A priori bounds and positive solutions for non-variational fractional elliptic systems

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Abstract
In this paper we study strongly coupled elliptic systems in non-variational form involving fractional Laplace operators. By mean of Liouville type theorems we establish a priori bounds of positive solutions for subcritical and superlinear non-linearities in a suitable sense. We then derive the existence of positive solutions through topological methods.

1 Introduction and main results

The present paper deals with a priori bounds and existence of positive solutions for elliptic systems of the form

\[
\begin{cases}
(-\Delta)^su = v^p & \text{in } \Omega \\
(-\Delta)^tv = u^q & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]

(1)

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), \(n \geq 2\), \(s, t \in (0, 1)\), \(p, q > 0\) and the fractional Laplace operator \((-\Delta)^s\) is defined as

\[
(-\Delta)^su(x) = C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,
\]

(2)
or equivalently,

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\[(−Δ)^s u(x) = −\frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) − 2u(x)}{|y|^{n+2s}} \, dy \]

for all \(x \in \mathbb{R}^n\), where P.V. denotes the principal value of the integral and

\[C(n, s) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1}\]

with \(\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n\). An alternative definition of the fractional Laplace operator \((−Δ)^s\) with zero Dirichlet boundary values on \(\partial Ω\) can be given in terms of the Dirichlet spectra of the Laplace operator. A basic property satisfied by \((−Δ)^s\) is

\[
\lim_{s \to 1} (−Δ)^s u = −Δ u
\]

pointwise in \(\mathbb{R}^n\) for all \(u \in C_0^\infty(\mathbb{R}^n)\), so \((−Δ)^s\) interpolates the Laplace operator in \(\mathbb{R}^n\).

Fractional Laplace operators have attracted attention in recent years for a great number of applications in Biology, Economy and Physics and, independently, for their nonlocal properties. A rather useful local description of the operator \((−Δ)^s\) in terms of the Laplace operator and of a Neumann type boundary condition was developed in the seminal paper [6]. Later, other theoretical contributions were given in the works by Brändle, Colorado, de Pablo and Sánchez [2], Cabré and Tan [5], Capella, D’avila, Dupaigne, and Sire [7] and Tan [36]. Thanks to these advances, the boundary fractional problem

\[
\begin{align*}
(−Δ)^s u &= u^p \quad \text{in } Ω \\
u &= 0 \quad \text{on } \partial Ω
\end{align*}
\]

has been widely studied on a smooth bounded domain \(Ω \subset \mathbb{R}^n\), \(n \geq 2\), \(s \in (0, 1)\) and \(p > 0\). Particularly, a priori bounds and existence of positive solutions for subcritical exponents \((p < \frac{n+2s}{n−2s})\) has been proved in [2, 5, 30] and nonexistence results has also been proved in [2, 30, 33, 36] for critical and supercritical exponents \((p ≥ \frac{n+2s}{n−2s})\).

Systems like (1) are strongly coupled vector extensions closely related to (3) which have been addressed for \(s = t = 1\) by several authors during the two last decades (we refer to the survey [10] and references therein) and also more recently for \(0 < s = t < 1\) in [8]. More specifically, a priori bounds and existence of positive solutions have been considered in these cases. In view of what is known for scalar equations and for systems of the type (1) with \(s = t = 1\), one expects that a priori
bounds depend on the values of the exponents \( p \) and \( q \). Indeed, the values \( p \) and \( q \) should be related to Sobolev embedding theorems.

A rather classical fact is that a priori bounds allow to establish existence of positive solutions for systems by mean of topological methods such as degree theory and Krasnoselskii’s index theory. For a list of works concerning with non-variational elliptic systems involving Laplace operators we refer to [1, 9, 14, 15, 24, 26, 32, 38], among others.

Our goal in this work is also establishing existence of positive classical solutions of non-variational strongly coupled systems of the type (1) by mean of a priori bounds for a family of exponents \( p \) and \( q \). By a classical solution of the system (1), we mean a couple \((u, v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2\) satisfying (1) in the usual classical sense.

Our main result is

**Theorem 1.1.** Let \( \Omega \) be a bounded domain of \( C^2 \) class in \( \mathbb{R}^n \). Assume that \( n \geq 2 \), \( s, t \in (0, 1) \), \( n > 2s + 1 \), \( n > 2t + 1 \), \( p, q \geq 1 \), \( pq > 1 \) and either

\[
\left(\frac{2s}{p} + 2t\right) \frac{p}{pq - 1} \geq n - 2s \quad \text{or} \quad \left(\frac{2t}{q} + 2s\right) \frac{q}{pq - 1} \geq n - 2t.
\]

(4)

Then, the system (1) admits, at least, one positive classical solution. Moreover, all such solutions are uniformly bounded in the \( L^\infty \)-norm by a constant that depends only on \( s, t, p, q \) and \( \Omega \).

**Remark 1.1.** Systems like (1) have been broadly studied when \( s = t = 1 \). In this case, it arises the well-known notions of superlinearity and criticality (under the form of critical hyperbole), see [22, 23, 28]. In particular, one knows (see [11, 13, 19]) that the system (1) always admits a positive classical solution provided that \( pq > 1 \) and

\[
\frac{1}{p + 1} + \frac{1}{q + 1} > \frac{n - 2}{n}.
\]

**Remark 1.2.** When \( 0 < s = t < 1 \) and \( p, q > 1 \), a priori bounds and existence of positive classical solutions of (1) have been derived in [3] provided that

\[
\frac{1}{p + 1} + \frac{1}{q + 1} > \frac{n - 2s}{n}.
\]

(5)

**Remark 1.3.** When \( 0 < s = t < 1 \) and \( pq > 1 \), the condition (4) implies (5).

The approach used in the proof of Theorem 1.1 is based on the blow-up method, firstly introduced by Gidas and Spruck in [18] to treat the scalar case and later extended to strongly coupled systems like (1) with \( s = t = 1 \) in [24] and then in
This method consists of a contradiction argument, which in turn relies on Liouville type results for equations or systems in the whole space $\mathbb{R}^n$ or in a half-space of it. Proving these last ones is usually the main obstacle in applying the Gidas-Spruck method.

For this purpose, we first shall establish Liouville type theorems for the system

$$\begin{cases}
(-\Delta)^s u = v^p & \text{in } G \\
(-\Delta)^t v = u^q & \text{in } G
\end{cases}$$

for $G = \mathbb{R}^n$ and $G = \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. In this latter one, we assume the Dirichlet condition $u = 0 = v$ on $\partial G$.

We recall a viscosity super-solution for the above system is a couple $(u, v)$ of continuous functions in $\mathbb{R}^n$ such that $u, v \geq 0$ in $\mathbb{R}^n \setminus G$ and for each point $x_0 \in G$ there exists a neighborhood $U$ of $x_0$ in $\overline{G}$ such that for any $\varphi, \psi \in C^2(U)$ satisfying $u(x_0) = \varphi(x_0)$, $v(x_0) = \psi(x_0)$, $u \geq \varphi$ and $v \geq \psi$ in $U$, the functions defined by

$$\bar{u} = \begin{cases}
\varphi & \text{in } U \\
u & \text{in } \mathbb{R}^n \setminus U
\end{cases} \quad \text{and} \quad \bar{v} = \begin{cases}
\psi & \text{in } U \\
v & \text{in } \mathbb{R}^n \setminus U
\end{cases}$$

satisfy

$$(-\Delta)^s \bar{u}(x_0) \geq v^p(x_0) \quad \text{and} \quad (-\Delta)^t \bar{v}(x_0) \geq u^q(x_0).$$

In a natural way, we have the notions of viscosity sub-solution and viscosity solution.

**Theorem 1.2.** Assume that $n \geq 2$, $s, t \in (0, 1)$, $n > 2s$, $n > 2t$, $p, q > 0$ and $pq > 1$. Then, the only non-negative viscosity super-solution of the system (6) with $G = \mathbb{R}^n$ is the trivial if and only if (4) holds.

**Theorem 1.3.** Assume that $n \geq 2$, $s, t \in (0, 1)$, $n > 2s + 1$, $n > 2t + 1$, $p, q \geq 1$ and $pq > 1$. If the condition (4) holds, then the only non-negative viscosity bounded solution of the system (6) with $G = \mathbb{R}^n_+$ is the trivial.

**Remark 1.4.** Non-existence results of positive solutions have been established for the scalar problem

$$(-\Delta)^s u = u^p \quad \text{in } G$$

in both cases $G = \mathbb{R}^n$ and $G = \mathbb{R}^n_+$ by assuming that $n > 2s$ and $1 < p < \frac{n+2s}{n-2s}$, see [14, 15, 32, 38] for $s = 1$ and [21, 27, 36] for $0 < s < 1$. 

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Remark 1.5. A number of works has focused attention on non-existence of positive solutions of (7) for $G = \mathbb{R}^n$ and $G = \mathbb{R}^n_+$ when $s = t = 1$ and $0 < s = t < 1$. We refer for instance to [1, 3, 12, 23, 24, 25, 29, 33, 34] for $s = t = 1$ and [27, 36] for $0 < s = t < 1$ and other references therein.

Several arguments have been employed in the proof of non-existence results of positive solutions of elliptic systems. Our approach is inspired on a powerful technique, based on maximum principles, developed by Quaas and Sirakov in [26] to treat systems involving different uniformly elliptic linear operators. Particularly, some maximum principles and related results for fractional operators due to Silvestre [31] and Quaas and Xia [27] as well as some auxiliary tools to be proved in the next section will be used in the proof of Theorem 1.2 and also in the rest of the work.

The paper is organized into four sections. In Section 2 we prove some key lemmas required in the proof of Theorem 1.2 which will be presented in Section 3. Section 4 is devoted to the proof of Theorem 1.3. In Section 5, we use Theorems 1.2 and 1.3 in order to prove Theorem 1.1.

2 Preliminary lemmas

We next present three lemmas which will be used in the proof of Theorem 1.2. Their proofs are inspired in the work of Felmer and Quaas [16] and adapted to fractional operators.

Throughout the paper, it is assumed that $p, q > 0$ and $pq > 1$. So, thanks to a suitable rescaling of $u$ and $v$, we can assume that $C(n,s) = 1$ and $C(n,t) = 1$.

Given a non-negative continuous function $u : \mathbb{R}^n \to \mathbb{R}$, define

$$m_u(r) = \min_{|x| \leq r} u(x)$$

for $r > 0$.

