Abstract: We carry on (in a self contained fashion) the study of the Alexander Conway invariant from the quantum field theory point of view started in [1]. We investigate for that purpose various aspects of WZW models on supergroups.

We first discuss in details $S$ and $T$ matrices for the $U(1, 1)$ super WZW model and obtain, for the level $k$ an integer, new finite dimensional representations of the modular group. These have the remarkable property that some of the $S$ matrix elements are infinite (we show how to properly handle such divergences). Moreover, typical and atypical representations as well as indecomposable blocks are mixed: truncation to maximally atypical representations, as advocated in some recent papers, is not consistent.

Using our approach, multivariable Alexander invariants for links in $S^3$ can now be fully computed by surgery. Examples of torus and cable knots are discussed. Consistency with classical results provides independent checks of the solution of the $U(1, 1)$ WZW model.

The main topological application of this work is the computation of Alexander invariants for 3-manifolds and more generally for links in 3-manifolds. Invariants of 3-manifolds themselves seem to depend trivially on the level $k$, but still contain interesting topological information. For Seifert manifolds for instance, they essentially coincide with the order (number of elements) of the first homology group. Examples of invariants of links in 3-manifolds are given. They exhibit interesting arithmetic properties.
1 Introduction

The first purpose of this paper is to carry on the study of the Alexander Conway invariant ($\Delta$) from the quantum field theory point of view that was started in [1]. The goal here, since the Alexander invariant is well known from the topology point of view, is to establish a dictionary between the topology and QFT results in that case, and then to proceed backwards to gain deeper understanding of say the Jones polynomial or the related 3-manifold invariants that are merely understood from the QFT (or combinatorial) point of view so far. The approach to that problem should a priori be straightforward. Since after some time the Alexander polynomial has been put in the right combinatorial framework by introduction of the quantum super group $U_q gl(1,1)$ [2, 3, 4], one simply has, following the seminal work [5], to consider the WZW model on the supergroup $U(1,1)$ and the corresponding Chern Simons theory. Difficulties are however met. Representation theory of superalgebras is still poorly understood in general. Despite various attempts [6, 7, 8, 9, 10, 11, 12], the case of affine superalgebras is also confusing, no good concept of integrability for instance having yet been exhibited. Moreover the $gl(1,1)$ case seems to present additional difficulties due to its balance between bosons and fermions (this balance is deeply related to the nature of the Alexander invariant[2, 13]). So far studies of WZW models on supergroups [12] had focused on (hoped to be consistent) truncated theories that involve

only "maximally atypical" representations. In our case however such a truncation would provide trivial invariants. The closer study of $U(1,1)$ WZW model we carry out in this paper shows that, in this problem at least, there is no possible truncation. All representations, including typical and atypical representations as well as indecomposable blocks, have to be considered, and infinite $S$ matrix elements send atypical to blocks. Infinites can be regularized so that, for the level $k$ an integer, a finite dimensional representation of the modular group is obtained. Multivariable Alexander Conway invariants can then be computed by surgery, with results agreeing with classical knot theory literature [14, 15]. Moreover invariants of links in 3-manifolds can also be computed, Casson invariants [17] based on super Chern Simons theories have been considered earlier in [16].

Besides, the knowledge of applications to topology proves to be of great help in the study of WZW models on supergroups. The second purpose of this paper is to exhibit unexpected features of such theories which we believe are generic and not peculiarities of $U(1,1)$.

The paper is organized as follows. In section 2 we recall some results of [1], give some formulas for characters and derive part of the $S$ matrix. The complete $S$ and $T$ matrices are derived in section 3 after consideration of the $U(1,1)$ WZW model on the torus. Sections 2 and 3 are oriented towards conformal field theory and may be skipped by the reader interested in applications to topology only. Basic properties of $S$ and $T$ matrices are discussed in section 4. Surgery is then used in section 5 to compute a variety of link invariants, including cable and torus links. Section 6 is devoted to some (preliminary) study of invariants of links in 3-manifolds and of invariants of 3-manifolds. We also present a discussion of some results from a three dimensional point of view, making partial contact with $U(1)$ Casson invariant and the results of [16]. Some conclusions are gathered in Section 7.

The first appendix makes some partial connection between our results and the combinatorial approach to 3-manifold invariants. The second appendix explores further relations between the Alexander Conway polynomial, Burau representation, and screening contours in the WZW model.
2 \( gl(1, 1)^{(1)} \) Representations and characters

2.1 Reminder

This subsection recalls some of the results of \[1\].

The algebra \( gl(1, 1) \) has two bosonic and two fermionic generators obeying the relations

\[
\{\psi, \psi^+\} = E, \psi^2 = (\psi^+)^2 = 0, [N, \psi^+] = \psi^+, [N, \psi] = -\psi
\]

\( E \) being central. Its irreducible representations have dimension one and two. In the one dimensional case they are parametrized by a single complex number \( n \), eigenvalue of the \( N \) generator. By convention we chose their unique state to be bosonic, and denote these representations by \((n)\) (in general superalgebra terminology \[18\] these are the "atypical"). In the two dimensional case they are parametrized by a pair of complex numbers \( e(\neq 0), n \) and one has

\[
E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, N = \begin{pmatrix} n-1 & 0 \\ 0 & n \end{pmatrix}, \psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \psi^+ = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}
\]

(2)

By convention we chose the state annihilated by \( \psi \) to be bosonic, and denote these representations by \((en)\) ("typical"). All other representations are four dimensional indecomposable ones ("blocks") with the schematic structure indicated in figure 1 (they are denoted by \((\hat{n})\)). The limit \( e \to 0 \) of \((en)\) produces an indecomposable representation ("pseudotypical"). Notice the tensor product

\[
(e_1, n_1) \otimes (e_2, n_2) = (e_1 + e_2, n_1 + n_2 - 1) \oplus (e_1 + e_2, n_1 + n_2)
\]

(3)

provided \( e_1 + e_2 \neq 0 \). In the study of the tensor product the second representation in the right hand side appears with the state annihilated by \( \psi \) being a fermion. We trade it for an equivalent representation where this state is a boson. This procedure may introduce some minus signs due to statistics effects, which is indicated by the \( \oplus \) symbol. Four dimensional representations are obtained by

\[
(e, n_1) \otimes (-e, n_2) = (\hat{n}), n = n_1 + n_2 - 1
\]

(4)

Notice the vanishing of the super dimension of these two (and four) dimensional representations.

The \( gl(1, 1)^{(1)} \) current algebra is defined in the usual way, and arises from quantization of the \( U(1, 1) \) WZW model (the level \( k \) is not quantized, while \( U(1, 1) \) can be considered as compact\[3\]). A noticeable fact is that \( gl(1, 1) \) has two casimirs, but a single combination of these emerges after conformal invariance requirement. The usual shift \( k \to k + c_v \) is replaced by \( 1/k^2 \) contributions to the stress energy tensor. For primary fields associated with one dimensional representations the conformal weight \( h \) vanishes. For primary fields associated with two dimensional representations one has

\[
h_{en} = (2n - 1)\frac{e}{2k} + \frac{e^2}{2k^2}
\]

(5)

In four dimensional indecomposable representations, the casimir is not diagonalizable. Therefore they have no associated primary field. The value of the conformal weight, defined eg by considering the trace of the casimir, vanishes.

\[3\]This compactity should not hide that the metric is indefinite and the corresponding CFT non unitary, with in fact a spectrum not bounded from below. The \( GL(1, 1) \) case behaves identically.
The current algebra has a free field representation involving a pair of bosonic fields $\phi_E, \phi_N$ with
\[
\langle \phi_E(z)\phi_N(w) \rangle = \langle \phi_N(z)\phi_E(w) \rangle = -\ln(z - w)
\]
and a pair of fermionic fields $\eta, \xi$ with
\[
\langle \xi(z)\eta(w) \rangle = \langle \eta(z)\xi(w) \rangle = \frac{1}{z - w}
\]
In particular the stress energy tensor reads
\[
L = -: \eta \partial \xi : -: \partial \phi_E \partial \phi_N : + \frac{i}{2\sqrt{k}} \partial^2 \phi_E
\]
and the central charge is $c = 0$. The primary fields $\Phi_{en}$ associated with $(en)$ representations are vertex operators
\[
V_{en} = \exp\frac{i}{\sqrt{k}}(\tilde{n}\phi_E + e\phi_N)
\]
Where we introduced for further convenience the notation
\[
\tilde{n} = n + \frac{e}{2k}
\]
(as often in this work the notations do not contain explicit reference to all the parameters involved, for the sake of simplicity. We hope this will not cause any confusion). There does not seem to be any simple free field representation associated with the block representations. The screening operator is
\[
\mathcal{J} = \eta \exp -\frac{i}{\sqrt{k}}\phi_E
\]
As was demonstrated in [1], there is no truncation of the operator algebra in the WZW model. If for the defining representation $e = 1, n = 1/2$, one therefore has to deal with all values of $e$ integer and $n$ half integer for the two and four dimensional representations, and all values of $n$ half integer for the one dimensional ones. Rational values of $k$ are special because then some of the conformal weights differ by integers and the $gl(1,1)^{(1)}$ representations can be regrouped in larger sets, maybe representations themselves of some bigger algebra. As we shall see these "extended representations" of $gl(1,1)^{(1)}$ will provide a finite representation of the modular group, and hence allow surgery computations of Alexander invariants for links and 3-manifolds. In the following we discuss the case $k$ half an odd integer only. Introduce for a while the notation
\[
\lambda_{en} = 2n - 1 + \frac{e}{k}, \mu_{en} = \frac{e}{2k}
\]
so that
\[
h = \lambda \mu
\]
Then
\[
h = h' \mod \text{integer if } \lambda = \lambda' + M, \mu = \mu' + N, M, N \text{ integer}, M + 2N = 0 \mod 2k
\]
\footnote{We do not restrict to the natural parity splitting for later notational convenience}
Conversely we chose (14) to define which conformal weights have to be regrouped into extended representations, such that a finite dimensional representation of the modular group will be obtained. The fundamental domain has volume

$$V^{-2} = 4k^2$$

(15)

It can be described by

$$e = 0, 1, \ldots, 2k - 1, n = 1/2, 1, 3/2, \ldots, k.$$  

For the $gl(1, 1)^{(1)}$ representations associated with the two and four dimensional representations of $gl(1, 1)$, the ordinary supercharacter defined by $\text{Str}(\exp 2i\pi \tau L_0)$ vanishes due to boson fermion symmetry. We are however interested in the evaluation of the $S$ matrix, which can be computed by considering any specialization of the characters. In the following we use the free field representation and obtain non vanishing results by introducing a background gauge field, or more simply by suppression of the zero mode of the $\eta\xi$ system.

We introduce

$$\zeta_{en} = \zeta_{\eta\xi} \zeta_{\phi E_\phi\xi} \exp[2i\pi \tau e(\bar{n} - 1/2)/k]$$

(16)

In this formula we can consider $\zeta_{\eta\xi}$ to be the partition function of the $\eta\xi$ system with periodic boundary conditions, the zero mode being subtracted

$$\zeta_{\eta\xi} = \text{Im} \tau \eta^2 \bar{\eta}^2$$

(17)

where $\eta$ is the usual Dedekind function. It is by itself a modular invariant, and cannot be factorized into the usual chiral antichiral form due to the $\text{Im} \tau$ term. Such factorization is easy to obtain by introducing a background gauge field $E$, which produces

$$\zeta_{\eta\xi} = \frac{\theta_1(E)}{\eta} \frac{\overline{\theta_1(E)}}{\bar{\eta}}$$

(18)

Similarly

$$\zeta_{\phi E_{\phi\xi}} = \frac{1}{\eta^2}$$

(19)

We set also, from figure 1

$$\zeta_n = \zeta_{0,n} - \zeta_{0,n+1}$$

(20)

To deal now with representations of $gl(1, 1)^{(1)}$ associated with one dimensional representations of $gl(1, 1)$, we consider the latter as infinite sums of two dimensional ones as in figure 2. We therefore have to take alternate sum of characters, and also to take into account the additional signs coming from statistics. The net result is

$$\zeta_n = \sum_{j=0}^{\infty} \zeta_{0,n+1+j} = -\sum_{j=0}^{\infty} \zeta_{0,n-j}$$

(21)

This sum is regularized by introduction of the background gauge field $E$. A similar formula appears in [13]. For $k$ generic we expect the above to be characters of irreducible representations of $gl(1, 1)^{(1)}$. When $k$ is half an odd integer, the screening operator $\mathcal{J}$ becomes local with respect to vertex operators $V_{en}$ for $e = 0 \text{ mod } 2k$. If $e$ does not satisfy this condition, since we have only one screening operator, which moreover is nilpotent, we expect the above formulas to be still characters.
of irreducible $gl(1,1)^{(1)}$ representations [20]. If $e = 0 \mod 2k$ the situation is very similar to what happens for $e = 0$. Consider for instance $e = -2pk$. Then one finds, using $J^\psi = i\sqrt{k}\partial\phi + k\partial\xi$

$$J^\psi_{-2p}V_{en} = 0$$

(22)

the corresponding state has therefore to be factored out of the free field Fock space. Its conformal weight is $h_{en} + 2p$ which coincides with $h_{e,n-1}$. As usual the state $V_{e,n-1}$ can be obtained from $V_{en}$ by acting successively with $J^\psi_0$ and $J$. Considering therefore Ker $\oint J$ in the free field Fock space based on $V_{en}$ should produce an irreducible representation of $gl(1,1)^{(1)}$. The picture of the corresponding cohomology is like the second diagram in figure 2 with representations $e = -2pk,n$ and the "BRS" operator $\oint J$ connecting $e = -2pk,n$ and $e = -2pk,n-1$. Disappearance of states in the Fock space based on $V_{e=-2pk,n}$ is associated with appearance of null states in the dual $V_{e=2pk,1-n}$ (first diagram in figure 2). The characters of irreducible representations are therefore

$$\zeta_{e=-2pk,n} = -\sum_{j=0}^{\infty} \zeta_{e=-2pk,n-j}, \text{ and } \zeta_{e=2pk,n} = \sum_{j=0}^{\infty} \zeta_{e=2pk,n+1+j}$$

(23)

Both expressions coincide for $p = 0$. For $k$ half an odd integer we define the characters $\chi$ of extended representations of $gl(1,1)^{(1)}$ as appropriate sums of the characters $\zeta$ over the two dimensional lattice [14]. Since the spectrum of dimensions is not bounded from below $\chi$ are naively infinite. They hopefully could be properly defined by a treatment analogous to the one of $\beta\gamma$ systems. In the following this divergence does not apparently cause difficulties. It is not at the origin of divergence of some $S$ matrix elements, as explained later.

