THE INTEGRAL MONODROMY OF ISOLATED QUASIHOMOGENEOUS SINGULARITIES

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Abstract. The integral monodromy on the Milnor lattice of an isolated quasihomogeneous singularity is subject of an almost untouched conjecture of Orlik from 1972. We prove this conjecture for all iterated Thom-Sebastiani sums of chain type singularities and cycle type singularities. The main part of the paper is purely algebraic. It provides tools for dealing with sums and tensor products of \(\mathbb{Z}\)-lattices with automorphisms of finite order and with cyclic generators. The calculations are involved. They use fine properties of unit roots, cyclotomic polynomials, their resultants and discriminants.

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1. Introduction and main results

Matrices in $GL(n, \mathbb{Z})$ arise in algebraic geometry as monodromy matrices. Usually it is not so difficult to control their eigenvalues and Jordan blocks, so their conjugacy classes with respect to $GL(n, \mathbb{C})$, but very difficult to understand their conjugacy classes with respect to conjugation by $GL(n, \mathbb{Z})$.

This paper gives some general algebraic tools which deal with block diagonal matrices whose blocks are companion matrices. And it shows their usefulness and applies them in a special situation in algebraic geometry, namely in the case of integral monodromy matrices of isolated quasihomogeneous singularities. We prove an old and beautiful, but almost untouched conjecture of Orlik [Or72, Conjecture 3.1] in a number of cases. They contain all invertible polynomials, so all iterated Thom-Sebastiani sums of chain type singularities and cycle type singularities.

We start with some notions. Then we formulate Orlik’s conjecture, and then the results for quasihomogeneous singularities. The algebraic tools are described only informally in the introduction. Instead of conjugacy classes of matrices, we work now with $\mathbb{Z}$-lattices with endomorphisms.

**Definition 1.1.** Let $H$ be a $\mathbb{Z}$-lattice of rank $n \in \mathbb{N}$, and let $h : H \to H$ be an endomorphism. The characteristic polynomial of $h$ is called $p_{H,h}$.

(a) The pair $(H, h)$ is a *companion block* if an element $a_0 \in H$ exists such that

$$H = \bigoplus_{j=0}^{n-1} \mathbb{Z} \cdot h^j(a_0).$$

Such an element $a_0$ is called a *generating element*.

(b) A sublattice $H^{(1)} \subset H$ is a *companion block* if it is $h$-invariant and the pair $(H^{(1)}, h)$ is a companion block (here and below, we write $h$ instead of $h|_{H^{(1)}}$).

(c) A *decomposition of $H$ into companion blocks* is a decomposition $H = \bigoplus_{i=1}^k H^{(i)}$ such that each $H^{(i)}$ is a companion block.
(d) A decomposition as in (c) is a standard decomposition into companion blocks if

\[ P_{H^{(k)}h} | P_{H^{(k-1)}h} | \cdots | P_{H^{(2)}h} | P_{H^{(1)}h}. \]  

(1.2)

(e) A companion block \((H, h)\) is called an Orlik block if \(h\) is an automorphism of finite order. Specializing (c) and (d), one obtains the notions decomposition into Orlik blocks and standard decomposition into Orlik blocks.

In (1.2), the tuple \((p_{H^{(1)}h}, \ldots, p_{H^{(k)}h})\) of characteristic polynomials is unique, see Remark 2.5 (iv).

A polynomial \(f \in \mathbb{C}[x_1, \ldots, x_n]\) is called quasihomogeneous if for some weight system \((w_1, \ldots, w_n)\) with \(w_i \in (0, 1) \cap \mathbb{Q}\) each monomial in \(f\) has weighted degree 1. It is called an isolated quasihomogeneous singularity if it is quasihomogeneous and the functions \(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\) vanish simultaneously only at \(0 \in \mathbb{C}^n\). Then the Milnor lattice \(H_{Mil} := H^{(n-1)}_{n,1}(f^{-1}(1), \mathbb{Z})\) (here \(H^{(n-1)}_{n,1}\) means the reduced homology in the case \(n = 1\) and the usual homology in the cases \(n \geq 2\)) is a \(\mathbb{Z}\)-lattice of some finite rank \(\mu \in \mathbb{N}\), which is called the Milnor number \([Mi68]\). It comes equipped with an automorphism \(h_{Mil}\) of finite order, the monodromy.

Orlik conjectured the following.

**Conjecture 1.2.** (Orlik’s conjecture \([Or72, Conjecture 3.1]\)) For any isolated quasihomogeneous singularity, the pair \((H_{Mil}, h_{Mil})\) admits a standard decomposition into Orlik blocks.

Conjecture 1.2 should not be confused with the weaker Conjecture 3.2 in \([Or72]\), which deals with the homology of the link \(f^{-1}(1) \cap S^{2n-1}\) of the singularity and which would follow from Conjecture 1.2. Section 13 discusses Conjecture 3.2 in \([Or72]\) and applications of both conjectures.

As an application of our algebraic results, we prove Conjecture 1.2 in the following cases. They surpass all known cases.

**Theorem 1.3.** (a) Conjecture 1.2 holds for the chain type singularities.

(b) \([HM20-1]\) Conjecture 1.2 holds for the cycle type singularities.

(c) If Conjecture 1.2 holds for a singularity \(f\) and a singularity \(g\), then Conjecture 1.2 holds also for the Thom-Sebastiani sum \(f + g = f(x_1, \ldots, x_{n_f}) + g(x_{n_f+1}, \ldots, x_{n_f+n_g})\).

(d) Conjecture 1.2 holds for all iterated Thom-Sebastiani sums of chain type singularities and cycle type singularities.
Part (a) follows from the combination of Theorem (2.11) in the paper [OR77] of Orlik and Randell, which we cite as Theorem 10.1 and our algebraic result Theorem 6.2. Theorem (2.11) in [OR77] gives for a chain type singularity an automorphism $h : H_{Mil} \to H_{Mil}$ such that $h_{Mil} = h^\mu$ and such that $(H_{Mil}, h)$ is a single Orlik block. The algebraic Theorem 6.2 starts with a single Orlik block $(H, h)$ and a number $\mu \in \mathbb{N}$ and gives a sufficient (and probably also necessary) condition for $(H, h^\mu)$ to admit a standard decomposition into Orlik blocks.

Part (b) follows from [HM20-1]. It builds on the paper [Co82] of Cooper, who worked on the conjecture, but made two serious mistakes, see section 11.

Part (c) follows from the basic result

$$(H_{Mil}, h_{Mil})(f + g) \cong (H_{Mil}, h_{Mil})(f) \otimes (H_{Mil}, h_{Mil})(g)$$  \hspace{1cm} (1.3)$$

of Sebastiani and Thom [ST71], from our algebraic result Theorem 9.10 and from Theorem 12.1. Theorem 9.10 states conditions under which the tensor product of two standard decompositions into Orlik blocks admits again a standard decomposition into Orlik blocks. Theorem 12.1 verifies these conditions in the cases of the quasihomogeneous singularities.

Part (d) is an immediate consequence of the parts (a), (b) and (c). The case of curve singularities (the case $n = 1$) is contained in Theorem 1.3 (d). For this case Michel and Weber claimed in the introduction of [MW86] to have a proof of Conjecture 1.2. In [He92] a few other cases, which are also contained in Theorem 1.3 (d), were checked by hand (using Coxeter-Dynkin diagrams). So, part (d) surpasses all cases in which Conjecture 1.2 was known before.

Theorem 9.10 builds on other algebraic results in the sections 3, 5, 7, 8 and 9.

Theorem 3.1 allows to improve a certain $\mathbb{Z}$-basis of a pair $(H, h)$ to a $\mathbb{Z}$-basis of a standard decomposition into companion blocks. Its proof is short and elementary. Theorem 3.1 is itself some evidence that standard decompositions into companion blocks are natural and arise more often than one might expect at first sight. It is used in the sections 6 and 8.

Theorem 5.1 compares different decompositions into companion blocks. The proof of Theorem 5.1 uses results on resultants which are recalled in section 4. Section 3 also collects results on cyclotomic polynomials (and their resultants and discriminants) which are partly proved here and partly in [He20].

These results are used in the proofs of Theorem 6.2 and Theorem 7.4. Their proofs are long, especially the one of Theorem 7.4. It takes
the whole section 8. Theorem 7.4 starts with two single Orlik blocks \((H^{(1)}, h^{(1)})\) and \((H^{(2)}, h^{(2)})\) and gives a sufficient (and probably also necessary) condition for their tensor product \((H^{(1)} \otimes H^{(2)}, h^{(1)} \otimes h^{(2)})\) to admit a standard decomposition into Orlik blocks. The proofs of the Theorems 6.2 and 7.4 combine Theorem 3.1 with the calculation of a certain determinant. The determinant calculations deal a lot with unit roots, cyclotomic polynomials, their resultants and their discriminants.

Section 9 builds on Theorem 7.4. It provides a condition on one Orlik block such that for any two Orlik blocks which satisfy such conditions, their tensor product admits a standard decomposition into Orlik blocks, see Theorem 9.9. Theorem 9.10 builds on this.

In section 10, chain type singularities are introduced, and Theorem 1.3 (a) is proved. In section 11, cycle type singularities are introduced and remarks on [HM20-1] and [Co82] are made. In section 12 the conditions in Theorem 9.10 are verified in the case of quasihomogeneous singularities. Section 13 recalls the second conjecture 3.2 in [Or72], it tells about applications of both conjectures of Orlik, and it formulates an open problem.

2. SOME NOTATIONS AND BASIC OBSERVATIONS

First, we fix some basic general notations.

**Notations 2.1.** \(\mathbb{N} := \{1, 2, 3, \ldots\}\), \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). The subset of \(\mathbb{N}\) of all prime numbers is denoted by \(\mathcal{P} \subset \mathbb{N}\).

For any \(m \in \mathbb{N}\) and any prime number \(p\) denote

\[v_p(m) := \max( k \in \mathbb{N}_0 \mid p^k \text{ divides } m).\]

Thus \(m = \prod_{p \text{ prime number}} p^{v_p(m)}\).

For any subset \(I \subset \mathbb{R}\) denote (especially for an interval \([r_1, r_2] \subset \mathbb{R}\))

\[\mathbb{Z}_I := \mathbb{Z} \cap I.\]

For a case discussion, whether or not a certain condition \((\text{Cond})\) holds, the Kronecker delta is generalized as follows,

\[\delta_{(\text{Cond})} := \begin{cases} 1 & \text{if } (\text{Cond}) \text{ holds}, \\ 0 & \text{if } (\text{Cond}) \text{ does not hold}. \end{cases}\]

Beyond this section, some general notations are given in the notations and definitions 4.1 4.4 6.1 7.1 9.1 9.3 and 9.4. The next notations fix our way to deal with matrices and bases.

**Notation 2.2.** Let \(R\) be a principal ideal domain, and let \(V\) a free \(R\)-module of rank \(n \in \mathbb{N}\). Let \(a = (a_1, \ldots, a_n)\) be an ordered \(R\)-basis of \(V\), and let \(b = (b_1, \ldots, b_k) \in R^k\) for some \(k \in \mathbb{N}\). Let \(f : V \to V\)
be an $R$-linear endomorphism. Then $M(a, f, b)$ denotes the matrix which expresses the elements $f(b_1), \ldots, f(b_k)$ as linear combinations of $a_1, \ldots, a_n$, namely

$$M(a, f, b) = (r_{ij})_{i=1}^n \in M_{n \times k}(R)$$

with

$$f(b_j) = \sum_{i=1}^n r_{ij} \cdot a_i. \quad (2.1)$$

We write this simultaneously for all $j$ as follows,

$$f(b) = a \cdot M(a, f, b). \quad (2.2)$$

If $b$ is also an $R$-basis of $V$, and if $c \in R^d$ is a tuple and $g : V \to V$ is a second endomorphism, then the calculation

$$f(g(c)) = f(b \cdot M(b, g, c)) = f(b) \cdot M(b, g, c)$$

shows

$$M(a, f \circ g, c) = M(a, f, b) \cdot M(b, g, c). \quad (2.3)$$

Especially, in the case $f = id$, we write $M(a, b) := M(a, id, b)$. If $b$ is also an $R$-basis of $V$, this is in $GL_n(R)$.

**Remark 2.3.** Let $R$ be a principal ideal domain. Because of the Notation 2.2, the following is clear. The isomorphism class of a pair $(V, h)$ with $V$ a free $R$-module of rank $n \in \mathbb{N}$ and $h : V \to V$ an endomorphism is equivalent to the conjugacy class

$$\{ M(a, h, a) \mid a \text{ an } R\text{-basis of } V \}$$

of matrices in $M_{n \times n}(R)$ with respect to $GL_n(R)$.

**Notation 2.4.** If $H$ is a $\mathbb{Z}$-lattice of some finite rank and $R$ is a principal ideal domain which contains $\mathbb{Z}$, then $H_R := H \otimes_{\mathbb{Z}} R$ is an $R$-lattice of the same rank. Then a $\mathbb{Z}$-linear endomorphism $h$ of $H$ extends to an $R$-linear endomorphism of $H_R$. In the case of $R = \mathbb{C}$, the generalized eigenspace with eigenvalue $\lambda \in \mathbb{C}$ of $h : H_C \to H_C$ is denoted by $H_\lambda$, so that $H_C = \bigoplus_{\lambda \text{ eigenvalue}} H_\lambda$.

**Remarks 2.5.** (i) Let $R$ be a principal ideal domain, and let $V$ be a finitely generated $R$-module. Then it is a basic theorem on such $R$-modules that $V$ is isomorphic to a direct sum of quotients

$$R^{k_1} \oplus \bigoplus_{j=1}^k \frac{R}{p_j R}$$

with $p_1, \ldots, p_k \in R - (R^* \cup \{0\})$, $p_k | p_{k-1} | \ldots | p_2 | p_1$, ($R^* := \{ \text{units in } R \}$). The numbers $k_1$ and $k$ are unique, and the elements $p_1, \ldots, p_k$ are unique up to multiplication by units, and they
are called elementary divisors. \( R^{k_1} \) is the free part, and \( \bigoplus_{j=1}^{k} R/p_j R \) is the torsion part of the sum.

(ii) Let \( (H, h) \) be a \( \mathbb{Z} \)-lattice of rank \( n \in \mathbb{N} \) and \( h : H \to H \) an endomorphism. Then \( H \) is a \( \mathbb{Z}[t] \)-module, where \( t \) acts as \( h \) on \( H \). The ring \( \mathbb{Z}[t] \) is not a principal ideal domain, but \( \mathbb{Q}[t] \) is. \( H_{\mathbb{Q}} \) is a \( \mathbb{Q}[t] \)-module. Part (i) applies. \( H_{\mathbb{Q}} \) is a torsion module of \( \mathbb{Q}[t] \), it is isomorphic to \( \bigoplus_{j=1}^{k} \mathbb{Z}[t]/p_j \mathbb{Q}[t] \) for unique unitary polynomials \( p_1, ..., p_k \in \mathbb{Q}[t] \) of degrees \( \geq 1 \), which satisfy \( p_k|p_{k-1}|...|p_2|p_1 \). They are the elementary divisors. In fact,

\[
p_{H,h} = p_1 \cdot \ldots \cdot p_k, \tag{2.4}
\]

and as this is unitary and in \( \mathbb{Z}[t] \), all \( p_j \) are in \( \mathbb{Z}[t] \).

(iii) \( (H, h) \) as in (ii) admits a standard decomposition into companion blocks if and only if a decomposition of \( H_{\mathbb{Q}} \) as in (ii) lifts from \( \mathbb{Q}[t] \) to \( \mathbb{Z}[t] \), so that \( H \) is isomorphic to \( \bigoplus_{j=1}^{k} \mathbb{Z}[t]/p_j \mathbb{Z}[t] \) as a \( \mathbb{Z}[t] \)-module. Orlik [Or72] gave this fact as a motivation for Conjecture 1.2.

(iv) The tuple of characteristic polynomials \( p_{H^{(i)},h} \) in (1.2) is unique, because over \( \mathbb{Q} \) they become the elementary divisors of the \( \mathbb{Q}[t] \)-module \( H_{\mathbb{Q}} \). They can also be understood in terms of the Jordan block structure of \( h \) on \( H_{\mathbb{C}} \). If \( \lambda_1, ..., \lambda_m \in \mathbb{C} \) are the different eigenvalues and if for \( \lambda_i \) there are Jordan blocks of sizes \( b_{i,1}, ..., b_{i,m(i)} \) with \( b_{i,1} \geq b_{i,2} \geq \ldots \geq b_{i,m(i)} \) \( (m(i) \geq 1) \) then

\[
k = \max(m(i) \mid i = 1, ..., m),
\]

\[
p_{H^{(i)},h} = \prod_{i: m(i) \geq j} (t - \lambda_i)^{b_{i,j}}. \tag{2.5}
\]

(v) Choose a unitary polynomial \( p \in \mathbb{C}[t] \) of some degree \( n \geq 1 \). Define \( H_{\mathbb{C}} := \mathbb{C}[t]/p \mathbb{C}[t] \) and \( h : H_{\mathbb{C}} \to H_{\mathbb{C}} \) as multiplication by \( t \). Consider the \( \mathbb{C} \)-basis \( a = (1, [t], ..., [t^{n-1}]) \) of \( H_{\mathbb{C}} \). The matrix \( M(a, h, a) \) is the companion matrix with characteristic polynomial \( p \),

\[
M(a, h, a) = \begin{pmatrix}
0 & -p_0 \\
1 & \ddots & \ddots \\
& \ddots & 0 & -p_{n-2} \\
& & 1 & -p_{n-1}
\end{pmatrix} \tag{2.6}
\]

where \( p(t) = t^n + p_{n-1}t^{n-1} + ... + p_1 t + p_0 \).

It is a regular matrix, that means, it has for each eigenvalue only one Jordan block. Of course, the characteristic polynomial of \( h \) on \( H_{\mathbb{C}} \) is \( p_{H_{\mathbb{C}},h} = p \), and it is also the minimal polynomial of \( h \) on \( H_{\mathbb{C}} \).
(vi) If in (v) \( p \in \mathbb{Z}[t] \) and \( H := \mathbb{Z}[t]/p\mathbb{Z}[t] \), then \((H, h)\) is a companion block, and \( p_{H,h} = p \).

(vii) Vice versa, if \((H, h)\) is a companion block of rank \( n \in \mathbb{N} \) and \( a_0 \in H \) is a generating element of it as in (1.1), then the matrix \( M(a, h, a) \) of \( h \) with respect to the \( \mathbb{Z} \)-basis \( a = (a_0, h(a_0), ..., h^{a-1}(a_0)) \) is the companion matrix with characteristic polynomial \( p_n \), the minimal polynomial, and then the two sublattices \( p \) is regular (only one Jordan block for each eigenvalue) then \((H, h)\) is a companion block, because of (vi) above, Up to isomorphism, it is the unique companion block with characteristic polynomial \( p \).

Definition 2.6. (a) Let \( p \in \mathbb{Z}[t] \) be a unitary polynomial of some degree \( n \geq 0 \). We denote \( H[p] := \mathbb{Z}[t]/p\mathbb{Z}[t] \), and \( h[p] : H[p] \to H[p] \) is the multiplication by \( t \). If \( n = 0 \) then \( H[p] = 0 \). If \( n \geq 1 \) then \((H[p], h[p])\) is a companion block, because of (vi) above, Up to isomorphism, it is the unique companion block with characteristic polynomial \( p \).

(b) Let \( M \subset \mathbb{N} \) be a finite subset. It defines a unitary polynomial \( p_M := \prod_{m \in M} \Phi_m \) (where \( \Phi_m \) is the cyclotomic polynomial, see the Notation 4.3 (iii)). Then \((H[p], h[p])\) is an Orlik block. It is also denoted \( \text{Or}(M) := (H[p], h[p]) \). \( M \) is the set of orders of eigenvalues of its monodromy. The Orlik block and \( M \) determine one another.

Remarks 2.7. (i) Let \( H \) be a \( \mathbb{Z} \)-lattice of rank \( n \in \mathbb{N} \). A sublattice \( H^{(1)} \subset H \) is primitive if the quotient \( H/H^{(1)} \) has no torsion, or, equivalently, if \( H^{(1)} = H \cap H_Q^{(1)} \), where \( H_Q^{(1)} \subset H_Q \) and \( H \subset H_Q \), or, equivalently, if a sublattice \( H_Q^{(2)} \subset H \) with \( H = H^{[1]} \oplus H^{[2]} \) exists. For any sublattice \( H^{(3)} \subset H \), a unique primitive sublattice \( H^{(4)} \subset H \) with \( H_Q^{(4)} = H_Q^{(3)} \) exists, namely \( H^{(4)} = H \cap H_Q^{(3)} \supset H^{(3)} \).

(ii) Let \((H, h)\) be a \( \mathbb{Z} \)-lattice and \( h : H \to H \) an endomorphism with characteristic polynomial \( p = p_1 \cdot p_2 \) with \( p_1, p_2 \in \mathbb{Z}[t] \) unitary of degrees \( \geq 1 \). Then \( p_1(h)p_2(h) = 0 \) and
\[
p_2(h)(H) \subset \ker(p_1(h) : H \to H).
\] (2.7)

The second sublattice is a kernel, so it is a primitive sublattice. If \( h \) is regular (only one Jordan block for each eigenvalue) then \( p \) is also the minimal polynomial, and then the two sublattices \( p_2(h)(H) \) and \( \ker(p_1(h) : H \to H) \) have the same rank, so the first has a finite index in the second. Often this index is \( > 1 \). But by Lemma 2.8 (b), equality holds if \((H, h)\) is a companion block.

Part (a) of Lemma 2.8 was stated and proved in [He20].

Lemma 2.8. Let \( p_1, p_2 \in \mathbb{Z}[t] \) be unitary polynomials of degrees \( \geq 1 \).
(a) $H^{[p_1,p_2]}$ contains a unique primitive sublattice which is $h_{[p_1,p_2]}$-invariant and such that the characteristic polynomial of $h_{[p_1,p_2]}$ on it is $p_1$. It is $(p_2)/(p_1p_2) \subset H^{[p_1,p_2]}$, and $(p_2)/(p_1p_2) \cong H^{[p_1]}$.

(b) Let $(H,h)$ be a companion block with characteristic polynomial $p_1p_2$. Then
\begin{equation}
p_2(h)(H) = \ker(p_1(h) : H \to H).
\end{equation}
Especially, $p_2(h)(H)$ is a primitive sublattice of $H$.

**Proof:** (a) [He20] Lemma 6.1] (not difficult).
(b) We can choose $(H,h) = (H^{[p_1,p_2]}, h_{[p_1,p_2]})$. Then
\[
p_2(h)(H) = (p_2)/(p_1p_2) = \ker((\text{multiplication by } p_1) : H^{[p_1,p_2]} \to H^{[p_1,p_2]}) = \ker(p_1(h) : H \to H).
\]
\[\square\]

### 3. A Frame for Constructing a Standard Decomposition into Companion Blocks

Let $(H,h)$ be a $\mathbb{Z}$-lattice of rank $n \in \mathbb{N}$ with an endomorphism $h : H \to H$. By Remark 2.3, the $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ as a $\mathbb{Q}[t]$-module is isomorphic to $\bigoplus_{j=1}^k H^{[p_j]}_{\mathbb{Q}}$ (see Definition 2.6 and Notation 2.3 for $H^{[p_j]}_{\mathbb{Q}}$), where $p_1, \ldots, p_k \in \mathbb{Z}[t]$ are the elementary divisors. They are unitary of degrees $\geq 1$ and satisfy $p_k | p_{k-1} | \ldots | p_2 | p_1$ and $p_{H,h} = p_1 \cdot \ldots \cdot p_k$. If $(H,h)$ admits a standard decomposition into companion blocks, that is isomorphic to $\bigoplus_{j=1}^k H^{[p_j]}$.

Theorem 3.1 gives a frame for constructing a standard decomposition into companion blocks. It builds on Lemma 2.8 (b).

**Theorem 3.1.** Let $(H,h)$ and $p_1, \ldots, p_k$ be as above. Consider $k$ elements $a_0^{[j]} \in H$ for $j \in \{1, \ldots, k\}$, and consider the elements $a_i^{[j]} := h^i(a_0^{[j]})$ for $i \in \mathbb{N}_0$. If the tuple of elements
\[
(a_0^{[1]}, a_1^{[1]}, \ldots, a_{\deg p_1-1}^{[1]}; a_0^{[2]}, a_1^{[2]}, \ldots, a_{\deg p_2-1}^{[2]}; \ldots, a_0^{[k]}, a_1^{[k]}, \ldots, a_{\deg p_k-1}^{[k]}).
\]
\[= (a_i^{[j]} \mid j = 1, \ldots, k, \ i = 0, \ldots, \deg p_j - 1) \quad (3.1)
\]
is a $\mathbb{Z}$-basis of $H$, then $(H,h)$ admits a standard decomposition $H = \bigoplus_{j=1}^k B^{[j]}$ into companion blocks $B^{[j]}$. The companion blocks can be chosen as follows. The first block $B^{[1]}$ with characteristic polynomial

$p_1$ is generated by $a_0^{[1]}$. The $j$-th block $B^{[j]}$ for $j \in \{2, \ldots, k\}$ with characteristic polynomial $p_j$ is generated by $a_0^{[j]} + b^{[j]}$ where $b^{[j]}$ is a certain element of the sum of the first $j-1$ blocks $B^{[1]}, \ldots, B^{[j-1]}$.

**Proof:** Suppose that the tuple in (3.1) is a $\mathbb{Z}$-basis of $H$. Then its first $\deg p_1$ elements $a_0^{[1]}, a_1^{[1]}, \ldots, a_{\deg p_1-1}^{[1]}$ generate a primitive sublattice $B^{[1]}$ of rank $\deg p_1$. It is $h$-invariant as $p_1$ is the minimal polynomial of $h$ on $H$ and $a_i^{[1]} = h^i(a_0^{[1]})$. So it is a companion block with characteristic polynomial $p_1$ and generator $a_0^{[1]}$. We define $b^{[1]} := 0$.

Now we proceed by induction on $j$ and suppose that for some $j \in \{2, \ldots, k\}$ the following holds. For any $l \in \{1, \ldots, j-1\}$ an element $b^{[l]} \in H$ has been constructed such that $a_0^{[l]} + b^{[l]}$ is a generator of a companion block

\[ B^{[l]} = \bigoplus_{i=0}^{\deg p_l-1} \mathbb{Z} \cdot h^i(a_0^{[l]} + b^{[l]}) \]  

(3.2)

with characteristic polynomial $p_l$ and such that $b^{[l]}$ is in the sublattice generated by the $l-1$ companion blocks $B^{[1]}, \ldots, B^{[l-1]}$ constructed before.

Then the sublattice generated by the first $j-1$ companion blocks is

\[ \bigoplus_{l=1}^{j-1} B^{[l]} = \bigoplus_{l=1}^{j-1} \bigoplus_{i=0}^{\deg p_l-1} \mathbb{Z} \cdot a_i^{[l]} . \]  

(3.3)

It is indeed a direct sum, and it is a primitive sublattice of $H$, both because the tuple in (3.1) is a $\mathbb{Z}$-basis of $H$.

We want to find an element $b^{[j]} \in \bigoplus_{l=1}^{j-1} B^{[l]}$ such that $a_0^{[j]} + b^{[j]}$ generates a companion block with characteristic polynomial $p_j$. We claim that it is sufficient to find an element $b^{[j]}$ with

\[ p_j(h)(a_0^{[j]} + b^{[j]}) = 0 \]  

(3.4)

That the tuple in (3.1) is a $\mathbb{Z}$-basis shows that we have a direct sum

\[ \left( \bigoplus_{l=1}^{j-1} B^{[l]} \right) \oplus \left( \bigoplus_{i=0}^{\deg p_j-1} \mathbb{Z} \cdot h^i(a_0^{[j]} + b^{[j]}) \right) . \]  

(3.5)

(3.4) and (3.5) show that the second summand in (3.5) is a companion block $B^{[j]}$ with characteristic polynomial $p_j$ and that it extends the sum of the earlier constructed companion blocks to a bigger primitive sublattice.

So it remains to find $b^{[j]} \in \bigoplus_{l=1}^{j-1} B^{[l]}$ with (3.4).
First we consider $H_Q$ and $p_j(h)(H_Q) \subset H_Q$. By Remark 2.5 (ii), $H_Q$ has a decomposition $H_Q = \bigoplus_{l=1}^k \tilde{B}_Q^{[l]}$ into $h$-invariant blocks $\tilde{B}_Q^{[l]}$ with characteristic and minimal polynomials $p_l$. For $l \geq j$ we have $p_l|p_j$ and thus $p_j(h)(\tilde{B}_Q^{[l]}) = 0$. Therefore

$$p_j(h)(H_Q) = \bigoplus_{l=1}^{j-1} p_j(h)(\tilde{B}_Q^{[l]})$$

(3.6)

and this has dimension $\sum_{l=1}^{j-1} (\deg p_l - \deg p_j)$. The subspace

$$\bigoplus_{l=1}^{j-1} p_j(h)(B_Q^{[l]}) = \bigoplus_{l=1}^{j-1} \ker\left(\frac{p_l}{p_j}(h) : B_Q^{[l]} \to B_Q^{[l]}\right)$$

(3.7)

has the same dimension, thus it is equal to $p_j(h)(H_Q)$. By Lemma 2.8 (b) $p_j(h)(B^{[l]})$ is for any $l \in \{1, \ldots, j-1\}$ a primitive sublattice of $B^{[l]}$. Therefore the sum $\bigoplus_{l=1}^{j-1} p_j(h)(B^{[l]})$ is a primitive sublattice of $\bigoplus_{l=1}^{j-1} B^{[l]}$. As this is a primitive sublattice of $H$, the sum $\bigoplus_{l=1}^{j-1} p_j(h)(B^{[l]})$ is a primitive sublattice of $H$. As its extension to $H_Q$ is equal to $p_j(h)(H_Q)$, it is itself equal to $p_j(h)(H_Q) \cap H$. We obtain

$$p_j(h)(H) \supset p_j(h)(\bigoplus_{l=1}^{j-1} B^{[l]}) = \bigoplus_{l=1}^{j-1} p_j(h)(B^{[l]})$$

$$= p_j(h)(H_Q) \cap H \supset p_j(h)(H),$$

thus

$$p_j(h)(H) = p_j(h)(\bigoplus_{l=1}^{j-1} B^{[l]}).$$

(3.8)

Therefore $b^{[l]} \in \bigoplus_{l=1}^{j-1} B^{[l]}$ with (3.4) exists. □

**Remark 3.2.** (i) Theorem 3.1 and its proof are evidence that standard decompositions into companion blocks are natural and arise more often than one might expect at first sight.

(ii) For any $j \in \{1, \ldots, k\}$, the $\mathbb{Z}$-lattice $V_j := \langle\text{the first } j \text{ lines in (3.1)}\rangle_{\mathbb{Z}}$ is an $h$-invariant primitive sublattice of $H$. The proof of Theorem 3.1 shows this and provides a $\mathbb{Z}$-sublattice $B_j$ with $V_j = B_j \oplus V_{j-1} = \bigoplus_{l=1}^{l} B_l$. 
4. Resultants and discriminants of cyclotomic polynomials

This section reviews resultants and discriminants in general and in the case of cyclotomic polynomials. This is needed in the determinant calculations in the proofs of Theorem 6.2 and Theorem 7.4. Essentially everything in this section is well-known. The review here is close to section 2 in [He20].

The resultant of two polynomials and the discriminant of one polynomial are very classical objects. One reference for the following definition is [vW71, §34].

**Definition 4.1.** (a) The *resultant* of two polynomials \( f = \sum_{i=0}^{m} f_i t^i \in \mathbb{C}[t] \setminus \{0\} \) and \( g = \sum_{j=0}^{n} g_j t^j \in \mathbb{C}[t] \setminus \{0\} \) of degrees \( \deg f = m, \deg g = n \) with \( m + n \geq 1 \) is \( \text{Res}(f, g) := \det A(f, g) \in \mathbb{C} \) where \( A(f, g) \in M_{(m+n) \times (m+n)}(\mathbb{C}) \) is the matrix

\[
A(f, g) = \begin{pmatrix}
       f_0 & 0 & \ldots & 0 & g_0 & 0 & \ldots & 0 \\
       f_1 & f_0 & \ddots & 0 & g_1 & g_0 & \ddots & 0 \\
       \vdots & f_1 & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
       \vdots & \vdots & \ddots & f_0 & \ddots & \ddots & \ddots & \vdots \\
       f_m & \ddots & \ddots & f_1 & g_m & \ddots & \ddots & g_0 \\
       0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
       \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
       0 & \ldots & 0 & f_m & 0 & \ldots & 0 & g_n
     \end{pmatrix}
\]

(4.1)

whose first \( n \) columns contain the coefficients of \( f \) and whose last \( m \) columns contain the coefficients of \( g \). In other words, it is the matrix with

\[
(f, tf, \ldots, t^{n-1}f, g, tg, \ldots, t^{m-1}g) = (1, t, \ldots, t^{m+n-1}) \cdot A(f, g).
\]

(4.2)

In the case \( m + n = 0 \) one defines \( \text{Res}(f, g) := 1 \).

(b) The *discriminant* of a polynomial \( f = \sum_{i=0}^{m} f_i t^i \in \mathbb{C}[t] \) of degree \( \deg f = m \geq 1 \) is \( \text{Discr}(f) := \text{Res}(f, \frac{df}{dt}) \in \mathbb{C} \).

The basic properties of the resultant and the discriminant are well known.

**Proposition 4.2.** (a) Let \( f \) and \( g \in \mathbb{C}[t] \) be as in definition 4.1 (a). Let \( a_1, \ldots, a_m \in \mathbb{C} \) and \( b_1, \ldots, b_n \in \mathbb{C} \) be the zeros of \( f \) and \( g \), so

\[
f = f_0 \cdot \prod_{i=1}^{m}(t - a_i), \quad g = g_0 \cdot \prod_{j=1}^{n}(t - b_j).
\]
Then
\[
\text{Res}(f, g) = f_0^n g_0^m \cdot \prod_{i=1}^{m} \prod_{j=1}^{n} (a_i - b_j) \quad (4.3)
\]
\[
= f_0^n \cdot \prod_{i=1}^{m} g(a_i) = (-1)^{m-n} g_0^m \cdot \prod_{j=1}^{n} f(b_j) \quad (4.4)
\]
\[
= (-1)^{m-n} \text{Res}(g, f), \quad (4.5)
\]
\[
\text{Res}(f, g) \neq 0 \iff \gcd(f, g)_{\mathbb{C}[t]} = 1. \quad (4.6)
\]

(b) If \( f, g, h \in \mathbb{C}[t] - \{0\} \) then
\[
\text{Res}(f, gh) = \text{Res}(f, g) \cdot \text{Res}(f, h). \quad (4.7)
\]

If \( f^{(1)}, \ldots, f^{(r)}, g^{(1)}, \ldots, g^{(s)} \in \mathbb{C}[t] - \{0\} \) then
\[
\text{Res}\left( \prod_{i=1}^{r} f^{(i)}, \prod_{j=1}^{s} g^{(j)} \right) = \prod_{i=1}^{r} \prod_{j=1}^{s} \text{Res}(f^{(i)}, g^{(j)}). \quad (4.8)
\]

(c) If \( f = f_0 \cdot \prod_{i=1}^{m} (t - a_i) \in \mathbb{C}[t] \) has degree \( m \geq 1 \) and zeros \( a_1, \ldots, a_m \in \mathbb{C} \) then
\[
\text{Discr}(f) = f_0^{2m-1} \cdot \prod_{(i,j) \in \{1, \ldots, m\}^2 \text{ with } i \neq j} (a_i - a_j). \quad (4.9)
\]

(4.3) is proved for example in [vW71, §35]. (4.4), (4.5), (4.6) and (4.7) follow from (4.3). (4.8) follows from (4.7) and (4.5). And (4.9) follows from (4.4) and \( \frac{d}{dt} f = \sum_{i=1}^{m} f_0 \cdot \prod_{j \neq i} (t - a_j) \).

We are mainly interested in \( \text{Res}(f, g) \) where \( f \) and \( g \) are unitary polynomials. We denote for \( k \in \mathbb{Z}_{\geq -1} \)
\[
\mathbb{C}_k[t] := \{ h \in \mathbb{C}[t] \mid \deg h \leq k \}, \quad (4.10)
\]
\[
\mathbb{Z}_k[t] := \mathbb{C}_k[t] \cap \mathbb{Z}[t]
\]
(so that \( \mathbb{C}_{-1}[t] = \mathbb{Z}_{-1}[t] = \{0\} \)). The following lemma is proved for example in [He20].

Lemma 4.3. [He20] Lemma 2.3 \([\text{Let } f, g \in \mathbb{Z}[t] \text{ be unitary polynomials of degrees } m = \deg f, n = \deg g. \text{ They generate an ideal } (f, g) \subset \mathbb{Z}[t] \text{ (here } \mathbb{Z}[t] \text{ is also considered as an ideal).} ]\)

(a)
\[
\mathbb{Z}_{n-1}[t] \cdot f + \mathbb{Z}_{m-1}[t] \cdot g = (f, g) \cap \mathbb{Z}_{m+n-1}[t]. \quad (4.11)
\]

(b) The \( \mathbb{Z} \)-lattice in (4.11) has rank \( m+n \) if and only if \( \text{Res}(f, g) \neq 0 \), and then
\[
|\text{Res}(f, g)| = \frac{\mathbb{Z}_{m+n-1}[t]}{(f, g) \cap \mathbb{Z}_{m+n-1}[t]} \in \mathbb{Z}_{>0}. \quad (4.12)
\]
|(c)| $|\text{Res}(f,g)| = 1 \iff (f,g) = \mathbb{Z}[t]. \quad (4.13)$

Now we turn to unit roots and cyclotomic polynomials.

**Notations 4.4.**

(i) In the proof of Theorem 4.5 (and only there), we will use the notation $[a]_m$ for the class of $a \in \mathbb{Z}$ in $\mathbb{Z}/m\mathbb{Z}$.

(ii) The order $\text{ord}(\lambda) \in \mathbb{N}$ of a unit root $\lambda \in S^1 \subset \mathbb{C}$ is the minimal $k \in \mathbb{N}$ with $\lambda^k = 1$. In the rest of this section, $\lambda$ denotes always a unit root. In the rest of this paper, $e(z)$ for $z \in \mathbb{C}$ denotes $e^{2\pi iz} \in \mathbb{C}$, so for example $e(r)$ for $r \in \mathbb{Q}$ is a unit root.

(iii) For $m \in \mathbb{N}$, the cyclotomic polynomial $\Phi_m$ is the polynomial

$$\Phi_m(t) := \prod_{\lambda : \text{ord}(\lambda) = m} (t - \lambda), \quad (4.14)$$

whose zeros are the $m$-th primitive unit roots. It is a unitary and irreducible polynomial in $\mathbb{Z}[t]$ of degree $\deg \Phi_m = \varphi(m) \in \mathbb{N}$, where $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is the Euler phi-function (see e.g. [Wa82, Ch 1,2]). Except for the irreducibility, this follows easily inductively from the formula

$$t^m - 1 = \prod_{k|m} \Phi_k. \quad (4.15)$$

Using this formula, one can compute the $\Phi_k$ inductively.

(iv) Recall (see e.g. [Wa82 Ch 1,2]) that $\mathbb{Z}[e(\frac{1}{m})]$ is the ring of the algebraic integers within $\mathbb{Q}[e(\frac{1}{m})]$ and that

$$\mathbb{Z}[e(\frac{1}{m})] \cap S^1 = \{ \pm e(\frac{k}{m}) \mid k \in \mathbb{Z} \}. \quad (4.16)$$

We will also use the norm

$$\text{Norm}_m : \mathbb{Z}[e(\frac{1}{m})] \rightarrow \mathbb{Z}, \quad g(e(\frac{1}{m})) \mapsto \prod_{\lambda : \text{ord}(\lambda) = m} g(\lambda). \quad (4.17)$$

An element of $\mathbb{Z}[e(\frac{1}{m})]$ has norm in $\{ \pm 1 \}$ if and only if it is a unit in $\mathbb{Z}[e(\frac{1}{m})]$. This and the calculation

$$\text{Norm}_{\text{ord}(\lambda)}(1 - \lambda) = \prod_{\kappa : \text{ord}(\kappa) = \text{ord}(\lambda)} (1 - \kappa) = \Phi_{\text{ord}(\lambda)}(1) \quad (4.18)$$

and Theorem 4.5 (a) imply Theorem 4.5 (b).

The following theorem collects relevant facts on unit roots, and it gives formulas for the resultants and the discriminants of cyclotomic polynomials. The parts (a), (b) and (d) and a part of part (c) are proved in [He20, Theorem 3.1]. Part (d) gives the resultants of the cyclotomic polynomials. It is the main result of [Ap70]. The proof of
it in [He20] is shorter than that in [Ap70]. We do not know a reference for part (e) and the rest of part (c), although they are certainly known. Therefore we provide proofs for them. Part (e) gives the discriminants of the cyclotomic polynomials.

**Theorem 4.5.** (a) $\Phi_m(1) = 1$ if $m \geq 2$ and $m$ is not a power of a prime number. $\Phi_{p^k}(1) = p$ if $p$ is a prime number and $k \in \mathbb{N}$.

(b) $1 - \lambda$ is a unit in $\mathbb{Z}[\lambda]$ if and only if $\text{ord}(\lambda)$ is not a power of a prime number and not equal to 1.

(c) Fix $m, n \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{N}$, a prime number $p$, and denote

$$\Lambda(m, n, p, k) := \varphi(p^k)^{-1} \cdot |\{(a, b) \in (\mathbb{Z}/m\mathbb{Z})^* \times (\mathbb{Z}/n\mathbb{Z})^* : \text{ord}(e(\frac{a}{m} - \frac{b}{n})) = p^k\}| \in \mathbb{N}_0.$$  

It is the multiplicity with which $e(\frac{a}{m} - \frac{b}{n})$ gives a fixed unit root of order $p^k$ if $(a, b)$ runs through $(\mathbb{Z}/m\mathbb{Z})^* \times (\mathbb{Z}/n\mathbb{Z})^*$.

(i) If neither $\frac{m}{n}$ nor $\frac{n}{m}$ is a power of a prime number, then

$$\Lambda(m, n, p, k) = 0.$$  

(ii) Suppose $\frac{m}{n} = q^l$ for a prime number $q$ and some $l \in \mathbb{N}$. Then

$$\Lambda(n, m, p, k) = \Lambda(m, n, p, k) = \begin{cases} 0 & \text{if } (p, k) \neq (q, v_p(m)), \\ \varphi(n) & \text{if } (p, k) = (q, v_p(m)). \end{cases}$$  

(iii) Suppose $m = n$. Then

$$\Lambda(m, m, p, k) = \begin{cases} 0 & \text{if } v_p(m) < k, \\ \varphi(m) \cdot \frac{p^2 - 1}{p - 1} & \text{if } v_p(m) = k, \\ \varphi(m) & \text{if } v_p(m) > k. \end{cases}$$  

(d) [Ap70] For $m, n \in \mathbb{N}$,

$$\text{Res}(\Phi_m, \Phi_n) = 0 \quad \text{if } m = n.$$  

$$\text{Res}(\Phi_m, \Phi_n) = 1 \quad \text{if neither } \frac{m}{n} \text{ nor } \frac{n}{m}$$

$$\text{is a power of a prime number.}$$  

$$\text{Res}(\Phi_{p^k}, \Phi_n) = \text{Res}(\Phi_n, \Phi_{p^k}) = p^{\varphi(n)} \quad \text{if } p \text{ is a prime number}$$

and $k \in \mathbb{N}$ and $(p, k, n) \neq (2, 1, 1)$.  

$$\text{Res}(\Phi_1, \Phi_2) = -\text{Res}(\Phi_2, \Phi_1) = 2.$$  

(e) For $m \in \mathbb{N}$, $\text{Discr}(\Phi_m) \in \mathbb{N}$. For any prime number $p$

$$v_p(\text{Discr}(\Phi_m)) = \begin{cases} 0 & \text{if } v_p(m) = 0, \\ (v_p(m) - \frac{1}{p - 1}) \cdot \varphi(m) & \text{if } v_p(m) \geq 1. \end{cases}$$
**Proof:** (a) and (b) and (d) [He20, Theorem 3.1].

(c) The parts (i) and (ii) are proved in [He20] as part of the proof of part (d). Therefore here we prove only part (iii).

Consider $m \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{N}$, and a prime number $p$. If $v_p(m) < k$ (for example if $v_p(m) = 0$) then the reduced denominator $\beta$ of $\frac{a-b}{m} \equiv a$ with $\gcd(\alpha, \beta) = 1$ is never equal to $p^k$. Thus then $\Lambda(m, m, p, k) = 0$.

Suppose now $k \leq v_p(m)$. If $a$ and $b$ run through $(\mathbb{Z}/m\mathbb{Z})^*$, then the classes $[a]_{m/p^k} \in \mathbb{Z}/(m/p^k)\mathbb{Z}$ and $[b]_{m/p^k} \in \mathbb{Z}/(m/p^k)\mathbb{Z}$ run both through $(\mathbb{Z}/(m/p^k)\mathbb{Z})^*$ with multiplicity $\varphi(m)/\varphi(m/p^k)$. Therefore the class $[a-b]_{m/p^k}$ in $\mathbb{Z}/(m/p^k)\mathbb{Z}$ is zero in

$$\frac{\varphi(m)}{\varphi(m/p^k)} \cdot \frac{\varphi(m)}{\varphi(m/p^k)} \cdot \varphi(m/p^k) = \frac{\varphi(m)^2}{\varphi(m/p^k)}$$

cases. This is the number of cases where the reduced denominator of $\frac{a-b}{m}$ divides $p^k$. Therefore the reduced denominator of $\frac{a-b}{m}$ is equal to $p^k$ in

$$\frac{\varphi(m)^2}{\varphi(m/p^k)} - \frac{\varphi(m)^2}{\varphi(m/p^k-1)}$$

= $\left\{ \begin{array}{ll} \varphi(p^k) \cdot \varphi(m) & \text{if } v_p(m) > k \\
\varphi(p^k) \cdot \varphi(m) \cdot \frac{p-2}{p-1} & \text{if } v_p(m) = k \end{array} \right.$, \quad (4.28)

cases. In these cases, there are $\varphi(p^k)$ possibilities for the reduced numerator of $\frac{a-b}{m}$. This shows (4.22).

(e) $\text{Discr}(\Phi_m) \in \mathbb{Z}$ because $\Phi_m \in \mathbb{Z}[t]$. And $\text{Discr}(\Phi_m) > 0$ because the zeros $a_i$ in formula (1.9) are here the primitive $m$-th unit roots, and with $\lambda$ also $\overline{\lambda}$ is such a unit root. By formula (1.9)

$$\text{Discr}(\Phi_m) = \prod_{(a,b) \in ((\mathbb{Z}/m\mathbb{Z})^*)^2} e\left(\frac{b}{m}\right) \cdot (1 - e\left(\frac{a-b}{m}\right)). \quad (4.29)$$

Recall (4.18) $\text{Norm}_{\text{ord}(\lambda)}(1 - \lambda) = \Phi_{\text{ord}(\lambda)}(1)$ for any unit root $\lambda$, and recall Theorem (4.5) (a). The right hand side of (4.29) can be seen as a product of unit roots and such norms $\text{Norm}_{\text{ord}(\lambda)}(1 - \lambda)$ for suitable $\lambda$.

Only $\text{ord}(\lambda) = p^k$ for $k \geq 1$ contributes to $v_p(\text{Discr}(\Phi_m))$. The precise amount of this contribution can be read off from formula (4.22). Thus

\[
\begin{align*}
  v_p(\text{Discr}(\Phi_m)) &= 0 & \text{if } v_p(m) = 0, \\
  v_p(\text{Discr}(\Phi_m)) &= \sum_{k=1}^{v_p(m)-1} \varphi(m) + \varphi(m) \cdot \frac{p-2}{p-1} & \text{if } v_p(m) > 0. \quad (4.30)
\end{align*}
\]

\[
= (v_p(m) - \frac{1}{p-1}) \cdot \varphi(m) & \text{if } v_p(m) > 0. \quad \square
\]
5. Different decompositions into companion blocks

Theorem 5.1 is after Theorem 3.1 our second structural result about decompositions of a \( \mathbb{Z} \)-lattice \( H \) with endomorphism \( h \) into companion blocks. Now the focus is on arbitrary decompositions, not just the standard decomposition. Theorem 5.1 has some similarity with the chinese remainder theorem. We work with the companion blocks \( H[p] \) for \( p \in \mathbb{Z}[t] \) unitary from Definition 2.6. Part (a) of Theorem 5.1 is Lemma 6.2 (b) in [He20], part (b) is new.

**Theorem 5.1.** (a) Let \( f, g \in \mathbb{Z}[t] \) be unitary polynomials with \( \text{Res}(f, g) \neq 0 \) (equivalent is \( \gcd(f, g) = 1 \)). Then

\[
H^{[fg]} \cong H^{[f]} \oplus H^{[g]} \iff |\text{Res}(f, g)| = 1. \tag{5.1}
\]

(b) Let \( f_1, f_2, f_3, f_4 \in \mathbb{Z}[t] \) be unitary polynomials with \( \text{Res}(f_i, f_j) \neq 0 \) for all \( i \neq j \). Then

\[
H^{[f_1f_2f_4]} \oplus H^{[f_2f_3]} \cong H^{[f_1f_3]} \oplus H^{[f_2f_3f_4]}
\]

\[
\iff |\text{Res}(f_1, f_4)| = |\text{Res}(f_2, f_4)| = 1. \tag{5.2}
\]

**Proof:** (a) Recall (4.13), \( |\text{Res}(f, g)| = 1 \iff (f, g) = \mathbb{Z}[t] \). By Lemma 2.8 (a), the ideals \( (g)/(fg) \) and \( (f)/(fg) \) in \( H^{[fg]} \) are isomorphic to the companion blocks \( H^{[f]} \) and \( H^{[g]} \), respectively, and they are the unique primitive sublattices in \( H^{[fg]} \) which are monodromy invariant and have the characteristic polynomials \( f \) and \( g \). Because of \( \gcd(f, g) = 1 \), their intersection is 0, so they form a direct sum within \( H^{[fg]} \). This sum is \( (f, g)/(fg) \subset \mathbb{Z}[t]/(fg) = H^{[fg]} \). Therefore \( H^{[fg]} \cong H^{[f]} \oplus H^{[g]} \) is equivalent to \( (f, g)/(fg) = H^{[fg]} \), and this is equivalent to \( (f, g) = \mathbb{Z}[t] \).

(b) In the case of a sum of companion blocks, the monodromy \( h \) of the sum is defined to be the sum of the monodromies of the single companion blocks.

Suppose that the isomorphism in the first line of (5.2) holds. Divide both sides by the kernel \( \ker(f_2(h)f_3(h)) \). Then one obtains an isomorphism

\[
H^{[f_1f_4]} \cong H^{[f_1]} \oplus H^{[f_4]}.
\]

Part (a) shows \( |\text{Res}(f_1, f_4)| = 1 \). The necessity of \( |\text{Res}(f_2, f_4)| = 1 \) is obtained analogously.

It remains to show that these two conditions are sufficient for the isomorphism. So suppose \( |\text{Res}(f_1, f_4)| = 1 = |\text{Res}(f_2, f_4)| \).

We identify \( H^{[f_1f_2f_4]} \oplus H^{[f_2f_3]} \) with \( H^{[f_1f_3f_4]} \times H^{[f_2f_3]} \). We denote its monodromy by \( h = (h_1, h_2) \), where \( h_1 = h_{[f_1f_3f_4]} \) is the monodromy of \( H^{[f_1f_3f_4]} \), and \( h_2 = h_{[f_2f_3]} \) is the monodromy of \( H^{[f_2f_3]} \). Let \( a_1 \in H^{[f_1f_3f_4]} \)
be a cyclic generator of it, and let $a_2 \in H^{[f_2 f_3]}$ be a cyclic generator of it.

$|\text{Res}(f_1, f_4)| = 1$ and $|\text{Res}(f_2, f_4)| = 1$ imply $|\text{Res}(f_1 f_2, f_4)| = 1$, and this implies the existence of polynomials $g_1, g_4 \in \mathbb{Z}[t]$ with $g_1 f_1 f_2 - g_4 f_4 = 1$. Observe that the following matrix has determinant 1 and thus is in $GL(2, \mathbb{Z}[t])$, and that its inverse is as follows,

$$
\begin{pmatrix}
g_1 f_1 & 1 \\
g_4 f_4 & f_2
\end{pmatrix}
\in GL(2, \mathbb{Z}[t]),
\quad
\begin{pmatrix}
g_1 f_1 & 1 \\
g_4 f_4 & f_2
\end{pmatrix}^{-1} =
\begin{pmatrix}
f_2 & -1 \\
g f_4 & g_1 f_1
\end{pmatrix}.
$$

Consider the elements

$$
b_1 := ((g_1 f_1)(h_1)(a_1), a_2) \in H^{[f_1 f_2 f_3]} \times H^{[f_2 f_3]},
$$

$$
b_2 := ((g_4 f_4)(h_1)(a_1), (f_2)(h_2)(a_2)) \in H^{[f_1 f_2 f_3]} \times H^{[f_2 f_3]}.
$$

$b_1$ is the generator of an Orlik block $B_1$ whose characteristic polynomial divides $f_2 f_3 f_4$. And $b_2$ is the generator of an Orlik block $B_2$ whose characteristic polynomial divides $f_1 f_3$.

It remains to show $B_1 + B_2 = H^{[f_1 f_2 f_3]} \times H^{[f_2 f_3]}$ because then by comparison of ranks one obtains $B_1 + B_2 = B_1 \oplus B_2$ and that the characteristic polynomial of $B_1$ is $f_2 f_3 f_4$ and the characteristic polynomial of $B_2$ is $f_1 f_3$. Thus it remains to show that $(a_1, 0)$ and $(0, a_2)$ are in $B_1 + B_2$. Calculate

$$
f_2(h)(b_1) - b_2
= ((f_2 g_1 f_1)(h_1)(a_1), (f_2)(h_2)(a_2))
- ((g_4 f_4)(h_1)(a_1), (f_2)(h_2)(a_2))
= (a_1, 0) \in B_1 + B_2
$$

and

$$
(-g_4 f_4)(h)(b_1) + (g_1 f_1)(h)(b_2)
= ((-g_4 f_4 g_1 f_1)(h_1)(a_1), (-g_4 f_4)(h_2)(a_2))
+ ((g_1 f_1 g_4 f_4)(h_1)(a_1), (g_1 f_1 f_2)(h_2)(a_2))
= (0, a_2) \in B_1 + B_2.
$$

Remark 5.2. We expect (but we did not prove it) that the following holds. Let $f_1, \ldots, f_a, g_1, \ldots, g_b \in \mathbb{Z}[t]$ be products of cyclotomic polynomials with no multiple roots. If an isomorphism

$$
\bigoplus_{i=1}^{a} H^{[f_i]} \cong \bigoplus_{j=1}^{b} H^{[g_j]}
$$

holds, then it can be deduced by repeated application of the rule (5.2) and adding to both sides the same Orlik blocks. This property would
say that the equivalence in (5.2) would be the most general rule for
going isomorphisms of sums of Orlik blocks.

6. WHEN DOES A POWER OF AN ORLIK BLOCK ADMIT A
STANDARD DECOMPOSITION INTO ORLIK BLOCKS?

Theorem 6.2 starts with one Orlik block \((H, h)\) and a number \(\mu \in \mathbb{N}\)
and gives a sufficient criterion for \((H, h^\mu)\) to admit a standard decom-
position into Orlik blocks. It will be crucial for proving Orlik’s Con-
jecture 1.2 in the case of the chain type singularities. The condition
will work with a graph whose set of vertices is the set \(M \subset \mathbb{N}\) of orders
of the eigenvalues of \(h : H_\mathbb{C} \rightarrow H_\mathbb{C}\). Theorem 6.2 is preceded by some
definitions and observations.

**Definition 6.1.** (a) Recall that \(\mathcal{P} \subset \mathbb{N}\) denotes the set of prime num-
bers. Consider the infinite directed graph \((\mathbb{N}, E)\) whose set of vertices
is \(\mathbb{N}\) und whose set \(E \subset \mathbb{N}^2\) of directed edges is defined as follows,
\[
E_p := \left\{ (m, n) \in \mathbb{N}^2 \mid \frac{m}{n} = p^k \text{ for some } k \in \mathbb{N} \right\}
\text{ for any } p \in \mathcal{P},
\]
\[
E := \bigcup_{p \in \mathcal{P}} E_p.
\]
An edge in \(E_p\) is called a \(p\)-edge.

(b) For any finite set \(M \subset \mathbb{N}\) consider the directed graph \((M, E(M))\)
which is the restriction of \((\mathbb{N}, E)\) to \(M\), so its set of directed edges is
\(E(M) = E \cap M^2\).

(c) For any \(\mu \in \mathbb{N}\) define the map
\[
\gamma_\mu : \mathbb{N} \rightarrow \mathbb{N},
\]
\[
m \mapsto \frac{m}{\gcd(m, \mu)}.
\]

(d) Consider a finite set \(M \subset \mathbb{N}\) and a number \(\mu \in \mathbb{N}\). The pair
\((M, \mu)\) is called \(sdiOb\)-sufficient \((sdiOb\ for \text{standard decomposition into}
Orlik blocks)\) if for any prime number \(p\) and any \(p\)-edge \((n_a, n_b)\) in
\(E_p(\gamma_\mu(M))\) at least one of the following two conditions holds.
\[
E_p \cap \left( (\gamma_\mu^{-1}(n_a) \cap M) \times \gamma_\mu^{-1}(n_b) \right) \subset E_p(M),
\]
\[
E_p \cap \left( \gamma_\mu^{-1}(n_a) \times (\gamma_\mu^{-1}(n_b) \cap M) \right) \subset E_p(M)
\]
( these two conditions are discussed in Remark 6.3 (iv)).

**Theorem 6.2.** Consider a finite non-empty set \(M \subset \mathbb{N}\), the corre-
sponding Orlik block \(\text{Or}(M) = H\) with monodromy \(h\), and a number
\(\mu \in \mathbb{N}\). Then \((H, h^\mu)\) admits a standard decomposition into Orlik blocks if \((M, \mu)\) is sdiOb-sufficient.

Before proving this theorem, we make some elementary observations.

**Remarks 6.3.** (i) We expect that *if and only if* holds in Theorem 6.2.

(ii) For any \(n \in \mathbb{N}\), the fiber \(\gamma^{-1}_\mu(n) \subset \mathbb{N}\) is finite and nonempty. It is

\[
\gamma^{-1}_\mu(n) = \{ n \cdot c \cdot \prod_{p \in \mathbb{P} : v_p(n) > 0} p^{v_p(\mu)} \mid c \text{ divides } \prod_{p \in \mathbb{P} : v_p(n) = 0} p^{v_p(\mu)} \}. \tag{6.5}
\]

Especially, if \(v_p(n) > 0\) for some prime number \(p\), then \(v_p(m) = v_p(n) + v_p(\mu)\) for any \(m \in \gamma^{-1}_\mu(n)\).

(iii) Consider \((n_a, n_b) \in \mathbb{N}^2\). If \(v_p(n_a) > v_p(n_b)\) for some prime number \(p\), then for any \((m_c, m_d) \in \gamma^{-1}_\mu(n_a) \times \gamma^{-1}_\mu(n_b)\)

\[
v_p(m_c) = v_p(n_a) + v_p(\mu) > v_p(n_b) + v_p(\mu) \geq v_p(m_d). \tag{6.6}
\]

Thus, if \((n_a, n_b)\) is no edge, then there is no edge in \(\gamma^{-1}_\mu(n_a) \times \gamma^{-1}_\mu(n_b)\). And if \((n_a, n_b)\) is a \(p\)-edge, then any edge in \(\gamma^{-1}_\mu(n_a) \times \gamma^{-1}_\mu(n_b)\) is a \(p\)-edge.

If \((n_a, n_b)\) is a \(p\)-edge and \(v_p(n_b) > 0\), then the map

\[
\gamma^{-1}_\mu(n_a) \rightarrow \gamma^{-1}_\mu(n_b) \quad \quad m \mapsto m \cdot \frac{n_b}{n_a} = m \cdot p^{-v_p(n_a) + v_p(n_b)},
\]

is a bijection and the set of \(p\)-edges in \(\gamma^{-1}_\mu(n_a) \times \gamma^{-1}_\mu(n_b)\) is

\[
\{(m, m \cdot \frac{n_b}{n_a}) \mid m \in \gamma^{-1}_\mu(n_a)\}. \tag{6.8}
\]

If \((n_a, n_b)\) is a \(p\)-edge and \(v_p(n_b) = 0\), then

\[
\gamma^{-1}_\mu(n_b) = \bigcup_{m \in \gamma^{-1}_\mu(n_a)} \{ m \cdot p^{-v_p(m) + k} \mid k \in \{0, \ldots, v_p(\mu)\}\}, \tag{6.9}
\]

and the set of \(p\)-edges in \(\gamma^{-1}_\mu(n_a) \times \gamma^{-1}_\mu(n_b)\) is

\[
\bigcup_{m \in \gamma^{-1}_\mu(n_a)} \{m\} \times \{ (m \cdot p^{-v_p(m) + k} \mid k \in \{0, \ldots, v_p(\mu)\} \}. \tag{6.10}
\]

(iv) In the case \(v_p(n_b) > 0\) says that for any \(m \in \gamma^{-1}_\mu(n_a) \cap M\) the number \(m \cdot \frac{n_b}{n_a}\) is in \(\gamma^{-1}_\mu(n_b) \cap M\), as then \((m, m \cdot \frac{n_b}{n_a})\) is the only \(p\)-edge in \(\{m\} \times \gamma^{-1}_\mu(n_b)\).

In the case \(v_p(n_b) = 0\) says that for any \(m \in \gamma^{-1}_\mu(n_a) \cap M\) the set \(\{m \cdot p^{-v_p(m) + k} \mid k \in \{0, \ldots, v_p(\mu)\}\}\) is a subset of \(\gamma^{-1}_\mu(n_b) \cap M\), as
then \( \{m\} \times \{\cdot \cdot \cdot p^{-v_p(m)+k} | k \in \{0, ..., v_p(\mu)\}\} \) is the set of \( p \)-edges in \( \{m\} \times \gamma^{-1}_{\mu}(n_b) \).

(6.4) says that for any \( m \in \gamma^{-1}_{\mu}(n_b) \cap M \) the number \( m \cdot p^{-v_p(m)+v_p(n_a)+v_p(\mu)} \) is in \( \gamma^{-1}_{\mu}(n_a) \cap M \), as then \( (m \cdot p^{-v_p(m)+v_p(n_a)+v_p(\mu)}, m) \) is the only \( p \)-edge in \( \gamma^{-1}_{\mu}(n_a) \times \{m\} \).

(v) In [He20], the group of automorphisms with eigenvalues in \( S^1 \) of an Orlik block \((H, h)\) was studied. Theorem 1.2 in [He20] gives a necessary and sufficient criterion for this group to be only \( \{\pm h^k | k \in \mathbb{Z}\} \). The criterion also uses the graph \((M, E(M))\). In fact, a \( p \)-edge \((m_a, m_b)\) here is called a \( p \)-edge there only if no \( m_c \notin \{m_a, m_b\} \) with \( m_b|m_c|m_a \) exists. The purpose of that restriction in [He20] was mainly to have graphs with not too many edges. The conditions in Theorem 1.2 in [He20] work also with \((M, E(M))\).

**Proof of Theorem 6.2**: Consider the pair \((H, h^\mu)\). Let \( p_1, ..., p_k \in \mathbb{Z}[i] \) be the unique unitary polynomials with \( p_{H,h^\mu} = p_1 \cdot ... \cdot p_k \) and \( p_k|p_{k-1}|...|p_2|p_1 \). If \((H, h^\mu)\) admits a standard decomposition into Orlik blocks, that is isomorphic to \( \bigoplus_{i=1}^k H[\nu_i] \).

We want to apply Theorem 3.1 to \((H, h^\mu)\) instead of \((H, h)\).

Let \( a_0 \in H \) be a generating element of the Orlik block \((H, h)\), so \( H = \bigoplus_{i=0}^{k-1} \mathbb{Z} \cdot h^i(a_0) \). Define

\[
\begin{align*}
    a_i &:= h^i(a_0) \quad \text{for } i \geq 0, \quad (6.11) \\
    a^{[j]}_i &:= a_{j-1} \quad \text{for } j \in \{1, ..., k\}, \quad (6.12) \\
    a^{[j]}_i &:= (h^\mu)^i(a^{[j]}_0) = a_{j-1+i} \quad \text{for } i \geq 0 \quad \text{and } j \in \{1, ..., k\}, \quad (6.13) \\
    a^{[i]} &:= (a_0^{[i]}, a_1^{[i]}, ..., a_{\deg p_j-1}^{[i]}). \quad (6.14)
\end{align*}
\]

We will show that the tuple

\[
\mathbf{a}^{dec} := (a^{[i]}_i | j \in \{1, ..., k\}, i \in \{0, ..., \deg p_j - 1\}) \quad (6.15)
\]

in (3.1) is a \( \mathbb{Z} \)-basis of \( H \) if and only if \((M, \mu)\) is sdiOb-sufficient. This and Theorem 3.1 show that \((H, h^\mu)\) admits a standard decomposition into Orlik blocks if \((M, \mu)\) is sdiOb-sufficient.

It remains to show that the tuple \( \mathbf{a}^{dec} \) is a \( \mathbb{Z} \)-basis of \( H \) if and only if \((M, \mu)\) is sdiOb-sufficient. The tuple

\[
\mathbf{a}^{st} := (a_0, a_1, ..., a_{\deg H_0}) \quad (6.16)
\]

is a \( \mathbb{Z} \)-basis of \( H \). The tuple \( \mathbf{a}^{dec} \) is a \( \mathbb{Z} \)-basis if and only if the matrix \( M(\mathbf{a}^{st}, \mathbf{a}^{dec}) \) with \( \mathbf{a}^{dec} = \mathbf{a}^{st} \cdot M(\mathbf{a}^{st}, \mathbf{a}^{dec}) \) (see the Notation 2.2) has...


determinant ±1. It remains to calculate this determinant. In order to do so, we consider also certain tuples of eigenvectors of \( h \) and \( h^\mu \).

Let \( \{\kappa_1, \kappa_2, \ldots, \kappa_{rk\ H}\} \subset \mathbb{C} \) be the set of eigenvalues of \( h \), ordered such that

\[
\text{ord}(\kappa_\alpha) \leq \text{ord}(\kappa_\beta) \quad \text{if} \quad \alpha < \beta, \quad (6.17)
\]

and let

\[
v^I = (v_1, v_2, \ldots, v_{rk\ H}) \quad (6.18)
\]

be the tuple of eigenvectors \( v_\alpha \in H_\mathbb{C} \) with

\[
h(v_\alpha) = \kappa_\alpha \cdot v_\alpha, \quad (6.19)
\]

\[
a_0 = \sum_{\alpha=1}^{rk\ H} v_\alpha. \quad (6.20)
\]

Then \( M(v^I, a^{st}) \) with \( a^{st} = v^I \cdot M(v^I, a^{st}) \) is the Vandermonde matrix

\[
M(v^I, a^{st}) = \begin{pmatrix}
1 & \kappa_1^1 & \cdots & \kappa_1^{rk\ H-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \kappa_{rk\ H}^1 & \cdots & \kappa_{rk\ H}^{rk\ H-1}
\end{pmatrix}. \quad (6.21)
\]

Let \( \{\lambda_1, \ldots, \lambda_{\deg p_1}\} \) be the set of eigenvalues of \( h^\mu \), ordered such that

\[
p_j(t) = \prod_{l=1}^{\deg p_j} (t - \lambda_l) \quad \text{for} \quad j \in \{1, \ldots, k\}. \quad (6.22)
\]

Define for \( \beta \in \{1, 2, \ldots, \deg p_1\} \) the index set \( A(\beta) \) by

\[
A(\beta) := \{\alpha \in \{1, 2, \ldots, rk\ H\} \mid \kappa_\alpha^\mu = \lambda_\beta\} =: \{\alpha(\beta, 1), \alpha(\beta, 2), \ldots, \alpha(\beta, |A(\beta)|)\}
\]

with \( \alpha(\beta, 1) < \alpha(\beta, 2) < \ldots < \alpha(\beta, |A(\beta)|) \).

