The mechanism of spin and charge separation in one dimensional quantum antiferromagnets

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Abstract

We reconsider the problem of separation of spin and charge in one dimensional quantum antiferromagnets. We show that spin and charge separation in one dimensional strongly correlated systems cannot be described by the slave boson or fermion representation within any perturbative treatment of the interactions between the slave holons and slave spinons. The constraint of single occupancy must be implemented exactly. As a result the slave fermions and bosons are not part of the physical spectrum. Instead, the excitations which carry the separate spin and charge quantum numbers are solitons. To prove this no-go result, it is sufficient to study the pure spinon sector in the slave boson representation. We start with a short-range RVB spin liquid mean-field theory for the frustrated antiferromagnetic spin-$\frac{1}{2}$ chain. We derive an effective theory for the fluctuations of the Affleck-Marston and Anderson order parameters. We show how to recover the phase diagram as a function of the frustration by treating the fluctuations non-perturbatively.

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I. INTRODUCTION

Since the discovery of high $T_c$ superconductivity, there has been a lot of interest in the $t-J$ model [1] for low dimensional lattices. It is known that for a linear chain, the $t-J$ model displays the phenomenon of spin and charge separation for strong enough electronic correlations [2–4] and belongs to the class of Luttinger liquids [5]. The spin and charge excitations which carry separately the elementary charge of the electron are solitons in a Luttinger liquid. This is most easily seen using a Jordan-Wigner representation of the spin-$\frac{1}{2}$ in the context of the Heisenberg chain. Much effort has been devoted to prove the existence of this phenomenon in higher dimensional $t-J$ models (most significantly in two dimensions in view of its possible relevance to high $T_c$ superconductivity).

One approach to this issue has been to think of the band electron as a point-like or local bound state of two constituents: one, the slave holon, carrying the electronic charge, the other, the slave spinon, carrying the electronic spin quantum number [6]. The slave holon and spinon constituents are held (glued) together by a strongly fluctuating gauge field. The hypothesis behind this picture is that, by some deconfining mechanism, it is favorable for the system to liberate the electron constituents, i.e., the energy cost for breaking the local bound state is finite (possibly zero). If this is so, it is then natural as a first step to approximate the interacting system of holons and spinons by free holons and free spinons with self-consistently determined renormalized kinetic energy scales and then to include perturbatively the gauge fluctuations. This is the content of all the mean-field theories for slave holons and spinons which freeze the strong gauge fluctuations between the holons and spinons in order to describe the separation of spin and charge. For lack of an alternative, this strategy has been widely used to implement spin and charge separation in two dimensions. On the other hand, this picture is not useful if the holons and spinons are always constrained to form bound states on the shortest possible scale, i.e., the lattice spacing.

The holon and spinon carry complementary statistics. In the slave boson scheme, the holon is a boson while the spinon is a fermion [7,8]. In the slave fermion scheme, the holon is
a fermion and the spinon is a boson\cite{9–11}. For the $t-J$ model, the mean-field predictions for both schemes differ qualitatively. This is not surprising since the slave boson scheme is most appropriate when the holes are moving in the background of a spin liquid whereas the slave fermion scheme is convenient in the presence of strong antiferromagnetic correlations. In one dimension and at the mean-field level neither the slave boson nor the slave fermion scheme describes a Luttinger liquid\cite{12}. Moreover, this failure of the mean-field theory in one dimension persists even if the self-energy corrections to the holon and spinon propagators due to the gauge fields are included (as shown by Feng et al\cite{12}). One might wonder then if vertex corrections due to the gauge fields are sufficient to restore the characteristic features of a Luttinger liquid and if not, how one recovers the Luttinger liquid.

In this paper, we show that it is not possible to obtain Luttinger liquid behavior by including perturbatively gauge fluctuations around the mean-field Ansatz for the holon-spinon system, the reason being that deconfinement is never allowed in one dimension. The separation of spin and charge in the one dimensional $t-J$ model is due to a topological mechanism which is qualitatively different from the mechanism of deconfinement. In one dimension, the charge elementary excitations and the spin elementary excitations are solitons. This no-go result is important in view of the fact that we have shown that a spin liquid proposed by Wen\cite{13} can support the deconfinement of the slave spinons in two-dimensional (and possibly higher) strongly correlated systems\cite{14}.

We are thus confronted with the following situation. On the one hand, spin and charge separation occurs in one dimensional strongly correlated electronic systems but it cannot be described by the deconfinement of local bound states of holons and spinons constituting the band electrons. On the other hand, the same local bound states of holons and spinons can break up in higher dimensional strongly correlated systems, thus achieving a different realization of spin and charge separation. It is not known at the present if an analogy of the topological mechanism of spin and charge separation is available in higher dimensional systems. It appears that dimensionality is crucial and it could well be that the one dimensional mechanism is not available in higher dimensions.
The essence of our no-go result is that the lower (space-time) critical dimension for the deconfinement of a pure gauge theory with a discrete symmetry group is $2 + 1$ \textsuperscript{[15,14]}. Hence, it applies to either the slave boson or slave fermion scheme. Moreover, it is sufficient to consider the pure spinon sector and to prove that the spinons can never deconfine. This we do by studying the Heisenberg chain for spin-$\frac{1}{2}$ with antiferromagnetic nearest and next-nearest neighbor couplings. We choose to represent the spin-$\frac{1}{2}$ by fermionic spinons. In other words, we use the slave boson representation of the $t - J_1 - J_2$ model at half-filling.

Our motivation for this choice is three-fold. First, in two dimensions, deconfinement of the slave spinons is made possible by the opening of an energy gap in the spinon spectrum as a result of frustration, i.e., if the spinon ground state describes a short-range spin liquid. Second, many properties of the frustrated Heisenberg chain for spin-$\frac{1}{2}$ are well understood. The exact ground state and low energy excitation spectrum are known in the limits $\frac{J_2}{J_1} = 0$ \textsuperscript{[16]}, $\frac{J_2}{J_1} = \frac{1}{2}$ \textsuperscript{[17]}, and $\frac{J_2}{J_1} = \infty$. Haldane has constructed the zero-temperature phase diagram as a function of not too large frustration $\frac{J_2}{J_1}$ \textsuperscript{[18]}. The parent $t - J_1 - J_2$ model has been studied numerically \textsuperscript{[19]}. Finally, the model close to the limit $\frac{J_2}{J_1} = 0$ resembles the double spin-$\frac{1}{2}$ chain problem which has recently received much attention \textsuperscript{[20]} in connection with the single rung $t - J$ ladder \textsuperscript{[21]}. Thus, the frustrated Heisenberg chain for spin-$\frac{1}{2}$ is an ideal model to understand the mechanism by which a separation of spin and charge unrelated to deconfinement is realized in the gauge field approach.

To construct a spin liquid, we choose in Sec. \textsuperscript{[II]} to rewrite the spin problem as a SU(2) lattice gauge theory coupling fermionic spinons with gauge fields. Our choice for a spin liquid ground state is given in Sec. \textsuperscript{[III]}. The mean-field theory for the spinons is described and an effective field theory which includes all the smooth fluctuations of the order parameters characterizing the spin liquid is derived. We treat the quantum fluctuations of the order parameters exactly in Sec. \textsuperscript{[IV]}. We show explicitly how all the predictions of the mean-field theory are modified by the quantum fluctuations. All the mean-field excitations are removed, quantum criticality for small frustration $\frac{J_2}{J_1}$ is restored. At criticality, the effective quantum field theory is shown to be the level $k = 1$ Wess-Zumino-Witten theory in agreement with
The gapless excitations in the critical regime are identified with the Jordan-Wigner topological excitations. We show in Sec. V how the second order phase transition from a gapless spin liquid to a dimerized phase is induced by frustration in our quantum field theory. Finally, we briefly discuss an effective quantum field theory for two weakly interacting antiferromagnetic Heisenberg chains.

II. THE EQUIVALENCE OF THE HEISENBERG MODEL FOR SPIN-$\frac{1}{2}$ TO A SU(2) LATTICE GAUGE THEORY

We start with the Heisenberg model for spin-$\frac{1}{2}$

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j,$$

where $\langle ij \rangle$ is an ordered pair of sites on an arbitrary lattice $\Lambda$, $J_{ij}$ are real coupling constants. The slave fermion (or spinon) representation of the spin-$\frac{1}{2}$ operators is

$$\vec{S}_i = \frac{1}{2} s_i^\dagger \vec{\sigma}_{\alpha\beta} s_i^\beta,$$

where $\vec{\sigma}$ is the Pauli matrices and the $s_i$’s obeying fermionic anticommutation relations. The spinon representation, Eq. (2.2), for the spin-$\frac{1}{2}$ degrees of freedom must be supplemented with any of the three constraints

$$\mathbb{1} = s_i^\dagger \delta_{\alpha\beta} s_i^\beta \iff 0 = s_i^\dagger s_i^\uparrow \iff 0 = s_i^\dagger s_i^\downarrow (2.3)$$

for all sites $i$ of the lattice.

From the fully symmetric tensor $\delta_{\alpha\beta}$ and the fully antisymmetric tensor $\epsilon_{\alpha\beta}$ of SU(2), the two bilinear forms

$$\chi_{ij}^\dagger = s_i^\dagger \delta_{\alpha\beta} s_j^\beta (2.4)$$

and

$$\eta_{ij}^\dagger = s_i^\dagger \epsilon_{\alpha\beta} s_j^\beta (2.5)$$
can be used to describe a singlet pairing of the two spin-$\frac{1}{2}$ located on site $i$ and $j$, respectively \cite{24,25}. Indeed, the identity

$$\vec{S}_i \cdot \vec{S}_j = -\frac{1}{4} \eta^\dagger_{ij} \eta_{ij} - \frac{1}{4} \chi^\dagger_{ij} \chi_{ij} + \frac{1}{4} \mathbb{I}$$

(2.6)

holds in the Hilbert space of one spinon per site. A spin liquid which, by definition, should not show any long range magnetic order, implies, in the spinon picture, the exponential decay with separation $|i - j|$ of the ground state expectation values $\langle \eta^\dagger_{ij} \rangle$ or $\langle \chi^\dagger_{ij} \rangle$.

The dynamics of these bilinear forms can be obtained from the partition function

$$Z = \int \mathcal{D}[\vec{a}_0] \int \mathcal{D}[s^\dagger] \mathcal{D}[s] e^{i \int dt \mathcal{L}'}$$

(2.7)

where the lattice Lagrangian is

$$L' = \sum_i s^*_{i\alpha} i \partial_t s_{i\alpha}$$

$$- \sum_i \left( \frac{1}{2} a^\dagger_{i\alpha t} \eta^*_{i\alpha t} + \frac{1}{2} a_{i\alpha t} \eta_{i\alpha t} + \frac{1}{2} a^3_{i\alpha t} (\chi^*_{i\alpha t} - 1) \right)$$

$$+ \sum_{\langle ij \rangle} \frac{J_{ij}}{4} \left( \eta^*_{ijt} \eta_{ijt} + \chi^*_{ijt} \chi_{ijt} \right).$$

(2.8)

The integration over the Lagrange multipliers

$$\begin{pmatrix}
  a^\dagger_{i\alpha t} \\
  a_{i\alpha t} \\
  a^3_{i\alpha t}
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{2} (a^1_{i\alpha t} - i a^2_{i\alpha t}) \\
  \frac{1}{2} (a^1_{i\alpha t} + i a^2_{i\alpha t}) \\
  a^3_{i\alpha t}
\end{pmatrix}$$

(2.9)

enforces the constraint of single occupancy on the spinon Hilbert space in a redundant way. But this redundancy allows for the mapping of Eq. (2.8) into the Lagrangian of a SU(2) lattice gauge theory \cite{25}, if a Hubbard-Stratanovich transformation with respect to the composite fields $\eta$ and $\chi$ is performed first.

With a particle-hole transformation of the spinons, our final Lagrangian will then take the form \cite{26,27}

$$L = \sum_i \psi^*_{i\alpha t} \left( i \partial_t - A_{i\alpha t} \right) \psi_{i\alpha t}$$

$$- \sum_{\langle ij \rangle} \frac{J_{ij}}{4} \left[ |\det W_{ijt}| + (\psi^*_{i\alpha t} W_{ijt} \psi_{j\beta t} + \psi^*_{j\beta t} W^\dagger_{ijt} \psi_{i\alpha t}) \right].$$

(2.10)
Here, the $\psi$’s are

$$\psi_{it} = \begin{pmatrix} s_{it\uparrow} \\ s_{it\downarrow} \end{pmatrix}. \tag{2.11}$$

The $A_\delta$’s belong to the fundamental representation of the su(2) Lie algebra

$$A_{\delta it} = \frac{1}{2} \vec{a}_{\delta it} \cdot \vec{\sigma}. \tag{2.12}$$

Finally, the $W$’s are $2 \times 2$ matrices of the form

$$W_{ijt} = \begin{pmatrix} -X_{ijt} - E_{ijt} \\ -E_{ijt}^* + X_{ijt}^* \end{pmatrix}, \tag{2.13}$$

which satisfy

$$W_{ijt} = W_{jit}^\dagger. \tag{2.14}$$

The entries $E$ and $X$ of the $W$’s are the Hubbard-Stratanovich degrees of freedom associated with the spinon bilinears $\eta$ and $\chi$, respectively.

The lattice Lagrangian in Eq. (2.10) is left unchanged by the local gauge transformations

$$\begin{align*}
\psi_{it} &\rightarrow \psi_{it}' = U_{it} \psi_{it}, \\
A_{\delta it} &\rightarrow A_{\delta it}' = U_{it} A_{\delta it} U_{it}^\dagger + (i \partial_t U_{it}) U_{it}^\dagger, \\
W_{ijt} &\rightarrow W_{ijt}' = U_{it} W_{ijt} U_{jt}^\dagger,
\end{align*} \tag{2.15}$$

for all $U_{it} \in SU(2)$. This local symmetry will be called a color symmetry. It is a different symmetry from the one generated by global spin rotations. Indeed, under the particle-hole transformation Eq. (2.11), the spin-$\frac{1}{2}$ operators of Eq. (2.2) are mapped into

$$\begin{align*}
S_i^1 &= + \frac{1}{2} \left( \psi_{i1}^\dagger \psi_{i2} + \psi_{i2} \psi_{i1} \right) \equiv + \frac{1}{2} \left( b_i^\dagger + b_i \right), \\
S_i^2 &= - \frac{1}{2} \left( \psi_{i1}^\dagger \psi_{i2}^\dagger - \psi_{i2} \psi_{i1} \right) \equiv - \frac{1}{2} \left( b_i^\dagger - b_i \right), \\
S_i^3 &= + \frac{1}{2} \left( \psi_i^\dagger \psi_i - 1 \right) \equiv + \frac{1}{2} \left( m_i - 1 \right). \tag{2.16}
\end{align*}$$

The bilinears $b$ and $m$ defined above are left unchanged by the local color transformation Eq. (2.13) and thus the Heisenberg interaction explicitly transforms like a color singlet when
expressed in terms of the $\psi$’s. Spin-spin correlations can be obtained from the generating functional

$$Z[\vec{J}] = \int D[W^\dagger] D[W] \int D[\vec{a}_o] \int D[\psi^\ast] D[\psi] \ e^{+i\int dt( L + \sum_i j_i \cdot \vec{s}_i )}$$

where the source term is written in terms of the bilinears $b$ and $m$ defined by Eq. (2.16).

