Completing the physical representation of quantum algorithms provides a retrocausal explanation of their speedup

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Abstract

In previous works, we showed that an optimal quantum algorithm can always be seen as a sum over classical histories in each of which the problem solver knows in advance one of the possible halves of the solution she will read in the future and performs the computation steps (oracle queries) still needed to reach it. Given an oracle problem, this retrocausal explanation of the speedup yields the order of magnitude of the number of oracle queries needed to solve it in an optimal quantum way. Presently, we provide a fundamental justification for the explanation in question and show that it comes out by just completing the physical representation of quantum algorithms. Since the use of retrocausality in quantum mechanics is controversial, showing that it answers the well accepted requirement of the completeness of the physical description should be an important pass.

1 Foreword

The quantum computational speedup is the fact that quantum algorithms solve the respective problems with fewer computation steps (oracle queries) than their best classical counterparts, sometimes demonstrably fewer than classically possible.

A paradigmatic example is the simplest instance of the quantum algorithm devised by Grover [1]. Bob, the problem setter, hides a ball in one of four drawers. Alice, the problem solver, is to locate it by opening drawers. In the classical case, Alice has to open up to three drawers, always one in the quantum case (the problem is an example of oracle problem and the operation of checking whether the ball is in a drawer is an example of oracle query).

Deutsch [2] commented his 1985 discovery of the seminal quantum speedup, of course allowed by quantum superposition and interference, with the state-
ment that computation is physical\(^1\). This is as simple as deep. The interplay between computing as a mental process and some "outside" physical process, like counting on the fingers, must be as ancient as the idea of computation itself. However, until the physical process has remained classical, its character has not enriched the idea of computation. The turning point comes with quantum computation. This time physical computation is richer than our idea of computation, as a matter of fact in ways that have yet to be well understood.

It should be noted that each quantum algorithm has been found by means of ingenuity. In mainstream literature, there is no fundamental explanation of the speedup, no unification of the various speedups (quadratic, exponential), no way of foreseeing the number of oracle queries needed to solve a generic oracle problem in an optimal quantum way.

Here we ascribe our limited understanding of the speedup to the fact that the physical representation of quantum algorithms arrived half done. Being limited to the input-output transformation typical of computation, it is physically incomplete. It consists of a unitary transformation followed by the final measurement required to read the solution. As well known, the complete representation of a quantum process must include the initial measurement, the unitary transformation of the state after measurement, and the final measurement.

Preliminary versions of the present retrocausal explanation of the speedup were already provided in the evolutionary approach [4 \textendash} 9]. In the present work we show that just completing the physical representation of quantum algorithms provides the explanation in question. This also clarifies the roles that time-symmetric and relational quantum mechanics play in it.

2 Extended summary

We are in the context of oracle computing. Let \( \{f_b(a)\} \) be a set of functions, with \( b \) the function identifier and \( a \) the argument of the function. Bob chooses one of these functions – i.e. a value of \( b \) – and gives Alice the black box that computes it. Alice knows the set of functions but not Bob’s choice. To her the function identifier \( b \) – from now on the problem setting – is hidden inside the black box. She is to find some characteristic of the function computed by the black box (e.g. the period in the case of a set of periodic functions) by performing function evaluations for appropriate values of \( a \). By the way, in literature, the black box is also called oracle and function evaluation oracle query.

The usual representation of quantum algorithms, limited to Alice’s problem-solving action, consists of the input-output transformation typical of computations. Alice works on a quantum register \( A \). The input state of this register is a pure quantum state independent of the value of \( b \) chosen by Bob. By performing function evaluations interleaved with other suitable transformations, she

\(^1\)In 1982, Feynman had pointed out that the classical simulation of a quantum process can require an amount of time \( \times \) physical resources exponentially higher than those involved in the quantum process itself [3].
unitarily sends the input state into an output state where register $A$ contains the solution of the problem, namely the characteristic of the function computed by the black box; in this summary we assume for simplicity that the solution, $s(b)$, is a one to one function of $b$. Then she acquires the solution by measuring the content of $A$. This representation has thus the form:

$$|0\rangle_A \xrightarrow{U_A} |s(b_c)\rangle_A \xrightarrow{M_A} |s(b_c)\rangle_A$$

In the input state of the quantum algorithm at time $t_1$, register $A$ contains a bit string of all zeroes – but any pure quantum state would do. $U_A$ is the unitary part of Alice’s action, it sends $|0\rangle_A$ into an output state where register $A$ contains the solution of the problem $s(b_c) - b_c$ being the value of $b$ chosen by Bob. $M_A$ is the measurement of the content of register $A$; the states before and after this measurement are the same; we call $t_{i+}$ the time of the state immediately after a measurement performed at time $t_i$.

This representation is physically incomplete in two related ways

(i) It lacks the initial measurement, whereas the complete representation of a quantum process must consist of an initial measurement, a unitary transformation of the measurement outcome, and a final measurement.

(ii) There is no physical representation of the value of $b$ chosen by Bob. This value is of course essential in determining the quantum process: it determines the result of function evaluations.

We restore both the initial measurement and the physical representation of the value of $b$ by extending the representation of the quantum algorithm to the process of setting the problem.

We add a possibly imaginary register $B$, under the control of Bob, that contains $b$. We assume that, initially, this register is in a maximally mixed state. Bob measures the content of this register selecting a problem setting, a value of $b$, at random. Then he unitarily changes the state after measurement into the state that encodes the desired problem setting. For simplicity, for the time being, we jump the transformation in question – we assume that Bob chooses the random outcome of the initial measurement.

The representation of the quantum algorithm is now:

$$\frac{1}{\sqrt{c_\sigma}} \sum_{b \in \sigma} e^{i\varphi_b} |b\rangle_B |0\rangle_A \xrightarrow{M_B} |b_c\rangle_B |0\rangle_A \xrightarrow{U_A} |b_c\rangle_B |s(b_c)\rangle_A \xrightarrow{M_A} |b_c\rangle_B |s(b_c)\rangle_A$$

To keep the usual ket vector representation of quantum algorithms, the maximally mixed state of register $B$ is represented as a dephased quantum superposition of all the possible values of $b$: $b$ ranges over the set of all the possible problem settings $\sigma$, of cardinality $c_\sigma$; the $\varphi_i$ are independent completely random phases (see what follows). $M_B$ is the measurement of the content of register $B$,
which selects a problem setting – here $b_c$ – at random (note that this measurement commutes with that of the content of $A$). At time $t_{1+}$ we have thus the new input state of the quantum algorithm; from here onwards the quantum states are those of the usual quantum algorithm but for the multiplication by $|b_c\rangle_B$.

We note that we are making a trivial use of the present random phase representation [10] of a maximally mixed state: we can always ignore the random character of the $\varphi_i$ but for the computation of the entropy of the quantum state, which is $n$ bit, with $n = \lg_2 c_\sigma$.

This extended representation is very similar to the usual one. However, there is an important consequence. Before going to it, it is convenient to make a step backward to explicit things that are usually given for understood.

As well known, the quantum state encapsulates everything that can be known about the quantum system between observations. To present ends, the question "known by whom?" is essential. We adopt the answer of the Copenhagen interpretation: known by the observer.

We still need to clarify who is the observer. The extended representation is certainly the representation to Bob, the problem setter, and any external observer: it tells all of them that the problem setting is completely undetermined at time $t_1$, is $b_c$ at time $t_{1+}$, etc. The point is that it cannot be the representation to Alice, the problem solver. The sharp state of register $B$ at time $t_{1+}$ would tell her that the problem setting chosen by Bob is $b_c$ (see the second state from the left of array [2]). It would tell her the solution of the problem before she begins her problem solving action: in fact we must think that the function $s(b)$ is known to Alice.

As is well known, to Alice, the value of $b$ chosen by Bob must be hidden inside the black box. This is why the box is called black in classical computer science. However, while in the classical case it suffices to keep this concealment in mind, to do the right thing at the right moment, now we are in the quantum case where, according to the present objective, all computational facts should be physical.

To represent the concealment physically, we must resort to the relational interpretation of quantum mechanics [11, 12]. According to it, a quantum state has meaning with respect to an observer, like in the Copenhagen interpretation. What the relational interpretation rejects is the notion of absolute, or observer-independent, state of a system. In equivalent terms, it rejects the notion of observer-independent values of physical quantities [12]. This notion "would be inadequate to describe the physical world beyond the $\hbar \to 0$ limit, in the same way in which the notion of observer-independent time is inadequate to describe the physical world beyond the $c \to \infty$ limit" [12].

