Covariant quark model of form factors in the heavy mass limit

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Abstract

We show that quark models of current matrix-elements based on the Bakamjian-Thomas construction of relativistic states with a fixed number of particles, plus the additivity assumption, are covariant in the heavy-quark limit and satisfy the full set of heavy-quark symmetry relations discovered by Isgur and Wise. We find the lower bound of $\rho^2$ in such models to be $3/4$ for ground state mesons, independently of any parameter. Another welcome property of these models is that in the infinite momentum limit the wave functions vanish outside the domain $0 \leq x \leq 1$. 

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1 Introduction

The necessity of a relativistic treatment of hadron center-of-mass motion in quark models is manifest in the calculation of form factors at high three-dimensional momentum transfer $\vec{q}$ and effort in this direction have been made since a long time. On the other hand, it has been realized rather recently that in QCD hadronic form factors must satisfy a set of remarkable relations in the limit where the mass of the active quark in the hadron is made heavy [1], the so-called heavy quark symmetries.

In the past few years ago [2], we have noticed that models based on a very simple treatment of hadron motion with Lorentz boost of spins and Lorentz contraction of spatial wave functions - which we have formulated in the 70's [3] - indeed present these symmetry properties in the heavy quark limit [2], [4]. This is not at all trivial because, as can be observed in the literature, most current models do not satisfy these properties [5] (or enforce them by hand). However our own models [2] [3] [4] have serious drawbacks. They are not covariant and they use, in addition to the basic assumptions of quark models, a series of approximations which are not settled in a well-defined framework. In addition, they do not show the expected behavior at large $\vec{q}^2$, which in turn is related to the fact that the null-plane limit of the wave functions do not vanish outside the domain $0 \leq x \leq 1$.

We will use here such a well-defined framework which in addition turns out to have the outstanding merit of being covariant in the heavy quark limit and maintain the scaling properties found in the na"ive models. The purpose of this letter is to show these properties of this new approach and to derive a bound on the widely discussed parameter $\rho^2$, the slope of the Isgur-Wise function. In addition, the class of models obtained in that way solves the above-mentioned problem at large $\vec{q}^2$, since, in the infinite momentum limit, the wave functions vanish outside the domain $0 \leq x \leq 1$, as will be proved elsewhere.

Let us say, before proceeding, a few general words on the method. Although this does not seem to be expected from the knowledge of field theory, it is a very old finding [6],[7],[8] that one can make an important step towards a fully relativistic theory with a fixed number of interacting constituents, with wave functions implementing a representation of the Poincaré group through the construction of the full set of generators, and with a rest-frame Hamiltonian (or mass operator) containing rather standard (non-relativistic looking) potentials depending on relative coordinates, two-dimensional Pauli spins, ... 

In this framework, the problem of knowing the relativistic wave function in motion in terms of the wave function at rest is solved exactly and in a rather simple manner, through a change of variables which is known once for all and does not depend on the interaction.

The whole solution relies on a complete separation between two types of variables, related to individual particle variables by explicit expressions which are simple at least in momentum space. On the one hand, we have global variables which
describe the whole system in analogy with the one-particle state. On the other hand, we have internal variables which somewhat generalise the relative variables of non-relativistic systems, a major property being that the two types of variables are commuting as operators.

The important point is then that one can construct the Poincaré generators as the ones of a free particle described by the global variables (total momentum $\vec{P}$ and relativistic center-of-mass position $\vec{R}$) and a mass which can be chosen arbitrarily provided it depends only on the internal variables.

This framework would be ideally suited to formulate relativistic quark models except for one serious drawback. There is no known covariant current operator, and in particular the usual one-quark free current, which corresponds to the basic additivity assumption of quarks models, is not covariant in general when sandwiched between hadron states. This failure entails that we have not a satisfactory relativistic model for transitions, which was however precisely the initial motivation for appealing to this treatment. We have then to return in general once more to the old discussions on a possible approximate covariance under certain particular conditions, or on the choice of a best reference frame.

One remarkable exception to this failure is the case of the heavy quark limit. Let us recall that this limit consists in considering systems containing one quark very heavy with respect to all the others, and moreover to consider transitions between such “heavy-light” hadrons, where it is the heavy quark which endows electroweak interaction through some external current. In this limit, we find that the above framework with the one-quark current gives a covariant model for transitions. It is then on this limit that we shall concentrate after a presentation of the general formalism, leaving the study of other situations for future discussions. We recall that the interest of this limit is not purely theoretical. It is believed to be roughly realized in the $B \to D^{(*)}l\nu$ semi-leptonic decays, where one is measuring in particular with increasing accuracy the slope $\rho^2$ at the no-recoil point, and where the knowledge of form factors is expected to improve much in the future. It is interesting that our lower bound on $\rho^2$ is rather close to the value estimated by CLEO II experiment.

On the theoretical side, we must emphasize that the model presented in this letter is not just one more quark model. It has the interest of embedding many recent attempts. Indeed, we find that certain recent models for mesons or baryons are explicitly based on the B-T formalism plus the free quark current [8], [13]. On the other hand, there are many approaches directly formulated on the null-plane, among which several [9], [10], [11], [12], can be shown to be the $P = \infty$ limit of the present approach. They will have the same heavy quark limit obtained in this letter, since we show this limit to be covariant. Also, according to our findings, it seems that the intuitive approach of Close and Wambach [15], directly formulated for the heavy quark limit, leads to the same final expressions. One consequence is that the lower bound on $\rho^2$ we have found is of general interest in that it will apply to a large class of proposed models.
Of course, this does not apply to all models encountered in the literature, and one may find seemingly similar approaches which differ by certain details which, in the end, reveal crucial [14]. We have also definitely different approaches leading to covariant and scaling models, like the one of Kaidalov [16].

