Existence of Traveling Fronts and Pulses in Lateral Inhibition Neuronal Networks with Sigmoidal Firing Rate Functions

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Abstract

The purpose of this work is to rigorously prove the existence of traveling waves in neural field models with lateral inhibition synaptic coupling types and sigmoidal firing rate functions. In the case of traveling fronts, we utilize theory of linear operators and the implicit function theorem on Banach spaces, providing a variation of the homotopy approach originally proposed by Ermentrout and McLeod (1992) in their seminal study of monotone fronts in neural field models. After establishing the existence of traveling fronts, we move to a well-studied singularly perturbed system with linear feedback. For the special case where the synaptic coupling kernel is a difference of exponential functions, we are able to combine our results for the front with theory of invariant manifolds in autonomous dynamical systems to prove the existence of fast traveling pulses that are comparable to singular homoclinical orbits. Finally, using a numerical approximation scheme, we derive the ubiquitous Evans function to study stability. A specific example is carried out, unifying our theoretical results with reasonable conjectures.

Key words. traveling waves, integro-differential equations, existence, stability, neural field models

AMS subject classifications. 35B25, 92C20

1 Introduction

Traveling waves are a novel neurophysiological pattern that researchers across disciplines are interested in studying. Their captivating appeal is that they are so transparently observed in experimentation, which breaks down into in vivo and in vitro types. The combination of advanced electrode recording technology, voltage-sensitive dyes [24, 43], and methods to pharmacologically block inhibition (such as delivering bicuculline, a GABA_A antagonist used in numerous applications [4, 38]) allows such patterns to be seen experimentally. Some examples of traveling waves include in the mammalian visual cortex [4, 37, 41, 49] (and during binocular rivalry [53]), primary somatosensory cortices of Wistar rats [25] and rodents [11, 44], and human [54] and guinea-pig [51] hippocampus.

Traveling waves are also known to be hallmark features of pathological disorders of the neocortex such as epilepsy. Using modern technology, grid electrodes can be implanted in pharmacologically resistant epilepsy patients for the purpose of continuous intracranial monitoring via electrocorticograms (ECoGs) [52]. Although neocortical propagation of focal seizures is especially nocuous, it is not perfectly understood in terms of spatiotemporal neural dynamics [52]. As a result, widespread studies, including the present one, typically model the brain with concessions in order to best approximate reality and improve our understanding.

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Since the human neocortex has approximately 20 billion neurons, \(0.15 \times 10^{15}\) synapses \([42]\), and neurons interact on the order of milliseconds, a common model is a neural field model, which coarse grains space and time, treating neurons as a continuum of objects that interact nonlocally. Neurons are treated as patches and their voltages as averages; firing rates are monotonic with respect to average voltage and do not have explicit dependence on time. The advantage of the present model is that it is extremely simplified and also amenable to heterogeneity \([7, 34]\) and stochasticity \([6]\) considerations.

In this paper, we consider a particular subset of the traveling wave problem that is seldom explored rigorously—existence of traveling fronts and pulses in neural field models with smooth Heaviside (sigmoidal) firing rates and lateral inhibition coupling types. Such coupling types are widely assumed to represent reality \textit{in vivo}, particularly in the visual cortex (see \([49]\) and references within). They do, however, impose mathematical difficulties that make traveling wave problems harder to study in neural field models—mainly, the fact that integral operators lose positivity. Nonetheless, such problems are worthy of exploration; experimentally, synchronized propagation has been found to occur even in the presence of meaningful levels of inhibition \([9]\). Our approach departs from most traveling wave studies where purely excitatory synaptic interactions are assumed.

Aside from the fact that traveling waves are propagations driven by excitation, a major motivating reason why previous authors frequently assume interactions are purely excitatory is because as discussed above, for experiments, inhibition can be blocked. For example, in \([25, 44, 51]\), cortical slices were prepared and recording electrodes were arranged for experimental purposes; traveling waves arise from almost exclusively excitatory interactions. In a controlled environment, experimental findings of wave speed are easier to compare to model predictions, providing valuable insight. On the other hand, the role of inhibition (such as long-range GABAergic projections) elicits a less straightforward conclusion \([49]\).

While inhibition generally prevents propagation, we note that all couplings in the present study are normalized in the following manner. Let \(K(x - y)\) be a kernel function to describe the weight (based on spatial positioning) that presynaptic patches at position \(y\) contribute to postsynaptic patches at position \(x\). In \([16]\), it is assumed that \(K \geq 0\) and

\[
\int_{\mathbb{R}} K(x) \, dx = 1 > 0. \tag{1.1}
\]

In the present study, we also assume \((1.1)\), but also that there exists \(M > 0\) such that \(K(\cdot) > 0\) on \((-M, M)\) and \(K(\cdot) < 0\) on \((-\infty, -M) \cup (M, \infty)\). Hence, roughly speaking, our coupling types can be understood as having strong local excitation with inhibitory tails.

When inhibition is present, the importance of \((1.1)\) being positive in traveling wave studies cannot be overstated. In fact, one finds from the literature that kernels with inhibition are likely to lead to dynamics that thwart propagation if \((1.1)\) does not hold. For example, standing waves \([2, 5, 8, 14, 35, 36, 45]\) are commonly of interest and arise when excitatory and inhibitory features somehow balance each other out. On the other hand, traveling waves exist in \([14, 39, 40, 57, 58, 59, 60]\) where \((1.1)\) is assumed, even though the kernels have a variety of inhibitory regions. Since we work with more realistic firing rate functions of the canonical sigmoidal type, the present study is in certain ways more difficult than the traveling wave studies cited in this paragraph since they assume firing rates are only of the Heaviside type. As a result, their methods take advantage of the fact that solutions have closed form; operator theory is not required to prove existence or uniqueness.

In contrast, our results for the front are heavy in the analysis of linear operators arising from Fréchet derivatives. Our techniques for proving existence of fronts are similar to and motivated by those in \([16]\), but our mathematical analysis is necessarily more delicate due to the presence of inhibitory regions. For the pulse, we reduce our system to an autonomous system of first order ODEs and apply geometric singular perturbation arguments, also seen in \([21]\). Even without closed form for solutions, such tools are available to us due to well-studied theory of invariant manifolds. We provide positive existence results to nonlinear problems that are assumed to approximate reality.
1.1 Model Equations

In this paper, we are interested in the existence of traveling front solutions to the following homogeneous neural field model [2, 16, 44]:

\[
  u_t + u = \int_{\mathbb{R}} K(x-y) S_\tau(u(y,t) - \theta) \, dy, \tag{1.2}
\]

and traveling pulse solutions to the following system with linear adaptation [44]:

\[
  u_t + u + q = \int_{\mathbb{R}} K(x-y) S_\tau(u(y,t) - \theta) \, dy \tag{1.3}
\]

\[
  q_t = \epsilon (u - \gamma q). \tag{1.4}
\]

Here, \( u = u(x,t) \) is the mean electric potential in the spatial patch at position \( x \) and time \( t \); \( K \) is a lateral inhibition coupling type as described in Section 1.2; \( \theta > 0 \) is the threshold of excitation for the network; \( S_\tau \in [0,1] \) is a monotonic smooth Heaviside function, representing the relative firing rate of a neural patch, which is described below. In (1.2), we assume there is no transmission delay between presynaptic patches \( y \) and postsynaptic patches \( x \). Such an assumption only makes the terms in the mathematical arguments simpler. See [44, 45] for the derivation of (1.2)-(1.4) and more background.

Along with biological motivation, the current work is motivated by the fact that in the case of fronts with single super threshold regions, existence and uniqueness [57] is relatively easy to establish when the kernel is of the lateral inhibition type and the firing rate is the Heaviside function. With smooth firing rates, existence, uniqueness, and stability results are quite extensive and notably, are malleable to a wide range of parameter choices. Naturally, one may believe that disrupting monotonicity by including negative regions for \( K \) may be a difficult, but solvable problem—particularly if the smooth firing rates share some similarities with the Heaviside ones.

Motivated by [13], we consider \( S_\tau \), which is a smoothed Heaviside defined by

\[
  S_\tau(u) = \begin{cases} 
  0 & u \leq 0, \\
  f(u, \tau) & 0 < u < \tau, \\
  1 & u \geq \tau,
  \end{cases} \tag{1.5}
\]

where \( f(u, \tau) \) is smooth and increasing in \( u \), \( L^1 \) continuous over \( u \) with respect to changes in \( \tau \), with \( f(0, \tau) = 0 \), \( f(\tau, \tau) = 1 \). We assume \( S_0 \) is the Heaviside step function in the sense that

\[
  \lim_{\tau \to 0^+} \frac{\partial f}{\partial u}(u, \tau) = \delta_0(u),
\]

the delta distribution. Functions like \( S_\tau \) are known as sigmoidal firing rate functions and are very commonly assumed, even in single cell studies.

The goals of our study are to analytically explore, through smooth continuation and repeated application of implicit function theorem, intervals of \( \tau \) where we are certain that traveling fronts exist in the scalar model and traveling pulses exist in the system when \( \epsilon \ll 1 \). Our strategy for the front is to build off of the homotopy argument in [15, 16], but for lateral inhibition kernels. Our starting point for continuation are fronts with Heaviside firing rates, occurring when \( \tau = 0 \).

Works such as [13, 33] have offered valuable numerical insight into the traveling wave problem in a similar setting, but aside from the landmark study in [16], the topic has since been largely unexplored. In the sense of rigorous mathematics, to the author’s knowledge, applying a homotopy in this paper’s setting—which leads to non-monotonic traveling fronts—is new. Our results are especially worthwhile because our assumptions are actually quite weak; the kernels we are studying are essentially the canonical definition of lateral inhibition kernels from the literature with no additional assumptions. The kernels we study are even more general than those in [57], which to the author’s knowledge, contains the most rigorous study of existence, uniqueness, and stability of traveling fronts in neural field models with Heaviside firing rates.
1.1.1 Fixed Points

Scalar Equation ($\epsilon = 0$)

Based on (1.5) and the fact that $K$ is normalized, we see from (1.2) that there are three fixed points in the model, $U \equiv 0$, $U \equiv \beta$, and $U \equiv 1$, where $\theta < \beta < \theta + \tau$. These points are solutions to the equation $U - S_C(U - \theta) = 0$. Moreover, since $f$ is increasing, our problem is of the bistable type. We wish to explore the existence of heteroclinic orbits connecting $U \equiv 0$ to $U \equiv 1$.

System ($0 < \epsilon \ll 1$)

When $0 < \epsilon \ll 1$ and $\gamma > 0$ is sufficiently small, equations (1.3)-(1.4) have precisely one fixed point, $(U, Q) = (0, 0)$, which is a saddle point. In this case, we are interested in homoclinical orbits connecting the fixed point to itself.

1.1.2 Firing Rates

Tracing back to the genesis of the pulse problem for the standard homogeneous neural field model, one finds that there are two broad flavors when it comes to firing rates. One approach is to only use firing rates that are single (or finite sums of) Heaviside functions with thresholds. The upside to this approach is that one can easily derive solutions formally and study complicated topics like existence, uniqueness, and stability with fairly simple mathematical objects. To see the downside, we recall that neural field models are continuum models where the voltage of neurons is understood as averages over small patches and the firing times of individual cells are not considered. As a result, it is unrealistic to assume the average voltage of a patch is truly all-or-none, even though such an assumption holds true for action potentials. Hence, methods for studying (1.2) with sigmoidal firing rates are desirable. Mathematically, the Heaviside firing rates also produce singularities that are undesirable for dynamical systems and singular geometric perturbation techniques.

Alternatively, smooth firing rates provide remedies to the problems above, but produce their own difficulties. Most of the issues are rooted in the fact that solutions do not have closed form so one is practically forced to invoke functional analysis and in particular, operator theory, as evident in the present study. Moreover, uniqueness and stability results are much harder to prove and even verify numerically.

The firing rates we consider are the middle ground between the two approaches above. We are able to remove some of the oversimplifications of the Heaviside function, while residing in a manageable mathematical environment. Like in the Heaviside case, our firing rates reach a maximum saturation point, but the jump up is smooth. Not only does such an approach find common ground in mathematics, it is also realistic biologically. Indeed, due to absolute refractory, neurons are known to be quiescent below a threshold and fire maximally above a different threshold [13].

1.2 Kernel Hypotheses

Also known as Mexican hat kernels, our kernel functions model lateral inhibition. We assume $K \in W^{2,1}(\mathbb{R})$ is continuous on $\mathbb{R}$, smooth on $\mathbb{R}$ except possibly at the origin, and symmetric. Furthermore, in all cases, we assume $K$ has the following typical properties for lateral inhibition kernels:

- $\int_{-\infty}^{0} K(x) \, dx = \int_{0}^{\infty} K(x) \, dx = \frac{1}{2}$, $|K(\cdot)| \leq C \exp(-\rho \cdot |\cdot|)$.
- There exists $M > 0$ such that $K(\cdot) > 0$ on $(-M, M)$ and $K(\cdot) < 0$ on $(-\infty, -M) \cup (M, \infty)$.

Symmetry is assumed for mathematical convenience rather than necessity; our main hypothesis in Section 1.3 can be slightly modified to account for asymmetric kernels.
1.3 Front Hypotheses

Fronts are assumed to be heteroclinical orbits crossing the thresholds $\theta + \delta \tau$ exactly once for $0 \leq \delta \leq 1$. Before stating our main hypothesis regarding the shape of $U_\tau$, define the following important parameters.

Define $\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)$ to be the positive constants that are unique solutions to the equations

\[
\int_{-\infty}^{-\sigma_1} K(x) \, dx = \int_{\sigma_1}^{\infty} K(x) \, dx = 0, \tag{1.6}
\]

\[
\left( \int_{-\infty}^{-M-\sigma_2} + \int_{-M}^{-M+\sigma_2} \right) K(x) \, dx = \theta < \frac{1}{2}, \tag{1.7}
\]

\[
\left( \int_{-\infty}^{-M-\sigma_3} + \int_{M}^{-M+\sigma_3} \right) K(x) \, dx = \theta + \tau < 1, \tag{1.8}
\]

respectively. With these constants defined, we make the following hypothesis used frequently throughout the paper.

$H_1(\theta, \tau)$: Suppose $U'_\tau > 0$ when $U_\tau \in [\theta, \theta + \tau]$ and $U^{-1}_\tau(\theta + \tau) - U^{-1}_\tau(\theta) \leq \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\}$.

In the case of Heaviside firing rates, it is easily proved that if $U_0$ crosses the threshold once, then $U'_0 > 0$ when $U_0 \in [0, 1]$. As we continuously change $\tau$, hypothesis $H_1(\theta, \tau)$ is quite reasonable and as we will see, is certainly satisfied for at least small $\tau$. We will show in Theorem 1.3 that hypothesis $H_1(\theta, \tau)$ is sufficient to assert a homotopy argument, proving the existence of fronts when $\tau$ is changed continuously.

Remark 1.1. Note that $\sigma_2(\theta)$ is increasing on $(0, \frac{1}{2})$, while $\sigma_3(\theta + \tau)$ is decreasing on $(0, 1)$. In turn, this means that if $\sigma_2(0) > \sigma_1$ and $\sigma_3(1) > \sigma_1$, then for any parameter choices, $\sigma_1 = \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\}$. The parameters $\sigma_2$ and $\sigma_3$ are mostly computed for technical reasons to prove Lemma 2.11. In general,

\[
\sigma_1 = \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\}
\]

holds.

Remark 1.2. When $K \geq 0$, the constants $\sigma_1, \sigma_2, \sigma_3$ all diverge to infinity and $H_1(\theta, \tau)$ reduces to the main result proved in [16] for the special case where the firing rate is of the smooth Heaviside type.

![Figure 1: Example of lateral inhibition kernel.](image-url)
1.4 Previous Results

In order to tie together our results and approach, we recall some relevant previous results.

1.4.1 Traveling Fronts

With Heaviside Firing Rates, Lateral Inhibition Kernels

In [57], the following result was proved. We state it somewhat explicitly since the fronts described in this theorem serve as the base point for our continuation argument.

**Theorem 1.1.** Suppose that $0 < \theta < \frac{1}{2}$ and $K$ is a lateral inhibition kernel with $\int_{-\infty}^{\infty} |x|K(x) \, dx \geq 0$. Then there exists a unique (modulo translation) traveling wave front solution $u(x, t) = U_0(z)$ to (1.2) such that $U_0(0) = \theta$, $U_0'(0) > 0$, $U_0(z) < \theta$ on $(-\infty, 0)$, and $U_0(z) > \theta$ on $(0, \infty)$. The front, which has closed form

$$ U_0(z) = \int_{-\infty}^{z} K(x) \, dx - \int_{-\infty}^{z} e^{-\frac{z-x}{\mu_0}} K(x) \, dx, $$

satisfies the reduced equation

$$ \mu_0 U_0' + U_0 = \int_{\mathbb{R}} K(z-x)H(U_0(x)-\theta) \, dx $$

with exponentially decaying limits

$$ \lim_{z \to -\infty} U_0(z) = 0, \quad \lim_{z \to \infty} U_0(z) = 1, \quad \lim_{z \to \pm \infty} U_0'(z) = 0. $$

The wave travels under the traveling coordinate $z = x + \mu_0 t$ at the unique wave speed $\mu_0 > 0$, where $\mu_0$ is the unique solution to

$$ \phi(\mu) := \int_{-\infty}^{0} e^{\frac{z}{\mu}} K(x) \, dx = \frac{1}{2} - \theta. $$

The method of proof relies on verifying that the formal solution in (1.9) satisfies the threshold and wave speed requirements.

In [26], Guo proved the existence and stability of traveling fronts in neural field models with nonsaturating linear gain firing rates and lateral inhibition kernels for the special case where $K$ is the difference of exponentials. Though rigorous, the technique in this work is much different from ours; the specific structure of $K$ and the firing rate allowed the author to reduce the problem to a high order local ODE. In contrast, our result does not depend on reducing the scalar nonlocal equation to a local one. Moreover, our firing rate is smooth and levels off to a saturated value.

With Nonnegative Kernels, Smooth Firing Rates

Our mathematical approach concerning the front are inspired by the pioneering study of Ermentrout and McLeod [16], where they invoked a homotopy argument to prove the existence of unique (modulo translation) monotone front solutions to (1.2) with smooth, increasing firing rates. A large part of their argument utilized the requirement that $K \geq 0$, and therefore, monotone fronts exist. Monotonicity, in particular, greatly simplifies the analysis for uniqueness and stability. Techniques that exploit monotonicity, such as assuming a comparison principle holds [3, 10], can be applied. Follow up studies, such as [15], have revealed interesting behavior when external stimulus evoke propagation.