Lemma 2.1. Let $s \in (0,1)$, $n > 2s$ and $u \neq 0$ be a non-negative viscosity supersolution of

$$(-\Delta)^s u = 0 \text{ in } \mathbb{R}^n. \quad (8)$$

Then, for each $R_0 > 1$ and $\sigma \in (-n,-n+2s)$, there exists a constant $C > 0$, independent of $u$, such that

$$m_u(r) \geq C m_u(R_0) r^\sigma \quad (9)$$

for $r > 0$.
for all \( r \geq R_0 \).

By a non-negative viscosity super-solution of the equation (8), we mean a non-negative continuous function \( u : \mathbb{R}^n \to \mathbb{R} \) satisfying the following property: each point \( x_0 \in \mathbb{R}^n \) admits a neighborhood \( U \) such that for any function \( \varphi \in C^2(U) \) with \( u(x_0) = \varphi(x_0) \) and \( u \geq \varphi \) in \( U \), the function defined by

\[
\begin{array}{ll}
\overline{u} = & \{ \varphi \text{ in } U \\
& u \text{ in } \mathbb{R}^n \setminus U \}
\end{array}
\]

satisfies

\[ (-\Delta)^s \overline{u}(x_0) \geq 0. \]

**Proof of Lemma 2.1**. Let \( R_0, \sigma \) and \( u \) be as in the above statement. Given \( R > R_0 \) and \( \varepsilon > 0 \), we consider the function

\[
w(r) = \begin{cases} 
\varepsilon^\sigma & \text{if } 0 < r \leq \varepsilon \\
r^\sigma & \text{if } \varepsilon \leq r
\end{cases}
\tag{10}
\]

We first assert that \((\Delta)^s w(r) < 0\) for all \( R_0 < r < R \) and \( \varepsilon > 0 \) small enough. In fact, for \( |x| = r \), we have

\[
2(\Delta)^s w(r) = - \int_{B_\varepsilon(-x)} \frac{\varepsilon^\sigma}{|y|^{n+2s}} \, dy - \int_{B_{\varepsilon}(x)} \frac{\varepsilon^\sigma}{|y|^{n+2s}} \, dy - \int_{B_\varepsilon(-x)} \frac{|x+y|^\sigma}{|y|^{n+2s}} \, dy
\]

\[
- \int_{B_{\varepsilon}(x)} \frac{|x-y|^\sigma}{|y|^{n+2s}} \, dy + 2 \int_{\mathbb{R}^n} \frac{|x|^\sigma}{|y|^{n+2s}} \, dy
\]

\[
= - \int_{\mathbb{R}^n} \frac{|x+y|^\sigma + |x-y|^\sigma - 2|x|^\sigma}{|y|^{n+2s}} \, dy
\]

\[
+ \left( \int_{B_\varepsilon(-x)} \frac{|x+y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}} \, dy + \int_{B_\varepsilon(x)} \frac{|x-y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}} \, dy \right)
\]

\[
= 2(\Delta)^s |x|^\sigma + \left( \int_{B_\varepsilon(-x)} \frac{|x+y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}} \, dy + \int_{B_\varepsilon(x)} \frac{|x-y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}} \, dy \right).
\]

Since \( R_0 > 1 \), the two last above integral converge uniformly to 0 for \( |x| > R_0 \) as \( \varepsilon \to 0 \).

On the other hand, using that \( R_0 > 1 \) and \( \sigma \in (-n, -n + 2s) \) and the fact that \( |x|^{-n+2s} \) is the fundamental solution of the fractional Laplace operator \((\Delta)^s \) (see
one easily checks that \((-\Delta)^s|x|^\sigma < 0\) for all \(|x| > R_0\), see [16]. Thus, the above claim follows for \(\varepsilon > 0\) small enough.

For such a parameter \(\varepsilon\) and \(|x| = r\), we set
\[
\varphi(x) = m_u(r/2) \frac{w(r) - w(R)}{w(\varepsilon) - w(R)}
\]
for all \(|x| < R\) and \(\varphi(x) = 0\) for \(|x| \geq R\). As can easily be checked, \((-\Delta)^s\varphi \leq 0\) for all \(R_0 < |x| < R\). Moreover, we have \(u(x) \geq \varphi(x)\) for \(|x| \leq R_0\) or \(|x| \geq R\), so that the Silvestre’s strong maximum principle [31] readily yields \(u(x) \geq \varphi(x)\) for all \(R_0 \leq |x| \leq R\). Finally, letting \(R \to \infty\) in this last inequality, we achieve the expected conclusion with \(C = \varepsilon^{-\sigma}\).

Our second auxiliary lemma is

**Lemma 2.2.** Let \(s \in (0, 1), n > 2s\) and \(u \neq 0\) be a non-negative viscosity supersolution of [3]. Then, there exist constants \(C > 0\) and \(R_0 > 0\), independent of \(u\), such that
\[
m_u(r/2) \leq C m_u(r)
\]
for all \(r \geq R_0\).

**Proof of Lemma 2.2.** Given \(r > 0\) and \(\varepsilon > 0\), set
\[
R = r \left[ \frac{\varepsilon}{1 + \varepsilon 2^{n-2s}} \right]^{1/(n-2s)},
\]
where \(\varepsilon\) is chosen such that \(R < r/2\).

Consider the functions
\[
w_r(\varphi) = \begin{cases} (R)^{-n+2s} & \text{if } 0 < \varphi \leq R \\ \varphi^{-n+2s} & \text{if } R \leq \varphi \leq 2r \\ (2r)^{-n+2s} & \text{if } \varphi \geq 2r \end{cases}
\]
and
\[
w(\varphi) = \begin{cases} (R)^{-n+2s} & \text{if } 0 < \varphi \leq R \\ \varphi^{-n+2s} & \text{if } R \leq \varphi \end{cases}
\]
Given a fixed function \(u\) as in the above statement, we define
\[
\varphi(x) = m_u(r/2) \frac{w_r(\varphi) - w(2r)}{w(R) - w(2r)}
\]
for $x$ with $|x| = r$. As a direct consequence, one has $u(x) \geq \varphi(x)$ for all $x$ with $|x| \leq r/2$ and $|x| \geq 2r$. Moreover, decreasing $\varepsilon$, if necessary, one gets

$$2(-\Delta)^s u_r(r) = -\int_{B_R(x)} \frac{r^{-n+2s}}{\varepsilon |y|^{n+2s}} + \frac{(2r)^{-n+2s}}{|y|^{n+2s}} dy - \int_{B_{2r}(x)} \frac{(2r)^{-n+2s}}{|y|^{n+2s}} dy$$

$$- \int_{B_{2r}(x) \setminus B_R(x)} \frac{|x-y|^{-n+2s}}{|y|^{n+2s}} dy + \int_{\mathbb{R}^n} \frac{|x|^{-n+2s}}{|y|^{n+2s}} dy \leq 0$$

for all $r/2 < \tau < 2r$. Thus, $(-\Delta)^s \varphi(x) \leq 0$ for all $x$ with $r/2 < |x| < 2r$.

Evoking the Silvestre’s maximum principle [31], we then deduce that $u(x) \geq \varphi(x)$ for all $x$ with $r/2 < |x| < 2r$. Lastly, we assert that this conclusion leads to

$$m_u(r) \geq \varepsilon m_u(r/2)(1 - 2^{-n+2s}).$$

In fact, we have

$$\varphi(x) = m_u(r/2) \geq \varepsilon m_u(r/2)(1 - 2^{-n+2s})$$

if $0 < |x| \leq R$, and

$$\varphi(x) = \varepsilon m_u(r/2) \frac{r^{-n+2s} - (2r)^{-n+2s}}{r^{-n+2s}} \geq \varepsilon m_u(r/2)(1 - 2^{-n+2s})$$

if $R < |x| \leq r$. So, the result follows with $C = (\varepsilon(1 - 2^{-n+2s}))^{-1}$ by minimizing $u$ on the closed ball $|x| \leq r$. ■

Our third lemma concerns with the behavior of fractional Laplace operators applied to the function $\Theta(x) = \log(1 + |x|)|x|^{-n+2s}$.

**Lemma 2.3.** Let $s \in (0, 1)$ and $n > 2s$. Then, there exists a constant $C_0 > 0$ such that

$$(-\Delta)^s \Theta(x) \leq C_0 |x|^{-n}$$

for all $x \neq 0$.

**Proof of Lemma 2.3.** Using that $|x|^{-n+2s}$ is the fundamental solution of $(-\Delta)^s$
(see [6]), one first has

\[-2(-\Delta)^s \Theta(x) = \int_{\mathbb{R}^n} \frac{\log(1 + |x - y|)|x - y|^{-n+2s}}{|y|^{n+2s}} \, dy \]

\[+ \int_{\mathbb{R}^n} \frac{\log(1 + |x + y|)|x + y|^{-n+2s}}{|y|^{n+2s}} \, dy - 2 \int_{\mathbb{R}^n} \frac{\log(1 + |x|)|x|^{-n+2s}}{|y|^{n+2s}} \, dy \]

\[= \int_{\mathbb{R}^n} \frac{(\log(1 + |x - y|) - \log(1 + |x|))|x - y|^{-n+2s}}{|y|^{n+2s}} \, dy \]

\[+ \int_{\mathbb{R}^n} \frac{(\log(1 + |x + y|) - \log(1 + |x|))|x + y|^{-n+2s}}{|y|^{n+2s}} \, dy \]

\[= \int_{\mathbb{R}^n} \left( \log \left( \frac{1 + |x - y|}{1 + |x|} \right) \right) \frac{1}{|y|^{n+2s}} \, dy \]

\[+ \int_{\mathbb{R}^n} \left( \log \left( \frac{1 + |x + y|}{1 + |x|} \right) \right) \frac{1}{|y|^{n+2s}} \, dy \]

\[= \int_{\mathbb{R}^n} r^{-n} \left( \log \left( \frac{1 + r |e_1 - z|}{1 + r} \right) \right) \frac{1}{|e_1 - z|^{n+2s}} \, dz \]

\[+ \int_{\mathbb{R}^n} r^{-n} \left( \log \left( \frac{1 + r |e_1 + z|}{1 + r} \right) \right) \frac{1}{|e_1 + z|^{n+2s}} \, dz , \]

where \( x = re_1 \) and \( z = y/r \). Note that there is no loss of generality in considering \( x = re_1 \), since \( \log(1 + |x|) \) and \( |x|^{-n+2s} \) are radially symmetric.

