Reflected Backward Stochastic Difference Equations with Finite State and their applications

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Abstract: In this paper, we first establish the reflected backward stochastic difference equations with finite state (FS-RBSDEs for short). Then we explore the Existence and Uniqueness Theorem as well as the Comparison Theorem by “one step” method. The connections between FS-RBSDEs and optimal stopping time problems are investigated and we also show that the optimal stopping problems with multiple priors under Knightian uncertainty is a special case of our FS-RBSDEs. As a byproduct we develop the general theory of g-martingales in discrete time with finite state including Doob-Mayer Decomposition Theorem and Optional Sampling Theorem. Finally, we consider the pricing models of American Option in both complete and incomplete markets.

Keywords: BSDE, RBSDE, Comparison Theorem, g-martingale, multiple prior martingale, Knightian uncertainty.

1 Introduction

The theory of backward stochastic differential equations (BSDEs) was first introduced by Pardoux and Peng [16]. Over the past twenty years, BSDEs are widely used in mathematical finance, stochastic control and other fields. Some people had studied discrete time approximation and Monte Carlo simulation of BSDEs, such as Briand et al. [2], Ma et al. [14], Bouchard and Touzi [1]. By analogy with the theory of BSDEs, Cohen and
Elliott [8] considered the backward stochastic difference equations related to discrete time and finite state processes as entities in their own right, not as approximations to the continuous case. In their paper, they obtained the corresponding Existence and Uniqueness Theorem of the solution and Comparison Theorem as well as other interesting properties. For deeper discussion, the readers may refer to [6] and [7].

The general theory of reflected backward stochastic differential equations (RBSDEs for short) was studied by El Karoui et al. [9]. They considered the case where the solution is forced to stay above a given stochastic process (called the obstacle) and introduced an increasing process which pushes the solution to remain above the obstacle. This important theory could be applied to the optimal stopping problem and some problems in finance markets [10]. So it is interesting to explore the reflected BSDEs in the framework of [8], as well as some applications in optimal stopping time problems in discrete time such as optimal stopping under Knightian uncertainty [21], pricing of American contingent claims and so on.

This paper is organized as follows. In section 2, the finite state reflected backward stochastic difference equations (FS-RBSDEs for short) are defined. We show that the Skorohod Lemma holds in this case. Moreover, the solution of the FS-RBSDEs corresponding to the value function of an optimal stopping time problem is proved. In section 3, we prove the corresponding Comparison Theorem. It yields the Existence and Uniqueness Theorem through the Comparison Theorem in [8] in section 4. In section 5, we show the solution of the FS-RBSDEs where \( f \) is a concave (or convex) function is the value function of a mixed optimal stopping stochastic control problem. In section 6, in order to study the optimal stopping problems in the framework of g-martingales, we first study the g-martingale, Doob-Mayer Decomposition Theorem and Optional Sampling Theorem, which were investigated in continuous time in [4], [17], [18]. We also give the connections between the FS-BSDEs and the multiple prior martingale under Knightian uncertainty. Consequently, the optimal stopping problem with multiple priors can be solved by computing a special kind of FS-RBSDEs. Finally we apply the above theory to study the pricing models of American option in complete and incomplete markets in section 7.
2 The Definition of FS-RBSDEs and the corresponding Skorohod Lemma

As in [8], we will consider some underlying discrete time, finite state process $X$ which can be always assumed to be essentially bounded and take values in the standard basis vector of $\mathbb{R}^m$, where $m$ is the number of states of the process. That is, for each $t \in \mathcal{N} = \{0, 1, 2, \ldots \}$, $X_t \in \{e_1, \ldots, e_m\}$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots 0)^t \in \mathbb{R}^m$, and $(\cdot)^*$ denotes the vector transposition.

Denote $\mathcal{F}_t$ is the completion of the $\sigma-$algebra generated by the process $X$ up to time $t$. Then we consider this problem in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$. We assume that $\mathcal{F}_0$ is the trivial $\sigma-$field and that $\mathcal{F}$ is the $\sigma-$field generated by the union of all $\mathcal{F}_t$, $t \in \mathcal{N}$.

Firstly let us introduce some useful notations. For each $t, t_0, t_1 \in \mathcal{N} = \{0, 1, 2, \ldots \}$,
\[
S^t = \{ \xi \text{ is an } \mathcal{F}_t \text{-adapted } \mathbb{R}\text{-valued r.v. and essentially bounded } \}; \\
L^n[t_0, t_1] = \{ \varphi_s, t_0 \leq s \leq t_1 \} \text{ is an } \mathcal{F}_s \text{-adapted } \mathbb{R}^n\text{-valued process and essentially bounded, } n = 1, 2, \ldots \}. \text{ For abbreviation, we let } \mathcal{L}^n := \mathcal{L}^n[0, T].
\]

Define $M_t := X_t - \mathbb{E}[X_t | \mathcal{F}_{t-1}]$ and $M_0 = 0$. $(M_t)$ is called the martingale difference process. Moreover, $X_t$ is essentially bounded which deduces $M_t$ is also essentially bounded.

We then have a representation of the process $X$ in the following form
\[X_t = \mathbb{E}[X_t | \mathcal{F}_{t-1}] + M_t \in \mathbb{R}^m.\]

The general form of backward stochastic difference equation was defined in [8], that is
\[Y_t = \xi + \sum_{t \leq u < T} f(u, Y_u, Z_u) - \sum_{t \leq u < T} Z_u M_{u+1}, \quad 0 \leq t \leq T. \quad (2.1)\]

where $T$ is a deterministic terminal time, $\xi \in \mathcal{S}^T$, $Y$ is an $\mathbb{R}$-valued stochastic process, $Z$ is an $\mathbb{R}^m$-valued stochastic process, the map $f : \Omega \times \{0, 1, \ldots, T\} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \in \mathcal{L}^1$.

From [8], we know that if $f$ satisfies the following two assumptions:
(A1) For any $Y$, if $Z^1 \sim_M Z^2$, then $F(\omega, t, Y_t, Z^1_t) = F(\omega, t, Y_t, Z^2_t)$ $P$-a.s. for all $t$;
(A2) For any $Z$, for all $t$, the map $Y_t \mapsto Y_t - F(\omega, t, Y_t, Z_t)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$ $P$-a.s.
Then for any \( \xi \in \mathcal{S}^T \), the BSDE (2.1) has a solution \((Y_t, Z_t)\). Moreover, this solution is unique up to indistinguishability for \( Y \) and \( Z \). For reader’s convenience, we recall the definition of \( Z^1 \sim_M Z^2 \). We write \( Z^1 \sim_M Z^2 \) if any case holds as follows:

\[
(i) \| Z^1 - Z^2 \|_M^2 = 0, \quad \text{where}
\]
\[
\| Z \|_M^2 := ETr \left[ \sum_{0 \leq u < T} Z_u \cdot E[M_{u+1} M_{u+1}^* | \mathcal{F}_u] \cdot Z_u^* \right] = \sum_{0 \leq u < T} T r E[(Z_u M_{u+1})(Z_u M_{u+1})^* | \mathcal{F}_u],
\]

\[
(ii) ETr[(Z^1_u - Z^2_u) M_{u+1} M_{u+1}^* (Z^1_u - Z^2_u)^* | \mathcal{F}_u] = 0, \quad \text{for all } u \in \{0, 1, \ldots, T-1\},
\]

\[
(iii) Z^1_t M_{u+1} = Z^2_t M_{u+1}, \quad \text{P-a.s. for all } u \in \{0, 1, \ldots, T-1\},
\]

\[
(iv) \sum_{0 \leq u < t} Z^1_u M_{u+1} = \sum_{0 \leq u < t} Z^2_u M_{u+1}, \quad \text{P-a.s. for all } t \in \{1, \ldots, T\}.
\]

Similarly we shall consider a FS-RBSDE based on \( M \). It is necessary to introduce some assumptions in advance.

Assumptions:

(H1) \( \xi \in \mathcal{S}^T \);

(H2) The map \( f : \Omega \times \{0, 1, \ldots, T\} \times R \times R^m \rightarrow R \) satisfies that for all \( y, z \in R \times R^m \), \( f(\cdot, y, z) \in \mathcal{L}^1 \);

(H3) The “obstacle” \( \{S_t, 0 \leq t \leq T\} \) is an \( \mathcal{F}_t \)-adapted real-valued process satisfying \( \{S_t^+\} \) is essentially bounded and \( S_T \leq \xi \) P-a.s..

**Definition 2.1.** A triple \((\xi, f, S)\) is called a standard data if it satisfies the above Assumptions (H1)-(H3).

**Definition 2.2.** A solution of our FS-RBSDE with terminal time \( T \) associated with standard data \((\xi, f, S)\) is a triple \((Y_t, Z_t, K_t), 0 \leq t \leq T\) of \( \mathcal{F}_t \) progressively adapted processes taking values in \( R \times R^m \times R \) satisfying

\[
(i) Y_t \in \mathcal{L}^1, Z_t \in \mathcal{L}^m, K_T \in \mathcal{S}^T;
\]

\[
(ii) Y_t = \xi + \sum_{t \leq u < T} f(u, Y_u, Z_u) + K_T - K_t - \sum_{t \leq u < T} Z_u^* M_{u+1}, \quad 0 \leq t \leq T;
\]

\[
(iii) Y_t \geq S_t \text{ P-a.s., } 0 \leq t \leq T;
\]

\[
(iv) \{K_t\} \text{ is increasing s.t. } K_0 = 0 \text{ and } \sum_{0 \leq t \leq T} (Y_t - S_t)(K_{t+1} - K_t) = 0.
\]
Intuitively, $K_{t+1} - K_t$ represents the amount of “push upwards” that we add to $-(Y_{t+1} - Y_t)$. Condition (iv) says the push is minimal, in the sense that we push only when the constraint is saturated, i.e. when $Y_t = S_t$. Notice that in a deterministic and continuous framework, this corresponds to the Skorohod problem [20]. Now we shall consider the classical Skorohod problem under the discrete time and finite state framework.

**Lemma 2.3.** Let $y(t)$ be a real-valued function on \{0, 1, ..., $T$\} such that $y(0) \geq 0$, there exists a unique pair of functions $(v(t), g(t))$ on \{0, 1, ..., $T$\} such that

- (i) $v(t) = y(t) + g(t)$;
- (ii) $v(t)$ is non-negative;
- (iii) $g(t)$ is increasing, vanishing at zero and

$$\sum_{1 \leq t \leq T} v(t)(g(t) - g(t-1)) = 0.$$

The function $g(t)$ is moreover given by

$$g(t) = \sup_{s \leq t} (-y(s) \lor 0).$$

**Proof.** We first claim that the pair $(g(t), v(t))$ defined by

$$g(t) = \sup_{s \leq t} (-y(s) \lor 0), v(t) = y(t) + g(t).$$

satisfies properties (i) through (iii).

To prove the uniqueness of the pair $(g(t), v(t))$, we suppose that $(g(t)', v(t)')$ is another pair which satisfies (i) through (iii). Then $v(t) - v(t)' = g(t) - g(t)'$ and note that $g(0) = g(0)' = 0$, consequently $v(0) - v(0)' = 0$, so we
Proposition 2.4. Let \( \{Y_t, Z_t, K_t\}, 0 \leq t \leq T \) be a solution of the above FS-RBSDE mentioned in Definition 2.2. Then for each \( t \in \{0, 1, ..., T\} \),

\[
K_T - K_t = \sup_{t \leq u \leq T} (\xi + \sum_{u \leq s < T} f(s, Y_s, Z_s) - \sum_{u \leq s < T} Z_s^* M_{s+1} - S_u^-).
\]

Proof. Set

\[
y_t = \xi + \sum_{T-t \leq s < T} f(s, Y_s, Z_s) - \sum_{T-t \leq s < T} Z_s^* M_{s+1} - S_{T-t}.
\]
Then $y_0 = \xi - S_T \geq 0$.

Notice that $Y_{T-t}(\omega) - S_{T-t}(\omega) = y_t + K_T(\omega) - K_{T-t}(\omega)$, so we know $(Y_{T-t}(\omega) - S_{T-t}(\omega), K_T(\omega) - K_{T-t}(\omega))$, $0 \leq t \leq T$ is the unique solution of the above Skorohod problem by applying Lemma 2.3, and

$$K_T - K_{T-t} = \sup_{0 \leq u \leq t} (\xi + \sum_{T-u \leq s < T} f(s, Y_s, Z_s) - \sum_{T-u \leq s < T} Z_s^* M_{s+1} - S_{T-u})^-.$$  

The result then follows immediately. $\square$

In the discrete time and finite state framework, we shall show that the solution $Y_t$ of the FS-RBSDE corresponds to the value of an optimal stopping time problem in the following proposition.

