A POLYHARMONIC MAASS FORM OF DEPTH 3/2 FOR $SL_2(\mathbb{Z})$

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Abstract. Duke, Imamoglu, and Tóth constructed a polyharmonic Maass form of level 4 whose Fourier coefficients encode real quadratic class numbers. A more general construction of such forms was subsequently given by Bruinier, Funke, and Imamoglu. Here we give a direct construction of such a form for the full modular group and study the properties of its coefficients. We give interpretations of the coefficients of the holomorphic parts of each of these polyharmonic Maass forms as inner products of certain weakly holomorphic modular forms and harmonic Maass forms. The coefficients of square index are particularly intractable; in order to address these, we develop various extensions of the usual normalized Peterson inner product using a strategy of Bringmann, Ehlen and Diamantis.

1. Introduction

We begin by discussing a polyharmonic Maass form of level 4. For $n \geq 0$, let $H(n)$ denote the Hurwitz class number. We have $H(0) = -1/12$ and $H(n) = 0$ for $n \equiv 1, 2 \pmod{4}$. Otherwise $H(n)$ is the number of positive definite quadratic forms of discriminant $-n$, counted with multiplicity equal to the inverse of the order of their stabilizer in $SL_2(\mathbb{Z})$.

Zagier [21] introduced the first example of what is known as a harmonic Maass form. For $y > 0$ let $\beta_k(y)$ denote the normalized incomplete gamma function

$$\beta_k(y) := \frac{\Gamma(1-k,y)}{\Gamma(1-k)} = \frac{y^{1-k}}{\Gamma(1-k)} \int_1^\infty t^{-k} e^{-yt} \, dt. \quad (1.1)$$

Zagier defined the function

$$\hat{Z}_-(\tau) := \sum_{n \geq 0} H(n)q^n + \frac{1}{8\pi \sqrt{y}} - \frac{1}{4} \sum_{n \neq 0} |n|^2 \beta_\frac{3}{2} (4\pi n^2 y) q^{-n^2} \quad (1.2)$$

(we use the notation $\hat{Z}_-$ to follow the notation of Duke, Imamoglu, and Tóth [12]). Here, and throughout, $\tau = x + iy$ and $q = e(\tau) = e^{2\pi i \tau}$. Zagier showed that the function $\hat{Z}_-(\tau)$ transforms like a modular form of weight $3/2$ on $\Gamma_0(4)$ and that

$$\xi_k \hat{Z}_- = -\frac{1}{16\pi} \theta,$$

where

$$\xi_k := 2iy^k \frac{\partial}{\partial \tau} \quad (1.3)$$

and $\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}$ is the usual theta function.
Suppose that $d$ is a non-square discriminant. If $d < 0$ then let $\omega_d$ be half the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, and if $d > 0$ let $\varepsilon_d$ be the fundamental unit in $\mathbb{Q}(\sqrt{d})$. Following [13, §2], define the regulator

$$R(d) := \begin{cases} 2\pi\omega_d^{-1} & \text{if } d < 0, \\ 2\log\varepsilon_d & \text{if } d > 0, d \neq \Box, \end{cases}$$

and the general Hurwitz function

$$h^*(d) := \frac{1}{2\pi} \sum_{\ell \mid d} R\left(\frac{d}{\ell^2}\right) h\left(\frac{d}{\ell^2}\right),$$

where $h(d)$ is the class number. Note that for $d < 0$ we have $h^*(d) = H(|d|)$. Define

$$\alpha(y) := \frac{\sqrt{y}}{4\pi} \int_0^\infty e^{-\pi y t^2} \frac{1}{t^{1/2}} \log(1 + t) \, dt.$$ 

Duke, Imamoḡlu, and Tóth [12, Theorem 4] (see also [13, (4.2)]) showed that there is a nonholomorphic modular form of weight $1/2$ on $\Gamma_0(4)$ whose Fourier expansion is

$$\hat{Z}^+ (\tau) := \sum_{d > 0, \ d \neq \Box} h^*(d) \frac{q^d}{\sqrt{d}} + \sum_{n > 0} a(n^2) q^{n^2} \frac{\sqrt{y}}{3} + \sum_{d < 0} h^*(d) \frac{\sqrt{|d|}}{\beta_1(4\pi|d|y)q^d}$$

$$- \frac{1}{4\pi} \log y + \sum_{n \neq 0} \alpha(4n^2 y)q^{n^2} - \frac{1}{\pi} \left( \frac{\zeta'(2)}{\zeta(2)} - \gamma + \log 4 \right)$$

and for which

$$\xi_+ \hat{Z}^+ = -2 \hat{Z}^-.$$ 

Here $\gamma$ denotes Euler’s constant, and we have corrected the value of the constant term using (5.4) and (2.24) of [12].

The form $\hat{Z}^+ (\tau)$ is defined through a limit of Poincaré series [12, (5.4)]. The coefficients $a(n^2)$ are particularly intractable since they correspond to poles of the Poincaré series, and they are not determined in [12].

Bruinier, Funke, and Imamoḡlu [9] introduced a general regularized theta lift which lifts weak Maass forms of weight zero to polyharmonic Maass forms of weight $1/2$. By polyharmonic we mean that the form is annihilated by repeated application of the operators $\xi_k$; a precise definition is in Section 2. Applying this lift to the constant function 1 produces a function $Z(\tau)$ which differs from $\hat{Z}^+ (\tau)$ by a constant multiple of $\theta(\tau)$, and which provides an interpretation of the mysterious coefficients of square index. After some computation using Theorem 4.2 and Remark 3.4 of [9] one can describe this form in such a way that every non-trivial coefficient has an interpretation in terms of the general Hurwitz function.

To state this result, we extend the definition of $R(d)$ as follows:

$$R(d) := \begin{cases} 2\pi\omega_d^{-1} & \text{if } d < 0, \\ 2\log\varepsilon_d & \text{if } d > 0, d \neq \Box, \\ 2\log \sqrt{d} & \text{if } d = \Box, \end{cases}$$

and define $h^*(d)$ via (1.5). Then the work of Bruinier, Funke, and Imamoḡlu implies the following. We note that there are a few typos in [9, Theorem 4.2]; details and a sketch of the computation which produces the following result are given in Section 6 below.
Theorem 1. There is a polyharmonic Maass form of weight $1/2$ and depth $3/2$ on $\Gamma_0(4)$ whose Fourier expansion is
\[
Z(\tau) := \sum_{d > 0} \frac{h^*(d)}{\sqrt{d}} q^d + \frac{\sqrt{y}}{3} + \sum_{d < 0} \frac{h^*(d)}{\sqrt{|d|}} \beta_2 \left(4\pi |d|y\right) q^d + \frac{\gamma - \log(16\pi y)}{4\pi} + \sum_{n \neq 0} \alpha(4n^2y)q^{n^2} \tag{1.9}
\]
and for which
\[
\xi_1 Z = -2\hat{Z}_-.
\]

The main result of Duke, Imamoglu, and Tóth [13] gives an interpretation of the coefficients $h^*(d)$ of $\hat{Z}_+$ as regularized inner products in the case when $d > 0$ is not a square. To describe the result, we recall that for each $d > 0$ there exists a unique weight $3/2$ weakly holomorphic modular form $g_d$ on $\Gamma_0(4)$ of the form
\[
g_d(\tau) = q^{-d} + \sum_{0 \leq n \equiv 0, 3(4)} B(d, n) q^n,
\]
where the $B(d, n)$ are integers and
\[
B(d, 0) = \begin{cases} 
-2 & \text{if } d = \Box, \\
0 & \text{otherwise}.
\end{cases}
\]

Proposition 4.1 of [13] gives the formula
\[
\langle g_d, \hat{Z}_- \rangle_{\text{reg}} = -3 \frac{h^*(d)}{4} \sqrt{d} \quad \text{if } d > 0 \text{ is not square.}
\]

Here $\langle \cdot, \cdot \rangle_{\text{reg}}$ is the usual regularized inner product. The integral defining this inner product does not converge when $d$ is square.

Motivated by recent work of Bringmann, Diamantis and Ehlen [7] we introduce a natural inner product $\langle \cdot, \cdot \rangle_4$ which extends $\langle \cdot, \cdot \rangle_{\text{reg}}$ and which allows us to treat the case when $d$ is square. We give the precise definition in Section 6. Letting
\[
\delta_{\Box}(d) = \begin{cases} 
1 & \text{if } d \text{ is square}, \\
0 & \text{otherwise},
\end{cases}
\]
we prove the following.

Theorem 2. For every positive discriminant $d$ we have
\[
\langle g_d, \hat{Z}_- \rangle_4 = -\frac{h^*(d)}{\sqrt{d}} + \delta_{\Box}(d) \left( \frac{\gamma - \log 4\pi}{2\pi} \right).
\]

Our main goal in this paper is to introduce and to study a polyharmonic Maass form analogous to $Z(\tau)$ on the full modular group. Let $p(n)$ denote the partition function, and let $spt(n)$ denote the number of smallest parts in the partitions of $n$. This function has been the object of much study (see, for example, [2, 5, 6, 14, 15], and the references in these papers). Let $\chi_{12}$ denote the Kronecker character for $\mathbb{Q}(\sqrt{3})$. If we define
\[
s(n) := spt(n) + \frac{1}{12} (24n - 1) p(n), \tag{1.10}
\]
then work of Bringmann [6] (see the next section for details) shows that, in analogy with (1.2), the generating function

\[ F(\tau) := \sum_{n=-\infty}^{\infty} s \left( \frac{n+1}{24} \right) q^{\frac{n^2}{24}} - \frac{1}{2} \sum_{n=1}^{\infty} \chi_{12}(n) \beta_n \left( \frac{\pi n^2 y}{6} \right) q^{\frac{n^2}{24}} \]  

(1.11)

is a harmonic Maass form of weight 3/2 on SL$_2(\mathbb{Z})$ with a certain multiplier. In particular we have

\[ \xi_{\frac{3}{2}} F = -\frac{\sqrt{6}}{4\pi} \eta, \]  

(1.12)

where \( \eta(\tau) := q^{\frac{1}{4\pi}} \prod_{n \geq 1} (1 - q^n) \) is Dedekind’s eta function. In analogy with (1.7) and (1.9) we introduce a polyharmonic Maass form \( H(\tau) \) in Theorem 3 below with

\[ \xi_{\frac{1}{2}} H = -2\sqrt{6} F. \]

The non-trivial coefficients of positive index are given by traces of a certain modular function of level 6 over geodesics on the modular curve. Those of negative index are given by the numbers \( s(n) \), which are known by recent work of the first two authors [2] to be traces of the same modular function over quadratic points in the upper half plane (see (2.4) below). Again, the terms of square index are intractable due to poles in the Poincaré series.

