String Cohomology of a Toroidal Singularity

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Abstract

We construct explicitly regular sequences in the semigroup ring $R = \mathbb{C}[K]$ of lattice points of the graded cone $K$. We conjecture that the quotients of $R$ by these sequences describe locally string-theoretic cohomology of a toroidal singularity associated to $K$. As a byproduct, we give an elementary proof of the result of Hochster that semigroup rings of rational polyhedral cones are Cohen-Macaulay.

1 Introduction

String cohomology vector space of a variety $X$ with Gorenstein toroidal singularities is a rather mysterious object. It is supposed to be a chiral ring of no less mysterious $N = (2, 2)$ superconformal field theory constructed from $X$ and it has known graded dimension. However, the space itself has not been identified so far in such generality. The goal of this paper is to present a candidate for the "contribution of a singular point" to this cohomology space.

The paper is organized as follows. Section 2 contains important preliminary results on the structure of lattice points of the graded cone $K$. Section 3 uses these results to show that some explicitly written sequences of elements of $R = \mathbb{C}[K]$ are regular in $R$ and in $R$-module $R^{\text{open}} = \mathbb{C}[K^{\text{open}}]$. It also contains the proof of an analog of Poincaré duality. It is worth mentioning that we give a short elementary proof of the theorem of Hochster [9]. Finally, the last section describes the relation of these results to Mirror Symmetry and string cohomology.

The author was inspired by recent preprints of Hosono [10] and Stienstra [12] who clarified the relationship between the solutions of GKZ hypergeometric system and Mirror Symmetry. The construction of this paper belongs to the A-side of Mirror Symmetry, any B-side construction should involve solutions of GKZ systems.

One of the basic ideas of the argument has the flavor of the theory of Gröbner bases, which the author learned from [3]. It also appears that it involves the large complex structure limit, see for example [11].

Author would like to thank Dave Bayer and Sorin Popescu for helpful remarks.
2 Decomposition of Cone Lattice Points

Let $N$ be a free abelian group of rank $r$. Let $K$ be a rational polyhedral cone inside $N \otimes \mathbb{R}$. We will assume that $K - K = N$ and $K \cap (-K) = \{0\}$. We will also assume that the cone $K$ is graded, that is there exists an element $\deg \in M = \text{Hom}(N, \mathbb{Z})$ such that the integer generators of all one-dimensional faces of the cone $K$ have degree 1. We will denote the interior of $K$ by $K^{\text{open}}$.

Another piece of data is a subset $\{e_i\}, i = 1, \ldots, d$ of the set of lattice points of degree 1 that lie in $K$. The only condition on the subset is that it includes the generators of all one-dimensional faces of $K$, that is

$$\sum_{i} R_{\geq 0} e_i = K.$$

We also choose a maximum regular triangulation $T$ based on these points $e_i$ and denote by $\psi$ a strictly convex function on $K$ which is linear on the simplices of triangulation $T$.

Our first goal is to construct a decomposition of the sets $K \cap N$ and $K^{\text{open}} \cap N$ into the disjoint union of sets $S_k$ of the form

$$S_k = b_k + \sum_{i \in I_k} \mathbb{Z}_{\geq 0} e_i$$

where $I_k$ is a simplex of triangulation $T$ of maximum dimension $r$ and $b_k$ is a lattice point inside $\sum_{i \in I_k} R_{\geq 0} e_i$.

To carry out the construction for a given cone $K$ we fix a generic vector $\xi \in N \otimes \mathbb{R}$ that lies in $K^{\text{open}}$. For every $I \in T$ of maximum dimension, we consider the coordinates of $\xi$ in $I$, that is we look at $\beta_{I,i}$, such that

$$\xi = \sum_{i \in I} \beta_{I,i} e_i.$$  

Because of the genericity of $\xi$, all $\beta$-s are non-zero. We introduce the sets $B_{I,\xi}$ and $B_{I,-\xi}$ as follows

