Research article

New subclass of analytic functions defined by $q$-analogue of $p$-valent Noor integral operator

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Abstract: In this paper, we introduce a certain subclass of analytic functions associated with $q$-analogue of $p$-valent Noor integral operator in the open unit disc. A variety of useful properties for this subclass are investigated including coefficient estimates and the familiar Fekete-Szegő type inequalities. Several known sequences of the main results are also highlighted.

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1. Introduction and definitions

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$ f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, \cdots \}), \quad (1.1) $$

which are $p$-valent and analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. We note that $\mathcal{A}(1) = \mathcal{A}$. For functions $f(z)$ given by (1.1) and $g(z)$ defined by

$$ g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \quad (p \in \mathbb{N}), \quad (1.2) $$
the convolution of \( f(z) \) and \( g(z) \) is defined by

\[
(f \ast g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g \ast f)(z).
\] (1.3)

For \( f \in \mathcal{A}(p) \) given by (1.1) and \( 0 < q < 1 \), the \( q \)-derivative of a function \( f(z) \) is given by (see [1, 6, 7])

\[
\mathcal{D}_{q,p} f(z) = \begin{cases} 
\frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\
\frac{f'(0)}{q} & \text{for } z = 0,
\end{cases}
\] (1.4)

provided that \( f'(z) \) exists. From (1.1) and (1.4), we deduce that

\[
\mathcal{D}_{q,p} f(z) = [p]_q z^{p-1} + \sum_{n=p+1}^{\infty} [n]_q a_n z^{n-1},
\] (1.5)

where

\[
[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0, \quad 0 < q < 1.
\] (1.6)

We note that

\[
\lim_{q \to 1^-} \mathcal{D}_{q,p} f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)
\]

for a function \( f \) which is differentiable in a given subset of \( \mathbb{C} \). Further, for \( p = 1 \), we have \( \mathcal{D}_{q,1} f(z) = \mathcal{D}_q f(z) \) (see [20]).

The \( q \)-number shift factorial for any non-negative integer \( n \) is defined by

\[
[n]_q! = \begin{cases} 
1 & \text{for } n = 0 \\
[1]_q[2]_q \cdots [n]_q & \text{for } n \in \mathbb{N}.
\end{cases}
\]

The Pochhammer \( q \)-generalized symbol for \( x > 0 \) and \( n \in \mathbb{N} \) is also

\[
[x, q]_n = \begin{cases} 
1 & \text{for } n = 0 \\
[x]_q[x + 1]_q \cdots [x + n - 1]_q & \text{for } n \in \mathbb{N},
\end{cases}
\]

and for \( x > 0 \), the \( q \)-gamma function is defined by

\[
\Gamma_q(x + 1) = [x]_q \Gamma_q(x) \quad \text{and} \quad \Gamma_q(1) = 1.
\]

For \( \lambda > -p \) (\( p \in \mathbb{N} \)), we define the function \( f_{\lambda+p-1,q}^{-1}(z) \) by

\[
f_{\lambda+p-1,q}^{-1}(z) * f_{\lambda+p-1,q}(z) = z^p + \sum_{n=p+1}^{\infty} \frac{[p + 1, q]_{n-p}}{\Gamma_q(n-p)} z^n,
\] (1.7)

where the function \( f_{\lambda+p-1,q}(z) \) is given by

\[
f_{\lambda+p-1,q}(z) = z^p + \sum_{n=p+1}^{\infty} \frac{[\lambda + p, q]_{n-p}}{\Gamma_q(n-p)} z^n.
\] (1.8)
It is clear that the function defined in (1.8) converges absolutely in $\mathcal{U}$. Using the idea of convolution we define the $q$-$p$-valent Noor integral operator $I_q^{\lambda+p-1} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ as follows:

$$I_q^{\lambda+p-1} f(z) = f_{\lambda+p-1,q}(z) * f(z) = z^p + \sum_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) a_n z^n,$$

where

$$\Phi_q(\lambda, p, n) = \frac{[p + 1, q]_{n-p}}{[\lambda + p, q]_{n-p}} (\lambda > -p, \ p \in \mathbb{N}).$$

