Kirigami, the Verifiable Art of Network Cutting

Tim Alberdingk Thijm  
Princeton University

Ryan Beckett  
Microsoft Research

Aarti Gupta  
Princeton University

David Walker  
Princeton University

Abstract. We introduce a modular verification approach to network control plane verification, where we cut a network into smaller fragments to improve the scalability of SMT solving. Users provide an annotated cut which describes how to generate these fragments from the monolithic network, and we verify each fragment independently, using the annotations to define assumptions and guarantees over fragments akin to assume-guarantee reasoning. We prove this modular network verification procedure is sound and complete with respect to verification over the monolithic network. We implement this procedure as Kirigami, an extension of NV [23] — a network verification language and tool — and evaluate it on industrial topologies with synthesized policies. We observe a 2–8x improvement in end-to-end NV verification time, with SMT solve time improving by up to 6 orders of magnitude.

1 Introduction

Networks have become incredibly vast and labyrinthine systems. To determine the best paths routers may use to forward traffic, networks typically run distributed routing protocols. Despite advances like software-defined networking, these protocols remain widely used. They are controlled by millions of lines of decentralized, low-level router configuration code. Operators must individually provision, maintain and reconfigure the network’s devices over time. This overwhelmingly complexity has led to many notable outages [42,47,49], with at times devastating pecuniary losses. More often than not, the culprits behind these incidents are subtle network misconfigurations.

In response, researchers have developed a variety of verification tools and techniques to catch errors before outages occur. Some [4,31,33–35,39,41,44] have targeted the network data plane, which is responsible for forwarding traffic from point A to point B. This work has produced scalable and performant methods for modeling the data plane and checking properties of how packets traverse it.

The data plane is produced by the network’s control plane, which uses the aforementioned routing protocols to decide which routes to use. Occasionally, these protocols may update their choice of routes — e.g., following a device failure — and recompute new paths. When this happens, the data plane is regenerated, and the user must
repeat any data plane analysis. Obscure control plane faults can lead to further issues, and manual verification by a human operator is an effort in locating what may be a minuscule typo within a gargantuan morass of router configurations.

To address this problem, researchers have developed another suite of tools to analyze the control plane [1, 6–8, 14, 16, 19, 21, 23, 50]. Control plane analyses consider what routes will be used by the data plane in given network environments, and check properties of the network in such environments. These tools can uncover bugs in real networks, but unfortunately tend not to scale as well as their data plane counterparts.

One branch of control plane verification, starting from Minesweeper [6], encodes a network as a Satisfiability Modulo Theories (SMT) formula and then asks an SMT solver [5] to check properties of the encoded network. While SMT-based verification has advantages over other approaches, such as its expressivity or applications in automated network repair [18], it nonetheless suffers from scalability issues. Prior work has explored using abstractions to resolve this problem, e.g., using symmetries in network topologies to compress networks [7, 21]. These abstractions offer some relief, but cannot always handle arbitrary non-symmetrical networks.

This paper offers another path forward in scaling SMT-based control plane verification, by being the first to leverage the inherent modularity of the control plane to cut a monolithic network into multiple fragments to be verified independently. Building on prior work on assume-guarantee verification of modular programs [20, 30], we present a novel technique for modular verification of control planes and implement it as Kirigami, an extension to the NV [23] network verification language and tool.

In a typical assume-guarantee verification approach, one can verify a safety property $P$ over a system of concurrent processes, by verifying each process independently, using assumptions over the environment in which it runs and guarantees over how it modifies this environment. The relationships between assumptions and guarantees (formulated as assume-guarantee rules) are then checked, which allows one to conclude that if all checks pass, then $P$ holds for the monolithic system. Our verification technique mirrors this idea: we verify a property over fragments (cf. processes) of the control plane, given assumptions over the rest of the network and guarantees over our fragments, to conclude that the monolithic network respects the property.

We start from an existing general model for distributed routing, the Stable Routing Problem (SRP) model [7]. In an SRP, each node of the network exchanges routes with its neighbors to compute a locally-stable solution. Like other work in control plane verification [1, 8, 19, 40], we focus on networks (i.e., SRPs) with unique solutions. We develop an SRP extension called “open SRPs”, in which a network receives routes along a set of input nodes and sends out routes along a different set of output nodes. We identify the solutions of our input nodes as our open SRP’s assumptions, and the solutions of our output nodes as its guarantees. We present a procedure $\text{CUT}$ which, given an interface — a mapping from a cut-set of edges to routes — cuts an open SRP $S$ into two smaller open SRPs $T_1$ and $T_2$ covering $S$, and where each cut edge is replaced by a route assumed in one SRP and guaranteed in the other. Interfaces can follow a network’s natural boundaries, e.g., a data center network interface might be cut according to its levels or hierarchy [2, 28, 29].
As with the traditional (closed) SRP, we can check that an open SRP satisfies a given safety property $P$ by verifying that $P$ holds for the SRP’s solutions. We prove that if $P$ holds on $T_1$ and $T_2$’s solutions, then it holds on $S$‘s. This is the basis for our modular network verification technique. Starting from a network $S$, an interface $I$, and a safety property $P$, we use Cut($S$, $I$) to obtain a set of $N$ open SRPs $T_1, \ldots, T_N$ that are verified independently. We verify $P$ and $T_i$’s guarantees for each open SRP $T_i$; if either the property or interface’s guarantees do not hold, we return a counterexample demonstrating the solution that does not satisfy $P$ or $I$. We believe this to be the first work to present a proven-correct general theory for automated modular verification of arbitrary properties, and where we check the correctness of the given interface: prior work on modular network verification like [31] considered specific architectures and properties without any guarantee of correctness.

As SMT-based verification time typically grows superlinearly with the size of the network [6], by verifying $P$ on each of the $N$ smaller open SRPs $T_i$, we can verify $P$ in a fraction of the time it takes to do so directly over the monolithic network $S$. Our experiments demonstrate that this modular verification technique works well for a variety of data center, random and backbone networks, with significant improvements in SMT solve time: we show for one set of fattree [2] benchmarks that verifying the fattree pod-by-pod cuts SMT time from 90 minutes to under 2 seconds; verifying every node individually reduces SMT time to around a hundredth of a second. Overall, while we are working on improving the engineering in NV for carving out partitions, we already see a 2–8x speedup in end-to-end NV verification time. We also observe that a modular approach assists in producing more localized errors and debugging feedback in the cases when verification fails.

In summary, we make the following contributions:

**A Theory of Network Fragments** We develop an extension of the Stable Routing Problem (SRP) model [7] for network fragments. Our extension provides a method to cut monolithic SRPs into a set of fragments. We define interfaces to cut SRPs and map the cut edges to annotations which then define assumptions and guarantees of our fragments. We prove that under these assumptions, if these guarantees hold, then a property that holds in every fragment also holds in the monolithic network.

**A Modular Network Verification Technique** We present a technique to decompose a monolithic network verification problem into multiple subproblems. We start from an SRP $S$, an interface $I$ and a property $P$, and cut $S$ into a set of fragments. We check each fragment’s guarantees and the given $P$ independently and report whether $P$ holds for $S$, or if $P$ or $I$ fail to hold. This enables a novel, modular approach to control plane verification based on assume-guarantee reasoning.

**Fast, Scalable and Modular SMT Verification** We implement Kirigami, a tool based on this theory, as an extension for NV, a network verification language and tool [23]. Using Kirigami, we improve on NV verification in terms of scalability and performance. SMT solve time using Kirigami is up to six orders of magnitude faster for a selection of NV benchmarks.
The Stable Routing Problem. A network is a graph with nodes \( V \) representing routers and edges \( E \) representing the links between them. A distributed control plane uses routing protocols to determine paths to routing destinations. Each router deploys its own local rules to broadcast routing announcements (or routes) and select a “best” route; the form of these rules varies with the protocol, but generally protocols focus on minimizing routing costs.

These elements — nodes and edges, a set of routes, and a set of rules to initialize, compare and broadcast them — form the basis for our control plane routing model, the SRP [7]. In a well-designed network, this exchange of routes eventually converges to a stable state, where no node may improve on its current best route by selecting another offered by a neighbor. A mapping from nodes to these stable routes is called a solution \( L \) to the SRP. While it is possible for routing to diverge (i.e., have no solution) or converge to multiple solutions, many typical networks have unique solutions (e.g., when routing costs strictly increase with distance to the destination [19,40]): we restrict our focus in this paper to such networks, like other work [1,8,19,40].

---

1 NV numbers nodes starting from 0\( n \); our example numbers Figure 1a from left to right, i.e., 0\( n \) is \( e_0 \), 4\( n \) is \( e_0 \), 12\( n \) is \( e_0 \) and so on. “0=4” refers to a bidirectional edge between 0\( n \) and 4\( n \).
An Example SRP. Let’s consider an SRP instance $S$ of a familiar fattree [2] data center network, as shown in Figure 1a. Routing in fattree networks typically follows a $\Lambda$ shape: traffic that starts from an edge switch ($e_0, \ldots, e_6, d$) travels up along a link to an aggregation switch ($a_0, \ldots, a_7$), then ascends from the pod to a core switch ($c_0, \ldots, c_3$) in the spine before descending back down into another pod. For this example, our routes will simply be the number of hops to some routing destination $d$. Initially $d$ will know a route with 0 hops to itself; the rest of the network starts with no route to $d$. Each node broadcasts its route to $d$ to all of its neighbors, incrementing the route by one hop. Nodes will then compare each received route with their current choice and select the one with the fewest hops. The unique solution $L_S(u)$ of a node $u$ in $S$ is thus the best route between $u$’s initial route and the transferred solutions of each of $u$’s neighbors. This toy policy elides the complexities of real routing protocols, which may have dozens of fields, each with particular semantics, but demonstrates all the basic elements of an SRP.

Verifying SRPs with NV [23]. We can verify properties of $S$’s solution to confirm our beliefs about $S$’s behavior. For instance, we may wish to check that every node’s route to $d$ is at most 4 hops. One verification tool we can use to do so is NV [23]. NV is a functional programming language for modeling control planes with an associated SMT verification engine. An NV program’s components map onto those of an SRP: it has a topology (nodes and edges); a type of routes (attribute); a function init to initialize routes; a function trans to broadcast routes; and finally a function merge to compare routes. Figure 1b presents a condensed NV program for Figure 1a.

Figure 1c demonstrates how to verify a safety property $P$ in NV, where $P$ holds iff $\forall u. L(u) \leq 4$. We define the solution (line 4) using init, trans and merge from Figure 1b. We then assert (line 6) that $P$ is true of this solution. When we supply Figure 1c to NV’s verification engine, NV encodes $S$ and $P$ as an SMT query, and confirms that $P$ holds for $L_S$. Encoding the network to SMT lets us reason about network states symbolically, avoiding state explosion when analyzing properties like fault tolerance or reasoning about routes arriving from outside the network.

Scaling Up SRP Verification. SMT-based verification is expressive, but has issues when it comes to scalability. Our evaluation in §7 shows that SMT verification scales superlinearly for larger fattrees with more complex policies: from 0.03 seconds for a 20-node network, to 1.41 seconds for an 80-node network, and 1833.66 seconds for a 320-node network! To verify the tens of thousands of switches in industrial fattree networks [31], we must find a way to scale this technique.

Suppose then that we took a large network and cut it into fragments (defined formally in §4), in order to verify a safety property $P$ on each fragment independently. In other words, if $P$ holds for every node in every fragment, then it holds for every node in the monolithic network; and otherwise, we want to observe real counterexamples as in the monolithic network. To achieve this goal, our cutting procedure must also summarize the network behavior external to each fragment.

