ON PARABOLIC AND ELLIPTIC EQUATIONS WITH SINGULAR OR DEGENERATE COEFFICIENTS

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Abstract. We study both divergence and non-divergence form parabolic and elliptic equations in the half space \( \{ x_d > 0 \} \) whose coefficients are the product of \( x_d^\alpha \) and uniformly nondegenerate bounded measurable matrix-valued functions, where \( \alpha \in (-1, \infty) \). As such, the coefficients are singular or degenerate near the boundary of the half space. For equations with the conormal or Neumann boundary condition, we prove the existence, uniqueness, and regularity of solutions in weighted Sobolev spaces and mixed-norm weighted Sobolev spaces when the coefficients are only measurable in the \( x_d \) direction and have small mean oscillation in the other directions in small cylinders. Our results are new even in the special case when the coefficients are constants, and they are reduced to the classical results when \( \alpha = 0 \).

1. Introduction and main results

In this paper, we study the existence, uniqueness, and regularity estimates of solutions in Sobolev spaces to a class of parabolic (and elliptic) equations in the upper half space, whose coefficients can be singular or degenerate on the boundary of the upper half space in a way which may not satisfy the classical Muckenhoupt \( A_2 \) condition.

Throughout the paper, let \( \Omega_T = (-\infty, T) \times \mathbb{R}^d_+ \) be a space-time domain, where \( T \in (-\infty, +\infty] \), \( \mathbb{R}_+ = (0, \infty) \), and \( \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+ \) is the upper-half space. Let \( (a_{ij}) : \Omega_T \rightarrow \mathbb{R}^{d \times d} \) be a matrix of measurable coefficients, which satisfies the following ellipticity and boundedness conditions: there is a constant \( \kappa \in (0, 1) \) such that

\[
\kappa |\xi|^2 \leq a_{ij}(t, x)\xi_i\xi_j \quad \text{and} \quad |a_{ij}(t, x)| \leq \kappa^{-1}
\]

for every \( \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d \) and \( (t, x) = (t, x', x_d) \in \Omega_T \). Here we do not impose the symmetry condition on \( (a_{ij}) \). Let \( \alpha \in (-1, \infty) \) be a fixed number. We investigate the conormal boundary value problem

\[
\begin{aligned}
x_d^\alpha (u_t + \lambda u) - D_i [x_d^\alpha (a_{ij}(t, x) D_j u - F_i)] &= \sqrt{\lambda} x_d^\alpha f \\
\lim_{x_d \rightarrow 0^+} x_d^\alpha (a_{ij}(t, x) D_j u - F_d) &= 0
\end{aligned}
\]

in \( \Omega_T \),

where \( F = (F_1, F_2, \ldots, F_d) : \Omega_T \rightarrow \mathbb{R}^d \) and \( f : \Omega_T \rightarrow \mathbb{R} \) are given measurable functions in suitable weighted Lebesgue spaces, and \( \lambda \geq 0 \) is a parameter. It is worth noting that the weight \( x_d^\alpha \) satisfies the Muckenhoupt \( A_2 \) condition only if
\( \alpha \in (-1,1) \). As a special case of our main results, for the model equation

\[
\begin{aligned}
\begin{cases}
x_d^\alpha u_t - \text{div}(x_d^\alpha (\nabla u - F)) = x_d^\alpha f \\
\lim_{x_d \to 0^+} x_d^\alpha (D_d u - F_d) = 0
\end{cases}
\end{aligned}
\tag{1.3}
\]

in the upper-half parabolic cylinder \( Q^+_T \) and for \( \alpha \in (-1, \infty) \), we obtain the local boundary weighted estimate

\[
\left( \int_{Q^+_T} [\|u\|^p + |Du|^p] x_d^\alpha \, dz \right)^{1/p} \leq N \left[ \int_{Q^+_2} [\|u\| + |Du|] x_d^\alpha \, dz \right]^{1/p} + N \left( \int_{Q^+_2} |F|^p x_d^\alpha \, dz \right)^{1/p} + N \left( \int_{Q^+_2} |F|^p x_d^\alpha \, dz \right)^{1/p}
\]

for every \( p \in (1, \infty) \), where \( p^* = [1, p] \) depending on \( \alpha, p, \) and \( d \) as in \([28, 22]\) below and \( N > 0 \) is a constant depending on \( d, \alpha, p, \) and \( p^* \). Equation (1.3) is related to the extension problem of the fractional heat operator (see, for example, \([27, 30, 1]\)) and our result in this special case is already new.

We also consider the parabolic equation in non-divergence form

\[
a_0(t,x)u_t - a_{ij}(t,x)D_{ij}u(t,x) - \frac{\alpha}{x_d} a_{dj}(t,x)D_j u(t,x) + \lambda c_0(t,x)u = f
\tag{1.4}
\]

in \( \Omega_T \) with the boundary condition

\[
\lim_{x_d \to 0^+} x_d^\alpha a_{dj}(t,x)D_j u(t,x', x_d) = 0,
\tag{1.5}
\]

where \( a_0, c_0 : \Omega_T \to \mathbb{R} \) are measurable functions satisfying

\[
\kappa \leq a_0(t,x), \quad c_0(t,x) \leq \kappa^{-1}, \quad (t,x) \in \Omega_T.
\tag{1.6}
\]

In this case, we impose an additional structural condition on the leading coefficients \( a_{ij} \):

\[
a_{dj}(t,x) = 0, \quad j = 1, 2, \ldots, d - 1,
\tag{1.7}
\]

or \( a_{dj} = \lambda_j a_{dd} \) for \( j = 1, 2, \ldots, d - 1 \) with constants \( \lambda_j \), which can be reduced to (1.7) after the change of variables \( y_j = x_j - \lambda_j x_d \) for \( j = 1, 2, \ldots, d - 1 \) and \( y_d = x_d \). We note that this condition is satisfied for a large class of equations. See, for instance, \([1, 3, 13, 15]\). Unlike (1.2), the equation (1.4) has extra coefficients \( a_0 \) and \( c_0 \). The main reason we introduce them in (1.4) is for convenience because in the proofs of main results for (1.4)-(1.5), we divide both sides of (1.4) by \( a_{dd} \) to use the hidden divergence structure of the equation. Nevertheless, with \( a_0 \) and \( c_0 \) the equation (1.4) is slightly more general. Of course, in view of (1.6), by dividing both sides of (1.4) by \( a_0 \) or \( c_0 \), one can always assume one of them to be the identity.

The interest of studying the class of equations (1.2) and (1.4) comes from both pure mathematics and applied problems. As examples, we refer the reader to \([1, 8]\) for problems about fractional heat and fractional Laplace equations, and \([13]\) for problems arising in mathematical finance. This paper is a continuation of \([11]\) and \([10]\). In \([11]\), we considered a class of parabolic equations in divergence form with a general weight

\[
a_0(x_d)u_t - \frac{1}{\mu(x_d)} D_t [\mu(x_d)(a_{ij} D_j u - F_i)] + \lambda u = f
\tag{1.8}
\]

in the half space \( \{x_d > 0\} \) with conormal boundary condition:

\[
\lim_{x_d \to 0^+} \mu(x_d)(a_{dj} D_j u - F_d) = 0.
\tag{1.9}
Here \((a_{ij})\) satisfies (1.1), \(a_0 \in [\kappa, \kappa^{-1}], \lambda \geq 0\), and the weight \(\mu\) satisfies the \(A_2\) condition and a relaxed \(A_1\) type condition away from the boundary. This, in particular, includes the \(A_2\) weights \(\mu(x_d) = x_d^\alpha\) for any \(\alpha \in (-1,1)\). We obtained the local and global weighted Calderón-Zygmund type estimates for (1.8)-(1.9) with respect to the weight \(\mu\), under the condition that the coefficients are only measurable in the \(x_d\) direction and have small mean oscillation in the other directions in small cylinders (partially VMO) with respect to the considered weight. The proofs in [11] carry over to systems under the usual strong ellipticity condition. In [10], we studied the corresponding non-divergence form scalar equations (1.1), where \(\alpha \in (-1,1)\) and \(a_0, c_0\) satisfy (1.6). Under the condition that \(a_0, a_{ij}, c_0\) are partially VMO respect to the weight \(x_d^\alpha\), we obtained weighted mixed-norm \(W^{1,2}_p\) estimates and solvability. The aim of this paper is to extend the results in [11, 10] to the full range of exponent \(\alpha \in (-1, \infty)\). It is worth noting that even for divergence form equations, in contrast to [11], the proofs below only work for scalar equations because the Moser iteration is used (cf. Lemma 4.3). For other related work in this direction, we refer the reader to the references in [11, 10].

The class of partially VMO coefficients was first introduced by Kim and Krylov [19, 20] for non-degenerate elliptic and parabolic equations in non-divergence form. Divergence form elliptic and parabolic equations with non-degenerate partially VMO coefficients were later studied in [6, 5]. This type of equations arises from the problems of linearly elastic laminates and composite materials, e.g., in homogenization of layered materials. See, for instance, [4]. We also refer the reader to [7, 8, 9] for extensions to second-order and higher-order systems with or without weights.

We apply a mean oscillation argument, which was used in [24] for non-degenerate parabolic equations with coefficients which are VMO in the space variables. In the case of partially VMO coefficients, the main difficulty is that, since they are merely measurable in \(x_d\), it is only possible to estimate the mean oscillation of \(D_d u\), not the full gradient \(Du\). Therefore, one needs to bound \(D_d u\) by \(D_{x'} u\). An idea in [6, 5] is to break the “symmetry” of the coordinates so that \(t\) and \(x_d\) are distinguished from \(x'\) by using a delicate re-scaling argument. Another idea is to estimate the mean oscillation of \(a_{dd} D_{x'} u\) instead of \(D_d u\), and apply a generalized Fefferman–Stein theorem established in [25]. In [7], a new method was developed, in which the key step is to estimate \(U := a_{ij} D_j u\) and \(D_i u, i = 1, \ldots, d - 1\), instead of the full gradient of \(u\). By using this argument, one was able to bypass the scaling argument mentioned above and greatly simplified the proof. In this paper, we adopt this method to singular/degenerate equations.

In our main results, Theorems 2.2, 2.4, and 2.7 below, we obtain the unique solvability (1.2) and (1.4)-(1.5) in weighted Sobolev spaces and mixed-norm weighted Sobolev spaces. Local boundary estimates for solutions of these equations are also obtained in Corollaries 2.6 and 2.9. To the best of our knowledge, these results are new even in the elliptic case and in the unmixed-norm case with constant coefficients \(a_{ij}, a_0,\) and \(c_0\).

The proofs of the main theorems are based on an idea in [7] mentioned above and the perturbation technique. To implement the method, we first consider equations whose coefficients depend only on \(x_d\) and prove various results on the existence, uniqueness, and regularity of solutions to this class of equations. For this, we establish the \(L_\infty\) estimate of weak solutions by applying the Moser iteration, and
then derive Lipschitz and Schauder type estimates. In particular, to estimate the $L_\infty$ norms of $D_j u$ and $\mathcal{U}$, we use a bootstrap argument. Schauder type estimates for elliptic equations similar to (2.8) were proved recently in [28] when the matrix $(a_{ij})$ is symmetric, Hölder in all variables, and satisfies a structural condition that the hyperplane $\{x_d = 0\}$ is invariant with respect to $(a_{ij})$, i.e., $a_{jd} = d_j = 0$ for $j = 1, \ldots, d - 1$. The proof in [28] uses a Liouville type theorem and a compactness argument. Our proof in Section 4 is more direct and works for more general operators. For the local estimates Corollaries 2.3 and 2.9, we prove a parabolic embedding (see Lemma 3.1) by using a generalized Hardy–Littlewood Sobolev inequality in [17], which seems to be new in the weighted setting and is of independent interest. We also remark that in contrast to the previous work such as [12, 2, 11] in which the $A_2$ weights are commonly assumed as the weighted Poincaré inequality is needed, we do not use the weighted Poincaré inequality in the proof. In fact, as pointed out in [25], when $\alpha \geq 1$, such inequality is not valid.

For simplicity, in this paper we choose not to consider lower-order terms. The results still hold for equations

$$x_d^a(u_t - b_i D_i u - c u + \lambda u) - D_j[ x_d^a(a_{ij} D_j u + \hat{b}_i u - F_i)] = \sqrt{x_d^a} f$$

and

$$a_0 u_t - a_{ij} D_{ij} u - \left(\frac{\alpha}{x_d} a_{dj} + b_i\right) D_j u(t, x) - c u + \lambda a_0 u = f,$$

where $b_i$, $\hat{b}_i$, and $c$ are bounded measurable functions. To see this, it suffices to move the terms $b_i D_i u$ and $c u$ to the right-hand side of the equations, absorb $\hat{b}_i u$ to $F_i$, and take a sufficiently large $\lambda$. See, for example, [24] for details. By using the weighted embedding results such as Lemma 3.1 below, it is also possible to consider unbounded lower-order coefficients. We refer the reader to the recent interesting work [18, 23, 26, 22] and the references therein.

The remaining part of the paper is organized as follows. In the next section, we introduce some notation and state the main results of the paper. In Section 3 we prove two weighted embedding results that are needed in the paper as well as a result on the existence and uniqueness of $L_2$-solutions. In Section 4 we study equations whose coefficients depend only on $x_d$. We prove the existence, uniqueness, and regularity estimates of solutions in $\mathcal{H}^1_T(\Omega_T, \mu)$ after we obtain the $L_\infty$ Lipschitz, and Schauder type estimates for solutions to homogeneous equations. Finally, in Section 5 we provide the proofs of Theorems 2.2 and 2.4 and Corollaries 2.3 and 2.9.

2. Notation and main theorems

2.1. Notation. For $r > 0$, $z_0 = (t_0, x_0)$ with $x_0 = (x_0', x_{0d}) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $t_0 \in \mathbb{R}$, we define $B_r(x_0)$ to be the ball in $\mathbb{R}^d$ of radius $r$ centered at $x_0$, $Q_r(z_0)$ to be the parabolic cylinder of radius $r$ centered at $z_0$:

$$Q_r(z_0) = (t_0 - r^2, t_0) \times B_r(x_0),$$

and $B^+_r(x_0)$ and $Q^+_r(z_0)$ to be the upper-half ball and cylinder of radius $r$ centered at $x_0$ and $z_0$, respectively.

$$B^+_r(x_0) = \{ x = (x_d, x') \in \mathbb{R}^d : x_d > 0, \ |x - x_0| < r \},$$

$$Q^+_r(z_0) = (t_0 - r^2, t_0) \times B^+_r(x_0).$$
When \( x_0 = 0 \) and \( t_0 = 0 \), for the simplicity of notation we drop \( x_0, z_0 \) and write \( B_t, B_t^+, Q_t, \) and \( Q_t^+ \), etc. We also define \( B'(z_0') \) and \( Q'(z_0') \) to be the ball and the parabolic cylinder in \( \mathbb{R}^{d-1} \) and \( \mathbb{R}^d \), where \( z_0' = (t_0, x_0') \).