In the representation to Alice, we postpone the projection of the quantum state due to the initial measurement until the end of the unitary part of her action. As is well known, the projection due to a quantum measurement can be postponed at will along a unitary transformation that follows it. The representation becomes:
Now the maximally mixed state of register $B$ remains unaltered throughout $M_B$, as the associated projection of the quantum state is postponed. The $n$ bit entropy of the state of register $B$ in the input state of the quantum algorithm (at time $t_{1+}$) represents Alice’s complete ignorance of the problem setting chosen by Bob. $U_A$ sends this input state into a mixture of tensor products, each the product of one of the possible problem settings and the corresponding solution (it maximally entangles the contents of registers $B$ and $A$). Eventually Alice measures the content of $A$, selecting the solution corresponding to the problem setting chosen by Bob. The measurement outcome cannot be predicted by Alice as usual, it is already known to Bob and any external observer.

Until now, the process of completing the physical representation of the quantum algorithm brought us to two time-symmetric and relational representations, one with respect to Bob and any external observer, the other with respect to Alice.

In either representation there is a unitary transformation between the initial and final measurement outcomes. There is a consequent ambiguity. The selection (determination) of the random outcome of the initial measurement and the corresponding outcome of the final measurement– of the pair $b$ and $s(b)$ – can be ascribed indifferently to the initial measurement of the content of $B$ or the final measurement of the content of $A$.

In the latter case, we should think that the projection of the quantum state associated with the initial measurement is removed. As a consequence, in either representation, the projection of the quantum state associated with the final measurement of the content of $A$ becomes that of the representation to Alice, namely:

\[
\frac{1}{\sqrt{c_\sigma}} \sum_{b \in \sigma} e^{i\varphi_b} |b\rangle_B |0\rangle_A \xrightarrow{U_A \dagger} \frac{1}{\sqrt{c_\sigma}} \sum_{b \in \sigma} e^{i\varphi_b} |b\rangle_B |s(b)\rangle_A \xrightarrow{M_B} |b_c\rangle_B |s(b_c)\rangle_A.
\]

We should advance this projection at the time of the initial measurement $t_1$. By this we mean propagating the two end states of it backward in time by the inverse of the unitary transformation that precedes it here by $U_A \dagger$. The projection becomes:

\[
\frac{1}{\sqrt{c_\sigma}} \sum_{b \in \sigma} e^{i\varphi_b} |b\rangle_B |0\rangle_A \rightarrow |b_c\rangle_B |0\rangle_A.
\]

We can see that the projection that would have been performed by the initial measurement is now performed back in time by the final measurement.

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The notion of advancing a state is taken from Wheeler-Feynman’s absorber theory [13] and Cramer’s transactional interpretation of quantum mechanics [14].
In [8,9], we represented the subject ambiguity by generically sharing the selection of the pair $b_c$ and $s(b_c)$ between the initial and final measurements. To this end we ascribed the selection of a generic $R$-th part of the information that specifies the solution to the final measurement, that of the complementary part to the initial measurement (in a quantum superposition of all the possible ways of doing it as clarified below). By comparing the consequent explanation of the speedup with a sample of optimal quantum algorithms, it turned out that the sharing had to be even.

In the present work we go the other way around. We assume to start with that the sharing is even. The reason is fundamental in character: any uneven sharing would not be we invariant under time-reversal; it would introduce a preferred direction of time, apparently with no reason in the present completely reversible context (keeping in mind that there is a unitary transformation between the initial and final measurement outcomes).

We note that this assumption is in line with (i) time-symmetric quantum mechanics [14 ÷ 19], which also excludes any preferred direction of time, and (ii) the logical [20] and physical [21 ÷ 23] reversibility of the computation process – physical reversibility implies indifference to time-reversal.

Evenly sharing the selection of the solution between the initial and final measurements is inconsequential in the representation of the quantum algorithm to Bob and any external observer. It would just say that part of the random outcome of the initial measurement has been selected back in time by the final measurement – in fact an unverifiable and inconsequential thing.

In the representation to Alice things go differently. The former input state to her (at time $t_{1+}$), of complete ignorance of the problem setting and the corresponding solution, is projected on a state of lower entropy where she knows one of the possible halves of the information that specifies them. The projection has the form:

$$\frac{1}{\sqrt{\sigma'}} \sum_{b \in \sigma} e^{i\varphi b} |bangle_B |0\rangle_A \rightarrow \frac{1}{\sqrt{\sigma'}} \sum_{b \in \sigma'} e^{i\varphi b} |b\rangle_B |0\rangle_A,$$

where $\sigma'$ is a suitable subset of $\sigma$.

Correspondingly, an optimal quantum algorithm should be seen as a sum over classical histories in each of which Alice, the problem solver, knows in advance one of the possible halves of the information about the solution she will read in the future and performs the function evaluations still needed to reach it.

We interpret Alice’s advanced knowledge from a metric standpoint. We assume that it gauges the distance, in number of function evaluations along the classical histories, between the initial state of the quantum algorithm and the solution. It would also be the distance covered by an optimal quantum algorithm. Of course a non-optimal quantum algorithm would take a longer route. As a matter of fact, all the oracle problems examined in this paper can be solved quantumly with any number of function evaluations above that of the optimal quantum algorithm.
Grover algorithm is optimal for any number of drawers [24 ÷ 26]. The present retrocausal model exactly explains the speedup of its four drawer instance, which requires just one function evaluation. Alice knows in advance that the ball is in one of 2 drawers. Opening either drawer allows her to locate it. When greater than one, the number of function evaluations required to find the solution along the classical histories is not univocally defined; it depends on the search criteria. However, for any reasonable criteria, the present retrocausal model gives the order of magnitude of the number of function evaluations required by Grover algorithm.

All the other optimal quantum algorithms examined in this paper are solved with just one function evaluation – their speedup is exactly explained by the present model.

Conversely, given an oracle problem, the model yields the order of magnitude of the number of function evaluations needed to solve it in an optimal quantum way. If this held for any oracle problem, as reasonable, it would solve the well known open problem of quantum query complexity.

In section 3 we develop our explanation of the speedup on Grover algorithm. In section 4 we generalize it to quantum oracle computing. In Section 5 we apply the generalization to the algorithms of Deutsch & Jozsa, Simon, and the Abelian hidden subgroup. In sections 6 and 7, we discuss the present theory of the speedup and provide the conclusions.

3 Grover algorithm

We show how the retrocausal explanation of the speedup comes from completing the physical representation of Grover algorithm. This algorithm is best suited to develop the present explanation, which is quantitative in character. Besides being optimal, it involves a number of function evaluations growing with problem size – most of the quantum algorithms discovered so far require just one function evaluation. We start with the four drawer instance of the algorithm.

3.1 Time symmetric and relational representations

Let the four drawers be numbered 00, 01, 10, 11 and b be the number of the drawer with the ball. Checking whether the ball is in drawer a amounts to computing the Kronecker function \( f_b(a) = \delta(b, a) \), which is 1 if \( b = a \) and 0 otherwise.

