Energy in Newtonian Gravity

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Received: 23 December 2021 / Accepted: 30 November 2022 / Published online: 12 December 2022 © The Author(s) 2022

Abstract
In Newtonian gravity it is a moot question whether energy should be localized in the field or inside matter. An argument from relativity suggests a compromise in which the contribution from the field in vacuum is positive definite. We show that the same compromise is implied by Noether’s theorem applied to a variational principle for perfect fluids, if we assume Dirichlet boundary conditions on the potential. We then analyse a thought experiment due to Bondi and McCrea that gives a clean example of inductive energy transfer by gravity. Some history of the problem is included.

Keywords Newtonian gravity · Localisation of energy · Tidal forces

1 Introduction
How is a local energy density to be defined in Newtonian gravity? This is a moot question. Traditionally, the two main contenders for a definition are that of Maxwell [1],
\[ E' = -\frac{1}{8\pi G} \partial_i \Phi \partial_j \Phi, \]
and an alternative where the energy is localized within the matter,
\[ E'' = -\frac{1}{2} \rho \Phi. \]

We will refer to the latter as Bondi’s energy density, because it was championed by him [2, 3]. Our conventions for the gravitational interaction are set by the equations that connect the gravitational potential \( \Phi \) to the mass density of matter \( \rho \),

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We have adopted the convention that the potential $\Phi$ is positive. At some points we may set the constant $G$ to 1. The formula using the Green function assumes that the matter density has compact support, so that we deal with an isolated system.

The total energy is obtained by integrating the energy density over all space, and comes out the same regardless of whether we use $E$ or $E'$. There is an analogous issue in electrostatics. Maxwell, who was thinking of force fields as emergent from a medium, insisted that the choice is important:

“I wish to be understood literally. All energy is the same as the mechanical energy, whether it exists in the form of motion or in that of elasticity ...The only question is, Where does it reside? ... On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves.” [1]

But Maxwell faced a problem with gravity. The energy density is negative definite while (in his view) energy is “essentially positive”. As a way out he suggested that a constant should be added to $E'$ to make it everywhere positive. But this would mean that the energy density of the medium must be huge where the gravitational field is weak. He concluded:

“As I am unable to understand in what way a medium can possess such properties, I cannot go any further in this direction in searching for the cause of gravitation.” [1]

It should be noticed that the notion of energy conservation was by no means uncontroversial at the time. Herschel regarded it as a verbal trick [4]. Much later, Mason and Weaver [5] argued that energy is a function of the configuration of the system as a whole, and that it is no more sensible to inquire about the location of energy than to declare that the beauty of a painting is distributed over the canvas in a specified manner.

Who is right: Maxwell, Bondi, or Mason and Weaver? For electromagnetism the question is often regarded as resolved (in favour of Maxwell) by the relativistic theory, and in particular by the way that the electromagnetic field couples to gravity. For gravity there is no external arbiter to make the decision.

In general relativity the total energy of an isolated system is well understood, but the localisation of energy is a tangled question indeed. The energy at a point can be argued away using the equivalence principle, but there are proposals for the energy located within some chosen closed surface. Some of these proposals, notably those of Hawking [6] and Penrose [7], suggest that the energy of a black hole is located within its event horizon. This is perhaps reminiscent of Bondi’s expression in the Newtonian case. Others, notably Lynden-Bell and Katz [8], have proposed expressions that are closer in spirit to that of Maxwell. To support their case Lynden-Bell and his coworkers considered a static gravitational field coupled to a perfect fluid. We then lose very little by assuming spherical symmetry as well, so that the formulas that follow should be familiar to most readers. To obtain a conserved current from the relativistic stress-energy tensor $T_{ab}$ we need to contract it with a timelike Killing vector $\xi^b$, and the energy

$$\nabla^2 \Phi = -4\pi G \rho \quad \Leftrightarrow \quad \Phi(\mathbf{x}, t) = G \int \frac{\rho(\mathbf{x'}, t)}{|\mathbf{x} - \mathbf{x'}|} d^3x'. \quad (3)$$
density gains an extra factor coming from the norm of the Killing vector field. Let \( t^a \) be a timelike unit vector and set

\[
T_{ab} = (\rho + p)t_a t_b + pg_{ab}
\]  