The space \( \bigoplus_{\alpha \in A(\beta)} \mathbb{C} \cdot v_\alpha \subset H_\mathbb{C} \) is the eigenspace with eigenvalue \( \lambda_\beta \) of \( h^\mu \). For any \( j \in \{1, \ldots, k\} \), the vector

\[
v^{III;j,\beta} := \sum_{\alpha \in A(\beta)} \kappa_\alpha^{j-1} \cdot v_\alpha \in H_\mathbb{C} \quad (6.24)
\]
is an eigenvector with eigenvalue \( \lambda_\beta \) of \( h^\mu \). It is useful as for \( i \geq 0 \) and \( j \in \{1, \ldots, k\} \)

\[
a^{[j]}_i = a_{j-1+i} = \sum_{\alpha=1}^{rkH} \kappa^{j-1+\mu+i}_{\alpha} \cdot v_\alpha
\]

\[
= \sum_{\beta=1}^{\deg p_1} \lambda_\beta^i \cdot \sum_{\alpha \in A(\beta)} \kappa^{j-1}_{\alpha} \cdot v_\alpha = \sum_{\beta=1}^{\deg p_1} \lambda_\beta^i \cdot v^{III,j,\beta}. \tag{6.25}
\]

Consider for \( j \in \{1, \ldots, k\} \) and \( \beta \in \{1, 2, \ldots, \deg p_1\} \) the following tuples of eigenvectors of \( h \) and/or \( h^\mu \),

\[
v^{II,\beta} := (v_{\alpha(\beta,1)}, v_{\alpha(\beta,2)}, \ldots, v_{\alpha(\beta,|A(\beta)|)}), \tag{6.26}
\]

\[
v^{II} := (v^{II,1}, \ldots, v^{II,\deg p_1}), \tag{6.27}
\]

\[
v^{III,\beta} := (v^{III,1,\beta}, \ldots, v^{III,|A(\beta)|,\beta}), \tag{6.28}
\]

\[
v^{III} := (v^{III,1,1}, \ldots, v^{III,\deg p_1}), \tag{6.29}
\]

\[
v^{IV,j} := (v^{IV,j,1}, v^{IV,j,2}, \ldots, v^{IV,j,\deg p_1}), \tag{6.30}
\]

\[
v^{IV} := (v^{IV,1}, \ldots, v^{IV,k}). \tag{6.31}
\]

\( v^{II,\beta} \) and \( v^{III,\beta} \) are \( \mathbb{C} \)-bases of the eigenspace with eigenvalue \( \lambda_\beta \) of \( h^\mu \). The base change matrix \( M(v^{II,\beta}, v^{III,\beta}) \) rewrites the relation \( (6.24) \). It is a Vandermonde matrix,

\[
M(v^{II,\beta}, v^{III,\beta}) = \begin{pmatrix}
1 & \kappa^{1}_{\alpha(\beta,1)} & \cdots & \kappa^{|A(\beta)|-1}_{\alpha(\beta,1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \kappa^{1}_{\alpha(\beta,|A(\beta)|)} & \cdots & \kappa^{|A(\beta)|-1}_{\alpha(\beta,|A(\beta)|)}
\end{pmatrix}. \tag{6.33}
\]

\( v^I, v^{II}, v^{III} \) and \( v^{IV} \) are \( \mathbb{C} \)-bases of \( H_\mathbb{C} \). The base change matrices \( M(v^I, v^{II}) \) and \( M(v^{III}, v^{IV}) \) are just permutation matrices. The entries of \( v^{IV,j} \) are linearly independent. Therefore the following rectangular \( (\deg p_1) \times (\deg p_j) \)-matrix is well defined. It rewrites the relations \( (6.25) \).

\[
M(v^{IV,j}, a^{[j]}) = \begin{pmatrix}
1 & \lambda^1_1 & \cdots & \lambda^{\deg p_j-1}_1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda^{\deg p_1}_1 & \cdots & \lambda^{\deg p_j-1}_{\deg p_1}
\end{pmatrix}. \tag{6.34}
\]

Now we want to describe the matrix \( M(v^{IV}, a^{dec}) \). Observe that \( v^{III,j,\beta} \) is in the case \( j > |A(\beta)| \) a linear combination of the entries of \( v^{III,\beta} \). Therefore the matrix \( M(v^{IV}, a^{dec}) \) is a block upper triangular matrix.
whose diagonal blocks are obtained from the matrices in (6.34) by cut-
ting off the lower lines. The diagonal blocks are the Vandermonde
matrices
\[
M[j] := \begin{pmatrix}
1 & \lambda_1^1 & \ldots & \lambda_1^{\deg p_j - 1} \\
\vdots & \vdots & & \vdots \\
1 & \lambda_1^{\deg p_j} & \ldots & \lambda_1^{\deg p_j - 1}
\end{pmatrix}.
\] (6.35)

Now the matrix \( M(a^{st}, a^{dec}) \) can be written as the product of matrices
\[
M(a^{st}, v^I)M(v^I, v^{II})M(v^{II}, v^{III})M(v^{III}, v^{IV})M(v^{IV}, a^{dec}). \] (6.36)
The absolute value of its determinant is
\[
| \det M(a^{st}, a^{dec}) | = | \det M(v^{II}, v^{III}) \cdot \det M(v^{IV}, a^{dec}) | \]
\[
= \left| \frac{\prod_{\beta=1}^{\deg p_1} \det M(v^{II,\beta}, v^{III,\beta}) \cdot \prod_{j=1}^{k} \det M[j]}{\det M(v^I, a^{st})} \right|. \] (6.37)
It is the absolute value of a quotient of determinants of Vandermonde
matrices. The determinants are
\[
\det M(v^I, a^{st}) = \prod_{1 \leq \alpha_1 < \alpha_2 \leq \rk H} (\kappa_{\alpha_2} - \kappa_{\alpha_1}), \] (6.38)
\[
\det M(v^{II,\beta}, v^{III,\beta}) = \prod_{\alpha_1, \alpha_2 \in A(\beta) : \alpha_1 < \alpha_2} (\kappa_{\alpha_2} - \kappa_{\alpha_1}), \] (6.39)
\[
\det M[j] = \prod_{1 \leq \beta_1 < \beta_2 \leq \deg p_j} (\lambda_{\beta_2} - \lambda_{\beta_1}). \] (6.40)

Only now we use that \( \lambda_\beta \) and \( \kappa_\alpha \) are unit roots. For \( n \in \gamma_\mu(M) \)
we denote the multiplicity as a zero of \( p_{H,h^\mu} \) of any unit root \( \lambda \) with
\( \ord(\lambda) = n \) by \( \psi(n) \in \mathbb{N}_0 \). It is
\[
\psi(n) = \varphi(n)^{-1} \cdot \sum_{m \in \gamma_\mu^{-1}(n) \cap M} \varphi(m) \] (6.41)
\[
= \max\{j \in \{1, \ldots, k\} \mid \Phi_n(p_j) \leq k\}. \] (6.42)
In view of the formulas (4.3) and (4.9), the determinants in (6.38) and (6.40) can be written as follows as products of resultants and discriminants,

\[
|\det M(v^I, a^{st})| = \prod_{m_c, m_d \in M: m_c > m_d} |\text{Res}(\Phi_{m_c}, \Phi_{m_d})|
\]

\[
\cdot \prod_{m \in M} \sqrt{|\text{Discr}(\Phi_m)|},
\]

(6.43)

\[
\prod_{j=1}^k |\det M[j]| = \prod_{n_a, n_b \in \gamma_\mu(M): n_a > n_b} |\text{Res}(\Phi_{n_a}, \Phi_{n_b})|^{\min(\psi(n_a), \psi(n_b))}
\]

\[
\cdot \prod_{n \in \gamma_\mu(M)} \sqrt{|\text{Discr}(\Phi_n)|}^{\psi(n)}.
\]

(6.44)

Thus we can rearrange \(|\det M(a^{st}, a^{dec})|\) as a product of the following factors in (6.45) and (6.46): For each pair \((n_a, n_b) \in \gamma_\mu(M)^2\) with \(n_a > n_b\)

\[
\prod_{(m_c, m_d) \in \gamma_\mu^{-1}(n_a) \times \gamma_\mu^{-1}(n_b) \cap M^2} |\text{Res}(\Phi_{m_c}, \Phi_{m_d})|^{\min(\psi(n_a), \psi(n_b))}
\]

\[
\prod_{n \in \gamma_\mu(M)} \sqrt{|\text{Discr}(\Phi_n)|}^{\psi(n)}.
\]

(6.45)

And for each \(n \in \gamma_\mu(M)\)

\[
\frac{\sqrt{|\text{Discr}(\Phi_n)|}^{\psi(n)}}{\prod_{\beta: \text{ord}(\lambda_\beta) = n} \prod_{\alpha_1, \alpha_2 \in A(\beta): \alpha_1 < \alpha_2} \mid \kappa_{\alpha_2} - \kappa_{\alpha_1}}
\]

(6.46)

We will now prove the following claims:

(A) Each factor of type (6.45) with \((n_a, n_b) \notin E(\gamma_\mu(M))\) is equal to 1.

(B) Each factor in (6.45) with \((n_a, n_b) \in E_p(\gamma_\mu(M))\) for some prime number \(p\) is a positive integer, and it is equal to 1 if and only if (6.3) or (6.4) is satisfied.

(C) Each factor of type (6.46) is equal to 1.

Together (A), (B) and (C) give that \(|\det M(a^{st}, a^{dec})|\) is equal to 1 if and only if \((M, \mu)\) is sdiOb-sufficient. Therefore the tuple in (3.13) with \(a_0^{[j]}\) as in (6.12) is a \(\mathbb{Z}\)-basis if and only if \((M, \mu)\) is sdiOb-sufficient.

It remains to prove the claims (A), (B) and (C).

Claim (A): If \((n_a, n_b)\) is not an edge, then \(\text{Res}(\Phi_{n_a}, \Phi_{n_b}) = 1\) because of (4.24). By Remark 6.3 (iii), then any pair \((m_c, m_d) \in \gamma_\mu^{-1}(n_a) \times \gamma_\mu^{-1}(n_b)\) is also not an edge, and again \(\text{Res}(\Phi_{m_c}, \Phi_{m_d}) = 1\) because of (4.24).
Claim (B): Let \((n_a, n_b)\) be a \(p\)-edge of \(\gamma_\mu(M)\). Then by Remark 6.3 (iii), also any edge in \(\gamma_\mu^{-1}(n_a) \cap \gamma_\mu^{-1}(n_b)\) is a \(p\)-edge. Therefore and by Theorem 4.5 (d), the numerator and the denominator in (6.45) are powers of \(p\),

\[
|\text{Res}(n_a, n_b)| = p^{\varphi(n_b)}, \quad (6.47)
\]

\[
|\text{Res}(m_c, m_d)| = \begin{cases} 1 & \text{if } (m_c, m_d) \notin E_p(M), \\ p^{\varphi(m_d)} & \text{if } (m_c, m_d) \in E_p(M). \end{cases} \quad (6.48)
\]

We claim

\[
\prod_{(m_c, m_d) \in E_p \cap (\gamma_\mu^{-1}(n_a) \cap M) \times \gamma_\mu^{-1}(n_b))} \frac{|\text{Res}(\Phi_{m_c}, \Phi_{m_d})|^{\psi(n_a)}}{|\text{Res}(\Phi_{n_a}, \Phi_{n_b})|^{\psi(n_b)}} = 1, \quad (6.49)
\]

\[
\prod_{(m_c, m_d) \in E_p \cap (\gamma_\mu^{-1}(n_a) \cap M) \times \gamma_\mu^{-1}(n_b) \cap M)} \frac{|\text{Res}(\Phi_{m_c}, \Phi_{m_d})|^{\psi(n_a)}}{|\text{Res}(\Phi_{n_a}, \Phi_{n_b})|^{\psi(n_b)}} = 1. \quad (6.50)
\]

The denominator in (6.49) is a multiple of the denominator in (6.45), and they are equal if and only if (6.3) holds. The denominator in (6.50) is a multiple of the denominator in (6.45), and they are equal if and only if (6.4) holds. Therefore the quotient in (6.45) is equal to 1 if and only if (6.3) or (6.4) hold. It remains to prove (6.49) and (6.50).

We use the Remarks 6.3 (iii) and (iv). First suppose \(v_p(n_b) > 0\). Then

\[
E_p \cap ((\gamma_\mu^{-1}(n_a) \cap M) \times \gamma_\mu^{-1}(n_b)) = \{ (m, m \cdot \frac{n_b}{n_a}) \mid m \in \gamma_\mu^{-1}(n_a) \cap M \}, \quad (6.51)
\]

\[
v_p(\text{denominator of } (6.49)) = \sum_{m \in \gamma_\mu^{-1}(n_a) \cap M} \varphi(m \cdot \frac{n_b}{n_a})
= \sum_{m \in \gamma_\mu^{-1}(n_a) \cap M} \varphi(m) \cdot \frac{\varphi(n_b)}{\varphi(n_a)} = \psi(n_a) \cdot \varphi(n_b)
= v_p(\text{numerator of } (6.49)). \quad (6.52)
\]

This shows (6.49) in the case \(v_p(n_b) > 0\).

Now suppose \(v_p(n_b) = 0\). Then

\[
E_p \cap ((\gamma_\mu^{-1}(n_a) \cap M) \times \gamma_\mu^{-1}(n_b)) = \bigcup_{m \in \gamma_\mu^{-1}(n_a) \cap M} \{ m \} \times \{ m \cdot p^{-v_p(m)+k} \mid k \in \{0, \ldots, v_p(\mu)\} \}, \quad (6.53)
\]
For pairs \((\alpha, p)\) of \((6.46)\) are positive integers. Fix a prime number.

\[
\kappa \text{ (see (4.18) and Theorem 4.5 (a)). Only pairs with } \Omega(\kappa_{\alpha_2}/\kappa_{\alpha_1}) = p^k \text{ for some } k \geq 1 \text{ and with } p^k|\mu \text{ give a contribution to } v_p((\text{numerator of } (6.46))^2). \text{ Its size for any } k \geq 1 \text{ is } 2\Lambda(m_2, m_1, p, k) \text{ if } m_2 \neq m_1 \text{ and } \Lambda(m_2, m_2, p, k) \text{ if } m_2 = m_1.
\]

This shows \((6.49)\) in the case \(v_p(n_1) = 0\).

In both cases, \(v_p(n_1) > 0 \) or \(v_p(n_1) = 0\), we have

\[
E_p \cap (\gamma_{\mu}^{-1}(n_1) \times (\gamma_{\mu}^{-1}(n_1) \cap M))
= \{(m \cdot p^{-v_p(m)}) + v_p(\mu), m) \mid m \in \gamma_{\mu}^{-1}(n_1) \cap M\}, \tag{6.55}
\]

\[
v_p(\text{denominator of } (6.50)) = \sum_{m \in \gamma_{\mu}^{-1}(n_1) \cap M} \varphi(m)
= \psi(n_1) \cdot \varphi(n_1) = v_p(\text{numerator of } (6.50)). \tag{6.56}
\]

This shows \((6.50)\).

**Claim (C):** The squares of the numerator and of the denominator of \((6.46)\) are positive integers. Fix a prime number \(p\). We will show

\[
v_p((\text{numerator of } (6.46))^2) = v_p((\text{denominator of } (6.46))^2). \tag{6.57}
\]

The second factor in the numerator is the most difficult part. It is the product of \(|\kappa_{\alpha_2} - \kappa_{\alpha_1}|\) over the pairs \((\alpha_2, \alpha_1)\) in the set

\[
\{(\alpha_2, \alpha_1) \mid 1 \leq \alpha_1 < \alpha_2 \leq \text{rk } H, \text{ord}(\kappa_{\alpha_2}^\mu) = n, (\kappa_{\alpha_2}/\kappa_{\alpha_1})^\mu = 1\}. \tag{6.58}
\]

For pairs \((\alpha_2, \alpha_1)\) in \((6.58)\) we denote \(m_2 := \text{ord}(\kappa_{\alpha_2})\) and \(m_1 := \text{ord}(\kappa_{\alpha_1})\). In order to understand their contribution to \(v_p((\text{numerator of } (6.46))^2)\), we have to consider Theorem 4.5 (c) and

\[
\text{Norm}_{\text{ord}(\kappa)}(1 - \kappa) = \begin{cases} q & \text{if ord}(\kappa) = q^l \text{ for some } q \in \mathcal{P}, l \geq 1, \\ 1 & \text{else} \end{cases}, \tag{6.59}
\]

(see \((4.18)\) and Theorem 4.5 (a)). Only pairs with \(\Omega(\kappa_{\alpha_2}/\kappa_{\alpha_1}) = p^k\) for some \(k \geq 1\) and with \(p^k|\mu\) give a contribution to \(v_p((\text{numerator of } (6.46))^2)\). Its size for any \(k \geq 1\) is \(2\Lambda(m_2, m_1, p, k)\) if \(m_2 \neq m_1\) and \(\Lambda(m_2, m_2, p, k)\) if \(m_2 = m_1\).
If \( v_p(n) > 0 \), pairs \( (\alpha_2, \alpha_1) \) in (6.58) with \( m_2 \neq m_1 \) give no contribution as \( v_p(m_2) = v_p(m_1) (= v_p(n) + v_p(\mu)) \) by Remark 6.3 (ii), and thus neither \( m_2/m_1 \) nor \( m_1/m_2 \) is a power of \( p \). Pairs \( (\alpha_2, \alpha_1) \) in (6.58) with \( m_2 = m_1 \) and \( \text{ord}(\kappa_{\alpha_2}/\kappa_{\alpha_1}) = p^k \) satisfy \( k \leq v_p(\mu) < v_p(m_2) \) because of \( (\kappa_{\alpha_2}/\kappa_{\alpha_1})^\mu = 1 \). Therefore and because of the third line of (4.22), the contribution of the square of the second factor in the numerator is in the case \( v_p(n) > 0 \)

\[
\sum_{m \in \gamma_{\mu}^{-1}(n) \cap M : v_p(m) > 0} \sum_{k=1}^{v_p(m)-1} (\sum_{k=1}^{(\varphi(m) + \varphi(m) \cdot \frac{p-2}{p-1})}) = \sum_{m \in \gamma_{\mu}^{-1}(n) \cap M : v_p(m) > 0} (v_p(m) - \frac{1}{p-1}) \cdot \varphi(m). \tag{6.61}
\]

If \( v_p(n) = 0 \), the pairs \( (\alpha_2, \alpha_1) \) in (6.58) with \( m_2 = m_1 \) give no restriction on \( k \), as anyway \( k \leq v_p(m_2) \leq v_p(\mu) \). Here any \( k \in \{1, ... , v_p(m_2)\} \) arises. Thus the pairs \( (\alpha_2, \alpha_1) \) in (6.58) with \( m_2 = m_1 \) give in the case \( v_p(n) = 0 \) the contribution

\[
\sum_{m \in \gamma_{\mu}^{-1}(n) \cap M : v_p(m) > 0} (v_p(m) - \frac{1}{p-1}) \cdot \varphi(m). \tag{6.61}
\]

If \( v_p(n) = 0 \), the pairs \( (\alpha_2, \alpha_1) \) in (6.58) with \( m_2 \neq m_1 \) give a contribution only if \( m_2/m_1 \) is a power of \( p \). This contribution for all \( (\alpha_2, \alpha_1) \) with fixed \( m_2 \) and \( m_1 = \varphi(m_1) \). It is the same as the contribution of the part with \( m_c = m_2 \) and \( m_d = m_1 \) of the second factor in the denominator of (6.46). Thus these contributions cancel.

In the case \( v_p(n) > 0 \) by Remark 6.3 (ii), any \( m \in \gamma_{\mu}^{-1}(n) \) satisfies \( v_p(m) = v_p(n) + v_p(\mu) \). Therefore \( \gamma_{\mu}^{-1}(n)^2 \cap E_p(M) = \emptyset \), and because of (6.48) the second factor in the denominator gives no contribution at all,

\[
v_p \left( \prod_{m_c, m_d \in \gamma_{\mu}^{-1}(n) \cap M : m_c > m_d} \text{Res}(\Phi_{m_c}, \Phi_{m_d}) \right) = 0. \tag{6.62}
\]

We are left with the contributions of the first factors of the numerator and the denominator of (6.46) and with (6.60) in the case \( v_p(n) > 0 \) and with (6.61) in the case \( v_p(n) = 0 \).

Consider the case \( v_p(n) > 0 \). Then (4.27) gives

\[
v_p(\text{Discr}(\Phi_n) \psi(n)) = (v_p(n) - \frac{1}{p-1}) \cdot \psi(n) \cdot \phi(n). \tag{6.63}
\]
and
\[
v_p \left( \prod_{m \in \gamma_{n-1}(n) \cap M} \text{Discr}(\Phi_m) \right)
= \sum_{m \in \gamma_{n-1}(n) \cap M} \left( v_p(m) - \frac{1}{p-1} \right) \cdot \varphi(m)
= \sum_{m \in \gamma_{n-1}(n) \cap M} \left( v_p(n) + v_p(\mu) - \frac{1}{p-1} \right) \cdot \varphi(m)
= (v_p(n) + v_p(\mu) - \frac{1}{p-1}) \cdot \psi(n) \cdot \varphi(n)
= \text{(the contributions in (6.60) and (6.63)).} \quad (6.64)
\]
This shows (6.57) in the case \( v_p(n) > 0 \).

Consider the case \( v_p(n) = 0 \). Then (4.27) gives \( v_p(\text{Discr}(\Phi_n))) = 0 \) and
\[
v_p \left( \prod_{m \in \gamma_{n-1}(n) \cap M} \text{Discr}(\Phi_m) \right)
= \sum_{m \in \gamma_{n-1}(n) \cap M : v_p(m) > 0} \left( v_p(m) - \frac{1}{p-1} \right) \cdot \varphi(m)
= \text{(the contribution in (6.61)).} \quad (6.65)
\]
This shows (6.57) in the case \( v_p(n) = 0 \). \qed

7. When does the tensor product of two Orlik blocks admit a standard decomposition into Orlik blocks?

Theorem 7.4 starts with two Orlik blocks \((G, g)\) and \((H, h)\) and gives a sufficient criterion for \((G \otimes H, g \otimes h)\) to admit a standard decomposition into Orlik blocks. It will be crucial for the Thom-Sebastiani sums of singularities. The condition will work with the sets \( M \subset \mathbb{N} \) and \( N \subset \mathbb{N} \) of orders of eigenvalues of \( g : G_\mathbb{C} \to G_\mathbb{C} \) and \( h : H_\mathbb{C} \to H_\mathbb{C} \). Theorem 7.4 is preceded by some definitions and observations. This section has similarities with section 6. Though the statement and the proof are more involved.

Definition 7.1. (a) Denote by \( \mu(\mathbb{C}) \subset S^1 \) the group of all unit roots. Denote by \( \mathbb{Z}[\mu(\mathbb{C})] \) the group ring with elements \( \sum_{j=1}^t a_j [\lambda_j] \) where \( a_j \in \mathbb{Z} \) and \( \lambda_j \in \mu(\mathbb{C}) \) and with multiplication \( [\lambda_1][\lambda_2] = [\lambda_1 \lambda_2] \).
The trace and the degree of an element are
\[
\text{tr} \left( \sum_{j=1}^{l} a_j \lambda_j \right) := \sum_{j=1}^{l} a_j \lambda_j \in \mathbb{C}, \quad (7.1)
\]
\[
\deg \left( \sum_{j=1}^{l} a_j \lambda_j \right) := \sum_{j=1}^{l} a_j \in \mathbb{Z}. \quad (7.2)
\]
The trace map \( \text{tr} : \mathbb{Z}[\mu(\mathbb{C})] \to \mathbb{C} \) and the degree map \( \deg : \mathbb{Z}[\mu(\mathbb{C})] \to \mathbb{Z} \) are ring homomorphisms.

(b) The divisor of a unitary polynomial \( f = \prod_{j=1}^{l} (t - \lambda_j) \) with \( \lambda_j \in \mu(\mathbb{C}) \) is
\[
\text{div}(f) := \sum_{j=1}^{l} [\lambda_j] \in \mathbb{Z}[\mu(\mathbb{C})]. \quad (7.3)
\]
The divisor of an endomorphism \( F : H_{\mathbb{C}} \to H_{\mathbb{C}} \) of a finite dimensional complex vector space \( H_{\mathbb{C}} \) with characteristic polynomial \( f \) is
\[
\text{div}(F) := \text{div}(f). \]
Then \( \deg(f) = \deg(\text{div}(f)) \).

(c) For two unitary polynomials \( f = \prod_{j=1}^{l} (t - \lambda_j) \) and \( g = \prod_{i=1}^{k} (t - \kappa_i) \) with \( \lambda_j, \kappa_i \in \mu(\mathbb{C}) \), define the new unitary polynomial \( f \otimes g \) with zeros in \( \mu(\mathbb{C}) \) by
\[
f \otimes g := \prod_{j=1}^{l} \prod_{i=1}^{k} (t - \lambda_j \kappa_i). \quad (7.4)
\]
Then
\[
\text{div}(f \otimes g) = \text{div}(f) \cdot \text{div}(g), \quad (7.5)
\]
\[
\text{tr} \left( \text{div}(f \otimes g) \right) = \text{tr}(\text{div}(f)) \cdot \text{tr}(\text{div}(g)), \quad (7.6)
\]
\[
\deg(f \otimes g) = \deg(f) \cdot \deg(g). \quad (7.7)
\]

(d) For \( m \in \mathbb{N} \) define
\[
\Lambda_m := \text{div}(t^m - 1), \quad \Psi_m := \text{div}(\Phi_m). \quad (7.8)
\]
Of course, then \( \Lambda_m = \sum_{d|m} \Psi_d \), \( \deg(\Lambda_m) = m \), \( \deg(\Psi_m) = \varphi(m) \).

(e) Define two maps \( \beta \) and \( \delta \),
\[
\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad \beta(m, n) := \prod_{p \in P: v_p(m) = v_p(n) > 0} p^{v_p(m)}, \quad (7.9)
\]
\[ \delta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}_0, \quad (7.10) \]
\[
\delta(m, n, l) := \begin{cases} 
\varphi(\gcd(m, n)) \cdot \prod_{p \in \mathcal{P}, 0 = v_p(c) < v_p(\beta(m, n))} \frac{p^{p-2} - p^{-2}}{p-1} & \text{if } \operatorname{lcm}(m, n) = c \cdot l \text{ with } c|\beta(m, n), \\
0 & \text{else}.
\end{cases}
\]

Lemma 7.2.
\[
\Lambda_m \cdot \Lambda_n = \gcd(m, n) \cdot \Lambda_{\operatorname{lcm}(m, n)} \quad \text{for } m, n \in \mathbb{N}, \quad (7.11)
\]
\[
[\lambda] \cdot \Lambda_m = \Lambda_m \quad \text{for } \lambda \in \mu(\mathbb{C}) \text{ with } \operatorname{ord}(\lambda)|m, \quad (7.12)
\]
\[
\Psi_m \cdot \Psi_n = \sum_{l \in \mathbb{N}} \delta(m, n, l) \cdot \Psi_l. \quad (7.13)
\]

Proof: (7.11) and (7.12) are obvious. (7.13) follows from the special cases
\[
\Psi_m \cdot \Psi_n = \Psi_{m \cdot n} \quad \text{if } \gcd(m, n) = 1, \quad (7.14)
\]
\[
\Psi_{p^k} \cdot \Psi_{p^{k'}} = \varphi(p^{k'}) \cdot \Psi_{p^k} \quad \text{for } p \in \mathcal{P} \text{ and } k > l \geq 0, \quad (7.15)
\]
\[
\Psi_{p^k} \cdot \Psi_{p^{k'}} = \frac{p^{2p-2} - p^{-2}}{p-1} \cdot \varphi(p^k) \cdot \Psi_{p^k} + \varphi(p^k) \cdot \sum_{l=0}^{k-1} \Psi_{p^l} \quad (7.16)
\]
for \( p \in \mathcal{P} \) and \( k > 0 \),
which follow easily from Theorem 4.5 (c). \( \square \)

Definition 7.3. (a) For a prime number \( p \) define the projection
\[
\pi_p : \mathbb{N} \rightarrow \mathbb{N}, \quad m \mapsto m \cdot p^{-v_p(m)}. \quad (7.17)
\]
Then \( \pi_p(\mathbb{N}) = \{m \in \mathbb{N} \mid v_p(m) = 0\} \).

(b) Now fix two finite non-empty sets \( M, N \subset \mathbb{N} \). Then
\[
\left( \sum_{m \in M} \Psi_m \right) \cdot \left( \sum_{n \in N} \Psi_n \right) = \sum_{(m, n, l) \in M \times N \times \mathbb{N}} \delta(m, n, l) \cdot \Psi_l = \sum_{l \in \mathbb{N}} \chi(l) \cdot \Psi_l = \sum_{l \in L} \chi(l) \cdot \Psi_l \quad (7.18)
\]
where
\[
\chi(l) := \sum_{(m, n) \in M \times N} \delta(m, n, l) \quad \text{for } l \in \mathbb{N} \quad (7.19)
\]
is the multiplicity of \( \Psi_l \) in the product \( \left( \sum_{m \in M} \Psi_m \right) \cdot \left( \sum_{n \in N} \Psi_n \right) \) and
\[
L := L(M, N) := \{l \in \mathbb{N} \mid \chi(l) > 0\} \quad (7.20)
\]
is the set of numbers \( l \in \mathbb{N} \) such that \( \Psi_l \) turns up in this product.
(c) Fix two finite non-empty sets $M, N \subset \mathbb{N}$. For each choice of a prime number $p$, it will be useful to decompose $\chi(l)$ into pieces as follows. For $p \in \mathcal{P}$ and $(m_0, n_0) \in \pi_p(M) \times \pi_p(N)$ and $l \in L$ define

$$\chi_{p,m_0,n_0}(l) := \sum_{(m,n) \in (M \cap \pi_p^{-1}(m_0)) \times (N \cap \pi_p^{-1}(n_0))} \delta(m,n,l). \quad (7.21)$$

Then

$$\chi(l) = \sum_{(m_0,n_0) \in \pi_p(M) \times \pi_p(N)} \chi_{p,m_0,n_0}(l). \quad (7.22)$$

(d) The pair $(M, N)$ of finite non-empty subsets of $\mathbb{N}$ is called $sdiOb$-sufficient ($sdiOb$ for standard decomposition into Orlik blocks) if for any prime number $p$ and any $p$-edge $(l_a, l_b) \in E_p(L)$ at least one of the following two conditions holds:

$$\chi_{p,m_0,n_0}(l_b) \leq \chi_{p,m_0,n_0}(l_a) \text{ for any } (m_0, n_0) \in \pi_p(M) \times \pi_p(N), \quad (7.23)$$

$$\chi_{p,m_0,n_0}(l_b) \geq \chi_{p,m_0,n_0}(l_a) \text{ for any } (m_0, n_0) \in \pi_p(M) \times \pi_p(N) \quad (7.24)$$

(these conditions will be discussed in Lemma 7.6).

**Theorem 7.4.** Consider two Orlik blocks $(G, g)$ and $(H, h)$. Let $M$ and $N \subset \mathbb{N}$ be the finite sets of orders of $g : G_C \rightarrow G_C$ respectively $h : H_C \rightarrow H_C$. Then $(G \otimes H, g \otimes h)$ admits a standard decomposition into Orlik blocks if $(M, N)$ is $sdiOb$-sufficient.

Theorem 7.4 will be proved in Section 8. Here in Section 7 we make the Remarks 7.5, we make the property $sdiOb$-sufficient explicit in Lemma 7.6, and we give the Examples 7.7.

**Remarks 7.5.** (i) We expect that if and only if holds in Theorem 7.4.

(ii) Of course,

$$\text{div}(g) = \sum_{m \in M} \Psi_m, \quad \text{div}(h) = \sum_{n \in N} \Psi_n, \quad (7.25)$$

$$\text{div}(g \otimes h) = \text{div}(g) \cdot \text{div}(h) = \sum_{l \in L} \chi(l) \cdot \Psi_l. \quad (7.26)$$
(iii) For any \( l \in \mathbb{N} \), the set of pairs \( (m, n) \in \mathbb{N}^2 \) with \( \delta(m, n, l) > 0 \) is infinite and is as follows,

\[
\{(m, n) \in \mathbb{N}^2 \mid \delta(m, n, l) > 0\} = \{(m, n) \in \mathbb{N}^2 \mid \text{one has for any prime number } p:\right.
\begin{align*}
&\text{either } v_p(n) < v_p(m) = v_p(l), \\
&\text{or } v_p(m) < v_p(n) = v_p(l), \\
&\text{or } v_p(m) = v_p(n) \begin{cases} \\
\geq v_p(l) & \text{if } p \geq 3 \\
> v_p(l) & \text{if } p = 2 \text{ and } v_p(m) = 0, \\
> v_p(l) & \text{if } p = 2 \text{ and } v_p(m) > 0. \\
\end{cases}
\end{align*}
\] (7.27)

(iv) In section 6, the two conditions (6.3) and (6.4) for sdiOb-sufficiency of a pair \((M, \mu)\) were formulated only in terms of existence of \( p \)-edges. This was made explicit in Remark 6.3 (iv). The two conditions (7.23) and (7.24) for sdiOb-sufficiency of a pair \((M, N)\) can also be made explicit by necessary and sufficient conditions. But these are more involved. We do not give all details. We will need only the sufficient conditions in part (a) of Lemma 7.6 and the special case where everything is vanishing in part (b).

**Lemma 7.6.** Consider the data in Remark 7.3 (d), so two finite non-empty sets \( M, N \subset \mathbb{N} \), a prime number \( p \) and numbers \( m_0 \in \pi_p(M), n_0 \in \pi_p(N), l_a, l_b \in L \) with \((l_a, l_b) \in E_p(L)\). Write \( k_a := v_p(l_a) > k_b := v_p(l_b) \geq 0 \). Define two finite sets of exponents \( K_{M,p,m_0} \) and \( K_{N,p,n_0} \subset \mathbb{N}_0 \) (they are non-empty because of \( m_0 \in \pi_p(M) \) and \( n_0 \in \pi_p(N) \)) by

\[
\begin{align*}
M \cap \pi_p^{-1}(m_0) &= \{p^k m_0 \mid k \in K_{M,p,m_0}\}, \\
N \cap \pi_p^{-1}(n_0) &= \{p^k n_0 \mid k \in K_{N,p,n_0}\}.
\end{align*}
\] (7.28)

(a) Then

\[
k_a \in K_{M,p,m_0} - K_{N,p,n_0} \text{ or } k_a \in K_{N,p,n_0} - K_{M,m_0} \\
\Rightarrow \chi_{p,m_0,n_0}(l_a) \leq \chi_{p,m_0,n_0}(l_a). \tag{7.29}
\]

\[
k_a \notin K_{M,p,m_0} \cup K_{N,p,n_0} \text{ or } k_a \in K_{M,p,m_0} \cap K_{N,p,n_0} \\
\Rightarrow \chi_{p,m_0,n_0}(l_a) \geq \chi_{p,m_0,n_0}(l_a). \tag{7.30}
\]

(b) If \( \delta(m_0, n_0, \pi_p(l_0)) = 0 \), then \( \chi_{p,m_0,n_0}(l_a) = \chi_{p,m_0,n_0}(l_b) = 0 \).