As to our previous model [2], it comes out that it could be defined, at the start, as an approximation to the present one, which would consist mainly in neglecting internal momenta with respect to the quark masses. However, although this could seem quite natural in a quark model, it is found to lead finally to drastic differences with the present one, especially at large \( \vec{q}^2 \), on which we will comment elsewhere.

2 Relativistic quark models for currents in the Bakamjian-Thomas formalism.

In the present section, we will not yet take the infinite mass limit. The \( n \)-particle Hilbert space, which is naturally the tensor product of the \( n \) individual one-particle Hilbert spaces, is made of functions \( \Psi_{s_1, ..., s_n}(\vec{p}_1, ..., \vec{p}_n) \) of the so-called one-particle variables, spins \( \vec{S}_i \) and momenta \( \vec{p}_i \). Assuming that particle 1 is the active particle, the additivity hypothesis means that the current density operator in the \( n \)-particle Hilbert space will be the tensor product of the current density operator on particle 1 by the identity operators on particles 2, ..., \( n \). This writes:

\[
\langle \Psi' | O | \Psi \rangle = \int \frac{d\vec{p}_1'}{(2\pi)^3} \frac{d\vec{p}_1}{(2\pi)^3} \prod_{i=2}^{n} \frac{d\vec{p}_i}{(2\pi)^3} \sum_{s_1', s_1, s_2, ..., s_n} \Psi_{s_1', s_2, ..., s_n}(\vec{p}_1', \vec{p}_2, ..., \vec{p}_n)^* O(\vec{p}_1', \vec{p}_1)s_1', s_1 \Psi_{s_1, s_2, ..., s_n}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n)
\]

where from now on the primes will denote the final states, and \( O(\vec{p}', \vec{p})_{s', s} \) is the matrix element between one-particle states:

\[
O(\vec{p}', \vec{p})_{s', s} = \langle \vec{p}', s' | O | \vec{p}, s \rangle
\]

The one-particle states that we use are defined, including their normalisation, by eq. (6) below.

Let us describe the essentials (for our purpose) of the B-T model, which is a way to implement exact Poincaré group transformations for a finite, fixed, number of interacting particles. In order to define the Poincaré transformations, one introduces another set of variables, namely, total momentum \( \vec{P} \), internal momenta \( \vec{k}_1, ..., \vec{k}_n \) (\( \sum \vec{k}_i = 0 \)) and internal spins \( \vec{S}_i' \). The unitary transformation which relates the previous wave functions to the wave functions \( \Psi_{s_1, ..., s_n}^{int}(\vec{P}, \vec{k}_2, ..., \vec{k}_n) \), depending on the internal variables, is the following:

\[
\Psi_{s_1, ..., s_n}(\vec{p}_1, ..., \vec{p}_n) = \sqrt{\frac{\Sigma p_0}{M_0}} \prod_{i=1}^{n} \sqrt{\frac{k_0^i}{p_0}} \sum_{s_1', ..., s_n'}
\]

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(\prod_{i=1}^{n} D_i(R_i)_{s_i,s'_i} \Psi^\nu_{s'_1...s'_n}(\Sigma \vec{k}_1, \vec{k}_2, ... \vec{k}_n))

where (on the right), the vectors \( \vec{k}_i \), the 0-components \( k^0_i \) and \( p^0_i \), \( M_0 \), and the Wigner rotations \( \mathbf{R}_i \) are functions of the \( \vec{p}_i \) defined as follows:

\[
\begin{align*}
  p^0_i &= \sqrt{\vec{p}_i^2 + m^2_i}, & M_0 &= \sqrt{(\Sigma p_j)^2}, \\
  k_i &= B^{-1}_{\Sigma p_i} p_i, & R_i &= B^{-1}_{p_i} B_{\Sigma p_j} B_{k_i}
\end{align*}
\]

(notations : \( B_{p} \) is the boost \( (\sqrt{\vec{p}^2}, \vec{0}) \to p \), \( D_i(\mathbf{R}) \) is the matrix of the rotation \( \mathbf{R} \) for the spin \( S_i \)). Let us stress some virtues of unitarity: starting from an orthonormal set of internal wave functions, one gets an orthonormal set of wave functions in any frame.

The Poincaré generators are then defined as \( \vec{P} \) for the space translations and,

\[
\begin{align*}
  H &= \bar{P}^0 = \sqrt{\bar{P}^2 + M^2} \\
  \bar{J} &= -i \bar{P} \times \frac{\partial}{\partial \bar{P}} + \bar{S}, & \bar{S} &= \sum_{i=1}^{n} \bar{S}'_i - i \sum_{i=2}^{n} \bar{k}_i \times \frac{\partial}{\partial \bar{k}_i} \\
  \bar{K} &= -i \frac{1}{2} [\bar{P}^0, \frac{\partial}{\partial \bar{P}}] + - \frac{\bar{P} \times \bar{S}}{\bar{P}^0 + M}
\end{align*}
\]

where, in order to satisfy the Poincaré commutators, the sole requirement is that the mass operator \( M \) depends only on the internal variables and is invariant by rotation. Namely, \( M \) must commute with \( \bar{P}, \frac{\partial}{\partial \bar{P}} \) and \( \bar{S} \). It is important to realize that, in the interacting case, we have to deal simultaneously with two different mass operators, \( M_0 \) of equation (4), and the true mass operator \( M \) which appears in (5) and contains the interaction.

The non-interacting case corresponds to \( M = M_0 = \sum_{i=1}^{n} \sqrt{k^2_i + m^2_i} \) and, through the transformation (3), the generators (5) reduce then to the sum of the one-particle free generators, which is the main virtue of (3).

