Plasmastatic model of toroidal trap “Galatea-belt”

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Abstract. Magnetic galatea-traps for thermonuclear plasma confinement with current carrying conductors immersed into the plasma volume, are represented by an example of the toroidal trap “The Belt” with two circular conductors. Numerical models of equilibrium plasma and field configurations are investigated in straightened into cylinder analogues of some toroidal galateas in a series of works by the authors. This paper presents a plasmastatic model of configurations in the toroidal variant of “The Belt” in terms of a boundary problem with the Grad-Shafranov equation. Distinctions of their geometry and quantitative characteristics from the cylindrical analogues and their dependence of parameters are determined in computation.

1. Introduction
The work deals with mathematical simulations and computation of physical processes in the magnetic trap for plasma confinement research. The traps are a basic element of plasma systems in many programs aimed at controlled nuclear fusion realization, because the magnetic field is the only accessible materials for keep the hot thermonuclear plasma. Because of some established reasons a considerable part of elaborations and researches treat with the toroidal traps, for example the well-known tokamak and stellarator. In the first one the magnetic field is induced mainly by the toroidal electric current in the plasma, the field of the second one owes its origin to a system of conductors outside the plasma volume. It is of interest a promising class of traps where the current carrying conductors are immersed into the plasma, but don’t immediately contact with the hot one. The magnetic field geometry is more complicated and varied in this case, that allows to expect a more effective plasma confinement. A special attention to such traps was concentrated in a series of works by A.I. Morozov, who named them “Galateas” (see [1,2] with references). The galatea-traps are supposed also to be toroidal, but some researches of general or conceptual problems in theoretical and computational plasma physics admit to replace the torus by its straightened into the cylinder analogue. The mathematical means become more simple in this case.

Investigation of magnetic traps include as an essential part the mathematical modeling and computation using the highly efficient computer technique up to day. Plasma, treated there, may be considered dense enough for dealing with the models in continuous media mechanics terms, i.e., magnetogasdynamics. The plasma confinement time, needed for nuclear fusion reaction expected, is significantly greater than characteristic ones of low-scaled plasma processes. Therefore equilibrium plasma and field configurations in the trap, more exactly, their plasmastatic models form our investigation object.
"The Belt" is a simple and visual example of the galatea-traps. It is a plasma torus with two circular conductors, suggested by analogy with a cylindrical device in the General Physics Institute Russian Academy of Sciences for investigation of current sheets by the group of S.I. Syrovatskii followers [3]. Numerical simulation of “The Belt” and its differences from the current sheets are also considered in the cylindrical geometry in the series of papers by the authors (see. [4, 5] with references).

A plasmastatic model of equilibrium configurations in the toroidal variant of “The Belt” is developed in this paper. We put a question of the differences between toroidal and cylindrical configurations and their quantitative characteristics, depending on the trap parameters. Some responses are obtained in computation.

2. Plasmastatic model. Setting of the problem and means of its solving.

The mathematical apparatus of plasmastatics is based on the magnetogasdynamics equations. In our case (the velocity $v \equiv 0$ and the time dependence $t \partial / \partial t \equiv 0$ absent) only the equilibrium one of them is nontrivial

$$\nabla p = \frac{1}{r} j \times H,$$

where $p, j, H$ are respectively the plasma pressure, electric current density and magnetic field strength. Together with the Maxwell’s equations

$$j = \frac{c}{4\pi} \text{rot} H; \ \ \ \text{div} H = 0,$$

they form a closed system for three unknown functions, that determine the equilibrium configuration required in the giving space region with the given boundary conditions.

Boundary value problems with the equations (1)-(2) are generally rather complicated. But they may be essentially simplified, given a symmetry, in particular in axially symmetric problems on “The Belt” configurations in torus, where $\partial / \partial \varphi \equiv 0$ in the cylindrical coordinates $(r, \varphi, z)$.

It is convenient to express the the $(r,z)$- components of solenoidal vectors $H$ and $j$ by their flux functions $\Psi$ and $I$

$$rH_r = - \frac{\partial \Psi}{\partial z}; \ \ \ rH_z = \frac{\partial \Psi}{\partial r}; \ \ \ rj_r = - \frac{\partial I}{\partial z}; \ \ \ rj_z = \frac{\partial I}{\partial r}, \ \ \ I = \frac{c}{4\pi} rH_\varphi$$

It follows from (1),(3), firstly, that the magnetic force lines $\Psi(r,z) = $ const and electric current stream lines $I(r,z) = $ const are isobars ($p = $ const), i.e., the three function $p, \Psi, I$ are interdependent:

$$p = p(\Psi); \ \ \ I = I(\Psi).$$

Secondly, the function $\Psi$ satisfies the Grad-Shafranov equation [6,7]

$$\Delta^* \Psi + 4\pi r^2 \frac{dp}{d\Psi} + \left( \frac{4\pi}{c} \right)^2 I \frac{dI}{d\Psi} = 0,$$

where

$$\Delta^* \Psi = r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + \frac{\partial^2 \Psi}{\partial z^2}.$$

Its low minor order terms include two functions (4), that must be given from additional requirements, relating with trap configurations.

Boundary value problems on the galatea-trap configurations must be considered, strictly speaking, in the torus cross section without conductors immersed. But we don’t want to operate with a multiply connected regions and admit one more simplification: the conductor current is
given by a function \( j_{ex}^\xi (r, z) \) localized in their cross sections, the equation (1) contains only the plasma current \( j^{pl} \), and the whole current \( j = j^{pl} + j_{ex} \) takes part in the Maxwell’s equations (2). The equation (5) obtains in this case the additional term \( 4\pi r j_{ex}^\xi / c \) in the left hand side.

In the specific case of a toroidal trap “The Belt” with two circular conductors, those cross-section centers are situated at \( z = \pm z_0, r = r_0 \) the current \( j_{ex} \) is given in the form

\[
j_{ex} = \sum_{k=1}^{2} j_0 \exp \left( -\frac{(r - r_0)^2 + (z - z_k)^2}{r_c^2} \right),
\]

where \( z_1 = z_0, z_2 = -z_0 \), and \( r_c \) is the conventional conductor radius. The factor \( j_0 \) is chosen by means of the condition

\[
\int \int j_{ex}^\xi (r, z) dr dz = J_c,
\]

where \( J_c \) is the given current value in each conductor, and the integral is taken around its vicinity.

There is no “toroidal” magnetic field \( H_\phi \) in the trap, thus \( I(\Psi) \equiv 0 \) in the equation (5). The relation \( p(\Psi) \) is chosen, starting from the desire to dispose plasmas in the trap, in particular, to insulate the conductor from it. We can gain such effect, concentrating plasma around the magnetic field singular point and hence along the separatrix \( \Psi = \Psi_0 \) passing through this point (the magnetic field topology in cylindrical and toroidal “Belt” variants are presented at the Fig. 1a and 2a). The function \( p(\Psi) \) must be nonmonotone, have the maximum in this point and decrease rapidly in the both sides from the separatrix. For example,

\[
p = p_0 \exp \left( -\frac{(\Psi - \Psi_0)^2}{q^2} \right)
\]

Boundary value problem with equation (5), including refinement made above, are considered in the rectangular (for simplicity sake) space region

\[
R_1 < r < R_2; \quad -Z < z < Z.
\]

The simplest boundary conditions correspond to the impermeable for the magnetic field boundary \( H_n = 0 \), where \( n \) is normal direction, i.e. \( \Psi = \text{const} \). For definiteness put

\[
\Psi = 0; \quad \text{at} \quad (r, z) \in \Gamma,
\]

where \( \Gamma \) is region (8) boundary. More complex conditions, taking into account magnetic transparent boundaries, are considered in [8].

The problem is considered in dimensionless variables, using units, composed from the given constants. The units of length, field strength, flux function, current density and pressure are respectively

\[
r_u = z_0; \quad H_u = \frac{2J_c}{e z_0}; \quad \Psi_u = H_u r_u; \quad j_u = \frac{c H_u}{4\pi r_u}; \quad p_u = \frac{H_u^2}{4\pi}.
\]

where \( z_0 \) is the half-distance between the conductors, \( J_c \) - the current value in the each one. In these units, the Grad-Shafranov equation(5) is

\[
\nabla^2 \Psi + r^2 \frac{dp}{d\Psi} + r j_{ex}^\xi (r, z) = 0
\]

The conductor coordinates and the constant \( j_0 \) in Eq. (7) are

\[
z_1 = 1, \quad z_2 = -1, \quad j_0 = \frac{2}{r_c}.
\]
Figure 1. Equilibrium configuration in the cylindrical “Belt” variant a) magnetic field \((\Psi (r, z) = \text{const})\), b) plasma pressure \((p = \text{const})\).

Formula (7) and the region (8) boundary are the same, but with parameters, related to the units (10).

