Iteration Complexity of Randomized Primal-Dual Methods for Convex-Concave Saddle Point Problems

E. Yazdandoost Hamedani*, A. Jalilzadeh*, N. S. Aybat*, U. V. Shanbhag*

Industrial & Manufacturing Engineering Department,
The Pennsylvania State University, PA, USA.

Emails: evy5047, azj5286, nsa10, udaybag@psu.edu

Abstract

In this paper we propose a class of randomized primal-dual methods to contend with large-scale saddle point problems defined by a convex-concave function \( \mathcal{L}(x, y) \triangleq \sum_{i=1}^{m} f_i(x_i) + \Phi(x, y) - h(y) \). We analyze the convergence rate of the proposed method under the settings of mere convexity and strong convexity in \( x \)-variable. In particular, assuming \( \nabla_y \Phi(\cdot, \cdot) \) is Lipschitz and \( \nabla_x \Phi(\cdot, y) \) is coordinate-wise Lipschitz for any fixed \( y \), the ergodic sequence generated by the algorithm achieves the \( O(m/k) \) convergence rate in a suitable error metric where \( m \) denotes the number of coordinates for the primal variable. Furthermore, assuming that \( \mathcal{L}(\cdot, y) \) is uniformly strongly convex for any \( y \), and that \( \Phi(\cdot, y) \) is linear in \( y \), the scheme displays convergence rate of \( O(m/k^2) \). We implemented the proposed algorithmic framework to solve kernel matrix learning problem, and tested it against other state-of-the-art solvers.

I. Introduction

Let \( (\mathcal{X}_i, \| \cdot \|_{\mathcal{X}_i}) \) for \( i \in \mathcal{M} \triangleq \{1, 2, \ldots, m\} \) and \( (\mathcal{Y}, \| \cdot \|_{\mathcal{Y}}) \) be finite dimensional, normed vector spaces such that \( \mathcal{X}_i = \mathbb{R}^{n_i} \) for \( i \in \mathcal{M} \). Let \( x = [x_i]_{i \in \mathcal{M}} \in \prod_{i \in \mathcal{M}} \mathcal{X}_i \triangleq \mathcal{X} = \mathbb{R}^n \) where \( n \triangleq \sum_{i \in \mathcal{M}} n_i \). In this paper, we study the following saddle point (SP) problem:

\[
(P) : \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \mathcal{L}(x, y) \triangleq \sum_{i \in \mathcal{M}} f_i(x_i) + \Phi(x, y) - h(y),
\]

where \( f_i : \mathcal{X}_i \to \mathbb{R} \cup \{+\infty\} \) is convex for all \( i \in \mathcal{M} \) and \( h : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\} \) is convex (possibly nonsmooth) and \( \Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) is convex in \( x \) and concave in \( y \) and satisfies certain differentiability assumptions – see Assumption \textbf{II.1}. Our study is motivated by large-scale...
problems with a *coordinate-friendly* structure [1], i.e., the number of operations to compute the partial-gradient $\nabla_x \Phi(x, y)$ should be order of $n_i/n \approx 1/m$ fraction of the full-gradient $\nabla_x \Phi(x, y)$ computation. Our objective is to design an efficient first-order randomized block-coordinate primal-dual method to compute a saddle point of the structured convex-concave function $\mathcal{L}$ in [1], and to investigate its convergence properties under mere and strong convexity settings.

There are many problems arising in machine learning that may be cast as [1]: a particularly important example is the regularized convex optimization problem with nonlinear constraints, i.e.,

$$\min_{x=[x_i]_{i \in \mathcal{M}} \in \mathcal{X}} \rho(x) \triangleq \sum_{i \in \mathcal{M}} f_i(x_i) + g(x) \quad \text{s.t. } G(x) \in -\mathcal{K},$$

(2)

where $\mathcal{K} \subseteq \mathcal{Y}^*$ is a closed convex cone in the dual space $\mathcal{Y}^*$, $f_i : \mathcal{X}_i \to \mathbb{R} \cup \{+\infty\}$ is a convex (possibly nonsmooth) regularizer function for $i \in \mathcal{M}$; $g : \mathcal{X} \to \mathbb{R}$ is a smooth convex function having a coordinate-wise Lipschitz continuous gradient with constant $L_i(g)$ for each coordinate $i \in \mathcal{M}$, and $G : \mathcal{X} \to \mathcal{Y}^*$ is a smooth $\mathcal{K}$-convex, coordinate-wise Lipschitz function having a coordinate-wise Lipschitz Jacobian with constants $C_i(G)$ and $L_i(G)$ for each $i \in \mathcal{M}$, respectively. This problem can be written as a special case of [1] by setting $\Phi(x, y) = g(x) + \langle G(x), y \rangle$ and $h(y) = \mathbb{I}_{\mathcal{K}^*}(y)$ is the indicator function of $\mathcal{K}^*$, where $\mathcal{K}^* \subseteq \mathcal{Y}$ denotes the dual cone of $\mathcal{K}$.

**Related Work.** Many real-life problems arising in machine learning, signal processing, image processing, finance, etc., can be formulated as a special case of the SP problem in [1]. As briefly discussed above, convex optimization problems with nonlinear constraints in the form of (2) can be reformulated as a SP problem. The advantages of such formulation lies in utilizing the corresponding dual variables in order to boost the primal convergence while controlling the constraints at the same time. Recently, there have been several work proposing efficient algorithms to solve convex-concave SP problems [2], [3], [4], [5], [6].

In a large-scale setting, the computation of full-gradient and/or prox operator might be prohibitively expensive; hence, presenting a strong motivation for using the partial-gradient and/or separable structure of the problem at each iteration of the algorithm. Therefore, the computation may be broken into smaller pieces; thereby, inducing tractability per iteration, at the cost of slower convergence in terms of total iteration complexity. There has been a vast body of work on randomized block-coordinate descent schemes for primal optimization problems [7], [8], [9], [10], [11]; but, there are far fewer studies on randomization of block coordinates for algorithms designed for solving SP problems. Many recent work on primal-dual algorithms are motivated
from the perspective of regularized empirical risk minimization (ERM) of linear predictors arising in machine learning [12], [13], [14], [15], and ERM can be formulated as an SP problem:

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \sum_{i \in \mathcal{M}} f_i(x_i) + \langle A_i x_i, y \rangle - h(y).
\]

(3)

It is clear that this problem is a special case of (1) by setting \(\Phi(x, y) = \sum_{i \in \mathcal{M}} \langle A_i x_i, y \rangle\), which is both bilinear and separable in \(x_i\)-coordinates. Recently, Chambolle et al. [15] considered (3), where they propose a primal-dual algorithm with arbitrary sampling of coordinates, an extension of a previous work [2], and show that \(L(\bar{x}_k, y^*) - L(x^*, \bar{y}_k)\) converges to zero with \(O(m/k)\) rate when \(\sum_{i \in \mathcal{M}} f_i(x_i)\) is merely convex and with \(O(m/k^2)\) rate when each \(f_i\) is strongly convex, where \(\{(\bar{x}_k, \bar{y}_k)\}_k\) is a weighted ergodic average sequence.

Furthermore, some recent studies has been carried out on block-coordinate ADMM-type algorithms [16], [17]. In [17], randomized primal-dual proximal block coordinate algorithm is proposed for solving \(\min \{g(x) + q(y) + \sum_{i=1}^{M} f_i(x_i) + \sum_{j=1}^{N} r_j(y_j) : \sum_{i=1}^{M} A_i x_i + \sum_{j=1}^{N} B_j y_j = b\}\) where \(g\) and \(q\) are differentiable convex functions, and \(\{u_i\}_{i=1}^{m_x}\) and \(\{r_j\}_{j=1}^{m_y}\) are closed convex functions with efficient proximal mappings. Assuming coordinate-wise Lipschitz differentiability of \(g\) and \(q\), \(O(1/(1 + \gamma k))\) convergence rate is shown under mere convexity assumption, where \(\gamma = \frac{m_x'}{m_x} = \frac{m_y'}{m_y}\) and \(m_x' (m_y')\) denotes the number of \(x_i\) \((y_j)\) coordinates updated at each iteration.

Majority of the previous work on block-coordinate algorithms for SP problems require a bilinear coupling term \(\Phi\) in the saddle point problem formulation [18], [19], [20] – having a similar structure with the formulation in (3). However, only a few of the existing methods [21], [22], [23] can handle the more general framework discussed in this paper, i.e., \(\Phi\) may not be bilinear and/or may not be separable in \(\{x_i\}_{i \in \mathcal{M}}\) coordinates.

In a recent work, He and Monteiro [5] have considered a bilinear SP problem and proposed a hybrid proximal extragradient (HPE) method which can compute an \(\epsilon\)-saddle point \((x_\epsilon, y_\epsilon)\) within \(O(1/\epsilon)\) iterations. Later, Kolossoski and Monteiro [6] propose another HPE-type method to solve SP problems as in (1) over bounded sets – the dual optimal set may be unbounded and/or it may not be trivial to get an upper bound on an optimal dual solution for problems of the form (2). Indeed, HPE-type method in [6] can be seen as an inexact proximal point method, each prox subinclusion (outer step) is solved using an accelerated gradient method (inner steps).

Nemirovski [21] studied \(\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y)\) as a variational inequality problem and proposed a prox-type extra-gradient based method, Mirror-prox, assuming that \(\mathcal{X} \subset \mathcal{X}\) and \(\mathcal{Y} \subset \mathcal{Y}\) are convex compact sets, \(\Phi(x, y)\) is differentiable, and \(F(x, y) = [\nabla_x \Phi(x, y)^\top, -\nabla_y \Phi(x, y)^\top]^\top\)
is Lipschitz with constant $L$. It is shown that the ergodic iterate sequence converges with $O(L/k)$ rate where in each iteration, $F$ is computed twice and a projection onto $X \times Y$ is computed with respect to general (Bregman) distance. Later Mirror-prox has been extended to exploit problems in the strongly convex setting; in particular, a multi-stage method that repeatedly calls Mirror-prox is proposed in Juditky and Nemirovski [24], and $O(1/k^2)$ rate is shown for the method when $\Phi(x, y)$ is strongly convex in $x$, $Y$ is a compact set and $\nabla_y \Phi(x, y)$ is independent of $y$, i.e., $L_{yy} = 0 - \Phi$ is linear in $y$. In a more recent paper, He et al. [25] extended the original Mirror-prox method [21] to solve the composite form, $\min_{x \in X} \max_{y \in Y} f(x) + \Phi(x, y) - h(y)$, with $O(L/k)$ convergence rate where $f$ and $h$ are possibly nonsmooth convex functions with simple prox maps with respect to a general (Bregman) distance. In all three papers, the primal and dual step-sizes are at most $1/L$.