$$B_{I,\xi} = \{ b \in I \cap N, \text{ such that } b = \sum_{i \in I} \gamma_i e_i \text{ with } 0 < \gamma_i \leq 1 \text{ if } \beta_{I,i} < 0$$

and $0 \leq \gamma_i < 1 \text{ if } \beta_{I,i} > 0 \}.$

$$B_{I,-\xi} = \{ b \in I \cap N, \text{ such that } b = \sum_{i \in I} \gamma_i e_i \text{ with } 0 < \gamma_i \leq 1 \text{ if } \beta_{I,i} > 0$$

and $0 \leq \gamma_i < 1 \text{ if } \beta_{I,i} < 0 \}.$

Proposition 2.1 In the above notations the following statements hold.

(a) The set $K \cap N$ is the disjoint union of sets $b + \sum_{i \in I} \mathbb{Z}_{\geq 0} e_i$ taken over all $I \in T$ of maximum dimension and all $b \in B_{I,\xi}$.

(b) The set $K^{\text{open}} \cap N$ is the disjoint union of sets $b + \sum_{i \in I} \mathbb{Z}_{\geq 0} e_i$ taken over all $I \in T$ of maximum dimension and all $b \in B_{I,-\xi}$. 

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Proof. (a) If \( n \in K \cap N \), consider \( n + \epsilon \xi \) for very small \( \epsilon > 0 \). It lies in \( \sum_{i \in I} R_{\geq 0} e_i \) for some maximum simplex \( I \in T \). Therefore, we have

\[ n + \epsilon \xi = \sum_{i \in I} \alpha_i(\epsilon) e_i \]

and

\[ n = \sum_{i \in I} (\alpha_i(\epsilon) - \epsilon \beta_{I,i}) e_i \]

where \( \alpha_i(\epsilon) > 0 \). Notice that \( (\alpha_i(\epsilon) - \epsilon \beta_{I,i}) \) are independent of \( \epsilon \). Therefore, they are always nonnegative. Moreover, they are positive for such \( i \) that \( \beta_{I,i} < 0 \). This easily implies that \( n \in b + \sum_{i \in I} Z_{\geq 0} e_i \) for some \( b \in B_{I,\xi} \). Conversely, if \( n \in b + \sum_{i \in I} Z_{\geq 0} e_i \) with \( b \in B_{I,\xi} \), then for small \( \epsilon > 0 \), the vector \( n + \epsilon \xi \) lies in \( \sum_{i \in I} R_{\geq 0} e_i \), which determines \( I \) uniquely. Besides, there are clearly no intersections between \( b_1 + \sum_{i \in I} Z_{\geq 0} e_i \) and \( b_2 + \sum_{i \in I} Z_{\geq 0} e_i \) for different \( b_1 \) and \( b_2 \) from \( B_{I,\xi} \). The proof of (a) could now be finished by observation that if \( n + \epsilon \xi \) lies in some \( \sum_{i \in I} R_{\geq 0} e_i \) for small \( \epsilon \), then \( n \) lies in \( K \).

(b) The proof is completely analogous. We use the fact that \( n - \epsilon \xi \) lies in one of \( \sum_{i \in I} R_{> 0} e_i \) if and only if \( n \) lies in \( K^{\text{open}} \). \( \square \)

**Corollary 2.2** Let us introduce polynomials

\[ S(t) = (1 - t)^r \sum_{n \in K \cap N} t^\deg(n) \quad \text{and} \quad T(t) = (1 - t)^r \sum_{n \in K^{\text{open}} \cap N} t^\deg(n). \]

Then

\[ S(t) = \sum_{I,b \in B_{I,\xi}} t^{\deg(b)}, \quad T(t) = \sum_{I,b \in B_{I,-\xi}} t^{\deg(b)}. \]

**Proof.** Follows immediately from the above proposition. \( \square \)

Notice that the standard duality formula

\[ S(t) = t^r T(t^{-1}) \]

follows immediately from this corollary together with the definitions of \( B_{I,\xi} \) and \( B_{I,-\xi} \).