In the usual Grover algorithm, the number of the drawer that Alice wants to open a (the argument of function evaluation) is contained in a register \( A \) of basis vectors \( |00\rangle_A, |01\rangle_A, |10\rangle_A, |11\rangle_A \). This register, under the control of Alice, will eventually contain the solution of the problem. A register \( V \) (as value), of basis vectors \( |0\rangle_V, |1\rangle_V \), is meant to contain the result of function evaluation, modulo 2 added to its former content for logical reversibility. By the way, one could do without this register: transformations remain unitary also without it – but they are simpler to describe with it.
As anticipated, the value of \( b \) is not represented physically. Let us assume it is \( b = 01 \). The usual Grover algorithm is limited to the unitary transformation of the input state:

\[
|\gamma\rangle = \frac{1}{\sqrt{2}} (|00\rangle_A (|0\rangle_V - |1\rangle_V)
\]

into the output state

\[
\mathcal{I}_A U_f H_A |\gamma\rangle = \frac{1}{\sqrt{2}} (|01\rangle_A (|0\rangle_V - |1\rangle_V),
\]

where register \( A \) contains the solution of the problem – namely the number of the drawer with the ball 01. \( H_A \) is the Hadamard transform on register \( A \). It transforms \(|00\rangle_A \) into \( \frac{1}{\sqrt{2}} (|00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A) \). \( U_f \) is function evaluation, thus performed in quantum parallelism [2] for all the possible values of \( a \). It leaves the state of register \( V \), \( \frac{1}{\sqrt{2}} (|0\rangle_V - |1\rangle_V) \), unaltered when \( a \neq 01 \) and thus \( \delta = 0 \); it changes it into \( -\frac{1}{\sqrt{2}} (|0\rangle_V - |1\rangle_V) \) when \( a = 01 \) and \( \delta = 1 \) (modulo 2 addition of 1 changes \(|0\rangle_V \) into \(|1\rangle_V \) and vice-versa). The transformation \( \mathcal{I}_A \), applying to register \( A \), is the so called inversion about the mean [1]: a rotation of the basis of \( A \) makes the information acquired with function evaluation accessible to measurement. We do not need to go into further detail: all we need to know of the quantum algorithm is already there.

Eventually Alice acquires the solution by measuring the content of \( A \), namely the observable \( \hat{A} \) of eigenstates the basis vectors of register \( A \) and eigenvalues (correspondingly) 00, 01, 10, 11.

Now we extend the representation of the quantum algorithm to the process of choosing the number of the drawer with the ball \( b \). We need to add a possibly imaginary register \( B \) that contains \( b \). This register, under the control of Bob, has basis vectors \(|00\rangle_B, |01\rangle_B, |10\rangle_B, |11\rangle_B \). We assume that the initial state of register \( B \) is a mixture of all the possible drawer numbers. To keep the usual ket vector representation of quantum algorithms, we represent it as a dephased quantum state superposition:

\[
|\psi\rangle_B = \frac{1}{2} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B).
\]

The \( \varphi_i \) are independent completely random phases. The use we make of the random phase representation [10] of a maximally mixed state is trivial. Because of the character of the unitary transformations involved (see below), we can see \(|\psi\rangle_B \) as it were a pure quantum state, like the \( \varphi_i \) were fixed phases. Only when we need to compute its von Neumann entropy we need to remember their random character. The von Neumann entropy of \(|\psi\rangle_B \) is 2 bit. By the way, the usual density operator, namely \( \frac{1}{4} (|00\rangle_B \langle 00|_B + |01\rangle_B \langle 01|_B + |10\rangle_B \langle 10|_B + |11\rangle_B \langle 11|_B) \), is the average over all the \( \varphi_i \) of \(|\psi\rangle_B \langle \psi|_B \).

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The fact that Alice opens a single drawer for a quantum superposition of all the possible drawer numbers is of course the key to achieve the speedup, but it does not provide any quantitative explanation of it in general.
The initial state of the three registers, at time $t_0$, is then:

$$|\psi\rangle_S = \frac{1}{2\sqrt{2}} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B) |00\rangle_A (|0\rangle_V - |1\rangle_V).$$

(4)

Let the observable $\hat{B}$, of eigenstates the basis vectors of register $B$ and eigenvalues (correspondingly) 00, 01, 10, and 11, be the content of register $B$ (by the way, $\hat{A}$ and $\hat{B}$ commute). At time $t_0$ Bob measures $\hat{B}$ selecting a drawer number at random, say 10. The state after measurement at time $t_0+$ is thus:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |10\rangle_B |00\rangle_A (|0\rangle_V - |1\rangle_V).$$

(5)

Then, by the unitary transformation $U_B$, he sends it into the state that encodes the desired problem setting, say 01. At time $t_1$, after $U_B$, the state is thus:

$$U_B |\psi\rangle = \frac{1}{\sqrt{2}} |01\rangle_B |00\rangle_A (|0\rangle_V - |1\rangle_V).$$

(6)

$U_B$ can be for example:

$$U_B \equiv |11\rangle \langle 00|_B + |10\rangle \langle 01|_B + |01\rangle \langle 10|_B + |00\rangle \langle 11|_B,$$

it is simpler to think that it changes zeros into ones and ones into zeros.

State (6) is the input state of the quantum algorithm – with the ball hidden by Bob in drawer 01. Alice unitarily sends it into the output state:

$$\Im A U_f H_A U_B |\psi\rangle = \frac{1}{\sqrt{2}} |01\rangle_B |01\rangle_A (|0\rangle_V - |1\rangle_V),$$

(7)

reached at time $t_2$. Note that the unitary part of Alice’s action does not change the content of register $B$, namely the problem setting chosen by Bob. In fact $B$ is the control register of the function evaluation transformation $U_f$: each basis vector of $B$ (each value of $b$) naturally affects the result of the computation of $f_b(a)$ while remaining unaltered throughout it. The other unitary transformations of Alice’s action, $H_A$ and $\Im A$, do not apply to $B$.

Eventually Alice acquires the solution of the problem by measuring $\hat{A}$. Note that this measurement leaves the quantum state unaltered; there is thus a unitary transformation between the initial and final measurement outcomes.

We can see that the representation of the quantum algorithm, now extended to the process of choosing the problem setting, is still physically incomplete. It is with respect to Bob, the problem setter, and any external observer, it cannot be with respect to Alice, the problem solver. State (6), with register $B$ in the sharp state $|01\rangle_B$, would tell her that the number of the drawer with the ball is 01 – it would tell her the solution of the problem before she opens any drawer.

As well known, to Alice the value of $b$ must be hidden inside the black box that computes $\delta(b,a)$. To represent this concealment physically, we resort to the relational interpretation of quantum mechanics [11, 12].
To Alice, we postpone the projection of the quantum state due to the initial measurement until the end of the unitary part of her problem-solving action; we should correspondingly \textit{retard} – i.e. propagate forward in time [13, 14] – the two end states of the projection by the unitary part in question.

As a consequence, the input state to Alice, at time $t_1$, remains the initial state (4); in fact $U_B$, being a unitary transformation applying to register $B$, leaves the maximally mixed state of this register unaltered. The 2 bit entropy of the state of register $B$ in the input state of the quantum algorithm to Alice represents her complete ignorance of the problem setting.

The unitary part of Alice’s action sends this input state into the output state:

$$\exists_A U_f H_A U_B |\psi\rangle = \frac{1}{2\sqrt{2}} \left( e^{i\phi_0} |00\rangle_B |00\rangle_A + e^{i\phi_1} |01\rangle_B |01\rangle_A + e^{i\phi_2} |10\rangle_B |10\rangle_A + e^{i\phi_3} |11\rangle_B |11\rangle_A \right) (\langle 0|_V - \langle 1|_V). \tag{8}$$

We can see that each possible problem setting is multiplied by the corresponding solution. The final measurement of the content of register $A$ projects the output state (8) on state (7), which is thus common to the representation to Bob and any external observer and to the representation to Alice. The measurement outcome is unpredictable to Alice as usual – not to Bob and any external observer who already know that the number of the drawer with the ball is $b = 01$.

By the way, we note that the projection of the quantum state due to the initial measurement, postponed until the end of the quantum algorithm, in the present case of Grover algorithm coincides with that due to the final measurement.

### 3.2 Sharing the selection of the random outcome of the initial measurement between the initial and final measurements

We provide two ways of sharing the selection of the random outcome of the initial measurement between the initial and final measurements: the synthetic way has a more evident physical meaning, the analytical way is more general and can be applied also to quantum oracle computing.

#### 3.2.1 Synthetic method

We reduce the complete measurements, of $\hat{B}$ and $\hat{A}$, to partial measurements, of $\hat{B}_i$ and $\hat{A}_j$, such that they: (i) together select without redundancies the random outcome of the initial measurement and (ii) evenly contribute to the selection of the solution – evenly reduce the entropy of the reduced density operator of register $A$ in the output state. Note that we are applying Occam’s razor; we give up the condition that everything is selected by the measurement performed first, not the economic condition that there are no redundant selections.

For example, one way of sharing compatible with (i) and (ii) is assuming that the initial measurement of $\hat{B}$ reduces to that of $\hat{B}_0$ (the content of the left cell...
of register $B$) and the final measurement of $\hat{A}$ to that of $\hat{A}_1$ (the content of the right cell of register $A$). Note that the outcomes of the complete measurements should be left unaltered, we should only share their selections between the two partial measurements.

