Abstract

Beyond Standard Model physics frequently connects flavor symmetry with a discrete group. If the discrete symmetry arises spontaneously from a gauge theory, one can maintain compatibility with quantum gravity and avoid anomalies. We provide an example of such a model with the Standard Model gauge group extended to $SU(3)_C \times SU(2)_L \times U(1)_Y \times SU(2)_{T'}$ where the binary tetrahedral flavor group $T'$ is embedded in $SU(2)_{T'}$. Quark and lepton masses and mixing angles are fit to data, where lepton mixing angles are shifted from tribimaximal values by the addition of scalar VEVs to agree with the experimental data.
I. INTRODUCTION

Flavor models of elementary particles have had to evolve as new data becomes available. As the data becomes more precise, the models become more sophisticated. The usual model building practice is to extend the standard model (SM) with a discrete symmetry which is used to fit the data. But variations abound, from extending a supersymmetric SM, to discrete group extended grand unified models (For reviews see [1–5]), to top-down fully gauged theories where the gauge group is sufficiently large to accommodate both the GUT and flavor symmetries [6]. Here we take a minimalist approach and look for the smallest fully gauged model that can explain all the data.

One of the simplest and most natural flavor models is the SM extended by the discrete group $T'$ [1, 7–13], where the one and two dimensional irreducible representations (irreps) accommodate the quarks, while the leptons fit naturally into one and three dimensional irreps. For a phenomenological discussion and recent summary of the data, see e.g., [27]. The current challenge is to fit the most recent neutrino data with a $T'$ model. A shortcoming of nearly all discrete flavor models is their lack of compliance with gravity [15], i.e., gravity breaks discrete global symmetries. But since gravity does not interfere with gauge symmetries, gauging a discrete symmetry by embedding it in a gauge group is a way to avoid this problem. But one still has to contend with discrete [16–20] or continuous chiral gauge anomalies. Our minimalist approach then leads us to gauge $T'$ flavor. The smallest continuous group that contains $T'$ is $SU(2)$, so this is what we will attempt below. Various complications arise, but we will be able to deal with them as we go along.

We take the simplified renormalizable $T'$ extension of the standard model of [10] and augment it in two ways. First, we add scalar singlets, that will acquire VEVs and shift the predictions of tribimaximal (TBM) mixing and of the Cabibbo angle from previous models to be more in line with current experimental values. Second, after adding a few fermions, the $T'$ group is embedded into a gauged $SU(2)$ group we will call $SU(2)_{T'}$. This averts problems with gravity and chiral anomalies that can arise from adding discrete groups to the standard model. It also provides an elegant description of the discrete symmetry as a residue of a gauge group acting at higher scale. Finally we summarize how $SU(2)_{T'}$ can be broken directly to $T'$ with a VEV for a particular scalar multiplet.

The next section contains the lepton sector particle assignments, plus the assignments
for the scalar fields that enter the lepton Yukawa Lagrangian at the $T'$ scale. Section III contains similar information for the quark sector; in Section IV we discuss tribimaximal (TBM) mixing, where a $T'$ triplet Higgs gets a vacuum expectation value (VEV). Since there is currently tension between the data and TBM predictions, we add $T'$ scalar singlets with VEVs to shift TBM predictions in Section V, where we show our new fit is in agreement with all lepton data. Section VI focuses on the quark sector, where the new scalar singlet VEVs now contribute to quark mixing.

It is the above described $T'$ model we gauge to $SU(2)_{T'}$, and describe in Section VII, where various additional particles need to be added to avoid all chiral anomalies. Section VIII describes the spontaneous symmetry breaking (SSB) from $SU(2)_{T'}$ to $T'$, and Section IX contains our conclusions and plans for further work. An appendix collects all the $T'$ group theory needed for this paper.

II. LEPTON SECTOR LAGRANGIAN AT THE $T'$ SCALE

We begin by reviewing the lepton sector just above the $T'$ scale. Because none of the leptons will be in even dimensional irreducible representations (irreps), this sector is equivalent to an $A_4$ model [21, 22]. We have also given the model a $Z_2$ symmetry in order to disallow certain terms in the Lagrangian. This $Z_2$ will also be gauged.

The standard model leptons are assigned to the following irreps [10] of $T' \times Z_2$ (and of $A_4 \times Z_2$):

\[
\begin{pmatrix}
\nu_{\tau} \\
\tau^- \\
\nu_{\mu} \\
\mu^- \\
\nu_e \\
e^-
\end{pmatrix}_L \begin{cases}
L_L(3, 0) & \begin{array}{l}
\tau^-_{R} (1_2, 1) \\
\mu^-_{R} (1_2, 1) \\
e^-_{R} (1_3, 1)
\end{array}
\end{cases}
\begin{array}{c}
N^{(1)}_R \\
N^{(2)}_R \\
N^{(3)}_R
\end{array}
R
\begin{cases}
N_R(3, 0)
\end{cases}
\]

(1)

where $N_R$ is a $T'$ triplet of right handed neutrinos. In addition we will need the following
scalars:

\[
H_3(3, 0) = (H_3^1, H_3^2, H_3^3)
\]
\[
H'_3(3, 1) = (H'_3^1, H'_3^2, H'_3^3)
\]
\[
H_{11}(1_1, 0)
\]
\[
H_{12}(1_2, 0)
\]
\[
H_{13}(1_3, 0)
\]  

Where the subscripts correspond to the \(T'\) irrep where the scalars live.

Aside: Note that here and and below we use a different notation from [10] which used a multiplicative form for the \(Z_2\) charges, i.e., \(\pm 1\). Since we will be concerned with discrete and continuous chiral gauge anomalies, we use additive \(Z_2\) charges, i.e., integers mod 2, to be consistent with most of the literature. When we later embed \(Z_2\) in a \(U(1)\) we will use integer charges.

