SHARPENED STRICHARTZ ESTIMATES AND BILINEAR RESTRICTION FOR THE MASS-CRITICAL QUANTUM HARMONIC OSCILLATOR

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ABSTRACT. We develop refined Strichartz estimates at \( L^2 \) regularity for a class of time-dependent Schrödinger operators. Such refinements begin to characterize the near-optimizers of the Strichartz estimate, and play a pivotal part in the global theory of mass-critical NLS. On one hand, the harmonic analysis is quite subtle in the \( L^2 \)-critical setting due to an enormous group of symmetries, while on the other hand, the spacetime Fourier analysis employed by the existing approaches to the constant-coefficient equation are not adapted to nontranslation-invariant situations, especially with potentials as large as those considered in this article.

Using phase space techniques, we reduce to proving certain analogues of (adjoint) bilinear Fourier restriction estimates. Then we extend Tao’s bilinear restriction estimate for paraboloids to more general Schrödinger operators. As a particular application, the resulting inverse Strichartz theorem and profile decompositions constitute a key harmonic analysis input for studying large data solutions to the \( L^2 \)-critical NLS with a harmonic oscillator potential in dimensions \( \geq 2 \). This article builds on recent work of Killip, Visan, and the author in one space dimension.

1. Introduction

In this article, we prove sharpened forms of the Strichartz inequality for the linear Schrödinger equation in nontranslation-invariant settings with \( L^2 \) initial data. Recall that solutions to the linear constant-coefficient Schrödinger equation
\[
i\partial_t u = -\frac{1}{2} \Delta u, \quad u(0, \cdot) = u_0 \in L^2(\mathbb{R}^d),
\]
satisfy the Strichartz inequality [Str77]
\[
\|u\|_{L^2_t L^{2(d+2)}_{x} (\mathbb{R} \times \mathbb{R}^d)} \leq C \|u(0, \cdot)\|_{L^2(\mathbb{R}^d)}.
\]
On the other hand, it is also known if \( u \) a solution that comes close to saturating this inequality, then it must exhibit some “concentration”; see [CK07, MV98, MVV99, BV07]. Such inverse theorems may be equivalently formulated as a refined estimate
\[
\|u\|_{L^2_{t,x}} \lesssim \|u\|_X \|u(0, \cdot)\|_{L^2(\mathbb{R}^d)}^{1-\theta},
\]
where the norm \( X \) is weaker than the right side of (2) but measures the “microlocal concentration” of the solution. We pursue analogues of such refinements when the right side of (1) is replaced by a more general Schrödinger operator \(-\frac{1}{2} \Delta + V(t, x)\).

Inverse theorems for the Strichartz inequality have provided a key harmonic analysis input to the study of the \( L^2 \)-critical NLS
\[
i\partial_t u = -\frac{1}{2} \Delta u \pm |u|^4 u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d),
\]
so termed because the rescaling \( u \mapsto u_{\lambda}(t, x) := \lambda^{d/2} u(\lambda^2 t, \lambda x) \) preserves both the equation (1) and the mass \( M[u] := \|u(t)\|_{L^2(\mathbb{R}^d)}^2 = \|u(0)\|_{L^2(\mathbb{R}^d)}^2 \). Indeed, they are the building block for the profile decompositions that underlie the Bourgain-Kenig-Merle concentration compactness and rigidity method by identifying potential blowup scenarios for nonlinear solutions with large
data. Using this method, the large data global regularity problem for (4) was recently settled by Dodson [Dod16a, Dod16b, Dod12, Dod15], building on earlier work of Killip, Visan, Tao, and Zhang [KTV09, KVZ08, TVZ07]. For further discussion of this equation we refer the interested reader to the lecture notes [KV13].

The large group of symmetries for the inequality (2) is a significant obstruction to characterizing its near-optimizers. Besides translation and scaling symmetry, both sides are also invariant under Galilei transformations

\[ u \mapsto u_{\xi_0}(t, x) := e^{i[(x, \xi_0) - \frac{\xi_0}{2}|\xi_0|^2]t}u(t, x - t\xi_0), \quad \xi_0 \in \mathbb{R}^d. \]

Equivalently, the estimate is invariant under translations in both position and frequency.

This last symmetry emerges only at $L^2$ regularity and creates an additional layer of complexity. In particular, while the Littlewood-Paley decomposition is extremely well-adapted to higher Sobolev regularity variants of (2), such as the $\dot{H}^{1}$-critical estimate

\[ \|u\|_{L^2_{t,x}} \lesssim \|\nabla u(0)\|_{L^2(\mathbb{R}^d)}, \]

it is useless for inverting the mass-critical estimate because one has no a priori knowledge of where the solution is concentrated in frequency. Instead, the mass-critical refinements cited above combine spacetime Fourier-analytic arguments with restriction theory for the paraboloid.

In physical applications, one is naturally led to consider variants of the mass-critical equation (4) with external potentials, such as the harmonic oscillator

\[ i\partial_t u = \left(-\frac{1}{2}\Delta + \sum_j \omega_j^2 x_j^2 \right) u \pm |u|^4 u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d). \]

For instance, the cubic equation (with a $|u|^2u$ nonlinearity) has been proposed as a model for Bose-Einstein condensates in a laboratory trap [Zha00], and in two space dimensions the critical Sobolev space for this equation (that is, the space preserved by the scaling symmetry $u \mapsto \lambda u(\lambda^2 t, \lambda x)$ of the equation) is $L^2$. Although scaling symmetry is broken, one nonetheless expects solutions with highly concentrated initial data to be approximated, for short times, by solutions to the scale-invariant equation (4). Less obviously, the equation is invariant under “generalized” Galilei boosts, as documented in Lemma 2.1 below, where the spatial and frequency parameters act together in a more complicated fashion on the solutions; in the constant coefficient setting, this reduces to the usual independent space translation and Galilei boost symmetries.

With an eye on the mass-critical Cauchy problem, this article develops inverse Strichartz theorems for the equation

\[ i\partial_t u = \left(-\frac{1}{2}\Delta + V \right) u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d), \]

for a large class of (real-valued) potentials $V(t, x)$ that merely obey similar bounds as the harmonic oscillator and possibly also depend on time. As this equation is not remotely constant-coefficient and is therefore ill-suited to Fourier analysis, we use physical space and microlocal techniques, including integration by parts, wavepacket decompositions, and analysis of the bicharacteristics. The case of one space dimension was treated in a previous joint work with Killip and Visan [JKV]; in this paper we generalize to higher dimensions.

1.1. A slightly more general setup. We begin by considering time-dependent, real-valued symbols $a(t, x, \xi)$ such that

\[ |\partial_t^\alpha \partial_x^\beta a| \leq c_{\alpha, \beta} \text{ for all } |\alpha| + |\beta| \geq 2. \]
For such a symbol, write $a^w(t, x, D)$ for its Weyl quantization. Let $U(t, s)$ denote its unitary propagator on $L^2(\mathbb{R}^d)$, so that $u := U(t, s)u_s$ is the solution to the equation

$$(D_t + a^w(t, x, D))u = 0, \quad u(s, \cdot) = u_s \in L^2(\mathbb{R}^d),$$

Evolution equations of the form (7) were studied by Koch and Tataru [KT05a]. This general framework encompasses several interesting situations:

- Schrödinger Hamiltonians with scalar potentials $a = \frac{1}{2}|\xi|^2 + V(t, x)$, where $V$ is drawn from the class $\mathcal{V}$ of potentials defined by the condition that there exist constants $M_2, M_3, \cdots < \infty$ such that $\|\partial_\xi^a V\|_{L^\infty} \leq M_\alpha < \infty$ for all $|\alpha| \geq 2$.

- Electromagnetic-type symbols $a = \frac{1}{2}|\xi|^2 + b(x, \xi) + V(t, x)$, where the first order symbol $b(x, \xi)$ is real and satisfies $|\partial_\xi^\alpha \partial_\xi^\beta b| \leq c_{\alpha \beta}$ for all $|\alpha| + |\beta| \geq 1$, and $V \in \mathcal{V}$ is a scalar potential as before.

- The frequency 1 portion of the Laplacian on a curved background.

We will not comment further on the last example since this article concerns the Galilei-invariant situation, which is incompatible with a priori frequency localization.

Crucially, we also want a characteristic curvature condition:

**Hypothesis 1.** The Hessian $a_{\xi \xi}$ is uniformly nondegenerate:

$$|\det a_{\xi \xi}| = 1 + O(\varepsilon) \text{ and } \|a_{\xi \xi}\| = 1 + O(\varepsilon).$$

for some small $\varepsilon > 0$.

The preceding hypothesis imply that the equation (7) satisfies a local-in-time dispersive estimate:

**Lemma 1.1.** If the symbol $a$ satisfies the bounds (6) as well has Hypothesis 1, there exists $T_0 > 0$ such that the propagator $U(t, s)$ for the evolution equation (7) satisfies the estimate

$$\|U(t, s)\|_{L^1_t L^\infty_x} \lesssim |t - s|^{-d/2} \text{ for all } |t - s| \leq T_0.$$

Hence, the solutions to (7) satisfy local-in-time Strichartz estimates

$$(8) \quad \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d) \lesssim |I| \|u_s\|_{L^2(\mathbb{R}^d)}$$

for any compact time interval $I$, and for all Strichartz exponents $(q, r)$ satisfying $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(q, r, d) \neq (2, \infty, 2)$.

**Proof.** That the curvature condition implies the dispersive estimate is shown in Koch-Tataru [KT05a, KT05b]. Standard arguments (see Ginibre-Velo [GV95] and also Keel-Tao [KT98]) then yield the Strichartz estimates. \qed

In fact, it suffices to choose the time increment $T_0$ so that

$$T_0 \leq 1, \quad T_0 \|a_{\xi \xi}\| + T_0^2 \|a_{\xi \xi}\| \leq \eta,$$

where $\eta = \eta(d)$ is a small parameter depending only on the dimension.

**Remark.** The concrete cases of scalar potentials and magnetic potentials were studied much earlier by Fujiwara and Yajima, respectively, who proved the dispersive bound using Fourier integral parametrices [Fuj79, Yaj91].

We seek inverse theorems for (8) in analogy to the Euclidean setting. Our refined norm measures concentration in the solution by testing it against scaled, modulated, and translated wavepackets.
For \( z_0 = (x_0, \xi_0) \), define the phase space shift operator \( \pi(z_0)f = e^{i(x-x_0,\xi_0)}f(x-x_0) \). Also define the unitary rescaling operators \( S_\lambda f(x) = \lambda^{-d/2}f(\lambda^{-1}x) \). Set

\[
\psi(x) = c_d e^{-\frac{|x|^2}{d}}, \quad \psi_{x_0,\xi_0} = \pi(x_0,\xi_0)\psi, \quad c_d = 2^{-d/2}\pi^{-3d/4}.
\]

Our first result states that such an estimate of the form (3) would follow from a suitable bilinear \( L^p \) estimate.

**Hypothesis 2.** There exist \( T_0 > 0 \) and \( 1 < p < \frac{4d+2}{d} \) such that the following holds: if \( f, g \in L^2(\mathbb{R}^d) \) have frequency supports in sets of diameter \( \lesssim N \) which are separated by distance \( \gtrsim N \), then

\[
\|U_\lambda f(t)U_\lambda g(t)\|_{L^p_t([-T_0,T_0] \times \mathbb{R}^d)} \lesssim N^{-\delta} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)},
\]

for all \( s \in [-1,1] \) and all \( 0 < \lambda \leq 1 \), where \( U_\lambda^s(t) = U_\lambda^s(t,0) \) are the propagators for the time-translated and rescaled symbols \( a_s^\lambda := \lambda^2a(s+\lambda^2t,\lambda x,\lambda^{-1}\xi) \).

In the translation-invariant case, estimates of this form are called (adjoint) bilinear Fourier restriction estimates and were exploited by Bégout-Vargas to obtain mass-critical Strichartz refinements in dimension 3 and higher [BV07] (the results in dimensions 1 and 2, due to Carles-Keraani, Merle-Vega, and Moyua-Vargas-Vega utilized linear restriction estimates [CK07, MV98, MVV99]). We make an analogous connection in the present, variable-coefficient setting:

**Theorem 1.2.** If Hypotheses 1 and 2 hold, then \( 0 < \theta < 1 \) such that for all initial data \( u_0 \in L^2(\mathbb{R}^d) \), the solution \( u \) to the equation (7) satisfies

\[
\|u\|_{L^\frac{2(d+2)}{d}([-1,1] \times \mathbb{R}^d)} \lesssim \left( \sup_{0<\lambda\leq 1, \, |t|\leq 1, \, (x_0,\xi_0)\in T^*\mathbb{R}^d} \left| \langle S_\lambda \psi_{x_0,\xi_0}, u(t) \rangle_{L^2(\mathbb{R}^d)} \right| \right)^\theta \|u_0\|_{L^2(\mathbb{R}^d)}^{1-\theta}.
\]

Note that the generality of our hypotheses forces us to state the estimates locally in time. Indeed, for most potentials the left side of the Strichartz estimate (11) is infinite if one takes the norm over all of \( \mathbb{R} \times \mathbb{R}^d \); for instance, the harmonic oscillator potential admits periodic-in-time solutions. Nonetheless, our methods do yield (a new proof of) a global-in-time refined Strichartz estimate

\[
\|u\|_{L^\frac{2(d+2)}{d}([0,\infty) \times \mathbb{R}^d)} \lesssim \left( \sup_{\lambda>0, \, t \in \mathbb{R}, \, (x_0,\xi_0)\in T^*\mathbb{R}^d} \left| \langle S_\lambda \psi_{x_0,\xi_0}, u(t) \rangle_{L^2(\mathbb{R}^d)} \right| \right)^\theta \|u_0\|_{L^2(\mathbb{R}^d)}^{1-\theta},
\]

for solutions to the constant coefficient equation (1).

In applications to PDE, such a refined estimate is usually interpreted in the framework of concentration compactness, and yields profile decompositions via repeated application of the following

**Lemma 1.3.** Assume the estimate (11) holds. Let \( u_n := U(t)f_n \) be a sequence of linear solutions with initial data \( u_n(0) = f_n \in L^2(\mathbb{R}^d) \) such that \( \|f_n\|_{L^2(\mathbb{R}^d)} \leq A < \infty \) and \( \|u_n\|_{L^\frac{2(d+2)}{d}} \geq \varepsilon > 0 \).

Then, after passing to a subsequence, there exist parameters

\[
\{(\lambda_n, t_n, x_n, \xi_n)\} \subset (0,1] \times [-1,1] \times \mathbb{R}^d_x \times \mathbb{R}^d_\xi
\]

and a function \( 0 \neq \phi \in L^2(\mathbb{R}^d) \) such that

\[
\pi(x_n,\xi_n)^{-1}S_{\lambda_n}^{-1}u_n \rightharpoonup \phi \text{ in } L^2
\]

\[
\|\phi\|_{L^2} \gtrsim \varepsilon \left( \frac{\varepsilon A}{\theta} \right)^{1-\theta}.
\]

Further,

\[
\|f_n\|_{L^2}^2 - \|f_n - U(t_n)^{-1}S_{\lambda_n} \pi(x_n,\xi_n)S_{\lambda_n} \phi\|_{L^2}^2 - \|U(t_n)^{-1}S_{\lambda_n} \pi(x_n,\xi_n)S_{\lambda_n} \phi\|_{L^2}^2 \to 0.
\]
Proof. By the estimate (11), there exist \(\lambda_n, t_n, x_n, \xi_n\) such that
\[
|\langle S_{\lambda_n} \psi_{x_n, \xi_n}, U(t_n)f_n \rangle| = |\langle \psi, \pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} U(t_n)f_n \rangle| \gtrsim \varepsilon^{\frac{1-\theta}{A}}.
\]
The sequence \(\pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} U(t_n)f_n\) is bounded in \(L^2\), and therefore converges weakly in \(L^2\) to some \(\phi\) after passing to a subsequence. The lower bound on \(\|\phi\|_{L^2}\) is immediate, while
\[
\|f_n\|_{L^2}^2 - \|f_n - U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi\|_{L^2}^2 - \|U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi\|_{L^2}^2
\]
\[
= 2 \text{Re} \langle f_n - U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi, U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi \rangle
\]
\[
= 2 \text{Re} \langle \pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} U(t_n)f_n - \phi, \phi \rangle \to 0.
\]

\(\square\)

In the second part of this paper, we verify Hypothesis 2 for scalar potentials.

