Effective Sine(h)-Gordon-like equations for pair-condensates composed of bosonic or fermionic constituents

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Abstract

An effective coherent state path integral for super-symmetric pair condensates is investigated with specification on the nontrivial coset integration measure. The non-Euclidean integration measure prevents straightforward classical equations and solutions of the independent, anomalous field variables which follow from variations of the actions in the exponential phase weight factors. We examine a transformation with a suitable super-Jacobi matrix for the change of coset integration measure to 'flat' Euclidean path integration fields of pair condensates. The independent parameter fields of the super-symmetric anomalous terms are given by those of the \( \text{Osp}(S, S|2L)/U(L|S) \) coset super-manifold. The described, effective coherent state path integral of pair condensates is obtained by a gradient expansion after a Hubbard-Stratonovich transformation (HST) of the original path integral with super-fields of bosonic and fermionic atoms. A modified HST of bosonic and fermionic super-fields converts the original path integral into one with 'Nambu' doubled, super-symmetric self-energies. Due to the addition of source fields, we consider a spontaneous symmetry breaking of the total \( \text{Osp}(S, S|2L) \) super-group to the \( \text{Osp}(S, S|2L)/U(L|S) \otimes U(L|S) \) coset decomposition with the super-unitary \( U(L|S) \) group as the invariant subgroup of the background density field. The nontrivial coset integration measure, determined by the square root \((\text{SDET}(\hat{G}_{\text{Osp}/U}))^{1/2}\) of the super-determinant of the \( \text{Osp}(S, S|2L)/U(L|S) \) coset metric tensor \( \hat{G}_{\text{Osp}/U} \), is eliminated by the 'inverse square root' of the coset metric tensor \((\hat{G}_{\text{Osp}/U})^{-1/2}\) as the appropriate super-Jacobi matrix; this results into Euclidean path integration variables for the pair condensate fields. A diagonal construction of the coset metric tensor \( \hat{G}_{\text{Osp}/U} \) allows a straightforward application of the super-Jacobi matrix \( (\hat{G}_{\text{Osp}/U})^{-1/2} \) to the pair condensate fields of the \( \text{Osp}(S, S|2L)/U(L|S) \) coset super-manifold which also appears with this coset metric tensor in the gradients and kinetic energies of the actions. Therefore, we acquire a considerable simplification of the effective coherent state path integral in terms of anomalous, 'Euclidean path integration variables'. In analogy and in the sense of statistical thermodynamics, the particular property of being a 'state variable' is verified for the modulus of eigenvalues of the coset order-parameter matrix for anomalous fields. On the contrary, the phase values of eigenvalues of the anomalous coset order-parameter matrix depend on the chosen time contour path or previous history of transformation fields and therefore correspond to the path dependent 'heat' or 'work' variables of thermodynamics in a transferred sense. According to the transformation to Euclidean, anomalous path integration variables, first order variations of fields can be performed for classical equations with inclusion of second and higher even order variations for universal fluctuations determined by the coset metric tensor \( \hat{G}_{\text{Osp}/U} \). Furthermore, we mention how to extend finite order gradient expansions to infinite order by using a suitable integral representation for the logarithm of an operator and similarly for its inverse.

Keywords: super-symmetry, spontaneous symmetry breaking, nonlinear sigma model, coherent state path integral, Keldysh time contour, many-particle physics.

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1 Introduction

1.1 Variation of classical actions in coherent state path integrals with nontrivial integration measures

Generating functionals, as coherent state path integrals, allow various kinds of approximations or even exact solutions apart from being representations of many-particle quantum mechanics \[1\]-\[4\]. Coherent state path integrals can be examined by Monte Carlo methods according to appropriate importance sampling and stationary phase filtering or even by exact solutions in the case of integrable systems \[1\] \[2\]. At zero temperature they consist of a weight factor, usually an exponential phase comprising a classical action, so that certain classical field configurations can contribute a dominant part in the weighting with the exponential. These dominant contributions are usually obtained by the stationary phase or first order variation of the actions in the exponent of the weight factor. In principle this variation can be extended to second or even higher order variations around the solutions of classical fields from the first order variation.

However, this process of variations for approximating by classical solutions becomes nontrivial in the case of non-Euclidean integration measures of the field variables. In this case one can apply the method of steepest descent for a polynomial-like integration measure with the exponential of classical actions \[5\] or one transforms the whole factor of the 'integration measure' to its 'exponential(logarithm(integration measure))' form so that it has an equivalent weight as the classical action terms in the exponents. This method is straightforward; but, one can also try to determine a more sophisticated transformation of the path field variables so that the nontrivial integration measure is eliminated for Euclidean integration variables by inclusion of an additional Jacobi-determinant. The functional dependence of the classical actions with the original fields is then altered by the corresponding Jacobi-matrix of the new Euclidean integration field variables. Both methods, the method of steepest descent (with 'exponential(logarithm(integration measure))') or the removal of nontrivial integration measures by transformations with a suitable Jacobi-matrix, are in general far from being equivalent. In spite of the 'exponential(logarithm(integration measure))' form, the fields of the nontrivial integration measure in the variations of steepest descent method contribute in a different manner than the fields of the actions weighted by exponentials. In the case of the considered coherent state path integral \[6\], it is inevitable that the 'exponential(logarithm(integration measure))' in the method of steepest descent has its first contributions from second and all higher even order variations of the fields whereas the main other actions have only non-vanishing terms in odd-numbered order of variations with the independent fields on the time contour. Therefore, the simple method of steepest descent causes inconsistent treatment in the case of nontrivial path integration measures and their variations on the time contour in comparison to the variations on the time contour dependent fields in the actions of the exponentials.

We illustrate this problem in analogy to a multidimensional integral \[Z[\vec{x}], (\vec{x} = \{x^1, \ldots, x^N\})\], with an action \[A[\vec{x}]\]

\[
Z[\vec{x}] = \int d[\vec{x}] \sqrt{\det(\hat{g}(\vec{x}))} \exp \{i A[\vec{x}]\},
\]

where the Euclidean integration measure \[d[\vec{x}]\] is modified by the square root of a metric tensor \[\hat{g}_{ij}(\vec{x})\]
as the nontrivial integration measure

\[ (ds)^2 = dx^i \hat{g}_{ij}(\vec{x}) \, dx^j . \]  

The transformation to Euclidean variables \( d[\vec{y}] \) is related to the inverse square root of the metric tensor \( \hat{g}^{-1/2}(\vec{x}) \) as the appropriate Jacobi matrix where the symmetry of the metric tensor allows a decomposition into orthogonal matrices \( \hat{O}_{ij}(\vec{x}) \) and real eigenvalues \( \hat{\lambda}^k(\vec{x}) \)

\[
(ds)^2 = dx^i \hat{g}_{ij}(\vec{x}) \, dx^j = dx^i \hat{O}_{ik}^T(\vec{x}) \hat{\lambda}^k(\vec{x}) \hat{O}_{kj}(\vec{x}) \, dx^j =
\]

\[
= dx^i \left( \hat{O}^T(\vec{x}) \cdot \hat{\lambda}^{1/2}(\vec{x}) \right)_k \left( \hat{\lambda}^{1/2}(\vec{x}) \cdot \hat{O}(\vec{x}) \right)_kJ dx^j = dy^k \, dy_k = dy^k \, dy^k ;
\]

\[
dy^j = \left( \hat{\lambda}^{1/2}(\vec{x}) \cdot \hat{O}(\vec{x}) \right)_j^i \, dx^i ;
\]

\[
\hat{O}_{ji}(\vec{x}) \, dx^i = \left( \hat{\lambda}^{-1/2}(\vec{x}) \cdot d\vec{y} \right)_j^i ;
\]

\[
\implies y^j = y^j(\vec{x}) \implies x^i = x^i(\vec{y}) .
\]

This yields with the additional Jacobi matrix \( \hat{J}^i_k = (\partial x^i/\partial y^k) = (\hat{O}^T(\vec{x}) \cdot \hat{\lambda}^{-1/2}(\vec{x}))^i_k \) Euclidean integration variables \( \vec{y} \) for \( Z[\vec{x}(\vec{y})] \)

\[
\hat{J}^i_k = \frac{\partial x^i}{\partial y^k} = \left( \hat{O}^T(\vec{x}) \cdot \hat{\lambda}^{-1/2}(\vec{x}) \right)_k^i ;
\]

\[
\det(\hat{J}^i_k) = \det \left[ \left( \hat{O}^T(\vec{x}) \cdot \hat{\lambda}^{-1/2}(\vec{x}) \right)_k^i \right] = \det[\hat{g}^{-1/2}(\vec{x})] = \left( \det[\hat{g}(\vec{x})] \right)^{-1/2} ;
\]

\[
Z[\vec{x}(\vec{y})] = \int d[\vec{y}] \det[\hat{g}^{-1/2}(\vec{x})] \sqrt{\det(\hat{g}(\vec{x}))} \exp \{ i \, A[\vec{x}(\vec{y})] \} ;
\]

\[
= \ Z'[\vec{y}] = \int d[\vec{y}] \exp \{ i \, A'[\vec{y}] \} ;
\]

\[
A'[\vec{y}] = A[\vec{x}(\vec{y})] ; \quad Z'[\vec{y}] = Z[\vec{x}(\vec{y})] .
\]

The functional Taylor expansion of the action is then achieved straightforwardly where 'classical equations' are determined in a transferred sense from the vanishing of the first order variation which can be improved by Gaussian integrals of the second order variation for fluctuations around the 'classical solutions'. In order to obtain the transformation to Euclidean fields \( \vec{y} \), it is of particular importance that the metric tensor \( \hat{g}_{ij}(\vec{x}) \) can be diagonalized to the eigenvalues \( \hat{\lambda}^k(\vec{x}) \). By taking the (inverse) square root of eigenvalues \( \sqrt{\hat{\lambda}^k(\vec{x})} \), one acquires the (inverse) square root of the metric tensor \( \sqrt{\hat{g}(\vec{x})} \) \( \pm 1/2 \) in combination with the orthogonal matrix \( \hat{O}_{ji}(\vec{x}) \) as the eigenvectors.

In this paper we investigate an analogous problem, but in the more involved context of a supersymmetric coherent state path integral \([6]\) with a nontrivial integration measure which also contains anti-commuting integration fields. The measure is given as the square root of the super-determinant \([\text{SDET}(\hat{G}_{\text{osp}}/\text{U})]^{1/2} \) of the metric tensor \( \hat{G}_{\text{osp}/\text{U}} \) in a coset decomposition \( \text{Osp}(S, S[2L])/\text{U}(L|S) \otimes \text{U}(L|S) \)
of the ortho-symplectic super-group Osp($S, S|2L$) with the super-unitary $U(L|S)$ group as subgroup [7]-[17]. The independent field degrees of freedom of the final effective actions are restricted to the anomalous molecular- and BCS- pair condensates in a spontaneous symmetry breaking (SSB) [18, 19] with the coset decomposition $Osp(S, S|2L)/U(L|S) \otimes U(L|S)$. We briefly describe the steepest descent method by exponentiating and taking the logarithm of the coset integration measure; furthermore, a detailed account is outlined for the transformation to a Euclidean coherent state path integration measure with the inverse square root of the metric tensor $(\hat{G}_{Osp/U})^{-1/2}$ as the appropriate super-Jacobi-matrix. The latter transformation completely removes the coset integration measure and yields nontrivial classical field dependence in the actions, but results in simple Euclidean path integration measures of the independent fields. According to the Euclidean path integration measure, the variations with the classical fields in the actions of the exponents allow a consistent treatment of the non-equilibrium time contour integrals in the coherent state path integrals [20]-[25].

One may expect that all the transformations of the anomalous, coset fields only involve spatially and time-like local expressions as one transforms to the ‘flat’ Euclidean path integration fields with the super-Jacobi matrix given by the inverse square root $(\hat{G}_{Osp/U})^{-1/2}$ of the coset metric tensor $\hat{G}_{Osp/U}$. However, we verify that one has to take into account previous values or time contour histories in the case of phase-valued transformations of the eigenvalues of the coset order-parameter matrix; this is in contrast to the absolute values of eigenvalues of the order-parameter matrix which only yield local space-time expressions in the transformations. Therefore, the absolute values of eigenvalues of the coset order-parameter are similar to ‘state variables’ in the sense of thermodynamics; on the contrary, the phase values of the eigenvalues of the coset order-parameter matrix require the detailed previous time contour history in order to achieve the transformed, Euclidean path integration variables. In consequence, one can compare the transformation of the phases of the coset eigenvalues with the path-dependent ’work’ or ’heat’ variables of thermodynamics in a transferred sense. This observed property of our transformations to Euclidean fields is in accordance with other models, as the transition from incoherent to coherent laser light, where the phase of the laser light is treated separately (as e.g. in a phase diffusion model) or in analogy to a second order phase transition for the laser threshold [30]-[37].

Section 1.2 is devoted to the issue of finite versus infinite order gradient expansion of a (super-)determinant. Finite order gradient expansions have the advantage to be related to known, integrable, classical Sine(h)-Gordon-like equations; however, as one only takes into account gradually varying spatial gradients of coset matrices, it turns out that the ‘inverse’ of these slowly altering gradients is inevitably involved yielding also strongly varying fields in coordinate space. Therefore, we point out a suitable integral representation for the logarithm and similarly for the inverse of an operator [48] so that infinite order gradients are considered in a reliable manner [62, 45]. In sections 1.3, 1.4 general properties of super-matrices are reviewed for the considered case of super-symmetric coherent state path integrals with ‘Nambu’ doubled super-matrices for the self-energy (compare Refs. [18, 19] for the doubling of fields and see Refs. [7]-[17] for more details concerning super-groups with their super-algebras). We define the underlying Hamiltonian with the combination of Bose- and Fermi-operators and introduce symmetry breaking source fields for a coherent BEC-wavefunction and for coherent molecular- and BCS- pair condensates (compare Refs. [26]-[29] for similar cases in many-body theory). In Ref. [6] the various steps and the analysis of super-symmetries $Osp(S, S|2L)/U(L|S)$ are outlined for the transformation to a coherent state path integral with the ‘Nambu’ doubled self-energy $\delta\tilde{\Sigma}_{ab}^{\alpha\beta}(\vec{x}, t_p) \tilde{K}$
taking values in the ortho-symplectic super-algebra $\text{osp}(S, S|2L)$. A gradient expansion, combined with a coset decomposition $\text{Osp}(S, S|2L)/U(L|S) \otimes U(L|S)$, reduces the independent angular momentum field degrees of freedom to anomalous terms whose effective Sine(h)-Gordon-like actions are determined by a real, scalar self-energy density as background field for effective coupling constants. Since the present paper aims at the removal of the nontrivial coset integration measure $(\hat{G}_{\text{Osp}/U})^{1/2}$ by a transformation with the inverse square root of the coset metric tensor $(\hat{G}_{\text{Osp}/U})^{-1/2}$, we also trace out the detailed parametrization of the self-energy as an exact element of the ortho-symplectic algebra $\text{osp}(S, S|2L)$ in section 2.1. The $\text{osp}(S, S|2L)$ self-energy generator is separated by a coset decomposition into $u(L|S)$ density terms as subalgebra and into $\text{osp}(S, S|2L)/u(L|S)$ related anomalous molecular- and BCS- terms whose nontrivial integration measure is briefly outlined in section 2.2. Section 2.3 contains the effective actions of the coset matrices for pair condensates following from the gradient expansion with averaging of coupling parameters according to the background self-energy density field. In section 2.4 we apply a scaling to dimensionless fields and parameters of the actions in the exponentials of coherent state path integrals with non-Euclidean path integration measure. After general symmetry considerations in section 3.1, sections 3.2.1 to 3.2.3 finally encompass the suitable transformations with the inverse square root of the metric tensor $(\hat{G}_{\text{Osp}/U})^{-1/2}$ of $\text{Osp}(S, S|2L)/U(L|S)$ in order to replace the nontrivial coset integration measure by Euclidean path integration measures of the independent fields. In section 3.3 diagonal elements of coset matrices as in $\hat{T}^{-1}(\vec{x}, t_p)$ ($\partial \hat{T}(\vec{x}, t_p)$) are related to the diagonal elements of the new transformed field variables for anomalous field degrees of freedom having Euclidean path integration measures. Furthermore, we describe the problem for the 'path-dependent' phase values of the coset order-parameter matrix where one has to include nonlocal time contour dependent histories for the transformation to Euclidean fields. Section 4.1 comprises the variations of the effective actions for classical field equations with the Euclidean path integration variables. In section 4.2 a brief summary is included how the transformations to Euclidean coherent state path fields effect the observables following from differentiation with respect to the source fields. We also point out again for the possible extensions of the few classical integrable systems to chaotic cases which may be classified in terms of r-s matrices and symmetry breaking extensions of quantum groups [53]-[60].

1.2 Finite versus infinite order gradient expansion of determinants

There always appears the problem whether the restriction to a finite order gradient expansion is sufficient for considering a functional determinant in the $\text{det}(\hat{O}) = \text{exponential}\{\text{trace logarithm}(\hat{O})\}$ kind. As one takes only terms with derivatives for stable, static energy configurations in $3(+1)$ spatial dimensions, one has to expand from second up to fourth order gradients so that one cannot scale the particular configuration to arbitrary small or large sizes in the three dimensional coordinate space integrations over the static Hamiltonian density (‘Derrick’s theorem’ [61]). The spatially two-dimensional case is expected to contribute to the Goldstone modes in a SSB with second order gradients as a lowest order approximation. Since we reduce the expansion up to second order gradients in the present paper, we have only extracted the Goldstone modes of the SSB $\text{Osp}(S, S|2L)/U(L|S) \otimes U(L|S)$. Following Ref. [6], one can straightforwardly continue with an expansion to higher order gradients according to the rules and principles of chapter 4 in [6]. However, one has to decide which transport coefficients, composed of the background field, should remain from the gradient expansion as the unsaturated operators act to the
right or left with commutator relations of the Green functions. The ambiguity, caused by possible partial integrations between background field coefficients and coset matrices, can be diminished by applying Ward identities for gauge transformations of the coset generator; however, there does not remain a unique Lagrangian of finite order gradients because the Ward identities may also be used under various approximations. In the present paper we concentrate on the nontrivial coset integration measure which is transformed to Euclidean path integration variables for straightforward classical approximations in the lowest, finite order gradient expansions.

These lowest, finite order gradient expansions of the coset matrices have the particular property to be related to the Sine(h)-Gordon-like equations which allow for various integrable, classical solutions. Nevertheless, we point out in Refs. [62, 45] how to circumvent the problem of finite order gradients by using the integral representation (1.10) for the logarithm of an operator which is also similarly applied for the inverse of an operator (compare e.g. [48])

$$
(\ln \hat{O}) = \left( \int_0^{+\infty} dv \, \frac{\exp\{-v \, \hat{1}\} - \exp\{-v \, \hat{O}\}}{v} \right) ;
$$

(1.10)

$$
(\hat{O}^{-1}) = \left( \int_0^{+\infty} dv \, \exp\{-v \, \hat{O}\} \right) .
$$

(1.11)

As we use the particular form (1.12) for the gradient operator $\left( \hat{1} + \delta \hat{H}(\hat{T}^{-1}, \hat{T}) \, \langle \hat{H} \rangle^{-1} \right)$ with mean field approximation $\langle \ldots \rangle$ for the anomalous doubled, one-particle operator $\hat{H}$, we obtain the relation $\hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T} \, \langle \hat{H} \rangle^{-1}$ which results for vanishing source term $\hat{J} \equiv 0$ into relation (1.14) for the superdeterminant

$$
\delta \hat{H}(\hat{T}^{-1}, \hat{T}) = \hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T} - \langle \hat{H} \rangle ;
$$

(1.12)

$$
\hat{O} = \left( \hat{1} + \left( \delta \hat{H}(\hat{T}^{-1}, \hat{T}) + \bar{J}(\hat{T}^{-1}, \hat{T}) \right) \langle \hat{H} \rangle^{-1} \right)
$$

(1.13)

$$
\hat{O} = \hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T} \, \langle \hat{H} \rangle^{-1} + \bar{J}(\hat{T}^{-1}, \hat{T}) \langle \hat{H} \rangle^{-1} ;
$$

$$
\mathcal{A}_{SDet}[\hat{T}, \langle \hat{H} \rangle; \hat{J} \equiv 0] = \frac{1}{2} \, \text{tr STR} \ln \left[ T^{-1} \langle \hat{H} \rangle \, \hat{T} \, \langle \hat{H} \rangle^{-1} \right]
$$

(1.14)

$$
= \frac{1}{2} \int_0^{+\infty} dv \, \text{tr STR} \left[ \exp\{-v \, \hat{1}\} - \exp\{-v \, \hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T} \, \langle \hat{H} \rangle^{-1} \} \right].
$$

If one assumes slowly varying finite order gradients of $\hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T}$, one will also obtain unintended, extraordinary large spatial and time-like variations with $\hat{T} \, \langle \hat{H} \rangle^{-1} \, \hat{T}^{-1}$ according to the additional trace operation on the logarithm. In order to circumvent this problem, we use in Refs. [62, 45] the particular integral representation (1.10, 1.11) for the logarithm of an operator (and similarly for the inverse) which gives a simple representation of the logarithm with the (coset matrix weighted) combination of $\langle \hat{H} \rangle$ and its inverse $\langle \hat{H} \rangle^{-1}$ in an exponential. One can emphasize this point by a gauge transformation of the coset decomposition so that the mean field operator $\langle \hat{H} \rangle$ is simplified to pure spatial gradient operators (compare Ref. [62]).

If we suppose finite, positive eigenvalues for the total operator $\hat{O} = \hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T} \, \langle \hat{H} \rangle^{-1}$, the inverse factorials $1/n!$ of $\exp\{-v \, \hat{T}^{-1} \langle \hat{H} \rangle \, \hat{T} \, \langle \hat{H} \rangle^{-1} \}$ cause a meaningful expansion and convergence instead of
a pure logarithm $\ln(\hat{T}^{-1} \langle \hat{H} \rangle \hat{T} \langle \hat{H} \rangle^{-1})$ with reciprocal integer numbers in the expansion. Therefore, one can also rely on the integral representations \((1.10,1.11)\) for the logarithm and for the inverse of an operator $\hat{0}$ and can apply these relations for reducing the path integral part with coset matrices $\hat{T}(\vec{x},t_p) = \exp\{-\hat{Y}(\vec{x},t_p)\}$ and coset generator $\hat{Y}(\vec{x},t_p)$ for molecular and BCS-pair condensates. We can even choose the eigenbasis of the mean field approximated, one-particle operator $\langle \hat{H} \rangle$ or of its anomalous doubled version $\langle \hat{0} \rangle$ instead of the $d+1$ dimensional coordinate representation. This particular matrix representation for $\hat{T}$ in terms of the eigenbasis of $\langle \hat{0} \rangle$ allows to calculate observables as correlation functions of anomalous super-field combinations $\langle \psi_{\vec{x},\alpha}(t_p) \psi_{\vec{x}',\beta}(t'_p) \rangle$, density terms $\langle \psi_{\vec{x},\alpha}^+(t_p) \psi_{\vec{x}',\beta}(t_p) \rangle$ and eigenvalue correlations of molecular and BCS-terms.

1.3 Bosonic and fermionic operators with their coherent state field representations

In this section we briefly define the super-fields from their corresponding bosonic and fermionic operators. We introduce the complex conjugation, super-transposition, super-trace and super-determinant in analogy to the operations on ordinary matrices \([7]-[17]\). The basic constituents are determined by super-fields $\psi_{\vec{x},\alpha}(t)$ which have $L(=2l+1)$ odd-numbered bosonic and $S(=2s+1)$ even-numbered fermionic angular momentum degrees of freedom. The summations over these bosonic and fermionic angular momentum degrees of freedom are abbreviated by the first greek letters $\alpha$, $\beta$, $\gamma$... in bilinear or quartic relations of the super-fields. We specify these $N = L + S$ component super-fields in \((1.15)\) with internal bosonic vector $b_{\vec{x}}(t) = \{b_{\vec{x},m}(t)\}$ and internal Grassmann-valued, fermionic vector $\tilde{\alpha}_{\vec{x}}(t) = \{\alpha_{\vec{x},r}(t)\}$

\[
\begin{align*}
\alpha, \beta, \ldots &= -l, \ldots, +l; \ -s, \ldots, +s; \ l = 0, 1, 2, \ldots; \quad s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots; \\
N &= L + S; \quad L = 2l + 1; \quad S = 2s + 1; \\
\psi_{\vec{x},\alpha}(t) &= \begin{pmatrix} \tilde{b}_{\vec{x}}(t) \\ \tilde{\alpha}_{\vec{x}}(t) \end{pmatrix}; \quad \tilde{b}_{\vec{x}}(t) = \{b_{\vec{x},m}(t)\} = \{b_{\vec{x},-l}(t), \ldots, b_{\vec{x},+l}(t)\}^T; \\
\psi_{\vec{x},\alpha}^+(t) &= \begin{pmatrix} \tilde{b}_{\vec{x},-l}^*(t), \ldots, \tilde{b}_{\vec{x},+l}^*(t); \ \tilde{\alpha}_{\vec{x},-s}^*(t), \ldots, \tilde{\alpha}_{\vec{x},+s}^*(t) \end{pmatrix}.
\end{align*}
\]

These coherent super-fields are applied on the non-equilibrium time contour to define the coherent state path integral following from the Hamilton operator \((1.17)\) with combined Bose- and Fermi-operators in \((1.16)\)

\[
\hat{H}(\hat{\psi}^+, \hat{\psi}, t) = \sum_{\vec{x}} \sum_{\alpha} \hat{\psi}_{\vec{x},\alpha}^+ \hat{h}(\vec{x}) \hat{\psi}_{\vec{x},\alpha} + \sum_{\vec{x},\vec{x}'} \sum_{\alpha,\beta} \hat{\psi}_{\vec{x}',\beta}^+ \hat{V}_{\vec{x},\vec{x}'} \hat{\psi}_{\vec{x},\alpha} \hat{\psi}_{\vec{x}',\beta}^+ + \ldots
\]

\[\hat{H}(\hat{\psi}^+, \hat{\psi}, t) = \sum_{\vec{x}} \sum_{\alpha} \hat{\psi}_{\vec{x},\alpha}^+ \hat{h}(\vec{x}) \hat{\psi}_{\vec{x},\alpha} + \sum_{\vec{x},\vec{x}'} \sum_{\alpha,\beta} \hat{\psi}_{\vec{x}',\beta}^+ \hat{V}_{\vec{x},\vec{x}'} \hat{\psi}_{\vec{x},\alpha} \hat{\psi}_{\vec{x}',\beta}^+ + \ldots
\]

\[\hat{H}(\hat{\psi}^+, \hat{\psi}, t) = \sum_{\vec{x}} \sum_{\alpha} \hat{\psi}_{\vec{x},\alpha}^+ \hat{h}(\vec{x}) \hat{\psi}_{\vec{x},\alpha} + \sum_{\vec{x},\vec{x}'} \sum_{\alpha,\beta} \hat{\psi}_{\vec{x}',\beta}^+ \hat{V}_{\vec{x},\vec{x}'} \hat{\psi}_{\vec{x},\alpha} \hat{\psi}_{\vec{x}',\beta}^+ + \ldots
\]

\[\hat{H}(\hat{\psi}^+, \hat{\psi}, t) = \sum_{\vec{x}} \sum_{\alpha} \hat{\psi}_{\vec{x},\alpha}^+ \hat{h}(\vec{x}) \hat{\psi}_{\vec{x},\alpha} + \sum_{\vec{x},\vec{x}'} \sum_{\alpha,\beta} \hat{\psi}_{\vec{x}',\beta}^+ \hat{V}_{\vec{x},\vec{x}'} \hat{\psi}_{\vec{x},\alpha} \hat{\psi}_{\vec{x}',\beta}^+ + \ldots
\]

\[\hat{H}(\hat{\psi}^+, \hat{\psi}, t) = \sum_{\vec{x}} \sum_{\alpha} \hat{\psi}_{\vec{x},\alpha}^+ \hat{h}(\vec{x}) \hat{\psi}_{\vec{x},\alpha} + \sum_{\vec{x},\vec{x}'} \sum_{\alpha,\beta} \hat{\psi}_{\vec{x}',\beta}^+ \hat{V}_{\vec{x},\vec{x}'} \hat{\psi}_{\vec{x},\alpha} \hat{\psi}_{\vec{x}',\beta}^+ + \ldots
\]
Aside from the one-particle operator $\hat{h}(\vec{x})$ (1.18) with kinetic energy, trap- and chemical potential $u(\vec{x})$, $\mu_0$, we include a short-ranged quartic interaction potential $V_{[\vec{x}-\vec{z}]}$ with super-symmetry between Bose- and Fermi-particles. These two operator terms obey a global super-unitary invariance $U(L|S)$ so that a super-symmetry results between bosonic and fermionic angular momentum degrees of freedom. We assume that this super-symmetry may be achieved by appropriate tuning of Feshbach resonances with similar effective masses and similar properties concerning the trap potential [38]-[42].

$$\hat{h}(\vec{x}) = \frac{\hat{p}^2}{2m} + u(\vec{x}) - \mu_0;$$

(1.18)

$$j_{\psi;N}(\vec{x},t) = \left\{ j_{\psi;BL}(\vec{x},t); j_{\psi;FS}(\vec{x},t) \right\}^T.$$  

(1.19)

Apart from the $U(L|S)$ symmetry breaking source field $j_{\psi;0}(\vec{x},t)$ (1.19) for a coherent BEC wavefunction, we specialize on the investigation of super-symmetric pair condensates which are created by the $N \times N = (L + S) \times (L + S)$ super-symmetric source matrix $\tilde{j}_{\psi;N \times N}(\vec{x},t)$ in (1.17). The boson-boson pair condensate terms are denoted by a $L \times L$ symmetric operator matrix $\hat{c}_{\vec{x},L \times L}$ (1.20) and the fermion-fermion pair condensates (acting as a boson in its entity) are marked by the anti-symmetric operator matrix $\hat{f}_{\vec{x},L \times S}$ (1.21). The fermion-boson mixed operator (1.22) and its transpose (1.23) are abbreviated by $\hat{\eta}_{\vec{x},S \times L}$ and $\hat{\eta}^T_{\vec{x},L \times S}$ and have fermionic properties as an entity, due to the composition of a boson and fermion operator

$$\hat{c}_{\vec{x},L \times L} = \{ \hat{b}_{\vec{x},m} \} = \{ \hat{b}_{\vec{x},m} \}; \quad \hat{c}^{T}_{\vec{x},mn} = \hat{c}_{\vec{x},mn}; \quad m, n = -l, \ldots, +l;$$

(1.20)

$$\hat{f}_{\vec{x},S \times S} = \{ \hat{\alpha}_{\vec{x},r} \} = \{ \hat{\alpha}_{\vec{x},r} \}; \quad \hat{f}^{T}_{\vec{x},rr'} = -\hat{f}_{\vec{x},r'r}; \quad r, r' = -s, \ldots, +s;$$

(1.21)

$$\hat{\eta}_{\vec{x},S \times L} = \{ \hat{\alpha}_{\vec{x},m} \} = \{ \hat{\alpha}_{\vec{x},m} \}; \quad r = -s, \ldots, +s; \quad m = -l, \ldots, +l;$$

(1.22)

$$\hat{\eta}^{T}_{\vec{x},L \times S} = \{ \hat{\alpha}^{T}_{\vec{x},mr} \} = \{ \hat{\alpha}^{T}_{\vec{x},mr} \}; \quad m = -l, \ldots, +l; \quad r = -s, \ldots, +s.$$  

(1.23)

The $N \times N$ source matrix $\tilde{j}_{\psi;N \times N}(\vec{x},t)$ (1.24) has to respect the symmetry properties of the super-symmetric, paired terms in (1.20,1.23) and therefore has a symmetric even sub-matrix $\hat{j}_{BL;L \times L}(\vec{x},t)$ for the boson-boson pair condensates (1.24) and an anti-symmetric even sub-matrix $\hat{j}_{FS;S \times S}(\vec{x},t)$ for the corresponding fermion-fermion paired terms (1.26). Furthermore, Grassmann or anti-commuting fields $\hat{j}_{q;S \times L}(\vec{x},t)$, $\hat{j}^{T}_{q;L \times S}(\vec{x},t)$ (1.27) generate the boson-fermion $\hat{\eta}^{T}_{\vec{x},L \times S}$ (1.23) or fermion-boson $\hat{\eta}_{\vec{x},S \times L}$ (1.22) pair condensates in a super-trace relation. Appropriate signs have to be taken into account due to the property of a super-trace in the fermion-fermion part of a super-matrix

$$\tilde{j}_{\psi;N \times N}(\vec{x},t) = \begin{pmatrix} \hat{j}_{BL;L \times L}(\vec{x},t) & \hat{\eta}^{T}_{q;L \times S}(\vec{x},t) \\ \hat{j}_{q;S \times L}(\vec{x},t) & \hat{j}_{FS;S \times S}(\vec{x},t) \end{pmatrix};$$

(1.24)

$$\hat{j}_{B;L \times L}(\vec{x},t) \in \mathbb{C}_{\text{even}}; \quad \hat{j}^{T}_{B;mn}(\vec{x},t) = \hat{j}_{B;mn}(\vec{x},t);$$

(1.25)
\[ \hat{J}_{F:S \times S}(\vec{x}, t) \in \mathbb{C}_{\text{even}} ; \quad \hat{J}_{F:rr'}(\vec{x}, t) = -\hat{J}_{F:rr'}(\vec{x}, t) ; \quad \hat{J}_{F:rr}(\vec{x}, t) = 0 ; \]  
\[ \hat{J}_{N:S \times L}(\vec{x}, t) = \{ j_{\eta;mr}(\vec{x}, t) \} \in \mathbb{C}_{\text{odd}} ; \quad m = -l, \ldots, +l ; \]  
\[ \hat{J}_{N:L \times S}(\vec{x}, t) = \{ j_{\eta;mr}^+(\vec{x}, t) \} \in \mathbb{C}_{\text{odd}} ; \quad r = -s, \ldots, +s . \]  

In the following we define the complex conjugation (1.28) of Grassmann variables and the super-transposition (1.30)(1.31), the super-trace (1.32), the super-hermitian conjugation (1.33) and the super-determinant (1.34) of graded- or super-matrices as \( \hat{N}_1, \hat{N}_2 \). The complex conjugation of a product (\( \xi_1 \ldots \xi_i \ldots \xi_n \)) of anti-commuting variables \( \xi_i \) changes these \( n \)-factors to its reversed order with complex conjugated, odd numbers \( \xi_i^* \). This definition provides the combination \( \xi_i^* \xi_i \) of a Grassmann number \( \xi_i \) and its complex conjugate \( \xi_i^* \) with the property of an even, real (but nilpotent) variable

\[ (\xi_1 \ldots \xi_i \ldots \xi_n)^* = \xi_n^* \ldots \xi_i^* \ldots \xi_1^* ; \quad (\xi_i^*)^* = \xi_i ; \quad (\xi_i^* \xi_i)^* = (\xi_i^*)^* = \xi_i^* \xi_i . \]  

Super-matrices as \( \hat{N}_1, \hat{N}_2 \) consist of the even boson-boson blocks \( \hat{c}_1, \hat{c}_2 \) and the even fermion-fermion blocks \( \hat{f}_1, \hat{f}_2 \). Sub-matrices of anti-commuting variables are placed in the non-diagonal fermion-boson blocks \( \hat{\chi}_1, \hat{\chi}_2 \) and boson-fermion blocks \( \hat{\eta}_1^T, \hat{\eta}_2^T \). Under a super-transposition ‘s’t of the super-matrices \( \hat{N}_1, \hat{N}_2 \), the even parts \( \hat{c}_1, \hat{c}_2 \) and \( \hat{f}_1, \hat{f}_2 \) are transposed in the manner of ordinary matrices whereas the fermion-boson blocks \( \hat{\chi}_1, \hat{\chi}_2 \) and boson-fermion blocks \( \hat{\eta}_1^T, \hat{\eta}_2^T \) are exchanged with transposition and with the inclusion of an additional minus sign in the resulting fermion-boson blocks \( -\hat{\eta}_1, -\hat{\eta}_2 \). This definition of super-transposition preserves the property of ordinary matrices to be reversed under transposition in a product of matrices (1.31)

\[ \hat{N}_1 = \left( \begin{array}{c} \hat{c}_1 \\ \hat{\chi}_1 \end{array} \right) \hat{\eta}_1^T \hat{f}_1 ; \quad \hat{\eta}_2^T \hat{f}_2 ) ; \quad \hat{N}_2 = \left( \begin{array}{c} \hat{c}_2 \\ \hat{\chi}_2 \end{array} \right) \hat{\eta}_1^T \hat{f}_1 \hat{\eta}_2^T \hat{f}_2) . \]  

The super-trace 'str' of a super-matrix \( \hat{N} \) comprises the traces of the even boson-boson part \( \hat{c} \) and the even fermion-fermion part \( \hat{f} \). However, an additional minus sign has to be included in the trace of the fermion-fermion part so that the cyclic invariance of a product of super-matrices is maintained in a super-trace relation as in a trace with the product of several ordinary matrices

\[ \text{str}[\hat{N}] = \text{str} \left( \begin{array}{c} \hat{c} \\ \hat{\chi} \end{array} \right) \hat{\eta}_1^T \hat{f}) = \text{tr}[\hat{c}] - \text{tr}[\hat{f}] ; \quad \text{str}[\hat{N}_1 \hat{N}_2] = \text{str}[\hat{N}_2 \hat{N}_1] . \]  

The super-hermitian conjugation (1.33) of super-matrices \( \hat{N}_1, \hat{N}_2 \) does not involve additional minus signs as the super-transposition (1.30)(1.31). In comparison to (1.31), the property \( (\hat{N}_1 \hat{N}_2)^+ = \hat{N}_2^+ \hat{N}_1^+ \) (reversal of a product under super-hermitian conjugation) is already contained without additional minus signs because the complex conjugation (1.28) of a product of anti-commuting numbers is defined with an exchange of the factors to its reversed order

\[ \hat{N}_1^+ = \left( \begin{array}{c} \hat{c}_1^* \\ \hat{\eta}_1^+ \end{array} \right) \hat{\chi}_1^+ \hat{f}_1^+ ) ; \quad \hat{N}_2^+ = \left( \begin{array}{c} \hat{c}_2^* \\ \hat{\eta}_2^+ \end{array} \right) \hat{\chi}_2^+ \hat{f}_2^+ \right) ; \quad (\hat{N}_1 \hat{N}_2)^+ = \hat{N}_2^+ \hat{N}_1^+ . \]
The definition of a super-determinant 'sdet' for a super-matrix \( \hat{N} \) is generalized from the relation
\[
\det(\hat{M}) = \exp\{\text{tr} \ln(\hat{M})\}
\]
of ordinary matrices \( \hat{M} \). The ordinary trace relation 'tr' in the exponent
is generalized with the super-trace 'str' \((1.32)\) for super-matrices \( \hat{N} \), consisting of the even boson-
fermion, fermion-fermion blocks \( \hat{c}_{L \times L}, \hat{f}_{S \times S} \) and the odd fermion-boson, boson-fermion blocks \( \hat{\chi}_{S \times L}, \hat{\eta}^{T}_{L \times S} \). Using the properties of a super-trace 'str' (as cyclic invariance), one can transform the generalized
relation \( \text{sdet}(\hat{N}_{N \times N}) = \exp\{\text{str} \ln(\hat{N}_{N \times N})\} \) \((1.34)\) to ordinary \( L \times L \) and \( S \times S \) determinants where the
determinant \( \det(\hat{f}_{S \times S}) \) of the even fermion-fermion section appears in the denominator because of the
additional negative sign in the fermion-fermion section of a super-trace

\[
sdet(\hat{N}_{N \times N}) = \exp\left\{ \text{str} \ln \left( \begin{pmatrix} \hat{c}_{L \times L} & \hat{\eta}^{T}_{L \times S} \\ \hat{\chi}_{S \times L} & \hat{f}_{S \times S} \end{pmatrix} \right) \right\} \quad \text{or} \quad (1.34)
\]

\[
= \exp\left\{ \text{str} \ln \left( \begin{pmatrix} \hat{c}_{L \times L} & 0 \\ 0 & \hat{f}_{S \times S} \end{pmatrix} \right) \left( \begin{pmatrix} \hat{1}_{L \times L} & \hat{c}_{L \times L}^{-1} \hat{\eta}^{T}_{L \times S} \\ \hat{f}_{S \times S}^{-1} \hat{\chi}_{S \times L} & 1_{S \times S} \end{pmatrix} \right) \right\}
\]

\[
= \frac{\det(\hat{c}_{L \times L} - \hat{\eta}^{T}_{L \times S} \hat{f}_{S \times S}^{-1} \hat{\chi}_{S \times L})}{\det(\hat{f}_{S \times S})}.
\]

In the case of a product of super-matrices, the property \( \text{sdet}(\hat{N}_1 \hat{N}_2) = \text{sdet}(\hat{N}_1) \text{sdet}(\hat{N}_2) \) for the
factorization of the super-determinant holds in a similar manner as in the case with ordinary matrices
because of the cyclic invariance in the super-trace \((1.32)\).

