The study of quantum correlations has traditionally focused on entanglement [1]. It is generally believed that entanglement is a necessary resource for quantum computers to outperform their classical counterparts. Indeed, it has been shown that for the setting of pure-state computation, the amount of entanglement present must grow with the system size for an exponential speed-up to occur [2]. In the context of mixed-state quantum information processing, however, there are computational and communication feats which are seemingly impossible to achieve with a classical computer, and yet can be attained with a quantum computer using little or no entanglement (e.g. [3, 4]). For example, the Deterministic Quantum Computation with one Qubit (DQC1) model is believed to estimate the trace of a unitary matrix exponentially faster than any classical algorithm, yet with vanishing entanglement during the computation [5]. A second example is the ability for certain bipartite quantum systems to contain a large amount of “locked” classical correlations, which can then be “unlocked” with a disproportionately small amount of classical communication [4]. This task is impossible classically, yet the quantum states involved are separable, that is, unentangled. This raises the crucial question about which, if not entanglement, is the fundamental resource enabling such feats.

One plausible explanation is associated with the presence in (generic [6]) quantum states of correlations which have nonclassical signatures that go beyond entanglement. Indeed, much attention has recently been devoted to understanding and quantifying such correlations for this very reason [6, 10, 16]. In particular, the separable quantum states of the systems involved in DQC1 and the locking protocol have been shown to possess non-zero amounts of such correlations [5, 17], as measured by the quantum discord [7]. The latter strives to capture nonclassical correlations beyond entanglement and has recently received operational interpretations in terms of the quantum state merging protocol [18], but is unfortunately not a faithful measure [19]. A more accurate quantification of nonclassical correlations is provided by the so-called relative entropy of quantumness (REQ) [8, 14–13], defined as the distance, in terms of relative entropy, between a multipartite quantum state and the closest strictly classically correlated state (see Definition [1]). Such a measure is faithful [11], symmetric under permutation of the subsystems, and enables a unified approach to the quantification of classical, separable and entangled correlations [10].

More generally, the role of nonclassical correlations in quantum information tasks remains unclear. While all entangled states are known to be useful for information processing [20], the fundamental question of whether the same holds for all nonclassically correlated (separable) states stays open. This raises the question: Is there a setting in which general nonclassical correlations produce a physically relevant effect that distinguishes them from purely classical ones?

In this Letter, we answer the question in the affirmative by demonstrating a protocol which in some sense activates the nonclassicality present in any multipartite quantum system, leading to the creation of entanglement. We then show that the REQ of any system state input to our protocol is precisely the minimum distillable entanglement generated between the system and local ancillae via the protocol. This result renders the REQ both an operational and faithful nonclassicality measure. According to our framework, all and only the quantumly correlated states are shown to possess an entanglement potential that makes them readily useful for better-than-classical information processing. Finally, we prove limits on nonclassical correlations for separable and pure entangled states in any dimension, while, perhaps surprisingly, these bounds can be exceeded by mixed entangled states.

Our results apply to general multipartite states, adopting the following definition of classicality [14].
FIG. 1. (Color online) Activation protocol for \( n = 3 \).

**Definition 1 (Strictly Classically Correlated Quantum State).** Given a set of \( n \) \( d \)-dimensional qudit systems, let \( B_i \) denote an orthonormal basis in \( \mathbb{C}^d \) for the \( i \)th system consisting of vectors \( |B_i(k)\rangle \) for \( 0 \leq k \leq d - 1 \), and let \( B \) denote an orthonormal basis \( \{|B(k)\rangle = |B_1(k_1)\rangle|B_2(k_2)\rangle\cdots|B_n(k_n)\rangle\} \) for the entire space \( (\mathbb{C}^d)^{\otimes n} \) formed by taking tensor products of all elements in bases \( \{B_i\}_{i=1}^n \). Then, an \( n \)-qudit state \( \rho \) is strictly classically correlated—or simply classical—if there exists a basis \( B \) with respect to which \( \rho \) is diagonal. Such states correspond to the embedding of a multipartite classical probability distribution into the quantum formalism.

**Activation protocol.**—We now describe our protocol for activation of nonclassical correlations. The scheme is somewhat inspired by the quantum optics setup of [21], where one attempts to quantify nonclassicality of a single field mode (defined there as the state deviation from a mixture of coherent states) by reducing the problem to quantifying the two-mode entanglement that can be generated from the field using linear optics, auxiliary classical (coherent) states, and ideal photodetectors. Similarly, we may expect that mapping the (still not-well-understood) nonclassicality of multipartite correlations into “more familiar” bipartite entanglement allows one to employ tools from entanglement theory [1] to interpret and quantify general nonclassical correlations.