In [22], which is closely related to our work, a convex minimization problem with functional constraints, $\min \{ f(x) + g(x) : Ax = b, G(x) \leq 0 \}$, is considered where $g$ and component functions of $G$ are Lipschitz differentiable convex functions, and $f$ is a proper closed convex function (possibly nonsmooth). In [22], a primal-dual method based on linearized augmented Lagrangian method (LALM) is proposed and a sublinear convergence rate in terms of suboptimality and infeasibility is shown for LALM. Moreover, when the function $f(x)$ has a separable structure, $\sum_{i \in M} f_i(x_i)$, a randomized block-coordinate variant of LALM (BLALM) with a convergence rate of $O(1/(1 + k/m))$ is proposed. The problem considered in [22] is clearly a special case of (1); however, BLALM cannot deal with (1) when $\Phi$ is not linear in $y$. Coordinatewise Lipschitz constants $\{L_i\}_{i \in M}$ of the smooth part of the augmented Lagrangian determine the block-coordinate step sizes. As it is mentioned in [22], computing $L_i$ requires the knowledge of a bound on both the primal and dual variables; these bounds are usually loose which typically leads to very small step-sizes compared to ours in the same setting. To remedy this issue, [22] introduced an adaptive line-search to determine the primal step-size.

| Paper                        | MC | SC | # of prox per iter. | General $\Phi$ | Random blocks | Rate for MC, SC |
|------------------------------|----|----|---------------------|----------------|---------------|----------------|
| Chambolle et al. [15]        | ✓  | ✓  | 2                   | ✓              | ✓            | $O(m/k), O(m/k^2)$ |
| Xu [22]                      | ✓  | ✓  | 2                   | ✓              | ✓            | $O(m/(m+k))$, $-$ |
| Dang & Lan [18]              | ✓  | ✓  | 2                   | ✓              | ✓            | $O(m/k), O(m/k^2)$ |
| Juditsky & Nemirovski [24]   | ✓  | ✓  | 4                   | ✓              | ✓            | $O(1/k), O(1/k^2)$ |
| This paper                   | ✓  | ✓  | 2                   | ✓              | ✓            | $O(m/k), O(m/k^2)$ |

| TABLE I: Comparison of different methods in Merely Convex (MC) and Strongly Convex (SC) settings. In convergence rates, $k$ denotes the iteration counter. It is worth emphasizing that the work-per-iteration for [24] is $O(m)$ while it is $O(1)$ for the others. |
**Application.** Many real-life problems arising in machine learning, signal processing, image processing, finance, etc., can be formulated as a special case of (1). In the following, we will briefly mention some of these applications as special cases of our problem setting: i) in robust optimization the problem data lies in a given ambiguity set, e.g., the objective function involves uncertain data and the aim is then to minimize the worst case (maximum) value of the objective function, which naturally leads to saddle-point problems [26], [27]; ii) distance metric learning formulation proposed in [28] is a convex optimization problem over positive semidefinite matrices subject to nonlinear convex constraints; iii) kernel matrix learning for transduction problem can be cast as an SDP or a QCQP [29], [30]; iv) training ellipsoidal machines [31] requires solving nonlinear SDPs. Problems in the aforementioned applications are typically large-scale and standard primal-dual methods do not scale well with the problem dimension and their iterations are memory expensive; therefore, the advantages of randomized block-coordinate schemes will be evident as problem dimension increases in terms of the efficiency of work required per-iteration.

**Contribution.** In this paper we studied large-scale SP problems with a more general structure: the coupling function is neither bilinear nor separable. To efficiently handle large-scale SP problems, we proposed a randomized block-coordinate primal-dual algorithm with a momentum term and equipped with Bregman distance functions that can generalize previous methods such as [15]. These type of schemes are the method of choice for the SP problems with a coordinate-friendly structure so that the computational tasks performed on each block coordinate at each iteration are significantly cheaper compared to full-gradient computations.

Suppose a saddle point of (1), \((x^*, y^*)\), exists. Under Lipschitz continuity of \(\nabla_y \Phi(\cdot, \cdot)\) and coordinate-wise Lipschitz continuity of \(\nabla_x \Phi(x, \cdot)\), we proved convergence rate of \(O(m/k)\) and \(O(m/k^2)\) for merely and strongly convex settings in terms of Lagrangian error metric \(E[\mathcal{L}(\bar{x}^k, y^*) - \mathcal{L}(x^*, \bar{y}^k)]\) and solution error \(E[\|x^k - x^*\|^2]\), respectively, where \(\{(\bar{x}^k, \bar{y}^k)\}_k\) is ergodic average sequence of the iterates. To the best of our knowledge, our proposed method is the only randomized block-coordinate primal-dual algorithm that can handle general SP problems as in (1), and our rate results achieve the best rates known for our setting.
II. RANDOMIZED ACCELERATED PRIMAL-DUAL ALGORITHM

Definition 1. Let \( f(x) \triangleq \sum_{i \in M} f_i(x_i) \) and \( M = \{1, \ldots, m\} \), and define \( U_i \in \mathbb{R}^{n \times n} \) for \( i \in M \) such that \( I_n = [U_1, \ldots, U_m] \), where \( I_n \) denotes the \( n \times n \) identity matrix.

Let \( \varphi_Y : Y \to \mathbb{R} \) be a differentiable on an open set containing \( \text{dom} \ h \). Suppose \( \varphi_Y \) is 1-strongly convex w.r.t. \( \| \cdot \|_Y \). Let \( D_Y : Y \times Y \to \mathbb{R}_+ \) be a Bregman distance function corresponding to \( \varphi_Y \), i.e., \( D_Y(y, \tilde{y}) \triangleq \varphi_Y(y) - \varphi_Y(\tilde{y}) - \langle \nabla \varphi_Y(\tilde{y}), y - \tilde{y} \rangle \). The dual space of \( Y \) is denoted by \( Y^* \), and \( \| \cdot \|_{Y^*} : Y^* \to \mathbb{R} \) such that \( \| y' \|_{Y^*} \triangleq \max \{ \langle y', y \rangle : \| y \|_Y \leq 1 \} \) denotes the dual norm. Similarly, for each \( i \in M \), given an arbitrary norm \( \| \cdot \|_{X_i} \) on \( X_i \), define \( \| \cdot \|_{X_i^*} : X_i^* \to \mathbb{R} \) and \( D_{X_i^*} : X_i \times X_i \to \mathbb{R} \) for some \( \varphi_{X_i} \) that is differentiable and 1-strongly convex w.r.t. \( \| \cdot \|_{X_i} \).

Assumption II.1. Suppose \( f_i : X_i \to \mathbb{R} \cup \{+\infty\} \) is a closed convex function and its convexity modulus w.r.t. \( \| \cdot \|_{X_i} \) is \( \mu_i \geq 0 \) for all \( i \in M \) and \( h : Y \to \mathbb{R} \cup \{+\infty\} \) is a closed convex function. Moreover, suppose that \( \{f_i\}_{i \in M}, h \) and \( \Phi : X \times Y \to \mathbb{R} \) satisfy the following assumptions:

(i) for any fixed \( y \in Y \), \( \Phi(x, y) \) is convex and coordinate-wise Lipschitz differentiable in \( x \), and for all \( i \in M \), there exists \( L_{x_i} \geq 0 \) such that for any \( x \in X \) and \( v \in X_i \),

\[
\| \nabla_{x_i} \Phi(\bar{x} + U_i v, y) - \nabla_{x_i} \Phi(\bar{x}, y) \|_{X_i^*} \leq L_{x_i} \| v \|_{X_i},
\]

(ii) for any fixed \( \bar{x} \in X \), \( \Phi(\bar{x}, y) \) is concave and differentiable in \( y \), and there exists \( L_{y_i} \geq 0 \) and \( L_{y x_i} > 0 \) for all \( i \in M \) such that for any \( y, \tilde{y} \in Y \) and \( v \in X_i \) for \( i \in M \),

\[
\| \nabla_y \Phi(\bar{x} + U_i v, \tilde{y}) - \nabla_y \Phi(\bar{x}, y) \|_{Y^*} \leq L_{y_i} \| y - \tilde{y} \|_Y + L_{y x_i} \| v \|_{X_i},
\]

(iii) for any \( i \in M \), \( \arg\min_{x_i \in X_i} \{ tf_i(x_i) + \langle s, x_i \rangle + D_{X_i}(x_i, \bar{x}_i) \} \) can be computed efficiently for any \( \bar{x}_i \in \text{dom} f_i \), \( s \in X_i^* \) and \( t > 0 \). Similarly, \( \arg\min_{y \in Y} \{ th(y) + \langle s, y \rangle + D_Y(y, \tilde{y}) \} \) is easy to compute for any \( \tilde{y} \in \text{dom} h \), \( s \in Y^* \) and \( t > 0 \).

Remark II.1. Consider the constrained optimization problem in (2). If a bound on a dual optimal solution is known\(^1\), i.e., for some dual optimal \( y^* \in K^* \), we are given \( B > 0 \) such that \( \| y^* \|_Y \leq B \), then (2) can be written as a special case of (1) by setting \( \Phi(x, y) = g(x) + \langle G(x), y \rangle \) and \( h(y) = \mathbb{I}_{K^* \cap B}(y) \), where \( B \triangleq \{ y \in Y : \| y \|_Y \leq B \} \). Moreover, the Lipschitz constants for \( \Phi \) can be set as follows: \( L_{y y} = 0 \), \( L_{y x_i} = C_i(G) \) and \( L_{x_i x_i} = L_i(g) + L_i(G)B \) for \( i \in M \).

\(^1\)A bound on dual optimal set can be computed if a Slater point is known – see [23, Lemma 8]
Next we will define some diagonal matrices to simplify the notation in the rest of the paper:

\[ M \triangleq \text{diag}([\mu_i]_{i \in \mathcal{M}}), \quad L_{xx} = \text{diag}([L_{x_i x_i}]_{i \in \mathcal{M}}), \quad L_{yx} = \text{diag}([L_{y x_i}]_{i \in \mathcal{M}}). \]  

(6)

Note that for any \( y \in \mathcal{Y} \) and \( i \in \mathcal{M} \), (4) implies

\[ 0 \leq \Phi(\bar{x} + U_i v, y) - \Phi(\bar{x}, y) - \langle \nabla_{x_i} \Phi(\bar{x}, y), v \rangle \leq \frac{1}{2} L_{x_i x_i} \| v \|_{\chi_i}^2, \quad \forall v \in \chi_i, \; \bar{x} \in \chi. \]  

(7)

Similarly (5) and concavity of \( \Phi(x, y) \) in \( y \) imply that for any \( x \in \chi \),

\[ 0 \geq \Phi(x, y) - \Phi(x, \bar{y}) - \langle \nabla_y \Phi(x, \bar{y}), y - \bar{y} \rangle \geq -\frac{1}{2} L_{yy} \| y - \bar{y} \|_{\chi_i}^2, \quad \forall y, \bar{y} \in \mathcal{Y}. \]  

(8)

We next state the algorithm being proposed to solve (1) when given an arbitrary norm \( \| \cdot \|_{\chi_i} \) on \( \chi_i \) and Bregman function \( D_{\chi_i} \) as in Definition 1 for all \( i \in \mathcal{M} \). The proposed Algorithm 1 consists of a single loop primal-dual steps. After initialization of parameters, a dual ascent step is taken in the direction of \( \nabla_y \Phi \) with a momentum term in terms of \( \nabla_y \Phi \) which is a novel algorithmic approach to gain acceleration – it generalizes the commonly used extrapolation step when the function \( \Phi \) is bilinear\(^2\). Afterwards, a primal block-coordinate descent step is taken in the negative direction of \( \nabla_{x_i} \Phi \) for a uniformly chosen random block coordinate.

**Algorithm 1 Randomized Accelerated Primal-Dual (RAPD) Algorithm**

1: **Input:** \( \{x_i^k\}_{i \in \mathcal{M}}, \sigma^k, \theta^k \}_{k \geq 0}, (x_0, y_0) \in \chi \times \mathcal{Y}
2: \( (x_{-1}, y_{-1}) \leftarrow (x_0, y_0) \)
3: **for** \( k \geq 0 \) **do**
4: \( s^k \leftarrow \nabla_y \Phi(x^k, y^k) + m \theta^k \left( \nabla_y \Phi(x^k, y^k) - \nabla_y \Phi(x^{k-1}, y^{k-1}) \right) \)
5: \( y^{k+1} \leftarrow \arg\min_y \left\{ h(y) - \langle s^k, y - y^k \rangle + \frac{1}{\sigma^k} D_{\chi_i}(y, y^k) \right\} \)
6: \( \text{Choose } i_k \in \mathcal{M} \text{ uniformly at random} \)
7: \( x_i^{k+1} \leftarrow \begin{cases} \arg\min_x \{ f_i(x) + \langle \nabla_{x_i} \Phi(x^k, y^{k+1}), x \rangle + \frac{1}{\tau_i^k} D_{\chi_i}(x_i, x_i^k) \} & \text{if } i = i_k \\ x_i^k & \text{otherwise} \end{cases} \)
8: **end for**

Note that when \( \| \cdot \|_{\chi_i} = \| \cdot \| \) and \( \varphi_{\chi_i}(x_i) = \frac{1}{2} \| x_i \|^2 \), the \( x_i \)-subproblem in Line 7 of RAPD can be written as \( \arg\min_x \{ f_i(x) + \langle \nabla_{x_i} \Phi(x^k, y^{k+1}), x \rangle + \frac{1}{2\tau_i^k} \| x_i - x_i^k \| \} \) for \( i = i_k \).