### 2.2 The naive $S$ matrix

For $k$ half an odd integer we expect the set of $\chi$’s to provide a finite dimensional representation of the modular group. Let us start the computation of the $S$ matrix elements from the above expressions. In what follows we discard the $\eta\xi$ contributions which are easily evaluated and affect the results by phases only, and we concentrate on the bosonic contributions. Using Poisson formula one finds readily

$$S_{en}^{e'n'} = iV\exp\left[-\frac{2i\pi}{k} \left(e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)\right)\right]$$

(24)

The usual $\sqrt{\tau}$ term that arises from Poisson formula is compensated by a similar contribution from the $1/\eta^2$ term in (14). From this, and using the expression (24) one obtains

$$S_{en}^{\tilde{n}'} = -V\exp(-2i\pi en'/k)/2\sin(\pi e/k)$$

(25)

while it is likely from boson fermion symmetry that

$$S_{en}^{n'} = 0$$

(26)

The $S$ matrix is defined here by $\chi_\alpha(-1/\tau) = \sum S^\alpha_\beta \chi_\beta(\tau)$

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Using (20) one can find also

\[ S_{e'n'}^n = -2V \sin(\pi e'/k) \exp(-2i\pi e'n/k) \]  

(27)

and using (21) (with an infinitesimal background gauge field that goes to zero at the end of the computation)

\[ S_{e'n'}^n = V \exp(-2i\pi e'n/k)/2\sin(\pi e'/k) \text{ "naive"} \]  

(28)

The label "naive" means this formula will be reconsidered by a proper treatment of the characters associated with one dimensional representations of \( gl(1,1) \) in the next section. Notice that this element can be non zero because the representation of \( gl(1,1) \) with vanishing superdimension appears as an upperindex.

3 The \( GL(1,1) \) WZW model on a torus

To obtain the entire \( S \) matrix is a delicate matter; we shall see in particular that some of its elements are infinite (such a result can be guessed from eg (28) as \( e \to 0 \)). In order to control and sometimes regularize things, it is necessary to work on the torus \([21]\). The manipulations we shall carry out now should appear more transparent from the Chern Simons point of view, see section 4.

We consider the situation of figure 3, with a state \(|en>\) propagating in the \( \tau \) direction, and insertion of a representation \(|e_0n_0>\) in \( z_1 \), of its conjugate \(|-e_0,1-n_0>\) in \( z_2 \). Two conformal blocks contribute, which can be computed by putting the free field representatives

\[ V_{e_0n_0}(z_1) = \exp\left(\frac{i}{\sqrt{k}}(\tilde{n_0}\phi_E + e_0\phi_N)\right) \]

\[ \xi V_{-e_0,1-n_0}(z_2) = \xi\exp\left(\frac{i}{\sqrt{k}}\left((1-\tilde{n_0})\phi_E - e_0\phi_N\right)\right) \]  

(29)

together with the screening operator \([11],\mathcal{J}\). We consider first integration on the contour shown on figure 3. The zero modes of the \( \eta\xi \) system are cancelled by the fermionic operators introduced in this construction, and we obtain the partition function, taking only the chiral contribution from the bosonic degrees of freedom

\[ \zeta_{en}(z_1,z_2;\tau) = \mathcal{N}(en) \int_C \zeta_{en}(z_1,z_2,w;\tau)dw = \]  

(30)

\[ \mathcal{N}(en)\text{Im}\eta^{\frac{1}{k}} \int_{C_1} \left[ \theta_1(z_1-w)/\theta_1'(0) \right]^{-e_0/k} \left[ \theta_1(z_2-w)/\theta_1'(0) \right]^{e_0/k} \left[ \theta_1(z_1-z_2)/\theta_1'(0) \right]^{-2e_0(\tilde{n_0}-1/2)/k} \]

\[ \exp\left[ \frac{2i\pi}{k}\tau e(\tilde{n} - 1/2) + \frac{2i\pi}{k} e(\tilde{n}_0 z_1 + (1-\tilde{n_0})z_2 - w) + \frac{2i\pi}{k}(\tilde{n} - 1/2)e_0(z_1 - z_2) \right] dw \]  

(31)

where \( \mathcal{N} \) is a normalization factor to be determined. For the case under study we should in principle sum such expressions over the lattice \([14]\) before performing a modular transformation. In order
to make things a little easier for notations we shall not do so, and deal for a while with continuous Fourier transforms. The results of the correct procedure are similar up to the introduction of the discrete lattice and multiplication of the $S$ matrix elements by a factor $1/2$ due to the fact that $e$ labels take integer values, $n$ labels half integer ones. Also one must treat a little more carefully the one and four dimensional representations, corresponding to $e' = 0$ in the integral. With these precautions we find

$$
\zeta_{en}(z_1/\tau, z_2/\tau, w/\tau; -1/\tau) = 
\left( \int de'dn' \frac{i}{k} \exp - \frac{2i\pi}{k}[e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)] \right)^{-1/2} \zeta_{e'n'}(z_1, z_2, w; \tau)
$$

However if on the left hand side the contour of integration for the screening charge looks like in figure 3 (with $\tau$ there being equal to $-1/\tau$), then on the right hand side we have integration on the contour called $C_1$ in figure 4, resulting in an integral $I_1$. We have to reexpress it in terms of integration along the contours $C_2, C_4$, giving rise to integrals $I_2, I_4$. First notice that

$$I_1 - I_2 + I_5 - I_3 + I_4 = 0 \tag{33}$$

because the total contour has no singular points inside. Now notice that

$$\zeta_{e'n'}(z_1, z_2, w + 1; \tau) = \exp(-2i\pi e'/k)\zeta_{e'n'}(z_1, z_2, w; \tau) \tag{34}$$

Similarly

$$\zeta_{e'n'}(z_1, z_2, w + \tau; \tau) = \zeta_{e',n'-1}(z_1, z_2, w; \tau) \tag{35}$$

or

$$I_2(e', n') = I_4(e', n' - 1) \tag{36}$$

Therefore

$$I_1 = \frac{I_4(e', n') - I_4(e', n' - 1) - I_5(e', n' - 1)}{\exp(-2i\pi e'/k) - 1} \tag{37}$$

In the limit $e_0 \to 0$, we can neglect $I_5$. In that limit indeed the screening operator does not pick up a phase when it goes around the $(-e_0, 1 - n_0)$ insertion, and the contributions of the down left and up right parts of $C_5$ cancel each other. Therefore

$$I_1 = \frac{I_4(e', n') - I_4(e', n' - 1)}{\exp(2i\pi e'/k) - 1}, \quad e_0 \to 0 \tag{38}$$

and we can rewrite

$$\mathcal{N}(en) \int_C \zeta_{en}(-w/\tau; -1/\tau) dw =$$

$$\left( \int de'dn' \mathcal{N}(en) \frac{i}{k} \exp - \frac{2i\pi}{k}[e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)] \right)^{-1/2} \zeta_{e'n'}(w; \tau) \int_{C_4} \zeta_{e',n'-1}(w; \tau) dw =$$

$$\left( \int de'dn' \frac{i}{k} \exp - \frac{2i\pi}{k}[e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)] \right)^{-1/2} \zeta_{e'n'}(w; \tau) \int_{C_4} \zeta_{e',n'-1}(w; \tau) dw =$$
\[ \frac{\mathcal{N}(en)}{\mathcal{N}(e'n')} \int_{C_4} \zeta_{e'n'}(w; \tau)dw \]  

Therefore if we choose
\[ \mathcal{N}(en) = \frac{1}{\exp(-2i\pi e/k) - 1} \]  

we obtain
\[ \zeta_{en}(-1/\tau) = \int de'dn' \frac{i}{k} \exp \left(-\frac{2i\pi}{k}[e'(\bar{n} - 1/2) + e(\bar{n}' - 1/2)]\right) \zeta_{e'n'}(\tau) \]  

and we recover the \( S \) matrix element \([24]\). The fact that \( e_0 \to 0 \) limit has been taken is implicit in the above notation since the dependence on \( z_1, z_2 \) has been suppressed. Also notice that the continuum treatment does not handle the case \( e' = 0 \) completely. In addition to \( \zeta_{e'n'}(e' \neq 0) \) the modular transformation maps \( \zeta_{en} \) also to one and four dimensional representations, as already discussed above.

Comparison with \([27]\) shows now that
\[ \zeta_{e'n}^{-1}(z_1, z_2; \tau) = \int_{C_1} \zeta_{0,n'+1}(z_1, z_2, w; \tau)dw \]  

We now argue that
\[ \zeta_{e'n}(z_1, z_2; \tau) \propto \int_{C_1} \zeta_{0,n'+1}(z_1, z_2, w; \tau) \]  

up to a normalization factor that we shall determine. First we notice that the integrand is periodic in \( w \to w + 1 \) if \( w \) is not moved between the insertion points of \((e_0n_0)\) and \((-e_0, 1 - n_0)\) belonging to the same torus. Therefore we can move \( C_0^4 \) up to \( C_0^8 \) (see figure 5) and
\[ \int_{C_0^4} \zeta_{0,n'+1}(z_1, z_2, w; \tau)dw = \int_{C_0^8} \zeta_{0,n'+1}(z_1, z_2, w; \tau)dw = \int_{C_0^6} \zeta_{0,n'}(z_1, z_2, w; \tau)dw \]  

Now using the fact that
\[ -\int_{C_4} \zeta_{0n'}(z_1, z_2, w; \tau)dw + \int_{C_4'} \zeta_{0n'}(z_1, z_2, w; \tau)dw = 0 \]  

we can add and subtract these integrals to get
\[ \int_{C_0^4} \zeta_{0,n'+1}(z_1, z_2, w; \tau)dw = \sum_{j=0}^{\infty} \left( \int_{C_0^4} - \int_{C_4} \right) \zeta_{0,n' - j}(z_1, z_2, w; \tau)dw \]  

Now (figure 6)
\[ \left( \int_{C_0} - \int_{C_4} \right) \zeta_{0,n' - j}(z_1, z_2, w; \tau)dw = \left( \int_{C_7} - \int_{C_8} \right) \zeta_{0,n' - j}(z_1, z_2, w; \tau)dw \]  

Now (figure 6)
However
\[ \zeta_{0,n'-j}(z_1, z_2, w - 1; \tau) = \exp(-2i\pi e_0/k)\zeta_{0,n'-j}(z_1, z_2, w; \tau) \]  
(48)
because in that case the point \( w \) is moved between the insertions of \((e_0n_0)\) and \((-e_0, 1 - n_0)\). Therefore
\[ \int_{C_s} \zeta_{0,n'-j}(z_1, z_2, w; \tau)dw = \exp(-2i\pi e_0/k)\int_{C_s} \zeta_{0,n'-j}(z_1, z_2, w; \tau)dw \]  
(49)
and we can write
\[ \int_{C_s} \zeta_{0,n'+1}(z_1, z_2, w; \tau)dw = [1 - \exp(-2i\pi e_0/k)] \sum_{j=0}^{\infty} \int_{C_s} \zeta_{0,n'-j}(z_1, z_2, w; \tau)dw \]  
(50)
We now set
\[ \zeta_n(z_1, z_2; \tau) = -\exp(-i\pi e_0/k) \sum_{j=0}^{\infty} \int_{C_s} \zeta_{0,n'-j}(z_1, z_2, w; \tau)dw \]  
(51)
\[ = -\frac{\exp(-i\pi e_0/k)}{1 - \exp(-2i\pi e_0/k)} \int_{C_s} \zeta_{0,n'+1}(z_1, z_2, w; \tau)dw \]  
(52)
Formulas (12) and (51,52) provide explicit expressions for the characters of four and one dimensional representations. We can now carry out the modular transformation
\[ \zeta_n(z_1/\tau, z_2/\tau; -1/\tau) = -\frac{\exp(-i\pi e_0/k)}{1 - \exp(-2i\pi e_0/k)} \tau^{-2e_0(n_0-1/2)} i \frac{k}{\bar{k}} \int d\tilde{n}' \int_{C_s} \zeta_{n'}(z_1, z_2, w; \tau)dw \]  
(53)
(the other terms will be analyzed later on) ie from (12)
\[ \zeta_n(z_1/\tau, z_2/\tau; -1/\tau) = -\frac{\exp(-i\pi e_0/k)}{1 - \exp(-2i\pi e_0/k)} \tau^{-2e_0(n_0-1/2)} i \frac{k}{\bar{k}} \int d\tilde{n}' \zeta_{n'}^\wedge(1, z_2, w; \tau) \]  
(54)
If we take the limit \( e_0 \to 0 \) in that expression, the prefactor diverges, meaning that the corresponding \( S \) matrix element diverges (the origin of this divergence lies in the fact that atypical representations must be considered as infinite sums of typical ones). We shall write symbolically
\[ S_n^\wedge = -iV\exp(-i\pi e_0/k) = -\frac{V}{2\sin(\pi e_0/k)} = -\frac{V}{2\epsilon} \]  
(55)
The meaning of the insertions necessary to regularize this expression will be explained in the discussion of link invariants later on. In a similar fashion we find
\[ \zeta_n(z_1/\tau, z_2/\tau; -1/\tau) = \exp(i\pi e_0/k)[1 - \exp(-2i\pi e_0/k)]\tau^{-2e_0(n_0-1/2)} i \frac{k}{\bar{k}} \int d\tilde{n}' \zeta_{n'}(z_1, z_2; \tau) \]  
(56)
and therefore
\[ S_{\hat{n}}' = -2V\sin(\pi e_0/k) = -2V\epsilon \] (57)

In the following we shall refer to the character (30) as \( \zeta_{\text{en}}/I \), because we need to introduce still another specie of character. It is obtained by a definition similar to (30), but without normalization factors, and with the contour \( C_7 \) of figure 6, corresponding to the other available conformal block. We refer to this new specie as \( Z_{\text{en}}/II \). The fact that there are two conformal blocks on the torus for insertion of \((e_0n_0), (-e_0,1-n_0)\) is because the product of these two representations, although giving only \((\tilde{n})\) at the classical level of \( gl(1,1) \) tensor product, produces also \((0)\) representation at the quantum level (see the study of four point functions in [1]). The contour \( C_7 \) is unchanged in modular transformation, and we therefore have immediately
\[ S_{\text{en}}' = S_{\text{en}}' = iV\exp -\frac{2i\pi}{k}[\epsilon'(\tilde{n}-1/2) + \epsilon(n'-1/2)] \] (58)

