SYMmetric functions in superspace

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Abstract. We construct a generalization of the theory of symmetric functions involving functions of commuting and anticommuting (Grassmannian) variables. These new functions, called symmetric functions in superspace, are invariant under the diagonal action of the symmetric group acting on the sets of commuting and anticommuting variables. We first obtain superspace analogues of a number of standard objects and concepts in the theory of symmetric functions: partitions, monomials, elementary symmetric functions, completely symmetric functions, power sums, involutions, generating functions, Cauchy formulas, and scalar products. We then consider a one-parameter extension of the combinatorial scalar product. It provides the natural setting for the definition of a family of “combinatorial” orthogonal Jack polynomials in superspace. We show that this family coincides with that of “physical” Jack polynomials in superspace that were previously introduced by the authors as orthogonal eigenfunctions of a supersymmetric quantum mechanical many-body problem. The equivalence of the two families is established by showing that the “physical” Jack polynomials are also orthogonal with respect to the combinatorial scalar product. This equivalence is also directly demonstrated for particular values of the free parameter.

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1. Introduction

Grassmannian variables refer to anticommuting variables \( \{\theta_i\} \), that is, to variables obeying the relation
\[
\theta_i \theta_j = -\theta_j \theta_i , \tag{1.1}
\]
and in particular,
\[
\theta_i^2 = 0 . \tag{1.2}
\]
In the first two subsections of this introduction, we have tried to review the origin of Grassmannian variables in mathematics and physics. Aside from its intrinsic interest, our aim in doing so is to show that our approach to superpolynomials is well established within the conceptual framework of supersymmetry. However, these first two subsections can be safely skipped. For the interested readers, we stress that no background in quantum field theory is assumed; we have simply tried to give a flavor of the underlying physics through simple illustrations. The introduction pertaining to the present work starts in subsection 1.3.

1.1. Grassmannian variables in physics and mathematics. As suggested by their name, the introduction of Grassmannian variables goes back to Grassmann in the framework of his theory of extension (1844), an ancestor of vector analysis (see e.g. [1], chapter 3).\(^1\) More precisely, Grassmann introduced in this context basis elements \( \{e_x, e_y, e_z\} \) and a new outer product satisfying
\[
e_x e_y = -e_y e_x, \quad e_x e_z = -e_z e_x, \quad e_y e_z = -e_z e_y, \quad e_x e_y = e_y e_x = e_z e_x = 0 . \tag{1.3}
\]
In the modern reinterpretation of this construction, \( e_x, e_y, e_z \) are replaced by \( dx, dy, dz, \) and the product is antisymmetrized (it corresponds to the wedge product); the anticommuting character of the variables is thus recaptured by a new product structure.

Anticommuting quantities \textit{per se} have been rediscovered in physics. In 1928, Jordan and Wigner realized that in order to implement the Pauli exclusion principle (i.e., two electrons cannot share...
the same set of quantum numbers), fermionic fields had to be quantized with modes subject to anticommutation relations instead of the usual commutation relations. This amounted to introduce operators $b_n$ (where $n$ is an integer labeling a normal mode) subject to the following rules:

$$\{b_n, b_m\} := b_n b_m + b_m b_n = \delta_{n+m,0}$$

(1.4)

the exclusion principle being then a consequence of $b_n^2 = 0$ for $n \neq 0$. (By contrast, the usual commutation relations would take the form $[a_n, a_m] = \delta_{n+m,0}$ where $a_n$ is an (ordinary) operator, i.e., a harmonic oscillator mode). Within this scheme, the new anticommuting quantities that were introduced were operators and not variables.

The early development of both quantum theory and quantum field theory relied entirely on the canonical quantization method, i.e., the lifting of a classical structure at the operator level and the decomposition of the operators in modes subject to a simple commutation or anticommutation relation according to their statistics (that of bosons or fermions respectively). The advent of Feynman path integration as an alternative method of quantization (a method, we stress, that involves variables instead of operators) posed the problem of integrating Grassmannian variables. It was first pointed out in [4] that the analog of the (bosonic) multidimensional Gaussian integral

$$I_b = \int \left( \prod_i dx_i \right) e^{\sum_{i,j} x_i b_{ij} x_j} \propto (\text{det} \ b)^{-1/2}$$

(1.5)

(with integration over all space), would have to take the following form for Grassmannian variables

$$I_f = \int \left( \prod_i d\theta_i \right) e^{\sum_{i,j} \theta_i f_{ij} \theta_j} \propto (\text{det} \ f)^{1/2}$$

(1.6)

This prompted the contribution of Berezin to the theory of Grassmannian algebra (whose results are summarized in [5]), and in particular, his proposal for the basic integral relations:

$$\int d\theta = 0, \quad \int d\theta \ \theta = 1$$

(1.7)

which ensured (1.6). With (left) differentiation defined in the natural way,

$$\frac{\partial}{\partial \theta_i} \theta_j \theta_k = \delta_{ij} \theta_k - \delta_{ik} \theta_j$$

(1.8)

it is readily seen that the Berezin integration of a Grassmannian variable is essentially equivalent to its differentiation:

$$\int d\theta [\alpha(x) + \theta \beta(x)] = \beta(x) = \frac{\partial}{\partial \theta} [\alpha(x) + \theta \beta(x)]$$

(1.9)

This work on Grassmannian algebras is at the root of the Lie algebra extensions with fermionic generators (see e.g., the review [4]).

Quite interestingly, at about the same time, another graded algebra arose in physics in a totally different setting. The superconformal (or Ramond-Neveu-Schwarz) algebra [7] is a graded version of the Virasoro algebra that appears when fermionic degrees of freedom are inserted in the dual model (at the time, an alternative to quantum field theory and subsequently understood as a string theory). Gervais and Sakita showed that this graded superconformal algebra comes from a sort of supergauge transformation, a transformation involving anticommuting parameters and which, in retrospect, precisely reflects the supersymmetric invariance of the dual model on the two-dimensional world-sheet [8].
1.2. **Supersymmetry, superfields and superspace.** Supersymmetry is certainly one of the most spectacular and profound ideas that has emerged from theoretical physics over the last thirty years [9]. This is a symmetry that relates bosons and fermions. And in the context of a quantum field theory, it corresponds to a fermionic symmetry that changes the statistics of the fields. Schematically, and for a one-dimensional space, this transformation takes the form

\[ \delta B(x) = \eta F(x), \quad \delta F(x) = \eta \partial_x B(x), \]  

(1.10)

where \( \eta \) is an anticommuting constant (\( \eta^2 = 0 \)) and where \( B \) and \( F \) are respectively bosonic and fermionic fields.

The discovery of supersymmetry within the context of four-dimensional quantum field theory, at a time when confidence in quantum field theory had been resurrected after a period of doubts, created a highly favorable situation for the rapid expansion of this area. To illustrate the depth and significance of these early developments, it suffices to mention the astonishing observation that whenever the supersymmetric transformation is localized (i.e., the parameter \( \eta \) is no longer regarded as a constant), supergravity emerges automatically. Gravity can thus be viewed as a consequence of supersymmetry [10].

In the study of supersymmetric quantum field theories, an important technical tool was introduced by Salam and Strathdee [11]: the concept of superspace.\(^3\) In its simplest setting (still keeping the illustrative formulas within the context of a one-dimensional space), a bosonic field \( B(x) \) and its fermionic partner \( F(x) \) are collected together within a superfield \( \Phi(x, \theta) \), regarded as a function of \( x \) and a new anticommuting space variable \( \theta \). Since \( \theta^2 = 0 \), the Taylor expansion of the superfield in the \( \theta \) variable contains only two terms, the two “component fields”

\[ \Phi(x, \theta) = F(x) + \theta B(x). \]  

(1.11)

The supersymmetric transformation can now be interpreted geometrically within superspace, the space described by the doublet \((x, \theta)\), as a simple translation of the form

\[ x \to x - \eta \theta, \quad \theta \to \theta + \eta. \]  

(1.12)

With

\[ \delta \Phi = \Phi(x - \eta \theta, \theta + \eta) - \Phi(x, \theta) = \theta \eta \partial_x F(x) + \eta B(x), \]  

(1.13)

and by setting

\[ \delta \Phi = \delta F(x) + \theta \delta B(x) \]  

(1.14)

we recover [1.10].\(^4\) Note that if we denote the supersymmetry transformation generator by \( Q \), i.e., if we set \( \delta \Phi = \eta \{ Q, \Phi \} \), we easily obtain the superspace differential realization \( Q = \partial_\theta - \theta \partial_x \).

Superfields (or, for the present matter, superfunctions) and superspace are the concepts we wanted to introduce before formulating the objectives of the present work.

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\(^2\)This is a sample transformation. One could have chosen the analogous transformation in which the derivative acts on \( F(x) \) instead of \( B(x) \). The proper choice is dictated by the relative dimension of the bosonic and fermionic fields.

\(^3\)It would be fair to indicate that superspace was actually first discovered in the context of dual models in [12]. Note that in one of the pioneering papers on supersymmetry (the second reference in [9]), the authors constructed a fermionic field theory whose translation invariance was enlarged to accommodate translations of the form \( x_\nu \to x_\nu + \eta \psi_\nu \) (where \( \nu \) is a space-time index and \( \psi_\nu \) is a fermionic field). In a sense, they investigated a field theory defined in a larger space: to the usual variables \( x_\nu \) describing the four dimensional Minkowski space, they added new fermionic coordinates \( \psi_\nu(x) \). The difference was that these \( \psi_\nu(x) \) were the very dynamical fields in terms of which the theory was defined.

\(^4\)Observe that \( \delta \) commutes with \( \theta \) since it is bosonic: \( \Phi \) and \( \delta \Phi \) have the same statistics. For our illustrative example, we chose to construct a fermionic superfield. Another option would have been to set \( \Phi = B + \theta F \), which is bosonic (the product of two fermionic variables is bosonic, i.e., \( \theta F \) is bosonic like \( B \)). This would have led to modified transformation rules: \( \delta B(x) = \eta \partial_x F(x) \) and \( \delta F(x) = \eta B(x) \).
1.3. Symmetric superpolynomials. Our program here is to lay down the foundation of the theory of symmetric polynomials in superspace. The superspace we are interested in is the superextension of the Euclidean space in $N$ variables, which we shall denote $E^{|N}$ and whose coordinates will be $(x_1, \ldots, x_N; \theta_1, \ldots, \theta_N)$, with $x_ix_j = x_jx_i$, $x_i\theta_j = \theta_jx_i$ and $\theta_i\theta_j = -\theta_j\theta_i$. Superfunctions, or functions in superspace, are thus functions of two types of variables. For instance, all superfunctions in $E^{2|2}$ are combinations of the following expressions

$$f_0(x_1, x_2), \quad \theta_1f_1(x_1, x_2) + \theta_2f_2(x_1, x_2), \quad \theta_1\theta_2f_3(x_1, x_2)$$

where the $f_i$'s stand for arbitrary functions of $x_1$ and $x_2$. Superfunctions of the second type are fermionic (alternatively said to be odd) while those of the first and third types are bosonic (even).

The immediate question we have to address is the following: what is the meaning of a symmetric superfunction? Observe that there are two copies of the symmetric group at our disposal: the usual one, acting on the commuting variables $x_i$, spanned by the exchange operators $K_{ij}$ defined by

$$K_{ij}x_j = x_iK_{ij}$$

and another one acting on the anticommuting variables $\theta_i$, generated by the new exchange operators $\kappa_{ij}$:

$$\kappa_{ij}\theta_j = \theta_i\kappa_{ij}$$

We stress that a symmetric superfunction is not a function invariant under each type of symmetry transformation. It is a function invariant under the action of the diagonal subgroup of the tensor product of these two copies of the symmetric group, i.e., superfunctions are invariant under the simultaneous interchange of $(x_i, \theta_i)$ and $(x_j, \theta_j)$. In other words, a symmetric superfunction $f$ satisfies the condition

$$K_{ij}\kappa_{ij}f = f$$

Examples of symmetric superpolynomials in $E^{2|2}$ are

$$x_1^2x_2^2, \quad \theta_1x_1^4 + \theta_2x_2^4, \quad \theta_1x_2^2 + \theta_2x_1^2, \quad \theta_1\theta_2(x_1^3x_2 - x_1x_2^3).$$

The enforced interconnection between the transformation properties of the bosonic and the fermionic variables is a direct consequence of the definition of a supersymmetric transformation as a translation in superspace. This is also what makes the resulting object most interesting and novel. In particular, it ensures that the resulting symmetric superpolynomials are completely different from the “supersymmetric polynomials” previously considered in the literature.

Recall that what is called a supersymmetric polynomial (see e.g., [16]) is first of all a doubly symmetric polynomial in two distinct sets of ordinary (commuting) variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, i.e., invariant under independent permutations of the $x_i$'s and the $y_i$'s. It is said to be supersymmetric if, in addition, it satisfies the following cancelation condition: by substituting $x_1 = t$ and $y_1 = t$, the polynomial becomes independent of $t$. An example of a generating function for such polynomials is

$$\prod_{i=1}^m(1 - qx_i)\prod_{j=1}^n(1 - qy_j)^{-1} = \sum_{r \geq 0} p(r)(x, y)q^r$$

This generating function is known to appear in the context of classical Lie superalgebras (as a superdeterminant) [17]. Actually most of the work on supersymmetric polynomials is motivated by its connection with superalgebras. For an example of such an early work, see [18]. More precisions and references are also available in [19, 20].
The key differences between these supersymmetric polynomials and our symmetric superpolynomials should be clear. In our case, we symmetrize two sets of variables with respect to the diagonal action of the symmetric group. And moreover, one of the two sets is made out of Grassmannian variables.

1.4. Toward a theory of symmetric polynomials in superspace. The first step in the elaboration of a theory of symmetric polynomials in superspace is the introduction of a proper labeling for bases of the ring of symmetric superpolynomials, that is, a superversion of partitions. With this concept in hand, the construction of the superextension of the symmetric monomial basis (symmetric monomial basis for short) is rather immediate. From there on, there are two natural routes that can be followed.

1.4.1. Orthogonal symmetric superpolynomials from Cauchy superf ormulas. The first approach amounts to extend to superspace the other classical symmetric functions. This could be done via the extension of their generating functions which, for the elementary $e_n$, homogeneous $h_n$ and power sum $p_n$ symmetric functions are respectively given by [21]:

$$
\sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 + x_i t) , \quad \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t} , \quad \sum_{n \geq 1} p_n t^n = \prod_{i \geq 1} x_i t (1 - x_i t)^{-1} .
$$

(1.21)

The basis elements are generated from the product of these functions:

$$
f_\lambda = f_{\lambda_1} \cdots f_{\lambda_n}
$$

(1.22)

where $\lambda$ denotes a partition and $f$ is any of $e$, $h$ or $p$. Recall also the Cauchy formula

$$
\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda z_\lambda^{-1} p_\lambda(x) t\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y)
$$

(1.23)

where $s_\lambda$ stands for the Schur functions and, for $\lambda = (1^{m_1} 2^{m_2} \cdots)$, $z_\lambda = \prod_i m_i!$. From this, a "combinatorial" scalar product can be defined:

$$
\langle\langle s_\lambda | s_\mu \rangle\rangle = \langle\langle m_\lambda | h_\mu \rangle\rangle = \delta_{\lambda,\mu} \quad \langle\langle p_\lambda | p_\mu \rangle\rangle = z_\lambda \delta_{\lambda,\mu} .
$$

(1.24)

Note that given the bases $e_\lambda$ and $m_\lambda$, we can recover the definition of the partition conjugation $\lambda'$ by enforcing that $e_{\lambda'}$ has the triangular decomposition $e_{\lambda'} = m_\lambda + \text{smaller terms}$ when expanded in the monomial basis.

The idea is to lift all this structure to superspace.

1.4.2. Physical construction of the Jack superpolynomials. Another line of attack is to start with a superspace extension of a general class of superpolynomials out of which all other simple bases can be extracted. A sufficiently rich basis for this purpose is the one given by the superanalogs of the Jack polynomials [22]. Recall that the ordinary Jack polynomials depend upon a free parameter, denoted $\beta$, and that various interesting bases are recovered in the appropriate limits: the symmetric monomial basis when $\beta \to 0$, the elementary basis (up to conjugation) when $\beta \to \infty$ and the Schur polynomials when $\beta = 1$.

The first difficulty with this approach is to have a well-defined way of generating the proper superextension of the Jack polynomials. But here again, the pathway is dictated by supersymmetry, or more precisely, by the consideration of a problem in supersymmetric quantum mechanics [23].

7In that vein, it is interesting to point out that it is precisely within the framework of supersymmetric quantum mechanics that supersymmetry first played a significant role in mathematics with the influential contribution of Witten in Morse theory [25] (see [26] for more on the relation between mathematics and supersymmetry).
Recall that the Jack polynomials $J_\lambda$ are uniquely characterized by their triangular decomposition in the monomial basis together with the following eigenfunction property:

$$H_0 J_\lambda = \left\{ \sum_i (x_i \partial x_i)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial x_i - x_j \partial x_j) \right\} J_\lambda = \epsilon_\lambda J_\lambda. \tag{1.25}$$

It turns out that $H_0$ is exactly the Hamiltonian for the trigonometric Calogero-Moser-Sutherland (tCMS) model without the contribution of its ground-state wave function. The tCMS model is a completely integrable quantum $N$-body problem that has a unique supersymmetric extension \[30\]. Jack superpolynomials are thus naturally defined from the supersymmetric tCMS eigenvalue problem \[31\] \[32\] \[33\].

Quantum mechanics is a special quantum field theory in 0 + 1 (no space and one time) dimension. For a non-supersymmetric $N$-body problem, we introduce $N$ position operators $\hat{x}_i$ ($i = 1, \cdots N$) and their canonical conjugate $\hat{p}_i$, the momenta operators, subject to the commutation relations

$$[\hat{x}_j, \hat{p}_k] = [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{p}_k] = i\hbar \delta_{jk} \tag{1.26}$$

(setting $\hbar = 1$). In the Schrödinger picture, we work with a differential realization of these variables: $\hat{x}_j$ is replaced by the ordinary variable $x_j$ while $\hat{p}_j$ is replaced by $-i\partial x_j$. Now given a Hamiltonian, how do we supersymmetrize it? The signature of a supersymmetric system is the presence of a fermionic conserved operator, conserved in the sense that it commutes with the Hamiltonian. But it is clearly impossible to construct a fermionic operator without having fermionic coordinates. The first step thus amounts to introduce Grassmannian partners to our phase space variables (i.e., positions and momenta). Call them $\hat{\theta}_i$ and $\hat{\bar{\theta}}_i$, with $i = 1, \cdots N$. We enforce the canonical anticommutation relations;

$$\{\hat{\theta}_j, \hat{\theta}_k\} = \{\hat{\bar{\theta}}_j, \hat{\bar{\theta}}_k\} = 0, \quad \{\hat{\theta}_j, \hat{\bar{\theta}}_k\} = \delta_{jk}. \tag{1.27}$$

Again, it is convenient to work with a differential realization: $\hat{\theta}_i \rightarrow \theta_i$ and $\hat{\bar{\theta}}_i \rightarrow \bar{\theta}_i$. Now, we still have to specify a procedure for constructing the supersymmetric Hamiltonian. This turns out to be rather simple. Introduce two fermionic quantities:

$$Q = \sum_{i=1}^{N} \theta_i A_i(x), \quad Q^\dagger = \sum_{i=1}^{N} \bar{\theta}_i A_i^\dagger(x), \tag{1.28}$$

with $A_i$ and $A_i^\dagger$ yet to be determined and set

$$H = \{Q, Q^\dagger\} = H_0 + H_1, \tag{1.29}$$

where $H_0 = H(\theta_i = 0)$. In other words, we impose that $H_0$, the part of the Hamiltonian independent of the fermionic variables, be equal to the non-supersymmetric Hamiltonian that we are trying to supersymmetrize. This fixes $A_i$ and $A_i^\dagger$, which in turn specifies $H_1$, and thus $H$. The construction ensures that both $Q$ and $Q^\dagger$ are conserved, e.g.,

$$[Q, \{Q, Q^\dagger\}] = Q(QQ^\dagger + Q^\dagger Q) - (QQ^\dagger + Q^\dagger Q)Q = 0 \tag{1.30}$$

since $Q^2 = (Q^\dagger)^2 = 0$.

---

The terms “superanalogs of Jack polynomials”, “super-Jack polynomials” and “Jack superpolynomials” have also been used in the literature for somewhat different polynomials. In \[35\], superanalogs of Jack polynomials designated the eigenfunctions of the CMS Hamiltonian constructed from the root system of the Lie superalgebra $su(m, N - m)$ (recall that to any root system corresponds a CMS model \[30\]). (The same objects are called super-Jack polynomials in \[37\].) But we stress that such a Hamiltonian does not contain anticommuting variables, so that the resulting eigenfunctions are quite different from our Jack superpolynomials. Notice also that in \[31\] \[32\], we used the term “Jack superpolynomials” for eigenfunctions of the stCMS model that decompose triangularly in the supermonomial basis. However, these are not necessarily orthogonal. The construction of orthogonal Jack superpolynomials was presented in \[33\] and from now on, when we refer to “Jack superpolynomials”, we refer to the orthogonal ones.
We have thus a clear procedure for supersymmetrizing the Jack polynomial eigenvalue problem, that is, for obtaining the supersymmetric tCMS (called stCMS for short) model. That this indeed leads to orthogonal superpolynomials that decompose trianularly in the supermonomial basis has been established in [33].

Now the orthogonality just alluded to is with respect to the so-called physical scalar product:

$$\langle A(x, \theta)|B(x, \theta)\rangle_{\beta,N} = \prod_{1 \leq j \leq N} \frac{1}{2\pi i} \oint_{x_j} \! \! \! \frac{dx_j}{x_j} \int d\theta_j \theta_j \prod_{1 \leq k, l \leq N} \left(1 - \frac{x_k}{x_l}\right)^\beta A(\bar{x}, \bar{\theta}) B(x, \theta),$$

(1.31)

where the “bar conjugation” is defined as

$$\bar{x}_j = 1/x_j \quad \text{and} \quad (\theta_{i_1} \cdots \theta_{i_m})\theta_{i_1} \cdots \theta_{i_m} = 1.$$

(1.32)

1.4.3. Physical vs combinatorial Jack superpolynomials. We construct in this article a superspace extension of the classical bases $m_\lambda, e_\lambda, h_\lambda$ and $p_\lambda$ by standard combinatorial methods. But we had also previously constructed a one-parameter family of orthogonal polynomials in superspace reducing to the Jack polynomials when the fermionic variables are equal to zero. The question is thus whether these constructions are arbitrary or somehow belong to the “proper” superspace extension of symmetric function theory. We will give two reasons why we believe the latter holds.

The first reason has already been mentioned at the beginning of the previous subsection: the aforementioned combinatorial bases are recovered as special cases of the Jack polynomials in superspace just as they are in the non-supersymmetric case.

A second and stronger reason comes from observing that the $\beta$-deformation of the combinatorial supersymmetric scalar product can be used to provide another definition of the Jack polynomials in superspace. Let us recall that there exists a one-parameter deformation of the scalar product (1.24) between the $p_\lambda$’s. The usual Jack polynomials $J_\lambda$ can be defined purely combinatorially by enforcing orthogonality with respect to this deformed scalar product, in addition with a triangularity requirement (i.e., the $J_\lambda$’s decompose triangularly in the monomial basis $\{m_\lambda\}$ with respect to the dominance ordering). Quite remarkably, the Jack polynomials are also orthogonal with respect to the physical scalar product induced by the CMS model (which is simply (1.31) without the $\theta$ dependence). In this sense, one could say that the physical and combinatorial scalar products are compatible, as the physical and combinatorial definitions give rise to the very same objects.

It occurs that it is rather immediate to $\beta$-deform the superspace extension of (1.24). The question is thus whether the physical Jack superpolynomials, eigenfunctions of the stCMS quantum many-body problem, are also orthogonal with respect to this combinatorial product. We will show in this article that this is indeed the case. We thus end up with the remarkable conclusion that our two lines of approach for building a theory of symmetric functions in superspace, the combinatorial and physical ones, yield the very same objects.

