Topology of the support
of the two-dimensional random walk

F. van Wijland, S. Caser, and H.J. Hilhorst
Laboratoire de Physique Théorique et Hautes Energies
Bâtiment 211
Université de Paris-Sud
91405 Orsay cedex, France

Abstract
We study the support (i.e. the set of visited sites) of a $t$ step random walk on a two-dimensional square lattice in the large $t$ limit. A broad class of global properties $M(t)$ of the support is considered, including, e.g., the number $S(t)$ of its sites; the length of its boundary; the number of islands of unvisited sites that it encloses; the number of such islands of given shape, size, and orientation; and the number of occurrences in space of specific local patterns of visited and unvisited sites. On a finite lattice we determine the scaling functions that describe the averages $\overline{M}(t)$ on appropriate lattice size dependent time scales. On an infinite lattice we first observe that the $M(t)$ all increase with $t$ as $\sim t/\log k t$, where $k$ is an $M$ dependent positive integer. We then consider the class of random processes constituted by the fluctuations around average $\Delta M(t)$. We show that to leading order as $t$ gets large these fluctuations are all proportional to a single universal random process $\eta(t)$, normalized to $\overline{\eta^2}(t) = 1$. For $t \to \infty$ the probability law of $\eta(t)$ tends to that of Varadhan’s renormalized local time of self-intersections. An implication is that in the long time limit all $\Delta M(t)$ are proportional to $\Delta S(t)$.  

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1 Introduction and summary

The reason for the multiple connections between the random walk and many questions of current and of permanent interest in science is evidently mathematical: The random walk models the action of the Laplace operator, which is common to all those problems. Indeed, important early results on the random walk are due to mathematicians. Among them, the famous Polya theorem asserts that in spatial dimensions \( d \leq 2 \) a random walk is certain to return to its initial position, whereas in \( d > 2 \) it will escape to infinity. The random walk in \( d = 2 \) is therefore at its critical dimension for return to the origin. Many features associated with this fact make the random walk in two dimensions particularly interesting. For example, the probability distribution of the time interval \( \tau_0 \) between two successive visits of the walk to its initial position decays for large intervals as \( \sim 1/\tau_0 \log^2 \tau_0 \), so that very long excursions occur away from the point of departure. Excellent recent monographs on random walk theory have been written by Weiss and by Hughes.

In this work we consider the simple random walk on a two-dimensional square lattice: The walk starts at time \( t = 0 \) at the origin \( \mathbf{x} = \mathbf{0} \) and steps at \( t = 1, 2, 3, \ldots \) with equal probability to one of the four neighboring lattice sites. Our investigation focuses on the statistical properties of the support of the walk at time \( t \), i.e., of the set of sites that have been visited during the first \( t \) steps.

At any given time \( t \) the support can be visualized as a set of black sites (the visited ones) in a lattice of otherwise white (unvisited) sites, as shown in Fig. 1. The set of unvisited sites is divided into components that are disjoint (i.e., not connected via any nearest neighbor link) and that we call islands. In the course of time existing islands will be reduced in size and single-site islands will eventually be destroyed; the number of islands increases each time that a step of the walk cuts an existing island, or the outer region surrounding the support, into disjoint components.

Questions about the statistical properties of the islands are natural and our interest in them was raised by a simulation study by Coutinho, Coutinho-Filho, Gomes, and Nemirovsky. These authors investigated the evolution of the number of islands \( I(t) \) and several of their properties on finite lattices of up to \( 1200^2 \) sites. In a short report we showed that an analytic calculation is possible, both on finite and infinite lattices, for some (not all) of the quantities considered by Coutinho et al., and that good agreement between theory and computer simulation is obtained. In this work we present a full account of most of the results announced in Ref. and consider a great many related questions. We are, in particular, led to consider the infinite lattice again.

It appears that the number of islands \( I(t) \) is but one member of a much wider class of observables, generically to be denoted as \( M(t) \), with closely
related properties. This class includes the total number $S(t)$ of sites in the support as well as the total length $E(t)$ of the boundary of the support (on an infinite lattice the boundary length is the sum of the external perimeter and the perimeters of the islands enclosed).

The main variable that characterizes the support is its total number of sites $S(t)$, sometimes called its range, which was first studied by Dvoretzky and Erdős [6]. In the limit of large times $t$ it has the average value [6, 7, 8, 9]

$$S(t) \sim \frac{\pi t}{\log 8t}$$  \hspace{1cm} (1.1)

and the root mean square deviation [10, 11]

$$\Delta S^2(t)^{1/2} \sim A\frac{\pi\log\in\forall\log}{\log\in\forall\log}$$  \hspace{1cm} (1.2)

where $A = \infty\exists\in\forall\ldots$ and where we write $\Delta S \equiv S - S$. The typical support is known to be far from spherical, and its principal moments of inertia and asphericity have been studied [12, 13]. Expressions exist [14] also for its span, that is, the smallest rectangular box that it fits in.

The question of calculating the average $S(t)$ becomes, in general dimension and in an appropriately taken continuum limit, the celebrated Wiener sausage problem: What is the volume swept out in time $t$ by a $d$-dimensional Brownian sphere of finite radius? The Wiener sausage appears naturally in certain applications of the random walk such as, for example, the study by Kac and Luttinger [15] of Bose-Einstein condensation in the presence of impurities. Historically this continuum problem precedes its lattice counterpart. It was considered as early as 1933 by Leontovitsh and Kolmogorov [16] and is still today an active subject of investigation in probability theory ([17]; see [18] for a recent overview) and in mathematical physics [19].

The islands in the support of a lattice random walk have their counterpart in the connected components into which a two-dimensional Brownian motion path divides the plane. These have been studied by Mountford [17] and Le Gall [17], and, most recently, by Werner [21], who determines, among other things, how many there are larger than a given size $\epsilon$, in the limit of $\epsilon \to 0$. The outer boundary of the Brownian motion path has been considered very recently by Lawler [22].

We characterize as follows the class of observables $M(t)$ to be studied below. For each lattice site $x$ we introduce an occupation number $m(x, t)$, equal to 0 if site $x$ is visited by the walker before or at time $t$, and equal to 1 if it is not. Let now $A_1$ and $A_2$ be disjoint finite sets of lattice vectors. Then the product

$$\prod_{a_1 \in A_1} m(x + a_1, t) \prod_{a_2 \in A_2} [1 - m(x + a_2, t)]$$  \hspace{1cm} (1.3)
codes for a specific spatial pattern $\alpha \equiv (A_1, A_2)$ of white (unvisited) and black (visited) sites. When $x$ runs through the lattice, the product (1.3) equals 1 when the pattern $\alpha$ is encountered and 0 otherwise; hence this product summed on all $x$ represents the total number $N_\alpha(t)$ of occurrences in space of the pattern $\alpha$. The observables that this work deals with are these pattern numbers $N_\alpha(t)$ and their linear combinations $M(t)$. It is easy to see that suitably chosen $M(t)$ may represent, e.g., the total number $S(t)$ of sites in the support; the total boundary length $E(t)$ of the support; the total number $I(t)$ of islands of unvisited sites enclosed by the support; or the total number $I_\beta(t)$ of islands of a given type $\beta$ (where type stands for shape, size, and orientation). Our main results concerning the observables $M(t)$ are of two kinds.

(i) Average behavior on a finite lattice. On a finite lattice of $N$ sites the averages $\overline{M}(t)$ approach their limiting values on the time scale $t \sim N \log^2 N$, whereas their main variation occurs on the earlier time scale $t \sim N \log N$. We calculate the scaling functions that describe the time dependence on both time scales. Several of the observables $M(t)$ of interest just mentioned are treated as examples. A comparison is made, where possible, with the simulations by Coutinho et al. [4]

(ii) Average behavior and fluctuations on the infinite lattice. On an infinite lattice the fluctuating properties of the support manifest a universality that can be described as follows. Let $M$ and $M'$ be two linear combinations of pattern numbers. Then, with the same notation as before for deviations from average, we show by explicit calculation that asymptotically for $t \to \infty$

$$\Delta M(t) \Delta M'(t) \simeq 4 A^\xi \langle || + \infty \rangle (||' + \infty \rangle \langle || + \infty \rangle \langle ||' + \infty \rangle \frac{\pi^k \log k + 1}{8t}$$

Here $k$ and $k'$ are nonnegative integers that depend on $M$ and $M'$, respectively, and are called their order; $m_k$ and $m'_k$ are known proportionality constants; and the same number $A$ appears that was first encountered in the study of the variance of $S(t)$. The above equations imply that the normalized deviations from average

$$\eta_M(t) = \frac{\log 8t \ \Delta M(t)}{(k + 1) \overline{M}(t)}$$

have a correlation matrix $\eta_M(t) \eta_{M'}(t)$ whose elements all equal unity. This can be true only if all $\eta_M$ are equal to a single random variable to be called $\eta(t)$. As a consequence the random variables $\Delta M(t)$ are, to leading order as $t \to \infty$, all proportional to $\eta(t)$. Explicitly,

$$\Delta M(t) \simeq (k + 1) m_k \frac{\pi^k t}{\log k + 2} A \eta(\omega)$$

(vi) Average behavior on an infinite lattice. On an infinite lattice the observables $M(t)$ are tractable.
where $\eta(t)$ is universal (independent of $M$). Hence what seemed to be a large number of independent fluctuating degrees of freedom of the support is hereby reduced, to leading order as $t \to \infty$, to only a single degree of freedom.

We now briefly summarize the contents of the successive sections. Since all quantities of interest are, in the end, expressed in terms of the random walk Green function, we collect in Sec. 2 the basic knowledge required about this function. In Sec. 3 we express the total boundary length $E(t)$ and the total number of islands $I(t)$ in terms of the occupation numbers and discuss the wider class of observables $M(t)$ of which they are examples. In Secs. 4.1 and 4.2 we calculate the time dependent averages $\overline{M}(t)$ of such observables. In Secs. 4.3–4.6 the general expression for the result, Eq. (4.16), is analyzed for large times and explicitly worked out for several examples both on the infinite and the finite lattice. In Secs. 5.1 and 5.2 we show, for the infinite lattice, how to calculate the correlation between two observables $M(t)$ and $M'(t)$. In Sec. 5.3 we arrive at the final results concerning the random variable $\eta(t)$. In the discussion in Sec. 6 we present various comments, compare where possible our lattice results to their continuum analogs, and speculate about further connections between several quantities.