In order to deduce (2.4), the authors of [2] used a theta lift of Bruinier-Funke [10]. In a similar way, the theta lift of [9] could be used to deduce Theorem 3 below. Here we compute the expansion directly from the limit definition. Of course, most of the difficulty comes from the coefficients of square index.

We remark that this leads to a proof of the algebraic formula (2.4) which does not involve the theta lift. We also remark that a similar argument applied to a suitable modification of the limit definition [12, (5.4)] of \( \tilde{Z}_+(\tau) \) produces the expansion (1.9) without recourse to the theta lift. We give a brief discussion in Section 6. We also define an appropriate inner product and give an analog of Theorem 2 in this context.

In the next section we give some background and describe the form \( H(\tau) \). Section 3 contains the limit definition of the form \( H(\tau) \) as well as some technical results on convergence issues (which are somewhat subtle). In Sections 4 and 5 we compute the coefficients of non-square and square index, respectively. In Section 7 we discuss inner products, and we provide an interpretation for the inner product of the form \( F \) with members of a family of weakly holomorphic modular forms of weight 3/2 on SL$_2(\mathbb{Z})$. In Section 6 we sketch the derivation of (1.9) using the current methods, and we give a proof of Theorem 2.

2. A Polyharmonic Maass Form of Depth 3/2 for SL$_2(\mathbb{Z})$

Let \( \text{spt}(n) \) denote the number of smallest parts in the partitions of \( n \), and let \( s(n) \) and \( F(\tau) \) be defined as in (1.10) and (1.11). Let \( \eta(\tau) \) be the Dedekind eta function, and define the multiplier \( \chi \) by

\[ \eta(\gamma \tau) = \chi(\gamma) \sqrt{c \tau + d} \eta(\tau) \quad \text{for} \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}). \]  

(2.1)

After work of Bringmann [6] (see Section 3 of [2]), we know that \( F(\tau) \) is a harmonic Maass form of weight 3/2 on SL$_2(\mathbb{Z})$ with multiplier \( \chi \). Let \( E_{2k}(\tau) \) denote the Eisenstein series of weight \( 2k \) on SL$_2(\mathbb{Z})$, and let \( f(\tau) \) be the modular function on \( \Gamma_0(6) \) given by

\[ f(\tau) = \frac{1}{24} E_4(\tau) - 4E_4(2\tau) - 9E_4(3\tau) + 36E_4(6\tau) \]  

\[ = q^{-1} + 12 + 77q + \ldots. \]  

(2.2)
For \( n \equiv 1 \pmod{24} \) define
\[
Q_n := \left\{ ax^2 + bxy + cy^2 : b^2 - 4ac = n, \ 6 \mid a > 0, \ b \equiv 1 \pmod{12} \right\}.
\]
The group
\[
\Gamma := \Gamma_0(6)/\{\pm 1\}
\] (2.3)
acts on this set, and for squarefree \( n \), the class number \( h(n) \) is the size of \( \Gamma \backslash Q_n \) (see [17, Section I]).

If \( 0 > n \equiv 1 \pmod{24} \) then for each \( Q \in Q_n \) let \( \tau_Q \) denote the root of \( Q(\tau, 1) \) in the upper-half plane \( \mathbb{H} \). In [2] (see the end of Section 3) the first two authors proved the algebraic formula
\[
12 s \left( \frac{1- n}{24} \right) = \sum_{Q \in \Gamma \backslash Q_n} f(\tau_Q).
\] (2.4)
This together with some analytic considerations leads to a transcendental formula for \( \text{spt}(n) \) (Theorem 1 of [2]) which is analogous to Rademacher’s formula for \( p(n) \).

For an indefinite quadratic form \( Q \) with non-square discriminant, let \( C_Q \) denote the geodesic in the upper half plane connecting the roots of \( Q \), modulo the stabilizer of \( Q \). For positive non-square \( n \equiv 1 \pmod{24} \), define
\[
\text{Tr}_n(f) := \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash Q_n} \int_{C_Q} f(\tau) \frac{d\tau}{Q(\tau, 1)}.
\] (2.5)
The situation is more difficult when \( n \) is square. In this case the geodesics \( C_Q \) are infinite and the integrals in (2.5) are divergent. In analogy with [3, 4] we will define dampened versions of the function \( f(\tau) \) in Section 5. For each quadratic form with square discriminant we will construct a function \( f_Q(\tau) \) by subtracting off the constant term and the exponentially growing terms in the Fourier expansion of \( f \) at the cusps corresponding to roots of \( Q(\tau, 1) \). See (5.7) for the precise definition. For each indefinite form \( Q \) with non-square discriminant we define \( f_Q(\tau) := f(\tau) \) to ease notation. Then we can make the uniform definition
\[
\text{Tr}_n(f) := \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash Q_n} \int_{C_Q} f_Q(\tau) \frac{d\tau}{Q(\tau, 1)}.
\] (2.6)
for every positive \( n \equiv 1 \pmod{24} \).

We follow [18] in defining polyharmonic Maass forms. Let \( \Delta_k \) denote the weight \( k \) hyperbolic Laplacian
\[
\Delta_k := -\xi_{2-k} \xi_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\] (2.7)
Let \( m \) be a non-negative integer. We say that a real analytic function \( h : \mathbb{H} \to \mathbb{C} \) is a polyharmonic Maass form of weight \( k \) and depth \( m \) on \( G \subseteq \text{SL}_2(\mathbb{Z}) \) with multiplier \( \nu \) if
\[
\Delta_k^m h = 0
\]
and if
\[
h(\gamma \tau) = \nu(\gamma) (\text{ct} + d)^k h(\tau) \quad \text{for all } \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G.
\] (2.8)
We define half-integral depths by assigning weakly holomorphic modular forms a depth of \( \frac{1}{2} \) since they are annihilated by \( \xi_k \), which is essentially “half” of \( \Delta_k \). We say that \( h \) is polyharmonic of depth \( m + \frac{1}{2} \) if \( \Delta_k^m h \) is holomorphic.
Recall the definitions (1.6) (1.1) and (2.1) of $\alpha, \beta_k,$ and $\chi$. By [11, §8.2] we have
\[
\beta_k(y) = 1 - y^{1-k} \gamma^*(1-k, y),
\] (2.9)
where $\gamma^*(1-k, z)$ is an entire function of $z$, given by
\[
\gamma^*(1-k, z) = \frac{1}{\Gamma(1-k)} \int_0^1 t^{-k} e^{-zt} dt \quad \text{for } k < 1.
\] (2.10)
We use the principal branch of the square root, with the convention that
\[
\sqrt{-y} = i \sqrt{y} \quad \text{for } y > 0.
\]
This gives a definition of $\beta_{\frac{1}{2}}(-y)$ for $y > 0$.

In analogy with Theorem 1 we will prove

**Theorem 3.** The function
\[
H(\tau) = -i q^{\frac{1}{2}} + \sum_{0<n\equiv 1(24)} \text{Tr}_n(f) q^{\frac{n}{24}} + 12 \sum_{n\geq 1} \frac{\chi_{12}(n)}{n} \beta_{\frac{1}{2}}(\frac{\pi n}{6}) q^{\frac{n^2}{24}}
\]
\[+ i \beta_{\frac{1}{2}}\left(\frac{-\pi y}{6}\right) q^{\frac{1}{2}} + \sum_{0>n\equiv 1(24)} \frac{\text{Tr}_n(f)}{\sqrt{|n|}} \beta_{\frac{1}{2}}\left(\frac{\pi n}{6}y\right) q^{\frac{n^2}{24}} + 24 \sum_{n\geq 1} \chi_{12}(n) \alpha\left(\frac{n^2 y}{6}\right) q^{\frac{n^2}{24}}
\] (2.11)
is a polyharmonic Maass form of weight $\frac{1}{2}$ and depth $\frac{3}{2}$ on $\text{SL}_2(\mathbb{Z})$ with multiplier $\chi$. Moreover, we have $\xi_{\frac{3}{2}} H(\tau) = -2\sqrt{6} F(\tau)$ and $\Delta_{\frac{3}{2}} H(\tau) = -\frac{3}{\pi} \eta(\tau)$.

3. **Definition of the Polyharmonic Maass Form $H(\tau)$**

In this section we define the polyharmonic Maass form $H(\tau)$ as a limit of Poincaré series. This requires the Whittaker functions $M_{\kappa, \mu}(y)$ and $W_{\kappa, \mu}(y)$ (see [11, §13.4]) and the Bessel functions $J_\nu(x)$, $I_\nu(x)$, and $K_\nu(x)$ (see [11, Chapter 10]). Let $\Gamma_\infty := \{ \pm \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \} \subseteq \text{SL}_2(\mathbb{Z})$.

For $\text{Re}(s) > 1$ we define the weight $\frac{1}{2}$ Poincaré series
\[
P(\tau, s) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})} \overline{\chi}(\gamma)(c\tau + d)^{-\frac{s}{2}} \mathcal{M}(\text{Im} \gamma, s) e_{24}(\text{Re} \gamma \tau),
\]
where $e_m(x) := e\left(\frac{x}{m}\right)$ and
\[
\mathcal{M}(y, s) := \left(\frac{\pi y}{6}\right)^{-\frac{s}{2}} M_{\frac{s-\frac{1}{2}}{2}}\left(\frac{\pi y}{6}\right).
\]

Define the generalized Kloosterman sum
\[
A_c(n) := \sum_{\substack{d \mod c \\ (d,c) = 1}} e^{\pi i s(d,c)} e\left(-\frac{dn}{c}\right),
\] (3.1)
where $s(d, c)$ is the Dedekind sum
\[
s(d, c) := \sum_{r=1}^{c-1} \frac{r}{c} \left(\frac{dr}{c} - \left[\frac{dr}{c}\right] - \frac{1}{2}\right).
\]
The papers [1] and [4] used the notation
\[ K(m, n; c) := \sum_{d \mod c^*} e^{\pi i s(d, c)} e\left(\frac{\overline{d}m + dn}{c}\right); \] (3.2)
we have
\[ A_c(n) = K(0, -n, c). \] (3.3)
By Proposition 8 of [1] we have the Fourier expansion
\[ P(\tau, s) = \mathcal{M}(y, s)e_{24}(x) + \sum_{n=1}^{24} a(n, s)\mathcal{W}(y, s)e_{24}(nx) \]
where
\[ \mathcal{W}(y, s) := \left(\frac{\pi|y|}{6}\right)^{-\frac{1}{4}} W_{\frac{\text{sgn}\, n}{4}, s}\left(\frac{\pi|y|}{6}\right) \]
and
\[ a(n, s) = \frac{2\pi \Gamma(2s)}{|n|^{\frac{1}{4}} \Gamma(s + \frac{\text{sgn}\, n}{4})} \sum_{c>0} A_c\left(\frac{-n}{24}\right) \frac{J_{2s-1}\left(\frac{\pi \sqrt{n}}{6c}\right)}{c} \] \( n > 0, \)
\[ \frac{J_{2s-1}\left(\frac{\pi |n|}{6c}\right)}{c} \] \( n < 0. \) (3.4)

Lehmer [19, Theorem 8] proved the sharp Weil-type bound \( |A_c(n)| \leq 2^{\omega_0(c)} \sqrt{c} \), where \( \omega_0(c) \) is the number of distinct odd primes dividing \( c \). Using this together with the estimates (3.10), (3.16), (3.21) and (3.22) below, one can show that \( P(\tau, s) \) has an analytic continuation to \( \text{Re}(s) > 3/4 \) (this can also be seen using the estimates in Lemma 5 below).