It is useful to notice that, if \( M \) stands for the mass of a particle and \( \vec{S} \) for its spin, formulae (5) gives precisely the one-particle free generators when (as assumed here) the moving spin states are defined by

\[
|\vec{P}, \mu\rangle = \sqrt{\frac{M}{P_0}} B_{\mu} |\vec{0}, \mu\rangle
\]

\[
\langle P^0 = \sqrt{\vec{P}^2 + M^2}, \quad \langle P', \mu'|P, \mu\rangle = (2\pi)^{3} \delta(P' - \vec{P}) \delta_{\mu', \mu}
\]

The generators follows from the finite transformations, which are given by :

\[
\Lambda |\vec{P}, \mu\rangle = \sqrt{\frac{(\Lambda P)^0}{P_0}} \sum_{\mu} D(\mathbf{B}_{\Lambda P}^{-1} \mathbf{B}_{P})_{\mu', \mu} |\Lambda \vec{P}, \mu'\rangle
\]
where $D(\mathbf{R})$ is the matrix of the rotation $\mathbf{R}$ for the spin $S$.

First it is then easy to see why (5) satisfy the Poincaré Lie algebra, since the calculation of the commutators in (5) is the same as in the one-particle free case, due to the commutativity of the mass operator $M$ with $\vec{P}$, $\frac{\partial}{\partial P^0}$ and $\vec{S}$.

Next the finite transformations generated by (5) are just given (on eigenstates of $M$ and $\vec{P}$) by (7), except that, instead of acting on the spin $\vec{S}$, the Wigner rotation $\vec{B}_M^A \vec{B}_P$ applies now to the internal spins $\vec{S}_i^A$ and the internal momenta $\vec{k}_i$. In fact they are directly given by (7) on eigenstates of $M$, $\vec{P}$, $S^2$, $S_z$.

We are now in position to construct the wave functions of moving bound states. Let $\varphi_{s_1,\ldots,s_n}(k_2,\ldots,k_n)$ be an eigenstate of $M$, $S^2$, $S_z$ (in the Hilbert space reduced with respect to $\vec{P}$). The corresponding (generalized) eigenstate of $M$, $\vec{P}$, $S^2$, $S_z$ with $\vec{P} = 0$ writes:

$$\Psi_{s_1,\ldots,s_n}(\vec{Q},k_2,\ldots,k_n) = (2\pi)^3 \delta(\vec{Q}) \varphi_{s_1,\ldots,s_n}(k_2,\ldots,k_n) \quad (8)$$

The moving wave function $\Psi(\vec{P}),\text{int}$ is obtained by applying the boost $\vec{B}_P$ ($P^0 = \sqrt{\vec{P}^2 + M^2}$) to the wave function (8). Since we start from $\vec{P} = 0$, we may use eq. (6), and the result is simply:

$$\Psi_{s_1,\ldots,s_n}(\vec{Q},k_2,\ldots,k_n) = (2\pi)^3 \delta(\vec{Q} - \vec{P}) \varphi_{s_1,\ldots,s_n}(k_2,\ldots,k_n) \quad (9)$$

The wave function in the one-particle variables, for a bound state of momentum $\vec{P}$, is then obtained from (3):

$$\Psi_{s_1,\ldots,s_n}(p_1,\ldots,p_n) = \sqrt{\frac{\Sigma p_i^0}{M_0}} \left( \prod_{i=1}^n \frac{\sqrt{k_i^0}}{\sqrt{p_i^0}} \right) \sum_{s'_1,\ldots,s'_n} \left( \prod_{i=1}^n D_i(R_i)_{s_i,s'_i} \right) (2\pi)^3 \delta(\Sigma p_i - \vec{P}) \varphi_{s'_1,\ldots,s'_n}(k_2,\ldots,k_n) \quad (10)$$

When $\vec{P} = 0$, this reduces to:

$$\Psi_{s_1,\ldots,s_n}(p_1,\ldots,p_n) = (2\pi)^3 \delta(\Sigma p_i) \varphi_{s_1,\ldots,s_n}(p_2,\ldots,p_n) \quad (11)$$

showing that $\varphi$ is just the rest-frame internal bound state wave function.

Introducing the momentum eigenstates given by (10) in formula (1), one gets:

$$\langle \vec{P}'|O|\vec{P} \rangle = \int \left( \prod_{i=2}^n \frac{dp_i}{(2\pi)^3} \right) \sqrt{\frac{\Sigma p_i^0 \Sigma p_i^0}{M_0 M_0}} \left( \prod_{i=1}^n \frac{\sqrt{k_i^0 k_i^0}}{\sqrt{p_i^0 p_i^0}} \right) \sum_{s_1,\ldots,s_n} \sum_{s'_1,\ldots,s'_n} \varphi_{s_1,\ldots,s_n}(k_2,\ldots,k_n)^* \quad (12)$$

$$[D_i'(R_i^{-1} O(\vec{P}_1,\vec{p}_1) D_1(\mathbf{R}_1)] s'_1,s_1 \left[ \prod_{i=2}^n D_i(R_i^{-1} \mathbf{R}_i)_{s'_i,s_i} \right] \varphi_{s_1,\ldots,s_n}(k_2,\ldots,k_n)$$

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where (on the right) the quantities \( k_i, p_0^i, M^0, R_i \) are the functions of \( \vec{p}_2 \ldots \vec{p}_n \) given by (4) with \( \vec{p}_1 = \vec{P} - \sum_{i=2}^n \vec{p}_i \), and the analogous primed quantities are also given by (4) replacing \( \vec{p}_1 \) by \( \vec{p}_1' = \vec{P}' - \sum_{i=2}^n \vec{p}_i \) and \( m_1 \) by \( m_1' \).