**Proposition 2.5.** Let \{(Y_t, Z_t, K_t), 0 \leq t \leq T\} be a solution of the above FS-RBSDE mentioned in Definition (2.2). Then for each $t \in \{0, 1, ..., T\}$,

$$Y_t = \text{ess sup} \ E\left[ \sum_{t \leq s < \theta} f(s, Y_s, Z_s) + S_{\theta 1_{\{\theta < T\}}} + \xi 1_{\{\theta = T\}} | F_t \right],$$

where $\mathcal{J}$ is the set of all stopping times dominated by $T$, and $\mathcal{J}_t = \{\theta \in \mathcal{J}; t \leq \theta \leq T\}$.

**Proof.** Choosing a stopping time $\theta \in \mathcal{J}_t$, then from equation (2.2) we have

$$Y_t = Y_{\theta} + \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + K_\theta - K_t - \sum_{t \leq u < \theta} Z_u^* M_{u+1}, \quad 0 \leq t \leq T.$$  

taking the conditional expectation follows that

$$Y_t = E[Y_{\theta} + \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + K_\theta - K_t | F_t] \geq E[\sum_{t \leq u < \theta} f(u, Y_u, Z_u) + S_{\theta 1_{\{\theta < T\}}} + \xi 1_{\{\theta = T\}} | F_t].$$

In order to get the reversed inequality, we shall define

$$D_t = \inf\{t \leq u \leq T; Y_u = S_u\}$$

and $D_t = T$ if $Y_u > S_u, t \leq u \leq T$. 

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Now the condition \( \sum_{0 \leq s \leq T}(Y_t - S_t)(K_{t+1} - K_t) = 0 \) implies that \( K_s - K_{s-1} = 0, t + 1 \leq s \leq D_t \).

Then
\[
K_{D_t} - K_t = \sum_{t+1 \leq s \leq D_t} (K_s - K_{s-1}) = 0, \quad 0 \leq t \leq T.
\]

Consequently we have
\[
Y_t = E[Y_{D_t} + \sum_{t \leq u < D_t} f(u, Y_u, Z_u) + K_{D_t} - K_t | \mathcal{F}_t] =
E[Y_{D_t} + \sum_{t \leq u < D_t} f(u, Y_u, Z_u) | \mathcal{F}_t] \leq ess \sup_{\theta \in J_t} E[\sum_{t \leq u < \theta} f(u, Y_u, Z_u) + S_{\theta} 1_{\{\theta < T\}} + \xi 1_{\{\theta = T\}} | \mathcal{F}_t].
\]

Then the result follows immediately. \( \square \)

**Remark 2.6.** Denote \( L_t = \sum_{t \leq u < T} Z_u^* M_{u+1} \). We consider the special case where \( f = C, S_T = \xi \geq 0 \), it follows from the above two propositions that
\[
Y_0 = E[\xi + CT + K_T] = E[\xi + \sup_t (S_t + L_t - C(T - t) - \xi)^+].
\]

Since \( S_T = \xi \), then it is easy to check that
\[
Y_0 = \sup_{\theta \in J_0} E[S_{\theta}] = E[\sup_{0 \leq t \leq T} (S_t + L_t - C(T - t))].
\]

And when \( C = 0 \), we get a special result that
\[
Y_0 = \sup_{\theta \in J_0} E[S_{\theta}] = E[\sup_{0 \leq t \leq T} (S_t + L_t)].
\]

### 3 Comparison Theorem

Given \( \mathcal{F}_t \), let \( Q_t \) denote the set of indices of possible values of \( X_{t+1} \), i.e.
\[
Q_t := \{ i : P(X_{t+1} = e_i | \mathcal{F}_t) > 0 \}.
\]

This set can be thought of as an \( \mathcal{F}_t \)-adapted random variable. In the following context.
**Theorem 3.1** (Comparison Theorem). Suppose we have two FS-RBSDEs associated with standard data $(\xi^1, f^1, S^1)$ and $(\xi^2, f^2, S^2)$ respectively. Suppose $(Y^1, Z^1, K^1)$ and $(Y^2, Z^2, K^2)$ are associated solutions, and the following conditions also hold:

(i) $\xi^1 \geq \xi^2$, P-a.s.;
(ii) $f^1(\omega, t, Y^1_t, Z^1_t) \geq f^2(\omega, t, Y^2_t, Z^2_t)$ P-a.s. for all times $t \in \{0, 1, ..., T\}$;

(iii) $S^1_t \geq S^2_t$, P-a.s.;
(iv) $f^1(\omega, t, Y^2_t, Z^2_t) - f^1(\omega, t, Y^1_t, Z^1_t) \geq \min_{i \in \mathbb{Q}_t} ((Z^1_t - Z^2_t)^* (e_i - E[X_{t+1} | \mathcal{F}_t]))$, P-a.s. for all times $t \in \{0, 1, ..., T\}$;
(v) if $Y^1_t - f^1(\omega, t, Y^1_t, Z^1_t) \geq Y^2_t - f^1(\omega, t, Y^2_t, Z^2_t)$ P-a.s. for all times $t \in \{0, 1, ..., T\}$, then $Y^1_t \geq Y^2_t$ P-a.s.

Then it is true that

$$Y^1_t \geq Y^2_t \quad P-a.s..$$

**Proof.** For $t = T$, it is clear that $Y^1_t - Y^2_t = \xi^1 - \xi^2 \geq 0$ P-a.s.

For an arbitrary $0 \leq t < T$, suppose we know $Y^1_{t+1} - Y^2_{t+1} \geq 0$ P-a.s.. Then by (2.2), we have

$$Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^2(t, Y^2_t, Z^2_t) + (Z^1_t - Z^2_t)^* M_{t+1} - (K^1_{t+1} - K^1_t)$$
$$+ (K^2_{t+1} - K^2_t) = Y^1_{t+1} - Y^2_{t+1} \geq 0.$$

Since $M_{t+1} = X_{t+1} - E[X_{t+1} | \mathcal{F}_t]$, and we assume that $X_{t+1}$ takes values in the basis vectors $e_i$. So

$$Y^1_t - Y^2_t - (K^1_{t+1} - K^1_t) + (K^2_{t+1} - K^2_t)$$
$$\geq f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^2_t, Z^2_t) - \min_{i \in \mathbb{Q}_t} ((Z^1_t - Z^2_t)^* (e_i - E[X_{t+1} | \mathcal{F}_t])).$$

Hence by assumptions (ii) and (iv), we obtain

$$Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^1(t, Y^2_t, Z^2_t) - (K^1_{t+1} - K^1_t) + (K^2_{t+1} - K^2_t)$$
$$\geq f^1(t, Y^2_t, Z^2_t) - f^2(t, Y^2_t, Z^2_t) + f^1(t, Y^2_t, Z^1_t) - f^1(t, Y^2_t, Z^2_t)$$
$$- \min_{i \in \mathbb{Q}_t} ((Z^1_t - Z^2_t)^* (e_i - E[X_{t+1} | \mathcal{F}_t])) \geq 0. \quad (3.2)$$
Since on \( \{ Y_1^1 < Y_2^2 \} \), \( S_2^2 < S_1^1 \leq Y_1^1 < Y_2^2 \), i.e. \( S_2^2 < Y_2^2 \). Then \( K_{t+1}^2 - K_t^2 = 0 \), so
\[
Y_t^1 - Y_t^2 - f^1(t, Y_t^1, Z_t^1) + f^1(t, Y_t^2, Z_t^1) \geq 0.
\]

Then by assumption (v), the above inequality implies
\[
Y_t^1 \geq Y_t^2 \quad P-a.s.,
\]
which is a contradiction.

So
\[
Y_t^1 \geq Y_t^2 \quad P-a.s.
\]

By backward induction, we know the statement is true. \( \square \)

**Remark 3.2.** Note the assumption (v), if the map \( y - f(y, z) \) is strictly increasing in \( y \), the theorem also holds.

**Corollary 3.3.** Suppose \((\xi^1, f^1, S)\) and \((\xi^2, f^2, S)\), \((Y_1^1, Z_1^1, K_1^1)\) and \((Y_2^2, Z_2^2, K_2^2)\) satisfy the assumptions in Theorem 3.1, if we also know \( Y_1^1 = Y_2^2 \), then \( K_1^1 \leq K_2^2 \) \( P \)-a.s. for all \( t \in \{0, 1, ..., T\} \), and \( K_1^1 - K_2^2 \) is decreasing in \( t \). Moreover, if we also have \( \xi^1 = \xi^2, f^1 = f^2 \) \( P \)-a.s., then \( K_1^1 = K_2^2 \) \( P \)-a.s. for all \( t \in \{0, 1, ..., T\} \).

**Proof.** By inequality (3.2), we know
\[
K_{t+1}^1 - K_t^1 \leq K_{t+1}^2 - K_t^2.
\]

Then \( K_1^1 - K_2^2 \) is decreasing in \( t \).

More again, we know \( K_0^1 = K_0^2 = 0 \), So \( K_1^1 \leq K_2^2 \) \( P \)-a.s..

Because \( K_1^1 - K_2^2 \) is decreasing in \( t \), we have
\[
K_2^1 - K_2^2 \leq K_1^1 - K_2^2 \leq 0.
\]

So
\[
K_2^1 \leq K_2^2.
\]

Then by induction we know \( K_1^1 \leq K_2^2 \) \( P \)-a.s..

Moreover, if we also have \( \xi^1 = \xi^2, f^1 = f^2 \) \( P \)-a.s., then it is easy to see that \( K_2^2 \leq K_1^1 \) \( P \)-a.s. Thus we have the result. \( \square \)
Remark 3.4. Under the assumptions of Corollary 3.3, if the notion \( \geq \) of any of the assumptions (i), (ii), (iv) in Theorem 3.1 becomes \( > \), then \( K_1^t < K_2^t \) a.s. for \( t \in \{0, 1, ..., T\} \), and \( K_1^t - K_2^t \) is strictly decreasing in \( t \). In this case, by (3.2) we have \( K_1^{t+1} - K_2^{t+1} > K_1^t - K_2^t \geq 0 \), so there holds \( Y_1^2 = S_1^2 \). Consequently we obtain \( S_1^t = S_2^t \) because of \( Y_1^t = Y_2^t = S_2^t \leq S_1^t \leq Y_1^1 \).

We now show a counterexample to state that Theorem 3.1 fails when one of Assumptions (iv) does not hold.

Example 3.5. For simplicity, suppose \( T = 1 \). Consider a pair of FS-RBSDEs associated with standard data \( (\xi_1, f, S_1) \) and \( (\xi_2, f, S_2) \) respectively which satisfy the Existence and Uniqueness Theorem 4.3 in the following section. Moreover, let \( \xi_1 = \xi_2, f_1 = f_2 = f \) and \( S_1 = S_2 \), then we have \( Y_1^0 = Y_2^0, K_0^1 = K_0^2 \) and \( K_1^1 = K_1^2 \) P-a.s.. Suppose Assumption (iv) of Theorem 3.1 does not hold, particularly we have

\[
f(\omega, 0, Y_1^2, Z_1^1) - f(\omega, 0, Y_1^2, Z_2^2) < \min_{i \in \mathbb{Q}} \{(Z_0^1 - Z_0^2)^*(e_i - E[X_1 | \mathcal{F}_0])\}.
\]

Then we have

\[
0 = Y_1^1 - Y_1^2 = Y_0^1 - Y_0^2 - f(0, Y_0^1, Z_1^1) + f(0, Y_0^2, Z_2^2) + (Z_0^1 - Z_0^2)^*M_1 - (K_1^1 - K_0^1) + (K_2^1 - K_0^2)
\]

\[
> Y_0^1 - Y_0^2 - f(0, Y_0^1, Z_1^1) + f(0, Y_0^2, Z_0^1) - (K_1^1 - K_0^1) + (K_1^2 - K_0^2).
\]

It follows that

\[
0 = (K_1^1 - K_0^1) - (K_1^2 - K_0^2) > Y_0^1 - Y_0^2 - f(0, Y_0^1, Z_0^1) + f(0, Y_0^2, Z_0^1).
\]

Thus, we have \( Y_0^1 < Y_0^2 \), which is a contradiction.

4 Existence and Uniqueness Theorem

In this section, we will prove the Existence and Uniqueness Theorem of the solution of FS-RBSDE in which the map \( y - f(\cdot, y, z) \) is strictly increasing and continuous in \( y \), basing on approximation via penalization in [9] as well as the comparison theorem mentioned in [8].