Our activation protocol can be thought of as a game between an adversary and \( n \) players, where the \( n \) players together aim to generate an entangled state between a system \( A \) they control and an ancillary system \( A' \), and the adversary’s goal is to thwart their efforts by locally rotating each subsystem of \( A \) before system and ancilla undergo a pre-defined interaction. More precisely, the protocol proceeds as follows (see Fig. 1). We consider \( n \) players \( P_i \), each controlling a system-ancilla pair of qudits \( (A_i, A'_i) \). We indicate by \( A \) the joint register \( A_1, \ldots, A_n \) (“system”), and by \( A' \) the joint register \( A'_1, \ldots, A'_n \) (“ancilla”). The initial state of the total \( 2n \) qudits is a tensor product \( \rho_{A:A'} = \rho_A \otimes |0\rangle\langle 0|^{\otimes n} \). For a given \( \rho_A \), an adversary is first allowed to apply a local unitary \( U_i \) of his choice to each \( A_i \). With the adversary’s turn complete, each player \( P_i \) now lets their subsystem \( A_i \) (control qudit) interact with the corresponding ancillary party \( A'_i \) (target qudit) via a CNOT gate \( C_{A_i:A'_i} \), whose action on the computational basis states \( |j\rangle|j'\rangle \) of \( \mathbb{C}^d \otimes \mathbb{C}^d \) is defined as \( C|j\rangle|j'\rangle = |j\rangle|j' \oplus j\rangle \), with \( \oplus \) denoting addition modulo \( d \).

The final state of system plus ancilla is

\[
\tilde{\rho}_{A:A'} = V(\rho_A \otimes |0\rangle\langle 0|^{\otimes n})V^\dagger,
\]

with \( V = C_{A:A'} \cdot (U_A \otimes \mathbb{I}_{A'}) \), \( U_A = \otimes_{i=1}^n U_i \), and \( C_{A:A'} = \otimes_{i=1}^n C_{A_i:A_i'} \). We ask: At the end of the protocol, have the \( n \) players succeeded in generating bipartite entanglement across the split \( A : A' \), and, if so, how much entanglement was created? It is natural to expect that the answer will depend on the initial state \( \rho_A \) of the \( n \)-qudit system. From a physical perspective, our aim is to understand precisely how the nature and amount of correlations between the parts \( A_i \) of the system \( A \) affects the entanglement that can be created with an ancilla \( A' \) via the paradigmatic entangling operation — the CNOT; we consider the worst case scenario with respect to the choice of the control bases. We then find the following.

**Theorem 1.** The (initial) state \( \rho_A \) of an \( n \)-qudit system is strictly classically correlated if and only if there exists some adversarial choice of local unitaries \( U_A \) such that the (final) state \( \tilde{\rho}_{A:A'} \) output by the activation protocol is separable across the system-ancilla split.

In other words, the system always becomes (for any choice of \( U_A \)) entangled with the ancilla as a result of the activation protocol, if and only if the input state of the system is nonclassically correlated. This establishes a qualitative equivalence between multipartite nonclassical correlations among components of a quantum system, and bipartite entanglement between the system and an ancilla, and settles the issue of the usefulness of nonclassical correlations in (even separable) quantum states for quantum primitives: Any kind of multipartite nonclassicality initially present in \( A \) is a resource for information processing that can always be activated, or mapped into bipartite entanglement across the \( A : A' \) split. While a direct proof of this result is quite straightforward (see Appendix A [23]), in the following we show a more powerful result that promotes the equivalence between nonclassicality and entanglement to a quantitative relationship.

**Quantifying nonclassicality.**—Having run the activation protocol, we now proceed to the next logical step: Namely, we wish to quantify the entanglement generated in the \( A : A' \) split whenever \( A \) is initially in a nonclassically correlated state. The present framework is general enough to allow us to uncover a full zoology of nonclassicality measures, as each choice of a different entanglement monotone [23] we adopt (at the output) leads in principle to a unique nonclassicality measure (for the input state), the association stemming exactly from the activation protocol. More precisely, let \( E \) denote some entanglement measure of choice and \( \tilde{\rho}_{A:A'} \) the final system-ancilla state as in Eq. (1), and define by

\[
Q_E(\rho_A) := \min_{U_A} E_{A:A'}(\tilde{\rho}_{A:A'})
\]
the minimum entanglement generated across the $A : A'$ split over all choices of adversarial local unitaries $U_A$. We call $Q_E(\rho_A)$ the minimum entanglement potential of $\rho_A$ with respect to $E$. As a consequence of Theorem 1, $Q_E$ is a measure of nonclassical correlations in the multipartite state $\rho_A$, for every entanglement monotone $E$.

In fact, the condition $Q_E(\rho_A) = 0$ perfectly characterizes the set of classically correlated states $\rho_A$ if $E$ is a faithful entanglement measure (i.e., if $E$ vanishes only for separable states). However, even certain non-faithful entanglement measures can be plugged in to obtain a faithful measure of nonclassical correlations [19].