As said before, eq. (12) does not give a in general a covariant model for the current matrix elements. The reason is that the one-particle operator \( O \) is covariant with respect to the free one-particle Lorentz transformations, while the transformations (5) depend on the interaction. However, it turns out that (12) becomes covariant in the heavy mass limit \( m_1, m_1' \to \infty \). This is a general result which requires only the covariance of \( O \) for free one-particle transformations. We leave the proof to a later publication and restrict in this letter to the case of mesons, namely systems of two spin 1/2 particles. Furthermore, in accordance with heavy mass limit ideas in QCD, we assume spin-independent interaction, and satisfy in this case Isgur-Wise scaling.

We consider only the ground states, pseudoscalar and vector mesons.

3 Formulation in terms of Dirac matrices.

Under the assumption of spin independent forces, the pseudoscalar and vector wave functions write

\[
\varphi_{s_1, s_2}(\vec{k}_2) = \frac{i}{\sqrt{2}} (\sigma_2)_{s_1, s_2} \varphi(\vec{k}_2) \tag{13}
\]

\[
\varphi_{s_1, s_2}(\vec{k}_2) = \frac{i}{\sqrt{2}} ((\vec{e}.\vec{\sigma}) \sigma_2)_{s_1, s_2} \varphi(\vec{k}_2)
\]

where \( \varphi(\vec{k}_2) \) is only required to be invariant by rotation, and \( \vec{e} \) is the rest frame polarisation vector of the vector meson. The spin sums in eq. (12) reduce to the trace of a 2 \times 2 matrix. For example, an harmonic oscillator potential leads to:

\[
\varphi(\vec{k}_2) = (2\pi)^{3/2} \left( R/\sqrt{\pi} \right)^{3/2} e^{-R^2\vec{k}_2^2/2}
\]

Using the relation \( \sigma_2 D(R) \sigma_2 = D(R^{-1})^t \), one gets:

\[
\langle \vec{P}', \vec{e}'|O|\vec{P}, \vec{e} \rangle = \int \frac{d\vec{p}_2}{(2\pi)^3} \sqrt{\frac{\Sigma p_2^0 \Sigma p_1^0 \sqrt{k_2^0 k_1^0}}{\sqrt{p_1^0 p_1^0}} M_0 M_0} \sqrt{\frac{k_2^0 k_1^0}{p_2^0}} \frac{1}{2} \text{Tr}[((\vec{e}'.\vec{\sigma})^t D(R_1^{-1}) O(\vec{p}_1', \vec{p}_1) D(R_1)(\vec{e}.\vec{\sigma}) D(R_2^{-1} R'_2)] \varphi'(\vec{k}_2)^* \varphi(\vec{k}_2) \tag{14}
\]

for the matrix element between vector mesons. The other matrix elements are obtained by omitting \( (\vec{e}.\vec{\sigma}) \) or \( (\vec{e}'.\vec{\sigma}) \) or both under the trace.

Next, this formula can be written in a more familiar form, involving the 4 \times 4 Dirac matrices instead of the 2 \times 2 Pauli matrices. Between vector mesons, one gets

\[
\langle \vec{P}', \vec{e}'|O|\vec{P}, \vec{e} \rangle = \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} F(\vec{p}_2, \vec{P}', \vec{P}) \tag{15}
\]
\[
\frac{1}{16} \text{Tr}[O(m_1 + \tilde{p}_1)(1 + \gamma_5)\gamma_5\bar{u}(m_2 + \tilde{p}_2)\gamma_5\bar{u}'(1 + \gamma_5)(m_1' + \tilde{p}_1')] \varphi'(k_2^*')\varphi(k_2)
\]

\[
F(\vec{p}_2, \vec{P}_1, \vec{P}) = \frac{\sqrt{u^0 u^0}}{p_1^0 p_1^0} \frac{\sqrt{k_1^0}}{\sqrt{k_1^0 + m_1'}} \frac{\sqrt{k_2^0}}{\sqrt{k_2^0 + m_2}} \frac{\sqrt{k_2^0}}{\sqrt{k_2^0 + m_2}} (16)
\]

In (15), the following additional notations are used. The unit 4-vectors \( u \) and \( u' \):

\[
u = p_1 + p_2, \quad u' = \frac{p_1' + p_2}{M_0'}
\]

The 4-vectors \( \epsilon_u \) and \( \epsilon'_u \) are related to the polarisation 4-vectors \( \epsilon = B_p(0, \vec{e}) \) and \( \epsilon' = B_{p'}(0, \vec{e}') \) by

\[
\epsilon_u = B_u B_{p'}^{-1} \epsilon, \quad \epsilon'_u = B_u B_{p'}^{-1} \epsilon'
\]

Moreover, the \( O \) under the trace stands for the Dirac matrix appropriate to the current considered, for example \( \gamma_\mu, \gamma_\mu\gamma_5 \), etc. The matrix elements for other mesons are obtained by omitting \( \gamma_5 \bar{u} \) or \( \gamma_5 \bar{u}' \) or both under the trace.

Let us describe the main steps to deduce (15, 16) from (14). \( O(p', \vec{p})_{s', s} \) is the matrix element of \( \gamma^0 O \) between spinors of the form

\[
\sqrt{m}p^0 B_p \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \vec{p}.\vec{\sigma} \end{pmatrix}
\]

where the boost \( B_p \) also stands for its matrix in the Dirac representation:

\[
B_p = \frac{m + \gamma^0}{\sqrt{2m(p^0 + m)}}
\]

The 2 × 2 Pauli matrix in (14) may be considered as a 4 × 4 Dirac matrix with only its upper left 2 × 2 block non vanishing. The matrix \( O(p', \vec{p}) \) then writes:

\[
O(p', \vec{p}) = \frac{\sqrt{m_1 m_1}}{\sqrt{p_1^0 p_1^0}} \frac{1 + \gamma_0}{2} B_{p'}^{-1} O B_{p_1} \frac{1 + \gamma_0}{2} (21)
\]