With the above content, the most general lepton sector Yukawa Lagrangian is:

\[
\mathcal{L}_l = Y_\tau L_L H_3^\dagger + Y_\mu L_L \mu_R H_3^\dagger + Y_e L_L e_R H_3^\dagger + L_L N_R (Y_2 H_{12} + Y_y H_{13} + Y_z H_{11} + Y_{TB} H_3) + m_N N_R N_R
\]  

The proper choice of VEVs for \(H_3\) and \(H'_3\) lead to values for the charged masses and the TBM mixing matrix. Giving VEVs to the singlets will break \(T'\) to \(Q\), the group of unit quaternions, and shift the TBM matrix closer to experimentally compatible values.

### III. QUARK SECTOR LAGRANGIAN AT THE \(T'\) SCALE

The main advantage of a \(T'\) flavor model is that it is the discrete group of smallest order with a sufficiently diverse set of irreps that can be used to model both the quark and lepton sectors. Specifically, it has even-dimensional irreps that can also be used to economically describe the quark sector, as we will now summarize [10]. The standard model quarks are assigned to the following irreps:

\[
\begin{pmatrix}
t \\
b \\
c \\
s \\
u \\
d \\
\end{pmatrix}_{L} Q_L \quad (1_1, 0) \quad \begin{pmatrix} t_R \\ b_R \\ c_R \\ u_R \\ s_R \\ d_R \end{pmatrix}_{R} C_R \quad (1_2, 1) \quad \begin{pmatrix} 2_1 \end{pmatrix}_L \quad \begin{pmatrix} 1 \end{pmatrix}_R \quad (2_2, 1) \quad \begin{pmatrix} 2_3 \end{pmatrix}_L \quad (2_3, 0).
\]
In addition to the scalars listed above, we add one more singlet:

\[ H_{13}'(13, 1) \]  

(5)

Hence the most general quark sector Yukawa Lagrangian is then:

\[
\mathcal{L}_q = Y_t Q_L t_R H_{11} + Y_b Q_L b_R H_{1_{3'}} + Q_L C_R (Y_C H_{3'} + Y_C' H_{1_{3'}}) + Q_L S_R (Y_S H_3 + Y_S' H_{1_2})
\]  

(6)

We see that a constraint on our model is that the VEVs of \( H_{3}, H_{3}', H_{1_2}, \) and \( H_{11} \) must have values that are simultaneously compatible with the experiment data for both the quark and lepton sector.

IV. TBM MIXING FROM \( T' \)

Before we derive our experimentally compatible PMNS matrix \([23]\), we show that just below the \( T' \) energy scale where only \( T' \) triplets have VEVs, the neutrinos exhibit the familiar TBM mixing pattern \([24]\). Using the Clebsch-Gordan coefficients for \( T' \) detailed in the Appendix, we find that the term \( m_{N} N_{R} N_{R} \) from equation (3) gives a mass matrix for right handed neutrinos:

\[
M_N = \begin{pmatrix}
  m_N & 0 & 0 \\
  0 & 0 & m_N \\
  0 & m_N & 0 
\end{pmatrix}
\]  

(7)

Similarly, we construct the Dirac mass matrix associated with the term \( Y_{TB} L_L N_R H_3 \) of the lepton Lagrangian:

\[
M_D = \begin{pmatrix}
  e N_1 & e N_2 & e N_3 \\
  \mu N_1 & \mu N_2 & \mu N_3 \\
  \tau N_1 & \tau N_2 & \tau N_3 
\end{pmatrix} = Y_{TB} \begin{pmatrix}
  v_2 & -v_1 & 0 \\
  -v_3 & 0 & v_1 \\
  0 & v_3 & -v_2 
\end{pmatrix}
\]  

(8)

Where \((v_1, v_2, v_3)\) is the VEV of the scalar \( H_3 \). The Majorana mass matrix is given by:

\[
M_\nu = M_D M_N^{-1} M_D^T
\]  

(9)

The rows of the Majorana mixing matrix are the normalized eigenvectors of this mass matrix, we find that for a VEV of \(< H_3 >= V(1, 1, -2)\), (where \( V \) is some constant), we
recover the TB mixing matrix in the form

\[
U_{TBM} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
-\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\
-\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}}
\end{pmatrix}.
\] (10)

Currently TBM is excluded at the 5\(\sigma\) level. For a different perspective see [25].

V. SHIFTED TBM MIXING

Our next step is to augment this matrix using VEVs for the additional scalars \(H_{12}, H_{13}\), and \(H_{11}\). (For an alternative perturbation theory approach see [26].) Including these in the model introduces the terms \(L_L N_R (Y_x H_{12} + Y_y H_{13} + Y_z H_{11})\) into the Lagrangian. These terms have a mass matrix:

\[
M_{xyz} = \begin{pmatrix}
-x & z & y \\
y & x & z \\
z & -y & -x
\end{pmatrix}
\] (11)

Where \(x, y,\) and \(z\) represent \(Y_x < H_{12} >, Y_y < H_{13} >,\) and \(Y_z < H_{11} >\) respectively. Our Dirac mass matrix is now

\[
M_{D'} = M_D + M_{xyz} = Y_{TB} \begin{pmatrix}
v_2 - x' & -v_1 + z' & y' \\
-v_3 - y' & x' & v_1 + z' \\
z' & v_3 - y' & -v_2 - x'
\end{pmatrix}
\] (12)

where \(x' = \frac{x}{Y_{TB}}, y' = \frac{y}{Y_{TB}},\) and \(z' = \frac{z}{Y_{TB}}.\) The Majorana mixing matrix, \(U,\) is obtained the same way as before. The fit parameters \(x', y',\) and \(z'\) can now be varied to shift the entries of \(U\) from their TBM mixing values closer to current experimental values. The present 3\(\sigma\) experimental ranges of the magnitudes of the matrix elements are given below [27–29]:

\[
\begin{pmatrix}
0.799 \leftrightarrow 0.844 & 0.516 \leftrightarrow 0.582 & 0.141 \leftrightarrow 0.156 \\
0.242 \leftrightarrow 0.494 & 0.467 \leftrightarrow 0.678 & 0.639 \leftrightarrow 0.774 \\
0.284 \leftrightarrow 0.521 & 0.490 \leftrightarrow 0.695 & 0.615 \leftrightarrow 0.754
\end{pmatrix}
\] (13)

