THE INTEGRALS IN GRADSHTEYN AND RYZHIK.
PART 3: COMBINATIONS OF LOGARITHMS AND EXPONENTIALS.

VICTOR H. MOLL

Abstract. We present the evaluation of a family of exponential-logarithmic integrals. These have integrands of the form \( P(e^{tx}, \ln x) \) where \( P \) is a polynomial. The examples presented here appear in sections 4.33, 4.34 and 4.35 in the classical table of integrals by I. Gradshteyn and I. Ryzhik.

1. Introduction

This is the third in a series of papers dealing with the evaluation of definite integrals in the table of Gradshteyn and Ryzhik [2]. We consider here problems of the form

\[
\int_0^{\infty} e^{-tx} P(\ln x) \, dx,
\]

where \( t > 0 \) is a parameter and \( P \) is a polynomial. In future work we deal with the finite interval case

\[
\int_a^b e^{-tx} P(\ln x) \, dx,
\]

where \( a, b \in \mathbb{R}^+ \) with \( a < b \) and \( t \in \mathbb{R} \). The classical example

\[
\int_0^{\infty} e^{-x} \ln x \, dx = -\gamma,
\]

where \( \gamma \) is Euler’s constant is part of this family. The integrals of type (1.1) are linear combinations of

\[
J_n(t) := \int_0^{\infty} e^{-tx} (\ln x)^n \, dx.
\]

The values of these integrals are expressed in terms of the gamma function

\[
\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} \, dx
\]

and its derivatives.

Date: May 14, 2007.
1991 Mathematics Subject Classification. Primary 33.
Key words and phrases. Integrals.
2. The evaluation

In this section we consider the value of $J_n(t)$ defined in (1.4). The change of variables $s = tx$ yields

\begin{equation}
J_n(t) = \frac{1}{t} \int_0^\infty e^{-s} (\ln s - \ln t)^n \, ds.
\end{equation}

Expanding the power yields $J_n$ as a linear combination of

\begin{equation}
I_m := \int_0^\infty e^{-x} (\ln x)^m \, dx, \quad 0 \leq m \leq n.
\end{equation}

An analytic expression for these integrals can be obtained directly from the representation of the gamma function in (1.5).

**Proposition 2.1.** For $n \in \mathbb{N}$ we have

\begin{equation}
\int_0^\infty (\ln x)^n x^{s-1} e^{-x} \, dx = \left( \frac{d}{ds} \right)^n \Gamma(s).
\end{equation}

In particular

\begin{equation}
I_n := \int_0^\infty (\ln x)^n e^{-x} \, dx = \Gamma^{(n)}(1).
\end{equation}

**Proof.** Differentiate (1.5) $n$-times with respect to the parameter $s$. \hfill \Box

**Example 2.2.** Formula 4.331.1 in [2] states that

\begin{equation}
\int_0^\infty e^{-\mu x} \ln x \, dx = -\frac{\delta}{\mu}
\end{equation}

where $\delta = \gamma + \ln \mu$. This value follows directly by the change of variables $s = \mu x$ and the classical special value $\Gamma'(1) = -\gamma$. The reader will find in chapter 9 of [1] details on this constant. In particular, if $\mu = 1$, then $\delta = \gamma$ and we obtain (1.3):

\begin{equation}
\int_0^\infty e^{-x} \ln x \, dx = -\gamma.
\end{equation}

The change of variables $x = e^{-t}$ yields the form

\begin{equation}
\int_{-\infty}^\infty t \, e^{-t} e^{-t} \, dt = \gamma.
\end{equation}

Many of the evaluations are given in terms of the polygamma function

\begin{equation}
\psi(x) = \frac{d}{dx} \ln \Gamma(x).
\end{equation}

Properties of $\psi$ are summarized in Chapter 1 of [4]. A simple representation is

\begin{equation}
\psi(x) = \lim_{n \to \infty} \left( \ln n - \sum_{k=0}^n \frac{1}{x+k} \right),
\end{equation}

from where we conclude that

\begin{equation}
\psi(1) = \lim_{n \to \infty} \left( \ln n - \sum_{k=1}^n \frac{1}{k} \right) = -\gamma,
\end{equation}

\begin{footnote}{1}The table uses $C$ for the Euler constant.\end{footnote}
this being the most common definition of the Euler’s constant $\gamma$. This is precisely the identity $\Gamma'(1) = -\gamma$.

The derivatives of $\psi$ satisfy

$$(2.11) \quad \psi^{(m)}(x) = (-1)^{m+1} m! \zeta(m+1, x),$$

where

$$(2.12) \quad \zeta(z, q) := \sum_{n=0}^{\infty} \frac{1}{(n + q)^z}$$

is the Hurwitz zeta function. This function appeared in [3] in the evaluation of some logarithmic integrals.

**Example 2.3.** Formula 4.335.1 in [2] states that

$$\int_0^{\infty} e^{-\mu x} \ln(x)^2 \, dx = \frac{1}{\mu} \left( \frac{\pi^2}{6} + \delta^2 \right), \tag{2.13}$$

where $\delta = \gamma + \ln \mu$ as before. This can be verified using the procedure described above: the change of variable $s = \mu x$ yields

$$\int_0^{\infty} e^{-\mu x} \ln(x)^2 \, dx = \frac{1}{\mu} \left( I_2 - 2I_1 \ln \mu + I_0 \ln^2 \mu \right), \tag{2.14}$$

where $I_n$ is defined in (2.4). To complete the evaluation we need some special values: $\Gamma(1) = 1$ is elementary, $\Gamma'(1) = \psi(1) = -\gamma$ appeared above and using (2.11) we have

$$\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left( \frac{\Gamma'(x)}{\Gamma(x)} \right)^2. \tag{2.15}$$