(4)

\[
\varepsilon^a = \sqrt{-g} t^a , \quad -g_{tt} = 1 - 2\Phi.
\]  

(5)

We can build an energy density \( \mu \) and take its Newtonian limit,

\[
\mu = -T^a_{\phantom{a}b} \varepsilon^b = \rho \sqrt{1 - 2\Phi} \approx \rho (1 - \Phi) = \rho + 2\varepsilon''.
\]  

(6)

Maxwell’s very large constant appears here in the guise of the mass density \( \rho \), but the binding energy is twice as large as he may have expected. To make the total energy come out as being equal to the mass plus the Newtonian energy we add a term coming from the gravitational field itself. We can then define a ‘true’ Newtonian energy density \( \mathcal{E} \) through [8, 9]

\[
\rho + \mathcal{E} = \rho + 2\varepsilon'' - \mathcal{E}' = \rho (1 - \Phi) + \frac{1}{8\pi} \partial_i \Phi \partial_i \Phi.
\]  

(7)

Maxwell’s sign problem has evaporated in the sense that the energy density is now positive definite outside matter, although the argument rests on the assumption that the gravitational field is static.

This argument did not lead to a consensus. Frauendiener and Szabados [10] use post-Newtonian corrections to Newton’s theory to argue that the energy density integrated over large volumes, enclosing all the matter, should be a monotonically decreasing function of the volume. So the question is indeed moot.

As far as we know the expression \( \mathcal{E} = 2\varepsilon'' - \mathcal{E}' \) for the energy density was first advocated by Ohanian [11]. The question continues to attract interest [12]. Our first aim here is to see how one can argue in favour of the energy density \( \mathcal{E} \) from within the Newtonian theory itself, at the same time dropping the assumption of static fields. We will do this by appealing to Noether’s theorem. In view of its hundreth anniversary this theorem has attracted interest from philosophers of science recently [13], and indeed Dewar and Weatherall used it to study the energy concept in Newtonian gravity [14]. However, since they used an external matter source they were unable to address the question that we consider in Sect. 2. We add that other aspects of their paper have given rise to illuminating discussions [15]. In particular the Newton–Cartan formulation of the theory has been a subject for these discussions. This is an interesting topic about which we have nothing to say.

In Sect. 3 we go on to consider an interesting example of energy transport in Newtonian gravity, in rather more detail than was offered in the original paper by Bondi and McCrea [16]. In Sect. 4 we draw attention to the fact that the energy density proposed in Sect. 2 plays no role in the concrete setting of Sect. 3. The question then becomes one of the pragmatic advantages of the various definitions.
2 An Energy Density from Noether’s Theorem

In Newtonian gravity we need matter to provide dynamics. A fluid described by a mass density \( \rho \) and a velocity field \( v_i \) is appropriate. These variables obey mass conservation

\[
\partial_i \rho + \partial_j (\rho v_j) = 0
\]

as well as the Euler equation

\[
\rho \frac{d}{dt} v_i = \rho (\partial_i v_i + v_j \partial_j v_i) = \partial_j \tau_{ij} + \rho \partial_i \Phi.
\]

Here \( \tau_{ij} \) is the stress tensor, and the last term is the gravitational body force. In the spirit of Maxwell the latter can be regarded as being due to gravitational stress, but for this we refer to Synge [17]. For simplicity we will set \( \tau_{ij} \) to zero and assume that the velocity field is irrotational. Thus our matter consists of irrotational gravitating dust, but we comment briefly on more general cases at the end. That the velocity field is irrotational means that there exists a velocity potential \( \lambda \),

\[
\partial_i v_j - \partial_j v_i = 0 \quad \Leftrightarrow \quad v_i = -\partial_i \lambda.
\]

The potential is essential in order to find an action integral from which Euler’s equations can be derived [18–20].

