Relativistic extensions of MOND using metric theories of gravity with curvature-matter couplings and their applications to the accelerated expansion of the Universe without dark components.

Ernesto Barrientos, Tula Bernal and Sergio Mendoza

1 Instituto de Astronomía, Universidad Nacional Autónoma de México, AP 70-264, Ciudad de México 04510, México
2 Universidad Autónoma Chapingo, Km. 38.5 Carretera México-Texcoco, Texcoco 56230, Estado de México, México

ABSTRACT

We discuss the advantages of using metric theories of gravity with curvature-matter couplings in order to construct a relativistic generalisation of the simplest version of Modified Newtonian Dynamics (MOND), where Tully-Fisher scalings are valid for a wide variety of astrophysical objects. We show that these proposals are valid at the weakest perturbation order for trajectories of massive and massless particles (photons). These constructions can be divided into local and non-local metric theories of gravity with curvature-matter couplings. Using the simplest two local constructions in a FLRW universe for dust, we show that there is no need for the introduction of dark matter and dark energy components into the Friedmann equation in order to account for type Ia supernovae observations of an accelerated universe at the present epoch.

Key words: gravitation; cosmology: theory, dark energy, dark matter, cosmological parameters; supernovae: general

1 INTRODUCTION

The concordance cosmological model requires the addition of two dark components into the energy-momentum tensor in order to balance the Hilbert-Einstein field equations, i.e. in order that Einstein’s tensor is $8\pi G/c^4$ times the energy-momentum tensor, with $c$ the speed of light and $G$ Newton’s constant of gravity. These dark matter and dark energy components are in very good agreement with a wide variety of cosmological observations (see White et al. 1993; Ostriker & Steinhardt 1995; Riess et al. 1998; Perlmutter et al. 1999; Tegmark et al. 2004; Bertone et al. 2005; Cole et al. 2005; Eisenstein et al. 2005; Komatsu et al. 2011; Bennett et al. 2013; Planck Collaboration et al. 2018, among others). However, the lack of any direct or indirect detection of dark matter and the more complicated understanding of the nature of the dark energy, have opened up different paths to explore this issue in the last decades.

One such exploration path, which is the one we are going to use in this article, is to assume that the balance of the Hilbert-Einstein action (see Landau & Lifshitz 1986):

$$S = -\frac{c^4}{16\pi G} \int R \sqrt{-g} \, d^4 x - \frac{1}{c} \int \mathcal{L}_{\text{matt}} \sqrt{-g} d^4 x,$$

be investigated. The immediate generalisation consists in building a more general Einstein tensor through a generalisation of the Hilbert action (see e.g. Landau & Lifshitz 1986):

$$S = -\frac{c^4}{16\pi G} \int R \sqrt{-g} \, d^4 x - \frac{1}{c} \int \mathcal{L}_{\text{matt}} \sqrt{-g} d^4 x,$$

by allowing for Ricci’s scalar $R$ to become a general $f(R)$ function (Capozziello & Faraoni 2011):

$$S = -\mathcal{K} \int f(R) \sqrt{-g} \, d^4 x - \frac{1}{c} \int \mathcal{L}_{\text{matt}} \sqrt{-g} d^4 x,$$

where $g := \det [g_{\mu\nu}]$ is the determinant of the metric tensor $g_{\mu\nu}$, and $\mathcal{K}$ is a coupling constant. Such kind of models are the so-called $f(R)$-theories.

In the previous equations, $\mathcal{L}_{\text{matt}}$ is the standard matter Lagrangian, which is connected to the energy-momentum tensor $T_{\mu\nu}$ through the relation:

$$T_{\mu\nu} := -\frac{2c}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_{\text{matt}})}{\delta g^{\mu\nu}}.$$
The introduction of a general $f(R)$ function into the gravitational action $S_g$ (first term on the right-hand side of equation (2)) produces fourth-order differential field equations when null variations with respect to the metric tensor are carried out (see e.g. Capozziello & Faraoni 2011).

In the non-relativistic regime, the first modification to Newton’s gravity proposed in order to explain the rotation curves and the Tully-Fisher relation of galaxies was the Modified Newtonian Dynamics (MOND) (Milgrom 1983a,b). The main idea of MOND is essentially an assumption that Kepler’s third law of planetary motion is not correct at all scales. This is based on a wide range of observations from dynamical studies of binary stars, spiral galaxies and dynamical pressure supported systems such as globular clusters, elliptical galaxies and clusters of galaxies (Hernandez et al. 2014; Durazo et al. 2017). The key ingredient to understand is that for the case of circular orbits, Kepler’s third law of planetary motion $v \propto \sqrt{M/r}$ (where $v$ is the velocity or the velocity dispersion at radius $r$ of a system with internal mass $M$) changes to a Tully-Fisher behaviour $v \propto M^{1/4}$ with no dependence on the radial distance $r$ at sufficiently large distances, where the equivalent Newtonian acceleration reaches a value of $a_0 \approx 1.2 \times 10^{-10}$ m/s$^2$. For the case of a test particle orbiting in circular motion about a point mass $M$ (like a planet orbiting about the Solar System), Newton’s basic formula for gravity is obtained using the fact that the acceleration a exerted by gravity on this test particle is given by the centrifugal force per unit mass: $a = v^2/r \propto M/r^2$. Following the same procedure when Kepler’s third law is changed by the Tully-Fisher law, then $a = v^2/r \propto \sqrt{M}/r$, in the so-called “deep-MOND” regime. The constant of proportionality is calibrated with e.g. the dynamics of rotation curves in spiral galaxies and can be written as (Mendoza 2015):

$$a = -G_M \sqrt{M/r} = -\frac{\sqrt{a_0GM}}{r} \tag{4}$$

In this sense, the MONDian acceleration $a_0$ serves as a way to choose the regime of gravity: the Newtonian regime or the deep MONDian regime or some transition function in between. Modifications of gravity at the non-relativistic regime using this approach were constructed by Mendoza et al. (2011) and it was shown using theoretical grounds and calibrations to Solar System dynamics that the transition zone occurred quite abruptly. Using this approach in globular clusters, Hernandez & Jiménez (2012) and Hernandez et al. (2013) showed very precisely that the transition occurs in a very abrupt manner and so there can only be a full Newtonian regime or a deep MONDian regime with an abrupt transition between one and the other occurring at accelerations $a_0$.

Moreover, in the search of a fundamental theory of MOND, Milgrom first noticed the coincidence relation: $2\pi a_0 \approx cH_0$, with $H_0$ the Hubble constant at the present epoch (Milgrom 1983a). The Hubble radius can be written as $R_H = c/H_0$ and the Hubble mass is given by $M_H = c^2/H_0G$. From these two relations we have $a \approx GM_H/R_H^2 \approx 2\pi a_0$ (see Milgrom 2008; Bernal et al. 2011a). Thus, the Newtonian gravitational acceleration at the present time

$1$ As in Newtonian gravity, the constant of proportionality $G_M$ in equation (4) requires to be a fundamental constant of nature. Its value is $G_M \approx 8.95 \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$, but since MOND’s basic formula was originally constructed to be a modification to inertia, then it is tradition to introduce an “ill defined” MONDian acceleration $a_0 := GM_G^2/G$ into the fundamental constants scenario and forget about $G_M$ (Mendoza 2015).

is approximately MONDian. In this sense, we should develop a MONDian relativistic treatment to understand the dynamics of the Universe without dark components. Also, it has been proposed that MOND’s acceleration constant is a fundamental constant that arises from astrophysical phenomenology (Mendoza et al. 2011; Bernal et al. 2011a).

A full potential non-relativistic theory of MOND was constructed by Bekenstein & Milgrom (1984) using an AQUA/daratic Lagrangian (AQUAL) which happens to predict an external field effect (Famaey & Binney 2005). Since an external field effect is not present in the galaxy with wide open binaries motion not following Newton’s law of gravitation (Hernandez et al. 2012, 2019), then one can no rely on all the results such approach present.

Bekenstein (2004) constructed a complicated relativistic theory using tensor, vector and scalar fields, which at the weak field limit of approximation converges to MOND. The mathematical complexities of the theory and the many problems it had presented to fit various astrophysical scenarios (see e.g. Ferreras et al. 2009, and references therein) required thoughts on generalisations even at the most fundamental aspects of gravitation: the action.

