativity is that the Weyl curvature $\Psi$ enters here as a Lagrange multiplier. This enables one to consider a simple class of modifications of action (2) in which the cosmological constant is allowed to become a function of the two $\mathfrak{su}(2)$ invariants of $\Psi$, which are $\text{Tr} \, \Psi^2$ and $\text{Tr} \, \Psi^3$:

$$\Lambda = \Lambda \left( \text{Tr} \, \Psi^2, \text{Tr} \, \Psi^3 \right) \text{ in action (2).}$$

This change, in fact, naturally arises as a modification of the classical action (2) by quantum corrections [5].

On the classical level, the “cosmological function” $\Lambda \left( \text{Tr} \, \Psi^2, \text{Tr} \, \Psi^3 \right)$ can be chosen arbitrarily: any choice gives a consistent theory propagating just two degrees of freedom, as is the case in general relativity [6]. Therefore, one can use the opening freedom to specify it in a desirable way. First of all, in accord with the preceding reasoning, we would like this function to vary insignificantly in space-time regions of sufficiently high values of the Weyl curvature, such as those encountered in solar neighborhood, in order to comply with the solar-system experiments which show no significant deviation from the general-relativistic predictions. Next, we can also reasonably assume that this function is smooth in the neighborhood of zero curvatures. The limit of this function as $\Psi \rightarrow 0$ is to be associated with the value of the cosmological constant observed in the supernovae type Ia experiment. Indeed, in the ideal cosmological background, the quantity $\Psi$ vanishes due to space-time symmetry; in the real universe, its value will be close to zero and will determine the effective value of cosmological constant via (3). The function $\Lambda \left( \text{Tr} \, \Psi^2, \text{Tr} \, \Psi^3 \right)$ will then smoothly interpolate
between these two regions of small and large values of $\Psi$; see Fig. 1.

Many of the physical implications of our theory can be deduced from the spherically symmetric vacuum solution, which was obtained in our paper \cite{7}. Notably, the solution respects the analog of the Birkhoff theorem saying that it is static. Due to spherical symmetry, the traceless symmetric $3 \times 3$ matrix $\Psi$ is described by a single function $\psi(r)$:

$$\Psi^{ij} = \psi(r) \left( 3 \frac{x^i x^j}{r^2} - \delta^{ij} \right),$$  \hspace{1cm} (4)

where $x^i$ are the Euclidean spatial coordinates, and $r^2 = \sum_i (x^i)^2$. In this case, the cosmological function $\Lambda$ of two invariants becomes a function of $\psi$ only, and its derivative with respect to $\psi$, which we denote by $\Lambda_\psi$, represents a dimensionless quantity characterizing the deviation of the theory from the general-relativistic behavior: the condition $|\Lambda_\psi| \ll 1$ implies the validity of general relativity.

One of the consequences of the specific form of the cosmological function $\Lambda$ described above is that an isolated spherically symmetric body in our theory is surrounded by a region of approximate validity of general relativity, in which the Weyl curvature is large, and the condition $|\Lambda_\psi| \ll 1$ is well satisfied. At large distances from such a body, the value of $\psi$ tends to zero, and the derivative $\Lambda_\psi$ becomes small again since this function smoothly depends on the two invariants which are, respectively, quadratic and cubic in $\psi$. Thus, the solution is asymptotically general-relativistic at large radial distances as well as at small ones. However, at intermediate distances at which $\Lambda_\psi$ is not much smaller than unity, the theory can significantly deviate from general relativity. It is this region around a central body that we call halo of modified gravity.

The two-form field $B$ in the theory under consideration bears information about the space-time metric. It can be deduced from the requirement that the two-form $B$ is self-dual in this metric and that its volume form coincides with $\frac{1}{3} \text{Tr} (B \wedge B)$. This leads to the expression for the metric in the form

$$ds^2 = \frac{1}{\sqrt{1 - \Lambda_\psi/3}} \left[ f^2(r) dt^2 - g^2(r) dr^2 - r^2 d\Omega^2 \right],$$  \hspace{1cm} (5)

where the functions $f(r)$ and $g(r)$ are specified by

$$g^2 = \left( \frac{1 - \Lambda_\psi/3}{1 + \Lambda_\psi/6} \right) g^2_*, \hspace{1cm} f^2 = \left( \frac{1 - \Lambda_\psi/3}{1 + \Lambda_\psi/6} \right) Z^2(r) g_*^{-2}$$  \hspace{1cm} (6)
and
\[ g_s^{-2} = 1 - \frac{r_s(r)}{r} - \frac{1}{3} \Lambda(r)r^2, \quad r_s(r) = \frac{r_s^{(\infty)}}{Z(r)}. \] (7)

The relation between \( \psi(r) \) and effective Schwarzschild radius \( r_s(r) \) is usual:
\[ \psi(r) = \frac{r_s(r)}{2r^3}, \] (8)

and the function \( Z(r) \) is implicitly expressed through \( \psi(r) \) as follows:
\[ Z(\psi) = \exp \left( \int_{0}^{\psi} \frac{\Lambda_\psi(\psi')d\psi'}{6\psi'} \right). \] (9)

The constant \( r_s^{(\infty)} \) then corresponds to the Schwarzschild radius at spatial infinity.

To be more specific, consider a concrete example of the function \( \Lambda_\psi \):
\[ \Lambda_\psi = \Lambda_0 + \frac{3\alpha}{\ell^2} \log \left[ 1 + \left( \ell^2 \psi \right)^2 \right], \] (10)

where \( \ell \) is a constant of dimension length, and \( \alpha \) is a dimensionless parameter. If \( \alpha \ll 1 \), then \( |\Lambda_\psi| \ll 1 \) everywhere, and the halo of modified gravity is absent. However, there is also a theoretical upper bound \( \Lambda_\psi < 3 \) which, if violated, results in unwanted singularities in the theory, as is clear from (5), (6). As the maximum value of \( \Lambda_\psi \) for function (10) is equal to \( 3\alpha \), it is interesting to consider the values \( \alpha \lesssim 1 \). Then the halo of modified gravity...
is characterized by the magnitudes of $\ell^2 \psi$ in the range $(6\alpha)^{-1} \lesssim \ell^2 \psi \lesssim 6\alpha$. This relation determines the radial extension of the halo for every object, which, in view of (8), is given by

$$(12\alpha Z_0)^{-1/3} \lesssim \frac{r}{r_c} \lesssim (3\alpha)^{1/3}, \quad r_c = \left(r_s^{(\infty)} \ell^2\right)^{1/3},$$

(11)

where

$$Z_0 = \lim_{\psi \to \infty} Z(\psi) = e^{\alpha\pi/2}.$$  

(12)

Taking the maximal values of $\alpha \approx 1$, we get an estimate

$$0.26 \lesssim \frac{r}{r_c} \lesssim 1.44,$$

(13)

which describes the relative extension of the halo of modified gravity in our specific example (see Fig. 2 for its depiction).

Looking at (6) and (7), one can notice two interesting features characterizing the spherically symmetric solution: (i) the effective Schwarzschild radius $r_s$ becomes distance dependent, being inversely proportional to $Z(r)$; (ii) the $g_{00}$ component of the metric acquires an additional redshift/blueshift factor $Z^2(r)$. Note that, for any reasonable choice of the $\Lambda$-function, the function $Z(r)$ is monotonically decreasing with distance, asymptotically approaching unity at large distances.

The first effect can be a modified-gravity substitute for dark matter in galaxies. Indeed, it means that any object weighed from infinity, i.e., together with its halo, is much more massive than how it looks from the inside of its halo. However, this is precisely the phenomenon motivating introduction of dark matter in galaxies and galaxy clusters. The theory under consideration exhibits this dark-matter phenomenon as a consequence of modification of gravity.

To give an estimate of parameters that would be necessary here, we take $\alpha \lesssim 1$, obtaining the effect of gravitational mass increase by a factor $Z_0 = e^{\alpha\pi/2} \approx 4.8$ — typical for the dark-to-luminous matter ratios in galaxies. If needed, a larger mass “magnification” is possible by choosing a different form of the $\Lambda$-function. A typical representative of a situation requiring dark matter is a spiral galaxy like our Milky Way, of mass $M_g \sim 10^{11} M_\odot$, in which deviations from Newton’s behavior (flat rotation curves) begin at distance $r_g \approx 3$ kpc from the center. Then relation (11) gives us the estimate

$$\ell \approx \sqrt{\frac{12 r_g^3}{r_s}} \approx 5.7 \text{ Mpc},$$

(14)
where \( r_s = 2GM_g \approx 10^{-2}\text{pc} \) is the Schwarzschild radius associated with the galaxy mass contained within the radius \( r_g \). Of course, this estimate is rather crude. The real situation will be more complicated because the halo of modified gravity for a galaxy, formed by many stars, will not have a spherical shape. However, it demonstrates that, in the theory under investigation, the usual halo of dark matter might in principle be replaced by the halo of modified gravity, to the same effect.

The described mechanism of mass “magnification” implies that the values of the local “cosmological constant” should be different deep inside and outside of the halo of modified gravity. Turning again to our example (10), we see that its value near the sources of gravity can be many orders of magnitude larger than the cosmological value \( \Lambda_0 \). This is true for our estimate (14) of the parameter \( \ell \), which gives

\[
\Lambda \simeq \Lambda_0 + \frac{3\alpha}{\ell^2} \simeq 10^6 \Lambda_0 .
\]  

(15)

The current upper bounds on the cosmological constant in the solar system are \( \Lambda < 10^{10}\Lambda_0 \), much higher than (15), but the future experiments can improve the precision and possibly be capable of detecting the cosmological constant of this magnitude [8].

Another feature of the spherically symmetric vacuum solution that we mentioned above is an unusual redshift/blueshift factor in the \( g_{00} \) metric coefficient, which is seen in equation (6). The presence of this factor, in particular, implies that the a photon emitted from a region of high curvature will get blueshifted as it travels through the halo into a region of low curvature. An increase in the photon energy occurs exactly in the same proportion as an increase of the gravitational mass of the central body, since one and the same factor \( Z(r) \) controls the redshift/blueshift in (6) and the effective Schwarzschild mass in (7). What makes this effect practically unobservable is the circumstance that photons are usually emitted and detected in regions of high curvature, so that their initial blueshifts are compensated by subsequent redshifts of exactly the same magnitude.

The effects of increase of gravitational mass and photon’s energy can be unified in a simple physical picture. As any entity propagates through space from a region of high curvature (and high local cosmological constant) into a region of low curvature (and low local cosmological constant), it is essentially sliding along the “potential well” formed by the “cosmological function” \( \Lambda \). This causes its gravitating energy to increase, and this is what we observe: massive bodies develop the low-curvature halos of modified gravity
enhancing their weight, and photons get blueshifted in exactly the same proportion.

The theory under consideration with a variable curvature-dependent cosmological “constant” exhibits many other interesting physical effects, which remained beyond the scope of this short essay. Its basic direct suggestion, which we elaborated upon above, is that the value of the cosmological constant in the solar system may be many orders of magnitude larger than the value measured in cosmology. Whether this possibility is realized in Nature can only be decided by experiment. However, the described theory of gravity already represents an exciting playground for the idea that the dark-matter phenomenon is an effect of modified gravity.

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