We begin by recalling how the variational principle works. In the action the velocity potential appears as a Lagrange multiplier imposing conservation of mass:

\[
S_0[\rho, v_i, \lambda, \Phi] = \int \left[ \frac{1}{2} \rho v_j v_i - \lambda (\partial_i \rho + \partial_j (\rho v_j)) + \rho \Phi - \frac{1}{8\pi} \partial_i \Phi \partial_j \Phi \right] d^4x.
\]

Varying the action with respect to the velocity field \( v_i \) we recover Eq. (10), which is an algebraic equation for \( v_i \) that can be inserted in the action. The result is

\[
S_1[\rho, \lambda, \Phi] = \int \left[ \rho \partial_i \lambda - \frac{1}{2} \rho \partial_i \lambda \partial_j \lambda + \rho \Phi - \frac{1}{8\pi} \partial_i \Phi \partial_j \Phi \right] d^4x.
\]

A total time derivative was added. This amounts to a canonical transformation in the matter sector, and does not affect our later arguments. Total space derivatives are important and will be discussed soon. Variation with respect to \( \Phi \) returns Poisson’s equation (3), and variation with respect to \( \rho \) gives

\[
\partial_i \lambda - \frac{1}{2} \partial_j \lambda \partial_i \lambda + \Phi = 0.
\]

Taking the gradient of this equation and making use of eq. (10) yields

\[
-\partial_i \partial_i \lambda + \partial_j \lambda \partial_j \lambda = \partial_i \Phi \quad \Rightarrow \quad \partial_j v_j + v_j \partial_j v_i = \partial_i \Phi.
\]

This is the equation of motion for irrotational dust.
The velocity potential \( \lambda \) has no direct physical interpretation since it is defined only up to a constant. In a Galilei invariant model we insist on the mass superselection rule, that is we insist that all observables Poisson commute with the total mass [20]

\[
M = \int \rho \, d^3x. \tag{15}
\]

Clearly \( \{M, \lambda\} = 1 \), so indeed \( \lambda \) is not an observable while its gradient is.

Before we apply Noether’s theorem we must address the issue of possible surface terms that can be added to the action. In infinite space they play no role in deriving the equations of motion, but we will consider a finite spacetime region \( V \) with an enclosing surface placed in vacuum [21]. The action amended with a surface term is

\[
S[\rho, \lambda, \Phi] = S[\rho, \lambda, \Phi] + \frac{a}{4\pi} \int \partial_i(\Phi \partial_i \Phi) d^4x
\]

\[
= \int \left[ \rho \partial_i \lambda - \frac{1}{2} \rho \partial_i \lambda \partial_i \lambda + \rho \Phi - \frac{1}{8\pi} \partial_i \Phi \partial_i \Phi + \frac{a}{4\pi} \partial_i(\Phi \partial_i \Phi) \right] d^4x. \tag{16}
\]

Here \( a \) is a parameter to be fixed. Assuming that the field equations hold we find that

\[
\delta S = \int_V \left[ \partial_i(\delta \lambda \rho) + \partial_i \left( \rho \nu_i \delta \lambda + \frac{a - 1}{4\pi} \partial_i \Phi \delta \Phi + \frac{a}{4\pi} \Phi \partial_i \Phi \right) \right] d^4x. \tag{17}
\]