Let
\[ c(s) := \frac{(2s - 1)\Gamma(s - \frac{1}{4})\Gamma(2s - \frac{1}{2})(2^{2s-\frac{s}{4}} - 1)(3^{2s-\frac{s}{4}} - 1)\zeta(4s - 1)}{6^{s-\frac{3}{4}}\pi^{2s}\Gamma(2s)}. \] (3.5)
We define \( \mathcal{H}(\tau) \) as follows:
\[ \mathcal{H}(\tau) := \frac{6}{\pi} \lim_{s \to \frac{3}{4}^+} \left( c(s)P(\tau, s) - \frac{\eta(\tau)}{(s - \frac{3}{4})}\right). \] (3.6)
Let \( \chi_{12} \) denote the Kronecker character for \( \mathbb{Q}(\sqrt{3}) \) (with the convention that \( \chi_{12}(n) := 0 \) if \( n \not\in \mathbb{Z} \)). The goal of this section is to prove the following.

**Proposition 4.** (1) The limit defining \( \mathcal{H}(\tau) \) exists.
(2) The function \( \mathcal{H}(\tau) \) is a polyharmonic Maass form of weight \( \frac{1}{2} \) and depth \( \frac{3}{2} \) on \( \text{SL}_2(\mathbb{Z}) \) with multiplier \( \chi \).
(3) We have \( \xi_\frac{3}{4} \mathcal{H}(\tau) = -2\sqrt{6}F(\tau) \) and \( \Delta_\frac{3}{2} \mathcal{H}(\tau) = -\frac{3}{\pi} \eta(\tau) \).
(4) We have
\[ \mathcal{H}(\tau) = -i \left(1 - \beta_{\frac{1}{2}}\left(\frac{-\pi y}{6}\right)\right) q^{\frac{\tau}{2}} + \frac{2}{\sqrt{\pi}} \sum_{0<n=1(24)} a(n, \frac{3}{4}) q^{\frac{\tau n}{2}} + 2 \sum_{0>n=1(24)} a(n, \frac{3}{4}) \beta_{\frac{1}{2}}\left(\frac{\pi |n| y}{6}\right) q^{\frac{\tau n}{2}} \]
\[ + \frac{6}{\pi} \sum_{n=1(24)} \lim_{s \to \frac{3}{4}} \left( c(s)a(n, s)\mathcal{W}(y, s) - \frac{\chi_{12}(\sqrt{n})}{s - \frac{3}{4}} e^{\frac{\pi n y}{4\tau}}\right) e_{24}(nx). \] (3.7)
We require the following lemma, whose proof we leave to the end of this section.
Lemma 5. There exists a positive constant $A$ such that the following are true.

1. For $n < 0$ we have
   \[ a(n, s) = O \left( |n|^A \exp \left( \frac{\pi \sqrt{|n|}}{6} \right) \right) \]  
   uniformly for $s \in \left[ \frac{3}{4}, 1 \right]$.

2. For $n > 0$ we have
   \[ a(n, s) = \chi_{12}(\sqrt{n}) \frac{3}{\sqrt{\pi}} \cdot \frac{1}{s - \frac{3}{4}} + O(n^A) \]  
   uniformly for $s \in \left( \frac{3}{4}, 1 \right]$ if $n$ is square, and for $s \in \left[ \frac{3}{4}, 1 \right]$ if $n$ is not square.

For the remainder of the section we let $A$ denote a positive constant whose value is allowed to change at each occurrence. We assume that $\tau$ is in a fixed compact subset of the upper half plane (in particular, that $y$ is bounded away from 0). The implied constants in the estimates which follow will depend on the particular choice of compact subset.

This estimate can be derived from the sentence which follows (13.19.3) of [11]:

\[ W_n(y, s) \ll |n|^A \exp \left( -\frac{\pi |n| |y|}{12} \right), \quad s \in \left[ \frac{3}{4}, 1 \right]. \]  

(3.10)

Proof of Proposition 4. Using [11, (13.14.31), (13.18.2), (13.18.5), and (13.14.32)], we have the evaluations

\[ W_n(y, \frac{3}{4}) = \begin{cases} e^{-\frac{\pi ny}{12}} & \text{if } n > 0, \\ \sqrt{\pi} \beta_2 \left( \frac{\pi |n| y}{6} \right) e^{-\frac{\pi ny}{12}} & \text{if } n < 0, \end{cases} \]  

(3.11)

\[ M(y, \frac{3}{4}) = -i\sqrt{\pi} 2 \left( 1 - \beta_1 \left( \frac{-\pi y}{6} \right) \right) e^{-\frac{\pi ny}{12}}. \]  

(3.12)

We also have

\[ c\left( \frac{3}{4} \right) = \frac{\sqrt{\pi}}{3}. \]  

(3.13)

This gives the first term in (3.7). If $n$ is not square then Lemma 5 and (3.10) give

\[ a(n, s)W_n(y, s) \ll |n|^A \exp \left( -\frac{\pi |n| y}{12} + \frac{\pi \sqrt{|n|}}{6} \right), \quad s \in \left[ \frac{3}{4}, 1 \right]. \]  

This justifies the interchange which gives the first two sums in (3.7).

For $n > 0$, a straightforward computation using the integral representation [16, (9.222.1)]

\[ W_{\frac{1}{2}, \frac{1}{2}}(y) = \frac{y e^{-y/2}}{\Gamma(s - \frac{1}{4})} \int_0^\infty e^{-yt} t^{s - \frac{3}{4}} (1 + t)^{s - \frac{3}{4}} dt \]  

(3.14)

shows that $\frac{\partial}{\partial s} W_n(y, s)$ also satisfies the bound (3.10). Expanding at $s = 3/4$ using this fact together with (3.9), (3.10) and (3.13) gives

\[ c(s)a(n, s)W_n(y, s) - \chi_{12}(\sqrt{n}) \frac{\sqrt{\pi}}{s - \frac{3}{4}} e^{-\frac{\pi ny}{12}} \ll n^A \exp \left( -\frac{\pi ny}{12} \right), \quad s \in \left( \frac{3}{4}, 1 \right]. \]  

This justifies the interchange in the sum containing the square coefficients, and gives assertions (1) and (4).
The transformation properties of $H(\tau)$ are inherited from those of $P(\tau, s)$ and $\eta(\tau)$. From (1) we have
\[
\lim_{s \to \frac{3}{4}} (s - \frac{3}{4}) c(s) P(\tau, s) = \eta(\tau).
\]
A computation involving [11, (13.14.1)] shows that $\Delta \frac{1}{2} P(\tau, s) = (s - \frac{1}{4})(\frac{3}{4} - s)P(\tau, s)$; from this we conclude that $\Delta \frac{1}{2} H(\tau) = -\frac{3}{\pi^6} \eta(\tau)$ (to interchange the limit and the derivatives requires uniform convergence, which follows as above). From (2.7) and (1.12) we have
\[
\xi_{\frac{1}{2}}(\xi_{\frac{1}{2}} H(\tau)) = \xi_{\frac{1}{2}}(-2\sqrt{6} F(\tau)).
\]
Now $\xi_{\frac{1}{2}} H(\tau)$ is a weak harmonic Maass form of weight $3/2$ and multiplier $\chi$ on $\text{SL}_2(\mathbb{Z})$. Using (2.9) and (2.10) we find that the only exponentially growing term in its expansion is
\[
\xi_{\frac{1}{2}}(i\beta \left(-\frac{\pi}{6}\right)q^{\frac{1}{24}}) = \frac{1}{\sqrt{6}} q^{-\frac{1}{24}}.
\]
From (1.11) we conclude that $\xi_{\frac{1}{2}} H(\tau) = -2\sqrt{6} F(\tau)$, since there are no modular forms of weight $3/2$ on $\text{SL}_2(\mathbb{Z})$ with multiplier $\chi$ which are holomorphic both on $\mathbb{H}$ and at $\infty$. This finishes the proof of the proposition. □

We turn to the proof of Lemma 5.

Proof of Lemma 5. Define
\[
S(n, x) := \sum_{c \leq x} A_c \left(\frac{1-n}{24}\right).c
\]
By Theorem 3 of [2], we have the asymptotic formula
\[
S(n, x) = \chi_{12}(\sqrt{n})\frac{12\sqrt{3}}{\pi^2} x^{\frac{1}{2}} + O_n(x^{\frac{1}{2}+\epsilon})
\]
for any $\epsilon > 0$. While the $n$-dependence in the error term is not given explicitly in [2], a straightforward modification of the proof (following arguments given in, e.g., [20]) shows that error term depends at worst polynomially on $n$. Taking $\epsilon = \frac{1}{12}$, we conclude that
\[
S(n, x) = \chi_{12}(\sqrt{n})\frac{12\sqrt{3}}{\pi^2} x^{\frac{1}{2}} + O(|n|^{\frac{1}{2}} x^{\frac{1}{2}}). \tag{3.15}
\]
Suppose first that $n > 0$. We require the facts that
\[
|J_\nu(y)| \leq \left|\frac{1}{2} y^\nu\right|, \quad \nu \geq -\frac{1}{2}, \tag{3.16}
\]
\[
J'_\nu(y) = -J_{\nu+1}(y) + \frac{\nu}{y} J_\nu(y), \tag{3.17}
\]
and that $J_\nu(y)$ decays at $\infty$ [11, (10.14.4), (10.6.2), (10.14.1), (10.17.3)]. Combining (3.16) and (3.17), we obtain, for $t \geq 1$,
\[
\left[J_{2s-1}\left(\frac{\pi \sqrt{n}}{6t}\right)\right]' \ll \frac{n^A}{t^{2s}}. \tag{3.19}
\]
Then, for \( s \in (3/4, 1] \), partial summation together with (3.16) and (3.17) gives

\[
\sum_{c>0} \frac{A_c}{c} \left( \frac{\pi \sqrt{n}}{6c} \right) J_{2s-1} \left( \frac{\pi \sqrt{n}}{6c} \right) = - \int_1^\infty S(n, t) \left[ J_{2s-1} \left( \frac{\pi \sqrt{n}}{6t} \right) \right] \, dt.
\]

