State Space Formulas for a Suboptimal Rational Leech Problem I: Maximum Entropy Solution

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Abstract. For the strictly positive case (the suboptimal case) the maximum entropy solution \( X \) to the Leech problem \( G(z)X(z) = K(z) \) and \( \|X\|_\infty = \sup_{|z| \leq 1} \|X(z)\| \leq 1 \), with \( G \) and \( K \) stable rational matrix functions, is proved to be a stable rational matrix function. An explicit state space realization for \( X \) is given, and \( \|X\|_\infty \) turns out to be strictly less than one. The matrices involved in this realization are computed from the matrices appearing in a state space realization of the data functions \( G \) and \( K \). A formula for the entropy of \( X \) is also given.

Mathematics Subject Classification (2010). Primary 47A57; Secondary 47A68, 93B15, 47A56.

Keywords. Leech problem, stable rational matrix functions, commutant lifting theorem, state space representations, algebraic Riccati equation.

1. Introduction

Let \( G \) and \( K \) be matrix-valued \( H^\infty \) functions on the open unit disc \( \mathbb{D} \) of sizes \( m \times p \) and \( m \times q \), respectively, and let \( T_G \) and \( T_K \) denote the corresponding block lower triangular Toeplitz operators,

\[
T_G : \ell^2_+ (\mathbb{C}^p) \to \ell^2_+ (\mathbb{C}^m), \quad T_K : \ell^2_+ (\mathbb{C}^q) \to \ell^2_+ (\mathbb{C}^m).
\]

A \( p \times q \) matrix-valued \( H^\infty \) function \( X \) is called a solution to the Leech problem associated with \( G \) and \( K \) whenever

\[
G(z)X(z) = K(z) \quad (z \in \mathbb{D}) \quad \text{and} \quad \|X\|_\infty = \sup_{z \in \mathbb{D}} \|X(z)\| \leq 1.
\]  

The Leech problem is an example of a metric constrained interpolation problem, the first part of (1.1) is the interpolation condition, and the second part is the metric constraint. In a note dating from 1971/1972, only published recently [18], see also [17], Leech proved that the problem is solvable if and only if the operator \( T_GT_G^* - T_KT_K^* \) is nonnegative. Later the Leech theorem
was derived as a corollary of more general results; see, e.g., [19, page 107], [8, Section VIII.6]), and [2, Section 4.7].

Now assume in addition that $G$ and $K$ are rational. In other words, assume that $G$ and $K$ are stable rational matrix functions. In that case, if the Leech problem associated with $G$ and $K$ is solvable, one expects the problem to have a stable rational matrix solution as well. However, a priori this is not clear, and the existence of rational solutions was proved only recently in [20] by reducing the problem to polynomials, in [16] by adapting the lurking isometry method used in [3], and in [11] by using a state space approach.

In the present paper $G$ and $K$ are also stable rational matrix functions. We assume additionally that the operator $T_G T_G^* - T_K T_K^*$ is strictly positive. It is then known from commutant lifting theory that the Leech problem has a unique maximum entropy solution, that is, the (unique) solution $X$ to the Leech problem associated with $G$ and $K$ for which the quantity

$$
\mathcal{E}(X) = \frac{1}{2\pi} \int_0^{2\pi} \ln \det[I_q - X(e^{i\omega})^* X(e^{i\omega})]d\omega
$$

(1.2)

is maximal. In this paper we show that this maximum entropy solution is a stable rational matrix function, we derive an explicit formula for this solution and a formula for its entropy $\mathcal{E}(X)$; see Theorem 1.2 below. When $T_G T_G^* - T_K T_K^*$ is only non-negative, the maximum entropy solution still exists but the problem whether or not it is rational remains open.

To prove the above mentioned results, we use the fact, well-known from mathematical systems theory (see, e.g., Chapter 1 of [7] or Chapter 4 in [4]), that rational matrix functions admit state space realizations. For our $G$ and $K$ this means that the matrix function $[G \ K]$ admits a representation of the following form:

$$
[G(z) \ K(z)] = [D_1 \ D_2] + zC(I_n - zA)^{-1} [B_1 \ B_2].
$$

(1.3)

Here $I_n$ is the $n \times n$ identity matrix, $A$ is an $n \times n$ matrix, and $B_1, B_2, C, D_1$ and $D_2$ are matrices of appropriate sizes. Moreover, since $G$ and $K$ are stable rational matrix functions, $G$ and $K$ have no pole in the closed unit disc, and therefore we may assume that matrix $A$ is stable, that is, $A$ has all its eigenvalues in the open unit disc. The realization (1.3) is called minimal if there exists no realization of $[G \ K]$ as in (1.3) with ‘state matrix’ $A$ of smaller size than the one in the given realization. In that case the order $n$ of $A$ is called the McMillan degree of $[G \ K]$. If the realization (1.3) is minimal, then the matrix $A$ is automatically stable and the observability operator $W_{obs}$, which is defined by

$$
W_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell_+^2(\mathbb{C}^m),
$$

(1.4)