The reason for introducing such characters is the study of the remaining terms in (53). It is difficult to use the form (52) because of the vanishing normalization factor \( N(0n') \). Let us therefore consider the form (51). By invariance of the contour \( C_7 \) we find
\[ \zeta_n(-z_1/\tau,-z_2/\tau;-1/\tau) \]
\[ = -\sum_{j=0}^{\infty} \int d\epsilon' d\epsilon \frac{i}{k} \exp -[2i\pi\epsilon'(n-j-1/2)/k] \int_{C_7} \frac{\zeta_{\epsilon'}(z_1, z_2, w; \tau)dw}{k} \] (59)
\[ \text{ie} \]
\[ S_{\text{en}}' = 0 \] (60)

and
\[ S_{\text{en}}' = V\exp(-2i\pi\epsilon'n/k)/2\sin(\pi\epsilon'/k) \] (61)

By similar methods one finds
\[ S_{\hat{n}}' = 0 \] (62)

and
\[ S_{\text{en}}' = 2V\sin(\pi\epsilon/k)\exp(2i\pi\epsilon'n'/k) \] (63)

and
\[ S_{\hat{n}}' = 0 \] (64)

Finally one has also
\[ S_{\hat{n}}' = 0, S_n' = 0 \] (65)

Notice that the fusion of type II representations does not lead to new block representations, which is why we restrict to one specie of \( (\tilde{n}) \) representations [1]. These \( S \) matrix elements are invariant under the translations [14]. The correct procedure using discrete sums over the lattice

\[ \text{One has } (en/II), (-en'/II) = -2(n+n'-1) + (n+n') + (n+n'-2) \]
shows that they are the $S$ matrix elements for the characters of extended representations $\chi$. For convenience we summarize our results in the following table

\[
S_{en/I}^{e' n'/I} = iV \exp -\frac{2i\pi}{k} [e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)], \quad S_{en/I}^{e' n'/I} = 0
\]

\[
S_{en/I}^{e n} = -V \exp(-2i\pi e n'/k)/2\sin(\pi e/k), \quad S_{en/I}^{e n} = 0
\]

\[
S_{en/I}^{e n} = iV \exp -\frac{2i\pi}{k} [e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)], \quad S_{en/I}^{e n} = 0
\]

\[
S_{en/I}^{e n} = 2V \sin(\pi e/k)\exp(-2i\pi e n'/k)
\]

\[
S_{n}^{e n/I} = -2V \sin(\pi e'/k)\exp(-2i\pi e n'/k), \quad S_{n}^{e n/I} = 0
\]

\[
S_{n}^{e n/I} = 0, \quad S_{n}^{e n/I} = 2V \sin(\pi e/k)\exp(-2i\pi e n'/k)
\]

\[
S_{n}^{e n/I} = 0, \quad S_{n}^{e n/I} = V \exp(-2i\pi e n'/k)/2\sin(\pi e'/k)
\]

\[
S_{n}^{e n/I} = \infty (\text{reg.} : -V/2\epsilon), \quad S_{n}^{e n/I} = 0
\] (66)

where we have set

\[
\sin \pi e_0/k = \epsilon
\] (67)

For $S$ matrix elements which remain finite as $e_0 \to 0$, we have taken this limit. For an ordinary WZW model, the partition functions corresponding to $\zeta_{en/I}$ and $\zeta_{en/I}$ would both tend towards the same character $\zeta_{en}$ as $e_0 \to 0$, and the effect of the insertions disappear. In our case such a limit cannot be taken because divergences occur. To control them we need to keep a small but non-zero value of $e_0$. Hence strictly speaking we are not dealing with characters but conformal blocks, and this implies a doubling of the species. This regularization has a simple geometrical meaning from the point of view of Chern Simons theories which we shall consider in section 5.

Let us now discuss a bit more the $(\hat{n})$ representations. Using the normalization \[13\] we can study the $e' \to 0$ limit of \[17\] which becomes

\[
I_1 = -\frac{k}{2i\pi} [\partial_{e'} I_4(0, n') - \partial_{e'} I_4(0, n' - 1) - \partial_{e'} I_5(0, n' - 1)]
\] (68)

while from the definitions

\[
\frac{k}{2i\pi} \partial_{e'} \zeta_{en'}(z_1, z_2, w, \tau) = \tau(\tilde{n}' - 1/2)\zeta_{en'}(z_1, z_2, w, \tau) + [\tilde{n}_0 z_1 + (1 - \tilde{n}_0 z_2)]\zeta_{en'}(z_1, z_2, w; \tau) - w\zeta_{en'}(z_1, z_2, w; \tau)
\] (69)

Now recall that $\zeta_{en}(z_1, z_2, w; \tau)$ is periodic under $w \to w + 1$ provided $w$ is not moved between $z_1$ and $z_2$. Therefore the combination of integrations

\[
\int_{C_4} - \int_{C_2} - \int_{C_3}
\]
leads to a vanishing contribution of the second term in the right hand side of (69). As for the first term its contribution to $I_1$ is easily evaluated to be
\[ \tau \int_{C_4} \zeta_{0w'}(z_1, z_2, w; \tau) dw \] (70)
and due to (52) it vanishes as $e_0 \to 0$. Therefore the main contribution comes from the last term in (69). Nevertheless (70) is important because it indicates a non naive behaviour of $\zeta_n^\wedge$ in the transformation $\tau \to \tau + 1$. We find
\[ T_n^\wedge = -2iV \sin(\pi e_0/k)\exp(-i\pi/6) \] (71)
We write next the complete $T$ matrix (recall we have discarded the $\eta \xi$ contributions so the apparent central charge is $c = 2$)
\[ T_{en/I}^{en/I} = \exp(-i\pi/6)\exp[2i\pi e(\bar{n} - 1/2)/k], \quad T_{en/I}^{\text{others}} = 0 \]
\[ T_{en/II}^{en/II} = \exp(-i\pi/6)\exp[2i\pi e(\bar{n} - 1/2)/k], \quad T_{en/II}^{\text{others}} = 0 \]
\[ T_n^\wedge = \exp(-i\pi/6), \quad T_n^{\wedge'} = 0 \text{ (reg. : } -2ieV\exp(-i\pi/6)), \quad T_n^{\text{others}} = 0 \]
\[ T_n = \exp(-i\pi/6), \quad T_n^{\text{others}} = 0 \] (72)
Instead of remembering that the above results apply to conformal blocks with insertion of operators with small charge $\pm e_0$, a more axiomatic, easier to use, point of view is to consider the partition functions as genuine characters. One then needs to assume that quantization has led to a doubling of species: while in $gl(1, 1)$ ($en$) and $(\bar{n})$ representations form a closed set in tensor product, quantization introduces also $(n)$ and the splitting $(en) \to (en/I), (en/II)$. Then one takes the limit $e_0 \to 0$ so that some $S$ matrix elements vanish, others become infinite, and the presence of $e$ in the above formulas is merely giving a rule for multiplying $0 \times \infty$. To check whether this is consistent we now discuss in details the properties of this $S$ matrix.

4 Properties of the $S$ matrix

4.1 $S^2 = C$

Let us now consider the square of the $S$ matrix. Start with a representation $(en/I)$ and apply $S$ twice. We have various possible sequences
\[ (i) \quad en/I \to e'n'/I \to e''n''/I \]
\[ (ii) \quad en/I \to e'n'/I \to \bar{n}' \]
\[ (iii) \quad en/I \to \bar{n}' \to e''n''/I \]
\[ (iii) \quad en/I \to \bar{n}' \to n'' \]
\[ ^7\text{The } T \text{ matrix is defined by the conventions } \chi_a(\tau + 1) = \sum T_a^b \chi_b(\tau) \]
Sequences (i) and (iii) contribute as

$$( S^2 )_{en/I}^{e''n''/I} = -V^2 \sum_{e' n'} \exp \left( \frac{2i\pi}{k} e' \tilde{n} (\tilde{n} + \tilde{n}' - 1) + (e + e'') (\tilde{n}' - 1/2) \right)$$

$$+ V^2 \sum_{n'} \frac{\sin(\pi e' / k)}{\sin(\pi e / k)} \exp \left( \frac{2i\pi}{k} n'(e + e'') \right)$$

(73)

where the * sum means the value $e' = 0$ is excluded (and of course $e, e''$ are non zero). Because the sum over $n'$ is unrestricted it imposes $e + e'' = 0$, and the second sum then contributes just what is missed in the first sum by the * restriction to provide a function $-\delta_{n+n'-1}$ (modulo the lattice $(14)$ (the dependence of $\tilde{n}$ upon $e$ does not influence the results of Fourier sums which can be as well evaluated as if $\tilde{n}$ was an independent variable). The sequence (ii) does not contribute since there the sum over $n$ imposes $e = 0$, which is excluded. The sequence (iii) vanishes as $\epsilon \to 0$.

Therefore

$$( S^2 )_{en/I}^{-e,1-n/I} = -1, \ 0 \text{ otherwise}$$

(74)

The minus sign occurs for the reason that in the tensor product of representations $(en)$ and $(-e, 1-n)$ both with the upper state being a boson, the $gl(1,1)$ invariant state generated is a fermion. In our conventions where one dimensional representations are made of a bosonic state, we trade it for a boson, and need therefore a minus sign in the definition of charge conjugation. Starting with a representation $(en/II)$ we find similarly

$$( S^2 )_{en/II}^{-e,1-n/II} = -1, \ 0 \text{ otherwise}$$

(75)

Consider now the action of $S^2$ on a representation $\tilde{n}$. Again four possible sequences can occur

(i) $\tilde{n} \to e'n'/I \to \tilde{n}'$

(ii) $\tilde{n} \to e'n'/I \to e''n''/I$

(iii) $\tilde{n} \to n' \to \tilde{n}'$

(iii) $\tilde{n} \to n' \to e''n''/II$

Sequences (i) and (iii) combine to give

$$( S^2 )_{\tilde{n}}^{\tilde{n}} = V^2 \sum_{e' n'} \exp \left( \frac{2i\pi}{k} e' (n + n'') \right)$$

$$+ V^2 \sum_{n'} 1$$

(76)

As before the second term provides the missing $e' = 0$ contribution in the first sum to give $\delta_{n+n''}$. It is therefore essential to consider it to insure $S^2 = C$. Sequence (ii) does not contribute because the sum over $n'$ imposes $e'' = 0$, which is excluded. Sequence (iii) does not contribute in the limit $\epsilon \to 0$ where the first $S$ matrix element vanishes. Therefore

$$( S^2 )_{\tilde{n}}^{-n} = 1, \ 0 \text{ otherwise}$$

(77)
The result for \((n)\) representations is entirely similar

\[(S^2)^{-n}_n = 1, \; 0 \text{ otherwise} \quad (78)\]

With the above rules to take conjugates of two dimensional representations, one checks that indeed \((\tilde{n})\) is the conjugate of \((\tilde{n})\), and similarly \((-n)\) the conjugate of \((n)\), so \(S^2 = C\) is established.

Let us summarize the action of charge conjugation for completeness

\[(en) \to -(-e,1-n), \; (\tilde{n}) \to (\tilde{n}), \; (n) \to (-n) \quad (79)\]

In the Chern Simons point of view, taking conjugate of the representation carried by a strand is like keeping this representation but changing the strand orientation. One checks that \(SC = CS\) and \(SC = S^*\) where \(*\) means complex conjugation.

### 4.2 \((ST)^3 = C\)

The computation of \((ST)^3\) can be done using the above formulas. As before the dependence of \(\tilde{n}\) upon \(e\) does not modify naive results obtained by treating it as an independent variable. Let us discuss for instance the matrix element between \((en)\) and \((e'n')\). Several sequences can contribute

\[(i)\] \[en/I \to e_1 n_1/I \to e_2 n_2/I \to e'n'/I\]
\[(ii)\] \[en/I \to \tilde{n}_1 \to e_2 n_2/I \to e'n'/I\]
\[(iii)\] \[en/I \to e_1 n_1/I \to \tilde{n}_2 \to e'n'/I\]

where the arrows represent action with \(ST\). As above the sequences \((ii), (iii)\) complete the missing terms in \((i)\) to restore the entire sum of the discrete fourier transform, and one finds

\[
[(ST)^3]^{-e,1-n/I}_{en/I} = -1, \; 0 \text{ otherwise} \quad (80)
\]

An identical result is obtained with the representations of type \(II\), the role of \((\tilde{n})\) representations as intermediates being played by \((n)\) representations. The infinitesimal branching of \((\tilde{n})\) to \((n')\) in \(T\) and infinitely large branching of \((n')\) to \((\tilde{n}'\tilde{n})\) in \(S\) produces also sequences involving only \((\tilde{n})\) and \((n)\) representations. For instance

\[n \to \frac{V}{2\epsilon} \tilde{n}_1 \to V^2 n_2 - \frac{iV^2}{2\epsilon} \tilde{n}_2 \to iV^3 n'\]

(we did not write the additional phase factor \(\exp(-3i\pi/6)\) coming from \(T\) transformations). Again such sequences just compensate for the missing terms in the sums over two dimensional representations to reproduce fourier transforms and one gets in every case

\[(ST)^3 = C \quad (81)\]
Acting on $\mathfrak{gl}(1,1)^{(1)}$ extended characters, "doubled" by the above regularization procedure, we have therefore obtained a finite dimensional representation of the modular group for $k$ half an odd integer.

In [1], the $S$ matrix elements $S_{en/I}^{en/I}$ had been correctly obtained using Verlinde formula (we shall discuss this again later). We had not treated the block representations in an independent way but rather using the pseudotypical $(0n/I)$ representations, which thanks to (20) amounts to the same thing. Some of the $S$ matrix elements involving representations $(n)$ were only deduced from consistency of Verlinde formula, and the whole $S$ matrix was not obtained. This is now completed, and will allow us later on to compute links and 3-manifold invariants by surgery.