It is important to realize that the gauge degrees of freedom in Eq. (2.10) are not independent from the fermionic degrees of freedom. For example, neither $A_o$ nor the SU(2) factor of $W_{ij}$ possess a lattice version of the kinetic energy. Their presence is simply a device to project the Fock space $\mathcal{F}$ which is generated cyclically from the vacuum state $|0 >_\psi$ defined by

$$\psi_{ia} |0 >_\psi = 0, \ \forall i \in \Lambda, \ a = 1, 2,$$

onto the physical Hilbert space which is the tensorial product over all sites $i$ of the vector spaces spanned by the color singlet states $|0 >_{\psi_i}$ and $b_i^{\dagger} |0 >_{\psi_i}$ (see [28]).

III. MEAN-FIELD THEORY AND FLUCTUATIONS AROUND IT

A. The mean-field Ansatz

For the rest of the paper, we specialize to a linear chain $\Lambda$ with nearest, $J_1$, and next-nearest, $J_2$, neighbor antiferromagnetic couplings. We try the translationally invariant Ansatz [13][14]

$$\vec{A}_{\text{ort}} = \frac{1}{2} a_{o}^{\dagger},$$

$$\vec{W}_{ijt} = \begin{cases} 
-X \sigma^3 & \text{if } j = i + 1, \\
-\text{Re } E \sigma^1 - \text{Im } E \sigma^2 & \text{if } j = i + 2, \\
0 & \text{otherwise.}
\end{cases}$$

The mean-field spinon spectrum $\pm |\vec{\xi}_k|$ is then given by
\[ \xi_k^1 = -\frac{a_0^1}{2} + \frac{J_2}{2} \text{Re } E \cos 2k, \]
\[ \xi_k^2 = \frac{J_2}{2} \text{Im } E \cos 2k, \] (3.2)
\[ \xi_k^3 = \frac{J_1}{2} X \cos k. \]

The label \( k \) denotes a reciprocal vector in the Brillouin zone \( \Omega \). Points of special significance in the Brillouin zone are the nodes of the mean-field spectrum, i.e., those points \( k^* \in \Omega \) with
\[ |\xi_k^*| = 0. \] (3.3)

For example, if the only non-vanishing mean-field parameter is \( X \), then there are two nodes at \( \pm \frac{\pi}{2} \). Nodes are absent from the mean-field spectrum whenever the mean-field parameters \( X \) and \( E \) are simultaneously non-vanishing \[29\]. The values of the mean-field parameters are determined by the saddle-point equations.

The saddle-point equations are
\[ 0 = + \frac{1}{|\Lambda|} \sum_{k \in \Omega} \frac{\xi_k^1}{|\xi_k|}, \]
\[ \text{Re } E = + \frac{1}{|\Lambda|} \sum_{k \in \Omega} \frac{\xi_k^1}{|\xi_k|} \cos 2k, \]
\[ \text{Im } E = - \frac{1}{|\Lambda|} \sum_{k \in \Omega} \frac{\xi_k^2}{|\xi_k|} \cos 2k, \]
\[ X = + \frac{1}{|\Lambda|} \sum_{k \in \Omega} \frac{\xi_k^3}{|\xi_k|} \cos k, \] (3.4)

where \( |\Lambda| \) is the (even) number of sites in the chain and \( \Omega \) is the Brillouin Zone. They can be solved analytically in the two limits \( \frac{J_1}{J_2} = 0 \) and \( \frac{J_2}{J_1} = 0 \). In the former case, the mean-field solution is
\[ a_0^1 = 0, \quad E = 0, \quad X = \frac{2}{\pi}. \] (3.5)

The mean-field excitation spectrum is gapless at the two \textit{discrete} locations \( \pm \frac{\pi}{2} \) of the Brillouin zone. In the latter case, the mean-field solution is
\[ a_0^1 = 0, \quad \text{Re } E = \frac{2}{\pi}, \quad \text{Im } E = 0, \quad X = 0. \] (3.6)
The mean-field excitation spectrum is gapless at the four *discrete* locations $\pm \frac{\pi}{4}$ and $\pm \frac{3\pi}{4}$ of the Brillouin zone. The doubling of the nodes in the excitation spectrum when $J_1/J_2 = 0$ is to be expected since in this limit the Heisenberg model effectively decouples into two independent Heisenberg models with antiferromagnetic nearest-neighbor interactions $J_2$ between the even and odd lattice sites, respectively. To sum up, both limits yield the same mean-field excitation spectrum namely that of a one dimensional tight-binding gas of fermions at *half-filling* and describe one dimensional versions of the Baskaran-Zou-Anderson (BZA) state [7].

The qualitative features of the mean-field solutions for finite $J_2/J_1$ can be understood in view of the nature of the phase space in one dimension. In one dimension, the gapless modes associated to nodes in the mean-field spectrum have a dispersion relation which can be linearized in the close vicinity of the nodes since the nodes are isolated points in the Brillouin zone [30]. This situation is in contrast to the one in two spatial dimensions where gapless modes can have a dispersion relation which is quadratic in reciprocal space (as is the case with the BZA state for which the nodes form lines) or linear (as is the case with the flux state for which the nodes are isolated points [24]). Another consequence of the restrictive nature of phase space is that there is no genuine flux phase since it is not possible to inclose flux *locally* in one spatial dimension.

This observation on the nature of the gapless modes in one spatial dimension, allows us to understand qualitatively the mean-field solution for finite values of $J_2/J_1$. Without loss of generality, we only need to consider the effect of an arbitrary small perturbation of the limit $J_2/J_1 = 0$. The unperturbed mean-field spectrum is

$$\omega_k = -\frac{J_1}{2} X \cos k, \quad -\pi \leq k \leq +\pi, \quad X = \frac{2}{\pi}. \quad (3.7)$$

Let us see first whether an infinitesimal value of $J_2/J_1$ can induce a non-vanishing value for $\text{Im} \ E$ alone. In other words, is there a solution to the mean-field equation

$$|X| = -\frac{J_2}{J_1} \int_{-\pi}^{+\pi} \frac{dk}{2\pi} \frac{\cos^2(2k)}{\cos^2 k + \left(\frac{J_2}{J_1} \frac{\text{Im} \ E}{X}\right)^2 \cos^2(2k)}. \quad (3.8)$$
The answer is negative due to the minus sign on the right-hand side. However, one immediately sees that the mean-field equation

$$|X| = \frac{J_2}{J_1} \int_{-\pi}^{+\pi} \frac{dk}{2\pi} \frac{\left(-\frac{a_1^1}{J_2} \text{Re} E + \cos 2k\right) \cos 2k}{\sqrt{\cos^2 k + \left(\frac{a_1^1}{J_1} + \frac{J_2 \text{Re} E}{J_1^2} \cos 2k\right)^2}}$$

(3.9)

can be approximately solved, since the integral on the right-hand side is dominated by the two contributions in the range $\pm \frac{\pi}{2} - \varepsilon \leq k \leq \pm \frac{\pi}{2} + \varepsilon$, to yield

$$\frac{2\pi |X|}{\frac{a_0^1}{J_1} \text{Re} E + \frac{J_2}{J_1}} \approx 2 \ln \left[1 + \frac{1}{\varepsilon^2} \left(\frac{a_1^1}{J_1} + \frac{J_2 \text{Re} E}{J_1^2}\right)^2 + 1 \right] \left[1 + \frac{1}{\varepsilon^2} \left(\frac{a_1^1}{J_1} + \frac{J_2 \text{Re} E}{J_1^2}\right)^2 - 1 \right].$$

(3.10)

One verifies that Eq. (3.9) has the solution

$$a_0^1 = 0, \quad \left(\frac{\text{Re} E}{X}\right)^2 \approx \varepsilon^2 \left(4 \frac{J_2}{J_1}\right)^{-1} \exp \left(-\frac{\pi |X|}{\frac{J_2}{J_1}}\right),$$

(3.11)

for any infinitesimal value of $\frac{J_2}{J_1}$.

We thus see that the linearity of the unperturbed mean-field spectrum in the vicinity of the nodes allows for the existence of simultaneous non-vanishing values of the mean-field parameters $X$ and $\text{Re} E$ for any infinitesimal $\frac{J_2}{J_1}$. Conversely, the same analysis holds infinitesimally close to the limit $\frac{J_1 J_2}{J_2 J_1} = 0$. It is natural to extrapolate that the simultaneous condensation of $\text{Re} E$ and $X$ takes place at the mean-field level for any finite $\frac{J_2}{J_1}$. We have solved numerically the saddle-point equations for $0 \leq \frac{J_2}{J_1} \leq 10$ in the thermodynamic limit. The numerical solution confirms that $\text{Re} E$ and $X$ approach monotonically $\frac{2}{\pi}$ and 0, respectively, as $\frac{J_2}{J_1}$ is increased from 0 to 10. The mean-field parameter $|a_0^1|$ reaches a maxima around $\frac{J_2}{J_1} = 1$ and quickly decreases. The mean-field spectrum has a gap within the numerical precision for any finite amount of frustration $\frac{J_2}{J_1}$. The gap only closes when the mean-field Ansatz approaches the one dimensional BZA states.

The opening of a gap in the mean-field excitation spectrum of our spin liquid for any amount of frustration is a dramatic signal of the failure of the mean-field theory in view of the argument given by Haldane [18] that the frustrated one dimensional spin-$\frac{1}{2}$ antiferromagnet
remains gapless for $0 \leq \frac{J_2}{J_1} < \left( \frac{J_2}{J_1} \right)_c = \frac{1}{6}$. Our mean-field theory only predicts quantum criticality in the absence of frustration. However, even in this limit the mean-field theory is highly unreliable [14]. Indeed, the constraint of single occupancy which is needed to establish the equivalence between the spinon model and the Heisenberg model is not satisfied locally but only on average in the mean-field approximation. Consequently, one should not believe the mean-field prediction for, say, staggered spin-spin correlations.

We are going to construct below an effective low energy theory which includes enough of the dynamics ignored by the mean-field approximation so as to recover the quantum criticality for small frustration $\frac{J_2}{J_1}$ and insure equivalence with the low energy sector of the Heisenberg model. The restoration of criticality does not imply that the mean-field excitations just above the mean-field gap have coalesced into a gapless branch of “dressed” mean-field excitations once the fluctuations around the mean-field have been included. Rather, the mean-field single-particle excitations have completely disappeared from the spectrum and the gapless branch describes totally different excitations, namely topological excitations (solitons).

**B. Low energy fluctuations around the mean-field Ansatz**

To begin with, we need some notation. Let $\Lambda_e$ and $\Lambda_o$ be the sublattices of even and odd sites, respectively,

$$\Lambda_e = \{i \in \Lambda | i \mod 2 = 0\},$$

$$\Lambda_o = \{i \in \Lambda | i \mod 2 = 1\}. \tag{3.12}$$

Define for any given even site $i \in \Lambda_e$

$$f^1_i = \psi_i, \quad f^2_i = \psi_{i+1},$$

$$A^1_i = A_{oi}, \quad A^2_i = A_{o(i+1)},$$

$$M^1_i = W_{i(i+1)}, \quad M^2_i = W_{(i+1)(i+2)}.$$
\[ Q_i^1 = W_{i(i+2)}, \quad Q_i^2 = W_{(i+1)(i+3)}, \]
\[ U_i^1 = U_i, \quad U_i^2 = U_{i+1}. \]  
(3.13)

The Lagrangian of Eq. (2.10) is now given by
\[
L = \sum_{i \in \Lambda_e} \left[ f_{it}^{1\dagger} (i \partial_t - A_{it}^1) f_{it}^1 + f_{it}^{2\dagger} (i \partial_t - A_{it}^2) f_{it}^2 \right] 
- \frac{J_1}{4} \sum_{i \in \Lambda_e} \left[ \frac{1}{2} \text{tr} \left( M_{it}^1 M_{it}^{1\dagger} \right) + \frac{1}{2} \text{tr} \left( M_{it}^2 M_{it}^{2\dagger} \right) \right] 
- \frac{J_1}{4} \sum_{i \in \Lambda_e} \left( f_{it}^{1\dagger} M_{it}^1 f_{it}^1 + f_{it}^{2\dagger} M_{it}^2 f_{it}^1 \right) H.c. 
- \frac{J_2}{4} \sum_{i \in \Lambda_e} \left[ \frac{1}{2} \text{tr} \left( Q_{it}^1 Q_{it}^{1\dagger} \right) + \frac{1}{2} \text{tr} \left( Q_{it}^2 Q_{it}^{2\dagger} \right) \right] 
- \frac{J_2}{4} \sum_{i \in \Lambda_e} \left( f_{it}^{1\dagger} Q_{it}^1 f_{it}^1 + f_{it}^{2\dagger} Q_{it}^2 f_{it}^1 \right) H.c. \]  
(3.14)

Consider now the gauge transformation \((\sigma^0)\) is the two by two unit matrix\)
\[
\psi_i \rightarrow U_i \psi_i, \quad \forall i \in \Lambda, \]  
(3.15)

where
\[
U_i = (i)^{-i} \begin{cases} 
\sigma^3 & \text{if } i \in \Lambda_e, \\
\sigma^0 & \text{if } i \in \Lambda_o. 
\end{cases} \]  
(3.16)

Under this gauge transformation the mean-field Ansatz Eq. (3.1) becomes \((i \mod 4 = 0)\)
\[
\bar{A}_i^1 \rightarrow -\frac{1}{2} a_{\delta}^1 \sigma^1, \\
\bar{A}_i^2 \rightarrow +\frac{1}{2} a_{\delta}^1 \sigma^1, \\
\bar{M}_i^1 \rightarrow -iX \sigma^0, \\
\bar{M}_i^2 \rightarrow -iX \sigma^0, \\
\bar{Q}_i^1 \rightarrow (-\text{Re } E \sigma^1 - \text{Im } E \sigma^2), \\
\bar{Q}_i^2 \rightarrow (+\text{Re } E \sigma^1 + \text{Im } E \sigma^2), \]  
(3.17)

