Abstract

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$, and let $\theta$ be a closed smooth real $(1, 1)$-form representing a big and nef cohomology class. We introduce a metric $d_p$, $p \geq 1$, on the finite energy space $E^p(X, \theta)$, making it a complete geodesic metric space.

Keywords Kähler manifolds · Pluripotential theory · Finite energy classes · Complete metric space

1 Introduction

Finding canonical (Kähler-Einstein, cscK, extremal) metrics on compact Kähler manifolds is one of the central questions in differential geometry (see [13, 41, 42] and the references therein). Given a Kähler metric $\omega$ on a compact Kähler manifold $X$, one looks for a Kähler potential $\varphi$ such that $\omega_\varphi := \omega + dd^c \varphi$ is “canonical”. Mabuchi introduced a Riemannian structure on the space of Kähler potentials $\mathcal{H}_\omega$. As shown by Chen [15] $\mathcal{H}_\omega$ endowed with the Mabuchi $d_2$ distance is a metric space. Darvas [21] showed that its metric completion coincides with a finite energy class of plurisubharmonic functions introduced by Guedj...
and Zeriahi [36]. Other Finsler geometries $d_p$, $p \geq 1$, on $\mathcal{H}_\omega$ were studied by Darvas [20] and they lead to several spectacular results related to a longstanding conjecture on existence of cscK metrics and properness of K-energy (see [6, 16–18, 29]). Employing the same technique as in [29] and extending the $L^1$-Finsler structure of [20] to big and semipositive classes via a formula relating the Monge-Ampère energy and the $d_1$ distance, Darvas [22] established analogous results for singular normal Kähler varieties. Motivated by the same geometric applications, the $L^p$ ($p \geq 1$) Finsler geometry in big and semipositive cohomology classes was constructed in [32] via an approximation method.

In this note we extend the main results of [20, 32] to the context of big and nef cohomology classes. Assume that $X$ is a compact Kähler manifold of complex dimension $n$ and let $\theta$ be a smooth closed real $(1, 1)$ form representing a big & nef cohomology class. Fix $p \geq 1$.

**Main Theorem** The space $\mathcal{E}^p(X, \theta)$ endowed with $d_p$ is a complete geodesic metric space.

For the definition of $\mathcal{E}^p(X, \theta)$, $d_p$ and relevant notions we refer to Section 2. When $p = 1$ Main Theorem was established in [26] in the more general case of big cohomology classes using the approach of [22]. Here, we use an approximation argument as in [32] with an important modification due to the fact that generally potentials in big cohomology classes are unbounded. Interestingly, this modification greatly simplifies the proof of [32, Theorem A].

**Organization of the Note** We recall relevant notions in pluripotential theory in big cohomology classes in Section 2. The metric space $(\mathcal{E}^p, d_p)$ is introduced in Section 3 where we prove Main Theorem. In case $p = 1$ we show in Proposition 3.18 that the distance $d_1$ defined in this note and the one defined in [26] do coincide.

### 2 Preliminaries

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. We use the following real differential operators $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$. We briefly recall known results in pluripotential theory in big cohomology classes, and refer the reader to [5, 12, 24–27] for more details.

#### 2.1 Quasi-plurisubharmonic Functions

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic (or quasi-psh) if it is locally the sum of a psh function and a smooth function. Given a smooth closed real $(1, 1)$-form $\theta$, we let $\text{PSH}(X, \theta)$ denote the set of all integrable quasi-psh functions $u$ such that $\theta + dd^c u \geq 0$, where the inequality is understood in the sense of currents. A function $u$ is said to have analytic singularities if locally $u = \log \sum_{j=1}^{N} |f_j|^2 + h$, where the $f_j$’s are holomorphic and $h$ is smooth.

The De Rham cohomology class $\{\theta\}$ is Kähler if it contains a Kähler potential, i.e., a function $u \in \text{PSH}(X, \theta) \cap C^\infty(X, \mathbb{R})$ such that $\theta + dd^c u > 0$. The class $\{\theta\}$ is nef if $\{\theta + \epsilon \omega\}$ is Kähler for all $\epsilon > 0$. It is pseudo-effective if the set $\text{PSH}(X, \theta)$ is non-empty, and big if $\{\theta - \epsilon \omega\}$ is pseudo-effective for some $\epsilon > 0$. The ample locus of $\{\theta\}$, which will be denoted by $\text{Amp}(\theta)$, is the set of all points $x \in X$ such that there exists $\psi \in \text{PSH}(X, \theta - \epsilon \omega)$ with analytic singularities and smooth in a neighborhood of $x$. It was shown in [11, Theorem 3.17] that $\{\theta\}$ is Kähler if and only if $\text{Amp}(\theta) = X$. 
Throughout this note we always assume that \( \{ \theta \} \) is big and nef. Typically, there are no bounded functions in PSH\((X, \theta)\), but there are plenty of locally bounded functions as we now briefly recall. By the bigness of \( \{ \theta \} \) there exists \( \psi \in \text{PSH}(X, \theta - \varepsilon \omega) \) for some \( \varepsilon > 0 \). Regularizing \( \psi \) (by [30, Main Theorem 1.1]) we can find a function \( u \in \text{PSH}(X, \theta - \frac{\varepsilon}{2} \omega) \) smooth in a Zariski open set \( \Omega \) of \( X \). Roughly speaking, \( \theta u \) locally behaves as a Kähler form on \( \Omega \). As shown in [11, Theorem 3.17], \( u \) and \( \Omega \) can be constructed in such a way that \( \Omega \) is the ample locus of \( \{ \theta \} \).

If \( u \) and \( v \) are two \( \theta \)-psh functions on \( X \), then \( u \) is said to be less singular than \( v \) if \( v \leq u + C \) for some \( C \in \mathbb{R} \), while they are said to have the same singularity type if \( u - C \leq v \leq u + C \), for some \( C \in \mathbb{R} \). An \( \theta \)-psh function \( u \) is said to have minimal singularities if it is less singular than any other \( \theta \)-psh function. An example of a \( \theta \)-psh function with minimal singularities is

\[
V_{\theta} := \sup\{u \in \text{PSH}(X, \theta) \mid u \leq 0\}.
\]

For a function \( f : X \to \mathbb{R} \), let \( f^* \) denote its upper semicontinuous regularization, i.e.,

\[
f^*(x) := \limsup_{X \ni y \to x} f(y).
\]

Given a measurable function \( f \) on \( X \) we define

\[
P_{\theta}(f) := (x \mapsto \sup\{u(x) \mid u \in \text{PSH}(X, \theta), u \leq f\})^*.
\]

**Essential Supremum** For \( u, v \) quasi-psh functions, the function \( u - v \) is defined almost everywhere on \( X \) (off the set where \( v = -\infty \)). By abuse of notation we let \( \sup_X (u - v) \) denote the essential supremum of \( u - v \). By basic properties of plurisubharmonic functions we have

\[
u - \sup_X (u - v) \leq v \leq u + \sup_X (v - u), \text{ on } X.
\]

We will need the following result on regularity of quasi plurisubharmonic envelope due to Berman [4].

**Theorem 2.1** Let \( f \) be a continuous function such that \( dd^c f \leq C \omega \) on \( X \), for some \( C > 0 \). Then \( \Delta_\omega(P_{\theta}(f)) \) is locally bounded on \( \text{Amp}(\theta) \), and

\[
(\theta + dd^c P_{\theta}(f))^n = 1_{\{P_{\theta}(f) = f\}}(\theta + dd^c f)^n. \tag{2.1}
\]

If \( \theta \) is moreover Kähler then \( \Delta_\omega(P_{\theta}(f)) \) is globally bounded on \( X \).

If \( f = \min(u, v) \) for \( u, v \) quasi-psh then \( f \) is upper semicontinuous on \( X \) and there is no need to take the upper semicontinuous regularization in the definition of \( P(u, v) := P_{\theta}(\min(u, v)) \). The latter is the largest \( \theta \)-psh function lying below both \( u \) and \( v \), and is called the rooftop envelope of \( u \) and \( v \) in [28].