**Proof:** Write \( l_0 := \pi_p(l_0) = \pi_p(l_b) \). Then \( l_a = l_0 \cdot p^{k_a}, l_b = l_0 \cdot p^{k_b} \). For \( m = m_0 \cdot p^{k_1} \in M \cap \pi_p^{-1}(m_0) \) (so \( k_1 \in K_{M,p,m_0} \)) and \( n = n_0 \cdot p^{k_2} \in N \cap \pi_p^{-1}(n_0) \) (so \( k_2 \in K_{N,p,n_0} \)) and \( l = l_0 \cdot p^k \in L \)

\[
\delta(m, n, l) = \delta(m_0, n_0, l_0) \cdot \delta(p^{k_1}, p^{k_2}, p^k). \tag{7.31}
\]
If \( \delta(m_0, n_0, l_0) = 0 \), then \( \chi_{p,m_0,n_0}(l_a) = \chi_{p,m_0,n_0}(l_b) = 0 \). This proves part (b). And (7.29) and (7.30) hold trivially in this case.

Suppose now \( \delta(m_0, n_0, l_0) \neq 0 \). Then for \( l = l_0 \cdot p^k \in L \) as above.

\[
\frac{\chi_{p,m_0,n_0}(l)}{\delta(m_0, n_0, l_0)} = \sum_{(k_1,k_2)\in K_{M,p,m_0}\times K_{N,p,n_0}} \delta(p^{k_1}, p^{k_2}, p^k)
\]

\[
= \delta(k\in K_{M,p,m_0}) \cdot \sum_{k_2\in K_{N,p,n_0}; k_2<k} \varphi(p^{k_2}) + \delta(k\in K_{N,p,n_0}) \cdot \sum_{k_1\in K_{M,p,m_0}; k_1<k} \varphi(p^{k_1})
\]

\[
+ \delta(k\in K_{M,p,m_0}\cap K_{N,p,n_0}\cap \mathbb{N}) \cdot \varphi(p^{k_2}) \cdot \frac{p-2}{p-1} + \delta(k\in K_{M,p,m_0}\cap K_{N,p,n_0}\cap \{0\})
\]

\[
+ \sum_{k_1\in K_{M,p,m_0}\cap K_{N,p,n_0}; k_1>k} \varphi(p^{k_1}).
\]

The following notations will be used to rewrite the difference \((\chi_{p,m_0,n_0}(l_a) - \chi_{p,m_0,n_0}(l_b))/\delta(m_0, n_0, l_0)\) in (7.39).

\[
A_{M,1} := \sum_{k_1\in K_{M,p,m_0}; k_1<k_b} \varphi(p^{k_1}),
\]

\[
A_{N,1} := \sum_{k_2\in K_{N,p,n_0}; k_2<k_b} \varphi(p^{k_2}),
\]

\[
A_{M,2} := \sum_{k_1\in K_{M,p,m_0}; k_b<k_1<k_a} \varphi(p^{k_1}),
\]

\[
A_{N,2} := \sum_{k_2\in K_{N,p,n_0}; k_b<k_2<k_a} \varphi(p^{k_2}),
\]

\[
A_3 := \sum_{k_1\in K_{M,p,m_0}\cap K_{N,p,n_0}; k_b<k_1<k_a} \varphi(p^{k_1}).
\]
The case $k_a \in K_{M,p,m_0} - K_{N,p,n_0}$ is treated analogously. This shows (7.29).

Now we prove (7.30). If $k_a \notin K_{M,p,m_0} \cup K_{N,p,n_0}$, then (7.39) is

$$\chi_{p,m_0,n_0}(l_a) - \chi_{p,m_0,n_0}(l_b)$$

$$\delta(m_0, n_0, l_0)$$

$$= (\delta(k_a \in K_{M,p,m_0}) - \delta(k_b \in K_{M,p,m_0})) \cdot A_{N,1}$$

$$+ (\delta(k_a \in K_{N,p,n_0}) - \delta(k_b \in K_{N,p,n_0})) \cdot A_{M,1}$$

$$+ (\delta(k_a \in K_{M,p,m_0} \cap K_{N,p,n_0} \cap \{0\}) + \delta(k_a \in K_{N,p,n_0} \cap K_{M,p,m_0})) \cdot \varphi(p^{k_b})$$

$$- \delta(k_b \in K_{M,p,m_0} \cap K_{N,p,n_0} \cap \{0\}) \cdot (p - 2)p^{k_b} - \delta(k_b \in K_{M,p,m_0} \cap K_{N,p,n_0} \cap \{0\})$$

$$+ \delta(k_a \in K_{M,p,m_0}) \cdot A_{N,2} + \delta(k_a \in K_{N,p,n_0}) \cdot A_{M,2} - A_3$$

$$- \delta(k_a \in K_{M,p,m_0} \cap K_{N,p,n_0}) \cdot p^{k_a-1}.$$
Suppose $k_a \in K_{M,p,m_0} \cap K_{N,p,n_0}$. Then (7.39) is
\[
(1 - \delta(k_0 \in K_{M,p,m_0})) \cdot A_{N,1} + (1 - \delta(k_0 \in K_{N,p,n_0})) \cdot A_{M,1}
\]
\[+ \delta(k_0 \in (K_{M,p,m_0} \cup K_{N,p,n_0}) \cap \{0\}) \cdot (p - 1) p^{k_b - 1} + \delta(k_0 \in K_{M,p,m_0} \cap K_{N,p,n_0} \cap \{0\}) \cdot p^{k_b - 1}
\]
\[+ \delta(k_0 \in (K_{M,p,m_0} \cup K_{N,p,n_0}) \cap \{0\}) \cdot (A_{N,2} + A_{M,2} - A_3) - p^{k_a - 1}
\]
(7.44)
\[\leq p^{k_b} + \left( \sum_{k_1 \in K_{M,p,m_0} \cup K_{N,p,n_0}; k_b < k_1 < k_a} \varphi(p^{k_1}) \right) - p^{k_a - 1}
\]
\[\leq p^{k_b} + (p^{k_a - 1} - p^{k_b}) - p^{k_a - 1} \leq 0.
\]

Part (a) is proved. \(\square\)

**Examples 7.7.** (i) The Milnor lattice with monodromy \((H_{Mil}, h_{Mil})\) of an $A_\mu$ singularity $x_1^{\mu+1}$ in one variable is a single Orlik block with set
\[
M = \{m \in \mathbb{N} | m| (\mu + 1)\} - \{1\}
\]
(7.45)
of orders of eigenvalues of the monodromy. This is well known. It also follows from Theorem 1.3 (a) and from the fact that all eigenvalues have multiplicity 1 and the set of their orders is $M$. For any prime number and any $m_0 \in \pi_p(M)$
\[
K_{M,p,m_0} = \begin{cases} 
\mathbb{Z}_{[0,v_p(\mu+1)]} & \text{if } m_0 \neq 1, \\
\mathbb{Z}_{[1,v_p(\mu+1)]} & \text{if } m_0 = 1 
\end{cases}
\]
(7.46)
(where $[1,0] = \emptyset$ and $\mathbb{Z}_{[1,0]} = \emptyset$).

(ii) We consider an $A_\mu$-singularity $x_1^{\mu+1}$ as in (i) and an $A_\nu$-singularity $x_2^{\nu+1}$ with set $N$ of orders of eigenvalues of its monodromy. $N$ and $K_{N,p,n_0}$ for $n_0 \in \pi_p(N)$ are as in (7.45) and (7.46), with $\mu$ replaced by $\nu$. We will show with Theorem 7.4 that the Thom-Sebastiani sum $A_\mu \otimes A_\nu$, i.e. $x_1^{\mu+1} + x_2^{\nu+1}$, satisfies Orlik’s conjecture, i.e. its Milnor lattice with monodromy admits a standard decomposition into Orlik blocks.

It is the tensor product of the Milnor lattices with monodromies of the two $A$-type singularities \([ST71]\), so the tensor product of Orlik blocks with sets $M$ and $N$. In order to apply Theorem 7.4 we have to show that (7.23) or (7.24) holds for any prime number $p$ and any $p$-edge $(l_a, l_b) \in E_p(L)$.

Then $k_a := v_p(l_a) > k_b := v_p(l_b) \geq 0$. The shape (7.46) of $K_{M,p,m_0}$ and analogously for $K_{N,p,n_0}$ shows that the properties ($k_a \in K_{M,p,m_0}$ or not) and ($k_a \in K_{N,p,n_0}$ or not) are independent of the choice of $m_0 \in \pi_p(M)$ and $n_0 \in \pi_p(N)$. Therefore the hypotheses in (7.29) and
are independent of the choice of \( m_0 \in \pi_p(M) \) and \( n_0 \in \pi_p(N) \). This shows that (7.23) or (7.24) holds for any prime number \( p \) and any \( p \)-edge \((l_a, l_b) \in E_p(L)\). Therefore Theorem 7.4 applies. The Thom-Sebastiani sum \( x_1^{n_0+1} + x_2^{n_0+1} \) satisfies Orlik’s conjecture.

This example is a very special case of Theorem 1.3 (d). But we find it instructive to see the sdiOb-sufficiency condition at work in a simple case.

(iii) The following is a small abstract example of a pair \((M, N)\) which is not sdiOb-sufficient. Consider \( M := \{3\} \) and \( N := \{2, 3\} \). We list some relevant data:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|}
 l \in L & \delta(3, 2, l) & \delta(3, 3, l) & \chi_{2,3,1}(l) & \chi_{2,3,3}(l) & \chi_{3,1,2}(l) & \chi_{3,1,1}(l) \\
1 & 0 & 2 & 0 & 2 & 0 & 2 \\
3 & 0 & 1 & 0 & 1 & 0 & 1 \\
6 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

The 3-edge \((3, 1)\) satisfies (7.24). But the 2-edge \((6, 3)\) satisfies neither (7.23) nor (7.24). The pair \((M, N)\) is not sdiOb-sufficient. Theorem 7.4 does not apply. Remark 7.5 (i) even claims that the tensor product \( H^{[\Phi_3]} \otimes H^{[\Phi_2 \Phi_3]} \) of the two Orlik blocks \( H^{[\Phi_3]} \) and \( H^{[\Phi_2 \Phi_3]} \) does not admit a standard decomposition into Orlik blocks. This is true. It can be proved with Theorem 5.1 and Example 7.7 (ii) as follows.

\[
\begin{align*}
H^{[\Phi_3]} \otimes H^{[\Phi_2 \Phi_3]} & \cong H^{[\Phi_3]} \otimes (H^{[\Phi_2]} \oplus H^{[\Phi_3]}) \\
& \cong H^{[\Phi_3]} \otimes H^{[\Phi_2]} \oplus H^{[\Phi_3]} \otimes H^{[\Phi_3]} \\
& \cong H^{[\Phi_3]} \oplus (H^{[\Phi_3 \Phi_1]} \oplus H^{[\Phi_3 \Phi_1]}) \\
& \cong H^{[\Phi_3 \Phi_1]} \oplus H^{[\Phi_3 \Phi_1]} \\
& \neq H^{[\Phi_6 \Phi_3 \Phi_1]} \oplus H^{[\Phi_1]} \\
& \neq H^{[\Phi_6 \Phi_3 \Phi_1]} \oplus H^{[\Phi_1]} \\
& \neq H^{[\Phi_6 \Phi_3 \Phi_1]} \oplus H^{[\Phi_1]}
\end{align*}
\]

So, \( H^{[\Phi_3]} \otimes H^{[\Phi_2 \Phi_3]} \) is isomorphic to the non-standard decomposition \( H^{[\Phi_6 \Phi_3 \Phi_1]} \oplus H^{[\Phi_3 \Phi_1]} \) into Orlik blocks, but not to the standard decomposition \( H^{[\Phi_6 \Phi_3 \Phi_1]} \oplus H^{[\Phi_1]} \) into Orlik blocks.
(iv) Here we present a small extension of the example in (iii) which is an $sdiOb$-sufficient pair $(M, N)$, namely $M = \{3\}$ and $N = \{1, 2, 3\}$. So Theorem 7.4 applies and shows that $H^{[\Phi_3]} \otimes H^{[\Phi_1, \Phi_2, \Phi_3]}$ admits a standard decomposition into Orlik blocks. We list some relevant data:

$$L = \{1, 3, 6\}, \quad E_2(L) = \{(6, 3)\}, \quad E_3(L) = \{(3, 1)\},$$

$$\pi_2(M) = \{3\}, \quad \pi_3(M) = \{1\}, \quad \pi_2(N) = \{1, 3\}, \quad \pi_3(N) = \{2, 1\},$$

| $l$ | $\delta(3, 1, l)$ | $\delta(3, 2, l)$ | $\delta(3, 3, l)$ | $\chi_{2,3,1}(l)$ | $\chi_{2,3,3}(l)$ | $\chi_{3,1,2}(l)$ | $\chi_{3,1,1}(l)$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 0               | 0               | 0               | 0               | 2               | 0               | 2               |
| 3   | 1               | 0               | 1               | 1               | 1               | 0               | 2               |
| 6   | 0               | 1               | 0               | 1               | 0               | 1               | 0               |

The 3-edge $(3, 1)$ satisfies (7.24) and (7.23). The 2-edge $(6, 3)$ satisfies (7.24).

8. Proof of Theorem 7.4

This section is devoted to the proof of Theorem 7.4. The proof is similar to the one of Theorem 6.2, especially the beginning. But the later part is much more involved. We will use all the notations of section 7.

Consider the pair $(G \otimes H, g \otimes h)$. Formula (7.26) for $\text{div}(g \otimes h)$ implies that the characteristic polynomial of $g \otimes h$ is

$$p_{G \otimes H, g \otimes h} = \prod_{l \in L} \Phi_l^{\chi(l)}. \quad (8.1)$$

Let $p_1, \ldots, p_{\chi_0} \in \mathbb{C}[t]$ be the unique unitary polynomials with $p_{G \otimes H, g \otimes h} = p_1 \cdot \ldots \cdot p_{\chi_0}$ and $p_{\chi_0} | p_{\chi_0-1} \ldots | p_2 | p_1$ and $p_1$ the minimal polynomial of $g \otimes h$. Here

$$\chi_0 = \max_{l \in L} \chi(l) \quad \text{and} \quad p_k = \prod_{l \in L: \chi(l) \geq k} \Phi_l \in \mathbb{Z}[t]. \quad (8.2)$$

If $G \otimes H$ admits a standard decomposition into Orlik blocks, that is isomorphic to $\bigoplus_{k=1}^{\chi_0} H^{[p_k]}$.

We want to apply Theorem 8.1 to $(G \otimes H, g \otimes h)$ instead of $(H, h)$.

Let $a_0$ and $b_0$ be generating elements of the Orlik blocks $(G, g)$ and $(H, h)$, so $G = \bigoplus_{i=0}^{r_k G-1} \mathbb{Z} \cdot g^i(a_0)$ and $H = \bigoplus_{j=0}^{r_k H-1} \mathbb{Z} \cdot h^j(b_0)$. Define

$$a_i := g^i(a_0) \quad \text{for } i \geq 0, \quad (8.3)$$

$$b_j := h^j(b_0) \quad \text{for } j \geq 0, \quad (8.4)$$

$$a := (a_0, a_1, \ldots, a_{r_k G-1}) \quad \text{a } \mathbb{Z}-\text{basis of } G, \quad (8.5)$$

$$b := (b_0, b_1, \ldots, b_{r_k H-1}) \quad \text{a } \mathbb{Z}-\text{basis of } H. \quad (8.6)$$
Then the tuple
\[ C^{st} := a \otimes b \]
\[ := (a_0 \otimes b_0, a_0 \otimes b_1, \ldots, a_0 \otimes b_{rkH-1}, a_1 \otimes b_0, a_1 \otimes b_1, \ldots, a_1 \otimes b_{rkH-1}, \ldots, a_{rkG-1} \otimes b_0, a_{rkG-1} \otimes b_1, \ldots, a_{rkG-1} \otimes b_{rkH-1}), \]  
(8.7)
is a \( \mathbb{Z} \)-basis of \( G \otimes H \).

Observe \((g \otimes h)^k(a_i \otimes b_j) = a_{i+k} \otimes b_{j+k}\). Consider the tuples
\[ C^{dec,i} := (a_i \otimes b_0, a_{i+1} \otimes b_1, \ldots, a_{i+degp_{i+1}-1} \otimes b_{degp_{i+1}-1}) \]  
(8.8)
for \( i \in \mathbb{Z}_{[0, \chi_0-1]} \),
\[ C^{dec} := (C^{dec,0}, C^{dec,1}, \ldots, C^{dec,\chi_0-1}). \]
(8.9)
We will show that \( C^{dec} \) is a \( \mathbb{Z} \)-basis of \( G \otimes H \) if and only if \((M,N)\) is sdiOb-sufficient. This and Theorem 3.1 show that \((G \otimes H, g \otimes h)\) admits a standard decomposition into Orlik blocks if \((M,N)\) is sdiOb-sufficient.

It remains to show that the matrix \( M(C^{st}, C^{dec}) \) with \( C^{dec} = C^{st} \cdot M(C^{st}, C^{dec}) \) has determinant \( \pm 1 \) if and only if the pair \((M,N)\) is sdiOb-sufficient. We will calculate this determinant up to the sign. We will show
\[ \det M(C^{st}, C^{dec}) = (\pm 1) \cdot \prod_{p \in \mathbb{P}} \prod_{(l_a, l_b) \in E_p(L)} p^{\varphi(l_b)} \Xi(l_a, l_b), \]  
(8.10)
where for \((l_a, l_b) \in E_p(L)\)
\[ \Xi_1(l_a, l_b) := \min(\chi(l_a), \chi(l_b)) \in \mathbb{N}_0, \]
\[ \Xi_2,p(l_a, l_b) := \sum_{(m_0, n_0) \in \pi_p(M) \times \pi_p(N)} \min(\chi_{p, m_0, n_0}(l_a), \chi_{p, m_0, n_0}(l_b)) \in \mathbb{N}_0, \]
\[ \Xi(l_a, l_b) := \Xi_1(l_a, l_b) - \Xi_2,p(l_a, l_b) \in \mathbb{N}_0. \]  
(8.11)
Obviously \( \Xi(l_a, l_b) = 0 \) if and only if \((7.23)\) or \((7.24)\) holds. Therefore \( \det M(C^{st}, C^{dec}) = \pm 1 \) if and only if the pair \((M, N)\) is sdiOb-sufficient.

As in the proof of theorem 6.2, for the calculation of the determinant, we will consider also certain tuples of eigenvectors of \( g, h \) and \( g \otimes h \). Let \( \{\kappa_1, \kappa_2, \ldots, \kappa_{rkG}\} \) be the set of eigenvalues of \( g \), and let \( \{\lambda_1, \lambda_2, \ldots, \lambda_{rkH}\} \) be the set of eigenvalues of \( h \), in both sets the indices are chosen such that
\[ \text{ord}(\kappa_\alpha) \leq \text{ord}(\kappa_\beta) \quad \text{and} \quad \text{ord}(\lambda_\alpha) \leq \text{ord}(\lambda_\beta) \quad \text{if} \ \alpha < \beta. \]  
(8.12)
Then \( a_0 \) determines a basis of eigenvectors \( u = (u_1, ..., u_{rkG}) \) of \( G_{\mathbb{C}} \), and \( b_0 \) determines a basis of eigenvectors \( v = (v_1, ..., v_{rkH}) \) of \( H_{\mathbb{C}} \), with

\[
a_0 = \sum_{\alpha=1}^{rkG} u_\alpha, \quad g(u_\alpha) = \kappa_\alpha \cdot u_\alpha \quad (8.13)
\]

\[
b_0 = \sum_{\beta=1}^{rkH} v_\beta, \quad h(v_\beta) = \lambda_\beta \cdot v_\beta. \quad (8.14)
\]

The base change matrices

\[
M(u, a) = \begin{pmatrix}
1 & \kappa^1_1 & \cdots & \kappa^{rkG-1}_1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & \kappa^1_{rkG} & \cdots & \kappa^{rkG-1}_{rkG}
\end{pmatrix} \quad (8.15)
\]

and

\[
M(v, b) = \begin{pmatrix}
1 & \lambda^1_1 & \cdots & \lambda^{rkH-1}_1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda^1_{rkH} & \cdots & \lambda^{rkH-1}_{rkH}
\end{pmatrix} \quad (8.16)
\]

are Vandermonde matrices. The tuple

\[
C^I := u \otimes v := (u_1 \otimes v_1, u_1 \otimes v_2, ..., u_1 \otimes v_{rkH}, u_2 \otimes v_1, u_2 \otimes v_2, ..., u_2 \otimes v_{rkH}, \ldots, u_{rkG} \otimes v_1, u_{rkG} \otimes v_2, ..., u_{rkG} \otimes v_{rkH}),
\]

is a \( \mathbb{C} \)-basis of \( G_{\mathbb{C}} \otimes H_{\mathbb{C}} \). And the base change matrix with \( C^{st} \) is

\[
M(C^I, C^{st}) = M(u, a) \otimes M(v, b). \quad (8.18)
\]

Let \( \{\mu_1, \mu_2, ..., \mu_{\deg p_1}\} \) be the set of eigenvalues of \( g \otimes h \). For \( \gamma \in \mathbb{Z}_{[1, \deg p_1]} \) define

\[
C(\gamma) := \{(\alpha, \beta) \in \mathbb{Z}_{[1,rkG]} \times \mathbb{Z}_{[1,rkH]} | \kappa_\alpha \lambda_\beta = \mu_\gamma\} \quad (8.19)
\]

\[
=:\bigcup_{k=1}^{C(\gamma)} \{(\alpha(\gamma, k), \beta(\gamma, k))\}.
\]

The eigenvalues \( \mu_1, ..., \mu_{\deg p_1} \) of \( g \otimes h \) are indexed such that

\[
\chi_0 = |C(1)| \geq |C(2)| \geq ... \geq |C(\deg p_1)| \geq 1. \quad (8.20)
\]

For any \( \gamma \in \mathbb{Z}_{[1, \deg p_1]} \), the space \( \bigoplus_{k=1}^{C(\gamma)} \mathbb{C} \cdot u_{\alpha(\gamma, k)} \otimes v_{\beta(\gamma, k)} \subset G_{\mathbb{C}} \otimes H_{\mathbb{C}} \) is the eigenspace with eigenvalue \( \mu_\gamma \) of \( g \otimes h \). For any \( \gamma \in \mathbb{Z}_{[1, \deg p_1]} \) and
any $i \geq 0$, the vector
\[
  w_{i,\gamma} := \sum_{k=1}^{\lvert C(\gamma) \rvert} \kappa_{\alpha(\gamma,k)}^i \cdot u_{\alpha(\gamma,k)} \otimes v_{\beta(\gamma,k)}
\]  
(8.21)
is an eigenvector with eigenvalue $\mu_\gamma$ of $g \otimes h$. It is useful as for $j \geq 0$

\[
  a_{i+j} \otimes b_j = \sum_{\alpha=1}^{rkG \cdot rkH} \sum_{\beta=1}^{deg_p} \kappa_{\alpha}^{i+j} \lambda_{\beta}^j \cdot u_{\alpha} \otimes v_{\beta}
\]  
\[
  = \sum_{\gamma=1}^{deg_p} \mu_{\gamma}^j \cdot w_{i,\gamma},
\]  
(8.22)

Consider for $\gamma \in \mathbb{Z}_{\lbrack 1, \deg p_1 \rbrack}$ and $i \in \mathbb{Z}_{\lbrack 0, \chi_0 - 1 \rbrack}$ the following tuples of eigenvectors of $g \otimes h$,

\[
  C^{I,\gamma} := (u_{\alpha(\gamma,1)} \otimes v_{\beta(\gamma,1)}, \ldots, u_{\alpha(\gamma,\lvert C(\gamma) \rvert)} \otimes v_{\beta(\gamma,\lvert C(\gamma) \rvert)}),
\]  
(8.23)

\[
  C^{I} := (C^{I,1}, \ldots, C^{I,\deg p_1}),
\]  
(8.24)

\[
  C^{III,\gamma} := (w_{0,\gamma}, \ldots, w_{\lvert C(\gamma) \rvert - 1, \gamma}),
\]  
(8.25)

\[
  C^{III} := (C^{III,1}, \ldots, C^{III,\deg p_1}),
\]  
(8.26)

\[
  C^{V,i} := (w_{i,1}, w_{i,2}, \ldots, w_{i,\deg p_1}),
\]  
(8.27)

\[
  C^{IV,i} := (w_{i,1}, w_{i,2}, \ldots, w_{i,\deg p_{i+1}}),
\]  
(8.28)

\[
  C^{IV} := (C^{IV,1}, \ldots, C^{IV,\chi_0}).
\]  
(8.29)

$C^{I,\gamma}$ and $C^{III,\gamma}$ are $\mathbb{C}$-bases of the eigenspace in $G_C \otimes H_C$ with eigenvalue $\mu_\gamma$ of $g \otimes h$. The base change matrix $M(C^{I,\gamma}, C^{III,\gamma})$ rewrites the relation (8.21). It is the Vandermonde matrix

\[
  M(C^{I,\gamma}, C^{III,\gamma}) = \begin{pmatrix}
  1 & \kappa_{\alpha(\gamma,1)}^1 & \ldots & \kappa_{\alpha(\gamma,1)}^{\lvert C(\gamma) \rvert - 1} \\
  \vdots & \vdots & & \vdots \\
  1 & \kappa_{\alpha(\gamma,\lvert C(\gamma) \rvert)}^1 & \ldots & \kappa_{\alpha(\gamma,\lvert C(\gamma) \rvert)}^{\lvert C(\gamma) \rvert - 1}
\end{pmatrix}
\]  
(8.30)

$C^I$, $C^{II}$, $C^{III}$ and $C^{IV}$ are $\mathbb{C}$-bases of $G_C \otimes H_C$. The base change matrices $M(C^I, C^{II})$ and $M(C^{III}, C^{IV})$ are just permutation matrices. The entries of $C^{V,i}$ are linearly independent. Therefore the following rectangular $(\deg p_1) \times (\deg p_{i+1})$-matrix is well defined. It rewrites the relations (8.22).

\[
  M(C^{V,i}, C^{dec,i}) = \begin{pmatrix}
  1 & \mu_1^1 & \ldots & \mu_{\deg p_{i+1}}^{\deg p_{i+1} - 1} \\
  \vdots & \vdots & & \vdots \\
  1 & \mu_1^{\deg p_1} & \ldots & \mu_{\deg p_1}^{\deg p_{i+1} - 1}
\end{pmatrix}
\]  
(8.31)
Now we want to describe the base change matrix \( M(\mathcal{C}^I, \mathcal{C}^\text{dec}) \). Observe that \( w_i, \gamma \) is in the case \( \gamma > \deg p_i + 1 \) a linear combination of the entries of \( \mathcal{C}^{III, \gamma} \). Therefore the matrix \( M(\mathcal{C}^I, \mathcal{C}^\text{dec}) \) is a block upper triangular matrix whose diagonal blocks are obtained from the matrices in (8.31) by cutting off the lower lines. The diagonal blocks are the Vandermonde matrices

\[
M^{[i]} := \begin{pmatrix}
1 & \mu^1_i & \cdots & \mu^{\deg p_i + 1 - 1}_i \\
: & : & & : \\
1 & \mu^1_{\deg p_i + 1} & \cdots & \mu^{\deg p_i + 1 - 1}_{\deg p_i + 1}
\end{pmatrix} \quad \text{for } i \in \mathbb{Z}_{[0, \chi_0 - 1]}. \quad (8.32)
\]

Now the matrix \( M(\mathcal{C}^I, \mathcal{C}^\text{dec}) \) can be written as the product of matrices

\[
M(\mathcal{C}^I, \mathcal{C}^I) M(\mathcal{C}^I, \mathcal{C}^{III}) M(\mathcal{C}^{III}, \mathcal{C}^{IV}) M(\mathcal{C}^{IV}, \mathcal{C}^\text{dec}). \quad (8.33)
\]

The absolute value of its determinant is

\[
|\det M(\mathcal{C}^I, \mathcal{C}^\text{dec})| = \left| \frac{\det M(\mathcal{C}^{III}, \mathcal{C}^{IV}) \cdot \det M(\mathcal{C}^{IV}, \mathcal{C}^\text{dec})}{\det M(\mathcal{C}^I, \mathcal{C}^I)} \right| = \left| \prod_{\gamma = 1}^{\deg p_i} \det M(\mathcal{C}^{III, \gamma}) \cdot \prod_{i = 0}^{\chi_0 - 1} M^{[i]} \right| \cdot (M(u, a)^{rk H} \cdot M(v, b)^{rk G}). \quad (8.34)
\]

Here we used the equality \( \det(A \otimes B) = (\det A)^b \cdot (\det B)^a \) for square matrices \( A \in M_{a \times a}(\mathbb{C}) \) and \( B \in M_{b \times b}(\mathbb{C}) \) (which becomes obvious if one looks at Jordan normal forms).

The quotient in (8.34) is a quotient of determinants of Vandermonde matrices. The determinants are

\[
\det M(u, a) = \prod_{1 \leq \alpha_1 < \alpha_2 \leq \rk G} (\kappa_{\alpha_2} - \kappa_{\alpha_1}), \quad (8.35)
\]

\[
\det M(v, b) = \prod_{1 \leq \beta_1 < \beta_2 \leq \rk H} (\lambda_{\beta_2} - \lambda_{\beta_1}), \quad (8.36)
\]

\[
\det M(C^{III, \gamma}, C^{III, \gamma}) = \prod_{1 \leq \alpha_1 < \alpha_2 \leq |C(\gamma)|} (\kappa_{\alpha(\gamma, \kappa_2)} - \kappa_{\alpha(\gamma, \kappa_1)}), \quad (8.37)
\]

\[
\det M^{[i]} = \prod_{1 \leq \alpha_1 < \alpha_2 \leq \deg p_i + 1} (\mu_{\alpha_2} - \mu_{\alpha_1}). \quad (8.38)
\]
Only now we use that \( \kappa_\alpha, \lambda_\beta, \mu_\gamma \) are unit roots and how they are related. For \( \det M(u, a) \) we obtain with the formulas (4.3) and (4.9)

\[
| \det M(u, a) | = \prod_{m_c, m_d \in M: m_c > m_d} | \text{Res}(\Phi_{m_c}, \Phi_{m_d}) | \\
\cdot \prod_{m \in M} \sqrt{\text{Discr} \Phi_m}. \quad (8.39)
\]

Now choose a prime number \( p \). The exponent of \( p \) in \( | \det M(u, a) |^2 \in \mathbb{N} \) is because of (4.3) and (4.9) and Theorem 4.5 (d) and (e)

\[
v_p(| \det M(u, a) |^2) = 2 \sum_{(m_c, m_d) \in E_p(M)} \varphi(m_d) \\
+ \sum_{m \in M: v_p(m) \geq 1} \varphi(m) \cdot (v_p(m) - \frac{1}{p - 1}). \quad (8.40)
\]

Analogously, we obtain for \( \det M(v, b) \) and \( \prod_{i=0}^{\chi_0-1} \det M[i] \)

\[
v_p(| \det M(v, b) |^2) = 2 \sum_{(n_c, n_d) \in E_p(N)} \varphi(n_d) \\
+ \sum_{n \in N: v_p(n) \geq 1} \varphi(n) \cdot (v_p(n) - \frac{1}{p - 1}), \quad (8.41)
\]

\[
v_p\left( \prod_{i=0}^{\chi_0-1} | \det M[i] | \right)^2 = 2 \sum_{(l_a, l_b) \in E_p(L)} \varphi(l_b) \cdot \min(\chi(l_b), \chi(l_a)) \\
+ \sum_{l \in L: v_p(l) \geq 1} \varphi(l) \cdot (v_p(l) - \frac{1}{p - 1}) \cdot \chi(l). \quad (8.42)
\]

As the squares of the absolute values of the other factors in the quotient in (8.34) are positive integers, the square of the absolute value of the factor \( \prod_{\gamma=1}^{\deg p_1} \det M(C^{II, \gamma}, C^{III, \gamma}) \) is a positive rational number, and the value \( v_p(| \prod_{\gamma=1}^{\deg p_1} \det M(C^{II, \gamma}, C^{III, \gamma})|^2) \) is a well defined integer. It is the most difficult part. We will discuss it below.
In order to prove [8.10], we have to show for any prime number \( p \)
\[
-rkH \cdot \left( 2 \sum_{(m_c,m_d) \in E_p(M)} \varphi(m_d) + \sum_{m \in M; v_p(m) \geq 1} \varphi(m) \cdot (v_p(m) - \frac{1}{p-1}) \right)
- rkG \cdot \left( 2 \sum_{(n_c,n_d) \in E_p(N)} \varphi(n_d) + \sum_{n \in N; v_p(n) \geq 1} \varphi(n) \cdot (v_p(n) - \frac{1}{p-1}) \right)
+ 2 \sum_{(l_a,l_b) \in E_p(L)} \varphi(l_b) \cdot \Xi_{2,p}(l_a,l_b) + \sum_{l \in L; v_p(l) \geq 1} \varphi(l) \cdot (v_p(l) - \frac{1}{p-1}) \cdot \chi(l)
+ v_p\left( \prod_{\gamma=1}^{\deg p_1} \det M(C^{II,\gamma}, C^{III,\gamma}) \right)^2 = 0. \tag{8.43}
\]

Now we discuss the most difficult part. Consider a difference \( \kappa_{\alpha(\gamma,k_2)} - \kappa_{\alpha(\gamma,k_1)} \) in [8.37]. Denote for a moment
\[
m_1 := \text{ord}(\kappa_{\alpha(\gamma,k_1)}), \quad m_2 := \text{ord}(\kappa_{\alpha(\gamma,k_2)}),
\]
\[
n_1 := \text{ord}(\lambda_{\beta(\gamma,k_1)}), \quad n_2 := \text{ord}(\lambda_{\beta(\gamma,k_2)}),
\]
\[
l := \text{ord}(\mu_\gamma), \quad \nu := \frac{\kappa_{\alpha(\gamma,k_1)}}{\kappa_{\alpha(\gamma,k_2)}}.
\]

Theorem 4.5 (a) and (b) give
\[
\text{Norm}_{\text{ord} \nu}(1 - \nu) = \begin{cases} 
\pm q & \text{if \ ord \nu is a (positive) power of a prime number } q \\
\pm 1 & \text{else.} \end{cases} \tag{8.44}
\]

Therefore the difference \( \kappa_{\alpha(\gamma,k_2)} - \kappa_{\alpha(\gamma,k_1)} = \kappa_{\alpha(\gamma,k_2)} \cdot (1 - \nu) \) makes a contribution to \( v_p\left( \prod_{\gamma=1}^{\deg p_1} \det M(C^{II,\gamma}, C^{III,\gamma}) \right)^2 \) only if \( \text{ord} \nu \) is a (positive) power of \( p \). By Theorem [4.5] (c)(i), this holds only if \( \pi_p(m_1) = \pi_p(m_2) \). Because of \( \kappa_{\alpha(\gamma,k)} \lambda_{\beta(\gamma,k)} = \mu_\gamma \) for any \( k \in \mathbb{Z}_{[1,|C(\gamma)|]} \), we have
\[
\frac{\kappa_{\alpha(\gamma,k_1)}}{\kappa_{\alpha(\gamma,k_2)}} = \frac{\lambda_{\beta(\gamma,k_2)}}{\lambda_{\beta(\gamma,k_1)}}. \tag{8.45}
\]

Therefore also \( \pi_p(n_1) = \pi_p(n_2) \) is needed. The following lemma makes the sizes of the possible contributions precise.

