Lattice Erasure Codes of Low Rank with Noise Margins

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Abstract—Lattice codes of low rank are considered for an additive Gaussian noise channel with erasures. The objective is to minimize the error probability with no erasures subject to rate and power constraints while ensuring that the error probability under an allowable erasure pattern is bounded from above. Allowable erasure patterns considered here are those of fixed cardinality. Bounds on performance are derived, and several constructions are investigated for lattices in dimension four. It is shown that the problem can be viewed as a simultaneous ellipsoid packing problem, a generalization of the well known sphere packing problem.

Index terms Lattices, Erasure Codes, Rotation Matrices, Sphere Packing, Ellipsoid Packing

I. INTRODUCTION

We consider lattice codes for transmission over a noisy erasure channel. A message is mapped by an encoder/modulator to a vector $x = (x_1, x_2, \ldots, x_n) \in \Lambda$, where $\Lambda$ is a rank-$m$ lattice in $\mathbb{R}^n$, $m \leq n$. Codeword $x$ is transmitted over a channel. The output of the channel is $y = x + z$, where $z = (z_1, z_2, \ldots, z_n)$ and the $z_i$’s are iid $\mathcal{N}(0,1)$ noise random variables, independent of $x$. Components of $y$ may be erased. For $S \subseteq \{1, 2, \ldots, n\}$, let $y_S$ be the vector of un-erased symbols seen by the decoder. As an example, with $n = 4$, $S = \{2, 4\}$ and $y = (a, b, c, d)$, $y_S = (b, d)$. The decoder, which is assumed to know the locations of the erased symbols, estimates the message based on $y_S$ with a probability of error denoted $P_e(S)$. The probability of error with no erasures is denoted $P_e$. It is assumed that $S$ lies in a given collection of subsets $\mathcal{S}$.

The broad objective is to understand the tradeoff between $P_e$ and $P_e(S)$, $S \in \mathcal{S}$, subject to the power constraint $\Pi$ i.e. $E[X'X] \leq \Pi$, where $E[\cdot]$ is the expectation assuming that each codeword is used with equal probability. Here we consider the special case where $\mathcal{S}$ consists of all $k$-subsets of $\{1, 2, \ldots, n\}$ and $m = k$ (low rank lattice).

Problem setup is in Sec II. Prior work is in Sec. III. Two performance bounds are presented in Sec. IV, constructions for codes in dimension $n = 4$ are presented in Sec. V and compared to the derived bounds in Sec. VI which also describes our interpretation as a simultaneous ellipsoid packing problem. A summary is in Sec. VII.

II. PROBLEM SETUP

Relevant definitions for lattices are followed by a description of the modulation system, and observations about performance. A full rank lattice in $\mathbb{R}^m$ is the set of all integer linear combinations of $m$ linearly independent basis vectors $\{v_i, i = 1, 2, \ldots, m\}$. The $m \times m$ matrix of column vectors $V = (v_i, i = 1, 2, \ldots, m)$ is the generator matrix of the lattice, and the lattice with generator matrix $V$ is referred to as $\Lambda_V$. The Gram matrix of a lattice $\Lambda_V$ is $G(\Lambda_V) = V'V$, where $V'$ is the transpose of $V$. The determinant of $\Lambda_V$ is defined in terms of the determinant of its Gram matrix by $\det \Lambda := \det(G(\Lambda))$. We define $\rho(\Lambda)$ to be the radius of the largest inscribed sphere in a Voronoi cell of a full-rank lattice $\Lambda \subseteq \mathbb{R}^m$ and its packing density is by definition

$$\Delta(\Lambda) = \nu_m \rho_m^m / \sqrt{\det \Lambda}$$

(1)

where $\nu_m$ the volume of a unit-radius Euclidean ball in $\mathbb{R}^m$. We denote by $\Delta_m(\text{opt})$, the largest packing density that can be achieved by any lattice in $\mathbb{R}^m$. The problem of finding lattices that maximize the packing density is a well known problem in number theory and geometry, see e.g. [2], [4].