### 1.4 The super-symmetric coherent state path integral

In the remainder we consider the time contour integral \((1.35)\) with the time variable \( t_p \) on the two
branches \( p = \pm \) for the time development of the Hamiltonian \((1.17)\) in forward \( \int_{-\infty}^{\infty} dt_+ \ldots \) and back-
ward \( \int_{-\infty}^{\infty} dt_- \ldots \) direction \([20]-[25]\). The negative sign of the backward propagation \( \int_{-\infty}^{\infty} dt_- \ldots = -\int_{-\infty}^{\infty} dt_- \ldots \) will be frequently taken into account by the time contour metric-symbol \( \eta_{p=\pm} = p = \pm \)

\[
\int_{C} dt_p \ldots = \int_{-\infty}^{\infty} dt_+ \ldots + \int_{-\infty}^{\infty} dt_- \ldots = \int_{-\infty}^{\infty} dt_+ \ldots - \int_{-\infty}^{\infty} dt_- \ldots \quad (1.35)
\]

\[
= \sum_{p=\pm} \int_{-\infty}^{\infty} dt_p \eta_p \ldots \quad (\eta_{p=\pm} = \pm).
\]

The corresponding coherent state path integral \( Z[\hat{\varphi}, \hat{\psi}, \hat{\bar{\psi}}] \) \([20]-[25]\) of the Hamiltonian \((1.17)\) with its
symmetry breaking source fields \( j_{\varphi,N}(\vec{x},t) \) \((1.19)\) and \( \hat{\psi}_{N \times N}(\vec{x},t) \) \((1.24)\) is given in relation
\((1.35)\) with inclusion of the time contour integrals \((1.35)\) in the exponentials

\[
Z[\hat{\varphi}, \hat{\psi}, \hat{\bar{\psi}}] = \int d[\psi_{\vec{x},p}(t_p)] \exp\left\{ -\frac{i}{\hbar} \int_{C} dt_p \sum_{\vec{x}} \sum_{\alpha} \psi^{*}_{\vec{x},\alpha}(t_p) \hat{H}_{p}(\vec{x},t_p) \psi_{\vec{x},\alpha}(t_p) \right\} \quad (1.36)
\]

\[
\times \exp\left\{ -\frac{i}{\hbar} \int_{C} dt_p \sum_{\vec{x},\vec{\varphi}} \sum_{\alpha,\beta} \psi^{*}_{\vec{\varphi},\beta}(t_p) \psi^{*}_{\vec{x},\alpha}(t_p) V_{[\vec{\varphi}-\vec{x}]}(t_p) \psi_{\vec{x},\alpha}(t_p) \psi_{\vec{\varphi},\beta}(t_p) \right\}
\]
Due to a missing potential for disorder with an ensemble average, the super-symmetric coherent state fields $\hat{\psi}_{x,\alpha}(t_p)$ only couple on a single specific branch of the time contour without any combinations between forward '+' and backward '-' propagation (compare with the Refs. [13]-[15] in the case of disorder). The one-particle operator $\hat{h}(\vec{x})$ (1.18) is completed to $\hat{H}_{p}(\vec{x},t_p)$ with the time contour derivative $\dot{E}_p = i\hbar \partial / \partial t_p$ and the imaginary time contour increment $\varepsilon \in (- \varepsilon_\infty, \varepsilon > 0_+)$ for appropriate convergence properties of Green functions with propagation according to suitable time directions:

$$\hat{H}_{p}(\vec{x},t_p) = -\dot{E}_p - \varepsilon \in \hat{h}(\vec{x}) = -i\hbar \partial / \partial t_p - i\varepsilon + \frac{\hat{p}^2}{2m} + u(\vec{x}) - \mu_0 . \quad (1.37)$$

A further source matrix $T_{\beta,\beta,x,\alpha}(t_p^2)$ is incorporated in the coherent state path integral $Z[\hat{\psi}, j_\psi, \tilde{j}_\psi \psi]$ (1.36) because it combines 'Nambu' doubled coherent state fields $\Psi_{x,\alpha}^{a=1/2}(t_p) = \{\psi_{x,\alpha}(t_p) ; \psi_{x,\alpha}^*(t_p)\}^T$ (18) (19). Therefore, it is possible to generate anomalous terms as $\langle \psi_{x,\alpha}(t_p) \psi_{x,\alpha}(t_p) \rangle$ by a single differentiation of $Z[\hat{\psi}, j_\psi, \tilde{j}_\psi \psi]$ (1.36) with respect to $\delta_{\beta,\beta,x,\alpha}(t_p, t_p)$. Furthermore, one has to distinguish between source fields $j_\psi_{x,\alpha}(\vec{x},t_p)$, $\tilde{j}_\psi_{x,\alpha}(\vec{x},t_p)$ with a dependence on the time contour branch 'p = ±' for generating observables by differentiation of (1.36) and the corresponding 'condensate seed' fields $\tilde{j}_{\psi,\alpha}(\vec{x},t)$, $\tilde{j}_{\psi,\alpha}(\vec{x},t)$ for SSB [26]-[29]. The latter 'condensate seeds' follow by setting the corresponding time branches to equivalent finite values (1.38, 1.39); this has to be performed at the final end of calculations with $Z[\hat{\psi}, j_\psi, \tilde{j}_\psi \psi]$ (1.36) after the prevailing observables have been determined by differentiation with respect to $j_\psi_{x,\alpha}(\vec{x},t_p)$, $\tilde{j}_\psi_{x,\alpha}(\vec{x},t_p)$ or with respect to $T_{\beta,\beta,x,\alpha}(t_p^2)$ (1.36). The last source matrix $T_{\beta,\beta,x,\alpha}(t_p^2)$ has then to be set to zero with remaining finite 'condensate seeds' $j_{\psi,\alpha}(\vec{x},t)$, $\tilde{j}_{\psi,\alpha}(\vec{x},t)$ for the creation of a coherent BEC-wavefunction and for the creation of pair condensates with super-symmetry on the coset space $\mathrm{Osp}(S,S|2L)/\mathrm{U}(L|S)$

$$j_{\psi,\alpha}(\vec{x},t_p) := j_{\psi,\alpha}(\vec{x},t) \neq 0 ; \quad \text{'condensate seed' for } \langle \psi_{x,\alpha}(t_p) \rangle ; \quad (1.38)$$
$$\tilde{j}_{\psi,\alpha}(\vec{x},t_p) := \tilde{j}_{\psi,\alpha}(\vec{x},t) \neq 0 ; \quad \text{'condensate seed' for } \langle \psi_{x,\alpha}(t_p) \psi_{x,\alpha}(t_p) \rangle . \quad (1.39)$$

The even coherent state fields $\hat{c}_{LXL}(\vec{x},t_p)$, $\hat{f}_{SXS}(\vec{x},t_p)$ and odd coherent fields $\hat{\eta}_{SXL}(\vec{x},t_p)$, $\hat{\eta}_{SXS}(\vec{x},t_p)$ (1.39), corresponding to the operators $\hat{c}_{LXL}$, $\hat{f}_{SXS}$ and $\hat{\eta}_{SXL}$, $\hat{\eta}_{SXS}$ (1.20) (1.23), involve supersymmetric combinations of anomalous terms as $\langle \psi_{x,\beta}(t_p) \psi_{x,\alpha}(t_p) \rangle$ so that a 'Nambu'-doubling (1.40)
\[ \Psi_{\alpha, \alpha}^{(1/2)}(t_p) = \{ \psi_{\alpha, \alpha}(t_p); \psi_{\alpha, \alpha}^*(t_p) \}^T \]

of coherent state fields has to be taken into account in transformations of \( Z[\hat{J}, \hat{j}_\psi, \hat{j}_\psi] \)

\[ \Psi_{\alpha, \alpha}^{(1/2)}(t_p) = \left\{ \begin{array}{c} \psi_{\alpha, \alpha}(t_p) \\ \psi_{\alpha, \alpha}^*(t_p) \end{array} \right\}^T \]

According to the presence of anomalous terms, an order-parameter \( \hat{\Phi}_{N, S} \)

\[ \text{In comparison to the} \ N \times N = (L + S) \times (L + S) \text{super-matrices} \ N_1, N_2 \text{in Eqs. (1.29-1.34), we have to consider} 2N \times 2N \text{super-matrices} \ \hat{\Phi}_{ab} \ \text{consisting of four} \ N \times N = (L + S) \times (L + S) \text{sub-super-matrices} \ \hat{\Phi}_{11}^{ab}, \ \hat{\Phi}_{22}^{ab} \ \text{and} \ \hat{\Phi}_{12}^{ab}, \ \hat{\Phi}_{21}^{ab}. \]

Each of the four sub-super-matrices is composed of an even \( L \times L \) (S×S) boson-boson (fermion-fermion) block and the two odd parts in the \( N \times L \) fermion-boson and \( L \times N \) boson-fermion blocks. Consequently, we have to generalize operations as the super-transposition 'st' of \( N \times N = (L + S) \times (L + S) \) super-matrices \( \hat{\Phi}_{11}^{ab}, \ \hat{\Phi}_{22}^{ab} \) and \( \hat{\Phi}_{12}^{ab}, \ \hat{\Phi}_{21}^{ab}. \)

The total, even, \( 2L \times 2L \) boson-boson, \( 2S \times 2S \) fermion-fermion sections and the total, odd, \( 2S \times 2L \) fermion-boson, \( 2L \times 2S \) boson-fermion sections are consistently split into four parts, respectively, and are distributed to four \( N \times N \) sub-super-matrices. The super-hermitian conjugation (1.44) of \( 2N \times 2N \) super-matrices \( \hat{\Phi}_{ab} \) follows by taking the super-hermitian conjugate of the block diagonal parts \( \hat{\Phi}_{11}^{ab}, \ \hat{\Phi}_{22}^{ab} \) as in (1.33) and also of \( \hat{\Phi}_{12}^{ab}, \ \hat{\Phi}_{21}^{ab}; \) in addition the latter super-hermitian conjugated, off-diagonal \( N \times N \) blocks have to exchange their places

\[ \left( \hat{\Phi}_{ab}^{\alpha \beta} \right)^{ST} = \begin{pmatrix} \hat{\Phi}_{11}^{ab} & \hat{\Phi}_{12}^{ab} \\ \hat{\Phi}_{21}^{ab} & \hat{\Phi}_{22}^{ab} \end{pmatrix}^{ST} = \begin{pmatrix} \hat{\Phi}_{11}^{ab}^{\alpha \beta} & \hat{\Phi}_{12}^{ab}^{\alpha \beta} \\ \hat{\Phi}_{21}^{ab}^{\alpha \beta} & \hat{\Phi}_{22}^{ab}^{\alpha \beta} \end{pmatrix}^{ST}; \]

\[ \text{STR}_{a, \alpha, b, \beta} \left[ \hat{\Phi}_{ab}^{\alpha \beta} \right] = \text{str}_{\alpha, \beta} \left[ \hat{\Phi}_{11}^{\alpha \beta} \right] + \text{str}_{\alpha, \beta} \left[ \hat{\Phi}_{22}^{\alpha \beta} \right] = \sum_{m=-l}^{m=l} \hat{\Phi}_{mm}^{11} - \sum_{r=-s}^{r=s} \hat{\Phi}_{rr}^{11} + \sum_{m=-l}^{m=l} \hat{\Phi}_{mm}^{22} - \sum_{r=-s}^{r=s} \hat{\Phi}_{rr}^{22}; \]

\[ \left( \hat{\Phi}_{ab}^{\alpha \beta} \right)^+ = \begin{pmatrix} \hat{\Phi}_{11}^{\alpha \beta} & \hat{\Phi}_{12}^{\alpha \beta} \\ \hat{\Phi}_{21}^{\alpha \beta} & \hat{\Phi}_{22}^{\alpha \beta} \end{pmatrix}^+ = \begin{pmatrix} \hat{\Phi}_{11}^{\alpha \beta}^+ & \hat{\Phi}_{12}^{\alpha \beta}^+ \\ \hat{\Phi}_{21}^{\alpha \beta}^+ & \hat{\Phi}_{22}^{\alpha \beta}^+ \end{pmatrix}. \]
In a similar manner the super-determinant 'sdet(\(\hat{N}\))' is extended to a super-determinant 'SDET(\(\hat{\phi}^{ab}_{\alpha\beta}\))' (1.45) of \(2N \times 2N\) super-matrices by substituting the super-trace 'str' (1.32) in relation (1.34) with the super-trace 'STR' (1.43) of \(2N \times 2N\) super-matrices, having symmetries as the dyadic product (1.41) of 'Nambu' doubled coherent state fields

\[
\text{SDET} \left( \hat{\phi}^{ab}_{\alpha\beta} \right) = \exp \left\{ \text{STR} \ln \left( \hat{\phi}^{ab}_{\alpha\beta} \right) \right\}.
\]  

In Ref. [6] we describe in detail how to transform the coherent state path integral \(Z[\hat{\jmath}, \hat{\psi}, \hat{\jmath}_\psi]\) (1.36) with 'Nambu' doubled super-fields, doubled one-particle and interaction parts to doubled self-energies using Hubbard-Stratonovich transformations (HST) [16]. The properties of a Osp\((S, S|2L)\) self-energy field are analyzed in general and require anti-hermitian anomalous terms in the self-energy matrix \(\delta \Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p)\) for an appropriate parametrization\(^3\). Additionally we have to incorporate a real, scalar background field \(\sigma_D^{(0)}(\vec{x}, t_p)\) as a self-energy density term for \(\sum_{\alpha=1}^{N=L+S} \psi_{\vec{x},\alpha}(t_p) \psi_{\vec{x},\alpha}(t_p)\) in the HST transformations. The \(U(L|S)\) density terms, as subgroup of Osp\((S, S|2L)\) in \(\delta \Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p)\), only contribute as 'hinge'-fields in the spontaneous symmetry breaking according to the coset decomposition Osp\((S, S|2L)\otimes U(L|S)\). After a complete 'Nambu' doubling and suitable HST's of \(Z[\hat{\jmath}, \hat{\psi}, \hat{\jmath}_\psi]\) (1.36), we obtain in Ref. [6] a coherent state path integral \(Z[\hat{\jmath}, \hat{\jmath}_\psi, \nu \hat{\jmath}_\psi]\) (1.46) which depends on the real, scalar self-energy density \(\sigma_D^{(0)}(\vec{x}, t_p)\) as background field and on the \(2N \times 2N\) super-symmetric self-energy \(\delta \Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p)\) with anti-hermitian anomalous terms \((\delta \Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p))^+ = -\delta \Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p)\) in the off-diagonals \((a \neq b)\). Corresponding to the short-ranged interaction potential \(V_{|\vec{x}-\vec{x}'|}\), the spatially nonlocal self-energies, resulting from the HST's, are approximated to their local form with an effective, constant interaction parameter \(V_0\). This approximation is justified by the assumption that the strong oscillations lead to a cancellation of phases for exceeding interaction range. Introducing the \(2N \times 2N\) 'Nambu' doubled super-matrix \(\mathcal{M}_{\vec{x},\alpha;\vec{x}',\beta}(t_p, t_p')\) (1.47), we achieve in Ref. [6] the coherent state path integral \(Z[\hat{\jmath}, \hat{\jmath}_\psi, \nu \hat{\jmath}_\psi]\) (1.46) where the source fields \(\hat{\jmath}_{\psi\alpha}(\vec{x}, t_p), \hat{\jmath}_{\psi\alpha\beta}(\vec{x}, t_p)\) are converted to their 'Nambu' doubled form \(J^\theta_{\psi\alpha}(\vec{x}, t_p) = \{ \hat{\jmath}_{\psi\alpha}(\vec{x}, t_p); \hat{\jmath}_{\psi\alpha}(\vec{x}, t_p) \} (1.48)\) and to \(\nu J^\sigma_{\psi\alpha\beta}(\vec{x}, t_p)\) (1.49,1.50). We have also to include various 'Nambu' metric tensors \(K^a = \{(\hat{1}_{LXL} , \hat{1}_{SXS})^{a=1}; (\hat{1}_{LXL}, -\hat{1}_{SXS})^{a=2}\}\), \(\tilde{K}^a = \{ (\hat{1}_{LXL}, \hat{1}_{SXS})^{a=1}; (-\hat{1}_{LXL}, \hat{1}_{SXS})^{a=2}\}\) and \(\hat{I}^a = \{(\hat{1}_{LXL}, \hat{1}_{SXS})^{a=1}; (\hat{1}_{LXL}, \hat{1}_{SXS})^{a=2}\}\). The parametrization and propagation of the self-energy fields in the exponentials are confined to the ortho-symplectic super-group Osp\((S, S|2L)\)

\[
Z[\hat{\jmath}, \hat{\jmath}_\psi, \nu \hat{\jmath}_\psi] = \int d[\sigma_D^{(0)}(\vec{x}, t_p)] \times \exp \left\{ \frac{i}{2\hbar V_0} \int_C dt_p \sum_{\vec{x}} \left[ \sigma_D^{(0)}(\vec{x}, t_p) \sigma_D^{(0)}(\vec{x}, t_p) \right] \times \left[ \delta \Sigma(\vec{x}, t_p) - \nu \hat{\jmath}_\psi(\vec{x}, t_p) \right] \times \delta \Sigma(\vec{x}, t_p) - \nu \hat{\jmath}_\psi(\vec{x}, t_p) \right\}
\]

\(^3\)In the remainder the tilde '\(\hat{\cdot}\)' of \(\delta \Sigma_{2N \times 2N}\) refers to a self-energy with anti-hermitian anomalous terms \(\nu \delta \Sigma_{12}^{N \times N}; \nu \delta \Sigma_{11}^{N \times N}\) in comparison to \(\delta \Sigma_{2N \times 2N}\) with hermitian pair condensates \(\delta \Sigma_{12}^{N \times N}, \delta \Sigma_{11}^{N \times N}; \delta \Sigma_{22}^{N \times N} = (\delta \Sigma_{12}^{N \times N})^+\). We mark the 'Nambu' doubled self-energy \(\delta \Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p)\) with a '\(\hat{\cdot}\)' in order to distinguish from the total sum \(\Sigma_{ab}^{\alpha\beta}(\vec{x}, t_p)\).

\(2.1\) has to be added to the self-energy for taking values within the ortho-symplectic super-algebra Osp\((S, S|2L)\).
\[ \times \left\{ \text{SDET} \left[ \tilde{M}_{x,\alpha;\beta}^{ab}(t_p, t'_q) \right] \right\}^{-1/2} \times \]

\[ \times \exp \left\{ \frac{i}{2\hbar} \Omega \int_C dt_p \, dt'_q \sum_{x, x'} N_x \, J_{\psi;\alpha}^{ab}(x', t'_q) \hat{I} \tilde{K} \tilde{M}_{x',\beta;\alpha}^{-1;bc}(t'_q, t_p) \hat{I} \, J_{\psi;\alpha}^{a}(x, t_p) \right\} ; \tag{1.46} \]

\[ \tilde{M}_{x,\alpha;\beta}^{ab}(t_p, t'_q) = \delta_{x, x'} \, \eta_p \, \delta_{t_p, t'_q} \left[ \left( \hat{H}_p(x, t_p) + \delta^{(0)}(x, t_p) \right) \right. \]

\[ \left. + \left( \delta^{11}(x, t_p) \quad \delta^{12}(x, t_p) \right) \right]_{\alpha\beta}^{ab} \]

\[ \hat{K}_{\alpha\beta} + i \hat{K} \eta_p \left( \frac{\hat{M}_{x,\alpha;\beta}^{ab}(t_p, t'_q)}{\Omega N_x} \right) \eta_q \hat{K} \hat{K} ; \tag{1.47} \]

\[ J_{\psi;\alpha}^{0(=1/2)}(x, t_p) = \left\{ \frac{\hat{j}_{\psi;\alpha}(x, t_p)}{a=1} \quad \frac{\hat{j}_{\psi;\alpha}^*(x, t_p)}{a=2} \right\}^T ; \tag{1.48} \]

\[ \hat{j}_{\psi;\alpha}^{a,b}(x, t_p) = \left( \frac{\hat{\gamma}_+ \hat{j}_{\psi;\alpha}(x, t_p)}{a=1} \quad \frac{\hat{\gamma}_- \hat{j}_{\psi;\alpha}(x, t_p)}{a=2} \right) ; \tag{1.49} \]

\[ \hat{j}_{\psi;\alpha}^{a,b}(x, t_p) = \left( \frac{\hat{j}_{B,L \times L}(x, t_p)}{B \times B} \quad \frac{\hat{j}_{B,F,S}(x, t_p)}{F \times S} \right) ; \tag{1.50} \]

\[ \hat{j}_{B,L \times L}^{T}(x, t_p) = \hat{j}_{B,L \times L}(x, t_p) ; \quad \hat{j}_{B,F,S}^{T}(x, t_p) = -\hat{j}_{F,S \times S}(x, t_p) . \]

Apart from the self-energy density \( \sigma^{(0)}_D(x, t_p) \) as background field in (1.46,1.47), the self-energy super-matrix \( \delta^{11}(x, t_p) \tilde{K} \) only enters into the coherent state path integral (1.46,1.47) with independent field degrees of freedom confined to the parameters of the ortho-symplectic \( \text{osp}(S,S|2L) \) superalgebra. It has to be noted that the self-energy super-matrix \( \delta^{11}(x, t_p) \) has to include the appropriate metric \( \tilde{K} \) in order to become an exact element of \( \text{osp}(S,S|2L) \)

\[ \delta^{11}(x, t_p) \tilde{K} = \left( \begin{array}{cc} \delta^{11}(x, t_p) & \delta^{11}(x, t_p) \\ \delta^{22}(x, t_p) & \delta^{22}(x, t_p) \end{array} \right) \tilde{K} ; \tag{1.51} \]

\[ \tilde{K} = \left\{ \begin{array}{c} \hat{1}_{L \times L} \quad \hat{1}_{S \times S} \\ -\hat{1}_{L \times L} \quad \hat{1}_{S \times S} \end{array} \right\} ; \quad \tilde{\kappa}_{N \times N} = \left\{ \begin{array}{c} -\hat{1}_{L \times L} \quad \hat{1}_{S \times S} \\ \hat{1}_{L \times L} \quad \hat{1}_{S \times S} \end{array} \right\} . \tag{1.52} \]

The density terms \( \delta^{11}(x, t_p), \delta^{22}(x, t_p) \), referring to the super-unitary \( U(L|S) \) group, are eliminated in combination of the coset decomposition with the gradient expansion and have the effect of 'hinge' functions in a SSB. According to the symmetry examination in Ref. [6], the coset decomposition \( \text{Osp}(S,S|2L)/U(L|S) \otimes U(L|S) \) requires anti-hermitian anomalous terms \( \delta^{12}(x, t_p) = i \, \delta^{12}(x, t_p) \).
\[ \delta \Sigma_{\alpha \beta}^{21}(\vec{x}, t_p) = \iota \delta \Sigma_{\alpha \beta}^{21}(\vec{x}, t_p), \quad \delta \Sigma_{\alpha \beta}^{21}(\vec{x}, t_p) = (\delta \Sigma_{\alpha \beta}^{21}(\vec{x}, t_p))^+ \] in order to obtain the correct number of independent field degrees of freedom as the independent parameters of \text{Osp}(S, S|2L). The gradient expansion of the super-matrix \( \tilde{N}_{x, \alpha ; x, \beta}^{ab}(t_p, t'_q) \) \((1.47)\) with the coset fields in \( \tilde{T}(\vec{x}, t_p) \) results in effective actions \( A'_{N-1}[\tilde{T}; J_\psi] \), \( A'_{N_0}[\tilde{T}; J_\psi] \), \( A'_{N+1}[\tilde{T}] \) \((1.53)\) which can be classified according to a parameter \( N = \hbar \Omega N_x \) \((\Omega = 1/\Delta t, N_x = (L/\Delta x)^d)\), denoting the total number of spatial points on an underlying grid and specifying the maximum possible energy \( \hbar \Omega \) corresponding to the discrete time steps \( \Delta t \). The nontrivial coset integration measure is indicated by \( d[\tilde{T}] = d[\tilde{T}(\vec{x}, t_p)] \) in \( Z[\tilde{\gamma}, J_\psi, i\tilde{J}_\psi] \) \((1.53)\) (see section 2.2). The source action \( A_{J_\psi}[\tilde{T}] \), following from \( i \tilde{J}_\psi^{a \neq b}(\vec{x}, t_p) \) \((1.49,1.50)\), is independent from gradients and the background field \( \sigma_D^{(0)}(\vec{x}, t_p) \) and can be simplified by using properties of Vandermonde matrices \([47]\). The action, resulting for the additional source matrix \( \tilde{J}_\psi^{a \neq b}(\vec{x}, t_p) \) within up to second order of the gradient expansion, is denoted by \( A'[\tilde{T}; \tilde{\gamma}] \) and is further investigated in section 4.2.

\[
Z[\tilde{\gamma}, J_\psi, i\tilde{J}_\psi] = \int d[\tilde{T}^{-1}(\vec{x}, t_p)] d[\tilde{T}(\vec{x}, t_p)] \exp \left\{ i A_{J_\psi}[\tilde{T}] \right\} \times \exp \left\{ -A'_{N-1}[\tilde{T}; J_\psi] - A'_{N_0}[\tilde{T}; J_\psi] - A'_{N+1}[\tilde{T}] \right\} \times \exp \left\{ -A'[\tilde{T}; \tilde{\gamma}] \right\}.
\]

It remains to identify the various classical actions in \((1.53)\) with the nontrivial coset integration measure which has to be replaced by Euclidean path integration fields. This is accomplished in sections 2.2 and 2.3 however, we have in advance to describe the precise parameters of anomalous fields \( \tilde{T}(\vec{x}, t_p) \) following from the total self-energy matrix \( \delta \Sigma_{\alpha \beta}^{ab}(\vec{x}, t_p) \) \((1.51,1.52)\) in the coset decomposition \text{Osp}(S, S|2L)/U(L|S) \( \otimes U(L|S) \) (section 2.1).

Instead of the gradient expansion for effective actions in \( Z[\tilde{\gamma}, J_\psi, i\tilde{J}_\psi] \) \((1.53)\), we remark an alternative solution \((1.51)\) of the original coherent state path integral \((1.46)\) with super-matrix \( \tilde{N}_{x, \alpha ; x, \beta}^{ab}(t_p, t'_q) \) \((1.47)\). This solution follows from the functional variation of \((1.46)\) with respect to the self-energy super-matrix \( \delta \Sigma_{\alpha \beta}^{ab}(\vec{x}, t_p) \) \((1.51)\) as a osp\((S, S|2L)\) super-generator

\[
0 \equiv \frac{\Delta \left( Z[\tilde{\gamma}, J_\psi, i\tilde{J}_\psi] \right)}{\Delta \left( \delta \Sigma(\vec{x}, t_p) \right) \tilde{K}} \bigg|_{ab}^{\tilde{\gamma}, \alpha \beta} = \left< \frac{\iota \eta_p}{\mathcal{N}} \right> \left[ \delta \Sigma(\vec{x}, t_p) - i \tilde{J}_\psi^{ab}(\vec{x}, t_p) \right] \tilde{K}^{ab}_{\gamma, \alpha \beta} \bigg|_{ab}^{\tilde{\gamma}, \alpha \beta} + \delta \Sigma(\vec{x}, t_p) \tilde{K}^{ab}_{\gamma, \alpha \beta} \bigg|_{ab}^{\tilde{\gamma}, \alpha \beta}.
\]

One can apply continued fractions of \( \delta \Sigma_{\alpha \beta}^{ab}(\vec{x}, t_p) \) \((1.51)\) for solving the mean field equation \((1.54)\) (compare Ref. [43]). This process considerably simplifies in case of spatial symmetries and under restriction to stationary solutions. We note that the matrix \( \tilde{N}_{x, \alpha ; x, \beta}^{ab}(t_p, t'_q) \) \((1.47)\) is not only of central importance for the gradient expansion with the anomalous terms, but also for the saddle point equation \((1.54)\) because it consists of the background field \( \sigma_D^{(0)}(\vec{x}, t_p) \) apart from the self-energy super-matrix.
\[ \delta \Sigma^{ab}_{\alpha\beta}(\vec{x}, t_p) \tilde{K} \quad (\text{1.51}) \]. The exact mean field equation (1.54) can be approximated by averaging of the inverse \( \cong_{\vec{x};\alpha;\beta}^{-1}(t_p, t'_q) \) of the super-matrix (1.47) as one-point Green function with the background field \( \sigma_D^{(0)}(\vec{x}, t_p) \). This averaging process of \( \cong_{\vec{x};\alpha;\beta}^{-1}(t_p, t'_q) \) by \( \sigma_D^{(0)}(\vec{x}, t_p) \) becomes itself more accessible by taking a saddle point solution for the background density field in order to approximate (1.54).