**Lemma 8.1.** Fix a prime number \( p \). Fix \( m_1, m_2 \in M, n_1, n_2 \in N \) and \( l \in L \) with \( \pi_p(m_1) = \pi_p(m_2) \) and \( \pi_p(n_1) = \pi_p(n_2) \) and \( \max(v_p(m_1), v_p(m_2)) > 0 \). The contribution of all differences \( \kappa_{\alpha(\gamma,k_2)} - \kappa_{\alpha(\gamma,k_1)} \) in [8.37] with
\[
\text{ord } \gamma = l, \quad \text{ord } \kappa_{\alpha(\gamma,k_i)} = m_i, \quad \text{ord } \lambda_{\beta(\gamma,k_i)} = n_i \tag{8.46}
\]
to \(v_p(|\prod_{\gamma=1}^{\deg p_1} \det M(C^{\gamma, \gamma}, C^{\gamma, \gamma})|^2)\) is as follows. We can suppose \(m_1 \leq m_2\) (by exchanging them if necessary). If \(m_1 = m_2\) we can suppose \(n_1 \leq n_2\). We have the following ten cases. The contribution has always the shape \(\varphi(l) \cdot \delta(m_1, n_1, l) \cdot (a \text{ factor } F)\).

| Case | the factor \(F\) |
|------|----------------|
| (C1) \(v_p(m_1) < v_p(m_2) = v_p(n_2) > v_p(n_1)\) | 2 |
| (C2a) \(v_p(m_1) < v_p(m_2) < v_p(n_1) = v_p(n_2) = v_p(l)\) | 2 |
| (C2b) \(v_p(m_1) < v_p(n_2) < v_p(m_2) = v_p(l)\) | 2 |
| (C3) \(v_p(m_1) < v_p(m_2) = v_p(n_1) = v_p(l) > v_p(n_2)\) | \((\varphi(p^{v_p(n_2)})\) |
| (C4a) \(v_p(m_1) < v_p(m_2) = v_p(n_1) = v_p(n_2) = v_p(l)\) | \((\varphi(p^{v_p(n_1)})\) |
| (C4b) \(v_p(n_1) < v_p(m_1) = v_p(m_2) = v_p(l)\) | \((\varphi(p^{v_p(n_1)})/p^{v_p(n_1)})\) |
| (C5a) \(1 \leq v_p(m_1) = v_p(m_2) < v_p(n_1) = v_p(n_2) = v_p(l)\) | \((v_p(m_1) - 1/p^{v_p(n_1)})\) |
| (C5b) \(1 \leq v_p(n_1) = v_p(n_2) < v_p(m_1) = v_p(m_2) = v_p(l)\) | \((v_p(n_1) - 1/p^{v_p(n_1)})\) |
| (C6) \(v_p(m_1) = v_p(m_2) = v_p(n_1) = v_p(n_2) > v_p(l)\) | \((v_p(m_1) - 1/p^{v_p(n_1)})\) |
| (C7) \(v_p(m_1) = v_p(m_2) = v_p(n_1) = v_p(n_2) = v_p(l) > 0\) | \((v_p(m_1) - 1/p^{v_p(n_1)})\) |

(In the case (C1) \(v_p(l) \leq \max(v_p(m_1), v_p(n_1))\) with equality if \(v_p(m_1) \neq v_p(n_1)\).)

**Proof:** \(\kappa_{\alpha(\gamma, k_1)} \lambda_{\beta(\gamma, k_1)} = \mu\) implies \(\delta(m_i, n_i, l) > 0\) for \(i \in \{1, 2\}\). With Remark 7.3 (iii), which describes the set \(\{(m, n) \in \mathbb{N}^2 | \delta(m, n, l) > 0\}\), one obtains easily that only the ten cases in the lemma are possible (assuming \(m_1 \leq m_2\), and assuming \(n_1 \leq n_2\) in the case \(m_1 = m_2\)).

Write for \(m \in \mathbb{N}\)

\[Z_m := Z_{[0, m-1]} \quad \text{and} \quad Z_m^* := \{a \in Z_m | \gcd(a, m) = 1\}. \quad (8.47)\]

The contribution of all differences \(\kappa_{\alpha(\gamma, k_2)} - \kappa_{\alpha(\gamma, k_1)}\) in (8.37) with (8.46) to \(v_p(|\prod_{\gamma=1}^{\deg p_1} \det M(C^{\gamma, \gamma}, C^{\gamma, \gamma})|^2)\) is

\[
2 \sum_{k \geq 1} \frac{1}{\varphi(p^k)} \left| \left\{ (a_1, b_1, a_2, b_2, c_1, c_2) : \right. \right.
\]

\[
\left. \left. \left( a_1 < a_2 \text{ if } m_1 = m_2 \text{ and } n_1 = n_2 \right) , \right. \right.
\]

\[
\left. \frac{a_1}{m_1} + \frac{b_1}{n_1} \equiv \frac{c_1}{l} \equiv \frac{a_2}{m_2} + \frac{b_2}{n_2} \mod \mathbb{Z} \right) ,
\]

\[
\left. \frac{a_1}{m_1} - \frac{a_2}{m_2} \equiv \frac{c_2}{p^k} \equiv \frac{b_2}{n_2} - \frac{b_1}{n_1} \mod \mathbb{Z} \right) \}. \quad (8.48)
\]
This follows with the identifications
\[ \kappa_{\alpha(\gamma,k)} = e^{2\pi i \frac{a_j}{m_j}}, \quad \lambda_{\beta(\gamma,k)} = e^{2\pi i \frac{-b_j}{m_j}}, \quad \gamma = e^{2\pi i \frac{c_j}{p^k}}, \quad \frac{\kappa_{\alpha(\gamma,k_1)}}{\kappa_{\alpha(\gamma,k_2)}} = e^{2\pi i \frac{c_{12}}{p^k}} \]

from the fact that the norm in (8.44) has \( \varphi(\text{ord } \nu) = \varphi(p^k) \) factors which together give a factor \( \varphi \) from the fact that the norm in (8.44) has \( \varphi(\text{ord } \nu) = \varphi(p^k) \) factors.

In the other cases, the condition which together give a factor \( \varphi \) from the fact that the norm in (8.44) has \( \varphi(\text{ord } \nu) = \varphi(p^k) \) factors.

It remains to calculate the number in (8.48) in each of the ten cases. For an arbitrary fixed \( m \) and some \( n \), the set
\[ \{(a_1, b_1) \in \mathbb{Z}_{m_1}^* \times \mathbb{Z}_{n_1}^* \mid \frac{a_1}{m_1} \equiv \frac{b_1}{n_1} \mod \mathbb{Z} \} \tag{8.49} \]

has \( \delta(m_1, n_1, l) \) elements. This follows from (7.13) \( \Psi_m \cdot \Psi_n = \sum_{l \geq 1} \delta(m, n, l) \Psi_l \). Therefore the set
\[ \{(a_1, b_1, c_1) \in \mathbb{Z}_{m_1}^* \times \mathbb{Z}_{n_1}^* \times \mathbb{Z}_l^* \mid \frac{a_1}{m_1} + \frac{b_1}{n_1} \equiv \frac{c_1}{l} \mod \mathbb{Z} \} \tag{8.50} \]

has \( \varphi(l) \cdot \delta(m_1, n_1, l) \) elements.

Now fix for a moment such numbers \( a_1, b_1 \) and \( c_1 \). If for some \( k \geq 1 \) and some \( c_2 \in \mathbb{Z}_{p^k}^* \) elements \( a_2 \in \mathbb{Z}_{m_2}^* \) and \( b_2 \in \mathbb{Z}_{n_2}^* \) with
\[ \frac{a_1}{m_1} - \frac{a_2}{m_2} \equiv \frac{c_2}{p^k} \equiv \frac{b_2}{n_2} - \frac{b_1}{n_1} \mod \mathbb{Z} \tag{8.51} \]

exist, they are uniquely determined by these equations, respectively by
\[ a_2 \equiv a_1 \frac{m_2}{m_1} - c_2 \frac{m_2}{p^k} \mod m_2 \mathbb{Z}, \tag{8.52} \]
\[ b_2 \equiv b_1 \frac{n_2}{n_1} + c_2 \frac{n_2}{p^k} \mod n_2 \mathbb{Z}. \tag{8.53} \]

The questions are whether they exist, and if yes, whether \( a_2 \) is in \( \mathbb{Z}_{m_2}^* \) and \( b_2 \) is in \( \mathbb{Z}_{n_2}^* \).

Because of \( m_2 = m_1 \cdot p^{v_p(m_2)} - v_p(m_1) \), \( a_2 \) is in \( \mathbb{Z}_{m_2}^* \) if and only if \( l \leq v_p(m_2) \). In the cases with \( m_1 < m_2 \), \( a_2 \) is in \( \mathbb{Z}_{m_2}^* \) if and only if \( k = v_p(m_2) \). In the cases with \( m_1 = m_2 \) and \( k < v_p(m_2) \), \( a_2 \) is in \( \mathbb{Z}_{m_2}^* \). In the cases with \( m_1 = m_2 \) and \( k = v_p(m_2) \), we need \( a_1 \neq c_2 \pi_p(m_1) \mod p \mathbb{Z} \). The set of triples \( (a_1, b_1, c_1) \) in (8.50) which satisfy this, has \( \varphi(l) \cdot \delta(m_1, n_1, l) \cdot \frac{p^k}{p-1} \) elements.

Similarly, \( n_2 = n_1 \cdot p^{v_p(n_2)} - v_p(n_1) \), but here also \( n_1 > n_2 \) is possible. If \( n_1 \leq n_2 \), then \( b_2 \) is in \( \mathbb{Z}_{n_2}^* \) if and only if \( k \leq v_p(n_2) \). In the cases with \( n_1 < n_2 \), \( b_2 \) is in \( \mathbb{Z}_{n_2}^* \) if and only if \( k = v_p(n_2) \). In the cases with \( n_1 = n_2 \) and \( k < v_p(n_2) \), \( b_2 \) is in \( \mathbb{Z}_{n_2}^* \). In the cases with \( n_1 = n_2 \) and \( k = v_p(n_2) \),
we need $b_1 \neq -c_2 \pi_p(n_1) \bmod p\mathbb{Z}$. The set of triples $(a_1, b_1, c_1)$ in (8.50) which satisfy this, has $\varphi(l) \cdot \delta(m_1, n_1, l) \cdot \frac{\pi_2}{p-1}$ elements.

Now consider the cases with $n_1 > n_2$. Then $b_2 \in \mathbb{Z}_{p^k}^*$ if and only if $k = v_p(n_1)$ and $v_p(b_2 + c_2 \pi_p(n_2)) = k - v_p(n_2)$. We claim that the set of triples $(a_1, b_1, c_1)$ in (8.50) which satisfy this, has $\varphi(l) \cdot \delta(m_1, n_1, l) \cdot \frac{\varphi(p^{v_p(n_2)})}{\varphi(p^2)}$ elements. The claim is a consequence of the following facts:

(1) The projection from the set in (8.50) to $\mathbb{Z}_{p^k}^*$ which sends a triple $(a_1, b_1, c_1)$ to the class of $b_1$ in $\mathbb{Z}_{p^k}^*$ is surjective and all fibers have the same size.

(2) For a fixed $\gamma \in \mathbb{Z}_{p^k}^*$, the set $\{\beta \in \mathbb{Z}_{p^k}^* \mid v_p(\beta + \gamma) = k - v_p(n_2)\}$ has $\varphi(p^{v_p(n_2)})$ elements.

Fact (2) follows from $\Lambda(p^k, p^k, p, v_p(n_2)) = \varphi(p^k)$ in Theorem 4.5 (c)(iii) and from the factor $\varphi(p^{v_p(n_2)})^{-1}$ in the definition (1.19) of $\Lambda(p^k, p^k, p, v_p(n_2))$.

Now the ten cases in Lemma 8.1 can be obtained by combining the results of the discussion above, with some extra arguments for (C6) and (C7). We discuss the cases separately.

(C1): $v_p(m_1) < v_p(m_2)$ gives the unique $k = v_p(m_2)$ and $a_2 \in \mathbb{Z}_{m_2}^*$. Then automatically $k = v_p(n_2) > v_p(n_1)$, and thus also $b_2 \in \mathbb{Z}_{n_2}^*$. As $c_2 \in \mathbb{Z}_{p^k}^*$ was fixed arbitrarily, the number in (8.48) is 2 times the number of elements of the set in (8.50), so the factor $F$ is 2.

(C2a): $v_p(m_1) < v_p(m_2)$ gives the unique $k = v_p(m_2)$ and $a_2 \in \mathbb{Z}_{m_2}^*$. Then automatically $k < v_p(n_1) = v_p(n_2)$, and thus also $b_2 \in \mathbb{Z}_{n_2}^*$. As $c_2 \in \mathbb{Z}_{p^k}^*$ was fixed arbitrarily, the number in (8.48) is 2 times the number of elements of the set in (8.50), so the factor $F$ is 2.

(C2b): Analogous to (C2a).

(C3): $v_p(m_1) < v_p(m_2)$ gives the unique $k = v_p(m_2)$ and $a_2 \in \mathbb{Z}_{m_2}^*$. Then automatically $k = v_p(n_1) > v_p(n_2)$. For $b_2 \in \mathbb{Z}_{n_2}^*$ we need the set of triples $(a_1, b_1, c_1)$ which satisfy $v_p(b_1 + c_2 \pi_p(n_2)) = k - v_p(n_2)$. Their number is $\varphi(l) \cdot \delta(m_1, n_1, l) \cdot \frac{\varphi(p^{v_p(n_2)})}{\varphi(p^2)}$. As $c_2 \in \mathbb{Z}_{p^k}^*$ was fixed arbitrarily, the number in (8.48) is 2 times this number, so the factor $F$ is $2 \cdot \frac{\varphi(p^{v_p(n_2)})}{\varphi(p^2)}$.

(C4a): $v_p(m_1) < v_p(m_2)$ gives the unique $k = v_p(m_2)$ and $a_2 \in \mathbb{Z}_{m_2}^*$. Then automatically $k = v_p(n_1) = v_p(n_2)$. For $b_2 \in \mathbb{Z}_{n_2}^*$ we need the set of triples $(a_1, b_1, c_1)$ which satisfy $b_1 \neq -c_2 \pi_p(n_1) \bmod p\mathbb{Z}$. Their number is $\varphi(l) \cdot \delta(m_1, n_1, l) \cdot \frac{\pi_2}{p-1}$. As $c_2 \in \mathbb{Z}_{p^k}^*$ was fixed arbitrarily, the number in (8.48) is 2 times this number, so the factor $F$ is $2 \cdot \frac{\pi_2}{p-1}$.

(C4b): Analogous to (C4a).
(C5a): \( v_p(m_1) = v_p(m_2) \) gives \( k \in \mathbb{Z}_{[1,v_p(m_1)]} \). For any \( k \in \mathbb{Z}_{[1,v_p(m_1)-1]} \), \( a_2 \in \mathbb{Z}_{m_2}^* \) and \( b_2 \in \mathbb{Z}_{n_2}^* \) hold automatically. For \( k = v_p(m_2) < v_p(n_2) \), \( b_2 \in \mathbb{Z}_{n_2}^* \) holds automatically, but for \( a_2 \in \mathbb{Z}_{m_2}^* \) we need the set of triples \((a_1,b_1,c_1)\) in \((8.50)\) with \( a_1 \not\equiv c_2 \pi_p(m_1) \mod p\mathbb{Z} \). Their number is \( \varphi(l) \cdot \delta(m_1,n_1,l) \cdot \frac{p-2}{p-1} \). As \( m_1 = m_2 \) and \( n_2 = n_1 \), the condition \( a_1 < a_2 \) in \((8.48)\) cancels the factor 2 in \((8.48)\). As \( c_2 \in \mathbb{Z}_p^* \) was fixed arbitrarily, the number in \((8.48)\) is \( \varphi(l) \cdot \delta(m_1,n_1,l) \cdot ((v_p(m_1) - 1) \cdot 1 + \frac{p-2}{p-1}) \), so the factor \( F \) is \( v_p(m_1) - \frac{1}{p-1} \).

(C5b): Analogous to (C5a).

(C6) and (C7): \( v_p(m_1) = v_p(m_2) = v_p(n_1) = v_p(n_2) \) gives \( k \in \mathbb{Z}_{[1,v_p(m_1)]} \). For any \( k \in \mathbb{Z}_{[1,v_p(m_1)-1]} \), \( a_2 \in \mathbb{Z}_{m_2}^* \) and \( b_2 \in \mathbb{Z}_{n_2}^* \) hold automatically. As in the case (C5a), these \( k \) give the contribution \( v_p(m_1) - 1 \) to the factor \( F \). Now consider \( k = v_p(m_1) \). Then we need for \( a_2 \in \mathbb{Z}_{m_2}^* \) and \( b_2 \in \mathbb{Z}_{n_2}^* \) the set of triples \((a_1,b_1,c_1)\) in \((8.50)\) with \( a_1 \not\equiv c_2 \pi_p(m_1) \mod p\mathbb{Z} \) and \( b_1 \not\equiv -c_2 \pi_p(n_1) \mod p\mathbb{Z} \). The relation in \((8.50)\) implies (for \( k = v_p(m_2) \))

\[
(a_1 \pi_p(n_1) + b_1 \pi_p(m_1)) \pi_p(l) \equiv c_1 \pi_p(m_1) \pi_p(n_1) \cdot p^{k-v_p(l)} \mod p^k \mathbb{Z}. \quad (8.54)
\]

Therefore the condition \( b_1 \not\equiv -c_2 \pi_p(n_1) \mod p\mathbb{Z} \) is equivalent to the condition

\[
a_1 \pi_p(l) \not\equiv c_1 \pi_p(m_1) p^{k-v_p(l)} + c_2 \pi_p(m_1) \pi_p(l) \mod p\mathbb{Z}. \quad (8.55)
\]

Of course, the condition \( a_1 \not\equiv c_2 \pi_p(m_1) \mod p\mathbb{Z} \) is equivalent to the condition

\[
a_1 \pi_p(l) \not\equiv c_2 \pi_p(m_1) \pi_p(l) \mod p\mathbb{Z}. \quad (8.56)
\]

In the case (C6) we have \( k > v_p(l) \), so then the conditions \((8.55)\) and \((8.56)\) coincide. Then \( k = v_p(m_1) \) gives the same contribution \( \frac{p-2}{p-1} \) to the factor \( F \) as in the case (C5a). Then the factor \( F \) is as in the case (C5a). In the case (C7) we have \( k = v_p(l) \), so then the conditions \((8.55)\) and \((8.56)\) exclude for the class of \( a_1 \) in \( \mathbb{Z}_p^* \) two different numbers. Then the set of triples in \((8.50)\) which satisfy \((8.55)\) and \((8.56)\) is \( \varphi(l) \cdot \delta(m_1,n_1,l) \cdot \frac{p-3}{p-1} \). Then \( k = v_p(m_1) \) gives the contribution \( \frac{p-3}{p-1} \) to the factor \( F \). Then the factor \( F \) is \((v_p(m_1) - 1) \cdot 1 + \frac{p-3}{p-1} = v_p(m_1) - \frac{2}{p-1} \). \( \square \)
The following notations will be useful.

\[ T := \{(m, n, l) \in M \times N \times L \mid \delta(m, n, l) > 0\}, \] (8.57)

\[ \pi_{p,3} : \mathbb{N}^3 \to \mathbb{N}^3, \ (m, n, l) \mapsto (\pi_p(m), \pi_p(n), \pi_p(l)), \] (8.58)

\[ T_0 := \pi_{p,3}(T), \] (8.59)

\[ K_{t_0} := \{(k_1, k_2, k) \in K_{M,p,m_0} \times K_{N,p,n_0} \times K_{L,p,l_0} \mid \delta(p^{k_1}, p^{k_2}, p^{k}) > 0\} \] for \( t_0 = (m_0, n_0, l_0) \in T_0. \) (8.60)

Observe that for any \( t = (m, n, l) \in \mathbb{N}^3 \)

\[ \delta(t) = \delta(\pi_{p,3}(t)) \cdot \delta(p^{\pi_p(m)}, p^{\pi_p(n)}, p^{\pi_p(l)}). \] (8.61)

Thus for \( t = (m, n, l) \in T \) and \( t_0 = \pi_{p,3}(t) \in T_0 \) we have \( \delta(t_0) > 0 \) and \((v_p(m), v_p(n), v_p(l)) \in K_{t_0}.

In the next lemma, we split each summand in (8.43) into a sum of pieces over \( T. \) Afterwards we will show that the sum of all pieces over each fiber of the map \( \text{id} \times \text{id} \times \pi_p : T \to M \times N \times \pi_p(L) \) is already equal to 0. That implies (8.43).

**Lemma 8.2.** The seven summands in (8.43) can be written as sums of pieces over \( T \) as follows.

\[ -\text{rk} \ H \cdot 2 \sum_{(m_d, m_d) \in E_p(M)} \varphi(m_d) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(1)}, \]

\[ -\text{rk} \ H \cdot \sum_{m \in M : \pi_p(m) \geq 1} \varphi(m) (v_p(m) - \frac{1}{p - 1}) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(2)}, \]

\[ -\text{rk} \ G \cdot 2 \sum_{(n_d, n_d) \in E_p(N)} \varphi(n_d) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(3)}, \]

\[ -\text{rk} \ G \cdot \sum_{n \in N : \pi_p(n) \geq 1} \varphi(n) (v_p(n) - \frac{1}{p - 1}) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(4)}, \]

\[ 2 \sum_{(l_a, l_b) \in E_p(L)} \varphi(l_a) \cdot \Xi_2 \varphi(l_a, l_b) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(5)}, \]

\[ \sum_{l \in L : \pi_p(l) \geq 1} \varphi(l)(v_p(l) - \frac{1}{p - 1}) \cdot \chi(l) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(6)}, \]

\[ v_p \left( \prod_{\gamma=1}^{\text{deg} \ p_1} \det M(C^{II, \gamma}, C^{III, \gamma}) \right) = \sum_{t \in T} \varphi(l) \delta(t) \cdot A_t^{(7)}, \]
where the summands $A^{(i)}_t$ are as follows. Here $(m, n, l) := t$, $(m_0, n_0, l_0) := \pi_{p,3}(t)$ and $(k_1, k_2, k) := (v_p(m), v_p(n), v_p(l))$.

\[
\begin{align*}
A^{(1)}_t &= -2 \cdot |K_{M,p,m_0} \cap \mathbb{Z}_{>k_1}|, \\
A^{(2)}_t &= -\delta_{(k_1 > 0)} \cdot (k_1 - \frac{1}{p-1}), \\
A^{(3)}_t &= -2 \cdot |K_{N,p,n_0} \cap \mathbb{Z}_{>k_2}|, \\
A^{(4)}_t &= -\delta_{(k_2 > 0)} \cdot (k_2 - \frac{1}{p-1}), \\
A^{(5)}_t &= \delta_{(k > 0)} \cdot (k - \frac{1}{p-1}), \\
A^{(6)}_t &= A^{(1)}_t + A^{(2)}_t + A^{(3)}_t + A^{(4)}_t + A^{(5)}_t + A^{(6)}_t
\end{align*}
\]

and

\[
\begin{align*}
A^{(5,1)}_t &= 2|K_{M,p,m_0} \cup K_{N,p,n_0} - (K_{M,p,m_0} \cap K_{N,p,n_0}) \cap \mathbb{Z}_{>\max(k_1,k_2)}|, \\
A^{(5,2)}_t &= \delta_{(k_1 > \max(k_1,k_2) \in K_{M,p,m_0} \cap K_{N,p,n_0})} \cdot \frac{2}{p-1}.
\end{align*}
\]

and

\[
A^{(7)}_t = \sum_{(C) \in \{(C1), (C2a), (C2b), (C3), (C4a), (C4b), (C5a), (C5b), (C6), (C7)\}} A^{(C)}_t
\]

with $A^{(C)}_t$ as follows.

\[
\begin{align*}
A^{(C1)}_t &= 2 \cdot |K_{M,p,m_0} \cap K_{N,p,n_0} \cap \mathbb{Z}_{>\max(k_1,k_2)}|, \\
A^{(C2a)}_t &= 2 \cdot |K_{M,p,m_0} \cap \mathbb{Z}_{[k_1+1,k_2-1]}|, \\
A^{(C2b)}_t &= 2 \cdot |K_{N,p,n_0} \cap \mathbb{Z}_{[k_2+1,k_1-1]}|, \\
A^{(C3)}_t &= A^{(C1)}_t, \\
A^{(C4a)}_t &= \delta_{(k_2 > k_1, k_2 \in K_{M,p,m_0})} \cdot 2 \cdot \frac{p-2}{p-1}, \\
A^{(C4b)}_t &= \delta_{(k_1 > k_2, k_1 \in K_{N,p,n_0})} \cdot 2 \cdot \frac{p-2}{p-1}.
\end{align*}
\]
\[ A_t^{(C5a)} = \delta_{(1 \leq k_1 < k_2)} \cdot (k_1 - \frac{1}{p-1}), \quad (8.77) \]
\[ A_t^{(C5b)} = \delta_{(1 \leq k_2 < k_1)} \cdot (k_2 - \frac{1}{p-1}), \quad (8.78) \]
\[ A_t^{(C6)} = \delta_{(k_1 = k_2 > k)} \cdot (k_1 - \frac{1}{p-1}), \quad (8.79) \]
\[ A_t^{(C7)} = \delta_{(k_1 = k_2 = k \geq 1)} \cdot (k_1 - \frac{2}{p-1}). \quad (8.80) \]

**Proof:** First (8.62)–(8.66) are proved. Observe

\[ \text{rk} G = \sum_{m \in M} \varphi(m), \quad \text{rk} H = \sum_{n \in N} \varphi(n), \quad (8.81) \]
\[ \varphi(m) \varphi(n) = \sum_{l \in L} \varphi(l) \delta(m, n, l) \quad \text{for} \ (m, n) \in M \times N, \quad (8.82) \]

where (8.82) follows from (7.13). Now one finds

\[
\begin{align*}
\text{rk} H \cdot \sum_{(m, p^a, m) \in E_p(M)} \varphi(m) \\
= \left( \sum_{n \in N} \varphi(n) \right) \left( \sum_{m \in M} \varphi(m) \cdot |K_{M, p, \pi_p(m)} \cap \mathbb{Z}_{>p}(m)| \right) \\
= \sum_{(m, n, l) \in T} \varphi(l) \delta(m, n, l) \cdot |K_{M, p, \pi_p(m)} \cap \mathbb{Z}_{>p}(m)|.
\end{align*}
\]

This shows (8.62). Similar calculations show (8.63)–(8.65). With (7.19),

\[ \chi(l) = \sum_{(m, n) \in M \times N} \delta(m, n, l), \]

one sees also (8.66) immediately.

Now we will prove (8.67)–(8.69). It is more difficult. The second equality sign below uses Lemma (7.6) (a). There we sum over \( t_0 = (m_0, n_0, l_0) \in T_0 \). After the third equality sign below, we split on purpose \( \mathbb{Z}_{>k} = \mathbb{Z}_{[k+1, \max(k_1, k_2)]} \cup \mathbb{Z}_{\max(k_1, k_2)}, \) as that will allow some simplification later. Note that for \( (k_1, k_2, k) \in K_{t_0} \) the inequality
max\(k_1, k_2 > k\) implies \(k_1 = k_2\) and \(\delta(p_{k_1}, p_{k_2}, p^k) = \varphi(p^k)\).

\[
\sum_{(l_a, l_b) \in E_p(L)} \varphi(l_b) \cdot \Xi_{2,p}(l_a, l_b) \\
= \sum_{t_0 \in T_0} \sum_{k \in K_{L,p,l_0}} \varphi(l_0 p^k) \min(\chi_{p,m_0,n_0}(l_0 p^{k_1}), \chi_{p,m_0,n_0}(l_0 p^{k_2})) \\
= \sum_{t_0 \in T_0} \sum_{k \in K_{L,p,l_0}} \varphi(l_0 p^k) \cdot \left(\chi_{p,m_0,n_0}(l_0 p^k) \right)
\]

\[
\begin{aligned}
&= \sum_{t_0 \in T_0} \varphi(l_0) \delta(t_0) \cdot \sum_{k \in K_{L,p,l_0}} \varphi(p^k) \cdot \left(\right) \\
&\sum_{(k_1, k_2) \in K_{M,p,m_0} \times K_{N,p,n_0}} \left[ \delta(p_{k_1}, p_{k_2}, p^k) \right. \\
&\left. \cdot \left| \left( (K_{M,p,m_0} \cup K_{N,p,n_0}) - (K_{M,p,m_0} \cap K_{N,p,n_0}) \right) \cap \mathbb{Z}_{>k} \right| \right. \\
&\left. \left. + \delta(k_1 = k_2 > k) \cdot \varphi(p_{k_1}) \right. \\
&\left. \left. \cdot \left| \left( (K_{M,p,m_0} \cup K_{N,p,n_0}) - (K_{M,p,m_0} \cap K_{N,p,n_0}) \right) \cap \mathbb{Z}_{[k+1,k_1]} \right| \right. \\
&\left. \right] \\
&\left. + \sum_{k_3 \in K_{L,p,l_0} - (K_{M,p,m_0} \cup K_{N,p,n_0}) : k_3 > k} \sum_{k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_4 > k_3} \varphi(p^{k_4}) \\
&\left. + \sum_{k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_3 > k} \sum_{k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_4 > k_3} \varphi(p^{k_4}) \varphi(p^{k_3}) \frac{p - 2}{p - 1} \right. \\
&\left. + \sum_{k_1 \in K_{M,p,m_0} : k_1 < k_3} \varphi(p^{k_1}) + \sum_{k_2 \in K_{N,p,n_0} : k_2 < k_3} \varphi(p^{k_2}) \right) \\
&= (8.83)
\end{aligned}
\]

The first three lines in this formula \((8.83)\) give \(\sum_{t \in T} \varphi(l(t)) \cdot \frac{1}{2} A_t^{(5,1)}\).

We will show that the last five lines give \(\sum_{t \in T} \varphi(l(t)) \cdot \frac{1}{2} A_t^{(5,2)}\).
First we take care of line eight of (8.83). It is

$$\sum_{t_0 \in T_0} \varphi(l_0) \delta(t_0) \cdot \sum_{k \in K_{L,p,l_0}} \varphi(p^k) \left( \sum_{k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_3 > k} \sum_{k_1 \in K_{M,p,m_0} : k_1 < k_3} \varphi(p^{k_1}) + \sum_{k_2 \in K_{N,p,n_0} : k_2 < k_3} \varphi(p^{k_2}) \right)$$

$$= \sum_{t_0 \in T_0} \varphi(l_0) \delta(t_0) \cdot \sum_{k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_3 > 0} \sum_{k=0}^{k_3-1} \varphi(p^k) \cdot \left( \sum_{k_1 \in K_{M,p,m_0} : k_1 < k_3} \varphi(p^{k_1}) + \sum_{k_2 \in K_{N,p,n_0} : k_2 < k_3} \varphi(p^{k_2}) \right)$$

$$= \sum_{t \in T} \varphi(l_0) \delta(t) \cdot \delta_{k_1 \neq k_2, \max(k_1, k_2) \in K_{M,p,m_0} \cap K_{N,p,n_0}} \cdot \frac{1}{p-1}. \quad (8.84)$$

Here we use $K_{L,p,l_0} \supset \mathbb{Z}_{[0,k_3]}$ if $k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0}$, we identify $\max(k_1, k_2)$ with $k_3$, we identify $l$ with $l_0 p^{k_3}$, and we use

$$\sum_{k=0}^{k_3-1} \varphi(p^k) = p^{k_3-1} = \varphi(p^{k_3}) \cdot \frac{1}{p-1}. \quad (8.85)$$

Now we simplify the lines four to seven in (8.83). We will rename $k_1 = k_2$ in the lines four and five as $k_4$, and write in the lines six and seven the sum over $k_3$ as $|Z_{k[1,k_4-1]} - (K_{M,p,m_0} \cup K_{N,p,n_0})|$ respectively as $|Z_{k[1,k_4-1]} \cap K_{M,p,m_0} \cap K_{N,p,n_0}|$. Here we use $K_{L,p,l_0} \supset \mathbb{Z}_{[0,k_4]}$ for $k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0}$. Then we obtain for the lines four to seven of formula (8.83)

$$\sum_{t_0 \in T_0} \varphi(l_0) \delta(t_0) \cdot \left( \sum_{k \in K_{L,p,l_0}} \varphi(p^k) \cdot \sum_{k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_4 > k} \varphi(p^{k_4}) \cdot \left[ |Z_{k[1,k_4-1]}| + \frac{p-2}{p-1} \right] \right)$$

$$= \sum_{t_0 \in T_0} \varphi(l_0) \delta(t_0) \cdot \left( \sum_{k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_3 > 0} \varphi(p^{k_3}) \cdot \sum_{k=0}^{k_4-1} \varphi(p^k) \left[ k_4 - \frac{1}{p-1} - k \right] \right)$$

$$= \sum_{t_0 \in T_0} \varphi(l_0) \delta(t_0) \sum_{k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0} : k_4 > 0} \varphi(p^{k_4}) \cdot \left[ p^{k_4-1} - \frac{1}{p-1} \right]. \quad (8.86)$$
Here we used
\[ \sum_{k=0}^{k_1-1} \varphi(p^k) = p^{k_1-1}, \quad \sum_{k=1}^{k_1-1} p^k = \frac{(k_1 - 1)p^{k_1} - k_1p^{k_1-1} + 1}{(p - 1)^2}. \quad (8.87) \]

On the other hand,
\[
\sum_{t \in T} \varphi(l)\delta(t) \cdot \delta(k_1=k_2, k>0) \cdot \frac{1}{p-1} = \sum_{t_0 \in T_0} \varphi(l_0)\delta(t_0) \frac{1}{p-1} \sum_{k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0}} \sum_{k_4>0} \varphi(p^{k_4})\delta(p^{k_4}, p^{k_4}, p^k)
\]
\[
= \sum_{t_0 \in T_0} \varphi(l_0)\delta(t_0) \sum_{k_4 \in K_{M,p,m_0} \cap K_{N,p,n_0}} \varphi(p^{k_4}) \cdot \frac{p^{k_4-1}-1}{p-1}
\]
\[= \text{the last line of } (8.86). \]

Therefore the lines four to eight of formula (8.83) are equal to
\[ \sum_{t \in T} \varphi(l)\delta(t) \cdot \delta(k_1\not=k_2, \max(k_1,k_2) \in K_{M,p,m_0} \cap K_{N,p,n_0}) \cdot \frac{1}{p-1} + \sum_{t \in T} \varphi(l)\delta(t) \cdot \delta(k_1=k_2, k>0) \cdot \frac{1}{p-1} = \frac{1}{2} \sum_{t \in T} \varphi(l)\delta(t) \cdot A_1^{(5,2)}. \]

The formulas (8.67)–(8.69) are proved.

