Schauder estimates for stationary and evolution equations associated to stochastic reaction-diffusion equations driven by colored noise

Davide A. Bignamini and Simone Ferrari

Abstract: We consider stochastic reaction-diffusion equations with colored noise on the space of real-valued and continuous functions on a compact subset of $\mathbb{R}^d$ for $d = 1, 2, 3$. We prove Schauder-type estimates, which will depend on the color of the noise, for the stationary and evolution problems associated with the corresponding transition semigroup.

1. Introduction

The theory of Schauder regularity estimates for equations driven by differential operators with bounded coefficients was developed throughout the 20th century: see for example, [1, 2, 3, 4, 5]. In the late 1990s, a new interest in Schauder regularity estimates for stationary and evolution equations driven by differential operators with unbounded coefficients began to develop: see for instance, [6, 7, 8, 9, 10]. Besides their obvious analytic interest, one of the main motivations for this research is the relationship between second-order elliptic operators and stochastic differential equations, for example in problems such as uniqueness in law, pathwise uniqueness, and uniqueness of the martingale problem for stochastic partial differential equations: see for instance, [11, 12, 13, 14].

This article is devoted to the study of Schauder regularity estimates for stationary and evolution equations driven by a second-order differential operator associated with a stochastic reaction-diffusion equation. Let $\mathcal{O}$ be an open and bounded subset of $\mathbb{R}^d$, with $d = 1, 2, 3$. Let $L^2(\mathcal{O})$ be the space of square integrable functions with respect to the $d$-dimensional Lebesgue measure on $\mathcal{O}$, with the usual quotient with respect to the equality Lebesgue almost everywhere, and let $C(\overline{\mathcal{O}})$ be the space of continuous functions on $\overline{\mathcal{O}}$ endowed with the...
uniform norm. Let $A : \text{Dom}(A) \subseteq C(\overline{\mathcal{O}}) \to C(\overline{\mathcal{O}})$ be the realization of the Laplacian operator with Dirichlet or Neumann boundary conditions. Let $F : C(\overline{\mathcal{O}}) \to C(\overline{\mathcal{O}})$ be the Nemytskii operator given by

$$F(x)(\xi) := b(\xi, x(\xi)), \quad x \in C(\overline{\mathcal{O}}), \; \xi \in \overline{\mathcal{O}},$$

where $b : \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is a smooth enough function having polynomial growth with respect to the second variable (see Hypotheses 1.1(d)). Let $\{W(t)\}_{t \geq 0}$ be a $L^2(\mathcal{O})$-cylindrical Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see formula (2.1)). We consider the following stochastic reaction-diffusion equation:

$$
\begin{cases}
  dX(t, x) = \left[AX(t, x) + F(X(t, x))\right]dt + (-A)^{-\gamma/2}dW(t), & t > 0; \\
  X(0, x) = x \in C(\overline{\mathcal{O}}),
\end{cases}
$$

(1.1)

where $\gamma \in [0, 1)$. For every $x \in E := \overline{\text{Dom}(A)}$ (the closure is taken in $C(\overline{\mathcal{O}})$, with respect to the uniform norm), under suitable assumptions (see Hypotheses 1.1), Equation (1.1) has a unique $E$-valued pathwise continuous mild solution $\{X(t, x)\}_{t \geq 0}$ (see Definition 2.1) and we can define the transition semigroup $\{P(t)\}_{t \geq 0}$ given by

$$
(P(t)\varphi)(x) := \mathbb{E}[\varphi(X(t, x))] := \int_{\Omega} \varphi(X(t, x)(\omega))\mathbb{P}(d\omega), \quad x \in E, \; t \geq 0,
$$

(1.2)

where $\varphi : E \to \mathbb{R}$ is a bounded and Borel function. We denote by $\text{BUC}(E)$ the space of real valued, bounded, and uniformly continuous functions on $E$. Using the same arguments as in [15, Proposition 3.3] it is possible to prove that $\{P(t)\}_{t \geq 0}$ is a weakly continuous semigroup on $\text{BUC}(E)$, so we denote by $N : \text{Dom}(N) \subseteq \text{BUC}(E) \to \text{BUC}(E)$ its weak generator, namely the unique closed operator such that

$$u := R(\lambda, N)\varphi = \int_{0}^{+\infty} e^{-\lambda t}P(t)\varphi dt, \quad \lambda > 0, \; \varphi \in \text{BUC}(E).$$

(1.3)

In the case $d = 1, \mathcal{O} = [0, 1]$ and $\gamma = 0$, the authors of [15] prove maximal Schauder regularity estimates for the function $u$ given by (1.3): if $\varphi$ is $\alpha$-Hölder continuous for some $\alpha \in (0, 1)$, then $u$ is twice Fréchet differentiable with bounded and continuous derivatives and with $\alpha$-Hölder continuous second order derivative, namely $u$ gains two degrees of regularity. Even in the case $F \equiv 0$, if $\gamma > 0$ in (1.1), then the function $u$ does not gain two degrees of regularity as in [15], see for instance [16, Section 5.1]. The main purpose of this article is to show how the Schauder regularity results for the function $u$ depends on the constant $\gamma \in [0, 1)$ (the color of the noise driving (1.1)), in particular the greater $\gamma$ the less Schauder regularity improving for $u$ we get. Clearly in the case $\gamma = 0$ and $d = 1$ we recover the same results of [15]. Moreover, we will prove Schauder regularity estimates for the mild solution of the following evolution equation

$$
\begin{cases}
  \frac{d}{dt}v(t, x) = Nv(t, x) + g(t, x), & T > 0, \; t \in (0, T], \; x \in E; \\
  v(0, x) = f(x), & x \in E,
\end{cases}
$$

(1.4)

where $f, g$ belong to suitable Hölder spaces. These Schauder results will be similar to the ones in [1, 2, 3], where they were proved for evolution equations driven by a second-order operator with bounded coefficients in $\mathbb{R}^d$.

We stress that the coloring of the noise in (1.1) (meaning, in this case, the presence of the operator $(-A)^{-\gamma/2}$ in front of $dW(t)$, with $\gamma > 0$), is not an arbitrary choice. Indeed, if $d = 2, 3$, the addition of the color is necessary to guarantee the existence and uniqueness of a
pathwise continuous mild solution for (1.1), even in this case $F \equiv 0$ (see [17, Remark 6.1.1] and [18, Section 4.2]).

For (1.1) (and even for other types of semilinear stochastic partial differential equations), we were unable to find results in the literature that link the color of the noise with the Hölder regularity of the solutions of the stationary and evolution equations, so the results of this article are the first of their kind for stochastic partial differential equations of this type. We underline that the results of this article are based on the estimates contained in [17, Section 6.5] and on the techniques presented in [19, 16] where analogous results were obtained for stationary and evolution equations associated with linear stochastic equations, namely when $F \equiv 0$ in (1.1). However, we note that in the linear case, the transition semigroup associated with the stochastic partial differential equation has a Mehler representation formula that makes proofs much easier.

Moreover, we believe that the results contained in this article may be the starting point to extend the results of [20] to a more general framework.

To give a more complete view of the results found in the literature, we wish to point out that in [21, 22] Schauder regularity estimates along suitable directions are studied, in [16] the case of pseudo-differential operators is considered, in [23, 24, 25] the Gross Laplacian and some of its perturbations are taken into account, and in [19] the non-autonomous linear case is investigated. For other related results, see also [26, 27, 28].

Before rigorously stating the hypotheses and the main results of this article, it is necessary to recall some standard definitions and fix the notation.

**Notations.** All the integrals appearing in the article are meant in the sense of Bochner, see [29].

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two real Banach spaces equipped with the norms $\| \cdot \|_{\mathcal{K}_1}$ and $\| \cdot \|_{\mathcal{K}_2}$, respectively. For $k \in \mathbb{N}$ we denote by $\mathcal{L}^{(k)}(\mathcal{K}_1; \mathcal{K}_2)$ the space of continuous multilinear mappings from $\mathcal{K}_1^k$ to $\mathcal{K}_2$, if $k = 1$ we simply write $\mathcal{L}(\mathcal{K}_1; \mathcal{K}_2)$, while if $\mathcal{K}_1 = \mathcal{K}_2$ we write $\mathcal{L}^{(k)}(\mathcal{K}_1)$.