The reason is that the output state $\tilde{\rho}_{A:U_A}$ has the so-called maximally correlated form [24] between $A$ and $A'$; namely, $\tilde{\rho}_{A:U_A} = \sum_k \tilde{\rho}_M^k |k\rangle |l\rangle_A \otimes |k\rangle |l\rangle_{A'}$ with $\rho_M^k = (\mathcal{B}(k)|\rho_A|\mathcal{B}(l))_A = U_{A'}^k |k\rangle$ and $|k\rangle = |k_1\rangle |k_2\rangle \cdots |k_n\rangle$. In particular, let us consider the non-faithful (as it vanishes on so-called bound entangled states) but physically motivated distillable entanglement $E_D$ [23] as a bipartite entanglement monotone. We find that the $A : A'$ distillable entanglement of $\tilde{\rho}_{A:U_A}$ is equal to $E_D(\tilde{\rho}_{A:U_A}) = S(\tilde{\rho}_{A:U_A}) = S(\tilde{\rho}_{A}) - S(\tilde{\rho}_A)$, where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy of a state $\rho$. In the first equality we used the results of [25] about distillable entanglement for maximally correlated states—for which it happens to coincide with the relative entropy of entanglement [26]. The second equality is justified by the fact that $\tilde{\rho}_{A:U_A}$ is the state resulting from local projective measurements in the local bases $|B\rangle$ on $\rho_A$ and is unitarily equivalent to $\rho_A$, while $\tilde{\rho}_{A:U_A}$ is obtained from $\rho_A$ via the activation protocol isometry, Eq. (1). Thus, the minimum distillable entanglement potential $Q_{E_D}(\rho_A)$ takes on the form $Q_{E_D}(\rho_A) = \min_B (S(\rho_B^A) - S(\rho_A))$, where the minimization is over the choice of the bases $B$. As proven in [10], this is an equivalent expression for the REQ.

$$Q(\rho_A) = \min_{\text{classical } \sigma_A} S(\rho_A || \sigma_A),$$

where the relative entropy is defined as $S(\rho || \sigma) = \text{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$ and the minimization is over all strictly classically correlated states $\sigma_A$. We have thus proven that the REQ quantifying general nonclassical correlations between the $n$ subsystems $A_i$ of $A$ is exactly equal to the minimum bipartite distillable entanglement potential—or, equivalently, to the minimum relative entropy of entanglement potential—generated between the system $A$ and the ancillary register $A'$.

This finding immediately provides a clearcut operational interpretation for the REQ, a quantity whose original definition was purely geometric [Eq. (5)], which then emerges as a mathematically sound and physically motivated measure of nonclassical correlations for arbitrary quantum states, quantifying equivalently the resource power of such correlations for (distillable) entanglement generation. Incidentally, since the REQ is faithful [11], this yields a proof of Theorem 1.

Other nonclassicality measures can be induced by different entanglement monotones. Choosing e.g. the “negativity” $N$ [27] as an entanglement measure, one obtains $Q_N(\rho_A) = (\min \sum_{i \neq j} |\langle i \rangle_{A'B}^j |^2)/2$ as a quantifier of nonclassical correlations (see Appendix B [23] for details), directly related to the off-diagonal coherences of the density matrix of the system, minimized over all local bases.

**Nonclassicality versus mixedness and entanglement.**—Equipped with a faithful and operational measure of nonclassical correlations, the REQ $Q \equiv Q_{E_0}$, we can investigate the interplay between nonclassicality, entanglement and mixedness of general states $\rho_A$. For the sake of simplicity, from now on we restrict to the bipartite case $A_1 = A, A_2 = B$. We begin with a few simple but general observations following from the definition of $Q$.

For pure states $\rho_{AB} = |\psi\rangle \langle \psi|$, the quantumness $Q$ reduces to the von Neumann entropy of entanglement $S(\rho_A) = S(\rho_B)$ [13], and is thus at most equal to $\log_2 d$. On the other hand, for arbitrary mixed $\rho_{AB}$, we have that $Q(\rho_{AB})$ is at most $2 \log_2 d$, since from Eq. (3) one has $Q(\rho_{AB}) \leq S(\rho_{AB} || \rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = I(\rho_{AB})$, where $I$ denotes the mutual information, a measure of total correlations. From this and the results of [28], one realizes that for a separable state a bound $Q(\rho_{AB}^{sep}) \leq \log_2 d$ holds. In Appendix C [22], we prove in fact that this inequality is always sharp for separable states, i.e., the bound $\log_2 d$ cannot be exactly saturated for separable nonclassical states, while it is instead trivially reached by pure maximally entangled states $|\psi\rangle = \sum_{j=1}^{d^2-1} |j\rangle |j\rangle$. Almost all separable states thus possess nonclassical correlations [6], but not to a maximal extent (as already observed in the particular cases of two-qubit [29] and two-mode Gaussian states [16]). However, with increasing $d \to \infty$ we find quite surprisingly that the upper bound on the REQ of separable states becomes asymptotically tight, in the sense that separable states exist such that $Q(\rho_{AB}^{sep})/\log_2 d \to 1$. Even more intriguingly, we can show that the upper bound on general mixed bipartite states $\rho_{AB}$ is also asymptotically tight, in the sense that families of mixed states exist such that in the limit $d \to \infty$, their quantumness converges to the maximum, $Q(\rho_{AB})/\log_2 d \to 2$. More precisely, in Appendix D [22] we prove the following two results using techniques from Refs. [30, 31]. Let $m = \lfloor \log_2 d \rfloor$.