In previous studies, it has been shown that in order to build a relativistic version of MOND using $f(R)$ theories in the pure metric approach or in the Palatini formalism or even with the inclusion of torsion, curvature-matter couplings need to be introduced into the gravitational action (Mendoza et al. 2011; Bernal et al. 2011b; Mendoza et al. 2013; Barrientos & Mendoza 2016, 2017, 2018). Bertolami et al. (2007) showed that for a particular generalisation of the $f(R)$ theories in the metric approach, by coupling the $f(R)$ function with the Lagrangian density of matter $\mathcal{L}_{\text{mat}}$, an extra-force arises which in the weak field limit can be connected with MOND’s acceleration and possibly explain the Pioneer anomaly.

Working with a static spherically symmetric space-time, Bernal et al. (2011b) constructed a gravitational $f(R)$ action in the pure metric approach which depends explicitly on the central point mass $M$ of the problem. This model became very successful explaining the non-relativistic Tully-Fisher weak field limit and turned out to be coherent when explaining deflection of light on individual, groups and clusters of galaxies (Mendoza et al. 2013).

The same model was also applied to reproduce the dynamical masses of 12 Chandra X-ray galaxy clusters, through the fourth-order metric coefficients, roughly extending the model to mass distributions (Bernal et al. 2019). Such theory was also able to describe a correct acceleration of the local Universe, without the introduction of dark matter and dark energy components, through type Ia supernovae (SNe Ia) observations, for the functional mass $M$ taken as the causally connected mass to a given fundamental observer moving with the Hubble flow, i.e. it was taken to be the Hubble mass (Carranza et al. 2013).

Since the mass function $M$ depends on the amount of matter at any given point for a given fundamental observer, the action is non-local (Tong 2007; Alexeev 2017; Maggiore & Mancarella 2014; Hehl & Mashhoon 2009; Blome et al. 2010). Although, in principle, there is nothing wrong with the construction of a non-local theory of gravity, we have got adapted to the idea of locality in physics, and gravitational theory is no exception. The search of a local relativistic theory of gravity in the full MONDian regime with a pure metric approach was carried on with success by Barrientos & Mendoza (2018), where the bending of light was successfully reproduced on such regime. Attempts on the construction of relativistic theories of MOND in the metric-affine formalism and with torsion were carried out by Barrientos & Mendoza (2016) and Barrientos & Mendoza (2017) respectively. The metric-affine one
being non-local. In these two last cases, the resulting equations are quite cumbersome and lead to no-simple paths for their applicability in astrophysical and cosmological environments.

In summary, all previous results signal that curvature-matter couplings are required with a more general action:

\[ S = \int F(R, \mathcal{L}_{\text{mat}}) \sqrt{-g} \, d^4x, \]

as described in Harko & Lobo (2010); Bertolami et al. (2007): Allemandi et al. (2005), all generalisations of the pioneer work of Goenner (1984). In the previous equation, \( F \) is a general function of the Ricci scalar \( R \) and the matter Lagrangian \( \mathcal{L}_{\text{mat}} \). The null variations of action (5) with respect to the metric tensor yield the following field equations (Harko & Lobo 2010):

\[ F_{\alpha\beta}R_{\alpha\beta} + (g_{\alpha\beta} \nabla^\mu \nabla_\mu - \nabla_\alpha \nabla_\beta) F_R - \frac{1}{2} (F - \mathcal{L}_{\text{mat}} F_{\mathcal{L}_{\text{mat}}}) g_{\alpha\beta} = \frac{1}{2} F_{\mathcal{L}_{\text{mat}}} T_{\alpha\beta}, \]

with \( T_{\alpha\beta} \) Ricci’s tensor and the trace given by:

\[ F_{\alpha\beta} R + 3\Delta F_R - 2 (F - \mathcal{L}_{\text{mat}} F_{\mathcal{L}_{\text{mat}}}) = \frac{1}{2} F_{\mathcal{L}_{\text{mat}}} T, \]

where \( T := T^\mu_\mu \) is the trace of the energy-momentum tensor, \( F_R := \partial F/\partial R \) and \( F_{\mathcal{L}_{\text{mat}}} := \partial F/\partial \mathcal{L}_{\text{mat}} \). The motion of test free particles is expressed through the following non/geodesic equation:

\[ u^\alpha \nabla_\alpha u^\beta = \frac{Du^\beta}{ds} = \frac{d^2 x^\beta}{ds^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = f^\beta, \]

with \( \Gamma^\mu_{\nu\rho} \) representing the Levi-Civita connection, \( ds \) the interval and the curvature-matter extra force vector \( f^\beta \) is given by the following relation:

\[ f^\beta := \left( \frac{g_{\lambda\mu}}{u^\lambda} - u^\beta u^\lambda \right) \nabla_\lambda \left( F_{\mathcal{L}_{\text{mat}}} \frac{d\mathcal{L}_{\text{mat}}}{d\rho} \right), \]

where \( \mathcal{L}_{\text{mat}} \) is the matter Lagrangian which will be further discussed in Section 2. The extra force \( f^\beta \) generated by the curvature-matter coupling is orthogonal to the velocity \( u^\beta \), since \( u_\alpha \nabla_\alpha u^\beta = 0 \).

In this article, we show that by using the pure metric approach of an extended gravitational action with curvature-matter couplings it is possible to build an infinite number of non-local and local relativistic descriptions of MOND as their weak-field limit. Since the interpretation of the mass function \( M \) might be only evident in systems with high degrees of symmetry (such as spherically symmetric ones and/or isotropic spaces), we chose what appears to be the two simplest proposals and test their validity in an expanding Friedmann-Lemaître-Robertson-Walker (FLRW) Universe, with no dark matter, nor dark energy components.

In Section 2 we describe the simplest assumptions we can make for the Lagrangian and show that there are an infinite number of relativistic theories that can yield MOND in the weak-field regime. In Section 3 we discuss the use of the cosmographic parameters as a model-independent way to constrain the cosmological observables. In Section 4 we show the field equations in a cosmological scenario using the FLRW metric with curvature-matter couplings, in view of the constraints to the cosmographic parameters for two specific models. In Section 5 we show the results of fitting the two models chosen with SNe Ia observations. Finally, in Section 7 we discuss our results and present our conclusions.

## 2 General Metric Curvature-Matter Coupling

The field equations (6) for a general function \( F(\chi, \xi) \), where \( \chi := \alpha R \) and \( \xi := \mathcal{L}_{\text{mat}} \lambda \), with \( \alpha \) and \( \lambda \) coupling constants which make \( \xi \) and \( \chi \) dimensionless quantities, are given by (Barrientos & Mendoza 2018):

\[ \alpha F_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (F - F_\xi) \xi + \alpha (g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) F_\chi = \frac{F_\xi T_{\mu\nu}}{2\lambda}, \]

with \( F_\xi := \partial F/\partial \chi \), \( F_\xi := \partial F/\partial \xi \) and the Laplace-Beltrami operator \( \Delta := \nabla^\mu \nabla_\mu \). The last equation can be written as

\[ R_{\mu\nu} = \frac{1}{\alpha F_\chi} \left[ \frac{F_\xi T_{\mu\nu}}{2\lambda} - \frac{1}{2} g_{\mu\nu} F_\xi \chi + T^\text{curv}_{\mu\nu} \right], \]

where we have defined

\[ T^\text{curv}_{\mu\nu} := \frac{1}{2} g_{\mu\nu} F_\xi - \alpha (g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) F_\chi. \]

In order to construct a relativistic theory which converges to MOND in its weak-field limit, we follow the procedures shown in Bernal et al. (2011b); Barrientos & Mendoza (2017, 2018), and as such we assume that the function \( F(\chi, \xi) \) has the following form:

\[ F(\chi, \xi) = \chi^p \xi^q + \xi^q, \]

where \( p, u \) and \( v \) are unknown real numbers.