In order to complete the proof we just need to find a constant \( C_0 > 0 \) such that

\[
\int_{\mathbb{R}^n} \frac{\log \left( \frac{1 + r |e_1 - z|}{1 + r} \right) |e_1 - z|^{-n+2s} + \log \left( \frac{1 + r |e_1 + z|}{1 + r} \right) |e_1 + z|^{-n+2s}}{|z|^{n+2s}} \, dz \geq -C_0 . \quad (12)
\]

For this purpose, we write for \( \rho > 0, \gamma \in [0,1) \) and \( r \geq 0 \),

\[
\log \left( \frac{1 + r |e_1 - z|}{1 + r} \right) |e_1 - z|^{-n+2s} = g(|e_1 - z|, \gamma) \quad (13)
\]

and

\[
\log \left( \frac{1 + r |e_1 + z|}{1 + r} \right) |e_1 + z|^{-n+2s} = g(|e_1 + z|, \gamma) , \quad (14)
\]
where

\[ g(\rho, \gamma) = \rho^{-n+2s} \log(1 + \gamma(\rho - 1)) \]

and

\[ \gamma = \frac{r}{1 + r}. \]

Consider first \( B_1 = \{ z : |z + e_1| \leq 1/2 \} \) and note that \( g(|e_1 - z|, \gamma) \) is bounded in \( B_1 \), while \( g(|e_1 + z|, \gamma) \) has a singularity at \( -e_1 \in B_1 \). Then, for some constants \( C > 0 \), independent of \( \gamma \), we have

\[
\int_{B_1} \frac{|g(|e_1 + z|, \gamma)|}{|z|^{n+2s}} \, dz = \int_{B_1/2(0)} \frac{|g(|z|, \gamma)|}{|z - e_1|^{n+2s}} \, dz \leq -C \int_0^{1/2} g(\rho, \gamma) \rho^{n-1} \, d\rho \\
\leq -C \int_0^{1/2} \rho^{2s-1} \log(\rho) \, d\rho \leq C.
\]

Since \( 1 + \gamma(\rho - 1) \geq \rho \) as \( \gamma \in [0,1) \), the integral in (12), when considered over \( B_1 \), is bounded below by a constant independent of \( r \). In a similar way, the conclusion follows for the set \( B_2 = \{ z : |z - e_1| \leq 1/2 \} \).

On the set \( B_3 = \{ z : |z| \geq 2 \} \), for some constant \( C > 0 \), independent of \( \gamma \), we have

\[ |g(|e_1 - z|, \gamma) + g(|e_1 + z|, \gamma)| \leq C|z|^{-2n} \log(|z|). \]

Thus, the integral in (12), when considered over \( B_3 \), is also bounded below by a constant independent of \( r \).

It then remains to analyze the behavior of the integral over \( B_4 = \{ z : |z| \leq 1/2 \} \). For each fixed \( r \geq 0 \) and \( \gamma \in [0,1) \), define \( f_r : \mathbb{R}^n \rightarrow \mathbb{R} \) given by \( f_r(z) = g(|e_1 + z|, \gamma) + g(|e_1 - z|, \gamma) \). Using that \( f_r(0) = 0 \) and \( D(f_r(0)) = 0 \), the Taylor formula provides

\[
f_r(z) = z^t \cdot \int_0^1 (1 - \rho) D^2(f_r(\rho z)) \, d\rho \cdot z,
\]

where all derivatives are taken only with respect to the variable \( z \). Thus, the estimate of the integral (12) over \( B_4 \) follows if we can show that
\[
\left| \frac{\partial^2 f_r(z)}{\partial z_i \partial z_j} \right| \leq C
\]

for all \(|z| \leq 1/2\), where \(C > 0\) is a constant independent of \(r\).

On the other hand, a straightforward computation gives
\[
\frac{d}{d\rho} g(\rho, \gamma) = (-n + 2s)\rho^{-n+2s-1} \log(1 + \gamma(\rho - 1)) + \frac{\gamma\rho^{-n+2s}}{1 + \gamma(\rho - 1)}
\]
and
\[
\frac{d^2}{d\rho^2} g(\rho, \gamma) = (-n + 2s)(-n + 2s - 1)\rho^{-n+2s-2} \log(1 + \gamma(\rho - 1))
\]
\[+ \frac{2\gamma(-n + 2s)\rho^{-n+2s-1}}{1 + \gamma(\rho - 1)} - \frac{\gamma^2 \rho^{-n+2s}}{(1 + \gamma(\rho - 1))^2}.
\]
Then, one easily checks that
\[
|\frac{d}{d\rho} g(\rho, \gamma)|, \ \left| \frac{d^2}{d\rho^2} g(\rho, \gamma) \right| \leq C
\]
for all \(1/2 \leq \rho \leq 3/2\) and \(\gamma \in [0, 1)\), where \(C\) is a constant independent of \(\rho\) and \(\gamma\).
So, for certain bounded functions \(D_{ij}\) and \(d_{ij}\) in \(B_4\), we have
\[
\frac{\partial^2 f_r(z)}{\partial z_i \partial z_j} = \frac{d^2}{d\rho^2} g(|e_1 + z|, \gamma) D_{ij} + \frac{d}{d\rho} g(|e_1 + z|, \gamma) d_{ij}
\]
and (16) follows.

Finally, joining the above estimates on the four sets \(B_i\), one gets (12) as desired.

3 Proof of Theorem 1.2

We organize the proof of Theorem 1.2 into two stages, according to the sufficiency and necessity of the assumption (4).

Proof of the sufficiency of (4). We analyze separately two different cases:

(I) \(\left(\frac{2s}{p} + 2t\right) \frac{p}{pq-1} > n - 2s\) or \(\left(\frac{2t}{q} + 2s\right) \frac{q}{pq-1} > n - 2t\);

(II) \(\left(\frac{2s}{p} + 2t\right) \frac{p}{pq-1} = n - 2s\) or \(\left(\frac{2t}{q} + 2s\right) \frac{q}{pq-1} = n - 2t\).
We first assume the situation (I). Let \((u, v)\) be a non-negative viscosity super-solution of the system\((\text{II})\) with \(G = \mathbb{R}^n\) and \(\eta : [0, +\infty) \to \mathbb{R}\) be a \(C^\infty\) cutoff function satisfying \(0 \leq \eta \leq 1\), \(\eta\) is non-increasing, \(\eta(r) = 1\) if \(0 \leq r \leq 1/2\) and \(\eta(r) = 0\) if \(r \geq 1\). Clearly, there exists a constant \(C > 0\) such that \((-\Delta)^s\eta(|x|) \leq C\) and \((-\Delta)^t\eta(|x|) \leq C\).

Choose \(R_0 > 0\) as in Lemma 2.2 for \(s\) and \(t\), simultaneously, and consider the functions

\[
\xi_u(x) = m_u(R_0/2)\eta(|x|/R_0) \quad \text{and} \quad \xi_v(x) = m_v(R_0/2)\eta(|x|/R_0).
\]

For some constant \(C_0 > 0\), independent of \(R_0, u\) and \(v\), we have

\[
(-\Delta)^s(\xi_u(x)) \leq C_0 \frac{m_u(R_0/2)}{R_0^{2s}} \quad \text{and} \quad (-\Delta)^t(\xi_v(x)) \leq C_0 \frac{m_v(R_0/2)}{R_0^{2t}}.
\]

Moreover, \(\xi_u(x) = 0 \leq u(x)\) if \(|x| > R_0\) and \(\xi_u(x) = m_u(R_0/2) \leq u(x)\) if \(|x| \leq R_0/2\). Similarly, \(\xi_v(x) = 0 \leq v(x)\) if \(|x| > R_0\) and \(\xi_v(x) = m_v(R_0/2) \leq v(x)\) if \(|x| \leq R_0/2\). Thus, the functions \(u - \xi_u\) and \(v - \xi_v\) attain their global minimum values at points \(x_u\) and \(x_v\) with \(|x_u| < R_0\) and \(|x_v| < R_0\), respectively.

Now let \(\varphi(x) := \xi_u(x) - u(x) + u(x_u)\) and \(\psi(x) := \xi_v(x) - v(x_v) + v(x_v)\). Note that \(\varphi(x_u) = u(x_u), \psi(x_v) = v(x_v), u(x) \geq \varphi(x)\) and \(v(x) \geq \psi(x)\) for all \(x \in B(0, R_0)\). Let \(\overline{w}_u\) and \(\overline{w}_v\) be defined as in (7) with \(U = B(0, R_0)\). Since \((u, v)\) is a viscosity super-solution of \((\text{II})\), one has

\[
(-\Delta)^s(\overline{w}_u)(x_u) \geq u^p(x_u) \quad \text{and} \quad (-\Delta)^t(\overline{w}_v)(x_v) \geq v^q(x_v). \quad (17)
\]

We now assert that

\[
(-\Delta)^s(\overline{w}_u)(x_u) \leq (-\Delta)^s(\xi_u)(x_u) \quad \text{and} \quad (-\Delta)^t(\overline{w}_v)(x_v) \leq (-\Delta)^t(\xi_v)(x_v).
\]

In fact, note that \(w_u(x) := \overline{w}_u(x) - \xi_u(x) \geq 0\) for all \(x \in \mathbb{R}^n\) and \(x_u\) is a global minimum point of \(w_u\). Thus, we have \((-\Delta)^s(w_u)(x_u) \leq 0\) and thus the first inequality follows. The other inequality also follows in an analogous way. Therefore, from (17), one gets

\[
m^q_u(R_0) \leq u^q(x_u) \leq C_0 \frac{m_v(R_0/2)}{R_0^{2t}} \quad \text{and} \quad m^p_u(R_0) \leq v^p(x_u) \leq C_0 \frac{m_u(R_0/2)}{R_0^{2s}}. \quad (18)
\]

Applying Lemma 2.2 in the above inequalities, one then derives

\[
m_u(R_0) \leq C_1 \frac{C_0}{R_0^{\frac{2s}{\tau} + 2t + \frac{2s}{\tau}}} \quad \text{and} \quad m_v(R_0) \leq C_2 \frac{C_0}{R_0^{\frac{2t}{\tau} + 2s + \frac{2t}{\tau}}}.
\]

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We now consider the case (I). It suffices to assume that \((\frac{2s}{p} + 2t) \frac{p}{pq - 1} > n - 2s\), since the argument is analogous for the second inequality in (I). Choose \(-n < \sigma_1 < -n + 2s\) such that
\[
(\frac{2s}{p} + 2t) \frac{p}{pq - 1} + \sigma_1 > 0.
\]
By Lemma 2.1 we have
\[
m_u(r) \leq m_u(R_0) \leq \frac{C}{r^{(\frac{2s}{p} + 2t) \frac{p}{pq - 1} + \sigma_1}}
\]
for all \(r \geq R_0 \geq 1\). Therefore, \(m_u(r)\) goes to 0 as \(r \to +\infty\), providing the contradiction \((u, v) = (0, 0)\).