We denote by $\text{BUC}(\mathcal{K}_1; \mathcal{K}_2)$ the space of bounded and uniformly continuous functions from $\mathcal{K}_1$ to $\mathcal{K}_2$. If $\mathcal{K}_2 = \mathbb{R}$ we simply write $\text{BUC}(\mathcal{K}_1)$. We denote by $C^\alpha_b(\mathcal{K}_1; \mathcal{K}_2)$ (Lip$_b(\mathcal{K}_1; \mathcal{K}_2)$, respectively) the subspace of $\text{BUC}(\mathcal{K}_1; \mathcal{K}_2)$ of the $\alpha$-Hölder (Lipschitz, respectively) continuous functions. The spaces $C^\alpha_b(\mathcal{K}_1; \mathcal{K}_2)$ and Lip$_b(\mathcal{K}_1; \mathcal{K}_2)$ are Banach spaces if endowed with the norms

$$
\| f \|_{C^\alpha_b(\mathcal{K}_1; \mathcal{K}_2)} := \| f \|_{\infty} + [f]_{C^\alpha_b(\mathcal{K}_1; \mathcal{K}_2)}; \quad \| f \|_{\text{Lip}_b(\mathcal{K}_1; \mathcal{K}_2)} := \| f \|_{\infty} + [f]_{\text{Lip}_b(\mathcal{K}_1; \mathcal{K}_2)},
$$

where $[\cdot]_{C^\alpha_b(\mathcal{K}_1; \mathcal{K}_2)}$ and $[\cdot]_{\text{Lip}_b(\mathcal{K}_1; \mathcal{K}_2)}$ denote the standard seminorms on $C^\alpha_b(\mathcal{K}_1; \mathcal{K}_2)$ and Lip$_b(\mathcal{K}_1; \mathcal{K}_2)$, respectively. If $\mathcal{K}_2 = \mathbb{R}$ we simply write $C^\alpha_b(\mathcal{K}_1)$ and Lip$_b(\mathcal{K}_1)$.

For every $k \in \mathbb{N}$ and every $k$-times Fréchet differentiable function $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ we denote by $D^k f(x)$ the $k$-th Fréchet derivative operator of $f$ at $x \in \mathcal{K}_1$. In particular $D^k f(x)$ belongs to $\mathcal{L}^{(k)}(\mathcal{K}_1; \mathcal{K}_2)$. We denote the Gâteaux derivative operator by $D_G$. For some standard results concerning differentiability in infinite dimension we refer to [30, Chapter 7].

For $k \in \mathbb{N}$, we denote by $\text{BUC}^k(\mathcal{K}_1)$ the space of bounded, uniformly continuous, and $k$-times Fréchet differentiable functions $f : \mathcal{K}_1 \rightarrow \mathbb{R}$ such that $D^i f \in \text{BUC}(\mathcal{K}_1; \mathcal{L}^{(i)}(\mathcal{K}_1; \mathbb{R}))$, for $i = 1, \ldots, k$. We denote by $\| \cdot \|_{\text{BUC}^k(\mathcal{K}_1)}$ the standard norm on $\text{BUC}^k(\mathcal{K}_1)$. For $k \in \mathbb{N}$ and
\( \alpha \in (0, 1) \), we denote by \( C_{b}^{\alpha+\beta}(K_{1}) \) the subspace of \( \text{BUC}^{k}(K_{1}) \) of functions \( f : K_{1} \rightarrow \mathbb{R} \) such that \( \mathcal{D}^{k}f \in C_{b}^{\alpha}(K_{1};L^{(k)}(K_{1};\mathbb{R})) \). We endow \( C_{b}^{\alpha+\beta}(K_{1}) \) with the norm
\[
\|f\|_{C_{b}^{\alpha+\beta}(K_{1})} := \|f\|_{\text{BUC}^{k}(K_{1})} + \|\mathcal{D}^{k}f\|_{C_{b}^{\alpha}(K_{1};L^{(k)}(K_{1};\mathbb{R}))}.
\]
For \( \beta > 0 \) and \( \beta \notin \mathbb{N} \), let \( [\beta] \) and \( \{\beta\} \) be the integer and the fractional part of \( \beta \), respectively. We denote by \( C_{b}^{[\beta]+\{\beta\}}(K_{1}) \) the space \( C_{b}^{[\beta]+\{\beta\}}(K_{1}) \).

The Zygmund space \( Z^{1}(K_{1};K_{2}) \) is the subspace of \( \text{BUC}(K_{1};K_{2}) \) consisting of functions \( f : K_{1} \rightarrow K_{2} \) such that
\[
[f]_{Z^{1}(K_{1};K_{2})} := \sup_{h \in K_{1} \setminus \{0\}, x \in K_{1}} \frac{\|f(x + 2h) - 2f(x + h) + f(x)\|_{K_{2}}}{\|h\|_{K_{1}}}
\]
is finite. The space \( Z^{1}(K_{1};K_{2}) \) is a Banach space if endowed with the norm \( \|f\|_{Z^{1}(K_{1};K_{2})} := \|f\|_{\infty} + [f]_{Z^{1}(K_{1};K_{2})} \). If \( K_{2} = \mathbb{R} \), then we simply write \( Z^{1}(K_{1}) \). If \( k \geq 2 \), then we consider the Zygmund spaces \( Z^{k}(K_{1}) \) defined as
\[
Z^{k}(K_{1}) := \{ f \in \text{BUC}^{k-1}(K_{1}) | \mathcal{D}^{k-1}f \in Z^{1}(K_{1};C^{0}(K_{1};\mathbb{R})) \},
\]
endowed with the norm \( \|f\|_{Z^{k}(K_{1})} := \|f\|_{\text{BUC}^{k-1}(K_{1})} + \|\mathcal{D}^{k-1}f\|_{Z^{1}(K_{1};C^{0}(K_{1};\mathbb{R}))} \). We recall that every Lipschitz continuous function belongs to \( Z^{1}(K_{1};K_{2}) \), but, even in the case \( K_{1} = K_{2} = \mathbb{R} \), there are bounded and continuous functions not belonging to \( Z^{1}(K_{1};K_{2}) \) (see, for example, [31]).

**Hypotheses and main results.** In this article, we will work in the same framework of [17, Chapter 6] which we recall in the following hypotheses.

**Hypotheses 1.1.** Let \( \mathcal{O} \) be an open and bounded subset of \( \mathbb{R}^{d} \), with \( d = 1, 2, 3 \) and let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space.

(a) \( \{W(t)\}_{t \geq 0} \) is a \( L^{2}(\mathcal{O}) \)-cylindrical Wiener process defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \).

(b) \( A : \text{Dom}(A) \subseteq C(\overline{\mathcal{O}}) \rightarrow C(\overline{\mathcal{O}}) \) is the realization of the Laplacian operator with Dirichlet or Neumann boundary conditions. Throughout the paper we let \( E = \text{Dom}(A) \), where the closure is taken in \( C(\overline{\mathcal{O}}) \) with respect to the uniform norm.

(c) There exists \( \gamma \in [0, 1) \) such that for every \( p \geq 1 \) and \( T > 0 \) the mapping \( \omega \mapsto W_{A}(\cdot)(\omega) \) belongs to \( \text{L}^{p}((\Omega, \mathcal{F}, \mathbb{P}); C([0, T]; E)) \), where
\[
W_{A}(t) := \int_{0}^{t} e^{(t-s)A}(-A)^{-\gamma/2}dW(s), \quad t \geq 0.
\]

(d) \( F : C(\overline{\mathcal{O}}) \rightarrow C(\overline{\mathcal{O}}) \) is the Nemitskii operator defined as
\[
F(x)(\xi) := b(\xi, x(\xi)), \quad x \in C(\overline{\mathcal{O}}), \quad \xi \in \overline{\mathcal{O}},
\]
where \( b : \overline{\mathcal{O}} \times \mathbb{R} \rightarrow \mathbb{R} \) is such that, for every \( \xi \in \overline{\mathcal{O}} \), the mapping \( z \mapsto b(\xi, z) \) belongs to \( C^{3}(\mathbb{R}) \) and the mappings \( D_{z}^{j}b : \overline{\mathcal{O}} \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous, for \( j = 0, 1, 2, 3 \). Moreover, there exists an integer \( m \geq 0 \) such that
\[
\sup_{\xi \in \overline{\mathcal{O}}} \sup_{z \in \mathbb{R}} \frac{|D_{z}^{j}b(\xi, z)|}{1 + |z|^{\max(0, 2m+1-j)}} < +\infty, \quad j = 0, 1, 2, 3.
\]
Here, by $D_z^0 b$ we mean the function $b$. Furthermore, if $m \geq 1$, then there exist $a > 0$, $c \in \mathbb{R}$ and $\theta \geq 0$ such that

$$\sup_{\xi \in \Omega} (b(\xi, z + h) - b(\xi, z))h \leq -ah^{2(m+1)} + c(1 + |z|^\theta), \quad z, h \in \mathbb{R}. \quad (1.5)$$

**Remark 1.2.** If $\Omega$ is a smooth enough bounded and open subset of $\mathbb{R}^d$, and $A : \text{Dom}(A) \subseteq C(\Omega) \to C(\Omega)$ is the realization of the Laplacian operator with Dirichlet (Neumann, respectively) boundary conditions, then $E = C_0(\Omega)$, the space of continuous function on $\overline{\Omega}$ with null trace at the boundary ($E = C(\overline{\Omega})$, respectively), see [32, Corollaries 3.1.21(ii) and 3.1.24(ii)].