(used \( \gamma_0 B_p \gamma_0 = B_{p'}^{-1} \)). Also we replace the Wigner rotations in (14) by their expressions (4) as products of three boosts. The resulting expressions contain matrices sandwiched between \( B_u \) and \( B_u^{-1} \) and between \( B_{u'} \) and \( B_{u'}^{-1} \). They are reduced using

\[
B_u (1 + \gamma_0) B_{k_2}^{-1} B_u^{-1} = (1 + \gamma_0) \frac{m_2 + \gamma_2}{\sqrt{2m_2(k_2^0 + m_2)}}
\]
and similar other formulae. (22) is obtained from the relation $B_u \frac{B_u^{-1}}{x} = \gamma \cdot B_u^x$, which simply expresses that the $\gamma^\mu$ matrices are forming a 4-vector, since from (20) we have

$$
(1 + \gamma^0)B_{k_2}^{-1} = (1 + \gamma^0) \frac{m_2 + k_2}{\sqrt{2m_2(k_2^0 + m_2)}}
$$

(23)

and $u = B_u(1, \vec{0}), p_2 = B_u k_2$. Finally, we use

$$
B_u(\vec{e} \cdot \vec{\sigma})\gamma^0B_u^{-1} = \gamma_{5}^\mu u, \quad B_u'(\vec{e}' \cdot \vec{\sigma})\gamma^0B_u'^{-1} = \gamma_{5}^{\mu'} u'
$$

(24)

4 Heavy mass limit: covariance and Isgur-Wise scaling.

We now consider the heavy mass limit of (15, 16). This limit is defined as $m_1, m'_1 \rightarrow \infty$ with $v' = \frac{P'}{M'}$ and $v = \frac{P}{M}$ fixed ($M'$ and $M$ are the final and initial meson masses). It is also assumed that $\frac{M}{m_1} \rightarrow 1, \frac{M'}{m'_1} \rightarrow 1$. It is then found that the integrand in (15) has a limit for fixed integration variable $\vec{p}_2$. We have

$$
\frac{p_1}{m_1} \rightarrow v, \quad \frac{p'_1}{m'_1} \rightarrow v',
$$

$$
u \rightarrow v, \quad u' \rightarrow v'
$$

$$
\epsilon_u \rightarrow \epsilon_v = \epsilon, \quad \epsilon'_u \rightarrow \epsilon'_v = \epsilon'
$$

$$
\frac{k_1^0}{m_1} \rightarrow 1, \quad \frac{k'_1^0}{m'_1} \rightarrow 1
$$

$$
k_2 \rightarrow B^{-1}_v p_2, \quad k'_2 \rightarrow B'^{-1}_v p_2
$$

(25)

Also,

$$
(B^{-1}_v p_2)^0 = p_2 v, \quad (B'^{-1}_v p_2)^0 = p_2 v'
$$

(26)

by invariance of the scalar product since $B^{-1}_v v = (1, \vec{0}), B'^{-1}_v v' = (1, \vec{0})$. The limit of (15, 16) is therefore

$$
\langle \vec{P}', \epsilon | O | \vec{P}, \epsilon \rangle = \frac{1}{2} \frac{1}{\sqrt{v^0 v'^0}} \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{(p_2 v')(p_2 v)}}{\sqrt{(p_2 v' + m_2)(p_2 v + m_2)}}
$$

$$
\frac{1}{4} \text{Tr}[O \gamma_5 \hat{\epsilon}(1 + \hat{\epsilon})(\vec{p}_2 + m_2)(1 + \hat{\epsilon}') \gamma_5 \hat{\epsilon}'] \varphi'(B'^{-1}_v p_2)^* \varphi(B^{-1}_v p_2)
$$

(27)

Now, this expression is explicitly a covariant function of the 4-vectors $P'$ and $P$ (or $v'$ and $v$) because we have the integral with the invariant measure $\frac{d\vec{p}_2}{p_2^0}$ of a covariant function of $p_2, v'$ and $v$. The only point perhaps not immediately apparent is the invariance of $\varphi'(B'^{-1}_v p_2)^* \varphi(B^{-1}_v p_2)$. In fact, $\varphi'(\vec{k})$ and $\varphi(\vec{k})$ being invariant by rotation
are function of $\vec{k}^2$ and, according to (26), we have

\[
(B_{\vec{v}^{-1}p_2})^2 = (p_2.v')^2 - m_2^2, \quad (B_{\vec{v}^{-1}p_2})^2 = (p_2.v)^2 - m_2^2 \quad (28)
\]

Therefore $\varphi'(B_{\vec{v}^{-1}p_2})^* \varphi(B_{\vec{v}^{-1}p_2})$ is a Lorentz scalar.

Using its covariance properties, eq. (27) can be reduced further. Indeed, (27) can apparently be expressed (for all Dirac matrices $O$) in term of three independent form factors $A, B, B'$, defined by :

\[
A(v'.v) = \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \sqrt{(p_2.v')(p_2.v)} \varphi'(B_{\vec{v}^{-1}p_2})^* \varphi(B_{\vec{v}^{-1}p_2}) \phi_{v' p_2} \phi_{v p_2} \quad (29)
\]

After integration however, the expression $(1 + \phi)(\phi_2 + m_2)(1 + \phi')$ becomes

\[
(1 + \phi')(B'\phi' + B\phi + m_2A)(1 + \phi') = (B + B' + m_2A)(1 + \phi)(1 + \phi')
\]

and, in fact, only the combination $\xi = B + B' + m_2A$ appears. We obtain finally the standard scaling formula:

\[
\langle \vec{P}', \epsilon'| O | \vec{P}, \epsilon \rangle = \frac{1}{2} \frac{1}{\sqrt{\epsilon_0^{\epsilon_0}}} \frac{1}{4} \text{Tr}[O\gamma_5\phi(1 + \phi')(1 + \phi')\gamma_5\phi^*] \varphi(v'.v) \quad (30)
\]

with the Isgur-Wise function $\xi$ given by :

\[
\xi(v'.v) = \frac{1}{v'.v + 1} \int \frac{d\vec{p}_2}{(2\pi)^3} \sqrt{(p_2.v')(p_2.v)} \varphi'(B_{\vec{v}^{-1}p_2})^* \varphi(B_{\vec{v}^{-1}p_2}) \phi_{v' p_2} \phi_{v p_2} \quad (31)
\]

The matrix element with one or two pseudoscalar mesons are obtained from (30) by omitting $\gamma_5\phi$ or $\gamma_5\phi^*$ or both under the trace.

One may notice that

\[
\xi(1) = \int \frac{d\vec{p}_2}{(2\pi)^3} \varphi'(\vec{p}_2)^* \varphi(\vec{p}_2) \quad (32)
\]

and that flavor independent forces and heavy mass limit entail $\varphi' = \varphi$, so that $\xi(1) = 1$.

Although they do not give a general expression like (27) and provide their results only in a particular frame, by gathering the various indications given in the
paper by Close and Wambach [15], we apparently end up with precisely this expression; we have also checked that their expression of the slope $\rho^2$, given for harmonic oscillator wave functions, coincides with ours. An important advantage of our approach is that we derive it from the general Bakamjian-Thomas formalism and we demonstrate the covariance and scaling properties of the result.

Let us write the vector $(O = V^\mu = \gamma^\mu)$ and axial $(O = A^\mu = \gamma^\mu\gamma_5)$ matrix elements with an initial pseudoscalar, following from (30):

\[
\sqrt{\nu_0 v_0} \langle \vec{P}' | V^\mu | \vec{P} \rangle = \frac{1}{2} (v'^\mu + v^\mu) \xi(v'.v) \tag{33}
\]

\[
\sqrt{\nu_0 v_0} \langle \vec{P}', \epsilon' | V^\mu | \vec{P} \rangle = -\frac{i}{2} \sum_{\nu\rho\sigma} \epsilon^{\mu\rho\sigma} v'_\nu v_\rho \epsilon'^\tau \xi(\tau'(v',v))
\]

\[
\sqrt{\nu_0 v_0} \langle \vec{P}' | A^\mu | \vec{P} \rangle = 0
\]

\[
\sqrt{\nu_0 v_0} \langle \vec{P}', \epsilon | A^\mu | \vec{P} \rangle = \frac{1}{2} \left[ (v'.v + 1) \epsilon'^\mu - (v.v') \epsilon^\mu \right] \xi(v'.v)
\]

The corresponding form factors are

\[
f_+(q^2) = V(q^2) = A_2(q^2) = A_0(q^2) = \frac{M' + M}{2\sqrt{M'M}} \xi(v'.v) \tag{34}
\]

\[
f_0(q^2) = A_1(q^2) = \frac{M' + M}{2\sqrt{M'M}} \left[ 1 - \frac{q^2}{(M' + M)^2} \right] \xi(v'.v)
\]

\[
(v'.v = \frac{M'^2 + M^2 - q^2}{2M'M})
\]

5 Lower bound on the $\rho^2$ slope.

Let us now consider the slope $\rho^2 = -\xi'(1)$ of the Isgur-Wise function given by (31). We establish here the following optimal lower bound of $\rho^2$:

\[
\rho^2 > \frac{3}{4} \tag{35}
\]

Eq. (31) is of the form

\[
\xi(v'.v) = \int \frac{d\vec{p}}{p^0} F(p,v',v,v') \tag{36}
\]

with

\[
F(x', x, y) = \frac{1}{(2\pi)^3} \sqrt{x'x} \frac{x' + x + m_2(y + 1)}{(x' + m_2)(x + m_2)} f(x'^2 - m_2^2) f(x^2 - m_2^2)
\]

\[
f(\vec{k}^2) = \varphi(\vec{k}) \tag{37}
\]
The integral in (36) depends only on the scalar product $v'.v$ because the integration measure $d\vec{p}/p^0$ is Lorentz invariant. Expanding (36) around $v'.v = 1$, we find

$$\xi'(1) = -\int \frac{d\vec{p}}{p^0} \left[ \frac{1}{3} p^2 \partial_1 \partial_2 F(p^0, p^0, 1) - \partial_3 F(p^0, p^0, 1) \right]$$

(38)

And using eq. (37) for $F$, the following formula for the slope is obtained:

$$\rho^2 = \frac{1}{3} \int \frac{d\vec{p}}{(2\pi)^3} \left[ \nabla^2 F(p^0, \varphi(\vec{p})) \right]^2$$

$$+ \int \frac{d\vec{p}}{(2\pi)^3} \left[ \frac{2}{3} \frac{m_2^2}{4 (p^0)^2} - \frac{1}{3} \frac{m_2}{p^0 + m_2} \right] \varphi(\vec{p})^* \varphi(\vec{p})$$

(39)

It is obvious that $\rho^2 > 0$. In fact this expression is a positive quadratic form of the wave function $\varphi$, and the best lower bound of $\rho^2$ is given by the greatest lower bound $B$ of the spectrum of the following corresponding self-adjoint operator $T$:

$$T = -\frac{1}{3} p^0 \Delta p^0 + \frac{2}{3} + \frac{1}{4} \frac{m_2^2}{(p^0)^2} - \frac{1}{3} \frac{m_2}{p^0 + m_2}$$

(40)

It is easily seen that the lower bound is obtained in the S-wave subspace. Indeed, the reduction $T_L$ of $T$ to the $L$-orbital eigenspace, acting on the radial functions $f(p)$ related to $\varphi(\vec{p}) = \sqrt{p^0} f(\varphi(\vec{p})) Y^M_L(\hat{\vec{p}})$, writes

$$T_L = -\frac{1}{3} p^0 \frac{d^2}{dp^2} p^0 + \frac{L(L+1)}{3} \frac{(p^0)^2}{p^2} + \frac{2}{3} + \frac{1}{4} \frac{m_2^2}{(p^0)^2} - \frac{1}{3} \frac{m_2}{p^0 + m_2}$$

(41)

and, due to the "centrifugal barrier", we have $T_L \leq T_{L'}$ if $L < L'$.