**Theorem 1.4.** Consider a Schrödinger operator of the form \(H(t) = -\frac{1}{2} \Delta + V(t, x)\), where \(V \in \mathcal{V}\).

Suppose \(S_1, S_2 \subset \mathbb{R}_d^2\) are subsets of Fourier space with \(\text{diam}(S_j) \leq N\) and \(\text{dist}(S_1, S_2) \geq cN\) for some \(c > 0\). There exists a constant \(\eta = \eta(c) \geq 0\) such that if \(T_0 > 0\) satisfies
\[
T_0 \leq 1 \quad \text{and} \quad T_0^2 \|\partial_x^2 V\|_{L^\infty} < \eta,
\]
then for any \(f, g \in L^2(\mathbb{R}^d)\) with \(\text{supp}(\hat{f}) \subset S_1\) and \(\text{supp}(\hat{g}) \subset L^2(\mathbb{R}^d)\), the corresponding linear solutions \(u = U(t, 0)f\) and \(v = U(t, 0)g\) satisfy the estimate
\[
\|uv\|_{L^q([-T_0, T_0] \times \mathbb{R}^d)} \lesssim \varepsilon N^{d-\frac{d+1}{q} + \varepsilon} \|f\|_{L^q} \|g\|_{L^2} \quad \text{for all} \quad \frac{d+3}{d+1} \leq q < \frac{d+2}{d}
\]
for any \(\varepsilon > 0\), \(N \geq 1\), and \(V \in \mathcal{V}\).

For \(V = 0\), the above estimate was conjectured by Klainerman and Machedon without the epsilon loss, and first proved by Wolff for the wave equation [Wol01] and subsequently by Tao [Tao03] for the Schrödinger equation (both with the epsilon loss). Strictly speaking, the time truncation is not present in the original formulations of those estimates, but may be easily removed by a rescaling and limiting argument.

Finally, while this article makes no attempt to address general magnetic potentials, we do consider a simple but physically relevant case:

**Theorem 1.5.** The conclusion of the previous theorem holds for \(H(t) = -\frac{1}{2} (\nabla - iA)^2 + V(t, x)\), where \(A = A_jdx^j\) is a 1-form whose components are linear in the space variables (i.e. the vector potential for a uniform magnetic field).

We remark that the \(L^p\) estimate (10) does not hold for all symbols satisfying the bounds (6) and Hypothesis 1. For instance, it was observed by Vargas [Var05] that when \(U(t) = e^{it\partial_x \partial_y} \) is the “nonelliptic” Schrödinger propagator in two space dimensions (thus \(a = \xi_x \xi_y\)), the bilinear restriction estimate (6) can fail unless the frequency supports of the two inputs are not only disjoint but also separated in both Fourier coordinates. In fact, in this context the refined estimate (11) is false as stated; to have a correct formulation, one should enlarge the symmetry group on the right side to include the hyperbolic rescalings \(u(x, y) \mapsto u(\mu x, \mu^{-1} y)\); see the work of Rogers and Vargas [RV06].

While there is seemingly no qualitative difference between the elliptic and nonelliptic propagators at the level of bicharacteristics—and indeed the dispersive estimates hold equally well for both—note that the propagators have radically different behavior in terms of oscillations in time. If one compares the travelling wave solutions
\[
e^{i[x\xi_x + y\xi_y - \frac{1}{2} \xi_x^2 + \xi_y^2 \varepsilon]}, \quad e^{i[x\xi_x + y\xi_y - \xi_x \xi_y \varepsilon]},
\]
then for any
it is evident that unlike in the elliptic case, two solutions to the nonelliptic equation which are well-separated in spatial frequency need not decouple in time.

The point we wish to drive home is that while the dispersive and Strichartz estimates follow directly from properties of the classical Hamiltonian flow, an inverse Strichartz estimate depends more subtly on the temporal oscillations of the quantum evolution, which is connected to the bilinear decoupling estimates.

1.2. The main ideas. We follow the general approach introduced in one space dimension [JKV] and briefly review that here. Suppose one has initial data \( u_0 \in L^2 \) such that the corresponding solution \( u \) has nontrivial Strichartz norm. Then, we need to identify a bubble of concentration in \( u \), which is characterized by several parameters that reflect the underlying symmetries in the problem. In the \( L^2 \)-critical setting, the relevant parameters consist of a significant length scale \( \lambda_0 \) as well as the position \( x_0 \), frequency \( \xi_0 \), and time \( t_0 \) when concentration occurs.

The existing Strichartz refinements for the constant-coefficient equation were generally proved in a two-stage process. First, one uses spacetime Fourier analysis (including restriction estimates) to identify a cube \( Q \) in Fourier space accounting for a significant portion of the spacetime norm of \( u \), which reveals the frequency center \( \xi_0 \) and scale \( \lambda_0 \) of the concentration. For example, Begout-Vargas [BV07] first establish an estimate of the form

\[
\|e^{it\Delta}f\|_{L^2(\mathbb{R}^d)} \lesssim \left( \sup_{Q \text{ dyadic cubes}} |Q|^{1-\frac{d}{2}} \int_Q |\hat{f}(\xi)|^p d\xi \right)^\frac{1}{p} \|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{1}{p}}
\]

Then, the time \( t_0 \) and position \( x_0 \) are recovered via a separate physical-space argument. These arguments ultimately rely on the fact that when \( V = 0 \), the equation is diagonalized by the Fourier transform.

For variable-coefficient equations, it is much more natural to regard position \( x_0 \) and frequency \( \xi_0 \) together as a point in phase space, which propagates along the bicharacteristics for the equation. Following the approach in [JKV], we work in the physical space and first isolate a significant time interval \([t_0 - \lambda_0^2, t_0 + \lambda_0^2]\), which also suggests a characteristic scale \( \lambda_0 \). Then, we recover \( x_0 \) and \( \xi_0 \) via phase space techniques.

In the present context, the first part of the argument carries over essentially unchanged from one space dimension, and is quickly reviewed in Section 3; however, the ensuing phase space analysis in higher dimensions is rather more involved and occupies the bulk of this article.

1.3. An application to mass-critical NLS. As mentioned, this article was originally motivated by the large data global regularity problem for the mass-critical quantum harmonic oscillator

\[
(13) \quad i\partial_t u = \left( -\frac{1}{2}\Delta + \sum_j \omega_j^2 x_j^2 \right) u \pm |u|^4 u.
\]

By spectral theory, the Cauchy problem for (13) is naturally posed in the “harmonic” Sobolev spaces

\[
u_0 \in \mathcal{H}^s := \{ u_0 \in L^2 : ( -\Delta + \sum_j \omega_j^2 |x|^2)^{s/2}, \ u_0 \in L^2 \}
\]

Global existence for data in the “energy” space \( \mathcal{H}^1 \) was studied by Zhang [Zha05]. More recently, Poiret, Robert, and Thomann established probabilistic wellposedness in two space dimensions for all subcritical cases \( 0 < s < 1 \), as well as for other supercritical problems [PRT14].

It is well-known that the isotropic case \( \omega_j \equiv \frac{1}{2} \) may be “trivially” solved; \( u \) is a solution of (4) iff its Lens transform

\[
\mathcal{L}u(t,x) := \frac{1}{(\cos t)^{d/2}} u(t,\frac{x}{\cos t}) e^{-\frac{|x|^2 \tan t}{2}}
\]
solves (13) with the same initial data. However, this trick relies on algebraic cancellations that no longer hold for more general harmonic oscillators. For further discussion of the nonlinear harmonic oscillator as well as its connection with the Lens transform, consult the article of Carles [Car11].

To solve (13) at critical regularity for large data, the concentration compactness and rigidity approach is much more promising. Experience has shown that constructing suitable profile decompositions is a principal challenge when implementing this strategy for dispersive equations with broken symmetries (e.g. loss of translation-invariance). See for instance [Jao16] for the energy-critical variant of the quantum harmonic oscillator, as well as [IPS12, KVZ], and the references therein, for other energy-critical NLS on noneuclidean domains. Therefore, this article removes an important obstruction (and we believe the only major roadblock) to the deterministic large data analysis of (13) at the critical regularity $s = 0$.

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2. Preliminaries

2.1. Notation and invariances. In the sequel we often exploit the invariance of the assumptions on the symbols with respect to time translation to simplify notation, and write $U^{s_0}(t) := U(t, s_0)$, $U(t) := U^0(t)$. It is also convenient that the class of symbols is preserved by phase space translations in the following sense:

**Lemma 2.1.** If $U(t, s)$ is the propagator for the symbol $a$ and $\sigma \mapsto z^{\sigma} = (x^{\sigma}, \xi^{\sigma})$ is a bicharacteristic, then

$$U(t, s)\pi(z_0) f = e^{i(\phi(t, z_0) - \phi(s, z_0))} \pi(z_0^s) U^{s_0}(t, s),$$

where $U^{s_0}$ is the propagator for the equation

$$[D_t + (a^{s_0})^w(t, x, D)]u = 0,$$

the phase is defined by

$$\phi(t, z_0) = \int_0^t \langle a_{z_0}(t, z_0^t), \xi^s_0 \rangle - a(z_0^t) d\tau,$$

and $a^{s_0}$ is the transformed symbol

$$a^{s_0}(t, z) = a(t, z_0^t + z) - \langle x, a(x, z_0^t) \rangle - \langle \xi, \xi(x, z_0^t) \rangle - a(z_0^t),$$

which also satisfies the preceding hypotheses.

This is a direct computation which is done in Koch-Tataru [KT05a]. As special cases, we see that symbols of the form $a = \frac{1}{2}|\xi|^2 + \langle A(t, x), \xi \rangle + \omega_{jk}(t)x^j x^k$ are themselves invariant under the mapping $a \mapsto a^{s_0}$, where $A = A_j dx^j$ is a 1-form whose components are linear functions of the space variables, as considered in Theorem 1.5.

2.2. Classical flow estimates. We collect some elementary properties of the classical Hamiltonian flow

$$\begin{cases}
\dot{x} = a\xi, & x(0) = y \\
\dot{\xi} = -a x, & \xi(0) = \eta.
\end{cases}$$

For a point $z = (x, \xi)$ in phase space, let $\sigma \mapsto z^{\sigma} = (x^{\sigma}, \xi^{\sigma})$ denote the bicharacteristic emanating from $(x, \xi)$. Write $(y, \eta) \mapsto (x^t(y, \eta), \xi^t(y, \eta))$ for the flow map.

We recall the standard Gronwall estimates for the linearization.
Lemma 2.2. Suppose $|t||\partial^2 a|_{L^\infty} \leq 1$. Then
\[
\frac{\partial x^t}{\partial t} = \int_0^t a_{\xi\xi} \, dt + O(t^2\|a_{\xi\xi}\|a_{\xi\xi}\|) + O(t^3\|a_{\chi\gamma}\|a_{\xi\xi}\|^2)
\]
\[
\frac{\partial \xi^t}{\partial t} = I + O(t\|a_{\xi\xi}\|) + O(t^2\|a_{\chi\gamma}\|a_{\xi\xi}\|)
\]
(14)
\[
\frac{\partial x^t}{\partial y} = I + O(t\|a_{\chi\gamma}\|) + O(t^2\|a_{\chi\gamma}\|a_{\xi\xi}\|)
\]
\[
\frac{\partial \xi^t}{\partial y} = \int_0^t -a_{\chi\gamma} \, dt + O(t^2\|a_{\chi\gamma}\|a_{\xi\xi}\|) + O(t^3\|a_{\chi\gamma}\|^2\|a_{\xi\xi}\|)
\]

These lead to the following integrated estimates:
\[
x_1^t - x_2^t = x_1^s - x_2^s + [I + O(\varepsilon)](t - s)(\xi_1^t - \xi_2^t)
\]
\[
+ O(|t - s|\|a_{\chi\gamma}\|)(|x_1^s - x_2^s| + |t - s|\|\xi_1^s - \xi_2^s|) + O(|t - s|^2\|a_{\chi\gamma}\|)(|x_1^s - x_2^s| + |t - s|\|\xi_1^s - \xi_2^s|).
\]
(15)
\[
\xi_1^t - \xi_2^t = \xi_1^s - \xi_2^s + O(|t - s|\|a_{\chi\gamma}\|)|x_1^s - x_2^s| + O(|t - s|^2\|a_{\chi\gamma}\|)(|x_1^s - x_2^s| + |t - s|\|\xi_1^s - \xi_2^s|).
\]

Corollary 2.3. If $|x_1^s - x_2^s| \leq r$, then $|x_1^t - x_2^t| \geq Cr$ whenever $\frac{2Cr}{|\xi_1^s - \xi_2^s|} \leq |t - s| \leq T_0$.

In other words, two particles colliding with sufficiently large relative velocity will only interact once in the time window of interest.

We record a technical lemma, first proved in the 1d case [JKV, Lemma 2.2], which will be used later.

Lemma 2.4. There exists a constant $C = C(\|\partial^2 a\|) > 0$ so that if $Q_{\eta} = (0, \eta) + [-1, 1]^2d \subset T^*\mathbf{R}^d$ and $r \geq 1$, then
\[
\bigcup_{|t - t_0| \leq \min(|\eta|^{-1}, 1)} \Phi(t)^{-1}(z_0^t + rQ_\eta) \subset \Phi(t_0)^{-1}(z_0^{t_0} + CrQ_\eta).
\]

In other words, if the bicharacteristic $z^t$ starting at $z \in T^*\mathbf{R}^d$ passes through the cube $z_0^t + rQ_\eta$ in phase space during some time window $|t - t_0| \leq \min(|\eta|^{-1}, 1)$, then it must lie in the dilate $z_0^{t_0} + CrQ_\eta$ at time $t_0$.

Proof. If $z \in \Phi(t)^{-1}(z_0^t + rQ_\eta)$, by definition we have $|x^t - x_0^t| \leq r$ and $|\xi^t - \xi_0^t - \eta| \leq r$. Assuming that $|\eta| \geq 1$, the estimates (15) imply that
\[
|x_0^{t_0} - x_0^t| \leq r + |\eta|^{-1}(|\eta| + r) + O(|\eta|^{-1}\|\partial^2 a\|)(r + |\eta|^{-1}(|\eta| + r)) + O(|\eta|^{-2}\|\partial^2 a\|)(r + |\eta|^{-1}(|\eta| + r)) \leq Cr
\]
\[
|\xi_0^{t_0} - \xi_0^t - \eta| \leq r + O(|\eta|^{-1}\|a_{\chi\gamma}\|)(r + |\eta|^{-1}\|a_{\chi\gamma}\|) + O(|\eta|^{-2}\|a_{\chi\gamma}\|)(|\eta| + r) + O(|\eta|^{-3}\|a_{\chi\gamma}\|^2)(|\eta| + r) \leq Cr.
\]

The case $|\eta| < 1$ is similar. □
2.3. Wavepackets. Let $R \geq 1$ be a scale. To each $z_0 = (x_0, \xi_0)$ in classical phase space with bicharacteristic $\gamma_{z_0}(t) = (x^t_0, \xi^t_0)$, we associate a spacetime “tube”

$$T_{z_0} := \{(t, x) : |x - x^t_0| \leq R^{1/2}, \ |t| \leq R\}.$$  

For such a tube $T$, let $z(T) = (x(T), \xi(T))$ denote the corresponding initial point in phase space.

