STRONGLY SINGULAR MASAS
IN TYPE II₁ FACTORS

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Abstract

In this paper we introduce and study strongly singular maximal abelian self–adjoint
subalgebras of type II₁ factors. We show that certain elements of free groups and
of non–elementary hyperbolic groups generate such masas, and these also give new
examples of masas for which Popa’s invariant δ(·) is 1. We also explore the connection
between Popa’s invariant and strong singularity.

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1 Introduction

The study of maximal abelian self-adjoint subalgebras (masas) in a von Neumann algebra $\mathcal{M}$ has a long and rich history. Various types of masas have been identified and investigated, often categorized by their groups of normalizing unitaries. At one end of the spectrum are the regular or Cartan masas; these have sufficiently many normalizing unitaries to generate $\mathcal{M}$. At the other end are the singular masas; the only normalizing unitaries of such a masa $\mathcal{A}$ are the unitaries of $\mathcal{A}$, [4]. In this paper we introduce a new class of singular masas, which we call strongly singular. These are defined by an inequality relating the distance between $\mathcal{A}$ and a unitary conjugate $u\mathcal{A}u^*$ to the distance of $u$ to $\mathcal{A}$, also allowing us to introduce a new invariant $\alpha(\mathcal{A})$ for masas, taking values in $[0, 1]$ (definitions are contained in the second section). Part of our original motivation was the observation that a reverse inequality between these quantities is always valid (Proposition 2.1).

From the point of view of the inner automorphism group, a Cartan masa $\mathcal{A}$ is flexible in that any two projections in $\mathcal{A}$ with equal trace can be switched by an inner automorphism of $\mathcal{M}$ which leaves $\mathcal{A}$ invariant, [15]. For a singular masa, any inner automorphism of $\mathcal{M}$ which leaves $\mathcal{A}$ invariant has trivial action on the masa, but this takes little account of the other unitaries in $\mathcal{M}$. Popa’s invariant $\delta(\mathcal{A})$ of a masa $\mathcal{A}$, [16], is a measure of this rigidity in terms of partial isometries in $\mathcal{M}$ whose initial and final projections are orthogonal in $\mathcal{A}$. Strong singularity is intended to develop a rigidity condition on $\mathcal{A}$ which reflects the perturbations of the masa by the inner automorphisms of $\mathcal{M}$. This condition is compatible with the isometric action of the inner automorphism group of $\mathcal{M}$ on the natural metric space of the masas in $\mathcal{M}$.

We now describe the contents of the paper. The second section contains definitions and some preliminary results, while the third section presents some examples of strongly singular masas. The main results here are that the masas arising from the generators of free groups (Corollary 3.4) and, more generally, from prime elements of Gromov’s non–elementary I.C.C. hyperbolic groups, [10], (Theorem 3.6) are strongly singular. The Infinite Conjugacy Class condition is included to ensure that the resulting von Neumann algebras are factors, [11, p.126]. This condition for a non-elementary hyperbolic group is equivalent to the group being torsion free. The techniques also show that these masas satisfy $\delta(\mathcal{A}) = 1$. The singular
masa in the hyperfinite type $II_1$ factor constructed by Tauer, [22], was the only previous example where $\delta(\cdot)$ could be exactly determined, [16], although Popa had shown that every type $II_1$ factor contains a masa with $\delta(\cdot) \geq 10^{-4}$, [16]. By tightening the argument given by Popa, we can improve this bound to $1/58$, but it seems difficult to obtain any estimate close to $1$. We also give an example of a strongly singular masa in the hyperfinite type $II_1$ factor whose Popa invariant is $1$ (Corollary 3.8).

Our examples of strongly singular masas have conditional expectations which satisfy a multiplicative condition, which we use to define an asymptotic homomorphism in the fourth section. The main result (Theorem 4.7) is that masas whose conditional expectations are asymptotic homomorphisms all satisfy $\delta(\mathcal{A}) = 1$. In the last section of the paper, we relate strong singularity to Popa’s invariant, and we prove that every masa satisfies $\delta(\mathcal{A}) \geq \alpha(\mathcal{A}) / \sqrt{5}$, (Theorem 5.3). In particular, $\delta(\mathcal{A}) \geq 1 / \sqrt{5}$ for strongly singular masas. Intuitively, it would seem reasonable that these two invariants should be equal, or at least mutually dominating. However, we have been unable to obtain a reverse inequality of the form $\delta(\mathcal{A}) \leq c \cdot \alpha(\mathcal{A})$ for some constant $c > 0$.

We conclude by mentioning that a significant part of our work has been motivated by the papers of Sorin Popa on masas, [15, 16, 17, 18], and particularly by the results on orthogonality in [17]. We also thank Pierre de la Harpe for pointing out an error in an earlier version of the paper.
2 Preliminaries

In this section we present some definitions and notation which we will use subsequently. In order to motivate the definition of an \( \alpha \)-strongly singular masa, we will first prove an easy inequality concerning unitary conjugates of subalgebras.

Throughout we denote the operator norm on a type \( II_1 \) factor \( \mathcal{M} \) by \( \| \cdot \| \), while \( \| \cdot \|_2 \) denotes the norm \((\text{tr} (x^*x))^{1/2}\) induced by the unique normalized trace. We write \( L^2(\mathcal{M}, \text{tr}) \) for the Hilbert space completion of \( \mathcal{M} \) in \( \| \cdot \|_2 \). A linear map \( \phi: \mathcal{M} \to \mathcal{M} \) may be viewed as having range in \( L^2(\mathcal{M}, \text{tr}) \). If it is then bounded, we denote its norm by \( \| \phi \|_\infty,2 \). If it is also bounded as a map on \( L^2(\mathcal{M}, \text{tr}) \), we write \( \| \phi \|_2 \) for this norm. We reserve \( \| \phi \| \) for the norm when \( \mathcal{M} \) has the operator norm for both range and domain. For each von Neumann subalgebra \( \mathcal{N} \), there is a unique trace preserving conditional expectation \( E_N: \mathcal{M} \to \mathcal{N} \), and it is contractive for each of the norms \( \| \cdot \|, \| \cdot \|_2 \) and \( \| \cdot \|_\infty,2 \). Moreover, if \( \phi, \psi: \mathcal{M} \to \mathcal{M} \) are linear maps, then the inequalities
\[
\| \phi \psi \|_\infty,2 \leq \| \phi \|_2 \| \psi \|_\infty,2, \quad \| \phi \|_2 \| \psi \|_\infty,2
\] (2.1)
are immediate from the definitions. We note, for future reference, one important property of conditional expectations: \( E_N \) is an \( \mathcal{N} \)-bimodule map, \( \mathbb{23} \).