Hypotheses 1.1 are required to obtain existence, uniqueness, and some regularity properties for the mild solution of (1.1) listed in Section 2. The following is a simple example of function $b$ that verifies Hypothesis 1.1(d):

$$b(\xi, x) := -C_{2m+1}(\xi)x^{2m+1} + \sum_{k=0}^{2m} C_k(\xi)x^k, \quad \xi \in \overline{\Omega}, \ x \in \mathbb{R},$$

where $C_0, \ldots, C_{2m}$ are continuous functions from $\overline{\Omega}$ to $\mathbb{R}$ and $C_{2m+1} : \overline{\Omega} \to \mathbb{R}$ is a continuous and strictly positive function. If $d = 1$ and $\Omega$ is any open bounded interval, then Hypothesis 1.1(c) is verified for every $\gamma \in [0, 1)$, while if $d = 2, 3$ then Hypothesis 1.1(c) may not be verified for some $\gamma \in [0, 1)$. However, if $\Omega$ is “regular” enough, then there exists $\gamma_\Omega > 0$ such that Hypothesis 1.1(c) holds true for every $\gamma > \gamma_\Omega$. The constant $\gamma_\Omega$ depends on the dimension $d$ and on the boundary of $\Omega$. We refer to [18, Section 4.2] for proof of the following proposition, we remark that in our case we can choose the parameter $\alpha$, appearing in [18, Section 4.2], to be arbitrarily small.

**Proposition 1.3.** Let $d = 1, 2, 3$. If $A$ is a realization of the Laplacian operator with Dirichlet boundary conditions, then Hypothesis 1.1(c) holds true in the following cases:

$$\gamma > \frac{2d - 3}{2}, \quad \Omega := \{x \in \mathbb{R}^d | \|x\|_{\mathbb{R}^d} \leq 1\};$$

$$\gamma > \frac{d - 2}{2}, \quad \Omega := [0, \pi]^d.$$

In view of Proposition 1.3, we can deduce some simple examples to which the main results of this article apply. However, our results cannot be applied to $\Omega = \{x \in \mathbb{R}^3 | \|x\|_{\mathbb{R}^3} \leq 1\}$. Indeed in this case, by Proposition 1.3, the condition in Hypothesis 1.1(c) is verified if $\gamma > 3/2$, but we do not know of a technique to obtain the estimates needed for our computations in the case $\gamma \geq 1$ (we are referring to estimates (2.7) and (2.9), see Remark 2.8).

The following is one of the main results of the paper.

**Theorem 1.4.** Assume Hypotheses 1.1 hold true and let $\lambda > 0$. Let either $\alpha \in (0, 1)$ and $f \in C^0_b(E)$ or $\alpha = 0$ and $f \in \text{BUC}(E)$. We denote by $u = R(\lambda, N)f$ the function introduced in (1.3).
(i) If $\alpha + 2/(1 + \gamma) \in (1, 2) \cup (2, 3)$, then $u$ belongs to $C^{\alpha+2/(1+\gamma)}_b(E)$ and there exists $C = C(\lambda, \gamma, \alpha) > 0$ such that

$$\|u\|_{C^{\alpha+2/(1+\gamma)}_b(E)} \leq C\|f\|_{C^\alpha_b(E)}.$$ 

(ii) If $\alpha + 2/(1 + \gamma) = 2$, then $u$ belongs to $Z^2(E)$ and there exists $C = C(\lambda) > 0$ such that

$$\|u\|_{Z^2(E)} \leq C\|f\|_{C^\alpha_b(E)}.$$ 

Here, we have used the convention that if $\alpha = 0$, then $\|\cdot\|_{C^\alpha_b(E)} := \|\cdot\|_{\infty}$.

In the case $\alpha \in (0, 1)$, $\gamma = 0$ and $\mathcal{O} = [0, 1]$, Theorem 1.4 is the result contained in [15, Theorem 4.1], and it constitutes an extension of the results in [15] to the case $\alpha = 0$. In the other cases, Theorem 1.4 is a new result for stationary equations associated to stochastic reaction-diffusion equations of the type (1.1).

Theorem 1.4 shows that the solution $u$, introduced in (1.3), gains $2/(1 + \gamma)$ degrees of regularity. We note that the greater $\gamma$ the greater the regularity of the trajectories of the stochastic convolution $\{W_A(t)\}_{t \geq 0}$, while the function $u$ given by (1.3) gains less regularity.

For stochastic partial differential equations with linear drifts (or at most with Lipschitz continuous non-linearity in the drift), it is possible to prove that, in the case $\gamma > 0$, the function $u$ given by (1.3) gains two degrees of regularity (as in the case $\gamma = 0$), but only along “suitable directions” (see, e.g., [33, 34, 21, 22, 16, 35]). A result similar to the ones of [21, 22, 16] for (1.1) is an open problem that we plan to discuss in future works.

At the cost of some technical modifications in the proofs, we believe that it is possible to prove the same results contained in this article in the more abstract setting considered in [36, 37, 38], but this goes beyond our aims.

We point out that in the linear case, namely if $F \equiv 0$ in (1.1), the regularity of the function $u$ given by (1.3) was studied in the case where $f$ is bounded and continuous (see [19, 16]), while in Theorem 1.4, when $\alpha = 0$, we assume that $f$ is bounded and uniformly continuous. In this setting, this variation is due to some technical issues arising in the computations needed to get (2.10). In particular, working on the space $BUC(E)$ is essential to use two preliminary results that are fundamental for the proof of (2.10): the interpolation result in [15, Appendix A] and the estimates in [17, Proposition 6.4.1]. In the linear case, the Mehler representation formula of the transition semigroup $\{P(t)\}_{t \geq 0}$ allow more direct calculation to obtain (2.10) and so it is possible to work on $C^\alpha_b(E)$ instead of $BUC(E)$.

We will also study the regularity of the mild solution of (1.4), namely the function

$$v(t, x) := (P(t)f)(x) + \int_0^t (P(t-s)g(s, \cdot))(x)ds, \quad T > 0, \ t \in [0, T], \ x \in E. \quad (1.6)$$

To state our main result, we need to introduce the following Banach space.

**Definition 1.5.** Let $Y$ be a Banach space with norm $\|\cdot\|_Y$. For every $\alpha \in (0, 1)$, $\beta \geq 0$ and $T > 0$ we define $BUC^{0,\beta}[0, T] \times E; Y)$ as the set of continuous functions $g : [0, T] \times E \rightarrow Y$, that are separately uniformly continuous (namely, the mappings $t \mapsto g(t, x)$ and $x \mapsto g(t, x)$ are uniformly continuous for every $x \in E$ and $t \in [0, T]$ respectively) and such that

$$\|g\|_{BUC^{0,\beta}[0, T] \times E; Y)} := \sup_{t \in [0, T]} \|g(t, \cdot)\|_{BUC^\beta(E; Y)} < +\infty.$$ 

If $Y = \mathbb{R}$ we write $BUC^{0,\beta}[0, T] \times E)$.
For every $T > 0$ and $\beta \geq 0$ the space $BUC^{0,\beta}([0, T] \times E; Y)$ is a Banach space if endowed with the norm $\| \cdot \|_{BUC^{0,\beta}([0,T] \times E; Y)}$. Our main result is a regularity improving property for the function given by (1.6).

**Theorem 1.6.** Assume Hypotheses 1.1 hold true. Let $T > 0$, $\alpha \in [0, 1)$, $g \in BUC^{0,\alpha}([0, T] \times E)$ and consider the function $v$ given by (1.6).