We incorporate these summaries into the traditional SRP model by generalizing it to open SRPs. Open SRPs extend the SRP model by designating some nodes as input nodes and some others as output nodes. Input and output nodes are annotated with routes representing solutions assumed on the inputs and guaranteed on the outputs. We
express these annotations using an interface: a mapping from each cut edge to a route annotation. Given an open SRP $S$ and an interface $I$, we cut $S$ into open SRP fragments, where each fragment identifies assumptions on its inputs and guarantees on its outputs.

**Cutting Down Fattrees.** We will now move on to demonstrating this idea for Figure 1. Let’s cut each pod of our network into its own fragment $T_{p0}$ through $T_{p3}$, leaving the spine nodes as a fifth fragment $T_{spines}$. Figures 2a and 2b show pod 0 and the spines of Figure 1 as open SRPs $T_{p0}$ and $T_{spines}$, respectively. In $T_{p0}$, we assume routes from the spines and check guarantees on $a_0$ and $a_1$. An assumption in one fragment will be guaranteed by another (and vice-versa): we assume $a_0$ has a route of 3 hops in $T_{spines}$ and check that it has a route of 3 hops in $T_{p0}$.

**Verifying Network Fragments.** In modular verification, we perform an independent verification query for each fragment: we encode the open SRP and property, along with an assumptions formula assuming a state of the inputs and a guarantees formula to check on the state of the outputs. We then submit every query to our solver and ask if the network has a solution where, under the given assumptions, either the property is false (as before) or the guarantees formula does not hold. Our solver then searches for a counterexample demonstrating a concrete violation of the property or our guarantees. Guarantee violations provide evidence of possible bugs in our network implementation or mistakes in our beliefs, in the same way that property violations do.

Let us consider our fattree network again. Suppose we misconfigured $a_6$ to black hole (silently drop) traffic, leading nodes $a_0$, $a_2$ and $a_4$ to re-route via the other nodes $a_1$, $a_3$, $a_5$ in their respective pods. Our interface maps $c_0a_0$ to 2, so we check that $L_{spines}(c_0) = 2$ when verifying $T_{spines}$. Due to our bug, this check fails and our solver returns a counterexample: because $c_0$ must reroute, $L_{spines}(c_0) = 6$. We can then modify our network configuration to fix the bug, and re-run verification to confirm that our guarantees and property hold for all fragments. We prove in §4 that this implies that $P$ holds for the monolithic network. Our guarantees can thus be thought of as a specification of the desired network behavior along its cut points, in addition to $P$.

**Verifiable Network Cutting with Kirigami.** As part of our work, we implemented an extension Kirigami to NV for cutting and verifying networks. Figure 3 shows an NV file with two new functions, partition and interface. partition assigns each node to a fragment, while interface adds assertions that check that a route $x$ along a cross-fragment edge is equal to the specified annotation, e.g., that the route from $0n$ to $4n$ is 2...
include "fat.nv"

let partition node = match node with
  | 0n | 1n | 2n | 3n -> 0 (* spines *)
  | 10n | 11n | 18n | 19n -> 4 (* p3 *)

let interface edge x = match edge with
  | 0~_ | 1~_ | 2~_ | 3~_ -> x = 2
  | 4~_ | 5~_ | 6~_ | 7~_ | 8~_ | 9~_ -> x = 3
  | 10~_ | 11~_ -> x = 1

let sol = solution { init = init; trans = trans; merge = merge;
  interface = interface }

assert foldNodes (fun node route acc -> acc && route <= 4) sol true

Fig. 3: An NV program which cuts Figure 1 into pods (some node and edge cases not shown).

hops. Under the hood, we cut the network using partition to generate our fragments, and then annotate the cuts using interface; verification can then proceed as described.

A Cut Above the Rest. Pod-based cuts suit our high-level understanding of fattrees, but we can consider many other cuts. We could cut Figure 1 so that every node is in its own fragment. Verifying a single node in SMT is extremely cheap, and hence leads to significant performance improvements. The corresponding NV program resembles Figure 3, except every node maps to its own fragment and we annotate every edge.

Next Steps. The rest of the paper proceeds as follows. §3 presents prior work formalizing SRPs, and §4 presents our extensions for cutting SRPs, with proofs of soundness and completeness of our procedure. We present our SMT checking procedure in §5, and the implementation of our theory in §6 as Kirigami, an extension of NV. We evaluate Kirigami in §7. We discuss related work in §8, and future work in §9.

3 Background on the Stable Routing Problem

We summarize prior work [7] on the Stable Routing Problem (SRP) network model. Many components of this model resemble routing algebras used for reasoning about convergence of routing protocols [11,26,48], but SRPs also include a network topology for reasoning about properties such as reachability between nodes.

An SRP instance $S$ is a 6-tuple $(V, E, R, \text{init}, \oplus, \text{trans})$, defined as follows.

Topology. $V$ is a set of nodes and $E \subseteq V \times V$ is a set of directed edges between them. We write $uv$ for an edge from node $u$ to node $v$. Edges may not be self-loops: $\forall v \in V, vv \notin E$.

Routes. $R$ is a set of routes that describe the fields of routing messages. For example, when modeling BGP, $R$ might represent a tuple of an integer local preference, a set of community tags, and a sequence of AS numbers representing the AS path [9,46].
Node Initialization. The initialization function \( \text{init} : V \rightarrow R \) describes the initial route of each node. When modeling single destination routing, \( \text{init} \) may map a destination node \( d \) to some initial route \( r_d \), and all other nodes to a null route; in multiple destination routing, we may have many initial routes.

Route Update. The merge function \( \oplus : R \times R \rightarrow R \) defines how to compare and merge routes. \( \oplus \) represents updates of a node’s selected route: we assume \( \oplus \) is associative and commutative, i.e., the order in which a sequence of routes are merged does not matter.

Route Transfer. The transfer function \( \text{trans} : E \times R \rightarrow R \) describes how routes are modified between nodes. Given an edge \( uv \) and a route \( r \) from node \( u \), \( \text{trans}(uv, r) \) determines the route received at \( v \).

Solutions. A solution \( L : V \rightarrow R \) is a mapping from nodes to routes. Intuitively, a solution is defined such that each node is locally stable, i.e., it has no incentive to deviate from its currently chosen neighbors. Nodes compute their solution via message exchange, where each node in the SRP advertises its chosen route to each of its neighbors. Formally, an SRP solution \( L \) satisfies the constraint:

\[
L(v) = \text{init}(v) \oplus \bigoplus_{uv \in E} \text{trans}(uv, L(u))
\]  

where \( \bigoplus \) is the sequence of \( \oplus \) operations on each transferred route \( \text{trans}(uv, L(u)) \) from each neighbor \( u \) of \( v \). These received routes are merged with \( v \)'s initial value \( \text{init}(v) \).

A solution may determine an SRP’s forwarding behavior or another decision-making procedure, as shown in [7]. We omit discussing forwarding behavior in this work to focus on a general SRP definition without restricting ourselves only to forwarding.

4 Cutting SRPs

We now introduce our original contributions, starting with open SRPs. We define a fragment relation between a smaller open SRP and a larger one, and define a \text{CUT} procedure to decompose one open SRP into a partition of two fragments. We prove soundness and completeness of partition solutions with respect to the larger SRP’s solution.

Notation. We introduce some notation in this section that may be unfamiliar. \( \text{dom}(f) \) is the domain of the function \( f \), and \( f|_X \) is the restriction of \( f \) to \( X \subseteq \text{dom}(f) \). We use subscripts to specify SRP components, e.g., \( \text{init}_{S} \) refers to SRP \( S \)'s init component.
Open SRPs. An open SRP generalizes our earlier SRP definition to include assumptions and guarantees. An open SRP instance $S$ is an 8-tuple $(V, E, R, \text{init}, \oplus, \text{trans}, \text{ass}, \text{guar})$.

The first six elements are defined exactly as for regular (closed) SRPs. The final two elements, $\text{ass}$ (“assumptions”) and $\text{guar}$ (“guarantees”), are partial functions $(V \rightarrow R)$ mapping mutually disjoints subsets $V^{\text{in}}, V^{\text{out}} \subseteq V$ to routes. We use $V^{\text{in}}$ (input nodes) as a shorthand for dom($\text{ass}$) and $V^{\text{out}}$ (output nodes) as a shorthand for dom($\text{guar}$). All nodes that are neither input nor output nodes are “base nodes” $V^{\text{base}}$. A closed SRP is an open SRP where $V^{\text{in}} = V^{\text{out}} = \emptyset$. Going forward, we assume an open SRP whenever we write “SRP”, except when the distinction is relevant.

Input nodes must be source nodes (in-degree = 0). Hence, they act as auxiliary nodes, indicating where a fixed incoming route “arrives” from outside the SRP, as specified by the assumptions $\text{ass}$. Output nodes correspondingly mark where routes “depart” the SRP, per the guarantees $\text{guar}$. We do not require any connectivity properties of output nodes: we think of them as simply identifying an outgoing route we wish to guarantee, but without requiring the SRP to tell us whither it is announcing that route.\(^2\) Figure 4 illustrates this concept with some example open SRPs and cuts.

Definition 1 (Open SRPs). An open SRP instance $S = (V, E, R, \text{init}, \oplus, \text{trans}, \text{ass}, \text{guar})$ respects the following properties:

- $V = V^{\text{in}} \cup V^{\text{out}} \cup V^{\text{base}}$ and $V^{\text{in}}, V^{\text{out}}, V^{\text{base}}$ are pairwise-disjoint;
- $\text{ass} : V^{\text{in}} \rightarrow R$ and $\text{guar} : V^{\text{out}} \rightarrow R$; and
- $\forall v \in V^{\text{in}}. \text{in-degree}(v) = 0$.

Open SRP Solutions. A mapping $L$ is a solution to an open SRP iff:

$$L(u) = \text{init}(u) \oplus \bigoplus_{v \in E} \text{trans}(vu, L(v)) \quad \forall v \notin V^{\text{in}} \quad (2)$$

$$L(u) = \text{ass}(u) \quad \forall v \in V^{\text{in}} \quad (3)$$

$$L(u) = \text{guar}(u) \quad \forall v \in V^{\text{out}} \quad (4)$$

Note that Equations (2) and (4) both apply for all outputs $v \in V^{\text{out}}$. Solutions for open SRPs resemble closed SRP solutions, with the addition of constraints based on the values of $\text{ass}$ and $\text{guar}$. For any input node $u$, its assumption $\text{ass}(u)$ determines the node’s solution directly; for an output node $u$, its solution $L(u)$ must be consistent with both the right-hand side of (2) and the right-hand side of (4). Hence, if $\exists u \in V^{\text{out}}. \text{init}(u) \oplus \bigoplus_{v \in E} \text{trans}(vu, L(v)) \neq \text{guar}(u)$, there is no solution to the open SRP. As with closed SRPs, we restrict our focus to open SRPs with unique solutions.