We first consider the case when \( n \) is square. From (3.4), (3.20) and (3.15) we obtain

\[
a(n, s) = -\chi_2(\sqrt{n}) \frac{24\sqrt{3} \Gamma(2s)}{n^{3/4} \pi} \int_1^\infty \sqrt{t} \left[ J_{2s-1} \left( \frac{\pi \sqrt{n}}{6t} \right) \right] \, dt + O(n^A).
\]

Integrating by parts using (3.16) and (3.18), and then using the evaluation [11, (10.22.43)], the integral in the last line becomes

\[
-\frac{1}{2} \int_1^\infty t^{-\frac{1}{2}} J_{2s-1} \left( \frac{\pi \sqrt{n}}{6t} \right) \, dt + O(1) = -\frac{1}{2} \int_0^\infty t^{-\frac{1}{2}} J_{2s-1} \left( \frac{\pi \sqrt{n}}{6t} \right) \, dt + O(1)
\]

\[
= -\frac{1}{8} \sqrt{\frac{\pi}{3}} \frac{n^{3/4} \Gamma(s - \frac{3}{4})}{\Gamma(s + \frac{3}{4})} + O(1),
\]

from which we obtain

\[
a(n, s) = \chi_2(\sqrt{n}) \frac{3}{\sqrt{\pi}} \cdot \frac{\Gamma(2s)\Gamma(s - \frac{3}{4})}{\Gamma(s + \frac{1}{4})\Gamma(s + \frac{3}{4})} + O(n^A).
\]

From the expansion \( \frac{\Gamma(2s)\Gamma(s - \frac{3}{4})}{\Gamma(s + \frac{1}{4})\Gamma(s + \frac{3}{4})} = \frac{1}{s - 3/4} + O(1) \) for \( s \in (3/4, 1] \), we obtain (3.9) in this case.

If \( n \) is not square, then (3.20) holds for \( s \in [3/4, 1] \), and the result follows from (3.15).

The case when \( n < 0 \) is similar. We require the facts that for fixed \( \nu > 0 \) we have

\[
I_\nu(x) \ll \frac{x^\nu}{\Gamma(\nu + 1)} \quad \text{as } x \to 0,
\]

that

\[
I_\nu(x) \ll \frac{e^x}{\sqrt{x}} \quad \text{as } x \to \infty,
\]

that

\[
I'_\nu(x) = I_{\nu+1}(x) + \frac{\nu}{x} I_\nu(x),
\]

and that \( I_\nu(x) \) is increasing as a function of \( x \) [11, (10.30.1), (10.30.4), (10.29.2), (10.37)]. We break the integral analogous to (3.20) at \( t = \sqrt{|n|} \). The first part of the integral is \( O(|n|^A \exp(\pi \sqrt{|n|}/6)) \). Using (3.21) we find that the second part is \( O(|n|^A) \). This gives (3.8). \( \square \)

4. Poincaré series and the coefficients of nonsquare index

In this section we relate the coefficients \( a(n, \frac{3}{4}) \) of non-square index appearing in Proposition 4 to the traces \( \text{Tr}_n(f) \) defined in (2.5). We first define a function \( f(\tau, s) \) in terms of a Maass-Poincaré series which, analytically continued, specializes to \( f(\tau) \) at \( s = 1 \). For \( r \mid 6 \), define the Atkin-Lehner matrix \( W_r \) by

\[
W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}, \quad W_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad W_6 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}.
\]

Then

\[
W_dW_{d'} = W_{dd'},
\]

(4.1)
Recall that \( \Gamma = \Gamma_0(6)/\{\pm 1\} \). Following Section 4 of [2] we define
\[
f(\tau, s) := \sum_{r|6} \mu(r) \sum_{\gamma \in \Gamma \setminus \Gamma_0} \phi_s(\Im\gamma W_r \tau) e(-\Re\gamma W_r \tau), \quad \Re(s) > 1, \tag{4.2}
\]
where \( \mu \) is the Möbius function and
\[
\phi_s(y) := 2\pi \sqrt{y} I_{s-\frac{1}{2}}(2\pi y).
\]
The function \( f(\tau, s) \) satisfies
\[
f(\gamma \tau, s) = f(\tau, s) \quad \text{for all } \gamma \in \Gamma_0(6)
\]
and
\[
f(W_r \tau, s) = \mu(r) f(\tau, s) \quad \text{for } r \mid 6. \tag{4.3}
\]
We also have
\[
\Delta_0 f(\tau, s) = s(1-s) f(\tau, s). \tag{4.4}
\]
As shown in [2, §4], the function \( f(\tau, s) \) has an analytic continuation to \( \Re(s) > \frac{3}{4} \) and
\[
f(\tau, 1) = f(\tau).
\]
In the next section we will need the Fourier expansion of \( f(\tau, s) \). By Proposition 5 of [2] we have
\[
f(\tau, s) = 2\pi \sqrt{y} I_{s-\frac{1}{2}}(2\pi y) e(-x) + a_s(0) y^{1-s} + 2\sqrt{y} \sum_{n \neq 0} a_s(n) K_{s-\frac{1}{2}}(2\pi |n| y) e(nx), \tag{4.5}
\]
where
\[
a_s(0) = \frac{2\pi s+1}{(s-\frac{1}{2}) \Gamma(s)} \sum_{r|6} \mu(r) \sum_{\frac{c}{r} \neq 0; 0 (6/r) \equiv 0 (c,r) \equiv 1} k(-\overline{r}, 0; c) \frac{1}{(c\sqrt{r})^{2s}}. \tag{4.6}
\]
Here \( k(a, b; c) \) is the ordinary Kloosterman sum. Exact formulas for the coefficients \( a_s(n), n \neq 0 \), are given in [2], but we will need only the crude estimate
\[
a_s(n) \ll e^{6\pi \sqrt{n}} \quad \text{uniformly for } s \in [1, \frac{3}{2}],
\]
from [2, (4.5)]. The constant coefficient \( a_s(0) \) simplifies in the following way.

**Lemma 6.** For \( \Re(s) > \frac{1}{2} \) we have
\[
a_s(0) = \frac{4\pi^{s+1}}{(2s-1) \Gamma(s)(2s-1)(3s-1) \zeta(2s)}. \tag{4.7}
\]

*Proof.* Evaluating the Kloosterman sums in (4.6) gives
\[
\sum_{0 < c \equiv 0 (6/r) \equiv 1 (c,r) = 1} k(-\overline{r}, 0; c) \frac{1}{(c\sqrt{r})^{2s}} = \frac{\mu(\frac{6}{r})^s}{6^{2s}} \sum_{c > 0; (c, 6) = 1} \frac{\mu(c)}{c^{2s}}, \tag{4.8}
\]
where we have replaced \( c \) by \( \frac{6c}{r} \) and used the fact that \( \mu(\frac{6c}{r}) = 0 \) unless \( (c, \frac{6}{r}) = 1 \). Therefore
\[
\sum_{r|6} \mu(r) \sum_{0 < c \equiv 0 (6/r) \equiv 1 (c,r) = 1} k(-\overline{r}, 0; c) \frac{1}{(c\sqrt{r})^{2s}} = \frac{1}{6^{2s}} \sum_{r|6} r^s \sum_{c > 0; (c, 6) = 1} \frac{\mu(c)}{c^{2s}} = \frac{1}{(2s-1)(3s-1) \zeta(2s)},
\]
and the lemma follows. \( \square \)
We will write the coefficients $a(n, \frac{3}{4})$ in terms of the traces $\text{Tr}_n(f)$ by modifying the proof of Proposition 7 of [4]. We begin by noting that $f(\tau, s)$ is related to the function $P_1(\tau, s)$ defined in (2.7) of that paper by

$$P_1(\tau, 2s - \frac{1}{2}) = \frac{2\Gamma(s + \frac{1}{4})}{\sqrt{\pi\Gamma(s - \frac{1}{4})}} f(\tau, 2s - \frac{1}{2}).$$

Suppose first that $n < 0$. Following the proof of [4, Proposition 7], we find that

$$|n|^{-\frac{1}{4}} \sum_{Q \in \Gamma \setminus \mathbb{Q}_n} P_1(\tau_Q, s) = \frac{2\sqrt{2\pi\Gamma(\frac{1}{2})}}{\Gamma(\frac{7}{4})} |n|^{-\frac{1}{4}} \sum_{Q \in \Gamma \setminus \mathbb{Q}_n} \left( \frac{12}{b} \right) a^{-\frac{1}{2}} I_{s-\frac{1}{4}} \left( \frac{\pi \sqrt{|n|}}{a} \right) e\left(\frac{b}{2a}\right).$$

Then the argument which follows [4, (5.15)] shows that

$$|n|^{-\frac{1}{4}} \sum_{Q \in \Gamma \setminus \mathbb{Q}_n} P_1(\tau_Q, s) = \frac{4\sqrt{\pi\Gamma(\frac{1}{2})}}{\Gamma(\frac{7}{4})} |n|^{-\frac{1}{4}} \sum_{c > 0} \frac{A_c(\frac{1-n}{2c})}{c} I_{s-\frac{1}{4}} \left( \frac{\pi \sqrt{|n|}}{6c} \right).$$

It follows that

$$a(n, \frac{3}{4}) = \frac{1}{2\sqrt{|n|}} \sum_{Q \in \Gamma \setminus \mathbb{Q}_n} f(\tau_Q) \quad \text{if } n < 0. \quad (4.9)$$

For nonsquare $n > 0$ we can apply [4, Proposition 7] directly, and we find that

$$a(n, \frac{3}{4}) = \frac{1}{4\sqrt{\pi}} \sum_{Q \in \Gamma \setminus \mathbb{Q}_n} \int_{C_Q} f(\tau) \frac{d\tau}{Q(\tau, 1)} \quad \text{if } n > 0. \quad (4.10)$$