4.3 Metric and unitarity of the $S$ matrix

In the computation of link invariants a crucial role is played by the partition function of Chern Simons theory on $S^2 \times S^1$ with two Wilson lines carrying different representations. This defines a metric for $\mathfrak{gl}(1,1)^{(1)}$ extended representations that we shall now determine (figure 7). We start by the convention

$$<en/I|e'n'/I> = \delta_{e+e'}\delta_{n+n'-1}$$  \hspace{1cm} (82)

($\hat{n}$) representations appearing as combinations of $(0n/I)$ representations we can set also

$$<\hat{n}|\hat{n'}> = -2\delta_{n+n'} + \delta_{n+n'+1} + \delta_{n+n'-1}$$  \hspace{1cm} (83)

$$<en/I|\hat{n'}> = 0$$  \hspace{1cm} (84)

Considering $(n)$ as infinite sum of $(0n/I)$ representations we find also

$$<\hat{n}|n'> = \delta_{n+n'}$$  \hspace{1cm} (85)

$$<en/I|n'> = 0$$  \hspace{1cm} (86)

We now have to set the following result that looks a priori surprising

$$<n|n'> = 0$$  \hspace{1cm} (87)

This implies in particular that for the sphere with two insertions of the fields associated with one dimensional representations of $\mathfrak{gl}(1,1)$, the Hilbert space of the Chern Simons theory has vanishing dimension, or that there is no conformal block. Such result can be derived from the computation of the four point functions done in [1]. In this reference in particular

$$<\Phi_{en_1}(\infty)\Phi_{-e,n_2}(z)\Phi_{e',n_3}(0)\Phi_{-e',n_4}(1)>$$

was determined. It involves two conformal blocks called $F^1$ and $F^2$ such that only $F^1 - F^2$ does not have logarithm at infinity. The latter combination is interpreted as representing the contribution of a field associated with a four dimensional block representation of $\mathfrak{gl}(1,1)$ in the product $\Phi_{en}(\infty)\Phi_{-e,1-n}(z)$. The other independent block say $F^1$ does contain a logarithm at

---

This is true in the limit $\epsilon \to 0$ only

15
infinity and represents, added generically to the above, the contribution of the field associated with a one dimensional representation of \( gl(1,1) \) in this product. Now the physical correlator was found to be

\[
G \propto (F^1 - F^2)(F_{\bar{1}} - F_{\bar{2}}) - \left[ F^1(F_{\bar{1}} - F_{\bar{2}}) + F_{\bar{1}}(F_1 - F_2) \right]
\]

In this expression the conformal blocks for four dimensional representations are coupled together and with the conformal blocks of one dimensional representations. The latter however are not coupled, in agreement with (87).

Various manipulations lead to the complete set of scalar products

\[
< en/I|e'n'/I > = \delta_{e+e'} \delta_{n+n'-1} = < en/I|e'n'/II >, < en/II|e'n'/II > = 0
\]

\[
< en/I|n' > = < en/II|\hat{n}' > = 0
\]

\[
< en/I|n' > = < en/II|\hat{n}' > = 0
\]

\[
< n|n' > = 0, < n|\hat{n}' > = \delta_{n+n'}
\]

\[
< \hat{n}|\hat{n}' > = -2\delta_{n+n'} + \delta_{n+n'+1} + \delta_{n+n'-1} \quad (88)
\]

This defines a metric \( g \) which is real symmetric. Using it one can in particular lower indices. Notice that the metric is not given by the complex conjugation matrix. We now have to check that the scalar product is invariant under modular transformations i.e

\[
< S \rho_i | S \rho_j > = < \rho_i | \rho_j > \quad (89)
\]

Let us discuss a few examples. Consider first

\[
< n|n' > = 0
\]

and apply \( S \) to both states. Under \( S \) a representation \( (n) \) branches to representations \( (e_1 n_1/II) \) and \( (\hat{n}_1) \). Because the first are mutually orthogonal, as are the first and the second, we get

\[
< Sn|Sn' > = \sum_{n_1,n_2} < \hat{n}_1|\hat{n}_2 > = 0
\]

since the trace of \( [33] \) vanishes. Consider now

\[
< n|\hat{n}' > = \delta_{n+n'}
\]

One finds

\[
< Sn|S\hat{n}' > = -V^2 \sum_{c_1 c_2 n_1 n_2} \exp \left\{ 2i\pi (-ne_1 + n' e_2)/k \right\} \sin(\pi e_1/k) \sin(\pi e_2/k) \delta_{c_1 + c_2} \delta_{n_1 + n_2 - 1}
\]

\[
+ V^2 \sum_{n_1n_2} \delta_{n_1 + n_2}
\]

as usual the last sum contributes the missing part in the first one to reproduce \( \delta_{n+n'} \) (modulo the lattice \( [14] \)). Other cases are checked in a similar fashion.
5 Link invariants and the Verlinde formula

Here as in [1] we assume that the relation between Chern Simons and WZW theories established in the compact, non graded case [5, 22] extends here. Call generically $Z$ the partition function of the Chern Simons theory for some link in $S^3$. According to [1], introduce

$$\Box' = \prod_{\text{crossings}} \exp -i\pi \epsilon [e'(\tilde{n} - 1/2) + e(\tilde{n}' - 1/2)] Z,$$

where $\epsilon$ is the sign of a crossing where representations $(en)$ and $(e'n')$ cross (independently of their type) \(^9\). Then the following holds

$$\Box' = IV \Delta,$$

where $\Delta$ is the multivariable Alexander Conway polynomial, whose precise axiomatic definition can be found in [1], with parameters $t_i = q^{e_i}, q = \exp(-i\pi/k)$ \(^10\).

5.1 Hopf links

As a first example we consider two linked circles, one carrying a representation $(en/I)$ and the other $(n')$. Suppose we make modular transformation on the circle carrying $(en/I)$. We find then $S^2 \times S^1$ with two parallel loops, one carrying the images of $(en/I)$ and the other $(n')$. Only the $(\tilde{n}'')$ representations couple to $(n')$ among the images of $(en/I)$. Due to $<n|\tilde{n}''> = \delta_{n+n'}$ one finds

$$Z = -V \exp(2i\pi en'/k)/2\sin(\pi e/k)$$

(92)

We can also make modular transformation on the circle carrying $(n')$. Only the $(e''n''/II)$ representations couple to $(en/I)$ among the images of $(n')$. Due to $<e''n''/II|en/I> = \delta_{e+e''}\delta_{n+n''-1}$ one finds

$$Z = V \exp(2i\pi en'/k)/2\sin(-\pi e/k)$$

(93)

coinciding with the above result. This common value is of course equal to $S_{en/I,n'}$.

As a second example we consider again two linked circles, one carrying $(\tilde{n})$ and the other $(n')$. Making modular transformation on the circle carrying $(\tilde{n})$ we find a vanishing result since $(\tilde{n})$ branches only to $(e''n''/I)$ and $(n'')$, both having vanishing scalar product with $(n')$. Making modular transformation on the circle that carries $(n')$, only the branching from $(n')$ to $(\tilde{n}'')$ contributes. However since the trace of $[e3]$ vanishes, the total evaluation is still zero. This gives the value of $S_{n,n''}$.

As a third example consider again two linked circles, one carrying $(en/I)$ and the other $(\tilde{n})$. Suppose we make modular transformation on the circle carrying $(en/I)$. Only the branching to $(\tilde{n}'')$ contributes. One finds

$$Z = -\frac{V}{2\sin\pi e/k} \{-2\exp(2i\pi en'/k) + \exp[2i\pi e(n' - 1)/k] + \exp[2i\pi e(n' + 1)/k]\}$$

\(^9\)(if one strand carries a four dimensional representation, this formula can still be used by considering it as the concatenation of two two dimensional ones)

\(^10\)The reader may notice some changes of sign with respect to [1]. This is because in the present paper we use the CFT conventions for $1, \tau$ instead of the more usual knot theoretic conventions that invert the torus meridian.
ie
\[ Z = 2V \sin(\pi e/k) \exp(2i\pi en'/k) \] (94)

If we make modular transformation on \( \hat{n}' \), only the branching to \( (e''n''/I) \) contributes. One finds
\[ Z = 2V \sin(\pi e/k) \exp(2i\pi en'/k) \]
as above. We notice that such value could as well be recovered by using the \( U_{qgl}(1,1) \) formalism.

Since in [1] the universal \( R \) matrix was obtained, one has simply to evaluate it when acting in the product of a block representation and a two dimensional one, and take properly normalized trace.

Corresponding to table (66) we now give the entire set of Hopf links (figure 8) partition functions
\[ S_{en/I,n'} = -V \exp(2i\pi en'/k)/2\sin(\pi e/k) \]
\[ S_{en/II,n'} = 0 \]
\[ S_{n,n'} = \infty \] (reg. : \(-V/2\epsilon\)) (95)

the others being deduced by symmetry \( S_{\rho_i \rho_j} = S_{\rho_j \rho_i} \). From the formal point of view where the regularization is forgotten and \( \epsilon \) is interpreted as coding the multiplication rules of \( 0 \times \infty \), we see that some link invariants have to be set equal to infinity. As was commented in [1] this is rather natural from the point of view of the axiomatic construction of the multivariable Alexander polynomial (see also the appendix).

### 5.2 Geometrical meaning of the regularization

It is now time to comment more on these results by looking at the geometrical meaning of the regularization introduced above. The partition function \( \mathcal{Z}_{en/I}(z_1, z_2; \tau) \) corresponds to the situation depicted in figure 9. We have a solid torus \( S^2 \times S^1 \) with a Wilson loop carrying the representation \( (en) \). Attached to this main line by trivalent vertices are two dotted lines with opposite orientation carrying \( (e_0 n_0) \) which cut the surface of the torus at points \( z_1 \) and \( z_2 \). The partition function \( \mathcal{Z}_{en/I}(z_1, z_2; \tau) \) corresponds to the situation depicted in figure 10. The difference is that there is a single dotted line carrying \( (e_0 n_0) \), that intersects the surface of the torus at \( z_1 \) and \( z_2 \). This identification relies on the free field computation of the four point functions. One finds in particular that for the contour of integration going between the two intermediate vertices the block obtained is \( \mathcal{F}^1 - \mathcal{F}^2 \), with no logarithm. It must therefore correspond to the topology of figure 10. In figure 9, which represents the "generic" situation, braiding of \( z_1 \) and \( z_2 \) results in a non trivial operation, which we correlate with the presence of logarithms.

We now see that the scalar product \( \langle en/I | e'n'/I \rangle \) corresponds to figure 11, \( \langle en/I | e'n'/II \rangle \) to figure 12. In both case we still have to let \( e_0 \to 0 \) to get the partition function of \( S^2 \times S^1 \).
with two strands carrying two dimensional representations. On the other hand if we consider \( <en/II|e'n'/II > \), this is represented as figure 13. Such invariant can be computed eg by surgery, and one gets in \( S^3 \) two linked strands plus a dotted loop. This is a split link whose Alexander invariant vanishes.

We also can understand the value of \( S_{n,n'} \). With regularization, this is in fact the invariant for the link of figure 14 where the two circles carry \( (n) \) and \( (n') \) representations and are connected by a pair of lines carrying \( (e_0 n_0) \) and \( (-e_0, 1 - n_0) \). This is the same as having a single loop carrying \( (e_0 n_0) \), with partition function \( Z = V/2\sin(\pi e_0/k) \) that diverges as \( e_0 \to 0 \).

To check further the consistency, especially the role of the anomalous transformation, let us discuss the example of two unlinked circles in \( S^3 \), one carrying \( (en/I) \) and the other \( (e'n'/I) \).

Suppose first \( e + e' \neq 0 \). Consider a solid torus \( D^2 \times S^1 \) containing these two lines. The state on its surface is

\[
|e + e', n + n' - 1/I > -|e + e', n + n'/I >
\]

To compute the invariant we introduce a third line carrying a one dimensional representation as in figure 15 and make a modular transformation. Since the modular matrix for \( en/I \to \hat{n}'' \) does not depend on \( n \), the contributions of the two states cancel and we get a vanishing result.

Suppose now \( e + e' = 0 \). Then the state on the surface of the torus containing the two loops is \( \hat{n} + n' \). By modular transformation it branches only to states orthogonal to \( (n'') \) representations, so we again get zero.

Now consider a Hopf link with one strand carrying \( (en) \), the other \( (-e, n') \). Interpret this as the result of \( \tau \to \tau + 1 \) on the preceding case. Under this operation

\[
|\hat{n} + n' > \to \exp(-i\pi/6) \left(|\hat{n} + n' > -2i\epsilon V \sum_{n''} |n'' > \right)
\]

Now apply \( S \), and compute scalar product with the second torus containing a one dimensional representation. Only the infinitesimal amount of \( |n'' > \) above contributes, however the corresponding \( S \) matrix element is infinitely large so we get a scalar product simply equal to \( iV \exp(-i\pi/6) \). We still have to correct for the change of framing in each of the circles when \( \tau \to \tau + 1 \), which results in a phase factor \( \exp[-2i\pi(h_{en} + h_{-en'})] \), and the change of framing of \( S^3 \). Hence

\[
Z = iV \exp -2i\pi(e^2/k^2 + (n - n')e/k] = S_{en/I, -en'/I}
\]

as it should be.

### 5.3 Fusion rules

The fusion rules \( ^{11} \) can be immediately deduced from the above analysis and \( ^{11} \). In fusion, two dotted lines extremities associated with one of each representation must be attached, so two dotted lines extremities remain free (see figure 16). We see in particular that a representation \( (\hat{n}) \) has regulators with similar position as \( (en/I) \).

\[
(en/I) \cdot (e'n'/I) = (e + e', n + n' - 1/I) - (e + e', n + n'/I)
\]

\( ^{11} \)We use the convention \( \Phi_a \cdot \Phi_b = \sum N_{ab}^c \Phi_c \)
\[(en/II) \cdot (e' n'/II) = (e + e', n + n' - 1/II) - (e + e', n + n'/II)\]
\[(en/I) \cdot (e' n'/II) = (e + e', n + n' - 1/I) - (e + e', n + n'/I)\]
\[(en/I) \cdot (-e, n'/I) = (n + n'/I) - (en/I) \cdot (e' n'/II) = (e + e', n + n' - 1/I) - (e + e', n + n'/I)\]
\[(en/I) \cdot (-e, n'/II) = -2(n + n' - 1) + (n + n') + (n + n' - 2)\]
\[(en/I) \cdot (-\hat{e}, n'/II) = (n + n'/II) - (en/I) \cdot (e' n'/II) = (e + e', n + n' - 1/I) - (e + e', n + n'/I)\]
\[(\hat{e} \cdot (n') = -2(n + n'/II) + (e, n + n' + 1/I) + (e, n + n' - 1/I)\]
\[(\hat{e} \cdot (\hat{n}') = -2(n + n'/II) + (n + n'/II - 1) + (n + n'/II - 1)\]
\[(\hat{e} \cdot (n') = -2(n + n'/II) + (n + n'/II - 1) + (n + n'/II - 1)\]
\[(\hat{e} \cdot (n') = -2(n + n'/II) + (n + n'/II - 1) + (n + n'/II - 1)\]
\[(\hat{e} \cdot (n') = -2(n + n'/II) + (n + n'/II - 1) + (n + n'/II - 1)\]
\[\text{where } (\ast n) \text{ means any representation with a } N \text{ number equal to } n.\]

### 5.4 An alternative basis

To describe the fusion rules it sometimes is more convenient to introduce the linear combinations
\[|en/1 > = |en/I >, \ |en/2 > = |en/I > - |en/II >\]

The second partition function can be obtained as in section 3, by integrating the screening operator along the contour \(C_6\), and one has
\[< en/1 | e' n'/2 > = 0, \ < en/2 | e' n'/2 >= -\delta_{e+e'} \delta_{n+n'-1}\]

Fusion of \((en/2)\) and \((-e, n'/2)\) produces a representation \((n + n'/2)\), setting \((\hat{n}) = (\hat{n}/1)\), and one has
\[\zeta_{\hat{n},2} = \zeta_{\hat{n}} + 2\zeta_n - \zeta_{n+1} - \zeta_{n-1}\]

together with
\[< \hat{n}/1 | \hat{n}'/2 > = 0, \ < \hat{n}/2 | \hat{n}'/2 > = 2\delta_{n+n'} - \delta_{n+n'-1} - \delta_{n+n'+1}\]

Fusion rules do not mix species of type 1 and type 2. They look identical for these two species up to signs.

\[\text{Notice that there is no contribution from } (n + n' - 1), \text{ although this representation appears in the operator product } \Phi_{en}\Phi_{-e,n'}. \text{ This is an example of the possible differences between operator products and fusion rules in the Verlinde sense, which is manifest also in some of the } N \text{ coefficients being greater than one. An argument in favor of this fusion rule is that the log no log coupling in the four point function is a purely quantum effect that disappears in the large } k \text{ limit. It is consistent with all further computations.}\]
5.5 Factorization

We must discuss briefly the factorization formulas that play a key role in the computation of links and 3-manifolds invariants \[5\]. The case of \( gl(1,1) \) presents some difficulties related to the fact, explained above, that for a sphere \( S^2 \) with a representation \( (n) \) and its conjugate \( (1-n) \) inserted the space of conformal blocks has vanishing dimension. Hence we cannot a priori establish a connection between the invariant of say the sphere \( S^3 \) with two split links \( L_1, L_2 \) and the product of invariants of \( S^3 \) with either \( L_1 \) or \( L_2 \). To solve this difficulty, as in \[1\], we can introduce a regularization by considering the link of figure 17 where the dotted lines carry again \( e_0, n_0 \).