The proof of Theorem 2.1 can be found in [4]. In the Kähler case, Theorem 2.1 was also surveyed in [23]. For convenience of the reader, and per recommendation of the referee, we briefly recall the arguments here.

**Proof of Theorem 2.1** We first assume that \( f \) is smooth and fix \( \varepsilon \in (0, 1] \). By nefness of \( \{ \theta \} \), the form \( \eta := \theta + \varepsilon \omega \) represents a Kähler class.

Fix \( \beta > 1 \) and let \( u_{\beta} \in \text{PSH}(X, \eta) \cap C^\infty(X) \) be the unique smooth function such that

\[
(\eta + dd^c u_{\beta})^n = e^{\beta(u_{\beta} - f)} \omega^n. \tag{2.2}
\]
The existence (and smoothness) of \( u_\beta \) follows from Aubin [1] and Yau [42].

By [4, Theorem 1.1], \( u_\beta \) converges uniformly to \( P_\theta(f) \) along with a uniform estimate for \( dd^c u_\beta \). The proof of [4, Theorem 1.2] actually establishes a Laplacian estimate for \( u_\beta \) independent of \( \varepsilon \) and \( \beta \).

We fix \( \psi \in \text{PSH}(X, \theta) \) such that \( \sup_X \psi = 0 \), \( \psi \) is smooth in \( \Omega \), the ample locus of \( \{ \theta \} \) and \( \theta + dd^c \psi \geq a_\omega \), where \( a > 0 \) is a small constant. Note that \( \psi \) and \( a \), whose existence follows from the bigness of \( \{ \theta \} \) as recalled in Section 2.1, are independent of \( \varepsilon \).

Consider

\[ H := \log \text{Tr}_\omega (\eta + dd^c u_\beta) - A(u_\beta - \psi), \]

defined on \( \Omega \), where \( A > 0 \) is a constant to be specified later. Then, \( H \) is smooth on \( \Omega \) and tends to \(-\infty\) on the boundary of \( \Omega \). Let \( x \in \Omega \) be a point where \( H \) attains its maximum in \( \Omega \). Setting \( \omega' := \eta + dd^c \psi \), it follows from [14, Lemma 2.2] (which is an improvement of [40]) that

\[ \Delta_{\omega'} \log \text{Tr}_{\omega'} (\omega') \geq \frac{\Delta_{\omega'}(\beta(u_\beta - f))}{\text{Tr}_{\omega'} (\omega')} - B \text{Tr}_{\omega'} (\omega) - An + Aa \text{Tr}_{\omega'} (\omega), \]

where \(-B\) is a negative lower bound for the holomorphic bisectional curvature of \( \omega \). In the remainder of this paragraph we carry all computations at the point \( x \). By the maximum principle, we have

\[ 0 \geq \Delta_{\omega'} H \geq \beta - \beta \frac{\text{Tr}_{\omega'} (\eta + dd^c f)}{\text{Tr}_{\omega'} (\omega')}, \]

Let \( C_1 \geq 0 \) be a constant such that \( \theta + \omega + dd^c f \leq e^{C_1} \omega \). Then, choosing \( A = B/a \), we arrive at

\[ 0 \geq (\beta - An) - \beta \frac{neC_1}{\text{Tr}_{\omega'} (\omega')}. \]

Thus, for \( \beta \geq 2An \) we have

\[ \text{Tr}_{\omega'} (\omega') \leq \frac{\beta neC_1}{\beta - An} \leq 2neC_1. \]

Let also \( \rho_0 \) be the unique \( \theta \)-psh function with minimal singularities such that

\[ (\theta + dd^c \rho_0)^n = C_3 \omega^n, \sup_X \rho_0 = 0, \]

for a uniform normalization constant \( C_3 = C(\theta, \omega) > 0 \). The existence of \( \rho_0 \) follows from [5, 12]. By [12, Theorem 4.1] we obtain a lower bound for \( \rho_0 \):

\[ \rho_0 \geq V_0 - C(\theta, \omega). \]

Since \( \rho_0 \leq f - \inf_X f \) we have that \( \rho_0 + \inf_X f + (\log C_3)/\beta \) is a subsolution to the Monge-Ampère equation defining \( u_\beta \), (2.2). By [24, Lemma 2.5] and the fact that \( V_0 \geq \psi \), we have that

\[ u_\beta \geq \rho_0 + \inf_X f + (\log C_3)/\beta \geq \psi - C_4, \]

where \( C_4 > 0 \) depends on \( \theta, \omega, \inf_X f \). From this and (2.3), we thus obtain

\[ H(x) \leq \log(2ne^{C_1}) + AC_4. \]

We finally have, for all \( \beta \geq 2nA \),

\[ \text{Tr}_\omega (\eta + dd^c u_\beta) \leq C_5 e^{-A\psi} \text{ on } \Omega. \]

Letting \( \beta \to +\infty \) and noting that \( u_\beta \) converges uniformly to \( P_{\theta+\varepsilon \omega}(f) \), we obtain

\[ \Delta_{\omega}(P_{\theta+\varepsilon \omega}(f)) \leq C_6 e^{-A\psi}, \]
where $C_6$ depends on $B, a, C_1, \inf_X f$. Letting $\varepsilon \to 0$ we arrive at
\[
\Delta_\omega(p_\theta(f)) \leq C_6 e^{-A\psi}.
\]

We finally remove the smoothness assumption on $f$. Assume that $f$ is a continuous function such that $ddc f \leq C \omega$. We approximate $f$ by smooth functions $f_j$ such that $ddc f_j \leq (C + 1) \omega$. This is possible thanks to Demailly [30]. Then, the previous steps yield
\[
\Delta_\omega(p_\theta(f_j)) \leq C' e^{-A\psi},
\]
where $C' > 0$ depends only on $C, B, a, \inf_X f, \theta, \omega$. Letting $j \to +\infty$ we arrive at the conclusion. Having the Laplacian bound, one can then argue as in [37, Theorem 9.25] to get (2.1), completing the proof of Theorem 2.1.

2.2 Non-pluripolar Monge-Ampère Products

Given $u_1, \ldots, u_p$ $\theta$-psh functions with minimal singularities, $\theta_{u_1} \wedge \cdots \wedge \theta_{u_p}$, as defined by Bedford and Taylor [2, 3] is a closed positive current in $\text{Amp}(\theta)$. For general $u_1, \ldots, u_p \in \text{PSH}(X, \theta)$, it was shown in [12] that the non-pluripolar product of $\theta_{u_1}, \ldots, \theta_{u_p}$, that we still denote by
\[
\theta_{u_1} \wedge \cdots \wedge \theta_{u_p},
\]
is well-defined as a closed positive $(p, p)$-current on $X$ which does not charge pluripolar sets. For a $\theta$-psh function $u$, the non-pluripolar complex Monge-Ampère measure of $u$ is simply $\theta^n u := \theta_{u_1} \wedge \cdots \wedge \theta_{u_p}$.

If $u$ has minimal singularities then $\int_X \theta^n u$, the total mass of $\theta^n u$, is equal to $\int_X \theta_{V_\theta}$, the volume of the class $[\theta]$ denoted by $\text{Vol}(\theta)$. For a general $u \in \text{PSH}(X, \theta)$, $\int_X \theta^n u$ may take any value in $[0, \text{Vol}(\theta)]$. Note that $\text{Vol}(\theta)$ is a cohomological quantity, i.e., it does not depend on the smooth representative we choose in $[\theta]$.