The surface terms must vanish if the action is to be used to derive the field equations in the bounded region. The total time derivative is not relevant for our problem, but the total divergence is. We have assumed that \( \rho = 0 \) on the spatial part of the boundary, but we still have to decide what boundary conditions to impose on \( \Phi \). This is the question of how we control the system [21], and an answer will fix the parameter \( a \). The two most obvious choices give

\[
\text{Dirichlet} \quad \Rightarrow \quad \delta \Phi_{\text{bdry}} = 0 \quad \Rightarrow \quad a = 0
\]

\[
\text{Neumann} \quad \Rightarrow \quad n^i \delta \partial_i \Phi_{\text{bdry}} = 0 \quad \Rightarrow \quad a = 1, \tag{18}
\]

where the vector \( n^i \) is normal to the boundary. In electrostatics it would be a straightforward task for the experimentalist to impose Dirichlet conditions on the electrostatic potential. For gravity the question is not so easily settled, but Dirichlet conditions do seem to be a natural choice. We will return to discuss this issue once we have discussed a concrete example of gravitational energy transport. Meanwhile we simply assume that a suitable choice has been made, so that the parameter \( a \) is fixed.

We are now ready to apply Noether’s theorem. Consider a rigid time translation,

\[
\delta \lambda = \epsilon \partial_i \lambda \quad \delta \rho = \epsilon \partial_i \rho \quad \delta \Phi = \epsilon \partial_i \Phi \quad \Rightarrow \quad \delta \mathcal{L} = \epsilon \partial_i \mathcal{L}, \tag{19}
\]

where \( \mathcal{L} \) is the integrand of the action integral. Noether’s theorem follows from the observation that for these variations we have the alternative expression.
\[ \delta S = \int \partial_i (\varepsilon \mathcal{L}) \, d^4x. \] (20)

The equality of the two expressions for \( \delta S \) implies the local conservation law

\[
\partial_i \left( \frac{1}{2} \rho \partial_i \lambda \partial_i \lambda \right) + (a - 1) \rho \Phi + \frac{1 - 2a}{8\pi} \partial_i \Phi \partial_i \Phi \right) \\
+ \partial_i \left( \rho v_i \partial_i \lambda + \frac{a - 1}{4\pi} \partial_i \Phi \partial_i \Phi + \frac{a}{4\pi} \Phi \partial_i \partial_i \Phi \right) = 0. \tag{21}
\]

The energy density (including the kinetic energy of matter) can be read off from the first term. If Dirichlet conditions are assumed we set \( a = 0 \) and find that the energy density is

\[ \mathcal{E} = \frac{1}{2} \rho v_i v_i - \rho \Phi + \frac{1}{8\pi} \partial_i \Phi \partial_i \Phi = \frac{1}{2} \rho v_i v_i - \mathcal{E}' + 2\mathcal{E}''. \] (22)

This is not a decision between Maxwell and Bondi, it is a compromise in the sense that energy is partly localized within matter and partly spread throughout the vacuum. It bears the mark of a good compromise since it agrees with the answer arrived at by Lynden-Bell and Katz, as in Eq. (7). And the point we wanted to make is precisely that with suitable assumptions this answer can be derived from within the Newtonian theory itself.

However, had we chosen to impose Neumann rather than Dirichlet boundary conditions we would have ended up with Maxwell’s expression for the energy density. That Noether’s theorem leads to different expressions for the energy depending on the boundary conditions we impose should, in fact, not be surprising. The situation is interestingly analogous to how internal and free energy appear in thermodynamics, depending on how the system is controlled [21].

The second term in the local conservation law (21) gives the local energy flux. We will return to it in Sect. 4 below. Meanwhile we observe that if we use the field equations to clear the conservation law of time derivatives we obtain

\[ \partial_i \left( \frac{1}{2} \rho v_i v_i \right) + \partial_i \left( v_i \frac{1}{2} \rho v_j v_j \right) = \rho v_i \partial_i \Phi. \] (23)

This is of course indisputable—the local kinetic energy changes due to work done by the gravitational field—but any gravitational contribution to the energy density has disappeared.

We have given the argument for irrotational dust. The entire argument clearly goes through if we add a pressure term to the equations. The restriction to irrotational flow can be dropped too, if we make use of Clebsch potentials. An economical choice is that due to Seliger and Whitham [18],

\[ v_i = -\partial_i \lambda - a \partial_i \beta. \] (24)

An additional pair of Clebsch potentials is needed to handle general flows globally [22]. Applying Noether’s theorem to the action proposed by Seliger and Whitham results in the same expressions for the local energy density and the local energy.
transport as the ones we just derived, once they have been expressed in terms of \( \rho \), \( v_i \), and \( \Phi \). For this reason we do not give the details here.

