On the Total Perimeter of Homothetic Convex Bodies
in a Convex Container∗

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Tuesday 1st March, 2022

Abstract

For two planar convex bodies, $C$ and $D$, consider a packing $S$ of $n$ positive homothets of $C$ contained in $D$. We estimate the total perimeter of the bodies in $S$, denoted $\text{per}(S)$, in terms of $\text{per}(D)$ and $n$. When all homothets of $C$ touch the boundary of the container $D$, we show that either $\text{per}(S) = O(\log n)$ or $\text{per}(S) = O(1)$, depending on how $C$ and $D$ “fit together,” and these bounds are the best possible apart from the constant factors. Specifically, we establish an optimal bound $\text{per}(S) = O(\log n)$ unless $D$ is a convex polygon and every side of $D$ is parallel to a corresponding segment on the boundary of $C$ (for short, $D$ is parallel to $C$). When $D$ is parallel to $C$ but the homothets of $C$ may lie anywhere in $D$, we show that $\text{per}(S) = O((1 + \text{esc}(S)) \log n / \log \log n)$, where $\text{esc}(S)$ denotes the total distance of the bodies in $S$ from the boundary of $D$. Apart from the constant factor, this bound is also the best possible.

Keywords: convex body, perimeter, maximum independent set, homothet, Ford disks, traveling salesman, approximation algorithm.

1 Introduction

A finite set $S = \{C_1, \ldots, C_n\}$ of convex bodies is a packing in a convex body (container) $D \subset \mathbb{R}^2$ if the bodies $C_1, \ldots, C_n \in S$ are contained in $D$ and they have pairwise disjoint interiors. The term convex body above refers to a compact convex set with nonempty interior in $\mathbb{R}^2$. The perimeter of a convex body $C \subset \mathbb{R}^2$ is denoted $\text{per}(C)$, and the total perimeter of a packing $S$ is denoted $\text{per}(S) = \sum_{i=1}^{n} \text{per}(C_i)$. Our interest is estimating $\text{per}(S)$ in terms of $n$. In this paper, we consider packings $S$ that consist of positive homothets of a convex body $C$. We start with an easy general bound for this case.

Proposition 1. For every pair of convex bodies, $C$ and $D$, and every packing $S$ of $n$ positive homothets of $C$ in $D$, we have $\text{per}(S) \leq \rho(C, D) \sqrt{n}$, where $\rho(C, D)$ depends on $C$ and $D$. Apart from this multiplicative constant, this bound is the best possible.

Our goal is to substantially improve the dependence of $\text{per}(S)$ on $n$ in two different scenarios, motivated by applications to the traveling salesman problem with neighborhoods (TSPN). In

* A preliminary version of this paper appeared in Proceedings of the 17th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX 2013), Berkeley, CA, 2013, LNCS 8096, pp. 96–109.
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Sections 3–4, we prove tight bounds on \( \text{per}(S) \) in terms of \( n \) when all homothets in \( S \) touch the boundary of the container \( D \) (see Fig. 1). In Section 5, we prove tight bounds on \( \text{per}(S) \) in terms of \( n \) and the total distance of the bodies in \( S \) from the boundary of \( D \). Specifically, for two convex bodies, \( C \subset D \subset \mathbb{R}^2 \), let the escape distance \( \text{esc}(C) \) be the distance between \( C \) and the boundary of \( D \) (Fig. 2, right); and for a packing \( S = \{C_1, \ldots, C_n\} \) in a container \( D \), let \( \text{esc}(S) = \sum_{i=1}^{n} \text{esc}(C_i) \).

Figure 1: Left: a packing of disks in a rectangle container, where all disks touch the boundary of the container. Right: a convex body \( C \) in the interior of a trapezoid \( D \) at distance \( \text{esc}(C) \) from the boundary of \( D \). The trapezoid \( D \) is parallel to \( C \): every side of \( D \) is parallel and “corresponds” to a side of \( C \).

Homothets touching the boundary of a convex container. We would like to bound \( \text{per}(S) \) in terms of \( \text{per}(D) \) and \( n \) when all homothets in \( S \) touch the boundary of \( D \) (see Fig. 1). Specifically, for a pair of convex bodies, \( C \) and \( D \), let \( f_{C,D}(n) \) denote the maximum perimeter \( \text{per}(S) \) of a packing of \( n \) positive homothet of \( C \) in the container \( D \), where each element of \( S \) touches the boundary of \( D \). We would like to estimate the growth rate of \( f_{C,D}(n) \) as \( n \) goes to infinity. We prove a logarithmic upper bound \( f_{C,D}(n) = O(\log n) \) for every pair of convex bodies, \( C \) and \( D \).

**Proposition 2.** For every pair of convex bodies, \( C \) and \( D \), and every packing \( S \) of \( n \) positive homothets of \( C \) in \( D \), where each element of \( S \) touches the boundary of \( D \), we have \( \text{per}(S) \leq \rho(C,D) \log n \), where \( \rho(C,D) \) depends on \( C \) and \( D \).

The upper bound \( f_{C,D}(n) = O(\log n) \) is asymptotically tight for some pairs \( C \) and \( D \), and not so tight for others. For example, it is not hard to attain an \( \Omega(\log n) \) lower bound when \( C \) is an axis-aligned square, and \( D \) is a triangle (Fig. 2, left). However, \( f_{C,D}(n) = \Theta(1) \) when both \( C \) and \( D \) are axis-aligned squares. We start by establishing a logarithmic lower bound in the simple setting where \( C \) is a circular disk and \( D \) is a unit square.

**Theorem 1.** The total perimeter of \( n \) pairwise disjoint disks lying in the unit square \( U = [0,1]^2 \) and touching the boundary of \( U \) is \( O(\log n) \). Apart from the constant factor, this bound is the best possible.

We determine \( f_{C,D}(n) \) up to constant factors for all pairs of convex bodies of bounded description complexity. We show that either \( f_{C,D} = \Theta(\log n) \) or \( f_{C,D}(n) = \Theta(1) \) depending on how \( C \) and \( D \) “fit together”. To distinguish these cases, we need the following definitions.

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1. Throughout this paper, \( \log x \) denotes the logarithm of \( x \) to base 2.
2. A planar set has **bounded description complexity** if its boundary consists of a finite number of algebraic curves of bounded degrees.
Definition of “parallel” convex bodies. For a direction vector $d \in S$ and a convex body $C$, the supporting line $\ell_d(C)$ is a directed line of direction $d$ such that $\ell_d(C)$ is tangent to $C$, and the closed halfplane on the left of $\ell_d(C)$ contains $C$. If $\ell_d(C) \cap C$ is a nondegenerate line segment, we refer to it as a side of $C$.