Firstly, we recall the Comparison Theorem in [8] which is very useful for the following context.
Theorem 4.1. Suppose we have two discrete time and finite state BSDEs associated with standard data \((\xi^1, f^1)\) and \((\xi^2, f^2)\) respectively. Suppose \((Y^1, Z^1)\) and \((Y^2, Z^2)\) are associated solutions, and the following conditions also hold:

(i) \(\xi^1 \geq \xi^2\) \(P\)-a.s.;
(ii) \(f^1(\omega, t, Y^2_t, Z^2_t) > f^2(\omega, t, Y^2_t, Z^2_t)\), \(P\)-a.s. for all times \(t \in \{0, 1, ..., T\}\);

(iii) \(f^1(\omega, t, Y^2_t, Z^1_t) - f^1(\omega, t, Y^2_t, Z^1_t) \geq \min_{i \in \mathbb{Q}} \{ |Z^1_t - Z^2_t| \} (e_i - E[X_{t+1} | \mathcal{F}_t])\) \(P\)-a.s. for all times \(t \in \{0, 1, ..., T\}\);
(iv) \(Y^1_t - f^1(\omega, t, Y^1_t, Z^1_t) \geq Y^2_t - f^1(\omega, t, Y^1_t, Z^1_t)\) \(P\)-a.s. for all times \(t \in \{0, 1, ..., T\}\), then \(Y^1_t \geq Y^2_t\).

Then it is true that \(Y^1_t \geq Y^2_t\) \(P\)-a.s.

Corollary 4.2. Considering the BSDE \((2.1)\), suppose \((\xi^1, f^1)\) and \((\xi^2, f^2)\), \((Y^1, Z^1)\) and \((Y^2, Z^2)\) satisfy the assumptions in theorem 4.1, if we also know any strict inequality holds as follows:

(i) \(\xi^1 > \xi^2\) \(P\)-a.s.,
(ii) \(f^1(\omega, t, Y^2_t, Z^2_t) > f^2(\omega, t, Y^2_t, Z^2_t)\), \(P\)-a.s. for all times \(t \in \{0, 1, ..., T\}\);

(iii) \(f^1(\omega, t, Y^1_t, Z^1_t) - f^1(\omega, t, Y^1_t, Z^1_t) > \min_{i \in \mathbb{Q}} \{ |Z^1_t - Z^2_t| \} (e_i - E[X_{t+1} | \mathcal{F}_t])\) \(P\)-a.s. for all times \(t \in \{0, 1, ..., T\}\);

Then it is true that \(Y^1_t > Y^2_t\) \(P\)-a.s.

Proof. By the above theorem, we have \(Y^1_t \geq Y^2_t\) \(P\)-a.s.. In this case, for a given \(t\), by the same argument as used to (3.2), we obtain

\[ Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^1(t, Y^2_t, Z^1_t) > 0. \]

It follows that \(Y^1_t \neq Y^2_t\) \(P\)-a.s.. Then we get the desired result. \(\Box\)

Theorem 4.3. (Existence and Uniqueness Theorem) Suppose we have a FS-RBSDE associated with standard data \((\xi, f, S)\). Moreover the map \(f\) satisfies the following two assumptions:

(i) for any \(Y_t\), if \(Z^1_t \sim_M Z^2_t\), then \(f(t, Y_t, Z^1_t) = f(t, Y_t, Z^2_t)\) \(P\)-a.s. for all times \(t \in \{0, 1, ..., T\}\).

(ii) The map \(y - f(\cdot, y, z)\) is strictly increasing in \(y\).

Then there exists a solution \(\{Y_t, Z_t, K_t\}, 0 \leq t \leq T\) of FS-RBSDE with standard data \((\xi, f, S)\) and terminal time \(T\). Moreover this solution is unique up to indistinguishability for \(Y\) and \(\sim_M\) for \(Z\).
Proof. If we know $Y_T = \xi$, we can begin with the time $t = T - 1$ and solve the one step FS-RBSDE to obtain $Y_{T-1}$. Similarly, if we know $Y_{T-1}$, then we can solve $Y_{T-2}$ (if $T \geq 2$) by one step method. Thus, piecing together all the one-step solutions, we can obtain the solution at any time $t \in \{0, 1, ..., T\}$. Without loss of generality, we only consider the following one step FS-RBSDE as follows:

$$Y_t = Y_{t+1} + f(t, Y_t, Z_t) + K_{t+1} - K_t - Z_t^* M_{t+1}. \quad (4.1)$$

For each $n = 1, 2, ..., $ define

$$k_t^n = n \sum_{0 \leq u < t} (Y_u - S_u)^-. \quad (4.2)$$

Then $k_t^n$ is increasing in $n$ and $k_t^n = 0$, $k_{t+1}^n - k_t^n = n(Y_t - S_t)^-.$

**1) Existence** We have divided the proof into two steps. In the first step, we shall construct a sequence backward stochastic difference equations and prove the corresponding solutions which converges to the solution of (4.1); in the second step, we shall prove the solution obtained in step (1) satisfies all the conditions of Definition 2.2.

Step 1. Considering the following sequence of general backward stochastic difference equations:

$$Y_t^n = Y_{t+1} + f(t, Y_t^n, Z_t^n) + n(Y_t^n - S_t)^- - (Z_t^n)^* M_{t+1}, \quad n \in \mathbb{Z}^+. \quad (4.2)$$

After taking conditional expectation for (4.2), we thus get

$$Y_t^n = E[Y_{t+1}|\mathcal{F}_t] + f(t, Y_t^n, Z_t^n) + n(Y_t^n - S_t)^-, \quad n \in \mathbb{Z}^+. \quad (4.3)$$

Hence,

$$(Z_t^n)^* M_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t].$$

By Martingale Representation Theorem in [12], there exists a unique $Z_t$ up to equivalence $\sim_{M_{t+1}}$ such that the above equation is satisfied for an arbitrary $n$. Using this $Z_t$, (4.3) can be rewritten as follows:

$$Y_t^n = Y_{t+1} + f(t, Y_t^n, Z_t) + n(Y_t^n - S_t)^- - Z_t^* M_{t+1}, \quad n \in \mathbb{Z}^+. \quad (4.4)$$
Let $f_n(t, y, z) = f(t, y, z) + n(y - S_t)^-$, then $y - f_n(t, y, z)$ is strictly increasing and continuous in $y$, hence it is also bijective. By theorem 4.1, (4.3) has a unique solution $(Y^n_t, Z_t)$. Clearly we have

(i) $f_{n+1}(t, y, z) \geq f_n(t, y, z) \quad \forall(y, z)$;

(ii) As $Z^n_t = Z^{n+1}_t = Z_t, f(t, Y_t, Z^n_t) - f(t, Y_t, Z^{n+1}_t) = 0 = \min_{i \in \Omega_t} \{[Z^n_t - Z^{n+1}_t] \ast (e_i - E[X_{t+1}])\};

(iii) As the map $y - f_n(t, y, z)$ is strictly increasing, we get the following truth: if

$$y_1 - f_{n+1}(t, y_1, z) \geq y_2 - f_n(t, y_2, z),$$

then $y_1 \geq y_2$ P-a.s..

Therefor, by the Comparison Theorem 4.1 , we get that $Y^{n+1}_t \geq Y^n_t$ P-a.s.. Hence

$$Y^n_t \uparrow Y_t \quad P-a.s..$$

Moreover from (4.3), if $Y^n_t \geq S_t$, then $Y^n_t = E[Y_{t+1}|\mathcal{F}_t] + f(t, Y^n_t, Z^n_t)$ which means $Y^n_t$ is essentially bounded; on the other hand, if $Y^n_t < S_t$, then $Y^n_t = E[Y_{t+1}|\mathcal{F}_t] + f(t, Y^n_t, Z^n_t) + \frac{nS_n}{n+1}$ which also means $Y^n_t$ is essentially bounded. Thus, from Fatou’s Lemma

$$E|Y_t| \leq \lim_{n \to \infty} E|Y^n_t| < \infty.$$

On the other hand, by (4.3) we have

$$|k^n_{t+1} - k^n_{t+1}| \leq |f(t, Y^{n+1}_t, Z_t) - f(t, Y^n_t, Z_t)| + |Y^{n+1}_t - Y^n_t|, \quad \forall p \in \mathcal{N}.$$

Since $f$ is continuous in $y$, and $Y^n_t \uparrow Y_t$ P-a.s.. This gives

$$|k^n_{t+1} - k^n_{t+1}| \to 0, \quad p \to +\infty.$$

Consequently there exists a adapted process $K_{t+1}$ such that $k^n_{t+1} \to K_{t+1}$, as $n \to \infty$. Denote $K_0 = k^n_0 = 0, \forall n \in \mathcal{N}.$

Then let $n \to \infty$, (4.4) becomes

$$Y_t = Y_{t+1} + f(t, Y_t, Z_t) + K_{t+1} - K_t - Z^*_t M_{t+1}.$$
Step 2. It is easy to know that $E | K_{t+1} | < +\infty$, so we can find a triple $(Y_t, Z_t, K_t)$ which satisfies $(i)$ and $(ii)$ of Definition 2.2. It remains to check $(iii)$ and $(iv)$.

First of all, $K_t$ is increasing as $k^n_t$ is increasing and $K_0 = 0$. As $(Y^n_t - S_t)(k^n_{t+1} - k^n_t) = n(Y^n_t - S_t)(Y^n_t - S_t)^- = -n[(Y^n_t - S_t)^-]^2 \leq 0$, then

$$(Y_t - S_t)(K_{t+1} - K_t) \leq 0.$$ 

On the other hand,

$$(Y_t^{n+1} - S_t)^- \leq (Y_t^n - S_t)^-.$$ 

By (4.3) we also have

$$(Y^n_t - S_t)^- = \frac{Y^n_t - E[Y_{t+1}\mathcal{F}_t] - f(t, Y^n_t, Z_t)}{n}.$$ 

This clearly forces

$$(Y^n_t - S_t)^- \downarrow 0, \quad (Y_t - S_t)^- = \lim_{n \to +\infty} (Y^n_t - S_t)^- = 0.$$ 

It follows that $Y_t \geq S_t$, hence

$$(Y_t - S_t)(K_{t+1} - K_t) \geq 0 \quad P-a.s.$$ 

So we obtain $(Y_t - S_t)(K_{t+1} - K_t) = 0$ P-a.s., as desired.

(2)Uniqueness. At last, we shall prove the uniqueness of the solution obtained in step 1. Suppose there exist two different solutions $(Y_t, Z_t, K_t)$ and $(Y'_t, Z'_t, K'_t)$ of FS-RBSDE (4.1). Without loss of generality, suppose $Y_t > Y'_t$. Then $Y_t > Y'_t \geq S_t$, it follows that $K_{t+1} - K_t = 0$. Then (4.1) can be simplified to:

$$Y_t = Y_{t+1} + f(t, Y_t, Z_t) - Z_t^* M_{t+1}.$$ 

On the other hand,

$$Y'_t = Y'_{t+1} + f(t, Y'_t, Z'_t) + K'_{t+1} - K'_t - Z'^n_t M_{t+1}.$$ 

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By theorem 4.1, we have \( Y_t \leq Y'_t \) P-a.s., which is a contradiction. Similarly, the case \( Y_t < Y'_t \) is not true yet. So \( Y_t = Y'_t \) P-a.s.

More again, by the corollary (3.3), we have \( K_t = K'_t, K_{t+1} = K'_{t+1} \).

Consequently we have

\[
Z^*_t M_{t+1} = Z^*_t M_{t+1} = Y_{t+1} - E[Y_{t+1} | \mathcal{F}_t] + K'_{t+1} - K'_t - E[K'_{t+1} - K'_t | \mathcal{F}_t] = Y_{t+1} - E[Y_{t+1} | \mathcal{F}_t] + K'_{t+1} - K_t - E[K_{t+1} - K_t | \mathcal{F}_t].
\]

Then by Martingale Representation Theorem, we have \( Z_t \sim M Z'_t \). □

**Remark 4.4.** Note the assumption (ii) in Theorem 4.3, if the map \( f \) is decreasing and continuous in \( y \), the theorem holds naturally.