For \( p \in (1, \infty), -\infty < S < T < +\infty, \) and \( D \subset \mathbb{R}^d_+ \), let \( L_p((S, T) \times D, \mu) \) be the weighted Lebesgue space consisting of measurable function \( g \) on \( (S, T) \times D \) such that its norm
\[
\|g\|_{L_p((S, T) \times D, \mu)} = \left( \int_{(S, T) \times D} |g(t, x)|^p \mu(dz) \right)^{1/p} < \infty,
\]
where \( \mu(dz) = x_0^2 \, dx \, dt \). For \( p, q \in (1, \infty) \) and the weights \( \omega_0 = \omega_0(t) \) and \( \omega_1 = \omega_1(x) \), we define \( L_{q,p}(\Omega_T, \omega \, d\mu) \) to be the weighted mixed-norm Lebesgue space on \( \Omega_T \) equipped with the norm
\[
\|f\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} = \left( \int_0^T \left( \int_{\mathbb{R}^d_+} |f(t, x)|^p \omega_1(x) \mu(dx) \right)^{q/p} \omega_0(t) \, dt \right)^{1/q},
\]
where \( \omega(t, x) = \omega_0(t) \omega_1(x) \). We also define
\[
\mathcal{H}^{-1}_{q,p}((S, T) \times D, \omega \, d\mu) = \{ g : g = D_t F_t + F_0/d + f \, \text{ for some } f \in L_{q,p}((S, T) \times D, \omega \, d\mu) \}
\]
and
\[
\mathcal{H}^1_{q,p}((S, T) \times D, \omega \, d\mu) = \{ g : g, Dg \in L_{q,p}((S, T) \times D, \omega \, d\mu), g_t \in \mathcal{H}^{-1}_{q,p}((S, T) \times D, \omega \, d\mu) \},
\]
which are equipped with the norms
\[
\|g\|_{\mathcal{H}^{-1}_{q,p}((S, T) \times D, \omega \, d\mu)} = \inf \{ \|F\|_{L_{q,p}((S, T) \times D, \omega \, d\mu)} + \|f\|_{L_{q,p}((S, T) \times D, \omega \, d\mu)} : g = D_t F_t + F_0/d + f \}
\]
and
\[
\|g\|_{\mathcal{H}^1_{q,p}((S, T) \times D, \omega \, d\mu)} = \|g\|_{L_{q,p}((S, T) \times D, \omega \, d\mu)} + \|Dg\|_{L_{q,p}((S, T) \times D, \omega \, d\mu)} + \|g_t\|_{\mathcal{H}^{-1}_{q,p}((S, T) \times D, \omega \, d\mu)}.
\]
When \( p = q \), we simply write \( \mathcal{H}^1_{p}((\Omega_T, \omega \, d\mu) = \mathcal{H}^1_{p,p}((\Omega_T, \omega \, d\mu) \). Similar notation are also used for other spaces. When \( \omega \equiv 1 \), we have \( L_{q,p}((\Omega_T, \omega \, d\mu) = L_{q,p}(\Omega_T, \mu) \), and similarly for other function spaces.

We say that \( u \in \mathcal{H}^1_{q,p}((S, T) \times D, \omega \, d\mu) \) is a weak solution of \((1.2)\) in \((S, T) \times D\) if
\[
\int_{(S,T) \times D} (-u \partial_t \varphi + \lambda u \varphi) \mu(dz) + \int_{(S,T) \times D} (a_{ij} D_j u - F_i) D_i \varphi \mu(dz) = \lambda^{1/2} \int_{(S,T) \times D} f(z) \varphi(z) \mu(dz)
\]
for any \( \varphi \in C_0^\infty((S, T) \times (D \cup (\overline{D} \cap \partial \mathbb{R}^d))). \)

We use the notation \( a_+ = \max\{a, 0\} \) and \( a_- = \max\{-a, 0\} \) for \( a \in \mathbb{R} \) so that \( a = a_+ - a_- \). Finally, for a set \( \Omega \subset \mathbb{R}^{d+1} \) and any integrable function \( f \) on \( \Omega \) with
respect to some Borel measure \( \omega \), we write
\[
\int_{\Omega} f \omega(dz) = \frac{1}{\omega(\Omega)} \int_{\Omega} f \omega(dz), \quad \text{where} \quad \omega(\Omega) = \int_{\Omega} \omega(dz).
\]

2.2. Main theorems. As in \cite{11,10}, we impose the following partially VMO condition on the leading coefficients.

Assumption 2.1 \((\gamma_0, R_0)\). For any \( r \in (0, R_0] \) and \( z_0 = (z_0', x_d) \in \mathbb{R}^d \times \mathbb{R}_+ \), we have
\[
\sup_{i,j} \int_{Q_r^+(z_0)} |a_{ij}(t, x) - [a_{ij}]_{r, z_0}(x_d)| \mu(dz) \leq \gamma_0,
\]
where \( \mu(dz) = x_d^3 \mu dt dx \), \([a_{ij}]_{r, z_0}(x_d)\) is the average of \( a_{ij} \) with respect to \((t, x')\) in \( Q_r^+(z_0')\):
\[
[a_{ij}]_{r, z_0}(x_d) = \int_{Q_r^+(z_0')} a_{ij}(t, x', x_d) dx' dt.
\]

In the special case that the coefficients \( (a_{ij}) \) only depend on the \( x_d \) variable, no regularity assumption is required on them as Assumption 2.1 \((\gamma_0, R_0)\) is always satisfied.

Our first main result is about the existence, uniqueness, and global regularity estimates of solutions to the divergence form equation \((1.2)\).

Theorem 2.2. Let \( \alpha \in (-1, \infty) \), \( \kappa \in (0, 1) \), \( R_0 \in (0, \infty) \), and \( p \in (1, \infty) \). Then there exist \( \gamma_0 = \gamma_0(d, \kappa, \alpha, p) \in (0, 1) \) and \( \lambda_0 = \lambda_0(d, \kappa, \alpha, p) \geq 0 \) such that the following assertions hold. Suppose that \((1.1)\) and Assumption 2.1 \((\gamma_0, R_0)\) are satisfied. If \( u \in H^1_p(\Omega_T, \mu) \) is a weak solution of \((1.2)\) for some \( \lambda \geq \lambda_0 R_0^{-2} \), \( f \in L_p(\Omega_T, \mu) \), and \( F \in L_p(\Omega_T, \mu)^d \), then we have
\[
||Du||_{L_p(\Omega_T, \mu)} + \sqrt{\lambda}||u||_{L_p(\Omega_T, \mu)} \leq N||F||_{L_p(\Omega_T, \mu)} + N||f||_{L_p(\Omega_T, \mu)},
\]
where \( N = N(d, \kappa, \alpha, p) > 0 \). Moreover, for any \( \lambda > \lambda_0 R_0^{-2} \), \( f \in L_p(\Omega_T, \mu) \), and \( F \in L_p(\Omega_T, \mu)^d \), there exists a unique weak solution \( u \in H^1_p(\Omega_T, \mu) \) to \((1.2)\).

In the next result, we give a local boundary estimate in a half cylinder. Consider
\[
\left\{ \begin{array}{ll}
x_d^2 u_t - D_i(x_d^2(a_{ij} D_j u - F_i)) = x_d^2 f & \text{in } Q^*_2, \\
\lim_{x_d \to 0^+} x_d^2(a_{ij} D_j u - F_i) = 0 & \text{in } Q^*_2.
\end{array} \right.
\]
Let \( p \in [1, \infty) \) and \( p^* \in [1, p) \) satisfy
\[
\left\{ \begin{array}{ll}
(d + 2 + \alpha_+) / p^* \leq 1 + (d + 2 + \alpha_+) / p & \text{when } p^* > 1 \\
(d + 2 + \alpha_+) / p^* < 1 + (d + 2 + \alpha_+) / p & \text{when } p^* = 1,
\end{array} \right.
\]
if \( d \geq 2 \) or \( \alpha = 0 \), and
\[
\left\{ \begin{array}{ll}
(4 + \alpha_+) / p^* \leq 1 + (4 + \alpha_+) / p & \text{when } p^* > 1 \\
(4 + \alpha_+) / p^* < 1 + (4 + \alpha_+) / p & \text{when } p^* = 1,
\end{array} \right.
\]
if \( d = 1 \) and \( \alpha \neq 0 \). Note that the condition on \( p^* \) is used in a weighted parabolic Sobolev embedding result. See Lemma 3.1 below.
Corollary 2.3. Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, $1 < p_0 < p < \infty$, and $p^* \in [1, p)$ satisfy (2.3) and (2.4). Then there exists $\gamma_0 = \gamma_0(d, \kappa, \alpha, p_0, p) \in (0, 1)$ such that the following assertion holds. Suppose that (1.1) and Assumption 2.1 ($\gamma_0, R_0$) are satisfied. If $u \in \mathcal{H}^1_{p_0}(Q^+_2, \mu)$ is a weak solution of (2.2), $F \in L_p(Q^+_2, \mu)^d$, and $f \in L_{p^*}(Q^+_2, \mu)$, then $u \in \mathcal{H}^1_{p^*}(Q^+_1, \mu)$ and
\[
\|u\|_{L_p(Q^+_1, \mu)} + \|Du\|_{L_p(Q^+_1, \mu)} \\
\leq N\|u\|_{L_1(Q^+_2, \mu)} + N\|Du\|_{L_1(Q^+_2, \mu)} + N\|F\|_{L_{p^*}(Q^+_2, \mu)} + N\|f\|_{L_{p^*}(Q^+_2, \mu)},
\]
where $N = N(d, \kappa, \alpha, p_0, p, p^*, R_0) > 0$.

We conjecture that for any $d \geq 1$ and $\alpha \in (-1, \infty)$, the above corollary still holds when $p^*$ satisfies (2.3).

Next, we state the main results for non-divergence form equations. Besides the regularity assumption on $(a_{ij})$ as in Assumption 2.1, we impose similar conditions on the coefficients $a_0$ and $c_0$. 

Theorem 2.4. Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, $p, q, K \in (1, \infty)$, $\omega_0 \in A_p(\mathbb{R}^d, \mu)$, $\omega_1 \in A_p(\mathbb{R}^d, \mu)$, and $\omega = \omega_0(t)\omega_1(x)$, such that
\[
[\omega_0]_{A_p(\mathbb{R})} \leq K, \quad [\omega_1]_{A_p(\mathbb{R}^d, \mu)} \leq K.
\]
Then there exist
\[
\gamma_0 = \gamma_0(d, \kappa, \alpha, p, q, K) \in (0, 1) \quad \text{and} \quad \lambda_0 = \gamma_0(d, \kappa, \alpha, p, q, K) \geq 0,
\]
such that the following assertions hold. Suppose that (1.1) and Assumption 2.1 ($\gamma_0, R_0$) are satisfied. If $u \in \mathcal{H}^1_{q,p}(\Omega_T, \omega d\mu)$ is a weak solution of (1.2) for some $\lambda \geq \lambda_0 R_0^{-2}$, $f \in L_{q,p}(\Omega_T, \omega d\mu)$, and $F \in L_{q,p}(\Omega_T, \omega d\mu)^d$, then we have
\[
\|Du\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \sqrt{\lambda}\|u\|_{L_{q,p}(\Omega_T, \omega d\mu)} \\
\leq N\|F\|_{L_{q,p}(\Omega_T, \omega d\mu)} + N\|f\|_{L_{q,p}(\Omega_T, \omega d\mu)},
\]
where $N = N(d, \kappa, \alpha, p, q, K) > 0$. Moreover, for any $\lambda > \lambda_0 R_0^{-2}$, $f \in L_{q,p}(\Omega_T, \omega d\mu)$, and $F \in L_{q,p}(\Omega_T, \omega d\mu)^d$, there exists a unique weak solution $u \in \mathcal{H}^1_{q,p}(\Omega_T, \omega d\mu)$ to (1.2).
Assumption 2.5 ($\gamma_0, R_0$). For any $r \in (0, R_0]$ and \( z_0 = (z'_0, x_d) \in \mathbb{R} \times \mathbb{R}_+^d \), we have
\[
\sup_{i,j} \int_{Q^+_r(z_0)} |a_{ij}(t, x) - [a_{ij}]_{r, z_0}(x_d)| \mu(dz) \leq \gamma_0,
\]
\[
+ \int_{Q^+_r(z_0)} \left( |a_0(t, x) - [a_0]_{r, z_0}(x_d)| + |c_0(t, x) - [c_0]_{r, z_0}(x_d)| \right) \mu(dz) \leq \gamma_0,
\]
where \([a_{ij}]_{r, z_0}(x_d), [a_0]_{r, z_0}(x_d), [c_0]_{r, z_0}(x_d)\) are respectively the average of \(a_{ij}, a_0,\) and \(c_0\) with respect to \((t, x')\) in \(Q^+_r(z_0)\) as defined in Assumption 2.1.

We also need the following definition which is used in a weighted Hardy inequality (cf. [10, Lemma 2.2]).

Definition 2.6. Let \(\alpha \in (-1, \infty)\) and \(p \in (1, \infty)\), we say that the weight \(\omega : \mathbb{R}_+ \to \mathbb{R}_+\) is in \(M_p(\mu)\) if
\[
[\omega]_{M_p(\mu)} = \sup_{r > 0} \left( \int_r^\infty y^{-p(\alpha+1)} \omega(y) \mu(dy) \right)^{\frac{1}{p-1}} < \infty,
\]
where \(\mu(dy) = y^\alpha \, dy\) for \(y \in \mathbb{R}_+\).

Define \(W^{1,2}_{q,p}(\Omega_T, \omega \, d\mu)\) to be the weighted mixed-norm Sobolev space equipped with the norm
\[
\|u\|_{W^{1,2}_{q,p}(\Omega_T, \omega \, d\mu)} = \|u\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} + \|u_t\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} + \|D^2u\|_{L_{q,p}(\Omega_T, \omega \, d\mu)}.
\]
When \(p = q\) and \(\omega \equiv 1\), we write \(W^{1,2}(\Omega_T, \mu) = W^{1,2}_{1,1}(\Omega_T, d\mu)\). A function \(u \in W^{1,2}_{q,p}(\Omega_T, \omega \, d\mu)\) is said to be a strong solution to (1.4) if it satisfies the equation almost everywhere. Our main result for the non-divergence form equation (1.4)–(1.5) is the following theorem.

