Flatlandia : a flat spacetime description of gravitation

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Abstract

The proof that a consistent theory of gravity cannot be constructed in a flat spacetime rests on the assumption that atoms be equal in every conditions. However special relativity and the principle of equivalence impose that atoms are equal only when relatively at rest and in equivalent gravitational positions. This is further backed up by implementing in QM the mass variation for moving objects and for bodies in a gravitational field.

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1 Introduction

It is usually said (see e.g. [1]) that the gravitational redshift proves that gravity curves spacetime.

According to the argument by Einstein himself special relativity (SR) is enough to prove that space-time is NOT flat.

Indeed by considering two atoms at different heights (see Fig.1), the photon emitted at \( h = z_2 - z_1 \), because of the energy mass equivalence, is received at the ground with an increased angular frequency

\[
\omega_A \simeq \omega_{A'}(1 + gh/c^2) \tag{1}
\]

thus implying

\[
\Delta \tau_{A'} > \Delta \tau_A \tag{2}
\]

i.e. that time "runs quicker" higher up.
Figure 1: The clock at the height $z_2$ runs quicker than the one at $z_1$. Therefore the two parallel sides at the different heights have unequal length, irrespective of (possible different) inclinations of the sides which allow the information transfer. Thus one concludes that **the inadequacy of the Minkowski space to account for reality be an experimental fact.**

The crucial point in this argument is however the **assumption** that the clocks (i.e. the atoms) be equal at different places.

Now, whereas there is little doubt that in the same conditions atoms are indeed equal because of quantization, already relatively moving ones are not, because of the mass increase due to SR. And there is no guarantee that at the $10^{-15}$ per meter level for gravitational differences at the earth surface the same might not be so!

What we are going to show is that the **quantized constituents of our standard clocks (and rods) i.e. the atoms are not equal.** Thus SR and the equivalence principle imply that a flat space-time description is legitimate as also backed up in a QM description.

## 2 Gravitation, relativity and QM

That space and time of an event form an invariant four vector is by now almost common place. This allows to relate phenomena in two different inertial frames, with the well known phenomena of time dilation and length contraction.

Things get more complicated in the presence of gravitation since, trivially, even if relatively at rest two different point are not equivalent. How can one **relate only locally well defined variables** using just special relativity, working hence in a flat space?

Let us follow the arguments of the work by Schiff [2], based on SR and the equivalence principle, using slightly different notations. (This work has been criticized as simplistic in [3], [4] which have been countercriticized in [5]. The problem will be commented upon in the following.)

Given two atoms in the gravitational field of a mass M, at $A = R$ and $A' = R' = R + h$, the standard of comparison (proper time) can only be provided by an atom at rest in a place free of gravitational effects at A” i.e. at $\infty$. [2]
To attain that situation they must be imparted a suitable velocity

\[ v_A^2 = 2GM/R > v_{A'}^2 = 2GM/R' \]  \hspace{1cm} (3)

where the previous relations are simply based on the trivial non relativistic energy (N.R.) conservation to obtain the escape velocity of a mass \( m \) in a standard Newtonian potential generated by \( M \)

\[ m, v^2/2 - GMm/r = 0 \] \hspace{1cm} (4)

and on the equivalence principle. *Thus the first atom, as seen from the standard one in \( A'' \) will live longer due to its velocity. And the same happens for the one at \( A' \), which however, because of the smaller velocity required to reach \( A'' \) where the comparison can be made, will live less.*

This is traditionally expressed as "times runs quicker at places of lower gravitational field!"

The previous argument can be quantified by using

\[ t_{A''} = \frac{\tau_A}{\sqrt{1-v_A^2/c^2}} = \frac{\tau_{A'}}{\sqrt{1-v_{A'}^2/c^2}} \] \hspace{1cm} (5)

i.e.

\[ \tau_A < \tau_{A'} \] \hspace{1cm} (6)

and Eq. (5) trivially transforms by use of the equivalence principle into

\[ \frac{\omega_A}{\omega_{A'}} = \sqrt{\frac{1-2GM/c^2R}{1-2GM/c^2R'}} \] \hspace{1cm} (7)

This expression just reproduces in elementary terms the famous red-violet shift of GR.

Expanding around the point \( A \) where \( g \) is defined (we are now specializing to the Earth case) one gets the usual result

\[ \omega_A \simeq \omega_{A'}(1 + gh/c^2) \] \hspace{1cm} (8)

It is worth stressing that whereas from our anthropomorphic point of view \( A' \) is more energetic than \( A \) (by naively applying the previous argument it might seem to live longer with respect to \( A \) since it would attain it with a non zero velocity), the reverse happens since the comparison has to be made with respect to a "gravitation free" clock (the one at \( \infty \))!