To build the coupling constants \( \alpha \) and \( \lambda \), we use Buckingham’s II theorem of dimensional analysis (cf. Sedov 1959) in the following way. Since we are interested in a relativistic action for MOND, we choose \( c, G \) and \( a_0 \) as our independent variables. The key feature of MOND is the introduction of a fundamental acceleration constant \( a_0 = 1.2 \times 10^{-10} \text{ms}^{-2} \), as a cut-off scale into gravitational phenomenon (see e.g. Mendoza et al. 2011; Mendoza 2015). The dimensions of these three constants are given by

\[ [c] = lt^{-1}, \quad [a_0] = lt^{-2} \quad \text{and} \quad [G] = l^3t^{-2}m^{-1}, \]

where \( l, t \) and \( m \) stand for dimensions of length, time and mass, respectively. It is straightforward to show that it is not possible to write \( c, G \) and \( a_0 \) as a combination of the other two and therefore they are useful choices to take as independent dimensional variables. Since the dimensions of \( R \) and \( \mathcal{L}_{\text{mat}} \) are

\[ [R] = l^{-2} \quad \text{and} \quad [\mathcal{L}_{\text{mat}}] = l^{-1}mt^{-2}, \]

then by demanding that

\[ [R] = [c]^a [G]^b [a_0]^d \quad \text{and} \quad [\mathcal{L}_{\text{mat}}] = [c]^A [G]^B [a_0]^D, \]

we obtain \( a = -4, b = A = 0, d = 2, B = -1 \) and \( D = 2 \). Thus, we can see that the coupling constants \( \alpha \) and \( \lambda \) are given by

\[ \alpha = k \frac{c^4}{a_0^3} \quad \text{and} \quad \lambda = \frac{\kappa c^2}{G}, \]
The total energy density takes the value $e = \rho c^2$ for a dust dominated universe with $p = 0$, and $e = -(\kappa - 1)p$ for a radiation dominated Universe with $\kappa = 4/3$ and a cosmological vacuum dominated Universe with $\kappa = 0$ (see e.g. Longair 2008). In the present article we are only interested in cosmological applications for a dust dominated universe and so:

$$T_{\mu\nu} = \rho c^2 u_\mu u_\nu. \quad (18)$$

In general terms, the expression for the matter Lagrangian is a controversial expression, even for the case of a perfect fluid (see e.g. Avelino & Azevedo 2018; Minazzoli & Harko 2012, and references therein). However, during recent years it became clear that for the case of dust it takes the following form as noted by Hawking & Ellis (1973; Bertolami et al. 2008)²:

$$\mathcal{L}_{\text{mat}} = \rho c^2 \quad (19)$$

Throughout this article, we will work with a dust dominated Universe. Therefore, we will use the energy-momentum tensor of a perfect fluid with pressure $p = 0$ which leads to relation (18) and equation (19).

To order of magnitude, the trace of the field equations (10) takes the following form:

$$F_\mu \chi + (F - F_\xi \chi) + \alpha \Delta F_\chi \sim \frac{F_i T_i}{\lambda}. \quad (20)$$

Using function (13), the last expression turns into:

$$\chi^{p-1} \xi^{u} + \xi^{v} + \alpha \Delta(\chi^{p-1} \xi^{u}) \sim \left(\chi^{p-1} \xi^{u-1} + \xi^{v-1}\right)T_i. \quad (21)$$

For the case of dust $T = \mathcal{L}_{\text{mat}}$ and so, the last equation is expressed as:

$$\alpha \Delta(\chi^{p-1} \xi^{u}) \sim \chi^{p} \xi^{u} + \xi^{v}. \quad (22)$$

We now explore the previous equation to perturbation order of powers of $c^{-1}$ (order $O(1)$)³. To do so, note that the metric coefficients at second perturbation order are given by (see e.g. Mendoza & Olmo 2014)⁴:

$$g_{00} = (0) g_{00} + (2) g_{00} = 1 + \frac{2\phi}{c^2},$$

$$g_{ij} = (0) g_{ij} + (2) g_{ij} = \delta_{ij} \left(-1 + \frac{2\phi}{c^2}\right), \quad (23)$$

$$g_{0i} = 0,$$

where $\phi$ is the non-relativistic scalar gravitational potential. There is not need to expand the metric beyond a second order in powers of $c^{-1}$ since in the weak field limit of a relativistic theory, the dynamics of massive particles is determined by the $O(2)$ time-component of the metric, while the deflection of light is determined by the $O(2)$ radial one (Will 1993, 2006).

With the metric (23), the Ricci scalar is given by: $R = -2\nabla^2\phi/c^2$. Using these results and the coupling constants given in (17), the perturbation orders of the terms in equation (22) are given by:

$$\mathcal{O}^{(2(p+u+1))} \mathcal{O}^{(2(p+u))} \mathcal{O}(p-u)$$

$$\alpha \Delta(\chi^{p-1} \xi^{u}) \sim \chi^{p} \xi^{u} + \xi^{v}. \quad (24)$$

Since $(p+u+1) > (p+u)$, the coefficients are chosen to satisfy the relation $p + u + 1 = \nu$. Using this and, since to order of magnitude, $\nabla \sim r^{-1}$, the trace (22) of the field equations can be expressed as:

$$\alpha p R^{p-1} \mathcal{L}_{\text{mat}} \sim \frac{\mathcal{L}_{\text{mat}}}{\lambda^p}. \quad (25)$$

To order of magnitude, the acceleration $a = |\nabla \phi| \sim \phi/r$, and for a point mass $M$ the matter Lagrangian (19) takes the form $\mathcal{L}_{\text{mat}} = \rho c^2 \sim M/r^3$. With these results we obtain:

$$\alpha p R^{p-1} \mathcal{L}_{\text{mat}} \sim \frac{(MG)^{p-u} \phi^{v-u}}{a_0^{2(p+u)} a_3^{3(p-u)}}. \quad (26)$$

The powers of $c$ must be the same in both sides of last equation. This is achieved when $v = p + u + 1$ and so, the expression for the acceleration takes the following form:

$$a = \frac{(MG)^{(p+1)}((p+1)-3)}{a_0^{p+1-a_0+1}(p+1)}/(p+1) \quad (27)$$

The acceleration in the so-called “deep-MOND” regime is $a = \sqrt{M G a_0}/r$ (Milgrom 1983a,b). In order to obtain the correct power for all the variables involved in the last equation ($M$, $G$, $a_0$ and $r$) and the MOND-like limit, the value $p = -3$ is found. For this value, the relation between $v$ and $u$ is $v = u - 2$. Since there is not another constriction, we conclude that there exist an infinite number of models which yield the MONDian basic formula.

Amongst all those models, we are interested in two cases: the first one was introduced by Barrientos & Mendoza (2018), where the authors were looking for an action with the matter Lagrangian alone in the matter sector, as is traditionally done, i.e. $v = 1$, in

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² For the case of dust it turns out that the choice $\mathcal{L}_{\text{mat}} = \rho c^2$, or $\mathcal{L}_{\text{mat}} = -p$, the latter obtained by (Schutz 1970; Brown 1993), yield the same field equations of general relativity: the Hilbert-Einstein field equations. However, the choice $\mathcal{L}_{\text{mat}} = -p$ may not be useful when matter-curvature couplings are taken into account and so, the value $\mathcal{L}_{\text{mat}} = \rho c^2$ is the correct choice (Bertolami et al. 2008).

³ We use the notation $O(n)$ for $n = 0, 1, 2, \ldots$ meaning $O(c^0), O(c^{-1}), O(c^{-2}), \ldots$, respectively, following the well known notation of Will (1993).

⁴ In general terms, when using isotropic coordinates, the space components of the metric $g_{ij} = \delta_{ij} (1 - 2\phi/c^2)$, but as shown by Mendoza et al. (2013); Mendoza (2015); Mendoza & Olmo (2014) the first Parametrised Post Newtonian (PPN) Parameter $\gamma = 1$ when studying the bending of light of individual, groups and clusters of galaxies under the assumption of a non-relativistic scalar MONDian potential.
Relativistic extensions of MOND: cosmology without dark components

The consequence $u = 3$ and so $F(\chi, \xi) = \chi^{-3} \xi^3 + \xi$. The second option is given by the choice $u = 0$ and $v = -2$, i.e. $F(\chi, \xi) = \chi^{-3} + \xi^{-2}$. This last model represents a more traditional point of view where the curvature and matter are independent parts of the action (see Table 1).

