Electron mean free path dependence of the vortex surface impedance

M Checchin1,2, M Martinello1,2, A Grassellino1, A Romanenko1 and J F Zasadzinski2

1 Fermi National Accelerator Laboratory, Batavia, IL 60510, USA
2 Department of Physics, Illinois Institute of Technology, Chicago, IL 60616, USA

E-mail: checchin@fnal.gov

Received 14 September 2016, revised 2 December 2016
Accepted for publication 8 December 2016
Published 17 January 2017

Abstract
In the present study the radio-frequency complex response of trapped vortices in superconductors is calculated and compared to experimental data previously published. The motion equation for a magnetic flux line is solved assuming a bi-dimensional and mean-free-path-dependent Lorentzian-shaped pinning potential. The resulting surface resistance shows the unprecedented bell-shaped trend as a function of the mean-free-path observed in our previous experimental work. We demonstrate that such bell-shaped trend of the surface resistance as a function of the mean-free-path may be described as the interplay of the two limiting regimes of the surface resistance, for low and large mean-free-path values: pinning and flux-flow regimes respectively. Since the possibility of defining the pinning potential at different locations from the surface and with different strengths, we discuss how the surface resistance is affected by different configurations of pinning sites. By tackling the frequency dependence of the surface resistance, we also demonstrate that the separation between pinning- and flux-flow-dominated regimes cannot be determined only by the depinning frequency. The dissipation regime can be tuned either by acting on the frequency or on the mean-free-path value.

Keywords: RF superconductivity, vortex, vortex dynamics, vortex dissipation, mean free path dependence

(Some figures may appear in colour only in the online journal)

1. Introduction

When the superconductive transition is performed in the presence of an external magnetic field, magnetic flux can be trapped in the superconducting materials as energetically stable flux quanta in the mixed state of type-II superconductors, or as magnetic flux pinned at defects in the Meissner state of type-I and type-II superconductors. In some circumstances, magnetic flux lines can penetrate in the Meissner state without the need of pinning sites and, as a consequence of the demagnetization effect, exist in the so-called intermediate state. Independently of their nature, any trapped magnetic flux structure can introduce dissipation in both dc and radio-frequency (rf) domains [1–4].

By controlling the pinning force of the superconducting material, it is possible to eliminate the vortex dc dissipation, enabling superconductors to transport very high currents, without any dissipation, up to the depinning current. By exceeding the depinning current, the dissipation due to vortices motion takes place because of the viscous drag force [5].

On the other hand, in rf applications the vortex dissipation cannot be avoided most of the time. Even if pinned, the vortex flux line dissipates power because of the oscillation induced by the rf currents. Therefore, the trapped flux problem is of critical importance for superconducting accelerating cavities, especially when high quality factors are needed for their implementation in continuous wave accelerators.

With the discovery of the nitrogen doping treatment [6], unprecedented high quality factors (Q-factors) are achievable.
in SRF niobium cavities. The presence of nitrogen as an interstitial impurity in Nb allows for high Q-factors by decreasing the temperature dependent part of the surface impedance [7]. On the other hand, N-doped cavities show higher dissipation per unit of magnetic flux trapped than standard cavities. It was indeed observed [8–10] that the vortex-related resistance per unit of trapped magnetic field is a function of the electron mean-free-path, and therefore of the cavity thermal history. N-doped cavities operating at 1.3 GHz fall exactly in the mean-free-path region where the vortex dissipation is increased.

Two different approaches to describe the flux motion exist. The first one assumes a point-like description of a rigid vortex [11–15], where the pinning force is usually introduced as a restoring force independent of the distance of the pinning point from the rf surface.

The second approach assumes the flux line as a mono-dimensional object that possess a certain tension [3]. Usually when this description is assumed the pinning force is disregarded [16, 17] and the vortex response is calculated in absence of pinning.

If both vortex line tension and pinning force are considered, the analytic form of the latter must possess the dependence on the distance from the rf surface, complicating substantially the problem. A clever way to partially overcome the problem is achieved by introducing Dirichlet boundary conditions at the pinning site, as described in [18]. The main drawback though is the impossibility of describing the problem as a function of the pinning strength, which is one of the most important parameters to characterize the vortex dynamics in a superconductor as experimentally observed [4].

It is worth mentioning that, in case the vortex is generated by the penetration of the rf field from the surface, and it is therefore parallel with respect to this latter, the defect-related pinning force can be disregarded [19], and the attractive force between the vortex and the surface [20] can be assumed as the unique restoring force.

In this paper, we propose a different approach. The motion equation that describes the vortex displacement is defined in such a way to be different for every point of the flux line, and hence dependent on the distance z from the rf surface. To some extent, our approach is similar to the point-like description of the vortex response, but it differs from it substantially in terms of pinning force description. The analytic form of the pinning potential adopted is dependent on the distance z from the rf surface, therefore also the vortex line displacement is function of the distance from the rf surface. Differently than the point-like description where the vortex is extremely rigid and remains straight through the entire material thickness, our approach describes the motion of a flexible mono-dimensional vortex line.

The most noticeable point of our approach is its simplicity and excellent description of the experimental data acquired for SRF cavities [8–10]. The pinning force values used in order to interpolate the experimental data are in good agreement with the pinning force of niobium found in literature [21, 22]. The predicted values of depinning frequency also agree with those for niobium found in literature [23].

Introducing a clear dependence of the pinning force on the electron mean-free-path l, the experimentally observed bell-shaped trend of the trapped flux surface resistance as a function of the mean-free-path [8–10], can be explained as the interplay of the resistivity calculated in flux-flow and pinning regimes.