Lattice $\Lambda_V \subseteq \mathbb{R}^m$ is embedded in $\mathbb{R}^n$, $n \geq m$ as follows. Let $\Phi = (\phi_i, i = 1, 2, \ldots, m)$ with $i$th column $\phi_i$ be an $n \times m$ matrix with orthogonal column vectors. Then

$$\Lambda_{\Phi V} = \{\Phi V u, u \in \mathbb{Z}^m\}$$

(2)

is a rank-$m$ lattice in $\mathbb{R}^n$ with generator matrix $\Phi V$. We will refer to $\Lambda_V$ as the mother lattice.

Our modulation system uses a subset of $\Lambda_{\Phi V}$ with $2^n R$ points as a codebook, where $R$ is the transmission rate. Assuming that $R$ is large and for given power $\Pi$, the error probability $P_e$ is determined to first-order by $\Delta(\Lambda_{\Phi V})$ and is invariant to $\Phi$. However, $P_e(S)$ is not invariant to $\Phi$. Our objective in this paper is to optimize $\Phi$ so as to minimize $\max_{S \in \mathcal{S}} P_e(S)$. We interpret this as the problem of maximizing the minimum distance under any of the allowed erasure patterns.

1Lattice related definitions are in Sec II.

2Often the subscript $V$ will be dropped when it is obvious.
III. APPLICATIONS AND PRIOR WORK

Our initial motivation came from the cross-layer coding study in [3]. Error and erasure coding finds applications in magnetic and optical recording systems [5], and more recently in efficient uncoordinated medium access in networks of sensors [9]. Prior work on erasure coding for continuous alphabet channels includes [10]. More recent examples are [8], and a source coding problem [6], where the squared error distortion measure is used. Lattice rotations have been considered for fading channels in [1]. The common feature with prior work is the search for an optimal rotation matrix. However, unlike previous approaches our formulation highlights the importance of the packing density and leads to the introduction of the packing volume contraction ratio as a measure of performance. We also provide a novel interpretation as a simultaneous ellipsoid packing problem, a generalization of the classical sphere packing problem [2].

IV. BOUNDS

For $S \subset \{1, 2, \ldots, n\}$, $|S| = k$ let $\Lambda_S$ be the (child) lattice obtained by retaining only those coordinates that are in $S$ or equivalently $\Lambda_S$ is the projection of $\Lambda$ into the subspace $C_S := \text{Span}(e_i, i \in S)$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^t$ is the $i$th unit vector in $\mathbb{R}^n$. For any $k$-subset $S \subset \{1, 2, \ldots, n\}$, we denote by $\Phi_S$ the $k \times m$ submatrix obtained by extracting from $\Phi$ the $k$ rows identified by $S$. The generator matrix for $\Lambda_S$ is $\Phi_S \Lambda$ and its Gram matrix $G(\Lambda_S) = V^t \Phi_S \Phi_S V$.

Remark 1. Even though the problem as formulated is for a rank $m$ lattice in $\mathbb{R}^n$ with $n - k$ erasures, $k \leq m \leq n$, henceforth we will assume that $m = k$.

Remark 2. The modulation codebook is a subset of $\Lambda_{SV}$. Assume that rate $R$ and power $P$ are fixed. The error probability $P_e$ is determined by the packing density of $\Lambda_V$, and is independent of $\Phi$. On the other hand, $P_e(S)$ is determined by $\Phi$, more specifically by the 'shrinkage' in the minimum distance of lattice $\Lambda_S$, relative to that of $\Lambda_V$. Thus the crux of the problem is to find $\Phi$ which is simultaneously good for all erasure patterns. This is intuitively clear, but mathematical justification must await lengthier exposition.

In order to capture the notion of 'shrinkage' mathematically, we define the packing volume contraction ratio by

$$\beta_S = (\rho(\Lambda_S)/\rho(\Lambda_V))^k. \quad (3)$$

Further, let $\rho_{\text{min}} = \min_S \rho(\Lambda_S)$ and $\beta_{\text{min}} = \min_S \beta_S$.

A. Determinant Upper Bound

We will use symbols $\bar{x}$, $x^\#$ to denote the arithmetic mean and geometric mean, respectively, of the real numbers $x_i$ over some index set. The following theorem develops one of two bounds presented in this paper.