1.5. Organization of the article. The article is organized as follows. Section 2 first introduces the concept of superpartition. Then relevant results concerning the Grassmann algebra and symmetric superpolynomials are reviewed. A simple interpretation of the later, in terms of differentials forms, is also given. This section also includes the definition of supermonomials and a formula for their products.

Section 3 gives the superspace analog of the well known elementary symmetric functions, completely symmetric functions and power-sum bases. The generating function for each of them is displayed. Determinantal formulas that generalize classical formulas describing basic relations between the basis elements are presented. Furthermore, orthogonality and duality relations are established. In the final subsection, we present a one-parameter deformation of the scalar product, the duality transformation and the homogeneous basis.
Section 4 starts with a review of basic facts concerning our previous (physical) construction of Jack polynomials in superspace. These functions are then linked to the combinatorial theory of symmetric superpolynomials elaborated in Section 3 in two different and independent ways. First, it is shown that the physical Jack superpolynomials are also orthogonal with respect to the combinatorial product introduced in Section 3.5. And later, it is shown that in non-trivial limiting cases (i.e., special values of the free parameter or particular superpartitions), the physical Jack superpolynomials reduce to the symmetric superfunctions constructed previously in Section 3.

We finally present, in the conclusion, some natural extensions of this work. In particular, we give a precise conjecture concerning the existence of (combinatorial) Macdonald superpolynomials.

As already indicated, this work concerns, to a large extent, a generalization of symmetric function theory. In laying down its foundation, we generalize a vast number of basic results from this theory which can be found for instance in [21] and [38] (Chap. 7). Clearly, the core of most of our derivations is bound to be a variation around the proofs of these older results. We have chosen not to refer everywhere to the relevant “zero-fermionic degree” version of the stated results. But we acknowledge our debt in that regard to these two classic references. For the results pertaining specifically to the Jack polynomials, we have relied heavily on the seminal paper [22] without complete credit in the bulk of the paper, again to avoid overquoting.

2. Foundations

2.1. Superpartitions. We recall that a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) of \( n \), also written as \( \lambda \vdash n \), is an ordered set of integers such that: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \) and \( \sum_{i=1}^{r} \lambda_i = n \). A particular juxtaposition of two partitions gives a superpartition.

**Definition 1.** A superpartition \( \Lambda \) in the \( m \)-fermion sector is a sequence of non-negative integers separated by a semicolon such that \( \Lambda = (\Lambda^a; \Lambda^s) \), and such that the remaining sequence is a usual partition with \( m \) distinct parts, and such that the remaining sequence is a usual partition. That is,

\[
\Lambda := (\Lambda_1, \ldots, \Lambda_m; \Lambda_{n+1}, \ldots, \Lambda_N),
\]

where \( \Lambda_i > \Lambda_{i+1} \geq 0 \) for \( i = 1, \ldots, m - 1 \) and \( \Lambda_j \geq \Lambda_{j+1} \geq 0 \) for \( j = m + 1, \ldots, N - 1 \).

Given \( \Lambda = (\Lambda^a; \Lambda^s) \), the partitions \( \Lambda^a \) and \( \Lambda^s \) are respectively called the antisymmetric and the symmetric components of the superpartition \( \Lambda \). The **bosonic and fermionic degrees** of \( \Lambda \) are \( |\Lambda| = \sum_{i=1}^{N} \Lambda_i \) and \( \overline{\Lambda} = m \), respectively. Note that, in the zero-fermion sector, the semicolon is usually omitted and \( \Lambda \) reduces then to \( \Lambda^a \).

We say that the ordered set \( \Lambda \) in (2.1) is a superpartition of \( (n|m) \) if \(|\Lambda| = n \) and \( \overline{\Lambda} = m \); in symbols, this is written as \( \Lambda \vdash (n|m) \). The set composed of all superpartitions of \( (n|m) \) is denoted \( \text{SPar}(n|m) \). When the fermionic degree is zero, we recover standard partitions: \( \text{SPar}(n|0) = \text{Par}(n) \).

We also define

\[
\text{SPar}(n) := \bigcup_{m \geq 0} \text{SPar}(n|m) \quad \text{and} \quad \text{SPar} := \bigcup_{m,n \geq 0} \text{SPar}(n|m),
\]

with \( \text{SPar}(0|0) = \emptyset \) and \( \text{SPar}(0|1) = \{(0; 0)\} \). For example, we have

\[
\text{SPar}(3|2) = \{ (3, 0; 0), (2, 1; 0), (2, 0; 1) (1, 0; 2), (1, 0; 1, 1) \}.
\]

Notice that \( \text{SPar}(n|m) \) is empty for all \( n < m(m - 1)/2 \).

We introduce the operator \( \ell : \text{SPar} \to N \) that gives the length of a superpartition as

\[
\ell(\Lambda) := \overline{\Lambda} + \ell(\Lambda^a) \quad \text{for} \quad \ell(\Lambda^s) := \text{Card}\{\Lambda_i \in \Lambda^s : \Lambda_i > 0\}.
\]

\(^9\text{From now on, superscripts } a \text{ and } s \text{ refer respectively to strictly decreasing and decreasing sequences of non-negative integers.}\)
With this definition, $\ell((1, 0; 1, 0)) = 2 + 2 = 4$ (i.e., a zero-entry in $\Lambda^a$ contributes to the length of $\Lambda$). To every superpartition $\Lambda$, we can also associate a unique partition $\Lambda^*$ obtained by deleting the semicolon and reordering the parts in non-increasing order. For instance,

$$(5, 2, 1, 0; 6, 5, 5, 2, 2, 1, 0) = (6, 5, 5, 2, 2, 2, 1, 1).$$

(2.5)

From this, we can introduce another notation for superpartitions. A superpartition $\Lambda = (\Lambda^a; \Lambda^s)$ can be viewed as the partition $\Lambda^*$ in which every part of $\Lambda^a$ is circled. If a part $\Lambda^a_j = b$ is equal to at least one part of $\Lambda^s$, then we circle the leftmost $b$ appearing in $\Lambda^*$. We shall use $C[\Lambda]$ to denote this special notation. For instance,

$$\Lambda = (3, 1, 0; 4, 3, 2, 1) \iff C[\Lambda] = (4, \overline{3}, 3, 3, 2, \overline{1}, \overline{1}, 0).$$

(2.6)

This allows us to introduce a diagrammatic representation of superpartitions. To each $\Lambda$, we associate a unique diagram, denoted by $D[\Lambda]$. It is obtained by first drawing the Ferrer’s diagram associated to $C[\Lambda]$, that is, by drawing a diagram with $C[\Lambda]_1$ boxes in the first row, $C[\Lambda]_2$ boxes in the second row and so forth, all rows being left justified. If, in addition, the integer $C[\Lambda]_j = b$ is circled, then we add a circle at the end of the $b$ boxes in the $j$-th row. For example,

$$D[3, 1, 0; 4, 3, 2, 1] =$$

(2.7)

The conjugate of a superpartition $\Lambda$, denoted by $\Lambda'$, is obtained by interchanging the rows and the columns in the diagram; in matrix notation, we have $D[\Lambda'] = (D[\Lambda])^t$ if $t$ stands for the transpose operation. Hence, $(3, 1, 0; 4, 3, 2, 1)' = (6, 4, 1; 3)$ since

$$
\begin{pmatrix}
\text{\textcircled{1}} & \text{\textcircled{2}} & \text{\textcircled{3}} \\
\text{\textcircled{4}} & \text{\textcircled{5}} & \text{\textcircled{6}} \\
\text{\textcircled{7}} & \text{\textcircled{8}} & \text{\textcircled{9}}
\end{pmatrix}^t =
\begin{pmatrix}
\text{\textcircled{1}} & \text{\textcircled{4}} & \text{\textcircled{7}} \\
\text{\textcircled{2}} & \text{\textcircled{5}} & \text{\textcircled{8}} \\
\text{\textcircled{3}} & \text{\textcircled{6}} & \text{\textcircled{9}}
\end{pmatrix}
$$

(2.8)

Recall that, for any partition $\lambda$, the conjugation can be defined by $\lambda_j' = \operatorname{Card}\{k : \lambda_k \geq j\}$. Thus, in symbols, the conjugation of $\Lambda \in \text{SPar}(n|m)$ reads

$$\Lambda' = (\Lambda'^a; \Lambda'^s),$$

(2.9)

where

$$\Lambda'^s = (\Lambda'^a \setminus \Lambda'^a)\dagger \quad \text{and} \quad \Lambda'^a_j = \operatorname{Card}\{k : \Lambda_k > \Lambda_{m+1-j}\},$$

(2.10)

with $\lambda^\dagger$ standing for the partition obtained by reordering the parts of $\lambda$ non-increasingly. Obviously, the conjugation of any superpartition $\Lambda$ satisfies

$$(\Lambda')' = \Lambda \quad \text{and} \quad (\Lambda^*)' = (\Lambda^*)^*.$$

(2.11)

Remark 2. The description of a superpartition in terms of partition with some parts circled makes clear that overpartitions are special cases of superpartitions. Indeed, overpartitions are circled superpartitions (with the circle replaced by an overbar) that do not contain a possible circled zero.
If we denote by $s_N(n|m)$ the number of superpartitions $\Lambda \in \operatorname{SPar}(n|m)$ such that $\ell(\Lambda) \leq N$, then this connection makes clear that their generating function is

$$
\sum_{n,m,p \geq 0} s_{m+p}(n|m) z^m y^p q^n = \frac{(-z;q)_{\infty}}{(yq;q)_{\infty}} \quad \text{with} \quad (a;q)_n := \prod_{n \geq 0} (1 - aq^n)
$$

(2.12)

To complete this subsection, we consider the natural ordering on superpartitions, which is defined in terms of the Bruhat order on compositions. Recall that a composition of $n$ is simply a sequence of non-negative integers whose sum is equal to $n$; in symbols $\mu = (\mu_1, \mu_2, \ldots) \in \operatorname{Comp}(n)$ iff $\sum \mu_i = n$ and $\mu_i \geq 0$ for all $i$. The Bruhat ordering on compositions is defined as follows. Given a composition $\lambda$, we let $\lambda^+$ denote the partition obtained by reordering its parts in non-increasing order. Now, $\lambda$ can be obtained from $\lambda^+$ by a sequence of permutations. Among all permutations $w$ such that $\lambda = w\lambda^+$, there exists a unique one, denoted $w_\lambda$, of minimal length. For two compositions $\lambda$ and $\mu$, we say that $\lambda \geq \mu$ if either $\lambda^+ > \mu^+$ in the usual dominance ordering or $\lambda^+ = \mu^+$ and $w_\lambda \leq w_\mu$ in the sense that the word $w_\lambda$ is a subword of $w_\mu$ (this is the Bruhat ordering on permutations of the symmetric group). Recall that for two partitions $\lambda$ and $\mu$ of the same degree, the dominance ordering is: $\lambda \geq \mu$ iff $\lambda_1 + \ldots + \lambda_k \geq \mu_1 + \ldots + \mu_k$ for all $k$.

Let $\Lambda$ be a superpartition of $(n|m)$. Then, to $\Lambda$ is associated a unique composition of $n$, denoted by $\Lambda^*$, obtained by replacing the semicolon in $\Lambda$ by a comma. We thus have $\operatorname{SPar}(n) \subset \operatorname{Comp}(n)$, which leads to a natural Bruhat ordering on superpartitions.

**Definition 3.** Let $\Lambda, \Omega \in \operatorname{SPar}(n|m)$. The Bruhat order, denoted by $\leq$, is such that $\Omega \leq \Lambda$ if $\Omega^* \leq \Lambda^*$.

We need two refinements of the previous order, namely the $S$ and $T$ orders (the origin of these names will become clearer in the following lines).

**Definition 4.** Let $\Lambda, \Omega \in \operatorname{SPar}(n|m)$. The $S$ and $T$ orders are respectively defined as follows:

$$
\begin{align*}
\Omega \leq_S \Lambda & \text{ if either } \Omega = \Lambda \text{ or } \Omega^* < \Lambda^*, \\
\Omega \leq_T \Lambda & \text{ if either } \Omega = \Lambda \text{ or } \Omega^* = \Lambda^* \text{ and } \Omega^* < \Lambda^*.
\end{align*}
$$

(2.13)

In order to describe other characterizations of these orders, we need the following operators on compositions (or superpartitions):

$$
\begin{align*}
S_{ij}(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & = \left\{ \begin{array}{ll}
(\ldots, \lambda_i - 1, \ldots, \lambda_j + 1 \ldots) & \text{if } \lambda_i - \lambda_j > 1, \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & \text{otherwise},
\end{array} \right.
\\
T_{ij}(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & = \left\{ \begin{array}{ll}
(\ldots, \lambda_j, \ldots, \lambda_i, \ldots) & \text{if } \lambda_i - \lambda_j > 0, \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) & \text{otherwise}.
\end{array} \right.
\end{align*}
$$

(2.14)

**Lemma 5.**[31][32] Let $\lambda$ and $\omega$ be two compositions of $n$. Then, $\lambda^+ > \omega^+$ iff there exists a sequence $\{S_{i_1,j_1}, \ldots, S_{i_k,j_k}\}$ such that

$$
\omega^+ = S_{i_1,j_1} \ldots S_{i_k,j_k} \lambda^+.
$$

(2.15)

Similarly, $\lambda^+ = \omega^+$ and $\lambda > \omega$ iff there exists a sequence $\{T_{i_1,j_1}, \ldots, T_{i_k,j_k}\}$ such that

$$
\omega = T_{i_1,j_1} \ldots T_{i_k,j_k} \lambda.
$$

(2.16)

This last property can be translated for superpartitions as: $\Lambda >_T \Omega$ iff $D[\Omega]$ can be obtained by moving step by step in the down-left direction the circles of $D[\Lambda]$.

At this stage, we are in a position to establish the fundamental property relating conjugation and Bruhat order which is that the Bruhat order is anti-conjugate (in the sense of the following proposition).

---

[10] The $S$ order is the precisely the ordering introduced in [31] but it differs from the more precise ordering of [32], called there $\leq^*$. In [33], it is called the $h$ ordering. See also appendix B of [34].
Proposition 6. Let $\Lambda, \Omega \in \text{SPar}(n|m)$. Then
\[ \Lambda \geq \Omega \iff \Omega' \geq \Lambda'. \] (2.17)

Proof. It suffices to prove the result for the $S$ and $T$ orderings. The case $\Lambda >_S \Omega$, that is, $\Lambda > \Omega$ and $\Lambda^* \neq \Omega^*$, is a well-known result on partitions (see for instance (1.11) of [21]).

We now consider $\Lambda >_T \Omega$, that is, $\Lambda > \Omega$ with $\Lambda^* = \Omega^*$. Lemma 5 tells us that in this case $D[\Omega]$ can be obtained by moving step by step in the down-left direction the circles of $D[\Lambda]$. Under transposition, this amounts to saying that $D[\Omega']$ can be obtained by moving step by step in the up-right direction the circles of $D[\Lambda']$, or equivalently that $D[\Omega']$ can be obtained by moving step by step in the down-left direction the circles of $D[\Omega']$. Hence,
\[ \Lambda >_T \Omega \iff \Lambda' <_T \Omega'. \] (2.18)
We have thus proved $\Lambda > \Omega \Rightarrow \Lambda' < \Omega'$. Since conjugation is an involution, the claim holds. \qed

Consider for instance $\Lambda = (3, 0; 4, 1)$ and $\Omega = (2, 0; 4, 2) = S_{14} \Lambda <_S \Lambda$. We find that $\Lambda' = (3, 1; 2, 2)$ and $\Omega' = (3, 1; 3, 1)$ so that $\Lambda' = S_{34} \Omega'$, i.e. $\Omega' >_S \Lambda'$. Consider also $\Gamma = (1, 0; 4, 3) = T_{14} \Lambda <_T \Lambda$, which gives $\Gamma' = (3, 2; 2, 1) >_T \Lambda' = T_{24} \Omega'$.

Remark 7. Notice that we could have introduced as an alternative ordering the dominance ordering on superpartitions, denoted by $\leq_D$ and defined as follows: $\Omega \leq_D \Lambda$ if either $\Omega^* < \Lambda^*$ or $\Omega^* = \Lambda^*$ and $\Omega_1 + \ldots + \Omega_k \leq \Lambda_1 + \ldots + \Lambda_k, \forall k$. The usefulness of this ordering in special contexts lies in its simple description in terms of inequalities. However, it is not the proper generalization of the dominance order on partitions because it is not anti-conjugate as will be illustrated later by an example. In fact, it is not as strict as the Bruhat ordering (i.e., more superpartitions are comparable in this order than in the Bruhat ordering). This follows from the second property of Lemma 5 which obviously implies that for superpartitions, the Bruhat ordering is a weak subposet of the dominance ordering, that is, $\Lambda \geq \Omega \Rightarrow \Lambda \geq_D \Omega$. However the converse is not true. For instance, if $\Lambda = (5, 2, 1; 4, 3, 3)$ and $\Omega = (4, 3, 0; 5, 3, 2, 1)$ we easily verify that $\Lambda >_D \Omega$. But $\Lambda' = (5, 4, 0; 6, 2, 1)$ and $\Omega' = (6, 2, 1; 5, 4)$, so that $\Omega' \not>_D \Lambda'$.

Proposition 6 implies that the two orderings need to be distinct. This corrects a loose implicit statement in [31] concerning the expected equivalence of these two orderings.

2.2. Ring of symmetric superpolynomials. Let $\mathcal{B} = \{B_j\}$ and $\mathcal{F} = \{F_j\}$ be the formal and infinite sets composed of all bosonic (commutative) and fermionic (anticommutative) quantities respectively:
\[ [B_j, B_k] = [B_j, F_k] = [F_j, F_k] = 0. \] (2.19)
Thus, $\mathcal{S} = \mathcal{B} \oplus \mathcal{F}$ is $\mathbb{Z}_2$-graded over any ring $\mathbb{A}$ when we identify $0\mathcal{S}$ with $\mathcal{B}$ and $1\mathcal{S}$ with $\mathcal{F}$. The degree of any element $s$ of $\mathcal{S}$, written $\hat{s}(s)$, is defined via
\[ \hat{s}(s) = \begin{cases} 0, & s \in \mathcal{B}, \\ 1, & s \in \mathcal{F}. \end{cases} \] (2.20)

Consequently, $\mathcal{S}$ possesses a parity operator (involution) $\hat{}$ defined by
\[ \hat{s}(s) = (-1)^{\hat{s}(s)} s \] (2.21)
and satisfying
\[ \hat{s}^2 = 1, \quad \hat{s}(st) = \hat{s}(t)\hat{s}(t), \quad \hat{s}(as + bt) = a\hat{s}(s) + b\hat{s}(t), \] (2.22)
for all $a, b \in \mathbb{A}$ and $s, t \in \mathcal{S}$. The second relation implies that the product of two elements of $\mathcal{S}$ belongs to $\mathcal{B}$, i.e., “the product of two fermions is a boson”.

As an example of such a structure, we consider the Grassmann algebra over a ring $\mathbb{A}$, denoted $\mathcal{G}_M(\mathbb{A})$. It is a non-commutative algebra with identity $1 \in \mathbb{A}$, generated by the $M$ elements
that belong to the ring of Grassmannian variables $G_M$, with product defined in \[11\].

Every element $g$ in $G_N(A)$ can be decomposed as $g = 0^g + 1^g$, with \[11\]

\[ g = a(1 - \epsilon) + \sum_{k=\epsilon \mod 2, k>0, \quad 1 \leq i_1 < \cdots < i_k \leq M} \alpha^{i_1 \cdots i_k} \theta_1 \cdots \theta_k , \quad (2.23) \]

where $\epsilon = 0, 1$ and with the constants $a$ and $\alpha^{i_1 \cdots i_k}$ belonging to $A$. The dimension of this Grassmann algebra is thus $\sum_{j=0}^{N} \frac{N!}{j!} = 2^N$. When $A$ is a field, the subalgebra $0^g G_M(A)$ is also a field in which the inverse is defined by (writing $0^g = a + f$)

\[ (0^g)^{-1} = \frac{1}{a + f} = \frac{1}{a} \sum_{n \geq 0} (-1)^n \left( \frac{f}{a} \right)^n . \quad (2.24) \]

Note that, due to the nilpotency of the $\theta_j$'s, the term $(a + f)^{-1}$ is finite for all $M < \infty$. For instance

\[ (1 - \theta_1 \theta_2 - \theta_3 \theta_4)^{-1} = 1 + \theta_1 \theta_2 + \theta_3 \theta_4 + 2\theta_1 \theta_2 \theta_3 \theta_4 . \quad (2.25) \]

Before going further, we define another involution on the Grassmann algebra:

\[ \overleftarrow{\theta_j_1 \cdots \theta_j_m} := \overrightarrow{\theta_j_m \cdots \theta_j_1} , \quad (2.26) \]

where

\[ \overrightarrow{\theta_j_1 \cdots \theta_j_m} := \theta_j_1 \cdots \theta_j_m . \quad (2.27) \]

In words, the operator \[\overleftarrow{\cdot}\] reverses the order of the anticommutative variables while \[\overrightarrow{\cdot}\] is simply the identity map. \[12\] Using induction, we get

\[ \overrightarrow{\theta_j_1 \cdots \theta_j_m} = (-1)^{m(m-1)/2} \overleftarrow{\theta_j_1 \cdots \theta_j_m} . \quad (2.28) \]

This result immediately implies the following simple properties.

**Lemma 8.** Let $\{\theta_1, \ldots, \theta_M\}$ and $\{\phi_1, \ldots, \phi_M\}$ be two sets of Grassmannian variables. Then

\[ \overleftarrow{\theta_j_1 \cdots \theta_j_m} \phi_j_1 \cdots \phi_j_m = \overrightarrow{\theta_j_1 \cdots \theta_j_m} \overleftarrow{\phi_j_1 \cdots \phi_j_m} = \overrightarrow{\theta_j_1 \cdots \theta_j_m} \overrightarrow{\phi_j_1 \cdots \phi_j_m} \quad (2.29) \]

and

\[ \overleftarrow{\theta_j_1 \cdots \theta_j_m} \phi_j_1 \cdots \phi_j_m \overleftarrow{\phi_j_1 \cdots \phi_j_m} = \overrightarrow{\phi_j_1 \cdots \phi_j_m} \overleftarrow{\theta_j_1 \cdots \theta_j_m} . \quad (2.30) \]

Now, let $x = \{x_1, \ldots, x_N\} \subset \mathcal{P}$ and $\theta = \{\theta_1, \ldots, \theta_M\} \subset \mathcal{F}$. The superpolynomial algebra over a unital ring $A$, denoted by $\mathcal{P}^N|\mathcal{M}(A)$ or by $A[x_1, \ldots, x_N, \theta_1, \ldots, \theta_M]$, is the Grassmann algebra $G_M$ over the ring $\mathcal{P}_N^N$ of polynomials in $x$. Recall that $\mathcal{P}^N$ is also a graded, unital and commutative algebra over any unital ring $A$.

Every $f(x) \in \mathcal{P}_N^N = \bigoplus_{n \geq 0} \mathcal{P}_{(n)}$ can be written as

\[ f(x) = \sum_{n \geq 0} f^{(n)}(x) , \quad f^{(n)}(x) = \sum_{\alpha \in N^{|\alpha|} \cdot n} a_\alpha x^\alpha \in \mathcal{P}_{(n)} , \quad (2.31) \]

where $x^\alpha = x_1^{i_1} \cdots x_N^{i_N}$ and $a_\alpha \in A$. Note that $|\alpha| = \sum_{i=1}^{N} a_i$ is the degree of the weak composition $\alpha$; it also corresponds to the degree of $f^{(n)}(x)$. Every superpolynomial $f(x, \theta)$ in $\mathcal{P}^N|\mathcal{M}(A)$ possesses a bosonic and a fermionic part, i.e.,

\[ f(x, \theta) = 0f(x, \theta) + 1f(x, \theta) , \quad (2.32) \]

\[ \text{It should be stressed that the following sum is formal since physically it makes no sense to add fermionic and bosonic quantities. More precisely, an equality between two elements of $g$ is an equality between the $0^g$ and $1^g$ parts of these elements, much like the real and imaginary parts of complex equations.} \]

\[ \text{The explicit use of } \overrightarrow{\cdot} \text{ is not essential. Nevertheless, it will make many formulas more symmetric and transparent.} \]
where the components of $f$ are similar to those given in \[\mathcal{P}^N\], apart from the fact that the $a_{i_1 \cdots i_k}$'s now belong to $\mathcal{P}^N$. From decompositions (2.31) and (2.32), it is obvious that $\mathcal{P}^{N|M}$ is bi-graded with respect to the bosonic and fermionic degrees, that is,

$$\mathcal{P}^{N|M} = \bigoplus_{n,m \geq 0} \mathcal{P}^{N|M}_{(n|m)}.$$  

(2.33)

Each submodule $\mathcal{P}^{N|M}_{(n|m)}$ is finite dimensional. It is composed of all homogeneous superpolynomials $f(x, \theta)$ with degrees $n$ and $m$ in $x$ and $\theta$, respectively. We shall write $\deg f = (n|m)$ for any such a superpolynomial $f$.

Pure fermionic polynomials (i.e., elements of $\mathcal{P}^{N|M}_{(n|m)}$ with $m$ odd) have nice properties. As an example, consider the following proposition that shall be useful in the subsequent sections.

**Proposition 9.** Let $\tilde{f} = \{\tilde{f}_0, \tilde{f}_1, \ldots\}$ and $\tilde{g} = \{\tilde{g}_0, \tilde{g}_1, \ldots\}$ be two sequences of fermionic polynomials parametrized by non-negative integers. Let also

$$\tilde{f}_\mu := \tilde{f}_{\mu_1} \tilde{f}_{\mu_2} \cdots$$ and $$\tilde{g}_\mu := \tilde{g}_{\mu_1} \tilde{g}_{\mu_2} \cdots$$

where $\mu$ belongs to $\text{Par}_\Lambda(n)$, the set of partitions of $n$ with strictly decreasing parts. Then

$$\exp \left[ \sum_{n=0}^{M-1} \tilde{f}_n \tilde{g}_n \right] = \sum_{n=0}^{M(M-1)/2} \sum_{\mu \in \text{Par}_\Lambda(n)} \sum_{\mu_1 + \mu_2 = \mu} \tilde{f}_\mu \tilde{g}_\mu.$$  

(2.34)

**Proof.** Due to the fermionic character of $\tilde{f}$ and $\tilde{g}$, we have

$$\exp \left[ \sum_{n=0}^{M-1} \tilde{f}_n \tilde{g}_n \right] = \prod_{0 \leq n \leq M-1} (1 + \tilde{f}_n \tilde{g}_n) = 1 + \sum_{0 \leq n \leq M-1} \tilde{f}_n \tilde{g}_n + \sum_{0 \leq m < n \leq M-1} \tilde{f}_m \tilde{g}_m \tilde{f}_n \tilde{g}_n + \ldots$$  

(2.35)

Since every term in the last equality can be reordered by Lemma 8, the proof follows. \hfill \Box

We finally consider the symmetric superpolynomials. We specialize to the case in which the number of bosonic and fermionic variables is the same, i.e., $N = M$ and $\mathcal{P} := \mathcal{P}^{N|N}$. The algebra of symmetric superpolynomials over the ring $\mathcal{A}$, denoted by $\mathcal{A}^{S_N}(\mathcal{A})$ or by $\mathcal{A}[x_1, \ldots, x_N, \theta_1, \ldots, \theta_N]^{S_N}$, is a subalgebra of $\mathcal{P}$. As mentioned in the introduction, $\mathcal{P}^{S_N}$ is made out of all $f(x, \theta) \in \mathcal{P}$ invariant under the diagonal action of the symmetric group $S_N$.

To be more explicit, we introduce $K_{ij}$ and $\kappa_{ij}$, two distinct and Abelian superpolynomial realizations of the transposition $(i, j) \in S_N$:

$$K_{ij} f(x_i, x_j, \theta_i, \theta_j) = f(x_j, x_i, \theta_i, \theta_j), \quad \kappa_{ij} f(x_i, x_j, \theta_i, \theta_j) = f(x_i, x_j, \theta_j, \theta_i),$$

(2.37)

for all $f \in \mathcal{P}$. Note that $\kappa_{ij}$ has the following realization

$$\kappa_{ij} := 1 - (\theta_i - \theta_j)(\partial_{\theta_i} - \partial_{\theta_j}).$$

(2.38)

Since every permutation is generated by products of elementary transpositions $(i, i+1) \in S_N$, we can define symmetric superpolynomials as follows.

**Definition 10.** A superpolynomial $f(x, \theta) \in \mathcal{P}$ is symmetric if and only if

$$K_{i,i+1} f(x, \theta) = f(x, \theta) \quad \text{where} \quad K_{i,i+1} := \kappa_{i,i+1} K_{i,i+1}$$

(2.39)

for all $i \in \{1, 2, \ldots, N - 1\}$. 

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But, every monomial \( \theta_J = \theta_{j_1} \cdots \theta_{j_m} \) is completely antisymmetric, that is,
\[
\kappa_{ik} \theta_J = \begin{cases} -\theta_J, & \text{if } i, k \in J, \\ \theta_J, & \text{if } i, k \not\in J. \end{cases}
\tag{2.40}
\]
This observation immediately implies the following result.

**Lemma 11.** Let \( f(x, \theta) \in \mathcal{P} \) be expressed as:
\[
f(x, \theta) = \sum_{m \geq 0} \sum_{1 \leq j_1 < \cdots < j_m \leq N} f^{j_1, \ldots, j_m}(x) \theta_{j_1} \cdots \theta_{j_m}. \tag{2.41}
\]
If \( f(x, \theta) \) is symmetric, then each polynomial \( f^{j_1, \ldots, j_m}(x) \) is completely antisymmetric in the set of variables \( y := \{x_{j_1}, \ldots, x_{j_m}\} \) and completely symmetric in the set of variables \( x \setminus y \).

### 2.3. Geometric interpretation of superpolynomials.

Symmetric functions can be interpreted as symmetric 0-forms \( f \) acting on a manifold: \( K_{ij} f(x) = f(x) \) where \( x \) is a local coordinate system. Similarly, symmetric superfunctions in the \( p \)-fermion sector can be interpreted as symmetric \( p \)-forms \( f^p \) acting on the same manifold: \( \tilde{K}_{ij} f^p(x) = f^p(x) \). Thus, the set of all symmetric super-functions is in correspondence with the completely symmetric de Rham complex. This geometric point of view is briefly explained in this subsection. (Note that none of our results relies on this observation.)

We consider a Riemannian manifold \( \mathcal{M} \) of dimension \( N \) with metric \( g_{ij} \) and let \( x = \{x^1, \ldots, x^N\} \) denote a coordinate system on \( U \subset \mathcal{M} \). Let \( T\mathcal{M} \) be the tangent bundle on which we choose an orthonormal coordinate frame \( \mathbf{e} = \{\partial_1, \ldots, \partial_N\} \). As usual, \( \mathbf{e}^* = \{dx^1, \ldots, dx^N\} \) denotes the dual basis to \( \mathbf{e} \) belonging to the cotangent bundle \( T^*\mathcal{M} \): \( dx^i(\partial_j) = \delta^i_j \). The set of all \( p \)-form fields on \( \mathcal{M} \) is a vector space denoted by \( \bigwedge^p \). Each \( p \)-form can be written as
\[
\alpha^p(x) = \sum_{1 \leq j_1 < \cdots < j_p \leq N} \alpha_{j_1, \ldots, j_p}(x) dx^{j_1} \wedge \cdots \wedge dx^{j_p}, \tag{2.42}
\]
where the exterior (wedge) product is antisymmetric: \( dx^i \wedge dx^j = -dx^j \wedge dx^i \). Let \( d \) be the exterior differentiation on forms, whose action is
\[
d \alpha^p(x) = \sum_{1 \leq k, j_1, \ldots, j_p \leq N} [\partial_{x_k} \alpha_{j_1, \ldots, j_p}(x)] dx^k \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p}. \tag{2.43}
\]
This operation is used to define the de Rham complex of \( \mathcal{M} \):
\[
0 \longrightarrow \mathbb{R} \longrightarrow \bigwedge^0 \longrightarrow \bigwedge^1 \longrightarrow \bigwedge^2 \longrightarrow \bigwedge^N \longrightarrow 0. \tag{2.44}
\]

In order to represent our Grassmannian variables \( \theta^i \) and \( \theta^{i \dagger} \) in terms of forms, we introduce the two operators \( \hat{e}_{dx^i} \) and \( i_{\partial_{x_k}} \), where \( \hat{e}_\alpha \) and \( i_\nu \) respectively stand for the left exterior product by the form \( \alpha \) and the interior product (contraction) with respect to the vector field \( \nu \). These operators satisfy a Clifford (fermionic) algebra
\[
\{ \hat{e}_{dx^i}, i_{\partial_{x_k}} \} = \delta^i_k \quad \text{and} \quad \{ \hat{e}_{dx^i}, \hat{e}_{dx^j} \} = 0 = \{ i_{\partial_{x_j}}, i_{\partial_{x_k}} \}. \tag{2.45}
\]
This implies that the \( \theta^i \)'s and \( \theta^{i \dagger} \), as operators, can be realized as follows:
\[
\theta^i \sim \hat{e}_{dx^i} \quad \text{and} \quad \theta^{i \dagger} \sim g^{ik} i_{\partial_{x_k}}, \tag{2.46}
\]
that is,
\[
dx^j \sim \theta^j(1) \quad \text{and} \quad \theta^{i \dagger} \sim g^{jk} \partial_{x_k} \quad \text{where} \quad \partial_{x_k} := \frac{\partial}{\partial x^k}. \tag{2.47}
\]
Note that introducing the Grassmannian variables as operators is needed to enforce the wedge product of the forms \( dx_j \). Moreover, if \( \alpha^p \) is a generic \( p \)-form field and \( \tilde{\pi} : \bigwedge^p \rightarrow \bigwedge^p \) is the operator defined by
\[
\tilde{\pi}_p := \theta^j \partial_{x_j} = \theta^j g_{jk} \theta^{k \dagger} \quad \Rightarrow \quad \tilde{\pi}_p \alpha^p = p \alpha^p, \tag{2.48}
\]
then $\hat{\Pi}_p := (-1)^{\bar{s}_p}$ is involutive. (Manifestly, $\hat{\Pi}_p = \hat{\Pi}$, the parity operator introduced previously.) This involution is also an isometry in the Hilbert space scalar product. The operator $\hat{\Pi}_p$ induces a natural $\mathbb{Z}_2$ grading in the de Rham complex.

The construction of the symmetric de Rham complex, denoted SRham, is obtained as follows. We make a change of coordinates: $x \rightarrow f(x)$, where $f = \{f^n\} := \{f^1, \ldots, f^N\}$ is an $N$-tuple of symmetric and independent functions of $x$. For instance, $f^n$ could be an elementary symmetric function $e_n$, a complete symmetric function $h_n$, or a power sum $p_n$ (see Section 3). This implies a change of basis in the cotangent bundle: $dx \rightarrow df(x)$. Explicitly,

$$df^n = \sum_i (\partial_i f^n)(x) dx_i \sim \hat{f}^n = \sum_i (\partial_i f^n)(x) \theta_i.$$  \hspace{1cm} (2.49)

In other words, $df$ is a new set of “fermionic” variables invariant under any permutation of the $x_j$'s.\textsuperscript{13}

These remarks explicitly show that symmetric polynomials in superspace can be interpreted as symmetric differential forms. We stress that the diagonal action of the symmetric group $S_N$ comes naturally in the geometric perspective. Note finally that for a Euclidean superspace (relevant to our context), the position (upper or lower) of the indices does not matter.

### 2.4. Supermonomial basis.

The symmetric supermonomial function, denoted by $m_\Lambda = m_\Lambda(x, \theta)$, is the superanalog of the monomial symmetric function. It is defined as follows [31].

**Definition 12.** To each superpartition $\Lambda \in SPar(n|m)$, we associate a superpolynomial $m_\Lambda \in \mathcal{P}^S_{(n|m)}$, called symmetric supermonomial, defined by

$$m_\Lambda = \sum_{\sigma \in S_N} \theta^{\sigma(1,\ldots,m)} x^{\alpha(\Lambda)},$$  \hspace{1cm} (2.50)

where the prime indicates that the summation is restricted to distinct terms, and where

$$x^{\alpha(\Lambda)} = x_1^{\Lambda_{s(1)}} \cdots x_m^{\Lambda_{s(m)}} x_{m+1}^{\Lambda_{s(m+1)}} \cdots x_N^{\Lambda_{s(N)}}$$

and

$$\theta^{\sigma(1,\ldots,m)} = \theta_{\sigma(1)} \cdots \theta_{\sigma(m)}.$$  \hspace{1cm} (2.51)

Obviously, the previous definition can be replaced by the following:

$$m_\Lambda = \frac{1}{n_\Lambda !} \sum_{\sigma \in S_N} K_\sigma \left( \theta_1 \cdots \theta_m x^{\Lambda} \right)$$  \hspace{1cm} (2.52)

for

$$n_\Lambda ! = n_{\Lambda^s} ! := n_{\Lambda^s(0)!} n_{\Lambda^s(1)!} n_{\Lambda^s(2)!} \cdots,$$  \hspace{1cm} (2.53)

where $n_{\Lambda^s(i)}$ indicates the number of $i$'s in $\Lambda^s$, the symmetric part of $\Lambda = (\Lambda^s; \Lambda^a)$. Moreover, $K_\sigma$ stands for $K_{i_1,i_1+1} \cdots K_{i_m,i_m+1}$ when the element $\sigma$ of the symmetric group $S_N$ is written in terms of elementary transpositions, i.e., $\sigma = \sigma_{i_1} \cdots \sigma_{i_m}$. Notice that a symmetric supermonomial $m_\Lambda$, with $\Lambda \vdash (n|m)$, belongs to $\mathcal{P}_{(n|m)}(\mathbb{Z})$, the module of superpolynomials of degree $(n|m)$ with integer coefficients.

**Theorem 13.** The set $\{m_\Lambda\}_{\Lambda \vdash (n|m)} := \{m_\Lambda : \Lambda \in SPar(n|m)\}$ is a basis of $\mathcal{P}^S_{(n|m)}(\mathbb{Z})$.

**Proof.** Each superpolynomial $f(x, \theta)$ of degree $(n|m)$, with $N$ variables and with integer coefficients, can be expressed as a sum of monomials of the type $\theta_{j_1} \cdots \theta_{j_m} x^\mu$, with coefficient $a_{\mu}^{j_1 \cdots j_m} \in \mathbb{Z}$, and where $\mu$ is a composition of $N$. Let $\Omega^a$ be the reordering of the entries $(\mu_1, \ldots, \mu_{3m})$, and let $\Omega^s$ be the reordering of the remaining entries of $\mu$. Because the superpolynomial $f(x, \theta)$ is also symmetric,\textsuperscript{13}

\hspace{1cm} 

\textsuperscript{13}Of course, this change of basis is well defined in $U \subset M$ if and only if the Jacobian determinant $J(f, x)$ is not zero. For any standard basis of the symmetric function space (i.e., the $e_n$'s, $h_n$'s or $p_n$'s), we easily verify that the Jacobian is, up to a sign, equal to the Vandermonde determinant. Thus, the change of coordinates is well defined for any locus $U$ such as the following one: $L_U := \{x(p) : x^1(p) > x^2(p) > \ldots ; \forall p \in U\}$. 

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\( f(x, \theta) \) is by definition invariant under the action of \( K_\sigma \), for any \( \sigma \in S_N \). Therefore, \( a^{1,2, \ldots, m}_\mu \) must be equal, up to a sign, to the coefficient \( a^{1,2, \ldots, m}_\Omega \) of \( \theta_1 \cdots \theta_m x^\Omega \) in \( f(x, \theta) \), where \( \Omega = (\Omega^1, \Omega^2) \). Note that from Lemma 11, \( \Omega^2 \) needs to have distinct parts, which means that \( \Omega \) is a superpartition. This gives that \( f(x, \theta) - a^{1,2, \ldots, m}_\Lambda m_\Lambda \) does not contain any monomial that is also a monomial of \( m_\Omega \), since otherwise it would need by symmetry to contain the monomial \( \theta_1 \cdots \theta_m x^\Omega \).

Now, consider any total order on superpartitions, and let \( \Lambda \) be the highest superpartition in this order such that there is a monomial of \( m_\Lambda \) appearing in \( f(x, \theta) \). By the previous argument, \( f(x, \theta) - a^{1,2, \ldots, m}_\Lambda m_\Lambda \) is a symmetric superpolynomial such that no monomial belonging to \( m_\Lambda \) appears in its expansion. Since no monomial of \( m_\Lambda \) appears in any other monomial of \( m_\Omega \), for \( \Omega \neq \Lambda \), the proof follows by induction.

**Corollary 14.** The set \( \{m_\Lambda \}_\Lambda := \{m_\Lambda : \Lambda \in S\text{Par}\} \) is a basis of \( \mathcal{P}^{S_N}(\mathbb{Z}) \).

This corollary implies that \( \mathcal{P}^{S_N}(\mathbb{Z}) \) could also be defined as the free \( \mathbb{Z} \)-module spanned by the set of symmetric supermonomials.

To end this section, we give a formula for the expansion coefficients of the product of two supermonomials in terms of supermonomials. This furnishes an illustrative example of what could be called supercombinatorics. In this kind of calculation, the standard counting of combinatorial objects (e.g., tableaux) is affected by signs resulting from the reordering of fermionic variables (represented by circles in the supertableaux).

**Definition 15.** Let \( \Lambda \in S\text{Par}(n|m) \), \( \Omega \in S\text{Par}(n'|m') \) and \( \Gamma \in S\text{Par}(n+n'|m+m') \). In each box or circle of \( D[\Lambda] \), we write a letter \( a \). In its \( i \)-th circle (the one corresponding to \( \Lambda_i \)), we add the label \( i \) to the letter \( a \). We do the same process for \( D[\Omega] \) replacing \( a \) by \( b \). We then define \( T[\Lambda, \Omega; \Gamma] \) to be the set of distinct fillings of \( D[\Gamma] \) with the letters of \( D[\Lambda] \) and \( D[\Omega] \) obeying the following rules:

1. The circles of \( D[\Gamma] \) can only be filled with labeled letters (an \( a_i \) or a \( b_j \));
2. Each row of the filling of \( D[\Lambda] \) is reproduced in a single and distinct row of the filling of \( D[\Gamma] \);
   in other words, rows of \( D[\Lambda] \) cannot be split and two rows of \( D[\Lambda] \) cannot be put within a single row of \( D[\Gamma] \);
3. Rule 2 also holds when \( D[\Lambda] \) is replaced by \( D[\Omega] \);
4. In each row, the unlabeled \( a \)'s appear to the left of the unlabeled \( b \)’s.

For instance, there are three possible fillings of \((2,1,0;1^3)\) with \((1,0;1)\) and \((0,2,1^2)\):

\[
\begin{array}{ccc}
\text{b} & \text{b} & (2) \\
\text{a} & \text{a} & (1) \\
\text{a} & \\
\text{b} & \\
\text{b} & (1) \\
\end{array}
\quad
\begin{array}{ccc}
\text{b} & \text{b} & (2) \\
\text{a} & \text{a} & (1) \\
\text{a} & \\
\text{b} & \\
\text{b} & (1) \\
\end{array}
\quad
\begin{array}{ccc}
\text{b} & \text{b} & (2) \\
\text{a} & \text{a} & (1) \\
\text{a} & \\
\text{b} & \\
\text{b} & (1) \\
\end{array}
\]

(2.54)

There are also three possible fillings of \((3,1,0;1^2)\) with \((1,0;1)\) and \((0,2,1^2)\):

\[
\begin{array}{ccc}
\text{a} & \text{b} & (1) \\
\text{b} & \text{a} & (2) \\
\text{a} & \\
\text{b} & \\
\text{b} & (1) \\
\end{array}
\quad
\begin{array}{ccc}
\text{a} & \text{b} & (1) \\
\text{b} & \text{a} & (2) \\
\text{a} & \\
\text{b} & \\
\text{b} & (1) \\
\end{array}
\quad
\begin{array}{ccc}
\text{a} & \text{b} & (1) \\
\text{b} & \text{a} & (2) \\
\text{a} & \\
\text{b} & \\
\text{b} & (1) \\
\end{array}
\]

(2.55)

**Definition 16.** Let \( T \in T[\Lambda, \Omega; \Gamma] \), with \( \Sigma = m \) and \( \Sigma' = m' \). The weight of \( T \), denoted by \( \hat{w}(T) \), corresponds to the sign of the permutation needed to reorder the content of the circles in the filling of \( D[\Gamma] \) so that from top to bottom they read as \( a_1 \cdots a_m b_1 \cdots b_{m'} \).
In the example \[257\], each term has weight \(-1\) (odd parity). The oddness of these fillings comes from the transposition that is needed to reorder \(a_1\) and \(a_2\). In the second example \[258\], the two first fillings are even while the last filling is odd due to the needed transposition of \(a_2\) and \(b_1\). As we shall see in the next proposition, the two previous sets lead respectively to the coefficients of 
\[m_{(2,1,0;1)} \] and 
\[m_{(3,1,0;1)} \] in the product of \(m_{(1,0;1)} \) and \(m_{(0,2,1;1)} \), that is,
\[m_{(1,0;1)} m_{(0,2,1;1)} = \left(\frac{-3}{1-\ell+1} \right) \times m_{(2,1,0;1)} + \left(\frac{1}{1+1+1} \right) \times m_{(3,1,0;1)} + \text{other terms}. \] (2.56)

**Proposition 17.** Let \(m_\Lambda\) and \(m_\Omega\) be any two supermonomials. Then
\[m_\Lambda m_\Omega = \sum_{\Gamma \in \text{SPar}} N^\Gamma_{\Lambda,\Omega} m_{\Gamma}, \] (2.57)
where the integer \(N^\Gamma_{\Lambda,\Omega} = (-1)^{\ell} \prod N^\Gamma_{N,\Lambda}\) is given by
\[N^\Gamma_{\Lambda,\Omega} := \sum_{T \in T[\Lambda,\Omega;\Gamma]} \hat{w}[T]. \] (2.58)

**Proof.** From the symmetry property in Definition [12], the coefficient \(N^\Gamma_{\Lambda,\Omega}\) is simply given by the coefficient of \(\theta_{1,\ldots,m+p} x^\Gamma\) in \(m_\Lambda m_\Omega\). The terms contributing to this coefficient correspond to all distinct permutations \(\sigma\) and \(w\) of the entries of \(\Lambda\) and \(\Omega\) respectively such that
\[\Gamma = (\Lambda_{\sigma(1)} + \Omega_{w(1)}, \ldots, \Lambda_{\sigma(N)} + \Omega_{w(N)})\], (2.59)
where the entries of \(\Lambda^a\) and \(\Omega^a\) are distributed among the first \(m+p\) entries (no two in the same position). But this set is easily seen to be in correspondence with the fillings in \(T[\Lambda,\Omega;\Gamma]\) when realizing that labeled letters simply give the positions of the fermions in \(C[\Gamma]\) (the circled version of \(\Gamma\)). The only remaining problem is thus the ordering of the fermions. In \[259\], from the definition of monomial symmetric functions, the sign of the contribution is equal to the sign of the permutation needed to reorder the fermionic entries of \(\Lambda^a\) and \(\Omega^a\) that are distributed among the first \(m+p\) entries so that they correspond to \((\Lambda^a, \Omega^a)\). But this is simply the sign of the permutation that reorders the circled entries in the corresponding filling of \(D[\Gamma]\) such that they read as \(a_1 \ldots a_m b_1 \ldots b_p\). \(\square\)

### 3. Generating functions and multiplicative bases

In the theory of symmetric functions, the number of variables is usually irrelevant, and can be set for convenience to be equal to infinity. In a similar way, we shall consider from now on that, unless otherwise specified, the number of \(x\) and \(\theta\) variables is infinite, and denote the ring of symmetric superfunctions as \(\mathfrak{S}_\infty\).

