POWER MOMENTS OF KLOOSTERMAN SUMS

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Abstract. In this paper we give an essential treatment for power moments of Kloosterman sums over the residue class ring \(\mathbb{Z}/q\mathbb{Z}\) of \(q\) elements. When \(q\) is a large enough power of a prime, we prove concrete formulas using computations with Igusa zeta functions. As a consequence, we show that there are explicit formulas for power moments of Kloosterman sums whenever \(q\) is powerful.

1. Introduction

Let \(q \geq 3\) be a positive integer and \(\zeta = e^{2\pi i/q}\) the \(q\)-th root of unity. For arbitrary integers \(u\) and \(v\), the classical Kloosterman sums are defined by

\[
K(u, v; q) = \sum_{0 < x \leq q} \zeta^{ux + vx},
\]

where \(\bar{x}\) denotes the multiplicative inverse of \(x\) in \((\mathbb{Z}/q\mathbb{Z})^*\).

The following fundamental properties of \(K(u, v; q)\) are well-known, see Salié [17, §1] and Esterman [4, p. 91]:

(1) \(K(u, v; q)\) is real;
(2) \(K(u, v; q) = K(v, u; q)\);
(3) \(K(u, v; q) = K(1, uv; q)\) when \((u, q) = 1\) and \(v\) is arbitrary;
(4) given integers \(v, q_1, q_2\) with \((q_1, q_2) = 1\), there exist integers \(v_1\) and \(v_2\) such that \(v \equiv v_1q_1^2 + v_2q_2^2 \pmod{q}\), and

\[
K(u, v; q_1q_2) = K(u, v_1; q_1)K(u, v_2; q_2).
\]

The study of Kloosterman sums \(K(u, v; q)\) is reduced by (4) to the prime-power case \(K(u, v; p^m)\), where \(m \geq 1\) is an integer and \(p\) is a prime. Indeed, the following is known at least to Esterman [4, p. 91].

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Let \( q = p_1^{m_1}p_2^{m_2} \cdots p_s^{m_s} \) with \( p_1, p_2, \ldots, p_s \) distinct primes. Given \( v \), we can determine \( v_1, v_2, \ldots, v_s \) such that for all \( u \) we have
\[
K(u, v; q) = \prod_{i=1}^{s} K(u, v_i; p_i^{m_i}).
\]

For a fixed \( v \), the individual sum \( K(1, v; q) \) has been studied extensively in the literature. An important problem is to understand the distribution of the Kloosterman sum \( K(1, v; q) \) as \( v \) varies. This problem is equivalent to understanding the \( n \)-th power moment of the Kloosterman sums defined by
\[
S_n(q) = \sum_{u=1}^{q} K(u, 1; q)^n = \sum_{v=1}^{q} K(1, v; q)^n,
\]
where \( n \) is a positive integer. Ideally, one would like to have an explicit formula for the \( n \)-th moment \( S_n(q) \). This is indeed the case for small \( n \), as one checks directly that
\[
S_1(q) = 0, \quad S_2(q) = \varphi(q)q,
\]
where \( \varphi(q) \) is the Euler function. As \( n \) grows, one expects that the moments \( S_n(q) \) become increasingly complicated. This is indeed the case for general \( q \). A surprising consequence of our work is that there are also explicit formulas for \( S_n(q) \) for higher \( n \) if \( q \) is powerful compared to \( \log_p n \). We state two results in this direction. For a prime \( p \), we let \( v_p(q) \) denote the \( p \)-adic valuation of \( q \).

**Theorem 1.** Let \( n \geq 1 \) be odd. If \( q \) has a prime factor \( p \) such that \( v_p(q) > \log_p n + 1 \), then \( S_n(q) = 0 \).

**Theorem 2.** Let \( n \geq 2 \) be even. If \( q \) is odd and each prime factor \( p \) of \( q \) satisfies \( v_p(q) > \log_p \frac{n}{2} + 1 \), then
\[
S_n(q) = \prod_{p | q} \left( \frac{n - 1}{2} - 1 \right) \frac{p - 1}{p} p^{(\frac{n}{2}+1)v_p(q)}.
\]

If \( q \) is even, there is a similar result that we do not state here for simplicity.

Here we take the point of view of fixing \( n \) and letting \( q \) vary. This yields uniform formulas for sufficiently powerful \( q \). Of course, one could also think about fixing \( q \) and letting \( n \) vary; in that case our formulas are valid for finitely many \( n \).

**Remark.** Motivated by applications in cryptography and coding theory [1], it is a challenging problem to give a sharp lower bound for the
quantity

\[ M_q = \max_{0 \leq u \leq q-1} |K(u, 1; q)|. \]

To get a good clean lower bound for \( M_q \), it is essential to get a clean explicit formula for the higher moments \( S_n(q) \). (Indeed, by the definition of \( S_n(q) \), one gets the inequality \( S_n(q) \leq qM_q^n \).) Our result fits exactly this purpose when \( q \) is powerful. We hope to study this type of applications in a more systematic way in a future paper.

The study of the power moment \( S_n(q) \) can be easily reduced to the case when \( q \) is a prime power. For any integer \( q \geq 3 \) and \( v \) with \((v, q) = 1\), let \( q = \prod_{i=1}^{s} p_i^{m_i} \), and

\[ u \equiv u_i \pmod{p_i^{m_i}}, \quad 1 \leq i \leq s. \]

It is clear that when \( u_i \) runs through a complete residue system modulo \( p_i^{m_i} \), \( u \) runs through a complete residue system modulo \( q \). Also note that

\[ K(u, v_i; p_i^{m_i}) = K(u_i, v_i; p_i^{m_i}), \]

where the \( v_i \) are determined by \( v \) as before. Then one can easily verify the identity

\[
S_n(q) = \sum_{u=1}^{q} \prod_{i=1}^{s} K(u, v_i; p_i^{m_i})^n \\
= \prod_{i=1}^{s} \sum_{u=1}^{p_i^{m_i}} K(u_i, v_i; p_i^{m_i})^n \\
= \prod_{i=1}^{s} S_n(p_i^{m_i}).
\]

So it suffices to study the power moments for Kloosterman sums for prime power modulus \( q = p^r \). The case when \( q = p \) is a prime (and thus \( r = 1 \)) has been studied extensively in the literature from different points of views, see Robba [16], Katz [13], Evans [5], and Fu-Wan [6][7] and the references listed there. The goal of this paper is to try to understand \( S_n(p^r) \) when \( r > 1 \). Our main result is the following explicit formula for \( S_n(p^r) \) if \( r \) is suitably large.

**Theorem 3.** Let \( n \geq 1 \) be odd. If \( p \) is odd and \( r > \log_p n + 1 \), then \( S_n(p^r) = 0 \). If \( p = 2 \), then \( S_n(2^r) = 0 \) for all \( r > 1 \).

**Theorem 4.** Let \( n \geq 2 \) be even. If \( p \) is odd and \( r > \log_p \frac{n}{2} + 1 \), then

\[ S_n(p^r) = \left( \frac{n-1}{\frac{n}{2} - 1} \right) p - 1 - p^{\left( \frac{n}{2} + 1 \right) r}. \]
If $p = 2$ and $r > \log_2 n + 2$, we have

$$S_n(2^r) = \left(\frac{n-1}{n^2 - 1}\right)2^{r-2}2^{(\frac{r}{2}+1)r}.$$ 

The bounds on $r$ in the conditions of the theorems are optimal. Indeed, we observed in several experiments that the clean formulas above for $S_n(p^r)$ do not hold when the lower bound on $r$ is not satisfied. When $n$ gets large compared to $p^r$, the behaviour of the moments becomes clearly more complicated, as illustrated by the following expressions, that we derived for $S_n(p^2)$. The ‘correction terms’ depend on how $n$ behaves mod $p$. (When $n$ is even, we do not provide an explicit proof, but it is similar to the proof when $n$ is odd.)

**Proposition 5.** Let $n$ and $p$ be odd and $p^2 > n$. Then

$$S_n(p^2) = -p^{n+1} \sum_{0 \leq i \leq n-2, \, 2i \in \{n-2,n-4\} \mod p} \left(\frac{n-2}{i}\right);$$

Let $n$ be even and $p > \max\{2, \sqrt{\frac{n}{2}}\}$. Then

$$S_n(p^2) = \left(\frac{n-1}{n^2 - 1}\right)\frac{p-1}{p}p^{n+2} - p^{n+1} \sum_{0 \leq i \leq n-2, \, i \in \{\frac{n}{2}-1, \frac{n}{2}+2\} \mod p} \left(\frac{n-2}{i}\right),$$

where $\sum^*$ means the two obvious terms $i = \frac{n}{2} - 1$ and $\frac{n}{2} - 2$ are excluded.