(i) If $\alpha + 2/(1 + \gamma) \in (1, 2) \cup (2, 3)$ and $f \in C_b^{\alpha+2/(1+\gamma)}(E)$, then $v$ belongs to the space $BUC^{0,\alpha+2/(1+\gamma)}([0, T] \times E)$ and there exists $C = C(T, \gamma, \alpha) > 0$ such that

$$\|v\|_{BUC^{0,\alpha+2/(1+\gamma)}([0,T] \times E)} \leq C \left( \|f\|_{C_b^{\alpha+2/(1+\gamma)}(E)} + \|g\|_{BUC^{0,\alpha}([0,T] \times E)} \right).$$

(ii) If $\alpha + 2/(1 + \gamma) = 2$ and $f \in Z^2(E)$, then $v$ belongs to $BUC^{0,1}([0, T] \times E)$, for every $t \in [0, T]$ the mapping $x \mapsto \mathcal{D}v(t, x)$ belongs to $Z^1(E; \mathcal{L}(E; \mathbb{R}))$ and there exists $C = C(T) > 0$ such that

$$\sup_{t \in [0, T]} \|\mathcal{D}v(t, \cdot)\|_{Z^1(E; \mathcal{L}(E; \mathbb{R}))} \leq C \left( \|f\|_{Z^2(E)} + \|g\|_{BUC^{0,\alpha}([0,T] \times E)} \right).$$

2. Preliminaries results

In this section, we recall some preliminaries results that are crucial in the proofs of the main theorems of this article.

2.1. The transition semigroup

We recall that a $L^2(\Omega)$-cylindrical Wiener process $\{W(t)\}_{t \geq 0}$ is defined as

$$W(t) := \sum_{k=1}^{+\infty} \beta_k(t) e_k, \quad t \geq 0,$$

where $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $\{\beta_1(t)\}_{t \geq 0}, \ldots, \{\beta_k(t)\}_{t \geq 0}, \ldots$ are independent real Brownian motions. The convergence of the series in (2.1) is meant in the space $L^2((\Omega, \mathcal{F}, \mathbb{P}); C([0, T]; \mathcal{K}))$, where $\mathcal{K}$ is an appropriate separable Hilbert space such that $L^2(\Omega)$ is continuously embedded in $\mathcal{K}$. We refer to [39, Section 4.1.2] for an overview of cylindrical Wiener processes.

**Definition 2.1.** Assume that Hypotheses 1.1 hold true. For every $x \in E$ we call mild solution of (1.1) any $E$-valued process $\{X(t, x)\}_{t \geq 0}$ such that, for every $t \geq 0$, it holds

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s, x))ds + W_A(t), \quad \mathbb{P}\text{-a.s.},$$

where $\{W_A(t)\}_{t \geq 0}$ is the stochastic convolution process defined in Hypothesis 1.1(c).

We summarize the results in [17, Proposition 6.2.2 and Theorem 6.2.3] in the following proposition.

**Proposition 2.2.** Assume that Hypotheses 1.1 hold true. For every $x \in E$ the stochastic partial differential Equation (1.1) has a unique mild solution such that the mapping $\omega \mapsto X(\cdot, x)(\omega)$
belongs to $L^p((\Omega, \mathcal{F}, \mathbb{P}); C([0, T]; E))$, for every $T > 0$ and $p \geq 1$. Moreover, for every $t > 0$, it holds
\[
\sup_{x \in E} \|X(t, x)\|_E \leq k(t)t^{-1/2m}, \quad \mathbb{P}\text{-a.s.,}
\] (2.2)
where the random variable $k(t)$ is defined, $\mathbb{P}\text{-a.s.}, as k(t) := c(1 + \|W_A(t)\|_E^{\max(\theta/(1+2m), 1)})$, and $c$, $\theta$, $m$ are the constants appearing in Hypothesis 1.1(d). Furthermore, there exists a constant $\eta \in \mathbb{R}$ such that for every $t > 0$ and $x, y \in E$ it holds
\[
\|X(t, x) - X(t, y)\|_E \leq e^\eta \|x - y\|_E, \quad \mathbb{P}\text{-a.s.}
\] (2.3)

We remark that if $m \geq 1$ in Hypothesis 1.1(d), then (1.5) is needed to obtain (2.2).

We need to recall a regularity result for the mild solution $\{X(t, x)\}_{t \geq 0}$. The following proposition summarizes the results in [17, Propositions 6.3.3 and 6.4.1].

**Proposition 2.3.** Assume that Hypotheses 1.1 hold true. For every $T > 0$ the mapping $x \mapsto X(\cdot, x)$ is three times Gâteaux differentiable as a mapping from $E$ to $L^2((\Omega, \mathcal{F}, \mathbb{P}); C([0, T]; E))$. Moreover, for every $j = 1, 2, 3, h_1, \ldots, h_j \in E$ and $T > 0$ it holds
\[
\sup_{x \in E} \|P(t)^{\frac{j}{2}}X(t, x)(h_1, \ldots, h_j)\|_E \leq c(T) \prod_{i=1}^{j} \|h_i\|_E,
\] (2.4)
and
\[
\sup_{x \in E} \int_0^t \|(-A)^{y/2}P(t)^{\frac{j}{2}}X(s, x)(h_1, \ldots, h_j)\|_{L^2(\Omega)}^2 ds \leq c_j(t) t^{1-y} \prod_{i=1}^{j} \|h_i\|_E^2,
\]
for a suitable constant process $c(T)$ and increasing processes $c_1(t), c_2(t)$ and $c_3(t)$ having finite moments of every order.

### 2.2. Regularity of the transition semigroup

Using the mild solution $\{X(t, x)\}_{t \geq 0}$ of (1.1), we define the family of linear and continuous operators $\{P(t)\}_{t \geq 0}$ on $\text{BUC}(E)$ as
\[
(P(t)\varphi)(x) := \mathbb{E}[\varphi(X(t, x))] := \int_\Omega \varphi(X(t, x)(\omega)) \mathbb{P}(d\omega), \quad \varphi \in \text{BUC}(E), \ x \in E, \ t \geq 0.
\] (2.5)

Arguing as in [15, Proposition 3.3], it is possible to prove that $\{P(t)\}_{t \geq 0}$ is a weakly continuous semigroup on $\text{BUC}(E)$ and define its weak generator (in the sense of [40] and [17, Definition B.1.5]), that is, the unique closed operator $N : \text{Dom}(N) \subseteq \text{BUC}(E) \to \text{BUC}(E)$ such that
\[
R(\lambda, N)\varphi = \int_0^{+\infty} e^{-\lambda t}P(t)\varphi dt, \quad \lambda > 0, \ \varphi \in \text{BUC}(E).
\]

Exploiting (2.3) and the integral representation of the transition semigroup $\{P(t)\}_{t \geq 0}$, given in (2.5), we obtain the following result.

**Proposition 2.4.** Assume that Hypotheses 1.1 hold true and let $\alpha \in (0, 1)$. If $t \geq 0$, then
\[
\|P(t)\|_{L(\text{BUC}(E))} \leq 1, \quad \|P(t)\|_{L(C_0^\alpha(E))} \leq e^{\alpha t}, \quad \|P(t)\|_{L(\text{Lip}_p(E))} \leq e^{\eta t},
\]
where $\eta \in \mathbb{R}$ is the constant appearing in (2.3).
The following proposition is a consequence of Proposition 2.3.

**Proposition 2.5.** (Theorems 6.5.1 of [17]). Assume that Hypotheses 1.1 hold true. If \( t > 0 \) and \( \varphi \in \text{BUC}(E) \), then \( P(t)\varphi \in \text{BUC}^3(E) \) and for every \( x, h \in E \)

\[
\mathcal{D}P(t)\varphi(x)h = \frac{1}{t} \mathbb{E} \left[ \varphi(X(t,x)) \int_0^t \langle (-A)^{\gamma/2} \mathcal{D}G X(s,x)h, dW(s) \rangle \right].
\]  

(2.6)

Moreover, for every \( j = 1, 2, 3 \), there exists a constant \( C_j > 0 \) such that for every \( t > 0 \), \( x \in E \), and \( \varphi \in \text{BUC}(E) \) it holds

\[
\|\mathcal{D}^jP(t)\varphi(x)\|_{\mathcal{L}^{(\gamma)}(\mathbb{R};\mathbb{R})} \leq C_j (\min\{1, t\})^{-j(1+\gamma)/2} \|\varphi\|_{\text{BUC}(E)}.
\]  

(2.7)

Combining Proposition 2.3 and (2.6), we obtain the following result.

