On a jump-diffusion process driven by the asymmetric Laplace distribution for stock price models

Mabel Eruore Adeosun¹,²* and Olabisi Oreofe Ugbebor²

¹Mathematics and Statistics Department, Osun State College of Technology, Esa-Oke, Nigeria
²Mathematics Department, University of Ibadan, Ibadan, Nigeria

*Corresponding author e-mail: maberuore74@gmail.com

Abstract. In this paper, a generalized jump-diffusion process driven by the Asymmetric Laplace (AL) Distribution for stock price modeling was proposed. The probability density function was derived for the dynamics of the log-returns when the random process of the jump amplitude obeys the AL distribution. Based on the derived density function, a Lévy-Khintchine formula for the process was obtained, which proved useful for the computation of moments of the process. Hence, the Asymmetric Laplace jump diffusion model can be useful for modeling of stock price processes with empirical features like discontinuous paths, asymmetry and high peaks found in the empirical distribution of most financial data.

Keywords: Stock price modeling, Density function, Lévy-Khintchine formula, Moments

1. Introduction
The observations made on the features of the normal distribution in the Black-Scholes (B-S) model in [1] and [2] as compared to the empirical distribution of the log-returns of asset price, show wide deviants. Hence, better models are needed to represent the dynamics of the stock price process in reality. We are aware that so many attempts have been made by researchers to modify the B-S model so as to accommodate realistic features. These include the addition of a jump term to the Geometric Brownian Motion (GBM) in [3], resulting to the jump-diffusion models. The dynamics of the jump-diffusion process in its generalized form is given as:

\[ dS_t = \mu S_t dt + \sigma S_t dW_t + J(Q_t) dN_t \]  

(1)

where \( S_0 > 0 \), \( \mu \) is the mean return rate of the diffusive process, \( \sigma \) is the diffusive volatility, \( W_t \) is a standard Brownian motion, \( N_t \) is a Poisson process with respect to the filtration \( F \) having a constant jump rate \( \lambda \); and \( J(Q_t) \) is a non-constant jump amplitude (random jump process), where also, the random variables \( W_t, N_t \) and \( J(Q_t) \) are independent.

In the works of Duffie, Pan and Singleton(2000) in [4]; Hanson and Westman (2002) in [5]; Merton (1976) in [3]; Kou (2002) in [6]; Synowiec (2008) in [7] and most recently in Lau et al (2019) in [8], the model in (1) above have been extensively studied. The distribution of the jumps that occur in any price process may not only be symmetric according to Lau et al (2019) in [8], it could also be skewed in both the upward and downward “jumps”. In this paper, we propose a jump-diffusion
process for stock price modeling, whose jump process obeys the Asymmetric Laplace (AL) distribution—this is a generalization of the Kou model. The AL distribution is a skewed-family of the Laplace distribution, which was proposed by Kozubowski and Podgorski (2000) in [9] and has been extensively studied by Kotz et al (2001) in [10].

The paper is organized as follows: the second section contains the basic definitions and concepts on the asymmetric Laplace distribution and the jump-diffusion process in general. In section three, the Asymmetric Laplace jump-diffusion model is proposed, given the density function of the dynamics of its log-returns and the Lévy- Khintchine formula for computing the moments of the process. The fourth section entails a concluding remark.

2. Basic Definition and Concepts
In this section we shall give some basic definitions and concepts on the Asymmetric Laplace distribution and the generalized jump-diffusion models for stock price.

2.1 Basic Definitions
The Asymmetric Laplace distribution henceforth named as $AL^*(\mu, \sigma^2, \kappa)$ distribution is a three parameter skewed Laplace distribution with a location and scale parameter which are respectively $\mu$ and $\sigma$, its skewness is indexed by the parameter $\kappa > 0$.

**Definition 2.1:** The Density of the $AL^*(\mu, \sigma^2, \kappa)$ Distribution (see [9] and [10])
The $AL^*(\mu, \sigma^2, \kappa)$ distribution has a probability density function given as:

$$f(x; \mu, \sigma, \kappa) = \frac{\sqrt{\kappa}}{\sigma} \left\{ \begin{array}{ll}
\exp \left( -\frac{\sqrt{\kappa}}{\sigma} (x - \mu) \right), & x \geq \mu \\
\exp \left( \frac{\sqrt{\kappa}}{\sigma} (x - \mu) \right), & x < \mu
\end{array} \right. \quad (2)$$

where $\kappa > 0, -\infty < \mu < \infty (\mu \in \mathbb{R})$.