In our preliminary study [24], we already demonstrated that the trapped flux surface resistance is a function of the mean-free-path and presents a maximum around 70 nm. In any case, because of a too crude approximation assumed in the calculation of the surface resistance and the adoption of a mono-dimensional pinning potential, our previous approach suggested that the dynamic contribution to the vortex dissipation generates largely overestimated values of surface resistance. Based on that, our past understanding was that the surface resistance is of pure static origin, and therefore related only to the normal-conducting nature of the vortex core. By means of the more rigorous approach adopted in this study, we verify that the trapped flux surface impedance can be described totally by the dynamic oscillation of vortices given in [8–10].

Since vortices are usually multiple-pinned in the material, we take into account also situations where more than one pinning point per vortex is present. We will demonstrate that, in the case of SRF cavities, the position of the pinning point, their number and their strength are of extreme importance in order to fully describe the experimental data.

We do also demonstrate that the transition between pinning and flux-flow regimes may be obtained not only by crossing the depinning frequency [11], but also by tuning the mean-free-path value of the superconducting material.

2. Single vortex resistivity

When trapped at the rf surface, the magnetic flux experiences a force generated by the interaction with the Eddy currents induced by the oscillating rf field. The rf current density \( \vec{j}(t) \) exercises a force on the magnetic flux quantum \( \phi_0 \) in the vortex, accordingly to the Lorentz force. The magnetic force acting on a single vortex per unit of length \( f_L \) is:

\[
 f_L = |\vec{j} \times \vec{\phi}_0 \vec{u}_n| = j_0 \phi_0 \sin \theta e^{i\omega t-\zeta / \lambda},
\]

where \( j_0 \) is the rf current, \( \theta \) the angle between \( \vec{j} \) and the normal to the rf surface \( \vec{u}_n \), \( \omega \) the rf angular frequency and \( \lambda = \lambda_0 \sqrt{1 + (\xi_0 / \bar{f})} \), with \( \lambda_0 \) the penetration depth (London penetration depth), \( \xi_0 \) the coherence length and \( \bar{f} \) the mean-free-path.

We can write the motion equation of a single vortex subjected to the Lorentz force as follows:

\[
 M \ddot{x} = \dot{f}_L + f_e + f_p,
\]

with \( M \) being the inertial mass of the vortex per unit of length as defined by Bardeen and Stephen [25]:

\[
 M = \frac{2\pi^2 \hbar c^3}{\phi_0} B_{c2}(T) \sin^2 \alpha,
\]

where \( B_{c2} = B_{c2}(0)(1 - (T/T_c)^2) \) is the upper critical field.
\[ \tan \alpha = \frac{e \phi_0 \tau}{m c_s^2} \]  

where \( \phi_0 \) is the magnetic flux quantum, \( e \) and \( m \) are the charge and mass of the electron, respectively, and \( \tau = 1/\nu_f \) is the electron relaxation time, with \( \nu_f \) the Fermi velocity.

The other forces acting on the vortex are \( f_p \) the pinning force and \( f_d \) the viscous drag force. In our description we are neglecting instead the Magnus force and the interaction between vortices, since we assume \( H \ll H_c \).

The viscous drag force is defined as \( f_d = -\eta \dot{\mathbf{v}} \) where \( \eta \) is the vortex drag coefficient per unit of length.

Bardeen and Stephen [25] describes the resistance related to the viscous drag force as generated by the Joule dissipation due to the normal-conductive currents induced in the vortex core while the flux is moving.

For the purpose of our work, we will adopt the Bardeen and Stephen [25] definition, since our area of interest is in the limit \( T \ll T_c \). The viscous drag coefficient per unit of length is then defined as:

\[ \eta(l, T) = \frac{\phi_0 B_2(T)}{\rho_n}, \]  

where \( \rho_n \) is the normal conducting resistivity.

The pinning force description is extremely complicated and related to the nature of the pinning sites [2, 28]. In the present study, we do not discriminate between different types of pinning centers, thus we assume an idealistic description of the pinning potential.

The ideal mono-dimensional pinning potential is a function of the effective coherence length \( \xi = (1/\xi_0 + 1/l)^{-1} \) and is described by an inverse Lorentzian function [29] (as shown in figures 1(b)). Since the vortex is a linear object, the pinning site will be localized at a certain distance \( q \) from the rf surface. The associated pinning potential has centroid at such pinning position and approaches zero in every direction far from it.

By adopting a bi-dimensional Lorentzian potential and by limiting the pinning interaction along the oscillation direction (x), we are able to simplify the problem significantly.

In order to solve analytically the motion equation of the vortex, we transform it to a linear equation in \( x \) by expanding the pinning potential to the second order with respect to \( x \) (as shown in figure 1(b)).

\[ U_p(x, z, t) = -\sum_{i=0}^n \frac{U_{0i} \xi^2}{\xi^2 + x^2 + (z - q_i)^2}, \]  

where \( U_{0i} \) is the potential depth per unit of length of the \( i \)th pinning point, while the sum accounts for multiple pinning potentials centered at the \( q_i \)th positions acting on the same vortex.
A single pinning potential is then parabolic along the direction of oscillation and Lorentzian along z. The tri-dimensional representation of the pinning potential considered is plotted in figure 1(a), while in figure 1(b) an example of an oscillating flux line subjected to two potentials in series is shown. Figure 1(b) shows also the sections of the pinning potential along x and z, and the positions of the two pinning points \((q_1, q_2)\).

The pinning force per unit of length is then defined as:

\[
f_p(x, z, l) = -\frac{\partial}{\partial x} U_p = -\sum_{j=0}^{N} \frac{2U_0 \xi^2}{(\xi^2 + (z - q_j)^2)^2} x = -p(z, l) x,
\]

with pinning constant \(p(z, l)\).