**Proposition 2.6.** Assume that Hypotheses 1.1 hold true. For every \( j = 1, 2, 3 \) there exists a constant \( K_j > 0 \) such that for every \( t > 0 \), \( x \in E \), and \( \varphi \in \text{BUC}(E) \) it holds

\[
\|\mathcal{D}^jP(t)\varphi(x)\|_{\mathcal{L}^{(\gamma)}(\mathbb{R};\mathbb{R})} \leq K_j (\min\{1, t\})^{-j(1+\gamma)/2} \|\varphi\|_{\text{BUC}(E)}.
\]  

(2.8)

Exploiting Proposition 2.4 and proceeding as in [15, Proposition 3.5 and (3.10)] we get the following result.

**Proposition 2.7.** Assume that Hypotheses 1.1 hold true. For every \( j = 1, 2, 3 \) there exists a constant \( K'_j > 0 \) such that for every \( t > 0 \), \( x \in E \), and \( \varphi \in \text{Lip}_b(E) \) it holds

\[
\|\mathcal{D}^jP(t)\varphi(x)\|_{\mathcal{L}^{(\gamma)}(\mathbb{R};\mathbb{R})} \leq K'_j (\min\{1, t\})^{-j(1+\gamma)/2} \|\varphi\|_{\text{Lip}_b(E)}.
\]  

(2.9)

**Remark 2.8.** Estimates (2.7), (2.8), and (2.9) are deduced directly from [17, Proposition 6.4.1]. The use of the Hölder inequality in that proof imposes the condition \( \gamma < 1 \). We do not know of any alternative proofs of [17, Proposition 6.4.1] that do not exploit the Hölder inequality and thus allow us to consider the case \( \gamma \geq 1 \).

We refer to [15, Appendix A] for a proof of the following interpolation result.

**Theorem 2.9.** Assume that Hypotheses 1.1 hold true and let \( r \in (0, 1) \). Up to an equivalent renorming, it holds \( \text{(BUC}(E), \text{Lip}_b(E))_{r,\infty} = C^r_b(E) \).

By (2.7), (2.9), Theorem 2.9 and [31, Theorem 1.12 in Chapter 5], we obtain the following proposition.

**Proposition 2.10.** Assume that Hypotheses 1.1 hold true and let \( \alpha \in (0, 1) \). There exists \( c_\alpha > 0 \) such that for every \( j = 1, 2, 3 \), \( \varphi \in C^\alpha_b(E) \), \( t > 0 \) and \( x \in E \)

\[
\|\mathcal{D}^jP(t)\varphi(x)\|_{\mathcal{L}^{(\gamma)}(\mathbb{R};\mathbb{R})} \leq c_\alpha (\min\{1, t\})^{-j(1+\gamma)/2} \|\varphi\|_{C^\alpha_b(E)}.
\]  

(2.10)

### 3. Proofs of the main results

This section is devoted to the proof of Theorems 1.4 and 1.6.
Proof of Theorem 1.4. Due to some technical complications, we split the proof in two cases: the case \( \sigma = 0 \), that is, \( f \in \text{BUC}(E) \), and the case \( \sigma \in (0, 1) \).

The case \( \sigma = 0 \). We start by proving that if \( \gamma \in [0, 1) \), then \( u \in \text{BUC}^1(E) \). By (2.7) (with \( j = 1 \)) we get that for every \( x, h \in E \) and \( t > 0 \)

\[
|P(t)f(x + h) - P(t)f(x) - DP(t)f(x)h| = \left| \int_0^1 (DP(t)f(x + \sigma h) - DP(t)f(x))h \, d\sigma \right| \\
\leq 2C_1 \min\{1, t\}^{-(1+\gamma)/2}\|h\|_E \|f\|_{\infty}.
\]  

(3.1)

Hence, by (3.1) and the dominated convergence theorem we get that \( u \) is Fréchet differentiable and for every \( \lambda > 0 \) and \( x \in E \) it holds

\[
Du(x) = \int_0^{+\infty} e^{-\lambda t} DP(t)f(x) \, dt.
\]

Exploiting the same arguments used in the proof of (3.1), we obtain for every \( x \in E \) and \( \lambda > 0 \)

\[
\|Du(x)\|_{\mathcal{L}(E; \mathbb{R})} \leq C_1 \left( \frac{2}{1 - \gamma} + \frac{1}{\lambda} \right) \|f\|_{\infty}.
\]  

(3.2)

The fact that \( u \in \text{BUC}^1(E) \) follows by [21, Proposition A.5] (with \( \mathcal{M} = E , d_{\mathcal{M}} = \|\cdot\|_E \), \( Y = (0, +\infty) \), \( \mu \) is the measure \( e^{-\lambda t} dt \) and \( Z = \mathcal{L}(E; \mathbb{R}) \)).

To prove (i), let \( x, h \in E \) be such that \( \|h\|_E \geq 1 \). By (3.2), it holds

\[
\|Du(x + h) - Du(x)\|_{\mathcal{L}(E; \mathbb{R})} \leq 2C_1 \left( \frac{2}{1 - \gamma} + \frac{1}{\lambda} \right) \|f\|_{\infty}
\]

\[
\leq 2C_1 \left( \frac{2}{1 - \gamma} + \frac{1}{\lambda} \right) \|h\|_{E}^{(1-\gamma)/(1+\gamma)} \|f\|_{\infty}.
\]  

(3.3)

Let now \( x, h \in E \) be such that \( \|h\|_E < 1 \), and consider the functions \( a_{1,\gamma}, b_{1,\gamma} : E \rightarrow \mathbb{R} \) defined as

\[
a_{1,\gamma}(x) := \int_0^{\|h\|_E^{2/(1+\gamma)}} e^{-\lambda t} DP(t)f(x) \, dt; \quad b_{1,\gamma}(x) := \int_0^{+\infty} e^{-\lambda t} DP(t)f(x) \, dt.
\]

Observe that by (2.7) (with \( j = 1 \)) it holds

\[
\|a_{1,\gamma}(x + h) - a_{1,\gamma}(x)\|_{\mathcal{L}(E; \mathbb{R})} \leq \int_0^{\|h\|_E^{2/(1+\gamma)}} e^{-\lambda t} \|DP(t)f(x + h) - DP(t)f(x)\|_{\mathcal{L}(E; \mathbb{R})} \, dt
\]

\[
\leq 2C_1 \int_0^{\|h\|_E^{2/(1+\gamma)}} t^{-(1+\gamma)/2} \, dt = \frac{4C_1}{1 - \gamma} \|h\|_{E}^{(1-\gamma)/(1+\gamma)} \|f\|_{\infty}.
\]  

(3.4)

Furthermore, by (2.7) (with \( j = 2 \)) and the fact that there exists \( K(\gamma, \lambda) > 0 \) such that, for every \( t > 0 \), it holds \( t^{(1+\gamma)} e^{-\lambda t} \leq K(\gamma, \lambda) \) we get

\[
\|b_{1,\gamma}(x + h) - b_{1,\gamma}(x)\|_{\mathcal{L}(E; \mathbb{R})}
\]

\[
\leq \int_0^{+\infty} e^{-\lambda t} \|DP(t)f(x + h) - DP(t)f(x)\|_{\mathcal{L}(E; \mathbb{R})} \, dt
\]

\[
\leq \int_0^{+\infty} e^{-\lambda t} \left\| \int_0^1 D^2 P(t)(x + \sigma h)(h, \cdot) \, d\sigma \right\|_{\mathcal{L}(E; \mathbb{R})} \, dt
\]
Now combining (3.3), (3.4), and (3.5) we get (i).