The material is organized as follows. In Section 2, we relate the power moment of Kloosterman sums for prime-power modulus ($q = p^r$) to the number of solutions of a certain equation over $(\mathbb{Z}/p^r\mathbb{Z})^*$. The latter can be naturally studied in the framework of Igusa’s zeta functions. When $n$ is odd, the associated hypersurface is fortunately non-singular in characteristic zero (although it may be singular when reduced modulo $p^r$). In this case a generalized Hensel lemma can be used to show that the sequence $S_n(p^r)$ ($r = 1, 2, \cdots$) stabilizes to zero when $r$ is larger than an explicit constant depending on $p$ and $n$. To obtain the optimal condition in the theorem, we have to work a little harder and resort to a more delicate analysis. When $n$ is even, the hypersurface is unfortunately singular in characteristic zero and the problem becomes significantly deeper. We have to use an elaborate calculation of the Igusa zeta function via an explicit embedded resolution of singularities of the hypersurface. In Section 3, we establish the link with the Igusa zeta function. In Section 4, we work out the detailed calculation in the case $p$ is odd. In Section 5, we deal with the case $p = 2$. To our pleasant surprise, the final results turn out to be quite nice and we always get a simple explicit formula for $S_n(p^r)$ when $r$ is suitably large.
2. Power moments for prime-power moduli

Let \( q = p^r, \) \( p \) prime, \( r \geq 2 \) a positive integer. Let \( \psi_q : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{C} \) be the additive character \( x \mapsto e^{2\pi i x/q} \). We denote the \( n \)-th power moment of the classical Kloosterman sums modulo \( q \) by

\[
S_n(q) = \sum_{\lambda \in \mathbb{Z}/q\mathbb{Z}} \left( \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^*} \psi_q(x + \lambda/x) \right)^n, \quad n \in \mathbb{Z}^+.
\]

Expanding the inner power and using the orthogonal property of additive characters, we have

\[
S_n(q) = \sum_{\lambda \in \mathbb{Z}/q\mathbb{Z}} \sum_{x_1, \ldots, x_n \in (\mathbb{Z}/q\mathbb{Z})^*} \psi_q(x_1 + \cdots + x_n + \lambda \left( \frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) )
\]

\[
= q \sum_{\frac{1}{x_1} + \cdots + \frac{1}{x_n} \equiv 0 \pmod{q}} \psi_q(x_1 + \cdots + x_n)
\]

\[
= q \sum_{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} \equiv 0 \pmod{q}} \sum_{x_n \in (\mathbb{Z}/q\mathbb{Z})^*} \psi_q(x_n (x_1 + \cdots + x_{n-1} + 1))
\]

\[
= q^2 \sum_{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} \equiv 0 \pmod{q}} \sum_{x_n \in \mathbb{Z}/q\mathbb{Z}} \psi_q(x_n (x_1 + \cdots + x_{n-1} + 1))
\]

\[
- q \sum_{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} \equiv 0 \pmod{q}} \sum_{x_n \in \mathbb{Z}/(q/p)\mathbb{Z}} \psi_{q/p}(x_n (x_1 + \cdots + x_{n-1} + 1))
\]

\[
= q^2 W_n(q) - q \cdot \frac{q}{p} \sum_{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} \equiv 0 \pmod{q}} \sum_{x_1 + \cdots + x_{n-1} + 1 \equiv 0 \pmod{q/p}} 1
\]

\[
= q^2 W_n(q) - q^2/p \cdot (*).
\]

Here and in the sequel

\[
W_n(p^k) = \sum_{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} \equiv 0 \pmod{p^k}} \sum_{x_1 + \cdots + x_{n-1} + 1 \equiv 0 \pmod{p^k}} 1
\]
for \( k \geq 1 \), and \((*)\) is the similar sum on the last but one line above. We first relate \((*)\) to \( W_n(q/p) \). If \( x_1, \ldots, x_{n-1} \) satisfies the system of congruences

\[
\begin{aligned}
    x_1 + \cdots + x_{n-1} + 1 &\equiv 0 \pmod{q/p} \\
    \frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + 1 &\equiv 0 \pmod{q},
\end{aligned}
\]

and we write \( x_i \to x_i - \frac{q}{p} y_i \), \( y_i \in \mathbb{F}_p \), then the second mod \( q \) congruence could be written as

\[
\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + 1 \equiv 0 \pmod{q}.
\]

Noting that \((q/p)^2 \equiv 0 \pmod{q}\) for \( q \neq p \), we have

\[
\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + 1 + \frac{q}{p} \left( \frac{y_1}{x_1^2} + \cdots + \frac{y_{n-1}}{x_{n-1}^2} \right) \equiv 0 \pmod{q},
\]

implying

\[
\frac{p}{q} \left( \frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + 1 \right) + \left( \frac{y_1}{x_1^2} + \cdots + \frac{y_{n-1}}{x_{n-1}^2} \right) \equiv 0 \pmod{p}.
\]

Thus for fixed \( x_1, \ldots, x_{n-1} \), the number of solutions in \((y_1, \ldots, y_{n-1}) \in \mathbb{F}_p^{n-1}\) equals \( p^{n-2} \). That is,

\[
(*) = \sum \frac{1}{x_1 + \cdots + x_{n-1} + 1} \equiv 0 \pmod{q/p}
\]

Similarly one easily computes \( S_n(p) \) in terms of \( W_n(p) \). We summarize.

**Theorem 6.** If \( q = p^r \) with \( r \geq 2 \), then

\[
S_n(q) = q^2 W_n(q) - q^2 p^{n-3} W_n(q/p) = q^2 \left( W_n(q) - p^{n-3} W_n(q/p) \right).
\]

If \( q = p \), then

\[
S_n(p) = p^2 W_n(p) - (p - 1)^{n-1} + (-1)^n).
\]

For small \( n \), explicit formulas for \( S_n(q) \) can be derived directly. For \( n = 1 \), it is clear that \( S_1(q) = 0 \) for all \( q \). For \( n = 2 \), one checks that \( W_2(q) = 1 \), and thus

\[
S_2(q) = q^2 \left( 1 - \frac{1}{p} \right).
\]

For \( n = 3 \), one checks that \( W_3(q) = 1 + \left( \frac{q}{3} \right) \) if \( p > 2 \) (and zero if \( p = 2 \)). This gives for \( r \geq 2 \) and all \( p \) that

\[
S_3(q) = 0.
\]
Thus, we shall assume that $n \geq 4$ below. (The case $n = 4$ may also be do-able directly. At any rate, it follows from the explicit formulas in later sections).

Solving $x_{n-1}$ from the congruence $x_1 + \cdots + x_{n-1} + 1 \equiv 0 \pmod{q}$ and substituting it into the congruence

$$\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}} + 1 \equiv 0 \pmod{q},$$

one finds that $W_n(q)$ is the number of $(\Z/q\Z)^*$-solutions of the equation

$$(1 + x_1 + x_2 + \cdots + x_{n-2})(1 + \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-2}}) = 1.$$

Let $V_n(q)$ denote the number of $(\Z/q\Z)^*$-solutions of the zero set of the Laurent polynomial

$$(1) \quad g(x_1, \ldots, x_{n-1}) = (x_1 + x_2 + \cdots + x_{n-1})(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}) - 1.$$

Replacing $x_i$ by $x_i x_{n-1}$ in $g(x_1, \ldots, x_{n-1})$ for $1 \leq i \leq n-2$, one checks that

$$W_n(q) = \frac{1}{\phi(q)} V_n(q),$$

where $\phi(q) = \phi(p^r) = p^{r-1}(p - 1)$ is the Euler function. With this new notation, the first part of the previous theorem can be restated as follows.

**Theorem 7.** If $q = p^r$ with $r \geq 2$, then

$$S_n(q) = \frac{q^2}{\phi(q)} (V_n(q) - p^{n-2} V_n(q/p)).$$

We now consider lifting solutions mod $q/p$ to solutions mod $q$, and then relate $V_n(q/p)$ to $V_n(q)$. If $g(x_1, \ldots, x_{n-1})$ has no singular toric solution modulo $p$, then the Hensel lemma gives the recursive formula $V_n(q) = p^{n-2} V_n(q/p)$ and thus $S_n(q) = 0$. We now check when the Laurent polynomial $g$ has no singular toric solution modulo $p$.

