Influence of quark masses and strangeness degrees of freedom on inhomogeneous chiral phases

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Introduction

QCD phase diagram (standard picture):

$\langle \bar{q}q \rangle$, $\langle q\bar{q} \rangle$ constant in space

How about non-uniform phases?
Introduction

- QCD phase diagram (standard picture):

- assumption: $\langle \bar{q}q \rangle, \langle qq \rangle$ constant in space
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- assumption: $\langle \bar{q}q \rangle$, $\langle qq \rangle$ constant in space

- How about non-uniform phases?
Introducing

NJL model, homogeneous phases only

\[ \langle \bar{q}q \rangle = 0 \]

\[ \langle \bar{q}q \rangle = \text{const.} \]

[D. Nickel, PRD (2009)]

Inhomogeneous phase rather robust under model extensions and variations

[D. Nickel, PRL (2009)]

This talk: Influence of strange quarks and bare quark masses

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NJL model, including inhomogeneous phase

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\[ \langle \bar{q}q \rangle = \text{const.} \]

inhom.

[D. Nickel, PRD (2009)]

[1st-order phase boundary completely covered by the inhomogeneous phase!]

Critical point \( \rightarrow \) Lifshitz point

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[MB, S. Carignano, PPNP (2015)]

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This talk:
Influence of strange quarks (and bare quark masses)
Digression: Localized quark matter

- Particular 1D modulation (most favored solution known so far):

\[
\langle \bar{q}q \rangle(z) \propto \sqrt{\nu} \Delta \text{sn}(\Delta z|\nu) \rightarrow \begin{cases} 
\sqrt{\nu} \Delta \sin(\Delta z) & \text{for } \nu \rightarrow 0 \\
\Delta \tanh(\Delta z) & \text{for } \nu \rightarrow 1
\end{cases}
\]

\[\mu = 345 \text{ MeV}\]

If it was 3D (but it isn’t yet):

Smooth transition from uniform quark matter to localized “baryons”!

- Revisit chiral solitons! [Alkofer, Reinhardt, Weigel; Goeke et al.; Ripka; ...]
Including strange quarks

2-flavor NJL: CP $\rightarrow$ LP

Is this also true in QCD?

No proof yet, but similar picture from QCD Dyson-Schwinger studies

If true, would it still hold for 3 flavors?

3-flavor QCD with very small quark masses:

CP reaches $T$-axis?

$\Rightarrow$ LP reaches $T$-axis

chance to be studied on the lattice!

Here: Ginzburg-Landau study of CP and LP for 3-flavor NJL
Motivation

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[D. Müller et al. PLB (2013)]
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- CP reaches $T$-axis

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- chance to be studied on the lattice!

[from de Forcrand et al., POSLAT 2007]
Motivation

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- If true, would it still hold for 3 flavors?
- 3-flavor QCD with very small quark masses:
  - CP reaches $T$-axis
  - LP reaches $T$-axis
    - chance to be studied on the lattice!

- Here: Ginzburg-Landau study of CP and LP for 3-flavor NJL

[Diagram from de Forcrand et al., POSLAT 2007]
Ginzburg-Landau analysis

- Expansion of the thermodynamic potential:

\[
\Omega[\Delta] = \Omega[0] + \frac{1}{V} \int d^3 x \left\{ a_2 |\Delta(\vec{x})|^2 + a_{4,a}(\vec{x})|\Delta|^4 + a_{4,b}|\nabla \Delta(\vec{x})|^2 + \ldots \right\}
\]

- \(\Delta(\vec{x})\): order parameter function, \(a_n = a_n(T, \mu)\): GL parameters
Ginzburg-Landau analysis

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- \( \Delta(\vec{x}) \): order parameter function, \( a_n = a_n(T, \mu) \): GL parameters

- case 1: \( a_{4,a}, a_{4,b} > 0 \)
  - \( a_2 > 0 \) \( \Rightarrow \) restored phase
Ginzburg-Landau analysis

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- \( \Delta(\vec{x}) \): order parameter function, \( a_n = a_n(T, \mu) \): GL parameters

- **case 1:** \( a_{4,a}, a_{4,b} > 0 \)
  - \( a_2 < 0 \) \( \Rightarrow \) hom. broken phase

- **case 2:** \( a_{4,a} < 0, a_{4,b} > 0 \)
  - 1st-order phase transition at \( a_2 > 0 \)

- **case 3:** \( a_{4,a}, b < 0 \)
  - inhomogeneous phase possible

- Tricritical point (CP): \( a_2 = a_{4,a} = 0 \)
- Lifshitz point (CP): \( a_2 = a_{4,a} = b = 0 \)
- 2-flavor NJL: \( a_{4,a} = a_{4,b} \Rightarrow \text{CP} = \text{LP} \) [Nickel, PRL (2009)]
Expansion of the thermodynamic potential:

\[ \Omega[\Delta] = \Omega[0] + \frac{1}{V} \int d^3x \left\{ a_2 |\Delta(\vec{x})|^2 + a_{4,a}(\vec{x})|\Delta|^4 + a_{4,b}|\vec{\nabla}\Delta(\vec{x})|^2 + \ldots \right\} \]

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Ginzburg-Landau analysis

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\[\Rightarrow \text{tricritical point (CP): } a_2 = a_{4,a} = 0\]
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- case 3: \( a_{4,b} < 0 \)
  - inhomogeneous phase possible

- 2-flavor NJL: \( a_{4,a} = a_{4,b} \) \[ \Rightarrow \text{CP} = \text{LP} ! \]  
  [Nickel, PRL (2009)]
3-flavor NJL model

- **Lagrangian:** \( \mathcal{L} = \bar{\psi} (i \partial - \hat{m}) \psi + \mathcal{L}_4 + \mathcal{L}_6 \)
  - fields and bare masses: \( \psi = (u, d, s)^T \), \( \hat{m} = \text{diag}_f(0, 0, m_s) \)
  - 4-point interaction: \( \mathcal{L}_4 = G \sum_{a=0}^{8} \left[ (\bar{\psi} \tau_a \psi)^2 + (\bar{\psi} i \gamma_5 \tau_a \psi)^2 \right] \)
  - 6-point ('t Hooft) interaction: \( \mathcal{L}_6 = -K \left[ \det_f \bar{\psi} (1 + \gamma_5) \psi + \det_f \bar{\psi} (1 - \gamma_5) \psi \right] \)
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- Lagrangian: \( \mathcal{L} = \bar{\psi}(i\hat{\partial} - \hat{m})\psi + \mathcal{L}_4 + \mathcal{L}_6 \)
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- Mean fields:
  - light sector: \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \frac{S}{2}, \quad \langle \bar{u}i\gamma_5u \rangle = -\langle \bar{d}i\gamma_5d \rangle \equiv \frac{P}{2} \)
    \( \Rightarrow \langle \bar{\psi}_e\psi_e \rangle \equiv \langle \bar{u}u \rangle + \langle \bar{d}d \rangle = S, \quad \langle \bar{\psi}_e\gamma_5\tau_3\psi_e \rangle \equiv \langle \bar{u}i\gamma_5u \rangle - \langle \bar{d}i\gamma_5d \rangle = P \)
  - strange sector: \( \langle \bar{s}s \rangle \equiv S_s, \quad \langle \bar{s}i\gamma_5s \rangle = 0 \)
  - no flavor-nondiagonal mean fields
  - allow for inhomogeneities: \( S = S(\vec{x}), \quad P = P(\vec{x}), \quad S_s = S_s(\vec{x}) \)
Mean-field Thermodynamic Potential

\[ \Omega_{MF}(T, \mu) = -\frac{T}{V} \text{Tr} \log (i\partial + \mu \gamma^0 - \hat{M}) + \frac{1}{V} \int d^3x \, \mathcal{V}(\vec{x}) \]

- dressed “masses”: \[ \hat{M}_{u,d}(\vec{x}) = -(2G - KS_s(\vec{x})) \left( S(\vec{x}) \pm i\gamma_5 P(\vec{x}) \right) \]

\[ \hat{M}_s(\vec{x}) = m_s - 4GS_s(\vec{x}) + \frac{1}{2} K \left( S^2(\vec{x}) + P^2(\vec{x}) \right) \]

- “potential field”: \[ \mathcal{V}(\vec{x}) = G \left( S^2(\vec{x}) + P^2(\vec{x}) + 2S_s(\vec{x}) \right) - KS_s(\vec{x}) \left( S^2(\vec{x}) + P^2(\vec{x}) \right) \]
Mean-field Thermodynamic Potential

\[ \Omega_{\text{MF}}(T, \mu) = -\frac{T}{V} \text{Tr} \log \left( i \hat{\mathcal{D}} + \mu \gamma^0 - \hat{M} \right) + \frac{1}{V} \int d^3 x \mathcal{V}(\vec{x}) \]

- dressed “masses”:
  \[ \hat{M}_{u,d}(\vec{x}) = -(2G - KS_s(\vec{x}))(S(\vec{x}) \pm i\gamma^5 P(\vec{x})) \]
  \[ \hat{M}_s(\vec{x}) = m_s - 4GS_s(\vec{x}) + \frac{1}{2} K \left( S^2(\vec{x}) + P^2(\vec{x}) \right) \]

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- \( K = 0 \): light and strange sectors decouple!

\[ \hat{M}_{u,d} = -2G \left( S \pm i\gamma^5 P \right), \quad \hat{M}_s(\vec{x}) = m_s - 4GS_s; \quad \mathcal{V} = G \left( S^2 + P^2 \right) + 2GS_s \]
Mean-field Thermodynamic Potential

\[ \Omega_{MF}(T, \mu) = -\frac{T}{V} \text{Tr} \log \left( i\hat{\theta} + \mu \gamma^0 - \hat{M} \right) + \frac{1}{V} \int d^3x \, V(\vec{x}) \]

- dressed “masses”:
  \[ \hat{M}_{u,d}(\vec{x}) = -(2G - K S_s(\vec{x})) \left( S(\vec{x}) \pm i\gamma_5 P(\vec{x}) \right) \]

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- “potential field”:
  \[ V(\vec{x}) = G \left( S^2(\vec{x}) + P^2(\vec{x}) + 2S_s(\vec{x}) \right) - KS_s(\vec{x}) \left( S^2(\vec{x}) + P^2(\vec{x}) \right) \]

\[ K = 0: \quad \text{light and strange sectors decouple!} \]

\[ \hat{M}_{u,d} = -2G \left( S \pm i\gamma_5 P \right), \quad \hat{M}_s(\vec{x}) = m_s - 4GS_s; \quad V = G \left( S^2 + P^2 \right) + 2GS_s \]

- Chiral density wave ansatz for the light sector:
  \[ S(\vec{x}) = \phi_0 \cos(\vec{q} \cdot \vec{x}), \quad P(\vec{x}) = \phi_0 \sin(\vec{q} \cdot \vec{x}), \quad S_s = \phi_s = \text{const} \]

\[ \Rightarrow \quad \hat{M}_{u,d} = \Delta e^{\pm i\gamma_5 \vec{q} \cdot \vec{x}}, \quad \Delta \equiv -(2G - K\phi_s)\phi_0, \]

\[ M_s = \text{const}., \quad V = \text{const}. \]

consistent with the literature [Moreira et al., PRD (2014)]
Ginzburg-Landau expansion

- Difficulty at $m_s \neq 0$: No $SU(3)_L \times SU(3)_R$ restored solution
- $m_u = m_d = 0$
  $\Rightarrow$ Expand about two-flavor restored solution $S = P = 0$:

\[
\Omega_{MF}[S, P, S_s] = \Omega_{MF}[0, 0, S_s^{(0)}] + \frac{1}{V} \int d^3x \Omega_{GL}[S(\vec{x}), P(\vec{x}), X(\vec{x})]
\]

- strange condensate: $S_s(\vec{x}) = S_s^{(0)} + X(\vec{x})$
- $S_s^{(0)}$: homogeneous solution of the gap equation for $S = P = 0$ at given $T$ and $\mu$
- Expand $\Omega_{GL}$ in $S$, $P$ and $X$, and their gradients.
Ginzburg-Landau potential

Define: \( \Delta_\ell = -2G(S + iP) \), \( \Delta_s = -4GX \)

\([\Delta_i] = \text{(mass)} \rightarrow \) counting scheme: \( \mathcal{O}(\vec{V}) = \mathcal{O}(\Delta_i) \)
Define: \[ \Delta_\ell = -2G(S + iP), \quad \Delta_s = -4GX \]

\[ [\Delta_i] = \text{(mass)} \quad \rightarrow \quad \text{counting scheme: } O(\vec{\nabla}) = O(\Delta_i) \]

Resulting structure:

\[
\Omega_{GL} = a_2 |\Delta_\ell|^2 + a_{4,a} |\Delta_\ell|^4 + a_{4,b} |\vec{\nabla} \Delta_\ell|^2 \\
+ b_1 \Delta_s + b_2 \Delta_s^2 + b_3 \Delta_s^3 + b_{4,a} \Delta_s^4 + b_{4,b} (\vec{\nabla} \Delta_s)^2 \\
+ c_3 |\Delta_\ell|^2 \Delta_s + c_4 |\vec{\nabla} \Delta_\ell|^2 (\vec{\nabla} \Delta_s)^2 \\
+ O(\Delta_i^5) 
\]
Define: \( \Delta_\ell = -2G(S + iP), \quad \Delta_s = -4GX \)

\[ [\Delta_i] = \text{(mass)} \quad \Rightarrow \quad \text{counting scheme: } O(\vec{\nabla}) = O(\Delta_i) \]

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+ c_3 |\Delta_\ell|^2 \Delta_s + c_4 |\vec{\nabla} \Delta_\ell|^2 (\vec{\nabla} \Delta_s)^2 + O(\Delta_i^5)
\]

Stationarity condition:

\[
\frac{\partial \Omega_{GL}}{\partial \Delta_s} \bigg|_{\Delta_\ell=\Delta_s=0} = 0 \quad \Leftrightarrow \quad b_1 = 0
\]
Ginzburg-Landau potential

Define: $\Delta_\ell = -2G(S + iP)$, $\Delta_s = -4GX$

$[\Delta_i] = \text{(mass)}$ → counting scheme: $O(\vec{\nabla}) = O(\Delta_i)$

Resulting structure:

$$\Omega_{GL} = a_2|\Delta_\ell|^2 + a_{4,a}|\Delta_\ell|^4 + a_{4,b}|\vec{\nabla}\Delta_\ell|^2 + b_2\Delta_s^2 + b_3\Delta_s^3 + b_{4,a}\Delta_s^4 + b_{4,b}(\vec{\nabla}\Delta_s)^2$$

$$+ c_3|\Delta_\ell|^2\Delta_s + c_4|\vec{\nabla}\Delta_\ell|^2(\vec{\nabla}\Delta_s)^2 + O(\Delta_i^5)$$

Stationarity condition: $\frac{\partial\Omega_{GL}}{\partial\Delta_s}|_{\Delta_\ell=\Delta_s=0} = 0 \iff b_1 = 0$
Ginzburg-Landau potential

- Define: \( \Delta_\ell = -2G(S + iP), \quad \Delta_s = -4GX \)
  
  \([\Delta_i] = \text{(mass)} \rightarrow \text{counting scheme: } O(\bar{\nabla}) = O(\Delta_i)\)

- Resulting structure:

  \[
  \Omega_{GL} = a_2 |\Delta_\ell|^2 + a_{4,a} |\Delta_\ell|^4 + a_{4,b} |\bar{\nabla}\Delta_\ell|^2 \\
  + b_2 \Delta_s^2 + b_3 \Delta_s^3 + b_{4,a} \Delta_s^4 + b_{4,b} (\bar{\nabla}\Delta_s)^2 \\
  + c_3 |\Delta_\ell|^2 \Delta_s + c_4 |\bar{\nabla}\Delta_\ell|^2 (\bar{\nabla}\Delta_s)^2 + O(\Delta_i^5)
  \]

- Stationarity condition:

  \[
  \frac{\partial \Omega_{GL}}{\partial \Delta_s} \bigg|_{\Delta_\ell=\Delta_s=0} = 0 \iff b_1 = 0
  \]

  \[
  \Rightarrow \quad M_s^{(0)} = m_s - 16N_c G T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{M_s^{(0)}}{(i\omega_n + \mu)^2 - \bar{p}^2 - M_s^{(0)}^2}
  \]

  \[
  (= \text{gap equation for } M_s^{(0)} \equiv \hat{M}_s|_{S=P=X=0} = m_s - 4GS_S^{(0)})
  \]
Eliminating the strange condensate

- Extremizing $\Omega_{MF}$ w.r.t. $\Delta_s(\bar{x})$
  - \[ \text{Euler-Lagrange equation} \quad \frac{\partial \Omega_{GL}}{\partial \Delta_s} - \partial_i \frac{\partial \Omega_{GL}}{\partial \partial_i \Delta_s} = 0 \]
  - $\Delta_s = -\frac{c_3}{2b_2} |\Delta_\ell|^2 + O(|\Delta_\ell|^4)$
Eliminating the strange condensate

- Extremizing $\Omega_{MF}$ w.r.t. $\Delta_s(\vec{x})$

  $\rightarrow$ Euler-Lagrange equation

  $$\frac{\partial \Omega_{GL}}{\partial \Delta_s} - \partial_i \frac{\partial \Omega_{GL}}{\partial \partial_i \Delta_s} = 0$$

  $\Leftrightarrow$ $\Delta_s = -\frac{c_3^2}{2b_2} |\Delta_\ell|^2 + O(|\Delta_\ell|^4)$

- Insert into $\Omega_{GL}$:

  $$\Omega_{GL} = a_2 |\Delta_\ell|^2 + (a_{4,a} - \frac{c_3^2}{4b_2}) |\Delta_\ell|^4 + a_{4,b} |\vec{\nabla} \Delta_\ell|^2 + O(\Delta_\ell^6)$$

Eliminating the strange condensate

- Extremizing $\Omega_{MF}$ w.r.t. $\Delta_s(\vec{x})$

  $\rightarrow$ Euler-Lagrange equation
  \[
  \frac{\partial \Omega_{GL}}{\partial \Delta_s} - \partial_i \frac{\partial \Omega_{GL}}{\partial \partial_i \Delta_s} = 0
  \]

  $\Leftrightarrow$ \[
  \Delta_s = - \frac{c_3}{2b_2} |\Delta_\ell|^2 + O(|\Delta_\ell|^4)
  \]

- Insert into $\Omega_{GL}$:

  \[
  \Omega_{GL} = a_2 |\Delta_\ell|^2 + \left( a_{4,a} - \frac{c_3^2}{4b_2} \right) |\Delta_\ell|^4 + a_{4,b} |\vec{\nabla} \Delta_\ell|^2 + O(\Delta_\ell^6)
  \]

- Critical and Lifshitz points:
  - CP: $a_2 = a_{4,a} - \frac{c_3^2}{4b_2} = 0$
  - LP: $a_2 = a_{4,b} = 0$
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CP and LP don’t coincide anymore!
Relevant GL coefficients (no guarantee yet!):

\[
\begin{align*}
a_2 &= \frac{1}{4G} (1 + 2\delta) + (1 + \delta)^2 \quad 4N_c \frac{1}{V_4} \sum \frac{1}{p^2} + \frac{K}{2G^2} \quad N_c \frac{1}{V_4} \sum \frac{M_s^{(0)}}{p^2 - M_s^{(0)}^2} \\
a_{4,a} &= (1 + \delta)^4 \quad 2N_c \frac{1}{V_4} \sum \frac{1}{p^4} + \frac{K^2}{32G^4} \quad N_c \frac{1}{V_4} \sum \frac{p^2 + M_s^{(0)}^2}{[p^2 - M_s^{(0)}^2]^2} \\
a_{4,b} &= (1 + \delta)^2 \quad 2N_c \frac{1}{V_4} \sum \frac{1}{p^4} \\
c_3 &= \frac{K}{2G^2} \left[ \frac{1}{8G} + (1 + \delta) \quad 2N_c \frac{1}{V_4} \sum \frac{1}{p^2} + N_c \frac{1}{V_4} \sum \frac{p^2 + M_s^{(0)}^2}{[p^2 - M_s^{(0)}^2]^2} \right] \\
\delta &\equiv -\frac{K}{2G} S_s^{(0)}, \quad \frac{1}{V_4} \sum \equiv T \sum_n \int \frac{d^3p}{(2\pi)^3}
\end{align*}
\]

Abbreviations:

\[
\begin{align*}
\delta &\equiv -\frac{K}{2G} S_s^{(0)}, \\
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a_2 = \frac{1}{4G} (1 + 2\delta) + (1 + \delta)^2 4N_c \frac{1}{V_4} \sum \frac{1}{p^2} + \frac{K}{2G^2} N_c \frac{1}{V_4} \sum \frac{M_s^{(0)}}{p^2 - M_s^{(0)} - 2}
\]

\[
a_{4,a} = (1 + \delta)^4 2N_c \frac{1}{V_4} \sum \frac{1}{p^4} + \frac{K^2}{32G^4} N_c \frac{1}{V_4} \sum \frac{p^2 + M_s^{(0)2}}{\left[p^2 - M_s^{(0)2}\right]^2}
\]

\[
a_{4,b} = (1 + \delta)^2 2N_c \frac{1}{V_4} \sum \frac{1}{p^4}
\]

\[
c_3 = \frac{K}{2G^2} \left[\frac{1}{8G} + (1 + \delta) 2N_c \frac{1}{V_4} \sum \frac{1}{p^2} + N_c \frac{1}{V_4} \sum \frac{p^2 + M_s^{(0)2}}{\left[p^2 - M_s^{(0)2}\right]^2}\right]
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Interesting limits:

\[
K = 0 \quad \Rightarrow \quad \delta = 0 \quad \Rightarrow \quad CP = LP
\]
Discussion

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- \( K = 0 \quad \Rightarrow \quad \delta = 0 \quad \Rightarrow \quad \text{CP=LP} \)

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- Numerical survey of the general case still to be done.
Finite bare quark masses

- What is the effect of nonzero $m_u$ and $m_d$?
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- Andersen, Kneschke, PRD (2018):
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$m_{u,d} = 0, 5$ MeV, 10 MeV
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- Can we investigate this more systematically within GL?
Ginzburg-Landau analysis with nonzero bare masses

- No restored phase $\Rightarrow$ Expand about arbitrary homogeneous $\Delta_0$:

  $\Omega_{GL} = a_1(\Delta - \Delta_0) + a_2(\Delta - \Delta_0)^2 + a_3(\Delta - \Delta_0)^3 + a_4,\Delta_0(\Delta - \Delta_0)^4 + a_4,\Delta(\nabla \Delta)^2 + \ldots$

  - Extremum $\Rightarrow$ gap equation: $a_1(T, \mu) = 0$ (partially fixes $\Delta_0(T, \mu)$)
Ginzburg-Landau analysis with nonzero bare masses

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  - Extremum ⇒ gap equation: $a_1(T, \mu) = 0$ (partially fixes $\Delta_0(T, \mu)$)

- Critical endpoint
  - left spinodal: $a_2 = 0, a_3 < 0$
  - right spinodal: $a_2 = 0, a_3 > 0$

  ⇒ CEP: $a_2 = a_3 = 0$
Ginzburg-Landau analysis with nonzero bare masses

No restored phase \( \Rightarrow \) Expand about arbitrary homogeneous \( \Delta_0 \):
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Critical endpoint

- left spinodal: \( a_2 = 0, \ a_3 < 0 \)
- right spinodal: \( a_2 = 0, \ a_3 > 0 \)

\( \Rightarrow \) CEP: \( a_2 = a_3 = 0 \)

“Lifshitz point” = upper corner of the inhomogeneous phase?

- @ CEP: We find \( a_{4,b} < 0 \) \( \Rightarrow \) The CEP is inside the inhomogeneous phase.
- No point with \( a_2 = a_{4,b} = 0 \) \( \Rightarrow \) No point with \( \vec{\nabla} \Delta = 0 \) at the phase boundary

\( \Rightarrow \) Further investigations necessary
Ginzburg-Landau analysis with nonzero bare masses

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  \[ \Omega_{GL} = a_1(\Delta - \Delta_0) + a_2(\Delta - \Delta_0)^2 + a_3(\Delta - \Delta_0)^3 + a_4,a(\Delta - \Delta_0)^4 + a_4,b(\vec{\nabla} \Delta)^2 + \ldots \]
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- “Lifshitz point” = upper corner of the inhomogeneous phase?
  - @ CEP: We find $a_4,b < 0$ ⇒ The CEP is inside the inhomogeneous phase.
  - No point with $a_2 = a_4,b = 0$ ⇒ No point with $\vec{\nabla} \Delta = 0$ at the phase boundary

  ⇒ Further investigations necessary

Ongoing work: Determine phase boundary via $1 - \Pi_{\sigma, \pi}(\omega = 0, \vec{q}) = 0$
Conclusions

- Ginzburg-Landau analysis of the effect of strangeness and bare quark masses on the inhomogeneous chiral phase in NJL

- strange quarks: CP and LP no longer agree

- nonzero $m_{u,d}$ (very preliminary):
  - CEP inside the inhomogeneous phase
  - No LP-like point with $\vec{\nabla} \Delta = 0$

- Detailed numerical study to be done.