Full counting statistics of persistent current

A. Komnik\textsuperscript{1} and G. W. Langhanke\textsuperscript{1}

\textsuperscript{1}Institut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg, Philosophenweg 19, D-69120 Heidelberg, Germany

(Dated: May 11, 2014)

We develop a method for calculation of charge transfer statistics of persistent current in nanostructures. We consider a simply connected one-dimensional system (a wire) with arbitrary interactions and develop a procedure for the calculation of FCS of persistent currents when the wire is closed into a ring via a weak link. For the non-interacting system we derive a general formula in terms of the two-particle Green’s functions. We show that, contrary to the conventional tunneling contacts, the resulting cumulant generating function has a doubled periodicity as a function of the counting variable. We apply our general formula to short tight-binding chains and show that the FCS perfectly reproduces the known evidence for the persistent current. Its second cumulant turns out to be maximal at the switching points and vanishes identically at zero temperature. Furthermore, we consider a system with an embedded Anderson impurity and employing a self-energy approximation find an overall suppression of persistent current as well as of its noise.

PACS numbers: 73.23.Ra, 72.10.Fk, 73.63.-b

Persistent current (PC) in ring-shaped nanostructures is one of the most fascinating phenomena in mesoscopic physics.\textsuperscript{1, 2} Despite enormous amount of work invested in its study there are still numerous aspects, which are yet not fully understood. Perhaps the most interesting are the issues of how electronic correlations affect the PC and whether it is subject to fluctuations in clean systems without disorder.\textsuperscript{3, 4} A quite natural extension of the latter topic is the question about the full counting statistics (FCS) of persistent current, which, to the best of our knowledge, has not been addressed yet.

In this letter we report the progress along several lines. First of all we develop a general approach, which allows for the evaluation of PC of arbitrarily shaped structure which is closed to a ring via single or several weak links, and which can be applied also to interacting systems. It turns out that for quadratic Hamiltonians one can even derive a closed formula for the FCS using the Green’s functions (GFs) of the original open structure. We generalize this method to the calculation of the FCS and using it we discuss the charge transfer statistics of non-interacting as well as interacting systems and find that in case of a wire with an Anderson impurity there is a suppression of PC as well as of its noise at least for not too strong interactions.

The canonical definition of the PC is based on the observation that if the partition function \( Z \) of a given system depends on the magnetic flux \( \Phi \) piercing it, then the associated conjugate quantity is the finite charge current, so that (we set \( \hbar = e = c = 1 \) throughout the paper)

\[
I_{PC} = -\frac{\partial F}{\partial \Phi},
\]

where \( F \) is the free energy of the system. There are numerous ways of computing \( I_{PC} \), ranging from quantum statistical to scattering methods. We would like not to use them and ask the following question: Let us suppose that we have a simply connected system, e. g. a quantum wire with open boundaries, which we would like to shape into a ring via a weak link, connecting two arbitrary points of the wire. Can we derive an expression for PC using the GFs of the original open system? In order to answer this question one can proceed as follows. Let \( H_0[\psi(x)] \) be the Hamiltonian describing the electronic degrees of freedom in an open simply connected wire (we concentrate on 1D wires, a generalization to higher-dimensional systems can be done along the same lines). It is closed to a ring by tunneling between the points \( x = 0 \) and \( x = L \) with the amplitude \( \gamma_0 \). Then the full system can be described by the following Hamiltonian,

\[
H = H_0[\psi(x)] + \gamma_0 e^{-i\phi} \psi(0) \psi(L) + \gamma_0 e^{i\phi} \psi(0) \psi(L),
\]

where \( \phi = 2\pi \Phi/\Phi_0 \) describes the magnetic flux enclosed by the wire measured in units of the flux quantum \( \Phi_0 = h/e \). In the following we consider a situation when

\[
Z = \frac{1}{Z_0} e^{-\beta \psi(0) \psi(L) + \beta \psi(0) \psi(L)}
\]

\[
Z_0 = \int dx_1 \int dx_2 \cdots \int dx_L e^{-\beta \psi(x_1) \psi(x_2) \cdots \psi(x_L) + \beta \psi(x_1) \psi(x_2) \cdots \psi(x_L)}
\]
the partition function can be written down as a functional integral over the local fields $\phi_0(n) = \psi_n(0)$ and $\phi_L(n) = \psi_n(L)$, where $n$ denotes the respective Matsubara component,