## 5 FS-RBSDE and optimal stopping time problems

In [9], El.Karoui et al. shew that the solution \( \{Y_t, 0 \leq t \leq T\} \) of the general reflected backward stochastic differential equation where \( f \) is a concave (or convex) function of \((y, z)\) is the value function of a mixed optimal stopping stochastic control problem. In our framework, we also have these properties. At first, we show the solution of the FS-RBSDE where \( f \) is a given stochastic process is the value function of an optimal stopping time problem, then to the case where \( f \) is a linear function of \((y, z)\). In the last case, \( f \) is a concave (or convex) function of \((y, z)\), \( \{Y_t, 0 \leq t \leq T\} \) will be a value function of a mixture of an optimal stopping time problem and a classical optimal stochastic control problem. Note that we only should consider the above problems in “one step” FS-RBSDE because of the Existence and Uniqueness Theorem as well as the properties of the discrete time, i.e.

\[
Y_t = Y_{t+1} + f(t, Y_t, Z_t) - Z^*_t M_{t+1} + K_{t+1} - K_t, 0 \leq t < T. \tag{5.1}
\]

Throughout this section, we maintain the Assumption \((H4)\): if \( Z^1 \sim_M Z^2 \), it is true that \( Z^1 = Z^2 \) P-a.s.. By proposition 2.5, we have the following properties without proof.
Proposition 5.1. Suppose that \( f \in \mathcal{L}^1 \) does not depend on \((y, z)\); that is, it takes the form
\[
f(t, y, z) = \alpha_t,
\]
where \( \{\alpha_t \in \mathcal{L}^1; 0 \leq t \leq T\} \) takes values in \(\mathbb{R}\). Then the unique solution \( \{(Y_t, Z_t, K_t), 0 \leq t < T\} \) of the FS-RBSDE (2.2) with the coefficient \( f \) satisfies
\[
Y_t = \text{ess sup}_{\theta \in \mathcal{J}_t} E[\sum_{t \leq s < \theta} \alpha_s + S_\theta I_{\{\theta < T\}} + \xi I_{\{\theta = T\}}|\mathcal{F}_t].
\]
where \( \mathcal{J}_t \) is defined in proposition 2.5. Moreover, if we only consider (5.1) which can give
\[
Y_t = S_t \vee (\alpha_t + E[Y_{t+1}|\mathcal{F}_t]).
\]

Remark 5.2. Here is another way of stating the Proposition 5.1 using the Definition 2.2 directly. Denote \(\rho_t = \alpha_t + E[Y_{t+1}|\mathcal{F}_t]\) which can be computed easily. After taking the conditional expectation for (5.1), we have \(Y_t = \rho_t + E[K_{t+1} - K_t|\mathcal{F}_t]\).

If \(\rho_t > S_t\), by (iv) of Definition 2.2, it follows that \(K_{t+1} - K_t = 0\). Then \(Y_t = \rho_t\).

On the other hand, if there holds \(\rho_t = S_t\), then \(Y_t - S_t = E[K_{t+1} - K_t|\mathcal{F}_t]\). More again by (iv) of Definition 2.2, it is true that \(K_{t+1} - K_t = 0\). Thus, we have \(Y_t = \rho_t = S_t\).

At last, if \(\rho_t < S_t\), it must hold that \(K_{t+1} - K_t > 0\). Then by (iv) of Definition 2.2, we have \(Y_t = \rho_t\).

To sum up, \(Y_t = S_t \vee \rho_t\), i.e. \(Y_t = S_t \vee (\alpha_t + E[Y_{t+1}|\mathcal{F}_t])\).

Proposition 5.3. Under \((H4)\), suppose \( f \in \mathcal{L}^1 \) be a linear function of \((y, z)\); that is, it takes the form
\[
f(t, y, z) = \alpha_t + \beta_t y + \langle \gamma_t, z \rangle,
\]
where \( \{\alpha_t, \beta_t, \gamma_t, 0 \leq t \leq T\} \) are essentially bounded and progressively adapted process with in \(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m\) and \(\beta_t \neq 1\) P-a.s.. Then a solution \( \{(Y_t, Z_t, K_t), 0 \leq t < T\} \) of the FS-RBSDE (5.1) with the coefficient \( f \) satisfies
\[
Y_t = S_t \vee (\alpha_t + \beta_t Y_t + \langle \gamma_t, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]), \quad (5.2)
\]
where \(Z_t\) satisfies that \(Z_t^* M_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t]\) from the proof of Theorem 4.3.
Remark 5.4. Actually, after observing (5.2), we only need to solve the equation
\[ Y_t = \alpha_t + \beta_t Y_t + \gamma_t Z_t + E[Y_{t+1} | \mathcal{F}_t] \]
i.e. \((1 - \beta_t)Y_t = \alpha_t + \gamma_t Z_t + E[Y_{t+1} | \mathcal{F}_t]. \) If the obtained solution \( Y_t > S_t \) \( P \)-a.s., this solution is desired; otherwise, \( Y_t = S_t \) \( P \)-a.s..

Remark 5.5. If there holds \( \beta_t = 1 \) \( P \)-a.s., then we have
\[ y(t, z) = -\alpha_t - \gamma_t Z_t \]
which does not satisfy the assumption (ii) of the Existence and Uniqueness Theorem in \([8]\), which leads that there does not exist a unique solution. So we limit that \( \beta_t \neq 1 \) \( P \)-a.s..

We now suppose that for each fixed \((\omega, t)\), \( f(t, y, z) \) is a concave function of \((y, z)\). Define the conjugate function \( F(t, \beta, \gamma) \) as follows. For each \((\omega, t, \beta, \gamma) \in \Omega \times \{0, 1, ..., T\} \times \mathbb{R} \times \mathbb{R}^m, \)
\[ F(\omega, t, \beta, \gamma) = \sup_{(y, z)} (f(t, y, z) - \beta y - \gamma z) \]
\[ D_F^t(\omega) = \{ (\beta, \gamma) \in \mathbb{R} \times \mathbb{R}^m; F(\omega, t, \beta, \gamma) \text{is essentially bounded} \}. \]

It follows that \( f(t, y, z) = \inf_{(\beta, \gamma) \in D_F^t} \{ F(t, \beta, \gamma) + \beta y - \gamma z \} \) and the infimum can be achieved at \((\beta', \gamma') \in D_F^t\), the set \( D_F^t \) is essentially bounded. We shall denote the unique solution \((Y_t^{\beta, \gamma}, Z_t^{\beta, \gamma}, K_t^{\beta, \gamma}); 0 \leq t \leq T\) and \((Y_t, Z_t, K_t); 0 \leq t \leq T\) of the FS-RBSDEs with the coefficient \( f^{\beta, \gamma}(t, y, z) = F(t, \beta_t, \gamma_t) + \beta_t y - \gamma_t z \) and \( f(t, y, z) \) respectively. Consequently we have
\[ f(t, Y_t, Z_t) = F(t, \beta', \gamma') + \beta' Y_t + \gamma' Z_t \quad P - \text{a.s., a.e.}; \]
\[ (Y_t, Z_t, K_t) = (Y_t^{\beta', \gamma'}, Z_t^{\beta', \gamma'}, K_t^{\beta', \gamma'}) \quad P - \text{a.s., } 0 \leq t \leq T. \]

We can then deduce an interpretation of \( Y_t = Y_t^{\beta', \gamma'} \) as the value functions of optimization problems.

Theorem 5.6. Under (H4), for each \((\beta_t, \gamma_t) \in D_F^t\) and \(|\beta_t| < 1 \) \( P \)-a.s. We have
\[
Y_t^{\beta, \gamma} = S_t \lor (F(t, \beta_t, \gamma_t) + \beta_t Y_t^{\beta, \gamma} + \gamma_t Z_t + E[Y_{t+1} | \mathcal{F}_t]) \\
Y_t = S_t \lor (F(t, \beta_t, \gamma_t) + \beta_t Y_t + \gamma_t Z_t + E[Y_{t+1} | \mathcal{F}_t]).
\]
where $Z_t$ satisfies that $Z_t^* M_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t]$ from the proof of Theorem 4.3. Moreover,

\[
Y_t = \text{ess inf}_{(\beta, \gamma) \in D_t^F} Y_t^{\beta, \gamma} \\
= \text{ess inf}_{(\beta, \gamma) \in D_t^F} (S_t \vee (F(t, \beta_t, \gamma_t) + \beta_t Y_t^{\beta, \gamma} + \langle \gamma_t, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t])) \\
= S_t \vee \text{ess inf}_{(\beta, \gamma) \in D_t^F} (F(t, \beta_t, \gamma_t) + \beta_t Y_t^{\beta, \gamma} + \langle \gamma_t, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]).
\]

In other words, $Y_t$ is the value function of a minimax control problem, and the triple $(\beta', \gamma', D_t)$, where $D_t = \inf \{t \leq s \leq T; Y_s = S_s\}$ is optimal.

**Proof.** The first part of the statement follows from Proposition 5.3.

Moreover, from the Comparison Theorem 3.1, we have

\[
Y_t \leq Y_t^{\beta, \gamma}, \quad \forall (\beta, \gamma) \in D_t^F.
\]

On the other hand,

\[
Y_t = Y_t^{\beta', \gamma'} \geq \inf_{(\beta, \gamma) \in D_t^F} Y_t^{\beta, \gamma},
\]

which immediately deduces

\[
Y_t = \text{ess inf}_{(\beta, \gamma) \in D_t^F} Y_t^{\beta, \gamma}.
\]

At last, it is easy to see that ess inf and $\vee$ can be interchanged. \qed

**Remark 5.7.** If $f$ is a convex function of $(y, z)$, we only need replace $\text{ess inf}_{(\beta, \gamma) \in D_t^F} \ldots \vee$ by $\text{ess sup}_{(\beta, \gamma) \in D_t^F} \ldots \vee$.

Actually, there are many processes $X_t$ satisfying the Assumption (H4), such as the random walk and basis vector.

**Example 5.8.** Suppose $X_t$ be a standard random walk, i.e. $X_t = \sum_{i=0}^{t} \epsilon_i$, where $\{\epsilon_i\}$ is a stochastic oscillator sequence which is independent and can only be $\pm 1$ with equal probability $1/2$. As is known, $X_t$ is a martingale. Then $M_t = X_t - E[X_t|\mathcal{F}_{t-1}] = \epsilon_t$.

In this case, if $Z^1 \sim_M Z^2$, i.e. $Z^1 \epsilon_1 = Z^2 \epsilon_1$, it is true that $Z^1 = Z^2$ P-a.s..
6 g-Martingale Theory in discrete time and finite state space

In this section, in order to study the optimal stopping problems in the framework of g-martingales in discrete time and finite state space, we first study the g-martingale, Doob-Mayer Decomposition Theorem and Optional Sampling Theorem, which were investigated in continuous time in [4], [13], [17] and [18]. We also explore the connections between minimum expectation and g-expectation which is important for computing the multiple prior expected rewards of an agent.

Riedel [21] has considered a theory of optimal stopping and multiple prior envelope when the expected payoff is evaluated by \( \inf_{P \in Q} E^P[X_{\tau}] \). Here, we give the connections between the FS-BSDEs and the multiple prior martingale under Knightian uncertainty. Consequently, the optimal stopping problem with multiple priors can be solved by computing a special kind of FS-RBSDEs to obtain the multiple prior envelope.

Firstly, we should establish the theory of g-expectation and g-martingale in our framework.

6.1 g-Expectations and g-Martingales

Peng [17] introduced the notions of g-expectations and conditional g-expectation as well as g-martingale via the general BSDEs, and he also proved a general nonlinear Doob-Mayer Decomposition Theorem for g-super-martingales in [18]. This section is aim to study the g-martingale and Doob-Mayer Decomposition Theorem in our framework.

As in [17], we give the following notion of “g-expectation” via BSDE (2.1). In the remainder of this section we also assume \( f \) satisfies

\[
f(0,0,t) = 0, \forall t \geq 0.
\]

(A3)

**Definition 6.1.** Given the finite time horizon \( T \), for each \( \xi \in S^T \), suppose the map \( f \) satisfies the assumptions (A1) - (A3) and \((Y_t,Z_t)\) is the solution of BSDE (2.1). We call \( G_{0,T}(\xi) \) defined by \( G_{0,T}(\xi) := Y_0 \) the g-expectation of the random variable \( \xi \) generated by function \( f \).

From the definition of g-expectation, we can define the conditional g-expectation as follows.
Theorem 6.2. Given the finite time horizon $T$, for each $\xi \in \mathcal{S}^T$. Then there exists a $\eta \in \mathcal{S}^r$ such that

$$G_0,T(1_A\xi) = G_0,r(1_A\eta), \forall A \in \mathcal{F}_r; r \in \{0, 1, ..., T\}.$$

Moreover, $\eta$ coincides with $Y_r$—the value of the solution of BSDE (2.1) at time $r$. We then call $\eta$ the conditional $g$-expectation of $\xi$ under $\mathcal{F}_r$ in the time sequence $\{r, r+1, ..., T\}$ and write it as $G_{r,T}(\xi)$. Under the assumptions (iii) and (iv) of Theorem 4.1, $\eta$ is unique.