Note that one can also introduce non-locality in such a way that the coupling parameter $a$ is a function of the mass function $M^2$, which from now on we will assume to be a power law of the form $M^3$ and so from (13) the function $F \propto M^3 R^p L^q_{\text{matt}} + L_{\text{matt}}$. In general terms, the mass $M$ can be thought of as the causal mass around any given point in space (for example, the mass within a Hubble radius for a fundamental observer in cosmological applications). For this particular situation and following the same procedure as with the local approach and setting for simplicity $\alpha = 1$ it follows from the trace of the field equations that $R^{p-1} \propto M^{1-u-q} / p^{1-3u}$ and so:

$$a \propto \frac{M^{(1-u-q)/(p-1)}}{r^{2(p-3q)/(p-1)}} \quad (28)$$

In this case, the choice $p = 6q - 3$ and $u = 3 - 4q$ yields MOND for any arbitrary value of $q$. The choice $u = 0$, i.e. $q = 3/4$, yields $p = 3/2$, which corresponds to the first metric extension of MOND built by Bernal et al. (2011b). All these results are summarised in Table 1. Due to the complicated nature of the function $M$ in the non-local proposal and since it is tradition to work with locality in physical constructions of gravity, in what follows we assume locality.

3 COSMOGRAPHY

In order to constrain the cosmological observables in a model-independent way, one can appeal to cosmography, i.e. using Taylor expansions of the scale factor $a(t)$ with respect to the cosmic time $t$ in order to have a distance-redshift relation that only depends on the FLRW metric (Weinberg 2008). In this way, the Taylor expansion is given by

$$H = \frac{1}{a} \frac{da}{dt}, \quad (29)$$
$$q = -\frac{1}{a} \frac{d^2a}{dt^2} H^{-2}, \quad (30)$$
$$j = \frac{1}{a} \frac{d^3a}{dt^3} H^{-3}, \quad (31)$$
$$s = \frac{1}{a} \frac{d^4a}{dt^4} H^{-4}, \quad (32)$$
$$l = \frac{1}{a} \frac{d^5a}{dt^5} H^{-5}, \quad (33)$$

which are called the Hubble, deceleration, jerk, snap and lerk parameters, respectively. Their values at the present epoch $t_0$ are denoted by a subscript 0. All models can be characterised by these parameters: when $H_0 > 0$ produces an expanding universe and $H_0 < 0$ a contracting one; $q_0 < 0$ gives an accelerating expansion, $q_0 > 0$ a deceleration one and $q_0 = 0$ yields zero acceleration; $H_0 = 0$ and $q_0 = 0$ represent a static universe. If the Taylor expansion is truncated to lower orders significant deviations for $z \gtrsim 1$ appear.

We use the standard definition of the density parameter $\Omega$:

$$\Omega := \frac{8\pi G\rho}{3H^2}, \quad (34)$$

such that in standard cosmology $\Omega = \Omega_m + \Omega_\Lambda + \Omega_\nu$, with $\Omega_m$ the matter density parameter, $\Omega_\Lambda$ the effective mass density attributed to dark energy, and $\Omega_\nu$ the curvature density parameter. In order to explain the accelerated expansion of the Universe without dark energy, we assume $\Omega_\Lambda = 0$, and reproduce the SN1a observations with the extra terms coming from the generalised gravitational action (5) with curvature-matter couplings. Moreover, the relativistic extensions we are dealing with converge to a MOND-like behaviour in the weak-field limit of the theory. As explained in Section 2, we recover MONDIian accelerations that explain the astrophysical observations for systems at accelerations $a < a_0$, without the inclusion of dark matter. In this case, we assume that we are dealing with $\Omega_m = \Omega_{bar}$, the baryonic contribution to the matter density parameter only.

For a universe with $\Omega_\nu \neq 0$, the cosmographic parameters are given by (Li et al. 2020):
\[ q_0 = \frac{3}{2} \Omega_m \Omega_{\Lambda} - 1, \quad (35) \]
\[ j_0 = 1 - \Omega_{\Lambda}, \quad (36) \]
\[ s_0 = 1 - \frac{9}{2} \Omega_m + \Omega_{\Lambda} - \Omega_{\Lambda} \left( 2 - \frac{3}{2} \Omega_m \right), \quad (37) \]
\[ l_0 = 1 + 3 \Omega_m + \frac{27}{2} \Omega_{\Lambda}^2 + \Omega_{\Lambda} - \Omega_{\Lambda} \left( 2 - 9 \Omega_m \right). \quad (38) \]

For the standard \( \Lambda \)CDM model (\( \Omega_{\Lambda} = 0 \)) the cosmographic parameters reduce to:
\[ q_0 = \frac{3}{2} \Omega_m - 1, \quad (39) \]
\[ j_0 = 1, \quad (40) \]
\[ s_0 = 1 - \frac{9}{2} \Omega_m, \quad (41) \]
\[ l_0 = 1 + 3 \Omega_m + \frac{27}{2} \Omega_m^2. \quad (42) \]

It is possible to use diverse estimations of the free parameter \( \Omega_m \) to obtain the values of the cosmographic parameters. From the last results of Planck collaboration based in a \( \Lambda \)CDM cosmology we have: \( H_0 = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad \Omega_m = 0.315 \pm 0.007 \) and \( \Omega_{\Lambda} = 0.0007 \pm 0.0019 \) (Planck Collaboration et al. 2018). Thus, observations are in agreement with a spatially flat universe. With these results, the predictions for the \( \Lambda \)CDM model are: \( q_0 = -0.527 \pm 0.0105, \quad j_0 = 1, \quad s_0 = -0.417 \pm 0.0315 \) and \( l_0 = 3.284 \pm 0.0631 \).

4 COSMOLOGY

There are many applications of extended theories of gravity to cosmology (see e.g. Nojiri et al. 2017, and references therein), but very few of them introduce curvature-matter couplings in the gravitational action (cf. Lobo & Harko 2012) and in any case they are only used to deal with no dark energy. We are interested in curvature-matter couplings with no dark matter, nor dark energy for which the weak-field limit of approximation converges to MOND deep regime. To do so, we proceed in the following manner.

For the cosmological applications we are interested in, we use a FLRW metric given by:
\[ ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right], \quad (43) \]
where \( \kappa \) is the curvature of the universe and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the angular displacement, with \( \theta \) and \( \phi \) the polar and azimuthal angles, respectively. From hereafter we adopt the convention: \( x^0 = ct \) and \( g_{00} = 1 \).

Using equations (18) and (19), the 00 component of \( T_{\mu \nu}^{\text{curv}} \) (equation (12)) takes the value:
\[ T_{00}^{\text{curv}} = \frac{1}{2} \left( F - \chi F_x \right) - \frac{3 \alpha}{c^2} H \dot{F}_x, \quad (44) \]
where \( \dot{()} \) means derivative with respect to the time coordinate \( t \) and \( H \) is the Hubble parameter defined as \( H := \dot{a}/a \).\(^6\)

In order to compute the 00-field equation from (11), Ricci’s scalar \( R \) and the 00-component of Ricci’s tensor \( R_{\mu \nu} \) for the FLRW metric (43) are required. Such quantities are given by
\[ R = -6 \left( \frac{a \ddot{a} + \dot{a}^2 + \kappa c^2}{c^2 a^2} \right), \quad (45) \]
\[ R_{00} = -\frac{3 \dot{a}}{a}. \quad (46) \]

Therefore, from the 00-component of equations (11) the Hubble parameter is
\[ H^2 = \frac{c^2}{3 \alpha F_x} \left[ \rho c^2 F_x + \frac{F - \chi F_x - \xi F_x - 3H \alpha}{c^2 F_x} - \frac{\kappa c^2}{a^2} \right], \quad (47) \]
Notice that for the spatial field equations required, due to the symmetry, we have only one independent equation, so we choose \( i = j = 1 \). In order to obtain the expression for this component, \( T_{11}^{\text{curv}} \) and \( R_{11} \) are needed. Such tensor components are given by
\[ T_{11}^{\text{curv}} = \frac{a^2}{1 - \kappa a^2} \left[ \frac{1}{2} \left( F - \chi F_x \right) + \frac{\alpha}{c^2} \left( F_x + 2H \dot{F}_x \right) \right], \quad (48) \]
and
\[ R_{11} = 2 \kappa c^2 + a \dot{a} + 2 \dot{a}^2 \frac{c^2}{c^2 (1 - \kappa a^2)}. \quad (49) \]
Substituting these relations into equation (11), the following equation for the 11 radial-component of the field equations is obtained:
\[ \frac{-\kappa c^2 + 2a \dot{a} + \dot{a}^2}{a^2 c^2} = \frac{1}{\alpha F_x} \left[ \frac{PF_x}{2} - \frac{F - \chi F_x - \xi F_x}{2} - \frac{3H \alpha}{c^2 F_x} \right], \quad (50) \]