2.3 The Energy Classes

From now on, we fix $p \geq 1$. Recall that for any $\theta$-psh function $u$ we have $\int_X \theta^n u \leq \text{Vol}(\theta)$. We denote by $\mathcal{E}(X, \theta)$ the set of $\theta$-psh functions $u$ such that $\int_X \theta^n u = \text{Vol}(\theta)$. We let $\mathcal{E}^p(X, \theta)$ denote the set of $u \in \mathcal{E}(X, \theta)$ such that $\int_X |u - V_\theta|^p \theta^n u < +\infty$. For $u, v \in \mathcal{E}^p(X, \theta)$ we define
\[
I_p(u, v) := I_{p, \theta}(u, v) := \int_X |u - v|^p (\theta^n u + \theta^n v).
\]
It was proved in [34, Theorem 1.6] that $I_p$ satisfies a quasi triangle inequality:
\[
I_{p, \theta}(u, v) \leq C(n, p)(I_{p, \theta}(u, w) + I_{p, \theta}(v, w)), \ \forall u, v, w \in \mathcal{E}^p(X, \theta).
\]
In particular, applying this for $w = V_\theta$ and using Theorem 2.1, we obtain $I_{p, \theta}(u, v) < +\infty$, for all $u, v \in \mathcal{E}^p(X, \theta)$. Moreover, it follows from the domination principle [24, Proposition 2.4] that $I_p$ is non-degenerate:
\[
I_{p, \theta}(u, v) = 0 \implies u = v.
\]

2.4 Weak Geodesics

Geodesic segments connecting Kähler potentials were first introduced by Mabuchi [38], Semmes [39] and Donaldson [33] independently realized that the geodesic equation can be reformulated as a degenerate homogeneous complex Monge-Ampère equation. The best
regularity of a geodesic segment connecting two Kähler potentials is known to be $C^{1,1}$ (see [8, 15, 19]).

In the context of a big cohomology class, the regularity of geodesics is very delicate. To avoid this issue, we follow an idea of Berndtsson [7] considering geodesics as the upper envelope of subgeodesics (see [24]).

For a curve $[0, 1] \ni t \mapsto u_t \in \text{PSH}(X, \theta)$, we define

$$X \times D \ni (x, z) \mapsto U(x, z) := u_{|\log |z|}(x),$$

where $D := \{z \in \mathbb{C} \mid 1 < |z| < e\}$. We let $\pi : X \times D \to X$ be the projection on $X$.

**Definition 2.2** We say that $t \mapsto u_t$ is a subgeodesic if $(x, z) \mapsto U(x, z)$ is a $\pi^*\theta$-psh function on $X \times D$.

**Definition 2.3** For $\varphi_0, \varphi_1 \in \text{PSH}(X, \theta)$, we let $S_{[0, 1]}(\varphi_0, \varphi_1)$ denote the set of all subgeodesics $[0, 1] \ni t \mapsto u_t$ such that $\limsup_{t \to 1} u_t \leq \varphi_0$ and $\limsup_{t \to 0} u_t \leq \varphi_1$.

Let $\varphi_0, \varphi_1 \in \text{PSH}(X, \theta)$. We define, for $(x, z) \in X \times D$,

$$\Phi(x, z) := \sup\{U(x, z) \mid U \in S_{[0, 1]}(\varphi_0, \varphi_1)\}.$$

The curve $t \mapsto \varphi_t$ constructed from $\Phi$ via (2.4) is called the weak Mabuchi geodesic connecting $\varphi_0$ and $\varphi_1$.

Geodesic segments connecting two general $\theta$-psh functions may not exist. If $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$, it was shown in [24, Theorem 2.13] that $P(\varphi_0, \varphi_1) \in \mathcal{E}^p(X, \theta)$. Since $P(\varphi_0, \varphi_1) \leq \varphi_1$, we obtain that $t \mapsto \varphi_t$ is a curve in $\mathcal{E}^p(X, \theta)$. Each subgeodesic segment is in particular convex in $t$:

$$\varphi_t \leq (1 - t) \varphi_0 + t \varphi_1, \quad \forall t \in [0, 1].$$

Consequently, the upper semicontinuous regularization (with respect to both variables $x, z$) of $\Phi$ is again in $S_{[0, 1]}(\varphi_0, \varphi_1)$, hence so is $\Phi$. In particular, if $\varphi_0, \varphi_1$ have minimal singularities, then the geodesic $\varphi_t$ is Lipschitz on $[0, 1]$ (see [24, Lemma 3.1]):

$$|\varphi_t - \varphi_s| \leq |t - s| \sup_X |\varphi_0 - \varphi_1|, \quad \forall t, s \in [0, 1].$$

### 2.5 Finsler Geometry in the Kähler Case

Darvas [20] introduced a family of distances in the space of Kähler potentials

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X, \mathbb{R}) \mid \omega_\varphi > 0\}.$$

**Definition 2.4** Let $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$. For $p \geq 1$, we set

$$d_p(\varphi_0, \varphi_1) := \inf\{\ell_p(\psi) \mid \psi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\},$$

where $\ell_p(\psi) := \int_0^1 \left(\frac{1}{V} \int_X |\dot{\psi}_t|^p \omega^n_{\dot{\psi}_t}\right)^{1/p} dt$ and $V := \text{Vol}(\omega) = \int_X \omega^n$.

It was then proved in [20, Theorem 1] (generalizing Chen’s original arguments [15]) that $d_p$ defines a distance on $\mathcal{H}_\omega$, and for all $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$,

$$d_p(\varphi_0, \varphi_1) = \left(\frac{1}{V} \int_X |\dot{\psi}_t|^p \omega^n_{\dot{\psi}_t}\right)^{1/p}, \quad \forall t \in [0, 1].$$


where \( t \to \varphi_t \) is the Mabuchi geodesic (defined in Section 2.4). It was shown in [20, Lemma 4.11] that (2.6) still holds for \( \varphi_0, \varphi_1 \in \text{PSH}(X, \omega) \) with \( dd^c \varphi_i \leq C \omega, \ i = 0, 1 \), for some positive constant \( C \).

By [9, 30], potentials in \( \mathcal{E}^p(X, \omega) \) can be approximated from above by smooth Kähler potentials. As shown in [21], the metric \( d_p \) can be extended for potentials in \( \varphi_0, \varphi_1 \in \mathcal{E}^p(X, \omega) \):

\[
\text{if } \varphi_k \text{ are smooth strictly } \omega\text{-psh functions decreasing to } \varphi_i, \ i = 0, 1; \text{ then, the limit } d_p(\varphi_0, \varphi_1) := \lim_{k \to +\infty} d_p(\varphi_k, \varphi_k) \text{ exists and it is independent of the approximants. By [20, Lemmas 4.4 and 4.5], } d_p \text{ defines a metric on } \mathcal{E}^p(X, \omega) \text{ and } (\mathcal{E}^p(X, \omega), d_p) \text{ is a complete geodesic metric space.}
\]

3 The Metric Space \((\mathcal{E}^p(X, \theta), d_p)\)

The goal of this section is to define a distance \( d_p \) on \( \mathcal{E}^p(X, \theta) \) and prove that the space \((\mathcal{E}^p(X, \theta), d_p)\) is a complete geodesic metric space. We follow the strategy in [32], approximating the space of “Kähler potentials” \( \mathcal{H}_\theta \) by regular spaces. Throughout this note we will use the notation

\[
\omega_\varepsilon := \theta + \varepsilon \omega, \ \varepsilon > 0.
\]

By nefness of \( \theta \), \( \omega_\varepsilon := \theta + \varepsilon \omega \) represents a Kähler cohomology class for any \( \varepsilon > 0 \). Note that \( \omega_\varepsilon \) is not necessarily a Kähler form but there exists a smooth potential \( f_\varepsilon \in C^\infty(X, \mathbb{R}) \) such that \( \omega_\varepsilon + dd^c f_\varepsilon \) is a Kähler form. For notational convenience we normalize \( \theta \) so that \( \text{Vol}(\theta) = \int_X \theta^n = 1 \) and we set \( V_\varepsilon := \text{Vol}(\omega_\varepsilon) \).

Typically there are no smooth potentials in \( \text{PSH}(X, \theta) \) but the following class contains plenty of potentials sufficiently regular for our purposes:

\[
\mathcal{H}_\theta := \{ \varphi \in \text{PSH}(X, \theta) \mid \varphi = P_\theta (f), \ f \in C(X, \mathbb{R}), \ dd^c f \leq C(f)\omega \}.
\]

Here \( C(f) \) denotes a positive constant which depends only on \( f \). Note that any \( u = P_\theta(f) \in \mathcal{H}_\theta \) has minimal singularities because, for some constant \( C > 0 \), \( V_\theta - C \) is a candidate defining \( P_\theta(f) \). The following elementary observation will be useful in the sequel.

