Binary Classification Under $\ell_0$ Attacks for General Noise Distribution

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Abstract

Adversarial examples have recently drawn considerable attention in the field of machine learning due to the fact that small perturbations in the data can result in major performance degradation. This phenomenon is usually modeled by a malicious adversary that can apply perturbations to the data in a constrained fashion, such as being bounded in a certain norm. In this paper, we study this problem when the adversary is constrained by the $\ell_0$ norm; i.e., it can perturb a certain number of coordinates in the input, but has no limit on how much it can perturb those coordinates. Due to the combinatorial nature of this setting, we need to go beyond the standard techniques in robust machine learning to address this problem. We consider a binary classification scenario where $d$ noisy data samples of the true label are provided to us after adversarial perturbations. We introduce a classification method which employs a nonlinear component called truncation, and show in an asymptotic scenario, as long as the adversary is restricted to perturb no more than $\sqrt{d}$ data samples, we can almost achieve the optimal classification error in the absence of the adversary, i.e. we can completely neutralize adversary’s effect. Surprisingly, we observe a phase transition in the sense that using a converse argument, we show that if the adversary can perturb more than $\sqrt{d}$ coordinates, no classifier can do better than a random guess.

1 Introduction

It is well-known that machine learning models are susceptible to adversarial attacks that can cause classification error. These attacks are typically in the form of a small norm-bounded perturbation to the input data that are carefully designed to incur misclassification – e.g. they can be form of an additive $\ell_p$-bounded perturbation for some $p \geq 0$ [BCM+13, SZS+14, GSS14, CW17, MMS+18].

There is an extensive body of prior work studying adversarial machine learning, most of which have focused on $\ell_2$ and $\ell_\infty$ attacks [ACW18, MGMP, BC21, BJC20]. To train models that are more robust against such attacks, adversarial training is the state-of-the-art defense method. However, the success of the current adversarial training methods is mainly based on empirical evaluations [MMS+18]. It is therefore imperative to study the fundamental limits of robust machine learning under different classification settings and attack models.

In this paper, we focus on the important case of $\ell_0$-bounded attacks that has been less investigated so far. In such attacks, given an $\ell_0$ budget $k$, an adversary can change $k$ entries of the input vector...
in an arbitrary fashion – i.e. the adversarial perturbations belong to the so-called ℓ₀ ball of radius k. In contrast with ℓₚ-balls (p ≥ 1), the ℓ₀-ball is non-convex and non-smooth. Moreover, the ℓ₀ ball contains inherent discrete (combinatorial) structures that can be exploited by both the learner and the adversary. As a result, the ℓ₀-adversarial setting bears various challenges that are absent in common ℓₚ-adversarial settings. In this regard, it has recently been shown that any piece-wise linear classifier, e.g. a feed-forward deep neural network with ReLu activations, completely fails in the ℓ₀ setting \[SSRD19\].

Perturbing only a few components of the data or signal has many real-world applications including natural language processing \[JJZS19\], malware detection \[GPM+16\], and physical attacks in object detection \[LSK19\]. There have been several prior works on ℓ₀-adversarial attacks including white-box attacks that are gradient-based, e.g. \[CW17, PMJ+16, MMDF19\], and black-box attacks based on zeroth-order optimization, e.g. \[SRBB18, CAS+20\]. Defense strategies against ℓ₀-bounded attacks have also been proposed, e.g. defenses based on randomized ablation \[LF20\] and defensive distillation \[PMW+16\]. None of the above works have studied the fundamental limits of the ℓ₀-adversarial setting theoretically. In our prior work, we have studied the ℓ₀-adversarial setting for the case of Gaussian mixture model \[DHP21\]. In this paper, we generalize our results to the case of binary classification with general noise distribution.

The goal of this paper is to characterize the optimal classifier and the corresponding robust classification error as a function of the adversary’s budget \(k\). More precisely, we focus on the binary classification setting with general but i.i.d. noise distributions, where the input is generated according to the following model: \(x_i = y\mu + z_i\), where \(y \in \{−1, 1\}\) is the true label, \(z_i\) is a zero-mean i.i.d. random noise process, and \(\mu\) is its mean vector. We seek to find the robust classification error of the optimal classifier in this setting. In other words, we would like to study “how robust” we can design a classifier given a certain budget for an ℓ₀ adversary. Specifically, we consider the asymptotic regime that the dimension of the input gets large, and ask the following fundamental question: What is the maximum adversary’s budget for which the optimal error in the absence of an adversary (standard error) can still be achieved and how does this limit scale with the input’s dimension?

The main contributions of the paper to answer the above questions are as follows.

- We prove an achievability result by introducing a classifier and characterizing its performance. Our proposed classification method finds the likelihood of each data sample, and applies truncation by removing a few of the largest and a few of the smallest values. This truncation phase effectively removes the “outliers” present in the input due to adversarial modification. We have shown in a previous work \[DHP21\] that truncation is effective to robustify against ℓ₀ attacks in a Gaussian mixture setting. The present work shows the effectiveness of this method in a much broader setting for general noise distributions.

- We prove a converse result by finding a lower bound on the optimal robust error, and show that the two bounds asymptotically match as the dimension \(d \to \infty\), hence our proposed classification method is optimally robust against such adversarial attacks. The key idea behind the converse proof is to use techniques from the optimal transport theory and studying the asymptotic behavior of the maximal coupling between the data distribution under the two labels +1 and −1. We use such a coupling to design a strategy for the adversary by making the distribution “look almost the same” under the two labels, hence removing the information about the true label.

- Surprisingly, we observe a phase transition for the optimal robust error in terms of the adversary’s budget. Roughly speaking, we observe that if the adversary’s budget is below \(\sqrt{d}\), we can asymptotically achieve the optimal standard error which corresponds to the case where there is no adversary, while if the adversary’s budget is above \(\sqrt{d}\), no classifier can do better than a random guess. In other words, we can totally compensate for the presence of the adversary as
long as its budget is below $\sqrt{d}$ and achieve a performance as if there were no adversary. On the other hand, above this threshold $\sqrt{d}$, the adversary can perturb the data in such a way that the information about the true label is lost and hence no classifier can do better than a random guess. Consequently, there is no trade-off between robustness and accuracy in this setting.

We close this section by introducing some notation. We denote the set of integers $\{1, \ldots, n\}$ by $[n]$. $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt$ denotes the complementary CDF of a standard normal distribution. $\mathcal{N}(\mu, \sigma^2)$ denotes a real-valued normal distribution with mean $\mu$ and variance $\sigma^2$. $\xrightarrow{\text{dist}}$ and $\xrightarrow{\text{prob}}$ denote convergence in distribution and convergence in probability, respectively. $X \sim p(.)$ means that the random variable $X$ has distribution $p(.)$. We use the boldface notation for vectors in the Euclidean space, e.g. $x \in \mathbb{R}^d$.

2 Problem Formulation

We consider the binary classification setting where the true label is $Y \sim \text{Unif}\{\pm 1\}$ and conditioned on a realization $y$, $d$ independent real-valued data samples $x_1^{(d)}, \ldots, x_d^{(d)}$ are generated such that $x_i^{(d)} = y\mu_d + z_i$. Here, $\mu_d \in \mathbb{R}$ is the conditional expectation of $x_i^{(d)}$ given $y = 1$ and $z_1, \ldots, z_d$ are i.i.d. samples of a zero-mean real-valued noise distribution which has a density $q(.)$. We consider a high-dimensional setting where the dimension $d \to \infty$, and $\mu_d$ can depend on the data dimension $d$. However, we assume that the noise density $q(.)$ is fixed and known. Note that since the $\ell_0$ norm is invariant under scalar multiplication, we can arbitrarily normalize the quantities, and this assumption is made without loss of generality. We denote the vector of the input data samples by $x^{(d)} = (x_i^{(d)} : i \in [d])$. Throughout this paper, the superscript $(d)$ emphasizes the dependence on the dimension $d$. However, we may drop it from the notations whenever the dimension is clear from the context. A classifier is a measurable function $C : x \mapsto \{\pm 1\}$ which predicts the true label from the input $x$. We consider the 0-1 loss $\ell(C; x, y) := 1[ C(x) \neq y]$ as a metric for discrepancy between the prediction of the classifier on the input $x$ and the true label $y$.