4.1 “Weak” curvature-matter coupling model

Let us assume the following form for the function \( F(\chi, \xi) \):
\[ F(\chi, \xi) = f(\chi) + g(\xi), \quad (51) \]
this is, a “weak” curvature-matter coupling, as curvature \( \chi \) and matter \( \xi \) appear independently into the gravitational action and in the resulting field equations. With this, the Hubble parameter (47) turns into:
\[ H^2 = \frac{c^2}{3 \alpha F_x} \left[ \rho c^2 F_x + \frac{f + g - \chi f_x - \xi g_x - 3H \alpha}{c^2 f_x} \right] - \frac{\kappa c^2}{a^2}, \quad (52) \]
where \( f_x := df/d\chi \) and \( g_x := dg/d\xi \). By considering a dust universe it follows that \( T_{00} = \mathcal{L}_{\text{mat}} = \rho c^2 \). Thus \( \xi = \rho c^2/\lambda \) and so, the previous equation reduces to
\[ H^2 = \frac{c^2}{3 \alpha F_x} \left[ \frac{f + g - \chi f_x - 3H \alpha}{c^2 f_x} \right] - \frac{\kappa c^2}{a^2}. \quad (53) \]
Given the results from Planck Collaboration et al. (2018) mentioned in the previous section, from now on we will assume a flat Universe with \( \kappa = 0 \).

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Relativistic extensions of MOND: cosmology without dark components

Let us assume a power-law distribution for the functions \( F = \chi^\gamma \) and \( g = \xi^\delta \), with \( \gamma \) and \( \beta \) real constants, so that

\[
F(\chi, \xi) = \chi^\gamma + \xi^\delta. \tag{54}
\]

With this choice, equation (53) becomes

\[
H^2 = \frac{c^2}{3\alpha \gamma \chi^{\gamma-1}} \left[ \frac{(1 - \gamma) \chi^\gamma + \xi^\delta}{2} - 3H\alpha \gamma d \chi^{\gamma-1} \right]. \tag{55}
\]

In order to simplify the previous expression, two cosmographic parameters are particularly useful: the deceleration \( q \) and the jerk \( j \) given by equations (30) and (31) respectively. With such definitions, Ricci’s scalar \( R \) shown in relation (45) and its time derivative \( \dot{R} \) for a flat universe take the following form:

\[
R = -\frac{6H^2}{c^2}(1 - q) \quad \text{and} \quad \dot{R} = RH \left( \frac{j - q - 2}{1 - q} \right). \tag{56}
\]

With all this, it follows that:

\[
\frac{d}{dt} (\chi^{\gamma-1}) = \chi^{\gamma-1} H(\gamma - 1) \left( \frac{j - q - 2}{1 - q} \right). \tag{57}
\]

After substitution of this last result and expressions (56) into equation (55), we obtain the following relation between the Hubble parameter \( H \) and the matter density \( \rho \):

\[
H^2 = \frac{(q - 1)^{1 - \gamma} \gamma^2 (\gamma + \beta)}{6\gamma^2 \chi^{\gamma-1} \rho^\beta}, \tag{58}
\]

where the function \( Z \) is defined as

\[
Z := 1 + (\gamma - 1) \left( \frac{q - 1}{\gamma} + \frac{j - q - 2}{1 - q} \right). \tag{59}
\]

After using the definition of the density parameter \( \Omega \) (equation (34)) and the coupling constants given in (17), equation (58) can be simplified to yield a modified Friedmann equation for the weak curvature-matter model:

\[
H^2 = \frac{\alpha_0}{c^2} \left[ (q - 1)^{1 - \gamma} \frac{\chi^{\gamma-1}}{6\gamma^2 \chi^{\gamma-1}} \left( \frac{3\Omega}{8\pi} \right)^\beta \right]^{1/2(\gamma - \beta)}. \tag{60}
\]

By using equations (30), (31) and (54), the 11-component of the field equations (50) can be written as:

\[
\frac{H^2}{c^2} (2q - 1) = \frac{1}{\alpha_0 \gamma \chi^{\gamma-1}} \left\{ \frac{(\beta - 1) \xi^\delta + (\gamma - 1) \xi^\delta}{2} \right\}^{1/2(\gamma - \beta)}.
\]

In order to compute \( \dot{f}_X \), the following expressions are useful:

\[
\dot{H} = -H^2(1 + q), \tag{62}
\]

\[
\dot{q} = -H(j - q - 2q^2), \tag{63}
\]

\[
\ddot{H} = H^3(2 + 3q + j), \tag{64}
\]

\[
\dot{j} = H(s + 2j + 3q), \tag{65}
\]

\[
\dddot{H} = -H^2(6 + 12q + 4j + 3q^2 - s), \tag{66}
\]

and so, using the previous relations and equation (57) we obtain:

\[
\dot{f}_X = \frac{\gamma (\gamma - 1)}{1 - q} \chi^{\gamma-1} H^2 A, \tag{67}
\]

where:

\[
A := (j - q - 2) \left[ (\gamma - 1) \left( \frac{j - q - 2}{1 - q} \right) - 1 - q \right] + s - q + 3j(1 + q) - 2q^2 + (j - q - 2)(q + 2q^2 - j). \tag{68}
\]

Substitution of this last result with the definitions of \( \chi \) and \( \xi \), together with equation (57), into relation (61) yield:

\[
\frac{H^2}{c^2} \left[ 2q - 1 + \frac{1 - \gamma}{1 - q} (A + 2(j - q - 2)) + \frac{3}{\gamma} (\gamma - 1)(1) \right] = \frac{\beta - 1}{2a\gamma} \left( \frac{\rho c^2}{\lambda} \right)^\alpha \left[ \frac{6\alpha H^2}{c^2}(q - 1) \right]^{1/\gamma}. \tag{69}
\]

In order to simplify this relation, we rewrite equation (58) as

\[
\frac{6Z H^2}{c^2} = \frac{1}{\alpha_0 \gamma} \left( \frac{\rho c^2}{\lambda} \right)^\alpha \left[ \frac{6\alpha H^2}{c^2}(q - 1) \right]^{1/\gamma}. \tag{70}
\]

Substitution of this relation into (69) yields:

\[
2q - 1 + \frac{1 - \gamma}{1 - q} A + \frac{3}{\gamma} (\gamma - 1)(1 - q) + 3(1 - \beta) Z = 0. \tag{71}
\]

The previous equation involves exclusively the cosmographic parameters \( q, j \) and \( s \). Thus, we can express the snap parameter \( s \) as a function of the deceleration \( q \) and jerk \( j \) parameters, i.e. \( s = s(q, j) \), decreasing the number of free parameters to fit with SNe Ia observations as explained in Section 5.

4.1.1 Geodesic equation

The contravariant form of the field equations (10) for the weak curvature-matter model given in (51) is:

\[
\alpha f_X R^\mu{}^\nu - \frac{1}{2} g^\mu{}^\nu f + \alpha (g^\mu{}^\nu \nabla^\mu \nabla^\nu - \nabla^\mu \nabla^\nu) f_X = \frac{1}{2} g^\mu{}^\nu (g - e g t^2) + \frac{e T^\mu{}^\nu}{2\lambda}. \tag{72}
\]

Taking the covariant divergence of the last equation and bearing in mind that \( \nabla_\mu f = f_\mu \partial_\mu \chi \) and \( \nabla_\mu g = g_\mu \partial_\mu \xi \), we obtain:

\[
-\frac{1}{2} g^\mu{}^\nu \xi \nabla^\rho g_\xi + \nabla_\mu g_\xi T^\mu{}^\nu + \frac{g_\xi \nabla_\rho g_\xi T^\mu{}^\nu}{2\lambda} = \alpha f_X \nabla_\mu R^\mu{}^\nu + \nabla_\mu f_\xi R^\mu{}^\nu + \alpha g^\mu{}^\nu (\partial_\mu \Delta - \Delta \nabla^\mu) f_X - \frac{1}{2} g^\mu{}^\nu f_\xi \partial_\mu \chi. \tag{73}
\]