In the next section we will use the following result. Let us fix a lattice element \( n \in K \). We look for all possible ways of representing \( n \) in the form

\[ n = b + \sum_{i=1}^d k_i e_i \]

where \( k_i \) are non-negative integers and \( b \in \cup B_{I,\xi} \). The decomposition of \( K \cap N \) above gives us one such representation

\[ n = b_0 + \sum_{i \in I_0} l_i e_i. \]

We claim that it has special properties with respect to the convex function \( \psi \).
Proposition 2.3 If \( n = b + \sum_{i=1}^{d} k_i e_i \) then for small \( \epsilon > 0 \)

\[
\psi(n + \epsilon \xi) \geq \psi(b + \epsilon \xi) + \sum_{i=1}^{d} k_i \psi(e_i)
\]

and equality holds if and only if \( b = b_0, \ k_i = l_i \) for \( i \in I_0, \ k_i = 0 \) for \( i \notin I_0 \).

Proof. The inequality is the basic property of the convex function \( \psi \). Equality holds if and only if there exists a maximum simplex \( I \) such that the cone \( \sum_{i \in I} R_{\geq 0} e_i \) contains \( b + \epsilon \xi \) and all \( e_i \) for which \( k_i \) are non-zero. Therefore, \( n + \epsilon \xi = \sum_{i \in I} R_{\geq 0} e_i \) and the proof of Proposition 2.1(a) shows that \( I = I_0 \). Because of \( b + \epsilon \xi = \sum_{i \in I_0} R_{\geq 0} e_i \), the lattice element \( b \) lies in \( b_1 + \sum_{i \in I_0} Z_{\geq 0} e_i \) for some \( b_1 \in B_{I_0, \xi} \). Because the union of such sets is disjoint, we have \( b = b_1 \). So \( b \in B_{I, \xi} \) and therefore \( b \) must equal \( b_0 \).

We will also need a similar statement for \( K^{open} \).

Proposition 2.4 Consider a lattice element \( n \) in \( K^{open} \). If \( n = b + \sum_{i=1}^{d} k_i e_i \) for some \( b \in \cup B_{I,-\xi} \) then for small \( \epsilon > 0 \)

\[
\psi(n - \epsilon \xi) \geq \psi(b - \epsilon \xi) + \sum_{i=1}^{d} k_i \psi(e_i)
\]

and equality holds if and only if \( b = b_0, \ k_i = l_i \) for \( i \in I_0, \ k_i = 0 \) for \( i \notin I_0 \), where \( n = b_0 + \sum_{i \in I_0} l_i e_i \) is given by Proposition 2.1(b).

Proof. The proof of this proposition is completely analogous to the proof of the previous one. \( \square \)

3 Regular Sequences

Let us fix a basis \( m_1, \ldots, m_r \) of the vector space \( M \otimes \mathbb{C} \) where \( M = \text{Hom}(N, \mathbb{Z}) \). We introduce the semigroup ring \( R = \mathbb{C}[K] \) and for every \( n \in K \) we denote the corresponding element in \( R \) by \( x^n \). We also introduce \( r \) elements of \( R \) by the formula

\[
Z_{j} = \sum_{i=1}^{d} e_{j_i} < m_{j_i}, e_i > e^{2\pi i a_i x_i}.
\]

Here \( a_i \) are some numbers assigned to the lattice elements \( e_i \) and the elements in \( R \) that correspond to \( e_i \) are denoted by \( x_i \). These \( Z_j \)-s act on \( R \) itself, and also on \( R \)-module \( R^{open} = \mathbb{C}[K^{open}] \), which is an ideal in \( R \).

The goal of this section is to show that for a generic choice of \( a_i \) the sequence \( Z_1, Z_2, \ldots, Z_r \) is regular on both \( R \) and \( R^{open} \). The following proposition is crucial.

Proposition 3.1 Denote by \( Z \) the ideal generated by \( Z_1, \ldots, Z_r \). Then the following statements hold for generic \( a_i \).

(a) Images of \( x^b \) for \( b \in \cup B_{I,\xi} \) generate \( R/ZR \) as \( \mathbb{C} \)-vector space.