**Lemma 3.1** If \( u, v \in \mathcal{H}_\theta \) then \( P_\theta(u, v) \in \mathcal{H}_\theta \).

**Proof** Set \( h = \min(f, g) \in C^0(X, \mathbb{R}) \), where \( f, g \in C^0(X, \mathbb{R}) \) are such that \( u = P_\theta(f) \) and \( v = P_\theta(g) \) and \( dd^c f \leq C \omega, dd^c g \leq C \omega \). Then, \( -h = \max(-f, -g) \) is a \( C(\omega)\)-psh function on \( X \), hence \( dd^c (-h) + C \omega \geq 0 \). \( \square \)

3.1 Defining a Distance \( d_p \) on \( \mathcal{H}_\theta \)

By Darvas [20], the Mabuchi distance \( d_{p, \omega} \) is well defined on \( \mathcal{E}^p(X, \omega) \) when the reference form \( \omega \) is a Kähler form. With the following observation, we show that such a distance behaves well when we change the Kähler representative in \( \{ \omega \} \).

**Proposition 3.2** Let \( \omega_f := \omega + dd^c f \in \{ \omega \} \) be another Kähler form. Then, given \( \varphi_0, \varphi_1 \in \mathcal{E}^p(X, \omega) \), we have

\[
d_{p, \omega}(\varphi_0, \varphi_1) = d_{p, \omega_f}(\varphi_0 - f, \varphi_1 - f).
\]
Proof Let $\varphi_t$ be the Mabuchi geodesic (with respect to $\omega$) joining $\varphi_0$ and $\varphi_1$ and let $\varphi_t^f$ be the Mabuchi geodesic (with respect to $\omega_f$) joining $\varphi_0 - f$ and $\varphi_1 - f$. We claim that $\varphi_t^f = \varphi_t - f$. Indeed, $\varphi_t - f$ is an $\omega_f$-subgeodesic connecting $\varphi_0 - f$ and $\varphi_1 - f$. Hence, $\varphi_t - f \leq \varphi_t^f$. On the other hand, $\varphi_t^f + f$ is a candidate defining $\varphi_t$, thus $\varphi_t^f + f \leq \varphi_t$, proving the claim.

Assume $\varphi_0, \varphi_1$ are Kähler potentials. By (2.6) we have

$$Vd_{\omega \eta}(\varphi_0, \varphi_1) = \int_X |\varphi_0|^p (\omega + dd^c \varphi_0)^n = \int_X \left( \lim_{t \to 0^+} \frac{(\varphi_t - f) - (\varphi_0 - f)}{t} \right)^p (\omega_f + dd^c (\varphi_0 - f))^n = \int_X |\varphi_0|^p (\omega_f + dd^c (\varphi_0 - f))^n = Vd_{\omega \eta}(\varphi_0 - f, \varphi_1 - f).$$

The identity for potentials in $\mathcal{E}^p(X, \omega)$ follows from the fact that the distance $d_{\omega \eta}$ between potentials $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \omega)$ is defined as the limit $\lim_j d_{\omega \eta}(\varphi_0, j, \varphi_1, j)$, where $\{\varphi_i, j\}$ is a sequence of smooth strictly $\omega$-psh functions decreasing to $\varphi_i$ for $i = 0, 1$.

Thanks to the above proposition, we can then define the Mabuchi distance with respect to any smooth $(1, 1)$-form $\eta$ in the Kähler class $[\omega]$:

$$d_{\omega \eta}(\varphi_0, \varphi_1) := d_{\omega \eta_f}(\varphi_0 - f, \varphi_1 - f), \quad \varphi_0, \varphi_1 \in \mathcal{E}^p(X, \eta), \quad (3.1)$$

where $\eta_f = \eta + dd^c f$ is a Kähler form. Proposition 3.2 reveals that the definition is independent of the choice of $f$.

We next extend the Pythagorean formula of [20, 21] for Kähler classes.

Lemma 3.3 If $[\eta]$ is Kähler and $u, v \in \mathcal{E}^p(X, \eta)$ then

$$d_{\omega \eta}(u, v) = d_{\omega \eta}(u, P_{\omega \eta}(u, v)) + d_{\omega \eta}(v, P_{\omega \eta}(u, v)).$$

Proof By [20, Corollary 4.14] and (3.1), we have

$$d_{\omega \eta}(u, v) = d_{\omega \eta_f}(u - f, P_{\omega \eta_f}(u - f, v - f)) + d_{\omega \eta_f}(v - f, P_{\omega \eta_f}(u - f, v - f)).$$

The conclusion follows observing that $P_{\omega \eta}(u - f, v - f) = P_{\eta}(u, v) - f$. □

The following results play a crucial role in the sequel.

Lemma 3.4 Let $\varphi = P_{\theta}(f), \psi = P_{\theta}(g) \in \mathcal{H}_\theta$. Set $\varphi_{\varepsilon} := P_{\omega_{\varepsilon}}(f)$ and $\psi_{\varepsilon} := P_{\omega_{\varepsilon}}(g)$. Then,

$$\lim_{\varepsilon \to 0} I_{\omega_{\varepsilon}}(\varphi_{\varepsilon}, \psi_{\varepsilon}) = I_{\omega}(\varphi, \psi).$$

Proof Observe that $|\varphi_{\varepsilon} - \psi_{\varepsilon}| \to |\varphi - \psi|$ almost everywhere on $X$ (in fact this holds off a pluripolar set) and they are uniformly bounded:

$$|\varphi_{\varepsilon} - \psi_{\varepsilon}| \leq \sup_X |f - g|.$$

Indeed, take a point $x \in X$ such that $\varphi(x) > -\infty$ and $\psi(x) > -\infty$. Recall that $\omega_{\varepsilon} := \theta + \varepsilon \omega \geq \theta$ and $\{\omega_{\varepsilon}\}$ is increasing in $\varepsilon$. Therefore, $\varphi_{\varepsilon}$ decreases to a $\theta$-psh function on $X$ as $\varepsilon \to 0$. The latter must be $\varphi$. We thus have that $\varphi_{\varepsilon}(x) \to \varphi(x)$ and $\psi_{\varepsilon}(x) \to \psi(x)$ as $\varepsilon \to 0$. Also, $\psi_{\varepsilon} + \sup_X |f - g|$ is a candidate defining $\varphi_{\varepsilon}$; hence, the claimed bound follows.
By Lemma 3.5 below and Lebesgue’s dominated convergence theorem,
\[
\lim_{\varepsilon \to 0} \int_X |\varphi_\varepsilon - \psi_\varepsilon|^p (\omega_\varepsilon + dd^c\varphi_\varepsilon)^n = \int_X |\varphi - \psi|^p (\theta + dd^c\varphi)^n.
\]
Similarly, the other term in the definition of $I_{p,\omega_\varepsilon}$ also converges to the desired limit. \hfill \Box

**Lemma 3.5** Let $\varphi = P_0(f) \in \mathcal{H}_\theta$. For $\varepsilon > 0$ we set $\varphi_\varepsilon = P_{\omega_\varepsilon}(f)$ and write
\[
(\omega_\varepsilon + dd^c\varphi_\varepsilon)^n = \rho_\varepsilon \omega^n; \quad (\theta + dd^c\varphi)^n = \rho \omega^n.
\]
Then, $\varepsilon \mapsto \rho_\varepsilon$ is increasing, uniformly bounded and $\rho_\varepsilon \to \rho$ pointwise on $X$.

**Proof** Define, for $\varepsilon > 0$, $D_\varepsilon := \{x \in X \mid \varphi_\varepsilon(x) = f(x)\}$. Since $\{\varphi_\varepsilon\}$ is increasing and $\varphi_\varepsilon \leq f$, $\{D_\varepsilon\}$ is also increasing. We set $D := \cap_{\varepsilon > 0} D_\varepsilon$. Then, $D = \{x \in X \mid \varphi(x) = f(x)\}$.

For $\varepsilon' > \varepsilon > 0$, it follows from Theorem 2.1 that
\[
(\omega_\varepsilon + dd^c\varphi_\varepsilon)^n = \mathbb{I}_{\{\varphi_\varepsilon = f\}}(\omega_\varepsilon + dd^c f)^n \leq \mathbb{I}_{\{\varphi_\varepsilon = f\}}(\omega_{\varepsilon'} + dd^c f)^n \leq (\omega_{\varepsilon'} + dd^c\varphi_{\varepsilon'})^n.
\]
Here we use the fact that $0 \leq \omega_\varepsilon + dd^c f \leq \omega_{\varepsilon'} + dd^c f$ on $D_\varepsilon$. This proves the first statement. The second statement follows from the bound $dd^c f \leq C\omega$. We now prove the last statement. If $x \in D$, using $(\theta + dd^c f) \leq C'\omega$, we can write
\[
\rho_\varepsilon(x)\omega^n = (\theta + \varepsilon\omega + dd^c f)^n \leq (\theta + dd^c f)^n + O(\varepsilon)\omega^n = (\rho(x) + O(\varepsilon))\omega^n.
\]
Hence, $\rho_\varepsilon(x) \to \rho(x)$. If $x \notin D$ then $x \notin D_\varepsilon$ for $\varepsilon > 0$ small enough, hence $\rho_\varepsilon(x) = 0 = \rho(x)$. \hfill \Box

**Lemma 3.6** Let $\varphi_j = P_0(f_j) \in \mathcal{H}_\theta$, for $j = 0, 1$. Let $\varphi_i$ (resp. $\varphi_{i,\varepsilon}$) be weak Mabuchi geodesics joining $\varphi_0$ and $\varphi_1$ (resp. $\varphi_{0,\varepsilon} = P_{\omega_\varepsilon}(f_0)$ and $\varphi_{1,\varepsilon} = P_{\omega_\varepsilon}(f_1)$). Then, we have the following pointwise convergence
\[
\mathbb{I}_{\{\varphi_{0,\varepsilon} = f_0\}}|\dot{\varphi}_{0,\varepsilon}|^p \to \mathbb{I}_{\{\varphi_0 = f_0\}}|\dot{\varphi}_0|^p.
\]

**Proof** Since $P_{\omega_\varepsilon}(f_j) \geq P_0(f_j)$, $j = 0, 1$, it follows from the definition that $\varphi_{i,\varepsilon} \geq \varphi_i$ (the curve $\varphi_i$ is a candidate defining $\varphi_{i,\varepsilon}$ for any $\varepsilon > 0$). Set $D_\varepsilon = \{\varphi_{0,\varepsilon} = f_0\}$ and $D = \{\varphi_0 = f_0\}$. Then, $D_\varepsilon$ is increasing and $\cap_{\varepsilon > 0} D_\varepsilon = D$ since $\varphi_0 \leq \varphi_{0,\varepsilon} \leq f_0$. If $x \in D$ then, for all small $s > 0$,
\[
\dot{\varphi}_0(x) = \lim_{t \to 0} \frac{\varphi_t(x) - f_0(x)}{t} \leq \dot{\varphi}_{0,\varepsilon}(x) \leq \frac{\varphi_{s,\varepsilon}(x) - \varphi_{0,\varepsilon}(x)}{s},
\]
where in the last inequality we use the convexity of the geodesic in $t$. Letting first $\varepsilon \to 0$ and then $s \to 0$ shows that $\dot{\varphi}_{0,\varepsilon}(x)$ converges to $\dot{\varphi}_0(x)$. If $x \notin D$ then $x \notin D_\varepsilon$, for $\varepsilon > 0$ small enough. In this case the convergence we want to prove is trivial. \hfill \Box

**Theorem 3.7** Let $\varphi_0 := P_0(f_0), \varphi_1 := P_0(f_1) \in \mathcal{H}_\theta$ and let $\varphi_{i,\varepsilon} = P_{\omega_\varepsilon}(f_i), i = 0, 1$. Let $d_{p,\varepsilon}$ be the Mabuchi distance with respect to $\omega_\varepsilon$ defined in (3.1). Then,
\[
\lim_{\varepsilon \to 0} d_{p,\varepsilon}^2(\varphi_{0,\varepsilon}, \varphi_{1,\varepsilon}) = \int_X |\dot{\varphi}_0|^p (\theta + dd^c\varphi_0)^n = \int_X |\dot{\varphi}_1|^p (\theta + dd^c\varphi_1)^n,
\]
where $\varphi_i$ is the weak Mabuchi geodesic connecting $\varphi_0$ and $\varphi_1$. 
Compared with [32] our approach is slightly different. We also emphasize that by [31, Example 4.5], there are functions in $\mathcal{E}^p(X, \theta)$ which are not in $\mathcal{E}^p(X, \omega)$.

**Proof** Let $\varphi_{t, \varepsilon}$ denote the $\omega_{\varepsilon}$-geodesic joining $\varphi_{0, \varepsilon}$ and $\varphi_{1, \varepsilon}$. Set $D_{\varepsilon} = \{ \varphi_{0, \varepsilon} = f_0 \}$ and $D = \{ \varphi_0 = f_0 \}$. Combining (2.6) and Theorem 2.1, we obtain

$$V_{\varepsilon}d_{p, \varepsilon}^p(\varphi_{0, \varepsilon}, \varphi_{1, \varepsilon}) = \int_X |\dot{\varphi}_{0, \varepsilon}|^p(\omega_{\varepsilon} + dd^c \varphi_{0, \varepsilon})^n = \int_{D_{\varepsilon}} |\dot{\varphi}_{0, \varepsilon}|^p(\omega_{\varepsilon} + dd^c f_0)^n.$$ 

Since $|\varphi_{0, \varepsilon} - \varphi_{1, \varepsilon}| \leq \sup_X |f_0 - f_1|$ and $f_0 - f_1$ is bounded, (2.5) ensures that $\dot{\varphi}_{0, \varepsilon}$ is uniformly bounded. It follows from Lemma 3.5 and Lemma 3.6 that the functions $1_{D_{\varepsilon}}|\dot{\varphi}_{0, \varepsilon}|^p\rho_{\varepsilon}$ and $1_D|\dot{\varphi}_{0, \varepsilon}|^p\rho$ are uniformly bounded and $1_{D_{\varepsilon}}|\dot{\varphi}_{0, \varepsilon}|^p\rho_{\varepsilon}$ converges pointwise to $1_D|\dot{\varphi}_{0}|^p\rho$. We also observe that $V_{\varepsilon}$ decreases to $\text{Vol}(\theta) = 1$. Lebesgue’s dominated convergence theorem then yields

$$\lim_{\varepsilon \to 0} d_{p, \varepsilon}^p(\varphi_{0, \varepsilon}, \varphi_{1, \varepsilon}) = \int_D |\dot{\varphi}_0|^p(\theta + dd^c f_0)^n = \int_X |\dot{\varphi}_0|^p(\theta + dd^c \varphi_0)^n,$$

where in the last equality we use Theorem 2.1. This shows the first equality in the statement. The second one is obtained by reversing the role of $\varphi_0$ and $\varphi_1$.

**Definition 3.8** Assume that $\varphi_0 := P_\theta(f_0)$, $\varphi_1 := P_\theta(f_1) \in \mathcal{H}_\theta$. Let $d_{p, \varepsilon}$ be the Mabuchi distance with respect to $\omega_{\varepsilon} := \theta + \varepsilon \omega$ defined in (3.1). We define

$$d_p(\varphi_0, \varphi_1) := \lim_{\varepsilon \to 0} d_{p, \varepsilon}(\varphi_{0, \varepsilon}, \varphi_{1, \varepsilon}),$$

where $\varphi_{0, \varepsilon} := P_{\omega_{\varepsilon}}(f_0)$ and $\varphi_{1, \varepsilon} := P_{\omega_{\varepsilon}}(f_1)$.

The limit exists and is independent of the choice of $\omega$ as shown in Theorem 3.7.

**Lemma 3.9** $d_p$ is a distance on $\mathcal{H}_\theta$.

**Proof** The triangle inequality immediately follows from the fact that $d_{p, \varepsilon}$ is a distance. From [20, Theorem 5.5] we know that

$$d_{p, \varepsilon}^p(\varphi_{0, \varepsilon}, \varphi_{1, \varepsilon}) \geq \frac{1}{C} I_{p, \omega_{\varepsilon}}(\varphi_{0, \varepsilon}, \varphi_{1, \varepsilon}), \quad C > 0.$$ 

Also, by Lemma 3.4 we have

$$\lim_{\varepsilon \to 0} I_{p, \omega_{\varepsilon}}(\varphi_{0, \varepsilon}, \varphi_{1, \varepsilon}) = I_{p, \theta}(\varphi_0, \varphi_1).$$

It follows from the domination principle (see [10, 24]) that

$$I_{p, \theta}(\varphi_0, \varphi_1) = 0 \iff \varphi_0 = \varphi_1.$$ 

Hence, $d_p$ is non-degenerate.

**3.2 Extension of $d_p$ to $\mathcal{E}^p(X, \theta)$**

The following comparison between $I_p$ and $d_p$ was established in [20, Theorem 3] in the Kähler case.

**Proposition 3.10** Given $\varphi_0, \varphi_1 \in \mathcal{H}_\theta$ there exists a constant $C > 0$ (depending only on $n$) such that

$$\frac{1}{C} I_p(\varphi_0, \varphi_1) \leq d_p^p(\varphi_0, \varphi_1) \leq CI_p(\varphi_0, \varphi_1).$$
Proof By Darvas [20, Theorem 3] we know that
\[ \frac{1}{C} I_{p, \omega_\epsilon}(\varphi_{0, \epsilon}, \varphi_{1, \epsilon}) \leq d_{p, \epsilon}^{I_{p}}(\varphi_{0, \epsilon}, \varphi_{1, \epsilon}) \leq CI_{p, \omega_\epsilon}(\varphi_{0, \epsilon}, \varphi_{1, \epsilon}). \]

Letting \( \epsilon \) to zero and using Lemma 3.4 and Definition 3.8, we get (3.2).

Now, let \( \varphi_0, \varphi_1 \in E^p(X, \theta) \). Let \( \{ f_{i,j} \} \) be a sequence of smooth functions decreasing to \( \varphi_i, i = 0, 1 \). We then clearly have that \( \varphi_{i,j} := P_{\theta}(f_{i,j}) \in H^\theta \) and \( P_{\theta}(f_{i,j}) \downarrow \varphi_i \).

Lemma 3.11 The sequence \( d_p(\varphi_{0,j}, \varphi_{1,j}) \) converges and the limit is independent of the choice of the approximants \( f_{i,j} \).

Proof Set \( a_j := d_p(\varphi_{0,j}, \varphi_{1,j}) \). By the triangle inequality and Proposition 3.10, we have
\[ a_j \leq d_p(\varphi_{0,j}, \varphi_{0,k}) + d_p(\varphi_{0,k}, \varphi_{1,k}) + d_p(\varphi_{1,k}, \varphi_{1,j}) \]
\[ \leq a_k + C \left( I_{p}^{1/p}(\varphi_{0,j}, \varphi_{0,k}) + I_{p}^{1/p}(\varphi_{1,j}, \varphi_{1,k}) \right), \]
where \( C > 0 \) depends only on \( n, p \). Hence,
\[ |a_j - a_k| \leq C \left( I_{p}^{1/p}(\varphi_{0,j}, \varphi_{0,k}) + I_{p}^{1/p}(\varphi_{1,j}, \varphi_{1,k}) \right). \]

By [34, Theorem 1.6 and Proposition 1.9], it then follows that \( |a_j - a_k| \to 0 \) as \( j, k \to +\infty \). This proves that the sequence \( d_p(\varphi_{0,j}, \varphi_{1,j}) \) is Cauchy; hence, it converges.

Let \( \tilde{\varphi}_{i,j} = P_{\theta}(\tilde{f}_{i,j}) \) be another sequence in \( H^\theta \) decreasing to \( \tilde{\varphi}_i, i = 0, 1 \). Then, applying the triangle inequality several times, we get
\[ d_p(\varphi_{0,j}, \varphi_{1,j}) \leq d_p(\varphi_{0,j}, \tilde{\varphi}_0,j) + d_p(\tilde{\varphi}_0,j, \tilde{\varphi}_1,j) + d_p(\tilde{\varphi}_1,j, \varphi_{1,j}), \]
and thus
\[ |d_p(\varphi_{0,j}, \varphi_{1,j}) - d_p(\tilde{\varphi}_{0,j}, \tilde{\varphi}_{1,j})| \leq C \left( I_{p}^{1/p}(\varphi_{0,j}, \tilde{\varphi}_0,j) + I_{p}^{1/p}(\varphi_{1,j}, \tilde{\varphi}_1,j) \right). \]

It then follows again from [34, Theorem 1.6 and Proposition 1.9] that the limit does not depend on the choice of the approximants.

Given \( \varphi_0, \varphi_1 \in E^p(X, \theta) \), we then define
\[ d_p(\varphi_0, \varphi_1) := \lim_{j \to +\infty} d_p(P_{\theta}(f_{0,j}), P_{\theta}(f_{1,j})). \]

Proposition 3.12 \( d_p \) is a distance on \( E^p(X, \theta) \) and the inequalities comparing \( d_p \) and \( I_p \) on \( H^\theta \) (3.2) hold on \( E^p(X, \theta) \). Moreover, if \( u_j \in E^p(X, \theta) \) decreases to \( u \in E^p(X, \theta) \) then \( d_p(u_j, u) \to 0 \).

Proof By the definition of \( d_p \) on \( E^p(X, \theta) \) we infer that the comparison between \( d_p \) and \( I_p \) in Proposition 3.10 holds on \( E^p(X, \theta) \). From this and the domination principle [24], we deduce that \( d_p \) is non-degenerate. The last statement follows from (3.2) and [34, Proposition 1.9].

The next result was proved in [6, Lemma 3.4] for the Kähler case.

Lemma 3.13 Let \( u_t \) be the Mabuchi geodesic joining \( u_0 \in H^\theta \) and let \( u_1 \in E^p(X, \theta) \). Then,
\[ d_p^p(u_0, u_1) = \int_X |\dot{u}_0|^p (\theta + dd^c u_0)^n. \]
Proof  We first assume that \( u_0 \geq u_1 + 1 \). We approximate \( u_1 \) from above by \( u_1^j \in \mathcal{H}_\theta \) such that \( u_1^j \leq u_0 \), for all \( j \). Let \( u_1^j \) be the Mabuchi geodesic joining \( u_0 \) to \( u_1^j \). Note that \( u_1^j \geq u_t \) and that \( \dot{u}_t^j \leq 0 \). By Theorem 3.7,

\[
d_p^p(u_0, u_1^j) = \int_X (\dot{u}_0^j)^p \theta_{u_0}^n.
\]

Also, \( \dot{u}_0^j \) decreases to \( \dot{u}_0 \); hence, the monotone convergence theorem and Proposition 3.12 give

\[
d_p^p(u_0, u_1) = \int_X (\dot{u}_0)^p \theta_{u_0}^n < +\infty.
\]

In particular \( |\dot{u}_0|^p \in L^1(X, \theta_{u_0}^n) \).