1.4.2 Traveling Pulses

Background

In vitro experiments [11, 25], where pharmaceautical techniques can block inhibitory neurons, reveal that propagations in the cortex are often better described as traveling pulses. Pulses differ from
fronts in that they actually account for metabolic feedback. Over time, fronts lead to all neural patches becoming and staying excited, seen as heteroclinical orbits connecting resting to excited states. On the other hand, neurons involved in pulse ensembles experience rising and falling phases seen as homoclinical orbits. Pinto and Ermentrout [44, 45] were the first to observe that a singular perturbation problem involving linear adaptation can be set up to encapsulate such behavior, matching experimental findings. Without rigorous proof, they introduced the concept of matching inner and outer layers for neural field equations. Since the results in [16] establish the existence of fronts when \( K \geq 0 \), they were able to construct singular homoclinical orbits and argue that real pulses were close to singular pulses for \( \epsilon \ll 1 \). We are interested in utilizing this very construction in the case where the kernel is the difference of exponentials. In Section 3, we rigorously recall their construction and prove the existence of pulses using geometric perturbation theory; our approach is inspired by [21], but the technical details are quite different.

Previous Partial Analytical Results

Although Heaviside firing rates are not the focus of the current work, we point out that an elegant proof of the existence of pulse solutions to (1.3)-(1.4) can be found in [46], for the special case where \( K \geq 0 \) is increasing (decreasing) on the left (right) half planes. In such a problem, pulse solutions have closed form. Hence, one may set assumptions on \( K \) and perform challenging analysis with relatively simple tools.

When firing rates are smooth or piecewise linear, numerical studies indicate the existence of pulses are quite ubiquitous. Unfortunately, rigorous mathematical analysis of such a problem is very difficult. Our inclination may be to believe that the presence of fronts and backs is enough to assume that fast pulses exist. However, this is far from straightforward; the main reason is that (1.3)-(1.4) have dynamics that are nonlocal. There have been promising existence results [22] for the nonlocal Fitzhugh Nagumo model

\[
\begin{align*}
    u_t + u + w &= [J * u] + f(u), \\
    w_t &= \epsilon(u - \gamma w),
\end{align*}
\]

but the nonlinearities acting on \( u \) occur outside the nonlocal term. To the author’s knowledge, a proof of the existence of pulses with a general kernel function and smooth firing rate is still an open problem for (1.3)-(1.4).

One may observe that for special choices of \( K \), equations (1.3)-(1.4) in fact reduce to higher order, local problems. Faye [21] and then Hastings [28] exploited such a property on a model with synaptic depression

\[
\begin{align*}
    u_t + u &= \int_{\mathbb{R}} J(x-y)q(y,t)S(u(y,t)) \, dy, \\
    q_t &= \epsilon(1 - q(x,t) - \beta q(x,t)S(u(x,t))),
\end{align*}
\]

with the choice \( J(x) = \frac{b}{2} e^{-b|x|} \). Faye used geometric perturbation theory and Hastings used classical ODE techniques to prove the existence of pulses. With the same choice of \( J \), Ermentrout et al. [15] used shooting techniques to numerically solve (1.3)-(1.4) for the correct wave speed that leads to homoclinical orbits. A convenient part of their study is that depending on how one orients forward time, either the stable or unstable manifold of the unique fixed point is one dimensional. For the kernel

\[
K(x) = Ae^{-a|x|} - Be^{-b|x|},
\]

which we study in Section 3, all stable and unstable manifolds are of more than one dimension.

With these results in mind, we wish to provide positive solutions to various related problems. In Section 2, we establish the existence of fronts with general lateral inhibition kernels and smooth Heaviside firing rates. In Section 3, we invoke geometric perturbation theory to prove the existence of pulses when \( K \) is the difference of exponentials.
To the author’s knowledge, proving the existence of traveling pulses arising from lateral inhibition kernels is an unsolved problem in the case of equations (1.3)-(1.4). We do note that in the case where the firing rate is a nonsaturating piecewise linear gain function and \( K \) is a difference of exponentials, the existence (resp. nonexistence) of pulses has been established for the scalar equation (1.2) when \( K \) is asymmetric (resp. symmetric) [27]. But the same problem has not been explored for the system (1.3)-(1.4).

1.5 Main Results

In this section, we state the main theorems that we prove carefully in Sections 2 and 3. In Section 4, we invoke a numerical method for approximating solutions and performing stability analysis. In Section 5, we work through an example, providing the reader with a picture of our results.

1.5.1 Fronts

The first theorem rigorously shows that fronts persist under \( L^1 \) perturbations of the Heaviside firing rate. We use the implicit function theorem to prove the claim.

**Theorem 1.2** (Perturbation of Heaviside). Suppose \( 0 < \theta < \frac{1}{2} \) and \((U_0, \mu_0)\) is a unique front solution pair to

\[
\mu_0 U_0' + U_0 = \int_\mathbb{R} K(z-y) S_0(U_0(y) - \theta) \, dy,
\]

with \( S_0(u-\theta) = H(u-\theta) \), \( K \) satisfying the assumptions in Section 1.2, and \((U_0, \mu_0)\) satisfying all of the conditions of Theorem 1.1. Assume \( \{S_r\}_{r>0} \) is a family of smooth firing rate functions continuously deformed in \( \|\cdot\|_1 \) by \( \tau \) defined by (1.5). Then there exists small \( \epsilon_0 > 0 \) such that for \( \tau \in [0, \epsilon_0) \), there exists a solution pair \((U_\tau, \mu_\tau)\) satisfying

\[
\mu_\tau U_\tau' + U_\tau = \int_\mathbb{R} K(z-y) S_\tau(U_\tau(y) - \theta) \, dy,
\]

The solution \( U_\tau \in C^2(\mathbb{R}) \) is translation invariant, travels under the coordinate \( z = x + \mu_\tau t \) with wave speed \( \mu_\tau > 0 \), satisfies hypothesis \( H_1(\theta, \tau) \), and has the limits

\[
\lim_{z \to -\infty} U_\tau(z) = 0, \quad \lim_{z \to \infty} U_\tau(z) = 1, \quad \lim_{z \to \pm \infty} U_\tau^{(j)}(z) = 0, \quad \text{for} \ j = 1, 2.
\]

For the next theorem, define \( \Lambda(\tau, \theta) : (0, 1-\theta) \times (0, \frac{1}{2}) \to \mathbb{R} \) by

\[
\Lambda(\tau, \theta) := \frac{1}{2} - (\theta + \tau) + \int_0^\tau S_r(u) \, du.
\]

For fixed \( \theta \), if \( \Lambda \) has zeros when \( \tau \in (0, 1-\theta) \), let

\[
\tau_0(\theta) = \inf\{\tau \in (0, 1-\theta) : \Lambda(\tau, \theta) = 0\}. \quad (1.20)
\]

If \( \Lambda \) does not have zeros when \( \tau \in (0, 1-\theta) \), set \( \tau_0(\theta) = 1 - \theta \).

We will show in Lemma 2.9 that if waves exist, their speeds are positive when \( \tau < \tau_0 \). Following up our perturbation result, we use the Arzelá–Ascoli and Bolzano–Weierstrass theorems to show that under hypothesis \( H_1 \), solutions may be passed through limits of subsequences, leading to new solutions. We can then use these new solutions as starting points to apply the implicit function theorem again; this process repeats and cannot possibly break down as long as \( H_1 \) continues to hold.

**Theorem 1.3** (Continuation Criteria). Suppose \( \tau < \tau_0(\theta) \) and for all \( \delta \in (\tau, \tau) \), there exists a solution \((U_\delta, \mu_\delta, S_\delta)\) with hypothesis \( H_1(\theta, \delta) \) holding. Then there exists a solution \((U_\tau, \mu_\tau, S_\tau)\) also satisfying \( H_1(\theta, \tau) \). Hence, as \( \tau \) is changed continuously, a necessary requirement for existence to fail at \( \tau \) is for there to exist some \( \tau^* < \tau \) such that there does not exist a solution satisfying \( H_1(\theta, \tau^*) \).

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Theorem 1.4. Suppose \( (0 < \theta < 1/2) \) in order for the waves to be leftward traveling. If we assume \( 1/2 < \theta < 1 \), we could obtain rightward traveling fronts. By symmetry of \( K \), there is no difficulty in studying such a starting point using similar techniques. Hence, we omit the analysis for waves with negative wave speeds only to avoid the tedious repetition of similar ideas.

1.5.2 Pulses

Our other flavor of results are related to the traveling pulse. We combine our results for the front with geometric perturbation theory to prove the existence of pulse solutions to (1.3)-(1.4) when \( K \) is of the form

\[
K(x) := K_1(x) - K_2(x) = Ae^{-a|x|} - Be^{-b|x|}
\]

with \( A > B > 0, a > b > 0 \), and constants chosen to normalize \( K \). By pulse solutions, we are referring to homoclinical orbits about the unique saddle fixed point that cross each threshold \( \theta + \delta \tau \) exactly twice—via dynamics that resemble the fast front and fast back (a backwards traveling front) respectively.

The singularly perturbed system

\[
\begin{align*}
\epsilon u_t + u + q &= \int_\mathbb{R} (K_1(x-y) - K_2(x-y)) S_\tau(u(y,t) - \theta) dy, \\
q_t &= \epsilon(u - \gamma q),
\end{align*}
\]

for \( 0 < \epsilon \ll 1, \gamma > 0 \), has fixed points when \( q = \frac{a}{\gamma} \) and \( q = S_\tau(u - \theta) - u \). Assuming \( \gamma \) is sufficiently small, as discussed in Section 1.1.1, there exists exactly one fixed point, \((u,q) = (0,0)\). We wish to solve (1.21)-(1.22) with solutions of the form \((u(x,t), q(x,t)) = (U(z), Q(z))\), with \( z = x + \mu(\epsilon)t; \mu(\epsilon) \) is the corresponding wave speed. We are seeking solutions that are locally excited, which was first defined in [35] for the case of standing waves. We modify the definition below.

Definition 1.1. We say a traveling pulse solution \((U,Q)\) is locally excited if there exists two sets of constants, \( \{\eta_\delta\}_{0 \leq \delta \leq 1} \) and \( \{\kappa_\delta\}_{0 \leq \delta \leq 1} \), both strictly increasing in \( \delta \) with \( \eta_1 = \max\{\eta_\delta\} < \min\{\kappa_\delta\} = \kappa_0 \) such that for each \( \delta \), \( U(\cdot) < \theta + \delta \tau \) on \((-\infty, \eta_\delta) \cup (\kappa_{1-\delta}, \infty) \) and \( U(\cdot) > \theta + \delta \tau \) on \((\eta_\delta, \kappa_{1-\delta})\). Furthermore, \((U(\pm \infty), Q(\pm \infty)) = (0,0)\).

In Section 3, we prove the following theorem, which utilizes the existence results from Section 1.5.1.

Theorem 1.4. Suppose \( 0 < \theta < 1/2 \), \( \tau < \tau_0(\theta) \), and when \( \epsilon = 0 \), front solutions to equation (1.2) exist and satisfy the properties in Theorems 1.2 and 1.3. Then for \( \epsilon > 0 \) sufficiently small, there exists a locally excited traveling pulse solution to equations (1.21)-(1.22).

2 Existence of Fronts

Before moving forward, we recall and improve the unique wave speed calculation result with Heaviside firing rates from (1.11) in Theorem 1.1 ([57]). By doing so, we greatly expand the number of lateral inhibition kernels that we may discuss throughout the paper, including kernels with larger negative tails.

Lemma 2.1. Suppose \( K \) is any lateral inhibition kernel and \( 0 < \theta < 1/2 \). Then there exists a unique solution \( \mu_0 > 0 \) to the system

\[
\begin{align*}
\phi(\mu) &= \int_{-\infty}^{0} e^{\frac{s}{\mu}}K(s) \, ds = \frac{1}{2} - \theta, \\
\phi'(\mu) &= \frac{1}{\mu} \int_{-\infty}^{0} |s| e^{\frac{s}{\mu}}K(s) \, ds > 0.
\end{align*}
\]

Therefore, the requirement \( \int_{-\infty}^{0} |s|K(s) \, dx \geq 0 \) may be removed from Theorem 1.1 ([57]), where our continuation process begins.
Proof. A simple calculation shows

\[ \phi'(\mu) = \frac{1}{\mu} \int_{-\infty}^{0} s e^{\frac{s^2}{2}} K(s) \, ds. \]  

(2.2)

\[ \phi''(\mu) = - \frac{2}{\mu} \phi'(\mu) + \frac{1}{\mu^2} \int_{-\infty}^{0} s^2 e^{\frac{s^2}{2}} K(s) \, ds. \]  

(2.3)

If \( \phi'(\mu_0) = 0 \) for some \( \mu_0 \), then \( \phi''(\mu_0) = \frac{1}{\mu_0} \int_{-\infty}^{0} s^2 e^{\frac{s^2}{2}} K(s) \, ds < \frac{M}{\mu_0} \int_{-\infty}^{0} s e^{\frac{s^2}{2}} K(s) \, ds = 0. \) Hence, by the second derivative test, \( \phi \) must have a local maximum when \( \mu = \mu_0 \). Taking the limit of \( \phi' \) as \( \mu \to \infty \), it is clear that the sign of \( \phi' \) for large \( \mu \) is determined by that of \( \int_{-\infty}^{0} s K(s) \, ds \). Positive values imply \( \phi \) is strictly increasing and negative values imply \( \phi \) has one local maximum. Since

\[ \lim_{\mu \to 0^+} \phi(\mu) = 0, \quad \lim_{\mu \to \infty} \phi(\mu) = \frac{1}{2}, \]

clearly there exists a unique \( \mu_0 > 0 \) such that \( \phi(\mu_0) = \frac{1}{2} - \theta \) and \( \phi'(\mu_0) > 0. \) \hfill \( \Box \)

With Lemma 2.1 complete, we now know that at our starting point when \( \tau = 0 \), traveling fronts exist and are unique for any lateral inhibition kernel. This fact may be helpful for proving uniqueness in the case where \( \tau \ll 1 \). We proceed with our first main goal which is to show that we may perturb \( \tau \) and solutions persist.

**Functional Spaces**

In this section, our main tool is implicit function theorem on Banach spaces. However, we note that the objects we wish to perturb, namely \( S_\tau, U_\tau \), and \( \mu_\tau \), do not perturb in the strict “open ball” sense. In particular, since any fronts we obtain are translation invariant, it easily follows that we cannot truly use the implicit function theorem in its classical form. The workaround, first proven in [16] and also later used in [3, 15], is to use properties of eigenfunctions to adjoint operators to fix the translation in order to establish the existence of a solution pair \( (U_\tau, \mu_\tau) \). Uniqueness is then a matter of showing any other solution pairs \( (\hat{U}_\tau, \hat{\mu}_\tau) \) have the property \( \mu_\tau = \hat{\mu}_\tau \) and \( \hat{U}_\tau \) is a translate of \( U_\tau \). Hence, as we proceed with our existence arguments, which inevitably require continuous Fréchet derivatives, we interchangeable understand our Fréchet derivatives as partial Gateaux derivatives acting on convex sets in admissible directions [1], defined below.

**Definition 2.1.** ([1]) Let \( C \subset X \) be a convex subset of a Banach space \( X \). We say that a vector \( h \in X \) is admissible for \( x \in C \) if and only if \( x + h \in C \).

Furthermore, define for \( \tau \geq 0 \),

\[ E(U_\tau, \mu_\tau) := \{ U \in C^2(\mathbb{R}) : \langle \mu_\tau (U - U_\tau)' + (U - U_\tau), \psi^*_\tau \rangle_{L^2(\mathbb{R})} = 0 \}, \]

(2.4)

with limits

\[ \bar{U}(-\infty) = 0, \quad \bar{U}(\infty) = 1, \quad \bar{U}^{(k)}(\pm \infty) = 0, \quad \text{for } k = 1, 2. \]

Here \( \psi^*_\tau = \psi^*_\tau(U_\tau, \mu_\tau) \) is the unique (modulo constant multiples) solution to the linearized adjoint problem, described below; the role of \( \psi^*_\tau \) will be apparent in Section 2.1.1. Clearly, \( E \) is a convex subset of the Banach space \( (C^2_0(\mathbb{R}), \| \cdot \|_{2, \infty}) \). The space of admissible directions

\[ A(U_\tau, \mu_\tau) := \{ U_1 - U_2 : U_1, U_2 \in E(U_\tau, \mu_\tau) \} \]

(2.5)

forms a subspace of the Banach space \( (C^2_0(\mathbb{R}), \| \cdot \|_{2, \infty}) \). We also note that any collection of firing rates \( \{ S_\tau \}_{\tau \geq 0} \) continuously deformed by \( \tau \) and defined by (1.5) forms a subset of \( L^1(\mathcal{T}) \), where \( \mathcal{T} \supset [0, 1] \) is a large interval that contains all possible values of \( U \) of interest.
2.1 Applying the Implicit Function Theorem

In this section, we outline how to apply the implicit function theorem on convex subsets of Banach spaces.

**Step 1: Define Mapping.** Define

\[ F(U, \mu, S_\tau)(z) := \mu U' + U - \int_{-\infty}^{\infty} K(z-y)S_\tau(U(y) - \theta) dy. \]  

(2.6)

Since \( K \) is continuous, smooth everywhere except possibly at the origin, it follows from the technique in [26],

\[ F'(U, \mu, S_\tau)(z) = \mu U'' + U' - \frac{d}{dz} \left[ \left( \int_{-\infty}^{z} + \int_{z}^{\infty} \right) K(z-y)S_\tau(U(y) - \theta) dy \right] \]

\[ = \mu U'' + U' - \int_{-\infty}^{\infty} K'(z-y)S_\tau(U(y) - \theta) dy. \]

Therefore, if \( U \in C^2(\mathbb{R}) \), then \( F \in C^1(\mathbb{R}) \), even when \( \tau = 0 \). In such a case, by Theorem 1.1 ([57]) and Lemma 2.1, we know that modulo translation, for each \( K_0 \), there exists a unique solution \((U_0, \mu_0)\) to

\[ F(U_0, \mu_0, S_0)(z) = 0. \]

2.1.1 Properties of Fréchet Derivatives and Adjoints

Taking the Fréchet derivative of \( F(U, \mu, S_\tau) \) with respect to \( U \) only, we have \( DF_U : C^2_0(\mathbb{R}) \rightarrow C^1_0(\mathbb{R}) \) given by

\[ DF_U(U, \mu, S_\tau)(h_u)(z) := \mu h''_u + h_u - \int_{-\infty}^{\infty} K(z-y)S_\tau'(U(y) - \theta) h_u(y) dy \]

(2.7)

Using a technique similar to [16, Theorem 4.2], we prove the following lemma:

**Lemma 2.2.** Suppose \( F(U_\tau, \mu_\tau, S_\tau) \equiv 0 \) and the solution \( U_\tau \) satisfies hypothesis \( H_1(\theta, \tau) \). Then the function \( DF_U[U_\tau, \mu_\tau, S_\tau](h_u)(z) \) has a simple eigenvalue at \( \lambda = 0 \) for \( \tau \geq 0 \). Namely, by translation invariance, \( U_\tau' \) is the only eigenfunction.