Since \( (\Delta \nabla_\mu - \nabla_\mu \Delta) \phi = R_{\mu\nu} \nabla^\nu \phi \) Koivisto (2006); Pani et al. (2013), the previous equation becomes

\[
\alpha f_X \nabla_\mu \left( R^\mu{}^\nu - \frac{1}{2} g^\mu{}^\nu R \right) = \frac{1}{2\lambda} (T^\mu{}^\nu \nabla_\mu g_\xi - g_\xi \nabla_\mu T^\mu{}^\nu) - \frac{1}{2} g^\mu{}^\nu \xi \nabla_\mu g_\xi. \tag{74}
\]

Using Bianchi’s identities, or equivalently the null divergence
of Einstein’s tensor, the left-hand side of the previous equation vanishes. Thus, the conservation equation is given by

$$\nabla_\mu T^{\mu\nu} = (T^{\mu\nu} - g^{\mu\nu} L_{\text{mat}}) \nabla_\mu \ln g_\xi. \quad (75)$$

We are interested in the 00 time-component of the last expression. Since the energy-momentum and the metric tensors are diagonal, and $T^{00} = L_{\text{mat}} = \rho c^2$ for dust, the right-hand side of equation (75) is zero. Thus, we effectively have a conserved quantity, which is a desirable characteristic of our theory:

$$\nabla_\mu T^{\mu0} = 0. \quad (76)$$

With the FLRW metric (43), this conservation equation yields:

$$\dot{\rho} + 3H\rho = 0, \quad (77)$$

and so:

$$\rho(t) = C a^{-3}(t), \quad (78)$$

where $C$ is an integration constant. Note that equation (78) is an expected result for a dust universe where matter is conserved.

### 4.2 “Strong” curvature-matter coupling model

Inspired in the article by Barrientos & Mendoza (2018), we explore a “strong” curvature-matter coupling for the function $F(\chi, \xi)$:

$$F(\chi, \xi) = \chi^\gamma \xi^\beta + \xi. \quad (79)$$

In this case, the matter Lagrangian $\xi$ appears as a linear independent term in addition to the “strong” coupling term between curvature $\chi$ and matter $\xi$ as a multiplication. With this function, equation (47) for the Hubble parameter in the dust case and curvature $\kappa = 0$ is given by:

$$H^2 = \frac{c^2}{3\alpha_\gamma \alpha_\chi^\gamma - 1} \frac{1}{\xi^\beta} \left( \frac{8\pi G\alpha}{c^2} \rho + \frac{1}{2}(1 - \gamma)\chi^\gamma \xi^\beta \right) \quad (80)$$

In order to manipulate the last expression, the following standard assumptions are made:

$$\rho = \rho_0 \left( \frac{a}{a(t_0)} \right)^\tau \quad \text{and} \quad a = a(t_0) \left( \frac{t}{t_0} \right)^\sigma, \quad (81)$$

for real constants $\tau$ and $\sigma$. The value $\tau = -3$ follows from equation (78). By taking the derivative with respect to the time coordinate we have

$$\dot{\rho} = \tau \rho H \quad \text{and} \quad \dot{H} = -\frac{H^2}{\sigma}. \quad (82)$$

Following the same procedure as the one used in Subsection 4.1, we use equations (56) for $\dot{R}$ and $\dot{R}$ and equations (82) to obtain the time derivative of $\chi$:

$$\frac{d}{dt} \left( \chi^{-1-\gamma} \xi^\beta \right) = \chi^{-1-\gamma} \xi^\beta \left( \gamma - 1 \right) \left( \frac{j - q}{1 - q} \right) + \beta \tau \quad (83)$$

Substitution of this last result and expressions (56) into equation (80) yields a relation between the Hubble parameter and the matter density:

$$H^{2\gamma} = \left[ 6(q - 1) \right]^{1 - \gamma} \frac{8\pi G}{3\gamma Z''} \frac{\lambda^\beta \alpha^{1 - \gamma}}{\beta^{1 - \beta}} \quad (84)$$

where

$$Z' := 1 + (1 - \gamma) \left( \frac{1 - q}{1 - q} \right) + \tau \beta. \quad (85)$$

With the values for the coupling constants $\alpha$ and $\lambda$ given in (17), together with the density parameter $\Omega$ defined in (34), the Hubble parameter (84) reduces to:

$$H = \frac{a_0}{c} \left[ 6(q - 1) \right]^{1 - \gamma} \frac{8\pi k^{1 - \gamma} k^{\beta}}{3\gamma Z''} \left( \frac{3\Omega}{8\pi} \right)^{1/2(\gamma + \beta - 1)} \quad (86)$$

With this function, the space 11-component of the field equations (50) can be written as:

$$\frac{H^2}{c^2} (2q - 1) = \frac{1}{\alpha \gamma \chi^{-1-\gamma} \xi^\beta} \left( \frac{1}{2} \left( \beta + \gamma - 1 \right) \chi^\gamma \xi^\beta \right) + \frac{\alpha}{c^2} \left( \frac{\dot{f}_\chi + 2H f_\chi}{2} \right), \quad (87)$$

where $\dot{f}_\chi$ is given by equation (83). The second time derivative $\ddot{f}_\chi$ is given by:

$$\frac{d^2}{dt^2} \left( \chi^{-1-\gamma} \xi^\beta \right) = \chi^{-1-\gamma} \xi^\beta H^2 B, \quad (88)$$

where:

$$B := \left[ \left( \gamma - 1 \right) \left( \frac{j - q - 2}{1 - q} \right) + \beta \tau \right] \left[ \left( \gamma - 1 \right) \left( \frac{j - q - 2}{1 - q} \right) \right] + \beta \tau - 1 - q + \left( \frac{\gamma - 1}{1 - q} \right) s + 3j + 3jq - q - 2q^2 + \left( \frac{j - q - 2}{1 - q} \right). \quad (89)$$

Substitution of equations (88) and (83) into (87) yields:

$$\frac{3}{\gamma} \left( \beta + \gamma - 1 \right) (q - 1) + 1 - 2q + B + 2 \left( \gamma - 1 \right) \left( \frac{j - q - 2}{1 - q} \right) + \beta \tau \right] = 0. \quad (90)$$

The last expression is the functional $s = s(q, j)$ equation for the strong curvature-matter coupling model. Unlike its analogous for the addition model in equation (71), note that in order to obtain relation (90) the time 00-component of the field equations was not required.

### 5 TYPE IA SUPERNOVAE BEST FIT

By precise mappings of the distance-redshift relation to redshifts $z \lesssim 1.3$, type Ia supernovae (SNe Ia) remain one of the most robust
probes for the accelerating expansion of the Universe at late times. In order to cosmologically test the models presented in Section 4, we used the distance-redshift data from the Supernova Cosmology Project (SCP) Union 2.1 (Suzuki et al. 2012).

Assuming that SNe Ia events form a homogeneous class for distance estimation, having on average the same intrinsic luminosity for all redshifts, a reasonably linear model for the distance modulus to construct the Hubble diagram is given by the following relation (Peebles 1993)⁷:

$$\mu(z) = 5 \log \left( \frac{H_0 d_L(z)}{c} \right) - 5 \log h(z) + 42.3856,$$

where $H_0$ is the Hubble constant at the present epoch, $h$ is the normalised Hubble constant defined by $h := H_0/(100 \, \text{km/s/Mpc})$, and $d_L(z)$ is the luminosity distance (Visser 2005) given by:

$$d_L(z) = \frac{c}{H_0} \left[ \frac{1}{2} (1 - q_0) z^2 - \frac{1}{6} (1 - q_0 - 3q_0^2 + j_0) z^3 - \frac{1}{24} (2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0j_0 + s_0) z^4 + \ldots \right],$$

for a flat universe with curvature $\kappa = 0$, and $H_0, q_0, j_0$ and $s_0$ the cosmographic parameters evaluated at the present epoch. It is important to mention that equations (91) and (92) are entirely model-independent. Therefore, with the observational data for the redshift and the modulus of the parameter $\mu$ from SCP Union 2.1 data, we test the models (54) and (79) to calibrate their corresponding free parameters.