Finally, assume the situation (II). In a similar way, we analyze only the equality \((\frac{2s}{p} + 2t) \frac{p}{pq - 1} = n - 2s\). Let \((u, v)\) be non-negative viscosity super-solution of (6) with \(G = \mathbb{R}^n\). We begin by proving that for certain \(C > 0\) and \(R_0 > 0\), we have
\[
m_u(r) \geq C m_u(R_0) r^{-n+2s} \tag{20}
\]
for all \(r \geq R_0\). Indeed, by Lemma 2.1 and (18), for any \(-n < \sigma < -n + 2s\), we have
\[
(\Delta)^s u(x) \geq v^p(x) \geq m_v(r)^p \geq C(m_u(2r))^{pq_r^{2tp}} \geq C(m_u(R_0))^{pq_r^{\sigma pq + 2tp}} \tag{21}
\]
for all \(x\) with \(|x| = r \geq R_0\).

Now consider the function
\[
w(r) = \begin{cases} 
\frac{\sigma}{r^{n+2s}} & \text{if } 0 < r \leq \varepsilon \\
\frac{\sigma}{r^{n+2s}} & \text{if } \varepsilon \leq r 
\end{cases} \tag{22}
\]
where \(0 < \varepsilon < R_0/2\). Since \(|x|^{-n+2s}\) is the fundamental solution of the fractional Laplace operator \((\Delta)^s\) (see [3]), we have
\[
2(\Delta)^s w(r) = \left( \int_{B_\varepsilon(x)} \frac{|x + y|^{-n+2s} - \varepsilon^{-n+2s}}{|y|^{n+2s}} \, dy + \int_{B_\varepsilon(x)} \frac{|x - y|^{-n+2s} - \varepsilon^{-n+2s}}{|y|^{n+2s}} \, dy \right),
\]
where \(|x| = r\). It is clear that \(|y| \geq |x|/2\) whenever \(|x| \geq R_0\) and \(y \in B_\varepsilon(x)\). Thus,
\[
\int_{B_\varepsilon(x)} \frac{|x - y|^{-n+2s} - \varepsilon^{-n+2s}}{|y|^{n+2s}} \, dy \leq \frac{C}{r^{n+2s}}
\]
for some constant $C > 0$ and then, by symmetry of the integrals, one obtains

$$2(-\Delta)^s w(r) \leq \frac{C}{r^{n+2s}}.$$  

For fixed $R_1 > R_0$, we define the comparison function

$$\varphi(x) = m_u(R_0)\frac{w(r) - w(R_1)}{w(\varepsilon) - w(R_1)}$$

for all $x$ with $|x| < R_1$ and $\varphi(x) = 0$ for $|x| \geq R_1$. As can easily be checked,

$$(-\Delta)^s \varphi(x) \leq \frac{C_1}{|x|^{n+2s}}$$  

for all $x$ with $R_0 < |x| < R_1$. On the other hand, since $n = pq(n - 2s) - 2tp$, we can choose $\sigma \in (-n, -n + 2s)$ such that $-\sigma pq - 2tp < n + 2s$. Then, using (21) and (23), one gets

$$(-\Delta)^s \varphi(x) \leq \frac{C_1}{|x|^{n+2s}} \leq \frac{C_1}{|x|^{-\sigma pq - 2tp}} \leq (-\Delta)^s u(x)$$

for all $x$ with $R_0 < |x| < R_1$ and $u(x) \geq \varphi(x)$ for $|x| \leq R_0$ or $|x| \geq R_1$, so that the Silvestre’s maximum principle [31] readily yields $u(x) \geq \varphi(x)$ for all $R_0 \leq |x| \leq R_1$. Finally, letting $R_1 \to +\infty$ in this last inequality, the claim (20) follows.

In the sequel, we split the proof into two cases according to the value of $-n + 2s$. The first one corresponds to $-n + 2s \in (-n, -1]$. In this range, note that the function $\Theta$, defined above Lemma 2.3, is decreasing for all $r > 0$, with a singularity at the origin if $-n + 2s \in (-n, -1)$ and bounded if $-n + 2s = -1$. For $0 < \varepsilon < R_0/2$, we define the function

$$w(r) = \begin{cases} 
\Theta(\varepsilon) & \text{if } 0 < r \leq \varepsilon \\
\Theta(r) & \text{if } \varepsilon < r 
\end{cases}.$$  

Using Lemma 2.3 for any $r \geq R_0$ and $x$ with $|x| = r$, we have

$$(-\Delta)^s w(r) \leq \int_{B_r(x)} \frac{\log(1 + |x - y|)|x - y|^{n+2s} - \log(1 + \varepsilon)|y|^{n+2s}}{|y|^{n+2s}} dy + \frac{C}{r^n}$$

$$\leq \frac{C \varepsilon^{2s}}{r^{n+2s}} + \frac{C}{r^n} \leq \frac{C}{r^n}$$

for all $r \geq R_0$ and some constant $C > 0$ independent of $r$.

Let $\varphi$ be defined as above for $R_1 > R_0$. Again, we have $\varphi(x) \leq u(x)$ for all $x$ with $|x| \leq R_0$ or $|x| \geq R_1$. Moreover,
for all \( x \) with \( R_0 < |x| < R_1 \). From (21), one also has
\[
(-\Delta)^s u(x) \geq C(m_u(R_0))^{pq}(-n+2s)^{pq+2tp} = \frac{C}{|x|^n}
\]
for \( r \geq R_0 \). By Silvestre’s maximum principle [31], we derive
\[ u(x) \geq \varphi(x) \]
for all \( R_0 < |x| < R_1 \). Letting \( R_1 \to +\infty \) in this inequality, one obtains
\[
u(x) \geq C \log(1 + |x|)
\]
for all \( x \) with \( |x| = r \) large enough. But this contradicts the positivity of \( u \).

It still remains the situation when \(-n + 2s \in (-1, 0)\). In this case, the function \( \Theta(r) \) is increasing near the origin and decreasing for \( r \) large, with exactly one maximum point, say at \( r_0 > 0 \). Consider the function
\[
w(r) = \begin{cases}
\Theta(r_0) & \text{if } 0 < r \leq r_0 \\
\Theta(r) & \text{if } r_0 < r
\end{cases}
\]
Again, one defines the comparison function for \( R_0 > 1 \) and \( R_0/2 > r_0 \) as in Lemma 2.2
\[
\varphi(x) = m_u(R_0) \frac{w(r) - w(R_1)}{w(r_0) - w(R_1)}
\]
for \( |x| < R_1 \) and \( \varphi(x) = 0 \) for \( |x| \geq R_1 \), where \( R_1 > R_0 \). It is clear that \( \varphi(x) \leq u(x) \)
for all \( x \) with \( |x| \leq R_0 \) or \( |x| \geq R_1 \). In addition,
\[
(-\Delta)^s \varphi(x) \leq \frac{C}{|x|^n}
\]
for all \( x \) with \( R_0 < |x| < R_1 \). Lastly, using Lemma 2.3 and the fact that \( \Theta \) is increasing
in \((0, r_0)\) and decreasing for \( r \geq r_0 \), the proof proceeds exactly as before and again
we achieve the contradiction \( u = 0 \). This concludes the proof of sufficiency. ■

**Proof of the necessity of (4).** Assume that the condition (4) fails. In other words, we have

\[
(-\Delta)^s \varphi(x) \leq \frac{C}{|x|^n}
\]
\[
\left(\frac{2s}{p} + 2t\right) \frac{p}{pq - 1} < n - 2s \quad \text{and} \quad \left(\frac{2t}{q} + 2s\right) \frac{q}{pq - 1} < n - 2t.
\tag{26}
\]

Consider the functions
\[
u(x) = \frac{A}{(1 + |x|)|2s_{k1}} \quad \text{and} \quad v(x) = \frac{B}{(1 + |x|)|2t_{k2}},
\tag{27}
\]

where
\[k_1 = \frac{t + sp}{t(pq - 1)} \quad \text{and} \quad k_2 = \frac{s + tq}{s(pq - 1)}.
\]

The basic idea is to prove that \((u, v)\) is a positive radial super-solution of \((6)\) with \(G = \mathbb{R}^n\) for a suitable choice of positive constants \(A\) and \(B\).

Firstly, we assert that the inequalities
\[
\frac{1}{(1 - a + |ae_1 + y|)|2s_{k1}|} + \frac{1}{(1 - a + |ae_1 - y|)|2s_{k1}|} \leq \frac{1}{|e_1 + y|_{2s_{k1}}} + \frac{1}{|e_1 - y|_{2s_{k1}}}
\tag{28}
\]

and
\[
\frac{1}{(1 - a + |ae_1 + y|)|2t_{k2}|} + \frac{1}{(1 - a + |ae_1 - y|)|2t_{k2}|} \leq \frac{1}{|e_1 + y|_{2t_{k2}}} + \frac{1}{|e_1 - y|_{2t_{k2}}}
\tag{29}
\]
hold for all \(a \in [0, 1), b \geq 0\) and \(y \in \mathbb{R}\). In fact, consider the function \(f(a, b, y)\) given by
\[
f(a, b, y) = (1 - a + (a + b)^2 + y^2)^{1/2})^{-2\alpha} + (1 - a + (a - b)^2 + y^2)^{1/2})^{-2\alpha}
\]
\[ - ((1 + b)^2 + y^2)^{-\alpha} - ((1 - b)^2 + y^2)^{-\alpha}
\]
where \(\alpha > 0\). One easily checks that
\[
\frac{\partial f}{\partial a}(a, b, y) = \frac{-2\alpha}{(1 - a + (a + b)^2 + y^2)^{1/2})^{2\alpha+1}} \left(-1 + \frac{a + b}{((a + b)^2 + y^2)^{1/2}}\right)
\]
\[+ \frac{-2\alpha}{(1 - a + (a - b)^2 + y^2)^{1/2})^{2\alpha+1}} \left(-1 + \frac{a - b}{((a - b)^2 + y^2)^{1/2}}\right) \geq 0
\]
and \(f(1, b, y) = 0\) for all \(a \in [0, 1), b \geq 0\) and \(y \in \mathbb{R}\). In particular, \(f(a, b, y) \leq 0\) for all \(a \in [0, 1), b \geq 0\) and \(y \in \mathbb{R}\).
For \( a = r/(1 + r) \) and \( x \) with \( r = |x| \), we then have

\[
\frac{1}{(1 + |x + y|)^{2\alpha}} + \frac{1}{(1 + |x - y|)^{2\alpha}} - \frac{2}{(1 + |x|)^{2\alpha}}
\]

\[
= \frac{1}{(1 + |x|)^{2\alpha}} \left\{ \frac{1}{(1 - a + |ae_1 + \overline{y}|)^{2\alpha}} + \frac{1}{(1 - a + |ae_1 - \overline{y}|)^{2\alpha}} - 2 \right\}
\]

\[
\leq \frac{1}{(1 + |x|)^{2\alpha}} \left\{ \frac{1}{|e_1 + \overline{y}|^{2\alpha}} + \frac{1}{|e_1 - \overline{y}|^{2\alpha}} - 2 \right\},
\]

where \( \overline{y} = \frac{1}{1+r}Py \), being \( P \) an appropriate rotation matrix.