5. The square-indexed coefficients and the proof of Theorem 3

To describe the square-indexed coefficients, we define the dampened functions $f_Q(\tau)$ which appear in (2.6). Suppose that $n \equiv 1 \pmod{24}$ is square and let $Q \in \mathbb{Q}_n$. Then $Q(x, y) = 0$ has two rational roots which correspond to cusps $a_1$ and $a_2$ in $\mathbb{P}^1(\mathbb{Q})$, and $C_Q$ is defined as the geodesic connecting $a_1$ and $a_2$. For each $i$ there is a unique $r_i$ and a unique $\gamma_i \in \Gamma_{\infty} \setminus \Gamma$ such that

$$\gamma_i W_{r_i} a_i = \infty.$$

Following the method of [4] (see (3.13) of that paper) we define

$$\tilde{f}_Q(\tau, s) := \sum_{r \mid 6} \mu(r) \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \phi_s(\text{Im} \gamma W_{r_i} \tau) e(-\text{Re} \gamma W_{r_i} \tau). \quad (5.1)$$

With the notation of [4] we have

$$P_{1, Q}(\tau, 2s - \frac{1}{2}) = \frac{2\Gamma(s + \frac{1}{4})}{\sqrt{\pi\Gamma(s - \frac{1}{4})}} \tilde{f}_Q(\tau, 2s - \frac{1}{2}).$$

Let $\Gamma^*$ be the group generated by $\Gamma$ and the Atkin-Lehner involutions $W_r$ for $r \mid 6$. Matrices $\gamma = (a \ b \ c \ d) \in \Gamma^*$ act on $Q \in \mathbb{Q}_n$ by

$$\gamma Q(x, y) = Q(dx - by, -cx + ay).$$

Then for $\gamma \in \Gamma^*$ we have

$$\gamma \tau_Q = \tau_{\gamma Q} \quad (5.2)$$
and
\[ \frac{d(\gamma \tau)}{\gamma Q(\gamma \tau, 1)} = \frac{d\tau}{Q(\tau, 1)}. \]  
(5.3)

For \( \sigma \in \Gamma_0(6) \) we have
\[ \tilde{f}_{\sigma Q}(\sigma \tau, s) = \tilde{f}_Q(\tau, s), \]  
(5.4)

and, for \( r | 6 \), (4.1) gives
\[ \tilde{f}_{WQ}(W_r \tau, s) = \mu(r) \tilde{f}_Q(\tau, s). \]  
(5.5)

As will be seen in the proof of Proposition 7 below, the function \( a(n, s) \) has a pole at \( s = \frac{3}{4} \) which arises from integrating the constant term in the Fourier expansion of \( \tilde{f}_Q(\tau, s) \).

Motivated by this, we define
\[ f_Q(\tau, s) := \tilde{f}_Q(\tau, s) - a_s(0)y^{1-s}. \]  
(5.6)

In particular, with \( f_Q(\tau) := f_Q(\tau, 1) \) and \( \tilde{f}_Q(\tau) := \tilde{f}_Q(\tau, 1) \), Lemma 6 gives
\[ f_Q(\tau) = \tilde{f}_Q(\tau) - 12. \]  
(5.7)

The next result gives the evaluation of the coefficients of square index.

**Proposition 7.** Suppose that \( n \equiv 1 \pmod{24} \) is square. Let \( \text{Tr}_n(f) \) and \( h^*(n) \) be defined by (2.6) and (1.5). Then
\[ c(s)a(n, s) = \chi_{12}(\sqrt{n}) \frac{2\pi}{\sqrt{n}} \chi_{12}(\sqrt{n})h^*(n) + \frac{\pi}{6} \text{Tr}_n(f) + O(s - \frac{3}{4}) \]  
(5.8)

uniformly for \( s \in \left( \frac{3}{4}, 1 \right) \).

To prove the proposition we will need the following lemma, which describes a set of representatives for \( \Gamma \backslash Q_n \) when \( n \) is square. We omit the proof, as it follows along the same lines as the proof of [3, Lemma 3].

**Lemma 8.** Suppose that \( n = b^2 \) with \( (b, 6) = 1 \). Then
\[ \Gamma \backslash Q_n \cong \{ W_r[0, b, c] : c \pmod{b} \}, \]  
(5.9)

where
\[ r = \begin{cases} 1 & \text{if } b \equiv 1 \pmod{12}, \\ 2 & \text{if } b \equiv 7 \pmod{12}, \\ 3 & \text{if } b \equiv 5 \pmod{12}, \\ 6 & \text{if } b \equiv 11 \pmod{12}. \end{cases} \]

**Proof of Proposition 7.** From (5.4) of [4] and (3.4) we find that
\[ a(n, s) = \frac{\Gamma(2s)}{2\sqrt{\pi} \Gamma(s - \frac{3}{4})} \sum_{Q \in \Gamma \backslash Q_n} \int_{C_Q} \tilde{f}_Q(\tau, 2s - \frac{1}{2}) \frac{d\tau}{Q(\tau, 1)}. \]  
(5.10)

Write \( n = b^2 \) and let \( r \in \{1, 2, 3, 6\} \) be as in Lemma 8. By Lemma 8, (5.1), (5.2), (5.5), and the fact that \( \mu(r) = \chi_{12}(\sqrt{n}) \) we have
\[ \sum_{Q \in \Gamma \backslash Q_n} \int_{C_Q} \tilde{f}_Q(\tau, 2s - \frac{1}{2}) \frac{d\tau}{Q(\tau, 1)} = \chi_{12}(\sqrt{n}) \sum_{c \pmod{b}} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6} + i\infty} \tilde{f}_{[0, b, c]}(\tau, 2s - \frac{1}{2}) \frac{d\tau}{b\tau + c}. \]
Let \( g = (b, c) \), write \( b = gb' \) and \( c = gc' \), and choose \( \gamma_c = \left( \frac{u}{6w}, -\frac{v}{b'} \right) \in \Gamma_0(6) \). Then \( \gamma_c W_6 \infty = -\frac{c'}{b'} \). We replace \( \tau \) by \( \gamma_c W_6 \tau \). Since \( \gamma_c W_6 [0, -b', v] = [0, b', c'] \), (5.2) and (5.3) give

\[
\int_{-\frac{c'}{b'}}^{-\frac{c'}{b'} + i\infty} f_{[0,b,c]}(\tau, 2s - \frac{1}{2}) \frac{d\tau}{b\tau + c} = \frac{1}{g} \int_{\frac{b'}{b'} + i\infty}^{b'} f_{[0,b,-v]}(\tau, 2s - \frac{1}{2}) \frac{d\tau}{b'\tau - v}.
\]

It follows that

\[
\int_{-\frac{c'}{b'}}^{-\frac{c'}{b'} + i\infty} f_{[0,b,c]}(\tau, 2s - \frac{1}{2}) \frac{d\tau}{b\tau + c} = \frac{1}{b} \int_{\frac{b'}{b'}}^{b'} \left( f_{[0,b,c]}(-\frac{c'}{b'} + iy, 2s - \frac{1}{2}) + f_{[0,b,-v]}(\frac{c}{b'}, iy, 2s - \frac{1}{2}) \right) \frac{dy}{y}. \tag{5.11}
\]

The cusps \( a_1, a_2 \) associated to \( Q = [0, b, c] \) are given by \( a_1 = \infty \) and \( a_2 = -\frac{c'}{b'} \), so we have

\[
\gamma_1 W_{r_1} = \left( \frac{1}{0}, \frac{2}{1} \right) \quad \text{and} \quad \gamma_2 W_{r_2} = \left( \frac{b'}{c'}, \frac{v}{c'} \right), \tag{5.12}
\]

for some \( w \in \mathbb{Z} \). Thus, by (5.1) and (4.5) we have the Fourier expansion

\[
f_{[0,b,s]}(-\frac{c'}{b'} + iy, s) = \sum a_s(0) y^{1-s} - \frac{2\pi}{\sqrt{b'y}} I_{s + \frac{1}{2}} \left( \frac{2\pi}{b'y} \right) e \left( -\frac{w}{b'} \right) + 2\sqrt{y} \sum_{n \neq 0} a_s(n) K_{s + \frac{1}{2}}(2\pi |n|y) e \left( -\frac{c'n}{b'} \right). \tag{5.13}
\]

The contribution from the constant term \( a_s(0) y^{1-s} \) of \( f_{[0,b,s]}(\tau, s) \) to the right-hand side of (5.11) equals

\[
\frac{2}{b} a_{2s - \frac{1}{2}}(0) \int_{\frac{b'}{b'}}^{b'} y^{\frac{3}{2} - 2s} \frac{dy}{y} = \frac{1}{b} \frac{6^{s - \frac{3}{2}} (b')^{2s - \frac{3}{2}} a_{2s - \frac{1}{2}}(0)}{s - \frac{3}{4}}. \tag{5.14}
\]

By the estimates (4.4) and (4.5) of [2], which are valid uniformly for \( s \in [\frac{3}{4}, 1] \), we have

\[
2\sqrt{y} \sum_{n \neq 0} a_{2s - \frac{1}{2}}(n) K_{2s - 1}(2\pi |n|y) e \left( -\frac{c'n}{b'} \right) \ll e^{-\pi y} \quad \text{as} \quad y \to \infty. \tag{5.15}
\]

Then, using that \( I_s(y) \ll y^s \) uniformly for \( s \in [\frac{3}{4}, 1] \) as \( y \to 0 \), we conclude that

\[
f_{[0,b,c]}(-\frac{c'}{b'} + iy, 2s - \frac{1}{2}) \ll y^{\frac{1}{2} - 2s} \quad \text{as} \quad y \to \infty.
\]

It follows that the contribution from \( f_{[0,b,s]}(\tau, s) \) to the right-hand side of (5.11) converges uniformly for \( s \in [\frac{3}{4}, 1] \).

Therefore

\[
\frac{1}{b} \int_{\frac{b'}{b'}}^{b'} \left( f_{[0,b,c]}(-\frac{c'}{b'} + iy, 2s - \frac{1}{2}) + f_{[0,b,-v]}(-\frac{v}{b'} + iy, 2s - \frac{1}{2}) \right) \frac{dy}{y} = \frac{1}{b} \int_{\frac{b'}{b'}}^{b'} \left( f_{[0,b,c]}(-\frac{c'}{b'} + iy) + f_{[0,b,-v]}(-\frac{v}{b'} + iy) \right) \frac{dy}{y} + O(s - \frac{3}{4})
\]

\[
= \int_{-\frac{c'}{b'}}^{-\frac{c'}{b'} + i\infty} f_{[0,b,c]}(\tau) \frac{d\tau}{b\tau + c} + O(s - \frac{3}{4}),
\]
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where, in the last line, we have reversed the calculations from above. From (3.5) and (4.7) we have

\[ \frac{\Gamma(2s)}{2\sqrt{\pi} \Gamma(s - \frac{3}{4})} 6^{s - \frac{3}{4}} c(s) a_{2s - \frac{3}{4}}(0) = 1. \]

With (5.14) and (3.13) this gives

\[ c(s)a(n, s) = \frac{\chi_{12}(\sqrt{n})}{\sqrt{n}} \sum_{c \mod \sqrt{n}} \left( \frac{\sqrt{n}}{(c, \sqrt{n})} \right)^{2s - \frac{3}{2}} \frac{1}{s - \frac{3}{4}} + \frac{1}{12} \sum_{Q \in \Gamma \setminus Q_n} \int_{C_Q} f_Q(\tau) \frac{d\tau}{Q(\tau, 1)} + O(s - \frac{3}{4}). \]

We have the series expansion

\[ x^{2s - \frac{3}{2}} = 1 + 2 \log x (s - \frac{3}{4}) + O(s - \frac{3}{4})^2. \]

Using Lemma 8 and (1.5) we see that for square \( n > 0 \), we have

\[ h^*(n) = \pi \sum_{c \mod \sqrt{n}} \log \left( \frac{\sqrt{n}}{c, \sqrt{n}} \right) \sum_{[a,b,c] \in \Gamma \setminus Q_n/\ell^2} \chi_{12}(\sqrt{n}) e^{-\pi ny/6} \frac{\log y}{6} \right). \]

The proposition follows.