More generally, for a positive integer $k$, let $x = (x_1, \ldots, x_{n-1})$ be a singular toric solution of $g = 0$ modulo $p^k$. That is,

$$g(x_1, \ldots, x_{n-1}) = \frac{\partial g}{\partial x_1}(x) \equiv \cdots \equiv \frac{\partial g}{\partial x_{n-1}}(x) \equiv 0 \pmod{p^k}.$$

For $1 \leq i \leq n-1$, we deduce that

$$\frac{\partial g}{\partial x_i}(x) = (\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}}) - \frac{x_1 + \cdots + x_{n-1}}{x_i} \equiv 0 \pmod{p^k}.$$
It follows that
\[ x_1^2 \equiv x_2^2 \equiv \cdots \equiv x_{n-1}^2 \pmod{p^k}. \]

For \( k \leq 2 \) or \( p > 2 \), this implies that \( x_i \equiv \pm x_0 \pmod{p^k} \) for some \( x_0 \in (\mathbb{Z}/p^k\mathbb{Z})^* \). For \( p = 2 \) and \( k \geq 3 \), we have the slightly weaker congruence \( x_i \equiv \pm x_0 \pmod{2^{k-1}} \) for some \( x_0 \in (\mathbb{Z}/2^{k-1}\mathbb{Z})^* \). Let
\[
m_+ = \#\{1 \leq i \leq n-1 \mid x_i \equiv x_0 \pmod{p^k}\},
m_- = \#\{1 \leq i \leq n-1 \mid x_i \equiv -x_0 \pmod{p^k}\}
\]
if \( k \leq 2 \) or \( p > 2 \), and analogously mod \( 2^{k-1} \) if \( p = 2 \) and \( k \geq 3 \). Then, at this singular point, for \( p > 2 \) or \( k \leq 2 \), we have
\[
g(x_1, \ldots, x_{n-1}) \equiv (m_+ - m_-)^2 - 1 \equiv 0 \pmod{p^k},
\]
and for \( p = 2 \) and \( k \geq 3 \), we have
\[
g(x_1, \ldots, x_{n-1}) \equiv (m_+ - m_-)^2 - 1 \equiv 0 \pmod{2^{k-1}}.
\]

Since \( m_+ + m_- = n - 1 \), it follows that for \( p = 2 \) and \( k \geq 3 \), we have
\[
(2m_+ - (n-1))^2 - 1 = (2m_+ - n)(2m_+ - (n-2)) \equiv 0 \pmod{2^{k-1}}.
\]

For \( p > 2 \) or \( k \leq 2 \), we have
\[
(2m_+ - (n-1))^2 - 1 = (2m_+ - n)(2m_+ - (n-2)) \equiv 0 \pmod{p^k}.
\]

This shows that if \( p^k > n \) (with \( p > 2 \) or \( k \leq 2 \)), \( g \) has no singular toric solutions modulo \( p^k \). In the case \( k = 1 \) and \( n \) odd, this is impossible if either \( p > n \) or \( p = 2 \). We obtain the following.

**Theorem 8.** Let \( n \) be odd. Assume that \( p > n \) or \( p = 2 \). If \( r \geq 2 \), then
\[
V_n(p^r) = p^{n-2} \cdot V_n(p^{r-1}), \quad S_n(p^r) = 0.
\]

**Question 9.** Is there a direct proof of the statement \( S_n(p^r) = 0 \) in Theorem 8? When will we have \( K(\lambda_1) = -K(\lambda_2) \)?

This theorem shows that for odd \( n \), the case \( p = 2 \) is completely settled. We now assume that \( n \) is odd and \( p \) is also odd in the rest of this section. We show that the above arguments can be refined to settle the more general case for odd \( n \) and \( p^2 > n \). For this, we first show that the singular points in \( \mathbb{F}_p \) never lift to points modulo \( p^2 \) and hence never lift to points modulo \( p^r \) with \( r \geq 2 \). Let \( (x_1, \ldots, x_{n-1}) \) be a singular point modulo \( p \). As above, we can take \( x_i \equiv \pm x_0 \pmod{p} \). This singular solution modulo \( p \) lifts to a solution modulo \( p^2 \) only if
\[
g(x_1, \ldots, x_{n-1}) \equiv (2m_+ - n)(2m_+ - (n-2)) \equiv 0 \pmod{p^2}.
\]
This is not possible as \( n \) is odd, \( 0 \leq m_+ \leq n - 1 \) and \( p^2 > n \). This implies that for \( r \geq 3 \), we still have
\[
V_n(p^r) = p^{n-2}V_n(p^{r-1}), \quad S_n(p^r) = 0.
\]
For \( r = 2 \), let \( N(n,p) \) denote the number of singular solutions modulo \( p \). Taking \( x_0 = x_{n-1} \), one finds that
\[
N(n,p) = (p - 1) \sum_{0 \leq i \leq n-2, \ i \in \{\frac{n}{2} - 1, \frac{n}{2} - 2\} \ mod \ p} \binom{n-2}{i},
\]
where \( i \) corresponds to \( m_+ - 1 \). We deduce that
\[
V_n(p^2) = p^{n-2}(V_n(p) - N(n,p)), \quad S_n(p^2) = -\frac{p^{n+1}}{p-1}N(n,p).
\]

We summarize.

**Theorem 10.** Let \( pn \) be odd and \( p^2 > n \). If \( r \geq 3 \), then \( S_n(p^r) = 0 \). If \( r = 2 \), then
\[
S_n(p^2) = -\frac{p^{n+1}}{p-1}N(n,p).
\]

Note that the last sum is zero in the case \( p > n \) and \( n \) odd, consistent with the previous theorem. The first part of the theorem can be further improved as follows.

**Theorem 11.** If \( np \) is odd and \( r > \log_p n + 1 \), then \( S_n(p^r) = 0 \).

**Proof.** Our assumption implies that \( p^{r-1} > n \). We have shown that for odd \( pn \) with \( p^{r-1} > n \), the Laurent polynomial \( g = 0 \) has no singular solutions modulo \( p^{r-1} \). That is, there are no integers \( x_i \) prime to \( p \) such that \( x = (x_1, \ldots, x_{n-1}) \) satisfies
\[
g(x) \equiv \frac{\partial g}{\partial x_1}(x) \equiv \cdots \equiv \frac{\partial g}{\partial x_{n-1}}(x) \equiv 0 \pmod{p^{r-1}}.
\]
This means that any solution \( x = (x_1, \ldots, x_{n-1}) \) counted in \( V_n(p^{r-1}) \) must satisfy the inequality
\[
\text{ord}_p\{\frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x)\} \leq r - 2.
\]
We claim that any solution \( x = (x_1, \ldots, x_{n-1}) \) counted in \( V_n(p^{r-1}) \) satisfies the stronger inequality
\[
k_x := \text{ord}_p\{\frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x)\} < \frac{r - 1}{2}.
\]
Otherwise, let \( x = (x_1, \ldots, x_{n-1}) \) be a solution counted in \( V_n(p^{r-1}) \) satisfying
\[
0 < \frac{r-1}{2} \leq k_x \leq r - 2.
\]
Let \( y_i = x_i + p^{k_x} z_i \), where \( z_i \in \mathbb{Z} \). Since \( 2k_x \geq r - 1 \) and \( k_x = \text{ord}_p \{ \frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x) \} \), the Taylor expansion shows that
\[
g(y_1, \ldots, y_{n-1}) \equiv g(x_1, \ldots, x_{n-1}) \equiv 0 \pmod{p^{r-1}}.
\]
Now, reducing \( x \) modulo \( p^{k_x} \), we see that \( \{ y_1, \ldots, y_{n-1} \} \) is a singular solution modulo \( p^{k_x} \). One has as before the congruence
\[
y_1^2 \equiv y_2^2 \equiv \cdots \equiv y_{n-1}^2 \pmod{p^{k_x}}.
\]
This implies that \( y_i \equiv \pm y_0 \pmod{p^{k_x}} \) for some \( 0 \leq y_0 < p^{k_x} \). We choose \( y_i \) such that \( y_i = \pm y_0 \) for all \( 1 \leq i \leq n - 1 \). Let
\[
m'_+ = \# \{ 1 \leq i \leq n - 1 \mid y_i = y_0 \}, \quad m'_- = \# \{ 1 \leq i \leq n - 1 \mid y_i = -y_0 \}.
\]
Then
\[
0 \equiv g(y_1, \ldots, y_{n-1}) = (m'_+ - m'_-)^2 - 1 \pmod{p^{r-1}}.
\]
This implies that
\[
(2m'_+ - n)(2m'_+ - (n - 2)) \equiv 0 \pmod{p^{r-1}}.
\]
It contradicts our assumption that \( p^{r-1} > n \) and \( n \) is odd. The claim is proved.