**Remark 1**
The probability density function given in (2) above can be represented as

$$f(x) = \|p_k \alpha_1 \exp \left( -\eta_1 (x - \mu) \right) 1_{[\mu, \infty)} (x) + q_k \alpha_2 \exp \left( \alpha_2 (x - \mu) \right) 1_{(-\infty, \mu)} (x) \quad (3)$$

where, $p_k = \frac{1}{1 + k^2}$, $q_k = \frac{\kappa}{(1 + k^2)}$, $\alpha_1 = \frac{\sqrt{\kappa}}{\sigma}$, $\alpha_2 = \frac{\sqrt{\kappa}}{\sigma k}$, $\|p_k + q_k \| > 0, \|p_k + q_k \| = 1$

2.1.1. The Jump-diffusion model. Now, let $S_t$ be the price process of the stock, which satisfies the Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathbb{P}, \mathbb{F})$. Consider that the dynamics of $S_t$ given in eqn. (1) above

The random jump process above can be expressed as:

$$\int_{t_1}^{t+\Delta t} J(Q_t) dN_t = \sum_{i=1}^{D_N} J(Q_i), \quad (4)$$
given that the $Q_i$'s are i.i.d random variables, the probability density function of $N_t$ is
The solution of eqn (1) can be obtained by the Itô’s formula for jump diffusion and the log-returns is obtained as:

$$\Delta(\ln S_t) = (\mu - \frac{1}{2} \sigma^2) \Delta t + \sigma \Delta W_t + J(Q_t) \Delta N_t$$  \hspace{1cm} (6)

3. A Jump-Diffusion Model driven by the \( AL^* (\mu, \sigma^2, \kappa) \) Distribution

A generalized double exponential Jump-diffusion process is presented here. Below we give the density of the process as well as the Lévy-Khintchine formula for the jump-diffusion process driven by the \( AL^* (\mu, \sigma^2, \kappa) \) distribution.

3.1 The density of the \( AL^* (\mu, \sigma, k) \) Jump Diffusion Process

In the next theorem, we shall state the density of the jump diffusion process of the log-returns \( \Delta(\ln S_t) \) when \( Q_t \sim AL^*(\mu, \sigma^2, \kappa) \).

**Theorem 1**

The probability density function of the Jump Diffusion (ALJD) process in (6) is given by

$$f_{\Delta(\ln S_t)}(x) = \left(1 - \frac{1}{\sqrt{2\pi} \Delta t} \varphi \left(\frac{x - (\mu - \frac{1}{2} \sigma^2) \Delta t}{\sigma \sqrt{\Delta t}}\right)\right) \lambda \Delta t \left[\varphi_k(\alpha_1 \hat{\kappa}(1) \Phi_\alpha(\mu_j)) + \varphi_k(\alpha_2 \hat{\kappa}(2) \Phi_\alpha(-\mu_j))\right]$$ \hspace{1cm} (7)

where, \( \varphi(\cdot) \) is a normal density, \( \hat{\kappa}(1) = e^{\left(\frac{2a_1^2 \mu + a_1^2 \sigma^2 \Delta t}{2}\right)} e^{-(x-(\mu-\sigma^2)\Delta t)\alpha_1} \), and

$$\hat{\kappa}(2) = e^{\left(\frac{2a_2^2 \mu + a_2^2 \sigma^2 \Delta t}{2}\right)} e^{(x-(\mu-\sigma^2)\Delta t)\alpha_2}.$$  \hspace{1cm} (8)

and

$$\Phi_\alpha(\mu_j) \quad \text{and} \quad 1 - \Phi_\alpha(-\mu_j) = \Phi_\alpha(-\mu_j)$$

are the cumulative normal distributions of \( y \sim N(y; (x - \mu_d \Delta t) - \alpha_1 \sigma^2 \Delta t, 2\sigma^2 \Delta t) \) and

\( y \sim N(y; (x - \mu_d \Delta t) + \alpha_2 \sigma^2 \Delta t, 2\sigma^2 \Delta t) \) respectively at \( y = \mu_j \) and \( y = -\mu_j \)

**Proof**

The density can be derived with the concept of convolution of densities using:

$$f_{\Delta(\ln S_t)}(x) = (1 - \lambda \Delta t) f_{\chi}(x) + \lambda \Delta t f_{\chi_1 + q_1}(x)$$  \hspace{1cm} (8)

For, \( f_{\chi}(x) = \frac{1}{\sqrt{2\pi} \sigma^2 \Delta t} \exp \left(\frac{(x-(\mu-\frac{1}{2} \sigma^2)\Delta t)^2}{2\sigma^2 \Delta t}\right) \) and \( f_{\chi_1 + q_1}(x) \) given in eqn. (3) above, gives the desired result. Q.E.D.
3.1.1. The Lévy- Khintchine formula for the jump-diffusion \( AL^* (\mu, \sigma^2, \kappa) \) process. The Lévy-Khintchine formula for the \( AL^* (\mu, \sigma^2, \kappa) \) jump-diffusion process is given in the theorem below.