From (1.9), we can easily get the identity

$$q^\lambda \mathcal{D}_{q,p}(I_q^{\lambda+p} f(z)) = [\lambda + p]_q I_q^{\lambda+p-1} f(z) - [\lambda]_q I_q^{1+p} f(z).$$

We note that:

(i) For $p = 1$, we have the $q$-Noor integral operator $I_q^1 f(z)$ ($f \in \mathcal{A}$) which was introduced and studied by Arif et al. [4];

(ii) $\lim_{q \rightarrow 1^-} I_q^{\lambda+p-1} f(z) = I^{\lambda+p-1} f(z)$ which is the $p$-valent Noor integral operator (see [11]);

(iii) Taking $p = 1$ and letting $q \rightarrow 1^-$ in (1.9), we obtain Noor integral operator for univalent functions (see [13, 14]);

(iv) For $\lambda = 1$, we have $I_q^p f(z) = f(z)$ and for $\lambda = 0$, we have

$$I_q^{p-1} f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{[p + 1, q]_{n-p}}{[1, q]_{n-p}} a_n z^{n} = z^p + \sum_{n=p+1}^{\infty} \frac{[n]_q}{[p]_q} a_n z^{n} = \mathcal{D}_{q,p} f(z),$$

$$\lim_{q \rightarrow 1^-} I_q^{p-1} f(z) = I^{p-1} f(z) = z + \sum_{n=p+1}^{\infty} \frac{\left( \frac{n}{p} \right) a_n z^{n}}{p}.$$

By using the operator $I_q^{\lambda+p-1} f(z)$ we define the subclass $ST_q(\lambda, p, k, b)$ of $\mathcal{A}(p)$ as follows:

**Definition 1.1.** Let $k \geq 0, \lambda > -p, \ p \in \mathbb{N}, \ b \in \mathbb{C}^*$, then $0 < q < 1$. A function $f \in \mathcal{A}(p)$ is said to be in the class $ST_q(\lambda, p, k, b)$ if it satisfies

$$\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{1}{[p]_q} \frac{\mathcal{D}_{q,p} \left( I_q^{\lambda+p-1} f(z) \right)}{I_q^{\lambda+p-1} f(z)} - 1 \right) \right\} > k \left( \frac{1}{b} \left( \frac{1}{[p]_q} \frac{\mathcal{D}_{q,p} \left( I_q^{\lambda+p-1} f(z) \right)}{I_q^{\lambda+p-1} f(z)} - 1 \right) \right), \ (z \in \mathcal{U}).$$

We note that:

(1) $\lim_{q \rightarrow 1^-} ST_q(1, p, k, 1 - \frac{a}{p}) = ST(p, k, \alpha) = \left\{ f \in \mathcal{A}(p) : \text{Re} \left( \frac{z f(z)}{f(z)} - \alpha \right) > k \left| \frac{z f(z)}{f(z)} - p \right|, \ 0 \leq \alpha < p, \ z \in \mathcal{U} \right\}$ (see [19]);

(2) $\lim_{q \rightarrow 1^-} ST_q(0, p, k, 1 - \frac{a}{p}) = \mathcal{UST}(p, k, \alpha) = \left\{ f \in \mathcal{A}(p) : \text{Re} \left( 1 + \frac{z f(z)}{f(z)} - \alpha \right) > k \left| 1 + \frac{z f(z)}{f(z)} - p \right|, \ 0 \leq \alpha < p, \ z \in \mathcal{U} \right\}$ (see [19]).
2. Geometric interpretation

