LATTICE HOMOLOGY, FORMALITY, AND PLUMBED L-SPACE LINKS

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Abstract. We define a link lattice complex for plumbed links, generalizing constructions of Ozsváth, Stipsicz and Szabó, and of Gorsky and Némethi. We prove that for all plumbed links in rational homology 3-spheres, the link lattice complex is homotopy equivalent to the link Floer complex as an $A_\infty$-module. Additionally, we prove that the link Floer complex of a plumbed L-space link is a free resolution of its homology. As a consequence, we give an algorithm to compute the link Floer complexes of plumbed L-space links, in particular of algebraic links, from their multivariable Alexander polynomial.

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1. Introduction

1.1. Overview. Lattice homology is a combinatorial invariant of plumbed 3-manifolds defined by Némethi [Ném05, Ném08], see also [Ném22, Chapter 11]. Némethi’s construction is a formalization of earlier work of Ozsváth and Szabó [OS03], which computes the Heegaard Floer homology groups of many plumbed 3-manifolds. If $Y$ is a plumbed 3-manifold, we write $\mathbb{HF}(Y)$ for its lattice homology, which is a module over the power series ring $\mathbb{F}[[U]]$.

Work of Némethi, Ozsváth, Stipsicz and Szabó [Ném05, Ném08, OS03, OSS14b] proves that $\mathbb{HF}(Y) \cong HF^-(Y)$ for many important families of plumbed 3-manifolds. More recently, the third author proved the isomorphism in general [Zem21b].

Given a knot $K$ in $S^3$, Ozsváth–Szabó [OS04a], and Rasmussen [Ras03] defined a refinement of Heegaard Floer homology, called knot Floer homology. There are several equivalent formulations of this invariant. For our purposes, it is most convenient to consider knot Floer homology as taking the form of a free, finitely generated chain complex $\mathcal{CFK}(K)$ over the 2-variable polynomial ring $\mathbb{F}[U, V]$.

Ozsváth and Szabó also defined a version of Heegaard Floer theory for links in 3-manifolds [OS08a]. For a link $L \subseteq S^3$, we will focus on the description of the link Floer complex as a finitely generated free chain complex $\mathcal{CFL}(L)$ over the polynomial ring $\mathbb{F}[U_1, \ldots, U_\ell, V_1, \ldots, V_\ell]$, where $\ell = |L|$. For our purposes, it is also helpful to consider the knot and link Floer complexes over the completed ring $\mathbb{F}[[U_1, \ldots, U_\ell, V_1, \ldots, V_\ell]]$.

Their construction also applies more generally when $L$ is a link in a rational homology 3-sphere $Y$. In this case, we denote the link Floer complex $\mathcal{CFL}(Y, L)$.

A relative version of lattice homology for plumbed knots is defined by Ozsváth, Stipsicz and Szabó [OSS14a]. Modulo notational differences, their version of knot lattice homology is analogous to the complex $\mathcal{CFK}(K)$. They proved that for a plumbed L-space knot in $S^3$, link lattice homology coincides with the knot Floer complex $\mathcal{CFK}(K)$.

Gorsky and Némethi [GN15] defined a relative version of lattice homology for L-space links. (Note, their construction does not require the link to be plumbed). They constructed a spectral sequence from their version of link lattice homology to a version of link Floer homology, which is the quotient complex $\mathcal{CFL}(L)/(v_1, \ldots, v_\ell)$, and proved that it degenerates for all algebraic links.
In particular, their version of link lattice homology is isomorphic (as a graded group) to the version of link Floer homology for algebraic links in $S^3$

In our paper, we construct a new version of link lattice homology. Our link lattice complex is more closely related to the construction of Ozsváth, Stipsicz and Szabó [OSS14a], and is modeled on the full link Floer complex $CFL(L)$. We use this link lattice complex to study plumbed L-space links, a family which includes all algebraic links in $S^3$.

1.2. The link lattice complex. Suppose that $L$ is a plumbed link in a plumbed 3-manifold $Y$. Such a pair $(Y, L)$ is presented by a weighted graph $\Gamma$, whose vertices are partitioned into two sets

$$V_\Gamma = V_G \cup V_\ell.$$

The vertices $V_G$ are equipped with integral weights. The vertices in $V_\ell$ have no weights, and we refer to them as arrow vertices. Unless specified explicitly otherwise, we will always assume that $\Gamma$ is a tree.

From the tree $\Gamma$, we obtain a partitioned link $L_\Gamma = L_G \cup L_\ell$ in $S^3$. This link may be described as a connected sum of Hopf links, with one unknotted component for each vertex of $\Gamma$, and one clasp for each edge of $\Gamma$. The manifold $Y$ is the result of surgery on $L_G$, with framing $\Lambda$ obtained from the weights. Inside of $Y \cong S^3(G)$, the link $L$ is identified with $L_\Gamma$.

Given a plumbing tree $\Gamma$ presenting a plumbed link $(Y, L)$, we will construct a chain complex $CFL(\Gamma, V_\ell)$.

Given an orientation of $L_\ell$, we will equip the chain complex $CFL(\Gamma, V_\ell)$ with the structure of a module over the ring $\mathbb{F}[[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_\ell, \mathcal{V}_\ell]]$, where $\ell = |V_\ell|$, as well as with a Maslov grading and a $\mathbb{Q}$-valued Alexander grading.

It is helpful to view $CFL(\Gamma, V_\ell)$ as an $A_\infty$-module over $\mathbb{F}[[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_\ell, \mathcal{V}_\ell]]$ with only $m_1$ and $m_2$ non-vanishing. We note that the complex $CFL(\Gamma, V_\ell)$ is not free over this ring.

A central result of the paper is the following:

**Theorem 1.1.** Suppose that $\Gamma$ is a plumbing link diagram which is a tree, and write $(Y, L)$ for the associated 3-manifold and link. If $Y$ is a rational homology sphere, then $CFL(Y, L)$ is homotopy equivalent to $CFL(\Gamma, V_\ell)$ as an absolutely graded $A_\infty$-module over $\mathbb{F}[[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_\ell, \mathcal{V}_\ell]]$.

See Theorem 5.1 for further details. In the above, we are writing $CFL(Y, L)$ for the full link Floer complex $CFL(Y, L)$ completed over the power series ring $\mathbb{F}[[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_\ell, \mathcal{V}_\ell]]$.

1.3. Algebraic and plumbed L-space links. We recall that a rational homology 3-sphere $Y$ is called an L-space if

$$\widehat{HF}(Y, s) \cong \mathbb{F}$$

for each $s \in \text{Spin}^c(Y)$. A link $L \subset S^3$ is called an L-space link if all sufficiently positive surgeries are L-spaces. We note that since Dehn surgery does not depend on the string orientation, the property of being an L-space is independent of string orientations.

An important family of plumbed links in $S^3$ are algebraic links, which are the links of complex plane curve singularities. According to Gorsky and Némethi [GN16], algebraic links in $S^3$ are L-space links.

There is a useful characterization of L-space links in terms of the link Floer complex. If $L \subset S^3$, then $L$ is an L-space link if and only if the homology group $HF(L)$ is torsion free as an $\mathbb{F}[U]$-module, where $U$ acts by $\mathcal{U}_i \mathcal{V}_j$ for any $i$. (Since $\mathcal{U}_i \mathcal{V}_i$ and $\mathcal{U}_j \mathcal{V}_j$ have chain homotopic actions for all $i$ and $j$, the definition is independent of the choice of $i$).

Ozsváth and Szabó [OS05] proved the knot Floer complex of an L-space knot is a staircase complex. A very natural question is whether an analog of Ozsváth and Szabó’s result holds for L-space links. L-space links have also been extensively studied in the literature, see e.g. [BG18, CL20, GH17, GLM20, GN15, Liu17, Liu19, Liu21]. Despite the interest in L-space links and many interesting results concerning them, there is to date no result which characterizes the structure of
the link Floer complex of an L-space link in parallel with Ozsváth and Szabó’s result for L-space knots.

We prove the following:

**Theorem 1.2.** Suppose that $L \subseteq S^3$ is a plumbed L-space link. Then the link Floer complex $CFL(L)$ is homotopy equivalent to a free resolution of its homology. Equivalently, the complex $CFL(L)$ is homotopy equivalent to its homology $HFL(L)$ as an $A_\infty$-module over $\mathbb{F}[U_1, V_1, \ldots, U_\ell, V_\ell]$, where we equip $HFL(L)$ with the $A_\infty$-module structure which has $m_j = 0$ unless $j = 2$.

![Figure 1.1. The knot and link Floer complexes of $T(3,4)$, $T(2,2)$ and $T(2,4)$.](image)

Each dot denotes a generator in a free basis. The horizontal direction indicates the grading of the free resolution.

**Remark 1.3.**

1. Theorem 1.2 is a natural generalization of the result of Ozsváth and Szabó for L-space knots because the staircase complex of an L-space knot is easily seen to be a free resolution of its homology over $\mathbb{F}[\mathcal{U}, \mathcal{V}]$.

2. Theorem 1.2 also holds more generally for plumbed L-space links $L$ in plumbed 3-manifolds $Y$ which are themselves L-spaces. For details, see Section 6.

3. A $dg$-module $M$ over a ring $A$ such that $(H_*(M), m_2)$ is homotopy equivalent to $M$ as an $A_\infty$-module over $A$ is called formal. Hence, we may restate Theorem 1.2 by saying that plumbed L-space links have formal link Floer complexes.

We do not know whether Theorem 1.2 holds for non-plumbed L-space links. We state the following open question:

**Question 1.4.** Are there non-plumbed L-space links $L \subseteq S^3$ for which $CFL(L)$ is not homotopy equivalent to a free-resolution of $HFL(L)$?

From Theorem 1.2, we obtain an algorithm to compute the link Floer complex of a plumbed L-space link. Namely, we observe that for an L-space link $L$, the $\mathbb{F}[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_\ell, \mathcal{V}_\ell]$-module $HFL(L)$ is completely determined by the $H$-function (or equivalently the $d$-invariants of large surgeries). According to [GN15], the $H$-function of an L-space link in $S^3$ is determined by the multivariable
Alexander polynomial of $L$. After determining $\mathcal{HFL}(L)$, one may compute $\mathcal{CFL}(L)$ by computing its free resolution. Finding such a resolution is algorithmic, see e.g. [Pee11], and may be done using computer algebra software such as Macaulay2 [GS]. We carry this out for $T(3,3)$ and $T(4,4)$ in Section 7.

**Corollary 1.5.** The link Floer complex of a plumbed L-space link $L$ in $S^3$ is computable from the multivariable Alexander polynomials of the link $L$ and its sublinks.

**Remark 1.6.** Computing free resolutions is in general a challenging task. For many purposes (e.g. taking tensor products), it is more practical to understand the homology group $\mathcal{HFL}(L)$ and use the fact that $\mathcal{CFL}(L)$ is homotopy equivalent as an $A_\infty$-module to $\mathcal{HFL}(L)$ with only $m_2$ non-trivial. Often the homology group $\mathcal{HFL}(L)$ has a much more concise description than its free resolution. For example in Section 7, we give a simple description of the module $\mathcal{HFL}(T(n,n))$. We describe free resolutions for $T(3,3)$ and $T(4,4)$, which are considerably larger in complexity.

We note that for algebraic links in $S^3$ (which are particular examples of plumbed L-space links) the fact that the link Floer complex is determined by the Alexander polynomial may be seen indirectly using the fact that the Alexander polynomial of an algebraic link determines the link. See work of Yamamoto [Yam84]. This is not the case for algebraic links in other 3-manifolds, or other plumbed L-space links in $S^3$: see [CDGZ20] and also Proposition 3.20. Nonetheless, our techniques give a concrete algorithm for computing the link Floer complex based on its Alexander polynomial. Although foundational, Yamamoto’s work does not give a practical algorithm for computing link Floer complexes of algebraic links in $S^3$.

1.4. **Gorsky and Némethi’s link lattice homology.** We also consider Gorsky and Némethi’s link lattice homology, which is defined for all L-space links. If $L$ is an L-space link, then they described a chain complex $\mathcal{K}(L)$, which they called the link lattice complex. They proved that if $L$ is an algebraic link, then

$$H_*(\mathcal{K}(L)) \cong H_*(\mathcal{CFL}(L)/(\mathcal{V}_1, \ldots, \mathcal{V}_\ell)),$$

as graded groups.

In our paper, we give an alternate perspective on the complex $\mathcal{K}(L)$. Namely, we show that $\mathcal{K}(L)$ is homotopy equivalent as an $A_\infty$-module to the derived tensor product

$$\mathcal{K}(L) \simeq \mathcal{HFL}(L) \otimes_{F[\mathcal{V}_1, \ldots, \mathcal{V}_\ell]} F[\mathcal{V}_1, \ldots, \mathcal{V}_\ell]/(\mathcal{V}_1, \ldots, \mathcal{V}_\ell)$$

where we equip $\mathcal{HFL}(L)$ with the $A_\infty$-module structure which has only $m_2$ non-trivial. As a corollary of Theorem 1.2, we obtain the following result:

**Corollary 1.7.** If $L$ is a plumbed L-space link, then Gorsky and Némethi’s link lattice complex $\mathcal{K}(L)$ is homotopy equivalent to $\mathcal{CFL}(L)/(\mathcal{V}_1, \ldots, \mathcal{V}_\ell)$ as a graded chain complex.

We note Gorsky and Némethi only prove the isomorphism in Equation (1.1) at the level of graded groups. Our proof of Corollary 1.7 improves on their result additionally because it equips $\mathcal{K}(L)$ with an $A_\infty$-module structure over $F[\mathcal{V}_1, \ldots, \mathcal{V}_\ell]$ and proves the isomorphism at the level of $A_\infty$-modules.

1.5. **Structure of the paper.** Section 2 describes background on $A_\infty$-modules and the homological perturbation lemma. We give a basic example from knot Floer theory of two complexes with isomorphic homology groups, such that one of them is chain homotopy equivalent to its homology (regarded as a chain complex with trivial differentials), while the other one is not.

Next, we recall some topological background in Section 3. We provide the definitions of plumbed manifolds and plumbed links. We state the result of Gorsky and Némethi that a plumbed link in a rational analytic singularity is an L-space link [GN16]. In particular, we show that plumbed L-space links are a natural generalization of algebraic links in $S^3$. In Section 4, we define our link lattice complex. We describe the gradings and the $F[\mathcal{V}_1, \ldots, \mathcal{V}_\ell]$-module structure.
Section 5 proves the equivalence of the link lattice complex and the link Floer complex, stated above in Theorem 1.1. This is the main technical result of the paper.

In Section 6 we focus on plumbed L-space links. Using Theorem 5.1 and the homological perturbation lemma, we prove Theorem 1.2, which states that the link Floer complex of a plumbed L-space link is a free resolution of its homology. As a consequence, we show that the link Floer complex of a plumbed L-space link is computable from its multivariable Alexander polynomial. See Theorem 6.7. Additionally, we describe how our link lattice complex recovers the theory described by Gorsky and Némethi in the case of plumbed L-space links. See Theorem 6.10.

Section 7 describes some examples. We describe the complex of a plumbed L-space link is computable from its multivariable Alexander polynomial. See Theorem 7.1. Additionally, we describe how our link lattice complex recovers the theory described by Gorsky and Hom. Next, we compute $CFL(T(3,3))$ and $CFL(T(4,4))$ by explicitly constructing free resolutions.

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2. Algebraic background

In this section we recall the notion of an $A_\infty$-module (Subsection 2.1). Then, in Subsection 2.2, we state the homological perturbation lemma in $A_\infty$-category. In Subsection 2.3, we consider the homological perturbation lemma in the context of free-resolutions of modules. In Subsection 2.4, we present two complexes over $\mathbb{F}[\mathbb{Z}, \mathbb{Y}]$ with isomorphic homology, but which are not chain homotopy equivalent. These two complexes appear naturally in knot Floer homology.

2.1. $A_\infty$-modules. Throughout the paper, we make use of the category of $A_\infty$-modules. The motivation is that $A_\infty$-module structures may be transferred along homotopy equivalences of groups. Suppose that $A$ is a ring that is an algebra over a field $k$. Given a finitely generated chain complex $C$ over $A$, in very general circumstances, one may pick a homotopy equivalence over $k$ between $C$ and $H_*(C)$. Given such a homotopy equivalence, the homotopy transfer lemma (cf. [Kad82]) equips $H_*(C)$ with the structure of an $A_\infty$-module over $A$, such that $C$ and $H_*(C)$ are homotopy equivalent as $A_\infty$-modules. Note that unless $A$ is a field, it is rarely the case that $C$ and $H_*(C)$ are homotopy equivalent as dg-modules over $A$. When the homotopy equivalence between $C$ and $H_*(C)$ is suitably simple, the $A_\infty$-module maps on $H_*(C)$ are usually computed using a version of the homological perturbation lemma, stated in Lemma 2.3.

We now recall the basics of $A_\infty$-modules. We mostly follow the notation of Lipshitz, Ozsváth and Thurston [LOT15, LOT18].

Let $A$ be an associative algebra with unit over a ring $k$. We will assume that $k$ has characteristic zero. We write $\mu_2$ for the multiplication on $A$.

Definition 2.1. A left $A_\infty$-module $A M$ over $A$ is a left $k$-module equipped with $k$-module maps 

$$m_{i+1}: A^\otimes j \otimes_k M \to M,$$

such that for each $n$ and any $a_1, \ldots, a_n \in A$, $x \in M$, the following holds.

$$\sum_{i=0}^{n} m_{n-i+1}(a_n, a_{n-1}, \ldots, a_{i+1}, m_{i+1}(a_i, \ldots, a_1, x)) + \sum_{k=1}^{n-1} m_n(a_n, a_{n-1}, \ldots, \mu_2(a_{k+1}, a_k), \ldots, a_1, x) = 0.$$

Lipshitz, Ozsváth and Thurston refer to $A_\infty$-modules as type-$A$ modules, in contrast to type-$D$ modules, which we now introduce.
Definition 2.2. A right type-D module $N^A$ over $A$ is a right $k$-module $N$, together with a $k$-linear structure map
\[ \delta^i : N \to N \otimes_k A, \]
such that
\[ (\text{id}_N \otimes \mu_2) \circ (\delta^1 \otimes \text{id}_A) \circ \delta^1 = 0. \]

2.2. The homological perturbation lemma. It is a general fact that $A_\infty$-algebra structures may be transferred along homotopy equivalences of the chain complex underlying an $A_\infty$-algebra. This was proved by Kadeishvili [Kad82]. Homological perturbation theory gives concrete formulas for the resulting $A_\infty$-module structure under certain restrictions on the chain homotopy equivalence. See [KS01, Theorem 3]. An exposition of the technique may be found in Ph.D. thesis of Lefèvre–Hasegawa [LH03].

Lemma 2.3. Suppose that $A$ is an associative algebra over a ground ring $k$, $AM = (M, m_j)$ is an $A_\infty$-module over $A$, $(Z, \partial)$ is a chain complex over $k$, and that we have three maps of left $k$-modules
\[ i : Z \to M, \quad \pi : M \to Z \quad \text{and} \quad h : M \to M \]

satisfying the following:
1. $i$ and $\pi$ are chain maps.
2. $\pi \circ i = \text{id}_Z$.
3. $i \circ \pi = \text{id}_M + \partial(h)$, where $\partial(h) := m_1 \circ h + h \circ m_1$.
4. $\partial \circ i = 0$.
5. $\pi \circ h = 0$.
6. $h \circ h = 0$.

Then there are $A_\infty$-module structure maps $m^Z_j$ on $Z$, as well as $A_\infty$-module morphisms
\[ i_j : A^Z \to AM, \quad \pi_j : AM \to A^Z \quad \text{and} \quad h_j : AM \to AM \]
satisfying $m^Z_1 = \partial, i_1 = i, \pi_1 = \pi$ and $h_1 = h$, and such that the relations (1)–(6) are also satisfied as $A_\infty$-module morphisms.

Remark 2.4. It is important to note that the maps $i, \pi, h$ in the assumption of homological perturbation lemma are required to be only $k$-module maps, not necessarily $A$-module maps. We refer the interested reader to [LH03, Section 1.4] for a detailed proof.

The extended $A_\infty$-module maps in the homological perturbation lemma have a concrete description below. The structure maps on $Z$ are given by the diagrams shown in Figure 2.1. Therein $m_{\geq 1}$ denote the $A_\infty$ structure maps of $M$, with $m_1$ excluded, and $\Delta$ is the comultiplication on the tensor algebra $T^*A$.

2.3. Free resolutions and $A_\infty$-actions. In this section, we describe a useful relation between free resolutions and $A_\infty$-module structures. We assume that $A$ is an algebra over $F = \mathbb{Z}/2$. Suppose that $(AM, m_j)$ is a type-$A$ module which has $m_j = 0$ for $j \neq 2$. That is, $M$ is a left $A$-module in the ordinary sense. A free resolution of $M$ is a collection of free $A$-modules $(F_i, f_i)_{i \in N}$ and $A$-linear maps, which form an exact sequence of the following form:
\[ \cdots \to F_i \xrightarrow{f_i} F_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0. \]

For such a free resolution, write $\mathcal{F}$ for the chain complex which is the direct sum of the $F_i$.

By definition, a free resolution $\mathcal{F}$ is quasi-isomorphic to $AM$, since the canonical projection map from $\mathcal{F}$ to $M$ is a chain map which induces an isomorphism on homology. In the category of $A_\infty$-modules, quasi-isomorphisms are always invertible as $A_\infty$ morphisms. An exposition of this principle may be found in [Kel01, Section 4]. In our present case, it is also helpful to construct explicitly the homotopy equivalence, since we will use it and similar homotopy equivalences later.

Proposition 2.5. Suppose that $\mathcal{F}$ is a free resolution of a left $A$-module $AM$. Then $\mathcal{F}$ and $AM$ are homotopy equivalent as $A_\infty$-modules over $A$. 
\[
\hat{m}(a, z) = a z \Delta m > 1 \Delta h m > 1 
\]

\[
\hat{i}(a, z) = a z \Delta m > 1 \Delta h m > 1 
\]

\[
\hat{\pi}(a, x) = a x \Delta h m > 1 \Delta h m > 1 
\]

\[
\hat{h}(a, x) = a x \Delta h m > 1 \Delta h m > 1 
\]

Figure 2.1. The maps appearing in the homological perturbation lemma for \(A_\infty\)-modules.