(b) Images of \( x^b \) for \( b \in \cup B_{I,-\xi} \) generate \( R^{open}/ZR^{open} \) as \( \mathbb{C} \)-vector space.
Proof. (a) We introduce the ring \( R_1 = \mathbb{C}[x_1, \ldots, x_d] \) and consider \( R \) and \( R/ZR \) as \( R_1 \)-modules. Proposition \ref{prop:module_properties} implies that these \( R_1 \)-modules are generated by \( x^b, \ b \in \cup B_{I, \xi} \). Therefore, for each \( q \) we have a surjective map

\[
\oplus_{b \in \cup B_{I, \xi}} R_1[x^b] \rightarrow R/ZR \rightarrow 0
\]

of \( R_1 \)-modules.

The kernel of map (1) contains generators of two types.

- **Binomial relations.** Whenever we have an identity in the lattice \( N \)

\[
n = b_1 + \sum_{i=1}^{d} k_{i1} e_i = b_2 + \sum_{i=1}^{d} k_{i2} e_i
\]

we have a generator of the form

\[
\prod_{i=1}^{d} x_i^{k_{i1}} [x^{b_1}] - \prod_{i=1}^{d} x_i^{k_{i2}} [x^{b_2}].
\]

- **Linear relations.** We have generators \( Z_j r_1 [x^b] \) for \( j = 1, \ldots, d, \ b \in \cup B_{I, \xi}, \ r_1 \in R_1 \).

It is enough to show that \( \oplus \mathbb{C}[x^b] \) maps surjectively on the part of \( R/ZR \) of degree less than some fixed big number \( D \). Really, it is enough to show that any element of form \( x_i[x^b] \) can be re-expressed as \( \sum_b \alpha_b [x^b] \) modulo above relations, and degrees of \( x^b \) are less than \( r \).

Let us pick a parameter \( q \) and choose

\[
e^{2\pi i a_i} = q^{\psi(e_i)}.
\]

We will also make the following change of variables for each non-zero \( q \). We introduce

\[
(x_i)_{\text{new}} = q^{\psi(e_i)} x_i, \ [x^b]_{\text{new}} = q^{\psi(b+\xi)} [x^b]
\]

where \( \epsilon \) is chosen to be small enough to fit in Proposition \ref{prop:change_of_variables} for all \( n \) of degree less than \( D \). Then we rewrite the generators of the kernel of map (1) in terms of new variables.

- **Binomial relations.** Whenever we have an identity in the lattice \( N \)

\[
n = b_1 + \sum_{i=1}^{d} k_{i1} e_i = b_2 + \sum_{i=1}^{d} k_{i2} e_i
\]

we have a generator of the form

\[
q^{\psi(b+\xi)+\sum_i k_{i1} \psi(e_i)-\psi(b_2+\xi)-\sum_i k_{i2} \psi(e_i)} \prod_{i=1}^{d} (x_i)_{\text{new}}^{k_{i1}} [x^{b_1}]_{\text{new}} - \prod_{i=1}^{d} (x_i)_{\text{new}}^{k_{i2}} [x^{b_2}]_{\text{new}}.
\]

- **Linear relations.** We have generators

\[
Z_j r_1 [x^b]_{\text{new}} = \sum_{i=1}^{d} < m_j, e_i > (x_i)_{\text{new}} r_1 [x^b]_{\text{new}}.
\]
Among the binomial relations, we will pick only the ones where \( n = b_1 + \sum_{i=1}^d k_i \epsilon_i \) is given by the decomposition of Proposition 2.1. Then, by Proposition 2.3, the power of \( q \) is positive, unless \( b_2 = b_1, \ k_2 = k_1 \).