We say that a convex polygon (container) $D$ is parallel to a convex body $C$ when for every direction $d \in S$ if $\ell_d(D) \cap D$ is a side of $D$, then $\ell_d(C) \cap C$ is also a side of $C$. Figure 2 (right) depicts a square $D$ parallel to a convex hexagon $C$. For example, every positive homothet of a convex polygon $P$ is parallel to $P$; and all axis-aligned rectangles are parallel to each other.

Classification. We generalize the lower bound construction in Theorem 1 to arbitrary convex bodies, $C$ and $D$, of bounded description complexity, where $D$ is not parallel to $C$.

Theorem 2. Let $C$ and $D$ be two convex bodies of bounded description complexity. For every packing $S$ of $n$ positive homothets of $C$ in $D$, where each element of $S$ touches the boundary of $D$, we have $\text{per}(S) \leq \rho(C,D) \log n$, where $\rho(C,D)$ depends on $C$ and $D$. Apart from the factor $\rho(C,D)$, this bound is the best possible unless $D$ is a convex polygon parallel to $C$.

If $D$ is a convex polygon parallel to $C$, and every homothet of $C$ in a packing $S$ of $n$ homothets touches the boundary of $D$, then it is not difficult to see that $\text{per}(S)$ is bounded from above by an expression independent of $n$.

Proposition 3. Let $C$ and $D$ be convex bodies such that $D$ is a convex polygon parallel to $C$. Then every packing $S$ of $n$ positive homothets of $C$ in $D$, where each element of $S$ touches the boundary of $D$, we have $\text{per}(S) \leq \rho(C,D)$, where $\rho(C,D)$ depends on $C$ and $D$.

Total distance form the boundary of a convex container. In the general case, when the homothets of $C$ can be in the interior of the container $D$, we improve the dependence on $n$ of the general bound Proposition 1 by using the escape distance, namely the total distance of the homothets of $C$ from the boundary of $D$. Combining the bound in Proposition 1 with inequality (2) yields the following bound.

Proposition 4. For every pair of convex bodies, $C$ and $D$, and every packing $S$ of $n$ positive homothets of $C$ in $D$, we have $\text{per}(S) \leq \rho(C,D)(\text{esc}(S) + \log n)$, where $\rho(C,D)$ depends on $C$ and $D$.

By Theorem 2, the logarithmic upper bound in terms of $n$ is the best possible when $D$ is not parallel to $C$. When $D$ is a convex polygon parallel to $C$, we derive the following upper bound for $\text{per}(S)$, which is also asymptotically tight in terms of $n$. 
Theorem 3. Let $C$ and $D$ be two convex bodies such that $D$ is a convex polygon parallel to $C$. For every packing $S$ of $n$ positive homothets of $C$ in $D$, we have

$$\per(S) \leq \rho(C, D) (\per(D) + \esc(S)) \frac{\log n}{\log \log n},$$

where $\rho(C, D)$ depends on $C$ and $D$. For every $n \geq 1$, there exists a packing $S$ of $n$ positive homothets of $C$ in $D$ such that $\per(S) \geq \sigma(C, D) (\per(D) + \esc(S)) \frac{\log n}{\log \log n}$, where $\sigma(C, D)$ depends on $C$ and $D$.

Related Previous Work. We consider the total perimeter $\per(S)$ of a packing $S$ of $n$ homothets of a convex body $C$ in a convex container $D$ in Euclidean plane. Other variants have also been considered: (1) If $S$ is a packing of $n$ arbitrary convex bodies in $D$, then it is easy to subdivide $D$ by $n - 1$ near diameter segments into $n$ convex bodies of total perimeter close to $\per(D) + 2(n - 2)diam(D)$. Glazaryn and Morić [11] have recently proved that this lower bound is the best possible when $D$ is a square or a triangle. For an arbitrary convex body $D$, they prove an upper bound of $\per(S) \leq 1.22 \per(D) + 2(n - 1)diam(D)$. (2) If all bodies in $S$ are congruent to a convex body $C$, then $\per(S) = n \per(C)$, and bounding $\per(S)$ from above reduces to the classical problem of determining the maximum number of interior-disjoint congruent copies of $C$ that fit in $D$ [3, Section 1.6].

Considerations of the total surface area of a ball packing in $\mathbb{R}^3$ also play an important role in a strong version of the Kepler conjecture [3, 13].

Motivation. In the Euclidean Traveling Salesman Problem (ETSP), given a set $S$ of $n$ points in $\mathbb{R}^d$, one wants to find a closed polygonal chain (tour) of minimum Euclidean length whose vertex set is $S$. The Euclidean TSP is known to be NP-hard, but it admits a PTAS in $\mathbb{R}^d$, where $d \in \mathbb{N}$ is constant [2]. In the TSP with Neighborhoods (TSPN), given a set of $n$ sets (neighborhoods) in $\mathbb{R}^d$, one wants to find a closed polygonal chain of minimum Euclidean length that has a vertex in each neighborhood. The neighborhoods are typically simple geometric objects (of bounded description complexity) such as disks, rectangles, line segments, or lines. TSPN is known to be NP-hard; and it admits a PTAS for certain types of neighborhoods [16], but is hard to approximate for others [6].

For $n$ connected (possibly overlapping) neighborhoods in the plane, TSPN can be approximated with ratio $O(\log n)$ by an algorithm of Mata and Mitchell [15]. See also the survey by Bern and Eppstein [4] for a short outline of this algorithm. At its core, the $O(\log n)$-approximation relies on the following early result by Levcopoulos and Lingas [14]: every (simple) rectilinear polygon $P$ with $n$ vertices, $r$ of which are reflex, can be partitioned into rectangles of total perimeter $O(\per(P) \log r)$ in $O(n \log n)$ time.

A natural approach for finding a solution to TSPN is the following [7, 9] (in particular, it achieves a constant-ratio approximation for unit disks): Given a set $S$ of $n$ neighborhoods, compute a maximal subset $I \subseteq S$ of pairwise disjoint neighborhoods (i.e., a packing), compute a good tour for $I$, and then augment it by traversing the boundary of each set in $I$. Since each neighborhood in $S \setminus I$ intersects some neighborhood in $I$, the augmented tour visits all members of $S$. This approach is particularly appealing since good approximation algorithms are often available for pairwise disjoint neighborhoods [16]. The bottleneck of this approach is the length increase incurred by extending a tour of $I$ by the total perimeter of the neighborhoods in $I$. An upper bound $\per(I) = o(OPT(I) \log n)$ would immediately imply an improved $o(\log n)$-factor approximation ratio for TSPN.

