Leading Logarithms in Field Theory

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Abstract

We consider the Sudakov form factor in effective theories and we show that one can derive correctly the double logarithms of the original, high-energy, theory. We show that in effective theories it is possible to separate explicitly soft and hard dynamics being these two regimes related to velocity conserving and to velocity changing operators respectively. A new effective theory is sketched which extracts the leading collinear singularities of the full theory amplitudes. Finally, we show how all leading logarithmic effects in field theory can be obtained by means of simple effective theories, where they correspond to a renormalization effect.
1 Introduction

The treatment of logarithmic singularities in local field theories is well understood. Logarithmic ultraviolet divergencies are treated with the renormalization procedure. In particular regions within the range of the momenta of the particles, however, additional logarithmic corrections do appear in the perturbative expansion. These additional infrared logarithms come from the region of small-momenta of the amplitudes and have therefore a dynamical origin. The appearance of these logarithms has been analyzed in detail long ago [1] in the context of perturbative evaluation of form factor in Quantum Electrodynamics. These logarithms are the usually called Sudakov double logarithms.

In this work we propose a new way to look at the Sudakov double logarithms in field theory by considering effective theories. Effective theories are a powerful tool to describe complicated dynamical processes. They are simpler than the original theories and give the same predictions in a given energy range. In general, to reproduce the whole dynamics of the original theory it is necessary to use a tower of effective theories, covering interval by interval the entire energy scale. In this paper we extremize this idea. We use two effective theories: the Heavy Quark Effective Theory (HQET) [2] and the Large Energy Effective Theory (LEET) [3].

The paper is organized as follows. In sec.2 we show that all double logarithms of Sudakov form factors can be extracted by a combined use of the Heavy Quark Effective Theory and the Large Energy Effective Theory. We also note that the Sudakov double logarithms in the effective theory do not represent a dynamical effect but are simply a renormalization. They are associated with the renormalization constants of operators of dimension three which change the velocity of the effective quark. These operators have the form

\[ O(x) = h^\dagger_{\nu}(x) h_{\nu}(x). \]  

We consider in sec.3 velocity changing operators of dimension four which, by covariance, have the form

\[ O'(x) = h^\dagger_{\nu}(x) D_{\mu} h_{\nu}(x). \]  

These operators are related to hard QCD interactions, while the operators in eq.(1) are related to hard external interactions such as \(\gamma\)-exchange or weak decays. The effective theory provides a natural and explicit separation of soft and hard dynamics, associated respectively with velocity conserving and velocity changing interactions.

In sec.4 we describe a new effective theory which allows the extraction of leading collinear logarithms, which we call Collinear Effective Field Theory. It is a variant of the Large Energy Effective Theory, in which spin fluctuations are taken into account.

In sec.5 we show that all leading logarithmic effects in field theory can be extracted by means of simple effective theories where they correspond to a renormalization effect.

Sec.6 contains the conclusions and an outlook at future developments.

In appendix A we evaluate the Altarelli-Parisi kernel \(P_{qq}\) in the Collinear Effective Field Theory.
2 Sudakov Form Factors

In this section we discuss the Sudakov form factors in the effective theories and their relation with those ones in the full theory. We consider the (HQET) \[^2\] and the (LEET) \[^3\] effective theory. The propagator of the HQET, which we may call *massive eikonal*, is given by

\[\text{i}S_v(k) = \frac{1 + \hat{v}}{2} \frac{i}{v \cdot k + i\epsilon}, \quad v^2 = 1,\] (3)

while the propagator of the LEET, which may be called *massless eikonal*, is given by

\[\text{i}S_n(k) = \frac{\hat{n}}{2} \frac{i}{n \cdot k + i\epsilon}, \quad n^2 = 0\] (4)

(see sec.\[^4\] for a derivation). The interaction vertices are derived noting that

\[-igt_a \frac{1 + \hat{v}}{2} \gamma_\mu \frac{1 + \hat{v}}{2} = -igt_a v_\mu \frac{1 + \hat{v}}{2},\] (5)

\[-igt_a \frac{\hat{n}}{2} \gamma_\mu \frac{\hat{n}}{2} = -igt_a n_\mu \frac{\hat{n}}{2}.\] (6)

It is therefore possible, as far as QCD interactions are concerned, to omit the spin structure of the effective propagators,

\[\text{i}S_v(k) \rightarrow \frac{i}{v \cdot k + i\epsilon},\] (7)

\[\text{i}S_n(k) \rightarrow \frac{i}{n \cdot k + i\epsilon},\] (8)

and use simplified vertices of the form

\[V_v = -igt_a v_\mu,\] (9)

\[V_n = -igt_a n_\mu.\] (10)

In general, the massive propagator extracts the infrared singularity of the amplitudes, while the massless propagator extracts the leading infrared times collinear singularity. The difference originates from the fact that in the massless case the ‘scale’ \(v^2\) is missing; we send \(v^2 \rightarrow 0\), so the \(\log v^2\), vanishing in the massive theory, becomes singular in the massless one.

There are three possible effective theory form factors, for:

(a) light to light transition \((ll)\), \(n^2 = 0, n'^2 = 0\);

(b) heavy to heavy transition \((hh)\), \(v^2 = 1, v'^2 = 1\);

(c) heavy to light transition \((hl)\), \(v^2 = 1, n^2 = 0\).

The case of the light to heavy transition is identical to \((c)\).

In general, the effective theory form factors are logarithmically divergent in the ultraviolet region due to the absence of quadratic terms in energy denominators. This has to be contrasted with the case of the full theory form factors, which, for example in the case of the vector or the axial vector current are ultraviolet finite because of conservation or partial conservation of the current. Even in the cases in which the full form factors are ultraviolet divergent, the UV
divergence has a complete different meaning in the two theories [3, 4], as will be clear from
the discussion. We work throughout the paper at the one-loop order.

Ultraviolet divergences of effective theory amplitudes are regulated with Dimensional Regularization (DR) [3], while soft divergences are regulated with off-shell external states or with
a non-vanishing gluon mass \( \lambda > 0 \). This implies that the unit of mass \( \mu \) has to be considered
as an ultraviolet scale; it would be replaced by the ultraviolet cut-off \( \Lambda_{UV} \) if we were dealing
with the bare effective theory: \( \mu \sim \Lambda_{UV} \).