Pick a basis of \( \oplus_b (R_1[x^b]_{\text{new}})_{\deg < D} \) that consists of the products of monomials in \( R_1 \) and \( [x^b]_{\text{new}} \). For every \( q \) we can introduce a matrix \( A(q) \) which describes the map to \( \oplus_b (R_1[x^b]_{\text{new}})_{\deg < D} \) from the direct sum

\[
\oplus_b C[x^b]_{\text{new}} \oplus_{\text{binomial}} C[\text{binomial}] \oplus_{j,b,r_i} CZ_j r_i [x^b]_{\text{new}}
\]

where the direct sum is over the binomial relations that we have just picked and \( r_i \) are chosen to be monomials in \( x_{\text{new}} \) of degree less than \( D \).

To show that the vector space \( \oplus_b C[x^b]_{\text{new}} \) surjects onto \( (R/Z R)_{\deg < D} \), it is enough to demonstrate that the matrix \( A(q) \) has full rank. Notice, that we have picked relations in such a way that \( A(q) \) has a limit \( A(0) \) as \( q \to 0 \). Therefore, it will be enough to show that \( A(0) \) has full rank.

The binomial relations become monomial in the limit \( q \to 0 \) and hence the image of \( A(0) \) contains all basis elements of \( \oplus_b (R_1[x^b]_{\text{new}})_{\deg < D} \) except, perhaps, the elements of the form \( \prod_{i \in I} (x_i)_{\text{new}}^b [x^b]_{\text{new}} \) for \( b \in B_{I,\xi} \). However, if we use the linear relations, we can express \( (x_i)_{\text{new}}, i \in I \) in terms of other \( (x_i)_{\text{new}} \), which shows that all the basis elements except for \( [x^b]_{\text{new}} \) themselves are in the image of \( A(0) \). And since \( [x^b]_{\text{new}} \) are also included in the image of \( A(0) \) by construction, we have the desired surjectivity of \( A(0) \), which finishes the proof of (a).

The proof of (b) is completely analogous. \( \square \)

From now on we assume that \( a_i \) are generic. It is easy now to prove that \( Z_1, \ldots, Z_r \) form a regular sequence on \( R \) and \( R_{\text{open}} \). We thus reprove for graded cones the result of Hochster [9] which states that \( R \) is Cohen-Macaulay.

**Proposition 3.2** The sequence \( Z_1, \ldots, Z_r \) is regular on \( R \) and \( R_{\text{open}} \). Thus \( R_{\text{open}} \) is a Cohen-Macaulay module over the Cohen-Macaulay ring \( R \).

**Proof.** Let us show that \( Z_1, \ldots, Z_r \) is regular on \( R \). For every two power series \( f(t) \) and \( g(t) \) we say that \( f(t) > g(t) \) if the first non-zero coefficient of \( f(t) - g(t) \) is positive.

For each \( k = 0, \ldots, r \) we denote

\[
f_k(t) = \sum_{l \geq 0} t^l \text{dim}_C(R/(Z_1, \ldots, Z_k)R)_{\deg = l}.
\]

The exact sequence

\[
R/(Z_1, \ldots, Z_k)R \to R/(Z_1, \ldots, Z_k)R \to R/(Z_1, \ldots, Z_{k+1})R \to 0
\]

implies that

\[
f_{k+1}(t) \geq f_k(t).
\]

On the other hand, the fact that \( \oplus_{b \in B_{I,\xi}} C[x^b] \) surjects onto \( R/ZR \) implies that the power series (in fact, it is a polynomial) \( f_r(t) \) is less or equal to \( \sum_{I, b \in B_{I,\xi}} t^{\deg(b)} \) which is equal to \( (1 - t)^r f_0(t) \) by Corollary 2.2. Therefore, all intermediate inequalities are equalities, which shows that the above sequences are exact on the left.

The same argument works for \( R_{\text{open}} \). \( \square \)
Remark 3.3 Theorem of Hochster could be proved in full generality using our methods. Really, for any cone $K$ we can pick points $e_i$ on one-dimensional faces that lie in the same hyperplane $\deg = 1$ for some $\deg \in M \otimes \mathbb{Q}$. Then the only difference is that $\deg(n)$ is allowed to take values in $\frac{1}{l} \mathbb{Z}$ for some $l$, which also requires the use of fractional powers of $t$. However, this does not present any problems, because the integrality of $\deg(n)$ was never used.