Fragments. We now introduce a fragment relation between two open SRPs. We may think of an open SRP as composed of fragments of smaller open SRPs, where each fragment represents its connection to the rest of the larger SRP with assumptions and guarantees. Consider the series of open SRPs in Figure 4. One can think of $T$ (Figure 4b) as a fragment representing part of $S$ (Figure 4a). To go from $S$ to $T$, we can cut $e$ off from

\(^2\) This is partly a design choice: we could have also attached auxiliary nodes to output nodes to indicate where these routes are going, but we found this definition the most straightforward.
Definition 2 (Fragments). Let S and T be open SRPs. T is a fragment of S when:

\[ V_T \subseteq V_S \]
\[ E_T = \{ uv \mid u \in V_T, v \in V_T, uv \in E_S, v \notin V^m_T \} \]
\[ R_T = R_S \]
\[ \ominus_T = \ominus_S \]
\[ \text{init}_T = \text{init}_S|_{V_T} \]
\[ \text{trans}_T = \text{trans}_S|_{E_T} \]

(5) (6) (7)

\[ V^{in}_T = (V^{in}_S \cup \{ v \mid uv \in E_S, u \notin V_T \}) \cap V_T \]
\[ V^{out}_T = (V^{out}_S \cup V^{in}_T \cup \{ u \mid uv \in E_S, v \notin V_T \}) \cap V_T \]
\[ \forall u \in (V^{in}_S \cap V^{out}_S) \text{. ass}_T(u) = \text{ass}_S(u) \]
\[ \forall u \in (V^{out}_S \cap V^{out}_S) \text{. guar}_T(u) = \text{guar}_S(u) \]

(8) (9) (10) (11)

Informally, the fragment T is made up of a subgraph of S over nodes V_T, conserving all edges from E_T between them, except any edges into input nodes (5). Routing and routing functions of S are as before (6) or restricted over T’s topology (7). Finally, T designates nodes whose neighbors have been cut as inputs (8) (summarizing the network “outside” T) or outputs (9) (communicating a summary to the “external” network), while preserving any assumptions (10) and guarantees (11) inherited from S.

Interfaces and Cutting SRPs. The fragment relation leaves unspecified how a smaller SRP’s assumptions and guarantees summarize its parent’s routes. We now consider how to cut an SRP S into two fragments T_1 and T_2, where T_1 and T_2 cover S and replicate its behavior with the help of their assumptions and guarantees. We do so by selecting a cut-set C \subseteq E of edges in S and annotating each cut edge uv with a route that describes the solution transferred from u to v. We call this annotated cut-set an interface I.

Definition 3 (Interface). Let S be an SRP and let C \subseteq E be a cut-set partitioning V_S. I : C \rightarrow R_S is an interface if it maps every element uv of C to a route I(uv) in R_S.

We now define a CUT procedure. Given an SRP S and an interface I, CUT(S, I) returns a partition of two SRP fragments, T_1 and T_2. Nodes along the cut edges are annotated with assumptions and guarantees. We can recursively CUT an SRP into arbitrarily many fragments. We elide the structural details of how CUT divides the nodes of S between T_1 and T_2 for now; curious readers should see Appendix A.

What is most important about CUT is that it defines T_1 and T_2 to have equal assumptions and guarantees along each cut edge. For each edge uv in our interface I, CUT(S, I) adds a guarantee guar(u) = I(uv) in T_1 and an assumption ass(u) = I(uv) in T_2 (or vice-versa). By requiring this equality, we rely on the stability of an open SRP’s solution to avoid the issue of circularity in assumptions. The edge uv now shows a route I(uv) “arriving” in T_2 after “departing” from T_1. As u’s solution is both assumed in one fragment and guaranteed in the other, we refer to it as an input-output node. We illustrate this idea in Figure 5, which shows how an interface defines assumptions and guarantees for input-output nodes c_0 and a_0 from Figure 2.
Definition 4 (Input-output nodes). Let $T_1$ and $T_2$ be two open SRPs with a set $(V_1^{in} \cap V_2^{out}) \cup (V_2^{in} \cap V_1^{out})$ of shared nodes. A node $u$ in this set is an input-output node iff $\text{ass}_1(u) = \text{guar}_2(u)$ or $\text{ass}_2(u) = \text{guar}_1(u)$.

Because CUT produces input-output nodes, we can reason over the solutions of both fragments separately, using the assumptions and guarantees of our input-output nodes to confirm that the solutions coincide along our cut. We can now define a partition as a relation between $T_1$ and $T_2$ and $S$.

Definition 5 (Partition). Let $S$, $T_1$ and $T_2$ be open SRPs. $(T_1, T_2)$ is a partition of $S$ when (i) $T_1$ and $T_2$ are both fragments of $S$, (ii) $V_1 \cup V_2 = V_S$ and $E_1 \cup E_2 = E_S$, (iii) every input node in $T_1$ or $T_2$ that is not an input node in $S$ is an input-output node.

We present the full definition of a partition — which includes some corner cases for when two fragments share the same input node — in Appendix A: these details are not required to understand our theorems. We prove that our CUT procedure always produces a partition, and subsequently prove that if $T_1$ and $T_2$ are a partition of $S$, then the joined solutions of $T_1$ and $T_2$ are a solution of $S$ (soundness); and that if $S$ has a solution, then there always exists an interface $I$ that given to CUT produces a partition of two fragments $T_1$ and $T_2$ such that the solution of $S$ is a solution (when appropriately restricted) for $T_1$ and $T_2$.

Definition 6 (CUT). Let $S$ be an SRP and let $I$ be an interface over $S$. Given $S$ and $I$, $\text{CUT}(S, I) = (T_1, T_2)$, where $T_1$ and $T_2$ are a partition of $S$ such that $\forall uv \in \text{dom}(I)$, $u$ is an input-output node between $T_1$ and $T_2$.

Correctness. We now prove theorems on the relationships between the SRP’s solution and the solutions of its CUT-produced fragments. By showing that the fragments’ solutions are the same as the monolithic SRP’s, we can use the fragments in place of the monolithic SRP during verification of a property $P$. Proofs can be found in Appendix A.

We start by proving that the solutions of the fragments $T_1, T_2$ are a solution to the monolithic SRP $S$: each node of $S$ is mapped to its fragment solution, with $S$’s input nodes mapping to their expected assumptions.

Theorem 1 (CUT is Sound). Let $S$ be an open SRP, and let $I$ be an interface over $S$. Let $\text{CUT}(S, I) = (T_1, T_2)$. Suppose $T_1$ has a unique solution $L_1$ and $T_2$ has a unique solution $L_2$. Consider a mapping $L^S_1 : V_S \rightarrow \mathbb{R}$, defined such that:

$$\forall v \in V_1, L^S_1(v) = L_1(v)$$
$$\forall v \in V_2, L^S_1(v) = L_2(v)$$
$$\forall v \in V_S^{in}, L^S_1(v) = \text{ass}_S(v)$$
Then \( L_S' \) is a solution of \( S \).

We can also always find a suitable interface \( I \) to cut \( S \), such that \( T_1 \) and \( T_2 \) have the same solution as \( S \) for each node: we simply annotate each cut edge \( uv \) with the solution \( L_S(u) \), which would be the solution transferred from \( u \) to \( v \) in \( S \).

**Theorem 2 (CUT is Complete).** Let \( S \) be an open SRP, and let \( I \) be an interface over \( S \). Let \( \text{CUT}(S, I) = (T_1, T_2) \). Assume \( S \) has a unique solution \( L_S \). Assume that \( \forall uv \in \text{dom}(I), I(uv) = L_S(u) \). Consider the following two mappings \( L_1' : V_1 \rightarrow R \) and \( L_2' : V_2 \rightarrow R \), defined such that:

\[
\forall v \in V_1, L_1'(v) = L_S(v) \\
\forall v \in V_2, L_2'(v) = L_S(v)
\]

Then \( L_1' \) is a solution for \( T_1 \) and \( L_2' \) is a solution for \( T_2 \).

Finally, our proof of soundness implies that any property that holds over the solutions of our fragments will hold over the solutions of our monolithic network.

**Corollary 1 (CUT Preserves Properties).** Let \( S \) be an open SRP, and let \( I \) be an interface over \( S \). Let \( \text{CUT}(S, I) = (T_1, T_2) \). Let \( P_1, P_2 \) be formulas such that \( P_1 = \forall v \in V_1, Q(v) \) and \( P_2 = \forall v \in V_2, Q(v) \), where \( Q \) is a predicate on \( L(v) \). Assume \( S \) has a unique solution \( L_S \), and that \( T_1 \) has a solution \( L_1 \) and \( T_2 \) has a solution \( L_2 \). Then if \( P_1 \) holds on \( T_1 \) and \( P_2 \) holds on \( T_2 \), \( P_1 \land P_2 \) holds on \( S \).

## 5 Checking Fragments in SMT

We now present our three-step modular verification methodology: (i) given an SRP \( S \) and an interface \( I \), produce \( N \) fragments using \( \text{CUT}(S, I) \), as defined in §4; then (ii) encode each fragment to SMT and check its guarantees and a safety property \( P \) under the given assumptions; and (iii) if any guarantees fail, let the user refine \( I \) or correct network bugs. By our theoretical results, when our SMT solver verifies \( P \) for these smaller fragments, we can conclude that it would have verified \( P \) for the monolithic SRP.

**Creating Interfaces.** For now, we treat our interfaces as given, meaning they function similarly to user-provided annotations in an annotation checking tool such as Dafny [37]. Hence, our checking algorithm acts as an analogous tool to verify beliefs about the network. In this sense, interfaces are user-provided specifications to the verifier.

Another way to create interfaces is to infer them: starting from a small amount of given information, say the initial route to a single destination, we could infer routes through the rest of the network. While we do not yet consider interface inference, we believe it is a fruitful direction for future work, and discuss doing so in §9.

**The Fragment Checking Algorithm.** Algorithm 1 shows how we cut an SRP and check the three constraints on open SRP solutions (described in §4) on each of the fragments. We start in the \text{CHECK} procedure on line 1.6. \text{CHECK} calls \text{CUT}(S, I) to cut \( S \) into fragments, and then calls \text{SOLVE} (line 1.1) on each fragment, reporting any \text{SAT} result it receives back from the solver. \text{SOLVE} encodes (2) on line 1.2, (3) on line 1.3.
Algorithm 1 The fragment checking algorithm.

1: proc \textsc{Solve}(fragment \textit{T}, property \textit{P})
2: \hspace{1em} \textbf{N} \leftarrow \text{ENCODE}(\textit{T}) \; \triangleright \text{closed SRP } \mathcal{L}_T \text{ constraints (1)}
3: \hspace{1em} \textbf{A} \leftarrow \bigwedge_{\textbf{u} \in \text{V}_I} \mathcal{L}_T(\textbf{u}) = \text{ass}_T(\textbf{u}) \; \triangleright \text{ass constraints (3)}
4: \hspace{1em} \textbf{G} \leftarrow \bigwedge_{\textbf{u} \in \text{V}_I} \mathcal{L}_T(\textbf{u}) = \text{guar}_T(\textbf{u}) \; \triangleright \text{guar constraints (4)}
5: \hspace{1em} \textbf{return} \text{ASKSAT}(\textbf{A} \land \textbf{N} \land \neg (\textbf{G} \land \textit{P}))

6: proc \textsc{Check}(SRP \textit{S}, property \textit{P}, interface \textit{I})
7: \hspace{1em} \textbf{T}_1, \ldots, \textbf{T}_\textbf{N} \leftarrow \text{CUT}(\textit{S}, \textit{I})
8: \hspace{1em} \textbf{for} \textbf{i} \leftarrow 1, \textbf{N} \textbf{do}
9: \hspace{2em} \textbf{r} \leftarrow \textsc{Solve}(\textbf{T}_i, \textit{P})
10: \hspace{2em} \textbf{if} \textbf{r} \neq \text{UNSAT} \textbf{then}
11: \hspace{3em} \textbf{return} \textbf{r}
12: \hspace{1em} \textbf{return} \text{UNSAT}

and (4) on line 1.4. Since we are interested in knowing if \textit{G} or \textit{P} are ever violated, our final formula is the conjunction of ENCODE(\textit{T}) and \textbf{A} with the negation of \textit{G} \land \textit{P} (line 1.5). ASKSAT asks our solver if this formula is satisfiable, and returns either SAT with a model, or UNSAT. This model will be a quasi-solution \mathcal{L}_T to \textit{T} where the ENCODE(\textit{T}) and \textbf{A} constraints hold, but \exists \textbf{u} \in \text{V}_I, \mathcal{L}_T(\textbf{u}) \neq \text{guar}_T(\textbf{u}) (guarantee violation) or \exists \textbf{u} \in \text{V}_I, \neg \text{P}(\textbf{u}) (property violation). Otherwise, if the solver returns UNSAT, then either \textit{S} has no solution or the guarantees and property always hold.