Theorem 2.7. Let \(\alpha \in (-1, \infty), \kappa \in (0, 1), R_0 \in (0, \infty), p, q, K \in (1, \infty).\) Let \(\omega_0 \in A_q(\mathbb{R}), \omega_1 \in A_p(\mathbb{R}^{d-1}), \omega_2 \in A_p(\mathbb{R}_+, \mu) \cap M_p(\mu)\), and \(\omega(t, x) = \omega_0(t)\omega_1(x')\omega_2(x_d)\), such that
\[
[\omega_0]_{A_q(\mathbb{R})} \leq K, \quad [\omega_1]_{A_p(\mathbb{R}^{d-1})} \leq K, \quad [\omega_2]_{A_p(\mathbb{R}_+, \mu)} \leq K, \quad [\omega_2]_{M_p(\mu)} \leq K.
\]
Then there exist
\[
\gamma_0 = \gamma_0(d, \kappa, \alpha, p, q, K) \in (0, 1) \quad \text{and} \quad \lambda_0 = \lambda_0(d, \kappa, \alpha, p, q, K) \geq 0
\]
such that the following assertions hold. Suppose that (1.1), (1.6), (1.7) and Assumption 2.5 \((\gamma_0, R_0)\) are satisfied. If \(u \in W^{1,2}_{q,p}(\Omega_T, \omega \, d\mu)\) is a strong solution of (1.4)–(1.5) with \(f \in L_{q,p}(\Omega_T, \omega \, d\mu)\) and \(\lambda \geq \lambda_0R_0^{-2}\), then
\[
\|u_t\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} + \|D^2u\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} + \|Du/x_d\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} + \sqrt{\lambda}\|Du\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} + \lambda\|u\|_{L_{q,p}(\Omega_T, \omega \, d\mu)} \leq N\|f\|_{L_{q,p}(\Omega_T, \omega \, d\mu)},
\]
where \(N = N(d, \kappa, \alpha, p, q, K) > 0\). Moreover, for any \(f \in L_{q,p}(\Omega_T, \omega \, d\mu)\) and \(\lambda > \lambda_0R_0^{-2}\), there is a unique strong solution \(u \in W^{1,2}_{q,p}(\Omega_T, \omega \, d\mu)\) of (1.4)–(1.5).

Remark 2.8. As a typical example, in Theorem 2.7 we can take the power weight \(\omega_2(x_d) = x_d^\beta\). It is easily seen that for any \(\beta \in (-\alpha - 1, (\alpha + 1)(p - 1))\), we have \(\omega_2 \in A_p(\mathbb{R}_+, \mu) \cap M_p(\mu)\). In the special case when \(\alpha = 0\) and \(\beta \in (-1, p - 1)\), a similar result was proved in [21] when the coefficients are measurable in the
time variable and have small mean oscillations in the spatial variables, by using a different argument.

Once Lipschitz and Schauder estimates in Section 4 and Theorem 2.2 are proved, Theorem 2.7 can be proved by using the same argument as in [10]. To keep the paper within a reasonable length, we skip the proof of Theorem 2.7 and refer the reader to [10] for details.

Similarly to Corollary 2.8, we also obtain the following local boundary estimate for solutions of (1.4) in $Q_2^+$.

**Corollary 2.9.** Let $\alpha \in (-1, \infty)$, $\kappa \in (0,1)$, $R_0 \in (0, \infty)$, and $1 < p_0 < p < \infty$. Then there exists $\gamma_0 = \gamma_0(d, \kappa, \alpha, p_0, p) \in (0, 1)$ such that the following assertion holds. Suppose that (1.1), (1.6), (2.7), and Assumption 2.1 (\(\gamma_0, R_0\)) are satisfied. If $u \in W^{1,2}_p(Q_2^+, \mu)$ is a strong solution of

$$
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u - a_{ij}(x)D_{ij}u + c_0u &= f \\
\lim_{x_d \to 0^+} x_d^\alpha a_{ij}D_{ij}u &= 0
\end{array} \right. & \quad \text{in } Q_2^+,
\end{align*}
$$

and $f \in L_p(Q_2^+, \mu)$, then we have $u \in W^{1,2}_p(Q_2^+, \mu)$ and

$$
\|u\|_{W^{1,2}_p(Q_2^+, \mu)} \leq N\|u\|_{W^{1,2}_p(Q_2^+, \mu)} + N\|f\|_{L_p(Q_2^+, \mu)},
$$

where $N = N(d, \kappa, \alpha, p_0, p, R_0) > 0$.

Using the above results for parabolic equations, we can directly derive similar results for elliptic equations by viewing solutions to elliptic equations as steady state solutions of the corresponding parabolic equations. See, for example, the proofs of [24] Theorem 2.6 and [10] Theorem 1.2. We only present here a result of the local boundary estimate for weak solutions. Consider

$$
\begin{align*}
\left\{ \begin{array}{ll}
D_i(x_d^\alpha a_{ij}(x)D_ju - F_i) &= x_d^\alpha f \\
\lim_{x_d \to 0^+} x_d^\alpha (a_{ij}(x)D_{ij}u - F_d) &= 0
\end{array} \right. & \quad \text{in } B_2^+,
\end{align*}
$$

where $a_{ij} : B_2^+ \to \mathbb{R}$, $F = (F_1, F_2, \ldots, F_d) : B_2^+ \to \mathbb{R}^d$ and $f : B_2^+ \to \mathbb{R}$ are given measurable functions. In this time-independent case, (1.1) and Assumption 2.1 can be stated similarly. For each $p \in (1, \infty)$, suppose that $\bar{p} \in [1, p)$ satisfies

$$
\begin{align*}
(d + \alpha+)/\bar{p} &\leq 1 + (d + \alpha+)/p \quad \text{when } \bar{p} > 1, \\
(d + \alpha+)/\bar{p} &\leq 1 + (d + \alpha+)/p \quad \text{when } \bar{p} = 1.
\end{align*}
$$

For $\Omega \subset \mathbb{R}^d$, $W^{1,p}_p(\Omega, \mu)$ denotes the weighted Sobolev space consisting of all measurable functions $u : \Omega \to \mathbb{R}$ such that $u, Du \in L_p(\Omega, \mu)$.

**Corollary 2.10.** Let $\alpha \in (-1, \infty)$, $\kappa \in (0,1)$, $R_0 \in (0, \infty)$, $1 < p_0 < p < \infty$, and $\bar{p} \in [1, p)$ satisfy (2.8). Then there exists $\gamma_0 = \gamma_0(d, \kappa, \alpha, p_0, \bar{p}) \in (0, 1)$ such that the following assertion holds. Suppose that (1.1) and Assumption 2.1 (\(\gamma_0, R_0\)) are satisfied. If $u \in W^{1,2}_p(B_2^+, \mu)$ is a weak solution of (2.8), $F \in L_p(B_2^+, \mu)^d$, and $f \in L_{\bar{p}}(B_2^+, \mu)$, then $u \in W^{1,2}_p(B_2^+, \mu)$ and

$$
\|u\|_{W^{1,2}_p(B_2^+, \mu)} \leq N\|u\|_{W^{1,2}_p(B_2^+, \mu)} + N\|F\|_{L_p(B_2^+, \mu)} + N\|f\|_{L_{\bar{p}}(B_2^+, \mu)},
$$

where $N = N(d, \kappa, \alpha, p_0, \bar{p}, R_0) > 0$. 
The proof of Corollary 2.10 is similar to that of Corollary 2.3 by using the corresponding weighted embedding inequality. See Remark 3.2 (ii). Therefore, we also omit it.

3. Weighted Sobolev inequalities and \( L_2 \)-solutions

Our first result in this section is a weighted parabolic embedding lemma which will be used in the proof of Corollary 2.3. The range of \( q^* \) below is optimal when \( d \geq 2 \). However, when \( d = 1 \), we impose a slightly stronger condition. In view of the classical parabolic Sobolev embedding when \( \alpha = 0 \), we conjecture that this condition can be relaxed.

**Lemma 3.1** (Weighted parabolic imbedding). Let \( \alpha \in (-1, \infty) \) and \( q, q^* \in (1, \infty) \) satisfy

\[
\begin{cases}
(d + 2 + \alpha_+)/q \leq 1 + (d + 2 + \alpha_+)/q^* & \text{if } d \geq 2 \\
(4 + \alpha_+)/q \leq 1 + (4 + \alpha_+)/q^* & \text{if } d = 1.
\end{cases}
\]

(3.1)

Then for any \( v \in \mathcal{H}_q^1(Q^+_{2}, \mu) \), we have

\[
\|v\|_{L_q^\tau(Q^+_{2}, \mu)} \leq N\|v\|_{\mathcal{H}_q^1(Q^+_{2}, \mu)},
\]

(3.2)

where \( N = N(d, \alpha, q, q^*) > 0 \) is a constant. The result still holds when \( q^* = \infty \) and the inequality in (3.1) is strict.

**Proof.** Note that the case when \( d = 1 \) follows by considering \( v(t, x_1) = v(t, x_1, x_2) \) with a dummy variable \( x_2 \) and using the result when \( d = 2 \). Hence, we only need to prove (3.2) when \( d \geq 2 \). It suffices to consider the case when \( q^* > q \). Without loss of generality, we may assume that

\[
v_t = D_iG_i + G_0/x_d + g
\]

in \( Q^+_{2} \) in the weak sense and

\[
\|v\|_{L_q^\tau(Q^+_{2}, \mu)} + \|Dv\|_{L_q^\tau(Q^+_{2}, \mu)} + \|G\|_{L_q^\tau(Q^+_{2}, \mu)} + \|g\|_{L_q^\tau(Q^+_{2}, \mu)} \leq 1,
\]

where \( G = (G_0, G_1, \ldots, G_d) \). Let \( \tilde{Q} = Q_{1/2}(0, 0, \ldots, 0, 3/2) \) and \( \psi \in C_0^\infty(\tilde{Q}) \) with unit integral. For any \( (t, x) \in Q^+_{2} \), by the fundamental theorem of calculus,

\[
v(t, x) - c = \int_{\tilde{Q}} \int_0^1 \left( v_i(t(1 - \theta^2) + s\theta^2, x(1 - \theta) + y\theta)2\theta(s-t)
\right.
\]

\[
+ (Dv)(t(1 - \theta^2) + s\theta^2, x(1 - \theta) + y\theta) \cdot (y-x) \big) \psi(s, y) d\theta ds dy
\]

\[
:= I_1 + I_2,
\]

(3.4)

where

\[
c = \int_{\tilde{Q}} v(s, y) \psi(s, y) ds dy.
\]

Let \( \dot{x} = x(1 - \theta) + y\theta \) and \( \tau = t(1 - \theta^2) + s\theta^2 \). Clearly,

\[
(|\dot{x} - x|^2 + |\tau - t|)^{1/2} = (|x - y|^2 + |t - s|)^{1/2}\theta \leq N\theta.
\]

(3.5)

It then follows from (3.3) that

\[
I_1 = 2 \int_{\tilde{Q}} \int_0^1 \left( g(\tau, \dot{x}) + (D_iG_i)(\tau, \dot{x}) + \frac{G_0(\tau, \dot{x})}{x_d} \right) \theta(s - t) \psi(s, y) d\theta ds dy.
\]

(3.6)
Since \( y \in \tilde{Q} \), we have \( y_d \geq 1 \) and thus
\[
|x - \hat{x}| = |x - y|/\theta \leq N \theta y_d \leq N \hat{x}_d \quad \text{and} \quad x_d \leq N \hat{x}_d. \tag{3.7}
\]
Moreover,
\[
(D_i G_i)(\tau, \hat{x}) = D_y G_i(\tau, \hat{x}) \theta^{-1}.
\]
Therefore, from (3.6) and integration by parts, we deduce
\[
|I_1| \leq N \int_Q \int_0^1 (|g(\tau, \hat{x})| + |G(\tau, \hat{x})| + |G_0(\tau, \hat{x})||x - \hat{x}|^{-1} \theta - |Dv(\tau, \hat{x})|) \, |s - t| \, d\theta \, ds \, dy. \tag{3.8}
\]
Combining (3.4) and (3.8), we obtain
\[
|v(t, x) - c| 
\leq N \int_Q \int_0^1 (|g(\tau, \hat{x})| + |G(\tau, \hat{x})| + |G_0(\tau, \hat{x})||x - \hat{x}|^{-1} \theta - |Dv(\tau, \hat{x})|) d\theta \, ds \, dy
\leq N \int_Q \int_0^1 \theta^{-d-2}(|g(\tau, \hat{x})| + |G(\tau, \hat{x})|)
+ |G_0(\tau, \hat{x})||x - \hat{x}|^{-1} |Dv(\tau, \hat{x})| d\theta \, d\tau \, d\hat{x}
\leq N \int_Q \int_0^1 \chi_{\{\|x - \hat{x}\| < |t - \tau| \}} \chi_{\{x_d \leq N \hat{x}_d \}} \, d\tau \, d\hat{x},
\]
where we used \( dy = \theta^{-d} \, d\hat{x}, \, d\tau = \theta^{-2} \, ds \), (3.5), and (3.7) in the third inequality.
We apply Young’s inequality for convolutions with respect to the time variable to get that for any \( x \in B_{\hat{x}}^2 \),
\[
\|v(\cdot, x) - c\|_{L_{\infty}((-4, 0))} \leq N \int_{B_{\hat{x}}^2} (|g(\cdot, \hat{x})| L_{q_{((-4, 0))}} d\hat{x})
+ |Dv(\cdot, \hat{x})| L_{q_{((-4, 0))}} |x - \hat{x}|^{-d-1+2/\ell} \chi_{\{x_d \leq N \hat{x}_d \}} \, d\hat{x}, \tag{3.9}
\]
where \( \ell \in (1, \infty) \) satisfies
\[
1/\ell + 1/q = 1 + 1/q^* \tag{3.10}
\]
and \( \ell d > 2 \) which always holds because \( \ell > 1 \) and \( d \geq 2 \). Similarly, using Young’s inequality in \( x' \) to get that for any \( x_d \in (0, 2) \),
\[
\|v(\cdot, x_d) - c\|_{L_{\infty}(Q_{\hat{x}}^2)} \leq N \int_0^2 (|g(\cdot, \hat{x}_d)| L_{q_{(Q_{\hat{x}}^2)}} |x_d - \hat{x}_d| + |G(\cdot, \hat{x}_d)| L_{q_{(Q_{\hat{x}}^2)}}
+ |Dv(\cdot, \hat{x}_d)| L_{q_{(Q_{\hat{x}}^2)}} |x_d - \hat{x}_d|^{-(d+1)/(1-1/\ell)} \chi_{\{x_d \leq N \hat{x}_d \}} \, d\hat{x}_d, \tag{3.11}
\]
where we used \( (d+1 - 2/\ell)\ell > d - 1 \) which holds true as \( \ell > 1 \). In the sequel, we discuss two cases: \( \alpha \geq 0 \) and \( \alpha \in (-1, 0) \).
**Case I:** \( \alpha \geq 0 \). We first consider the case when \( q^* < \infty \). Multiplying both sides of
by $x_d^{\alpha/q^*}$, we get
\[
x_d^{\alpha/q^*} \|v(\cdot, x_d) - c\|_{L^q(Q^*_d)}
\leq N x_d^{\alpha/q^*} \int_0^2 \left( \|g(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} |x_d - \hat{x}_d|^{-(d+1)/(1-\ell)} \chi_{\{x_d \leq N \hat{x}_d\}} \right) \hat{d} \hat{x}_d
\leq N \int_0^2 \left( \|g(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} |x_d - \hat{x}_d| + \|G(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} \right) \hat{d} \hat{x}_d
\leq N \int_0^2 \left( \|g(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} |x_d - \hat{x}_d|^{-(d+1)/(1-\ell)/q} \right) \hat{d} \hat{x}_d.
\]

Since both $x_d$ and $\hat{x}_d$ are bounded, we can apply the Hardy–Littlewood–Sobolev inequality for fractional integration in $x_d$ to obtain
\[
\|v - c\|_{L^q(Q^*_d)} \leq N \|g\|_{L^q(Q^*_d)} + N \|G\|_{L^q(Q^*_d)} + N \|Dv\|_{L^q(Q^*_d)} \tag{3.12}
\]
provided that
\[
(d + 1)(1 - 1/\ell) - \alpha/q^* + \alpha/q \leq 1 + 1/q^* - 1/q.
\]
From (3.10), we see that this condition is equivalent to (3.1).