Substituting equations (1), (7) and the viscous drag force with \(\eta\) equal to equation (5) in equation (2), we get the motion equation of a single vortex:

\[
\dot{x}(z, t, l) + \alpha \ddot{x}(z, t, l) + \beta^2 x(z, t, l) = \gamma e^{i\omega t},
\]

which corresponds to a driven-damped oscillator second order differential equation, with \(\alpha = \eta/M, \beta^2 = p/M, \gamma = j\phi_0 \sin \theta/M \) and \(j = j_0 e^{-z/\lambda}\).

The solution of such a differential equation is:

\[
x(z, t, l) = (A_1 + jA_2)e^{i\omega t},
\]

where:

\[
A_1(z, l) = \frac{j\phi_0 \sin \theta (p - M\omega^2)}{(p - M\omega^2)^2 + (\eta \omega)^2},
A_2(z, l) = -\frac{j\phi_0 \sin \theta \eta \omega}{(p - M\omega^2)^2 + (\eta \omega)^2}.
\]

We can now calculate the average (active) dissipated power \(\langle P \rangle\) and the reactive power \(\langle Q \rangle\) per unit of volume as:

\[
\langle P(z, l) \rangle = \frac{1}{T} \int_0^T \text{Re} \{F_l(t) \dot{x}(z, t, l)\} dt,
\]

\[
\langle Q(z, l) \rangle = \frac{1}{T} \int_0^T \text{Im} \{F_l(t) \ddot{x}(z, t, l)\} dt,
\]

where \(T\) is the rf period \((T = 2\pi/\omega), F_l = jB_l \sin \theta\) is the Lorentz force per unit of volume, with \(B_l\) the vortex magnetic field.

Solving the two integrals for the real and imaginary parts of Lorentz force and vortex velocity \(\dot{x}(z, t, l)\), we obtain the apparent power per unit of volume:

\[
\langle S(z, l) \rangle = \langle P \rangle + j \langle Q \rangle = \frac{\omega \phi_0 B_s \sin^2 \theta}{2[(p - M\omega^2)^2 + (\eta \omega)^2]} \left[\eta \omega + i(p - M\omega^2)\right]^2 = \frac{1}{2} \rho j^2,
\]

where \(\rho\) is the complex vortex resistivity.

Since \(B_s = \phi_0/\pi \xi^2\), where \(\pi \xi^2\) is the vortex core area, the complex vortex resistivity \(\rho\) is then equal to:

\[
\rho(z, l) = \rho_1 + j \rho_2 = \frac{\omega \phi_0^2 \sin^2 \theta}{\pi \xi^2 \left[1(p - M\omega^2)^2 + (\eta \omega)^2\right]} \left[\eta \omega + i(p - M\omega^2)\right],
\]

where \(\rho_1\) and \(\rho_2\) respectively describe the resistive and reactive behavior of the vortex when subjected to a rf current.

2.1. Flux-flow regime

In the large mean-free-path limit, the vortex dissipation is described by the flux-flow regime. In such a limit, we can neglect the pinning force \((p \sim 0)\) since the pinning potential becomes very shallow driven by the longer coherence length.

On the other hand, the viscous coefficient (equation (5)) is larger because of its dependence on the normal-state resistivity (the longer \(l\), the smaller \(\rho\) and the larger \(\eta\)). Therefore, the main force acting on the vortex is the viscous drag force. Since very small compared to \(\eta\), the vortex’s inertial mass can be neglected as well.

Neglecting inertial and pinning terms, we can rewrite the motion equation (equation (8)) as:

\[
\alpha \ddot{x}(z, t, l) = \gamma e^{i\omega t}.
\]

This first order differential equation can be easily solved by meaning of the ansatz used above (equation (9)). Solving the equation, we obtain an imaginary coefficient:

\[
A_2(l) = -\frac{j\phi_0 \sin \theta}{\omega \eta},
\]

and calculating the apparent power in equation (12), we get a purely real resistivity:

\[
\rho_1(l) = \frac{\phi_0^2 \sin^2 \theta}{\pi \xi^2 \eta}.
\]

Such a definition of the flux-flow resistivity is equivalent to the result obtained by Kim et al [5] and by Marcon et al [13] (real part). We should also notice that the same form of \(\rho_1\) can be obtained by neglecting the term \((p - M\omega^2)^2\) in equation (13).

This result suggests that the vortex dissipation for large mean-free-path values is independent on the frequency and depends only on the mean-free-path.

Moreover, as the purity of the material increases \((l)\) increases) the resistivity decreases since it is inversely proportional to the viscous coefficient \(\eta\). For big enough mean-free-path values, the resistivity is minimized, and the dissipation introduced by the vortex oscillation negligible.

2.2. Pinning regime

In the limit of small mean-free-path values, we can define the pinning regime. Since \(\eta\) decreases with decreasing \(l\), the viscous drag force is negligible in the small \(l\) limit. On the contrary, the pinning force is larger due to a shorter coherence length which increases the steepness of the pinning potential.
In this regime, the main force acting on the vortex is the pinning force, and the vortex inertial mass may also be neglected.