**Refining Interfaces.** If every fragment returns UNSAT, by Corollary 1, we conclude that if there exists a solution to each fragment, then \textit{P} and \textit{G} hold and the interface is correct. On the other hand, if any fragment returns SAT, we must determine why our property or guarantees were violated. For example, in §2, we considered if our interface correctly captured the intended network behaviour, but a bug in the network policy led to a guarantee violation. If the reverse were true — the network was configured correctly, but our interface is incorrect — we must refine our interface to correct it.

By Theorem 1, we know that any incorrect interface will not define a solution in \textit{T}_1 and \textit{T}_2, meaning our guarantee constraint in \textsc{Solve} fails and a counterexample is returned. This counterexample may then inform a new interface we can provide in a successive run of \textsc{Check}. Returning to our fattree fragments in Figure 2, suppose our interface provided the incorrect annotation \textit{I}(a_0c_0) = 1. This generates an unattainable guarantee \text{guar}_0(a_0) = 1, meaning we can reach \textit{d} in one hop from \textit{a}_0. \textsc{Solve}(\textit{T}_p0, \textit{P}) returns SAT, providing \text{L}_0(a_0) = 3 as a counterexample which violates this guarantee. We can then create a new interface with \textit{I}(a_0c_0) = 3 and re-run verification: if no further annotations are incorrect, then \textsc{Solve}(\textit{T}_p0, \textit{P}) will report UNSAT.

6 Implementation

Our Kirigami extension adds partition and interface functions to the NV language: when a user runs NV on a file that declares these functions, NV cuts the SRP into a set of fragments as part of a partitioning step. Each fragment is generated as described by
Definition 6 of CUT. Most of the partitioning step deals with restricting the monolithic init and trans functions. We create \( N \) copies of the initial NV file and traverse the AST of each to update any references to the topology. This implementation is currently not optimized and performs redundant work, which can be improved to reduce overhead when partitioning large policies.

Beyond assigning nodes and edges to fragments using partition and interface, Kirigami also decomposes the properties we wish to test. Many useful end-to-end properties can be expressed as predicates over individual node solutions, including reachability, path length, waypointing, black holes and fault tolerance [6]. These properties can be decomposed into separate assertions over the nodes of each fragment: if no fragment reports a property violation, we can then conclude that the property holds for the monolithic network as well, as proven in Corollary 1.

Kirigami’s SMT encoding follows Algorithm 1, using NV’s monolithic SRP encoding as the encoding function ENCODE. As discussed above, the solver returns a SAT or UNSAT response for each fragment to the user. Any fragments that return SAT provide a solution violating the guarantees or properties, allowing us to determine if the violation indicates a problem with our network policy or interface.

7 Evaluation

We evaluated Kirigami on a variety of NV benchmarks representing fattree, random and Internet topologies.\(^3\) Our questions focus on the scalability and performance of Kirigami in comparison to NV, specifically: (i) does Kirigami improve on NV verification time across topologies and properties, and (ii) how do different cuts impact Kirigami performance? We consider two metrics for verification time: the maximum time reported to verify an SMT query encoding the monolithic network or fragment using the Z3 [12] SMT solver;\(^4\) and the “total time” of NV, which is the time taken by NV’s pipeline of network transformations, partitioning (for cut networks), encoding to SMT and solving every query sequentially.

We ran each benchmark on a computing cluster node with a 2.4GHz processor and up to 24GB of memory per benchmark. Each benchmark tested verification of either the monolithic network or a cut network, and we ran each benchmark for 5 trials and took the average time. We used two timeouts: an 8-hour timeout on NV as a whole and a 2-hour timeout on Z3. The NV timeout prevented fragments from spending too long partitioning or solving multiple Z3 queries, while the Z3 timeout also ensured that benchmarks did not spend too long solving any single fragment’s SMT query.

Fattrees. To evaluate Kirigami’s performance for fattrees, we made use of the shortest path policy \( \text{SP} \) and valley-free policy \( \text{FAT} \) described in [23], along with an original fault-tolerance policy \( \text{MAINT} \). \( \text{MAINT} \) extends \( \text{SP} \) by requiring that nodes avoid routing through a non-destination node \( \text{down} \) which is currently down for maintenance: routes advertised by \( \text{down} \) will be dropped. We encode \( \text{down} \) as a symbolic value, meaning that we check that routing bypasses the down node for all concrete choices of \( \text{down} \).

\(^3\) All of our benchmarks are available or adapted from those at [22].
\(^4\) We take the maximum query time as each fragment SMT query is independent of the others and hence could be parallelized on a multi-processor platform or a cluster of servers.
As in [23], we parameterize fattree designs by $k$, the number of pods: we vary the topology size from $k = 4$ (20 nodes) to $k = 20$ (500 nodes) to assess scalability. Furthermore, we consider four different cuts of our fattree networks:

- **Vertical**: creates 2 fragments, each with half the spines and half the pods;
- **Horizontal**: creates 3 fragments: the pod containing the routing destination, the spines, and all the other pods;
- **Pods**: creates $k + 1$ fragments (given $k$ pods): the spines and each pod in their own fragment; and
- **Full**: creates $|V|$ fragments (given $|V|$ nodes), with every node in its own fragment.

We generate interfaces automatically using shortest paths algorithms, irrespective of the kind of cut. For SP, a standard shortest-paths computation is sufficient; for FAT, we track the level of a route to block valleys [17, 43]; and for MAINT, we use Yen’s 2-shortest paths algorithm [51]: this gives us the shortest and second-shortest path (taken if $\text{down}$ lies on the shortest path) to the destination from each node. For our interface, we then assign a route to each cut edge depending on the value of $\text{down}$.

We compare the SMT verification time for monolithic benchmarks versus their cut counterparts in Figure 6 for each of our policies. We plot the number of nodes in the monolithic benchmark against the maximum time spent by Z3 solving the SMT queries: for monolithic networks, there is only a single query, while for cut networks, we have
$N$ queries. Note that SMT time is shown on a logarithmic scale. All three policies show extreme improvements in SMT time as the number of fragments grows. The maximum SMT time for a full cut fragment of our largest SP network considered is six orders of magnitude faster than the baseline monolithic time. The FAT policy’s SMT encoding is most complex, leading to Z3 timeouts for the monolithic FAT16 and FAT20 benchmarks.

We compare total time (i.e., with no parallelization) in Figure 7 for the largest fattree benchmarks of each of our three policies for different cuts. The relationship between cuts is similar for the smaller benchmarks. We distinguish SMT time from non-SMT (partitioning, encoding, etc.) time, and see that across policies, SMT time takes up the majority of total time for the monolithic benchmarks and vertically-cut benchmarks, but other operations then dominate for the remaining cuts. This leads the horizontal and pods cuts to perform the best overall relative to the monolithic benchmark, completing 2–8x faster. The FAT20 full cut benchmark times out partway through solving due to the time spent partitioning and encoding, but all other cuts complete before the NV timeout. As mentioned above, NV’s partitioning step is under-optimized: hence, we consider slowness outside SMT to be surmountable following improvements to Kirigami’s partitioning and NV’s encoding steps.

**Random Networks.** We also assess Kirigami on random networks. We generate topologies of $N$ nodes using the Erdős–Rényi–Gilbert model [13,24], where each edge has independent probability $p$ of being present. To assess scalability, we vary $N$ and $p$ in our experiments according to a parameter $x$ where $N = 2^x$ and $p = 2^{2-x}$ for $x \in [4,12]^5$. We use a shortest-path policy based on SP for these networks. Our interfaces are generated by a shortest paths algorithm and cut the network fully.

We show the SMT solve times for these benchmarks in Figure 8. As expected, monolithic verification hits our Z3 timeout at $N = 256$; fully partitioning allows us to verify all larger benchmarks in under 6 minutes over all SMT queries, with no individual query taking longer than a minute.

**Backbone Networks.** To assess Kirigami more fully, we expand our evaluation to backbone network topologies from the Internet Topology Zoo [36]. We consider three networks: a 41-node topology B41, a 174-node topology B174 and 754-node topology B754. B41 is an educational network with a more clustered topology: its policy uses shortest-path routing where routes transiting [17] through AS customers or peers is disallowed. The larger topologies are less structured and hence use standard shortest-

---

As our topologies are not always fully connected, we expect NV to return property violations as appropriate, and otherwise for all checks to pass.
path routing as in SP. We use a graph partitioning tool, hMETIS [32], to compute $N$ fragments of each topology. The computed fragments minimize the number of edges cut between fragments, and capture clustering behavior of the topology, while keeping fragments as close in graph order as possible. We consider $N = 2, 4, 7, 41$ for B41, $N = 2, 4, 20, 174$ for B174, and $N = 2, 4, 8, 25, 75, 754$ for B754.

We show that larger cuts lead to greater reductions in SMT solve time for these benchmarks in Figure 9. As for fattrees, the lowest total times tend to be lowest for larger non-full cuts ($N = 20$ for B174 and $N = 75$ for B754).

8 Related Work

**Data Plane Analysis.** Much prior work has analyzed properties of the network data plane [4, 31, 33–35, 39, 41, 44]. These tools operate on snapshots of the data plane — representing the global forwarding state at a single point in time — and verify that forwarding properties are satisfied.

Our approach most closely resembles the work of Jayaraman et al. on SECGURU and RCDC [31]. SECGURU verifies reachability using invariants it infers from specific data center topologies: our work develops a formal theory to verify arbitrary properties and invariants as specified by a user’s interface, provides a framework for doing so automatically and instead focuses on the control plane.

Another relevant work is that of Plotkin et al. [44]. They demonstrate the use of bisimulations to relate simpler networks and formulas to more complex ones, improving verification scalability. Modular verification is recognized as a viable direction but left as future work; we focus on using modular verification in the control plane.

**Control Plane Analysis.** Our open SRP model builds on prior work on formal models of control planes: in particular, the SRP model of Bonsai [7], which presents a topology with an attached routing algebra. Unlike other prior work [11, 25, 26], we ignore questions of network convergence and assume a unique solution exists to our network.

Many control plane verification tools address scalability by abstracting routing behaviors, rather than modularizing the network. Abstraction necessarily loses precision, which can limit the properties or networks considered. Bonsai [7] and Origami [21] are perhaps closest to our work in that they seek to compress large concrete networks to smaller abstract networks which soundly approximate the original. Both tools use abstraction refinement to find abstract networks and use a similar formal model to our
own. Compression requires similar forwarding behavior across multiple nodes of the network; our cutting approach avoids this restriction.

ShapeShifter [8] checks control plane reachability by simulating the network using abstract routes determined by abstract interpretation. They define an asynchronous network semantics for routing; we instead model the network’s converged state using open SRPs. ShapeShifter’s abstractions sacrifice precision, unlike our technique.