3 Tweedledum and Tweedledee

The arguments in Sect. 2 were rather formal. It is interesting to see how they fare in a concrete problem. We do know that energy is being transported by gravity within the solar system, in a way that is well described by Newtonian theory. As an example, tidal friction within the Earth–Moon system causes the Moon to move away from the Earth. But in this case it is not easy to pinpoint where the energy ends up. A more dramatic example is provided by the tidal heating of Io, one of the moons of Jupiter [23]. A simpler example is called for here. Bondi and McCrea invented a thought experiment in which tidal forces give rise to a net energy transport even though the gravitational field returns to its initial state after some energy has been transmitted [16]. The experiment concerns two mutually gravitating bodies in elliptical orbits around each other. Bondi later named them Tweedledum and Tweedledee [2], although in the original paper they were referred to as the receiver (R) and the transmitter (T). They have spherical outlines, but their mass distributions can be changed between prolate and oblate with the axial direction orthogonal to the orbital plane. If they both turn oblate the gravitational attraction between them grows. If one of them turns oblate and the other prolate the attraction can be kept constant, and this is the key to the whole idea since it will allow them to stay on their elliptical orbits throughout the duration of the experiment. The changes of shape are controlled by some machinery powered by batteries external to the system that we will describe, which goes to say that we will study an open system. Tidal forces tend to make a body oblate, and when this happens work is done on the body. Conversely, work is done by the body when it turns prolate. This is how transmission of energy can occur. The question of how energy is stored in the gravitational field is avoided because, as far as the gravitational field is concerned, the process is cyclic and we will compute the energy transmitted during a full cycle.

The original paper is very brief, and we feel that it may be useful to tell the story using equations. Full calculational details are given elsewhere [24]. The twins are modelled as spheres with radii \( r_0 \) and total mass \( M \) in both cases. Their mass densities have time dependent quadrupole moments \( Q_R = Q_R(t) \) and \( Q_T = Q_T(t) \) that can be freely prescribed. In coordinate systems with origos at the centres of the bodies

\[
\rho_R = \begin{cases} 
\frac{3}{4\pi} \frac{M}{r_0^3} + \frac{35}{3\pi} \frac{Q_R(t)}{r_0^5} r^2 P_2(\cos \theta) , & r < r_0 \\
0 , & r > r_0
\end{cases}
\]  

(25)

\[
\rho_T = \begin{cases} 
\frac{3}{4\pi} \frac{M}{r_0^3} + \frac{35}{3\pi} \frac{Q_T(t)}{r_0^5} r^2 P_2(\cos \theta) , & r < r_0 \\
0 , & r > r_0
\end{cases}
\]  

(26)
where $P_2$ is a Legendre polynomial and $\theta$ is the angle against the normal to the orbital plane. A body is prolate if its quadrupole moment is positive.

For the calculations to follow we need some formulas for the spherical harmonics $Y_{\ell,m}$. We recall that

$$P_{\ell}(x) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell,0}(x), \quad (27)$$

where the functions depend only on the direction of the vector in the argument. The direction cosine of the vector is taken relative an axis orthogonal to the orbital plane. For vectors $x, y$ with lengths related by $y < x$ the translation theorem states that

$$\frac{Y_{\ell,m}(x - y)}{|x - y|^{\ell + 1}} = \sum_{\ell', m'} C_{\ell,\ell'}^{m,m'} y^{\ell'} \frac{y_{\ell',m + m'}(x) Y_{\ell',m'}(y)}{x^{\ell + \ell' + 1}} \quad (28)$$

where the expression for the coefficients $C_{\ell,\ell'}^{m,m'}$ is somewhat unwieldy. Because we are going to integrate against the monopole-quadrupole mass densities we need only two terms in each of two special cases,

$$\frac{1}{|x - y|} = 1 + \frac{y^2}{x^3} P_2(x) P_2(y) + \ldots \quad (29)$$

$$\frac{P_2(x - y)}{|x - y|^3} = \frac{P_2(x)}{x^3} + \frac{6y^2}{x^5} P_4(x) P_2(y) + \ldots. \quad (30)$$