Theorem 2 shows that this approach cannot beat the $O(\log n)$ approximation ratio for most types of neighborhoods (e.g., circular disks). In the current formulation, Proposition 2 yields
the upper bound $\text{per}(I) = O(\log n)$ assuming a convex container, so in order to use this bound, a tour of $I$ needs to be augmented into a convex partition; this may increase the length by a $\Theta(\log n / \log \log n)$-factor in the worst case \[8, 13\]. For convex polygonal neighborhoods, the bound $\text{per}(I) = O(1)$ in Proposition \[3\] is applicable after a tour for $I$ has been augmented into a convex partition with parallel edges (e.g., this is possible for axis-aligned rectangle neighborhoods, and an axis-aligned approximation of the optimal tour for $I$). The convex partition of a polygon with $O(1)$ distinct orientations, however, may increase the length by a $\Theta(\log n)$-factor in the worst case \[13\]. Overall our results show that we cannot beat the current $O(\log n)$ ratio for TSPN for any type of homothetic neighborhoods if we start with an arbitrary independent set $I$ and an arbitrary near-optimal tour for $I$.

2 Preliminaries: A Few Easy Pieces

Proof of Proposition \[1\] Let $\mu_i > 0$ denote the homothety factor of $C_i$, i.e., $C_i = \mu_i C$, for $i = 1, \ldots, n$. Since $S$ is a packing we have $\sum_{i=1}^n \mu_i^2 \cdot \text{area}(C) \leq \text{area}(D)$. By the Cauchy-Schwarz inequality we have $(\sum_{i=1}^n \mu_i)^2 \leq n \sum_{i=1}^n \mu_i^2$. It follows that

$$\text{per}(S) = \sum_{i=1}^n \text{per}(C_i) = \text{per}(C) \sum_{i=1}^n \mu_i \leq \text{per}(C) \sqrt{n} \left( \sum_{i=1}^n \mu_i^2 \right) \leq \text{per}(C) \sqrt{\frac{\text{area}(D)}{\text{area}(C)}} \sqrt{n}.$$ 

Set now $\rho(C, D) := \text{per}(C) \sqrt{\frac{\text{area}(D)}{\text{area}(C)}}$, and the proof of the upper bound is complete.

For the lower bound, consider two convex bodies, $C$ and $D$. Let $U$ be a maximal axis-aligned square inscribed in $D$, and let $\mu C$ be the largest positive homothet of $C$ that fits into $U$. Note that $\mu = \mu(C, D)$ is a constant that depends on $C$ and $D$ only. Subdivide $U$ into $\lceil \sqrt{n} \rceil^2$ congruent copies of the square $\frac{1}{\lceil \sqrt{n} \rceil} C$. Let $S$ be the packing of $n$ copies of $\frac{1}{\lceil \sqrt{n} \rceil} C$ (i.e., $n$ translates), with at most one in each square $\frac{1}{\lceil \sqrt{n} \rceil} U$. The total perimeter of the packing is $\text{per}(S) = n \mu \frac{1}{\lceil \sqrt{n} \rceil} \text{per}(C) = \Theta(\sqrt{n})$, as claimed. \[\square\]

Proof of Proposition \[2\] Let $S = \{C_1, \ldots, C_n\}$ be a packing of $n$ homothets of $C$ in $D$ where each element of $S$ touches the boundary of $D$. Observe that $\text{per}(C_i) \leq \text{per}(D)$ for all $i = 1, \ldots, n$. Partition the elements of $S$ into subsets as follows. For $k = 1, \ldots, \lceil \log n \rceil$, let $S_k$ denote the set of homothets $C_i$ such that $\text{per}(D)/2^k < \text{per}(C_i) \leq \text{per}(D)/2^{k-1}$; and let $S_0$ be the set of homothets $C_i$ of perimeter less than $\text{per}(D)/2^\lceil \log n \rceil$. Then the sum of perimeters of the elements in $S_0$ is $\text{per}(S_0) \leq n \text{per}(D)/2^{\lceil \log n \rceil} \leq \text{per}(D)$, since $S_0 \subseteq S$ contains at most $n$ elements altogether.

For $k = 1, \ldots, \lceil \log n \rceil$, the diameter of each $C_i \in S_k$ is bounded above by

$$\text{diam}(C_i) < \text{per}(C_i)/2 \leq \text{per}(D)/2^k. \quad (1)$$

Consequently, every point of a body $C_i \in S_k$ lies at distance at most $\text{per}(D)/2^k$ from the boundary of $D$, denoted $\partial D$. Let $R_k$ be the set of points in $D$ at distance at most $\text{per}(D)/2^k$ from $\partial D$. Then

$$\text{area}(R_k) \leq \text{per}(D) \frac{\text{per}(D)}{2^k} = \frac{(\text{per}(D))^2}{2^k}. \quad (2)$$
Since $S$ consists of homothets, the area of any element $C_i \in S_k$ is bounded from below by

$$\text{area}(C_i) = \text{area}(C) \left( \frac{\text{per}(C_i)}{\text{per}(C)} \right)^2 \geq \text{area}(C) \left( \frac{\text{per}(D)}{2^k \text{per}(C)} \right)^2.$$ (3)

By a volume argument, [2] and [3] yield

$$|S_k| \leq \frac{\text{area}(R_k)}{\min_{C_i \in S_k} \text{area}(C_i)} \leq \frac{(\text{per}(D))^2 / 2^k}{\text{area}(C)(\text{per}(D))^2 / (2^k \text{per}(C))^2} = \frac{(\text{per}(D))^2}{\text{area}(C)} \cdot 2^k.$$

Since for $C_i \in S_k$, $k = 1, \ldots, \lfloor \log n \rfloor$, we have $\text{per}(C_i) \leq \text{per}(D)/2^{k-1}$, it follows that

$$\text{per}(S_k) \leq |S_k| \cdot \frac{(\text{per}(D))^2}{2^{k-1}} \leq 2 \frac{(\text{per}(C))^2}{\text{area}(C)} \text{per}(D).$$

Hence the sum of perimeters of all elements in $S$ is bounded by

$$\text{per}(S) = \sum_{k=0}^{\lfloor \log n \rfloor} \text{per}(S_k) \leq \left( 1 + 2 \frac{(\text{per}(C))^2}{\text{area}(C)} \lfloor \log n \rfloor \right) \text{per}(D),$$

as required. \hfill \square

**Proof of Proposition 3.** Let $\rho'(C)$ denote the ratio between $\text{per}(C)$ and the length of a shortest side of $C$. Recall that each $C_i \in S$ touches the boundary of polygon $D$. Since $D$ is parallel to $C$, the side of $D$ that supports $C_i$ must contain a side of $C_i$. Let $a_i$ denote the length of this side.

$$\text{per}(S) = \sum_{i=1}^{n} \text{per}(C_i) = \sum_{i=1}^{n} a_i \frac{\text{per}(C_i)}{a_i} \leq \rho'(C) \sum_{i=1}^{n} a_i \leq \rho'(C) \text{per}(D).$$

Set now $\rho(C, D) := \rho'(C) \text{per}(D)$, and the proof is complete. \hfill \square

**Proof of Proposition 4.** The proof is similar to that of Proposition 2 with a few adjustments. Let $S = \{C_1, \ldots, C_n\}$ be a packing of $n$ homothets of $C$ in $D$. Note that $\text{per}(C_i) \leq \text{per}(D)$ for all $i = 1, \ldots, n$. Partition the elements of $S$ into subsets as follows. Let

$$S^\text{in} = \{C_i \in S : \text{per}(C_i) \leq \text{esc}(C_i)\} \text{ and } S^\text{bd} = S \setminus S^\text{in}.$$ 

For $k = 1, \ldots, \lfloor \log n \rfloor$, let $S_k$ denote the set of homothets $C_i \in S^\text{bd}$ such that $\text{per}(D)/2^k \leq \text{per}(C_i) \leq \text{per}(D)/2^{k-1}$; and let $S_0$ be the set of homothets $C_i \in S^\text{bd}$ of perimeter at most $\text{per}(D)/2^\lfloor \log n \rfloor$.