A scale-$R$ wavepacket concentrated at $z_0$ is a function $\phi_{z_0}(x)$ such that $\phi_{z_0}$ and $\hat{\phi}_{z_0}$ are concentrated respectively in the regions $|x - x_0| \leq R^{1/2}$ and $|\xi - \xi_0| \leq R^{-1/2}$, and which are smooth and rapidly decreasing on the $R^{1/2}$ and $R^{-1/2}$ scale, respectively:

$$|(R^{1/2}\partial_ x)^k \phi_{z_0}| \lesssim_{k,N} \langle \frac{|x - x_0|}{R^{1/2}} \rangle^{-N}, \quad |(R^{-1/2}\partial_ \xi)^k \hat{\phi}_{z_0}| \lesssim_{k,N} \langle \frac{|\xi - \xi_0|}{R^{-1/2}} \rangle^{-N} \quad \forall k, N \geq 0.$$  

We now recall the wavepacket decomposition of a function in $L^2$. For the first part of this article, it is technically somewhat more convenient to use a continuous decomposition. Later on in Section 6.1, we switch to a discrete version which is more common in the restriction theory literature.

To keep things simple, we work at unit scale since that is all we shall need for the moment. For a function $f \in L^2(\mathbb{R}^d)$, its Bargmann transform or FBI transform is a function $Tf \in L^2(T^*\mathbb{R}^d)$ defined as

$$Tf(z) = \langle f, \psi_z \rangle_{L^2(\mathbb{R}^d)} = c_d \int_{\mathbb{R}^d} e^{i(x-y, \xi)} e^{-\frac{|y-x|^2}{2}} f(y) \, dy.$$  

We have a Plancherel identity $\|Tf\|_{L^2(T^*\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$. Dualizing, one sees that for any wavepacket coefficients $F(z) \in L^2(T^*\mathbb{R}^d)$, one has

$$\|T^* F\|_{L^2} = \left\| \int_{T^*\mathbb{R}^d} F(z) \psi_z \, dz \right\|_{L^2(\mathbb{R}^d)} \leq \|F\|_{L^2}.$$  

Indeed, $TT^*$ is the projection onto the image of $L^2(\mathbb{R}^d)$ under $T$.

Lemma 2.5. If $\phi_{z_0}$ is a scale-1 wavepacket, and $U(t)$ is the propagator for the equation (7), then $U(t)\Phi_{z_0}$ is a scale-1 wavepacket concentrated at $\gamma_{z_0}(t)$ for all $|t| = O(1)$.

Proof sketch. By Lemma 2.1, we reduce to the case $z_0 = 0$ and also ensure that the symbol $a(t, x, \xi)$ vanishes to second order at $(x, \xi) = (0, 0)$ in addition to satisfying the bounds (6). Then it suffices to show that propagator $U(t)$ for such symbols maps Schwartz functions to Schwartz functions on unit time scales. This is done using weighted Sobolev estimates as in [KT05a, Section 4].

Using this lemma, we can resolve any Schrödinger solution into a continuous superposition of wavepackets

$$U(t)f = \int_{T^*\mathbb{R}^d} \langle f, \psi_z \rangle U(t) \psi_z \, dz.$$  

It will be very important later on to exploit not just the spacetime localization of $U(t)\psi_z$, but also its temporal phase as described in Lemma 2.1.

3. Choosing a length scale

We begin with the following fundamental lemma from [JKV, Proposition 3.1], which is obtained by a variant of the usual $TT^*$ derivation of the Strichartz estimates. While that article concerned just Schrödinger operators with scalar potentials, the proof works equally well in the current, more general setting.
Proposition 3.1. Suppose $U(t, s)$ satisfies a local in time dispersive estimate as in Lemma 1.1. Let $(q, r)$ be Strichartz exponents (i.e. satisfying the conditions in that Lemma) with $2 < q < \infty$. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies $\|f\|_{L^2(\mathbb{R}^d)} = 1$ and

$$\|U(t) f\|_{L^q_t L^r_x([-1, 1] \times \mathbb{R}^d)} \geq \varepsilon.$$ 

Then there is a time interval $J \subset [-1, 1]$ such that

$$\|U(t, s) f\|_{L^{q-1}_t L^r_x(J \times \mathbb{R}^d)} \gtrsim |J|^{\frac{1}{q(q-1)}} \langle f \|_{L^2(\mathbb{R}^d)}^\frac{2}{d}.$$ 

Equivalently,

$$\|U(t, s) f\|_{L^q_t L^r_x} \lesssim \left( \sup_{J \subset [-1, 1]} |J|^{-\frac{1}{q(q-1)}} \|U(t, s) f\|_{L^{q-1}_t L^r_x(J \times \mathbb{R}^d)} \right)^{1-\frac{2}{q}} \|f\|_{L^2(\mathbb{R}^d)}.$$ 

Note that by pigeonholing we may always assume that $|J| \leq T_0$, where $T_0$ is the time increment selected in (9).

Now let $(q, r)$ be the Strichartz exponents determined by the conditions $\frac{2}{q} + \frac{d}{r} = \frac{d}{q}$ and $q - 1 = r$. It is easy to see that $2 < r < \frac{2(d+2)}{d} < q < \infty$.

For each $J = [s - \mu, s + \mu] \subset [-1, 1]$, we write

$$U(t, s) f = \left( \frac{T_0}{\mu} \right)^{d/4} \tilde{U} \left( \frac{T_0}{\mu} (t - s), 0 \right) \tilde{f} \left( \frac{\sqrt{\mu}}{T_0} x \right), \quad \tilde{f} = \left( \frac{\mu}{T_0} \right)^{d/4} f \left( \frac{\mu}{T_0} x \right),$$

where $\tilde{U}(t, s)$ is the propagator for the rescaled equation $(D_t + \tilde{a}^w) \tilde{u} = 0$, and

$$\tilde{a}(t, x, \xi) := \frac{\mu}{T_0} a \left( s + \frac{\mu}{T_0} t, \sqrt{\frac{\mu}{T_0}} x, \sqrt{\frac{T_0}{\mu}} \xi \right).$$

Changing variables, we obtain

$$|J|^{-\frac{1}{q(q-1)}} \|U(t, s) f\|_{L^{q-1}_t L^r_x(J \times \mathbb{R}^d)} = \|\tilde{U}(t) \tilde{f}\|_{L^{q-1}_t L^r_x([-T_0, T_0] \times \mathbb{R}^d)}.$$ 

By interpolating with $L^2_t L^r_x([-T_0, T_0] \times \mathbb{R}^d)$, which is bounded by unitarity, we see that Theorem 1.2 would follow if we prove that for some $2 < q_0 < \frac{2(d+2)}{d}$ and $0 < \theta < 1$, the scale-1 refined estimate

$$\|U^{\lambda}_s(t) f\|_{L^q([-T_0, T_0] \times \mathbb{R}^d)} \lesssim (\sup_z |\langle \psi_z, f \rangle|)^\theta \|f\|_{L^2}^{1-\theta}.$$ 

holds for all $s \in [-1, 1]$, $0 < \lambda \leq 1$, where the notation $U^{\lambda}_s(t)$ is as in Hypothesis 2.

Over the next two sections we establish

Proposition 3.2. If Hypothesis 2 holds, then so does the estimate (16).

4. A REFINED BILINEAR $L^2$ ESTIMATE

In one space dimension, the symmetric Strichartz space is $L^0_t L^2_x$, and in our previous work [JKV] we pursued (16) with $q_0 = 4$. This is a particularly convenient Lebesgue exponent as one can view the estimate as a bilinear $L^2$ estimate and exploit orthogonality.

Such a direct approach breaks down in higher dimensions. Since $2 < \frac{2(d+2)}{d} \leq 4$, the left side of (16) could well be infinite when $q_0 = 4$. To obtain a refined linear $L^{q_0}$ estimate for $q_0 < \frac{2(d+2)}{d}$, we shall begin by interpreting it as a refined bilinear $L^{q_0/2}$ estimate, but use dyadic decomposition and interpolation between two microlocalized estimates:

- A localized refined bilinear $L^2$ estimate (“refined” in the sense of exhibiting a sup over wavepacket coefficients) with some loss in the frequency separation of the inputs.
- A bilinear $L^p$ estimate for some $p < \frac{d+2}{d}$ which yields gains in the frequency separation.
In one dimension we effectively took $p = 2 < \frac{6}{2}$ and melded both steps together without any localization. As just mentioned, however, a "global" estimate is not possible in higher dimensions.

We discuss the $L^2$ component in this section.

**Proposition 4.1.** If $f = \int f_z \psi_z \, dz$ and $g = \int g_z \psi_z \, dz$ are $L^2(\mathbb{R}^d)$ initial data with corresponding Schrödinger evolutions $u = \int u_z \, dz$ and $v = \int v_z \, dz$, then for some $\alpha = \alpha(d)$ and $1 < p < 2$, we have

$$\left\| \int_{|\xi_1 - \xi_2| \sim N} u_z v_z \, dz \right\|_{L^2([-T_0, T_0] \times \mathbb{R}^d)} \lesssim N^\alpha \left( \sup_z |f_z|^{1/p'} \|f_z\|_{L^2_z}^{1/p} \right) \left( \sup_z |g_z|^{1/p'} \|g_z\|_{L^2_z}^{1/p} \right)$$

To begin the proof, square the left side and expand

$$\int f_z g_z \chi_{\mathbb{R}^d} K_N(z_1, z_2, z_3, z_4) \, dz_1 dz_2 dz_3 dz_4,$$

where $K_N := K\chi_{[\xi_1 - \xi_2| \sim N, |\xi_3 - \xi_4| \sim N}$, and

$$K(z_1, z_2, z_3, z_4) = \langle U(t)\psi_z, U(t)\psi_z \rangle_{L^2_z([\xi_3 - \xi_4] \times \mathbb{R}^d)}.$$

The estimate would follow if we could show that

$$N^{-\alpha} (z_1 - z_2)^2 (z_3 - z_4)^\theta |K_N(z)|$$

is a bounded operator on $L^2_{z_1, z_2}$ for some $\theta > 0$, as Young’s inequality would then imply

$$\left\| \int u_z \, dz \right\|_{L^4_z}^2 \lesssim \left( \int |f_z g_z|^{2} (z_1 - z_2)^{-2\theta} \, dz_1 dz_2 \right)^{1/2} \left( \int |g_z|^{2} (z_3 - z_4)^{-2\theta} \, dz_3 dz_4 \right)^{1/2} \lesssim \sup_z |f_z|^{2/p'} \sup_z |g_z|^{2/p} \|f\|_{L^2_z}^{2/p} \|g\|_{L^2_z}^{2/p} \text{ for some } 1 < p < 2.$$

In view of the crude bound $|K(z)| \lesssim \min_{j,k} (z_j - z_k)^{-1}$, which is just a consequence of the spacetime supports of the wavepackets, the weighted estimate (18) would follow from

**Lemma 4.2.** The localized kernel $K_N$ satisfies

$$\|K_N|^{1-\delta}_{L^2_{z_1, z_2} \to L^2_{z_3, z_4}} \lesssim N^\alpha,$$

where $\alpha$ is a constant depending only on the dimension.

**Proof of Lemma 4.2.** In view of the unit scale spatial localization of the wavepackets, we may further truncate the kernel to the phase space region

$$R = \{ |x_1 - x_2| \leq 4|\xi_1 - \xi_2|, \ |x_3 - x_4| \leq 4|\xi_3 - \xi_4| \}.$$n

For instance, if $|x_1^s - x_2^s| \geq 4|\xi_1 - \xi_2|$, and $|t - s| \leq T_0$ with the parameter $\eta$ in (9) chosen sufficiently small,

$$|x_1 - x_2| \geq (1 - |t - s|^2 \|\partial_x^2 V\|_{L^\infty} \|e^{(t-s)^2 \|\partial_x^2 V\|_{L^\infty}}\)|x_1^s - x_2^s| - ((|t - s| + |t - s|^3 \|\partial_x^2 V\|_{L^\infty} e^{(t-s)^2 \|\partial_x^2 V\|_{L^\infty}})|\xi_1 - \xi_2| \geq 1/2|x_1^s - x_2^s| - 3/2|t - s| |\xi_1^s - \xi_2^s| \geq 1/8|x_1^s - x_2^s|,$n

therefore $|K_N(1 - \chi_R)| \lesssim_M (x_1 - x_2)^{-M} (x_3 - x_4)^{-M} N^{-M}$ for any $M > 0$. Thus it suffices to prove that

$$\|K_N \chi_R\|_{L^2 \to L^2} \lesssim N^\alpha.$$
An estimate of this flavor was proved in the 1d case [JKV]. We shall argue similarly, but the proof is somewhat simpler since we aim for a cruder bound at this stage, completely ignoring temporal oscillations, and defer the more delicate analysis to the bilinear $L^p$ estimate.

Partition the 4-particle phase space $(T^*\mathbb{R}^d)^4$ according to the degree of physical interaction between the particles. Let

$$E_0 = \{ \vec{z} \in (T^*\mathbb{R}^d)^4 : \min_{|t| \leq T_0} \max_{j,k} |x_j^t - x_k^t| \leq 1 \},$$

$$E_k = \{ \vec{z} \in (T^*\mathbb{R}^d)^4 : 2^{k-1} < \min_{|t| \leq T_0} \max_{j,k} |x_j^t - x_k^t| \leq 2^k \},$$

and decompose the kernel $K_N = \sum_{k \geq 0} K_N \chi_{E_k}$. Then we have the following pointwise bound

$$|K(\vec{z})| \lesssim_M 2^{-k_M} \frac{(\xi_1^{(\vec{z})} + \xi_2^{(\vec{z})} - \xi_3^{(\vec{z})} - \xi_4^{(\vec{z})}) - M}{(\xi_1^{(\vec{z})} - \xi_2^{(\vec{z})}) + (\xi_3^{(\vec{z})} - \xi_4^{(\vec{z})})}, \quad \vec{z} \in E_k,$$

where $t(\vec{z})$ is a time minimizing the “mutual distance” $\max_{i,j} |x_i^t - x_j^t|$. Further, the additional localization to $R$ implies, by the estimates (15), that

$$|\xi_1 - \xi_2 - (\xi_1 - \xi_2)| \lesssim \frac{1}{10} |\xi_1 - \xi_2|$$

$$|\xi_3 - \xi_4 - (\xi_3 - \xi_4)| \lesssim \frac{1}{10} |\xi_3 - \xi_4|$$

for all $|t| \leq T_0$. In particular $|\xi_1^{(\vec{z})} - \xi_2^{(\vec{z})}| \sim |\xi_3^{(\vec{z})} - \xi_4^{(\vec{z})}| \sim N$; thus, while the $\xi_j^t$ may vary rapidly with time if $x_j^t$ are extremely far from the origin, the relative frequencies retain the same order of magnitude.

Assuming the bound (19) for the moment, we apply Schur’s test to complete the proof of Lemma 4.2. Fix $(z_3, z_4)$ belonging to the projection $E_k \rightarrow T^*\mathbb{R}^d_{z_3} \times T^*\mathbb{R}^d_{z_4}$, define

$$E_k(z_3, z_4) = \{(z_1, z_2) : (z_1, z_2, z_3, z_4) \in E_k\},$$

and let $t_1$ be the time minimizing $|x_3^{t_1} - x_4^{t_1}| \leq 2^k$. For any $(z_1, z_2) \in E_k(z_3, z_4)$, the mutual distance $\max_{j,k} |x_j^t - x_k^t|$ between $x_1^t, x_2^t, x_3^t, x_4^t$ is minimized in the time window

$$I = \{ t : |t - t_1| \lesssim \min(1, \frac{2^k}{|\xi_3 - \xi_4|}) \},$$

as for all other times we have $|x_3^t - x_4^t| \gg 2^k$ (Corollary 2.3).

We estimate the size of the level sets of $|K|$. For a momentum $\xi \in \mathbb{R}^d$, let $Q_\xi = (0, \xi) + [-1, 1]^d \times [-1, 1]^d \subset T^*\mathbb{R}^d$ denote the unit phase space box centered at $(0, \xi)$, and write $\Phi^t = \Phi(t, 0)$ for the propagator on classical phase space relative to time 0 for the Hamiltonian $h(x, \xi) = \frac{1}{2}|\xi|^2 + V(t, x)$. For $\mu_1, \mu_2 \in \mathbb{R}^d$, define

$$Z_{\mu_1, \mu_2} = \bigcup_{t \in I} (\Phi^t \otimes \Phi^t)^{-1} \left( \frac{z_3 + z_4}{2} + 2^k Q_{\mu_1} \right) \times \left( \frac{z_3 + z_4}{2} + 2^k Q_{\mu_2} \right).$$

This set is depicted schematically in Figure 1 when $k = 0$, and corresponds to the pairs of wave packets $(z_1, z_2) \in E_m(z_3, z_4)$ with momenta $(\mu_1, \mu_2)$ relative to the wavepackets $(z_3, z_4)$ at the “collision time” $t(\vec{z})$.

We note that $E_k(z_3, z_4) \subset \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}^d} Z_{\mu_1, \mu_2}$, and recall the following estimate from the 1d paper, whose proof we reproduce below for convenience:

**Lemma 4.3.**

$$|Z_{\mu_1, \mu_2}| \lesssim 2^{4dk} \max(1, |\mu_1|, |\mu_2|) |I|.$$
\[ Z_i^t = \begin{align*} &\frac{z_i^t + z_i^t}{2} + Q_{\mu_i} \\
&Z_3^t = \frac{z_3^t + z_3^t}{2} + Q_{\mu_3} \\
&Z_4^t = \frac{z_4^t + z_4^t}{2} + Q_{\mu_4} \end{align*} \]

**Figure 1.** \( Z_{\mu_1, \mu_2} \) comprises all \((z_1, z_2)\) such that \(z_1^t\) and \(z_2^t\) belong to the depicted phase space boxes for \(t\) in the interval \(I\).

**Proof.** Without loss assume \(|\mu_1| \geq |\mu_2|\). Partition the interval \(I\) into subintervals of length \(|\mu_1|^{-1}\) if \(\mu_1 \neq 0\) and in subintervals of length 1 if \(\mu_1 = 0\). For each \(t'\) in the partition, Lemma 2.4 implies that for some constant \(C > 0\) we have

\[
\bigcup_{|t-t'| \leq \min(1, |\mu_1|^{-1})} \Phi(t)^{-1}\left(\frac{z_3^t + z_4^t}{2} + 2kQ_{\mu_1}\right) \subset \Phi(t')^{-1}\left(\frac{z_3^{t'} + z_4^{t'}}{2} + C2^kQ_{\mu_1}\right) \\
\bigcup_{|t-t'| \leq \min(1, |\mu_1|^{-1})} \Phi(t)^{-1}\left(\frac{z_3^t + z_4^t}{2} + 2kQ_{\mu_2}\right) \subset \Phi(t')^{-1}\left(\frac{z_3^{t'} + z_4^{t'}}{2} + C2^kQ_{\mu_2}\right),
\]

and so

\[
\bigcup_{|t-t'| \leq \min(1, |\mu_1|^{-1})} (\Phi(t) \otimes \Phi(t'))^{-1}\left(\frac{z_3^{t} + z_4^{t}}{2} + 2kQ_{\mu_1}\right) \times \left(\frac{z_3^{t'} + z_4^{t'}}{2} + 2kQ_{\mu_2}\right) \\
\subset (\Phi(t') \otimes \Phi(t'))^{-1}\left(\frac{z_3^{t'} + z_4^{t'}}{2} + C2^kQ_{\mu_1}\right) \times \left(\frac{z_3^{t'} + z_4^{t'}}{2} + C2^kQ_{\mu_2}\right).
\]

By Liouville’s theorem, the right side has measure \(O(2^{4dk})\) in \((T^dR^d)^2\). The claim follows by summing over the partition. \(\square\)

For each \((z_1, z_2) \in E_k(z_3, z_4) \cap Z_{\mu_1, \mu_2}\), we have by definition \(z_j^{t(\xi)} \in \frac{z_j^{t(\xi)} + z_j^{t(\xi)}}{2} + 2kQ_{\mu_j}\), thus

\[
\begin{align*}
\xi_1^{t(\xi)} + \xi_2^{t(\xi)} - \xi_3^{t(\xi)} - \xi_4^{t(\xi)} &= \mu_1 + \mu_2 + O(2^k) \\
\xi_1^{t(\xi)} - \xi_2^{t(\xi)} &= \mu_1 - \mu_2 + O(2^k)
\end{align*}
\]

Hence when \((z_1, z_2) \in Z_{\mu_1, \mu_2}\), for any \(M\) we have

\[
|K(\xi)| \lesssim_M 2^{-Mk} \frac{\langle \mu_1 + \mu_2 \rangle^{-M}}{\langle |\mu_1 - \mu_2| \rangle + |\xi_3^{t(\xi)} - \xi_4^{t(\xi)}|}. \tag{21}
\]
To apply Schur’s test, we combine the estimates (20), (21), and evaluate
\[
\int |K_N(z_1, z_2, z_3, z_4)|^{1 - \delta} \chi_{E_k}(\bar{z}) \, dz_1 \, dz_2 \leq \sum_{\mu_1, \mu_2} \int |K_N^{1 - \delta} \chi_{E_k}| \, dz_2 \, dz_2 \\
\lesssim M 2^{-Mk} \sum_{|\mu_1 - \mu_2| \leq N + 2^k} 2^{-Mk} (\mu_1 + \mu_2)^{-M} \\
\lesssim N^{d_2 - (M - d)k}.
\]
For fixed $z_1, z_2$, the integral over $z_3$ and $z_4$ is estimated the same way. This concludes the proof of Lemma 4.2, modulo some remarks on the crucial pointwise bound (19).

To obtain that estimate, we use Lemma 2.1 to write
\[
K(z) = \int e^{i\Phi} \prod_{j=1}^{4} U_{t_j, x_j}(t) \psi(x - x_j^t) \, dx dt,
\]
\[
\Phi(t, x; z) = \sum_j \sigma_j \left[ \langle x - x_j^t, \xi_j^t + \phi(t, x_0, \xi_0) \right]
\]
where $\sigma = (+, +, -, -)$, and we denote $\prod_j c_j := c_1 c_2 c_3 c_4$.

It is convenient to partition the integral further, writing
\[
U_{t, x}(t) \psi(x - x_j^t) = \sum_{\ell_j \geq 0} U_{t, x}(t) \psi(x - x_j^t) \theta_{\ell_j}(x - x_j^t),
\]
where $\sum_{\ell_j \geq 0} \theta_{\ell_j}$ is a partition of unity with $\theta_{\ell_j}$ supported on the dyadic annulus of radius $\sim 2^{\ell_j}$. For $z \in E_k$, only the terms
\[
K_k(z) := \int e^{i\Phi} \prod_{j=1}^{4} U_{t_j, x_j}(t) \psi(x - x_j^t) \theta_{\ell_j}(x - x_j^t) \, dx dt
\]
with $\ell^* := \max_j \ell_j \geq k$ will be nonzero.

By Lemma 2.2, the integral is supported on the spacetime region
\[
\{(t, x) : |t - t(z)| \leq \min \left( 1, \frac{2^{\ell^*}}{\max_{i,j} |\xi_i^{t(z)} - \xi_j^{t(z)}|} \right) \quad \text{and} \quad |x - x_j^t| \leq 2^{\ell_j} \},
\]
and for all such $t$ we have
\[
|x_j^t - x_k^t| \leq 2^{\ell^*}, \quad |\xi_j^t - \xi_k^t - (\xi_j^{t(z)} - \xi_k^{t(z)})| \leq 2^{\ell^*}.
\]

Integrating by parts in $x$, we may produce as many factors of $|\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|^{-1}$ as desired and freeze $t = t(z)$ to obtain
\[
|K_k(z)| \lesssim M 2^{-\ell^*M} \frac{\langle \xi_1^{t(z)} + \xi_2^{t(z)} - \xi_3^{t(z)} - \xi_4^{t(z)} \rangle^{-M}}{\langle |\xi_1^{t(z)} - \xi_2^{t(z)}| + |\xi_3^{t(z)} - \xi_4^{t(z)}| \rangle} \quad \text{for any} \quad M \geq 0,
\]
and the bound (19) follows upon summing over $\ell^*$.

\[\square\]

5. PROOF OF THEOREM 1.2

We prove Proposition 3.2 and hence Theorem 1.2. Begin with a Whitney decomposition of
\[
(R^d \times R^d) \setminus \{(\xi, \xi) : \xi \in R^d\} = \bigcup_{N \in 2^Z} \bigcup_{Q \in Q_N} Q,
\]
where $Q_N$ is the set of dyadic cubes in $\mathbb{R}^d \times \mathbb{R}^d$ with diameter $\sim N$ and distance $\sim N$ to the diagonal. For each $Q \in Q_N$, its characteristic function factorizes $\chi_N^Q(\xi_1, \xi_2) = \chi_N^{Q,1}(\xi_2)\chi_N^{Q,2}(\xi_2)$, where $\chi_N^{Q,j}$ are characteristic functions of $d$-dimensional cubes of width $N$. Then we can decompose

$$1(\xi_1, \xi_2) = \chi_0(\xi_1, \xi_2) + \sum_{N \geq 1} \sum_{Q \in Q_N} \chi_N^{Q,1}(\xi_1)\chi_N^{Q,2}(\xi_2),$$

where $\chi_0(\xi_1, \xi_2)$ is supported on the set $|\xi_1 - \xi_2| \lesssim 1$.

Now suppose $u$ and $v$ are linear solutions with initial data $f = \int f_z \psi_z$ and $g = \int g_z \psi_z$, respectively, where $f_z = (f, \psi_z)$ and $g_z = (g, \psi_z)$. It follows from Hypothesis 2 that in fact

$$\sum_{Q \in Q_N} \int_Q u_z v_z d\xi_1 d\xi_2 \lesssim N^{-\delta} \|f_z\|_{L^2} \|g_z\|_{L^2}.$$  \hspace{1cm} (22)

for each $N \geq 1$. Indeed, note that for each cube $Q$, the integral has a product structure

$$\int_Q u_z v_z d\xi_1 d\xi_2 = \left( \int u_z \chi_N^{Q,1}(\xi_1) d\xi_1 \right) \left( \int v_z \chi_N^{Q,2}(\xi_2) d\xi_2 \right) = U(t) \left[ \int f_z \chi_N^{Q,1}(\psi_z) \xi_1 , \xi_1 d\xi_1 \right] U(t) \left[ \int g_z \chi_N^{Q,2}(\xi_2) \psi_z \xi_2 dx \xi_2 \right].$$

By the rapid decay of the wavepackets, we may harmlessly insert frequency cutoffs $\chi_N^{Q,j}(D)$, where $\chi_N^{Q,j}$ are slightly fattened versions of $\chi_N^{Q,j}$ and still have supports separated by distance $\sim N$, and apply Hypothesis 2 to estimate

$$\left\| \int_Q u_z v_z d\xi_1 d\xi_2 \right\|_{L^q} \lesssim N^{-\delta} \left\| \int f_z \chi_N^{Q,1}(\xi_1) d\xi_1 \right\|_{L^2} \left\| \int g_z \chi_N^{Q,2}(\xi_2) d\xi_2 \right\|_{L^2} \lesssim N^{-\delta} \|f_z\|_{L^2} \|g_z\|_{L^2} \chi(\xi)\|_{L^2}.$$

The left side of (22) is therefore bounded by

$$\sum_{Q \in Q_N} N^{-\delta} \|f_z \chi_N^{Q,1}(\xi)\|_{L^2} \|g_z \chi_N^{Q,2}\chi(\xi)\|_{L^2} \leq N^{-\delta} \left( \sum_{Q \in Q_N} \|f_z \chi_N^{Q,1}(\xi)\|^2_{L^2} \right)^{1/2} \left( \sum_{Q \in Q_N} \|g_z \chi_N^{Q,2}(\xi)\|^2_{L^2} \right)^{1/2} \lesssim N^{-\delta} \|f_z\|_{L^2} \|g_z\|_{L^2}.$$

We decompose the product

$$uv = \int u_z v_z \chi_0(\xi_1, \xi_2) d\xi_1 d\xi_2 + \sum_{N \geq 1} \sum_{Q \in Q_N} \int_Q u_z v_z d\xi_1 d\xi_2,$$

and estimate each group of terms in $L^q$ for $q$ between $p$ and 2. To estimate the sum over $Q_N$ we interpolate between the $L^p$ and $L^2$ bounds. Writing $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$, we have

$$\left\| \sum_{Q \in Q_N} \int_Q u_z v_z d\xi_1 d\xi_2 \right\|_{L^q} \leq \left\| \sum_{Q \in Q_N} \int_Q u_z v_z d\xi_1 d\xi_2 \right\|_{L^p}^{1-\theta} \left\| \sum_{Q \in Q_N} \int_Q u_z v_z d\xi_1 d\xi_2 \right\|_{L^2}^{\theta} \lesssim N^{-\delta(1-\theta) + \frac{\theta}{p}} \left[ \left( \sup_z \|f_z \psi_z\|^{1/p'} \sup_z \|g_z \psi_z\|^{1/p'} \right)^{1-\theta} \left( \|f\|_{L^2} \|g\|_{L^2} \right)^{1-\theta + \frac{\theta}{p}}$$

and for $q$ sufficiently close to $p$ (hence $\theta$ sufficiently small) the exponent of $N$ is negative.
Theorem 6.1. Suppose we will almost always just consider the propagator from time 0 and write for any \( f, g \) then for any \( V \) for some \( \Theta \), where \( \frac{1}{q} = 1 - \theta' + \frac{d}{q} \).

We sum in \( N \) to conclude that

\[
\| uv \|_{L^q} \lesssim \left( \sup_z \| f \|_{L^2} \| g \|_{L^2} \right)^{\theta} \left( \| f \|_{L^2} \| g \|_{L^2} \right)^{\frac{d}{q} - \frac{d}{q} - 1}
\]

for some \( \theta \in (1, \frac{d + 2}{d}) \). Taking \( u = v \) we obtain Proposition 3.2.

6. The restriction-type estimate

In the remainder of this article, we prove Theorem 1.4.

We shall systematically use certain notation. For \( N \geq 1 \) and a potential \( V \), we consider the rescaled potentials \( V_N(t, x) := N^{-2}V(N^{-2}t, N^{-1}x) \). Let \( U(t, s) \) and \( U_N(t, s) \) denote the propagators for the corresponding Schrödinger operators \( H(t) := -\frac{1}{2}\Delta + V \) and \( H_N(t) := -\frac{1}{2}\Delta + V_N \); note the change in convention from the first part of this article. We will often use the letter \( U \) to write the propagators for different potentials \( V \in \mathcal{V} \); this ambiguity will not cause any serious issue, however, since all the estimates we shall need are valid uniformly over \( \mathcal{V} \).

In a similar vein, in view of the time translation-invariance of our assumptions on the potential, we will almost always just consider the propagator from time 0 and write \( U(t) := U(t, 0), U_N(t) := U_N(t, 0) \).

In the sequel, the letter \( C \) will denote a constant, depending only on the dimension \( d \), which may change from line to line.