In this gauge, the average \(\frac{1}{2}(\bar{A}_i^1 + \bar{A}_i^2)\) vanishes while the difference \(\frac{1}{2}(\bar{A}_i^1 - \bar{A}_i^2)\) does not.
Similarly, the average \(\frac{1}{2}(\bar{Q}_i^1 + \bar{Q}_i^2)\) vanishes while the difference \(\frac{1}{2}(\bar{Q}_i^1 - \bar{Q}_i^2)\) does not.
To obtain a naive continuum limit, we define first

\[ v_F = \frac{J_1X\bar{\epsilon}}{4}, \quad \bar{\epsilon} = 2\epsilon, \quad X = \frac{2}{\pi}, \tag{3.18} \]

where \( \epsilon (\bar{\epsilon}) \) is the lattice spacing on \( \Lambda (\Lambda_e) \). The sum over the lattice points \( i \in \Lambda_e \) is approximated by an integral:

\[ \sum_{i \in \Lambda_e} \bar{\epsilon} \rightarrow \int dx. \tag{3.19} \]

The second step consists in identifying the slow or smooth variables. For example, the fermionic site variable \( f^\alpha_i \) becomes a field \( u_x^\alpha \) which is smooth over the scale \( \bar{\epsilon} \), i.e., one assumes that

\[ f^\alpha_{(i+2)a} = \sqrt{\epsilon} \left[ u_{ax}^\alpha + \bar{\epsilon} \partial_x u_{ax}^\alpha + \mathcal{O}(\epsilon^2) \right]. \tag{3.20} \]

The upper index \( \alpha = 1, 2 \) will be associated below with the components of a Dirac spinor in two space-time dimensions. The lower index \( a = 1, 2 \) is the SU(2) color index. Finally, \( x \) labels the continuous spatial co-ordinate. In the sector implementing the local constraint of single occupancy, one assumes the smooth variables to be

\[ A^1_i = v_F (A^0_x + \bar{\phi}^0 + \phi^0_x), \tag{3.21} \]
\[ A^2_i = v_F (A^0_x - \bar{\phi}^0 - \phi^0_x). \tag{3.22} \]

In the presence of frustration \( J_2/J_1 \), the uniform fluctuating field \( A^0_x \) does not pick up an expectation value in our mean-field theory in contrast to the staggered fluctuation \( \phi^0_x \). The use of the upper index is motivated below where we show that \( A^0_x \) can be interpreted as the scalar component of a non-Abelian gauge field. The vector component of the non-Abelian gauge field comes from the uniform fluctuations of the nearest-neighbor links provided one uses the non-linear parametrization

\[ M^1_i = -iX [1 + \epsilon (\bar{\rho}_x + \rho_x)] e^{-i(A^0_x + \phi^0_x)}, \tag{3.23} \]
\[ M^2_i = -iX [1 + \epsilon (\bar{\rho}_x - \rho_x)] e^{-i(A^0_x - \phi^0_x)}. \tag{3.24} \]
Besides the uniform and staggered su(2) fluctuating fields $A^1_x$ and $\phi^1_x$, respectively, there are uniform and staggered determinant fluctuations $\rho_x$ and $\rho_x$, respectively. Finally, on the next-nearest neighbor links, we will distinguish between the color singlet fluctuations $r^\alpha_x$, $\alpha = 1, 2$, from the su(2) fluctuation $R^\alpha_x$, $\alpha = 1, 2$, by choosing the linear parametrization

$$Q^1_i = v_F \left[ -i r^1_x \sigma^0 + \bar{R}^1_x + \bar{R}^1_x \right], \quad (3.25)$$

$$Q^2_i = v_F \left[ -i r^2_x \sigma^0 + \bar{R}^2_x + \bar{R}^2_x \right]. \quad (3.26)$$

Only $R^\alpha_x = \bar{R}^\alpha_x + \bar{R}^\alpha_x$ picks up an expectation value $\bar{R}^\alpha$ in the presence of frustration at the mean-field level. Separating the $r$’s from the $R$’s parallels separating the determinant fluctuations from the su(2) fluctuations on the nearest-neighbor links. Integration over the $r$’s and $\rho$’s will turn out to generate a crucial interaction which induces Umklapp processes [5], without which our continuum theory cannot capture the departure from criticality (dimerization) as the frustration reaches a critical value [18].

We also need to infer the gauge transformation law of the fluctuating fields. We restrict ourself to SU(2) gauge transformation which are uniform within the unit cell labelled by even sites:

$$U^1_{it} = U^2_{it} \equiv G_{xt}, \quad \forall i \in \Lambda_e. \quad (3.27)$$

To the same order in the lattice spacing, the gauge transformation law for the fluctuating fields is

$$A_\mu \to G A_\mu G^{-1} + (i \partial_\mu G)G^{-1}, \quad \mu = 0, 1,$$

$$\phi_\mu \to G \phi_\mu G^{-1} - (1 - \delta_{0\mu}) (i \partial_\mu G)G^{-1}, \quad \mu = 0, 1,$$

$$\bar{R}^\alpha \to G \bar{R}^\alpha G^{-1}, \quad \alpha = 1, 2,$$

$$\rho \to \rho, \quad \rho \to \rho,$$

$$r^\alpha \to r^\alpha, \quad \alpha = 1, 2. \quad (3.28)$$

Collecting terms of lowest order in $\bar{\epsilon}$ yields the effective Lagrangian density.
\[ \mathcal{L} = \sum_{\alpha=1,2} v_F \left[ \left( \frac{\partial_{\perp}}{v_F} - A^0 \right) u^\alpha + u^{\alpha\dagger}(-1)^\alpha \left( \bar{\phi}^0 + \phi^0 \right) u^\alpha \right] \]

\[ - v_F \left[ \frac{X}{2\epsilon^2} + \frac{X}{\epsilon} g + \frac{X}{2} \left( g^2 + \rho^2 \right) \right] \]

\[ + v_F \left[ u^{\dagger\dagger} (i\partial_t + A^1) u^2 + u^{\dagger\dagger} (i\partial_t + A^1) u^1 + u^{\dagger\dagger} \bar{\rho} u^2 - u^{\dagger\dagger} \rho u^1 \right] \]

\[ - v_F \frac{J_2 J_1 X}{16} \sum_{\alpha=1,2} \frac{1}{2} \text{tr} \left[ R^\alpha R^{\alpha\dagger} \right] - v_F \frac{J_2}{4} \left[ u^{\dagger\dagger} \left( R^1 + R^{1\dagger} \right) u^1 + u^{\dagger\dagger} \left( R^2 + R^{2\dagger} \right) u^2 \right] \]

\[ - v_F \frac{J_2 J_1 X}{16} \sum_{\alpha=1,2} (r^\alpha)^2 + v_F \frac{J_2}{4} \left[ u^{\dagger\dagger}_x i r^1_x u^1_{x+\epsilon} + u^{\dagger\dagger}_x i r^2_x u^2_{x+\epsilon} + \text{H.c.} \right]. \] (3.29)

We have displayed the fields spatial dependency whenever necessary to indicate that the limiting procedure \( \bar{\epsilon} \to 0 \) should be treated with special care. Indeed, some terms appear to vanish due to the Pauli principle if the continuum limit is taken without caution. Notice that the uniform determinant fluctuation \( \rho \) and the su(2) staggered fluctuation \( \phi_1 \) of \( M_1 \) and \( M_2 \) have decoupled from the other dynamical degrees of freedom to lowest order in \( \bar{\epsilon} \).

We will ignore them completely from now on. We will also ignore the irrelevant additive constants in Eq. (3.29).

In view of the linearity of the mean-field spectrum in the limit \( J_1 J_2 = 0 \), we should be able, in this limit, to recast Eq. (3.29) in the form of a relativistic theory in two space-time dimensions. To stress this point, we introduce a new set of Pauli matrices \( \tau \) and define the gamma matrices

\[ \gamma^0 = \tau^2, \quad \gamma^1 = -i \tau^3, \quad \gamma^5 = \tau^1, \] (3.30)

which satisfy the usual algebra

\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}, \quad g^{00} = -g^{11} = 1, \quad g^{01} = +g^{10} = 0. \] (3.31)

We also use the notation

\[ D_\mu = \partial_\mu + iA_\mu, \quad \partial_\mu \equiv \left( \frac{1}{v_F} \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) \equiv (\partial_0, \partial_1) \] (3.32)

for the covariant derivative. We can then rewrite Eq. (3.29) as

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_2^\epsilon. \] (3.33)
We have separated the contributions $\mathcal{L}_0$ and $\mathcal{L}_1$ which are invariant under proper Lorentz transformation in two space-time dimensions and are given by

$$
\mathcal{L}_0 = v_F \bar{u} \, i \gamma^\mu D_\mu \, u,
$$

and

$$
\mathcal{L}_1 = -v_F \left( \bar{u} \, i \gamma^5 \phi_0 \, u + \bar{u} \, u \, \rho + \frac{X}{2} \rho^2 \right),
$$

respectively, from the contribution

$$
\mathcal{L}_2 = -v_F \bar{u} \, i \gamma^5 \bar{\phi}_0 \, u
$$

$$
- v_F \frac{J_2 J_1 X}{16} \sum_{a=1,2} \frac{1}{2} \text{tr} \left[ R^a R^{a\dagger} \right]
$$

$$
- v_F \frac{J_2}{4} \bar{u} \, \gamma^0 \left( \frac{R^1 + R^{1\dagger} + R^2 + R^{2\dagger}}{2} \right) u
$$

$$
- v_F \frac{J_2}{4} \bar{u} \, i \gamma^5 \left( \frac{R^1 + R^{1\dagger} - R^2 - R^{2\dagger}}{2} \right) u,
$$

and the contribution

$$
\mathcal{L}^\epsilon_2 = -v_F \frac{J_2 J_1 X}{16} \sum_{a=1,2} (r^a)^2
$$

$$
+ iv_F \frac{J_2}{4} \sum_{a=1,2} \left( u_x^a \bar{u}_{x+\epsilon}^a \, r_x^a - \text{H.c.} \right),
$$

which both describe the interactions due to the frustration $\frac{J_2}{J_1}$. In the absence of fluctuations, the low energy sector of the mean-field theory is correctly described by

$$
\tilde{\mathcal{L}} = v_F \left[ \bar{u} \, i \gamma^\mu \partial_\mu \, u + \bar{u} \, i \gamma^5 \left( \frac{a_0^1}{2} + \frac{J_2}{2} \text{Re} \, E \right) \sigma^1 \, u \right].
$$

The contribution Eq. (3.36) to the low energy effective theory does not possess a full relativistic invariance. On the other hand, we know that the Heisenberg model in the limit of small frustration $\frac{J_2}{J_1}$ is in the quantum critical regime and therefore should have its low energy sector described by a field theory characterized by a gapless spectrum and relativistic invariance [18, 22]. That our field theory fails to do so on both account is an artifact of the mean-field theory. We are now going to show that the quantum theory constructed from
Eq. (3.33) in the limit of sufficiently small frustration is equivalent to a relativistic quantum field theory with no mass gap. In other words, we are first going to show that the quantum fluctuations restore quantum criticality by removing any mean-field gap associated to the condensation of \( \phi_0 \) and the \( R \)'s. We then show that the spinon interactions induced by the fluctuations \( \rho \) and \( r^\alpha \) are irrelevant for small enough frustration \( \frac{J_2}{J_1} \).

C. Quantum fluctuations around the mean-field Ansatz

The quantum theory for the fluctuations around the mean-field Ansatz is constructed from the partition function

\[
Z = \int \mathcal{D}\mu_b \int \mathcal{D}\mu_f \, e^{+i \int dt \mathcal{L}},
\]

where the bosonic and fermionic integration measures are

\[
\mathcal{D}\mu_b = \mathcal{D}[r^1, r^2] \, \mathcal{D}[R^1, R^2] \, \mathcal{D}[\phi_0] \, \mathcal{D}[A_\mu] \frac{\mathcal{D}[\rho]}{V^{-1}} \mathcal{D}[\rho],
\]

\[
\mathcal{D}\mu_f = \mathcal{D}[\bar{u}, u],
\]

respectively, and \( \mathcal{L} \) is the Lagrangian density of Eq. (3.33). The measure for the fields \( \phi_0 \) and \( A_\mu \) is the measure for the Lie algebra \( \text{su}(2) \). The measure for the fields \( r^{1(2)} \) and \( R^{1(2)} \) is the product of the measure for real scalar fields with the \( \text{su}(2) \) measure. Finally, the measure for \( \rho \) is the measure for real scalar fields. The factor \( V^{-1} \) serves to remind us that the gauge for the \( A_\mu \) has to be fixed.

We now show the important result that the fields \( R^{1(2)} \) decouple from all other fields in the partition function Eq. (3.39). The \( R \)'s belong to the Lie algebra \( \text{su}(2) \) by construction. Consequently, the Hermitean linear combinations

\[
B_+ \equiv R^1 + R^{1\dagger} + R^2 + R^{2\dagger} = B_+^\dagger,
\]

\[
B_- \equiv R^1 + R^{1\dagger} - R^2 - R^{2\dagger} = B_+^\dagger,
\]

also belong to the Lie algebra \( \text{su}(2) \). If we choose to integrate over the fields \( A_0 \) and \( \phi_0 \) before integrating over the \( R \)'s, then
\[ \mathcal{A}_0 \rightarrow \mathcal{A}_0 - \frac{J_2}{8} B_+, \quad \phi_0 \rightarrow \phi_0 - \tilde{\phi}_0 - \frac{J_2}{8} B_- , \quad (3.42) \]

is a well defined shift of integration variable which decouples the \( R \)'s from the other fields and eliminate any mean-field gap.

Quantum fluctuations of the uniform and staggered components of the Lagrange multipliers which enforce the constraint of single occupancy, thus remove the mean-field gap due to the condensation of \( \phi_0 \) and the \( R \)'s for any amount of frustration \( \frac{J_2}{J_1} \) and restore the full relativistic invariance of \( \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \). We want to see how these and the remaining quantum fluctuations of \( \rho \) and \( \mathcal{A}_1 \) affect physical observables like the spin-spin correlation functions. We will treat the interaction \( \mathcal{L}_2^\xi \) separately. To this end and without loss of generality, we will only include the dynamics contained in the partition function

\[ Z = \int \mathcal{D}[\phi_0] \int \mathcal{D}[\mathcal{A}_\mu] \int \mathcal{D}[\bar{u}, u] \ e^{+i \int dt \left( \mathcal{L}_0 + \mathcal{L}_1' \right)} , \quad (3.43) \]

\[ \mathcal{L}_0 = \bar{u} \ i \gamma^\mu \left( \partial_\mu + i \mathcal{A}_\mu \right) u , \quad (3.44) \]

\[ \mathcal{L}_1' = -\bar{u} \ i \gamma^5 \phi_0 u + \frac{1}{2 \piX} (\bar{u} u)^2 . \quad (3.45) \]

We have set \( \nu_F = 1 \) and performed the Gaussian integration over \( \rho \). The mean-field parameter \( X \) is the solution of the saddle-point equations in the limit \( \frac{J_2}{J_1} = 0 \).