For simplicity’s sake, let us consider a vector form factor in the full theory; we could
consider an axial vector current as well. In this way we do not need a regulator for the
ultraviolet region. We regulate soft divergences of full theory for m factors with
\( DR \). \( \mu \) therefore has to be considered as an infrared scale, such as a light parton mass or the virtuality
of external states. We will see that this complementary use of \( DR \) does not cause any confusion
once the physical meaning of the regulators is understood. Again for simplicity let us consider
the form factors in the space-like region; in this way we avoid imaginary contributions to form
factors (the well known \( i\pi \) terms), which are irrelevant in this contest.

\[ 2.1 \quad \text{Massless Case} \]

Let us start with the simplest case, the on-shell form factor of a massless
quark [7, 8]:

\[
F(Q^2) = \gamma_\mu \left[ 1 - 2 \frac{C_F \alpha_S}{4\pi} \left( \frac{Q^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \frac{1}{1-2\epsilon} \left( \frac{1}{(-\epsilon)^2} + \frac{1}{2-\epsilon} + 1 \right) \right]
\]

\[
\approx \gamma_\mu \left[ 1 - 2 \frac{C_F \alpha_S}{4\pi} \left( \frac{Q^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \left( \frac{1}{\epsilon^2} + \frac{3}{2} \frac{1}{\epsilon} \right) + \ldots \right]
\]

\[
\rightarrow \gamma_\mu \left[ 1 - \frac{C_F \alpha_S}{4\pi} \left( \log^2 \frac{Q^2}{4\pi \mu^2} - 3 \log \frac{Q^2}{4\pi \mu^2} \right) \right] \quad \text{(11)}
\]

where

\[
q = p' - p \quad \text{(12)}
\]

is the momentum transfer and

\[
Q^2 = -q^2 = 2p \cdot p' > 0 \quad \text{(13)}
\]

is the virtual mass squared (the hard scale of the process). \( \epsilon = 2-n/2 \), and \( n \) is the space-time
dimension. \( C_F = (N^2 - 1)/(2N) = 4/3 \) in QCD is the Casimir operator in the fundamental
representation of \( SU(N) \). We see the appearance of \( 1/(-\epsilon) \) poles, of infrared nature.

In the last line of eq.(11) only the terms dependent on the momentum transfer \( Q^2 \), which
are finite for \( \epsilon \to 0 \), have been kept.

It is interesting to look also at the expression for the form factor when the \( \hat{k} \) terms \( (k \) is
the gluon momentum) in the numerators of Dirac propagators have been omitted:

\[
F'(Q^2) = \gamma_\mu \left[ 1 - 2 \frac{C_F \alpha_S}{4\pi} \left( \frac{Q^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \frac{1}{(-\epsilon)^2} \right]
\]

\footnote{Technically, it is easier to regulate soft divergences of massless lines with a virtuality of the external state, while massive lines are most conveniently regulated with a gluon mass.}
\[ \rightarrow \gamma_\mu \left[ 1 - \frac{C_F \alpha_S}{4\pi} \log^2 \frac{Q^2}{4\pi \mu^2} \right]. \quad (14) \]

We see that the leading double $1/\epsilon^2$ pole has the same coefficient in $F(Q^2)$ and $F'(Q^2)$, while the simple $1/\epsilon$ pole has a different coefficient in the two cases: it is $3/2$ in $F$ and $0$ in $F'$. The blocks of $\Gamma$-functions are identical. As a consequence, the double collinear logarithm, $\log^2 Q^2/\mu^2$, is the same in the two cases, while the single collinear logarithm, $\log Q^2/\mu^2$, has a different coefficient. This effect occurs also when one considers the form factor of a particle with a different spin like the gluon. With respect to the quark form factor the structure of the leading double logarithms remains unchanged provided that $C_F \rightarrow C_A$. On the contrary the coefficient of the single logarithms, of collinear origin, changes reflecting the presence of a particle with a different spin \[9\]. We conclude that the single collinear logarithm is sensitive to the spin fluctuations: we will see that the same phenomenon occurs in the LEET and this is the key point in the construction of an effective theory to extract collinear logarithms (see sec.[4]).

Let us consider now the effective vertex \((a)\), in which the quarks are taken massless:

\[
\begin{align*}
F_{ll} &= 1 - 2 \frac{C_F \alpha_S}{4\pi} \left(\frac{4\pi \mu^2 n \cdot n'}{2 n \cdot p n' \cdot p'}\right)^\epsilon \Gamma(1 + \epsilon) \Gamma(1 - \epsilon) \frac{1}{\epsilon^2} \\
&\rightarrow 1 - \frac{C_F \alpha_S}{4\pi} \log^2 \left(\frac{2 \mu n \cdot \mu n'}{n \cdot p n' \cdot p'}\right) + O\left(\log \frac{2 \mu n \cdot \mu n'}{n \cdot p n' \cdot p'}\right) \quad (15)
\end{align*}
\]

Infrared and collinear divergences are regulated by taking the effective quarks off-shell, $n \cdot p \neq 0$, $n' \cdot p' \neq 0$. There is a double $1/\epsilon^2$ pole, of ultraviolet nature.

Note that the effective diagram does not contain any scale (unlike the full theory vertex which depends on $Q^2$), so the theory can develop a single mass ratio and consequently a single kind of logarithm.

The double logarithm of the full theory, $\log^2 Q^2/\mu^2$, is reproduced by the effective vertex \((a)\),

\[ \log^2 \frac{Q^2}{\mu^2} \Leftrightarrow \log^2 \left(\frac{2 \mu n \cdot \mu n'}{n \cdot p n' \cdot p'}\right). \quad (16) \]

We interpret the quark momenta as

\[ p = \mu n, \quad p' = \mu n', \quad (17) \]

and we match the infrared regulator $\mu$ of the full theory amplitude with the off-shellness of the effective theory lines

\[ n \cdot p, \quad n' \cdot p' \sim \mu \quad (18) \]

To sum up, the massless form factor of the full theory is correctly reproduced to Double Logarithmic Approximation (DLA) by the effective vertex \((a)\).