We now prove Theorem 3.

**Proof of Theorem 3.** After Proposition 4 it remains only to show that \( H(\tau) \) has Fourier expansion (2.11). For the nonsquare coefficients, this follows from Proposition 4 and (4.9)–(4.10). For square \( n > 0 \), the \( n \)-th term of \( H(\tau) \) is given by

\[ \frac{6}{\pi} \lim_{s \to \frac{3}{4}^+} \left( c(s)a(n, s)W_n(y, s) - \frac{\chi_{12}(\sqrt{n})}{s - \frac{3}{4}} e^{-\pi ny/6} \right) e_{24}(nx). \]

We also require a short lemma.

**Lemma 9.** If \( n > 0 \) then

\[ \frac{\partial}{\partial s} W_n(y, s) \bigg|_{s = \frac{3}{4}} = 4\pi e^{-\pi ny/6} \alpha \left( \frac{ny}{6} \right). \]

**Proof.** By [16, (9.222.1)] we have the integral representation

\[ W_{s - \frac{3}{4}}(y) = \frac{y^s e^{-\frac{y}{2}}}{\Gamma(s - \frac{3}{4})} \int_0^\infty e^{-yt} t^{-\frac{3}{4}} (1 + t)^{s - \frac{3}{4}} dt. \]

Differentiating under the integral sign, we find that

\[ \frac{\partial}{\partial s} W_{s - \frac{3}{4}}(y) \bigg|_{s = \frac{3}{4}} = \frac{y^s e^{-\frac{y}{2}}}{\sqrt{\pi}} \left( \log y - \psi \left( \frac{1}{2} \right) \right) \]

\[ + \frac{y^s e^{-\frac{y}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-yt} t^{-\frac{3}{4}} \log t dt + \frac{y^s e^{-\frac{y}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-yt} t^{-\frac{3}{4}} \log(1 + t) dt, \]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function. By [16, (4.352.1)] we have

\[ \int_0^\infty e^{-yt} t^{-\frac{3}{4}} \log t dt = \sqrt{\frac{\pi}{y}} \left( \psi \left( \frac{1}{2} \right) - \log y \right), \]

from which the lemma easily follows.

We now prove Theorem 3.

**Proof of Theorem 3.** After Proposition 4 it remains only to show that \( H(\tau) \) has Fourier expansion (2.11). For the nonsquare coefficients, this follows from Proposition 4 and (4.9)–(4.10). For square \( n > 0 \), the \( n \)-th term of \( H(\tau) \) is given by
By Lemma 9 we have the Taylor expansion
\[
W_n(y, s) = W_n(y, \frac{3}{4}) + \frac{\partial}{\partial y} W_n(y, s) \bigg|_{y=\frac{3}{4}} (s - \frac{3}{4}) + O(s - \frac{3}{4})^2
\]
\[
eq e^{-\frac{ny}{12}} \left[ 1 + 4\pi \alpha \left( \frac{ny}{6} \right) (s - \frac{3}{4}) + O(s - \frac{3}{4})^2 \right].
\]
This, together with Proposition 7, shows that the expression (5.18) equals
\[
\left( \frac{12}{\sqrt{n}} \chi_{12}(\sqrt{n}) h^*(n) + \text{Tr}_n(f) + 24 \chi_{12}(\sqrt{n}) \alpha \left( \frac{ny}{6} \right) \right) q^{\frac{2}{3}}.
\]
Theorem 3 follows. 

6. Theorems 1 and 2

We briefly sketch how Theorem 1 can be deduced from Theorem 4.2 of [9]. Let \( L \) be the lattice of Example 2.1 of [9] with \( N = 1 \), and let \( h \in L/L \cong \mathbb{Z}/2\mathbb{Z} \) denote the non-trivial element. With \( H_h(\tau, 1) \) as in [9, Theorem 4.2] we have the relation
\[
Z(\tau) = \frac{1}{2} (H_L(4\tau, 1) + H_{L+h}(4\tau, 1)).
\]
(6.1)
We note that there are a few errors in [9] which should be corrected as follows. First, the term \((\frac{1}{2} \log 2 + 1)\gamma\) in Lemmas 8.5 and 8.6, and in the definition of \( F(t) \) in Theorem 4.1 should be changed to \((\log 2 + \frac{1}{2} \gamma)\) (see (8.10)–(8.12) of [9]). With the corrected definition of \( F(t) \), we have
\[
F(2\sqrt{\pi y} m) = -2\pi \alpha(4m^2y).
\]
In particular, Remark 4.3 no longer applies. Second, the constant terms in Theorem 4.2 (the first and last lines of the formula for \( H_h(\tau, 1) \)) should be multiplied by \( \delta_{h,0} \).

Theorem 1 can also be deduced directly in analogy with Sections 3–5 from the definition of \( Z(\tau) \) as a limit. Since this computation is quite involved, we give a sketch here. Let
\[
c'(s) := \frac{2^{4s-1} \Gamma(s + \frac{1}{4}) \Gamma(s - \frac{1}{4})^2 \zeta(2s - \frac{1}{2}) \zeta(4s - 1)}{\pi^{s+\frac{1}{4}} \Gamma(2s - 1) \zeta(4s - 2)},
\]
(6.2)
and note that \( c'(\frac{3}{4}) = \frac{4\pi}{3} \). With \( P_0^+(\tau, s) \) as in Section 5 of [12] we define
\[
Z(\tau) := \frac{1}{4\pi} \lim_{s \to \frac{3}{4}} \left( c'(s) P_0^+(\tau, s) - \frac{\theta(\tau)}{s - \frac{3}{4}} \right).
\]
(6.3)
By (2.24) of [12], the contribution from the constant term of \( P_0^+(\tau, s) \) equals
\[
\frac{1}{4\pi} \lim_{s \to \frac{3}{4}} \left( \frac{2^{2s-\frac{1}{2}} c'(s) b_0(0, s)}{(2s - 1) \Gamma(2s - \frac{1}{2})} y^{\frac{3}{4}-s} - \frac{1}{s - \frac{3}{4}} \right) = \frac{\gamma - \log(16\pi y)}{4\pi}.
\]
(6.4)
For \( d > 0 \) let \( \mathcal{Q}_d := \{[a, b, c] : b^2 - 4ac = d\} \) and let \( \Gamma_1 := \text{PSL}_2(\mathbb{Z}) \). Let \( b_0(d, s) \) denote the \( d \)-th coefficient of \( P_0^+(\tau, s) \) (see [12, (2.20–21)]). For non-square \( d \), the function \( b_0(d, s) \) is analytic at \( s = \frac{3}{4} \), so the coefficients of non-square index in \( Z(\tau) \) agree with the corresponding coefficients of \( \tilde{Z}_+(\tau) \).

Suppose that \( d > 0 \) is a square. By (4.5) of [3] we have
\[
b_0(d, \frac{5}{4} + \frac{1}{4}) = \frac{2^{1-2s} \pi^{s+\frac{1}{4}} \Gamma(s)}{\zeta(s) \Gamma(\frac{5}{4})^2} \sum_{Q \in \mathcal{Q}_1 \setminus \mathcal{Q}_d} \int_{C_Q} G_{0,Q}(\tau, s) \frac{\sqrt{d} \, d\tau}{Q(\tau, 1)},
\]
(6.5)
where \( G_{0,Q}(\tau, s) \) is a dampened version of the Eisenstein series \( G_0(\tau, s) \) defined in §4 of [12]. As in Lemma 8 we have
\[
\Gamma_1 \backslash \mathbb{Q}_d \cong \{ [0, b, c] : c \text{ mod } b \} \quad \text{for } d = b^2.
\]

Following the proof of Proposition 7 above, we find that
\[
b_0(\tau, \frac{\tau}{2} + \frac{1}{4}) = \frac{2^{2-2s} \pi^{\frac{s+1}{2}} \Gamma(s) c_0(0, s)}{\zeta(s) \Gamma(\frac{s}{2})^2} \sum_{c \text{ mod } b} \left( \frac{b}{(b, c)} \right)^{s-1} \frac{1}{s-1} + O(s-1),
\]
where \( c_0(0, s) = \sqrt{\pi} \frac{\Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s) \zeta(2s)} \) is the coefficient of \( y^{1-s} \) in \( G_0(\tau, s) \). We replace \( s \) by \( 2s - \frac{1}{2} \) in (6.7) and multiply by \( c'(s) \). Since
\[
c'(s) = \frac{2^{2-4s} \pi^{s+\frac{1}{2}} \Gamma(2s-\frac{1}{2}) c_0(0, 2s - \frac{1}{2})}{\zeta(2s-\frac{1}{2}) \Gamma(s-\frac{1}{4})^2} = 2 \Gamma(s + \frac{1}{4}),
\]
we find, using (5.16), that
\[
c'(s) b_0(\tau, s) = \frac{2 \sqrt{d} \Gamma(s + \frac{1}{4})}{s - \frac{1}{4}} + 4 \pi \Gamma(s + \frac{1}{4}) h^*(d) + O(s - \frac{3}{4}).
\]

Using the Taylor expansion
\[
(4\pi dy)^{-\frac{1}{2}} W_{\frac{s}{2}, s-\frac{1}{2}} (4\pi |d| y) e(dx) = q^d + 4\pi \alpha(4dy) q^d (s - \frac{3}{4}) + O((s - \frac{3}{4})^2),
\]
we find that the \( d \)-th term in the Fourier expansion of \( \mathbf{Z}(\tau) \) equals
\[
\frac{1}{4\pi} \lim_{s \to \frac{1}{4}} \left( c'(s) b_0(\tau, s) d^{-\frac{1}{2}} \Gamma(s + \frac{1}{4})^{-1} (4\pi y)^{-\frac{1}{2}} W_{\frac{s}{2}, s-\frac{1}{2}} (4\pi dy) e(dx) - \frac{2q^d}{s - \frac{1}{4}} \right) = 2 \alpha(4dy) q^d + d^{-\frac{1}{2}} h^*(d) q^d.
\]

Theorem 1 follows.