**Proposition 2.1.** Let \( \mathcal{M} \) be a type \( II_1 \) factor and let \( \mathcal{A} \) be a von Neumann subalgebra. For any unitary \( u \in \mathcal{M} \),
\[
\| E_A - E_{uAu^*} \|_{\infty,2} \leq 4 \| u - E_A(u) \|_2. \tag{2.2}
\]
**Proof.** For any \( x \in \mathcal{M} \),
\[
E_{uAu^*}(x) = uE_A(u^*xu)u^*, \tag{2.3}
\]
and so
\[
\| (E_A - E_{uAu^*})(x) \|_2 = \| E_A(x) - uE_A(u^*xu)u^* \|_2
\]
\[
= \| E_A(x)u - uE_A(u^*xu) \|_2. \tag{2.4}
\]
To estimate \( \| E_A - E_{uAu^*} \|_{\infty,2} \), we may replace \( x, \| x \| \leq 1, \) in \( (2.4) \) by unitaries \( w \), and we may further assume that \( w \) has the form \( uv \) for some unitary \( v \in \mathcal{M} \). Thus it suffices to estimate
\[
\| E_A(uv)u - uE_A(vu) \|_2 \tag{2.5}
\]
as $v$ ranges over the unitary group of $\mathcal{M}$.

Write $a = \mathbb{E}_A(u) \in \mathcal{A}$, $b = (I - \mathbb{E}_A)(u)$. Then $u = a + b$, and $\|a\|_2^2 + \|b\|_2^2 = 1$. Thus (2.5) becomes

$$
\|\mathbb{E}_A(av + bv)u - u\mathbb{E}_A(va + vb)\|_2 \\
\leq \|\mathbb{E}_A(bv)u\|_2 + \|u\mathbb{E}_A(vb)\|_2 + \|\mathbb{E}_A(av)(a + b) - (a + b)\mathbb{E}_A(va)\|_2 \\
\leq 2\|b\|_2 + \|\mathbb{E}_A(av)b\|_2 + \|b\mathbb{E}_A(va)\|_2 + \|a\mathbb{E}_A(v)a - a\mathbb{E}_A(v)\|_2 \\
\leq 4\|b\|_2 \\
= 4\|u - \mathbb{E}_A(u)\|_2. 
$$

(2.6)

The result follows by taking the supremum over all unitaries $v$ in (2.6).

It is natural to ask whether a reverse inequality of the form

$$
\|\mathbb{E}_{uAu^*} - \mathbb{E}_A\|_{\infty, 2} \geq \alpha\|u - \mathbb{E}_A(u)\|_2
$$

(2.7)

can hold for some $\alpha > 0$ and for all unitaries $u \in \mathcal{M}$. Of course some restrictions on this question must be made, because (2.7) forces any normalizing unitary of $\mathcal{A}$ to lie in $\mathcal{A}$. This rules out abelian algebras which are not maximal, regular and semi-regular masas, and any algebra $\mathcal{A}$ for which $\mathcal{A}' \not\subseteq \mathcal{A}$. Thus any masa which satisfies (2.7) is automatically singular. We will show subsequently that many singular masas satisfy such an inequality, and this suggests the following terminology.

**Definition 2.2.** A masa $\mathcal{A}$ in a type $II_1$ factor $\mathcal{M}$ is said to be $\alpha$-strongly singular if (2.7) holds. When $\alpha = 1$, we say that $\mathcal{A}$ is strongly singular. We let $\alpha(\mathcal{A})$ denote the supremum of all numbers $\alpha$ for which (2.7) is valid. We note that strong singularity and $\alpha(\cdot)$ can be defined in this way for any von Neumann subalgebra.

It is clear, from Proposition 2.1, that $\alpha(\mathcal{A})$ takes its value in $[0, 4]$. Since we will construct examples where $\alpha(\mathcal{A}) = 1$, the following result gives the optimal upper estimate on $\alpha(\cdot)$.

**Proposition 2.3.** If $\mathcal{A}$ is a masa in a type $II_1$ factor $\mathcal{M}$, then $\alpha(\mathcal{A}) \leq 1$.

**Proof.** Let $u \in \mathcal{M}$ be any unitary, and regard $\mathbb{E}_A$ and $\mathbb{E}_{uAu^*}$ as projections in $B(L^2(\mathcal{M}), \text{tr})$. Both are positive operators, so the inequality

$$
\|\mathbb{E}_{uAu^*} - \mathbb{E}_A\|_2 \leq \max\{\|\mathbb{E}_{uAu^*}\|_2, \|\mathbb{E}_A\|_2\} = 1
$$

(2.8)
follows by applying states to this difference. Then \( \|E_{uA^*} - E_A\|_{\infty,2} \leq 1 \), since \( \|\cdot\|_2 \geq \|\cdot\|_{\infty,2} \) by taking \( \psi = I \) in (2.4).

Now choose a projection \( p \in A \), \( \text{tr}(p) = 1/2 \), and choose a partial isometry \( v \in M \) such that

\[
vv^* = p, \quad v^*v = p^\perp. \tag{2.9}
\]

The element \( u = v + v^* \) is a unitary in \( M \) satisfying \( upu = p^\perp \) or, equivalently, \( pu = up^\perp \). Then \( pE_A(u) = E_A(u)p^\perp \), which forces \( E_A(u) = 0 \), since these operators commute. Thus

\[
\|u - E_A(u)\|_2 = \|u\|_2 = 1. \tag{2.10}
\]

This choice of unitary shows that the inequality in (2.7) fails for each \( \alpha > 1 \), and it follows that \( \alpha(A) \leq 1 \).