It is interesting to note that the blocks of $\Gamma$-functions entering in $F$ (or $F'$) and in $F_{ll}$ do coincide up to first order included, i.e. up to subleading terms:

\[ \frac{\Gamma(1 + \epsilon) \Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \simeq \Gamma(1 + \epsilon)^2 \Gamma(1 - \epsilon) \simeq \Gamma(1 + \epsilon) = 1 - \gamma_E \epsilon + O(\epsilon^2) \quad (19) \]

where $\gamma_E \simeq 0.57$ is the Euler constant.
2.2 Massive Case

Let us consider now the on-shell form factor of a massive quark [10]:

\[
F(Q^2) = \gamma_\mu - \frac{C_F \alpha_s}{4\pi} \left[ \gamma_\mu \frac{1 + \xi^2}{1 - \xi} \left( \log^2 \frac{1}{\xi} - \frac{\pi^2}{3} - 4Li_2(-\xi) + 4 \log \frac{1}{\xi} \log(1 + \xi) \right) 
- 3 \frac{(1 + 2/3 \xi + \xi^2)}{1 - \xi^2} \log \frac{1}{\xi} + 2 \left( \frac{1 + \xi^2}{1 - \xi} \log \frac{1}{\xi} - 1 \right) \log \frac{m^2}{\lambda^2} \right] + 2 \sigma_{\mu\nu} q_\nu \frac{\xi}{m} \log \frac{1}{\xi} \right] \tag{20}
\]

where \( Li_2(z) \) is the dilogarithm function defined by [11]

\[
Li_2(z) = - \int_0^z \frac{\log(1-x)}{x} \, dx = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \tag{21}
\]

and \( \xi \) is defined by the relation

\[
\frac{(1 - \xi)^2}{\xi} = \frac{Q^2}{m^2}. \tag{22}
\]

We have regulated (for convenience) the infrared divergence with a gluon mass \( \lambda > 0 \). Radiative corrections generate structures not present in the tree level form factor (the \( \sigma_{\mu\nu} \)-term), unlike to what happens in the massless case.

The asymptotic behaviour of the form factor for

\[
Q^2 \to \infty, \tag{23}
\]

since

\[
\frac{1}{\xi} \sim \frac{Q^2}{m^2}, \tag{24}
\]

is obtained taking the limit

\[
\xi \to 0^+. \tag{25}
\]

We select terms according to the degree of singularity in this limit. The leading terms are the double logarithms, \( \log^2 1/\xi \), the sub-leading terms are the single logarithms, \( \log 1/\xi \), the further sub-leading terms are constants. Powers of \( \xi \sim m^2/Q^2 \) in this framework are exponentially small contributions (the so-called power-suppressed corrections), because

\[
\xi = e^{-\log 1/\xi}, \quad \text{and} \quad \log \frac{1}{\xi} \to \infty. \tag{26}
\]

We have:

\[
F(Q^2) = \gamma_\mu \left[ 1 - \frac{C_F \alpha_s}{4\pi} \left( \log^2 \frac{Q^2}{m^2} + 2 \log \frac{Q^2}{m^2} \log \frac{m^2}{\lambda^2} + 
- 3 \log \frac{Q^2}{m^2} - 2 \log \frac{m^2}{\lambda^2} - \frac{\pi^2}{3} + (\text{pow. corr.}) \right) \right] \tag{27}
\]

where, for completeness, also the subleading logs and the finite term have been written. The magnetic form factor is suppressed by a power of \( \xi \) and therefore can be neglected to logarithmic accuracy.
The two double logarithms in eq. (27) can be extracted by means of two effective theories. The ‘semihard’ double logarithm, \( \log^2 \frac{Q^2}{m^2} \) (it is related to loop momenta \( k \) in the range \( m^2 < k^2 < Q^2 \)), is reproduced by the effective vertex \((a)\) in which the quarks are taken massless,

\[
\log^2 \frac{Q^2}{m^2} \iff \log^2 \left( \frac{2 \mu n \cdot \mu n'}{n \cdot p' n' \cdot p'} \right). \tag{28}
\]

We interpret the quark momenta as for the massless case before,

\[
p = \mu n, \quad p' = \mu n', \tag{29}
\]

and the quark mass \( m \) as the infrared cutoff in the effective theory \((a)\)

\[
n \cdot p, \quad n' \cdot p' \sim m. \tag{30}
\]

The ‘semisoft’ double logarithm, \( \log \frac{Q^2}{m^2} \log \frac{m^2}{\lambda^2} \), is reproduced by the vertex \((b)\), in which the quark is treated as an infinite mass particle \([12, 13]\):

\[
F_{hh} = 1 - \frac{C_F \alpha_S}{4\pi} \left( \frac{4\pi \mu^2}{\lambda^2} \right)^\epsilon \Gamma(1 + \epsilon) \frac{1}{\epsilon} \left( v \cdot v' r(v \cdot v') - 1 \right)
\]

\[
\rightarrow 1 - \frac{C_F \alpha_S}{4\pi} \left(2(v \cdot v' r(v \cdot v') - 1) \log \frac{\mu^2}{\lambda^2} \right)
\]

\[
\simeq 1 - \frac{C_F \alpha_S}{4\pi} \left[2 \log(2v \cdot v') \log \frac{\mu^2}{\lambda^2} - 2 \log \frac{\mu^2}{\lambda^2} + \ldots \right] \tag{31}
\]

where \( r(x) = 1/\sqrt{x^2 - 1} \log(x + \sqrt{x^2 - 1}) \) is the usually called cusp anomalous dimension \([12]\) or velocity dependent anomalous dimension \([13]\). We have regulated the infrared divergence with a gluon mass \( \lambda \), as in the full amplitude. There is a \textit{simple} \( 1/\epsilon \) pole, of ultraviolet nature, unlike the case \((a)\), because collinear singularities are absent. We have the correspondence:

\[
\log \frac{Q^2}{m^2} \log \frac{m^2}{\lambda^2} \iff \log(2v \cdot v') \log \frac{\mu^2}{\lambda^2} \tag{32}
\]

According to eq. (13) we have indeed:

\[
\frac{Q^2}{m^2} \sim 2v \cdot v' \tag{33}
\]

We identify the renormalization scale \( \mu \) of the effective vertex (or, equivalently, the ultraviolet cutoff \( \Lambda_{UV} \) of the bare effective theory) with the mass in the original vertex:

\[
\mu^2 \sim m^2. \tag{34}
\]

Therefore we see that the whole form factor can be reproduced in DLA by means of two effective theories, each one describing a different dynamical range. It is to remark that the quark mass \( m \) acts as an \textit{infrared} cutoff for the \textit{massless} effective vertex \((a)\),

\[
k^2 < m^2 \quad (HQET), \tag{35}
\]

while it acts as an \textit{ultraviolet} cutoff for the \textit{massive} effective vertex \((b)\),

\[
k^2 > m^2 \quad (LEET). \tag{36}
\]
We see a realization of the idea of the tower of effective theories, each one describing a different energy range, discussed in the introduction: the HQET describes dynamics below the quark mass, while the LEET describes dynamics above the quark mass.