Theorem 1. (Determinant Bound) Given a mother lattice $\Lambda_V$ and orthonormal basis $\Phi$, let $\beta^\#$ and $\Delta^\#$ be respectively, the geometric mean of the volume contraction ratios and packing densities of the child lattices $\Lambda_S$, taken over all $k$-subsets of $\{1, 2, \ldots, n\}$. Then

$$(\beta^\# \Delta(\Lambda_V))^2 \leq \frac{(\Delta^\#)^2}{\binom{n}{k}}. \quad (4)$$

Equality holds if and only if all child lattices have equal determinants.

Proof. The packing densities of the mother lattice $\Lambda_V$ and child lattice $\Lambda_S$ are related by the following identity

$$\Delta^2(\Lambda_V) \beta_S^2 = \Delta^2(\Lambda_S) \frac{\det \Lambda_S}{\det \Lambda_V}. \quad (5)$$

Compute the geometric mean of both sides over the collection of $k$-subsets $S$ to get

$$\Delta^2(\Lambda_V)(\beta^\#)^2 = (\Delta^\#)^2 \left( \prod_{S=1}^n \frac{\det \Lambda_S}{\det \Lambda_V} \right) \binom{n}{k}. \quad (6)$$

From the arithmetic-geometric mean inequality it follows that

$$\Delta^2(\Lambda_V)(\beta^\#)^2 \leq (\Delta^\#)^2 \frac{1}{\binom{n}{k}} \left( \sum_{S} \det \Lambda_S \frac{\det \Lambda_S}{\det \Lambda_V} \right) \binom{n}{k} \quad (7)$$

and equality holds if and only if $\det \Lambda_S$ is a constant with respect to $S$. However

$$\sum_{S} \det \Lambda_S = \sum_{S} \det(G(\Lambda_S)) \quad \sum_{S} \det((\Phi_S \Lambda)^t) \det(\Phi_S \Lambda) \quad (a) \quad \det((\Phi_S \Lambda)^t) = \det(\Lambda_S) \quad \det \Lambda_V, \quad (8)$$

where in (a) we have used the Cauchy-Binet formula, see e.g. [7]. The remainder of the proof follows directly. □

The following corollary is immediate.

Corollary 1. Given mother lattice $\Lambda_V$ and orthonormal basis $\Phi$, let $\beta_{\text{min}}$ be the minimum volume contraction ratio of the child lattices $\Lambda_S$, taken over all $k$-subsets of $\{1, 2, \ldots, n\}$, and $\Delta^\#$ the geometric mean of the packing densities. Then

$$(\beta_{\text{min}} \Delta(\Lambda_V))^2 \leq \frac{(\Delta^\#)^2}{\binom{n}{k}} \leq \frac{\Delta_k(\text{opt})^2}{\binom{n}{k}}. \quad (9)$$

Equality holds in the left inequality iff all the contraction ratios are equal and all the child lattices have equal determinants. Equality holds in the right inequality iff all child lattices achieve the optimal packing density in dimension $k$. 

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B. Trace Upper Bound

Theorem 2. (Trace Bound) For an \((n, k)\) code, the volume contraction ratio is bounded as

\[
\beta_{min}^{4/k} \leq \beta_S \leq \frac{k}{n}. \tag{10}
\]

Equality holds if the shortest vector of each lattice \(\Lambda_S\) is the image of a single shortest vector in \(\Lambda_V\).