With the choice \( \alpha = sk_1 \) and \( \alpha = tk_2 \), we derive (28) and (29), respectively. Using these inequalities, we find

\[
(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{A}{(1 + |x - y|)^{2sk_1}|y|^{n+2s}} + \frac{A}{(1 + |x + y|)^{2sk_1}|y|^{n+2s}}
\]

\[
- \frac{2A}{(1 + |x|)^{2sk_1}|y|^{n+2s}} \, dy
\]

\[
\geq -\frac{1}{2} \frac{A}{(1 + |x|)^{2sk_1+1}} \int_{\mathbb{R}^n} \frac{|e_1 + y|^{-2sk_1} + |e_1 - y|^{-2sk_1} - 2}{|y|^{n+2s}} \, dy
\]

\[
= \frac{c_1 A}{(1 + |x|)^{2sk_1+1}}
\]

and

\[
(-\Delta)^t v(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{B}{(1 + |x - y|)^{2tk_2}|y|^{n+2t}} + \frac{B}{(1 + |x + y|)^{2tk_2}|y|^{n+2t}}
\]

\[
- \frac{2B}{(1 + |x|)^{2tk_2}|y|^{n+2t}} \, dy
\]

\[
\geq -\frac{1}{2} \frac{B}{(1 + |x|)^{2tk_2+1}} \int_{\mathbb{R}^n} \frac{|e_1 + y|^{-2tk_2} + |e_1 - y|^{-2tk_2} - 2}{|y|^{n+2t}} \, dy
\]

\[
= \frac{c_2 B}{(1 + |x|)^{2tk_2+1}}.
\]

Since \( pq > 1 \), there exist constants \( k_1 \) and \( k_2 \) such that \( 2s(k_1 + 1) = 2tk_2p \) and \( 2t(k_2 + 1) = 2sk_1q \). Thanks to (26), it readily follows that \( k_1 \) and \( k_2 \) are positive, \( 2sk_1 < n - 2s \) and \( 2tk_2 < n - 2t \). These last two conditions guarantee the positivity of the above constants \( c_1 \) and \( c_2 \).

On the other hand, we have
\[ (-\Delta)^s u(x) - v^p(x) \geq \frac{c_1 A}{(1 + |x|)^{2s(k_1+1)}} - \frac{B^p}{(1 + |x|)^{2tk_2p}} = \frac{c_1 A - B^p}{(1 + |x|)^{2s(k_1+1)}} \]

and

\[ (-\Delta)^t v(x) - u^q(x) \geq \frac{c_2 B}{(1 + |x|)^{2t(k_2+1)}} - \frac{A^q}{(1 + |x|)^{2sk_1q}} = \frac{c_2 B - A^q}{(1 + |x|)^{2t(k_2+1)}} \]

for all \( x \in \mathbb{R}^n \). Finally, the assumption \( pq > 1 \) also allows us to choose \( A = (c_1 c_2)^{\frac{p}{pq-1}} > 0 \) and \( B = (c_1 c_2)^{\frac{q}{pq-1}} > 0 \) so that the right-hand side of the above inequalities are equal to zero. This concludes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

The first tool to be used in the proof of Theorem 1.3 is the following result whose proof is based on the method of moving plane.

Proposition 4.1. Let \((u, v)\) be a positive viscosity bounded solution of

\[
\begin{aligned}
(-\Delta)^s u &\geq v^p & \text{in } \mathbb{R}^n_+ \\
(-\Delta)^t v &\geq u^q & \text{in } \mathbb{R}^n_+ \\
u = v = 0 & \text{ on } \partial \mathbb{R}^n_+
\end{aligned}
\]

Assume \( p, q \geq 1 \). Then, \( u \) and \( v \) are strictly increasing in \( x_n \)-direction.

Proof of Proposition 4.1. Let \( \Sigma := \{(\overline{\sigma}, x_n) \in \mathbb{R}^n_+ : 0 < x_n < \mu\} \) and \( T_\mu := \{(\overline{\sigma}, x_n) \in \mathbb{R}^n_+ : x_n = \mu\} \). For \( x = (\overline{\sigma}, x_n) \in \mathbb{R}^n \), we denote \( u_\mu(x) = u(x_\mu) \), \( w_{\mu,u}(x) = u_\mu(x) - u(x) \), \( v_\mu(x) = v(x_\mu) \) and \( w_{\mu,v}(x) = v_\mu(x) - v(x) \), where \( \mu > 0 \) and \( x_\mu = (\overline{\sigma}, 2\mu - x_n) \) for all \( (\overline{\sigma}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \). For any subset \( A \) of \( \mathbb{R}^n \), we write \( A_\mu = \{x_\mu : x \in A\} \), the reflection of \( A \) with respect to \( T_\mu \).

We next divide the proof into two steps.

First step: We here prove that if \( \mu > 0 \) is small enough, then \( w_{\mu,u} > 0 \) and \( w_{\mu,v} > 0 \) in \( \Sigma_\mu \). For this purpose, we define

\[ \Sigma_{\mu,u} = \{x \in \Sigma_\mu : w_{\mu,u}(x) < 0\} \text{ and } \Sigma_{\mu,v} = \{x \in \Sigma_\mu : w_{\mu,v}(x) < 0\} \].

We first show that \( \Sigma_{\mu,u} \) is empty if \( \mu \) is small enough. Indeed, assume for a contradiction that \( \Sigma_{\mu,u} \) is not empty and define
\[ w^1_{\mu,u}(x) = \begin{cases} w_{\mu,u}(x) & \text{if } x \in \Sigma_{\mu,u}^- \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Sigma_{\mu,u}^- \end{cases} \quad (31) \]

and

\[ w^2_{\mu,u}(x) = \begin{cases} 0 & \text{if } x \in \Sigma_{\mu,u}^- \\ w_{\mu,u}(x) & \text{if } x \in \mathbb{R}^n \setminus \Sigma_{\mu,u}^- \end{cases} \quad (32) \]

It is clear that \( w^1_{\mu,u}(x) = w_{\mu,u}(x) - w^2_{\mu,u}(x) \) for all \( x \in \mathbb{R}^n \). For each \( \mu > 0 \), we now assert that

\[ (-\Delta)^s w^2_{\mu,u}(x) \leq 0 \text{ for all } x \in \Sigma_{\mu,u}^- . \quad (33) \]

In fact, from the definition of \((\Delta)^s\), we have

\[
(-\Delta)^s w^2_{\mu,u}(x) = \int_{\mathbb{R}^n} \frac{w^2_{\mu,u}(x) - w^2_{\mu,u}(y)}{|x-y|^{n+2s}} \, dy - \int_{\mathbb{R}^n \setminus \Sigma_{\mu,u}^-} \frac{w^2_{\mu,u}(y)}{|x-y|^{n+2s}} \, dy \\
= - \int_{\Sigma_{\mu,u}^- \setminus \Sigma_{\mu,u} \cup (\Sigma_{\mu,u} \setminus \Sigma_{\mu,u})_{\mu}} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}} \, dy \\
- \int_{(\mathbb{R}^n \setminus \Sigma_{\mu,u}) \cup (\mathbb{R}^n \setminus \Sigma_{\mu,u})_{\mu}} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}} \, dy \\
= -A_1 - A_2 - A_3 
\]

for all \( x \in \Sigma_{\mu,u}^- \).

We next estimate separately each of these integrals.

Firstly, note that \( w_{\mu,u}(y_{\mu}) = -w_{\mu,u}(y) \) for all \( y \in \mathbb{R}^n \) and \( w^2_{\mu,u}(y) \geq 0 \) in \( \Sigma_{\mu} \setminus \Sigma_{\mu,u}^- \).