Let \( A \) be a masa in a type \( II_1 \) factor \( M \), and let \( v \) be a non-zero partial isometry in \( M \) such that \( p = vv^* \) and \( q = v^*v \) are orthogonal projections in \( A \). Define \( \delta(vAv^*, A) \) by

\[
\delta(vAv^*, A) = \sup\{\|x - E_A(x)\|_2 : x \in vAv^*, \|x\| \leq 1\}. \tag{2.11}
\]

Then \( \delta(A) \) is the largest number \( \lambda \) for which the inequality

\[
\delta(vAv^*, A) \geq \lambda\|v^*v\|_2 \tag{2.12}
\]

holds for all such partial isometries (see [16]). Since any element \( x \in vAv^* \) satisfies \( x = pxp \), it is clear that

\[
\delta(vAv^*, A)^2 \leq \sup\{\|pxp\|_2^2 : x \in M, \|x\| \leq 1\} \leq \text{tr}(p) = \text{tr}(q) = \|q\|_2^2 = \|v^*v\|_2^2. \tag{2.13}
\]

It follows from (2.13) that \( \delta(A) \leq 1 \). This was stated in [16], but we have included a proof for the reader’s convenience.
3 Strong singularity in discrete group factors

In this section we present some examples of strongly singular masas and subfactors arising from discrete groups. When \( \Gamma \) is a discrete I.C.C. group (each element other than the identity has an infinite conjugacy class) the resulting von Neumann algebra \( VN(\Gamma) \), represented on \( \ell^2(\Gamma) \), is a type \( II_1 \) factor. Each element of the group is a unitary in \( VN(\Gamma) \) and thus generates an abelian von Neumann subalgebra. Since the principal examples of type \( II_1 \) factors arise from discrete groups, our examples of strongly singular masas will be generated by elements of groups. The first two lemmas give key technical results which will be needed for our main theorems. The common hypotheses for the first three results are taken from [17]. Note that the I.C.C. hypothesis is inessential for the proofs, and is only included to ensure that the associated von Neumann algebras are factors.

**Lemma 3.1.** Let \( G \) be an infinite subgroup of a countable discrete I.C.C. group \( \Gamma \) with the property that \( xGx^{-1} \cap G = \{e\} \) for all \( x \in \Gamma \setminus G \), let \( \mathcal{M} = VN(\Gamma) \), and let \( \mathcal{N} = VN(G) \). For each set of elements \( u_i \in \mathbb{C} \Gamma \), \( 1 \leq i \leq n \), the equations

\[
E\mathcal{N}(u_sgut) = E\mathcal{N}(u_s)gE\mathcal{N}(u_t), \quad 1 \leq s, t \leq n, \tag{3.1}
\]

are satisfied by all but a finite number of \( g \in G \).

**Proof.** Each \( u_i \) is a finite linear combination of group elements, so it suffices to prove, for fixed \( h, k \in \Gamma \), that the equation

\[
E\mathcal{N}(hgk) = E\mathcal{N}(h)gE\mathcal{N}(k) \tag{3.2}
\]

is satisfied by all but a finite number of \( g \in G \). The modular properties of \( E\mathcal{N} \) show that (3.2) always holds when either \( h \) or \( k \) is in \( G \), so we may assume that both elements are not. In this case the right hand side of (3.2) is 0, and so we only need establish that \( hgk \in G \) for only finitely many \( g \in G \). If \( g_1 \) and \( g_2 \) are two such elements, then \( hg_1g_2^{-1}h^{-1} \in G \). The hypotheses then imply that \( g_1 = g_2 \), showing that (3.2) fails for at most one \( g \in G \). \( \square \)

**Lemma 3.2.** Let \( G \) be an infinite subgroup of a countable discrete I.C.C. group \( \Gamma \) with the property that \( xGx^{-1} \cap G = \{e\} \) for all \( x \in \Gamma \setminus G \), let \( \mathcal{M} = VN(\Gamma) \), and let \( \mathcal{N} = VN(G) \). If \( u, v \in \mathcal{M} \), then

\[
\|(I - E\mathcal{N})(uE\mathcal{N}(\cdot)v)\|_{2,2}^2 \geq \text{tr}[E\mathcal{N}(E\mathcal{N} \cap \mathcal{M}(u^*u))(E\mathcal{N}(vv^*) - E\mathcal{N}(v)E\mathcal{N}(v)^*)]
\]

\[
- \|(I - E\mathcal{N} \cap \mathcal{M})(u^*u)\|_2(\|vv^*\|_2 + \|E\mathcal{N}(v)E\mathcal{N}(v)^*\|_2). \tag{3.3}
\]


If \( u \) is a unitary, then

\[
\| (I - \mathbb{E}_\mathcal{N})(u \mathbb{E}_\mathcal{N}(\cdot)v) \|_{2,\infty}^2 \geq \text{tr}[\mathbb{E}_\mathcal{N}(vv^*) - \mathbb{E}_\mathcal{N}(v)\mathbb{E}_\mathcal{N}(v)^*].
\] (3.4)

**Proof.** If we can prove \((3.3)\) for \( u, v \in \mathbb{C}\mathcal{G} \), then the \( \| \cdot \|_2 \)-norm continuity of conditional expectations and the Kaplansky density theorem will show that it holds generally. Thus we assume that \( u, v \in \mathbb{C}\mathcal{G} \). Then, by Lemma \[3.1\], we may choose \( g \in \mathcal{G} \) so that

\[
\mathbb{E}_\mathcal{N}(ugv) = \mathbb{E}_\mathcal{N}(u)g\mathbb{E}_\mathcal{N}(v), \quad \mathbb{E}_\mathcal{N}(u^*ugvv^*) = \mathbb{E}_\mathcal{N}(u^*u)g\mathbb{E}_\mathcal{N}(vv^*). \tag{3.5}
\]

For this choice of \( g \),

\[
\| (I - \mathbb{E}_\mathcal{N})(u \mathbb{E}_\mathcal{N}(\cdot)v) \|_{2,\infty}^2 \geq \| (I - \mathbb{E}_\mathcal{N})(ugv) \|_2^2
\]

\[
= \| ugv \|_2^2 - \| \mathbb{E}_\mathcal{N}(ugv) \|_2^2
\]

\[
= \text{tr}(u^*ugvv^*g^{-1}) - \| \mathbb{E}_\mathcal{N}(ugv) \|_2^2
\]

\[
= \text{tr}(\mathbb{E}_\mathcal{N}(u^*ugvv^*)g^{-1}) - \| \mathbb{E}_\mathcal{N}(ugv) \|_2^2
\]

\[
= \text{tr}(\mathbb{E}_\mathcal{N}(u^*u)g\mathbb{E}_\mathcal{N}(vv^*)g^{-1}) - \| \mathbb{E}_\mathcal{N}(ugv) \|_2^2, \tag{3.6}
\]

where the last equality follows from \((3.3)\), since \( g^{-1} \in \mathcal{N} \). Now write

\[
a = \mathbb{E}_{\mathcal{N} \cap \mathcal{M}}(u^*u), \quad b = (I - \mathbb{E}_{\mathcal{N} \cap \mathcal{M}})(u^*u). \tag{3.7}
\]