The sum of perimeters of the elements in $S^\text{in}$ is $\text{per}(S^\text{in}) \leq \text{esc}(S^\text{in}) \leq \text{esc}(S)$. We next consider the elements in $S^\text{bd}$. The sum of perimeters of the elements in $S_0$ is $\text{per}(S_0) \leq n \text{per}(D)/2^\lfloor \log n \rfloor \leq \text{per}(D)$, since $S_0 \subseteq S$ contains at most $n$ elements altogether.

For $k = 1, \ldots, \lfloor \log n \rfloor$, the diameter of each $C_i \in S_k$ is bounded above by $\text{diam}(C_i) < \text{per}(C_i)/2 \leq \text{per}(D)/2^k$. Observe that every point of a body $C_i \in S_k$ lies at distance at most $\text{esc}(C_i) + \text{diam}(C_i) \leq \text{per}(C_i) + \text{diam}(C_i) \leq 1.5 \text{per}(C_i) \leq 3 \text{per}(D)/2^k$ from $\partial D$. Let now $R_k$ be the set of points in $D$ at distance at most $3 \text{per}(D)/2^k$ from $\partial D$. Then

$$\text{area}(R_k) \leq \text{per}(D) \frac{3 \text{per}(D)}{2^k} = \frac{3 (\text{per}(D))^2}{2^k}.$$
Analogously to the proof of Proposition 2, a volume argument yields
\[ |S_k| \leq 3 \frac{(\text{per}(C))^2}{\text{area}(C)} \cdot 2^k. \]

It follows that
\[ \text{per}(S_k) \leq |S_k| \cdot \frac{\text{per}(D)}{2^k-1} \leq 6 \frac{(\text{per}(C))^2}{\text{area}(C)} \text{ per}(D). \]

Hence the sum of perimeters of all elements in \( S \) is bounded by
\[ \text{per}(S) \leq \text{esc}(S) + \left( 1 + 6 \frac{(\text{per}(C))^2}{\text{area}(C)} \left\lceil \log n \right\rceil \right) \text{ per}(D), \]
as required.

\[ \square \]

3 Disks Touching the Boundary of a Square: Proof of Theorem 1

Let \( S \) be a set of \( n \) interior-disjoint disks in the unit square \( U = [0,1]^2 \) that touch the boundary of \( U \). From Proposition 2, we deduce the upper bound \( \text{per}(S) = O(\log n) \), as required. To prove the matching lower bound, it remains to construct a packing of \( O(n) \) disks in the unit square \( U \) such that every disk touches the \( x \)-axis, and the sum of their diameters is \( \Omega(\log n) \). We present two constructions attaining this bound: (i) an explicit construction in Subsection 3.1 which will be generalized in Section 4 and (ii) a greedy disk packing.

3.1 An Explicit Construction

For convenience, we use the unit square \([-\frac{1}{2}, \frac{1}{2}] \times [0,1]\) for our construction. To each disk we associate its vertical projection interval (on the \( x \)-axis). The algorithm greedily chooses disks of monotonically decreasing radii such that (1) every diameter is \( 1/16^k \) for some \( k \in \mathbb{N} \); and (2) if the projection intervals of two disks overlap, then one interval contains the other.

For \( k = 0, 1, \ldots, \lfloor \log_{16} n \rfloor \), denote by \( S_k \) the set of disks of diameter \( 1/16^k \), constructed by our algorithm. We recursively allocate a set of intervals \( X_k \subseteq [-\frac{1}{2}, \frac{1}{2}] \) to \( S_k \), and then choose disks in \( S_k \) such that their projection intervals lie in \( X_k \). Initially, \( X_0 = [-\frac{1}{2}, \frac{1}{2}] \), and \( S_0 \) contains the disk of diameter 1 inscribed in \([-\frac{1}{2}, \frac{1}{2}] \times [0,1] \). The length of each maximal interval \( I \subseteq X_k \) will be a multiple of \( 1/16^k \), so \( I \) can be covered by projection intervals of interior-disjoint disks of diameter \( 1/16^k \) touching the \( x \)-axis. Every interval \( I \subseteq X_k \) will have the property that any disk of diameter \( 1/16^k \) whose projection interval is in \( I \) is disjoint from any (larger) disk in \( S_j \), \( j < k \).

Consider the disk \( Q \) of diameter 1, centered at \((0, \frac{1}{2})\), and tangent to the \( x \)-axis (see Fig. 3). It can be easily verified that:

(i) the locus of centers of disks tangent to both \( Q \) and the \( x \)-axis is the parabola \( y = \frac{1}{2}x^2 \); and
(ii) any disk of diameter \( 1/16 \) and tangent to the \( x \)-axis whose projection interval is in \( I_1(Q) = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}] \) is disjoint from \( Q \).

Indeed, the center of any such disk is \((x_1, \frac{1}{10})\), for \( x_1 \leq -\frac{5}{16} \) or \( x_1 \geq \frac{5}{16} \), and hence lies below the parabola \( y = \frac{1}{2}x^2 \). Similarly, for all \( k \in \mathbb{N} \), any disk of diameter \( 1/16^k \) and tangent to the \( x \)-axis whose projection interval is in \( I_k(Q) = [-\frac{1}{2^k}, -\frac{1}{2^{k+1}}] \cup [\frac{1}{2^{k+1}}, \frac{1}{2^k}] \) is disjoint from \( Q \). For an arbitrary disk \( D \) tangent to the \( x \)-axis, and an integer \( k \geq 1 \), denote by \( I_k(D) \subseteq [-\frac{1}{2}, \frac{1}{2}] \) the pair of intervals corresponding to \( I_k(Q) \); for \( k = 0 \), \( I_k(D) \) consists of only one interval.
projection intervals are contained in \( S_0 \) selected disks in the unit square \([0, 1] \times [0, 1]\). Assume that we have already defined the intervals in \( X_{k-1} \), and selected disks in \( S_{k-1} \). Let \( X_k \) be the union of the interval pairs \( I_{k-j}(D) \) for all \( D \in S_j \) and \( j = 0, 1, \ldots, k-1 \). Place the maximum number of disks of diameter \( 1/16^k \) into \( S_k \) such that their projection intervals are contained in \( X_k \). For a disk \( D \in S_j \) \( (j = 0, 1, \ldots, k-1) \) of diameter \( 1/16^j \), the two intervals in \( X_{k-j} \) each have length \( \frac{1}{2} \cdot \frac{1}{16^j} = \frac{8^{k-j}}{2} \cdot \frac{1}{16^j} \), so they can each accommodate the projection intervals of \( \frac{8^{k-j}}{2} \) disks in \( S_k \).