For the general case we can find a constant \( C > 0 \) such that \( u_1 \leq u_0 + C \) since \( u_1 \) has minimal singularities. Let \( w_t \) be the Mabuchi geodesic joining \( u_0 \) and \( u_1 - C - 1 \). Note that \( w_t \leq u_1^j \) since \( w_1 = u_1 - C - 1 < u_1 \leq u_1^j \) and \( w_0 = u_0 = u_0^j \) and \( \dot{w}_t \leq 0 \). It then follows that

\[
\dot{w}_0 \leq \dot{u}_0^j \leq u_1^j - u_0 \leq (u_1^j - V_0) + (V_0 - u_0) \leq \sup_x u_1^j + \sup_x (V_0 - u_0) \leq C_1,
\]

for a uniform constant \( C_1 > 0 \). In the second inequality above, we use the fact that the Mabuchi geodesic \( u_1^j \) connecting \( u_0 \) to \( u_1^j \) is convex in \( t \), while in the last inequality we use the fact that \( u_0 \) has minimal singularities.

The previous inequalities then yield \( |\dot{u}_0^j|^p \leq C_2 + 2^{p-1} |\dot{w}_0|^p \), where \( C_2 \) is a uniform constant. On the other hand by Theorem 3.7, we have

\[
d_p^p(u_0, u_1^j) = \int_X |\dot{u}_0^j|^p \theta_{u_0}^n.
\]

We claim that \( |\dot{u}_0^j|^p \) converges a.e. to \( |\dot{u}_0|^p \). Indeed, the convergence is pointwise at points \( x \) such that \( u_1(x) > -\infty \), but the set \( \{u_1 = -\infty\} \) has Lebesgue measure zero. Also, the above estimate ensures that \( |\dot{u}_0^j|^p \) are uniformly bounded by \( 2^{p-1} |\dot{w}_0|^p + C_2 \) which is integrable with respect to the measure \( \theta_{u_0}^n \) since \( \int_X |\dot{w}_0|^p \theta_{u_0}^n = d_p^p(u_0, u_1 - C - 1) < +\infty \). Proposition 3.12 and Lebesgue’s dominated convergence theorem then give the result. \( \square \)

**Proposition 3.14** If \( u, v \in \mathcal{E}^p(X, \theta) \) then

(i) \( d_p^p(u, v) = d_p^p(u, P_\theta(u, v)) + d_p^p(v, P_\theta(u, v)) \) and

(ii) \( d_p(u, \max(u, v)) \geq d_p(v, P_\theta(u, v)) \).

We recall that from [24, Theorem 2.13] \( P_\theta(u, v) \in \mathcal{E}^p(X, \theta) \). The identity in the first statement is known as the Pythagorean formula and it was established in the Kähler case by Darvas [20]. The second statement was proved for \( p = 1 \) in [26] using the differentiability of the Monge-Ampère energy. As will be shown in Proposition 3.18, our definition of \( d_1 \) and the one in [26] do coincide.

Proof To prove the Pythagorean formula, we first assume that \( u = P_\theta(f), v = P_\theta(g) \in \mathcal{H}_\theta \). Set \( u_\varepsilon := P_{\omega_\varepsilon}(f), v_\varepsilon := P_{\omega_\varepsilon}(g) \). It follows from Lemma 3.3 that

\[
d_{p,\varepsilon}(u_\varepsilon, v_\varepsilon) = d_{p,\varepsilon}(u_\varepsilon, P_{\omega_\varepsilon}(u_\varepsilon, v_\varepsilon)) + d_{p,\varepsilon}(v_\varepsilon, P_{\omega_\varepsilon}(u_\varepsilon, v_\varepsilon))
= d_{p,\varepsilon}(u_\varepsilon, P_{\omega_\varepsilon}(\min(f, g))) + d_{p,\varepsilon}(v_\varepsilon, P_{\omega_\varepsilon}(\min(f, g))).
\]
where in the last identity we use that $P_{\omega_\varepsilon}(u_\varepsilon, v_\varepsilon) = P_{\omega_\varepsilon}(\min(f, g))$. It follows from Lemma 3.1 that $dd^c \min(f, g) \leq C \omega$. Applying Theorem 3.7 we obtain (i) for this case. To treat the general case, let $u_j = P_\theta(f_j)$, $v_j = P_\theta(g_j)$ be sequences in $\mathcal{H}_\theta$ decreasing to $u, v$. By Lemma 3.1, $P_\theta(u_j, v_j) = P_\theta(\min(f_j, g_j)) \in \mathcal{H}_\theta$ and it decreases to $P_\theta(u, v)$. Then, (i) follows from the first step and Proposition 3.12 since

$$|d_p(u_j, v_j) - d_p(u, v)| \leq d_p(u_j, u) + d_p(v, v_j).$$

To prove the second statement, in view of Proposition 3.12, we can assume that $u = P_\theta(f)$, $v = P_\theta(g) \in \mathcal{H}_\theta$. By Lemma 3.1 we have

$$d_p(u, \max(u, v)) = \int_X |\hat{u}|^n \theta^n,$$

where $\hat{u}$ is the Mabuchi geodesic joining $u_0 = u$ to $u_1 = \max(u, v)$. Let $\phi_t$ be the Mabuchi geodesic joining $\phi_0 = P_\theta(u, v)$ to $\phi_1 = v$. We note that $0 \leq \phi_0 \leq v - P(u, v)$. Indeed, $\phi_0 \geq 0$ since $\phi_0 \leq \phi_1$ while the second inequality follows from the convexity in $t$ of the geodesic. Using this observation and the fact that $\phi_t \leq u_t$, we obtain

$$1_{\{P(u, v) = u\}} \phi_0 \leq 1_{\{P(u, v) = u\}} u_0,$$

and

$$1_{\{P(u, v) = v\}} \phi_0 = 0.$$

Since $P_\theta(u, v) = P_\theta(\min(f, g))$ with $dd^c \min(f, g) \leq C \omega$, Theorem 2.1, Theorem 3.7, and [34, Lemma 4.1] then yield

$$d_p(P_\theta(u, v), v) = \int_X |\hat{\phi}_0|^n \theta^n \leq \int_{\{P(u, v) = u\}} |\hat{\phi}_0|^n (\theta + dd^c u)^n \leq \int_{\{P(u, v) = u\}} \hat{u}_0^n (\theta + dd^c u)^n \leq d_p(u, \max(u, v)).$$

**Remark 3.15** By Proposition 3.14 we have a “Pythagorean inequality” for max:

$$d_p(u, \max(u, v)) + d_p(v, \max(u, v)) \geq d_p(u, v), \forall u, v \in \mathcal{E}^p(X, \theta).$$

### 3.3 Completeness of $(\mathcal{E}^p(X, \theta), d_p)$

In the sequel we fix a smooth volume form $dV$ on $X$ such that $\int_X dV = 1$.

**Lemma 3.16** Let $u \in \mathcal{E}^p(X, \theta)$ and let $\phi$ be a $\theta$-psh function with minimal singularities, $\sup_X \phi = 0$ satisfying $\theta^n_\phi = dV$. Then, there exist uniform constants $C_1 = C_1(n, \theta)$ and $C_2 = C_2(n) > 0$ such that

$$|\sup_X u| \leq C_1 + C_2 d_p(u, \phi).$$

**Proof** Using the Hölder inequality and [35, Proposition 2.7], we obtain

$$|\sup_X u| \leq \int_X |u - \sup_X u| dV + \int_X |u| dV \leq A + \left(\int_X |u|^p dV\right)^{1/p} \leq A + \left(\|u - \phi\|_{L^p(dV)} + \|\phi\|_{L^p(dV)}\right).$$

By Proposition 3.12,

$$\int_X |u - \phi|^p dV = \int_X |u - \phi|^p \theta^n_\phi \leq I_p(u, \phi) \leq C(n) d_p(u, \phi).$$
Combining the above inequalities we get the conclusion. \( \square \)

**Theorem 3.17** The space \( (\mathcal{E}^p(X, \theta) , d_p) \) is a complete geodesic metric space which is the completion of \( (\mathcal{H}_\theta , d_p) \).