We may therefore concentrate on (41) with $L = 0$. Numerical integration of the ordinary differential equation $T_0 f = B' f$ (with initial condition $f(0) = 0, f'(0) = 1$) gives a strong indication that the lower bound of the spectrum is

$$B = \frac{3}{4}.$$  

(42)

Indeed, one finds that every $B' \geq \frac{3}{4}$ is a generalized eigenvalue and that the (non normalisable) eigenfunction is oscillating for $B' > \frac{3}{4}$ and non oscillating for $B' = \frac{3}{4}$.

For a proof of (42), we use a unitary transformation $U : L^2([0, \infty[) \rightarrow L^2([0, \infty[)$ which has the virtue of converting $T_L$ into an ordinary Schrödinger operator. $U$ is just the following change of variable:

$$(Uf)(x) = \sqrt{m_2 \cosh(x)} f(m_2 \sinh(x)),$$

$$(U^{-1}f)(p) = \frac{1}{\sqrt{p^0}} f(\text{Arccosh} p/m_2)$$

(43)
The operator $p^0 d/dp$ becomes:

$$U p^0 \frac{d}{dp} U^{-1} = \frac{d}{dx} - \frac{1}{2} \tanh(x)$$

(44)

and the operator $T_0$ (eq. (41) with $L = 0$) becomes

$$U T_0 U^{-1} = \frac{3}{4} + \frac{1}{3} \left( - \frac{d^2}{dx^2} - \frac{1}{1 + \cosh(x)} \right)$$

(45)

Here we can see that $B \leq \frac{3}{4}$ because the (always positive) expectation value of the operator $-d^2/dx^2$ on a spreading function $f_\lambda(x) = \sqrt{\lambda} f(\lambda x)$, $\lambda \to 0$, ($||f_\lambda|| = ||f|| = 1$) goes to 0. On the other hand we have $B \geq \frac{3}{4}$ because the operator between parenthesis in (45) is positive, as can be seen from the following identity:

$$- \frac{d^2}{dx^2} - \frac{1}{1 + \cosh(x)} = \left( \frac{d}{dx} + \frac{1}{\sinh(x)} \right) \left( - \frac{d}{dx} + \frac{1}{\sinh(x)} \right)$$

(46)

Finally, one sees as follows that the lower bound $B = 3/4$ of $\rho^2$ cannot be attained. If $\rho^2 = 3/4$, the expectation value of (46) vanishes on the normalized function $f(x)$ which corresponds to $\varphi(\vec{p})$. This implies $(- \frac{d}{dx} + \frac{1}{\sinh(x)}) f(x) = 0$. However, the solutions $f(x) = c \tanh(\frac{x}{2})$ of this equation are not normalizable.

The bound (35) is obviously in agreement with Bjorken’s lower bound $[17] \rho^2 > 1/4$. It does not contradict Voloshin’s upper bound $[18] \rho^2 < 0.75 \pm 0.15$, but not much room is left and a careful study of the relativistic generalisation of Thomas-Reiche-Kuhn sum rule in our model should be performed. Neither does it contradict de Rafael-Taron’s $[19]$ conservative estimate $\rho^2 < 1.7$ not to speak of their rigorous bound $\rho^2 < 6.0$.

Let us emphasize that the bound $B = 3/4$ applies to a large class of quark models, which at least in this heavy quark limit show very remarkable properties, and is independent of particular parameters of the models. Whence the interest of testing it experimentally. But this is not an easy task. Let us recall that $\rho^2$ is not directly comparable to the $\hat{\rho}^2$ measured by CLEO II experiment $[21]$, from which it should differ by QCD radiative corrections and $1/m_c$ corrections $[20]$. The QCD radiative corrections enhance $\hat{\rho}^2$ versus $\rho^2$; the $1/m_c$ corrections are not known from exact QCD, and cannot be safely deduced from our model in which they are not covariant. The central value of the present CLEO II data: $\hat{\rho}^2 = 0.87 \pm 0.12 \pm 0.08$ $[21]$, is compatible with our lower bound, although not far from it. But one must keep in mind the above-mentioned corrections. A further improvement of data may prove very instructive.

The wave functions that saturate the lower bound are, for example, of the type:

$$\varphi(\vec{p}) \propto \sqrt{\epsilon} (p^0)^{-(1.5+\epsilon)}$$

(47)

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for which $\rho^2 \to 0.75$ when $\epsilon \to 0$.