A function $f \in \mathcal{A}(p)$ is in the class $\mathcal{ST}_q(\lambda, p, k, b)$ if

$$1 + \frac{1}{b} \left( \frac{z D_{q,p} \left( I_q^{1+p-1} f(z) \right)}{[p]_q I_q^{1+p-1} f(z)} - 1 \right)$$

takes all the values in the conic domain $\Omega_k = p_k(\mathcal{U})$, where

$$\Omega_k = \left\{ u + iv : u > k \sqrt{(u-1)^2 + v^2} \right\},$$

or, equivalently,

$$1 + \frac{1}{b} \left( \frac{z D_{q,p} \left( I_q^{1+p-1} f(z) \right)}{[p]_q I_q^{1+p-1} f(z)} - 1 \right) < p_k(z), \quad \Omega_k = p_k(\mathcal{U}). \tag{2.1}$$

The boundary $\partial \Omega_k$ of the above set when $k = 0$ becomes the imaginary axis, when $0 < k < 1$ a hyperbola, when $k = 1$ a parabola and an ellipse when $1 < k < \infty$. The functions $p_k(z)$ are defined by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi} \log \left( \frac{1 + \sqrt{1-v^2}}{1-v} \right), & k = 1, \\ 1 + \frac{1}{1-k^2} \cos \left( \frac{1}{\pi} \log \left( \frac{1 + \sqrt{1-v^2}}{1-v} \right) \right) - \frac{k^2}{1-k^2}, & 0 < k < 1, \\ 1 + \frac{1}{k^2-1} \sin \left( \frac{\pi}{2 \pi(t)} \int_0^{\pi \left( \frac{\pi}{2 \pi(t)} \right)} \frac{dx}{\sqrt{1-x^2}} \right) + \frac{k^2}{k^2-1}, & 1 < k < \infty, \end{cases} \tag{2.2}$$

where $u(z) = \frac{z - \sqrt{1-v^2}}{1-v}$ ($0 < t < 1$, $z \in \mathcal{U}$), $t$ is chosen such that $k = \cosh \left( \frac{\pi R(t)}{4R(t)} \right)$, $R(t)$ is the Legendre’s complete elliptic integral of the first kind, and $R’(t)$ is complementary integral of $R(t)$ (see [9, 10, 18]).

By giving a specific value to the parameters $q, \lambda, p, k,$ and $b$ in the class $\mathcal{ST}_q(\lambda, p, k, b)$, we get a lot of new and known subclasses studied by various others, for example,

1. $\mathcal{ST}_q(\lambda, 1, k, b) = \mathcal{ST}_q(\lambda, k, b) = \left\{ f \in \mathcal{A} : 1 + \frac{1}{b} \left( \frac{z D_q(f(z))}{f(z)} - 1 \right) < p_k(z), \quad z \in \mathcal{U} \right\};$

2. $\mathcal{ST}_q(\lambda, 1, k, 1) = \mathcal{ST}_q(\lambda, k, 1) = \left\{ f \in \mathcal{A} : \frac{z D_q(f(z))}{f(z)} < p_k(z), \quad z \in \mathcal{U} \right\};$

3. $\mathcal{ST}_q(\lambda, p, k, 1 - \frac{a}{[p]_q}) = \mathcal{ST}_q(\lambda, p, k, \alpha) = \left\{ f \in \mathcal{A} \left( p \right) : \frac{1}{[p]_q - \alpha} \left( \frac{z D_q(f(z))}{f(z)} - \alpha \right) < p_k(z), \quad 0 \leq \alpha < [p]_q, \quad z \in \mathcal{U} \right\};$

4. $\mathcal{ST}_q(\lambda, p, k, 1 - \frac{a}{[p]_q}) \cos \gamma e^{-iy} = \mathcal{ST}_q\gamma(\lambda, p, k, \alpha) = \left\{ f \in \mathcal{A} \left( p \right) : e^{iy} \frac{z D_q(f(z))}{f(z)} < ([p]_q - \alpha) \cos \gamma p_k(z) + \alpha \cos \gamma + i[p]_q \sin \gamma, \quad 0 \leq \alpha < [p]_q, \quad y \in \mathcal{U} \right\};$