If we neglect the quantum fluctuations of the bosonic fields, it appears that the scaling dimensions of physical observables could depend continuously on the value of \( X \) as the action of Eq. (3.43) then reduces to a variant of the Thirring model [31] in the presence of background fields. This will turn out not to be the case as we show in the next section.

**IV. NON-ABELIAN BOSONIZATION**

We have constructed a theory for the quantum fluctuations around the mean-field Ansatz of Eq. (3.1) given by Eq. (3.43). The action in Eq. (3.43) resembles the action for the SU(2) Thirring model except for the presence of the gauge fields. It is known that the four fermion interaction is a marginal operator which changes the anomalous dimensions of the fermionic
correlation functions in a *pure* fermionic theory \[^{[32]}\]. Such an interaction is crucial to derive the correct anomalous dimensions of the staggered magnetization in the Jordan-Wigner representation of the Heisenberg chain for spin-$\frac{1}{2}$. However, in our case this interaction has highly undesirable consequences since the change in the anomalous dimensions is a function of the mean-field parameter $X$, and not of a physical parameter like the anisotropy in the Jordan-Wigner approach.

In this section, we *first* consider the quantum theory with action $\mathcal{L}_0$. We show how the quantum gauge fluctuations can be treated *exactly* using *non-Abelian bosonization*. As a by product, the physical states can be constructed explicitly and the anomalous dimensions of the uniform and staggered magnetizations extracted from the quantum theory with the action $\mathcal{L}_0$ are the correct one. We then show that the quantum theory with action $\mathcal{L}_0$ is stable with respect to the perturbation $\mathcal{L}_1'$ in the sense that the additional interactions are irrelevant *and* do not modify the values of the anomalous dimensions for the uniform and staggered magnetizations.

**A. Non-Abelian bosonization of the quantum critical theory**

We want to construct the physical states and calculate the anomalous dimensions of the uniform and staggered magnetizations from

$$Z_0 = \int \mathcal{D}[A_\mu] \int \mathcal{D}[\bar{u}, u] \ e^{+i \int dt \ \mathcal{L}_0}, \quad (4.1)$$

$$\mathcal{L}_0 = \bar{u} i \gamma^\mu \left( \partial_\mu + i A_\mu \right) u. \quad (4.2)$$

The partition function $Z_0$ describes the critical theory in the limit of small frustration and in the absence of the perturbation $\mathcal{L}_1'$. It is equivalent to the partition function for the infinitely strong coupling limit of $QCD_2$ with $su(2)$ color gauge fields. The role of the $su(2)$ gauge fields $A_\mu$ is to project the spinon Fock space onto the subspace of color singlet states which is defined by the constraint

$$\bar{J}_\mu \equiv \bar{u} \gamma^\mu \frac{\bar{\sigma}}{2} u = 0. \quad (4.3)$$
The effect of the perturbations due to the fluctuations of the staggered gauge field $\phi_0$ and of the current-current interaction will be studied in the next subsection.

Our strategy is to establish an equivalence between the partition function Eq. (4.1) and the partition function of a non-interacting theory to be described below which is explicitly conformally invariant [33–35]. We can then borrow general results from two-dimensional conformal field theory to show explicitly how all the single particle mean-field excitations are removed from the singlet sector of the Fock space. Our conformal field theory turns out to be a special example of a coset model (specifically a conformal field theory on the homogeneous space $U(2)/SU(2)$ [36,33]). This allows us to construct the physical states from the energy-momentum tensor of $Z_0$. Finally, the correct anomalous dimensions for the uniform and staggered magnetizations are recovered. This last result makes explicit the SU(2) spin dynamical symmetry which is hidden in $Z_0$.

The details of the construction of the equivalent theory can be found in appendix B. The idea is to use the vector and axial symmetry of the Lagrangian density Eq. (4.2) together with the property (unique to two space-time dimensions) that $\gamma^\mu \gamma^5 = \epsilon^{\mu\nu} \gamma_\nu$, to decouple the gauge fields from the spinons in the Lagrangian density. The resulting action $S_1$ describes free Dirac spinons. However, because $Z_0$ in Eq. (4.1) only shares the vector gauge invariance of the Lagrangian density, a non-trivial change in the fermionic measure under the transformation decoupling the spinons from the gauge fields induces a Wess-Zumino-Witten contribution $S_2$ [37,38]. The Wess-Zumino-Witten field depends only on the gauge fields and its kinetic energy is negative definite. Karabali and Schnitzer [33] have shown that the theory of free Dirac spinons and a Wess-Zumino-Witten action with negative definite kinetic energy is well defined provided a contribution $S_3$ needed to fix the gauge is also accounted for. We will see that the role of the sectors $S_2$ and $S_3$ associated to the gauge fields is to remove unphysical states from the spinon Fock space. To put it differently, the gauge fields implement, up to some scale defined by the effective low energy theory $Z_0$, the Gutzwiller projection necessary for the equivalence between the parent lattice theory for the spinons Eq. (2.8) and the Heisenberg model.
The quantum critical theory Eq. (4.1) is equivalent to the *conformally invariant* theory

\[ Z_0 = \int D\mu_b \int D\mu_f e^{+i(S_1 + S_2 + S_3)}, \quad (4.4) \]

\[ S_1 = \int \frac{dx^+dx^-}{2} (u^+_+ i\partial_- u^+_- + u^-_+ i\partial_+ u^-_-), \quad (4.5) \]

\[ S_2 = -(1 + 2c_v) W_- [\tilde{G}_-], \quad (4.6) \]

\[ S_3 = \int \frac{dx^+dx^-}{2} \left[ \text{tr} \left( \beta^a_+ \sigma^a \textbf{i} \partial_- \sigma^b \alpha^b_+ \right) + \text{tr} \left( \beta^a_- \sigma^a \textbf{i} \partial_+ \sigma^b \alpha^b_- \right) \right]. \quad (4.7) \]

The measures are defined in Eqs. (B32) and (B33) and the relationships between the fields in Eq. (4.4) and the Dirac spinons and gauge fields of Eq. (4.1) are given by Eqs. (B35), (B36), (B38). Eq. (4.4) describes three sectors: the free spinon sector \((S_1)\), the Wess-Zumino-Witten sector \((S_2)\) with *negative level* \(k = -(1 + 2 \times 2) = -5\), and the ghost sector \((S_3)\). The three sectors are individually conformally invariant and do not interact with each other. Hence, we can construct the entire Fock space by taking the tensor product of the eigenstates of each individual sectors. However, not all states constructed in this way are physical due to the existence of states with negative definite norm coming from the Wess-Zumino-Witten and ghost sectors or, equivalently, due to the constraint Eq. (4.3).

We begin with the counting of the physical degrees of freedom in the partition function Eq. (4.4). The action of Eq. (4.4) has a traceless (conformal invariance) energy-momentum tensor. Its light-cone components \(T^0_{\pm}\) are the sum of pairwise commuting energy-momentum tensors corresponding to the spinon, gauge, and ghost sectors, respectively:

\[ T^0_{\pm} = T^1_{\pm} + T^2_{\pm} + T^3_{\pm}. \quad (4.8) \]

The algebras obeyed by the two copies \(T^0_n, n = 0, 1, 2, 3\) are the Virasoro algebras with central charges \(C_n, n = 0, 1, 2, 3\). The Virasoro central charges (which count the degrees of freedom) are related by

\[ C_0 = C_1 + C_2 + C_3. \quad (4.9) \]

Since \(S_1\) describes 2 free Dirac fermions, one has \(C_1 = 2\). According to Knizhnik and Zamolodchikov [40], the Virasoro central charge associated to a Wess-Zumino-Witten action of level \(k\) is
Thus, in our case, $C_2 = 5$. Finally, the Virasoro central charge for the ghost sector is negative and given by

$$C_3 = -2 \dim SU(2) = -6.$$  

(4.11)

The central charge $C_0$ for the critical theory Eq. (4.1) is therefore

$$C_0 = 2 + 5 - 6 = 1.$$  

(4.12)

Since the central charge $C_0$ counts the number of physical degrees of freedom, we see that the gauge and ghost sectors reduce the number $C_1 = 2$ of mean-field degrees of freedom.

To stress this point more strongly, and to draw a connection with coset conformal theories which will be very useful to us, one rewrites the central charge as

$$C_0 = 2 - \frac{1}{1 + 2(2^2 - 1)} + 0.$$  

(4.13)

The motivation for this arithmetic game is that the energy-momentum tensors of a large class of quantum field theories can be constructed from currents obeying Kac-Moody algebras. For example, consider the two dimensional quantum currents with the + chiral component

$$j^a_+ \equiv \frac{j^a_0 + j^a_1}{2}, \quad a = 1, \cdots, n,$$  

(4.14)

obeying the equal-time algebra

$$[j^a_+(x), j^b_+(y)] = i f^{abc} j^c_+(x) \delta(x - y) + k \frac{i}{2\pi} \delta^{ab} \delta'(x - y).$$  

(4.15)

The numbers $f^{abc}$ are the structure constants of SU($n$). The term which is proportional to the spatial derivative of the delta function is called the Schwinger term. It arises from the quantum nature of the currents and is multiplied by the integer number $k$. The algebra defined by Eq. (4.15) is called a Kac-Moody algebra of level $k$. It was shown long time ago how to interpret the right-hand side of
in order for \( T_+ \) to describe the energy-momentum tensor of a local quantum-field theory (see appendix C for details). Here, \( c_v \delta^{aad'} = f^{abc} f^{a'd'b} \) is the quadratic Casimir invariant in the adjoint representation of SU(\( n \)). Assume now that to each central charge on the right-hand side of Eq. (4.13), there corresponds an energy-momentum tensor which can be built from appropriate currents like in Eq. (4.16). The first number on the right-hand side would coincide with the central charge of the energy-momentum tensor \( T_{\pm}^{U(2)} \) which is built from currents obeying a U(2) Kac-moody algebra of level 1. The second (negative) number on the right-hand side would coincide with the central charge of the energy-momentum tensor \( T_{\pm}^{SU(2)} \) which is built from currents obeying a SU(2) Kac-moody algebra of level 1. The zero is meant to remind us that energy-momentum tensors can have vanishing central charges and we allow for this possibility by denoting with \( T'_{\pm} \) the corresponding energy-momentum tensor. Eq. (4.13) is then very suggestive of the decomposition

\[
T_{\pm}^0 = T_{\pm}^{U(2)} - T_{\pm}^{SU(2)} + T'_{\pm} \approx T_{\pm}^{U(2)/SU(2)} + T'_{\pm}. \tag{4.17}
\]

In our problem, we have the U(1) color singlet current Eq. (B16) and the SU(2) color current Eq. (B17) at disposal. We show in appendix C that the color singlet current obeys an Abelian Kac-Moody algebra for two fermion species and that the color SU(2) current obeys a level one SU(2) Kac-Moody algebra so that \( T_{\pm}^1 \) can be identified with \( T_{\pm}^{U(2)} \) and \( T_{\pm}^{SU(2)} \) can indeed be constructed. The construction of \( T'_{\pm} \) has been done by Karabali and Schnitzer and it involves the color and ghost currents. They have investigated the nature of the second “equality” sign of Eq. (4.17). The issue is delicate since the theory defined by \( T_{\pm}^0 \) does not have a positive definite metric. It is sufficient here to interpret Eq. (4.17) as the removal of all unphysical states induced by the spinon representation. The physical Hilbert space is to be constructed from the unitary representation of the Kac-Moody algebra and associated Virasoro algebra generated by the color singlet currents Eq. (B16). Details can be found in the work of Karabali and Schnitzer but, for our purpose, the important point
is that the states of the physical Hilbert space defined by $Z_0$ are (vector) gauge singlets. In other words, the mean-field one-particle excitations of Eq. (3.38) have been entirely projected out of the physical spectrum.

The conformal invariance of Eq. (4.4) implies that for each components of the uniform and staggered magnetizations there exists two conformal weights [41]. To see this, recall that pairs of conformal weights determine the transformation law of the primary fields under conformal transformations. From the knowledge of the conformal weights one easily extracts the scaling dimensions (anomalous dimensions) of the primary fields. So, if we can show that the magnetizations are products of one primary field from each different sectors, and hence primary fields themselves, one can calculate their conformal weights and their scaling dimensions. The primary fields of Eq. (4.4) are the Dirac spinons $u'$, the Wess-Zumino-Witten fields $\tilde{G}_-$, and the ghosts $\beta',\alpha'$. By inspection of Eq. (A5)

$$\vec{M}_{+i} = \frac{1}{2} \left( \vec{S}_i + \vec{S}_{i+1} \right) \propto \frac{1}{2} \left( \vec{J}_+ + \vec{J}_- \right),$$

(4.18)

for the uniform magnetization $\vec{M}_{+i}$ and by inspection of Eq. (A7)

$$\vec{M}_{-i} = \frac{1}{2} \left( \vec{S}_i - \vec{S}_{i+1} \right) \propto \frac{1}{2} \left( \vec{K}_+ - \vec{K}_- \right),$$

(4.19)

for the staggered magnetization $\vec{M}_{-i}$, it is apparent that the uniform magnetization is invariant under all chiral transformations of Eq. (B14,B15) while the staggered magnetization is only invariant under the diagonal subgroup of vector gauge transformations. Hence, the sequence of chiral transformations which decouples the spinons from the gauge fields results in

$$\vec{J}_\pm = \frac{1}{2} \left( \begin{array}{c} -u_{\pm 1}^t u_{\pm 2}^t + u_{\pm 2}^t u_{\pm 1}^t \\ +i(u_{\pm 1}^t u_{\pm 2}^t - u_{\pm 2}^t u_{\pm 1}^t) \\ +(u_{\pm 1}^t u_{\pm 2}^t + 2u_{\pm 2}^t u_{\pm 2}^t) \end{array} \right),$$

(4.20)

and

$$\vec{K}_{++} = \frac{1}{2} \left( \begin{array}{c} -(u_{\pm b}^t \tilde{G}_{-1 b}^* u_{\pm 2}^t + u_{\pm 2}^t u_{\pm b}^t \tilde{G}_{-1 b}^*) \\ +i(u_{\pm b}^t \tilde{G}_{-1 b}^* u_{\pm 2}^t - u_{\pm 2}^t u_{\pm b}^t \tilde{G}_{-1 b}^*) \\ +(u_{\pm b}^t \tilde{G}_{-1 b}^* u_{\pm 1}^t + u_{\pm 2}^t \tilde{G}_{-2 b}^* u_{\pm 2}^t) \end{array} \right).$$

(4.21)
The magnetizations are now solely expressed in terms of the primary fields \( u' \) and \( \tilde{G}_- \) of the conformal field theory Eq. (4.4) \[35\]. One can therefore associate two conformal weights to each component of the magnetizations. We use a vector notation for the conformal weights: \( (\Delta_+, \Delta_-) [\tilde{J} (\vec{K})] \). In turn, the scaling dimensions of the magnetizations are the sum of the conformal weights in the + and − chiral sectors:

\[
\Delta [\tilde{J} (\vec{K})] = \Delta_+ [\tilde{J} (\vec{K})] + \Delta_- [\tilde{J} (\vec{K})].
\] (4.22)

Using Eq. (B41) of appendix \[3\], we find that the mean-field prediction for the scaling dimensions of the uniform magnetization is unchanged by the quantum fluctuations of the gauge fields and are

\[
\Delta (\vec{J}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\] (4.23)

The mean-field prediction for the scaling dimensions of the staggered magnetization are, however, changed by the quantum fluctuations of the gauge fields. The gauge fields fluctuations effectively reduce the mean-field predictions for the scaling dimensions to yield

\[
\Delta (\vec{K}) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.
\] (4.24)

According to Eq. (4.23) (4.24), all three components of the vector for the scaling dimensions of the uniform (staggered) magnetization are the same as required by the spin SU(2) invariance. They, moreover, agree with the scaling dimensions derived from the Jordan-Wigner representation of the Heisenberg chain \[32\]. The isotropy is not surprising in view of the fact that our mean-field Ansatz respects the spin symmetry. However, the mean-field theory overestimates the scaling dimension of the staggered magnetization. The quantum gauge fluctuations are needed to restore the correct scaling dimension of the staggered magnetization.