**Proof.** Without loss of generality, assume \( U_\tau^{-1}(\theta) = 0 \). Differentiating \( F(U_\tau, \mu_\tau, S_\tau) \) and choosing \( h_u = U_\tau' \), it is easy to see \( U_\tau' \) is an eigenfunction when \( \lambda = 0 \). To show simplicity, we take advantage of the fact that the operator

\[ h_u \mapsto \frac{1}{\mu_\tau} \int_{-\infty}^{\infty} e^{\frac{x}{\mu_\tau}} DF_U[U, \mu, S_\tau](h_u)(x) dx \]

(2.8)

has the same null space as \( DF_U \). Therefore, a series of calculations reveals that neutral eigenfunctions solve the equation

\[ h_u(z) = \frac{1}{\mu_\tau} \int_{-\infty}^{0} e^{\frac{s}{\mu_\tau}} \int_{0}^{U_\tau^{-1}(\theta+s)} K(z+s-y)S_\tau'(U_\tau(y) - \theta) h_u(y) dy ds, \]

(2.9)

where hypothesis \( H_1(\theta, \tau) \) allows us to simplify the \( y \) bounds. Consider if there was another eigenfunction, \( \zeta_\tau(z) \). Then for any constant \( c \), we have

\[ U_\tau'(z) + c\zeta_\tau(z) = \frac{1}{\mu_\tau} \int_{-\infty}^{0} e^{\frac{s}{\mu_\tau}} \int_{0}^{U_\tau^{-1}(\theta+s)} K(z+s-y)S_\tau'(U_\tau(y) - \theta)(U_\tau'(y) + c\zeta_\tau(y)) dy ds, \]

(2.10)
Consider the domain of \(0 \leq z \leq U_{\tau}^{-1}(\theta + \tau)\), where \(U_{\tau}'(z) > 0\). Then in the integral,
\[-\sigma_1 \leq -U_{\tau}^{-1}(\theta + \tau) \leq z - y \leq U_{\tau}^{-1}(\theta + \tau) \leq \sigma_1.\]

By the definition of \(\sigma_1\), all integrand terms are nonnegative except possibly \(U_{\tau}'(y) + c\zeta_{\tau}(y)\). But since \(U_{\tau}'(y) > 0\) in the range of integration, there exists \(c_0\) such that \(U_{\tau}'(y) + c_0\zeta_{\tau}(y) \geq 0\), with the set of equality being nonempty.

Let \(z_0\) be such a point where \(U_{\tau}'(z_0) + c_0\zeta_{\tau}(z_0) = 0\). Plugging back into (2.10) at \(z = z_0\), the left hand side is zero and the right hand side consists of integrals of nonnegative functions, positive on sets of positive measure. Obviously, the right hand side is positive if \(U_{\tau}' \neq -c_0\zeta_{\tau}\) and we arrive at contradiction. Hence, \(U_{\tau}' \equiv -c_0\zeta_{\tau}\).

The result holds when \(\tau = 0\) and \(S_{\tau}^0\) is the delta distribution. For if \(\zeta_0(0) \neq 0\), then by choosing \(c_0 = \frac{-U_{\tau}'(0)}{\zeta_0(0)}\), the right hand side of (2.10) is always zero, forcing \(U_{\tau}' \equiv -c_0\zeta_0\).

The following lemma is motivated by [15, Theorem 2.2], but we show the details behind why the result holds for lateral inhibition kernels and smooth Heaviside firing rates.

Lemma 2.3. For \(\tau \geq 0\), the adjoint operator \(D^*F_U[U_{\tau}, \mu_{\tau}, S_{\tau}](h_u)(z)\) has a simple eigenvalue at \(\lambda = 0\). The corresponding eigenfunction \(\psi_{\tau}^*\) is of one sign.

Proof. For the simplicity argument, we apply a technique inspired by a similar approach for integral equations in [16, Theorem 4.3]. Using the Arzelá–Ascoli theorem, we see that restricted to compact sets, the operator
\[
G_1(h_u)(z) := \int_0^{U_{\tau}^{-1}(\theta+\tau)} K(z - y)S_{\tau}'(U_{\tau}(y) - \theta)h_u(y) \, dy
\]
(2.11)
is compact. Considering that \(K\) is exponentially bounded, it follows that for a uniformly bounded set of functions \(\{h_u\}\), mappings converge to zero uniformly outside of compact sets. By sequential compactness, clearly \(G_1\) is compact. Moreover, the operator \(G_2 : C_0^2(\mathbb{R}) \to C_0^1(\mathbb{R})\) defined by
\[
G_2(h_u)(z) := \mu_{\tau}h_u^1 + h_u
\]
(2.12)
is invertible. Noting that \(DF_U = G_2 - G_1\), it follows that \(DF_U(h_u) = 0\) if and only if \(h_u = cU_{\tau}'\) if and only if \((G^{-1}G_1 - \mathcal{I})(cU_{\tau}') = 0\). The simplicity of \(\lambda = 0\) for \(D^*F_U\) then follows from the compactness of \(G_2^{-1}G_1\) and the Fredholm Alternative, seen in the following argument: \(G_2^*\) is invertible, \(G_1^*\) is compact with \(D^*F_U(\psi) = (G_2^* - G_1^*)(\psi) = 0\) if and only if \((G_2^{-1}G_1^* - \mathcal{I})(\psi) = 0\). Since the operator \(G_2^{-1}G_1\) has a simple eigenvalue at \(\lambda = 1\), the Fredholm Alternative implies \((G_2^{-1}G_1)^* = G_1^*G_2^{-1}\) has a simple eigenvalue at \(\lambda = 1\) as well; denote \(\phi\) as the corresponding unique eigenfunction. We have
\[
G_1^*G_2^{-1}(\phi) = \phi
\]
if and only if
\[
G_2^{-1}G_1^*(G_2^{-1}(\phi)) = G_2^{-1}(\phi),
\]
which means \(\psi = G_2^{-1}(\phi)\) is the unique eigenfunction associated with the eigenvalue \(\lambda = 1\) of \(G_2^{-1}G_1^*\). Hence, \(\lambda = 0\) is a simple eigenvalue of \(D^*F_U\).

The solutions to \(D^*F_U(\psi) = 0\) are solutions to the equation
\[
D^*F_U(\psi) = -\mu\psi' + \psi - S_{\tau}'(U_{\tau}(z) - \theta)\int_{\mathbb{R}} K(z - y)\psi(y) \, dy = 0.
\]
(2.13)
Suppose \(\tau > 0\). Since \(U_{\tau}\) is assumed to cross each threshold \(\theta + \delta\tau\) exactly once and we may translate so that \(U_{\tau}^{-1}(\theta) = 0\), equation (2.13) simplifies to
\[-\mu\psi' + \psi = 0
\]
(2.14)
Given (2.15), the solution is of the form 
\[ \psi(z) = \psi(0) e^{\frac{z}{\mu(z)}}. \] (2.15)

For \( z \geq z_\tau := U^{-1}_\tau(\theta + \tau) \), equation (2.13) also simplifies to (2.14) since \( \mathcal{S}'_\tau(U_\tau(z) - \theta) = 0 \). In this domain, the solution blows up, forcing \( \psi \equiv 0 \). When \( 0 \leq z \leq z_\tau \), the solution is given by
\[ \psi(z) = \frac{1}{\mu_\tau} \int_{\sigma}^{\tau_\tau} e^{\frac{z}{\mu(z)}} \mathcal{S}'_\tau(U_\tau(x) - \theta) \left( \int_{-\infty}^{0} + \int_{0}^{\tau_\tau} \right) K(x - y)\psi(y) dydx. \] (2.16)

Given (2.15), the \( y \) integrals can be written as
\[ \psi(0) \int_{-\infty}^{0} e^{\frac{y}{\mu_\tau}} K(x - y) dy + \int_{0}^{\tau_\tau} K(x - y)\psi(y) dy \]
\[ = \frac{\psi(0)}{\mu_\tau} \int_{-\infty}^{0} e^{\frac{y}{\mu_\tau}} \left( \int_{y}^{0} K(x - y') dy' \right) dy + \int_{0}^{\tau_\tau} K(x - y)\psi(y) dy \] (2.17)

Given that \( 0 \leq x \leq z_\tau \), it follows from hypothesis \( H_1(\theta, \tau) \) that \( 0 \leq x \leq \sigma_1 < M \). Therefore, the integrand of the first term in (2.18) is positive so the sign of the term is entirely determined by \( \psi(0) \).

In the second term, we have the bounds \(-M < -\sigma_1 \leq x - y \leq \sigma_1 < M \) so \( K(x - y) \geq 0 \). In summary, after integrating with respect to \( x \), the solution is of the form
\[ \psi(z) = \psi(0) h(z) + \int_{0}^{\tau_\tau} J(y, z)\psi(y) dy \]

with \( h \geq 0, J \geq 0 \). It follows from [47] that if \( \psi(0) \geq 0 \), then \( \psi \geq 0 \) on \([0, z_\tau] \) (and therefore, on \((-\infty, z_\tau])\). If \( \psi(0) \leq 0 \), then obviously \(-\psi \geq 0 \). But in fact, if \( \psi(0) = 0 \), then
\[ \int_{0}^{\tau_\tau} J(y, 0)\psi(y) dy = 0, \]

forcing \( \psi \equiv 0 \). Therefore, \( \psi \geq 0 \) if \( \psi(0) > 0 \) and \( \psi \leq 0 \) if \( \psi(0) < 0 \). If \( \tau = 0 \), the result holds since \( \psi(z) = \psi(0) e^{\frac{z}{\mu(z)}} H(z) \) from [15].

**Lemma 2.4.** Assuming \( \tau \geq 0 \), hypothesis \( H_1(\theta, \tau) \), and without loss of generality, \( \psi^*_\tau \geq 0 \), the adjoint solution satisfies

1. \( \langle U'_\tau, \psi^*_\tau \rangle_{L^2(\mathbb{R})} > 0 \),
2. \( \langle \mu_\tau U''_\tau + U'_\tau, \psi^*_\tau \rangle_{L^2(\mathbb{R})} > 0 \).

**Proof.** (i) Since \( \psi^*_\tau \equiv 0 \) on \([z_\tau, \infty)\), we split the inner product as
\[ \langle U'_\tau, \psi^*_\tau \rangle_{L^2(\mathbb{R})} = \langle U'_\tau, \psi^*_\tau \rangle_{L^2((-\infty, 0])} + \langle U'_\tau, \psi^*_\tau \rangle_{L^2([0, z_\tau])}. \]

It is suffices to show both inner products are positive. By hypothesis \( H_1(\theta, \tau) \), we have \( U_\tau(\cdot) < \theta \) on \((-\infty, 0) \) and \( U'_\tau(\cdot) > 0 \) on \([0, z_\tau] \), where we recall \( U_\tau(0) = \theta \) and \( U_\tau(z_\tau) = \theta + \tau \). Obviously \( \langle U'_\tau, \psi^*_\tau \rangle_{L^2([0, z_\tau])} > 0 \) when \( \tau > 0 \) and vanishes when \( \tau = 0 \). For the other term,
\[ \langle U'_\tau, \psi^*_\tau \rangle_{L^2((-\infty, 0])} = \int_{-\infty}^{0} e^{\frac{z}{\mu(z)}} U'_\tau(z) dz \]
\[ = \psi^*_\tau(0) \left( \theta - \frac{1}{\mu_\tau} \int_{-\infty}^{0} e^{\frac{z}{\mu(z)}} U_\tau(z) dz \right) \]
\[ > \psi^*_\tau(0) \left( \theta - \frac{\theta}{\mu_\tau} \int_{-\infty}^{0} e^{\frac{z}{\mu(z)}} dz \right) = 0. \]

Hence \( \langle U'_\tau, \psi^*_\tau \rangle_{L^2(\mathbb{R})} > 0 \).
(ii) Differentiating the equation \( F[U_\tau, \mu_\tau, S_\tau](z) = 0 \),

\[
\mu_\tau U''_\tau + U'_\tau = \int_0^{z_\tau} K(z - y)S'_\tau(U_\tau(y) - \theta)U'_\tau(y) \, dy.
\]

Therefore,

\[
\langle \mu_\tau U''_\tau + U'_\tau, \psi^*_\tau \rangle = \int_0^{z_\tau} S'_\tau(U_\tau(y) - \theta)U'_\tau(y) \left( \int_{-\infty}^0 + \int_0^{z_\tau} \right) K(z - y)\psi^*_\tau(z) \, dz \, dy.
\]

We see that \(-y \geq -z_\tau \geq -\sigma_1\). The \(z\) integrals can be written as

\[
\psi^*_\tau(0) \int_{-\infty}^0 e^{\frac{z}{\mu_\tau}} K(z - y) \, dz + \int_0^{z_\tau} K(z - y)\psi^*_\tau(z) \, dz = \psi^*_\tau(0) \int_{-\infty}^0 e^{\frac{z}{\mu_\tau}} K(z) \, dz = \psi^*_\tau(0) \left( \frac{1}{\mu_\tau} - \theta \right) > 0
\]

by Lemma 2.1.

\[\square\]

**Step 2: Continuous Fréchet Derivatives and Banach Space Isomorphisms.** In order to correctly apply the implicit function theorem on convex sets, we must show near \((U_\tau, \mu_\tau, S_\tau)\), the partial Gateaux derivatives in all admissible directions are continuous in operator norm and \(DF_{U, \mu}[U_\tau, \mu_\tau, S_\tau] : A(U_\tau, \mu_\tau) \times \mathbb{R} \rightarrow C_0^1(\mathbb{R})\) is a Banach space isomorphism. In this step, assume \((U_\tau, \mu_\tau, S_\tau)\) are solutions to \(F[U_\tau, \mu_\tau, S_\tau] \equiv 0\) for \(\tau \geq 0\).

**Lemma 2.5.** For \(\tau \geq 0\), the following mappings are continuous.

(i) \((U, \mu, S_\tau) \mapsto DF_U[U, \mu, S_\tau]\) with respect to the norms \(\|\cdot\|_{1,\infty} + \|\cdot\|_1 \rightarrow \|\cdot\|_{1,\infty}\) at and near \((U_\tau, \mu_\tau, S_\tau)\) in the operator norm.

(ii) \((U, \mu, S_\tau) \mapsto DF_{\mu}[U, \mu, S_\tau]\) with respect to the norms \(\|\cdot\|_{2,\infty} + \|\cdot\|_1 \rightarrow \|\cdot\|_{1,\infty}\) at and near \((U_\tau, \mu_\tau, S_\tau)\) in the operator norm.

(iii) For fixed \((U, \mu), S_\tau \mapsto F[U, \mu, S_\tau]\) with respect to the norms \(\|\cdot\|_1 \rightarrow \|\cdot\|_{1,\infty}\).

**Proof.** The proof is straightforward with the possible exception of case (i) when \(S_\tau\) is the Heaviside function. See Appendix A. \(\square\)

Related to the technique in [16, Theorem 4.4], the following lemma precisely defines our Banach space isomorphisms at each perturbation.

**Lemma 2.6.** For \(\tau \geq 0\), the Gateaux derivative \(DF_{U, \mu}[U_\tau, \mu_\tau, S_\tau] : A(U_\tau, \mu_\tau) \times \mathbb{R} \rightarrow C_0^1(\mathbb{R})\) is a Banach space isomorphism. Hence, by the bounded inverse theorem for continuous linear operators on Banach spaces, \(DF_{U, \mu}^{-1}[U_\tau, \mu_\tau, S_\tau] : C_0^1(\mathbb{R}) \rightarrow A(U_\tau, \mu_\tau) \times \mathbb{R}\) is a continuous, bijective linear operator.

**Proof.** With \(g \in C_0^1(\mathbb{R})\), we wish to solve

\[
DF_{U, \mu}[U_\tau, \mu_\tau, S_\tau](h_u, h_\mu)(z) = DF_U[U_\tau, \mu_\tau, S_\tau](h_u)(z) + DF_{\mu}[U_\tau, \mu_\tau, S_\tau](h_\mu)(z)
\]

\[
= \mu_\tau h'_u(z) + h_u(z) - \int_{\mathbb{R}} K(z - y)S'_\tau(U_\tau(y) - \theta)h_u(y) \, dy + h_\mu U'_\tau(z)
\]

\[
= g(z)
\]

(2.21)
uniquely for $h_u \in A(U_\tau, \mu_\tau)$, $h_\mu \in \mathbb{R}$. By the definition of adjoint with eigenvalue $\lambda = 0$,

$$\langle DF_U[U_\tau, \mu_\tau, S_\tau](h_u), \psi_\tau^* \rangle = \langle h_u, D^*F_U[U_\tau, \mu_\tau, S_\tau](\psi_\tau^*) \rangle = 0,$$

which means $h_\mu$ is uniquely determined by

$$h_\mu = \frac{\langle g, \psi_\tau^* \rangle}{\langle U_\tau', \psi_\tau^* \rangle},$$

with denominator nonzero by Lemma 2.4 (i). The problem reduces to solving

$$DF_U[U_\tau, \mu_\tau, S_\tau](h_u)(z) = g(z) - h_\mu U'_\tau(z) \in C^1_0(\mathbb{R}) \cap \{\psi_\tau^* \}^\perp \quad (2.22)$$

uniquely for $h_u \in A(U_\tau, \mu_\tau)$. As we saw from Lemma 2.3, $DF_U = G_2 - G_1$ with $G_1$ compact, $G_2$ invertible, and the solution to $DF_U[U_\tau, \mu_\tau, S_\tau](h_u) = 0$ is a solution to $(G_2^{-1}G_1 - \mathcal{I})(h_u) = 0$ with $G_2^{-1}G_1$ compact. The solution is given by $h_u^* = cU'_\tau$ and since

$$\langle \mu_\tau U''_\tau + U'_\tau, \psi_\tau^* \rangle \neq 0$$

by Lemma 2.4 (ii), it follows that $h_u^* \in A(U_\tau, \mu_\tau)$ if and only if

$$\langle \mu_\tau h''_u + h_u^*, \psi_\tau^* \rangle = 0$$

if and only if $c = 0$. Rewriting (2.22) in the form

$$(G_2^{-1}G_1 - \mathcal{I})(h_u)(z) = -G_2^{-1}(g(z) - h_\mu U'_\tau(z)),$$

denote the right hand side by $\xi$. Since $\xi \in C^2_0(\mathbb{R})$ satisfies

$$\mu_\tau \xi' + \xi = h_\mu U'_\tau(z) - g(z)$$

by the definition of $G_2$ and the right hand side is orthogonal to $\psi_\tau^*$ by (2.22), obviously the left hand side is as well, proving $\xi \in A(U_\tau, \mu_\tau)$ and $(G_2^{-1}G_1 - \mathcal{I}) : A(U_\tau, \mu_\tau) \rightarrow A(U_\tau, \mu_\tau)$. Since this operator restricted to $A(U_\tau, \mu_\tau)$ has trivial null space, it follows from the Fredholm Alternative that a solution $h_u \in A(U_\tau, \mu_\tau)$ exists and is unique. Applying the bounded inverse theorem, the result follows.