2 The random walk Green function

2.1 Definition

A random walker starts at time $t = 0$ at the origin $x = 0$ of a square lattice and steps at each instant of time $t = 1, 2, 3, \ldots$ with probability $\frac{1}{4}$ to one of its four nearest-neighbor sites. The Green function $G(x, t)$ denotes the probability that at time $t$ the walker is at lattice site $x$, and

$$
\hat{G}(x, z) = \sum_{t=0}^{\infty} z^t G(x, t) \quad (2.1)
$$

its generating function. In this section we collect in concise form those properties of the generating function that will be needed later. An elementary calculation gives, for a finite periodic lattice of $L \times L = N$ sites,

$$
\hat{G}(x, z) = \frac{1}{N} \sum_{\mathbf{q}} \frac{e^{-i\mathbf{q} \cdot x}}{1 - \frac{1}{2} z (\cos q_1 + \cos q_2)} \quad (2.2)
$$

where $\mathbf{q} = (q_1, q_2) = 2\pi(\kappa_1, \kappa_2)/L$ with the $\kappa_i$ running through the values $0, 1, 2, \ldots, L - 1$. In the limit of an infinite lattice expression (2.2) becomes

$$
\hat{G}(x, z) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dq_1 \int_{-\pi}^{\pi} dq_2 \frac{e^{-i\mathbf{q} \cdot x}}{1 - \frac{1}{2} z (\cos q_1 + \cos q_2)} \quad (2.3)
$$

At each point in this work it will be clear whether we are discussing the finite or the infinite lattice; in some cases we shall denote corresponding quantities
in the two geometries by the same symbol, as for example in Eqs. (2.2) and (2.3), and not explicitly indicate their $N$ dependence on a finite lattice.

### 2.2 Expansion near $z = 1$

The long time behavior of the physical quantities of interest is determined by the behavior of $\hat{G}(x, z)$ in the complex plane near $z = 1$. Expressions (2.2) and (2.3) both have the property that for $z \to 1$ the function $\hat{G}(x, z)$ diverges. In order to study this divergence it is convenient to write

$$\hat{G}(x, z) = \hat{G}(0, z) - g(x, z)$$

where on the RHS the term $g(x, z)$ contains all the $x$ dependence and remains finite for $z \to 1$. We shall discuss $\hat{G}(0, z)$ and $g(x, z)$ separately.

#### 2.2.1 The function $\hat{G}(0, z)$

**Finite lattice.** Expression (2.2) has a simple pole as a function of $z$ whenever one of the denominators inside the sum on $q$ vanishes. This leads to a sequence of poles on the real axis for $z \geq 1$, of which the first one is located exactly at $z = 1$. The interval between two successive poles is of $O(N^{-1})$ and contains a zero. Upon setting $x = 0$ in Eq. (2.2) and expanding each term for small $1 - z$ one gets

$$\hat{G}(0, z) = \frac{1}{N(1 - z)} + a(N) - a_1(N)(1 - z) + O((1 - z)^2)$$

in which the coefficients are functions of $N$ that in the limit $N \to \infty$ behave as

$$a(N) = \frac{1}{\pi} \log cN + O(N^{-1})$$

$$a_1(N) = \frac{c_1}{N} + O(\log N)$$

with $c = 1.8456...$ and $c_1 = 0.06187...$

The expansion in Eq. (2.3) represents well the behavior of $\hat{G}(0, z)$ near the pole at $z = 1$, but is certainly not valid on approach of the next pole. It can be used, however, to determine the location of the zero $z = z_0$ between the first two poles. Upon solving Eq. (2.5) in successive orders for $z_0$ one finds

$$z_0 = 1 + \frac{1}{N a(N)} \left[ 1 - \frac{c_1}{a^2(N)} + \cdots \right]$$

From Eqs. (2.7) and (2.6) we conclude that when $N \to \infty$ this zero is separated from the pole at $z = 1$ by a distance only of $O(1/N \log N)$. All other zeros of $\hat{G}$ are separated from $z = 1$ by a distance of at least $O(N^{-1})$. Due to its exceptional proximity to $z = 1$ the zero $z_0$ plays a special role in the
long-time behavior of the random walk on finite lattices. This fact was first noted by Weiss, Havlin, and Bunde \cite{24} and has also been exploited \cite{25} in the study of the covering time of a finite lattice by a random walk.

Infinite lattice. When \( N \to \infty \) the poles of \( \hat{G} \) densify to a branch cut and the expansion near \( z = 1 \) is \cite{26}

\[
\hat{G}(0, z) = \frac{1}{\pi} \log \frac{8}{1 - z} + \mathcal{O}\left((1 - z) \log(1 - z)\right)
\] (2.8)

Later on in this work we shall also for brevity denote \( \hat{G}(0, z) \) as \( G_0(z) \).

\subsection{2.2.2 The function \( g(x, z) \)}

Expanding \( g(x, z) \) for finite \( N \) around \( z = 1 \) gives

\[
g(x, z) = g_N(x) + g_N'(x)(1 - z) + \mathcal{O}((1 - z)^2)
\] (2.9)

where we now explicitly indicate that the expansion coefficients are \( N \)-dependent. We shall set

\[
g(x) = \lim_{N \to \infty} g_N(x).
\] (2.10)

Spitzer \cite{27} shows how to calculate the \( g(x) \) for \( x \) close to the origin. Letting \( e_1 \) and \( e_2 \) denote the unit vectors we have the following values:

\[
\begin{align*}
g(0) &= 0 & g(2e_1) &= 4 - 8/\pi \\
g(e_1) &= 1 & g(2e_1 + e_2) &= 8/\pi - 1 \\
g(e_1 + e_2) &= 4/\pi
\end{align*}
\] (2.11)

When combining preceding results we see that for finite \( N \) the quantity \( \hat{G}(x, z) \) has the expansion

\[
\hat{G}(x, z) \simeq \frac{1}{N(1 - z)} + a(N) - g_N(x) + \mathcal{O}(1 - z)
\] (2.12)

whose second term behaves for large \( N \) as

\[
a(N) - g_N(x) \simeq \frac{1}{\pi} \log cN - g(x) + \cdots
\] (2.13)

with the dots representing terms that vanish as \( N \to \infty \).

\subsection{2.3 Scaling limit}

In our study of the fluctuations in Sec.\[3\] we shall also need \( \hat{G}(x, z) \) in the scaling limit \( z \to 1 \), \( x \to \infty \) with \( x^2(1 - z) \) fixed. The behavior in this limit is \cite{3}

\[
\hat{G}(x, z) \simeq \frac{2}{\pi} K_0(2x(1 - z)^{1/2})
\] (2.14)

where \( K_0 \) is the modified Bessel function of order zero. As is well-known, for \( z \to 1 \) the dominant contribution to the sum (2.1) comes from values of \( t \) that are of \( \mathcal{O}((1 - z)^{-1}) \), and therefore the scaling limit corresponds to focusing on distances \( x \) of \( \mathcal{O}(\sqrt{t}) \).
3 Islands and other observables

3.1 Islands

The representation of Fig. 1 divides the lattice into a black area and white ones. We introduce for each lattice site $\mathbf{x}$ the occupation number

$$m(\mathbf{x}, t) = \begin{cases} 
1 & \text{if $\mathbf{x}$ has not yet been visited at time $t$} \\
0 & \text{otherwise}
\end{cases} \quad (3.1)$$

At this point it may seem more natural to work with the equivalent occupation numbers $n(\mathbf{x}, t) \equiv 1 - m(\mathbf{x}, t)$, but definition (3.1) will soon prove to be more convenient.

If starting at an arbitrary element, one follows the boundary such that the white sites are on the left and the black ones on the right, then one will return to the point of departure after having turned either through an angle $2\pi$ (if that part of the boundary encloses an island in the support) or through an angle $-2\pi$ (if it is the outer boundary of the support). The number of islands is therefore obtained from the number of turns in the boundary, by adding those of Fig. 2a with weight $1$ and those of Fig. 2b with weight $-1$. The diagrams of Fig. 2c correspond to two turns of $\pi/2$ and therefore count with weight factor $1/2$. This procedure counts the outer boundary with weight $-1$ and therefore $2$ has to be added to obtain the final result. It is now easy to construct the expression for the total number of islands in terms of the occupation variables $m(\mathbf{x}, t)$. We have to shift a $2 \times 2$ window across the lattice, check all $2 \times 2$ local site configurations and add all those that are of the types of Fig. 2 with the proper weights. Let $\mathbf{x}$ denote the lower lefthand site in the $2 \times 2$ window. Then, for example, the expression

$$m(\mathbf{x}, t) [1 - m(\mathbf{x} + \mathbf{e}_1, t)] [1 - m(\mathbf{x} + \mathbf{e}_2, t)] m(\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2, t) \quad (3.2)$$

(which is of type [1.3]) equals 1 or 0 according to whether the local $2 \times 2$ configuration is or is not equal to the diagram of Fig. 2c. Writing down analogous expressions for all other diagrams, summing these, and subsequently summing them on $\mathbf{x}$ yields $I(t)$ as a linear combination of pattern numbers. After rearranging terms one finds

$$I(t) = 2 + \sum_{\mathbf{x}} \left[ m(\mathbf{x}, t) - m(\mathbf{x}, t)m(\mathbf{x} + \mathbf{e}_1, t) - m(\mathbf{x}, t)m(\mathbf{x} + \mathbf{e}_2, t) \\
+ m(\mathbf{x}, t)m(\mathbf{x} + \mathbf{e}_1, t)m(\mathbf{x} + \mathbf{e}_2, t)m(\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2, t) \right] \quad (3.3)$$

This expression is at the basis of all calculations concerning islands on the infinite lattice. Finite lattice calculations require a further remark, which is made below Eq. (4.44).
3.2 Other observables

We are now interested in other variables whose values can be obtained by the "window" method. To see the general form of these variables, let $A$ be a finite subset of lattice vectors. The set $\{x + a \mid a \in A\}$, obtained by translating $A$ by a vector $x$, will be written as $x + A$. The variable

$$m_{x+A}(t) = \prod_{a \in A} m(x + a, t)$$

(3.4)

is equal to unity if at time $t$ all the sites of this set are white (unvisited), and is zero otherwise. The sum variable

$$M_A(t) = \sum_x m_{x+A}(t)$$

(3.5)

counts the total number of wholly white sets in the lattice that can be obtained from $A$ by a translation.

In the remainder we shall also wish to take for $A$ the empty set $\emptyset$. In that case the right hand side of Eq. (3.4) should be assigned the value unity and Eq. (3.5) shows that $M_{\emptyset}(t)$ is the total number of lattice sites.