We assume that an adversary is allowed to perturb the input $x$ within the $\ell_0$ ball of radius $k$:

$$B_0(x^{(d)}, k) := \{x^{(d)} \in \mathbb{R}^d : \|x^{(d)} - x^{(d)}\|_0 \leq k\},$$

where $\|x^{(d)}\|_0 := \sum_{i=1}^{d} 1[x_i^{(d)} \neq 0]$. Effectively, the adversary can change at most $k$ data samples. The parameter $k$ is called the adversary’s budget. Similar to the above, whenever the dimension $d$ is clear from the context, we may denote the adversary’s perturbed data samples as $x' = (x'_i : i \in [d])$. In this setting, the robust classification error (or robust error for short) associated to a classifier $C$ is defined to be

$$\mathcal{L}_\mu^{(d)}(C, k) := \mathbb{E}_{\text{max}} \ell(C; x', y),$$

(1)

where the expectation is taken with respect to the above mentioned distribution parametrized by $d, \mu_d$, and $q$. The optimal robust classification error (or optimal robust error for short) is defined by optimizing the robust error over all possible (measurable) classifiers:

$$\mathcal{L}^{(d)}_{\mu_d, q}(k) := \inf_C \mathcal{L}_\mu^{(d)}(C, k).$$

(2)

In words, $\mathcal{L}^{(d)}_{\mu_d, q}(k)$ is the minimum error that any classifier can achieve in the presence of an adversary with an $\ell_0$ budget $k$. In other words, no classifier can obtain a robust error smaller than $\mathcal{L}^{(d)}_{\mu_d, q}(k)$ in
this setting. Whenever the problem parameters are clear from the context, we may drop them from
the notation and write \( \mathcal{L}(d) \) or \( \mathcal{L}(c, k) \), and \( \mathcal{L}^{(d)}(k) \) or \( \mathcal{L}^{*}(k) \).

In the absence of the adversary, or equivalently when \( k = 0 \), \( \mathcal{L}^{*}(0) \) reduces to the \textit{optimal standard error},
which is optimal Bayes error of estimating \( Y \) upon observing the noisy samples \( x_1, \ldots, x_d \). In
order to fix the baseline, specifically to have a meaningful asymptotic discussion as \( d \to \infty \),
we assume that \( \mu_d \) is such that the optimal standard error \( \mathcal{L}^{(d)}_{\mu_d q}(0) \) remains constant as \( d \to \infty \).
As we will see later (see Theorem 2 in Section 3.1), this is achieved when \( \mu_d = c/\sqrt{d} \) for some \( c > 0 \). Motivated by
this, we study the setting where \( \mu_d = c/\sqrt{d} \) for some constant \( c > 0 \) throughout this paper.
When \( \mu_d = c/\sqrt{d} \) when \( c < 0 \), similar results still hold after substituting \( c \) with \( |c| \).

3 Main Results

In order to prove our main results, we need the following assumptions on the noise distribution \( q(.) \).
We will show later (see Section 3.4) that all of these assumptions are satisfied for a large class of
distributions, including the exponential family of distributions with polynomial exponents, e.g. the
normal distribution.

Assumption 1. We have \( q(z) > 0 \) for all \( z \in \mathbb{R} \), \( q(.) \) is three times continuously differentiable,
and
\[
\int_{-\infty}^{\infty} q'(z)dz = \int_{-\infty}^{\infty} q''(z)dz = 0,
\]
where \( q'(.) \) and \( q''(.) \) denote the first and second derivatives of \( q(.) \). Furthermore, the location family
of distributions
\[
q(z; \theta) := q(z - \theta),
\]
parameterized by \( \theta \in \mathbb{R} \) has well-defined and finite Fisher information \( \{I_q(\theta)\}_{\theta \in \mathbb{R}} \).

The Fisher information of the parametric family of distributions \( q(z; \theta) \) where \( z, \theta \in \mathbb{R} \) is defined
to be
\[
I_q(\theta) := \int \left( \frac{\partial}{\partial \theta} \log q(z; \theta) \right)^2 q(z; \theta)dz.
\]
See, for instance, [LC06] for more details. Since \( q(z; \theta) = q(z - \theta) \) is a location family, it turns out
that \( I_q(\theta) \) is independent of \( \theta \). The common value, which we denote by \( I_q \) by an abuse of notation,
is given by
\[
I_q := \int_{-\infty}^{\infty} \left( \frac{q'(z)}{q(z)} \right)^2 dz.
\]

Assumption 2. There exists \( \zeta > 0 \) such that
\[
\mathbb{E}_{Z \sim q(.)} \left[ \sup_{t \in [Z-\zeta,Z+\zeta]} \left| \frac{d^3}{dt^3} \log q(t) \right| \right] < \infty.
\]

Assumption 3. There exist \( \zeta > 0 \) such that
\[
\mathbb{E}_{Z \sim q(.)} \left[ \sup_{t \in [Z-\zeta,Z+\zeta]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 \right] < \infty.
\]
Assumption 4. There exist constants $\gamma > 0$ and $C_4 > 0$ such that
\[
\lim_{d \to \infty} P \left( \max_{1 \leq i \leq d} \left| \frac{d}{dz} \log q(Z_i) \right| > C_4 (\log d)^\gamma \right) = 0,
\]
where $Z_i$ are i.i.d. with distribution $q(.)$.

The following theorem formalizes the phase transition we discussed previously, i.e. if adversary’s budget is orderwise below $\sqrt{d}$, we can totally compensate for its presence, while if adversary’s budget is orderwise above $\sqrt{d}$, no classifier can do better than a random guess. As we discussed previously, we assume that $\mu_d = c/\sqrt{d}$ for a constant $c > 0$ to ensure that the standard error is asymptotically constant (see Theorem 2 in Section 3.1).

Theorem 1. Assume that $\mu_d = c/\sqrt{d}$ for some constant $c > 0$, and the assumptions 1-4 are satisfied for the noise density $q(.)$. Then, if $k_d$ is a sequence of adversary’s $t_0$ budget, then we have

1. If $\limsup_{d \to \infty} \log_d k_d < 1/2$, there exists a sequence of classifiers $C_{k_d}^{(d)}$ such that
\[
\limsup_{d \to \infty} \mathcal{L}_{\mu_d,q}(C_{k_d}^{(d)}, k_d) - \mathcal{L}_{\mu_d,q}^{*}(0) = 0.
\]
In other words, the excess risk of this sequence of classifiers as compared to the optimal standard error (when there is no adversary) converges to zero.

2. If $\liminf_{d \to \infty} \log_d k_d > 1/2$, we have
\[
\liminf_{d \to \infty} \mathcal{L}_{\mu_d,q}^{*}(k_d) \geq 1/2.
\]
In other words, no classifier can asymptotically do better than a random guess.

The proof of this result, which is given in Appendix D, essentially follows from Theorems 3 and 4. More precisely, in Section 3.2, we prove an achievability result by introducing a sequence of robust classifiers in the sub-$\sqrt{d}$ regime (first part of the theorem), while in Section 3.3, we prove a converse result by introducing a strategy for the adversary in the super-$\sqrt{d}$ regime which perturbs the data in such a way that the information about the true label is asymptotically removed (second part of the theorem). See [DHP22] for a complete proof of Theorem 1.

3.1 Asymptotic Standard Error

Recall that in the absence of the adversary, or equivalently when adversary’s budget $k$ is zero, the optimal robust error $\mathcal{L}_{\mu_d,q}^{*}(0)$ reduces to the optimal Bayes error of estimating $Y$ upon observing the noisy samples $x_1, \ldots, x_d$. With an abuse of notation, we write $\mathcal{L}_{\mu_d,q}^{*}(d)$ (or $\mathcal{L}^*$ for short) for this optimal Bayes error. Our goal in this section is to find the appropriate scaling of $\mu_d$ with $d$ such that $\mathcal{L}_{\mu_d,q}^{*}(d)$ converges to a constant as $d \to \infty$.

In order to characterize $\mathcal{L}^*$, note that since there is no adversary, and the prior on $Y$ is uniform, the optimal Bayes classifier is the maximum likelihood estimator that computes the likelihood
\[
\sum_{i=1}^{d} \tilde{x}_i^{(d)} \quad \text{where} \quad \tilde{x}_i^{(d)} := \log \frac{q(x_i^{(d)} - \mu_d)}{q(x_i^{(d)} + \mu_d)},
\]
where $x_i^{(d)}$ are the noisy samples.
and returns the estimate \( \hat{y} \) of \( y \) as

\[
\hat{y} = \begin{cases} 
1 & \sum_{i=1}^{d} \tilde{x}_i^{(d)} > 0 \\
-1 & \text{otherwise.}
\end{cases} \quad (9)
\]

The following Theorem 2 shows that if \( \mu_d = c/\sqrt{d} \), then the optimal Bayes error converges to a constant. The proof of Theorem 2 is given in Appendix A.