By rescaling, we equivalently need to prove

**Theorem 6.1.** Suppose \( S_1, S_2 \subset \mathbb{R}^d_\xi \) are subsets of Fourier space with \( \text{diam}(S_j) \leq 1 \) and \( \text{dist}(S_1, S_2) \geq c > 0 \). There exists a constant \( \eta = \eta(c) \geq 0 \) such that if \( T_0 > 0 \) satisfies

\[
T_0 \leq \frac{1}{2} \quad \text{and} \quad T_0^2 \| \partial_x^2 V \|_{L^\infty} < \eta,
\]

then for any \( f, g \in L^2(\mathbb{R}^d) \) with \( \text{supp}(f) \subset S_1 \) and \( \text{supp}(g) \subset L^2(\mathbb{R}^d) \), the corresponding Schrödinger solutions \( u_N = U_N(t, 0) f \) and \( v_N = U_N(t, 0) g \) satisfy the estimate

\[
\| u_N v_N \|_{L^q([-T_0 N^2, T_0 N^2] \times \mathbb{R}^d)} \lesssim \varepsilon \ N^\varepsilon \| f \|_{L^2} \| g \|_{L^2}
\]

for all \( d + 3 < q \leq \frac{d + 2}{d} \) for any \( \varepsilon > 0 \) and \( N \geq 1 \).
We use the induction on scales method pioneered by Wolff [Wol01] and adapted by Tao to the paraboloid [Tao03]. Our proof is modeled closely on Tao’s; the main differences are as follows:

- The induction scheme (section 6.3) is complicated by the fact that frequency is not conserved. Indeed, modulo negligible tails the frequency is transported along the bicharacteristics, and one cannot directly apply an induction hypothesis which involves the assumptions on the frequency supports at time 0 to a spacetime ball at a later time, for the distance between the supports could shrink.
- The roughness of \( V \) in time demands more care in the bilinear \( L^2 \) estimate (section 6.6) and in the ensuing tube combinatorics, as one obtains weaker decay from temporal oscillations.
- In the Kakeya-type estimate, the key combinatorial lemma (Lemma 6.10, the analogue of Lemma 8.1 in Tao) allows for tubes that bend. We also need to be slightly more precise to compensate for the weaker decay in the \( L^2 \) bound.

6.1. Wavepacket decomposition. While the first part of this paper employed continuous wavepacket decompositions, in the sequel we shall use a discrete version which is more conventional in the restriction theory literature and more convenient for the combinatorial arguments involved.

**Lemma 6.2.** Let \( u = U_N(t,0)f \) be a linear Schrödinger wave with \( \text{supp}(\hat{f}) \subset S_1 \). For each \( 1 \leq R \leq N^2 \), there exists a collection of tubes \( T \) and a decomposition

\[
u = \sum_{T \in T} a_T \phi_T,
\]

into \( R \times (R^{1/2})^d \) wave packets with the following properties:

- Each \( T \in T \) has the property that \( (x(T), \xi(T)) \in R^{1/2}Z^d \times R^{-1/2}Z^d \).
- Each wavepacket \( \phi_T \) is a free Schrödinger wave localized near the bicharacteristic \( (x(T)^t, \xi(T)^t) \), i.e. which satisfies the pointwise bounds

\[
|(R^{1/2} \partial_x)^k \phi_T(t)| \lesssim_{k,M} \left( \frac{x-x(T)^t}{R^{1/2}} \right)^{-M}, \quad \|(R^{-1/2} \partial_t)^k \hat{\phi}_T(t)| \lesssim_{k,M} \left( \frac{\xi-\xi(T)^t}{R^{-1/2}} \right)^{-M} \quad \forall k, M \geq 0.
\]

Moreover, \( \hat{\phi}_T[0] \) is supported in a \( R^{-1/2} \) neighborhood of \( \xi(T) \in S_1 \).
- Frame property: the complex coefficients \( a_T \) obey the \( \ell^2 \) bound

\[
\sum_{T} |a_T|^2 \lesssim \|f\|_{L^2}^2.
\]

Moreover, for any subcollection of tubes \( T' \subset T \) and complex numbers \( a_T \), one has

\[
\| \sum_{T \in T'} a_T \phi_T \|_{L^2}^2 \lesssim \sum_{T \in T'} |a_T|^2.
\]

Thus, the wavepacket \( \phi \) is essentially supported in spacetime on the tube \( T_{z_0} \), and we shall often emphasize this fact by writing \( \phi_T \).

**Proof sketch.** We outline the main steps as this procedure is fairly standard; consult for instance Lemma 4.1 in [Tao03]. Begin with partitions of unity \( 1 = \sum_{x_0 \in \mathbb{Z}^d} \eta(x-x_0) \) and \( 1 \sum_{\xi_0 \in \mathbb{Z}^d} \chi(\xi-\xi_0) \) such that \( \chi \) and \( \eta \) are compactly supported. By rescaling and quantizing, we obtain a pseudodifferential partition of unity

\[
1 = \sum_{(x_0, \xi_0) \in R^{1/2} \mathbb{Z}^d \times R^{-1/2} \mathbb{Z}^d} \eta \left( \frac{x-x_0}{R^{1/2}} \right) \chi(R^{1/2}(D-\xi_0)),
\]
which is used to decompose the initial data
\[ f = \sum_{(x_0, \xi_0)} \eta \left( \frac{x-x_0}{R^{1/2}} \right) \chi(R^{1/2}(D - \xi_0)) f. \]

The propagation estimates then follow from

\textbf{Lemma 6.3.} If \( \phi_{z_0} \) is a scale-R wavepacket concentrated at \( z_0 \), and \( U_N(t) \) is the propagator for \( H(t) = -\frac{1}{2}\Delta + V_N \), then \( U_N(t) \) is a scale-R wavepacket concentrated at \( z_0 \) for all \( |t| \leq R \).

\textit{Proof.} By rescaling we reduce to \( R = 1 \) and replace \( V \) by \( V_{N/R^{1/2}} \). Then the symbol \( a = \frac{1}{2} |\xi|^2 + V_{N/R^{1/2}}(t, x) \) satisfies the estimates (6), and we can appeal to Lemma 2.5. \( \square \)

6.2. Space localization. We claim it suffices to prove Theorem 6.1 with the spacetime norm restricted to a box \( \Omega_N := [-T_0 N^2, T_0 N^2] \times [-CT_0N^2, CT_0N^2]^d \) centered at the origin, where \( C \) is a large multiple of \( T \). In the ensuing discussion we see the advantage of considering the class of potentials \( \mathcal{V} \) at once.

\textbf{Proposition 6.4.} With the hypotheses and notation of Theorem 6.1, there exists \( C > 0 \) such that
\begin{equation}
\|u_N v_N\|_{L^2_x L^2_t(\Omega_N)} \lesssim \varepsilon \|f\|_{L^2} \|g\|_{L^2}
\end{equation}
for any \( \varepsilon > 0 \).

To recover Theorem 6.1, we appeal to approximate finite speed of propagation.

Begin by partitioning physical space \( \mathbf{R}^d = \bigcup_{j \in \mathbf{Z}^d} T_0 N^2 Q_j \) into cubes of width \( T_0 N^2 \), where \( Q_j \) denotes the cube centered at \( T_0 N^2 j \in T_0 N^2 \mathbf{Z}^d \). We decompose \( u := u_N \) and \( v := v_N \) into \( N^2 \times (N)^d \) wavepackets and group the terms in the product according to their initial positions. Write
\[ u = \sum_T a_T \phi_T = \sum_{j \in \mathbf{Z}^d} \sum_{T \in T_j} u_T, \]
\[ v = \sum_T b_T \phi_T' = \sum_{j' \in \mathbf{Z}^d} \sum_{T' \in T'_j} v_{T'}, \]
where \( T_j = \{T \in T : x(T) \in Q_j \} \) and similarly for \( T'_j \). Then we estimate by the triangle inequality
\[ \|uv\| \leq \sum_{k \geq 0} \left\| \sum_{|j-j'| \sim 2^k} \sum_{T \in T_j, T' \in T'_{j'}} u_T v_{T'} \right\|. \]

For the kth sum, note from Lemma 2.2 that if \( (x_1, \xi_1) := (x(T), \xi(T)) \) and \( (x_2, \xi_2) := (x(T'), \xi(T')) \), we have
\[ |x_1 - x_2| \geq (1 - t^2 \|\partial_x^2 V_N\|_{L^\infty} e^{t^2 \|\partial_x^2 V_N\|_{L^\infty}}) |x_1 - x_2| - (|t| + |t|^3 \|\partial_x^2 V_N\|_{L^\infty} e^{t^2 \|\partial_x^2 V_N\|_{L^\infty}}) |\xi_1 - \xi_2| \]
\[ \geq (1 - T_0^2 \|\partial_x^2 V\|_{L^\infty} e^{T_0^2 \|\partial_x^2 V\|_{L^\infty}}) |x_1 - x_2| - T_0 N^2 (1 + T_0^2 \|\partial_x^2 V\|_{L^\infty} e^{T_0^2 \|\partial_x^2 V\|_{L^\infty}}) |\xi_1 - \xi_2| \]
\[ \geq (1 - \eta) |x_1 - x_2| - T_0 N^2 (1 + \eta) |\xi_1 - \xi_2|. \]

So if \( |x_1 - x_2| \geq 2^k T_0 N^2 \) and \( \eta \) is chosen small enough, we obtain \( |x_1^t - x_2^t| \gtrsim 2^k T_0 N^2 \). As each wavepacket \( \phi_T \) decays rapidly on the \( N \) spatial scale away from its tube \( T \), one has
\[ \left\| \sum_{|j-j'| \sim 2^k} \sum_{T \in T_j, T' \in T'_{j'}} u_T v_{T'} \right\| \lesssim 2^{-100dk} N^{-100d} \|f\|_{L^2} \|g\|_{L^2}. \]
Now
\begin{equation}
\left\| \sum_{|j-j'| \leq 1} \sum_{T \in T_j, T' \in T'_{j'}} u_T v_{T'} \right\| \leq \sum_{m \in \mathbb{Z}^d + \mathbb{Z}^d} \left( \sum_{T \in T_j, T' \in T'_{j'}} \left| a_T \right|^2 \right)^{1/2} \left( \sum_{T' \in T'_{j'}} \left| b_{T'} \right|^2 \right)^{1/2}.
\end{equation}

For a fixed pair \((j, j')\) in the above sum, the wavepackets \(u_T\) and \(v_{T'}\) are localized initially to the spatial cubes \(Q_j\) and \(Q_{j'}\) which are separated by distance \(\lesssim T_0 N^2\). Translating the initial data of both \(\sum_{T \in T_j} u_T\) and \(\sum_{T' \in T'_{j'}} v_{T'}\), by the midpoint \(x_0 = \frac{i j + j'}{2} T_0 N^2\) of \(Q_j\) and \(Q_{j'}\), applying Lemma 2.1, and writing \(\tilde{u}, \tilde{v}\) for the Schrödinger solutions for the transformed potential \(V(x_0, 0)\), the norm becomes
\[ \left\| \sum_{T \in T_j, T' \in T'_{j'}} \tilde{u}_T \tilde{v}_{T'} \right\|, \]
where the initial positions \(x(T)\) and \(x(T')\) of the tubes now belong to the translated cubes \(\tilde{Q}_j := Q_j - \frac{i j + j'}{2} T_0 N^2\), \(\tilde{Q}_{j'} := -\frac{i j + j'}{2} T_0 N^2\), which are now distance \(\lesssim T_0 N^2\) from the origin (note however that the tubes in \(T_j\) are not simply translates of those in \(T_{j'}\)).

By finite speed of propagation, the norm outside \(\Omega_N := [-T_0 N^2, T_0 N^2] \times [-CT_0 N^2, CT_0 N^2]^d\) is negligible for \(C\) large enough:
\[ \left\| \sum_{T \in T_j, T' \in T'_{j'}} \tilde{u}_T \tilde{v}_{T'} \right\|_{L^\infty([-T_0 N^2, T_0 N^2] \times [-CT_0 N^2, CT_0 N^2]^d)} \lesssim N^{-100} \left( \sum_{T \in T_j} \left| a_T \right|^2 \right)^{1/2} \left( \sum_{T' \in T'_{j'}} \left| b_{T'} \right|^2 \right)^{1/2} \]
\[ \lesssim N^{-100} \left( \sum_{T \in T_j} \left| a_T \right|^2 \right)^{1/2} \left( \sum_{T' \in T'_{j'}} \left| b_{T'} \right|^2 \right)^{1/2} . \]

Using Proposition (6.4) inside \(\Omega_N\), we may bound the right side of (26) by
\[ \sum_{m \in \mathbb{Z}^d} \sum_{|j-j'| \leq 1} \left| a_T \right|^2 \left| b_{T'} \right|^2 \left( \sum_{T \in T_j} \left| a_T \right|^2 \right)^{1/2} \left( \sum_{T' \in T'_{j'}} \left| b_{T'} \right|^2 \right)^{1/2} \]
\[ \lesssim N^\epsilon \left( \sum_{T \in T_j} \left| a_T \right|^2 \right)^{1/2} \left( \sum_{T' \in T'_{j'}} \left| b_{T'} \right|^2 \right)^{1/2} \]
\[ \lesssim N^\epsilon \left\| f \right\|_{L^2} \left\| g \right\|_{L^2} . \]

6.3. **Induction on scales.** In the presence of a potential, the frequency support of a solution is not preserved. This causes a slight issue when trying to improve the exponent in a local restriction estimate over spacetime balls \(Q_R(t_Q, x_Q)\) centered away from \(t = 0\), as by time \(t_Q\) the frequency supports of the two inputs could have expanded and grown closer than assumed at \(t = 0\). We deal with this by a subdivision and rescaling argument, which again exploits the fact that we consider the entire class of potentials \(\mathcal{V}\).

In this section, we explicitly display the dependence of the propagator on the potential, and write \(U_N^V(t) = U_N^V(t, 0)\) for the propagator with potential \(V_N\).

Let \(\mathcal{H}(\alpha)\) denote the following hypothesis:

Suppose \(S_j \subset \mathbb{R}^d\) satisfy \(\text{diam}(S_j) \leq 1\) and \(\text{dist}(S_1, S_2) \geq c\). For each \(N \geq 1\) and for all potentials \(V \in \mathcal{V}\):
Therefore, we have

\[ \|U_N^V(t)fU_N^V(t)g\|_{L^2(\Omega_N)} \lesssim_\alpha R^\alpha \|f\|_{L^2} \|g\|_{L^2} \]

for all \( f, g \in L^2(\mathbb{R}^d) \) with \( \hat{f}, \hat{g} \) supported in the dilates \((1 + \frac{\epsilon}{100})S_1 \) and \((1 + \frac{\epsilon}{100})S_2 \), respectively.

- The estimate

\[ \|U_N^V(t)fU_N^V(t)g\|_{L^2(\Omega_N)} \lesssim_\alpha N^\alpha \|f\|_{L^2} \|g\|_{L^2} \]

holds for all \( f, g \in L^2(\mathbb{R}^d) \) with \( \hat{f}, \hat{g} \) supported in \( S_1 \) and \( S_2 \), respectively.

The purpose of dilating the frequency supports in the first part of the induction hypothesis is to accommodate the (small but nonzero) enlargement of Fourier supports resulting from the wavepacket decompositions. This technicality is needed for our induction argument and arises in other contexts, such as in Tao’s work on the cone [?] where he introduces the notion of “margin” to address the same issue.

We prove:

**Proposition 6.5.** If \( IH(\alpha) \) holds, then \( IH(\max((1 - \delta)\alpha, C\delta) + \varepsilon) \) holds for all \( 0 < \delta, \varepsilon \ll 1 \).

By choosing \( \delta \) and \( \varepsilon \) sufficiently small depending on \( \alpha \), we can always arrange that \( \max((1 - \delta)\alpha, C\delta) + C\varepsilon < \alpha - c\alpha^2 \) for some absolute constant \( c \), and Proposition 6.4 follows.

For \( 1 \leq R \leq N^2 \), fix a ball \( Q_R \subset \Omega_N \) with center \((t_Q, x_Q)\) and width \( R \). Given \( IH(\alpha) \), we wish to prove

\[ \|U_N^V(t)fU_N^V(t)g\|_{L^2(Q_R)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^2} \|g\|_{L^2}. \]

We begin by estimating how much the Fourier supports can shift under the flow.

**Lemma 6.6.** Suppose the initial data \( f, g \) satisfy \( \text{supp}(\hat{f}) \subset (1 + \frac{\epsilon}{100})S_1 \) and \( \text{supp}(\hat{g}) \subset (1 + \frac{\epsilon}{100})S_2 \). There exist decompositions \( u(t_Q) = f_1 + f_2 \) and \( v(t_Q) = g_1 + g_2 \), with the following properties:

- \( \hat{f}_1 \) and \( \hat{g}_1 \) are supported in sets \( \tilde{S}_1, \tilde{S}_2 \) with \( \text{diam}(\tilde{S}_j) \leq 2 \text{diam}(S_j) \) and \( \text{dist}(\tilde{S}_1, \tilde{S}_2) \geq \frac{1}{2} \text{dist}(S_1, S_2) \).
- \( \|f_2\|_{L^2} \lesssim N^{-100\delta} \|f\|_{L^2} \) and \( \|g_2\|_{L^2} \lesssim N^{-100\delta} \|g\|_{L^2} \).