Proof. Upon summing over all \(k\)-subsets \(S\) we obtain

\[
\sum_S G(\Lambda_S) = \sum_S V^t \Phi_S^t \Phi_S V
\]

holds if follows immediately.

\[
= V^t \left( \sum_S \Phi_S^t \Phi_S \right) V
\]

\[
= V^t \left( \frac{n-1}{-1} \right) \Phi \Phi V
\]

By definition the smallest packing radius of any child lattice, \(\rho_{min}\), satisfies

\[
\rho_{min}^2 = \rho^2(\Lambda_S) \leq (1/2)u^t G(\Lambda_S) u \tag{12}
\]

for any non-zero \(u \in \mathbb{Z}^k\) and any \(k\)-subset \(S\). Upon averaging over subsets \(S\) we obtain the upper bound

\[
\frac{\rho^2(\Lambda_S)}{2}\leq \sum_S u^t G(\Lambda_S) u \]

\[
= \frac{1}{2(n-k)} u^t \sum_S G(\Lambda_S) u
\]

\[
= \frac{(n-1)}{2(n-k)} u^t G(\Lambda_V) u. \tag{13}
\]

Equality holds if \(\Phi_S^t \Phi_S V u\) is the shortest vector in \(\Lambda_S\) for all \(S\). Thus

\[
\rho^2(\Lambda_S) \leq \frac{k}{n} \rho^2(\Lambda_V) \tag{14}
\]

and (10) follows immediately.

V. Analysis of Some \((4, k)\) Codes

We construct \(\Phi\) for \(n = 4\) for various values of \(k\) and various mother lattices \(\Lambda_V\).

A. \((4, 1)\)

Let \(\Phi^t = (\begin{array}{c} a \\ a \\ a \\ a \end{array})\), \(a = 1/2\). We obtain \(\beta_{min}^2 = 1/4\). The trace and determinant upper bounds yield \(\beta^2 \leq 1/4\). Hence this construction is optimal.

B. \((4, 2)\)

With \(\Lambda_V = \mathbb{Z}^2\), \(a = 1/\sqrt{3}\) and

\[
\Phi^t = \left( \begin{array}{cccc} a & a & a & 0 \\ a & -a & 0 & 0 \end{array} \right)
\]

we obtain six child lattices with Gram matrices

\[
\begin{pmatrix} 2a^2 & 0 & a^2 & 0 \\ 0 & 2a^2 & a^2 & 0 \\ a^2 & 2a^2 & a^2 & 0 \\ 0 & 0 & a^2 & a^2 \end{pmatrix}, \begin{pmatrix} a^2 & -a^2 & 2a^2 & 0 \\ -a^2 & a^2 & 0 & 2a^2 \\ 2a^2 & -a^2 & a^2 & 0 \\ 0 & 2a^2 & a^2 & a^2 \end{pmatrix} \tag{15}
\]

All six lattices are similar to \(\mathbb{Z}^2\). The first one has shortest vector of square length \(2a^2\) and all the others have square length \(a^2\). This code achieves \(\beta_{min}^2 = \beta_{min} = 1/3\), \(\beta^2 = 21/9/3 = 0.374\).

The trace upper bound is \(\beta_{min} \leq \beta_S \leq 1/2\), regardless of the mother lattice, while the determinant upper bound depends on the mother lattice, and is \(\sqrt{2}/3 = 0.4714\) for \(\Lambda_V = \mathbb{Z}^2\) and \(1/\sqrt{6} = 0.4082\) for \(\Lambda_V = A_2\) (hexagonal lattice). Thus the determinant bound is tighter than the trace bound but greater than \(1/3\). This construction does not meet the trace bound or the determinant bound with equality. However with \(\Lambda_V = \mathbb{Z}^2\), the following theorem shows this to be the best \(\beta_{min}\) possible. A limited computer-based search has failed to reveal any improvements in \(\beta^2\), but analytic verification is as yet missing.

Theorem 3. Let \(\Phi\) be a \(4 \times 2\) matrix with orthonormal columns, so that \(\Phi^t \Phi = I\). Let \(\Lambda_V = \mathbb{Z}^2\). Let \(\Lambda_S = \Phi_S \Lambda_V\) where \(S\) is a \(2\)-subset of \(\{1, 2, 3, 4\}\). Let \(r_S\) be the length of the shortest non-zero vector in \(\Lambda_S\). Then \(r_S^2 \leq 1/3\) for at least one of the six \(2\)-subsets of \(\{1, 2, 3, 4\}\).