**Theorem 2.** Assume that assumptions 1 and 2 are satisfied for the noise density \( q(.) \). Then, if \( \mu_d = \frac{c}{\sqrt{d}} \) for some constant \( c > 0 \), we have

\[
\lim_{d \to \infty} \mathcal{L}^{(d)}_{\mu_d, q} = \bar{\Phi}(c/\sqrt{d}).
\]

Furthermore, in this case, as \( d \to \infty \), conditioned on \( Y = +1 \), the log likelihood \( \sum_{i=1}^{d} \tilde{x}_i^{(d)} \) converges in distribution to a normal \( \mathcal{N}(2c^2 I_q, 4c^2 I_q) \) where \( I_q \) was defined in (5) and is the Fisher information associated to the location family defined in (4). Moreover, conditioned on \( Y = -1 \), \( \sum_{i=1}^{d} \tilde{x}_i^{(d)} \) converges in distribution to a normal \( \mathcal{N}(-2c^2 I_q, 4c^2 I_q) \).

**Remark 1.** As we will see in Appendix A, if \( c < 0 \), we need to replace \( \bar{\Phi}(c/\sqrt{d}) \) by \( \bar{\Phi}(|c|/\sqrt{d}) \) in the above theorem.

### 3.2 Achievability: Upper Bound on the Optimal Robust Error

In this section, we introduce a classifier and study its robustness against \( \ell_0 \) adversarial perturbations. Recall that if \( k \) is the adversary’s budget, the input to the classifier is \( x' = (x'_1, \ldots, x'_d) \) which is different from the original sequence \( x_1, \ldots, x_d \) in at most \( k \) coordinates. Recall from Section 3.1 that in the absence of the adversary, the optimal Bayes classifier is the maximum likelihood estimator based on \( \sum_{i=1}^{d} \tilde{x}_i \), as was defined in (8). Motivated by this, we define

\[
\tilde{x}_i^{(d)} := \log \frac{q(x'_i^{(d)} - \mu_d)}{q(x'_i^{(d)} + \mu_d)}.
\]

Note that if \( \tilde{x}^{(d)} \) denotes the vector \( \{ \tilde{x}_i^{(d)} : i \in [d] \} \), since \( \| \tilde{x}^{(d)} - x^{(d)} \|_0 \leq k \), we have

\[
\| \tilde{x}^{(d)} - \tilde{x}^{(d)} \|_0 \leq k. \quad (11)
\]

We define the truncated classifier \( \mathcal{C}^{(d)}_k \) as follows. Given a vector \( u = (u_i : i \in [d]) \in \mathbb{R}^d \) and an integer \( k \geq 0 \), we define the truncated summation \( \text{TSum}_k(u) \) to be the summation of coordinates in \( u \) except for the top and bottom \( k \) coordinates. More precisely, let \( s = (s_i : i \in [d]) = \text{sort}(u) \) be obtained by sorting the coordinates of \( u \) in descending order. We then define

\[
\text{TSum}_k(u) := \sum_{i=k+1}^{d-k} s_i.
\]

(12)

When \( k = 0 \), this indeed reduces to the normal summation. Motivated by (11), we replace \( \sum_{i=1}^{d} \tilde{x}_i^{(d)} \) with its robustified version \( \text{TSum}_k(\sum_{i=1}^{d} \tilde{x}_i^{(d)}) \) and define

\[
\mathcal{C}^{(d)}_k(x^{(d)}) := \begin{cases} 
+1 & \text{TSum}_k(\tilde{x}^{(d)}) > 0 \\
-1 & \text{otherwise.}
\end{cases}
\]

(13)
This method essentially removes the “outliers” introduced by the adversary into the data.

The following theorem shows that this classifier is asymptotically robust against adversarial attacks with $k_d$ budget of at most $\sqrt{d}$. A matching lower bound is provided in Section 3.3. The proof of Theorem 3 below is given in Appendix B.

**Theorem 3.** Assume that Assumptions 1-4 are satisfied for the noise density $q(\cdot)$, and $\mu_d = c/\sqrt{d}$ for some $c > 0$. Then if $k_d$ is a sequence of adversary’s budgets so that $k_d < d^{1/2 - \epsilon}$ for some $\epsilon > 0$, then we have

$$\limsup_{d \to \infty} L^{(d)}_{\mu_d, q}(C_{k_d}) \leq \Phi(c\sqrt{I_q}).$$

In particular, we have

$$\limsup_{d \to \infty} L^{(d)}_{\mu_d, q}(C_{k_d}) - L^{* (d)}_{\mu_d, q} = 0.$$ (15)

Note that $L^{* (d)}_{\mu_d, q}$, as was defined in Section 3.1 above, is the optimal Bayes error in an ideal scenario where there is no adversary, and $L^{(d)}_{\mu_d, q}(C_{k_d}) - L^{* (d)}_{\mu_d, q}$ is the excess error of our truncated classifier with respect to this ideal scenario. In fact, (15) implies that our truncated classifier is asymptotically optimal in the specified regime of adversary’s budget. The truncated classifier manages to compensate for the presence of the adversary, and performs as if there is no adversary.

**Remark 2.** As we will see in Appendix B, if $c < 0$, we need to replace $\Phi(c\sqrt{I_q})$ by $\Phi(|c|\sqrt{I_q})$ in the above theorem.

### 3.3 Converse: Lower Bound on the Optimal Robust Error

In this section, we provide a lower bound on the optimal robust error. In Section 3.2, we observed that roughly speaking, if adversary’s budget is below $\sqrt{d}$, we can asymptotically compensate for its effect and recover the Bayes optimal error, as if no adversary is present. In this section, we show that, roughly speaking, if adversary’s budget is above $\sqrt{d}$, no classifier can asymptotically do better than a random guess, resulting in a robust error of $1/2$. We do this by introducing an attack strategy for the adversary. In this strategy, the adversary with a sufficiently large budget, perturbs the input data in such a way that all the information about the true label $Y$ is lost, resulting in a perturbed data which has a vanishing correlation with the true label. The proof of Theorem 4 below is given in Appendix C.

**Theorem 4.** Assume that Assumptions 1 and 3 are satisfied for the noise density $q(\cdot)$, and $\mu_d = c/\sqrt{d}$ for some $c > 0$. Then, if $k_d$ is a sequence of adversary’s budgets so that $k_d > d^{1/2 + \epsilon}$ for some $\epsilon > 0$, we have

$$\liminf_{d \to \infty} L^{* (d)}_{\mu_d, q}(k_d) \geq 1/2.$$  

### 3.4 Exponential Family of Distributions

In this section, we show that the Assumptions 1-4 are all satisfied for a large class of distributions, namely the exponential family of noise distributions of the form

$$q(z) = \frac{\exp(\psi(z))}{A},$$ (16)

where

$$\psi(z) = -a_{2n}z^{2n} + a_{2n-1}z^{2n-1} + \ldots + a_1z + a_0,$$

is a polynomial in $z$ with even degree $2n > 0$ such that $a_{2n} > 0$. Here, $A := \int_{-\infty}^{\infty} \psi(z)dz$ is the normalizing constant. Note that since $\psi(\cdot)$ has an even degree with a negative leading coefficient, we have $A < \infty$. 

Theorem 5. Assumptions 1-4 are all satisfied for the density \( q(.) \) of the form (16).

4 Conclusion

We studied the binary classification problem in the presence of an adversary constrained by the \( \ell_0 \) norm. We introduced a robust classification method which employs truncation on the log likelihood. We showed that this classification method can asymptotically compensate for the presence of the adversary as long as adversary’s budget is orderwise below \( \sqrt{d} \). Moreover, we showed a phase transition through a converse argument in the sense that no classifier can asymptotically do better than a random guess if adversary’s budget is orderwise above \( \sqrt{d} \).