To do so, we substitute equation (92) into (91). In this way the distance modulus turns into a function of $z, H_0, q_0, j_0$ and $s_0$. Thus, initially the expression has four free parameters since $z$ is given by the observations. From the 00-component of Friedmann’s equations for each proposal, we have a relation for $H_0$ as a function of $\Omega_0, q_0$ and $j_0$ (see equations (60) and (86)). Note that we still have four free parameters since we have only changed $H_0$ for $\Omega_0$. But from the space 11-component of Friedmann’s equations (see (71) and (90)) we can obtain an expression for $s_0$ as a function of $q_0$ and $j_0$ in this manner the number of unknown free parameters is diminished to three, $q_0, j_0$ and $\Omega_0$.

In order to compute the values for the free parameters, a fit was performed using the free software gnuplot (www.gnuplot.info). First, the value for the constants $c$ and $a_0$ are given. Since the interest lies in the functions $F(\chi, \xi) = \chi^{-3} \xi^{3} + \xi$ and $F(\chi, \xi) = \chi^{-3} + \xi^{-2}$, the set of values: $\gamma = -3$, $\beta = 3$ and $\gamma = -3$, $\beta = -2$ are introduced for the corresponding model. According to equations (60) and (86), the coupling constants $k^2 k' k^{3}$ and $k^2 k^{3}$ must be given for each model too. With all this, the functions $Z_0 = Z_0(q_0, j_0), H_0 = H_0(q_0, j_0, \Omega_0), s_0 = s_0(q_0, j_0), d_L = d_L(z, q_0, j_0)$ and $\mu = \mu(z, q_0, j_0, \Omega_0)$ can now be computed. Initial values for $q_0, j_0$ and $\Omega_0$ must be specified for every fit. We used the fit function command in gnuplot for the calibration of the cosmographic parameters $q, j$ and the density parameter $\Omega_0$ given in our model. This function uses non-linear and linear least squares methods.

With the previous information gnuplot’s fit function returns the information of the number of iterations employed for the converged fit, the correlation matrix between the parameters, the best fit value for the parameters and its asymptotic standard error, the final sum of the squares of residuals (SSR), the root mean squares of residuals and the $p$-value for the $\chi$-square distribution.

6 RESULTS

6.1 “Weak” curvature-matter coupling model

The best fit results from the SNe Ia observations for the model $F(\chi, \xi) = \chi^{-3} + \xi^{-2}$ for the cosmographic parameters and the correlation matrix are shown in Table 2. The set of initial values for this fit was: $q_0 = -0.8, j_0 = 0.5$ and $\Omega_0 = 0.8$. The relation between $k$ and $k'$ is given in equation (A12).

A graph of $\mu$ vs. $z$ for the observational data is shown in Figure (1).

Notice that from the best fit $q_0 < 1$, meaning that the Universe

| $q_0$ | $j_0$ | $\Omega_0$ |
|---|---|---|
| 1.000 | -0.846 | 1.000 |
| -0.997 | 0.886 | 1.000 |

Table 2: Left: Best fit results for the “weak” curvature-matter coupling model (see Subsection 4.1), which corresponds to case (ii) of Table 1. The deceleration $q_0$, jerk $j_0$ and density parameter $\Omega_0$ are shown with their corresponding errors. Right: Correlation matrix for the best fit values reported. The SSR for this model is: 563.621.

Figure 1. Apparent magnitude $\mu$ vs. redshift $z$ Hubble diagram from the Union 2.1 SNe Ia data (dots with their corresponding error bars) and the best fit from our addition model (case (i) in Table 1): $F(\chi, \xi) = \chi^{-3} + \xi^{-2}$. The solid line represents the distance modulus $\mu(z)$ from the best fit to the data of the model; the dashed lines represent the maximum and minimum errors of the fit.

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By performing an analogous procedure as the one in subsection 6.1 for the “strong” curvature-matter coupling model \( F(\chi, \xi) = \chi^{-3}\xi^3 + \xi \), the values of the cosmographic parameters for this function are displayed in Table 3, with their corresponding correlation matrix of the fit. The initial values for the parameters in this fit were: \( q_0 = -0.3 \), \( j_0 = 0.5 \) and \( \Omega_0 = 1.0 \). The relation between \( k \) and \( k' \) is the following: \( k^2k'^3 = 9/4\pi^2 \approx 8.9 \times 10^{-4} \) (see the Erratum at the end of the article by Barrientos & Mendoza 2018, in https://arxiv.org/abs/1808.01386).

Table 3: Left: Best-fit results for the “strong” curvature-matter coupling model (see Subsection 4.2) corresponding to case (i) of Table 1. The deceleration \( q_0 \), jerk \( j_0 \) and density parameter \( \Omega_0 \) are shown with their corresponding errors. Right: Correlation matrix for the best-fit values reported. The SSR for this model is: 639.521.

The resulting value for the Hubble parameter within this proposal is given by:

\[
H_0 = 70.363944^{+5.955497}_{-5.530341} \text{ km s}^{-1}\text{Mpc}^{-1}.
\]  

**7 DISCUSSION**

As mentioned in Section 1, the construction of a relativistic theory of MOND, or more precisely a relativistic theory that converges in the weak field limit to the Tully-Fisher law, has so far been a complicated issue. Motivated by general relativity, it has always been desirable to build a pure metric relativistic theory of MOND with no extra (scalar, vector or tensor) fields. In this article we have shown how this is possible in an infinite number of ways by allowing curvature-matter couplings in the gravitational action. Specifically, we showed that it is possible to have an infinite number of local and non-local Lagrangians with curvature-matter couplings for which their weak field limit of approximation reproduce the Tully-Fisher law (deep MOND regime) and it is coherent with the deflection of massless (light) particles observed in individual, groups and clusters of galaxies. These general results were presented in Table 1 and for the cosmological applications presented in this article, we decided to work only with a local theory of gravity in which two simplest models were studied: (a) a weak curvature-matter coupling and (b) a strong curvature-matter coupling. Of course it would be possible to use any non-local model for cosmological applications (see e.g. the work by Carranza et al. 2013, where the non-local model example presented in Table 1 was applied to the accelerated expansion of the Universe fitting SNIa observations), but since it is tradition in gravitational studies to work with locality we decided to follow that path.

It is important to note that in Barrientos & Mendoza (2016, 2017) the feasibility of having the connection \( \Gamma \), both symmetric (Palatini) and non-symmetric (torsion), as an independent field responsible for the MONDian behaviour of the gravitational phenomenon in the correct scale was proposed. Despite the positive results found in those works, the cosmological consequences of this approach were not explored in this article because of the following facts. In studies of the pure metric formalism, the term responsible for the MONDian behaviour is that which contains derivatives of the Ricci scalar (the fourth-order derivative term of the metric). This term does not appear in the Palatini metric-affine formalism. In this approach that term comes from the conformal relation between the Levi-Civita metric and the general one. This fact makes the resultant field equations in the Jordan frame very complex. Also, an extra linear assumption about the transformation from the Jordan frame to the Einstein one is necessary to achieve the desirable expression in the weak field limit. On the other hand, in the torsion formalism the field equations only apply for the symmetric part of the Ricci’s tensor. The disadvantages mentioned previously for the Palatini metric-affine formalism also appear in the torsion approach. In order to avoid this, derivatives of the matter lagrangian were introduced in the matter sector of the action. The meaning of these derivatives is not clear and yield an unwanted complicated field equations. Therefore, the study of matter-curvature coupling theories can be safely restricted to the pure metric formalism.

The conception of dark energy as responsible for the accelerated expansion of the Universe gained momentum among other ideas because its success in fitting type Ia supernovae observations. In Perlmutter et al. (1999) an analysis between 46 type Ia supernovae and general relativity through a Friedmann model with dark
energy was performed. In that work, the cosmographic parameters \( q_0 \), \( j_0 \) and \( s_0 \) were written as functions of the matter density parameter \( \Omega_m \) and the dark energy density parameter \( \Omega_\Lambda \) (see e.g. Section 3). Using equation (92), a theoretical function for the distance modulus \( \mu(z) \) is then built where \( \Omega_m \) and \( \Omega_\Lambda \) are the only two free parameters of the theory. It was found that in order to reproduce the observed modulus distance of those 46 type Ia supernovae, the values for the density parameters must have the following values: \( \Omega_m \approx 0.3 \) and \( \Omega_\Lambda \approx 0.7 \). In the work presented in this article we made a similar analysis, using a single matter density parameter \( \Omega_m \). Its particular functional form as function of the cosmographic parameters is cumbersome but can directly be obtained form equation (60) and (86) for each model. In any case, as mentioned in Section 5, it is possible to obtain all cosmographic parameters by fitting only three of them. Our choice was to fit \( \omega_m \), \( q_0 \) and \( j_0 \). As shown in Section 5, the found values for the cosmographic parameters of our two models are in good agreement with the values expected from the cosmological concordance model (Planck Collaboration et al. 2018). Therefore, the weak and strong curvature-matter coupling models studied in this work are feasible candidates to explain the accelerated expansion of the Universe without the introduction of any dark component.