**Theorem 2.** Define the following random separable state:

$$\sigma_{AB} = \frac{1}{dm} \sum_{i=1,\ldots,d^2} |i\rangle \langle i| \otimes \left( U_i |i\rangle \langle i| U_i^\dagger \right),$$

with unitaries $U_i$ drawn independently from the Haar measure. Then, $S(\sigma_{AB}) \leq \log_2 d + \log_2 m$, while on the other hand, for $d$ sufficiently large and with high probability, $S(\sigma_{AB}^{\text{sep}}) \geq 2 \log_2 d - \text{const.}$, for all $B$. Hence, $Q(\sigma_{AB}) \geq \log_2 d - O(\log_2 \log_2 d)$.

**Theorem 3.** Define the following random state: For a system of dimension $m$, let $\rho_{AB} = \text{Tr}_C |\psi\rangle \langle \psi|_{ABC}$, where $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^m$ is uniformly distributed (with probability induced by the Haar measure). Then, $S(\rho) \leq \log_2 m$, while on the other hand, for $d$ sufficiently large and with high probability, $S(\rho^B) \geq \log_2 d - \log_2 \log_2 m$. These results show that, first, there are separable states that
asymptotically (in \( d \)) are as nonclassical as the most nonclassical pure state (which is the maximally entangled state); second, mixed entangled states can be twice as nonclassical as pure entangled states. Both entanglement and mixedness are required to “break the barrier” of \( \log_2 d \), thus showing that entanglement by itself is not the strongest form of nonclassicality.

Conclusions.— The study of general nonclassical correlations is currently a burgeoning area, but in many ways such correlations are still not well-understood. Our activation protocol lends new insight into the nature of these correlations by furnishing them, in full generality, with a new operational meaning in terms of resources for entanglement generation. Furthermore, we have reduced the problem of quantifying nonclassicality to the more familiar setting of quantifying entanglement, for which a multitude of tools for analysis are already known (see e.g. [1]). As an added bonus, we have obtained an alternative operational interpretation for the relative entropy of quantumness measure \([8, 10]\). Finally, with respect to the latter, we have demonstrated that, remarkably, there exist mixed entangled quantum states whose nonclassical correlations are stronger than those of pure entangled states. Further investigation on the nature and the structure of nonclassical correlations, following the programme laid by this Letter, may trigger novel developments in quantum technology and shed light on foundational aspects of quantum theory.

Note added.—After completion of this Letter, we became aware of some related results by Streltsov et al. [32], who showed that the quantumness of correlations (as measured e.g. by the quantum discord) is also related to the minimum entanglement generated between system and apparatus in a partial measurement process. In light of those results, our findings can be understood also as dealing with the interplay between system-apparatus entanglement and nonclassicality of correlations when realizing local measurements.

We thank F. Brandão, N. Brunner, D. Bruß, H. Kampermann, D. Leung and A. Streltsov for discussions. We acknowledge support by NSERC, QuantumWorks, CIFAR, Ontario Centres of Excellence, the Spanish government (program FIS2008-01236/FIS), the Catalan government (program 2009SGR-0985), the United Kingdom EPSRC, the European Commission, the ERC, the Philip Leverhulme Trust, the Royal Society, and the Integrated Project QESSENCE. M. P. was supported by the Austrian Science Fund (FWF) through the Lise Meitner program while at the University of Innsbruck.

[1] R. Horodecki et al., Rev. Mod. Phys. 81, 865 (2009).
[2] R. Jozsa and N. Linden, Proc. Roy. Soc. A 459, 2011 (2003).
[3] E. Knill and R. Laflamme, Phys. Rev. Lett. 81, 5672 (1998).
[4] D. P. DiVincenzo et al., Phys. Rev. Lett. 92, 067902 (2004).
[5] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. 100, 050502 (2008).
[6] A. Ferraro et al., Phys. Rev. A 81, 052318 (2010).
[7] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001); L. Henderson and V. Vedral, J. Phys. A: Math. Gen. 34, 6899 (2001).
[8] M. Horodecki et al., Phys. Rev. A 71, 062307 (2005).
[9] B. Groisman, S. Popescu, and A. Winter, Phys. Rev. A 72, 032317 (2005).
[10] K. Modi et al., Phys. Rev. Lett. 104, 080501 (2010).
[11] B. Groisman, D. Kenigsberg, and T. Mor, arXiv:quant-ph/0703103.
[12] A. SaiToh, R. Rahimi, and M. Nakahara, Phys. Rev. A 77, 052101 (2008); S. Luo, ibid. 77, 022301 (2008).
[13] S. Bravyi, Phys. Rev. A 67, 012313 (2003).
[14] M. Piani, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 100, 090502 (2008).
[15] M. Piani et al., Phys. Rev. Lett. 102, 250503 (2009).
[16] G. Adesso and A. Datta, Phys. Rev. Lett. 105, 030501 (2010).
[17] A. Datta and S. Gharibian, Phys. Rev. A 82, 067902 (2009).
[18] D. Cavalcanti et al., Phys. Rev. A 83, 032324 (2011); V. Madhok and A. Datta, ibid. 83, 032323 (2011).
[19] A measure of nonclassical correlations is faithful when it vanishes iff a state is strictly classically correlated.
[20] L. Masanes, Phys. Rev. Lett. 96, 15051 (2006); M. Piani and J. Watrous, ibid. 102, 250501 (2009).
[21] J. K. Asbóth, J. Calsamiglia, and H. Ritsch, Phys. Rev. Lett. 94, 173602 (2005).
[22] See Supplemental material, EPAPS no. XXXX, for additional proofs and details.
[23] M. B. Plenio and S. Virmani, Quant. Inf. Comp. 7, 1 (2007).
[24] E. M. Rains, IEEE Trans. Inf. Theory 47, 2921 (2001).
[25] T. Hiroshima and M. Hayashi, Phys. Rev. A 70, 030302 (2004).
[26] V. Vedral et al., Phys. Rev. Lett. 78, 2275 (1997); V. Vedral and M. B. Plenio, Phys. Rev. A 57, 1619 (1998).
[27] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[28] M. A. Nielsen and I. Kempe, Phys. Rev. Lett. 86, 5184 (2001).
[29] A. Al-Qasimi and D. F. V. James, Phys. Rev. A 83, 032101 (2011).
[30] P. Hayden et al., Comm. Math. Phys. 250, 371 (2004).
[31] P. Hayden, D. Leung, and A. Winter, Comm. Math. Phys. 265, 95 (2006).
[32] A. Streltsov, H. Kampermann, and D. Bruss, Phys. Rev. Lett. 106, 160401 (2011).
Supplemental Material