We prove by induction on \( k \) that the length of \( X_k \) is \( \frac{1}{2} \), and so the sum of the diameters of the disks in \( S_k \) is \( \frac{1}{2} \), \( k = 1, 2, \ldots, \lceil \log_{16} n \rceil \). The interval \( X_0 = [-\frac{1}{2}, \frac{1}{2}] \) has length 1. The pair of intervals \( X_1 = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}] \) has length \( \frac{1}{2} \). For \( k = 2, \ldots, \lceil \log_{16} n \rceil \), the set \( X_k \) consists of two types of (disjoint) intervals: (a) The pair of intervals \( I_1(D) \) for every \( D \in S_{k-1} \) covers half of the projection interval of \( D \). Over all \( D \in S_{k-1} \), they jointly cover half the length of \( X_{k-1} \). (b) Each pair of intervals \( I_{k-j}(D) \) for \( D \in S_{k-j} \), \( j = 0, 1, \ldots, k-2 \), has half the length of \( I_{k-j-1}(D) \). So the sum of the lengths of these intervals is half the length of \( X_{k-1} \); although they are disjoint from \( X_{k-1} \). Altogether, the sum of lengths of all intervals in \( X_k \) is the same as the length of \( X_{k-1} \). By induction, the length of \( X_{k-1} \) is \( \frac{1}{2} \), hence the length of \( X_k \) is also \( \frac{1}{2} \), as claimed. This immediately implies that the sum of diameters of the disks in \( \bigcup_{j=0}^{\lceil \log_{16} n \rceil} S_k \) is \( 1 + \frac{1}{2} \lceil \log_{16} n \rceil \). Finally, one can verify that the total number of disks used is \( O(n) \). Write \( K = \lceil \log_{16} n \rceil \). Indeed, \( |S_0| = 1 \), and \( |S_k| = |X_k|/16^{-k} = 16^k/2 \), for \( k = 1, \ldots, K \), where \( |X_k| \) denotes the total length of the intervals in \( X_k \). Consequently, \( |S_0| + \sum_{k=1}^{K} |S_k| = O(16^K) = O(n) \), as required.

3.2 A Greedy Disk Packing

The following simple greedy algorithm produces a packing \( S_n \) of \( n \) disks in the unit square \( U = [0, 1]^2 \) with all disks touching the boundary of \( U \) and whose total perimeter is \( \Omega(\log n) \). For \( i = 1 \) to \( n \), let \( C_i \) be a disk of maximum radius that lies in \( U \setminus \bigcup_{j<i} C_j \) and intersects \( \partial U \), and let \( S_n = \{ C_1, \ldots, C_n \} \); refer to Fig. 4 (left). The radius of \( C_1 \) is \( 1/2 \), the radii of \( C_2, \ldots, C_5 \) are \( 3-2\sqrt{2} \), etc. We use Apollonian circle packings \cite{12} to derive the lower bound \( \text{per}(S_n) = \Omega(\log n) \).

We now consider a greedy algorithm in a slightly different setting, applicable to Apollonian circles. For \( r_1, r_2 > 0 \), we construct a set \( F_n(r_1, r_2) \) of \( n \) disks by the following greedy algorithm. Let \( A_1 \) and \( A_2 \) be two tangent disks of radii \( r_1 \) and \( r_2 \) that are also tangent to the \( x \)-axis from
Recall that the first two disks in Proposition 5.

Figure 4: Left: A greedy packing of $n = 7$ disks in $[0, 1]^2$. Right: Ford disks visible in the window $[0, 1]^2$.

**Proposition 5.** $\text{per}(S_n) \geq \text{per}(F_n(1/2, 3 - 2\sqrt{2}))$.

*Proof.* Recall that the first two disks in $S_n$ have radii $1/2$ and $3 - 2\sqrt{2}$. Let $I$ be the line segment between the tangency points of $A_1$ and $A_2$ with the bottom side of $[0, 1]^2$. Because of the greedy strategy, all disks in $S_n$ that intersect segment $I$ are in $F_n(1/2, 3 - 2\sqrt{2})$. The radius of every disk in $S_n \setminus F_n(1/2, 3 - 2\sqrt{2})$ is at least as large as any disk in $F_n(1/2, 3 - 2\sqrt{2}) \setminus S_n$. Therefore, there is a one-to-one correspondence between $S_n$ and $F_n(1/2, 3 - 2\sqrt{2})$ such that each disk in $S_n$ corresponds to a disk of the same or smaller radius in $F_n(1/2, 3 - 2\sqrt{2})$.

Given two tangent disks of radii $r_1$ and $r_2$ that are also tangent to the $x$-axis, there is a unique disk tangent to both these disks and the $x$-axis, and its radius $r_3$ satisfies $r_3^{-1/2} = r_1^{-1/2} + r_2^{-1/2}$. Observe that $r_3 = r_3(r_1, r_2)$ is a continuous and monotonically increasing function of both variables, $r_1$ and $r_2$. Therefore, if $r_1 \leq r_1'$ and $r_2 \leq r_2'$, then

$$\text{per}(F_n(r_1, r_2)) \leq \text{per}(F_n(r_1', r_2')).$$  \hfill (4)

This observation allows us to bound $\text{per}(S_n)$ from below by the perimeter of a finite subfamily of Ford disks.

The Ford disks $\square_{p,q}$ are a packing of an infinite set of disks in the halfplane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, where each disk is tangent to the $x$-axis from above. Every pair $(p, q) \in \mathbb{N}^2$ of relative prime positive integers defines a Ford circle $C_{p,q}$ of radius $1/(2q^2)$ centered at $(p/q, 1/(2q^2))$; see Fig. 4 (right). The Ford disks $C_{p,1}$ have the largest radius $1/2$; all other Ford disks have smaller radii and each is tangent to two larger Ford disks. Hence, the set of the $n$ largest Ford disks that touch the unit segment $[0, 1]$ is exactly $F_n(1/2, 1/2)$.