The next step we can take is to vary the parameters \(x', y',\) and \(z'\) from -1 to 1 (within a reasonable precision), and find the values for which the least accurate elements’ error is minimized. We find that to the nearest hundredth, this minimum is obtained at \((x', y', z') = \ldots\)
(0.32, −0.26, −0.40) with maximum error of 1.922σ. Explicitly, these values correspond to a mixing matrix:

\[
\begin{pmatrix}
-0.829 & 0.539 & 0.148 \\
0.289 & 0.640 & -0.712 \\
0.478 & 0.548 & 0.686
\end{pmatrix}
\] (14)

which can be compared to the experimental numbers in eq. (13).

The errors relative to experiment are given below in units of σ:

\[
\begin{pmatrix}
1.029 & 0.910 & 0.228 \\
1.882 & 1.919 & 0.224 \\
1.922 & 1.313 & 0.083
\end{pmatrix}
\] (15)

In addition to minimizing the error of the least accurate entry we can minimize the average error of the matrix elements.

Looping over all possible values of \(x', y', z'\) (again to the nearest hundredth) minimizes the mean error at \((0.32, -0.27, -0.045)\), with a value of 0.870 σ. Our mixing matrix is now

\[
\begin{pmatrix}
-0.822 & 0.549 & 0.149 \\
0.298 & 0.638 & -0.710 \\
0.485 & 0.539 & 0.688
\end{pmatrix}
\] (16)

with errors

\[
\begin{pmatrix}
0.090 & 0.0351 & 0.199 \\
1.671 & 1.869 & 0.151 \\
2.091 & 1.558 & 0.167
\end{pmatrix}
\] (17)

From both these perspectives on error analysis, our \(T'\) model extended with a pair of scalar singlets agrees with the current experimental data which provides a significant improvement over the simple TBM model with singlet VEVs on the same order as the triplet VEVs, i.e., without introducing a new length scale.

We can also examine our fitting with a contour plot. Our error values are the most sensitive to changes in \(x'\) so we hold it constant at 0.32 and allow our parameters \(y'\) and \(z'\) to vary between -1 and 0 as shown in the plots below: (The parameter range \((y', z') > 0\) gives very high error values so it is not shown.)
FIG. 1. Contour plot with $x'$ fixed at 0.32 of maximum mixing matrix error relative to experimental data, where values are in units of $\sigma$. 
FIG. 2. Contour plot with $x'$ fixed at 0.32 of average mixing matrix error relative to experimental data, where values are in units of $\sigma$.

We see from these plots that there are relatively small ranges, but still without fine tuning beyond an order of magnitude, for our parameters that give us maximum error and average error less that $2\sigma$ and $1\sigma$ respectively.

VI. QUARK MIXING

As shown in [10], one can derive a reasonable prediction for the Cabbibo angle from the Lagrangian in equation (6). We rederive this result here for our basis and then augment the value when an additional scalar has a VEV. We also find the mass matrices for the first two generations of up and down type quarks from the terms $Q_L C_R H_3'$ and $Q_L S_R H_3$ respectively. From the discussion above we know that $H_3$ must have a VEV of the form $V(1,1,-2)$. To
give the correct masses to the charged leptons $H_{3'}$ must have a VEV:

$$< H_{3'} > = \left( \frac{m_{\tau}}{Y_{\tau}}, \frac{m_{\mu}}{Y_{\mu}}, \frac{m_e}{Y_e} \right)$$  (18)

Using these values along with the Clebsch-Gordan coefficients found in the Appendix, we obtain a mass matrix $U$ for the up-type quarks:

$$U = Y_C \begin{pmatrix} m_{\mu} Y_{\mu} & -\frac{1}{\sqrt{2}} m_e Y_e \\ -\frac{1}{\sqrt{2}} m_e Y_e & m_e Y_e \end{pmatrix}$$  (19)

Because $m_{\tau} > m_{\mu} >> m_e$ we can, to lowest order, set $m_e = 0$ which gives a diagonal mass matrix $U$. For our down-type mass matrix, $D$ we obtain:

$$D = Y_S \begin{pmatrix} -1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{pmatrix}$$  (20)

The mixing matrix for the first two quark generations, (the upper left corner of the CKM matrix) is $W = K_u^T K_d$. Where $K_u$ and $K_d$ are the unitary matrices that diagonalize the Hermitian squares of $U$ and $D$ respectively. Since $U$ is already diagonal, $K_u$ is just the identity matrix and we have:

$$DD^T = Y_S^2 \begin{pmatrix} \frac{3}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 9/2 \end{pmatrix}$$  (21)

$$W = K_d = \begin{pmatrix} 0.976 & 0.218 \\ -0.218 & 0.976 \end{pmatrix}$$  (22)

This gives an expression for the Cabibbo angle $\Theta$

$$\tan(2\Theta) = \left( \frac{\sqrt{3}}{3} \right) \rightarrow \sin(\Theta) = .218$$  (23)

in agreement with [10].

We can see that this prediction roughly approximates the measured value of $\sin \Theta \approx .225$. But given the high experimental precision of the Cabibbo angle, this prediction disagrees with experiment by over twelve standard deviations. Some of this variation can be explained by the fact that we have not included mixing with the third family in this simplified model.
We could include third family mixing by adding the terms to the Lagrangian in equation (6):

\[ \mathcal{L}_q' = Y_{tcu} Q_L C_R H_{23} + Y_{bsd} Q_L S_R H_{22} + Y_{cut} Q_L t_R H_{21} + Y_{sdb} Q_L b_R H_{23}, \]

but this introduces at least six more free parameters into the theory, and since we know such contributions to be very small we ignore them in the present analysis. Instead we can shift our prediction to well within the one sigma range by giving a VEV to the scalar \( H_{13} \) in the quark Lagrangian in eq.(6). With this additional term, the down-type quark matrix becomes:

\[ D' = Y_S \left( \begin{array}{c} \frac{-1}{\sqrt{2}}(-1 - \frac{Y_{3s} < H_{12} >}{Y_S}) \\
\frac{1}{\sqrt{2}} \left( m_e Y_e - \frac{Y_{3s} < H_{12} >}{Y_S} \right) \end{array} \right). \]  

(25)

We can similarly give a VEV to \( H_{13}' \), which will give an up-type matrix:

\[ U' = Y_C \left( \begin{array}{c} \frac{-1}{\sqrt{2}} \left( m_e Y_e + \frac{Y_{3c} < H_{13} >}{Y_C} \right) \\
- \frac{1}{\sqrt{2}} \left( \frac{m_e Y_e}{Y_C} - \frac{Y_{3c} < H_{13} >}{Y_C} \right) \end{array} \right). \]  

(26)

But because the diagonal elements of \( U' \) are so large, this will not have a significant effect. Similar to what the neutrino sector above, we can vary the value \( \frac{Y_{3s} < H_{12} >}{Y_S} \) in order to get a more accurate prediction for the Cabibbo angle. Because we are only varying one parameter we can find the optimal value to a much higher precision. In fact setting the value to \( \frac{Y_{3s} < H_{12} >}{Y_S} = .8792 \) gives us a mixing matrix:

\[ W = \begin{pmatrix} 0.974464 & -0.224545 \\
0.224545 & 0.974464 \end{pmatrix}. \]  

(27)

The values for \( W_{ud} \) and \( W_{us} \), and thus the prediction for the Cabibbo angle, are almost identical to those found from the latest experimental fit [29]:

\[ |W| = \begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 \\
0.22438 \pm 0.00044 & 0.97359 \pm 0.00010 \end{pmatrix}. \]  

(28)

Specifically, our errors are (again in units of \( \sigma \)):

\[ \begin{pmatrix} 0.0379 & 0.0562 \\
0.3744 & 7.9435 \end{pmatrix}. \]  

(29)

The \( W_{cd} \) prediction is also well within 1\( \sigma \). The prediction for \( W_{cs} \) is quite a bit off, but this is to be expected, or at least not surprising, given our neglect of third family mixing effects.
VII. $T'$ EMBEDDING IN $SU(2)$

As explained in the Introduction, it is often desirable to embed discrete symmetries into continuous gauge groups at higher energy scales. The remainder of this paper will focus on generalizing our $T'$ model to a gauged $SU(2)_{T'}$ flavor theory.

There are three main tasks needed for our gauge group embedding. First, we must identify which $SU(2)_{T'}$ representations our $T'$ particles can fall into. This is easily accomplished by examining the branching rules from Table IV. New particles will have to be introduced to fill out these $SU(2)_{T'}$ irreps, as a full theory cannot contain incomplete group representations. Second, we must ensure our theory is anomaly free. This involves checking that our representations satisfy certain sum rules on their quantum numbers (see e.g., [30]). Again we will see we must add more particles to the theory in order to cancel all anomalies. Finally, we formulate a scalar Lagrangian where we can find a particular vacuum expectation value that breaks $SU(2)_{T'}$ down stepwise to $T'$ [31–33], then to $Q$, etc. and eventually to nothing.

A. $SU(2)$ Multiplets

Table I shows the results of embedding the $T'$ irreps of equations (1) and (4) into $SU(2)_{T'}$. Each row shows the particle in their $SU(2)_{T'}$ multiplet, and each column gives the representation of the constituent particles under the specified gauge group.
In order to complete the various irreps of $SU(2)_{T'}$, we have to include a number of new particles. Specifically we have added three new leptons: $(a, b, c)$, and eight new quarks: \(X, \alpha, \beta, \gamma, i, j, k, l\).

Our next step is to check our theory for anomalies. With the current irreps, the only anomaly that does not cancel is $SU(2)_{T'} \times SU(2)_{T'} \times U(1)_Y$. To cancel this anomaly and avoid disrupting other cancellations, we add the multiplets listed in Table II to the theory. Note that this is not the only way to do the embedding, but it is the most straightforward and economical embedding we have found.
TABLE II. Additional particles needed for cancellation of chiral anomalies

| Particles     | $SU(3)_C$ irrep | $SU(2)_L$ irrep | $U(1)_Y$ charge | $SU(2)_{T'}$ irrep |
|---------------|-----------------|-----------------|-----------------|-------------------|
| $(a_1, a_2, a_3, a_4, a_5)$ | 1               | 1               | -2              | 5                 |
| $(n_1, n_2)$ | 1               | 1               | 2               | 2                 |
| $m_1$        | 1               | 1               | 2               | 1                 |
| $m_1'$       | 1               | 1               | 2               | 1                 |
| $m_1''$      | 1               | 1               | 2               | 1                 |

With that we have a complete fermion sector for the theory. Although we have had to add many new particles, all of them can be made sufficiently heavy such that they are only relevant at very high energy scales.

B. $Z_2$ anomaly cancellation

In the above formulation we have canceled all anomalies that come about due to the addition of the $T'$ symmetry to the standard model. However, recall that we also included an extra $Z_2$ symmetry in order to forbid certain unwanted Lagrangian terms. This $Z_2$ can be embedded in an extra $U(1)_{Z_2}$ symmetry that breaks at an arbitrary scale independent of the $SU(2)_{T'}$ breaking. We detail the charge assignments for an example anomaly-free $SU(3) \times SU(2)_L \times U(1)_Y \times SU(2)_{T'} \times U(1)_{Z_2}$ theory below in Table III. Notice we have added an SM singlet $SU(2)_{T'}$ 4 with $Z_2$ charge $-1$ and fourteen fermions that are trivial singlets under everything but $U(1)_Y \times U(1)_{Z_2}$. Five of them, the $E$s have charge $(2,1)$ and the other five, the $F$s have charge $(-2,0)$ under this group, the remaining four have $U(1)_Y$ charge $\pm 10$ and $U(1)_{Z_2}$ charge 0, 1 or -1.

There is significant freedom in assigning $U(1)_{Z_2}$ charges to existing particles as they reduce to particles with identical $Z_2$ charges modulo 2. So even though this example has involved adding many extra particles, a less baroque model may be possible.
TABLE III. Example charge assignments for $U(1)_{Z_2}$ anomaly cancellation [16, 17]

| Particles | $SU(3)_C$ irrep | $SU(2)_L$ irrep | $U(1)_Y$ charge | $SU(2)_{T'}$ irrep | $U(1)_{Z_2}$ charge |
|-----------|----------------|----------------|-----------------|--------------------|---------------------|
| $((\nu_\tau, \tau), (\nu_\mu, \mu), (\nu_e, e))_L$ | 1 | 2 | -1 | 3 | 0 |
| $\tau^c_R$ | 1 | 1 | 2 | 1 | 1 |
| $(A, B, C)_L$ | 1 | 1 | -2 | 3 | 0 |
| $(\mu, e, A, B, C)^c_R$ | 1 | 1 | 2 | 5 | -1 |
| $(N_1, N_2, N_3)_R$ | 1 | 1 | 0 | 3 | 0 |
| $((c, s), (u, d))_L$ | 3 | 2 | $\frac{1}{3}$ | 2 | 0 |
| $(t, b)_L$ | 3 | 2 | $\frac{1}{3}$ | 1 | 0 |
| $t^c_R$ | 3 | 1 | $-\frac{4}{3}$ | 1 | 0 |
| $(X, b, \alpha, \beta, \gamma)^c_R$ | 3 | 1 | $\frac{2}{3}$ | 5 | 1 |
| $X_L$ | 3 | 1 | $-\frac{2}{3}$ | 1 | -1 |
| $(\alpha, \beta, \gamma)_L$ | 3 | 1 | $-\frac{2}{3}$ | 3 | 0 |
| $(c, u, i, j)^c_R$ | 3 | 1 | $-\frac{4}{3}$ | 4 | -1 |
| $(s, d, k, l)^c_R$ | 3 | 1 | $\frac{2}{3}$ | 4 | 0 |
| $i_L$ | 3 | 1 | $\frac{4}{3}$ | 1 | 0 |
| $j_L$ | 3 | 1 | $\frac{4}{3}$ | 1 | 0 |
| $k_L$ | 3 | 1 | $-\frac{2}{3}$ | 1 | 0 |
| $l_L$ | 3 | 1 | $-\frac{2}{3}$ | 1 | 0 |
| $(a_1, a_2, a_3, a_4, a_5)$ | 1 | 1 | -2 | 5 | 0 |
| $(n_1, n_2)$ | 1 | 1 | 2 | 2 | 0 |
| $m_1$ | 1 | 1 | 2 | 1 | 1 |
| $m_1'$ | 1 | 1 | 2 | 1 | 1 |
| $m_1''$ | 1 | 1 | 2 | 1 | 1 |
| $(b_1, b_2, b_3, b_4)$ | 1 | 1 | 0 | 4 | -1 |
| $5 \times E$ | 1 | 1 | 2 | 1 | 1 |
| $5 \times F$ | 1 | 1 | -2 | 1 | 0 |
| $g_1$ | 1 | 1 | -10 | 1 | 1 |
| $g_2$ | 1 | 1 | -10 | 1 | -1 |
| $h_1$ | 1 | 1 | 10 | 1 | 0 |
| $h_2$ | 1 | 1 | 10 | 1 | 0 |
VIII. SPONTANEOUS SYMMETRY BREAKING

Our final step is to provide the spontaneous breaking of $SU(2)_T \rightarrow T'$. We have already performed this analysis in a previous paper [33], so will only summarize the results here. To have this spontaneous symmetry breaking we must include a scalar multiplet of $SU(2)_T$ that contains a trivial singlet of $T'$. Looking at the branching rules of table IV, we see the smallest available irrep for this purpose is the 7. The 7 can be real or complex, but for simplicity we choose a real multiplet with scalar potential

$$V_7 = -m^2 T_{abc} T_{abc} + \lambda (T_{abc} T_{abc})^2 + \kappa T_{abd} T_{abe} T_{fgf} T_{fgd}, \quad (30)$$

where $T$ is a traceless, symmetric, $3 \times 3 \times 3$ tensor, $\lambda$ and $\kappa$ are the scalar quartic coupling constants, and the indices $a, b, ...$ run from 1 to 3.

Spontaneous breaking to $T'$ occurs when the potential is minimized and the scalar is given a Vacuum Expectation Value (VEV) in a particular direction. For the real 7 this VEV is [33]:

$$V = \sqrt{\frac{3m^2}{2(3\lambda + \kappa)}} [0, 0, 0, 0, 0, 0, 0, 1]. \quad (31)$$

After breaking $SU(2)_T$, the 7 real scalars reduce to their $T'$ irreps with mass eigenvalues

| Value       | Multiplicity |
|-------------|--------------|
| 0           | 3            |
| $4m^2$      | 1            |
| $\frac{8m^2\kappa}{5(3\lambda + \kappa)}$ | 3            |

which contains the three requisite Goldstone bosons that get eaten by the $SU(2)_T$ gauge bosons. To ensure a stable minimum the coupling constants must satisfy the constraints $3\lambda + \kappa > 0$ and $\kappa > 0$. Clearly there is a substantial region of parameter space where this pattern of SSB is the stable minimum of the potential in eq.(30).

This 7 is obviously not the only scalar in the theory as more scalars are needed to construct Yukawa terms at the $SU(2)_T$ scale. However, we omit the full scalar Lagrangian
in this paper because we will not be exploring its complete phenomenology at present. We are assuming that the coupling of the 7 to the other scalars is sufficiently weak that the breaking to \( T' \) is not destabilized. The analysis of a specific example of this type of VEV stability can be found in [33].

IX. CONCLUSION

We have extended the basic \( T' \) flavor model to fit the current best available quark and lepton mass and mixing angle data. More specifically, we have constructed an extended but fairly simple, renormalizable \( T' \) model that predicts neutrino mixing parameters within 2\( \sigma \) of experiment, as well as a Cabibbo angle well within 1\( \sigma \). This has required the addition of \( T' \) scalar singlets with VEVs. Once our new \( T' \) model was fixed, we then extended it further by embedding it in \( SU(2)_{T'} \) such that the entire model was fully gauged. This avoided all problems with gauge and gravity mixed anomalies at the expense of adding a number of new fermions to the lepton and quark sectors. The additional fermions were not necessarily the minimal set, as there are many possible choices, so what we have provided is a proof of principle that fully gauged flavor models can be found to fit all current flavored data. It still remains quite challenging to find a full gauge unification of flavor, but it is perhaps not unreasonable to hope that one could eventually find a top-down GUT flavor model that reduces to a product gauge model of the type we have discussed here.

Besides the \( T' \) model discussed here, gauged \( A_4 \) models [34–36] have also appeared, but there remains a long list of discrete groups \( S_4, A_5, Q_6, O', I', T_7, \Delta(27) \) and \( PSL(2, 7) \) that are easy to obtain from breaking \( SU(2) \) or \( SU(3) \). So it appears possible to gauge some if not all of the models based on these groups [37–45].

There is still more to explore within our present model; in particular the phenomenology of the new scalar singlets and the additional fermions required for anomaly cancellation. The phenomenology of the \( SU(2)_{T'} \) scalar 7 would also benefit from further study, but we leave these topics for future work. Beyond this specific model, it would be preferable to avoid \( Z_n \) factors by either reassigning irreps of SM states, or by using different initial non-abelian discrete groups. This would simplify the anomaly cancelation and hence minimize the introduction of extra fermionic states. We plan to search for such models in the future.
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Appendix A: Useful Information About the Binary Tetrahedral Group $T'$

1. $T'$ Character Table

| Dimension | $C_1$ | $C_2$ | $4C_3$ | $6C_4$ | $4C_5$ | $4C_6$ | $4C_7$ |
|-----------|-------|-------|--------|--------|--------|--------|--------|
| 11        | 1     | 1     | 1      | 1      | 1      | 1      | 1      |
| 12        | 1     | 1     | $\omega^2$ | $\omega^4$ | 1      | $\omega^2$ | $\omega^4$ |
| 13        | 1     | 1     | $\omega^4$ | $\omega^2$ | 1      | $\omega^4$ | $\omega^2$ |
| 21        | 2     | -2    | -1     | -1     | 0      | 1      | 1      |
| 22        | 2     | -2    | $\omega^5$ | $\omega$ | 0      | $\omega^2$ | $\omega^4$ |
| 23        | 2     | -2    | $\omega$ | $\omega^5$ | 0      | $\omega^4$ | $\omega^2$ |
| 3         | 3     | 3     | 0      | 0      | -1     | 0      | 0      |

Where $\omega = e^{\frac{2\pi i}{6}}$.

2. Kronecker Products of $T'$ Irreps

| Dimension | 11 | 12 | 13 | 21 | 22 | 23 | 3 |
|-----------|----|----|----|----|----|----|---|
| 11        | 11 | 12 | 13 | 21 | 22 | 23 | 3 |
| 12        | 12 | 13 | 11 | 22 | 23 | 21 | 3 |
| 13        | 13 | 11 | 12 | 23 | 21 | 22 | 3 |
| 21        | 21 | 22 | 23 | 11 + 3 | 12 + 3 | 13 + 3 | 21 + 22 + 23 |
| 22        | 22 | 23 | 21 | 12 + 3 | 13 + 3 | 11 + 3 | 21 + 22 + 23 |
| 23        | 23 | 21 | 22 | 13 + 3 | 11 + 3 | 12 + 3 | 21 + 22 + 23 |
| 3         | 3  | 3  | 3  | $21 + 22 + 23$ | $21 + 22 + 23$ | $21 + 22 + 23$ | $11 + 12 + 13 + 3 + 3$ |
3. Decomposition of SU(2) Irreps to $T'$ Irreps

| SU(2) | Dynkin Index | $T'$ |
|-------|--------------|------|
|       |              | 11 2 13 21 22 23 3 |
| 1     | 0            | 1 0 0 0 0 0 0 0     |
| 2     | 1            | 0 0 0 1 0 0 0 0     |
| 3     | 4            | 0 0 0 0 0 0 0 1     |
| 4     | 10           | 0 0 0 0 1 1 0 0     |
| 5     | 20           | 0 1 1 0 0 0 0 1     |
| 6     | 35           | 0 0 0 1 1 1 0 0     |
| 7     | 56           | 1 0 0 0 0 0 0 2     |
| 8     | 84           | 0 0 0 2 1 1 0 0     |
| 9     | 120          | 1 1 1 0 0 0 0 2     |