The number of islands $I(t)$ was initially found in Sec. 3.1 as a linear combination of pattern numbers, which led to the final expression (3.3). The representation of an observable as a linear combination of pattern numbers is in general nonunique, but expression (3.3) is a unique member of the class of variables $M(t)$ that are of the form

$$M(t) = \sum_A \mu_A M_A(t)$$

(3.6)

with arbitrary coefficients $\mu_A$. Various other quantities of potential interest can be expressed this way. Some of these, with their coefficients $\mu_A$, have been listed in Table I. The best known example is the total number $S(t)$ of sites in the support,

$$S(t) = \sum_x \left[1 - m(x, t)\right]$$

(3.7)

which has $\mu_A = \pm 1$ for $A = \emptyset$ and $A = \{0\}$, respectively, and $\mu_A = 0$ otherwise. Another example is the total boundary length $E(t)$ between the visited and unvisited lattice sites (that is, the total number of pairs of neighboring sites of which one is white and one black). It can be expressed as

$$E(t) = \sum_x \left[m(x, t)(1 - m(x + e_1, t) + (1 - m(x, t))m(x + e_1, t)
+ m(x, t)(1 - m(x + e_2, t) + (1 - m(x, t))m(x + e_2, t))\right]$$

$$= \sum_x \left[4m(x, t) - 2m(x, t)m(x + e_1, t) - 2m(x, t)m(x + e_2, t)\right]$$

(3.8)
We note that one has the relation

\[ \sum_A \mu_A = 0 \tag{3.9} \]

for the three observables \( S, E, \) and \( I, \) but that

\[ \sum_{A \neq \emptyset} \mu_A = 0 \tag{3.10} \]

only for \( E \) and \( I, \) but not for \( S. \) Property (3.9) is required if the sum on \( x \) in Eq. (3.5) is to have a finite limit when the lattice size \( N \) tends to infinity. The property (3.10) is easily traced back to the fact that for \( E \) and \( I \) the window selects patterns consisting of both white and black sites, whereas for \( S \) it selects only black sites. When Eq. (3.9) holds, Eq. (3.10) is equivalent to \( \mu_\emptyset = 0. \)

### 4 Averages of observables

#### 4.1 Relation to first passage time probabilities

We shall now calculate in a unified way averages and fluctuations of quantities \( M(t) \) of type (3.6). The approach of this subsection will serve as the basis for the developments to follow. Using Eqs. (3.4)–(3.6) we see that the averages \( M(t) \) are linear combinations of expressions of the type

\[
m_{x+A}(t) = \prod_{a \in A} m(x+a,t) = 1 - \sum_{\tau=0}^t f_{x+A}(\tau) = 1 - \sum_{\tau=0}^t \sum_{a \in A} f_{x+A}(x+a,\tau) \tag{4.1}
\]

in which the last two transformations make sense only when \( A \neq \emptyset; \) \( f_{x+A}(\tau) \) is the probability that the walker’s first visit to any site of the set \( x + A \) takes place at time \( \tau; \) and \( f_{x+A}(x+a,\tau) \) is the probability that it takes place at time \( \tau \) and that it concerns the specific site \( x + a. \) Upon averaging Eq. (3.5), using Eq. (4.1), and passing to generating functions we find, for all nonempty \( A, \)

\[
\hat{M}_A(z) = -\frac{1}{1-z} \sum_x \left[ \sum_{a \in A} f_{x+A}(x+a,z) - 1 \right] \tag{4.2}
\]

For the empty set one derives directly that

\[
\hat{M}_\emptyset(z) = N/(1-z) \tag{4.3}
\]
We now sum the $\hat{M}_A(z)$ given by Eqs. (4.2) and (4.3) on all $A$ with coefficients $\mu_A$. Using Eq. (3.9) and introducing for all $A \neq \emptyset$

$$\hat{F}_A(a, z) = \sum_x \hat{f}_{x+A}(x + a, z) \quad (4.4)$$

yields

$$\hat{M}(z) = -\frac{1}{1 - z} \sum_{A \neq \emptyset} \mu_A \sum_{a \in A} \hat{F}_A(a, z) \quad (4.5)$$

With this formula we have reduced the generating function of the average of interest, $\overline{M}(t)$, to the quantities $\hat{F}_A$ which are closely related to first passage times, but still unknown. We shall now proceed to determine $\hat{F}_A$.

### 4.2 Solving the first passage time probabilities

Standard random walk theory [27, 2] relates the first passage probabilities $\hat{f}$ to the Green function $\hat{G}$ by

$$\hat{G}(x + a, z) = \sum_{a' \in A} \hat{f}_{x+A}(x + a')\hat{G}(a - a', z) \quad (4.6)$$

for all $a \in A$. With the aid of Eq. (4.4) we find for $\hat{F}_A$ the equation

$$\sum_{a' \in A} \hat{F}_A(a', z)\hat{G}(a - a', z) = \frac{1}{1 - z} \quad (4.7)$$

for all $a \in A$. This is a matrix equation for the $\hat{F}_A$ whose dimension is the number $|A|$ of sites in the set $A$. This equation possesses special properties which are best exhibited by converting it to the shorthand notation

$$\gamma_{aa'}(z) = g(a - a', z)/G_0(z) \quad \mathcal{F}_a = (1 - z)G_0(z)\hat{F}_A(a, z) \quad (4.8)$$

Eq. (4.7) then becomes

$$\sum_{a' \in A} (1 - \gamma_{aa'}(z))\mathcal{F}_{a'} = 1 \quad (4.9)$$

for all $a \in A$. If we denote by $\gamma^{(A)}$ the matrix of elements $\gamma_{a,a'}$ with $a$, $a' \in A$, then in matrix notation

$$(J - \gamma^{(A)}(z))\mathcal{F} = \mathbf{j} \quad (4.10)$$

where $J$ and $\mathbf{j}$ are the matrix and vector, respectively, of dimension $|A|$, whose elements all equal 1. As shown in Eq. (4.5), we only need the sum of the components of $\mathcal{F}$. Formal inversion gives

$$\sum_{a \in A} \mathcal{F}_a = \sum_{a \in A} \sum_{a' \in A} [(J - \gamma^{(A)}(z))^{-1}]_{aa'} \quad (4.11)$$
In a later stage we shall wish to take the limit $z \to 1$. In view of Eq. (4.8) and the known behavior of $G_0(z)$ this implies that $\gamma^{(A)}(z) \to 0$, so that, except when $|A| = 1$, the matrix inverse $(J - \gamma^{(A)}(z))^{-1}$ in Eq. (4.11) ceases to exist. We therefore now convert that equation to a form more suitable for taking that limit. In the appendix it is shown that

$$\sum_{a \in A} F_a = \frac{1}{1 - \gamma_A} \quad (4.12)$$

where

$$\gamma_A^{-1}(z) \equiv \sum_{a \in A} \sum_{a' \in A} [(\gamma^{(A)}(z))^{-1}]_{aa'} \quad (4.13)$$

Upon coming back to the original notation, but now with the abbreviation

$$g_A(z) \equiv \gamma_A(z)G_0(z) \quad (4.14)$$

we find from Eq. (4.12) the solution of Eq. (4.7) in the form

$$\sum_{a \in A} \hat{F}_A(a, z) = \frac{1}{(1 - z)(G_0(z) - g_A(z))} \quad (4.15)$$

When $|A| = 1$, in which case we may take $A = \{0\}$, one deduces directly from (4.7) that (4.15) holds with $g_{\{0\}}(z) = 0$. For $A$ of diameter not too large, as is the case in many examples of interest, the quantity $g_A(z)$ is easily expressed explicitly in terms of the $g(a - a', z)$. If after substitution of expression (4.15) in Eq. (4.5) we sum on $A$, use Eq. (3.9), and transform back to the time domain, we get

$$\mathcal{M}(t) = -\frac{1}{2\pi i} \oint \frac{dz}{z^{t+1}} \frac{1}{(1 - z)^2} \sum_{A \neq \emptyset} \mu_A G_0(z) - g_A(z) \quad (4.16)$$

where the integral runs counterclockwise around the origin. This result is still fully exact and applies to both finite and infinite lattices.

### 4.3 Long-time behavior of averages. Infinite lattice

The starting point for the analysis of this section is Eq. (4.16). On an infinite lattice the asymptotic behavior of $\mathcal{M}(t)$ as $t \to \infty$ is determined by the $z \to 1$ behavior of the integrand. This behavior follows from expression (2.8) for $G_0(z)$ and from Eqs. (4.14) and (4.13) which together determine $g_A(z)$. We shall satisfy ourselves with retaining the leading $z \to 1$ behavior and corrections that are of relative order of negative powers of $\log(1 - z)$. The terms neglected are of relative order $1 - z$, apart from logarithmic factors. This means that in Eq. (4.16) we may replace $G_0(z)$ with $\pi^{-1} \log(8/(1 - z))$ and the $g_A(z)$ with their values at $z = 1$, which for brevity we shall denote by $g_A$. The coefficient $g_A$ appears in potential theory and is the two-dimensional lattice analog of the electrostatic capacity of the set $A$; its properties have been reviewed by Spitzer [27].
We expand the summand in Eq. (4.16) in inverse powers of \( \log(8/(1-z)) \). Using the above results we so obtain, asymptotically for \( t \to \infty \),

\[
\bar{M}(t) \simeq \sum_{n=0}^{\infty} m_n \mathcal{J}_{n+1,2}(t)
\]  

(4.17)

where the coefficients \( m_n \) are determined by the observable \( M \) according to

\[
m_n = - \sum_{A \neq \emptyset} \mu_A g_A^n
\]  

(4.18)

for \( n = 0, 1, 2, \ldots \) and where

\[
\mathcal{J}_{n\ell}(t) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+\ell+1}} \frac{1}{(1-z)^\ell} \log^{n+1} \frac{8}{1-z}
\]  

(4.19)

The \( t \to \infty \) behavior of the integrals (4.19) has been studied, in particular, by Henyey and Seshadri [28] for the case \( \ell = 2 \). A generalization of their result is

\[
\mathcal{J}_{n\ell}(t) \simeq \frac{\pi^{n+\ell-1}}{\log^n 8t} \sum_{m=0}^{\infty} (-1)^m \binom{m+n-1}{m} \frac{b_{m\ell}}{\log^m 8t}
\]  

(4.20)

with coefficients

\[
b_{m\ell} = \frac{d^m}{dx^m} \Gamma(x+\ell) \bigg|_{x=0}
\]  

(4.21)

We have \( b_{0\ell} = 1/(\ell-1)! \) and shall also need explicitly below

\[
\begin{align*}
b_{12} &= -1 + C & = -0.422... \\
b_{13} &= -\frac{2}{3} + \frac{1}{2} C & = -0.461... \\
b_{22} &= 2 - \frac{1}{6} \pi^2 - 2C + C^2 & = -0.466... \\
b_{23} &= \frac{7}{4} - \frac{1}{12} \pi^2 - \frac{3}{2} C + \frac{1}{2} C^2 & = 0.227...
\end{align*}
\]  

(4.22)

where \( C = 0.577215... \) denotes Euler’s constant. Combining Eqs. (4.17) and (4.20) we find for \( \bar{M}(t) \) an asymptotic expansion in inverse powers of \( \log t \),

\[
\bar{M}(t) \simeq \frac{\pi t}{\log 8t} m_0 \\
+ \frac{\pi t}{\log^2 8t} [-b_{12} m_0 + \pi m_1] \\
+ \frac{\pi t}{\log^3 8t} [b_{22} m_0 - 2\pi b_{12} m_1 + \pi^2 m_2] \\
+ \frac{\pi t}{\log^4 8t} [-b_{23} m_0 + 3\pi b_{22} m_1 - 3\pi^2 b_{12} m_2 + \pi^3 m_3] \\
+ \ldots
\]  

(4.23)

where we have set \( b_{02} = 1 \). We recall that whereas the \( b_{m\ell} \) are numerical coefficients, the \( m_n \) defined by Eq. (4.18) are specific for the observable \( M \).
Eq. (4.23) shows that the leading asymptotic behavior is \( \sim t / \log 8t \) for observables that have \( m_0 \neq 0 \), and \( \sim t / \log^2 8t \) for those that have \( m_0 = 0 \) but \( m_1 \neq 0 \). In view of the definition of \( m_0 \) and the discussion at the end of Sec. 3.2, observables with \( m_0 \neq 0 \) involve only black patterns and will be called, for short, "black" observables, whereas those with \( m_0 = 0 \) will be called "black-and-white" observables.

### 4.4 Examples

It is now easy to derive results for many examples of interest by applying the formulas of Sec. 4.3.

**Example 1.1.** We take for \( M \) the total number \( S \) of sites in the support. The first column of coefficients \( \mu_A \) in Table I shows that the only nonzero term in the sum (4.18) is due to the set \( A = \{0\} \). Since this set has \( \mu_A = -1 \) and \( g_A = 0 \), the only nonzero coefficient produced by Eq. (4.18) is \( m_0 = 1 \).

From Eq. (4.23) we then have
\[
S(t) \simeq \frac{\pi t}{\log 8t} \left[ 1 - \frac{b_{12}}{\log 8t} + \frac{b_{22}}{\log^2 8t} - \frac{b_{32}}{\log^3 8t} + \cdots \right] \quad (4.24)
\]

The numerical values of the coefficients of the first two subleading terms are given in Eq. (4.22) and agree with those of Torney [11].

**Example 1.2.** Next we take for \( M \) the total boundary length \( E \) between the white and black areas. In the second column of coefficients \( \mu_A \) in Table I shows that only two sets \( A \) enter, with the pair \( (\mu_A, g_A) \) equal to \((4, 0)\) and \((-4, \frac{1}{2})\). Eq. (4.18) then leads to \( m_0 = 0 \) and \( m_n = 2^{2^{-n}} \) for \( n = 1, 2, \ldots \), after which Eq. (4.23) gives
\[
E(t) \simeq \frac{2\pi^2 t}{\log^2 8t} \left[ 1 + \frac{\varepsilon_1}{\log 8t} + \frac{\varepsilon_2}{\log^2 8t} + \cdots \right] \quad (4.25)
\]

with the coefficients
\[
\varepsilon_1 = \frac{1}{3} \pi - 2b_{12} = 2.41636...
\]
\[
\varepsilon_2 = \frac{1}{3} \pi^2 - \frac{3}{2} \pi b_{12} + 3b_{22} = 3.06116...
\]

**Example 1.3.** Now take for \( M \) the total number \( I \) of islands. Using the third column of coefficients \( \mu_A \) in Table I as input in Eq. (4.18) we find that \( m_0 = 0 \) and
\[
m_n = 2^{1^{-n}} - [(\pi + 2)/2\pi]^n \quad (4.27)
\]
for \( n = 1, 2, \ldots \). Substituting as in the previous examples we find
\[
I(t) \simeq \frac{1}{2} \pi (\pi - 2) \frac{t}{\log^2 8t} \left[ 1 + \frac{\varepsilon_1}{\log 8t} + \frac{\varepsilon_2}{\log^2 8t} + \cdots \right] \quad (4.28)
\]
Analytic expressions for the coefficients $\iota_i$ are easily found with the aid of the preceding formulas but are of little interest. The numerical values of the first two of them are

\[
\begin{align*}
\iota_1 &= -2.087... \\
\iota_2 &= -21.304...
\end{align*}
\]

In the leading asymptotic behavior of both $\overline{E}(t)$ and $\overline{I}(t)$ an extra factor $1/\log 8t$ appears compared to that of $\overline{S}(t)$ as a consequence of $m_0$ being zero, i.e., of $E(t)$ and $I(t)$ being black-and-white observables. In Sec. 4.5 we shall come back to $\overline{T}(t)$ and also make a comparison with the numerical simulations by Coutinho et al. [4].

Example 1.4. Let $\beta$ be a specific type of island, where type indicates shape, size and orientation. Let $I_\beta(t)$ be the observable that counts the total number of islands of that type. According to the preceding discussion we must have that

\[
\overline{I}_\beta(t) \simeq \frac{1}{2} \pi (\pi - 2) f_\beta \frac{t}{\log^2 8t} \quad \text{as} \quad t \to \infty
\]  

for some proportionality constant $f_\beta$, even though we cannot expect the approach to this asymptotic behavior to be uniform in $\beta$. By summing Eq. (4.30) on all $\beta$ and comparing to Eq. (4.28) we conclude that the average number of islands of type $\beta$ represents, as $t \to \infty$, a fixed fraction $f_\beta$ of the average total number of islands. We have calculated the fraction $f_1$ of islands that are single isolated sites and the fraction $f_2$ of "dimer" islands, consisting of two neighboring sites, with the result

\[
\begin{align*}
\frac{f_1}{f_2} &= \frac{\pi (\pi^3 - 7\pi^2 + 14\pi - 4)}{(\pi - 1)(\pi^2 - 6\pi + 4)} = 0.560079... \\
\frac{f_2}{f_1} &= 0.073557...
\end{align*}
\]

The main effort goes into the calculation of the necessary coefficients $g_A$. Those needed for $I_1$ are listed in Table I. We do not present the 22 coefficients needed for $I_2$, nor the final analytic expression for $f_2$. Since the dimers may have two orientations, the islands of sizes 1 and 2 represent a fraction $f_1 + 2f_2 = 0.707193...$ of all islands.

With these results in hand we return to the total boundary length considered in Example 1.2. On an infinite lattice one can write $E(t) = E_{\text{ext}}(t) + E_{\text{int}}(t)$, where $E_{\text{ext}}(t)$ is the external perimeter of the support and $E_{\text{int}}(t)$ the total perimeter of the islands enclosed by it. In Example 1.2. we determined only the average of their sum; determining $E_{\text{ext}}(t)$ and $E_{\text{int}}(t)$ separately is a much more difficult problem that we have not seen how to solve by the present method. A rigorous lower bound for $E_{\text{int}}(t)$ is nevertheless easily obtained by adding up the perimeters of the single-site and dimer islands, and taking into account that all other islands have a perimeter at
least equal to 8. This gives $E_{\text{int}}(t) > 0.4965\ldots E(t)$. By arguments different from ours Lawler \[22\] shows that in fact for $t \to \infty$ the external perimeter increases only as $E_{\text{ext}}(t) \sim t^{2/3}$, so that $E_{\text{int}}(t) \simeq E(t)$.

**Example 1.5.** Let finally $M(t)$ be equal to the pattern number $N_\alpha(t)$ obtained by summing the expression (1.3) over all sites $x$. If neither $A_1$ nor $A_2$ is empty, then the pattern $\alpha$ is composed of both white and black sites and has $m_0 = 0$. Hence the average total number of these patterns is obtained by setting $m_0 = 0$ in Eq. (4.23) and increases with $t$ as

$$\overline{N}_\alpha(t) \simeq m_1 \frac{\pi^2 t}{\log^2 8t} \quad (4.32)$$

where $m_1$ is $\alpha$ dependent. The black-and-white patterns considered here necessarily lie on the boundary of the support; comparison of Eqs. (4.32) and (4.25) shows that the *number per unit of boundary length* of such patterns tends to a fixed value as $t \to \infty$.

### 4.5 Long-time behavior of averages. Finite lattice

For a finite lattice of $N$ sites, after an initial increase with time identical to what happens on the infinite lattice, we must expect deviations from the infinite lattice behavior to appear on a characteristic $N$ dependent time scale $\tau(N)$ that will tend to infinity when $N$ does. For larger times all black observables will level off and tend to a constant times $N$, and all black-and-white observables will pass through a maximum value, then bend down and asymptotically approach zero. The expansion that we shall look for in the case of a finite lattice will therefore involve a determination of this time scale.

We shall not attempt in this case a full asymptotic expansion as for the infinite lattice, but only determine the leading asymptotic behavior. Our starting point is again Eq. (4.16), in which, when $z \to 1$, using Eqs. (2.5) and (2.6), we may substitute

$$G_0(z) - g_A(z) = \frac{1}{N(1 - z)} + \frac{1}{\pi} \log cN - g_A + \cdots \quad (4.33)$$

where as before $g_A$ stands for $g_A(1)$ and the dots denote higher order terms. As for the infinite lattice, our procedure will be to bring the summation on $A$ outside the integration on $z$ and shift the integration path around the poles of the integrand. Therefore we should now discuss these poles. All required knowledge about the behavior of the various quantities involved has been collected in Sec. 2.2. First, it follows from Eq. (4.33) and the definition (4.18) of $m_0$ that for $z \to 1$

$$\sum_{A \neq \emptyset} \mu_A \frac{1}{G_0(z) - g_A(z)} \simeq -Nm_0(1 - z) \quad (4.34)$$

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Hence the integrand of Eq. (4.16) has a simple pole at $z = 1$ with residue $Nm_0$. Secondly, as is clear from the discussion in Sec. 2.2, this integrand has special simple poles for $z = z_A$, where

$$z_A = 1 + \frac{\pi}{N \log cN} \left[ 1 + \frac{\pi g_A}{\log cN} + \mathcal{O}\left(\frac{1}{\log^2 N}\right) \right].$$

(4.35)

These are the only poles at a distance of $\mathcal{O}(1/N \log N)$ from $z = 1$, all the other ones being at least at distances of $\mathcal{O}(1/N)$. Therefore, on time scales that are at least of $\mathcal{O}(N \log N)$, the other poles will contribute vanishingly to the result.