Considering the complex resistivity defined in equation (13), we can obtain a form of \( \rho \) that describes the vortex impedance in the pinning regime for intermediate values of mean-free-path. As already discussed, for small \( l \) values the dominant contribution is the pinning force. Therefore, neglecting the inertial term \( M \omega^2 \) and rewriting the denominator as \( \pi \xi_0^2 \rho_p^2 \), we can define the complex resistivity as:

\[
\rho(z, l) = \frac{\omega \phi_0^3 \sin^2 \theta}{\pi \xi_0^2 \rho_p^2} [\eta \omega + i \rho]. \tag{17}
\]

In such an intermediate regime, the vortex response is both of resistive and reactive nature. The decreasing of the \( \rho^2 \) with \( l \) is more rapid than the increasing of \( \eta \), thus \( \rho_1 \) increases with \( l \). Also, in the pinning regime \( \rho_1 \) is proportional to \( \omega^2 \).

On the opposite, as the mean-free-path decreases, the real part decreases with it until it becomes negligible. Such a limiting condition is well described by the vortex motion equation where both the inertial and viscous contributions are neglected:

\[
\beta^2 x(z, t, l) = \gamma e^{i\omega t}. \tag{18}
\]

Solving for \( x \) and comparing with the ansatz in equation (9), we obtain a pure real coefficient:

\[
A_1(z, l) = \frac{j \phi_0 \sin \theta}{\rho}, \tag{19}
\]

and the apparent power in equation (12) is totally reactive. Thus, the reactive response of the vortex oscillation is purely reactive:

\[
\rho_2(z, l) = \frac{\omega \phi_0^3 \sin^2 \theta}{\pi \xi_0^2 \rho_p^2}. \tag{20}
\]

Such a result suggests that for low mean-free-paths, a vortex interacting with the oscillating field does not contribute to active power dissipation, its response is purely reactive. Moreover, in the limit of very small mean-free-paths, the pinning constant (\( \rho \)) increases substantially and minimizes the reactive response as well.

3. Vortices surface impedance

Now that the vortex resistivity is known, the surface impedance for a single vortex is calculated assuming the classic definition:

\[
Z_1(l) = \frac{E_0(0)}{\int_0^\infty \dot{j}_0(z) \, dz} = \left[ \int_0^\infty \frac{e^{-z/\lambda}}{\rho(z, l)} \, dz \right]^{-1}, \tag{21}
\]

where \( \dot{j}_0(z) = j_0 e^{-z/\lambda} \).

In the non-local description, a quantized flux line is represented as a modulation of the order parameter of the superconductor that tends to zero at the center of the vortex, and approaches to its finite value far from it \[27\].

Differently, in the local description \[33\], the vortex is described as a normal conducting core with dimension of the order of the coherence length \( \xi_n \), with superconducting currents spinning around it and screening the magnetic flux confined inside. In this scenario, when a finite value of magnetic field \( B \) is applied to the superconductor during the transition, \( N \) vortices are created, and each of them carries a magnetic flux quantum \( \phi_0 \) through an area \( \pi \xi_0^2 \), i.e. \( N \phi_0 = AB \), where \( A \) is the normal conducting area that experiences the magnetic field at transition.

We can therefore extend the single vortex resistance defined in equation (21) to a multi-vortex resistance multiplying by the fraction of the area occupied by the trapped vortices \( N \pi \xi_0^2 / A = \pi \xi_0^2 B / \phi_0 \).

Now, we should consider that most likely there will be a certain distribution of pinning potentials in the material. In order to take into account that, we define the probability density of finding the pinning point at the position \( q_i \) as \( \Gamma(q_i) \), and the probability density that a pinning potential has strength \( \phi(i) \) and centroid \( q^0 \):

\[
\Gamma(q_i) = B_i e^{-\frac{(q_i - q^0)^2}{2\sigma^2}}, \tag{22}
\]

\[
B_i = \frac{\sqrt{2\pi} \sigma_i}{\text{Erf} \left( \frac{q_i - q^0}{\sqrt{2\pi} \sigma_i} \right) + \text{Erf} \left( \frac{q^0 - q_i}{\sqrt{2\pi} \sigma_i} \right)}, \tag{23}
\]

with \( q^0 \) the maximum extension of the integration domain over all the possible positions \( q_i \) defined as \( q_i + 5 \sigma_i \). While the probability density \( \Lambda(U_0) \) has variance \( \sigma_{U_0} \) and centroid \( U_0^0 \):

\[
\Lambda(U_0) = C_i e^{-\frac{(U_0 - U_0^0)^2}{2\sigma^2}}, \tag{24}
\]

\[
C_i = \frac{1}{\text{Erf} \left( \frac{U_0^0 - U_0^0}{\sqrt{2\pi} \sigma_i} \right) - \text{Erf} \left( \frac{U_0^0 - U_0^0}{\sqrt{2\pi} \sigma_i} \right)}, \tag{25}
\]

where \( U_0^0 \) and \( U_0^0 \) are the extremes of the integration domain, set as \( U_0^0 \pm 5 \sigma_{U_0} \). If \( U_0^0 - 5 \sigma_{U_0} < 0 \), then \( U_0^0 = 0 \), since no negative pinning strength values are allowed.

The vortices surface impedance weighted over pinning point position and strength distributions, for a given trapped field \( B \) is defined as:

\[
Z(l) = \frac{\pi \xi_0^2 B}{\phi_0} \int_0^{U_0^0} \int_0^{U_0^0} \cdots \int_0^{U_0^0} \frac{[\Gamma(q_i) \Lambda(U_0)]}{\rho(z, l)} \int_0^{U_0^0} e^{-z/\lambda} \, dz \, dU_0 \, dq_0 \cdots dU_0 \, dq_n. \tag{26}
\]
with \( L \) being the cavity wall thickness.

In order to simplify the interpretation of the simulation results, we adopt a Dirac-\( \delta \) distribution for both \( \Gamma(q) \) and \( \Lambda(U_0) \) instead of the more realistic Gaussian probability density function—and a single pinning point with \( q_0 = 20 \text{ nm} \) and \( U_0 = 1.1 \text{ MeV m}^{-1} \), in all the simulation performed (if not differently specified). All the other parameters used are reported in Table 1.