Then,

\[
A_1 = \int_{(\Sigma_{\mu} \setminus \Sigma_{\mu,u}^-) \cup (\Sigma_{\mu,u} \setminus \Sigma_{\mu,u})_{\mu}} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}} \, dy \\
= \int_{\Sigma_{\mu} \setminus \Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}} \, dy + \int_{\Sigma_{\mu,u} \setminus \Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y)}{|x-y_{\mu}|^{n+2s}} \, dy \\
= \int_{\Sigma_{\mu} \setminus \Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}} \left( \frac{1}{|x-y_{\mu}|^{n+2s}} - \frac{1}{|x-y_{\mu}|^{n+2s}} \right) \, dy \geq 0 ,
\]

since \( |x-y_{\mu}| > |x-y| \) for all \( x \in \Sigma_{\mu,u}^- \) and \( y \in \Sigma_{\mu} \setminus \Sigma_{\mu,u}^- \).
In order to discover the sign of $A_2$ we observe that $u = 0$ in $\mathbb{R}^n \setminus \mathbb{R}_+^n$ and $u_\mu = 0$ in $(\mathbb{R}^n \setminus \mathbb{R}_+^n)_\mu$, so we have

$$A_2 = \int_{(\mathbb{R}^n \setminus \mathbb{R}_+^n) \cup (\mathbb{R}^n \setminus \mathbb{R}_+^n)_\mu} \frac{w_{\mu,u}(y)}{|x - y|^{n+2s}} \, dy$$

$$= \int_{\mathbb{R}^n \setminus \mathbb{R}_+^n} \frac{u_\mu(y)}{|x - y|^{n+2s}} \, dy - \int_{(\mathbb{R}^n \setminus \mathbb{R}_+^n)_\mu} \frac{u(y)}{|x - y|^{n+2s}} \, dy$$

$$= \int_{\mathbb{R}^n \setminus \mathbb{R}_+^n} u_\mu(y) \left( \frac{1}{|x - y|^{n+2s}} - \frac{1}{|x - y_\mu|^{n+2s}} \right) \, dy \geq 0,$$

since $u_\mu \geq 0$ in $\mathbb{R}^n \setminus \mathbb{R}_+^n$ and $|x - y_\mu| > |x - y|$ for all $x \in \Sigma^{-}_{\mu,u}$ and $y \in \mathbb{R}^n \setminus \mathbb{R}_+^n$. Finally, since $w_{\mu,u} < 0$ in $\Sigma^{-}_{\mu,u}$, we have

$$A_3 = \int_{\Sigma^{-}_{\mu,u}} \frac{w_{\mu,u}(y)}{|x - y|^{n+2s}} \, dy = \int_{\Sigma^{-}_{\mu,u}} \frac{w_{\mu,u}(y)}{|x - y_\mu|^{n+2s}} \, dy = - \int_{\Sigma^{-}_{\mu,u}} \frac{w_{\mu,u}(y)}{|x - y_\mu|^{n+2s}} \, dy \geq 0.$$

Hence, the claim (33) follows.

Using now (33), for any $x \in \Sigma^{-}_{\mu,u}$, one has

$$(-\Delta)^s w^1_{\mu,u}(x) = (-\Delta)^s w_{\mu,u}(x) = (-\Delta)^s u_\mu(x) - (-\Delta)^s u(x)$$

$$= v^p_\mu(x) - v^p(x) = \frac{v^p_\mu(x) - v^p(x)}{v_\mu(x) - v(x)} w_{\mu,v}(x).$$

Define

$$\varphi_v(x) = \frac{v^p_\mu(x) - v^p(x)}{v_\mu(x) - v(x)}$$

for $x \in \Sigma^{-}_{\mu,u}$.

Since $p \geq 1$, we have $\varphi_v \in L^\infty(\Sigma^{-}_{\mu,u})$ and $\varphi_v w_{\mu,v}$ is continuous. In addition, since $w^1_{\mu,u} = 0$ in $\mathbb{R}^n \setminus \Sigma^{-}_{\mu,u}$, by Theorem 2.3 of [27], one gets

$$\|w^1_{\mu,u}\|_{L^\infty(\Sigma^{-}_{\mu,u})} \leq CR(\Sigma^{-}_{\mu,u})^{2s} \|\varphi_v w_{\mu,v}\|_{L^\infty(\Sigma^{-}_{\mu,u})},$$

where $R(\Sigma^{-}_{\mu,u})$ is the smallest positive constant $R$ such that

$$|B_R(x) \setminus \Sigma^{-}_{\mu,u}| \geq \frac{1}{2} |B_R(x)|.$$
for all \( x \in \Sigma_{\mu,u} \). Besides, we have

\[
\varphi_w w_{\mu,v}(x) = v^p(x) - v_\mu^p(x) \leq 0 \text{ in } \Sigma_{\mu} \setminus \Sigma_{\mu,v}^-
\]

and

\[
\varphi_w w_{\mu,v}(x) = v^p(x) - v_\mu^p(x) > 0 \text{ in } \Sigma_{\mu,v}^-.
\]

Let \( \Sigma_{\mu}^- = \Sigma_{\mu,u}^- \cap \Sigma_{\mu,v}^- \). Then, from (34), one derives

\[
\|w_{\mu,u}^1\|_{L^\infty(\Sigma_{\mu,u}^-)} \leq CR(\Sigma_{\mu,u}^-)^{2s} \|\varphi_w w_{\mu,v}\|_{L^\infty(\Sigma_{\mu}^-)} \leq CR(\Sigma_{\mu,u}^-)^{2s} \|\varphi_w\|_{L^\infty(\Sigma_{\mu}^-)} \|w_{\mu,v}\|_{L^\infty(\Sigma_{\mu}^-)} \leq CR(\Sigma_{\mu,u}^-)^{2s} \|w_{\mu,v}\|_{L^\infty(\Sigma_{\mu}^-)},
\]

where in the last inequality we use the condition \( p \geq 1 \).

Similar to (31) and (32), we define

\[
w_{\mu,v}^1(x) = \begin{cases} w_{\mu,v}(x) & \text{if } x \in \Sigma_{\mu,v}^- \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Sigma_{\mu,v}^- \end{cases} \tag{35}
\]

and

\[
w_{\mu,v}^2(x) = \begin{cases} 0 & \text{if } x \in \Sigma_{\mu,v}^- \\ w_{\mu,v}(x) & \text{if } x \in \mathbb{R}^n \setminus \Sigma_{\mu,v}^- \end{cases} \tag{36}
\]

and argue in a completely analogous way with the aid of the assumption \( q \geq 1 \) to obtain

\[
\|w_{\mu,v}^1\|_{L^\infty(\Sigma_{\mu,v}^-)} \leq CR(\Sigma_{\mu,v}^-)^{2s} \|w_{\mu,u}\|_{L^\infty(\Sigma_{\mu}^-)}.
\]

Thus,

\[
\|w_{\mu,u}\|_{L^\infty(\Sigma_{\mu,u}^-)} \leq CR(\Sigma_{\mu,u}^-)^{2s} R(\Sigma_{\mu,v}^-)^{2t} \|w_{\mu,u}\|_{L^\infty(\Sigma_{\mu,u}^-)}
\]

and

\[
\|w_{\mu,u}^1\|_{L^\infty(\Sigma_{\mu,u}^-)} \leq CR(\Sigma_{\mu,u}^-)^{2s} R(\Sigma_{\mu,v}^-)^{2t} \|w_{\mu,v}^1\|_{L^\infty(\Sigma_{\mu,v}^-)}.
\]

Now choosing \( \mu \) small enough so that \( C^2 R(\Sigma_{\mu,u}^-)^{2s} R(\Sigma_{\mu,v}^-)^{2t} < 1 \), we conclude that \( \|w_{\mu,u}\|_{L^\infty(\Sigma_{\mu,u}^-)} = 0 \), so \( |\Sigma_{\mu,u}^-| = 0 \). Since \( \Sigma_{\mu,u}^- \) is open, we deduce that \( \Sigma_{\mu,u}^- \) is empty, which is a contradiction. Therefore, we get \( w_{\mu,u} \geq 0 \) in \( \Sigma_{\mu} \) for \( \mu > 0 \) small enough. Similarly, one gets \( w_{\mu,v} \geq 0 \) in \( \Sigma_{\mu} \) for \( \mu > 0 \) small enough too. Moreover, since the
functions $u$ and $v$ are positive in $\mathbb{R}^n_+$ and $u = v = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^n_+$, it follows that $w_{\mu,u}$ and $w_{\mu,v}$ are positive in $\{x_n = 0\}$ and then, by continuity, $w_{\mu,u} \not= 0$ and $w_{\mu,v} \not= 0$ in $\Sigma_\mu$.

In order to complete the proof of this step, we assert that if $w_{\mu,u} \geq 0$, $w_{\mu,v} \geq 0$, $w_{\mu,u} \not= 0$ and $w_{\mu,v} \not= 0$ in $\Sigma_\mu$ with $\mu > 0$, then $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in $\Sigma_\mu$. Indeed, we have

$$(-\Delta)^s w_{\mu,u}(x) = v_\mu^p(x) - v^p(x) \geq 0 \text{ in } \Sigma_\mu$$

and

$$(-\Delta)^t w_{\mu,v}(x) = u_\mu^q(x) - u^q(x) \geq 0 \text{ in } \Sigma_\mu.$$ 

Since $w_{\mu,u} \geq 0$, $w_{\mu,v} \geq 0$, $w_{\mu,u} \not= 0$ and $w_{\mu,v} \not= 0$ in $\Sigma_\mu$, by the Silvestre’s strong maximum principle, the conclusion follows.

**Second step:** Define

$$\mu^* = \sup \{\mu > 0 : w_{\nu,u} > 0, w_{\nu,v} > 0 \text{ in } \Sigma_\nu \text{ for all } 0 < \nu < \mu\}.$$ 

It is clear that $\mu^* > 0$ and $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in $\Sigma_\mu$ for all $0 < \mu < \mu^*$, so that $u$ and $v$ are strictly increasing in $x_n$-direction. Indeed, for $0 < x_n < \overline{x}_n < \mu^*$, let $\mu = \frac{x_n + \overline{x}_n}{2}$. Since $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in $\Sigma_\mu$, we have

$$0 < w_{\mu,u}(x', x_n) = u_\mu(x', x_n) - u(x', x_n) = u(x', \overline{x}_n) - u(x', x_n)$$

and

$$0 < w_{\mu,v}(x', x_n) = v_\mu(x', x_n) - v(x', x_n) = v(x', \overline{x}_n) - v(x', x_n),$$

so that $u(x', \overline{x}_n) > u(x', x_n)$ and $v(x', \overline{x}_n) > v(x', x_n)$, as claimed. Thus, the proposition is proved if we are able to show that $\mu^* = +\infty$.

Suppose for a contradiction that $\mu^*$ is finite. Now choose $\varepsilon_0 > 0$ small enough such that the operators $(-\Delta)^s - \varphi_v$ and $(-\Delta)^t - \varphi_u$ satisfies the strong maximum principle in the domain $\Sigma_{\mu^*+\varepsilon_0} \setminus \Sigma_{\mu^*+\varepsilon_0}$, see [27]. Here we use that $\varphi_u(x) = \frac{w_\mu(x) - u^q(x)}{w_\mu(x) - u(x)}$ and $\varphi_v(x) = \frac{w_\mu(x) - v^p(x)}{w_\mu(x) - v(x)}$ can be taken small in the $L^\infty$-norm, since $p, q > 1$. Therefore, $w_{\mu^*+\varepsilon_0,u} > 0$ and $w_{\mu^*+\varepsilon_0,v} > 0$ in $\Sigma_{\mu^*+\varepsilon_0}$, providing a contradiction. \(\blacksquare\)
Proposition 4.2. Let $p, q > 0$. If the system
\[
\begin{cases}
(\Delta)^s u = v^p & \text{in } \mathbb{R}_+^n \\
(\Delta)^t v = u^q & \text{in } \mathbb{R}_+^n \\
u = v = 0 & \text{on } \partial\mathbb{R}_+^n
\end{cases}
\tag{37}
\]
has a positive viscosity bounded solution, then the same system has a positive viscosity solution in $\mathbb{R}^{n-1}$.