Proof. There is a canonical projection map \(\pi: \mathcal{F} \rightarrow M\), which is just \(f_0\) on \(F_0\), and 0 on every other summand. We pick any map of graded \(\mathcal{F}\)-vector spaces \(i: M \rightarrow F_0\) such that \(f_0 \circ i = \text{id}_M\). The map \(i\) induces a direct sum splitting (of \(\mathcal{F}\)-vector spaces) \(\mathcal{F}_0 = \text{im} f_1 \oplus \text{im} i\). The map \(f_1: F_1 \rightarrow \text{im} f_1\) is surjective, so we pick a section of \(\mathcal{F}\)-vector space maps, which we denote by \(h_0\). The map \(h_0\) induces a splitting of \(F_1\) into \(\text{im} h_0 \oplus \text{im} f_2\). We define a map \(h_1: \text{im} f_2 \rightarrow F_2\) to be a splitting of the map \(f_2\). We proceed in this manner to split the entire free resolution to obtain a diagram

\[
\cdots \rightarrow \text{im} f_{i+1} \oplus \text{im} h_{i-1} \rightarrow \text{im} f_i \oplus \text{im} h_{i-2} \rightarrow \cdots \rightarrow \text{im} f_2 \oplus \text{im} h_0 \rightarrow \text{im} f_1 \oplus \text{im} i \rightarrow M \rightarrow 0.
\]

Clearly the maps \(i\) and \(\pi\) are chain maps. Furthermore, \(\pi \circ i = \text{id}_M\), and

\[
i \circ \pi = \text{id}_{\mathcal{F}} + [\partial, h]
\]

where \(h\) is the direct sum of the \(h_i\). Additionally,

\[
h \circ i = 0 \quad h \circ h = 0 \quad \text{and} \quad \pi \circ h = 0.
\]

In particular, the homological perturbation lemma induces an \(A_\infty\)-module structure on \(M\), which we denote by \(\hat{A}\mathcal{M}\), such that \(\hat{A}\mathcal{F}\) and \(\hat{A}\mathcal{M}\) are homotopy equivalent as \(A_\infty\)-modules. We claim that the higher actions on \(\hat{A}\mathcal{M}\), vanish. Indeed, the map \(h\) always maps \(F_i\) to \(F_{i+1}\), while \(m_2^\mathcal{F}\) preserves the index \(F_i\), and \(\pi\) is only non-vanishing on \(F_0\). A quick inspection of the left-most map in Figure 2.1 shows that \(m_j = 0\) for \(j > 2\).

From the claim it follows that \(\hat{A}\mathcal{M} = \hat{A}\mathcal{M}\), completing the proof. \(\square\)

The following is a helpful restatement of the above result:
Corollary 2.6. Suppose \( \mathcal{A} \) is an algebra over \( \mathbb{F} = \mathbb{Z}/2 \). Let \( \mathcal{A}M \) be an \( A_\infty \)-module over \( \mathcal{A} \), and let \( \mathcal{F} \) be the total complex of a free resolution of \( H_*(\mathcal{M}) \). Then \( \mathcal{A}M \) is homotopy equivalent to \( \mathcal{A} \mathcal{F} \) as an \( A_\infty \)-module if and only if \( \mathcal{A}M \) is homotopy equivalent as an \( A_\infty \)-module to \( H_*(\mathcal{M}) \), equipped with vanishing \( m_1 \) and vanishing \( m_j \) for \( j > 2 \).

2.4. Example of non-formal chain complexes. As we mentioned in Subsection 2.1 if \( \mathcal{A} \) is a field, then any finitely generated chain complex is homotopy equivalent to its homology. If \( \mathcal{A} \) is a PID, then it is not hard to show that any finitely generated free chain complex is quasi-isomorphic to its homology. For general rings, this is not always the case. In this section, we illustrate the case \( \mathcal{A} = \mathbb{F}[\mathcal{U}, \mathcal{V}] \) with examples from the theory of knot Floer homology.

We consider the two complexes \( C \) and \( D \) shown below:

\[
C = u \oplus \frac{\mathcal{V}}{y} \quad \text{and} \quad D = \frac{\mathcal{V}}{y} \oplus \frac{\mathcal{U}}{z} \quad \text{and} \quad a \quad b
\]

For appropriate choices of gradings, there is an isomorphism \( H_*(C) \cong H_*(D) \), since both are isomorphic to \( \mathbb{F}[\mathcal{U}, \mathcal{V}] \oplus \mathbb{F} \), where \( \mathbb{F} \) has vanishing action of \( \mathcal{U} \) and \( \mathcal{V} \). On the other hand, it is easy to see that \( C \) and \( D \) are not homotopy equivalent over \( \mathbb{F}[\mathcal{U}, \mathcal{V}] \). For example, if we tensor both with the module \( \mathbb{F}[\mathcal{U}, \mathcal{V}]/(\mathcal{U}, \mathcal{V}) \) and take homology, we obtain vector spaces of different rank over \( \mathbb{F} \).

We now equip both \( H_*(C) \) and \( H_*(D) \) with \( A_\infty \)-actions by applying the homological perturbation lemma. Since \( C \) is a free resolution of its homology, the induced \( A_\infty \)-action has only \( m_2 \) non-trivial.

For \( D \), the homology is the \( \mathbb{F} \)-span of \( \mathcal{U}^i \mathcal{V}^j (\mathcal{U} a + \mathcal{V} c) \) and \( b \), where \( i, j \in \mathbb{N} \). We may define a homotopy equivalence of chain complexes over \( \mathbb{F} \) with \( D \) and the complex \( \mathbb{F}[\mathcal{U}, \mathcal{V}] \oplus \mathbb{F} \) (with vanishing differential). Write \( x \) for the generator of \( \mathbb{F}[\mathcal{U}, \mathcal{V}] \) and write \( y \) for the generator of \( \mathbb{F} \).

The inclusion map \( I \) is the obvious one. We define a projection map \( \Pi \) by setting

\[
\Pi(\mathcal{U}^i \mathcal{V}^j a) = \begin{cases} 
\mathcal{U}^{i-1} \mathcal{V}^j x & \text{if } i > 0 \\
0 & \text{otherwise.}
\end{cases}
\]

The map \( \Pi \) sends \( b \) to the generator of \( \mathbb{F} \). The map \( \Pi \) vanishes on multiples of \( c \). The maps \( \Pi \) and \( I \) are clearly chain maps, and \( \Pi \circ I = \text{id} \). There is also a homotopy \( H : D \to D \), which vanishes on multiples of \( a \) and \( c \), and acts on multiples of \( b \) by the formula

\[
H(\mathcal{U}^i \mathcal{V}^j b) = \begin{cases} 
\mathcal{U}^{i-1} \mathcal{V}^j c & \text{if } i > 0 \\
\mathcal{V}^{j-1} a & \text{if } i = 0.
\end{cases}
\]

It is straightforward to see that \( I \circ \Pi = \text{id} + [\partial, H] \). Furthermore, \( H \circ I, H \circ \Pi \) and \( \Pi \circ H \) vanish. In particular, the maps \( I, \Pi \) and \( H \) induce an \( A_\infty \)-module structure on \( H_*(D) \). We compute easily that \( m_3(\mathcal{U}, \mathcal{V}, y, x) = x \).

Remark 2.7. In Heegaard Floer theory, it is common to also consider the ring \( \mathbb{F}[\mathcal{U}, \mathcal{V}]/\mathcal{U} \mathcal{V} \) (see, e.g., [DHST21]). A similar computation as above shows that \( H_*(C/\mathcal{U} \mathcal{V}) \cong H_*(D/\mathcal{U} \mathcal{V}) \).

3. Plumbled manifolds and plumbled links

The goal of this section is to recall notions like plumbled manifolds and resolution graphs. More importantly, we describe links, for which our methods work.

3.1. Review of plumbled manifolds. To set up the notation, we recall the constructions of 3-manifolds via plumbing. We refer the reader to [Neu81] for a detailed exposition.

Suppose \( G \) is a finite graph. We let \( V_G \) be the set of its vertices. We assume that each \( v \in V_G \) has an associated weight \( \lambda_v \in \mathbb{Z} \). From \( V_G \) we construct a real four-manifold \( X_G \), as follows. For each \( v \in V_G \), we take \( T_v \), the oriented disk bundle over \( S^2 \) with Euler number \( \lambda_v \). The manifold
\(X_G\) is obtained by taking a disjoint union of all the \(T_v\) and gluing them using the following recipe. Whenever two vertices \(v, v' \in V_G\) are connected by an edge \(e\), we trivialize the bundles \(T_v\) and \(T_{v'}\) over chosen disks in the base. Then, we glue these bundles together by an orientation-preserving diffeomorphism that swaps the base and the fiber. Refer to [GS99, Example 4.6.2] or [Neu81] for more details.

By convention, if \(G\) is not connected, we take a boundary connected sum of manifolds \(X_{G_i}\) corresponding to connected components \(G_i\) of \(G\).

**Definition 3.1.** The manifold \(X_G\) is called the *plumbed 4-manifold* associated with \(G\). The boundary \(Y_G = \partial X_G\) is the *plumbed 3-manifold* associated with \(G\).

It is possible to encode a class of links in a plumbed manifold. To this end, suppose \(\Gamma\) is a graph with vertices partitioned into two sets \(V_G \cup V_\uparrow\). We call \(V_\uparrow\) the *arrow* vertices, and we call \(V_G\) the *non-arrow* vertices. We do not add weights to \(V_\uparrow\).

**Remark 3.2.** From a topological perspective, it is most natural to require each vertex of \(V_\uparrow\) to have valence 1. However, in the combinatorial construction of link lattice homology, we do not need to make this assumption.

We write \(G \subseteq \Gamma\) for the full subgraph spanned by the non-arrow vertices. The vertices \(V_\uparrow\) determine a link \(L_\uparrow\) in \(Y_G\) as follows. Suppose \(v \in V_\uparrow\) is adjacent to a non-arrow vertex \(w \in V_G\). We let \(L_v\) be a circle fiber of the \(S^1\)-bundle \(\partial T_w \to S^2\), such that the projection of \(L_v\) onto \(S^2\) is disjoint from all the disks used to plumb the disk bundles of other non-arrow vertices. If more than one arrow vertex is adjacent to the same non-arrow vertex \(w\), we require each of the corresponding components of \(L_\uparrow\) to be fibers of \(\partial T_w \to S^2\) over distinct points.

We define \(L_\uparrow\) as the union of the circle fibers \(L_v\) ranging over \(v \in V_\uparrow\).

**Definition 3.3.** The link \(L_\uparrow \subseteq Y_G\) is called the *plumbed link* associated with \(\Gamma\). We say that a link \(L \subseteq Y\) is a *plumbed link* if there exists an arrow-decorated plumbing graph \(\Gamma\) and a diffeomorphism \((Y, L) \cong (Y_G, L_\uparrow)\).

**Remark 3.4.** One can also consider more general plumbings of disk bundles over higher genus surfaces. To do so, one considers a plumbing graph where each vertex \(v \in V_G\) is assigned an additional weight, corresponding to the genus of the base space of disk bundle. However, the resulting manifold \(Y_G\) is not a rational homology sphere if at least one surface has positive genus.

In the present paper, we are mostly concerned with rational homology spheres, so we restrict the discussion to the case where all surfaces are spheres. We refer to [Neu81] for more details.

**Definition 3.5.** Suppose \(G\) is a plumbing tree with no arrow vertices. We define the **incidence matrix** \(Q_G\), as follows. The diagonal entries are the weights associated to vertices, while the off-diagonal terms are 1 or zero, depending on whether the two vertices are connected by an edge.

By construction, \(Q_G\) represents the intersection form on \(X_G\). As the intersection form on \(X_G\) determines the homology of \(Y_G\), we have:

**Lemma 3.6.** There is an isomorphism \(H_1(Y_G; \mathbb{Z}) \cong \text{coker} Q_G\). In particular, \(Y_G\) is a rational homology sphere if and only if \(\det Q_G \neq 0\).

There is a well known description of \((Y_G, L_\uparrow)\) in terms of Dehn surgery. See [GS99, Example 4.6.2]. We form a partitioned link \(L_\Gamma = L_G \cup L_\uparrow\) in \(S^3\), as follows. For each vertex of \(\Gamma\), we add an unknotted component to \(L_\Gamma\), and for each edge, we add a clasp between the corresponding components. The link \(L_\Gamma\) may alternatively be described as an iterated connected sum of positive Hopf links. The weights on the vertices in \(V_G\) determine an integral framing \(\Lambda\) on \(L_G\), as in Definition 3.5. Then

\[
(Y_G, L_\uparrow) \cong (S^3_\Lambda(L_G), L_\uparrow).
\]

We remark a slight abuse of notation: \(L_\uparrow\) denotes both the link in \(S^3\) (as a part of \(L_\Gamma \subseteq S^3\)) and its image in the plumbed manifold \(Y_G\).
3.2. Plumbed manifolds and resolutions of analytic singularities. One of the main motivations for introducing plumbed manifolds comes from resolutions of singularities. We give now a short account on plumbed manifolds obtained from surface singularities. We refer the reader to introductory lectures of Némethi [Ném99], or to [Ném22, Section 3.3], [Loo13], [NS12, Chapter 4] for more details and references.

First we focus on absolute case corresponding to graphs with no arrow vertices. Later on we discuss the relative case of embedded resolutions, leading to plumbed links.

Suppose $(X, x_0)$ is (a germ of) a normal complex analytic surface. The word ‘normal’ refers to the property of the local ring $\mathcal{O}_{x_0}(X)$ being integrally closed, see [Har77, Exercise I.3.7]. It implies, among other things, that $x_0$ is an isolated singular point. See [Lau71, Ném99] for more details. The surface $(X, x_0)$ can be analytically embedded into $(\mathbb{C}^N, 0)$ for $N$ sufficiently large. Let $B_\varepsilon$ be a ball in $\mathbb{C}^N$ with center at 0 and radius $\varepsilon > 0$. It is known, see [Mil68], that the diffeomorphism type of the intersection $L_X := X \cap \partial B_\varepsilon$ is independent of $\varepsilon$ and of the embedding of $X$ into $\mathbb{C}^N$, provided $\varepsilon > 0$ is small enough. Moreover, the pair $(B_\varepsilon, X \cap B_\varepsilon)$ is topologically a cone over $(\partial B_\varepsilon, L_X)$. The space $L_X$ is a smooth real 3-dimensional manifold. We call it the *link of the surface singularity* $(X, x_0)$.

We stress that we study local behavior of $X$ near $x_0$. From the perspective of algebraic geometry, this is emphasized by saying that $X$ is a germ of a surface. The reader unfamiliar with this notion, might assume that we replace $X$ by $X \cap B_\varepsilon$, where $B_\varepsilon$ is as above.

The manifold $L_X$ admits another description. We let $(\tilde{X}, E)$ be a resolution of $(X, x_0)$, that is, a smooth complex analytic surface together with a map $\pi: (\tilde{X}, E) \to (X, x_0)$, which is one-to-one except on $\pi^{-1}(x_0) = E$. Now $E = \sum E_i$ is a union of smooth complex curves (Riemann surfaces) intersecting transversally. Each of the $E_i$ is assigned a number $\lambda_i$ which is its self-intersection. The curves $E_i$ are referred to as the *exceptional components* of the map $\pi$.

With the resolution we can assign two objects. One is the *dual graph* $G_X$ of the resolution. Its vertices correspond to divisors $E_i$. Each vertex is assigned to a weight $\lambda_i$. There is an additional weight of a vertex by the genus of $E_i$ (in the present paper we will consider only the case where each of the $E_i$ is a sphere, compare Remark 3.4). We add $|E_i \cap E_j|$ edges between vertices $v_i$ and $v_j$, when $i \neq j$. We add no self-edges. There is a matrix $Q_G$ associated with $G_X$ as in Definition 3.5. We have the following result.

**Proposition 3.7.**

(a) The link $L_X$ of singularity $(X, x_0)$ is diffeomorphic to $Y_{G_X}$.

(b) $L_X$ is a rational homology sphere if each of the $E_i$ is a sphere and $G_X$ is a tree.

The manifold $L_X$ determines the graph $G_X$ up to a precisely described equivalence relation; see [Neu81]. The way the resolution is constructed implies that $Q_G$ is negative definite. A deep theorem of Grauert [Gra62] shows that this characterizes links of analytic singularities among all plumbed links.

**Theorem 3.8.** If $Q_G$ is negative definite, then $Y_G$ is a link of an analytic singularity.

We stress that the statement is far from true if the word ‘analytic’ is replaced by ‘algebraic’. Moreover, in general, there is no uniqueness. While $G_X$ determines the diffeomorphism type of $L_X$, there might be analytically different singularities with the same link $L_X$. There is a vast research area concerning which invariants of $X$ depend on the link $L_X$, and which depend on the analytic structure. We refer the reader to the book [Ném22]. Examples of invariants depending on the analytic structure of $X$ include geometric genus $p_g$ (see [Ném22, Section 6.8]), the embedding dimension (minimal $N$ for which $X$ embeds into $\mathbb{C}^N$, see [Ném22, Example 6.7.17]), and the Hilbert-Samuel function [Ném22, Section 5.1.40]).

3.3. Embedded resolutions. We now consider embedded singularities, which are pairs of analytic spaces $Z \subseteq X$, with a point $x_0 \in Z$, where $X$ and $Z$ are possibly singular at $x_0$. We restrict
our attention to the case where $\dim_{\mathbb{C}} X = 2$ and $\dim_{\mathbb{C}} Z = 1$. References include [EN85, Pic01], [Ném22, Section 2.2], as well as a brief overview in [NS12, Section 4.3].

**Definition 3.9.** An embedded singularity is a triple $(X, Z, x_0)$, where $(X, x_0)$ is a (germ of a) normal complex analytic surface and $Z \subseteq X$ is a complex analytic curve passing through $x_0$.

**Example 3.10.** If $X = \mathbb{C}^2$, an embedded singularity is precisely a plane curve singularity.

Embed $X$ analytically in $\mathbb{C}^N$ with $x_0$ mapped to 0. Take a small ball $B_\varepsilon$ around $x_0$ in $\mathbb{C}^N$ as above. For sufficiently small $\varepsilon > 0$, the triple $(B_\varepsilon, X \cap B_\varepsilon, Z \cap B_\varepsilon)$ is topologically a cone over $(\partial B_\varepsilon, L_X, L_Z)$, where $L_X$ and $L_Z$ are, respectively, intersections of $X$ and $Z$ with $\partial B_\varepsilon$. The diffeomorphism type of the pair $(L_X, L_Z)$ depends on neither the choice of $\varepsilon$ nor the choice of embedding.

**Definition 3.11.** The pair $(L_X, L_Z)$ is called the link of the embedded singularity.

**Example 3.12.** Suppose $(X, x_0) = (\mathbb{C}^2, 0)$, and $Z$ is a plane algebraic curve passing through 0. Then, $L_X = S^1$, and the link $L_Z$ is precisely the algebraic link in the ordinary sense.

Since the study of singularities is local, we consider only the germ of the singularity. We note that, by definition, $(X, Z)$ and $(X \cap B_\varepsilon, Z \cap B_\varepsilon)$ have the same germ. In particular, we may and will assume that $(X, Z)$ is a topologically a cone over $(L_X, L_Z)$.

We can recover $(L_X, L_Z)$ from an embedded resolution. By an embedded resolution of $(X, Z, x_0)$ we mean the triple $(\tilde{X}, \tilde{Z}, E)$ together with a proper analytic map $\pi: (\tilde{X}, \tilde{Z}, E) \to (X, Z, x_0)$ with the following conditions

- $\pi: \tilde{X} \to X$ is one-to-one away from $E$. In particular, the restriction $\pi|_\tilde{Z}$ is one-to-one away from $\tilde{Z} \cap E$;
- $\tilde{X}$ is a smooth surface and $\tilde{Z}$ is a smooth complex curve;
- Each algebraic component of $E$ is a projective (that is, closed) smooth complex curve;
- The union $E \cup \tilde{Z}$ has only transverse double points as singularities;

As in the non-embedded case, the smooth complex curves whose union in $E$ are referred as the exceptional components.

Given the embedded resolution, we can create a dual graph of the resolution. The construction is in two steps. First, out of $E$, we construct the graph $G_X$ as above. Next, if $v_i \in V_{G_X}$ and $E_i$ is the corresponding component of $E$, we adjoin $|E_i \cap \tilde{Z}|$ arrow vertices to $v_i$. We denote the resulting graph $\Gamma_{X, Z}$. Recall that we work locally (topologically, we have replaced $X$ by $X \cap B_\varepsilon$).

Therefore, $\tilde{Z}$ is the union of disks, each intersecting the graph $E$ precisely at one point. That is, every arrow vertex of $\Gamma_{X, Z}$ corresponds to a connected component of $\tilde{Z}$.

**Proposition 3.13.** The pair $(Y_{G_X}, L_1)$ is diffeomorphic to $(L_X, L_Z)$.

Our next aim is to explain the relative analog of Grauert’s Theorem 3.8. As the statement is slightly technical, we give some extra explanation. Let $g: X \to \mathbb{C}$ be a reduced analytic map such that $g^{-1}(0) = Z$. Here, reduced means that $g$ is not divisible by a square of a non-invertible analytic function on $X$. Then, $g$ induces an analytic map $\tilde{g}: \tilde{X} \to \mathbb{C}$ via $\tilde{g} = g \circ \pi$. Let $v$ be a vertex of $\Gamma := \Gamma_{X, Z}$. The vertex $v$ corresponds either to an exceptional component $E_v$ (if $v$ is a non-arrow vertex), or to a component $\tilde{Z}_v$ of $\tilde{Z}$, if $v$ is an arrow vertex. In both cases, $\tilde{g}$ vanishes on that component. We let $m_v > 0$ denote the order of vanishing. This quantity is called the multiplicity of the vertex $v$. Note that since $g$ is reduced, $m_v = 1$ for all arrow vertices; see [NS12, Section 4.3.2].

The multiplicities and the weights satisfy the following compatibility relation (see [NS12, Equation (4.1.5)]):

$$\lambda_v m_v + \sum_{w \in V_v} m_w = 0,$$

for each non-arrow vertex $v \in V_G$, where $V_v$ denote the set of all vertices in $V_{\Gamma}$ adjacent to $v$. Note that (3.1), together with the condition $m_v = 1$ for all arrow vertices, determines uniquely all other
multiplicities. However, unless $Q_G$ is unimodular, the multiplicities need not be integral. If that is the case, such a plumbed link cannot be realized as an embedded link of an analytic singularity.

We now explain a relative analog of Grauert’s Theorem 3.8. For reference, see [Pic01, Corollaire 5.5].

**Proposition 3.14.** Let $\Gamma$ be a graph with vertices $V = V_G \cup V_\ell$ such that the $Q_G$ is negative definite. If assigning multiplicity $1$ to each arrow vertex of $\Gamma$ leads to integral positive multiplicities on all vertices of $V_G$ via the compatibility relation (3.1), then $(Y_G, L_{\Gamma})$ is a link of an embedded analytic singularity.

3.4. **Rationality.** Suppose $(X, x_0)$ is an analytic singularity. We define the geometric genus $p_g = h^1(O_X)$; see [Ném99, Section 2]. Many properties of geometric genus are given in various chapters of [Ném22]. The definition of $p_g$ does not depend on the choice of resolution. Geometric genus is an invariant of the analytic structure of $X$; there are known examples of singularities with the same link, but different geometric genus, see [Ném99, Paragraph 4.8]. Put differently, in general $p_g$ cannot be read off from the combinatorics of the resolution graph $G$.