The scaling dimension of the uniform magnetization is left unchanged by the quantum gauge fluctuations as the uniform magnetization generates a continuous symmetry, namely
the spin SU(2) symmetry. It is tempting to identify the energy-momentum tensor $T^{U(2)/SU(2)}$ in Eq. (4.17) with the energy-momentum tensor constructed from the currents $\vec{J}_\pm$ through the Sugawara construction [42]. This is done explicitly in appendix C where we also show that the currents $\vec{J}_\pm$ satisfy a level one Kac-Moody algebra. Hence, the spin SU(2) symmetry of the Heisenberg chain appears as a dynamical symmetry of the critical theory Eq. (4.1) and we recover the results of Affleck and Haldane on a (non-Abelian) bosonization scheme of the Heisenberg chain which makes explicit the spin symmetry [22].

Having shown how the quantum gauge fluctuations restore quantum criticality and the correct scaling dimensions of the mean-field theory for a one dimensional spin liquid, we now turn to the effect of the perturbation $L'_1$ on the conformal field theory Eq. (4.4). We will show in the next subsection that the constraints imposed by the quantum fluctuations of the gauge fields and of $\phi_0$ guaranty all the results derived thus far even in the presence of the perturbation $L'_1$.

### B. Perturbations of the critical theory

We want to see how the staggered fluctuations of the Lagrange multipliers enforcing the constraint of one spinon per site and the staggered fluctuations of the determinant on the nearest-neighbor links modify our previous results. We are going to show that the scaling dimensions are still given by the scaling dimensions of the unperturbed conformal field theory Eq. (4.4) if and only if the fluctuations of $\rho$ are always treated together with the fluctuations of $\phi_0$.

We consider the partition function $Z$ given by Eq. (3.43). The perturbation $L'_1$ has two effects. First, in addition to the constraints

$$\bar{J}^\mu = \bar{u} \gamma^\mu \frac{\bar{u}}{2} u = 0, \quad (4.25)$$

there are the constraints

$$\bar{u} i \gamma_5 \bar{\sigma} u = 0, \quad (4.26)$$

27
due to the quantum fluctuations of the staggered gauge fields $\phi_0$. Second, there is a quartic
spinon interaction $(\bar{u}u)^2$ which can be rewritten

$$(\bar{u}u)^2 = \frac{1}{3} (\bar{u} i\gamma_5 \vec{\sigma} u)^2 - \frac{4}{3} \vec{J}^\mu \cdot \vec{J}_\mu.$$  

(4.27)

This quartic interaction is caused by staggered fluctuations of the determinant of the nearest-
neighbor link $W_{i(i+1)}$ variable. Alone, it would change the scaling dimensions of the staggered
magnetization. However, the quartic interaction vanishes identically on the states annihi-
lated by the left-hand sides of Eq. (4.25) and (4.26).

To better understand the role of the constraints Eq. (4.26) recall that the Lagrangian
density $L_0$ of our critical theory is invariant under local chiral transformations Eq. (B14,B15)
and global chiral transformations Eq. (B12,B13). The perturbation $L'_1$ lowers the symmetry
of $L_0$ down to the vector subgroup together with the discrete subgroup

$$u \rightarrow -ie^{i\pi/2}\gamma_5 u.$$  

(4.28)

This is not surprising since the pure axial symmetry of $L_0$ has no counterpart in the rewriting
of the Heisenberg model as a lattice gauge theory. The discrete chiral symmetry simply
indicates that our mean-field Ansatz does not break the translational invariance by one
lattice spacing as would be the case for an Ansatz with long range antiferromagnetic order.
To see this last point, it is instructive to look at the origin of the constraints Eq. (4.26) and
Eq. (4.25) on the unit cell labelled by $i \in \Lambda_e$.

The Fock space spanned by the two spinons $s_{i\sigma}$ and $s_{(i+1)\sigma}$ where $\sigma = \uparrow, \downarrow$, is 16 dimen-
sional. It can be decomposed as the direct sum of the Hilbert spaces $\mathcal{H}^n$ with total spinon
occupation number $n$ ranging from 0 to 4. The physical subspace of the Fock space is the
four dimensional Hilbert space $\mathcal{H}_{phy}^2$ with one spinon per site. It is to be distinguished from
its complementary (with respect to $\mathcal{H}^2$) subspace $\mathcal{H}_{unphy}^2$ which has two spinons on either
one of the two sites.

There are two operators which will characterize uniquely the physical states of the Fock
space if one requires the physical states to be annihilated by these operators, namely
\[ \hat{O}_+^3 = (s_i^\dagger s_i - 1) + (s_{i+1}^\dagger s_{i+1} - 1), \quad (4.29) \]

and

\[ \hat{O}_-^3 = (s_i^\dagger s_i - 1) - (s_{i+1}^\dagger s_{i+1} - 1). \quad (4.30) \]

However, this choice is not unique since

\[ \hat{O}_+ = s_{i\uparrow}^\dagger s_{i\downarrow}^\dagger + s_{(i+1)\uparrow}^\dagger s_{(i+1)\downarrow}^\dagger, \quad (4.31) \]

and

\[ \hat{O}_- = s_{i\uparrow}^\dagger s_{i\downarrow}^\dagger - s_{(i+1)\uparrow}^\dagger s_{(i+1)\downarrow}^\dagger, \quad (4.32) \]

or \( \hat{O}_\pm \) perform the same task.

With the choice of Eq. (3.30) for the gamma matrices and by retracing all the steps relating the Dirac spinons to the original spinons of Eq. (2.2), one can relate the local constraints in the continuum to constraints on the states of the unit cell \( i \in \Lambda_e \). For example,

\[ \bar{u} \gamma^{13} \sigma u = u^\dagger \tau^1 \sigma u \]

\[ = + f^{1\dagger} \sigma^3 f^2 + f^{2\dagger} \sigma^3 f^1 \]

\[ \propto -i \left( \psi_i^\dagger \sigma^3 \sigma^3 \psi_{i+1} + \psi_{i+1}^\dagger \sigma^3 \sigma^3 \psi_i \right) \]

\[ \propto -i \left( \psi_i^\dagger \psi_{i+1} + \psi_{i+1}^\dagger \psi_i \right) \]

\[ = -i \left( s_{i\uparrow}^\dagger s_{(i+1)\uparrow}^\dagger + s_{i\downarrow}^\dagger s_{(i+1)\downarrow}^\dagger + \text{H.c.} \right), \quad (4.33) \]

tells us that the constraint on the space component of the color current \( \vec{J}_1 = 0 \) is related to spin currents in the unit cell \( i \in \Lambda_e \). Similarly,

\[ \bar{u} \gamma^0 \vec{\sigma} u \propto \left( - \left[ \hat{O}_- + \hat{O}_\uparrow^\dagger \right] \right) \]

\[ + i \left[ \hat{O}_- - \hat{O}_\uparrow^\dagger \right] \]

\[ \hat{O}_+^3 \]

and

29
\[ \bar{u} \gamma^1 \vec{\sigma} u \propto \begin{pmatrix} -[\hat{O}_+ + \hat{O}_+] \\ +i[\hat{O}_+ - \hat{O}_+] \\ \hat{O}_3 \end{pmatrix} \] (4.35)

relate the original constraints and the two time-like constraints of the continuum theory.

We see that the role of the staggered fluctuations \( \phi_0 \) is to insure that there are as many spinons on either basis of the unit cell \( i \in \Lambda_e \), whereas the role of the uniform scalar gauge fluctuation is to insure that there is an average of two spinons per unit cell. The lattice counterpart of the discrete symmetry Eq. (4.28) is simply the exchange

\[ c_{i\uparrow} \rightarrow +c_{(i+1)\uparrow}, \quad c_{i\downarrow} \rightarrow -c_{(i+1)\downarrow}. \] (4.36)

In other words, multiplication of \( u \) by \( \gamma^5 \) amounts to an interchange of the upper and lower components of \( u \), which on the lattice implies an interchange of even and odd sites. There, is an additional flipping of the spins, due to the particle-hole transformation Eq. (2.11) and a spin dependent sign change due to the gauge transformation Eq. (3.16).

In summary, our quantum critical theory correctly describes the low energy sector of the Heisenberg chain for small frustration although it only treats the constraint of spinon single occupancy on average over the unit cell \( \Lambda_e \). Beyond this microscopic scale, the constraint of single occupancy is exactly satisfied. By rewriting the quantum critical theory as a coset conformal theory, the dynamical spin \( SU(2) \) symmetry manifests itself explicitly and contact is made with the non-Abelian bosonization scheme of Affleck and Haldane for quantum spin chains [22]. All the one particle mean-field excitations, the spinons, have been projected out of the physical Hilbert space. The gapless modes carrying spin-\( \frac{1}{2} \) are topological excitations (solitons) which change the boundary conditions. In two space-time dimensions, it costs an infinite amount of energy to break the gauge singlet bound states of spinons carrying integer spin quantum number and deconfinement is not possible as a mechanism for spin and charge separation.
V. DIMERIZATION

In the previous two sections, we have shown that the quantum theory $\mathcal{L}_0 + \mathcal{L}_1' + \mathcal{L}_2$ in Eq. (3.33) is insensitive to any frustration $\frac{J_2}{J_1}$. However, we know from the exact ground state and the low lying excitations of the Heisenberg chain when $\frac{J_2}{J_1} = \frac{1}{2}$ that criticality cannot hold for all values of the frustration $\frac{J_2}{J_1}$ [17]. Haldane [18] has argued that criticality of the system at $\frac{J_2}{J_1} = 0$ subsists up to a critical value $(\frac{J_2}{J_1})_c = \frac{1}{6}$. The existence of a critical value for the frustration has been confirmed numerically [15]. Haldane’s analysis starts with the representation of the Heisenberg chain in terms of Jordan-Wigner fermions (solitons). He shows that, as the frustration $\frac{J_2}{J_1}$ is switched on, an Umklapp interaction which is marginally irrelevant initially becomes relevant for a finite value of the frustration and drives the system into a massive phase characterized by long range dimer order. First, we want to see if the perturbation $\mathcal{L}_2'$, Eq. (3.37), drives the system away from the level $k = 1$ Wess-Zumino-Witten fixed point and into a phase with dimer long range order. Second, we derive the critical theory in the limit $\frac{J_2}{J_1} = 0$ and investigate what are the perturbations induced by a small frustration $\frac{J_2}{J_1}$ which can drive the system towards dimerization. This issue is of relevance to the problem of two weakly coupled Heisenberg chains.

A. Relevant perturbation around $\frac{J_2}{J_1} = 0$

It is tempting, on the basis of Haldane’s argument, to believe that Umklapp processes for the spinons are responsible for dimerization. However, one needs to be careful with this analogy. Indeed, the relationship between Umklapp processes for the Jordan-Wigner fermions, which are the gauge invariant spin-$\frac{1}{2}$ gapless modes of topological character in the critical theory, and Umklapp processes for the spinons is not obvious.

Umklapp processes for the spinons are scattering events in which a pair of excitation close to the Fermi point $+k_F = +\frac{\pi}{2}$ is annihilated and a pair of excitations close to the Fermi point $-k_F$ is created or vice et versa. Moreover, all participants to this scattering
event have the *same* color quantum number. Such processes are consistent with momentum conservation at half-filling but explicitly *break* the chiral symmetry of \( \mathcal{L}_0 \), Eq. (3.34). For example, in terms of the chiral components of our Dirac spinors [see Eq. (1.8)], two possible Umklapp processes result from the interactions

\[
u_{-ax}^* \nu_{-a(x+\epsilon)} u_{+ax} u_{+a(x+\epsilon)}, \quad a = 1 \text{ or } 2.
\] (5.1)

Umklapp processes for the spinons are *consistent* with local vector gauge invariance since they all are induced by interactions of the form

\[
\left(u_{-ax}^* \delta_{ab} u_{+bx}\right) \left(u_{+c(x+\epsilon)}^* \delta_{cd} u_{+d(x+\epsilon)}\right),
\] (5.2)

\[
\left(u_{+ax}^* \delta_{ab} u_{-bx}\right) \left(u_{+c(x+\epsilon)}^* \delta_{cd} u_{-d(x+\epsilon)}\right),
\] (5.3)

or, equivalently, by \((\bar{u} u)^2\) and \((\bar{u} i \gamma^5 u)^2\). Notice that to obtain a representation of the Umklapp process in a quantum field theory, one needs to account for singularities associated with multiplication of quantum fields at the same point, e.g., with the procedure of point splitting given in appendix C.

The only gauge invariant fluctuating fields at our disposal are the nearest neighbor staggered determinant fluctuations \( \rho \) and the next-nearest neighbor fluctuations \( r^\alpha \), \( \alpha = 1, 2 \). Integrating over \( \rho \) and \( r^\alpha \) induces Umklapp interactions for the spinons with *coupling strength* \( v_F \frac{1}{2X} \), and \( v_F \frac{1}{2X} \times 4 \frac{J_2}{J_1} \), respectively. What about their relative sign?