Now a length \( l \) is respectively seen by the two meters as

\[ \Delta z = l\sqrt{1-2GM/c^2R} \text{ and as } \Delta z' = l\sqrt{1-2GM/c^2R'} \]. Thus

\[ \frac{\Delta z_A}{\Delta z_{A'}} = \sqrt{\frac{1-2GM/c^2R}{1-2GM/c^2R'}} \] \hspace{1cm} (9)

i.e. \( \Delta z_A < \Delta z_{A'} \) which means that the quicker \( A \), i.e. the closer to \( M \), the shorter the measured length.

Notice that this is in line with Eq. (7) from which for the respective wavelengths \( \lambda \)

\[ \frac{\lambda_A}{\lambda_{A'}} = \sqrt{\frac{1-2GM/c^2R}{1-2GM/c^2R'}} \] \hspace{1cm} (10)
The reason to stress this obvious fact lies in the critiques to the approach by Schiff mentioned above. Whereas his "elementary" derivation of time variation in the gravitational field has been accepted, it has not been so for space intervals. It has been argued that the Schwarzschild metric cannot be reproduced with so too simple arguments. Now, whereas it is clear that a distinctive feature of gravitation i.e. non linearity cannot be reproduced by SR and has to be introduced e.g. by the effective mass approach in the expression for $v^2$ (as discussed later), it is equally obvious that the variation in the frequency of a radiation implies an opposite variation in the wavelength with which one measures space intervals. This is incidentally the standard definition of the meter.

*Let us underline once more that the use of SR implies that $c$ is constant and that the equivalence principle has been advocated!* In this connection let us also parenthetically remember the related gravity-induced neutron quantum interference [6] in connection with the identity of gravitational and inertial mass.

Notice that loosely speaking by such a procedure one has disposed of the gravity. In other words one cannot let the emitted photon vary a second time its frequency in the (by now no longer existing) gravitational field.

![Figure 2: The atom A is brought to the height $z$. In the comparison of A and A' attention must be paid to the fact whereas from an anthropomorphic point of view the energy of A' is bigger than that of A, since the comparison must be made with respect to a gravity free region i.e. at $\infty$, the reverse happens. Therefore the angular frequency and "meter" at A' are respectively smaller and bigger that at A. The equivalence principle has been used.](image)

A QM realization of the previous SR predictions is immediately obtained by using the standard expression for the quantizes energy levels

$$\hbar \omega \simeq \Delta E \propto m \alpha^2 / \hbar^2 \propto \alpha / a_0$$

(11)
so that, by the above argument

$$\hbar \omega \propto \frac{m \alpha^2}{\hbar^2} \Rightarrow \frac{(m \alpha^2)/\hbar^2}{\sqrt{1-v^2/c^2}} = \frac{(m \alpha^2)/\hbar^2}{\sqrt{1-2GM/c^2R}}$$  \hspace{1cm} (12)$$

We then see that the conclusion, mentioned in the introduction, that space time is curved, apparently based on SR, is indeed in contradiction with it, since it does not take into account the mass variation of moving bodies which is probably its most distinctive feature!

Now the reluctance to accept the simple SR reproduction of the Schwarzschild metric is probably due to the often not enough appreciated fact that already SR is, even if disguised, a dynamical theory.

As a matter of fact the if the system $S'$ has a velocity $v$ with respect to $S$ some work must have been done to bring it to such a kinematical situation.

Indeed the famous $\gamma$ factor can be reexpressed as follows

$$\sqrt{1-v^2/c^2} = \sqrt{1-\frac{m\Delta v^2}{mc^2}} = \sqrt{1-2 \int F \cdot dr/mc^2}$$  \hspace{1cm} (13)$$

where no mention is made of the necessary force. In the gravitational case that is made explicit since the effect cannot be put under the rug, gravitation neither being a short range force nor a "neutralizable" one because of the existence of opposite charges like e.m.

In conclusion the usual sentence "atoms are all equal" should turn into atoms are equal only when relatively at rest in gravitationally equivalent places.

3 On the equivalence principle

Let us now investigate higher order terms in the coupling constant $G$ i.e. non linearity.