Proof of Proposition 4.2. Let $(u, v)$ be a positive bounded solution of (37), that is there exists a constant $M$ such that $0 < u \leq M$ and $0 < v \leq M$ in $\mathbb{R}_+^n$. In the strip $\Sigma_1 = \{x \in \mathbb{R}^n : 0 < x_n < 1\}$, we set
\[
u_k(x', x_n) = u(x', x_n + k) \quad \text{and} \quad v_k(x', x_n) = v(x', x_n + k).
\]
Note that $(u_k, v_k)$ solves the system (37) in $\Sigma_1$ for each integer $k \geq 1$. In addition, $0 < u_k \leq M$ and $0 < v_k \leq M$ in $\Sigma_1$. Thus,
\[
(\Delta)^s u_k \leq M^p \quad \text{and} \quad (\Delta)^s u_k \geq 0 \quad \text{in } \Sigma_1,
\]
\[
(\Delta)^t v_k \leq M^q \quad \text{and} \quad (\Delta)^t v_k \geq 0 \quad \text{in } \Sigma_1.
\]
Then, by Theorem 2.6 of [27], for any $\Omega' \subset \subset \Sigma_1$ and $0 < \beta < 1$, there exists a constant $C > 0$ such that $u_k, v_k \in C^\beta(\Omega')$ and
\[
\|u_k\|_{C^\beta(\Omega')} \leq C\left\{\|u_k\|_{L^\infty(\Sigma_1)} + M^p\right\}
\]
and
\[
\|v_k\|_{C^\beta(\Omega')} \leq C\left\{\|v_k\|_{L^\infty(\Sigma_1)} + M^q\right\}.
\]
So, the sequences $\{u_k\}$ and $\{v_k\}$ are bounded in $C^\beta(\Omega')$ and then, up to a subsequence, $\{u_k\}$ and $\{v_k\}$ converge uniformly on compact subset of $\Sigma_1$ to functions $\overline{u}$ and $\overline{v}$, respectively. By Theorem 2.7 of [27], $(\overline{u}, \overline{v})$ satisfies
\[
\begin{cases}
(\Delta)^s \overline{u} = \overline{v}^p & \text{in } \Sigma_1 \\
(\Delta)^t \overline{v} = \overline{u}^q & \text{in } \Sigma_1
\end{cases}
\tag{38}
\]
in the viscosity sense. The strict monotonicity provided in Proposition 4.1 guarantees that $(\overline{u}, \overline{v})$ is positive and independent of the $x_n$-variable.
On the other hand, the definition of \((-\Delta)^s\) gives
\[
(-\Delta)^s \overline{u}(x) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{\overline{u}(x') - \overline{u}(y')}{(|x' - y'|^2 + (x_n - y_n)^2)^{\frac{n+2s}{2}}} \, dy_n \, dy' \\
= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{\overline{u}(x') - \overline{u}(x' - y')}{|y'|^2 + (y_n)^2 \frac{n}{2s}} \, dy_n \, dy'.
\]

Let \(y_n = |y'| \tan \theta\), where \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\), then
\[
(-\Delta)^s \overline{u}(x) = \int_{\mathbb{R}^{n-1} - \frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\overline{u}(x') - \overline{u}(x' - y')}{|y'|^{n-1+2s}} (\cos \theta)^{n-2+2s} \, d\theta \, dy' \\
= \int_{\mathbb{R}^{n-1}} \frac{\overline{u}(x') - \overline{u}(x' - y')}{|y'|^{n-1+2s}} \, dy' \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta)^{n-2+2s} \, d\theta
\]

and
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta)^{n-2+2s} \, d\theta = 2 \int_{0}^{\frac{\pi}{2}} (\cos \theta)^{n-2+2s} \, d\theta < +\infty,
\]
since \(n - 2 + 2s > 0\). This means that the \(n\)-dimension fractional Laplace operator is actually \((n - 1)\)-dimension, and we have
\[
\begin{cases} 
(-\Delta)^s \overline{u} = \overline{u}^q \text{ in } \mathbb{R}^{n-1} \\
(-\Delta)^s \overline{u} = \overline{u}^p \text{ in } \mathbb{R}^{n-1}
\end{cases}
\]  

(39)

Finally, Theorem 1.3 follows directly from Theorem 1.2 and Proposition 4.2.

5 Proof of Theorem 1.1

The proof of the part of existence is an application of degree theory for compact operators in cones. This theory, essentially developed by Krasnoselskii, has often been used to show that certain operators admit fixed points. We are going to use an extension of Krasnoselskii results (see for instance [26]). The applicability of this
theory relies on a priori bounds in $L^\infty$ of solutions of certain systems related to (1) to be obtained through blow-up techniques by invoking Theorems 1.2 and 1.3.

We begin by stating the above-mentioned abstract tool.

**Proposition 5.1.** Let $K$ be a closed cone with non-empty interior in a Banach space $X$ and let $T : K \to K$ and $H : [0, \infty) \times K \to K$ be continuous compact operators such that $T(0) = 0$ and $H(0, x) = T(x)$ for all $x \in K$. Assume there exist $\theta_0 > 0$ and $0 < r < R$ such that

(i) $x \neq \theta T(x)$ for all $0 \leq \theta \leq 1$ and $x \in K$ such that $\|x\| = r$;

(ii) $H(\theta, x) \neq x$ for all $\theta \geq \theta_0$ and $x \in K$ with $\|x\| \leq R$,

(iii) $H(\theta, x) \neq x$ for all $\theta \in [0, +\infty)$ and $x \in K$ with $\|x\| = R$.

Then, $T$ has a fixed point $x_0 \in K$ such that $r \leq \|x_0\| \leq R$.

Here $X$ denotes the Banach space \{$(u, v) \in C(\Omega) \times C(\Omega) : u, v = 0$ on $\partial\Omega$\} endowed with the norm

$$
\|(u, v)\| := \max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\}.
$$

and $K = \{u \in X : u, v \geq 0$ in $\Omega\}$. It is clear that solving (1) is equivalent to finding a fixed point in $K$ of the operator $T : K \to K$ given by

$$
T(u, v)(x) := S(v^p, u^q)
$$

for $x \in \Omega$, where for any $(f, g) \in K$ we define $S(f, g)$ as the solution of the Dirichlet problem

\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega \\
(-\Delta)^t v = g & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]  

Using that $\Omega$ is $C^2$ class, by Lemma 6.1 of [26], the operator $S$ is well defined, linear, continuous and compact. Thus, one easily deduces that the operator $T$ is well defined, continuous and compact. In addition, we have $T(0, 0) = 0$.

We also define $H : [0, \infty) \times K \to K$ as

$$
H(\theta, u, v) = S((v + \theta)^p, (u + \theta)^q).
$$

Clearly, $H$ is well defined, continuous and compact too.

First we show that the condition (i) of Proposition 5.1 is satisfied. This is the content of the following lemma:
Lemma 5.1. Assume that $s, t \in (0, 1)$ and $pq > 1$. Then, there exists a constant $r > 0$ such that for any $\theta \in [0, 1]$, the system

$$
\begin{cases}
  (-\Delta)^s u = \theta v^p & \text{in } \Omega \\
  (-\Delta)^t v = \theta u^q & \text{in } \Omega \\
  u = v = 0 & \text{on } \partial \Omega
\end{cases}
$$

has no classical solution $(u, v) \in K$ with $\|(u, v)\| = r$.

Proof of Lemma 5.1. We argue by contradiction. Let $\{(\theta_k, u_k, v_k)\}_{k \in \mathbb{N}}$ be a sequence of triples with $\theta_k \in [0, 1]$ and $(u_k, v_k) \in K$ satisfying (41) such that $\|u_k\|_{L^\infty(\Omega)}, \|v_k\|_{L^\infty(\Omega)} \to 0$ as $k \to +\infty$. Since $pq > 1$, we choose $\gamma$ such that

$$
\frac{1}{q} < \gamma < p
$$

and set $a_k = \|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)}^{\gamma}$. Define

$$
z_k = \frac{u_k}{a_k} \quad \text{and} \quad w_k = \frac{v_k}{a_k^{1/\gamma}}.
$$

We then have

$$
(-\Delta)^s z_k = \frac{\theta_k}{a_k} v_k^p \quad \text{and} \quad (-\Delta)^t w_k = \frac{\theta_k}{a_k^{1/\gamma}} u_k^q.
$$

Note that $\|z_k\|_{L^\infty(\Omega)} + \|w_k\|_{L^\infty(\Omega)}^{\gamma} = 1$,

$$
\left| \frac{\theta_k}{a_k} v_k^p \right| \leq \|v_k\|_{L^\infty(\Omega)}^{p-\gamma} \to 0 \quad \text{and} \quad \left| \frac{\theta_k}{a_k^{1/\gamma}} u_k^q \right| \leq \|u_k\|_{L^\infty(\Omega)}^{q-1/\gamma} \to 0
$$

uniformly for $x \in \Omega$. So, one easily deduces that $(z_k, w_k)$ converges uniformly to some couple $(z, w)$ satisfying $\|z\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}^{\gamma} = 1$ and

$$
\begin{cases}
  (-\Delta)^s z = 0 & \text{in } \Omega \\
  (-\Delta)^t w = 0 & \text{in } \Omega \\
  z = w = 0 & \text{on } \partial \Omega
\end{cases}
$$

But by uniqueness, we have $(z, w) = (0, 0)$, providing a contradiction. ■

The condition (ii) of Proposition 5.1 follows from the following lemma:
Lemma 5.2. Assume that \( s, t \in (0, 1) \), \( p, q \geq 1 \) and \( pq > 1 \). Then, there exists a constant \( \theta_0 > 0 \) such that for any \( \theta \geq \theta_0 \) the system

\[
\begin{cases}
(-\Delta)^s u = (v + \theta)^p & \text{in } \Omega \\
(-\Delta)^t v = (u + \theta)^q & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]

has no classical solution \((u, v) \in K\).