Let us comment about subleading one-loop effects. The single infrared logarithm, \( \log \frac{\mu^2}{\lambda^2} \), is also reproduced by the effective vertex (b) \([12]\):

\[
\log \frac{m^2}{\lambda^2} \iff \log \frac{\mu^2}{\lambda^2}.
\] (37)

The extraction of the other one-loop subleading effect, the single collinear logarithm, \( \log \frac{Q^2}{m^2} \), cannot be performed with the LEET or the HQET, i.e. with the propagators in eq.(3) or (4). A new, more complicated, effective theory is required (see sec.4). We end up this section by observing that the double and the single collinear logarithms are the same in the massless and massive full theory form factors,

\[
\log^2 \frac{Q^2}{m^2} - 3 \log \frac{Q^2}{m^2} \iff \log^2 \frac{Q^2}{4\pi\mu^2} - 3 \log \frac{Q^2}{4\pi\mu^2},
\] (38)

provided one makes the identification \( 4\pi\mu^2 = m^2 \).

### 2.3 Massive to Massless Case

Let us consider now the form factor for the transition of a quark with mass \( m \) and momentum \( p \) with \( p^2 = m^2 \) into a massless quark with momentum \( p' \) with \( p'^2 = 0 \) \([14]\). Following ref.\([8]\) we have:

\[
F(Q^2) = \gamma_\mu - \frac{C_F\alpha_S}{4\pi} \left( \frac{M^2}{4\pi\mu^2} \right)^{-\epsilon} \Gamma(1+\epsilon) \left[ \gamma_\mu \left( \frac{1}{1-2\epsilon}(1-\xi)F(1+\epsilon,1,1-\epsilon,\xi) \right) \right.
\]

\[
\left. + \frac{1-2\epsilon}{(1-\epsilon)} F(\epsilon,1,2-\epsilon,\xi) \frac{1}{-\epsilon} - \frac{3}{2} \frac{1-2/3}{1-2\epsilon} \frac{1}{-\epsilon} \right] - 2 \frac{1}{(1-\epsilon)(1-2\epsilon)} F(1+\epsilon,1,2-\epsilon,\xi) \left[ \gamma_\mu \left( \frac{1}{1-2\epsilon}(1-\xi)F(1+\epsilon,1,1-\epsilon,\xi) \right) + \frac{p_\mu}{m} \frac{-1}{1-\epsilon/2} F(1+\epsilon,1,3-\epsilon,\xi) \right.
\]

\[
\left. + \frac{p'_\mu}{m} \left( \frac{1}{(1-\epsilon)(1-\epsilon/2)} F(1+\epsilon,2,3-\epsilon,\xi) \right) \right] + \frac{1}{4} \frac{1}{(1-\epsilon)(1-2\epsilon)} F(1+\epsilon,1,2-\epsilon,\xi) \right]
\] (39)

where

\[
\xi = \frac{q^2}{m^2} = 1 - \frac{2p\cdot p'}{m^2}
\] (40)

and \( F(a,b,c,\xi) = _2F_1(a,b,c;\xi) \) is the hypergeometric function, defined by the series \([11]\):

\[
_2F_1(a,b,c;\xi) = 1 + \frac{a}{1\ c} z + \frac{a(a+1)}{1\ 2\ c(c+1)} z^2 + \ldots
\] (41)

\[\text{In the massive case there are power corrections of the form } (m^2/Q^2)^n, \text{ which are absent in the massless case.}\]
DR regulates the collinear divergence related to the massless line and the infrared divergences, i.e. \( \mu \) can be thought of as the light quark mass or the gluon mass. Radiative corrections produce new structures with respect to the tree level (the \( p_\mu \) and \( p'_\mu \) terms), as it happens in the massive case.

There are two different dynamical situations, according to the condition

\[
(\alpha) \quad p \cdot p' \gg m^2; \\
(\beta) \quad p \cdot p' \ll m^2.
\]

(42)
(43)

To understand the physical meaning of these inequalities, let us take the massive quark at rest, \( p = (m, \vec{0}) \). Condition (\( \alpha \)) means \( E \gg m \), i.e. a light quark energy \( E \) much larger than the heavy quark mass \( m \). This means that a large energy \( E - m = E(1 - \epsilon) \), \( \epsilon \ll 1 \), is ‘pumped’ inside the system by the current. In case (\( \alpha \)) we have:

\[
F(Q^2) \simeq \gamma_\mu \left[ 1 - \frac{C_F \alpha_S}{4\pi} \left( \log^2 \frac{2p \cdot p'}{4\pi \mu^2} - \frac{1}{2} \log^2 \frac{m^2}{4\pi \mu^2} + \ldots \right) \right]
\]

(44)

Due to the increased difficulty, let us limit ourselves to study the dependence of the form factor on the 4-velocities, i.e. we drop the logarithms squared of the mass ratio \( m/\mu \) (these, together with subleading two-loop effects, will be treated in a future work [15]):

\[
F(Q^2) = \gamma_\mu \left[ 1 - \frac{C_F \alpha_S}{4\pi} \log^2 2v \cdot n + (mass \ logs) \right]
\]

(45)

Since the momentum transfer \( Q^2 = 2p \cdot p' - m^2 \) is much larger than the heavy quark mass squared \( m^2 \),

\[
Q^2 \simeq 2p \cdot p' \gg m^2,
\]

(46)

we can consider both quarks massless. The double logarithm is reproduced by the effective vertex (\( a \)) in eq.(15),

\[
\log^2 2v \cdot n \iff \log^2 2n \cdot n',
\]

(47)

with the (natural) identification

\[
v \cdot n \sim n \cdot n'.
\]

(48)

Condition (\( \beta \)) means on the contrary \( E \ll m \), i.e. most of the rest energy of the heavy quark \( m - E = m(1 - \epsilon) \), \( \epsilon \ll 1 \), is absorbed by the external current. We have in this case:

\[
F(Q^2) \simeq \gamma_\mu \left[ 1 - \frac{C_F \alpha_S}{4\pi} \left( \log^2 \frac{2p \cdot p'}{4\pi \mu^2} + \log^2 \frac{2p \cdot p'}{m^2} - \frac{1}{2} \log^2 \frac{m^2}{4\pi \mu^2} \right) \right]
\]

\[
= \gamma_\mu \left[ 1 - \frac{C_F \alpha_S}{4\pi} 2 \log^2 2v \cdot n + (mass \ logs) \right]
\]

(49)

Note the additional factor two of case (\( \beta \)) with respect to case (\( \alpha \)) in the coefficient of the logarithm squared of the 4-velocities product.