The gravitational interaction energy is traditionally given by the Newtonian expression

$$E_G = U = -\frac{GMm}{r}$$  \hspace{1cm} (14)$$

However, since energy is the source of gravitation, the mass (energy) $m$ is not the mass the second object possesses at $\infty$. The interaction renormalizes it, to the first order, in a space dependent way \[7\] \[8\]

$$m \Rightarrow m' = m_0(1 - \frac{GM}{c^2r})$$  \hspace{1cm} (15)$$

where $m_0$ stands for the bare mass (without the influence of M), so that, relying again on Einstein’s mass-energy equivalence, the effective potential energy is

$$U' = -(GMm_0/r)/(1 - r'_S/r)$$  \hspace{1cm} (16)$$

where

$$r'_S = GM/c^2$$  \hspace{1cm} (17)$$

which represents a low brow implementation of the non linearity of gravitation [?] which account for the "two additional crucial tests of GR" \[7\] and for the higher order
corrections in the derivation of the "effective vector formulation" of gravitation \[11\]. Only when the effective mass varies because of a distance variation, can its effect be appreciated.

Let us first underline that obviously self energy effects do not intervene in the atomic electromagnetic interaction.

Indeed in Eq.(12) manifestly \(m_i\) enters. If one would allow for the substitution of the gravitational effective mass one would obtain

\[
\hbar \omega \propto \left(\frac{m \alpha}{\hbar^2}\right) \left(1 - \frac{GM}{c^2 R}\right) = \frac{m \alpha^2}{\hbar^2} (1 - \frac{GM}{c^2 R}) \quad (18)
\]

In other words there would be no frequency shift in the gravitational field!

The very fact that the difference between these two interactions is not just a matter of different coupling constants is probably fundamental in allowing to measure the expansion of the Universe.

Let us then come to the principle of equivalence. If we write a post Newtonian equation of motion, correct to \(O(v^4/c^4)\)

\[
\frac{d}{dt}(mv/\sqrt{1-v^2/c^2}) = -GMm \frac{r}{r^3} (1 - 2r'/r) \quad (19)
\]

whose solution is

\[
v^2/c^2 = 2GM/c^2r(1 - GM/c^2r) \quad (20)
\]

\(m\) appearing on both sides of the equation of motion represents at the same time the SR rest mass \(m_0\) and the "gravitation free" mass at \(\infty\). In this respect the equivalence principle holds. If however the corrective self energy bracket is incorporated in the gravitational effective mass in a purely Newtonian language then the two of course differ.

Therefore one might conceive of improving consistently Eq.(7) via the second order effects of Eq.(20), with an ensuing tiny differences with respect to the GR value.

In particular the wavelength \(\lambda^*\) of a photon emitted at the surface at \(R^*\) a receding star of mass \(M^*\) when arriving at the earth would be given by

\[
z + 1 = \frac{\lambda}{\lambda^*} = \sqrt{\frac{1 + v/c}{1 - v/c}} \sqrt{\frac{(1 - 2GM_E/c^2 R_E)(1 - GM_E/c^2 R_E)}{(1 - 2GM^*/c^2 R^*)(1 - GM^*/c^2 R^*)}} \quad (21)
\]

the first term representing the ordinary relativistic Doppler effect, whereas the second the gravitational effect. Needless to say this time the velocity is the real one and not the artifact used in the previous paragraph to determine via SR alone the frequency change of an atom at rest in the gravitational field. In traditional terms the photon emitted by the star looses energy to reach \(\infty\) and is then violet shifted falling in the gravitational field of the earth (this second effect being negligible). This might somewhat alter the standard Hubble' treatment and will be considered elsewhere.

Let us conclude by considering the system moon-earth, for simplicity in a circular orbit of radius \(R_M\), in the field of the sun at a distance \(S\) from the earth. Clearly whereas the moon effective mass with respect to the earth does not vary, it does as regards the sun and the difference when it is aligned in between and outward is \(2GM_S/c^2 R_S / 2GM_E/c^2 R_E\).

This necessary correction in a Newtonian approach is included via "space time curvature" in GR and is now automatically accounted for in a consistent (to this order) treatment of the atoms Eq.s(12)(20) our bodies are made of.
Thus luckily all kind of ”clocks”measure the same time!

4 Rotating frames

a) a horizontal Pound Rebka experiment

To further clarify the previous considerations, let us take the elementary case of an atom which has been brought to move on a circular orbit of radius R with velocity v.