\(^2\)Majority of the existing methods use past iterates to gain momentum, e.g., \([2, 4]\) use the momentum term \((1 + \theta^k)x^k - \theta^k x^{k-1}\) – this iteration can be recovered by our method when \( \Phi \) is bilinear.
III. RATE ANALYSIS

In this section we discuss the convergence properties of RAPD algorithm in Theorem III.1 which is the main result of this paper, and we provide an easy-to-read convergence analysis considering a general step-size rule. In particular, our analysis assumes conditions on \(\{([\tau_k^i]_{i\in\mathcal{M}}, \sigma_k^i, \theta_k^i)\}_{k\geq 0}\), as stated in Assumption III.1 in the rest, \(E[\cdot]\) denotes the expectation operation and the conditional expectation is denoted by \(E^k[\cdot] \triangleq E[\cdot | \mathcal{F}_{k-1}]\), where \(\mathcal{F}_{k-1} \triangleq \{x^0, i_0, \ldots, i_{k-1}\}\) for \(k \geq 1\).

For the sake of simplicity of the proof, Theorem III.1 is proven for the case where \(\|\cdot\|_{\mathcal{X}_i}\) is the Euclidean norm \(\|\cdot\|\) and \(D_{\mathcal{X}_i}(x_i, \bar{x}_i) = \frac{1}{2} \|x_i - \bar{x}_i\|^2\); however, the analysis for Part I, considering the merely convex setting, can simply be extended to the more general case employing a Bregman function \(D_{\mathcal{X}_i}\) as in Definition 1 for some \(i \in \mathcal{M}\).

**Definition 2.** Given a diagonal matrix \(D = \text{diag}(\{d_i\}_{i\in\mathcal{M}})\) for some \(d_i \geq 0\) for \(i \in \mathcal{M}\), define \(\|\cdot\|_D : \mathcal{X} \to \mathbb{R}\) such that \(\|x\|_D \triangleq (\sum_{i\in\mathcal{M}} d_i \|x_i\|^2)^{\frac{1}{2}}\).

**Definition 3.** For any \(k \geq 0\), given \(\{\tau_k^i\}_{i\in\mathcal{M}} \subset \mathbb{R}^{++}\), define \(T_k \triangleq \text{diag}(\{\frac{1}{\tau_k^i}\}_{i\in\mathcal{M}})\).

**Assumption III.1.** (Step-size Condition) For any \(k \geq 0\), the step-sizes \(\{\tau_k^i\}_{i\in\mathcal{M}} \subset \mathbb{R}^{++}\), \(\sigma_k > 0\) and momentum parameter \(\theta_k \in \left[1 - \frac{1}{m}, 1\right]\) satisfy the following conditions: \(\theta^0 = t^0 = 1\) and

\[
\begin{align*}
T_k &\succeq L_{xx} + \frac{L_{yy}^2}{\alpha^{k+1}}, \quad \frac{1}{\sigma^k} \geq m\theta^k(\alpha^k + \beta^k) + \frac{mL_{yy}^2}{\beta^{k+1}}, \\
t^k(T_k + M) &\succeq t^{k+1}(T^{k+1} + (1 - \frac{1}{m})M), \quad \frac{t^k}{\sigma^k} \geq \frac{t^{k+1}}{\sigma^{k+1}}, \quad t^{k+1}\sigma^{k+1} = t^k,
\end{align*}
\]

for some \(\{\alpha^k\}_{k\geq 0} \subset \mathbb{R}^{++}\) and \(\{\beta^k\}_{k\geq 0} \subset \mathbb{R}^{++}\).

Next, as the main result of this paper, in Theorem III.1 we discuss the convergence properties of the proposed method under mere convexity and strong convexity assumptions on \(\{f_i\}_{i\in\mathcal{M}}\). Following the main result, we also provide some corollaries and remarks in this section, and finally give an easy-to-read proof of Theorem III.1 in Section III-A.

**Theorem III.1.** Let \(z^* = (x^*, y^*)\) be a saddle point of problem (I). Suppose Assumption (II.1) holds and let \(\{x_k^i, y_k^i\}_{k\geq 0}\) be a sequence generated by RAPD, stated in Algorithm [I] then the following two assertions hold for RAPD:

\[\text{We define } 0^2/0 = 0 \text{ which may arise when } L_{yy} = 0.\]
(Part I) Suppose \( t^k = \theta^k = 1 \) for all \( k \geq 0 \), and the step-size sequences, \( \{\sigma^k\}_{k \geq 1} \) and \( \{\tau^k\}_{k \geq 1} \) for \( i \in \mathcal{M} \), satisfy Assumption III.7. Then for all \( K \geq 1 \), the ergodic average iterates \( \bar{x}^K = \frac{1}{K} \sum_{k=0}^{K-1} x^{k+1} \) and \( \bar{y}^K = \frac{1}{K} \sum_{k=0}^{K-1} y^{k+1} \) satisfy the following error bound:

\[
E[\mathcal{L}(\bar{x}^K, y^*) - \mathcal{L}(x^*, \bar{y}^K)] \leq \frac{m}{K} \Delta_1(x^0, y^0),
\]

\[
\Delta_1(x^0, y^0) \triangleq \frac{1}{2} \|x^* - x^0\|^2_{\mathcal{T}_0} + \frac{1}{m\sigma^0} + \frac{1}{m}(1 - \frac{1}{m})L_{yy}D_\mathcal{Y}(y^*, y^0) + (1 - \frac{1}{m})(\mathcal{L}(x^0, y^*) - \mathcal{L}(x^*, y^*)).
\]

In particular, given arbitrary \( \alpha > 0 \) and \( c_\tau, c_\sigma \in (0, 1) \), the result in (10) holds when \( \tau_i^k = \tau_i^0 \triangleq c_\tau (L_{x_i} x_i + L_{x_i}^2 / \alpha)^{-1} \) for \( i \in \mathcal{M} \), \( \sigma^k = \sigma^0 \triangleq c_\sigma (m(\alpha + 2L_{yy}))^{-1} \) and \( \theta^k = 1 \) for all \( k \geq 0 \). Moreover, if a saddle point for (1) exists and \( c_\tau, c_\sigma \in (0, 1) \), then the actual sequence \( \{x^k, y^k\}_{k \geq 0} \) converges to a random saddle point \( (x^*, y^*) \) almost surely such that \( 0 \leq E[\mathcal{L}(\bar{x}^K, y^*) - \mathcal{L}(x^*, \bar{y}^K)] \leq O(m/K) \). Finally, for \( c_\sigma \in (0, 1) \) one also has \( E[D_\mathcal{Y}(y^*, y^K)] \leq \frac{c_\sigma}{1 - c_\sigma} \frac{\Delta_1(x^0, y^0)}{m\sigma} \).

(Part II) Suppose \( \mu_i > 0 \) for all \( i \in \mathcal{M} \), i.e., \( M \succ 0 \), and \( L_{yy} = 0 \), i.e., \( \Phi \) is linear in \( y \). If \( \{t^k\}_k \), \( \{\theta^k\}_k \), \( \{\sigma^k\}_k \) and \( \{\tau^k\}_k \) for \( i \in \mathcal{M} \) satisfy Assumption III.1 then for any \( K \geq 1 \),

\[
E[\frac{1}{2} \|x^{K+1} - x^*\|^2_{\mathcal{T}_0 + (1 - \frac{1}{\mu})M}] \leq \frac{m}{K} \Delta_2(x^0, y^0),
\]

\[
\Delta_2(x^0, y^0) \triangleq \frac{1}{2} \|x^* - x^0\|^2_{\mathcal{T}_0 + (1 - \frac{1}{\mu})M} + \frac{1}{m\sigma^0}D_\mathcal{Y}(y^*, y^0) + (1 - \frac{1}{m})(\mathcal{L}(x^0, y^*) - \mathcal{L}(x^*, y^*)).
\]

In particular, the result in (11) holds when \( \theta^0 = t^0 = 1 \) and for all \( k \geq 0 \),

\[
\tau_i^k \leftarrow \left( \frac{\mu_i}{m} (1 + \frac{1}{\tau}) - \mu_i \right)^{-1}, \quad \theta^{k+1} \leftarrow \frac{1}{\sqrt{1 + \tau}}, \quad \sigma^{k+1} \leftarrow \frac{\sigma^k}{\theta^{k+1}}, \quad \tau^{k+1} \leftarrow \tau^{k+1} \tau^k,
\]

where \( \tau^0 = \min_{i \in \mathcal{M}} \frac{\mu_i}{m} (L_{x_i} x_i + L_{x_i}^2 / \alpha + (1 - \frac{1}{m})\mu_i)^{-1} \), \( \sigma^0 = c_\sigma / (m\alpha) \) for any \( \alpha > 0 \) and \( c_\sigma \in (0, 1] \). Moreover, for this specific parameter choice, \( \tau_i^K / t^K = \Theta(1/K^2) \) for all \( i \in \mathcal{M} \) which clearly implies that \( E\left[ \|x^K - x^*\|^2 \right] \leq O(m/K^2) \). Finally, for \( c_\sigma \in (0, 1) \) one also has \( E[D_\mathcal{Y}(y^*, y^K)] \leq \frac{c_\sigma}{1 - c_\sigma} \frac{\Delta_2(x^0, y^0)}{m\sigma} \).

Proof. The proof is provided at the end of Section III-A.

Remark III.1. The schemes with the ability of employing general prox terms based on Bregman distance functions can exploit the special structure of the problem and potentially lead to almost dimension-free complexity results; hence, methods using general prox terms involve those using \( \| \cdot \|_{x_i} = \| \cdot \| \) and \( D_{x_i}(x_i, \bar{x}_i) = \frac{1}{2} \| x_i - \bar{x}_i \|^2 \) for \( i \in \mathcal{M} \) as a special case – see [33], [27].
Our analysis for Part I will continue to hold even if we use a Bregman function for each coordinate \( x_i \), in which case \( x_i \)-update requires computing a general prox subproblem (line 7 RAPD).

The Bregman function should have the following separable form:  
\[
D_\theta(x, \bar{x}) = \sum_{i \in M} D_{\theta_i}(x_i, \bar{x}_i),
\]
for any \( x, \bar{x} \in \mathcal{X} \).

**Remark III.2.** For Part I of Theorem III.1 fix \( \theta^k = t^k = 1 \) for all \( k \geq 0 \). Then, for any \( \alpha > 0 \) and \( c_\tau, c_\sigma \in (0, 1] \), the specific step-size sequences given in Part I of Theorem III.1 satisfy Assumption III.1 together with \( \alpha^k = \alpha \) and \( \beta^k = L_{yy} \) for \( k \geq 0 \).

For Part II of Theorem III.1 for any \( c_\sigma \in (0, 1] \), the parameter sequences generated as in (12), similar to [15], satisfy Assumption III.1 when \( t^k = \frac{\sigma^k}{\sigma_i^p} \), \( \alpha^k = \frac{c_\sigma}{m^\alpha \sigma^k} \) and \( \beta^k = 0 \) for \( k \geq 0 \).

