1. Introduction

In this paper, we consider the economic problems faced by a market designer who wants to produce student matchings (or object allocations) that are responsive to agents’ preferences and leave the smallest number of them unmatched (that is, have maximum cardinality among individually rational matchings).\(^1\)

One of the main motivations for studying this problem is the fact that, in practice, market designers often make adaptations to standard procedures with the objective of preventing agents from being left unmatched. The real life use of allocation mechanisms in school choice procedures, for example, often consists of using a standard mechanism, such as the Gale-Shapley deferred acceptance (Gale and Shapley, 1962), followed by some additional procedure to assign the students who were left unmatched into some school. These secondary steps or other ad-hoc methods for filling up the remaining seats, however, result in the loss of the properties of the mechanism that was used in the first place, such as fairness and strategy-proofness (Dur and Kesten, 2019). In this paper we start instead from the presumption that the market designer has the objective of leaving the minimum number of students unmatched. While this objective is not attainable via a strategy-proof mechanism, we propose mechanisms that produce maximum matchings and are efficient or satisfy a novel fairness criterion, when students are non-strategic. We also show that they satisfy desirable characteristics in equilibrium, and increase the cardinality of the matching as the proportion of truthful agents increases.

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\(^1\)Even though our entire analysis translates naturally to most unit-demand discrete assignment problems, we will use the framing of school allocation throughout the paper.
2. Related Literature

While algorithms for finding maximum matchings are well-known (Kuhn, 1955; Berge, 1957), the research on the incentives induced by the use of these procedures is limited, and typically rely on random mechanisms. One exception is Afacan and Dur (2018), which follows-up to this paper and shows that no strategy-proof and individually rational mechanism systematically matches more students than either of Boston, Gale-Shapley deferred acceptance, and serial dictatorship mechanisms. Krysta et al. (2014) consider the problem of producing maximal matchings in a house allocation problem. They show that there is no mechanism that is deterministic, maximal, and strategy-proof, and provide instead a random mechanism that is strategy-proof and yields approximately maximal outcomes. Bogomolnaia and Moulin (2015) evaluate the trade-off between maximality and envy-freeness, a notion of fairness that is stronger than the ones we consider in this paper. Bogomolnaia and Moulin (2004) consider the random assignment when agents have dichotomous preferences. When that is the case, Pareto efficiency is equivalent to maximality of the matching, and moreover, since agents are indifferent between all “acceptable” allocations, maximality doesn’t result in incentive problems even in deterministic mechanisms. Noda (2018) studies the matching size achieved by strategy-proof mechanisms in a general model of matching with constraints.

Finding the matching with maximum cardinality subject to some constraints is also a problem that is explored in the literature. Irving and Manlove (2010) consider the problem of finding stable matchings with maximum cardinality when priorities have ties, which is known to be an NP-hard problem, and present heuristics for finding them. Ashlagi et al. (2020) also consider object assignment problems under distributional constraints. The authors show that variants of serial dictatorship and Probabilistic Serial (Bogomolnaia and Moulin, 2001) mechanisms assign at least as many agents as one can match under the constraints, while the violations of the constraints are relatively small.

Assignment maximization has been the primary objective in the organ exchange literature, as it means the maximum number of transplants. This literature was initiated by the seminal work on kidney exchange of Roth et al. (2004). In a subsequent study, in order to accommodate several physical and geographical restrictions in operating transplants, Roth et al. (2005) introduce the idea of pairwise kidney exchange where exchanges can only be made between two pairs. They suggest implementing the priority-based maximal matching algorithm from the combinatorial optimization literature (Korte and Vygen, 2011). The first stages of both EAM and FAM are adaptations of the priority-based maximal matching algorithm. Some other studies on organ exchanges include Sönmez et al. (2018), Andersson and Kratz (2018), Chun et al. (2018), Ergin et al. (2017), Nicoló and Rodríguez-Alvarez (2017), and Ergin et al. (2018).

Refugee reassignment is another real-world application in which maximality might be a primary design objective. Andersson and Ehlers (2018) study the problem of finding housing for refugees once they have been granted asylum. The authors propose an easy-to-implement mechanism that finds an efficient stable maximum matching. They show that such a matching guarantees that housing is efficiently provided to a maximum number of refugees and that no unmatched refugee-landlord pair prefers each other.