Proof. Let $(y, z)$ be the solution of BSDE (2.1). For $\forall A \in \mathcal{F}_r$, let $(\bar{y}, \bar{z})$ be the solution of the following BSDE:

$$\bar{y}_t = \xi + \sum_{t \leq s < T} f(s, \bar{y}_s, \bar{z}_s) - \sum_{t \leq s < T} \bar{z}_s^* M_{s+1}. \quad (6.1)$$

Multiply $1_A$ on both sides of BSDE (2.1) and then observe $y_t 1_A$ on $\{r, r+1, ..., T\}$. Note there exists the relation $f(t, 1_A y, 1_A z) = f(t, y, z) 1_A, \forall t \geq 0$ because of the assumption (A3). Immediately, by the uniqueness of the solution of BSDE (6.1), we have

$$y_s 1_A = \bar{y}_s, \forall s \in \{r, r+1, ..., T\}. \quad (6.2)$$

Define $\eta := y_r$, then $\eta \in \mathcal{S}^r$ obviously. By the definition of $G_0,T(1_A\xi)$ and from (6.2), we have

$$G_0,T(1_A\xi) = \bar{y}_0 = G_0,r(\bar{y}_r) = G_0,r(y_r 1_A) = G_0,r(\eta 1_A).$$

It remains to prove $\eta$ is unique. Assume that there exists another $\beta \in \mathcal{S}^r$ such that for any $A \in \mathcal{F}_r$,

$$G_0,r(\eta 1_A) = G_0,r(\beta 1_A). \quad (6.3)$$

But $P(\eta \neq \beta) > 0$. We can choose $A = \{\eta \neq \beta\}$, then it follows from corollary 4.2 that $G_0,r(\eta 1_A) \neq G_0,r(\beta 1_A)$, which is contrary to (6.3). The proof is complete. \qed

Definition 6.3. Under the assumptions (iii) and (iv) of Theorem 4.1, a real valued adapted process $\{U_t\}$ is called $g$-martingale (resp. super-martingale, sub-martingale ), if $U_t \in \mathcal{S}^t$, $\forall t, s \in \{0, 1, ..., T\}, t \leq s$,

$$G_{t,s}(U_s) = U_t (\text{resp.} \leq U_t, \geq U_t) P - a.s..$$

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It is easy to check the following properties of $G_s(\cdot)$ (see [4] for more details).

**Corollary 6.4.** Any $g$-martingale $\{U_t\}$ has the following basic properties:

1. For each $0 \leq t_1 \leq t_2 \leq t_3 \leq T$, $G_{t_1,t_3}(U_{t_3}) = G_{t_2,t_3}(G_{t_1,t_2}(U_{t_2}))$;
2. (comparison theorem) If $U_1^1 \leq U_2^2$, then $G_{t,s}(U_1^1) \leq G_{t,s}(U_2^2)$ for $0 \leq t \leq s \leq T$;
3. For any $A \in \mathcal{F}_t$, $G_{t,s}(1_A) = 1_A G_{t,s}(1)$, $0 \leq t \leq s \leq T$;
4. Let $\{A_i\}_{i=1}^n \subset \mathcal{F}_0$ be a partition of $\Omega$, then for any $U_T^i \in \mathcal{S}, i = 1, 2, ..., n$, we have

$$G_{t,T}(\sum_{i=1}^n U_T^i 1_{A_i}) = \sum_{i=1}^n G_{t,T}(U_T^i 1_{A_i}), t \in \{t_0, t_0 + 1, ..., T\}.$$

To prove the Doob-Mayer Decomposition Theorem, we need show a general type of $g$-martingale $G_{s,t}(\cdot; K)$ induced by $G_{s,t}(\cdot)$ for each given process $K \in \mathcal{D}(0, t)$ which often represents a dividend or a consumption process in finance. Here $\mathcal{D}(0, t) = \{\{\varphi_t, 0 \leq t \leq T\} \text{ is an } \mathcal{F}_t\text{-predictable real-valued process and essentially bounded}\}$. For convenience, we would consider stopping times $0 \leq \sigma \leq \tau \leq T$ instead of deterministic times $s$ and $t$. The corresponding backward stochastic difference equation becomes as follows:

$$Y_s = X + K_\tau - K_{s \wedge \tau} + \sum_{s \wedge \tau \leq u < \tau} f(u, Y_u, Z_u) - \sum_{s \wedge \tau \leq u < \tau} Z_u^s M_{u+1}, 0 \leq s \leq T. \tag{6.4}$$

where $X \in \mathcal{S}^\tau, K \in \mathcal{D}(0, t)$. Then we have the following theorem:

**Theorem 6.5.** For each given process $K \in \mathcal{D}(0, t)$, assume assumptions (A1) - (A3), there exists a unique solution $(Y_s^{\tau,X,K}, Z_s^{\tau,X,K}), s \in \{0, 1, ..., \tau\}$ of BSDE (6.4).

**Proof.** Define

$$\bar{Y}_s := Y_s + K_s,$$

$$\bar{f}(s, y, z) := f(s, y - K_s, z) 1_{\{0,1,\ldots,\tau\}}(s).$$

Considering the following equivalent BSDE

$$\bar{Y}_s = X + K_\tau + \sum_{s \leq u < T} \bar{f}(u, \bar{Y}_u, Z_u) - \sum_{s \leq u < T} Z_u^s M_{u+1}, 0 \leq s \leq T. \tag{6.5}$$
It is clear that $\tilde{Y}_s \equiv X + K_s$, $Z_s \equiv 0$ on $\{\tau, \tau + 1, ..., T\}$. Since $(X + K_\tau, \tilde{f})$ satisfies the assumptions (A1)-(A3), then BSDE (6.4) has a unique solution $(\tilde{Y} - K, Z)$.

We denote $G_{\sigma,\tau}(X; K) := Y_{\sigma,X,K}^{\tau,K}$ and $G_{\sigma,\tau}(X) := G_{\sigma,\tau}(X; 0)$. Here we will introduce the notion of $G(\cdot; A)$-martingale.

**Definition 6.6.** Let $K \in D(0, t)$ be given. A process $Y \in L^1[t_0, t_1]$ is said to be a $G(\cdot; A)$-martingale (resp. $G(\cdot; A)$-super-martingale, $G(\cdot; A)$-sub-martingale) on $\{t_0, t_0 + 1, ..., t_1\}$ if for each $t_0 \leq s \leq t \leq t_1$, we have

$$G_{s,t}(Y_t; A) = Y_s (\text{resp.} \leq Y_s, \geq Y_s), P-a.s..$$

**Theorem 6.7** (Doob-Mayer Decomposition Theorem). Let $U$ be a $g$-super-martingale. Then there exists a process $A \in D(0, t)$ with $A_0 = 0$ such that $U_t = G_{t,T}(U_T; A), \forall t \in \{0, 1, 2, ..., T\}$ i.e. $U$ is a $G(\cdot; A)$-martingale.

In order to prove the above theorem, we need to introduce a sequence BSDEs of the following form: for $n = 1, 2, ...$

$$u^n_t = G_{t,T}(U_T; n \sum_{0 \leq s < t} (U_s - u^n_s)^+),$$

i.e.

$$u^n_t = U_T + \sum_{t \leq s < T} f(s, u^n_s, z^n_s) - \sum_{t \leq s < T} (z^n_s)^* M_{s+1} + n \sum_{t \leq s < T} (U_s - u^n_s)^+. \quad (6.6)$$

which has the following useful property.

**Lemma 6.8.** For each $n = 1, 2, ...$, we have $u^n_t \uparrow U_t$, $dt \times dP$-a.e..

**Proof.** Considering the “one step” BSDEs as follows:

$$u^n_t = U_{t+1} + n(U_t - u^n_t)^+ + f(u^n_t, z^n_t) - (z^n_t)^* M_{t+1}, n \in \mathcal{Z}^+. \quad (6.7)$$

After taking conditional expectation for (6.7), we get

$$u^n_t = E[U_{t+1} | \mathcal{F}_t] + n(U_t - u^n_t)^+ + f(u^n_t, z^n_t), n \in \mathcal{Z}^+.$$

It follows that

$$(z^n_t)^* M_{t+1} = U_{t+1} - E[U_{t+1} | \mathcal{F}_t].$$
By martingale representation theorem, there exists a unique $Z_t$ such that the above equation is satisfied for an arbitrary $n$. Then the equation (6.7) can be rewritten as follows:

$$u^n_t = U_{t+1} + n(U_t - u^n_t)^+ + f(u^n_t, Z_t) - Z^*_t M_{t+1}, n \in \mathbb{Z}^+. \quad (6.8)$$

On the set $\{U_t < u^n_t\}$, the equation (6.8) becomes $u^n_t = U_{t+1} + f(u^n_t, Z_t) - Z^*_t M_{t+1}$, then we have

$$u^n_t = E[U_{t+1} + f(u^n_t, Z_t) | F_t] = E[G_{t,t+1}(U_{t+1}) | F_t] \leq U_t,$$

which is a contradiction. So it is true that $u^n_t \leq U_t, dt \times dP$-a.e..

By Comparison Theorem 4.1, it is easy to verify that $u^1_t \leq u^2_t \leq \ldots \leq U_t$. Similar to the discussion of $Y^n_t$ in theorem 4.3, we also have that $u^n_t$ is essentially bounded, consequently $n(U_t - u^n_t)^+$ is also essentially bounded for an arbitrary $n$. Let $n \to \infty$, we have $u^n_t \uparrow U_t, dt \times dP$-a.e..  

Proof of Theorem 6.7. We only need show that the theorem is true when $T = t + 1$ by the existence and uniqueness theorem. Define $A^n_t := n \sum_{0 \leq s < t}(U_s - u^n_s)^+ = n \sum_{0 \leq s < t}(U_s - u^n_s)$, which is predictable and essentially bounded for an arbitrary $n$, and non-decreasing in $t$. Then $u^n_t$ has the following expression

$$u^n_t = U_{t+1} + A^n_{t+1} - A^n_t + f(u^n_t, Z_t) - Z^*_t M_{t+1}, n \in \mathbb{Z}^+. \quad (6.9)$$

We rewrite (6.6) in the following forward version:

$$u^n_t = u^n_0 - A^n_t - \sum_{0 \leq s < t} f(u^n_s, Z_t) - \sum_{0 \leq s < t} Z^*_s M_{t+1}, n \in \mathbb{Z}^+. \quad (6.10)$$

It follows that

$$A^n_t = -u^n_t + u^n_0 - \sum_{0 \leq s < t} f(u^n_s, Z_t) - \sum_{0 \leq s < t} Z^*_s M_{t+1}, n \in \mathbb{Z}^+. \quad (6.11)$$
Let $n \to \infty$ and by lemma 6.8, we have $A^n_t$ is convergent. Denote $A := \lim_{n \to \infty} A^n_t$.

Let $n \to \infty$, then (6.9) becomes

$$U_t = U_{t+1} + A_{t+1} - A_t + f(U_t, Z_t) - Z^*_t M_{t+1}, n \in \mathbb{Z}^+.$$ 

Which means $U_t = G_{t, t+1}(U_{t+1}, A)$, consequently the result is true.

**Remark 6.9.** This theorem can be applied in finance that a $g$-super-martingale $U$ can be equivalent to the dynamical evaluation of the sum of an increasing process $A$ and the “final payoff” by one step method, where $A$ can be dividend or consumption process.

### 6.2 Optional Sampling Theorem for $G_{\sigma, \tau}(\cdot)$

We now consider the situation where the times $s$ and $t$ in $G_{s, t}(\cdot)$ is replaced by stopping times $0 \leq \sigma, \tau \leq T$ instead of deterministic times $s$ and $t$. We shall develop a generalization version of the optional sampling theorem for $g$-super and $g$-sub-martingale in our framework. Firstly, we will define $G_{\sigma, \tau}(\cdot)$.

For a given $U \in \mathcal{S}_\tau$, we can solve the following BSDE step by step

$$Y_\sigma = U + \sum_{\sigma \wedge \tau \leq u < \tau} f(u, Y_u, Z_u) - \sum_{\sigma \wedge \tau \leq u < \tau} Z^*_u M_{u+1}.$$ 

Then define $G_{\sigma, \tau}(U) := Y_\sigma$.