According to SR its mean life is increased due to the well known time dilation factor depending on the velocity. On the other hand it is subjected to an acceleration. How can we reconcile the two facts? Simply because

\[ v^2 = \omega^2 R^2 = (\omega^2 R) R = aR \]  

(22)

where \( a \) stands for the acceleration and \( \omega \) for the angular velocity.

Thus time \( \tau \) at radius \( R \) is connected to \( t \) measured in a rotation (acceleration) free point i.e. at \( R = 0 \) by

\[ t = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\tau}{\sqrt{1 - \frac{aR}{c^2}}} \]  

(23)

This is reminiscent of the gravitational case where \( v^2 = 2gh \) and \( t = \tau/\sqrt{1 - 2gh/c^2} \), the difference of the factor 2 simply coming from the constant vs. linear acceleration field. Indeed, exactly in the same way, the work done against the centrifugal force for the mass \( m \) to reach the center, where the comparison can be made, is

\[ \Delta K = \Delta \left( \frac{mv^2}{2} \right) = \int m\omega^2 r \cdot dr = \int F \cdot dr \]  

(24)

The previous statement thus easily follows since only the inertial mass enters.

One could thus easily envisage a ” horizontal Pound Rebka experiment” with the standard difference in the emission frequency of the moving (accelerated) atom. To reproduce their result one would simply need, for \( R = 1m \), a period \( T \simeq .3 \) sec.

b) is space curved?

Consider again a rotating disk.

The tangential meter of the comoving observer A’ will get Lorentz contracted because of special relativity as

\[ R'_t = R \sqrt{1 - \frac{v^2}{c^2}} \]  

(25)

but, at the same time, the radial acceleration will contract the radial meter

\[ R' = R \sqrt{1 - \frac{aR}{c^2}} \]  

(26)

Thus operationally the observer A’ will find no difference with respect to the laboratory observer A who made the same measurement with the disk at rest and Euclidian geometry remains valid . Otherwise one would have this problem also for atoms!
The Ehrenfest [10] paradox is thus disposed of.

The connection between gravitational and rotating systems is thus reconfirmed up to a point: the tangential (orthogonal to the acceleration) velocity in the second case.

In the previous considerations the standard N.R. kinematics of Eq.(22) has been used, reproducing however GR results. Apart from speculations about the (non) existence of rigid bodies for \( R \geq c/\omega \), we are in a sense in a kinematical situation since no indication is given of the force that should impart such an acceleration. Thus the essential feature which makes gravitation non linear i.e. self energy is of course absent in the rotating frame dynamics.

c) clock synchronization : a pseudo problem?

The argument about space time curvature goes further when considering clock synchronization. This time the point is not just an academic one. Indeed it is known that two clocks (satellites) on the same orbit in the equatorial plane around the earth in opposite directions are desynchronized when they meet after one revolution by a quantity proportional to the angular velocity of the earth. Is this delay due to GR? Has therefore anything to do with ”space-time curvature“?

From the previous equations one immediately gets that a photon traveling along the circumference arrives at the starting point after a time \( \Delta T = \frac{2\pi R/c}{\sqrt{1 - v^2/c^2}} \) (27)

thus non synchronization is a sort of a trivial unavoidable effect which will be further commented upon later on in connection with the Sagnac effect. And is nothing else than what we encounter in the twin paradox. When we come back to the same point (having necessarily accelerated !), where only the direct comparison can be made, the twins are ”desynchronized“: they do not indicate the same time.

Coming to physics let us remark that the gravitational clock precession arises already because of the traditional Coriolis force: trivially the standard Newtonian equation has now two distinct roots. To get the right value, which coincides with the GR prediction, one has however to add the gravitomagnetic force. In the end, even in this case, the flat spacetime picture coexists therefore peacefully with the much more elaborated curved spacetime formulation.

Of course if one considers just one satellite period as seen from the earth one has to add the SR Thomas precession, whereas in the case of the two counterrotating satellites this effect cancels out and, on the contrary, the precession and the regression add up.

Let us finally come to frame dragging and geodetic precession.

In [14] it has been shown that the orbital revolution of the earth around the spin S of a planar gyroscope in a satellite at a distance R from its center causes the latter to rotate with an angular velocity of precession given by

\[ \Omega_{geo}/\omega_{orb} = 3/2(GM_T/(c^2R)) \] (28)

i.e. essentially the usual (weak) strong field parameter in a flat Minkowski spacetime.