The double logarithm in the last line of eq.(15) can be extracted by means of the effective vertex (\( c \)), in which the initial quark is treated as an infinite mass quark while the final quark is taken as an effective massless quark:

\[
F_{hl} = 1 - \frac{C_F \alpha_S}{4\pi} \left( \frac{4\pi \mu^2 (v \cdot n)^2}{n \cdot p^2} \right)^\epsilon \Gamma(1 + \epsilon) \Gamma(1 + 2\epsilon) \Gamma(1 - 2\epsilon) \frac{1}{\epsilon^2}
\]

9
\[
\rightarrow \quad 1 - \frac{C_F\alpha_S}{4\pi} \frac{1}{2} \log^2 \left( \frac{\mu^2(v \cdot n)^2}{n \cdot p^2} \right) \\
= \quad 1 - \frac{C_F\alpha_S}{4\pi} 2 \log^2 v \cdot n + \text{(mass logs)} \quad (50)
\]

There is a double, ultraviolet $1/\epsilon^2$ pole coming from the product of the infrared singularities with the collinear singularity related to the massless line. We have regulated the soft divergences taking only the massless quark off-shell, $n \cdot p \neq 0$ (this is indeed sufficient to render the integral finite for $\epsilon < 0$). The power of $\mu$ (which is 2) does not match the number of velocities (which is 4), so it is impossible to associate a power of $\mu$ with each 4-velocity, $n$ and $v$, as we did in case (a).

As we said, we considered for illustrative purposes the case of a vector current, i.e. a conserved current. In the case of a general form factor, the ultraviolet divergence also can be extracted with an effective theory [5]: we just set to zero all the masses and external momenta in the full diagrams and we integrate from the $UV$ cutoff $\Lambda_{UV}$ down to the largest scale in the diagram (it can be either a mass or a momentum transfer).

### 2.4 Dynamical Logarithms and Renormalization Logarithms

The effective lagrangian can be written in a covariant form introducing a field for each velocity as [2]

\[
\mathcal{L}(x) = \int \frac{d^3v}{2v_0} h_v^\dagger(x) iv \cdot D h_v(x) + \int \frac{d^3n}{2n_0} h_n^\dagger(x) in \cdot D h_n(x). \quad (51)
\]

$\mathcal{L}(x)$ contains all the operators diagonal in the velocity with dimension $d$ less or equal to the canonical one $n = 4$ (the field dimensions are $d_h = 3/2$ and $d_A = 1$ and mass operators of the form $h_v^\dagger h_v$, $h_n^\dagger h_n$ can be set equal to zero [13]).

All the operators $O(x)$ with dimension

\[
d = 3 \quad (52)
\]

can be written in a covariant form as

\[
O(x) = \sum_{i,j}^2 \int \frac{d^3v_i}{2v_0} \frac{d^3v_j}{2v_0} c(v_i \cdot v_j) h_{v_i}^\dagger(x) h_{v_j}(x), \quad (53)
\]

where we defined for notational simplicity $v_1 = v$, $v_2 = n$. The effective theory form factors $(a)$, $(b)$ and $(c)$ correspond formally to the renormalization constants $c(v_i \cdot v_j)$ of the operators in eqs. (53) [12].

If we define the renormalization constants as

\[
c_R(v \cdot v^\prime) = \frac{c_B(v \cdot v^\prime)}{Z_{hh}}, \\
c_R(n \cdot n^\prime) = \frac{c_B(n \cdot n^\prime)}{Z_{ll}}, \\
c_R(v \cdot n) = \frac{c_B(v \cdot n)}{Z_{hl}}, \quad (54)
\]

\[
\text{(mass logs)}
\]
we have in the $MS$ scheme (according to eqs. (15), (31) and (50)):

\[
Z_{hh} = 1 - 2 C_F \frac{\alpha_S}{4\pi} \frac{1}{\epsilon} (v \cdot v' r(v \cdot v') - 1),
\]

\[
Z_{ll} = 1 - 2 C_F \frac{\alpha_S}{4\pi} \frac{1}{\epsilon^2} + \ldots,
\]

\[
Z_{hl} = 1 - C_F \alpha_S \frac{1}{4\pi} \frac{1}{\epsilon^2} + \ldots.
\]

Note that the light-to-light and the heavy-to-light renormalization constants contain a double ultraviolet logarithm at the one-loop level, a novel feature in field theory (in $DR$ we have that $1/\epsilon_{UV} \rightarrow \log \Lambda^2_{UV}/\mu^2$, where $\mu$ is the renormalization point). The coefficient of the double pole in the heavy-to-light vertex is one-half of that one in the light-to-light vertex because the collinear singularity is not coupled to the heavy leg.

Therefore we have that the double logarithms of the full theory, dynamical logarithms, correspond to renormalization logarithms of local operators of dimension three in the effective theory.

We argue that subleading terms can be extracted by means of a combined use of the $HQET$ and a new effective theory (the collinear effective field theory ($CEF T$)) to be discussed in the next section. The justification of this conjecture is the following. As it is well known, subleading effects involve both the single infrared logarithm, $\alpha_S \log m^2/\lambda^2$, and the single collinear logarithm $\alpha_S \log Q^2/m^2$ [17]. The former is well reproduced by the $HQET$ as we have seen in sec.2.2 [12], while the latter is well reproduced by the $CEF T$, as we show explicitly in a particular case in the appendix A.