**Lemma 6.10.** Let $\tau$ be stopping times take values on $\{0, 1, \ldots, T\}$ and $(U_t)$ be a $g$-super-martingale. Then for each $t \in \{0, 1, \ldots, T\}$, we have

$$G_{t, \tau}(U_\tau) \leq U_{\tau \wedge t}, \text{ P-a.s..} \quad (6.10)$$

**Proof.** We first consider the case where $t \geq T-1$. Note that $\tau = \tau 1_{\{\tau \leq T-1\}} + T 1_{\{\tau = T\}}$ and the fact that $1_{\{\tau \leq T-1\}}$ and $1_{\{\tau = T\}}$ are $\mathcal{F}_t$-adapted.

Applying Corollary 6.4(4), we have

$$G_{t, \tau}(U_\tau) = G_{t, \tau}(U_T 1_{\{\tau = T\}} + U_\tau 1_{\{\tau \leq T-1\}})$$

$$= 1_{\{\tau = T\}} G_{t, T}(U_T) + U_\tau 1_{\{\tau \leq T-1\}}$$

$$\leq 1_{\{\tau = T\}} U_t + U_\tau 1_{\{\tau \leq T-1\}} = U_{\tau \wedge t}.$$
Particularly, we have \( G_{T-1,\tau}(U_\tau) \leq U_{\tau \land (T-1)} \). Since \( \tau \land i \) is valued in \( \{0, 1, ..., i\} \), we can repeatedly use the above result to check the case where \( t = i, \ i < T - 1 \), i.e.

\[
G_t,\tau(U_\tau) = G_{t, T-1}(G_{T-1,\tau}(U_\tau)) \leq G_{t, \tau \land (T-1)}(U_{\tau \land (T-1)}) \leq \cdots \leq G_{t, \tau \land i + 1}(U_{\tau \land (i+1)}) \leq U_{\tau \land i} = U_{\tau \land t},
\]

which complete the proof.

In (6.10), we replace the time \( t \) by a stopping time \( \sigma \), we have the following more general proposition.

**Proposition 6.11.** Let \( \tau \) and \( \sigma \) be stopping times take values on \( \{0, 1, ..., T\} \) and \((U_t)\) be a \( g \)-super-martingale. Then for each \( t \in \{0, 1, ..., T\} \), we have

\[
G_{\sigma,\tau}(U_\tau) \leq U_{\tau \land \sigma}, \ P - a.s..
\]

**Proof.** From (6.10), we have

\[
G_{\sigma,\tau}(U_\tau) = \sum_{i=0}^{T} G_{i,\tau}(U_\tau) 1\{\sigma = i\} \leq \sum_{i=0}^{T} (U_{\tau \land i}) 1\{\sigma = i\} = U_{\tau \land \sigma}.
\]

From proposition 6.11, we also have the following Optional Sampling Theorem for \( g \)-martingale in our framework.

**Theorem 6.12.** *(Optional Sampling Theorem)* Let \( \tau \) and \( \sigma \) be stopping times take values on \( \{0, 1, ..., T\} \) such that \( \sigma \leq \tau \) and \((U_t)\) be a \( g \)-martingale (resp. \( g \)-super-martingale, \( g \)-sub-martingale). Then for each \( t \in \{0, 1, ..., T\} \), we have

\[
G_{\sigma,\tau}(U_\tau) = U_\sigma (\text{resp.} \leq U_\sigma, \geq U_\sigma), \ P - a.s..
\]

### 6.3 Applications to multiple prior martingale under Knightian uncertainty

It is possible to discuss more details about \( g \)-martingale and \( g \)-expectation as in [17], but we will not develop this point here. We now focus our attention on the theory of \( g \)-martingale under a multiple prior framework.
In order to solve the Knightian uncertainty problem in the sense that the distribution considered is not exactly known, Frank Riedel [21] developed a theory of optimal stopping along the classical lines extending suitable results from usual martingale theory to the nonlinear multiple prior expectation operator, which works as long as the set of priors is time consistent. The method can be used in the fields of Microeconomics, Operations Research and Finance and so on. He defined the multiple prior martingale \((U_t)\) if it satisfies

\[
U_t = \text{ess inf}_{p \in \Lambda} E^p[U_{t+1}|\mathcal{F}_t],
\]

where \(\Lambda\) is the set of time-consistent priors, but we can not solve the essential infimum easily. In this subsection, we will transfer this computing problem into solving a kind of BSDE, which can be solved by some numerical methods. In [13], Li et al. discussed how to use BSDE based on Brownian Motion to compute one kind of the minimum expectation based on [3], and now we want to study the connection between minimum expectation and g-expectation in our framework.

We denote by \(\mathcal{Q}\) the set of all probability measures \(Q \sim P\). For any \(Q \in \mathcal{Q}\), let \(W_t := E\left[\frac{dQ}{dP}|\mathcal{F}_t\right]\), then \(W_t\) is a martingale and by Martingale Representation Theorem in [3], there exists an adapted process \(z\) such that \(W_t = 1 + \sum_{0 \leq s < t} z_s M_{s+1}\).

Let \(\theta_t := \frac{z_t}{W_t}\), then \(W_t = \prod_{0 \leq s < t} (1 + \theta_s M_{s+1})\), and so

\[
\frac{dQ}{dP} = W_T = \prod_{0 \leq s < T} (1 + \theta_s M_{s+1}),
\]

which means there exists an adapted process \(\{\theta_t\}\) such that \(\frac{dQ}{dP}\) can be generated by (6.11) for any \(Q \in \mathcal{Q}\). We denote \(Q^\theta\) by the probability measure generated by \(\{\theta_t\}\). To guarantee the multiple prior martingale is well-defined, we shall consider the following probability measure set \(\mathcal{B}\):

\[
\mathcal{B} = \{Q^\theta \in \mathcal{Q} : \text{the adapted process}\{\theta_t\}\text{generating} Q^\theta \text{satisfies} \sup_{0 \leq t \leq T} |\theta_t| \leq k\},
\]

where \(k > 0\) is a given constant.

**Definition 6.13.** Suppose \(\xi \in \mathcal{S}^T\), let

\[
G(\xi) := \inf_{Q \in \mathcal{B}} \{E_Q[\xi]\}; \quad G(\xi|\mathcal{F}_t) := \text{ess inf}_{Q \in \mathcal{B}} \{E_Q[\xi|\mathcal{F}_t]\}, 0 \leq t \leq T.
\]
Then we call $G(\xi)$ and $G(\xi|\mathcal{F}_t)$ minimum expectation and minimal conditional expectation respectively of $\xi$ about $\mathcal{B}$. Similarly, we can define the corresponding maximum expectation and minimal conditional expectation respectively.

**Remark 6.14.** For any $Q \in \mathcal{B}$, $\frac{dQ}{dP}$ is essentially bounded because $M_t$ is essentially bounded and (6.13), we have $E_Q[\xi] < \infty$, so the Definition 6.13 is well defined.

Next we will give the main results of this section. Suppose $\xi \in \mathcal{S}^T$, $k$ be the constant in (6.13), and $(y_t, z_t)$ be the solution of the following BSDE:

$$y_t = \xi - \sum_{t \leq s < T} k|z_s| - \sum_{t \leq s < T} z_s^* M_{s+1}, 0 \leq t \leq T. \quad (6.13)$$

Or equivalently,

$$y_t = y_{t+1} - k|z_t| - z_t^* M_{t+1}, y_T = \xi, 0 \leq t \leq T.$$ 

Then we have the following connection between minimum expectation and $g$-expectation.

**Theorem 6.15.** Under (H4), suppose $\xi \in \mathcal{S}^T$, $k$ is the constant in (6.13), $M_{t+1} = X_{t+1} - E[X_{t+1}|\mathcal{F}_t]$ where $\text{var}(X_{t+1}|\mathcal{F}_t) = 1$. Then we have

$$G(\xi) = G_{0,T}^{-k}(\xi), G(\xi|\mathcal{F}_t) = G_{t,T}^{-k}(\xi),$$

where $G_{\cdot, \cdot}^{-k}(\cdot)$ denotes the corresponding solution of BSDE generated by $f(t, y, z) = -k|z|$ and $\text{var}(X_{t+1}|\mathcal{F}_t) := E[X_{t+1}^2|\mathcal{F}_t] - (E[X_{t+1}|\mathcal{F}_t])^2$.

**Proof.** For a given $\xi \in \mathcal{S}^T$, by the existence and uniqueness theorem in [8], we know BSDE (6.13) has a unique solution $(y_t, z_t)$. Let $a_s = -k \text{sgn} z_s$, then (6.13) can be rewritten as

$$y_t = \xi - \sum_{t \leq s < T} z_s^* \tilde{M}_{s+1}, 0 \leq t \leq T,$$

where $\tilde{M}_{s+1} = M_{s+1} - a_s$.

In our framework, we can divide the equation into many “one step” equations, so we only need consider the times $t$ and $t+1$, i.e.

$$y_t = y_{t+1} - z_t^* \tilde{M}_{t+1}, y_T = \xi, 0 \leq t < T,$$

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In this case, suppose $Q^a$ is the probability measure generated by $\{a_t\}$. Define $\frac{dQ^a_{\cdot}}{dP_{\cdot}}|_{\mathcal{F}_{t+1}} := 1 + a_t M_{t+1}$ as the corresponding “one step” Radon-Nikodym derivative from $t$ to $t+1$. We can show that $\tilde{M}_{t+1}$ is a martingale difference process under $Q^a$ as follows.

\[
X_{t+1} - E^{Q^a}[X_{t+1}|\mathcal{F}_t] = X_{t+1} - E[X_{t+1}(1 + a_t M_{t+1})|\mathcal{F}_t] \\
= X_{t+1} - E[X_{t+1}|\mathcal{F}_t] - a_t E[X_{t+1}(X_{t+1} - E[X_{t+1}|\mathcal{F}_t])|\mathcal{F}_t] \\
= X_{t+1} - E[X_{t+1}|\mathcal{F}_t] - a_t \tilde{M}_{t+1}.
\]

Thus,

\[
E_{Q^a}[\tilde{M}_{t+1}|\mathcal{F}_t] = 0.
\]

Consequently, $E_{Q^a}[z^*_t \tilde{M}_{t+1}|\mathcal{F}_t] = 0$.

So we have $y_t = E_{Q^a}[y_{t+1}|\mathcal{F}_t] \geq ess \inf_{Q \in \mathcal{B}} E_{Q}[y_{t+1}|\mathcal{F}_t]$.

On the other hand, suppose $Q^\theta \in \mathcal{B}$ which is generated by $\{\theta_t\}$. Then consider the following linear BSDE:

\[
y^\theta_t = \xi - \sum_{t \leq s < T} \theta_s z_s - \sum_{t \leq s < T} z^*_s M_{s+1}, 0 \leq t \leq T. \tag{6.14}
\]

Or equivalently, we have

\[
y^\theta_t = y^\theta_{t+1} - \theta_t z_t - z^*_t M_{t+1}, y^\theta_T = \xi, 0 \leq t \leq T. \tag{6.15}
\]

Similarly to the above method, we solve the above equation by “one step” method and obtain $y^\theta_t = E_{Q^\theta}[y^\theta_{t+1}|\mathcal{F}_t]$. Moreover, note that $-\theta_t z \geq -k|z|, \forall (z, t) \in \mathbb{R}^m \times \{0, 1, ..., T\}$. Using the Comparison Theorem to equations (6.13) and (6.14), we have

\[
y^\theta_t \geq y_t.
\]

Thus by the Existence and Uniqueness Theorem and (6.15), we have

\[
E_{Q^\theta}[y^\theta_{t+1}|\mathcal{F}_t] = E_{Q^\theta}[\xi|\mathcal{F}_t] \geq y_t.
\]
Then we obtain
\[
\text{ess inf}_{Q \in \mathcal{B}} E_Q[y_{t+1}|\mathcal{F}_t] \geq y_t.
\]

Which means \(y_t = \text{ess inf}_{Q \in \mathcal{B}} E_Q[\xi|\mathcal{F}_t]\). Especially, let \(t = 0\), we have \(y_0 = \text{ess inf}_{Q \in \mathcal{B}} E_Q[\xi]\) i.e. \(G(\xi) = G^{-k}_{0,T}(\xi), G(\xi|\mathcal{F}_t) = G^{-k}_{t,T}(\xi)\).

**Corollary 6.16.** The condition \(\text{var}(X_{t+1}|\mathcal{F}_t) = 1\) in theorem 6.15 guarantees that \(\tilde{M}_t\) is also a martingale difference process. There are many processes satisfying this condition, such as standard random walk and standard basis vector.