When $q^* = \infty$, we have $\ell = p = q/(q-1)$. Thus, if the inequality (3.1) is strict, we also get (3.12) by using Hölder’s inequality. From (3.12) and the definition of $c$, we easily get (3.2).

**Case II**: $\alpha \in (-1, 0)$. For the case $q^* < \infty$, we will apply the generalized Hardy–Littlewood–Sobolev inequality (see [17] Theorem 6 or [29] Theorem 6) to conclude (3.12), which gives (3.2). Indeed, in terms of the notation in [17] Theorem 6, we choose
\[
r = q, \quad s = \frac{q}{q^* - 1}, \quad h = \frac{\alpha}{q^*}, \quad k = -\frac{\alpha}{q^*}, \quad \lambda = 2 - \frac{1}{s} - \frac{1}{r}.
\]
Then it is easily seen that the conditions in there are satisfied. Let
\[
f(\hat{x}_d) = \left( \|g(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} + \|G(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} + \|Dv(\cdot, \cdot, \hat{x}_d)\|_{L^q(Q^*_d)} \right) \chi_{(0,2)}(\hat{x}_d).
\]
As both $x_d$ and $\hat{x}_d$ are bounded, if
\[
(d + 1)(1 - 1/\ell) \leq \lambda - h - k,
\]
then by (3.11) we see that for any $g \in L_s((0,2))$
\[
\left\| \int_0^2 x_d^{\alpha/q^*} \|v(\cdot, x_d) - c\|_{L^q(Q^*_d)} g(x_d) \, dx_d \right\|
\leq N \left\| \int_0^2 f(\hat{x}_d) \hat{x}_d^h |x_d - \hat{x}_d|^{-h-k} \, dx_d \hat{x}_d \right\|
\leq N \left( \int_0^2 f^q(\hat{x}_d) \hat{x}_d^{h+1} \, dx_d \right)^{1/q} \|g\|_{L^s((0,2))}.
\]
Lemma 3.3. Let \( \alpha \in (-\infty, \infty) \), \( l_0 = \frac{d + \alpha_+ + 2}{d + \alpha_+} \) if \( d + \alpha_+ > 2 \) and \( l_0 \in (1, 2) \) be any number if \( d + \alpha_+ \leq 2 \). Then there exists a constant \( N = N(d, l_0, \alpha) \) such that
\[
\left( \int_{Q^2_d(z_0)} |u(t,x)|^{2l_0} \mu(dz) \right)^{1/l_0} \leq N \sup_{t \in (t_0 - r^2, t_0)} \int_{B^+_r(x_0)} |u(t,x)|^2 \mu(dx) + N r^2 \int_{Q^2_d(z_0)} |Du(t,x)|^2 \mu(dz),
\]
for every \( z_0 = (t_0, x_0) \in \mathbb{R}^{d+1} \), \( r > 0 \), and
\[
u \in L_\infty((t_0 - r^2, t_0); L_2(B^+_r(x_0), \mu)) \cap L_2((t_0 - r^2, t_0); W^1_2(B^+_r(x_0), \mu)).
\]
Proof. Let \( \Gamma = (t_0 - r^2, t_0) \), and let \( \kappa_0 = \frac{2}{2 - l_0} \in (1, \infty) \). By Remark 3.2 (ii) (see also [28, Theorem 2.4]) and after rescaling, we have the following weighted Sobolev inequality:
\[
\left( \int_{B^+_r(x_0)} |u(t,x)|^{\kappa_0} \mu(dx) \right)^{1/\kappa_0} \leq N r \left( \int_{B^+_r(x_0)} |Du(t,x)|^2 \mu(dx) \right)^{1/2} + N \left( \int_{B^+_r(x_0)} |u(t,x)|^2 \mu(dx) \right)^{1/2},
\]
which is (3.11) when \( \alpha < 0 \). When \( q^* = \infty \) and the inequality (3.14) is strict, we also have (3.12) by using Hölder’s inequality. The lemma is proved. \( \square \)

Remark 3.2. (i) In view of the additional factors in the \( g \) terms in (3.9) and (3.11), it is possible to relax the integrability condition on \( g \) in Lemma 3.1 we only need \( g \in L_q(Q^2_d, \mu) \), where \( q \in (1, q) \) satisfies
\[
(d + 2 + \alpha_+)/q \leq 2 + (d + 2 + \alpha_+)/q^*.
\]
when \( d \geq 2 \). However, this will not be used in the proofs of our main results.
(ii) In the time-independent case, (3.9) is not needed. Therefore, with a minor modification of the proof, we also have the embedding:
\[
\|u\|_{L_{q^*}(B^+_2, \mu)} \leq N \|u\|_{W^1_q(B^+_2, \mu)}
\]
for all \( u \in W^1_q(B^+_2, \mu) \) with \( q, q^* \in (1, \infty) \) satisfying
\[
(d + \alpha_+)/q \leq 1 + (d + \alpha_+)/q^*.
\]
The result still holds when \( q > d + \alpha_+ \) and \( q^* = \infty \). See [16, Theorem 6] for a different proof in a more general setting.

We also need a weighted parabolic embedding result for functions in the energy space, which will be used in the proof of Lemma 4.3 when we apply the Moser iteration.

Lemma 3.3. Let \( \alpha \in (-\infty, \infty) \), \( l_0 = \frac{d + \alpha_+ + 2}{d + \alpha_+} \) if \( d + \alpha_+ > 2 \) and \( l_0 \in (1, 2) \) be any number if \( d + \alpha_+ \leq 2 \). Then there exists a constant \( N = N(d, l_0, \alpha) \) such that
where \( N = N(d, l_0, \alpha) > 0 \). This together with Hölder’s inequality gives
\[
\int_{B_1^+(x_0)} |u(t, x)|^{2l_0} \mu(dx) \\
\leq \left( \int_{B_1^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{1-\frac{2}{l_0}} \left( \int_{B_1^+(x_0)} |u(t, x)|^{l_0} \mu(dx) \right)^{\frac{2}{l_0}} \\
\leq N \left( \sup_{t \in \Gamma} \int_{B_1^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{1-\frac{2}{l_0}} \\
\cdot \left( \int_{B_1^+(x_0)} |Du(t, x)|^2 \mu(dx) + \int_{B_1^+(x_0)} |u(t, x)|^2 \mu(dx) \right).
\]

Now by integrating with respect \( t \) on \( \Gamma \) and using Young’s inequality, we obtain
\[
\int_{Q_T^+(x_0)} |u(t, x)|^{2l_0} \mu(dz) \\
\leq N \left( \sup_{t \in \Gamma} \int_{B_1^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{l_0} + N \left( \int_{Q_T^+(x_0)} |Du(t, x)|^2 \mu(dz) \right)^{l_0}.
\]

The lemma is then proved. \( \square \)

Finally, we conclude this section with the following useful result on the existence and uniqueness of \( L_2 \)-solutions of a class of equations that are slightly more general than (1.2). The result is considered as a special case of Theorem 2.2 when \( p = 2 \), but no regularity requirements are imposed on the coefficients.

**Lemma 3.4.** Let \( \alpha \in (-1, \infty) \), \( \lambda > 0 \), and let \( (a_{ij}) \), \( a_0 \), and \( c_0 \) be measurable functions defined on \( \Omega_T \) such that (1.1) and (1.6) are satisfied. Then for each \( F \in L_2(\Omega_T, \mu)^d \) and \( f \in L_2(\Omega_T, \mu) \), there exists a unique weak solution \( u \in H^2_0(\Omega_T, \mu) \) to
\[
\begin{align*}
  \left\{ \begin{array}{l}
    x_0^\alpha (a_0(t, x)u_t + \lambda c_0(t, x)u) - D_t(x_0^\alpha [a_{ij}(t, x)D_j u - F_i]) = \sqrt{\lambda} x_0^\alpha f \\
    \lim_{x_0 \to 0^+} x_0^\alpha (a_{ij}(t, x)D_j u - F_i) = 0 
  \end{array} \right.
\end{align*}
\]  
(3.16)
in \( \Omega_T \). Moreover,
\[
\|Du\|_{L_2(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_2(\Omega_T, \mu)} \leq N \|F\|_{L_2(\Omega_T, \mu)} + N \|f\|_{L_2(\Omega_T, \mu)},
\]  
(3.17)
where \( N = N(\kappa) \).

**Proof.** We first prove the a priori estimate (3.17). Let \( u \in H^2_0(\Omega_T, \mu) \) be a weak solution of (3.16). By multiplying the equation (3.16) with \( u \) and using integration by parts and (1.1), we obtain
\[
\sup_{t \in (-\infty, T)} \int_{B^+_1} |u(t, x)|^2 \mu(dx) + \int_{\Omega_T} |Du|^2 \mu(dz) + \lambda \int_{\Omega_T} |u(z)|^2 \mu(dz) \\
\leq N \int_{\Omega_T} |F(z)||Du(z)| \mu(dz) + N \lambda^{1/2} \int_{\Omega_T} |f(z)||u(z)| \mu(dz).
\]

Then by Young’s inequality, we obtain (3.17).

From (3.17), we see that the uniqueness follows. Now, to prove the existence of solution, for each \( k \in \mathbb{N} \), let
\[
\hat{Q}_k = (-k^2, \min\{k^2, T\}) \times B_k^+.
\]  
(3.18)
We consider the equation
\[ x_d^2(a_0 u_t + \lambda c_0 u) - D_i(x_d^2(a_{ij} D_j u - F_i)) = \lambda^{1/2} x_d^2 f \quad \text{in } \hat{Q}_k \]
with the boundary conditions
\[ u = 0 \quad \text{on } \partial_{p} \hat{Q}_k \setminus \{ x_d = 0 \} \quad \text{and} \quad \lim_{x_d \to 0^+} x_d^2(a_{dj} D_j u - F_d) = 0. \]
where \( \partial_{p} \hat{Q}_k \) is the parabolic boundary of \( \hat{Q}_k \). By Galerkin's method, for each \( k \), there exists a unique weak solution \( u_k \in H^1_\lambda(\hat{Q}_k, \mu) \) to (3.19) and (3.20). By taking \( u_k = 0 \) on \( \Omega_T \setminus \hat{Q}_k \), we also have
\[ \sup_{t \in (-\infty, T)} \| u_k(t, \cdot) \|_{L^2(\mathbb{R}^d, \mu)} + \| Du_k \|_{L^2(\Omega_T, \mu)} + \lambda^{1/2} \| u_k \|_2(\Omega_T, \mu) \]
\[ \leq N \| F \|_{L^2(\Omega T, \mu)} + N \| f \|_{L^2(\Omega T, \mu)}. \]

By the weak compactness, there is a subsequence which is still denoted by \( \{ u_k \} \) and \( u \in H^1_\lambda(\Omega_T, \mu) \) such that
\[ u_k \rightharpoonup u, \quad Du_k \rightharpoonup Du \]
weakly in \( L^2(\Omega_T, \mu) \). By taking the limit in the weak formulation of solutions, it is easily seen that \( u \) is a weak solution of (1.2). The lemma is proved. \( \square \)

4. Equations with simple coefficients

Throughout this section, let \( \overline{\mu}_{ij} : \mathbb{R}_+ \to \mathbb{R}^{d \times d} \) be measurable functions, which satisfy the ellipticity and boundedness conditions: there is a constant \( \kappa \in (0, 1) \) such that
\[ \kappa |\xi|^2 \leq \overline{\mu}_{ij}(x_d) \xi_i \xi_j, \quad \text{and} \quad |\overline{\mu}_{ij}(x_d)| \leq \kappa^{-1}, \quad \forall \xi \in \mathbb{R}^d, \quad x_d \in \mathbb{R}_+. \]
Let \( \overline{a}_0, \overline{c}_0 : \mathbb{R}_+ \to \mathbb{R} \) be measurable functions satisfying
\[ \kappa \leq \overline{a}_0(x_d), \quad \overline{c}_0(x_d) \leq \kappa^{-1} \quad \text{for } x_d \in \mathbb{R}_+. \]
We study (1.2) in which the coefficients \( a_{ij} \) are replaced with \( \overline{\mu}_{ij} \). More precisely, we consider
\[ \begin{cases}
  x_d^2(\overline{a}_0(x_d) u_t + \lambda \overline{c}_0(x_d) u) - D_i(x_d^2(\overline{\mu}_{ij}(x_d) D_j u - F_i)) = \sqrt{\lambda} x_d^2 f \\
  \lim_{x_d \to 0^+} x_d^2(\overline{a}_{dj}(x_d) D_j u - F_d) = 0
\end{cases} \]
in \( \Omega_T \). The above equation is slightly different from (1.2) as there are coefficients \( \overline{\mu}_0 \) and \( \overline{c}_0 \) instead of the identity. We do not need this generality for the proofs of our main results for the divergence form equation (1.2). However, the results below for (4.3) are needed in the proofs of the main results for the non-divergence form equation (1.3) as in [10].

The main result of this section is the following theorem, which is a weak version of Theorem 2.2.

**Theorem 4.1.** Let \( \alpha \in (-1, \infty), \ p \in (1, \infty), \) and \( \lambda > 0 \). Suppose that (1.11) and (1.12) are satisfied. Then for each \( F \in L_p(\Omega T, \mu)^d \) and \( f \in L_p(\Omega T, \mu) \), there exists a unique solution \( u \in H^1_p(\Omega T, \mu) \) of (4.3). Moreover,
\[ \| Du \|_{L_p(\Omega T, \mu)} + \sqrt{\lambda} \| u \|_{L_p(\Omega T, \mu)} \leq N \| F \|_{L_p(\Omega T, \mu)} + N \| f \|_{L_p(\Omega T, \mu)}, \]
where \( N = N(d, \alpha, \kappa, p) \).
The rest of the section is devoted to the proof of this theorem. We need some preliminaries to prove it.

4.1. Lipschitz and Schauder estimates for homogeneous equations. Let $\lambda \geq 0$, $z_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$ and $r > 0$. We study \((4.3)\) in $Q^+_r(z_0)$ when $F = 0$, $f = 0$, i.e., the homogeneous parabolic equation

$$- x^j \partial_j (\varphi_0(x) u_t + \lambda \varphi_0(x) u) + D_i (x^j \partial_j (x d) D_j u) = 0$$

in $Q^+_r(z_0)$ with the homogenous conormal boundary condition

$$x^j \partial_j (x d) D_j u = 0 \quad \text{if} \quad B_r(z_0) \cap \partial \mathbb{R}^d_+ \neq \emptyset.$$

Our goal is to derive Lipschitz and Schauder estimates for homogeneous equations. We begin with the following lemma.