We see how things go in the two representations of the quantum algorithm, starting with that to Bob and any external observer. Here the measurement of $\hat{B}_0$ at time $t_0$, selecting the left digit of the number contained in register $B$, must select the 1 of the outcome of the initial measurement $b = 10$. This projects the initial state (4) on:

$$|\xi\rangle = \frac{1}{2} (e^{i\phi_2} |10\rangle_B + e^{i\phi_3} |11\rangle_B) |00\rangle_A (|0\rangle_V - |1\rangle_V).$$

At time $t_2$, state (9) has evolved into:

$$\mathcal{A}_A U_f H_A U_B |\xi\rangle = \frac{1}{2} (e^{i\phi_2} |01\rangle_B |01\rangle_A + e^{i\phi_3} |00\rangle_B |00\rangle_A) (|0\rangle_V - |1\rangle_V).$$

Then the measurement of $\hat{A}_1$, selecting the right digit of the number of the drawer with the ball 01 contained in register $A$, projects state (10) on the original output state (7). Advancing the two ends of this projection by the inverse of $\mathcal{A}_A U_f H_A U_B$ projects state (9) on the original state after the initial measurement (5).

Summing up, the two partial measurements rebuild the selection of the random outcome of the initial measurement and that of the final measurement while leaving the original quantum algorithm to Bob and any external observer unaltered. Moreover, the measurement of $\hat{A}_1$ selects one of the two bits that specify the solution, that of $\hat{B}_0$ – retarded at the time of the final measurement – the other bit. Therefore the selection of the solution evenly shares between the two partial measurements as required.

In the quantum algorithm with respect to Bob and any external observer, evenly sharing all selections between the initial and final measurements only says that the right digit of the random outcome of the initial measurement has been selected back in time by the final measurement, an unverifiable thing. Retrocausality is inconsequential in this representation, which is the usual one up to the representation of Bob’s choice.

Things change dramatically in the representation with respect to Alice. Here the measurement of $\hat{B}_0$ at time $t_0$ selects the 1 of the random outcome of the initial measurement 10 without altering the original quantum algorithm; in fact the projection of the quantum state associated with it is postponed until the end of the algorithm. We can go directly to the output state to Alice (5), at time $t_2$, when the measurement of $\hat{A}_1$ acquires the right digit of $a = 01$. This projects state (5) on:

$$|\chi\rangle = \frac{1}{2} (e^{i\phi_1} |01\rangle_B |01\rangle_A + e^{i\phi_3} |11\rangle_B |11\rangle_A) (|0\rangle_V - |1\rangle_V).$$

Also now we should propagate this projection backward in time until it selects, at time $t_{0+}$, the right digit of the outcome of the initial measurement, namely the 0 of 10 – the 1 was selected by the measurement of $\hat{B}_0$. 

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What is interesting to present ends is the value of this backward propagation at time $t_1$, immediately after $U_B$ and before the unitary part of Alice’s action $\mathcal{Z}_A U_f H_A$. We should advance the two ends of the projection of the output state (8) on state (11) by the inverse of $\mathcal{Z}_A U_f H_A$. The result is the projection of the input state (4) on the state:

$$H_A^\dagger U_f^\dagger \mathcal{Z}_A^\dagger |\chi\rangle = \frac{1}{2} (e^{i\phi_1} |01\rangle_B + e^{i\phi_3} |11\rangle_B) |00\rangle_A (|0\rangle_V - |1\rangle_V).$$

(12)

This is an outstanding consequence. State (12), the input state to Alice under the assumption that the selection of the random outcome of Bob’s measurement evenly shares between the initial and final measurements, tells her, before she performs any function evaluation, that the number of the drawer with the ball is either $b = 01$ or $b = 11$. We will say that Alice knows in advance that $b \in \{01, 11\}$. We take this as a metric notion: advanced knowledge of half of the problem setting, or of the corresponding half of the solution, gauges the distance (in number of function evaluations along the classical histories) of the input state from the solution – we will further discuss this point in sections 3.3 and 6.3.1.

### 3.2.2 Analytical method

Throughout this section, we put ourselves in the representation of the quantum algorithm with respect to Alice.

The analytical method is a way of calculating Alice’s advanced knowledge that avoids the necessity of knowing the unitary part of Alice’s action. It hinges on the following points.

a) Since in the output state the content of register $A$ is a bijective function of that of register $B$, a partial measurement of the content of $A$ can always be represented as a partial measurement of the content of $B$. We can thus replace the measurement of the generic $\hat{A}_j$ in the output state by that of the generic $\hat{B}_j$.

b) The reduced density operator of register $B$ and any projection thereof remain unaltered throughout the unitary part of Alice’s action, as we will see in a moment.

In the random phase representation, the reduced density operator of register $B$ is the trace over registers $A$ and $V$ of the overall state of the three registers. In the input state of Alice’s problem solving action (13), it has evidently the form:

$$|\psi\rangle_B = \frac{1}{2} (e^{i\phi_0} |00\rangle_B + e^{i\phi_1} |01\rangle_B + e^{i\phi_2} |10\rangle_B + e^{i\phi_3} |11\rangle_B).$$

(13)

This form remains unaltered throughout the unitary part of Alice’s action – namely until state (8). This is because each basis vector of $B$ – thus also any superposition thereof – remains unaltered through it (see the comment to equation (8)).
Measuring $\hat{B}_1$ – now replacing $\hat{A}_1$ – in the output state selects the right digit of 01, projecting $|\psi\rangle_B$ on

$$\frac{1}{\sqrt{2}} \left( e^{i\phi_1} |01\rangle_B + e^{i\phi_3} |11\rangle_B \right).$$

(14)

Advancing this projection at time $t_1$, the time of the input state, by $H_A^\dagger U_f^\dagger \mathcal{Z}_A^\dagger$, leaves it unaltered.

Only the interpretation of the projection changes. Advanced at time $t_1$, the projection of (13) on (14) becomes the projection of the state of register $B$ in the input state of the quantum algorithm, of complete ignorance of the problem setting, on the state of lower entropy that represents Alice’s advanced knowledge.

Note that the projection in question can be obtained more simply by thinking of measuring $\hat{B}_1$ in the input state (4). In fact any measurement on the content of register $B$ can be seen as a projection of $|\psi\rangle_B$ and as such, for the purpose of the present calculation, can be moved at will along the unitary part of Alice’s action.

It is also convenient to move the measurement of $\hat{B}_0$ (the content of the left cell of register $B$) from the initial state at time $t_0$ to the input state at time $t_1$. To Alice, these two states are identical. It suffices to ask that the measurement selects the left digit of the problem setting chosen by Bob, namely of 01, no more of the random outcome of the initial measurement 10.

We end up with the problem of splitting the measurement of $\hat{B}$ in the input state to Alice into two partial measurements, of $\hat{B}_i$ and $\hat{B}_j$, such that they (I) together select without redundancies the problem setting chosen by Bob and (II) evenly contribute to the selection of the solution – evenly reduce the entropy of the reduced density operator of register $A$ in the output state.

At this point we have lost the memory of which was the measurement performed in the output state. Therefore, once satisfied conditions (I) and (II), the measurement of either $\hat{B}_i$ or $\hat{B}_j$ projects the reduced density operator of register $B$ on an instance of Alice’s advanced knowledge. We say for short it projects $\sigma$ – the set of all the possible problem settings – on such an instance.

We will see in section 4 that this way of calculating Alice’s advanced knowledge can be applied as it is to any oracle problem.

### 3.3 Sum over classical histories

We need to reconcile the notion of Alice’s advanced knowledge of half of the information that specifies the solution with the fact that such a half can be taken in a plurality of ways; in other words we need to symmetrize the notion for all the possible ways of taking half of the information. Moreover, we need an operational interpretation of the notion.

We kill two birds with one stone by resorting to Feynman’s path integral formulation of quantum mechanics [27]. We see an optimal quantum algorithm (its time-symmetric representation with respect to Alice) as a sum over classical
histories in each of which Alice knows in advance one of the possible halves of the information about the solution and performs the function evaluations needed to find the other half. An example of history is:

\[ e^{i\varphi_1} |01\rangle_B |00\rangle_A |0\rangle_V \xrightarrow{H} e^{i\varphi_1} |01\rangle_B |11\rangle_A |0\rangle_V \xrightarrow{U_f} e^{i\varphi_1} |01\rangle_B |11\rangle_A |0\rangle_V \xrightarrow{\sqrt{3}} e^{i\varphi_1} |01\rangle_B |01\rangle_A |0\rangle_V. \]

The left-most state is one of the elements of the input state superposition (4). The state after each arrow is one of the elements of the quantum superposition generated by the unitary transformation of the state before the arrow (with the exception of function evaluation which leaves the state sharp); the transformation in question is specified above the arrow.

In the history we are dealing with, the problem setting is \( b = 01 \). Register \( B \) is correspondingly in \( |01\rangle_B \) throughout Alice’s action, which of course does not change the problem setting chosen by Bob.

Alice performs function evaluation for \( a = 11 \). The content of register \( A \) in the second and third state is correspondingly \( |11\rangle_A \); the basis vectors of both \( B \) and \( A \) always remain unaltered through \( U_f \). The state of register \( V \) also remains unaltered here since the result of function evaluation is \( \delta (01, 11) = 0 \).

Alice’s advanced knowledge must be that \( b \) belongs to \{01, 11\}. In fact the subset of \( \sigma \) that represents it must always comprise the problem setting chosen by Bob, namely \( b = 01 \); the second element must be \( b = 11 \) given that Alice tries function evaluation for \( a = 11 \). Since the state of register \( V \) remains unaltered through \( U_f \), she knows that the result of function evaluation is zero and thus that the ball must be in drawer 01 – this with just one function evaluation.

Summing up, the sum over classical histories picture exactly explains the speedup of the present four drawer instance of Grover algorithm. In the next section we discuss the case of more than four drawers.

### 3.4 Grover algorithm with \( N > 4 \)

We go to the general case of \( N = 2^n \) drawers – \( n \) is the number of bits that specify the drawer number. With Grover algorithm, the number of function evaluations (drawer openings) required to find the solution is in general:

\[ k (n) = \frac{\pi}{4 \arcsin 2^{-n/2}} - \frac{1}{2} \approx \frac{\pi}{4} 2^{n/2}. \]

Let us call \( \mathbb{N} (n) \) the number of function evaluations foreseen by the present retrocausal model of the speedup in the case of \( 2^n \) drawers. \( \mathbb{N} (n) \) has been defined as the number of function evaluations needed to classically reach the solution given the advanced knowledge of one of the possible halves of the information that specifies it. We should distinguish between the two cases: \( n = 2 \) and \( n > 2 \).

As we have seen, for \( n = 2 \), we have \( \mathbb{N} (2) = k (2) = 1 \). In words, the present retrocausal model exactly explains the speedup of the four drawers instance of Grover algorithm. The explanation is no more exact when \( n > 2 \). While
the definition of Alice’s advanced knowledge – as half of the information that
specifies the solution – remains unaltered, that of $N(n)$ becomes not univocal.
It gets dependent on the criteria adopted for searching the solution.

We provide a few examples. Given the advanced knowledge of $n/2$ of the
bits that specify the solution, we could define $N(n)$ as:

(i) The number of function evaluations required to have the a-priori certainty
of finding the solution through an exhaustive classical search (never using a
second time the same argument for function evaluation). In this case we would
have $N(n) = 2^{n/2} - 1$ function evaluations, against the $k(n) \simeq \frac{\pi}{4}2^{n/2}$ of the
optimal Grover algorithm.

(ii) The average number of function evaluations required to classically reach
the solution under an exhaustive randomly ordered search. One can guess that
in this case $N(n)$ would be a bit smaller than $k(n)$.

(iii) The average number of function evaluations required to classically reach
the solution under a completely random search, etc.

Perhaps $N(n)$ could be defined more precisely by relating it to the structure
of the sum over classical histories. This should be for further study. For the
time being, we make reference to the exhaustive classical search with a-priori
certainty of finding the solution. This always gives the order of magnitude of
the number of function evaluations required by Grover algorithm.

By the way, all the above holds for Grover algorithm, which is optimal in
character. We should also note that there is always a quantum algorithm that
solves Grover’s problem with any number of function evaluations provided it is
not smaller than the minimum number required by Grover algorithm. This is
the revision of Grover algorithm devised by Long [25, 26], which can be tuned
to solve Grover problem with any number of function evaluations equal to or
above the minimum required by Grover algorithm.

The two things go together as follows. We should keep in mind the assump-
tion that Alice’s advanced knowledge gauges the distance, in number of function
evaluations along the classical histories, between the initial state of the quan-
tum algorithm and the solution. Of course it is also the distance covered by
the optimal quantum algorithm; non-optimal quantum algorithms can take any
longer route.

4 Quantum oracle computing

Until now, the retrocausal model of the speedup has been used to explain the
speedup of a known quantum algorithm. Now we show how to use it to calculate
the number of function evaluations needed to solve an oracle problem in an
optimal quantum way.

To start with, we assume that the solution is a bijective function of the
problem setting like in section 3.2. To calculate Alice’s advanced knowledge
(hence the number of function evaluations), we need to find the pairs $B_i$ and

\footnote{The algorithm devised by Long always yields the solution with absolute certainty, Grover
algorithm, with absolute certainty, only for $n = 2$.}
\( \hat{B}_j \) satisfying conditions (I) and (II) of section 3.2.2. We follow the analytical method of section 3.2.2, which dispenses us from knowing the unitary part of Alice’s action. We only need to know the state of registers \( B \) and \( A \) in the input and output states of the time-symmetric representation of the quantum algorithm with respect to Alice. For any quantum algorithm that solves the problem, these have the form:

\[
|\psi\rangle_I = \frac{1}{\sqrt{c_\sigma}} \sum_{b \in \sigma} e^{i\varphi_b} |b\rangle_B |0\rangle_A \quad \text{and} \quad |\psi\rangle_O = \frac{1}{\sqrt{c_\sigma}} \sum_{b \in \sigma} e^{i\varphi_b} |b\rangle_B |s(b)\rangle_A,
\]

where \( b \) and \( s(b) \) are respectively the setting and the solution of the problem; \( b \) ranges over the set of the problem settings \( \sigma \) of cardinality \( c_\sigma \). Note that \(|\psi\rangle_I\) and \(|\psi\rangle_O\) are written uniquely on the basis of the pairs \( b \) and \( s(b) \), namely of the oracle problem; there is no need of knowing the unitary transformation in between it.

Summing up, we can calculate Alice’s advanced knowledge, and thus the number of function evaluations required to solve the problem in an optimal quantum way, solely on the basis of the problem itself.

We show that this method of calculation can be applied as it is also in the case that the solution is a many to one function of the problem setting. To satisfy condition (I) and (II), the measurements of \( \hat{B}_i \) and \( \hat{B}_j \) must acquire two complementary halves of the information that specifies the solution. However, they must also acquire information about the problem setting that is not in the solution. It is the information that identifies the problem setting chosen by Bob among the "many" settings that correspond to the solution in question.

This raises the following objection. Since Alice (in each classical history) knows in advance the information acquired by the measurement of either \( \hat{B}_i \) or \( \hat{B}_j \), she also knows in advance information that is not in the solution. Given that Alice knows in advance part of the information she will acquire in the future, her knowing in advance information that is not in the solution might seem in contradiction with the fact that she measures only the solution. The way out is that, by measuring the solution, she also triggers the projection of the quantum state due to the initial measurement of \( \hat{B} \), which cannot be retarded beyond the unitary part of her action. Therefore, by measuring the solution, she necessarily acquires both the solution and the problem setting chosen by Bob.

From now on, we call this method of calculating Alice’s advanced knowledge and thus the number of function evaluations required to solve an oracle problem in an optimal quantum way the advanced knowledge rule. In the following, we apply it to a variety of oracle problems.

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5By the way, there can always be such a unitary transformation because the output \( b, s(b) \) conserves the memory of the input \( b \).
5 Deutsch&Jozsa, Simon, and the Abelian hidden subgroup algorithms

We apply the advanced knowledge rule to compute the number of function evaluations required to solve the oracle problems addressed by the other major quantum algorithms, which are all optimal. We will always obtain the number required by the real quantum algorithm.