Our SMT encoding is inspired by Minesweeper [6], although we do not consider packet forwarding (only routing) and Minesweeper cannot perform modular verification. Plankton [45] uses explicit-state model checking to check a comparable set of properties to Minesweeper and use a network semantics similar to ShapeShifter’s. They avoid state explosion using heuristic reductions that work well for the networks considered. Our approach is more general and avoids explicitly exploring network states by using SMT. Other control plane analyses also do not consider modularizing the network, and many are more restrictive than our approach: either limited to specific network properties [1, 14, 19] or to specific protocols [50].

**Modular Verification.** As mentioned above, our work borrows from the compositional verification technique of assume-guarantee reasoning [3, 15, 20]. Such reasoning has been widely used in software, hardware and reactive systems [15, 27, 30]. While [38] applies assume-guarantee reasoning in network congestion control, assume-guarantee appears to be under-explored in analyzing routing. Instead of modeling processes, we model network fragments, whose shared environment is their input and output nodes. By requiring a partition’s assumptions and guarantees to be equal, our reasoning avoids the common pitfall of circularity by relying on the stability of an open SRP’s solution.

### 9 Discussion & Future Work

**Choosing Cuts.** This paper answers two major questions about network partitions: first, given a network cut, can we verify properties of a monolithic network using its fragments? Second, does verifying the fragments scale better than verifying the monolithic network? We leave unanswered a third critical question: where should we cut?

As we saw in our evaluation, verification time is inversely proportional to the number of fragments. However, this introduces a tradeoff: for every edge our interface cuts, we must supply another annotation. How easy it may be to annotate a given edge depends on many factors: who manages the network (e.g., private organizations vs. the internet), how policy is determined along the edge, etc. Nonetheless, by making these factors explicit using interfaces, we make it easier to understand the monolithic behavior of legacy networks, thereby improving their safety and long-term robustness.

**Future Directions.** Our theoretical framework provides a foundation for two promising avenues for future work: abstracting our interface annotations and inferring interfaces automatically. Using abstract annotations — annotating with a set of routes, e.g., $I(uv) = \{n \mid 0 \leq n \leq 4\}$, instead of a concrete route $I(uv) = 4$ — we could potentially simplify the task of annotating the network by over-approximating the set of routes we assume. Interface inference, perhaps building on our manual refinement process in §5,
could further simplify this task: we hypothesize a viable technique might use counterexamples to automatically refine our initial interface as a series of verification passes [10].

**Conclusion.** We demonstrate that scalability in control plane verification can be achieved by leveraging networks’ inherent modularity. We prove that we can verify a property of a network by verifying it independently across fragments of the original, and present a procedure to do so. We implemented this procedure in NV as Kirigami and show that it succeeds in verifying NV benchmarks with dramatic improvements in SMT time.

**Acknowledgements.** This work was supported in part by the National Science Foundation awards NeTS 1704336 and FMitF 1837030, and Facebook Research Award on “Network control plane verification at scale.” We would like to thank Todd Millstein for his contributions in our early discussions. Our evaluation was substantially performed using the Princeton Research Computing resources at Princeton University, which is a consortium of groups led by the Princeton Institute for Computational Science and Engineering (PICSciE) and the Office of Information Technology’s Research Computing.

**References**

1. Abhashkumar, A., Gember-Jacobson, A., Akella, A.: Tiramisu: Fast multilayer network verification. In: 17th USENIX Symposium on Networked Systems Design and Implementation (NSDI 20). pp. 201–219 (2020), https://www.usenix.org/system/files/nsdi20-paper-abhashkumar.pdf
2. Al-Fares, M., Loukissas, A., Vahdat, A.: A scalable, commodity data center network architecture. In: SIGCOMM (2008). https://doi.org/10.1145/1402946.1402967
3. Alur, R., Henzinger, T.A.: Reactive modules. Formal methods in system design **15**(1), 7–48 (1999). https://doi.org/10.1023/A:1008739929481
4. Anderson, C.J., Foster, N., Guha, A., Jeannin, J.B., Kozen, D., Schlesinger, C., Walker, D.: NetKAT: Semantic foundations for networks. In: POPL (2014). https://doi.org/10.1145/2578855.2535862
5. Barrett, C., Tinelli, C.: Satisfiability modulo theories. In: Handbook of model checking, pp. 305–343. Springer (2018). https://doi.org/10.1007/978-3-319-10575-8_11
6. Beckett, R., Gupta, A., Mahajan, R., Walker, D.: A general approach to network configuration verification. In: SIGCOMM (August 2017). https://doi.org/10.1145/3098822.3098834
7. Beckett, R., Gupta, A., Mahajan, R., Walker, D.: Control plane compression. In: Proceedings of the 2018 Conference of the ACM Special Interest Group on Data Communication. pp. 476–489. SIGCOMM ’18, ACM, New York, NY, USA (2018). https://doi.org/10.1145/3230543.3230583
8. Beckett, R., Gupta, A., Mahajan, R., Walker, D.: Abstract interpretation of distributed network control planes. Proceedings of the ACM on Programming Languages **4**(POPL), 1–27 (2019). https://doi.org/10.1145/3371110
9. Chandra, R., Traina, P., Li, T.: BGP communities attribute. rfc 1997, RFC Editor (1996), https://www.rfc-editor.org/rfc/rfc1997.txt, https://www.rfc-editor.org/rfc/rfc1997.txt
10. Clarke, E.M., Grumberg, O., Jha, S., Lu, Y., Veith, H.: Counterexample-guided abstraction refinement. In: CAV. pp. 154–169 (2000). https://doi.org/10.1007/10722167_15
11. Daggit, M.L., Gurney, A.J., Griffin, T.G.: Asynchronous convergence of policy-rich distributed Bellman-Ford routing protocols. In: Proceedings of the 2018 Conference of the ACM Special Interest Group on Data Communication. pp. 103–116. ACM (2018). https://doi.org/10.1145/3230543.3230561
12. De Moura, L., Bjørner, N.: Z3: An efficient SMT solver. In: TACAS (March 2008). https://doi.org/10.1007/978-3-540-78800-3_24
13. Erdös, P., Rényi, A.: On random graphs i. Publicationes Mathematicae (Debrecen) 6, 290–297 (1959), https://www.renyi.hu/~p_erdos/1959-11.pdf
14. Fayaz, S.K., Sharma, T., Fogel, A., Mahajan, R., Millstein, T., Sekar, V., Varghese, G.: Efficient network reachability analysis using a succinct control plane representation. In: OSDI (2016), https://www.usenix.org/system/files/conference/osdi16/osdi16-fayaz.pdf
15. Flanagan, C., Qadeer, S.: Thread-modular model checking. In: International SPIN Workshop on Model Checking of Software. pp. 213–224. Springer (2003). https://doi.org/10.1007/3-540-44829-2_14
16. Fogel, A., Fung, S., Pedrosa, L., Walraed-Sullivan, M., Govindan, R., Mahajan, R., Millstein, T.: A general approach to network configuration analysis. In: NSDI (October 2015), https://www.usenix.org/system/files/conference/nsdi15/nsdi15-paper-fogel.pdf
17. Gao, L.: On inferring autonomous system relationships in the internet. IEEE/ACM Transactions on networking 9(6), 733–745 (2001), https://ieeexplore.ieee.org/abstract/document/974527
18. Gember-Jacobson, A., Akella, A., Mahajan, R., Liu, H.H.: Automatically repairing network control planes using an abstract representation. In: Proceedings of the 26th Symposium on Operating Systems Principles. pp. 359–373 (2017). https://doi.org/10.1145/3132747.3132753
19. Gember-Jacobson, A., Viswanathan, R., Akella, A., Mahajan, R.: Fast control plane analysis using an abstract representation. In: SIGCOMM (August 2016). https://doi.org/10.1145/2934872.2934876
20. Giannakopoulou, D., Namjoshi, K.S., Păsăreanu, C.S.: Compositional reasoning. In: Handbook of Model Checking, pp. 345–383. Springer (2018). https://doi.org/10.1007/978-3-319-10575-8_12
21. Giannarakis, N., Beckett, R., Mahajan, R., Walker, D.: Efficient verification of network fault tolerance via counterexample-guided refinement. In: International Conference on Computer Aided Verification. pp. 305–323. Springer (2019). https://doi.org/10.1007/978-3-030-25543-5_18
22. Giannarakis, N., Loehr, D., Beckett, R., Walker, D.: Nv source code (2019), https://github.com/NetworkVerification/nv
23. Giannarakis, N., Loehr, D., Beckett, R., Walker, D.: NV: An intermediate language for verification of network control planes. In: Proceedings of the 41st ACM SIGPLAN Conference on Programming Language Design and Implementation. p. 958–973. PLDI 2020, Association for Computing Machinery, New York, NY, USA (2020). https://doi.org/10.1145/3385412.3386019
24. Gilbert, E.N.: Random graphs. The Annals of Mathematical Statistics 30(4), 1141–1144 (1959), https://www.jstor.org/stable/2237458
25. Griffin, T.G., Shepherd, F.B., Wilfong, G.: The stable paths problem and interdomain routing. IEEE/ACM Trans. Networking 10(2) (2002), https://ieeexplore.ieee.org/abstract/document/993304
26. Griffin, T.G., Sobrinho, J.L.: Metarouting. In: SIGCOMM. pp. 1–12 (August 2005). https://doi.org/10.1145/1080091.1080094
27. Grumberg, O., Long, D.E.: Model checking and modular verification. ACM Transactions on Programming Languages and Systems (TOPLAS) 16(3), 843–871 (1994). https://doi.org/10.1145/177492.177725
28. Guo, C., Lu, G., Li, D., Wu, H., Zhang, X., Shi, Y., Tian, C., Zhang, Y., Lu, S.: BCube: A high performance, server-centric network architecture for modular data centers. In: SIGCOMM (2009). https://doi.org/10.1145/1592568.1592577
29. Guo, C., Wu, H., Tan, K., Shi, L., Zhang, Y., Lu, S.: Dcell: A scalable and fault-tolerant network structure for data centers. In: SIGCOMM (2008). https://doi.org/10.1145/1402958.1402968
30. Henzinger, T.A., Qadeer, S., Rajamani, S.K.: You assume, we guarantee: Methodology and case studies. In: International Conference on Computer Aided Verification. pp. 440–451. Springer (1998). https://doi.org/10.1007/BFb0028765
31. Jayaraman, K., Bjørner, N., Padhye, J., Agrawal, A., Bhargava, A., Bissonnette, P.A.C., Foster, S., Helwer, A., Kasten, M., Lee, I., Namdharı, A., Niaz, H., Parkhi, A., Pinnamraju, H., Power, A., Raje, N.M., Sharma, P.: Validating datacenters at scale. In: Proceedings of the ACM Special Interest Group on Data Communication. p. 200–213. SIGCOMM ’19, Association for Computing Machinery, New York, NY, USA (2019). https://doi.org/10.1145/3341302.3342094
32. Karypis, G., Aggarwal, R., Kumar, V., Shekhar, S.: Multilevel hypergraph partitioning: applications in VLSI domain. IEEE Transactions on Very Large Scale Integration (VLSI) Systems 7(1), 69–79 (1999). https://doi.org/10.1109/92.748202, https://ieeexplore.ieee.org/abstract/document/748202
33. Kazemian, P., Chang, M., Zeng, H., Varghese, G., McKeown, N., Whyte, S.: Real time network policy checking using header space analysis. In: NSDI. pp. 99–112 (April 2013), https://www.usenix.org/system/files/conference/nsdi13/nsdi13-final8.pdf
34. Kazemian, P., Varghese, G., McKeown, N.: Header space analysis: Static checking for networks. In: NSDI (April 2012), https://www.usenix.org/system/files/conference/nsdi12/nsdi12-final8.pdf
35. Khurshid, A., Zou, X., Zhou, W., Caesar, M., Godfrey, P.B.: Veriflow: Verifying network-wide invariants in real time. In: NSDI (April 2013), https://www.usenix.org/system/files/conference/nsdi13/nsdi13-final100.pdf
36. Knight, S., Nguyen, H.X., Falkner, N., Bowden, R., Roughan, M.: The internet topology zoo. IEEE Journal on Selected Areas in Communications 29(9), 1765–1775 (2011). https://doi.org/10.1109/JSAC.2011.1111002, https://ieeexplore.ieee.org/abstract/document/6027859
37. Leino, K.R.M.: Dafny: An automatic program verifier for functional correctness. In: International Conference on Logic for Programming Artificial Intelligence and Reasoning. pp. 348–370. Springer (2010). https://doi.org/10.1007/978-3-642-17511-4_20
38. Lomuscio, A., Strulo, B., Walker, N., Wu, P.: Assume-guarantee reasoning with local specifications. In: International conference on formal engineering methods. pp. 204–219. Springer (2010). https://doi.org/10.1007/978-3-642-16901-4_15
39. Lopes, N.P., Bjørner, N., Goddefroid, P., Jayaraman, K., Varghese, G.: Checking beliefs in dynamic networks. In: NSDI (2015), https://www.usenix.org/system/files/conference/nsdi15/nsdi15-paper-lopes.pdf
40. Lopes, N.P., Rybalchenko, A.: Fast BGP simulation of large datacenters. In: International Conference on Verification, Model Checking, and Abstract Interpretation. pp. 386–408. Springer (2019), https://web.ist.utl.pt/nuno.lopes/pubs/fastplane-vmc19.pdf
41. Mai, H., Khurshid, A., Agarwal, R., Caesar, M., Godfrey, P.B., King, S.T.: Debugging the data plane with anteater. In: SIGCOMM (2011). https://doi.org/10.1145/2043164.2018470
42. McCarthy, K.: BGP super-blunder: How verizon today sparked a 'cascading catastrophic failure' that knackered cloudflare, amazon, etc. https://www.theregister.com/2019/06/24/verizon_bgp MISconfiguration_cloudflare/ (2019)
43. Pepelnjak, I.: Valley-free routing in data center fabrics. https://blog.ipspace.net/2018/09/valley-free-routing-in-data-center.html (2018)
44. Plotkin, G.D., Bjørner, N., Lopes, N.P., Rybalchenko, A., Varghese, G.: Scaling network verification using symmetry and surgery. In: POPL (January 2016). https://doi.org/10.1145/2914770.2837657
45. Prabhu, S., Kheradmand, A., Godfrey, B., Caesar, M.: Predicting network futures with plankton. In: Proceedings of the First Asia-Pacific Workshop on Networking. pp. 92–98. APNet’17 (August 2017). https://doi.org/10.1145/3106989.3106991