Let
\[
k_0 = \max_x k_x = \max_x \text{ord}_p \{ \frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x) \},
\]
where \( x = (x_1, \ldots, x_{n-1}) \) runs over all solutions counted in \( V_n(p^{r-1}) \). The above claim shows that \( k_0 < (r - 1)/2 \), that is, \( r \geq 2(k_0 + 1) \). A more general Hensel lemma (see [18]) implies that for \( s \geq 2(k_0 + 1) - 1 \), we have
\[
V_n(p^s) = V_n(p^{2(k_0+1)-1}p^{(n-2)(s-2(k_0+1)+1)}).
\]
This is done by applying the general Hensel lemma only to those solutions modulo \( p^{k_0+1} \) which can be lifted to solutions modulo \( p^s \). Thus, for \( s \geq 2(k_0 + 1) \), we still have
\[
V_n(p^s) = p^{n-2}V_n(p^{s-1}).
\]
Since \( r \geq 2(k_0+1) \), we can take \( s = r \) and conclude that \( S_n(p^r) = 0 \). \( \square \)

When \( n \) is even, it turns out that the Laurent polynomial \( g \) always has singular toric solutions modulo \( p^k \). This makes the determination of \( S_n(p^r) \) via \( V_n(p^r) \) much more difficult. In the last two sections we solve the problem using algebraic and geometric techniques from the study
of Igusa zeta functions. First we explain the link with our problem in the next section.

3. Relation with Igusa zeta functions

In this section we describe the hypersurface in (1) rather as the zero set of the polynomial

\[ h = \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}} \right) x_1 x_2 \cdots x_{n-1}. \]

Classically one studies the behavior of the \( V_n(p^r) \) through the properties of its generating series. We put

\[ P(t) = \left( \frac{p-1}{p} \right)^{n-1} + \sum_{r \geq 1} V_n(p^r) \left( p^{-(n-1)} t \right)^r \in \mathbb{Q}[[t]], \]

where the constant \( \left( \frac{p-1}{p} \right)^{n-1} \) and the factor \( p^{-(n-1)} \) are the standard conventions, in order to relate \( P(t) \) in a natural way with the Igusa zeta function of \( h \). Igusa [9] proved that \( P(t) \) is in fact a rational function in \( t \) through the study of that zeta function. We will obtain information about the poles of \( P(t) \) by studying the Igusa zeta function of \( h \), and then use the precise description of the \( V_n(p^r) \) in terms of the poles (and their orders) of \( P(t) \), as calculated in [18].

We introduce the version of the Igusa zeta function that we will use. We denote by \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \) the field of \( p \)-adic numbers and the ring of \( p \)-adic integers, respectively, by \( | \cdot | \) the standard \( p \)-adic norm on \( \mathbb{Q}_p \) and by \( dx \) the standard Haar measure on \( \mathbb{Q}_p^k \). For \( a \in \mathbb{Z}_p \) we denote by \( \bar{a} \) its image in \( \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p \), and similarly for \( a \in \mathbb{Z}_p^k \) and \( W \subset \mathbb{Z}_p^k \) we use the notation \( \bar{a} \) and \( \bar{W} \) for their images in \( \mathbb{F}_p^k \).

**Definition 12.** Let \( f \in \mathbb{Z}_p[x_1, \ldots, x_k] \) and let \( W \) be a residual subset of \( \mathbb{Z}_p^k \), that is, a disjoint union of residue classes mod \( p \). Then the Igusa zeta function associated to \( f \) and \( W \) is

\[ Z_W(f; s) = \int_W |f(x)|^s dx, \]

where \( s \in \mathbb{C} \) with \( \Re(s) > 0 \).

Igusa [9] showed in fact that \( Z_W(f; s) \) is a rational function in \( p^{-s} \), using an embedded resolution of singularities of \( f \). Because of this result one considers \( Z_W(f; s) \) as a function in \( t = p^{-s} \) and writes \( Z_W(f; t) \)
for it. In fact $Z_W(f; t)$ contains the same information as the so-called Poincaré series

$$P_W(f; t) = p^{-k}#W + \sum_{r \geq 1} V(f, W; p^r) (p^{-k}t)^r \in \mathbb{Q}[[t]],$$

where $V(f, W; p^r)$ is the number of $k$-tuples $x \in (\mathbb{Z}/p^r \mathbb{Z})^k$ satisfying $f(x) \equiv 0 \pmod{p^r}$ and such that the image of $x$ in $\mathbb{F}_p^k$ belongs to $\bar{W}$. Note that the constant term is just the measure of $W$. More precisely one has the relation

$$(2) \quad P_W(f; t) = \frac{p^{-k}#W - tZ_W(f; t)}{1 - t}$$

by a straightforward adaptation of the proof of the standard case [11, Theorem 8.2.2]. Note that, since $\mathbb{Z}/p^r \mathbb{Z} \cong \mathbb{Z}/p^{r-1} \mathbb{Z}$, we have that $P(t) = P(\mathbb{Z}/p^{r-1} \mathbb{Z})$.

We now recall two techniques to compute the Igusa zeta function.

The $p$-adic stationary phase formula. We assume that at least one of the coefficients of $f$ does not belong to $p\mathbb{Z}_p$. (This can always be achieved by dividing $f$ by a suitable power of $p$.) Then we denote by $\bar{f}$ the non-zero polynomial over $\mathbb{F}_p$ obtained by reducing all the coefficients of $f$ modulo $p$.

Denote by $S$ the subset of all $\bar{a}$ in $\bar{W}$ such that $\bar{f}(\bar{a}) = 0$ and $(\partial \bar{f}/\partial x_i)(\bar{a}) = 0$ for all $i \in \{1, \ldots, k\}$, and by $S$ its preimage in $\mathbb{Z}_p^k$. Then [10][11, Theorem 10.2.1]

$$(3) \quad Z_W(f; t) = p^{-k} (#W - N) + p^{-k} (N - \#S) \left(\frac{p-1}{p-1} - t\right) + \int_S |f(x)|^s dx,$$

where $N$ is the number of zeroes of $\bar{f}$ in $\bar{W}$.

Resolution of singularities. Let $\sigma : X \rightarrow \mathbb{Q}_p^k$ be an embedded resolution of singularities of $f$, where $X$ is a non-singular algebraic variety over $\mathbb{Q}_p$, $\sigma$ is a projective birational morphism, the inverse image of $\{f = 0\}$ has simple normal crossings and $\sigma$ is an isomorphism outside that inverse image. Thus the irreducible components $E_i, i \in I$, of $\sigma^{-1}\{f = 0\}$ are nonsingular hypersurfaces, intersecting transversely.

Note that at most $k$ different components $E_i$ contain a given point of $X$. For $i \in I$ we denote by $N_i$ and $\nu_i - 1$ the multiplicities of $E_i$ in the divisor of $\sigma^*f$ and of $\sigma^*(dx_1 \wedge \cdots \wedge dx_k)$, respectively. Then $Z_W(f; t)$ can be written as a rational function in $t$ with denominator $\prod_{i \in I}(1 - p^{-\nu_i}t^{N_i})$, see [11, Theorem 8.2.1]. More precisely, $Z_W(f; t)$ is a sum of rational functions with denominator $\prod_{i \in J}(1 - p^{-\nu_i}t^{N_i})$, where
the $E_i, i \in J$, have a nonempty intersection (and hence $\# J \leq k$). Note that, by (2), the same is then true for $P_W(f; t)$.