Proof. For this proof we will write

\[
\Phi^t = \left( \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{array} \right) \tag{16}
\]

Our proof is by contradiction. Suppose that the shortest non-zero vector in all \(\Lambda_S\) has square length \(r^2 > 1/3\). Now \(x^2_1 < r^2/2\), in at most one position \(i\), else the shortest would be smaller than \(r^2\). The same is true for \(y^2_i = (x_i - y_i)^2 = (x_i + y_i)^2\). Hence there exists one position \(i\), without loss of generality \(i = 1\), such that \(x^2_1 \geq r^2/2\) and \(y^2_1 \geq r^2/2\) and \((x_1 - y_1)^2 \geq r^2/2\). It follows that (i) \(x_1, y_1\) must be of opposite sign and (ii) \((x_1 + y_1)^2 < r^2/2\). (i) is true because if \(x_1, y_1\) are of the same sign and \((x_1 - y_1)^2 \geq r^2/2\) then \(|y_1| \geq \sqrt{2r}\) or \(|x_1| \geq \sqrt{2r}\). Assuming \(x^2_1 \geq \sqrt{2r}\), we have \(r^2 \leq x^2_1 + x^2_4 = 1 - x^2_2 - x^2_3 \leq 1 - 2r^2\) which contradicts the hypothesis that \(r^2 > 1/3\). (ii) is true for if not \(|x_1| \geq \sqrt{2r}\) or \(|y_1| \geq \sqrt{2r}\), and by the same argument as in (i) \(r^2\) cannot exceed \(1/3\).

Since \((x_1 + y_1)^2 < r^2/2\), it follows that \((x_1 + y_1)^2 \geq r^2/2\) for positions \(i = 2, 3, 4\). In at least one of these positions, say \(i = 2\), \(x^2_2 \geq r^2/2\) and \(y^2_2 \geq r^2/2\). Now \(x_2, y_2\) must be of the same sign and \((x_2 - y_2)^2 < r^2/2\), by a proof similar to that used before. Thus for \(i = 3, 4\), \((x_i + y_i)^2 \geq r^2/2\) and \((x_i - y_i)^2 \geq r^2/2\). Again for \(i = 3, 4\), both \(x^2_i \geq r^2/2\) and \(y^2_i \geq r^2/2\) cannot hold for the same \(i\), hence, either \(x^2_3 < r^2/2\), \(y^2_3 \geq r^2/2\) and \(x^2_4 \geq r^2/2\), \(y^2_4 < r^2/2\) or \(x^2_4 < r^2/2\), \(y^2_4 \geq r^2/2\) and
We assume the first case. The proof for the other case is similar.

We have already proved that \(x_2\) and \(y_2\) are of the same sign. Assume they are both positive (if not reverse signs of all elements of \(\Phi\)). Further, assume that \(y_2 > x_2\) and consider positions \(i = 2, 3\) (if \(x_2 > y_2\), then the same proof applies but for positions \(i = 2, 4\)). We now break up the proof into two cases:

**Case 1:** \((y_3 \text{ and } x_3 \text{ of the same sign})\): Either (a) \((y_3 - x_3)^2 \geq r^2\) or (b) \((y_3 - x_3)^2 < r^2\). If (a) then \(|y_3| - |x_3| \geq r\) and \(y_3^2 + y_3^2 \geq x_3^2 + x_3^2 + r^2 \geq 2r^2\). Thus \(r^2 \leq y_3^2 + y_3^2 = 1 - (y_3^2 + y_3^2) \leq 1 - 2r^2\), hence \(r^2 \leq 1/3\).

If (b) then let \((y_3 - x_3)^2 = r^2 - \varepsilon^2\) (\(0 < \varepsilon^2 \leq r^2/2\)). Thus \(|y_3| - |x_3| = \sqrt{r^2 - \varepsilon^2}\) and it follows that \(y_3^2 \geq x_3^2 + r^2 - \varepsilon^2\). Since \((y_3 - x_3)^2 + (y_3 - x_3)^2 \geq r^2\) it follows that \((y_3 - x_3)^2 \geq r^2\) and that \(|y_3| \geq |x_3| + \varepsilon\) and thus \(y_3^2 \geq x_3^2 + r^2 - \varepsilon^2\). Thus \(y_3^2 + y_3^2 \geq x_3^2 + x_3^2 + r^2 - \varepsilon^2 = x_3^2 + x_3^2 + r^2 \geq 2r^2\). But then \(r^2 \leq y_3^2 + y_3^2 = 1 - (y_3^2 + y_3^2) \leq 1 - 2r^2\) and it follows that \(r^2 \leq 1/3\).