**Remark 6.17.** If we want to know whether an adapted stochastic process \((U_t)\) is a multiple prior martingale, we just need compute \(G^{-k}_{t,T}(U_{t+1})\) and verify the equality \(U_t = G^{-k}_{t,t+1}(U_{t+1})\). The theorem 6.15 then gives us a method to compute the multiple prior martingale.

### 6.4 Applications to optimal stopping problems in a multiple prior framework

Actually, Riedel [21] considered the optimal stopping problem under ambiguity for ambiguity-averse agents. This problem can be formulated as follows:

\[
\text{maximize } \inf_{P \in \mathcal{C}} E^P U_\tau \quad \text{over all stopping times } \tau \leq T
\]

for a finite horizon \(T < \infty\), where \(\mathcal{C}\) is the set of priors, \((U_t)_{t \in \mathbb{N}}\) is an essentially bounded and adapted process.

To solve the above problem, Riedel [21] studied a complete solution using the multiple prior Snell envelope \(\bar{U}\) defined by \(\bar{U}_T = U_T\) and

\[
\bar{U}_t = \max\{X_t, \text{ess inf}_{P \in \mathcal{C}} E^P [\bar{U}_{t+1}|\mathcal{F}_t]\}, t \in \{0, 1, \ldots, T - 1\}.
\]

Moreover, Riedel claimed that this approach works as long as \(\mathcal{C}\) is time-consistent which can ensure that the prior constructed in some way belongs to the original set of priors. A more challenging work is how we can obtain the multiple prior envelope and solve it by numerical method. Motivated by the method of solving minimum expectation used in the last section,
we study the connection between a special kind of FS-RBSDEs and multiple prior envelope. Then the problem can be transformed into solving the corresponding FS-RBSDEs as follows:

\[
\begin{aligned}
\bar{U}_t &= \bar{U}_{t+1} - k|Z_t| - Z^*_t M_{t+1} + K_{t+1} - K_t \\
\bar{U}_T &= U_T, \bar{U}_t \geq U_t \\
(U_t - U_t)(K_{t+1} - K_t) &= 0
\end{aligned}
\] (6.16)

Thus, by Theorem 4.3, FS-RBSDEs (6.16) has a unique solution \((\bar{U}_t, Z_t, K_t)\). Then we have the main results as follows.

Actually, we can state that \(\mathcal{B}\) is time-consistent as its own.

**Theorem 6.18.** \(\mathcal{B}\) is time-consistent.

**Proof.** Suppose \(\forall Q^a \in \mathcal{B}\) be the probability measure generated by \(\{a_t\}\).

As in the proof of Theorem 6.15, we can define \(\frac{dQ^a}{dP}||_{\mathcal{F}_{t+1}} := 1 + a_t M_{t+1}\) as the corresponding “one step” Radon-Nikodym derivative from \(t\) to \(t + 1\). Then

\[
\frac{dQ^a}{dP} = \prod_{0 \leq s < T} (1 + a_s M_{s+1}) = \prod_{0 \leq s < T} \frac{dQ^a}{dP}||_{\mathcal{F}_{s+1}},
\]

and

\[
\frac{dQ^a}{dP}||_{\mathcal{F}_{s}} = \prod_{t \leq u < s} (1 + a_u M_{u+1}) = \prod_{t \leq u < s} \frac{dQ^a}{dP}||_{\mathcal{F}_{u+1}}, \quad 0 \leq t < s \leq T.
\]

Thus, for \(\forall Q^b \in \mathcal{B}\), let \((p_t)\) and \((q_t)\) be the density process of \(Q^a\) resp. \(Q^b\) with respect to \(P\), i.e.

\[
p_t = \frac{dQ^a}{dP}||_{\mathcal{F}_t}, \quad q_t = \frac{dQ^b}{dP}||_{\mathcal{F}_t}, \quad 0 < t \leq T.
\]

Fix some stopping time \(\tau\). Define a new probability measure \(R\) for \(0 < t \leq T\) as follows:

\[
\frac{dR}{dP}||_{\mathcal{F}_t} = \begin{cases} 
    p_t & \text{if } 0 < t \leq \tau \\
    \frac{p_t q_t}{q_\tau} & \text{else}
\end{cases}
\]

The task is now to verify that \(R\) belongs to \(\mathcal{B}\) as well. The proof falls naturally into two cases:
If $t \leq \tau$, the result is obviously true;

(2) If else, we have

$$\frac{dR}{dP}\big|_{\mathcal{F}_t} = \prod_{0 \leq u < \tau} (1 + a_u M_{u+1}) \cdot \prod_{\tau \leq u < t} (1 + b_u M_{u+1}).$$

Define $c_t = (a_0, a_1, ..., a_{\tau-1}, b_{\tau}, ..., b_{t-1})$, then $\sup_{0 \leq t \leq T} |c_t| \leq k$.

Thus, we have $R \in \mathcal{B}$.

**Theorem 6.19.** Under Assumption (H4), suppose $U_T \in \mathcal{S}^T$, $k$ is the constant in (6.13), and $M_{t+1} = X_{t+1} - E[X_{t+1}|\mathcal{F}_t]$ where $\text{var}(X_{t+1}|\mathcal{F}_t) = 1$. Then the solution $\bar{U}_t$ of FS-RBSDEs is the multiple prior Snell envelope of $U$.

**Proof.** By Proposition (5.3), we know $\bar{U}_t = U_t \lor (-k|Z_t| + E[\bar{U}_{t+1}|\mathcal{F}_t])$ where $Z^*_t M_{t+1} = \bar{U}_{t+1} - E[\bar{U}_{t+1}|\mathcal{F}_t]$. Then consider the following BSDE:

$$y_t = \bar{U}_{t+1} - k|z_t| - z^*_t M_{t+1}.$$ 

It follows that $y_t = -k|Z_t| + E[\bar{U}_{t+1}|\mathcal{F}_t]$, where $z^*_t M_{t+1} = \bar{U}_{t+1} - E[\bar{U}_{t+1}|\mathcal{F}_t]$. Then by Assumption (H4), we have $Z_t = z_t$ $P$-a.s. i.e. $y_t = -k|Z_t| + E[\bar{U}_{t+1}|\mathcal{F}_t]$.

Moreover, by Theorem 6.15, we have $y_t = \text{ess inf}_{P \in B} E^P[\bar{U}_{t+1}|\mathcal{F}_t]$.

Thus, we have $\bar{U}_t = U_t \lor (\text{ess inf}_{P \in B} E^P[\bar{U}_{t+1}|\mathcal{F}_t])$.

Naturally, by the above theorem and Proposition 2.5 as well as some properties of FS-RBSDEs, we can obtain the following useful results:

(i) $ar{U}$ is the smallest multiple prior super-martingale with respect to $\mathcal{B}$ that dominates $U$,

(ii) $ar{U}$ is the value process of the following optimal stopping problem under ambiguity, i.e.

$$\bar{U}_t = \text{ess sup}_{\tau \in \mathcal{J}_t} \text{ess inf}_{P \in B} E^P[U_\tau|\mathcal{F}_t].$$

(iii) an optimal stopping rule can be given by

$$\tau^* = \inf\{t \geq 0 : \bar{U}_t = U_t\}.$$
We will consider a simple example in the condition where the distribution is not exactly known. It has been discussed in [15] (see example 10.2.2) and [5] (see example 5.1). Here we reconsider this problem in our framework.

Example 6.20. Suppose someone owns an asset, whose value process is governed as follows:

\[
\begin{aligned}
Y_{t+1} - Y_t &= bY_t + \sigma Y_t (M_{t+1} - M_t) \\
Y_0 &= y > 0,
\end{aligned}
\]  

(6.17)

where \(b, \sigma\) are given, \(M_t\) is a martingale difference process generated by some 1-dimensional stochastic process \(X_t\), where \(\text{var}(X_{t+1} | F_t) = 1\), i.e. \(M_t = X_t - E[X_t | F_{t-1}]\) and \(M_0 = 0\). It also satisfies Assumption (H4). For simplicity, we let the interest rate is 0 and \(b > 0\).

We aim to find the optimal time \(\tau^* \in \{0, 1, ..., T\}\) to sell this asset. If there does not exist any uncertainty, the risk only comes from the martingale difference process. The problem can be formulated as follows:

\[
\sup_{0 \leq \tau \leq T} E[Y_\tau].
\]

From (6.17), we know

\[
E[Y_{t+1} - Y_t] = bE[Y_t].
\]

By \(Y_0 = y > 0\), we have \(Y_1 \geq Y_0 > 0\) P-a.s.. Then by induction we get \(Y_{t+1} \geq Y_t > 0\) P-a.s.. So the optimal time is \(\tau^* = T\), which implies that the owner is better hold this asset until the time \(T\).

Now if there exists uncertainty in this problem, which can be represented by a family probability measures:

\[
\frac{dQ^\theta}{dP} := \prod_{0 \leq t < T} 1 + \theta_t M_{t+1}.
\]

Where \(\theta\) is a predictable process taking values in \([-1, 1]\).

If this asset owner is a ambiguity averse decision maker, this model can be formulated as:

\[
\sup_{\tau \in \{0, 1, ..., T\}} \inf_{\theta} E_Q^\theta [Y_\tau].
\]
By the above theory, we know this model can be transferred to solve the following DF-RBSDE:

\[
\begin{cases}
\bar{U}_t = \bar{U}_{t+1} - |Z_t| - Z^*_t M_{t+1} + K_{t+1} - K_t \\
\bar{U}_T = Y_T, \bar{U}_t \geq Y_t \\
(\bar{U}_t - Y_t)(K_{t+1} - K_t) = 0.
\end{cases}
\]

By the Proposition 2.5, we know \( \tau^* = \inf\{t \leq u \leq T; \bar{U}_u = Y_u\} \) and \( \tau^* = T \) if \( \bar{U}_u > Y_u, t \leq u \leq T \).

7 Applications to American Contingent Claims

In a complete market, it is well-known that the price of an American option corresponds to the solution of reflected BSDEs which are based on the Brownian motion in the continuous situation, where the information flow is generated by the Brownian motion [11]. Actually, the transactions occurred in the finance market are discrete though many researchers consider the problem in the continuous framework because of some conveniences. So it is challenging to explore the valuation problem of American option in the discrete time and finite state case, where the pricing of stock can be derived by some martingale difference process in stead of Brownian motion. Then the problem of pricing of American option can be attributed to the solutions of FS-RBSDEs based on \( \{M_u, 0 \leq u \leq T\} \) in the discrete time and finite state space which is not yet limited to the Brownian motion or random walk. In particular, \( M_t \) can be generated by a stock price process \( \{S_t\} \), i.e. \( M_t = S_t - E[S_t|\mathcal{F}_{t-1}] \) and \( M_0 = 0 \).

7.1 The model of pricing of American options in a dynamically complete market

Throughout this section, we maintain the Assumption (H4) holds. For guarantee of dynamically complete market, we make the following standard assumptions:

- The short rate \( r \) is a predictable and bounded process which is generally nonnegative.
- The stock appreciation rates \( b = (b^1, b^2, ..., b^n)^* \) is a predictable and bounded process.
• The volatility matrix $\sigma = (\sigma^{i,j})$ is a predictable and bounded process which has full rank a.s for all $t \in \{0, 1, ..., T\}$ and the corresponding inverse matrix is also a bounded process.

• There exists a predictable and bounded process vector $\theta$ named risk premium, such that

$$b_t - r_t \mathbf{1} = \sigma_t \theta_t, \, dt \times dP - a.e..$$

where $\mathbf{1}$ is the vector whose every component is 1.

We start with the classical setup for discrete time asset pricing: the basic securities consist of $m+1$ assets $\{S^i_t; 0 \leq t \leq T, i = 0, 1, ..., m\}$, one of which is a non-risky asset with price process as follows:

$$S^0_{t+1} - S^0_t = r_t S^0_t,$$

where $r_t$ is the interest rate. Other $m$ risky asset (the stocks) are traded discretely, of which the price process $S^i_t$ for one share of $i$th stock is governed by the linear difference equation

$$S^i_{t+1} - S^i_t = S^i_t[b^i_t + \sum_{j=1}^m \sigma^{i,j}_t M^j_{t+1}], \, i = 1, ..., m.$$

where $M_t = (M^1_t, M^2_t, ..., M^m_t)$* is a martingale difference sequence on $R^m$.