**Proposition 6.** $\text{per}(F_n(1/2, 1/2)) = \Omega(\log n)$.

*Proof.* For a positive integer $Q$, the number of Ford disks of radius at least $1/(2Q^2)$ touching the unit segment $[0, 1]$ is $1 + \sum_{q=1}^{Q} \varphi(q)$, where $\varphi(.)$ is Euler’s totient function, i.e., the number positive
integers less than or equal to \( q \) that are relatively prime to \( q \). It is known [1, Theorem 3.7, p. 62] that
\[
\sum_{q=1}^{Q} \varphi(q) = \frac{3}{\pi^2} Q^2 + O(Q \log Q).
\]
Hence, for a suitably large \( Q = \Theta(\sqrt{n}) \), there exists exactly \( n \) Ford disks of radius at least \( \frac{1}{2Q^2} \) that touch \([0,1]\). Let \( F_n(1/2, 1/2) \) be the subset of these \( n \) Ford disks. Then we have
\[
\text{per}(F_n) = \sum_{q=1}^{Q} \varphi(q) \cdot 2\pi \cdot \frac{1}{2q^2} = \pi \sum_{q=1}^{Q} \frac{\varphi(q)}{q^2}.
\]
It is also known [1, Exercise 6, p. 71] that
\[
\sum_{q=1}^{Q} \frac{\varphi(q)}{q^2} = \frac{6}{\pi^2} \ln Q + O\left(\frac{\log Q}{Q}\right).
\]
Using this estimate, we have
\[
\text{per}(F_n) = \pi \left( \sum_{q=1}^{Q} \frac{\varphi(q)}{q^2} \right) = \Omega(\log Q) = \Omega(\log \sqrt{n}) = \Omega(\log n),
\]
as claimed.

The bounds in Propositions [5,6] in conjunction with (4) yield
\[
\text{per}(S_n) \geq \text{per}(F_n(1/2, 3 - 2\sqrt{2})) \geq \text{per}(F_n(3 - 2\sqrt{2}, 3 - 2\sqrt{2}))
\]
\[
= \Omega(\text{per}(F_n(1/2, 1/2))) = \Omega(\log n).
\]

When \( C \) is a disk and the container \( D \) is any other convex body, the above argument goes through and shows that a greedy packing \( S_n \) has total perimeter \( \text{per}(S) = \Omega(\log n) \), where the constant of proportionality depends on \( D \). However, when \( C \) is not a circular disk, the theory of Apollonian circles does not apply.

### 4 Homothets Touching the Boundary: Proof of Theorem [2]

The upper bound \( \text{per}(S) = O(\log n) \) follows from Proposition [2]. It remains to construct a packing \( S \) of perimeter \( \text{per}(S) = \Omega(\log n) \) for given \( C \) and \( D \). Let \( C \) and \( D \) be two convex bodies with bounded description complexity. We wish to argue analogously to the case of disks in a square. Therefore, we choose an arc \( \gamma \subset \partial D \) that is smooth and sufficiently “flat,” but contains no side parallel to a corresponding side of \( C \). Then we build a hierarchy of homothets of \( C \) touching the arc \( \gamma \), so that the depth of the hierarchy is \( O(\log n) \), and the homothety factors decrease by a constant between two consecutive levels.

We choose an arc \( \gamma \subset \partial D \) as follows. If \( D \) has a side with some direction \( \mathbf{d} \in \mathbb{S} \) such that \( C \) has no parallel side of the same direction \( \mathbf{d} \), then let \( \gamma \) be this side of \( D \). Otherwise, \( \partial D \) contains an algebraic curve \( \gamma_1 \) of degree 2 or higher. Let \( q \in \gamma_1 \) be an interior point of this curve such that \( \gamma_1 \) is twice differentiable at \( q \). Assume, after a rigid transformation of \( D \) if necessary, that \( q = (0,0) \) is the origin and the supporting line of \( D \) at \( q \) is the \( x \)-axis. By the inverse function theorem, there is an arc \( \gamma_2 \subseteq \gamma_1 \), containing \( q \), such that \( \gamma_2 \) is the graph of a twice differentiable function of \( x \).
Then the tangent line of $D$ no segments (sides).

Finally, let $I$ be an interval in $\mathbb{R}$. Let $Q$ and every homothety factor $\rho > 0$, such that for every point $p \in I$ and every homothety factor $\rho$, $0 < \rho < \rho_0$, the polynomials

$$\alpha_p(x) = A|x - x_p|^\kappa + s_p(x - x_p), \quad \text{and} \quad \beta_p(x) = B|x - x_p|^\kappa + s_p(x - x_p),$$

separate $\gamma$ from the convex body $Q_p = (\rho C)_p$.

Similarly to the proof of Theorem 4, the construction is guided by nested projection intervals. Let $Q = (\rho C)_p$ be a homothet of $C$ that lies in $D$ and is tangent to $\gamma$ at point $p \in \gamma$. Denote by $I(Q)$ the vertical projection of $Q$ to the $x$-axis. For $k = 1, \ldots$, we recursively define disjoint intervals or interval pairs $I_k(Q) \subset I(Q)$ of length $|I_k(Q)| = |I(Q)|/2^k$. During the recursion, we maintain the invariant that the set $J_k(Q) = I(Q) \setminus \bigcup_{j<k} I_j(Q)$ is an interval of length $|I(Q)|/2^{k-1}$ that contains $x_p$. Assume that $I_1(Q), \ldots, I_{k-1}(Q)$ have been defined, and we need to choose $I_k(Q) \subset J_k(Q)$. If $x_p$ lies in the central one quarter of $J_k(Q)$, then let $I_k(Q)$ be a pair of intervals that consists of the left and right quarters of $J_k(Q)$. If $x_p$ lies to the left (right) of the central one quarter of $J_k(Q)$, then let $I_k(Q)$ be the right (left) half of $J_k(Q)$. It is now an easy matter to check (by induction on $k$) that $|x - x_p| \geq |I(Q)|/8^k$ for all $x \in I_k(Q)$. Consequently,

$$\beta_p(x) - \alpha_p(x) \geq (B - A) \cdot \left(\frac{|I(Q)|}{8^k}\right)^\kappa$$

for all $x \in I_k(Q)$. There is a constant $\mu > 0$ such that a homothet $\mu k Q$ with arbitrary projection interval in $I_k(Q)$ fits between the curves $\alpha_p$ and $\beta_p$. Refer to Fig. 5. Therefore we can populate the region between curves $\alpha_p$ and $\beta_p$ and above $I_k(Q)$ with homothets $\rho Q$, of homothety factors $\mu^k/2 < \rho \leq \mu^k$, such that their projection intervals are pairwise disjoint and cover $I_k(Q)$. By translating these homothets vertically until they touch $\gamma$, they remain disjoint from $Q$ and preserve their projection intervals. We can now repeat the construction of the previous section and obtain $\lceil \log_{2/\mu^k} n \rceil$ layers of homothets touching $\gamma$, such that the total length of the projections of the homothets in each layer is $\Theta(1)$. Consequently, the total perimeter of the homothets in each layer is $\Theta(1)$, and the overall perimeter of the packing is $\Theta(\log n)$, as required. \qed
5 Bounds in Term of the Escape Distance: Proof of Theorem 3