**Case 2:** \((y_3 \text{ and } x_3 \text{ are of opposite signs})\): Either (a) \((y_3 + x_3)^2 \geq r^2\) or (b) \((y_3 + x_3)^2 < r^2\). If (a) then \(|y_3| + |x_3| \geq r\) and due to opposite signs \(|y_3| - |x_3| \geq r\) from which \(y_3^2 \geq x_3^2 + r^2\). This implies \(r^2 \leq 1/3\).

If (b) then let \((y_3 + x_3)^2 = r^2 - \varepsilon^2\). Since \(y_3 + x_3 \geq 0\) and \(y_3 - x_3 \leq 0\) we obtain child lattices with Gram matrices given in terms of \(d = 2c^2 + b^2, c = 2b^2 + c^2, f = -c^2 - 2bc, g = c^2 + 2bc, h = 3bc, i = -b^2 - 2bc\).

In each child lattice has determinant 4 and shortest vector with square length \(d = (9/4 - 1/\sqrt{2}) = 1.5429\) and packing density \(\Delta = \pi e^{2/2}/12 = 0.5017\). This results in \(2^{2/k} = d/3 = 0.5143\).

**VI. Numerical Results**

Numerical results are presented for \((4, k)\) codes, \(k = 2, 3\) in Fig. 1. Here \(\beta_{2/k}\) is plotted as a function of the packing density of the mother lattice. The determinant bound has been obtained using optimal and the cubic child lattices. We have also plotted the trace bound.

Observe that in the \((n, k) = (4, 3)\) case there is a significant gap between the best possible construction and the upper bounds. In the \((4, 3)\) case performance
close to the determinant bound is achieved by setting the mother lattice to be the cubic lattice. Also with the cubic lattice as the mother lattice, since the trace bound is lower than the determinant bound, this is proof that it is impossible to simultaneously achieve the packing density of $D_3$ when the mother lattice is the cubic lattice. Geometrically, the ability to decode correctly, post-erasure, with iid Gaussian noise is determined by the columns of $\Phi$, a noise sphere is transformed into an ellipsoid. To see this consider the noise sphere $\|z\|^2 \leq r^2$ in subspace $C_S$. Setting $z = \Phi_S y$, this leads to the noise ellipsoid $B_\Sigma(r) = \{y : \|\Phi_S y\|^2 \leq r^2\}$. The packing of the noise ellipsoids $B_\Sigma(\rho_{\min})$, by the mother lattice $\Lambda_V$ is shown in the six panels in Fig. 2(a), one for each subspace. Fig. 2(b) shows the star body $B(2\rho_{\min}) = \bigcup_{S \subseteq S} B_\Sigma(2\rho_{\min})$, which is the union of the six ellipses, two of which are circles. Also shown are points of the lattice $\Lambda_V$, which in this case is an admissible lattice for $B(2\rho_{\min})$. Observe that $\mathbb{Z}^2$ is simultaneously good as a packing for all six erasure configurations, i.e. for each of the bodies $B_\Sigma(\rho_{\min})$. Also, $\Lambda_V$ is simultaneously critical for five of the six bodies $B_\Sigma(2\rho_{\min})$ (notice that one circle does not touch any of the lattice points).

The reader is referred to [4] for definitions of star bodies and packing, admissible and critical lattices.

Fig. 1. Bounds on $\beta^{2/k}$ and values obtained from the construction.

Fig. 2. Packings (a) and the star body (b) derived from noise spheres in each of the six subspaces for the $(4, 2)$ code with $\Lambda_V = \mathbb{Z}^2$.

VII. Summary

The problem of code design for the erasure channel with additive noise has been considered. Constructions and performance bounds are provided along with an interpretation as a simultaneous ellipsoid packing problem.

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