We turn to the proof of Theorem 2. Recall (see [22] for example) that for each positive discriminant \( d \) there exists a unique weight \( \frac{d}{2} \) weakly holomorphic modular form \( g_d \) on \( \Gamma_0(4) \) of the form
\[
g_d(\tau) = q^{-d} + \sum_{0 \leq n = 0, 3(4)} B(d, n) q^n,
\]
where the \( B(d, n) \) are integers and \( B(d, 0) = -2\delta_{d<0}(d) \). The first three forms are
\[
g_1 = q^{-1} - 2 + 248 q^3 - 492 q^4 + 4119 q^7 + \ldots,
\]
\[
g_4 = q^{-4} - 2 - 26752 q^3 - 143376 q^4 - 8288256 q^7 + \ldots,
\]
\[
g_5 = q^{-5} + 0 + 85995 q^3 - 565760 q^4 + 52756480 q^7 + \ldots.
\]

To follow the notation in [13], we define
\[
g_0(\tau) := \tilde{\mathbf{Z}}_+^*(\tau).
\]

Duke, Imamoğlu and Tóth [13, Prop. 4.1] proved that for any positive non-square discriminant \( d \) we have
\[
\langle g_d, g_0 \rangle_{\text{reg}} = \frac{3}{4} \frac{h^*(d)}{\sqrt{d}},
\]
where
\[ \langle g_d, g_0 \rangle_{\text{reg}} := \lim_{Y \to \infty} \int_{\mathcal{F}_Y(4)} g_d(\tau) \overline{g_0(\tau)} y^2 \frac{dx dy}{y^2} \]
and \( \mathcal{F}_Y(4) \) is a fundamental domain for \( \Gamma_0(4) \) truncated by removing \( Y \)-neighborhoods of the cusps.

To define an inner product in the case when \( d \) is square, we adopt the strategy in a recent paper of Bringmann, Diamantis, and Ehlen [7]. In that paper a general regularization of the inner product of two weakly holomorphic modular forms was given by introducing extra terms in two auxiliary variables. In this section and the next, we will adapt this strategy in only the generality we need; in particular we require only one of the auxiliary variables for each application.

The language of vector valued forms is most convenient in this section. Suppose that \( f(\tau) = \sum_{n=0} a(n) q^n \) is a weakly holomorphic modular form of weight \( \frac{3}{2} \) on \( \Gamma_0(4) \) in the plus-space, and write \( f(\tau) = f^e(\tau) + f^o(\tau) \), where these denote the sums over even and odd indices respectively. To \( f(\tau) \) we associate the vector valued form
\[ \tilde{f}(\tau) := f^e(\tau/4)e_0 + f^o(\tau/4)e_1 \]
Then \( \tilde{f} \) transforms in weight \( \frac{3}{2} \) with respect to a certain representation of \( \text{Mp}_2(\mathbb{Z}) \) (see e.g. [7, §4.3] for details). Similarly, we write
\[ \tilde{g}_d(\tau) := g_d^e(\tau/4)e_0 + g_d^o(\tau/4)e_1 \quad \text{and} \quad \tilde{Z}(\tau) := Z^e(\tau/4)e_0 + Z^o(\tau/4)e_1. \]

Suppose that \( d > 0 \) is not square. Let \( \mathcal{F}_Y \) be the standard fundamental domain for \( \text{SL}_2(\mathbb{Z}) \), truncated at height \( Y \). A computation as in [7, §4.3], using Lemma 3.2 of [13] shows that
\[ \langle g_d, g_0 \rangle_{\text{reg}} = \frac{3}{4} \lim_{Y \to \infty} \int_{\mathcal{F}_Y} \tilde{g}_d(\tau) \cdot \overline{\tilde{g}_0(\tau)} y^2 \frac{dx dy}{y^2}. \]
(6.9)
(We note that the constant \( \frac{3}{4} \) appears incorrectly as \( \frac{3}{2} \) in the corresponding computation of [7, §4.3]; this arises from the fact that the relationship \( \xi_{\frac{1}{2}}(2G) = g \) in that section should read \( \xi_{\frac{1}{2}}(2G) = g \).

For any \( d > 0 \), we define
\[ I(g_d, g_0; s) := \lim_{Y \to \infty} \int_{\mathcal{F}_Y} \tilde{g}_d(\tau) \cdot \overline{\tilde{g}_0(\tau)} y^{\frac{3}{2}-s} \frac{dx dy}{y^2}. \]
(6.10)
We will show that \( I(g_d, g_0; s) \) is defined for \( \text{Re}(s) \) sufficiently large, and that it has a meromorphic continuation to a neighborhood of \( s = 0 \). We may therefore define the extended inner product
\[ \langle g_d, g_0 \rangle_4 := \text{CT}_{s=0} \left( I(g_d, g_0; s) \right) \]
(6.11)
as the constant term in the Laurent expansion at \( s = 0 \). By (6.9) we have
\[ \langle g_d, g_0 \rangle_4 = \frac{4}{3} \langle g_d, g_0 \rangle_{\text{reg}} \quad \text{if} \quad d \text{ is not square;}
\]
this also follows from the computations below.

To show that the definition makes sense, we truncate at \( y = 1 \) to obtain
\[ \int_{\mathcal{F}_Y} \tilde{g}_d(\tau) \cdot \overline{\tilde{g}_0(\tau)} y^{\frac{3}{2}-s} \frac{dx dy}{y^2} = \int_{\mathcal{F}_1} \tilde{g}_d(\tau) \cdot \overline{\tilde{g}_0(\tau)} y^{\frac{3}{2}} \frac{dx dy}{y^2} + \int_{1}^{Y} \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{g}_d(\tau) \cdot \overline{\tilde{g}_0(\tau)} y^{\frac{3}{2}-s} \frac{dx dy}{y^2}. \]
Using Fourier expansions and integrating term by term, the second integral becomes

\[ \int_1^Y \sum_{n>0} H(n) B(d, n) e^{-\pi n y} y^{-\frac{1}{2} - s} \, dy - \frac{\delta_\nabla(d) \sqrt{y}}{4} \int_1^Y \beta_{\frac{1}{2}}(\pi dy) e^{\pi dy} y^{-\frac{1}{2} - s} \, dy \]

\[ + \delta_\Box(d) \int_1^Y \left( \frac{1}{6} - \frac{1}{2\pi \sqrt{y}} \right) y^{-\frac{1}{2} - s} \, dy. \quad (6.12) \]

If \( \{c(n)\} \) are the coefficients of a weakly holomorphic modular form or a mock modular form, then by [8, Lemma 3.4] we have the estimate

\[ c(n) \ll e^{C \sqrt{n}} \quad \text{for some } C \text{ as } n \to \infty. \quad (6.13) \]

By [11, (8.11.2)] we have

\[ \beta_k(y) \ll y^{-k} e^{-y} \quad \text{as } y \to \infty. \quad (6.14) \]

We also have the crude estimate \( H(n) \ll n^{1+\epsilon} \).

It follows that the integral defining \( I(g_d, g_0; s) \) converges for \( \Re(s) > \frac{1}{2} \). In the region of convergence we have

\[ I(g_d, g_0; s) = \lim_{Y \to \infty} \left[ \int_{\mathcal{F}_Y} \tilde{g}_d(\tau) \cdot \overline{g_0(\tau)} y^{\frac{3}{2} - s} \, dy - \delta_\Box(d) \int_1^Y \left( \frac{1}{6} - \frac{1}{2\pi \sqrt{y}} \right) y^{-\frac{1}{2} - s} \, dy \right] 
\]

\[ + \delta_\Box(d) \int_1^\infty \left( \frac{1}{6} - \frac{1}{2\pi \sqrt{y}} \right) y^{-\frac{1}{2} - s} \, dy. \quad (6.15) \]

By the discussion above, the first term is holomorphic in a neighborhood of \( s = 0 \). The second term has a meromorphic continuation to \( s = 0 \). This justifies the definition (6.11), and shows that we have

\[ \langle g_d, g_0 \rangle_s = \lim_{Y \to \infty} \left[ \int_{\mathcal{F}_Y} \tilde{g}_d(\tau) \cdot \overline{g_0(\tau)} y^{\frac{3}{2} - s} \, dy - \delta_\Box(d) \left( \frac{\sqrt{Y} - 1}{3} - \frac{\log Y}{2\pi} \right) \right] - \frac{\delta_\Box(d)}{3}. \quad (6.16) \]

Since \( \xi_{\frac{1}{2}} Z = -2g_0 \), we have

\[ \xi_{\frac{1}{2}} Z^o(\tau/4) = -g_0(\tau/4), \quad \xi_{\frac{1}{2}} Z^o(\tau/4) = -g_0(\tau/4). \]

If \( g \) is holomorphic, then by Stokes’ theorem (see, e.g., [13, Lemma 3.2], noting that the identity there should read \( d\tau d\sigma = -2i \, dx \, dy \)) we have

\[ \int_{\mathcal{F}_Y} g(\tau) h(\tau) \, d\tau d\sigma = - \int_{\partial_{\mathcal{F}_Y}} g(\tau) h(\tau) \, d\tau. \quad (6.17) \]

Therefore

\[ \int_{\mathcal{F}_Y} \tilde{g}_d(\tau) \cdot \overline{g_0(\tau)} y^{\frac{3}{2} - s} \, dy = - \int_{\mathcal{F}_Y} \left( g^o_d \left( \frac{\tau}{4} \right) Z^o \left( \frac{\tau}{4} \right) + g^o_d \left( \frac{\tau}{4} \right) Z^o \left( \frac{\tau}{4} \right) \right) y^{-\frac{1}{2}} \, dx \, dy \]

\[ = \int_{\partial_{\mathcal{F}_Y}} \left( g^o_d \left( \frac{\tau}{4} \right) Z^o \left( \frac{\tau}{4} \right) + g^o_d \left( \frac{\tau}{4} \right) Z^o \left( \frac{\tau}{4} \right) \right) \, d\tau \]

\[ = - \int_{-\frac{1}{2} + iY}^{\frac{1}{2} + iY} \left( g^o_d \left( \frac{\tau}{4} \right) Z^o \left( \frac{\tau}{4} \right) + g^o_d \left( \frac{\tau}{4} \right) Z^o \left( \frac{\tau}{4} \right) \right) \, d\tau. \]
Using the Fourier expansions (1.9) and (6.8), we find that
\[
\int_{\mathcal{F}} \overline{\tilde{g}_d(\tau)} \cdot \overline{\tilde{g}_0(\tau)} g_3^{\frac{3}{2}} \frac{dxdy}{y^2} = -\frac{h^*(d)}{\sqrt{d}} + \delta(d) \left( \frac{\sqrt{Y}}{3} + \frac{\gamma - \log 4\pi Y}{2\pi} - \alpha(dY) \right) - \sum_{n>0} B(d,n) \frac{h^*(-n)}{\sqrt{n}} \beta_{\frac{1}{2}}(\pi nY). \quad (6.18)
\]

Since \( e^{-x} \leq x^{-1} \) for all \( x > 0 \), we have \( \alpha(Y) \to 0 \) as \( Y \to \infty \). By (6.16), (6.18), (6.13), and (6.14), we have
\[
\langle g_d, g_0 \rangle_4 = -\frac{h^*(d)}{\sqrt{d}} + \delta(d) \frac{\gamma - \log 4\pi}{2\pi},
\]
and Theorem 2 follows.