It is very important to mention once more that the matter density parameter \( \Omega \) defined in equation (34) correspond to the standard definition used in cosmological studies using general relativity. Furthermore, it follows that in standard cosmology the sum of all density parameters (baryonic plus dark matter, dark energy, curvature and radiation) is equal to one when evaluated at the present epoch (Longair 2008). This result is a direct consequence of the standard density parameters definitions directly motivated by the standard Friedmann equation. The use of a more general gravitational action with curvature-matter couplings in this article yields a more extended gravitational field equations (6) and in consequence the Friedmann equations (60) and (86) obtained for our two particular strong and weak models greatly differ from the standard Friedmann equation. This means that in order to keep the definition of e.g. the matter density parameter we would have to do it as it was done in standard cosmology: it will be the ratio of the matter density to the critical density that closes the universe in the absence of any other density parameters. For simplicity and in the spirit of avoiding confusion, we preferred to keep the standard definition of the density parameter with the immediate consequence that \( \Omega_m \) will not be one for a flat dust universe without dark components. Furthermore, the \( \Omega_m \) value will also be model dependent as it can be seen from the results reported in Section 5.

Equations (60) and (86) can be written as:

\[
H = \frac{a_0}{c} \Omega_{\text{eff}},
\]

(95)

where the definition of \( \Omega_{\text{eff}} \) is model dependent but its numerical value should not. For the results given by equations (93) and (94), \( \Omega_{\text{eff}} \) has the following values:

\[
\Omega_{\text{eff}} = 5.630465 \pm 1.803125 \quad \text{and} \quad \Omega_{\text{eff}} = 5.699477 \pm 0.452638 \quad \text{for the weak and strong curvature-matter coupling models respectively. These expressions are in great agreement with the empirical relation: } H_0 = 2\pi a_0/c, \text{ reported by Milgrom (1983a). Note that the "correct" definition of the density parameter at the present epoch for each model would have to be: } \Omega_{\text{correct}} = a_0 \Omega_{\text{eff}}/c H_0, \text{ which has a value equals to } 1 \text{ according to equation (95).}
\]

The fact that our obtained values for \( H_0 \) are within the range of the current inferred values, give us the certainty that the calibrated values of our three free parameters \( q_0 \), \( j_0 \) and \( \Omega_\Lambda \) are reliable.

Since both weak and strong curvature-matter coupling models presented in this article have the same number of parameters, then the best fitting model is the one for which the SSR is minimal. From the results of Table 2 and 3 it follows that the best fitting model is the one with weak curvature-matter coupling. However the advantage between one and the other is minimal and in principle both reproduce quite well SNIa observations.

In summary, if one requires to build a local or non-local relativistic theory of MOND in the pure metric formalism it is necessary to introduce curvature-matter couplings into the action. The infinite possibilities that arise with this choice need to be compared with more astronomical and cosmological observations in order to find a "unique" correct theory. Also, an equivalent Parametrised Post-Newtonian formalism needs to be developed at least in the deep-MOND regime and more comparisons with cosmological phenomenology are to be performed.

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APPENDIX A: MOND AS THE WEAK-FIELD LIMIT OF

\[
F(\chi, \xi) = \chi^{-3} + \xi^{-2}
\]

In Section 4, we proposed the function \( F(\chi, \xi) = \chi^{-3} + \xi^{-2} \) and in Section 5 we fit data for \( \gamma = -3 \) and \( \beta = -2 \). These values are justified because in the weak-field limit of the theory a MONDian behaviour for the acceleration is obtained.

Lett’s start with the field equations of the theory given by equation (10). Taking the trace, substituting \( F(\chi, \xi) = \chi^{-3} + \xi^{-2} \) and using the definitions of \( \xi \) and \( \chi \), we obtain the following equation:

\[
(\gamma - 2)\alpha^{-2} \gamma R^\gamma + 3\alpha^{-2} \gamma \Delta R^{\gamma - 1} = \frac{2(1 - \beta)}{\lambda^\beta} L_{\text{mat}}^\beta + \beta \frac{T^\beta}{2 \lambda^\beta} L_{\text{mat}}^{\beta - 1},
\]

(A1)

due for dust \( L_{\text{mat}} = T \), the previous equation turns into:

\[
\frac{c(2\gamma)}{c(2(\gamma + 1))} (\gamma - 2\alpha^{-2} \gamma R^\gamma + 3\alpha^{-2} \gamma \Delta R^{\gamma - 1}) = \frac{c(2\beta)}{(2 - 3\beta \lambda^\beta} L_{\text{mat}}^\beta),
\]

(A2)

where the order in \( c^{-1} \) is shown. Since \( 2(\gamma + 1) \) is always greater than \( 2\gamma \), the following choice seems natural:

\[
\gamma + 1 = \beta.
\]

(A3)

Therefore, the equation for the dominant order in \( c \) is:

\[
3\alpha^{-2} \gamma \Delta R^{\gamma - 1} = \frac{1 - 3\gamma}{2\gamma} \left( \frac{L_{\text{mat}}}{\lambda} \right)^{\gamma + 1}.
\]

(A4)
The weak-field limit for the previous equations at order of magnitude is given by:
\[ a \sim \left(\frac{GM}{\sigma_0^2 r^{(\gamma+1)/(\gamma-1)}}\right) r^{-2(\gamma+1)/(\gamma-1)}, \]
(A5)
in order to recover the MONDian acceleration \( a = (GMa_0)^{1/2} r^{-1} \) the following value for \( \gamma \) is found:
\[ \gamma = -3 \rightarrow \beta = -2. \]
(A6)

Now, we will perform an analogous approach as the one made by Barrientos & Mendoza (2018) in order to find the values of the constants \( k \) ans \( k' \). Direct substitution of the parameters \( \gamma \) and \( \beta \), as well as of \( R = -2\nabla^2\phi/c^2 \) in equation (A2) for the order of interest yields to:
\[ \frac{9(a_0 G)^2}{5 - 2k^2 c^2} \nabla^2(\nabla^2 \phi)^{-4} = \frac{1}{\rho}, \]
(A7)
where the matter Lagrangian for dust has been employed. For a point-mass source in spherical symmetry:
\[ \rho = \frac{M\delta(r)}{4\pi r^2}, \quad \text{and} \quad \nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right). \]
(A8)

With these assumptions, equation (A7) turns into:
\[ \frac{9}{5 - 2k^2 c^2 \pi^2} \left( \frac{M a_0 G}{4\pi} \right)^2 \frac{d}{dr} \left[ \frac{d}{r^2} \frac{d}{dr} (\nabla^2 \phi)^{-4} \right] = \frac{r^6}{[\delta(r)]^3} \delta(r), \]
(A9)

Integrating with respect to \( r \) and using the Dirac’s delta function as: \( \delta(r = 0) = \lim_{\epsilon \to 0} (2\pi r)^{-1} \) (Gspooner 2008), the previous equation is given by:
\[ - \frac{9}{5 - 2k^2 c^2 \pi^2} (Ma_0 G)^2 \frac{d}{dr} (\nabla^2 \phi)^{-4} = r^7. \]
(A10)

Performing two more integrations with respect to \( r \) and since the acceleration is defined as: \( a = -\nabla \phi \), the final expression for the acceleration is:
\[ - \left[ \frac{72}{5 - 2k^2 c^2 \pi^2} \right]^{1/4} (Ma_0 G)^{1/2} \frac{d}{r} = a. \]
(A11)

In order to obtain the MONDian expression for acceleration:
\[ a = -\sqrt{G a_0 M/r}, \]
the following relation for \( k \) and \( k' \) holds:
\[ k^3 k'^2 = \frac{72}{5 - 2k^2 \pi^2} \approx -2.3 \times 10^{-5}. \]
(A12)

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