**Remark III.3.** Suppose that \( \Phi(x, y) \) in (1) has a separable structure in blocks of \( x \), i.e., \( \Phi(x, y) = \sum_{i \in M} \Phi_i(x_i, y_i) \), and \( \mathbb{P} \) is a discrete distribution over \( M \) such that \( \mathbb{P}(i) = p_i > 0 \) for \( i \in M \) and \( \sum_{i \in M} p_i = 1 \) for some \( \{p_i\}_{i \in M} \). The results of Theorem III.1 continue to hold with slightly different \( O(1) \) constants depending on \( \mathbb{P} \), if at each iteration \( k \geq 1 \), one samples index \( i_k \in M \) (line 6 of RAPD) according to \( \mathbb{P} \).

Consider the constrained optimization problem in (2) and suppose that a dual bound \( B > 0 \) as in Remark II.1 is known. The primal-dual iterate sequence generated by RPDA applied to SP formulation with \( \Phi(x, y) = g(x) + \langle G(x), y \rangle \) and \( h(y) = \mathbb{I}_{K^* \cap B}(y) \) converges as in (10) and as in (11) under the premises of Part I and Part II, respectively, when the algorithm parameter sequences \( \{\theta^k\}_k, \{\sigma^k\}_k \) and \( \{\tau^k_i\}_k \) for \( i \in M \) are chosen as described in Part I and Part II of Theorem III.1 and when the constants are set as \( L_{yy} = 0, L_{yx_i} = C_i(G) \) and \( L_{xix_i} = L_i(G) + B_Li(G) \) for \( i \in M \), where \( B = \{y : \|y\|_Y \leq B\} \), \( C_i(G) \) and \( L_i(G) \) denote the coordinate-wise Lipschitz constants of \( G \) and the Jacobian of \( G \), respectively, for all \( i \in M \).

In particular, when \( g(x) = \sum_{i \in M} g_i(x_i) \) and \( G(x) = \sum_{i \in M} G_i(x_i) \), Remark III.3 implies that if one samples index \( i_k \in M \) (line 6 of RAPD) according to \( \mathbb{P} \) at each iteration \( k \geq 1 \), then for any \( \alpha > 0 \), the RAPD iterate sequence generated using \( \theta^k = 1 \) and

\[
\tau^k_i = \left( L_i(G) + B_Li(G) + C_i(G)^2/(p_i m\alpha) \right)^{-1}, \quad i \in M, \quad \sigma^k = \frac{1}{m\alpha}, \quad k \geq 0
\]

satisfies the following error bound on suboptimality and infeasibility for all \( K \geq 1 \):

\[
\max \left\{ E \left[ \rho(\bar{x}^K) - \rho(x^*) \right], \quad E \left[ d_{\mathbb{P}}^2(K, \sum_{i \in M} G_i(x^*_K)) \right] \right\} \leq \frac{1}{K} \left( \frac{1}{2} \|x^* - x^0\|_{\mathbb{P}^{-1}T}^2 + m\alpha D_{\mathbb{P}}(y^*, y^0) \right), \quad (13)
\]
where \( \mathbf{P} = \text{diag} ([p_i]_{i \in \mathcal{M}}) \), \( \mathbf{T} = \text{diag} ([\frac{1}{\tau_i}]_{i \in \mathcal{M}}) \), and \( d_{-\mathcal{K}}(\mathbf{u}) = \| \mathbf{u} - \Pi_{-\mathcal{K}}(\mathbf{u}) \| \).

Moreover, if \( f_i + g_i \) in (2) is strongly convex with modulus \( \mu_i > 0 \) for each \( i \in \mathcal{M} \), then Remark III.3 implies that when \( i_k \in \mathcal{M} \) is sampled according to \( \mathbb{P} \) at each iteration \( k \geq 1 \), the RAPD iterate sequence generated using the update rule:

\[
\tau_{ik}^k \leftarrow \left( \mu_i p_i (1 + \frac{1}{\tau_{ik}}) - \mu_i \right)^{-1}, \quad \theta_{k+1} \leftarrow \frac{1}{\sqrt{1 + \tau_k}}, \quad \sigma_{k+1} \leftarrow \frac{\sigma_k}{\theta_{k+1}}, \quad \tilde{\tau}_{k+1} \leftarrow \theta_{k+1} \tilde{\tau}_k,
\]

where \( \tilde{\tau}_0 = \min_{i \in \mathcal{M}} \mu_i p_i \left( L_i(g) + B L_i(G) + C_i(G)^2 \right) / \max \left( 1 - p_i, \mu_i \right) \) and \( \sigma_0 = 1 / (m \alpha) \) for any \( \alpha > 0 \), satisfies the following error bound on the primal iterates for all \( K \geq 1 \),

\[
\mathbf{E} \left[ \| x^k - x^* \|^2 \right] \leq \mathcal{O}(m/K).
\]

### A. Convergence proof

In order to prove the main result stated in Theorem III.1 we first state a one-step bound on the Lagrangian error metric in the following lemma which is the main building block of deriving rate results in Theorem III.1 for the proposed RAPD method.

**Lemma III.2.** (One-step analysis) Let \( \{ x^k, y^k \}_{k \geq 0} \) be the sequence generated by RAPD, stated in Algorithm 1 initializing from arbitrary \( x^0 \in \mathcal{X} \) and \( y^0 \in \mathcal{Y} \). Suppose Assumption II.1 holds and the parameter sequence \( \{ \{ \tau^k_i \}_{i \in \mathcal{M}}, \sigma^k, \theta^k, t^k \}_{k \geq 0} \) satisfies Assumption III.1. Then for any \( z = (x, y) \in \mathcal{X} \times \mathcal{Y} \), the following inequality holds for any \( k \geq 0 \),

\[
\begin{align*}
\mathbf{E}^k[\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1})] & \leq Q^k(z) - \mathbf{E}^k[R^{k+1}(z)] + (m - 1)(H^k(z) - \mathbf{E}^k[H^{k+1}(z)]), \\
& \quad + (m - 1)(1 - \theta^k) (f(x^k) - f(x) + \phi(x^k, y) - \phi(x, y)) \quad (14a) \\
Q^k(z) & \triangleq \left[ \frac{m}{2} \left\| x - x^k \right\|^2_{\mathbf{T}^k + (1 - \frac{1}{m}) \mathbf{M}} + \left( \frac{1}{\sigma^k} + (m - 1)(1 - \theta^k) L_{yy} \right) D_y(y, y^k) \right] \\
& \quad + \theta^k \left< r^k, y^k - y \right> + \theta^k m \left[ \frac{L_y^2}{\beta_y} D_y(y^k, y^{k-1}) + \left( \frac{m}{2} \left\| x^k - x^{k-1} \right\|^2_{L^2_x} + \frac{\beta_k}{\alpha_k} \left\| x^k - x^{k-1} \right\|_{L^\infty_x} \right) \right] \quad (14b) \\
R^{k+1}(z) & \triangleq \left[ \frac{m}{2} \left\| x - x^{k+1} \right\|^2_{\mathbf{T}^k + \mathbf{M}} + \frac{1}{\sigma^k} D_y(y, y^{k+1}) + \left< r^{k+1}, y^{k+1} - y \right> \right] \\
& \quad + \frac{m}{2} \left\| x^{k+1} - x^k \right\|^2_{\mathbf{T}^k - \mathbf{L}^\infty_x} + \left( \frac{1}{\sigma^k} - \theta^k \frac{m(\alpha_k + \beta_k)}{\beta_k} \right) D_y(y^{k+1}, y^k) \left( x^{k+1}, y^{k+1} \right) \right], \quad (14c)
\end{align*}
\]

where \( H^k(z) \triangleq f(x^k) - f(x) + \phi(x^k, y^k) - \phi(x, y) \) and \( r^k \triangleq \nabla_y \phi(x^k, y^k) - m \nabla_y \phi(x^{k-1}, y^{k-1}) \) for any \( k \geq 0 \).

**Proof.** We define an auxiliary sequence \( \{ \tilde{x}^k \}_{k \geq 1} \subseteq \mathcal{X} \) such that for \( k \geq 0 \),

\[
\tilde{x}_{i}^{k+1} \triangleq \arg\min_{x_i \in X_i} f_i(x_i) + \left< \nabla_{x_i} \phi(x^k, y^{k+1}), x_i \right> + \frac{1}{2 \tau_i} \left\| x_i - x_i^k \right\|^2, \quad i \in \mathcal{M}. \quad (15)
\]
The auxiliary sequence \( \{\tilde{x}^k\}_{k \geq 1} \) is never actually computed in the implementation of the algorithm \( \text{RPDA} \) and it is defined for analyzing the convergence behavior of \( \{x^k\}_{k \geq 1} \subseteq \mathcal{X} \). We will use three-point inequality for Bregman functions given in Lemma \( \text{V.1} \) provided in the appendix.

For \( k \geq 0 \), when Lemma \( \text{V.1} \) is applied to the \( \tilde{x}_i \)-subproblem in (15) and the \( y \)-subproblem in Line 5 of \( \text{RAPD} \) algorithm, we obtain two inequalities that hold for any \( y \in \mathcal{Y} \) and \( x \in \mathcal{X} \):

\[
\begin{align*}
    h(y^{k+1}) - \langle s_k, y^{k+1} - y \rangle &\leq h(y) + \frac{1}{\sigma^k} \left[ D_{\mathcal{Y}}(y, y^k) - D_{\mathcal{Y}}(y, y^{k+1}) - D_{\mathcal{Y}}(y^{k+1}, y^k) \right], \\
    f_i(\tilde{x}^{k+1}_i) + \langle \nabla x, \Phi(x^k, y^{k+1}), \tilde{x}^{k+1}_i - x_i \rangle + \frac{\mu_i}{2} \left\| x_i - \tilde{x}^{k+1}_i \right\|^2
    &\leq f_i(x) + \frac{1}{2\tau_i^k} \left[ \left\| x_i - x_i^k \right\|^2 - \left\| x_i - \tilde{x}^{k+1}_i \right\|^2 - \left\| \tilde{x}^{k+1}_i - x_i^k \right\|^2 \right], \quad \forall i \in \mathcal{M}.
\end{align*}
\]  

(16)

(17)

Note that by invoking (7), we may bound the inner product in (17) as follows,

\[
\begin{align*}
    \langle \nabla x, \Phi(x^k, y^{k+1}), \tilde{x}^{k+1}_i - x_i \rangle &\geq \Phi(x^k + U_i(\tilde{x}^{k+1}_i - x_i^k), y^{k+1}) - \Phi(x^k, y^{k+1}) \\
    &\quad - \frac{1}{2} J_{x,x_i} \left\| \tilde{x}^{k+1}_i - x_i^k \right\|^2 + \langle \nabla x, \Phi(x^k, y^{k+1}), x_i^k - x_i \rangle.
\end{align*}
\]  

(18)

Next, we define two auxiliary sequences for \( k \geq 0 \):

\[
\begin{align*}
    A^{k+1} &\triangleq \frac{1}{\sigma^k} D_{\mathcal{Y}}(y, y^k) - \frac{1}{\sigma^k} D_{\mathcal{Y}}(y, y^{k+1}) - \frac{1}{\sigma^k} D_{\mathcal{Y}}(y^{k+1}, y^k), \\
    B^{k+1} &\triangleq \frac{1}{2} \left\| x - x^k \right\|^2_{\mathcal{T}^k} - \frac{1}{2} \left\| x - x^{k+1} \right\|^2_{\mathcal{T}^{k+1} + \mathcal{M}} - \frac{1}{2} \left\| x^{k+1} - x^k \right\|^2_{\mathcal{T}^k}.
\end{align*}
\]  