**Proof.** Begin with a decomposition of \( u = U_N^Vf \) and \( v = U_N^Vg \) into \( N^2 \times (N)^d \) wavepackets:

\[ u = \sum_{T \in \mathcal{T}_1} a_T \phi_T, \quad v = \sum_{T \in \mathcal{T}_2} b_T \phi_T. \]

By the spatial localization (24), we may ignore \( u \) and \( v \) the wavepackets whose tubes \( T \in \mathcal{T}_j \) do not intersect \( 2Q_N := [-T_0N^2, T_0N^2] \times [-2CT_0N^2, 2CT_0N^2] \), as the portion of the sum involving those terms contributes at most \( O(N^{-10\delta}) \|f\|_{L^2} \|g\|_{L^2} \). Thus there are \( O(N^{2d}) \) remaining wavepackets.

Suppose \( \phi_{T_1} \) and \( \phi_{T_2} \) are wavepackets in the decomposition for \( u \).

Let \( (x_1^T, \xi_1^T) \) and \( (x_2^T, \xi_2^T) \) be bicharacteristics with \( |x_1|, |x_2| \leq 2CT_0N^2 \). By Lemma (2.2), for \( |t| \leq T_0N^2 \) we have

\[ |\xi_1^t - \xi_2^t - (\xi_1 - \xi_2)| \leq T_0N^{-2} \|\partial_x^2 V\|_{L^\infty}(2CT_0N^2 + T_0N^2|\xi_1 - \xi_2|)e^{T_0\|\partial_x^2 V\|_{L^\infty}} \leq T_0 \|\partial_x^2 V\|_{L^\infty}(2CT_0 + T_0|\xi_1 - \xi_2|)e^{T_0\|\partial_x^2 V\|_{L^\infty}} \leq 10\eta. \]

Therefore, we have \( |\xi_1^t - \xi_2^t| \leq 1 + 10\eta \) if \( \xi_1, \xi_2 \) both belong to \( S_1 \) or \( S_2 \), while \( |\xi_1^{t_Q} - \xi_2^{t_Q}| \geq \frac{9}{10}c \) if \( \xi_1 \in (1 + \frac{\epsilon}{100})S_1 \) and \( \xi_2 \in (1 + \frac{\epsilon}{100})S_2 \).
Consequently, if
\[(29) \quad S_j^i := \{\xi^i : \xi \in (1 + \frac{1}{10^i})S_j, \quad |x| \leq CT_0N^2\}\]
denotes the frequencies for the wavepackets at time \(t\), then \(\text{diam}(S_j^i) \leq \frac{2}{\sqrt{10}}\text{diam}(S_j)\) and \(\text{dist}(S_{j_1}^i, S_{j_2}^i) \geq \frac{n}{10}\text{dist}(S_1, S_2)\). Now let \(\tilde{S}_j\) denote \(O(N^{-9/10})\) neighborhoods of \(S_j^i\), and decompose
\[u(t_Q) = f_1 + f_2, \quad v(t_Q) = g_1 + g_2,\]
where \(\hat{f}_1\) is supported on \(\tilde{S}_1\) and \(\hat{f}_2\) on the complement, and similarly for \(g_1, g_2\). For \(N\) large enough we certainly have \(\text{dist}(\tilde{S}_1, \tilde{S}_2) \geq \frac{\sqrt{2}}{2}\text{dist}(S_1, S_2)\). The estimates in the second bullet point now follow from the rapid decay of each wavepacket from its central frequency on the \(N^{-1}\) scale (the estimates (24) with \(R = N^{1/2}\)). \(\square\)

From here the argument proceeds as follows:

- After discarding a negligible portion of \(f_Q := u(t_Q)\) and \(g_Q := v(t_Q)\) according to the lemma, we have that \(\text{supp}(\hat{f}_Q) \subset \tilde{S}_1\) and \(\text{supp}(\hat{g}_Q) \subset \tilde{S}_2\). After translating \(Q_R\) in spacetime to be centered at \((t, x) = (0, 0)\) using Lemma 2.1, we wish to prove
  \[\|U_NV f_Q U_N^* g_Q\|_{L^{\frac{1+4}{1+3}}(Q_R)} \lesssim \varepsilon R^{\frac{5}{10}} R^{\max((1-\delta)\alpha, C\delta)} \|f_Q\|_{L^2} \|g_Q\|_{L^2}.\]

- We translate \(\tilde{S}_1\) and \(\tilde{S}_2\) via Lemma 2.1 to be on opposite sides of the origin, and rescale slightly to push apart the initial Fourier supports to distance \(\geq c\). The rescaled equation will have a slightly larger potential \(V_{4N/5}\). However, one still has \(4N/5 \geq R^{1/2}\), whence
  \[V_{4N/5} = \tilde{V}_{R^{1/2}} \text{ for some other } \tilde{V} \in \mathcal{V}.\]

By partitioning \(f_Q\) and \(g_Q\) in Fourier space we may reduce the diameter of their Fourier supports to less than \(\text{diam}(S_j)\).

It follows from this discussion that to prove the inductive step (27) for arbitrary \(Q_R \subset \Omega_N\), it suffices to consider the case \(Q_R = \Omega_N, R = N^2\), for all \(N\) (that is, just the second bullet point in our induction hypothesis). We focus on this in the sequel.

From here on the argument closely follows that of Tao, and we adopt the following notation: we write \(A \lesssim B\) if \(A \lesssim_{\varepsilon} N^\varepsilon B\) for all \(N \gg 1\) and for all \(\varepsilon > 0\).

To reiterate, we want to prove
\[(30) \quad \|U_N V f U_N^* g\|_{L^{\frac{4+1}{4+3}}(\Omega_N)} \lesssim N^{2\max((1-\delta)\alpha, C\delta)} \|f\|_{L^2} \|g\|_{L^2}\]
assuming \(\text{supp}(\hat{f}) \subset S_1\) and \(\text{supp}(\hat{g}) \subset S_2\) with \(\text{diam}(S_j) \leq 1\) and \(\text{dist}(S_1, S_2) \geq c\).

As a preliminary remark, we note that by partitioning the Fourier supports of \(f\) and \(g\), it suffices to assume that in fact
\[(31) \quad \text{diam}(S_j) < \frac{\min(1, c)}{100},\]
so that the differences in \(S_2 - S_1\) are all nearly the same; this will be convenient for Lemma 6.7 below. Moreover, by a Galilei boost we may assume that \(S_1\) and \(S_2\) are symmetrically placed with respect to the origin.

Normalize \(f\) and \(g\) in \(L^2\), and decompose
\[u := U_N^* f = \sum_T a_T \phi_T, \quad v := U_N^* g = \sum_T b_T \phi_T\]
as in the proof of Lemma 6.6. As in that discussion, we discard all but the \(O(N^{2d})\) wavepackets that intersect \(2\Omega_N\). We also throw away the terms where \(|a_T| = O(N^{-100d})\) or \(|b_T| = O(N^{-100d})\), as that portion of the product can be bounded using the estimates (24) and Cauchy-Schwartz.
Consequently, in the decompositions of \( u \) and \( v \) we only consider the tubes \( T \) with \( N^{-100d} \lesssim |a_T|, |b_T| \lesssim 1 \). Partitioning the interval \([N^{-100d}, 1]\) into log \( N \) dyadic groups, we may further restrict to the tubes with \( |a_T| \sim \gamma_1 \) and \( |b_T| \sim \gamma_2 \) for dyadic numbers \( N^{-100d} \lesssim \gamma_1, \gamma_2 \lesssim 1 \). Let \( T_1, T_2 \) be the tubes for \( u \) and \( v \), respectively with this property. It therefore suffices to prove

\[
\left\| \sum_{T_1 \in T_1} \phi_{T_1} \sum_{T_2 \in T_2} \phi_{T_2} \right\|_{L^{\frac{d+3}{2}}(\Omega_N)} \lesssim (N^2(1-\delta)\alpha + N^2C\delta)\#T_1^{1/2}\#T_2^{1/2}
\]

(we have absorbed the complex phases into the wavepackets).

We have in effect reduced to considering the region of phase space \( \{(x, \xi) : |x| \lesssim T_0N^2, |\xi| \lesssim 1\} \), where the potential makes only a bounded perturbation to the Euclidean flow. For if \( |x^s| \lesssim T_0N^2 \) and \( |t-s| \lesssim T_0N^2 \), one has

\[
|x^t| \lesssim T_0N^2
\]

\[
|x^t - x^s| \leq \int_s^t |\partial_x(V_N)(\tau, x^\tau)| \, d\tau \lesssim T_0^2 \|\partial_x^2 V\|_{L^\infty} \lesssim \eta.
\]

Thus if \( \xi \in S_j \), then \( \xi^t \) belongs to a small neighborhood of \( S_j \) provided that \( \eta \) is sufficiently small; in particular, if \( \xi_1, \xi_2 \) are initially separated, then

\[
|\xi^t_1 - \xi^t_2| \sim 1
\]

for all choices of initial positions \( |x_j| \lesssim T_0N^2 \). Note, however, that later on we shall use a much stronger form of near-constancy of frequencies for pairs of colliding wavepackets.

6.4. Coarse scale decomposition. Following Tao, for small \( \delta > 0 \) we decompose \( \Omega_N = \bigcup_{B \in B'} B \) into \( O(N^{2d}) \) smaller balls of radius \( N^{2(1-\delta)} \), and estimate

\[
\left\| \sum_{T_1 \in T_1} \sum_{T_2 \in T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{d+3}{2}}(\Omega_N)} \lesssim \sum_{B \in B} \left\| \sum_{T_1 \in T_1} \sum_{T_2 \in T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{d+3}{2}}(B)}.
\]

Let \( \sim \) be a relation between tubes and balls to be specified later. Estimate the norm by the local part

\[
\sum_{B \in B} \left\| \sum_{T_1 \sim B} \sum_{T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{d+3}{2}}(B)}
\]

and the global part

\[
\sum_{B \in B} \left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{d+3}{2}}(B)}.
\]

We use the induction hypothesis to estimate the local term by

\[
(33) \lesssim \sum_{B \in B} N^{2(1-\delta)\alpha} \left( \sum_{T_1 \sim B} 1 \right)^{1/2} \left( \sum_{T_2 \sim B} 1 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{T_1 \in T_1} \#\{B : T_1 \sim B\} \right)^{1/2} \left( \sum_{T_2 \in T_2} \#\{B : T_2 \sim B\} \right)^{1/2}
\]

\[
\lesssim 1
\]

if the relation \( \sim \) is chosen so that each \( T \) is associated to \( \lesssim 1 \) balls. Note that this step is why we needed to slightly enlarge the Fourier supports in the induction hypothesis, as \( \text{supp}(\partial T_1(0)) \) is not quite contained in \( S_1 \).

Heuristically, a judicious choice of \( \sim \) allows one to avoid the worst interactions that would otherwise occur in the bilinear \( L^2 \) estimate if one were to natively interpolate between \( L^1 \) and \( L^2 \). As a simple example, if all the tubes were to intersect in a single ball \( B \), it would be advantageous to bound \( L^{\frac{d+3}{2}}(B) \) directly using the inductive hypothesis rather than attempt to estimate \( L^2(B) \).
It remains to bound the global piece (34), which we do by interpolating between \( L^1 \) and \( L^2 \).

By Cauchy-Schwartz and mass conservation,

\[
\sum_B \left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \right\|_{L^1(B)} \lesssim \sum_B \left( \left\| \sum_{T_1 \sim B} \phi_T \right\|_{L^2(B)} + \left\| \sum_{T_2 \sim B} \phi_T \right\|_{L^2(B)} \right) \left( \left\| \sum_{T_1 \sim B} \phi_T \right\|_{L^2(B)} + \left\| \sum_{T_2 \sim B} \phi_T \right\|_{L^2(B)} \right) \lesssim N^{2\delta} N^2 \#T_1^{1/2} \#T_2^{1/2}.
\]

In the remaining sections we prove

\[
\left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_T \right\|_{L^2(B)} \lesssim N^{-\frac{d+1}{2}} N^{C\delta} \#T_1^{1/2} T_2^{1/2}.
\]

6.5. **Fine scale decomposition.** Cover \( \Omega_N = \bigcup_{q \in \mathbf{q}} q \) by a finitely overlapping collection \( \mathbf{q} \) of balls of radius \( N \). It suffices to show

\[
\sum_{q \in \mathbf{q} \subseteq 2B} \left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \lesssim N^{-(d-1)} N^{C\delta} \#T_1 \#T_2.
\]

We adopt the following notation from Tao. Fix \( q \in \mathbf{q} \) and let \( \mu_1, \mu_2, \lambda_1 \) be dyadic numbers.

- \( T_j(q) \) is the set of tubes \( T \in T_j \) such that \( T \cap N^\delta q \neq \emptyset \).
- \( T_j^{-B}(q) = \{ q \in T_j(q) : T \sim B \} \).
- \( q(\mu_1, \mu_2) \) is the set of balls \( q \) such that \( \#\{ T_j \in T_j(q) : T \cap N^\delta q \neq \emptyset \} \sim \mu_j \).
- \( \lambda(T_j, \mu_1, \mu_2) \) is the number of \( (N^\delta \text{ neighborhoods of}) \) balls \( q(\mu_1, \mu_2) \) that \( T \) intersects.
- \( T_j[\lambda_1, \mu_1, \mu_2] \) is the set of tubes \( T \in T_j \) such that \( \lambda(T_j, \mu_1, \mu_2) \sim \lambda_1 \).

Pigeonholing dyadically in \( \mu_1, \mu_2 \), and \( \lambda_1 \), it suffices to show

\[
\sum_{q \in \mathbf{q}(\mu_1, \mu_2) \subseteq 2B} \left\| \sum_{T_1 \in T_1^{-B}(q) \cap T_j[\lambda_1, \mu_1, \mu_2]} \sum_{T_2 \in T_j(q)} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \lesssim N^{C\delta} N^{-(d-1)} \#T_1 \#T_2.
\]

6.6. **The \( L^2 \) bound.** Fix a ball \( q = q(t, x_q) \in \mathbf{q}(\mu_1, \mu_2) \) centered at \( (t, x_q) \). Suppose want to estimate an expression of the form

\[
\left\| \sum_{T_1} \sum_{T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2.
\]

There are two main points to keep in mind:

- Only tubes that intersect \( N^\delta q \) will make a nontrivial contribution; that is, tubes whose bicharacteristics \( (x^t, \xi^t) \) satisfy \( |x^t - x_q| \leq N^{1+\delta} \).
- To decouple the contributions of tubes that all overlap near \( \mathbf{q} \), one needs to exploit oscillation in space and time. While Tao employs the spacetime Fourier transform, we remain in physical space and integrate by parts in space and time. Some caution is needed here due to the limited time regularity of the phase, which ultimately allows one to integrate by parts just once in time. Although the resulting decay is weaker than what Tao obtains, it turns out to be just enough provided that we slightly strengthen the analogue of Tao’s main combinatorial estimate for tubes (estimate (42) below).

Upon expanding out the \( L^2 \) norm and integrating by parts, we shall obtain terms of the form

\[
(N|\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|)^{-1}, \quad (N||\xi_1^t - \xi_2^t|^2 - |\xi_3^t - \xi_4^t|^2|)^{-1},
\]

\[
(N||\xi_1^t - \xi_2^t|^2 + |\xi_3^t - \xi_4^t|^2|)^{-1}, \quad (N|\xi_1^t - \xi_2^t|)^{-1}.
\]
where \((x_j^q, \xi_j^q)\) are bicharacteristics with \(|x_j^q - x_q| \leq N^{1+\delta}\). Since, by Lemma 2.2, the relative frequencies \(\xi_j^q - \xi_k^q\) vary by at most \(O(N^{-2+2\delta})\) during the \(O(N^{1+\delta})\) time window when the wavepackets intersect the ball \(N^\delta q\), we can freeze \(t = t_q\) above.