\section{The classical Lagrangian for anomalous terms}

\subsection{The independent field variables for the super-symmetric pair condensates}

According to super-group properties of \( \text{Osp}(S, S|2L) \), the sum of self-energies \( \sigma_D^{(0)} \) \( \hat{1}_{2N \times 2N} \) and \( \delta \Sigma^{2N \times 2N \times 2N \times 2N}_{\vec{x};\alpha;\beta} \) \( \tilde{K} \) is factorized into density terms \( \delta \Sigma_{D;N \times N}^{11}, \delta \Sigma_{D;N \times N}^{22} \) (2.6, 2.7) and super-matrices \( \hat{T}(\vec{x}, t_p) \) (2.2, 2.5) for pair condensates within the coset decomposition \( \text{Osp}(S, S|2L)/U(L|S) \otimes U(L|S) \) (2.1). The self-energy density matrices \( \delta \Sigma_{D;N \times N}^{11}, \delta \Sigma_{D;N \times N}^{22} \) \( \kappa \) are related to the super-unitary group \( U(L|S) \) and only act as 'hinge' fields for the SSB and the gradient expansion. The background self-energy density \( \sigma_D^{(0)}(\vec{x}, t_p) \) is invariant under \( U(L|S) \) subgroup transformations and has therefore to be considered as the invariant ground or vacuum state in the SSB of \( \text{Osp}(S, S|2L) \) with \( U(L|S) \) as subgroup

\[ \Sigma^{2N \times 2N}_{\vec{x};\alpha;\beta}(\vec{x}, t_p) \tilde{K} = \sigma_D^{(0)}(\vec{x}, t_p) \hat{1}_{2N \times 2N} + \delta \Sigma^{2N \times 2N}_{\vec{x};\alpha;\beta}(\vec{x}, t_p) \tilde{K} = \]

\[ \begin{pmatrix} \sigma_D^{(0)}(\vec{x}, t_p) \hat{1}_{N \times N} + \delta \Sigma_{D;N \times N}^{11}(\vec{x}, t_p) & 0 \\ 0 & \sigma_D^{(0)}(\vec{x}, t_p) \tilde{k} + \delta \Sigma_{D;N \times N}^{22}(\vec{x}, t_p) \end{pmatrix} \tilde{K} \hat{T}^{-1}(\vec{x}, t_p) \]

\[ = \sigma_D^{(0)}(\vec{x}, t_p) \hat{1}_{2N \times 2N} + \hat{T}_{2N \times 2N}(\vec{x}, t_p) \left( \begin{array}{cc} \delta \Sigma_{D;N \times N}^{11}(\vec{x}, t_p) & 0 \\ 0 & \delta \Sigma_{D;N \times N}^{22}(\vec{x}, t_p) \end{array} \right) \hat{T}^{-1}_{2N \times 2N}(\vec{x}, t_p). \]

The independent field degrees of freedom for pair condensates, originally defined by \( \tau \delta \Sigma_{\alpha;\beta}^{12}, \tau \delta \Sigma_{\alpha;\beta}^{21} \) in (1.51), are described by the \( \text{osp}(S, S|2L)/u(L|S) \) super-generator \( \hat{Y}_{2N \times 2N}(\vec{x}, t_p) \) (2.3) or its exponential form \( \hat{T}_{2N \times 2N}(\vec{x}, t_p) = \exp\{-\hat{Y}_{2N \times 2N}(\vec{x}, t_p)\} \) (2.2) for the coset manifold \( \text{Osp}(S, S|2L)/U(L|S) \). The super-generator \( \hat{Y}_{2N \times 2N}(\vec{x}, t_p) \) (2.3) consists of the sub-generators \( \hat{X}_{N \times N}(\vec{x}, t_p) \) (2.4) and its super-hermitian conjugate \( \tilde{\kappa} \hat{X}^+_{N \times N} \) in the off-diagonal blocks \((a \neq b)\). They are itself composed of the even complex symmetric matrix \( \hat{c}_{D;L \times L}(\vec{x}, t_p) \) for molecular condensates and the even complex anti-symmetric matrix \( \hat{f}_{D;S \times S}(\vec{x}, t_p) \) for BCS terms (2.5). The Grassmann valued field degrees of freedom for anomalous terms are given by the matrix \( \hat{\eta}_{D;S \times L} \) and its transpose \( \hat{\eta}_{D;L \times S}^T \) in the fermion-boson and boson-fermion blocks

\[ \hat{T}^{ab}_{\alpha;\beta}(\vec{x}, t_p) = \exp \left\{ \tau \left( \begin{array}{cc} 0 & \hat{X}_{N \times N}(\vec{x}, t_p) \\ \tilde{\kappa} \hat{X}^+_{N \times N}(\vec{x}, t_p) & 0 \end{array} \right) \right\} \]

\[ = \exp \left\{ -\hat{Y}_{2N \times 2N}(\vec{x}, t_p) \right\} \].
\[
\hat{Y}_{\alpha\beta}(\vec{x}, t_p) = \begin{pmatrix}
0 & \hat{X}_{\alpha\beta}(\vec{x}, t_p) \\
\widetilde{\kappa}_{N\times N} \hat{X}_{\alpha\beta}^+(\vec{x}, t_p) & 0
\end{pmatrix}^{ab}; \\
\hat{X}_{\alpha\beta}(\vec{x}, t_p) = \begin{pmatrix}
-\hat{c}_{D;L\times L}(\vec{x}, t_p) & \hat{\eta}_{DL;LS}^T(\vec{x}, t_p) \\
-\hat{\eta}_{DL;LS}(\vec{x}, t_p) & \hat{f}_{DL;LS}^T(\vec{x}, t_p)
\end{pmatrix}; \\
\hat{c}_{D;L\times L}^T(\vec{x}, t_p) = \hat{c}_{D;L\times L}(\vec{x}, t_p); \\
\hat{f}_{DL;LS}^T(\vec{x}, t_p) = -\hat{f}_{DL;LS}(\vec{x}, t_p). 
\]

This parametrization with \(\hat{Y}_{2N\times2N}(\vec{x}, t_p)\) takes into account the exact structure of symmetry breaking source terms for super-symmetric pair condensates which have been introduced into the original Hamiltonian \((1.17)\) for the coherent state path integrals. The \(U(L|S)\) self-energy density matrices \(\delta \hat{\Sigma}_{D;N\times N}^{11}(\vec{x}, t_p)\) and its super-transposed copy \(\delta \hat{\Sigma}_{D;N\times N}^{22}(\vec{x}, t_p)\) contain the independent field degrees following from the dyadic product density parts \(\psi_{\vec{x}\alpha}(t_p) \otimes \psi_{\vec{x}\beta}^*(t_p)\) and \(\psi_{\vec{x}\alpha}^*(t_p) \otimes \psi_{\vec{x}\beta}(t_p)\) within the HST transformations

\[
\delta \hat{\Sigma}_{D;N\times N}^{11}(\vec{x}, t_p) = \begin{pmatrix}
\delta \hat{B}_{D;L\times L}(\vec{x}, t_p) & \delta \hat{\chi}_{D;L\times L}^+(\vec{x}, t_p) \\
\delta \hat{\chi}_{D;S\times L}^-(\vec{x}, t_p) & \delta \hat{F}_{D;S\times S}(\vec{x}, t_p)
\end{pmatrix}; \\
\delta \hat{\Sigma}_{D;N\times N}^{22}(\vec{x}, t_p) = \begin{pmatrix}
\delta \hat{B}_{D;L\times L}^T(\vec{x}, t_p) & \delta \hat{\chi}_{D;L\times L}^T(\vec{x}, t_p) \\
\delta \hat{\chi}_{D;S\times L}(\vec{x}, t_p) & -\delta \hat{F}_{D;S\times S}(\vec{x}, t_p)
\end{pmatrix}; \\
\delta \hat{B}_{D;L\times L}^T(\vec{x}, t_p) = \delta \hat{B}_{D;L\times L}(\vec{x}, t_p); \\
\delta \hat{F}_{D;S\times S}^T(\vec{x}, t_p) = \delta \hat{F}_{D;S\times S}(\vec{x}, t_p); \\
\delta \hat{\Sigma}_{D;N\times N}^{11}(\vec{x}, t_p) = -\left(\delta \hat{\Sigma}_{D;N\times N}^{22}(\vec{x}, t_p) \kappa\right)^{st}. 
\]

The even density terms of the boson-boson, fermion-fermion blocks \((2.8)\) are given by hermitian matrices \(\delta \hat{B}_{D;L\times L}(\vec{x}, t_p), \delta \hat{F}_{D;S\times S}(\vec{x}, t_p)\). The odd density terms in the fermion-boson, boson-fermion sections are determined by \(\delta \hat{\chi}_{D;S\times L}(\vec{x}, t_p)\) and its super-hermitian conjugate \(\delta \hat{\chi}_{D;L\times S}^+(\vec{x}, t_p)\). These self-energy densities \(\delta \hat{\Sigma}_{D;N\times N}^{11}(\vec{x}, t_p), \delta \hat{\Sigma}_{D;N\times N}^{22}(\vec{x}, t_p)\) or \(\delta \hat{\Sigma}_{D;2N\times2N}^{aa}(\vec{x}, t_p)\) act as 'hinge' fields in the SSB and can be factorized to real \(N = L + S\) eigenvalues \(\delta \lambda_{N\times N}\) or its 'Nambu' doubled form \(\delta \hat{\lambda}_{2N\times2N}\) which comprise the maximal abelian Cartan subalgebra of rank \(N\) for the \(U(L|S)\) or \(\text{Osp}(S, S|2L)\) super-group \((2.10)\). The remaining field degrees of freedom \(\hat{Q}_{N\times N}(\vec{x}, t_p), \hat{Q}_{N\times N}^2(\vec{x}, t_p)\) for the self-energy density matrices have their parameters within the ladder operators of the super-unitary algebra \(u(L|S)\). Since \(N = L + S\) real parameter fields are already contained in the eigenvalues \(\delta \hat{\lambda}_{N\times N}(\vec{x}, t_p)\), the diagonal values of \(\hat{B}_{D;mm} = 0\) \((m = 1, \ldots, L)\) and \(\hat{F}_{D;ii} = 0\) \((i = 1, \ldots, S)\) have to vanish in the generators of \(\hat{\hat{Q}}_{N\times N}(\vec{x}, t_p)\) \((2.14), \hat{Q}_{N\times N}^2(\vec{x}, t_p)\) \((2.15)\). In consequence the ladder operators with their independent fields only remain from the super-unitary \(U(L|S)\) algebra within the eigenvector matrices \(\hat{Q}_{N\times N}, \hat{Q}_{N\times N}^2\) of the block diagonal self-energy densities \(\delta \hat{\Sigma}_{D;2N\times2N}^{aa}(\vec{x}, t_p)\) \(\tilde{\kappa}\)

\[
\delta \hat{\Sigma}_{D;2N\times2N}(\vec{x}, t_p) \tilde{\kappa} = \hat{Q}_{2N\times2N}(\vec{x}, t_p) \delta \hat{\lambda}_{2N\times2N}(\vec{x}, t_p) \hat{Q}_{2N\times2N}(\vec{x}, t_p); \\
\delta \hat{\lambda}_{2N\times2N}(\vec{x}, t_p) = \delta \hat{\lambda}_{\alpha}(\vec{x}, t_p) = \text{diag}\{\delta \lambda_{N\times N}(\vec{x}, t_p); -\delta \lambda_{N\times N}(\vec{x}, t_p)\}; \\
\delta \lambda_{N\times N}(\vec{x}, t_p) = \delta \hat{\lambda}_{\alpha}(\vec{x}, t_p) = \{\delta \lambda_{B;1}, \ldots, \delta \lambda_{B;m}, \ldots, \delta \lambda_{B;L}; \delta \lambda_{F;i}, \ldots, \delta \lambda_{F;S}\}; \\
\hat{Q}_{2N\times2N}(\vec{x}, t_p) = \begin{pmatrix}
\hat{Q}_{N\times N}^{11} & 0 \\
0 & \hat{Q}_{N\times N}^{22}
\end{pmatrix}; \\
(\hat{Q}_{N\times N}^{22})^{st} = \hat{Q}_{N\times N}^{11} + \hat{Q}_{N\times N}^{11} \tilde{\kappa} = \hat{Q}_{N\times N}^{11,11}, \\
(\hat{Q}_{N\times N}^{22})^{st} = \hat{Q}_{N\times N}^{11}. 
\]
2.1 The independent field variables for the super-symmetric pair condensates

\[ \hat{Q}^{11}_{N\times N}(\vec{x}, t_p) = \exp \left\{ i \left( \hat{B}_{D:L\times L} \hat{\omega}^{+}_{D:L\times S} \hat{\mathcal{F}}_{D:S\times S} \right) \right\} ; \hat{B}_{D:L\times L} = \hat{B}_{D:L\times L} ; \]

\[ \hat{Q}^{22}_{N\times N}(\vec{x}, t_p) = \exp \left\{ i \left( -\hat{B}_{T:L\times L} \hat{\omega}^{T}_{D:L\times S} \hat{\mathcal{F}}^{+}_{D:S\times S} \right) \right\} ; \hat{\mathcal{F}}^{+}_{D:S\times S} = \hat{\mathcal{F}}_{D:S\times S} ; \]

\[ \hat{B}_{D:mm} = 0 \quad (m = 1, \ldots, L) ; \quad \hat{\mathcal{F}}_{D:ii} = 0 \quad (i = 1, \ldots, S). \]

In analogy one can diagonalize the off-diagonal blocks \( \hat{X}_{N\times N}(\vec{x}, t_p), \vec{\kappa}, \hat{X}^{+}_{N\times N}(\vec{x}, t_p) \) of the super-generator \( \hat{Y}_{2N\times 2N}(\vec{x}, t_p) \) for the pair condensates \((2.16,2.20)\). The diagonal blocks of \( \hat{Y}_{DD:2N\times 2N} \) in the anomalous sectors \((2.16,2.17)\) consist of the diagonal matrices \( \hat{X}_{DD:2N\times 2N}, \vec{\kappa}, \hat{X}^{+}_{DD:2N\times 2N} \) which are rotated with the parameters \( \hat{\xi}_{D:L\times L}, \hat{\xi}_{D:S\times S}, \hat{\xi}^{+}_{D:S\times S} \) by the matrices \( \hat{P}^{11}_{N\times N}, \hat{P}^{22}_{N\times N} \) (or \( \hat{P}^{aa}_{2N\times 2N} \)) \((2.21,2.23)\) to the generators \( \hat{X}_{N\times N}(\vec{x}, t_p), \vec{\kappa}, \hat{X}^{+}_{N\times N}(\vec{x}, t_p) \) \((2.3,2.4)\). The complex L parameters \( \vec{m} \) within the diagonal matrix \( \hat{\mathcal{F}}_{L\times L} \) describe the anomalous molecular terms and are factorized into its modulus \( |\vec{m}| \) and phase \( \varphi_{\vec{m}} \) \((2.19)\). The parameters \( \hat{\mathcal{F}}_{S\times S}(\vec{x}, t_p) \) \((2.20)\) of the fermionic degrees have to consider the anti-symmetric form \( \hat{\mathcal{F}}_{S\times S} \) of BCS terms \((2.5)\) so that one has to introduce the quaternion algebra \((2.25)\) with anti-symmetric Pauli matrix \( (\tau_{2})_{\mu\nu} \) and complex field variables \( \vec{f} \) as corresponding anti-symmetric eigenvalues for the BCS terms \((2.20)\). The rotation matrices \( \hat{P}^{aa}_{2N\times 2N} \) \((2.21,2.23)\) include the remaining field degrees of freedom of \( \hat{X}_{N\times N}(\vec{x}, t_p), \vec{\kappa}, \hat{X}^{+}_{N\times N}(\vec{x}, t_p) \) \((2.4)\) with even hermitian matrices \( \hat{\xi}_{D:L\times L}, \hat{\xi}_{D:S\times S} \) \((2.24,2.25)\) for the boson-boson, fermion-fermion parts where L diagonal real parameters \( (S/2) \) complex parameters with quaternion \( (\tau_{2})_{\mu\nu} \) have to be excluded from the ladder operators. They are already contained within the L complex eigenvalues \( \vec{m} \) or \( S/2 \) complex anti-symmetric quaternion eigenvalues \( (\tau_{2})_{\mu\nu} \vec{f} \) \(^4\).

\[ \hat{Y}_{2N\times 2N}(\vec{x}, t_p) = \hat{P}^{11}_{2N\times 2N}(\vec{x}, t_p) \hat{Y}_{DD:2N\times 2N}(\vec{x}, t_p) \hat{P}^{22}_{2N\times 2N}(\vec{x}, t_p) \]

\[ = \left( \begin{array}{cc} \hat{P}^{22}_{2N\times 2N} & \hat{P}^{11}_{2N\times 2N} \hat{X}_{DD} \hat{P}^{aa}_{2N\times 2N} \end{array} \right) ; \]

\[ \hat{Y}_{DD:2N\times 2N}(\vec{x}, t_p) = \left( \begin{array}{cc} \vec{\kappa} \hat{X}_{DD:2N\times 2N}^{+}(\vec{x}, t_p) & 0 \\ 0 & \hat{\mathcal{F}}_{S\times S}(\vec{x}, t_p) \end{array} \right) ; \]

\[ \hat{X}_{DD:2N\times 2N}(\vec{x}, t_p) = \left( \begin{array}{cc} -\hat{\mathcal{F}}_{L\times L}(\vec{x}, t_p) & 0 \\ 0 & \hat{\mathcal{F}}_{S\times S}(\vec{x}, t_p) \end{array} \right) ; \]

\[ \hat{\mathcal{F}}_{L\times L}(\vec{x}, t_p) = \text{diag}\{ \vec{m}(\vec{x}, t_p), \ldots, \vec{m}(\vec{x}, t_p), \ldots, \vec{m}(\vec{x}, t_p) \} ; \]

\[ \vec{m}(\vec{x}, t_p) = |\vec{m}(\vec{x}, t_p)| \exp\{i \varphi_{\vec{m}(\vec{x}, t_p)}\} ; \quad (\vec{m}(\vec{x}, t_p) \in \mathbb{C}_{\text{even}}) ; \]

\(^4\)The range of indices for the angular momentum degrees of freedom for the fermions and bosons is adapted from \(-s, \ldots, +s \) and \(-l, \ldots, +l \) to the range \(1, \ldots, S = 2s + 1 \) and \(1, \ldots, L = 2l + 1 \). Furthermore, two notations for the index range of the fermions are used in parallel in the remainder : (i) The first notation labels the angular momentum degrees of freedom from \(i, j = 1, \ldots, S = 2s + 1 \) especially in the density parts. (ii) The second one regards the quaternionic structure of the fermion-fermion parts concerning the anomalous sectors and has a \( 2 \times 2 \) block matrix structure with \( \mu, \nu = 1, 2 \) and \( r, r' = 1, \ldots, S/2 \) so that e.g. \( \delta \lambda_{F:2(r-1)+\mu} \) corresponds to \( \delta \lambda_{F:2(r-1)+\mu} \).
\[ \hat{f}_{S \times S}(\vec{x}, t_p) = \text{diag}\{ (\tau_2)_{\mu \nu} \hat{f}_1(\vec{x}, t_p), \ldots, (\tau_2)_{\mu \nu} \hat{f}_r(\vec{x}, t_p), \ldots, (\tau_2)_{\mu \nu} \hat{f}_{S/2}(\vec{x}, t_p) \}; \] 

(2.20)

\[ \hat{P}_{2N \times 2N}(\vec{x}, t_p) = \begin{pmatrix} \hat{P}_{N \times N}^{11}(\vec{x}, t_p) & 0 \\ 0 & \hat{P}_{N \times N}^{22}(\vec{x}, t_p) \end{pmatrix}; \quad (\hat{P}_{N \times N}^{22})^* = \hat{P}_{N \times N}^{11} = \hat{P}_{N \times N}^{11-} = \hat{P}_{N \times N}^{11-}; \] 

(2.21)

\[ \hat{P}_{N \times N}^{11}(\vec{x}, t_p) = \exp \left\{ i \left( \hat{e}_{D:LL}(\vec{x}, t_p) \hat{g}_{D:LL}^+(\vec{x}, t_p) \right) \right\}; \] 

(2.22)

\[ \hat{P}_{N \times N}^{22}(\vec{x}, t_p) = \exp \left\{ i \left( -\hat{e}_{D:LL}^T(\vec{x}, t_p) \hat{g}_{D:LL}^T(\vec{x}, t_p) \right) \right\}; \] 

(2.23)

\[ \hat{e}_{D:LL}(\vec{x}, t_p) = \hat{e}_{D:LL}^+(\vec{x}, t_p); \quad \hat{e}_{D:mm}(\vec{x}, t_p) = 0; \quad (m = 1, \ldots, L); \] 

(2.24)

\[ \hat{g}_{D:LL}(\vec{x}, t_p) = \hat{g}_{D:LL}^+(\vec{x}, t_p); \quad \hat{g}_{D:mm}(\vec{x}, t_p) = 0; \quad (r = 1, \ldots, S/2), (\mu, \nu = 1, 2); \] 

(2.25)

\[ \hat{g}_{D:mm}(\vec{x}, t_p) = \sum_{k=0}^3 (\tau_k)_{\mu \nu} \hat{g}_{D:mm}^{(k)}(\vec{x}, t_p); \quad \left( \hat{g}_{D:mm}(\vec{x}, t_p) \right)^* = \hat{g}_{D:mm}^{(k)}(\vec{x}, t_p). \] 

(2.26)

This can be accomplished by a gauge transformation \((2.28)\) of \(\hat{P}_{N \times N}^{aa}(\vec{x}, t_p)\) with a diagonal (quaternion diagonal) matrix \(\hat{P}_{N \times N}^{aa}(\vec{x}, t_p)\) which has only non-vanishing matrix elements \(\hat{e}_{D:mm} \neq 0, \hat{g}_{D:mm} \neq 0\) along the diagonals, just in opposite to \(\hat{P}_{D;DD}^{aa}(\vec{x}, t_p)\) \((2.24, 2.25)\). These diagonal, real \(\hat{e}_{D:mm}\) and hermitian \(2 \times 2\) elements \(\hat{g}_{D:mm}^{(k)}\) \((2.28)\) have to depend on the off-diagonal parameters of the ladder operators in \(\hat{P}_{N \times N}^{11}, \hat{P}_{N \times N}^{22}\) and have to be chosen with suitable dependence in such a manner that the block diagonal, gauge transformed super-matrices \(\hat{P}_{N \times N}^{aa} = \hat{P}_{D;DD}^{aa} \hat{P}_{N \times N}^{aa}\) \((2.28)\) fulfill the property of \(\sum_{\beta=1}^{N=L+S} (\partial_{\alpha \beta} \hat{P}_{\beta \alpha}^{aa}) \hat{P}_{\beta \alpha}^{1-aa} = 0 \) \((2.28)\). One has to take into account the quaternion algebra for the fermion-fermion parts in order to achieve \(\sum_{\alpha=1}^{N=L+S} (\partial_{\alpha \beta} \hat{P}_{\beta \alpha}) \hat{P}_{\beta \alpha}^{1-aa} = 0\) for diagonal elements \(\alpha\) (with ‘\(\alpha\)’ denoting a quaternion matrix element in the fermion-fermion section !)

\[ \hat{P}_{N \times N}^{aa}(\vec{x}, t_p) \rightarrow \hat{P}_{N \times N}^{aa}(\vec{x}, t_p) = \hat{P}_{D;DD}^{aa} \hat{P}_{N \times N}^{aa}(\vec{x}, t_p) \hat{P}_{N \times N}^{aa}(\vec{x}, t_p); \] 

(2.28)

\[ \hat{P}_{D;DD}^{aa}(\vec{x}, t_p) = \exp \left\{ i \left( \hat{e}_{D:mm}(\vec{x}, t_p) 0 \\ 0 \hat{g}_{D:mm}(\vec{x}, t_p) \right) \right\}; \] 

(2.29)

\[ \hat{P}_{D;DD}^{aa}(\vec{x}, t_p) = \exp \left\{ -i \left( \hat{e}_{D:mm}(\vec{x}, t_p) 0 \\ 0 \hat{g}_{D:mm}(\vec{x}, t_p) \right) \right\}; \] 

(2.30)
The coset integration measure of $Osp(S,S|2L)/U(L|S) \otimes U(L|S)$

The coset decomposition $Osp(S,S|2L)/U(L|S) \otimes U(L|S)$, as described in section 2.1 for the self-energy super-matrix $\delta \hat{\Sigma}^{ab}_{\alpha \beta}(\vec{x},t_p)$, involves a nontrivial integration measure. The corresponding super-Jacobi matrix of this transformation follows from the square root $\hat{G}^{1/2}_{Osp/U}$ of the $Osp(S,S|2L)/U(L|S)$ metric tensor $\hat{G}_{Osp/U}$ for the invariant length $(ds_{Osp/U}(\delta \lambda))^2$ (2.36). We neglect anomalies, which are caused by the anticommuting variables, and introduce the super-determinant $SDET(\hat{G}^{1/2}_{Osp/U}) = (SDET(\hat{G}^{1/2}_{Osp/U}))^{1/2}$ of this super-Jacobi-matrix $\hat{G}^{1/2}_{Osp/U}$ for the change of integration measure from $d[\sigma_D^{(0)}(t_p)] d[\delta \Sigma_{2N \times 2N}(t_p) \hat{K}]$ to $d[\sigma_D^{(0)}(t_p)] d[dQ \hat{Q}^{-1} \delta \lambda] d[T^{-1} d\hat{T}; \delta \lambda]$. This change of integration measure can be particularly obtained by diagonalizing the metric tensor $\hat{G}_{Osp/U}$ with the eigenvalues $\hat{c}_{L \times L}(\vec{x},t_p)$, $\hat{t}_{S \times S}(\vec{x},t_p)$, $(\hat{c}_m(\vec{x},t_p), \hat{t}_r(\vec{x},t_p))$ (2.16-2.20) and eigenvector matrices $\hat{P}_{N \times N}^{ab}(\vec{x},t_p)$ (2.21,2.25,2.28,2.34) of the coset matrices $\hat{X}_{N \times N}(\vec{x},t_p)$, $\hat{K} \hat{X}^+_{N \times N}(\vec{x},t_p)$ (2.22,2.25) for the independent anomalous terms

\[
\begin{align*}
(ds_{Osp/U}(\delta \lambda))^2 &= -2 \text{str}_{\alpha,\beta} \left( (\hat{T}_0^{-1} d\hat{T}_0)^{12} (\hat{T}_0^{-1} d\hat{T}_0)^{21} \delta \hat{\lambda} + \delta \hat{\lambda}_\alpha \right)^2 ; \\
(\hat{T}_0^{-1} d\hat{T}_0)^{ab}_{\alpha \beta} &= (\hat{P} \hat{T}^{-1} d\hat{P} \hat{T}^{-1})^{ab}_{\alpha \beta} - (\hat{P} \hat{Q}^{-1} d\hat{Q} \hat{P}^{-1})^{ab}_{\alpha \beta} ; \\
d[\sigma_D^{(0)}(t_p)] d[\delta \Sigma_{2N \times 2N}(t_p) \hat{K}] &= d[\sigma_D^{(0)}(t_p)] d[dQ(t_p) \hat{Q}^{-1}(t_p) ; \delta \lambda(t_p)] d[T^{-1}(t_p) d\hat{T}(t_p) ; \delta \hat{\lambda}(t_p)] .
\end{align*}
\]

Repeated application of Eq. (2.39) (for the variation $\delta \hat{B}$ of general generators $\hat{B}$ in the exponent) allows to simplify the combination $(\hat{P} \hat{T}^{-1} d\hat{T} \hat{P}^{-1})^{ab}_{\alpha \beta}$ of coset matrices to a relation (2.40) with their eigenvalues $\hat{X}_{DD}$, $\hat{K} \hat{X}^+_{DD}$, $\hat{Y}_{DD}$ for the pair condensates by an additional integration over a parameter $v \in [0,1]$ with $\hat{Y}_{DD}$ [38, 6]

\[
\exp \{ \delta \hat{B} \} \exp \{ - \hat{B} \} = - \int_0^1 dv \exp \{ v \hat{B} \} \delta \hat{B} \exp \{ - v \hat{B} \} ;
\]

(2.39)
\[(\hat{\Phi} \hat{T}^{-1} d\hat{T} \hat{\Phi}^{-1})_{\alpha\beta} = (\hat{\Phi} \exp\{\hat{Y}\} d(\exp\{-\hat{Y}\}) \hat{\Phi}^{-1})_{\alpha\beta} \quad (2.40)\]

\[-\int_0^1 dv \left( \exp\{v \hat{Y}_{DD}\} \right) d\hat{Y}' \exp\{-v \hat{Y}_{DD}\} \right)_{\alpha\beta} ;

\[d\hat{Y}' = \hat{\Phi} d\hat{Y} \hat{\Phi}^{-1} ; \quad \text{str}[d\hat{Y}' d\hat{Y}'] = \text{str}[d\hat{Y} d\hat{Y}] . \quad (2.41)\]

The integration measure of the \(\hat{\Phi}\), \(\hat{\Phi}^{-1}\) rotated, independent coset elements \(d\hat{Y}'\) \[^5\] is equivalent to the original anomalous terms within matrix \(d\hat{Y}\). Therefore, one can perform the integrations over the parameter \(v \in [0,1]\) with the eigenvalues \(\hat{Y}_{DD}\) of \(\hat{Y}\) straightforwardly to obtain the metric tensor \(\hat{G}_{Osp/U}\) for \((ds_{Osp/U}(\hat{\lambda}))^2\) \[^2.36, 2.38\]. In the following we list the results for the integration measure in terms of the diagonal coset metric tensor, determined by \(\tau_m, \hat{T}_r\), and specify in relations (2.42-2.45) the integration measure for the block diagonal self-energy densities (2.6-2.15) \(\delta\hat{\Sigma}_{D;2N \times 2N} \hat{K}\) in the coset decomposition

\[\delta\hat{\Sigma}_{D;2N \times 2N} \hat{K} = \hat{Q}_{2N \times 2N} \delta\hat{\Lambda}_{2N \times 2N} \hat{Q}_{2N \times 2N} ; \quad (2.42)\]

\[d[\hat{d}\hat{Q}^{-1}; \delta\hat{\Lambda}] = d[\delta\hat{\Sigma}_D \hat{K}] ; \quad (2.43)\]

\[d[\hat{d}\hat{Q}^{-1}; \delta\hat{\Lambda}] = d[\delta\hat{\Sigma}_D \hat{K}] = \prod_{\{\hat{x}_i, \hat{t}_p\}} \left\{ 2^{(L+S)/2} \left( \prod_{m=1}^{L} d(\delta\hat{\Lambda}_{B;mm}) \right) \left( \prod_{i=1}^{S} d(\delta\hat{\Lambda}_{F;ii}) \right) \right\} \times \quad (2.44)\]

\[\times \left\{ \prod_{m=1}^{L} \prod_{n=m+1}^{L} \left( \frac{4}{2^t} \left( \frac{d(\hat{Q}^{11} \hat{Q}^{11,-1})_{BB;mn}}{\hat{B}_{mm}} \right) \delta\hat{\Lambda}_{B;m} \delta\hat{\Lambda}_{B;m} \right) \right\} \times \quad (2.44)\]

\[\times \left\{ \prod_{i=1}^{S} \prod_{i'=i+1}^{S} \left( \frac{4}{2^t} \left( \frac{d(\hat{Q}^{11} \hat{Q}^{11,-1})_{FF;ii'}}{\hat{F}_{ii'}} \right) \delta\hat{\Lambda}_{F;ii'} \delta\hat{\Lambda}_{F;ii'} \right) \right\} \times \quad (2.44)\]

\[\times \left\{ \prod_{m=1}^{L} \prod_{i'=1}^{L} \left( \frac{1}{4} \left( \frac{d(\hat{Q}^{11} \hat{Q}^{11,-1})_{BF;mm}}{\hat{B}_{mm}} \right) \delta\hat{\Lambda}_{B;mm} \delta\hat{\Lambda}_{B;mm} \right) \right\} ; \quad (2.44)\]

\[d[\delta\hat{\Sigma}_D \hat{K}] = d[\hat{d}\hat{Q}^{-1}; \delta\hat{\Lambda}] = \prod_{\{\hat{x}_i, \hat{t}_p\}} \left\{ 2^{(L+S)/2} \left( \prod_{m=1}^{L} d(\delta\hat{\Lambda}_{B;mm}) \right) \left( \prod_{i=1}^{S} d(\delta\hat{\Lambda}_{F;ii}) \right) \right\} \times \quad (2.45)\]

\[\times \left\{ \prod_{m=1}^{L} \prod_{n=m+1}^{L} \left( \frac{4}{2^t} \left( \frac{d(\delta\hat{B}^*_{mm})}{\hat{B}_{mm}} \right) \delta\hat{\Lambda}_{B;mm} \right) \right\} \times \quad (2.45)\]

\[\times \left\{ \prod_{i=1}^{S} \prod_{i'=i+1}^{S} \left( \frac{4}{2^t} \left( \frac{d(\delta\hat{F}^*_{ii'})}{\hat{F}_{ii'}} \right) \delta\hat{\Lambda}_{F;ii'} \right) \right\} \times \quad (2.45)\]

\[\times \left\{ \prod_{m=1}^{L} \prod_{i'=1}^{L} \left( \frac{1}{4} \left( \frac{d(\delta\hat{\chi}^*_{D;mm})}{\hat{\chi}_{D;mm}} \right) \delta\hat{\Lambda}_{D;mm} \right) \right\} . \quad (2.45)\]

\[^5\]In the remainder \(\hat{\Phi}\), \(\hat{\Phi}^{-1}\) transformed coset elements are marked by an additional prime "'", as e. g. for \(d\hat{Y}' = \hat{\Phi} d\hat{Y} \hat{\Phi}^{-1}\).
2.2 The coset integration measure of $Osp(S,S|2L)/U(L|S) \otimes U(L|S)$

One has to consider that the original invariant length \((ds_{Osp/U(\hat{\delta}\lambda)})^2\) \((2.36)\) of the coset integration measure \(d[\hat{T}^{-1} d\hat{T}; \delta\lambda] \ (2.38,2.46)\) also incorporates the eigenvalues \(\delta\lambda\alpha\) of the density terms. However, the eigenvalues \(\delta\lambda\alpha\) of the densities factorize into a polynomial \(\Psi(\delta\lambda) \ (2.47)\) and separate from the coset integration \(d[\hat{T}^{-1} d\hat{T}]\) which solely depends on the field variables of \(X_{N\times N}, \tilde{K}_{X_{N\times N}}\) weighted by functions of their eigenvalues \(\tau_m, \tilde{f}_r\)

\[
d[\hat{T}^{-1} d\hat{T}; \delta\lambda] = \Psi(\delta\lambda) \ d[\hat{T}^{-1} d\hat{T}]; \tag{2.46}
\]

The actual coset integration measure \(d[\hat{T}^{-1} d\hat{T}]\) is listed in relation \((2.48)\) where the polynomial \(\Psi(\delta\lambda) \ (2.47)\) has been isolated and been shifted to the action terms, which are determined by integrations over the self-energy densities \(\delta\Sigma_{D,2N\times 2N} \tilde{K} \ (2.42,2.45)\). After their removal by integration, these self-energy densities or 'hinge' fields of the SSB yield the action term \(A_{\hat{T}} \Psi[\hat{T}]\) of the 'condensate seeds' with the source matrix \(i \hat{J}^{ab}_{\Psi;0\beta}(\tilde{\tau},t)\) for the pair condensates

\[
d[\hat{T}^{-1} d\hat{T}] = \prod_{\{\tilde{\tau},t\}} \left[ \prod_{m=1}^{L} \left( \frac{\hat{d}_{D,mm} \wedge \hat{d}_{D,mm}}{2} \frac{2 \sin \left( \frac{2}{|\tilde{c}_m|} \right)}{|\tilde{c}_m|} \right) \right]
\]

\[
\times \left[ \prod_{m=1}^{L} \prod_{n=m+1}^{L} \left( \frac{\hat{d}_{D,mm} \wedge \hat{d}_{D,mm}}{2} \frac{2 \sin \left( \frac{2 (|\tilde{c}_m| + |\tilde{c}_n|)}{|\tilde{c}_m| + |\tilde{c}_n|} \right)}{|\tilde{c}_m| + |\tilde{c}_n|} \right) \right]
\]

\[
\times \left[ \prod_{r=1}^{S/2} \frac{\hat{d}_{D,rr} \wedge \hat{d}_{D,rr}}{2} \frac{2 \sinh \left( \frac{2 |\tilde{f}_r|}{|\tilde{f}_r|} \right)}{|\tilde{f}_r|} \right]
\]

\[
\times \left[ \prod_{r=r+1}^{S/2} \prod_{r'=r+1}^{S/2} \prod_{k=0}^{3} \left( \frac{\hat{d}_{D,rr} \wedge \hat{d}_{D,rr}}{2} \frac{4 \sinh \left( \frac{|\tilde{f}_r| + |\tilde{f}_{r'}|}{|\tilde{f}_r| + |\tilde{f}_{r'}|} \right)}{|\tilde{f}_r| + |\tilde{f}_{r'}|} \right) \right]
\]

\[
\times \left[ \prod_{m=1}^{L} \prod_{r'=1}^{S/2} \frac{\hat{d}_{D,rr'} \wedge \hat{d}_{D,rr'}}{2} \frac{2 \sinh \left( \frac{|\tilde{f}_{r'}| + |\tilde{c}_m|}{|\tilde{f}_{r'}| + |\tilde{c}_m|} \right)}{|\tilde{f}_{r'}| + |\tilde{c}_m|} \right] \cdot \left( \frac{\hat{d}_{D,rr'} \wedge \hat{d}_{D,rr'}}{2} \frac{2 \sinh \left( \frac{|\tilde{f}_{r'}| - |\tilde{c}_m|}{|\tilde{f}_{r'}| - |\tilde{c}_m|} \right)}{|\tilde{f}_{r'}| - |\tilde{c}_m|} \right)^2 \right].\tag{2.48}
\]
2.3 Effective action for pair condensates with coupling coefficients of the background field