An exception is the case of rational singularities, which are characterized by the property that $p_g = 0$; see [Ném22, Section 7.1]. Given a graph $G$, we can determine, whether it represents a rational singularity; this result is due to Artin [Art66], see also [Ném99, Theorem 3.8] and [Ném22, Theorem 7.1.2]. For instance, if $Y_G$ is a link of a rational singularity, then $b_1(Y_G) = 0$.

By studying the relation between $p_g$ and combinatorial invariants of $X_G$ encoded by $G$, Némethi proves the following groundbreaking result:

**Theorem 3.15 ([Ném17]).** The singularity $(X, x_0)$ is rational if and only if $L_X$ is an L-space.

The proof of Theorem 3.15 can also be done using recently proved equivalence of lattice and Heegaard Floer homology [Zem21b]. As the first step, one uses Némethi’s theorem stating that if $G$ is negative definite, then $X$ is a rational singularity if and only if the reduced lattice homology of $Y_G$ is zero; see [Ném05,Ném08]. Next, one refers to [Zem21b] to show the reduced lattice homology of $Y_G$ vanishes if and only if $Y_G$ is an L-space.

**Remark 3.16.** To the best of our knowledge, Theorem 3.15 characterizes only graphs representing an L-space among graphs with negative definite incidence matrix $Q_G$. There are indefinite graphs representing L-spaces. For example, if $G$ is a linear plumbing such that $Q_G$ is non-degenerate, then $Y_G$ is a lens space, regardless of whether $Q_G$ is definite or not.

3.5. **Algebraic links and L-space links.** Let us recall the following definition.

**Definition 3.17.** Let $L$ be a link in a rational homology 3-sphere. We say $L$ is an L-space link if all sufficiently large positive surgeries on $L$ are L-spaces.

Though usually one considers L-space links in $S^3$ or an integer homology sphere (see e.g. [Liu17, GN16]), we note that the same definition can be applied to links in rational homology 3-spheres. Suppose $K$ is a rationally null-homologous knot in $Y$, then Morse framings on $K$ can be identified with an affine $\mathbb{Z}$ subspace of $\mathbb{Q}$ by taking the intersection number of the framing (viewed as a parallel longitude of $K$) with a rational Seifert surface. Hence, large surgeries on rationally null-homologous links are surgeries with Morse framings which are sufficiently large in $\mathbb{Q}^3$ with respect to this identification.

In [GN16] Gorsky and Némethi studied which plumbed links in plumbed manifolds are L-space links. Their main result is that an algebraic link in $S^3$ is an L-space link. Their proof works in a more general setting, leading to the following statement, which is given in [GN16, Theorem 12] and the ensuing discussion.

**Proposition 3.18.** Suppose $\Gamma$ is a graph such that $Q_G$ is negative definite and $Y_G$ is a link of a rational singularity. Then $L_{\Gamma}$ is an L-space link.
A careful analysis of their proof reveals that the assumption on multiplicities as in Proposition 3.14 is never used. That is, Proposition 3.18 does not require that the link $L_1$ be a link of an analytic singularity. Proposition 3.18 only makes a restriction on the graph $G$.

It is well-known that an algebraic knot is determined by its Alexander polynomial. A natural question is whether this result generalizes to plumbed L-space links. The following well-known fact is due to Yamamoto.

**Proposition 3.19** (see [Yam84]). *Two algebraic links in $S^3$ with the same Alexander polynomial are equal.*

The result of Yamamoto relies on the classification of algebraic links in $S^3$, due to Zariski [Zar32]; we refer to [EN85] for this characterization. In particular, this result does not admit direct generalizations to links in other 3-manifolds. We give now a few counterexamples for some naive attempts to generalize the result.

**Proposition 3.20.**

(a) There exist non-isotopic plumbed L-space links in $S^3$ with the same Alexander polynomial;

(b) There exist non-isotopic knots that are links of embedded analytic surface singularities with the same Alexander polynomial.

**Proof.** Item (a) is classical. We know that the $(2, 3)$-cable on the positive trefoil is a plumbed knot, and it is an L-space knot by [Hed09]. However, its Alexander polynomial is the same as that of the $T(3, 4)$ torus knot.

Item (b) expands on results of Campillo, Delgado and Gussein-Zade [CDGZ20]. In fact, in [CDGZ20, Example 3.2] there are two knots in the Poincaré sphere with the same Alexander polynomial, represented by two plumbing diagrams. An explicit algorithm described in [EN85, Chapter 20] transforms these plumbing diagrams into graph links, which are presented in Figure 3.1. Using the algorithm of [Neu83], we compute the signature functions of these links, and we present them in Figure 3.2, omitting straightforward calculations. The signatures are different, so the knots are different, in fact, the knots are even not concordant.

**□**

4. **Link lattice homology**

In this section, we recall some basics about Heegaard Floer homology, and subsequently define our link lattice complex.
4.1. Background on link Floer homology. To fix the notation and terminology, we give some necessary background on link Floer homology. We assume some familiarity with basics of Heegaard Floer homology and its refinements for knots, see [OS04b, OS04a, OS08a, Ras03].

Let $L$ be an $\ell$-component link in a 3-manifold $Y$. Recall [OS08a, Section 3.5] that the pair $(Y, L)$ can be encoded in a multi-pointed Heegaard link diagram $(\Sigma, \alpha, \beta, w, z)$, as follows:

1. $\Sigma$ is a closed oriented genus $g$ surface;
2. $\alpha = \{\alpha_1, \ldots, \alpha_{g+\ell-1}\}$ and $\beta = \{\beta_1, \ldots, \beta_{g+\ell-1}\}$ are collections of simple closed curves on $\Sigma$. The curves $\alpha_i$ are pairwise non-intersecting. Also, the curves $\beta_i$ are pairwise non-intersecting. Moreover, $\alpha$ and $\beta$ each span a $g$-dimensional subspace of $H_1(\Sigma; \mathbb{Z})$;
3. $w = \{w_1, \ldots, w_\ell\}$, $z = \{z_1, \ldots, z_\ell\}$. Each component of $\Sigma \setminus \alpha$ (respectively of $\Sigma \setminus \beta$) contains a single point of $w$ and a single point of $z$.

It is not hard to see there exists a Heegaard link diagram for any pair $(Y, L)$. Furthermore, any two diagrams can be connected by sequence of Heegaard moves for link diagrams. See [OS08a, Theorem 4.7].

Given a Heegaard link diagram, we consider Lagrangian tori

$$T_\alpha = \alpha_1 \times \cdots \times \alpha_{g+\ell-1}, \quad T_\beta = \beta_1 \times \cdots \times \beta_{g+\ell-1}$$

in the symmetric product $\text{Sym}^{g+\ell-1}(\Sigma)$. The link Floer chain complex, $\text{CFL}(Y, L)$, is a free chain complex over

$$\mathcal{R}_\ell = \mathbb{F}[\mathcal{H}_1, \mathcal{V}_1, \ldots, \mathcal{H}_\ell, \mathcal{V}_\ell]$$

generated by intersection points $x \in T_\alpha \cap T_\beta$ with the differential counting pseudo-holomorphic curves in $\text{Sym}^{g+\ell-1}(\Sigma)$ via:

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \left(\# \mathcal{M}(\phi)/\mathbb{R}\right) \mathcal{H}^{n_1}_1(\phi) \cdots \mathcal{H}^{n_\ell}_\ell(\phi) \mathcal{V}_1^{m_1}(\phi) \cdots \mathcal{V}_\ell^{m_\ell}(\phi) y.$$

Here the sum is taken over all homotopy classes $\pi_2(x, y)$ of maps $\phi$ of a unit disk $D \subseteq C$ to $\text{Sym}^{g+\ell-1}(\Sigma)$, where $\phi(-1) = x$, $\phi(1) = y$, $\phi(\partial D \cap \{\text{im}(z) \leq 0\}) \subseteq T_\alpha$, $\phi(\partial D \cap \{\text{im}(z) \geq 0\}) \subseteq T_\beta$. Here, $\mu(\phi)$ denotes the Maslov index of the class $\phi$. The space $\mathcal{M}(\phi)$ consists of all pseudo-holomorphic curves representing the class $\phi$, for a generic 1-parameter family of almost complex structures on $\text{Sym}^{g+\ell-1}(\Sigma)$. For $x \in \Sigma \setminus (\alpha \cup \beta)$, we denote by $n_x(\phi)$ the intersection number of $\{x\} \times \Sigma^{g+\ell-2} \subseteq \text{Sym}^{g+\ell-1}(\Sigma)$ with $\phi(D)$. We refer to [OS04b] for more details.

There is a map $s_w$ from $T_\alpha \cap T_\beta$ to the set of Spin$^c$ structures on $Y$. The component of the map $\partial$ from $x$ to $y$ can be non-trivial only if $s_w(x) = s_w(y)$. That is to say, the chain complex $\text{CFL}(Y, L)$ splits as a direct sum over complexes $\text{CFL}(Y, L, s)$, for $s \in \text{Spin}^c(Y)$.

There is completed version of $\text{CFL}(Y, L)$ regarded as a module over the ring of power series

$$\mathcal{R}_\ell := \mathbb{F}[\mathcal{H}_1, \mathcal{V}_1, \ldots, \mathcal{H}_\ell, \mathcal{V}_\ell],$$

namely we set

$$\text{CFL}(Y, L) := \text{CFL}(Y, L) \otimes_{\mathcal{R}_\ell} \mathcal{R}_\ell.$$

In other words, $\text{CFL}(Y, L)$ has the same generators as $\text{CFL}(Y, L)$ and the same differential, except that we work over a larger ring. The completed version appears in the surgery formula.

When $Y$ is a rational homology 3-sphere, the link Floer homology groups have several gradings. Firstly, there is a $\mathbb{Q} \times \mathbb{Q}$-valued Maslov bigrading, denoted $\text{gr}_w, \text{gr}_z$, as well as a $\mathbb{Q}^\ell$-valued Alexander grading $A$. Furthermore

$$\text{gr}_w(\mathcal{H}_i) = (-2, 0), \quad \text{gr}_w(\mathcal{V}_i) = (0, -2) \quad \text{and} \quad A(\mathcal{V}_i) = -A(\mathcal{H}_i) = e_i,$$

where $e_i$ is the standard $i$-th coordinate vector in $\mathbb{Q}^\ell$. Note also that

$$\text{gr}_w - \text{gr}_z = 2 \sum_{i=1}^\ell A_i.$$
Remark 4.1. In this paper, we normalize $\text{gr}_w$ so that the isomorphism

$$H_*((\text{CFL}(Y, L))/\langle \gamma_1 - 1, \ldots, \gamma_\ell - 1 \rangle) \cong H\text{F}^{-}(Y)$$

is grading preserving. In the above, we are writing $H\text{F}^{-}(Y)$ for the Heegaard Floer homology computed with a singly pointed Heegaard diagram for $Y$. We make a similar normalization for $\text{gr}_v$. Equivalently, our grading convention is that $H\text{F}^{-}$ of a 3-manifold is invariant under adding extra basepoints as a graded module; compare [OS08a, Section 6.1]. We note that some authors normalize the Maslov gradings so that $H_*((\text{CFL}(Y, L))/\langle \gamma_1 - 1, \ldots, \gamma_\ell - 1 \rangle)$ is isomorphic instead to $H\text{F}^{-}(Y)[(\ell - 1)/2]$.

4.2. Lattice homology. We recall the definition of lattice homology [Ném08]. We use the notation of Ozsváth, Stipsicz and Szabó [OSS14b], since our construction of link lattice homology is slightly easier to describe using their notation. Let $G$ be a plumbing tree, and write $V_G$ for the vertices of $G$. Write $\mathbb{P}(V_G)$ for the power set of $V_G$ (i.e. the set of all subsets of $V_G$). The lattice complex is the $\mathbb{F}[\llbracket U \rrbracket]$ module

$$\text{CFL}(G) := \prod_{[K, E] \in \text{Char}(G) \times \mathbb{P}(V_G)} \mathbb{F}\llbracket U \rrbracket \otimes \langle [K, E] \rangle,$$

where $\text{Char}(G) \subseteq H^2(X_G, \mathbb{Z})$ denotes the set of characteristic elements of $H^2(X_G, \mathbb{Z})$ on the 4-manifold $X_G$.

We now define the differential on $\text{CFL}(G)$. Note that each vertex $v \in V_G$ determines an element of $H^2(X_G)$, which is the base space of the disc bundle $T_v$ used in the plumbing construction. One first defines

$$2f(K, I) = \left( \sum_{v \in I} K(v) \right) + \left( \sum_{v \in I} v \right) : \left( \sum_{v \in I} v \right).$$

We set $g(K, E) = \min \{ f(K, I) : I \subseteq E \}$. Next, one defines

$$A_v(K, E) = g(K, E - v) \quad B_v(K, E) = \min \{ f(K, I) : v \in I \subseteq E \}.$$

Set

$$a_v(K, E) = A_v(K, E) - g(K, E) \quad b_v(K, E) = B_v(K, E) - g(K, E).$$

The differential on $\text{CFL}(G)$ is defined by the formula,

$$\partial[K, E] = \sum_{v \in E} U^{a_v(K, E)} \otimes [K, E - v] + \sum_{v \in E} U^{b_v(K, E)} \otimes [K + 2v^*, E - v], \quad (4.1)$$

extended linearly over $\mathbb{F}[\llbracket U \rrbracket]$. We will refer to the first summand in (4.1) as the $A$-terms in the differential, and we will refer to the second summand as the $B$-terms of the differential.

4.3. The link lattice complex. We now suppose that $\Gamma$ is a plumbing tree, whose vertex set is partitioned into two sets:

$$V_\uparrow = V_G \cup V_\updownarrow.$$

Recall that the components of $V_G$ are equipped with a framing, while those of $V_\uparrow$ are not.

The vertices $V_\uparrow$ determine a link $L_\uparrow$ in the 3-manifold $Y_G$. We assume that each component of $L_\uparrow$ is rationally null-homologous in $Y_G$. This occurs, for example, when the incidence matrix $Q_G$ is non-singular.

To define the link lattice complex, we first pick a framing on the components of $V_\uparrow$ arbitrarily. In Proposition 4.10, we will show that the choice of framing on $V_\uparrow$ does not affect the link lattice complex.

We define the link lattice complex $\text{CFL}(\Gamma, V_\uparrow)$ as the quotient of $\text{CFL}(\Gamma)$ by the subspace generated over $\mathbb{F}[\llbracket U \rrbracket]$ by tuples $[K, E]$ where $V_\downarrow \not\subseteq E$. Equivalently, we may view $\text{CFL}(\Gamma, V_\uparrow)$ as being generated by $[K, E]$ where $V_\downarrow \subseteq E$, equipped with quotient complex differential. We think of the differential on $\text{CFL}(\Gamma, V_\uparrow)$ as being given by the same formula as Equation (4.1), except with the sums being taken over only $v \in E \cap V_G$. 

4.4. The module structure. Recall that
\[ R_\ell = \mathbb{F}[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_\ell, \mathcal{V}_\ell]. \]
We now describe the action of \( R_\ell \) on link lattice homology, where \( \ell = |V_\ell| \). Write \( V_\ell = \{v_1, \ldots, v_\ell\} \).
For each \( i \in \{1, \ldots, \ell\} \), there is an induced element \( \mu_i^* \in H_2(X; \mathbb{F}_2) \cong H^2(X; \mathbb{F}_2) \). This element is dual to the class \( \nu_i \in H_2(X) \) in the sense that \( \mu_i^*(\nu_i) = 1 \), and \( \mu_i^*(w) = 0 \) for all \( w \in V_\ell \setminus \{v_i\} \).
The class \( \mu_i^* \) is represented by the co-core disk of the 2-handle corresponding to \( v_i \).

Define the quantities:
\[ \delta^+_i(K, E) = g(K + 2\mu_i^*, E) - g(K, E) \quad \text{and} \quad \delta^-_i(K, E) = g(K - 2\mu_i^*, E) - g(K, E). \]

An easy computation shows that
\[ f(K \pm 2\mu_i^*, I) = f(K, I) \pm 1 \]
if \( v_i \in I \), and \( f(K \pm 2\mu_i^*, I) = f(K, I) \) if \( v_i \notin I \). In particular, we have that
\[ \delta^+_i(K, E) \in \{0, 1\} \quad \text{and} \quad \delta^-_i(K, E) \in \{0, -1\} \]
for all \( i \).

For \( i \in \{1, \ldots, \ell\} \), we define
\[ \mathcal{U}_i \cdot [K, E] = \begin{cases} U[K - 2\mu_i^*, E] & \text{if } \delta^-_i(K, E) = 0 \\ [K - 2\mu_i^*, E] & \text{if } \delta^-_i(K, E) = -1, \end{cases} \]
and
\[ \mathcal{V}_i \cdot [K, E] = \begin{cases} U[K + 2\mu_i^*, E] & \text{if } \delta^+_i(K, E) = 1 \\ [K + 2\mu_i^*, E] & \text{if } \delta^+_i(K, E) = 0. \end{cases} \]

We extend \( \mathcal{U}_i \) and \( \mathcal{V}_i \) to the entire link lattice complex by declaring them to be \( \mathbb{F}[U] \) equivariant. Equivalently, we set
\[ \mathcal{U}_i \cdot [K, E] = U^{g(K - 2\mu_i^*, E) - g(K, E) + 1}[K - 2\mu_i^*, E] \quad \text{and} \quad \mathcal{V}_i \cdot [K, E] = U^{g(K + 2\mu_i^*, E) - g(K, E)}[K + 2\mu_i^*, E]. \]

**Lemma 4.2.** If \( v_i \in V_\ell \), then the endomorphisms \( \mathcal{U}_i \) and \( \mathcal{V}_i \) are chain maps.

**Proof.** The differential of \( CFL(G, V_\ell) \) is given by modifying (4.1) to sum over only \( v \in V_G \). Clearly the summands of \( \partial \mathcal{V}_i[K, E] \) are in bijection with the summands of \( \partial \mathcal{U}_i[K, E] \). It remains to show that the powers of \( U \) coincide. We consider the A-terms of the differential first. The power of \( U \) from \( \mathcal{V}_i \partial [K, E] \) of \([K + 2\mu_i^*, E - v]\) is
\[ g(K + 2\mu_i^*, E - v) - g(K, E - v) + g(K, E - v) - g(K, E) \]
whereas the power of \( U \) from \( \partial \mathcal{V}_i[K, E] \) is
\[ g(K + 2\mu_i^*, E - v) - g(K + 2\mu_i^*, E) + g(K + 2\mu_i^*, E) - g(K, E). \]
These are obviously equal. Similarly, for the B-terms of the differential we use the equality
\[ B_v(K, E) = (K(v) + v \cdot v)/2 + g(K + 2v^*, E - v). \]
From here, the argument is similar to the case of type-A terms. We note that
\[ (K + 2\mu_i^*)(v) + v \cdot v = K(v) + v \cdot v \]
if \( v \neq v_i \). For the B-terms of the differential, the \( U \) power of the term from \( \partial \mathcal{V}_i[K, E] \) is
\[ B_v(K + 2\mu_i^*, E) - g(K + 2\mu_i^*, E) + g(K + 2\mu_i^*, E) - g(K, E). \]
The U-power from \( \mathcal{V}_i \partial [K, E] \) is
\[ g(K + 2v^* + 2\mu_i^*, E - v) - g(K + 2v^*, E - v) + B_v(K, E) - g(K, E). \]
The difference between these terms is \(((K + 2\mu_i^*)(v) + v \cdot v)/2 - (K(v) + v \cdot v)/2\) which vanishes by (4.5).
The claim about the map \(\mathcal{U}_t\) follows from essentially the same logic. □

**Lemma 4.3.**

(1) For each \(i\), we have \(\mathcal{U}_i \gamma_i = \gamma_i \mathcal{U}_t = U\).
(2) For all \(i, j\), the commutators \([\mathcal{U}_i, \gamma_j], [\mathcal{U}_i, \mathcal{U}_j]\) and \([\gamma_i, \gamma_j]\) vanish.

**Proof.** All of the stated relations are easily derived from Equation (4.4). □

**Lemma 4.4.** The action of \(\mathcal{R}_t\) on \(\text{CFL}(\Gamma, V_\Gamma)\) extends to an action of the ring of power series

\[
\mathcal{R}_t = \mathbb{F}[\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_t, \gamma_i].
\]

**Proof.** Due to the fact that we used direct products in the definition of the lattice complex, it is sufficient to show that for each \(y = U^n \otimes [K, E]\), there are at most finitely many monomials \(a \in \mathcal{R}_t\) and generators \(x = U^n \otimes [K', E]\) such that \(a \cdot x = y\). If \(a = \mathcal{U}_1^{i_1} \ldots \mathcal{U}_t^{i_t} \gamma_1^{j_1} \ldots \gamma_t^{j_t} \in \mathcal{R}_t\) is a monomial, we define \(\nu(a) \in H^2(X_\Gamma)\) by the formula

\[
\nu(a) = (j_1 - i_1)\mu_{v_1} + \cdots + (j_t - i_t)\mu_{v_t},
\]

Monomials in \(\mathcal{R}_t\) are equipped with a \(U\)-weight

\[
w_U(a) = \min(i_1, j_1) + \cdots + \min(i_t, j_t).
\]

A generator \(U^n \otimes [K, E]\) also has a \(U\)-weight \(w_U(U^n \otimes [K, E]) = n\). It is straightforward to verify that \(w_U(a \cdot x) \geq w_U(a) + w_U(x)\) for any \(a\) and \(x\). In particular, if \(y = U^n \otimes [K, E]\) is fixed, then there are only finitely many possible \(U\) weights of \(a\) and \(x\) such that \(y = a \cdot x\). Monomials in \(\mathcal{R}_t\) also have an Alexander grading \(A(a) = (j_1 - i_1, \ldots, j_t - i_t)\). Since \(H^2(X_\Gamma)\) is torsion free and of rank \(|L_\Gamma|\), we have that \(\nu(a) = \nu(a')\) if and only if \(A(a) = A(a')\). Also, if \(a \cdot [K', E] = U^n \otimes [K, E]\), then it must be the case that \(K' = K - 2\nu(a)\).

Let \(s \in \{1, \ldots, t\}\). Note that if \(A_s(a) = j_s - i_s\) is sufficiently negative, then \(\gamma_s[K - 2\nu(a), E] = U[K - 2\nu(a) + 2\mu_{v_s}, E]\) by the observation that \((K - 2\nu(a))(v_s) < 0\) and the minima involving \(g(K - 2\nu(a), E)\) and \(g(K - 2\nu(a) + 2\mu_{v_s}, E)\) will both be attained at some \(I \subseteq E\) containing \(v_s\). A similar argument shows that if \(A_s(a)\) is sufficiently positive, then \(\gamma_s[K - 2\nu(a), E] = U[K - 2\nu(a) - 2\mu_{v_s}, E]\).