\textbf{Lemma 2.7.} If $(\overline{U}, \overline{\mu}, S_\tau) \in E(U_\tau, \mu_\tau) \times \mathbb{R} \times \{S_\tau\}_{\tau \geq 0}$, then $F[\overline{U}, \overline{\mu}, S_\tau] \in C^1_0(\mathbb{R})$.

\textbf{Proof.} By Step 1, certainly at least $F[\overline{U}, \overline{\mu}, S_\tau] \in C^1(\mathbb{R})$. By dominated convergence theorem,

$$F[\overline{U}, \overline{\mu}, S_\tau](\pm \infty) = \overline{\mu}\overline{U}'(\pm \infty) + \overline{U}(\pm \infty) - S_\tau(\overline{U}(\pm \infty) - \theta) \int_\mathbb{R} K(y) \, dy \quad (2.23)$$

and

$$F'[\overline{U}, \overline{\mu}, S_\tau](\pm \infty) = \overline{\mu}\overline{U}''(\pm \infty) + \overline{U}'(\pm \infty) - S_\tau(\overline{U}'(\pm \infty) - \theta) \int_\mathbb{R} K'(y) \, dy \quad (2.24)$$

The result follows by the definition of $S_\tau$ and $E(U_\tau, \mu_\tau)$.

Now that we have carefully defined our spaces, we may easily apply the Implicit function theorem.
Lemma 2.8. For fixed $\tau \geq 0$ with $U_\tau$ satisfying hypothesis $H_1(\theta, \tau)$, there exists $\epsilon_0 > 0$ and $\delta_0 > 0$ sufficiently small such that for $(U, \bar{m}, S_\tau) \in E(U_\tau, \mu_\tau) \times \mathbb{R} \times \{S_\tau\}_{\tau \geq 0}$ with $\|U - U_\tau\|_\infty + |\bar{m} - \mu_\tau| < \epsilon_0$ and $\|S_\tau - S_\tau\|_1 < \delta_0$, the mapping $N : E(U_\tau, \mu_\tau) \times \mathbb{R} \times \{S_\tau\}_{\tau \geq 0} \to E(U_\tau, \mu_\tau) \times \mathbb{R}$ defined by

$$N[U, \bar{m}, S_\tau] := (U, \bar{m}) - DF_{U, \mu}[U_\tau, \mu_\tau, S_\tau](F[U, \bar{m}, S_\tau])$$

has a unique fixed point $N(U_\tau, \mu_\tau, S_\tau) = (U_\tau, \mu_\tau)$. In particular, we may write $F[U(S_\tau), \mu(S_\tau), S_\tau] = 0$ for $\|S_\tau - S_\tau\|_1 < \delta_0$, with $(U(S_\tau), \mu(S_\tau))$ continuous in the norm $\|\cdot\|_2 + |\cdot|$ with respect to changes in $S_\tau$ in the norm $\|\cdot\|_1$.

Proof. The proof is a standard application of Banach fixed-point theorem. See Appendix A.

We are now able to finish up the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.8, it follows that for $0 < \epsilon_0 \ll 1$, equation (1.17) is satisfied for a (not necessarily globally unique) pair $(U_\tau, \mu_\tau)$, with $U_\tau$ unique in $E(U_0, \mu_0)$, for all $\tau \leq \epsilon_0$. Based on how $E(U_\tau, \mu_\tau)$ is defined, there are no difficulties verifying the conclusions of Theorem 1.2. In particular, since the (unique) Heaviside solution satisfies $U_0' > 0$ when $U_0 \in [0, 1]$, we can always restrict $\epsilon_0$ in order for hypothesis $H_1(\theta, \tau)$ to be satisfied. The wave speed $\mu_\tau$ is positive since $\mu_0$ is positive.

2.2 Continuation Process

In Section 2.1, we established that fronts exist with smooth firing rate functions, but there is no description of the perturbation amount $\epsilon_0$ whatsoever. Therefore, prior to the estimates in this subsection, it seems plausible that we may only be able to perturb the Heaviside function smoothly, but insignificantly. We must discuss sufficient criteria for when our continuation process may proceed.

2.2.1 Positive Wave Speed

We recall from Section 1.5,

$$\Lambda(\tau, \theta) := \frac{1}{2} - (\theta + \tau) + \int_0^\tau S_\tau(u) \, du. \quad (1.19)$$

For fixed $\theta$, if $\Lambda$ has zeros when $\tau \in (0, 1 - \theta)$, let

$$\tau_0(\theta) = \inf\{\tau \in (0, 1 - \theta) : \Lambda(\tau, \theta) = 0\}. \quad (2.26)$$

Otherwise, let $\tau_0(\theta) = 1 - \theta$. We start by showing that if solutions exist and satisfy hypothesis $H_1(\theta, \tau)$, then the wave speeds are positive if $\Lambda(\tau, \theta) > 0$.

Lemma 2.9. Under hypothesis $H_1(\theta, \tau)$, the wave speed $\mu_\tau$ is positive if $\Lambda(\tau, \theta) > 0$.

Proof. Without loss of generality, assume $U_\tau^{-1}(\theta) = 0$. Reworking the argument in [16], the wave speed has the formula

$$\mu_\tau = \frac{\int_{0}^{U_\tau^{-1}(\theta + \tau)} (-U_\tau(y) + S_\tau(U_\tau(y) - \theta)) S_\tau'(U_\tau(y) - \theta) U_\tau'(y) \, dy}{\int_{0}^{U_\tau^{-1}(\theta + \tau)} U_\tau'(x)^2 S_\tau'(U_\tau(x) - \theta) \, dx}. \quad (2.27)$$

Clearly the denominator is positive so the proof comes down to showing the numerator of (2.27) equals $\Lambda(\tau, \theta)$. 

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By the definition of $S_\tau$ and since $U'_\tau(\cdot) > 0$ on $[0, U^{-1}_\tau(\theta + \tau)]$, a simple calculation shows the numerator of (2.27)

$$
\begin{align*}
= & \int_{\theta}^{\theta + \tau} (-u + S_\tau(u - \theta)) S'_\tau(u - \theta) \, du \\
= & 1 - (\theta + \tau) - \int_{\theta}^{\theta + \tau} (-1 + S'_\tau(u - \theta)) S_\tau(u - \theta) \, du \\
= & 1 - (\theta + \tau) + \int_{\theta}^{\theta + \tau} S_\tau(u - \theta) \, du - \frac{1}{2} \int_{\theta}^{\theta + \tau} [S_\tau(u - \theta)^2]' \, du \\
= & \frac{1}{2} - (\theta + \tau) + \int_{0}^{\tau} S_\tau(u) \, du.
\end{align*}
$$

The result follows immediately. \hfill \Box

### 2.2.2 Proof of Continuation

In this subsection, we use the exact definitions of $\sigma_1$, $\sigma_2$, $\sigma_3$ in hypothesis $H_1(\theta, \tau)$ in order to prove

**Theorem 1.3** (Continuation Criteria). Suppose $\tau < \tau_0(\theta)$ and for all $\delta \in [\tau, \tau)$, there exists a solution $(U_\delta, \mu_\delta, S_\delta)$ with hypothesis $H_1(\theta, \delta)$ holding. Then there exists a solution $(U_\tau, \mu_\tau, S_\tau)$ also satisfying $H_1(\theta, \tau)$. Hence, as $\tau$ is changed continuously, a necessary requirement for existence to fail at $\tau$ is for there to exist some $\tau^* < \tau$ such that there does not exist a solution satisfying $H_1(\theta, \tau^*)$.

That is, we show solutions exist under limits as $\tau \to \bar{\tau}$. The technique breaks down into two main parts. The first part, also seen in [3, 16], is to show the corresponding solution sets

$$
\mathcal{U} := \{U_\delta : \delta \in [\tau, \bar{\tau})\}, \quad \mathcal{M} := \{\mu_\delta : \delta \in [\tau, \bar{\tau})\},
$$

satisfy the requirements to apply the Arzelà–Ascoli and Bolzano–Weierstrass theorems respectively. Therefore, for a subsequence $\{\delta_n\}$ with $\delta_n \to \bar{\tau}$, the limiting solution $(U_{\delta_n}, \mu_{\delta_n}, S_{\delta_n}) \to (U_\tau, \mu_\tau, S_\tau)$ exists. The second part is to show $U_\tau$ satisfies hypothesis $H_1(\theta, \bar{\tau})$. In all cases, we fix the translation so that $U_\tau^{-1}(\theta) = 0$ for all $\delta$.

Since we have not proven uniqueness (although we suspect it holds true), for a given $\delta$, there may be multiple solution choices for $(U_\delta, \mu_\delta, S_\delta)$ that satisfy hypothesis $H_1(\theta, \delta)$. Therefore, the set $\mathcal{U} \times \mathcal{M}$ is understood to consist of one arbitrary pair for each $\delta$. In a similar manner, $(U_\tau, \mu_\tau, S_\tau)$ may not be unique, even though it is a unique limit of some subsequence.

**Lemma 2.10.** For the sets $\mathcal{U}$ and $\mathcal{M}$, the following properties hold.

(i) $\mathcal{U}$ is bounded in the norm $\|\cdot\|_{2,\infty}$.

(ii) There exists positive constants $C_L$ and $C_U$ such that $C_L \leq \mu_\delta \leq C_U$ for all $\mu_\delta \in \mathcal{M}$.

(iii) $\mathcal{U}$ is equicontinuous.

(iv) For all $U_\delta \in \mathcal{U}$, the limits

$$
U_\delta(-\infty) = 0, \quad U_\delta(\infty) = 1, \quad U_\delta^{(k)}(\pm\infty) = 0, \quad \text{for } k = 1, 2,
$$

hold uniformly in $\delta$.

**Proof.** (i) For all $\delta$,

$$
|U_\delta^{(j)}(z)| = \left| \int_{-\infty}^{\infty} e^s \int_{\mathbb{R}} K^{(j)}(z + \mu_\delta s - y) S_\delta(y - \theta) \, dy \, ds \right| \leq \left| K^{(j)} \right|_1
$$

for $j = 0, 1, 2$. 

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(ii) To show the existence of \( C_L \), it suffices to show in equation (2.27), the numerator is bounded from below and the denominator is bounded from above for all \( \delta \). The numerator is bounded from below since \( \tau < \tau_0(\theta) \) and \( \tau_0(\theta) \) is where the wave speed is zero. The denominator is bounded from above since it may be written as

\[
\int_0^\delta U_\delta'(U_\delta^{-1}(\theta + u))S_\delta'(u) \, du \leq \|U_\delta'\|_\infty \int_0^\delta S_\delta'(u) \, du \leq \|K'\|_1
\]

by (i).

For the existence of \( C_U \), we see that the only way the claim could not hold is if there exists a sequence \( \{U_{\delta_n}\} \) such that the denominator in (2.27), which is the integral of nonnegative functions, is zero in the limit. By assumption, all \( U_{\delta_n} \) satisfy hypothesis \( H_1(\theta, \delta_n) \). Therefore, for each \( \delta_n \), we have \( U_{\delta_n}' > 0 \) when \( U_{\delta_n} \in [\theta, \theta + \delta_n] \) and \( U_{\delta_n}^{-1}(\theta + \delta_n) \leq \min\{\sigma_1, \sigma_2, \sigma_3\} \). It follows that the only way the denominator in (2.27) could converge to zero is if \( U_{\delta_n}' \to 0 \). But this can only occur if \( U_{\delta_n}^{-1}(\theta + \delta_n) \to \infty \), which is clearly impossible. Hence, \( M \) is bounded from below away from zero and bounded above.

(iii) Let \( z^* \) be fixed. Then for \( z \) near \( z^* \),

\[
|U_{\delta}(z) - U_{\delta}(z^*)| \leq \frac{1}{\mu_\delta} \int_{-\infty}^0 e^{y\delta} \int_{\mathbb{R}} |K(z + s - y) - K(z^* + s - y)|S_\delta(U_{\delta}(y) - \theta) \, dy \, ds
\]

\[
\leq \frac{1}{\mu_\delta} \int_{-\infty}^0 e^{y\delta} \int_{\mathbb{R}} |K(z + s - y) - K(z^* + s - y)| \, dy \, ds.
\]

As \( z \to z^* \), above converges to zero, uniformly in \( (U_{\delta}, \mu_\delta) \) since \( C_L \leq \mu_\delta \leq C_U \) by (ii). A similar proof applies in the first and second derivatives of \( U_{\delta} \).

(iv) We write \( U_{\delta} \) in the form

\[
U_{\delta}(z) = \int_{-\infty}^0 e^{s} \int_{-\infty}^\infty K(y)S_\delta(U_{\delta}(z + \mu_\delta s - y) - \theta) \, dy \, ds = \int_{-\infty}^0 e^{s} \int_{-\infty}^{z+\mu_\delta s-U_{\delta}^{-1}(\theta)} K(y)S_\delta(U_{\delta}(z + \mu_\delta s - y) - \theta) \, dy \, ds = \int_{-\infty}^0 e^{s} \int_{-\infty}^{z+\mu_\delta s} K(y)S_\delta(U_{\delta}(z + \mu_\delta s - y) - \theta) \, dy \, ds.
\]

As \( z \to -\infty \), the results are trivial. For the limits as \( z \to \infty \), we notice that outside of large compact sets \( K_y, K_s \), the integrals with respect to \( y \) and \( s \) are as small as we desire, uniform in \( (U_{\delta}, \mu_\delta) \), since \( S_\delta \leq 1 \) and \( C_L \leq \mu_\delta \leq C_U \). Now let \( K_y, K_s \) be fixed. For each \( \delta \), we have \( S_\delta(U_{\delta}(\eta) - \theta) = 1 \) for \( \eta \geq U_{\delta}^{-1}(\theta + \delta) \). But we also have the bound

\[
U_{\delta}^{-1}(\theta + \delta) \leq \sup_{\delta} \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \delta)\} < \infty.
\]

Therefore, there exists a large number \( T \), uniform in \( (U_{\delta}, \mu_\delta) \), such that for all \( z \geq T \),

\[
U_{\delta}(z) \approx \int_{K_s} e^{s} \int_{K_y} K(y) \, dy \, ds \approx 1,
\]

with the error being uniform in \( (U_{\delta}, \mu_\delta) \). Therefore, \( U_{\delta}(\infty) = 1 \) uniformly. The uniform limits \( U_{\delta}(k)(\infty) = 0 \) for \( k = 1, 2 \) follow by the same argument but replacing \( K \) with its derivatives.

\[\blacksquare\]

By the Arzelá-Ascoli and Bolzano-Weierstrass theorems, there exists a subsequence \( \{\delta_n\} \) with \( \delta_n \to \tau \) such that \( (U_{\delta_n}, \mu_\delta, S_{\delta_n}) \to (U_{\tau}, \mu_{\tau}, S_{\tau}) \) exists with respect to the norm \( \|\cdot\|_{2,\infty} + |\cdot| + \|\cdot\|_1 \).
By applying the dominated convergence theorem, we can show that \((U_\tau, \mu_\tau, \sigma_\tau)\) is a solution in that it solves \(F[U_\tau, \mu_\tau, \sigma_\tau](z) = 0\). Finally, we complete the proof of Theorem 1.3 by showing hypothesis \(H_1(\theta, \tau)\) is satisfied for the limiting solution. Ultimately, we show that \(U_\tau\) is strictly increasing in the threshold region.

At the very least, the desired asymptotics hold and \(U'_\tau \geq 0\) when \(U_\tau \in [\theta, \theta + \tau]\). Define
\[
\alpha := \max\{z : U_\tau(z) = \theta\},
\beta := \min\{z : U_\tau(z) = \theta + \tau\}.
\]

By the translations \(U_{\delta_n}^{-1}(\theta) = 0\), we must have \(\alpha = 0\). Note that \(U'_\tau \geq (\neq) 0\) on \([0, \beta]\) and \(U'_\tau = 0\) for any points where \(U_\tau \in \{\theta, \theta + \tau\}\) outside of \([0, \beta]\). Certainly \(\beta \leq \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\}\) since the claim holds for all \(\delta\). It suffices to show that \(U'_\tau(z) > 0\) on \([0, \beta]\), \(U_\tau(z) < \theta\) on \((-\infty, 0)\), and \(U_\tau(z) > \theta + \tau\) on \((\beta, \infty)\). Therefore, \(U_\tau\) is invertible when \(U_\tau \in [\theta, \theta + \tau]\), \(\beta = U_{\delta_n}^{-1}(\theta + \tau)\), and hypothesis \(H_1(\theta, \tau)\) is satisfied.

The most challenging task is verifying the threshold conditions outside of \([0, \beta]\). In general, the difficulty is that we require information about the wave shape when closed form is not available. In particular, the region
\[(-M, -M + \beta) \cup (M, M + \beta) \subset [0, \beta]^c\]
is the most difficult because it is in this region where very precise information about \(U_\tau\) is required in order to determine if \(U_\tau\) has critical points. This is where the parameters \(\sigma_2, \sigma_3\) provide value in the estimate \(\beta \leq \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\}\) since this bound guarantees \(U_\tau \notin \{\theta, \theta + \tau\}\) inside this region. We prove this fact in the proceeding lemma.

**Lemma 2.11.** (i) Suppose \(\beta \leq \sigma_2(\theta) < M\) and \(U_\tau\) has a critical point at \(z = z_* \in (-M, -M + \beta)\). Then \(U_\tau(z_*) < \theta\).

(ii) Suppose \(\beta \leq \sigma_3(\theta + \tau) < M\) and \(U_\tau\) has a critical point at \(z = z^* \in (M, M + \beta)\). Then \(U_\tau(z^*) > \theta + \tau\).