5.1 Deutsch&Jozsa algorithm

In Deutsch&Jozsa problem, Bob chooses a function out of the set of all the constant and balanced functions (with the same number of zeroes and ones) \( f_{b}(a) : \{0,1\}^{n} \rightarrow \{0,1\} \). Array (15) gives the tables of four of the eight functions for \( n = 2 \):

\[
\begin{array}{cccccc}
   a & f_{0000}(a) & f_{1111}(a) & f_{0011}(a) & f_{1100}(a) & \ldots \\
   00 & 0 & 1 & 0 & 1 & \ldots \\
   01 & 0 & 1 & 0 & 1 & \ldots \\
   10 & 0 & 1 & 1 & 0 & \ldots \\
   11 & 0 & 1 & 1 & 0 & \ldots \\
\end{array}
\]  

Note that we use the table of the function – the sequence of function values for increasing values of the argument – as the suffix of the function. Alice knows the set of functions but not Bob’s choice and is to find whether the function chosen by Bob is constant or balanced by computing \( f_{b}(a) \) for appropriate values of \( a \). Classically, this requires in the worst case a number of function evaluations exponential in \( n \). It requires just one function evaluation with the quantum algorithm devised by Deutsch&Jozsa [28].

We use the advanced knowledge rule twice. First to explain the speedup of Deutsch&Jozsa algorithm. Then to compute the number of function evaluations required to solve Deutsch&Jozsa problem in an optimal quantum way; in this latter case we must ignore Deutsch&Jozsa algorithm.

We start with the first case. The time-symmetric representation of Deutsch&Jozsa algorithm with respect to Alice is:

\[
|\psi\rangle_I = \frac{1}{4} \left( e^{i\varphi_0} |0000\rangle_B + e^{i\varphi_1} |1111\rangle_B + e^{i\varphi_2} |0011\rangle_B + e^{i\varphi_3} |1100\rangle_B + \ldots \right) |00\rangle_A \left( |0\rangle_V - |1\rangle_V \right),
\]

\[
H_A U_f H_A |\psi\rangle_I = \frac{1}{4} \left[ \left( e^{i\varphi_0} |0000\rangle_B - e^{i\varphi_1} |1111\rangle_B \right) |00\rangle_A + \left( e^{i\varphi_2} |0011\rangle_B - e^{i\varphi_3} |1100\rangle_B \right) |10\rangle_A + \ldots \right]

\left( |0\rangle_V - |1\rangle_V \right).
\]

In general register \( B \) is \( 2^n \) qubit, register \( A \) is \( n \) qubit. \( H_A \) is the Hadamard transform on register \( A \), \( U_f \) is function evaluation. In the output state, register \( A \) contains the pre-solution \( s(b) \): the function is constant if \( s(b) \) is all zeros, balanced otherwise. Measuring \( A \) in the output state yields \( s(b) \).
To compute Alice’s advanced knowledge, we should split in all the possible ways the initial measurement of $B$ into two partial measurements, of $\hat{B}_i$ and $\hat{B}_j$, satisfying conditions (I) and (II) of section 3.2.2.

Given the problem setting of a balanced function, there is only one pair of partial measurements of the content of register $B$ compatible with these conditions. With problem setting, say, $b = 0011$, $\hat{B}_i$ must be the content of the left half of register $B$ and $\hat{B}_j$ that of the right half. The measurement of $\hat{B}_i$ yields all zeros, that of $\hat{B}_j$ all ones.

In fact, a partial measurement yielding both zeroes and ones would provide enough information to identify the solution – the fact that $f_b$ is balanced. Then the cases are two. If the other partial measurement does not contain both zeros and ones, it would not identify the solution; this would violate the requirement that the two partial measurements evenly contribute to the selection of the solution. If it did, the two partial measurements would be redundant with one another.

Moreover, given that either partial measurement must yield all zeroes or all ones, it must concern the content of half register. Otherwise either the requirement of even contribution to the selection of the solution would be violated or the problem setting would not be completely determined, as readily checked.

One can see that, with $b = 0011$, the measurement of $\hat{B}_i$, performed alone, projects $\sigma$ (the set of all the possible problem settings) on the subset $\{0011, 0000\}$, that of $\hat{B}_j$ on $\{0011, 1111\}$. Either subset represents an instance of Alice’s advanced knowledge.

The case of the problem setting of a constant function is analogous. The only difference is that there are more pairs of partial measurements that satisfy conditions (I) and (II) – see [7].

There is a shortcut to finding the subsets of $\sigma$ that represent Alice’s advanced knowledge. Here the problem setting – the bit string $b$ – is the table of the function chosen by Bob. For example $b = 0011$ is the table $f_b(00) = 0, f_b(01) = 0, f_b(10) = 1, f_b(11) = 1$. We call good half table any half table in which all the values of the function are the same. One can see that good half tables are in one-to-one correspondence with the subsets in question. For example, the good half table $f_b(00) = 0, f_b(01) = 0$ corresponds to the subset $\{0011, 0000\}$, is the identical part of the two bit-strings in it. Thus, given a problem setting, i. e. an entire table, either good half table, or identically the corresponding subset of $\sigma$, is a possible instance of Alice’s advanced knowledge.

Because of the structure of tables, given the advanced knowledge of a good half table, the entire table and thus the solution can be identified by performing just one function evaluation for any value of the argument $a$ outside the half table.

Summing up, the advanced knowledge rule explains the fact that Deutsch&Jozsa algorithm requires just one function evaluation – it explains the algorithm’s exponential speedup. By the way, for the fact of requiring just one function evaluation, this quantum algorithm is necessarily optimal.

One can see that the present analysis, like the notion of good half table,
holds unaltered for \( n > 2 \).

We show how to apply the advanced knowledge rule to compute the number of function evaluations required to solve Deutsch\&Jozsa problem in an optimal quantum way, of course without knowing Deutsch\&Jozsa algorithm. Now \( s(b) \) is the solution of the original problem: 0 if the function is constant and 1 if balanced. The input and output states of the registers \( B \) and \( A \) of any quantum algorithm that solves the problem must be respectively:

\[
|\psi_I\rangle = \frac{1}{2\sqrt{2}} (e^{i\phi_0} |0000\rangle_B + e^{i\phi_1} |1111\rangle_B + e^{i\phi_2} |0011\rangle_B + e^{i\phi_3} |1100\rangle_B + \ldots) |0\rangle_A
\]

\[
|\psi_O\rangle = \frac{1}{2\sqrt{2}} [(e^{i\phi_0} |0000\rangle_B - e^{i\phi_1} |1111\rangle_B) |0\rangle_A + (e^{i\phi_2} |0011\rangle_B - e^{i\phi_3} |1100\rangle_B) |1\rangle_A + \ldots].
\]

Note that \( |\psi_I\rangle \) and \( |\psi_O\rangle \) have been written uniquely on the basis of the oracle problem, namely of the pairs \( b \) and \( s(b) \).

One can readily see that Alice’s advanced knowledge, and thus the number of function evaluations required to solve the problem in an optimal quantum way, can be computed exactly as in the case of Deutsch\&Jozsa algorithm. All results are the same.

### 5.2 Simon and the Abelian hidden subgroup algorithms

Simon problem consists in finding the "period" (up to bitwise modulo 2 addition) of a periodic function \( f_b(a) : \{0, 1\}^n \rightarrow \{0, 1\}^{n-1} \) — see [7] for details. Array (16) gives the tables of four of the six functions for \( n = 2 \):

| \( a \) | \( f_{0011}(a) \) | \( f_{1100}(a) \) | \( f_{0101}(a) \) | \( f_{1010}(a) \) | ... |
|------|----------------|----------------|----------------|----------------|-----|
| 00   | 0              | 1              | 0              | 1              | ... |
| 01   | 0              | 1              | 1              | 0              | ... |
| 10   | 1              | 0              | 0              | 1              | ... |
| 11   | 1              | 0              | 1              | 0              | ... |

We note that each value of the function appears exactly twice in each table; thus 50% of the rows plus one always identify the period. Alice is to find the period of the function by performing function evaluation for appropriate values of \( a \).

In present knowledge, a classical algorithm requires a number of function evaluations exponential in \( n \). The quantum part of Simon algorithm [29] solves with just one function evaluation the hard part of this problem, which is finding a bit string orthogonal to the period. See [7] for further detail.