The effective actions (2.49) of coset matrices $\hat{T}(\vec{x}, t_p)$ (2.2) with their independent fields for anomalous terms in the super-generator $\hat{Y}(\vec{x}, t_p)$ follow from a gradient expansion of the super-matrix (1.47) $\tilde{M}_{x,\alpha;\beta}^a(t_p, t'_q)$ in the coherent state path integral $Z[\hat{J}, J_\psi, \dot{J}_\psi]$ (1.46). It is of central importance that the coset decomposition allows a factorization of the integration measure into density terms and fields for the pair condensates. In the following we give the result of the gradient expansion combined with the coset decomposition and classify the effective, remaining actions for anomalous fields $\hat{Y}(\vec{x}, t_p)$ (2.2) according to the parameter $N = \hbar/\Lambda (\hbar = 1/\Delta t, \Lambda = (L/\Delta x)^d)$. This parameter $N$ arises e. g. in the course of the gradient expansion of the super-determinant when one has to introduce an appropriate integration for the discrete spatial and time-like points on an underlying grid with intervals $\Delta x$ and $\Delta t$

$$Z[\hat{J}, J_\psi, \dot{J}_\psi] = \int d[\hat{T}^{-1}(\vec{x}, t_p) \ d\hat{T}(\vec{x}, t_p)] \ \exp \left\{ i \ A_{j,\psi} \ [\hat{T}] \right\} \times \exp \left\{ -A_{N^{-1}} [\hat{T}; J_\psi] - A_{N^0} [\hat{T}; J_\psi] - A_{N^+} [\hat{T}] \right\} \times \exp \left\{ -A' [\hat{T}; \hat{J}] \right\}$$

The effective action $A_{N^{-1}} [\hat{T}; J_\psi]$ (2.49) of order $N^{-1}$ is composed of the gradients (2.50) with super-matrices $\hat{Z} = \hat{T} \hat{S} \hat{T}^{-1}$ (2.51), following from the expansion of the super-determinant, and the gradients $\hat{\partial}_i \hat{T}^{-1}$, resulting from the expansion of the inverse super-matrix $\tilde{M}_{x,\alpha;\beta}^{-1,ab}(t_p, t'_q)$ (1.47) with the 'Nambu' doubled source fields $J^a(x,t_p) \ldots J^a(x,t_p)$. The combination of coset matrices $\hat{T}, \hat{T}^{-1}$ to the super-matrix $\hat{Z} = \hat{T} \hat{S} \hat{T}^{-1}$ with metric $\hat{S} = \{ \hat{S}^{a\beta} \delta_{a\beta} \}$ ($\hat{S}^{11} = +1; \hat{S}^{22} = -1$) (2.52) completely restricts the 'Nambu' doubled super-trace 'STR' (1.43) to terms of the pair condensates in the off-diagonal blocks of $\left( \hat{T}^{-1} \ (\hat{\partial}_i \hat{T}) \right)^{\alpha \neq \beta}$ (2.53) with super-trace 'str' (1.32)

$$\tilde{\partial}_i := \frac{\hbar}{\sqrt{2m}} \frac{\partial}{\partial x^i} ;$$

$$\hat{Z}(\vec{x}, t_p) = \hat{T}(\vec{x}, t_p) \hat{S} \hat{T}(\vec{x}, t_p) ;$$

$$\hat{S} = \{ \hat{S}^{a\beta} \delta_{a\beta} \} = \{ +1_{N \times N} ; -1_{N \times N} \} ;$$

$$\text{STR}_{a,\alpha \neq \beta} \left[ (\tilde{\partial}_i \hat{Z}(\vec{x}, t_p)) (\tilde{\partial}_j \hat{Z}(\vec{x}, t_p)) \right] =$$

$$= -4 \sum_{a = 1,2} \text{str}_{a,\alpha \beta} \left[ \left( \hat{T}^{-1}(\vec{x}, t_p) \ (\tilde{\partial}_i \hat{T}(\vec{x}, t_p)) \right)^{\alpha \neq \beta} \left( \hat{T}^{-1}(\vec{x}, t_p) \ (\tilde{\partial}_j \hat{T}(\vec{x}, t_p)) \right)^{\beta \neq \alpha} \right].$$

The effective coupling functions $c^{ij}(\vec{x}, t_p)$ (2.55, 2.58) for gradients of $\hat{Z}$ and $d^{ij}(\vec{x}, t_p)$ (2.59) for $(\tilde{\partial}_i \hat{T}) \hat{T}^{-1}$ are achieved from the action of 'unsaturated' gradient operators ' $\tilde{\partial}_i$ ' onto Green functions of the
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background field $\sigma_D^{(0)}(\vec{x},t_p)$ and by the average $\langle \ldots \rangle_{\sigma_D^{(0)}}$ with the corresponding background functional $Z[j_\psi; \sigma_D^{(0)}]$ (see Eqs. (2.74),(2.75) and Ref. [3] with chapter 4 and appendix B)

$$
A'_{N^{-1}}[\hat{T};J_\psi] = \frac{1}{4N} \int_C \frac{dt_p}{\hbar} \sum_{\vec{x}} c^{ij}(\vec{x},t_p) \ \text{STR}\left[(\tilde{\partial}_i \tilde{Z}(\vec{x},t_p)) \left(\tilde{\partial}_j \tilde{Z}(\vec{x},t_p)\right)\right] +
$$

$$
- \frac{1}{N} \int_C \frac{dt_p}{\hbar} \sum_{\vec{x}} \sum_{a,b=1,2} \sum_{\alpha,\beta=1} d^{ij}(\vec{x},t_p) \times
$$

$$
\times \frac{J_{\psi,\beta}(\vec{x},t_p)}{N} \left(\hat{I} \tilde{K} \left(\tilde{\partial}_i \hat{T}(\vec{x},t_p)\right) \hat{T}^{-1}(\vec{x},t_p) \left(\tilde{\partial}_j \hat{T}(\vec{x},t_p)\right) \hat{T}^{-1}(\vec{x},t_p) \hat{I}\right)^{ba} \frac{J_{\psi,\alpha}(\vec{x},t_p)}{N};
$$

$$
c^{ij}(\vec{x},t_p) = c^{(1),ij}(\vec{x},t_p) + c^{(2),ij}(\vec{x},t_p);
$$

$$
\bar{v}(\vec{x},t_p) = \bar{u}(\vec{x}) + \bar{\sigma}_D^{(0)}(\vec{x},t_p) = \frac{u(\vec{x}) + \sigma_D^{(0)}(\vec{x},t_p)}{N};
$$

$$
c^{(1),ij}(\vec{x},t_p) = -2 \left(\tilde{\partial}_i \tilde{\partial}_j \bar{v}(\vec{x},t_p)\right)_{\sigma_D^{(0)}} - \delta_{ij} \sum_{k=1}^d \left(\tilde{\partial}_k \tilde{\partial}_k \bar{v}(\vec{x},t_p)\right)_{\sigma_D^{(0)}};
$$

$$
c^{(2),ij}(\vec{x},t_p) = 2 \left(\tilde{\partial}_i \bar{v}(\vec{x},t_p) \left(\tilde{\partial}_j \bar{v}(\vec{x},t_p)\right)\right)_{\sigma_D^{(0)}} - \delta_{ij} \sum_{k=1}^d \left(\tilde{\partial}_k \bar{v}(\vec{x},t_p)\right)^2_{\sigma_D^{(0)}};
$$

$$
d^{ij}(\vec{x},t_p) = 2 \left(3 \left(\tilde{\partial}_i \bar{v}(\vec{x},t_p)\right) \left(\tilde{\partial}_j \bar{v}(\vec{x},t_p)\right) - \left(\tilde{\partial}_k \tilde{\partial}_k \bar{v}(\vec{x},t_p)\right)\right)_{\sigma_D^{(0)}}.
$$

The action $A'_{N^0}[\hat{T};J_\psi]$ of order $N^0$ does not contain coupling parameters as $c^{ij}(\vec{x},t_p)$, $d^{ij}(\vec{x},t_p)$ of the background field apart from the average of $\langle \sigma_D^{(0)}(\vec{x},t_p) \rangle_{\sigma_D^{(0)}}$ for an effective potential which modifies the trap potential $u(\vec{x})$. It is also composed of a part following from the gradient expansion of the super-determinant and a part for the coherent BEC wavefunction with the ’Nambu’ doubled bilinear source fields $J_{\psi,\alpha}^{(0)}(\vec{x},t_p) \ldots J_{\psi,\alpha}^{(a)}(\vec{x},t_p)$

$$
A'_{N^0}[\hat{T};J_\psi] = -\frac{1}{2} \int_C \frac{dt_p}{\hbar} \sum_{\vec{x}} \left\{ \text{STR}\left[\hat{T}^{-1}(\vec{x},t_p) \hat{S} \left(\hat{E}_p \hat{T}(\vec{x},t_p)\right) + \hat{T}^{-1}(\vec{x},t_p) \left(\tilde{\partial}_i \tilde{\partial}_i \hat{T}(\vec{x},t_p)\right)\right]\right\} +
$$

$$
+ \left(\frac{u(\vec{x}) - \mu_0 - \varepsilon_p + \langle \sigma_D^{(0)}(\vec{x},t_p) \rangle_{\sigma_D^{(0)}}}{\hbar} \right) \text{STR}\left[\left(\hat{T}^{-1}(\vec{x},t_p)\right)^2 - \hat{1}_{2N \times 2N}\right]\right\} +
$$

$$
- \frac{1}{2} \int_C \frac{dt_p}{\hbar} \sum_{\vec{x}} \sum_{a,b=1,2} \sum_{\alpha,\beta=1} \frac{\tilde{J}^{+a,b}_{\psi,\beta}(\vec{x},t_p)}{N} \left[\hat{I} \tilde{K} \left(\tilde{\partial}_i \hat{T}(\vec{x},t_p)\right) \hat{T}^{-1}(\vec{x},t_p) +
$$

$$
+ \hat{T}(\vec{x},t_p) \hat{S} \hat{T}^{-1}(\vec{x},t_p) \left(\hat{E}_p \hat{T}(\vec{x},t_p)\right) \hat{T}^{-1}(\vec{x},t_p) +
$$

$$
- 2 \left(\tilde{\partial}_i \hat{T}(\vec{x},t_p)\right) \hat{T}^{-1}(\vec{x},t_p) \left(\tilde{\partial}_i \hat{T}(\vec{x},t_p)\right) \hat{T}^{-1}(\vec{x},t_p) \hat{I}\right]^{ba} \frac{J_{\psi,\alpha}^{(a)}(\vec{x},t_p)}{N}.
$$
The action $\mathcal{A}_{N+1}'[\hat{T}]$ (2.61) of order $N+1$ does not involve any gradients and has completely different properties for the variation of classical field solutions, due to the additional metric $\eta_p$ in the time contour integral

$$
\mathcal{A}_{N+1}'[\hat{T}] = \frac{N}{2} \int_C \frac{dt_p}{\hbar} \eta_p \sum_{\vec{x}} \text{STR} \left[ \left( \hat{T}^{-1}(\vec{x}, t_p) \right)^2 - \hat{1}_{2N \times 2N} \right].
$$

(2.61)

According to the additional contour metric $\eta_p$ in (2.61), the two branches of the time contour integral in $\mathcal{A}_{N+1}'[\hat{T}]$ are added whereas the two branches of time contour integrals in $\mathcal{A}_{N-1}'[\hat{T}; J\psi]$ (2.54), $\mathcal{A}_{N0}'[\hat{T}; J\psi]$ (2.60) are subtracted. Therefore, the variation $\delta \hat{Y}(\vec{x}, t_p)$ (2.62, 2.64) of classical fields in $\mathcal{A}_{N+1}'[\hat{T}]$ (2.61) has its first contribution in the second order variation with $\delta \hat{Y}(\vec{x}, t_p)$ for the independent, anomalous fields $y_\kappa(\vec{x}, t_p)$ in $\hat{Y}(\vec{x}, t_p) = y_\kappa(\vec{x}, t_p) \hat{Y}(\kappa)$ with coset super-generators $\hat{Y}(\kappa)$ (concerning variation of actions with contour time integrals in coherent state path integrals see Refs. [49, 50]).

$$
\hat{Y}(\vec{x}, t_p = \pm) = \hat{Y}(\vec{x}, t) + \delta \hat{Y}(\vec{x}, t_p = \pm) = \hat{Y}(\vec{x}, t) \pm \frac{1}{2} \delta \hat{Y}(\vec{x}, t);
$$

(2.62)

$$
\hat{Y}(\vec{x}, t_p = \pm) = y_\kappa(\vec{x}, t_p = \pm) \hat{Y}(\kappa);
$$

(2.63)

$$
y_\kappa(\vec{x}, t_p = \pm) = y_\kappa(\vec{x}, t) + \delta y_\kappa(\vec{x}, t_p = \pm) = y_\kappa(\vec{x}, t) \pm \frac{1}{2} \delta y_\kappa(\vec{x}, t).
$$

(2.64)

The variations of $\mathcal{A}_{N-1}'[\hat{T}; J\psi]$ (2.54), $\mathcal{A}_{N0}'[\hat{T}; J\psi]$ (2.60) already contribute in first order of $\delta \hat{Y}(\vec{x}, t_p = \pm)$ (2.62, 2.64) and allow for classical field solutions following from first order variations to a stationary phase in the coherent state path integral (2.49). The second and all higher even order variations of $\mathcal{A}_{N+1}'[\hat{T}]$ (2.61) modify these classical, first order variational solutions of $\mathcal{A}_{N-1}'[\hat{T}; J\psi]$, $\mathcal{A}_{N0}'[\hat{T}; J\psi]$ and can be regarded as general fluctuation terms with universal properties, entirely determined by the symmetries of the coset decomposition for the anomalous fields. This property of contributing only from second and higher even order variations with $\delta \hat{Y}(\vec{x}, t_p = \pm)$ also holds for the coset integration measure (2.48) and causes the inconsistent treatment in comparison to the other main actions $\mathcal{A}_{N-1}'[\hat{T}; J\psi]$ (2.54), $\mathcal{A}_{N0}'[\hat{T}; J\psi]$ (2.60). The transformation with the inverse square root $\hat{G}_{\text{Osp/U}}^{-1/2}$ of the coset metric tensor removes this artificial problem and yields Euclidean path integration measures for the independent, anomalous fields. However, one obtains a different dependence of the pair condensate fields in the actions $\mathcal{A}_{N-1}'[\hat{T}; J\psi]$ (2.54), $\mathcal{A}_{N0}'[\hat{T}; J\psi]$ (2.60), according to the transformation with the super-Jacobi matrix $\hat{G}_{\text{Osp/U}}^{-1/2}$. The functional dependence of anomalous fields is also changed by this transformation in the action term $\mathcal{A}_{N+1}'[\hat{T}]$ (2.61) which, however, cannot be eliminated as the coset integration measure (2.48). In spite of the transformation with the inverse square root of the coset metric tensor, the action term $\mathcal{A}_{N+1}'[\hat{T}]$ (2.61) only allows non-vanishing variations with $\delta \hat{Y}(\vec{x}, t_p = \pm)$ in second and higher even orders and modifies the classical solutions of the first order variations of $\mathcal{A}_{N-1}'[\hat{T}; J\psi]$, $\mathcal{A}_{N0}'[\hat{T}; J\psi]$ by universal fluctuations. These universal fluctuations are solely determined by the coset decomposition $\text{Osp}(0|2L)/U(L|S) \otimes U(L|S)$, due to the absence of any background field dependencies.

The pair condensate action term $\mathcal{A}_{J\psi}[\hat{T}]$ (2.65) in $Z[\hat{J}, J\psi, \hat{J}\psi]$ (2.49) arises from the integration of the quadratic self-energy density action $\mathcal{A}_2[\hat{T}, \delta \hat{\Sigma}_D; \hat{J}, \hat{J}\psi]$ (2.66) over the independent density fields of $\delta \hat{\Sigma}_D \hat{K}$ (2.45) with inclusion of the polynomial $\mathfrak{P}(\delta \hat{\lambda})$ (2.47) of the eigenvalues $\delta \hat{\lambda}$ (2.11, 2.12) for the
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density super-matrices $\delta \Sigma_{D}^{11}$ or $\delta \Sigma_{D}^{22}$ $\tilde{\kappa}$ (2.6.29). The eigenvalues $\delta \lambda_{\alpha}$ can be discerned as the parameters or variables of the $U(L|S)$ related density terms $\delta \Sigma_{D}^{11}$, $\delta \Sigma_{D}^{22}$ $\tilde{\kappa}$ (2.6.2.15) within the characteristic eigenvalue equations of the super-determinants (2.6.7, 2.6.8) so that the integration measure $d[\delta \Sigma_{D} \tilde{K}]$ (2.4.25) with additional polynomial $\Psi(\delta \lambda(\delta \Sigma_{D} \tilde{K}))$ (2.4.7) can be used to specify the effective pair ’condensate seed’ action $A_{i_{\psi\psi}}[\bar{T}]$ (2.6.3). Alternatively, the factorization of $\delta \Sigma_{D} \tilde{K}$ into eigenvalues $\delta \lambda_{\alpha}$ and eigenvectors $\hat{Q}_{\alpha\beta}^{aa}$ (2.6.1.15) can be applied to determine the action $A_{i_{\psi\psi}}[\bar{T}]$ (2.6.5) after integration of $A_{2}[\bar{T}, \hat{Q}^{-1} \delta \lambda \hat{Q}; i_{\psi\psi}]$ (2.6.9) over the eigenvalues and eigenvectors with inclusion of $\Psi(\delta \lambda)$ (2.6.7).

The latter method of factorization with eigenvalues allows to disentangle the integrations with properties of Vandermonde matrices and Gaussian weights for orthogonal Hermite- (or related Laguerre) polynomials (47).

$$\exp \left\{ i A_{f_{\psi\psi}}[\bar{T}] \right\} = \int d[\delta \Sigma_{D}(\bar{x}, t_{p}) \tilde{K}] \; \Psi(\delta \lambda(\bar{x}, t_{p})) \exp \left\{ i A_{2}[\bar{T}, \delta \Sigma_{D}; i_{\psi\psi}] \right\} =$$ (2.6.5)

$$= \int d[\hat{Q}(\bar{x}, t_{p}) \hat{Q}^{-1}(\bar{x}, t_{p}) \delta \lambda(\bar{x}, t_{p})] \; \Psi(\delta \lambda(\bar{x}, t_{p})) \exp \left\{ i A_{2}[\bar{T}, \hat{Q}^{-1} \delta \lambda \hat{Q}; i_{\psi\psi}] \right\} ;$$

$$A_{2}[\bar{T}, \delta \Sigma_{D}; i_{\psi\psi}] = \frac{1}{4h V_{0}} \int_{C} dt_{p} \sum_{\bar{x}} \left\{ \text{STR} \left[ \delta \Sigma_{D;2N \times 2N}(\bar{x}, t_{p}) \tilde{K} \delta \Sigma_{D;2N \times 2N}(\bar{x}, t_{p}) \tilde{K} \right] + \right. \right.$$

$$- 2 \; \text{STR} \left[ i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \tilde{K} \hat{T}(\bar{x}, t_{p}) \delta \Sigma_{D;2N \times 2N}(\bar{x}, t_{p}) \tilde{K} \hat{T}^{-1}(\bar{x}, t_{p}) \right] + \right.$$

$$+ \; \text{STR} \left[ i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \tilde{K} i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \tilde{K} \right] \right\} ;$$

$$\text{sdet} \left\{ \delta \Sigma_{D;11}^{11} - \delta \lambda \; \delta \alpha_{\beta} \right\} = 0 ; \quad \text{sdet} \left\{ \delta \Sigma_{D;11}^{22} \tilde{\kappa} - (- \delta \lambda) \; \delta \alpha_{\beta} \right\} = 0 ;$$ (2.6.7)

$$\delta \Sigma_{D;11}^{11} = - \left( \delta \Sigma_{D;11}^{22} \right)^{st} ;$$ (2.6.8)

$$A_{2}[\bar{T}, \hat{Q}^{-1} \delta \lambda \hat{Q}; i_{\psi\psi}] =$$ (2.6.9)

$$= \frac{1}{4h V_{0}} \int_{C} dt_{p} \sum_{\bar{x}} \text{STR} \left[ (\delta \Sigma(\bar{x}, t_{p}) - i \hat{J}_{\psi\psi}(\bar{x}, t_{p})) \tilde{K} \left( \delta \Sigma(\bar{x}, t_{p}) - i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \right) \tilde{K} \right]$$

$$= \frac{1}{4h V_{0}} \int_{C} dt_{p} \sum_{\bar{x}} \left\{ 2 \text{str} \left[ (\delta \lambda_{N \times N}(\bar{x}, t_{p}))^{2} \right] + \right. \right.$$

$$- 2 \; \text{STR} \left[ i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \tilde{K} \hat{T}(\bar{x}, t_{p}) \hat{Q}^{-1}(\bar{x}, t_{p}) \delta \lambda(\bar{x}, t_{p}) \hat{Q}(\bar{x}, t_{p}) \hat{T}^{-1}(\bar{x}, t_{p}) \right] + \right.$$

$$+ \; \text{STR} \left[ i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \tilde{K} i \hat{J}_{\psi\psi}(\bar{x}, t_{p}) \tilde{K} \right] \right\} .$$

It remains to outline the averaging procedure $\langle \ldots \rangle_{\sigma_{D}^{(0)}}$ of the coupling coefficients $c^{ij}(\bar{x}, t_{p})$, $d^{ij}(\bar{x}, t_{p})$ with the generating functional $Z[j_{\psi}; \sigma_{D}^{(0)}]$ (2.7.4) of the background field $\sigma_{D}^{(0)}(\bar{x}, t_{p})$. Aside from the
quadratic term of \( \sigma^{(0)}(\vec{x}, t_p) \) following from the HST’s, the Hamilton operator \( \hat{\mathcal{H}}[\sigma^{(0)}] \) (2.70) specifies the determinant and the coherent BEC-wavefunction parts with the source fields \( j_{\psi;0}(\vec{x}, t_p) \) in \( Z[j_{\psi}; \sigma^{(0)}] \) (2.74). If the number \( L = 2l + 1 \), \( (l = 0, 1, 2, \ldots) \) of bosonic angular momentum degrees of freedom exceeds those of fermionic angular momentum degrees of freedom \( (L > S = 2s + 1), (s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots) \), the determinant of the operator \( \hat{\mathcal{H}}[\sigma^{(0)}] \) (2.70) appears in the denominator \( \det(\hat{\mathcal{H}}[\sigma^{(0)}]) \) with a power \( (L - S) > 0 \). In this extraordinary case \( (L - S) > 0 \) combined with attractive interactions \( V_0 < 0 \), the background generating functional \( Z[j_{\psi}; \sigma^{(0)}] \) (2.74) may describe the experimentally observable, considerable increase of the coherent bosonic BEC-wavefunctions towards the collapse, due to the appearance of effective zero eigenvalues of \( \hat{\mathcal{H}}[\sigma^{(0)}] \) in the propagation with \( \det(\hat{\mathcal{H}}[\sigma^{(0)}]) \) in \( Z[j_{\psi}; \sigma^{(0)}] \) [51, 52]. We list in relations (2.70-2.73) the Hamilton operator \( \hat{\mathcal{H}}[\sigma^{(0)}] \) (2.70), its corresponding Green function \( \hat{g}^{(0)}[\sigma^{(0)}] \) (2.71) and the definitions of the trace ‘tr’ (2.72) and unit operator ‘\( \hat{1} \)’ (2.73) in the considered Hilbert space with the complete set of states concerning spatial points and the contour time (compare with Ref. [6], chapter 4.2)

\[
\hat{\mathcal{H}}[\sigma^{(0)}] = \left( \dot{\hat{\rho}} - \hat{E}_p - i \varepsilon_p + \frac{\hat{\vec{p}}^2}{2m} + u(\vec{r}) - \mu_0 \right) + \hat{\sigma}^{(0)}; \quad \hat{g}^{(0)}[\sigma^{(0)}] = \left( \hat{\mathcal{H}}[\sigma^{(0)}] \right)^{-1} = \left( \dot{\hat{\rho}} - \hat{E}_p - i \varepsilon_p + \frac{\hat{\vec{p}}^2}{2m} + u(\vec{r}) - \mu_0 \right) + \hat{\sigma}^{(0)}; \quad \text{tr} = \int \frac{dt_p}{\hbar} \eta_p \sum_{\vec{x}} N \langle \vec{x}, t_p | \ldots | \vec{x}, t_p \rangle \quad \text{tr} \left( \ldots \right) \quad \dot{\hat{1}} = \int \frac{dt_p}{\hbar} \frac{\dot{\rho}_p}{\hbar} \sum_{\vec{x}} N | \vec{x}, t_p \rangle \langle \vec{x}, t_p | \quad \dot{\hat{1}} \quad \text{Z}[j_{\psi}; \sigma^{(0)}] = \int d[\sigma^{(0)}(\vec{x}, t_p)] \exp \left\{ \frac{i}{2\hbar V_0} \int_C dt_p \sum_{\vec{x}} \sigma^{(0)}_D(\vec{x}, t_p) \sigma^{(0)}_D(\vec{x}, t_p) \right\} \exp \left\{ - (L - S) \text{tr} \left[ \ln \left( \dot{\hat{\rho}} - \hat{E}_p - i \varepsilon_p + \frac{\hat{\vec{p}}^2}{2m} + u(\vec{r}) - \mu_0 \right) + \hat{\sigma}^{(0)} \right] \right\} \times \exp \left\{ \sum_{\alpha=1}^{N=L+S} \frac{1}{N} \langle j_{\psi;\alpha} | \hat{\rho} \hat{g}^{(0)}[\sigma^{(0)}] \hat{\rho} | j_{\psi;\alpha} \rangle \right\}
\]

The averaging procedure \( \langle \ldots \rangle_{\sigma^{(0)}} \) (2.75) for the coupling coefficients \( c^{ij}(\vec{x}, t_p) \), \( d^{ij}(\vec{x}, t_p) \) has therefore to be performed with the generating function \( Z[j_{\psi}; \sigma^{(0)}] \) (2.74) of the background field \( \sigma^{(0)}(\vec{x}, t_p) \) according
to the following relation

\[
\left\langle \left( \text{functional of } \sigma_D^{(0)}(\vec{x}, t_p) \text{ with gradient terms} \right) \right\rangle_{\sigma_D^{(0)}} = \tag{2.75}
\]

\[
= \int d[\dot{\sigma}_D^{(0)}(\vec{x}, t_p)] \exp \left\{ \frac{i}{2\hbar V_0} \int_C dt_p \sum_{\vec{x}} \sigma_D^{(0)}(\vec{x}, t_p) \sigma_D^{(0)}(\vec{x}, t_p) \right\} \times \exp \left\{ - (L - S) \text{tr} \ln \left( \hat{\eta} \left( - \hat{E}_p - i \varepsilon_p + \frac{\hat{p}_p^2}{2m} + u(\vec{x}) - \mu_0 + \sigma_D^{(0)}(\vec{x}, t_p) \right) \right) \right\} \times \left( \text{functional of } \sigma_D^{(0)}(\vec{x}, t_p) \text{ with gradient terms} \right)
\]

Instead of functional averaging by $Z[j_\psi; \hat{\sigma}_D^{(0)}]$ according to Eq. (2.75), one can also apply a saddle point equation or first order variation with the background field in order to obtain a mean field solution for $\sigma_D^{(0)}(\vec{x}, t_p)$. This mean field solution can then be substituted for the functional dependence of the coupling coefficients $c^{ij}(\vec{x}, t_p)$, $d^{ij}(\vec{x}, t_p)$ on the background field according to the defining relations (2.55, 2.59). One can expect a good approximation by this mean field solution because the generating function $Z[j_\psi; \hat{\sigma}_D^{(0)}]$ is only determined by the background field $\sigma_D^{(0)}(\vec{x}, t_p)$ which is itself related to the difference of boson-boson and fermion-fermion densities due to the U(1) symmetries in $Z[j_\psi; \hat{\sigma}_D^{(0)}]$.

\[
0 \equiv \frac{i}{V_0} \frac{1}{N} \sigma_D^{(0)}(\vec{x}, t_p) - (L - S) \left[ - \hat{E}_p - i \varepsilon_p + \frac{\hat{p}_p^2}{2m} + u(\vec{x}) - \mu_0 + \sigma_D^{(0)}(\vec{x}, t_p) \right]^{-1}_{\vec{x}, \vec{x}}(t_p + \delta t_p, t_p + \delta t_p') + \frac{i}{\hbar} \int_C \sum_{\alpha=1}^{N=L+S} \sum_{\vec{g}_1, \vec{g}_2} \left( \text{functional of } \sigma_D^{(0)}(\vec{x}, t_p) \text{ with gradient terms} \right)
\]

\[
\times \quad \dot{j}_{\psi, \alpha}(\vec{g}_2, t^{(2)}_{\vec{q}_2}) \quad \left[ - \hat{E}_p - i \varepsilon_p + \frac{\hat{p}_p^2}{2m} + u(\vec{x}) - \mu_0 + \sigma_D^{(0)}(\vec{x}, t_p) \right]^{-1}_{\vec{g}_2, \vec{x}}(t_{\vec{q}_2}, t_p + \delta t_p') \times
\]

\[
\times \quad \left[ - \hat{E}_p - i \varepsilon_p + \frac{\hat{p}_p^2}{2m} + u(\vec{x}) - \mu_0 + \sigma_D^{(0)}(\vec{x}, t_p) \right]^{-1}_{\vec{x}, \vec{g}_1}(t_p + \delta t_p, t^{(1)}_{\vec{q}_1}) \quad j_{\psi, \alpha}(\vec{g}_1, t^{(1)}_{\vec{q}_1}). \tag{2.76}
\]

### 2.4 Scaling of physical parameters and quantities to dimensionless values and fields

A typical property of classical equations concerns the re-scaling of physical parameters and quantities to dimensionless values and fields, as e.g. in the Gross-Pitaevskii or nonlinear Schrödinger equation. In the considered, effective coherent state path integral $Z[\hat{\mathcal{A}}, \hat{\mathcal{J}}]$ (2.49), we therefore scale the actions $\mathcal{A}_{N+1}^'(T; J_{\psi})$ (2.54), $\mathcal{A}_{N}^'(T; J_{\psi})$ (2.60) and $\mathcal{A}_{N+1}^'(T)$ (2.61) with the pair condensate fields in dependence on discrete spatial and time-like coordinates to dimensionless values. In consequence, one can perform the first and higher order variations of the actions (under the presumed Euclidean path integration fields)
in classical correspondence to the coherent state path integral which represents many-body quantum mechanics. We list in Eq. (2.77) the parameters $\Omega$ and $N_x$, which indicate the maximum energy $\hbar \Omega$ and number of discrete spatial points, and combine them to the parameter $N$ which inevitably appears with the space and time contour integrals. Furthermore, we have to scale all energy parameters and potentials to dimensionless quantities with the parameter $N$

$$\Omega = 1/\Delta t, \quad N_x = (L/\Delta x)^d \rightarrow N = h\Omega N_x;$$

$$\varepsilon_p, \mu_0, V_0, u(\vec{x}) \rightarrow \tilde{\varepsilon}_p = \varepsilon_p/N, \quad \tilde{\mu}_0 = \mu_0/N, \quad \tilde{V}_0 = V_0/N, \quad \tilde{u}(\vec{x}) = u(\vec{x})/N;$$

$$\sigma^{(0)}(\vec{x}, t_p) \rightarrow \tilde{\sigma}^{(0)}(\vec{x}, t_p) = \sigma^{(0)}(\vec{x}, t_p)/N.$$  

(2.77)

(2.78)

(2.79)

The dimensionless, scaled quantities are denoted by the additional symbol ‘$\tilde{\cdot}$’ above the corresponding, original physical parameter or physical quantity symbol. Similarly, the contour time derivative $\dot{t}_p = t_0 \hat{t}_p$, the contour time derivative $\dot{E}_p = N \tilde{E}_p$ and the spatial coordinates $\vec{x} = x_0 \hat{x}$ with their gradients $\partial_i = \hat{\partial}_i/x_0$ are scaled by the parameters $t_0 = h/N, x_0 = t_0 \cdot 1/(2N)^{1/2}$ to dimensionless quantities

$$\dot{E}_p = \hbar \frac{\partial}{\partial t_p}; \quad \hbar \omega_p \rightarrow \frac{\partial}{\partial t_p} = \dot{E}_p/N = \dot{\bar{E}}_p = \dot{\bar{t}}_p; \quad \bar{\omega}_p = \hbar \omega_p/N;$$

$$t_0 = h/N; \quad \frac{\hbar^2}{2m} \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} \rightarrow \left( \frac{\hbar^2}{2N} \right) \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} = x_0^2 \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} = \tilde{\vec{x}} = \vec{x}/x_0;$$

$$d^4x/L^d \rightarrow (x_0/L)^d d^4\tilde{x}; \quad x_0 = \left( \frac{\hbar^2}{2m} \right)^{1/2} t_0 \left( \frac{1}{(2N)^{1/2}} \right)^{1/2}.$$  

(2.80)

(2.81)

(2.82)

(2.83)

Application of (2.77-2.83) for the re-scaling of $A'_{N-1}[\tilde{T}; J_\psi]$ (2.54), $A'_{N\psi}[\tilde{T}; J_\psi]$ (2.60), $A'_{N+1}[\tilde{T}]$ (2.61) yields the action $A^{(d)}[\check{Z}, \check{T}; \check{J}_\psi]$ (2.84) with Lagrangian $\mathcal{L}^{(d)}[\check{Z}, \check{T}; \check{J}_\psi]$ (2.85) for the anomalous fields in the coset super-generator $\check{Y}(\vec{x}, \hat{t}_p)$. The action term $A_{J_{\psi\psi}}[\tilde{T}]$ in (2.84) creates these anomalous fields from the vacuum state through the ‘condensate seed matrix’ $J_{\psi\psi;\alpha\beta}^{a\neq b}(\vec{x}, t_p)$. However, instead of a detailed creation process by $J_{\psi\psi;\alpha\beta}^{a\neq b}(\vec{x}, t_p)$, we simply assume suitable, initial conditions of the pair condensate fields in $\check{Y}(\vec{x}, \hat{t}_p)$ whose dynamics are determined by the action $A^{(d)}[\check{Z}, \check{T}; \check{J}_\psi]$ (2.84) or $\mathcal{L}^{(d)}[\check{Z}, \check{T}; \check{J}_\psi]$ (2.85)

$$Z[\check{\Lambda}, J_\psi, i\hat{J}_{\psi\psi}] = \int d[\tilde{T}^{-1}(\vec{x}, \hat{t}_p) d\tilde{T}(\vec{x}, \hat{t}_p)] \exp \left\{ i A_{J_{\psi\psi}}[\tilde{T}] \right\} \times \exp \left\{ - A'[\tilde{T}; \hat{\check{\Lambda}}] \right\}$$

$$\times \exp \left\{ - A^{(d)}[\check{Z}, \check{T}; \check{J}_\psi] \right\};$$

$$A^{(d)}[\check{Z}, \check{T}; \check{J}_\psi] = \int_C d\hat{t}_p \int d^{d}\tilde{x} \left( \frac{x_0}{L} \right)^d \mathcal{L}^{(d)}[\check{Z}, \check{T}; \check{J}_\psi].$$  

(2.84)

(2.85)

The action $A'[\tilde{T}; \hat{\check{\Lambda}}]$ in $Z[\check{\Lambda}, J_\psi, i\hat{J}_{\psi\psi}]$ (2.84) specifies the observable quantities by differentiation with respect to $\hat{J}_{\psi\psi;\alpha\beta}^{a\neq b}(t_p, t'_p)$ (compare section 4.2) which has afterwards to be set to zero. Therefore, $A'[\tilde{T}; \hat{\check{\Lambda}}]$ cannot effect the dynamics of the pair condensate fields as the action $A^{(d)}[\check{Z}, \check{T}; \check{J}_\psi]$ (2.85).
Relation (2.86) finally contains the complete, re-scaled Lagrangian $\mathcal{L}^{(d)}[\tilde{Z}, \tilde{T}; \tilde{J}_\psi]$ whose dependencies on pair condensates have to be modified by the transformation of the super-Jacobi matrix $\tilde{G}_{\text{Osp}/U}^{-1/2}$ from the coset metric tensor; in consequence, the nontrivial coset integration measure $d[\tilde{T}^{-1}(\tilde{x}, t_p) d\tilde{T}(\tilde{x}, t_p)]$ (2.48) in (2.51) is eliminated for Euclidean path integration fields as the new, independent anomalous field variables in $Z[\tilde{J}, J_\psi, i\tilde{J}_\psi]$ (2.54) (2.55) (see following section 3).

$$\mathcal{L}^{(d)}[\tilde{Z}, \tilde{T}; \tilde{J}_\psi] = \frac{\epsilon^{ij}(\tilde{x}, \tilde{t}_p)}{4} \text{STR} \left[ \left( \partial_i \tilde{Z}(\tilde{x}, \tilde{t}_p) \right) \left( \partial_j \tilde{Z}(\tilde{x}, \tilde{t}_p) \right) \right] +$$

$$- \frac{1}{2} \text{STR} \left[ \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \tilde{S} \tilde{E}_p \tilde{T}(\tilde{x}, \tilde{t}_p) \right] \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \left( \partial_i \tilde{T}(\tilde{x}, \tilde{t}_p) \right) \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \left( \partial_j \tilde{T}(\tilde{x}, \tilde{t}_p) \right) +$$

$$- \frac{1}{2} \left( \tilde{u}(\tilde{x}) - \tilde{\mu}_0 - \tilde{\varepsilon}_p + \left( \tilde{\sigma}_D^{(0)}(\tilde{x}, \tilde{t}_p) \right) \right) \text{STR} \left[ \left( \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \right)^2 - \hat{I}_{2N\times2N} \right] +$$

$$\times \left[ \tilde{J}^{+,b}_{\psi;\alpha}(\tilde{x}, \tilde{t}_p) \left[ \tilde{I} \tilde{K} \left( \partial_i \tilde{T}(\tilde{x}, \tilde{t}_p) \right) \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \left( \partial_j \tilde{T}(\tilde{x}, \tilde{t}_p) \right) \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \tilde{I} \right]^{ba}_{\beta\alpha} J^a_{\psi;\alpha}(\tilde{x}, \tilde{t}_p) +$$

$$- \frac{1}{2} \left[ \tilde{J}^{+,b}_{\psi;\alpha}(\tilde{x}, \tilde{t}_p) \left[ \tilde{I} \tilde{K} \tilde{T}(\tilde{x}, \tilde{t}_p) \tilde{S} \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \left( \tilde{E}_p \tilde{T}(\tilde{x}, \tilde{t}_p) \right) \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \tilde{I} \right]^{ba}_{\beta\alpha} J^a_{\psi;\alpha}(\tilde{x}, \tilde{t}_p) +$$

$$+ \frac{\eta_p}{2} \text{STR} \left[ \left( \tilde{T}^{-1}(\tilde{x}, \tilde{t}_p) \right)^2 - \hat{I}_{2N\times2N} \right] .$$

3 Classical field equations with Euclidean path integration variables

3.1 General symmetry considerations for the transformation to Euclidean variables

It is the aim of this section to transform the '\(\hat{\mathcal{P}}\), \(\hat{\mathcal{P}}^{-1}\)' rotated derivative \(\hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1}\) involving sine-(sinh-) functions of eigenvalues to a Euclidean form (\(\partial \hat{\mathcal{Z}}^{ab}_{\alpha\beta}\)) (3.1) (compare appendix A and C in Ref. [6]). The general symbolic derivative '\(\partial\)', appearing in this section, representatively replaces a partial, spatial gradient '\(\partial_i\)' or time contour derivative '\(\partial_{t_p}\)', a variation symbol '\(\delta\)' for stationary phases or a total derivative '\(d\)'. The transformed 'Nambu' doubled super-matrix (\(\partial \hat{\mathcal{Z}}^{ab}_{\alpha\beta}\)) (3.1) of Euclidean form is composed of four sub-super-matrices with densities (\(\partial \hat{\mathcal{Y}}^{11}_{\alpha\beta}\), (\(\partial \hat{\mathcal{Y}}^{22}_{\alpha\beta}\) and anomalous terms (\(\partial \hat{X}_{\alpha\beta}\), \(\tilde{\kappa} (\partial \hat{X}^{+}_{\alpha\beta})\))

$$\left( \partial \hat{\mathcal{Z}}^{ab}_{\alpha\beta} \right) = - \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)_{\alpha\beta}^{ab} = \left( \begin{array}{c} \left( \partial \hat{\mathcal{Y}}^{11}_{\alpha\beta} \right) \\ \tilde{\kappa} \left( \partial \hat{X}^{+}_{\alpha\beta} \right) \end{array} \right) \left( \begin{array}{c} \left( \partial \hat{\mathcal{Y}}^{22}_{\alpha\beta} \right) \\ \partial \hat{X}_{\alpha\beta} \end{array} \right) .$$

Apart from the dependence of the densities (\(\partial \hat{\mathcal{Y}}^{11}_{\alpha\beta}\), (\(\partial \hat{\mathcal{Y}}^{22}_{\alpha\beta}\) on the pair condensate fields in (\(\partial \hat{X}_{\alpha\beta}\), \(\tilde{\kappa} (\partial \hat{X}^{+}_{\alpha\beta})\), the 'Nambu' doubled super-matrix (\(\partial \hat{\mathcal{Z}}^{ab}_{\alpha\beta}\)) (3.1) has similar symmetries between matrix
entries as the self-energy $\delta \tilde{\Sigma}_{ab}^{\alpha \beta}(\bar{x}, \bar{t}) \tilde{K}$ and is also confined to taking values within the ortho-symplectic super-algebra $\text{osp}(S,S|2L)$. The super-matrix $(\partial \hat{X}_{\alpha \beta})$ and its super-hermitian conjugate $\tilde{\kappa} (\partial \hat{X}_{\alpha \beta}^{+})$ consist of the symmetric, complex, even matrices $(\partial \hat{b}_{mn})$, $(\partial \hat{b}_{mn}^{+})$ for the molecular pair condensates and the anti-symmetric, complex, even matrices $(\partial \hat{a}_{\tau \mu, r'}(\nu))$, $(\partial \hat{a}_{\tau \mu, r'}^{+}(\nu))$ for the BCS pair condensate terms $(3.3)$. This is in accordance with the $N \times N$ super-matrices $\hat{X}_{N \times N}$, $\tilde{\kappa} \hat{X}_{N \times N}^{+}$ and their symmetric, complex, even boson-boson and fermion-fermion parts $\hat{f}_{D;S \times S}$, $\hat{f}_{D;S \times S}^{+}$ $(2.4)$ (respectively their complex conjugates $\tilde{\kappa} \hat{f}_{D;S \times S}$, $\tilde{\kappa} \hat{f}_{D;S \times S}^{+}$ $(2.4)$). The odd parts $\tilde{\eta}_{D;S \times L}$, $\tilde{\eta}_{D;L \times S}^{T}$ $(2.4)$ (and their super-hermitian $\tilde{\kappa} (\partial \hat{X}_{\alpha \beta}^{+})$) (compare footnote 4 for the indexing and numbering of the anomalous fermion-fermion parts by $2 \times 2$ quaternion elements)

$$(\partial \hat{X}_{\alpha \beta}) = -\left(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1}\right)_{\alpha \beta}^{12} = \begin{pmatrix}
-(\partial \hat{b}_{mn}) & (\partial \hat{c}_{mn}^{+}(r_{\tau \mu}, \nu)) \\
-(\partial \hat{c}_{mn}(\nu)) & (\partial \hat{a}_{\tau \mu, r'}(\nu))
\end{pmatrix}_{\alpha \beta} \quad (3.2)$$

$$(\partial \hat{b}_{mn}) = (\partial \hat{b}_{mn}^{T}), \quad (\partial \hat{a}_{\tau \mu, r'}(\nu)) = -(\partial \hat{a}_{\tau \mu, r'}^{T}) = \sum_{k=0}^{3} (\tau_{k})_{\mu \nu} (\partial \hat{a}_{\tau \mu, r'}^{(k)}) \quad (3.4)$$

The density part $(\partial \hat{\eta}_{11}^{\alpha \beta})$ has in its entity as a $N \times N$ super-matrix anti-hermitian properties, with the even, hermitian boson-boson and fermion-fermion matrices $(\partial \hat{d}_{mn})$, $(\partial \hat{d}_{\tau \mu, r'}^{+}(\nu))$ $(3.6)$. The odd parts of $(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1})^{11}$ (similar to $\delta \hat{X}_{D;S \times L}$, $\delta \hat{X}_{D;L \times S}$ of $\delta \hat{\Sigma}_{11}^{D;N \times N}$) are represented by $(\partial \hat{\xi}_{\tau \mu}^{+}(\nu))$, $(\partial \hat{\xi}_{\tau \mu, r'}^{+}(\nu))$ so that one obtains an anti-hermitian property of $(\partial \hat{\eta}_{11}^{\alpha \beta})$ because of the total imaginary factor

$$(\partial \hat{\eta}_{11}^{\alpha \beta}) = -\left(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1}\right)_{\alpha \beta}^{11} = -\begin{pmatrix}
(\partial \hat{d}_{mn}) & (\partial \hat{\xi}_{\tau \mu}^{+}(\nu)) \\
(\partial \hat{\xi}_{\tau \mu}^{+}(\nu)) & (\partial \hat{g}_{\tau \mu, r'}^{+}(\nu))
\end{pmatrix}_{\alpha \beta} \quad (3.5)$$

$$(\partial \hat{\eta}_{11}^{\alpha \beta}) = -\left(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1}\right)_{\alpha \beta}^{22} = -\begin{pmatrix}
(\partial \hat{d}_{mn}) & (\partial \hat{\xi}_{\tau \mu}^{+}(\nu)) \\
(\partial \hat{\xi}_{\tau \mu}^{+}(\nu)) & (\partial \hat{g}_{\tau \mu, r'}^{+}(\nu))
\end{pmatrix}_{\alpha \beta} \quad (3.7)$$

According to the coset decomposition, both parts, density $(3.5)$ and anomalous terms $(3.7)$ of $(\partial \hat{\eta}_{11}^{\alpha \beta})$ depend on the original, independent anomalous fields in $\hat{X}_{N \times N}$, $\tilde{\kappa} \hat{X}_{N \times N}^{+}$ and on the '22' $N \times N$ super-matrix $(\partial \hat{\eta}_{22}^{\alpha \beta})$ $(3.8)$ is related by super-transposition 'st' $(1.29)$ $\tilde{\kappa}$ to the '11' $N \times N$ density super-matrix $(\partial \hat{\eta}_{11}^{\alpha \beta})$ $(3.5)$ with inclusion of an additional minus sign. It is composed of the same variables with $(\partial \hat{d}_{\tau \mu, L \times L})$, $(\partial \hat{g}_{\tau \mu, S \times S})$ and also contains the same odd parts $(\partial \hat{\xi}_{\tau \mu}^{+}(\nu))$, $(\partial \hat{\xi}_{\tau \mu, r'}^{+}(\nu))$ apart from the additional super-transposition

$$(\partial \hat{\eta}_{22}^{\alpha \beta}) = -\left(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1}\right)_{\alpha \beta}^{22} = -\begin{pmatrix}
(\partial \hat{d}_{mn}) & (\partial \hat{\xi}_{\tau \mu}^{+}(\nu)) \\
(\partial \hat{\xi}_{\tau \mu}^{+}(\nu)) & (\partial \hat{g}_{\tau \mu, r'}^{+}(\nu))
\end{pmatrix}_{\alpha \beta} \quad (3.7)$$

$$(\partial \hat{\eta}_{22}^{\alpha \beta}) = -(\partial \hat{\eta}_{11}^{\alpha \beta}) = -\left(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1}\right)_{\alpha \beta}^{22, st} = \left(\hat{\rho} \hat{T}^{-1}(\partial \hat{T}) \hat{\rho}^{-1}\right)_{\alpha \beta}^{11} \quad (3.8)$$
3.2 Removal of the coset integration measure and transformation to Euclidean integration variables

In this subsection we record the transformation of $(\hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1})^a_b$ in terms of the eigenvalues $\hat{Y}_{DD}$, $\hat{X}_{DD}$, $\kappa \hat{X}_{DD}^+$ of the independent anomalous fields $(\partial \hat{c}_{D,L\times L})$, $(\partial \hat{f}_{D,S\times S})$, $(\partial \hat{h}_{D,S\times L})$, $(\partial \hat{g}_{D,L\times S})$ of the coset decomposition. Using relation (3.9) for the derivative of an exponential of a matrix, it remains to multiply the rotated derivative $(\partial \hat{\mathcal{P}}^\prime) = \hat{\mathcal{P}} (\partial \hat{Y}) \hat{\mathcal{P}}^{-1}$ by the diagonal anomalous matrices in the various block parts of $\exp\{\pm v \hat{Y}_{DD}\}$ with eigenvalues $\tau_m$ and quaternion eigenvalues $(\tau_2)_{\mu \nu} \mathcal{F}_r$ in terms of the eigenvalues $\hat{Y}_{DD}$, $\hat{X}_{DD}$, $\kappa \hat{X}_{DD}^+$ of the independent anomalous fields $(\partial \hat{c}_{D,L\times L})$, $(\partial \hat{f}_{D,S\times S})$, $(\partial \hat{h}_{D,S\times L})$, $(\partial \hat{g}_{D,L\times S})$ of the coset decomposition. Using relation (3.9) for the derivative of an exponential of a matrix, it remains to multiply the rotated derivative $(\partial \hat{\mathcal{P}}^\prime) = \hat{\mathcal{P}} (\partial \hat{Y}) \hat{\mathcal{P}}^{-1}$ by the diagonal anomalous matrices in the various block parts of $\exp\{\pm v \hat{Y}_{DD}\}$ with eigenvalues $\tau_m$ and quaternion eigenvalues $(\tau_2)_{\mu \nu} \mathcal{F}_r$.

After straightforward integration (3.10) of (hyperbolic) trigonometric functions with parameter $v \in [0,1]$, one acquires the dependence of $(\partial \hat{X}_{\alpha \beta})$, $\kappa (\partial \hat{X}_{\alpha \beta}^+)$ on the independent rotated variables within $(\partial \hat{c}_{D,L\times L})$, $(\partial \hat{f}_{D,S\times S})$, $(\partial \hat{h}_{D,S\times L})$, $(\partial \hat{g}_{D,L\times S})$ of the coset decomposition. Therefore, the densities $(\partial \hat{y}_{\alpha \beta}^{11})$, $(\partial \hat{y}_{\alpha \beta}^{22})$ of the independent anomalous fields $(\partial \hat{X}_{\alpha \beta})$, $\kappa (\partial \hat{X}_{\alpha \beta}^+)$ of the coset decomposition. In the following subsections 3.2.1, 3.2.2 and 3.2.3, we apply the results for $(\partial \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1})^a_b$ of Ref. [6] with appendix C in order to transform to Euclidean fields. These are separated into the split even boson-boson, fermion-fermion and odd fermion-boson, boson-fermion parts.

3.2.1 Boson-boson part of the transformation to Euclidean integration variables of pair condensate fields

In this subsection we record the transformation of $(\hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1})^a_b$ to $(\partial \hat{b}_{mn})$, $(\partial \hat{b}_{mn}^*)$ ($a \neq b$) and corresponding density parts $(\partial \hat{a}_{mn})$ ($a = b$) and have furthermore to distinguish between the diagonal
(m = n) and off-diagonal (m ≠ n) matrix elements of transformations which are restricted to the total, even boson-boson part

\[
- \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)^{12}_{\alpha\beta} = (\partial \hat{\mathcal{X}})_{\alpha\beta} = \begin{pmatrix}
- (\partial \hat{b}_{mn}) & (\partial \hat{c}^T_{m,n \rho \nu}) \\
- (\partial \hat{\xi}_{m,n}) & (\partial \hat{a}_{\rho \mu \nu})
\end{pmatrix}_{\alpha\beta};
\]

(3.12)

\[
(\partial \hat{b}_{mn}) = (\partial \hat{b}^T_{mn}); \quad (\partial \hat{a}_{\rho \mu \nu}) = - (\partial \hat{a}^T_{\rho \mu \nu});
\]

(3.13)

\[
- \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)^{11}_{\alpha \beta} = (\partial \hat{\mathcal{Y}})_{\alpha \beta} = \begin{pmatrix}
(\partial \hat{\xi}_{mn}) & (\partial \hat{c}^+_{m,n \rho \nu}) \\
(\partial \hat{\xi}_{m,n}) & (\partial \hat{g}_{\rho \mu \nu})
\end{pmatrix}_{\alpha \beta};
\]

(3.14)

\[
(\partial \hat{c}^+_{mn}) = (\partial \hat{a}_{mn}) \quad \text{and} \quad (\partial \hat{g}^+_{\rho \mu \nu}) = (\partial \hat{g}_{\rho \mu \nu}).
\]

(3.15)

In relations (3.16-3.19) we give the detailed transformation (3.16-3.17) from the diagonal, rotated, anomalous molecular condensates (\(\partial \hat{c}'_{D:mm}\), (\(\partial \hat{c}^*_{D:mm}\)) to Euclidean fields (\(\partial \hat{b}_{mn}\), (\(\partial \hat{b}^*_{mn}\)) and also determine the reverse transformations (3.18) from (\(\partial \hat{b}_{mn}\), (\(\partial \hat{b}^*_{mn}\)) to the original fields (\(\partial \hat{c}'_{D:mm}\), (\(\partial \hat{c}^*_{D:mm}\)) in the coset decomposition. The back transformation (3.18) allows to calculate the change of integration measure from \(d\hat{c}'_{D:mm} \wedge d\hat{c}^*_{D:mm}\) or in equivalence from the un-rotated fields \(d\hat{c}_{D:mm} \wedge d\hat{c}^*_{D:mm}\) to the diagonal, Euclidean elements \(d\hat{b}_{mn} \wedge d\hat{b}^*_{mn}\)

\[
- \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)^{12}_{BB:mm} = \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)^{21,*}_{BB:mm} = - (\partial \hat{b}_{mn}) = \left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) \left( \frac{1}{2} - \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) \rho^{2 \phi_m} \left( \partial \hat{c}'_{D:mm} \frac{2 |\tau_m|}{\sin(2 |\tau_m|)} \right)
\]

\[
\left( \frac{\partial \hat{b}_{mn}}{(\partial \hat{c}'_{D:mm})} \right) = \begin{pmatrix}
\left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) e^{2 \phi_m} \left( \frac{1}{2} - \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) & \left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) \\
\left( \frac{1}{2} - \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) & \left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right)
\end{pmatrix}
\]

(3.17)

\[
\left( \frac{\partial \hat{c}'_{D:mm}}{(\partial \hat{c}^*_{D:mm})} \right) = \begin{pmatrix}
\left( \frac{1}{2} - \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) e^{2 \phi_m} \left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) & \left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) \\
\left( \frac{1}{2} + \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right) & \left( \frac{1}{2} - \frac{\sin(2 |\tau_m|)}{4 |\tau_m|} \right)
\end{pmatrix}
\]

(3.18)

\[
d\hat{c}'_{D:mm} \wedge d\hat{c}^*_{D:mm} = d\hat{c}_{D:mm} \wedge d\hat{c}^*_{D:mm} = d\hat{b}_{mn} \wedge d\hat{b}^*_{mn} \frac{2 |\tau_m|}{\sin(2 |\tau_m|)}.
\]

(3.19)

The matrix in (3.17) with eigenvalues \(\tau_m \) (2.19) represents the square root of the coset metric tensor \(\hat{G}^{1/2}_{Osp/U}\) concerning the boson-boson part, however, in its diagonalized form with eigenvalues \(\tau_m\).

The inverse transformation (3.18) of Euclidean fields (\(\partial \hat{b}_{mn}\), (\(\partial \hat{b}^*_{mn}\)) to (\(\partial \hat{c}'_{D:mm}\), (\(\partial \hat{c}^*_{D:mm}\)) therefore contains the inverse square root of the coset metric tensor \(\hat{G}^{-1/2}_{Osp/U}\), as already mentioned in the introduction. Both kinds of metric tensors, \(\hat{G}^{1/2}_{Osp/U}\) and \(\hat{G}^{-1/2}_{Osp/U}\), are considerably simplified because
of the inclusion of the transformation from $\check{X}_{N \times N}$, $\check{\kappa}$ $\check{X}_{N \times N}^+$ to their eigenvalues in $\check{X}_{DD}$, $\check{\kappa}$ $\check{X}_{N \times N}^+$ with eigenvectors $\hat{\phi}_{\alpha \beta}^{aa}, \hat{\phi}_{\alpha \beta}^{-1,aa}$, (2.16-2.35). Since the transformation of $(\partial \hat{b}_{mn})$, $(\partial \hat{b}_{mn}^*)$ (3.18) is composed of the inverse square root of the metric tensor $\hat{G}^{-1/2}$, the corresponding integration measure $\text{SDET}(\hat{G}^{-1/2}) = (\text{SDET}(\hat{G}_{\text{Osp/U}}))^{-1/2}$ cancels the nontrivial integration measure originally introduced for the coset decomposition of $\delta \Sigma_{ab}(\bar{x}, t_\mu) \tilde{K} \tilde{\delta} \Sigma_D_{2N \times 2N} \tilde{K} \tilde{T}^{-1}$ (compare section 2.2). According to appendix C of Ref. [4], we can also give the transformation of the diagonal elements of the boson-boson density part $(\hat{T}^{-1} (\partial \hat{T}) \hat{T}^{1})^{11}_{BB;mn}$. It yields with the sub-metric tensor $(\hat{G}_{\text{Osp/U}})_{BB;mn}$ (3.18) for the relations between $(\partial \hat{b}_{mn})$, $(\partial \hat{b}_{mn}^*)$ and $(\partial \hat{c}_{D;mn})$, $(\partial \hat{c}_{D;mn}^*)$, the diagonal density elements $t (\partial \hat{d}_{mn})$ (3.20) in terms of $\tan(|\check{\tau}_m|)$ of the eigenvalues $\check{\tau}_m$ (2.19) and in terms of the diagonal, Euclidean anomalous fields $(\partial \hat{b}_{mm})$, $(\partial \hat{b}_{mm}^*)$

$$- (\hat{T}^{-1} (\partial \hat{T}) \hat{T}^{1})^{11}_{BB;mn} = (\hat{T}^{-1} (\partial \hat{T}) \hat{T}^{1})^{22}_{BB;mm} = t (\partial \hat{d}_{mm}) =$$

$$= - \left( \frac{\sin (|\check{\tau}_m|)}{2 |\check{\tau}_m|} \right)^2 \left( (\partial \hat{c}_{D;mn}^*) e^{i \check{\varphi}_m} - (\partial \hat{c}_{D;mn}^*) e^{-i \check{\varphi}_m} \right)$$

$$= \frac{1}{2} \tan (|\check{\tau}_m|) \left( (\partial \hat{b}_{mm}) e^{-i \check{\varphi}_m} - (\partial \hat{b}_{mm}^*) e^{i \check{\varphi}_m} \right) .$$

We transfer the results of the transformation to Euclidean fields from the diagonal boson-boson parts to the case of off-diagonal matrix elements $(m \neq n)$ (3.21) and introduce the coefficients $A_{BB}$, $B_{BB}$ (3.22, 3.23), depending on the modulus of eigenvalues $|\check{\tau}_m|$, $|\check{\tau}_n|$ for specification of $(\partial \hat{b}_{m \neq n})$, $(\partial \hat{b}_{m \neq n}^*)$.

The transformation (3.21, 3.22) determines the off-diagonal matrix elements of the boson-boson part of $(\hat{G}_{\text{Osp/U}})_{BB;m \neq n}$ whereas relation (3.25) describes the back transformation from $(\partial \hat{b}_{m \neq n})$, $(\partial \hat{b}_{m \neq n}^*)$ to the original, $\hat{\phi}_{\alpha \beta}^{aa}, \hat{\phi}_{\alpha \beta}^{-1,aa}$, rotated fields $(\partial \hat{c}_{D;m \neq n})$, $(\partial \hat{c}_{D;m \neq n}^*)$ of the coset decomposition. The diagonalized forms of $\hat{G}^{1/2}_{\text{Osp/U}}$, $\hat{G}^{-1/2}_{\text{Osp/U}}$ (3.24, 3.25) are simplified by the coefficients $A_{BB}$, $B_{BB}$ which are defined by the relations (3.22, 3.23) of the eigenvalues $|\check{\tau}_m|$, $|\check{\tau}_n|$. Note that the limit process $|\check{\tau}_n| \rightarrow |\check{\tau}_m|$ reproduces the results (3.16, 3.19) of diagonal elements $(m = n)$ for the boson-boson part of the metric tensor. This holds in particular for the case of the integration measure (3.28) which attains the identical form of (3.19) in case of the limit process $|\check{\tau}_n| \rightarrow |\check{\tau}_m|$. The integration measure (3.28) for $d\hat{b}_{m \neq n}$, $d\hat{b}_{m \neq n}^*$ results into the Euclidean form after substitution into the original coset integration (2.48) for $d\hat{c}_{D;m \neq n}$, $d\hat{c}_{D;m \neq n}^*$ of the coset decomposition $\text{Osp}(S, S|2L)/U(L|S) \otimes U(L|S)$

$$- (\hat{T}^{-1} (\partial \hat{T}) \hat{T}^{1})^{12}_{BB;mm} = (\hat{T}^{-1} (\partial \hat{T}) \hat{T}^{1})^{21,*}_{BB;mm}$$

$$= - (\partial \hat{b}_{mm}) = - A_{BB} (\partial \hat{c}_{D;mn}^*) - e^{i(\varphi_m + \varphi_n)} B_{BB} (\partial \hat{c}_{D;mn}^*) ;$$

$$A_{BB} = \frac{|\check{\tau}_m| \cos (|\check{\tau}_m|) \sin (|\check{\tau}_m|) - |\check{\tau}_m| \cos (|\check{\tau}_m|) \sin (|\check{\tau}_m|)}{|\check{\tau}_m^2 - |\check{\tau}_n|^2} ;$$

$$B_{BB} = \frac{|\check{\tau}_m| \cos (|\check{\tau}_m|) \sin (|\check{\tau}_m|) - |\check{\tau}_m| \cos (|\check{\tau}_m|) \sin (|\check{\tau}_m|)}{|\check{\tau}_m^2 - |\check{\tau}_n|^2} ;$$

3.2 Removal of the coset integration measure and transformation to Euclidean integration variables
According to appendix C of Ref. [6], we state the corresponding boson-boson density part for off-diagonal matrix elements $\nu (\partial \hat{d}_{mn})$ (3.29) by using coefficient functions $C_{BB}, B_{BB}$ (3.30), $D_{BB}$ (3.32) of the eigenvalues $|\bar{c}_m|, |\bar{c}_n|$ (2.19). Combining the transformation (3.25) with (3.29), the coefficients $A_{BB}, B_{BB}$ and $C_{BB}, D_{BB}$ allow to reduce the dependence of $(\hat{\Phi} \hat{T}^{-1} (\partial \hat{T}) \hat{T}^{-1})_{BB: m \neq n}^{11}$ onto the Euclidean variables $(\partial \hat{b}_{m \neq n}), (\partial \hat{b}_{m \neq n}^*)$ of anomalous terms for final relation (3.33).

\[
\begin{align*}
\left( \frac{\partial \hat{b}_{mn}}{\partial \hat{c}_{mn}^*} \right) & = \left( e^{-i(\varphi_m + \varphi_n)} B_{BB} e^{i(\varphi_m + \varphi_n)} A_{BB} \right) \left( \frac{\partial \hat{c}_{D:mn}}{\partial \hat{c}_{D:mn}^*} \right); \\
\left( \frac{\partial \hat{c}_{D:mn}}{\partial \hat{c}_{D:mn}^*} \right) & = \frac{1}{A_{BB}^2 - B_{BB}^2} \left( A_{BB} \left( -e^{-i(\varphi_m + \varphi_n)} B_{BB} e^{i(\varphi_m + \varphi_n)} A_{BB} \right) \left( \frac{\partial \hat{b}_{mn}}{\partial \hat{b}_{mn}^*} \right); \\
A_{BB} & = \frac{1}{2} \left( \frac{|\bar{c}_m| - |\bar{c}_n|}{\sin (|\bar{c}_m| - |\bar{c}_n|)} + \frac{|\bar{c}_m| + |\bar{c}_n|}{\sin (|\bar{c}_m| + |\bar{c}_n|)} \right); \\
B_{BB} & = \frac{1}{2} \left( \frac{|\bar{c}_m| + |\bar{c}_n|}{\sin (|\bar{c}_m| + |\bar{c}_n|)} - \frac{|\bar{c}_m| - |\bar{c}_n|}{\sin (|\bar{c}_m| - |\bar{c}_n|)} \right); \\
d\hat{c}_{D:mn} \wedge d\hat{c}_{D:mn}^* & = d\hat{c}_{D:mn} \wedge d\hat{c}_{D:mn}^* \frac{|\bar{c}_m| + |\bar{c}_n|}{\sin (|\bar{c}_m| + |\bar{c}_n|)} \frac{|\bar{c}_m| - |\bar{c}_n|}{\sin (|\bar{c}_m| - |\bar{c}_n|)}.
\end{align*}
\]

In case of a limit process $|\bar{c}_n| \rightarrow |\bar{c}_m|$, we obtain from Eq. (3.33) the result (3.20) for the diagonal density elements $(m = n)$.
3.2 Removal of the coset integration measure and transformation to Euclidean integration variables

3.2.2 Fermion-fermion part of the transformation to Euclidean integration variables of pair condensate fields

In analogy to the boson-boson part, we list corresponding results of the transformation to Euclidean field variables for the fermion-fermion parts (appendix C in Ref. [6]). However, one has to exchange the ordinary matrix elements of the boson-boson parts by matrix elements of the quaternion algebra with standard $2 \times 2$ Pauli matrices and the $2 \times 2$ unit matrix \( (3.34) \). In the case of quaternionic, diagonal elements of the pair condensates, one has to restrict to the quaternion element with Pauli matrix \((\tau_2)_{\mu\nu} (\partial \hat{a}_{rr}^{(2)} \hat{a}_{rr}^{(2)})\) due to the anti-symmetry of \((\hat{p} \hat{T}^{-1} (\partial \hat{T}) \hat{p}^{-1})_{\mu\nu}^{12}\) and of \((\partial \hat{a}_{rr}^{(2)} \hat{a}_{rr}^{(2)}) = (\tau_2)_{\mu\nu} (\partial \hat{a}_{rr}^{(2)})\) for the BCS pair condensate terms

\[
\left( \hat{\hat{p}} \hat{T}^{-1} (\partial \hat{T}) \hat{\hat{p}}^{-1} \right)_{\mu\nu}^{12} = \left( \hat{\hat{p}} \hat{T}^{-1} (\partial \hat{T}) \hat{\hat{p}}^{-1} \right)_{\mu\nu}^{21,+} = \left( \partial \hat{a}_{rr}^{(2)} \right)_{\mu\nu} (\partial \hat{a}_{rr}^{(2)}) = (\tau_2)_{\mu\nu} (\partial \hat{a}_{rr}^{(2)}) \tag{3.34}
\]

The complementary diagonal forms of $G_{\text{Osp}/U}^{1/2}, \tilde{G}_{\text{Osp}/U}^{-1/2}$ with diagonal elements of the fermion-fermion section follow from the transformations of \((\partial \tilde{\hat{f}}_{\mu\nu}^{(2)}), (\partial \tilde{\hat{f}}_{\mu\nu}^{(2)*})\) \((2.24, 2.5)\) to \((\partial \hat{a}_{rr}^{(2)}), (\partial \hat{a}_{rr}^{(2)*})\) and vice versa \((3.35, 3.36)\). The change of integration measure is given in \((3.37)\) and yields with the original coset integration measure of \(\text{Osp}(S, S[2L]/U(L,S) \otimes U(L,S))\) \((2.48)\) Euclidean integration variables \(d\hat{a}_{rr}^{(2)} \wedge d\hat{a}_{rr}^{(2)*}\)

\[
\left( \begin{array}{c}
(\partial \hat{a}_{rr}^{(2)}) \\
(\partial \hat{a}_{rr}^{(2)*})
\end{array} \right) = \left( \begin{array}{c}
\left( \frac{1}{2} + \frac{\sinh(2 |\hat{T}_r|)}{4 |\hat{T}_r|} \right) e^{i \phi_r} \\
\frac{1}{2} - \frac{\sinh(2 |\hat{T}_r|)}{4 |\hat{T}_r|}
\end{array} \right) \tag{3.35}
\]

\[
\left( \begin{array}{c}
(\partial \tilde{\hat{f}}_{\mu\nu}^{(2)}) \\
(\partial \tilde{\hat{f}}_{\mu\nu}^{(2)*})
\end{array} \right) = \left( \begin{array}{c}
\left( \frac{1}{2} + \frac{|\hat{T}_r|}{\sinh(2 |\hat{T}_r|)} \right) e^{i \phi_r} \\
\frac{1}{2} - \frac{|\hat{T}_r|}{\sinh(2 |\hat{T}_r|)}
\end{array} \right) \tag{3.36}
\]

\[
d\tilde{\hat{f}}_{\mu\nu}^{(2)} \wedge d\tilde{\hat{f}}_{\mu\nu}^{(2)*} = d\hat{a}_{rr}^{(2)} \wedge d\hat{a}_{rr}^{(2)*} = \frac{2 |\hat{T}_r|}{\sinh(2 |\hat{T}_r|)} \tag{3.37}
\]

We quote the result \((3.38)\) for the quaternionic, diagonal densities \((\hat{p} \hat{T}^{-1} (\partial \hat{T}) \hat{p}^{-1})_{\mu\nu}^{11}\) in terms of the coset fields \((\partial \tilde{\hat{f}}_{\mu\nu}^{(2)})\) \((2.24, 2.5)\) according to appendix C in Ref. [6]. Incorporating the transformation \((3.36)\) with diagonal sub-metric tensor \(\tilde{G}_{\text{Osp}/U}^{-1/2}\), we obtain the diagonal density elements \(\iota \left( \partial \hat{g}_{\mu\nu} \right)\) of the fermion-fermion part in dependence on \(\tanh(|\hat{T}_r|)\) and the Euclidean fermion-fermion pair condensate fields \((\partial \hat{a}_{rr}^{(2)}), (\partial \hat{a}_{rr}^{(2)*})\)

\[
- \left( \hat{p} \hat{T}^{-1} (\partial \hat{T}) \hat{p}^{-1} \right)_{\mu\nu}^{11} = \left( \hat{p} \hat{T}^{-1} (\partial \hat{T}) \hat{p}^{-1} \right)_{\mu\nu}^{22,T} = \iota \left( \partial \hat{g}_{\mu\nu} \right) = \delta_{\mu\nu} \left( \frac{\sinh(|\hat{T}_r|)}{2 |\hat{T}_r|} \right)^2 \left( \partial \hat{f}_{\mu\nu}^{(2)*} e^{i \phi_r} - \partial \hat{f}_{\mu\nu}^{(2)} e^{-i \phi_r} \right) \tag{3.38}
\]
\[ = -\delta_{\mu\nu} \frac{1}{2} \tanh (|\tilde{f}_{rr}|) \left( (\partial \tilde{a}^{(2)}_{rr}) e^{-i \phi_{rr}} - (\partial \tilde{a}^{(2)*}_{rr}) e^{i \phi_{rr}} \right). \]