Carrying the integral out but retaining only the contribution of the poles at $z = 1$ and $z = z_A$ leads to

$$\overline{M}(t) \simeq Nm_0 + N \sum_{A \neq \emptyset} \mu_A \exp \left[ -\frac{\pi t}{N \log cN} \left( 1 + \frac{\pi g_A}{\log cN} + \mathcal{O}\left(\frac{1}{\log^2 N}\right) \right) \right].$$

(4.36)

The first term in this equation is not present for the black-and-white observables, which have $m_0 = 0$. We shall discuss these observables first. Two different time scales are of interest.

1. **The main regime**, in which an observable takes values of the same order as its maximum value. To focus on this regime we scale time as

$$\tau = \frac{\pi t}{N \log cN}$$

(4.37)

and take the limit $N \to \infty$, $t \to \infty$ at $\tau > 0$ fixed. In this "$\tau$-limit" Eq. (4.36) leads directly to

$$\overline{M}(t) \simeq \frac{\pi N}{\log cN} m_1 \tau e^{-\tau}$$

(4.38)

with $m_1$ defined by Eq. (4.18).

2. **The long-time regime**, in which the observable approaches its final value. It is now appropriate to scale time as

$$\sigma = \frac{\pi t}{N \log^2 cN}$$

(4.39)

with $\sigma > 0$. In this "$\sigma$-limit" Eq. (4.36) gives

$$\overline{M}(t) \simeq N(cN)^{-\sigma} \sum_{A \neq \emptyset} \mu_A e^{-\pi g_A \sigma}$$

(4.40)

One may notice that the result (4.38) is recovered from Eq. (4.40) if one sets $\sigma = \tau / \log cN$ and expands the terms inside the sum on $A$ to first order in $1/\log cN$. 

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For black observables Eq. (4.36) with the scaling of Eq. (4.37) leads directly to

\[ \mathcal{M}(t) \simeq N m_0 (1 - e^{-\tau}) \]  

(4.41)

This decay law depends on the type of the black pattern only through the prefactor \( m_0 \).

### 4.6 Examples

We consider the same examples as in Sec. 4.4 but now on the finite lattice.

**Example 2.1.** Let \( M = S \). Substituting \( m_0 = 1 \) in Eq. (4.36) we see that on a finite lattice of \( N \) sites the average number of unvisited sites decays as

\[ N - \mathcal{S}(t) \simeq N e^{-\tau} \]  

(4.42)

a result first obtained by Weiss et al. [24]. It is also valid in the long-time regime, where it can be written as \( N (cN)^{-\sigma} \). In this regime, since \( \sigma > 0 \), the unvisited sites constitute an infinitesimally small fraction of all lattice sites.

**Example 2.2.** Let now \( M = E \). Using in Eq. (4.36) the appropriate coefficients \( \mu_A \) from Table I we obtain

\[ \mathcal{E}(t) \simeq \frac{2\pi N}{\log cN} \tau e^{-\tau} \]

\[ \mathcal{E}(t) \simeq 4N (cN)^{-\sigma} (1 - e^{-\pi\sigma/2}) \]  

(4.43)

in the main and long-time regimes, respectively. The prefactor \( N (cN)^{-\sigma} \) in the second one of these equations is the average number of unvisited sites found in the preceding example. If these sites were randomly distributed, then since they are infinitely dilute, each of them would have four visited neighbors and the result for \( \mathcal{E}(t) \) would be only \( 4N (cN)^{-\sigma} \). Hence the factor \( 1 - e^{-\pi\sigma/2} \) represents nontrivial correlations due to unvisited sites clustering together.

**Example 2.3.** Let \( M = I \). Using in Eq. (4.36) the coefficients \( \mu_A \) from Table I appropriate to this case, we find

\[ \mathcal{T}(t) \simeq \frac{(\pi/2 - 1)N}{\log cN} e^{-\tau} \]

\[ \mathcal{T}(t) \simeq N (cN)^{-\sigma} (1 - 2e^{-\pi\sigma/2} + e^{-(\pi+2)\sigma/2}) \]  

(4.44)

for the main regime and the long-time regime, respectively. Again, the factor in parentheses in the last equation is due to correlations in the positions of the unvisited sites.

This example requires the following remark. The expression of Eq. (3.3) for the number of islands \( I(t) \) is correct only for the infinite lattice. For
the finite lattice with periodic boundary conditions in both directions one should add $-1$ when the support closes onto itself around the torus in one of the directions and $-2$ when it does so in both directions. In the latter case the first term in Eq. (3.3) is absent and one obtains the expression that we used to derive Eqs. (4.28) and (4.44). The extra term $-2$ is of course of no importance in the main regime, but it needs to be taken into account in the long-time regime to ensure that $\mathcal{T}(t)$ vanishes when $t$ tends to infinity.

If we neglect the difference between averages of ratios and ratios of averages, then we have from Eqs. (4.42), (4.43), and (4.44) that the quantities

$$\frac{[N - S(t)]}{\mathcal{T}(t)} \simeq 1/(1 - 2e^{-\pi \sigma/2} + e^{-(\pi+2)\sigma/2})$$

$$\frac{\mathcal{E}(t)}{\mathcal{T}(t)} \simeq 4(1 - e^{-\pi \sigma/2})/(1 - 2e^{-\pi \sigma/2} + e^{-(\pi+2)\sigma/2})$$

(4.45)

represent the average area and the average perimeter, respectively, of an island. In the main time regime the expression for the average area simplifies to

$$\frac{[N - S(t)]}{\mathcal{T}(t)} \simeq \frac{2 \log e^{N \pi}}{\pi - 2} t^{-1}$$

(4.46)

It is now possible to make a comparison with the simulations by Coutinho et al. [4], carried out on finite square lattices of up to $N = 1200^2$ sites with periodic boundary conditions. These authors were interested in the "fragmentation" of the finite lattice into islands and the way the average number and size of the islands eventually tend to zero. The comparison leads to the following conclusions.

(i) In an early time regime, which for a lattice of $600^2$ sites corresponds to $t$ less than $\approx 0.5 \times 10^6$, the finite lattice size still plays no role and our Eq. (4.28) for $\mathcal{T}(t)$ agrees within error bars with the simulation data shown in Ref. [4] for $N = 600^2$. These data are for $t > 0.05 \times 10^6$; for the agreement to be of this quality, not only the leading order term but also the two correction terms in Eq. (4.28) have to be taken into account.

(ii) In the early time regime and in the main regime, where $\mathcal{T}(t)$ passes through its maximum, our long time expansion Eq. (4.44) overestimates the numerical values of $\mathcal{T}(t)$ by up to 50%; in the long time regime, when $\mathcal{T}(t)$ starts to decay, the agreement with the simulation data becomes rapidly better and stays good all the way up to $t \approx 19 \times 10^6$, where with a large probability no unvisited sites are left.

(iii) Coutinho et al. in their simulation find the time dependent average island size on a lattice of $N$ sites to be a function only of the scaling variable $\sigma$. Our expression (4.45) confirms this result. The scaling function is not, however, the power law that it was thought to be in Ref. [4], but the inverse of a sum of exponentials given in Eq. (4.45). According to our Eq. (4.46) a power law appears only in the main regime and has the exponent $-1$. In the long time regime Eq. (4.45) leads to an apparent exponent with larger absolute value.
In the preceding discussion we have compared the numerical data for the average island size \( \frac{N - S(t)}{I(t)} \) to the theoretical result (4.45) for \( \frac{N - S(t)}{T(t)} \) that we were able to calculate. We have not been able to calculate directly the average island size. Nor have we been able to calculate still another quantity determined in the simulation of Ref. [4], viz. the time dependent "diversity" of the island sizes, defined as the number of different sizes that occur at any given time.

5 Fluctuations and correlations

5.1 Relation to a first passage time problem

The original determination by Dvoretzky and Erdős [1] of the average number \( S(t) \) of lattice sites in the support of a random walk was followed only much later [10, 11] by a calculation of the root-mean-square deviation of this quantity from its average. Yet that calculation was important, because the result, exhibited in our Eq. (1.2), shows that in the limit \( t \to \infty \) the probability distribution of \( S(t) \) becomes infinitely narrow, even though only logarithmically slowly with \( t \).

It is now natural to ask if the more general observables whose averages we studied in Sec. 4 also have infinitely narrow distributions for \( t \to \infty \). Without much extra effort it will be possible to calculate also the cross-correlations. We therefore consider two observables of the form (3.6), namely

\[
M(t) = \sum_A \mu_A M_A(t), \quad M'(t) = \sum_B \mu'_B M_B(t)
\]  

which have \( \sum_A \mu_A = \sum_B \mu'_B = 0 \), and focus on

\[
\Delta M(t) \Delta M'(t) \equiv M(t) M'(t) - \overline{M(t)} \overline{M'(t)}
\]

In this section we confine our analysis to the infinite lattice.

The first step will be to find an expression for the generating function \( \hat{C}_{MM'} \) defined by

\[
\hat{C}_{MM'}(z) = \sum_{t=0}^{\infty} z^t \overline{M(t) M'(t)}
\]

When working out the RHS of Eq. (5.3) with the aid of Eqs. (5.1) and (3.5) we encounter averages of products \( m_{x+A}(t) m_{y+B}(t) \) where \( x \) and \( y \) are arbitrary lattice vectors. It is then convenient to write \( y = x + r \) and to define \( U(r) \) as the union of \( A \) and \( r + B \).

\( A \) and \( B \) are finite sets, small in most applications, and we may locate them near the origin without loss of generality. For \( r \) not too small, therefore, \( U(r) \) will be the union of disjoint sets \( A \) and \( r + B \), and its number of elements will be \( |U| = |A| + |B| \).
With this notation we get

\[ M(t)M'(t) = \sum_{A,B} \mu_A \mu'_B \sum_r M_{U(r)}(t) \]  

(5.4)

The observable \( M_{U(r)} \) that occurs here is exactly of the form (3.5), with \( A \) replaced by \( U(r) \).