Corollary 3.4 Surjective maps of Proposition 3.1 are isomorphisms.

Proof. It follows from the proof of Proposition 3.2 that graded dimensions of these spaces are the same, so surjectivity implies bijectivity. \[\square\]

Remark 3.5 Regularity of the sequence $Z$ was used in a special case without proof in the paper [7]. In the later correction note [8] the result is stated explicitly, but the proof is inadequate.

Because of the duality $S(t) = t^r T(t^{-1})$, we have $\dim_{\mathbb{C}}(R_{\text{open}}/Z R_{\text{open}})_{\deg = r} = 1$. We denote by $\varphi$ a surjective map $R_{\text{open}}/Z R_{\text{open}} \to \mathbb{C}$ which sends $(R_{\text{open}}/Z R_{\text{open}})_{\deg < r}$ to zero. Then we have a pairing

$$(R/Z R) \otimes \mathbb{C} (R_{\text{open}}/Z R_{\text{open}}) \to \mathbb{C}$$

which maps $x \otimes y$ to $\varphi(xy)$.

Proposition 3.6 (Poincaré Duality) The pairing

$$(R/Z R) \otimes \mathbb{C} (R_{\text{open}}/Z R_{\text{open}}) \to \mathbb{C}$$

is non-degenerate.

Proof. We need to show that for every element $x \in R_{\text{open}}/Z R_{\text{open}}$ the principal $R$-submodule it generates inside $R_{\text{open}}/Z R_{\text{open}}$ is non-zero at degree $r$. Let us pick a homogeneous $x$ whose principal submodule is zero in degree $r$, which has the highest degree (less than $r$) among all $x$ with this property. Denote by $R_{>0}$ the maximum ideal in $R$. For every homogeneous $y \in R_{>0}$ the principal submodule of $xy$ is zero in degree $r$, but $xy$ has a higher degree, so it must be zero. This implies that there is a non-trivial homomorphism from $\mathbb{C} = R/R_{>0}$ to $R_{\text{open}}/Z R_{\text{open}}$, which maps $1$ to $x$. Since the top element certainly provides us with a homomorphism $\mathbb{C} \to R_{\text{open}}/Z R_{\text{open}}$, it suffices to show that

$$\text{Hom}^R(\mathbb{C}, R_{\text{open}}/Z R_{\text{open}}) \cong \mathbb{C}.$$ 

Now we use the well-known result (see, for example, [5]) that $R_{\text{open}}$ is the canonical module for $R$. Hence

$$\text{Ext}^i_R(\mathbb{C}, R_{\text{open}}) \cong 0, \ i \neq r, \ \text{Ext}_r^R(\mathbb{C}, R_{\text{open}}) \cong \mathbb{C},$$

which is a standard property of canonical modules, see [4]. Now it can be easily deduced from the Koszul complex associated to $Z$ and $R_{\text{open}}$ that

$$\text{Hom}^R(\mathbb{C}, R_{\text{open}}/Z R_{\text{open}}) \cong \text{Ext}_r^R(\mathbb{C}, R_{\text{open}}) \cong \mathbb{C},$$

which completes the proof. \[\square\]
4 Relation to Mirror Symmetry and Other Comments

Now it is time to explain the title of the paper. String-theoretic cohomology of a variety $X$ with toroidal Gorenstein singularities is supposed to be the chiral ring of the corresponding $N = (2, 2)$ superconformal field theory. It is still not clear how to construct such a theory for varieties with most general singularities of this type. However, graded dimension of string-theoretic cohomology vector spaces was suggested by Batyrev and Dais in their paper [2]. It was later verified in [1] that this definition of string-theoretic Hodge numbers is compatible with mirror duality of Calabi-Yau hypersurfaces and complete intersections in Gorenstein toric Fano varieties.