Instead of determining the relative sign from the field theory, it is instructive to go back to the original pure spinon lattice theory, Eq. (2.8). Recall that the quartic interactions originate from

\[
\chi_{ij}^* = s_{i\uparrow}^* s_{j\uparrow} + s_{i\downarrow}^* s_{j\downarrow},
\] (5.4)

\[
\eta_{ij}^* = s_{i\uparrow}^* s_{j\downarrow} - s_{i\downarrow}^* s_{j\uparrow},
\] (5.5)

or equivalently from

\[
\chi_{ij}^* = \psi_{i1}^* \psi_{j1} - \psi_{j2}^* \psi_{i2},
\] (5.6)

\[
\eta_{ij}^* = \psi_{i1}^* \psi_{j2} + \psi_{j1}^* \psi_{i2},
\] (5.7)
where $\psi_{i1} = s_{i\uparrow}$, $\psi_{i2} = s_{i\downarrow}^*$. Umklapp processes originate solely from the Affleck-Marston order parameter $\chi_{ij}^*$ since

$$\chi_{ij}^* \chi_{ij} = \psi_{i1}^* \psi_{j1}^* \psi_{i1} \psi_{j1} + \psi_{i2}^* \psi_{j2}^* \psi_{i2} \psi_{j2} + \psi_{i1}^* \psi_{j2}^* \psi_{j1} \psi_{i2} + \psi_{i2}^* \psi_{j1}^* \psi_{i1} \psi_{j2},$$

(5.8)

whereas the color index of the annihilated pair of spinons always differ in

$$\eta_{ij}^* \eta_{ij} = \psi_{i1}^* \psi_{j2} \psi_{i1} \psi_{j2} + \psi_{i2}^* \psi_{j1} \psi_{i2} \psi_{j1} + \psi_{j1}^* \psi_{j2}^* \psi_{i1} \psi_{i2} + \psi_{j1}^* \psi_{j2} \psi_{i1} \psi_{i2}.$$

(5.9)

Thus, Umklapp processes for the spinons are caused by the interaction $\chi_{ij}^* \chi_{ij}$ through

$$K_{ij} = \psi_{i1}^* \psi_{j1}^* \psi_{i1} \psi_{j1} + \psi_{i2}^* \psi_{j2}^* \psi_{i2} \psi_{j2}.$$

(5.10)

We need the continuum limit of $K_{ij}$ when $j$ is either a nearest or next-nearest neighbor of $i$. To take the continuum limit, we use the variables $f_{1i}$ and $f_{2i}$ defined in Eqs. (3.13), perform the gauge transformation Eq. (3.15), and rewrite the interaction in terms of the chiral components of the smooth fields $u_{\alpha a x}^\ast$, $\alpha = 1, 2$, $a = 1, 2$. In this way, we extract the Umklapp terms

$$-\frac{\epsilon^2}{4} \sum_{a=1,2} \left[ u_{a x}^* u_{a(x+\epsilon)}^* u_{+a x} u_{+a(x+\epsilon)} + (- \leftrightarrow +) \right]$$

from $K_{i(i+1)}$ and $K_{(i+1)(i+2)}$. On the other hand, one extracts

$$+\frac{\epsilon^2}{4} \sum_{a=1,2} \left[ u_{-a x}^* u_{-a(x+\epsilon)}^* u_{+a x} u_{+a(x+\epsilon)} + (- \leftrightarrow +) \right]$$

from $K_{i(i+2)}$ and $K_{(i+1)(i+3)}$. We obtain the important result that a given Umklapp process is induced by the interaction $\chi_{i(i+1)}^* \chi_{i(i+1)}$ as well as by the interaction $\chi_{i(i+2)}^* \chi_{i(i+2)}$ but with couplings of opposite sign. The magnitude of the Umklapp coupling will thus be proportional to $|1 - 4 \frac{J_2}{J_1}|$.

The instability of the critical theory $L_0$ towards Umklapp processes for the spinons follows at once if we can show that Umklapp processes can be induced by a gauge invariant
perturbation of $L_0$ which is marginally irrelevant when $\frac{J_2}{J_1} < \frac{1}{4}$, and is marginally relevant when $\frac{J_2}{J_1} > \frac{1}{4}$.

As we have shown in the Sec. IV A, the critical theory described by $L_0$ is equivalent to a level $k = 1$ Wess-Zumino-Witten theory constructed from the level one Kac-Moody SU(2) currents $\vec{J}_{\pm}$. These currents are color singlets. Their SU(2) algebra is related to the underlying spin symmetry of the problem. The only relevant perturbation to the level $k = 1$ Wess-Zumino-Witten theory which respects the diagonal chiral invariance and the discrete chiral invariance Eq. (4.28) is $\vec{J}_+ \cdot \vec{J}_-$. It turns out that this interaction is marginally relevant or irrelevant depending on the sign of its coupling constant [22]. In appendix D, we relate the Umklapp interaction for spinons to $\vec{J}_+ \cdot \vec{J}_-$ with the help of Eq. (D15). The crucial point is that both interactions have the same action on the physical states, since they only differ by the gauge invariant interactions $(\bar{u} i \gamma^5 u)^2$ and $\vec{J}_+ \cdot \vec{J}_-$, which are both constrained to annihilate the physical states of the theory.

In the absence of frustration $\frac{J_2}{J_1}$, we have shown in Sec. IV B that Umklapp processes for the spinons are irrelevant at the fixed point corresponding to the level $k = 1$ Wess-Zumino-Witten theory. These Umklapp processes result from the fluctuations of the staggered determinant $\rho$. We know from the construction of the continuum limit that the fluctuations $r^\alpha$, $\alpha = 1, 2$, only induce Umklapp processes ($L_2^\rho$ vanishes to lowest order in $\bar{c}$), and since they come with a sign opposite to that due to the $\rho$’s they are marginally relevant perturbations to the critical theory. The competition between the nearest and next-nearest Umklapp processes yield an instability of the critical theory for $(\frac{J_2}{J_1})_c = \frac{1}{4}$. For frustration larger than the critical one, the order parameter $\rho$ develops spontaneously an expectation value corresponding to the onset of dimerization. Note that our critical value for the frustration differs numerically from Haldane’s. This is to be expected since this number is not universal but depends on the short distance cutoff used.

We have thus recovered qualitatively the analysis of Haldane [18] by using a spinon representation of the Heisenberg chain and starting from a mean-field theory around a
spin liquid. A necessary ingredient to this reconstruction is to implement exactly the local constraint of single occupancy in the spinon Fock space. Another necessary ingredient is to treat the Affleck-Marston and Anderson order parameters on an equal footing. Together, these two ingredients amount to a non-perturbative treatment of the SU(2) color symmetry.

B. An effective field theory around the limit $J_2 = 0$

The frustrated spin-$\frac{1}{2}$ chain is solvable when $J_2 = 0$, being equivalent to two independent antiferromagnetic Heisenberg chains with nearest neighbor interaction $J_2$. An interesting question is what is the effect of the frustration $J_2$, i.e., is it a relevant perturbation or is it irrelevant up to some critical value $(\frac{J_1}{J_2})_c$. It has been argued for the closely related problem of the spin-$\frac{1}{2}$ ladder that any interaction across the rung is relevant and induces dimerization [20]. We have found in Sec. III A that our mean-field theory predicts a short-range RVB state in the presence of any infinitesimal frustration. A short-range RVB state ($\bar{\rho} = 0$) certainly does not carry long range dimer order ($\bar{\rho} \neq 0$). However, it can be argued that it is unstable towards dimerization. Here, we want to point out, on the basis of the symmetry of a field theory for the fluctuations around the short-range BZA spin liquid, that other instabilities are present as well (three besides the instability towards dimerization). We also show that the mechanism restoring criticality for small $\frac{J_1}{J_2}$ is not present in our field theory.

We begin by writing down the field theory at criticality. It is constructed from two species of spinons $u$ and $v$ (one for each chains). We only need to replace Eq. (3.20) by

$$f_{(i+4)a}^{e\alpha} = \sqrt{2\bar{\epsilon}} \left[ u_{ax}^{\alpha} + 2\bar{\epsilon}\partial_x u_{ax}^{\alpha} + O(\bar{\epsilon}^2) \right],$$

$$f_{(i+4)a}^{o\alpha} = \sqrt{2\bar{\epsilon}} \left[ v_{ax}^{\alpha} + 2\bar{\epsilon}\partial_x v_{ax}^{\alpha} + O(\bar{\epsilon}^2) \right],$$

(5.13)

(5.14)

where $i \mod 4 = 0$, and $f_{ia}^{e1} = \psi_{ia}, f_{ia}^{o1} = \psi_{(i+1)a}, f_{ia}^{e2} = \psi_{i(a+2)}, f_{ia}^{o2} = \psi_{i(a+3)}$, $a = 1, 2$ being the color index. Our critical theory depends on twice as many slow variables as the single chain critical theory. We borrow the notation from Sec. III E for the slow bosonic variables adding only an upper index $e$ and $o$ where necessary. The relevant kinetic scale is
The critical theory is described by two independent level $k = 1$ Wess-Zumino-Witten theories with gauged Lagrangian density

$$\mathcal{L}_0 = v_F \bar{u} i \gamma^\mu \left( \partial_\mu + i A_\mu^a \right) u + v_F \bar{v} i \gamma^\mu \left( \partial_\mu + i A_\mu^a \right) v. \quad (5.15)$$

The action has a local color $SU(2) \times SU(2)$ chiral symmetry. There exists within each chain an irrelevant perturbation due to staggered fluctuations

$$\mathcal{L}_1 = -v_F \left[ \bar{u} i \gamma^5 \phi_0^e u + \bar{u} u \rho^e + \frac{\text{Re} E}{2} (\rho^e)^2 \right]$$

$$- v_F \left[ \bar{v} i \gamma^5 \phi_0^o v + \bar{v} v \rho^o + \frac{\text{Re} E}{2} (\rho^o)^2 \right]. \quad (5.16)$$

Besides the enlarged gauge symmetry, our critical field theory for the two chains has a new feature compared to the single chain problem. There exists an additional global $U(2)$ flavor symmetry. For example, the transformation $u \rightarrow \frac{1}{\sqrt{2}}(u + v)$, $v \rightarrow \frac{1}{\sqrt{2}}(u - v)$, leaves $\mathcal{L}_0 + \mathcal{L}_1$ unchanged. Interactions due to the frustration $\frac{J_1}{J_2}$ break this flavor symmetry and dynamical off diagonal mass generation can induce dimerization, or other types of order [14].

This situation is not unlike the one we encountered in our study of the frustrated Heisenberg model on a square lattice, which, in the limit of very small $\frac{J_1}{J_2}$, resembles two weakly coupled unfrustrated planar antiferromagnet [14]. Since flavor $U(2)$ has four generators, we expect that there will be a competition between four independent interactions to destroy criticality. In particular, dynamical mass generation in the channel $u_a^* \delta_{ab} v_b^*$ induces dimerization. A difference between this field theory and the one close to the limit $\frac{J_1}{J_2} = 0$ is that any mean-field gap triggered by the frustration $\frac{J_1}{J_2}$ cannot be removed anymore by a simple shift of integration variables as we did in Eq. (3.42). We leave it to future work for a more detailed study of this theory.

VI. CONCLUSIONS

The concept of Luttinger liquid, which appears to apply to a large class of interacting one dimensional electronic systems, is characterized by the striking phenomenon of spin
and charge separation. Our goal has been to understand the relationship between this phenomenon and the separation of spin and charge quantum numbers predicted by some mean-field theories for slave holons and spinons. Here, the slave holon and spinon refer to a picture of the electron in term of a local bound state of a pair of holon and spinon with the electronic spin and charge quantum numbers carried separately by the spinon and holon, respectively.

Whereas it is easy to show that mean-field theories for slave spinons and holons do not describe Luttinger liquids, there have been attempts to recover the properties of the Luttinger liquid by including perturbatively fluctuations of the gauge fields. We have shown explicitly in this paper how, starting from a mean-field theory for a short-range RVB spin liquid, one recovers the spin-$\frac{1}{2}$ sector of a Luttinger liquid. A sufficient and necessary condition for this reconstruction is to include the strong fluctuations of the gauge fields constraining the spinons and to treat them non-perturbatively.

In the case of the frustrated Heisenberg chain for spin-$\frac{1}{2}$, the quantum critical fixed point is described in the language of the slave spinons by a gauged Wess-Zumino-Witten theory for the group $U(2)/SU(2)$. Since this quantum field theory is equivalent to a level $k = 1$ Wess-Zumino-Witten theory constructed from currents obeying a level $k = 1$ SU(2) Kac-Moody algebra, and having shown that these currents are the infinitesimal generators of the spin symmetry of the Heisenberg model, we have recovered Affleck and Haldane’s description of quantum criticality in the spin-$\frac{1}{2}$ chain. This equivalence is a quantum field theory implementation of the Gutzwiller projection. The non-perturbative effects of the gauge fields are to wipe out any spurious mean-field gap and to eliminate altogether the spinons from the spectrum by reducing the Virasoro central charge at mean-field from $C_1 = 2$ to the physical Virasoro central charge $C_0 = 1$.

Umklapp processes for the spin-$\frac{1}{2}$ topological excitations of Luttinger liquids can be relevant perturbations. We have shown that Umklapp processes for the slave spinons can also be relevant perturbations to the quantum critical fixed point. Our description of the onset of dimerization in the slave spinon representation makes it clear that one needs to
treat on an equal footing the Affleck-Marston and Anderson order parameters in order to
detect instabilities of the Luttinger liquid.

The essence of the failure of a slave boson (fermion) scheme to capture some sort of
separation of spin and charge in any “simple” way (say at the Gaussian level around a given
mean-field Ansatz) is due to the fact that the lower critical space-time dimension for the
deconfinement transition in a pure gauge symmetry with discrete symmetry is 3. On the
other hand, it is not known if the mechanism for spin and charge separation in Luttinger
liquids generalizes to higher dimensions. If it does, it will be unrelated to the mechanism of
deconfinement of slave holons and spinons.

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Materials Research Laboratory of the University of Illinois.