Upper bound. Let $S = \{C_1, \ldots, C_n\}$ be a packing of $n$ homothets of a convex body $C$ in a container $D$ such that $D$ is a convex polygon parallel to $C$. For each element $C_i \in S$, $\text{esc}(C_i)$ is the distance between a side of $D$ and a corresponding side of $C_i$. For each side $a$ of $D$, let $S_a \subseteq S$ denote the set of $C_i \in S$ for which $a$ is the closest side of $D$ (ties are broken arbitrarily). Since $D$ has finitely many sides, it is enough to show that for each side $a$ of $D$, we have

$$\text{per}(S_a) \leq \rho_a(C, D) (\text{per}(D) + \text{esc}(S)) \frac{\log |S_a|}{\log \log |S_a|},$$

where $\rho_a(C, D)$ depends on $a$, $C$ and $D$ only.

Suppose that $S_a = \{C_1, \ldots, C_n\}$ is a packing of $n$ homothets of $C$ such that $\text{esc}(C_i)$ equals the distance between $C_i$ and side $a$ of $D$. Assume for convenience that $a$ is horizontal. Let $c \subseteq \partial C$ be the side of $C$ corresponding to the side $a$ of $D$. Let $\rho_1 = \text{per}(C)/|c|$, and then we can write $\text{per}(C) = \rho_1|c|$. Refer to Fig. 6 (left).

Denote by $b \subset c$ the line segment of length $|b| = |c|/2$ with the same midpoint as $c$. Since $C$ is a convex body, the two vertical lines though the two endpoints of $b$ intersect $C$ in two line segments denoted $h_1$ and $h_2$, respectively. Let $\rho_2 = \min(|h_1|, |h_2|)/|b|$, and then $\min(|h_1|, |h_2|) = \rho_2|b|$. By convexity, every vertical line that intersects segment $b$ intersects $C$ in a vertical segment of length at least $\rho_2|b|$. Note that $\rho_1$ and $\rho_2$ are constants depending on $C$ and $D$. For each homothet $C_i \in S_a$, let $b_i \subset \partial C_i$ be the homothetic copy of segment $b \subset \partial C$.

![Diagram](image)

Figure 6: Left: A convex body $C$ with a horizontal side $c$. The segment $b \subset c$ has length $|b| = |c|/2$, and the vertical segments $h_1$ and $h_2$ are incident to the endpoints of $b$. Right: Two homothets, $C_i$ and $C_j$, in a convex container $D$. The vertical projections of $b_i$ and $b_j$ onto the horizontal side $a$ are $\text{proj}_i$ and $\text{proj}_j$.

Put $\lambda = 2[\log n/\log \log n]$. Partition $S_a$ into two subsets $S_a = S_{\text{far}} \cup S_{\text{close}}$ as follows. For each $C_i \in S_a$, let $C_i \in S_{\text{close}}$ if $\text{esc}(C_i) < \rho_2|b_i|/\lambda$, and $C_i \in S_{\text{far}}$ otherwise. For each homothet $C_i \in S_{\text{close}}$, let $\text{proj}_i \subseteq a$ denote the vertical projection of segment $b_i$ onto the horizontal side $a$ (refer to Fig. 6 right). The perimeter of each $C_i \in S_a$ is $\text{per}(C_i) = \rho_1|c_i| = 2\rho_1|b_i| = 2\rho_1|\text{proj}_i|$. We have

$$\text{per}(S_{\text{far}}) = \sum_{C_i \in S_{\text{far}}} \text{per}(C_i) = \sum_{C_i \in S_{\text{far}}} 2\rho_1|b_i| \leq \sum_{C_i \in S_{\text{far}}} 2\rho_1 \frac{\text{esc}(C_i) \lambda}{\rho_2} \leq 2\rho_1 \frac{\text{esc}(S)}{\rho_2} \lambda. \quad (6)$$

It remains the estimate $\text{per}(S_{\text{close}})$ as an expression of $\lambda$.

$$\sum_{C_i \in S_{\text{close}}} \text{per}(C_i) = 2\rho_1 \sum_{C_i \in S_{\text{close}}} |\text{proj}_i|. \quad (7)$$

Define the depth function for every point of the horizontal side $a$ by

$$d : a \to \mathbb{N}, \quad d(x) = |\{C_i \in S_{\text{close}} : x \in \text{proj}_i\}|.$$
That is, \( d(x) \) is the number of homothets such that the vertical projection of segment \( b_i \) contains point \( x \). For every positive integer \( k \in \mathbb{N} \), let
\[
I_k = \{ x \in a : d(x) \geq k \},
\]
that is, \( I_k \) is the set of points of depth at least \( k \). Since \( S_{\text{close}} \) is finite, the set \( I_k \subseteq a \) is measurable. Denote by \( |I_k| \) the measure (total length) of \( I_k \). By definition, we have \(|a| \geq |I_1| \geq |I_2| \geq \ldots \). A standard double counting for the integral \( \int_{x \in a} d(x) \, dx \) yields
\[
\sum_{C_i \in S_{\text{close}}} |\text{proj}_i| = \sum_{k=1}^{\infty} |I_k|. \quad (8)
\]

If \( d(x) = k \) for some point \( x \in a \), then \( k \) segments \( b_i \), lie above \( x \). Each \( C_i \in S_{\text{close}} \) is at distance \( \text{esc}(C_i) < \rho_2|b_i|/\lambda \) from \( a \). Suppose that \( \text{proj}_i \) and \( \text{proj}_j \) intersect for \( C_i, C_j \in S_{\text{close}} \) (Fig. 6, right). Then one of them has to be closer to \( a \) than the other: we may assume w.l.o.g. \( \text{esc}(C_j) < \text{esc}(C_i) \). Now a vertical segment between \( b_i \subset C_i \) and \( \text{proj}_i \subset a \) intersects \( b_j \). The length of this intersection segment satisfies \( \rho_2|b_j| \leq \text{esc}(C_i) < \rho_2|b_i|/\lambda \). Consequently, \( |b_j| < |b_i|/\lambda \) (or, equivalently, \( |\text{proj}_j| < |\text{proj}_i|/\lambda \)) holds for any consecutive homothets above point \( x \in a \). In particular, for the \( k \)-th smallest projection containing \( x \in a \), we have \( |\text{proj}_k| \leq |a|/\lambda^{k-1} = |a|\lambda^{-k} \).

We claim that
\[
|I_k| \leq |a|\lambda^{-k} \quad \text{for } k \geq \lambda + 1. \quad (9)
\]
Suppose, to the contrary, that \( |I_k| > |a|\lambda^{-k} \) for some \( k \geq \lambda + 1 \). Then there are homothets \( C_i \in S_{\text{close}} \) of side lengths at most \(|a|/\lambda^{k-1} \), that jointly project into \( I_k \). Assuming that \( |I_k| > |a|\lambda^{-k} \), it follows that the number of these homothets is at least
\[
\frac{|a|\lambda^{-k}}{|a|\lambda^{1-k}} = \lambda^{-1} = \left(2 \left\lceil \frac{\log n}{\log\log n} \right\rceil \right)^2 \left\lceil \frac{\log n}{\log\log n} \right\rceil^{-1} > n,
\]
contradicting the fact that \( S_{\text{close}} \subseteq S \) has at most \( n \) elements. Combining (7), (8), and (9), we conclude that
\[
\text{per}(S_{\text{close}}) = 2\rho_1 \sum_{k=1}^{\infty} |I_k| \leq 2\rho_1 \left( \lambda |I_1| + \sum_{k=\lambda+1}^{\infty} |I_k| \right) \leq 2\rho_1 \left( \lambda + \sum_{j=1}^{\infty} \frac{1}{\lambda^j} \right) |a|
\leq 2\rho_1 (\lambda + 1) \text{per}(D). \quad (10)
\]

Putting (6) and (10) together yields
\[
\text{per}(S_a) = \text{per}(S_{\text{close}}) + \text{per}(S_{\text{far}}) \leq 2\rho_1 \left( (\lambda + 1) \text{per}(D) + \frac{\text{esc}(S)}{\rho_2} \lambda \right)
\leq \rho(C, D) (\text{per}(D) + \text{esc}(S)) \lambda = \rho(C, D) (\text{per}(D) + \text{esc}(S)) \frac{\log n}{\log\log n},
\]
for a suitable \( \rho(C, D) \) depending on \( C \) and \( D \), as required; here we set \( \rho(C, D) = 2\rho_1 \max(2, 1/\rho_2) \).

**Lower bound for squares.** We first confirm the given lower bound for squares, i.e., we construct a packing \( S \) of \( O(n) \) axis-aligned squares in the unit square \( U = [0,1]^2 \) with total perimeter \( \Omega((\text{per}(U) + \text{esc}(S)) \log n/\log\log n) \).
Let $n \geq 4$, and put $\lambda = \lfloor \log n / \log \log n \rfloor / 2$. We arrange each square $C_i \in S$ such that per($C_i$) = $\lambda \text{esc}(C_i)$. We construct $S$ as the union of $\lambda$ subsets $S = \bigcup_{j=1}^{\lambda} S_j$, where $S_j$ is a set of congruent squares, at the same distance from the bottom side of $U$.

Let $S_1$ be a singleton set consisting of one square of side length $1/4$ (and perimeter 1) at distance $1/\lambda$ from the bottom side of $U$. Let $S_2$ be a set of $2\lambda$ squares of side length $1/(4 \cdot 2\lambda)$ (and perimeter $1/(2\lambda)$), each at distance $1/(2\lambda^2)$ from the bottom side of $U$. Note that these squares lie strictly below the first square in $S_1$, since $1/(8\lambda) + 1/(2\lambda^2) < 1/\lambda$. The total length of the vertical projections of the squares in $S_2$ is $2\lambda \cdot 1/(8\lambda) = 1/4$.

Similarly, for $j = 3, \ldots, \lambda$, let $S_j$ be a set of $(2\lambda)^{j-1}$ squares of side length $\frac{1}{4(2\lambda)^j}$ (and perimeter $1/(2\lambda)^j$), each at distance $1/(2^j\lambda^j)$ from the bottom side of $U$. These squares lie strictly below any square in $S_{j-1}$, and the total length of their vertical projections onto the $x$-axis is $(2\lambda)^{j-1} \cdot \frac{1}{4(2\lambda)^j} = 1/4$.

The number of squares in $S = \bigcup_{j=1}^{\lambda} S_j$ is

$$\sum_{j=1}^{\lambda} (2\lambda)^{j-1} = \Theta((2\lambda)^{\lambda}) = O(n).$$

The total distance from the squares to the boundary of $U$ is

$$\text{esc}(S) = \sum_{j=1}^{\lambda} (2\lambda)^{j-1} \cdot \frac{1}{2^j\lambda^j} = \lambda \frac{1}{\lambda} = 1.$$ 

The total perimeter of all squares in $S$ is

$$4 \cdot \sum_{j=1}^{\lambda} \frac{1}{4} = \lambda = \Omega\left(\frac{\log n}{\log \log n}\right) = \Omega\left((\text{per}(U) + \text{esc}(S)) \frac{\log n}{\log \log n}\right),$$

as required.

**General lower bound.** We now establish the lower bound in the general setting. Given a convex body $C$ and a convex polygon $D$ parallel to $C$, we construct a packing $S$ of $O(n)$ positive homothets of $C$ in $D$ with total perimeter $\Omega((\text{per}(D) + \text{esc}(S)) \log n / \log \log n)$.

Let $a$ be an arbitrary side of $D$. Assume w.l.o.g. that $a$ is horizontal. Let $U_C$ be the minimum axis-aligned square containing $C$. Clearly, we have $\frac{1}{2} \text{per}(U_C) \leq \text{per}(C) \leq \text{per}(U_C)$. We first construct a packing $S_U$ of $O(n)$ axis-aligned squares in $D$ such that for each square $U_i \in S_U$, esc($U_i$) equals the distance from the horizontal side $a$. We then obtain the packing $S$ by inscribing a homothet $C_i$ of $C$ in each square $U_i \in S_U$ such that $C_i$ touches the bottom side of $U_i$. Consequently, we have $\text{per}(S) \geq \text{per}(S_U)/2$ and $\text{esc}(S) = \text{esc}(S_U)$, since esc($C_i$) = esc($U_i$) for each square $U_i \in S_U$.

It remains to construct the square packing $S_U$. Let $U(a)$ be a maximal axis-aligned square contained in $D$ such that its bottom side is contained in $a$. $S_U$ is a packing of squares in $U(a)$ that is homothetic with the packing of squares in the unit square $U$ described previously. Put $\rho_1 = \text{per}(U(a))/\text{per}(U) = \text{per}(U(a))/4$. We have $\text{per}(S) \geq \frac{1}{4} \rho_1 \Omega\left((\text{per}(U) + \text{esc}(S)) \frac{\log n}{\log \log n}\right)$, or

$$\text{per}(S) \geq \rho(C, D) \left((\text{per}(D) + \text{esc}(S)) \frac{\log n}{\log \log n}\right),$$

where $\rho(C, D)$ is a factor depending on $C$ and $D$, as required. \(\square\)
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