**Proof.** (i) From the equation
\[\mu_\tau U'_\tau + U_\tau = \int_{\mathbb{R}} K(z - y)S_\tau(U_\tau(y) - \theta) \, dy,\]
if \(U'_\tau(z_*) = 0\) for some \(z_*\), then
\[U_\tau(z_*) = \int_0^\infty K(z_* - y)S_\tau(U_\tau(y) - \theta) \, dy\]
\[= \int_{-\infty}^{z_* - \beta} K(y) \, dy + \int_{z_* - \beta}^{z_*} K(y)S_\tau(U_\tau(z_* - y) - \theta) \, dy\]
\[< \int_{-\infty}^{-M - \beta} K(y) \, dy + \int_{-M - \beta}^{z_*} K(y) \, dy\]
\[< \int_{-\infty}^{-M - \sigma_2(\theta)} K(y) \, dy + \int_{-M + \beta}^{z_*} K(y) \, dy\]
\[< \int_{-\infty}^{-M - \sigma_2(\theta)} K(y) \, dy + \int_{-M}^{z_*} K(y) \, dy = \theta\]
by the definition of \(\sigma_2\).

(ii) Similar to (i), if \(U'_\tau(z^*) = 0\) for some \(z^*\), then
\[U_\tau(z^*) > \int_{-\infty}^{-M - \beta} K(y) \, dy + S_\tau(U_\tau(z^* - M) - \theta) \int_{z^* - \beta}^{z^*} K(y) \, dy\]
\[> \int_{-\infty}^{-M - \beta} K(y) \, dy + \int_{-M}^{M + \beta} K(y) \, dy\]
\[> \int_{-\infty}^{-M - \sigma_3(\theta, \tau)} K(y) \, dy + \int_{M}^{M + \sigma_3(\theta, \tau)} K(y) \, dy = \theta + \tau\]
by the definition of $\sigma_3$. □

The remaining region we have not dealt with yet is

$$(-\infty, -M) \cup (-M + \beta, M) \cup (M + \beta, \infty),$$

but it is easy to see the possible behavior of $U'_{\tau}$ on these intervals. Define

$$h(z) := U'_{\tau}(z)e^{z},$$

which notably, has the same sign as $U'_{\tau}$. The following lemma will account for the behavior of $U'_{\tau}$ on all three intervals.

**Lemma 2.12.** (i) The function $h$ is strictly decreasing on $(-\infty, -M) \cup (M + \beta, \infty)$.

(ii) The function $h$ is strictly increasing on $(-M + \beta, M)$.

**Proof.** A simple calculation shows

$$h'(z) = e^{z} \mu_{\tau} \int_{0}^{\beta} K(z - y)S_{\tau}'(U_{\tau}(y) - \theta)U'_{\tau}(y) dy.$$

Recalling that $K(\cdot) < 0$ on $(-\infty, -M) \cup (M, \infty)$ and $K(\cdot) > 0$ on $(-M, M)$, it can easily be seen that the integrand is negative for the regions described in (i) and positive for the one in (ii). □

We are finally ready to complete the proof of Theorem 1.3 with one final lemma.

**Lemma 2.13.** The limiting solution $U_{\tau}$ satisfies hypothesis $H_{1}(\theta, \tau)$.

**Proof.** We track the solution on $\mathbb{R}$ and show that all threshold requirements are met.

On $(-\infty, -M)$: Starting with $U_{\tau}(-\infty) = 0$, by Lemma 2.12 (i), $U_{\tau}$ decreases.

On $(-M, -M + \beta)$: As discussed, the only way $U_{\tau} = \theta$ can occur is at a local maximum. But by Lemma 2.11 (i), any possible local maximums that occur on this interval stay below $\theta$.

On $(-M + \beta, M)$: The function $h$ is increasing by Lemma 2.12 (ii) and since $U'_{\tau}(\cdot) \geq 0$ on $[0, \beta] \subset [0, M]$, there are only two possibilities. If $h(-M + \beta) < 0$, then $h$ changes signs once, from negative to positive; therefore, $U_{\tau}$ has exactly one critical point, a local minimum, for some $z_{*} \in (-M + \beta, 0)$. On the other hand, if $h(-M + \beta) \geq 0$, then $U_{\tau}$ is strictly increasing. In either case, we may conclude that $U'_{\tau} > 0$ on $[0, M] \cup [0, \beta]$ and $U_{\tau}$ crosses each threshold exactly once.

On $(M, M + \beta)$: Any possible local minimums stay above $\theta + \tau$ by Lemma 2.11 (ii).

On $(M + \beta, \infty)$: The function $h$ is decreasing by Lemma 2.12 (i), leaving only two possibilities. If $h(M + \beta) < 0$, then we must have $U'_{\tau} < 0$. Otherwise, since $h(\infty) = -\infty$, $h$ must change signs exactly once, from positive to negative. Therefore, $U_{\tau}$ has exactly one critical point, a local maximum, in which $U'_{\tau} < 0$ thereafter. In either case, we see that $U_{\tau}(\infty) = 1$ and $U_{\tau} > \theta + \tau$ on this interval.

In conclusion, $U_{\tau}$ is strictly increasing through all thresholds. Therefore, hypothesis $H_{1}(\theta, \tau)$ is satisfied. □

Combining all lemmas in this subsection, we have completed the proof of Theorem 1.3. The work in this section rigorously established the existence of fronts under continuous changes in $\tau$. We will use these solutions as building blocks for the next section, as we move from the scalar equation to the singular system.

**Remark 2.1.** In each iteration, there is no reason why we cannot also deform $K$ to $K$ with $\|K - K\|_{1,1}$ small; we have left this idea out of the discussion because it requires calculating $\sigma_1, \sigma_2, \sigma_3$ for each $K$, cluttering the mathematical arguments while providing similar insight.
3 Existence of Pulses

In this section, our goal is to build up the proper theory to prove Theorem 1.4. We recall that we wish to prove the existence of locally excited pulse solutions when \( K \) is of the form
\[
K(x) := K_1(x) - K_2(x) = Ae^{-a|x|} - Be^{-b|x|}
\]
with \( A > B > 0 \), \( a > b > 0 \), and constants chosen to normalize \( K \). We first build some necessary background originally discussed in [44].

3.1 Singular Construction

Our goal is to show we may construct a locally excited pulse for \( \epsilon \ll 1 \) that is very close to the singular pulse when \( \epsilon = 0 \), which we will now describe.

3.1.1 Fast and Slow Systems

Writing (1.21)-(1.22) in terms of the traveling coordinate and augmenting a trivial equation for the wave speed, we obtain
\[
\begin{align*}
\mu U' + U + Q &= \int_{\mathbb{R}} (K_1(z - y) - K_2(z - y))S_\tau(U(y) - \theta) \, dy, \\
\mu Q' &= \epsilon(U - \gamma Q), \\
\mu' &= 0.
\end{align*}
\]
\[
\text{Let } V_i := \int_{\mathbb{R}} K_i(z - y)S_\tau(U(y) - \theta) \, dy \text{ for } i = 1, 2. \text{ Then by taking Fourier transforms and using properties of convolutions, the system breaks down into the autonomous system}
\begin{align*}
U' &= \frac{1}{\mu}(V_1 - V_2 - U - Q), \\
V_1' &= W_1, \\
W_1' &= a^2 V_1 - 2AaS_\tau(U - \theta), \\
V_2' &= W_2, \\
W_2' &= b^2 V_2 - 2BbS_\tau(U - \theta), \\
Q' &= \frac{\epsilon}{\mu}(U - \gamma Q), \\
\mu' &= 0.
\end{align*}
\]
We call this the fast system since when \( \epsilon \to 0 \), the \( Q \) coordinate is fixed, but the flow on the others occurs in \( O(1/\epsilon) \) time. On the other hand, let \( t = \epsilon z \). Then rewriting (3.1)-(3.3) and letting \( \epsilon \to 0 \), we obtain
\[
\begin{align*}
0 &= \frac{1}{\mu}(V_1 - V_2 - U - Q), \\
0 &= W_1, \\
0 &= a^2 V_1 - 2AaS_\tau(U - \theta), \\
0 &= W_2, \\
0 &= b^2 V_2 - 2BbS_\tau(U - \theta), \\
\dot{Q} &= \frac{1}{\mu}(U - \gamma Q), \\
\dot{\mu} &= 0.
\end{align*}
\]
In this system, which we call the slow system, we see that we that \( Q \) now has flow occurring in \( O(1/\epsilon) \) time, but it only has flow on the set defined by (3.11)-(3.15) and (3.17).
3.1.2 Critical, Stable, and Unstable Manifolds

We call the set \( (3.11)-(3.15) \) and \( (3.17) \) the critical manifold, which breaks down into cleaner form when we consider the definition of \( S_r \). Since \( S_r(U - \theta) = 0 \) for \( U < \theta \) and \( S_r(U - \theta) = 1 \) for \( U > \theta + \tau \), we may define the left and right branches of the critical manifold respectively by

\[
\mathcal{M}_L(Q, \mu) = \{ (U, V_1, W_1, V_2, W_2, Q, \mu) = (-Q, 0, 0, 0, 0, Q, \mu) \},
\]

\[
\mathcal{M}_R(Q, \mu) = \{ (U, V_1, W_1, V_2, W_2, Q, \mu) = (1 - Q, \frac{2A}{a}, 0, \frac{2B}{b}, 0, Q, \mu) \}.
\]

A crucial assumption needed in order to apply the Exchange Lemma [30], which is proved when the system is written in Fenichel normal form [23], is that the critical manifold is normally hyperbolic. Such an assumption is sufficient to conclude that the critical manifold [23, 29] and its stable and unstable manifolds persist for \( \epsilon \) small.

In order to see why this is true in our case, we set \( \epsilon = 0 \) and linearize the right hand side of \( (3.4)-(3.10) \) about points on the critical manifold, noting that the result is the same on both branches where \( S'_r = 0 \). For normal hyperbolicity, the eigenvalues of the corresponding Jacobian should have \( \lambda = 0 \) as an eigenvalue with multiplicity two (for the two slow variables) and all of the others should satisfy \( \text{Re}(\lambda) \neq 0 \). Indeed, this is the case, and the corresponding nontrivial eigenvalues are \( \lambda_+ \in \{a, b\} \), \( \lambda_- \in \{-\frac{1}{\mu}, -a, -b\} \).

The values of the eigenvalues reveal that for fixed \( Q \) and \( \mu \), each branch has two dimensional unstable and three dimensional stable fast foliations. Fixing just \( Q \) and taking the union over \( \mu \) in small intervals close to \( \mu = \mu_{\text{front}} \), we see that these become three dimensional center-unstable and four dimensional center-stable manifolds respectively. The most important point is the fixed point \( \tilde{0} \), located on \( \mathcal{M}_L \). It follows from [23] that when \( \epsilon \) is small, the critical, stable, and unstable manifolds perturb so the center-unstable and center-stable manifolds of \( \tilde{0} \) are three and four dimensions respectively. We seek values of \( \mu \) where these manifolds intersect, forming a homoclinical orbit.

3.1.3 Singular Homoclinical Orbit

The following \( \epsilon \to 0 \) construction was first outlined in [44] for the case where \( K \geq 0 \). As we saw in Section 2, for \( \tau \) sufficiently small and \( Q = 0 \), there exists a front connecting \( U \equiv 0 \) to \( U \equiv 1 \) with \( U' > 0 \) when \( U \in [\theta, \theta + \tau] \). In reference to the autonomous system, we see this as a fast jump connecting \( (U, V_1, W_1, V_2, W_2, Q, \mu) = (0, 0, 0, 0, 0, \mu_{\text{front}}) \) to \( (U, V_1, W_1, V_2, W_2, Q, \mu) = (1, \frac{2A}{a}, 0, \frac{2B}{b}, 0, 0, \mu_{\text{front}}) \). After this fast jump from \( \mathcal{M}_L \) to \( \mathcal{M}_R \), we see that in slow time, as \( \epsilon \to 0 \), the flow on \( \mathcal{M}_R \) is in the direction of increasing \( Q \) and decreasing \( U \), as seen by the equation

\[
\dot{Q} = \frac{1}{\mu} (U - \gamma Q).
\]

Since it can be shown that \( \tilde{0} \) is a stable fixed point on \( \mathcal{M}_L \) under slow flow, it would be satisfying if we can demonstrate the existence of some \( Q_0 \) where a fast jump from \( \mathcal{M}_R \) to \( \mathcal{M}_L \), seen as a traveling wave back, occurs at the same wave speed as the front. Then the slow flow on \( \mathcal{M}_L \) returns the trajectory back to \( \tilde{0} \), completing the homoclinical orbit.

Indeed, as was shown in [44] for neural fields (and is a well known feature of the Fitzhugh Nagumo equations due to a cubic nullcline), this can easily be achieved if the nullcline \( Q = -U + S_r(U - \theta) \) is odd symmetric about its inflection point. Hence, we assert the following hypothesis.

(H3) For fixed \( \tau \), the firing rate \( S_r(u) \) is odd symmetric about the point \( (\frac{\tau}{2}, S_r(\frac{\tau}{2})) \). Specifically, for all \( U \),

\[
S_r \left( \frac{\tau}{2} + U \right) - \frac{1}{2} = - \left( S_r \left( \frac{\tau}{2} - U \right) - \frac{1}{2} \right).
\]

Under hypothesis (H3) we prove the existence of a back that travels at the same speed as the front.
Lemma 3.1. Let $\varepsilon = 0$ and consider the equation

$$\mu U' + U + Q_0 = \int_{\mathbb{R}} K(z-y)S_\tau(U(y) - \theta)\,dy. \quad (3.18)$$

Then assuming hypothesis (H3), the choices $U = U_b = 2\theta + \tau - U_f$, $\mu = \mu_f$, $Q_0 = 1 - (2\theta + \tau)$ solve equation (3.18). The solution $U_b$ has the property $U_b' < 0$ when $U_b \in [\theta, \theta + \tau]$. In the limits, $U_b(-\infty) = 2\theta + \tau$ and $U_b(\infty) = 2\theta + \tau - 1$.

Proof. It suffices to show the choices given solve (3.18) since the properties follow from the properties of the front. Plugging in and using (H3),

$$\mu_f U'_b = -\mu_f U'_f = U_f - \int_{\mathbb{R}} K(z-y)S_\tau(U_f(y) - \theta)\,dy$$

$$= 2\theta + \tau - U_b - \int_{\mathbb{R}} K(z-y)\left[ S_\tau\left(\frac{\tau}{2} - U_b(y) + \theta + \frac{\tau}{2}\right) + \frac{1}{2} - \frac{1}{2} \right]\,dy$$

$$= 2\theta + \tau - U_b - \frac{1}{2} - \int_{\mathbb{R}} K(z-y)\left[ S_\tau\left(\frac{\tau}{2} + U_b(y) - \theta - \frac{\tau}{2}\right) - \frac{1}{2} \right]\,dy$$

$$= -U_b - (1 - (2\theta + \tau)) + \int_{\mathbb{R}} K(z-y)S_\tau(U_b(y) - \theta)\,dy.$$

In conclusion, when $\varepsilon \to 0$, we may construct a singular homoclinical orbit $\mathcal{S}_0 := \mathcal{F}_f \cup \mathcal{M}_R \cup \mathcal{F}_b \cup \mathcal{M}_L$, consisting of two fast and slow pieces respectively; the fast jumps travel at equal speed. See Figure 2.

Figure 2: Phase space portrait of $\mathcal{S}_0$ in $(u,q)$ space. The red pieces represent the fast front and back respectively. The blue pieces represent the slow left and right critical manifolds respectively. The arrows show that $\mathcal{F}_f$ and $\mathcal{F}_b$ may retrace their own paths in $(u,q)$ space while jumping to the next critical manifold; this is caused by the lack of monotonicity of front solutions.
3.2 Proof of Theorem 1.4

The singular homoclinical orbit $\mathcal{S}_0$ is only a formal construction; its existence does not prove the existence of homoclinical orbits $\mathcal{S}_\epsilon$ for $\epsilon$ small. However, in the case of autonomous systems, there are some exceptionally valuable tools for proving such a persistence exists. One landmark result of Jones and Kopell is the $k + 1$ Exchange Lemma [30], a tool that allows one to track the closeness of $k + 1$ dimensional shooting manifolds to stable and unstable manifolds of different dimensions.

To illustrate our predicament, denote $\Sigma_0$ to be the shooting unstable manifold about the fixed point $\bar{0}$ when $\epsilon = 0$. When taken as a union over $\mu$ close to $\mu_f$, this locally invariant manifold is three dimensional. By [23], when $\epsilon \ll 1$, the manifold $\Sigma_\epsilon$ persists and is also three dimensional; hence $k = 2$. Consider the critical manifold $\mathcal{M}_R$, which is two dimensional when seen as a union over $Q$ and $\mu$. However, for $Q$ and $\mu$ near the takeoff point of the back (when $\mu = \mu_f$, $Q_0 = 1 - (2\theta + \tau)$), the unstable manifold $W_0^{cu}(\mathcal{M}_R)$ is four dimensional. Since our ultimate goal is to show $\mathcal{S}_\epsilon$ stays close to $\mathcal{S}_0$, the very difficult problem is comparing invariant manifolds of different dimensions.

The Exchange Lemma provides an elegant solution to our problem. Since the proof is technical and involves converting coordinates to Fenichel normal form [23], we will state a slightly less technical version of it in Lemma 3.2. See [30] for more details.

Transversality Hypotheses

(i) For fixed $Q_0 = 0$ (value is chosen based on the fixed point $\bar{0}$), the manifolds $\Sigma_0$ and $W_0^{cs}(\mathcal{M}_R)$ intersect transversely along the front when $\mu = \mu_f$.

(ii) For fixed $\mu = \mu_f$, the manifolds $W_0^{cu}(\mathcal{M}_R)$ and $W_0^{cs}(\mathcal{M}_L)$ intersect transversely along the back when $Q_0 = 1 - (2\theta + \tau)$.

See [30, Section 4] for background on why these conditions are sufficient.

Lemma 3.2. (i) Let $\mathcal{B}_\delta$ be a neighborhood around $\mathcal{M}_R$ written in Fenichel coordinates and assume hypothesis (i). Then there exists a point $q \in \partial \mathcal{B}_\delta$ close to the point $p \in \partial \mathcal{B}_\delta \cap W_0^{cs}(\mathcal{M}_R)|_{Q=0}$ such that $\mathcal{S}_\epsilon$ passes through $q$, stays close to $\mathcal{M}_R$, and exits $\mathcal{B}_\delta$ at a point $\bar{p} \in \partial \mathcal{B}_\delta \cap \Sigma_\epsilon$ close to the point $\bar{q} \in \partial \mathcal{B}_\delta \cap W_0^{cu}(\mathcal{M}_R)|_{Q=1-(2\theta+\tau)}$. This exit point is reached at a time $O(\frac{1}{\epsilon})$.

(ii) In addition to the exit point $\bar{q}$ being close to $\bar{p}$, their tangent spaces, $T_{\bar{q}}(\Sigma_\epsilon)$ and $T_{\bar{p}}(W_0^{cu}(\mathcal{M}_R)|_{Q=1-(2\theta+\tau)})$, are also close. In other words, $\bar{q}$ is $C^1 - O(\epsilon)$ close to $\bar{p}$.