We can now show (ii) (observe that $\gamma = 0$). If $x, h \in E$ are such that $\|h\|_E \geq 1$, then by (3.2), it holds

$$
\|D u(x + 2h) - 2D u(x + h) + D u(x)\|_{L(E; R)} \leq 4C_1 \left( 2 + \frac{1}{\lambda} \right) \|f\|_E
$$

$$
\leq 4C_1 \left( 2 + \frac{1}{\lambda} \right) \|h\|_E \|f\|_E. \quad (3.6)
$$

Now we are going to study the case $\|h\|_E < 1$. Consider the functions $a, b : E \to \mathbb{R}$ defined, for every $x, h \in E$, as

$$
a_1(x) := \int_0^{\|h\|_E^2} e^{-\lambda t} D P(t)f(x) dt; \quad b_1(x) := \int_0^{+\infty} e^{-\lambda t} D P(t)f(x) dt.
$$

For every $x, h \in E$ with $\|h\|_E < 1$ and $t > 0$, by (2.7) (with $j = 1$ and $\gamma = 0$), we have

$$
\|a_1(x + 2h) - 2a_1(x + h) + a_1(x)\|_{L(E; R)}
\leq \int_0^{\|h\|_E^2} e^{-\lambda t} \|D P(t)f(x + 2h) - 2D P(t)f(x + h) + D P(t)f(x)\|_{L(E; R)} dt
\leq 8C_1 \|h\|_E \|f\|_E. \quad (3.7)
$$

Before proceeding with $b_1$ we need some intermediate estimates. First of all observe that for every $x, h \in E$ and $t > 0$

$$
D P(t)f(x + 2h) - D P(t)f(x + h) = \int_0^1 D^2 P(t)f(x + (1 + \sigma)h)(h, \cdot) d\sigma; \quad (3.8)
$$

$$
D P(t)f(x + h) - D P(t)f(x) = \int_0^1 D^2 P(t)f(x + \sigma h)(h, \cdot) d\sigma. \quad (3.9)
$$

So combining (3.8) and (3.9) we get

$$
D P(t)f(x + 2h) - 2D P(t)f(x + h) + D P(t)f(x)
\leq \int_0^1 \int_0^1 D^3 P(t)f(x + (\tau + \sigma)h)(h, h, \cdot) d\tau d\sigma. \quad (3.10)
$$

By (2.7) (with $j = 3$ and $\gamma = 0$), (3.10) and recalling that there exists $K(\lambda) > 0$ such that for every $t > 0$, it holds $t^{3/2} e^{-\lambda t} \leq K(\lambda)$, we get
\[ \|b_1(x + 2h) - 2b_1(x + h) + b_1(x)\|_{L(E;\mathbb{R})} \leq \int_{\|h\|_E^2}^{+\infty} e^{-\lambda t} \|DP(t)f(x + 2h) - 2DP(t)f(x + h) + DP(t)f(x)\|_{L(E;\mathbb{R})} dt \leq 2C_3K(\lambda)\|h\|_E\|f\|_\infty. \] (3.11)

Combining (3.6), (3.7), and (3.11) we get (ii).

The case \( \alpha \in (0, 1) \). The case (i) where \( 0 < \alpha < 2\gamma/(1 + \gamma) \) and the case (ii) where \( \alpha = 2\gamma/(1 + \gamma) \) can be obtained by reasoning as above. We will therefore focus on the case (i) where \( \alpha > 2\gamma/(1 + \gamma) \). We start by showing that if \( \gamma \in [0, 1) \) and \( \alpha > 2\gamma/(1 + \gamma) \), then \( u \in \text{BUC}^2(E) \). By (2.10) (with \( j = 2 \)) we get that for every \( x, h \in E \) and \( t > 0 \)

\[ \|DP(t)f(x + h) - DP(t)f(x) - D^2P(t)f(x)(h, \cdot)\|_{L(E;\mathbb{R})} \leq 2\epsilon_\alpha \left( \min\{1, t\} \right)^{-(2-\alpha)(1+\gamma)/2} \|h\|_E \|f\|_{C_0^\gamma(E)}. \] (3.12)

Hence, by (3.12) and the dominated convergence theorem, we get that \( u \) is two times Fréchet differentiable and for every \( \lambda > 0 \) and \( x \in E \) it holds

\[ D^2u(x) = \int_{0}^{+\infty} e^{-\lambda t} D^2P(t)f(x) dt. \]

Exploiting the same arguments used in the proof of (3.12), we obtain

\[ \|D^2u(x)\|_{L^2(E;\mathbb{R})} \leq \epsilon_\alpha \left( \frac{2}{2 - (2-\alpha)(1+\gamma)} + \frac{1}{\lambda} \right) \|f\|_{C_0^\gamma(E)}. \]

The fact that \( u \in \text{BUC}^2(E) \) follows by [21, Proposition A.5] (with \( M = E, d_M = \|\cdot\|_E, \ Y = (0, +\infty), \mu \) is the measure \( e^{-\lambda t} dt \) and \( Z = \mathcal{L}(E;\mathbb{R}) \)).

Now let \( \lambda > 0 \) and \( x, h \in E \) such that \( \|h\|_E < 1 \) and consider the functions

\[ a_{1,\gamma}(x) := \int_0^{\|h\|_E^{2/(1+\gamma)}} e^{-\lambda t} DP(t)f(x) dt; \quad b_{1,\gamma}(x) := \int_{\|h\|_E^{2/(1+\gamma)}}^{+\infty} e^{-\lambda t} DP(t)f(x) dt. \]

By (2.10) (with \( j = 1 \)) it holds

\[ \|a_{1,\gamma}(x + h) - a_{1,\gamma}(x)\|_{L(E;\mathbb{R})} \leq 2\epsilon_\alpha \|f\|_{C_0^\gamma(E)} \int_0^{\|h\|_E^{2/(1+\gamma)}} t^{-(1-\alpha)(1+\gamma)/2} dt \]

\[ = \frac{4\epsilon_\alpha}{2 - (1 - \alpha)(1 + \gamma)} \|h\|_E^{\alpha + (1-\gamma)/(1+\gamma)} \|f\|_{C_0^\gamma(E)}. \] (3.13)

Using (2.10) (with \( j = 2 \)) and arguing in the same way as in the proof of (3.4), we get that there exists \( K = K(\lambda, \gamma, \alpha) > 0 \) such that

\[ \|b_{1,\gamma}(x + h) - b_{1,\gamma}(x)\|_{L(E;\mathbb{R})} \leq (1 + K)\|h\|_E \|f\|_{C_0^\gamma(E)} \int_{\|h\|_E^{2/(1+\gamma)}}^{+\infty} t^{-(2-\alpha)(1+\gamma)/2} dt \]

\[ = \frac{2(1 + K)}{(2 - \alpha)(1 + \gamma) - 2} \|h\|_E^{\alpha + (1-\gamma)/(1+\gamma)} \|f\|_{C_0^\gamma(E)}. \] (3.14)

Combining (3.13) and (3.14), we get the thesis, since the case \( \|h\|_E \geq 1 \) can be obtained using arguments similar to those used in the proof of (3.3) and (3.6).
In order to study the regularity of the mild solution of (1.4), namely the function
\[ \nu(t, x) := (P(t)f)(x) + \int_0^t (P(t-s)g(s, \cdot))(x)ds, \quad T > 0, \ t \in [0, T], \ x \in E, \] (3.15)
we will analyze the regularity of the two summands on the right hand side of (3.15). In the
next proposition, we show that the mapping \( x \mapsto (P(t)f)(x) \) has the same regularity of \( f \). The
proof follows the same ideas of the proof of [16, Lemma 3.5], using the integral representation
(1.2), the chain rule and estimate (2.4).

**Proposition 3.1.** Assume that Hypotheses 1.1 hold true and let \( T > 0 \).

(i) For every \( \beta \in (0, 3) \) and \( f \in C^\beta_b(E) \), the mapping \( (t, x) \mapsto (P(t)f)(x) \) belongs to the space
\( \text{BUC}^{0, \beta}([0, T] \times E) \).

(ii) For \( k = 1, 2 \) and \( f \in Z^k(E) \), the mapping \( (t, x) \mapsto (P(t)f)(x) \) belongs to the space
\( \text{BUC}^{0,k-1}([0, T] \times E) \) and there exists \( K = K(T) > 0 \) such that
\[ \sup_{t \in [0, T]} \|D^{k-1}P(t)f\|_{Z^k(E; L^{(k-1)}(E; \mathbb{R}))} \leq K \|f\|_{Z^k(E)}. \]
Here \( D^0P(t)f \) denotes the function \( P(t)f \) and \( L^{(0)}(E; \mathbb{R}) \) is \( \mathbb{R} \).

Now we prove the regularity improving property for the second summand in the right-hand side of (3.15).