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A Proof of Theorem 2

Proof of Theorem 2. To simplify the discussion and to avoid considering multiple cases, it turns out that it is more convenient to assume that the constant $c$ can be negative. Therefore, for the rest of the proof, we assume that $\mu_d = c/\sqrt{d}$ where $c \in \mathbb{R}$ and $c \neq 0$. Note that even for negative $c$, the maximum likelihood estimator in (9) is still the optimal Bayes estimator. Therefore

$$
\mathcal{L}^{(d)}_{\mu_d,q} = \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} x_i \leq 0 \mid Y = +1 \right) + \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} x_i > 0 \mid Y = -1 \right) = \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} \log \frac{q(z_i)}{q(z_i + 2\mu)} \leq 0 \right) + \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} \log \frac{q(z_i - 2\mu_d)}{q(z_i)} > 0 \right). \tag{17}\n$$

We focus on the first term. Using $\mu_d = c/\sqrt{d}$, we may write

$$
\sum_{i=1}^{d} \log \frac{q(z_i)}{q(z_i + 2\mu)} = \frac{c}{\sqrt{d}} \sum_{i=1}^{d} \mu_d \log \frac{q(z_i)}{q(z_i + 2\mu_d)}. \tag{18}\n$$

From Assumption 1, we know that $q(.)$ is positive everywhere and three times continuously differentiable, hence $\log q(.)$ is three times continuously differentiable. Therefore, writing the Taylor expansion, we get

$$
\log q(z_i + 2\mu_d) = \log q(z_i) + 2\mu_d \frac{d}{dz} \log q(z_i) + \frac{4\mu_d^2}{2} \frac{d^2}{dz^2} \log q(z_i) + \frac{8\mu_d^3}{6} \frac{d^3}{dz^3} \log q(z_i + \epsilon_i),
$$

where $|\epsilon_i| < 2|\mu_d| = 2|c|/\sqrt{d}$. Note that $\epsilon_i$ is random and only depends on $z_i$. Substituting this into (18), we get

$$
\sum_{i=1}^{d} \log \frac{q(z_i)}{q(z_i + 2\mu)} = \frac{2c}{\sqrt{d}} \sum_{i=1}^{d} \frac{d}{dz} \log q(z_i) =: T_1
$$

$$
-\frac{2c\mu_d}{\sqrt{d}} \sum_{i=1}^{d} \frac{d^2}{dz^2} \log q(z_i) =: T_2
$$

$$
-\frac{4c\mu_d^2}{3\sqrt{d}} \sum_{i=1}^{d} \frac{d^3}{dz^3} \log q(z_i + \epsilon_i) =: T_3
$$

We now study each of the three terms individually.

$T_1$: Denoting $\frac{d}{dz} q(z)$ by $q'(z)$, we have

$$
\mathbb{E}_{z \sim q(.)} \left[ \frac{d}{dz} \log q(z) \right] = \mathbb{E}_{z \sim q(.)} \left[ \frac{q'(z)}{q(z)} \right] = \int_{-\infty}^{\infty} q'(z) q(z) dz = \int_{-\infty}^{\infty} q'(z) dz = 0, \tag{20}\n$$

where last equality uses Assumption 1. On the other hand,

$$
\left. \frac{d}{d\theta} \log q(z - \theta) \right|_{\theta=0} = -\frac{q'(z - \theta)}{q(z - \theta)} \bigg|_{\theta=0} = -\frac{q'(z)}{q(z)} = -\frac{d}{dz} \log q(z).
$$
Therefore,
\[
\mathbb{E}_{z \sim q(\cdot)} \left[ \left( \frac{d}{dz} \log q(z) \right)^2 \right] = \mathbb{E}_{z \sim q(\cdot)} \left[ \left( \frac{d}{d\theta} \log q(z - \theta) \right)^2 \right] \bigg|_{\theta=0} = \mathbb{E}_{z \sim q(\cdot)} \left[ \left( \frac{d}{d\theta} \log q(z; \theta) \right)^2 \right] \bigg|_{\theta=0} = \mathcal{I}_q,
\]
where \( \mathcal{I}_q \) is the Fisher information associated to the location family of distributions \( q(z; \theta) \) defined in (4). Note that Assumption 1 ensures that \( \mathcal{I}_q \) is well-defined and finite. Therefore, combining this with (20) and using the central limit theorem, we realize that
\[
\frac{1}{\sqrt{d}} \sum_{i=1}^{d} \frac{d}{dz} \log q(z_i) \overset{\text{dist}}{\rightarrow} \mathcal{N}(0, \mathcal{I}_q).
\]
Consequently
\[
T_1 = \frac{-2c}{\sqrt{d}} \sum_{i=1}^{d} \frac{d}{dz} \log q(z_i) \overset{\text{dist}}{\rightarrow} \mathcal{N}(0, 4c^2 \mathcal{I}_q).
\]
\( T_2: \) Since \( \mu_d = c/\sqrt{d} \), we have
\[
T_2 = \frac{-2c^2}{d} \sum_{i=1}^{d} \frac{d^2}{dz^2} \log q(z_i).
\]
On the other hand, note that
\[
\mathbb{E}_{z \sim q(\cdot)} \left[ \frac{d^2}{dz^2} \log q(z) \right] = \mathbb{E}_{z \sim q(\cdot)} \left[ \frac{\partial^2}{\partial \theta^2} \log q(z - \theta) \right] \bigg|_{\theta=0}.
\]
Using Assumption 1 and [LC06, Lemma 5.3], we have
\[
\mathcal{I}_q = \mathcal{I}_q(0) = \mathbb{E}_{z \sim q(\cdot)} \left[ \left( \frac{\partial}{\partial \theta} \log q(z - \theta) \right)^2 \right] \bigg|_{\theta=0} = -\mathbb{E}_{z \sim q(\cdot)} \left[ \frac{\partial^2}{\partial \theta^2} \log q(z - \theta) \right] \bigg|_{\theta=0}.
\]
Substituting this into (23), we get
\[
\mathbb{E}_{z \sim q(\cdot)} \left[ \frac{d^2}{dz^2} \log q(z) \right] = -\mathcal{I}_q.
\]
Since \( \mathcal{I}_q < \infty \) from Assumption 1, using the law of large numbers in (22), we realize that
\[
\lim_{d \to \infty} T_2 = 2c^2 \mathcal{I}_q \quad \text{a.s.}
\]
\( T_3: \) Since \( \mu_d = c/\sqrt{d} \), we may bound \( T_3 \) as follows:
\[
|T_3| = \left| \frac{-4c^3}{3\sqrt{d}} \sum_{i=1}^{d} \frac{d^3}{dz^3} \log q(z_i + \epsilon_i) \right|
\]
Moreover, since the left hand side of (24) is precisely \( \sum_{i=1}^{d} \tilde{x}_i^{(d)} \) when \( Y = +1 \), we realize that conditioned on \( Y = +1 \), the log likelihood \( \sum_{i=1}^{d} \tilde{x}_i^{(d)} \) converges in distribution to a normal \( N(2c^2I_q, 4c^2I_q) \). Likewise, (25) implies that conditioned on \( Y = -1 \), \( \sum_{i=1}^{d} \tilde{x}_i^{(d)} \) converges in distribution to a normal \( N(-2c^2I_q, 4c^2I_q) \). This completes the proof. \( \square \)
B Proof of Theorem 3

The following lemma will be useful in our analysis.

Lemma 1 (Lemma 1 in [DHP21]). Given $x, x', \omega \in \mathbb{R}^d$, for integer $k$ satisfying $\|x - x'\|_0 \leq k < d/2$, we have
\[
\langle w, x' \rangle_k - \langle w, x \rangle \leq 8k \|\omega \otimes x\|_{\infty}.
\]
In particular, for $w$ being the all-one vector, we have
\[
\left| \text{TSum}_k(x') - \sum_{i=1}^{d} x_i \right| \leq 8k \|x\|_{\infty}.
\]

Proof of Theorem 3. It turns out that in order to simplify the discussion and to avoid considering multiple cases, it is more convenient to allow $c$ to be negative. Therefore, in this proof we assume that $\mu_d = c/\sqrt{d}$ where $c \in \mathbb{R}$ and $c \not= 0$. Note that we still stick to the definition of $\bar{x}^{(d)}$ in (10) and $\mathcal{C}_k^{(d)}$ in (13). We have

\[
\begin{aligned}
\mathcal{L}_{\mu_d,q}(\mathcal{C}_k^{(d)}, k_d) &= \mathbb{E} \left[ \max_{x^{(d)} \in B_0(x^{(d)}, k_d)} \| \mathcal{C}_k^{(d)}(x^{(d)}') \neq y \| \right] \\
&= \mathbb{E} \left[ \| \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \mathcal{C}_k^{(d)}(x^{(d)}') \neq y \| \right] \\
&= \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \mathcal{C}_k^{(d)}(x^{(d)}') \neq y \right) \\
&= \frac{1}{2} \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \text{TSum}_k(\bar{x}^{(d)}) \leq 0 \mid Y = +1 \right) \\
&\quad + \frac{1}{2} \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \text{TSum}_k(\bar{x}^{(d)}) \geq 0 \mid Y = -1 \right) \\
\end{aligned}
\]

Note that for $x^{(d)} \in B_0(x^{(d)}, k_d)$, we have $\|\bar{x}^{(d)} - \bar{x}'^{(d)}\|_0 \leq 0$. Therefore, using Lemma 1, we have
\[
\left( \sum_{i=1}^{d} \bar{x}_i^{(d)} \right) - 8k_d \|\bar{x}^{(d)}\|_{\infty} \leq \text{TSum}_k(\bar{x}^{(d)}) \leq \left( \sum_{i=1}^{d} \bar{x}_i^{(d)} \right) + 8k_d \|\bar{x}^{(d)}\|_{\infty} \quad \forall x^{(d)} \in B_0(x^{(d)}, k_d).
\]