In figure 2, we plot the real part of the vortices surface impedance normalized to the trapped magnetic field \( B \) as a function of the electron mean-free-path \( l \), calculated from the real part of the resistivity defined in equation (13). In the same plot we show also the two limits for clean and dirty materials (flux-flow and pinning regimes), calculated with equations (16) and (17), respectively.

In the inset of figure 2, the imaginary part of the surface impedance is reported. Its behavior is roughly opposite to the real part, but with absolute value about two orders of magnitude lower than the latter, and therefore negligible.

The most noticeable feature shown in figure 2 is the presence of a peak in the surface resistance around 70 nm. For large mean-free-paths, the surface resistance follows perfectly the flux-flow result and decreases with the cleanliness of the material.

On the opposite, when the the mean-free-path decreases, the surface resistance deviates substantially from the flux-flow regime, approaching to the pinning regime.

Starting in the large mean-free-path region (flux-flow regime) and moving towards small mean-free-path values (pinning regime) both the pinning constant \( p \) and the viscous drag coefficient \( \eta \) are subjected to a substantial variation. In particular, \( p \) increases driven by the decreased coherence length, while \( \eta \) decreases because of the lower normal-state conductivity.

The decreasing of the surface resistence in the pinning regime can be explained by the vanishing of the real part of the resistivity for small mean-free-path values, as discussed in section 2.2. For very dirty materials, the drag coefficient can be neglected, and the vortices response to a rf field is purely reactive.

For large mean-free-paths instead, the situation is the opposite. As discussed in section 2.1, for very large values of mean-free-path the resistivity is purely real, but the drag coefficient is so large that the vortices response is weak and the surface impedance tends to zero—less movable vortices dissipate less.

The peak is then generated by the interplay of flux-flow and flux pinning regimes. In other words, we observe the progressing variation of the vortices response from purely reactive—low mean-free-paths, to purely resistive—large mean-free-paths.

### 3.1. Pinning strength dependence

The pinning strength \( U_0 \) is a parameter that modifies the pinning constant \( p \), and therefore the pinning force (equation (7)). In order to visualize the effect of different \( U_0 \) on the surface resistance, we plot in figure 3, the real part of the surface impedance as a function of the mean-free-path for increasing values of pinning strength \( U_0 \). As shown, the surface resistance peak is decreased in height and its position shifted to larger mean-free-path values for increasing values of \( U_0 \).

Since \( p \) increases linearly with \( U_0 \), we expect that the pinning force is becoming larger and larger for increasing \( U_0 \). Such a variation affects the maximum of the surface resistance and shifts it towards larger mean-free-path values.

A larger pinning force implies a wider mean-free-path range within which the pinning regime is favorable than the flux-flow regime. In such a scenario, larger values of \( l \) are needed to decrease the pinning force and make it negligible with respect to the viscous drag force. This different balance of the forces in play results in a higher mean-free-path onset for the flux-flow regime, and the surface resistance peak is shifted to larger values.

---

**Table 1.** Parameters values used in the simulations for niobium.

| Parameter   | Value       | Reference |
|-------------|-------------|-----------|
| \( \xi_0 \) | \( 38 \times 10^{-9} \text{ m} \) | [30]     |
| \( \lambda_0 \) | \( 39 \times 10^{-9} \text{ m} \) | [30]     |
| \( B_{\text{c2}}(0) \) | \( 442 \text{ mT} \) | [31]     |
| \( \nu_f \) | \( 1.37 \times 10^6 \text{ m s}^{-1} \) | [32]     |
| \( n \) | \( 5.56 \times 10^{28} \text{ m}^{-3} \) | [32]     |
| \( f \) | \( 1.3 \times 10^9 \text{ Hz} \) |           |
| \( T \) | \( 1.5 \text{ K} \) |           |
| \( L \) | \( 3 \text{ mm} \) |           |
3.2. Multiple pinning

We now analyze the situation in which multiple pinning points are present per single flux line. From now on we should consider that the vortex oscillation is extended beyond the rf layer, where the rf currents are present. An example of vortex oscillation is reported in figure 4(a), where the vortex displacement normalized to the current density amplitude at the surface $j_0$ is plotted as a function of the depth $z$ for successive time instants (with $T = 2\pi/\omega$ the rf period).

In the simulation of figure 4(a), two pinning points with same strength $U_0 = 1.1$ MeV m$^{-1}$ are assumed at 80 nm and 200 nm respectively. As shown, the vortex displacement is not rigid. Depending on the local pinning force and current density level, it has different oscillation amplitudes. The vortex displacement is minimum at the pinning sites, while it increases far from them. It is worth mentioning also that, at the same temporal instant, different sections of the vortex may assume displacements of opposite sign, because of the vortex complex response.

As a result of such flexible vortex oscillation, the power absorption is dependent on the whole fraction of an oscillating vortex. The active power is indeed dependent on the distance from the rf layer as shown by the $z$ dependence of equation (12). In figure 4(b), the active power per unit of volume normalized with respect to $j_0^2$ (for the vortex displacement shown in figure 4(a)) is reported. In correspondence of the pinning points (arrows), the dissipation is decreased of about four orders of magnitude because of the stronger local restoring force. Far from the pinning points, the vortex line has more freedom and the dissipation is indeed larger.

The surface resistance dependence on the distance and number of pinning points from the rf surface is reported in figure 5(a). The real part of the vortices surface impedance (equation (24)) was calculated considering again a Dirac-$\delta$ distribution profile, but considering one, two or three pinning points.