#### 3.1. Elementary symmetric superfunctions

Let \(J = \{j_1, \ldots, j_r\}\) with \(1 \leq j_1 < j_2 < j_3 \cdots\) and let \(#J := \text{Card}\ J\). The \(n\)-th bosonic and fermionic elementary symmetric superfunctions, for \(n \geq 1\), are defined respectively by
\[e_n := \sum_{J; \#J = n} x_{j_1} \cdots x_{j_n} \quad \text{and} \quad \tilde{e}_n := \sum_{i \geq 1, \#J = n} \sum_{i \not\in J} \theta_i x_{j_1} \cdots x_{j_n} \] (3.1)

In addition, we impose
\[e_0 = 1 \quad \text{and} \quad \tilde{e}_0 = \sum_i \theta_i. \] (3.2)

So, in terms of supermonomials, we have
\[e_n = m_{(1^n)} \quad \text{and} \quad \tilde{e}_n = m_{(0:1^n)}. \] (3.3)
We introduce two parameters: $t \in \mathcal{B}$ and $\tau \in \mathcal{F}$. It is easy to verify that the generating function for the elementary superfunctions is

$$E(t, \tau) := \sum_{n=0}^{\infty} t^n (e_n + \tau e_n) = \prod_{i=1}^{\infty} (1 + tx_i + \tau \theta_i).$$

(3.4)

Actually, to go from the usual generating function $E(t) := E(t, 0)$ to the new one, one simply replaces $x_i \rightarrow x_i + \tau \theta_i$ and redefines $\tau' = \tau/t$, an operation that makes manifest the invariance of $E(t, \tau)$ under the simultaneous interchange of the $x_i$’s and the $\theta_i$’s.

From an analytic point of view, the fermionic elementary superfunctions are obtained by exterior differentiation:

$$\tilde{e}_{n-1}(x, \theta) \sim \tilde{e}_{n-1}(x, dx) = d e_n(x),$$

(3.5)

for all $n \geq 1$. How can we explain that the generating function (3.4) leads precisely to the fermionic elementary superfunctions that are obtained by the action of the exterior derivative of the elementary symmetric function? The rationale for this feature turns out to be rather simple. Indeed, let $\tau := t dt$ and define $D$ to act on a function $f(x, t)$ as a tensor-product derivative:

$$D f := dt \wedge df.$$  

(3.6)

In consequence, we formally have

$$(1 + tx_i + \tau \theta_i) \sim (1 + D)(1 + tx_i) \quad \text{and} \quad E(t, \tau) \sim (1 + D) E(t),$$

(3.7)

which is the desired link.

In order to obtain a new basis of the symmetric superpolynomial algebra, we associate, to each superpartition $\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_{\ell})$ of $(n|m)$, a superpolynomial $e_{\Lambda} \in \mathcal{P}_{(n|m)}^{S_{\infty}}$ defined by

$$e_{\Lambda} := \prod_{i=1}^{m} \tilde{e}_{\Lambda_i} \prod_{j=m+1}^{\ell} e_{\Lambda_j},$$

(3.8)

Note that the product of anticommutative quantities is always done from left to right: $\prod_{i=1}^{N} F_i := F_1 F_2 \cdots F_N$. We stress that the ordering matters in the fermionic sector since for instance

$$e_{3,4,1} = \tilde{e}_3 \tilde{e}_4 e_1 = \tilde{e}_4 \tilde{e}_3 e_1.$$  

(3.9)

**Theorem 18.** Let $\Lambda$ be a superpartition of $(n|m)$ and $\Lambda'$ its conjugate. Then

$$\tilde{e}_\Lambda = \tilde{e}_{\Lambda_m} \cdots \tilde{e}_{\Lambda_1} e_{\Lambda_{m+1}} \cdots e_{\Lambda_N} = m_{\Lambda'} + \sum_{\Omega < \Lambda'} N^{\Omega}_{\Lambda} m_{\Omega},$$

(3.10)

where $N^{\Omega}_{\Lambda}$ is an integer. Hence, $\{ e_{\Lambda} : \Lambda \vdash (n|m) \}$ is a basis of $\mathcal{P}_{(n|m)}^{S_{\infty}}(\mathbb{Z})$.

**Proof.** We first observe that $\tilde{e}_\Lambda = (-1)^{m(m-1)/2} e_{C[\Lambda]}$, where $C[\Lambda]$ denotes as usual the partition $\Lambda^*$ in which fermionic parts of $\Lambda$ are identified by a circle. Then, assuming that we work in $N$ variables, the monomials $\theta_j x^\nu$ that appear in the expansion of $e_{C[\Lambda]}$ are in correspondence with the fillings of $D[\Lambda']$ with the letters $1, \ldots, N$ such that:

1. the non-circled entries in the filling of $D[\Lambda']$ increase when going down in a column;
2. if a column contains a circle, then the entry that fills the circle cannot appear anywhere else in the column.

The correspondence follows because the reading of the $i$-th column corresponds to one monomial of $e_{\Lambda_i}$ (or $\tilde{e}_{\Lambda_i}$). To be more specific, if the reading of the column is $j_1, \ldots, j_{\Lambda_i}$ (with a possible extra letter $a$), it corresponds to the monomial $x_{j_1} \cdots x_{j_{\Lambda_i}}$ (or $\theta_a x_{j_1} \cdots x_{j_{\Lambda_i}}$). The first condition ensures that we do not count the permutations of $x_{j_1} \cdots x_{j_{\Lambda_i}}$ as distinct monomials. The second one ensures that in the fermionic case, the index of $\theta_a$ is distinct from the index of the variables $x_{j_1}, \ldots, x_{j_{\Lambda_i}}$. 

Now, to obtain the coefficient $N^\Omega_\Lambda$, it suffices to compute the coefficient of $\theta_{C[\Omega]} x^{C[\Omega]}$ in $e_{C[\Lambda]}$, where $\theta_{C[\Omega]}$ represents the product of the fermionic entries of $C[\Omega]$ read from left to right. Note that this coefficient has the same sign as $\theta_{(1, \ldots, m)} x^\Omega$ in $m_\Omega$ and there is thus no need to compensate by a sign factor. The monomials contributing to $N^\Omega_\Lambda$ are therefore fillings of $D[\Lambda']$ (obeying the two conditions given above) with the letter $i$ appearing $C[\Omega]_i$ times in non-circled cells with one additional time in a circled cell if $C[\Omega]_i$ is fermionic. We will call the set of such fillings $T^{(\omega)[\Omega; \Lambda']}$.

Finally, given a filling $T \in T^{(\omega)[\Omega; \Lambda']}$, we read the content of the circles from top to bottom and obtain a word $a_1 \ldots a_m$. The sign of the permutation needed to reorder this word such that it be increasing gives the weight associated to the filling $T$, denoted this time $\bar{w}[T]$. The weight of $T$ is the sign needed to reorder the monomial associated to $T$ so that it corresponds to $\theta_{C[\Omega]} x^{C[\Omega]}$ up to a factor $(-1)^{m(m-1)/2}$. This is because to coincide with the product in $e_{C[\Lambda]}$ being done columnwise, we would have to read from bottom to top. Reading from top to bottom provides the $(-1)^{m(m-1)/2}$ factor needed to obtain the coefficient in $e_\Lambda$ instead of in $e_{C[\Lambda]}$. We thus have

$$N^\Omega_\Lambda = \sum_{T \in T^{(\omega)[\Omega; \Lambda']}} \bar{w}[T]. \tag{3.11}$$

We now use this equation to prove the theorem.

First, it is easy to convince ourselves that there is only one element in $T^{(\omega)[\Lambda'; \Lambda']}$ and that it has a positive weight. Because the rows of $D[\Lambda']$ and $C[\Lambda']$ coincide, for the filling to have increasing rows, we have no choice but to put the $C[\Lambda']_i$ letters $i$ in the $i$-th row of $D[\Lambda']$. In the case when $C[\Lambda']_i$ is fermionic, the extra $i$ has no choice but to go in the circle in row $i$ of $D[\Lambda']$ for no two $i$’s to be in a same column. For example, given $\Lambda' = (3,1;2,1)$ filling $D[\Lambda']$ with the letters of $C[\Lambda']$ leads to:

```
 1 1 1 1
 2 2
 3 3
 4
```

This explains the first term in (3.10).

Second, let $\omega = \Omega^*$ and $\lambda = \Lambda^*$. If $\Omega \not\leq_S \Lambda'$, a filling of $\Omega$ by $\Lambda$ is obviously impossible because we would need to obtain in particular (forgetting about the circles) a filling of the type $T^{(\omega)[\omega; \Lambda']}$, which would contradict the well known fact that the theorem holds in the zero-fermion case.

Finally, we suppose that $\Lambda^* = \Omega^*$ and $\Omega \not\geq_T \Lambda'$. From Lemma 6 this implies that there is at least one circle in $D[\Lambda']$, let’s say in row $i$, lower than its counterpart in $D[\Omega]$. Since to fill $D[\Lambda']$ with $C[\Omega]$, the non-circled entries are filled with a row of 1’s, then a row of 2’s and so on, we would need to be able to put an entry $j < i$ in the circle in row $i$ of $D[\Lambda']$. But this cannot happen since it would create a column with two $j$’s.

Note that for the various examples that we have worked out, the coefficients $N^\Omega_\Lambda$ are non-negative. So we may surmise that a stronger version of the theorem, where $N^\Omega_\Lambda$ is a non-negative integer, holds.

The linear independence of the $e_\Lambda$’s in $\mathcal{B}^{SN}$ implies that the first $N$ bosonic and fermionic elementary superfunctions are algebraically independent over $\mathbb{Z}$. Symbolically,

$$\mathbb{Z}[x_1, \ldots, x_N, \theta_1, \ldots, \theta_N]^{SN} \equiv \mathbb{Z}[e_1, \ldots, e_N, \tilde{e}_0, \ldots, \tilde{e}_{N-1}], \tag{3.13}$$

which can be interpreted as the fundamental theorem of symmetric superpolynomials.
3.2. Complete symmetric superfunctions and involution. The $n$-th bosonic and fermionic complete symmetric superfunctions are given respectively by

\[ h_n := \sum_{\lambda \vdash n} m_\lambda \quad \text{and} \quad \hat{h}_n := \sum_{\Lambda \vdash (n+1)} (\Lambda_1 + 1) m_\Lambda, \tag{3.14} \]

From the explicit form of $h_n(x)$, namely, $\sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_n} x_{i_1} \cdots x_{i_n}$, we see that its fermionic partner is again generated by the action of $d$ in the form-representation:

\[ \tilde{h}_{n-1}(x, \theta) \sim \tilde{h}_{n-1}(x, dx) = d h_n(x) \quad \text{for all} \quad n \geq 1. \tag{3.15} \]

The generating function for complete symmetric superpolynomials is

\[ H(t, \tau) := \sum_{n=0}^{\infty} t^n(h_n + \tau \tilde{h}_n) = \prod_{i=1}^{\infty} \frac{1}{1 - tx_i - \tau \theta_i}. \tag{3.16} \]

To prove (3.16), one simply uses the inversion of even elements in the Grassmann algebra, given in (2.21), which gives

\[ \frac{1}{1 - tx_i - \tau \theta_i} = \sum_{n \geq 0} (tx_i + \tau \theta_i)^n = \sum_{n \geq 0} [(tx_i)^n + n \tau \theta_i (tx_i)^{n-1}] . \tag{3.17} \]

From relations (3.14) and (3.16), we get

\[ H(t, \tau) E(-t, -\tau) = 1. \tag{3.18} \]

By expanding the generating functions in terms of $e_n$, $\hat{e}_n$, $h_n$ and $\hat{h}_n$ in the last equation, we obtain recursion relations, of which the non-fermionic one is a well known formula.

**Lemma 19.** Let $n \geq 1$, then

\[ \sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0. \tag{3.19} \]

Let $n \geq 0$, then

\[ \sum_{r=0}^{n} (-1)^r (e_r \hat{h}_{n-r} - \hat{e}_r h_{n-r}) = 0. \tag{3.20} \]

Note that the second relation can be obtained from the first one by the action of $d$ (representing, as usual, $\theta_i$ as $dx_i$).

We consider a homomorphism $\tilde{\omega} : \mathcal{P}^S(\mathbb{Z}) \to \mathcal{P}^S(\mathbb{Z})$ defined by the following relations:

\[ \tilde{\omega} : e_n \mapsto h_n \quad \text{and} \quad \tilde{\omega} : \hat{e}_n \mapsto \hat{h}_n. \tag{3.21} \]

**Theorem 20.** The homomorphism $\tilde{\omega}$ is an involution, i.e., $\tilde{\omega}^2 = 1$. Equivalently, we have

\[ \tilde{\omega} : h_n \mapsto e_n \quad \text{and} \quad \tilde{\omega} : \hat{h}_n \mapsto \hat{e}_n. \tag{3.22} \]

**Proof.** This comes from the application of transformation (3.21) to the recursions appearing in Lemma 19 followed by the comparison with the original recursions. Explicitly:

\[ 0 = \sum_{r=0}^{n} (-1)^r \tilde{\omega}(e_r) \tilde{\omega}(h_{n-r}) = (-1)^n \sum_{r=0}^{n} (-1)^r \tilde{\omega}(h_r) h_{n-r}, \tag{3.23} \]

which implies $\tilde{\omega}(h_r) = e_r$. Similarly, we have

\[ 0 = \sum_{r=0}^{n} (-1)^r \left( \tilde{\omega}(e_r) \tilde{\omega}(\hat{h}_{n-r}) - \tilde{\omega}(\hat{e}_r) \tilde{\omega}(h_{n-r}) \right) = (-1)^{n-1} \sum_{r=0}^{n} (-1)^r \left( e_r \hat{h}_{n-r} - \hat{\omega}(\hat{h}_r) h_{n-r} \right), \tag{3.24} \]

leading to $\tilde{\omega}(\hat{h}_r) = \hat{e}_r$. \qed
Now, let

\[ h_A := \prod_{i=1}^{n} \tilde{h}_A \cdot \prod_{j=\Delta+1}^{\ell(A)} h_{\Lambda_j}. \]  

(3.25)

Equation (3.24) and Theorem 20 immediately give a bijection between two sets of multiplicative superpolynomials:

\[ \hat{\omega}(e_A) = h_A \quad \text{and} \quad \hat{\omega}(h_A) = e_A. \]  

(3.26)

We have thus obtained another \( \mathbb{Z} \)-basis for the algebra of symmetric superpolynomials.

**Corollary 21.** The set \( \{ h_A : \Lambda \vdash (n|m) \} \) is a basis of \( S^\infty_{(n|m)}(\mathbb{Z}) \).

Finally, Lemma 19 allows us to write determinantal expressions for the elementary symmetric superfunctions in terms of the complete symmetric superfunctions and vice versa using the homomorphism \( \hat{\omega} \).

**Proposition 22.** For \( n \geq 1 \), we have

\[
\begin{pmatrix}
 h_1 & h_2 & h_3 & \ldots & h_{n-1} & h_n \\
 1 & h_1 & h_2 & \ldots & h_{n-2} & h_{n-1} \\
 0 & 1 & h_1 & \ldots & h_{n-3} & h_{n-2} \\
 & & & \ddots & & \vdots \\
 0 & 0 & 0 & \ldots & 1 & h_1
\end{pmatrix}
\]

(3.27)

For \( n \geq 0 \), we have

\[
\begin{pmatrix}
 \hat{h}_0 & \hat{h}_1 & \hat{h}_2 & \ldots & \hat{h}_{n-1} & \hat{h}_n \\
 n & (n+1)h_1 & (n+2)h_2 & \ldots & (2n-1)h_{n-1} & 2nh_n \\
 0 & n-1 & nh_1 & \ldots & (2n-3)h_{n-2} & (2n-2)h_{n-1} \\
 & & & \ddots & \vdots & \vdots \\
 0 & 0 & n-2 & \ldots & (2n-5)h_{n-3} & (2n-4)h_{n-2} \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 0 & 0 & 0 & \ldots & 1 & 2h_1
\end{pmatrix}
\]

(3.28)

**Proof.** The first relation is well known to be a simple application of Cramer’s rule to the linear system coming from Lemma 19 \( h = eH \), where

\[
\begin{pmatrix}
 h_1 \\
 h_2 \\
 h_3 \\
 \vdots \\
 h_N
\end{pmatrix}
\quad
\begin{pmatrix}
 e_1 \\
 e_2 \\
 e_3 \\
 \vdots \\
 e_N
\end{pmatrix}
\quad
\begin{pmatrix}
 1 & h_1 & h_2 & h_3 & \ldots \\
 0 & -1 & -h_1 & -h_2 & \ldots \\
 0 & 0 & 1 & h_1 & \ddots \\
 0 & 0 & 0 & -1 & \ddots \\
 \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

(3.29)

To obtain the other determinant, we use the second formula of Lemma 19 to obtain the linear system:

\[
\hat{h} \bar{H} = \bar{e} \bar{H},
\]

(3.30)
where $H$ is as given above, and where

$$\hat{h} = \begin{pmatrix} \tilde{h}_0 & \tilde{h}_1 & \tilde{h}_2 & \ldots & \tilde{h}_{n-1} & \tilde{h}_n \\ \tilde{h}_1 & \tilde{h}_2 & \ldots & \tilde{h}_{n-1} & \tilde{h}_n \\ \vdots & \vdots & & \vdots & \vdots \\ \tilde{h}_{n-1} & \vdots & \ldots & \tilde{h}_2 & \tilde{h}_1 \end{pmatrix} \quad \tilde{e} = \begin{pmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \vdots \\ \tilde{e}_{n-1} \end{pmatrix} \quad H = \begin{pmatrix} 1 & -e_1 & -e_2 & \ldots \\ 0 & 1 & -e_1 & e_2 & \ldots \\ 0 & 0 & 1 & -e_1 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \end{pmatrix}.$$ (3.31)

Using the coadjoint formula for the inverse of a matrix, and the determinantal expression for $h_n$ obtained by applying the homomorphism $\hat{\omega}$ on the determinant of $e_n$ given above, it is not hard to see that the $(i, j)$-th component of the inverse of $H$ is simply $h_{j-i}$. We are thus led to the matrix relation:

$$\hat{h} = \hat{e} H,$$ (3.32)

where

$$\hat{H}_{i,j} = (-1)^{i+j} \sum_{k=0}^{i-j} h_{i-j-k} h_k = -\hat{H}_{i+1,j+1}.$$ (3.33)

If we set $H_i = \sum_{k=0}^i h_{i-k} h_k$, the matrix $\hat{H}$ can be expressed in a convenient form as

$$\hat{H} = \begin{pmatrix} 1 & H_1 & H_2 & H_3 & \ldots \\ 0 & -1 & -H_1 & -H_2 & \ldots \\ 0 & 0 & 1 & H_1 & \ldots \\ 0 & 0 & 0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix}.$$ (3.34)

Using Cramer’s rule and then multiplying rows 2, 3, …, $n$ of the resulting determinant by $n, n-1, \ldots, 1$ respectively, we obtain

$$\tilde{e}_n = \frac{1}{n!} \begin{vmatrix} \tilde{h}_0 & \tilde{h}_1 & \tilde{h}_2 & \ldots & \tilde{h}_{n-1} & \tilde{h}_n \\ n & nH_1 & nH_2 & \ldots & nH_{n-1} & nH_n \\ 0 & n-1 & (n-1)H_1 & \ldots & (n-1)H_{n-2} & (n-1)H_{n-1} \\ 0 & 0 & n-2 & \ldots & (n-2)H_{n-3} & (n-2)H_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & H_1 \end{vmatrix}.$$ (3.35)

We will finally show that by manipulating determinant $\text{(3.23)}$, we obtain determinant $\text{(3.35)}$. Let $R_i$ be row $i$ of determinant $\text{(3.23)}$. If $R_2 \rightarrow R_2 + R_3 h_1 + \cdots + R_n h_{n-2}$ in this determinant, the second row becomes that of determinant $\text{(3.35)}$ due to the simple identity (for $i = 1, \ldots, n$)

$$nH_i = (n+i)h_i + (n+i-2)h_{i-1}h_1 + \cdots + (n+i-2i+2)h_{i-1}h_{i-1} + (n+i-2i)h_i.$$ Doing similar operations on the lower rows, the two determinants are seen to coincide. \hfill $\square$

3.3. Superpower sums. We define the $n$-th bosonic and fermionic superpower sums as follows:

$$p_n := \sum_{i=1}^n x_i^n = m(n) \quad \text{and} \quad \tilde{p}_n := \sum_{i=1}^n \theta_i x_i^n = m_{(n,0)}.$$ (3.36)

Note that this time we will set $p_0 = 0$. Obviously,

$$n \tilde{p}_{n-1}(x, \theta) \sim n \tilde{p}_{n-1}(x, dx) = dp_n(x)$$ (3.37)

for all $n \geq 1$. 

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Proceeding as in the complete symmetric superfunctions case, we introduce products of power sums:

\[ p_{\lambda} := \prod_{i=1}^{\lambda} \tilde{p}_{\lambda_i} \prod_{j=\lambda+1} \tilde{p}_{\lambda_j} \]  

(3.38)

Also, we find that the generating function for superpower sums is

\[ P(t, \tau) := \sum_{n \geq 0} t^n p_n + \tau \sum_{n \geq 0} (n + 1) t^n \tilde{p}_n = \sum_{i=1}^{\infty} \frac{tx_i + \tau \theta_i}{1 - tx_i - \tau \theta_i} . \]  

(3.39)

One directly verifies

\[ H(t, \tau) P(t, \tau) = (t \partial_t + \tau \partial_{\tau}) H(t, \tau) \] 

(3.40)

and

\[ E(t, \tau) P(-t, -\tau) = -(t \partial_t + \tau \partial_{\tau}) E(t, \tau) . \] 

(3.41)

These expressions lead after some manipulations to the following recursion relations.

**Lemma 23.** Let \( n \geq 1 \). Then

\[ h_n = \sum_{r=1}^{n} \tilde{p}_r h_{n-r} , \quad n e_n = \sum_{r=1}^{n} (-1)^{r+1} \tilde{p}_r e_{n-r} . \] 

(3.42)

Let \( n \geq 0 \), and recall that \( p_0 = 0 \). Then

\[ (n + 1) \tilde{h}_n = \sum_{r=0}^{n} [ p_r \tilde{h}_{n-r} + (r + 1) \tilde{p}_r h_{n-r} ] , \] 

(3.43)

\[ (n + 1) \tilde{e}_n = \sum_{r=0}^{n} (-1)^{r+1} [ p_r \tilde{e}_{n-r} - (r + 1) \tilde{p}_r e_{n-r} ] . \] 

(3.44)

**Theorem 24.** Let \( \tilde{\omega} \) be the involution defined in (3.21). Then, for \( n > 0 \),

\[ \tilde{\omega} : p_n \mapsto (-1)^{n-1} p_n \quad \text{and} \quad \tilde{p}_{n-1} \mapsto (-1)^{n-1} \tilde{p}_{n-1} \] 

(3.45)

or, equivalently,

\[ \tilde{\omega}(p_{\lambda}) = \omega_{\lambda} p_{\lambda} \quad \text{with} \quad \omega_{\lambda} := (-1)^{|\lambda| + \lambda - \ell(\lambda)} . \] 

(3.46)

**Proof.** We use Lemma 23 and proceed as in the proof of Theorem 20. \( \square \)

**Proposition 25.** For \( n \geq 1 \), we have

\[
\begin{pmatrix}
  e_1 & 2e_2 & 3e_3 & \cdots & ne_n \\
  1 & e_1 & e_2 & \cdots & e_{n-1} \\
  0 & 1 & e_1 & \cdots & e_{n-2} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & e_1 \\
\end{pmatrix}, \quad n! e_n =
\begin{pmatrix}
p_1 & p_2 & \cdots & p_{n-1} & p_n \\
1 & p_1 & \cdots & p_{n-2} & p_{n-1} \\
0 & 2 & \cdots & p_{n-3} & p_{n-2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & n-1 & p_1 \\
\end{pmatrix}.
\]

(3.47)

For \( n \geq 0 \), we have

\[
\begin{pmatrix}
  \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \cdots & \tilde{e}_n \\
  1 & e_1 & e_2 & \cdots & e_n \\
  0 & 1 & e_1 & \cdots & e_{n-1} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & e_1 \\
\end{pmatrix}, \quad n! \tilde{e}_n =
\begin{pmatrix}
  \tilde{p}_0 & \tilde{p}_1 & \cdots & \tilde{p}_{n-1} & \tilde{p}_n \\
  n & p_1 & \cdots & p_{n-2} & p_{n-1} \\
  0 & n-1 & \cdots & p_{n-3} & p_{n-2} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 & p_1 \\
\end{pmatrix}.
\]

(3.48)

Similar formulas for the complete symmetric superfunctions are obtained by using the involution \( \tilde{\omega} \).
Proposition 27. Let introduce a bilinear form, \( \langle \langle \cdot, \cdot \rangle \rangle \), that is, in addition to bilinearity, we have 
\[
\|f \|_p^2 = \sum_{\Lambda} f_{\Lambda}^2 z_{\Lambda} \quad \text{for} \quad z_{\Lambda} := \prod_{k \geq 1} \left[ k^{n_{\Lambda}(k)} n_{\Lambda'}(k)! \right].
\]

Proof. The proof is similar to that of Proposition 25. \( \square \)

The explicit formulas presented in Proposition 25 establish the correspondence between the sets \( \{p_n, \tilde{p}_{n-1}\} \) and \( \{c_n, \tilde{c}_{n-1}\} \). This implies, in particular, that \( e_{\Lambda} = \sum_{\Omega} c_{\Lambda\Omega} p_{\Omega} \) for uniquely determined coefficients \( c_{\Lambda\Omega} \in \mathbb{Q} \). Note that \( c_{\Lambda\Omega} \) is not necessarily an integer since, for instance, \( e_2 = \tilde{p}_1^2 / 2 - \tilde{p}_2 \). Theorem 18 and Proposition 25 thus imply the following result.