Since \( ga = ag \), we may apply \( \mathbb{E}_\mathcal{N} \) to conclude that \( g \) and \( \mathbb{E}_\mathcal{N}(a) \) commute. Thus

\[
\text{tr}(\mathbb{E}_\mathcal{N}(u^*u)g\mathbb{E}_\mathcal{N}(vv^*)g^{-1}) = \text{tr}(\mathbb{E}_\mathcal{N}(a + b)g\mathbb{E}_\mathcal{N}(vv^*)g^{-1})
\]

\[
= \text{tr}(g\mathbb{E}_\mathcal{N}(a)\mathbb{E}_\mathcal{N}(vv^*)g^{-1}) + \text{tr}(\mathbb{E}_\mathcal{N}(b)g\mathbb{E}_\mathcal{N}(vv^*)g^{-1})
\]

\[
= \text{tr}(\mathbb{E}_\mathcal{N}(a)\mathbb{E}_\mathcal{N}(vv^*)) + \text{tr}(\mathbb{E}_\mathcal{N}(b)g\mathbb{E}_\mathcal{N}(vv^*)g^{-1})
\]

\[
\geq \text{tr}(\mathbb{E}_\mathcal{N}(a)\mathbb{E}_\mathcal{N}(vv^*)) - \| b \|_2 \| vv^* \|_2. \tag{3.8}
\]

We now estimate the last term in \((3.6)\). By \((3.3)\),

\[
\| \mathbb{E}_\mathcal{N}(ugv) \|_2^2 = \| \mathbb{E}_\mathcal{N}(u)g\mathbb{E}_\mathcal{N}(v) \|_2^2
\]

\[
= \text{tr}(\mathbb{E}_\mathcal{N}(v)^*g^{-1}\mathbb{E}_\mathcal{N}(u)^*\mathbb{E}_\mathcal{N}(u)g\mathbb{E}_\mathcal{N}(v))
\]

\[
\leq \text{tr}(\mathbb{E}_\mathcal{N}(v)^*g^{-1}\mathbb{E}_\mathcal{N}(u^*u)g\mathbb{E}_\mathcal{N}(v))
\]

\[
= \text{tr}(\mathbb{E}_\mathcal{N}(a)\mathbb{E}_\mathcal{N}(v)\mathbb{E}_\mathcal{N}(v)^*) + \text{tr}(g^{-1}\mathbb{E}_\mathcal{N}(b)g\mathbb{E}_\mathcal{N}(v)\mathbb{E}_\mathcal{N}(v)^*)
\]

\[
\leq \text{tr}(\mathbb{E}_\mathcal{N}(a)\mathbb{E}_\mathcal{N}(v)\mathbb{E}_\mathcal{N}(v)^*) + \| \mathbb{E}_\mathcal{N}(v)\mathbb{E}_\mathcal{N}(v)^* \|_2 \| b \|_2. \tag{3.9}
\]
Using (3.8) and (3.9), (3.6) becomes

\[
\|(I - E_N)(uE_N(\cdot)v)\|_\infty^2 \geq \text{tr}[E_N(a)(E_N(vv^*) - E_N(v)E_N(v)^*)] \\
- \|b\|_2(\|vv^*\|_2 + \|E_N(v)E_N(v)^*\|_2).
\]  (3.10)

Replacing \(a\) and \(b\) from (3.7) gives (3.3).

If \(u\) is a unitary, (3.4) follows immediately from (3.3) by replacing \(u^*u\) with 1.

**Theorem 3.3.** Let \(G\) be an infinite subgroup of a countable discrete I.C.C. group \(\Gamma\) with the property that \(xGx^{-1} \cap G = \{e\}\) for all \(x \in \Gamma \setminus G\), let \(\mathcal{M} = VN(\Gamma)\), and let \(\mathcal{N} = VN(G)\).

(i) If \(u\) is a unitary in \(\mathcal{M}\), then

\[
\|u - E_N(u)\|_2 \leq \|E_{uNu^*} - E_N\|_\infty,2;
\]  (3.11)

(ii) If \(G\) is abelian, then \(\mathcal{N}\) is a strongly singular masa satisfying

\[
\alpha(\mathcal{N}) = \delta(\mathcal{N}) = 1.
\]  (3.12)

**Proof.** (i) For \(x \in \mathcal{M}\), \(\|x\| \leq 1\), we have

\[
\|E_{uNu^*} - E_N\|_\infty,2^2 \geq \|E_{uNu^*}(uxu^*) - E_N(uxu^*)\|_2^2 \\
= \|uE_N(x)u^* - E_N(uxu^*)\|_2^2 \\
\geq \|(I - E_N)[uE_N(x)u^* - E_N(uxu^*)]\|_2^2 \\
= \|(I - E_N)(uE_N(x)u^*)\|_2^2.
\]  (3.13)

Taking the supremum in (3.13) over \(x\) implies that

\[
\|E_{uNu^*} - E_N\|_\infty,2^2 \geq \|(I - E_N)(uE_N(\cdot)u^*)\|_\infty,2^2.
\]  (3.14)

Applying (3.4) with \(v = u^*\) gives

\[
\|E_{uNu^*} - E_N\|_\infty,2^2 \geq 1 - \|E_N(u)\|_2^2 = \|u - E_N(u)\|_2^2,
\]  (3.15)

which proves (3.11).

(ii) Assume now that \(G\) is abelian. The estimate in (3.11) shows that \(\mathcal{N}\) is a strongly singular masa in \(\mathcal{M}\) (although, \textit{a priori}, it was not clear that \(\mathcal{N}\) was maximal). Thus, \(\alpha(\mathcal{N}) = 1.\)
We now estimate \( \delta(A) \). Let \( p \) and \( q \) be orthogonal projections in \( A \) which are equivalent in \( VN(\Gamma) \). We may choose a nilpotent partial isometry \( v \in VN(\Gamma) \) such that \( p = vv^* \) and \( q = v^*v \). Then
\[
E_N(v) = E_N(pvq) = pqE_N(v) = 0. \tag{3.16}
\]
By (3.3) and (3.16),
\[
\| (I - E_N)vE_N(\cdot) v^* \|^2 \geq \text{tr}(q) = \text{tr}(p) = \|vv^*\|^2, \tag{3.17}
\]
since \( v^*v \in N' \cap M = N \). It follows that \( \delta(N) \geq 1 \), and since \( \delta(N) \leq 1 \) is always true, equality is immediate.