Concerning the off-diagonal, anomalous fields of the fermion-fermion sections, one has to apply all four quaternion elements \((\partial \tilde{a}^{(k)}_{rr'})\) with \((\tau_0)_{\mu\nu}, (\tau_1)_{\mu\nu}, (\tau_2)_{\mu\nu}, (\tau_3)_{\mu\nu}\) and \((\partial \tilde{a}^{(k)}_{rr'}) = -(\partial \tilde{a}^{(k)}_{rr'})\) being anti-symmetric for \(k = 0, 1, 3\) and being symmetric \((\partial \tilde{a}^{(2)}_{rr'}) = (\partial \tilde{a}^{(2)}_{rr'})\) for \(k = 2\)

\[ - \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)^{12}_{\mu\nu, r'r'} = - \left( \hat{\mathcal{P}} \hat{T}^{-1} (\partial \hat{T}) \hat{\mathcal{P}}^{-1} \right)^{21+}_{\mu\nu, r'r'} = \sum_{k=0}^{3} (\tau_k)_{\mu\nu} (\partial \tilde{a}^{(k)}_{rr'}). \quad (3.39) \]

We abbreviate various terms of eigenvalues \(|\tilde{f}_{rr}|, |\tilde{f}_{rr'}|\) \((2.20)\) by the coefficients \(A_{FF} \quad (3.40), B_{FF} \quad (3.41)\) and have to distinguish between quaternion elements of \((\tau_0)_{\mu\nu}\) and \((\tau_k)_{\mu\nu}\) \(k=1,2,3\) for the transformation of the original, anomalous coset fields \((\partial \hat{f}_{Drr'})^{(k)} \), \((\partial \hat{f}_{Drr'}^{(k)*}) \quad (2.2, 2.3)\) to their Euclidean correspondents

\[
A_{FF} = \frac{|\tilde{f}_{rr}| \cosh (|\tilde{f}_{rr}|) \sinh (|\tilde{f}_{rr}|) - |\tilde{f}_{rr'}| \cosh (|\tilde{f}_{rr'}|) \sinh (|\tilde{f}_{rr'}|)}{|\tilde{f}_{rr}|^2 - |\tilde{f}_{rr'}|^2}; \quad (3.40) \\
B_{FF} = \frac{|\tilde{f}_{rr}| \cosh (|\tilde{f}_{rr}|) \sinh (|\tilde{f}_{rr'}|) - |\tilde{f}_{rr'}| \cosh (|\tilde{f}_{rr'}|) \sinh (|\tilde{f}_{rr'}|)}{|\tilde{f}_{rr}|^2 - |\tilde{f}_{rr'}|^2}; \quad (3.41) \\

\begin{align*}
\left( \begin{array}{c} (\partial \tilde{a}^{(0)}_{rr}) \\ (\partial \tilde{a}^{(0)*}_{rr}) \end{array} \right) &= \left( \begin{array}{c} A_{FF} \\ e^{-i(\phi_{rr}+\phi_{rr'})} B_{FF} \end{array} \right) \left( \begin{array}{c} e^{i(\phi_{rr}+\phi_{rr'})} B_{FF} \\ A_{FF} \end{array} \right); \\
\left( \begin{array}{c} (\partial \tilde{f}^{(0)}_{Drr'}) \\ (\partial \tilde{f}^{(0)*}_{Drr'}) \end{array} \right) &= \frac{1}{A_{FF}^2 - B_{FF}^2} \left( \begin{array}{c} A_{FF} \\ -e^{-i(\phi_{rr}+\phi_{rr'})} B_{FF} \end{array} \right) \left( \begin{array}{c} -e^{i(\phi_{rr}+\phi_{rr'})} B_{FF} \\ A_{FF} \end{array} \right); \\
\left( \begin{array}{c} (\partial \tilde{a}^{(k)}_{rr}) \\ (\partial \tilde{a}^{(k)*}_{rr}) \end{array} \right)_{k=1,2,3} &= \left( \begin{array}{c} A_{FF} \\ -e^{-i(\phi_{rr}+\phi_{rr'})} B_{FF} \end{array} \right) \left( \begin{array}{c} e^{-i(\phi_{rr}+\phi_{rr'})} B_{FF} \\ A_{FF} \end{array} \right); \\
\left( \begin{array}{c} (\partial \tilde{f}^{(k)}_{Drr'}) \\ (\partial \tilde{f}^{(k)*}_{Drr'}) \end{array} \right)_{k=1,2,3} &= \frac{1}{A_{FF}^2 - B_{FF}^2} \left( \begin{array}{c} A_{FF} \\ -e^{-i(\phi_{rr}+\phi_{rr'})} B_{FF} \end{array} \right) \left( \begin{array}{c} e^{i(\phi_{rr}+\phi_{rr'})} B_{FF} \\ A_{FF} \end{array} \right); \\
\end{align*}

\[ \frac{A_{FF}^2 - B_{FF}^2}{A_{FF}^2 - B_{FF}^2} = \frac{1}{2} \left( \begin{array}{c} |\tilde{f}_{rr}|-|\tilde{f}_{rr'}| \\ |\tilde{f}_{rr}|-|\tilde{f}_{rr'}| \sinh (|\tilde{f}_{rr}|) \sinh (|\tilde{f}_{rr'}|) \end{array} \right); \quad (3.46) \\
\frac{B_{FF}^2 - A_{FF}^2}{A_{FF}^2 - B_{FF}^2} = \frac{1}{2} \left( \begin{array}{c} |\tilde{f}_{rr}|-|\tilde{f}_{rr'}| \\ |\tilde{f}_{rr}|-|\tilde{f}_{rr'}| \sinh (|\tilde{f}_{rr}|) \sinh (|\tilde{f}_{rr'}|) \end{array} \right). \quad (3.47) \]

Taking the determinants of the \(\hat{G}_{1/2}^{-1}\) coset sub-metric transformations \(3.43, 3.45\), we acquire the integration measure \(\text{SDET}(\hat{G}_{1/2}^{-1})\) of the particular, diagonalized fermion-fermion parts \(3.48\) which are eliminated with the sub-metric tensor parts \(\text{SDET}(\hat{G}_{Osp/U})\) \(1/2 \quad (2.48)\) of the original coset decomposition to Euclidean integration variables \(d\tilde{a}^{(k)}_{rr'} \land d\tilde{a}^{(k)*}_{rr'}\), \(k = 0, 1, 2, 3\)

\[ d\tilde{f}^{(k)}_{Drr'} \land d\tilde{f}^{(k)*}_{Drr'} = d\tilde{f}^{(k)}_{Drr'} \land d\tilde{f}^{(k)*}_{Drr'} = d\tilde{a}^{(k)}_{rr'} \land d\tilde{a}^{(k)*}_{rr'} \quad \frac{|\tilde{f}_{rr}| + |\tilde{f}_{rr'}|}{\sinh (|\tilde{f}_{rr}|) \sinh (|\tilde{f}_{rr'}|)} \quad \frac{|\tilde{f}_{rr}| - |\tilde{f}_{rr'}|}{\sinh (|\tilde{f}_{rr}|) \sinh (|\tilde{f}_{rr'}|)}. \quad (3.48) \]
3.2 Removal of the coset integration measure and transformation to Euclidean integration variables

The off-diagonal density parts 

\(-\left(\hat{\Phi} \hat{T}^{-1} (\partial \hat{T}) \hat{\Phi}^{-1}\right)_{r \neq r'}^{11} = \imath (\partial \hat{g}_{r \mu}, r' \nu)\) (3.49) are given as quaternion matrix elements and by coefficients \(C_{FF}\) (3.51), \(D_{FF}\) (3.52) according to appendix C in Ref. 6.

\[
-\left(\hat{\Phi} \hat{T}^{-1} (\partial \hat{T}) \hat{\Phi}^{-1}\right)_{r \neq r'}^{11} \equiv \left(\hat{\Phi} \hat{T}^{-1} (\partial \hat{T}) \hat{\Phi}^{-1}\right)_{r \neq r'}^{22} \equiv \imath (\partial \hat{g}_{r \mu}, r' \nu) = 3.49
\]

\[
= -\left(\tau_2\right)_{\mu \nu} \left[ e^{-i \phi_{r'}} \sinh (\hat{f}_{r'}) \left( \partial \hat{a}_{r \mu}^{(0)} \right) + e^{i \phi_r} \sinh (\hat{f}_r) \left( \partial \hat{a}_{r' \nu}^{(0)*} \right) \right] +
\]

\[
- \sum_{k=1,2,3} (\hat{m}_k)_{\mu \nu} \left[ e^{-i \phi_{r'}} \sinh (\hat{f}_{r'}) \left( \partial \hat{a}_{r \mu}^{(k)} \right) - e^{i \phi_r} \sinh (\hat{f}_r) \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right]
\]

\[
= -\frac{1}{2} \left(\tau_2\right)_{\mu \nu} \left\{ \tanh \left(\frac{|\hat{f}_r| + |\hat{f}_{r'}|}{2}\right) \left[ e^{-i \phi_{r'}} \left( \partial \hat{a}_{r \mu}^{(0)} \right) + e^{i \phi_r} \left( \partial \hat{a}_{r' \nu}^{(0)*} \right) \right] +
\]

\[
- \tanh \left(\frac{|\hat{f}_r| - |\hat{f}_{r'}|}{2}\right) \left[ e^{-i \phi_{r'}} \left( \partial \hat{a}_{r \mu}^{(0)} \right) - e^{i \phi_r} \left( \partial \hat{a}_{r' \nu}^{(0)*} \right) \right] \right\} +
\]

\[
- \frac{1}{2} \sum_{k=1,2,3} (\hat{m}_k)_{\mu \nu} \left\{ \tanh \left(\frac{|\hat{f}_r| + |\hat{f}_{r'}|}{2}\right) \left[ e^{-i \phi_{r'}} \left( \partial \hat{a}_{r \mu}^{(k)} \right) + e^{i \phi_r} \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right] +
\]

\[
- \tanh \left(\frac{|\hat{f}_r| - |\hat{f}_{r'}|}{2}\right) \left[ e^{-i \phi_{r'}} \left( \partial \hat{a}_{r \mu}^{(k)} \right) - e^{i \phi_r} \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right] \right\}
\]

\[
= -\frac{1}{2} \sum_{k=0}^3 (\hat{\tau}_k \tau_2)_{\mu \nu} \left\{ \tanh \left(\frac{|\hat{f}_r| + |\hat{f}_{r'}|}{2}\right) \left[ e^{-i \phi_{r'}} \left( \partial \hat{a}_{r \mu}^{(k)} \right) + e^{i \phi_r} \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right] +
\]

\[
- \left(\tau_2\right)_{\mu \nu} \left[ e^{-i \phi_{r'}} \sinh (\hat{f}_{r'}) \left( \partial \hat{a}_{r \mu}^{(0)} \right) + e^{i \phi_r} \sinh (\hat{f}_r) \left( \partial \hat{a}_{r' \nu}^{(0)*} \right) \right] +
\]

\[
- \sum_{k=1,2,3} (\hat{m}_k)_{\mu \nu} \left[ e^{-i \phi_{r'}} \sinh (\hat{f}_{r'}) \left( \partial \hat{a}_{r \mu}^{(k)} \right) - e^{i \phi_r} \sinh (\hat{f}_r) \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right]
\]

\[
= \left(\hat{\tau}_k \tau_2\right)_{\mu \nu} \left\{ \tanh \left(\frac{|\hat{f}_r| + |\hat{f}_{r'}|}{2}\right) \left[ e^{-i \phi_{r'}} \left( \partial \hat{a}_{r \mu}^{(k)} \right) + e^{i \phi_r} \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right] +
\]

\[
- \left(\tau_2\right)_{\mu \nu} \left[ e^{-i \phi_{r'}} \sinh (\hat{f}_{r'}) \left( \partial \hat{a}_{r \mu}^{(0)} \right) + e^{i \phi_r} \sinh (\hat{f}_r) \left( \partial \hat{a}_{r' \nu}^{(0)*} \right) \right] +
\]

\[
- \sum_{k=1,2,3} (\hat{m}_k)_{\mu \nu} \left[ e^{-i \phi_{r'}} \sinh (\hat{f}_{r'}) \left( \partial \hat{a}_{r \mu}^{(k)} \right) - e^{i \phi_r} \sinh (\hat{f}_r) \left( \partial \hat{a}_{r' \nu}^{(k)*} \right) \right]
\]
3.2.3 Fermion-boson and boson-fermion parts of the transformation to Euclidean integration variables of pair condensate fields

In the case of transformations in the odd fermion-boson and boson-fermion sections, we have to consider the two quaternion elements \((\tau_0)_{\mu\nu}\) and \((\tau_2)_{\mu\nu}\) and cite the result \((3.55,3.56)\) for \((\hat{\mathcal{T}} _{-1} (\partial \hat{T} ) \hat{\mathcal{T}} _{-1})^{12}_{FB;\mu\nu,n}\) from appendix C of Ref. [3]. The coefficients \(A_{FB}, B_{FB}\) abbreviate the relations \((3.57,3.58)\) for the eigenvalues \(|\tau_n| (2.19), |\tau_r| (2.20); (\kappa, \mu, \nu = 1, 2), (r, r' = 1, \ldots, S/2), (m, n = 1, \ldots, L)\)

\[-(\partial \hat{c}_{\mu,n}) = -(\hat{\mathcal{T}} _{-1} (\partial \hat{T} ) \hat{\mathcal{T}} _{-1})^{12}_{FB;\mu,n} = (\hat{\mathcal{T}} _{-1} (\partial \hat{T} ) \hat{\mathcal{T}} _{-1})^{21+}_{BF;n\tau'} (3.55)\]

\[-(\partial \hat{c}^{*}_{\mu,n}) = -A_{FB} (\tau_0)_{\mu\nu} (\partial \hat{\eta}^{*}_{D;\tau,n}) - B_{FB} e^{i(\phi_+ + \phi_\eta)} (\tau_2)_{\mu\nu} (\partial \hat{\eta}^{*}_{D;\tau,n}) ; (3.56)\]

\[A_{FB} = \frac{|\tau_n| \cosh (|\tau_r|) \sinh (|\tau_n|) + |\tau_r| \cosh (|\tau_n|) \sinh (|\tau_r|)}{|\tau_n|^2 + |\tau_r|^2} ; (3.57)\]

\[B_{FB} = \frac{|\tau_n| \cosh (|\tau_n|) \sinh (|\tau_r|) - |\tau_r| \cosh (|\tau_n|) \sinh (|\tau_r|)}{|\tau_n|^2 + |\tau_r|^2} . (3.58)\]

The diagonalized, \(4 \times 4\) sub-metric tensors \(\hat{G}^{1/2}_{\text{osp}/U}, \hat{G}^{-1/2}_{\text{osp}/U}\) of the fermion-boson, boson-fermion sections are described in relations \((3.59,3.60)\) by using the coefficients \(A_{FB}, B_{FB}\) for abbreviating relations \((3.61,3.62)\)

\[
\begin{pmatrix}
(\partial \hat{c}_{\mu,n}) \\
(\partial \hat{c}^{*}_{\mu,n})
\end{pmatrix} = \begin{pmatrix}
A_{FB} (\tau_0)_{\mu\nu} & B_{FB} e^{i(\phi_+ + \phi_\eta)} (\tau_2)_{\mu\nu} \\
-B_{FB} e^{-i(\phi_+ + \phi_\eta)} (\tau_2)_{\mu\nu} & A_{FB} (\tau_0)_{\mu\nu}
\end{pmatrix} \begin{pmatrix}
(\partial \hat{\eta}^{*}_{D;\tau,n}) \\
(\partial \hat{\eta}^{*}_{D;\tau,n})
\end{pmatrix} ; (3.59)
\]

\[
\begin{pmatrix}
(\partial \hat{c}^{*}_{\mu,n}) \\
(\partial \hat{c}^{*}_{\mu,n})
\end{pmatrix} = \begin{pmatrix}
\tilde{A}_{FB} (\tau_0)_{\mu\nu} & -\tilde{B}_{FB} e^{i(\phi_+ + \phi_\eta)} (\tau_2)_{\mu\nu} \\
\tilde{B}_{FB} e^{-i(\phi_+ + \phi_\eta)} (\tau_2)_{\mu\nu} & \tilde{A}_{FB} (\tau_0)_{\mu\nu}
\end{pmatrix} \begin{pmatrix}
(\partial \hat{\eta}^{*}_{D;\tau,n}) \\
(\partial \hat{\eta}^{*}_{D;\tau,n})
\end{pmatrix} ; (3.60)
\]

\[
\tilde{A}_{FB} = \frac{A_{FB}}{A_{FB}^2 + B_{FB}^2} = \frac{1}{2} \left( \frac{|\tau_n| - i |\tau_r|}{\sinh (|\tau_n| + i |\tau_r|)} + \frac{|\tau_n| + i |\tau_r|}{\sinh (|\tau_n| - i |\tau_r|)} \right) ; (3.61)
\]

\[
\tilde{B}_{FB} = \frac{B_{FB}}{A_{FB}^2 + B_{FB}^2} = \frac{i}{2} \left( \frac{|\tau_n| - i |\tau_r|}{\sinh (|\tau_n| - i |\tau_r|)} - \frac{|\tau_n| + i |\tau_r|}{\sinh (|\tau_n| + i |\tau_r|)} \right) . (3.62)
\]

The integration measure \((3.63)\) follows from the 'inverse' of the determinant of transformation \((3.60)\) where the eigenvalue \(|\tau_n|\) of the boson-fermion part fits into the hyperbolic sinh-function with the eigenvalue \(|\tau_r|\) of the fermion-fermion section by using an imaginary factor. In consequence, the original, odd, anomalous coset fields \(d\hat{c}^{*}_{D;\tau,n}, d\hat{c}^{*}_{D;\tau,n}, d\hat{c}^{*}_{D;\tau,n}\) are substituted by the odd Euclidean fields \(d\hat{c}^{*}_{D;\tau,n}, d\hat{c}^{*}_{D;\tau,n}, d\hat{c}^{*}_{D;\tau,n}\) in combination of the coset integration measure \((2.48)\)

\[d\hat{c}^{*}_{D;\tau,n} d\hat{c}^{*}_{D;\tau,n} d\hat{c}^{*}_{D;\tau,n} = (3.63)\]
\[\begin{align*}
&= \left\{ d\hat{\zeta}_{r,1,n}^* d\hat{\zeta}_{r,1,n} \left( \frac{\sinh \left( |\vec{f}_r| + i |\vec{e}_n| \right)}{|\vec{f}_r| + i |\vec{e}_n|} \right) \left( \frac{\sinh \left( |\vec{f}_r| - i |\vec{e}_n| \right)}{|\vec{f}_r| - i |\vec{e}_n|} \right) \right\} \times \\
&\times \left\{ d\hat{\zeta}_{r,2,n}^* d\hat{\zeta}_{r,2,n} \left( \frac{\sinh \left( |\vec{f}_r| + i |\vec{e}_n| \right)}{|\vec{f}_r| + i |\vec{e}_n|} \right) \left( \frac{\sinh \left( |\vec{f}_r| - i |\vec{e}_n| \right)}{|\vec{f}_r| - i |\vec{e}_n|} \right) \right\}.
\end{align*}\]

According to Ref. [6] with appendix C, we list the odd density part (3.64) for the fermion-boson, boson-fermion sections by introducing the coefficients \(C_{FB}\) (3.65), \(D_{FB}\) (3.66) as abbreviation

\[-\left( \hat{\phi} \hat{T}^{-1} \left( \partial \hat{T} \right) \hat{\phi}^{-1} \right)_{11}^{FB,r,\mu,n} = -\left( \hat{\phi} \hat{T}^{-1} \left( \partial \hat{T} \right) \hat{\phi}^{-1} \right)_{22,T}^{FB,n,r,\mu} = i \left( \partial \hat{\xi}_{r,\mu,n} \right) = \ (3.64)\]

\[C_{FB} = \frac{|\vec{f}_r| - |\vec{e}_n| \cos (|\vec{e}_n|) \cosh (|\vec{f}_r|) - |\vec{e}_n| \sin (|\vec{e}_n|) \sinh (|\vec{f}_r|)}{|\vec{e}_n|^2 + |\vec{f}_r|^2}; \quad \ (3.65)\]

\[D_{FB} = \frac{|\vec{f}_r| - |\vec{e}_n| \cos (|\vec{e}_n|) \cosh (|\vec{f}_r|) + |\vec{e}_n| \sin (|\vec{e}_n|) \sinh (|\vec{f}_r|)}{|\vec{e}_n|^2 + |\vec{f}_r|^2}. \quad \ (3.66)\]

We apply (3.60) with the \(A_{FB}, B_{FB}\) coefficients (3.57, 3.58), together with (3.64) and coefficients \(C_{FB}, D_{FB}\) (3.65, 3.66), and finally obtain relation (3.67) for the odd density parts. One achieves a dependence on the odd, anomalous Euclidean fields \((\partial \hat{\zeta}_{r,1,n}^*), (\partial \hat{\zeta}_{r,1,n}), (\partial \hat{\zeta}_{r,2,n}^*), (\partial \hat{\zeta}_{r,2,n})\) for the odd, fermion-boson, boson-fermion density parts (3.64) in combination of the eigenvalues \(|\vec{f}_r|, |\vec{e}_n|\) (2.19, 2.20) appearing with \(\tanh((|\vec{f}_r| \pm i |\vec{e}_n|)/2)\)

\[-\left( \hat{\phi} \hat{T}^{-1} \left( \partial \hat{T} \right) \hat{\phi}^{-1} \right)_{FB,r,\mu,n}^{11} = i \left( \partial \hat{\xi}_{r,\mu,n} \right) = \ (3.67)\]

\[= \frac{e^{-i \varphi_n} \sin (|\vec{e}_n|) (\tau_0)_{\mu \kappa} (\partial \hat{\zeta}_{r,n}) \left( \tau_2 \right)_{\mu \kappa} (\partial \hat{\zeta}_{r,n})}{\cosh (|\vec{f}_r|) \cosh (|\vec{e}_n|)} + \frac{e^{i \varphi_n} \sinh (|\vec{e}_n|) (\tau_0)_{\mu \kappa} (\partial \hat{\zeta}_{r,n})}{\cosh (|\vec{f}_r|) \cosh (|\vec{e}_n|)} + \frac{e^{-i \varphi_n} \sin (|\vec{e}_n|) (\tau_0)_{\mu \kappa} (\partial \hat{\zeta}_{r,n})}{\cosh (|\vec{f}_r|) \cosh (|\vec{e}_n|)} + \frac{e^{i \varphi_n} \sinh (|\vec{e}_n|) (\tau_0)_{\mu \kappa} (\partial \hat{\zeta}_{r,n})}{\cosh (|\vec{f}_r|) \cosh (|\vec{e}_n|)}. \]

3.3 Eigenvalues of cosets for anomalous terms and their transformed, Euclidean correspondents

In section 3.2 with subsections 3.2.1, 3.2.2, 3.2.3, we have used the results of appendix C in Ref. [6] for relations (3.9, 3.11) in order to transform the original, coset fields in \((\partial \hat{X}_{N \times N}), \tilde{\kappa} (\partial \hat{X}^+_{N \times N})\) (2.2, 2.3) and in \((\hat{\phi} \hat{T}^{-1} \left( \partial \hat{T} \right) \hat{\phi}^{-1})_{\alpha \beta}^{a b}\) to Euclidean integration variables \((\partial \hat{\zeta}_{\alpha \beta}^{a b})\) (3.1) depending on \((\partial \hat{X}_{\alpha \beta}), \tilde{\kappa} (\hat{\kappa})\) (3.2, 3.4). (In this section we have to specialize on the total derivative \(\partial\) for the pair condensate path field variables in place of the general symbolic derivative \(\partial\) of section 3.2. The general symbolic derivative \(\partial\) has been applied as abbreviation for partial, spatial or time-contour-like gradients \(\partial_t, \partial_t^\prime, \partial_t^\prime\).
and 'δ'-variations of fields for classical equations or total derivatives 'd' for the independent path fields of the integration measure.) However, apart from the dependence on Euclidean integration variables \((d\tilde{b}_{mn}), (d\tilde{a}_{r\mu,r'})\), \((d\tilde{c}_{r\mu,n})\), \((\pm \text{c.c.})\) \[3.23,3.34\] there also appear the eigenvalues \(\overline{\tau}_m\) \[2.19\], \(\overline{f}_r\) \[2.21\] of the original coset decomposition for anomalous fields. Their dependence has to be determined in terms of the new, independent Euclidean pair condensate fields \((d\tilde{b}_{mn}), (d\tilde{a}_{r\mu,r'})\), \((d\tilde{c}_{r\mu,n})\), \((\pm \text{c.c.})\). According to section 3.2, we therefore list again relations \[3.68,3.75\] which have all been calculated in terms of the anomalous Euclidean fields \((d\tilde{X}_{\alpha\beta})\) and their super-hermitian conjugate \(\tilde{\kappa}\) \(\left(d\tilde{X}^+_{\alpha\beta}\right)\)

\[
(d\tilde{Z}^{ab}_{\alpha\beta}) = -(\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{ab}_{\alpha\beta} = \left(\begin{array}{cc}
(d\tilde{Y}^{11}_{\alpha\beta}) & (d\tilde{X}_{\alpha\beta}) \\
\tilde{\kappa} & (d\tilde{X}^+_{\alpha\beta})
\end{array}\right),
\]

\[3.68\]

\[
(d\tilde{X}_{\alpha\beta}) = -(\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{12}_{\alpha\beta} = \left(\begin{array}{cc}
-(d\tilde{b}_{mn}) & -(d\tilde{c}_{m,r'}) \\
-(d\tilde{c}_{r\mu,n}) & -(d\tilde{a}_{r\mu,r'})
\end{array}\right)_{\alpha\beta},
\]

\[3.69\]

\[
(\tilde{\kappa}) (d\tilde{X}^+_{\alpha\beta}) = -(\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{21}_{\alpha\beta} = \left(\begin{array}{cc}
(d\tilde{b}^+_{mn}) & (d\tilde{c}_{m,r'}) \\
(d\tilde{c}^+_{r\mu,n}) & (d\tilde{a}^+_{r\mu,r'})
\end{array}\right)_{\alpha\beta},
\]

\[3.70\]

\[
(d\tilde{b}_{mn}) = (d\tilde{b}^T_{mn}); \quad (d\tilde{a}_{r\mu,r'}) = -(d\tilde{a}^T_{r\mu,r'});
\]

\[3.71\]

\[
(d\tilde{y}^{11}_{\alpha\beta}) = -(\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{11}_{\alpha\beta} = \left(\begin{array}{cc}
(d\tilde{a}_{mn}) & (d\tilde{c}_{m,r'}) \\
(d\tilde{c}_{r\mu,n}) & (d\tilde{a}_{r\mu,r'})
\end{array}\right)^{11}_{\alpha\beta},
\]

\[3.72\]

\[
(d\tilde{y}^{11}_{\alpha\beta})^+ = -(d\tilde{y}^{11}_{\alpha\beta}); \quad (d\tilde{d}^+_{mn}) = (d\tilde{d}_{mn}); \quad (d\tilde{g}^+_{r\mu,r'}) = (d\tilde{g}_{r\mu,r'});
\]

\[3.73\]

\[
(d\tilde{y}^{22}_{\alpha\beta}) = -(\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{22}_{\alpha\beta} = \left(\begin{array}{cc}
(d\tilde{d}^T_{mn}) & -(d\tilde{d}_{m,r'}) \\
(d\tilde{c}^T_{r\mu,n}) & (d\tilde{g}^T_{r\mu,r'})
\end{array}\right)^{22}_{\alpha\beta},
\]

\[3.74\]

\[
(d\tilde{y}^{22}_{\alpha\beta})^{st} = -(d\tilde{y}^{22}_{\alpha\beta}) = -(\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{22,st}_{\alpha\beta} = \left(\begin{array}{cc}
(d\tilde{d}^T_{mn}) & -(d\tilde{d}_{m,r'}) \\
(d\tilde{c}_{r\mu,n}) & (d\tilde{g}_{r\mu,r'})
\end{array}\right)^{22,st}_{\alpha\beta} = (\hat{P} \hat{T}^{-1} \ (d\hat{T}) \ \hat{P}^{-1})^{11}_{\alpha\beta}.
\]

\[3.75\]

Since the coset eigenvalues \(\overline{\tau}_m\) \[2.19\], \(\overline{f}_r\) \[2.20\] only take part in the transformed actions of Euclidean path integration variables with their total values of the moduli \(|\overline{\tau}_m|\), \(|\overline{f}_r|\) and phases \(\varphi_m, \varphi_r\), but without any derivatives 'd', we have to relate the coset eigenvalues \(\overline{\tau}_m, \overline{f}_r\) to the total derivative 'd' to the Euclidean pair condensate variables. The causal structure of the original time development operators, which result with over-complete sets of states at every time step into coherent state path integrals as an underlying lattice theory, also leads to a natural time ordering in our case \[2.84,2.86\]. The causal structure of \[2.84,2.86\] is determined by the time contour, due to the two branches of forward and backward propagation. At each slice of the time contour propagation along the coherent state path generating function, we choose independent, Euclidean pair condensate path fields where the independence of the Euclidean, anomalous path fields at every time slice refers to the spatial distribution and internal indices of the super-matrices. After having chosen the set of independent spatial fields at every time slice, the exponential phases with the actions assign a weight to the particular chosen sets of Euclidean fields according to the quadratic couplings of the kinetic energies and composed densities from anomalous variables. Therefore, the total derivative 'd' in \[3.68,3.75\], which relates the coset eigenvalues \(\overline{\tau}_m, \overline{f}_r\) to the total values of anomalous, Euclidean fields, only contains the partial time
3.3 Eigenvalues of cosets for anomalous terms and their transformed, Euclidean correspondents

contour derivative, corresponding to chosen time contour paths for every spatial point of an underlying lattice field theory

\[
\begin{align*}
(d|\vec{c}_m(t_p)|) &= \frac{\partial|\vec{c}_m(t_p)|}{\partial t_p} dt_p; \\
(d|\vec{f}_r(t_p)|) &= \frac{\partial|\vec{f}_r(t_p)|}{\partial t_p} dt_p;
\end{align*}
\] (3.76)

The spatial vector $\vec{x}$ is omitted in relations (3.76) because the Euclidean integration variables of the generating function (2.84-2.86) are determined by time contour paths with spatially independent points which could be abbreviated by an additional index '$\vec{x}' apart from the indices $m = 1, \ldots, L$ or $r = 1, \ldots, S/2$ for the angular momentum degrees of freedom. In consequence, it suffices to calculate the partial time contour derivatives of relations (3.76) in dependence on the Euclidean, pair condensate, integration variables of $\hat{X}_{\alpha\beta}$ and $\kappa \hat{X}^+_{\alpha\beta}$. In fact, it turns out that the differential, absolute values $(d|\vec{c}_m(t_p)|), (d|\vec{f}_r(t_p)|)$ can be integrated to their total values $|\vec{c}_m(t_p)|, |\vec{f}_r(t_p)|$, according to the property of a total derivative for a state variable, whereas the phase values $\varphi_m(t_p), \phi_r(t_p)$ involve the detailed time contour path or the past time contour history of the Euclidean, pair condensate path variables in order to perform the time contour integrals of the phases in (3.76).