Assuming

\[
\lim_{e_0 \to 0} Z \left( \begin{array}{c} \end{array} \right) = Z \left( \begin{array}{c} \end{array} \right)
\]

we deduce from the same arguments as in \[3\]

\[
Z(S^3; L_1, L_2) = \frac{Z(S^3; L_1) \times Z(S^3; L_2)}{Z(S^3)}
\]

(102)

where \( Z(S^3) \) is obtained from the invariant of \( S^3 \) with a loop carrying \( (e_0n_0) \), as \( e_0 \to 0 \). Using the above results we know that this quantity is infinite, and its regularized value is \( V/2\epsilon \), as \( S_{nn'} \).

Therefore set

\[
Z(S^3) = \infty (\text{reg.}: V/2\epsilon)
\]

(103)

Now if both \( L_1 \) and \( L_2 \) had finite invariants, the invariant of their disconnected sum vanishes, a result well known in the theory of the Alexander Conway invariant. Suppose now that \( L_1 \) is say a loop carrying a representation \( (n) \) (a shadow component \[1\]). On the one hand we expect then

\[
Z(S^3; L_1, L_2) = Z(S^3; L_2)
\]

on the other hand we get from the factorization formula

\[
Z(S^3; L_1, L_2) = \frac{Z(S^3; L_2) \times \infty}{\infty}
\]

This indefinite ratio is well determined in the regularized version since then \( Z(S^3; L_1) = \infty (\text{reg.}: V/2\epsilon) = Z(S^3) \), and it is simply equal to \( Z(S^3; L_2) \), as desired.

5.6 Verlinde formula

We can now consider the Verlinde formula \[23, 24\], which from the knot theory point of view, is the consistency equation for computing invariants of links as in figure 18. Provided there is only one conformal block for the sphere \( S^2 \) with insertion of a representation \( \rho_i \) and its conjugate one has

\[
S_{\rho_i\rho_j} S_{\rho_i\rho_k} / S_{0\rho_i} = \sum_m S_{\rho_i\rho_m} N_{\rho_i\rho_k}^m
\]

(104)

Let us consider this formula for various cases of representation \( \rho_i \).
First suppose \( \rho_i = (n_i) \), \( \rho_j \) and \( \rho_k \) are two dimensional representations. Then the left member vanishes, corresponding to the vanishing of a split link as discussed above. If \( \rho_j \) and \( \rho_k \) are of type \( II \), the right member vanishes term by term. If \( \rho_j \) and \( \rho_k \) are of type \( I \), two terms contribute. Suppose \( \rho_j = (e_j n_j / I) \) and \( \rho_k = (e_k n_k / I) \). Then the terms \( \rho_m = (e_j + e_k, n_j + n_k+1) \) and \( \rho_m = (e_j + e_k, n_j + n_k - 1) \) contribute, with opposite \( N \) coefficients. However \( S_{e' n'/I,n} \) does not depend on \( n' \). Therefore the two terms add to zero. Still for \( \rho_i = (n_i) \), let us mention the case where \( \rho_j = (n_j) \), and say \( \rho_k = (e_k n_k) \). In that case the two infinities in the left hand side, regularized by the above \( \epsilon \), cancel and one gets simply \( S_{e_i n_i / I,n_i} \). On the right hand side only one term contributes, \( \rho_m = (e_k, n_j + n_k / I) \). But by formulas (105),

\[
S_{e_i n_i / I,n_i} = S_{e_k, n_j + n_k / I,n_i},
\]

so both things coincide again.

The most interesting situation arises when all of them are two dimensional representations: \( e_i n_i / I \), \( e_j n_j / I \), \( e_k n_k / I \). The left hand side reads

\[
2 V \sin(\pi e_i / k) \exp \frac{2i\pi}{k} [e_i(\tilde{n}_j + \tilde{n}_k - 1) + (e_j + e_k)(\tilde{n}_i - 1/2)]
\]

Let us now consider the right hand side. Because of graded fusion coefficients it reads

\[
i V \exp \frac{2i\pi}{k} [e_i(\tilde{n}_j + \tilde{n}_k - 3/2) + (e_j + e_k)(\tilde{n}_i - 1/2)]
\]

\[-i V \exp \frac{2i\pi}{k} [e_i(\tilde{n}_j + \tilde{n}_k - 1/2) + (e_j + e_k)(\tilde{n}_i - 1/2)]
\]

which is

\[
i V [\exp(-i\pi e_i / k) - \exp(i\pi e_i / k)] \exp \frac{2i\pi}{k} [e_i(\tilde{n}_j + \tilde{n}_k - 1) + (e_j + e_k)(\tilde{n}_i - 1/2)]
\]

equal to the left hand side.

Therefore the presence of sine functions, which is a characteristic of the multivariable Alexander Conway polynomial, occurs in some part of the formalism due to infinite sums of exponential functions, in other parts because of graded fusion rules.

Let us emphasize that the Verlinde formula is not expected to hold if \( \rho_i = (\tilde{n}) \). In that case indeed the space of conformal blocks of \( S^2 \) with \( (\tilde{n}) \) and its conjugate inserted has dimension greater than one. Factorization arguments to cut the loop that carries \( (\tilde{n}) \) can therefore not be used. For the link represented in figure 19 for instance, one finds a vanishing invariant. Similarly Verlinde formula is not expected to hold if \( \rho_i = (e n / 11) \) since then this space of conformal blocks has vanishing dimension.

Let us also discuss briefly the Verlinde operators. The one corresponding to the \( b \) cycle, which we denote \( V_{\rho_i}^{1,0} \) has its matrix elements readily evaluated

\[
[V_{\rho_i}^{1,0}]_{\rho_j}^{\rho_i} = N_{\rho_i \rho_j}^{\rho_k}
\]

(105)
where the fusion coefficients \( N \) have been given above. The one corresponding to the \( a \) cycle \( V^0_{\rho_i} \) is then obtained by modular transformation

\[
V^0_{\rho_i} = S^{-1}V^1_{\rho_i}S
\]

The operator corresponding to a torus knot of type \((p, q)\) \([15]\) can be obtained via the formulas

\[
V^{p, q}_{\rho_i} = S^{-1}V^{-q, p}_{\rho_i}S, \quad V^{p, q}_{\rho_i} = T^{-1}V^{p, q+p}_{\rho_i}T
\]

### 5.7 Cabling in Alexander theory

The composition rules of \( gl(1, 1) \) representations allow simple derivation of the cabling formulas for Alexander invariants. Consider first some knot \( K \) carrying some representation \((e_1 n_1)\). Double it by running parallel to the original strand another one carrying \((e_2 n_2)\) (figure 20). The partition function of this system writes, due to (90)

\[
Z = \left[ \exp(2i\pi wh_{e_1 + e_2, n_1 + n_2 - 1}) - \exp(2i\pi wh_{e_1 + e_2, n_1 + n_2}) \right] \square'_{e_1 + e_2}(K)
\]

where \( \square'_{e_1 + e_2}(K) \) is the invariant of \( K \) carrying a representation with \( E \) number \( e_1 + e_2 \), \( w \) is the writhe, sum of the signs of the crossings of \( K \). On the other hand, also by (90),

\[
Z = \exp(2i\pi w) \left[ (n_1 + n_2 - 1)(e_1 + e_2)/k + (e_1 + e_2)^2/2k^2 \right] \square'_{e_1, e_2}(double of K)
\]

from which one deduces immediately

\[
\square'_{e_1, e_2}(double of K) = -2i \sin\pi w(e_1 + e_2)/k \square'_{e_1 + e_2}(K)
\]

It is interesting to consider the limit \( e_1 + e_2 \to 0 \). In that case it is well known (this can be deduced from the skein relation for the ordinary Alexander Conway polynomial) \([14]\) that

\[
\square'_{e_1 + e_2}(K) \approx -\frac{V}{2\sin\pi(e_1 + e_2)/k}
\]

and therefore one finds that for a double knot with same \( E \) numbers but opposite orientations (figure 21)

\[
\square' = iVw, \quad \Delta = w
\]

One can also allow the second strand to wind around the first one. If the total twist (figure 22) is \( t \) one finds, owing to the well known formula \([14, 15]\)

\[
l = w + t
\]

where \( l \) is the linking number, the general result for a twisted double

\[
\square'_{e_1 e_2}(twisted double of K) = -2i \sin\pi l(e_1 + e_2)/k \square'_{e_1 + e_2}(K)
\]

These formulas generalize easily to an \( n \) cable (figure 23)

\[
\square'_{e_1, e_2, ..., e_n}(n \text{ cable of } K) = \left( -2i \sin\pi w \sum e_i/k \right)^{n-1} \square'_{\sum e_i}(K)
\]
5.8 Invariants of torus links

Consider first a torus knot of type \((p, q)\) \((p, q\) being coprimes). We first claim that

\[
V_{en/I}^{p,q} |_{e'n'/I} = - \exp \left[ 2i \frac{q}{p} (h_{e' + pe, n' + pn} - h_{e', n'}) \right] |e' + pe, n' + pn/I > + \left[ (en) \rightarrow (e, n - 1) \right]
\]

(113)

By applying \(T\) (which is trivial) and \(S\), one checks that each of the two terms in this sum has the right behaviour \([107]\). The precise combination can be found by considering the case \(p = 1, q = 0\). Now to obtain action of \(V^{p,q}\) on a one dimensional representation we use formula \([2]\). Due to the minus sign in the above formula, only \(p\) terms remain and we get

\[
V_{en/I}^{p,q} |_{0} = \sum_{j=0}^{p-1} \exp \left[ 2i \frac{q}{p} h_{pe, pn-j} \right] |pe, pn - j/I >
\]

(114)

Using the value of \(S_{0}^{en/I}\) we get the partition function

\[
Z(S^3, (p, q)) = - \frac{V}{2 \sin \pi pe/k} \frac{\sin \pi p q e/k}{\sin \pi q e/k} \exp (2i \pi p q h_{en})
\]

(115)

and therefore we get the general result for a torus knot, after appropriate framing correction,

\[
\Delta = - \frac{\sin \pi p q e/k}{2i \sin \pi p e/k \sin \pi q e/k}
\]

(116)

in agreement with known formulas \([13]\).

We can with the same formalism deal with torus links of type \((ap, aq), q > 1\). In that case \([13]\) becomes

\[
V_{en/I}^{ap,aq} |_{e'n'/I} = (-)^{a} \sum_{j=0}^{a} \sum_{j=0}^{a} (-1)^{j} \left( \begin{array}{c}
\begin{array}{c}
a \\
j
\end{array}
\end{array} \right) \exp \left[ 2i \frac{q}{p} (h_{e' + ape, n' + apn - pj} - h_{e', n'}) \right] |e' + ape, n' + apn - pj/I >
\]

(117)

the combination of representations being found by considering the case \(p = 1, q = 0\). We get therefore

\[
\Delta = (-2i)^{a-2} \frac{(\sin \pi ape/k)^a}{\sin(\pi ape/k) \sin(\pi q e/k)}
\]

(118)

6 3-Manifold Invariants

6.1 \(X_k \times S^1\)

We have already encountered the following

\[
Z(S^3) = \infty, \; Z(S^2 \times S^1) = 0
\]