We apply the advanced knowledge rule directly to the calculation of the number of function evaluations required to solve Simon problem in an optimal quantum way. We will see further below that this also explains the speedup of Simon algorithm.

The input and output states of the registers \( B \) and \( A \) of any quantum algorithm that solves the problem must be respectively:

\[
|\psi_I\rangle = \frac{1}{2\sqrt{2}} (e^{i\phi_0} |0000\rangle_B + e^{i\phi_1} |1111\rangle_B + e^{i\phi_2} |0011\rangle_B + e^{i\phi_3} |1100\rangle_B \ldots) |0\rangle_A,
\]

\[
|\psi_O\rangle = \frac{1}{2\sqrt{2}} [(e^{i\phi_0} |0000\rangle_B - e^{i\phi_1} |1111\rangle_B) |0\rangle_A + (e^{i\phi_2} |0011\rangle_B - e^{i\phi_3} |1100\rangle_B) |1\rangle_A + \ldots].
\]
\[ |\psi\rangle_I = \frac{1}{\sqrt{6}} (e^{i\varphi_0} |0011\rangle_B + e^{i\varphi_1} |1100\rangle_B + e^{i\varphi_2} |0101\rangle_B + e^{i\varphi_3} |1010\rangle_B + \ldots) |00\rangle_A, \]

\[ |\psi\rangle_O = \frac{1}{\sqrt{6}} \left[ (e^{i\varphi_0} |0011\rangle_B + e^{i\varphi_1} |1100\rangle_B) |01\rangle_A + (e^{i\varphi_2} |0101\rangle_B + e^{i\varphi_3} |1010\rangle_B) |10\rangle_A + \ldots \right]. \]

In the output state, the \(2^n\) qubit register \(B\) contains the problem setting \(b\) and the \(n\) qubit register \(A\) the corresponding solution of the problem \(s(b)\), namely the period of the function \(f_b\).

Here a good half table, which represents an instance of Alice’s advanced knowledge like in Deutsch&Jozsa algorithm, is any half table where the values of the function are all different from one another (so that the period cannot be identified) – see [7]. Since 50% of the rows plus one identify the period, this can always be found by performing just one function evaluation for any value of the argument \(a\) outside the good half table.

Summing up, the advanced knowledge rule says that Simon’s problem can be solved with just one function evaluation. This also explains the exponential speedup of Simon algorithm. Of course, according to the advanced knowledge rule, also finding a bit string orthogonal to the period requires just one function evaluation (knowing the period amounts to know all the bit strings orthogonal to it).

The present analysis, like the notion of good half table, holds unaltered for \(n > 2\). It should also apply to the generalized Simon’s problem and to the Abelian hidden subgroup problem. In fact, the corresponding algorithms are essentially Simon algorithm. In the Abelian hidden subgroup problem, the set of functions \(f_b : G \rightarrow W\) map a group \(G\) to some finite set \(W\) with the property that there exists some subgroup \(S \leq G\) such that for any \(a, c \in G\), \(f_b(a) = f_b(c)\) if and only if \(a + S = c + S\). The problem is to find the hidden subgroup \(S\) by computing \(f_b(a)\) for the appropriate values of \(a\). Now, a large variety of problems solvable with a quantum speedup can be reformulated in terms of the Abelian hidden subgroup problem. Among these we find: the seminal Deutsch problem, finding orders, finding the period of a function (thus the problem solved by the quantum part of Shor factorization algorithm), discrete logarithms in any group, hidden linear functions, self shift equivalent polynomials, Abelian stabilizer problem, graph automorphism problem [30].

6 Discussion

As we have moved into uncharted waters, it is worth discussing at some length the present retrocausal explanation of the speedup.

6.1 Positioning

The present explanation of the speedup relies on three areas of research that had remained separate until now: (i) time-symmetric quantum mechanics, (ii) relational quantum mechanics, and (iii) quantum computation.
The notion that the complete description of the quantum computation process must include a forward propagation from the initial measurement and a backward propagation from the final one, central to the present explanation of the speedup, has clearly been inspired by time-symmetric quantum mechanics \([13 \div 19]\). It is a variation of its standard form in the particular case that there is a unitary transformation between the initial and final measurement outcomes and that the initial measurement is performed in a maximally mixed state. We represent the indifference of ascribing the selection of the random outcome of the initial measurement to the initial or final measurement by sharing it evenly between the two. This particular formalization of the time-symmetric picture has been inspired by the work of Dolev and Elitzur on the non-sequential behavior of the wave function highlighted by partial measurement \([16]\).

The role played by relational quantum mechanics \([11, 12]\) is equally essential. In the representation of the quantum algorithm with respect to Bob (the problem setter) and any external observer, which is close to the usual representation, retrocausality is without consequences. In the representation with respect to Alice (the problem solver) it explains the speedup.

For what concerns the study of the speed up, the present approach is orthogonal to the mainstream ones. As far as we know, no other approach resorts to the notion of retrocausality.

There are various studies on the relationship between speedup and other fundamental quantum features such as entanglement and discord \([31 \div 34]\). However, until now, these studies could not provide a common explanation to the various speedups. Quoting from \([32]\): *The speedup appears to always depend on the exact nature of the problem while the reason for it varies from problem to problem.*

The present retrocausal interpretation of the speedup, instead, quantitatively justifies both the quadratic and a variety of exponential speedups. Moreover, given an oracle problem, the advanced knowledge rule foresees the order of magnitude of the number of function evaluations required to solve it in an optimal quantum way – this at least in the very diverse cases examined.

For completeness, we mention the other main approaches to the study of the speedup: (i) Quantum computer science. Its aim is to find where quantum complexity classes, such as BQP and QMA, lie with respect to classical complexity classes such as P, NP, PP, etc. – see \([35]\) for an example. (ii) Tree size complexity. A measure of the complexity of the multiqubit state is shown to be related to the speedup of a variety of quantum algorithms \([36]\). (iii) Contextually based arguments, which address the relation between speedup and the contextual character of quantum mechanics \([37]\). The present approach would be in competition with these others in providing some estimate of the number of function evaluations required to solve an oracle problem in an optimal quantum way. Its promise, providing an order of magnitude estimate for any oracle problem, would be unparalleled by the other approaches.

We should also mention a work of Morikoshi \([38]\) that might be related to our own. It shows that Grover algorithm violates an information theoretic temporal Bell inequality. The present notion that Alice knows in advance – in
each classical history – half of the information about the solution she will read in the future, a form of temporal nonlocality, is likely related to the violation in question. This should be for further study.

The present interpretation of the speedup is also in line with those explanations of quantum nonlocality that resort to the notion of retrocausality – see [39, 40]. Causality zigzagging back and forth between the measurements of two entangled observables is a common feature.

6.2 Grover’s anticipation

Interestingly, Grover anticipated the need for a simple explanation of the speedup [41]. Quoting his words: It has been proved that the quantum search algorithm cannot be improved at all, i.e. any quantum mechanical algorithm will need at least $O\left(\sqrt{N}\right)$ steps to carry out an exhaustive search of $N$ items [4] [5]. Why is it not possible to search in fewer than $O\left(\sqrt{N}\right)$ steps? The arguments used to prove this are very subtle and mathematical. What is lacking is a simple and convincing two line argument that shows why one would expect this to be the case.

The present "two line argument" would be that the quantum search algorithm is a sum over classical histories in each of which the problem solver knows in advance one of the possible halves of the information that specifies the solution she will read in the future and performs the function evaluations still needed to reach it.

That the advanced knowledge of half of the information about the number of the drawer with the ball explains the quadratic speedup of Grover algorithm is almost tautological. The important thing is that this is the seed of a more general notion. We have seen that knowing in advance half of the information about the solution explains as well the major exponential speedups and seems to answer a fundamental time-reversal symmetry.

6.3 Criticism of retrocausality

We discuss the present interpretation of the speedup at the light of the criticism typically moved to the use of retrocausality in physics.

6.3.1 Bell’s criticism

Quoting from [42]: As Bell was well aware, the dilemma [of non locality] can be avoided if the properties of quantum systems are allowed to depend on what happens to them in the future, as well as in the past. Like most researchers interested in these issues, however, Bell felt that the cure would be worse than the disease – he thought that this kind of “retrocausality” would conflict with free will, and with assumptions fundamental to the practice of science. (He said that when he tried to think about retrocausality, he “lapsed into fatalism”). See also [29, 30].
We compare the present interpretation of the speedup with Bell’s observations.