Now we prove the formulas for the ten parts \(A_1^{(C)}\) of \(A_1^{(T)}\). Lemma 8.1 will be applied. All formulas except the formula for \(A_1^{(C3)}\) are immediate consequences of it. In all ten cases in Lemma 8.1 except the case \((C3)\), we put \((m,n,l)_{\text{Lemma 8.2}} = (m_1, n_1, l)_{\text{Lemma 8.4}}\) so that \((k_1, k_2, k)_{\text{Lemma 8.2}} = (v_p(m_1), v_p(n_1), v_p(l))_{\text{Lemma 8.4}}\). We sum over the possible pairs \((m_2, n_2)_{\text{Lemma 8.1}}\).

It rests to prove formula (8.74) for \(A_1^{(C3)}\). Here we put \((m,n)_{\text{Lemma 8.2}} := (m_1, n_2)_{\text{Lemma 8.4}}\) so that \((k_1, k_2, k)_{\text{Lemma 8.2}} = (v_p(m_1), v_p(n_2))_{\text{Lemma 8.4}}\). We sum over the possible values \(k_3 = v_p(m_2)_{\text{Lemma 8.7}}\). They form the set \(K_{M,p,m_0} \cap K_{N,p,n_0} \cap \mathbb{Z}_{\max(k_1,k_2)}\). We need a case discussion.

The case \(k_1 < k_2\): Then \(k = k_2\) and
\[
\left( \varphi(p^{v_p(l)})\delta(p^{v_p(m_1)}, p^{v_p(n_1)}, p^{v_p(l)}) \cdot \frac{\varphi(p^{v_p(n_2)})}{\varphi(p^{v_p(n_1)})} \right)_{\text{Lemma 8.1}}
\]
\[= \varphi(p^{k_2})\delta(p^{k_2}, p^{k_2}, p^{k_2}) \cdot \frac{\varphi(p^{k_2})}{\varphi(p^{k_2})}
\]
\[= \varphi(p^{k_2}) \varphi(p^{k_1}) = \varphi(p^k)\delta(p^{k_1}, p^{k_2}, p^k). \]
Summing over \( k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0} \cap \mathbb{Z}_{>\max(k_1,k_2)} \) gives formula (8.74) for \( A_t^{(C_3)} \).

The case \( k_1 > k_2 \): Analogous to the case \( k_1 < k_2 \).

The case \( k_1 = k_2 = 0 \): Analogous to the case \( k_1 < k_2 \).

The case \( k_1 = k_2 > 0 \): Then \( k \in \mathbb{Z}_{[0,k_1]} \) and

\[
\left( \varphi(p^{v_p(l)}) \delta(p^{v_p(m_1)}, p^{v_p(n_1)}, p^{v_p(l)}) \cdot \frac{\varphi(p^{v_p(n_2)})}{\varphi(p^{v_p(n_1)})} \right) \text{Lemma 8.1}
\]

\[
= \varphi(p^{k_1}) \delta(p^{k_1}, p^{k_3}, p^{k_3}) \cdot \frac{\varphi(p^{k_2})}{\varphi(p^{k_3})} = \varphi(p^{k_1}) \varphi(p^{k_1})
\]

\[
= \varphi(p^{k_1}) \cdot \left( \sum_{k=0}^{k_1-1} \varphi(p^k) + \varphi(p^{k_1}) \frac{p-2}{p-1} \right)
\]

\[
= \sum_{k=0}^{k_1} \varphi(p^k) \cdot \delta(p^{k_1}, p^{k_2}, p^{k_3}).
\]

Summing over \( k_3 \in K_{M,p,m_0} \cap K_{N,p,n_0} \cap \mathbb{Z}_{>\max(k_1,k_2)} \) gives formula (8.74) for \( A_t^{(C_3)} \). This finishes the proof of Lemma 8.2.

Lemma 8.2 makes it now easy to prove (8.43), i.e. that its left hand side vanishes.

\[
A_t^{(1)} + A_t^{(C2a)} + A_t^{(C4a)}
\]

\[
= -2 \cdot |K_{M,p,m_0} \cap \mathbb{Z}_{>\max(k_1,k_2)}| + \delta(k_2 > k_1, k_2 \in K_{M,p,m_0}) \cdot \frac{-2}{p-1},
\]

\[
A_t^{(3)} + A_t^{(C2b)} + A_t^{(C4b)}
\]

\[
= -2 \cdot |K_{N,p,n_0} \cap \mathbb{Z}_{>\max(k_1,k_2)}| + \delta(k_1 > k_2, k_1 \in K_{N,p,n_0}) \cdot \frac{-2}{p-1},
\]

\[
A_t^{(5,1)} + A_t^{(C1)} + A_t^{(C3)}
\]

\[
= 2 \cdot |K_{M,p,m_0} \cap \mathbb{Z}_{>\max(k_1,k_2)}| + 2 \cdot |K_{N,p,n_0} \cap \mathbb{Z}_{>\max(k_1,k_2)}|.
\]

Therefore

\[
A_t^{(\text{part 1})} := \sum_{(j) \in \{(1), (3), (5,1), (C1), (C2a), (C2b), (C3), (C4a), (C4b)\}} A_t^{(j)} \quad (8.88)
\]

\[
= \left( \delta(k_2 > k_1, k_2 \in K_{M,p,m_0}) + \delta(k_1 > k_2, k_1 \in K_{N,p,n_0}) \right) \cdot \frac{-2}{p-1}.
\]

For the other terms, we make a case discussion.
The case $k_1 > k_2$: Then $k = k_1$ and

\[
A_t^{(2)} + A_t^{(6)} = 0, \\
A_t^{(4)} + A_t^{(C5b)} = 0, \\
A_t^{(C5a)} = A_t^{(C6)} = A_t^{(C7)} = 0, \\
A_t^{(5,2)} + A_t^{(\text{part 1})} = 0.
\]

Therefore then $\sum_{\text{all possible (j)}} A_t^{(j)} = 0$, and (8.43) is true.

The case $k_1 < k_2$: Analogous to the case $k_1 > k_2$.

The case $k_1 = k_2 = 0$: Then $k = k_1$ and

\[
A_t^{(j)} = 0 \quad \text{for (j) } \in \{(2), (4), (6), (5, 2), (C5a), (C5b), (C6), (C7), (\text{part 1})\}.
\]

Also then $\sum_{\text{all possible (j)}} A_t^{(j)} = 0$, and (8.43) is true.

The case $k_1 = k_2 > 0$: Then $k \in \mathbb{Z}_{[0,k_1]}$ and

\[
0 = A_t^{(C5a)} = A_t^{(C5b)} = A_t^{(\text{part 1})}, \\
A_t^{(\text{part 2})} := A_t^{(2)} + A_t^{(4)} + A_t^{(6)} + A_t^{(5,2)} + A_t^{(C6)} + A_t^{(C7)} = -k_1 + \delta_{(k_1 > k)} \cdot \frac{1}{p - 1} + \delta_{(k > 0)} \cdot (k + \frac{1}{p - 1}).
\]

This sum does not vanish for a single $k$. But summing over $k \in \mathbb{Z}_{[0,k_1]}$, we obtain 0:

\[
\sum_{k=0}^{k_1} \varphi(p^k) \delta(p^{k_1}, p^{k_2}, p^k) \cdot A_t^{(\text{part 2})} = \varphi(p^{k_1}) \sum_{k=0}^{k_1} \varphi(p^k)(1 - \delta_{(k_1 = k)}) \frac{1}{p - 1} \cdot A_t^{(\text{part 2})} = \varphi(p^{k_1}) \left(-k_1 + \frac{1}{p - 1} + \sum_{k=1}^{k_1-1} p^{k-1}((-k_1 + k)(p - 1) + 2) + p^{k_1-1}(p - 2) \frac{1}{p - 1}\right) = \varphi(p^{k_1}) \cdot 0 = 0.
\]
Here we used (8.87). Therefore then for any fixed \((m, n, l_0) \in M \times N \times \pi_p(L)\) the sum
\[
\sum_{k \in K_{L,p,l_0}} \varphi(l_0 p^k) \delta(m, n, l_0 p^k) \cdot \sum_{\text{all possible } (j)} A^{(j)}_{(m,n,l_0 p^k)}
\]
vanishes. Thus (8.43) is true also in the case \(k_1 = k_2 > 0\). This finishes the proof of Theorem 7.4. \(\square\)

**Remark 8.3.** At the heart of the proof of Theorem 7.4 in this section is the proof that the set \(C_{dec}\) in (8.9) is a \(\mathbb{Z}\)-basis of \(G \otimes H\) in the special case \(M,N \subset \{ p^k \mid k \in \mathbb{N}_0 \}\) for some prime number \(p\) if the pair \((M,N)\) is sdiOb-sufficient. Our proof for this case is long and tedious. It would be desirable to have an elegant or short proof for this case.

### 9. A Compatibility Condition for Sets of Orders of Eigenvalues of Orlik Blocks

This section proposes and discusses a condition for finite sets \(M \subset \mathbb{N}\) of orders of eigenvalues of Orlik blocks. It is given in Definition 9.4 (c). It has a number of good properties, which will be given below in Lemma 9.5, Theorem 9.6, Lemma 9.8, Theorem 9.9, Theorem 9.10 and Lemma 9.12. Theorem 9.9 contains the statement that \(\text{Or}(M) \otimes \text{Or}(N)\) admits a standard decomposition into Orlik blocks if the finite sets \(M\) and \(N \subset \mathbb{N}\) satisfy this condition. This is prepared by several definitions.

**Definition 9.1.** (a) An **excellent order** \(\succ\) on a set \(\mathbb{Z}_{[0,s(\succ)]}\) for some bound \(s(\succ) \in \mathbb{N}_0\) is a strict order (so transitive and for all \(a, b \in \mathbb{Z}_{[0,s(\succ)]}\) either \(a = b\) or \(a \succ b\) or \(b \succ a\)) which is determined by the set
\[
S(\succ) := \{ k \in \mathbb{Z}_{[0,s(\succ)]} \mid k \succ 0 \} \quad (9.1)
\]
in the following way:
\[
\begin{align*}
\succ &\text{ equals } \succ \text{ on } S(\succ) \cup \{0\}, \\
\succ &\text{ equals } < \text{ on } \mathbb{Z}_{[0,s(\succ)]} - S(\succ). \\
(9.2)
\end{align*}
\]
\((S(\succ) = \emptyset\) is allowed.) The maximal element of \(\mathbb{Z}_{[0,s(\succ)]}\) with respect to \(\succ\) is called \(s^+(\succ)\), so \(s^+(\succ) \succ k\) for any other element \(k \in \mathbb{Z}_{[0,s(\succ)]}\).

(b) The trivial excellent order is \(\succ_0\) with \(s(\succ_0) := 0\), so it is the empty order on \(\mathbb{Z}_{[0,s(\succ_0)]} = \{0\}\) (and, of course \(S(\succ_0) = \emptyset\)).

(c) The tensor product of two excellent orders \(\succ_1\) and \(\succ_2\) is the excellent order \(\succ_1 \otimes \succ_2\) with
\[
\begin{align*}
s(\succ_1 \otimes \succ_2) &:= \max(s(\succ_1), s(\succ_2)) \quad \text{and} \\
S(\succ_1 \otimes \succ_2) &:= (S(\succ_1) \cup S(\succ_2)) - (S(\succ_1) \cap S(\succ_2)). \quad (9.4)
\end{align*}
\]
Examples 9.2. (i) The excellent order $\succ_1$ with $s(\succ_1) = 7$ and $S(\succ_1) = \{6, 4, 1\}$ is given by

$$s^+(\succ_1) = 6 \succ_1 4 \succ_1 1 \succ_1 0 \succ_1 2 \succ_1 3 \succ_1 5 \succ_1 7.$$ 

(ii) The excellent order $\succ_2$ with $s(\succ_2) = 6$ and $S(\succ_2) = \{6, 5, 2, 1\}$ is given by

$$s^+(\succ_2) = 6 \succ_2 5 \succ_2 2 \succ_2 1 \succ_2 0 \succ_2 3 \succ_2 4.$$ 

(iii) The excellent order $\succ_3 := (\succ_1 \otimes \succ_2)$ for $\succ_1$ and $\succ_2$ in (i) and (ii) satisfies $s(\succ_3) = 7$, $S(\succ_3) = \{5, 4, 2\}$ and is given by

$$5 \succ_3 4 \succ_3 2 \succ_3 0 \succ_3 3 \succ_3 6 \succ_3 7.$$ 

(iv) For any excellent order $\succ$, the tensor product with the trivial excellent order is $\succ$ itself, $\succ \otimes \succ_0 = \succ$.

Definition 9.3. (a) A path in a finite directed graph $(V,E)$ (so $V$ is a finite non-empty set and $E \subset V \times V$) is a tuple $(v_1,\ldots,v_l)$ for some $l \in \mathbb{Z}_{\geq 2}$ with $v_j \in V$ and $(v_j,v_{j+1}) \in E$ for $j \in \mathbb{Z}_{[1,l-1]}$. It is a path from $v_1$ to $v_l$, so with source $v_1$ and target $v_l$.

(b) A finite directed graph $(V,E)$ has a center $v_V \in V$ if it has no path from any vertex to itself and if it has at least one path from $v_V$ to any other vertex $v \in V$. (The center is unique, which justifies the notation $v_V$.)

(c) Consider a tuple $(\succ_p)_{p \in P}$ of excellent orders for a finite set $P \subset \mathcal{P}$ of prime numbers. It defines a finite directed graph $(V,E_V)$ with center $v_V$ as follows. Its set $V = V((\succ_p)_{p \in P})$ of vertices is the quadrant in $\mathbb{N}$

$$V := \{ \prod_{p \in P} p^{k_p} \mid k_p \in \mathbb{Z}_{[0,s(\succ_p)]} \}. \quad (9.5)$$

Its set of edges $E_V = E((\succ_p)_{p \in P})$ is the set

$$E_V := \bigcup_{p \in P} E_{V,p} \quad (9.6)$$

$$E_{V,p} := \{ (m_a,m_b) \in V \times V \mid \pi_p(m_a) = \pi_p(m_b), v_p(m_a) \succ_p v_p(m_b) \}.$$ 

The edges in $E_{V,p}$ are called $p$-edges. So, the underlying undirected graph coincides with the undirected graph which underlies $(V,E(V))$ (defined in Definition 6.1 (c)). But the directions of edges might have changed. The graph $(V,E_V)$ is obviously centered with center

$$v_V = \prod_{p \in P} p^{s^+(\succ_p)}. \quad (9.7)$$
Definition 9.4. (a) Let $\succ$ be an excellent order on the set $\mathbb{Z}_{[0,s(\succ)]}$. A set $K \subset \mathbb{N}_0$ is subset compatible with $\succ$ if a bound $k_K \in \mathbb{Z}_{[0,s(\succ)]}$ with

$$K = \{k \in \mathbb{Z}_{[0,s(\succ)]} \mid k \succ k_K\} \quad (9.8)$$

exists or if $K = \mathbb{Z}_{[0,s(\succ)]}$. $(k_K = s(\succ))$ gives $K = \emptyset$, which is allowed.

(b) For a finite non-empty set $M \subset \mathbb{N}$, let

$$\mathcal{P}(M) := \{p \in \mathcal{P} \mid M \neq \pi_p(M)\} \quad (9.9)$$

$$= \{p \in \mathcal{P} \mid \exists m \in M \text{ with } v_p(m) > 0\}$$

be the set of prime numbers which turn up as factors of some numbers in $M$.

(c) A finite non-empty set $M \subset \mathbb{N}$ is compatible with a tuple $(\succ_p)_{p \in \mathcal{P}}$ of excellent orders for a finite set $P \supset \mathcal{P}(M)$ of prime numbers if

$$M \subset V((\succ_p)_{p \in \mathcal{P}}) \quad (9.10)$$

and if for any prime number $p \in \mathcal{P}(M)$ and any $m_0 \in \pi_p(M)$ the set $K_{M,p,m_0}$ (with $\pi_p^{-1}(m_0) \cap M = \{m_0 \cdot p^k \mid k \in K_{M,p,m_0}\}$, see (7.25) ) is subset compatible with $\succ_p$. (So, here the $\succ_p$ for $p \in P - \mathcal{P}(M)$ are irrelevant. But $P \supset \mathcal{P}(M)$ instead of $P = \mathcal{P}(M)$ will be useful.)

(d) A map $\chi : \mathbb{N} \rightarrow \mathbb{N}_0$ with finite support $\text{supp}(\chi) := \{m \in \mathbb{N} \mid \chi(m) \neq 0\}$ is compatible with a tuple $(\succ_p)_{p \in \mathcal{P}}$ of excellent orders for a finite set $P \supset \mathcal{P}(\text{supp}(\chi))$ of prime numbers if

$$\text{supp}(\chi) \subset V((\succ_p)_{p \in \mathcal{P}}) \quad (9.11)$$

and if for any edge $(m_a, m_b) \in E_V$

$$\chi(m_a) \geq \chi(m_b). \quad (9.12)$$

(e) A covering of a map $\chi : \mathbb{N} \rightarrow \mathbb{N}_0$ with finite support is a tuple $(M_1, \ldots, M_l)$ $(l \in \mathbb{N}_0)$ of finite non-empty sets $M_j \subset \mathbb{N}$ with

$$\chi(m) = |\{j \in \{1, \ldots, l\} \mid m \in M_j\}| \text{ for any } m \in \mathbb{N}. \quad (9.13)$$

Here obviously $l \geq \max(\chi(m) \mid m \in \mathbb{N}_0) =: l_\chi$. In the case $\text{supp}(\chi) = \emptyset$ we have $l = 0$ and an empty tuple. The standard covering of $\chi$ is the tuple $(M_1^{(st)}, \ldots, M_l^{(st)})$ with

$$M_j^{(st)} = \{m \in \text{supp}(\chi) \mid \chi(m) \geq j\} \text{ for } j \in \{1, \ldots, l\}. \quad (9.14)$$

It is the unique covering with $M_1 \supset \ldots \supset M_l$, and it satisfies $M_1^{(st)} = \text{supp}(\chi)$.

(f) Let $\chi : \mathbb{N} \rightarrow \mathbb{N}_0$ have finite support, let $P \supset \mathcal{P}(\text{supp}(\chi))$ be a finite set of prime numbers, and let $(\succ_p)_{p \in \mathcal{P}}$ be a tuple of excellent
Lemma 9.5. A covering \((M_1, ..., M_l)\) of \(\chi\) is called compatible with \((\succ_p)_{p\in P}\) if each set \(M_j\) is compatible with \((\succ_p)_{p\in P}\).

The following lemma expresses the compatibility conditions in Definition 9.4 (c) and (d) in a different way, and it shows their relationship.

**Lemma 9.5.** (a) Let \(M \subset \mathbb{N}\) be a finite non-empty set, let \(P \supset \mathcal{P}(M)\) be a finite set of prime numbers, and let \((\succ)_{p\in P}\) be a tuple of excellent orders with (9.11). (Recall the definition of \((V,E_V,v_V)\) in Definition 9.3 (c).) The following three conditions are equivalent:

(i) \(M\) is compatible with \((\succ)_{p\in P}\).

(ii) \((M,E_V \cap M \times M)\) is a directed graph with center \(v_V\) (so \(v_V \in M\)), and if \(M\) contains the target of a path in \(V\), it contains all vertices in this path.

(iii) If \(m_b \in M\) and \((m_a,m_b) \in E_V\) then \(m_a \in M\).

(b) Let \(\chi: \mathbb{N} \to \mathbb{N}_0\) be a map with finite support, let \(P \supset \mathcal{P}(\text{supp}(\chi))\) be a finite set of prime numbers, and let \((\succ)_{p\in P}\) be a tuple of excellent orders with (9.11). The following three conditions are equivalent.

(i) \(\chi\) is compatible with \((\succ)_{p\in P}\).

(ii) \(\chi\) has a covering \((M_1, ..., M_l)\) which is compatible with \((\succ_p)_{p\in P}\).

(iii) The standard covering of \(\chi\) is compatible with \((\succ)_{p\in P}\).

**Proof:**

(a) (i)\(\Rightarrow\)(iii): Suppose \(m_b \in M\) and \((m_a,m_b) \in E_{V,p}\) for some prime number \(p \in P\). Then \(\pi_p(m_a) = \pi_p(m_b)\) and

\[
v_p(m_a) \succ_p v_p(m_b) \in K_{M,p,\pi_p(m_b)} = K_{M,p,\pi_p(m_a)}.
\]

As \(M\) is compatible with \((\succ_q)_{q\in P}\), also \(v_p(m_a) \in K_{M,p,\pi_p(m_a)}\). Therefore \(m_a \in M\).

(iii)\(\Rightarrow\)(ii): Apply (iii) several times.

(ii)\(\Rightarrow\)(i): Consider a prime number \(p \in \mathcal{P}(M)\), an element \(m_0 \in \pi_p(M)\), any number \(k_1 \in K_{M,p,m_0}\) and any number \(k_2 \in \mathbb{Z}_{[0,s(\succ_p)]}\) with \(k_2 \succ_p k_1\). We have to show \(k_2 \in K_{M,p,m_0}\).

Define \(m_i := m_0 \cdot p^{k_i}\) for \(i \in \{1,2\}\). Then \(m_1 \in M\), and we have to show \(m_2 \in M\). But obviously \((m_2, m_1)\) is a path in \((V,E_V)\). (ii) applies and yields \(m_2 \in M\).

(b) The case \(\text{supp}(\chi) = \emptyset\) is trivial. We suppose \(\text{supp}(\chi) \neq \emptyset\).

(i)\(\Rightarrow\)(iii): Let \(M_j^{(st)}\) be one of the sets in the standard covering of \(\chi\). It is sufficient to prove property (iii) in (a) for \(M_j^{(st)}\). Suppose \(m_b \in M_j^{(st)}\) and \((m_a,m_b) \in E_V\). Because of (9.12) and \((m_a,m_b) \in E_V\), \(\chi(m_a) \geq \chi(m_b)\). By definition of \(M_j^{(st)}\), then \(m_a \in M_j^{(st)}\).
(iii) ⇒ (ii): Trivial.
(ii) ⇒ (i): Let \((M_1, ..., M_l)\) be a covering of \(\chi\) which is compatible with \((\succ_p)_{p \in P}\). Let \((m_a, m_b) \in E_V\). We have to show \(\chi(m_a) \geq \chi(m_b)\). Consider one of the sets \(M_j\) with \(m_b \in M_j\). By hypothesis it is compatible with \((\succ_p)_{p \in P}\). Part (a) and \(m_b \in M_j\) give \(m_a \in M_j\). Therefore
\[
\chi(m_a) = |\{i \in \{1, ..., l\} \mid m_a \in M_i\}| \geq |\{i \in \{1, ..., l\} \mid m_b \in M_i\}| = \chi(m_b).
\]

\(\square\)

Theorem 9.6 compares different decompositions into Orlik blocks.

**Theorem 9.6.** Let \((\succ_p)_{p \in P}\) be at tuple of excellent orders for a finite set \(P\) of prime numbers, and let \(\chi : \mathbb{N} \to \mathbb{N}_0\) be a map with finite support which is compatible with \((\succ_p)_{p \in P}\). Let \((M_1(1), ..., M_l(1))\) and \((M_1(2), ..., M_l(2))\) be two coverings of \(\chi\) which are both compatible with \((\succ_p)_{p \in P}\). Then the corresponding sums of Orlik blocks are isomorphic,
\[
\bigoplus_{i=1}^{l_1} \text{Or}(M_i^{(1)}) \cong \bigoplus_{j=1}^{l_2} \text{Or}(M_j^{(2)}),
\]
and \(l_1 = l_2 = \chi(\max(\chi(m) \mid m \in \mathbb{N}))\).

**Proof:** At the end of the proof, we will apply Theorem 5.1 (b).

But before that, the main work is the discussion how coverings of \(\chi\) which are compatible with \((\succ_p)_{p \in P}\) can look like. There is not so much freedom.

Let \((M_1^{(st)}, ..., M_{l_{\chi}}^{(st)})\) be the standard covering of \(\chi\). For each \(j \in \mathbb{Z}_{[1,l_{\chi}]\}}\) consider the set
\[
M_j^{(st)} - M_{j+1}^{(st)} = \{m \in \text{supp}(\chi) \mid \chi(m) = j\}.
\]
as a subgraph of \((V, E_V)\) and denote its components by \(M_{j,i}^{(st)}\) for \(i \in \mathbb{Z}_{[1,c_j]}\) for some \(c_j \in \mathbb{N}\). We define a directed graph \((V^{(\text{comp})}, E^{(\text{comp})})\) with set of vertices
\[
V^{(\text{comp})} := \{M_{j,i}^{(st)} \mid j \in \mathbb{Z}_{[1,l_{\chi}]}, i \in \mathbb{Z}_{[1,c_j]}\}\]
and set of edges
\[
E^{(\text{comp})} := \{(v_1, v_2) \in V^{(\text{comp})} \times V^{(\text{comp})} \mid \text{v_1 \neq v_2, an edge in } E_V \text{ from a vertex in } v_1 \text{ to a vertex in } v_2 \text{ exists}\}.
\]
Furthermore, we define the map
\[
\chi^{(\text{comp})} : V^{(\text{comp})} \to \mathbb{N}_0, \quad M_{j,i}^{(st)} \mapsto j.
\]

(9.15)
By Lemma 9.5 (b) and by hypothesis, the standard covering is compatible with \((\succ_p)_{p \in P}\). Therefore for any edge \((v_1, v_2) \in E^{(\text{comp})}\),
\[
\chi^{(\text{comp})}(v_1) \geq \chi^{(\text{comp})}(v_2).
\] In fact, for any edge \((v_1, v_2) \in E^{(\text{comp})}\) even
\[
\chi^{(\text{comp})}(v_1) > \chi^{(\text{comp})}(v_2)
\] holds. Because if on the contrary \(\chi^{(\text{comp})}(v_1) = \chi^{(\text{comp})}(v_2)\), then \(v_1\) and \(v_2\) would be subsets of the subgraph \(M_j^{(\text{st})} - M_{j+1}^{(\text{st})}\) where \(j := \chi^{(\text{comp})}(v_1)\), and because of \((v_1, v_2) \in E^{(\text{comp})}\) an edge from a vertex in \(v_1\) to a vertex in \(v_2\) would exist, so they would not be different components.

\(M_1^{(\text{st})} = \text{supp}(\chi)\) is a directed graph with center \(v_V\). Therefore and because of (9.19), also \((V^{(\text{comp})}, E^{(\text{comp})})\) is a directed graph with center. Its center is the component \(M_{l_{\chi,1}}^{(\text{st})}\), which contains \(v_V\), and which is the only component of \(M_1^{(\text{st})}\).

The following claim makes the shape of any covering \((M_1, ..., M_{l_{\chi}})\) of \(\chi\) which is compatible with \((\succ_p)_{p \in P}\) explicit.

**Claim 1:** A covering \((M_1, ..., M_{l_{\chi}})\) of \(\chi\) is compatible with \((\succ_p)_{p \in P}\) if and only if the following holds:

- \(l_1 = l_{\chi}\), and a tuple \((V_1, ..., V_{l_{\chi}})\) of subsets of \(V^{(\text{comp})}\) exists such that
  \[
  M_j = \bigcup_{v \in V_j} v \subset \text{supp}(\chi),
  \]
  \[
  \chi^{(\text{comp})}(v) = |\{j \in \mathbb{Z}_{[1,l_{\chi}]} | v \in V_j\}|,
  \]
and such that each subset \(V_j\) is as a subgraph of \(V^{(\text{comp})}\) a directed graph with center \(M_j^{(\text{st})}\) and contains all vertices of a path in \(V^{(\text{comp})}\) whose target it contains.

(Examples are given in Examples 9.7.)

**Proof of Claim 1:** \(\Rightarrow\): Suppose that \((M_1, ..., M_{l_{\chi}})\) is a covering of \(\chi\) which is compatible with \((\succ_p)_{p \in P}\). First we will prove (9.20) and (9.21) (with \(l_1\) instead of \(l_{\chi}\)).

Consider an edge \((m_a, m_b) \in E_V\) with \(m_a\) and \(m_b\) in the same component \(M_{j,i}^{(\text{st})}\). Then

\[
|\{k \in \mathbb{Z}_{[1,l_{\chi}]} | m_a \in M_k\}| = \chi(m_a) = \chi^{(\text{comp})}(M_{j,i}^{(\text{st})}) = j
\]
\[
= \chi(m_b) = |\{k \in \mathbb{Z}_{[1,l_{\chi}]} | m_b \in M_k\}|.
\]

Let \(M_k\) be one subset which contains \(m_b\). Because \(M_k\) is compatible with \((\succ_p)_{p \in P}\) and because of Lemma 9.3 (a), also \(m_a \in M_k\). Therefore \(m_a\) and \(m_b\) are elements of the same sets \(M_k\).
Because $M_{j,i}^{(st)}$ is a connected subgraph of $(V, E_V)$, this implies that these sets $M_k$ contain all of $M_{j,i}^{(st)}$ and that all other sets $M_k$ do not intersect $M_{j,i}^{(st)}$.

Therefore a tuple $(V_1, ..., V_{l_i})$ of subsets of $V^{(comp)}$ with (9.20) and (9.21) (with $l_i$ instead of $l_k$) exists.

Suppose that $v_2 \in V_j$ for some $j \in \mathbb{Z}_{[1, l_i]}$ and that $(v_1, v_2) \in E^{(comp)}$ is an edge. Then an edge $(m_1, m_2) \in E_V$ with $m_1 \in v_1$ and $m_2 \in v_2 \subset M_j$ exists. $M_j$ is compatible with $(\succ_p)_{p \in P}$. Lemma 9.5 (a) implies $m_1 \in M_j$. Therefore $v_1 \in V_j$.

This shows that $V_j$ contains any vertex $v \in V^{(comp)}$ such that a path to a vertex $v_2 \in V_j$ exists. And especially, therefore $V_j$ is a directed subgraph of $V^{(comp)}$ with center $M_{l_i}^{(st)}$.

Any set $V_j$ contains the vertex $M_{l_i}^{(st)}$, and $\chi^{(comp)}(M_{l_i}^{(st)}) = l_\chi$. With (9.21) this gives $l_1 = l_\chi$.

$\Leftarrow$: (9.20) and (9.21) show that $(M_1, ..., M_{l_\chi})$ is a covering of $\chi$. It remains to show that each set $M_j$ is compatible with $(\succ_p)_{p \in P}$.

Let $(m_a, m_b) \in E_V$ be an edge and $m_b \in M_j$ for some $j \in \mathbb{Z}_{[1, l_\chi]}$. It is sufficient to show $m_a \in M_j$, because then one can apply Lemma 9.5 (a) (iii) $\Rightarrow$ (i).

Let $v_a \in V^{(comp)}$ respectively $v_b \in V_j$ be the vertex of $V^{(comp)}$ which contains $m_a$ respectively $m_b$.

$\chi(m_a) < \chi(m_b)$ is impossible because $\chi$ is compatible with $(\succ_p)_{p \in P}$.

If $\chi(m_a) = \chi(m_b)$, then $v_a = v_b$, and $m_a \in M_j$ because of (9.20).

If $\chi(m_a) > \chi(m_b)$, then $(v_a, v_b) \in E^{(comp)}$. Then the hypothesis on $V_j$ implies $v_a \in V_j$. And $m_a \in M_j$ because of (9.20). This finishes the proof of Claim 1. $\square$

**Claim 2:** (a) Let $(M_1, ..., M_{l_\chi})$ be a covering of $\chi$ which is compatible with $(\succ_p)_{p \in P}$, and which satisfies $M_1 \not\subseteq M_2$ and $M_2 \not\subseteq M_1$. Then also $(M_1 \cup M_2, M_1 \cap M_2, M_3, ..., M_{l_\chi})$ is a covering of $\chi$ which is compatible with $(\succ_p)_{p \in P}$.

(b) Iterating the procedure in (a), one can go from any covering of $\chi$ which is compatible with $(\succ_p)_{p \in P}$ to the standard covering and thus also to any other covering of $\chi$ which is compatible with $(\succ_p)_{p \in P}$.

**Proof of Claim 2:** (a) Because of (9.20) and (9.21) it is clear that $(M_1 \cup M_2, M_1 \cap M_2, M_3, ..., M_{l_\chi})$ is a covering of $\chi$ and that it satisfies (9.20) and (9.21). The corresponding tuple of subsets of $V^{(comp)}$ is $(V_1 \cup V_2, V_1 \cap V_2, V_3, ..., V_{l_\chi})$. It is also clear that $V_1 \cup V_2$ and $V_1 \cap V_2$ are directed subgraphs of $V^{(comp)}$ with center $M_{l_\chi}^{(st)}$ and that $V_1 \cup V_2$...
respectively \( V_1 \cap V_2 \) contains all vertices of a path in \( V^{(\text{comp})} \) whose target it contains.