Our “fairness for unassigned students” is a weakening of the usual stability of Gale and Shapley (1962), therefore, the current study is also related to the surging literature on the weakening of stability in different ways. Among others, Dur et al. (2018), Afacan et al. (2017), Morrill and Ehlers (2018), and Troyan et al. (2018) are recent papers from that literature.

3. Model

A school choice problem consists of a finite set of students $I = \{i_1, ..., i_n\}$, a finite set of schools $S = \{s_1, ..., s_m\}$, a strict priority structure for schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is a linear order over $I$, a capacity vector $q = (q_{s_1}, ..., q_{s_m})$, and a profile of strict preference of students $P = (P_i)_{i \in I}$, where $P_i$ is student $i$’s preference relation over $S \cup \{\emptyset\}$ and $\emptyset$ denotes the option of being unassigned. We
denote the set of all possible preferences for a student by $\mathcal{P}$. Let $R_i$ denote the at-least-as-good-as preference relation associated with $P_i$, that is: $sR_is' \leftrightarrow sP_is'$ or $s = s'$. A school $s$ is acceptable to $i$ if $sP_i\emptyset$, and unacceptable otherwise. Let $Ac(P_i) = \{c \in S: cP_i\emptyset\}.$

In the rest of the paper, we consider the tuple $(I, S, \succ, q)$ as the commonly known primitive of the problem and refer to it as the market. We suppress all those from the problem notation and simply write $P$ to denote the problem. A matching is a function $\mu : I \rightarrow S \cup \{\emptyset\}$ such that for any $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A student $i$ is assigned under $\mu$ if $\mu(i) \neq \emptyset$. For any $k \in I \cup S$, we denote by $\mu_k$ the assignment of $k$. Let $|\mu|$ be the total number of students assigned under $\mu$.

A matching $\mu$ is individually rational if, for any student $i \in I$, $\mu_iR_i\emptyset$. A matching $\mu$ is non-wasteful if for any school $s$ such that $sP_i\mu_s$ for some student $i \in I$, $|\mu_s| = q_s$. A matching $\mu$ is fair if there is no student-school pair $(i, s)$ such that $sP_i\mu_i$, and for some student $j \in \mu_s$, $i \succ_s j$. A matching $\mu$ is stable if it is individually rational, non-wasteful, and fair.

In the rest of the paper, we will consider only individually rational matchings. Therefore, whenever we refer to a matching, unless explicitly stated, we refer to an individually rational matching. Let $\mathcal{M}$ be the set of matchings.

A matching $\mu$ dominates another matching $\mu'$ if, for any student $i \in S$, $\mu_iR_i\mu'_i$, and for some student $j$, $\mu_jP_j\mu'_j$. A matching $\mu$ is efficient if it is not dominated by any other matching. We say that a matching $\mu$ size-wise dominates another matching $\mu'$ if $|\mu| > |\mu'|$. A matching $\mu$ is maximal if it is not size-wise dominated.

A mechanism $\psi$ is a function from $\mathcal{P}^{[I]}$ to $\mathcal{M}$. A Mechanism $\psi$ is strategy-proof if there exist no problem $P$, and student $i$ with a false preference $P'_i$ such that $\psi_i(P'_i, P_{-i})P_i\psi_i(P)$.

A mechanism $\psi$ size-wise dominates another mechanism $\phi$ if, for any problem $P$, $\phi(P)$ does not size-wise dominate $\psi(P)$, while, for some problem $P'$, $\psi(P')$ size-wise dominates $\phi(P')$. A mechanism $\psi$ is maximal if it is not size-wise dominated by any other mechanism.

We start our analysis by first observing that none among four well-known mechanisms commonly used and considered for the kind of allocation problems that we are considering — deferred-acceptance (DA), top trading cycles (TTC), Boston (BM), and serial dictatorship (SD) — is maximal.

**Proposition 1.** None of DA, TTC, BM, and SD is maximal.

**Proof.** Let $I = \{i_1, i_2\}$ and $S = \{a, b\}$, each with unit quota. Let $P_{i_1} : a, b, \emptyset$ and $P_{i_2} : a, \emptyset$. The priorities are such that agent $i_1$ has the top priority at object $a$. Then, the DA, TTC, and BM outcomes are the same. If we write $\mu$ for their outcome, then $\mu_{i_1} = a$ and $\mu_{i_2} = \emptyset$. Likewise, for SD, let us consider the ordering where agent $i_1$ comes first. Then, the SD outcome is the same as $\mu$. This shows that none of these mechanisms is maximal because the matching $\mu'$ where $\mu'_{i_1} = b$ and $\mu'_{i_2} = a$ is individually rational and matches more agents than $\mu$. \hfill $\Box$

Given the lack of maximality of the well-known mechanisms, in the rest of the paper, we introduce two maximal mechanisms and study their properties.