We have seen that retrocausality is without consequences in the quantum algorithm with respect to Bob and any external observer. It instead affects the quantum algorithm with respect to Alice. In the four drawers instance of Grover algorithm, the projection induced by the final measurement of $\hat{A}_1$ in state $\langle 8 \rangle$, propagating backward in time, at time $t_1$ projects the input state to her $\langle 4 \rangle$ on:

$$H_A^1 U_f^\dagger 3_A^1 |\chi\rangle = \frac{1}{2} (e^{i\varphi_1} |01\rangle_B + e^{i\varphi_3} |11\rangle_B) |00\rangle_A (|0\rangle_V - |1\rangle_V),$$

At first sight, one may think that this projection, selecting the right digit of the number of the drawer with the ball, restricts back in time Bob’s freedom of choice to choosing between $b = 01$ and $b = 11$.

However, this would require that the projection in question is free – random – and independent of Bob’s choice. This is not the case. In spite of the fact that the measurement of $\hat{A}_1$ is performed in the mixture of tensor products $\langle 8 \rangle$, the associated projection of the quantum state occurs deterministically on the right digit of the number of the drawer with the ball already chosen by Bob at time $t_1$. We should keep in mind that we are in the quantum algorithm with respect to Alice. The outcome of the final measurement is unpredictable to her, not to Bob and any external observer who already know the number of the drawer with the ball.

Summing up, advancing the projection in question does not restrict Bob’s choice but only Alice’s ignorance of it.

We face now the other Bell’s observation, about the possible conflict of retrocausality with assumptions fundamental to the practice of science. Let us refer again to the four drawer instance of Grover algorithm. The fact that Alice, in each classical history, knows in advance half of the information about the number of the drawer with the ball she will read in the future is an obvious candidate to conflict. Obvious questions are: (i) Alice is an abstract entity, what does it mean that ”she knows”? (ii) At time $t_1$ Alice has done nothing yet, what tells her this information? (iii) Is it information sent back in time?

The answer to question (i) is that we are at a fundamental level where knowing is doing [43]. Alice, the problem solver, ”knows” from an operational standpoint, as far as the solution can be reached with a correspondingly reduced number of function evaluations. In other words we are talking of the metric of quantum computation. Alice’s advanced knowledge in the input state of the quantum algorithm gauges, in number of function evaluations, the distance of this state from the solution.

About question (ii), Alice is told by the backward propagation of the projection induced by the final measurement of $\hat{A}_1$. On its way to selecting, at time $t_{0+}$, part of the random outcome of the initial measurement, at the intermediate time $t_1$ it projects the original input state to Alice, of complete ignorance of the problem setting and consequently the solution, on one of lower entropy where she knows part of them – in the operational way discussed above.
Question (iii) is whether Alice’s advanced knowledge – in each classical history – of part of the information about the solution she will read in the future implies that information is sent back in time. The same question applies to the related fact that part of the random outcome of the initial measurement is selected back in time by the final measurement. Of course there is information sent back in time along the classical histories, but our answer is negative as long as it is not measurable.

At time $t_1$, immediately before the beginning of Alice’s problem-solving action, the information in question would be in register $B$. It would correspond to a reduction of the entropy of the state of this register, in fact to a reduction of Alice’s ignorance of the problem setting.

By definition, the observer Alice cannot perform any measurement of the content of register $B$ at time $t_1$. If she did, she would destroy the physical context that originates her advanced knowledge.

Bob and any external observer instead do measure the content of register $B$ at time $t_0$ and could repeat the measurement at time $t_1$. At time $t_{0+}$, they see a completely random measurement outcome and have no way of saying whether part of it is selected back in time by the final measurement. At time $t_1$, they would see the problem setting freely chosen by Bob. In either case, no information coming from the future can be identified in the measurement outcome.

We would like to add a common sense consideration. The idea that Alice, in each classical history, knows in advance part of what she will read in the future might anyway conflict with our sense of physical reality. Our advice for the time being would be to stick to the quantum computation context, where Alice’s advanced knowledge has a precise meaning and admits an apparently harmless metric interpretation – anyway no longer harmful than the speedup itself.

### 6.3.2 Redundancy of the retrocausal interpretation

A natural question is whether the time-symmetric and relational interpretations of quantum mechanics are necessary to derive the present results. An upstream question is of course whether these interpretations are necessary to quantum mechanics. Apropos of the latter question we cite the following positions.

Rovelli drew an analogy between relational quantum mechanics and special relativity; in both cases physical quantities must be related to the observer. After noting the revolutionary impact of the famous 1905 Einstein’s paper, Rovelli [12] writes: *The formal content of special relativity, however is coded into the Lorentz transformations, written by Lorentz, not by Einstein, and before 1905. So, what was Einstein’s contribution? It was to understand the physical meaning of the Lorentz transformations. (And more, but this is what is of interest here). We could say – admittedly in a provocative manner – that Einstein’s contribution to special relativity has been the interpretation of the theory, not its formalism: the formalism already existed.*
Elitzur expressed a similar position about the time symmetric interpretations of quantum mechanics [44]: even if they were pure interpretations, adding nothing to the formalism, they did and could allow to see things that would be otherwise very difficult to see.

We believe that the retrocausal interpretation of the speedup lends itself to similar considerations.

The number of function evaluations required to solve an oracle problem in an optimal quantum way, presumably given in the order of magnitude by the advanced knowledge rule, should also be implicit in the mathematics of unitary transformations, as follows.

In the most general case, the transformation that represents the unitary part of Alice’s problem solving action is (in the appropriate Hilbert space) a sequence of function evaluations, each preceded and followed by a suitable unitary transformation.

In principle, these transformations could be seen as the unknowns of the problem of finding the optimal quantum algorithm. They should have variable matrix elements up to the unitarity of the transformation.

For a given number of function evaluations, we should find the values of the matrix elements that maximize the probability of finding the solution in the final measurement; then repeat the procedure each time with that number increased by one; stop when the probability in question reaches one.

We would have obtained analytically the number of function evaluations required by the optimal quantum algorithm. In present assumptions, the order of magnitude of this number is given in a synthetic way by the advanced knowledge rule.

However, the analytic way is likely impracticable. In this case the synthetic one, based on the time-symmetric and relational interpretations of quantum mechanics, could provide a stunning shortcut; it does in all the cases examined.

7 Conclusions

The principle of the present explanation of the speedup is simple. Let us use again the simplifying assumption that the solution is a one to one function of a problem setting that is directly the random outcome of the initial measurement.

The selection of the problem setting and the corresponding solution can be performed indifferently by the initial measurement of the problem setting or the final measurement of the solution. We share it evenly between the two – any uneven sharing would introduce a preferred direction of time apparently unjustified in the present fully reversible context.

In the representation of the quantum algorithm to Bob, the problem setter, and any external observer, this says that half of the information that specifies the random outcome of the initial measurement has been selected by the final measurement, an inconsequential thing.

In that to Alice, the problem solver, it projects the input state of the quantum algorithm to her, of complete ignorance of the problem setting and the
solution, on one of lower entropy where she knows one of the possible halves of the information that specifies them. An optimal quantum algorithm turns out to be a sum over classical histories in each of which Alice knows in advance one of the possible halves of the information about the solution she will read in the future and performs the function evaluations still needed to reach it.

We interpret Alice's advanced knowledge in a metric way. It would gauge the distance, in number of function evaluations along the classical histories, of the input state of the quantum algorithm from the solution.

Conversely, given an oracle problem, the number of function evaluations required to solve it in an optimal quantum way is that of a classical algorithm that knows in advance half of the information about the solution of the problem.

Summing up, although just a physical interpretation of the mathematics of quantum algorithms, the present explanation of the speedup has potentially important practical consequences. Until now there was no fundamental explanation of the speedup, no unification of the quadratic and exponential speedups, no solution to the so-called quantum query complexity problem. The subject explanation provides a fundamental, quantitative justification of all kinds of speedup and promises to solve the problem in question.

The present form of quantum retrocausality, which cannot be measured but can explain the higher than classical efficiency of a quantum process, might be interesting from the foundational standpoint.

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