4. $T'$ Clebsch-Gordan Coefficients

For our basis we take the tensor products in section 5 of [2] with $p = i$, $p_1 = -1$, and $p_2 = 1$.

\[
\begin{align*}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2(2')} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2(2')} &= \left( \frac{x_1y_2 - x_2y_1}{\sqrt{2}} \right)_{1'} \oplus \left( \frac{1}{\sqrt{2}} (x_1y_2 + x_2y_1) \right)_{3} \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'_{2(2)} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'_{2(2')} &= \left( \frac{x_1y_2 - x_2y_1}{\sqrt{2}} \right)_{1''} \oplus \left( \frac{1}{\sqrt{2}} (x_1y_2 + x_2y_1) \right)_{3} \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}''_{2(2)} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}''_{2(2')} &= \left( \frac{x_1y_2 - x_2y_1}{\sqrt{2}} \right)_{1'} \oplus \left( \frac{1}{\sqrt{2}} (x_1y_2 + x_2y_1) \right)_{3}
\end{align*}
\]
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\otimes
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= [x_1 y_1 + x_2 y_3 + x_3 y_2]_1
\]
\[
\oplus [x_3 y_3 - (x_1 y_2 + x_2 y_1)]_1' \oplus [(x_2 y_2 - (x_1 y_3 + x_3 y_1)]_1''
\]
\[
\begin{pmatrix}
2x_1 y_1 - x_2 y_3 - x_3 y_2 \\
-2x_3 y_3 - x_1 y_2 - x_2 y_1 \\
-2x_2 y_2 - x_1 y_3 - x_3 y_1
\end{pmatrix}_3
\]
\[
\oplus
\begin{pmatrix}
x_2 y_3 - x_3 y_2 \\
x_1 y_2 - x_2 y_1 \\
x_3 y_1 - x_1 y_3
\end{pmatrix}_3,
\quad (A4)
\]

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\otimes
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
-\sqrt{2} x_2 y_2 + x_1 y_1 \\
-\sqrt{2} x_1 y_3 - x_2 y_1
\end{pmatrix}_{2,2',2''}
\]
\[
\oplus
\begin{pmatrix}
\sqrt{2} x_2 y_3 + x_1 y_2 \\
-\sqrt{2} x_1 y_1 - x_2 y_2
\end{pmatrix}_{2,2'',2'}
\]
\[
\oplus
\begin{pmatrix}
-\sqrt{2} x_2 y_1 + x_1 y_3 \\
\sqrt{2} x_1 y_2 - x_2 y_3
\end{pmatrix}_{2'',2,2'}
\quad ,\quad (A5)
\]

\[
(x)_1' (1'') \otimes
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
xy_1 \\
xy_2
\end{pmatrix}_{2,2',2''}, \quad (A6)
\]

\[
(x) \otimes
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
xy_3 \\
x y_1 \\
-x y_2
\end{pmatrix}_3,
\quad (x)_1'' \otimes
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = \begin{pmatrix}
xy_2 \\
x y_3 \\
-x y_1
\end{pmatrix}_3\quad .\quad (A7)
\]
[1] P. H. Frampton and T. W. Kephart, Int. J. Mod. Phys. A 10, 4689 (1995) [arXiv:hep-ph/9409330].

[2] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu and M. Tanimoto, Prog. Theor. Phys. Suppl. 183, 1 (2010) [arXiv:1003.3552 [hep-th]].

[3] G. Altarelli and F. Feruglio, Rev. Mod. Phys. 82, 2701 (2010) doi:10.1103/RevModPhys.82.2701 [arXiv:1002.0211 [hep-ph]].

[4] S. F. King and C. Luhn, Rept. Prog. Phys. 76, 056201 (2013) doi:10.1088/0034-4885/76/5/056201 [arXiv:1301.1340 [hep-ph]].

[5] S. F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, New J. Phys. 16, 045018 (2014) doi:10.1088/1367-2630/16/4/045018 [arXiv:1402.4271 [hep-ph]].

[6] C. H. Albright, R. P. Feger and T. W. Kephart, Phys. Rev. D 93, no. 7, 075032 (2016) doi:10.1103/PhysRevD.93.075032 [arXiv:1601.07523 [hep-ph]].

[7] A. Aranda, C. D. Carone and R. F. Lebed, Phys. Lett. B 474, 170 (2000) doi:10.1016/S0370-2693(99)01497-5 [hep-ph/9910392].

[8] M. C. Chen and K. T. Mahanthappa, Phys. Lett. B 652, 34 (2007) doi:10.1016/j.physletb.2007.06.064 [arXiv:0705.0714 [hep-ph]].

[9] P. H. Frampton and T. W. Kephart, JHEP 0709, 110 (2007) [arXiv:0706.1186 [hep-ph]].

[10] P. H. Frampton, T. W. Kephart and S. Matsuzaki, Phys. Rev. D 78, 073004 (2008) doi:10.1103/PhysRevD.78.073004 [arXiv:0807.4713 [hep-ph]].

[11] P. H. Frampton, C. M. Ho, T. W. Kephart and S. Matsuzaki, Phys. Rev. D 82, 113007 (2010) doi:10.1103/PhysRevD.82.113007 [arXiv:1009.0307 [hep-ph]].

[12] A. Natale, Nucl. Phys. B 914, 201 (2017) doi:10.1016/j.nuclphysb.2016.11.006 [arXiv:1608.06999 [hep-ph]].