Using this in (29), we get
\[
\begin{aligned}
\mathcal{L}_{\mu_d,q}(\mathcal{C}_k^{(d)}, k_d) &\leq \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} \bar{x}_i^{(d)} \leq 8k_d \|\bar{x}^{(d)}\|_{\infty} \mid Y = +1 \right) + \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} \bar{x}_i^{(d)} > -8k_d \|\bar{x}^{(d)}\|_{\infty} \mid Y = -1 \right) \\
&\quad + \frac{1}{2} \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \text{TSum}_k(\bar{x}^{(d)}) \geq 0 \mid Y = -1 \right) \\
&\quad + \frac{1}{2} \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \text{TSum}_k(\bar{x}^{(d)}) \leq 0 \mid Y = +1 \right) \\
&= \frac{1}{2} \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \text{TSum}_k(\bar{x}^{(d)}) \geq 0 \mid Y = -1 \right) \\
&\quad + \frac{1}{2} \mathbb{P} \left( \exists x^{(d)} \in B_0(x^{(d)}, k_d) : \text{TSum}_k(\bar{x}^{(d)}) \leq 0 \mid Y = +1 \right) \\
&\quad + \frac{1}{2} \mathbb{P} \left( \sum_{i=1}^{d} \bar{x}_i^{(d)} \leq 8k_d \|\bar{x}^{(d)}\|_{\infty} \mid Y = +1 \right) \\
\end{aligned}
\]

We study each of the two terms separately.

Conditioned on $Y = +1$, we have
\[
\bar{x}_i^{(d)} = \log \left( \frac{q(z_i)}{q(z_i + 2\mu_d)} \right) = \frac{c}{\sqrt{d} \mu_d} \log \left( \frac{q(z_i)}{q(z_i + 2\mu_d)} \right).
\]

Using the Taylor expansion, we get
\[
\log q(z_i + 2\mu_d) - \log q(z_i + \epsilon_i) = 2\mu_d \frac{d}{dz} \log q(z_i) + 2\mu^2 \frac{d^2}{dz^2} \log q(z_i + \epsilon_i),
\]

13
where $|\epsilon_i| < 2|\mu_d| = 2|c|/\sqrt{d}$. Using this in (31), since $\mu_d = c/\sqrt{d}$, we get
\[
\tilde{x}_i^{(d)} = -\frac{2c}{\sqrt{d}} \frac{d}{dz} \log q(z_i) - \frac{2c^2}{d} \frac{d^2}{dz^2} \log q(z_i + \epsilon_i).
\]
Consequently,
\[
\|\tilde{x}^{(d)}\|_\infty \leq \frac{2|c|}{\sqrt{d}} \max_{1 \leq i \leq d} \left| \frac{d}{dz} \log q(z_i) \right| =: T_1
\]
\[
+ \frac{2c^2}{d} \max_{1 \leq i \leq d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right| =: T_2.
\] (32)

For $T_1$, note that using Assumption 4, there are constants $\gamma > 0$ and $C_4 > 0$ such that
\[
\lim_{d \to \infty} \mathbb{P} \left( T_1 > \frac{2cC_4 (\log d)^\gamma}{\sqrt{d}} \right) = 0.
\]
This in particular implies that, since $k_d \leq d^{1/\epsilon} - \epsilon$, we have
\[
k_d T_1 \xrightarrow{d \to \infty} 0.
\] (33)

For $T_2$, let $d$ be large enough so that with the constant $\zeta$ in Assumption 3, we have $2|\mu_d| = 2|c|/\sqrt{d} < \zeta$. For such $d$, we may write
\[
T_2 \leq \frac{2c^2}{d} \max_{1 \leq i \leq d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right| \sup_{1 \leq i \leq d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 1/2
\]
\[
= \frac{2c^2}{d} \left( \max_{1 \leq i \leq d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 \right)^{1/2}
\]
\[
= \frac{2c^2}{\sqrt{d}} \left( \frac{1}{d} \sum_{1 \leq i \leq d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 \right)^{1/2}
\]
\[
\leq \frac{2c^2}{\sqrt{d}} \left( \frac{1}{d} \sum_{i=1}^{d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 \right)^{1/2}.
\]

Note that from Assumption 3, we have
\[
\lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \sup_{t \in [z_i - c, z_i + c]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 = \mathbb{E}_{Z \sim q} \left[ \sup_{t \in [z - c, z + c]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 \right] < \infty \quad \text{a.s.}
\]

Thereby, since $k_d < d^{1/\epsilon} - \epsilon$, we have
\[
\lim_{d \to \infty} k_d T_2 = 0 \quad \text{a.s.}
\] (34)

Combining this with (33) and substituting into (32), we realize that conditioned on $Y = +1$, $k_d \|\tilde{x}^{(d)}\|_\infty$ converges to zero in probability as $d \to \infty$. On the other hand, from Theorem 2, we know that conditioned on $Y = +1$, $\sum_{i=1}^{d} \tilde{x}_i^{(d)}$ converges in distribution to a normal $N(2c^2 I_q, 4c^2 I_q)$. Consequently,
we have

\[
\lim_{d \to \infty} P \left( \sum_{i=1}^{d} x_i^{(d)} \leq 8k_d \| x^{(d)} \|_\infty \bigg| Y = +1 \right) = \lim_{d \to \infty} P \left( \sum_{i=1}^{d} \log \frac{q(z_i)}{q(z_i) + 2\mu_d} \leq 8k_d \max_{1 \leq i \leq d} \left| \log \frac{q(z_i)}{q(z_i) + 2\mu_d} \right| \right) = P \left( \mathcal{N} \left( 2(c^2 \mathcal{I}_q, 4c^2 \mathcal{I}_q) \right) \leq 0 \right) = \Phi \left( |c| \sqrt{\mathcal{I}_q} \right).
\]

Conditioned on \( Y = -1 \), we have

\[
\tilde{x}_i^{(d)} = \log \frac{q(z_i - 2\mu_d)}{q(z_i)}.
\]

Therefore,

\[
P \left( \sum_{i=1}^{d} \tilde{x}_i^{(d)} > -8k_d \| \tilde{x}^{(d)} \|_\infty \bigg| Y = -1 \right) = P \left( \sum_{i=1}^{d} \log \frac{q(z_i - 2\mu_d)}{q(z_i)} > -8k_d \max_{1 \leq i \leq d} \left| \log \frac{q(z_i - 2\mu_d)}{q(z_i)} \right| \right) = P \left( \sum_{i=1}^{d} \log \frac{q(z_i)}{q(z_i) - 2\mu_d} < 8k_d \max_{1 \leq i \leq d} \left| \log \frac{q(z_i)}{q(z_i) - 2\mu_d} \right| \right)
\]

Comparing this with (35), we realize that by replacing \( c \) with \(-c\) in the above discussion for \( Y = +1 \), we have

\[
\lim_{d \to \infty} P \left( \sum_{i=1}^{d} \tilde{x}_i^{(d)} > -8k_d \| \tilde{x}^{(d)} \|_\infty \bigg| Y = -1 \right) = \lim_{d \to \infty} P \left( \sum_{i=1}^{d} \log \frac{q(z_i)}{q(z_i) - 2\mu_d} < 8k_d \max_{1 \leq i \leq d} \left| \log \frac{q(z_i)}{q(z_i) - 2\mu_d} \right| \right) = P \left( \mathcal{N} \left( 2(c^2 \mathcal{I}_q, 4c^2 \mathcal{I}_q) \right) \leq 0 \right) = \Phi \left( |c| \sqrt{\mathcal{I}_q} \right).
\]

Combining this with (35) and substituting back in (30), we get

\[
\limsup_{d \to \infty} c_{\mu_d q}^{(d)}(C_{k_d}^{(d)}, k_d) \leq \Phi \left( |c| \sqrt{\mathcal{I}_q} \right),
\]

which completes the proof. \( \square \)

### C Proof of Theorem 4

Consider the set of all joint distributions of random variables \((X_+, X_-)\) where the marginal distribution of \(X_+\) is the same as the distribution of \(Z + \mu_d\) where \(Z \sim q(.)\), and the marginal distribution of \(X_-\) is the same as the distribution of \(Z - \mu_d\). In other words, we consider the set of all couplings of \(Z + \mu_d\) and \(Z - \mu_d\). In fact, the marginal distribution of \(X_+\) is the same as that of a data sample conditioned on \(Y = +1\), and the marginal distribution of \(X_-\) is the same as that of a data samples conditioned on \(Y = -1\). Fix a maximal coupling \((X_+, X_-)\) in this set, which is defined to be a coupling that maximizes \(P(X_+ = X_-)\), or equivalently minimizes \(P(X_+ \neq X_-)\).\(^1\) We use such a maximal coupling to design an effective strategy for the adversary. Note that maximal coupling is intuitively relevant to adversarial