(119)
the last result occurring because there is no conformal block on the sphere with one dimensional representations inserted, or \(< n|n' > = 0\). Consider now the torus \(S^1 \times S^1\). Its Hilbert space contains twice as many fields as the volume of the fundamental domain due to the doubling of representations explained above. We therefore expect
\[
Z(S^1 \times S^1 \times S^1) = 2V^{-2}
\] (120)
Consider now a 3-manifold of the form \(M = X_h \times S^1\) where \(X_h\) is a surface of genus \(h\). It can be obtained by surgery on the system of loops of figure 24. For an ordinary theory like \(SU(n)\), as discussed in [25], one finds the result
\[
\sum_i (S_{0\rho_i})^{1-h} (S_{0\rho_i})^{1-h}
\] (121)
Suppose we forget for a while subtleties attached with vanishing values of the \(e\) number and compute this sum by taking the expressions of the \(S\) matrix elements established in the case of two dimensional representations and \(e \neq 0\). Then one finds for \((en/1)\)
\[
V^{2-2h} \sum_{en} \left[ \exp(i\pi e/k) - \exp(-i\pi e/k) \right]^{2h-2} = V^{-2h} \left( \frac{h-1}{2h-2} \right)
\] (122)
and, up to a sign, an identical result for \((en/2)\) representations (this combinatorial factor can also be obtained by counting paths on the \(gl(1,1)\) Bratteli diagram [1]). Therefore we expect
\[
Z(X_h \times S^1) = \frac{(2h-2)!}{[(h-1)]^2} \left[ 1 + (-)^{h-1} \right] V^{-2h}
\] (123)
It is better to explicitly check this result by manipulating our various representations and taking correctly into account modular properties. We will first consider \(S^1 \times S^1 \times S^1\). This three dimensional torus can be produced from \(S^3\) by surgery on the link of figure 25. Let us first do surgery on loop 1, to get \(S^2 \times S^1\) with loops 2 and 3 linked in it as in figure 26. Now instead of performing an actual surgery on loop 2 we can consider it as a Wilson loop carrying all possible representations \(\rho\) along it, each one coming with a factor \(S_{0\rho}\). We then have to compute a trace of Chern Simons evolution operator between the states in sections 1 and 2. Suppose first \(\rho\) is of type \((en/II)\). There is then only one conformal block for sections 1 and 2. One can therefore using factorization arguments perform the calculation by glueing caps as in figure 27 and dividing by the scalar product of caps themselves, which is clearly \(S^3\) with the unknot carrying \((en/II)\). We finally perform surgery on loop 3 to obtain an \(S^2 \times S^1\) that contains two Wilson loops carrying \((en/II)\) with opposite orientations. Therefore we find the contribution of a given \((en/II)\) representation
\[
\frac{S_{0,en/II}^{en/II}}{S_{0,en/II}} < en/II|C|en/II >
\]
Unfortunately this is an undeterminate since the denominator vanishes as well as the scalar product. To get a finite result we must change the basis. A natural basis to consider is the representations \(|en/1>, |en/2>\). In that case one has \(S_{0}^{en/1} = -S_{0}^{en/2} = S_{0,en/1} = S_{0,en/2}\) and thus one finds a
factor of 2 per representation (en/II). Therefore the contribution of $\rho$ irreducible to $Z(S^3 \times S^1 \times S^1)$ is

$$2 \times 2k(2k - 1)$$

Let us now discuss the contribution originating from $\rho = (\tilde{\rho})$ an indecomposable block. The situation is more complicated because now there are two conformal blocks for sections 1 and 2. One block is logarithmless, let us call it $|1>$, and the other contains logarithms, let us call it $|2>$. (it is well defined only up to addition of $|1>$). The partition function of $S^2 \times S^1$ as in figure 26 can be expressed as the trace of some matrix $M$. Let us replace the loop carrying an indecomposable block representation by two parallel loops of opposite orientation, one carrying $(e,n_1/I)$ and the other $(e,n_2/I)$. Then the “body” looks as figure 28 while there are two possible caps extracting the conformal block $|1>$ (figure 29) or $|2>$ (figure 30). Now $<1|1>=<\text{cap1}|\text{cap1}>$ and $<2|2>$ are the Alexander polynomials of $S^3$ with two disconnected loops, which are known to vanish. Similarly $<1|2>=<\text{cap1}|\text{cap2}>$ is the Alexander polynomial of $S^3$ with the unknot, with value $S_{0, en/I}$. Let us consider now the quantity

$$\frac{<1|M|2>}{<1|2>} = M_{22} = \frac{<2|M|1>}{<2|1>} = M_{11}$$

The numerator of this equation is the partition function of the body of figure 28 with cap 1 glued on top and cap 2 on the bottom. The result is presented in figure 31 (dotted lines originating from regularization). By factorization we get

$$Z(\text{figure31}) = \frac{Z(\text{figure32})S_{0, en/I}}{Z(S^3)}$$

The factor $Z(S^3)$ that diverges when $\epsilon \to 0$ will just compensate for the $S^3$ generated when putting $(\tilde{\rho})$ on loop 2. We are thus left with computing $Z(\text{figure32})$. By surgery on loop 3 we get $S^2 \times S^1$ with only one conformal block, so

$$M_{22} = 1$$

Therefore

$$M_{11} = M_{22} = 1, \text{ Tr}M = 2$$

so each block representation contributes a factor 2. We thus confirm by this explicit computation the doubling phenomena.

As for $X_h \times S^1$, which can be obtained by surgery on the loops of figure 24, a similar computation can be done.

### 6.2 Seifert Manifolds

We now would like to discuss Seifert manifolds

$$X(p_1/q_1, \ldots, p_n/q_n)$$

where $p_i$ and $q_i$ are coprimes. Recall that $X$ is obtained from $S^2 \times S^1$ by removing $n$ disjoint solid tori $D_i \times S^1$ ($D_1, \ldots, D_n$ disjoint disks in $S^2$) and gluing them back after twisting the boundary

$$26$$
by certain \( SL(2, \mathbb{Z}) \) matrices \( M_1, \ldots, M_n \). The matrix \( M_i = M(p_i, q_i) \) has first column \( \left( \begin{array}{c} p_i \\ q_i \end{array} \right) \). The particular choice of \( M_i \) affects the 2-framing, but not the diffeomorphism type of the resulting manifold (see [21]).

Let us first consider the Lens space \( L(q, p) = X(p/q) \). First of all notice that in a basis \( |n>, |\tilde{n}'> \) we can write

\[
S = \begin{pmatrix} 0 & -2 \epsilon \\ -1/2 \epsilon & 0 \end{pmatrix}
\]

and

\[
T = \exp(-i \pi/6) \begin{pmatrix} 1 & -2 \epsilon i \\ 0 & 1 \end{pmatrix}
\]

therefore

\[
T^p S = i \exp(-i \pi/6) \begin{pmatrix} p & 2 \epsilon i \\ i/2 \epsilon & 0 \end{pmatrix}
\]

in the sense that

\[
T^p S \left( a|n> + b|\tilde{n}'> \right) = i \exp(-i \pi/6) V \sum_{mm'} \frac{i}{2 \epsilon} a|\tilde{m}n> + (2 \epsilon b + p)|m'>
\]

Let us now write a continued fraction expansion

\[
p/q = a_r - \frac{1}{a_{r-1} - \frac{1}{\ldots - \frac{1}{a_1}}} \]

then chose for \( M_i \)

\[
M_i = \begin{pmatrix} a_r & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{r-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix}
\]

with

\[
\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} = T^a S
\]

Now let us suppose for the moment that \( T, S \) transformations restricted to the space of \( |n>, |\tilde{n}'> \) give the invariant of the Lens space. In that case, starting with a representation \( |0> \) and making appropriate surgery gives

\[
i^r \exp(-i \pi/6 \sum_{j=1}^{r} a_j) \prod_{j=r}^{1} \begin{pmatrix} a_j & 2 \epsilon i \\ i/2 \epsilon & 0 \end{pmatrix} \begin{pmatrix} V \\ 0 \end{pmatrix}
\]

from which one deduces

\[
Z(L(q, p)) = i^r \exp(-i \pi/6 \sum_{j=1}^{r} a_j) \frac{V i q}{2 \epsilon}
\]

It is easy to check that apart from \( |n>, |\tilde{n}'> \), the other representations indeed do not contribute to the final result due to summation over all possible \( n \) numbers. Recall that \( L(0, 1) = S^2 \times S^1, L(1, 0) = S^3 \), in agreement with already established results.
For a Seifert manifold \( X(p_1/q_1,\ldots,p_n/q_n) \) one finds immediately
\[
Z(X(p_1/q_1,\ldots,p_n/q_n)) = \prod_{i=1}^{n} \frac{i r_i}{2e} \exp\left(-\frac{i \pi}{6} \sum_{j=1}^{r_i} a_{ij}\right) V^{2\epsilon[p_1 p_2 \ldots p_{n-1} q_n + \text{permutations}]} \tag{136}
\]

Strictly speaking we still have to let \( \epsilon \to 0 \) so all these invariants become infinite. However the ratio \( Z(X)/Z(S^3) \) remains well defined. The necessary corrections for framing are discussed in details in [28]. After they are taken into account, the invariant obtained depends only trivially on the level \( k \) by an overall scale. The non trivial content is \( k \) independent and reduces essentially to the absolute value of
\[
p_1 p_2 \ldots p_{n-1} q_n + \text{permutations} \tag{137}
\]
which is nothing but the order (number of elements) of the first homology group of \( X \) (with the convention that zero indicates infinite order).

Recall that \( X(-2/1, 3/1) \approx S^3 \) and
\[
X(p_1/q_1,\ldots,p_n/q_n, 0/1) = \text{disconnected sum of } L(-p_i, q_i) \tag{138}
\]
both results in agreement with (136).

### 6.3 A three dimensional point of view

We now wish to discuss the divergence of the \( S^3 \) invariant from the 3 dimensional Chern Simons point of view. Recall that in the large \( k \) limit, \( Z(S^3) \) can be estimated using the saddle points of the Chern Simons integral, given by flat connections. For \( S^3 \) there is only the trivial connection, so one gets for an arbitrary Lie group \( G \)
\[
Z(S^3) \propto \frac{1}{\text{Vol}(G)}, \quad k \to \infty \tag{139}
\]
(It is necessary to divide by the volume of the group as the connection is reducible and is invariant under global gauge transformations). In the \( SU(n) \) case for instance one has
\[
\text{Vol}(SU(n)) = \prod_{p=2}^{p=n} \frac{2\pi^p}{\Gamma(p)} \tag{140}
\]
in agreement with
\[
S_{00} \approx (2\pi)^{n(n-1)/2} \frac{1}{k^{(n-1)(n+1)/2}} \frac{\Gamma(n-1)\ldots\Gamma(2)}{\sqrt{n}} \tag{141}
\]
Notice that if one naively lets \( n \to 0 \) in that equation one finds \( S_{00} \to \infty \).

We now discuss the volume of \( U(1,1) \). The most direct way to find the appropriate measure is to study characters. We write an element \( g \) of \( U(1,1) \) as
\[
g = \exp(x \varepsilon N + x_n E) \exp(x \psi \bar{\psi} + x \psi^+ \bar{\psi}^+) \tag{142}
\]
Consider first a representation \((en)\). Then one has
\[
\text{ch}_{en}(g) = \text{Str}_{en}(g) = \exp[i(x_n e + x_e (n-1/2))] \left[ e^{-ix_e/2} (1 - \frac{e}{2} x_{\psi^+} x_{\psi}) - e^{ix_e/2} (1 + \frac{e}{2} x_{\psi^+} x_{\psi}) \right]
\] (143)

We discard now the \(x_{\psi^+} x_{\psi}\) contribution for simplicity, i.e., restrict to \(g\) belonging to the maximal torus. Then
\[
\text{ch}_{en}(g) = -2i \exp[i(x_n e + x_e (n-1/2))\sin(x_e/2)]
\] (144)
and also
\[
\text{ch}_{n}(g) = -4e^{ix_e n}\sin^2(x_e/2)
\] (145)
\[
\text{ch}_{n}(g) = e^{ix_e n}
\] (146)

The range of parameters is \(x_e \in [0, 4\pi], x_n \in [0, 2\pi]\). It is a very natural assumption that the representations \((en)\) and \((-e, 1-n)\) are conjugate. Calling \(d\mu\) the measure of integration for class functions we require therefore
\[
<en|e' n'> = \int d\mu \text{ch}_{en}(g)\text{ch}_{e'n'}(g) = \delta_{e+e'} \delta_{n+n'-1}
\] (147)
so
\[
d\mu = -\frac{dx_e dx_n}{4\sin^2(x_e/2)}
\] (148)

This form is the naive extension of well-known results in the theory of ordinary Lie groups where the measure for class functions contains \(\sin^2\) from each of the roots. In the supergroup case fermionic roots, due to the standard formulas for change of variables in Grassman integration [27], provide the inverse of such terms [28]. Now for \(<n|n'>\) we find
\[
<n|n'> = -\int dx_e dx_n \frac{e^{ix_e (n+n')}}{4\sin^2(x_e/2)}
\] (149)

To give a meaning to this integral we give to \(x_e\) a small imaginary part so that
\[
<n|n'> = 2\pi \int_0^{4\pi} dx_e e^{ix_e (n+n')} \sum_{j=0}^{\infty} e^{-i(2j+1)x_e} e^{-2jc}
\] (150)

and each term gives vanishing contribution once integrated due to periodicity. Therefore we recover \(<n|n'> = 0\) as claimed earlier in the text. Other scalar products are recovered in a similar fashion. In particular we see now that
\[
\text{Vol}(U(1,1)) \propto \int d\mu \propto <0|0> = 0
\] (151)

hence reaching the remarkable result that \(U(1,1)\) has a vanishing volume.

Once plotted in the large \(k\) expansion, this vanishing volume explains the divergence of the \(S^3\) invariant. More generally such divergence should occur for any 3-manifold that admit a discrete set of flat connections. The divergence comes from the flat connections that commute with the
entire $gl(1,1)$, i.e. are of the form $A^\mu = A^\mu_N E$. In that case the covariant derivative with respect to $A$ is the same as the ordinary derivative, and the study of fluctuations around the saddle point involves only Gaussian integrals. The corresponding determinants in numerator and denominator cancel up to signs due to boson-fermion symmetry. The result is then identical to the invariant that would be obtained from super $IU(1)$, and as argued in [16] it should be nothing but the $U(1)$ Casson invariant. Indeed the $U(1)$ Casson invariant counts the maps from $\Pi_1(M)$ into $U(1)$ (up to conjugacy). Since $U(1)$ is commutative this counts in fact maps from $\Pi_1,\Pi_1(M)$ into $U(1)$. But this number of maps has to coincide with the number of elements of $H_1$ since the latter is commutative.

6.4 Invariants of links in 3-manifolds

We have in principle at our disposal all the necessary ingredients to compute invariants of links in 3-manifolds. As an example we consider again the lens space $L(q,p)$. To put a knot in it we start with $S^2 \times S^1$, remove a solid torus $D \times S^1$ that contains a Wilson loop carrying the representation $(en/I)$, and glue it back after twisting the boundary by a $SL(2,\mathbb{Z})$ matrix with first column $\begin{pmatrix} p \\ q \end{pmatrix}$. We suppose moreover

$$p/q = a_2 - \frac{1}{a_1}$$

To compute the corresponding invariant we follow the general strategy, i.e. apply $T^{a_2}S T^{a_1}S$ to $|en/I>$ and evaluate scalar product of the final state with $|n = 0>$. One finds the result

$$Z(L(q,p),K) = -i V^2 \exp\left(-i \frac{\pi}{6} (a_1 + a_2)\right) \sum_{n'} \sum_{\epsilon' \neq 0} \frac{\exp \left[ 2i\pi \epsilon' (\tilde{n} - 1/2) - e' (\tilde{n} - 1/2) - e (\tilde{n}' - 1/2) \right]}{2 \sin \pi e' / k}$$

The result depends therefore on the number of solutions in the fundamental domain of the equation (for the unknown $\epsilon'$)

$$qe' = e \mod 2k$$

If there is no solution the invariant vanishes. Suppose there is a single solution of the form $\epsilon' = e/q$. Then the invariant is

$$Z(L(q,p),K) = -i V \exp\left(-i \frac{\pi}{6} (a_1 + a_2)\right) \frac{\exp \left[ 2i\pi \frac{e}{k} (\tilde{n} - 1/2) \right]}{2 \sin \pi e / k q}$$

In the limit $e = e_0 \to 0$ one finds then

$$Z(L(q,p),K) \to -\exp\left(-i \frac{\pi}{6} (a_1 + a_2)\right) \frac{V q}{2 e}$$

in agreement with the above computation of the invariant of an "empty" Lens space. In the case of $L(1,0) = S^3$ one also recovers the invariant of the unknot is $S^3$. In general the result depends on arithmetic properties of $e$ and $q$.
7 Conclusion

The study of the WZW on $GL(1,1)$ presented here, together with the results of [1], show a complicated and rather unexpected pattern. Although we are not totally happy with our derivations ($gl(1,1)^{(1)}$ characters for instance need a more complete and rigorous analysis), believe the results to be correct. In particular they reproduce nicely known all known features of the multivariable Alexander Conway polynomial. Moreover we have strong indication that the present pattern extends to other super groups, eg $SU(n,m)$ with mixtures of typical, atypical and indecomposable blocks, and that the picture presented in [12] is too simple.