Moreover, suppose the portfolio process is $H = (H_0, H_1, ..., H_m)$ which is self-financing. Then the value process $V = (V_0, V_1, ..., V_t)$ can be formulated as follows:

$$V_t = H_0(t)S^0_t + \sum_{i=1}^m H_i(t)S^i_t = H_0(t+1)S^0_t + \sum_{i=1}^m H_i(t+1)S^i_t.$$

So

$$V_{t+1} - V_t = H_0(t+1)(S^0_{t+1} - S^0_t) + \sum_{i=1}^m H_i(t+1)(S^i_{t+1} - S^i_t)$$

$$= r_t V_t + \sum_{i=1}^m H_i(t+1)S^i_t(b^i_t - r_t + \sum_{j=1}^m \sigma^{i,j}_t M^j_{t+1})$$

$$= r_t V_t + \pi^*_t \sigma_t(M_{t+1} + \theta_t), \quad (7.1)$$

where $\pi_t = (\pi^1_t, ..., \pi^m_t), \, \pi^i_t = H_i(t)S^i_t$. Note that the equation (7.1) satisfies the assumptions of the Existence and Uniqueness Theorem, so there exists
a unique solution \((V_t, \pi_t)\) which means any essentially bounded contingent claim is attainable in a dynamically consistent market. From (7.1), we can solve \(V_t\) easily.

As is known, the European Option can be viewed as the non-negative contingent claim, and the pricing of European contingent claims can be formulated in terms of backward stochastic differential equations, even in an imperfect market. We also have the corresponding representation of the European contingent claims in our framework which is more realistic.

In a finance market, whatever perfect or imperfect, suppose we only know the contingent claim \(\xi\) is attainable (or marketable), we want to know the value process \(V_t\) and the portfolio \(H_t\), then we can use the following backward method.

Because \(V_T = \xi\), we can firstly solve

\[
\xi = H_0(T)S_T^0 + \sum_{i=1}^m H_i(T)S_T^i.
\]

More again, notice that \(H\) is predictable, so we can obtain \(H(T)\); on the other hand, \(H\) is self-financing which means

\[
V_{T-1} = H_0(T)S_{T-1}^0 + \sum_{i=1}^m H_i(T)S_{T-1}^i.
\]

Then we can get \(V_{T-1}\). Now using the same backward method, we can solve \(V_0\) finally (see more details and examples in [19].

Let us consider the valuation problem of an American contingent claim \(\{\xi_t, 0 \leq t \leq T\}\) the holder can exercise only once at any time between \(\{0, 1, ..., T\}\) and anyone’s actions can not affect market prices. The key problem is to determine value \(V_t\) of this option, that is, the value process \(V = \{V_t; t = 0, 1, \cdots T\}\). As is known, this kind of claim can not be hedged by a general self-financing portfolio, and so it is necessary to introduce self-financing super-strategies with a cumulative consumption process.

**Definition 7.1.** A self-financing super-strategy is a vector process \((V, \pi, C)\), where \(V\) is the market value (or wealth process), \(\pi\) is the portfolio process which is bounded, and \(C\) is the cumulative consumption process, such that

\[
V_{t+1} - V_t = r_t V_t + \pi_t^s \sigma_t [\theta_t + M_{t+1}] - (C_{t+1} - C_t),
\]

where \(C\) is an increasing, right-continuous, adapted process with \(C_0 = 0\).
Given a payoff process $\{\xi_t; t \in \{0, 1, ..., T\}\}$, a super-strategy is called a super-hedging strategy if there holds

$$V_t \geq \xi_t, t \in \{0, 1, ..., T\}, P-a.s..$$

The smallest endowment to finance a super-hedging strategy is the price of the American option which could be greater than the price of $\xi_\tau$ for any stopping time $\tau \leq T$.

According to the Existence and Uniqueness Theorem of FS-RBSDEs, we can prove the existence of a minimal essentially bounded super-hedging strategy associated with an essentially bounded payoff.

**Theorem 7.2.** In a dynamically consistent complete market, consider an essentially bounded payoff process $\xi$ with $\lim_{t \to T} \xi_t \leq \xi_T$, a.s.. The American price process $Y$ is associated with an essentially bounded super-hedging strategy, that is there exists a unique essentially bounded process $(\phi, K)$ such that for $t \in \{0, 1, ..., T\}$,

$$Y_{t+1} - Y_t = r_t Y_t + \phi_t^* \sigma_t [\theta_t + M_{t+1}] - (K_{t+1} - K_t).$$

Moreover $K$ satisfies the minimality condition $\sum_{0 \leq t < \tau} (Y_t - \xi_t)(K_{t+1} - K_t) = 0$.

The American price is also the maximum $X^*$ of the European price process associated with an exercise at the stopping time $\tau$ before $T$, that is

$$Y_t = X^*_t = \underset{\tau \in J_t}{\text{ess sup}} X_t(\tau, \xi_\tau).$$

The stopping time $D_t = \inf\{t \leq s \leq T, Y_s = \xi_s\}$ is optimal, that is $Y_t = X_t(D_t, \xi_{D_t})$, where $X_t(\tau, \xi_\tau)$ denote the European price at $t$ dominated by $\tau$ and the terminal value is $\xi_\tau$.

**Proof.** The existence and uniqueness follows from Theorem 4.3.

Moreover, given a stopping time $\tau \in \{t, t+1, ..., T\}$, let us consider a super-hedging strategy $(Y, \phi, K)$ and calculate the variation of $Y$ between $t$ and $\tau$.

$$Y_t = E[- \sum_{t \leq s < \tau} (r_s Y_s + (\phi_s)^* \sigma_s) + Y_\tau + K_\tau - K_t | \mathcal{F}_t]$$

$$= E[- \sum_{t \leq s < \tau} (r_s Y_s + (\phi_s)^* \sigma_s) + \xi_\tau | \mathcal{F}_t].$$
By Comparison Theorem on \( \{t, t + 1, \ldots, T\} \), \( Y \) dominates the price process for the contingent claim with exercise at time \( \tau \), \( X(\tau, \xi_\tau) \). Hence,

\[
Y_t \geq X_t^* = \text{ess sup}_{\tau \in J_t} X(\tau, \xi_\tau).
\]

On the other hand, we can choose an optimal elementary stopping time in order to get the reversed inequality. Define

\[
D_t = \inf\{t \leq s \leq T; Y_s = \xi_s\}.
\]

and \( D_t = T \) if \( Y_u > S_u, t \leq u \leq T \).

Similar to the discussion in Proposition 2.5 and by uniqueness of BSDEs, for \( s \in \{t, t + 1, \ldots, D_t\} \), we have

\[
Y_s = E[- \sum_{s \leq u < D_t} (r_u Y_u + (\phi_u)^* \sigma_u) + Y_{D_t} + K_{D_t} - K_s|F_s]
= E[- \sum_{s \leq u < D_t} (r_u Y_u + (\phi_u)^* \sigma_u) + Y_{D_t}|F_s]
= X_s(D_t, \xi_{D_t})
\leq \text{ess sup}_{\tau \in J_t} E[- \sum_{s \leq u < D_t} (r_u Y_u + (\phi_u)^* \sigma_u) + \xi_\tau 1_{\{\tau < T\}} + \xi_T 1_{\{\tau = T\}}|F_s].
\]

Then the result follows immediately. \( \square \)

In particular, \( M_t \) can be generated by a stock price process \( \{S_t\} \), i.e. \( M_t = S_t - E[S_t|F_{t-1}] \) and \( M_0 = 0 \). In this case, we will give a detailed example to show the pricing model of the European Option in a dynamically consistent complete market.

**Example 7.3.** For simplicity, suppose \( T = 1, m = 1, \Omega = \{\omega_1, \omega_2\}, r = 0, \) and \( Y^B \) is the wealth process of non-risky asset. Then \( M_t = S_1 - E[S_t|F_0] = \Delta S - E[\Delta S|F_0] \) and \( M_0 = 0 \), where \( \Delta S = S_1 - S_0 \). Suppose we have \( Y_1 = \xi \) given by

\[
\xi(\omega) = \begin{cases} 
6, & \omega = \omega_1 \\
8, & \omega = \omega_2.
\end{cases}
\]

Moreover, we have the stock price as follows:
Then the pricing model is given by

\[
Y_1 = Y_0 + \Delta Y = Y_0 + \frac{Y_0 - Y^B}{S_0} \Delta S
\]

\[
= Y_0 + \frac{Y_0 - Y^B}{S_0}(M_1 + E[\Delta S|\mathcal{F}_0])
\]

\[
= Y_0 + Z_0M_1 + Z_0E[S_1 - S_0|\mathcal{F}_0].
\]

where \(Z_0 = \frac{Y_0 - Y^B}{S_0}\), which is the shares of the stock at time 0.

It follows that

\[
Y_0 = Y_1 - Z_0M_1 - Z_0E[S_1 - S_0|\mathcal{F}_0].
\]

(7.2)

The risk neutral probability is \(Q = (1/2, 1/2)\), under which we have \(E_Q[S_1 - S_0|\mathcal{F}_0] = 0\), and

\[
M_1(\omega) = \begin{cases} 
-1, & \omega = \omega_1 \\
1, & \omega = \omega_2
\end{cases}
\]

Then (7.2) becomes

\[
Y_0 = Y_1 - Z_0M_1.
\]

Taking conditional expectation for (7.3) gives

\[
Y_0 = E_Q[Y_1|\mathcal{F}_0] = E_Q[\xi|\mathcal{F}_0] = 7
\]

\[
Z_0M_1 = Y_1 - Y_0 = \begin{cases} 
-1, & \omega = \omega_1 \\
1, & \omega = \omega_2
\end{cases}
\]

Then \(Z_0 = 1\), which gives \(Y^B = 2\). Thus, the portfolio is \((2, 7)\).

Next, we will consider the valuation problem of an American contingent claim \(\{\xi_t\}\) in the following example in a dynamically consistent complete market.
Example 7.4. Suppose the conditions in Example 7.4 also hold here. Given a process \( \{\vartheta_0, \vartheta_1\} \) as follows: \( \vartheta_0 = 1 \), and
\[
\vartheta_1 = \begin{cases} 
5, & \omega = \omega_1 \\
7, & \omega = \omega_2.
\end{cases}
\]

By Theorem 7.3, there exists a unique essentially bounded process \((\phi, K)\) such that
\[
Y_0 = Y_1 - Z_0 M_1 - Z_0 E\left[S_1 - S_0 | \mathcal{F}_0\right] + K_1 - K_0,
\]
which can be rewritten as
\[
Y_0 = Y_1 - Z_0 M_1 + K_1 - K_0.
\]

Moreover \( K \) satisfies the minimality condition \((Y_0 - \vartheta_0)(K_1 - K_0) = 0\).
By the proof of Theorem 4.3, we have
\[
Z_0 M_1 = Y_1 - E[Y_1 | \mathcal{F}_0] = \begin{cases} 
-1, & \omega = \omega_1 \\
1, & \omega = \omega_2.
\end{cases}
\]

It also follows \( Z_0 = 1 \). And
\[
Y_0 = E[Y_1 | \mathcal{F}_0] + K_1 - K_0 = 7 + K_1 - K_0.
\]

Then
\[
Y_0 - \vartheta_0 = 6 + K_1 - K_0 > K_1 - K_0 \geq 0.
\]
(7.3)

So we get \( K_1 - K_0 = 0 \).
Then we have \( Y_0 = 7 \) and \( Y^B = 2 \).
Otherwise, if we let \( \vartheta_0 = 8 \), then (7.3) can be rewritten as
\[
0 \leq Y_0 - \vartheta_0 = -1 + K_1 - K_0 < K_1 - K_0.
\]

More again by the minimality condition, we have \( Y_0 = \vartheta_0 = 8 \).
Actually, the method used in above method is corresponding to Remark 5.2.
7.2 The model of pricing of American Options in an incomplete market

If American Options are considered in an incomplete market, there is more than one equivalent martingale measure, and then we have to face with the multiple prior set; alternatively, some people may want to assess the risk of an option by studying the optimal stopping problems under coherent risk measures, they again have a multiple prior setting. In this case, people want to know the minimum and maximum price of the option, i.e. minimum and maximum conditional expectation in mathematical terminology.

Considering an investor who exercises an American Option that pays off \( U_t = F(t, S_t) \) when exercised at time \( t \), we aim to solve the following problem:

\[
\text{maximize } \inf_{P \in \mathcal{P}} E^P U_\tau \text{ over all stopping times } 0 \leq \tau \leq T.
\]

By Theorem 6.19, we can solve a special kind FS-BSDE to obtain the multiple prior Snell envelope of \( U \), which is the desired solution.

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