Harmonic oscillator wave functions,

$$\varphi(\vec{p}) = (2\pi)^{3/2} \left( R/\sqrt{\pi} \right)^{3/2} e^{-R^2\vec{p}^2/2},$$

which are most commonly used, yield $\rho^2$ well above the bound 0.75. Indeed we found that the lower bound for harmonic oscillator wave functions is 1.208. In the limit $m^2R^2 = 0$ they give $\rho^2 = 5/4$. The minimum 1.208 is obtained for $m^2R^2 \approx 0.05$ which is far below the standard quark model parameters, say, $m^2 \simeq 0.1$ GeV$^2$, $R^2 \simeq 6$ GeV$^{-2}$, leading to $m^2R^2 \simeq 0.6$. Using the latter values we obtain $\rho^2 \approx 1.37$ which is, as expected, very similar to Close and Wambach’s [15]: $\rho^2 \simeq (1.19)^2$. In the weak coupling limit $m^2R^2 \to \infty$ one gets

$$\rho^2 = \frac{m^2R^2}{2} + 1 + O\left(\frac{1}{m^2R^2}\right)$$

and the Isgur-Wise function takes the very simple form in the large $m^2R^2$:

$$\xi(v.v') = \frac{2}{1 + v.v'\sqrt{v.v'}} \exp[-m^2R^2(v.v' - 1)/2] \left(1 + O\left(\frac{1}{m^2R^2}\right)\right)$$

We have checked numerically that expression (50) is a surprisingly good approximation to the Isgur-Wise function obtained with the standard quark model parameters: $m^2 \simeq 0.1$ GeV$^2$, $R^2 \simeq 6$ GeV$^{-2}$, although $m^2R^2$ does not look so large. This amusingly simple formula is not phenomenologically very useful since, as already stressed, the harmonic oscillator wave functions yield a $\rho^2$ much above the lower bound and above experiment.

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References

[1] N. Isgur and M. Wise, Phys. Lett. B232 (1990) 113, Phys. Lett. B237 (1990) 527; Phys. Rev. Lett. 66 (1991) 1130; D42 (1990) 2388; see also M.B. Voloshin and M.A. Shifman, Sov. J. Nucl. Phys. 45 (1987) 292; Sov. J. Nucl. Phys. 47 (1988) 511; H.D. Politzer and M. Wise, Phys. Lett. B206 (1988) 681; Phys. Lett. B208 (1988) 504; E. Eichten, Nucl. Phys. B (Proc.Suppl.) 4 (1988) 170; H. Georgi, Nucl. Phys. B361, 339 (1991); J.D. Bjorken, Invited Talk given at
25th Int. Conf. on High Energy Physics, Singapore, Aug 2-8, 1990, Published in Singapore H. E. Phys. 1990; J.D. Bjorken, Lectures given at 18th Annual SLAC Summer Inst. on Particle Physics, Stanford, CA, Jul 16-27, 1990, Published in SLAC Summer Inst. 1990; H. Georgi and F. Uchiyama, Phys. Lett. B238 (1990) 395; H. Georgi and M. Wise, Phys. Lett. B243 (1990) 279.

[2] A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Désintégration faible des quarks lourds, Gif lectures 1991, Editions IN2P3, Paris, p.96; A. Le Yaouanc and O. Pène, Third Workshop on the Tau-Charm Factory, 1-6 June 1993, Marbella, Spain, Editions frontieres, p. 275, hep-ph 9309230; R. Aleksan, A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Beauty 94 Workshop, April 1994, Mont Saint Michel, France, Nucl. Inst. and Meth. A351,15 (1994), LPTHE Orsay 94/53, hep-ph 9406334; R. Aleksan, A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Rev. D 51, 6235 (1995); A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Talk delivered at the at the Journées sur les projets de Physique Hadronique, Société Française de Physique, Super-Besse (France), 12-14 janvier 1995.

[3] P. Andreadis et al. Ann. of Phys. 88 (1974) 242; A. Le Yaouanc et al. Phys. Rev. D9, 2636 (1974); D15, 844 (1977); A. Amer et al., Phys. Lett. 81B (1979) 48; M.B. Gavela, Phys. Lett. 83B (1979) 367; M.B. Gavela, Doctoral thesis univ. Paris-Sud (LPTHE), July 1779.

[4] A detailed account of this older model is in preparation.

[5] R. Aleksan, A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Rev. D 51, 6235 (1995).

[6] B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).

[7] H. Osborn, Phys. Rev. 176, 1514 (1968).

[8] B. D. Keister and W. N. Polyzou, Adv. Nucl. Phys. 20, 225 (1991) (extensive review).

[9] M. Terent’ev, Sov. J. Nucl. Phys. 24,106 (1976)

[10] F. Cardarelli, I. L. Grach, I. M. Narodetskii, E. Pace, G. Salme, and S. Simula, Phys. Lett. B332, 1 (1994).

[11] W. Jaus, Phys. Rev. D41, 3394 (1990).

[12] S. Capstick and B. D. Keister, Phys. Rev. D 51, 3598 (1995).

[13] A. Szczepaniak, C-R Ji and S. R. Cotanch, North Carolina State University at Raleigh, report (1995).
[14] Z. Dziembovski, C. J. Martoff and P. Żyła, Phys. Rev. D 50, 5613 (1994)

[15] F. E. Close and A. Wambach, Nucl. Phys. B412, 169 (1994).

[16] A. Dubin and A. Kaidalov, Sov. Journal of Nucl. and Atomic Physics 56 237 (1993), find the Isgur-Wise scaling relations in an approach based on Feynman diagrams.

[17] J.D. Bjorken, Invited talk given at Les Rencontres de la Vallée d’Aoste, La Thuile, Italy, March 18-24, 1990, Published in La Thuile Rencontres 1990.

[18] M.B. Voloshin Phys. Rev. D46 , 3062 (1992).

[19] E. de Rafael and J. Taron Phys. Rev. D50, 373 (1994).

[20] M. Neubert, talk delivered at XXXth Rencontre de Moriond, Electroweak Interactions and Unified Theories, Les Arcs, Savoie France, March 11-18 1995, Ed. Frontières.

[21] T. E. Browder, Glasgow Proceedings of the XXVII Int. Conf. on High Energy Phys., Glasgow, July 1994, Institute of Physics Publishing, Bristol, vol. II, p. 1117. We do not consider here the other estimate of CLEOII, R. K. Kutschke, ibidem, p. 1113, which uses another definition of $\rho^2$. 