**Theorem 3.2.** Assume Hypotheses 1.1 hold true. Let \( T > 0, \alpha \in [0, 1), g \in \text{BUC}^{0,\alpha}([0, T] \times E) \)
and consider the function \( \tilde{\nu} \) given by
\[ \tilde{\nu}(t, x) := \int_0^t (P(s)g(t-s, \cdot))(x)ds, \quad t \in [0, T], \ x \in E. \]

(i) If \( \alpha + 2/(1 + \gamma) \notin \mathbb{N} \), then \( \tilde{\nu} \) belongs to \( \text{BUC}^{0,\alpha+2/(1+\gamma)}([0, T] \times E) \) and there exists
\( C = C(T, \gamma, \alpha) > 0 \) such that
\[ \|\tilde{\nu}\|_{\text{BUC}^{0,\alpha+2/(1+\gamma)}([0, T] \times E)} \leq C \|g\|_{\text{BUC}^{0,\alpha}([0, T] \times E)}. \]

(ii) If \( \alpha + 2/(1 + \gamma) = 2 \), then \( \tilde{\nu} \) belongs to \( \text{BUC}^{0,1}([0, T] \times E) \), for every \( t \in [0, T] \) the mapping
\( x \mapsto D\tilde{\nu}(t, x) \) belongs to \( Z^1(E; L(E; \mathbb{R})) \) and there exists \( C = C(T) > 0 \) such that
\[ \sup_{t \in [0, T]} \|D\tilde{\nu}(t, \cdot)\|_{Z^1(E; L(E; \mathbb{R}))} \leq C \|g\|_{\text{BUC}^{0,\alpha}([0, T] \times E)}. \]

**Proof.** We will just prove the case \( \alpha = 0 \) since the remaining cases follow similarly. Observe that for every \( \gamma \in [0, 1) \), by Proposition 2.4 it holds \( \sup_{t \in [0, T]} \|\tilde{\nu}(t, \cdot)\|_\infty \leq T\|g\|_{\text{BUC}^{0,0}([0, T] \times E)}. \) If \( \gamma \in [0, 1) \) then by (2.7) (with \( j = 1 \)) we get that for every \( x, h \in E \) and \( t \in [0, T] \)
\[ \left| \tilde{\nu}(t, x + h) - \tilde{\nu}(t, x) - \int_0^t DP(s)g(t-s, \cdot)(x)hds \right| \]
\[ = \left| \int_0^t \int_0^1 (DP(s)g(t-s, \cdot)(x + \sigma h) - DP(s)g(t-s, \cdot)(x))hds d\sigma \right| \]
\[
\leq 2C_1 \left( \int_0^t \left( \min \{1, s \} \right)^{-\left(1+\gamma\right)/2} ds \right) \|h\|_E \|g\|_{BUC^{0,0}(\{0,T\} \times E)} \\
= 2C_1 \left( \frac{2}{1-\gamma} + (T-1)\chi_{(1,+\infty)}(T) \right) \|h\|_E \|g\|_{BUC^{0,0}(\{0,T\} \times E)},
\]

where \(\chi_{(1,+\infty)}\) denotes the characteristic function of the interval \((1, +\infty)\). By (3.16), for every \(t \in [0, T]\), the function \(x \mapsto \tilde{v}(t, x)\) is Fréchet differentiable in \(E\) and by [21, Proposition A.5] (with \(\mathcal{M} = E, d_\mathcal{M} = \|\cdot\|_E, Y = (0, t), \mu\) is the measure \(dt\) and \(Z\) is \(\mathcal{L}(E; \mathbb{R})\)) it belongs to \(BUC^1(E)\). Using the same arguments as in (3.16) we get

\[
\sup_{t \in [0, T]} \|D\tilde{v}(t, \cdot)\|_{\mathcal{L}(E; \mathbb{R})} \leq C_1 \left( \frac{2}{1-\gamma} + (T-1)\chi_{(1,+\infty)}(T) \right) \|h\|_E \|g\|_{BUC^{0,0}(\{0,T\} \times E)}. \tag{3.17}
\]

So \(\tilde{v}\) belongs to \(BUC^{1,1}([0, T] \times E)\).

Let us prove (i). For every \(t \in [0, T]\) and \(x, h \in E\) with \(\|h\|_E < 1\) we consider

\[
a_{1,\gamma}(t, x) := \int_0^t \min \{t, \|h\|_E^{2/(1+\gamma)} \} \mathcal{D} \mathcal{P}(s) g(t-s, \cdot)(x) ds; \\
b_{1,\gamma}(t, x) := \int_0^t \min \{t, \|h\|_E^{2/(1+\gamma)} \} \mathcal{D} \mathcal{P}(s) g(t-s, \cdot)(x) ds.
\]

Now observe that by (2.7) (with \(j = 1\)) we get

\[
\|a_{1,\gamma}(t, x+h) - a_{1,\gamma}(t, x)\|_{\mathcal{L}(E; \mathbb{R})} \leq 2C_1 \left( \int_0^t \min \{t, \|h\|_E^{2/(1+\gamma)} \} s^{-(1+\gamma)/2} ds \right) \|g\|_{BUC^{0,0}(\{0,T\} \times E)} \\
\leq \frac{4C_1}{1-\gamma} \|h\|_E^{(1-\gamma)/(1+\gamma)} \|g\|_{BUC^{0,0}(\{0,T\} \times E)}. \tag{3.18}
\]

Observe that \(b_{1,\gamma}(t, x+h) - b_{1,\gamma}(t, x) = 0\) if \(\|h\|_E^{2/(1+\gamma)} \geq t\). So if \(\|h\|_E^{2/(1+\gamma)} < t\), by (2.7) (with \(j = 2\)), we get

\[
\|b_{1,\gamma}(t, x+h) - b_{1,\gamma}(t, x)\|_{\mathcal{L}(E; \mathbb{R})} = \left\| \int_0^t \int_0^1 \mathcal{D}^2 \mathcal{P}(s) g(t-s, \cdot)(x+\sigma h)(h, \cdot) d\sigma ds \right\|_{\mathcal{L}(E; \mathbb{R})} \\
\leq C_2 \|h\|_E \left( \int_0^1 s^{-(1+\gamma)} ds + \chi_{(1, +\infty)}(T) \int_1^T ds \right) \|g\|_{BUC^{0,0}(\{0,T\} \times E)} \\
\leq C_2 \|h\|_E \left( \int_0^\infty s^{-(1+\gamma)} ds + \chi_{(1, +\infty)}(T) \int_1^T \frac{1}{\|h\|_E^{2/(1+\gamma)} ds} \right) \|g\|_{BUC^{0,0}(\{0,T\} \times E)} \\
= C_2 \left( \frac{1}{\gamma} + (T-1)\chi_{(1,+\infty)}(T) \right) \|h\|_E^{(1-\gamma)/(1+\gamma)} \|g\|_{BUC^{0,0}(\{0,T\} \times E)}. \tag{3.19}
\]

Combining (3.18) and (3.19) we get (i), since the case \(\|h\|_E \geq 1\) can be treated using arguments similar to those used in the proof of (3.3), and exploiting (3.17).

Now we prove (ii). For every \(t \in [0, T]\) and \(x, h \in E\) with \(\|h\|_E < 1\) we consider

\[
a_1(t, x) := \int_0^t \min \{t, \|h\|_E^{2} \} \mathcal{D} \mathcal{P}(s) g(t-s, \cdot)(x) ds; \quad b_1(t, x) := \int_0^t \min \{t, \|h\|_E^{2} \} \mathcal{D} \mathcal{P}(s) g(t-s, \cdot)(x) ds.
\]
Now observe that by (2.7) (with \(j = 1\)) we get
\[
\|a_1(t, x + 2h) - 2a_1(t, x + h) + a_1(t, x)\|_{\mathcal{L}(E; \mathbb{R})}
\leq 4C_1 \left( \int_0^{\min(t, \|h\|_E^2)} s^{-1/2} ds \right) \|g\|_{\text{BUC}^0,0([0,T] \times E)}
\leq 8C_1 \|h\|_E \|g\|_{\text{BUC}^0,0([0,T] \times E)}. \tag{3.20}
\]
Observe that \(b_1(t, x + 2h) - 2b_1(t, x + h) + b_1(t, x) = 0\) if \(\|h\|_E^2 \geq t\). So if \(\|h\|_E^2 < t\), by (2.7) (with \(j = 3\)), we get
\[
\|b_1(t, x + 2h) - 2b_1(t, x + h) + b_1(t, x)\|_{\mathcal{L}(E; \mathbb{R})}
\leq C_3 \|h\|_E^2 \left( \int_0^{1} s^{-3/2} ds + \chi_{(1,\infty)}(T) \int_1^{T} ds \right) \|g\|_{\text{BUC}^0,0([0,T] \times E)}
\leq C_3 \|h\|_E^2 \left( \int_0^{+\infty} s^{-3/2} ds + \chi_{(1,\infty)}(T) \int_1^{T} \frac{1}{\|h\|_E} ds \right) \|g\|_{\text{BUC}^0,0([0,T] \times E)}
= C_3 \left( 2 + (T - 1) \chi_{(1,\infty)}(T) \right) \|h\|_E \|g\|_{\text{BUC}^0,0([0,T] \times E)}. \tag{3.21}
\]
Combining (3.20) and (3.21), we get (ii). Indeed, the case \(\|h\|_E \geq 1\) can be obtained using arguments similar to those used in the proof of (3.6), and taking (3.17) into account.