Corollary 26. The set \( \{p_{\Lambda} : \Lambda \vdash (n|m)\} \) is a basis of \( \mathcal{S}_{(n|m)}^N(\mathbb{Q}) \) if either \( n \leq N \) and \( m = 0 \) or \( n < N \) and \( m > 0 \). In particular, the set \( \{p_{\Lambda} : \Lambda \vdash (n|m)\} \) is a basis of \( \mathcal{S}_{(n|m)}^\infty(\mathbb{Q}) \).

The power sums will play a fundamental role in the remainder of the article. For this reason, we will consider, from now on, only symmetric superpolynomials defined over the rational numbers (or any greater field):
\[
\mathcal{S}_{\infty} := \mathcal{S}_{(n|m)}^\infty(\mathbb{Q}).
\]

3.4. Orthogonality. Let \( n_{\Lambda}(i) \) denote the number of parts equal to \( i \) in the partition \( \Lambda \). We introduce a bilinear form, \( \langle \langle \cdot, \cdot \rangle \rangle : \mathcal{S}_{\infty} \times \mathcal{S}_{\infty} \to \mathbb{Q} \), defined by
\[
\langle \langle p_{\Lambda} | p_{\Omega} \rangle \rangle := z_{\Lambda} \delta_{\Lambda,\Omega}, \quad \text{for} \quad z_{\Lambda} := \prod_{k \geq 1} \left[ k^{n_{\Lambda}(k)} n_{\Lambda'}(k)! \right].
\]

Proposition 27. Let \( f \) and \( g \) be superpolynomials in \( \mathcal{S}_{\infty} \). Then \( \langle \langle f | g \rangle \rangle \) is a scalar product, that is, in addition to bilinearity, we have
\[
\begin{align*}
\langle \langle f | g \rangle \rangle &= \langle \langle g | f \rangle \rangle = \langle \langle g | f \rangle \rangle \quad \text{(symmetry)} \\
\langle \langle f | f \rangle \rangle &> 0 \quad \forall \, f \neq 0 \quad \text{(positivity)}.
\end{align*}
\]

Proof. The symmetry property is a consequence of Lemma 8. The positivity of the scalar product is proved as follows. By definition \( z_{\Lambda} > 0 \) and by virtue of Corollary 26 there is a unique decomposition \( f = \sum_{\Lambda} f_{\Lambda} p_{\Lambda} \). Therefore \( \langle \langle f | f \rangle \rangle = \sum_{\Lambda} f_{\Lambda}^2 z_{\Lambda} > 0 \). \( \square \)

Proposition 28. The involution \( \hat{\omega} \) is an isometry.

Proof. Given that \( \{p_{\Lambda}\}_\Lambda \) is a basis of \( \mathcal{S}_{\infty} \), for any symmetric polynomials \( f \) and \( g \), we have 
\[
\langle \langle \hat{\omega}f | \hat{\omega}g \rangle \rangle = \sum_{\Lambda,\Omega} f_{\Lambda} g_{\Omega} \langle \langle \hat{\omega}p_{\Lambda} | \hat{\omega}p_{\Omega} \rangle \rangle
\]
\[
= \sum_{\Lambda} f_{\Lambda} (-1)^{\Lambda(\Lambda)} x^{\lambda(\Lambda)} (-1)^{\Omega(\Omega)} x^{\lambda(\Omega)} \langle \langle p_{\Lambda} | p_{\Omega} \rangle \rangle
\]
\[
= \sum_{\Lambda} z_{\Lambda} f_{\Lambda} g_{\Lambda} = \langle \langle f | g \rangle \rangle,
\]
as claimed. \( \square \)

The following theorem is of particular importance since it gives Cauchy-type formulas for the superpower sums.

Theorem 29. Let \( K = K(x, \theta; y, \phi) \) be the bi-symmetric formal superfunction given by
\[
K := \prod_{i,j} \frac{1}{1 - x_i y_j - \theta_i \phi_j}.
\]

\[\text{(3.53)}\]
Then
\[ K = \sum_{\Lambda \in \text{Par}} z_\Lambda^{-1} p_\Lambda(x, \theta) \overleftarrow{p_\Lambda(y, \phi)}. \] (3.54)

**Proof.** We have:
\[
\prod_{i,j} \frac{1}{1 - x_iy_j - \theta_i\phi_j} = \exp \left\{ \sum_{i,j} \ln \left[ (1 - x_iy_j - \theta_i\phi_j)^{-1} \right] \right\} = \exp \left\{ \sum_{i,j} \sum_{n \geq 1} \left[ \frac{1}{n} (x_iy_j + \theta_i\phi_j)^n \right] \right\} \\
= \exp \left\{ \sum_{n \geq 1} \left[ \frac{1}{n} p_n(x) p_n(y) \right] + \sum_{n \geq 0} \left[ \tilde{p}_n(x, \theta) \tilde{p}_n(y, \phi) \right] \right\} \\
= \prod_{n \geq 1} \prod_{k_n \geq 0} \left[ p_n(x) p_n(y) \right] \exp \left[ \sum_{n \geq 0} \tilde{p}_n(x, \theta) \tilde{p}_n(y, \phi) \right].
\] (3.55)

Considering Proposition 30, we find
\[
\prod_{i,j} \frac{1}{1 - x_iy_j - \theta_i\phi_j} = \sum_{n \geq 0} \sum_{\lambda \in \text{Par}(n)} \sum_{\mu \in \text{Par}_+(m)} \left[ z_\Lambda^{-1} p_\Lambda(x) p_\Lambda(y) \overleftarrow{\tilde{p}_\mu(x, \theta) \tilde{p}_\mu(y, \phi)} \right].
\] (3.56)

This equation (together with Lemma 30) proves the theorem. \(\square\)

**Remark 30.** The inverse of the kernel satisfies:
\[
K(-x, -\theta; y, \phi)^{-1} = \prod_{i,j} (1 + x_iy_j + \theta_i\phi_j) = \sum_{\Lambda \in \text{Par}} \omega_\Lambda z_\Lambda^{-1} p_\Lambda(x, \theta) \overleftarrow{p_\Lambda(y, \phi)}. \] (3.57)

The proof of this result is similar to that of Theorem 29, apart from the presence of the coefficient \(\omega_\Lambda = (-1)^{|\Lambda| + \lambda_\Lambda - \ell(\Lambda)}\), which comes from the expansion of \(\ln(1 + x_iy_j + \theta_i\phi_j)\). This shows that
\[
K(x, \theta; y, \phi) = \omega(x, \theta) K(-x, -\theta; y, \phi)^{-1} = \omega(y, \phi) K(-x, -\theta; y, \phi)^{-1}
\] (3.58)

where \(\omega(x, \theta)\) indicates that \(\omega\) acts on the \((x, \theta)\) variables and similarly for \(\omega(y, \phi)\).

We now give two direct consequences of Theorem 29.

**Corollary 31.** \(K\) is a reproducing kernel in the space of symmetric superfunctions:
\[
\langle \langle K(x, \theta; y, \phi) \mid f(x, \theta) \rangle \rangle = f(y, \phi), \quad \text{for all } f \in \mathcal{S}_\infty.
\] (3.59)

**Proof.** If \(f \in \mathcal{S}_\infty\), there exist unique coefficients \(f_\Lambda\) such that \(f = \sum_\Lambda f_\Lambda p_\Lambda\). Hence,
\[
\langle \langle K(x, \theta; y, \phi) \mid f(x, \theta) \rangle \rangle = \sum_{\Omega, \Lambda} f_\Lambda \overleftarrow{p_\Omega(x, \theta) p_\Lambda(x, \theta)} \overleftarrow{p_\Lambda(y, \phi)} = \sum_{\Lambda} f_\Lambda p_\Lambda(y, \phi) = f(y, \phi),
\] (3.60)
as desired. \(\square\)

**Corollary 32.** We have
\[
h_n = \sum_{\Lambda \in \text{Par}(n|0)} z_\Lambda^{-1} p_\Lambda \quad \text{and} \quad e_n = \sum_{\Lambda \in \text{Par}(n|0)} z_\Lambda^{-1} \omega_\Lambda p_\Lambda,
\] (3.61)
\[
\tilde{h}_n = \sum_{\Lambda \in \text{Par}(n|1)} z_\Lambda^{-1} p_\Lambda \quad \text{and} \quad \tilde{e}_n = \sum_{\Lambda \in \text{Par}(n|1)} z_\Lambda^{-1} \omega_\Lambda p_\Lambda.
\] (3.62)
Proof. Using the definition of the generating function $E(t, \tau)$, we first make the following correspondence:

$$E(t, 0) = \sum_{n \geq 0} t^n e_n(x) = K(-x, 0; y, 0)^{-1} \big|_{y=\tau}.$$  \hfill (3.63)

Thus, from Theorem 29 and $p_\lambda(y)|_{y=(t,0,0,...)} = \tau|^{\lambda}$, we have

$$\sum_{n \geq 0} t^n e_n(x) = \sum_{\lambda \in \text{Par}} \tau|^{\lambda} \omega_\lambda z^{-1} p_\lambda(x) \quad \Longrightarrow \quad e_n = \sum_{\lambda \vdash n} z^{-1} \omega_\lambda p_\lambda.$$  \hfill (3.64)

Then, we observe that

$$\partial_x E(t, \tau) = \sum_{n=0}^{N-1} t^n \tilde{e}_n(x, \tau) = \partial_x K(x, \theta; y, \phi) \bigg|_{y=(t,0,0,...) \atop \phi=(-\tau,0,...)}.$$  \hfill (3.65)

Theorem 29 and

$$p_\lambda(y, \phi)|_{y=(t,0,0,...) \atop \phi=(-\tau,0,...)} = \begin{cases} \tau|^{\lambda} & \text{if } \lambda = 0, \\
-\tau|^{\lambda} & \text{if } \lambda = 1, \\
0 & \text{otherwise} \end{cases}$$  \hfill (3.66)

finally lead to

$$\sum_{n \geq 0} t^n \tilde{e}_n(x, \tau) = \sum_{\Lambda, \Delta = 1} z^{-1} \omega_\lambda p_\lambda(x, \theta) \quad \Longrightarrow \quad \tilde{e}_n = \sum_{\Lambda \in \text{Par}(n|1)} z^{-1} \omega_\lambda p_\lambda.$$  \hfill (3.67)

Note that the minus sign disappears since $\partial_x$ and $p_\lambda(x, \theta)$ anticommute when $\lambda = 1$. Similar formulas relating the superpower sums to the homogeneous symmetric superpolynomials are obtained using the involution $\hat{\omega}$.

\begin{lemma}
Let $\{u_\Lambda\}$ and $\{v_\Lambda\}$ be two bases of $\mathcal{S}_n^{\infty}$ (see [21]). Then

$$K(x, \theta; y, \phi) = \sum_{\Lambda} u_\Lambda(x, \theta) v_\Lambda(y, \phi) \quad \Longleftrightarrow \quad \left\langle u_\Lambda \mid v_\Lambda \right\rangle = \delta_{\Lambda, \Omega}.$$  \hfill (3.68)

\end{lemma}

Proof. The proof is identical to the one in the case without Grassmannian variables (see [21] I.4.6).

\begin{proposition}
Let $K$ be the superfunction defined in $\mathcal{S}_n^{\infty}$ (see [21]). Then,

$$K(-x, -\theta; y, \phi) = \prod_{i,j} (1 + x_i y_j + \theta_i \phi_j) \prod_i E(x_i, \theta_i).$$

\end{proposition}

Proof. We start with the definition of the generating function $E(t, \tau)$:

$$K(-x, -\theta; y, \phi)^{-1} = \prod_{i,j} (1 + x_i y_j + \theta_i \phi_j) = \prod_i E(x_i, \theta_i)$$

$$= \prod_i \left[ \sum_{n \geq 0} x_i^n e_n(y) + \theta_i \sum_{n \geq 0} x_i^n \tilde{e}_n(y, \phi) \right]$$

$$= \sum_{\epsilon_1, \epsilon_2, \cdots \in \{0,1\}, n_1, n_2, \cdots \geq 0} \prod_i \left( \theta_i \tau^{\epsilon_i} x_i^{n_i} e_{n_i}^{(\epsilon_i)}(y, \phi) \right)$$

$$= \sum_{\Lambda \in \text{Par}} m_\Lambda(x, \theta) e_\Lambda(y, \phi).$$  \hfill (3.70)

In the third line we have set $e_n^{(0)}(y, \phi) = e_n(y, \phi)$ and $e_n^{(1)}(y, \phi) = \tilde{e}_n(y, \phi)$. The fourth line follows by reordering the variables using Lemma 33. Using (3.65), we can recover $K(x, \theta; y, \phi)$ by acting with $\hat{\omega}(y, \phi)$ on $K(-x, -\theta; y, \phi)^{-1}$. The identity then follows from $\hat{\omega}(e_\Lambda) = h_\Lambda$.

The previous proposition and Lemma 33 have the following corollary.
Corollary 35. The supermonomials are dual to the complete symmetric superfunctions:

\[
\langle \hat{h}_\Lambda | m_\Omega \rangle = \delta_{\Lambda,\Omega}.
\]  

(3.71)

3.5. One-parameter deformation of the scalar product and the homogeneous basis. In this section, we introduce a natural one-parameter – called \( \beta \) – deformation of the scalar product, of the corresponding superspace kernel as well as the deformation of the homogeneous symmetric basis in superspace.

Let \( \mathcal{S}^{\Lambda}(\beta) \) stand for the algebra of symmetric superpolynomials with coefficients in \( \mathbb{Q}(\beta) \), i.e., rational functions of \( \beta \). We now introduce a mapping,

\[
\langle \cdot | \cdot \rangle_\beta : \mathcal{S}^{\Lambda}(\beta) \times \mathcal{S}^{\Lambda}(\beta) \to \mathbb{Q}(\beta)
\]

defined by

\[
\langle p_\alpha | p_\beta \rangle_\beta := z_\Lambda(\beta)\delta_{\Lambda,\Omega}, \quad \text{where} \quad z_\Lambda(\beta) := \beta^{-\ell(\Lambda)}z_\Lambda.
\]

(3.73)

This bilinear form can again be shown to be a scalar product.

We also introduce a homomorphism that generalizes the involution \( \hat{\omega} \). It is defined on the power sums as:

\[
\hat{\omega}_\alpha(p_n) = (-1)^{n-1} \alpha p_n \quad \text{and} \quad \hat{\omega}_\alpha(p_n) = (-1)^n \alpha \hat{p}_n,
\]

where \( \alpha \) is some unspecified parameter. This implies

\[
\hat{\omega}_\alpha(p_\Lambda) = \omega_\Lambda(\alpha) p_\Lambda \quad \text{with} \quad \omega_\Lambda(\alpha) := \alpha^{\ell(\Lambda)}(-1)^{\ell(\Lambda)}\hat{\omega}_\Lambda(\beta) - z_\Lambda(\beta)\omega_\Lambda(\beta).
\]

(3.75)

Notice that \( \hat{\omega}_1 \equiv \hat{\omega} \). This homomorphism is still self-adjoint, but it is now neither an involution (\( \hat{\omega}_\alpha^{-1} = \hat{\omega}_\alpha^{-1} \)) nor an isometry (\( \|\hat{\omega}_\alpha p_\Lambda\|^2 = z_\Lambda(\beta/\alpha^2) \)). Note also that

\[
z_\Lambda(\beta)\omega_\Lambda(\beta) = z_\Lambda\omega_\Lambda \quad \text{and} \quad z_\Lambda(\beta)^{-1}\omega_\Lambda(\beta^{-1}) = z_\Lambda^{-1}\omega_\Lambda.
\]

(3.76)

Theorem 36. With \( K^\beta \) given by

\[
K^\beta = \prod_{i,j} \frac{1}{1 - x_i y_j - \theta_i \phi_j},
\]

(3.77)

we have

\[
K^\beta = \sum_\Lambda z_\Lambda(\beta)^{-1} \frac{\hat{p}_\Lambda(x, \theta)}{p_\Lambda(y, \phi)}.
\]

(3.78)

Proof. Starting from

\[
\prod_{i,j} \frac{1}{1 - x_i y_j - \theta_i \phi_j} = \exp \left\{ \beta \sum_{i,j} \ln \left[ (1 - x_i y_j - \theta_i \phi_j)^{-1} \right] \right\},
\]

(3.79)

The above identity can be obtained straightforwardly proceeding as in the proof of Proposition 29.

Remark 37. The inverse of \( K^\beta \) satisfies:

\[
K(-x, -\theta; y, \phi)^{-\beta} = \prod_{i,j} (1 + x_i y_j + \theta_i \phi_j)^{\beta} = \sum_\Lambda z_\Lambda(\beta)^{-1} \omega_\Lambda \frac{\hat{p}_\Lambda(x, \theta)}{p_\Lambda(y, \phi)},
\]

(3.80)

which is obtained by using

\[
p_\Lambda(-x, -\theta) = (-1)^{\ell(\Lambda)+\sum} p_\Lambda(x, \theta) \quad \text{and} \quad z_\Lambda(-\beta) = (-1)^{\ell(\Lambda)}z_\Lambda(\beta).
\]

(3.81)

Notice also the simple relation between the kernel and its \( \beta \)-deformation

\[
K^\beta(x, \theta; y, \phi) = \hat{\omega}_\beta K(-x, -\theta; y, \phi)^{-\beta},
\]

(3.82)

where it is understood that \( \hat{\omega}_\beta \) acts either on \( (x, \theta) \) or on \( (y, \phi) \).
Corollary 38. \( K^\beta(x;\theta;y,\phi) \) is a reproducing kernel in the space of symmetric superfunctions with rational coefficients in \( \beta \):

\[
\langle K^\beta(x;\theta;y,\phi) \mid f(x;\theta) \rangle_\beta = f(y,\phi), \quad \text{for all} \quad f \in \mathcal{P}^{S\infty}(\beta).
\] (3.83)

We now introduce a \( \beta \)-deformation of the bosonic and fermionic complete homogeneous symmetric functions, respectively denoted as \( g_n(x) \) and \( \tilde{g}_n(x,\theta) \) (the \( \beta \)-dependence being implicit). Their generating function is

\[
G(t,\tau;\beta) := \sum_{n \geq 0} t^n g_n(x) + \tau \tilde{g}_n(x,\theta) = \prod_{i \geq 1} \frac{1}{(1 - tx_i - \tau \theta_i)^\beta}.
\] (3.84)

Clearly, \( g_n = h_n \) and \( \tilde{g}_n = \tilde{h}_n \) when \( \beta = 1 \). As usual, we define

\[
g_\Lambda := \prod_{i=1}^{\overline{3}} \tilde{g}_{\lambda_i} \prod_{j=\overline{3}+1}^{\Lambda} g_{\lambda_j}.
\] (3.85)

Proposition 39. We have

\[
K^\beta(x;\theta;y,\phi) = \sum_{\Lambda} m_\Lambda(x,\theta) g_\Lambda(y,\phi).
\]

Proof. We proceed as in the proof of Proposition 41.

\[
K^\beta = \prod_i G(x_i;\theta_i;\beta) = \prod_i \left( \sum_{n \geq 0} x_i^n g_n(y) + \theta_i \tilde{g}_n(y,\phi) \right)
= \sum_{\lambda_1,\lambda_2,\ldots \in \{0,1\}} \sum_{n_1,n_2,\ldots \geq 0} \prod_i \left( \theta_i x_i^{n_i} g_n^{(\lambda_i)}(y,\phi) \right),
\] (3.86)

where \( g_n^{(0)}(y,\phi) = g_n(y,\phi) \) and \( g_n^{(1)}(y,\phi) = \tilde{g}_n(y,\phi) \). By reordering the variables using Lemma 8 we get the desired result.

Corollary 40. We have

\[
g_n = \sum_{\Lambda \vdash (n|0)} z_\Lambda(\beta)^{-1} p_\Lambda \text{ and } \tilde{g}_n = \sum_{\Lambda \vdash (n|1)} z_\Lambda(\beta)^{-1} p_\Lambda.
\] (3.87)

Proof. On the one hand,

\[
G(t,0;\beta) = \sum_{n \geq 0} t^n g_n(x) = K^\beta(x,0;y,0)\big|_{y=(t,0,0,\ldots)}.
\] (3.88)

The previous proposition and Theorem 56 imply

\[
\sum_{n \geq 0} t^n g_n = \sum_{\lambda \in \text{Par}} t^{|\lambda|} z_\lambda(\beta)^{-1} p_\lambda \quad \implies \quad g_n = \sum_{\lambda \in \text{Par}(n)} z_\lambda(\beta)^{-1} p_\lambda.
\] (3.89)

On the other hand,

\[
\partial_\tau G(t,\tau;\beta) = t^n \tilde{g}_n(x,\theta) = K^\beta_+(x,\theta;y,\phi)\big|_{y=(t,0,0,\ldots)}\big|_{\phi=(\tau,0,0,\ldots)}.
\] (3.90)

Hence

\[
\sum_{n \geq 0} t^n \tilde{g}_n = \sum_{\lambda \vdash 1} t^{|\lambda|} z_\lambda(\beta)^{-1} p_\lambda \quad \implies \quad \tilde{g}_n = \sum_{\lambda \vdash \lambda(\beta)^{-1} p_\lambda},
\] (3.91)

as claimed.

\[29\]
Using the previous corollary and the relation \( np_{n-1} \sim dp_n \), it is easy to show that the fermionic superfunction \( \tilde{g}_{n-1} \) can be represented as the exterior derivative of \( g_n \):

\[
\tilde{g}_{n-1}(x, \theta) \sim dg_n(x) \quad \text{for} \quad n \geq 1.
\]

This is a direct extension of the \( \beta = 1 \) case. Also, applying \( \omega_{\beta-1} \) on equation and comparing with Corollary \ref{corollary32}, we get

\[
\beta_{\beta-1} g_n = e_n \quad \text{and} \quad \beta_{\beta-1} \tilde{g}_n = \tilde{e}_n.
\]

Lemma \ref{lemma33} can also be trivially generalized.

\[
K^{\beta}(x, \theta; y, \phi) = \sum_{\Lambda} \text{u}_{\Lambda}(x, \theta) \text{v}_{\Lambda}(y, \phi) \iff \left\langle \left\langle \text{u}_{\Lambda} \mid \text{v}_{\Lambda} \right\rangle \right\rangle_{\beta} = \delta_{\Lambda \Omega}.
\]

This equation, together with Proposition \ref{proposition34}, immediately implies the following.