APPENDIX A: UNIFORM AND STAGGERED MAGNETIZATIONS

In this appendix, we want to express the uniform and staggered magnetizations in terms
of the Dirac spinons $u$. The uniform and staggered magnetization are defined by

$$
\bar{M}_{\pm i} = \frac{1}{2}(\vec{S}_i \pm \vec{S}_{i+1}), \quad i \in \Lambda_e.
$$

(A1)

By combining Eq. (2.16) and the gauge transformation Eq. (3.15), one verifies that the
uniform magnetization becomes

$$
\bar{M}_{+i} \propto \frac{1}{4} \left( + u^\dagger_{\alpha} (u_{\beta}^\dagger \epsilon_{\alpha\beta}^\epsilon u_{\beta} \pm \text{H.c.}) 
- i(u^\dagger_{\alpha} \epsilon_{\alpha\beta}^\epsilon u_{\beta} - \text{H.c.}) 
+ u^\dagger_{\alpha} \delta_{\alpha\beta} u_{\beta} \right),
$$

(A2)
while the staggered magnetization becomes
\[
\vec{M}_{-i} \propto \frac{1}{4} \begin{pmatrix}
+ (u^\dagger_i \gamma^1 \epsilon^{\alpha \beta} u^\dagger_{\beta}) + \text{H.c.} \\
- i (u^\dagger_i \gamma^1 \epsilon^{\alpha \beta} u^\dagger_{\beta} - \text{H.c.}) \\
+ u^\dagger_i \gamma^1 \delta^{\alpha \beta} u^\dagger_{\beta}
\end{pmatrix}.
\] (A3)

Here,
\[
\epsilon^{\alpha \beta} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} = -\epsilon_{\alpha \beta}.
\] (A4)

The uniform magnetization has a very simple decomposition with respect to the two chiral sectors:
\[
\vec{M}_{+i} \propto \frac{1}{2} (\vec{J}_+ + \vec{J}_-),
\] (A5)

where
\[
\vec{J}_\pm = \frac{1}{2} \begin{pmatrix}
- (u^\dagger_{\pm1} u^\dagger_{\pm2} + u_{\pm2} u_{\pm1}) \\
+ i (u^\dagger_{\pm1} u^\dagger_{\pm2} - u_{\pm2} u_{\pm1}) \\
+ (u^\dagger_{\pm1} u_{\pm1} + u^\dagger_{\pm2} u_{\pm2})
\end{pmatrix}.
\] (A6)

On the other hand, the staggered magnetization mixes the two chiral sectors:
\[
\vec{M}_{-i} \propto \frac{1}{2} (\vec{K}_+ + \vec{K}_-),
\] (A7)

where ( \( \vec{K}_{-+} \) is obtained from \( \vec{K}_{+-} \) by exchanging – and + )
\[
\vec{K}_{+-} = \frac{1}{2} \begin{pmatrix}
- (u^\dagger_{+1} u^\dagger_{-2} + u_{-2} u_{+1}) \\
+ i (u^\dagger_{+1} u^\dagger_{-2} - u_{-2} u_{+1}) \\
+ (u^\dagger_{+1} u_{-1} + u^\dagger_{+2} u_{-2})
\end{pmatrix}.
\] (A8)

Here, we have chosen the chiral basis to be
\[
\gamma^0 = +\tau^2, \quad \gamma^1 = -i\tau^1, \quad \gamma^5 = -\tau^3.
\] (A9)
APPENDIX B: DECOUPLING OF THE GAUGE FIELDS FROM THE SPINONS

The Lagrangian density of Eq. (4.2) has two $U(2)=U(1)\times SU(2)$ symmetry: the axial and vector symmetry. The $U(1)$ symmetry are global:

\[ [U(1)]: \quad u \rightarrow e^{i\theta} u, \quad A_{\mu} \rightarrow e^{i\theta} A_{\mu}, \quad \theta \rightarrow \theta \]

\[ \theta = \theta_5 \gamma_5 \]

\[ [U(1)]_5: \quad u \rightarrow e^{i\theta_5 \gamma_5} u, \quad A_{\mu} \rightarrow e^{i\theta_5 \gamma_5} A_{\mu}. \]

The $SU(2)$ symmetry are local:

\[ [SU(2)]: \quad u \rightarrow U_\omega u, \quad A_{\mu} \rightarrow U_\omega A_{\mu} U_\omega^{-1} + (i\partial_{\mu} U_\omega) U_\omega^{-1}, \]

\[ [SU(2)]_5: \quad u \rightarrow U_{\omega_5} u, \quad A_{\mu} \rightarrow U_{\omega_5} A_{\mu} U_{\omega_5}^{-1} + (i\partial_{\mu} U_{\omega_5}) U_{\omega_5}^{-1}. \]

The quantum theory does not possess the full gauge invariance. The continuity equations for the vector and axial currents

\[ j^\mu = \bar{u} \gamma^\mu u, \quad \bar{j}^\mu = \bar{u} \frac{\sigma}{2} \gamma^\mu u, \]

\[ j_5^\mu = \bar{u} \gamma_5 \gamma^\mu u, \quad \bar{j}_5^\mu = \bar{u} \frac{\sigma}{2} \gamma_5 \gamma^\mu u, \]

cannot be satisfied simultaneously at the quantum level. This is so because it is impossible to construct a fermionic measure for the partition function which is simultaneously invariant under vector and axial gauge transformation [46]. We choose to work with a fermionic measure which is gauge invariant. We then use the unique property of two space-time dimensions that allows for the decoupling of the gauge sector from the spinon sector through a mixture of a vector and axial gauge transformation.

To carry out this program, it is advantageous to rewrite the Lagrangian density in terms of the light-cone coordinates:
\[ x^\pm = x^0 \pm x^1, \quad x^\pm = x_0 \pm x_1, \quad (B7) \]

the Weyl spinors:

\[ u_\pm = \frac{1}{2} (1 \pm \gamma_5) u, \quad (B8) \]

and the gauge fields light-cone components

\[ A^\pm = A^0 \pm A^1, \quad A^\pm = A_0 \pm A_1. \quad (B9) \]

In this basis, the chiral basis, the Lagrangian density is

\[ \mathcal{L}_0 = u_+^\dagger iD_- u_+ + u_-^\dagger iD_+ u_-, \quad (B10) \]

where

\[ D_\mp = \partial_\mp + iA_\mp. \quad (B11) \]

The global symmetry of the Lagrangian density are now

\[ [U(1)]_+ : \quad u_+ \rightarrow \theta^+_+ u_+ = e^{+i\theta^+_+} u_+, \]
\[ A_- \rightarrow \theta^+_+ A_- = A_-, \quad (B12) \]

\[ [U(1)]_- : \quad u_- \rightarrow \theta^-_- u_- = e^{-i\theta^-_-} u_-, \]
\[ A_+ \rightarrow \theta^-_- A_+ = A_+, \quad (B13) \]

whereas the local symmetry of the Lagrangian density are

\[ [SU(2)]_+ : \quad u_+ \rightarrow \omega^+_+ u_+ = U_{\omega^+_+} u_+, \]
\[ A_- \rightarrow \omega^+_+ A_- = U_{\omega^+_+} A_- U_{\omega^+_+}^{-1} + (i\partial_- U_{\omega^+_+}) U_{\omega^+_+}^{-1}, \quad (B14) \]

\[ [SU(2)]_- : \quad u_- \rightarrow \omega^_-_- u_- = U_{\omega^-_-} u_-, \]
\[ A_+ \rightarrow \omega^_-_- A_+ = U_{\omega^-_-} A_+ U_{\omega^-_-}^{-1} + (i\partial_+ U_{\omega^-_-}) U_{\omega^-_-}^{-1}. \quad (B15) \]

The classical Noether currents are
\[ j_+ = 2u_+^\dagger u_+ , \quad j_- = 2u_-^\dagger u_- , \]  
\[ \vec{J}_+ = u_+^\dagger \vec{\sigma} u_+ , \quad \vec{J}_- = u_-^\dagger \vec{\sigma} u_- . \]  
\[ (B16) \]

The first step towards decoupling the gauge fields from the spinons consists in parametrizing the gauge fields \( A_{\mp} \) of the Lie algebra by fields \( G_{\mp} \) of the Lie group according to
\[ A_- = +i(\partial_- G_-) G_-^{-1} , \]
\[ A_+ = +i(\partial_+ G_+) G_+^{-1} . \]
\[ (B18) \] \[ (B19) \]

The advantage of such a parametrization is that the transformation law obeyed by the fields \( G_{\mp} \) under local \([SU(2)]_+ \times [SU(2)]_- \) transformations is very simple, namely it amounts to multiplication from the left:
\[ G_- \rightarrow \omega_- G_- = U_{\omega_-} G_- , \]
\[ G_+ \rightarrow \omega_- G_+ = U_{\omega_+} G_+ . \]
\[ (B20) \] \[ (B21) \]

The disadvantage is that the relationship between \( A_{\mp} \in \text{su}(2) \) and \( G_{\mp} \in \text{SU}(2) \) is non-linear. It is also important to notice that the relationship is not one to one since multiplication from the right of any pair of solutions \( G_{\mp} \) by a pair of SU(2) valued matrices which do not depend on \( x^\mp \), respectively, is also an appropriate parametrization. Naturally, one must account for a bosonic Jacobian when one goes from the su(2) to the SU(2) measures:
\[ D [A_-; A_+] = D [G_-] \ Det \left( \nabla_- \right) D [G_+] \ Det \left( \nabla_+ \right) , \]
\[ (B22) \]

where
\[ \nabla_{\mp} \cdot = \partial_{\mp} \cdot + i \left[ +i(\partial_+ G_{\mp}) G_{\mp}^{-1} \right] \]
\[ (B23) \]

are the covariant derivatives in the adjoint representation.

Clearly, the representation of the gauge fields in terms of the Lie group valued fields relies on the assumption that the bosonic Jacobian \( \Det(\nabla_-) \ Det(\nabla_+) \) is non-vanishing. In other words, we cannot allow for vanishing eigenvalues of the covariant derivative in the
adjoint representation. This condition restricts the allowed gauge configurations to those for which the Dirac operator has no zero modes. The determinants of the covariant derivatives can be converted into Grassmann integrals

\[ \text{Det}(\nabla_-) = \int D[\beta_+, \alpha_+] e^{+i\frac{1}{2}\text{tr}(\beta_+^a \sigma^a \nabla_- \sigma^b \alpha_-^b)}, \tag{B24} \]

\[ \text{Det}(\nabla_+) = \int D[\beta_-, \alpha_-] e^{+i\frac{1}{2}\text{tr}(\beta_-^a \sigma^a \nabla_+ \sigma^b \alpha_+^b)}, \tag{B25} \]

The \( \beta \)'s and \( \alpha \)'s are the ghosts needed to fix the gauge. The ghosts must transform according to

\[ [\text{SU}(2)]_+ : \quad \alpha_+ \to \omega_+ \alpha_+ = U_{\omega_+} \alpha_+ U_{\omega_+}^{-1}, \]

\[ \beta_+ \to \omega_+ \beta_+ = U_{\omega_+} \beta_+ U_{\omega_+}^{-1}, \tag{B26} \]

\[ [\text{SU}(2)]_- : \quad \alpha_- \to \omega_- \alpha_- = U_{\omega_-} \alpha_- U_{\omega_-}^{-1}, \]

\[ \beta_- \to \omega_- \beta_- = U_{\omega_-} \beta_- U_{\omega_-}^{-1}, \tag{B27} \]

since the bosonic measure has the full \( \text{SU}(2)_+ \times \text{SU}(2)_- \) symmetry of the Lagrangian density.

The second step is to perform the local gauge transformation

\[ u_+ \to \tilde{u}_+ = G_{\omega_+}^{-1} u_+, \]

\[ u_- \to \tilde{u}_- = G_{\omega_-}^{-1} u_-, \]

\[ G_- \to \tilde{G}_- = G_{\omega_-}^{-1} G_-, \]

\[ G_+ \to \tilde{G}_+ = G_{\omega_+}^{-1} G_+ = 1, \tag{B28} \]

under which the \( \psi_+ \)'s, \( \beta_+ \)'s and \( \alpha_+ \)'s decouple from the bosonic sector. Since the fermionic measure is invariant under this gauge transformation one can integrate over the fields \( G_+ \).

We choose the factor \( V^{-1} \) of Eq. (4.1) to be the gauge volume, and we are left with the partition function

\[ Z = \int D[\tilde{G}_-] \int D[\tilde{\beta}_+; \tilde{\alpha}_+] \int D[\tilde{u}_+; \tilde{\alpha}_+] e^{+iS}. \tag{B29} \]

The third step is to apply the mixed vector and axial gauge transformation
\[\tilde{u}_+ \rightarrow u'_+ = \tilde{G}^{-1}_- \tilde{u}_+ = G^{-1}_- u_+,\]
\[\tilde{u}_- \rightarrow u'_- = \tilde{u}_- = G^{-1}_+ u_-,\]
\[\tilde{G}_- \rightarrow G'_- = \tilde{G}^{-1}_- \tilde{G}_- = 1,\]
\[\tilde{G}_+ \rightarrow G'_+ = \tilde{G}_+ = 1.\] (B30)

As promised, it fully decouples the fermions (spinons and ghosts) from the bosonic degrees of freedom. However, due to the axial component of the transformation, the spinon measure and the ghost measure change by a non-trivial Jacobian \[38,33,34\]:

\[Z = \int \mathcal{D}_{\mu_b} \int \mathcal{D}_{\mu_f} e^{+i(S_1 + S_2 + S_3)},\] (B31)
\[\mathcal{D}_{\mu_b} = \mathcal{D}[\tilde{G}_-],\] (B32)
\[\mathcal{D}_{\mu_f} = \mathcal{D}[\beta'_+; \alpha'_+; \beta'_-; \alpha'_-] \mathcal{D}[u'^+_+, u'_-; i \bar{\partial}_+ u'_-, u'_{-}].\] (B33)

The action is the sum of three independent sectors. The first sector is the sector for free Dirac spinons

\[S_1 = \int \frac{dx^+ dx^-}{2} (u'^+_+ i \bar{\partial}_- u'_+ + u'^-_+ i \bar{\partial}_+ u'_-),\] (B34)

where the relationship between the free spinons and the original spinons is

\[u'_+ = \tilde{G}^{-1}_- \tilde{u}_+ = G^{-1}_- u_+,\] (B35)
\[u'_- = \tilde{u}_- = G^{-1}_+ u_-.\] (B36)

The second sector results from the non-invariance of the fermionic measure under an axial transformation and is given by the Wess-Zumino-Witten action \[37\] with negative level \(-1 - 2c_v\) \[47\]

\[S_2 = -(1 + 2c_v) W_-[\tilde{G}_-].\] (B37)

Here, the Wess-Zumino-Witten action \(W_-\) depends on the gauge fields through the non-linear relation

\[\tilde{G}_- = G^{-1}_+ G_-,\] (B38)

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and is given by

\[ W_-[G] = \frac{1}{8\pi} \int dx^0 dx^1 \, \text{tr} \left[ \partial_\mu G \, \partial^\mu G^{-1} \right] \]

\[ + \frac{1}{12\pi} \int_{B, B=\partial S^2} dx^0 dx^1 dx^2 \, e^{\mu\nu\lambda} \, \text{tr} \left[ (\partial_\mu G)G^{-1}(\partial_\nu G)G^{-1}(\partial_\lambda G)G^{-1} \right] . \]  

Finally, the third sector is the ghost sector

\[ S_3 = \int \frac{dx^+ dx^-}{2} \left[ \text{tr} \left( \beta^a_+ \sigma^a \partial_- \sigma^b \alpha^b_+ \right) + \text{tr} \left( \beta^a_- \sigma^a \partial_+ \sigma^b \alpha^b_- \right) \right] . \]  

The partition function Eq. (B31) describes a two dimensional conformal field theory \[ \text{[1]} \] with the conformal weights \[ \text{[1],[2]} \]

\[ \left( \Delta_+, \Delta_- \right) \left( u'_+ \right) = \left( \frac{1}{2}, 0 \right) , \]

\[ \left( \Delta_+, \Delta_- \right) \left( u'_- \right) = \left( 0, \frac{1}{2} \right) , \]  

\[ \left( \Delta_+, \Delta_- \right) \left( \tilde{G}_- \right) = \left[ \begin{array}{c} \frac{N^2-1}{2N} \\ N+k \end{array} \right] \quad \text{for} \quad N=2, \quad k=0,1,\ldots,2N \]  

for the primary fields \( u' \) and \( \tilde{G}_- \).