### 3.1. A Class of Efficient Maximal Mechanisms

Given a problem $P$ and an enumeration of the students in $I (i_1, \ldots, i_n)$.

**Step 0.** Let $\xi^0 = \mathcal{M}$.

**Step 1.**

**Sub-step 1.1.** Define the set $\xi^1 \subseteq \xi^0$ as follows:

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2 Notice that the notions of size domination and maximality we use is in th set of agents (or nodes) involved in a matching. In most of the literature in graph theory, the cardinality of a matching is measured in the set of edges of the graph that are part of the matching. While when considering the set of edges there is a difference between maximal and maximum cardinality matchings, in our setup these are equivalent: maximal matchings are always maximum.

3 For the description of these mechanisms, the reader could refer to Abdulkadiroglu and Sönmez (2003).
\[ \xi^1 = \begin{cases} \{ \mu \in \xi^0 : \mu_{i_1} \neq \emptyset \} & \text{If } \exists \mu \in \xi^0 \text{ such that } \mu_{i_1} \neq \emptyset \\ \xi^0 & \text{otherwise} \end{cases} \]

In general, for every \( k \leq n \).

**Sub-step 1.k.** Define the set \( \xi^k \subseteq \xi^{k-1} \) as follows:

\[ \xi^k = \begin{cases} \{ \mu \in \xi^{k-1} : \mu_{i_k} \neq \emptyset \} & \text{If } \exists \mu \in \xi^{k-1} \text{ such that } \mu_{i_k} \neq \emptyset \\ \xi^{k-1} & \text{otherwise} \end{cases} \]

Step 1 ends with the selection of a matching \( \mu \in \xi^n \).

**Step 2.**

In general:

**Sub-step 2.k.** Let \( \tilde{\mu} \) be the matching obtained in the previous step of the procedure. If \( \tilde{\mu} \) does not admit an improving chain or cycle then the algorithm ends with the final outcome of \( \tilde{\mu} \). Otherwise, pick such a chain or cycle, and obtain a new matching by assigning each student in the chosen chain (cycle) to the school he prefers in the chain (cycle), and move to the next sub-step.

**Theorem 1.** Every EAM mechanism is maximal and efficient.

Notice, however, that while EAM mechanisms are maximal, they are not fair.

### 3.2. A Class of Maximal and Fair for Unassigned Students Mechanisms.

We say that a matching \( \mu \) is **fair for unassigned students** if there is no student-school pair \((i, s)\) where \( \mu_i = \emptyset \) and \( i \succ_s j \) for some \( j \in \mu_s \). A mechanism \( \psi \) is fair for unassigned students if, for any problem \( P \), \( \psi(P) \) is fair for unassigned students.

Below is a description of how each mechanism in this class works. Given a problem \( P \),

**Step 1.** Pick an EAM mechanism \( \psi \), and let \( \psi(P) = \mu \).

**Step 2.**

In general,

**Sub-step 2.k.** Let \( \tilde{\mu} \) be the matching obtained in the previous step. If \( \tilde{\mu} \) is fair for unassigned students, the algorithm terminates with the outcome \( \tilde{\mu} \). Otherwise, pick a student-school pair \((i, s)\) such that \( s \succ_i \emptyset, \tilde{\mu}_i = \emptyset \), and \( i \succ_s j \) for some \( j \in \tilde{\mu}_s \). Place student \( i \) at school \( s \), and let the lowest priority student in \( \tilde{\mu}_s \) be unassigned, while keeping everyone else’s assignment the same. Note that as in each sub-step the number of assigned students is preserved, \( \tilde{\mu} \) is maximal. Hence, we have \( |\tilde{\mu}_s| = q_s \). Let \( \hat{\mu} \) be the obtained matching, and move to the next sub-step.

As, in every sub-step, a higher priority student is placed at a school while a lower priority one is displaced from the school, and both the students and schools are finite, the algorithm terminates in finitely many rounds. The above procedure defines a class of mechanisms, each of which is associated with different selections of the first stage EAM mechanism as well as the student-school pairs in the course of Step 2. We refer them as “Fair Assignment Maximizing” (FAM) mechanisms.