(19a)

(19b)

By substituting (18) in (17), summing over \( i \in \mathcal{M} \), and then using the convexity of \( \Phi(x, y^{k+1}) \) in \( x \), we obtain

\[
f(\tilde{x}^{k+1}) + \sum_{i \in \mathcal{M}} \Phi(x^k + U_i(\tilde{x}^{k+1}_i - x_i^k), y^{k+1}) \leq \sum_{i \in \mathcal{M}} \left[ f(x) + \Phi(x^k, y^{k+1}) + (m - 1) \Phi(x^k, y^{k+1}) + \frac{1}{2} \left\| \tilde{x}^{k+1}_i - x_i^k \right\|^2_{\mathcal{L}_{xx}} + B^{k+1} \right].
\]  

(20)

Note that for any \( y \in \mathcal{Y} \), we have \( f(\tilde{x}^{k+1}) = m\mathbb{E}^k[f(x^{k+1})] - (m - 1)f(x^k) \) and

\[
\mathbb{E}^k[\Phi(x^{k+1}, y)] = \frac{1}{m} \sum_{i \in \mathcal{M}} \Phi(x^k + U_i(\tilde{x}^{k+1}_i - x_i^k), y).
\]

Therefore, multiplying (20) by \( \frac{1}{m} \) and adding \( f(x) \) to both sides, it can be simplified as follows:

\[
\begin{align*}
    \mathbb{E}^k[f(x^{k+1}) + \Phi(x^{k+1}, y^{k+1})] - f(x) - \frac{1}{m} \Phi(x, y^{k+1}) &\leq \\
    \left( 1 - \frac{1}{m} \right) [f(x^k) - f(x) + \Phi(x^k, y^{k+1})] + \frac{1}{2m} \left\| \tilde{x}^{k+1} - x^k \right\|^2_{\mathcal{L}_{xx}} + \frac{1}{m} B^{k+1}.
\end{align*}
\]  

(21)
First, multiplying (16) by $\frac{1}{m}$ and adding to (21), then adding $\mathbb{E}^k[\Phi(x^{k+1}, y)]/m$ to the both sides, and rearranging the terms leads to

\[
\frac{1}{m} \left( \mathbb{E}^k[f(x^{k+1})] - f(x) + \mathbb{E}^k[\Phi(x^{k+1}, y)] \right) + \frac{1}{m} \left( h(y^{k+1}) - h(y) - \Phi(x, y^{k+1}) \right) \\
\leq \frac{1}{m} \left( \mathbb{E}^k[\Phi(x^{k+1}, y)] - \mathbb{E}^k[\Phi(x^{k+1}, y^{k+1})] \right) + \frac{1}{2m} \| \tilde{x}^{k+1} - x^k \|^2_{L_{xx}} \\
+ \left( 1 - \frac{1}{m} \right) \left[ f(x^k) - f(x) + \Phi(x^k, y^{k+1}) \right] \\
- \left( 1 - \frac{1}{m} \right) \left( \mathbb{E}^k[f(x^{k+1})] - f(x) + \mathbb{E}^k[\Phi(x^{k+1}, y^{k+1})] \right) \\
+ \frac{1}{m} B^{k+1} + \frac{1}{m} A^{k+1} + \frac{1}{m} \langle s^k, y^{k+1} - y \rangle \\
\leq \frac{1}{m} \mathbb{E}^k[(\nabla_y \Phi(x^{k+1}, y^{k+1}), y - y^{k+1})] + \frac{1}{2m} \| \tilde{x}^{k+1} - x^k \|^2_{L_{xx}} \\
+ (1 - \frac{1}{m}) \left[ f(x^k) - f(x) + \Phi(x^k, y^{k+1}) - \Phi(x, y) \right] \\
- \left( 1 - \frac{1}{m} \right) \left( \mathbb{E}^k[f(x^{k+1})] - f(x) + \mathbb{E}^k[\Phi(x^{k+1}, y^{k+1})] - \Phi(x, y) \right) \\
+ \frac{1}{m} B^{k+1} + \frac{1}{m} A^{k+1} + \frac{1}{m} \langle s^k, y^{k+1} - y \rangle, \tag{22}
\]

where in the last inequality, we use the concavity for $\Phi(x^{k+1}, y)$ in $y$. Next, we bound $(\ast)$ in (22) by first splitting it into $\theta^k \in [0, 1]$ and $1 - \theta^k$ fractions, and then using concavity and (3):

\[
(\ast) = (1 - \frac{1}{m}) \theta^k \left[ f(x^k) - f(x) + \Phi(x^k, y^{k+1}) - \Phi(x, y) \right] \\
+ (1 - \frac{1}{m})(1 - \theta^k) \left[ f(x^k) - f(x) + \Phi(x^k, y^{k+1}) - \Phi(x, y) \right] \\
\leq (1 - \frac{1}{m}) \theta^k H^k(z) + (1 - \frac{1}{m})(1 - \theta^k) \left[ f(x^k) - f(x) + \Phi(x^k, y) - \Phi(x, y) + \frac{L_{yy}}{2} \| y - y^k \|^2 \right] \\
+ (1 - \frac{1}{m}) \theta^k \langle \nabla_y \Phi(x^k, y^k), y - y^k \rangle + (1 - \frac{1}{m}) \langle \nabla_y \Phi(x^k, y^k), y^{k+1} - y \rangle. \tag{23}
\]

Now before combining (23) with (22), we first simplify the summation of all inner product terms coming from both inequalities. Recall that $s^k = \nabla_y \Phi(x^k, y^k) + \theta^k q^k$ where $q^k \triangleq m(\nabla_y \Phi(x^k, y^k) - \nabla_y \Phi(x^{k-1}, y^{k-1}))$ and note that $r^k = q^k - (m - 1)\nabla_y \Phi(x^k, y^k)$ for all $k \geq 0$. Using these
definitions, we can rearrange the sum of all inner products in (23) and (22) as follows:

\[
\frac{1}{m} E^k \left[ \langle \nabla_y \Phi(x^{k+1}, y), y - y^{k+1} \rangle \right] + (1 - \frac{1}{m}) \theta^k \langle \nabla_y \Phi(x^k, y), y - y^k \rangle \\
+ (1 - \frac{1}{m}) \langle \nabla_y \Phi(x^k, y^k), y^{k+1} - y \rangle + \frac{1}{m} \langle s^k, y^{k+1} - y \rangle \\
= E^k \left[ \langle \frac{1}{m} \nabla_y \Phi(x^{k+1}, y^{k+1}) - \nabla_y \Phi(x^k, y^k), y - y^{k+1} \rangle \right] \\
+ (1 - \frac{1}{m}) \theta^k \langle \nabla_y \Phi(x^k, y), y - y^k \rangle + \frac{\theta^k}{m} \langle q^k, y^k - y \rangle + \frac{\theta^k}{m} \langle q^k, y^{k+1} - y^k \rangle \\
= -\frac{1}{m} E^k \left[ \langle r^{k+1}, y^{k+1} - y \rangle \right] + \frac{\theta^k}{m} \langle r^k, y^k - y \rangle + \frac{\theta^k}{m} \langle q^k, y^{k+1} - y^k \rangle. \tag{24}
\]

Hence, using (23) and (24) within (22), we obtain the following inequality:

\[
\frac{1}{m} E^k [\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1})] \leq \\
\frac{1}{m} \left( \frac{1}{2} \left\| \bar{x}^{k+1} - x^k \right\|_{L_{xx}}^2 + B^{k+1} + A^{k+1} \right) \\
- E^k \left[ \langle r^{k+1}, y^{k+1} - y \rangle \right] + \theta^k \left( \langle r^k, y^k - y \rangle + \theta^k \langle q^k, y^{k+1} - y^k \rangle \right) \\
+ (1 - \frac{1}{m})(1 - \theta^k) \left( f(x^k) - f(x) + \Phi(x^k, y) - \Phi(x, y) + L_{y y} D_y(y, y^k) \right) \\
+ \left( 1 - \frac{1}{m} \right) \left( \theta^k H^k(z) - E^k [H^{k+1}(z)] \right), \tag{25}
\]

where we also used the fact that \( D_y(y, \bar{y}) \geq \frac{1}{2} \| y - \bar{y} \|^2 \). Note that random variable \( x^{k+1} \)

is different from \( x^k \) only in the \( i_k \)-th block. Hence, for any \( \bar{x} \in \mathcal{X} \) and diagonal matrix \( D = \text{diag}(\{d_i\}_{i \in M}) \),

\[
E^k \left[ \left\| x^{k+1} - \bar{x} \right\|^2_D \right] = \frac{1}{m} \left[ \left\| \bar{x}^{k+1} - \bar{x} \right\|^2_D + (1 - \frac{1}{m}) \left\| x^k - \bar{x} \right\|^2_D \right]. \tag{26}
\]

Thus, one can conclude that \( \left\| \bar{x}^{k+1} - x^k \right\|^2_{L_{xx}} = m E^k \left[ \left\| x^{k+1} - x^k \right\|^2_{L_{xx}} \right] \) by setting \( \bar{x} = x^k \) and

\( D = L_{xx} \) in (26). Similarly, using (26), it is easy to verify that \( B^{k+1} \) defined in (19b) satisfies

\[
B^{k+1} = \frac{m}{2} \left\| x^k - x \right\|^2_{T^k + (1 - \frac{1}{m})M} - \frac{m}{2} E^k \left[ \left\| x^{k+1} - x \right\|^2_{T^{k+M}} + \left\| x^{k+1} - x^k \right\|^2_{T_k} \right]. \tag{27}
\]

Replacing (27) in (25), rearranging the terms and multiplying both sides by \( m \), we get the
following inequality:

\[
\mathbb{E}^k[\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1})] \\
\leq \left[ \frac{m}{2} \|x - x^k\|^2_{T^k + (1 - \frac{1}{m})M} + \frac{1}{\sigma_k^2} (m - 1)(1 - \theta_k) L_{yy} \right] \mathcal{D}_y(y, y^k) + \theta_k \langle r^k, y^k - y \rangle \\
- \mathbb{E}^k \left[ \frac{m}{2} \|x - x^{k+1}\|^2_{T^k + M} + \frac{1}{\sigma_k^2} \mathcal{D}_y(y, y^{k+1}) + \langle r^{k+1}, y^{k+1} - y \rangle \\
+ \frac{m}{2} \|x^{k+1} - x^k\|^2_{T^k - L_{xx}} + \frac{1}{\sigma_k^2} \mathcal{D}_y(y^{k+1}, y^k) \right] \\
+ (m - 1)(1 - \theta_k) \left( f(x^k) - f(x) + \Phi(x^k, y) - \Phi(x, y) \right) \\
+ (m - 1) \left( \theta_k H^k(z) - \mathbb{E}^k[H^{k+1}(z)] \right) + \theta_k \langle q^k, y^{k+1} - y \rangle.
\]

(28)

Finally, the last term (**) in (28) can be bounded using the fact that for any \( y \in \mathcal{Y} \), \( y' \in \mathcal{Y}^* \), and \( \eta > 0 \), we have \( \langle y', y \rangle \leq \frac{\eta}{2} \|y\|_Y^2 + \frac{1}{2\eta} \|y'\|_Y^2 \). Using the previous inequality together with (5) and the fact that \( \mathcal{D}_y(y, \bar{y}) \geq \frac{1}{2} \|y - \bar{y}\|_Y^2 \), one can obtain:

\[
(**) = m \langle \nabla_y \Phi(x^k, y^k) - \nabla_y \Phi(x^k, y^{k-1}), y^{k+1} - y^k \rangle \\
+ m \langle \nabla_y \Phi(x^k, y^{k-1}) - \nabla_y \Phi(x^{k-1}, y^{k-1}), y^{k+1} - y^k \rangle \\
\leq m \frac{L_{yy}^2}{\beta_k^2} \mathcal{D}_y(y^{k+1}, y^k) + m(\alpha_k + \beta_k) \mathcal{D}_y(y^{k+1}, y^k) + \frac{m}{2\alpha_k} \|x^k - x^{k-1}\|^2_{L_{xx}^*}
\]