**Definition 4.1** [2] Let $X = \cup_{i \in I} X_i$ be a stratified algebraic variety over $\mathbb{C}$ with at most Gorenstein toroidal singularities such that for any $i \in I$ the singularities of $X$ along the stratum $X_i$ of codimension $k_i$ are defined by a $k_i$-dimensional finite rational polyhedral cone $K_i$; that is $X$ is locally isomorphic to

$$\mathbb{C}^{\dim(X) - k_i} \times U_{K_i}$$

at each point $x \in X_i$ where $U_{K_i}$ is a $k_i$-dimensional affine toric variety which is associated with the cone $K_i$ (see [3]). Batyrev and Dais have introduced the polynomial

$$E_{st}(X; u, v) = \sum_{i \in I} E(X_i; u, v) \cdot S_{K_i}(uv)$$

where $E(X_i; u, v)$ are E-polynomials of Danilov and Khovanskii, see [6]. It is called the string-theoretic E-polynomial of $X$. If we write $E_{st}(X; u, v)$ in form

$$E_{st}(X; u, v) = \sum_{p, q} a_{p, q} u^p v^q,$$

then the numbers $h_{st}^{p, q}(X) = (-1)^{p+q} a_{p, q}$ are called the string-theoretic Hodge numbers of $X$.

Thus it seems that the "local" description of string cohomology should be provided by a vector space that has graded dimension $S(t)$. This is precisely what we have achieved in the previous sections. We now suggest that the vector space $R/ZR$ should be thought of as the local contribution of a toroidal singularity to string cohomology of the variety $X$. There also seems to be a nice notion of local string homology, which is provided by $R^{open}/ZR^{open}$. The existence of natural pairing that satisfies Poincaré duality provides a further justification of our terminology.

It is important to remark that we had to choose some numbers $a_i$ to facilitate the construction, and that the resulting spaces do depend on this choice. This is due to the fact that the superconformal field theory in question depends on both complex and Kähler parameters when $X$ is smooth. So it is natural to suggest that $a_i$ play the role of Kähler parameters here. This is best illustrated by the example
of hypersurfaces in mirror dual Gorenstein toric Fano varieties. To construct an $N = (2, 2)$ theory, we really need a pair of such hypersurfaces, and it roughly amounts to choosing coefficients $a_i$ for points on both dual reflexive polyhedra.

In general, if we choose consistently the numbers $a_i$ for integer points of degree one in cones $K_i$, we can define a (non-coherent) sheaf $\mathcal{A}$ of $\mathbb{C}$-algebras over $X$ as follows. Whenever the closure of one strata $X_i$ contains another strata $X_j$ there is a surjective map

$$\left(\mathbb{R}/\mathbb{Z}\mathbb{R}\right)_{K_j} \to \left(\mathbb{R}/\mathbb{Z}\mathbb{R}\right)_{K_i}.$$ 

We define the germ $\mathcal{A}_x$ over a point $x \in X_i$ to be $(\mathbb{R}/\mathbb{Z}\mathbb{R})_{K_i}$ and sections are, by definition, constant on each strata and compatible with the above maps. The sheaf $\mathcal{A}$ is naturally graded. One can hope to somehow use the cohomology of this sheaf to define string-theoretic cohomology vector spaces. The cohomology of $\mathcal{A}$ appears to be a viable candidate for string cohomology of $X$ when it is an orbifold. Unfortunately, in general the Hodge structure of $H^*(\mathcal{A})$ is not pure, which is certainly one of the properties to be expected of string cohomology.

As a side remark, the construction of this paper works not only for $R$ and $R^{\text{open}}$ but for some other ideals of $R$ that are associated with the choice of $\xi$ which is neither in $K$ nor in $-K$. It would be interesting to see whether these ideals have any additional nice properties.

The construction here has the flavor of the A-side of Mirror Symmetry. It would be very interesting to see the B-side construction, which is presumably more useful. It is very possible, that the spaces constructed here could be mapped to the spaces of solutions of GKZ hypergeometric systems and their duals. One may also try to define a flat connection on the vector bundle with fibers $\mathbb{R}/\mathbb{Z}\mathbb{R}$ over the space of parameters $a_i$. This is the direction of further research that the author plans to pursue.

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