(b) Iterating the procedure in (a), one can increase the set \( M_1 \) until it is \( M_1^{(st)} = \text{supp}(\chi) \). Then again iterating the procedure in (a), but leaving \( M_1 \) as it is, one can increase the set \( M_2 \) until it is \( M_2^{(st)} \). One can continue this until one reaches the standard covering of \( \chi \). \( \square \)

Claim 3: In the situation of part (a) of Claim 2,
\[
\bigoplus_{j=1}^{l_x} \text{Or}(M_j) \cong \text{Or}(M_1 \cup M_2) \oplus \text{Or}(M_1 \cap M_2) \oplus \bigoplus_{j=3}^{l_x} \text{Or}(M_j). \quad (9.22)
\]

Proof of Claim 3: We will apply Theorem 5.1 (b). Define
\[
f_1 := 1, \quad f_2 := \prod_{m \in M_2-M_1} \Phi_m, \quad f_3 := \prod_{m \in M_1 \cap M_2} \Phi_m, \quad f_4 := \prod_{m \in M_1-M_2} \Phi_m.
\]
Then
\[
H^{[f_1 f_3 f_4]} \cong \text{Or}(M_1), \quad H^{[f_2 f_4]} \cong \text{Or}(M_2),
\]
\[
H^{[f_1 f_3]} \cong \text{Or}(M_1 \cap M_2), \quad H^{[f_2 f_3 f_4]} \cong \text{Or}(M_1 \cup M_2).
\]

\(| \text{Res}(f_1, f_4) \) = \( \text{Res}(1, f_4) = 1 \) is trivially true. We want to show \(| \text{Res}(f_2, f_4) \) = 1. Suppose \(| \text{Res}(f_2, f_4) \) > 1. Because of (4.24) and (4.8) there is an edge in \( E(\mathbb{N}) \) which connects a vertex \( m_1 \in M_1 - M_2 \) with a vertex \( m_2 \in M_2 - M_1 \). Then either \( (m_1, m_2) \) or \( (m_2, m_1) \) is an edge in \( E_V \). For \( j \in \{1, 2\} \), let \( V_j \subset V^{(\text{comp})} \) be the subset with \( M_j = \bigcup_{v \in V_j} v_j \) and let \( v_j \in V_j \) be the vertex with \( m_j \in v_j \). Then \( (v_1, v_2) \) or \( (v_2, v_1) \) is an edge in \( E(\text{comp}) \), but \( v_1 \in V_1 - V_2 \) and \( v_2 \in V_2 - V_1 \), as \( m_1 \in M_1 - M_2 \) and \( m_2 \in M_2 - M_1 \). This is a contradiction to the property of \( V_1 \) and \( V_2 \) that they contain with the endpoint of a path in \( V^{(\text{comp})} \) also the source of the path. Because of this contradiction
\(| \text{Res}(f_2, f_4) \) = 1.

Theorem 5.1 (b) can be applied and gives
\[
\text{Or}(M_1) \oplus \text{Or}(M_2) \cong \text{Or}(M_1 \cup M_2) \oplus \text{Or}(M_1 \cap M_2). \quad (9.23)
\]
This implies (9.22) and finishes the proof of Claim 3. \( \square \)

Claim 3 and part (b) of Claim 2 show that any covering \( (M_1, ..., M_{l_\chi}) \) of \( \chi \) which is compatible with \( (\succ_p)_{p \in P} \) satisfies
\[
\bigoplus_{j=1}^{l_\chi} \text{Or}(M_j) \cong \bigoplus_{j=1}^{l_\chi} \text{Or}(M_j^{(st)}). \quad (9.24)
\]
This implies immediately also (9.15) (and there \( l_1 = l_2 = l_\chi \)). This finishes the proof of Theorem 9.6. \( \square \)
Examples 9.7. (i) Fix two prime numbers $p_1$ and $p_2$ and four numbers $k_1, k_2, k_3, k_4 \in \mathbb{N}$ with $k_1 < k_3$, $k_2 < k_4$, and define the following three rectangles of numbers,

$\begin{align*}
N_0 &:= \{p_1^{l_1}p_2^{l_2} \mid (l_1, l_2) \in \mathbb{Z}_{[0,k_1]} \times \mathbb{Z}_{[0,k_2]}\}, \\
N_1 &:= \{p_1^{l_1}p_2^{l_2} \mid (l_1, l_2) \in \mathbb{Z}_{[k_1+1,k_3]} \times \mathbb{Z}_{[0,k_2]}\}, \\
N_2 &:= \{p_1^{l_1}p_2^{l_2} \mid (l_1, l_2) \in \mathbb{Z}_{[0,k_1]} \times \mathbb{Z}_{[k_2+1,k_4]}\}.
\end{align*}$

Define a map $\chi : \mathbb{N} \to \mathbb{N}_0$ with finite support by

$$\chi(m) := \begin{cases} 
2 & \text{if } m \in N_0, \\
1 & \text{if } m \in N_1 \cup N_2, \\
0 & \text{if } m \in \mathbb{N} - (N_0 \cup N_1 \cup N_2).
\end{cases}$$

Then the standard covering of $\chi$ is given by $l_\chi = 2$ and

$$M_2^{(st)} = N_0 \subset M_1^{(st)} = N_0 \cup N_1 \cup N_2.$$ 

Consider the excellent orders $>_p$ and $>_p$ with

$\begin{align*}
s(>_p) &= k_3, \quad S(>_p) = \emptyset, \text{ so } >_p \text{ is } < \text{ on } \mathbb{Z}_{[0,k_3]}, \\
s(>_p) &= k_4, \quad S(>_p) = \emptyset, \text{ so } >_p \text{ is } < \text{ on } \mathbb{Z}_{[0,k_4]},
\end{align*}$

$\chi$ and its standard covering are compatible with the tuple $(>_p, >_p)$ of excellent orders. The only other covering of $\chi$ which is compatible with $(>_p, >_p)$, consists of

$$M_1 = N_0 \cup N_1, \quad M_2 = N_0 \cup N_2,$$

because

$$V^{(comp)} = \{M_2^{(st)}, M_1^{(st)}, M_1^{(st)}\} \text{ with } M_1^{(st)} = N_1.$$

(ii) Keep the data from (i). Define a new map $\tilde{\chi} : \mathbb{N} \to \mathbb{N}_0$ with finite support by

$$\tilde{\chi}(m) := \chi(m) + \delta_{(m=p_1^{k_1+1}p_2^{k_2+1})}.$$ 

Then its standard covering is given by $l_{\tilde{\chi}} = 2$ and

$$\tilde{M}_2^{(st)} = N_0, \quad \tilde{M}_1^{(st)} = N_0 \cup N_1 \cup N_2 \cup \{p_1^{k_1+1}p_2^{k_2+1}\}.$$ 

$\tilde{\chi}$ and its standard covering are compatible with the tuple $(>_p, >_p)$ of excellent orders. No other covering of $\tilde{\chi}$ is compatible with $(>_p, >_p)$, because the difference set $\tilde{M}_1^{(st)} - \tilde{M}_2^{(st)}$ has only one component, so

$$V^{(comp)} = \{\tilde{M}_2^{(st)}, \tilde{M}_1^{(st)} - \tilde{M}_2^{(st)}\}.$$ 

Now we turn to a situation which is motivated by Theorem 7.4.

Two finite non-empty sets $M \subset \mathbb{N}$ and $N \subset \mathbb{N}$ give rise to the map $\chi : \mathbb{N} \to \mathbb{N}_0$ defined in (7.19) with finite support $L$ (as in (7.20)).

First we consider a special case.
Lemma 9.8. Let \( \succ_1 \) and \( \succ_2 \) be two excellent orders. Let \( K_1 \subset \mathbb{N}_0 \) and \( K_2 \subset \mathbb{N}_0 \) be two finite non-empty sets such that \( K_j \) is subset compatible with \( \succ_j \) (Definition 9.4 (a)) for \( j \in \{1, 2\} \). Let \( p \) be a prime number. Define \( M := \{p^k \mid k \in K_1\} \) and \( N := \{p^k \mid k \in K_2\} \). Consider \( \chi : \mathbb{N} \to \mathbb{N}_0 \) as in \((7.19)\).

Then \( \chi \) is compatible with \( \succ_1 \otimes \succ_2 \) (considered as a tuple of excellent orders over \( P = \{p\} \)).

**Proof:** Define for \( j \in \{1, 2\} \)
\[
\begin{align*}
  f(K_j) &:= \begin{cases} 
    \max(k \in \mathbb{N}_0 \mid \mathbb{Z}_{[0,k]} \subset K_j) & \text{if } 0 \in K_j, \\
    -1 & \text{if } 0 \notin K_j,
  \end{cases} \\
  e(K_j) &:= \begin{cases} 
    \max(k \in \mathbb{N}_0 \mid \mathbb{Z}_{[0,k]} \cap K_j = \emptyset) & \text{if } 0 \notin K_j, \\
    -1 & \text{if } 0 \in K_j,
  \end{cases} \\
  k^{(j)} &:= \max(f(K_j), e(K_j)), \\
  k^{\max} &:= \max(k^{(1)}, k^{(2)}).
\end{align*}
\]

Then either \( f(K_j) \geq 0 \) and \( e(K_j) = -1 \) or \( f(K_j) = -1 \) and \( e(K_j) \geq 0 \). In both cases \( k^{(1)}, k^{(2)}, k^{\max} \in \mathbb{N}_0 \).

**Claim 1:** \( \chi \) has constant values on \( \{p^k \mid k \in \mathbb{Z}_{[0,k^{\max}]}\} \).

**Proof of Claim 1:** Here \( \chi(p^k) \) for any \( k \in \mathbb{N}_0 \) is given by a formula similar to \((7.33)\), namely
\[
\chi(p^k) = \sum_{(k_1, k_2) \in K_1 \times K_2} \delta(p^{k_1}, p^{k_2}, p^k) \quad (9.29)
\]
\[
\begin{align*}
  &= \delta(k \in K_1) \cdot \sum_{k_2 \in K_2 : k_2 < k} \varphi(p^{k_2}) + \delta(k \in K_2) \cdot \sum_{k_1 \in K_1 : k_1 < k} \varphi(p^{k_1}) \\
  &\quad + \delta(k \in K_1 \cap K_2 \cap \mathbb{N}) \cdot (p - 2)p^{k-1} + \delta(k \in K_1 \cap K_2 \cap \{0\}) + \sum_{k_1 \in K_1 \cap K_2 : k_1 > k} \varphi(p^{k_1}).
\end{align*}
\]

We can restrict to the two cases \( k^{\max} = f(K_1) \) and \( k^{\max} = e(K_1) \).

The case \( k^{\max} = f(K_1) \geq 0 \): We consider \( k \in \mathbb{Z}_{[0,k^{\max}]} \) and claim
\[
\chi(p^k) = \sum_{k_2 \in K_2 : k_2 \leq k^{\max}} \varphi(p^{k_2}) + \sum_{k_1 \in K_1 \cap K_2 : k_1 > k^{\max}} \varphi(p^{k_1}). \quad (9.30)
\]

We prove this in the two subcases \( k \notin K_2 \) and \( k \in K_2 \). In both cases \( \mathbb{Z}_{[0,k^{\max}]} \subset K_1 \), and \( \delta(k \in K_1) = 1 \), and \( \sum_{k_1 \in K_1 : k_1 < k} \varphi(p^{k_1}) = \delta(k > 0) \cdot p^{k-1} \).

The subcase \( k \notin K_2 \): Then only the first and the last summand in \((9.29)\) do not vanish. The last summand can be split into
\[
\sum_{k_2 \in K_2 : k < k_2 \leq k^{\max}} \varphi(p^{k_2}) + \sum_{k_1 \in K_1 \cap K_2 : k_1 > k^{\max}} \varphi(p^{k_1}).
\]
The first summand in (9.29) and this sum give (9.30).

The subcase \( k \in K_2 \): Then (9.29) takes the following shape, where we split again the last summand into the two pieces above,

\[
\chi(k) = \sum_{k_2 \in K_2: k_2 < k} \varphi(p^{k_2}) + \delta(k > 0) \cdot p^{k-1} + \delta(k > 0) \cdot (p - 2)p^{k-1} \\
+ \delta(k = 0) + \sum_{k_2 \in K_2: k < k \leq k^{max}} \varphi(p^{k_2}) + \sum_{k_1 \in K_1 \cap K_2: k_1 > k^{max}} \varphi(p^{k_1}).
\]

In the case \( k > 0 \), the second and third summand give \( \varphi(p^k) \). In the case \( k = 0 \), the fourth summand \( \delta(k = 0) \) gives \( \varphi(p^k) = 1 \). In both cases (9.30) is true.

The case \( k^{max} = e(K_1) \): We consider \( k \in \mathbb{Z}_{[0,k^{max}]} \) and claim

\[
\chi(p^k) = \sum_{k_1 \in K_1 \cap K_2: k_1 > k^{max}} \varphi(p^{k_1}). \tag{9.31}
\]

Here \( \mathbb{Z}_{[0,k^{max}]} \cap K_1 = \emptyset \) and especially \( k \notin K_1 \). Only the last summand in (9.29) does not vanish. It gives (9.31). This finishes the proof of Claim 1.

\( \blacksquare \)

**Claim 2:** For \( j \in \{1, 2\} \)

\[
K_j \cap \mathbb{Z}_{>k(j)} = S(_{>j}) \cap \mathbb{Z}_{>k(j)}. \tag{9.32}
\]

**Proof of Claim 2:** If \( k(j) = e(K_j) \) then \( K_j = K_j \cap \mathbb{Z}_{>k(j)} \) and especially \( 0 \notin K_j \). Then the subset compatibility of \( K_j \) with \( _{>j} \) shows \( K_j = S(_{>j}) \cap \mathbb{Z}_{>k(j)} \). Then (9.32) is clear.

If \( k(j) = f(K_j) \) then \( K_j \supset \mathbb{Z}_{[0,k(j)]} \) and \( k(j) + 1 \notin K_j \) and \( 0 \in K_j \). Then the subset compatibility of \( K_j \) with \( _{>j} \) shows \( K_j = (S(_{>j}) \cap \mathbb{Z}_{>k(j)}) \cup \mathbb{Z}_{[0,k(j)]} \). Also then (9.32) is clear. This finishes the proof of Claim 2.

\( \blacksquare \)

Write \( _{>3} := (_{>1} \otimes _{>2}) \). First we will show (9.11), then (9.12). By subset compatibility \( K_j \subset \mathbb{Z}_{[0, s(_{>j})]} \). By definition of \( \chi \),

\[
\text{supp}(\chi) \subset \{ p^k | k \leq \text{max}(K_1 \cup K_2) \} \\
\quad \subset \{ p^k | k \leq s(_{>3}) \} = V(_{>3}).
\]

This is (9.11). For the proof of (9.12), we have to show the following: For \( k_1, k_2 \in \mathbb{Z}_{[0, s(_{>3})]} \) with \( k_1 > k_2 \)

\[
k_1 > k_2 \quad \Rightarrow \quad \chi(p^{k_1}) \geq \chi(p^{k_2}) , \\
k_2 > k_1 \quad \Rightarrow \quad \chi(p^{k_2}) \geq \chi(p^{k_1}). \tag{9.33}
\]

We consider several cases.

The case \( k_1 \leq k^{max} \). Then Claim 1 implies \( \chi(p^{k_1}) = \chi(p^{k_2}) \). Then it does not matter whether \( k_1 > k_2 \) or \( k_2 > k_1 \).
The case $k_1 > k^{\text{max}}$ and $k_1 \in S(\succ_3)$: By the relation between $S(\succ_3)$ and $\succ_3$ then $k_1 \succ_3 k_2$. By definition of $S(\succ_3)$ in (9.4) and by (9.32) $k_1 \in (K_1 \cup K_2) - (K_1 \cap K_2)$. Lemma 7.6 (a) implies $\chi(p^{k_1}) \geq \chi(p^{k_2})$. This shows (9.33).

The case $k_1 > k^{\text{max}}$ and $k_1 \notin S(\succ_3)$: By the relation between $S(\succ_3)$ and $\succ_3$ then $k_2 \succ_3 k_1$. By definition of $S(\succ_3)$ in (9.4) and by (9.32) $k_1 \notin (K_1 \cup K_2) - (K_1 \cap K_2)$. Lemma 7.6 (a) implies $\chi(p^{k_2}) \geq \chi(p^{k_1})$. This shows (9.33). This finishes the proof of Lemma 9.8.

**Theorem 9.9.** Let $M \subset \mathbb{N}$ and $N \subset \mathbb{N}$ be two finite non-empty sets, let $P \supset \mathcal{P}(M) \cup \mathcal{P}(N)$ be a finite set of prime numbers, and let $(\succ_p^M)_{p \in P}$ and $(\succ_p^N)_{p \in P}$ be two tuples of excellent orders such that $M$ and $(\succ_p^M)_{p \in P}$ are compatible and $N$ and $(\succ_p^N)_{p \in P}$ are compatible.

Let $\chi : \mathbb{N} \to \mathbb{N}_0$ be as in (7.11) with finite support $L$. For any $p \in P$ and any $(m_0, n_0) \in \pi_p(M) \times \pi_p(N)$ define $\chi_{p,m_0,n_0} : L \to \mathbb{N}_0$ as in (7.21). Extend it to $\mathbb{N} - L$ with values 0. Write $\succ_L^p := \succ_p^M \otimes \succ_p^N$ for any $p \in P$.

(a) Any $\chi_{p,m_0,n_0}$ is compatible with the excellent order $\succ_L^p$ in the following sense (which was not considered in Definition 9.4 (d)):

\[(l_a, l_b) \in E_{V,p} \Rightarrow \chi_{p,m_0,n_0}(l_a) \geq \chi_{p,m_0,n_0}(l_b). \tag{9.34}\]

Here $V = V((\succ_q^L)_{q \in P})$, and $E_{V,p}$ in (9.6) is determined by $\succ_L^p$.

(b) $\chi$ is compatible with the tuple $(\succ_L^p)_{p \in P}$ of excellent orders.

(c) The pair $(M, N)$ is sdiOb-sufficient.

**Proof:** (a) Define $l_0 := \pi_p(l_a) = \pi_p(l_b)$. If $\delta(m_0, n_0, l_0) = 0$ then by Lemma 7.6 (b) $\chi_{p,m_0,n_0}(l_a) = \chi_{p,m_0,n_0}(l_b) = 0$, and (9.34) holds. Consider the case $\delta(m_0, n_0, l_0) > 0$. Define $K_1 := K_{M,p,m_0}$ and $K_2 := K_{N,p,n_0}$. Then for $k \in \mathbb{N}_0$

\[(\chi \text{ in Lemma 9.8}) (p^k) = \frac{\chi_{p,m_0,n_0}(l_0 \cdot p^k)}{\delta(m_0, n_0, l_0)} \tag{9.35}\]

Therefore Lemma 9.8 implies (9.34).

(b) We have to show for any edge $(l_a, l_b) \in E_V \chi(l_a) \geq \chi(l_b)$. There is a unique prime number $p$ with $(l_a, l_b) \in E_{V,p}$. By (7.22)

\[\chi = \sum_{(m_0, n_0) \in \pi_p(M) \times \pi_p(N)} \chi_{p,m_0,n_0}.\]

This and part (a) show $\chi(l_a) \geq \chi(l_b)$.

(c) We have to show for any prime number $p$ and any $p$-edge $(l_a, l_b) \in E_p(L)$ that (7.23) or (7.24) holds.
Either \((l_a, l_b) \in E_{V, p}\) or \((l_b, l_a) \in E_{V, p}\). In the case \((l_a, l_b) \in E_{V, p}\), (9.34) gives (7.23). In the case \((l_b, l_a) \in E_{V, p}\), (9.34) gives (7.24). □

**Theorem 9.10.** Let \(\chi_1 : \mathbb{N} \to \mathbb{N}_0\) and \(\chi_2 : \mathbb{N} \to \mathbb{N}_0\) be two maps with finite supports \(M = \text{supp}(\chi_1)\) and \(N = \text{supp}(\chi_2)\). Define a map \(\chi_3 : \mathbb{N} \to \mathbb{N}_0\) by

\[
\left( \prod_{m \in M} \Phi_{m}^{\chi_1(m)} \right) \otimes \left( \prod_{n \in N} \Phi_{n}^{\chi_2(n)} \right) = \prod_{l \in \mathbb{N}_0} \Phi_{l}^{\chi_3(l)}.
\]

(9.36)

\(\chi_3\) has finite support \(\text{supp}(\chi_3) =: L\).

Let \(P \supset \mathcal{P}(M) \cup \mathcal{P}(N)\) be a finite set of prime numbers. Let \((\succsim^M_p)_{p \in P}\) and \((\succsim^N_p)_{p \in P}\) be two tuples of excellent orders such that \(\chi_1\) is compatible with \((\succsim^M_p)_{p \in P}\) and \(\chi_2\) is compatible with \((\succsim^N_p)_{p \in P}\). Write \(\succsim^L_p := (\succsim^M_p \otimes \succsim^N_p)\) for any \(p \in P\).

(a) \(\chi_3\) is compatible with the tuple \((\succsim^L_p)_{p \in P}\) of excellent orders.

(b) Let \((M^{(st)}_1, ..., M^{(st)}_{i,1}), (N^{(st)}_1, ..., N^{(st)}_{i,2})\) and \((L^{(st)}_1, ..., L^{(st)}_{i,3})\) be the standard coverings of \(\chi_1\), \(\chi_2\) and \(\chi_3\). Then

\[
\left( \bigoplus_{i=1}^{l_1} \text{Or}(M^{(st)}_i) \right) \otimes \left( \bigoplus_{j=1}^{l_2} \text{Or}(N^{(st)}_j) \right) \cong \bigoplus_{k=1}^{l_3} \text{Or}(L^{(st)}_k),
\]

(9.37)

so the tensor product of sums of Orlik blocks on the left hand side admits a standard decomposition into Orlik blocks.

**Proof:** (a) For \((i, j) \in \mathbb{Z}_{[0,l_{i,1}]} \times \mathbb{Z}_{[0,l_{i,2}]}\) define a map \(\chi_{3,i,j} : \mathbb{N} \to \mathbb{N}_0\) by

\[
\left( \prod_{m \in M^{(st)}_i} \Phi_{m}^{\chi_1(m)} \right) \otimes \left( \prod_{n \in N^{(st)}_j} \Phi_{n}^{\chi_2(n)} \right) = \prod_{l \in \mathbb{N}_0} \Phi_{l}^{\chi_{3,i,j}(l)}.
\]

(9.38)

\(\chi_{3,i,j}\) has finite support.

By Lemma 9.5 (b) and the hypotheses on \(\chi_1\) and \(\chi_2\), the set \(M^{(st)}_i\) is compatible with \((\succsim^M_p)_{p \in P}\) and the set \(N^{(st)}_j\) is compatible with \((\succsim^N_p)_{p \in P}\).

By Theorem 9.9 (b), \(\chi_{3,i,j}\) is compatible with \((\succsim^L_p)_{p \in P}\).

Because of (9.38), \(\chi_3 = \sum_{(i,j)} \chi_{3,i,j}\). Therefore also \(\chi_3\) is compatible with \((\succsim^L_p)_{p \in P}\).

(b) Let \((L^{(st)}_1, ..., L^{(st)}_{i,3})\) be the standard covering of \(\chi_{3,i,j}\). By Lemma 9.5 (b) each set \(L^{(st)}_k\) is compatible with \((\succsim^L_p)_{p \in P}\).

By Theorem 9.9 (c), each pair \((M^{(st)}_i, N^{(st)}_j)\) is sdiOb-sufficient.
By Theorem 7.4, the tensor product $\text{Or}(M^{(st)}_i) \otimes \text{Or}(N^{(st)}_j)$ admits a standard decomposition into Orlik blocks. In other words,

$$\text{Or}(M^{(st)}_i) \otimes \text{Or}(N^{(st)}_j) \cong \bigoplus_{k=1}^{t_{x_3,i,j}} \text{Or}(L^{(i,j,st)}_k). \tag{9.39}$$

Because of $x_3 = \sum_{(i,j)} x_{3,i,j}$, the tuple

$$\bigl(L^{(i,j,st)}_k \mid (i,j) \in \mathbb{Z}[0,t_{x_1}] \times \mathbb{Z}[0,t_{x_2}], \ k \in \mathbb{Z}[0,t_{x_3,i,j}]\bigr)$$

is a covering of $x_3$. Because all sets in it are compatible with $(\succ)_{p \in \mathbb{P}}$, it is also compatible with $(\succ)_{p \in \mathbb{P}}$.

(9.39) gives the (possibly) non standard decomposition on the right hand side of (9.40) for the tensor product of sums of Orlik blocks on the left hand side of (9.40),

$$\left(\bigoplus_{i=1}^{t_{x_1}} \text{Or}(M^{(st)}_i)\right) \otimes \left(\bigoplus_{j=1}^{t_{x_2}} \text{Or}(N^{(st)}_j)\right) = \bigoplus_{(i,j,k)} \text{Or}(L^{(i,j,st)}_k). \tag{9.40}$$

We can apply Theorem 9.6. It says that the sum of Orlik blocks on the right hand side of (9.40) is isomorphic to the standard decomposition into Orlik blocks on the right hand side of (9.37).

Remarks 9.11. (i) In Theorem 9.9 the sdiOb-sufficiency of the pair $(M,N)$ in part (c) is weaker than part (a). The sdiOb-sufficiency demands only that for any fixed $p$-edge $(l_a,l_b) \in E_p(M)$ one has (7.23) or (7.24). Part (a) gives for fixed $k_a$ and $k_b$ and any $(l_a,l_b) \in E_p(M)$ with $v_p(l_a) = k_a$ and $v_p(l_b) = k_b$ the same alternative (7.23) or (7.24).

(ii) Therefore one might ask whether a weaker condition than the compatibility with tuples of excellent orders might also have good properties. Lemma 9.12 says in a precise sense that this is not the case, but that the compatibility with tuples of excellent orders is needed. Lemma 9.12 was the way how we found the excellent orders and the compatibilities with them.

(iii) Condition (ii) in Lemma 9.12 is via Theorem 7.4 and the (conjectural) Remark 7.3 (i) the condition that the tensor product of an Orlik block $\text{Or}(M)$ with the Milnor lattice of an arbitrary $A_{\mu}$-singularity admits a standard decomposition into Orlik blocks. So, it is a quite natural condition.

Lemma 9.12. Let $M \subset \mathbb{N}$ be a finite non-empty set. The following conditions are equivalent.
(i) For any $n_N \in \mathbb{N}$, the pair $(M, \{n \in \mathbb{N} | n|n_N\})$ is sdiOb-sufficient.

(ii) For any $n_N \in \mathbb{N}$, the pair $(M, \{n \in \mathbb{N} | n|n_N\} - \{1\})$ is sdiOb-sufficient.

(iii) A tuple $(\succ_p)_{p \in \mathcal{P}(M)}$ of excellent orders exists such that $M$ is compatible with it.

**Proof:** For any $n_N \in \mathbb{N}$, the sets $\{n \in \mathbb{N} | n|n_N\}$ and $\{n \in \mathbb{N} | n|n_N\} - \{1\}$ are compatible with suitable tuples of excellent orders. Therefore and by Theorem 9.9 (c), (iii) implies (i) and (ii).

(i)⇒(iii): Suppose that (i) holds. We will define for any prime number $p \in \mathcal{P}(M)$ an excellent order $\succ_p$ and then show that the set $M$ is compatible with the tuple $(\succ_p)_{p \in \mathcal{P}(M)}$ of excellent orders (Definition 9.4 (c)).

Fix a prime number $p \in \mathcal{P}(M)$. We will define an excellent order $\succ_p$ by fixing the number $s(\succ_p) \in \mathbb{N}_0$ and the set $S(\succ_p) \subset \mathbb{Z}_{[0, s(\succ_p)]}$. Then we have to show that for any $m_0 \in \pi_p(M)$ the set $K_{M,p,m_0}$ is subset compatible with $\succ_p$ (Definition 9.4 (a)).

Define the map $g_p : \pi_p(M) \to \mathbb{N}_0$ by

$$g_p(m_0) := \begin{cases} \max(k \in \mathbb{N}_0 | Z_{[0,k]} \subset K_{M,p,m_0}) & \text{if } 0 \in K_{M,p,m_0}, \\ \max(k \in \mathbb{N}_0 | Z_{[0,k]} \cap K_{M,p,m_0} = \emptyset) & \text{if } 0 \notin K_{M,p,m_0}. \end{cases}$$

Define $g_{p,\text{min}} := \min(g_p(m_0) | m_0 \in \pi_p(M))$. If the set

$$\{m_0 \in \pi_p(M) | g_p(m_0) = g_{p,\text{min}}, 0 \notin K_{M,p,m_0}\}$$

is not empty, choose an element of it and denote it by $\tilde{m}_0$. If the set in (9.42) is empty, the set

$$\{m_0 \in \pi_p(M) | g_p(m_0) = g_{p,\text{min}}, 0 \notin K_{M,p,m_0}\}$$

is not empty. Then choose an element of if and denote it by $\tilde{m}_0$. Define an excellent order $\succ_p$ by

$$s(\succ_p) := \max(v_p(m) | m \in M),$$

$$S(\succ_p) := \left\{ K_{M,p,\tilde{m}_0} \quad \text{if } 0 \notin K_{M,p,\tilde{m}_0}, \\ K_{M,p,\tilde{m}_0} - \{0\} \quad \text{if } 0 \in K_{M,p,\tilde{m}_0}. \right.$$ (9.44)

The inclusion $S(\succ_p) \subset \mathbb{Z}_{[0, s(\succ_p)]}$ is obvious from the definition of $s(\succ_p)$.

We will show for each $m_0 \in \pi_p(M)$

$$K_{M,p,m_0} \cap Z_{\succ_p(m_0)} = K_{M,p,\tilde{m}_0} \cap Z_{\succ_p(m_0)}. \quad (9.45)$$

Because of $g_p(m_0) \geq g_{p,\text{min}} = g_p(\tilde{m}_0)$, this is equivalent to $K_{M,p,m_0}$ being subset compatible with $\succ_p$. It is sufficient to show (9.45) for $m_0 \in \pi_p(M) - \{\tilde{m}_0\}$. 


We will need for this the assumption that (i) holds. We take that into account in the following way. Choose any \( k_N \in \mathbb{N}_0 \) and define

\[
\begin{align*}
  l_0 &:= n_0 := \prod_{q \in \mathcal{P}(M) - \{p\}} q^{1 + \max(v_q(m) \mid m \in M)}, \quad (9.46) \\
  n_N &:= n_0 \cdot p^{kN}, \\
  N &:= \{n \in \mathbb{N} \mid n | n_N\}.
\end{align*}
\]

Then \( v_p(n_N) = k_N, \pi_p(n_N) = n_0 \), and

\[
\delta(m_0, n_0, l_0) > 0 \quad \text{for any } m_0 \in \pi_p(M). \quad (9.47)
\]

A priori \( \chi_{p,m_0,n_0}/\delta(m_0, n_0, l_0) \) for any \( m_0 \in \pi_p(M) \) is as in \((7.33)\). But as in the proof of Lemma 9.8 it boils down to the following much simpler form, because of \( K_{N,p,n_0} = \mathbb{Z}_{[0,k_N]} \):

\[
\frac{\chi_{p,m_0,n_0}(l_0 \cdot p^k)}{\delta(m_0, n_0, l_0)} = \begin{cases}
  p^{kN} & \text{if } k > k_N \text{ and } k \in K_{M,p,m_0}, \\
  0 & \text{if } k > k_N \text{ and } k \notin K_{M,p,m_0}, \\
  \sum_{k_1 \in K_{M,p,m_0}, k_1 \leq k_N} \varphi(p^{k_1}) & \text{if } k \leq k_N.
\end{cases} \quad (9.48)
\]

The last sum is in \( \mathbb{Z}_{[0,p^{k_N}]}. \) Thus the quotient in \((9.48)\) has only 2 or 3 values, for fixed \( m_0 \) and varying \( k \).

Now consider any \( m_0 \in \pi_p(M) - \{\widehat{m}_0\}. \) We want to prove \((9.45)\).

**1st case:** Suppose \( k \in K_{M,p,\widehat{m}_0} \cap \mathbb{Z}_{>g_p(m_0)} \) and \( k \notin K_{M,p,m_0}. \) We want to arrive at a contradiction. We choose \( k_N := k - 1. \) We have \( K_{M,p,\widehat{m}_0} \notin \mathbb{Z}_{[0,k-1]} \), because \( g_p(m_0) \leq k - 1. \) \((9.48)\) for \( \widehat{m}_0 \) gives

\[
0 \leq \frac{\chi_{p,\widehat{m}_0,n_0}(l_0 \cdot p^{k-1})}{\delta(m_0, n_0, l_0)} < \frac{\chi_{p,\widehat{m}_0,n_0}(l_0 \cdot p^k)}{\delta(m_0, n_0, l_0)} = p^{-k}. \quad (9.49)
\]

We have \( K_{M,p,m_0} \cap \mathbb{Z}_{[0,k-1]} \neq \emptyset \), because \( g_p(m_0) \leq k - 1. \) \((9.48)\) for \( m_0 \) gives

\[
p^{-k} \geq \frac{\chi_{p,m_0,n_0}(l_0 \cdot p^{k-1})}{\delta(m_0, n_0, l_0)} > \frac{\chi_{p,m_0,n_0}(l_0 \cdot p^k)}{\delta(m_0, n_0, l_0)} = 0. \quad (9.50)
\]

The strict inequalities in \((9.49)\) and \((9.50)\) contradict the sd1Ob-sufficiency of the pair \((M, N)\) for the \( p \)-edge \((l_a, l_b) = (k, k - 1) \) (Definition \((7.3)\) (d)).

**2nd case:** Suppose \( k \in K_{M,p,m_0} \cap \mathbb{Z}_{>g_p(m_0)} \) and \( k \notin K_{M,p,\widehat{m}_0}. \) We exchange the roles of \( \widehat{m}_0 \) and \( m_0 \) in the 1st case, and we arrive in exactly the same way at a contradiction.

We have proved \((9.45)\).