The generating function \( \hat{C}_{MM'}(z) \) is now easily expressed in terms of the first passage probabilities defined in Sec. 3.2. The derivation is analogous to the one of Eq. (4.5) and the result is

\[ \hat{C}_{MM'}(z) = -\frac{1}{1-z} \sum_{A,B} \mu_A \mu'_B \sum_r \hat{F}_{U(r)}(u, z) \]

\[ = -\frac{1}{1-z} \sum_{A,B \neq \emptyset} \mu_A \mu'_B \sum_r \left[ \sum_{u \in U(r)} \hat{F}_{U(r)}(u, z) - \sum_{a \in A} \hat{F}_{A}(a, z) - \sum_{b \in B} \hat{F}_{B}(b, z) \right] \]  

(5.5)

in which the asterisk indicates that the term with \( A = B = \emptyset \) is excluded from the summation. The function \( \hat{F}_{U(r)}(u, z) \) in Eq. (5.5) should be determined from the linear system of equations

\[ \sum_{u' \in U(r)} \hat{F}_{U(r)}(u', z) \hat{G}(u - u', z) = \frac{1}{1-z} \]  

(5.6)

valid for all \( u \in U(r) \). Formally this equation is strictly analogous to Eq. (4.7), but the structure of the matrix involved is different. It depends on the parameter \( r \), which represents the distance between the two components of the set \( U(r) \).

5.2 Scaling limit and long time behavior

We shall be able to solve \( \hat{F}_{U(r)} \) from Eq. (5.6) only in the scaling limit \( z \to 1, r \to \infty \), with \( \xi^2 \equiv 4r^2(1-z) \) fixed. But since \( z \to 1 \) is exactly the limit of interest, this solution suffices provided the sum on \( r \) in Eq. (5.4) is dominated by values \( r \sim (1-z)^{-1/2} \). In the scaling limit we need to consider only the case where \( A \) and \( r + B \) are disjoint. Using the shorthand notation of Eq. (4.8) we find from Eq. (5.6)

\[ \sum_{a' \in A} (1 - \gamma_{aa'}(z)) F_{a'} + \sum_{b' \in B} \gamma_{a',r+b'}(z) F_{r+b'} = 1 \]

\[ \sum_{a' \in A} \gamma_{r+b,a'}(z) F_{a'} + \sum_{b' \in B} (1 - \gamma_{bb'}(z)) F_{r+b'} = 1 \]  

(5.7)

valid for all \( a \in A \) and \( b \in B \), with \( F \) now defined on the set \( U(r) \). In the scaling limit Eqs. (2.14) and (4.8) imply the behavior

\[ \gamma_{a,r+b}(z) \simeq 2\pi^{-1} K_0(2r\sqrt{1-z})/G_0(z) + \mathcal{O}(1-z) \]  

(5.8)
whose important feature is that to leading order no dependence on \(a\) or \(b\) appears. Furthermore
\[
\gamma_{aa'}(z) \simeq \frac{g(a - a')}{G_0(z)}, \quad \gamma_{bb'}(z) \simeq \frac{g(b - b')}{G_0(z)}
\]
and we shall henceforth suppress the argument of the \(\gamma\)'s.

In the scaling limit Eqs. (5.7) can then be written, in matrix notation, as
\[
\begin{pmatrix}
J[\alpha, \alpha] - \gamma(A) \\
\lambda J[\alpha, \beta] \\
J[\beta, \alpha] \\
\gamma(B)
\end{pmatrix}
\begin{pmatrix}
F
\end{pmatrix}
= j
\]
(5.10)

where \(\alpha = |A|, \beta = |B|, \gamma(A)\) and \(\gamma(B)\) are matrices as defined in Sec. 4.2, \(J[\ell, m]\) is the \(\ell \times m\) matrix with all elements equal to 1, \(j\) is the vector with all elements equal to 1, and
\[
\lambda(\xi, z) = 2\pi^{-1}K_0(\xi)/G_0(z)
\]
(5.11)

We can obtain from Eq. (5.10) a formal expression, analogous to Eq. (4.11), for the sum of the components of \(F\). In the appendix it is shown that this expression can be transformed to
\[
\sum_{u \in U(r)} F_u = \frac{2 - 2\lambda - \gamma_A - \gamma_B}{1 - \lambda^2 - \gamma_A - \gamma_B + \gamma_A\gamma_B}
\]
(5.12)

Upon substituting Eq. (5.12) in Eq. (5.5) and replacing the sum on \(r\) that occurs there by an integral on \(\xi\) we find
\[
\hat{C}_{MM'}(z) \simeq -\frac{\pi}{2(1 - z)^3G_0(z)} \sum_{A, B \neq \emptyset} \mu_A\mu'_B \int_{0}^{\infty} d\xi \xi I(\xi, z)
\]
(5.13)
in which the function \(I(\xi, z)\) is given by
\[
I = \frac{2 - 2\lambda - \gamma_A - \gamma_B}{1 - \lambda^2 - \gamma_A - \gamma_B + \gamma_A\gamma_B} - \frac{1}{1 - \gamma_A} - \frac{1}{1 - \gamma_B}
\]
(5.14)

It is useful to observe that the three quantities \(\gamma_A, \gamma_B, \text{ and } \lambda(\xi, z)\) are all of order \(G_0^{-1}(z)\).

The steps that follow are again analogous to the procedure of Sec. 4.3. We wish to expand \(I(\xi, z)\) in inverse powers of \(G_0(z)\) and substitute the result in Eq. (5.13). The \(\mu'_B\) that characterize the observable \(M'\) define coefficients \(m'_n\) analogous to the \(m_n\) of Eq. (4.18). In the expansion we encounter furthermore the coefficients
\[
a_n = \int_{0}^{\infty} d\xi \xi K_0^n(\xi)
\]
(5.15)
of which we shall need the explicit values \(a_1 = 1\) and \(a_2 = \frac{1}{2}\), as well as
\[
a_3 = -\frac{1}{2} \int_{0}^{1} d\xi \frac{\log\xi}{1 - \xi + \xi^2} = 0.58597...
\]
(5.16)
Finally, before working out this expansion it is useful to classify the observables $M$ according to their order. We shall say that $M$ is of order $k$ if

$$m_0 = m_1 = \cdots = m_{k-1} = 0 \quad \text{and} \quad m_k \neq 0$$

(5.17)

Observables of order $k = 0$ and $k = 1$ have occurred in the preceding sections, and physically interesting examples with $k \geq 2$ perhaps exist. Let now $k$ and $k'$ be the orders of $M$ and $M'$, respectively. We anticipate – as will be confirmed by the calculation – that we have to expand $\hat{C}_{MM'}(z)$ in Eq. (5.13) to order $1/G_0^{k+k'+4}(z)$. In view of Eqs. (5.17) and (4.18) only those terms in the expansion will survive the summation on $A$ and $B$ that contain at least a factor $\gamma_A^k$ and a factor $\gamma_B^{k'}$. Since there is one factor $1/G_0(z)$ outside the sum in Eq. (5.13), this leaves room for at most three factors $\lambda$. We have therefore found it convenient to begin by expanding $I = I'\lambda + I''\lambda^2 + I'''\lambda^3 + \cdots$ and then to determine the first three coefficients of this series in terms of $\gamma_A$ and $\gamma_B$. Then the $1/G_0(z)$ expansion of $\hat{C}_{MM'}(z)$ leads to

$$\hat{C}_{MM'}(z) \simeq \frac{1}{(1-z)^3G_0^{k+k'+2}(z)} \left[ 2a_1 m_km_{k'}' ight.$$  
$$-G_0^{-1}(z)\left(\pi^{-1}(k+k'+2)a_2m_km_{k'}' - 2a_1(m_{k}m_{k+1} + m_{k+1}m_{k'}') \right)$$
$$+ G_0^{-2}(z) \left( 8\pi^{-2}(k+1)(k'+1)a_3m_km_{k'}' \right.$$  
$$- 2\pi^{-1}(k+k'+3)a_2(m_{k}m_{k+1} + m_{k+1}m_{k'}')$$
$$+ 2a_1(m_{k}m_{k+1} + m_{k+1}m_{k+2} + m_{k+2}m_{k'})$$
$$+ \cdots \right]$$

(5.18)

The next step is to invert Eq. (5.13). After integrating on $z$ with the aid of Eq. (4.12), one finds

$$M(t)M'(t) \simeq \frac{\pi^{k+k'+2}4^2}{\log^{k+k'+2}8t} \left[ a_1m_km_{k'}' ight.$$  
$$+ \log^{8}t \left[ (k+k'+2)(2a_1b_{13} + a_2)m_{k}m_{k'}' + \pi a_1(m_{k}m_{k+1} + m_{k+1}m_{k'}') \right]$$
$$+ \log^{2}t \left[ \{(k+k'+2)(k+k'+3)(a_1b_{23} + 2a_2b_{13}) \right.$$  
$$+ 2(k+1)(k'+1)a_3, m_{k}m_{k'}' \right.$$  
$$- (k+k'+3)\pi(a_2 + 2a_1b_{13})(m_{k}m_{k+1} + m_{k+1}m_{k'}')$$
$$+ \pi^2a_1(m_{k}m_{k+1} + m_{k+1}m_{k+2} + m_{k+2}m_{k'}) \right.$$  
$$+ \cdots \right]$$

(5.19)
From Eq. (4.23) we deduce that when $M$ is of order $k$ its average is given by
\[
\overline{M}(t) \simeq \frac{\pi^{k+1} t}{\log^{k+1} 8t} \left[ m_k + \frac{1}{\log 8t} \left( - (k + 1)b_{12}m_k + \pi m_{k+1} \right) \right. \\
+ \left. \frac{1}{\log 8t} \left( \frac{1}{2}(k + 1)(k + 2)b_{22}m_k - \pi(k + 2)b_{12}m_{k+1} + \pi^2 m_{k+2} \right) \right.
+ \cdots \right] 
\]  
(5.20)

Eq. (5.20), its counterpart for $M'(t)$, and Eq. (5.19) now have to be combined in Eq. (5.2). Using the explicit expressions for the coefficients $a_1, a_2, b_{12}, b_{13}, b_{22},$ and $b_{23}$ leads to the desired correlation function. For $t \to \infty$ the two leading orders cancel in the subtraction in Eq. (5.2). The final result is
\[
\frac{1}{(k+1)(k'+1)} \frac{\Delta M(t) \Delta M'(t)}{\overline{M}(t)\overline{M'}(t)} \simeq \frac{A^c}{\log \forall \land} + \cdots 
\]  
(5.21)

where the dots indicate terms of higher order in $1/\log 8t$ and
\[
A^c = \Delta \sin - \frac{\pi^2}{\log} = \infty \forall \forall \infty. 
\]  
(5.22)

Eq. (5.21) contains as a special case the well-known result of Eq. (1.2) for the variance of $S(t)$, originally due to Jain and Pruitt [10], and rederived with the aid of a method more similar to ours by Torney [11].