**APPENDIX C: SUGAWARA CONSTRUCTION FOR A LEVEL ONE SPIN SU(2) KAC-MOODY ALGEBRA**

We consider the action Eq. (4.5) where the ' over the Dirac spinons has been dropped for brevity. We choose canonical quantization in a box of length \( L \) and we impose periodic boundary conditions. All products of operators are at equal time. The spatial coordinate is denoted \( x \). The Fourier convention is

\[ u_{\sigma ax} = \frac{1}{\sqrt{L}} \sum_{p \in \mathbb{Z}} e^{-i\frac{2\pi}{L}px} u_{\sigma ap}, \]  

where

\[ \sigma = -, +, \quad a = 1, 2, \quad 0 \leq x \leq L. \]  

The only non-vanishing anticommutators are
\{u_{\sigma ax}, u_{\sigma' a' x'}^\dagger\} = \delta_{\sigma,\sigma'} \delta_{a,a'} \delta_{x,x'},
(C3)

\{u_{\sigma ap}, u_{\sigma' a' p'}^\dagger\} = \delta_{\sigma,\sigma'} \delta_{a,a'} \delta_{p,p'}.
(C4)

The Hamiltonian is

\[ H = \int L_0 dx \left( u_{+ ax}^\dagger i \partial_1 u_{+ ax} - u_{- ax}^\dagger i \partial_1 u_{- ax} \right) \]
\[ = \frac{2\pi}{L} \sum_{p \in \mathbb{Z}} p \left( u_{+ ap}^\dagger u_{+ ap} - u_{- ap}^\dagger u_{- ap} \right). \]
(C5)

The momentum operator is

\[ P = \int L_0 dx \left( u_{+ ax}^\dagger i \partial_1 u_{+ ax} + u_{- ax}^\dagger i \partial_1 u_{- ax} \right) \]
\[ = \frac{2\pi}{L} \sum_{p \in \mathbb{Z}} p \left( u_{+ ap}^\dagger u_{+ ap} + u_{- ap}^\dagger u_{- ap} \right). \]
(C6)

The light-cone components of the Hamiltonian and momentum operators are

\[ \Theta_+ = + \int_0^L dx \ u_{+ ax}^\dagger i \partial_1 u_{+ ax} = + \frac{2\pi}{L} \sum_{p \in \mathbb{Z}} p \ u_{+ ap}^\dagger u_{+ ap}, \]
(C9)

\[ \Theta_- = - \int_0^L dx \ u_{- ax}^\dagger i \partial_1 u_{- ax} = - \frac{2\pi}{L} \sum_{p \in \mathbb{Z}} p \ u_{- ap}^\dagger u_{- ap}. \]
(C10)

The ground state of the Hamiltonian is the Fermi sea

\[ |\Psi_{fs}\rangle = \left( \prod_{p \leq 0} u_{+ 1 p}^\dagger u_{+ 2 p}^\dagger \right) \left( \prod_{p \geq 0} u_{- 1 p}^\dagger u_{- 2 p}^\dagger \right) |0\rangle. \]
(C11)

The only non-vanishing ground-state expectation value for bilinears in \( u \) is

\[ \langle \Psi_{fs} | u_{\sigma a(x-\epsilon)}^\dagger u_{\sigma' a'(x'+\epsilon)} \ |\Psi_{fs}\rangle = \delta_{\sigma,\sigma'} \delta_{a,a'} \delta_{x,x'} \]
\[ \begin{cases} \frac{-i}{4\pi \epsilon + i 0^+} & \text{if } \sigma = +, \\ \frac{i}{4\pi \epsilon - i 0^+} & \text{if } \sigma = -, \\ \end{cases} \]
(C12)

and its complex conjugate.

Normal ordering with respect to the Fermi sea is denoted by \( : \cdot : \). The commutator of the current

\[ u_{+1x}^\dagger u_{+1x} \]
(C13)

is defined through the point splitting procedure.
\[
\left[ :u^\dagger_{+1x}u_{+1x} : , :u^\dagger_{+1x'}u_{+1x'} : \right] = \lim_{\epsilon' \to 0} \left[ :u^\dagger_{+1(x-\epsilon)}u_{+1(x+\epsilon)} : , :u^\dagger_{+1(x'-\epsilon')}u_{+1(x'+\epsilon')} : \right].
\] (C14)

Wick theorem allows to express the right-hand side solely in terms of product of (possibly singular) complex functions and normal ordered products:

\[
\left[ :u^\dagger_{+1x}u_{+1x} : , :u^\dagger_{+1x'}u_{+1x'} : \right] = +\frac{i}{\pi} \delta'_{x,x'}.
\] (C15)

Here, the spatial derivative (with respect to \( x - x' \)) of the delta function is

\[
\delta'_{x,x'} = \lim_{\epsilon' \to 0} \frac{1}{2(\epsilon + \epsilon')} \left( \delta_{x-x'+\epsilon',0} - \delta_{x-x'-\epsilon',0} \right).
\] (C16)

The coefficient \( \frac{1}{\pi} \) depends on the choice of our conventions. Other commutators of interest (see appendix A and B for the normalization factors of the currents) are

\[
\left[ :\frac{j^a_{\sigma x}}{2} : , :\frac{j^{a'}_{\sigma' x'}}{2} : \right] = \sigma \frac{2i}{\pi} \delta_{\sigma,\sigma'} \delta'_{x,x'},
\] (C17)

\[
\left[ :\frac{J^a_{\sigma x}}{2} : , :\frac{J^{a'}_{\sigma' x'}}{2} : \right] = i\epsilon^{abc} :\frac{J^c_{\sigma x}}{2} : \delta_{\sigma,\sigma'} \delta_{x,x'} + \sigma \frac{i}{2\pi} \delta_{\sigma,\sigma'} \delta'_{x,x'},
\] (C18)

\[
\left[ :J^a_{\sigma x} : , :J^{a'}_{\sigma' x'} : \right] = i\epsilon^{abc} :J^c_{\sigma x} : \delta_{\sigma,\sigma'} \delta_{x,x'} + \sigma \frac{i}{2\pi} \delta_{\sigma,\sigma'} \delta'_{x,x'}.
\] (C19)

The factor \( \frac{1}{\pi} \) comes multiplied by \( \text{tr} \left( \sigma^0 \right) \) and \( \frac{2}{\pi} \) of the delta function is respectively. The algebras of Eqs. (C18) and (C19) are identical and equivalent to a level \( k = 1 \) SU(2) Kac-Moody algebra.

Similarly, one can show that

\[
\lim_{\epsilon \to 0} :\vec{J}_{+(x-\epsilon)} : \cdots :\vec{J}_{+(x+\epsilon)} : = +\frac{3}{2} :u^\dagger_{+1x}u^\dagger_{+2x}u_{+2x}u_{+1x} : + \frac{3}{4\pi} :u^\dagger_{+x} i\partial_x u_{+x} : ;
\] (C20)

\[
\lim_{\epsilon \to 0} \frac{\vec{J}_{+(x-\epsilon)}}{2} : \cdots :\vec{J}_{+(x+\epsilon)} : = -\frac{3}{2} :u^\dagger_{+1x}u^\dagger_{+2x}u_{+2x}u_{+1x} : + \frac{3}{4\pi} :u^\dagger_{+x} i\partial_x u_{+x} : ;
\] (C21)

\[
\lim_{\epsilon \to 0} \frac{\vec{j}_{+(x-\epsilon)}}{2} : \cdots :\vec{j}_{+(x+\epsilon)} : = +2 :u^\dagger_{+1x}u^\dagger_{+2x}u_{+2x}u_{+1x} : + \frac{1}{\pi} :u^\dagger_{+x} i\partial_x u_{+x} : .
\] (C22)

We have dropped vacuum expectation values on the right-hand side. Hence, the local generators \( T_+ \) of the energy momentum tensor \( \Theta_+ \) can be expressed solely in terms of point split and normal ordered current bilinears:

\[
:T_+ : = \frac{\pi}{2} :\frac{j_+}{2} : \cdots :\frac{j_+}{2} : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ :.
\] (C23)

\[
= \frac{2\pi}{3} :\vec{J}_+ : \cdots :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : + \frac{2\pi}{3} :\vec{J}_+ : ;
\] (C24)
where the vacuum expectation values have been dropped on the right-hand side. The same relation holds in the \(-\) chiral sector. This completes the Sugawara construction of the energy-momentum tensor in terms of currents.

Notice that one can rewrite
\[
\frac{2\pi}{3} = \frac{2\pi}{1+2} = \left[ \frac{2\pi}{k+c_v} \right]_{c_v=2}^{k=1},
\]
if one wants to stress the fact that the currents \(\vec{J}_\pm\) and \(\vec{J}_\mp\) obey a level \(k = 1\) Kac-Moody SU(2) algebra.

**APPENDIX D: USEFUL IDENTITIES**

We work with the chiral basis
\[
\gamma^0 = +\tau^2, \quad \gamma^1 = -i\tau^1, \quad \gamma_5 = -\tau^3.
\]
The components of the spinor \(u\) are \(u_{\sigma a}\) where \(\sigma = +, -\) refers to the Lorentz degrees of freedom and \(a = 1, 2\) refers to the color degrees of freedom. One has
\[
\gamma_5 u_{\sigma a} = -\sigma u_{\sigma a}, \quad a = 1, 2, \quad \sigma = +, -.
\]
As usual \(\bar{u}\) will denote \(u^\dagger \gamma^0\). We use repeatedly the identity
\[
\vec{\sigma}_{ab} \cdot \vec{\sigma}_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}
\]
when contracting color indices to express quartic interactions in terms of quadratic forms in
\[
K_{++} = u_{+a}^* \delta_{ab} u_{-b},
\]
\[
\dot{j}_+ = 2 u_{+a}^* \delta_{ab} u_{+b},
\]
\[
\vec{J}_+ = u_{+a}^* \vec{\sigma}_{ab} u_{+b},
\]
\[
K_{--} = u_{-a}^* \delta_{ab} u_{-b},
\]
\[
\dot{j}_- = 2 u_{-a}^* \delta_{ab} u_{-b},
\]
\[
\vec{J}_- = u_{-a}^* \vec{\sigma}_{ab} u_{-b},
\]

\[
\text{together with } K_{-+}, j_-, \text{ and } \vec{J}_- \text{ obtained by interchanging } + \text{ for } - \text{ and } - \text{ for } +. \text{ The } K's \text{ and } j's \text{ are singlets under vector gauge transformations. The } \vec{J}'s \text{ transform like the adjoint}
\]

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of color SU(2) under vector gauge transformations. Only the $j$’s are singlet under all chiral transformations. Only the $K$’s induce Umklapp processes. One finds

\[(\bar{u} u)^2 = -K_{+-} K_{+-} - \frac{1}{8} j_+ j_- - \frac{1}{2} \vec{J}_+ \cdot \vec{J}_- + (\leftrightarrow -), \]  
\[(\bar{u} i\gamma_5 \vec{\sigma} u)^2 = -3 K_{+-} K_{+-} - \frac{3}{8} j_+ j_- + \frac{1}{2} \vec{J}_+ \cdot \vec{J}_- + (\leftrightarrow -), \]  
\[(\bar{u} i\gamma^5 u)^2 = +K_{+-} K_{+-} - \frac{1}{8} j_+ j_- - \frac{1}{2} \vec{J}_+ \cdot \vec{J}_- + (\leftrightarrow -), \]  
\[(\bar{u} \vec{\sigma} u)^2 = +3 K_{+-} K_{+-} - \frac{3}{8} j_+ j_- + \frac{1}{2} \vec{J}_+ \cdot \vec{J}_- + (\leftrightarrow -). \]  

This results in the important identities

\[(\bar{u} u)^2 = \frac{1}{3} (\bar{u} i\gamma_5 \vec{\sigma} u)^2 - \frac{2}{3} [\vec{J}_+ \cdot \vec{J}_- + (\leftrightarrow -)], \]  
\[(\bar{u} i\gamma^5 u)^2 = \frac{1}{3} (\bar{u} \vec{\sigma} u)^2 - \frac{2}{3} [\vec{J}_+ \cdot \vec{J}_- + (\leftrightarrow -)]. \]  

The singlet current-current interaction can be expressed as

\[\frac{1}{8} j_+ j_- = 2 \vec{J}_+ \cdot \vec{J}_- - (u_{+1}^* u_{+1} - u_{+2}^* u_{+2} + \text{H.c.}), \]  

so that

\[K_{+-} K_{+-} + \frac{1}{8} j_+ j_- = 2 \vec{J}_+ \cdot \vec{J}_- + \sum_{a=1,2} u_{+ax}^* u_{-ax} u_{+a(x+\xi)}^* u_{-a(x+\xi)}. \]  

With the help of Eq. (D8) we obtain the second important identity relating the Umklapp interaction with the interactions \(\vec{J}_+ \cdot \vec{J}_-:\)

\[\sum_{a=1,2} [u_{+ax}^* u_{+a(x+\xi)}^* u_{-ax} u_{-a(x+\xi)} + (\leftrightarrow -)] = 4 \vec{J}_+ \cdot \vec{J}_- + \frac{1}{3} (\bar{u} i\gamma^5 u)^2 - \frac{1}{3} \vec{J}_+ \cdot \vec{J}_-. \]  

(D15)
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[29] It could be that in some pathological cases \(a^0_0\) is such that the gap closes at \(\pm \pi/2\) for
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