In particular, it follows from the above reasoning that the set of monomials \(a\) such that there is an \(i\) satisfying \(a \cdot U^i \otimes [K - 2\nu(a), E] = U^n \otimes [K, E]\) is bounded in \(U\)-weight and Alexander grading. However, it is easy to verify that the set of elements in \(\mathcal{R}_t\) in bounded Alexander grading and \(U\)-weight is finite, completing the proof. □

### 4.5. Maslov gradings.

If the intersection form of \(X_\Gamma\) is non-singular, then the lattice complex \(\text{CFL}(\Gamma, V_\Gamma)\) inherits a Maslov grading from \(\text{CFL}(\Gamma)\). We recall the formula

\[
gr(U^i \otimes [K, E]) = -2i + 2g(K, E) + |E| + \frac{1}{4}(K^2 - 3\sigma(X_\Gamma) - 2\chi(X_\Gamma)).
\] (4.6)

Compare [OSS14a, Section 2.3].

In the setting of link lattice homology, it is more natural to define the Maslov grading via the formula

\[
gr_w(U^i \otimes [K, E]) = -2i + 2g(K, E) + |E| - |V_\Gamma| + \frac{1}{4}(K^2_{X_G} - 3\sigma(X_G) - 2\chi(X_G)).
\] (4.7)

This grading is defined when the intersection form \(Q_G\) of \(X_G\) is non-singular. More generally, this grading may also be defined when \(Q_G\) is singular, as long as we restrict to torsion \(\text{Spin}^c\) structures on \(Y_G\).

**Lemma 4.5.** The differential \(\partial\) on \(\text{CFL}(\Gamma, V_\Gamma)\) decreases \(gr_w\) by 1. Furthermore, if \(v_i \in V_\Gamma\), then \(\gamma_i\) preserves \(gr_w\) and \(\mathcal{U}_i\) decreases \(gr_w\) by 2.
We now observe that the \( J \)-map is skew Alexander graded:

**Lemma 4.9.** The \( J \) map satisfies

\[
A(J([K, E])) = -A([K, E]),
\]

where \( A \) denotes the Alexander grading.
Therefore, $\text{Spin}^c$-structures. We describe how the link lattice complex $\text{CFL}(\Gamma, V_\Gamma)$ naturally splits over $\text{Spin}^c(Y_G)$ as a module over $R_\ell$. Recall the isomorphism

$$\text{Spin}^c(Y_G) \cong \text{Spin}^c(X_G)/H_2(X_G).$$

Also, the Chern class map $c_1 : \text{Spin}^c(X_G) \to \text{Char}(X_G)$ is an isomorphism of affine $H^2(X_G)$-sets (where $C \in H^2(X_G)$ acts on $\text{Char}(X_G)$ by $K \mapsto K + 2C$).

In the link lattice complex, we associate the generator $[K, E] \otimes U^i$ with the $\text{Spin}^c$ structure $[K, X_G]$, viewed as an element of $\text{Char}(X_G)/H_2(X_G)$.

Since the differential on $\text{CFL}(\Gamma, V_\Gamma)$ is constructed by modifying (4.1) to sum over only $v \in V_G$, this decomposition is preserved by $\partial$. The actions of $\mathscr{V}_i$ and $\mathfrak{V}_i$ also preserve this decomposition, because they change $K$ to $K \pm 2\mu_i^*$, and $\mu_i^* \in H^2(X_G)$ has trivial restriction to $H^2(X_G)$.

4.9. Independence from the framing on arrow components. We now show that our chain complex $\text{CFL}(\Gamma, V_\Gamma)$ is independent of the choice of framing on the arrow components, up to canonical isomorphism.

Suppose that $G$ is a weighted plumbing tree obtained by weighting the arrow vertices of $\Gamma$ by (any) integral weights, and using the weights from $\Gamma$ on $V_\Gamma$. We obtained a model of the link lattice complex in the previous section, which we denote by $\text{CFL}_G(\Gamma, V_\Gamma)$. In this section, we describe a canonical isomorphism

$$F_{G, G'} : \text{CFL}_G(\Gamma, V_\Gamma) \to \text{CFL}_{G'}(\Gamma, V_\Gamma)$$

for any two extensions $G$ and $G'$.

Let $L_\Gamma \subseteq S^3$ denote the link associated to $\Gamma$ as in Subsection 3.1. We write $L_G$ to denote $L_\Gamma$, equipped with the framing from $G$. Write $n$ for $|V_\Gamma|$. Following the notation of Manolescu and Oszvath [MO10], we define the linking lattice $\mathbb{H}(L_G)$ to be the affine $\mathbb{Z}^n$ subspace of $\mathbb{Q}^n$ consisting of vectors $s = (s_1, \ldots, s_n)$ such that $s_i \in \mathbb{Z} + \text{lk}(L_i, L_\Gamma - L_i)/2$. As sets, we clearly have

$$\mathbb{H}(L_G) = \mathbb{H}(L_{G'}).$$

The lattices $\mathbb{H}(L_G)$ and $\mathbb{H}(L_{G'})$ are distinguished by their natural actions of $H_2(X_G) \cong H_2(X_{G'}) \cong \mathbb{Z}^n$. The action of $H_2(X_G)$ on $\mathbb{H}(L_G)$ is as follows. Given $v \in V_\Gamma$, write $\lambda_v$ for the longitude of $K_v$ determined by the framing of $K_v$. We write $\mathbb{Z}^n \cong H_2(S^3 \setminus L_\Gamma)$, and the action of $v$ is by $\lambda_v \in \mathbb{Z}^n$.

Next, there is a canonical isomorphism

$$\Phi_G : \text{Char}(X_G) \to \mathbb{H}(L_G)$$

given by the formula

$$\Phi_G(K) = \left( \frac{K(v_1) + v_G \cdot v_1}{2}, \ldots, \frac{K(v_n) + v_G \cdot v_n}{2} \right).$$

In the above, we write $v_G = v_1 + \cdots + v_n \in H_2(X_G)$. Note that $\Phi_G$ is equivariant with respect to the action of $H_2(X_G)$.

Given two weight-extensions $G$ and $G'$ of $\Gamma$, we define the group isomorphism

$$F_{G, G'} : \text{CFL}_G(\Gamma, V_\Gamma) \to \text{CFL}_{G'}(\Gamma, V_\Gamma),$$

via the formula

$$F_{G, G'}([K, E] \otimes U^i) = [(\Phi_G^{-1} \circ \Phi_{G'})(K), E] \otimes U^i.$$
The map $F_{G, G'}$ is clearly an isomorphism of $\mathbb{F}[[U]]$-modules. In fact, we have the following:

**Proposition 4.10.** The map $F_{G, G'}$ is a $(\text{gr}_w, A)$-grading preserving chain isomorphism between $\mathbb{CFL}_G(\Gamma, V_\Gamma)$ and $\mathbb{CFL}_{G'}(\Gamma, V_\Gamma)$.

**Proof.** The differentials on $\mathbb{CFL}_G(\Gamma, V_\Gamma)$ and $\mathbb{CFL}_{G'}(\Gamma, V_\Gamma)$ are similar to Equation (4.1), except that we take the sum only over the vertices $v \in V_G$. Note that there are no summands in the differential for $v \in V_\Gamma$. We consider the restricted action of $H_2(X_G) \subseteq H_2(X_\Gamma)$ on $\text{Char}(X_G)$, $\mathbb{H}(L_G)$ and $\mathbb{H}(L_{G'})$. Since the framings of the vertices of $V_G$ coincide on $X_G$ and $X_{G'}$, the identification $\mathbb{H}(L_G) \cong \mathbb{H}(L_{G'})$ is equivariant with respect to the action of $H_2(X_G)$.

It is straightforward to verify that the map $\Phi_G$ is equivariant with respect to the action of $H_2(X_G)$. The same argument applies to show that $\Phi_{G'}^{-1}$ is equivariant as well.

Let $K \in \text{Char}(X_G)$ and write $K' = (\Phi_{G'}^{-1} \circ \Phi_G)(K)$. The remainder of the proof follows from the following two claims:

1. $K|_{X_G} = K'|_{X_G}$.
2. For all $E$, we have $f(K, E) = f(K', E)$.

We verify these two claims presently.

The first claim follows from the proof of [OSS14b, Lemma 4.6], which we repeat for the benefit of the reader using our present notation. Write $s = \Phi_G(K) = \Phi_{G'}(K')$. Since $H^2(X_G)$ is torsion free, it suffices to show that $K|_{X_G}(v) = K'|_{X_G}(v)$ for each $v \in V_G$. Note that if $v_i \in V_G$, then $K(v_i) = 2s_i - v_G \cdot v_i$. This quantity only depends on $s$, on the framing of $K_i$ (the link component represented by the vertex $v_i$), and on the linking numbers of $K_i$ with other link components. In particular, $K(v_i) = K'(v_i)$, completing the proof of the first claim.

We now consider the second claim. By definition,

$$2f(K, E) = K(v_E) + v_E \cdot v_E$$

where $v_E$ is the sum of $v$ for $v \in E$. We may rearrange the above expression to obtain

$$2f(K, E) = (v_E - v_G) \cdot v_E + \sum_{v \in E} (K(v) + v_G \cdot v) = (v_E - v_E) \cdot v_E + \sum_{v \in E} s_i.$$ 

The right-hand side depends only on $s$, the framing of $L_G$, and the linking numbers of the components of $L_G$, but not on the framings of $V_\Gamma$. This establishes the second claim.

From these considerations, it follows that $F_{G, G'}$ preserves the gr$_w$-grading, and is also a chain map.

We now establish that $F_{G, G'}$ preserves the Alexander grading. Suppose that $v_i \in V_\Gamma$. The corresponding component of the Alexander grading is half of

$$K(v_i - \hat{v}_i) + \sum_{v \in V_\Gamma} v \cdot (v_i - \hat{v}_i) = K(v_i) + \sum_{v \in V_\Gamma} v \cdot v_i - K(\hat{v}_i) - \sum_{v \in V_\Gamma} v \cdot \hat{v}_i$$

$$= K(v_i) + \sum_{v \in V_\Gamma} v \cdot v_i - \sum_{v \in V_\Gamma} v \cdot v_i - K(\hat{v}_i) - \sum_{v \in V_\Gamma} v \cdot \hat{v}_i.$$ 

The first two terms above sum to $2s_i$. The last three terms may be rewritten as follows:

$$- \sum_{v \in V_\Gamma} PD[v](v_i) - K(\hat{v}_i) - \sum_{v \in V_\Gamma} PD[v](\hat{v}_i).$$

In particular each term is the evaluation of an element of $H^2(X_\Gamma)$ on an element of $H_2(X_G; \mathbb{Q})$. We observe that $PD[v] \big|_{X_G}$, for $v \in \Gamma$, is independent of the framing on $V_\Gamma$. Furthermore, $K|_{X_G} = K'|_{X_G}$. Hence $A_i([K, E]) = A_i([K', E])$. 

\[ \square \]
4.10. Freeness of the lattice complex. The link lattice complex is not free over $F[[\mathcal{Y}_*]]$ since $\mathcal{Y}_* \mathcal{Y}_i = U$ for all $i$. Nonetheless, we prove that the link lattice complex is free over $F[[\mathcal{Y}_*]]$, for each index $i$.

Proposition 4.11. Suppose that $\Gamma$ is an arrow decorated plumbing graph with a chosen vertex $v_i \in V_\Gamma$. Write $\mathcal{Y}_i$ and $\mathcal{Y}_i'$ for the variables associated to $v_i$. Then the module $\text{CFL}(\Gamma, V_\Gamma)$ is a completion of a free module over $F[[\mathcal{Y}_*, \mathcal{Y}_*]]$.

Proof. We consider the set of generators $[K, E]$ modulo the equivalence relation generated by $[K, E] \sim [K + 2\mu_i^* E]$ for all $K$ and $E$. Equivalence classes may be identified with elements of $(\text{Char}(\Gamma) \times \mathbb{P}(V_G))/\mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on $\text{Char}(\Gamma)$ by $2\mu_i^*$.

Fix an equivalence class, and let $W$ denote the $F[U]$-span of the generators in this class. We may write $W \cong \bigoplus_{s \in \mathbb{Z}} W_s$ where each $W_s \cong F[U]$. We order the $W_s$ so that $\mathcal{Y}_i \cdot W_s \subseteq W_{s+1}$ and $\mathcal{Y}_i \cdot W_s \subseteq W_{s-1}$. We make the following claims, from which the result will follow fairly easily:

(f-1) For each $s$, exactly one of $\mathcal{Y}_i \cdot W_s \rightarrow W_{s+1}$ and $\mathcal{Y}_i \cdot W_{s+1} \rightarrow W_s$ will be multiplication by 1, and the other will be multiplication by $U$. This follows from the fact that $\mathcal{Y}_i \cdot \mathcal{Y}_i'$ acts by $U$.

(f-2) If $[K, E]$ is fixed, then $[K, E]$ is not in the image of $\mathcal{Y}_i^n$ or $\mathcal{Y}_i'^n$ for arbitrarily large $n$. This follows immediately from Lemma 4.4.

(f-3) If $\mathcal{Y}_i \cdot [K, E] = [K + 2\mu_i^* E]$, then $\mathcal{Y}_i \cdot [K + 2\mu_i^* E] = [K + 4\mu_i^* E]$. Similarly, if $\mathcal{Y}_i \cdot [K, E] = [K - 2\mu_i^* E]$ then $\mathcal{Y}_i \cdot [K - 2\mu_i^* E] = [K - 4\mu_i^* E]$.

We now prove claim (f-3), focusing on the argument for $\mathcal{Y}_i$ since the claim about $\mathcal{Y}_i'$ is similar. We recall from Section 4.4 that

$$f(K + 2\mu_i^* E) = \begin{cases} f(K, E) & \text{if } v_i \notin E \\ f(K, E) + 1 & \text{if } v_i \in E. \end{cases}$$

Hence, $\delta_i^+(K, E) = 0$ if and only if there is a $J \subseteq E$ such that $f(K, J) = g(K, E)$ and $v_i \notin J$. In particular, if $\mathcal{Y}_i \cdot [K, E] = [K + 2\mu_i^* E]$, then there exists such a $J$. Hence $g(K, E) = f(K, J) \geq g(K + 4\mu_i^* E) \geq g(K, E)$, so we have equality throughout. It follows that $\mathcal{Y}_i \cdot [K + 2\mu_i^* E] = [K + 4\mu_i^* E]$.

Note that the claim (f-3) implies there is no $[K, E]$ which is in the image of both $\mathcal{Y}_i$ and $\mathcal{Y}_i'$, since if $[K, E]$ were in the image of both, then the above claims show that $\mathcal{Y}_i \mathcal{Y}_i' [K - 2\mu_i^* E] = \mathcal{Y}_i [K, E] = [K - 2\mu_i^* E]$, which contradicts $\mathcal{Y}_i \mathcal{Y}_i' = U$.

Claims (f-1) and (f-2) imply that there exist generators $[K, E]$ in $W$ which are in the image of $\mathcal{Y}_i$, and there also exist generators which are in the image of $\mathcal{Y}_i'$.

From the above considerations, we obtain that there is a unique generator $[K, E]$ such that $\mathcal{Y}_i [K, E] = [K - 2\mu_i^* E]$ and $\mathcal{Y}_i' [K, E] = [K + 2\mu_i^* E]$. By (f-3) this $[K, E]$ must be a free generator of $W$ over $F[[\mathcal{Y}_*, \mathcal{Y}_*]]$.

4.11. Type-D modules over $\mathcal{K}$. If $G$ is a weighted plumbing tree (without arrow vertices) and $v$ is a distinguished vertex, we now describe how to view the lattice complex as a type-D module over the algebra $\mathcal{K}$, described by the third author [Zem21a].

We first recall the algebra $\mathcal{K}$ from [Zem21a]. It is an algebra over the idempotent ring $I \cong I_0 \oplus I_1$, where each $I_i \cong F$. We define

$$I_0 \cdot \mathcal{K} \cdot I_0 \cong F[\mathcal{Y}, \mathcal{Y}], \quad I_0 \cdot \mathcal{K} \cdot I_1 = 0 \quad \text{and} \quad I_1 \cdot \mathcal{K} \cdot I_1 \cong F[\mathcal{Y}, \mathcal{Y}, \mathcal{Y}^{-1}] .$$

Finally, $I_1 \otimes \mathcal{K} \otimes I_0$ is isomorphic to the direct sum of two copies of $F[\mathcal{Y}, \mathcal{Y}, \mathcal{Y}^{-1}]$, viewed as being generated by two distinguished elements $\sigma$ and $\tau$. These elements satisfy

$$\sigma \cdot \mathcal{Y} = U \mathcal{Y}^{-1} \cdot \sigma, \quad \sigma \cdot \mathcal{Y} = \mathcal{Y} \cdot \sigma$$

$$\tau \cdot \mathcal{Y} = \mathcal{Y}^{-1} \cdot \tau \quad \text{and} \quad \tau \cdot \mathcal{Y} = U \mathcal{Y} \cdot \tau.$$
where $U = \mathcal{U} \mathcal{V}$.

In this section, we describe how to construct a type-D module $\mathcal{X}(G)^K$ from the data of $\mathcal{CF}(G)$. The construction of $\mathcal{X}(G)^K$ may be equivalently described as a tensor product of the Hopf, merge and solid torus modules from [Zem21a], though we presently give a direct construction in terms of lattices. We define $\mathcal{X}(G)^K$ at the end of the section, after we prove several properties about the lattice complex.

Write $\mathcal{CF}_0(G)$ for the codimension 1 subcube of $\mathcal{CF}(G)$ generated by tuples $[K, E]$ where $v \in E$. Write $\mathcal{CF}_1(G)$ for the codimension 1 subcube generated by tuples $[K, E]$ where $v \notin E$. We view $\mathcal{CF}(G)$ as a mapping cone

$$\mathcal{CF}(G) \cong \text{Cone} \left( F^A_v + F^B_v : \mathcal{CF}_0(G) \to \mathcal{CF}_1(G) \right),$$

where $F^A_v$ and $F^B_v$ are the summands of the differential which are weighted by $U^{a_v(K,E)}$ and $U^{b_v(K,E)}$, respectively.

We observe that $\mathcal{CF}_0(G)$ is exactly the link lattice complex if we designate the special vertex $v$ as the sole arrow vertex. In particular, Proposition 4.11 implies that it is a completion of a free $\mathbb{F}[[\mathcal{U}, \mathcal{V}]]$ module (where $\mathcal{U}$ and $\mathcal{V}$ are the actions for $v$).

We may define actions of $\mathcal{U}$ and $\mathcal{V}$ also on $\mathcal{CF}_1(G)$, using the same formulas as in Section 4.4. We first observe that the formulas have a comparatively easier description than on $\mathcal{CF}_0(G)$:

**Lemma 4.12.** On $\mathcal{CF}_1(G)$, we have

$$\mathcal{U} \cdot [K, E] = U[K - 2\mu^*, E] \quad \text{and} \quad \mathcal{V} \cdot [K, E] = [K + 2\mu^*, E]$$

for all $K$ and $E$ such that $v \notin E$.

**Proof.** If $v \notin E$, then $f(K, E) = f(K \pm 2\mu^*, E)$, and hence

$$g(K, E) = g(K \pm 2\mu^*, E). \quad (4.9)$$

Both equations follow by applying this fact to the definition of $\mathcal{U}$ and $\mathcal{V}$ from Section 4.4. $\square$

As a consequence of the above, we may define an action of $\mathcal{V}^{-1}$ on $\mathcal{CF}_1(G)$ via the formula $\mathcal{V}^{-1} \cdot [K, E] = [K - 2\mu^*, E]$. As an additional consequence of Lemma 4.12, we have the following easy analog to Proposition 4.11:

**Corollary 4.13.** The complex $\mathcal{CF}_1(G)$ is the completion of a free module over $\mathbb{F}[[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}]]$.

Let $F^A_v$ denote the $A$-term of the differential which increments $v$. Write $F^B_v$ for the $B$-term of the differential which increments $v$.

**Lemma 4.14.** The map $F^A_v$ satisfies

$$F^A_v(\mathcal{U} \cdot x) = \mathcal{U} \cdot F^A_v(x) \quad \text{and} \quad F^A_v(\mathcal{V} \cdot x) = \mathcal{V} \cdot F^A_v(x)$$

for all $x \in \mathcal{CF}_0(G)$. Similarly,

$$F^B_v(\mathcal{U} \cdot x) = \mathcal{V}^{-1} \cdot F^B_v(x) \quad \text{and} \quad F^B_v(\mathcal{V} \cdot x) = \mathcal{V} \cdot F^B_v(x).$$

**Proof.** We begin with the map $F^A_v$. From direct computation,

$$\mathcal{V} \cdot F^A_v([K, E]) = U^{g(K+2\mu^*, E-v)-g(K,E)} \cdot [K+2\mu^*, E-v]$$

$$= F^A_v(\mathcal{V} \cdot [K, E]).$$

An entirely analogous argument shows that $[F^A_v, \mathcal{U}] = 0$.

Next, we consider the commutation of $F^B_v$ with $\mathcal{U}$. We compute that

$$F^B_v(\mathcal{U} \cdot [K, E]) = U^{g(K+2\mu^*, E-v)-g(K,E)+1+(K-2\mu^*)(v)/2+v^2/2} \cdot [K - 2\mu^* + 2\mu^*, E - v]$$

$$= \mathcal{V}^{-1} \cdot F^B_v([K, E]).$$

Going from the first line to the second, we are using Equation (4.9).
Next, we consider the commutation of \( F^B_v \) with \( \mathcal{Y} \). We compute
\[
F^B_v (\mathcal{Y} \cdot [K, E]) = U^{g(K+2u^*+2v^*, E-v)} - g(K, E) \cdot (K+2u^*) \cdot (v+2v^* \cdot E-v) \cdot [K+2v^*+v^*, E-v]
\]
\[
= U^\mathcal{Y} \cdot \left( U^{g(K+2u^*, E-v)} - g(K, E) \cdot [K+2v^*, E-v] \right)
\]
\[
= U^\mathcal{Y} \cdot F^B_v ([K, E]) .
\]

Going from the first line to the second, we use Equation (4.9). The proof is complete. \( \Box \)

We are now able to define the type-\( D \) module \( \mathcal{X}(G)^\mathcal{K} \). As a right \( \mathcal{I} \)-module, we write \( \mathcal{X}(G) = \mathcal{X}_0 \oplus \mathcal{X}_1 \), where each \( \mathcal{X}_\approx \) is a vector space over \( \mathbb{F} \). We define \( \mathcal{X}_0 \) to be the \( \mathbb{F} \) vector space generated by a free \( \mathbb{F}[\mathcal{Y}, \mathcal{Y}^\cdot] \)-basis of \( \mathcal{CF}_0(G) \) from Proposition 4.21. We define \( \mathcal{X}_1 \) to be the \( \mathbb{F} \) vector space generated by a free \( \mathbb{F}[\mathcal{Y}, \mathcal{Y}^\cdot, \mathcal{Y}^{-1}] \)-basis of \( \mathcal{CF}_1(G) \) from Corollary 4.13.

We now define the structure map
\[
\delta^1 : \mathcal{X}(G) \to \mathcal{X}(G) \otimes_{\mathcal{I}} \mathcal{K}.
\]
The construction is entirely analogous to the setting of the link surgery formula. See [Zem21a, Section 8.5] for a parallel construction. If \( x \) is a basis element of \( \mathcal{CF}_0(G) \) and \( \partial(x) \) has a summand of \( \mathcal{Y}^\cdot \mathcal{Y}^\cdot \mathcal{Y}^\cdot \cdot y \), where \( y \) is a basis element, then we define \( \delta^1(x) \) to have a summand of \( y \otimes \mathcal{Y}^\cdot \mathcal{Y}^\cdot \mathcal{Y}^\cdot \cdot y \).

We make a similar definition for basis elements of \( \mathcal{CF}_1(G) \). Next, if \( F^A_v(x) = \mathcal{Y}^\cdot \mathcal{Y}^\cdot \cdot y \), we declare \( \delta^1(x) \) to also have a summand of \( y \otimes \mathcal{Y}^\cdot \mathcal{Y}^\cdot \cdot y \). Similarly, if \( F^B_v(x) = \mathcal{Y}^\cdot \mathcal{Y}^\cdot \cdot y \), then we declare \( \delta^1(x) \) to have a summand of \( y \otimes \mathcal{Y}^\cdot \mathcal{Y}^\cdot \cdot y \). It is straightforward to verify that \( \mathcal{X}(G)^\mathcal{K} \) satisfies the type-\( D \) structure relations.

We note that the underlying vector space of \( \mathcal{X}(G)^\mathcal{K} \) is infinite dimensional, so completions play a subtle yet important role in the theory. We leave it to the reader to verify that the modules satisfy the Alexander coboundedness condition described in [Zem21a, Section 6].

5. THE EQUIVALENCE WITH LINK FLOER HOMOLOGY

In this section, we prove that link lattice homology and link Floer homology are isomorphic. The argument follows from similar logic to the case of 3-manifolds [OSS14b] [Zem21b].

**Theorem 5.1.** Suppose that \( \Gamma \) is an arrow-decorated plumbing tree with vertex set \( V_G \cup V_\gamma \), such that \( Y_G \) is a rational homology 3-sphere. For each \( s \in \text{Spin}^c(Y_G) \), there is an absolutely \( (\mathfrak{g}_\mathcal{Y}, A) \)-graded isomorphism of \( A_\infty \)-modules over \( \mathcal{R}_\mathcal{Y} \):
\[
\text{CFL}(\Gamma, V_\gamma, s) \cong \text{CFL}(Y_G, L_\gamma, s).
\]

Here, both \( \text{CFL} \) and \( \text{CFL} \) are equipped with the natural \( A_\infty \)-module structures which have only \( m_1 \) and \( m_2 \) non-trivial.

The proof of Theorem 5.1 is completed in Subsection 5.3. We now provide a sketch of the proof. We will use a relative version of the Manolescu–Ozsváth link surgery formula [MO10], which computes link Floer homology as a subcube of the full link surgery hypercube. This is stated in Theorem 5.2. From here, we follow the approach of [Zem21b] and view \( L_\gamma \) as a connected sum of Hopf links. Using a tensor product formula from [Zem21a] for the link surgery complex, one obtains a combinatorial model for the link surgery complex of \( L_\gamma \).

Following the approach of Ozsváth, Stipsicz and Szabó [OSS14b], one may identify the lattice complex with a simplified version of the link surgery hypercube obtained by taking the homology of the link surgery complex at each vertex of the cube \( \{0,1\}^\ell \), and using only the length 1 maps of the link surgery hypercube. In [Zem21b], the third author shows directly using the connected sum formula for the link surgery formula that the link surgery complex for \( L_\gamma \) is chain homotopy equivalent to this simplified model of the link surgery complex. When \( b_1(Y_G, \mathbb{Q}) = 0 \), we show that this homotopy equivalence induces a homotopy equivalence between and the link lattice complex and the corresponding quotient complex of the link surgery complex of \( L_\gamma \). We show that the morphisms in this homotopy equivalence are well-behaved with respect to the actions of \( \mathcal{Y}_i \) and \( \mathcal{Y}_i^{-1} \), and give a homotopy equivalence of \( A_\infty \)-modules.
5.1. The link surgery complex and sublinks. As a first step, we describe a refinement of the Manolescu and Ozsváth link surgery formula. If \( L \subseteq S^3 \) is a link equipped with integral framing \( \Lambda \), then Manolescu and Ozsváth construct a chain complex \( C_\Lambda(L) \) over \( \mathbb{F}[U] \) (defined in terms of the link Floer complex \( CFL(S^3, L) \), equipped with additional data) and prove that \( H_*(C_\Lambda(L)) \cong HF^-(S^3_\Lambda(L)) \).

There is a refinement of this result which can be used to compute link Floer homology, as follows. Suppose that \( M = J \cup L \subseteq S^3 \) is a partitioned link with \( |M| = n \) and \( |L| = \ell \). Equip \( J \) with a framing \( \Lambda \). We may extend \( \Lambda \) arbitrarily to a framing \( \Lambda' \) on all of \( M \) to obtain a link surgery complex \( C_{\Lambda'}(M) \) whose homology is \( HF^-(S^3_{\Lambda'}(M)) \).

The link surgery complex \( C_{\Lambda'}(M) \) is a hypercube of chain complexes, which means that it admits a natural filtration by the integral points of the cube \( \{0,1\}^n \), where \( n = |M| \). While more details are given in [MO10, Section 5], we give some necessary background and introduce the notation. For any \( \varepsilon \in \{0,1\}^n \), we consider the multiplicatively closed subset \( S_\varepsilon \subseteq R_n \) generated by \( \mathcal{V}_i \) such that \( \varepsilon(i) = 1 \). The complex \( C_\varepsilon \) is defined as the algebraic completion of the localization \( S^{-1}_\varepsilon \cdot CFL(M) \).

We remark that the original definition in [MO10] is seemingly different, though the equivalence with the above description follows from [Zem21a, Lemma 5.7].

If \( \varepsilon, \varepsilon' \in \{0,1\}^n \), we write \( \varepsilon \leq \varepsilon' \) if \( \varepsilon_i \leq \varepsilon'_i \) for all \( i \). We write \( \varepsilon < \varepsilon' \) if \( \varepsilon \leq \varepsilon' \) and \( \varepsilon \neq \varepsilon' \). If \( \varepsilon, \varepsilon' \in \{0,1\}^n \), Manolescu and Ozsváth construct a map \( D_{\varepsilon,\varepsilon'}: C_\varepsilon \to C_{\varepsilon'} \); see [MO10, Section 5].

The chain complex \( C_{\Lambda'}(M) \) is the direct sum of complexes \( C_\varepsilon \) and the differential is the sum of the internal differentials in \( C_\varepsilon \) and of the maps \( D_{\varepsilon,\varepsilon'} \).

We now describe a relative complex \( C_\Lambda(J, L) \). Note that each axis direction in \( \{0,1\}^n \) corresponds to a component of \( M \). Hence, we may consider the quotient complex \( C_\Lambda(J, L) \) obtained by quotienting the subcomplex of \( C_{\Lambda'}(M) \) consisting of those \( C_\varepsilon \) such that \( \varepsilon(i) = 1 \) for at least one index \( i \) corresponding to \( L \). Examining Manolescu and Ozsváth’s construction, it is evident that \( C_\Lambda(J, L) \) is independent of the framings of the \( L \) components.

Furthermore, the module \( C_\Lambda(J, L) \) has a natural action of the ring \( R_\ell \), corresponding to the variables for \( L \). The underlying spaces \( C_\varepsilon \) are preserved by this \( R_\ell \)-module structure. It follows from [Zem21a, Lemma 5.9] that the hypercube maps of \( C_\Lambda(J, L) \) commute with the action of \( R_\ell \), i.e. the action of the variables from \( L \). Note that in general the differential on \( C_\Lambda(J, L) \) will not commute with the actions of the variables from \( J \). The next result is important for our purposes:

**Theorem 5.2.** Suppose that \( M \subseteq S^3 \) is a link which is partitioned into two sublinks \( M = J \cup L \). Let \( \Lambda \) be an integral framing on \( J \) and write \( \ell = |L| \). Then there is a homotopy equivalence of chain complexes over \( R_\ell \):

\[
CFL(S^3_\Lambda(J), L) \simeq C_\Lambda(J, L).
\]

Furthermore, this isomorphism respects \( \text{Spin}^c \) structures under an isomorphism

\[
\text{Spin}^c(S^3_\Lambda(J)) \cong \mathbb{H}(M)/\langle \text{Span}(\mu^*_1, \ldots, \mu^*_\ell) \rangle + H_2(W_\Lambda(J)),
\]

where \( W_\Lambda(J) \) is the cobordism from \( S^3 \) to the surgery on the link \( J \).

The above result is a folklore result. We believe the techniques of [MO10] can be adapted in a straightforward manner to prove this theorem. Nonetheless, experts in the Heegaard Floer surgery formulas may recognize that although conceptually simple, a rigorous proof requires a substantial amount of bookkeeping because of the role of algebraic truncations in the surgery formula. A conceptually simple proof, avoiding truncations, will appear in [Zem].

5.2. Link lattice homology and the link surgery formula. We now describe how to recast the link lattice complex in terms of the link surgery complex. This is an adaptation of [OSS14b, Proposition 4.4] to our present context of links.

Construct an \( |L_G| \)-dimensional hypercube of chain complexes as follows: For \( \varepsilon \in \{0,1\}^{|L_G|} \), define \( Z_\varepsilon := H_*(C_\varepsilon) \), where \( C_\varepsilon \) is the corresponding submodule of \( C_\Lambda(L_G, L_\ell) \). If \( \varepsilon < \varepsilon' \), construct
Here, the Hopf link takes the following form:

\[ \delta_{\varepsilon, \varepsilon'} := \begin{cases} (D_{\varepsilon, \varepsilon'})_* & \text{if } |\varepsilon'| - |\varepsilon| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \]

Write \( Z = (Z, \delta_{\varepsilon, \varepsilon'})_{\varepsilon \in \{0,1\}^{d_G}} \). Clearly \( Z \) is a hypercube of chain complexes over \( \mathcal{R}_I \).

Compare the following to [OSS14b, Proposition 4.4]:

**Proposition 5.3.** Let \( \Gamma \) be an arrow-decorated plumbing tree. The hypercube \( Z = (Z, \delta_{\varepsilon, \varepsilon'})_{\varepsilon \in \{0,1\}^{d_G}} \) is isomorphic to the link lattice complex \( \mathcal{CFL}(\Gamma, V_\Gamma) \).

Before proving Proposition 5.3, we prove a technical lemma which is helpful for relating the \( \mathcal{R}_I \)-actions on \( \mathcal{CFL}(\Gamma, V_\Gamma) \) and \( Z \). We note that the following lemma is essentially implicit in the definition of lattice homology and also Ozsváth, Stipsicz, and Szabó’s construction of the spectral sequence (cf. [OSS14b, Proposition 4.4]), though we have been unable to find an exact reference which is suitable for our purposes. If \( L \subseteq S^3 \) and \( s \in \mathbb{H}(L) \), write

\[ d(L, s) = \max \{ \text{gr}_w(x) : x \in H, \mathfrak{A}^-(L, s), x \text{ is } U\text{-nontorsion} \}. \]

Here, \( \text{gr}_w \) denotes the internal Maslov grading from link Floer homology, and \( \mathfrak{A}^-(L, s) \subseteq \mathcal{CFL}(S^3, L) \) is a subcomplex corresponding to the Alexander grading \( s \).

**Lemma 5.4.** Let \( G \) be a forest of plumbing trees. If \( E \subseteq V_G \), write \( L_E \subseteq S^3 \) for the sublink of \( L_G \) containing exactly the components corresponding to vertices of \( E \). Let \( \phi_E : \text{Char}(X_G) \to \mathbb{H}(L_E) \) be the composition of the restriction map from \( \text{Char}(X_G) \) to \( \text{Char}(X_E) \) and the isomorphism from \( \text{Char}(X_E) \) to \( \mathbb{H}(L_E) \) in Equation (4.8). Then

\[ 2g(K, E) = d(L_E, \phi_E(K)). \]

**Proof.** Note that \( g(K, E) \) (computed in \( X_G \)) is the same as \( g(K|_{X_E}, E) \) (computed in \( X_E \)). Hence, we may assume without loss of generality that \( E = V_G \).

Next, we observe that the framings on the components of \( L_G \) play no role in the statement. Indeed, if \( K \in \text{Char}(X_G) \) and \( \Phi_G(K) = (s_1, \ldots, s_n) \), then Equation (4.8) implies that

\[ 2g(K, V_G) = \min_{I \subseteq V_G} \left( \sum_{i \in I} 2s_i - v_{G-I} \cdot v_I \right) \tag{5.1} \]

where \( v_{G-I} \) is the sum of \( v_i \) for \( i \in V_G \setminus I \) and \( v_I \) is similar. Our proof will be by induction on the number of vertices. If \( s \in \mathbb{H}(L_G) \), we will write \( 2g(L_G, s) \) for the quantity on the right-hand side of Equation (5.1). Similarly, if \( I \subseteq V_G \) we write

\[ 2f(s, I) = \sum_{i \in I} 2s_i - v_{G-I} \cdot v_I. \tag{5.2} \]

We claim that \( 2g(L_G, s) = d(L_G, s) \).

We begin with the case that \( L_G \) is an \( n \)-component unlink \( \text{U}_n \). In this case, the homology \( \mathcal{HFL}(\text{U}_n) \cong \bigoplus_{s \in \mathbb{H}(L_G)} \mathfrak{A}^-(\text{U}_n, s) \) is well known to be \( \mathbb{F}[\varphi_1, \varphi_{2}, \ldots, \varphi_n, \eta_n]/(\varphi_i \varphi_j - \varphi_j \varphi_i, i, j \in \{1, \ldots, n\}) \), with the class of 1 having \((\text{gr}_w, \text{gr}_z\text{-bigrading}) \ (0, 0) \). The claim is easily verified for \( \text{U}_n \) from this description.

We now assume that claim is true for any forest \( G \) of plumbing trees. We will prove the claim also holds for \( L_G \# H \), where \( H \) is the positive Hopf link. We recall that the complex of the positive Hopf link takes the following form:

\[
\begin{array}{c}
\text{a} - \varphi_{n+1} \to \text{b} \\
\downarrow_{\gamma_n} \uparrow \quad \uparrow_{\varphi_n} \\
\text{c} \leftarrow \varphi_{n+1} - \text{d}
\end{array}
\tag{5.3}
\]
The Alexander bigrading of $a, b, c, d$ are $(\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$, and $(-\frac{1}{2}, \frac{1}{2})$, respectively. The $(\text{gr}_w, \text{gr}_x)$-bigradings are $(-1, -1), (0, -2), (-2, 0)$, and $(-1, -1)$, respectively.

We assume that $L_n \subseteq H$ is the component where the connected sum is taken, and $L_{n+1} \subseteq H$ for the remaining component.

We recall that $\mathcal{CFL}(L_G)$, being a tensor product of an unlink complex and a collection of Hopf links, may be written as $F_s \to F_{s-1} \to \cdots \to F_0$, where each $F_i$ is a free $\mathbb{R}_n$-module and the homology $H\mathcal{F}(L_G)$ is supported in $F_0$. From this fact, it is straightforward to see that any homogeneously graded $\mathbb{F}[U]$-non-torsion cycle in $\mathcal{CFL}(H) \otimes_{\mathbb{F}[z_n, z_n]} \mathcal{CFL}(L_G)$ may be written as a sum of an odd number of terms of the form $\alpha \cdot b \otimes x$ or $\beta \cdot c \otimes x$, where $x \in \mathcal{CFL}(L_G)$ is an $\mathbb{F}[U]$-non-torsion cycle, and $\alpha, \beta \in \mathbb{F}[z_{n+1}, y_{n+1}]$.

Consider $s = (s_1, \ldots, s_{n+1}) \in \mathbb{H}(L_G \# H)$, where $s_{n+1} = \frac{1}{2} + \mathbb{Z}$ corresponds to the component $L_{n+1} \subseteq H$. We break the proof into two cases: $s_{n+1} > 0$, and $s_{n+1} < 0$.

We consider first the case that $s_{n+1} > 0$. In this case, $H, \mathcal{M}^-(L_G \# H, s)$ is generated over $\mathbb{F}[U]$ by the elements $\mathcal{Y}_{n+1}^{s_{n+1}+1/2} \cdot b \otimes x$ and $\mathcal{Y}_{n+1}^{-s_{n+1}+1/2} \cdot c \otimes x$, where $x \in \mathbb{F}[U]$-non-torsion. Write $s' = (s_1, \ldots, s_n)$. We obtain from the above argument that

$$d(L_G \# H, s) = \max\{\text{gr}_w(b) + d(L_G, s' - \frac{1}{2}e_n), \text{gr}_w(c) + d(L_G, s' + \frac{1}{2}e_n)\}$$

$$= \max\{d(L_G, s' - \frac{1}{2}e_n), -2 + d(L_G, s' + \frac{1}{2}e_n)\}$$

$$= d(L_G, s' - \frac{1}{2}e_n).$$

Let $I' \subseteq V_G$ be any subset. We may view $I'$ also as subset of $V_G \cup \{v_{n+1}\}$. We compute easily that

$$2f(s, I') = 2f(s' - \frac{1}{2}e_n, I'),$$

and

$$2f(s, I' \cup \{v_{n+1}\}) = \begin{cases} 2f(s' - \frac{1}{2}e_n, I') + 2s_{n+1} + 1 & \text{if } v_n \in I' \\ 2f(s' - \frac{1}{2}e_n, I') + 2s_{n+1} - 1 & \text{if } v_n \notin I', \end{cases}$$

Since $s_{n+1} > 0$, we observe from these equations that

$$f(s, I') = f(s' - \frac{1}{2}e_n, I') \leq f(s, I' \cup \{v_{n+1}\}).$$

for all $I' \subseteq V_G$ and $s \in \mathbb{H}(L_G)$. It follows easily that

$$g(L_G \# H, s) = g(L_G, s' - \frac{1}{2}e_n).$$

By induction $2g(L_G, s' - \frac{1}{2}e_n) = d(L_G, s' - \frac{1}{2}e_n)$, so from Equation (5.4) we obtain

$$2g(L_G \# H, s) = d(L_G \# H, s)$$

when $s_{n+1} > 0$.

We now consider the case $s_{n+1} < 0$. In this case, $H, \mathcal{M}^-(L_G \# H, s)$ is generated by elements of the form $\mathcal{Y}_{n+1}^{-s_{n+1}+1/2} \cdot b \otimes x$ and $\mathcal{Y}_{n+1}^{s_{n+1}-1/2} \cdot c \otimes x$. A similar argument to $s_{n+1} > 0$ case yields that

$$d(L_G \# H, s) = d(s' + \frac{1}{2}e_n) + 2s_{n+1} - 1.$$ (5.5)

One computes directly that

$$2f(s, I') = \begin{cases} 2f(s' + \frac{1}{2}e_n, I') - 2 & \text{if } v_n \in I' \\ 2f(s' + \frac{1}{2}e_n, I') & \text{if } v_n \notin I', \end{cases}$$

and

$$2f(s, I' \cup \{v_{n+1}\}) = 2f(s' + \frac{1}{2}e_n, I') + 2s_{n+1} - 1,$$

for all $I' \subseteq V_G$. Since $s_{n+1} < 0$, we have

$$2f(s, I' \cup \{v_{n+1}\}) = 2f(s' + \frac{1}{2}e_n, I') + 2s_{n+1} - 1 \leq 2f(s, I'),$$

where the last inequality comes from (5.6), so

$$2g(L_G \# H, s) = 2g(L_G, s' + \frac{1}{2}e_n) + 2s_{n+1} - 1.$$ (5.6)

Combined with Equation (5.5), we conclude that $2g(L_G \# H, s) = d(L_G \# H, s)$, completing the proof of Lemma 5.4.
Proof of Proposition 5.3. Note that [OSS14b, Proposition 4.4] proves the identification on the level of chain complexes of $\mathbb{F}[[U]]$-modules. It suffices to show that the decomposition respects the refined actions of the ring $R$. We recall the basics of their isomorphism. The lattice $\mathbb{H}(L_T)$ represents the set of Alexander gradings supported by the link Floer complex $\text{CFL}(L_T)$. If $L \subseteq L_T$, and $\varepsilon \in \{0, 1\}^n$ is the indicator function for the components of $L$, then we may write $\mathbb{A}_\varepsilon(L_T, s)$ for the subcomplex of $S^{* \cdot} \cdot \text{CFL}(L_T)$ in Alexander grading $s$. According to [OSS14b, Lemma 4.2], there is an isomorphism

$$H_n(\mathbb{A}_\varepsilon(L_T, s)) \cong \mathbb{F}[[U]].$$

Additionally, there is an isomorphism $\mathbb{H}(L_T) \to \text{Spin}^c(X_T)$ (stated in Equation (4.8)). This gives an isomorphism between $\mathcal{Z}$ and $\text{CFL}(\Gamma, \mathcal{V}_T)$ as $\mathbb{F}[[U]]$-modules. According to [OSS14b, Proposition 4.4], the differentials also coincide. It remains to show that the isomorphism respects the $R$-modules.

To disambiguate the actions, let us write $\mathbb{F}[[\mathcal{U}_i, \mathcal{V}_i, \ldots, \mathcal{U}_i, \mathcal{V}_i]]$ for the action we have described on $\text{CFL}(\Gamma, \mathcal{V}_T)$, and let us write $\mathbb{F}[[\mathcal{U}_i, \mathcal{V}_1, \ldots, \mathcal{U}_i, \mathcal{V}_1]]$ for the action induced by the identification $\text{CFL}(\Gamma, \mathcal{V}_T) \cong \mathcal{Z}$. As a first step, note that by definition $\mathcal{U}_i$ changes the Alexander grading $s \in \mathbb{H}(L_T)$ by $-e_i \in \mathbb{Z}^n$ (where $e_i$ is the unit vector with $i$-th component 1, and other components 0). It follows from Lemma 5.4, that $\mathcal{U}_i$ has the same $\text{gr}_w$-grading as $\mathcal{U}_i$, and similarly $\mathcal{V}_i$ has the same $\text{gr}_w$-gradings as $\mathcal{V}_i$. Clearly, if $[K, E]$ is a generator, then $\mathcal{U}_i \cdot [K, E] = U^\epsilon [K - 2\mu^\epsilon_i, E]$ and $\mathcal{U}_i \cdot [K, E] = U^\epsilon [K - 2\mu^\epsilon_i, E]$ for some $\epsilon, \epsilon' \in \{0, 1\}$. Since $\mathcal{U}_i$ and $\mathcal{U}_i$ have the same $\text{gr}_w$-grading as endomorphisms, we must have that $\epsilon = \epsilon'$, so that $\mathcal{U}_i$ and $\mathcal{U}_i$ have the same action. The same argument implies $\mathcal{V}_i = \mathcal{V}_i$. □

5.3. Proof of Theorem 5.1. In this subsection we give the proof of Theorem 5.1, though we delay the discussion about absolute gradings until Subsection 5.4. The main steps of the proof follow from [Zem21b, Section 5.2]. We provide a summary and highlight the necessary changes for our present setting.

By Theorem 5.2, we have a homotopy equivalence of chain complexes over $R_T$:

$$\text{CFL}(Y_G, L_T) \simeq C_A(L_G, L_T).$$

In particular, the above chain homotopy equivalence may be viewed as a homotopy equivalence of $A_\infty$-modules over $R_T$, where each module has $m_j = 0$ for $j > 2$. Above, we also constructed a hypercube $Z = Z_{\varepsilon, \delta, \sigma'}$ of $R_T$-modules by taking the homology of $C_A(L_G, L_T)$ at each cube point, and using only the length 1 maps from $C_A(L_G, L_T)$. By Proposition 5.3, we have a chain isomorphism

$$Z \simeq \text{CFL}(\Gamma, \mathcal{V}_T).$$

Hence, it suffices to construct an $A_\infty$-homotopy equivalence

$$Z \simeq C_A(L_G, L_T).$$

This homotopy equivalence follows from the same logic as [Zem21b, Section 6.2], which we summarize for the benefit of the reader.

We will write $C = (C_\varepsilon, \Phi_{\varepsilon, \varepsilon'})$ for $C_A(L_G, L_T)$. The underlying complex $C_\varepsilon$ at each vertex of $C_A(L_G, L_T)$ is obtained from $\text{CFL}(L_T)$ by localizing at the variables $\mathcal{V}_i$ such that $\varepsilon(i) = 1$, and then taking an appropriate completion. The complex $\text{CFL}(L_T)$ is a tensor product of Hopf link complexes. The Hopf link has a 2-step filtration (see Equation (5.3)), and hence $\text{CFL}(L_T)$ has a description as

$$\text{CFL}(S^3, L_G \cup L_T) \cong \left( F^m \longrightarrow F^{m-1} \longrightarrow \cdots \longrightarrow F^0 \right).$$

where each $F^i$ is a free $R_n$-module, $n = |V_T|$ and $m$ is the number of edges in $\Gamma$ (i.e. Hopf link components). Each $C_\varepsilon$ has a similar filtration, which we denote by $F^i_\varepsilon$. We call the superscript $i$ in $F^i$ the Hopf grading.

Since $L_T$ is an L-space link, each $Z_\varepsilon$ is a direct product of copies of $\mathbb{F}[[U]]$. Following [Zem21b, Proposition 6.2], there is a natural way to construct a homotopy equivalence between each $C_\varepsilon$ and...
We will also need to understand the map $\Pi$, which is given by constructed similarly to the maps $\Pi, \Phi^{\prime}$, maps $F$ in $[\text{Zem21b}, \text{Theorem 3.1}]$. The key property of the hypercube maps appearing in the model of the tensor product, where several terms have been deleted. (This simplified model appears in $[\text{Zem21b}, \text{Equation 3.2}]$. The complex $C_{\alpha}^\prime (L_G, L_\Gamma)$ is constructed by tensoring the link surgery complexes of Hopf links together using an algebraically simplified model of the tensor product, where several terms have been deleted. (This simplified model appears in $[\text{Zem21b}, \text{Theorem 3.1}]$). The key property of the hypercube maps appearing in $C_{\alpha}^\prime (L_G, L_\Gamma)$ are the maps $\Phi^{\prime}_{\varepsilon, \varepsilon^\prime}$ are non-increasing in the Hopf grading from (5.8) only when $|\varepsilon - \varepsilon^\prime|_{L_\Gamma} \leq 1$. Furthermore, when $|\varepsilon - \varepsilon^\prime|_{L_\Gamma} = 1$, the map $\Phi^{\prime}_{\varepsilon, \varepsilon^\prime}$ preserves the Hopf grading. Since $h_\varepsilon$ strictly
increases the Hopf grading and \( \pi_\varepsilon \) is non-vanishing only on the lowest Hopf grading, the composition in (5.9) will only be non-trivial when \(|\varepsilon - \varepsilon'|_{L_1} = 1\). Hence

\[
W' = Z.
\]

In particular, composing these homotopy equivalences, we obtain a homotopy equivalence of chain complexes

\[
CFL(Y_G, L_\uparrow) \simeq Z. \tag{5.11}
\]

It remains to show that the homotopy equivalence in (5.11) may be extended to a homotopy equivalence of \( A_\infty \)-modules over \( R_\ell \). There are two subclaims:

(R-1) The homotopy equivalence between \( C_\Lambda(L_G, L_\uparrow) \) and \( C'_\Lambda(L_G, L_\uparrow) \) may be taken to be \( R_\ell \)-equivariant.

(R-2) The homotopy equivalence \( C'_\Lambda(L_G, L_\uparrow) \simeq CFL\Gamma, V_\uparrow \) (from the homological perturbation lemma of hypercubes) extends to a homotopy equivalence of \( A_\infty \)-modules over \( R_\ell \).

We address (R-1) first. In [Zem21b, Corollary 4.8], it is shown that the simplified connected sum formula yielding \( C'_\Lambda(L_G, L_\uparrow) \) is valid as long as in forming \( L_\Gamma \) by an iterated connected sum, we never take the connected sum of two knot components which are both homologically essential after we surger on the other components of \( L_G \). In the case that \( b_1(Y_G) = 0 \), we always avoid this configuration (cf. [Zem21b, Lemma 6.5]). The homotopy equivalence between \( C_\Lambda(L_G, L_\uparrow) \) and \( C'_\Lambda(L_G, L_\uparrow) \) is concrete and obtained by relating the connected sum formula in [Zem21b, Equations (3.2)] and the simplified connected sum formula in [Zem21b, Theorem 3.1]. As described in [Zem21b, Corollary 4.8] relating these two models amounts to constructing a null-homotopy of an algebraically defined homology action on the link surgery formula. It is observed [Zem21b, Remark 4.3] that this null-homotopy may be taken to be \( R_\ell \)-equivariant.

We now address (R-2). Proposition 5.3 shows that \( Z \) is chain isomorphic as an \( R_\ell \)-module to \( CFL\Gamma, V_\uparrow \). It is sufficient to show that the homotopy equivalence \( C' \simeq Z \) of chain complexes is in fact a homotopy equivalence of \( A_\infty \)-modules over \( R_\ell \). To see this, it is in fact sufficient to show that the map \( \Pi'_\varepsilon : C' \to Z \) defined as in (5.10), commutes with the \( R_\ell \) action. This is sufficient because in the category of \( A_\infty \)-modules, quasi-isomorphisms are always invertible as \( A_\infty \)-morphisms. To establish the \( R_\ell \)-equivariance, we observe that the projection maps \( \pi_\varepsilon \) are themselves equivariant since they are merely quotient maps; compare Remark 5.5. Next, we examine the expression for \( \Pi'_\varepsilon \), as in (5.10). By considering the Hopf grading similarly to how we did with \( d_{\varepsilon, \varepsilon'} \) above, we observe that the maps \( \Pi'_{\varepsilon, \varepsilon'} \) are non-trivial only when \( \varepsilon = \varepsilon' \). In this case, the only non-vanishing contribution is from \( \pi_\varepsilon \), so \( R_\ell \)-equivariance is established, and the proof is complete. \( \square \)

**Remark 5.6.** By using the homological perturbation lemma, stated in Lemma 2.3, a concrete homotopy equivalence of \( A_\infty \)-modules between \( C'_\Lambda(L_G, L_\uparrow) \) and \( Z \) may be constructed. Indeed the hypercube maps \( \Pi', I' \) and \( H' \), constructed via the homological perturbation lemma for hypercubes, also satisfy the assumptions of the homological perturbation lemma for \( A_\infty \)-modules. These maps then induce an \( A_\infty \)-module structure on \( Z \) over the ring \( R_\ell \), which is homotopy equivalent to \( CFL(Y_G, L_\uparrow) \). By considering the Hopf grading, similarly to the above, we obtain only a non-trivial \( m_1 \) and \( m_2 \) on \( Z \). We observe also that the morphisms \( \Pi', H' \) and \( I' \) extend to \( A_\infty \)-morphisms \( \Pi'_j, H'_j \) and \( I'_j \). Hopf grading considerations show that \( \Pi'_j = 0 \) unless \( j = 1 \), in which case the only contribution is from \( \pi_\varepsilon \). We observe that \( I'_j \) may be non-trivial for \( 1 \leq j \leq |L_\Gamma| \).

### 5.4. Absolute gradings

In this section, we prove the subclaim of Theorem 5.1 concerning the absolute Maslov and Alexander gradings. Compare [OSS14b, Proposition 4.8].

We begin by stating formulas for the absolute gradings on the link surgery complex, and its subcube refinement for sublinks. Although likely known to experts, these formulas have not appeared in the literature except in special cases. For example, in the case of knots, the result is due to Ozsváth and Szabó [OS04a, Section 4]. Detailed proofs of the absolute grading formula will appear in [Zem].
If \( L \subseteq S^3 \) is a link with framing \( \Lambda \), let \( W_\Lambda(L) \) denote the standard 2-handle cobordism from \( S^3 \) to \( S^3_\Lambda \). If \( \varepsilon \in \{0, 1\}^n \), where \( n = |L| \), and \( s \in \mathbb{H}(L) \), write \( \mathcal{C}_\varepsilon(s) \subseteq \mathcal{C}_\Lambda(L) \) for the subspace of \( \mathcal{C}_\varepsilon \) which lies in internal Alexander grading \( s \). Finally, if \( s \in \mathbb{H}(L) \), write \( \mathfrak{s}_s \in \text{Spin}^c(W_\Lambda(L)) \) for the \( \text{Spin}^c \) structure which satisfies

\[
\frac{(c_1(\mathfrak{s}_s), \Sigma_i) - \Sigma \cdot \Sigma_i}{2} = -s_i
\]

for all \( i \in \{1, \ldots, n\} \). In the above, \( \Sigma_i \) denotes the core of the 2-handle attached along component \( K_i \subseteq L \), and \( \Sigma \) denotes the sum of all \( \Sigma_i \).

**Lemma 5.7 ([Zem]).** Suppose that \( L \subseteq S^3 \) is a link with framing \( \Lambda \), and that \( b_1(S^3_\Lambda(L)) = 0 \). The homotopy equivalence \( \text{CF}^-(S^3_\Lambda(L)) \simeq \mathcal{C}_\Lambda(L) \) is absolutely graded if we equip \( \mathcal{C}_\varepsilon(s) \subseteq \mathcal{C}_\Lambda(L) \) with the Maslov grading

\[
\text{gr} := \text{gr}_w + \frac{c_1(\mathfrak{s}_s)^2 - 2\chi(W_\Lambda(L)) - 3\sigma(W_\Lambda(L))}{4} + |L| - |\varepsilon|,
\]

where \( \text{gr}_w \) is the Maslov grading from \( \text{CFL}(L) \).

In [OS08b], Ozsváth and Szabó prove this formula in the context of knot surgery formula. Their main tool is computing the grading change of a surgery cobordism map, which they denote by \( f^*_\varepsilon \). A similar strategy to Ozsváth and Szabó’s proof of the grading formula is likely possible in the context of the link surgery formula. Nonetheless, algebraic truncations make writing a simple proof challenging.

There is also a relative version of the statement. Suppose that we have a partitioned link \( J \sqcup L \subseteq S^3 \) and that \( J \) is equipped with an integral framing \( \Lambda \). If \( K_i \) is a link component of \( J \sqcup L \), there is a class \( \Sigma_i \in H_2(W_\Lambda(J), \partial W_\Lambda(J)) \). If \( K_i \) is in \( J \), then \( \Sigma_i \) is the core of the corresponding 2-handle. If \( K_i \) is in \( L \), then \( \Sigma_i \) is an annulus, with boundary on the images of \( K_i \) in \( S^3 \) and \( S^3_\Lambda(J) \). We write \( \Sigma \) for the sum of the classes for all components of \( J \sqcup L \). If the component \( K_i \subseteq L \) becomes rationally null-homologous in \( S^3_\Lambda(J) \), we write \( \Sigma_i \) for the class in \( H_2(W_\Lambda(J); \mathbb{Q}) \) obtained by capping with a rational Seifert surface. Finally, if \( s \in \mathbb{H}(L) \), we write \( \mathfrak{s}_s \in \text{Spin}^c(W_\Lambda(J)) \) for the \( \text{Spin}^c \) structure which satisfies Equation (5.12) for all link components \( K_i \) in \( J \).

**Lemma 5.8 ([Zem]).** Suppose that \( J \sqcup L \subseteq S^3 \) is a partitioned link and \( \Lambda \) is an integral framing on \( J \) such that \( b_1(S^3_\Lambda(J)) = 0 \). Then the isomorphism \( \text{CF}(S^3_\Lambda(J), L) \simeq \mathcal{C}_\Lambda(J, L) \) from Theorem 5.2 is absolutely \( \text{gr}_w \)-graded if we equip \( \mathcal{C}_\varepsilon(s) \subseteq \mathcal{C}_\Lambda(J, L) \) with the Maslov grading

\[
\bar{\text{gr}}_w = \text{gr}_w + \frac{c_1(\mathfrak{s}_s)^2 - 2\chi(W_\Lambda(J)) - 3\sigma(W_\Lambda(J))}{4} + |J| - |\varepsilon|,
\]

where \( \text{gr}_w \) is the internal \( \text{gr}_w \)-grading on \( \text{CFL}(J \sqcup L) \). The isomorphism is absolutely graded with respect to the Alexander grading \( A = (A_1, \ldots, A_{|L|}) \) if we define \( A_i \) on \( \mathcal{C}_\varepsilon(s) \) via the formula

\[
A_i = s_i + \frac{c_1(\mathfrak{s}_s)^2, \Sigma_i) - \Sigma \cdot \Sigma_i}{2}.
\]

**Proposition 5.9.** If \( b_1(Y_G) = 0 \), the isomorphism from Theorem 5.1 respects the absolute Maslov and Alexander gradings.

**Proof.** As a first step, we consider the absolute case when there are no arrow vertices. We recall that we already constructed an isomorphism \( \Phi_G : \text{Char}(X_G) \to \mathbb{H}(L) \) in Equation (4.8). In the present case, it is straightforward to verify from the definitions that \( c_1(\mathfrak{s}_s) = -\Phi_G^{-1}(s) \). If \( s \in \mathbb{H}(L_G) \), then this is equivalent to

\[
c_1(\mathfrak{s}_s) = -K.
\]

Next, we recall that in Lemma 5.4, we identified the quantity \( g([K, E]) \) with the \( \text{gr}_w \)-grading of the generator of the tower in the \( s \)-graded subspace of the homology of \( \text{CFL}(L_G) \), localized at the \( \gamma_i \) variables for vertices in \( E \). Noting that \((-K)^2 = K^2\), we obtain Equation (4.6).
We now consider the case that there are arrow vertices. Suppose that Γ is a tree with \( V_\Gamma = V_G \cup V_I \). Lemma 5.8 computes the Maslov grading shift. If \( K \in \text{Char}(X_\Gamma) \) and \( s = \Phi_\Gamma(K) \), then we have similarly to Equation (5.13) that
\[
c_1(\delta_s^{L_G}) = -K|_{X_G}. \tag{5.14}
\]
In particular, the statement from Lemma 5.4 implies that the Maslov grading on \( \text{CFL}(S^3_L(G), L_\Gamma) \) coincides with the one defined in Equation (4.7).

We now consider the Alexander grading. We note that given \( s \in \mathbb{H}(L_\Gamma) \), there are two \( \text{Spin}^c \) structures of interest: \( \delta_s^{L_G} \in \text{Spin}^c(X_G) \) and \( \delta_s^{L_\Gamma} \in \text{Spin}^c(X_\Gamma) \). It is straightforward to verify that \( \delta_s^{L_G} = \delta_s^{L_\Gamma}|_{X_G} \).

If \( s = (s_1, \ldots, s_{|L_\Gamma|}) \in \mathbb{H}(L_\Gamma) \), and \( K_i \in L_\Gamma \), we view \( \mathcal{C}_A(L_G, L_\Gamma, s) \) as having internal Alexander grading \( A_i \) equal to \( s_i \). Lemma 5.8 implies that the isomorphism \( \text{CFL}(S^3_L(G), L_\Gamma) \simeq \mathcal{C}_A(L_G, L_\Gamma) \) is Alexander graded if we shift the internal Alexander grading \( s_i \) (for a component \( K_i \in L_\Gamma \)) of \( \mathcal{C}_A(L_G, L_\Gamma, s) \) by \( (c_1(\delta_s^{L_G}), \Sigma_i) - \Sigma \cdot \Sigma_i)/2 \).

By the definition of \( \Phi_\Gamma \), if \( \Phi_\Gamma(K) = s \), then
\[
s_i = \frac{(K, \Sigma_i) + \Sigma \cdot \Sigma_i}{2},
\]
where the pairings occur in \( X_\Gamma \). In particular, the generator \([K, E]\) in the lattice complex will be given \( i^{th} \) Alexander grading
\[
s_i + \frac{(c_1(\delta_s^{L_G}), \Sigma_i) - \Sigma \cdot \Sigma_i}{2}.
\]
By Equation (5.14), we may write the above as
\[
K(v_i - \hat{v}_i) + \sum_{v \in V_G} v \cdot (v_i - \hat{v}_i).
\]
We note that for \( v \in V_G \), the pairing \( v \cdot (v_i - \hat{v}_i) \) will vanish, and hence we can replace the sum in the above equation with a sum over only \( v \in V_\Gamma \). This recovers the formula for the Alexander grading stated in Section 4.6, so the proof is complete.

6. Plumber L-space links

In this section, we compute the link Floer complexes of plumbed L-space links. In Section 6.1, we prove that their complexes are formal (i.e. \( \text{CFL}(Y_G, L_\Gamma) \) is homotopy equivalent as an \( A_\infty \)-module to \( \text{HFL}(Y_G, L_\Gamma, m_j) \), where \( m_j = 0 \) unless \( j = 2 \)). Consequently, the chain complexes are also homotopy equivalent to free resolutions of their homology. In Section 6.2 we recall work of Gorsky and Némethi [GN15] which allows one to compute the module \( \text{HFL}(Y_G, L_\Gamma) \) when \( Y_G = S^3 \) and \( L_\Gamma \) is an L-space link. We extend their description to the case where \( Y_G \) is a rational homology L-space. Finally, in Section 6.3, we prove that our model of link lattice homology recovers the version of Gorsky and Némethi [GN15] in the case of plumbed L-space links.

6.1. Plumber L-space links and free resolutions. Suppose \( \Gamma \) is an arrow-decorated plumbing tree. Our goal is to show that the chain complex \( \text{CFL}(Y_G, L_\Gamma) \) is a free resolution of its homology, in particular, that the chain complex is determined by the homology up to chain homotopy equivalence.

In this section, we consider plumbings where \( L_\Gamma \) is an L-space link. We observe that this implies that \( Y_G \) is a rational homology 3-sphere, and furthermore is itself an L-space. In particular, by Proposition 3.18, the results in this subsection hold for links of embedded analytic singularities, as long as the underlying surface singularity is rational.

**Theorem 6.1.** Suppose that \( \Gamma \) is an arrow-decorated plumbing tree, with \( V_\Gamma = V_G \cup V_\Gamma \). Let \( L_\Gamma \subseteq Y_G \) be the associated link and assume that \( Y_G \) is a rational homology 3-sphere. If \( L_\Gamma \) is an L-space link, then \( \text{CFL}(Y_G, L_\Gamma) \) is a free resolution over \( \mathcal{R}_\ell \) of \( \text{HFL}(Y_G, L_\Gamma) \).
Proof. To simplify the notation, we assume that $Y_G$ is an integer homology 3-sphere. For rational homology 3-spheres, one may apply the same argument to each Spin$^c$ structure.

Next, we observe that it is sufficient to show that $\text{CFL}(Y_G, L_\ell)$ is homotopy equivalent to a free resolution of $H_{\text{FL}}(Y_G, L_\ell)$. This may be seen as follows: The $\mathcal{R}_\ell$-module $H_{\text{FL}}(Y_G, L_\ell)$ is finitely generated and hence admits a finitely generated free resolution over $\mathcal{R}_\ell$ by Hilbert’s syzygy theorem [Hil90]. See e.g. [Pee11, Theorem 15.2] for a modern exposition. Furthermore, it is straightforward to see that if $C$ and $C'$ are two free, finitely generated chain complexes over $\mathcal{R}_\ell$ which are both $(\text{gr}_w, \text{gr}_z)$-graded, then $C$ and $C'$ are homotopy equivalent over $\mathcal{R}_\ell$ if and only if $C \otimes_{\mathcal{R}_\ell} \mathcal{R}_\ell$ and $C' \otimes_{\mathcal{R}_\ell} \mathcal{R}_\ell$ are homotopy equivalent. Moreover, the completion of a free resolution of $H_{\text{FL}}(Y_G, L_\ell)$ will be a free resolution of $H_{\text{FL}}(Y_G, L_\ell)$, by similar reasoning. Hence, $\text{CFL}(Y_G, L_\ell)$ will be homotopy equivalent to a free resolution of $H_{\text{FL}}(Y_G, L_\ell)$ if and only if $\text{CFL}(Y_G, L_\ell)$ is homotopy equivalent to a free resolution of $H_{\text{FL}}(Y_G, L_\ell)$.

Since $L_\ell \subseteq Y_G$ is an L-space link, $H_{\text{FL}}(Y_G, L_\ell)$ is a torsion-free $\mathbb{F}[U]$-module. By Theorem 5.1, $\text{CFL}(Y_G, L_\ell)$ is homotopy equivalent to $\text{CFL}(\Gamma, V_\ell)$ as an $A_\infty$-module over $\mathcal{R}_\ell$.

The link lattice complex has a cube grading:

$$\text{CFL}(\Gamma, V_\ell) = \langle Z_n \xrightarrow{f_{n,n-1}} Z_{n-1} \rightarrow \cdots \rightarrow Z_1 \xrightarrow{f_{1,0}} Z_0 \rangle,$$

where $Z_n$ is spanned by $U^p[K, E]$ where $|E| = q$, $p \geq 0$. Furthermore, each $Z_i$ is itself an $\mathcal{R}_\ell$-module. (In particular, the action of $\mathcal{R}_\ell$ preserves the cube grading).

The homology of $\text{CFL}(\Gamma, V_\ell)$ is torsion free as an $\mathbb{F}[U]$-module, and the action of $\mathcal{R}_\ell$ preserves each $Z_\ell$. Moreover, if we localize the homology of $\text{CFL}(\Gamma, V_\ell)$ at the $\mathcal{R}_\ell$ and $V_\ell$ variables, we obtain a module which is isomorphic to $H_{\text{FL}}^\infty(S^3, U_\ell)$, which is isomorphic to

$$\mathbb{F}[U_1, \ldots, U_\ell][U_1^{-1}, \ldots, U_\ell^{-1}]/(U_i V_i - U_j V_j : i, j \in \{1, \ldots, \ell\}).$$

(i.e. the ring of infinite Laurent polynomials in $U_1, \ldots, U_\ell, V_1, \ldots, V_\ell$ with finite negative tails, modulo the relation that $U_i V_i = U_j V_j$ for all $i$ and $j$). Here $U_\ell$ denotes an $\ell$-component unlink. In particular, it follows that the homology of $\text{CFL}(\Gamma, V_\ell)$ is supported in a single $Z_i$.

Consider first the case that the homology is supported in cube grading $i = 0$. In this case, we may pick a splitting over $\mathbb{F}$ of the complex $\text{CFL}(\Gamma, V_\ell)$. Such a splitting determines a homotopy equivalence of $\text{CFL}(\Gamma, V_\ell)$ with its homology. We may apply the homological perturbation lemma to obtain an induced $A_\infty$-module structure over $\mathcal{R}_\ell$ on the homology $H_{\text{FL}}(\Gamma, V_\ell)$. A filtration argument like the one in the proof of Theorem 5.1 implies that the induced $A_\infty$-module structure on $Z_0/\text{im} Z_1 \cong H_{\text{FL}}(\Gamma, V_\ell)$ has $m_j = 0$ unless $j = 2$. By Corollary 2.6, this implies that $\text{CFL}(Y_G, L_\ell)$ is homotopy equivalent over $\mathcal{R}_\ell$ to a free resolution of its homology.

We now consider the case that $\text{CFL}(\Gamma, V_\ell)$ is supported at $Z_i$ for some $i \neq 0$. Our argument proceeds by induction, with the base case $i = 0$ covered above. We will use techniques described in Subsection 2.3.

As $i > 0$, in particular $H_*(Z_0) = 0$, so $f_{1,0}$ is surjective. We may pick a splitting $i_{0,1}$ of $f_{1,0}$ as a map of vector spaces, which induces a splitting of $Z_1$ as $Z_1 = Z_1^0 \oplus Z_1^1$ (where the direct sum is of $\mathbb{F}$-vector spaces), and $Z_1^1 = \ker(f_{1,0})$ and $Z_1^0 = \text{im}(i_{0,1})$. Note that $Z_1^1$ is not in general an $\mathcal{R}_\ell$-module, as it is the image of an $\mathbb{F}$-linear map, however $\ker(f_{1,0})$ is always an $\mathcal{R}_\ell$-submodule since $f_{1,0}$ is $\mathcal{R}_\ell$-equivariant.

There is a chain complex $Z'$ obtained by deleting $Z_1^1$ and $Z_0$, and the above maps determine a chain homotopy equivalence between $\text{CFL}(\Gamma, V_\ell)$ and $Z'$ as chain complexes over $\mathbb{F}$. Via the homological perturbation lemma for $A_\infty$-modules, we may equip $Z'$ with an $A_\infty$-module structure over $\mathcal{R}_\ell$ which is homotopy equivalent to $\text{CFL}(\Gamma, V_\ell)$. The map $h$ appearing in the homological perturbation lemma is the map $i_{0,1}$. The inclusion and projection maps are the obvious ones. Compare Section 2.3. Since $\ker(f_{1,0})$ is closed under the action of $\mathcal{R}_\ell$, the action on $Z'$ from the homological perturbation lemma is the standard one. This reduces the index at which the homology of $\text{CFL}(\Gamma, V_\ell)$ is supported. Proceeding by induction we reduce to the base case $i = 0$, completing the proof. \qed
6.2. Computing the Floer chain complex from Alexander polynomials. Gorsky and Nemethi [GN15] defined the $H$-function of oriented links in the 3-sphere as follows:

**Definition 6.2.** For an oriented link $L \subseteq S^3$ in the 3-sphere, we define $H_L(s)$ by declaring $-2H_L(s)$ to be the maximal $\text{gr}_w$-grading of a non-zero element in the free part of $H_\gamma(\mathfrak{A}^-(L, s))$ where $\mathfrak{A}^-(L, s)$ is the subcomplex of $\mathcal{CFL}(S^3, L)$ in Alexander grading $s$.

The $H$-function is a generalization of the $V$-function considered by Rasmussen [Ras03]. For algebraic links, it can be related to the semigroup counting function [GN15, Section 3.5].

In this subsection, we consider the $H$-function for oriented $\ell$-component links $L$ in rational homology spheres $Y$, generalizing Definition 6.2. We begin by defining a lattice $\mathbb{H}(Y, L)$, which is an affine space over $H_1(Y \setminus L, \mathbb{Z})$. The set $\mathbb{H}(Y, L)$ is a subspace of $\mathbb{Q}^\ell \times \text{Spin}^c(Y)$. The simplest definition of $\mathbb{H}(Y, L)$ is that it is the set of $(s, t) \in \mathbb{Q}^\ell \times \text{Spin}^c(Y)$ such that $\text{HF}_L(Y, L, t)$ is non-trivial in some Alexander grading $s' \in \mathbb{Q}^\ell$ satisfying $s - s' \in \mathbb{Z}^\ell$. Since $\text{HF}_L(Y, L, t) \neq 0$ for each $t$ when $Y$ is a rational homology 3-sphere, this construction gives a well-defined set $\mathbb{H}(Y, L)$. The action of an element $\gamma \in H_1(Y \setminus L)$ is given by

$$\gamma \cdot (s_1, \ldots, s_\ell, t) = (s_1 - \text{lk}(\gamma, K_1), \ldots, s_\ell - \text{lk}(\gamma, K_\ell), t + \text{PD}[i_\ast \gamma])$$

where $\text{lk}(\gamma, K_i) \in \mathbb{Q}$ is the rational linking number, and $i : Y \setminus L \to Y$ is inclusion.

A more topological description may be obtained by presenting $Y \setminus L$ as surgery on a link $L'$ in the complement of an $\ell$-component unlink in $S^3$. Such a presentation induces link cobordism $(W, \Sigma)$ from the complement of an $\ell$-component unlink to $Y \setminus L$ such that $W$ is a 2-handlebody, and $\Sigma$ consists of $\ell$ annuli, each of which cobounds an unknot component in $S^3$ and a knot component of $L$. If $t \in \text{Spin}^c(W)$, the fiber over $t$ under the map $\mathbb{H}(Y, L) \to \text{Spin}^c(Y)$ consists of $(s, t)$ where

$$s \in \left(\frac{c_1(t) \cdot \hat{\Sigma}_1}{2} - \frac{[\Sigma_1] \cdot |\Sigma|}{2}, \ldots, \frac{c_1(t) \cdot \hat{\Sigma}_{\ell}}{2} - \frac{[\Sigma_{\ell}] \cdot |\Sigma|}{2}\right) + \mathbb{Z}^\ell,$$

and $t' \in \text{Spin}^c(W)$ is any lift of $t \in \text{Spin}^c(Y)$. Here, we view $\Sigma$ as the union of $\Sigma_1, \ldots, \Sigma_{\ell}$. We write $|\Sigma|$ and $|\Sigma_i|$ for the induced classes in $H_2(W, \partial W; \mathbb{Z})$, and we write $[\Sigma_i]$ for the lifts under the map $H_2(W; \mathbb{Q}) \to H_2(W, \partial W; \mathbb{Q})$.

This may be seen to coincide with the definition in terms of Alexander gradings on $\text{HF}_L(Y, L)$ using a small modification of the cobordism argument from [Zem19, Section 5.5] (which is stated for integrally null-homologous links). See [HHSZ22, Section 3.2] for an exposition in the setting of rationally null-homologous knots. It is an easy consequence of the cobordism description of the Alexander grading that $\mathcal{CFL}(Y, L)$ is supported on $\mathbb{H}(Y, L)$.

**Lemma 6.3.** As affine spaces over $H_1(Y \setminus L; \mathbb{Z})$, there is an isomorphism $\mathbb{H}(Y, L) \cong H_1(Y \setminus L; \mathbb{Z})$.

**Proof.** This follows from the short exact sequence of affine spaces

$$0 \to \mathbb{Z}^\ell \to \mathbb{H}(Y, L) \to \text{Spin}^c(Y) \to 0$$

which is parallel to the short exact sequence of homology groups

$$0 \to \mathbb{Z}^\ell \to H_1(Y \setminus L; \mathbb{Z}) \to H_1(Y; \mathbb{Z}) \to 0.$$

If we pick a base element $(s, t) \in \mathbb{H}(Y, L)$, we obtain a map of affine spaces from $H_1(Y \setminus L; \mathbb{Z})$ to $\mathbb{H}(Y, L)$ which makes the natural diagram commute. By the five-lemma, we obtain that $\mathbb{H}(Y, L)$ and $H_1(Y \setminus L; \mathbb{Z})$ are isomorphic as affine spaces.

We are now ready to define the $H$-function of a link in a rational homology sphere.

**Definition 6.4.** For an oriented link $L \subseteq Y$ in a rational homology sphere $Y$ and $(s, t) \in \mathbb{H}(Y, L)$, we define the $H_L : \mathbb{H}(Y, L) \to \mathbb{Q}$ by saying that $-2H_L(s, t)$ is the maximal $\text{gr}_w$-grading of a non-zero element in the free part of $H_\gamma(\mathfrak{A}^-(L, s, t))$ where $\mathfrak{A}^-(L, s, t)$ is the subcomplex of $\mathcal{CFL}(Y, L, t)$ lying in Alexander grading $s$. 

The $H$-function of links in the 3-sphere can be computed from Alexander polynomials of the link and all sublinks [GN15, BG18]. In order to generalize the result to links in rational homology sphere, we first recall generalized Alexander polynomials. Friedl, Juhász and Rasmussen [FJR11, Theorem 1] prove that $HFL(Y, L)$ categorifies the Turaev torsion $\Delta(Y, L)$ of $Y \setminus L$, which we view as an element in $\mathbb{F}[H_1(Y \setminus L)]$, well-defined up to multiplication by monomials. When $Y$ is a rational homology 3-sphere, we refer to $\Delta(Y, L)$ as the generalized Alexander polynomial.

In our present setting, it is helpful to view $\Delta(Y, L)$ as taking values in the $\mathbb{F}[H_1(Y \setminus L)]$-module $\mathbb{F}[\mathbb{H}(Y, L)]$ instead of $\mathbb{F}[H_1(Y \setminus L)]$ itself, via the following formula:

$$
\Delta(Y, L) = \chi(\hat{HFL}(Y, L)) := \sum_{(s, t) \in \mathbb{H}(Y, L)} t^{(s, t)} \cdot \chi(\hat{HFL}(Y, L, s, t)) \in \mathbb{F}[\mathbb{H}(Y, L)].
$$

We may think of $\chi(\hat{HFL}(Y, L))$ as a normalized version of the Turaev torsion. Similarly to [OS08a, Proposition 8.1], it is not hard to see that the Euler characteristic of $HFL(Y, L)$ is symmetric with respect to the involution of $\mathbb{H}(Y, L)$ given by $(s, t) \mapsto (-s, t + PD[L])$ (cf. [Zem19, Proposition 8.3]).

To make more transparent connections with integer homology spheres, we refine the definition of $\Delta(Y, L)$. For $t \in \text{Spin}^c(Y)$, we set:

$$
\Delta(Y, L, t) = \chi(\hat{HFL}(Y, L, t)) := \sum_{(s, t) \in \mathbb{H}(Y, L)} t^s \cdot \chi(\hat{HFL}(Y, L, s, t)) \in \mathbb{F}[\mathbb{H}(Y, L)]. \quad (6.3)
$$

We recall that the graded Euler characteristics of $\hat{HFL}$ and $HFL^-$ are related by the formula

$$
\chi(HFL^-(Y, L)) = \frac{1}{\prod_{i=1}^\ell (1 - t_i)} \chi(\hat{HFL}(Y, L)).
$$

See [OS08a, Proposition 9.2].

For any sublink $L' \subseteq L$, define the natural forgetful map:

$$
\pi_{L, L-L'} : \mathbb{H}(Y, L) \to \mathbb{H}(Y, L - L')
$$

as follows. In the case that $L' = K_1$, the map $\pi_{L, L-L'}$ is given by the formula

$$
\pi_{L, L-L'}(s_1, \ldots, s_\ell, t) = \left( s_2 - \frac{\text{lk}(K_1, K_2)}{2}, \ldots, s_\ell - \frac{\text{lk}(K_1, K_\ell)}{2}, t \right),
$$

where $L = K_1 \cup \cdots \cup K_\ell$. For general $L'$, the formula is a composition of several maps of a similar form to the above. We refer reader to [MO10, Section 3.7] for explicit formulas of the forgetful map for links in $S^3$. Compare [BG18, Section 3].

Given $(s, t), (s', t') \in \mathbb{H}(Y, L)$, we say $(s, t) \geq (s', t')$ if and only if $t = t'$ and $s \geq s'$. That is, $s_i \geq s'_i$ for all $i$, where $s = (s_1, \ldots, s_\ell)$ and $s' = (s'_1, \ldots, s'_\ell)$.

**Lemma 6.5.** For an oriented L-space link $L \subseteq Y$ in a rational homology sphere with $t \in \text{Spin}^c(Y)$, the $H$-function $H_L$ satisfies:

$$
H_L(s, t) = \sum_{L' \subseteq L} (-1)^{|L'| - 1} \sum_{(s', t') \in \mathbb{H}(Y, L'), \pi_{L,L'}(s', t') \geq (s, t)} \chi(HFL^-(Y, L', s', t)). \quad (6.4)
$$

where $1 = (1, \ldots, 1)$. In particular, the $H$-function is determined by the generalized Alexander polynomials $\Delta(Y, L', t)$ of all sublinks $L' \subseteq L$.

**Proof.** The arguments of [GN15, Theorem 2.10] relating $H_L$ to $\chi(HFL^-(Y, L))$ in the case of links in $S^3$ can be repeated verbatim for the case of $H_L$ and $\chi(HFL^-(Y, L))$ to get (6.4), though here we follow the convention in [BG18, Theorem 3.15] and assume $HFL^-(Y, \emptyset) = 0$. The right-hand side of (6.4) is determined by $\chi(\hat{HFL}(Y, L'))$ for all sublinks $L'$, which can be computed from Alexander polynomials of $L'$ by (6.3). Therefore, the $H$-function is determined by the Alexander polynomials of all sublinks $L' \subseteq L$. 

$\square$
Example 6.6. We normalize the multivariable Alexander polynomial of the unlink $U$ in the 3-sphere to be 0, and the $H$-function for an $\ell$-component unlink in $S^3$ is the following:

$$H_{U_\ell}(s_1, \ldots, s_\ell) = \sum_i (|s_i| - s_i)/2.$$ 

If $L \subseteq S^3$ is the Hopf link in the 3-sphere, its Alexander polynomial $\Delta(t_1, t_2) = 1$ and $s_i \in \mathbb{Z} + 1/2$. By the formula in [BG18,Liu17], its $H$-function is the following:

$$H_L(s_1, s_2) = H_{U_2}(s_1 - 1/2, s_2 - 1/2)$$

where $U_2$ is the 2-component unlink. See [BG18, Sections 3.5 and 7.6] for more examples.

Theorem 6.7. Suppose that $\Gamma$ is an arrow-decorated plumbing tree, with $V_\Gamma = V_G \cup V_\Gamma$. Let $L_\Gamma \subseteq Y_G$ be the associated link and assume that $Y_G$ is a rational homology 3-sphere. If $L_\Gamma$ is an $L$-space link, then the full link Floer complex $\mathcal{CFL}(Y_G, L_\Gamma)$ is determined by the generalized Alexander polynomials of $L_\Gamma$ and its sublinks.

Proof. By Theorem 6.1, the chain complex $\mathcal{CFL}(Y_G, L_\Gamma)$ is a free resolution of $\mathcal{HF}(Y_G, L_\Gamma)$. In particular, $\mathcal{CFL}(Y_G, L_\Gamma)$ is determined by the homology group $\mathcal{HF}(Y_G, L_\Gamma)$ as an $R_\ell$-module. Therefore, it suffices to prove that the generalized Alexander polynomials determine the homology $\mathcal{HF}(Y_G, L_\Gamma)$. By Lemma 6.5, the $H$-function is determined by the generalized Alexander polynomials of $L_\Gamma$ and its sublinks, it remains to prove that the $H$-function determines the $R_\ell$-module structure of $\mathcal{HF}(Y_G, L_\Gamma)$.

There is a decomposition of $\mathbb{F}[U]$-modules

$$\mathcal{HF}(Y_G, L_\Gamma) = \bigoplus_{(s,t) \in \mathbb{E}(Y_G, L_\Gamma)} \mathcal{HF}(Y_G, L_\Gamma, s, t).$$

Since $L_\Gamma$ is an $L$-space link in $Y_G$, $\mathcal{HF}(Y_G, L_\Gamma, s, t) \cong \mathbb{F}[U]$ for all $(s,t) \in \mathbb{E}(Y_G, L_\Gamma)$. The $(\text{gr}_w, A)$-grading of the generator of $\mathcal{HF}(Y_G, L_\Gamma, s, t)$ is determined by the $H$-function. Hence, it suffices to see the $\mathcal{U}_i$ and $\mathcal{V}_i$ actions on $\mathcal{HF}(Y_G, L_\Gamma, t)$ are also determined by the $H$-function.

To see this, note that we may view $\mathcal{U}_i$ as restricting to a map

$$\mathcal{HF}(Y_G, L_\Gamma, s, t) \cong \mathbb{F}[U] \to \mathcal{HF}(Y_G, L_\Gamma, s - e_i, t) \cong \mathbb{F}[U]. \quad (6.5)$$

Since the map $\mathcal{V}_i$ goes in the opposite direction and $\mathcal{U}_i \mathcal{V}_i = U$, the map $\mathcal{U}_i$ is given by either multiplication by 1 or $U$, with respect to identifications in (6.5). We observe that the map $\mathcal{U}_i$ has $\text{gr}_w$-grading $-2$, and hence the choice of being $U$ or 1 is determined by the $\text{gr}_w$-gradings of the copies of $\mathbb{F}[U]$ in (6.5), which is encoded by the $H$-function. Similarly, the action of $\mathcal{V}_i$ is also determined by the $H$-function since $\mathcal{U}_i \mathcal{V}_i = U$, and the action of $\mathcal{V}_i$ is determined by the $H$-function. Therefore, for $L$-space links, the Alexander polynomials determine the full link Floer chain complexes. \hfill \Box

6.3. Comparison to Gorsky and Némethi’s link lattice homology. We now compare our chain complex $\mathcal{CFL}(\Gamma, V_\Gamma)$ to the definition of link lattice homology due to Gorsky and Némethi [GN15].

As a first step, we recall their definition. Let $L \subseteq S^3$ be a link of $\ell$ components. Write $\mathcal{G}(L)$ for the $\ell$-dimensional hypercube of chain complexes obtained by tensoring $\mathcal{CFL}(L)$ with the complexes

$$C_i := \text{Cone}(\mathcal{V}_i: R_\ell \to R_\ell), \ i = 1, \ldots, \ell.$$ 

Write $\mathcal{X}(L)$ for the (non-free) complex of $R_\ell$-modules obtained by tensoring the homology group $\mathcal{HFL}(L)$ with $C_1, \ldots, C_\ell$, over the ring $R_\ell$. If $L$ is an $L$-space link, Gorsky and Némethi define the link lattice complex to be the chain complex $\mathcal{X}(L)$. Gorsky and Némethi prove the following:

Theorem 6.8 ([GN15, Theorem 2.9]). If $L$ is an algebraic link, then

$$H_* \mathcal{X}(L) \cong H_* \left( \mathcal{CFL}(L) / (\mathcal{V}_1, \ldots, \mathcal{V}_\ell) \right).$$
Remark 6.9. As noted by Gorski and Némethi, the proof of Theorem 6.8 gives an isomorphism of graded \( \mathbb{F} \) vector spaces. There is also an action of \( \mathbb{F}[U] \) on \( H_\ast \mathcal{H}(L) \), arising from the action of \( \mathbb{F}[U] \) on \( \mathfrak{X}^-(L,s) \). In our present set-up, this action corresponds to the action of \( \mathcal{H}_i \) for any \( i \in \{1, \ldots, \ell\} \). Gorsky and Némethi prove that this action is trivial on \( H_\ast \mathcal{H}(L) \).

We now explain how our Theorem 6.1 quickly recovers Theorem 6.8, and we give the following improvement on Gorsky and Némethi’s result:

**Theorem 6.10.** If \( L \) is a plumbed L-space link, then there is an isomorphism of \( \mathbb{F}[\mathcal{H}_1, \ldots, \mathcal{H}_\ell] \)-modules

\[
H_\ast \mathcal{H}(L) \cong H_\ast \left( \mathbb{CFL}(L)/(\mathcal{H}_1, \ldots, \mathcal{H}_\ell) \right) .
\]

**Proof.** Note that \( C_i \) is a free resolution of \( \mathcal{H}_i \) as an \( \mathcal{H}_i \)-module. Furthermore, the \( \ell \)-dimensional cube-shaped complex \( C_1 \otimes \cdots \otimes C_\ell \) is a free resolution of \( \mathbb{F}[\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_\ell]/(\mathcal{H}_1, \ldots, \mathcal{H}_\ell) \).

We use the algebraic formalism of type-D and type-A modules of Lipshitz, Oszváth and Thurston [LOT18] [LOT15] which we use to give a small model of the derived tensor product of \( A_\infty \)-modules. We may view \( C_i \) as a type-D module \( \mathcal{H}_i \), whose underlying \( \mathbb{F} \) vector space has two generators, \( x_i \) and \( y_i \), and whose structure map \( \delta^1 \) is given by

\[
\delta^1(x_i) = y_i \otimes y_i.
\]

Similarly, the complex \( C_1 \otimes \cdots \otimes C_\ell \) naturally corresponds to a type-D module over \( \mathcal{H}_i \), whose underlying vector space is generated by the points of an \( \ell \)-dimensional cube. We write \( \mathcal{H}_i C_1, \ldots, \mathcal{H}_i \) for this cube-shaped complex.