Thus a spinning object, even in free fall, would experience an unescapable torque.
\[ \sum F = 0 \implies \sum M = 0 \]  \hspace{1cm} (29)

In other words free fall does not imply cancellation of dynamical effects, even if one might argue that a "real" (i.e. extended) gyroscope undergoes indeed tidal effects.

d) The Sagnac effect : absolute motion ?

The Sagnac effect has a long history, remarkable practical applications and has caused a considerable amount of discussions about its connection with Special and General Relativity.

In its standard form two counter propagating photon beams in a circular waveguide mounted on a disk are made to interfere after having traveled one circumference.

When the disk is put in rotation the interference figure is seen to shift by an amount proportional to the rotation velocity.

To clarify the previous point let us consider an interesting generalization where the same phenomenon has been observed in a linear moving waveguide, the details being given in [12].

In this case the two gamma rays proceeding along and in the opposite direction suffer a Doppler shift, which to the first order in the velocity \( v \) of the waveguide is given by

\[ \omega_{\pm} = \omega (1 \pm \frac{v}{c}) \]  \hspace{1cm} (30)

where \( \omega \) stands for the radiation angular frequency.

There is no doubt in the standard treatment of inertial systems that the propagation velocity \( c \) is the same in the two directions ! and that the time \( T \) each photon needs to travel the waveguide length \( L \) is given by \( T = L/c \).

Therefore a phase shift

\[ \Delta \phi = \Delta \omega \times T = \omega \times (2Lv/c^2) \]  \hspace{1cm} (31)

results.

Let us then apply the same considerations to the standard rotating case [13].

This time the two gamma rays proceeding in the clockwise and counterclockwise directions suffer a Doppler shift, which to the first order in the rotation angular velocity is given by

\[ \omega_{\pm} = \omega (1 \pm \frac{\Omega r}{c}) \]  \hspace{1cm} (32)

where \( \Omega \) stands for the angular velocity of the apparatus.

Therefore the photons arrive at the detector after a time \( T = 2\pi r/c \) with a phase difference

\[ \Delta \phi = \Delta \omega \times T = \omega (2\pi r/c) 2\pi r/c = \omega (2Lv/c^2) \]  \hspace{1cm} (33)

where \( L = 2\pi r \) is again the distance traveled along the circumference and \( v = \Omega r \). This result is generally presented in another form in terms of the disk area \( \Lambda \) as \( \omega \times (4\Lambda \Omega/c^2) \).

The reason to prefer the former expression is due to its similarity with the linear case, so to underline in both cases the role of the traveled distance.
Notice that the light velocity has been kept constant even in the accelerated case. In other words, in using the Doppler effect, the moving photon changes energy but propagates with a universal velocity $c$. For that reason we privilege the phase as the physical entity.

The similarity between the two formulas is manifest. In other words, the treatment of translations and rotation is the same.

The conclusion might seem surprising: even if in the rotating case we are in an accelerated frame and in the linear case apparently not, also in the latter case since the effect is due to a variation of the interference fringes, the interferometer is actually an accelerometer since it depends on velocity differences (with respect to $\Omega = 0$ in the circular case and to $v = 0$ in the linear one). More important in both cases accounted for just by SR!

As already stressed at the beginning in order to reach a relative velocity $v$ one of the two systems has to be accelerated!

5 Conclusions

In the present paper the role of SR in obtaining some GR results and in altering the curved spacetime picture has been underlined.

In all existing cases of weak gravitational fields the prediction coming from its plausible tensor nature can be simply and satisfactorily accounted for by an effective vector interaction [14] overcoming the much talked about causality violation of the traditional instantaneous Newtonian interaction. Indeed already at the level of classical e.m., a correct description in the Coulomb gauge for the instantaneous potential is commonplace as well as its Q.M. extension due to Fermi.

The precession of the perihelion of Mercury which had been regarded as a major problem for about a century, had also been accounted for by modifying Newton’s law and this was considered untenable because of the violation of Gauss theorem. However, as explained with elementary and rudimentary considerations in [7], the non-linearity of gravitation, or in other words the graviton self-coupling because of the mass-energy equivalence, have clarified why this is indeed the case. In simple terms the ”necessary” violation of Gauss theorem is just a reflection of the energy non-conservation in the sophisticated GR formulation.

Thus the fact of accounting for these facts by the explicit use of a non-linear term or by incorporating such an additional interaction in the metric, should dispose of the claim that space-time IS curved.

In a historical perspective a final remark: in spite of the work by Friedmann, GR has not predicted but only accommodated the expansion of the Universe, the piece of evidence having come from Hubble’s work!
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