**Proof** Let \( (\varphi_j) \in \mathcal{E}^p(X, \theta)^\mathbb{N} \) be a Cauchy sequence for \( d_p \). Extracting and relabelling we can assume that there exists a subsequence \( (u_j) \subseteq (\varphi_j) \) such that

\[
d_p(u_j, u_{j+1}) \leq 2^{-j}.
\]

Define \( v_{j,k} := P_\theta(u_j, \ldots, u_{j+k}) \) and observe that it is decreasing in \( k \). Also, by Proposition 3.14 (i) and the triangle inequality,

\[
d_p(u_j, v_{j,k}) = d_p(u_j, P_\theta(u_j, v_{j+1,k})) \leq d_p(u_j, v_{j+1,k}) \leq 2^{-j} + d_p(u_{j+1}, v_{j+1,k}).
\]

Hence,

\[
d_p(u_j, v_{j,k}) \leq \sum_{\ell=j}^{k-1} 2^{-\ell} \leq 2^{-j+1}.
\]

In particular \( I_p(u_j, v_{j,k}) \) is uniformly bounded from above. We then infer that \( v_{j,k} \) decreases to \( v_j \in \text{PSH}(X, \theta) \) as \( k \to +\infty \) and a combination of Proposition 3.12 and [34, Proposition 1.9] gives

\[
d_p(u_j, v_j) \leq 2^{1-j}, \forall j. \tag{3.3}
\]

Let \( \varphi \) be the unique \( \theta \)-psh function with minimal singularities such that \( \sup_X \varphi = 0 \) and \( \theta^n = dV \). By Lemma 3.16,

\[
|\sup_X v_j| \leq C_1 + C_2 d_p(v_j, \varphi) \leq C_1 + C_2 \left( d_p(v_j, u_1) + d_p(u_1, \varphi) \right)
\]

\[
\leq C_1 + C_2 \left( d_p(v_j, u_j) + d_p(u_j, u_1) + d_p(u_1, \varphi) \right)
\]

\[
\leq C_1 + C_2 \left( 4 + d_p(u_1, \varphi) \right).
\]

It thus follows that \( v_j \) increases a.e. to a \( \theta \)-psh function \( v \). By the triangle inequality we have

\[
d_p(\varphi_j, v) \leq d_p(\varphi_j, u_j) + d_p(u_j, v_j) + d_p(v_j, v).
\]

Since \( (\varphi_j) \) is Cauchy, \( d_p(\varphi_j, u_j) \to 0 \). By [34, Proposition 1.9] and Proposition 3.12, we have \( d_p(v_j, v) \to 0 \). These facts together with (3.3) yield \( d_p(\varphi_j, v) \to 0 \); hence, \( (\mathcal{E}^p(X, \theta) , d_p) \) is a complete metric space.

Also, any \( u \in \mathcal{E}^p(X, \theta) \) can be approximated from above by functions \( u_j \in \mathcal{H}_\theta \) such that \( d_p(u_j, u) \to 0 \) (Proposition 3.12). It thus follows that \( (\mathcal{E}^p(X, \theta) , d_p) \) is the metric completion of \( \mathcal{H}_\theta \).

Let now \( u_t \) be the Mabuchi geodesic joining \( u_0, u_1 \in \mathcal{E}^p(X, \theta) \). We are going to prove that, for all \( t \in [0, 1] \),

\[
d_p(u_t, u_s) = |t-s| d_p(u_0, u_1).
\]

We claim that for all \( t \in [0, 1] \),

\[
d_p(u_0, u_t) = t d_p(u_0, u_1) \quad \text{and} \quad d_p(u_1, u_t) = (1-t) d_p(u_0, u_1). \tag{3.4}
\]

We first assume that \( u_0, u_1 \in \mathcal{H}_\theta \). The Mabuchi geodesic joining \( u_0 \) to \( u_t \) is given by \( w_\ell = u_{t\ell}, \ell \in [0, 1] \). Lemma 3.13 thus gives

\[
d_p^p(u_0, u_t) = \int_X |\dot{w}_0|^p \theta^n_{u_0} = t^p \int_X |\dot{u}_0|^p \theta^n_{u_0} = t^p d_p^p(u_0, u_1),
\]

proving the first equality in (3.4). The second one is proved similarly.
We next prove the claim for \( u_0, u_1 \in \mathcal{E}(X, \theta) \). Let \( (u^j_i), i = 0, 1, j \in \mathbb{N} \), be decreasing sequences of functions in \( \mathcal{H}_\theta \) such that \( u^j_i \downarrow u_i \), \( i = 0, 1 \). Let \( u^j_i \) be the Mabuchi geodesic joining \( u^j_0 \) and \( u^j_1 \). Then, \( u^j_i \) decreases to \( u_t \). By the triangle inequality we have \(|d_p(u^j_0, u^j_t)| - d_p(u_0, u_t)| \leq d_p(u^j_0, u_0) + d_p(u_t, u^j_t)\). The claim thus follows from Proposition 3.12 and the previous step.

Now, if \( 0 < t < s < 1 \) then applying twice (3.4), we get
\[
d_p(u_t, u_s) = \frac{s - t}{s} d_p(u_0, u_s) = (s - t)d_p(u_0, u_1). \]

We end this section by proving that the distance \( d_1 \) defined by approximation (see Definition 3.8) coincides with the one defined in [26] using the Monge-Ampère energy.

**Proposition 3.18** Assume \( u_0, u_1 \in \mathcal{E}(X, \theta) \). Then,
\[
d_1(u_0, u_1) = E(u_0) + E(u_1) - 2E(P(u_0, u_1)).
\]

Here the Monge-Ampère energy \( E \) is defined as
\[
E(u) := \frac{1}{n + 1} \sum_{j=0}^{n} \int_X (u - V_\theta)_{\theta}^j \wedge \theta_{V_\theta}^{n-j}.
\]

**Proof** We first assume that \( u_0, u_1 \in \mathcal{H}_\theta \) and \( u_0 \leq u_1 \). Let \([0, 1] \ni t \mapsto u_t \) be the Mabuchi geodesic joining \( u_0 \) and \( u_1 \). By [24, Theorem 3.12], \( t \mapsto E(u_t) \) is affine, hence for all \( t \in [0, 1] \),
\[
\frac{E(u_t) - E(u_0)}{t} = E(u_1) - E(u_0) = \frac{E(u_1) - E(u_t)}{1-t}.
\]

Since \( E \) is concave along affine curves (see [5, 12], [26, Theorem 2.1]), we thus have
\[
\int_X \frac{u_t - u_0}{t} \theta^n_{u_0} \geq E(u_1) - E(u_0) \geq \int_X \frac{u_1 - u_t}{1-t} \theta^n_{u_1}.
\]

Letting \( t \to 0 \) in the first inequality and \( t \to 1 \) in the second one, we obtain
\[
\int_X \dot{u}_0 \theta^n_{u_0} \geq E(u_1) - E(u_0) \geq \int_X \dot{u}_1 \theta^n_{u_1}.
\]

By Theorem 3.7 we then have
\[
d_1(u_0, u_1) = \int_X \dot{u}_0 \theta^n_{u_0} = \int_X \dot{u}_1 \theta^n_{u_1} = E(u_1) - E(u_0).
\]

We next assume that \( u_0, u_1 \in \mathcal{H}_\theta \) but we remove the assumption that \( u_0 \leq u_1 \). By Lemma 3.1, \( P(u_0, u_1) \in \mathcal{H}_\theta \). By the Pythagorean formula (Proposition 3.14) and the first step, we have
\[
d_1(u_0, u_1) = d_1(u_0, P(u_0, u_1)) + d_1(u_1, P(u_0, u_1)) = E(u_0) - E(P(u_0, u_1)) + E(u_1) - E(P(u_0, u_1)).
\]

We now treat the general case. Let \( (u^j_i), i = 0, 1, j \in \mathbb{N} \) be decreasing sequences of functions in \( \mathcal{H}_\theta \) such that \( u^j_i \downarrow u_i \), \( i = 0, 1 \). Then, \( P(u^j_0, u^j_i) \downarrow P(u_0, u_1) \). By [26, Proposition 2.4], \( E(u^j_i) \to E(u_i) \), for \( i = 0, 1 \) and \( E(P(u^j_i, u^j_i)) \to E(P(u_0, u_1)) \) as \( j \to +\infty \). The result thus follows from Proposition 3.12, the triangle inequality, and the previous step. \( \square \)
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