$$Z = Z_0 \int \mathcal{D} \left[ \phi_0, \phi_L, \bar{\phi}_0, \bar{\phi}_L \right] \exp \left\{ -\frac{1}{\beta} \sum_n \left[ \bar{\phi}_0(n) G_{00}^{-1} \right. \right. \times \phi_0(n) + \bar{\phi}_L(n) G_{LL}^{-1} \phi_L(n) - \left. \left. \bar{\phi}_0(n) \left( G_{L0}^{-1} - \gamma_0 e^{i\varphi} \right) \right) \phi_0(n) \right\},$$

where $G_{00}$, $G_{LL}$, $G_{L0}$ and $G_{0L}$ are the corresponding two-particle Green’s functions. In the case of a quadratic action they can easily be evaluated by integrating out all degrees of freedom away from $x = 0, L$ using the standard methods. The GFs entering the above expression, can be shown to descend from the following two independent Matsubara GFs: (i) the local one $G_0(n) = -\langle T \phi_0(n) \bar{\phi}_0(n) \rangle$, which involves only fields at the contact points, in our case $x = 0, L$ (from now on we assume both GFs to be identical due to spatial inversion symmetry of the open system [7]); (ii) and the non-local one connecting both contact points $G_L(n) = -\langle T \phi_L(n) \phi_0(n) \rangle = -\langle T \phi_0(n) \phi_L(n) \rangle$. While the GFs (i) reflect the local density of states in the contact points of the weak link, the GF (ii) describes the single-particle propagation dynamics between the contact points along the wire. With the help of these definitions we find

$$G_{00} = G_{LL} = G_0 - G_L G_0^{-1} G_L$$
$$G_{0L} = G_{L0} = G_0 G_L^{-1} \left( G_0 - G_L G_0^{-1} G_L \right) = G_0 G_L^{-1} G_0.$$  

Using these relations we then immediately obtain the partition function for our composite system

$$\frac{Z}{Z_0} = \prod_n \left( 1 - \gamma_0^2 G_{00} G_0 + 2 \gamma_0 G_L \cos \varphi \right),$$

where $Z_0$ is the partition function of the open system ($\gamma_0 = 0$). The emerging PC is then given by

$$I_{PC} = \frac{1}{\beta} \sum_n \frac{2 \gamma_0 G_0 \sin \varphi}{1 - \gamma_0^2 G_{00} G_0 + 2 \gamma_0 G_L \cos \varphi}.$$ 

Needless to say, with the help of this relation one can recover all known results for the PC in non-interacting systems.

Now we would like to extend our formalism to the computation of the FCS in terms of the cumulant generating function (CGF) $\ln \chi(\lambda)$ [3]. There are different ways to obtain it, the procedure we want to implement is the one of Ref. [9]. It consists of multiplying every forward/backward tunneling term in the Hamiltonian by the factor $e^{1/2 \lambda/2}$, where $\lambda$ is the counting field, which carries opposite sign on the forward/backward Keldysh branch, see e.g. [10].

There is, however, one decisive detail, which makes our ring-shaped setup completely different from the orthodox tunneling systems, for which the FCS procedure was designed. In tunneling systems one usually considers the scattered particles as coming from and vanishing into the ‘infinity’ – the incoherent background of the electrodes. At least from the ideological point of view this is one of the reasons the counting procedure works, all tunneling events are distinguishable from each other. This is not so for a ring. The particles which are already counted once are coming back in coherent fashion, and there is a possibility that the whole procedure would not give meaningful results. This problem can, however, be circumvented by introduction of an additional very large particle bath – for instance just an additional electrode. We shall see later though, that this is not necessary and that the conventional FCS method works well even for very short tight-binding chains. Nonetheless, sometimes it is useful to keep the extra electrode as in some experiments it is explicitly present and used to measure the PC [11].

The FCS computation is easiest in the Keldysh representation, in which all fields have two components $\psi(x) = (\psi_-(x), \psi_+(x))$, describing the forward/backward $(\pm)$ time propagation. All GFs have then 4 different components,

$$G(x - x', t - t') = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix},$$

where $G^{--}$ and $G^{++}$ are the time-ordered and anti-time-ordered components, and $G^{-+}$ and $G^{+-}$ are the lesser and greater Keldysh GFs [12]. Using this language and assuming a quadratic action we can again integrate out all fields away from the contact points and end up with the following generalization of the partition function [3]:

$$Z = Z_0 \int \mathcal{D} \left[ \phi_0, \phi_L, \bar{\phi}_0, \bar{\phi}_L \right] \exp \left\{ -\frac{1}{2\pi} \int d\omega \right\} \left[ \bar{\phi}_0(\omega) G_{00}^{-1} \bar{\phi}_0(\omega) \right.ight.
$$

$$\times \left. \phi_0(\omega) + \bar{\phi}_L(\omega) G_{LL}^{-1} \phi_L(\omega) - \bar{\phi}_0(\omega) \left( G_{L0}^{-1} - \gamma_0 \right) \phi_0(\omega) \right\},$$

where the counting field enters the matrices

$$\gamma_0 = \begin{pmatrix} \gamma_0 e^{i\lambda/2 + \varphi} & 0 \\ 0 & -\gamma_0 e^{-i\lambda/2 - \varphi} \end{pmatrix}.$$ 

The analogous of (2) are now given by

$$G_{00} = G_{LL} = G_0 - G_L G_0^{-1} G_L$$
$$G_{0L} = G_{L0} = G_0 G_L^{-1} - G_0.$$  

The CGF is up to a prefactor given by the $\lambda$-dependent Keldysh partition function of the system, therefore performing the last remaining functional integrations in (3) we arrive at our principal result,

$$\ln \chi(\lambda) = T \int d\omega \ln \det \Lambda,$$

$$\Lambda = G_{00}^{-1} - (G_{0L}^{-1} - \gamma_0^* G_{00} G_{0L}^{-1} - \gamma_0).$$
where $T$ is the waiting time. It is instructive to compare it with the statistics of a simple structureless tunneling contact with the Hamiltonian

$$H = H_0[\psi_{1,2}(x)] + \gamma_0 \psi_1^*(0) \psi_2(0) + \gamma_0^2 \psi_1^*(0) \psi_1(0),$$

with $\psi_{1,2}(x)$ describing the electron fields of the right/left contact. The corresponding FCS is given by

$$\ln \chi(\lambda) = T \int d\omega \ln \det \left( G^{-1}_1 - \gamma_0^2 G_2 \gamma_0 \right),$$

where $G_i$ denotes the local GF on the respective electrode at the tunneling point. As $G_{00}$ in the wire geometry also describes the strictly local GF we have an immediate parallel $G_{00} \leftrightarrow G_1$ and $G_{00} \leftrightarrow G_2$. Therefore we can split the object $\Lambda = \Lambda_c + \Lambda_{ic}$ into the incoherent contribution which has the same shape as the matrix for the tunneling contact,

$$\Lambda_{ic} = G^{-1}_{00} - \gamma_0^2 G_{00} \gamma_0,$$

and the coherent part

$$\Lambda_c = \gamma_0^2 G_{00} G^{-1}_{0L} + G^{-1}_{0L} G_{00} \gamma_0,$$

which is absent in the tunneling contact case and which reflects the fact that in the case of the doubly connected system both contacts ‘talk’ to each other through the wire itself. Its presence has interesting consequences. While the conventional FCS of non-interacting systems turns out to be $2\pi$-periodic in $\lambda$, the FCS of PC has a doubled periodicity. This can be understood in the following way. The term $\Lambda_{ic}$ is of the order $\gamma_0^2$ and arises from the tunneling of the electron across the weak link and back (forward and backward propagation along the Keldysh contour). On the contrary, the coherent term $\Lambda_c$ is linear in $\gamma_0$. That means that the counted electron tunnels through the weak link on the forward (backward) path and returns back following the wire rather than tunneling back directly.

The individual cumulants of the charge transfer statistics are computed as usual using the prescription

$$C_n = (-i)^n \frac{\partial^n \ln \chi(\lambda)}{\partial \lambda^n} \bigg|_{\lambda = 0}.$$

Thus $I_{PC} = C_1/T$. We have tested the prediction for the PC found using $[9]$ and $[4]$ for a tight-binding chain with $N$ sites (we take odd $N$ in order so satisfy the symmetry requirements imposed on the Green’s functions), connected by hopping integral $\gamma$, which sets the energy scale of the system, and fixed chemical potential. We have compared it with the values for $I_{PC}$ for the continuum model of finite length given in $[14]$. In one case we have coupled the central chain site to an additional metallic electrode (bath) with some small hybridization $\Gamma$. We find an excellent agreement already for very short chains with $N = 3$ as long as the hybridization is the smallest energy scale $[15]$. Furthermore we considered a fully decoupled system in which $\Gamma = 0$, obtaining precisely the same results as the classical formulas and recovering all important details such as the parity effect and the $1/N$ dependence of the PC on the chain length $N$ and exponential suppression of $I_{PC}$ with growing temperature $[15, 16]$.\[15, 16]\n
![FIG. 2. Second cumulant $C_2$ measured in units of $\epsilon \gamma T / \Phi_0$ as a function of magnetic flux for different values of temperature, from above $T/\gamma = 1.6, 0.4, 0.1, 0.025, 0.00625$. The energy level width is kept at $\delta/\gamma = 0.025$ and $\gamma_0/\gamma = 0.9$.](image)