5. $\mathcal{ST}_q(1, p, k, b) = \mathcal{ST}_q(p, k, b) = \left\{ f \in \mathcal{A} \left( p \right) : 1 + \frac{1}{b} \left( \frac{z D_q(f(z))}{f(z)} - 1 \right) < p_k(z), \quad z \in \mathcal{U} \right\};$

6. $\mathcal{ST}_q(1, p, k, 1 - \frac{a}{[p]_q}) = \mathcal{ST}_q(p, k, \alpha) = \left\{ f \in \mathcal{A} \left( p \right) : \frac{1}{[p]_q - \alpha} \left( \frac{z D_q(f(z))}{f(z)} - \alpha \right) < p_k(z), \quad 0 \leq \alpha < [p]_q, \quad z \in \mathcal{U} \right\};$
\( (7) \ ST_q \left(1, p, k, \left(1 - \frac{a}{p} \right) \cos \gamma e^{-i\gamma} \right) = ST_q^\gamma(p, k, \alpha) = \)
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ [p] + \frac{1}{b} \left( \frac{z f''(z)}{f'(z)} \right) - [p] \right\} > 0, \ z \in \mathcal{U} \right\}, \]
\[ 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U}. \]

Also we note that:
\( (8) \ ST_q(\lambda, p, 0, b) = S_q(\lambda, p, b) = \)
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ [p] + \frac{1}{b} \left( \frac{z f''(z)}{f'(z)} \right) - [p] \right\} > 0, \ z \in \mathcal{U} \right\}, \]
\[ S_q(1, p, 0 - \frac{\alpha}{p}) = S_q(\lambda, p, 0) = \]
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ z \in \mathcal{U} \right\}, \]
\[ S_q(1, p, 0) = \]
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ z \in \mathcal{U} \right\}; \]
\( (9) \ ST_q(\lambda, p, 0, b) = S_q(\lambda, p, b) = \)
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}; \]
\[ S_q(\lambda, p, 0) = \]
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}; \]
\[ S_q(1, p, 0) = \]
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}; \]
\( (10) \ ST_q(\lambda, p, 0, b) = S_q(\lambda, p, b) = \)
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}; \]
\[ S_q(1, p, 0) = \]
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}; \]
\[ S_q(1, p, 0) = \]
\[ \left\{ f \in \mathcal{A}(p) : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < [p], \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}; \]
\( (11) \ lim_{q \to 1^-} ST_q(1, 1, k, 1 - \alpha) = ST(k, \alpha) = \)
\[ \left\{ f \in \mathcal{A} : \Re \left\{ \frac{z f''(z)}{f'(z)} - \alpha \right\} > k \left| \frac{z f''(z)}{f'(z)} - 1 \right|, \ 0 \leq \alpha < 1, \ z \in \mathcal{U} \right\}; \]
\( (12) \ lim_{q \to 1^-} ST_q(1, p, k, \left(1 - \frac{a}{p} \right) \cos \gamma e^{-i\gamma}) = ST^\gamma(p, k, \alpha) = \)
\[ \left\{ f \in \mathcal{A} : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > k \left| \frac{z f''(z)}{f'(z)} - 1 \right|, \ 0 \leq \alpha < p, \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}, \]
\[ \lim_{q \to 1^-} ST_q(0, p, k, \left(1 - \frac{a}{p} \right) \cos \gamma e^{-i\gamma}) = \lim_{q \to 1^-} \left\{ f \in \mathcal{A} : \Re \left\{ \frac{z f''(z)}{f'(z)} \right\} > k \left| \frac{z f''(z)}{f'(z)} - 1 \right|, \ 0 \leq \alpha < p, \ |\gamma| < \frac{\pi}{2}, \ z \in \mathcal{U} \right\}. \]