### 5.3 Conclusions

It is remarkable that the ratio on the RHS of Eq. (5.21) is universal: It is independent of the choice of the observables $M$ and $M'$. But we shall now see that Eq. (5.21) has consequences that reach far beyond this simple fact.

For $k = 0, 1, 2, \ldots$ we define for each observable $M$ of order $k$ its normalized deviation from average, $\eta_M$, by
\[
\eta_M(t) = \log 8t \frac{\Delta M(t)}{(k+1)A \overline{M}(t)} 
\]  
(5.23)

These variables satisfy to leading order
\[
\overline{\eta_M(t)} = 0, \quad \overline{\eta_M(t)\eta_{M'}(t)} = 1 \quad \text{for all} \quad M, M' 
\]  
(5.24)

It follows that for any two $M$ and $M'$ the difference $\eta_M(t) - \eta_{M'}(t)$ has zero variance, and therefore the $\eta_M(t)$ are all equal to a single random variable that we shall call $\eta(t)$. As a consequence we can relate the deviation from average $\Delta M(t)$ of any observable $M(t)$ of order $k$ to $\eta(t)$ by
\[
\Delta M(t) \simeq (k + 1) \frac{A}{\log \forall \land} \overline{M}(t) \eta(t) 
\]  
(5.25)

Upon writing down this equation for the special case $M = S$, using the explicit expression (4.24) for $S$, and eliminating $\eta(t)$, one finds
\[
\Delta M(t) \simeq (k + 1) \frac{\pi^k}{\log^k 8t} m_k \Delta S(t) 
\]  
(5.26)
This last equation embodies one of the main conclusions of this work: All the observables $M$ fluctuate around their averages in strict proportionality with the fluctuation of total number of sites $S(t)$ in the support. This conclusion applies, in particular, to the pattern numbers $N_{\alpha}(t)$, the total perimeter length $E(t)$ of the support, the total number $I(t)$ of islands enclosed by it, and the total number $I_{\beta}(t)$ of islands of a specific type $\beta$. We are not aware of any computer simulations that confirm Eq. (5.26), although they would be easy to carry out.

There is a still different and instructive way to formulate this conclusion. Let $M$ be an observable of order $k$ and $\rho$ a quantity that remains of $O(\infty)$ when $t \to \infty$. Eq. (5.20) can now be used to establish the asymptotic expansion in powers of $1/\log 8t$ of $\rho^{-1}M(\rho t)$. Upon comparison with Eq. (5.25) and choosing $\log \rho = -A\eta(\sqcup)$ one finds that all fluctuating observables $M(t)$ can be written to second order in the form

$$M(t) \simeq e^{A\eta(\sqcup)}\overline{M}(e^{-A\eta(\sqcup)}t)$$

(5.27)

In the mathematical literature on Brownian motion (for a review see Le Gall [18]) the quantity $-A\eta/(\in\pi)$ has appeared in the study of the asymptotic behavior of the volume of the Wiener sausage (where it is commonly denoted by the symbol $\gamma$) and is known as the renormalized local time of self-intersections, a concept introduced by Varadhan in an appendix to an article by Symanzik [29].

6 Discussion

In a preceding letter [3] a more general approach was presented to the much more restricted problem of how to calculate the average $\overline{T}(t)$. The total number of islands was written as $I(t) = C(t) - D(t)$, where the increments $\Delta C(t) = C(t) - C(t-1)$ and $\Delta D(t) = D(t) - D(t-1)$ are the numbers of islands created and destroyed, respectively, in the $t$th step (either $\Delta C(t)$ or $\Delta D(t)$ vanishes). Hence $C(t)$ and $D(t)$ are the total number of islands created and destroyed, respectively, up until time $t$. It was shown [3] that asymptotically for $t \to \infty$ to leading order

$$\overline{C}(t) \simeq \overline{D}(t) \simeq A\pi t/\log 8t$$

(6.1)

where $A = 0.1017\ldots$. In the difference $\overline{T}(t) = \overline{C}(t) - \overline{D}(t)$ the leading order (6.1) cancels and the result (4.28) appears. It does not seem possible to express $C(t)$ and $D(t)$ as observables of the type $M(t)$. The additional determination of $\overline{C}(t)$ and $\overline{D}(t)$ makes the calculation of Ref. [3] more involved. Nevertheless, the generating function found there (Eq. (5) of Ref. [3]) for $d^2\overline{T}(t)/dt^2$ is equivalent to the one for $\overline{T}(t)$ implied by Eq. (4.16) of this work.
The success of the study presented here is due to the generating function method, whose potential is fully exploited. We also run into what may be the limitations of this method. There are several quantities that appear naturally but whose averages and variances we do not see a way to determine. These include the total external perimeter discussed in Sec. 4.4 and the area of the islands enclosed by the support.

Many of the quantities discussed in the preceding chapters have close analogs in planar Brownian motion. We shall denote these analogs by the superscript \( B \). The Brownian motion analog of the support of the lattice random walk is the set of points \( S_B^b(t) \subset \mathbb{R}^2 \) that has been swept out in the time interval \( [0, t] \) by a disk of radius \( b \) performing Brownian motion with diffusion constant \( D \). The set \( S_B^b(t) \) is commonly called the Wiener sausage associated with the Brownian motion trajectory. The total area \( S_B^b(t) \) of this set is analogous to the number of sites \( S(t) \) in the support of the lattice random walk.

The diffusion constant \( D \) is defined so that the mean square displacement equals \( 4Dt \). It can be scaled away, but we shall keep it here to facilitate comparisons between results from different sources. In the mathematical literature one customarily sets \( D = \frac{1}{2} \), whereas the long-time, large-distance limit of the random walk of this work yields \( D = \frac{1}{4} \).

The asymptotic behavior of \( S_B^b(t) \) was first determined by Leontovitsh and Kolmogorov \[16\]. Berezhkovskii et al. give the complete asymptotic expansion

\[
S_B^b(t) \simeq \frac{4\pi D t}{\log \kappa D t b^{-2}} \sum_{m=0}^{\infty} \frac{(-1)^m b_{m2}}{\log^m \kappa D t b^{-2}}
\]

(6.2)

where \( \kappa \equiv 4e^{-2C} \) and the \( b_{m2} \) are as defined by Eq. (4.21). The difference \( \Delta S_B^b(t) \) was shown \[18\] to be a random variable such that the distribution of \( t^{-1} (\log t)^2 \Delta S_B^b(t) \) converges for \( t \to \infty \) to a limit distribution identical to the one of \( t^{-1} (\log t)^2 \Delta S(t) \).

Two remarks can be made about the relations obtained by differentiating Eq. (6.2) with respect to \( b \). First, let \( E_B^b(t) \) be the total boundary length of the Wiener sausage \( S_B^b(t) \). Assuming that \( \partial S_B^b \) is sufficiently regular we have

\[
E_B^b(t) = \frac{dS_B^b(t)}{db}
\]

(6.3)

Upon averaging, using Eq. (6.2) and setting \( b_{02} = 1 \), we find from Eq. (6.3)

\[
\overline{E_B^b(t)} \simeq \frac{8\pi D t b^{-1}}{\log^2 \kappa D t b^{-2}} \left[ 1 + \cdots \right]
\]

(6.4)

This expression has the same asymptotic time dependence as Eq. (1.25) for \( E(t) \), but the coefficients do not coincide. Secondly, differentiate once more and consider the dimensionless number

\[
J_B^b(t) \equiv -\frac{d^2 S_B^b(t)}{db^2}
\]

(6.5)
Its average behaves in the large $t$ limit as
\[ J_b^B(t) \simeq \frac{8\pi Dt b^{-2}}{\log^2 \kappa Dt b^{-2}} \] (6.6)

It is easy to show that if the boundary $\partial S_b^B(\cup)$ is sufficiently regular (having at least a tangent vector in each point), then the quantity (6.5) is equal to $2\pi (I_b^B(t) - 2)$, with $I_b^B(t)$ the number of islands. However, these regularity conditions are not satisfied here since the boundary has cusp points. Nevertheless, comparison of Eqs. (6.6) and (4.28) shows that $J_b^B(t)$ has the same asymptotic time dependence as $I(t)$ (but with a different coefficient).

We now discuss the relation of our work to results that have appeared in the mathematical literature. Throughout the comparison it should be borne in mind that whereas those results are rigorous, the ones of this paper have been obtained by the usual methods of mathematical physics.

Mountford [20] was the first to study the connected components of the complement of the Brownian path of a point (i.e., the case $b = 0$). For all times $t > 0$ the number of these components is infinite due to the presence of many small ones. However, one can ask for example what the number $C_\varepsilon(t)$ is of connected components with an area larger than a prescribed value $\pi \varepsilon^2$. Le Gall, strengthening the results due to Mountford, has shown that for almost all Brownian motion trajectories (with $D = \frac{1}{2}$) in a fixed time interval
\[ \lim_{\varepsilon \to 0} \frac{\varepsilon^2 \log^2 D t \varepsilon^{-2}}{4 Dt} C_\varepsilon(t) = 2 \] (6.7)

where we have used dimensional analysis to restore the variables $D$ and $t$. If one now assumes that $C_\varepsilon(t)$ is of the same order of magnitude as the total number $I_b^B(t)$ of islands in the Wiener sausage associated with the same Brownian motion trajectory executed by a disk of radius $\varepsilon$, then Eq. (6.7) has an asymptotic time dependence that agrees with the result of our Eq. (4.28).

Werner [21] considers the shape of the connected component containing a prescribed point not on the trajectory, and shows that for $t \to \infty$ the probability distribution of this quantity tends to a well-defined limit law. This result is possible only because of the scale invariance of Brownian motion and it would be cumbersome to formulate its lattice counterpart, even in the long-time, large-distance limit. The statement of our Eq. (4.30) about the number of islands $I_\beta(t)$ of given type $\beta$, and the associated result about its fluctuation $\Delta I_\beta(t)$ implied by the discussion at the end of Sec. 5, constitute the point of closest approach between this paper and Werner’s.