All non-classical correlations can be activated into distillable entanglement
Marco Piani, Sevag Gharibian, Gerardo Adesso, John Calsamiglia, Paweł Horodecki, and Andreas Winter

Proof of Theorem 1

Proof. The "if" part is trivial, as given a strictly classical correlated state one can choose $U_A$ to make the $n$-orthogonal spectral basis of

$$\rho_A = \sum_i \rho_i |B(i)\rangle\langle B(i)|$$

coincide with the computational basis, so that

$$\tilde{\rho}_{A : A'} = \sum_i \rho_i |i\rangle_A \otimes |i\rangle_{A'}.$$

As regards the "only if" part, let us consider the separable decomposition

$$\tilde{\rho}_{A : A'} = \sum_{\alpha} q_{\alpha} |\psi^{\alpha}\rangle \langle \psi^{\alpha}| \otimes |\phi^{\alpha}\rangle \langle \phi^{\alpha}|$$

which exists by hypothesis for some choice of $U_A$. Since the transformation (1) is unitary and invertible, there must exist a pure ensemble $\{q_{\alpha}, |\xi^{\alpha}\rangle_A\}$ such that $\rho_A = \sum_{\alpha} q_{\alpha} |\xi^{\alpha}\rangle \langle \xi^{\alpha}|$ and

$$V|\xi^{\alpha}\rangle_A |0\rangle_{A'} = |\psi^{\alpha}\rangle_A \otimes |\phi^{\alpha}\rangle_{A'}$$

(4)

Let us expand $|\xi^{\alpha}\rangle_A$ on the computational basis rotated by $U_A^\dagger$:

$$|\xi^{\alpha}\rangle_A = \sum_a c^\alpha_{a A} U_A^\dagger |a\rangle_A$$

and compute the action of $V$:

$$V|\xi^{\alpha}\rangle_A |0\rangle_{A'} = \sum_a c^\alpha_{a A} |a\rangle_A \otimes |a\rangle_{A'}$$

Imposing the factorization condition (4) we find that it must be $c^\alpha_{a A} = c^{\alpha}_{f (\alpha)} \delta_{a, f (\alpha)}$, $|c^{\alpha}_{f (\alpha)}| = 1$ for some $f(\alpha) \in \mathbb{N}^n$. Therefore,

$$\rho_A = \sum_{\alpha} q_{\alpha} \left( \sum_a c^\alpha_{a A} U_A^\dagger |a\rangle_A \right) \left( \sum_b c^\alpha_{b A} \langle b_A|U_A \right)$$

$$= \sum_{\alpha} q_{\alpha} \left( \sum_a c^{\alpha}_{f (\alpha)} \delta_{a, f (\alpha)} U_A^\dagger |a\rangle_A \right) \left( \sum_b c^{\alpha}_{f (\alpha)} \delta_{b, f (\alpha)} \langle b_A|U_A \right)$$

$$= \sum_{\alpha} q_{\alpha} U_A^\dagger |f(\alpha)\rangle_A \langle f(\alpha)|U_A.$$

As every $U_A^\dagger |f(\alpha)\rangle_A$ is part of one and the same orthogonal basis, we get the claim. □

Thus, every non-strictly classical state, either separable or entangled, does lead to the production of entanglement in the $A : A'$ cut, whatever the local rotation $U_A$. On the other hand, such production of entanglement can be excluded by a proper local rotation $U_A$ in the case of a strictly classically correlated state.
Negativity of quantumness