Proof of Lemma 5.2. Firstly, we define

\[
\lambda_1 := \inf \left\{ \int_{\Omega} \left| (-\Delta)^{s/2}u \right|^2 + \left| (-\Delta)^{t/2}v \right|^2 \, dx : (u, v) \in H_0^s(\Omega) \times H_0^t(\Omega), \int_{\Omega} u^+ v^+ \, dx = 1 \right\},
\]

where \( f^+ = \max\{f, 0\} \). As usual, it follows that \( \lambda_1 \) is positive and attained for some couple \((\varphi, \psi) \in H_0^s(\Omega) \times H_0^t(\Omega)\). Also, by the weak maximum principle, \( \varphi, \psi \geq 0 \) in \( \Omega \) and \( \varphi, \psi \neq 0 \) and, moreover, \((\varphi, \psi)\) satisfies

\[
\begin{cases}
(-\Delta)^s \varphi = \lambda_1 \psi & \text{in } \Omega \\
(-\Delta)^t \psi = \lambda_1 \varphi & \text{in } \Omega \\
\varphi = \psi = 0 & \text{on } \partial\Omega
\end{cases}
\]

On the other hand, by assumption, \( p > 1 \) or \( q > 1 \). If the first situation occurs, then for \( A \geq \lambda_1^2 \) there exists \( \theta_0 > 0 \) such that

\[(y + \theta)^p \geq A(y + \theta) > Ay \quad \text{and} \quad (y + \theta)^p \geq (y + \theta) > y\]

for all \( y \geq 0 \) and \( \theta \geq \theta_0 \).

Now let \( \theta \geq \theta_0 \) and \((u, v) \in K\) be a classical solution of \((42)\). Then, by the Silvestre’s strong maximum principle, we have \( u, v > 0 \) in \( \Omega \) and

\[
\begin{cases}
(-\Delta)^s u > Av & \text{in } \Omega \\
(-\Delta)^t v > u & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]

Using the above equations satisfied by \((\varphi, \psi)\), one obtains

\[
\lambda_1 \int_{\Omega} u \psi \, dx > A \int_{\Omega} v \varphi \, dx \quad \text{and} \quad \lambda_1 \int_{\Omega} v \varphi \, dx > \int_{\Omega} u \psi \, dx,
\]

so that \( A < \lambda_1^2 \), providing a contradiction. \( \blacksquare \)

Finally, the condition (iii) of Proposition 5.1 is a consequence of the following lemma:
Lemma 5.3. Assume that $\Omega$ is of $C^2$ class, $s, t \in (0, 1)$, $n > 2s + 1$, $n > 2t + 1$, $p, q \geq 1$, $pq > 1$ and (4) is satisfied. For each $\theta_0 > 0$ there exists a constant $C > 0$, depending only of $s, t, p, q$ and $\Omega$, such that for any classical solution $(u, v) \in K$ of the system (42) with $0 \leq \theta \leq \theta_0$, one has

$$
\|(u, v)\| \leq C.
$$

Proof of Lemma 5.3. Suppose for a contradiction that there exists a sequence $(u_k, v_k) \in K$ of solutions of (42) with $\theta = \theta_k \in [0, \theta_0]$ such that at least one of the sequence $(u_k)$ and $(v_k)$ tends to infinity in the $L^\infty$-norm.

Let $\beta_1 = \left(\frac{2s}{p} + 2t\right) \frac{p}{pq-1}$ and $\beta_2 = \left(\frac{2t}{q} + 2s\right) \frac{q}{pq-1}$. We set

$$
\lambda_k = \|u_k\|^{-\frac{1}{\beta_1}}_{L^\infty(\Omega)},
$$

if $\|u_k\|_{L^\infty(\Omega)}^{\beta_2} \geq \|v_k\|_{L^\infty(\Omega)}^{\beta_1}$, up to a subsequence, and $\lambda_k = \|v_k\|^{-\frac{1}{\beta_2}}_{L^\infty(\Omega)}$, otherwise. It suffices to assume the first of these two situations.

Note that $\lambda_k \to 0$ as $k \to +\infty$. Let $x_k \in \Omega$ be a maximum point of $u_k$. The functions

$$
z_k(x) = \lambda_k^{\beta_1} u_k(\lambda_k x + x_k) \quad \text{and} \quad w_k(x) = \lambda_k^{\beta_2} v_k(\lambda_k x + x_k)
$$

are such that $z_k(0) = 1$ and $0 \leq z_k, w_k \leq 1$ in $\Omega_k := \frac{1}{\lambda_k}(\Omega - x_k)$. Also, one checks that the functions $z_k$ and $w_k$ satisfy

$$
\begin{cases}
(\Delta)^s z_k = \left(\lambda_k^{(2s+\beta_1-p\beta_2)/p} w_k + \lambda_k^{(2s+\beta_1)/p} \theta_k\right)^p = \left(w_k + \lambda_k^{(2s+\beta_1)/p} \theta_k\right)^p \\
(\Delta)^t w_k = \left(\lambda_k^{(2t+\beta_2-q\beta_1)/q} z_k + \lambda_k^{(2t+\beta_2)/q} \theta_k\right)^q = \left(z_k + \lambda_k^{(2t+\beta_2)/q} \theta_k\right)^q
\end{cases}
$$

(43)

in the domain $\Omega_k$.

By compactness, modulo a subsequence, $(x_k)$ converges to some point $x_0 \in \overline{\Omega}$. Let

$$
d_k = \text{dist}(x_k, \partial \Omega).
$$

Two cases may occur as $k \to +\infty$:

(a) $\frac{d_k}{\lambda_k} \to +\infty$, modulo a subsequence still denoted as before, or

(b) $\frac{d_k}{\lambda_k}$ is bounded.
If (a) occurs, then \( \frac{1}{\lambda_k} B_{d_k}(0) \subset \Omega_k \) and \( \frac{d_k}{\lambda_k} \to +\infty \) as \( k \to +\infty \). So, \( (\Omega_k) \) tends to \( \mathbb{R}^n \) as \( k \to +\infty \). We recall that \( 0 \leq z_k, w_k \leq 1 \) in \( \Omega_k \). Thus, the right-hand side of (43) is bounded in \( L^\infty(\Omega_k) \), so by compactness, we deduce that, up to a subsequence, \( (z_k, w_k) \) converges to some function \((z, w)\) uniformly in compact sets of \( \mathbb{R}^n \). By Theorem 2.7 of [27], \((z, w)\) is a viscosity solution of (6) with \( G = \mathbb{R}^n \). Note also that \( z(0) = 1 \), since \( z_k(0) = 1 \) for all \( k \), and hence \((z, w) \neq (0, 0)\) and, by the Silvestre’s strong maximum principle, \( z, w > 0 \) in \( \mathbb{R}^n \). But this contradicts Theorem 1.2.

Assume now that (b) occurs, that is \( d_k \lambda_k^{-1} \) is bounded. In this case, up to a subsequence, we may assume that

\[
\frac{d_k}{\lambda_k} \to a \in [0, \infty). \tag{44}
\]

Assume for a moment that \( a > 0 \). After a suitable rotation of \( \mathbb{R}^n \) for each fixed \( k \), one concludes that \( (\Omega_k) \) converge to the half-space \( \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > -a\} \). Again, we have \( 0 \leq z_k, w_k \leq 1 \) in \( \Omega_k \) and then, by compactness, \((z_k, w_k)\) converges, modulo a subsequence, to some function \((z, w)\) uniformly in compact sets of \( \mathbb{R}^n_+ \). As before, \((z, w)\) is a viscosity bounded solution of (6) with \( G = \mathbb{R}^n_+ \). Furthermore, using that \( a > 0 \) and \( z_k(0) = 1 \) for all \( k \), one gets \( z(0) = 1 \), so that again \( z, w > 0 \) in \( \Omega \) and this contradicts Theorem 1.3.

The remainder of the proof consists in showing that \( a > 0 \). We argue by contradiction and assume that \( a = 0 \). The basic idea is to construct a barrier function \( h_k \) on \( \Omega_k \) for \( z_k \). For this purpose, we define

\[
h_k(x) = \left( e^{-\frac{d_k}{\lambda_k} x_n} - e^{x_n} \right) \sup_{\Omega_k} \frac{(w_k + \lambda_k^{(2s+\beta_1)/p} \theta_k)^p}{C_0},
\]

where \( C_0 \) is a positive constant such that

\[
(-\Delta)^s e^{x_n} = -\int_{\mathbb{R}^n} e^{(x_n+y_n)} + e^{(x_n-y_n)} - 2e^{x_n} \frac{y_n}{|y|^{n+2s}} dy \\
= -e^{x_n} \int_{\mathbb{R}^n} e^{y_n} + e^{-y_n} - 2 \frac{y_n}{|y|^{n+2s}} dy \leq -C_0 < 0
\]

for all \(-\frac{d}{\lambda_k} < x_n < 0\). Thus, from (43),

\[
(-\Delta)^s(h_k - z_k) \geq C_0 \sup_{\Omega_k} \left( \frac{(w_k + \lambda_k^{(2s+\beta_1)/p} \theta_k)^p}{C_0} - \frac{(w_k + \lambda_k^{(2s+\beta_1)/p} \theta_k)^p}{C_0} \right) \geq 0
\]

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in $\Omega_k$ and $z_k \leq h_k$ on $\partial \Omega_k$. Then, the weak maximum principle gives $z_k \leq h_k$ in $\Omega_k$.

In addition, there exist $C_1 > 0$ and $\delta > 0$ such that

$$|\nabla w_k(x)| \leq C_1$$

for all $x \in \Omega_k \cap \{x \in \mathbb{R}^n : x_n + \frac{d_k}{\lambda_k} \leq \delta\}$. Since $x_k \in \Omega$, we have $0 \in \Omega_k \cap \{x \in \mathbb{R}^n : x_n + \frac{d_k}{\lambda_k} \leq \delta\}$ for $k$ large enough. Finally,

$$1 = z_k(0) \leq h_k(0) \leq C_2 \left( e^{-\frac{d_k}{\lambda_k}} - 1 \right) \to 0$$

as $k \to \infty$, providing a contradiction. ■

Lastly, the conclusion of Theorem 1.1 follows readily from Lemmas 5.1, 5.2 and 5.3 applied to Proposition 5.1.

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