**Theorem 2.** Every FAM mechanism is fair for unassigned students and maximal.

**Proof.** Let \( \psi \) be a FAM mechanism, and \( \mu \) be the outcome of its first step. As \( \mu \) is the outcome of an EAM mechanism, and in Step 2 of \( \psi \), no student is assigned to one of his unacceptable choices, \( \psi \) is individually rational. Because \( \mu \) is maximal and the number of assigned students is preserved as \( |\mu| \) in the course of Step 2, \( \psi \) is maximal. Moreover, as \( \psi \) does not stop until no student-school pair violates fairness for unassigned students, \( \psi \) is fair for unassigned students as well.

### 4. Incentives and Equilibrium Analysis

In this section we show that the mechanisms in the classes EAM and FAM have surprisingly regular properties in terms of equilibrium outcomes. Consider the preference reporting game induced
by a mechanism $\psi$. At problem $P$, a preference submission $P' = (P'_i)_{i \in I}$ is a (Nash) equilibrium of $\psi$ if for every student $i$, $\psi_i(P') R_i \psi_i(P'_i, P'_{-i})$ for any $P'_i \in P$. Let $\Omega$ be the set of mechanisms that admit an equilibrium in any problem $P \in \mathcal{P}$\[1\]. In the rest of this section, we consider only the mechanisms in $\Omega$.

**Proposition 2.** Every EAM and FAM mechanism is in $\Omega$. Moreover, for any problem, an EAM mechanism has a unique equilibrium outcome that is equivalent to the outcome of the serial dictatorship where the student ordering is the same as that used in that EAM mechanism.

Proposition 2 shows, therefore, that equilibrium outcomes of EAM are not only Pareto efficient, but will match as many students as a commonly used strategy-proof mechanism.

Our next question is how mechanisms compare, in terms of the number of assignments, in equilibrium. For that, we define the concept of **size-wise domination in equilibrium**.

**Definition 1.** For a given market $(I, S, \succ, q)$, a mechanism $\psi$ size-wise dominates another mechanism $\phi$ in equilibrium if, for any problem $P$ and for every equilibria $P', P''$ under $\psi$ and $\phi$, respectively $|\psi(P')| \geq |\phi(P'')|$, and there exists a problem $P^*$ such that for every equilibria $\hat{P}, \bar{P}$ under $\psi$ and $\phi$, respectively $|\psi(\hat{P})| > |\phi(\bar{P})|.$

**Theorem 3.** In any market $(I, S, \succ, q)$, no EAM mechanism is size-wise dominated by an individually rational mechanism in equilibrium.

While we do not have a similar result to above for the FAM mechanisms, we are able to compare the number of assigned students under the FAM in equilibrium and the weakly dominant strategy equilibrium of the DA, which is truth-telling.

**Theorem 4.** Regarding the FAM mechanisms:

(i) For any problem $P$ and any stable matching for $P \mu^*$, for every equilibrium $P'$ of a FAM mechanism $\psi$, $|\psi(P')| \geq |\mu^*|.$

(ii) There exist a FAM mechanism $\psi$, problem $P$, and an equilibrium profile $P'$ of $\psi$ at $P$ such that $|\psi(P')| > |\mu^{**}|$, where $\mu^{**}$ is any stable matching for $P$.

One may interpret the results in this section as an indication that there isn’t much gain in using maximal mechanisms such as EAM and FAM, since when agents respond to their incentives, outcomes are similar to those produced by other non-maximal mechanisms. Below we show, however, that there are improvements in terms of the cardinality of the matching, as long as some fraction of the students are sincere.

**Proposition 3.** For any maximal mechanism $\psi$, problem $P$, and student $i$ with false preferences $P'_i$ such that $\psi_i(P'_i, P_{-i}) R_i \psi_i(P)$, we have $|\psi(P)| \geq |\psi(P'_i, P_{-i})|$. Moreover, there exist a problem $\hat{P}$ and student $i$ with false preferences $\hat{P}_i$ such that $\psi_i(\hat{P}_i, P_{-i}) R_i \psi_i(\hat{P})$ and $|\psi(\hat{P})| > |\psi(\hat{P}, \hat{P}_i)|$.

In a preference-reporting game induced by a maximal mechanism where the only active players are strategic students in the sense that the rest is always sincere, Proposition 3 leads to the following corollary.