46. Rekhter, Y., Li, T., Hares, S., et al.: A border gateway protocol 4 (BGP-4). RFC 4271, RFC Editor (2006), https://www.rfc-editor.org/rfc/rfc4271.txt, https://www.rfc-editor.org/rfc/rfc4271.txt

47. Sharwood, S.: Facebook rendered spineless by buggy audit code that missed catastrophic network config error. https://www.theregister.com/2021/10/06/facebook_outage_explained_in_detail/ (2021)

48. Sobrinho, J.a.L.: An algebraic theory of dynamic network routing. IEEE/ACM Trans. Netw. 13(5), 1160–1173 (October 2005). https://ieeexplore.ieee.org/abstract/document/1528502

49. Sverdlik, Y.: Microsoft: misconfigured network device led to azure outage. http://www.datacenterdynamics.com/content-tracks/servers-storage/microsoft-misconfigured-network-device-led-to-azure-outage/68312.fullarticle (2012)

50. Weitz, K., Woos, D., Torlak, E., Ernst, M.D., Krishnamurthy, A., Tatlock, Z.: Formal semantics and automated verification for the border gateway protocol. In: NetPL (March 2016), https://www.dougwoos.com/papers/bagpipe-netpl16.pdf

51. Yen, J.Y.: Finding the k shortest loopless paths in a network. Management Science 17(11), 712–716 (1971). https://doi.org/10.1287/mnsc.17.11.712
A Proofs

Cuts Form Partitions. We start by stating the formal definition of a cut from §4. For our formal definition of CUT, we add an additional structural restriction over our interface \(I\) to simplify some of our definitions. Essentially, we will require that, given an SRP \(S\), \(I\) cuts \(S\) along its base and output nodes. Formally, for an SRP \(S\), let the input-free graph of \(S\) be \((V_S \setminus V_S^{in}, \{uv \mid u, v \in V_S \setminus V_S^{in}\})\), i.e., the induced subgraph of \(S\)’s base and output nodes. If we cut the input-free graph into \((W_1, W_2)\), we then can assign the input nodes of \(S\) to the two fragments in order to cover \(S\): \(V_S^{in}\) is disjoint from \(W_1\) and \(W_2\) and \(V_S = W_1 \cup W_2 \cup V_S^{in}\). Any input node \(u \in V_S^{in}\) which has an edge \(uv\) to a node \(v\) in \(W_1\) (respectively \(W_2\)) is also an input node \(u \in V_i^{in}\) (respectively \(V_2^{in}\)). Importantly, if there exists \(u_1, v_1 \in W_1\) and \(u_2, v_2 \in E_S\), if \(v_1 \in W_1\) and \(v_2 \in W_2\), then \(u\) is a shared input in both \(T_1\) and \(T_2\), i.e., \(u \in V_1^{in} \cap V_2^{in}\).

Definition 7 (CUT). Let \(S\) be an SRP. Let \((W_1, W_2)\) be a cut of the input-free graph of \(S\) where \(C\) is a cut-set of edges \(\{uv \in E_S \mid (u \in W_1 \land v \in W_2) \lor (u \in W_2 \land v \in W_1)\}\). Let \(I\) be an interface over \(S\) such that \(\text{dom}(I)\) is equal to \(C\). Then \(\text{CUT}(S, I) = (T_1, T_2)\) where the following properties hold for \(i \in \{1, 2\}\):

\[
\begin{align*}
V_i^{in} &= \{u \mid u \in V_S^{in} \land \exists v \in W_i \land v \in E_S\} \cup \{u \mid \exists v \in \text{dom}(I) \land v \in W_i\} \\
V_i^{out} &= \{u \mid u \in W_i \land u \in V_S^{in}\} \cup \{u \mid \exists v \in \text{dom}(I) \land u \in W_i\} \\
V_i &= W_i \cup V_i^{in} \\
E_i &= \{uv \mid u, v \in W_i \land uv \in E_S\} \\
R_i &= R_S \\
\text{init}_i &= \text{init}_S | W_i \\
\oplus_i &= \oplus_S \\
\text{trans}_i &= \text{trans}_S | V_i \\
\text{ass}_i(u) &= \begin{cases} 
\text{ass}_S(u) & \text{if } u \in V_S^{in} \\
I(u) & \text{if } uv \in \text{dom}(I) \land v \in W_i
\end{cases} \\
\text{guar}_i(u) &= \begin{cases} 
\text{guar}_S(u) & \text{if } u \in (V_i^{out} \setminus V_i^{in}) \\
I(u) & \text{if } uv \in \text{dom}(I) \land v \notin W_i
\end{cases}
\end{align*}
\]

We now state the partition relation that summarizes the properties \(\text{CUT}(S, I)\) ensures.

Definition 8 (Partition). Let \(S\), \(T_1\), and \(T_2\) be open SRPs. \((T_1, T_2)\) is a partition of \(S\) when:

- \(T_1\) and \(T_2\) are both fragments of \(S\)
- \(V_1 \cup V_2 = V_S\) and \(E_1 \cup E_2 = E_S\)
- Input-output constraints: every input or output that is not inherited from the parent is an input-output node:
  - \(V_1^{in} \setminus V_2^{in} \subseteq V_2^{out}\)
Proof. Consider the input-free graph of $C_S$. Let $UT$ be a proof from the definition of $C_T$, and $T$ be inherited from $T_1$ and $T_2$. Our shared input constraint states that, if a node $u$ appears in both $T_1$ and $T_2$, then $u$ is either (i) an input-output node; or (ii) a shared input of both $T_1$ and $T_2$.

We prove that as defined, $CUT(S,I)$ is a partition of $S$. This is a straightforward proof from the definition of $CUT$, using some set identities to prove the properties of a partition.

**Theorem 3 (CUT Creates Partitions).** Let $S$ be an SRP, and let $I$ be an interface over $S$. Let $CUT(S,I) = (T_1, T_2)$. Then $(T_1, T_2)$ is a partition of $S$.

Proof. Consider the input-free graph of $S$, $(V_S \setminus V_S^{in}, \{uv \mid u, v \in V_S \setminus V_S^{in}\})$, such that $C = (W_1, W_2)$ cuts the input-free graph with $\text{dom}(I)$ as the cut-set of $C$.

It is trivial to see that based on the definition of $CUT$, $T_1$ and $T_2$ are both fragments of $S$: we hence proceed to prove the remaining properties of the partition relation below.

- **Shared input constraint**: a node shared by $T_1$ and $T_2$ is either an input into both fragments, or an input-output node: $V_1 \cap V_2 = (V_1^{in} \cap V_2^{in}) \cup (V_1^{in} \cup V_2^{in}) \setminus V_S^{in}$

The properties of a partition state everything we still need (beyond the properties of open SRPs and fragments) in order to prove that our $CUT$ procedure is correct. Our input-output constraints state that the input-output nodes must agree on their assumptions and guarantees, and that these nodes make up a subset of the respective input and output nodes in each sibling fragment (since some input and output nodes may be inherited from $S$). Our shared input constraint states that, if a node $u$ appears in both $T_1$ and $T_2$, then $u$ is either (i) an input-output node; or (ii) a shared input of both $T_1$ and $T_2$.

We prove that as defined, $CUT(S,I)$ is a partition of $S$. This is a straightforward proof from the definition of $CUT$, using some set identities to prove the properties of a partition.

**Theorem 3 (CUT Creates Partitions).** Let $S$ be an SRP, and let $I$ be an interface over $S$. Let $CUT(S,I) = (T_1, T_2)$. Then $(T_1, T_2)$ is a partition of $S$.

Proof. Consider the input-free graph of $S$, $(V_S \setminus V_S^{in}, \{uv \mid u, v \in V_S \setminus V_S^{in}\})$, such that $C = (W_1, W_2)$ cuts the input-free graph with $\text{dom}(I)$ as the cut-set of $C$.

It is trivial to see that based on the definition of $CUT$, $T_1$ and $T_2$ are both fragments of $S$: we hence proceed to prove the remaining properties of the partition relation below.

**Theorem 3 (CUT Creates Partitions).** Let $S$ be an SRP, and let $I$ be an interface over $S$. Let $CUT(S,I) = (T_1, T_2)$. Then $(T_1, T_2)$ is a partition of $S$.

Proof. Consider the input-free graph of $S$, $(V_S \setminus V_S^{in}, \{uv \mid u, v \in V_S \setminus V_S^{in}\})$, such that $C = (W_1, W_2)$ cuts the input-free graph with $\text{dom}(I)$ as the cut-set of $C$.

It is trivial to see that based on the definition of $CUT$, $T_1$ and $T_2$ are both fragments of $S$: we hence proceed to prove the remaining properties of the partition relation below.

**Theorem 3 (CUT Creates Partitions).** Let $S$ be an SRP, and let $I$ be an interface over $S$. Let $CUT(S,I) = (T_1, T_2)$. Then $(T_1, T_2)$ is a partition of $S$.
For the input-output constraints, we show one side: the other direction is symmetrical.

\[
V_1^\text{in} \setminus V_2^\text{in} = \{ u \mid \exists \nu \in \text{dom}(I). \ v \in W_1 \} \quad \text{by definition of } V_1^\text{in}
\]
\[
\subseteq V_2^\text{out} \quad \text{since dom}(I) \text{ is a cut-set}
\]
\[
V_1^\text{out} \setminus V_2^\text{out} = \{ u \mid \exists \nu \in \text{dom}(I). \ u \in W_1 \} \quad \text{by definition of } V_1^\text{out}
\]
\[
= \{ u \mid \exists \nu \in \text{dom}(I). \ v \in W_2 \} \quad \text{since dom}(I) \text{ is a cut-set}
\]
\[
\subseteq V_2^\text{in} \quad \text{by definition of } V_2^\text{in}
\]

Then \( V_1^\text{in} \setminus V_2^\text{in} \subseteq V_2^\text{out} \) and \( V_1^\text{out} \setminus V_2^\text{out} \subseteq V_2^\text{in} \). The other directional is symmetrical, swapping 1 and 2. Then the input-output constraints hold.

Finally, we can determine that the shared input constraint holds as follows:

\[
V_1 \cap V_2 = (W_1 \cup V_1^\text{in}) \cap (W_2 \cup V_2^\text{in}) \quad \text{definition of } V
\]
\[
= (W_1 \cap W_2) \cup (W_1 \cap V_2^\text{in}) \cup (V_1^\text{in} \cap W_2) \cup (V_1^\text{in} \cap V_2^\text{in}) \quad \text{distributivity}
\]
\[
= \emptyset \cup (W_1 \cap V_2^\text{in}) \cup (V_1^\text{in} \cup W_2) \cup (V_1^\text{in} \cup V_2^\text{in}) \quad \text{disjointness of Ws}
\]
\[
= (V_1^\text{in} \cap V_2^\text{in}) \cup (V_1^\text{in} \setminus W_2 \cup V_2^\text{in}) \cup (V_1^\text{in} \setminus (W_1 \cup V_2^\text{in})) \quad \text{commutativity, rewrite Ws}
\]
\[
= (V_1^\text{in} \cap V_2^\text{in}) \cup (V_2^\text{in} \setminus W_2 \setminus V_2^\text{in}) \cup (V_1^\text{in} \setminus W_1 \setminus V_2^\text{in}) \quad \text{set identity}
\]
\[
= (V_1^\text{in} \cap V_2^\text{in}) \cup (V_2^\text{in} \setminus V_2^\text{in}) \cup (V_1^\text{in} \setminus V_2^\text{in}) \quad \text{by } V_1^\text{in} \cap W = \emptyset
\]
\[
= (V_1^\text{in} \cap V_2^\text{in}) \cup (V_2^\text{in} \setminus V_1^\text{in}) \quad \text{factoring}
\]

Then all the partition relation constraints hold, so \( T_1, T_2 \) is a partition of \( S \).

**Correctness.** We now continue with a series of lemmas we will use in our proofs of soundness and completeness. As a reminder to readers, our theorems of soundness and completeness focus on demonstrating that the solutions of an open SRP’s fragments are the solution of the parent SRP (or vice-versa); we prove these theorems by making use of case analysis over the cases of an open SRP’s solution, as presented in §4. Loosely speaking, like the three subsets of an SRP’s nodes, these cases can be divided into (a) base node solutions (cf. closed SRP solutions); (b) input node solutions (equality to ass); and (c) output node solutions (the closed SRP solution plus equality to guar).

It is straightforward by the definitions of fragments that, if fragments inherit input and output nodes from their parent, then their solution will also be a solution in the parent; the more difficult cases involve using the closed SRP solution, and reasoning over the input-output nodes between the two fragments after a parent edge was cut.

Our three following lemmas help us through these difficult cases by proving properties of the nodes which are in both fragments of a partition. Lemma 1 starts by proving that any node in both fragments of a partition must be either an input node or an output node.

**Lemma 1 (Shared nodes are either inputs or outputs).** Let \( S, T_1, T_2 \) be open SRPs such that \( (T_1, T_2) \) is a partition of \( S \). Then \( V_1 \cap V_2 \subseteq V_1^\text{in} \cup V_2^\text{out} \) and \( V_1 \cap V_2 \subseteq V_1^\text{out} \cup V_2^\text{in} \).
Proof. \textbf{T}_1 \text{ case}. \quad V_1 \cap V_2 \subseteq V_1^{in} \cup V_1^{out}

\begin{align*}
V_1 \cap V_2 &= (V_1^{in} \cap V_2^{in}) \cup (V_1^{in} \cup V_1^{in}) \setminus V_1^{in} \\
&= (V_1^{in} \cap V_2^{in}) \cup (V_1^{in} \setminus V_2^{in}) \cup (V_2^{in} \setminus V_1^{in}) \\
&\subseteq (V_1^{in} \cap V_2^{in}) \cup V_1^{in} \cup (V_2^{in} \setminus V_1^{in}) \\
&\subseteq (V_1^{in} \cap V_2^{in}) \cup V_1^{in} \cup V_1^{out} \\
&\subseteq V_1^{in} \cup V_1^{in} \cup V_1^{out} \\
&\subseteq V_1^{in} \cup V_1^{out}
\end{align*}

by shared node division constraint

distribute \ \over \cup

by definition of \ \subseteq, \ \setminus

by \ \subseteq, \ \cap

by \ \cup \ idempotence

\textbf{T}_2 \text{ case}. \quad \text{Similar to the } \textbf{T}_1 \text{ case.}

We next prove an additional lemma about shared input nodes in Lemma 2: if a node is an input to both fragments, then it is also an input of the parent SRP.

\textbf{Lemma 2 (Shared Inputs are Inherited).} Let \( T_1, T_2, S \) be open SRPs such that \((T_1, T_2)\) is a partition of \( S \). Then \( V_1^{in} \cap V_2^{in} \subseteq V_2^{in} \).

\textbf{Proof.}

\begin{align*}
V_1^{in} \cap V_2^{out} &= \emptyset \\
\Rightarrow V_1^{in} \cap (V_2^{in} \setminus S^{in}) &= \emptyset \\
\Rightarrow (V_1^{in} \cap V_2^{in}) \setminus V_2^{in} &= \emptyset \\
\Rightarrow V_1^{in} \cap V_2^{in} &\subseteq V_2^{in}
\end{align*}

by definition of open SRPs

by input-output constraints

by \ \emptyset \ \cap (B \setminus C) = (A \cap B) \setminus C

by \ A \setminus B = \emptyset \Rightarrow A \subseteq B

We will use Lemma 2 in the following proof which now moves on to considering open SRP solutions directly by proving that, if the two fragments have solutions, then the solutions are equal for shared nodes.

\textbf{Lemma 3 (Shared Nodes have the Same Solutions).} Let \( T_1, T_2, S \) be open SRPs such that \((T_1, T_2)\) is a partition of \( S \). Assume \( T_1 \) has a solution \( L_1 \) and \( T_2 \) has a solution \( L_2 \). Then \( \forall v \in (V_1 \cap V_2), \ L_1(v) = L_2(v) \).

\textbf{Proof.} We want to show that \( \forall v \in (V_1 \cap V_2), \ L_1(v) = L_2(v) \). Recall the shared node division constraint:

\[ V_1 \cap V_2 = (V_1^{in} \cap V_2^{in}) \cup (V_1^{in} \cup V_2^{in}) \setminus V_1^{in} \]

Then, by substitution, we want to show:

\[ \forall v \in ((V_1^{in} \cap V_2^{in}) \cup (V_1^{in} \cup V_2^{in}) \setminus V_1^{in}), \ L_1(v) = L_2(v) \]

which we can split into two separate conjuncts:

\[ ((\forall v \in (V_1^{in} \cap V_2^{in}), L_1(v) = L_2(v)) \land (\forall v \in ((V_1^{in} \cup V_2^{in}) \setminus V_1^{in}), L_1(v) = L_2(v)) \]

\textbf{Case 1}: \( (\forall v \in (V_1^{in} \cap V_2^{in}), L_1(v) = L_2(v)) \). Consider an arbitrary \( v \) in \( V_1^{in} \cap V_2^{in} \). By Lemma 2, \( v \in (V_1^{in} \cap V_2^{in}) \rightarrow v \in V_2^{in} \). Then \( v \in V_1^{in} \cap V_2^{in} \cap V_3^{in} \). Then by the definition
of a fragment, \( \text{ass}_1(v) = \text{ass}_2(v) \) and \( \text{ass}_2(v) = \text{ass}_3(v) \). By transitivity and the definitions of \( L_1 \) and \( L_2 \), we then have \( L_1(v) = \text{ass}_3(v) \) and \( L_2(v) = \text{ass}_3(v) \). Then, again by transitivity, \( L_1(v) = L_2(v) \).

**Case 2:** \( \forall v \in ((V_1^{in} \cup V_2^{in}) \setminus V_S^{in}). L_1(v) = L_2(v) \). Recall that by the input-output constraints, we have the following:

\[
V_1^{in} \setminus V_S^{in} \subseteq V_2^{out} \tag{12}
\]

\[
V_2^{in} \setminus V_S^{in} \subseteq V_1^{out} \tag{13}
\]

\[
\forall v \in (V_1^{in} \setminus V_S^{in}). \text{ass}_1(v) = \text{guar}_2(v) \tag{14}
\]

\[
\forall v \in (V_2^{in} \setminus V_S^{in}). \text{ass}_2(v) = \text{guar}_1(v) \tag{15}
\]

We also have the following by the definition of \( L \):

\[
\forall v \in V_1^{in}. L_1(v) = \text{ass}_1(v) \tag{16}
\]

\[
\forall v \in V_2^{in}. L_2(v) = \text{ass}_2(v) \tag{17}
\]

\[
\forall v \in V_1^{out}. L_1(v) = \text{guar}_1(v) \tag{18}
\]

\[
\forall v \in V_2^{out}. L_2(v) = \text{guar}_2(v) \tag{19}
\]

Using the relationships between the sets, we can then substitute the equalities over solutions into Equations (14) and (15) to get the desired statement.

Since \( V_1^{in} \setminus V_S^{in} \subseteq V_2^{out} \) by (12), and \( V_1^{in} \setminus V_S^{in} \subseteq V_1^{in} \) by set identities, we can substitute \( L_2 \) for \( \text{guar}_2 \) (per (19)) and \( L_1 \) for \( \text{ass}_1 \) (per (16)) in Equation (14) to get a statement over solutions:

\[
\forall v \in (V_1^{in} \setminus V_S^{in}). L_1(v) = L_2(v)
\]

We can use the same reasoning with Equation (13) and Equations (18) and (17) to get another statement from Equation (15):

\[
\forall v \in (V_2^{in} \setminus V_S^{in}). L_2(v) = L_1(v)
\]

We can then rearrange the ground formulas by commutativity and conjoin the two statements to obtain:

\[
(\forall v \in (V_1^{in} \setminus V_S^{in}). L_1(v) = L_2(v)) \land (\forall v \in (V_2^{in} \setminus V_S^{in}). L_1(v) = L_2(v))
\]

Finally, we can rewrite the conjunction to instead be one formula over \((V_1^{in} \setminus V_S^{in}) \cup (V_2^{in} \setminus V_S^{in})\): factoring out the set difference gives us: \((\forall v \in ((V_1^{in} \cup V_2^{in}) \setminus V_S^{in}). L_1(v) = L_2(v))\), which was what was required.