We need the following lemmas in order to establish our main results.
Lemma 2.1. [8] Let $0 \leq k < \infty$ be fixed and let $p_k$ be defined by (2.2). If $p_k(z) = 1 + Q_1 z + Q_2 z^2 + \cdots$, then

$$Q_1 = \begin{cases} \frac{2A^2}{1 - z^2}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4 \sqrt{h^2(1 - z^2)^2}}, & 1 < k < \infty, \end{cases}$$

(2.3)

and

$$Q_2 = \begin{cases} \left(\frac{A^2 + 2}{3}\right) Q_1, & 0 \leq k < 1, \\ \frac{2}{3} Q_1, & k = 1, \\ \frac{4R'(t)(t^2 + 6t + 1) - \pi}{24 \sqrt{R(t)(1 + t)}} Q_1, & 1 < k < \infty, \end{cases}$$

(2.4)

where $A = \frac{2 \cos^{-1} k}{\pi}$, and $t \in (0, 1)$ is chosen such that $k = \cosh \left(\frac{\pi R'(t)}{R(t)}\right)$, where $R(t)$ is the Legendre’s complete elliptic integral of the first kind.

Lemma 2.2. [12] Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, i.e., let $h$ be analytic in $U$ and satisfies $\Re (h(z)) > 0$ $(z \in U)$, then

$$|c_2 - v c_1^2| \leq 2 \max \{1, |2v - 1|\} \ (v \in \mathbb{C}).$$

The result is sharp for a function given by

$$g(z) = \frac{1 + z^2}{1 - z^2} \ \text{or} \ g(z) = \frac{1 + z}{1 - z}.$$ 

Lemma 2.3. [12] If $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, then

$$|c_2 - v c_1^2| \leq \begin{cases} 2 - 4v & \text{if} \ \ v \leq 0, \\ 2 & \text{if} \ \ 0 \leq v \leq 1, \\ 4v - 2 & \text{if} \ \ v \geq 1, \end{cases}$$

(2.6)

where $v < 0$ or $v > 1$, the equality holds iff $h(z) = \frac{1 + z}{1 - z}$ or one of its rotations. If $0 < v < 1$, then he equality holds iff $h(z) = \frac{1 + z^3}{1 - z^3}$ or one of its rotations. If $v = 0$, then he equality holds iff $h(z) = \left(\frac{1 + \lambda}{2}\right) \frac{1 + z}{1 - z} + \left(\frac{1 - \lambda}{2}\right) \frac{1 - z}{1 + z} (0 \leq \lambda \leq 1)$ or one of its rotations. If $v = 1$, then he equality holds if and only if $g$ is reciprocal of one of the function such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp, and it can improved as follows when $0 < v < 1$:

$$|c_2 - v c_1^2| + v |c_1|^2 \leq 2 \ (0 \leq v \leq \frac{1}{2}),$$

and

$$|c_2 - v c_1^2| + (1 - v) |c_1|^2 \leq 2 \ (\frac{1}{2} \leq v \leq 1).$$
3. Main results

We shall assume throughout this paper, unless otherwise stated, that $0 \leq k < \infty$, $p \in \mathbb{N}$, $\lambda > -p$, $b \in \mathbb{C}^*$, $0 < q < 1$, $Q_1$ is given by (2.3) and $Q_2$ is given by (2.4), $\Phi_q(\lambda, p, n)$ is given by (1.10) and $z \in \mathcal{U}$.

**Theorem 3.1.** Let $f \in \mathcal{A}(p)$ be given by (1.1). If the inequality

$$
\sum_{n=p+1}^{\infty} \left( [n]_q - [p]_q \right) \Phi_q(\lambda, p, n) |a_n| \leq [p]_q |b|,
$$

(3.1)

holds, then $f \in ST_q(\lambda, p, k, b)$.