### 3 Hard and Soft Dynamics in the Effective Theory

It is natural to consider all the operators of dimension less or equal to the canonical one $n = 4$. The operators of dimension three have already been considered in the previous section: they are related to hard external interactions (such as $\gamma$-exchange or weak decays). The gauge invariant operators of dimension

\[
d = 4,
\]

can be written in a covariant form as

\[
O(x) = \sum_{i,j}^{1,2} \int \frac{d^3v_i}{2v_i^0} \frac{d^3v_j}{2v_j^0} d_\mu(v_i, v_j) h^i_{v_i}(x) D^\mu h_{v_j}(x)
\]

These operators have a clear physical meaning: they correspond to ‘cusps’ in space-time associated with ‘hard’ gluon emission or absorption, i.e. a gluon with momentum

\[
k \sim m, \ E.
\]

For example, the operator $h^i_{v_i} A_\mu h_{v_j}$ is associated with the absorption of a gluon with momentum $k = m(v' - v)$.

The effective theory provides a natural separation of soft and hard dynamics: these two regimes are related with the velocity conserving interactions in eq.(51) and with the velocity changing operators in eq.(57) respectively. In other words, an arbitrary $QCD$ diagram can be decomposed in effective theory diagrams, in which soft interactions are described by the
vertices in eq. (51), while hard interactions are described by the external operators in eq. (57). This decomposition requires the introduction of a scale $\mu$ separating hard momenta from soft momenta, similar in spirit to the factorization scale of Wilson's operator product expansion [18]. In general a QCD process contains a few hard subprocesses while it contains an arbitrary number of soft subprocesses. To soft processes, factorized with respect to the hard ones, then the resummation procedure can be also applied in order to obtain reliable perturbative results [19].

This difference is encoded in the effective theory because the interaction lagrangian, inserted any number of times, describes soft gluons, while external operators, inserted just a few times, describe hard effects.

4 An Effective Theory for Collinear Logarithms

We have seen in sec. 3 that the Sudakov region can be extracted with the HQET and the LEET. As it is well known, DLA Sudakov form factors are related to both soft and quasi-collinear emissions [20]:

$$\theta \ll 1, \quad k \ll E,$$

(59)

where $\theta$ is the gluon emission angle and $E$ and $k$ are the energies of the parent quark and the gluon respectively.

Collinear singularities, related to configurations with

$$\theta \ll 1, \quad k \sim E,$$

(60)

can also be extracted by means of an effective theory, a new effective theory which we call Collinear Effective Field Theory (CEFT), and whose elements are sketched in the following lines. In the propagator of a massless quark,

$$iS_F(P) = i \frac{\hat{P}}{P^2 + i\epsilon},$$

(61)

we separate the basic momentum from the fluctuation according to

$$P = En + k,$$

(62)

so that

$$iS_F(En + k) = \left(\frac{\hat{n}}{2} + \frac{\hat{k}}{2E}\right) \frac{i}{n \cdot k + k^2/2E + i\epsilon}.$$ 

(63)

$n^\mu$ is a light-like vector, $n^2 = 0$, normalized in such a way that $v \cdot n = 1$, where $v$ is a reference time-like vector, $v^2 = 1$, $v_0 > 0$. $E$ is the primordial (large) energy of the quark. There are two terms of order $1/E$:

$$\frac{k^2}{2E} \quad and \quad \frac{\hat{k}}{2E}.$$ 

(64)

They are subleading in the limit $E \to \infty$ and of the same order in $1/E$ but, as we are going to show, they have different dynamical meanings. The term $k^2/(2E)$, in the denominator,
describes both antiquark excitations and quark transverse momentum dynamics. We may write
\[ k^2 = (k_0 - k_z)(k_0 + k_z) - \vec{k}_T^2. \] (65)

Without any loss of generality we may take \( n = (1; 0, 0, 1) \). We have, close to the quark mass-shell, or equivalently at lowest order in \( 1/E \), \( k_0 \sim k_z \). If we neglect antiquark creation, a process very far from the quark mass-shell, we may simplify dynamics according to
\[ k^2 \sim -\vec{k}_T^2. \] (66)

Keeping the term \( k^2/(2E) \) in the approximation (66), we account for a transverse motion (relative to \( n \)) of the effective quark. The transverse motion is important, for example, in bound state dynamics [21].

The term \( k^2 \) is small both in the infrared region and in the collinear region. In the infrared region all the components \( k_\mu \) are small, so the square is small, while in the collinear region the individual components \( k_\mu \) are large but \( k_0 \sim |\vec{k}| \), so that \( k^2 \sim 0 \). Therefore we can neglect the term \( k^2/(2E) \) completely for the extraction of the infrared as well as the collinear region. This is a huge simplification of the dynamics: the propagator does not involve a sum over all the possible trajectories of the particle (the Feynman’s sum over histories), but only the classical path. Just for this reason, for example, the propagator of the \( LEET \) can be computed in closed form in an arbitrary gauge field:
\[ iS_n(x | 0) = \frac{\hat{n}}{2} \int d\tau \delta^{(4)}(x - n\tau) \int_0^\tau A_\mu dx^\mu. \] (67)

It is remarkable that both the leading infrared singularities and the leading collinear singularities are related to such a simple semiclassical dynamics.

Let us now turn to the correction \( \hat{k}/(2E) \) related to spin fluctuations of the quark. This term can be neglected in the infrared region in which all the components \( k_\mu \) are small, but cannot be neglected in the collinear region, in which the components are large and are projected over an arbitrary direction \( q_\mu \).

In the case of a massive quark the decomposition of the momentum \( P \) is done according to the formula
\[ P = mv + k, \] (68)
so we end up with a propagator of the form
\[ iS_F(mv + k) = \left( \frac{1 + \hat{v}}{2} + \frac{\hat{k}}{2m} \right) \frac{i}{v \cdot k + k^2/2m + i\epsilon}. \]
\[ = \frac{1 + \hat{v}}{2} \frac{i}{v \cdot k + i\epsilon} + O\left( \frac{1}{m} \right). \] (69)

In this case the collinear singularity is absent, so this propagator can extract only infrared logarithms. It is therefore consistent to neglect both kinds of \( 1/m \) corrections as far as logarithmic effects are concerned. To summarize, an effective propagator for the collinear region is derived neglecting only the term \( k^2/(2E) \) in the denominator of eq. (53):
\[ i\tilde{S}_F(k; E) = \left( \frac{\hat{n}}{2} + \frac{\hat{k}}{2E} \right) \frac{i}{n \cdot k + i\epsilon}. \] (70)
Unlike the LEET case (cf. eq. (4)), we keep the fluctuating momentum $k$ in the numerator in order to account for collinear but not soft configurations. The propagator (70) does still depend on the hard scale $E$, contrary to what happens to the LEET propagator (4). If the quark emits a longitudinal momentum fraction $1 - x$, such that $k \sim -(1 - x)En$, the propagator after the emission looks like

$$i\tilde{S}_F(k; E) \rightarrow x \frac{n}{2} \frac{i}{n \cdot k + i\epsilon},$$

and takes correctly into account the longitudinal momentum loss (the Sudakov region (59) corresponds to the limit $x \rightarrow 1$). In the appendix we describe a sample computation with the $CEFT$, the evaluation of the non-singlet Altarelli-Parisi kernel.

5 Leading Logarithms in Quantum Field Theory

Let us end up discussing what seems to us a general property of quantum field theories. We observe that all leading logarithmic effects in field theory can be extracted by means of a simple effective theory, where they correspond to a renormalization effect (i.e. they are not dynamical, they renormalize a parameter of the effective theory). To justify this statement, we just need to enumerate the known sources of logarithmic effects in field theory.

- Sudakov region (see eq.(59)). It has been discussed at length in the main body of the paper and can be extracted by means of the $HQET$ and the $LEET$;
- Collinear region (see eq.(60)). The discussion has been mostly qualitative; it requires the introduction of a new effective theory, the ($CEFT$), with the propagator given in eq.(70);
- Infrared region, $k \ll E$. Infrared singularities of form factors have been extracted and factorized with massive eikonal lines, i.e. with the $HQET$, by Korchemsky and Radyushkin in ref.[12], so we refer to these papers for the proof and for a full discussion;
- Hard region, $k^2 \sim \Lambda^2$, where $\Lambda$ is the ultraviolet cutoff. According to Wilson’s Renormalization Group Transformation [22], ultraviolet logarithmic divergences induce a renormalization of parameters of the (leading) effective hamiltonian.

6 Conclusions

The main conclusion of our work is that leading logarithmic effects in field theory do have a very simple dynamical origin. They do not involve the full complexity of quantum field theory and can be extracted by means of effective theories. The latter represent very simple kinematical configurations: particles which move with constant velocity along segments of a given broken line. We stressed the difference between the massive eikonal propagator and the massless eikonal propagator, and that the replacement of a Dirac propagator with one of these propagators depends on the energy scale under consideration. We have here discussed the various approximations involved in the construction of an effective theory. The quadratic term $k^2$ in energy denominators can be neglected as far as the extraction of the soft region (i.e. infrared e/o collinear) is concerned. Neglecting $k^2$ implies a dramatic simplification of the
dynamics: quantum fluctuations are suppressed and particles move along classical trajectories only. We have seen that this term is responsible both for pair creation and for transverse momentum dynamics. It cannot be neglected completely, but it can be simplified \((k^2 \rightarrow -\vec{k}_T^2)\) in the construction of an effective theory describing bound state dynamics \([21]\), i.e. effects which are somehow complementary to the logarithmic ones. For the collinear region the spin fluctuation \((\hat{k})\) in numerators of Dirac propagators has to be kept, while it can be neglected for the extraction of the infrared region.

We believe that the main conclusions of our analysis remain true when higher order corrections are taken into account.

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A Derivation of the Non-Singlet Splitting Function

As a sample computation with the CEFT we present in this appendix the derivation of the non-singlet splitting function $P_{qq}$ entering the Altarelli-Parisi equation.

We consider the collision of a massless quark $q$ with momentum $p_\mu$ with $p^2 = 0$, with a space-like photon with momentum $q_\mu$, $q^2 < 0$:

$$q + \gamma^* \rightarrow q + X,$$

where $X = 0$ or $g$ in the elastic channel or in the inelastic channel of order $\alpha_S$ respectively. Given $p$ and $q$ it is possible to construct two light-like vectors: $p$ itself and $\eta = xp + q$, where $x$ is the Bjorken variable, $x = -q^2/s$, and $s = 2p \cdot q$. The tree QCD diagram (the leading contribution to the elastic process), gives a rate:

$$L = 2\pi \delta((p + q)^2) \frac{1}{2} Tr \mathcal{O} = \frac{\pi}{s} \frac{\delta(1-x)}{x} Tr \mathcal{O}$$

where we defined

$$Tr \mathcal{O} = Tr\left[x\hat{p} \gamma_\rho (x\hat{p} + q) \gamma_\sigma\right].$$

The elastic process is used in the effective theory to define the vertex, i.e. the cusp angle from the kinematics. We replace the electromagnetic vertex of the full theory $\bar{q}(x)\gamma_\mu q(x)$, with an effective vertex of the form $\bar{h}_n'(x)\gamma_\mu h_n(x)$, where $h_n$ and $h_n'$ are fields of the CEFT. As the only possible choice, we take:

$$E_n = p, \quad E'_n' = \eta.$$  

Let us consider now single gluon emission to order $\alpha_S$. We take the gluon propagator in planar gauge with the gauge vector along $\eta$, so that the gluon polarization sum is

$$S_{\mu\nu}(k, \eta) = -g_{\mu\nu} + \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k}.$$  

With this gauge choice, collinear singularities are decoupled from the final quark leg, and the only relevant diagram is the (one-rung, cut) ladder diagram \cite{20, 23}. The ladder diagram is given in the full theory by

$$F = N \int d^4k \frac{\delta((p - k)^2) \delta((k + q)^2)}{(k^2)^2} X_{\rho\sigma},$$

and in the effective theory by

$$E = N \int d^4k \frac{\delta((p - k)^2) \delta(2\eta \cdot (k - xp))}{(2p \cdot (k - p))^2} X_{\rho\sigma},$$

where $N = \alpha_S C_F/(2\pi)$ is a normalization factor and $X_{\rho\sigma}$ is defined as

$$X_{\rho\sigma} = Tr\left[ \hat{p} \gamma_\nu \hat{k} \gamma_\sigma (\hat{k} + \hat{q}) \gamma_\rho \hat{k} \gamma_\mu \right] S^{\mu\nu}(k, \eta).$$

It encodes the effects of the spin structure of the gluon and of the quark for the process. $X_{\rho\sigma}$ is the same in full QCD and in the CEFT because the latter does not involve any approximation for the quark spin structure, but simply a different writing of the helicity sum (see eq.(74)).
The $\delta$-function of the emitted gluon is clearly the same in the two theories (we did nothing to the gluon field), while the $\delta$-function of the final quark and the propagator of the virtual quark are different. In this particular kinematical configuration however, since the gluon is on the mass shell, $(p - k)^2 = 0$, the quark propagators in the two theories actually coincide: $k^2 = (p + (k - p))^2 = 2p \cdot (k - p)$.