Curve $a$ considers only one pinning point in the whole vortex line. The surface resistance is approximately constant for a pinning point positioned 1–2 nm from the rf surface, it has a minimum for $q_0$ around 15 nm and it returns constant for $q_0 > 200$ nm. The effect of a pinning point is equivalent to a constraint on the flux line oscillation in the material. If the pinning point is too far from the rf surface ($q_0 \gg \lambda$), then the dissipation will reach its constant and maximum value since the oscillation is wider. Indeed, when the pinning point is far enough from the rf surface, the vortex oscillation is not perturbed by the presence of the pinning point, and the effect is equivalent to the condition when no pinning points are present at all.

If the pinning point is near the surface, the vortex oscillation amplitude is large on both sides of the pinning point (as shown in figure 4(a)) and the position of the pinning point will define the magnitude of the dissipation. In figure 5(a), the minimum of the surface resistance falls at about 15 nm, which is roughly comparable to half of the penetration dept for niobium with $l = 70$ nm ($\lambda \sim 47$ nm). In such condition the vortex is well constrained and the resistance minimized, since both the two sides of the flux line (above and below the pinning point) have restrained oscillation amplitudes.

Let us examine curve $b$ in figure 5(a). For such curve the coordinate $q_0$ corresponds to the position of the second pinning point, while the first one was assumed fixed at $q_{01} = 2$ nm from the surface (red dot on the curve). The first noticeable effect is the overall lowering of the surface resistance for all the values of $q_0$. When $q_0$ increases above 10 nm the surface resistance increases and approaches its constant value for $q_0 > 200$ nm.

Interesting to notice that the surface resistance plateau for $q_0 > 200$ nm of curve $b$ corresponds to the surface resistance value obtained for a single pinning point at 2 nm (green up triangle—$\square$—on curve $a$). This means that if $q_0$ is too large the second pinning point does not perturb the vortex behavior.

Adding a second fixed pinning point at $q_{02} = 20$ nm (red dots) and defining the abscissa as the position of the third pinning point, we obtain curve $c$ (figure 5(a)). Also this time the surface resistance value is lowered for all the $q_0$ values. As in the double pinning case, the surface resistance is constant above a certain threshold ($q_0 > 100$ nm), and approaches the values it would have if only two pinning points were present (green diamond—$\diamond$—on curve $b$).

In figure 5(b), the surface resistance as a function of the mean-free-path is instead shown. The simulation was performed considering one (curve $1$), two (curve $2$) and three (curve $3$) pinning points per vortex line, positioned at 2 nm, 20 nm and 2, 20 and 50 nm respectively.

Since the pinning point number and position play a role only in the low mean-free-path region, as expected, no variation of the mean-free-path dependence are shown in the flux-flow regime range. Noticeable variations of the trend are instead observable in the peak position and in the low mean-free-path region.

In case of multiple pinning, the peak changes position and moves towards larger mean-free-path values. Such a
phenomenon is a symptom of an overall larger pinning force, i.e. the flux-flow regime starts to take over at larger values of mean-free-path. Such a larger pinning force acts also on the peak height, which is lowered in case of multiple pinning. Therefore a larger number of pinning points assures lower resistance. The green points with same shape in figures 5(a) and 5(b) refers to surface resistance values calculated with the same parameters as reported in table 2.

3.3. Frequency dependence

Up to this point we have considered always 1.3 GHz as a constant frequency. Such a frequency was selected inasmuch as it is the most commonly used in SRF basic research and lots of experimental data are available. On the other hand though, the frequency dependence of the vortex surface impedance is of extreme importance.

Figure 6(a) shows the dependence of the real part of the vortices surface impedance as a function of the exciting frequency. In figure 6(b) the average power per unit of volume normalized to $j_0^2$ in instead reported. Both the simulations were performed assuming $l = 10$ nm, $U_0 = 1.1$ MeV and two pinning points at 80 and 200 nm.

Table 2. Pinning point positions of the green points in figure 5. For all the points $U_0 = 25$ meV and $l = 70$ nm.

| Point | Pinning points position |
|-------|-------------------------|
| ▲     | $q_{01} = 2$ nm         |
| ♦     | $q_{01} = 2$ nm, $q_{02} = 20$ nm |
| ▼     | $q_{01} = 2$ nm, $q_{02} = 20$ nm, $q_{03} = 50$ nm |

Figure 4. In (a) simulation of the vortex displacement normalized to $j_0$ as a function of the depth for different time instants in units of rf period $T$. In (b) the average power per unit of volume normalized to $j_0^2$ in instead reported. Both the simulations were performed assuming $l = 10$ nm, $U_0 = 1.1$ MeV and two pinning points at 80 and 200 nm.

Figure 5. In (a) simulations of the surface resistance as a function of the pinning point position are reported. For all three the curves $l = 70$ nm and $U_0 = 1.1$ MeV m$^{-1}$. Curve a considers only one variable pinning point position, curve b one fixed and one variable pinning point positions and curve c two fixed and one variable pinning point positions. In (b) the surface impedance as a function of the mean-free-path is reported in the condition of single (1), double (2) and triple (3) pinning per vortex flux line. The pinning points positions are reported in table 2.
frequency \( f \). As shown by Gittleman and Rosenblum [11], the vortices surface resistance as a function of \( f \) approaches two different limits: (i) the high frequency regime where the flux-flow dominates, and (ii) the low frequency regime where the pinning dominates.

These two regimes are separated by the so-called depinning frequency \( f_0 \), which corresponds to the frequency where the pinning force is equal to the viscous drag force or, in other words, to ‘the frequency where the absorption reaches half of its ideal dc value’ [11].