**Corollary 1.** Under any maximal mechanism, as the set of sincere students increases, in any problem, the number of students matched in equilibrium either stays the same or increases.
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Proofs.

\textit{Theorem 1.} We will use the following Lemma:

\textbf{Lemma.} A maximal matching $\mu$ is efficient if and only if it does not admit an improving chain or cycle.

\textbf{Proof.} "Only If" Part. Let $\mu$ be an efficient matching. If it admits an improving chain $\{i_1, \ldots, i_n, c_1, \ldots, c_{n+1}\}$, then we can define a new matching by assigning each agent $i_k$ to $c_{k+1}$ while keeping the assignments of the others the same. By the improving chain definition, that new matching dominates $\mu$, contradicting our starting supposition that $\mu$ is efficient. The same argument shows for the case of improving cycle.

"If" Part. Let $\mu$ be a maximal matching that does not admit improving chains or cycles. Assume for a contradiction that there exists a matching $\mu'$ that dominates $\mu$.

Let $W = \{i \in I : \mu'_i \neq \emptyset\}$. By the supposition, $W \neq \emptyset$. Note that for any student $i$ with $\mu_i = \emptyset$, we have $\mu'_i \neq \emptyset$. This, along with the maximality of $\mu$, implies that $|\mu'| = |\mu|$. Hence, for any student $i$ with $\mu_i = \emptyset$, $\mu'_i = \emptyset$.

Enumerate the students in $W = \{i_1, \ldots, i_n\}$ and write $\mu'_{i_k} = c_k$ for any $k = 1, \ldots, n$. If $|\mu_{c_k}| < q_{c_k}$ for some $k$, then the pair $\{i_k, c_k\}$ constitutes an improving chain, a contradiction.

Suppose that $|\mu_{c_k}| = q_{c_k}$ for any $k = 1, \ldots, n$. As $c_1$ does not have excess capacity at $\mu$, and $\mu'_{i_1} = c_1$, we have another student in $W$, say $i_2$, such that $\mu_{i_2} = c_1$. Then, consider student $i_2$, and as $c_2$ does not have excess capacity at $\mu$ and $\mu'_{i_2} = c_2$, we have another student in $W$, say $i_3$, such that $\mu_{i_3} = c_2$. If we continue to apply the same arguments to the other students in $W$, as $W$ is finite, we would eventually obtain an improving cycle, a contradiction.

\hfill \Box

Let now $\psi$ be an EAM mechanism, and $\mu$ and $\mu'$ be its first stage and final outcome, respectively. As students are not assigned to one of their unacceptable schools in Step 1 of $\psi$, $\mu$ is individually rational.

Assume for a contradiction that $\mu$ is not maximal and there exists $\mu'' \neq \mu$ such that $|\mu''| > |\mu|$. Let $\{i_1, \ldots, i_n\}$ be the agent-enumeration that is used under $\psi$.

As $|\mu''| > |\mu|$, there exists some agent $i_k \in I$ such that $\mu''_{i_k} \neq \emptyset$ and $\mu_{i_k} = \emptyset$. Let $i_k'$ be the first agent according to the above enumeration such that $\mu''_{i_k'} \neq \emptyset$ and $\mu_{i_k'} = \emptyset$. This means that for each $k < k'$, either $\mu_{i_k} = \emptyset$ or $\mu'_{i_k} = \emptyset$. Let $B(\mu, k') = \{i \in N : \mu_i \neq \emptyset \text{ for any } k < k'\}$. That is, it is set of agents who come before agent $i_k'$ in the above enumeration and are assigned under matching $\mu$.

Now consider agent $i_{k'}$. By the definition of $\psi$, $\mu_{i_{k'}} = \emptyset$ because it is not possible to match agent $i_{k'}$ to some of his acceptable objects while keeping all the agents in $B(\mu, k')$ assigned to one of their acceptable objects. This means that in order for agent $i_{k'}$ to receive one of his acceptable objects, one of the assigned agents under $\mu$ from $B(\mu, k')$ has to be unassigned. This arguments holds for each other agent who is assigned under $\mu''$, but not under $\mu$. This implies that $\mu$ is maximal. 

In Step 2 of $\psi$, new matchings are obtained by implementing improving chains and cycles (if any). By their definitions no student receives a worse school than his assignment $\mu$. This, along with the individual rationality of $\mu$, implies that $\mu'$ is maximal. The efficiency of $\mu'$ directly comes from the Lemma above.