The value

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \tag{2.16}$$

where $\zeta(z) = \zeta(z, 1)$ is the Riemann zeta function, comes directly from (2.11). Thus

$$\Gamma''(1) = \zeta(2) + \gamma^2. \tag{2.17}$$

Let $\mu = 1$ in (2.13) to produce

$$\int_0^{\infty} e^{-x} \ln(x)^2 \, dx = \zeta(2) + \gamma^2. \tag{2.18}$$

Similar arguments yields formula 4.335.3 in [2]:

$$\int_0^{\infty} e^{-\mu x} \ln(x)^3 \, dx = -\frac{1}{\mu} \left[ \delta^3 + \frac{1}{2} \pi^2 \delta - \psi''(1) \right], \tag{2.19}$$

where, as usual, $\delta = \gamma + \ln \mu$. The special case $\mu = 1$ now yields

$$\int_0^{\infty} e^{-x} \ln(x)^3 \, dx = -\gamma^3 - \frac{1}{2} \pi^2 \gamma + \psi''(1). \tag{2.20}$$

Using the evaluation

$$\psi''(1) = -2\zeta(3) \tag{2.21}$$

produces

$$\int_0^{\infty} e^{-x} \ln(x)^3 \, dx = -\gamma^3 - \frac{1}{2} \pi^2 \gamma - 2\zeta(3). \tag{2.22}$$
Problem 2.4. In [1], page 203, we introduced the notion of weight for some real numbers. In particular, we have assigned \( \zeta(j) \) the weight \( j \). Differentiation increases the weight by 1, so that \( \zeta'(3) \) has weight 4. The task is to check that the integral
\[
I_n := \int_0^\infty e^{-x} (\ln x)^n \, dx
\]
is a homogeneous form of weight \( n \).

3. A SMALL VARIATION

Similar arguments are now employed to produce a larger family of integrals. The representation
\[
\int_0^\infty x^{s-1} e^{-\mu x} \, dx = \mu^{-s} \Gamma(s),
\]
is differentiated \( n \) times with respect to the parameter \( s \) to produce
\[
\int_0^\infty (\ln x)^n x^{s-1} e^{-\mu x} \, dx = \left( \frac{d}{ds} \right)^n \mu^{-s} \Gamma(s) .
\]
The special case \( n = 1 \) yields
\[
\int_0^\infty x^{s-1} e^{-\mu x} \ln x \, dx = \frac{d}{ds} \mu^{-s} \Gamma(s) = \mu^{-s} (\Gamma'(s) - \ln \mu \Gamma(s)) = \mu^{-s} \Gamma(s) (\psi(s) - \ln \mu) .
\]
This evaluation appears as 4.352.1 in [2]. The special case \( \mu = 1 \) yields
\[
\int_0^\infty x^{s-1} e^{-x} \ln x \, dx = \Gamma'(s),
\]
that is 4.352.4 in [2].

Special values of the gamma function and its derivatives yield more concrete evaluations. For example, the functional equation
\[
\psi(x + 1) = \psi(x) + \frac{1}{x},
\]
that is a direct consequence of \( \Gamma(x + 1) = x \Gamma(x) \), yields
\[
\psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} .
\]
Replacing \( s = n + 1 \) in (3.3) we obtain
\[
\int_0^\infty x^{n} e^{-\mu x} \ln x \, dx = \frac{n!}{\mu^{n+1}} \left( \sum_{k=1}^{n} \frac{1}{k} - \gamma - \ln \mu \right) ,
\]
that is 4.352.2 in [2].

The final formula of Section 4.352 in [2] is 4.352.3
\[
\int_0^\infty x^{n-1/2} e^{-\mu x} \ln x \, dx = \frac{\sqrt{\pi} (2n - 1)!!}{2^n \mu^{n+1/2}} \left[ 2 \sum_{k=1}^{n} \frac{1}{2k - 1} - \gamma - \ln(4\mu) \right] .
\]
This can also be obtained from (3.3) by using the classical values
\[
\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^n} (2n - 1)!!
\]
\[
\psi(n + \frac{1}{2}) = -\gamma + 2 \left( \sum_{k=1}^{n} \frac{1}{2k - 1} - \ln 2 \right).
\]
The details are left to the reader.

Section 4.353 of [2] contains three peculiar combinations of integrands. The first two of them can be verified by the methods described above: formula 4.353.1 states
\[
\int_{0}^{\infty} (x - \nu)x^{\nu-1}e^{-x} \ln x \,dx = \Gamma(\nu),
\]
and 4.353.2 is
\[
\int_{0}^{\infty} (\mu x - n - \frac{1}{2})x^{n-\frac{1}{2}}e^{-\mu x} \ln x \,dx = \frac{(2n - 1)!!}{(2\mu)^n} \sqrt{\frac{\pi}{\mu}}.
\]

Acknowledgments. The author wishes to thank Luis Medina for a careful reading of an earlier version of the paper. The partial support of NSF-DMS 0409968 is also acknowledged.

References

[1] G. Boros and V. Moll. Irresistible Integrals. Cambridge University Press, New York, 1st edition, 2004.

[2] I. S. Gradshteyn and I. M. Ryzhik. Table of Integrals, Series, and Products. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

[3] V. Moll. The integrals in Gradshteyn and Ryzhik. Part 1: a family of logarithmic integrals. Scientia, 13:1–8, 2006.

[4] H. M. Srivastava and J. Choi. Series associated with the zeta and related functions. Kluwer Academic Publishers, 1st edition, 2001.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu