Quantum information approach to the ultimatum game

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Abstract

The paper is devoted to quantization of extensive games with the use of both the Marinatto-Weber and the Eisert-Wilkens-Lewenstein concept of quantum game. We revise the current conception of quantum ultimatum game and we show why the proposal is unacceptable. To support our comment, we present the new idea of the quantum ultimatum game. Our scheme also makes a point of departure for a protocol to quantize extensive games.

1 Introduction

During the last twelve years of research into quantum games the theory has been already extended beyond $2 \times 2$ games. Since majority of noncooperative conflict problems are described by games in extensive form, it is interesting to place extensive games in the quantum domain. Although there is still no commonly accepted idea of how to play quantum extensive games, we have proved in [4] that it is possible to use the framework [2] of strategic quantum game to get some insight into quantum extensive games. Namely, we have shown that a Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, a unit vector $|\psi_i\rangle \in \mathcal{H}$, the collection of subsets $\{U_j\}_{j=1,2,3}$ of $SU(2)$, and appropriately defined functionals $E_1$ and $E_2$ express the normal representation of a two stage sequential game. Moreover, it allows to get a result inaccessible in the game played classically. In this paper, the above-mentioned quantum computing description will be used to the two proposed variants of the ultimatum game [5]. It is a game in which two players take part. The first player proposes one of two proposals how to divide a fixed amount of good. Then the second player either accepts or rejects the proposal. In the first case, each player receives the part of goods according to player 1’s proposal. In the second case, the players receive nothing. A game-theoretic analysis shows that player 1 is in a better position. Since player 2’s rational move is to accept each proposal, player 1’s rational move is to make the best proposal for her. As we will show in this article, the Eisert-Wilkens-Lewenstein (EWL) approach [1] as well as Marinatto-Weber (MW) approach [7] can change the scenario of the ultimatum game significantly improving the strategic position of player 2. Our paper also provides an argument indicating that the previous idea [8] of quantum ultimatum game is not sufficient to describe the game in the quantum domain. We will explain that, in fact, the formerly proposed protocol does not quantize the ultimatum game but another $2 \times 2$ game. The last part of the paper is devoted to a form of a game
2 Preliminaries to game theory

Definitions in the preliminaries are based on [11]. This section starts with a definition of a finite extensive game.

Definition 2.1 Let the following components be given.

- A finite set \( N = \{1, 2, \ldots, n\} \) of players.
- A set \( H \) of finite sequences that satisfies the following two properties:
  1. the empty sequence \( \emptyset \) is a member of \( H \);
  2. if \((a_k)_{k=1,2,\ldots,K} \in H \) and \( K > 1 \) then \((a_k)_{k=1,2,\ldots,K-1} \in H \).

Each member of \( H \) is a history and each component of a history is an action taken by a player. A history \((a_1,a_2,\ldots,a_K) \in H \) is terminal if there is no \( a_{K+1} \) such that \((a_1,a_2,\ldots,a_K,a_{K+1}) \in H \). The set of actions available after the nonterminal history \( h \) is denoted \( A(h) = \{a: (h,a) \in H\} \) and the set of terminal histories is denoted \( Z \).

- The player function \( P: H \setminus Z \to N \cup \{c\} \) that points to a player who takes an action after the history \( h \). If \( P(h) = c \) then chance (the chance-mover) determines the action taken after the history \( h \).

- A function \( f \) that associates with each history \( h \) for which \( P(h) = c \) an independent probability distribution \( f(\cdot|h) \) on \( A(h) \).

- For each player \( i \in N \) a partition \( I_i \) of \( \{h \in H \setminus Z : P(h) = i\} \) with the property that for each \( I_i \) and for each \( h, h' \in I_i \) an equality \( A(h) = A(h') \) is fulfilled. Every information set \( I_i \) of the partition corresponds to the state of player’s knowledge. When the player makes move after certain history \( h \) belonging to \( I_i \), she knows that the course of events of the game takes the form of one of histories being part of this information set. She does not know, however, if it is the history \( h \) or the other history from \( I_i \).

- For each player \( i \in N \) a utility function \( u_i: Z \to \mathbb{R} \) which assigns a number (payoff) to each of the terminal histories.

A six-tuple \((N,H,P,f,\{I_i\},\{u_i\})\) is called a finite extensive game.

Our deliberations focus on games with perfect recall (although Def. 2.1 defines extensive games with imperfect recall as well) - this means games in which at each stage every player remembers all the information about a course of the game that she knew earlier (see [9] and [11] to learn about formal description of this feature).

The notions: action and strategy mean the same in static games, because the players choose their actions once and simultaneously. In the majority of extensive games a player can make her decision about an action depending on all the actions taken previously by herself and also by all the other players. In other words, players can make some plans of actions at their disposal such that these plans point out to a specific action depending on the course of a game. Such a plan is defined as a strategy in an extensive game.
Definition 2.2 A pure strategy $s_i$ of a player $i$ in a game $(N, H, P, f_c, \{I_i\}, \{u_i\})$ is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}$.

Like in the theory of strategic games, a mixed strategy $t_i$ of a player $i$ in an extensive game is a probability distribution over the set of player $i$’s pure strategies. Therefore, pure strategies are of course special cases of mixed strategies and from this place whenever we shall write strategy without specifying that it is either pure or mixed, this term will cover both cases. Let us define an outcome $O(s)$ of a strategy profile $s = (s_1, s_2, \ldots, s_n)$ in an extensive game without chance moves to be a terminal history that results when each player $i \in N$ follows the plan of $s_i$. More formally, $O(s)$ is the history $(a_1, a_2, \ldots, a_K) \in \mathcal{Z}$ such that for $0 \leq k < K$ we have $s_{P(a_1,a_2,\ldots,a_k)}(a_1, a_2, \ldots, a_k) = a_{k+1}$. If $s$ implies a history that contains chance moves, the outcome $O(s)$ is an appropriate probability distribution over histories generated by $s$.

Definition 2.3 Let an extensive game $\Gamma = (N, H, P, \{I_i\}, \{u_i\})$ be given. The normal representation of $\Gamma$ is a strategic game $(N, \{S_i\}, \{u_i'\})$ in which for each player $i \in N$:

- $S_i$ is the set of pure strategies of a player $i$ in $\Gamma$;
- $u_i' : \prod_{i \in N} S_i \to \mathbb{R}$ defined as $u_i'(s) := u_i(O(s))$ for every $s \in \prod_{i \in N} S_i$ and $i \in N$.

One of the most important notions in game theory is a notion of an equilibrium introduced by John Nash in [10]. A Nash equilibrium is a profile of strategies where the strategy of each player is optimal if the choice of its opponents is fixed. In other words, in the equilibrium none of the players has any reason to unilaterally deviate from an equilibrium strategy. A precise formulation is as follows:

Definition 2.4 Let $(N, S_i, \{u_i\}_{i \in N})$ be a strategic game. A strategy profile $(t_1^*, t_2^*, \ldots, t_n^*)$ is a Nash equilibrium (NE) if for each player $i \in N$ and for all $s_i \in S_i$:

$$u_i(t_i^*, t_{-i}^*) \geq u_i(s_i, t_{-i}^*) \quad \text{where} \quad t_{-i}^* = (t_1^*, \ldots, t_{i-1}^*, t_{i+1}^*, \ldots, t_n^*). \quad (1)$$

A Nash equilibrium in an extensive game with perfect recall is a Nash equilibrium of its normal representation, hence Def. 2.4 applies to strategic games as well as extensive ones.

3 The ultimatum game

The ultimatum game is a problem in which two players face a division of some amount $\mathcal{E}$ of money. The first player makes the second one a proposal of how to divide $\mathcal{E}$ between them. Then the second player has to decide either accept or reject that proposal. The acceptance means each player receives a part of $\mathcal{E}$ according to the first player’s proposal. If the second player rejects, each player receives nothing. Let us consider the a variant of the ultimatum game in which player 1 has two proposals to share $\mathcal{E}$: a fair division $u_f = (\mathcal{E}/2, \mathcal{E}/2)$ and unfair one $u_a = (\delta \mathcal{E}, (1 - \delta) \mathcal{E})$, where the $\delta$ is a fixed factor such that $1/2 < \delta < 1$. This problem is an extensive game with perfect information that takes the form:

$$\Gamma_1 = (\{1, 2\}, H, P, \{I_i\}, u) \quad (2)$$

with components defined as follows:

- $H = \{\emptyset, c_0, c_1, (c_0, d_0), (c_0, d_1), (c_1, e_0), (c_1, e_1)\}$;
\( P(\emptyset) = 1, \ P(c_0) = P(c_1) = 2; \)

- \( I_1 = \{\emptyset\}, \ I_2 = \{(c_0)\}, \{(c_1)\}\); 
- \( u(c_0, d_0) = (\mathcal{E}/2, \mathcal{E}/2), \ u(c_1, e_0) = (\delta\mathcal{E}, (1-\delta)\mathcal{E}), \ u(c_0, d_1) = u(c_1, e_1) = (0, 0) \).

The extensive and the normal representation of \( \Gamma_1 \) is shown in Figure 1. Equilibrium analysis of the normal representation gives us three pure Nash equilibria: \((c_0, d_0e_1), \ (c_1, d_0e_0)\) and \((c_1, d_1e_0)\). There are also mixed equilibria: a profile where player 1 chooses \( c_0 \) and player 2 chooses \( d_0e_0 \) with probability \( p \leq 1/(2\delta) \) and \( d_0e_1 \) with probability \( 1-p \), and a profile where player 1 decides to play \( c_1 \) and player 2 chooses any probability distribution over strategies \( d_0e_0 \) and \( d_1e_0 \). However, we can put these ones aside since both mixed equilibria do not contribute to the utility outcomes of \( \Gamma_1 \). They generate the same utility outcomes as the pure ones: \((\mathcal{E}/2, \mathcal{E}/2)\) and \((\delta\mathcal{E}, (1-\delta)\mathcal{E})\), respectively. The key feature that make the game \( \Gamma_1 \) so curious is that only equilibrium profile \((c_1, d_0e_0)\) with unfair outcomes \((\delta\mathcal{E}, (1-\delta)\mathcal{E})\) is a reasonable scenario among all the equilibria of the ultimatum game (many experiments show that people are inclined to choose fair division \((\mathcal{E}/2, \mathcal{E}/2)\), however we stick to the natural assumption of game theory that players are striving to maximize their payoffs). The strategy combination \((c_1, d_0e_0)\) is the unique equilibrium that is subgame perfect (the idea of subgame perfection is the well-known equilibrium refinement formulated by Selten [13]) i.e. it is a profile of strategies that induces a Nash equilibrium in every subgame (there are three subgames in \( \Gamma_1 \): the entire game, a game after the action \( c_0 \) and a game after the action \( c_1 \)). At the same time the subgame perfection rejects equilibria that are not credible. Let us consider the profile \((c_0, d_0e_1)\). Here, the strategy \( d_0e_1 \) of player 2 demands the action \( e_1 \) when player 1 chooses \( c_1 \). However, when \( c_1 \) occurs, a rational move of player 2 is \( e_0 \). Similar analysis shows that also \((c_1, d_1e_0)\) is not subgame perfect equilibrium. Although the notion of subgame perfection is related to the extensive form of a game, we can easily determine subgame perfect equilibria in any two stage extensive game with perfect information (or even in a wider class of extensive games) through an analysis of its normal representation. In the game \( \Gamma_1 \) an action taken by player 1 determines a subgame in which only player 2 makes a move. Thus subgame perfect equilibrium in \( \Gamma_1 \) is a Nash equilibrium with a property that a strategy of player 2 is the best response to every strategy of player 1 (i.e., a strategy that weakly dominates the others).
4 Criticism of the previous approach to quantum ultimatum game

A misrepresentation of the classical ultimatum game is the source of its incorrect quantum representation in [8]. The author describes the ultimatum problem as a $2 \times 2$ game and then applies the MW and the EWL schemes to construct the quantum game. However, as we have seen in Figure 1b, $2 \times 4$ is a minimal dimension allowing to represent the ultimatum game in normal form. A hypothetical case of the ultimatum game in which player 2 has only two strategies after an action taken by player 1 implies that player 2 is deprived of capability to make her move conditioned on the action of the first player. That is tantamount to an event where the players take their actions at the same time or one of the players chooses her action as the second but she does not have any information about an action taken by her opponent. It does not correspond to a description of the ultimatum game where the second player knows a proposal of her opponent and depending on the move of the first player she makes her action. Although the player 2 has only two actions: accept or reject in the two-proposal ultimatum game, in fact she has four pure strategies defined as her plans of an action at each of her information sets. Therefore, a $2 \times 2$ strategic game cannot depict the ultimatum game. Consequently, the MW and the EWL approach used for quantization of a $2 \times 2$ game cannot produce a quantum version of this game. Neither of these quantum realizations contains the classical ultimatum game.

5 The quantum ultimatum game obtained by quantization of the normal representation of the classical game

First, let us remind the protocol for playing quantum games defined in [4]. It is a six-tuple:

$$
\Gamma^\text{QI} = (H, N, |\psi_{\text{in}}\rangle, \xi, \{U_j\}, \{E_i\})
$$

(3)

where the components are defined as follows:

- $H$ is a complex Hilbert space $\bigotimes_{j=1}^m C^2$ with an orthonormal basis $B$.
- $N$ is a set of players with the property that $|N| \leq m$.
- $|\psi_{\text{in}}\rangle$ is the initial state of a system of $m$ qubits $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_m\rangle$.
- $\xi: \{1, 2, \ldots, m\} \to N$ is a surjective mapping. A value $\xi(j)$ indicates a player who carries out a unitary operation on a qubit $|\varphi_j\rangle$.
- For each $j \in \{1, 2, \ldots, m\}$ the set $U_j$ is a subset of unitary operators from $\text{SU}(2)$ that are available for a qubit $j$. A (pure) strategy of a player $i$ is a map $\tau_i$ that assigns a unitary operation $U_j \in U_j$ to a qubit $|\varphi_j\rangle$ for every $j \in \xi^{-1}(i)$. The final state $|\psi_{\text{fin}}\rangle$ when the players have performed their strategies on corresponding qubits is defined as:

$$
|\psi_{\text{fin}}\rangle := (\tau_1, \tau_2, \ldots, \tau_n)|\psi_{\text{in}}\rangle = \bigotimes_{i \in N} \bigotimes_{j \in \xi^{-1}(i)} U_j|\psi_{\text{in}}\rangle.
$$

(4)
For each \( i \in N \) the map \( E_i \) is a utility (payoff) functional that specifies a utility for the player \( i \). The functional \( E_i \) is defined by the formula:

\[
E_i = \sum_{|b\rangle \in B} v_i(b) |\langle b|\psi_{\text{fin}}\rangle|^2, \quad \text{where} \quad v_i(b) \in \mathbb{R}.
\] (5)

The above scheme is adapted for extensive games with two available actions at each information set so that we could use only qubits for convenience. Any game richer in actions can be transferred to quantum domain by using quantum objects of higher dimensionality.

The idea framed in [4] bases on identifying unitary actions taken on a qubit with actions taken in an information set of classical game. Therefore, three qubits are required to express the ultimatum game in quantum information language. Since the first player has one information set and the second player has two ones, player 1 performs a unitary operation on only one qubit and player 2 operates on the rest. Like in [8] we examine the two approaches: the MW approach and the EWL approach to quantizing \( \Gamma_1 \).

5.1 The MW approach

Let us consider the following six-tuple:

\[
\Gamma_1^{\text{MW}} = (\mathcal{H}, \{1, 2\}, |\psi_{\text{in}}\rangle, \xi, \{\{\sigma_0, \sigma_1\}_i\}, \{E_i\}),
\] (6)

where:

- \( \mathcal{H} \) is a Hilbert space \( \bigotimes_{j=1}^3 \mathbb{C}^2 \) with the computational basis states \( |x_1, x_2, x_3\rangle \), \( x_j = 0, 1 \);
- the initial state \( |\psi_{\text{in}}\rangle \) is a general pure state of three qubits:

\[
|\psi_{\text{in}}\rangle = \sum_{x \in \{0, 1\}^3} \lambda_x |x\rangle, \quad \text{where} \quad \lambda_x \in \mathbb{C} \quad \text{and} \quad \sum_{x \in \{0, 1\}^3} |\lambda_x|^2 = 1;
\] (7)
- the map \( \xi \) on \( \{1, 2, 3\} \) given by the formula: \( \xi(j) = \begin{cases} 1, & \text{if } j = 1; \\ 2, & \text{if } j \in \{2, 3\}. \end{cases} \)
- \( \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \);
- the payoffs functionals \( E_i, i = 1, 2 \), are of the form:

\[
E_1 = \frac{1}{2} \epsilon \sum_{x_3} |\langle 00, x_3|\psi_{\text{fin}}\rangle|^2 + \delta \epsilon \sum_{x_2} |\langle 1, x_2, 0|\psi_{\text{fin}}\rangle|^2;
\]
\[
E_2 = \frac{1}{2} \epsilon \sum_{x_3} |\langle 00, x_3|\psi_{\text{fin}}\rangle|^2 + (1 - \delta) \epsilon \sum_{x_2} |\langle 1, x_2, 0|\psi_{\text{fin}}\rangle|^2.
\] (8)

By definition of \( \xi \) in \( \Gamma_1 \), player 1 acts on the first qubit and treats the operators \( \sigma_0^1 \) and \( \sigma_1^1 \) as her strategies. Player 2 acts on the second and the third qubit, hence her pure strategies are \( \sigma_0^2 \otimes \sigma_0^3, \sigma_0^2 \otimes \sigma_1^3, \sigma_1^2 \otimes \sigma_0^3 \) and \( \sigma_1^2 \otimes \sigma_1^3 \) (the upper index denotes a qubit on which an operation is made). Let us determine for each profile \( (\sigma_{\kappa_1}^1, (\sigma_{\kappa_2}^2, \sigma_{\kappa_3}^3)) \), where \( \kappa_1, \kappa_2, \kappa_3 \in \{0, 1\} \), the corresponding expected utility \( E_i \) by using formulae (4)-(5).
and the specification of $\text{6}$. We illustrate it using as an example $E_i(\sigma^1, (\sigma^2, \sigma^3))$ for $i = 1, 2$. The initial state after the players choose the profile $(\sigma^1, (\sigma^2, \sigma^3))$ takes the form $|\psi_{in}\rangle = \sigma^1 \otimes \sigma^2 \otimes \sigma^3 |\psi_{in}\rangle$. Thus, we have:

$$|\psi_{in}\rangle = \sum_{x_1, x_2, x_3 \in \{0,1\}} \lambda_{x_1, x_2, x_3} |x_1, \overline{x}_2, x_3\rangle,$$

where $\overline{x}_2$ is the negation of $x_2$. Putting the final state $|\psi\rangle$ into the first of Eq. (8) we obtain:

$$E_i(\sigma^1, (\sigma^2, \sigma^3)) = \frac{1}{2} \mathcal{E}(|\lambda_{010}|^2 + |\lambda_{011}|^2) + \delta \mathcal{E}(|\lambda_{100}|^2 + |\lambda_{110}|^2).$$

(10)

Obviously, we have $(1 - \delta) \mathcal{E}$ instead of $\delta \mathcal{E}$ in the expected utility $E_2$. Therefore, the payoff vector $(E_1, E_2)$ is $u_1(|\lambda_{010}|^2 + |\lambda_{011}|^2) + u_4(|\lambda_{100}|^2 + |\lambda_{110}|^2)$ in that case. Payoff vectors $(E_1, E_2)$ for all possible profiles $(\sigma^1, (\sigma^2, \sigma^3))$ are placed in the matrix representation in Figure 2 (for convenience we convert binary indices $(x_1, x_2, x_3)$ of $\lambda_{x_1, x_2, x_3}$ to the decimal numeral system).

Let us examine the game in Figure 2 to answer to what degree passing to the quantum domain may influence the result of the game. Notice first that $\text{6}$ is indeed the quantum game in the spirit of the MW approach - the normal representation of $\Gamma_1$ can be obtained from $\Gamma_1^{\text{MW}}$ by putting $|\lambda_0|^2 = 1$ and $|\lambda_i|^2 = 0$ for $x = 1, 2, \ldots, 7$, i.e., if we put $|\psi_{in}\rangle = |000\rangle$. More generally: $\Gamma_1^{\text{MW}}$ coincides to a game isomorphic to the normal representation of $\Gamma_1$ if we put as $|\psi_{in}\rangle = |x_1, x_2, x_3\rangle$ any basis state. Then $\Gamma_1^{\text{MW}}$ is equal to $\Gamma_1$ up to the order of players' strategies. The game $\Gamma_1$ favors player 1 as we learnt in Section 3.

Thus, an interesting problem is to look for another form of the initial state $\text{7}$ that imply fairer solution unavailable in the game $\Gamma_1$. Let us study first:

$$|\psi_{in}\rangle = \frac{1}{2} (|000\rangle + |001\rangle + |100\rangle + |110\rangle).$$

(11)

Through the substitution $|\lambda_0|^2 = |\lambda_1|^2 = |\lambda_4|^2 = |\lambda_6|^2 = 1/4$ (the other squares of the moduli equal 0) to entries of the matrix representation in Figure 2 we obtain a game where the only reasonable equilibrium profile is $\sigma^1 \otimes \sigma^2 \otimes \sigma^3$ with corresponding expected utility vector $E = (E_1, E_2)$ equal $(u_1 + u_4)/2$. The other pure equilibria: $\sigma^1 \otimes \sigma^2 \otimes \sigma^3$ and $\sigma^1 \otimes \sigma^2 \otimes \sigma^1$ - both generating the utility outcome $(u_1 + u_4)/4$ are obviously worse for both players so they won’t be chosen. Moreover, $\sigma^1 \otimes \sigma^2 \otimes \sigma^3$ is an imitation of a subgame perfect equilibrium - the strategy of the second player $\sigma^2 \otimes \sigma^3$ is the best response to any strategy of the first player. To sum up, the initial state $\text{11}$ is beneficial to player 2 compared with the classical case. It turns out that the answer to the question: is there any $|\psi_{in}\rangle$ allowing to obtain a fair division of $\mathcal{E}$, is also positive. Let us consider any state of the form:

$$|\psi_{in2}\rangle = \sqrt{\frac{1}{2\delta'}}|000\rangle + \sqrt{\frac{1}{2\delta'}}|001\rangle,$$

where $\frac{1}{2} < \delta < \delta' < 1$.  

(12)
Once again the profile $\sigma_0^1 \otimes \sigma_0^2 \otimes \sigma_0^3$ constitutes a Nash equilibrium and the strategy of the second player $\sigma_0^2 \otimes \sigma_0^3$ weakly dominates her other strategies as a result of putting $|\lambda_0|^2 = 1/2\delta'$ and $|\lambda_1|^2 = 1 - 1/2\delta'$ in the game in Figure 2. Since there are no other profiles with that property, $\sigma_1^1 \otimes \sigma_0^2 \otimes \sigma_0^3$ is the most reasonable scenario that implies $E_{1,2}(\sigma_0^1 \otimes \sigma_0^2 \otimes \sigma_0^3) = \epsilon/2$. The superposition of the third qubit (the second qubit of player 2) is essential to obtain fair division result since it is impossible to achieve $\delta \epsilon$ by player 1 then. Therefore, the payoff $\epsilon/2$ becomes the most attractive for her now.

The conclusions we can draw from the analysis of the MW approach to the ultimatum game are as follows. First, the game $\Gamma_1^{MW}$ that begins with $|\psi_0^1\rangle$ discloses a game tree different from the one in Figure 1a). If there is a protocol for quantizing the extensive state is in the state $|\psi|_\text{qubits}$ as we have seen in the case (11). In particular, if each qubit of the initial superposition of a player’s qubit causes some limitation on players’ influence on their histories implies the normal representation specified by only these four payoff outcomes. However, the game $\Gamma_1^{MW}$ where the initial state take the form of (11) has five different outcomes. Notice, that is not irrelevant issue bearing in mind the fact that the bimatrix of a strategic game played classically as well as played by the MW protocol always have the same dimension.

The case where game begins with the state (12) is applied shows that even a separable initial state can influence significantly a result of $\Gamma_1$. It is not strange property. Any superposition of a player’s qubit causes some limitation on players’ influence on their qubits as we have seen in the case (11). In particular, if each qubit of the initial state is in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, no player can affect amplitudes of her qubit applying only $\sigma_0$ and $\sigma_1$ (measurement outcomes 0 and 1 on qubit occur with the same probability). Then the result of the game only depends on the initial state $|\psi_0^1\rangle = |+\rangle|+\rangle|+\rangle$.

5.2 The EWL approach

As we have seen, the two-element set of unitary operators is too simple in some cases. The two-parameter unitary operations used in the EWL protocol allow to avoid player’s powerlessness when she acts on $|+\rangle$, and generally each player can essentially affect amplitudes of the initial state. Thus, it is interesting to find a result of the ultimatum game played according to the EWL approach. Let the following six-tuple be given:

$$\Gamma_1^{EWL} = (\mathcal{H}, \{1, 2\}, |\psi_{000}\rangle, \xi, \{\{U(\theta, \beta)\}_1, \{E_1\}_{i_e}\}$$

where:

- $\mathcal{H}$ is a Hilbert space $\bigotimes_{j=1}^3 C^2$ with the basis $\{|\psi_{x_1, x_2, x_3}\rangle: x_j = 0, 1\}$ of entangled states defined as follows:

$$|\psi_{x_1, x_2, x_3}\rangle = \frac{|x_1, x_2, x_3\rangle + i|\overline{x_1}, \overline{x_2}, \overline{x_3}\rangle}{\sqrt{2}}$$

- the mapping $\xi$ is the same as in six-tuple (6);

- the player’s actions $\{U(\theta, \beta): \theta \in [0, \pi], \beta \in [0, \pi/2]\}$, studied, for example, in the paper [3], form an alternative to two-parameter unitary operations used in $\Gamma_1$. They are of the form:

$$U(\theta, \beta) = \begin{pmatrix}
\cos(\theta/2) & i e^{i\beta} \sin(\theta/2) \\
ie^{-i\beta} \sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}$$

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\[ E_i = \frac{1}{2} \langle x_3 | \psi_{00,0} \rangle + \frac{1}{2} \langle x_2 | \psi_{01,0} \rangle + \delta \langle x_2 | \psi_{01,0} \rangle \]

Each strategy \( U_1 \) of player 1 is simply \( U(\theta_1, \beta_1) \). The strategies of the second player are chosen in a manner similar to \( U_1^{MW} \) - they are tensor products \( U_2 \otimes U_3 = U(\theta_2, \beta_2) \otimes U(\theta_3, \beta_3) \). The final state \( |\psi_{000}\rangle \) corresponding to a profile \( \tau = ((\theta_1, \beta_1), (\theta_2, \beta_2, \theta_3, \beta_3)) \) is as follows:

\[ |\psi_{000}\rangle = U_1 \otimes U_2 \otimes U_3 |\psi_{000}\rangle = \frac{1}{\sqrt{2^3}} \sum_{x \in \{0,1\}^3} u_x |x\rangle, \quad (17) \]

where

\[ u_{x_1,x_2,x_3} = \sum_{j} e^{-i \sum x_j \theta_j} \prod_j \cos \left( \frac{x_j \pi - \beta_j}{2} \right) \]

and \( j = 1, 2, 3 \), \( x_j = 0, 1 \), and \( \bar{x}_j \) is negation of \( x_j \). Putting (16) and (17) into formula (5) we obtain the following expected payoff vector \( (E_1(\tau), E_2(\tau)) \):

\[ (E_1(\tau), E_2(\tau)) = \frac{u_t}{u_t} \left[ \begin{array}{c} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \left( \cos^2 \frac{\theta_3}{2} + \sin^2 \frac{\theta_3}{2} \cos^2 \beta_3 \right) \\ + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \left( \sin^2 \frac{\theta_3}{2} \sin^2 (\beta_1 + \beta_2 + \beta_3) + \cos^2 \frac{\theta_3}{2} \sin^2 (\beta_1 + \beta_2) \right) \end{array} \right] \]

\[ + u_u \left[ \begin{array}{c} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_3}{2} \left( \cos^2 \frac{\theta_2}{2} \cos^2 (\beta_1 + \beta_3) + \sin^2 \frac{\theta_2}{2} \cos^2 (\beta_1 + \beta_2) \right) \\ + \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_3}{2} \left( \sin^2 \frac{\theta_2}{2} \sin^2 (\beta_2 + \beta_3) + \cos^2 \frac{\theta_2}{2} \sin^2 \beta_3 \right) \end{array} \right]. \quad (19) \]

Let us check first that \( \Gamma_1^{EWL} \) generalizes the classical ultimatum game \( \Gamma_1 \). Pure strategies of the first player are represented by \( U(0,0) \) and \( U(\pi,0) \). Similarly, the set of strategies of the second player in \( \Gamma_1 \) is represented by a set \( \{U(\theta_2,0) \otimes U(\theta_3,0) : \theta_2, \theta_3 \in \{0, \pi\}\} \) since the set of profiles

\[ \{((\theta_1,0), (\theta_2,0, \theta_3,0)) : \theta_1, \theta_2, \theta_3 \in \{0, \pi\}\} \quad (20) \]

in (13) and the set of profiles

\[ \{(c_k, d_k, e_k) : k_1, k_2, k_3 \in \{0,1\}\} \quad (21) \]

in (2) generate the same payoffs. Equivalents of behavioral strategies of \( \Gamma_1 \) (i.e., independent probability distributions \( p, q \) and \( r \) over the actions \( c_k, d_k \) and \( e_k \), respectively, specified by players at their own information sets) can be found among unitary
strategies as well. If we restrict unitary actions to $U(\theta, 0)$, i.e., to profiles of the form $((\theta_1, 0), (\theta_2, 0, \theta_3, 0))$, $\theta_j \in [0, \pi]$, the right-hand side of Eq. (19) takes the form:

$$u_t \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + u_u \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_3}{2}.$$  

(22)

By substituting $p$ for $\cos^2(\theta_1/2)$, $q$ for $\cos^2(\theta_2/2)$, and $r$ for $\cos^2(\theta_3/2)$ we get the expected payoffs corresponding to any behavioral strategy profile $((p, 1-p), ((q, 1-q), (r, 1-r)))$ in $\Gamma_1$.

Let us examine an impact of the unitary strategies on a result of the EWL approach to $\Gamma_1$. In particular we ask the question if the unfair division $u_u$ or the fair division $u_t$ in $\Gamma_1^{EWL}$ is more probable. Notice, that the profile $((\theta_1, \beta_1), (\theta_2, \beta_2, \theta_3, \beta_3)) = ((\pi, 0), (0, 0, 0, 0))$ (corresponding to subgame perfect equilibrium $(c_1, d_0 e_0)$ in $\Gamma_1$) is not Nash equilibrium in $\Gamma_1^{EWL}$. The second player can gain by choosing, for example, $(\theta_2, \beta_2, \theta_3, \beta_3) = (\pi, \pi/2, \pi, 0)$ instead of $(0, 0, 0, 0)$. Then she obtains the fair division payoff. Moreover, for any other strategy of the first player $(\theta_1, \beta_1)$, player 2 can select, for instance, $(0, 0, 0, 0)$ to obtain a payoff being a mixture of $u_t$ and $u_u$. This proves that the unfair division $u_u$ cannot be a result in (13). The fair division $u_t$ in turn can be achieved through continuum of Nash equilibria. Let us denote by $\text{NE}(\Gamma_1^{EWL})$ the set of all Nash equilibria of $\Gamma_1^{EWL}$. An examination of (19) shows that:

$$\left\{ ((\pi, \beta_1), (\pi, \beta_2, \pi, \beta_3)) : \beta_2 + \beta_3 \leq \frac{\pi}{4}, \sum_{j=1}^{3} \beta_j = \frac{\pi}{2} \right\} \subset \text{NE}(\Gamma_1^{EWL})$$

(23)

as well as

$$\left\{ ((0, \beta_1), (0, \beta_2, \pi, 0)) : \beta_1, \beta_2 \in \left[0, \frac{\pi}{2}\right] \right\} \subset \text{NE}(\Gamma_1^{EWL}).$$

(24)

Moreover, all strategy profiles of these sets generate the payoff vector $u_t$ for any division factor $1/2 < \delta < 1$. To prove inclusion (23) let us consider any strategy $(\theta_1', \beta_1')$ of player 1 given that player 2’s strategy from (23) is fixed. Then for $\beta_2 + \beta_3 \leq \pi/4$ we have

$$E_1((\theta_1', \beta_1'), (\pi, \pi, \beta_2, \beta_3)) = \left[\frac{1}{2} \mathcal{E} \sin^2 \frac{\theta_1'}{2} \sin^2 (\beta_1' + \beta_2 + \beta_3) + \delta \mathcal{E} \cos^2 \frac{\theta_1'}{2} \sin^2 (\beta_2 + \beta_3)\right].$$

(25)

Since $\beta_2 + \beta_3 \leq \pi/4$, the maximum value of (25) is achieved if the second element of the sum is 0. It implies that the best response of player 1 is $\theta_1' = \pi$ and $\beta_1' = \pi/2 - \beta_2 - \beta_3$. The second player cannot gain by deviating as well because she always obtains no more than $\mathcal{E}/2$ in $\Gamma_1^{EWL}$. Therefore, each profile of set (23) indeed constitutes Nash equilibrium. Inclusion (21) can be proved in similar way. Notice that there are also Nash equilibria different from (23) and (24) that generate the payoff outcome $\mathcal{E}/2$ for both players. For example, a strategy profile $((\pi, \pi/4), (\pi, \pi/4, \pi/2, 0))$.

Intuitively, a huge number of fair solutions in $\Gamma_1^{EWL}$ being NE together with a lack of an equilibrium outcome $u_u$ favors the second player in comparison to the classical game $\Gamma_1$. However, it does not assure the second player the fair payoff $\mathcal{E}/2$ yet. Since the players choose their strategies simultaneously, they cannot coordinate them. If the first player unilaterally deviates from a strategy dictated by (21) and she plays a strategy being a part of (23) then both players receive nothing as we have $E_{1,2}((\pi, \beta_1), (0, \beta_2, \pi, 0)) = 0$ for all $\beta_1, \beta_2 \in [0, \pi/2]$. On the other hand, it turns out that the statement that each of these equilibria is equally likely to occur is not true.
Let us investigate which equilibria in $\Gamma_1$ are preserved in $\Gamma_{1}^{EWL}$ bearing in mind that the unitary strategies $U(\theta, 0)$ are quantum counterparts to classical moves in $\Gamma_1$. As we have seen there is no equilibrium profile in $\Gamma_{1}^{EWL}$ that allows the first player to gain $\delta\mathcal{E}$. Therefore, in particular, the unfair division equilibrium $(c_1, d_0e_0)$ of $\Gamma_1$ cannot be generated by any unitary operations $U(\theta, 0)$. However, each fair division equilibrium (pure or mixed) of $[2]$ can be reconstructed in $[13]$. The profile $((0, 0), (0, 0, \pi, 0))$ corresponding to the equilibrium $(c_0, d_0e_1)$ in $\Gamma_1$ is Nash equilibrium of $\Gamma_{1}^{EWL}$ since it is element of the set $[24]$. Next, the mixed equilibria mentioned in Section $[5]$ can be implemented in $\Gamma_{1}^{EWL}$ as follows: they are the profiles where the first player chooses $U(0, 0)$ and the second player chooses either $U(0, 0) \otimes U(0, 0)$ with probability $p \in [0, 1/2\delta]$ and $U(0, 0) \otimes U(\pi, 0)$ with probability $1 - p$, or in a language of behavioral strategies she just takes an operator from $\{U(0, 0) \otimes U(\theta, 0) : \theta \in [2\arccos(1/\sqrt{2\delta}), \pi/2]\}$. According to the concept of Schelling Point $[12]$ players tend to select a solution that is the most natural as well as the most distinctive among all possible choices. Therefore, if we assume that the players prefer the fair division, they choose a profile that is an equilibrium of both $\Gamma_1$ and $\Gamma_{1}^{EWL}$ among all equal equilibria of $\Gamma_{1}^{EWL}$. Since all these shared equilibria generate the same outcome, the pure equilibrium is the most natural and it ought to be chosen as the Schelling Point.

6 Extensive form of the quantum ultimatum game

In subsection $[5.1]$ we made observation that an extensive game and its quantum realization differ not only in utilities but also in game trees. Now, we are going to give the answer to the question how would a game tree of such quantum realization look like? Let us reconsider an extensive game form given by the game tree on Figure $[1]$, where the answer to the question how would a game tree of such quantum realization look like? Let us reconsider an extensive game form given by the game tree on Figure $[1]$, where the components $H$, $P$ and $L_i$ are derived from $\Gamma_1$, and the outcomes $O_{00}, O_{01}, O_{10}$ and $O_{11}$, are assigned to the terminal histories $(c_0, d_0), (c_0, d_1), (c_1, e_0)$ and $(c_1, e_1)$, respectively, instead of particular payoff values. Let us denote this problem as:

$$\Gamma_2 = (\{1, 2\}, H, P, L_i, O).$$

(26)

Then the tuple $\Gamma_{2}^{MW}$ associated with $\Gamma_2$ is derived from $\Gamma_{1}^{MW}$ and only the payoff functionals $E_i$ undergo appropriate modifications. Let us write $\Gamma_{2}^{MW}$ in the language of density matrices, for convenience. That is:

$$\Gamma_{2}^{MW} = (\mathcal{H}, \{1, 2\}, \rho_{\text{in}}, \xi, \{\sigma_0, \sigma_1\}_i, X),$$

(27)

where

- $\rho_{\text{in}}$ is a density matrix of the initial state $[7]$;
- the outcome operator $X$ is a sum of $X^0 + X^1$ defined as:

$$X^0 = O_{00}|00\rangle\langle 00| \otimes I + O_{01}|01\rangle\langle 01| \otimes I;$$

$$X^1 = O_{10}|11\rangle\langle 11| \otimes I \otimes |0\rangle\langle 0| + O_{11}|11\rangle\langle 11| \otimes I \otimes |1\rangle\langle 1|. $$

(28)

In this case, the density matrix $\rho_{\text{fin}}$ of the final state $|\psi_{\text{fin}}\rangle$ takes a form

$$\rho_{\text{fin}} = \sigma_{k_1}^{1} \otimes \sigma_{k_2}^{2} \otimes \sigma_{k_3}^{3} \rho_{\text{fin}} \sigma_{k_1}^{1} \otimes \sigma_{k_2}^{2} \otimes \sigma_{k_3}^{3}.$$

(29)

The outcome functionals $[5]$ are then equivalent to the following one:

$$E (\sigma_{k_1}, (\sigma_{k_2}, \sigma_{k_3}^{3})) = \text{tr} (X \rho_{\text{fin}}).$$

(30)
In order to give an extensive form to determine the final state $\rho_{\text{fin}}$ in $\Gamma_{2}^{\text{MW}}$ let us modify the way (29) of calculating the final state $\rho_{\text{fin}}$. To begin with, player 1 acts on the first qubit. Next, player 2 carries out a measurement on that qubit in the computational basis to find out what is a current state of the game. Then she performs an operation on either the second or the third qubit of the post-measurement state depending on whether the measurement outcome 0 or 1 has occurred. The operation of the second player ultimately defines the final state that is inserted to the formula (30). The procedure can be formalized as follows:

**Sequential procedure**

1. $\sigma_{\kappa_1}^1 \rho_{\text{in}} \sigma_{\kappa_1}^1 = \rho_{\kappa_1}$ the player 1 performs an operation $\sigma_{\kappa_1}^1$ on her qubit of the initial state $\rho_{\text{in}}$

2. $\frac{M_i \rho_{\kappa_1} M_i}{\text{tr}(M_i \rho_{\kappa_1})} = \rho_{\kappa_1,\iota}$, $p_{\kappa_1,\iota} = \text{tr}(M_i \rho_{\kappa_1,\iota})$ the player 2 prepares the measurement $\{M_0, M_1\}$ defined by $M_i = |i\rangle \langle i| \otimes I$, $\iota = 0, 1$ on the first qubit of the state $\sigma_{\kappa_1} \rho_{\text{in}} \sigma_{\kappa_1}$ (the probability of obtaining result $\iota$ is denoted by $p_{\kappa_1,\iota}$)

3. $\sum_{\iota} p_{\kappa_1,\iota} \sigma_{\kappa_2,\iota}^2 \rho_{\kappa_1,\iota} \sigma_{\kappa_2,\iota}^2 = \rho_{\text{fin}}$ if a measurement outcome $\iota$ occurs, the player 2 performs an operation $\sigma_{\kappa_2+\iota}$ on $\iota + 2$ qubit of the post-measurement state

It turns out that for any strategy profile $(\sigma_{\kappa_1}^1, (\sigma_{\kappa_2}^2, \sigma_{\kappa_3}^3))$ the final state $\rho_{\text{fin}}$ defined both by the formula (29) and by the sequential procedure determine the same outcome of the game $\Gamma_{2}^{\text{MW}}$.