**Proof.** Assume the inequality (3.1) holds. Let us assume that

$$
H(z) = 1 + \frac{1}{b} \left( \frac{1}{[p]_q} \frac{zD_{q,p}(I_{q}^{1+p-1}f(z))}{I_{q}^{1+p-1}f(z)} - 1 \right).
$$

We have

$$
|H(z) - 1| = \frac{1}{[p]_q |b|} \left| \sum_{n=p+1}^{\infty} \left( [n]_q - [p]_q \right) \Phi_q(\lambda, p, n) a_n z^{n-p} \right| 
\left| \frac{1}{1 + \sum_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) a_n z^{n-p}} \right|
\leq \frac{1}{[p]_q |b|} \frac{\sum_{n=p+1}^{\infty} \left( [n]_q - [p]_q \right) \Phi_q(\lambda, p, n) |a_n|}{1 - \sum_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) |a_n|}.
$$

Now consider

$$
k |H(z) - 1| - \Re (H(z) - 1) \leq (k + 1) |H(z) - 1| 
\leq \frac{(k + 1) \sum_{n=p+1}^{\infty} \left( [n]_q - [p]_q \right) \Phi_q(\lambda, p, n) |a_n|}{[p]_q |b| \left( 1 - \sum_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) |a_n| \right)}.
$$

The last inequality is bounded by 1 if (3.1) holds.

**Corollary 3.2.** If $f \in ST_q(\lambda, p, k, b)$, then

$$
|a_n| \leq \frac{[p]_q |b|}{(k + 1) \left( [n]_q - [p]_q \right) + [p]_q |b|} \Phi_q(\lambda, p, n) (n \geq p + 1).
$$

(3.2)

The inequality (3.2) is sharp for the function

$$
f(z) = z^p + \frac{[p]_q |b|}{(k + 1) \left( [n]_q - [p]_q \right) + [p]_q |b|} \Phi_q(\lambda, p, n) z^n (n \geq p + 1).
$$

(3.3)
Choosing $p = 1$ and $b = 1 - \alpha$, $0 \leq \alpha < 1$, in Theorem 3.1, we obtain the following corollary.

**Corollary 3.3.** Let $f \in \mathcal{A}$ be given by (1.1) with $p = 1$ and satisfy

$$
\sum_{n=2}^{\infty} \left( (k+1)\left( [n]_q - 1 \right) + (1 - \alpha) \right) \Phi_q(\lambda, 1, n) |a_n| \leq 1 - \alpha.
$$

Then $f \in ST_q(\lambda, k, \alpha)$.

Taking $b = 1 - \frac{\alpha}{[p]_q}$ ($0 \leq \alpha < [p]_q$) in Theorem 3.1, we obtain the following consequence.

**Corollary 3.4.** Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy

$$
\sum_{n=p+1}^{\infty} \left( (k+1)\left( [n]_q - [p]_q \right) + ([p]_q - \alpha) \right) \Phi_q(\lambda, p, n) |a_n| \leq [p]_q - \alpha.
$$

Then $f \in ST_q(\lambda, p, k, \alpha)$.

Letting $q \to 1^-$ in Theorem 3.1, we obtain the following corollary.

**Corollary 3.5.** Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy

$$
\sum_{n=p+1}^{\infty} \left\{ (k+1)(n - p) + p|b| \right\} \Phi_q(\lambda, p, n) |a_n| \leq p|b|.
$$

Then $f \in ST(\lambda, p, k, b)$.

Putting $b = \left( 1 - \frac{\alpha}{[p]_q} \right) \cos \gamma e^{-r} \ (0 \leq \alpha < [p]_q, |\gamma| < \frac{\pi}{2})$ in Theorem 3.1, we obtain the following consequence.

**Corollary 3.6.** Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy

$$
\sum_{n=p+1}^{\infty} \left( (k+1)\left( [n]_q - [p]_q \right) + ([p]_q - \alpha) \cos \gamma \right) \Phi_q(\lambda, p, n) |a_n| \leq ([p]_q - \alpha) \cos \gamma.
$$

Then $f \in ST_q(\lambda, p, k, \alpha)$.