**Corollary 41.** The set \( \{g_\Lambda\}_{\Lambda} \) constitutes a basis of \( \mathcal{P}_{S,N}^{S,N}(\beta) \) dual to that of the supermonomials, that is,

\[
\left\langle \left\langle g_\Lambda \mid m_\Omega \right\rangle \right\rangle_{\beta} = \delta_{\Lambda \Omega}.
\]

We shall need in the next section to make explicit the distinction between an infinite and a finite number of variables. Therefore, we also let

\[
\left\langle \left\langle \cdot \right\rangle \right\rangle_{\beta,N} : \mathcal{P}_{S,N}^{S,N}(\beta) \times \mathcal{P}_{S,N}^{S,N}(\beta) \rightarrow \mathbb{Q}(\beta)
\]

be defined by requiring that the bases \( \{g_\Lambda\}_{\ell(\Lambda) \leq N} \) and \( \{m_\Lambda\}_{\ell(\Omega) \leq N} \) be dual to each other:

\[
\left\langle \left\langle g_\Lambda \mid m_\Omega \right\rangle \right\rangle_{\beta,N} := \delta_{\Lambda \Omega},
\]

whenever \( \ell(\Lambda) \) and \( \ell(\Omega) \) are not larger than \( N \). From this definition, it is thus obvious that

\[
\left\langle \left\langle f^{(N)} \mid g^{(N)} \right\rangle \right\rangle_{\beta,N} = \left\langle \left\langle f \mid g \right\rangle \right\rangle_{\beta}.
\]

if \( f \) and \( g \) are symmetric superpolynomials of degrees not larger than \( N \), and if \( f^{(N)} \) and \( g^{(N)} \) are their respective restriction to \( N \) variables. This is because \( f \) and \( f^{(N)} \) (resp. \( g \) and \( g^{(N)} \)) then have the same expansion in terms of the \( g \) and \( m \) bases. Note that with this definition, we have that

\[
K^{\beta,N} = \sum_{\ell(\Lambda) \leq N} g_\Lambda(x, \theta) m_\Lambda(y, \phi),
\]

where \( K^{\beta,N} \) is the restriction of \( K^{\beta} \) to \( N \) variables and where \( (x, \theta) \) stand for \( (x_1, \ldots, x_N; \theta_1, \ldots, \theta_N) \) and \( (y, \phi) \) for \( (y_1, \ldots, y_N; \phi_1, \ldots, \phi_N) \).

We complete this section by displaying a relationship between the \( g \)-basis elements and the bases of monomials and homogeneous superpolynomials.

**Proposition 42.** Let \( n_\Lambda! := n_{\Lambda^\prime}(1)! n_{\Lambda^\prime}(2)! \cdots \), and

\[
\binom{\beta}{n} := \frac{\beta(\beta - 1) \cdots (\beta - n + 1)}{n!}, \quad (\beta)_n := \beta(\beta - 1) \cdots (\beta - n + 1).
\]

Then

\[
g_n = \sum_{\Lambda^\prime(n_0)} \prod_i \binom{\beta + \Lambda_i - 1}{\Lambda_i} m_\Lambda = \sum_{\Lambda^\prime(n_0)} (\beta)_{\ell(\Lambda)} n_\Lambda! h_\Lambda, \]

\[
\tilde{g}_n = \sum_{\Lambda^\prime(n_1)} (\beta + \Lambda_1) \prod_i \binom{\beta + \Lambda_i - 1}{\Lambda_i} m_\Lambda = \sum_{\Lambda^\prime(n_1)} (\beta)_{\ell(\Lambda)} n_\Lambda! h_\Lambda.
\]
Proof. We start with the generating function (3.84). The product on the right hand side can also be written as
\[
\prod_{i \geq 1} \sum_{k \geq 0} (\beta + k - 1) k (tx_i + \tau \theta_i)^k
\]
After some easy manipulations, this becomes
\[
\sum_{n \geq 0} t^n \left[ \prod_{\lambda \vdash n} \left( \beta + \lambda_i - 1 \right)^m_{\lambda_i} + \tau \sum_{\Lambda} (\beta + \Lambda_1) \prod_{i} \left( \beta + \Lambda_i - 1 \right)^m_{\Lambda_i} \right]
\]
and the first equality in the two formulas (3.101) and (3.102) are seen to hold.

To prove the remaining two formulas, we use the generating function of the homogeneous symmetric functions and proceed as follows:
\[
\prod_i (1 - tx_i - \tau \theta_i)^{-\beta} = \left( 1 + \sum_{m \geq 1} t^m h_m + \tau \sum_{n \geq 0} t^n \tilde{h}_n \right)^\beta
\]
\[
= \sum_{k \geq 0} \left( \beta \right)^k \left( \sum_{m \geq 1} t^m h_m + \tau \sum_{n \geq 0} t^n \tilde{h}_n \right)^k
\]
\[
= \sum_{n \geq 0} \left[ \sum_{\lambda \vdash n} \frac{(\beta)^{\ell(\lambda)}}{n_\lambda!} h_\lambda + \tau \sum_{m \geq 0} \frac{(\beta)^{\ell(\lambda) + 1}}{\lambda!} h_\lambda \sum_{n \geq 0} t^n \tilde{h}_n \right]
\]
From which the desired expressions can be obtained. □

4. Jack polynomials in superspace

The main goal of this section is to show that there exists a natural supersymmetric analog to the combinatorics of the Jack polynomials. Our main result is thus the following theorem, to be proved in section 4.2, as part of Corollary 56:

**Theorem 43.** There exists a basis \( \{ \hat{J}_\Lambda \} \) of \( \mathcal{P}^{S\infty} \) such that
\[
1) \quad \hat{J}_\Lambda = m_\Lambda + \sum_{\Omega < \Lambda} c_{\Lambda \Omega}(\beta)m_\Lambda \quad \text{(triangularity)};
2) \quad \langle \hat{J}_\Lambda | \hat{J}_\Omega \rangle_\beta \propto \delta_{\Lambda, \Omega} \quad \text{(orthogonality)}.
\] (4.1)

Quite remarkably, the resulting construction is completely equivalent to that of the Jack polynomials \( J_\Lambda \) in superspace (or Jack superpolynomials) that were defined in [33] from a physical eigenvalue problem. Before we establish this, it is probably appropriate to first review some relevant aspects of that article.

4.1. Characterizations of the physical Jack superpolynomials. We give the main properties of Jack superpolynomials. The results in the first part of this subsection can all be found in [33]. The section is completed with the presentation of two technical lemmas. All the results of this section are independent of those of section 3.
First, we define a scalar product in $\mathcal{P}$, the algebra of superpolynomials. Given

$$\Delta(x) = \prod_{1 \leq j < k \leq N} \frac{x_j - x_k}{x_j x_k}, \quad (4.2)$$

$\langle \cdot | \cdot \rangle_{\beta,N}$ is defined (for $\beta$ a positive integer) on the basis elements of $\mathcal{P}$ as

$$\langle \theta_I x^\lambda | \theta_J x^{\mu} \rangle_{\beta,N} = \begin{cases} \text{C.T.} \left[ \Delta^\beta(\bar{x}) \Delta^\beta(\bar{x}) x^\lambda \right] & \text{if } I = J, \\ 0 & \text{otherwise}. \end{cases} \quad (4.3)$$

where $\bar{x}_i = 1/x_i$, and where C.T.$[E]$ stands for the constant term of the expression $E$. (This is another form of the scalar product $(1.31)$. More precisely, the latter is the analytic deformation of the former for all values of $\beta$.) This gives our first characterization of the Jack superpolynomials.

**Proposition 44.** There exists a unique basis $\{J_A\}_A$ of $\mathcal{P}^S_N$ such that

1) $J_A = m_A + \sum_{\sigma < A} c_{\sigma}(\beta)m_\sigma$ (triangularity);

2) $\langle J_A | J_B \rangle_{\beta,N} \propto \delta_{A,B}$ (orthogonality). \quad (4.4)

In order to present the other characterizations, we need to introduce the Dunkl-Cherednik operators (for instance, see [40]):

$$D_j := x_j \partial_{x_j} + \beta \sum_{k < j} O_{jk} + \beta \sum_{k > j} O_{jk} - \beta(j - 1), \quad (4.5)$$

where

$$O_{jk} = \begin{cases} \frac{x_j}{x_j - x_k}(1 - K_{jk}), & k < j, \\ \frac{x_k}{x_k - x_j}(1 - K_{jk}), & k > j. \end{cases} \quad (4.6)$$

(recall that $K_{jk}$ is the operator that exchanges the variables $x_j$ and $x_k$). These operators can be used to define two families of operators that preserve the space $\mathcal{P}^S_{n|m}$:

$$H_r := \sum_{j=1}^N D_j^r \quad \text{and} \quad I_s := \frac{1}{(N-1)!} \sum_{\sigma \in S_N} K_\sigma \left( \theta_1 \partial_{\theta_1} D_1^s \right) K_\sigma^{-1}, \quad (4.7)$$

for $r \in \{1, 2, 3, \ldots, N\}$ and $s \in \{0, 1, 2, \ldots, N-1\}$ (recall this time that $K_\sigma$ is built out of the operators $K_{jk}$ that exchange $x_j \leftrightarrow x_k$ and $\theta_1 \leftrightarrow \theta_k$ simultaneously). These operators are mutually commuting when restricted to $\mathcal{P}^{S_N}$, that is

$$[H_r, H_s] f = [H_r, I_s] f = [I_r, I_s] f = 0 \quad \forall r, s, \quad (4.8)$$

where $f$ represents an arbitrary polynomial in $\mathcal{P}^{S_N}$. Since they are also symmetric with respect to the scalar product $\langle \cdot | \cdot \rangle_\beta$ and have, when considered as a whole, a non-degenerate spectrum, they provide our second characterization of the Jack superpolynomials.

**Proposition 45.** The Jack superpolynomials $\{J_A\}_A$ are the unique common eigenfunctions of the $2N$ operators $H_r$ and $I_s$, for $r \in \{1, 2, 3, \ldots, N\}$ and $s \in \{0, 1, 2, \ldots, N-1\}$.

We shall now define two operators that play a special role in our study.

$$H := H_2 + \beta(N-1)H_1 - \text{cst} \quad \text{and} \quad I := I_1, \quad (4.9)$$

where $\text{cst} = \beta N(1 - 3N - 2N^2)/6$. When acting on symmetric polynomials in superspace, the explicit form of $H$ is simply

$$H = \sum_i (x_i \partial_{x_i})^2 + \beta \sum_{i < j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_{x_i} - x_j \partial_{x_j}) - 2\beta \sum_{i < j} \frac{x_i x_j}{(x_i - x_j)^2} (1 - \kappa_{ij}). \quad (4.10)$$
The operator $H$ is the Hamiltonian of the stCMS model (see Section 1.4.2); it can be written in terms of two fermionic operators $Q$ and $Q^\dagger$ as

$$H = \{Q, Q^\dagger\},$$

where

$$Q := \sum_i \theta_i x_i \partial x_i \quad \text{and} \quad Q^\dagger := \sum_i \partial \theta_i \left( x_i \partial x_i + \beta \sum_{j \neq i} \frac{x_i + x_j}{x_i - x_j} \right),$$

so that $Q^2 = (Q^\dagger)^2 = 0$. Physically, $Q$ is seen as creating fermions while $Q^\dagger$ annihilates them. A state (superfunction) which is annihilated by the fermionic operators is called supersymmetric. In the case of superpolynomials, the only supersymmetric state is the identity.

**Remark 46.** The stCMS Hamiltonian $H$ has an elegant differential geometric interpretation as a Laplace-Beltrami operator. To understand this assertion, consider first the real Euclidean space $T^N$, where $T = [0, 2\pi]$. Then, set $x_j = e^{it_j}$ for $t_j \in T$, and identify the Grassmannian variable $\theta_i$ with the differential form $dt_i$. This allows us to rewrite the physical scalar product $\langle A(t, \theta) | B(t, \theta) \rangle_{\beta, N}$ as a Hodge-de Rham product involving complex differential forms, that is,

$$\langle A(t, \theta) | B(t, \theta) \rangle_{\beta, N} \sim \int_{T^N} A(t, dt) \wedge *B(t, dt),$$

where the bar denotes de complex conjugation and where the Hodge duality operator $*$ is formally defined by

$$A(t, dt) \wedge *B(t, dt) = C_{\beta, N} \prod_{i < j} \sin^{2\beta} \left( \frac{t_i - t_j}{2} \right) \sum_k \sum_{i_1 < \cdots < i_k} A_{i_1, \ldots, i_k} B_{i_1, \ldots, i_k} dt_1 \wedge \cdots \wedge dt_N,$$

for some constant $C_{\beta, N}$. Note that, in the last equation, the forms $A$ and $B$ are developed in a way similar to that of Eq. (4.12). Hence, we find that the fermionic operators $Q$ and $Q^\dagger$ can be respectively interpreted as the exterior derivative and its dual: $Q \sim -id$ and $Q^\dagger \sim id^*$. Thus

$$H = \Delta := d^* d.$$

In consequence, the Jack superpolynomials can be viewed as symmetric, homogeneous, and orthogonal eigenforms of a Laplace-Beltrami operator. This illustrates the known connection between supersymmetric quantum mechanics and differential geometry.

If the triangularity of the Jack superpolynomial $J_A$ with respect to the supermonomial basis is imposed, requiring that it be a common eigenfunction of $H$ and $I$ is sufficient to define it. This is our third characterization of the Jack superpolynomials.

**Theorem 47.** The Jack superpolynomials $\{J_A\}_A$ form the unique basis of $\mathcal{B}^{SN}(\beta)$ such that

$$H(\beta) J_A = \epsilon_A(\beta) J_A, \quad I(\beta) J_A = \epsilon_A(\beta) J_A \quad \text{and} \quad J_A = m_A + \sum_{\Omega < A} c_{\Omega A}(\beta) m_\Omega.$$ 

The eigenvalues are given explicitly by

$$\epsilon_A(\beta) = \sum_{j=1}^N \left[ (\Lambda_j^*)^2 + \beta (N + 1 - 2j) \Lambda_j^* \right],$$

$$\epsilon_A(\beta) = \sum_{i=1}^m \left[ \Lambda_i - \beta m (m - 1) - \beta \#_A \right],$$

where $\#_A$ denotes the number of pairs $(i, j)$ such that $\Lambda_i < \Lambda_j$ for $1 \leq i \leq m$ and $m + 1 \leq j \leq N$. 


When no Grassmannian variables are involved, that is when $\Lambda = 0$, our characterizations of the Jack superpolynomials specialize to known characterizations of the Jack polynomials that can be found for instance in [22]. There is however in the Jack polynomials’ case a more common characterization in which the scalar product appearing in Proposition 44 is replaced by the strictly combinatorial scalar product (3.73). As already announced, this more combinatorial characterization can be extended to the supersymmetric case. But before turning to the analysis of the behavior of $J^\Lambda$ with respect to the combinatorial scalar product, we present two lemmas concerning properties of the eigenvalues $\varepsilon^\Lambda(\beta)$ and $\varepsilon^{\Lambda}(\beta)$.

**Lemma 48.** Let $\Lambda \in \text{SPar}(n|m)$ and write $\lambda = \Lambda^*$. Let also $\varepsilon^\Lambda(\beta)$ and $\varepsilon^{\Lambda}(\beta)$ be the eigenvalues given in Theorem 47. Then

$$
\varepsilon^\Lambda(\beta) = 2 \sum_j j(\lambda'_j - \beta \lambda_j) + \beta n(N + 1) - n,
$$

$$
\varepsilon^{\Lambda}(\beta) = |\Lambda^a| - \beta |\Lambda'^a| - \beta \frac{m(m-1)}{2}.
$$

(4.18)

**Proof.** Using the well known identity ([21], Eq. 1.6),

$$
\sum_j (j - 1)\lambda_j = \sum_j \left( \frac{\lambda'_j}{2} \right),
$$

(4.19)

we obtain

$$
\sum_j \lambda_j^2 = \sum_j \lambda_j(\lambda_j - 1) + \sum_j \lambda_j = 2 \sum_j \left( \frac{\lambda_j}{2} \right) + n = 2 \sum_j \lambda_j^2 - n.
$$

(4.20)

Hence, we have

$$
\sum_j [\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j] = 2 \sum_j j(\lambda'_j - \beta \lambda_j) + \beta n(N + 1) - n,
$$

(4.21)

as desired. As for the second formula, we consider

$$
\#_\Lambda = \sum_{i=1}^m \#_{\Lambda_i},
$$

(4.22)

where $\#_{\Lambda_i}$ denotes the number of parts in $\Lambda^*$ bigger than $\Lambda_i$. But from the definition of the conjugation, we easily find that

$$
\#_{\Lambda_i} = \Lambda'_{m+1-i} + 1 - i,
$$

(4.23)

so that

$$
\#_\Lambda = \sum_{i=1}^m (\Lambda'_i + 1 - i) = |\Lambda'^a| + \frac{m(m-1)}{2},
$$

(4.24)

from which the second formula follows. $\square$

The following lemma is of interest by itself but it will also be used to establish the orthogonality of the Jack superpolynomials in the case where the superpartitions can be compared (cf. the discussion following Proposition 50).

**Lemma 49.** Let $\Lambda$ and $\Omega$ be two superpartitions of $(n|m)$. Then

$$
\Lambda >_S \Omega \implies \varepsilon^\Lambda \neq \varepsilon^\Omega \quad \text{and} \quad \Lambda >_T \Omega \implies \varepsilon^\Lambda \neq \varepsilon^\Omega.
$$

(4.25)

**Proof.** First, let $\Lambda >_S \Omega$, with $\omega = \Omega^*$ and $\lambda = \Lambda^*$. Then, suppose that $\omega = S_{ij} \lambda$ for some $i < j$. By conjugation, this implies that $\lambda' = S_{i'j'} \omega'$ for some $i' < j'$. More explicitly, we have

$$
\omega_i = \lambda_i - 1 \geq \omega_j = \lambda_j + 1 \quad \text{and} \quad \lambda'_{i'} = \omega'_{i'} - 1 \geq \lambda'_{j'} = \omega'_{j'} + 1.
$$

(4.26)
Since
\[ \sum_k k(\omega_k - \lambda_k) = i(\lambda_i - 1) + j(\lambda_j + 1) - i\lambda_i - j\lambda_j = j - i, \] (4.27)
it follows from the previous lemma that
\[ \varepsilon_{\Lambda} - \varepsilon_{\Omega} = 2(j' - i') + 2\beta(j - i). \] (4.28)
Since by supposition, we have \( i < j \) and \( i' < j' \), the difference is a first order polynomial in \( \beta \) with positive coefficients. From Lemma 4, when \( \Lambda > S \Omega \), we know that \( \omega \) can be obtained by successive applications of such \( S_{ij} \)'s on \( \lambda \). Therefore, when \( \Lambda > S \Omega \), we have that \( \varepsilon_{\Lambda} - \varepsilon_{\Omega} \) is in general a first order polynomial in \( \beta \) with positive coefficients, and thus \( \varepsilon_{\Lambda} \neq \varepsilon_{\Omega} \).

For the second case, let \( \Lambda > T \Omega \) be such that \( \Omega = T_{ij} \Lambda \) for some \( i \in \{1, \ldots, m\} \) and \( j \in \{m + 1, \ldots, N\} \). By conjugation, this implies that \( \Lambda' = T_{i'j'} \Omega' \) for some \( i' \in \{1, \ldots, m\} \) and \( j' \in \{m + 1, \ldots, N\} \), and we thus have
\[ \Omega_i = \Lambda_j < \Omega_j = \Lambda_i \quad \text{and} \quad \Lambda'_{i'} = \Omega'_{j'} < \Lambda'_{j'} = \Omega'_{i'}, \] (4.29)
which imply
\[ |\Omega|^2 < |\Lambda|^2 \quad \text{and} \quad |\Lambda'|^2 < |\Omega'|^2. \] (4.30)
Therefore, from the preceding lemma, \( \varepsilon_{\Lambda} - \varepsilon_{\Omega} \) is a first order polynomial in \( \beta \) with positive coefficients. But again, when \( \Lambda > T \Omega \), Lemma 4 assures us that \( \Omega \) can be obtained by successive applications of such \( T_{ij} \)'s on \( \Lambda \). This means that \( \varepsilon_{\Lambda} - \varepsilon_{\Omega} \) is in this case a first order polynomial in \( \beta \) with positive coefficients and we have \( \varepsilon_{\Lambda} \neq \varepsilon_{\Omega} \) as claimed. \( \Box \)

4.2. Orthogonality of the Jack superpolynomials. In terms of the combinatorial scalar product \( \langle \cdot | \cdot \rangle_{\beta} \), we can directly check the self-adjointness of our eigenvalue-problem defining operators, \( \mathcal{H} \) and \( \mathcal{I} \).

**Proposition 50.** The operators \( \mathcal{H} \) and \( \mathcal{I} \) defined in (4.3) are self-adjoint (symmetric) with respect to the combinatorial scalar product \( \langle \cdot | \cdot \rangle_{\beta} \) defined in (3.73).

**Proof.** We first rewrite \( \mathcal{H} \) and \( \mathcal{I} \) in terms of power sums. Since these differential operators are both of order two, it is sufficient to determine their action on the products of the form \( p_m p_n, \tilde{p}_m p_n \) and \( \tilde{p}_m \tilde{p}_n \). Direct computations give
\[
\mathcal{H} = \sum_{n \geq 1} \left[ n^2 + \beta n(N - n) \right] (p_n \partial_{p_n} + \tilde{p}_n \partial_{\tilde{p}_n}) + \beta \sum_{n,m \geq 1} \left[ (m + n)p_m p_n \partial_{p_{m+n}} + 2mp_m \tilde{p}_n \partial_{\tilde{p}_{m+n}} \right] \\
+ \sum_{n,m \geq 1} mn[p_{m+n} \partial_{p_n} \partial_{p_m} + 2\tilde{p}_{n+m} \partial_{\tilde{p}_n} \partial_{\tilde{p}_m}] \] (4.31)
and
\[
\mathcal{I} = \sum_{n \geq 0} \left[ (1 - \beta)(n\tilde{p}_n \partial_{\tilde{p}_n}) + \frac{\beta}{2} \sum_{m,n \geq 0} \tilde{p}_m \tilde{p}_n \partial_{\tilde{p}_m} \partial_{\tilde{p}_n} + \sum_{m \geq 0, n \geq 1} \left[ n \tilde{p}_{m+n} \partial_{\tilde{p}_n} \partial_{\tilde{p}_m} + \beta p_m \tilde{p}_n \partial_{\tilde{p}_{m+n}} \right] \right] \] (4.32)

Note that these equations are valid when \( N \) is either infinite or finite. In the latter case, the sums over the terms containing \( \tilde{p}_m \) and \( p_n \) are respectively restricted such that \( m \leq N - 1 \) and \( n \leq N \).

Then, letting \( A^\dagger \) denote the adjoint of a generic operator \( A \) with respect to the combinatorial scalar product (3.73), it is easy to check that
\[ \beta \tilde{p}_n^\dagger = n \partial_{\tilde{p}_n} \quad \text{and} \quad \beta p_n^\dagger = \partial_{p_n}. \] (4.33)
Hence, comparing the three previous equations, we obtain that \( \mathcal{H}^\dagger = \mathcal{H} \) and \( \mathcal{I}^\dagger = \mathcal{I} \). For these calculations, we recall that \( (ab)^\dagger = b^\dagger a^\dagger \) even when \( a \) and \( b \) are both fermionic. \( \Box \)
The eigenvalue problem solved in Theorem 47, together with Proposition 50 and Lemma 49 readily imply the orthogonality of the Jack superpolynomials with respect to the combinatorial scalar product in the special case where the two superpartitions can be compared with respect to the Bruhat order on compositions.

In order to extend this conclusion to all superpartitions, comparable or not, the most natural path consists in establishing the self-adjointness of all the operators $H_n$ and $I_n$. But proceeding as for $H$ and $I$ above, by trying to reexpress them in terms of $p_n$, $\tilde{p}_n$ and their derivatives, seems hopeless. An indirect line of attack is mandatory.