By definition,

\[
\mathcal{H}(L) := \mathcal{HFL}(L) R_{\mathcal{H}} \boxtimes R_{\mathcal{H}} C_1, \ldots, \mathcal{H}_i.
\]

By Theorem 6.1, if \( L \) is a plumbed L-space link, \( \mathcal{HFL}(L) \) and \( \mathbb{CFL}(L) \) are homotopy equivalent as \( A_\infty \)-modules over \( \mathcal{H}_i \). On the other hand, \( \mathcal{CFL}(L) \mathcal{R}_i \) is free over \( \mathcal{H}_i \), which translates to the fact that there is a type-D module \( \mathbb{CFL}(L) \mathcal{R}_i \) such that

\[
\mathbb{CFL}(L) \mathcal{R}_i \cong \mathbb{CFL}(L) \mathcal{R}_i \boxtimes \mathcal{H}_i \mathcal{R}_i \mathcal{R}_i.
\]

On the other hand, since \( \mathcal{H}_i C_1, \ldots, \mathcal{H}_i \) is a free resolution of \( \mathcal{H}_i/(\mathcal{H}_1, \ldots, \mathcal{H}_\ell) \), it follows that \( \mathcal{H}_i C_1, \ldots, \mathcal{H}_i \cong \mathcal{H}_i \mathcal{R}_i/(\mathcal{H}_1, \ldots, \mathcal{H}_\ell) \).

Putting these relations together, we obtain

\[
\mathcal{H}(L) = \mathcal{HFL}(L) \mathcal{R}_i \boxtimes \mathcal{H}_i C_1, \ldots, \mathcal{H}_i
\]

\[
\cong \mathbb{CFL}(L) \mathcal{R}_i \boxtimes \mathcal{H}_i C_1, \ldots, \mathcal{H}_i
\]

\[
= \mathbb{CFL}(L) \mathcal{R}_i \boxtimes \mathbb{CFL}(L) \mathcal{R}_i \mathcal{R}_i \boxtimes \mathcal{H}_i C_1, \ldots, \mathcal{H}_i
\]

\[
\cong \mathbb{CFL}(L) \mathcal{R}_i \boxtimes \mathcal{H}_i \mathcal{R}_i/(\mathcal{H}_1, \ldots, \mathcal{H}_\ell),
\]

concluding the proof. \( \square \)

7. The \( T(n,n) \) Torus Links

In this section, we build on the work of Gorsky and Hom [GH17] and describe the full link Floer complex of \( T(n,n) \). In Section 7.2 we describe the \( \mathcal{R}_n \)-module \( \mathcal{HFL}(T(n,n)) \) based on Gorsky and Hom’s computation of the \( H \)-function of \( T(n,n) \). By our Theorem 6.1, this \( \mathcal{R}_n \)-module contains equivalent information to \( \mathcal{CFL}(T(n,n)) \). In Sections 7.3 and 7.4, we compute explicit free resolutions of \( \mathcal{HFL}(T(3,3)) \) and \( \mathcal{HFL}(T(4,4)) \).

7.1. Generators and relations. In this section, we describe the generators and relations for the modules \( \mathcal{HFL}(L) \) when \( L \) is an L-space link. We focus on the case \( L \subseteq S^3 \) to simplify the notation, but this is not essential.

If \( s \in \mathbb{Z}(L) \), we write \( X_s \in \mathcal{HFL}(L) \) for the generator of the \( \mathbb{F} \)-module in Alexander grading \( s \).

By definition,

\[
A(X_s) = s \quad \text{and} \quad \text{gr}_s(X_s) = -2H_L(s).
\]
Lemma 7.1. Let $L$ be an $\ell$-component $L$-space link in $S^3$.

1. The $R_\ell$-module $\mathcal{HF}(L)$ admits a unique, minimal length generating set of homogeneously graded generators. This generating set consists of all $X_s$ which satisfy

$$H_L(s - e_i) - H_L(s) = 1, \quad \text{and} \quad H_L(s) - H_L(s + e_i) = 0$$

(7.1)

for all $i \in \{1, \ldots, \ell\}$. Here, $e_i$ denotes the unit vector $(0, \ldots, 1, \ldots, 0)$.

2. If $\{X_{s_1}, \ldots, X_{s_n}\}$ is the generating set above, then the kernel of the natural map

$$R_\ell^n \to \mathcal{HF}(L)$$

is spanned by the following generating set:

(a) All elements of the form

$$\alpha_{i,j}X_{s_i} + \beta_{i,j}X_{s_j}$$

where $\alpha_{i,j}, \beta_{i,j} \in R_\ell$ are non-zero monomials such that $\alpha_{i,j}X_{s_i}$ and $\beta_{i,j}X_{s_j}$ have the same Alexander and Maslov gradings, $\gcd(\alpha_{i,j}, \beta_{i,j}) = 1$, and $\min(\mu_U(\alpha_{i,j}), \mu_U(\beta_{i,j})) = 0$. Here, $\mu_U(a)$ denotes the minimal $j \geq 0$ such that $a$ is in the ideal $(\mathcal{U}_1Y_1, \ldots, \mathcal{U}_dY_d)$.

(b) For each $i, j$ and $s_k$ the element

$$(\mathcal{U}_iY_j + \mathcal{U}_jY_i)X_{s_k}.$$ 

Proof. Since $CFL(L)$ is a finitely generated $\mathcal{R}_\ell$-module, and $CFL(L)$ admits Maslov and Alexander gradings, the $R_\ell$-module $\mathcal{HF}(L)$ is spanned by finitely many homogeneously graded vectors $X_s$. Equation (7.1) is equivalent to the statement that $X_s$ is in the image of neither $\mathcal{U}_i$ nor $\mathcal{V}_j$ for any $i \in \{1, \ldots, \ell\}$, which any minimal length generating set must satisfy. Furthermore, none of these elements is in the $R_\ell$-span of the others, since if

$$X_{s_j} = \sum_{i \neq j} \alpha_iX_{s_i}$$

we may assume each $\alpha_iX_{s_i}$ has the same gradings as $X_{s_j}$. This implies that if $\alpha_i \neq 0$, then $X_{s_j} = \alpha_iX_{s_i}$, which is the generator of the $\mathbb{F}[U]$-tower in Alexander grading $s_j$. This implies that $\alpha_i = 1$ since none of the $X_{s_i}$ admit non-trivial factorizations.

The second statement is similar. Suppose that

$$\sum_{i=1}^n \alpha_iX_{s_i} = 0.$$

We may assume that each $\alpha_i$ is a monomial (or zero) and all $\alpha_iX_{s_i}$ have the same grading. In this case, $\sum_{i=1}^n \alpha_iX_{s_i} = 0$ if and only if $\#\{i : \alpha_i \neq 0\}$ is even. In particular, pairing elements of $\{i : \alpha_i \neq 0\}$ arbitrarily, we may write the sum $\sum_{i=1}^n \alpha_iX_{s_i}$ as an $R_\ell$-linear combination of elements of the form $\alpha_{i,j}X_{s_i} + \beta_{i,j}X_{s_j}$, where $\alpha_{i,j}X_{s_i}$ and $\beta_{i,j}X_{s_j}$ have the same Maslov and Alexander gradings and $\gcd(\alpha_{i,j}, \beta_{i,j}) = 1$. By adding elements of the form $(\mathcal{U}_iY_j + \mathcal{U}_jY_i)X_{s_k}$ (and possibly factoring out monomials) we may reduce to the case that $\min(\mu_U(\alpha_{i,j}), \mu_U(\beta_{i,j})) = 0$ and $\gcd(\alpha_{i,j}, \beta_{i,j}) = 1$, completing the proof. \qed

7.2. The type-A module of the $T(n,n)$ torus link. We now compute the Heegaard Floer module $\mathcal{HF}(T(n,n))$ of torus link $T(n,n)$ in the three sphere. Recall that $R_n = \mathbb{F}[\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_n, \mathcal{V}_n]$. Theorem 7.2. As an $R_n$-module, the group $\mathcal{HF}(T(n,n))$ has a unique minimal generating set. This generating set has $n$ generators, $X_1, \ldots, X_n$. The relations are spanned by the following

$$\prod_{i \in I_k} \mathcal{U}_iX_k = \left( \prod_{j \in \{1, \ldots, n\} \setminus I_k} \mathcal{V}_j \right) X_{k+1}$$

(7.2)

and

$$\mathcal{U}_i\mathcal{V}_jX_k = \mathcal{U}_j\mathcal{V}_iX_k.$$ (7.3)

Here $I_k$ runs through all subsets of the set $\{1, \ldots, n\}$ of length $k$ (so (7.2) has $\binom{n}{k}$ relations for each $k$), and in (7.3), $i, j$, and $k$ range from 1 to $n$. 

Proof. The proof is similar to the proof of Proposition 7.1. The generating set is constructed by considering the $\mathbb{F}[U]$-linear combination of elements of the form $\alpha_{i,j}X_{s_i} + \beta_{i,j}X_{s_j}$, where $\alpha_{i,j}X_{s_i}$ and $\beta_{i,j}X_{s_j}$ have the same Maslov and Alexander gradings and $\gcd(\alpha_{i,j}, \beta_{i,j}) = 1$. By adding elements of the form $\mathcal{U}_i\mathcal{V}_jX_k$ (and possibly factoring out monomials) we may reduce to the case that $\min(\mu_U(\alpha_{i,j}), \mu_U(\beta_{i,j})) = 0$ and $\gcd(\alpha_{i,j}, \beta_{i,j}) = 1$, completing the proof. \qed
Proof. Since the torus link $T(n, n)$ is an L-space link, by Lemma 7.1, it suffices to find all $X_s$ which satisfy
\[ H(s - e_i) - H(s) = 1 \quad \text{and} \quad H(s) - H(s + e_i) = 0, \tag{7.4} \]
where $H(s)$ is the $H$-function of the torus link $T(n, n)$ and $i \in \{1, \ldots, n\}$.

The $H$-function of the torus link $T(n, n)$ is computed in [GH17]. Its Alexander polynomial equals to
\[ \Delta(t_1, \ldots, t_n) = ((t_1 \cdots t_n)^{1/2} - (t_1 \cdots t_n)^{-1/2})^{n-2}, \]
which is symmetric in the variables $t_1, \ldots, t_n$. Then the $H$-function is also symmetric in $s_1, \ldots, s_n$. Therefore we consider the case that $s_1 \leq s_2 \leq \cdots \leq s_n$. By [GH17, Theorem 4.3]
\[ H(s_1, \ldots, s_n) = h(s_1 - \frac{n-1}{2}) + h(s_2 - \frac{n-1}{2} + 1) + \cdots + h(s_n - \frac{n-1}{2} + n - 1) \tag{7.5} \]
where $h(s) = (\lfloor |s| - s \rfloor)/2$. We note that $h(s)$ is the $H$-function of the unknot. It is not hard to compute that
\[ H(s_1, \ldots, s_n) = 0 \tag{7.6} \]
if $s_i \geq (n - 1)/2$ for all $i$, and
\[ H(s) = -(s_1 + s_2 + \cdots + s_n) \tag{7.7} \]
if $s_i \leq -(n - 1)/2$ for all $i$.

We first consider the diagonal vertices, that is, $s_1 = s_2 = \cdots = m$. If $m > (n - 1)/2$, then
\[ H(s_1, \ldots, s_n) = H(s_1 - 1, s_2, \ldots, s_n) = 0 \]
by (7.6), which does not satisfy (7.4). Similarly, if $m < -(n - 1)/2$, then
\[ H(s_1, \ldots, s_n) = H(s_1, \ldots, s_n + 1) + 1 \]
by (7.7), which also does not satisfy (7.4). Now we consider the $n$ diagonal vertices where $-(n - 1)/2 \leq m \leq (n - 1)/2$, that is,
\[ s_1 = s_2 = \cdots = s_n = \frac{n+1}{2} - k \]
for all integers $k$ between 1 and $n$. The corresponding generators are $X_1, \ldots, X_n$ with Alexander gradings
\[ \left( \frac{n+1}{2} - k, \frac{n+1}{2} - k, \ldots, \frac{n+1}{2} - k \right). \]

By a straightforward computation, (7.4) is satisfied by values of the $H$-function at these vertices. This implies that $X_1, \ldots, X_n$ cannot be factored through the actions of $\mathcal{W}_t$ and $\mathcal{V}_r$ on $\mathcal{HFL}(T(n, n))$.

We claim these diagonal vertices are the only ones where the $H$-function satisfies (7.4). It suffices to prove that the value of the $H$-function at all non-diagonal vertices does not satisfy (7.4). Recall that we assume $s_1 \leq s_2 \leq \cdots \leq s_n$. Suppose that $s_n = s$ is the maximal value among the $s_i$, and there are exactly $\lambda$ coordinates equal to $s$ where $\lambda < n$, i.e., $s_{n - 1} = s_{n - 2} = \cdots = s_n = s$, and $s_{n - \lambda} < s$.

If $s > (\lambda - 1) - (n - 1)/2$, then by (7.5)
\[ H(s_1, \ldots, s_{n - \lambda}, s_{n - \lambda + 1} - 1, \ldots, s_n) = H(s_1, \ldots, s_n) \]
which does not satisfy (7.4).

If $s \leq (\lambda - 1) - (n - 1)/2$, then by (7.5)
\[ H(s_1, \ldots, s_{n - \lambda} + 1, s_{n - \lambda + 1}, \ldots, s_n) + 1 = H(s_1, \ldots, s_n), \]
which also does not satisfy (7.4). By Lemma 7.1, $X_1, \ldots, X_n$ form a unique minimal generating set of $\mathcal{HFL}(T(n, n))$ over $\mathcal{R}_n$. Based on the values of the $H$-function, the generator $X_k$ has Maslov grading (i.e. $\text{gr}_n$-grading)
\[ -k(k - 1). \]
Based on the Alexander grading and Maslov grading of $X_k$, all relations between $X_1, \ldots, X_n$, the fact that the relations in (7.2) and (7.3) generate the space of relations follows from Lemma 7.1.

In general, a free resolution of the homology $\mathcal{HFL}(T(n,n))$ can be computed algorithmically, see [Pec11], or, for a concrete value of $n$, using a computer algebra system such as Macaulay2 [GS].

### 7.3. The free complex of the $T(3,3)$ torus link.

We present a free resolution of the torus link $T(3,3)$. The homology of the torus link $T(3,3)$ is generated by $X_1, X_2, X_3$ with the following relations:

$$\mathcal{U}_iX_1 \sim \prod_{j \in \{1,2,3\} \setminus \{i\}} \mathcal{V}_jX_2, \quad \mathcal{V}_iX_3 \sim \prod_{j \in \{1,2,3\} \setminus \{i\}} \mathcal{W}_jX_2;$$

$$\mathcal{U}_i \mathcal{V}_iX_k = \mathcal{W}_j \mathcal{V}_jX_k.$$ 

Then a free resolution of the homology is

$$0 \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0,$$

with the spaces $C_0, C_1, C_2, C_3$ and the maps $\partial_1, \partial_2, \partial_3$ defined as follows.

The space $C_0 = \mathcal{R}_3$ is generated by $X_1, X_2, X_3$. Take the space $C_1 = \mathcal{R}_3^3$ generated by $b_1, b_2, b_3$, $B_1, B_2, B_3$, and $Z_1, Z_2$. For symmetry, it is helpful to consider an extra variable $Z_3$ which satisfies $Z_3 = Z_1 + Z_2$; $Z_3$ is not a generator of $C_1$. The differential $\partial_1 : C_1 \to C_0$ is given by

$$\partial_1 b_1 = \mathcal{V}_iX_1 + \prod_{j \in \{1,2,3\} \setminus \{i\}} \mathcal{V}_jX_2, \quad \partial_1 B_1 = \mathcal{V}_iX_3 + \prod_{j \in \{1,2,3\} \setminus \{i\}} \mathcal{W}_jX_2,$$

$$\partial_1 Z_1 = \mathcal{V}_2 \mathcal{V}_2X_2 + \mathcal{W}_3 \mathcal{V}_3X_2, \quad \partial_1 Z_2 = \mathcal{V}_1 \mathcal{V}_1X_2 + \mathcal{W}_3 \mathcal{V}_3X_2$$

$$\partial_1 Z_3 = \partial_1(Z_1 + Z_2) = \mathcal{V}_1 \mathcal{V}_1X_2 + \mathcal{W}_2 \mathcal{V}_2X_2.$$

The link Floer homology of $T(3,3)$ is $\text{coker} \partial_1$. Indeed, the relations $\mathcal{U}_1 \mathcal{V}_1X_k = \mathcal{W}_j \mathcal{V}_jX_k$ for $k = 1$ and $k = 3$ follow from other relations. For example,

$$0 = \mathcal{V}_1(\mathcal{V}_1X_1 - \mathcal{V}_2 \mathcal{V}_3X_2) - \mathcal{V}_2(\mathcal{V}_2X_1 - \mathcal{V}_3 \mathcal{V}_3X_2) = (\mathcal{V}_1 \mathcal{V}_1 - \mathcal{V}_2 \mathcal{V}_2)X_1.$$ 

We define the module $C_2 = \mathcal{R}_3^3$ with generators $c_1, c_2, c_3, d_1, d_2, d_3$ and let $\partial_2$ be the differential

$$\partial_2 c_k = \mathcal{U}_i b_j + \mathcal{V}_j b_i + \mathcal{V}_k Z_k$$

$$\partial_2 d_k = \mathcal{V}_i B_j + \mathcal{V}_j B_i + \mathcal{W}_k Z_k.$$ 

Here $\{i, j, k\}$ ranges through all permutations of the set $\{1, 2, 3\}$. There is a relation between $c_1, \ldots, d_3$. That is, there is a module $C_3 = \mathcal{R}_3$ generated by $e$ with $\partial_3 : C_3 \to C_2$ given by

$$\partial_3 e = \mathcal{U}_1 c_1 + \mathcal{W}_2 c_2 + \mathcal{W}_3 c_3 + \mathcal{V}_1 d_1 + \mathcal{V}_2 d_2 + \mathcal{V}_3 d_3.$$ 

It can be checked either directly, or using a computer algebra system, that $\ker \partial_1 = \text{im} \partial_2$, $\ker \partial_2 = \text{im} \partial_3$ and $\ker \partial_3 = 0$. That is to say, the complex we constructed is a free resolution of $\text{coker} \partial_1$.

### 7.4. The free complex of the $T(4,4)$ torus link.

To stress the usefulness and power of Theorem 6.1, we show how to compute the link Floer chain complex of the $T(4,4)$ torus link. The computations can be done by hand. Acyclicity of the complex at gradings different than 0 has been verified with Macaulay [GS].

We can give a quick description of the free resolution of the $T(4,4)$ torus link. The free resolution has length four, as follows:

$$0 \to C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

Let $C_0$ be the free $\mathcal{R}_4$ module generated by $X_1, \ldots, X_4$. Consider the module $C_1 \cong \mathcal{R}_4^{20}$ generated by $Z_{ij}$ with $k = 2, 3, j = i + 1, a_1, \ldots, a_4, B_{ij}$ with $1 \leq i < j \leq 4$ and $A_1, \ldots, A_4$. As in the $T(3,3)$
By Theorem 7.2, the link Floer homology of $T(3,1)$ is coker $\partial_0$. We define now the module $C_2 \cong \mathbb{Z}^3$. It is generated by $a_{i,j}$ with $1 \leq i < j \leq 4$, and $v_{i,j}$ are $2$-generators. The indexing of $c$ and $C$-generators is a bit complex. We choose, $1 \leq k \leq 4$, and, $\{i,j\}$ is a subset of $\{1,2,3,4\}$ such that $j > i$. To obtain a small resolution, we reduce the number of $C$-generators by declaring that the $c$-generators may appear in the differential, we declare it to be the sum of the other two generators. However, another configuration of indices appears in the differential, we declare it to be the sum of the other two generators. Therefore, we have 8 $c$-generators and 8 $C$-generators.

For $C$-generators, for each three-element subset $\{i,j,k\}$ with $i < j$, we choose $g_i^j$ as generators. The third object is the same as before.

For computing $B$, we set $B_0 = B_{i,j}$. The map $\partial : C_i \to C_{i+1}$ is given by

$$\partial_B = \partial_B^i + \sum_{j \neq i} B_{i,j}$$

$$\partial_A = \partial_A^i + \sum_{j \neq i} A_{i,j}$$

We compute the boundary maps $\partial_B$ and $\partial_A$ for $B$ and $A$.

$$\partial_B = \delta_{B,x} + \sum_{j \neq i} B_{i,j}$$

$$\partial_A = \delta_{A,x} + \sum_{j \neq i} A_{i,j}$$

**Figure 7.1.** The complex $C_2(T(3,1))$ as a free resolution.
\[ \partial_2 c_{ij}^{\ell} = \gamma_i B_{i\ell} + \gamma_j B_{j\ell} + \gamma_k Z_{ij}^\ell, \]

where \( \ell = \{1, 2, 3, 4\} \setminus \{i, j, k\} \).

The module \( C_3 \cong R_3^4 \) is generated by \( a_{ijk}, A_{ijk}, B_{ij} \) where \( 1 \leq i < j < k \leq 4 \) with the map \( \partial_3: C_3 \to C_2 \) given by

\[
\begin{align*}
\partial_3 a_{ijk} &= \mathcal{U} a_{ijk} + \mathcal{W} a_{ki} + \mathcal{U} a_{ij} + \gamma_\ell (\gamma_i c_{ik}^\ell + \gamma_j c_{jk}^\ell + \gamma_k c_{kj}^\ell) \\
\partial_3 A_{ijk} &= \gamma_i A_{ijk} + \gamma_j A_{ijk} + \gamma_k A_{ijk} + \mathcal{U} (\gamma_i c_{ij}^k + \gamma_j c_{jk}^k + \gamma_k c_{kj}^k) \\
\partial_3 B_{ij} &= \gamma_k c_{ij}^k + \gamma_j c_{ij}^k + \mathcal{U} c_{ij}^{k\ell},
\end{align*}
\]

where we let \( \ell = \{1, 2, 3, 4\} \setminus \{i, j, k\} \).

The module \( C_4 \cong R_3^4 \) is generated by \( d_{1234}, D_{1234} \) with the map \( \partial_4 \) is given by

\[
\begin{align*}
\partial_4 (d_{1234}) &= \sum_i \mathcal{U} a_{ijk} + \sum_{i<j} \gamma_i \gamma_j B_{ij} \\
\partial_4 (D_{1234}) &= \sum_i \gamma_i A_{ijk} + \sum_{i<j, k<\ell} \sum_{\{i,j,k,\ell\} = \{1,2,3,4\}} \mathcal{U} \gamma_j B_{kl}.
\end{align*}
\]

In each of the two equations, where the sum runs over the indices \( i \), we let \( 1 \leq j < k < \ell \leq 4 \) be such that \( \{i, j, k, \ell\} \) is a permutation of \( \{1, 2, 3, 4\} \).

It can be verified that the above description of \( C_4 \) is acyclic except at resolution grading 0, with coker \( \partial_4 \) being the link Floer homology of \( T(4,4) \).

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