We are now able to give some examples of strongly singular masas, which also have the property that \( \delta(A) = 1 \).

**Corollary 3.4.** Let \( F_n, 2 \leq n \leq \infty \), denote the free group on \( n \) generators, let \( a \) be one of these generators and let \( A \) be the masa generated by \( a \). Then \( A \) is strongly singular and \( \delta(A) = 1 \).

**Proof.** Any generator satisfies the hypotheses of Theorem 3.3.

An element \( a \) in a discrete group \( \Gamma \) is prime, \([10, 12]\), if the equation \( a = b^n \) has only two solutions in \( \Gamma \): \( b = a \) and \( n = 1 \), or \( b = a^{-1} \) and \( n = -1 \). This says that \( a \) is not a proper power of some other group element. The following lemma is surely well known, but we do not know a reference.

**Lemma 3.5.** Let \( a \) be a prime element of a group \( \Gamma \). Then \( xGp(a)x^{-1} \cap Gp(a) = \{ e \} \) for all \( x \in \Gamma \setminus Gp(a) \) if and only if the normalizer \( N(Gp(a^p)) \) of \( Gp(a^p) \) is \( Gp(a) \) for all \( p \in \mathbb{N} \).

**Proof.** One direction is clear. Conversely, suppose that the hypotheses on the normalizers are fulfilled, but suppose that there is an \( x \in \Gamma \setminus Gp(a) \) such that, for some \( p \in \mathbb{N} \) and \( k \in \mathbb{Z} \setminus \{ 0 \} \), \( xa^px^{-1} = a^k \). Then
\[
(x^{-1}ax)a^p(x^{-1}a^{-1}x) = x^{-1}a^{k^2}a^{-1}x = a^{pk}, \tag{3.18}
\]
and so \( x^{-1}ax \in N(Gp(a^p)) \). Hence \( x^{-1}ax = a^r \) for some \( r \in \mathbb{Z} \), since one of \( pk, -pk \) is in \( \mathbb{N} \). Since \( a \) is prime, so too is \( x^{-1}ax \), forcing \( r = \pm 1 \). Thus \( x \) normalizes \( Gp(a) \), a contradiction which proves the result.
Our next examples of strongly singular masas include those of Corollary 3.4, and are based on a group theoretic result of Gromov, [10].

**Theorem 3.6.** Let $A$ be the abelian von Neumann algebra generated by a prime element $a$ in a non-elementary I.C.C. hyperbolic group $\Gamma$. Then $A$ is a strongly singular masa in $VN(\Gamma)$, and $\delta(A) = 1$.

*Proof.* By [10], (see also Theorem 8.30 of [9]), a prime element $a$ in a non–elementary hyperbolic group $\Gamma$ satisfies

$$N(Gp(a^p)) = Gp(a)$$

(3.19)

for all $p \in \mathbb{N}$. Lemma 3.5 then shows that the hypotheses of Theorem 3.3 are satisfied, and the result follows.

**Corollary 3.7.** Let $n < m$, let $F_m$ be the free group with generators $\{g_i\}_{i=1}^m$, and regard $F_n$ as a subgroup generated by $\{g_i\}_{i=1}^n$. If $M = VN(F_m)$ and $N = VN(F_n)$, then

$$\|u - E_N(u)\|_2 \leq \|E_N(u^*) - E_N\|_{\infty,2}$$

(3.20)

for all unitaries $u \in M$.

*Proof.* The subgroup $F_n$ of $F_m$ satisfies the hypotheses of Theorem 3.3.

The following corollary provides an example of a strongly singular masa $A$ in the hyperfinite type $II_1$ factor $R$, and it also has Popa invariant 1.

**Corollary 3.8.** In the hyperfinite type $II_1$ factor $R$, there exists a masa $A$ satisfying

$$\alpha(A) = \delta(A) = 1.$$  

(3.21)

*Proof.* Dixmier, [4, Theorem 1], and Popa, [17, Theorem 5.1], have both given examples of countable amenable discrete I.C.C. groups containing abelian subgroups which satisfy the hypotheses of Theorem 3.3, and the result is then immediate.

For the reader’s convenience, we briefly describe Dixmier’s example. Let $K$ be an infinite field that is the countable union of finite subfields (the algebraic closure of a finite field has this property). Let $\Gamma$ be the group of affine transformations of the linear space of dimension 1 over $K$, and let $G$ be the abelian subgroup of homotheties about 0. The calculations of [4, p.282] show that the hypotheses of Theorem 3.3 are satisfied.
Remark 3.9. Let $\mathcal{A}$ be a masa in a type $II_1$ factor $\mathcal{M}$, and let $\omega$ be a free ultrafilter on $\mathbb{N}$. Then $\mathcal{A}^\omega$ is a masa in $\mathcal{M}^\omega$ which is strongly singular when $\mathcal{A}$ also has this property. The proof is similar to Popa’s proof that $\delta(\mathcal{A}^\omega) = \delta(\mathcal{A})$ for a masa $\mathcal{A}$ in $\mathcal{M}$ ([16, Section 5.2]). There is also a version corresponding to a sequence of strongly singular masas, again following [10, Section 5.2]. □
4 Asymptotic homomorphism conditional expectations

In this section we introduce the notion of an asymptotic homomorphism for conditional expectations, and we show the certain abelian algebras arising from group elements have this property. We then discuss some applications.

**Definition 4.1.** Let $\mathcal{A}$ be an abelian von Neumann subalgebra of a type $II_1$ factor $\mathcal{M}$. The conditional expectation $E_A$ is an asymptotic homomorphism if there is a unitary $u \in A$ such that

$$\lim_{|k| \to \infty} \|E_A(xu^ky) - E_A(x)E_A(y)u^k\|_2 = 0$$

(4.1)

for all $x, y \in \mathcal{M}$.