(29)

which is true for any \( \alpha_k, \beta_k > 0 \) - (29) also holds for \( \beta_k = 0 \) when \( L_{yy} = 0 \). The desired result follows from using (29) within (28).

\[ \Box \]

**Proof of Theorem III.1** Here we prove Part I and Part II of the theorem.

(1) Let \( z = (x, y) \in \mathcal{X} \times \mathcal{Y} \) be an arbitrary point. Setting \( \theta_k = 1 \) within the result of Lemma III.2 we get the following inequality for \( k \geq 0 \):

\[
\mathbb{E}^k[\mathcal{L}(x^{k+1}, y) - \mathcal{L}(x, y^{k+1})] \leq Q^k(z) - \mathbb{E}^k[R^{k+1}(z)] + (m - 1)(H^k(z) - \mathbb{E}^k[H^{k+1}(z)]),
\]

(30a)

\[
Q^k(z) \triangleq \left[ \frac{m}{2} \|x - x^k\|^2_{T^k + (1 - \frac{1}{m})M} + \frac{1}{\sigma_k^2} \mathcal{D}_y(y, y^k) \right] \\
+ \langle r^k, y^k - y \rangle + \frac{m}{2\alpha_k} \|x^k - x^{k-1}\|^2_{L_{xx}^*} + \frac{mL_{yy}^2}{\beta_k^2} \mathcal{D}_y(y^k, y^{k-1})
\]

(30b)

\[
R^{k+1}(z) \triangleq \left[ \frac{m}{2} \|x - x^{k+1}\|^2_{T^k + M} + \frac{1}{\sigma_k^2} \mathcal{D}_y(y, y^{k+1}) + \langle r^{k+1}, y^{k+1} - y \rangle \right] \\
+ \frac{m}{2} \|x^{k+1} - x^k\|^2_{T^k - L_{xx}} + \left( \frac{1}{\sigma_k} - m(\alpha_k + \beta_k) \right) \mathcal{D}_y(y^{k+1}, y^k)
\]

(30c)

\[
H^k(z) \triangleq f(x^k) - f(x) + \Phi(x^k, y^k) - \Phi(x, y)
\]

(30d)

Now we take the unconditional expectation of (30a), and then step-size conditions (III.1) implies
that $\mathbb{E}[R^K(z) - Q^K(z)] \geq 0$, for $k \geq 1$; therefore, summing over $k = 0$ to $K - 1$ in (30a) and using Jensens’s inequality leads to

\[
K(\mathbb{E}[\mathcal{L}(\bar{x}^K, y) - \mathcal{L}(x, \bar{y}^K)]) \leq 
\]

\[
Q^0(z) - \mathbb{E}[R^K(z)] + (m - 1)(H^0(z) - \mathbb{E}[H^K(z)])
\]

\[
= \frac{m}{2} \|x - x^0\|^2_{\mathbb{T}^{+1-(1 - \frac{1}{m})M}} + \frac{1}{\sigma^0} \mathcal{D}_y(y, y^0) - \langle x^K, y^K - y \rangle - (m - 1) \langle \nabla_y \Phi(x^0, y^0), y^0 - y \rangle
\]

\[
- \mathbb{E} \left[ \frac{m}{2} \|x - x^K\|^2_{\mathbb{T}^{K-1+M}} + \frac{1}{\sigma_{K-1}} \mathcal{D}_y(y, y^K) + \frac{m}{2} \|x^K - x^{K-1}\|^2_{\mathbb{T}^{K-1-L_{xx}}} + \frac{mL^2_{yy}}{\beta_K} \mathcal{D}_y(y^K, y^{K-1}) \right]
\]

\[
+ \left( \frac{1}{\sigma_{K-1}} - m(\alpha_{K-1} + \beta_{K-1}) \right) \mathcal{D}_y(y^K, y^{K-1})
\]

\[
+ (m - 1)[H^0(z) - \mathbb{E}[H^K(z)]]. \tag{31}
\]

where in last equality we used the fact that initializing $x^0 = x^{-1}$ and $y^0 = y^{-1}$ implies $q^0 = 0$ and $\|x_0 - x^{-1}\|^2 = \mathcal{D}_y(y_0, y^{-1}) = 0$. Furthermore, using concavity of $\Phi(x^k, y)$ in $y$ and using (30c) for $k = K$ we have that,

\[
H^K(z) \geq f(x^K) - f(x) + \Phi(x^K, y) - \Phi(x, y) + \langle \nabla_y \Phi(x^K, y^K), y^K - y \rangle \tag{32}
\]

Combining the inner product term in (32) with (*** ) lead to $\langle q^K, y - y^K \rangle$ which can be bounded similar to (29) with $k = K$ and replacing $y^{K+1}$ with $y$ in (29). Now we can combine this upper bound with (31) to get

\[
K(\mathbb{E}[\mathcal{L}(\bar{x}^K, y) - \mathcal{L}(x, \bar{y}^K)]) \leq \frac{m}{2} \|x - x^0\|^2_{\mathbb{T}^{+1-(1 - \frac{1}{m})M}} + \frac{1}{\sigma^0} \mathcal{D}_y(y, y^0) - \langle x^K, y^K - y \rangle - (m - 1) \langle \nabla_y \Phi(x^0, y^0), y^0 - y \rangle
\]

\[
+ \left( \frac{1}{\sigma_{K-1}} - m(\alpha_{K-1} + \beta_{K-1}) \right) \mathcal{D}_y(y^K, y^{K-1})
\]

\[
+ (m - 1)[H^0(z) - \mathbb{E}[f(x^K) - f(x) + \Phi(x^K, y) - \Phi(x, y)]]. \tag{33}
\]

where in the last inequality we used the step-size condition (9a) for $k = K - 1$.

Finally, suppose $(x^*, y^*)$ is a saddle point of (1), and let $x = x^*$ and $y = y^*$ in (33). From
first-order optimality condition of (1) one can conclude that \(-\nabla x \Phi(x^*, y^*) \in \partial f(x^*)\) which then using convexity of \(f(x)\) and \(\Phi(x, y)\) in \(x\) we obtain,

\[ 0 \leq f(x^K) - f(x^*) + \langle \nabla_x \Phi(x^*, y^*), x^K - x^* \rangle \leq f(x^K) - f(x^*) + \Phi(x^K, y^*) - \Phi(x^*, y^*). \] (34)

Therefore, \(f(x^K) - f(x^*) + \Phi(x^K, y^*) - \Phi(x^*, y^*)\) is a non-negative term which can be dropped. Moreover, using Lipschitz bound \(L\) we have that

\[ H^0(z) - \langle \nabla_y \Phi(x^0, y^0), y^0 - y \rangle \leq f(x^0) - f(x) + \Phi(x^0, y) - \Phi(x, y) + \frac{L_{yy}}{2} \|y - y^0\|^2. \] (35)

The result immediately follows from combining (34) and (35) with (33).

Next, we will show the almost sure convergence of the actual iterate sequence for Part I. Lemma \(V.2\) in the appendix will be used to establish the convergence of the primal-dual iterate sequence. To fix the notation, suppose \(w : \Omega \to \mathbb{R}\) is a random variable, \(w(\omega)\) denotes a particular realization of \(w\) corresponding to \(\omega \in \Omega\) where \(\Omega\) denotes the sample space.

Suppose \(k^1 = 1, \theta^k = 1, \alpha^k = \alpha, \beta^k = L_{yy}\) for all \(k \geq 0\) for some \(\alpha > 0\) such that the conditions in (9a) hold with strict inequality. More precisely, since we choose \(c_r, c_\sigma \in (0, 1)\), it follows that for some \(\delta > 0\),

\[ T^0 \geq \delta I + L_{xx} + \frac{L_{yx}}{\alpha}, \quad \frac{1}{\sigma_0} \geq \delta + m(\alpha + 2L_{yy}). \] (36)

Suppose \(z^k = (x^#, y^#)\) is a saddle point \(L\) in (1), and set \(x = x^#\) and \(y = y^#\) in (30). Then, (36) implies the following inequalities:

\[ R^k(z^#) \geq Q^k(z^#) + \delta mD_X(x^k, x^{k-1}) + \delta D_Y(y^k, y^{k-1}), \] (37a)

\[ (m - 1)H^k(z^#) + R^k(z^#) \geq (m - 1)L^k_j(x^#, y^#) + \sum_{i \in M} \frac{1}{\tau_i^0} D_X(x_i^#, x_i^k) + \frac{1}{\sigma_0} - m(\alpha + L_{yy})D_Y(y^#, y^k) \geq \delta D_X(x^#, x^k) + \delta D_Y(y^#, y^k) \geq 0. \] (37b)

Using \(L(x^{k+1}, y^#) - L(x^#, y^{k+1}) \geq 0\), the inequalities (30a) and (37a) lead to

\[ \mathbb{E}^k[R^{k+1}(z^#) + (m - 1)H^{k+1}(z^#)] \leq \]

\[ R^k(z^#) + (m - 1)H^k(z^#) - \delta (mD_X(x^k, x^{k-1}) + D_Y(y^k, y^{k-1})). \] (38)

Let \(a^k = (m - 1)H^k(z^#) + R^k(z^#), b^k = \delta (mD_X(x^k, x^{k-1}) + D_Y(y^k, y^{k-1}))\), and \(c^k = 0\) for
\( k \geq 0 \), then Lemma \[ \text{V.2} \] implies that \( \lim_{k \to \infty} a^k \) exists almost surely. Therefore, (37b) implies that \( \{z^k(\omega)\} \) is a bounded sequence for any \( \omega \in \Omega \); hence, it has a convergent subsequence \( z^{k_n}(\omega) \to z^*(\omega) \) as \( n \to \infty \) for some \( z^*(\omega) \in X \times Y \) – note that \( k_n \) also depends on \( \omega \) which is omitted to simplify the notation. Define \( z^* = (x^*, y^*) \) such that \( z^* = [z^*(\omega)]_{\omega \in \Omega} \). Lemma \[ \text{V.2} \] also implies that \( \sum_{k=0}^{\infty} b^k < \infty \); hence, for any realization \( \omega \in \Omega \) and \( \epsilon > 0 \), there exists \( N_1(\omega) \) such that for any \( n \geq N_1(\omega) \), we have \( \max\{|z^{k_n}(\omega) - z^{k_n-1}(\omega)|, |z^{k_n}(\omega) - z^{k_n+1}(\omega)|\} < \frac{\epsilon}{z} \). Convergence of \( \{z^{k_n}(\omega)\} \) sequence also implies that there exists \( N_2(\omega) \) such that for any \( n \geq N_2(\omega), \|z^{k_n}(\omega) - z^*(\omega)\| < \frac{\epsilon}{z} \). Therefore, for \( \omega \in \Omega \), letting \( N(\omega) \overset{\Delta}{=} \max\{N_1(\omega), N_2(\omega)\} \), we conclude that \( \|z^{k_n+1}(\omega) - z^*(\omega)\| < \epsilon \), i.e., \( z^{k_n+1} \to z^* \) almost surely as \( n \to \infty \).