**Lemma 4.2** (Caccioppoli type inequality). Let $r > 0$, $z_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$, and $u \in H^1_1(Q^+_r(z_0), \mu)$ be a weak solution to \((4.5)\) and \((4.6)\). Then we have

$$\int_{Q^+_{r/2}(z_0)} |Du|^2 + \lambda |u|^2 \mu(dz) \leq N r^{-2} \int_{Q^+_r(z_0)} |u|^2 \mu(dz)$$

and

$$\int_{Q^+_{r/2}(z_0)} |u_t|^2 \mu(dz) \leq N r^{-2} \int_{Q^+_r(z_0)} (|Du|^2 + \lambda |u|^2) d\mu(dz),$$

where $N(d, \alpha, \kappa) > 0$.

**Proof.** The proof is more or less standard. For the first inequality, we test the equation with $u\zeta^2$, where $\zeta \in C^\infty_0$ is a smooth function, $\zeta = 1$ in $Q^+_{r/2}(z_0)$, and $\zeta = 0$ near the parabolic boundary $\partial \mu_r(z_0)$. For the second inequality, we test the equation with $u_t \zeta^2$, and then use the fact that $u_t$ satisfies the same equation as $u$ and the first inequality applied to $u_t$. See, for example, the proof of [7, Lemma 3.3]. We omit the details. \(\square\)

Next we prove the local boundedness of solutions of \((4.5)\) and \((4.6)\).

**Lemma 4.3** (Local boundedness estimate). Let $r > 0$, $z_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$, and $u \in H^1_1(Q^+_r(z_0), \mu)$ be a weak solution to \((4.5)\) and \((4.6)\). Then we have

$$\|u\|_{L^\infty(Q^+_{r/2}(z_0))} \leq N \left( \int_{Q^+_r(z_0)} |u(t, x)|^2 \mu(dz) \right)^{1/2},$$

where $N = N(d, \alpha, \kappa) > 0$.

**Proof.** We use the Moser iteration. For elliptic equations, similar argument was also used in [28]. By a scaling, we only need to prove the lemma when $r = 1$. For each $R, \rho \in (0, 1]$ with $\rho < R$, let $\phi \in C^\infty_0((t_0 - R^2, t_0 + R^2) \times B_r(x_0))$ be a cut-off function satisfying

$$\phi = 1 \quad \text{in} \quad Q_\rho(z_0), \quad 0 \leq \phi \leq 1, \quad \text{and} \quad |D\phi|^2 + |\partial_t \phi| \leq \frac{N(d)}{(R - \rho)^2} \quad \text{in} \quad Q_R(z_0).$$
Let $w = u_+$. For $\beta \geq 2$, using $\phi^2 w^{\beta - 1}$ as a test function for the equation \ref{eq:4.5} and using \ref{eq:4.11}, we obtain
\[
\frac{d}{dt} \int_{B_R^+(x_0)} a_0(x_d) w^{\beta} \phi^2 \mu(dx) + \frac{4\kappa(\beta - 1)}{\beta} \int_{B_R^+(x_0)} |D(w^{\beta/2})| \phi^2 \mu(dx) \\
\leq 2\beta \int_{B_R^+(x_0)} a_0(x_d) w^{\beta} |\phi_1| \mu(dx) + 4\kappa^{-1} \int_{B_R^+(x_0)} |D(w^{\beta/2})| |D\phi| w^{\beta/2} \mu(dx),
\]
where we used the fact that $\lambda \xi_0(x_d) w^{\beta - 1} \geq 0$. As $\beta \geq 2$, we have $\frac{\beta - 1}{\beta} \geq \frac{1}{2}$. It then follows that
\[
\frac{d}{dt} \int_{B_R^+(x_0)} a_0(x_d) w^{\beta} \phi^2 \mu(dx) + 2\kappa \int_{B_R^+(x_0)} |D(w^{\beta/2})| \phi^2 \mu(dx) \\
\leq 2\beta \int_{B_R^+(x_0)} a_0(x_d) w^{\beta} \phi_1 \mu(dx) + 4\kappa^{-1} \int_{B_R^+(x_0)} |D(w^{\beta/2})| |D\phi| w^{\beta/2} \mu(dx).
\]
By applying Young's inequality to the last term and then cancelling similar terms, we have
\[
\frac{d}{dt} \int_{B_R^+(x_0)} a_0(x_d) w^{\beta} \phi^2 \mu(dx) + \int_{B_R^+(x_0)} |D(w^{\beta/2})|^2 \phi^2 \mu(dx) \\
\leq N\beta \int_{B_R^+(x_0)} w^{\beta} (|\phi_1| + |D\phi|^2) \mu(dx),
\]
where $N = N(d, \kappa)$ and we used \ref{eq:4.12}. Integrating this estimate with respect to $t$ on $(t_0 - R^2, t_0)$ and using \ref{eq:4.12} again, we find that
\[
\sup_{t \in (t_0 - R^2, t_0)} \int_{B_R^+(x_0)} w^{\beta} \phi^2 \mu(dx) + R^2 \int_{Q_{\beta}(x_0)} |D(w^{\beta/2})|^2 \mu(dz) \\
\leq \frac{N(d, \kappa)\beta}{(R - \rho)^2} \int_{Q_{\beta}(x_0)} w^{\beta} \mu(dz).
\]
From this estimate and Lemma \ref{lem:5.3}, it follows that
\[
\left( \int_{Q_{\beta}(x_0)} w^{\beta_0} \mu(dz) \right)^{\frac{1}{\beta_0}} \leq \left( \frac{N}{R - \rho} \right)^{\frac{1}{\beta}} \beta^{\frac{1}{2}} \left( \int_{Q_{\beta}(x_0)} w^{\beta} \mu(dz) \right)^{\frac{1}{\beta}}. \tag{4.7}
\]
We now choose a sequence of radii
\[
\begin{align*}
    r_0 &= 1, & r_{k+1} &= \frac{r_k + 1/2}{2},
\end{align*}
\]
and a sequence of exponents
\[
\beta_0 = 2, \quad \beta_{k+1} = \beta_k l_0, \quad k = 0, 1, 2, \ldots,
\]
such that
\[
\lim_{k \to \infty} r_k = \frac{1}{2}, \quad \lim_{k \to \infty} \beta_k = \infty, \quad \text{and} \quad r_k - r_{k+1} = \frac{1}{2k+2}, \quad k = 0, 1, 2, \ldots
\]
By applying \ref{eq:4.7} with $R = r_k, \rho = r_{k+1} < R$, and $\beta = \beta_k$, we have
\[
\left( \int_{Q_{r_{k+1}}(x_0)} w^{\beta_{k+1}} \mu(dz) \right)^{\frac{1}{\beta_{k+1}}} \leq \left( 4N \right)^{\frac{1}{\beta_k}} 2^{2\beta_k} \beta_k^{\frac{1}{2}} \left( \int_{Q_{r_k}(x_0)} w^{\beta_k} \mu(dz) \right)^{\frac{1}{\beta_k}}. 
\]
By iterating this estimate, we obtain
\[
\left( \int_{Q_{k+1}^{+}(z_0)} w^{2k+1} \mu(dz) \right)^{1/(2k+1)} \leq M_k \left( \int_{Q_1^{+}(z_0)} w^2 \mu(dz) \right)^{1/2},
\] (4.8)
where
\[
M_k = (4N)^{\sum_{j=0}^{k} 2/\beta_j} 2^{\sum_{j=0}^{k} 2j/\beta_j} \prod_{j=0}^{k} \beta_j^{1/\beta_j}.
\]
As
\[
\sum_{j=0}^{\infty} 2/\beta_j < \infty, \quad \sum_{j=0}^{\infty} 2j/\beta_j < \infty, \quad \text{and} \quad \prod_{j=0}^{\infty} \beta_j^{1/\beta_j} < \infty,
\]
we conclude that \( \{M_k\}_k \) is convergent. Therefore, by sending \( k \to \infty \), we deduce from (4.8) that
\[
\|u_+\|_{L_\infty(Q_{1/2}^{+}(z_0))} \leq N \left( \int_{Q_1^{+}(z_0)} u_+^2(t,x) \mu(dz) \right)^{1/2}.
\]
With the same argument, we can get a similar estimate for \( u_- = \max\{-u,0\} \).

Hence,
\[
\|u\|_{L_\infty(Q_{1/2}^{+}(z_0))} \leq N \left( \int_{Q_1^{+}(z_0)} |u(t,x)|^2 \mu(dz) \right)^{1/2}.
\]

The lemma is proved. \( \square \)

We recall that for \( \beta \in (0,1] \) and each parabolic cylinder \( Q \subset \mathbb{R}^{d+1} \), the \( \beta \)-Hölder semi-norm of a function \( f \) in \( Q \) is defined as
\[
[f]_{C^{\beta/2,\beta}(Q)} = \sup_{(t,x),(s,y) \in Q} \frac{|f(t,x) - f(s,y)|}{|t-s|^{\beta/2} + |x-y|^{\beta}}.
\]
The following proposition is the key step of the proof.

**Proposition 4.4.** Let \( q \in (1,2] \), \( r > 0 \), \( z_0 \in \mathbb{R}^{d+1}_+ \), and \( u \in H_r^1(Q_{1/2}^{+}(z_0), \mu) \) be a weak solution to (4.5)–(4.10). Then we have
\[
\|Du\|_{L_\infty(Q_{1/2}^{+}(z_0))} + \sqrt{\lambda} \|u\|_{L_\infty(Q_{1/2}^{+}(z_0))} \leq N \left( \int_{Q_1^{+}(z_0)} (|Du|^q + \lambda^{q/2} |u|^q) \mu(dz) \right)^{1/q}
\] (4.9)
and
\[
[D_\alpha u]_{C^{1/2,1}(Q_{1/2}^{+}(z_0))} + [Du]_{C^{1/2,1}(Q_{1/2}^{+}(z_0))} + \sqrt{\lambda} \|u\|_{C^{1/2,1}(Q_{1/2}^{+}(z_0))} \leq Nr^{-1} \left( \int_{Q_1^{+}(z_0)} (|Du|^q + \lambda^{q/2} |u|^q) \mu(dz) \right)^{1/q},
\] (4.10)
where \( \mathcal{U} = \pi_0(x_0)D_j u \) and \( N = N(d,\alpha,\kappa,q) \).

**Proof.** First of all, whenever the lemma is proved for \( q = 2 \), the case \( q \in (1,2] \) follows by a standard iteration. See, for example, [13] pp. 80–82). Therefore, we only consider the case when \( q = 2 \). As before, we may assume that \( r = 1 \). The
bound of $\|u\|_{L_\infty(Q_{1/2}(z_0))}$ follows from Lemma 4.13. Since $D_{x'}u$ and $u_t$ satisfy the same equation as $u$, from Lemmas 4.13 again we have
\[
\|D_{x'}u\|_{L_\infty(Q_{1/2}(z_0))} \leq N \left( \int_{Q_{1/3}(z_0)} |D_{x'}u|^2 \, \mu(dz) \right)^{1/2}
\]
and
\[
u_t(u)_{L_\infty(Q_{1/2}(z_0))} \leq N \left( \int_{Q_{1/3}(z_0)} |u_t|^2 \, \mu(dz) \right)^{1/2}.
\]
To make this rigorous, we need to use the finite-difference quotient and pass to the limit. These together with Lemma 4.2 give
\[
\|D_{x'}u\|_{L_\infty(Q_{1/2}(z_0))} + \|u_t\|_{L_\infty(Q_{1/2}(z_0))} \leq N \left( \int_{Q_{1/3}(z_0)} (|Du|^2 + \lambda |u|^2) \, \mu(dz) \right)^{1/2}.
\]
Moreover, again from Lemma 4.2 we also have for any $i, j = 0, 1, 2, \ldots$ satisfying $i + j \geq 1$,
\[
\int_{Q_{1/2}(z_0)} \left( |\partial_i^j D_{x'}u|^2 + |\partial_i^j D_{x'}Du|^2 \right) \mu(dz) \leq N \int_{Q_{1/2}(z_0)} (|Du|^2 + \lambda |u|^2) \, \mu(dz),
\]
where $N = N(d, \kappa, i, j)$. Next we estimate $D_d u$. We first consider the boundary estimate and, without loss of generality, we take $z_0 = 0$. We use a bootstrap argument. Since $\mathcal{U} = \mathcal{D}_{x'}(x_d) D_{x'}u$, from the equation we have
\[
D_d(x_0^d \mathcal{U}) = x_0^d \left( \bar{a}_0(x_d) u_t + \lambda \bar{c}_0(x_d) u - \sum_{i=1}^{d-1} D_i(\bar{a}_{ij} D_{x'}u) \right).
\]
By using the boundary condition and Hölder’s inequality, we get for any $z \in Q_1^+$,
\[
x_0^d |\mathcal{U}| \leq N \int_0^{x_d} s^\alpha \left( |u_t(z', s)| + \lambda |u(z', s)| + |DD_{x'}u(z', s)| \right) \, ds \leq N \left( \int_0^{x_d} s^\alpha |u_t(z', s)|^2 + \lambda^2 |u(z', s)|^2 + |DD_{x'}u(z', s)|^2 \right)^{1/2} \left( \int_0^{x_d} s^\alpha \, ds \right)^{1/2}.
\]
Thus, when $x_d \in (0, 1/2]$, by the Sobolev embedding in the $z'$ variables, 4.12, and Lemma 4.2, for an integer $k \geq (d + 1)/4$,
\[
x_0^d |\mathcal{U}| \leq N \left( \int_0^{1/2} s^\alpha |u_t(z', s)|^2 + \lambda^2 |u(z', s)|^2 + |DD_{x'}u(z', s)|^2 \right)^{1/2} \left( \int_0^{x_d} s^\alpha \, ds \right)^{1/2} 
\]
\[
\leq N \left( \int_0^{1/2} s^\alpha \left( |u_t(z', s)|^2 W_{2, k}^{2k}(Q_{1/2}) + \lambda^2 |u(z', s)|^2 W_{2, k}^{2k}(Q_{1/2}) \right) ds \right)^{1/2} \left( \int_0^{x_d} s^\alpha \, ds \right)^{1/2} 
\]
\[
\leq N \left( \int_{Q_1^+} (|Du|^2 + \lambda |u|^2) \, \mu(dz) \right)^{1/2} x_0^d (\alpha + 1)^{1/2},
\]
which implies that

$$|U| \leq N \left( \int_{Q^{1}_{1/2}} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{\frac{1}{2}} \leq N \left( \int_{Q^{1}_{1/2}} (|Du|^{2+\alpha} + \lambda|D_x u|^2) \mu(dz) \right)^{\frac{1}{2}} x_d^{-(1-\alpha)/2} \ \text{in} \ Q^{+}_{1/2}. \quad (4.14)$$

This together with (4.11) gives

$$|Du| \leq N \left( \int_{Q^{1}_{1/2}} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{\frac{1}{2}} x_d^{-(1-\alpha)/2} \ \text{in} \ Q^{+}_{1/2}. \quad (4.15)$$

Since $D_x u$ satisfies the same equation, by a covering argument and Lemma 1.2, we have

$$|DD_{x'} u| \leq N \left( \int_{Q^{1}_{1/2}} (|DD_{x'} u|^2 + \lambda|D_{x'} u|^2) \mu(dz) \right)^{\frac{1}{2}} x_d^{-(1-\alpha)/2} \ \text{in} \ Q^{+}_{1/2}. \quad (4.16)$$

Now we plug (4.11) and (4.15) into (4.13) and use Lemmas 4.3 and 4.2 to get

$$|U| \leq N x_d^{-\alpha} \frac{1}{\alpha} \left( \int_{Q^{1}_{1/2}} (|Du|^{2+\alpha} + \lambda|u|^2) \mu(dz) \right)^{\frac{1}{2}} \ \text{in} \ Q^{+}_{1/2},$$

which improves (4.14). Repeating this procedure, in finite many steps, we get

$$|U| \leq N x_d \left( \int_{Q^{1}_{1/2}} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{\frac{1}{2}} \ \text{in} \ Q^{+}_{1/2}, \quad (4.17)$$

and therefore

$$|Du| \leq N \left( \int_{Q^{1}_{1/2}} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{\frac{1}{2}} \ \text{in} \ Q^{+}_{1/2}, \quad (4.18)$$

which gives (4.9) in this case.