For a proof see van Gelderen, who has an easily corrected misprint in his expression for $C_{\ell,\ell'}^{m,m'}$ [25].

Using Eq. (29) we can now calculate the gravitational potential outside the receiver as

$$\Phi_R(x, t) = G \int \frac{\rho_R(x', t)}{|x - x'|} d^3x' = \frac{GM}{r} + \frac{GQ_R}{r^3} P_2(x). \quad (31)$$

There is a similar formula for $\Phi_T$. We recall that, in Newtonian gravity, the total self-force on a body vanishes [26, 27], and we calculate the total work needed to place the receiver at a distance $D$ from the transmitter. Using a vector $D$ of length $D$ we need to calculate

$$V_R = -\int \rho_R \Phi_T d^3x = -\int \rho_R(x) \Phi_T(x + D) d^3x. \quad (32)$$

Appealing to Eq. (30) we find that

$$V_R = -\frac{GM^2}{D} + \frac{GM(Q_R + Q_T)}{2D^2} - \frac{9GQ_R Q_T}{4D^5}. \quad (33)$$

By the terms of the agreement between the twins Tweedledum can choose the quadrupole moment $Q_R = Q_R(t)$ at will, but Tweedledee must then adapt the function
\( Q_T = Q_T(t) \) in such a way that the gravitational force between them stays the same as in the two-body problem for spherical bodies. Thus we impose

\[
\frac{\partial V_T}{\partial D} = \frac{GM^2}{D^2} - \frac{3GM(Q_R + Q_T)}{2D^4} + \frac{45GQ_RQ_T}{4D^6} = \frac{GM^2}{D^2}.
\]

The solution is

\[
Q_T = -\frac{2MD^2Q_R}{2MD^2 - 15Q_R}.
\]

For later use we also record the differential form of the constraint,

\[
(15Q_T - 2MD^2)dQ_R + (15Q_R - 2MD^2)dQ_T - 4MD(Q_R + Q_T)dD = 0.
\]

We assumed that \( 2MD^2 > 15Q_R \). With this precise choice for the quadrupole moments the orbits of the twins are ellipses with a common focus at the midpoint of the line between them, and the mutual distance \( D = D(t) \) has a specified time dependence. It is a periodic function, and we assume that Tweedledum chooses a \( Q_R \) with the same periodicity.

We now wish to calculate the rate of work \( \dot{W}_R \) done on the receiver as his quadrupole moment is changing. Bondi and McCrea use an elegant shortcut for this purpose, but it is interesting to calculate it using field theory. For clarity we decide that \( \dot{W}_R \) is to be calculated in an inertial system where the centre of mass of the receiver is momentarily at rest, while \( \dot{W}_T \) is calculated in a system where the transmitter is momentarily at rest.

From Eq. (9) we see that the rate of gravitational work is

\[
\dot{W}_R = \int \rho_R v_i \partial_i \Phi_T \, d^3x.
\]