\(^{1}\)Note that \((Z + \mu_d, Z - \mu_d)\) where \(Z \sim q(.)\) is probably not the optimal coupling since \(P(X_+ \neq X_-) = 1\) unless \(\mu_d = 0\).
perturbations, since the adversary wants to change the data so that the samples conditioned on \( Y = +1 \) and \( Y = -1 \) “look almost the same”, so that the classifier can extract minimal or no information about the true label upon observing adversarially perturbed samples. Given such a maximal coupling, let

\[
W := \begin{cases} X_+ & \text{if } X_+ = X_-, \\ 0 & \text{otherwise.} \end{cases}
\]

Moreover, let \( Y \sim \text{Unif}(\pm 1) \) be independent from \((X_+, X_-, W)\) and define

\[
X := \begin{cases} X_+ & Y = +1 \\ X_- & Y = -1. \end{cases}
\]

It is easy to verify that \((X, Y)\) have the same joint distribution as our true feature vector-label pair, i.e. for \( a \in \mathbb{R} \) we have

\[
P(X \leq a | Y = +1) = P(X_+ \leq a) = P(Z + \mu_d \leq a),
\]

and

\[
P(X \leq a | Y = -1) = P(X_- \leq a) = P(Z - \mu_d \leq a).
\]

Keep in mind that the joint distribution of \((X_+, X_-, W, Y, X)\) depend on \( \mu_d \) and hence on \( d \). However, we do not make such a dependence explicit to simplify the notation.

Note that by definition, \( W \) is a function of \((X_+, X_-)\) and hence is independent from \( Y \). This suggests that \( W \) can be considered as a good candidate for the adversary’s perturbation, since the adversary would ideally like to perturb the data in a way that the information about the true label is removed. More precisely, given the true label \( y \) and data samples \((x_{i}^{(d)} : i \in [d])\), we generate the modified data samples \( w_{i}^{(d)} = (w_{i}^{(d)} : i \in [d]) \) such that \( w_{i}^{(d)} \) are conditionally independent conditioned on \( y \) and \( x_{i}^{(d)} \), and \( w_{i}^{(d)} \) is generated from the law of \( W \) conditioned on \( Y = y \) and \( X = x_{i}^{(d)} \). As we discussed above, \( W \) is independent from \( Y \), hence the modified samples \( w_{i}^{(d)} \) do not bear any information about the label \( y \), indicating that \( w_{i}^{(d)} \) is an ideal candidate for the adversary. However \( \|w_{i}^{(d)} - x_{i}^{(d)}\|_0 \) might be above the adversary’s budget \( k_d \). In order to address this, we define the final perturbed data vector \( x_{i}^{(d)} \) as follows:

\[
x_{i}^{(d)} = \begin{cases} w_{i}^{(d)} & \text{if } \|w_{i}^{(d)} - x_{i}^{(d)}\|_0 \leq k_d \\ x_{i}^{(d)} & \text{otherwise.} \end{cases}
\]

This ensures that indeed \( \|x_{i}^{(d)} - x_{i}^{(d)}\|_0 \leq k_d \). In fact, it turns out that if \( k_d \gg \sqrt{d} \), then \( \|w_{i}^{(d)} - x_{i}^{(d)}\|_0 \leq k_d \) with high probability. The following lemma will be later useful to make this statement precise. The proof of Lemma 2 below is given at the end of this section.

**Lemma 2.** Assume that the Assumptions 1 and 3 are satisfied and \( \mu_d = c/\sqrt{d} \) for some \( c > 0 \). Then for any \( \delta > 0 \) we have

\[
\lim_{d \to \infty} \frac{P(W \neq X | Y = +1)}{d^{-\frac{1}{2} + \delta}} = 0,
\]

and

\[
\lim_{d \to \infty} \frac{P(W \neq X | Y = -1)}{d^{-\frac{1}{2} + \delta}} = 0.
\]
Proof of Theorem 4. Assume that the adversary employs the above strategy to perturb the input samples. In order to obtain a lower bound for the optimal robust error $L^{(d)}_{\mu,d,q}(k_d)$, we consider any classifier $C$. Let $I$ be the indicator of the event $\|w^{(d)} - x^{(d)}\|_0 > k_d$. We assume that the classifier knows adversary’s strategy, and also observes $I$. This indeed makes the classifier stronger and results in a lower bound for the robust error. Note that if $I = 0$, we have $x^{(d)} = w^{(d)}$ is independent from $y$, and no classifier can do better than a random guess, resulting in an error $1/2$. In other words,

$$L^{(d)}_{\mu,d,q}(C,k_d) \geq \frac{1}{2} \mathbb{P}(I = 0) = \frac{1}{4} \mathbb{P}(I = 0|Y = +1) + \frac{1}{4} \mathbb{P}(I = 0|Y = -1).$$

Since this holds for any classifier $C$, we have

$$L^{(d)}_{\mu,d,q}(k_d) \geq \frac{1}{2} \mathbb{P}(I = 0) = \frac{1}{4} \mathbb{P}(I = 0|Y = +1) + \frac{1}{4} \mathbb{P}(I = 0|Y = -1). \quad (39)$$

Let $I_i, 1 \leq i \leq d$ be the indicator that $w^{(d)}_i \neq x^{(d)}_i$. Using the Markov inequality, we have

$$\mathbb{P}(I = 1|Y = +1) = \mathbb{P} \left( \sum_{i=1}^d I_i > k_d | Y = +1 \right) \leq \frac{d \mathbb{P}(W \neq X|Y = +1)}{k_d} \leq \frac{d \mathbb{P}(W \neq X|Y = +1)}{d^{1/2+\epsilon}} = \frac{\mathbb{P}(W \neq X|Y = +1)}{d^{-1/2+\epsilon}},$$

which goes to zero as $d \to \infty$ due to Lemma 2. Equivalently, $\mathbb{P}(I = 0|Y = +1) \to 1$ as $d \to \infty$. Similarly, $\mathbb{P}(I = 0|Y = -1) \to 1$ as $d \to \infty$. Using these in (39) we realize that $\liminf_{d \to \infty} L^{(d)}_{\mu,d,q}(k_d) \geq 1/2$ which completes the proof.

Proof of Lemma 2. Let $p_+$ and $p_-$ the distribution of $X_+$ and $X_-$, respectively. The total variation distance between $p_+$ and $p_-$ is defined to be

$$d_{TV}(p_+, p_-) := \sup_B |p_+(B) - p_-(B)|,$$

where the supremum is over all Borel sets in $\mathbb{R}$. It is well known that (see, for instance [BLM13, Lemma 8.1]) if $(X_+, X_-)$ is the optimal coupling that minimizes $\mathbb{P}(X_+ \neq X_-)$, we have

$$\mathbb{P}(X_+ \neq X_-) = d_{TV}(p_+, p_-).$$

We have

$$\mathbb{P}(W \neq X|Y = +1) \overset{(a)}= \mathbb{P}(W \neq X_+|Y = +1) \overset{(b)}= \mathbb{P}(W \neq X_+) \overset{(c)}\leq \mathbb{P}(X_+ \neq X_-) \overset{(d)}= d_{TV}(p_+, p_-) \overset{(e)}\leq \sqrt{\frac{1}{2} D(p_+||p_-)},$$

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where (a) uses the fact that by definition, conditioned on \( Y = +1 \), we have \( X = X_+ \); in (b) we use the fact that \( Y \) is independent from \((X_+, X_-)\) and \( W \) is a function of \((X_+, X_-)\); in (c) we use the definition of \( W \) to conclude that if \( X_+ = X_- \), we have \( W = X_+ \); and finally (d) uses Pinsker inequality (see, for instance, [BLM13, Theorem 4.19]) where \( D(p_+ \| p_-) \) is the Kullback–Leibler (KL) divergence between \( p_+ \) and \( p_- \). With an abuse of notation, we may use \( p_+ \) and \( p_- \) for the densities of \( X_+ \) and \( X_- \), respectively, so that \( q_+(x) = q(x - \mu_d) \) and \( q_-(x) = q(x + \mu_d) \). Therefore,

\[
D(q_+ \| q_-) = \int_{-\infty}^{\infty} q_+(x) \log \frac{q_+(x)}{q_-(x)} \, dx
\]

\[
= \int_{-\infty}^{\infty} q(x - \mu_d) \log \frac{q(x - \mu_d)}{q(x + \mu_d)} \, dx
\]

\[
= \int_{-\infty}^{\infty} q(z) \log \frac{q(z)}{q(z + 2\mu_d)} \, dz
\]