The role of hypothesis (ii) concerns the behavior of $\mathcal{S}_\epsilon$ on its travel towards $\mathcal{M}_{L,\epsilon}$. If true, by Lemma 3.2 (ii), this implies $\Sigma_\epsilon$ transversely intersects $W_\epsilon^{cs}(\mathcal{M}_{L,\epsilon})$ for $\mu(\epsilon)$ close to $\mu_f(\epsilon)$ and $\epsilon \ll 1$. But $\mathcal{M}_{L,\epsilon}$ is the slow time stable manifold of the fixed point when $\mu = \mu(\epsilon)$. Piecing everything together, one finds a unique value $\mu = \mu(\epsilon)$ that generates the pulse. In the next subsection, we show that hypothesis (i) and (ii) are satisfied, completing the proof of Theorem 1.4.

3.3 Proof of Transversality

A reasonable approach to proving two manifolds intersect transversely is to invoke $k$–forms, as was done in the proof of the Exchange Lemma [30]. They have also been used in specific pulse constructions such as single pulses in neural field models with synaptic depression when $K(x) = \frac{b}{2}e^{-b|x|}$ (see [21]), single pulses in the Fitzhugh Nagumo model [31], and multipulses in perturbed Hamiltonian systems [32]. Given that our problem is of high dimension, we will, however, prove transversality using a more standard approach, as was done by Szmolyan in [50]. In Lemma 3.3, we prove the claim by tracing through the methods in [50, Theorem 4.1] and plugging in the specific calculations of interest.

Lemma 3.3. (i) The manifolds $\Sigma_0$ and $W_0^{cs}(\mathcal{M}_R)$ intersect transversely at $\mu = \mu_f$.

(ii) When $\mu = \mu_f$, the manifolds $W_0^{cu}(\mathcal{M}_R)$ and $W_0^{cs}(\mathcal{M}_L)$ intersect transversely at $Q_0 = 1 - (2\theta + \tau)$.
Proof. (i) When \( \epsilon = Q = 0 \), we consider the system

\[
\begin{align*}
U' &= \frac{1}{\mu}(V_1 - V_2 - U), \quad (3.19) \\
V_1' &= W_1, \quad (3.20) \\
W_1' &= a^2V_1 - 2AaS_\tau(U - \theta), \quad (3.21) \\
V_2' &= W_2, \quad (3.22) \\
W_2' &= b^2V_2 - 2BbS_\tau(U - \theta), \quad (3.23) \\
\mu' &= 0. \quad (3.24)
\end{align*}
\]

Let \( T_p \) denote the tangent space at a point \( p \) on the heteroclinical orbit (seen as the front described in Section 3.1.3). Then by the definition of transversality, we must show for all \( p \),

\[
T_p(\Sigma_0) + T_p(W_{0}^{cs}(\mathcal{M}_R)) = \mathbb{R}^6.
\]

Counting dimensions, we know \( \dim(T_p(\Sigma_0)) = 3 \) and \( \dim(T_p(W_{0}^{cs}(\mathcal{M}_R))) = 4 \) so transversality occurs if and only if

\[
d := \dim(T_p(\Sigma_0) \cap T_p(W_{0}^{cs}(\mathcal{M}_R))) = 1.
\]

Denote \( x = (U, V_1, W_1, V_2, W_2)^T \) to be the fast variables (and \( \mu \) is the slow variable). If a point \( p = (x_0, \mu_f) \) is on the heteroclinical orbit, then by definition, vectors \( (dx, d\mu) \in T_p(\Sigma_0) \cap T_p(W_{0}^{cs}(\mathcal{M}_R)) \) if and only if \( (dx, d\mu)(z) \) solves the variational system on \( \mathbb{R} \)

\[
\begin{align*}
dx' &= D_x \Phi_z(x_0, \mu_f)dx + D_\mu \Phi_z(x_0, \mu_f)d\mu, \quad (3.25) \\
d\mu' &= 0, \quad (3.26)
\end{align*}
\]

with initial value \( (dx_0, d\mu_0) = (dx, d\mu)(0) \). Here, \( \Phi_z \) represents flow of the dynamical system, seen as the right hand sides of (3.19)-(3.23). A solution to this system must satisfy \( d\mu = \mu_0 \) and so system (3.25)-(3.26) has a solution if and only if \( D_\mu \Phi_z(x_0, \mu_f)d\mu_0 \) is in the range of the operator \( \mathcal{L} : C^2_0(\mathbb{R}, \mathbb{R}^5) \to C^1_0(\mathbb{R}, \mathbb{R}^5) \) defined by

\[
\mathcal{L}(dx) := dx' - D_x \Phi_z(x_0, \mu_f)dx.
\]

In a similar manner to the proof of Lemma 2.3, \( \mathcal{L} \) splits as \( \mathcal{L}(dx) = G_2(dx) - G_1(dx) \) with

\[
G_2(dx) = dx' - \begin{pmatrix}
-\frac{1}{\mu_f} & \frac{1}{\mu_f} & 0 & -\frac{1}{\mu_f} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & a^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & b^2 & 0
\end{pmatrix} dx,
\]

\[
G_1(dx) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-2AaS_\tau(U_f - \theta) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-2BbS_\tau(U_f - \theta) & 0 & 0 & 0 & 0
\end{pmatrix} dx.
\]

Here, \( G_2 \) is invertible and \( G_1 \) is compact. By very similar reasoning to Lemma 2.2, it is easy to see that \( \lambda = 0 \) is a simple eigenvalue with eigenfunction \( dx = x'_f \), the derivative of the front. By the Fredholm Alternative, equations (3.25)-(3.26) have a solution if and only if

\[
\langle dx_\ast^T \cdot D_\mu \Phi_z(x_0, \mu_f), d\mu_0 \rangle_{L^2(\mathbb{R})} = 0,
\]

where \( dx_\ast^T = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \) is the unique solution to the adjoint equation

\[
\mathcal{L}_\ast(dx_\ast) = dx'_\ast + [D_x \Phi_z(x_0, \mu_f)]^T dx_\ast = 0.
\]

25
The uniqueness of such a solution follows by emulating the proof of Lemma 2.3; in this case, $G_1$ and $G_2$ are matrix operators, but their relevant properties for applying the Fredholm Alternative are identical to those in Lemma 2.3. A simple calculation shows

$$D_\mu \Phi_z(x_0, \mu_f) = \left( -\frac{1}{\mu_f} U'_f, 0, 0, 0, 0 \right)^T,$$

so

$$\langle dx^T \cdot D_\mu \Phi_z(x_0, \mu_f), d\mu_0 \rangle_{L^2(\mathbb{R})} = -\frac{d\mu_0}{\mu_f} \langle \psi_1', U'_f \rangle_{L^2(\mathbb{R})}.$$ 

Therefore, equations (3.25)-(3.26) have a solution if and only if $d\mu_0 = 0$ or $\langle \psi_1', U'_f \rangle = 0$. At this point, we recall that transversality occurs if and only if $d = 1$. Since we see that one solution is given by $(dx, d\mu) = (x_f', 0)$, we are done if we can show $\langle \psi_1', U'_f \rangle \neq 0$. In Lemma B.1 in the appendix, it is shown that $\psi_1 = \psi^*_\tau$, where $\psi^*_\tau$ is the solution to $D^* F_{T_f}(\psi) = 0$ from Section 2.1.1. Then Lemma 2.4 (i) may be applied; since $\langle \psi_1', U'_f \rangle \neq 0$, the proof is complete.

(ii) By fixing $\mu = \mu_f = \mu_b$ and letting $Q$ be the free variable (justified since $\mu$ is fixed after the first jump), we replace the equation $\mu' = 0$ with $Q' = 0$ in equation (3.24). The exact same argument as in (i) may be adjusted with $D_\mu \Phi_z(x_0, \mu_f)d\mu_0$ replaced by

$$D_\mu \Phi_z(x_0, \mu_f)d\mu_0 = -\frac{d\mu_0}{\mu_b} (1, 0, 0, 0)^T.$$ 

Again, since one solution to the variational equations is $(dx, dQ) = (x_b', 0)$, where $x_b$ is the back, we see that $d = 1$ if and only if $\langle \psi_1, 1 \rangle = \langle \psi^*_\tau, 1 \rangle \neq 0$. But this is obviously true since $\psi^*_\tau$ is of one sign by Lemma 2.3.

By applying the Exchange Lemma, we have proven the existence of $\mathcal{S}_\epsilon$ close to $\overline{S}_0$ for $\epsilon \ll 1$. This completes the proof of Theorem 1.4 and the existence of traveling pulses is established.

**Remark 3.1.** Similar methods can be applied with lateral inhibition kernels such as

$$K(x) = Ce^{-a|x|(b + c|x|)},$$

where the problem also reduces to the high order local type.

## 4 Approximating with $N$ Heaviside Functions

We recall that the smoothed Heaviside function has the form

$$S_\tau(u) = \begin{cases} 0 & u \leq 0, \\ f(u, \tau) & 0 < u < \tau, \\ 1 & u \geq \tau. \end{cases} \quad (1.5)$$

We can partition the domain $[\theta, \theta + \tau]$ by $N + 1$ points and approximate $S_\tau$ by

$$S_{\tau,N}(u - \theta) := \sum_{k=0}^{N} \alpha_k H(u - (\theta + \Delta_k)), \quad (4.1)$$

where $\alpha_0 + \alpha_1 + \ldots + \alpha_N = 1$, $0 = \Delta_0 < \Delta_1 < \ldots < \Delta_N = \tau$, and $\|S_{\tau,N} - S_\tau\|_1 \rightarrow 0$.

By tracing through the proofs in Section 2, we see that there is no reason why we cannot approximate $S_\tau$ by $S_{\tau,N}$. All pertinent properties of the functional spaces still hold and by emulating the proof in Appendix A, we see that the Fréchet derivatives are continuous at points where $S_\tau$ is the sum of Heavisides (where Delta distributions arise). Moreover, the calculations of $\sigma_1, \sigma_2, \sigma_3$ are independent of $S_\tau$. We illustrate how we may implement approximations with closed form. From a numerical analysis perspective, our approach is nonrigorous in the sense that we do not provide estimates for how quickly $\|U_{\tau,N} - U_{\tau}\|_{2,\infty} + |\mu_{\tau,N} - \mu_{\tau}| \rightarrow 0$ as $N \rightarrow \infty$, provided solutions are unique.
4.1 Formal Solutions

Front

When $\epsilon = 0$, we may write

$$\mu_{\tau,N} U'_{\tau,N} + U_{\tau,N} = \int_{\mathbb{R}} K(z - y) S_{\tau,N}(U_{\tau,N}(y) - \theta) \, dy.$$  \hspace{1cm} (4.2)

Equation (4.2) has fixed points when $U_{\tau,N} = S_{\tau,N}(U_{\tau,N} - \theta)$. If $N$ is chosen to be sufficiently large, similar to the analysis in Section 1.1, (4.2) will have three fixed points: $U_{\tau,N} = 0$, $U_{\tau,N} = \beta_{\tau,N}$, and $U_{\tau,N} = 1$ for some $\beta_{\tau,N} \in (\theta, \theta + \tau)$. Fix the translation so that $U_{\tau,N}(0) = \theta$.

Applying hypothesis $H_1(\theta, \tau)$, we have $U'_{\tau,N}(\cdot) > 0$ on $[0, U_{\tau,N}^{-1}(\theta + \tau)]$. For each $k = 0, \ldots, N$, there exists $0 < z_1 < \ldots < z_k < \ldots < z_N$ such that $U_{\tau,N}(\cdot) < \theta + \Delta_k$ on $(-\infty, z_k)$, and $U_{\tau,N}(\cdot) > \theta + \Delta_k$ on $(z_k, \infty)$. Hence, a solution to (4.2) must reduce to

$$\mu_{\tau,N} U'_{\tau,N} + U_{\tau,N} = \sum_{k=0}^{N} \alpha_k \int_{z_k}^{\infty} K(z - x) \, dx$$

$$= \sum_{k=0}^{N} \alpha_k \int_{-\infty}^{z-z_k} K(x) \, dx,$$  \hspace{1cm} (4.3)

which is a simple first order, linear ODE, solved by

$$U_{\tau,N}(z) = \sum_{k=0}^{N} \alpha_k \left[ \int_{-\infty}^{z-z_k} K(x) \, dx - \int_{-\infty}^{z} \exp \left( \frac{x - z}{\mu_{\tau,N}} \right) K(x - z_k) \, dx \right],$$  \hspace{1cm} (4.4)

$$U'_{\tau,N}(z) = \frac{1}{\mu_{\tau,N}} \sum_{k=0}^{N} \alpha_k \int_{-\infty}^{z} \exp \left( \frac{x - z}{\mu_{\tau,N}} \right) K(x - z_k) \, dx.$$  \hspace{1cm} (4.5)

We have the scheme $U_{\tau,N}(z_k) = \theta + \Delta_k$ to set up $N+1$ equations with $N+1$ unknowns: $\mu_{\tau,N}, z_1, \ldots, z_N$.

Remark 4.1. Our scheme is related to a discrete version of the one in [13], where they approximated the function $\Delta : [0, \tau] \rightarrow \mathbb{R}$ such that $U(\Delta(\xi)) = \theta + \xi$.

Pulse

When $\epsilon > 0$, the pulse solutions we obtain in this paper are locally excited (see Definition 1.1). Write

$$\mu_{\tau,N} U'_{\tau,N} + U_{\tau,N} + Q_{\tau,N} = \int_{\mathbb{R}} K(z - y) S_{\tau,N}(U_{\tau,N}(y) - \theta) \, dy,$$  \hspace{1cm} (4.6)

$$\mu_{\tau,N} Q'_{\tau,N} = \epsilon (U_{\tau,N} - \gamma Q_{\tau,N}).$$  \hspace{1cm} (4.7)

Applying Definition 1.1, for each $k = 0, \ldots, N$, there exists

$$0 = \eta_0 < \eta_1 < \ldots < \eta_k < \ldots < \eta_N < \kappa_0 < \ldots < \kappa_k < \ldots < \kappa_N$$

such that $U_{\tau,N}(\cdot) < \theta + \Delta_k$ on $(-\infty, \eta_k) \cup (\kappa_{N-k}, \infty)$ and $U_{\tau,N}(\cdot) > \theta + \Delta_k$ on $(\eta_k, \kappa_{N-k})$. Hence, possible solutions solve the system

$$\mu_{\tau,N} \begin{pmatrix} U_{\tau,N} \\ Q_{\tau,N} \end{pmatrix}' = \begin{pmatrix} -1 & -1 \\ \epsilon & -\gamma \epsilon \end{pmatrix} \begin{pmatrix} U_{\tau,N} \\ Q_{\tau,N} \end{pmatrix} + \left( \sum_{k=0}^{N} \alpha_k \int_{z-\kappa_{N-k}}^{z-\eta_k} K(x) \, dx \right).$$  \hspace{1cm} (4.8)
Using standard methods, the solution is given by

\[ U_{\tau,N}(z) = \sum_{k=0}^{N} \alpha_k \left[ \frac{\gamma}{1 + \gamma} \int_{z - \kappa_{N-k}}^{z} K(x) \, dx - \int_{-\infty}^{z} C(x - z, \mu_{\tau,N}, \epsilon)(K(x - \eta_k) - K(x - \kappa_{N-k})) \, dx \right], \]

and

\[ Q_{\tau,N}(z) = \sum_{k=0}^{N} \alpha_k \left[ \frac{1}{1 + \gamma} \int_{z - \kappa_{N-k}}^{z} K(x) \, dx - \epsilon \int_{-\infty}^{z} D(x - z, \mu_{\tau,N}, \epsilon)(K(x - \eta_k) - K(x - \kappa_{N-k})) \, dx \right], \]

where

\[ C(x, \mu, \epsilon) = \frac{1}{\omega_1(\epsilon) - \omega_2(\epsilon)} \left[ \frac{1 - \omega_2(\epsilon)}{\omega_1(\epsilon)} \exp \left( \frac{\omega_1(\epsilon)}{\mu} x \right) - \frac{1 - \omega_1(\epsilon)}{\omega_2(\epsilon)} \exp \left( \frac{\omega_2(\epsilon)}{\mu} x \right) \right], \]

\[ D(x, \mu, \epsilon) = \frac{1}{\omega_1(\epsilon) - \omega_2(\epsilon)} \left[ -\frac{1}{\omega_1(\epsilon)} \exp \left( \frac{\omega_1(\epsilon)}{\mu} x \right) + \frac{1}{\omega_2(\epsilon)} \exp \left( \frac{\omega_2(\epsilon)}{\mu} x \right) \right], \]

\[ \omega_1(\epsilon) = \frac{1 + \gamma \epsilon + \sqrt{(1 - \gamma \epsilon)^2 - 4\epsilon}}{2}, \]

\[ \omega_2(\epsilon) = \frac{1 + \gamma \epsilon - \sqrt{(1 - \gamma \epsilon)^2 - 4\epsilon}}{2}. \]

This system has \(2(N + 1)\) equations \(U_{\tau,N}(\eta_k) = U_{\tau,N}(\kappa_{N-k}) = \theta + \Delta_k\) with \(2(N + 1)\) unknowns: \(\mu_{\tau,N}, \eta_1, \ldots, \eta_N, \kappa_0, \ldots, \kappa_N\).

### 4.2 Stability

Stability, or lack thereof, is a major topic of interest in dynamical systems. Roughly speaking, in the case of traveling waves, stability means that trajectories that at some point in time are close to a translate of a traveling wave stay close to some (possibly different) translate. Since the model we study is an approximation of reality, unstable waves are undesirable because they may just be ad hoc mathematical objects. They may only be valuable when we are able to understand why the solutions are unstable through a physically motivated bifurcation. Ultimately, we want to either prove waves are stable or provide insight as to why unstable waves are unstable.

In this subsection, we construct an Evans function [17, 18, 19, 20] in the case of firing rates \(S_{\tau,N}\), drawing an equivalence between the point spectrum of a linearized operator and zeros of a complex analytic function. Our construction will most closely resemble the one in [55], where an Evans function was derived in the case where \(S_f\) is a Heaviside function. In our case, the construction is analogous, but in the case of \(N\) Heaviside functions. See [56] for the case where \(N = 2\) and \(K \geq 0\). We restrict our attention to the front; the derivations in Section 4.1 can also be used to construct an Evans function for the pulse.