Now we show that for any \( \omega \in \Omega \), \( z^*(\omega) \) is indeed a saddle point of (1) by considering the optimality conditions for Line 5 and Line 6 of the RAPD Algorithm. In particular, fix an arbitrary \( \omega \in \Omega \) and consider the subsequence \( \{k_n\}_{n \geq 1} \). For all \( n \in \mathbb{Z}_+ \), one can conclude that

\[
\frac{1}{\sigma^0} \left( \nabla \varphi_Y(y_i^k(\omega)) - \nabla \varphi_Y(y_{i+1}^k(\omega)) \right) + s^k(\omega) \in \partial h(y_{i+1}^k(\omega)),
\]

\[
\frac{1}{\tau_{i_{k_n}}} \left( \nabla \varphi_X(x_i^k(\omega)) - \nabla \varphi_X(x_{i+1}^k(\omega)) \right) - \nabla x_i \Phi(x_i^k(\omega), y_{i+1}^k(\omega)) \in \partial f_i(x_{i+1}^k(\omega)).
\]

Note that the sequence of randomly chosen block coordinates in RPDA, i.e., \( \{i_{k_n}\}_{n \geq 1} \), is a Markov chain containing a single recurrent class. More specifically, the states are represented by \( \mathcal{M} \) and starting from state \( i \in \mathcal{M} \) the probability of eventually returning to state \( i \) is strictly positive for all \( i \in \mathcal{M} \). Therefore, for any \( i \in \mathcal{M} \), we can select a further subsequence \( \mathcal{K}^i \subseteq \{k_n\}_{n \in \mathbb{Z}_+} \) such that \( i_{\ell} = i \) for all \( \ell \in \mathcal{K}^i \). Note that \( \mathcal{K}^i \) is an infinite subsequence w.p. 1 and \( \{\mathcal{K}^i\}_{i \in \mathcal{M}} \) is a partition of \( \{k_n\}_{n \in \mathbb{Z}_+} \). For any \( i \in \mathcal{M} \), one can conclude from (39a) and (39b) that for all \( \ell \in \mathcal{K}^i \),

\[
\frac{1}{\sigma^0} \left( \nabla \varphi_Y(y_i^\ell(\omega)) - \nabla \varphi_Y(y_{i+1}^\ell(\omega)) \right) + s^\ell(\omega) \in \partial h(y_{i+1}^\ell(\omega)),
\]

\[
\frac{1}{\tau_i} \left( \nabla \varphi_X(x_i^\ell(\omega)) - \nabla \varphi_X(x_{i+1}^\ell(\omega)) \right) - \nabla x_i \Phi(x_i^\ell(\omega), y_{i+1}^\ell(\omega)) \in \partial f_i(x_{i+1}^\ell(\omega)).
\]

Moreover, since \( \mathcal{K}^i \subseteq \{k_n\}_{n \in \mathbb{Z}_+} \), we have that \( \lim_{\ell \in \mathcal{K}^i} z^\ell(\omega) = \lim_{\ell \in \mathcal{K}^i} z^{\ell+1}(\omega) = z^*(\omega) \), and using the fact that for any \( i \in \mathcal{M} \), \( \nabla \varphi_X \) and \( \nabla \varphi_Y \) are continuously differentiable on \( \text{dom} f_i \) and \( \text{dom} h \), respectively, it follows from Theorem 24.4 in [34] that by taking the limit of both sides of (40), we get \( 0 \in \nabla x_i \Phi(x_i^*(\omega), y_i^*(\omega)) + \partial f_i(x_i^*(\omega)) \), and \( 0 \in \partial h(y_i^*(\omega)) - \nabla y \Phi(x_i^*(\omega), y_i^*(\omega)) \), which implies that \( z^*(\omega) \) is a saddle point of (1) for any \( \omega \in \Omega \).

Finally, since (37) and (38) are true for any saddle point \( z^\# \), letting \( z^\# = z^* \) and invoking Lemma \[ \text{V.2} \] again, one can conclude that \( w^* = \lim_{k \to \infty} w^k \geq 0 \) exists almost surely, where
Therefore, these properties imply

\[ \text{implies} \ (1) \]

we may conclude that

\[ \text{E} \left[ \left( \frac{42a}{42k} \right) \right] \]

Let \( \mathcal{L}^k_f(x, y) = \frac{42a}{42k} f(x^k) - f(x) + \frac{42a}{42k} \Phi(x, y) \) and \( \mathcal{L}^k_h(x, y) = \frac{42a}{42k} \Phi(x, y) + \frac{42a}{42k} \Phi(x, y) \), which imply that

\[ \mathcal{L}^k_f(x, y) - \mathcal{L}^k(x, y) = \mathcal{L}^k_f(x, y) + \mathcal{L}^k_h(x, y) \]. Moreover, using (41) we obtain

\[ \mathcal{L}^k_f(x^k, y^k) \geq \frac{1}{2} \left\| x^k - x^{k+1} \right\|^2 \]

and similarly one can show \( \mathcal{L}^k(x^k, y^k) \geq 0 \).

Using \( \mathcal{L}_{yy} = 0 \) within (14) by setting \( x = x^* \) and \( y = y^* \) we obtain,

\begin{align}
\mathbb{E}[\mathcal{L}^{k+1}_f(x^*, y^*) + \mathcal{L}^{k+1}_h(x^*, y^*)] &\leq Q^k(z^*) - \mathbb{E}[R^{k+1}(z^*]) + (1 - \theta^k)(m - 1)\mathcal{L}^k_f(x^*, y^*) \\
&\quad + (m - 1)(\theta^k H^k(z) - \mathbb{E}[H^{k+1}(z)]) \tag{42a}
\end{align}

\begin{align}
Q^k(z) &\triangleq \left[ \frac{m}{2} \left\| x - x^k \right\|^2 + \frac{1}{\sigma^k} \mathcal{D}_y(y, y^k) + \left( \theta^k \alpha^k, y^k - y \right) \right. \\
&\quad + \frac{m\theta^k}{2\alpha^k} \left\| x^k - x^{k-1} \right\|^2 \left[ \mathcal{T}^{k+1} \left( \frac{1}{m} \right) \right] + \frac{1}{\sigma^k} \mathcal{D}_y(y, y^{k+1}) + \left( r^{k+1}, y^{k+1} - y \right) + \frac{m}{2} \left\| x^{k+1} - x^k \right\|^{2} \mathcal{T}^{k+1} - \mathcal{L}_{xx} \\
&\quad \left. \left( \frac{1}{\sigma^k} - m\theta^k \alpha^k \right) \mathcal{D}_y(y^{k+1}, y^k) \right] \tag{42b}
\end{align}

\begin{align}
R^{k+1}(z) &\triangleq \left[ \frac{m}{2} \left\| x - x^{k+1} \right\|^2 \mathcal{T}^{k+1} + \frac{1}{\sigma^k} \mathcal{D}_y(y, y^{k+1}) + \left( r^{k+1}, y^{k+1} - y \right) + \frac{m}{2} \left\| x^{k+1} - x^k \right\|^{2} \mathcal{T}^{k+1} - \mathcal{L}_{xx} \\
&\quad \left( \frac{1}{\sigma^k} - m\theta^k \alpha^k \right) \mathcal{D}_y(y^{k+1}, y^k) \right] \tag{42c}
\end{align}

\[ H^k(z) \triangleq f(x^k) - f(x) + \Phi(x^k, y^k) - \Phi(x, y). \tag{42d} \]

Now multiply (42a) by \( t^k \) and take unconditional expectation. From step-size conditions (9), one may conclude that \( \mathbb{E}[R^k(z) - Q^k(z)] \geq 0 \), for all \( k \geq 0 \). The fact that \( \theta^k \in [1 - \frac{1}{m}, 1] \) and \( \theta^0 = 1 \) implies \( (1 - \theta^k)(m - 1) \leq \theta^k \). Therefore, summing the result over \( k = 0 \) to \( K - 1 \) implies that

\begin{align}
\frac{t_{K-1}}{2} - \mathbb{E} \left[ \left\| x^K - x^\star \right\|^2 \right] &\leq t^{K-1} \mathbb{E} \left[ \mathcal{L}^K_f(x^K, y^k) \right] + \sum_{k=0}^{K-1} \mathbb{E} \left[ t^k \mathcal{L}^{k+1}_h(x^k, y^{k+1}) \right] \\
&\quad \leq t^0 Q^0(z^\star) - t^{K-1} \mathbb{E} \left[ R^K(z^\star) \right] \\
&\quad + (m - 1) \left( t^0 H^0(z^\star) - t^{K-1} \mathbb{E} \left[ H^K(z^\star) \right] \right). \tag{43}
\end{align}
Moreover, using concavity in $y$ and convexity in $x$ of $\Phi(x,y)$, and then (41) we can find a lower bound on $H^K(z^*)$,
\[
H^K(z^*) \geq f(x^K) - f(x^*) + \Phi(x^K, y^*) - \Phi(x^*, y^*) + \langle \nabla_y \Phi(x^K, y^K), y^K - y^* \rangle \\
\geq f(x^K) - f(x^*) + \langle \nabla_x \Phi(x^*, y^*), x^K - x^* \rangle + \langle \nabla_y \Phi(x^K, y^K), y^K - y^* \rangle \\
\geq \frac{1}{2} \|x^K - x^*\|^2_M + \langle \nabla_y \Phi(x^K, y^K), y^K - y^* \rangle. \tag{44}
\]

Hence, combining (44) with (43) we obtain,
\[
\frac{tK^{-1}}{2} E[\|x^K - x^*\|^2_M] \leq \\
\frac{mt^0}{2} \|x^* - x^0\|^2_{T_0+(1-\frac{1}{m})M} + \frac{t^0}{\sigma^0} D_y(y^*, y^0) - (m - 1)t_0 \langle \nabla_y \Phi(x^0, y^0), y^0 - y^* \rangle \\
+ tK^{-1} \langle r^K, y^* - y^K \rangle - tK^{-1} E\left[ \frac{m}{2} \|x^* - x^K\|^2_{T_{K^{-1}} + M} + \frac{1}{\sigma_{K-1}} D_y(y^*, y^K) \right] \\
+ \frac{m}{2} \|x^K - x^{K-1}\|^2_{T_{K-1} - L_{xx}} + \left( \frac{1}{\sigma_{K-1}} - m\theta K^{-1}\alpha K^{-1} \right) D_y(y^K, y^{K-1}) \\
+ (m - 1) \left( t^0 H^0(z^*) - tK^{-1} E \left[ \frac{1}{2} \|x^* - x^K\|^2_M + \langle \nabla_y \Phi(x^K, y^K), y^K - y^* \rangle \right] \right). \tag{45}
\]

Similar to (35) sum of the inner products in lines 3 and 5 of (45) is $tK^{-1} \langle q^K, y^* - y^K \rangle$ which can be bounded using (29) with $k = K$ and replacing $y^{K+1}$ with $y^*$ in (29), leading to the following inequality after multiplying both sides by $m$:
\[
\frac{mtK^{-1}}{2} E[\|x^K - x^*\|^2_M] \leq \\
\frac{mt^0}{2} \|x^* - x^0\|^2_{T_0+(1-\frac{1}{m})M} + \frac{t^0}{\sigma^0} D_y(y^*, y^0) - (m - 1)t_0 \langle \nabla_y \Phi(x^0, y^0), y^0 - y^* \rangle \\
+ (m - 1)t^0 H^0(z^*) - tK^{-1} E \left[ \frac{m}{2} \|x^* - x^K\|^2_{T_{K^{-1}} + M} + \frac{m}{2} \|x^{K-1} - x^K\|^2_{T_{K^{-1}} - L_{xx} - L_{xx}/\alpha K} \right] \\
+ \left( \frac{1}{\sigma_{K-1}} - m\theta K^{-1}\alpha K^{-1} \right) D_y(y^K, y^{K-1}) + \left( \frac{1}{\sigma_{K-1}} - m\theta K^{-1}\alpha K^{-1} \right) D_y(y^K, y^{K-1}) \tag{46}
\]
\[
\leq \frac{mt^0}{2} \|x^* - x^0\|^2_{T_0+(1-\frac{1}{m})M} + \frac{t^0}{\sigma^0} D_y(y^*, y^0) + (m - 1)t^0 H^0(z^*) - \langle \nabla_y \Phi(x^0, y^0), y^0 - y^* \rangle \\
- tK^{-1} E \left[ \frac{m}{2} \|x^* - x^K\|^2_{T_{K^{-1}} + M} + \frac{1}{\sigma_{K-1}} D_y(y^*, y^K) \right], \tag{47}
\]

where in the last inequality the step-size conditions (9a) for $k = K - 1$ is used. Therefore, using a similar argument as in (35) the assertion is proved.
IV. NUMERICAL EXPERIMENTS