Writing the Taylor expansion, we get

\[
\log q(z + 2\mu_d) = \log q(z) + 2\mu_d \frac{d}{dz} \log q(z) + 2\mu_d^2 \frac{d^2}{dz^2} \log q(z + \epsilon_z),
\]

where \(|\epsilon_z| < 2|\mu_d|\). Using this in (41), we get

\[
D(q_+ \| q_-) = -2\mu_d \int_{-\infty}^{\infty} q(z) \frac{d}{dz} \log q(z) \, dz
\]

\[
= T_1 - 2\mu_d^2 \int_{-\infty}^{\infty} q(z) \frac{d^2}{dz^2} \log q(z + \epsilon_z) \, dz.
\]

Observe that from Assumption 1, we have

\[
T_1 = -2\mu_d \int_{-\infty}^{\infty} q'(z) \, dz = 0.
\]

Moreover,

\[
|T_2| \leq \frac{2\epsilon^2}{d} \int_{-\infty}^{\infty} q(z) \left| \frac{d^2}{dz^2} \log q(z + \epsilon_z) \right| \, dz.
\]

Since \(|\epsilon_z| \leq 2|\mu_d|\) and \(\mu_d = c/\sqrt{d} \to 0\) as \(d \to \infty\), for \(d \) large enough we have \(|\epsilon_z| < \zeta\) for all \(z \in \mathbb{R}\), where \(\zeta\) is the constant in Assumption 3. Thereby, for \(d \) large enough, we have

\[
|T_2| \leq \frac{2\epsilon^2}{d} \int_{-\infty}^{\infty} q(z) \sup_{t \in [-\zeta, \zeta]} \left| \frac{d^2}{dt^2} \log q(t) \right| \, dt
\]

\[
= \frac{2\epsilon^2}{d} \mathbb{E}_{Z \sim q(\cdot)} \left[ \sup_{t \in [-\zeta, \zeta]} \left| \frac{d^2}{dt^2} \log q(t) \right| \right]
\]

\[
\leq \frac{2\epsilon^2}{d} \sqrt{\mathbb{E}_{Z \sim q(\cdot)} \left[ \sup_{t \in [-\zeta, \zeta]} \left| \frac{d^2}{dt^2} \log q(t) \right|^2 \right]}
\]

\[
=: \frac{\alpha}{d},
\]

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where $\alpha$ is the resulting constant, which is finite from Assumption 3. Using this together with (43) in (42), we realize that for $d$ large enough, we have $D(q_+\Vert q_-) \leq A/d$. Using this in (40), we realize that for $d$ large enough, we have

$$P(W \neq X|Y = +1) \leq \sqrt{\frac{\alpha}{2d}},$$

which implies

$$\lim_{d \to \infty} \frac{P(W \neq X|Y = +1)}{d^{-\frac{1}{2}+\delta}} = 0.$$

The proof of (38) is similar. This completes the proof.

\[\square\]

D \hspace{1em} Proof of Theorem 1

Note that if $\lim \sup_{d \to \infty} \log dk_d < 1/2$, there exists $\epsilon > 0$ such that for $d$ large enough, $\log dk_d < 1/2 - \epsilon$, or equivalently, $k_d > d^{1/2+\epsilon}$. Therefore, the first part of the theorem follows from Theorem 3. On the other hand, if $\lim \inf_{d \to \infty} k_d > 1/2$, there exists $\epsilon > 0$ such that for $d$ large enough, $\log dk_d > 1/2 + \epsilon$, or equivalently, $k_d > d^{1/2+\epsilon}$. Therefore, the second part of the theorem follows from Theorem 4.

E \hspace{1em} Proof of Theorem 5

Here, we prove that all the assumptions 1–4 are satisfied for the noise density $q(.)$ of the form (16). Before proving this, we need some lemmas. The proof of Lemmas 3, 4, and 5 below are given at the end of this section.

Lemma 3. Assume that a degree $n$ polynomial $p : \mathbb{R} \to \mathbb{R}$ is given. Given $\epsilon > 0$, we define $\tilde{p} : \mathbb{R} \to \mathbb{R}$ as follows

$$\tilde{p}(x) := \sup_{t \in [x-\epsilon, x+\epsilon]} |p(t)|,$$

Then, there exists a polynomial $r : \mathbb{R} \to \mathbb{R}$ with degree $n$, such that for all $x \in \mathbb{R}$, we have $\tilde{p}(x) \leq r(|x|)$.

Lemma 4. Given the noise density $q(.)$ as in (16), there exists a constant $c_1 > 0$ such that for all $t \geq c_1$, if $Z$ is a random variable with law $q(.)$, we have

$$P(|Z| \geq t) \leq \frac{2}{nAa_{2n}t^{2n-1}} \exp\left(-\frac{a_{2n}t^{2n}}{2}\right).$$

Lemma 5. Given the noise density $q(.)$ as in (16), there exists a constant $c_2 > 0$ such that

$$\lim_{d \to \infty} P\left(\max_{1 \leq i \leq d} |Z_i| > c_2 (\log d)^{1/2}\right) = 0,$$

where $(Z_i : i \geq 1)$ are i.i.d. random variables with law $q(.)$.

Proof of Theorem 5. As in (16), let

$$q(z) = \frac{\exp(\psi(z))}{A},$$

where

$$\psi(z) = -a_{2n}z^{2n} + a_{2n-1}z^{2n-1} + \ldots a_1z + a_0,$$
is a polynomial in \( z \) with even degree \( 2n > 0 \) such that \( a_{2n} > 0 \). We verify each of the four assumptions separately.

**Assumption 1:** It is straightforward to check that \( q(z) > 0 \) for all \( z \) and \( q(.) \) is three times continuously differentiable. In order to verify (3), note that

\[
q'(z) = \psi'(z) \frac{\exp(\psi(z))}{A},
\]

and

\[
\int_\infty^\infty |q'(z)|dz = \frac{1}{A} \int_\infty^\infty |\psi'(z)|\exp(\psi(z))dz < \infty,
\]

where the last step follows from the fact that \( \psi'(z) \) is a polynomial in \( z \) and \( \psi(z) \) is a polynomial with even degree and positive leading coefficient. This implies that

\[
\int_\infty^\infty q'(z) = \lim_{a \to \infty} \int_a^\infty q'(z)dz = \lim_{a \to \infty} q(a) - q(-a) = 0,
\]

since \( q(z) \to 0 \) as \( z \to \infty \) or \( z \to -\infty \). Furthermore,

\[
q''(z) = (\psi''(z) + (\psi'(z))^2) \frac{\exp(\psi(z))}{A}.
\]

Since \( (\psi''(z) + (\psi'(z))^2) \) is a polynomial in \( z \), similar to the above we have \( \int_\infty^\infty |q''(z)|dz < \infty \). Additionally, it is evident from (44) that \( q'(z) \to 0 \) as \( z \to \infty \) or \( z \to -\infty \). Therefore, we get \( \int_\infty^\infty q''(z)dz = 0 \) similar to the above. This establishes (3).

On the other hand, for the family of densities \( q(z; \theta) = q(z - \theta) \), we have

\[
\frac{\partial}{\partial \theta} \log q(z; \theta) = \frac{\partial}{\partial \theta} (\psi(z - \theta) - \log A) = -\psi'(z - \theta).
\]

Hence, recalling the definition of the Fisher information, we have

\[
\mathcal{I}(\theta) := \mathbb{E}_{z \sim q(z; \theta)} \left[ \left( \frac{\partial}{\partial \theta} \log q(z; \theta) \right)^2 \right] = \int_\infty^\infty (\psi'(z - \theta))^2 \exp(\psi(z - \theta))A dz < \infty,
\]

since \( (\psi'(z - \theta))^2 \) is a polynomial in \( z \). This means that the above quantity is well defined and finite, and hence the Fisher information \( \mathcal{I}(\theta) \) is well-defined and finite for all \( \theta \).

**Assumption 2** Note that \( \frac{d^2}{dt^2} \log q(t) \) is a polynomial in \( t \), therefore Lemma 3 implies that for \( \zeta > 0 \), there exists a polynomial \( r : \mathbb{R} \to \mathbb{R} \) such that

\[
\sup_{t \in [-\zeta, \zeta]} \left| \frac{d^3}{dt^3} \log q(t) \right| \leq r(\zeta).
\]

Therefore, since all the moments of \( q(.) \) are finite, and \( r(\cdot) \) is a polynomial, the expectation of the left hand side is finite.

**Assumption 3** Similar to the above case, since \( \sup_{t \in [-\zeta, \zeta]} |\frac{d^2}{dt^2} \log q(t)|^2 \) is bounded by a polynomial and all the finite moments of \( q(.) \) are finite, the expectation is indeed finite.