Write (1.2) in terms of the traveling coordinate \(z = x + \mu_{\tau,N} t\). Then

\[ P_t + \mu_{\tau,N} P_p + P = \int_{\mathbb{R}} K(z - y) S_{\tau,N}(P(y, t) - \theta) \, dy. \]

Letting \(p(z, t) = P(z, t) - U_{\tau,N}(z)\) and linearizing (4.11) about the wave front, we yield

\[ p_t + \mu_{\tau,N} p_z + p = \sum_{k=0}^{N} \frac{\alpha_k K(z - z_k)p(z_k, t)}{U'_{\tau,N}(z_k)}. \]

With \(p(z, t) = \exp(\lambda t)\psi(z)\), we produce the eigenvalue problem \(\mathcal{L}_N^V : C_0^1(\mathbb{R}) \to C_0(\mathbb{R})\) defined by

\[ \mathcal{L}_N^V \psi = -\mu_N \psi' - \psi + \sum_{k=0}^{N} \frac{\alpha_k K(z - z_k)\psi(z_k)}{U'_{\tau,N}(z_k)}. \]
Denote $\sigma(\mathcal{L}_N)$ to be the spectrum of $\mathcal{L}_N$, which is made up of the point spectrum and essential spectrum; the point spectrum is made up of eigenvalues. The proceeding definition, formulated by John Evans [17, 18, 19, 20] for the Nerve Axon equations, is how we define spectral stability.

**Definition 4.1.** A traveling wave solution to (1.2) is spectrally stable if the following conditions hold:

(i) The essential spectrum $\sigma_{\text{essential}}(\mathcal{L})$ lies entirely to the left of the imaginary axis.

(ii) There exists a positive constant $\kappa_0 > 0$ such that $\max\{\Re \lambda \mid \lambda \in \sigma_{\text{point}}(\lambda), \lambda \neq 0\} \leq -\kappa_0$.

(iii) The eigenvalue $\lambda = 0$ is algebraically simple.

Sandstede [48], using an elegant application of the spectral mapping theorem, proved that spectral stability implies nonlinear stability for neural field models with single Heaviside firing rates. We see no reason why a similar result would break down by replacing single Heaviside functions with finitely many Heaviside functions since all relevant compact operators remain compact.

**Essential Spectrum**

Since $\mathcal{L}_N$ is readily seen to split into compact and noncompact components respectively, the essential spectrum is derived from the intermediate eigenvalue problem $\mathcal{L}_N^\infty \psi = \lambda \psi$, where

$$
\mathcal{L}_N^\infty \psi := -\mu_0 \psi' - \psi, \quad (4.14)
$$

The solution to (4.14) is given by

$$
\psi_0(\lambda, z) = C(\lambda) \exp \left( -\frac{\lambda + 1}{\mu_0} z \right), \quad (4.15)
$$

Assuming $C(\lambda) \neq 0$, the solution $\psi_0(\lambda, z)$ will blow up as $z \to -\infty$ or $z \to +\infty$ unless $\Re(\lambda) = -1$. Therefore, the essential spectrum is the vertical line

$$
\sigma_{\text{essential}} = \{ \lambda \in \mathbb{C} \mid \Re(\lambda) = -1 \}, \quad (4.16)
$$

which safely stays entirely on the left half plane.

**Remark 4.2.** For a given solution $(U_\tau, \mu_\tau)$ corresponding with a smooth Heaviside firing rate $S_\tau$, the essential spectrum is the same since $S'_\tau(U_\tau(\cdot) - \theta)$ has compact support.

**Point Spectrum**

We consider the domain

$$
\Omega = \{ \lambda \in \mathbb{C} \mid \Re(\lambda) > -1 \}.
$$

The formal solution to the original eigenvalue problem is given by

$$
\psi(z) = C(\lambda) \exp \left( -\frac{\lambda + 1}{\mu_{\tau,N}} z \right) + \sum_{k=0}^{N} \frac{\alpha_k \psi(z_k)}{\mu_{\tau,N} U'_{\tau,N}(z_k)} \int_{-\infty}^{z} \exp \left( \frac{\lambda + 1}{\mu_{\tau,N}} (x - z) \right) K(x - z_k) \, dx, \quad (4.17)
$$

Since we require $C(\lambda) = 0$ to prevent blow up, we have compatibility conditions as solutions to the equation

$$
(I - A(\lambda)) \vec{\psi} = 0, \quad (4.18)
$$

where $\vec{\psi}^T = (\psi(z_0), ..., \psi(z_N))$ and $A$ is the matrix with entries

$$
a_{jk} = \frac{\alpha_k}{\mu_{\tau,N} U'_{\tau,N}(z_k)} \int_{-\infty}^{0} \exp \left( \frac{\lambda + 1}{\mu_{\tau,N}} x \right) K(x + z_j - z_k) \, dx = \frac{\alpha_k(\lambda + 1)}{\mu_{\tau,N}^2 U'_{\tau,N}(z_k)} \int_{-\infty}^{\lambda x} \exp \left( \frac{\lambda + 1}{\mu_N} x \right) \left[ \int_{x}^{0} K(y + z_j - z_k) \, dy \right] \, dx. \quad (4.19)
$$
Note that when $\lambda \in \Omega$ is real, all entries in $A$ are strictly positive since by hypothesis $H_1(\theta, \tau)$,

$$-\sigma_1 \leq -U_{\tau,N}^{-1}(\theta + \tau) \leq z_j - z_k \leq U_{\tau,N}^{-1}(\theta + \tau) \leq \sigma_1,$$

making \((4.19)\) positive for all $j, k$. Such an observation is sufficient to prove the following lemma.

**Lemma 4.1.** The operator $L_N$ has a simple eigenvalue at $\lambda = 0$, which by translation invariance, has $\psi = U'_{\tau,N}$ as an eigenfunction.

**Proof.** The proof ultimately follows Lemma 2.2, but for clarity, we prove the claim explicitly. By setting $C(0) = 0$, plugging in, and comparing \((4.17)\) to \((4.5)\), clearly $\lambda = 0$ is an eigenvalue with eigenfunction $\psi = U'_{\tau,N}$. We have from \((4.18)\),

$$(I - A(0))\vec{U}'_{\tau,N} = 0,$$

where $\vec{U}'_{\tau,N} = (U'_{\tau,N}(z_0), ..., U'_{\tau,N}(z_N))$. By hypothesis $H_1(\theta, \tau)$, all $z_k \in [0, U_{\tau,N}^{-1}(\theta + \tau)]$, where $U'_{\tau,N} > 0$. Therefore, all entries of $\vec{U}'_{\tau,N}$ are strictly positive. It follows from Perron-Frobenius theorem that $\alpha = 1$ is the unique eigenvalue of the positive matrix $A(0)$ with maximum modulus and most importantly, it is simple. \(\square\)

**Evans Function**

Similar to the $N = 2$ case in \([56]\), we have nontrivial solutions to \((4.18)\) if and only if $\det(I - A(\lambda)) = 0$. Hence we define our Evans function by

$$E_{\tau,N}(\lambda) := \det(I - A(\lambda)). \quad (4.20)$$

Note that the eigenvalues (and their algebraic multiplicities) of $L_N$ are equivalent to the zeros (and their multiplicities) of $E_{\tau,N}$ \([12]\). As was the case in \([12, 55, 56, 57]\), $E_N$ is complex analytic. We see this by observing that all matrix entries of $A$ are Laplace transforms. Also, it is easy to see $E_{\tau,N}(\lambda)$ is real if $\lambda$ is real and in $\Omega$,

$$\lim_{|\lambda| \to \infty} E_{\tau,N}(\lambda) = 1.$$

In turn, zeros of $E_{\tau,N}$ are isolated and contained in a ball $B_R \subset \Omega$. With $E_{\tau,N}$ constructed, a very difficult problem is to show $E_{\tau,N}$ has no zeros on the right half plane other than the simple root at $\lambda = 0$.

In the case where $N = 0$, $S_\tau = H$ and $K = Ae^{-a|x|} - Be^{-b|x|}$, front solutions to \((1.2)\) are unique and stable \([57]\); the Evans function $E_0$ has a simple zero at $\lambda = 0$ and one other zero $\lambda_-(a, A, b, B) < 0$. Hence, we suspect a related result holds for the firing rates in this study.

We are now in a position to state some open problems. Firstly, does an Evans function $E_\tau$ exist for front solutions $(U_\tau, \mu_\tau)$ corresponding with firing rates $S_\tau$? Can it be described as a limit of $E_{\tau,N}(\lambda; (U_{\tau,N}, \mu_{\tau,N}))$ as $N \to \infty$ for some sequence? If so, can we estimate the convergence rate? All of these questions are somehow related to uniqueness, which is also an open problem. We leave these questions for future research.

**5 Numerical Example**

In this final section, we combine all results from previous sections in order to compute solutions numerically for an example.
Firing Rate

Inspired by the work in [13], we define

\[ S_\tau(u - \theta) = \begin{cases} 
0 & u \leq \theta, \\
\int_0^{u-\theta} A(\tau) \exp \left( \frac{r}{x(x-\tau)} \right) \, dx & \theta < u < \theta + \tau, \\
1 & u \geq \theta + \tau. 
\end{cases} \quad (5.1) \]

Choose \( \theta = 0.1, r = 0.01; A(\tau) \) is a normalizing constant. Note that \( S_\tau \) is odd symmetric about its inflection point so our hypothesis needed to prove the existence of fast pulses holds true. In order to approximate \( S_\tau \) by \( S_{\tau,N} \), we choose \( N = 20 \) and partition \( [\theta, \theta + \tau] \) by \( N + 1 \) equally spaced points.

Recalling the results in Theorems 1.2 and 1.3, we have rigorously proven traveling fronts exist as long as hypothesis \( H_1(\theta,\tau) : \) Suppose \( U_\tau' > 0 \) when \( U_\tau \in [\theta, \theta + \tau] \) and \( U_\tau^{-1}(\theta + \tau) - U_\tau^{-1}(\theta) \leq \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\} \) holds. Hence, we suspect there is some value \( \tau^* < \tau_0(\theta) \) where \( H_1(\theta, \tau^*) \) breaks down thereafter.

Roughly speaking, if \( \tau^* \) is close to \( \tau_0(\theta) \), then \( H_1 \) must be a weak hypothesis. In this section, we only work through one example; certainly there are ways to choose the parameters so that \( H_1 \) has varying levels of restrictiveness for lateral inhibition kernels.

Kernel

Motivated by our proof of the existence of pulses, we choose

\[ K(x) = Ae^{-a|x|} - Be^{-b|x|}, \quad (5.2) \]

with \( A = 5, a = 0.5, B = 4, b = 0.4211 \); such a kernel is normalized. Recalling the definitions

\[ \int_{-\infty}^{-\sigma_1} K(x) \, dx = \int_{\sigma_1}^{\infty} K(x) \, dx = 0, \quad (1.6) \]

\[ \left( \int_{-\infty}^{-M-\sigma_2(\theta)} + \int_{-\infty}^{-M+\sigma_2(\theta)} \right) K(x) \, dx = \theta < \frac{1}{2}, \quad (1.7) \]

\[ \left( \int_{-\infty}^{-M-\sigma_3(\theta+\tau)} + \int_{-\infty}^{-M+\sigma_3(\theta+\tau)} \right) K(x) \, dx = \theta + \tau < 1, \quad (1.8) \]

we obtain

\[ \sigma_1 = 0.6497, \quad \sigma_2(0) = 1.9754 > \sigma_1, \quad \sigma_3(1) = 1.9754 > \sigma_1. \]

By Remark 1.1, it follows that for all \( \theta, \tau \),

\[ \min\{\sigma_1, \sigma_2(\theta), \sigma_3(\theta + \tau)\} = \sigma_1. \]

Therefore, \( \sigma_1 \) will be our cutoff point in hypothesis \( H_1(\theta, \tau) \).
Existence and Stability of Front

Starting with $\tau = 0$, we increased $\tau$ in small steps. At each step, we used equation (4.4) and the scheme $U_{\tau,N}(z_k) = \theta + \Delta_k$ in order to numerically solve for $N + 1 = 21$ unknowns. Then we compared $z_N = U_{\tau,N}^{-1}(\theta + \tau)$ to $\sigma_1$ in order to see if hypothesis $H_1(\theta, \tau)$ held after each step. We stopped increasing $\tau$ when $z_N$ approached $\sigma_1$. Numerical simulations suggest

$$\tau^* \approx 0.52.$$ 

See Figures 3 and 4.

![Figure 3](image1.png)

(a) Plots of $S_\tau$ after small increments. The firing rate $S_{\tau^*}$ is where hypothesis $H_1$ begins to fail; $S_{\tau_0}$ is where solutions that are monotone through the threshold region have zero wave speed. (b) Scatter plot of $z_N$ at different values of $\tau$. The dark circle is where $z_N \approx \sigma_1$ and $H_1$ begins to fail thereafter.

![Figure 4](image2.png)

Figure 4: Plot of $U_{\tau,N}$ when $\tau = \tau^*$, $N = 20$. The solution is a traveling front connecting $U_{\tau,N} \equiv 0$ to $U_{\tau,N} \equiv 1$. 
We show numerically that the front $U_{\tau^*,N}$ is stable. Recalling the derivations in Section 4.2 and the Evans function defined in equation (4.20), we plot $|E_{\tau,N}(\lambda)|$ for $\lambda \in \Omega$ with $\text{Re}(\lambda) \geq 0$, $\text{Im}(\lambda) \geq 0$ and show that $|E_{\tau,N}(\lambda)| > 0$, except at $\lambda = 0$. See Figure 5.

Figure 5: Plot of $|E_{\tau,N}(\lambda)|$ for $\lambda \in \Omega$ with $\text{Re}(\lambda) \geq 0$, $\text{Im}(\lambda) \geq 0$. Since the only zero is at $\lambda = 0$, the solution $U_{\tau^*,N}$ is stable.

Existence of Pulses

In Section 3, we proved the existence of a fast pulse when $K$ is of the form used in this example and when the front satisfies hypothesis $H_1(\theta, \tau)$. We only calculate the case where $\tau = \tau^*$, where it was shown above that a (stable) front exists. Pulses also exist for the cases where $\tau < \tau^*$ and $\epsilon \ll 1$, but we omit these solutions here.

Choose

$$\epsilon = 0.005, \quad \gamma = 0.001.$$  

Using equation (4.9) and the scheme $U_{\tau^*,N}(\eta_k) = U_{\tau^*,N}(\kappa_{N-k}) = \theta + \Delta_k$ outlined in Section 4.1, there are $2(N + 1) = 42$ equations and $2(N + 1) = 42$ unknowns, which we solve numerically. The resulting pulse solution can be seen in Figure 6 below.
Discussion

In the present study, we first applied the powerful homotopy technique in [16] in order to prove the existence and smooth continuation of traveling fronts in neural field models with lateral inhibition kernels and smooth Heaviside firing rates; our results expand the existence results in [57] by exploring the problem beyond just models with Heaviside firing rates. Furthermore, our requirement for continuation has only one main hypothesis regarding the wave shape; the kernels we studied are standard lateral inhibition kernels from the literature without additional assumptions. Can our hypotheses be even further relaxed? To the author’s knowledge, beyond the landmark study of monotone fronts in [16], our problem was previously unsolved in a setting as rigorous and general as ours. We hope that our methods provide insight into how to handle nonmonotone traveling fronts in neural field models.

In the singularly perturbed system (1.3)-(1.4), we then used our existence results for the front to prove the existence of fast traveling pulses using geometric singular perturbation theory. In particular, we invoked the Exchange Lemma [30] for the special case where the problem reduces to a higher order autonomous local ODE. In part, our results complement those in [21], but for the case of lateral inhibition kernels.

In Section 4, we introduced numerical methods to support our theory and help us gain insight into some difficult theoretical questions. For example, can an Evans function be derived to study stability? What is the exact shape of solutions? How do we verify our main continuation hypothesis $H_1$ using rigorous mathematical analysis? In Section 5, we worked through a numerical example with the intent of examining some of these questions. Our calculations provide support for our proven results; they also support our conjecture that solutions are stable.

Beyond the questions above, there are other fundamental concepts that this study has left entirely open. One such topic is wave speed analysis. While we provide a formula for the wave speed of the front and discuss why it is positive, our study lacks a careful discussion of the impact parameters have on wave speed. Finally, our front existence results would be more satisfying with a proof of uniqueness as well. Based on the uniqueness results in [10, 16] for monotone fronts and [57] for fronts arising from...
lateral inhibition kernels and Heaviside firing rates, the author conjectures that uniqueness is true. Ultimately, constructing sub and super solutions and setting up a comparison is nontrivial.

A Application of the Implicit Function Theorem

Proof of Lemma 2.5. We only prove the claim for case (i) at points \((U^*, \mu^*, S_0)\) in a neighborhood of \((U_0, \mu_0, S_0)\). The claim is trivial (using triangle and Hölder’s inequalities) when \(S_0\) is not the Heaviside function.

(i) For \(\tau \geq 0\), the function \((U, \mu, S_\tau) \mapsto DF_U[U, \mu, S_\tau]\) is continuous at \((U^*, \mu^*, S_0)\).

Proof. Let \(h \in C_0^2(\mathbb{R})\) with \(\|h\|_{2,\infty} = 1\). Then for all \(z\),

\[
|DF_U[U, \mu, S_\tau](h)(z) - DF_U[U^*, \mu^*, S_0](h)(z)|
\leq |DF_U[U, \mu, S_\tau](h)(z) - DF_U[U^*, \mu, S_\tau](h)(z)|
+ |DF_U[U^*, \mu, S_\tau](h)(z) - DF_U[U^*, \mu^*, S_\tau](h)(z)|
+ |DF_U[U^*, \mu^*, S_\tau](h)(z) - DF_U[U^*, \mu^*, S_0](h)(z)|
\]

(A.1)

\[
\text{If } \tau > 0,
(A.1) = \left| \mu(U - U^*)' + (U - U^*) - \int_{\mathbb{R}} K(z - y)(S'_0(U(y) - \theta) - S'_0(U^*(y) - \theta))h(y) \, dy \right|
\leq \mu|(U - U^*)'| + |U - U^*| + \max |S'_\tau(\cdot)| \|K\|_1 \|h\|_{2,\infty} \|U - U^*\|_{2,\infty} \to 0
\]

as \(\|U - U^*\|_{2,\infty} \to 0\) independent of \(h\) and \(z\).