The problem posed is solved numerically by means of iterative relaxation method. We have to do with the initial- boundary value problem for the parabolic equation

\[
\frac{\partial \Psi}{\partial t} = \Delta^* \Psi + r^2 \frac{dp}{d\Psi} + r j_{ex}(r, z) \tag{12}
\]

in assumption that its solution would approach to the required one with the time. Its difference analogue is solved by the well-known alternating direction method [9]. In this case the computation of the next \((n + 1-\text{th})\) “time”-step or iteration uses the value of nonlinear function \(p(\Psi)\) from the preceding \((n-\text{th})\) one.

A particular feature of the problem is the requirement: the parameter \(\Psi_0\) in eq. (7) must coincide with the value of the solution \(\Psi\) in the field singular point. We use the following suitable way to determine it: the value \(\Psi_0\) in computation of the \(n + 1-\text{th}\) step is the one of \(\Psi\) in the singular point on the \(n-\text{th}\) step, that is its maximum of the vertical line \(z = 0\). Thus

\[
(\Psi_0)^{n+1} = \max_r (\Psi(r, 0))^n \tag{13}
\]

The model of equilibrium configurations in a straight cylinder for comparison with our results below is constructed in a similar manner under the plane symmetry assumption: \(\partial / \partial z \equiv 0\) in the Cartesian coordinates. The dimensionless Grad-Shafranov type equation is

\[
\Delta \Psi + \frac{dp}{d\Psi} + j_{ex}(x, y) = 0 \tag{14}
\]

The magnetic field is presented by the flux function \(\Psi\),

\[
H_x = \frac{\partial \Psi}{\partial y}; H_y = -\frac{\partial \Psi}{\partial x}
\]

the independent variable space region is a rectangle \(|x| < X, |y| < Y\) (or a circle \(x^2 + y^2 < R^2[4]\)), the current in the conductors is distributed as (6), the plasma pressure is given in the form (7) with the maximum in the center and at the separatrix, passing through it.
3. Computation results.

In numerical solving of the problems in the torus as in the cylinder equilibrium configurations of plasma, magnetic field and electric current are obtained under some conditions. The examples of them in cylindrical and toroidal traps are presented at the fig 1, 2 by the magnetic force lines $\Psi = \text{const}$ and isobars $p = \text{const}$. The plasma is insulated from the conductors and concentrated around the field singular point. It has a form of curvilinear quadrangle with concave boundaries and thin branches along the separatrix surrounding the conductors. The electric current in plasma, orthogonal to the figure plane, vanishes in the configuration center and is concentrated at its boundaries, where the pressure gradient is maximal. It is positive (of conductor current direction) outside the separatrix and negative inside it, i.e. around the conductors. Interacting with the magnetic field of anti-clockwise direction, it promotes to confine plasma near the separatrix. The configuration in the straight cylinder are symmetric relating the center $x = y = 0$. The ones is the torus are deformed: the singular point at the fig.2 is situated farther than the conductors from the $z$- axis at the distance $\delta r$. Its value decreases with the growth of the current radius $r_0$ because the configuration geometry tends to the cylindrical one, when $r_0 \to \infty$.

The problem includes the dimensionless parameter $p_0$ in (7). Its physical meaning is the maximum plasma pressure, referred to the characteristic magnetic one $p_u$ (10). The equilibrium mentioned above is established as usual under the restriction

$$p_0 < p_0^c,$$  \hspace{1cm} (15)

where the critical value $p_0^c$ depends on the parameter $q$ in (7) and on the torus radius $r_0$. It restricts the capability of a trap with fixed field and current magnitude to confine the plasma with pressure desired. The values $p_0^c$ in the toroidal “Belt” exceed the ones in the cylinder and increase, when the radius $r_0$ takes away. The total current in plasma and its positive (outside the separatrix) and negative (inside it) parts are obtained in computation under the condition (15). They increase proportional the pressure $p_0$. The singular point shift $\delta r$ increases with $p_0$ about twice, when $0 < p_0 < p_0^c$. The parameter $\Psi_0$ in (7) that means the magnetic flux between the separatrix and outer boundary, also increases with $p_0$ but weekly.

Figure 2. Equilibrium configuration in the toroidal “Belt” variant a) magnetic field ($\Psi(r, z) = \text{const}$), b) plasma pressure ($p = \text{const}$)
The restriction of the type (15) takes place in a large class of the mathematical models of reaction and diffusion interaction processes. Their mathematical nature is connected with the solution existence and uniqueness in boundary problems with semilinear elliptic type equations and with spectral analysis of their linearized operators (see [10] with detailed references).

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