In Fig. 2 we plot the second cumulant computed for a decoupled 3-site tight-binding chain with finite very small energy level widening $\delta$. The generic feature is the rapid decay of $C_2$ towards lower temperatures until it ultimately vanishes at $T = 0$. This is to be expected since at zero temperature the system is in its ground state and all fluctuations are frozen out. This finding is compatible to the results of Refs. $[3, 17]$. Another important feature is the maximum located precisely at half period $\[\pi\]$. This is not surprising either since at precisely these points the PC changes sign, therefore the probability for current fluctuations is the highest. $C_3$ shows an interesting behavior as a function of temperature. In the case of tight-binding chains we find $C_3(T) \sim \alpha_1 - \alpha_2/[1 + (T/T^*)^2]$, where $\alpha_{1,2}$ are some model specific constants and $T^*$ is the energy scale set by $E_F$. It turns out to be very close to the characteristic temperature on which the exponential suppression of PC itself occurs.

Contrary to the second cumulant the third one does not vanish at zero temperature at least in the vicinity of the turning point $\Phi/\Phi_0 = 1/2$ or $\phi = \pi$, see Fig. 3 and turns out to have a singularity there. One interesting peculiarity is also the very strong dependence of the third cumulant on the coupling to the lead $\Gamma$, the asymptotic value of $C_3$ at $\Phi/\Phi_0 = 1/2$ approximately growing as $\sim 1/\Gamma$. While the strong dependence of the third cumulant on the effects of the environment has been found before even experimentally $[18]$, it is not clear yet whether this singularity can be explained by the same physical mechanism. $C_3$ shows a non-monotonous behavior as a function of temperature, see Fig. 3. It has a distinct maximum at intermediate temperatures and de-
The temperature dependence of the third cumulant measured in units of $e^2 \gamma T / 4 \Phi_0$ for $\gamma_0 / \gamma = 0.9$ in a coupled system with $\Gamma / \gamma = 0.05$.

Another interesting feature of $C_3$ is its exponential decay with the length of the system. A detailed study of these properties as well as of the higher order cumulants will be presented in [16]. The general tendency is that at zero temperature all even order cumulants vanish and all odd order objects show a behavior similar to that of the third one.

Now we would like to turn to interacting systems. Although it is not always possible to find a representation of the form [19], the general procedure of closing an open system via term [19] and subsequent evaluation of the generalized $\lambda$-dependent Keldysh partition function is still perfectly applicable. To illustrate that we consider a situation in which one of the chain sites is replaced by an Anderson impurity. Although the corresponding exact GFs are not known, there are plenty of powerful approximative techniques. In this introductory study we restrict ourselves to the approach using the second order self-energies [19], closed analytical expressions for which is e.g. presented in [20]. Fig. 5 shows the results for the PC. For growing $U$ the PC as well as the noise turn out to be suppressed while keeping their overall non-interacting shape as a function of the magnetic flux. The decrease of PC is in agreement with general expectation and can be understood in terms of the decreasing impurity density of states at the Fermi edge due to the formation of the Hubbard sidebands.

To summarize, we have discussed the full counting statistics of a persistent current flowing in a quantum wire closed to a ring via a weak link. We derive a simple formula for the FCS in terms of the Green’s functions of a disconnected system. We find that as a function of the counting field the CGF has a doubled periodicity as compared to the FCS of the nanostructures contacted by two independent electrodes. As an application we use the derived formula in order to address the charge statistics of a tight-binding chain with and without an embedded Anderson impurity. We find that the interactions turn out to suppress the persistent current as well as its noise. In the future it would be particularly interesting to address the FCS of exactly solvable interacting systems in ring geometries, such as Luttinger liquids [16], quantum impurity models [21], or consider the analytic properties of the CGF in the spirit of Ref. [22].

The authors would like to thank late A. O. Gogolin and P. Schmitteckert for many fruitful discussions. A. K. is supported by the Center of Quantum Dynamics of...
the University of Heidelberg.

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