If \(\tau = 0\),

\[
(A.1) = \left| \mu(U - U^*)' + (U - U^*) - \int_{\mathbb{R}} K(z - y)(S'_0(U(y) - \theta) - S'_0(U^*(y) - \theta))h(y) \, dy \right|
\leq (\mu + 1) \|U - U^*\|_{2,\infty}
+ \left| K(z - U^{-1}(\theta))h(U^{-1}(\theta)) - K(z - U^{*-1}(\theta))h(U^{*-1}(\theta)) \right|
+ \left| \frac{K(z - U^{-1}(\theta))h(U^{-1}(\theta))}{U'(U^{-1}(\theta))} - \frac{K(z - U^{*-1}(\theta))h(U^{*-1}(\theta))}{U'(U^{*-1}(\theta))} \right|
\leq (\mu + 1) \|U - U^*\|_{2,\infty}
+ \sup |K(\cdot)| \|h\|_{2,\infty} \left| \frac{U^{-1}(\theta) - U^{*-1}(\theta)}{U'(U^{-1}(\theta))} \right|
+ \|h\|_{2,\infty} \sup \left| \frac{K(\cdot - U^{-1}(\theta))}{U'(U^{-1}(\theta))} - \frac{K(\cdot - U^{*-1}(\theta))}{U'(U^{*-1}(\theta))} \right| \to 0
\]

as \(\|U - U^*\|_{2,\infty} \to 0\) independent of \(h\) and \(z\).

Moreover,

\[
(A.2) = |\mu - \mu^*||U^*(z)| \leq |\mu - \mu^*| \|U^*\|_{2,\infty} \to 0
\]
By Lemma 2.5 (iii), we may choose $\delta$ so that 

$$
\int_{U^{*-1}(\theta)} U^{*-1}(\theta+y) S'(y) (S'_r(y) - S'_0(y)) \frac{h(U^{*-1}(y+\theta))}{U^*(U^{*-1}(y+\theta))} dy \\
\leq C \|S_r - S_0\|_1 \to 0
$$

as $\tau \to 0^+$. Note that all inverse functions exist by the closeness of $U$ and $U^*$ to $U_0$.

Proof of Lemma 2.8.

Using standard methods like those in [1], we first show that $\epsilon_1$ and $\delta_1$ may be chosen so that for fixed $S_0$ with $\|S_r - S_0\|_1 < \delta_1$, the function $\mathcal{N}$ is a contraction mapping. Then we show we may choose $\epsilon_0$ so that $\mathcal{N}$ maps the ball $E(U_0, \mu_r) \times \mathbb{R} \cap B((U_0, \mu_r); \epsilon_0)$ into itself. We use the following notation:

$$
\|(U, \bar{\mu})\| = \|U\|_{2,\infty} + |\bar{\mu}|, \quad L := DF_{U, \mu}[U_0, \mu_r, S_0].
$$

For the first part, let $S_0$, $(U_1, \bar{\mu}_1)$, and $(U_2, \bar{\mu}_2)$ be fixed. Then

$$
\mathcal{N}[U_2, \bar{\mu}_2, S_r] - \mathcal{N}[U_1, \bar{\mu}_1, S_r] \\
= L^{-1} (L(U_2 - U_1, \bar{\mu}_2 - \bar{\mu}_1) - (F(U_2, \bar{\mu}_2, S_r) - F(U_1, \bar{\mu}_1, S_r))) \\
\leq \epsilon_1 \|L^{-1}\| \left(\|U_2 - U_1\|_{2,\infty} + |\bar{\mu}_2 - \bar{\mu}_1|\right).
$$

By the continuity claims in Lemma 2.5 (i) and (ii), choose $\delta_1 > 0, \epsilon_0 > 0$ small so that $\|S_r - S_0\|_1 < \delta_1, (\|U_2 - U_1\|_{2,\infty} + |\bar{\mu}_2 - \bar{\mu}_1|) < \epsilon_0$. Then $\epsilon_1 = \frac{1}{2\|L^{-1}\|}$ may be chosen independent of $(U_1, \bar{\mu}_1), (U_2, \bar{\mu}_2)$. 

For the second part, let $S_0$, $(U, \bar{\mu})$ be fixed. Then

$$
\mathcal{N}[U, \bar{\mu}, S_r] - \mathcal{N}[U_0, \mu_r, S_r] \\
= \mathcal{N}[U, \bar{\mu}, S_r] - \mathcal{N}[u_r, \mu_r, S_r] \\
\leq \|\mathcal{N}[U, \bar{\mu}, S_r] - \mathcal{N}[U_0, \mu_r, S_r]\| \\
+ \|\mathcal{N}[U_0, \mu_r, S_r]\| \\
\leq \epsilon_0.
$$

By the analysis for the contraction mapping in (A.4),

$$
\|\mathcal{N}[U, \bar{\mu}, S_r] - \mathcal{N}[U_0, \mu_r, S_r]\| \leq \frac{1}{2} (\|U - U_0\|_{2,\infty} + |\bar{\mu} - \mu_r|) \\
\leq \epsilon_0.
$$

By Lemma 2.5 (iii), we may choose $\delta_2$ so that

$$
\|\mathcal{N}[U_0, \mu_r, S_r]\| \leq \frac{\epsilon_0}{2}.
$$

By (A.5),

$$
\|\mathcal{N}[U_0, \mu_r, S_r]\| \leq \frac{\epsilon_0}{2}.
$$

By (A.6),

$$
\|\mathcal{N}[U_0, \mu_r, S_r]\| \leq \frac{\epsilon_0}{2}.
$$

By Lemma 2.5 (iii), we may choose $\delta_2$ so that

$$
\|\mathcal{N}[U_0, \mu_r, S_r]\| = \left\| L^{-1} (F(U_0, \mu_r, S_r) - F[U_0, \mu_r, S_r]) \right\| \\
\leq \frac{\epsilon_0}{2}.
$$

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when \( \| S_\tau - S_\tau \|_1 < \delta_2 \). Finally the choice \( \delta_0 = \min\{\delta_1, \delta_2\} \) guarantees \( \| N(U, \mu, S_\tau) - (U_\tau, \mu) \| \leq \epsilon_0 \). We may write \( F_1 = \psi_\tau \) as

\[
\psi_\tau = \frac{\psi_1}{\mu} - 2AaS_\tau(U_f - \theta) + 2BbS_\tau(U_f - \theta) = 0.
\]

Proof. Linearization and direct calculations show \( (3.30) \) is equivalent to

\[
\psi_1 = \psi_\tau^* - \frac{1}{\mu} \psi_\tau^* - \frac{S_\tau(U_f - \theta)}{\mu} p(z; \psi_\tau^*)
\]

and the first equation in \( (B.1) \) satisfies

\[
\psi_1' = \frac{1}{\mu} \psi_1 + 2AaS_\tau(U_f - \theta) \psi_3 + 2BbS_\tau(U_f - \theta) \psi_5.
\]

natural solutions for \( \psi_3 \) and \( \psi_5 \) are

\[
\psi_3 = -\frac{1}{2Aa\mu} p_1(z; \psi_\tau^*), \quad \psi_5 = \frac{1}{2Bb\mu} p_2(z; \psi_\tau^*).
\]

respectively, making the right hand sides of \( (B.4) \) and \( (B.5) \) equivalent. Indeed, system \( (B.1) \) then implies \( \psi_2 = -\psi_3' \) and \( \psi_4 = -\psi_5' \) so we have

\[
\psi_2' = \frac{1}{2Aa\mu} p_1(z; \psi_\tau^*), \quad \psi_4' = -\frac{1}{2Bb\mu} p_2(z; \psi_\tau^*).
\]

Recalling the form of \( K_i \) as exponentials, properties of Fourier transforms of convolutions shows \( p_1 \) and \( p_2 \) satisfy \( p_1'' = a^2 p_1 - 2Aa \psi_1 \) and \( p_2'' = b^2 p_2 - 2Bb \psi_1 \). A simple calculation shows

\[
\psi_2' = -\frac{1}{\mu} \psi_1 - a^2 \psi_3, \quad \psi_4' = -\frac{1}{\mu} \psi_1 - b^2 \psi_5
\]

are satisfied. Therefore, by choosing \( \psi_1 = \psi_\tau^* \), equation \( (B.1) \) is solved.

\[ \square \]

Remark B.1. In [15], the authors draw a related connection when \( K(x) = \frac{1}{\pi} e^{-|x|} \).
Competing Interests
The author declares that they have no competing interests.

References
[1] E. Accinelli, A generalization of the implicit function theorem, Appl. Math. Sci., 4 (2010), pp. 1289–1298.
[2] S. i. Amari, Dynamics of pattern formation in lateral-inhibition type neural fields, Biol. Cybernet., 27 (1977), pp. 77–87.
[3] P. W. Bates, P. C. Fife, X. Ren, and X. Wang, Traveling waves in a convolution model for phase transitions, Arch. Ration. Mech. Anal., 138 (1997), pp. 105–136.
[4] A. Benucci, R. A. Frazor, and M. Carandini, Standing Waves and Traveling Waves Distinguish Two Circuits in Visual Cortex, Neuron, 55 (2007), pp. 103–117.
[5] F. Botelho, J. Jamison, and A. Murdock, Single-pulse solutions for oscillatory coupling functions in neural networks, J. Dynam. Differential Equations, 20 (2008), pp. 165–199.
[6] P. Bressloff, Stochastic neural field theory, Encyclopedia of Computational Neuroscience, (2015), pp. 2891–2895.
[7] P. C. Bressloff, Traveling fronts and wave propagation failure in an inhomogeneous neural network, Phys. D, 155 (2001), pp. 83–100.
[8] ———, Weakly interacting pulses in synaptically coupled neural media, SIAM J. Appl. Math., 66 (2005), pp. 57–81.
[9] Y. Chagnac-Amitai and B. Connors, Horizontal spread of synchronized activity in neocortex and its control by gaba-mediated inhibition, J. Neurophysiol., 61 (1989), pp. 747–758.
[10] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differential Equations, 2 (1997), pp. 125–160.
[11] B. W. Connors and Y. Amitai, Generation of epileptiform discharges by local circuits in neocortex, Epilepsy: Models, Mechanisms and Concepts, (1993), pp. 388–424.
[12] S. Coombes and M. R. Owen, Evans Functions for Integral Neural Field Equations with Heaviside Firing Rate Function, SIAM J. Appl. Dyn. Syst., 3 (2004), pp. 574–600.
[13] S. Coombes and H. Schmidt, Neural fields with sigmoidal firing rates: Approximate solutions, Discrete Contin. Dyn. Syst. Ser. A, 28 (2010), pp. 1369–1379.
[14] A. J. Elvin, C. R. Laing, R. I. McLachlan, and M. G. Roberts, Exploiting the Hamiltonian structure of a neural field model, Phys. D, 239 (2010), pp. 537–546.
[15] G. B. Ermentrout, J. Z. Jalics, and J. E. Rubin, Stimulus-driven traveling solutions in continuum neuronal models with a general smooth firing rate function, SIAM J. Appl. Math., 70 (2010), pp. 3039–3064.
[16] G. B. Ermentrout and J. B. McLeod, Existence and uniqueness of travelling waves for a neural network, Proc. Roy. Soc. Edinburgh Sect. A, 123A (1993), pp. 461–478.
[17] J. W. Evans, Nerve Axon Equations: 1 Linear Approximations, Indiana Univ. Math. J., 21 (1972), p. 877.
[18] ——, Nerve Axon Equations: II Stability at Rest, Indiana Univ. Math. J., 22 (1972), p. 75.
[19] ——, Nerve Axon Equations: III Stability of the Nerve Impulse, Indiana Univ. Math. J., 22 (1972), p. 577.
[20] ——, Nerve Axon Equations: IV The Stable and the Unstable Impulse, Indiana Univ. Math. J., 24 (1975), pp. 1169–1190.
[21] G. Faye, Existence and Stability of Traveling Pulses in a Neural Field Equation with Synaptic Depression, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 2032–2067.
[22] G. Faye and A. Scheel, Existence of pulses in excitable media with nonlocal coupling, Adv. Math., 270 (2015), pp. 400–456.
[23] N. Fenichel and J. Moser, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J., 21 (1971), pp. 193–226.
[24] I. Ferezou, S. Bolea, and C. C. Petersen, Visualizing the cortical representation of whisker touch: voltage-sensitive dye imaging in freely moving mice, Neuron, 50 (2006), pp. 617–629.
[25] D. Golomb and Y. Amitai, Propagating neuronal discharges in neocortical slices: computational and experimental study, J. Neurophysiol., 78 (1997), pp. 1199–1211.
[26] Y. Guo, Existence and stability of traveling fronts in a lateral inhibition neural network, SIAM J. Appl. Dyn. Syst., 11 (2012), pp. 1543–1582.
[27] Y. Guo and A. Zhang, Existence and nonexistence of traveling pulses in a lateral inhibition neural network, Discrete Contin. Dyn. Syst. Ser. S, 21 (2016), pp. 1729–1755.
[28] S. Hastings, On travelling wave solutions of the hodgkin-huxley equations, Arch. Ration. Mech. Anal., 60 (1976), pp. 229–257.
[29] M. W. Hirsch, C. C. Pugh, and M. Shub, Invariant manifolds, Bull. Amer. Math. Soc., 76 (1970), pp. 1015–1019.
[30] C. Jones and N. Kopell, Tracking invariant manifolds with differential forms in singularly perturbed systems, J. Differential Equations, 108 (1994), pp. 64–88.
[31] C. Jones, N. Kopell, and R. Langer, Construction of the fitzhugh-nagumo pulse using differential forms, in Patterns and dynamics in reactive media, Springer, 1991, pp. 101–115.
[32] C. K. Jones, T. J. Kaper, and N. Kopell, Tracking invariant manifolds up to exponentially small errors, SIAM J. Appl. Math., 27 (1996), pp. 558–577.
[33] Z. P. Kilpatrick and P. C. Bressloff, Effects of synaptic depression and adaptation on spatiotemporal dynamics of an excitatory neuronal network, Phys. D, 239 (2010), pp. 547–560.
[34] Z. P. Kilpatrick, S. E. Folias, and P. C. Bressloff, Traveling pulses and wave propagation failure in inhomogeneous neural media, SIAM J. Appl. Dyn. Syst., 7 (2008), pp. 161–185.
[35] K. Kishimoto and S. Amari, Existence and stability of local excitations in homogeneous neural fields, J. Math. Biol., 7 (1979), pp. 303–318.
[36] C. R. Laing, W. C. Troy, B. Gutkin, and G. B. Ermentrout, Multiple bumps in a neuronal model of working memory, SIAM J. Appl. Math., 63 (2002), pp. 62–97.
[37] S.-H. Lee, R. Blake, and D. J. Heeger, Traveling waves of activity in primary visual cortex during binocular rivalry, Nat. Neurosci., 8 (2005), pp. 22–23.
A. G. Leventhal, Y. Wang, M. Pu, Y. Zhou, and Y. Ma, Gaba and its agonists improved visual cortical function in senescent monkeys, Science, 300 (2003), pp. 812–815.

G. Lv and M. Wang, Traveling waves of some integral-differential equations arising from neuronal networks with oscillatory kernels, J. Math. Anal. Appl., 370 (2010), pp. 82–100.

F. M. G. Magpantay and X. Zou, Wave fronts in neuronal fields with nonlocal post-synaptic azonal connections and delayed nonlocal feedback connections, Math. Biosci. Eng., 7 (2010), pp. 421–442.

I. Nauhaus, L. Busse, M. Carandini, and D. L. Ringach, Stimulus contrast modulates functional connectivity in visual cortex, Nat. Neurosci., 12 (2009), p. 70.

B. Pakkenberg, D. Pelvig, L. Marner, M. J. Bundgaard, H. J. G. Gundersen, J. R. Nyengaard, and L. Regeur, Aging and the human neocortex, Exp. Geront., 38 (2003), pp. 95–99.

C. C. Petersen, A. Grinvald, and B. Sakmann, Spatiotemporal dynamics of sensory responses in layer 2/3 of rat barrel cortex measured in vivo by voltage-sensitive dye imaging combined with whole-cell voltage recordings and neuron reconstructions, J. Neurosci., 23 (2003), pp. 1298–1309.

D. J. Pinto and G. B. Ermentrout, Spatially structured activity in synaptically coupled neuronal networks: I. traveling fronts and pulses, SIAM J. Appl. Math., 62 (2001), pp. 206–225.

D. J. Pinto, R. K. Jackson, and C. E. Wayne, Existence and stability of traveling pulses in a continuous neuronal network, SIAM J. Appl. Dyn. Syst., 4 (2005), pp. 954–984.

M. Rus, A note on the existence of positive solutions of fredholm integral equations, Fixed Point Theory, 5 (2004), pp. 369–377.

B. Sandstede, Evans functions and nonlinear stability of traveling waves in neuronal network models, Int. J. Bifurc. Chaos, 17 (2007), pp. 2693–2704.

T. K. Sato, I. Nauhaus, and M. Carandini, Traveling Waves in Visual Cortex, Neuron, 75 (2012), pp. 218–229.

P. Szmolyan, Transversal heteroclinic and homoclinic orbits in singular perturbation problems, J. Differential Equations, 92 (1991), pp. 252–281.

R. Traub, J. Jefferys, and R. Miles, Analysis of the propagation of disinhibition-induced after-discharges along the guinea-pig hippocampal slice in vitro., J. Physiol., 472 (1993), pp. 267–287.

F. B. Wagner, E. N. Eskandar, G. R. Cosgrove, J. R. Madsen, A. S. Blum, N. S. Potter, L. R. Hochberg, S. S. Cash, and W. Truccolo, Microscale spatiotemporal dynamics during neocortical propagation of human focal seizures, Neuroimage, 122 (2015), pp. 114–130.

H. R. Wilson, R. Blake, and S.-H. Lee, Dynamics of travelling waves in visual perception, Nature, 412 (2001), p. 907.

H. Zhang and J. Jacobs, Traveling theta waves in the human hippocampus, J. Neurosci., 35 (2015), pp. 12477–12487.
[55] L. Zhang, *On stability of traveling wave solutions in synaptically coupled neuronal networks*, Differential Integral Equations, 16 (2003), pp. 513–536.

[56] ———, *Traveling waves of a singularly perturbed system of integral-differential equations arising from neuronal networks*, J. Dynam. Differential Equations, 17 (2005), pp. 489–522.

[57] ———, *How do synaptic coupling and spatial temporal delay influence traveling waves in nonlinear nonlocal neuronal networks?*, SIAM J. Appl. Dyn. Syst., 6 (2007), pp. 597–644.

[58] L. Zhang, *Existence and uniqueness of wave fronts in neuronal network with nonlocal post-synaptic axonal and delayed nonlocal feedback connections*, Adv. Difference Equ., 2013 (2013), p. 243.

[59] L. Zhang and A. Hutt, *Traveling wave solutions of nonlinear scalar integral differential equations arising from synaptically coupled neuronal networks*, J. Appl. Anal. Comp., 4 (2014), pp. 1–68.

[60] L. Zhang, L. Zhang, J. Yuan, and C. Khalique, *Existence of wave front solutions of an integral differential equation in nonlinear nonlocal neuronal network*, in Abstr. Appl. Anal., vol. 2014, Hindawi Publishing Corporation, 2014.