In the flux-flow regime (above \( f_0 \)), the pinning force is negligible and the dissipation is governed by the oscillation of the vortex within the pinning potential. Below \( f_0 \) the viscous drag force can be neglected, so the dissipation decreases since the vortex resistivity starts to assume a pure imaginary form (see section 2.2).

In figure 6(b), the surface resistance peak dependence on the mean-free-path (the simulation is done assuming a single pinning point with \( q_0 = 20 \text{ nm} \)) is shown. When the frequency is increased the peak becomes higher and its position shifted to lower mean-free-path values. This happens because in the intermediate pinning regime (described by equation (17)) the real part of the resistivity increases with the frequency squared, therefore for the same value of \( l \) the surface resistance is larger for higher frequency values.

Such dependence on \( f^2 \) implies also that for high frequencies the pinning regime is sustained up to lower values of mean-free-path. On the contrary, the lower the frequency, the higher the mean-free-path values reachable in the condition pinning regime response. In other words, for lower frequencies, the onset of the flux-flow regime occurs at higher values of mean-free-path, i.e. the peak moves towards longer \( l \).

The simulations plotted in figure 6(a) show that the frequency dependence is a function also of the number of pinning points (and their position). In particular, curve 1 corresponds to a single pinning point \((q_0 = 2 \text{ nm})\), curve 2 to two pinning points \((q_0 = 2 \text{ nm} \text{ and } q_0 = 20 \text{ nm})\) and curve 3 to three pinning points \((q_0 = 2 \text{ nm}, \ q_0 = 20 \text{ nm} \text{ and } q_0 = 50 \text{ nm})\).

The number of pinning points modifies also the depinning frequency. The curves in figure 6(a) are indeed shifted as a function of the number of pinning points. The depinning frequency—calculated as the abscissa coordinate for which the surface resistance is equal to half of the value in the dc case (flux-flow regime) of the curves in figure 6(a)—is equal to about 0.64 GHz, 1.29 GHz and 1.57 GHz for one, two and three pinning points, respectively.

Experimental data obtained for Nb thin films [23] shows that the depinning frequency decreases as the thickness of the film increases, with \( f_0 \lesssim 1 \text{ GHz} \) for very clean \((l \gtrsim 500 \text{ nm})\) epitaxially grown thicker films \((\sim 160 \text{ nm})\), in agreement with our simulated \( f_0 \).

Figure 7 shows the depinning frequency dependence on the mean-free-path. The simulation was done considering \( U_0 = 1.1 \text{ MeV/m} \) and a single pinning point at \( q_0 = 20 \text{ nm} \).

As shown \( f_0 \) decreases with \( l \), implying that the smaller the pinning force, the smaller the depinning frequency. As discussed before, by varying the mean-free-path, it is then possible to tune the depinning frequency, and therefore to define the vortices response regime: pinning regime for \( f < f_0 \) and flux-flow regime for \( f > f_0 \). This study then highlights that the transition between pinning and flux-flow regimes can be obtained not only by changing the excitation frequency, but also modifying the material mean-free-path.

4. Model versus experimental data

Several TESLA type [34] SRF niobium cavities were rf tested at the Fermi National Accelerator Laboratory’s vertical test facility. The experimental procedures to measure the vortex
The data acquired presents a bell-shaped trend in agreement with the model prediction, with a maximum around 70 nm, and surface resistance decreasing for large and small mean-free-path values.

The $U_0$ values used to obtain a satisfactory description of the data are consistent with the experimental data for niobium obtained by Allen and Claassen [21] and Park et al. [22]. In the mean-free-path range 2–2000 nm, the pinning potential strength returns values of the maximum force per meter (the force per meter was assumed as equal to the maximum of $-\partial U_p(x)/\partial x$) equal to $8 \times 10^{-5}$–$4 \times 10^{-6}$ N m$^{-1}$. Such values are in perfect agreement with [21] ($10^{-5}$–$10^{-6}$ N m$^{-1}$) and slightly underestimated with respect to [22] ($\sim 10^{-4}$ N m$^{-1}$).

Because of the peculiar shape of TESLA cavities [34], the current density and the trapped flux directions can have an intersection angle lower than ninety degrees, therefore $\sin^2 \theta$ (in equation (13)) cannot be approximated to 1. It was indeed observed in several works [35–37] that the vortex-related resistance varies consistently with the Lorentz force dependence on the angle between the directions of the currents and the trapped flux. In order to get reasonable resistance values, we consider an average angle of about 18°, and $\sin^2 \theta$ is assumed equal to 0.1.

The simulated curves describe the experimental data satisfactorily even by considering a single pinning point per flux line. Large data scattering is observed in the peak area where the pinning force plays the central role in the vortex dissipation description.

All the cavities measured are prepared with different procedures in order to ensure a good variability of mean-free-path. It is therefore reasonable to assume that pinning potential strength and pinning site distributions ($\Gamma(q_0)$ and $\Lambda(U_0)$), as well as the number of pinning points might be different from cavity to cavity. Such aleatory difference between the various cavities studied may explain the data scattering, as well as the discrepancies with the theoretical model. It is indeed unlikely that one single simulated curve can describe exhaustively the data all in once.

Our model can in principle explain why between one and two orders of magnitude lower values of vortex-related surface resistance are observed in Nb on Cu SRF cavities [38], compared to our experimental results. Because of niobium thin films are more defective (e.g. porosity, columnar growth, etc), the pinning force might be substantially larger than in bulk Nb ($U_0$ and the number of pinning points might be larger). In such a scenario, the perfect reactive response of vortices might survive up to larger mean-free-path values and the peak shifted towards larger $l$, that have not been surveyed yet.