**Assumption 4** Note that

\[
\frac{d}{dz} \log q(z) = \psi'(z) = -2na_2nz^{2n-1} + \cdots + a_1.
\]
Therefore, for all \( z \in \mathbb{R} \),
\[
\frac{d}{dz} \log q(z) \leq \sum_{i=1}^{2n} |a_i| |z|^{i-1},
\]
and for all \((z_i : i \in [d])\),
\[
\max_{1 \leq i \leq d} \left| \frac{d}{dz} \log q(z_i) \right| \leq \sum_{i=1}^{2n} |a_i| \left( \max_{1 \leq i \leq d} |z_i| \right)^{i-1}.
\]
Note that if \( \max_{1 \leq i \leq d} |z_i| \leq c_2(\log d)^{1/2n} \) with \( c_2 \) being the constant from Lemma 5, then
\[
\max_{1 \leq i \leq d} \left| \frac{d}{dz} \log q(z_i) \right| \leq \sum_{i=1}^{2n} |a_i| (\log d)^{m/2n}.
\]
Observe that there exists a constant \( C_4 > 0 \) such that for \( d \) large enough, we have
\[
\sum_{i=1}^{2n} |a_i| (\log d)^{m/2n} \leq C_4 (\log d)^{2n-1} = C_4 (\log d)^{1-1/2n}.
\]
Combining this with the above argument, we realize that for \( d \) large enough
\[
\mathbb{P} \left( \max_{1 \leq i \leq d} \left| \frac{d}{dz} \log q(z_i) \right| > C_4 (\log d)^{1-1/2n} \right) \leq \mathbb{P} \left( \max_{1 \leq i \leq d} |z_i| > c_2(\log d)^{1/2n} \right),
\]
which converges to zero as \( d \to \infty \) from Lemma 5. This means that Assumption 4 holds with \( C_4 \) as above and \( \beta = 1 - 1/2n \).

**Proof of Lemma 3.** Let \( p(x) = a_n x^n + \cdots + a_1 x + a_0 \). Let \( p'(.) \) be the derivative of \( p(.) \). Since \( p'(.) \) is a polynomial of degree \( n - 1 \), it has at most \( n - 1 \) real roots. Consequently, there exist \( -\infty = a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = \infty \) where \( m \leq n \) and \( p(.) \) is monotone in \([\alpha_i, \alpha_{i+1}]\) for \( 0 \leq i < m \). Let
\[
A := \bigcup_{i=1}^{m-1} [\alpha_i - \epsilon, \alpha_i + \epsilon].
\]
Note that if \( x \notin A \), \( p(.) \) is monotone in \([x - \epsilon, x + \epsilon]\). Hence,
\[
\tilde{p}(x) \leq \max \{ |p(x - \epsilon)|, |p(x + \epsilon)| \} \leq |p(x - \epsilon)| + |p(x + \epsilon)| \quad \forall x \notin A. \quad (45)
\]
Furthermore, let
\[
B := \bigcup_{i=1}^{m-1} [\alpha_i - 2\epsilon, \alpha_i + 2\epsilon].
\]
Note that \( B \) is a compact set, and \( p(.) \) is continuous. Therefore, we may define
\[
\beta := \max_{x \in B} |p(x)|,
\]
and \( \beta < \infty \). Since for \( x \in A \), we have \([x - \epsilon, x + \epsilon] \subset B \), we may write
\[
\tilde{p}(x) = \sup_{t \in [x - \epsilon, x + \epsilon]} |p(t)| \leq \sup_{t \in B} |p(t)| = \beta \quad \forall x \in A. \quad (46)
\]
Combining this with (45), we realize that for all \( x \in \mathbb{R} \), we have
\[
\tilde{p}(x) \leq |p(x - \epsilon)| + |p(x + \epsilon)| + \beta
\]
\[
\leq \beta + \sum_{i=0}^{n} |a_i|(|x - \epsilon|^i + |x + \epsilon|^i)
\]
\[
\leq \beta + \sum_{i=0}^{n} 2|a_i|(|x|+|\epsilon|)^i
\]
\[
= r(|x|),
\]
where \( r(.) \) is a polynomial of degree \( n \). This completes the proof.

**Proof of Lemma 4.** Recalling the polynomial form of \( \psi(.) \) and the assumption that \( a_{2n} > 0 \), we realize that there exists \( c_1 > 0 \) such that if \( z > c_1 \), we have
\[
-2a_{2n}z^{2n} \leq \psi(z) \leq -\frac{a_{2n}}{2}z^{2n},
\]
and if \( z < -c_1 \), we have
\[
-\frac{a_{2n}}{2}z^{2n} \leq \psi(z) \leq -2a_{2n}z^{2n}.
\]
Thereby, if \( t \geq c_1 \), we have
\[
\mathbb{P}(Z \geq t) = \int_{t}^{\infty} \frac{1}{A} \exp(\psi(z)) \, dz
\]
\[
\leq \frac{1}{A} \int_{t}^{\infty} \exp\left(-\frac{a_{2n}}{2}z^{2n}\right) \, dz
\]
\[
= \frac{1}{A} \int_{t}^{\infty} \frac{n a_{2n} z^{2n-1}}{na_{2n}z^{2n-1}} \exp\left(-\frac{a_{2n}}{2}z^{2n}\right) \, dz
\]
\[
\leq \frac{1}{n A a_{2n} t^{2n-1}} \int_{t}^{\infty} -\frac{d}{dz} \exp\left(-\frac{a_{2n}}{2}z^{2n}\right) \, dz
\]
\[
= \frac{1}{n A a_{2n} t^{2n-1}} \exp\left(-\frac{a_{2n}}{2}t^{2n}\right).\]
Similarly, using (48), for \( t \geq c_1 \), we may write
\[
\mathbb{P}(Z \leq -t) = \int_{-\infty}^{-t} \frac{1}{A} \exp(\psi(z)) \, dz
\]
\[
\leq \frac{1}{A} \int_{-\infty}^{-t} \exp(-2a_{2n}z^{2n}) \, dz
\]
\[
= \frac{1}{A} \int_{-\infty}^{-t} \frac{4n a_{2n} z^{2n-1}}{4n a_{2n} z^{2n-1}} \exp(-2a_{2n}z^{2n}) \, dz
\]
\[
\leq \frac{-1}{4n A a_{2n} t^{2n-1}} \int_{-\infty}^{-t} -\frac{d}{dz} \exp(-2a_{2n}z^{2n}) \, dz
\]
\[
= \frac{1}{4n A a_{2n} t^{2n-1}} \exp(-2a_{2n}t^{2n})
\]
\[
\leq \frac{1}{n A a_{2n} t^{2n-1}} \exp\left(-\frac{a_{2n}}{2}t^{2n}\right).
\]
Combining (49) and (50) and using the union bound, we arrive at the desired result. \( \square \)
Proof of Lemma 5. Since $a_{2n} > 0$, we may choose $c_2$ large enough so that
\begin{equation}
\frac{a_{2n}}{2} c_2^{2n} > 1. \tag{51}
\end{equation}

Using the union bound, we get
\begin{equation}
P\left( \max_{1 \leq i \leq d} |Z_i| > c_2 (\log d)^{\frac{1}{2n}} \right) \leq d P\left( |Z| > c_2 (\log d)^{\frac{1}{2n}} \right), \tag{52}
\end{equation}
where $Z \sim q(.)$. Using Lemma 4, if $d$ is large enough so that $c_2 (\log d)^{\frac{1}{2n}} > c_1$, we have
\begin{equation}
P\left( |Z| > c_2 (\log d)^{\frac{1}{2n}} \right) \leq \frac{2}{nAa_{2n}c_2^{2n-1}(\log d)^{\frac{2n}{2n-1}}} \exp \left( -\frac{a_{2n}}{2} c_2^{2n} \log d \right). \tag{53}
\end{equation}

Using (53) in (52), we realize that for $d$ large enough,
\begin{align*}
P\left( \max_{1 \leq i \leq d} |Z_i| > c_2 (\log d)^{\frac{1}{2n}} \right) &\leq \frac{2d}{nAa_{2n}c_2^{2n-1}(\log d)^{\frac{2n}{2n-1}}} \exp \left( -\frac{a_{2n}}{2} c_2^{2n} \log d \right) \\
&= \frac{2}{nAa_{2n}c_2^{2n-1}(\log d)^{\frac{2n}{2n-1}}} \exp \left( -\left[ \frac{a_{2n}}{2} c_2^{2n} - 1 \right] \log d \right),
\end{align*}
which goes to zero as $d \to \infty$ due to (51). This completes the proof. \hfill \Box