We consider again the relations (3.9-3.11) for the variation of exponentials of matrices and obtain Eq. (3.77). However, the suitable choice of gauge (3.78) for diagonal elements of $(d\hat{P}) \hat{P}^{-1}$, described in Eqs. (2.26-2.35), gives rise to a vanishing of the diagonal commutator matrix elements $[\hat{Y}_{DD}, (d\hat{P}) \hat{P}^{-1}]_{\mu\alpha}^a$ between $\hat{Y}_{DD}$ (3.79) (the diagonal, original coset generator with eigenvalues $\vec{c}_m$, $\vec{f}_r$) and the diagonal (quaternion diagonal), vanishing elements of $(d\hat{P}) \hat{P}^{-1}$ (3.78). In consequence, eigenvalues $\vec{c}_m$ (quaternion eigenvalues $(\tau_2)_{\mu\nu} \vec{f}_r$) of $\hat{Y}_{DD}$ (3.79) are mapped onto the diagonal (quaternion diagonal) anomalous matrix elements in $(d\hat{X}_{BB:mm}) = -(d\hat{b}_{mm}), \kappa (d\hat{X}^+_{BB:mm}) = (d\hat{b}^*_{mm})$ and $(d\hat{X}_{FF,rr,rr}) = (d\hat{a}_{rr,rr})$, $\kappa (d\hat{X}^+_{FF,rr,rr}) = (d\hat{a}^*_{rr,rr})$ (3.80). The latter fields are also taken as the independent variables for the various diagonal elements of the densities (3.86-3.89) and block parts in $(d\hat{Y}_{BB:mm}) = t (d\hat{d}_{mm}), (d\hat{Y}_{FF,rr,rr}) = t (d\hat{d}^{(0)}_{rr}) \delta_{\mu\nu}$ and correspondingly in $(d\hat{Y}^{(2)}_{NN\times N})$: \((d\hat{d}_{rr,rr}) = \sum_{k=0}^3 (\tau_k)_{\mu\nu} (d\hat{d}^{(k)}_{rr,rr})\)

\[
\begin{align*}
-(d\hat{Z}_{ab}^{\alpha\beta}) &= \left( \delta \hat{P}^{-1} \delta \hat{P} \right)_{ab}^{\alpha\beta} = - \int_0^1 dv \ e^v \hat{Y}_{DD} \ \hat{P} \left( d \hat{P}^{-1} \hat{Y}_{DD} \ \hat{P} \right) e^{-v} \hat{Y}_{DD} \\
&= - \int_0^1 dv \ e^v \hat{Y}_{DD} \left( (d\hat{Y}_{DD}) + \left[ \hat{Y}_{DD}, (d\hat{P}) \hat{P}^{-1} \right] \right) e^{-v} \hat{Y}_{DD} \\
\left( (d\hat{P}) \hat{P}^{-1} \right)_{BB:mm} &\equiv 0; \quad \left( (d\hat{P}) \hat{P}^{-1} \right)_{FF,rr,rr} \equiv 0 \quad \left( \left[ \hat{Y}_{DD}, (d\hat{P}) \hat{P}^{-1} \right] \right)_{\alpha\alpha} \equiv 0; \\
(d\hat{Y}_{DD}) &= \left( 0 \ k \hat{d}_{DD} \ k \hat{X}_{DD} \ 0 \right); \quad (d\hat{X}_{DD}) = \left( -\left( d\vec{c}_m \right) \delta_{m,n} \ k \hat{d}_{DD} \ 0 \right) \left( (\tau_2)_{\mu\nu} \ k \hat{d}_{rr,rr} \right) \ 0 \ \\
(d\hat{Z}_{ab}^{BB:mm}) &= \int_0^1 dv \ e^v \hat{Y}_{DD} \ (d\hat{Y}_{DD}) e^{-v} \hat{Y}_{DD} \ \\
&= \int_0^1 dv \ \end{align*}
\] (3.77)

(3.78)

(3.79)

(3.80)
where one can also observe the additional negative sign of the fermion-fermion density element. The separation into real and imaginary parts of Eqs. (3.90-3.93) guides us to the relations (3.94-3.97)

\[
(d\hat{\gamma}_{ab}^{12})_{FF;\tau_{\mu},\nu} = \int_{0}^{1} dv \left( e^{v\hat{\gamma}_{DD}} (d\hat{\gamma}_{DD}) e^{-v\hat{\gamma}_{DD}} \right)_{FF;\tau_{\mu},\nu}^{ab}
\]

\[
(d\hat{\gamma}_{11}^{11})_{\alpha\beta} = \left( -\left( d\hat{g}_{mm} \right)_{\delta_{mn}} \left( \tau_{2} \right)_{\mu\nu} \left( d\hat{a}_{rr}^{(2)} \right)_{\delta_{rr'}} \right)_{\alpha\beta}^{11}
\]

\[
(d\hat{\gamma}_{22}^{22})_{\alpha\beta} = -i \left( \left( d\hat{d}_{mm} \right)_{\delta_{mn}} \left( \delta_{\mu\nu} \delta_{rr'} \right) \left( d\hat{g}_{rr}^{(0)} \right) \right)_{\alpha\beta}^{22}
\]

\[
(d\hat{a}_{mm}(t_{p}) - d\hat{a}_{mm}(-\infty)) = \int_{-\infty}^{t_{p}} dt' \left( \frac{\partial (d\hat{a}_{mm}(t'))}{\partial t'} \right); \quad \left( m = 1, \ldots, L = 2l + 1 \right)
\]

\[
(d\hat{g}_{rr}(t_{p}) - d\hat{g}_{rr}(-\infty)) = \int_{-\infty}^{t_{p}} dt' \left( \frac{\partial (d\hat{g}_{rr}(t'))}{\partial t'} \right); \quad \left( r = 1, \ldots, S/2 = s + 1/2 \right)
\]

According to the relations in section 3.2 and appendix C of Ref. [6], we can apply the known transformations (3.80-3.83) between eigenvalues \(\bar{c}_{m} \), \(\hat{f}_{r} \), and the new, independent Euclidean elements \((d\hat{b}_{mm}) \), \((d\hat{a}_{rr}^{(2)}) \) and as well the dependent, diagonal density terms \((d\hat{d}_{mm}) \), \((d\hat{g}_{rr}^{(0)}) \) in order to determine the functions (3.84-3.85, 3.88, 3.89) with following relations (3.90-3.93)

\[
(d\hat{b}_{mm}) = d\hat{b}_{mm} \left( e^{i\beta_{m}} \right) = (d\bar{c}_{m}) e^{i2\phi_{m}} \left( \frac{1}{2} - \frac{\sin \left( 2 \left| \bar{c}_{m} \right| \right)}{4 \left| \bar{c}_{m} \right|} \right) + (d\bar{c}_{m}) \left( \frac{1}{2} + \frac{\sin \left( 2 \left| \bar{c}_{m} \right| \right)}{4 \left| \bar{c}_{m} \right|} \right)
\]

\[
(d\hat{a}_{rr}^{(2)}) = d\hat{a}_{rr}^{(2)} \left( e^{i\alpha_{r}} \right) = (d\hat{f}_{r}) e^{i2\phi_{r}} \left( \frac{1}{2} - \frac{\sinh \left( 2 \left| \hat{f}_{r} \right| \right)}{4 \left| \hat{f}_{r} \right|} \right) + (d\hat{f}_{r}) \left( \frac{1}{2} + \frac{\sinh \left( 2 \left| \hat{f}_{r} \right| \right)}{4 \left| \hat{f}_{r} \right|} \right)
\]

\[
\left( d\hat{d}_{mm} \right) = \left( \left( d\bar{c}_{m} \right) \left( e^{-i\phi_{m}} \right) - \left( d\bar{c}_{m}^{*} \right) \left( e^{i\phi_{m}} \right) \right) \left( \frac{\sin \left( \left| \bar{c}_{m} \right| \right)}{2 \left| \bar{c}_{m} \right|} \right); \quad \hat{d}_{mm} \in \mathbb{R}
\]

\[
\left( d\hat{g}_{rr}^{(0)} \right) = \left[ \left( d\hat{f}_{r} \right) \left( e^{-i\phi_{r}} \right) - \left( d\hat{f}_{r}^{*} \right) \left( e^{i\phi_{r}} \right) \right] \left( \frac{\sinh \left( \left| \hat{f}_{r} \right| \right)}{2 \left| \hat{f}_{r} \right|} \right); \quad \hat{g}_{rr}^{(0)} \in \mathbb{R}
\]

The separation into real and imaginary parts of Eqs. (3.90-3.93) guides us to the relations (3.94-3.97) where one can also observe the additional negative sign of the fermion-fermion density element \((d\hat{g}_{rr}^{(0)})\)
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with respect to the boson-boson density element \((d\hat{b}_{mm})\). This additional negative sign is caused by the U(\(L|S\)) super-symmetry of the original super-symmetric density \(\psi_{\tilde{x},m}^\dagger(t_p) \psi_{\tilde{x},m}(t_p) + \psi_{\tilde{x},r\mu}^\dagger(t_p) \psi_{\tilde{x},r\mu}(t_p)\) which corresponds to the difference of boson-boson and fermion-fermion densities

\[
\begin{align*}
(d\hat{b}_{mm}) &= d(\hat{b}_{mm} | e^i\beta_m) = e^i\varphi_m \left[ (d|\bar{c}_m|) + i \frac{\sin(2|\bar{c}_m|)}{2} (d\varphi_m) \right]; \\
(d\bar{a}_{rr}^{(2)}) &= d(\bar{a}_{rr}^{(2)} | e^i\alpha_r) = e^i\phi_r \left[ (d|\bar{f}_r|) + i \frac{\sinh(2|\bar{f}_r|)}{2} (d\phi_r) \right]; \\
(d\hat{d}_{mm}) &= \left( \sin(|\bar{c}_m|) \right)^2 (d\varphi_m); \\
(d\bar{g}_{rr}^{(0)}) &= -\left( \sinh(|\bar{f}_r|) \right)^2 (d\phi_r).
\end{align*}
\]

We introduce new pair condensate integration variables \(\tilde{b}_{mm} = \bar{b}_{mm} | e^i\tilde{\beta}_m\) and perform integration measure preserving phase rotations with \(e^{-i}\varphi_m\), \(e^{-i}\tilde{\beta}_m\) and \(e^{-i}\phi_r\), \(e^{-i}\tilde{\alpha}_r\), respectively

\[
\begin{align*}
\tilde{b}_{mm} &= \bar{b}_{mm} | e^i\tilde{\beta}_m; \\
e^{-i}\tilde{\beta}_m (d\tilde{b}_{mm}) &= e^{-i}\varphi_m (d\bar{b}_{mm}) \implies (d\bar{b}_{mm}) \wedge (d\tilde{b}_{mm}) = (db_{mm},) \wedge (db_{mm}) + (d\bar{b}_{mm}) \wedge (d\bar{b}_{mm}) ; \\
e^{-i}\tilde{\alpha}_r (d\tilde{a}_{rr}^{(2)}) &= e^{-i}\phi_r (d\bar{a}_{rr}^{(2)}) \implies (d\bar{a}_{rr}^{(2)}) \wedge (d\tilde{a}_{rr}^{(2)}) = (d\bar{a}_{rr}^{(2)}) \wedge (d\bar{a}_{rr}^{(2)}) .
\end{align*}
\]

Accordingly, we can replace the diagonal, Euclidean, pair condensate integration variables of Eqs. \[(3.98)\] by \((d\bar{b}_{mm}), (d\bar{a}_{rr}^{(2)}), (+c.c.)\) and obtain new relations between the coset eigenvalues \(|\bar{c}_m|, \varphi_m, |\bar{f}_r|, \phi_r\) and the rotated Euclidean, anomalous elements \(\tilde{b}_{mm} = \bar{b}_{mm} | e^i\tilde{\beta}_m, \tilde{a}_{rr}^{(2)} = \bar{a}_{rr}^{(2)} | e^i\tilde{\alpha}_r\) of Eqs. \[(3.98)\]

\[
\begin{align*}
(d|\tilde{b}_{mm}|) + i |\tilde{b}_{mm}| (d\tilde{\beta}_m) &= (d|\bar{c}_m|) + i \frac{\sin(2|\bar{c}_m|)}{2} (d\varphi_m); \\
(d|\tilde{a}_{rr}^{(2)}|) + i |\tilde{a}_{rr}^{(2)}| (d\tilde{\alpha}_r) &= (d|\bar{f}_r|) + i \frac{\sinh(2|\bar{f}_r|)}{2} (d\phi_r).
\end{align*}
\]

In consequence of measure preserving phase rotations, the total derivatives \((d|\tilde{b}_{mm}|), (d|\bar{c}_m|)\) and \((d|\tilde{a}_{rr}^{(2)}|), (d|\bar{f}_r|)\) result between the absolute values of Euclidean, diagonal variables and the coset eigenvalues so that the absolute values of these transformations are related to path-independent 'state variables' of thermodynamics in a 'transferred sense'. (One can even substitute the contour time 't' by the inverse temperature '\(\tau\)' and the contour integrals by the inverse temperature path '0...\beta = 1/(KT)' of grand canonical statistical operators. In this case, thermodynamical state variables of the absolute values \(|\tilde{b}_{mm}(\tau)|, |\bar{c}_m(\tau)|\) and \(|\tilde{a}_{rr}^{(2)}(\tau)|, |\bar{f}_r(\tau)|\) can be identified after similar HST’s and coset decompositions of coherent state representations of grand canonical 'inverse temperature' development operators so that the analogy becomes exact.)

\[
|\bar{c}_m(\tilde{x}, t_p)| - |\bar{c}_m(\tilde{x}, -\infty_+)| \bigg|_{0} = |\tilde{b}_{mm}(\tilde{x}, t_p)| - |\tilde{b}_{mm}(\tilde{x}, -\infty_+)| \bigg|_{0};
\]
The phases \( \varphi_m, \phi_r \) of the complex coset eigenvalues \( \tau_m, \mathcal{T}_r \) are path-dependent with respect to the contour time \( t_p \) because they are not determined by total derivatives and therefore correspond to a kind of 'heat'- or 'work'-variables of thermodynamics, also in a transferred sense.

\[
\varphi_m(\tilde{x}, \tilde{t}_p) - \varphi_m(\tilde{x}, -\infty_+) = \int_{t_p}^{t_p} d\tilde{t}_q \frac{\partial \varphi_m(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q} = \int_{t_p}^{t_p} d\tilde{t}_q 2 \left| \bar{b}_{mm}(\tilde{x}, \tilde{t}_q) \right| \frac{\partial \bar{\beta}_m(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q}; \quad (3.104)
\]

\[
\phi_r(\tilde{x}, \tilde{t}_p) - \phi_r(\tilde{x}, -\infty_+) = \int_{t_p}^{t_p} d\tilde{t}_q \frac{\partial \phi_r(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q} = \int_{t_p}^{t_p} d\tilde{t}_q 2 \left| \bar{a}^{(2)}_{rr}(\tilde{x}, \tilde{t}_q) \right| \frac{\partial \bar{\alpha}_{rr}(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q}. \quad (3.105)
\]

The diagonal boson-boson and fermion-fermion densities are also path-dependent, due to the phases \( (d\varphi_m) \) and \( (d\phi_r) \), and are specified by following Eqs. (3.106-3.107), after substitution of (3.100-3.101) into (3.96-3.97)

\[
\hat{d}_{mm}(\tilde{x}, \tilde{t}_p) - \hat{d}_{mm}(\tilde{x}, -\infty_+) = \int_{t_p}^{t_p} d\tilde{t}_q \tan \left( \left| \bar{b}_{mm}(\tilde{x}, \tilde{t}_q) \right| \right) \frac{\partial \bar{\beta}_m(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q}; \quad (3.106)
\]

\[
\hat{g}_{rr}^{(0)}(\tilde{x}, \tilde{t}_p) - \hat{g}_{rr}^{(0)}(\tilde{x}, -\infty_+) = - \int_{t_p}^{t_p} d\tilde{t}_q \tanh \left( \left| \tilde{a}^{(2)}_{rr}(\tilde{x}, \tilde{t}_q) \right| \right) \frac{\partial \bar{\alpha}_{rr}(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q}. \quad (3.107)
\]

In summary, we have defined new integration variables \( \tilde{b}_{mm}(\tilde{x}, \tilde{t}_p), \tilde{a}^{(2)}_{rr}(\tilde{x}, \tilde{t}_p) \) for the diagonal matrix elements of Eqs. (3.82-3.83) and (3.86-3.87) so that, according to the contour time ordering, we choose partial derivatives of phases \( \partial \bar{\beta}_m/\partial \tilde{t}_q, \partial \bar{\alpha}/\partial \tilde{t}_q \) which determine the path-dependent phases \( \varphi_m(\tilde{x}, \tilde{t}_p) \), \( \phi_r(\tilde{x}, \tilde{t}_p) \) (3.104-3.105) and also the path-dependent diagonal density elements \( \hat{d}_{mm}(\tilde{x}, \tilde{t}_p), g_{rr}^{(0)}(\tilde{x}, \tilde{t}_p) \) (3.106-3.107). The absolute values of the coset eigenvalues \( |\bar{c}_m|, |\mathcal{T}_r| \) transform in a path-independent manner as corresponding 'state variables' and are equivalent to the absolute values \( |\tilde{b}_{mm}|, |\tilde{a}^{(2)}_{rr}| \) of the new, phase-rotated, Euclidean integration variables (3.98-3.101).

It remains to determine the block diagonal \( \hat{\Phi}, \hat{\Phi}^{-1} \) super-matrix of Eqs. (3.77-3.108) in terms of the anomalous parts \( (d\hat{\Phi}_{\alpha\beta}) \). Since we have accomplished definite relations between coset eigenvalues \( \hat{Y}_{DD} \) and the diagonal elements of pair condensates \( (\hat{\kappa}_{\alpha\beta}^{ab}) \) (also comprising the diagonal parts of densities), we can apply relations (3.77-3.78) or (3.108-3.109) in order to calculate \( (d\hat{\Phi}) \hat{\Phi}^{-1} \) in terms of \( (d\hat{\Phi}_{\alpha\beta}) \)

\[
-(d\hat{\Phi}_{\alpha\beta}) = \left( \hat{\Phi} \hat{\Phi}^{-1} (d\hat{\Phi}_{\alpha\beta}) \hat{\Phi}^{-1} \right) = - \int_0^1 dv \, e^v \hat{Y}_{DD} \hat{\Phi} \left( d\hat{\Phi}^{-1} \hat{Y}_{DD} \hat{\Phi} \right) \hat{\Phi}^{-1} e^{-v} \hat{Y}_{DD} \quad (3.108)
\]

\[
= - \int_0^1 dv \, e^v \hat{Y}_{DD} \left( (d\hat{Y}_{DD}) + [\hat{Y}_{DD}, (d\hat{\Phi}) \hat{\Phi}^{-1}] \right) e^{-v} \hat{Y}_{DD}
\]
\[
(d\hat{\mathcal{P}})\hat{\mathcal{P}}^{-1}\bigg|_{BB,mm} = 0; \quad (d\hat{\mathcal{P}})\hat{\mathcal{P}}^{-1}\bigg|_{FF,\nu,rv} = 0 \quad \left[\hat{Y}_{DD}, (d\hat{\mathcal{P}})\hat{\mathcal{P}}^{-1}\right]_{\alpha\alpha} = 0. \tag{3.109}
\]

This can be achieved by a separation of the block diagonal \(N \times N\) matrices \(\hat{\mathcal{P}}, \hat{\mathcal{P}}^{-1}\) into subsequent multiplications of matrices where each matrix factor only contains a generator for a single parameter or (single quaternion parameter for the fermion-fermion parts). As consequence, one has to treat only \(2 \times 2\) matrices (or \(2 \times 2\) quaternion-valued matrices) which connect the different parts of the \(N \times N\) ladder generators within the block diagonal \(N \times N\) super-matrices \(\hat{\mathcal{P}}, \hat{\mathcal{P}}^{-1}\). After the factorization of \((d\hat{\mathcal{P}})\hat{\mathcal{P}}^{-1}\) into single group parts with generators comprising only single parameters, we use again (3.108) in order to integrate over \(v \in [0, 1]\) within \(\exp\{\pm v \hat{Y}_{DD}\}\) and the commutator \([\hat{Y}_{DD}, (d\hat{\mathcal{P}})\hat{\mathcal{P}}^{-1}]\) with the coset eigenvalues. This is a straightforward procedure, but tedious task for general \(N \times N\) super-matrices; we have also to point out that the resulting relation between \((d\hat{\mathcal{P}})\hat{\mathcal{P}}^{-1}\) and \((d\hat{Z}_{ab}^{\phi})\) strongly depends on the details of the parametrization, as e. g. the chosen sequence of factors with generators having only a single parameter.

## 4 Classical field equations and observables

### 4.1 Variation for classical field equations with Euclidean integration variables

In consequence to the previous section 3.3 we rename the diagonal, Euclidean integration variables \(\tilde{b}_{mm}, \tilde{a}_{rr,rv} = (\tau_2)^{\mu\nu} \tilde{a}_{rr}^{(2)}\) (3.98, 3.99), which preserve the Euclidean integration measure of anomalous fields, to their original symbols \(b_{mm} = e^{i\beta_m} |b_{mm}|\) and \(\tilde{a}_{rr,rv} = (\tau_2)^{\mu\nu} a_{rr}^{(2)} = (\tau_2)^{\mu\nu} e^{i\alpha_c} |a_{rr}^{(2)}|\). The total, Euclidean integration measure therefore consists of the time contour path fields \(d\tilde{b}_{mm}, d\tilde{a}_{rr,rv}, d\tilde{c}_{rr,rv}\) or of the terms of the pair condensate matrices \((d\hat{X}_{a\beta}) = (d\hat{Z}_{a\beta}^{12}), \tilde{\kappa} (d\hat{X}_{a\beta}^+), (d\hat{Z}_{a\beta}^{21})\) (compare Eqs. 3.1, 3.3)

\[
\begin{align*}
&= d[(d\hat{X}_{a\beta}), \tilde{\kappa} (d\hat{X}_{a\beta}^+)] = \prod_{\{\hat{\tau}, \hat{\mu}\}} \left\{ \prod_{m=1}^{L} \prod_{n=1}^{L} \frac{(db_{mn}^\alpha \wedge db_{mn}^{\beta})}{2i} \right\} \times \prod_{r=1}^{S/2} \left\{ \left( \frac{d\tilde{a}_{rr} \wedge d\tilde{a}_{rr}^{(2)}}{2i} \right) \right\} \times \prod_{r'=r}^{S/2} \left\{ \left( \frac{d\tilde{c}_{rr,rv} \wedge d\tilde{c}_{rr,rv}^{(2)}}{2i} \right) \right\} \times \prod_{r=1}^{S/2} \left\{ \left( \frac{d\tilde{c}_{rr,rv} \wedge d\tilde{c}_{rr,rv}^{(2)}}{2i} \right) \right\} \times \prod_{r=1}^{S/2} \left\{ \left( \frac{d\hat{Z}_{a\beta}^{12} \wedge d\hat{Z}_{a\beta}^{21}}{2i} \right) \right\}.
\end{align*}
\tag{4.1}
\]

The coherent state path integral of Eqs. 2.84, 2.85 thus takes the form (4.2) where we have transformed the coset integration measure \(d[T^{-1}(\hat{\tau}, \hat{\mu}) (dT(\hat{\tau}, \hat{\mu}))]\) (2.48) in sections 3.2.1, 3.2.3 to the Euclidean correspondents of integration variables (4.1) for the anomalous pair condensate fields

\[
Z[\hat{\mathcal{J}}, \hat{\mathcal{J}}, \hat{\mathcal{J}}_{\phi\psi}] = \int d[(\hat{Z}_{a\beta}^{12}, (d\hat{Z}_{a\beta}^{21})] \exp \left\{ iA_{\phi\psi}[\hat{T}] \right\} \times \exp \left\{ -A'[\hat{T}; \hat{\mathcal{J}}] \right\} \times \tag{4.2}
\]
The action $A_{\psi}[^{\hat{T}}]$ in (4.2) generates the anomalous fields; however, we neglect the detailed process of generation of pair condensates, which depends on temperature, the trap potential and further special properties in the experiments, and assume initial conditions for the Euclidean, anomalous fields $\hat{Z}_{12}, \hat{Z}_{21}$. The second action $A'[\hat{T}; \hat{\theta}]$ in (4.2) determines the observables with the original source field $\bar{J}_{a\beta} \bar{x}_x (\bar{t}'_q, \bar{t}_p)$ which relates observables, obtained by differentiation, to the original coherent state path integral (1.36) of super-fields $\psi_{x,\alpha}(t_p), \psi^*_{x,\alpha}(t_p)$. Hence, one can track the original observables (1.36) composed of the super-fields $\psi_{x,\alpha}(t_p), \psi^*_{x,\alpha}(t_p)$ to the transformed generating function (4.2) with the Euclidean, pair condensate integration variables whose action $A^{(d)}[^{\hat{Z}; \hat{\psi}}]$ contains the coupling coefficients $\bar{c}^{ij}(\bar{x}, \bar{t}_p)$ (2.55–2.58) and $\bar{d}^{ij}(\bar{x}, \bar{t}_p)$ (2.59) of the background density field. In terms of the new, Euclidean, pair condensate fields, the action $A^{(d)}[^{\hat{Z}; \hat{\psi}}]$ (2.83, 2.86) is altered to relations (4.3, 4.4) which allow variations for classical field equations, avoiding inconsistencies of nontrivial integration measures

$$A^{(d)}[^{\hat{Z}; \hat{\psi}}] = \int_C d\bar{t}_p \int d\bar{x} \left( \frac{x_0}{L} \right)^d L^{(d)}[^{\hat{Z}; \hat{\psi}}];$$

$$L^{(d)}[^{\hat{Z}; \hat{\psi}}] = -\left( \bar{c}^{ij} + \frac{1}{2} \delta_{ij} \right) \sum_{a,b=1,2} \text{str} \left[ \left( \partial_i \hat{Z}_{a \alpha}^{ab} \right) \left( \partial_j \hat{Z}_{b \alpha}^{ba} \right) \right] - \frac{1}{2} \sum_{a=1,2} \text{str} \left[ \left( \partial_i \hat{Z}_{a \alpha}^{a} \right) \left( \partial_i \hat{Z}_{a \alpha}^{a} \right) \right]$$

- $\frac{1}{2} \left( \bar{\mu}_0 - \bar{\mu}_p - \bar{\nu}_p \right) + \left( \bar{d}_{\alpha}^{(0)}(\bar{x}, \bar{t}_p) \right) \left( \frac{1}{2} \sum_{m=1}^L \left[ \cos (2 |\bar{b}_mm|) - 1 \right] - 4 \sum_{r=1}^{S/2} \left[ \cosh (2 |\bar{\alpha}_r^{(2)}|) - 1 \right] \right) + \frac{1}{2} \text{STR} \left[ \exp \left\{ 2 \hat{Y}_{DD} \right\} \bar{S} \left( \partial_{\bar{t}_p} \hat{Z} \right) - \bar{t} \left( \bar{d}^{ij} - \frac{1}{2} \delta_{ij} \right) \bar{J}_{\bar{t}_p} \bar{I} \bar{K} \bar{\Phi}^{-1} \left( \partial_{\bar{t}_p} \hat{Z} \right) \bar{\Phi} \bar{I} \bar{J}_p + \bar{\mu}_p \bar{I} \bar{K} \bar{\Phi}^{-1} \exp \left\{ -2 \bar{Y}_{DD} \right\} \left( \partial_{\bar{t}_p} \hat{Z} \right) \bar{S} \bar{\Phi} \bar{I} \bar{J}_p + \bar{\nu}_p \right)$$

$$+ \frac{\bar{n}_p}{2} \left\{ 2 \sum_{m=1}^L \left[ \cos (2 |\bar{b}_mm|) - 1 \right] - 4 \sum_{r=1}^{S/2} \left[ \cosh (2 |\bar{\alpha}_r^{(2)}|) - 1 \right] \right\}.$$
\[ (\partial \hat{\chi}_{\alpha \beta}) = -\left( \hat{P} \hat{T}^{-1} (\partial \hat{T}) \hat{P}^{-1} \right)_{\alpha \beta}^{12} = \begin{pmatrix} -\left( \partial \hat{b}_{mn} \right) & \left( \partial \hat{\zeta}_{m,r' \nu} \right) \\ -\left( \partial \hat{\xi}_{r,m,n} \right) & \left( \partial \hat{a}_{r,m,r' \nu} \right) \end{pmatrix}_{\alpha \beta} ; \] (4.6)

\[ \bar{\kappa} \left( \partial \hat{\chi}^{+}_{\alpha \beta} \right) = -\left( \hat{P} \hat{T}^{-1} (\partial \hat{T}) \hat{P}^{-1} \right)_{\alpha \beta}^{21} = \begin{pmatrix} \left( \partial \hat{b}^{*}_{mn} \right) & \left( \partial \hat{\zeta}^{+}_{m,r' \nu} \right) \\ \left( \partial \hat{\xi}^{+}_{r,m,n} \right) & \left( \partial \hat{a}^{+}_{r,m,r' \nu} \right) \end{pmatrix}_{\alpha \beta} ; \] (4.7)

\[ (\partial \hat{b}_{mn}) = (\partial \hat{b}^{T}_{mn}) ; \quad (\partial \hat{a}_{r,m,r' \nu}) = - (\partial \hat{a}^{T}_{r,m,r' \nu}) = \sum_{k=0}^{3} (\tau_{k})_{\mu \nu} \left( \partial \hat{d}_{r,r' \nu}^{(k)} \right) ; \] (4.8)

density terms in dependence on the Euclidean pair condensate fields

\[ (\partial \hat{y}_{\alpha \beta}^{11}) = \begin{pmatrix} (\partial \hat{d}_{mn}) & (\partial \hat{\zeta}_{m,r' \nu}^{+}) \\ (\partial \hat{\xi}_{r,m,n}) & (\partial \hat{g}_{r,m,r' \nu}) \end{pmatrix}_{\alpha \beta} ; \] (4.9)

\[ (\partial \hat{y}_{\alpha \beta}^{11})^{+} = \begin{pmatrix} (\partial \hat{d}_{mn}) & (\partial \hat{\zeta}_{m,r' \nu}^{+}) \\ (\partial \hat{\xi}_{r,m,n}) & (\partial \hat{g}_{r,m,r' \nu}) \end{pmatrix}_{\alpha \beta} ; \] (4.10)

\[ (\partial \hat{y}_{\alpha \beta}^{22}) = \begin{pmatrix} (\partial \hat{d}_{mn}^{T}) & (\partial \hat{\zeta}_{m,r' \nu}^{T}) \\ (\partial \hat{\xi}_{r,m,n}^{T}) & (\partial \hat{g}_{r,m,r' \nu}^{T}) \end{pmatrix}_{\alpha \beta} ; \] (4.11)

\[ (\partial \hat{y}_{\alpha \beta}^{22})^{st} = - (\partial \hat{y}_{\alpha \beta}) = \begin{pmatrix} (\partial \hat{d}_{mn}^{T}) & (\partial \hat{\zeta}_{m,r' \nu}^{T}) \\ (\partial \hat{\xi}_{r,m,n}^{T}) & (\partial \hat{g}_{r,m,r' \nu}^{T}) \end{pmatrix}_{\alpha \beta} . \] (4.12)

Appropriately to sections 3.2 and 3.3, we can also relate the listed density terms \((4.9,4.12)\) to the anomalous, Euclidean integration variables with time contour path dependent phases \(\varphi_{m}(\vec{x}, \hat{t}_{p})\), \(\phi_{r}(\vec{x}, \hat{t}_{p})\)

\[ i \left( \partial \hat{a}_{nn} \right) = \frac{1}{2} \tan \left( \frac{|\hat{b}_{mn}| + |\hat{b}_{nn}|}{2} \right) \left[ e^{-1} \varphi_{n} \left( \partial \hat{b}_{mn} \right) - e^{i} \varphi_{m} \left( \partial \hat{b}_{mn}^{*} \right) \right] + \] (4.13)

\[ i \left( \partial \hat{a}_{r,m,r' \nu} \right) = - \frac{1}{2} \sum_{k=0}^{3} (\tau_{k})_{\mu \nu} \left\{ \tanh \left( \frac{|\hat{a}_{rr}| + |\hat{a}_{r' \nu}|}{2} \right) \left[ e^{-1} \phi_{r} \left( \partial \hat{a}_{rr}^{(k)} \right) - \left( -1 \right)^{k} e^{i} \phi_{r} \left( \partial \hat{a}_{r,r' \nu}^{(k)} \right) \right] + \right. \] (4.14)

\[ \hat{a}_{rr}^{(2)} \neq 0 ; \quad \hat{a}_{r,r' \nu}^{(k)} \equiv 0 \quad \text{for} \quad k = 0, 1, 3 \] (4.15)

\[ i \left( \partial \hat{\xi}_{r,m,n} \right) = \frac{1}{2} \tanh \left( \frac{|\hat{a}_{rr}| + i |\hat{b}_{mn}|}{2} \right) \left[ e^{i} \phi_{r} \left( \tau_{2} \right)_{\mu \kappa} \left( \partial \hat{\zeta}_{r,r, \nu}^{*} \right) + \right. \] (4.16)

\[ + \frac{1}{2} \tanh \left( \frac{|\hat{a}_{rr}| - i |\hat{b}_{mn}|}{2} \right) \left[ e^{i} \phi_{r} \left( \tau_{2} \right)_{\mu \kappa} \left( \partial \hat{\xi}_{r,r, \nu}^{*} \right) \right] \];

\[ \varphi_{m}(\vec{x}, \hat{t}_{p}) = \int_{-\infty}^{\hat{t}_{p}} d\hat{t}_{q} \frac{2 |\hat{b}_{mn}(\vec{x}, \hat{t}_{q}^{'})|}{\sin \left( 2 |\hat{b}_{mn}(\vec{x}, \hat{t}_{q}^{'})| \right)} \frac{\partial \beta_{m}(\vec{x}, \hat{t}_{q}^{'})}{\partial \hat{t}_{q}^{'}} ; \] (4.17)
According to the listed Eqs. (4.3–4.20), the Lagrangian $L^{(d)[\tilde{Z}; \tilde{J}_\psi]}$ is outlined for its various boson-boson, fermion-fermion and odd fermion-boson, boson-fermion parts of the super-matrices. As already mentioned in Ref. [6], the fermion-fermion density parts are always accompanied by a phase factor of $e^{\pm \tau} = -1$ relative to the corresponding boson-boson density part of super-matrices.

\[
\begin{align*}
\phi_r(\tilde{x}, \tilde{t}_p) &= \int_{-\infty}^{\tilde{t}_p} d\tilde{t}_q \frac{2 |\hat{a}_{tr}^{(2)}(\tilde{x}, \tilde{t}_q)|}{\sinh(2 |\hat{a}_{tr}^{(2)}(\tilde{x}, \tilde{t}_q)|)} \frac{\partial \alpha_r(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q}; \quad (4.18) \\
\hat{Y}_{DD}(\tilde{x}, \tilde{t}_p) &= \left( \begin{array}{cc} 0 & \hat{X}_{DD}(\tilde{x}, \tilde{t}_p) \\ \hat{\kappa} \hat{X}_{DD}(\tilde{x}, \tilde{t}_p) & 0 \end{array} \right); \quad (4.19) \\
\hat{X}_{DD}(\tilde{x}, \tilde{t}_p) &= \left( -|\hat{b}_{mm}(\tilde{x}, \tilde{t}_p)| e^{i \varphi_m(\tilde{x}, \tilde{t}_p)} \quad 0 \right) \left( \begin{array}{c} 0 \\ (\tau_2)_{\mu \nu} |\hat{a}_{rr}^{(2)}(\tilde{x}, \tilde{t}_p)| e^{i \phi_r(\tilde{x}, \tilde{t}_p)} \end{array} \right). \quad (4.20)
\end{align*}
\]
\[ + \frac{\eta_p}{2} \left\{ \sum_{m=1}^{L} \left[ \cos \left( 2 |\hat{b}_{mm}(\vec{x}, t_p)| - 1 \right) \right] - 4 \sum_{r=1}^{S/2} \left[ \cosh \left( 2 |\hat{a}_{rr}^{(2)}| - 1 \right) \right] \right\} ; \\
\hat{\eta}_{\psi}(\vec{x}, t_p) = \hat{\eta}_{11}(\vec{x}, t_p) \hat{\eta}_{\psi}(\vec{x}, t_p). \] (4.22)

It remains to perform the variation of \( \mathcal{L}^{(d)}[\hat{Z}; \hat{\eta}_{\psi}] \) with respect to the independent, Euclidean fields in \( \delta \hat{Z}_{12}^{12}, \delta \hat{Z}_{13}^{21} \) for classical equations and quadratic fluctuations and we thus tabulate the variations of the different terms occurring in (4.4) or its expanded version (4.21)

\[
\hat{Z}_{\alpha \beta}(\vec{x}, t_{p=\pm}) = \hat{Z}_{\alpha \beta}(\vec{x}, t) \pm \frac{1}{2} \delta \hat{Z}_{\alpha \beta}(\vec{x}, t) \quad (4.23)
\]
\[
\delta \hat{Z}_{\alpha \beta}(\vec{x}, t) = \delta \hat{Z}_{\alpha \beta}^{ab} \hat{Z}_{\alpha \beta}(\vec{x}, t) + \delta \hat{Z}_{\alpha \beta}^{a \neq b}(\vec{x}, t) 
\] (4.24)
\[
\delta \hat{Z}_{12}^{12}(\vec{x}, t) = \left( \begin{array}{c}
-\delta \hat{b}_{mn}(\vec{x}, t) \\
-\delta \hat{\xi}_{\tau \mu, n}(\vec{x}, t)
\end{array} \right) \left( \begin{array}{c}
\delta \hat{\xi}_{\tau \mu, \nu}(\vec{x}, t) \\
\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}(\vec{x}, t)
\end{array} \right) \quad (4.25)
\]
\[
\delta \hat{Z}_{13}^{21}(\vec{x}, t) = \tilde{\kappa} \left( \delta \hat{Z}_{12}^{12}(\vec{x}, t) \right)^{+} = \left( \begin{array}{c}
\delta \hat{b}_{mn}(\vec{x}, t) \\
\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}(\vec{x}, t)
\end{array} \right) \left( \begin{array}{c}
\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}(\vec{x}, t) \\
\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}(\vec{x}, t)
\end{array} \right) \quad (4.26)
\]
\[
\delta \hat{Z}_{11}^{11}(\vec{x}, t) = \iota \left( \begin{array}{c}
\delta \hat{d}_{mn}(\vec{x}, t) \\
\frac{\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}}{\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}}(\vec{x}, t)
\end{array} \right) \quad (4.27)
\]
\[
\delta \hat{Z}_{22}^{22}(\vec{x}, t) = -\iota \left( \begin{array}{c}
\delta \hat{d}_{mn}(\vec{x}, t) \\
\frac{\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}}{\delta \hat{\xi}_{\tau \mu, \nu}^{(k)}}(\vec{x}, t)
\end{array} \right) \quad (4.28)
\]
\[
\iota \left( \delta \hat{d}_{mn} \right) = \frac{1}{2} \tan \left( \left| \hat{b}_{mn} \right| \right) e^{-i \varphi m} (\delta \hat{b}_{mn}) - e^{i \varphi m} (\delta \hat{b}_{mn}) \quad (4.29)
\]
\[
\iota \left( \delta \hat{d}_{mn} \right) = \frac{1}{2} \tan \left( \frac{\left| \hat{b}_{mn} + |\hat{b}_{mn}|}{2} \right) e^{-i \varphi n} (\delta \hat{b}_{mn}) - e^{i \varphi n} (\delta \hat{b}_{mn}) \quad (4.30)
\]
\[
\iota \left( \delta \hat{d}_{mn} \right) = \frac{1}{2} \tan \left( \frac{\left| \hat{b}_{mn} - |\hat{b}_{mn}|}{2} \right) e^{-i \varphi n} (\delta \hat{b}_{mn}) + e^{i \varphi m} (\delta \hat{b}_{mn}) \quad (4.31)
\]
\[
\iota \left( \delta \hat{g}_{\mu \nu} \right) = \iota \left( \tau_{0} \right)_{\mu \nu} (\delta \hat{a}^{(0)}_{rr}) = \frac{1}{2} \delta_{\mu \nu} \tanh (|\delta \hat{a}_{rr}^{(2)}|) e^{-i \varphi} (\delta \hat{a}_{rr}^{(2)}) - e^{i \varphi} (\delta \hat{a}_{rr}^{(2)}) \quad (4.32)
\]
\[
\iota \left( \delta \hat{g}_{\mu \nu} r_{r}^{r} \right) = -\frac{1}{2} \sum_{k=0}^{3} \left( \tau_{k} \tau_{2} \right)_{\mu \nu} \frac{\tanh \left( \left| \hat{a}_{rr}^{(2)} \right| \right)}{2} \left( e^{-i \varphi} (\delta \hat{a}_{rr}^{(2)}) - e^{i \varphi} (\delta \hat{a}_{rr}^{(2)}) \right) \quad (4.33)
\]
The variation \( \delta(\exp\{2 \tilde{Y}_{DD}\}) \) of the coset eigenvalues also involves a variation of the phases \( \delta \varphi_m(\tilde{x}, \tilde{t}_p) \), \( \delta \phi_r(\tilde{x}, \tilde{t}_p) \) with respect to \( \delta \beta_m(\tilde{x}, \tilde{t}_p) \), \( \delta \alpha_r(\tilde{x}, \tilde{t}_p) \) apart from the variation of absolute values \( \delta |\tilde{b}_{mm}(\tilde{x}, \tilde{t}_p)| \), \( \delta |\tilde{a}_{rr}(\tilde{x}, \tilde{t}_p)| \). The first order variation of the last term in \([1.4.1.21]\) vanishes completely, due to the additional contour metric \( \eta_p \). This term begins to contribute from second and all higher even order variations with universal fluctuations which are entirely determined by the coset decomposition \( \text{Osp}(S, S[2L]/\text{U}(L|S) \otimes \text{U}(L|S)) \). In the following relations \([1.3.3.1.37]\) we arrange the various diagonal parts of the coset eigenvalue matrix \( \exp\{2 \tilde{Y}_{DD}\}_{ab} \) and also point out the non-Markovian, path dependent phases \( \varphi_m(\tilde{x}, t_p) \), \( \phi_r(\tilde{x}, t_p) \) determined by the contour time history of the anomalous, Euclidean fields \( \tilde{b}_{mm}(\tilde{x}, t_p) \), \( \tilde{a}_{rr}(\tilde{x}, t_p) \)

\[
\begin{align*}
\left( \exp\{2 \tilde{Y}_{DD}\}_{ab} \right)_{\alpha \beta}^{11} &= \left( \exp\{2 \tilde{Y}_{DD}\}_{ab} \right)_{\alpha \beta}^{22} = \\
&= \begin{pmatrix}
\cos(2 |\tilde{b}_{mm}(\tilde{x}, \tilde{t}_p)|) & \delta_{mn} \\
0 & \cosh(2 |\tilde{a}_{rr}(\tilde{x}, \tilde{t}_p)|) & \delta_{rr'} & \delta_{\mu \nu}
\end{pmatrix}_{\alpha \beta}
\end{align*}
\]

(4.34)

\[
\begin{align*}
\left( \exp\{2 \tilde{Y}_{DD}\}_{ab} \right)_{\alpha \beta}^{12} &= \\
&= \begin{pmatrix}
-\sin(2 |\tilde{b}_{mm}(\tilde{x}, \tilde{t}_p)|) e^{i \varphi_m(\tilde{x}, \tilde{t}_p)} \delta_{mn} & 0 \\
0 & \sinh(2 |\tilde{a}_{rr}(\tilde{x}, \tilde{t}_p)|) e^{i \phi_r(\tilde{x}, \tilde{t}_p)} \delta_{rr'}
\end{pmatrix}_{\alpha \beta}
\end{align*}
\]

(4.35)

\[
\begin{align*}
\left( \exp\{2 \tilde{Y}_{DD}\}_{ab} \right)_{\alpha \beta}^{21} &= \\
&= \begin{pmatrix}
\sin(2 |\tilde{b}_{mm}(\tilde{x}, \tilde{t}_p)|) e^{-i \varphi_m(\tilde{x}, \tilde{t}_p)} \delta_{mn} & 0 \\
0 & \sinh(2 |\tilde{a}_{rr}(\tilde{x}, \tilde{t}_p)|) e^{-i \phi_r(\tilde{x}, \tilde{t}_p)} \delta_{rr'}
\end{pmatrix}_{\alpha \beta}
\end{align*}
\]

(4.36)

\[
\begin{align*}
\varphi_m(\tilde{x}, t_p) &= \int_{-\infty}^{t_p} \frac{d\tilde{t}_q}{\sin(2 |\tilde{b}_{mm}(\tilde{x}, \tilde{t}_p)|)} \frac{\partial \beta_m(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q} ; \\
\phi_r(\tilde{x}, t_p) &= \int_{-\infty}^{t_p} \frac{d\tilde{t}_q}{\sinh(2 |\tilde{a}_{rr}(\tilde{x}, \tilde{t}_p)|)} \frac{\partial \alpha_r(\tilde{x}, \tilde{t}_q)}{\partial \tilde{t}_q} .
\end{align*}
\]

(4.37)

After partial integrations in \([1.4.1.21]\), we obtain the first order variation \( \delta \mathcal{L}^{(d)}(\tilde{Z}; \tilde{J}_\psi) \) of the Lagrangian in terms of the matrices \( (\delta \tilde{Z}^{ab}_{\alpha \beta}) \) and \( (\delta \exp\{2 \tilde{Y}_{DD}\})_{ab}^{\alpha \beta} \) and also list the part which starts to contribute after second order variations for universal fluctuations around the classical solutions within the coset decomposition \( \text{Osp}(S, S[2L]/\text{U}(L|S) \otimes \text{U}(L|S)) \)

\[
\delta \mathcal{L}^{(d)}(\tilde{Z}; \tilde{J}_\psi) = \text{STR}_{a \alpha, b \beta} \left[ (\delta \tilde{Z}^{ab}_{\alpha \beta}) \left( \tilde{\partial}_l \left( 2 \tilde{c}^{ij}(\tilde{x}, \tilde{t}) (\tilde{\partial}_j \tilde{Z}^{a \beta}_{\alpha \sigma}) (1 - \delta_{ab}) + (\tilde{\partial}_j \tilde{Z}^{b \alpha}_{\beta \sigma}) \right) \right) + \\
+ \left\{ i \left( \tilde{\partial}_l \left( \tilde{c}^{ij}(\tilde{x}, \tilde{t}) - \frac{1}{2} \tilde{\partial}_{ij} \right) (\tilde{\partial}_j \tilde{Z}) , \tilde{\partial} \tilde{I} \tilde{J}_\psi \otimes \tilde{J}^+ \tilde{J}_\psi \tilde{\partial}^{-1} \right) \right\} + \\
+ \frac{1}{2} \tilde{\partial}_l \left( \left( i \tilde{1}_{2N \times 2N} + \tilde{\partial} \tilde{I}^{-1} \tilde{J}_\psi \otimes \tilde{J}^+ \tilde{I} \tilde{K} \tilde{\partial}^{-1} \right) \exp\{2 \tilde{Y}_{DD}\} \tilde{S} \right) + \left( \frac{\partial \tilde{\varphi}}{\partial \tilde{Z}} \right) \tilde{\partial}^{-1} \times
\right]
\]

\[\]
4.1 Variation for classical field equations with Euclidean integration variables

\[
\times \left[ \left( (\hat{\partial}_i \hat{Z}) (\hat{\partial}_j \hat{Z}) \right) \left( \hat{d}^i - \frac{1}{2} \delta_{ij} \right) + \frac{1}{2} \exp \{ -2 \hat{Y}_{DD} \} \left( \hat{\partial}_{p,i} \hat{Z} \right) \hat{S} , \hat{P} \hat{I} \hat{J}_p \otimes \hat{J}_p^+ \hat{I} \hat{K} \hat{P}^{-1} \right] \right\}^{ba}_{\beta\alpha} + \\
- \frac{1}{2} \mathrm{STR} \left[ (\delta \exp \{2 \hat{Y}_{DD} \})^{ab}_{\alpha \beta} \left( \left\{ \hat{S} \left( \hat{\partial}_i \hat{Z} \right) + \hat{1}_{2N \times 2N} (\hat{u}(\tilde{x}) - \hat{\mu}_0 + \Re (\langle \hat{\sigma}_D^{(0)}(\tilde{x}, \hat{t}) \rangle_{D}^{(0)}) \right) \right] + \\
+ \left[ \hat{S} \left( \hat{\partial}_i \hat{Z} \right) \hat{P} \hat{I}^{-1} \hat{J}_p \otimes \hat{J}_p^+ \hat{I} \hat{K} \hat{P}^{-1} \right]^{ba}_{\beta\alpha} + \frac{\eta_0}{2} \mathrm{STR} \left[ (\delta \exp \{2 \hat{Y}_{DD} \}) \right] + \\
+ \frac{\eta_0}{2} \left( 1 + \epsilon_+ + \Im (\langle \hat{\sigma}_D^{(0)}(\tilde{x}, \hat{t}) \rangle) \right) \left( \delta \mathrm{STR} \left[ (\delta \exp \{2 \hat{Y}_{DD} \}) \right] \right).
\]

The following steps seem to be involved and complicated, but are straightforward in order to attain the first order variations for the classical equations with the independent, anomalous Euclidean fields \( \hat{b}_{m,n}(\tilde{x}, \hat{t}_p), \hat{a}_{r,r'}(\tilde{x}, \hat{t}_p), \hat{\delta}_{r,r',r''}(\tilde{x}, \hat{t}_p) \) \((k = 0, 1, 2, 3), \hat{\phi}_{r,n}(\tilde{x}, \hat{t}_p), +c.c.)\). One has to relate the variations of the matrices \( (\delta \hat{Z}_{a\beta}) \) and \( (\delta \exp \{2 \hat{Y}_{DD} \})^{ab}_{\alpha \beta} \) in (4.38) to these independent, anomalous, Euclidean fields where each of these finally defines a classical equation. At first we specify the variations of the coset eigenvalue matrix \( (\delta \exp \{2 \hat{Y}_{DD} \})^{ab}_{\alpha \beta} \) in terms of \( \hat{b}_{m,n}(\tilde{x}, \hat{t}_p) \) and \( \hat{a}_{r,r'}(\tilde{x}, \hat{t}_p) \) which introduce the \( \sin(h) \) and \( \cos(h) \)-functions of these diagonal elements into the first order variations to classical field equations.
At first we specialize on the variation with diagonal (quaternion diagonal) matrix elements \( \delta \hat{b}_{mn}^* (\tilde{x}, \tilde{t}) \), \( (\delta \hat{a}_{rr}^* (\tilde{x}, \tilde{t})) \) and have to extract these Euclidean fields from the variation within the matrices \( (\delta \hat{Z}_{\alpha \beta}^{ab}) \) and \( (\delta \exp\{2 \hat{Y}_{DD}\})_{\alpha \beta}^{ab} \) which appear in \( \delta \mathcal{L}^{(d)} [\hat{Z}; \hat{J}] \). As these fields \( \delta \hat{b}_{mn}^* (\tilde{x}, \tilde{t}) \), \( (\delta \hat{a}_{rr}^* (\tilde{x}, \tilde{t})) \) are separated from the various parts of the variations of the matrices \( (\delta \hat{Z}_{\alpha \beta}^{ab}) \) and \( (\delta \exp\{2 \hat{Y}_{DD}\})_{\alpha \beta}^{ab} \), one has to include coefficients \( \mathfrak{B}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \), \( \mathfrak{Y}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \) (derived from \( (\delta \hat{Z}_{\alpha \beta}^{ab}) \), \( (\delta \exp\{2 \hat{Y}_{DD}\})_{\alpha \beta}^{ab} \)) into the resulting field equation.

\[
\text{Variation with } (\delta \hat{b}_{mn}^* (\tilde{x}, \tilde{t}))
\]

\[
\mathfrak{B}_{mn}^{21} (\tilde{x}, \tilde{t}) = 1 ; \hspace{1cm} (4.46)
\]

\[
\mathfrak{B}_{mn}^{11} (\tilde{x}, \tilde{t}) = - \mathfrak{B}_{mn}^{22} (\tilde{x}, \tilde{t}) = - \frac{1}{2} \tan \left( |\hat{b}_{mn} (\tilde{x}, \tilde{t})| \right) e^{i \varphi_m (\tilde{x}, \tilde{t})} ; \hspace{1cm} (4.47)
\]

\[
\mathfrak{Y}_{mn}^{21} (\tilde{x}, \tilde{t}) = \left( e^{-i \varphi_m (\tilde{x}, \tilde{t})} \cos \left( 2 |\hat{b}_{mn} (\tilde{x}, \tilde{t})| \right) + 1 \right) e^{i \beta_m (\tilde{x}, \tilde{t})} ; \hspace{1cm} (4.48)
\]

\[
\mathfrak{Y}_{mn}^{11} (\tilde{x}, \tilde{t}) = \mathfrak{Y}_{mn}^{22} (\tilde{x}, \tilde{t}) = - \sin \left( 2 |\hat{b}_{mn} (\tilde{x}, \tilde{t})| \right) e^{i \beta_m (\tilde{x}, \tilde{t})} . \hspace{1cm} (4.49)
\]

According to the above coefficients \( \mathfrak{B}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \), \( \mathfrak{Y}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \) of \( (\delta \hat{Z}_{\alpha \beta}^{ab}) \) and \( (\delta \exp\{2 \hat{Y}_{DD}\})_{\alpha \beta}^{ab} \), we can simplify the resulting equation of the diagonal pair condensate fields in the boson-boson part.

\[
0 \equiv \sum_{a,b=1,2}^{(a \geq b)} \mathfrak{B}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \left[ \partial_t \left( \sum_{j=1}^2 \tilde{c}^{ij} (\tilde{x}, \tilde{t}) \left( \partial_{\tilde{J}_j} \hat{Z}_{\alpha \beta}^{a \geq b} \right) (1 - \delta_{ab}) + (\partial_{\tilde{J}_j} \hat{Z}_{\alpha \beta}^{b \geq a}) \right) + \right.
\]

\[
+ \left\{ i \left( \tilde{c}^{ij} (\tilde{x}, \tilde{t}) - \frac{1}{2} \delta_{ij} \right) \left( \partial_{\tilde{J}_j} \hat{Z} \right) \hat{\Phi} \hat{I} \hat{J}_0 \hat{J}_0^* \hat{I} \hat{K} \hat{\Phi}^{-1} \right\} \left[ \hat{J}_0 \hat{J}_0^* \hat{I} \hat{K} \hat{\Phi}^{-1} \right] \bigg]_{BB;mn}^{b \leq a} + + \]

\[
+ \frac{1}{2} \partial_t \left[ \left( \sum_{j=1}^2 \tilde{c}^{ij} (\tilde{x}, \tilde{t}) - \frac{1}{2} \delta_{ij} \left( \partial_{\tilde{J}_j} \hat{Z} \right) \hat{\Phi} \hat{I} \hat{J}_0 \hat{J}_0^* \hat{I} \hat{K} \hat{\Phi}^{-1} \right) \exp\{2 \hat{Y}_{DD}\} \hat{S} \right] \left( \left( \partial_{\tilde{J}_j} \hat{Z} \right) \hat{\Phi} \hat{I} \hat{J}_0 \hat{J}_0^* \hat{I} \hat{K} \hat{\Phi}^{-1} \right) \bigg]_{BB;mn}^{b \leq a} + + \]

\[
- \frac{1}{2} \sum_{a,b=1,2}^{(a \geq b)} \mathfrak{Y}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \left[ i \left( \hat{S} \left( \partial_{\tilde{J}_j} \hat{Z} \right) \hat{\Phi} \hat{I} \hat{J}_0 \hat{J}_0^* \hat{I} \hat{K} \hat{\Phi}^{-1} \right) \right]_{BB;mn}^{b \leq a}.
\]

Similarly, we specify coefficients \( \mathfrak{Y}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \), \( \mathfrak{Y}_{mn}^{a \geq b} (\tilde{x}, \tilde{t}) \) from varying \( (\delta \hat{Z}_{\alpha \beta}^{ab}) \) and \( (\delta \exp\{2 \hat{Y}_{DD}\})_{\alpha \beta}^{ab} \) for the quaternion diagonal elements of the BCS pair condensates within the fermion-fermion section.
4.1 Variation for classical field equations with Euclidean integration variables

The analogous list of coefficients is defined in relations (4.51-4.54) for the equations following by the variation with respect to \( \delta \hat{a}_{rr}^{(2)*}(\vec{x}, \tilde{t}) \)

\[
\delta \mathbf{S}_{r\mu, r\nu}^{(2)}(\vec{x}, \tilde{t}) = -\mathbf{S}_{r\mu, r\nu}^{(2)}(\vec{x}, \tilde{t}) = (\tau_2)_{\mu\nu} \left( \hat{a}_{rr}^{(2)*}(\vec{x}, \tilde{t}) \right)
\]

\[
\delta \mathbf{Y}_{r\mu, r\nu}^{(2)}(\vec{x}, \tilde{t}) = -\mathbf{Y}_{r\mu, r\nu}^{(2)}(\vec{x}, \tilde{t}) = (\tau_2)_{\mu\nu} \left( \hat{a}_{rr}^{(2)*}(\vec{x}, \tilde{t}) \right)
\]

These coefficients contain Sinh-, Cosh- and Tanh-functions instead of their trigonometric correspondents for the diagonal pair condensates within the boson-boson part and allow to disentangle the resulting matrix equation for the quaternion diagonal fermion-fermion section. Since one has to consider quaternion elements in the case of fermion-fermion parts, we have to perform a trace \( 'tr_{\mu\nu} ' \) over the \( 2 \times 2 \) quaternion matrices occurring in the coefficients and the other parts of the resulting field equation

\[
0 \equiv - \sum_{a,b=1,2} \sum_{\mu, \nu} tr \left[ \mathbf{S}_{r\mu, r\nu}^{(2)b}(\vec{x}, \tilde{t}) \left[ \partial_t \left( 2 \hat{c}^{i}(\vec{x}, \tilde{t}) \hat{Z}_{\beta\gamma}^{b,a}(\vec{x}, \tilde{t}) \right) \right] \right] + \\
+ \left\{ i \left( \hat{d}^{ij}(\vec{x}, \tilde{t}) - \frac{1}{2} \delta_{ij} \right) \left( \hat{\partial}_j \hat{Z} \right) , \hat{P} \hat{I} \hat{J}_\psi \otimes \hat{J}^*_\psi \hat{I} \hat{K} \hat{P}^{-1} \right\}
\]

In the case of off-diagonal variations \( \delta \hat{b}_{m\neq n}^{(k)}(\vec{x}, \tilde{t}) \), \( \delta \hat{a}_{rr}^{(2)*}(\vec{x}, \tilde{t}) \) in the anomalous boson-boson or fermion-fermion part, we can neglect variations \( \left( \delta \exp\{2 \hat{Y}_{DD}\} \right)_{\alpha\beta} \) of the coset eigenvalue matrix because these only consist of diagonal (quaternion diagonal) elements \( \hat{b}_{m\neq n}^{(k)}(\vec{x}, \tilde{t}) \) (\( \hat{a}_{rr}^{(2)*}(\vec{x}, \tilde{t}) \)) apart from the non-Markovian phases \( \varphi_{m}(\vec{x}, \tilde{t}), \phi_{r}(\vec{x}, \tilde{t}) \). Therefore, we have only to take into account coefficients \( \mathbf{S}_{m\neq n}(\vec{x}, \tilde{t}) \) arising from the variation with the matrix \( \delta \hat{Z}_{ab}^{\alpha\beta} \)

Variation with \( \left( \delta \hat{b}_{m\neq n}^{(k)}(\vec{x}, \tilde{t}) \right) \)
Mind the symmetry: 
\[ (\delta \hat{b}_{m,n}^*(\tilde{x}, \tilde{t})) = (\delta \hat{b}_{n,m}^*(\tilde{x}, \tilde{t})) \]
\[
\mathfrak{B}_{mn}^{21}(\tilde{x}, \tilde{t}) = 1; \\
\mathfrak{B}_{mn}^{11}(\tilde{x}, \tilde{t}) = -\mathfrak{B}_{nm}^{22}(\tilde{x}, \tilde{t}) = -\frac{e^t \dot{\varphi}_m(\tilde{x}, \tilde{t})}{2} \left[ \tan \left( \frac{|b_{mm}| + |b_{nn}|}{2} \right) + \tan \left( \frac{|b_{nn}| - |b_{nn}|}{2} \right) \right].
\]

Application of the above coefficients \( \mathfrak{B}_{mn}^{a;b}(\tilde{x}, \tilde{t}) \) finally allows to give the classical field equations for the off-diagonal anomalous, boson-boson part in abbreviated from which also includes the trigonometric functions

\[
0 \equiv \left\{ \sum_{a,b=1,2} \mathfrak{B}_{mn}^{a;b}(\tilde{x}, \tilde{t}) \left[ \partial_t \left( 2 \dddot{\delta}^i (\tilde{x}, \tilde{t}) \left( \partial_j \dot{Z}_{\alpha \beta} \right) \left( 1 - \delta_{ab} \right) + \left( \partial_j \dot{Z}_{\alpha \beta} \right) \right) \right] + \\
+ \left\{ i \left( \dddot{\delta}^i (\tilde{x}, \tilde{t}) - \frac{1}{2} \delta_{ij} \right) \left( \partial_j \dot{Z} \right) \right\} + \\
+ \frac{1}{2} \dddot{\delta}_i \left( (\partial_t \dot{Z}) (\partial_j \dot{Z}) \left( \dddot{\delta}^i - \frac{1}{2} \delta_{ij} \right) + e^{-2 \hat{Y}_{DD}} \right) \left( \partial_t \dot{Z} \right) \right\} + \\
+ \left\{ \text{entire above terms with } m \leftrightarrow n \right\}.
\]

In analogy we catalogue the coefficients \( \mathfrak{S}_{\mu r,\nu r'}^{21,\alpha}(\tilde{x}, \tilde{t}) \) of the variations \( \delta \hat{Z}_{\alpha \beta}^{ab} \) for the off-diagonal quaternion matrix elements within the fermion-fermion section. Since the variation with \( (\delta \exp \left\{ 2 \hat{Y}_{DD} \right\}) \) is restricted to the quaternion diagonal elements, we have only to introduce coefficients \( \mathfrak{S}_{\mu r,\nu r'}^{21,\alpha}(\tilde{x}, \tilde{t}) \) for the variations within \( (\delta \hat{Z}_{\alpha \beta}^{ab}) \)

\[
\text{Variation with } (\delta \hat{a}_{\mu r}^{+}(\tilde{x}, \tilde{t})) \text{ in } (\delta \hat{a}_{\nu r'}^{+}(\tilde{x}, \tilde{t})) = \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu \nu} \left( \delta \hat{a}_{r r'}^{(k)}(\tilde{x}, \tilde{t}) \right);
\]

Mind the symmetries:
\[
(\delta \hat{a}_{\mu r}^{+}(\tilde{x}, \tilde{t})) = - (\delta \hat{a}_{\nu r'}^{+}(\tilde{x}, \tilde{t})) ; \\
(\delta \hat{a}_{\mu r}^{(2)+}) = + (\delta \hat{a}_{\nu r'}^{(2)+}) ; \quad \text{but } (\delta \hat{a}_{r r'}^{(k)+}) = - (\delta \hat{a}_{r r'}^{(k)+}) \text{ for } k = 0, 1, 3 ; \\
\mathfrak{S}_{\mu r,\nu r'}^{21,(\alpha)}(\tilde{x}, \tilde{t}) = (\hat{\tau}_k)_{\mu \nu} ; \\
\mathfrak{S}_{\mu r,\nu r'}^{11,(\alpha)}(\tilde{x}, \tilde{t}) = - \mathfrak{S}_{\nu r,\mu r'}^{22,(\alpha)}(\tilde{x}, \tilde{t}) = \frac{(-1)^k}{2} (\hat{\tau}_2)_{\mu \nu} e^{t \phi_r(\tilde{x}, \tilde{t})} \times \\ \\
\times \left[ \tanh \left( \frac{|a_{r r'}^{(2)}| + |a_{r r'}^{(2)}|}{2} \right) + \tanh \left( \frac{|a_{r r'}^{(2)}| - |a_{r r'}^{(2)}|}{2} \right) \right].
\]

The above coefficients with hyperbolic trigonometric functions again reduce the field equations to a compact form which includes traces over the \( 2 \times 2 \) quaternion elements for the anomalous, Euclidean
4.1 Variation for classical field equations with Euclidean integration variables

field variables in the off-diagonal fermion-fermion parts

\[
0 \equiv \left\{ - \sum_{a,b=1,2} \sum_{\nu} \text{tr} \left[ \delta_{r \mu, r' \nu}^{a \geq b}(k) \left( \tilde{\hat{\xi}}^{i j}(\tilde{x}, \tilde{t}) \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \left( \tilde{\partial}_{j \mu} \tilde{Z}^{2 a b} + \left( \tilde{\partial}_{j \alpha} \tilde{Z}^{b 2 a} \right) \right) \right) \right. \right. \\
+ \left. \left. \left\{ \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right\} \right\} \right\} + \\
+ \frac{1}{2} \tilde{\partial}_{i j} \left( \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right) \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right) \right) + \frac{e^2}{2} \tilde{\partial}_{i j} \left( \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right) \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right) \right) \right) \\
\times \left( \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right) \left( \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \right) \right) \\
+ \left\{ \text{entire above terms with } r \leftrightarrow r' \text{ for } k = 2 \right\} \text{ or } \\
- \left\{ \text{entire upper terms with } r \leftrightarrow r' \text{ for } k = 0, 1, 3 \right\} \\
\right)
\]

Finally, we approach the variation with respect to the anti-commuting, anomalous fields \( \tilde{\delta}^{\ast}_{r \mu, r \mu}(\tilde{x}, \tilde{t}) \) and extract corresponding coefficients \( 3_{FB; r \mu, n}^{a \geq b, (\kappa)}(\tilde{x}, \tilde{t}) \), \( 3_{BF; r \mu, n}^{a \geq b, (\kappa)}(\tilde{x}, \tilde{t}) \) for the fermion-boson and boson-fermion parts which are derived from the matrix \( \delta_{a \beta}^{\tilde{Z}} \).

\[
\text{Variation with } \left( \tilde{\delta}^{\ast}_{r \mu, n}(\tilde{x}, \tilde{t}) \right) ; \left( \tilde{\delta}^{\ast}_{n \mu, r}(\tilde{x}, \tilde{t}) \right)
\]

\[
3_{FB; r \mu, n}^{21, (\kappa)}(\tilde{x}, \tilde{t}) = 1 ; \\
3_{FB; r \mu, n}^{11, (\kappa)}(\tilde{x}, \tilde{t}) = \frac{e^2 \phi_n(\tilde{x}, \tilde{t})}{2} \left[ \text{tanh} \left( \frac{|\tilde{a}_{r \tau}|}{2} \right) + \text{tanh} \left( \frac{|\tilde{b}_{n \mu}|}{2} \right) \right] \left( \tau_2 \right)_{\mu \kappa} ; \\
3_{FB; r \mu, n}^{22, (\kappa)}(\tilde{x}, \tilde{t}) = \frac{i}{2} \left[ \text{tanh} \left( \frac{|\tilde{a}_{r \tau}|}{2} \right) - \text{tanh} \left( \frac{|\tilde{b}_{n \mu}|}{2} \right) \right] \left( \tau_0 \right)_{\mu \kappa} ; \\
3_{BF; n \mu, r \mu}^{21, (\kappa)}(\tilde{x}, \tilde{t}) = 1 ; \\
3_{BF; n \mu, r \mu}^{11, (\kappa)}(\tilde{x}, \tilde{t}) = -i \frac{e^2 \phi_n(\tilde{x}, \tilde{t})}{2} \left[ \text{tanh} \left( \frac{|\tilde{a}_{r \tau}|}{2} \right) - \text{tanh} \left( \frac{|\tilde{b}_{n \mu}|}{2} \right) \right] \left( \tau_0 \right)_{\kappa \mu} ; \\
3_{BF; n \mu, r \mu}^{22, (\kappa)}(\tilde{x}, \tilde{t}) = -i \frac{e^2 \phi_n(\tilde{x}, \tilde{t})}{2} \left[ \text{tanh} \left( \frac{|\tilde{a}_{r \tau}|}{2} \right) + \text{tanh} \left( \frac{|\tilde{b}_{n \mu}|}{2} \right) \right] \left( \tau_2 \right)_{\kappa \mu} .
\]

The coefficients \( 3_{FB; r \mu, n}^{a \geq b, (\kappa)}(\tilde{x}, \tilde{t}) \), \( 3_{BF; n \mu, r \mu}^{a \geq b, (\kappa)}(\tilde{x}, \tilde{t}) \) are partially composed of compact and non-compact (hyperbolic) trigonometric functions and have to be summed over the two spin degrees of freedom with the rest of the field equation which finally takes values within the Grassmann sector of the super-symmetric matrices

\[
0 \equiv \sum_{a,b=1,2} \sum_{\mu=1,2} \left[ \tilde{\delta}_{r \mu, n}^{a \geq b, (\kappa)}(\tilde{x}, \tilde{t}) \left[ \tilde{\partial}_{i j}(\tilde{x}, \tilde{t}) \left( \tilde{\partial}_{j \mu} \tilde{Z}^{2 a b} + \left( \tilde{\partial}_{j \alpha} \tilde{Z}^{b 2 a} \right) \right) (1 - \delta_{a b}) \right] \right]
\]
4.2 Observable quantities in terms of coset fields and their corresponding, Euclidean variables

Apart from the gradient expansion with \([4.69]\), we have also to take into account the generating source field \(\mathcal{J}(\hat{T}^{-1}, \hat{T})\) \([4.70]\) whose second order expansion of the effective actions is listed in relation \([4.71]\). This generating source field \(\mathcal{J}(\hat{T}^{-1}, \hat{T})\) \([4.70]\) can be replaced by derivatives with respect to the pair condensates 'seeds' \(\hat{J}_{\psi \psi ; \alpha \beta}^{ab}(\vec{x}, t_p)\) \([2.63][2.64]\) of the action \(\mathcal{A}_{\hat{J}_{\psi \psi}}[\hat{T}]\) \([2.65][2.69]\) for observables which go beyond the second order expansion of \(\mathcal{J}(\hat{T}^{-1}, \hat{T})\) in relation \([4.71]\). However, the pair condensate 'seed' fields \(\hat{J}_{\psi \psi ; \alpha \beta}^{ab}(\vec{x}, t_p)\) \([2.63][2.64]\) of the action \(\mathcal{A}_{\hat{J}_{\psi \psi}}[\hat{T}]\) \([2.65][2.69]\) do not allow for generating density terms as the source field \(\mathcal{J}(\hat{T}^{-1}, \hat{T})\) \([4.70]\). In correspondence to chapter 4 of Ref. [6], one can perform the gradient expansion with \(\delta \mathcal{H}(\hat{T}^{-1}, \hat{T})\) \([4.69]\) and with the source term \(\mathcal{J}(\hat{T}^{-1}, \hat{T})\) \([4.70]\)

We have classified in relations \((4.46-4.68)\) the various classical field equations following from first order variations of the independent, Euclidean pair condensate fields. Since we have considered general angular momentum degrees of freedom of the boson-boson, fermion-fermion and the odd fermion-boson, boson-fermion parts, various coefficients [cf. Eqs. \((4.46-4.49), (4.51-4.54), (4.56-4.57), (4.59-4.60), (4.62-4.67)\)] have to be used as part of the variation within \(\delta \mathcal{E}_{ab}\) or \(\delta \exp\{2 \hat{Y}_{DP}\\}_{ab}^{ab}\). One achieves coupled super-symmetric matrix equations which are composed of \(\sin(h)\)-, \(\cos(h)\)- or \(\tan(h)\)-functions of the diagonal, Euclidean pair condensate fields so that these illustrate modifications of the well-known, integrable \(\text{Sine-Gordon}\) equations in 1+1 or 2+1 dimensions. These matrix equations correspond to the Gross-Pitaevskii equation in a transferred sense if one regards the coherent super-symmetric pair condensates in analogy to the coherent BEC-wavesfunction.
order to classify the various terms for the pair condensate observables or density related observables

\[
\delta \hat{H}(\hat{T}^{-1}, \hat{T}) = -\hat{\eta} \left( \hat{T}^{-1} \hat{S} (E_p \hat{T}) + \hat{T}^{-1} (\partial_i \hat{\eta} \hat{T}) + (\hat{T}^{-1} \hat{S} \hat{T} - \hat{S}) \hat{E}_p + 2 \hat{T}^{-1} (\hat{\eta} \hat{T}) \hat{\eta} \right) \tag{4.69}
\]

\[
\tilde{\sigma}_{ab}^{\alpha x, p, \beta} (\hat{T}^{-1}(t_p), \hat{T}(t_q)) = \hat{T}_{\alpha \alpha'}^{-1} (\hat{x}, t_p) \hat{I} \hat{K} \eta_p \frac{\hat{g}_{\alpha' x, p, \beta}^{\gamma}(t_p, t_q)}{\eta_q} \hat{K} \hat{I} \hat{K} \hat{T}_{\gamma \beta}(\hat{x}', t_q) \tag{4.70}
\]

\[
\mathcal{A}' [\hat{T}; \hat{\eta}] = -\frac{1}{4} \left\langle \text{Tr STR} \left[ \hat{\delta} (\hat{T}^{-1}, \hat{T}) \hat{G}(0)[\hat{\sigma}_D^{(0)}] \hat{\delta} (\hat{T}^{-1}, \hat{T}) \hat{G}(0)[\hat{\sigma}_D^{(0)}] \right] \right\rangle_{\hat{\sigma}_D^{(0)}} + \tag{4.71}
\]

The action \(\mathcal{A}' [\hat{T}; \hat{\eta}]\) also involves the averaging \(\langle \ldots \rangle_{\hat{\sigma}_D^{(0)}}\) over the background density field \(\sigma_D^{(0)}(\tilde{x}, t_p)\) with generating function \(Z[j_{\psi}]; \hat{\sigma}_D^{(0)}(0)\) (compare with \(2.74\)). However, we can simplify this averaging process by taking the classical field value which results from the saddle point equation outlined in \((2.76)\).

### 4.3 Outlook for relations between chaotic and integrable systems with modified r-s matrices

A particular property of the nonlinear sigma-model equations \((4.16)\) is the integrability for special dimensions, as 1+1 or even 2+1 \([53]-[60]\). These properties of integrability are determined by r-s matrix properties which can be investigated as quantum groups \([57]-[60]\). However, as already suggested in \([6]\), one can also try to classify chaotic systems as extensions of these r-s matrix bi-algebras in analogy of extensions of group or algebraic properties if one adds symmetry breaking generators or group elements to the classical equations. If one considers the general BCH-formulas for abstract time- or spatial development operators, one can always relate the multiplication of two exponentials \(\exp\{A\}, \exp\{B\}\) with some operators \(\hat{A}\) and \(\hat{B}\) to the exponential of commutator terms between these generators. If the commutator algebra between \(\hat{A}\) and \(\hat{B}\) is closed, one has specific, closed group properties which may be
related to integrable systems. If the commutator algebra of operators $\hat{A}$ and $\hat{B}$ has deviations from the prevailing, closed algebraic structures, it should be possible to refer the non-closed algebraic structure in the exponential to some chaotic behavior. According to these suggestions, it might be possible to examine chaotic systems as extensions of the few integrable classical equations determined by r-s matrix structures.

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