Combining Proposition 3.1 and Theorem 3.2, we obtain Theorem 1.6.

**Declarations**

**Acknowledgments**
The authors would like to express their gratitude to the anonymous referees for the meticulous review of the manuscript and for providing numerous valuable comments and suggestions that have significantly improved the paper.

**Disclosure statement**
No potential conflict of interest was reported by the author(s).

**Fundings**
The authors are members of GNAMPA (Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni) of the Italian Istituto Nazionale di Alta Matematica (INdAM).

**References**
[1] Kružkov, S. N., Castro, A., Lopes, M. (1975). Schauder type estimates, and theorems on the existence of the solution of fundamental problems for linear and nonlinear parabolic equations. *Dokl. Akad. Nauk SSSR*. 220(2): 60–64. Soviet Math. Dokl.
[2] Kružkov, S. N., Castro, A., Lopes, M. (1980). Mayoraciones de schauder y teorema de existencia de las soluciones del problema de Cauchy para ecuaciones parabolicas lineales y no lineales (i). Ciencias Matemáticas. 1: 55–76.

[3] Kružkov, S. N., Castro, A., Lopes, M. (1982). Mayoraciones de schauder y teorema de existencia de las soluciones del problema de Cauchy para ecuaciones parabolicas lineales y no lineales (ii). Ciencias Matemáticas. 3: 37–56.

[4] Ladyženskaja, O. A, Ural’ceva, N. (1968). Ural’ceva. Linear and Quasilinear Elliptic Equations. New York-London: Academic Press.

[5] Ladyženskaja, O. A., Solonnikov, V. A., Ural’ceva, N. (1968). Ural’ceva. Linear and quasilinear equations of parabolic type. In Translations of Mathematical Monographs, Vol 23. Providence, RI: American Mathematical Society.

[6] Cannarsa, P., Da Prato, G. (1994). Schauder estimates for Kolmogorov equations in Hilbert spaces. In Progress in Elliptic and Parabolic Partial Differential Equations, Vol. 350. Capri, Harlow: Longman, pp. 100–111.

[7] Da Prato, G. (2003). A new regularity result for Ornstein-Uhlenbeck generators and applications. J. Evol. Equ. 3(3): 485–498. doi:10.1007/s00028-003-0114-x.

[8] Da Prato, G., Lunardi, A. (1995). On the Ornstein-Uhlenbeck operator in spaces of continuous functions. J. Funct. Anal. 131(1):94–114. doi:10.1006/jfan.1995.1084.

[9] Lunardi, A. (1997). Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $bfR^n$. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 24(4): 133–164.

[10] Priola, E. (2009). Global Schauder estimates for a class of degenerate Kolmogorov equations. Studia Math. 194(2):117–153. doi:10.4064/sm194-2-2.

[11] Addona, D., Masiero, F., Priola, E. (2023). A BSDEs approach to pathwise uniqueness for stochastic evolution equations. J. Differ. Equ. 366: 192–248. doi:10.1016/j.jde.2023.04.014.

[12] Da Prato, G., Flandoli, F. (2010). Pathwise uniqueness for a class of SDE in Hilbert spaces and applications. J. Funct. Anal. 259(1): 243–267. doi:10.1016/j.jfa.2009.11.019.

[13] Da Prato, G., Flandoli, F., Priola, E., Röckner, M. (2013). Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. Ann. Probab. 41(5): 3306–3344.

[14] Zambotti, L. (2000). An analytic approach to existence and uniqueness for martingale problems in infinite dimensions. Probab. Theory Relat. Fields. 118(2): 147–168. doi:10.1007/s440-000-8012-6.

[15] Cerrai, S., Da Prato, G. (2012). Schauder estimates for elliptic equations in Banach spaces associated with stochastic reaction–diffusion equations. J. Evol. Equ. 12(1): 83–98. doi:10.1007/s00028-011-0124-0.

[16] Lunardi, A., Röckner, M. (2021). Schauder theorems for a class of (pseudo-)differential operators on finite- and infinite-dimensional state spaces. J. London Math. Soc. 104(2): 492–540. doi:10.1112/jlms.12436.

[17] Cerrai, S., Da Prato, G., Flandoli, F. (2012). Pathwise uniqueness for stochastic reaction-diffusion equations in Banach spaces with an Hölder drift component. Stoch. PDE: Anal. Comp. 1(3): 749–782. doi:10.1007/s40072-013-0016-0.

[18] Bignamini, D. A., Ferrari, S. (2023b). Schauder regularity results in separable Hilbert spaces. J. Differ. Equ. 370: 305–345. doi:10.1016/j.jde.2023.06.023.

[19] Cerrai, S., Lunardi, A. (2019). Schauder theorems for Ornstein-Uhlenbeck equations in infinite dimension. J. Differ. Equ. 267(12): 7462–7482. doi:10.1016/j.jde.2019.08.005.

[20] Cannarsa, P., Da Prato, G. (1996). Infinite-dimensional elliptic equations with Hölder-continuous coefficients. Adv. Differ. Equ. 1(3): 425–452.
[24] Priola, E. (1999). *Partial Differential Equations with Infinitely Many Variables*. Università degli Studi di Torino: Iris, AperTO.

[25] Priola, E., Zambotti, L. (2000). New optimal regularity results for infinite-dimensional elliptic equations. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* 3(8): 411–429.

[26] Athreya, S. R., Bass, R. F., Gordina, M., Perkins, E. A. (2006). Infinite dimensional stochastic differential equations of Ornstein–Uhlenbeck type. *Stoch. Process. Appl.* 116(3):381–406. doi:10.1016/j.spa.2005.10.001.

[27] Athreya, S. R., Bass, R. F., Perkins, E. A. (2005). Hölder norm estimates for elliptic operators on finite and infinite-dimensional spaces. *Trans. Amer. Math. Soc.*. 357(12): 5001–5029. doi:10.1090/S0002-9947-05-03638-X.

[28] Da Prato, G. (2012). Schauder estimates for some perturbation of an infinite dimensional Ornstein–Uhlenbeck operator. *DCDS-S*. 6(3): 637–647. doi:10.3934/dcdss.2013.6.637.

[29] Diestel, J., Uhl, J. J., Jr. (1977). Vector measures. In *Mathematical Surveys*, Vol. 15. Providence, RI 15: American Mathematical Society.

[30] Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V. (2011). Banach space theory. In *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. New York: Springer.

[31] Bennett, C., R., Sharples, (1998). Interpolation of operators. In *Pure and Applied Mathematics*, Vol. 129. Boston, MA: Academic Press, Inc.

[32] Lunardi, A. (1995). Analytic Semigroups and Optimal Regularity in Parabolic Problems. Basel AG, Basel: Birkhäuser/Springer.

[33] Bignamini, D. A., Ferrari, S. (2023a). Regularizing properties of (non-Gaussian) transition semigroups in Hilbert spaces. *Potential Anal.* 58(1): 1–35. doi:10.1007/s11118-021-09931-2.

[34] Bignamini, D.A., Ferrari, S. (2022). On generators of transition semigroups associated to semilinear stochastic partial differential equations. *J. Math. Anal. Appl.* 508(1): 125878. doi:10.1016/j.jmaa.2021.125878.

[35] Masiero, F. (2007). Regularizing properties for transition semigroups and semilinear parabolic equations in Banach spaces. *Electron. J. Probab*. 12(13): 387–419.

[36] Addona, D., Bandini, E., Masiero, F. (2020). A nonlinear Bismut-Elworthy formula for HJB equations with quadratic Hamiltonian in Banach spaces. *Nonlinear Differ. Equ. Appl*. 27(4): Article 37.

[37] Angiuli, L., Bignamini, D. A., Ferrari, S. (2023). Harnack inequalities with power $p \in (1, +\infty)$ for transition semigroups in Hilbert spaces. *Nonlinear Differ. Equ. Appl*. 30(1): Article 6.

[38] Bignamini, D. A. (2023). $L^2$-theory for transition semigroups associated to dissipative systems. *Stoch. PDE: Anal. Comp*. 11(3):988–1043. doi:10.1007/s40072-022-00253-x.

[39] Da Prato, G., Zabczyk, J. (2014). Stochastic equations in infinite dimensions. *Encyclopedia of Mathematics and its Applications*, Vol. 152. Cambridge: Cambridge University Press.

[40] Cerrai, S. (1994). A Hille-Yosida theorem for weakly continuous semigroups. *Semigroup Forum*. 49(1):349–367. doi:10.1007/BF02573496.