There is an explicit formula of Denef [3, Theorem 3.1], when a certain technical condition concerning the resolution $\sigma$ is satisfied. For the following notions we refer to [3] for more information. To an algebraic set $V$ over $\mathbb{Q}_p$ is associated its reduction $\mod p$, being an algebraic set over $\mathbb{F}_p$ and denoted by $\tilde{V}$. Also, to the map $\sigma$ one associates its reduction $\mod p$, being a morphism $\tilde{\sigma} : \tilde{X} \to \mathbb{F}_p^k$. When the restriction of $\sigma$ to $\sigma^{-1}W$ has good reduction $\mod p$ (see [3] for this notion), we have

$$Z_W(f; t) = p^{-k} \sum_{J \subseteq I} c_J \prod_{i \in J} \frac{(p - 1)p^{-\nu_i}t^{N_i}}{1 - p^{-\nu_i}t^{N_i}}, \tag{4}$$

where $c_J = \{ \tilde{x} \in \tilde{X} \mid \tilde{x} \in \tilde{E}_i$ if and only if $i \in J$, and $\tilde{\sigma}(\tilde{x}) \in \tilde{W} \}$.

Here, to simplify notation, we denote for a variety $\tilde{V}$ over $\mathbb{F}_p$ the set of its $\mathbb{F}_p$-rational points by the same symbol $\tilde{V}$.

In the next two sections we use these techniques to study $P(t)$ through the Igusa zeta function associated to $h$.

4. FORMULA FOR $S_n(p^n)$ WHEN $p$ IS ODD

We assume in this section that $n$ is even and $p$ is odd. In fact we determined already when there exist $\tilde{a}$ in $(\mathbb{F}_p^*)^{n-1}$ such that $\tilde{h}(\tilde{a}) = 0$ and $(\partial h / \partial x_i)(\tilde{a}) = 0$ for all $i \in \{1, \ldots, n-1\}$. We use the notation $m_+$ and $m_-$ as before. Replacing $x_0$ by $-x_0$ if necessary, we may assume that $0 \leq m_+ \leq n/2 - 1$. There exist such $\tilde{a}$ in $(\mathbb{F}_p^*)^{n-1}$ if and only if

$$2m_+ \equiv n \quad (\mod p) \quad \text{or} \quad 2m_+ \equiv n - 2 \quad (\mod p). \tag{5}$$

Since $n$ is even, (5) is equivalent to $m_+ \equiv \frac{n}{2} \quad (\mod p)$ or $m_+ \equiv \frac{n}{2} - 1 \quad (\mod p)$. When $p \geq \frac{n}{2} + 1$ this happens if and only if $m_+ = \frac{n}{2} - 1$. When $3 \leq p \leq \frac{n}{2}$ this happens for $m_+ = \frac{n}{2} - 1$ and for at least one other $m_+$, namely at least for $m_+ = \frac{n}{2} - p$.

We study the Igusa zeta function $Z(t) = Z_{(\mathbb{Z}_p^*)^{n-1}}(h; t)$. The hypersurface $h = 0$ in $(\mathbb{Q}_p^*)^{n-1}$ has singularities (of multiplicity 2) at $x_1^2 = x_2^2 = \cdots = x_{n-1}^2$. With a similar argument as above these are the points where each $x_i = \pm x_0$ for some $x_0 \in \mathbb{Q}_p^*$ and $\#\{1 \leq i \leq n - 1 \mid x_i = x_0\} = \frac{n}{2} - 1$. Hence the singular locus of $h = 0$ consists of $\binom{n - 1}{\frac{n}{2} - 1}$ disjoint copies of $\mathbb{Q}_p^*$. One obtains an embedded resolution $\sigma$ by blowing up with centres these lines; each exceptional component $E_i$ is the product of such a centre $Z_i$ with a $(n - 3)$-dimensional projective
space and has data \((N_i, \nu_i) = (2, n - 2)\). The strict transform \(E_0\) of \(\{h = 0\}\) has data \((N_0, \nu_0) = (1, 1)\). We describe now the intersection of a fixed \(E_i\) with the strict transform. One easily computes that the quadratic form

\[
q = \sum_{i=1}^{n-1} x_i^2 + \sum_{i<j}^{n-2} x_i x_j
\]

is the lowest degree term of a local equation of a transversal section of the hypersurface \(h = 0\) at a singular point. Consequently the intersection of \(E_i\) with \(E_0\) is the product of the centre with the projective variety determined by \(q = 0\).

Using Igusa’s result above, we see that \(Z(t)\) can be written as a rational function with denominator \((1 - p^{-1}t)(1 - p^{-n+2}t^2)\).

**First case: \(p \geq \frac{n}{2} + 1\).**

One can check that \(\sigma\) has good reduction mod \(p\), and hence we can apply Denef’s formula (4). Let \(N\) denote the number of zeroes of \(\bar{h}\) in \(\mathbb{F}_p^{n-1}\). We claim that \(p^{n-2} - 1\) is the number of points of \(E_i\) mapping to \((\mathbb{F}_p^{n-1})^\times\) by \(\bar{\sigma}\). Indeed, this is the product of \(p - 1\), being the number of points of \(\bar{Z}_i \cap (\mathbb{F}_p^{n-1})^\times\), and the number of points of projective \((n - 3)\)-space over \(\mathbb{F}_p\). Finally we denote by \(Q\) the number of points of \(\bar{E}_i \cap \bar{E}_0\) mapping to \((\mathbb{F}_p^{n-1})^\times\); it is the product of \(p - 1\) and the number of points on the projective variety determined by \(\bar{q} = 0\). Then Denef’s formula yields

\[
p^{n-1}Z(t) = (p - 1)^{n-1} - N + \left( N - \left( \frac{n-1}{2} \right)(p - 1)\right) \frac{(p - 1)p^{-1}t}{1 - p^{-1}t} \\
+ \left( \frac{n-1}{2} \right)(p^{n-2} - 1 - Q) \frac{(p - 1)p^{-n+2}t^2}{1 - p^{-n+2}t^2} \\
+ \left( \frac{n-1}{2} \right) Q \frac{(p - 1)^2 p^{-n+1}t^3}{(1 - p^{-1}t)(1 - p^{-n+2}t^2)}.
\]

More concretely, since \(Q\) is also 1 less than the number of points of the affine variety determined by \(\bar{q} = 0\), we have by [11, Theorem 9.2.1] that

\[
Q = (p^{\frac{n}{2} - 2} + 1)(p^{\frac{n}{2} - 1} - 1).
\]
Power Moments of Kloosterman Sums

General case: \( n \geq 6 \). It will turn out that we can write \( Z(t) \), applying decomposition in partial fractions, in the form

\[
A + \frac{B}{1 - p^{-1}t} + \frac{C}{1 - p^{-\frac{n}{2}+1}t}
\]

with \( A, B, C \) constants. (Note that one expects a priori a term of the form \( \frac{D + Et}{1 - p^{-n+2}t^2} \). However, this term simplifies.) A similar statement is then true for \( P(t) \), yielding a concrete description of the behavior of \( V_n(p^r) \) for \( r \geq 1 \).

We provide some details of this computation. Decomposing the last two terms of (7) in partial fractions yields, as contribution to \( \frac{1}{1 - p^{-n+2}t^2} \), the terms

\[
\left(\frac{n - 1}{n^2 - 1}\right)(p^n - 1 - Q)(p - 1) \quad \text{and} \quad -\left(\frac{n - 1}{n^2 - 1}\right)\frac{Q(p - 1)^2}{p^{n-4} - 1}(p^{n-4} + p^{-1}t),
\]

respectively. Adding, dividing by \( p^{n-1} \), plugging in the expression in (8) for \( Q \) and simplifying yields

\[
-\left(\frac{n - 1}{n^2 - 1}\right)(p - 1)^2(p^{\frac{n}{2}-1} - 1) \cdot \frac{1 + p^{-\frac{n}{2}+1}t}{p^{\frac{n}{2}+1}(p^{\frac{n}{2}-2} - 1)} + \frac{1}{1 - p^{-n+2}t^2},
\]

and indeed the last factor is equal to \( \frac{1}{1 - p^{-\frac{n}{2}+1}t} \).