Observe that there is a closely related weak limit that converges for all masas $\mathcal{A}$ in $\mathcal{M}$. Let $u$ be a unitary generating $\mathcal{A}$ and let LIM be a Banach limit on $\mathbb{Z}$. Then, for $x, z \in \mathcal{M}$, we claim that

$$\text{LIM} \left\langle n^{-1} \sum_{j=1}^n u^{-j}xu^j, z \right\rangle = \langle E_A(x), z \rangle.$$  

(4.2)

The left hand side of (4.2) defines a bounded map $\phi: \mathcal{M} \to \mathcal{M}$ by

$$\text{LIM} \left\langle n^{-1} \sum_{j=1}^n u^{-j}xu^j, z \right\rangle = \langle \phi(x), z \rangle,$$  

(4.3)

and the invariance of LIM shows that

$$\langle u\phi(x), z \rangle = \text{LIM} \left\langle n^{-1} \sum_{j=1}^n u^{-(j-1)}xu^{j-1}u, z \right\rangle = \langle \phi(x)u, z \rangle.$$  

(4.4)

Thus $\phi(x) \in A' \cap M = A$ for all $x \in M$, and since $\phi$ is trace preserving, it is clear that $\phi = E_A$. Also, for $x, y, z \in M$,

$$\text{LIM} n^{-1} \sum_{j=1}^n \langle u^{-j}E_A(xu^jy) - E_A(x)E_A(y), z \rangle$$

$$= \langle E_A(E_A(x)y) - E_A(x)E_A(y), z \rangle$$

$$= 0,$$  

(4.5)

where we have used the fact that $E_A$ is both normal and an $A$-bimodule map.
Theorem 4.2. Let $\Gamma$ be a discrete group, let $\mathcal{M} = VN(\Gamma)$, and let $\mathcal{A}$ be the abelian von Neumann algebra generated by a fixed element $g \in \Gamma$. If $g$ has the property that

$$\{k \in \mathbb{Z}: \ xg^ky \in Gp(g)\}$$

is finite for each pair $x, y \in \Gamma \setminus Gp(g)$, then $E_A$ is an asymptotic homomorphism.

Proof. For each $k \in \mathbb{Z}$ define a bounded bilinear map $\phi_k: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by

$$\phi_k(x, y) = E_A(xg^ky) - E_A(x)E_A(y)g^k$$

(4.6)

for $x, y \in \mathcal{M}$. We consider first the case where $x$ and $y$ are group elements in $\Gamma$. If either one is in $Gp(g)$ then the module properties of $E_A$ imply that $\phi_k(x, y) = 0$ for all $k \in \mathbb{Z}$.

Now suppose that $x, y \in \Gamma \setminus Gp(g)$. By hypothesis, there exists $K$ such that $xg^ky \notin Gp(g)$ for $|k| \geq K$, so both terms on the right hand side of (4.6) are 0, showing that $\phi_k(x, y) = 0$ for all $|k| \geq K$. It then follows that, for $x, y \in C\Gamma$, $\phi_k(x, y) = 0$ for $|k|$ sufficiently large.

The estimate

$$\|\phi_k(x, y)\|_2 \leq 2\|x\|_2\|y\|$$

(4.7)

for $x, y \in \mathcal{M}$, $k \in \mathbb{Z}$ is immediate from (4.6), so if $x \in \mathcal{M}$, $\{x_n\}_{n=1}^\infty \in C\Gamma$, $y \in C\Gamma$, and $\lim_{n \to \infty} \|x - x_n\|_2 = 0$, we obtain

$$\|\phi_k(x, y)\|_2 \leq \|\phi_k(x - x_n, y)\|_2 + \|\phi_k(x_n, y)\|_2$$

$$\leq 2\|x - x_n\|_2\|y\| + \|\phi_k(x_n, y)\|_2. \quad (4.8)$$

Thus, for each $n \geq 1$,

$$\lim_{|k| \to \infty} \|\phi_k(x, y)\|_2 \leq 2\|x - x_n\|_2\|y\|, \quad (4.9)$$

since $\phi_k(x_n, y) = 0$ for $k$ sufficiently large. Let $n \to \infty$ in (4.9) to see that $\lim_{|k| \to \infty} \|\phi_k(x, y)\|_2 = 0$ for $x \in \mathcal{M}$ and $y \in C\Gamma$. Equation (4.6) also gives the estimate

$$\|\phi_k(x, y)\|_2 \leq 2\|x\|_2\|y\|_2. \quad (4.10)$$

We then repeat the previous argument, this time in the second variable, to obtain

$$\lim_{|k| \to \infty}\|\phi_k(x, y)\|_2 = 0$$

for all $x, y \in \mathcal{M}$. This completes the proof. \qed
Corollary 4.3. Let $g$ be a prime element in a non-elementary I.C.C. hyperbolic group $\Gamma$, and let $A$ be the masa generated by $g$ in $VN(\Gamma)$. Then $E_A$ is an asymptotic homomorphism.

Proof. From the third section, $g$ satisfies the hypotheses of Theorem 4.2, and the result follows. □

Remark 4.4. We remind the reader that Corollary 4.3 applies, in particular, to the generators of free groups. □

We now consider some consequences of asymptotic homomorphisms. We will need the following inequality, which is close to Lemma 3.2 under different hypotheses.

Proposition 4.5. Let $A$ be an abelian von Neumann subalgebra of a type $II_1$ factor $M$. If $E_A$ is an asymptotic homomorphism, then

$$\|(I - E_A)(x E_A(\cdot)y)\|_{\infty,2}^2 \geq \text{tr}(E_A(x^* x E_A(yy^*)) - E_A(x)E_A(x)^*E_A(y)E_A(y)^*)$$

(4.11)

for all $x, y \in M$.