Making use of the conservation of mass, Eq. (8), this can be rewritten in two useful ways. Either as

\[
\dot{W}_R = \int [\partial_i(\rho_R v_i \Phi_T) - \Phi_T \partial_i(\rho_R v_i)] \, d^3x = \int \Phi_T \partial_i \rho_R \, d^3x
\]

where a surface term was discarded, or as

\[
\dot{W}_R = \int \rho_R \left( \frac{d}{dt} \Phi_T - \partial_i \Phi_T \right) \, d^3x = -\frac{d}{dt} V_R - \int \rho_R \partial_i \Phi_T \, d^3x
\]

where we made use of the material time derivative, and again used conservation of mass in the second step. The first way is slightly objectionable since it assumes that the mass distribution is smooth, whereas in fact we have chosen it to be discontinuous. This can be repaired by providing the bodies with a smooth skin. There is no such objection to the second way, and we will see that the two ways of calculation give the same result.
Pursuing the first way of calculation we note that in the chosen inertial system the time dependence in $\rho_R$ enters only through the function $Q_R(t)$. Using the translation theorem we find that

$$W_R = \int \partial_t \rho_R(x) \Phi_T(x + D) d^3x$$

$$= \frac{GM \dot{Q}_R}{2} \left( 1 - \frac{9Q_T}{2D^2} \right) = -GM \frac{MD^2 - 3Q_R}{2MD^2 - 15Q_R} \frac{\dot{Q}_R}{D^3}. \tag{40}$$

In the last step we used the constraint (35) between the quadrupole moments. Similarly

$$\dot{W}_T = -GM \frac{MD^2 - 3Q_T}{2MD^2 - 15Q_T} \frac{\dot{Q}_T}{D^3}. \tag{41}$$

Using the constraint (36) we find (after some calculation) that

$$\dot{W}_R + W_T = \dot{F}, \tag{42}$$

where

$$F = \frac{9GQ_RQ_T}{4D^3} - \frac{GM(Q_R + Q_T)}{2D^2} = V_R + \frac{GM^2}{D}. \tag{43}$$

If we integrate to find the total amount of work transmitted during a cycle we find

$$W_R + W_T = \int dF = 0. \tag{44}$$

Hence all of the energy transmitted by Tweedledee is received by Tweedledum.

For the second way of calculation it is convenient to choose an inertial system in which the transmitter is momentarily at rest. Then we are no longer calculating the same thing. Denoting the rate of work done on the receiver in an inertial system where the transmitter is momentarily at rest by $\dot{W}'$, we find

$$\dot{W}' + \frac{d}{dt} V_R = -\int \rho_R(x) \partial_t \Phi_T(x + D) d^3x = -\dot{W}_T. \tag{45}$$

Comparing the equations obtained, and recalling expression (33) for $V_R$, we see that they are fully consistent, once we observe that

$$\dot{W}' = \dot{W}_R + \frac{d}{dt} \left( \frac{GM^2}{D} \right). \tag{46}$$

The motion of the centre of mass explains the difference.

How much energy is being transmitted? With $Q_R = Q_R(t)$ we get
Tweedledum’s aim is to maximize this expression. We see that it can be made positive by letting the function $Q_R(t)$ lag behind the periodic function $D = D(t)$ so that its derivative is negative when $D$ is close to its minimum, and positive when $D$ is close to its maximum. This will ensure that the total amount of work on the receiver in a cycle is positive. A natural choice is to let both bodies be spherical at the moment of closest approach, and at the moment when they are at maximal distance from each other.

From the solution of the two-body problem we know that

$$D = D(\varphi(t)) = \frac{D_{\text{min}}(1 + e)}{1 + e \cos \varphi}$$

where $e$ is the eccentricity of the ellipses and $\varphi$ is an angular coordinate with respect to the centre of mass. For definiteness we let Tweedledum choose

$$Q_R = -Q^\text{max}_R \sin \varphi.$$

The work integral (47) is easily approximated if the bodies are small compared to the distance between them, that is if $Q_R << MD^2$. Then we obtain the positive result

$$W_R \approx -\frac{GM}{2} \int dQ_R = \frac{3\pi}{8} \frac{e(4 + e^2)}{(1 + e)^3} \frac{GMQ^\text{max}_R}{D^3_{\text{min}}}.$$

The function of $e$ that occurs here has a maximum at $e = 2/3$.

The full work integral (47) is best treated numerically. It can be seen that it is always positive and that it diverges as the limit $2D^2M = 15Q_R$ is approached [24]. Thus it is clear that the twins can achieve their aim of transmitting energy from the one to the other through the gravitational field.