In order to find the constant \( C \) in the expression (9) for \( P(t) \), we only need the similar constant in the expression for \( Z(t) \). Using (2) one easily derives that \( P(t) \) can be written in the form (9) with

\[
C = -\left(\frac{n - 1}{n^2 - 1}\right)(p - 1)^2.
\]

Looking at the main result in [18] and its proof, we have for all \( r \geq 1 \) that

\[
V_n(p^r) = Bp^{(n-2)r} + Cp^{\frac{n}{2}r}.
\]

We compute by Theorem 7 that

\[
S_n(p^r) = \frac{p^{2r+1}}{p - 1} \left( \frac{V_n(p^r)}{p^r} - p^{n-3}V_n(p^{r-1}) \right)
\]

\[
= \frac{p^{2r+1}}{p - 1} \left( Bp^{(n-3)r} + Cp^{\left(\frac{n}{2}-1\right)r} - p^{n-3}(Bp^{(n-3)(r-1)} + Cp^{\left(\frac{n}{2}-1\right)(r-1)}) \right)
\]

\[
= \frac{p^{2r+1}}{p - 1} C \left( p^{\left(\frac{n}{2}-1\right)r} - p^{\frac{n}{2}-2+\left(\frac{n}{2}-1\right)r} \right)
\]

\[
= C \frac{p(1 - p^{\frac{n}{2}-2})}{p - 1} p^{\left(\frac{n}{2}+1\right)r}
\]
for all \( r \geq 2 \). Plugging in (11) we obtain finally for all \( r \geq 2 \) that

\[
S_n(p^r) = \left(\frac{n-1}{n^2-1}\right) \frac{p-1}{p} p^{\left(\frac{n}{2}+1\right)r}.
\]

**Case \( n = 4 \).** In this special case a straightforward calculation simplifies (7) to

\[
Z(t) = \frac{p-1}{p^5} \cdot \frac{p^2(p^2 - 5p + 7) + p(p^2 - 2p - 5)t + (p^2 + p + 1)t^2}{(1 - p^{-1}t)^2},
\]

yielding

\[
P(t) = \frac{p-1}{p^5} \cdot \frac{p^2(p^2 - 2p + 1) + p(p^2 - 2p - 2)t + (p^2 + p + 1)t^2}{(1 - p^{-1}t)^2}.
\]

Decomposing \( P(t) \) in partial fractions now results in the form

\[
A + \frac{B}{1 - p^{-1}t} + \frac{C}{(1 - p^{-1}t)^2}
\]

with \( A, B, C \) constants, and more precisely \( C = 3^{\left(\frac{p-1}{2}\right)^2} \). In this case we have for all \( r \geq 1 \) by [18] that

\[
V_n(p^r) = ((r + 1)C + B) p^{2r}.
\]

(Note that there is a typo in [18] precisely at this point. On the last line of page 4 the numbers involving \( e \) must be augmented by 1.) By Theorem 7 we compute

\[
S_n(p^r) = \frac{p^{2r+1}}{p - 1} \left( \frac{V_n(p^r)}{p^r} - \frac{V_n(p^{r-1})}{p^{r-1}} \right)
\]

\[
= \frac{p^{2r+1}}{p - 1} \left( ((r + 1)C + B)p^r - p(rC + B)p^{r-1} \right)
\]

\[
= \frac{p^{2r+1}}{p - 1} Cp^r
\]

\[
= 3 \frac{p - 1}{p} p^{3r}.
\]

Note that this turns out to be exactly (13) when substituting \( n = 4 \).

**Second case: \( 3 \leq p \leq \frac{n}{2} \).**

(Hence \( n \geq 6 \).) We partition the \( \tilde{a} \) in \((\mathbb{F}_p^n)^{n-1}\) such that \( \tilde{h}(\tilde{a}) = 0 \) and \((\partial \tilde{h}/\partial x_i)(\tilde{a}) = 0\) for all \( i \in \{1, \ldots, n-1\}\) into the subsets \( \tilde{S}_1 \), corresponding to \( m_+ = \frac{n}{2} - 1 \), and \( \tilde{S}_2 \), corresponding to all other values
of $m_+$. Let $S_1$ and $S_2$ denote their preimages in $\mathbb{Z}_p^{n-1}$, respectively. The $p$-adic stationary phase formula (3) yields

$$p^{n-1}Z(t) = (p - 1)^{n-1} - N + (N - \#(\bar{S}_1 \cup \bar{S}_2)) \frac{(p - 1)p^{-1}t}{1 - p^{-1}t}$$

$$+ p^{n-1} \int_{S_1} |h(x)|^s dx + p^{n-1} \int_{S_2} |h(x)|^s dx.$$  \hspace{1cm} (16)

In fact, the restriction of $\sigma$ to $S_1$ still has good reduction mod $p$, and by Denef’s formula $p^{n-1} \int_{S_1} |h(x)|^s dx$ equals the sum of the last two terms in (7).

On the other hand, since $h = 0$ has no singular points in $S_2$, we can write $p^{n-1} \int_{S_2} |h(x)|^s dx$ as a rational function in $t$ with denominator $1 - p^{-1}t$. In general we cannot apply Denef’s formula here; in particular we have no control over the degree of the numerator. At any rate, decomposing $Z(t)$ and $P(t)$ in partial fractions, this time we can write $P(t)$ in the form

$$A_n'(t) + \frac{B_n'}{1 - p^{-1}t} + \frac{C_n}{1 - p^{-\frac{s}{2} + 1}t} \quad (n \geq 6),$$

where $A_n'(t) \in \mathbb{Q}[t]$, $B_n'$ is a constant and

$$C_n = - \frac{\binom{n-1}{\frac{s}{2}}}{{p^2(p^{\frac{s}{2} - 2} - 1)}}$$

as before. By [18] we still have similar expressions for $V_n(p^r)$ as in (12) and (15), but now they are only valid when $r$ is big enough, more precisely when $r > \deg A_n'(t)$. We conclude that $V_n(p^r)$ is still given by the formula in (13) when $r$ is big enough (with respect to $n$ and $p$).

We note that (13) is also valid for $n = 2$ and we summarize.

**Theorem 13.** Let $n$ be an even positive integer. Let $p$ be an odd prime number and $r \geq 2$. If $p \geq \frac{n}{2} + 1$, then we have for all $r \geq 2$ that

$$S_n(p^r) = \binom{n - 1}{\frac{n}{2} - 1} \frac{p - 1}{p} p^{(\frac{n}{2} + 1)r}.$$  \hspace{1cm} (17)

If $3 \leq p \leq \frac{n}{2}$, the same formula is valid for $r$ big enough (depending on $n$ and $p$).

For $3 \leq p \leq \frac{n}{2}$, the above theorem gives the precise information when $r$ is big enough. For small $r$, the problem is caused by the integration over $S_2$, corresponding to points which are non-singular over $\mathbb{Q}_p$, but become singular modulo $p$. This part can be handled as in the second section when counting $V_n(p^r)$. Combining the elementary method of
that section and the above Igusa zeta function calculation, we obtain the following additional results.

**Theorem 14.** Let \( n \) be an even positive integer and \( p > \max\{2, \sqrt{\frac{n}{2}}\} \). For \( r \geq 3 \), we have

\[
S_n(p^r) = \left( \frac{n}{2} - 1 \right) \frac{p - 1}{p} p^{\frac{r}{2+1}} r.
\]

For \( r = 2 \), we have

\[
S_n(p^2) = \left( \frac{n}{2} - 1 \right) \frac{p - 1}{p} p^{n+2} - p^{n+1} \sum_{0 \leq i \leq n-2, i \in \{\frac{n}{2} - 1, \frac{n}{2} - 2\} \mod p}^{*} \binom{n - 2}{i},
\]

where \( \sum^{*} \) means the two obvious terms \( i = \frac{n}{2} - 1 \) and \( \frac{n}{2} - 2 \) are excluded.

Note that the second term is zero if \( p \geq \frac{n}{2} + 1 \), consistent with the previous theorem.

**Theorem 15.** Let \( n \) be an even positive integer and \( p \) be odd. For \( r > \log_p \frac{n}{2} + 1 \), we have

\[
S_n(p^r) = \left( \frac{n}{2} - 1 \right) \frac{p - 1}{p} p^{\frac{r}{2+1}} r.
\]

Note that in the case \( n \) even and \( p \) odd, the modulo \( p^k \) singularity condition

\[
(2m_+ - (n - 1))^2 - 1 = (2m_+ - n)(2m_+ - (n - 2)) \equiv 0 \pmod{p^k}
\]

is equivalent to

\[
(m_+ - \frac{n}{2})(m_+ - (\frac{n}{2} - 1)) \equiv 0 \pmod{p^k}.
\]

Thus, we can replace the previous condition \( r - 1 > \log_p n \) by the slightly weaker condition \( r - 1 > \log_p \frac{n}{2} \).

**5. Formula for \( S_n(2^r) \)**

We still assume in this section that \( n \) is even. As before, the singular locus of the hypersurface \( h = 0 \) in \((\mathbb{Q}_2^*)^{n-1}\) consists of \( \binom{n-1}{\frac{n}{2} - 1} \) disjoint copies of \( \mathbb{Q}_2^* \), being the points where each \( x_i = \pm x_0 \) for some \( x_0 \in \mathbb{Q}_2^* \) and \( \#\{1 \leq i \leq n - 1 \mid x_i = x_0\} = \frac{n}{2} - 1 \). Blowing up with centres these lines yields an embedded resolution, and hence \( Z(t) \) can be written as a rational function with denominator \((1 - 2^{-1}t)(1 - 2^{-n+2}t^2)\). But this resolution has bad reduction \( \pmod{2} \).
General case: \( n \geq 6 \). We partition the integration domain \((1 + 2\mathbb{Z}_2)^{n-1}\) into (open and closed) pieces, where each piece contains at most one component of the singular locus. We can describe each such component \( Z_J \) with its equations

\[
x_i = x_{n-1} \quad (i \in J),
\]

\[
x_i = -x_{n-1} \quad (i \in \{1, \ldots, n - 2\} \setminus J),
\]

where \( J \) is a (uniquely determined) subset of \( \{1, \ldots, n - 2\} \) with cardinality \( \frac{n}{2} - 1 \) or \( \frac{n}{2} - 2 \). We consider the neighborhood \( U_J \) of \( Z_J \) given by

\[
x_i = x_{n-1} + 4\mathbb{Z}_2 \quad (i \in J) \quad \text{and} \quad x_i = -x_{n-1} + 4\mathbb{Z}_2 \quad (i \notin J).
\]

Clearly \( U_J \) and \( U_{J'} \) are disjoint if \( J \neq J' \). In order to compute

\[
\text{Int}_J = \int_{U_J} |h(x)|^s dx
\]

we perform the (measure preserving) coordinate change \( x_i = y_i + y_{n-1} \) \( (i \in J) \), \( x_i = y_i - y_{n-1} \) \( (i \notin J) \), \( x_{n-1} = y_{n-1} \), and we use the original description of the hypersurface. Then

\[
\text{Int}_J = \int_{(4\mathbb{Z}_2)^{n-2} \times (1 + 2\mathbb{Z}_2)} \left| \left( \sum_{i=1}^{n-2} y_i \pm y_{n-1} \right) \left( \sum_{i \in J} \frac{1}{y_i + y_{n-1}} + \sum_{i \notin J} \frac{1}{y_i - y_{n-1} + 1} \right) - 1 \right|^s dy,
\]

where in the first factor we have \( +y_{n-1} \) (resp. \( -y_{n-1} \)) if \( \# J = \frac{n}{2} - 1 \) (resp. \( \# J = \frac{n}{2} - 2 \)). We further simplify the integral by ‘eliminating’ the variable \( y_{n-1} \). More precisely we perform the (also measure preserving) coordinate change \( y_i = z_i z_{n-1} \) \( (i \in \{1, \ldots, n - 2\}) \), \( y_{n-1} = z_{n-1} \), yielding

\[
\text{Int}_J = \frac{1}{2} \int_{(4\mathbb{Z}_2)^{n-2}} \left| \left( \sum_{i=1}^{n-2} z_i \pm 1 \right) \left( \sum_{i \in J} \frac{1}{z_i + 1} + \sum_{i \notin J} \frac{1}{z_i - 1 + 1} \right) - 1 \right|^s dz,
\]

where we used that \( \int_{1 + 2\mathbb{Z}_2} d z_{n-1} = \frac{1}{2} \). We can multiply the function within \( |\cdot| \) with \( \prod_{i \in J} (z_i + 1) \prod_{i \notin J} (z_i - 1) \) (having norm 1 on the integration domain), in order to obtain a polynomial. A straightforward computation yields that

\[
\left( \sum_{i=1}^{n-2} z_i \pm 1 \right) \left( \sum_{i \in J} \frac{1}{z_i + 1} + \sum_{i \notin J} \frac{1}{z_i - 1 + 1} \right) - 1 \right) \prod_{i \in J} (z_i + 1) \prod_{i \notin J} (z_i - 1)
\]

is (up to sign) equal to

\[
2q(z) + g_{\geq 3}(z),
\]
where \( g_{\geq 3}(z) \) contains only terms of degree at least 3 and

\[
\begin{align*}
q(z) &= \sum_{i \in J} z_i^2 + \sum_{1 \leq i < j \leq n-2} z_i z_j \quad \text{if } \# J = \frac{n}{2} - 1, \\
q(z) &= \sum_{i \in J} z_i^2 + \sum_{1 \leq i < j \leq n-2} z_i z_j \quad \text{if } \# J = \frac{n}{2} - 2.
\end{align*}
\]

Substituting \( z_i = 4x_i \) for \( i = 1, \ldots, n-2 \) yields

\[
\text{Int}_J = \frac{1}{2} \cdot \frac{1}{4^{n-2}} \cdot 2^{-5s} \int_{(\mathbb{Z}_2)^{n-2}} |q(x) + 2g'_{\geq 3}(x)|^s dx,
\]

where \( g'_{\geq 3}(x) \) contains only terms of degree at least 3.

Note that the notation \( q \) is consistent with (6). In fact this last integrand has an isolated singularity in the origin, and blowing up at the origin yields an embedded resolution with good reduction mod 2 and we can use Denef’s formula. We can now proceed completely analogously as in the case \( p \geq \frac{n}{2} + 1 \). Comparing with the last two lines in (7), we claim that the contribution to \( \int_{(\mathbb{Z}_2)^{n-2}} |q(x) + 2g'_{\geq 3}(x)|^s dx \) involving \( \frac{1}{1-2^{-n+2}t^2} \) is

\[
\frac{1}{2^{n-2}} \left( (2^{n-2} - 1) - Q \frac{2^{-n+2}t^2}{1 - 2^{-n+2}t^2} + Q \frac{2^{-n+1}t^3}{(1 - 2^{-1}t)(1 - 2^{-n+2}t^2)} \right),
\]

where

\[
Q = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} - 1).
\]

In order to see this, we note the following.

(i) The only difference is the factor \( 2^{n-2} \) (versus \( p^{n-1} \)). Indeed, now only \( n - 2 \) variables are involved.

(ii) For (7) the centres \( Z_i \) were one-dimensional with \( p - 1 \) as number of points of their reduction mod \( p \), and our present situation can be considered as a ‘transversal section’ of the previous one. So we should a priori divide all ‘numbers of points’ by \( p - 1 \) to derive (20). But since here \( p - 1 = 1 \) this makes no difference.

(iii) The formula for \( Q \) in [11, Theorem 9.2.1] is still valid for \( p = 2 \) and for the two possible equations for \( q \).

Arguing further as in the case \( p \geq \frac{n}{2} + 1 \), we see (compare with (10)) that the contribution of (20) to \( \frac{1}{1-2^{-n+2}t^2} \) simplifies to

\[
\frac{(2^{\frac{n}{2}} - 1)}{2^{\frac{n}{2}}(2^{\frac{n}{2}} - 2)} \cdot \frac{1}{1 - 2^{-\frac{n}{2} + 1}t}.
\]

Combining this last expression with (19), we see that the total contribution to \( Z(t) \) involving \( \frac{1}{1-2^{-\frac{n}{2} + 1}t} \) of all the integration domains \( U_J \)
is

\[
\left(\frac{n-1}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{4^{n-2}} \cdot t^5 \cdot \left(-\frac{(2^{\frac{n}{2}-1} - 1)}{2^2(2^{\frac{n}{2}-2} - 1)} \cdot \frac{1}{1 - 2^{-\frac{n}{2}+1}t}\right)
\]

\[
= -\left(\frac{n-1}{2}\right) \frac{(2^{\frac{n}{2}-1} - 1)}{2^{5\frac{n}{2}-3}(2^{\frac{n}{2}-2} - 1)} \cdot \frac{t^5}{1 - 2^{-\frac{n}{2}+1}t}.
\]

Note that integrating \(|h|^s\) over \((1 + 2Z_2)^{n-1} \backslash \bigcup J\) will not contribute to a term involving \(\frac{1}{1 - 2^{-\frac{n}{2}+1}t}\) since \(h = 0\) is nonsingular there.