Proof. In proving (4.11), it clearly suffices to assume that $\|x\|, \|y\| \leq 1$. Now fix $\varepsilon > 0$. By the asymptotic homomorphism hypothesis we may choose a unitary $u \in A$ such that

$$\|E_A(xuy) - E_A(x)E_A(uy)u\|_2 < \varepsilon$$

(4.12)

and

$$\|E_A(x^* xuy y^*) - E_A(x^* x E_A(yy^*))u\|_2 < \varepsilon.$$

(4.13)

Then, using (4.12) and (4.13),

$$\|(I - E_A)(x E_A(\cdot)y)\|_{\infty,2}^2 \geq \|(I - E_A)(xuy)\|_2^2$$

$$= \|xuy\|_2^2 - \|E_A(xuy)\|_2^2$$

$$= \text{tr}(x^* xuy y^* u^*) - \|E_A(xuy)\|_2^2$$

$$= \text{tr}(E_A(x^* xuy y^*)u^*) - \|E_A(xuy)\|_2^2$$

$$\geq \text{tr}(E_A(x^* x)E_A(yy^*)) - \varepsilon - \|E_A(x)E_A(y)\|_2^2 - 2\varepsilon.$$ 

(4.14)

Since $E_A(x) = E_A(y)$ commute,

$$\|E_A(x)E_A(y)\|_2^2 = \text{tr}(E_A(x)E_A(x)^*E_A(y)E_A(y)^*),$$

(4.15)

and so (4.11) follows by substituting (4.13) into (4.14) and letting $\varepsilon \to 0$. □
Remark 4.6. With only minor modifications to the proof, (4.11) could be strengthened (under the same hypotheses) to

\[
\left\| (I - \mathbb{E}_A) \left( \sum_{j=1}^{n} x_j \mathbb{E}_A(\cdot) y_j \right) \right\|_{\infty,2}^2 \geq \\
\text{tr} \left( \sum_{i,j=1}^{n} \left[ \mathbb{E}_A(x_i^* x_j^*) \mathbb{E}_A(y_j y_i^*) - \mathbb{E}_A(x_i^*) \mathbb{E}_A(x_j) \mathbb{E}_A(y_j) \mathbb{E}_A(y_i^*) \right] \right).
\]

(4.16)

Note that, in the particular case of a singly generated subgroup, the following theorem gives the conclusions of Theorem 3.3 and the asymptotic homomorphism condition appeared implicitly in the proof of Lemma 3.2.

**Theorem 4.7.** Let \( A \) be an abelian von Neumann subalgebra of a type II\(_1\) factor \( M \). If \( \mathbb{E}_A \) is an asymptotic homomorphism, then \( A \) is a strongly singular masa with Popa invariant \( \delta(A) = 1 \).

**Proof.** Let \( u \in M \) be an arbitrary unitary. To show strong singularity, we will apply Proposition 4.5 with \( x = u \) and \( y = u^* \). Then

\[
\| \mathbb{E}_{uAu^*} - \mathbb{E}_A \|_{\infty,2}^2 = \| u \mathbb{E}_A(\cdot)^* u - \mathbb{E}_A(u \cdot u^*) \|_{\infty,2}^2 \\
\geq \| u \mathbb{E}_A(\cdot)^* u - \mathbb{E}_A(u \mathbb{E}_A(\cdot)^*) \|_{\infty,2}^2 \\
= \| (I - \mathbb{E}_A)(u \mathbb{E}_A(\cdot)^*) \|_{\infty,2}^2 \\
\geq 1 - \| \mathbb{E}_A(u) \mathbb{E}_A(u^*) \|_2^2 \\
\geq 1 - \| \mathbb{E}_A(u) \|_2^2 \\
= \| (I - \mathbb{E}_A)(u) \|_2^2.
\]

(4.17)

Thus \( A \) is strongly singular. We now estimate \( \delta(A) \).

Let \( v \) be a nilpotent partial isometry such that \( p = vv^* \) and \( q = v^* v \) are orthogonal projections in \( A \). From (3.16), \( \mathbb{E}_A(v) = 0 \), so the choices of \( x = v \) and \( y = v^* \) in (4.11) lead to

\[
\| (I - \mathbb{E}_A)(v \mathbb{E}_A(\cdot)^* v^*) \|_2^2 \geq \text{tr}(\mathbb{E}_A(v^* v) \mathbb{E}_A(v^* v)) \\
= \text{tr}(q) \\
= \text{tr}(p) \\
= \| vv^* \|_2^2.
\]

(4.18)

This proves that \( \delta(A) \geq 1 \), and the reverse inequality always holds. \( \square \)

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Remark 4.8. By inserting different powers of $u$ between elements of $\mathcal{M}$ in Definition 4.1 and letting the powers tend to $\infty$ successively, we can easily deduce a multivariable version as follows. If $\mathbb{E}_{\mathcal{A}}$ is an asymptotic homomorphism and $x_1, \ldots, x_{n+1} \in \mathcal{M}$, then

$$\lim_{k_1 \to \infty} \ldots \lim_{k_n \to \infty} \| \mathbb{E}_{\mathcal{A}}(x_1 u^{k_1} x_2 \ldots u^{k_n} x_{n+1}) - \mathbb{E}_{\mathcal{A}}(x_1) \ldots \mathbb{E}_{\mathcal{A}}(x_{n+1}) u^{k_1+\ldots+k_n} \|_2 = 0. \quad (4.19)$$

If we insisted on using the same power in all $n$ places, then we would have a type of freeness for $\mathbb{E}_{\mathcal{A}}$ and $\mathcal{A}$: for all $x_1, \ldots, x_{n+1} \in \mathcal{M}$,

$$\lim_{|k| \to \infty} \| \mathbb{E}_{\mathcal{A}}(x_1 u^k x_2 \ldots u^k x_{n+1}) - \mathbb{E}_{\mathcal{A}}(x_1) \ldots \mathbb{E}_{\mathcal{A}}(x_{n+1}) u^k \|_2 = 0. \quad (4.20)$$

Modifying the proof of Theorem 4.2 shows that if $g$ is a generator of the free group $\mathbb{F}_n$ with associated masa $\mathcal{A}$, then $\mathbb{E}_{\mathcal{A}}$ has this freeness property using $u = g$ in (4.20). We have not investigated this idea. \qed
5 Strong singularity and the Popa invariant

Recall that, for a masa $A$ in a type II$_1$ factor $M$, we defined $\alpha(A)$ to be the largest constant satisfying
\[
\|E_{uAu^*} - E_A\|_{\infty,2} \geq \alpha(A) \|(I - E_A)(u)\|_2
\] (5.1)
for all unitaries $u \in M$. In this section we obtain an inequality which links $\delta(A)$ and $\alpha(A)$. We will need the following two lemmas.

**Lemma 5.1.** Let $\phi, \psi : M \to M$ be linear maps on a type II$_1$ factor $M$, bounded in the $\| \cdot \|_{\infty,2}$-norm, and suppose that their ranges are orthogonal in $L^2(M, \text{tr})$. Then
\[
\|\phi \pm \psi\|_{\infty,2} \leq \sqrt{\|\phi\|_{\infty,2}^2 + \|\psi\|_{\infty,2}^2}.
\] (5.2)

**Proof.** This is an immediate consequence of
\[
\|h \pm k\| = \sqrt{\|h\|^2 + \|k\|^2}
\] (5.3)
for any pair of orthogonal vectors in a Hilbert space. 

