EXAMPLE OF A NON-LOG-CONCAVE
DUISTERMAAT-HECKMAN MEASURE

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Abstract. We construct a compact symplectic manifold with a Hamiltonian circle action for which the Duistermaat-Heckman function is not log-concave.

1. Introduction

Let $T$ be a torus and $\mathfrak{t}$ its Lie algebra. Let $(M, \omega)$ be a symplectic manifold with an action of $T$ and with a moment map

$$\Phi : M \to \mathfrak{t}^*.$$

Recall, this means that for every $\xi \in \mathfrak{t}$, if $\xi_M$ is the corresponding vector field on $M$, $\iota(\xi_M)\omega = -d <\Phi, \xi>.$

**Liouville measure** on $M$ associates to an open set $U$ the measure $\int_U \omega^n$ where $n$ is half the dimension of the manifold and where we integrate with respect to the symplectic orientation. The **Duistermaat-Heckman measure** [DH] on $\mathfrak{t}^*$ is the push-forward of Liouville measure via the moment map $\Phi$. If $T$ acts effectively, the Duistermaat-Heckman measure is absolutely continuous with respect to Lebesgue measure, and the density function on $\mathfrak{t}^*$ is called the **Duistermaat-Heckman function**.

If $M$ is compact, the image of $\Phi$ is a convex polytope [GS, At]. If, in addition, the dimension of $T$ is half the dimension of $M$ and $T$ acts effectively, the Duistermaat-Heckman function is equal to one on the convex polytope $\Phi(M)$ and zero outside [De]. This function is log-concave, i.e., its logarithm is concave. Moreover, if we restrict this action to a subgroup $H$ of $T$, the moment map for $H$ is the composition of the moment map for $T$ with the natural linear projection $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$. The Duistermaat-Heckman function for $H$ is the function $x \mapsto \text{vol}(\pi^{-1}(x) \cap \Phi(M))$ which associates to every point $x$ in $\mathfrak{h}^*$ the volume of the corresponding “slice” of the convex polytope $\Phi(M)$. This function is again log-concave [P7, Theorem 6].

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It was conjectured [Gi, Kn] that for any Hamiltonian torus action on a compact manifold, the Duistermaat-Heckman function is log-concave. This was proved for circle actions on four manifolds in [Ka, Remark 2.19], for coadjoint orbits in classical groups in [Ok], and for arbitrary Kähler manifolds in [Gr]. In this note we construct a counterexample to the conjecture; we construct a Hamiltonian circle action on a compact symplectic manifold for which Duistermaat-Heckman function is not log-concave. This construction came from investigating an example of Dusa McDuff of a 6-manifold with a circle valued moment map [MD]. I use her notation wherever possible.

Our conventions regarding factors of $2\pi$ etc. are irrelevant and will not be made explicit.

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## 2. The construction

Let $T^4$ be the four dimensional torus with periodic coordinates $x_i$, $1 \leq i \leq 4$, and let $\sigma_{ij} = dx_i \wedge dx_j$ and $\sigma_{1234} = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. Let $L$ be a complex Hermitian line bundle over $T^4$ with Chern class $[-\sigma_{14} - \sigma_{32}]$. Let $\Theta$ be a connection one-form with curvature $-\sigma_{14} - \sigma_{32}$. This means that $\Theta$ is defined on $L$ outside the zero section, that the restriction of $\Theta$ to a fiber of $L$ is $d\theta$ in polar coordinates on the fiber, and that $d\Theta$ is the pullback of $-\sigma_{14} - \sigma_{32}$ via the bundle map $L \to T^4$. Denote by the same letters $\sigma_{ij}, \sigma_{1234}$ the pullbacks of these forms to $L$. Let the function $\Phi : L \to \mathbb{R}$ be the norm squared, with respect to the fiberwise Hermitian metric on $L$. Consider the two-form

$$\omega = \sigma_{12} + \sigma_{34} + (2 - \Phi)\sigma_{14} + (3 - \Phi)\sigma_{32} + d\Phi \wedge \Theta \quad (1)$$

on $L$ minus its zero section. It is easy to check that $\omega$ is closed and that its top power is

$$\omega^3 = 6(1 + (2 - \Phi)(3 - \Phi))\sigma_{1234} \wedge d\Phi \wedge \Theta.$$

Since $\sigma_{1234} \wedge d\Phi \wedge \Theta \neq 0$ and since the function $(1 + (2 - s)(3 - s))$ is always positive, $\omega$ is symplectic.

The circle group acts on $L$ by fiberwise rotation. Let $\xi$ be the generating vector field. From [2] it is clear that $\iota(\xi)\omega = -d\Phi$, so $\Phi$ is a moment map for the circle action. The Duistermaat-Heckman function is a constant positive multiple of the function

$$\rho(s) = 1 + (2 - s)(3 - s). \quad (2)$$
This function decreases for $0 < s < 2.5$ and increases for $2.5 < s < \infty$, so it is not log-concave.

To make a compact example out of our noncompact one, we perform “Lerman cutting” \cite{Le}: choose any two numbers, $0 < A < 2.5$ and $2.5 < B < \infty$. “Lerman cutting” produces a compact symplectic manifold $(M, \omega)$ with a circle action and a moment map $\Phi : M \to [A, B]$ such that the preimages in $M$ and in $L$ of the open interval $(A, B)$ are equivariantly symplectomorphic. Consequently, the Duistermaat-Heckman functions are the same: for the compact manifold $M$ we get the function (2) restricted to the interval $A \leq s \leq B$, and this function is not log-concave.

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