In the interior case when $x_{0d} \geq 2r = 2$, the coefficients $\tilde{a}_{ij}(x_d) = x_d^2 \tilde{a}_{ij}(x_d)$ are nondegenerate in $Q^{2/3}_{2/3}(z_0)$ and independent of $x'$. By using the standard energy estimate (cf. [7, Lemma 3.5]), we also have

$$|Du| \leq N \left( \int_{Q^{1}_{1/2}} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{\frac{1}{2}} \ \text{in} \ Q^{1}_{1/2}(z_0). \quad (4.19)$$

Since in $Q^{1}_{1/2}(z_0)$, $x_d \sim x_{0d}$ so that $\mu(dz) \sim x_{0d}^{2} dz$, we also obtain (4.19) in the interior case. Moreover, (4.11) still holds in this case. When $x_{0d} \in (0, 2)$, (4.9) follows from a covering argument and the doubling property of $\mu$.

It remains to prove (4.19). By using (4.17) and (4.11), we obtain the bound of the third term on the left-hand side of (4.19). Since $D_x u$ and $u_t$ satisfy the same equation as $u$, from (4.9), (4.12), and Lemma 4.2, we have

$$\|DD_{x'} u\|_{L_{\infty}(Q^{1}_{1/2}(z_0))} + \|Du_t\|_{L_{\infty}(Q^{1}_{1/2}(z_0))} \leq N \left( \int_{Q^{1}_{1/2}(z_0)} (|DD_{x'} u|^2 + |Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} \leq N \left( \int_{Q^{1}_{1/2}(z_0)} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2}, \quad (4.18)$$
which yields
\[
[D_x u]_{C^{1/2}(Q_{r/2}^+(z_0))} + \|u\|_{L_\infty(Q_{r/2}^+(z_0))} + \|D_x u\|_{L_\infty(Q_{r/2}^+(z_0))}
\leq N \left( \int_{Q_{r/2}^+(z_0)} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2}.
\]

To estimate \(D_d u\), we again discuss two cases. In the boundary case when \(z_0 = 0\), from the equation we have
\[
D_d u = \tilde{a}_0 u_t + \lambda \tilde{\alpha}_0 u - \sum_{i=1}^{d-1} \tilde{a}_{ij} D_{ij} u - \alpha x^{-1}_d u,
\]
which together with (4.11), (4.16), (4.18), and Lemma 4.3 gives
\[
\|D_d u\|_{L_\infty(Q_{r/2}^+(z_0))} \leq N \left( \int_{Q_{r/2}^+(z_0)} (|D_u|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2}.
\]
In the interior case when \(x_0 \geq 2\), by (4.11), (4.16), (4.18), and Lemma 4.3 we still get (4.20). This completes the proof of (4.10) and thus the proposition. \(\square\)

From Lemma 3.4 and Proposition 4.4 we obtain the following solution decomposition.

**Proposition 4.5.** Let \(z_0 \in \overline{\Omega_T}\) and \(r > 0\). Suppose that \(F \in L^2(Q_{2r}^+(z_0), \mu)^d\), \(f \in L^2(Q_{2r}^+(z_0), \mu)\), and \(u \in \mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)\) is a weak solution of (4.3) in \(Q_{2r}^+(z_0)\). Then we can write
\[
u(t, x) = v(t, x) + w(t, x) \quad \text{in } Q_{2r}^+(z_0),
\]
where \(v\) and \(w\) are functions in \(\mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)\) and satisfy
\[
\int_{Q_{2r}^+(z_0)} |V|^2 \mu(dz) \leq N \int_{Q_{2r}^+(z_0)} \left( |F|^2 + |f|^2 \right) \mu(dz)
\]
and
\[
\|W\|_{L_\infty(Q_{2r}^+(z_0))} \leq N \int_{Q_{2r}^+(z_0)} |U|^2 \mu(dz) + N \int_{Q_{2r}^+(z_0)} \left( |F|^2 + |f|^2 \right) \mu(dz),
\]
where \(N = N(d, \kappa, \alpha)\) and
\[
V = |Dv| + \lambda^{1/2} |v|, \quad W = |Dw| + \lambda^{1/2} |w|, \quad U = |Du| + \lambda^{1/2} |u|.
\]

**Proof.** Let \(v \in \mathcal{H}_2^1(\Omega_T, \mu)\) be a weak solution of the equation
\[
x_d^2 (\overline{\alpha}_0(x_d)v_t + \lambda \overline{\alpha}_0(x_d)v) - D_i \left( x_d^2 (\overline{\alpha}_j(x_d)D_j v - F_i(z)\chi_{Q_{2r}^+(z_0)}(z)) \right)
= \lambda^{1/2} x_d^2 f(z)\chi_{Q_{2r}^+(z_0)}(z)
\]
in \(\Omega_T\) with the boundary condition
\[
\lim_{x_d \to 0^+} x_d^2 (\overline{\alpha}_j(x_d)D_j v - F_d(z)\chi_{Q_{2r}^+(z_0)}(z)) = 0.
\]
Then (4.21) follows from Lemma 3.4. Now let \(w = u - v\) so that \(w \in \mathcal{H}_2^1(Q_{2r}^+(z_0))\) is a weak solution of
\[
x_d^2 (\overline{\alpha}_0(x_d)w_t + \lambda \overline{\alpha}_0(x_d)w) - D_i \left( x_d^2 \overline{\alpha}_j(x_d)D_j w \right) = 0 \quad \text{in } Q_{2r}^+(z_0)
\]
with the boundary condition
\[
\lim_{x_d \to 0^+} x_d^2 \overline{\alpha}_j(x_d) D_j w = 0 \quad \text{if } B_{2r}(x_0) \cap \partial \Omega_+^d \neq \emptyset.
\]
By Proposition 4.3 and the triangle inequality, we get (4.22). The proof of the proposition is completed. 

4.2. Proof of Theorem 4.1. We are now ready to give the proof of Theorem 4.1.

Proof. When \( p = 2 \), Theorem 4.1 follows from Lemma 3.4. Therefore, we only need to consider the cases when \( p \in (2, \infty) \) and \( p \in (1, 2) \).

Case I: \( p \in (2, \infty) \). Let \( u \in H^1_{2, \text{loc}}(\Omega_T, \mu) \) be a weak solution of (4.23). It follows from Proposition 4.5 that for every \( z_0 \in \bar{\Omega}_T \) and \( \eta > 0 \), we have the decomposition

\[
u = v + w \quad \text{in} \quad Q^+_{\eta}(z_0),
\]

where \( v \) and \( w \) satisfy (4.21) and (4.22). Then (4.4) follows from the standard real variable argument. See, for example, [8]. We omit the details.

By (4.4), the uniqueness of solutions follows. Hence, it remains to prove the existence of the solution. Recall the definition (3.18). For \( k = 1, 2, \ldots \), let \( F^{(k)} = F(z)\chi_{Q_k}(z) \). Then \( F^{(k)} \in L^2(\Omega_T, \mu)^d \cap L_p(\Omega_T, \mu)^d \) and by the dominated convergence theorem, \( F^{(k)} \to F \) in \( L_p(\Omega_T, \mu) \) as \( k \to \infty \). Similarly, we define \( \{f^{(k)}\} \subset L^2(\Omega_T, \mu) \cap L_p(\Omega_T, \mu) \). Let \( u^{(k)} \in H^1_{2, \text{loc}}(\Omega_T, \mu) \) be the weak solution of the equation (4.3) with \( F^{(k)} \) and \( f^{(k)} \) in place of \( F \) and \( f \), respectively. The existence of \( u^{(k)} \) follows from Lemma 3.4. By the estimate (4.4), we have \( u^{(k)} \in H^1_{2, \text{loc}}(\Omega_T, \mu) \). Moreover, by the strong convergence of \( \{F^{(k)}\} \) and \( \{f^{(k)}\} \) in \( L_p(\Omega_T, \mu) \), we infer that \( \{u^{(k)}\} \) is a Cauchy sequence in \( H^1_{2, \text{loc}}(\Omega_T, \mu) \). Let \( u \in H^1_{2, \text{loc}}(\Omega_T, \mu) \) be its limit. Then, by passing to the limit in the weak formulation of solutions, it is easily seen that \( u \) is a solution to the equation (4.3).

Case II: \( p \in (1, 2) \). We use a duality argument. We first prove the estimate (4.4). Let \( q = p/(p - 1) \in (2, \infty) \) and let \( G \in L^q(\Omega_T, \mu)^d \) and \( g \in L^q(\Omega_T, \mu) \). We consider the adjoint problem in \( \mathbb{R} \times \mathbb{R}^d_+ \)

\[
\begin{cases}
x^a_d(-\tilde{a}_0 v_t + \lambda \tilde{c}_0 v) - D_i \left(x^a_d(\sigma_{ji}(x_d)D_j v - G_i \chi_{(-\infty, T)})\right) = \lambda^{1/2} x^a_d g \chi_{(-\infty, T)}, \\
\lim_{x_d \to 0^+} x^a_d (\sigma_{ji} D_j v - G_i \chi) = 0.
\end{cases}
\]

(4.23)

By Case I, there exists unique solution \( v \in H^1_q(\mathbb{R} \times \mathbb{R}^d_+, \mu) \) of the above equation, which satisfies

\[
\int_{\mathbb{R} \times \mathbb{R}^d_+} (|Dv|^q + \lambda^{q/2} |v|^q) \mu(dz) \leq N \int_{\Omega_T} (|G|^q + |g|^q) \mu(dz).
\]

(4.24)

Moreover, by the uniqueness of solutions, we have \( v = 0 \) for \( t \geq T \). It follows from the equations (4.3) and (4.22) that

\[
\int_{\Omega_T} (G \cdot \nabla u + \lambda^{1/2} gu) \mu(dz) = \int_{\Omega_T} (F \cdot \nabla v + \lambda^{1/2} f v) \mu(dz).
\]
Therefore, by Hölder’s inequality and (4.24),
\[
\left| \int_{\Omega_T} (G \cdot \nabla u + \lambda^{1/2} g u) \mu(dz) \right| \\
\leq \|F\|_{L_p(\Omega, \mu)} \|\nabla v\|_{L^q(\Omega_T, \mu)} + \lambda^{1/2} \|f\|_{L_p(\Omega_T, \mu)} \|v\|_{L^q(\Omega_T, \mu)} \\
\leq N \left( \|F\|_{L_p(\Omega, \mu)} + \|f\|_{L_p(\Omega_T, \mu)} \right) \left( \|G\|_{L_q(\Omega_T, \mu)} + \|g\|_{L_q(\Omega_T, \mu)} \right).
\]
From this last estimate and as G and g are arbitrary, we obtain (4.3).

It now remains to prove the existence of solution \( u \in H^1_\mu(\Omega_T, \mu) \). We proceed slightly differently from Case I and follow the argument in [9, Section 8]. For \( i = 1, 2, \ldots, d \) and \( k = 1, 2, \ldots, \), let
\[
F_i^{(k)} = \max(-k, \min(k, F_i)) \chi_{\hat{Q}_k}.
\]
Then \( F^{(k)} \in L^2(\Omega_T, \mu)^d \cap L^p(\Omega_T, \mu)^d \) and by the dominated convergence theorem, \( F^{(k)} \to F \) in \( L^p(\Omega_T, \mu) \) as \( k \to \infty \). Similarly, we define \( f^{(k)} \) in place of \( F^{(k)} \) and \( f^{(k)} \) in place of \( F \) and \( f \), respectively. As in Case I, it suffices to prove that \( u^{(k)} \in H^1_\mu(\Omega_T, \mu) \). Let us fix a \( k \in \mathbb{N} \). Because \( \mu \) is a doubling measure, there exists \( N_0 = N_0(\alpha, d) > 0 \) such that
\[
\mu(Q_{2r}) \leq N_0 \mu(Q_r), \quad \forall \ r > 0.
\]
Since \( u^{(k)} \in H^1_\mu(\Omega_T, \mu) \), by Hölder’s inequality,
\[
\|u^{(k)}\|_{L_p(\Omega_T \setminus \hat{Q}_k)} + \|Du^{(k)}\|_{L_p(\Omega_T \setminus \hat{Q}_k)} < \infty.
\]
Therefore, it remains to prove that \( \|u^{(k)}\|_{L_p(\Omega_T \setminus \hat{Q}_k)} < \infty \). To this end, for \( j \geq 0 \), let \( \eta_j \) be such that
\[
\eta_j \equiv 0 \quad \text{in } \hat{Q}_{2^j k}, \quad \eta_j \equiv 1 \quad \text{outside } \hat{Q}_{2^{j+1} k},
\]
and \( |D\eta_j| \leq C_0 2^{-j}, \quad |(\eta_j)_t| \leq C_0 2^{-2j} \), where \( C_0 \) is independent of \( j \). Observe that the supports of \( F^{(k)} \) and \( f^{(k)} \) are in \( \hat{Q}_k \), while the supports of \( \eta_j \) are all outside \( \hat{Q}_k \). Consequently, \( \eta_j F_i^{(k)} = \eta_j f^{(k)} = F_i^{(k)} D_i \eta_j = 0 \) for every \( i = 1, 2, \ldots, d \) and \( j = 0, 1, \ldots, \). Because of this, a simple calculation reveals that \( w^{(k,l)} := u^{(k)} \eta_l \in H^1_\mu(\Omega_T, \mu) \) is a weak solution of
\[
\begin{align*}
\left\{ x_d^\alpha (\eta_{0,l} w^{(k,l)}_i + \lambda \eta_{0,l} w^{(k,l)}_i) - D_i \left( x_d^\alpha (\eta_{ij,l} D_j w^{(k,l)}_i - F_i^{(k,l)}) \right) \right. \\
\left. \lim_{x_d \to 0^+} x_d^\alpha (\eta_{ij,l} D_j w^{(k,l)} - F_d^{(k,l)}) = 0 \right. \\
\end{align*}
\]
in \( \Omega_T \), where
\[
F_i^{(k,l)} = u^{(k)} \eta_l D_i \eta_l, \quad F_l^{(k,l)} = \lambda^{-1/2} (u^{(k)}(\eta_l)_t - \eta_{ij,l} D_j u^{(k,l)} D_i \eta_l).
\]
Now, by applying the estimate (3.17) to the above equation of \( w^{(k,l)} \), we have
\[
\|Du^{(k,l)}\|_{L^2(\Omega_T, \mu)} + \sqrt{\lambda} \|w^{(k,l)}\|_{L^2(\Omega_T, \mu)} \\
\leq N \|F^{(k,l)}\|_{L^2(\Omega_T, \mu)} + N \|f^{(k,l)}\|_{L^2(\Omega_T, \mu)},
\]
which implies that

$$\|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \leq N \left(2^{\gamma} \|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \lambda^{-1/2} 2^{-j} \|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \right)$$

$$+ \lambda^{-1/2} 2^{-j} \|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})}$$

$$\leq C 2^{-j} \left(\|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \right)$$

for every $j \geq 1$, where $C$ also depends on $\lambda$, but is independent of $j$. By iterating the last estimate, we obtain

$$\|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \leq C^j 2^{-j(j-1)/2} \left(\|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \right).$$