The negativity is an entanglement monotone, defined for a bipartite state \(\rho_{A:B}\) as \(N(\rho_{A:B}) = (||\rho_{A:B}^T|| - 1)/2\), with \(||X||_1 = \text{Tr}\sqrt{X^TX}\) the trace norm, and \(\rho_{A:B}^T\) the partially transposed state. Thanks to the maximally correlated form, even in the multipartite case it is easy to calculate the eigenvalues of \(\rho_{A:A'}\), which are given by \(\rho_{i,i}^B\), for all \(i\), and by the coherences \(|\rho_{i,j}^B|\) for \(i > j\) (understood lexicographic order). Thus, \(N(\hat{\rho}_{A:A'}) = (||\rho_{A:A'}^T|| - 1)/2 = \left(\sum_{i\neq j} |\rho_{i,j}^B|\right)/2\), and we obtain another (quantitative) proof that \(\hat{\rho}_{A:A'}\) is entangled for any rotation \(U_{A}\) if and only if \(\rho_{A}\) is not classical. Indeed, by definition a non-classical state has some non-vanishing coherence \(\rho_{i,j}^B\), for some \(i \neq j\), in any basis \(\mathcal{B}\). For a pure state \(|\psi\rangle_A = \sum_i \psi_i^B |\mathcal{B}(i)\rangle\), with \(\sum_i |\psi_i|^2 = 1\), one has \(\rho_{A:A'} = \sum_i \psi_i^B |\psi_i^B\rangle\langle\psi_i^B|\), so that \(\|ho_{A:A'}^T\| = \left(\sum_i |\psi_i|^2\right)^2\). In the bipartite case \(|\psi\rangle_{AB}\) one has \(\sum_i |\psi_i|^2 = \sum_i \epsilon_i \|\psi_i\|^2 = \sum_i \epsilon_i \|\hat{\rho}_{i,i}^B\| = \sum_i \sqrt{\lambda_i^B}\), with \(\hat{\rho}_{i,i}^B\) the matrix of coefficients, \(\|\cdot\|_\epsilon\) and \(\|\cdot\|_1\) the \(\epsilon_1\)-norm and trace norm, respectively, and \(|\psi\rangle = \sum_k \lambda_k^B |\alpha_k\rangle_A |\beta_k\rangle_B\) the Schmidt decomposition of \(|\psi\rangle\).

Thus, the negativity of quantumness \(Q_N(\rho_{AB})\), i.e., the minimum negativity of \(\hat{\rho}_{A:B:A':B'}\), is exactly equal to the standard negativity of \(|\psi\rangle_{AB}\). One can further consider the exemplary mixture \(\rho(\psi,p) = (1-p)\mathbb{I} + p|\psi\rangle\langle\psi|\), with \(\mathbb{I}/d^2\) the maximally mixed state. A straightforward calculation, taking again into account the maximally correlated structure of \(\hat{\rho}_{A:B:A':B'}\), leads to \(Q_N(\rho(\psi,p)) = pN(\psi)\). So, as already observed in, e.g., \(\textbf{11}\), \(\rho(\psi,p)\) is non-classical as long as \(p > 0\) and \(\psi\) is entangled.

Maximal non-classicality of separable states

Let us consider a separable state \(\rho_{AB} = \rho_{AB}^{sp} = \sum_m p_m |\alpha_m\rangle\langle\alpha_m| \otimes |\beta_m\rangle\langle\beta_m|\), with \(\{p_m\}\) a probability distribution, and \(|\alpha_m\rangle, |\beta_m\rangle\) arbitrary pure states. Then,

\[
Q(\rho_{AB}) = \min_{\mathcal{B}} \left( S(\rho_{AB}^B) - S(\rho_{AB}) \right)
\leq \min_{\mathcal{B}} \left( S(\rho_{AB}^B) - S(\rho_{A}) \right)
= \min_{\mathcal{B}} \left( S(\rho_{AB}^A) + \sum_i \langle B_A(i)|B_A(i)\rangle S(\sigma_i^B) - S(\rho_{A}) \right)
\leq \min_{\mathcal{B}} \sum_i p_i A S(\sigma_i^B),
\]

where

\[
\sigma_i^B = \frac{\langle B_A(i)|B_B(j)\rangle \rho_{AB} |B_A(i)B_B(j)\rangle}{\langle B_A(i)|\rho_{A}|B_A(i)\rangle} |B_B(j)\rangle |B_B(j)\rangle,
\]

and \(\{p_i\}\) are the eigenvalues of \(\rho_{A}\). The first inequality is due to the fact that for any separable state \(S(\rho_{AB}) \geq \max\{S(\rho_{A}), S(\rho_{B})\}\) \(\textbf{[28]}\). The second inequality comes from choosing as particular basis \(\mathcal{B}_{A}\) an eigenbasis of \(\rho_{A}\), so that \(S(\rho_{A}^B) = S(\rho_{A})\). Now, this upper bound is equal to \(\log_2 d\) only if \(\sigma_i^B\) is maximally mixed for all \(i\), that implies that \(\rho_{B}\) is also maximally mixed. Reversing the role of \(A\) and \(B\), we also find that \(\rho_{A}\) must be maximally mixed for \(Q(\rho_{AB})\) to be compatible with \(\log_2 d\). This means that the basis chosen in the second inequality is arbitrary, and we find that for the last line to be equal to \(\log_2 d\), it must be that \(\langle B_A(i)|B_B(j)\rangle |\rho_{AB} |B_A(i)B_B(j)\rangle = 1/d^2\) for all \(B_A, B_B\) and all \(i,j\). Thus it must be \(\rho_{AB} = \mathbb{I}/d^2\). But the latter state is classical. Thus, for any separable state that is not classical, we find that \(Q(\rho_{AB})\) is less than \(\log_2 d\), a value that is instead achieved by a maximally entangled state of \(A\) and \(B\).