4 Energy Transfer

Reading Bondi and McCrea behind a veil of hindsight, and knowing that no-one did more than Bondi to prove that gravitational waves are real and do carry energy away, it is easy to read their paper as an argument for the reality of gravitational waves. This is probably a misreading of history though, since Bondi approached that problem in the best scientific tradition, where nothing is taking for granted [28]. In 1957 we find him arguing against any glib analogy to the simpler theory of electrodynamics:

“The cardinal feature of electromagnetic radiation is that when radiation is produced the radiator loses an amount of energy which is independent of the location of the absorbers. With gravitational radiation, on the other hand, we still [in 1957] do not know whether a gravitational radiator transmits energy whether there is a near receiver or not.” [29]
Two years later, when Bondi and McCrea wrote their paper, the question of the reality of gravitational waves was still on their minds. But the electrodynamic analogy of their thought experiment is not to electromagnetic radiation, the analogy is to inductive energy transport in the near zone such as occurs in a transformer. If the receiver is not there the sender simply stores some energy in the magnetic field, and gets it back when the AC current is turned off.

However, our reason for revisiting the Bondi–McCrea example was that we wanted to see how the various candidates for a local gravitational energy density fare in a concrete example. Equation (21) makes it clear that each choice of energy density comes with a gravitational Poynting vector describing energy transport. Thus, outside matter, the energy density proposed by Lynden-Bell and Katz leads to the Poynting vector

$$S_i = -\frac{1}{4\pi} \partial_i \Phi \partial_j \Phi$$  \hspace{1cm} (51)

The energy densities preferred by Maxwell and by Bondi lead to the respective Poynting vectors

$$S'_i = \frac{1}{4\pi} \Phi \partial_i \partial_j \Phi$$  \hspace{1cm} (52)

$$S''_i = \frac{1}{8\pi} (\Phi \partial_i \partial_j \Phi - \partial_i \Phi \partial_j \Phi).$$  \hspace{1cm} (53)

In Sect. 2 we made the choice between them by deciding what boundary conditions to use in order to control the system. But this is not at all how the concrete problem is posed. The gravitational potential is controlled from inside the system by Tweedledum, and by his agreement with Tweedledee.

However, among the three, Bondi’s Poynting vector $S''_i$ enjoys the advantage that it is divergence free in vacuum. This lends a special significance to the flux integral

$$I = \int_S S''_i dS_i,$$  \hspace{1cm} (54)

where $S$ is a closed surface in vacuum. The surface can be freely deformed within the vacuum region without changing the value of the integral. If there is vacuum outside the surface it evaluates to zero. There is no net flux out of the surface, regardless of how the body inside the surface is changing its multipole moments. But if the surface divides two regions containing two distinct bodies it quantifies the amount of energy transferred between them [2]. This means that Bondi’s Poynting vector offers the best way of calculating the energy transferred between Tweedledum and Tweedledee. Nevertheless the necessary calculations are very complex. They are discussed in Ref. [24].

In conclusion, we have found that there is a sense in which Newtonian gravity prefers a local energy density that is in agreement with that of Lynden-Bell and Katz [8], but it is notable that this plays no useful role in the concrete discussion of energy transfer between Tweedledee and Tweedledum. There it seems much more
helpful to regard the energy as localized within the matter. In the end, the work done
on the body is independent of the way in which gravitational energy is localised
[30]. In general relativity the question of how to define a useful notion of quasi-
local energy has given rise to a large literature [31], while the definition of the total
energy of an isolated system is clear [32]. The warning to relativists is not to expect
a unique answer, rather we expect several different answers that are useful in differ-
ent ways. The question should be: What quasi-local energy expression is best suited
to describe energy transport in a given concrete situation?

Acknowledgements

It is a pleasure to thank the referees for their thoughtful comments.

Funding

Open access funding provided by Stockholm University.

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