**Lemma 5.2.** Let $A$ and $B$ be von Neumann subalgebras of a type II$_1$ factor $M$. Then the following inequality holds:
\[
\|E_{A \cap M}(I - E_{B \cap M})\|_{\infty,2} \leq 2\|(I - E_A)E_B\|_{\infty,2}.
\]

**Proof.** Let $x, y \in M$ with $\|x\|, \|y\|_2 \leq 1$. Write $w = E_{A \cap M}(y)$ and let $u \in U(B)$. Note that $w \in A'$ and $\|w\|_2 \leq 1$. Then
\[
|\langle E_{A \cap M}(x - uxu^*), y \rangle| = |\langle x - uxu^*, w \rangle|
= |\text{tr}((x - uxu^*)w^*)|
= |\text{tr}(xw^* - xu^*wu^*)|
\leq |\text{tr}(xw^* - xu^*E_A(u)w^*)| + |\text{tr}(xu^*w^*(u - E_A(u)))|.
\] (5.4)
Here we have used the module properties of conditional expectations and that $w^*$ and $E_A(u)$ commute. The last expression in (5.4) is no greater than

$$|\text{tr}(xu^*(u - E_A(u))w^*)| + |\text{tr}(xu^*w^*(u - E_A(u)))|$$

$$\leq 2\|(I - E_A)(u)\|_2$$

$$\leq 2\|(I - E_A)E_B\|_{\infty, 2}$$

(5.5)

since $u = E_B(u)$. The estimates (5.4) and (5.5) combine to yield

$$\|E_{A' \cap M}(x - u xu^*)\|_2 \leq 2\|(I - E_A)E_B\|_{\infty, 2},$$

(5.6)

letting $y$ vary over the unit ball of $L^2(M, \text{tr})$. Since

$$E_{B' \cap M}(x) \in \text{conv}\{uwu^*: u \in U(B)\},$$

(see [2, 15]), the last inequality gives

$$\|E_{A' \cap M}(I - E_{B' \cap M})(x)\|_2 \leq 2\|(I - E_A)E_B\|_{\infty, 2}.$$  

(5.7)

The result follows by letting $x$ vary over the unit ball of $M$ in (5.7).

Theorem 5.3. If $A$ is a masa in a type $II_1$ factor $M$, then

$$\delta(A) \geq \alpha(A)/\sqrt{5}.$$  

(5.8)

In particular, $\delta(A) \geq 1/\sqrt{5}$ for all strongly singular masas.

Proof. Let $v$ be a nilpotent partial isometry such that $p = vv^*$ and $q = v^*v$ are orthogonal projections in $A$, and define a unitary $u \in M$ by

$$u = v + v^* + 1 - p - q.$$  

(5.9)

Then

$$A = (1 - p - q)A + pA + qA,$$  

(5.10)

and

$$uAu^* = (1 - p - q)A + vAv^* + v^*A v.$$  

(5.11)
Thus
\[ E_A - E_{uA^*} = E_{pA} + E_{qA} - E_{vA^*} - E_{v^*A^*}. \]  
(5.12)

By the orthogonality of \( p \) and \( q \), and the modularity of \( E_A \),
\[ \|E_A - E_{uA^*}\|_{\infty,2}^2 = \|E_{pA} - E_{vA^*}\|_{\infty,2}^2 + \|E_{qA} - E_{v^*A^*}\|_{\infty,2}^2. \]  
(5.13)

The two terms on the right hand side of (5.13) are equal because the map \( v^*(\cdot)v \) implements an isometry from \( p\mathcal{M}p \) to \( q\mathcal{M}q \) in the norms \( \| \cdot \| \) and \( \| \cdot \|_2 \). Thus (5.13) becomes
\[ \|E_A - E_{uA^*}\|_{\infty,2}^2 = 2\|E_{pA} - E_{vA^*}\|_{\infty,2}^2. \]  
(5.14)

From (3.16), \( E_A(v) = 0 \), and so the definition of \( u \) gives
\[ E_A(u) = 1 - p - q. \]  
(5.15)

Thus
\[ \|u - E_A(u)\|_2^2 = 1 - 2\|E_A(u)\|_2^2 = \text{tr}(p + q) = 2\text{tr}(p) = 2\|vv^*\|_2^2. \]  
(5.16)

Then (5.14) and (5.16) give
\[ \|E_{pA} - E_{vA^*}\|_{\infty,2} \geq \alpha(A)\|vv^*\|_2. \]  
(5.17)

In the von Neumann algebra \( p\mathcal{M}p \), consider the masas \( pA \) and \( vA^* \). We may apply Lemma 5.2 to obtain
\[ \|E_{pA}(I - E_{vA^*})\|_{\infty,2} \leq 2\|(I - E_{pA})E_{vA^*}\|_{\infty,2}. \]  
(5.18)

Since \( E_{pA}(I - E_{vA^*}) \) and \( (I - E_{pA})E_{vA^*} \) have orthogonal ranges,
\[ \|E_{pA} - E_{vA^*}\|_{\infty,2} = \|E_{pA}(I - E_{vA^*}) - (I - E_{pA})E_{vA^*}\|_{\infty,2} \leq \sqrt{5}\|(I - E_{pA})E_{vA^*}\|_{\infty,2}, \]  
(5.19)

using (5.18) and Lemma 5.1. Note that these expectations are defined on \( p\mathcal{M}p \), but viewing them on \( \mathcal{M} \) by first applying \( E_{p\mathcal{M}p} \) does not change the inequality (5.19). We now combine (5.17) and (5.19) to obtain
\[ \|(I - E_{pA})E_{vA^*}\|_{\infty,2} \geq \alpha(A)\|vv^*\|_2/\sqrt{5}, \]  
(5.20)
which shows that \( \delta(A) \geq \alpha(A)/\sqrt{5}. \)

Remark 5.4. Theorem 5.3 raises several obvious questions. Can the factor of \( \sqrt{5} \) be removed from the inequality (5.18) in this theorem? Is there an inequality in the opposite direction? Is it possible that \( \delta(A) = \alpha(A) \) in general?
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