We now move onto the proof of soundness of \( \text{CUT} \), which states that if \( \text{CUT}(S, I) = (T_1, T_2) \), which have respective solutions \( L_1 \) and \( L_2 \), then there is a solution to the parent \( \text{SRP} \ S \) which is equal to both fragment solutions over all relevant nodes. We define this solution in the theorem statement, and then prove it satisfies the solution constraints for any node in \( S \), regardless of whether it is an input, an output or a base node.
**Theorem 4 (Cut is Sound).** Let $S$ be an open SRP, and let $I$ be an interface over $S$. Let $\text{CUT}(S, I) = (T_1, T_2)$. Suppose $T_1$ has a unique solution $L_1$ and $T_2$ has a unique solution $L_2$. Consider a mapping $L_S': V_S \rightarrow R$, defined such that:

\[
\forall v \in V_1. \ L_S'(v) = L_1(v) \tag{20}
\]
\[
\forall v \in V_2. \ L_S'(v) = L_2(v) \tag{21}
\]
\[
\forall v \in V_S^m. \ L_S'(v) = \text{ass}_S(v) \tag{22}
\]

Then $L_S'$ is a solution of $S$.

**Proof.** **Preliminaries.** By Theorem 3, we have that $(T_1, T_2)$ is a partition of $S$. Consider a mapping $L_S': V_S \rightarrow R$, defined as stated above. Even though none of the cases are over $V_S$, this defines $L_S'$ over all $V_S$, since $V_1 \cup V_2 = V_S$. No case is ever in conflict: by Lemma 3, $\forall v \in V_1 \cap V_2. \ L_1(v) = L_2(v)$, so Equations (20) and (21) both apply for all shared nodes; by Lemma 2 and the definition of fragments, if $v \in V_1^m \cap V_2^m$, then $\text{ass}_1(v) = \text{ass}_2(v) = \text{ass}_S(v)$, so Equation (22) holds for any shared inputs.

Our goal is to show that $L_S'$ is a solution for $S$ as stated in §4. We proceed by considering an arbitrary node $u$, and show that, for each node subset $u$ could belong to ($u \notin V_S^m, u \in V_S^m, u \in V_S^{\text{out}}$), $L_S'(u)$ is a solution for $u$.

**Case $u \notin V_S^m$.** Then we want to show that our mapping implies that $L_S'(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_S} \text{trans}(vu, L_S'(v))$. We have two cases to consider here, depending on if $u$ is in a single fragment $(V_1 \ominus V_2$, meaning either $V_1$ or $V_2$, the symmetric difference of $V_1$ and $V_2$), or whether $u \in (V_1 \cap V_2) \setminus V_S^m$.

**Sub-Case.** Suppose w.l.o.g. that $u \in V_1$. Then since $T_1$ has a solution $L_1$, we have that $L_1(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_1} \text{trans}(vu, L_1(v))$, since $u$ must be either a base node or an output node in $V_S^{\text{out}}$.

In either such case, we then also know that $u$ has the same neighbors in $T_1$ as in $S$, so $\{vu \mid vu \in E_1\} = \{vu \mid vu \in E_S\}$. By (20), we have that $L_S'(u) = L_1(u)$ that for each neighbor $v$, $L_S'(v) = L_1(v)$, so we can then substitute $L_S'$ for $L_1$ and the set of neighbors in $E_S$ for the set of neighbors in $E_1$, giving $L_S'(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_S} \text{trans}(vu, L_S'(v))$. Then this sub-case holds.

**Case $u \in (V_1 \cap V_2) \setminus V_S^m$.** By the shared input constraint, $(V_1 \cap V_2) \setminus V_S^m = (V_1^m \setminus V_S^m) \cup (V_2^m \setminus V_S^m)$. In other words, since $u \notin V_S^m$, it is an input-output node.

Suppose w.l.o.g. that $u \in V_1^m \setminus V_S^m$. Then $u \in V_2^{\text{out}}$ by the input-output constraints. Then since $T_1$ and $T_3$ have solutions, we have that

\[
L_1(u) = \text{ass}_1(u) \tag{23}
\]
\[
L_2(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_2} \text{trans}(vu, L_2(v)) \tag{24}
\]
\[
L_2(u) = \text{guar}_2(u) \tag{25}
\]

By these equations and the input-output constraints, we then have that $L_1(u) = \text{ass}_1(u) = \text{guar}_2(u) = L_2(u)$, so $L_1(u) = L_2(u)$. By (20) and (21), we also have that $L_S'(u) = L_1(u) = L_2(u)$. 


Now we wish to show that $\mathcal{L}_s'$ is a solution for $u$: since $u$ is not in $V_{in}^S$, we must show the non-input case $\mathcal{L}_s'(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_S} \text{trans}(vu, \mathcal{L}_s'(v))$ (we defer the output constraint $\mathcal{L}_s'(u) = \text{guar}_S(u)$ to the end of the proof).

As above, we start by observing that $u$ has the same in-neighbors in $S$ as in $T_2$, and that this encompasses all of its in-neighbors since $u \in V_{in}^1$, so it has no in-neighbors in $T_1$. Then $\{vu \mid vu \in E_2\} = \{vu \mid vu \in E_S\}$.

Next, by (21), we can substitute $\mathcal{L}_s'$ for $\mathcal{L}_2$ in (24). By the reasoning above, we can also substitute the set of in-neighbors of $u$ in $E_S$ for the set of in-neighbors of $u$ in $E_2$, leaving us with $\mathcal{L}_s'(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_S} \text{trans}(vu, \mathcal{L}_s'(v))$.

Then this case holds as well, and we have that $\mathcal{L}_s'(u)$ is a solution for $u \notin V_{in}^S$.

$u \in V_{in}^S$ Case. Since $\mathcal{L}_s'(u) = \text{ass}_S(u)$ by (22), this case immediately holds.

$u \in V_{out}^S$ Case. By the definition of a fragment, $\mathcal{L}_2(u) = \text{guar}_S(u)$ if $u \in V_1$ and $\mathcal{L}_2(u) = \text{guar}_S(u)$ if $u \in V_2$. Since $\mathcal{L}_s'(u) = \mathcal{L}_1(u)$ by (20) in the former case and $\mathcal{L}_s'(u) = \mathcal{L}_2(u)$ by (21) in the latter case, we have that $\mathcal{L}_s'(u) = \text{guar}_S(u)$, so this case holds.

Then for all three cases, $\mathcal{L}_s'$ is a solution for $S$.

This concludes our proof of soundness: we now know that the solutions of the fragments constitute a solution of the parent SRP. We now also prove completeness, meaning that any solution to the parent SRP is also a solution to the fragments, so long as the fragments’ inputs and outputs are annotated with assumptions and guarantees that match the parent SRP’s solution. The form of the proof is also by solution cases, this time over the fragment solution.

**Theorem 5 (Cut is Complete).** Let $S$ be an open SRP, and let $I$ be an interface over $S$. Let $\text{Cut}(S, I) = (T_1, T_2)$. Assume $S$ has a unique solution $\mathcal{L}_S$. Assume that $\forall uv \in \text{dom}(I). I(uv) = \mathcal{L}_S(u)$. Consider the following two mappings $\mathcal{L}_1' : V_1 \rightarrow R$ and $\mathcal{L}_2' : V_2 \rightarrow R$, defined such that:

$$\forall v \in V_1. \mathcal{L}_1'(v) = \mathcal{L}_S(v)$$

$$\forall v \in V_2. \mathcal{L}_2'(v) = \mathcal{L}_S(v)$$

Then $\mathcal{L}_1'$ is a solution for $T_1$ and $\mathcal{L}_2'$ is a solution for $T_2$.

**Proof.** By Theorem 3, we have that $(T_1, T_2)$ is a partition of $S$. Furthermore, by the definition of Cut and the assumption that every cut edge is annotated with the solution in $S$, we have the following equalities on $T_1$ and $T_2$’s inputs and outputs:

$$\forall u \in V_{1in}. \text{ass}_1(u) = \mathcal{L}_S(u)$$  \hspace{1cm} (26)

$$\forall u \in V_{2in}. \text{ass}_2(u) = \mathcal{L}_S(u)$$  \hspace{1cm} (27)

$$\forall u \in V_{1out}. \text{guar}_1(u) = \mathcal{L}_S(u)$$  \hspace{1cm} (28)

$$\forall u \in V_{2out}. \text{guar}_2(u) = \mathcal{L}_S(u)$$  \hspace{1cm} (29)

As the two cases are symmetric, w.l.o.g., we proceed by considering an arbitrary node $u$ in $T_1$. Then we have three cases to show, based on the three cases of $\mathcal{L}_1'(u)$.

$u \notin V_{1in}$ Case. By the fragment constraints, if $u \in V_{in}^S$ then if $u \in V_1$, then $u \notin V_{in}^S$. Then by the contrapositive, if $u \notin V_{in}^S$, then $u \notin V_{in}^S$. Then $\mathcal{L}_S(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_S} \text{trans}(vu, \mathcal{L}_S(v))$. 
By the fact that $T_1$ is a fragment, $\{vu \mid vu \in E_1\} = \{vu \mid vu \in E_S\}$. Then, by our definition of $L_1'(u)$, we can substitute $L_1'(u)$ for $L_S(u)$ to obtain: $L_1'(u) = \text{init}(u) \oplus \bigoplus_{vu \in E_1} \text{trans}(vu, L_1'(v))$. Then this case holds for $u$.

$u \in V^\text{in}_1$ Case. Then by (26), we have $L_S(u) = \text{ass}_1(u)$. Then by substitution, we have $L_1'(u) = \text{ass}_1(u)$. Then this case holds for $u$.

$u \in V^\text{out}_1$ Case. Then by (28), we have $L_S(u) = \text{guar}_1(u)$. Then by substitution, we have $L_1'(u) = \text{guar}_1(u)$. Then this case holds for $u$.

Then $T_1$ has a solution $L_1'$. By a symmetric proof using (27) and (29), $T_2$ has a solution $L_2'$.

An important corollary of our theorem of soundness is that, since the solutions of the fragments are a solution to the parent SRP, any property over solutions that holds on the fragments will also hold on the parent SRP.

**Corollary 2 (Cut Preserves Properties).** Let $S$ be an open SRP and let $I$ be an interface over $S$. Let $\text{CUT}(S, I) = (T_1, T_2)$. Let $P_1, P_2$ be formulas such that $P_1 = \forall v \in V_1. Q(v)$ and $P_2 = \forall v \in V_2. Q(v)$, where $Q$ is a predicate on $L(v)$. Assume $S$ has a unique solution $L_S$, and that $T_1$ has a solution $L_1$ and $T_2$ has a solution $L_2$. Then if $P_1$ holds on $T_1$ and $P_2$ holds on $T_2$, $P_1 \land P_2$ holds on $S$.

**Proof.** By Theorem 4, $\forall u \in V_1. L_1(u) = L_S(u)$ and $\forall u \in V_2. L_2(u) = L_S(u)$. Assume $P_1$ holds on $T_1$ and $P_2$ holds on $T_2$. Consider w.l.o.g. a node $u$ in $V_1$. Then $Q(u)$ holds in $T_1$. Since $L_1(u) = L_S(u)$, $Q(u)$ holds in $S$ as well. Then since $V_1 \cup V_2 = V_S$, $\forall v \in V_S. Q(v)$, and therefore $P_1 \land P_2$ holds on $S$. 