**Theorem 2**

The Lévy-Khintchine formula for the \( AL^* (\mu, \sigma^2, \kappa) \) jump-diffusion process is given as:

\[
\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \lambda \left( \frac{\mathbb{P}_k a_1 e^{iu\mu_j}}{a_1 - iu} - \frac{\mathbb{Q}_k a_2 e^{iu\mu_j}}{a_2 + iu} - 1 \right), \quad u \in \mathbb{R}
\]

(9)

**Proof:**

We recall here that for a Lévy Process \( X_t \), given its Levy triplet as \( (\mu, \sigma^2, \nu(dx)) \); an expression for the characteristic exponent \( \psi(u) = \log \phi_{X_t}(u), u \in \mathbb{R} \) where \( \phi_{X_t}(u) \) is the characteristic function of \( X_t \). Then,

\[
\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left( e^{iux} - 1 - iux1_{\{|x|\leq 1\}} \right) \nu(dx)
\]

(10)

The ALJD process has finite number of jumps in a finite period of time. Hence,

\[
\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left( e^{iux} - 1 \right) \nu(dx) : \quad \nu(dx) = \lambda f(x)
\]

(11)

Using eqn (3) in (11), the result follows Q.E.D.

**Remark 2**

Given the \( n^{th} \) cumulant of a characteristic exponent \( \psi(u) \) as:

\[
k_n = \frac{1}{i^n} \left. \frac{d^n \psi^{(n)}(u)}{du^n} \right|_{u=0}
\]

(12)

The mean and variance of the \( AL^* (\mu, \sigma^2, \kappa) \) jump-diffusion process are obtained and given as:

\[
\mathbb{E}(X_{\Delta t}^{ALJD}) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \lambda\Delta t \left( \frac{\mu_j + \frac{1}{\alpha_j}}{\alpha_j} \mathbb{P}_k - \left( \frac{\mu_j}{\alpha_j} \right)^2 \right)
\]

(13)

and

\[
\text{Var}(X_{\Delta t}^{ALJD}) = \sigma^2 \Delta t + \lambda\Delta t \left( \frac{\mu_j - 2\mu_j}{\alpha_j} \mathbb{P}_k - \left( \frac{\mu_j^2 - 2\mu_j}{\alpha_j^2} \right) \mathbb{Q}_k \right)
\]

(14)

3.2 The Relation between the Kou model and \( AL^* (\mu, \sigma, \kappa) \) Jump Diffusion Model

We give here, some similarities and differences between the Kou model and the \( AL^* (\mu, \sigma, \kappa) \) jump diffusion model. Both models are jump-diffusion models with finite number of jumps, such that the distributions of their jump-amplitude follow a skewed family of exponential distribution. However, in the Kou model, the distributions of the upward and downward jumps are skewed about a zero location parameter.
Owing to the rapid and unexpected changes in the market stock price, there is the need for a more generalized and robust process that can possibly capture some anomalies. Hence, the $AL^* (\mu, \sigma, k)$ jump diffusion process with jump-amplitude distribution given as a skewed distribution with a non-zero location parameter $\mu_j$ for both the upward and downward distribution of the jump amplitude. It is worthwhile to note that for $\mu_j = 0$ in eqns. (3), (7) (9), (13) and (14) will give us the result in the Kou model (see [6]).

The $AL^* (\mu, \sigma, k)$ jump diffusion model can be useful for modelling stock price process having more empirical features like discontinuous paths, asymmetry and high peaks found in the empirical distribution of stock price processes that cannot be captured by the existing stock models in literature.

4. Conclusion
In this work, a generalized jump-diffusion process which is driven by $AL^* (\mu, \sigma^2, \kappa)$ distribution for stock price modeling is proposed. The derived probability density function (pdf) gives a more robust density as compared to the density of the double exponential jump diffusion process in [6]. The expression for the Lévy-Khintchine formula of the process derived in this work proves useful for the computation of moments of the process. Hence, the asymmetric Laplace jump-diffusion process possesses analytical tractability and therefore is also useful for option pricing models when jumps are found in asset price process.

References
[1] F. Black, M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3) 637-654, 1973.
[2] M. E. Adeosun, S. O. Edeki, O. O. Ugbebor. Stochastic analysis of stock market price models: A case study of the Nigerian Stock Exchange (NSE). *WSEAS Transactions on Mathematics*, 14, 363-363. 2015.
[3] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2): 125-144, 1976.
[4] D. Du, J. Pan, K. Singleton. (2000). Transform analysis and asset pricing for a jump-diffusions. *Econometrica*, 68(6):1343–1376. 2002.
[5] F. B. Hanson, J.J. Westman. Stochastic analysis of jump-diffusions for financial log-return processes. In Stochastic theory and control, pages 169–183. Springer 2002.
[6] S. G. Kou. A jump-diffusion model for option pricing. *Management Science*, 48(8): 1086- 1101, 2002.
[7] D. Synowiec. Jump-diffusion models with constant parameters for financial log-return processes. *Computers & Mathematics with Applications*, 56(8), 2120-2127, 2008.
[8] K. J. Lau, Y. K.Goh, A.C. Lai. An empirical study on asymmetric jump diffusion for option and annuity pricing. *PloS one*, 14(5), e0216529, 2019.
[9] T. J. Kozubowski, K. Podgorski. Asymmetric Laplace distributions. *Mathematical Scientist*, 25(1), 37-46. 2000.
[10] S. Kotz, T.J. Kozubowski, K. Podgorski. The Laplace distribution and generalizations: A Revisit with Applications to Communications. *Economics, Engineering, and Finance*, 183. 2001.