(ii)\(\Rightarrow\)(iii): This is similar to the proof of (i)\(\Rightarrow\)(iii). \(\square\)
10. Chain type singularities

A chain type singularity is a quasihomogeneous singularity of the special shape

\[ f = f(x_1, \ldots, x_n) = x_1^{a_1+1} + \sum_{i=2}^{n} x_{i-1} x_i^{a_i} \tag{10.1} \]

for some \( n \in \mathbb{N} \) and some \( a_1, \ldots, a_n \in \mathbb{N} \). This quasihomogeneous polynomial has an isolated singularity. Define

\[ b_k := (a_1 + 1) \cdot a_2 \cdot \ldots \cdot a_k \quad \text{for} \quad k \in \{1, \ldots, n\}, \quad b_0 := 1. \tag{10.2} \]

The Milnor number \( \mu \), the weights and the characteristic polynomial are calculated for example in [HZ19, Corollary 4.3] (applying formulas in [MO70]). Here we need the following quite surprising result of Orlik and Randell. The function \( \chi : \mathbb{N} \to \{0, 1, \ldots, n+1\} \) with

\[ \chi(m) := \begin{cases} n + 1 & \text{if} \quad m \not| b_n, \\ \min(i \in \{0, 1, \ldots, n\} \mid m|b_i) & \text{if} \quad m|b_n. \end{cases} \tag{10.3} \]

will be useful.

**Theorem 10.1.** [OR77, Theorem (2.11)] For any chain type singularity \( f \) as in (10.1), an automorphism \( h : H_{\text{Mil}} \to H_{\text{Mil}} \) exists such that \((H_{\text{Mil}}, h)\) is an Orlik block, \( h_{\text{Mil}} = h^\mu \), and the set \( M \) of orders of the eigenvalues of \( h \) is as follows,

\[ M = \{ m \in \mathbb{N} \mid \chi(m) \equiv n \mod 2 \} \subset \{ m \in \mathbb{N} \mid m|b_n \}. \tag{10.4} \]

Theorem 1.3 (a) says that the pair \((H_{\text{Mil}}, h_{\text{Mil}})\) for each chain type singularity admits a standard Orlik decomposition. Here we give the proof. It is an easy application of Theorem 10.1 and Theorem 6.2.

**Proof of Theorem 1.3 (a):** We will show that the pair \((M, \mu)\) is sdiOb-sufficient. This and Theorem 6.2 imply that \((H_{\text{Mil}}, h_{\text{Mil}})\) admits a standard Orlik decomposition.

Consider a prime number \( p \) and a \( p \)-edge \((n_a, n_b) \in E_p(\gamma_{\mu}(M))\). Then because of (6.5)

\[ \gamma_{\mu}^{-1}(n_a) = \{ m_a^0 \cdot c \mid c \text{ divides } \prod_{q \in \mathcal{P}, v_q(n_a) = 0} q^{v_q(\mu)} \}, \tag{10.5} \]

where \( m_a^0 := n \cdot \prod_{q \in \mathcal{P}, v_q(n_a) > 0} q^{v_q(\mu)}. \tag{10.6} \)

We want to prove that the \( p \)-edge \((n_a, n_b)\) satisfies (6.3) if \( m_a^0 \notin M \) and that it satisfies (6.4) if \( m_a^0 \in M \).

But before we make an observation which is valid in both cases.
Observation 11.2: If \((m_c, m_d) \in \gamma^{-1}_\mu(n_a) \times \gamma^{-1}_\mu(n_b)\) is a \(p\)-edge and \(\chi(m_c) > \chi(m_c^0)\) then \(\chi(m_d) = \chi(m_c)\).

Proof of Observation 11.2: As \((m_c, m_d)\) is a \(p\)-edge, \(v_q(m_c) = v_q(m_d)\) for any prime number \(q \neq p\) and \(v_p(m_c) > v_p(m_d)\). The number \(\chi(m_c)\) is characterized by \(v_q(b_{\chi(m_c)})\) for any prime number \(q\) and \(v_r(b_{\chi(m_c)}) > v_r(b_{\chi(m_c)-1})\) for some prime number \(r\). Here \(r \neq p\) follows from \(v_r(b_{\chi(m_c)}) > v_r(b_{\chi(m_c^0)})\) (which follows from (10.5)). This shows \(v_q(m_d) \leq v_q(b_{\chi(m_c)})\) for any prime number \(q\) and \(v_r(m_d) = v_r(m_c) > v_r(b_{\chi(m_c)-1})\) for the given prime number \(r\). This implies \(\chi(m_d) = \chi(m_c)\). \(\Box\)

First suppose \(m_c^0 \notin M\). Then any \(m_c \in \gamma^{-1}_\mu(n_a) \cap M\) satisfies \(\chi(m_c) > \chi(m_c^0)\) and \(\chi(m_c) \equiv n \mod 2\). By the Observation 9.2, any \(m_d \in \gamma^{-1}_\mu(n_a)\) with \((m_c, m_d)\) a \(p\)-edge satisfies \(\chi(m_d) = \chi(m_c) \equiv n \mod 2\), thus \(m_d \in M\). This shows (6.3) for the \(p\)-edge \((n_a, n_b)\).

Second suppose \(m_c^0 \in M\). Consider a \(p\)-edge \((m_c, m_d) \in \gamma^{-1}_\mu(n_a) \times (\gamma^{-1}(n_b) \cap M)\). If \(\chi(m_c) > \chi(m_c^0)\) then \(\chi(m_c) = \chi(m_d)\) by the Observation 9.2, and thus \(\chi(m_c) = \chi(m_d) \equiv n \mod 2\), so \(m_c \in M\). If \(\chi(m_c) = \chi(m_c^0)\) then \(\chi(m_c) = \chi(m_c^0) \equiv n \mod 2\), and again \(m_c \in M\). So in both cases \(m_c \in M\). This shows (6.4) for the \(p\)-edge \((n_a, n_b)\). \(\Box\)

Remark 10.3. The proof gives that \((M, \bar{\mu})\) is sciOB-sufficient for any \(\bar{\mu} \in \mathbb{N}\). The proof does not use the following formula (10.7) for the Milnor number:

\[
\mu = \sum_{i=0}^{n} (-1)^i \cdot b_{n-i} = b_n - b_{n-1} + \ldots + (-1)^{n-1} b_1 + (-1)^n.
\]  

(10.7)

11. Cycle type singularities

A cycle type singularity is a quasihomogeneous singularity of the special shape

\[
f = f(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} x_i^{a_i} x_{i+1} + x_n^{a_n} x_1
\]  

(11.1)

for some \(n \in \mathbb{Z}_{\geq 2}\) and some \(a_1, \ldots, a_n \in \mathbb{N}\) which satisfy

for even \(n\) neither \(a_j = 1\) for all even \(j\) nor \(a_j = 1\) for all odd \(j\).

The following well known facts are proved for example in [HZ19, Lemma 4.1]. This polynomial is quasihomogeneous and has an isolated singularity. The Milnor number is \(\mu = \prod_{i=1}^{n} a_i\). The weights have the shape \((w_1, \ldots, w_n) = (\frac{a_1}{d}, \ldots, \frac{a_n}{d})\) with \(d = \mu - (-1)^n\) and \(v_1, \ldots, v_n \in \mathbb{N}\) (for even \(n\) \(v_1, \ldots, v_n > 0\) requires (11.2)). These numbers
satisfy $\gcd(v_1, d) = \ldots = \gcd(v_n, d)$. Define $b := d/\gcd(v_1, d) \in \mathbb{N}$. Then

$$p_{H_{\text{Mil}}, h_{\text{Mil}}} = (t^b - 1)^{\gcd(v_1, d)} \cdot (t - 1)^{(-1)^n}.$$  \hspace{1cm} (11.3)

Therefore Orlik’s conjecture says here the following.

$$(H_{\text{Mil}}, h_{\text{Mil}}) \cong \begin{cases} 
    (\gcd(v_1, d) - 1) H^{[tb - 1]} \\ + H^{[(tb - 1)/(t - 1)]} & \text{if } n \text{ is odd,} \\
    \gcd(v_1, d) H^{[tb - 1]} + H^{[t - 1]} & \text{if } n \text{ is even.}
\end{cases} \hspace{1cm} (11.4)

It is true by Theorem 1.3 (b). We proved Theorem 1.3 (b) in [HM20-1, Theorem 1.3], using algebraic topology and a spectral sequence and building on [Co82]. Cooper’s paper had the same aim. But it contains two serious mistakes. The second one leads for even $n$ to the (wrong) claim in [Co82] that $(H_{\text{Mil}}, h_{\text{Mil}})$ has a decomposition into Orlik blocks, but not a standard decomposition into Orlik blocks. See the introduction and the Remarks 5.1 in [HM20-1] for the relation of [HM20-1] to [Co82].

### 12. Quasihomogeneous singularities and their Thom-Sebastiani sums

Theorem 12.1 says that the exponent map $\chi^f_{\text{Mil}}$ of the characteristic polynomial $p^f_{\text{Mil}}$ of an isolated quasihomogeneous singularity $f$ comes equipped with a canonical compatible tuple of excellent orders. Together with Theorem 9.10 it implies Theorem 1.3 (c), namely that the Thom-Sebastiani sum of two singularities satisfies Orlik’s conjecture if the two singularities satisfy Orlik’s conjecture.

**Theorem 12.1.** Consider an isolated quasihomogeneous singularity $f(x_1, \ldots, x_n)$ with weight system $(w_1, \ldots, w_n) \in (\mathbb{Q} \cap (0, 1))^n$ with $w_j = \frac{s_j}{t_j}$ and $s_j, t_j \in \mathbb{N}$ with $\gcd(s_j, t_j) = 1$. The characteristic polynomial of the monodromy $h^f_{\text{Mil}}$ on its Milnor lattice $H^f_{\text{Mil}}$ is called here $p^f_{\text{Mil}}$. It gives rise to an exponent map $\chi^f_{\text{Mil}} : \mathbb{N} \to \mathbb{N}_0$ with finite support $M_f := \text{supp}(\chi^f_{\text{Mil}})$. By $p^f_{\text{Mil}} = \prod_{m \in \mathbb{N}} \Phi^f_{\text{Mil}}(m)$. The map $\chi^f_{\text{Mil}}$ is compatible with the tuple $(\succ^f_p)_p \in P(M_f)$ of excellent orders which is defined as follows,

$$s(\succ^f_p) := \max(v_p(m) \mid m \in M_f),$$

$$S(\succ^f_p) := \{ k \in \mathbb{Z}_{\{0,s(\succ^f_p)\}} \mid \{ j \in \{1, \ldots, n\} \mid p^k | t_j \} \text{ is odd}\}. \hspace{1cm} (12.1)$$

Before the proof, we make some remarks, show how Theorem 12.1 implies Theorem 1.3 (c), and cite two classical results Theorem 12.4 and Theorem 12.5 which will be needed in the proof of Theorem 12.1.
Remark 12.2. Consider two isolated quasihomogeneous singularities \( f(x_1, \ldots, x_{n_f}) \) and \( g(x_{n_f+1}, \ldots, x_{n_f+n_g}) \). Denote the tuples of excellent orders of \( f, g \) and \( f + g \) by \((\succ_p^f)_{p \in M_f}, (\succ_p^g)_{p \in M_g}\) and \((\succ_p^{f+g})_{p \in M_{f+g}}\). Then \( M_{f+g} \subset M_f \cup M_g \). Extend them to tuples of excellent orders \((\succ_p^f)_{p \in M_f \cup M_g}, (\succ_p^g)_{p \in M_f \cup M_g}\) and \((\succ_p^{f+g})_{p \in M_{f+g}}\) by copies of the trivial excellent order \( \succ_0 \) (Definition 9.1 (b)). Then (12.2) and the definition of the tensor product of two excellent orders (Definition 9.1 (c)) show immediately

\[
\succ_p^{f+g} = \succ_p^f \otimes \succ_p^g \quad \text{for } p \in P(M_f \cup M_g). \tag{12.3}
\]

Proof of Theorem 1.3 (c): Consider the data in Remark 12.2. Identify them as follows with the data in Theorem 9.10:

\[
\chi_{M_{f+g}}^f = \chi_1, \quad \succ_p^f = \succ_{p,M_f}, \quad \chi_{M_{f+g}}^g = \chi_2, \quad \succ_p^g = \succ_{p,N_g}.
\]

The basic result

\[
(H_{M_{f+g}}^{f+g}, h_{M_{f+g}}^{f+g}) \cong (H_{M_f}^{f}, h_{M_f}^{f}) \otimes (H_{M_g}^{g}, h_{M_g}^{g}) \tag{12.4}
\]

of Sebastiani and Thom [ST71] implies \( \succ_p^{f+g} \). And Remark 12.2 implies \( \succ_p^{f+g} = \succ_3 \).

Let \((M_{1}^{(st)}, \ldots, M_{k_1}^{(st)}), (N_{1}^{(st)}, \ldots, N_{k_2}^{(st)})\) and \((L_{1}^{(st)}, \ldots, L_{k_3}^{(st)})\) be the standard coverings of \( \chi_1, \chi_2 \) and \( \chi_3 \). The assumption that \( f \) and \( g \) satisfy Orlik’s conjecture says

\[
(H_{M_{f+g}}^{f}, h_{M_{f+g}}^{f}) \cong \bigoplus_{i=1}^{k_1} \text{Or}(M_i^{(st)}), \quad (H_{M_{f+g}}^{g}, h_{M_{f+g}}^{g}) \cong \bigoplus_{j=1}^{k_2} \text{Or}(N_j^{(st)}). \tag{12.5}
\]

Theorem 9.10 applies because of Theorem 12.1. Together (12.4), (12.5) and (9.37) in Theorem 9.10 (b) give

\[
(H_{M_{f+g}}^{f+g}, h_{M_{f+g}}^{f+g}) \cong \bigoplus_{k=1}^{k_3} \text{Or}(L_k^{(st)}), \tag{12.6}
\]

which is Orlik’s conjecture for \( f + g \).

\[\square\]

Remark 12.3. Consider an isolated quasihomogeneous singularity \( f \) as in Theorem 12.1. Write \( \chi := \chi_{M_{f+g}}^f \). Let \((M_{1}^{(st)}, \ldots, M_{k}^{(st)})\) be the standard covering of \( \chi \). We can show with some extra work which we will carry out in [HM20-2] the following: The compatibility of \( \chi \) with the tuple \((\succ_p^f)_{p \in P(M_f)}\) implies that each set \( M_{k}^{(st)} \) satisfies condition (I) in Theorem 1.2 in [He20], which is also Theorem 6.2 in [HZ19].
Therefore Theorem 12.1 and this implication solve the problems 6 and 7 in [HZ19]. And therefore

$$\text{Aut}_{S^1}(\text{Or}(M_k^{(st)})) = \{ \pm h_{[p_k]}^j \mid j \in \mathbb{Z} \},$$  \hspace{1cm} (12.7)

where $p_k := \prod_{m \in M_k^{(st)}} \Phi_m$, $\text{Or}(M_k^{(st)}) = (H[p_k], h_{[p_k]})$ (see Definition 2.6) and $\text{Aut}_{S^1}(\text{Or}(M_k^{(st)}))$ denotes the automorphisms of the Orlik block $\text{Or}(M_k^{(st)})$, whose eigenvalues are in $S^1$. We will discuss this in [HM20-2].

Milnor and Orlik [MO70] proved a formula for the characteristic polynomial of an isolated quasihomogeneous singularity. Recall the notations in Definition 7.1.

**Theorem 12.4.** [MO70] Consider an isolated quasihomogeneous singularity $f(x_1, ..., x_n)$ with weight system $(w_1, ..., w_n) \in (\mathbb{Q} \cap (0, 1))^n$ with $w_j = \frac{s_j}{t_j}$ and $s_j, t_j \in \mathbb{N}$ with $\gcd(s_j, t_j) = 1$, and with characteristic polynomial $p_{Mil}^f = \prod_{m \in \mathbb{N}} \Phi_m^{\chi_{Mil}(m)}$, where $\chi_{Mil}^f : \mathbb{N} \rightarrow \mathbb{N}_0$ has finite support $M_f := \text{supp} \chi_{Mil}^f$. The divisor $\text{div} p_{Mil}^f = \sum_{m \in M_f} \chi_{Mil}^f(m) \cdot \Psi_m$ is determined by the weights via the following formula,

$$\text{div} p_{Mil}^f = \prod_{j=1}^n \left( \frac{1}{s_j} \Lambda_{t_j} - \Lambda_1 \right).$$  \hspace{1cm} (12.8)

Kouchnirenko [Ko76] gave a characterization of the weight systems which allow quasihomogeneous polynomials with an isolated singularity at 0. Roughly these are the weight systems which allow sufficiently many monomials of weighted degree 1. His result was rediscovered and generalized. See [HK12] and [HM20-2] for references.

**Theorem 12.5.** [Ko76] Remarque 1.13 (ii) Let a weight system $(w_1, ..., w_n) \in (\mathbb{Q} \cap (0, 1))^n$ be given. For $J \subset \{1, ..., n\}$ and $q \in \mathbb{Q}_{\geq 0}$, denote

$$(\mathbb{N}_0^J)_q := \{ (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n \mid \alpha_j = 0 \text{ for } j \notin J, \sum_{j \in J} w_j \alpha_j = q \}. \hspace{1cm} (12.9)$$

The set of quasihomogeneous polynomials of weighted degree 1 with an isolated singularity at 0 is not empty if and only if the weight system satisfies the following condition (C2).

$$(C2) \hspace{1cm} \forall \ J \subset \{1, ..., n\} \text{ with } J \neq \emptyset \hspace{0.5cm} \exists K \subset \{1, ..., n\} \hspace{1cm} \text{ with } |K| = |J| \text{ and } \forall \ k \in K \ (\mathbb{N}_0^J)_1-w_k \neq \emptyset. \hspace{1cm} (12.10)$$
Proof of Theorem \[12.1\]: Consider the data in Theorem \[12.1\]. Write \(\chi := \chi_{\text{Mil}}^{f}\). Fix a prime number \(p \in \mathcal{P}(M_f)\) and two numbers \(m_a, m_b \in M_f\) with \((m_a, m_b) \in E_p(M_f)\). Because of Definition \[9.4\] (c), we have to show

\[\chi(m_a) \geq \chi(m_b) \quad \text{if} \quad v_p(m_a) \succ^f_p v_p(m_b),\] (12.11)

\[\chi(m_a) \leq \chi(m_b) \quad \text{if} \quad v_p(m_b) \succ^f_p v_p(m_a).\]

Because of the definition of the excellent order \(\succ^f_p\), this is equivalent to the claim (which has to be proved):

\[\chi(m_a) \geq \chi(m_b) \quad \text{if} \quad |\{j \in \{1, \ldots, n\} \mid p v_p(m_a) \mid t_j\}| \text{ is odd},\] (12.12)

\[\chi(m_a) \leq \chi(m_b) \quad \text{if} \quad |\{j \in \{1, \ldots, n\} \mid p v_p(m_a) \mid t_j\}| \text{ is even}.\]

Define the map \(\nu : \mathbb{N} \to \mathbb{Z}\) by

\[\nu(k) := \sum_{m : k|m} \chi(m) \cdot \mu_{\text{Moeb}}(m_k),\] (12.13)

where \(\mu_{\text{Moeb}} : \mathbb{N} \to \{0, 1, -1\}\) is the Moebius function with

\[\mu_{\text{Moeb}}(m) := \left\{ \begin{array}{ll}
(-1)^r & \text{if } m = p_1 \cdot \ldots \cdot p_r \text{ with } \\
0 & \text{else.}
\end{array} \right.\] (12.14)

It has finite support and satisfies

\[\text{div } p_{\text{Mil}}^f = \sum_{k \in \mathbb{N}} \nu(k) \cdot \Lambda_k, \quad \chi(m) = \sum_{k : m|k} \nu(k).\] (12.15)

Thus

\[\chi(m_b) - \chi(m_a) = \sum_{k : m_b|k, m_a|k} \nu(k) = \sum_{k : m_b|k, p v_p(m_a) \mid k} \nu(k).\] (12.16)

Formula \[12.8\] allows a good control on the map \(\nu\). Suppose that the weights \((w_1, \ldots, w_n)\) are numbered such that

\[\{j \in \{1, \ldots, n\} \mid p v_p(m_a) \mid t_j\} = \{1, \ldots, \tilde{n}\} \quad \text{for some } \tilde{n} \leq n.\] (12.17)

Formula \[12.8\] and formula \(\[7.11\] (\Lambda_a \cdot \Lambda_b = \gcd(a, b)\Lambda_{\text{lcm}(a, b)})\) tell

\[\sum_{k \in \mathbb{N}} \nu(k) \cdot \Lambda_k = \prod_{j=1}^{n} \left( \frac{1}{s_j} \Lambda_{t_j} - \Lambda_1 \right),\] (12.18)

\[\sum_{k : p v_p(m_a) \mid k} \nu(k) \cdot \Lambda_k = (-1)^{n-\tilde{n}} \cdot \prod_{j=1}^{\tilde{n}} \left( \frac{1}{s_j} \Lambda_{t_j} - \Lambda_1 \right).\] (12.19)

Now we claim that the shorter weight system \((w_1, \ldots, w_{\tilde{n}})\) satisfies Kouchnirenko’s condition (C2) in Theorem \[12.5\]. To prove this, start
with a subset $J \subset \{1, \ldots, \tilde{n}\}$ with $J \neq \emptyset$. The weight system $(w_1, \ldots, w_n)$ satisfies (C2). Therefore a set $K \subset \{1, \ldots, n\}$ with $|K| = |J|$ and $\forall k \in K (N_{0}^J)_{1-w_k} \neq \emptyset$ exists. For any $k \in K$ choose a tuple $(\alpha_1, \ldots, \alpha_n) \in (N_{0}^J)_{1-w_k}$, so $\sum_{j \in J} w_j \alpha_j = 1 - w_k$. As $J \subset \{1, \ldots, \tilde{n}\}$, $p_{\nu_p(m_0)} \nmid t_j$ for any $j \in J$. Therefore $p_{\nu_p(m_0)} \nmid t_k$. This shows $k \in \{1, \ldots, \tilde{n}\}$. Therefore $K \subset \{1, \ldots, \tilde{n}\}$. Thus the weight system $(w_1, \ldots, w_{\tilde{n}})$ satisfies Kouchnirenko’s condition (C2).

Theorem 12.5 gives us the existence of a quasihomogeneous polynomial $\tilde{f}$ with this weight system and an isolated singularity at 0. Write $\tilde{\chi} := \chi_{\tilde{f}}$ for the exponential map of its characteristic polynomial $p_{\tilde{f}}$. Define the map $\tilde{\nu} : \mathbb{N} \to \mathbb{Z}$ by (12.13) with $\chi$ replaced by $\tilde{\chi}$. Then

$$\text{div} p_{\tilde{f}} = \sum_{m \in \mathbb{N}} \tilde{\chi}(m) \cdot \Psi_m = \sum_{k \in \mathbb{N}} \tilde{\nu}(k) \cdot \Lambda_k. \quad (12.20)$$

By Theorem 12.4 this is equal to $\prod_{j=1}^{\tilde{n}} (\frac{1}{\nu_j} \Lambda_j - \Lambda_1)$, which is up to the sign $(-1)^{n-\tilde{n}}$ the right hand side of (12.19). Therefore

$$\text{supp}(\tilde{\nu}) \subset \{ k \in \mathbb{N} | p_{\nu_p(m_0)} \nmid k \}, \quad (12.21)$$

$$\tilde{\nu}(k) = (-1)^{n-\tilde{n}} \cdot \nu(k) \quad \text{for } k \in \text{supp}(\tilde{\nu}). \quad (12.22)$$

This and (12.16) show

$$\chi(m_b) - \chi(m_a) = (-1)^{n-\tilde{n}} \cdot \sum_{k : m_b | k} \tilde{\nu}(k)$$

$$= (-1)^{n-\tilde{n}} \cdot \tilde{\chi}(m_b) \in (-1)^{n-\tilde{n}} \cdot \mathbb{N}_0. \quad (12.23)$$

This implies the claim (12.12) which we had to prove.

13. LOOKING BACKWARD AND FORWARD

**Remarks 13.1.** (i) By Theorem 1.3, Orlik’s Conjecture [1.2] holds for any iterated Thom-Sebastiani sum of chain type singularities and cycle type singularities (and any quasihomogeneous singularity with the same weights as such a sum). Such sums are also called invertible polynomials. The Brieskorn-Pham singularities, $f = f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^{a_i}$ for some $n \in \mathbb{N}$ and some $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 2}$ are special cases, as the $A$-type singularities $x_i^{a_i}$ are special chain type singularities.

(ii) For each weight system in $n = 2$ variables which allows a quasihomogeneous singularity, a singularity of at least one of the following 3 types exists: Brieskorn-Pham singularity (type I), chain type singularity (type II), cycle type singularity (type III). This observation and these types are due to Arnold [AGV85]. Therefore and because of Theorem 13.3 (d), each quasihomogeneous curve singularity satisfies Orlik’s
conjecture. Michel and Weber claimed in the introduction of \cite{MW86} that they have a proof of this. In view of their techniques, this claim can be trusted. But the proof was not written.

(iii) In the case of \( n = 3 \) variables, Arnold distinguishes 7 types I, II, III, IV, V, VI and VII of weight systems which allow quasihomogeneous singularities (some weight systems are simultaneously of several types) \cite{AGV85} (see also e.g. \cite{HK12}). 5 of the 7 types arise from iterated Thom-Sebastiani sums of chain type singularities and cycle type singularities, the 2 types III and VI not. For each quasihomogeneous singularity with a weight system of one of the 5 types, Orlik’s conjecture holds by Theorem 1.3 (d). For each quasihomogeneous singularity with a weight system only of type III or VI, Orlik’s conjecture is open.

(iv) The isolated hypersurface singularities with modality \( \leq 2 \) were classified by Arnold 1972, see \cite{AGV85} for the results. Each of the families with modality \( \leq 2 \) which contains a quasihomogeneous singularity, contains especially an iterated Thom-Sebastiani sum of chain type singularities and cycle type singularities. Therefore for these families Orlik’s conjecture is true.

Remarks 13.2. (i) Orlik’s paper \cite{Or72} contains a second conjecture, which may not be confused with Conjecture 1.2. It is also often called Orlik’s conjecture. It is a consequence of Conjecture 1.2 and it is weaker than Conjecture 1.2. In the case of a quasihomogeneous singularity \( f \in \mathbb{C}[x_1, ..., x_n] \) in \( n \geq 3 \) variables, it predicts the homology of the link \( K := f^{-1} \cap S^{2n-1} \) of the singularity. The Wang sequence (see below (13.7)) tells how this homology looks like if Conjecture 1.2 is true. A first version was fixed by Orlik as Conjecture 3.2 in \cite{Or72}. With some additional arguments and Theorem 12.4, he gave a second more explicit version Conjecture 3.3 in \cite{Or72} which allows to determine the homology of \( K \) solely in terms of the weights of \( f \). Lemma 13.3 below recalls the first version. With Theorem 12.4 and Theorem 4.5 (a), it is not so difficult to derive the second more explicit version.

(ii) The links of some quasihomogeneous singularities give interesting examples of Sasakian structures. This was explored by Boyer, Galicki and others, see e.g. \cite{Bo08}. There the explicit version Conjecture 3.3 in \cite{Or72} is cited as Conjecture 19. In Theorem 27 in \cite{Bo08}, the links of 12 Brieskorn-Pham singularities and 2 chain type singularities in \( n = 5 \) variables are considered. Then the link has real dimension 7.

(iii) In \cite{Bo08} in Proposition 20, Conjecture 19 is claimed to be true for \( n \in \{3, 4\} \), for Brieskorn-Pham polynomials and for chain type singularities.
Our Theorem 1.3 (d) and Lemma 13.3 below give this Conjecture 19 now for all Thom-Sebastiani sums of chain type singularities and cycle type singularities. These contain the Brieskorn-Pham singularities and the chain type singularities, but not all singularities in $n = 3$ or $n = 4$ variables.

For $n = 4$, Boyer cites work of Galicki. For $n = 3$, Boyer cites [OW71]. See (iv) for that paper. But Arnold’s two types III and VI are open in the case $n = 3$. For Brieskorn-Pham singularities, [Ra75] gives the result. For chain type singularities, Boyer cites [OR77]. But that paper gives only our Theorem 10.1. One needs additionally our algebraic Theorem 6.2, see section 10 above.

(iv) The paper [OW71] and also [Or72] miss in the classification of quasihomogeneous singularities with $n = 3$ variables Arnold’s types III and VI. Proposition (3.1.2) and Theorem (3.1.4) in [OW71] list only the other 5 types. Proposition 2.2 in [Or72] even claims that each weight system which allows a quasihomogeneous singularity allows also a Thom-Sebastiani sum of chain type singularities and cycle type singularities. This is wrong for $n \geq 3$. Proposition 3.4 in [Or72] claims that Conjecture 3.3 in [Or72] (= Conjecture 19 in [Bo08]) is true for $n = 3$. It refers to [OW71]. Because there Arnold’s types III and VI are missed, for these types Conjecture 3.3 in [Or72] is open.

Lemma 13.3 shows how Conjecture 3.2 in [Or72] follows from Conjecture 1.2 (=Conjecture 3.1 in [Or72]).

**Lemma 13.3.** [Or72] Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a quasihomogeneous singularity with weight system $(w_1, \ldots, w_n)$ with $w_i = \frac{s_i}{t_i} \in \mathbb{Q} \cap (0, 1)$ and $s_i, t_i \in \mathbb{N}$, $\gcd(s_i, t_i) = 1$. Let $l := \chi_{\text{Mil}}^f(1) \in \mathbb{N}_0$ be the multiplicity of $\Phi_1$ as a factor of the characteristic polynomial $p_{\text{Mil}}^f$ of $\Phi_1$. Then

$$l = \sum_{s=0}^{n} \sum_{\{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}} (-1)^s \frac{w_{i_1} \cdot \ldots \cdot w_{i_s} \cdot \text{lcm}(t_{i_1}, \ldots, t_{i_s})}{}.$$  \hspace{1cm} (13.1)

Let $p_1, \ldots, p_k$ be the elementary divisors of the characteristic polynomial $p_{\text{Mil}}^f$, so they are products of cyclotomic polynomials with multiplicities $1$ and with

$$p_{\text{Mil}}^f = p_1 \cdot \ldots \cdot p_k, \quad p_k|p_{k-1}| \ldots |p_2|p_1.$$ \hspace{1cm} (13.2)

Then $k \geq l$ and

$$p_j(1) \begin{cases} = 0 & \text{for } j \in \{1, \ldots, l\}, \\ \in \mathbb{N} & \text{for } j \in \{l+1, \ldots, k\}, \end{cases} \quad \text{and} \quad p_k(1)|p_{k-1}(1)| \ldots |p_{l+1}(1).$$ \hspace{1cm} (13.3)
Now suppose that \( f \) satisfies Conjecture 1.2. Then the homology of the link \( K := f^{-1}(1) \cap S^{2n-1} \) is given by

\[
H_{n-1}(K, \mathbb{Z}) \cong \mathbb{Z}^l, \quad H_{n-2}(K, \mathbb{Z}) \cong \mathbb{Z}^l \oplus \bigoplus_{j=l+1}^{k} \frac{\mathbb{Z}}{p_j(1)\mathbb{Z}}.
\] (13.4)

**Proof:** Because of (7.11), formula (12.8) in Theorem 12.4 can be rewritten as the following sum with rational coefficients,

\[
\text{div } p_{Mil}^f = \sum_{s=0}^{n} \sum_{\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}} \frac{(-1)^s}{w_{i_1} \cdot \ldots \cdot w_{i_s} \cdot \text{lcm}(t_{i_1}, \ldots, t_{i_s})} \cdot \Lambda_{\text{lcm}(t_{i_1}, \ldots, t_{i_s})}.
\] (13.5)

Each \( \Lambda_m \) contains \( \Phi_1 \) with multiplicity 1. This shows (13.1). The uniqueness of the elementary divisors was discussed in Remark 2.5 (iv). Their definition gives immediately (13.3).

Now suppose that \( f \) satisfies Conjecture 1.2. That means

\[
\left( H_{Mil}, h_{Mil} \right) \cong \bigoplus_{j=1}^{k} \left( H[[p_j]], h[[p_j]] \right) = \bigoplus_{j=1}^{k} \left( \mathbb{Z}[t] \right)_{p_j\text{ mult. by } t}.
\] (13.6)

The Wang sequence connects in the case \( n \geq 3 \) the homology of the link \( K \) with the pair \( (H_{Mil}, h_{Mil}) \) of Milnor lattice and monodromy, see [Or72]. It gives the following short exact sequence:

\[
0 \to H_{n-1}(K, \mathbb{Z}) \to H_{Mil} \xrightarrow{h_{Mil}-\text{id}} H_{Mil} \to H_{n-2}(K, \mathbb{Z}) \to 0.
\] (13.7)

We find

\[
H_{n-1}(K, \mathbb{Z}) \cong \ker \left( h_{Mil} - \text{id} : H_{Mil} \to H_{Mil} \right) \cong \mathbb{Z}^l, \quad (13.8)
\]

\[
H_{n-2}(K, \mathbb{Z}) \cong \frac{H_{Mil}}{(h_{Mil} - \text{id})(H_{Mil})} \cong \mathbb{Z}^l \oplus \bigoplus_{j=1}^{k} \frac{\mathbb{Z}}{p_j(1)\mathbb{Z}}.
\] (13.9)

Here \( \mathbb{Z}/0\mathbb{Z} \cong \mathbb{Z} \), of course. \( \square \)

**Remarks 13.4.** (i) The first author’s interest in Orlik’s Conjecture 1.2 comes from his work on a Torelli conjecture for isolated hypersurface singularities ([He92] and many later papers). Proofs of this conjecture for many classes of singularities consist of two steps, a transcendent step, the calculation of a period map, and an algebraic step, the determination of the action of the automorphism group of the pair
(\(H_{\text{Mil}}, \text{Seifert form}\)) on the image of this period map. The Seifert form is a bilinear unimodal form \(H_{\text{Mil}} \times H_{\text{Mil}} \to \mathbb{Z}\) which determines the monodromy \(h_{\text{Mil}}\). In many cases, it is useful to determine first the automorphism group of the pair \((H_{\text{Mil}}, h_{\text{Mil}})\), and this becomes easier if this pair satisfies Orlik’s conjecture. By Remark 12.3 (which builds on Theorem 12.1 and [He20]), the automorphism group of an Orlik block \((H_{[p_j]}, h_{[p_j]})\), where \(p_j\) is as in Lemma 13.3 is simply \(\{\pm h_{[p_j]}^j | j \in \mathbb{Z}\}\).

(ii) It would be useful to generalize Orlik’s Conjecture 1.2 to a conjecture on the pair \((H_{\text{Mil}}, \text{Seifert form})\) for a quasihomogeneous singularity. But it is not at all clear how this generalization could look like. This is an interesting open question.

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