Although it may look a little surprising to find possibly infinite 3-manifold invariants, the emerging relation with homology (which is expected from the classical theory of Alexander invariants [15]) makes these infinities quite natural. Introduce $O(\mathcal{M})$ to be the order of the first homology group of the 3-manifold $\mathcal{M}$. Take the convention that this number is zero if the homology group is infinite. Then in all cases we considered the following holds

$$\frac{Z(\mathcal{M})}{Z(S^3)} \propto O(\mathcal{M})$$

(up to framing and normalization factors). Indeed for $\mathcal{M} = X_h \times S^1$, the order of the homology group is infinite and this ratio is zero. For $\mathcal{M}$ a Seifert manifold the result was obtained in the last section. Moreover, while $O(S^3) = 1$, with order one, if we put inside a knot $K$ carrying a representation with non vanishing $E$ number, the complement $S^3-K$ acquires a first homology group equal to $Z$, with order zero. This matches the fact that $Z(S^3)/Z(S^3) = 1$ while $Z(S^3,K)/Z(S^3) = 0$. Therefore our results are perfectly consistent with topology. On the basis of the discussions in the last section it is likely that (157) is generally true and coincides with the $U(1)$ Casson invariant, which can also be interpreted as a kind of Reidemeister torsion [29]. We hope to get back to the topological meaning of Alexander invariants of links in 3-manifolds and their relation with Reidemeister torsion.

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Appendix 1

The purpose of this appendix is to discuss briefly \( U_q \mathfrak{gl}(1,1) \) in the context of the general construction for 3-manifold invariants proposed in [30] for the non graded case. First of all introduce

\[
K = q^{(E-N)/2}, \quad L = q^{(E+N)/2}
\]  

one can then write the defining relation of \( U_q \mathfrak{gl}(1,1) \) as

\[
\{ \psi, \psi^+ \} = \frac{KL - (KL)^{-1}}{q - q^{-1}}
\]

and

\[
K\psi^{(+)K}^{-1} = q^{-1/2}\psi^{(+)}, \quad L\psi^{(+)}L^{-1} = q^{-(+)1/2}\psi^{(+)}
\]

The coproduct reads

\[
\Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm}, \quad \Delta(L^{\pm}) = L^{\pm} \otimes L^{\pm}
\]

\[
\Delta(\psi^{(+)}) = \psi^{(+) \otimes (KL)^{1/2}} + (KL)^{-1/2} \otimes \psi^{(+)}
\]

the counit

\[
\varepsilon(\psi) = \varepsilon(\psi^+) = 0, \quad \varepsilon(K) = \varepsilon(L) = 1
\]

and the antipode

\[
s(K) = K^{-1}, \quad s(L) = L^{-1}, \quad s(\psi^{(+)}) = -\psi^{(+)}
\]

The universal \( R \) matrix takes a very simple form

\[
R = \left[ 1 + (q - q^{-1})q^{(E \otimes 1 - 1 \otimes E)/2} \psi^{+} \otimes \psi \right] q^{-E \otimes N - N \otimes E} = \sum \alpha_i \otimes \beta_i
\]

where \( \alpha \) and \( \beta \) are obtained by expanding the exponentials. Introduce now the element

\[
u = \sum s(\beta_i)\alpha_i = q^{2EN} \left[ 1 + (q - q^{-1})q^{E}\psi^{+} \right]
\]

with

\[
s(u) = \sum \alpha_i s(\beta_i) = q^{2EN} \left[ 1 - (q - q^{-1})q^{-E}\psi^{+} \psi \right]
\]

and

\[
u^{-1} = \sum \beta_i s^2(\alpha_i)
\]

One has

\[
\varepsilon(u) = \varepsilon(s(u)) = 1
\]

Then for any element \( a \) in the algebra

\[
u au^{-1} = s^2(a)
\]

Set

\[
v^2 = us(u) = s(u)u = q^{4EN} \left[ 1 + (q - q^{-1})(q^{E}\psi^{+} - q^{-E}\psi^{+}) \right]
\]
Then $v$ is central. Now consider a pair of representations $\rho\rho'$ and define

$$S_{\rho\rho'} = \sum_{ij} \text{Str}_\rho(uv^{-1}\beta_i\alpha_j)\text{Str}_{\rho'}(uv^{-1}\alpha_i\beta_j)$$

It is easy to check that when $\rho$ or $\rho'$ (or both) is a two or four dimensional representation, this vanishes. Such a result is well known in the theory of the Alexander Conway polynomial where $S_{\rho\rho'}$ is (up to a framing correction) the "bare" invariant for a Hopf link (figure 8), which vanishes because the superdimension of $\rho$ and $\rho'$ is zero, or alternatively because the Alexander matrix has a zero mode. If one sticks to (171), a non vanishing result is obtained only for $\rho$ and $\rho'$ one dimensional, but it is trivial.

To get interesting Alexander invariants, a strand has to be "open". This means one has to take supertrace only on one of the representations, say $\rho'$. Let us therefore define

$$S_{\rho\rho'} = \sum_{ij} \text{Tr}_{(b \text{ in } \rho)}(uv^{-1}\beta_i\alpha_j)\text{Str}_{\rho'}(uv^{-1}\alpha_i\beta_j)$$

where the first factor is a trace restricted to bosonic states in $\rho$. Suppose $\rho = (en), \rho' = (e'n')$. Then

$$S_{\rho\rho'} = q^{2e+e'}q^{-2(e'\hat{m}+en')} \left(q^e - q^{-e}\right)$$

Now, as discussed in [1] to get link invariants when one opens a strand it is necessary to rescale the above result by a factor depending on the $e$ number of the strand that has been open, leading to

$$S_{en,e'n'} = \frac{iV}{q^{2e} - 1}S_{en,e'n'} = iV\exp\frac{2i\pi}{k}[e'(n - 1/2) + e(n' - 1/2)]$$

where we used $q = \exp(-i\pi/k)$. Suppose now $\rho'$ is a one dimensional representation ($n'$). Then one finds

$$S_{en,n'} = -V\exp(2i\pi en'/k)/2\sin(\pi e/k)$$

(If $\rho$ was one dimensional and $\rho' = (en)$ we would get an indeterminate form). Suppose $\rho'$ is a four dimensional representation ($\hat{n}'$). Then

$$S_{en,\hat{n}'} = 2V\sin(\pi e/k)\exp(2i\pi en'/k)$$

Up to the $1/k^2$ quantum corrections to the conformal weight of the fields, these formulas agree with what we derived above from modular transformations. $S_{nn'}$ is infinite because of the normalization factor $\lambda$, which on the other hand is necessary to provide link invariants. Not opening strands gives trivial invariants, while opening them forces us to treat also one dimensional representations ("shadow components" [1]) and to face divergences.

However the quantum group approach does not seem to provide any insight as to what are the type $II$ representations. This doubling still seems to be a purely quantum effect.
Appendix 2
In this Appendix we discuss in more details the computation of the Alexander polynomial for torus knots and links. Consider a strand which is wound $p$ times along the noncontractible cycle and $q$ times along the contractible cycle of a solid torus which belongs to $S^3$. If $p$ and $q$ are coprime and the torus is not knotted in $S^3$, then the strand forms a torus knot of the type $(p,q)$. More generally we can consider torus links with windings $(ap,aq)$ where $p$ and $q$ are coprimes. They correspond to a linked $(p,q)$ knots.

We regard a $(P,Q)$ torus knot (or link) as a Verlinde operator $V_{en}^{PQ}$. Here $|en>$ specifies a typical $gl(1,1)$ representation flowing along the strand. We assume that all typical representations appearing in this Appendix are of the type $I$ and that the framing of the strand goes parallel (or perpendicular) to the surface of the solid torus. We will derive the action of $V_{en}^{PQ}$ on the Hilbert space of Kac-Moody characters. This operator should satisfy the following identities when conjugated with modular transformations $S$ and $T$:

\[ SV_{en}^{PQ} S^{-1} = V_{en}^{-Q,P} \]
\[ TV_{en}^{PQ} T^{-1} = V_{en}^{P,Q+P} \] (177) (178)

It is easy to see that a repeated conjugation of $V_{en}^{PQ}$ with $T$ and $S$ can reduce it to $V_{en}^{a,0}$ (where $a$ is the greatest common divisor of $P$ and $Q$). The action of Verlinde's operator $V_{en}^{a,0}$ can be directly expressed through the fusion rules

\[ V_{en}^{1,0}|e_1, n_1> = |e_1 + e, n_1 + n - 1 > - |e_1 + e, n_1 + n > \] (179)

and generally

\[ V_{en}^{a,0}|e_1, n_1> = (-a) \sum_{j=0}^{a} \left( \begin{array}{c} j \\ a \end{array} \right) (-1)^j |e_1 + ae, n_1 + an - j > \] (180)

Our strategy is to propose a formula for $V_{en}^{PQ}|e_1, n_1>$ and then to check that it satisfies equations (177) and (178) as well (179) or (180) in the case when $Q = 0$. We will represent $V_{en}^{PQ}$ as a linear combination of the elementary operators $V_{en}^{e,n}$, whose action is defined as follows:

\[ V_{en}^{PQ}|e_1, n_1> = \exp \left[ 2\pi i \frac{P}{k}(h_{e_1+pe,n_1+pn} - h_{e_1,n_1}) \right] |e_1 + pe, n_1 + pn > \] (181)

where the conformal weights $h_{en}$ are given in the text. The operator $V_{en}^{PQ}$ satisfies both eqs. (177) and (178). Indeed,

\[ TV_{en}^{PQ} T^{-1}|e_1, n_1> = \exp \left[ -\frac{2\pi i}{k} e_1(n_1 - 1/2) \right] TV_{en}^{PQ}|e_1, n_1> \]
\[ = -\exp \left[ \frac{2\pi i}{k} \frac{P}{k}(e_1 + Pe)(n_1 + Pn - 1/2) - (Q + P)e_1(n_1 - 1/2) \right] T|e_1 + Pe, n_1 + Pn > \]
\[ = -\exp \left[ \frac{2\pi i}{k} \frac{P}{k}(e_1 + Pe)(n_1 + Pn - 1/2) - (Q + P)e_1(n_1 - 1/2) \right] |e_1 + Pe, n_1 + Pn > \]
\[ = V_{en}^{PQ+P}|e_1, n_1> \] (182)
and

\[ \sum_{\tilde{e}, n_2} \exp \left( \frac{2 \pi i}{k} \left[ e_1 (\tilde{n}_2 - 1/2) + e_2 (\tilde{n}_1 - 1/2) \right] \right) \sum_{\tilde{e}, n_1} \exp \left( \frac{2 \pi i}{k P} \left[ P e_1 (\tilde{n}_2 - 1/2) + Q (e_2 + P e) (n_2 + P n - 1/2) - Q e_2 (\tilde{n}_2 - 1/2) \right] \right) S |e_1, n_1> = \]

\[ \sum_{\tilde{e}, n_2} \exp \left( \frac{2 \pi i}{k P} \left[ P e_1 (\tilde{n}_2 - 1/2) + Q (e_2 + P e) (n_2 + P n - 1/2) - Q e_2 (\tilde{n}_2 - 1/2) \right] \right) \sum_{\tilde{e}, n_1} \exp \left( \frac{2 \pi i}{k P} \left[ P e_1 (\tilde{n}_2 - 1/2) + Q (e_2 + P e) (n_2 + P n - 1/2) - Q e_2 (\tilde{n}_2 - 1/2) \right] \right) S |e_1, n_1> \]

\( (183) \)

It is easy to check that the following linear combination of operators

\[ V_{en}^{PQ} = (-)^a \sum_{j=0}^{a} (-1)^j \binom{j}{a} V_{\tilde{e}, n}^{PQ} \]

satisfies \( (184) \). So we conclude that

\[ V_{en}^{PQ} |e_1, n_1> = \]

\[ (-)^a \sum_{j=0}^{a} (-1)^j \binom{j}{a} \exp \left( \frac{2 \pi i}{k P} \left[ h_{e_1 + P e, n_1 + P n - j} - h_{e_1, n_1} \right] \right) |e_1 + P e, n_1 + P n - \frac{P}{a} j > \]

\( (185) \)

If we want to have a standard framing for the torus link, then we should add an extra factor

\[ \exp \left( -2 \pi i P Q h_{en} \right) \]

\( (186) \)

to this formula. Note the relation

\[ V_{en}^{a, a} = (V_{en}^{PQ})^a \]

\( (187) \)

which has an obvious geometric interpretation. Another relation

\[ V_{en}^{a, a} = (-)^a \sum_{j=0}^{a} (-1)^j \binom{j}{a} V_{\tilde{e}, n}^{PQ} \]

\( (188) \)

is due to the cabling properties of the Alexander polynomial.
To calculate the Alexander polynomial of the \((P, Q)\) torus link we observe that

\[
V_{en}^{PQ}|0> = -\sum_{l=0}^{\infty} V_{en}^{PQ}|0, -l> = (-)^{a+1} \sum_{l=0}^{\infty} \sum_{j=0}^{a} (-1)^j \left( \frac{j}{a} \right)
\]

\[
\exp \left[ \frac{2\pi i Q}{k} Pe(Pn - l - \frac{P}{a}j - 1/2) \right] |Pe, Pn - l - \frac{P}{a}j >
\]  

By using the formula

\[
\sum_{j=0}^{a-1} (-1)^j \left( \frac{k}{a} \right) = (-1)^j \left( \frac{j}{a-1} \right)
\]

we can transform eq. (189) into the following form

\[
V_{en}^{PQ}|0> = (-)^{a+1} \sum_{j=0}^{a-1} (-1)^j \left( \frac{j}{a-1} \right) \sum_{l=0}^{a-1} \exp \left[ \frac{2\pi i}{k} Qe(Pn - l - \frac{P}{a}j - 1/2) \right] |Pe, Pn - l - \frac{P}{a}j >
\]  

The Alexander polynomial of the unknot carrying a representation \(|Pe, Pn - l - \frac{P}{a}j >\) is equal to 

\[
\frac{1}{2i \sin \frac{\pi}{k} Pe}.
\]

Since

\[
\sum_{l=0}^{a-1} \exp \left( -\frac{2\pi i}{k} Qe \right) = \frac{1 - \exp(-\frac{2\pi i}{k} PQ e)}{1 - \exp(-\frac{2\pi i PQ e}{ak})}
\]

and

\[
\sum_{j=0}^{a-1} (-1)^j \left( \frac{j}{a-1} \right) \exp \left( -\frac{2\pi i PQ e}{ak} \right) = \left[ 1 - \exp \left( -\frac{2\pi i PQ e}{ak} \right) \right]^{-1}
\]

then after adding an extra factor \([186]\) to correct the framing, we get the following expression for the Alexander polynomial of a torus link:

\[
\Delta = (-2i)^{a-2} \frac{(\sin \pi PQ e / ak)^a}{\sin(Pe/k) \sin(Qe/k)}
\]

in agreement with known results.