7. Regularized Inner Products for \( \text{SL}_2(\mathbb{Z}) \)

Here we prove an analogue of Theorem 2 for \( \text{SL}_2(\mathbb{Z}) \). Let \( F \) be the harmonic Maass form of weight 3/2 and multiplier \( \overline{\chi} \) on \( \text{SL}_2(\mathbb{Z}) \) defined in (1.11). We also introduce a natural infinite family of weakly holomorphic modular forms.

Lemma 10. For any integer \( d > 1 \) with \( d \equiv 1 \mod 24 \), there exists a unique weight \( \frac{3}{2} \) weakly holomorphic modular form \( h_d \) on \( \text{SL}_2(\mathbb{Z}) \) with multiplier \( \overline{\chi} \) such that
\[
h_d(\tau) = q^{-\frac{d}{24}} + \sum_{-1 \leq n \equiv 23 (24)} A(d,n) q^{\frac{n}{24}}.
\]
Furthermore, we have \( A(d,-1) = -\chi_{12}(\sqrt{d}) \).

Proof. Letting \( j(\tau) \) be the usual \( j \)-invariant and defining
\[
\Theta := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq},
\]
we have
\[
h_{25}(\tau) = -\frac{\Theta(j(\tau))}{\eta(\tau)} = q^{-\frac{25}{24}} + q^{-\frac{1}{24}} - 196882q^{\frac{23}{24}} - \cdots.
\]
The subsequent forms \( h_d(\tau) \) are constructed by multiplying \( h_{25}(\tau) \) by a suitable element of \( \mathbb{C}[j] \). For instance, the next two forms are
\[
h_{49}(\tau) = (j(\tau) - 745)h_{25}(\tau) = q^{-\frac{49}{24}} + q^{-\frac{1}{24}} - 21296875q^{\frac{23}{24}} - \cdots,
\]
\[
h_{73}(\tau) = (j(\tau)^2 - 1489j(\tau) + 357395)h_{25}(\tau) = q^{-\frac{73}{24}} - 842609326q^{\frac{23}{24}} - \cdots.
\]
The remaining claim follows from the fact that each \( h_d(\tau) \eta(\tau) \) is a weakly holomorphic modular form of weight 2 on \( \text{SL}_2(\mathbb{Z}) \). \( \square \)

We define in (7.1) an inner product \( \langle \cdot, \cdot \rangle_1 \) which extends the natural regularized inner product, and we will prove

Theorem 11. Let \( f \) be the modular function on \( \Gamma_0(6) \) defined in (2.2). Then for each positive integer \( d \equiv 1 \mod 24 \), we have
\[
\sqrt{24} \langle h_d, F \rangle_1 = -\text{Tr}_d(f) + \chi_{12}(\sqrt{d}) \left( \text{Tr}_1(f) - 12 \frac{h^*(d)}{\sqrt{d}} - i \right).
\]
The usual regularized inner product of $h_d$ and $F$ is given by

$$\langle h_d, F \rangle_{\text{reg}} = \lim_{Y \to \infty} \int_{\mathcal{F}_Y} h_d(\tau) \overline{F(\tau)} y^{\frac{3}{2}} e^{-wy} \frac{dxdy}{y^2}. $$

The computation below will show that $\langle h_d, F \rangle_{\text{reg}}$ exists if and only if $d$ is not square, and that

$$\sqrt{24} \langle h_d, F \rangle_{\text{reg}} = - \text{Tr}_d(f) \quad \text{if } d \text{ is not square}. $$

We again adopt the strategy from [7] to extend the inner product to the case when $d$ is square. Since there are exponentially growing terms in this case (see below) the extension differs from that of the last section.

Define

$$I(h_d, F; w) := \int_{\mathcal{F}} h_d(\tau) \overline{F(\tau)} y^{\frac{3}{2}} e^{-wy} \frac{dxdy}{y^2}. $$

The integral converges when $\Re w \gg 0$. We will show that there is an analytic continuation to $w = 0$; we then define

$$\langle h_d, F \rangle_1 := I(h_d, F; 0). \quad (7.1) $$

For $\Re w \gg 0$, we have

$$I(h_d, F; w) = \lim_{Y \to \infty} \left( \int_{\mathcal{F}_1} h_d(\tau) \overline{F(\tau)} y^{\frac{3}{2}} e^{-wy} dxdy + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{1}^{Y} h_d(\tau) \overline{F(\tau)} y^{\frac{3}{2}} e^{-wy} dxdy \right). $$

Integrating term by term (note that $s(0) = -\frac{1}{12}$) yields

$$\int_{1}^{Y} \int_{-\frac{1}{2}}^{\frac{1}{2}} h_d(\tau) \overline{F(\tau)} y^{\frac{3}{2}} e^{-wy} dxdy = \frac{\chi_{12}(\sqrt{d})}{12} \int_{1}^{Y} y^{-\frac{1}{2}} e^{(\frac{\pi}{6} - w)y} dy - \frac{\chi_{12}(\sqrt{d})\sqrt{d}}{2} \int_{1}^{Y} y^{-\frac{1}{2}} e^{\left(\frac{\pi}{6} - w\right)y} dy \frac{\beta_{\frac{1}{2}}(\frac{\pi}{6})}{\beta_{\frac{1}{2}}(\frac{\pi}{6})}$$

$$+ \frac{\chi_{12}(\sqrt{d})}{2} \int_{1}^{Y} y^{-\frac{1}{2}} e^{(\frac{\pi}{6} - w)y} \beta_{\frac{1}{2}}(\frac{\pi}{6}) dy + \sum_{n>0} A(d, n) s(n+\frac{1}{24}) \int_{1}^{Y} y^{-\frac{1}{2}} e^{(-\frac{\pi}{6} - w)y} dy. $$

By (6.13) and (6.14), we see that all but the first integral on the right side converge absolutely on $\Re w \geq 0$ as $Y \to \infty$.

For $\Re w \gg 0$, we have

$$I(h_d, F; w) = \lim_{Y \to \infty} \left( \int_{\mathcal{F}_Y} h_d(\tau) \overline{F(\tau)} y^{\frac{3}{2}} e^{-wy} dxdy - \frac{\chi_{12}(\sqrt{d})}{12} \int_{1}^{Y} y^{-\frac{1}{2}} e^{(\frac{\pi}{6} - w)y} dy \right)$$

$$+ \frac{\chi_{12}(\sqrt{d})}{12} \int_{1}^{\infty} y^{-\frac{1}{2}} e^{(\frac{\pi}{6} - w)y} dy. $$

Using (1.1), the last term is

$$\frac{\chi_{12}(\sqrt{d})}{12} \frac{\sqrt{\pi}}{\sqrt{w - \frac{\pi}{6}}} \beta_{\frac{1}{2}}(w - \frac{\pi}{6}). $$
so we have

\[
\langle h_d, F \rangle_1 = I(h_d, F; 0) = \lim_{Y \to \infty} \left( \int_{\mathcal{F}_Y} h_d(\tau) \overline{F(\tau)} \frac{dx dy}{y^2} - \frac{\chi_{12}(\sqrt{d})}{12} \int_1^Y y^{-\frac{1}{2}} e^{\frac{\pi y}{6}} dy \right)
- i \frac{\chi_{12}(\sqrt{d})}{\sqrt{24}} \beta_{\frac{1}{2}} \left( -\frac{\pi}{6} \right). \tag{7.2}
\]

We turn to the proof of Theorem 11. Since \(-\sqrt{24} F(\tau) = \xi_{\frac{1}{2}} H(\tau)\), arguing as above using (6.17) gives

\[
\int_{\mathcal{F}_Y} h_d(\tau) \overline{F(\tau)} \frac{dx dy}{y^2} = \frac{1}{\sqrt{24}} \int_{\partial \mathcal{F}_Y} h_d(\tau) H(\tau) d\tau = \frac{-1}{\sqrt{24}} \int_{\frac{1}{2} + iY}^{\frac{1}{2} + Y} h_d(\tau) H(\tau) d\tau.
\]

Integrating term by term gives

\[
\sqrt{24} \int_{\mathcal{F}_Y} h_d(\tau) \overline{F(\tau)} \frac{dx dy}{y^2} = - \text{Tr}_d(f) + \chi_{12}(\sqrt{d}) \left( \text{Tr}_1(f) - 12 \frac{h^*(d)}{\sqrt{d}} \right)
+ i \chi_{12}(\sqrt{d}) \left( \beta_{\frac{1}{2}} \left( -\frac{\pi Y}{6} \right) - 1 \right) - 24 \chi_{12}(\sqrt{d}) \left( \alpha \left( \frac{dY}{6} \right) - \alpha \left( \frac{\pi Y}{6} \right) \right)
- \sum_{n>0} A(d, n) \text{Tr}_{-n}(f) \frac{\beta_{\frac{1}{2}}}{\sqrt{n}} \left( \frac{\pi n Y}{6} \right). \tag{7.3}
\]

Using (2.9) and (2.10), we find that

\[
\beta_{\frac{1}{2}} \left( -\frac{\pi Y}{6} \right) - 1 = \beta_{\frac{1}{2}} \left( -\frac{\pi}{6} \right) - 1 - \frac{i}{\sqrt{6}} \int_1^Y y^{-\frac{1}{2}} e^{\frac{\pi y}{6}} dy. \tag{7.4}
\]

Theorem 11 follows from (7.2), (7.3), and (7.4).

In closing, we remark that by generalizing these arguments it would be possible to investigate inner products of larger families of forms in the spirit of [7] and [13].

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