Finally, by Hölder’s inequality, (4.25), and (4.27), we have

$$\|D u^{(k)}\|_{L^p(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^p(\hat{Q}_{2j+1})} \leq (\mu(\hat{Q}_{2j+1}))^{\frac{1}{p} - \frac{1}{2}} \left(\|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \right)$$

$$\leq N^j (\mu(\hat{Q}_{2j+1}))^{\frac{1}{p} - \frac{1}{2}} C^j 2^{\frac{j(j-1)}{2}} \left(\|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} \right).$$

Hence,

$$\|D u^{(k)}\|_{L^p(\Omega_T \setminus \hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^p(\Omega_T \setminus \hat{Q}_{2j+1})} = \sum_{j=1}^{\infty} \left(\|D u^{(k)}\|_{L^p(\hat{Q}_{2j+1})} + \sqrt{\lambda}\|u^{(k)}\|_{L^p(\hat{Q}_{2j+1})} \right)$$

$$\leq N \|D u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} + N \sqrt{\lambda}\|u^{(k)}\|_{L^2(\hat{Q}_{2j+1})} < \infty.$$
where \( v \) and \( w \) are functions in \( \mathcal{H}_v^1(Q^+_{2r}(z_0), \mu) \) that satisfy

\[
\int_{Q^+_{2r}(z_0)} |W|^2 \mu(dz) \leq N \int_{Q^+_{2r}(z_0)} |G|^2 \mu(dz) + N \int_{Q^+_{2r}(z_0)} |Dv|^q \mu(dz) \tag{5.1}
\]

and

\[
\|W\|^2_{L^\infty(Q^+_{2r}(z_0))} \leq N \int_{Q^+_{2r}(z_0)} |U|^2 \mu(dz) + N \int_{Q^+_{2r}(z_0)} |G|^2 \mu(dz),
\]

where

\[ V = |Dv| + \sqrt{\lambda}|v|, \quad W = |Dw| + \sqrt{\lambda}|w|, \quad U = |Du| + \sqrt{\lambda}|u|, \]

and \( N = N(d, \alpha, \kappa, q) \).

**Proof.** For \( i = 1, 2, \ldots, d \), let

\[ b_i(t, x) = \chi_{Q^+_{2r}(z_0)}(z)(a_{ij}(t, x) - [a_{ij}]_{2r, Q_d}(x_d))D_j u(t, x) - F_i(z)\chi_{Q^+_{2r}(z_0)}(z), \]

where \([a_{ij}]_{2r, Q_d}(x_d)\) is defined in Assumption 2.1. Observe that \( b_i \in L_2(\Omega_r, \mu) \). In particular, if \( r \in (0, R_0/2) \), it follows from Hölder’s inequality and Assumption 2.1 (\( \gamma_0, R_0 \)) that

\[
\int_{Q^+_{2r}(z_0)} |b(z)|^2 \mu(dz) \leq \left( \int_{Q^+_{2r}(z_0)} |a_{ij} - [a_{ij}]_{2r, Q_d}(x_d)|^2 \mu(dz) \right)^{\frac{2}{q}} \left( \int_{Q^+_{2r}(z_0)} |Du|^q \mu(dz) \right)^{\frac{2}{q}} + \int_{Q^+_{2r}(z_0)} |F|^2 \mu(dz).
\]

On the other hand, when \( r > R_0/2 \), as \( \text{spt}(u) \subset (s - (R_0r_0)^2, s + (R_0r_0)^2) \times \mathbb{R}_+^d \) and by the boundedness of \((a_{ij})\) in \( L_2 \), we have

\[
\int_{Q^+_{2r}(z_0)} |b(z)|^2 \mu(dz) \leq N(\kappa) \left( \int_{Q^+_{2r}(z_0)} \chi_{(s - (R_0r_0)^2, s + (R_0r_0)^2)}(t) \mu(dz) \right)^{\frac{q-2}{q}} \left( \int_{Q^+_{2r}(z_0)} |Du|^q \mu(dz) \right)^{\frac{2}{q}} + \int_{Q^+_{2r}(z_0)} |F|^2 \mu(dz)
\]

\[
\leq N \left( \frac{R_0r_0}{r} \right)^{\frac{2(q-2)}{q}} \left( \int_{Q^+_{2r}(z_0)} |Du|^q \mu(dz) \right)^{\frac{2}{q}} + N \int_{Q^+_{2r}(z_0)} |F|^2 \mu(dz)
\]

\[
\leq Nr_0 \left( \int_{Q^+_{2r}(z_0)} |Du|^q \mu(dz) \right)^{\frac{2}{q}} + N \int_{Q^+_{2r}(z_0)} |F|^2 \mu(dz).
\]
with some

Then we apply Proposition 4.4 to conclude that

\[ \lambda > (1.2). \]

We suppose that

Proof of Theorem 2.2. It suffices to consider the case

From this and (5.4), we obtain (5.2).

We first prove the a-priori estimate (2.1) for each weak solution

d sufficiently small depending on

\[ r > (1, 2) \] can be proved by using the duality argument as in the proof of Theorem 4.1. We suppose that \( \lambda > 0 \). Assume for a moment that

\[ \text{spt}(u) \subset (s - (R_0r_0)^2, s + (R_0r_0)^2) \times \mathbb{R}^d_+ \]

with some \( s \in (-\infty, T) \) and \( r_0 \in (0, 1) \). We claim that (2.1) holds if \( \gamma_0 \) and \( r_0 \) are sufficiently small depending on \( d, \alpha, \kappa, \) and \( p \). Let \( q \in (2, p) \) be fixed. Applying Proposition 5.1 for each \( r > 0 \) and \( z_0 \in \Gamma_T \), we can write

\[ u(t, x) = v(t, x) + w(t, x) \text{ in } Q_{2r}^+(z_0), \]
where \( v \) and \( w \) satisfy \((5.2)\) and \((5.2)\). Then it follows from the standard real variable argument, see [8] for example, that
\[
\|Du\|_{L^p(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L^p(\Omega_T, \mu)} \leq N(\gamma_0^{1-2/q} + r_0^{2-4/q}) \|Du\|_{L^p(\Omega_T, \mu)} + N\|F\|_{L^p(\Omega_T, \mu)} + N\|f\|_{L^p(\Omega_T, \mu)}
\]
for \( N = N(d, \alpha, \kappa, p) \). From this, and by choosing \( \gamma_0 \) and \( r_0 \) sufficiently small so that \( N(\gamma_0^{1-2/q} + r_0^{2-4/q}) < 1/2 \), we obtain \((2.1)\).

We now remove the additional assumption that \( \text{spt}(u) \subset (s - (R_0r_0)^2, s + (R_0r_0)^2) \times \mathbb{R}^d \) by using a partition of unity argument. Let
\[
\xi = \xi(t) \in C_0^\infty(-(R_0r_0)^2, (R_0r_0)^2)
\]
be a standard non-negative cut-off function satisfying
\[
\int_{\mathbb{R}} \xi^p(s) \, ds = 1, \quad \int_{\mathbb{R}} |\xi'(s)|^p \, ds \leq \frac{N}{(R_0r_0)^{2p}}.
\]
(5.5)
For any \( s \in (-\infty, \infty) \), let \( u^{(s)}(z) = u(z)\xi(t - s) \) for \( z = (t, x) \in \Omega_T \). Then \( u^{(s)} \in H^{1, p}_\mu(\Omega_T, \mu) \) is a weak solution of
\[
\begin{cases}
 x^2_d(u_i^{(s)}) + \lambda u^{(s)} - D_i(x^2_d(a_{ij}D_ju^{(s)} - F^{(s)}_i)) = \lambda^{1/2} x^2_d f^{(s)}
\end{cases}
\]
in \( \Omega_T \), where
\[
F^{(s)}(z) = \xi(t - s)F(z), \quad f^{(s)}(z) = \xi(t - s)f(z) + \lambda^{-1/2} \xi'(t - s)u(z).
\]
As \( \text{spt}(u^{(s)}) \subset (s - (R_0r_0)^2, s + (R_0r_0)^2) \times \mathbb{R}^d \), we can apply the estimate we just proved and infer that
\[
\|Du^{(s)}\|_{L^p(\Omega_T, \mu)} + \sqrt{\lambda} \|u^{(s)}\|_{L^p(\Omega_T, \mu)} \leq N\|F^{(s)}\|_{L^p(\Omega_T, \mu)} + N\|f^{(s)}\|_{L^p(\Omega_T, \mu)}.
\]
Integrating with respect to \( s \), we get
\[
\int_{\mathbb{R}} \left( \|Du^{(s)}\|^p_{L^p(\Omega_T, \mu)} + \lambda^{p/2} \|u^{(s)}\|^p_{L^p(\Omega_T, \mu)} \right) \, ds \leq N \int_{\mathbb{R}} \left( \|F^{(s)}\|^p_{L^p(\Omega_T, \mu)} + \|f^{(s)}\|^p_{L^p(\Omega_T, \mu)} \right) \, ds.
\]
(5.6)
It follows from the Fubini theorem and \((5.5)\) that
\[
\int_{\mathbb{R}} \|Du^{(s)}\|^p_{L^p(\Omega_T, \mu)} \, ds = \int_{\Omega_T} \int_{\mathbb{R}} |Du(z)|^p \xi^p(t - s) \, ds \, dz = \|Du\|^p_{L^p(\Omega_T, \mu)}.
\]
Similarly,
\[
\int_{\mathbb{R}} \|u^{(s)}\|^p_{L^p(\Omega_T, \mu)} \, ds = \|u\|^p_{L^p(\Omega_T, \mu)}, \quad \int_{\mathbb{R}} \|F^{(s)}\|^p_{L^p(\Omega_T, \mu)} \, ds = \|F\|^p_{L^p(\Omega_T, \mu)}.
\]
Since \( r_0 \) depends only on \( d, \alpha, \kappa, \) and \( p \), from the definition of \( f^{(s)} \), \((5.5)\), and the Fubini theorem, we have
\[
\left( \int_{\mathbb{R}} \|f^{(s)}\|^p_{L^p(\Omega, \mu)} \right)^{1/p} \leq N\|f\|_{L^p(\Omega_T, \mu)} + N R_0^2 \lambda^{1/2} \|u\|_{L^p(\Omega_T, \mu)}.
\]
for $N = N(d, \alpha, \kappa, p)$. Collecting these estimates, we infer from (5.6) that

$$
\|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_p(\Omega_T, \mu)} \\
\leq N \|F\|_{L_p(\Omega_T, \mu)} + N \|f\|_{L_p(\Omega_T, \mu)} + N R_0^{-2} \lambda^{-1/2} \|u\|_{L_p(\Omega_T, \mu)}
$$

with $N = N(d, \alpha, \kappa, p)$. Now we choose $\lambda_0 = 2N$. For $\lambda \geq \lambda_0 R_0^{-2}$, we have $N R_0^{-2} \lambda^{-1/2} \leq \sqrt{\lambda}/2$, and therefore

$$
\|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_p(\Omega_T, \mu)} \\
\leq N \|F\|_{L_p(\Omega_T, \mu)} + N \|f\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_p(\Omega_T, \mu)},
$$

which yields (1.2).

Finally, the solvability of solution $u \in H^1_p(\Omega_T, \mu)$ can be obtained by the method of continuity using the solvability of the equation

$$
\begin{align*}
x_d(u_t + \lambda u) - D_i \left(x_d^a(a_{ij} D_j (u \eta) - \tilde{F}_i)\right) &= \lambda^{1/2} x_d^a f \\
\lim_{x_d \to 0^+} x_d^a (D_d u - F_d) &= 0
\end{align*}
$$

in $\Omega_T$, which is proved in Theorem 4.1. The proof is now completed. □

### 5.2. Proof of Corollary 2.3

We adapt an idea in [22]. Let $\eta \in C^\infty_0((-4, 4) \times B_2)$ be such that $\eta \equiv 1$ on $Q_1$. A direct calculation yields that $u \eta \in H^1_{p_0}(\Omega_0, \mu)$ satisfies

$$
\begin{align*}
x_d(x_d^a ((u \eta)_t + \lambda u \eta) - D_i \left(x_d^a (a_{ij} D_j (u \eta) - \tilde{F}_i)\right)) &= x_d^a f \\
\lim_{x_d \to 0^+} x_d^a (a_{ij} D_j (u \eta) - \tilde{F}_d) &= 0
\end{align*}
$$

in $(-4, 0) \times \mathbb{R}^d_+$ (5.7) with the zero initial condition $(u \eta)(-4, \cdot) = 0$, where

$$
\tilde{F}_i = F_i \eta - a_{ij} u D_j \eta, \quad \tilde{f} = f \eta + \lambda u \eta + u_{tij} - a_{ij} D_i (D_j u - F_i),
$$

and $\lambda > \lambda_0 R_0^{-2}$.