This discussion motivates the following definition. For frequencies \(\xi_1\) and \(\xi_2\), define
\[
(37) \quad \pi(\xi_1, \xi_2) = \left\{ (\xi_j^q) : \right. \left. \xi_j^q \in S_1, \ |\xi_1 + \xi_j^q - (\xi_j^q)' - \xi_2^2| \leq N^{-1+C\delta} \right\},
\]
\[
|\xi_1 - \xi_j^q| - |(\xi_j^q)' - \xi_2^2| \sim 2^k N^{-1+C\delta},
\]
\[
|\xi_1 - \xi_j^q| - |(\xi_j^q)' - \xi_2^2| \leq N^{1+\delta} \text{ for some } \xi_2 \in S_2, \ |x_2| \leq 2CN^2, \ |x_j^q - x_q| \leq N^{1+\delta}\}
\]
This is a slight modification of Tao’s definition which reflects the time dependence of frequency in our context.

The following claim is evident from geometry.

**Lemma 6.7.** The set \(\pi(\xi_1, \xi_2)\) is contained in the hyperplane passing through \(\xi_1\) and orthogonal to \(\xi_2 - \xi_1\) (and is therefore transverse to \(\xi_2 - \xi_1\) if \(\xi_1\) and \(\xi_2\) are small perturbations of \(\xi_1\) and \(\xi_2\), respectively).

In view of the second remark above, we need to account more carefully for the contributions away from the “resonant set” \(\pi\).

For \(\xi_1, \xi_2\) and \(k > 0\), define the “time nonresonance” sets
\[
(38) \quad \pi_k(\xi_1, \xi_2) = \left\{ (\xi_j^q) : \xi_j^q \in S_1, \ |\xi_1 + \xi_j^q - (\xi_j^q)' - \xi_2| \leq N^{-1+C\delta} \right\},
\]
\[
|\xi_1 - \xi_j^q| - |(\xi_j^q)' - \xi_2| \sim 2^k N^{-1+C\delta},
\]
\[
|\xi_1 - \xi_j^q| - |(\xi_j^q)' - \xi_2| \leq N^{1+\delta} \text{ for some } \xi_2 \in S_2, \ |x_2| \leq 2CN^2, \ |x_j^q - x_q| \leq N^{1+\delta}\}
\]
and the “space nonresonance” set
\[
(39) \quad \pi^s(\xi_1, \xi_2) = \left\{ (\xi_j^q) : \xi_j^q \in S_1, \ |\xi_1 + \xi_j^q - (\xi_j^q)' - \xi_2| \geq N^{-1+C\delta} \right\},
\]
\[
|\xi_1 - \xi_j^q| - |(\xi_j^q)' - \xi_2| \leq N^{1+\delta} \text{ for all } \xi_2 \in S_2, \ |x_2| \leq 2CN^2, \ |x_j^q - x_q| \leq N^{1+\delta}\}
\]
An elementary computation shows that
\[
(40) \quad \text{dist}(\pi_k^1, \pi) \lesssim 2^k N^{-1+C\delta}.
\]
Indeed, writing \(\delta_1 := (\xi_j^q)' - \xi_1\) and \(\delta_2 := \xi_2 - \xi_2\), we have
\[
|\xi_1 - \xi_j^q| - |(\xi_j^q)' - \xi_2| = |\xi_1 - \xi_2 - \delta_2^2 - |\delta_1 + \xi_1 - \xi_2^2|\]
\[
= (\delta_1^2)^2 - |\delta_1^2 - |\xi_1 - \xi_2 - \delta_2^2||^2 - |\xi_1 - \xi_2^2|\]
\[
= O(N^{-1+C\delta}) + |\xi_1 - \xi_2 - \delta_2^2||^2 - |\delta_1^2 - |\xi_1 - \xi_2^2|\]
\[
= -2|\delta_1^2 - |\xi_1 - \xi_2 - \delta_2^2||^2 + O(N^{-1+C\delta}),
\]
where we decomposed \(\delta_j = |\delta_j| + \delta_j^\perp\) into the components parallel and orthogonal to \(\xi_1 - \xi_2\), and also used the constraint \(|\delta_1 - \delta_2| \leq N^{-1+C\delta}\). The claim follows upon recalling the initial hypothesis (31) that \(\text{diam}(S_j) \ll \text{dist}(S_1, S_2)\).

For \(q \in \mathbf{q}(\mu_1, \mu_2)\) with \(q \subset 2B\), define
\[
T^\infty_B(q, \lambda_1, \mu_1, \mu_2, \xi_1, \xi_2, k)
\]
to be the collection of tubes \(T \in T^B(q) \cap T[\lambda_1, \mu_1, \mu_2]\) whose frequency \(\xi(T)^q\) at time \(t_q\) belongs to the set \(\pi_k(\xi_1, \xi_2)\). Set
\[
(41) \quad \nu_k(q, \lambda_1, \mu_1, \mu_2) := \sup_{\xi_1 \in S_1, \ \xi_2 \in S_2} \#T^B(q, \lambda_1, \mu_1, \mu_2, \xi_1, \xi_2, k),
\]
where \(|x_j^q - x_q| + |(\xi_j^q)' - x_q| \lesssim N^{1+\delta}\).
Then, the analogue of Tao’s Lemma 7.1 is:

**Lemma 6.8.** For each \( q \in q(\mu_1, \mu_2) \), we have

\[
\left\| \sum_{T_1 \in T_1^q (q)} \sum_{\lambda_1, \mu_1, \mu_2} \phi_{T_1} \phi_{T_2} \right\|^{2}_{L^2(q)} \lesssim N^{C\delta} N^{-(d-1)} \sum_{k} 2^{-k} \nu_k(q, \lambda_1, \mu_1, \mu_2) \#(T_1^q (q) \cap T_1 [\lambda_1, \mu_1, \mu_2]) \#T_2 (q).
\]

**Proof.** For conciseness, set

\[
T_1' := T_1^q (q) \cap T_1 [\lambda_1, \mu_1, \mu_2] \\
T_2 := T_2 (q).
\]

Then the norm \( L^2(q) \) is bounded by the norm \( L^2(\eta_N dt \sigma) \), where \( \eta_N(t) \) is a smooth weight equal to 1 on \( |t-t_q| \leq N^{1+\delta} \) and supported in \( |t-t_q| \leq 2N^{1+\delta} \).

\[
\left\| \sum_{T_1' \in T_1'} \sum_{T_2 \in T_2} \phi_{T_1'} \phi_{T_2} \right\|^{2}_{L^2(\eta_N dt \sigma)} = \sum_{T_1' \in T_1'} \sum_{T_2 \in T_2} \langle \phi_{T_1'} \phi_{T_2}, \phi_{T_1'} \phi_{T_2} \rangle_{L^2(\eta_N dt \sigma)}.
\]

By the bounds (24), the integrand has magnitude \( N^{-2d} \) and is essentially supported on a \( N \times (N^d) \) set in space time (having magnitude \( O(N^{-100d}) \) on the complement of a \( N^{\delta} \) neighborhood). Thus we have the crude bound

\[
\langle \phi_{T_1'} \phi_{T_2}, \phi_{T_1'} \phi_{T_2} \rangle \lesssim N^{C\delta} N^{-2d} N^{d+1} = N^{C\delta} N^{-(d-1)}.
\]

On the other hand, we may integrate by parts to obtain a more refined bound.

**Lemma 6.9.** For each \( k_1, k_2, \ell \geq 0 \) and for all tubes \( T_1, T_2, T_3, T_4 \), we have

\[
\langle \phi_{T_1} \phi_{T_2}, \phi_{T_3} \phi_{T_4} \rangle \lesssim_{k_1, k_2} N^{C\delta} N^{-(d-1)} \min \left[ N^{-\ell} \xi_1 \xi_2 - \xi_3 \xi_4, N^{-1} \left| \xi_1 - \xi_2 \right|^{2} - \left| \xi_3 - \xi_4 \right|^{2} \right].
\]

**Proof.** The proof has a similar flavor to the earlier estimate (19), but here we aim for a more precise bound exploiting oscillations in both space and time.

Let \( z_j^f = (x_j^f, t_j^f) \) denote the bicharacteristic for \( \phi_{T_j} \), \( j = 1, 2, 3, 4 \). By Lemmas 2.1 and 6.2, we can write

\[
(41) \quad \langle \phi_{T_1} \phi_{T_2}, \phi_{T_3} \phi_{T_4} \rangle = \int e^{i\Psi} \phi_{T_1} \phi_{T_2} \overline{\phi_{T_3} \phi_{T_4}} \eta_N(t) dt \sigma,
\]

where \( \phi_j \) is a Schrödinger wave which satisfies

\[
(N \partial_x)^k \phi_j (t, x) \lesssim_{k, M} N^{-d/2} (N^{-1} (x-x_j^f))^{-M},
\]

and

\[
\Psi = \sum_{j=1}^{4} \sigma_j \left[ \langle x-x_j^f, \xi_j^f \rangle - \int_{0}^{\tau} \frac{1}{2} |\xi_j^f|^2 - V(\tau, x_j^f) \, d\tau \right], \quad \sigma = (+, +, -, -).
\]

Using the rapid decay of each \( \phi_j \), we may harmlessly (with \( O(N^{-100d}) \) error) localize \( \phi_j \) to a \( N^{\delta} \) neighborhood of the tube \( T_j \), so that \( \phi_j(t) \) is supported in a \( O(N^{1+\delta}) \) neighborhood of the classical path \( x_j^f \).

Then

\[
\partial_x \Psi = \sum_{j} \sigma_j \xi_j^f, \quad -\partial_t \Psi = \frac{1}{2} \sum_{j} \sigma_j |\xi_j^f|^2 + \sum_{j} \sigma_j \left[ V(t, x_j^f) + \langle x-x_j^f, \partial_t V(t, x_j^f) \rangle \right].
\]
The first bound in the statement of the lemma results from integrating by parts in $x$, as in the proof of (19), to get factors of $(N|\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|)^{-1}$; noting that since the relative frequencies $\xi_1^t - \xi_k^t$ vary by at most $O(N^{-2+2\delta})$ during the time window $|t - t_q| \leq O(N^{1+\delta})$ when $|x_j^t - x_q| \leq N^{1+\delta}$, we may freeze $t = t_q$.

As in our work in one space dimension (more specifically, the proof of Lemma 4.4 in [JKV]), instead of integrating by parts purely in time we use a vector field adapted to the average bicharacteristic for the four wavepackets $\phi_{T_j}$. Defining

$$x_t^t := \frac{1}{4} \sum_{j=1}^4 x_j^t, \quad \xi_t^t := \frac{4}{4} \sum_{j=1}^4 \xi_j^t,$$

$$L := \partial_t + \langle \xi^t, \partial_x \rangle,$$

we compute as in that paper that

$$-L\Psi = \frac{1}{2} \sum \sigma_j |\xi_j^t|^2 + \sum \sigma_j [V^*(t, x_j^t) + \langle x - x_j^t, \partial_x (V^*)(t, x_j^t) \rangle],$$

where

$$x_j^t := x_j^t - x_t^t, \quad \xi_j^t := \xi_j^t - \xi_t^t$$

denote the coordinates of $\phi_{T_j}(t)$ in phase space relative to $(x_t^t, \xi_t^t)$; see Figure 2. To pass to the third line we have, as before, frozen $t = t_q$ up to $O(N^{-2+2\delta})$ errors, and also used the estimates $|x_j^t| \leq \max_{j,k} |x_j^t - x_k^t| \lesssim N^{1+\delta}$, $|x - x_j^t| \lesssim N^{1+\delta}$ on the support of the integral (41).

At this point the phase $\Psi$ is too rough to integrate by parts, since that would require two time derivatives of $\Psi$ but the assumptions on $V$ only allow $\Psi$ to be differentiated once in time. However, we can decompose $\Psi = \Psi_1 + \Psi_2$, where $\Psi_2$ has two time derivatives and accounts for the majority of the oscillation of $e^{i\Psi}$; indeed, we define $\Psi_1$ and $\Psi_2$ via the ODE

$$-L\Psi_2 = \frac{1}{2} \sum \sigma_j |\xi_j^t|^2 = \frac{1}{4} \left( |\xi_1^t - \xi_2^t|^2 - |\xi_3^t - \xi_4^t|^2 \right) + O(N^{-2+2\delta}),$$

$$-L\Psi_1 = \sum \sigma_j \left[ V^*(t, x_j^t) + \langle x - x_j^t, \partial_x (V^*)(t, x_j^t) \rangle \right] = O(N^{-2+2\delta}).$$
Note also that the equation \( \frac{d}{dt} e^{\lambda t} = -\lambda e^{\lambda t} \) implies \( L^2 \psi_2 = O(N^{-2}) \). Integrating by parts, we obtain
\[
\begin{align*}
\text{RH (41)} &= \int e^{i \psi_2} e^{i \psi_1} \prod_j \phi_j \eta_N(t) \, dx dt = i \int e^{i \psi_2} \left\langle L, \frac{L \psi_2}{|L \psi_2|^2} \right\rangle e^{i \psi_1} \phi_1 \psi_3 \phi_4 \eta_N(t) \, dx dt \\
&= i \int e^{i \psi} \left[ \frac{L^2 \psi_2}{|L \psi_2|^2} + \left\langle L \psi_2, i L \psi_1 + L \right\rangle \right] \phi_1 \psi_3 \phi_4 \eta_N(t) \, dx dt,
\end{align*}
\]
and the second bound in the lemma follows.

Returning to the proof of Lemma 6.8, we decompose the sum
\[
\sum_{(T_1, T_2') \in T_1' \times T_2'} \left[ \sum_{T_1' \in T_1'} \sum_{T_2' \in T_2'} 0 \leq k \leq \log N \sum_{\lambda \in \mathcal{T}_{1,k}, \mu \in \mathcal{T}_{2,k}} \sum_{T, \lambda, \mu} \sum_{q, \lambda, \mu} \right],
\]
where \( T_1^s \) is the set of tubes in \( T_1' \) whose bicharacteristic \( ((x_1^r, (\xi_1^r)^r) \) satisfies \( (\xi_1^r)^r \in \pi^s(\xi_1^r, (\xi_2^r)^r), \) and we abbreviate
\[
T^s_{1,k} := T^s_{1,B}(q, \lambda_1, \mu_1, \mu_2, (\xi_1^r)^r, (\xi_2^r)^r, k).
\]
The contribution from the “space nonresonance” terms \( T_1^s \) is \( O(N^{-100d}) \).

Now consider the \( k \)th sum. Lemma 6.9 implies that
\[
|\langle \phi_{T_1} \phi_{T_2}, \phi_{T_1} \phi_{T_2}' \rangle| \lesssim N^{C\delta} N^{-(d-1)} 2^{-k}.
\]
For each \( T_1' \in T_1^s \) \((q, \lambda_1, \mu_1, \mu_2, (\xi_1^r)^r, (\xi_2^r)^r, k)\), the possible tubes \( T_2' \) correspond to the bicharacteristics \((x_1^r, x_2^r)\) such that
\[
|x_2^r - x_q| \leq N^{1+\delta}, \quad (\xi_1^r) + (\xi_2^r)^r - (\xi_1^r)^r = O(N^{-1+C\delta}).
\]
The preimage of this set under the time \( t_q \) Hamiltonian flow map is a \( (N^{1+C\delta})^d \times (N^{-1+C\delta})^d \) box, so there are \( O(N^{C\delta}) \) choices of tubes \( T_2 \). Therefore, the \( k \)th sum is at most
\[
N^{C\delta} N^{-(d-1)} 2^{-k} \nu_k \# T_1^s \# T_2',
\]
whereupon the sum over \( k \) is replaced by the supremum at the cost of a log \( N \) factor.