We use the Sudakov decomposition of the loop momentum $k$,

$$ k = \alpha \eta + \beta p + k_T, \quad (80) $$

where $k_T$ is the transverse momentum, orthogonal both to $p$ and $\eta$: $k_T \cdot p = k_T \cdot \eta = 0$. For notational simplicity, let us define $k_T^2 = \vec{k}_T^2$. The loop measure is

$$ d^4k = \frac{s}{2} \, d\alpha \, d\beta \, d^2k_T. \quad (81) $$

We use the notation of ref. [20] and we refer to this book for further details and for a physical discussion of DIS in full QCD. In terms of the Sudakov variables the full diagram reads

$$ F = N \frac{s}{2} \int d\alpha \, d\beta \, d^2k_T \frac{\delta(\alpha(1 - \beta)s + k_T^2) \, \delta(s(1 + \alpha)(\beta - x) - k_T^2)}{(k_T^2/(1 - \beta))^2}, \quad (82) $$

while the effective diagram reads

$$ E = N \frac{s}{2} \int d\alpha \, d\beta \, d^2k_T \frac{\delta(\alpha(1 - \beta)s + k_T^2) \, \delta(s(\beta - x))}{(k_T^2/(1 - \beta))^2}. \quad (83) $$

The only difference is the argument of the $\delta$-function of the final quark:

$$ QCD : \quad s \, (1 + \alpha) \, (\beta - x) - k_T^2, $$

$$ CEFT : \quad s \, (\beta - x). \quad (84) $$

It is well known that the collinear singularity (the collinear log) is related to the quasi-real gluon emission, so that

$$ \beta \sim x \sim 1, \quad \alpha \ll 1, \quad \frac{k_T^2}{s} \ll 1. \quad (85) $$

In this region the two integrands look the same. More explicitly, if we perform the integrations over $\alpha$ and $\beta$, we are left with identical expressions as far as the collinear singularity is concerned:

$$ F = E = \frac{N}{2s} \int \frac{d^2k_T}{k_T^2} \, (1 - x) \, \frac{X_{\rho\sigma}}{k_T^2} + (\text{terms with no coll. log.}). \quad (86) $$

It is to note that in full QCD one makes a set of approximations to extract the logarithmic structure. In the CEFT these approximations are ‘automatic’ in the sense that they are already built-in in the lagrangian. The evaluation of $X_{\rho\sigma}$ gives

$$ X_{\rho\sigma} = 2 \, k_T^2 \, \frac{1 + x^2}{x \, (1 - x)^2} \, Tr \, O + O(k_T^4) \quad (87) $$

Summing the tree diagram with the effective diagram of order $\alpha_S$, we recover the famous one-loop expression for the parton density:

$$ L + E = \left[ \delta(1 - x) + \frac{C_F \alpha_S}{2\pi} \, P_{qq} \, \log \frac{Q^2}{\mu^2} \right] \sigma_0 \quad (88) $$
where $\sigma_0$ is the hard cross section in the lowest order, $\sigma_0 = \pi/(x s) \, Tr \, O$ and

$$P_{qq} = \frac{1 + x^2}{1 - x}. \tag{89}$$

It is interesting to compute $X_{\rho\sigma}$ also in the LEET. We have in this case:

$$X_{\rho\sigma}^{(LEET)} = Tr \left[ \hat{p} \gamma_\nu \hat{p} \gamma_\sigma \hat{\eta} \gamma_\rho \hat{\eta} \gamma_\mu \right] S^{\mu\nu}(k, \eta)$$

$$= 2 \, k_T^2 \, \frac{2}{x(1 - x)^2} \, Tr \, O. \tag{90}$$

The latter expression coincides with $X_{\rho\sigma}$ in eq.(87) in the limit $x \to 1$, i.e. in the Sudakov region. The LEET kernel is

$$P_{qq}^{LEET} = \frac{2}{1 - x}, \tag{91}$$

so that the difference with the ‘right’ CETF (or QCD) kernel is

$$P_{qq}^{(CETF)} - P_{qq}^{(LEET)} = \frac{1 + x^2}{1 - x} - \frac{2}{1 - x} = -(1 + x) \tag{92}$$

The integral of the difference over $x$ gives:

$$\int_0^1 dx \, (-)(1 + x) = -\frac{3}{2}, \tag{93}$$

i.e. an infrared finite term. This coefficient accounts for the single collinear logarithm, $\log Q^2/m^2$. We also observe that if we take into account the longitudinal momentum loss in the trace of the LEET, i.e. if we make the replacement $\hat{p} \to x\hat{p}$ in the numerators of quark propagators, the trace changes into:

$$X_{\rho\sigma}^{(LEET)} \to Tr \left[ \hat{p} \gamma_\nu x\hat{p} \gamma_\sigma \hat{\eta} \gamma_\rho x\hat{\eta} \gamma_\mu \right] S^{\mu\nu}(k, \eta)$$

$$= 2 \, k_T^2 \, \frac{2x^2}{x(1 - x)^2} \, Tr \, O. \tag{94}$$

In the latter case the kernel is

$$\tilde{P}_{qq} = \frac{2 \, x^2}{1 - x}, \tag{95}$$

i.e. it comes out smaller then the ‘true’ one (89). The correct kernel is the average of (91) and (95). This may be interpreted by saying that spin fluctuations are so important in collinear dynamics that their effect cannot be represented with any kind of effective constant helicity sum, such as $\hat{p}$ or $x\hat{p}$. In this particular case, the ‘right kernel’ (89) is the average of the LEET kernel (91) with the effective kernel (95) in which strict longitudinal momentum loss, $\hat{p} \to x\hat{p}$ has been assumed.

To summarize, we have shown that the CETF reproduces in a very simple way the kernel $P_{qq}$ of full QCD.
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