As in the previous cases our final aim is to determine \(C_n\) in the description of \(P(t)\) as

\[
(21) \quad A_n(t) + \frac{B_n}{1 - 2^{-\frac{n}{2}+1}t} + \frac{C_n}{1 - 2^{-\frac{n}{2}+1}t},
\]

where \(A_n(t) \in \mathbb{Q}[t]\), and \(B_n\) and \(C_n\) are constants. Since \(\frac{t^5}{1 - 2^{-\frac{n}{2}+1}t}\) is the sum of a polynomial and \(\frac{2^\frac{n}{2}-5}{1 - 2^{-\frac{n}{2}+1}t}\), we conclude that, when writing \(Z(t)\) in the form (21), the constant \(C_n\) is equal to

\[
-\left(\frac{n-1}{2}\right) \frac{(2^{\frac{n}{2}-1} - 1)}{2^2(2^{\frac{n}{2}-2} - 1)}.
\]

Then, using as before (2), one easily derives that \(P(t)\) can be written in the form (21) with

\[
C_n = -\left(\frac{n-1}{2} - 1\right) \frac{2^\frac{n}{2}-3}{(2^{\frac{n}{2}-2} - 1)}.
\]

We conclude as before, by using Theorem 7, that

\[
S_n(2^r) = C_n \cdot 2(1 - 2^{\frac{n}{2}-2})2(\frac{n}{2}+1)^r
\]

\[
= \left(\frac{n-1}{2} - 1\right) 2^{\frac{n}{2}-2}2(\frac{n}{2}+1)^r
\]

when \(r\) is big enough (depending on \(n\)).

\textbf{Case }\(n = 4\). Then the polynomial \(h\) is simply \((x_1+x_2)(x_1+x_3)(x_2+x_3)\)
and one can compute in an elementary way that

\[
Z(t) = \frac{t^3(1 - t + t^2)}{2^4(2-t)^2},
\]

and hence

\[
P(t) = \frac{4 + t^2 + t^3 + t^5}{2^4(2-t)^2}.
\]

And then

\[
V_4(2^r) = \frac{3}{2}(r-3)2^{2r} \quad \text{for } r > 3
\]
and
\[ S_4(2^r) = 3 \cdot 2^{3r} \quad \text{for } r > 4. \]

We note again that this formula for \( S_4(2^r) \) is compatible with the formula for \( n \geq 6 \), which is also compatible with the formula for \( n = 2 \) by the remark in section 4. We summarize.

**Theorem 16.** Let \( n \geq 2 \) be an even positive integer. Then
\[ S_n(2^r) = \left( \frac{n}{2} - 1 \right) 2^{\frac{n}{2} - 2(\frac{n}{2} + 1)r} \]
when \( r \) is big enough (depending on \( n \)).

Again, this result can be made more precise by using the elementary method to remove the integration of \(|h|s\) over \((1 + 2\mathbb{Z}_2)^{n-1} \cup J\cup J\). We can use the ideas in the proof of Theorem 11, but in order to obtain an optimal bound, we need more subtle arguments.

**Theorem 17.** Let \( n \geq 2 \) be an even positive integer. For \( r > \log_2 n + 2 \), we have
\[ S_n(2^r) = \left( \frac{n}{2} - 1 \right) 2^{\frac{n}{2} - 2(\frac{n}{2} + 1)r}. \]

**Proof.** Recall that for small \( r \), the problem is caused by points which are non-singular over \( \mathbb{Q}_2 \), but become singular modulo 2. Also, we saw that the singular locus of \( g = 0 \) in \((\mathbb{Q}_2^*)^{n-1} \) consists of the points \((x_1, \ldots, x_{n-1})\) where each \( x_i = \pm x_0 \) for some \( x_0 \in \mathbb{Q}_2^* \) and \(#\{1 \leq i \leq n-1 \mid x_i = x_0\} = \frac{n}{2} - 1.\)

Consider odd integers \( x_i \) such that \( x = (x_1, \ldots, x_{n-1}) \) satisfies \( g(x) \equiv 0 \pmod{2^{r-1}} \), but such that \( x \pmod{2} \) does not lift to a singular solution over \( \mathbb{Z}_2 \). We claim that \( x \) satisfies the inequality
\[ k_x := \text{ord}_2 \{ \frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x) \} < \frac{r}{2}. \]
Otherwise, suppose that
\[ 0 < \frac{r}{2} \leq k_x. \]

Let \( y_i = x_i + 2^{\min(k_x, r-1)-1}z_i \), where \( z_i \in \mathbb{Z} \). As before, we want to argue using the Taylor expansion. In this case, an easily verified but important fact is that all second partial derivatives \( \frac{\partial^2 g}{\partial x_i \partial x_j}(x) \) are congruent to 0 modulo 2 (using only that the \( x_i \) are odd).

Since \( 2 \min(k_x, r-1)-1 \geq r-1 \) and \( k_x = \text{ord}_2 \{ \frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x) \} \), the Taylor expansion shows that
\[ g(y_1, \cdots, y_{n-1}) \equiv g(x_1, \cdots, x_{n-1}) \equiv 0 \pmod{2^{r-1}}. \]
Because $x$ is clearly a singular solution modulo $2^{\min(k, r-1)}$, one has as in the fourth section the congruence
\[ x_1^2 \equiv x_2^2 \equiv \cdots \equiv x_{n-1}^2 \pmod{2^{\min(k, r-1)}}, \]
implying that $x_i \equiv \pm y_0 \pmod{2^{\min(k, r-1)}}$ for some $y_0$ satisfying $0 \leq y_0 < 2^{\min(k, r-1)}$. We choose $y_i$ such that $y_i = \pm y_0$ for all $1 \leq i \leq n-1$. Let
\[ m'_+ = \# \{1 \leq i \leq n-1 \mid y_i = y_0\}, \quad m'_- = \# \{1 \leq i \leq n-1 \mid y_i = -y_0\}, \]
where $0 \leq m'_+ \leq \frac{n}{2} - 1$. Then
\[
0 \equiv g(y_1, \ldots, y_{n-1}) = (m'_+ - m'_-)^2 - 1 \pmod{2^{r-1}}.
\]
This implies that
\[
(2m'_+ - n)(2m'_+ - (n-2)) \equiv 0 \pmod{2^{r-1}},
\]
which is equivalent to
\[
(m'_+ - \frac{n}{2})(m'_+ - (\frac{n}{2} - 1)) \equiv 0 \pmod{2^{r-3}}.
\]
Our assumption that $2^{r-2} > n$, or equivalently, $2^{r-3} > \frac{n}{2}$, then implies that $m'_+ = \frac{n}{2} - 1$. This contradicts the condition that we imposed on $x \pmod{2}$. The claim is proved.

As in the proof of Theorem 11, we want to conclude using some Hensel lemma. Let
\[
k_0 = \max_{x} k_x = \max_{x} \text{ord}_2 \{ \frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_{n-1}}(x) \},
\]
where $x = (x_1, \ldots, x_{n-1})$ runs over all solutions modulo $2^{r-1}$ such that $x \pmod{2}$ does not lift to a singular solution over $\mathbb{Z}_2$. The above claim shows that $k_0 < r/2$.

If $k_0 < (r-1)/2$, the general Hensel lemma implies, as in the proof of Theorem 11, that each such solution modulo $2^{r-1}$ lifts to exactly $2^{n-2}$ solutions modulo $2^r$, and so on. And then the contribution to $S_n(2^r)$ is zero.

When $k_0 = (r-1)/2$ (implying that $r$ is odd), we cannot invoke the statement of the Hensel lemma, but in this case we can adapt its classical proof with Taylor expansions, using again the crucial fact that all second partial derivatives $\frac{\partial^2 g}{\partial x_i \partial x_j}(x)$ are congruent to $0$ modulo $2$.

More precisely, let $x$ be a solution modulo $2^{r-1}$ as above with moreover $k_x = (r-1)/2$. Let $y_i = x_i + 2^{(r-1)/2}z_i$, where $z_i \in \mathbb{Z}$. Looking at the Taylor expansion, requiring that $g(y_1, \ldots, y_{n-1}) \equiv 0 \pmod{2^r}$ yields a non-trivial linear relation modulo $2$ between $z_1, \ldots, z_{n-1}$, that is, a
non-trivial linear relation between their first digits. Continuing this way we can still conclude that each such solution $x$ modulo $2^{r-1}$ lifts to exactly $2^{n-2}$ solutions modulo $2^r$, resulting again in a zero contribution to $S_n(2^r)$. (As usual for the formal argument one has to start with solutions modulo $2^{(r-1)/2}$ which can be lifted to solutions modulo $2^{r-1}$.)

\[\square\]

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