We derived equation (187) by using inductive arguments. Now we present a direct derivation of eqs. (185) and (191) which also makes contact with some results of [2] and with the Burau matrix representation of [31] discussed in [1]. We will calculate a scalar product which is equal to a matrix element of \(V_{en}^{PQ}\) with both indices lowered:

\[
< e_2, n_2 | V_{en}^{PQ} | e_1, n_1 > = \left[ V_{en}^{PQ} \right]_{(e_2, n_2)(e_1, n_1)}
\]

This scalar product is equal to the invariant of a manifold \(S^1 \times S^2\) obtained by glueing together two solid tori, one of which has a Wilson line with representation \((e_2, n_2)\) inside, while the other
one has a Wilson line with representation \((e_1, n_1)\) inside and a torus link \((P, Q)\) with representation \((e, n)\) on its surface.

An \(S^2\) section of the whole manifold \(S^1 \times S^2\) in drawn in figure 33. As we know, an invariant of the link in such a manifold is equal to the supertrace of the braiding matrix \(B_Q\), which shifts the \((e, n)\) vertices cyclically by \(Q\) positions. The supertrace should be taken over the space of all conformal blocks. A typical conformal block in the free-field representation is drawn in figure 34. The \(n\) screening operators are integrated along the contours \(C_i\). Let \(\mathcal{V}_m\) denote the \(\left(\begin{array}{c} m \\ P \end{array}\right)\)-dimensional space of conformal blocks with \(m\) screening operators. The \(N\)-charge \(n_2\) in these blocks is fixed by the anomalous charge conservation:

\[
\begin{align*}
n_2 &= m - n_1 - Pn + 1 \\
\end{align*}
\]

(196)

The action of the braiding matrix \(B_Q\) consists of shifting the contours \(C_i\)

\[
B_Q : \ C_i \mapsto C_{i+Q}
\]

(197)

and multiplicated by a phase factor

\[
\exp \left[ 2\pi i \frac{Q}{P} (h_{e_2 n_2} - h_{e_1 n_1}) \right]
\]

(198)

Apart from this factor, the matrices \(B_Q\) \((0 \leq Q < P)\) form a group isomorphic to \(Z_p\).

To simplify our discussion we take the case when \(P = p\) and \(Q = q\) are coprime (our arguments can be extended to a general case). The action of \(Z_p\) in \(\mathcal{V}_m\) \((1 \leq m \leq p - 1)\) splits this space into the \(\frac{(p-1)!}{m!(p-m)!}\) \(p\)-dimensional invariant subspaces. The eigenvalues of \(B_q\) in these subspaces are \(p\)th order roots of unity up to the common factor \(\left[ \frac{2\pi i}{P} (h_{e_2 n_2} - h_{e_1 n_1}) \right]\) (198), therefore

\[
\text{Str}_{\mathcal{V}_m} B_q = 0 \text{ for } 1 \leq m \leq p - 1
\]

(199)

However \(\mathcal{V}_1\) and \(\mathcal{V}_p\) are 1-dimensional representations of \(Z_p\), therefore in those spaces the supertrace of \(B_q\) should be equal to the factor \(\left[ \frac{2\pi i}{P} (h_{e_2 n_2} - h_{e_1 n_1}) \right]\). In fact, the supertrace in \(\mathcal{V}_p\) gets an extra factor of \((-1)^p\) due to its fermionic parity, also the operator \(B_q\) itself gets a factor \((-1)^{(p-1)q}\), because it permutes fermionic screenings. Since \(p\) and \(q\) are coprime, the product of both factors is always equal to \(-1\). Thus we see that

\[
< e_2, n_2 | V^{pq}_{en} | e_1, n_1 > =
\]

\[
- \exp \left[ 2\pi i \frac{P}{Q} (h_{n_2 e_2} - h_{e_1 n_1}) \right] (\delta_{e_2 e_1} + pe \delta_{1-n_2 n_1} + p - e_{e_2 e_1} + pe \delta_{1-n_2 n_1} + p - p)
\]

(200)

which is equivalent to eq. (198) if we put there \(a = 1\).

Now we turn to the case when \(V^{pq}_{en}\) acts on a vacuum representation \(|0\rangle\), i.e. the operator \(V_{e_1 n_1}\) is removed from figure 34. Let us first recall that according to \[1\] the action in \(\mathcal{V}_1\) of the braiding of two neighboring operators \(V_{en}\), one of which is screened (see figure 35), is given by Burau matrices. Thus in the free-fermion representation of the Alexander polynomial developed in \[2\], unscreened operators \(V_{en}\) correspond to the strands carrying a vacuum state, while the
screened ones correspond to a one-fermion state flowing along the strand. Indeed, the screening operators are similar to free fermions: they are mutually local and fermionic, which means that their permutation inside a correlator causes a change in the overall sign.

Let us introduce a normalized matrix $B'_q$:

$$B'_q = \exp \left( -2\pi i \frac{p}{q} h_{pe,pn} \right) B_q$$

(201)

whose property is that it acts trivially in $V_0$. It is easy to see that the action of $B'_q$ in $V_m$ is isomorphic to its action in $\Lambda^m V_1$ ($\Lambda^m$ denotes the $m$th antisymmetric tensor power). Therefore (see [2] for details)

$$\det_{V_1} [I - sB'_q] = \sum_{m=0}^{p} s^m \text{Str}_{\Lambda^m V_1} B'_q$$

(202)

here $s$ is a complex variable, $I$ is an identity matrix and obviously $\text{Str}_{\Lambda^m V_1} = (-1)^m \text{Tr}_{\Lambda^m V_1}$.

The formula (202) requires an important modification. The screening contours of figure 34 do not produce the valid conformal blocks when the operator $V_{i_1,n_1}$ is absent, as in this case. The valid screening contours are those of figure 36: $C_{ij} = C_i - C_j$. This means that $V_1$ should be factored over the one-dimensional subspace generated by the block with the screening contour $C = \sum_{i=1}^{p} C_i$ and the appropriate factorizations should also be made in other spaces $\Lambda^m V_1$. The determinant in eq. (202) should therefore be taken over the factor-space $V'_1$, then we will get the supertraces of $B'_q$ over the valid conformal block spaces $\Lambda^m V'_1$ in the r.h.s. of eq. (202).

A conformal block in $V_1$ with the screening contour $C$ is an eigenvector of $B'_q$ with an eigenvalue equal to 1, so a factorization over that vector is equivalent to the regularization of the braiding determinant for Alexander polynomial considered in [2]. The remaining eigenvalues of $B'_q$ in $V_1$ are the $p$th order roots of unity $\exp(2\pi il/p)$ ($1 \leq l \leq p - 1$) up to a phase factor $\exp(2\pi i qe)$. So by using the simple algebraic relation

$$\prod_{l=1}^{p-1} (x - \exp(2\pi il/p)) = \frac{\prod_{l=0}^{p-1} (x - \exp(2\pi il/p))}{x - 1} = \frac{x^p - 1}{x - 1} = \sum_{l=0}^{p-1} x^l$$

(203)

we conclude that

$$\det_{V'_1} [I - sB'_q] = \sum_{m=0}^{p-1} \left[ \exp(-2\pi i qe) \right]^m$$

(204)

$$\text{Str}_{V'_m} = \exp \left( \frac{2\pi i}{k} [qe(pn - 1/2) - qme] \right)$$

(205)

Formula (205) leads directly to eq. (191) taken at $a = 1$.

Finally we briefly comment on a special case when the operator $V_{en}^{pq}$ acts on a state $| - pe, n_1 >$. The resulting state has its $E$-charge equal to zero. In fact, this state is a linear combination of block representations:

$$V_{en}^{pq} | - pe, n_1 > = \sum_{n_2} A_{n_2} | \hat{n}_2 >$$

(206)
To find the coefficients $A_{n_2}$ we calculate the scalar product $< n_2 | V_{e,n}^p | -p e, n_1 >$, that is, the vertex $V_{e_2,n_2}$ in figure 34 becomes $V_{n_2}$. Note that

$$< n_2 | V_{e,n}^p | -p e, n_1 > = A_{n_2}$$

(207)

The operator $V_{n_2}$ has zero $E$-charge, so it is local with respect to the screening current $J$ and it has no poles in the operator product expansion with that current. Hence the conformal block corresponding to the contour $C'$ in figure 37 is again equal to zero. It is easy to see out the other hand that

$$\int_{C'} J \, dz = 2i \sin \left( \frac{\pi}{k} \epsilon \right) \sum_{i=1}^{p} \int_{C_i} J \, dz$$

(208)

This means that similarly to the previous case, the sum of conformal blocks corresponding to the contours $C_i$ is equal to zero. The repetition of the arguments that lead us to eq. (205) produces the following result

$$V_{e,n}^p | -p e, n_1 > = \exp \left( -2\pi i \frac{p}{q} h_{-p e, n_1} \right) \sum_{m=0}^{p-1} | n_1 + \tilde{m} - m - 1 >$$

(209)

Note that if we assume naively that $| \tilde{n} > = | 0, n > - | 0, n + 1 >$, then eq. (209) is reduced to eq. (204). In fact, the formula (203) requires corrections of order $\epsilon$, but we will not comment on their derivation here.
**Figure Captions**

Figure 1: Schematic representation of a "block representation" in $\text{gl}(1,1)$ where arrows indicate the action of the fermionic generators. By convention the state with lowest $n$ number is bosonic ($b$).

Figure 2: A one dimensional representation ($n$) must be considered as an infinite sum of "pesudo typical" representations ($e = 0, n'$).

Figure 3: To compute $S$ matrix elements we consider the $U(1,1)$ WZW model on a torus with a representation ($en$) running along $\tau$ and insertion of ($e_0n_0$) and its conjugate. A first block can be obtained by turning to free field representation and integrating the screening operator along the indicated dotted contour.

Figure 4, Figure 6: Various possible contours of integration for the screening operator.

Figure 5: Set of contours for computing $\zeta_n$.

Figure 7: By definition the metric $< \rho_i | \rho_j >$ is the partition function of $S^2 \times S^1$ with two parallel Wilson lines carrying respectively $\rho_i$ and $\rho_j$.

Figure 8: A Hopf link made of two linked loops (by convention we take positive crossings).

Figure 9: Schematic representation of an $S^2 \times S^1$ containing a representation ($en/I$). Dotted lines indicate the position of the regulators.

Figure 10: Schematic representation of an $S^2 \times S^1$ containing a representation ($en/II$).

Figure 11: The scalar product of two representations of type $I$. A non vanishing result is obtained as $\epsilon \to 0$.

Figure 12: The scalar product of a representation of type $I$ and a representation of type $II$. Again a non vanishing result is obtained as $\epsilon \to 0$.

Figure 13: The scalar product of two representations of type $II$. A detached loop is obtained, that leads to a vanishing invariant.

Figure 14: The regularized value of $S_{nn'}$ corresponds to a Hopf link with two loops carrying ($n$) and ($n'$) and connected by a pair of dotted lines. It amounts to a single dotted loop, with invariant going as $1/\epsilon$.

Figure 15: A situation with two unlinked circles is equivalent to a situation where a loop carrying a one dimensional representation connects them.

Figure 16: Schematic representation of fusion $I.I, I.II, II.II$.

Figure 17: The factorization formula for two disconnected links can be established if one connects them with a pair of dotted lines.

Figure 18: Verlinde formula expresses the consistency in calculating the invariant of this link.

Figure 19: When $\rho_i$ is a four dimensional indecomposable block, the Hilbert space obtained by cutting the corresponding loop by an $S^2$ has dimension greater than one, and the factorization formula cannot be used.

Figure 20: A double of the trefoil.

Figure 21: Double where stands carry the same representation and have opposite orientations.

Figure 22: A twisted double of the trefoil.
Figure 23: A three cable of the trefoil.
Figure 24: Surgery on the indicated system of loops produces the manifold $X_h \times S^1$.
Figure 25: Surgery on this link produces $S^1 \times S^1 \times S^1$.
Figure 26: After surgery on loop 1, $S^2 \times S^1$ is obtained.
Figure 27: The partition function of figure 26 is evaluated by factorization and leads to considering figure 27.
Figure 28: The case where $\rho = (\hat{n})$ is studied by considering instead a pair of Wilson lines with opposite orientations that carry $(en/I)$.
Figure 29: Cap 1 extracts the conformal block $|1\rangle$ without logarithms.
Figure 30: Cap 2 extracts a conformal block $|2\rangle$ with logarithms.
Figure 31, 32: Link configurations used in the computation of $TrM$.
Figure 33: A section $S^2$ of the manifold $S^2 \times S^1$ obtained by gluing two solid tori, one of which has a Wilson line with representation $(e_2n_2)$ inside, while the other has a Wilson line with representation $(e_1n_1)$ inside and a torus link $(p,q)$ with representation $(en)$ on its surface.
Figure 34: A typical conformal block in the free field representation, where dotted lines represent screening contours.
Figure 35: Braiding of two vertex operators, one of which is screened, is represented by the Burau matrix $[1]$.
Figure 36: When the operator $V_{e_1n_1}$ is absent, the correct screening contours are the $C_{ij} = C_i - C_j$.
Figure 37: The conformal block corresponding to the contour $C'$ vanishes.
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