Let $q = p/(p-1)$, and $G = (G_1, \ldots, G_d), g \in C^\infty_0(Q^+_1)$ satisfying

$$
\|G\|_{L_q(Q^+_1, \mu)} = \|g\|_{L_q(Q^+_1, \mu)} = 1.
$$

By Theorem 2.2 there is a weak solution $v \in H^1_q((-4, 0) \times \mathbb{R}^d_+, \mu)$ to

$$
\begin{align*}
-x_d^a ((v_t + \lambda v) - D_i \left(x_d^a (a_{ij} D_j (v \eta) - G_i)\right)) &= \sqrt{\lambda} x_d^a g \\
\lim_{x_d \to 0^+} (x_d^a (a_{ij} D_j (v \eta) - G_d)) &= 0
\end{align*}
$$

in $(-4, 0) \times \mathbb{R}^d_+$ (5.8) with the zero terminal condition $v(0, \cdot) = 0$ and it satisfies

$$
\sqrt{\lambda} \|v\|_{L_q((-4, 0) \times \mathbb{R}^d_+, \mu)} + \|Dv\|_{L_q((-4, 0) \times \mathbb{R}^d_+, \mu)} \leq N. \tag{5.9}
$$

Testing (5.7) and (5.8) with $v$ and $u \eta$, respectively, we get

$$
\int_{Q^+_1} (\nabla u \cdot G + \sqrt{\lambda} u g) \, d\mu(z) = \int_{Q^+_1} (\nabla v \cdot F + v f) \, d\mu(z),
$$

which together with Hölder’s inequality gives

$$
\begin{align*}
\int_{Q^+_1} (\nabla u \cdot G + \sqrt{\lambda} u g) \, d\mu(z) \\
&\leq \|Dv\|_{L_q(Q^+_2, \mu)} \|F\|_{L_p(Q^+_2, \mu)} + \|v\|_{L_q(Q^+_2, \mu)} \|f\|_{L_p(Q^+_2, \mu)}, \tag{5.10}
\end{align*}
$$

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where $q^* = p^*/(p^* - 1)$. From (5.3), we see that $v \in \mathcal{H}^1_\lambda(Q^+_z; \mu)$ satisfies
\[
\begin{cases}
-x^\nu_t - D_i \left( x^\nu_\alpha \left( a_j D_j v - G_i \right) \right) = x^\nu_\beta \tilde{g} \\
\lim_{x^\nu_\alpha \to 0^+} x^\nu_\alpha \left( a_j D_j v - G_d \right) = 0
\end{cases}
\text{ in } Q_2,
\]
where $\tilde{g} = -\lambda v + \sqrt{\lambda} g$. When $\alpha \neq 0$, by (2.13)-(2.14), $q^*$ satisfies the condition (3.1) in Lemma 3.1. Then by using Lemma 3.1 and (5.9), we get
\[
\|v\|_{L^q(Q^+_z; \mu)} \leq N \|v\|_{L^q(Q^+_z; \mu)} + N \|Du\|_{L^q(Q^+_z; \mu)} + N \|v_t\|_{H^{-1}_\lambda(Q^+_z; \mu)} \leq N + N \|G\|_{L^q(Q^+_z; \mu)} + N \|\tilde{g}\|_{L^q(Q^+_z; \mu)} \leq N \sqrt{\lambda},
\]
(5.11)
When $\alpha = 0$, by the usual unweighted parabolic Sobolev embedding, we still get (5.11). It then follows from (5.10), (5.9), (5.11), and the arbitrariness of $G$ and $g$ that
\[
\begin{align*}
\|Du\|_{L^p(Q^+_z; \mu)} + \sqrt{\lambda} \|u\|_{L^p(Q^+_z; \mu)} & \leq N \|\tilde{F}\|_{L^p(Q^+_z; \mu)} + N \sqrt{\lambda} \|\tilde{f}\|_{L^p(Q^+_z; \mu)} \\
& \leq N \|F\|_{L^p(Q^+_z; \mu)} + N \|u\|_{L^p(Q^+_z; \mu)} + N \sqrt{\lambda} \|f\|_{L^p(Q^+_z; \mu)} \\
& + N \sqrt{\lambda} (\lambda + 1) \|u\|_{L^p(Q^+_z; \mu)} + N \sqrt{\lambda} \|Du\|_{L^p(Q^+_z; \mu)},
\end{align*}
\]
(5.12)
where $N$ is independent of $\lambda$. Recall that $p^* < p$. Finally, from (5.12) we conclude (2.21) by using Hölder’s inequality and a standard iteration argument for a sufficiently large $\lambda$. See, for example, [14, pp. 80–82]. The corollary is proved.

5.3. Proof of Theorem 2.24
It follows from Corollary 2.3 and Proposition 4.2 that for any $q_0 \in (1, 2)$, if $v \in \mathcal{H}^1_{p_0}(Q^+_z(z_0), \mu)$ is a weak solution of (4.3)-(4.6), we have
\[
[D_x v]_{C^{1,2;1}(Q^+_z(z_0))} + |\nabla u|_{C^{1,2;1}(Q^+_z(z_0))} + \sqrt{\lambda} |u|_{C^{1,2;1}(Q^+_z(z_0))} \leq N \tau^{-1} \left( \int_{Q^+_z(z_0)} |Du|^{q_0} + \lambda^{\theta/2} |v|^{q_0} \mu(dz) \right)^{1/q_0},
\]
(5.13)
where $V = \partial_d (x_d D_j v).$ By using (5.13), Theorem 2.2 and a decomposition argument as in the proof of Proposition 5.1, we have the following mean oscillation estimate: if $\text{spt}(u) \subset (s - (R_0 r_0)^2, s + (R_0 r_0)^2) \times \mathbb{R}^d_+$ for some $s \in \mathbb{R}$, then for any $\tau \leq 30$ and $z_0 \in \Omega_\tau$,
\[
\begin{align*}
\int_{Q^+_z(z_0)} |D_x u - (D_x u)_{Q^+_z(z_0)}| & + |u - (u)_{Q^+_z(z_0)}| + \sqrt{\lambda} |u - (u)_{Q^+_z(z_0)}| \mu(dz) \\
& \leq N \tau^{-d+2+\alpha_+ \gamma_0} \left( \int_{Q^+_z(z_0)} |Du|^{q_0} \mu(dz) \right)^{1/q_0} \\
& + N \tau^{-d+2+\alpha_+ \gamma_0} \left( \int_{Q^+_z(z_0)} |F|^{q_0} + |\sqrt{\lambda} f|^{q_0} \mu(dz) \right)^{1/q_0} \\
& + N \tau \left( \int_{Q^+_z(z_0)} |Du|^{q_0} + \lambda^{\theta/2} |v|^{q_0} \mu(dz) \right)^{1/q_0} \\
& + N \tau^{-d+2+\alpha_+ \gamma_0} \left( \int_{Q^+_z(z_0)} |Du|^{q_0 \nu_2} \mu(dz) \right)^{1/q_0},
\end{align*}
\]
where \( \nu_1 \in (1, \infty) \), \( \nu_2 = \nu_1/(\nu_1 - 1) \), and \( \mathcal{U} = a_{dj} D_j u \). Here we used the notation
\[
(g)_{Q^+_{\omega}(z_0)} = \int_{Q^+_{\omega}(z_0)} g(z) \mu(dz)
\]
for a function \( g \) defined in \( Q^+_{\omega}(z_0) \). The a-priori estimate \( \text{(2.6)} \) then follows from the mean oscillation estimate, the reverse Hölder’s inequality for \( A_p \) weights, the weighted mixed-norm Fefferman–Stein type theorems on sharp functions, and the weighted mixed-norm Hardy–Littlewood maximal function theorem. See, for instance, Corollary 2.6, 2.7, and Section 7 of \( \textbf{[9]} \) for details. The solvability in weighted mixed-norm Sobolev spaces then follows from the estimate \( \text{(2.6)} \) and an approximate argument by using the solvability result in Theorem \( \textbf{2.2} \). We omit the details and refer the reader to \( \textbf{[9]} \) Section 8).

5.4. Proof of Corollary 2.9. We first assume that
\[
(d + 3 + \alpha_+)/p_0 < 1 + (d + 3 + \alpha_+)/p.
\] (5.14)

Let \( \eta \in C_0^\infty((-4, 4) \times B_2) \) be an even function with respect to \( x_d \) such that \( \eta \equiv 1 \) on \( Q_1 \). A direct calculation yields that \( w := u\eta \in W^{1,2}_{p_0}(\Omega_0, \mu) \) satisfies
\[
\begin{aligned}
\begin{cases}
\alpha w_t - a_{ij} D_{ij} w - \frac{\alpha}{x_d} a_{dd} D_d w + \lambda c_0 w = \tilde{f} \\
\text{in } (-4, 0) \times \mathbb{R}^d
\end{cases}
\end{aligned}
\]
with the zero initial condition \( w(-4, \cdot) = 0 \), where
\[
\tilde{f} = f \eta + (a_0 \eta_t - a_{ij} D_{ij} \eta - a a_{dd} D_d \eta / x_d + (\lambda - 1) c_0 \eta) u \\
- (a_{ij} + a_{ji}) D_j \eta D_j u,
\]
\( \lambda > \lambda_0 R_0^{-2} \) is a fixed number, and \( \lambda_0 \) is the constant from Theorem \( \textbf{2.7} \) with \( q = p \) and \( \omega \equiv K = 1 \). It follows from Lemma \( \textbf{(5.11)} \) and \( \textbf{(5.14)} \) that
\[
\|u\|_{L_p(Q^+_{\omega}, \mu)} + \|D u\|_{L_p(Q^+_{\omega}, \mu)} \leq N \|u\|_{W^{1,2}_{p_0}(Q^+_{\omega}, \mu)}.
\] (5.16)

By using Theorem \( \textbf{2.7} \) with \( q = p \) and \( \omega \equiv K = 1 \), \( \text{(5.15)} \) has a unique solution \( v \in W^{1,2}_{p_0}(\Omega_0, \mu) \). Since \( \tilde{f} \) is compactly supported, as in Case II of the proof of Theorem \( \textbf{4.1} \) we have \( v \in W^{1,2}_{p_0}(\Omega_0, \mu) \). Now by the uniqueness of \( W^{1,2}_{p_0}(\Omega_0, \mu) \)-solutions to \( \text{(5.15)} \), we conclude that \( u\eta = v \in W^{1,2}_{1,2}(\Omega_0, \mu) \). Furthermore, by Theorem \( \textbf{2.7} \) and \( \textbf{(5.16)} \),
\[
\|u\|_{W^{1,2}_{p_0}(Q^+_{\omega}, \mu)} \leq N \|	ilde{f}\|_{W^{1,2}_{p_0}(Q^+_{\omega}, \mu)} \leq N \|f\|_{W^{1,2}_{p_0}(Q^+_{\omega}, \mu)} + N \|u\|_{W^{1,2}_{p_0}(Q^+_{\omega}, \mu)};
\]
which, together with Hölder’s inequality and a standard iteration argument, yields \( \text{(2.7)} \) under the additional condition \( \textbf{(5.14)} \).

Finally, for general \( p \in (p_0, \infty) \), the result follows from an induction argument by taking a sequence of increasing exponents \( p_j, j = 1, \ldots, n \), such that \( p_n = p \) and
\[
(d + 3 + \alpha_+)/p_{j-1} < 1 + (d + 3 + \alpha_+)/p_j
\]
for \( j = 1, \ldots, n \).
REFERENCES

[1] A. Banerjee and N. Garofalo. Monotonicity of generalized frequencies and the strong unique continuation property for fractional parabolic equations. *Adv. Math.*, 336:149-241, 2018.

[2] D. Cao, T. Mengesha, T. Phan. Weighted $W^{1,p}$-estimates for weak solutions of degenerate and singular elliptic equations, *Indiana University Mathematics Journal*, 67(6):2225–2277, 2018.

[3] C. L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–260, 2007.

[4] M. Chipot, D. Kinderlehrer, and G. Vergara-Caffarelli. Smoothness of linear laminates. *Arch. Ration. Mech. Anal.*, 96(1):81–96, 1986.

[5] Hongjie Dong. Solvability of parabolic equations in divergence form with partially BMO coefficients. *J. Funct. Anal.*, 258(7):2145–2172, 2010.

[6] Hongjie Dong and Doyoon Kim. Elliptic equations in divergence form with partially BMO coefficients. *Arch. Ration. Mech. Anal.*, 196(1):25–70, 2010.

[7] Hongjie Dong, Doyoon Kim. Parabolic and elliptic systems in divergence form with variably partially BMO coefficients. *SIAM J. Math. Anal.* 43(3):1075–1098, 2011.

[8] Hongjie Dong, Doyoon Kim. Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains. *J. Funct. Anal.* 261(11):3279–3327, 2011.

[9] Hongjie Dong and Doyoon Kim. On $L^p$-estimates for elliptic and parabolic equations with $A_p$ weights. *Trans. Amer. Math. Soc.* 370(7):5081–5130, 2018.

[10] Hongjie Dong and Tuoc Phan. Weighted mixed-norm $L^p$-estimates for elliptic and parabolic equations in non-divergence form with singular degenerate coefficients. *Rev. Mat. Iberoam.*, accepted. [arXiv:1811.06393]

[11] Hongjie Dong and Tuoc Phan. Regularity theory for parabolic equations with singular degenerate coefficients, [arXiv:1802.09294]

[12] E. B. Fabes, C. E. Kenig, R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[13] Paul M. N. Feehan and Camelia A. Pop. Schauder a priori estimates and regularity of solutions to boundary-degenerate elliptic linear second-order partial differential equations. *J. Differential Equations*, 256(3):895–956, 2014.

[14] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993, ISBN 37643-28797.

[15] V. V. Grushin. On a class of elliptic pseudodifferential operators degenerate on a submanifold. (Russian) *Mat. Sb. (N.S.)* 84 (126) 1971 163–195.

[16] P. Hajłasz. Sobolev spaces on an arbitrary metric space, *Potential Anal.*, 5:403–415, 1996.

[17] G. H. Hardy and J. E. Littlewood. Some properties of fractional integrals. I. *Math. Z.* 27(1):565–606, 1928.

[18] Byungsoo Kang and Hyunseok Kim. On $L^p$-resolvent estimates for second-order elliptic equations in divergence form. *Potential Anal.* 50(1):107–133, 2019.

[19] Doyoon Kim and N. V. Krylov. Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others. *SIAM J. Math. Anal.*, 39(2):489–506, 2007.

[20] Doyoon Kim and N. V. Krylov. Parabolic equations with measurable coefficients. *Potential Anal.*, 26(4):345–361, 2007.

[21] Doyoon Kim, Hongjie Dong, and Hong Zhang. Neumann problem for non-divergence elliptic and parabolic equations with BMO coefficients in weighted Sobolev spaces. *Discrete Contin. Dyn. Syst.* 36(9):4895–4914, 2016.

[22] Doyoon Kim, Seungjin Ryu, and Kwan Woo. Parabolic equations with unbounded lower-order coefficients in Sobolev spaces with mixed norms. preprint, 2020.

[23] Hyunseok Kim and Tai-Peng Tsai. Existence, Uniqueness, and Regularity Results for Elliptic Equations with Drift Terms in Critical Weak Spaces. *SIAM J. Math. Anal.* 52(2):1146–1191, 2020.

[24] N. V. Krylov. Parabolic and elliptic equations with VMO coefficients. *Comm. Partial Differential Equations*, 32(1-3):453–475, 2007.

[25] N. V. Krylov. Second-order elliptic equations with variably partially VMO coefficients. *J. Funct. Anal.*, 257(6):1695–1712, 2009.

[26] N. V. Krylov. Elliptic equations with VMO $a, b \in L_d$, and $c \in L_{d/2}$. [arXiv:2004.01778]
[27] K. Nyström and O. Sande. Extension properties and boundary estimates for a fractional heat operator. *Nonlinear Anal.*, 140:29–37, 2016.

[28] Yannick Sire, Susanna Terracini, and Stefano Vita. Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions [arXiv:1904.02143](https://arxiv.org/abs/1904.02143).

[29] E. M. Stein and Guido Weiss. Fractional integrals on n-dimensional Euclidean space. *J. Math. Mech.*, 7:503–514, 1958.

[30] P. R. Stinga and J. L. Torrea, Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation. *SIAM J. Math. Anal.*, 49(5):3893–3924, 2017.

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