Recently Nb$_3$Sn vortex surface impedance data for SRF cavities became available [39], showing comparable vortex surface resistance with respect to dirty niobium. In principle one would expect higher vortex surface resistance in the flux-flow regime because of the higher resistivity of Nb$_3$Sn in the normal-conducting state. On the other hand though, Nb$_3$Sn was shown to posses high pinning force at the grain boundaries [40–42], which would be in agreement with an overall
suppressed vortex surface resistance, as experimentally observed because of a larger $U_0$.

5. Conclusions

In this paper we proposed the explicit description of the vortex-related surface impedance as a function of the mean-free-path at rf and microwave frequencies.

We approached the problem by assuming a bi-dimensional pinning potential dependent on the electrons mean-free-path and by solving the single-vortex motion equation. Differently than previous works, we found an explicit dependence of the vortices surface resistance on the mean-free-path, and we studied the dependence of the surface resistance as a function of the number, disposition and strength of the pinning points in the material.

The experimental data observed for different SRF niobium cavities at 1.3 GHz can be explained exhaustively by the interplay of the limiting responses of the surface resistance for low and large values of mean-free-path: the pinning and the flux-flow regime, respectively.

Because of the different thermal history of every cavity, the experimental data shows some scattering: the pinning position and strength distributions may indeed be different. This means that the model here presented does provide us an average description of the experimental data.

The bell-shape trend experimentally observed is generated by the variation of the vortices response from totally resistive at large values of mean-free-path, to totally reactive for small $l$.

In the pinning regime (small $l$), the pinning force is governing the vortices response since, for small $l$, the viscous drag force is negligible. Hence, in absence of dissipative mechanisms, the response is totally reactive and the surface resistance tends to zero.

As the mean-free-path increases, the viscous drag force increases driven by the decreasing of the normal-state resistivity, meanwhile the pinning force becomes negligible because of a larger coherence length. Consequently, the surface resistance follows increasing.

Above a certain mean-free-path value threshold—defined by frequency, position, number and strength of pinning points—the flux-flow regime takes over and the surface resistance assumes its maximum value. For larger values of $l$, the surface resistance decreases driven by the increment of the drag force—the surface resistance in the flux-flow regime is inversely proportional to the viscous drag coefficient.

Such behavior of the surface resistance with the mean-free-path is extremely important since it highlights the mean-free-path as another parameter needed to tune the dissipation regime other than the frequency.

We have shown that the position, strength and number of the pinning sites can modify substantially the vortex surface resistance and the depinning frequency, shifting and affecting the maximum of surface resistance as a function of the mean-free-path.

Pinning sites arising nearby the rf surface affects the vortices response lowering the surface resistance. On the contrary, when the pinning site is far enough from the rf surface its presence does not perturb anymore the vortex oscillation and the resistance approaches to its constant and maximum value.

The frequency dependence shows the typical trend expected, but observing the surface resistance as a function of the mean-free-path, we showed that larger frequencies shift the peak to small $l$ values and increase the maximum height, because of the different balance of forces—larger frequencies allows the real part of the resistivity to grow faster when moving from low to large values of mean-free-path, enhancing the peak maximum and shifting it to lower mean-free-path values.

Acknowledgments

The authors want to acknowledge Professor Y Shilinov and Professor A Gurevich for their important suggestions and corrections. This work was supported by the United States Department of Energy, Offices of High Energy and Nuclear Physics and by the DOE HEP Early Career grant of Dr A - Grassellino, and DOE NP Early Career grant of Dr A Romanenko. Fermilab is operated by Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the United States Department of Energy.

References

[1] Huebener R P and Clem J R 1974 Rev. Mod. Phys. 46 409
[2] Huebener R P 2001 Magnetic Flux Structures in Superconductors (Berlin: Springer)
[3] Brandt E H 1995 Rep. Prog. Phys. 58 1465
[4] Campbell A M and Evetts J E 1972 Adv. Phys. 21 199
[5] Kim Y B, Hempstead C F and Strnad A R 1965 Phys. Rev. 139 A1163
[6] Grassellino A, Romanenko A, Sergatskov D A, Melnychuk O, Trenikhina Y, Crawford A C, Rowe A, Wong M, Khabibouline F and Barkov F 2013 Supercond. Sci. Technol. 26 102001
[7] Mattis D C and Bardeen J 1958 Phys. Rev. 111 412
[8] Martinello M, Grassellino A, Checchin M, Romanenko A, Melnychuk O, Sergatskov D A, Posen S and Zasadzinski J 2016 Appl. Phys. Lett. 109 062601
[9] Martinello M, Checchin M, Grassellino A, Melnychuk O, Posen S, Romanenko A, Sergatskov D A and Zasadzinski J 2015 Trapped flux surface resistance analysis for different surface treatments Proc. 17th Int. Conf. on RF Superconductivity p 115 MOPB015
[10] Martinello M, Checchin M, Grassellino A, Melnychuk O, Posen S, Romanenko A, Sergatskov D A and Zasadzinski J 2016 Tailoring surface impurity content to maximize Q-factors of superconducting resonators Proc. 7th Int. Particle Accelerator Conference p 2258 WEPMR003
[11] Gittleman J I and Rosenblum B 1966 Phys. Rev. Lett. 16 734
[12] Coffey M W and Clem J R 1991 Phys. Rev. Lett. 67 386
[13] Marcon R, Fastampa R, Giura M and Silva E 1991 Phys. Rev. B 43 2394
[14] Rabinowitz M 1971 J. Appl. Phys. 42 88

11