[13] C. D. Carone, S. Chaurasia and S. Vasquez, Phys. Rev. D 95, no. 1, 015025 (2017) doi:10.1103/PhysRevD.95.015025 [arXiv:1611.00784 [hep-ph]].

[14] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler and T. Schwetz, JHEP 1701, 087 (2017) doi:10.1007/JHEP01(2017)087 [arXiv:1611.01514 [hep-ph]].

[15] L. M. Krauss and F. Wilczek, Phys. Rev. Lett. 62, 1221 (1989) doi:10.1103/PhysRevLett.62.1221
[16] L. E. Ibanez and G. G. Ross, Nucl. Phys. B 368, 3 (1992). doi:10.1016/0550-3213(92)90195-H
[17] L. E. Ibanez and G. G. Ross, Phys. Lett. B 260, 291 (1991). doi:10.1016/0370-2693(91)91614-2
[18] C. Luhn and P. Ramond, JHEP 0807, 085 (2008) doi:10.1088/1126-6708/2008/07/085 [arXiv:0805.1736 [hep-ph]].
[19] M. Fallbacher, Nucl. Phys. B 898, 229 (2015) doi:10.1016/j.nuclphysb.2015.07.004 [arXiv:1506.03677 [hep-th]].
[20] J. Talbert, Phys. Lett. B 786, 426 (2018) doi:10.1016/j.physletb.2018.10.025 [arXiv:1804.04237 [hep-ph]].
[21] E. Ma and G. Rajasekaran, Phys. Rev. D 64, 113012 (2001) doi:10.1103/PhysRevD.64.113012 [hep-ph/0106291].
[22] K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552, 207 (2003) doi:10.1016/S0370-2693(02)03153-2 [hep-ph/0206292].
[23] B. Pontecorvo, Soviet Physics JETP. 7: 172. 1958; Z. Maki, M. Nakagawa, S. Sakata, Progress of Theoretical Physics. 28 (5): 870 (1962).
[24] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002) doi:10.1016/S0370-2693(02)01336-9 [hep-ph/0202074].
[25] M. H. Rahat, P. Ramond and B. Xu, Phys. Rev. D 98, no. 5, 055030 (2018) doi:10.1103/PhysRevD.98.055030 [arXiv:1805.10684 [hep-ph]].
[26] D. A. Eby, P. H. Frampton and S. Matsuzaki, Phys. Lett. B 671, 386 (2009) doi:10.1016/j.physletb.2008.11.074 [arXiv:0810.4899 [hep-ph]].
[27] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler and T. Schwetz, JHEP 1701, 087 (2017) doi:10.1007/JHEP01(2017)087 [arXiv:1611.01514 [hep-ph]].
[28] NuFIT 3.2 (2018), www.nu-fit.org.
[29] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018)
[30] A. Bilal, arXiv:0802.0634 [hep-th].
[31] C. Luhn, JHEP 1103, 108 (2011) doi:10.1007/JHEP03(2011)108 [arXiv:1101.2417 [hep-ph]].
[32] A. Merle and R. Zwicky, JHEP 1202, 128 (2012) doi:10.1007/JHEP02(2012)128 [arXiv:1110.4891 [hep-ph]].
[33] B. L. Rachlin and T. W. Kephart, JHEP 1708, 110 (2017) doi:10.1007/JHEP08(2017)110 [arXiv:1702.08073 [hep-ph]].
[34] J. Berger and Y. Grossman, JHEP 1002, 071 (2010) doi:10.1007/JHEP02(2010)071 [arXiv:0910.4392 [hep-ph]].

[35] Y. Grossman and W. H. Ng, Phys. Rev. D 91, no. 7, 073005 (2015) doi:10.1103/PhysRevD.91.073005 [arXiv:1404.1413 [hep-ph]].

[36] S. F. King and Y. L. Zhou, arXiv:1809.10292 [hep-ph].

[37] L. L. Everett and A. J. Stuart, Phys. Lett. B 698, 131 (2011) doi:10.1016/j.physletb.2011.02.054 [arXiv:1110.6233 [hep-ph]].

[38] C. S. Chen, T. W. Kephart and T. C. Yuan, PTEP 2013, no. 10, 103B01 (2013) doi:10.1093/ptep/ptt071 [arXiv:1403.6136 [hep-ph]].

[39] C. Luhn, S. Nasri and P. Ramond, Phys. Lett. B 652, 27 (2007) doi:10.1016/j.physletb.2007.06.059 [arXiv:0706.2341 [hep-ph]].

[40] J. Kile, M. J. Prez, P. Ramond and J. Zhang, Phys. Rev. D 90, no. 1, 013004 (2014) doi:10.1103/PhysRevD.90.013004 [arXiv:1403.6136 [hep-ph]].

[41] V. V. Vien, Mod. Phys. Lett. A 29, 28 (2014) doi:10.1142/S0217732314501399 [arXiv:1508.02585 [hep-ph]].

[42] V. V. Vien and H. N. Long, arXiv:1609.03895 [hep-ph].

[43] V. V. Vien, A. E. Creamo Hernandez and H. N. Long, Nucl. Phys. B 913, 792 (2016) doi:10.1016/j.nuclphysb.2016.10.010 [arXiv:1601.03300 [hep-ph]].

[44] P. M. Ferreira, W. Grimus, L. Lavoura and P. O. Ludl, JHEP 1209, 128 (2012) doi:10.1007/JHEP09(2012)128 [arXiv:1206.7072 [hep-ph]].

[45] G. Chen, M. J. Prez and P. Ramond, Phys. Rev. D 92, no. 7, 076006 (2015) doi:10.1103/PhysRevD.92.076006 [arXiv:1412.6107 [hep-ph]].