It remains to show that
\[
\sum_{q \in q(\mu_1, \mu_2), q \subset 2B} 2^{-k} \nu_k(q, \lambda_1, \mu_1, \mu_2) \# (T^s_{1,B}(q) \cap T_1[\lambda_1, \mu_1, \mu_2]) \# T_2(q) \lesssim N^{C\delta} \# T_1 \# T_2.
\]

### 6.7. Tube combinatorics for general potentials

This section begins exactly as in [Tao03, Section 8]. We define the relation \( \sim \) between tubes and radius \( N^{2(1-\delta)} \) balls. For a tube \( T \in T_1[\lambda_1, \mu_1, \mu_2], B(T, \lambda_1, \mu_1, \mu_2) \) be the ball \( B \in \mathcal{B} \) that maximizes
\[
\# \{ q \in q(\mu_1, \mu_2) : T \cap N^q \neq \phi; q \cap B \neq \phi \}.
\]
As \( T \) intersects about \( \lambda_1 \) (neighborhoods of) \( q \in q(\mu_1, \mu_2) \) in total and there are \( O(N^{2\delta}) \) many balls in \( \mathcal{B} \), \( B(T, \lambda_1, \mu_1, \mu_2) \) must intersect at least \( N^{-2\delta} \lambda_1 \) of those balls.

Declare \( T \sim_{\lambda_1, \mu_1, \mu_2} B' \) if \( T \in T_1[\lambda_1, \mu_1, \mu_2] \) and \( B' \subset 10B(T, \lambda_1, \mu_1, \mu_2) \). Finally, for \( T \in T_1 \) set \( T \sim B \) if \( T \sim_{\lambda_1, \mu_1, \mu_2} B \) for some \( \lambda_1, \mu_1, \mu_2 \). Evidently \( T \sim B \) for at most \( \log N \) balls.

We can similarly define the relation between tubes in \( T_2 \) and balls in \( \mathcal{B} \).
Now we begin the proof of (42). We have
\[
\sum_{q \in q(\mu_1, \mu_2)} \#(\mathbf{T}_1[\lambda_1, \mu_1, \mu_2] \cap \mathbf{T}_1(q)) = \sum_{q \in q(\mu_1, \mu_2)} \sum_{T \in \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]} 1_{T_1 \cap N^\delta q \neq 0} = \sum_{T \in \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]} \sum_{q \in q(\mu_1, \mu_2)} 1_{T_1 \cap N^\delta q \neq 0} \lesssim \sum_{T \in T_1} \lambda_1 \lesssim \lambda_1 \# \mathbf{T}_1.
\]
On the other hand, by definition \(\# \mathbf{T}_2(q) \lesssim \mu_2\). The claim (42) would therefore follow if we could show that
\[
(43) \quad \nu_k(q_0, \lambda_1, \mu_1, \mu_2) \lesssim 2^k N^{C\delta} \frac{\# \mathbf{T}_2}{\lambda_1 \mu_2}
\]
for all \(q_0 \in q(\mu_1, \mu_2)\) such that \(q_0 \subset 2B\).

Fix \(\xi_1 \in S_1, \xi_2 \in S_2\) and a ball \(q_0 = q_0(t_q, x_q)\). By the definition of \(\nu_k\), we need to show that
\[
\# \mathbf{T}_1^{\xi_1} = \mathbf{T}_1^{\xi_1}(q_0, \lambda_1, \mu_1, \mu_2, \xi_1, \xi_2, k) \lesssim 2^k N^{C\delta} \frac{\# \mathbf{T}_2}{\lambda_1 \mu_2}.
\]
For brevity write \(T_1' := T_1^{\xi_1}(q_0, \lambda_1, \mu_1, \mu_2, \xi_1, \xi_2, k)\).

Fix \(T_1 \in T_1'\). Since \(T_1 \approx B\), the ball \(10B(T_1, \lambda_1, \mu_1, \mu_2)\) has distance \(\gtrsim N^{2(1-\delta)}\) from \(2B\). Thus
\[
\# \{ q \in q(\mu_1, \mu_2) : T_1 \cap N^\delta q \neq 0, \text{ dist}(q, q_0) \gtrsim N^{2(1-\delta)} \} \gtrsim N^{-2\delta} \lambda_1.
\]
As each \(q \in q(\mu_1, \mu_2)\) intersects at approximately \(\mu_2\) tubes in \(T_2\),
\[
\# \{ (q, T_2) \in q(\mu_1, \mu_2) \times T_2 : T_1 \cap N^\delta q \neq 0, T_2 \cap N^\delta q \neq 0, \text{ dist}(q, q_0) \gtrsim N^{2(1-\delta)} \} \gtrsim N^{-2\delta} \lambda_1 \mu_2.
\]
Therefore
\[
\# \{ (q, T_1, T_2) \in q \times T_1' \times T_2 : T_1 \cap N^\delta q \neq 0, T_2 \cap N^\delta q \neq 0, \text{ dist}(q, q_0) \gtrsim N^{-2\delta} N^2 \} \gtrsim N^{-2\delta} \lambda_1 \mu_2 \# T_1' \gtrsim N^{-2\delta} \lambda_1 \mu_2 \# T_1'.
\]
The analogue of Tao’s Lemma 8.1 is:

**Lemma 6.10.** For each \(T_2 \in T_2\),
\[
\# \{ (q, T_1) \in q \times T_1' : T_1 \cap N^\delta q, T_2 \cap N^\delta q \neq 0, \text{ dist}(q, q_0) \gtrsim N^{-2\delta} N^2 \} \lesssim 2^k N^{C\delta}.
\]
**Proof.** We estimate in two steps.

- For any tubes \(T_1 \in T_1'\) and \(T_2 \in T_2\), the intersection \(N^\delta T_1 \cap N^\delta T_2\) is contained in a ball of radius \(N^{C\delta}\).
- The number of tubes \(T_1 \in T_1'\) such that \(T_1\) intersects \(N^\delta T_2\) at distance \(\gtrsim N^{-2\delta} N^2\) from \(q_0\) bounded above by \(2^k N^{C\delta}\).

The first is evident from transversality. Hence we turn to the second claim.

In Tao’s situation, the tubes in \(T_1'\) are all constrained to lie in a \(O(N^{-1+C\delta})\) neighborhood of a spacetime hyperplane transverse to the tube \(T_2\) (basically because of Lemma 6.7), and there are \(O(N^{C\delta})\) many such tubes that intersect \(T_2\) at distance \(\gtrsim N^{-2\delta} N^2\) from \(q_0\). The extra \(2^k\) factor results from the fact that we allow the tubes to deviate from that hyperplane by distance \(2^k N^{-1+C\delta}\). Also, we need to argue microlocally since our tubes are curved.

Fix a tube \(T_2 \in T_2\) with bicharacteristic \(x \mapsto (x_2, \xi_2)\). Then, the tubes \(T_1 \in T_1'\) such that \(N^\delta T_1 \cap N^\delta T_2\) are characterized by the property that
\[
| x(T_1)^\xi - x_2^{\xi_2} | \lesssim N^{1+\delta} \text{ for some } |t - t_q| \gtrsim N^{-2\delta} N^2.
\]
which yields this map is well-defined by Lemma 2.2, which asserts that the “exponential map” \(\exp\).

Lemma 6.11.

Proof. By a slight abuse of notation, write \((x'(y, \eta), \xi'(y, \eta))\) for the bicharacteristic passing through \((y, \eta)\) at time \(t = t_q\) instead of \(t = 0\). Both claims will ultimately be consequences of Lemma 2.2, which yields

\[
x'_2 = x'(x_0, \zeta_{x_0}(t)), \quad \xi'q(x_0, \zeta_{x_0}(t)) = \zeta_{x_0}(t),
\]

\[
\xi' = \xi'(x_0, \zeta_{x_0}(t)) + \frac{\partial x'}{\partial \zeta_{x_0}} \dot{\zeta}_{x_0}(t)
\]

\[
= \xi'(x_0, \zeta_{x_0}(t)) + (t - t_q)(I + O(\eta)) \dot{\zeta}_{x_0}(t),
\]

therefore

\[
\dot{\zeta}_{x_0}(t) = (t - t_q)^{-1} (I + O(\eta)) (\xi'q - \xi'(x_0, \zeta_{x_0}(t))).
\]
We claim that
\[ \text{dist}(\zeta_{x_0}(t), S_{t_q}^{t_q}) \lesssim N^{-1+C\delta}. \]
(44)
Otherwise, as \(|t-t_q| \gtrsim N^{2(1-\delta)}\), for any ray \((x_1^t, \xi_1^t)\) with \(\xi_1 \in S_1\) and \(|x_1^t - x_q| \leq N^{1+\delta}\), Lemma 2.2 would imply that
\[ |x_1^t - x_2^t| \gtrsim |t-t_q| N^{-1+C\delta} - N^{1+\delta} \gtrsim N^{1+C\delta}, \]
therefore the claim holds if (as we may assume) there is at least one \(T_1 \in T_1'\) emanating from \(q_0\) that passes near \(T_2\).

By Lemma 6.7 and the near-constancy (32) of the frequency variable, the covector \(\xi_1^t - \xi^t(x_0, \zeta_{x_0}(t))\) belongs a small perturbation of the difference set \(S_2 - S_1\), which are in turn uniformly transverse to the hypersurface \(\pi\).

In view of this lemma, the fiber of \(\Sigma_{t_q}\) in \(T_{x_q}^{*}R^d\) is contained in a curved “frequency tube”
\[ \Theta(x_0) := \bigcup_t B(\zeta_{x_0}(t), N^{-1+C\delta}). \]
As the basepoint \(x_0\) varies in \(N^{1+\delta}\) neighborhood of \(x_q\), Lemma 2.2 implies that the curve \(\zeta_{x_0}(t)\) shifts by at most \(O(N^{-1+3\delta})\). Hence, the tubes \(\Theta(x_0)\) are all contained in a dilate of \(\Theta(x_q)\), which we denote by
\[ \tilde{\Theta}(x_q) := \bigcup_t B(\zeta_{x_q}(t), N^{-1+C\delta}) \]
with a larger \(C\).
Therefore, \(\Sigma_{t_q}\) is contained in the region
\[ \{ (x, \xi) : |x - x_q| \leq N^{1+\delta}, \xi \in \pi_{x_q}^t \cap \tilde{\Theta}(x_q) \subset \{ \xi \in \tilde{\Theta}(x_q) : \text{dist}(\xi, \pi) \lesssim 2^k N^{-1+C\delta} \}\}, \]
depicted in Figure 4, where for the last containment we recall the estimate (40). Using the lemma for the central curve \(\zeta_{x_q}\), we count phase space boxes and find at most \(2^k N^{C\delta}\) corresponding tubes in \(T_1'\).

7. Remarks on magnetic potentials

We sketch the modifications needed to prove Theorem 1.5. Let \(U(t)\) be the propagator whose symbol has the form
\[ a = \frac{1}{2} |\xi|^2 + \langle A, \xi \rangle + V(t, x), \]
where \(A = A_j(t, x)dx^j\) and \(A_j\) are linear functions in the space variables and bounded in time.
• Easy computation shows that
  \[
  a^{z_0} = \frac{1}{2} |\xi|^2 + \langle A_{z_0}^{(1)}(t, x), \xi \rangle + \langle A_{z_0}^{(2)}(t, x), \xi^1 \rangle + V_{z_0}^{(1)}(t, x),
  \]
  where \( A_{z_0}^{(1)}(t, x) = A(t, x_0 + x) - A(t, x_0) \) and \( A_{z_0}^{(2)}(t, x) = A(t, x_0 + x) - \langle x, \partial_x A(t, x_0) \rangle - A(x_0) \),
and similarly for \( V \). Thus when \( A \) is linear, the first order component of the symbol is exactly “Galilei-invariant”,
preserved by the transformation \( a \mapsto a^{z_0} \) in Lemma 2.1.

• After rescaling, the inequality (12) takes the form
  \[
  \|U_N f U_N g\|_{L^{4+\delta}_x([-T_0,N^2,T_0,N^2] \times \mathbb{R}^3)} \lesssim \varepsilon \|f\|_{L^2} \|g\|_{L^2},
  \]
where \( U_N(t) \) is the propagator for the rescaled symbol
  \[
  a_N := N^{-2} a(N^{-2} t, N^{-2} x, N^{-2} \xi) = \frac{1}{2} |\xi|^2 + N^{-2} \langle A(x), \xi \rangle + N^{-2} V(N^{-2} t, N^{-2} x).
  \]

• Exploiting Galilei-invariance, we may reduce to a spatially localized estimate as in Proposition 6.4. Note that in the region of phase space corresponding to that estimate \( \{ x, \xi : |x| \leq N^2, |\xi| \leq 1 \} \), and over a \( O(N^2) \) time interval, both potential terms have strength \( O(1) \), although the magnetic term dominates near \( x = 0 \).

• Then, the rest of the previous proof can be mimicked with essentially no change except for Lemma 6.9. There, one argues essentially as before except the vector field \( L \) should be replaced by
  \[
  L := \partial_t + \langle \overline{a}_\xi(z_j^{(1)}), \partial_x \rangle,
  \]
where \( z_j = (x_j, \xi_j) \) and \( \overline{a}_\xi(z_j^{(1)}) = \frac{1}{4} \sum_k a_\xi(z_k^{(1)}) \). Then one finds that
  \[
  -L \Psi = \frac{1}{2} \sum_j \sigma_j |\xi_j|^2 + \sum_j \sigma_j \langle A(x_j^{(1)}), \xi_j \rangle + \sum_j \sigma_j \langle V(x_j^{(1)}), \xi_j \rangle + \sum_j \sigma_j \langle V(x_j^{(1)}), \xi_j \rangle + \langle x - x_j^{(1)}, \partial_x (V(x_j^{(1)})) (t, x_j^{(1)}) \rangle,
  \]
and decomposes as before \( \Psi = \Psi_1 + \Psi_2 \), where
  \[
  -L \Psi_1 = \frac{1}{2} \sum_j \sigma_j |\xi_j|^2 = |\xi_{j_1}^{(1)} - \xi_{j_2}^{(1)}|^2 - |\xi_{j_3} - \xi_{j_4}^{(1)}|^2 + O(N^{-1+\delta})
  \]
  \[
  -L \Psi_2 = \sum_j \sigma_j \langle A(x_j^{(1)}), \xi_j \rangle + \sum_j \sigma_j \langle V(x_j^{(1)}), \xi_j \rangle + \langle x - x_j^{(1)}, \partial_x (V(x_j^{(1)})) (t, x_j^{(1)}) \rangle
  \]
  \[
  = O(N^{-1+\delta}).
  \]
  Note that the error terms are larger than before (when we had \( O(N^{-2+2\delta}) \)) but still acceptable.

References

[BV07] P. Bégout and A. Vargas, Mass concentration phenomena for the \( L^2 \)-critical nonlinear Schrödinger equation, Trans. Amer. Math. Soc. 359 (2007), no. 11, 5257–5282. MR 2327030 (2008g:35190)

[Car11] R. Carles, Nonlinear schrödinger equation with time-dependent potential, Commun. Math Sci. 9 (2011), no. 4, 937–964.

[CK07] R. Carles and S. Keraani, On the role of quadratic oscillations in nonlinear Schrödinger equations. II. The \( L^2 \)-critical case, Trans. Amer. Math. Soc. 359 (2007), no. 1, 33–62 (electronic). MR 2247881 (2008a:35260)

[Dod12] B. Dodson, Global well-posedness and scattering for the defocusing, \( L^2 \)-critical nonlinear Schrödinger equation when \( d \geq 3 \), J. Amer. Math. Soc. 25 (2012), no. 2, 429–463. MR 2869023

[Dod15] ———, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state, Adv. Math. 285 (2015), 1589–1618. MR 3406535

[Dod16a] ———, Global well-posedness and scattering for the defocusing, \( L^2 \) critical, nonlinear Schrödinger equation when \( d = 1 \), Amer. J. Math. 138 (2016), no. 2, 531–569. MR 3483476

[Dod16b] ———, Global well-posedness and scattering for the defocusing, \( L^2 \)-critical, nonlinear Schrödinger equation when \( d = 2 \), Duke Math. J. 165 (2016), no. 18, 3435–3516. MR 3577369