Let us first recall that the conserved operators (4.7) can all be expressed in terms of the Dunkl-Cherednik operators defined in (4.5). The $D_i$ all commute among themselves:

\[ [D_i, D_j] = 0. \tag{4.34} \]

They are not quite invariant however, as

\[ D_i K_{i,i+1} - K_{i,i+1} D_{i+1} = \beta. \tag{4.35} \]

We will also need the following commutation relations:

\[ [D_i, x_i] = x_i + \beta \left( \sum_{j<i} x_i K_{ij} + \sum_{j>i} x_j K_{ij} \right), \tag{4.36} \]

while if $i \neq k$,

\[ [D_i, x_k] = -\beta x_{\text{max}(i, k)} K_{ik}. \tag{4.37} \]

The idea of the proof for the orthogonality is the following: in a first step, we show that the conserved operators $H_n$ and $I_n$ are self-adjoint with respect to the combinatorial scalar product and then we demonstrate that this implies the orthogonality of the $J_{\Lambda}'s$. The self-adjointness property is established via the kernel: showing that $F = F^\perp$ is the same as showing that

\[ F(x) K^{\beta, N} = F(y) K^{\beta, N}, \tag{4.38} \]

where $K^{\beta, N}$ is defined in Theorem 36 and where $F^{(x)}$ (resp. $F^{(y)}$) stands for the quantity $F$ in the variable $x$ (resp. $y$). In order to prove this for our conserved operators $H_n$ and $I_n$, we need to establish some results on the action of symmetric monomials in the Dunkl-Cherednik operators acting on the following expression:

\[ \tilde{\Omega} = \prod_{i=1}^N \frac{1}{1 - x_i y_i} \prod_{i,j=1}^N \frac{1}{1 - x_i y_j} \beta, \tag{4.39} \]

as well as some modification of $\tilde{\Omega}$. For that matter, we recall a result of Sahi [44]:

**Proposition 51.** The action of the Dunkl-Cherednik operators $D_j$ on $\tilde{\Omega}$ defined by (4.39) satisfies:

\[ D_j^{(x)} \tilde{\Omega} = D_j^{(y)} \tilde{\Omega}. \tag{4.40} \]

Before turning to the core of our argument, we establish the following lemma.

**Lemma 52.** Given a set $J = \{j_1, \ldots, j_k\}$, denote by $x_J$ the product $x_{j_1} \ldots x_{j_k}$. Suppose $x_J = K_\sigma x_I$ for some $\sigma \in S_N$ such that $K_\sigma F K_{\sigma^{-1}} = F$. Then

\[ \frac{1}{x_J} F^{(x)} x_J \tilde{\Omega} = \frac{1}{y_J} F^{(y)} y_J \tilde{\Omega} \quad \Longrightarrow \quad \frac{1}{x_J} F^{(x)} x_J \tilde{\Omega} = \frac{1}{y_J} F^{(y)} y_J \tilde{\Omega}. \tag{4.41} \]

\[ ^{14}\text{This corrects a misprint in eq. (25) of [33].} \]
Proof. The proof is straightforward and only uses the simple property $K^{(x)}_{\sigma} \tilde{\Omega} = K^{(y)}_{\sigma} \tilde{\Omega}$. To be more precise,

\[
\frac{1}{x_J} F^{(x)}_{x_J} \tilde{\Omega} = K^{(x)}_{\sigma} \frac{1}{x_I} F^{(x)}_{x_I} K^{(x)}_{\sigma-1} \tilde{\Omega} = K^{(y)}_{\sigma} K^{(x)}_{\sigma} \frac{1}{y_I} F^{(y)}_{y_I} \tilde{\Omega} = K^{(y)}_{\sigma} y_I F^{(y)}_{y_I} \tilde{\Omega},
\]

(4.42)

We are now ready to attack the main proposition.

**Proposition 53.** The mutually commuting operators $\mathcal{H}_n$ and $\mathcal{I}_n$ satisfy

\[
\mathcal{H}_n^{(x)} K^{\beta,N} = \mathcal{H}_n^{(y)} K^{\beta,N} \quad \text{and} \quad \mathcal{I}_n^{(x,\theta)} K^{\beta,N} = \mathcal{I}_n^{(y,\theta)} K^{\beta,N},
\]

(4.43)

with $K^{\beta,N}$ the restriction to $N$ variables of the kernel $K^\beta$ such as defined in Theorem 36.

Proof. We first expand the kernel as follows:

\[
K^{\beta,N} = K_0 \prod_{i,j} \left( 1 + \beta \frac{\theta_i \phi_j}{(1-x_i y_j)} \right) = K_0 \left\{ 1 + \beta e_1 \left( \frac{\theta_i \phi_j}{(1-x_i y_j)} \right) + \beta^N e_N \left( \frac{\theta_i \phi_j}{(1-x_i y_j)} \right) \right\}
\]

(4.44)

where $K_0$ stands for $K^{\beta,N}(x, y, 0, 0)$, i.e.,

\[
K_0 := \prod_{i,j=1}^N \frac{1}{(1-x_i y_j)^\beta},
\]

(4.45)

and where $e_\ell(u_{i,j})$ is the elementary symmetric function $e_\ell$ in the variables $u_{i,j}$ where

\[
u_{i,j} := \frac{\theta_i \phi_j}{(1-x_i y_j)} \quad i, j = 1, \ldots, N.
\]

(4.46)

(4.47)

Note that, in these variables, the maximal possible elementary symmetric function is $e_N$ given that $\theta_i^2 = \phi_i^2 = 0$. In the following, we will use the compact notation $I^- = \{1, \ldots, i-1\}$ and $I^+ = \{i, \ldots, N\}$ (and similarly for $J^\beta$), together with $w_{I^-} = w_1 \cdots w_{i-1}$ and $w_{I^+} = w_i \cdots w_N$.

The action of the operators on $K^\beta$ can thus be decomposed into their action on each monomial in this expansion. Now observe that $K_0$ is invariant under the exchange of any two variables $x$ or any two variables $y$. Therefore, if an operator $F$ is such that $K_0 F K_0^{-1} = F$ for all $\sigma \in S_N$, and such that

\[
F^{(\sigma,\theta)} v_{I^-} K_0 = F^{(\sigma,\theta)} v_{I^-} K_0 \quad \text{with} \quad v_1 := u_{i,i},
\]

(4.48)

for all $i = 1, \ldots, N + 1$, then we immediately have by symmetry that $F^{(\sigma,\theta)} K^\beta = F^{(\sigma,\theta)} K^\beta$. We will use this observation in the case of $\mathcal{H}_n$ and $\mathcal{I}_n$.

We first consider the case $F = \mathcal{H}_n$. Recall from (4.79) that $\mathcal{H}_n = p_n(D_x)$ is such that $K_0 p_n(K_0^{-1} = \mathcal{H}_n$ (see 33). Since $\mathcal{H}_n$ does not depend on the fermionic variables, we thus have to prove from the previous observation that

\[
\mathcal{H}_n^{(x)} \frac{1}{(1-xy)_{I^-}} K_0 = \mathcal{H}_n^{(y)} \frac{1}{(1-xy)_{I^-}} K_0,
\]

(4.49)

or equivalently

\[
\mathcal{H}_n^{(x)} (1-xy)_{I^+} \tilde{\Omega} = \mathcal{H}_n^{(y)} (1-xy)_{I^+} \tilde{\Omega},
\]

(4.50)

for all $i = 1, \ldots, N + 1$ (the case $i = N + 1$ corresponds to the empty product).

The underlying symmetry allows us to further simplify the problem by focusing on the terms

\[
y_{J^+} \mathcal{H}_n^{(x)} x_{J^+}, \tilde{\Omega} = x_{J^+} \mathcal{H}_n^{(y)} y_{J^+} \tilde{\Omega},
\]

(4.51)
for \( j \geq i \), or equivalently, on
\[
\frac{1}{x_{J+}} h_n^{(x)}(x_{J+}, \tilde{\Omega}) = \frac{1}{y_{J+}} h_n^{(y)}(y_{J+}, \tilde{\Omega}).
\]  
(4.52)

This follows from Lemma [52] which assures us that all the different terms can be obtained from these special ones.

Now, instead of analyzing the family \( \mathcal{H}_n = p_n(D_i) \), it will prove simpler to consider the equivalent family \( e_n(D_i) \). We will first show the case \( e_n(D_i) \), that is,
\[
\frac{1}{x_{J+}} D_1^{(x)} \cdots D_N^{(x)}(x_{J+}, \tilde{\Omega}) = \frac{1}{y_{J+}} D_1^{(y)} \cdots D_N^{(y)}(y_{J+}, \tilde{\Omega}).
\]  
(4.53)

Let us concentrate on the left hand side. We note that
\[
\frac{1}{x_{J+}} D_1^{(x)} \cdots D_N^{(x)}(x_{J+}, \tilde{\Omega}) = \frac{1}{x_{J+}} D_1^{(x)}(x_{J+}) \cdots \frac{1}{x_{J+}} D_N^{(x)}(x_{J+}, \tilde{\Omega}).
\]  
(4.54)

It thus suffices to study each term \( (x_{J+})^{-1} D J_{x_{J+}} \) separately. In each case we find that
\[
D_k x_{J+} = x_{J+} \tilde{D}_k.
\]  
(4.55)

The form of \( \tilde{D} \) depends upon \( j \) and \( k \). There are two cases:
\[
k < j : \quad \tilde{D}_k = D_k - \beta \sum_{\ell = j}^{N} K_{\ell, k},
\]
\[
k \geq j : \quad \tilde{D}_k = D_k + 1 + \beta \sum_{\ell = 1}^{j - 1} K_{\ell, k}
\]  
(4.56)

which can be easily checked using (4.38) and (4.37). We can thus write
\[
\frac{1}{x_{J+}} D_1^{(x)} \cdots D_N^{(x)}(x_{J+}, \tilde{\Omega}) = \tilde{D}_1^{(x)} \cdots \tilde{D}_N^{(x)}(\tilde{\Omega}).
\]  
(4.57)

Using proposition [51] and \( K_{ij}^{(x)} \tilde{\Omega} = K_{ij}^{(y)} \tilde{\Omega} \), the rightmost term \( \tilde{D}_N^{(x)} \) can thus be changed into \( \tilde{D}_N^{(y)} \).

Since it commutes with the previous terms (i.e., it acts on the variables \( y \) while the others act on \( x \)), we have
\[
\tilde{D}_1^{(x)} \cdots \tilde{D}_N^{(x)} \tilde{D}_N^{(y)} \tilde{\Omega} = \tilde{D}_1^{(y)} \tilde{D}_N^{(y)} \tilde{D}_1^{(x)} \cdots \tilde{D}_N^{(x)} \tilde{\Omega} = \tilde{D}_1^{(y)} \tilde{D}_N^{(y)} \tilde{D}_1^{(x)} \cdots \tilde{D}_N^{(x)} \tilde{\Omega} = \frac{1}{y_{J+}} D_1^{(y)}(y_{J+}) \cdots \frac{1}{y_{J+}} D_N^{(y)}(y_{J+}, \tilde{\Omega}) = \frac{1}{y_{J+}} D_1^{(y)}(y_{J+}) \cdots D_N^{(y)}(y_{J+}, \tilde{\Omega}),
\]  
(4.58)

which is the desired result.

At this point, we have only considered a single conserved operator, namely \( e_n(D_i) \). But by replacing \( D_i \) with \( D_i + t \) in \( e_n(D_i) \), we obtain a generating function for all the operators \( e_n(D_i) \). Since to prove \( e_n(D_i + t) K^{(x),N} = e_n(D_i + t) K^{(y),N} \) simply amounts to replacing \( \tilde{D}_i \) by \( \tilde{D}_i + t \) in the previous argument, we have completed the proof of \( \mathcal{H}_n^{(x)} K^{(x),N} = \mathcal{H}_n^{(y)} K^{(y),N} \).

For the case of \( \mathcal{I}_n \), we start with the expression given in (4.7), which readily implies that \( K_n \mathcal{I}_n \mathcal{K}_n^{-1} = \mathcal{I}_n \). Therefore, from the observation surrounding formula (4.48), and because the derivative \( \theta_t \partial \theta_t \) annihilates the \( K_0 \) term in the expansion of \( K^{(x),N} \), we only need to show that
\[
\mathcal{I}_n^{(x,\theta)} v_{I-} K_0 = \mathcal{T}_n^{(y,\phi)} v_{I-} K_0,
\]  
(4.59)

for \( i = 2, \ldots, N + 1 \). Up to an overall multiplicative factor, the only contributing part in \( \mathcal{I}_n \), when acting on \( v_{I-} \), is
\[
\mathcal{O}_n := D_1^a + K_{12} D_1^a K_{12} + \cdots K_{1,i-1} D_1^a K_{1,i-1}.
\]  
(4.60)
It thus suffices to show that
\[ O_n^{(x)} (1 - xy)_{j+} \tilde{\Omega} = O_n^{(y)} (1 - xy)_{j+} \tilde{\Omega}. \] (4.61)

Once more, we can use Lemma \ref{lem:commute} since \( O_n \) commutes with \( K_{k,\ell} \) for \( k, \ell \geq i \). Thus, we only need to check that for \( j \geq i \),
\[ \frac{1}{x_{j+}} O_n^{(x)} x_{j+} \tilde{\Omega} = \frac{1}{y_{j+}} O_n^{(y)} y_{j+} \tilde{\Omega}. \] (4.62)
Since the \( K_{1\ell} \)’s act trivially on the variables \( x_j \) for \( j > \ell \), the previous relation reduces to proving
\[ \frac{1}{x_{j+}} [\mathcal{D}_1^{(x)}(x)] x_{j+} \tilde{\Omega} = \frac{1}{y_{j+}} [\mathcal{D}_1^{(y)}(y)] y_{j+} \tilde{\Omega}. \] (4.63)
The left hand side takes the form
\[ \frac{1}{x_{j+}} [\mathcal{D}_1^{(x)}(x)] x_{j+} \tilde{\Omega} = \left\{ \frac{1}{x_{j+}} \mathcal{D}_1^{(x)}(x) \right\}^n \tilde{\Omega}. \] (4.64)
We then only have to evaluate \( (x_{j+})^{-1} \mathcal{D}_1^{(x)}(x) \). The result is given by the first case in (4.56) (since \( j > 1 \)). The proof is completed as follows
\[ \left\{ \frac{1}{x_{j+}} \mathcal{D}_1^{(x)}(x) \right\}^n \tilde{\Omega} = \left[ \mathcal{D}_1^{(x)}(x) \right]^n \tilde{\Omega} = \left[ \mathcal{D}_1^{(y)}(y) \right]^n \tilde{\Omega} = \frac{1}{y_{j+}} \left[ \mathcal{D}_1^{(y)}(y) \right] y_{j+} \tilde{\Omega}. \] (4.65)
\[ \hfill \square \]

As previously mentioned, the proposition has the following corollary.

**Corollary 54.** The operators \( \mathcal{H}_r \) and \( \mathcal{I}_s \) defined in (4.58) are self-adjoint (symmetric) with respect to the combinatorial scalar product \( \langle \cdot | \cdot \rangle_{\beta,\eta} \) given in (3.37).

This immediately gives our main result.

**Theorem 55.** The Jack superpolynomials \( \{ J_\Lambda \}_\Lambda \) are orthogonal with respect to the combinatorial scalar product, that is,
\[ \langle J_\Lambda | J_\Omega \rangle_{\beta} \propto \delta_{\Lambda,\Omega}. \] (4.66)

**Proof.** The fact that in \( N \) variables \( \langle J_\Lambda | J_\Omega \rangle_{\beta,\eta} \propto \delta_{\Lambda,\Omega} \) follows from the equivalence between the statement of the theorem and Corollary \ref{cor:orthogonality}. This equivalence follows from Proposition \ref{prop:prop}, which says that the Jack polynomials are the unique common eigenfunctions of the \( 2N \) operators appearing in Corollary \ref{cor:orthogonality}. Given that the expansion coefficients of the Jack superpolynomials in terms of supermonomials do not depend on the number of variables \( N \), the theorem then follows from Proposition \ref{prop:prop}.
\[ \hfill \square \]

**Corollary 56.** The following statements are direct consequences of the orthogonality property of the Jack polynomials in superspace.\textsuperscript{15}

1. There exists a basis \( \{ \tilde{J}_\Lambda \}_\Lambda \) of \( \mathcal{S}_\infty \) such that
   \begin{enumerate}
   \item \( \tilde{J}_\Lambda = m_\Lambda + \sum_{\Omega < \Lambda} \tilde{c}_{\Lambda\Omega}(\beta)m_\Lambda \) (triangularity);
   \item \( \langle \tilde{J}_\Lambda | \tilde{J}_\Omega \rangle_{\beta} \propto \delta_{\Lambda,\Omega} \) (orthogonality).
   \end{enumerate}
   (4.67)

2. Let \( K^\beta \) be the reproducing kernel defined in Proposition \ref{prop:prop}. Then,
   \[ K^\beta(x, \theta; y, \phi) = \sum_{\Lambda \in \mathcal{S}_\text{Par}} j_{\Lambda}^\beta(x, \theta)^{-1} \tilde{J}_\Lambda(x, \theta)^{-1} J_\Lambda(y, \phi) \] (4.68)
   where
   \[ j_{\Lambda}^\beta := \langle \tilde{J}_\Lambda | J_\Lambda \rangle_{\beta}. \] (4.69)

\textsuperscript{15}In fact, it can be shown that all statements of Corollary \ref{cor:orthogonality} and Theorem \ref{thm:orthogonality} below are not only consequences of Proposition \ref{prop:prop} but are equivalent to it.
From these definitions, we get 50 as

\[ J_\Lambda = \sum_{\Omega \geq \Lambda} u_{\Lambda \Omega}(\beta) g_{\Omega}, \quad \text{with} \quad u_{\Lambda \Lambda}(\beta) \neq 0. \]  (4.70)

**Proof.** It was shown in [33] that the operators \( \mathcal{H} \) and \( \mathcal{I} \) act triangularly on the supermonomial basis. Thus, \( \mathcal{H} \) and \( \mathcal{I} \) also act triangularly on the basis \( \{ J_\Lambda \}_\Lambda \). Furthermore, from Proposition 50, they are self-adjoint with respect to the combinatorial scalar product. Hence, we must conclude from the previous argument that \( \hat{J}_\Lambda \) is an eigenfunction of \( \mathcal{H} \) and \( \mathcal{I} \), from which Theorem 47 implies that \( \hat{J}_\Lambda = J_\Lambda \).

2. The proof is similar to that of Lemma 33 (see also Section VI.2 of [21]).

3. Suppose that \( \langle J_\Lambda | J_\Gamma \rangle_\beta \propto \delta_{\Lambda, \Omega} \), and let \( J_\Lambda = \sum_{\Omega \in \mathcal{S}} u_{\Lambda \Omega} g_{\Omega} \), where \( \mathcal{S} \) is some undefined set. If \( \Lambda \) is not the smallest element of \( \mathcal{S} \), then there exists at least one element \( \Gamma \) of \( \mathcal{S} \) that does not dominate any of its elements. In this case, we have

\[ \langle J_\Lambda | J_\Gamma \rangle_\beta = \sum_{\Omega \in \mathcal{S}} u_{\Lambda \Omega}(\beta) \sum_{\Delta \leq \Gamma} c_{\Gamma \Delta}(\beta) \langle g_{\Omega} | m_\Delta \rangle_\beta. \]  (4.71)

Since \( \Gamma \) does not dominate any element of \( \mathcal{S} \), the unique non-zero contribution in this expression is that of \( u_{\Lambda \Gamma}(\beta) c_{\Gamma \Gamma}(\beta) \langle g_{\Omega} | m_\Gamma \rangle_\beta = u_{\Lambda \Lambda}(\beta) \). Since this term is non-zero by supposition, we have the contradiction \( 0 = \langle J_\Lambda | J_\Gamma \rangle_\beta = u_{\Lambda \Lambda}(\beta) \neq 0 \).

\[ \square \]

4.3. **Duality.** In this subsection, we show that the homomorphism \( \tilde{\omega}_\beta \), defined in Eq. (4.74), has a simple action on Jack superpolynomials. To avoid any confusion, we make explicit the \( \beta \) dependence of the Jack superpolynomials by writing \( J_\Lambda^{(1/\beta)} \).\[16\]

**Proposition 57.** We have

\[ \mathcal{H}(\beta) \tilde{\omega}_\beta J_\Lambda^{(1/\beta)} = \epsilon_{\Lambda'}(\beta) \tilde{\omega}_\beta J_\Lambda^{(1/\beta)} \quad \text{and} \quad \mathcal{I}(\beta) \tilde{\omega}_\beta J_\Lambda^{(1/\beta)} = \epsilon_{\Lambda'}(\beta) \tilde{\omega}_\beta J_\Lambda^{(1/\beta)}. \]  (4.72)

**Proof.** Let us rewrite the special form of the operator \( \mathcal{H}(\beta) \) appearing in the proof of Proposition 50 as

\[ \mathcal{H}(\beta) = \sum_{n \geq 1} [n^2 + \beta n(N - n)] \hat{A}_n + \sum_{m, n \geq 1} (\beta \hat{B}_{m,n} + \hat{C}_{m,n}), \]  (4.73)

with

\[ \hat{A}_n = \frac{1}{\beta} \partial_{p_n} \partial_{\bar{p}_n}, \quad \hat{B}_{m,n} = (m + n) p_m p_n \partial_{p_{m+n}} + 2m p_m \partial_{\bar{p}_{m+n}}, \quad \hat{C}_{m,n} = mn (p_{m+n} \partial_{p_m} + 2\bar{p}_{m+n} \partial_{\bar{p}_n}). \]  (4.74)

From these definitions, we get

\[ \hat{\omega}_{1/\beta} \hat{A}_n = \hat{A}_n \hat{\omega}_{1/\beta}, \quad \hat{\omega}_{1/\beta} \hat{B}_{m,n} = -\frac{1}{\beta} \hat{B}_{m,n} \hat{\omega}_{1/\beta} \quad \text{and} \quad \hat{\omega}_{1/\beta} \hat{C}_{m,n} = -\beta \hat{C}_{m,n} \hat{\omega}_{1/\beta}. \]  (4.75)

These relations imply

\[ \hat{\omega}_{1/\beta} \mathcal{H}(\beta) \tilde{\omega}_\beta = \sum_{n \geq 1} [n^2 + \beta n(N - n)] \hat{A}_n - \sum_{m, n \geq 1} (\hat{B}_{m,n} + \beta \hat{C}_{m,n}) = (1 + \beta) N \sum_{n \geq 1} n \hat{A}_n - \beta \mathcal{H}(1/\beta). \]

\[ ^{16}\text{The rationale for this notation is to match the one used in [21] when } m = 0: J_\Lambda^{(1/\beta)}(x, \theta) = J_\Lambda^{(\alpha)}(x) = J_{\Lambda^*}^{(\alpha)}(x), \text{ where } \alpha = 1/\beta. \text{ (Similarly, in our previous works [31, 34], we denoted } J_\Lambda^{(1/\beta)} \text{ by } J_\Lambda(x; \theta; 1/\beta) \text{ to keep our definition similar to the usual form introduced by Stanley [22] as } J_\Lambda(x; \alpha) \text{ when } m = 0. \text{ We stress however, that when we need to make explicit the } \beta\text{-dependence of } j_\Lambda, \mathcal{H} \text{ and } \mathcal{I}, \text{ we write } j_\Lambda(\beta), \mathcal{H}(\beta)\text{and } \mathcal{I}(\beta) \text{ respectively.} \]
Now, considering \( \sum_{n \geq 1} n \hat{A}_n m_\Lambda = |\Lambda| m_\Lambda \) and Lemma 48 we obtain
\[
\hat{\omega}_{1/\beta} \mathcal{H}(\beta) \hat{\omega}_{\beta} J^{(\beta)}_\Lambda = \varepsilon_{\Lambda'}(\beta) J^{(\beta)}_\Lambda
\]
(4.76) as claimed. The relation involving \( \mathcal{I}(\beta) \) is proved in a similar way.

**Theorem 58.** The homomorphism \( \hat{\omega}_\beta \) is such that
\[
\hat{\omega}_{1/\beta} J^{(1/\beta)}_\Lambda \xrightarrow{\beta} j_\Lambda(\beta) \xleftarrow{\beta} J^{(\beta)}_{\Lambda'},
\]
(4.77) with \( j_\Lambda(\beta) \) such as defined in (4.69).

**Proof.** Let us first prove that \( \hat{\omega}_\beta J^{(1/\beta)}_\Lambda \propto J^{(1/\beta)}_\Lambda \). From the third point of Corollary 50 we know that
\[
J^{(1/\beta)}_\Lambda = \sum_{\Omega \geq \Lambda} u_{\Omega\Lambda}(\beta) g_\Omega.
\]
But Eq. 3.93 implies \( \hat{\omega}_{1/\beta} g_\Lambda = e_\Lambda \). Hence,
\[
\hat{\omega}_{1/\beta} J^{(1/\beta)}_\Lambda = \sum_{\Omega \geq \Lambda} u_{\Omega\Lambda}(\beta) e_\Omega = \sum_{\Omega \geq \Lambda} u_{\Omega\Lambda}(\beta) \sum_{\Gamma \leq \Omega} N^\Gamma_{\Omega} \frac{1}{m_\Gamma} = \sum_{\Gamma \leq \Lambda'} v_{\Lambda\Gamma}(\beta) \frac{1}{m_\Gamma},
\]
(4.78) where we have used (3.93), Theorem 18 and the fact that \( \Omega \geq \Lambda \iff \Omega' \leq \Lambda' \). Further, since \( N_{\Lambda'} = 1 \) and \( u_{\Lambda\Lambda}(\beta) \neq 0 \), we have \( v_{\Lambda\Lambda'} \neq 0 \). Now, from Proposition 51 \( \hat{\omega}_{1/\beta} (J^{(1/\beta)}_\Lambda) \) is an eigenfunction of \( \mathcal{H}(1/\beta) \) and \( \mathcal{I}(1/\beta) \) with eigenvalues \( \varepsilon_{\Lambda'}(1/\beta) \) and \( \varepsilon_{\Lambda}(1/\beta) \) respectively. The triangularity we just obtained ensures from Theorem 47 that \( \hat{\omega}_{1/\beta} (J^{(1/\beta)}_\Lambda) \) is proportional to \( J^{(\beta)}_{\Lambda'} \).

Again from Theorem 18 we know that \( m_\Lambda = (-1)^{m(m-1)/2} e_{\Lambda'} + \text{higher terms} \), so that
\[
J^{(1/\beta)}_\Lambda = (-1)^{m(m-1)/2} e_{\Lambda'} + \text{higher terms}.
\]
(4.79) Moreover, from Eq. 3.93, we get
\[
\hat{\omega}_\beta J^{(1/\beta)}_\Lambda = (-1)^{m(m-1)/2} e_{\Lambda'} + \text{higher terms}.
\]
(4.80) But the proportionality proved above implies
\[
\hat{\omega}_{1/\beta} J^{(1/\beta)}_\Lambda = A_\Lambda(\beta) \xleftarrow{\beta} J^{(\beta)}_{\Lambda'} = A_\Lambda(\beta) \xleftarrow{\beta} m_{\Lambda'} + \text{lower terms},
\]
(4.81) for some constant \( A_\Lambda(\beta) \). Finally, considering the duality between \( g_\Lambda \) and \( m_\Lambda \), we obtain
\[
(-1)^{m(m-1)/2} J_\Lambda(\beta) = \langle J^{(1/\beta)}_\Lambda | J^{(1/\beta)}_\Lambda \rangle_\beta = \langle \hat{\omega}_\beta J^{(1/\beta)}_\Lambda | \hat{\omega}_1 J^{(1/\beta)}_\Lambda \rangle_\beta = \langle (-1)^{m(m-1)/2} g_{\Lambda'} | A_\Lambda(\beta) m_{\Lambda'} \rangle_\beta = (-1)^{m(m-1)/2} A_\Lambda(\beta)
\]
(4.82) as desired.

4.4. Limiting cases. In Section 4.2, we have proved that the physical Jack superpolynomials are orthogonal with respect to the combinatorial scalar product. This provides a direct link with the material of Section 3. Other links, less general but more explicit, are presented in this section, from the consideration of \( J_\Lambda \) for special values of \( \beta \) or for particular superpartitions.

**Proposition 59.** For \( \Lambda = (n) \) or \( (n; 0) \), we have (using the notation of Proposition 72):
\[
J^{(n)} = \frac{n!}{(\beta + n - 1)n} g_n \quad \text{and} \quad J^{(n; 0)} = \frac{n!}{(\beta + n)n+1} \tilde{g}_n.
\]
(4.83)
Then, it is easy to get the limiting expressions of $J_{(n;0)}$ in $J_{(n;0)}$ needs to be equal to one, we obtain $(\beta + n)_{n+1} J_{(n;0)} = n! \tilde{g}_n$. The relation between $J_{(n)}$ and $g_n$ is proved in a similar way. □

**Corollary 60.** For $\Lambda = (n)$ or $(n;0)$, the combinatorial norm of $J_{\Lambda}$ is

\[
\langle\langle J_{(n)} | J_{(n)} \rangle \rangle_\beta = \frac{n!}{(\beta + n - 1)_n} \quad \text{and} \quad \langle\langle J_{(n;0)} | J_{(n;0)} \rangle \rangle_\beta = \frac{n!}{(\beta + n)_{n+1}}.
\]

**Proof.** Using the previous proposition, we get

\[
(n!)^2 \langle\langle g_n | g_n \rangle \rangle_\beta = (\beta + n - 1)_n^2 \langle\langle J_{(n)} | J_{(n)} \rangle \rangle_\beta,
\]

and

\[
(n!)^2 \langle\langle \tilde{g}_n | \tilde{g}_n \rangle \rangle_\beta = (\beta + n)_{n+1} \langle\langle J_{(n;0)} | J_{(n;0)} \rangle \rangle_\beta.
\]

From Proposition 52, we know that

\[
n! g_n = (\beta + n - 1)_n m_{(n)} + \ldots, \quad n! \tilde{g}_n = (\beta + n)_{n+1} m_{(n;0)} + \ldots,
\]

where the dots stand for lower terms in the Bruhat ordering. Thus, considering Corollary 41, we get

\[
\langle\langle g_n | g_n \rangle \rangle_\beta = \frac{(\beta + n - 1)_n}{n!}, \quad \langle\langle \tilde{g}_n | \tilde{g}_n \rangle \rangle_\beta = \frac{(\beta + n)_{n+1}}{n!}
\]

and the proof follows. □

**Theorem 61.** For $\beta = 0, 1$, or $\beta \to \infty$, the limiting expressions of $J_{\Lambda}^{(1/\beta)}$ are

\[
J_{\Lambda}^{(1/\beta)} \to \begin{cases} \frac{m_{\Lambda}}{e_{\Lambda'}} & \text{when } \beta \to 0, \\ \frac{e_{\Lambda'}}{m_{\Lambda}} & \text{when } \beta \to \infty. \end{cases}
\]

and

\[
J_{(1)}^{(1)} = h_n \quad \text{and} \quad J_{(1)}^{(0)} = \frac{1}{n+1} \tilde{h}_n.
\]

**Proof.** The case $\beta \to 0$ is a direct consequence of Theorem 47, given that $\mathcal{H}(\beta)$ and $\mathcal{I}(\beta)$ act diagonally on supermonomials in this limit. The second case is also obtained from the eigenvalue problem. Indeed, when $\beta \to \infty$, $\beta^{-1} \mathcal{H}(\beta)$ and $\beta^{-1} \mathcal{I}(\beta)$ behave as first order differential operators. Then, it is easy to get

\[
\lim_{\beta \to \infty} \frac{\mathcal{H}(\beta)}{\beta} e_{\Lambda'} = \left[ -2 \sum_j j \lambda_j + n(N-1) \right] e_{\Lambda'} \quad \text{where } \lambda = \Lambda^*
\]

and

\[
\lim_{\beta \to \infty} \frac{\mathcal{I}(\beta)}{\beta} e_{\Lambda'} = \left[ -|\Lambda'| - m(m-1) \right] e_{\Lambda'}.
\]

These are the eigenvalues of $J_{\Lambda}$ in the limit where $\beta \to \infty$ (cf. Lemma 45). The proportionality constant between $e_{\Lambda'}$ and $J_{\Lambda}$ is fixed by Theorem 48 and Theorem 47. We have thus

\[
\frac{e_{\Lambda'}}{e_{\Lambda'}} = \lim_{\beta \to \infty} \frac{J_{\Lambda}^{(1/\beta)}}{J_{\Lambda}^{(1/\beta)}}.
\]

Finally, we note that the property concerning $h_n$ and $\tilde{h}_n$ is an immediate corollary of Proposition 52. □
4.5. **Normalization.** In this subsection, $\tilde{m}_\Lambda$ shall denote the augmented supermonomial:

$$\tilde{m}_\Lambda = n_\Lambda! m_\Lambda,$$  \hspace{1cm} (4.93)

where $n_\Lambda!$ is such as defined in (2.53).

It is easy to see that the smallest superpartition of degree $(n|m)$ in the Bruhat ordering is

$$\Lambda_{\text{min}} := (\delta_m ; 1^{\ell_n,m}),$$ \hspace{1cm} (4.94)

where

$$\ell_n,m := n - |\delta_m|, \quad \delta_m := (m - 1, m - 2, \ldots, 0) \text{ and } |\delta_m| = \frac{m(m - 1)}{2}. \hspace{1cm} (4.95)$$

Now, let $c_{\Lambda_{\text{min}}}^\min(\beta)$ stand for the coefficient of $\tilde{m}_{\Lambda_{\text{min}}}$ in the monomial expansion of $J_\Lambda^{(1/\beta)}$. We shall establish a relation between this coefficient and the norm of the Jack superpolynomials $J_\Lambda$.

**Proposition 62.** The norm $j_\Lambda(\beta)$ defined in (4.67), with $\Lambda \vdash (n|m)$, is

$$j_\Lambda(\beta) = \beta^{-m-\ell_n,m} c_{\Lambda_{\text{min}}}^\min(\beta) \frac{c_{\Lambda'}^\min(1/\beta)}{c_{\Lambda'}^\min(1/\beta)} \hspace{1cm} (4.96)$$

**Proof.** One readily shows that

$$m_{\Lambda_{\text{min}}} = p_{\Lambda_{\text{min}}} + \text{higher terms}. \hspace{1cm} (4.97)$$

Since $m_{\Lambda_{\text{min}}}$ is the only supermonomial containing $p_{\Lambda_{\text{min}}}$, we can write

$$j_\Lambda^{(1/\beta)} = c_{\Lambda_{\text{min}}}^\min(\beta) p_{\Lambda_{\text{min}}} + \text{higher terms}. \hspace{1cm} (4.98)$$

Let us now apply $\hat{\omega}_{1/\beta}$ on this expression. Using Eq. (3.46) we get

$$\hat{\omega}_{1/\beta} j_\Lambda^{(1/\beta)} = \beta^{-m-\ell_n,m} (-1)^{m(m-1)/2} c_{\Lambda_{\text{min}}}^\min(\beta) p_{\Lambda_{\text{min}}} + \text{higher terms}. \hspace{1cm} (4.99)$$

But if we apply $\hat{\omega}_{1/\beta}$ on $J_\Lambda^{(1/\beta)}$ by using first Theorem 58 to write it as $(-1)^{m(m-1)/2} j_\Lambda(\beta) J_{\Lambda'}^{(\beta)}$ and expand $J_{\Lambda'}^{(\beta)}$ using (4.98), we get instead

$$\hat{\omega}_{1/\beta} J_\Lambda^{(1/\beta)} = j_\Lambda(\beta)(-1)^{m(m-1)/2} c_{\Lambda'}^\min(1/\beta) p_{\Lambda_{\text{min}}} + \text{higher terms}. \hspace{1cm} (4.100)$$

Here we have used the fact that $\Lambda_{\text{min}}$, being the smallest superpartition of degree $(n|m)$ in the Bruhat ordering, labels the smallest supermonomial in both the decomposition of $J_\Lambda$ and $J_{\Lambda'}$. The result follows from the comparison of the last two equations. \hfill $\Box$

The coefficient $c_{\Lambda_{\text{min}}}^\min(\beta)$ appears from computer experimentation to have a very simple form. We shall now introduce the notation needed to describe it. Recall that $D[\Lambda]$ is the diagram used to represent $\Lambda$. Given a cell $s$ in $D[\Lambda]$, let $a_\Lambda(s)$ be the number of cells (including the possible circle at the end of the row) to the right of $s$. Let also $\ell_\Lambda(s)$ be the number of cells (not including the possible circle at the bottom of the column) below $s$. Finally, let $\Lambda^c$, be the set of cells of $D[\Lambda]$ that do not appear at the same time in a row containing a circle and in a column containing a circle.

**Conjecture 63.** The coefficient $c_{\Lambda_{\text{min}}}^\min(\beta)$ of $\tilde{m}_{\Lambda_{\text{min}}}$ in the monomial expansion of $J_\Lambda^{(1/\beta)}$ is given by

$$c_{\Lambda_{\text{min}}}^\min(\beta) = \frac{1}{\prod_{s \in \Lambda^c} \left(a_\Lambda(s)/\beta + \ell_\Lambda(s) + 1\right)} \hspace{1cm} (4.101)$$

with $\Lambda_{\text{min}}$ and $\ell_{n,m}$ defined in (4.94) and (4.95) respectively.
For instance, if \( \Lambda = (3, 1, 0; 4, 2, 1) \), we can fill \( D[\Lambda] \) with the values \( (a_\Lambda(s)/\beta + \ell_\Lambda(s) + 1) \) corresponding to the cells \( s \in \Lambda^\circ \). This gives (using \( \gamma = 1/\beta \)):

\[
\begin{array}{ccc}
\gamma + 1 & \gamma + 2 & 1 \\
\gamma + 3 & 1 & \\
& 1 & \\
& & \\
\end{array}
\]

Therefore, in this case,

\[
\epsilon^{\text{min}}_\Lambda(\beta) = \frac{1}{(3/\beta + 5)(2/\beta + 3)(1/\beta + 2)(1/\beta + 1)(1/\beta + 3)}
\]

(4.103)

Even though the Jack superpolynomials cannot be normalized to have positive coefficients when expanded in terms of supermonomials, we nevertheless conjecture they satisfy the following integrality property.

**Conjecture 64.** Let

\[
J^{(1/\beta)}_\Lambda = c^{\text{min}}_\Lambda(\beta) \sum_{\Omega \leq \Lambda} \hat{c}_{\Lambda \Omega}(\beta) \hat{m}_\Omega.
\]

Then \( \hat{c}_{\Lambda \Omega} \) is a polynomial in \( 1/\beta \) with integral coefficients.

5. **Conclusion**

5.1. **Summary.** In this work, we have presented an extension of the theory of symmetric functions involving fermionic variables as well as the usual bosonic variables. Our construction being motivated by supersymmetric considerations, we enforce from the beginning an equal number of variables of each type. These variables can thus be regarded as the coordinates of an Euclidian superspace. Symmetric functions in superspace, or equivalently, superfunctions, are defined to be symmetric with respect to the diagonal action of the symmetric group.

Basically all essential objects in the theory of symmetric functions have been extended to superspace. If some of them had already been introduced in previous works of ours (such as superpartitions and supermonomials in [31, 32, 33] and power-sum superpolynomials in [42], section 2.5 17), most of these extensions are new.

The resulting theory of symmetric functions in superspace, exposed in Section 3, is quite elegant and appears to be rather rich. We have also pointed out an interesting connection between superpolynomials and de Rham complexes of symmetric \( p \)-forms.

The core results of the elementary theory of symmetric functions are known to have a one parameter (our \( \beta \)) deformation that leads to the combinatorial definition of the Jack polynomials. This deformation also has a superspace lift that turns out to be related to our previous construction of the Jack superpolynomials using an approach rooted in the solution of a supersymmetric integrable quantum many-body problem [31 33]. In special cases, namely, \( \beta = 0, \beta \to \infty \) and for \( \Lambda = (n; 0) \) or \((n)\), the physical \( J_\Lambda \) were shown to reduce to combinatorial symmetric superfunctions (cf. Theorem 61 and Proposition 59 respectively).

\[17\text{We have noticed in the meantime that the elementary and power-sum superpolynomials had also been introduced in the second reference of [30].}\]
The physical Jack superpolynomials were already known to be orthogonal with respect to the physical scalar product (1.31) or (4.3) (the first one being the analytic extension of the second one, which holds when \( \beta \) is a positive integer). Here, by relying on the integrability of the physical underlying quantum many-body problem, we have been able to prove that the physical Jack superpolynomials are also orthogonal with respect to the combinatorial scalar product (3.73).

At once, these two products are manifestly very different from each other. That the Jack superpolynomials are orthogonal with respect to both products is certainly remarkable.

Even in the absence of fermionic variables, the orthogonality of the Jack polynomials with respect to both scalar products is a highly non-trivial observation. In that case, one can provide a partial rationale for the compatibility between the two scalar products, by noticing their equivalence in the following two circumstances [23, 21]:

\[
\langle f | g \rangle_{\beta,N}^{\beta} = \langle \langle f | g \rangle \rangle_{\beta,N}^{\beta} \quad (5.1)
\]

(see e.g., [21] VI.9 remark 2) and

\[
\lim_{N \to \infty} \langle f | g \rangle_{\beta,N}^{\beta} = \langle \langle f | g \rangle \rangle_{\beta} \quad ,
\]

(see e.g., [21] VI.9 (9.9)) for \( f, g \), two arbitrary symmetric polynomials.

In superspace, this compatibility between the two products is even more remarkable since the limiting-case equivalences (5.1) and (5.2) are simply lost. This is most easily seen by realizing that, after integration over the fermionic variables, we obtain

\[
\langle p_{\lambda} \tilde{p}_n | p_{\mu} \tilde{p}_m \rangle_{\beta,N}^{\beta} = \langle p_{\lambda} | p_{\mu} p_{m-n} \rangle_{\beta,N}^{\beta} , \quad m > n ,
\]

(5.3)

and thus the super power-sums cannot be orthogonal for any value of \( N \) and \( \beta \). This shows that the connection between the two scalar products is rather intricate.

5.2. Outlook: Schur superpolynomials and supercombinatorics. A central chapter in the theory of symmetric functions concerns the Schur polynomials \( s_{\lambda} \). These are limiting case \( \beta = 1 \) of the Jack polynomials \( J_{\lambda}^{(\beta)} \). Their special importance lies in their deep representation theoretic interpretation: \( s_{\lambda} \) is a Lie-algebra character, being expressible as a sum of semistandard tableaux of shape \( \lambda \). This implies, in particular, that \( s_{\lambda} \) has a monomial expansion with non-negative integer coefficients.

Schur superpolynomials could similarly be defined from the Jack superpolynomials evaluated at \( \beta = 1 \):

\[
s_{\lambda}(x, \theta) = J_{\lambda}^{(1)}(x, \theta)
\]

(5.4)

This is a quite natural guess. But is there any special property that points toward this identification of \( J_{\Lambda}^{(1)} \) with the superspace generalization of the Schur polynomials? Unfortunately, we have not been able to pinpoint a genuine distinguishable feature of the Jack superpolynomials that would single out the special value \( \beta = 1 \). For instance, the supermonomial decomposition is not integral, as shown by the following example:

\[
s_{(2,1;0)} = m_{(2,1;0)} + \frac{1}{2} m_{(2,0;1)} - \frac{1}{8} m_{(1,0;2)} + \frac{1}{4} m_{(1,0,1,1)} .
\]

(5.5)

Similarly, the kernel \( K \) can be expanded as

\[
K(x, \theta; y, \phi) = \sum_{\Lambda \in \text{SFar}} j_{\lambda}(1)^{-1} s_{\lambda}(x, \theta) \frac{1}{s_{\lambda}(y, \phi)} .
\]

(5.6)

\[18\] Off hand, one could have contemplated a connection more intricate than \( \beta = 1 \) between the would-be Schur superpolynomials and the Jack superpolynomials, such as a limiting value of \( \beta \) that depends upon the fermionic degree, for instance, \( \beta = m + 1 \). This would provide a stable definition with respect to the product of two superpolynomials, under which the fermionic degree is additive. But physically, it is not natural to have a coupling constant that depends upon the fermionic degree, that is, upon a special property of the eigenfunctions.
However, the coefficients $j_\Lambda(1)$ are not equal to 1 as in the usual bosonic case. The only property of the Schur polynomials that is readily transposed to the superspace is the following one: the homogeneous superpolynomials decompose upward triangularly in terms of $s_\Lambda$’s. Indeed, since $J_\Lambda$ is upper triangular in the $g$-basis, the inverse is true (that is, $g_\Lambda$ is upper triangular in the $J$-basis) and the $\beta = 1$ version of this expansion is the announced property:

$$h_\Lambda(x, \theta) = \sum_{\Omega \geq \Lambda} v_{\Lambda, \Omega} s_\Lambda(x, \theta)$$  \hspace{1cm} (5.7)

For completeness, we mention that the proper superspace generalization of the classical definition of the Schur polynomials as a bialternant has not been found yet. (We point out in that regard that division by anticommuting variables is prohibited). Similarly, the Jacobi-Trudi identity, which expresses the Schur polynomials in terms of the $h_\Lambda$’s, has not been generalized. Notice that in all the instances where we have obtained a determinantal expression, we had at most one row or one column made out of fermionic quantities, something which cannot be the case for the sought for Jacobi-Trudi super-identity. To determine whether these properties are specific to the $m = 0$ sector or not requires further study.

Note finally that, off-hand, it appears unlikely that the Schur superpolynomials would be related to the representation theory of special Lie superalgebras since these theories do not involve Grassmannian variables. Actually, it could well be that for the Schur superpolynomials, the representation theoretic interpretation is simply lost.

In a different vein, with the introduction of superdiagrams, we expect a large number of results linked to “Ferrer-diagram combinatorics” to have nontrivial extensions to the supercase. Pieces of supercombinatorics have already been presented at the end of Section 2.4. The combinatorics of the diagrams $D[\Lambda]$ also enters in the formulation of the norm of the Jack superpolynomials. Another natural ground for supercombinatorics would be to find a Pieri formula for the Jack superpolynomials.

5.3. Outlook: Macdonald superpolynomials. In this work, we have heavily stressed the existence of a one-parameter (i.e., $\beta$) deformation of the scalar product as the key tool for defining Jack superpolynomials combinatorially. However, there also exists a two-parameter deformation ($t$ and $q$) of the combinatorial scalar product. Again, this has a natural lift to the superspace, namely

$$\langle \langle p_\Lambda | p_\Omega \rangle \rangle_{q,t} := z_\Lambda(q, t) \delta_{\Lambda, \Omega},$$  \hspace{1cm} (5.8)

where

$$z_\Lambda(q, t) = z_\Lambda \prod_{i=1}^m \frac{1 - q^{A_i} + t^{A_i}}{1 - t^{A_i}}, \hspace{1cm} m = \sum A_i.$$  \hspace{1cm} (5.9)

This reduces to the previous scalar product $\langle \langle \cdot | \cdot \rangle \rangle_\beta$ when $q = t^{1/\beta}$ and $t \to 1$. The generalized form of the reproducing kernel reads

$$\prod_{i,j} \frac{(tx_i y_j + t\theta_i \phi_j; q)_\infty}{(x_i y_j + \theta_i \phi_j; q)_\infty} = \sum_\Lambda z_\Lambda(q, t)^{-1} \langle \langle p_\Lambda(x, \theta) | p_\Lambda(y, \phi) \rangle \rangle_{q,t},$$  \hspace{1cm} (5.10)

with $(a; q)_\infty$ defined in (2.12).

Now, the scalar product (5.8) leads directly to a conjectured combinatorial definition of Macdonald superpolynomials.

**Conjecture 65.** In the space of symmetric superfunctions with rational coefficients in $q$ and $t$, there exists a basis $\{ M_\Lambda \}_\Lambda$, where $M_\Lambda = M_\Lambda(x, \theta; q, t)$, such that

1) $M_\Lambda = m_\Lambda + \sum_{\Omega \prec \Lambda} C_{\Lambda \Omega}(q, t)m_\Omega$ (triangularity);  \hspace{1cm} (5.11)

2) $\langle \langle M_\Lambda | M_\Omega \rangle \rangle_{q,t} \propto \delta_{\Lambda, \Omega}$ (orthogonality).
Note that in this context, the combinatorial construction cannot be compared with the physical one since the corresponding supersymmetric eigenvalue problem has not been formulated yet. In other words, the proper supersymmetric version of the Ruijsenaars-Schneider model is still missing.

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