Proofs of Theorems \(\textbf{2}\) and \(\textbf{3}\)

We will consider arbitrary local complete von Neumann measurements

\[
M = (|m_x\rangle\langle m_x|)_{x=1}^d, \quad N = (|n_y\rangle\langle n_y|)_{y=1}^d.
\]
on $A$ and $B$, respectively, and denote by $M$ also the completely positive trace-preserving projection associated to the measurement:

$$M(\sigma) = \sum_x |m_x\rangle\langle m_x| \sigma |m_x\rangle\langle m_x|,$$

and likewise for $N$. In the following let $m = \lceil (\log_2 d)^4 \rceil$.

**Proof of Theorem 2.** The state $\sigma$ is almost identical to the information locking states considered in [30][Thm. V.1, eq. (64)], except that there also $j$ is given in an extra register to $A$. There it is shown that — when $d$ is sufficiently large and with high probability — for any (projective) measurements on $A$ and $B$ with classical outputs $x$ and $y$, respectively,

$$I(x:y) \leq I(i:j:y) \leq \text{const.} \quad (5)$$

This is because with our choice for $m$ ($n$ in [30]), the parameter $\epsilon$ in the Eq. (66) of [30] can be chosen to scale as $1/\log_2 d$. The bound (5) must hold true also for our state, since all we do is remove $j$ before the measurement.

Note however, that $x$ and $y$ have maximal entropy $\log_2 d$, since $\sigma_A = \sigma_B = \mathbb{1}/d$ are both maximally mixed. That means that $S(\sigma^{AB}_A) \geq 2 \log_2 d - \text{const.}$ as claimed.

The upper bound on $S(\sigma_{AB})$ follows by observing that the rank of $\sigma_{AB}$ can be at most $dm$. \hfill \square

**Proof of Theorem 3.** The random state considered here is analysed in detail already in [31], and we may refer to that paper for technical results.

Since the rank of $\rho$ is bounded by $m$, the upper bound on $S(\rho)$ is clear.

On the other hand, let us analyze the measure concentration of the entropy $S((M \otimes N)\rho)$ — first only for a fixed pair $M$ and $N$. We use the elementary estimate

$$S((M \otimes N)\rho) \geq S_2((M \otimes N)\rho)$$

$$= -\log_2 \sum_{x,y=1}^d \langle \text{Tr}(M_x \otimes N_y \otimes \mathbb{1}) \rangle^2,$$

where $S_2(\sigma) = -\log_2(\text{Tr}\sigma^2)$ is the (quantum) Renyi entropy of order 2. Hence, invoking the convexity of $-\log$ and the unitary invariance of the distribution of $\psi$,

$$\mathbb{E}_\psi S((M \otimes N)\rho) \geq -\log_2 \sum_{x,y=1}^d \langle \text{Tr}(M_x \otimes N_y \otimes \mathbb{1}) \rangle^2$$

$$= -\log_2 \left( d^2 \mathbb{E}_\psi \langle \text{Tr}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes \mathbb{1}) \rangle^2 \right)$$

$$= -\log_2 \left( d^2 \text{Tr}(\mathbb{1} + F_{ABC:A'B'C'} \langle 00\rangle\langle 00|_{AA'} \otimes |00\rangle_{BB'} \otimes \mathbb{1}_{CC'}) \right)$$

$$= \log_2 \frac{d^2 m + 1}{m + 1} \geq 2 \log_2 d - \log \left( 1 + \frac{1}{m} \right).$$

The identity in the third line, $\text{Tr}(F_{C:D}X_C \otimes Y_D) = \text{Tr}(XY)$, where $F_{C:D}$ is the swap operator between $C$ and $D$, is a standard trick.

The Lipschitz constant of the entropy $S((M \otimes N)\rho)$ can be taken directly from the Appendix B of [31]: it is upper bounded by $\sqrt{3} \log_2 d^2$. Thus, by Levy’s Lemma:

$$\text{Pr} \left\{ S((M \otimes N)\rho) < 2 \log_2 d - \log \left( 1 + \frac{1}{m} \right) - \epsilon \right\} \leq \exp \left( -\frac{\epsilon^2}{32(\log_2 d)^2 d^2 m} \right). \quad (6)$$

for some constant $c > 0$.