\textit{Proposition 2.} Let $\psi$ be an EAM mechanism. The first student in Step 0 of the EAM obtains his top choice by reporting it as the only acceptable choice. By the same reasoning, the second student can obtain his top choice among the remaining schools with seats after considering the first student’s assignment by reporting that school as his only acceptable choice. Once we repeat the same arguments for every other student, we not only find an equilibrium of $\psi$, but also conclude that
it is the unique equilibrium outcome, which coincides with the outcome of serial dictatorship with the ordering being the same as that in Step 0 of ψ.

Let φ be a FAM mechanism. Let μ be a stable matching at P. Consider the preferences submission P′ under which for any student i, the only acceptable school is μi. Any unassigned student at μ reports no school acceptable at P′. It is easy to verify that φ(P′) = μ.

Next, we claim that P′ is an equilibrium submission under φ. Suppose for a contradiction that there exist a student i and P′ such that φ1(P′, P′) = s = φ1(P). For ease of writing, let φ1(P′, P′) = s. As μ is stable, |μs| = qs. This, along with the definition of P′ and φ1(P′, P′) = s, implies that there exists a student j \neq i such that μj = s and φj(P′, P′) = ∅. Moreover, from the stability of μ, we also have j \succ_i j. These altogether contradict the fairness for unassigned students of φ, showing that P′ is equilibrium of φ.

**Theorem 3.** We will use the following.

**Lemma.** Let ψ be an EAM and φ be an individually rational mechanism. In any market (I, S, \succ, q) and problem P, if |ψ(P′)| < |φ(P′)| where P′ and P′ are equilibria under ψ and φ, respectively, then there exists a student i such that ψi(P′) = φi(P′) P′ǎ.

**Proof.** In a market (I, S, \succ, q) and problem P, let |ψ(P′)| < |φ(P′)| where P′ and P′ are equilibria under ψ and φ, respectively. This implies that for some school s, |ψs(P′)| < |φs(P′)| ≤ qs. Hence, let i ∈ φs(P′) \ ψs(P′). By the individual rationality of φ and P′ being equilibrium under φ, we have sP′ǎ, where φi(P′) = s. As the unique equilibrium outcome of ψ coincides with the (truth telling) outcome of a SD mechanism (Proposition 5), we have ψ(P′) = SD(P). Hence, school s has an excess capacity under SD(P). Moreover, from above, ψi(P′) = SDi(P) ≠ s. Hence, by the non-wastefulness of SD, i must be matched to a school strictly better than s and therefore ψi(P′) = SDi(P) P′φi(P′) P′ǎ, which finishes the proof. □

Let now (I, S, \succ, q) be a market and ψ be an EAM mechanism. Assume for a contradiction that an individually rational mechanism φ size-wise dominates ψ in equilibrium. This in particular implies that for some problem P, |ψ(P′)| < |φ(P′)| for every equilibria P′ and P′ under ψ and φ, respectively. In what follows, we will fix one such pair P′, P′. We prove the result in two steps.

**Step 1.** By the Lemma above, there exists a student i such that ψi(P′) Pφi(P′) Pǎ. Let P̂ be the preference relation that keeps the relative rankings of the schools the same as under P̂, while reporting any school that is worse than ψi(P′) as unacceptable. In other words, P̂ truncates P̂ below ψi(P′). Let us write P̂ = (P̂, P̂). Recall that the unique equilibrium outcome of ψ always coincides with the truth telling outcome of a SD mechanism (Proposition 5). Moreover, by the construction of P̂, SD(P̂) = SD(P). This in turn implies that ψ(P′) = φ(P′) for every equilibrium P̂ under ψ in problem P̂.

We next consider problem P̂. If there exists no student j such that ψj(P′) P̂φj(P′) P̂ǎ for some equilibria P′ and P′ under ψ and φ, respectively, then we move to Step 2. Otherwise, we pick such student j. Note that because of the definition of P̂ states that any outcome below ψi(P′) is unacceptable for i and φ is individually rational, ψj(P′) P̂φj(P′) P̂ǎ cannot hold for j = i, therefore j \neq i. Then, as the same as above, let P̂ be the preference list that truncates P̂ below ψj(P′). Let us write P̂ = (P̂, P̂, P̂). By the same reason as above, ψ(P′) = P̂ for any equilibrium P̂ under ψ in problem P̂.

We next consider problem P̂. If there exists no student k such that ψk(P′) P̂φk(P′) P̂ǎ for some equilibria P′ and P′ under ψ and φ, respectively, then we move to Step 2. Otherwise, we pick such a student k. By the same reason as above, student k is different from both i and j. Then, we follow the same arguments above and obtain a new preference profile. In each iteration, we have to consider a different student. But then, since there are finitely many students, this case cannot hold forever. Hence, we eventually obtain a problem, say P̂, in which there exists no student h such that
ψ_h(\hat{P}^\rho)\hat{P}_h \phi_h \left( \hat{P}^{\rho'} \right) \hat{P}_h \emptyset \text{ for some equilibria } \hat{P}^\rho \text{ and } \hat{P}^{\rho'} \text{ under } \psi \text{ and } \phi, \text{ respectively, and move to Step 2. We also have } \psi(P) = \psi(\hat{P}^\rho) \text{ for any equilibrium } \hat{P}^\rho \text{ under } \psi \text{ in problem } \hat{P}. 

**Step 2.** By the Lemma above, in problem \( \hat{P} \), we have \( \left| \psi(\hat{P}^\rho) \right| \geq \left| \phi(\hat{P}^{\rho'}) \right| \) for any equilibria \( \hat{P}^\rho \text{ and } \hat{P}^{\rho'} \text{ under } \psi \text{ and } \phi, \text{ respectively. If it holds strictly for some equilibria, then we reach a contradiction. Suppose } \left| \psi(\hat{P}^\rho) \right| = \left| \phi(\hat{P}^{\rho'}) \right| \text{ for any equilibria } \hat{P}^\rho \text{ and } \hat{P}^{\rho'}. \)

We now claim that \( \hat{P}^\rho \) is an equilibrium under \( \phi \) in problem \( P \). Suppose it is not, and let student \( k \) have a profitable deviation, say \( \hat{P}_k \), from \( \hat{P}^{\rho'}_k \). This means that \( \phi_k(k, \hat{P}^{\rho'}_k) P_k \phi_k(\hat{P}^\rho) \). But then, by construction above, \( \hat{P}_k \) preserves the relative rankings under \( P_k \). This implies that \( \phi_k(k, \hat{P}^{\rho'}_k) P_k \phi_k(\hat{P}^\rho), \) contradicting \( \hat{P}^{\rho'} \) being an equilibrium under \( \phi \) in problem \( \hat{P}. \)

Recall that \( \psi(P) = \psi(\hat{P}^\rho). \) Hence, this, along with \( \left| \psi(\hat{P}^\rho) \right| = \left| \phi(\hat{P}^{\rho'}) \right| \) and our above finding, implies that in problem \( P, \left| \psi(P') \right| = \left| \phi(P') \right| \) where \( P' \) and \( \hat{P}^{\rho'} \) are equilibria under \( \psi \) and \( \phi \), respectively. Therefore, we constructed an equilibrium pair for problem \( P \) where \( \psi \) matches as many students as \( \phi \), contradicting our assumption that this does not hold in problem \( P. \)

**Theorem 4.** (i). First, by the rural hospital theorem (Roth, 1984), the number of assignments in any stable matching is the same as that of DA. Let \( \psi \) be a FAM mechanism. Assume for a contradiction that there exist a problem \( P \) and an equilibrium profile \( P' \) under \( \psi \) such that \( \left| \psi(P') \right| < \left| DA(P) \right| \). For ease of writing, let \( DA(P) = \mu \) and \( \psi(P') = \mu'. \)

We now claim that for some student \( i, \mu_i = s \) for some school \( s \) whereas \( \mu'_i = \emptyset \) and, moreover, \( |\mu'_s| < q_s \). To prove this claim, let us define \( W = \{ i \in I : \mu_i = s \text{ and } \mu'_i = \emptyset \} \). By our supposition that \( \left| DA(P) \right| > \left| \psi(P') \right| \), we have \( W \neq \emptyset \). Suppose that for each \( i \in W \) with \( \mu_i = s \) and \( |\mu'_s| = q_s \). But then this implies that \( \left| \mu' \right| \geq |\mu| \), contradicting our initial supposition, which finishes the proof of the claim.

Let \( i \in I \) such that \( \mu_i = s, \mu'_i = \emptyset, \) and \( |\mu'_s| < q_s \). Now, consider the following preferences \( P'' \):

\[
P''_k = \begin{cases} P'_k & \text{If } k \neq i \\ s, \emptyset & \text{If } k = i \end{cases}
\]

First, observe that there exists a (individually rational) matching at \( P'' \) that assigns \( |\mu'| + 1 \) many students (to see this, keep the assignment of everyone except student \( i \) the same as at \( \mu' \), and place student \( i \) at school \( s \)). Therefore, due to the maximality of \( \psi \), we have \( |\psi(P'')| \geq |\mu'| + 1 \). If student \( i \) is assigned to school \( s \) at \( \psi(P'') \) then this contradicts \( P'' \) being equilibrium under \( \psi \). Hence, \( \psi_i(P'') = \emptyset. \) But then, by the definition of \( P'' \), \( \psi(P'') \) is individually rational at \( P''. \) This, along with the maximality of \( \psi \), implies that \( |\psi(P')| \geq |\psi(P'')| \), contradicting our previous finding that \( |\psi(P')| \geq |\psi(P'')| + 1 \), which finishes the proof of the first part.

(ii). Let us consider \( I = \{ i, j, k, h \} \) and \( S = \{ a, b, c \} \), each with unit capacity. The preferences and the priorities are given below:

\[
P_i : a, b, \emptyset; P_j : c, a, \emptyset; P_k : c, a, \emptyset; P_h : c, \emptyset.
\]

\[\succ a : k, i, j, h; \succ b : k, h, j, i; \succ c : k, h, i, j.\]

Let \( \psi \) be a FAM mechanism with the student ordering \( k, j, i, h \). Mechanism \( \psi \) is such that it produces matching \( \mu \) at \( P \) where \( \mu_i = b, \mu_j = a, \mu_k = c, \) and \( \mu_h = \emptyset. \) For any \( P'' \in P \) with \( bP''a \), let \( \psi(P''_i, P''_{-i}) = \mu'' \) where \( \mu''_i = b, \mu''_j = a, \mu''_k = c, \text{ and } \mu''_h = \emptyset. \) Moreover, for any \( P'' \in P \) with \( \emptyset P'' b \), \( \psi(P''_i, P''_{-i}) = \mu'' \) where \( \mu''_i = b, \mu''_j = a, \mu''_k = c, \text{ and } \mu''_h = c. \) And, for any \( P''_h \in P \), let \( \psi(P''_{-h}, P''_h) = \mu. \)

Note that student \( j \) can never get school \( c \) under \( \psi \) by misreporting because otherwise student \( h \) would be unassigned, and he has higher priority at school \( c. \) It is immediate to see that the above matchings can be obtained in the course of FAM through particular selection. All of these show
that under $\psi$, truth-telling is an equilibrium at $P$, and $|\psi(P)| = 3$. On the other hand, $DA(P)$ is such that $DA_i(P) = a$, $DA_k(P) = c$, and $DA_h(P) = DA_j(P) = \emptyset$. Hence, $|\psi(P)| > |DA(P)|$, finishing the proof of the second part.

**Proposition 3.** Let $P' = (P'_i, P'_{-i})$, $\psi(P) = \mu$, and $\psi(P'_i, P'_{-i}) = \mu'$. Assume that $|\mu'| > |\mu|$. By our supposition, $\mu'_i P_i \mu_i$. This, along with the fact that $P_j = P'_j$ for each $j \neq i$, $\mu'$ is individually rational in problem $P$. But then, $|\mu'| > |\mu|$ contradicts the fact that $\mu$ is maximal in problem $P$.

Consider a problem where $\{i, j\} \subseteq N$, $\{a, b\} \subseteq S$, each with unit capacity. Let $P_i : a, \emptyset$, $P_j : a, \emptyset$, and each other student (if any) finds every school unacceptable. Without loss of generality, assume that the outcome of $\psi$ in that problem, say $\mu$, is such that $\mu_i = a$, and each other student is unassigned.

Consider a problem where $P'_i : a, b, \emptyset$, while each other student’s preferences are the same as above. Under the true preferences, $\psi$ produces $\mu'$ where $\mu'_i = b$, $\mu'_j = a$, and each other student is unassigned. However, student $i$ can misreport his preferences by submitting $P'_i$ above as, under this false profile, $\psi$ produces matching $\mu$ above. Finally, note that $|\mu'| > |\mu|$, finishing the proof.