**Proof.** Let density operator $\rho_{\text{in}}$ of a state (7) be given. Then the state $\rho_{\text{fin}}'$ after the third step of procedure can be expressed as:

$$
\rho_{\text{fin}}' = \sigma_{\kappa_2}^2 M_0 \rho_{\kappa_1} M_0 \sigma_{\kappa_2}^2 + \sigma_{\kappa_3}^3 M_1 \rho_{\kappa_1} M_1 \sigma_{\kappa_3}^3
= M_0 \sigma_{\kappa_2}^2 \rho_{\kappa_1} \sigma_{\kappa_2}^2 M_0 + M_1 \sigma_{\kappa_3}^3 \rho_{\kappa_1} \sigma_{\kappa_3}^3 M_1.
$$

(31)

Since $X^\kappa M_i = \delta_{\kappa_1} X^\kappa$, where $\delta_{\kappa_1}$ is the Kronecker’s delta, we obtain:

$$
\text{tr}(X \rho_{\text{fin}}') = \text{tr}(X^0 \sigma_{\kappa_2}^2 \rho_{\kappa_1} \sigma_{\kappa_2}^2 + X^1 \sigma_{\kappa_3}^3 \rho_{\kappa_1} \sigma_{\kappa_3}^3).
$$

(32)

Notice that operation $\sigma_1$ on the second (third) qubit of any state (7) does not influence the measurement of outcomes $O_{10}$ and $O_{11}$ ($O_{00}$ and $O_{01}$), because of the form of $X^1$ ($X^0$), which means that:

$$
\text{tr}(X^i \sigma_{\kappa_2}^2 \rho_{\kappa_1} \sigma_{\kappa_2}^2) = \text{tr}(X^i \sigma_{\kappa_2}^2 \sigma_{\kappa_3}^3 \rho_{\kappa_1} \sigma_{\kappa_3}^3) \text{ for } \iota = 0, 1.
$$

(33)

Inserting (33) into the formula (32) we get:

$$
\text{tr} (X \rho_{\text{fin}}') = \text{tr} \left( (X^0 + X^1) \left( \bigotimes_{j=1}^{3} \sigma_{\kappa_j}^j \rho_{\text{in}} \bigotimes_{j=1}^{3} \sigma_{\kappa_j}^j \right) \right).
$$

(34)

The right-hand side of (34) is equal the expected outcome given by formula (30). Thus, the two ways of determining the final state are outcome-equivalent.

We claim that performing quantum measurement is a more natural manner to play quantum games than observation of player’s actions taken previously - the way suggested
by games played classically. Since the result of a quantum game is determined by the measurement outcome of the final state instead of actions taken by players, each stage of the quantum game also ought to be set via a quantum measurement of a current state. Moreover, when we suppose the second player’s move dependence on actions of the first player in $\Gamma_2$ then it implies the same game tree as in Figure 1a). This way, however, stands in contradiction to the results in subsection 5.1 that tell us that the game trees must be different. Of course, if the initial state is $|000\rangle\langle000|$ (i.e., when game given by (27) boils down to a game (26)), observation of the course of the game played classically and with the use of quantum measurement coincide.

Let us study what a game tree corresponding to the game $\Gamma^\text{MW}_2$ is yielded by the above-mentioned procedure. According to the first step, the initial history $\emptyset$ is followed by two actions of the first player. Next, the measurement on the first qubit is made. The two possible measurement outcomes $\iota = 0, 1$ can be identified with two actions (following each player 1’s move) of a chance mover that are taken with probability $p_{\kappa_1,\iota}$. Finally the player 2 acts on $\iota + 2$ qubit of the state $\rho_{\iota}$ after each history associated with the outcome $\iota$. Therefore, all histories followed by given outcome $\iota$ constitutes an information set of player 2. Such description in a form of a game tree is illustrated in Figure 3. The outcomes $O'_{0,\kappa_1,\kappa_2}$ and $O'_{1,\kappa_1,\kappa_3}$ are determined by the following equations:

$$O'_{0,\kappa_1,\kappa_2} = \text{tr}(X\sigma_{\kappa_2}\rho_{\kappa_1,0}\sigma_{\kappa_2}), \quad O'_{1,\kappa_1,\kappa_3} = \text{tr}(X\sigma_{\kappa_3}\rho_{\kappa_1,1}\sigma_{\kappa_3}).$$

(35)

We have proved that the two approaches: (29) and the sequential one to calculate the final state are outcome-equivalent. Therefore, it should be expected that extensive forms of $\Gamma_2$ and $\Gamma^\text{MW}_2$ coincide when the initial state is a basis state. In fact, given $\rho_{\text{in}} = |000\rangle\langle000|$ the probabilities $p_{\kappa_1,\iota}$ are expressed by the formula $p_{\kappa_1,\iota} = \delta_{\kappa_1,\iota}$, where $\kappa_1, \iota \in \{0, 1\}$. Then, the available outcomes given by Eq. (35) are as follows: $O'_{0,00} = O_{00}$, $O'_{0,01} = O_{01}$, $O'_{1,10} = O_{10}$ and $O'_{1,11} = O_{11}$. By identifying $\sigma_{\kappa_1} := c_{\kappa_1}$, $\sigma_{\kappa_2} := d_{\kappa_2}$, $\sigma_{\kappa_3} := e_{\kappa_3}$ the extensive game in Figure 3 represents game $\Gamma_2$. 

Figure 3: The extensive game associated with the quantum realization $\Gamma^\text{MW}_2$. 

7 Conclusion

We have shown that our proposal extends the ultimatum game in the quantum area. Although proposed scheme is suitable only for a normal representation of the ultimatum game in which some features of corresponding game in extensive form are lost, it passes on valuable information about how passing to the quantum domain influences a course of extensive games. The dominant position of player 1, when the ultimatum game is played classically, can be weakened in the case of playing the game via both the MW approach and the EWL approach. Another thing worth noting is that the quantization significantly extends the game tree compared with classical case. It makes the normal representation to be more convenient way to analyze the game than the way of extensive form.

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