The rest of the proof is a net argument. If we consider $T$ possible basis pairs $M_t$, $N_t$ ($t = 1, \ldots, T$), by the union bound we have:

$$\text{Pr} \left\{ \exists t \ S((M_t \otimes N_t)\rho) < 2 \log_2 d - \log \left( 1 + \frac{1}{m} \right) - \epsilon \right\} \leq T \exp \left( -\frac{\epsilon^2}{32(\log_2 d)^2 d^2 m} \right). \quad (7)$$

We shall show that it is enough to consider

$$T \leq \left( \frac{\epsilon' d^{3/2} (\log_2 d)^2}{\epsilon^2} \right)^{4d^2}$$
basis pairs, for which the probability in eq. (7) is $\ll 1$ for our choice of $m = \lceil \log_2 d \rceil$ and $d$ large enough. Each local measurement is described by an orthonormal basis $\{|b_x\rangle\}_{x=1}^d$. If we think of the vectors as column vectors relative to some standard basis, we can arrange them in a $d \times d$ unitary matrix, the matrix rotating the standard basis to $\{|b_x\rangle\}_{x=1}^d$. Then, the claim is that on the unitary group $\mathcal{U}(d)$ with operator norm distance, there exists a $\delta$-net of

$$T_0 \leq \left( \frac{c' \delta^{-3/2}}{\delta} \right)^{2d^2}$$

elements (see the net estimate below). Choosing $\delta = \epsilon^2/(4 \log^2 d)$ we obtain $T$ as $T_0^2$ (that is, using $T_0$ elements $\{M_i\}$ and $T_0$ elements $\{N_i\}$).

Now let $M$ and $N$ be arbitrary product von Neumann projective measurements. We find $M_s$ and $N_t$ in the net such that the unitaries of $M$ and $M_s$ are $\delta$-close, and those of $N$ and $N_t$ likewise, which implies that $M \otimes N$ is $2\delta$-close to $M_s \otimes N_t$ or in other words the projection bases are related by a unitary $U$ that is $2\delta$-close to the identity $\mathbb{1}$. But then it holds in trace distance $\frac{1}{2}\|U \rho U^\dagger - \rho\|_1 \leq 2\delta$ (see [30] Lemma II.4, Eq. (17)), and hence

$$|S((M \otimes N)\rho) - S((M_s \otimes N_t)\rho)| = |S((M \otimes N)\rho) - S((M \otimes N)U \rho U^\dagger)|$$

$$\leq H_2(2\delta) + 2\delta \log_2 d^2$$

$$\leq (2\sqrt{2\delta} + 2\delta) \log_2 d^2 \leq 4\epsilon.$$ 

where $H_2$ denotes the binary entropy, and we have used the Fannes-Audenaert inequality [K. M. R. Audenaert, J. Phys. A: Math. Theor. 40, 8127 (2007)] and the estimate $H_2(x) \leq 2\sqrt{x(1-x)}$. Thus, we get directly

$$\text{Pr}\left\{ \exists M, N S((M \otimes N)\rho) < 2\log_2 d - \log \left(1 + \frac{1}{m}\right) - 5\epsilon \right\} \leq \left( \frac{c' \delta^{-3/2}(\log_2 d)^2}{\epsilon^2} \right)^{4d^2} \exp \left(-\frac{c^2}{32(\log_2 d)^2 d^2 m}\right),$$

and we’re done.

**Proof of the net estimate.** From [31] we know that one can find an $\eta$-net on the pure state vectors in $\mathbb{C}^d$ (w.r.t. the Euclidean norm) with at most $(\frac{\eta}{\epsilon})^{2d^2}$ elements. For each vector $|b_i\rangle$ in the given basis, find an $\eta$-close neighbour $|b_i'\rangle$. Of course this is not an orthogonal basis in general, so we perform an orthogonalisation inspired by the square-root measurement: define the operator $B = \sum_{i=1}^d |b_i'\rangle \langle b_i'|$ and let

$$|b_i''\rangle = B^{-1/2} |b_i'\rangle.$$ 

It is easy to see that if the $|b_i''\rangle$ are linearly independent, then this defines an orthonormal basis. Linear independence is equivalent to $B$ being invertible, which we also need for the above definition to make sense.

Now observe $\| |b_i'\rangle \langle b_i'| - |b_i\rangle \langle b_i| \| \leq 2\| |b_i'\rangle - |b_i\rangle \|_2 \leq 2\eta$, hence

$$\| B - \mathbb{1} \| = \left\| \sum_i (|b_i'\rangle \langle b_i'| - |b_i\rangle \langle b_i|) \right\| \leq \sum_i \| |b_i'\rangle \langle b_i'| - |b_i\rangle \langle b_i| \| \leq 2d\eta,$$

and consequently $\| B^{-1/2} - \mathbb{1} \| \leq \frac{2d\eta}{1-2d\eta}$.

Putting all this together, and assuming $2d\eta \leq 1/2$, we see that $\| |b_i\rangle - |b_i''\rangle \|_2 \leq (4d + 1)\eta$.

An elementary estimate now shows that for the corresponding unitary matrices $U$ and $U''$, $\| U - U'' \| \leq (4d + 1)\sqrt{d}\eta$. The number of different $U''$ encountered in this construction is bounded by the $d$-th power of the net size on vectors we started with, i.e. $T_0 \leq \left( \frac{\eta}{\epsilon} \right)^{2d^2}$.

Letting $\eta = \delta \frac{\epsilon}{c d^{3/2}}$ concludes the proof.