In this section, we implement our scheme on a kernel matrix learning problem and benchmark with off-the-shelf interior-point methods (IPM), randomize block coordinate method proposed in [22], and Mirror-prox [25]. We consider an example of kernel matrix learning problem with a relatively large data-set. We aim to benefit from breaking the computations into smaller tasks to reduce the computational complexity at each iteration. Suppose that labeled-data points consisting of training data \( \{a_i\}_{i \in S} \in \mathbb{R}^n \), corresponding labels \( \{b_i\}_{i \in S} \subset \{-1, +1\} \), and unlabeled test data points \( \{a_i\}_{i \in T} \) are given. Let \( m_{\text{tot}} = m_{\text{tr}} + m_{\text{t}} \). A kernel matrix \( K_\ell \) is defined using kernel functions \( \phi_\ell: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ell = 1, \ldots, N \) such that \( [K_\ell]_{ij} \equiv \phi_\ell(a_i, a_j) \) for \( i, j \in S \cup T \). Also, consider the partition of \( K_\ell = \begin{bmatrix} K_{tr, tr}^{\ell} & K_{tr, t}^{\ell} \\ K_{t, tr}^{\ell} & K_t^{\ell} \end{bmatrix} \) for all \( \ell \in \{1, \ldots, M\} \). Kernel matrix learning is a technique that minimizes the training error of an SVM as a function of \( K \) (for more details see [29]). We are interested in \( \ell_2 \)-norm soft margin SVM with relatively large dataset by considering the following generic formulation:

\[
\min_{K \in \mathbb{K}, \text{trace}(K) = c} \max_{x \geq 0, \langle b, x \rangle = 0} 2e^T x - x^T (G(K^{tr}) + \lambda I)x,
\]

(48)

where \( c > 0 \) and \( \lambda > 0 \) are model parameters, \( b = [b_i]_{i=1}^{m_{tr}} \) and \( G(K^{tr}) = \text{diag}(b) K^{tr} \text{diag}(b) \). One approach is by considering \( K \) as a combination of some given matrix kernel \( K_\ell \), i.e.,

\[
K = \sum_{\ell=1}^N y_\ell K_\ell, \quad y = [y_\ell]_{\ell=1}^N \geq 0; \quad \text{clearly, } K \succeq 0.
\]

Therefore (48) takes the following form:

\[
\min_{z, y, z, y \geq 0} \max_{x \geq 0, \langle b, x \rangle = 0} 2e^T x - \sum_{\ell=1}^M y_\ell x^T G(K_\ell^{tr})x - \lambda \|x\|_2^2 + z(b^T x),
\]

(49)

where \( \eta = [y_\ell]_{\ell=1}^M \) and \( r = [r_\ell]_{\ell=1}^M \) for \( r_\ell = \text{trace}(K_\ell) \). Note that (49) is a special case of (1) for \( f, \Phi \) and \( h \) chosen as follows: let \( y_\ell \equiv \frac{u r_\ell}{c} \) for each \( \ell \) and define \( h(y) = I_{\Delta}(y) \) where \( y = [y_\ell]_{\ell=1}^M \in \mathbb{R}^M \) and \( \Delta \) is an \( M \)-dimensional unit simplex; \( \Phi(x, y, z) = -2e^T x + \sum_{\ell=1}^M \frac{y_\ell}{r_\ell} x^T G(K_\ell^{tr})x + \lambda \|x\|_2^2 + z(b^T x) \), and \( f(x) = I_X(x) \) where \( X = \{x \in \mathbb{R}^{m_{tr}} : x \geq 0\} \).

In this experiment, we utilize SIDO0 [35] (6339 observations, 4932 features – we used half of the observations in data set) with three given kernel functions \((M = 3)\); polynomial kernel function \( \phi_1(a, \bar{a}) = (1 + a^\top \bar{a})^2 \), Gaussian kernel function \( \phi_2(a, \bar{a}) = \exp(-0.5(a - \bar{a})^\top (a - \bar{a})/0.1) \), and linear kernel function \( \phi_3(a, \bar{a}) = a^\top \bar{a} \) to compute \( K_1, K_2, K_3 \) respectively. We set \( \lambda = 1, c = \sum_{\ell=1}^3 r_\ell \), where \( r_\ell = \text{trace}(K_\ell) \) for \( \ell = 1, 2, 3 \). The kernel matrices are normalized as in [29]; thus, \( \text{diag}(K_\ell) = 1 \) and \( r_\ell = m_{\text{tot}} \) for each \( \ell \). We have sampled 80% of data for training test and consider \( m = 461 \) blocks each with size of \( n_i = 11 \). The reported results are the average values over 10 random replications. The experiment is performed on a machine running 64-bit
Windows 10 with Intel i7-8650U @2.11GHz and 16GB RAM. The algorithms are compared in terms of relative error for the solution \( \frac{\|x^k - x^*\|_2}{\|x^*\|_2} \), where \((x^*, y^*)\) denotes a saddle point for the problem (49). The purpose of this experiment is to benchmark our method against other methods in terms of empirical convergence. To this end, the optimal solution \((x^*, y^*)\) of the problem (49) is obtained using commercial optimization solver MOSEK through CVX [36].

Due to strong convexity, \( \|x^*\|_2 \) can be bounded depending on \( \lambda > 0 \) by a constant \( B > 0 \) and the Lipschitz constants can be computed as follows: Let \( \|\cdot\| \) denote the spectral norm; the Lipschitz constants defined in (4) and (5) can be set as

\[
L_{xy} \triangleq 6 \sqrt{3} B \max_{\ell=1,2,3} \left\{ \|G(K_{\ell})U_i\| + \frac{1}{m} \|U_i^T G(K_{\ell})U_i\| \right\}
\]

for any \( i \in M \) and \( L_{yy} = 0 \). The step-sizes of our methods are chosen according to Remark III.2 by choosing \( \alpha = \max_{i \in M} \{L_{xy_i}\} \). Since these constants are global parameters, they are multiplied by a factor of 0.1 to get larger steps.

Note that for block linearized augmented Lagrangian method (BLALM) we consider the primal formulation of (49), \( \min_{x,\gamma} \{\gamma - 2e^T x - \lambda \|x\|_2^2 : b^T x = 0, \ x \geq 0, \ x^T G(K_{\ell}^*) x \leq \frac{m}{\lambda} \gamma, \ \ell = 1, \ldots, N\} \). Let the Lipschitz constant of the smooth part of the augmented Lagrangian for the primal formulation be \( L_{\psi_i} \). As suggested in [22], we set \( 1/L_{\psi_i} \) as the primal step-size, and \( 1/(m\alpha) \) and \( 1/(2m\alpha) \) as the dual step-sizes corresponding to inequality and equality constraints respectively. Moreover, we compare with Mirror-prox method as well which is a non-randomize method that can handle general SP problems. It evaluates two proximal steps at each iterations with step-size \( 1/L \) where \( L \) is the overall Lipschitz constant of \( \Phi \).

We tested two different implementations of the RAPD algorithm: we will refer to the constant step version of RAPD, stated in Part I of the main result in Theorem III.1 as RAPD1; and we refer to the adaptive step version of RAPD stated in Part II of the main result, as RAPD2. We compare RAPD1, RAPD2 against BLALM, Mirror-prox and the widely-used interior point methods Sedumi and SDPT3. Sedumi and SDPT3 are second-order methods and have much better theoretical convergence rates compared to the first-order primal-dual methods.

In Fig. 1, the quality of relative solution accuracy \( \frac{\|x^k - x^*\|_2}{\|x^*\|_2} \) over time is under study. Let \( x^* \) be the solution of problem (49) which we computed using MOSEK in CVX with the best accuracy option. The problem in (49) is first solved by Sedumi and SDPT3 using their default setting. Let \( t_1 \) and \( t_2 \) denote the run time of Sedumi and SDPT3 in seconds, respectively. Next, the primal-dual methods RAPD1, RAPD2, BLALM and Mirror-prox are used to solve the problem for \( \max\{t_1, t_2\} \) seconds. We observe that RAPD2 has the best accuracy among the
Fig. 1: Comparison of run time complexity

compared first-order methods. The result can be interpreted as the effectiveness of implementing randomized methods (RAPD1, RAPD2 and BLALM) as oppose to non-randomized one (Mirror-prox). Generally, the computational complexity of IPMs are significantly more than a first-order methods. Here we observe that the three randomized first-order methods can achieve a better accuracy than SDPT3 and Sedumi in 800 and 1500 seconds; hence, using first-order methods can achieve low-to-medium accuracy faster.

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V. Appendix

Lemma V.1. Let $\mathcal{X}$ be a finite dimensional normed vector space with norm $\| \cdot \|_\mathcal{X}$, $f : \mathcal{X} \to \mathbb{R} \cup \{ +\infty \}$ be a closed convex function with convexity modulus $\mu \geq 0$ w.r.t. $\| \cdot \|_\mathcal{X}$, and $D : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a Bregman distance function corresponding to a strictly convex function $\phi : \mathcal{X} \to \mathbb{R}$ that is differentiable on an open set containing $\text{dom } f$. Given $\bar{x} \in \text{dom } f$ and $t > 0$, let

$$x^+ = \arg\min_{x \in \mathcal{X}} f(x) + tD(x, \bar{x}).$$

Then for all $x \in \mathcal{X}$, the following inequality holds:

$$f(x) + tD(x, \bar{x}) \geq f(x^+) + tD(x^+, \bar{x}) + tD(x, x^+) + \frac{\mu}{2} \| x - x^+ \|_\mathcal{X}^2. \tag{51}$$

Proof. This result is a trivial extension of Property 1 in [37]. The first-order optimality condition for (50) implies that $0 \in \partial f(x^+) + t\nabla_x D(x^+, \bar{x})$ – where $\nabla_x D$ denotes the partial gradient with respect to the first argument. Note that for any $x \in \text{dom } f$, we have $\nabla_x D(x, \bar{x}) = \nabla \phi(x) - \nabla \phi(\bar{x})$. Hence, $t(\nabla \phi(\bar{x}) - \nabla \phi(x^+)) \in \partial f(x^+)$. Using the convexity inequality for $f$, we get

$$f(x) \geq f(x^+) + t \langle \nabla \phi(\bar{x}) - \nabla \phi(x^+), x - x^+ \rangle + \frac{\mu}{2} \| x - x^+ \|_\mathcal{X}^2.$$ 

The result in (51) immediately follows from this inequality. \qed

Lemma V.2. [38] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and for each $k \geq 0$ suppose $a^k$ and $b^k$ are finite nonnegative $\mathcal{F}^k$-measurable random variables where $\{\mathcal{F}^k\}_{k \geq 0}$ is a sequence sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}^k \subset \mathcal{F}^{k+1}$ for $k \geq 0$. If $\mathbb{E}[a^{k+1} | \mathcal{F}^k] \leq a^k - b^k$, then then $a = \lim_{k \to \infty} a^k$ exists almost surely, and $\sum_{k=0}^{\infty} b^k < \infty$. 

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