Finally, we summarize the new results of this paper. On the one hand, we give the explicit asymptotic behavior as $t \to \infty$ for several new observables associated with the support of the lattice random walk. These include the total perimeter length, the total number of islands, and the total number of single-site and dimer islands. On an infinite lattice the asymptotic behaviors
all consist of a leading term multiplied by a series in inverse powers of $1/\log 8t$. The perimeter and the number of islands are also considered on a finite lattice, where scaling laws are obtained in terms of the time $t$ and the lattice size $N$, and a comparison with computer simulations by Coutinho et al. is possible.

On the other hand, there is the important general result of Sec. 5.3, stating that the pattern numbers all fluctuate in strict proportionality with one another and with the total number $S(t)$ of sites in the support. The fundamental fluctuating variable, called $\eta$ in this work, is the renormalized local time of self-intersections. This result strongly contributes to shape our picture of the support of the two-dimensional random walk: At any fixed time $t$, a large class of detailed properties is determined by the value of the single random variable $\eta$.

One of the open questions that can now be formulated is connected exactly with this random variable, which appears in our work as the time dependent variable $\eta(t)$. Existing results seem to concern exclusively its stationary distribution $P(\eta)$, which prevails in the limit $t \to \infty$. Our investigations point towards the interest of also studying the autocorrelation function $\eta(t)\eta(t')$ and possibly other time dependent properties. A second question that naturally comes up is: How does the picture in higher dimensions differ from the one found here in $d = 2$? It should certainly be expected that in sufficiently high dimension the pattern numbers become independent random variables. The mechanism by which this independence comes about seems worthy of further elucidation. We shall leave these and other questions for future work.

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A Appendix

In this appendix we prove the matrix algebra results we use to derive averages and correlations.

The following result is used in Sec. 4.2. Let $A$ denote an invertible matrix of dimension $\ell \times \ell$, with $\ell \geq 2$. In what follows $J^{[m,n]}$ stands for the $m \times n$ matrix whose elements are $J^{[m,n]}_{ij} = 1$. Define furthermore

$$g_A^{-1} = \sum_{i,j} (A^{-1})_{ij} = \text{Tr}(J^{[\ell,\ell]}A^{-1}) \quad (A.1)$$
We achieve our aim of finding a simplified form for

$$\Gamma_A \equiv \sum_{i,j} [(J^{[\ell,\ell]} + A)^{-1}]_{ij}$$  \hspace{1cm} (A.2)$$
by rewriting Eq. (A.2) as

$$\Gamma_A = \text{Tr} \left[ J^{[\ell,\ell]} A^{-1} (1 + J^{[\ell,\ell]} A^{-1})^{-1} \right]$$  \hspace{1cm} (A.3)$$
where 1 is the unit matrix.

An intermediate step of the demonstration consists in noting that

$$J^{[k,l]} C J^{[\ell,m]} D = g^{-1}_{C^{-1}} J^{[k,m]} D$$  \hspace{1cm} (A.4)$$
where C and D are square matrices of dimensions $\ell \times \ell$ and $m \times m$, respectively. In Eq. (A.3) we now expand the argument of the trace in powers of $J^{[\ell,\ell]} A^{-1}$. Iteration of Eq. (A.4) and use of Eq. (A.1) lead to

$$\text{Tr} \left[ (J^{[\ell,\ell]} A^{-1})^n \right] = g_A^{-n}$$  \hspace{1cm} (A.5)$$
whence

$$\Gamma_A = \frac{1}{1 + g_A}$$  \hspace{1cm} (A.6)$$

When $\ell = 1$ formula (A.6) remains valid for all $A \in \mathbb{R}$ (and in particular for $A = 0$) if for that case one supplements Eq. (A.1) with $g_A = A$.

A generalized version of the above result is needed in Sec. 5. It involves two invertible matrices $A$ and $B$ of dimensions $\ell \times \ell$ and $m \times m$, respectively, with $\ell, m \geq 2$. We now wish to find a simplified expression for

$$\Gamma_{AB}(\lambda) \equiv \sum_{i,j} \left[ \left( \begin{array}{cc} J^{[\ell,\ell]} + A & \lambda J^{[\ell,m]} \\ \lambda J^{[m,\ell]} & J^{[m,m]} + B \end{array} \right)^{-1} \right]_{ij}$$  \hspace{1cm} (A.7)$$
Upon using the decomposition

$$\left( \begin{array}{cc} J^{[\ell,\ell]} + A & \lambda J^{[\ell,m]} \\ \lambda J^{[m,\ell]} & J^{[m,m]} + B \end{array} \right) = \left( \begin{array}{cc} 1 + J^{[\ell,\ell]} A^{-1} & \lambda J^{[\ell,m]} B^{-1} \\ \lambda J^{[m,\ell]} A^{-1} & 1 + J^{[m,m]} B^{-1} \end{array} \right) \left( \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right)$$  \hspace{1cm} (A.8)$$
and writing the double sum in Eq. (A.7) as a trace with the aid of the matrix $J^{[\ell+m,\ell+m]}$, one finds

$$\Gamma_{AB}(\lambda) = \text{Tr} \left[ M(1)(1 + M(\lambda))^{-1} \right]$$  \hspace{1cm} (A.9)$$
where

$$M(\lambda) \equiv \left( \begin{array}{cc} J^{[\ell,\ell]} A^{-1} & \lambda J^{[\ell,m]} B^{-1} \\ \lambda J^{[m,\ell]} A^{-1} & J^{[m,m]} B^{-1} \end{array} \right)$$  \hspace{1cm} (A.10)$$
We now expand the argument of the trace in Eq. (A.9) in powers of $M(\lambda)$. Hence we are left with the calculation of $\text{Tr}(M(1)M(\lambda)^n)$, which is carried out by finding recursively in $n$ relations between the blocks of

$$M(1)M(\lambda)^n \equiv U_n \equiv \begin{pmatrix} X_n & Y_n \\ Z_n & T_n \end{pmatrix}$$  \hspace{1cm} (A.11)

Knowing that $U_{n+1} = U_n M(\lambda)$, one has

$$X_{n+1} = X_n J[\ell,\ell] A^{-1} + \lambda Y_n J[m,\ell] A^{-1}$$
$$Y_{n+1} = \lambda X_n J[\ell,m] B^{-1} + Y_n J[m,m] B^{-1}$$  \hspace{1cm} (A.12)

Hence, again with the use of Eq. (A.4), one proves recursively that $X_n$ and $Y_n$ are of the form

$$X_n = \xi_n J[\ell,\ell] A^{-1}$$
$$Y_n = \eta_n J[\ell,m] B^{-1}$$  \hspace{1cm} (A.13)

with initial values $\xi_0 = \eta_0 = 1$ and the recursion relations

$$\xi_{n+1} = g_A^{-1} \xi_n + \lambda g_B^{-1} \eta_n$$
$$\eta_{n+1} = \lambda g_A^{-1} \xi_n + g_B^{-1} \eta_n$$  \hspace{1cm} (A.14)

Upon setting $Z_n = \zeta_n J[m,\ell] A^{-1}$ and $T_n = \tau_n J[m,m] B^{-1}$ one finds in a similar way

$$\tau_{n+1} = g_B^{-1} \tau_n + \lambda g_A^{-1} \zeta_n$$
$$\zeta_{n+1} = \lambda g_B^{-1} \tau_n + g_A^{-1} \zeta_n$$  \hspace{1cm} (A.15)

with initial values $\tau_0 = \zeta_0 = 1$. Hence, using that $\text{Tr} X_n = g_A^{-1} \xi_n$ and $\text{Tr} T_n = g_B^{-1} \tau_n$, we have from Eqs. (A.9) and (A.10)

$$\Gamma_{AB}(\lambda) = \sum_{n=0}^{\infty} (-1)^n \text{Tr} U_n = \sum_{n=0}^{\infty} (-1)^n (g_A^{-1} \xi_n + g_B^{-1} \tau_n)$$  \hspace{1cm} (A.16)

By summing each of the equations (A.14)-(A.15) on all $n$ with weight $(-1)^n$ one obtains two sets of linear equations from which $\sum_{n=0}^{\infty} (-1)^n \xi_n$ and $\sum_{n=0}^{\infty} (-1)^n \tau_n$ may be solved. Substitution in Eq. (A.16) yields the final result

$$\Gamma_{AB}(\lambda) = \frac{2 - 2\lambda + g_A + g_B}{1 - \lambda^2 + g_A + g_B + g_A g_B}$$  \hspace{1cm} (A.17)

By redoing the calculation with $\ell$ or $m = 1$ one finds that this expression for $\Gamma_{AB}$ remains valid for those special cases.
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FIGURE CAPTIONS

**Figure 1.** A random walker makes $t$ random steps between the centers of neighboring squares. The squares visited are colored black and constitute the *support* of the walk. In the example of this figure the support encloses four islands of unvisited sites. The boundary of the support is represented by a heavy line.

**Figure 2.** Let the boundary of the support be oriented such that as one proceeds along it the white sites are on the left and the black ones on the right. Then in (a) the boundary turns through an angle $+\frac{1}{2}\pi$ and in (b) through an angle $-\frac{1}{2}\pi$. In (c) the black squares are (by convention) considered to separate the white squares from one another and the part of the boundary shown makes two turns through $+\frac{1}{2}\pi$. 

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| $A$                  | $\mu_A$ | $g_A$  |
|---------------------|---------|--------|
|                     | $S$     | $E$    | $I$   | $I_1$ |        |
| $\emptyset$        | 1       |        | -1   | -4    | -1     |
| $\{0\}$            | -1      | 4      | 1    | 1     | 0      |
| $\{0, e_1\}$       | -4      | -2     | -4   | 1/2   |        |
| $\{0, e_1, e_2, e_1 + e_2\}$ | -4 | -2 | -4 | 1/2 |        |
| $\{0, e_1, e_2\}$  |        | 4      |      | $\pi/2(\pi - 1)$ |        |
| $\{0, e_1, 2e_1\}$ |        | 2      |      | $\pi/4$ |        |
| $\{0, e_1, -e_1, e_2\}$ | -4 |      |      | $\pi(\pi - 6)/2(\pi^2 - 6\pi + 4)$ |        |
| $\{e_1, e_2, -e_1, -e_2\}$ | 1 |      | 1    |      | 1      |

Table I. The coefficients $\mu_A$ for the four observables $S, E, I$ and $I_1$ defined in the text; entries not shown are zero. Symmetry under rotations over $\pi/2$ has been exploited to reduce the length of the table. The coefficients in each column add up to zero. The parameter $g_A$ is given for all sets $A$ that occur; it is undefined for the empty set $\emptyset$. $A$ itself is defined only up to a translation.