Letting $q \to 1^-$ and putting $b = 1 - \frac{\alpha}{p}$ ($0 \leq \alpha < p$) and $\lambda = 1$ in Theorem 3.1, we obtain the following corollary (see also [19], Theorem 1, with $n = 0$).

**Corollary 3.7.** Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy

$$
\sum_{n=p+1}^{\infty} \left( (k+1)(n - p) + (p - \alpha) \right) |a_n| \leq p - \alpha.
$$

Then $f \in ST(p, k, \alpha)$.

Letting $q \to 1^-$ and putting $b = 1 - \frac{\alpha}{p}$ ($0 \leq \alpha < p$) and $\lambda = 0$ in Theorem 3.1, we obtain the following corollary.
Corollary 3.8. Let \( f \in A(p) \) be given by (1.1) and satisfy

\[
\sum_{n=p+1}^{\infty} \binom{n}{p} (k+1) (n-p) + (p-\alpha) |a_n| \leq p - \alpha.
\]

Then \( f \in UST(p, k, \alpha) \).

Taking \( p = 1 \) in Theorem 3.1, we obtain the following corollary.

Corollary 3.9. If a function \( f \in A \) has the form (1.1) (with \( p = 1 \)) and satisfy

\[
\sum_{n=2}^{\infty} (k+1) ([n]_q - 1) + |b| \Phi_q(\lambda, n) |a_n| \leq |b|.
\]

Then \( f \in ST_q(\lambda, k, b) \).

Theorem 3.10. If \( f \in ST_q(\lambda, p, k, b) \). Then

\[
|a_{p+1}| \leq \frac{[p]_q |b| Q_1}{q^p \Phi_q(\lambda, p, p+1)} = \frac{[p]_q |b| Q_1 [\lambda + p]_q}{q^p [p+1]_q},
\]

and for all \( n \geq 3 \)

\[
|a_{n+p-1}| \leq \frac{[p]_q |b| Q_1}{q^p [n-1]_q \Phi_q(\lambda, p, n+p-1)} \prod_{j=1}^{n-2} \left( 1 + \frac{[p]_q |b| Q_1}{q^p [j]_q} \right),
\]

where \( Q_1 \) is given by (2.3).

Proof. Let

\[
p(z) = 1 + \frac{1}{b} \left( \frac{z D_{q, p} I_q^{\lambda+p-1} f(z)}{I_q^{\lambda+p-1} f(z)} - 1 \right),
\]

where \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) is analytic in \( \mathcal{U} \) and it can be written as

\[
\sum_{n=p+1}^{\infty} ([n]_q - [p]_q) \Phi_q(\lambda, p, n) a_n z^n \leq [p]_q b \left( I_q^{\lambda+p-1} f(z) \right) \left( \sum_{n=1}^{\infty} c_n z^n \right).
\]

Comparing the coefficients of \( z^{n+p-1} \) on both sides of (3.6), we obtain

\[
([n+p-1]_q - [p]_q) \Phi_q(\lambda, p, n+p-1) a_{n+p-1} = [p]_q b \left( c_1 \Phi_q(\lambda, p, n+p-1) a_{n+p-2} + \cdots + c_{n-1} \right).
\]

Taking the absolute value on both sides and using \( |c_n| \leq Q_1 (n \geq 1) \) (see [18]), we obtain
The result is true for $n = p - 1$. Using (3.8), we obtain
\[
|a_{n+p-1}| \leq \frac{[p]_q|b|Q_1}{q^p[n-1]_q\Phi_q(\lambda, p, n + p - 1)} \times \left\{ 1 + \Phi_q(\lambda, p, p + 1) |a_{p+1}| + \cdots + \Phi_q(\lambda, p, n + p - 2) |a_{n+p-2}| \right\}. \tag{3.7}
\]
We apply the mathematical induction on (3.7), so for $n = 2$, we have
\[
|a_{p+1}| \leq \frac{[p]_q|b|Q_1}{q^p[2]_q\Phi_q(\lambda, p, p + 2)} \left( 1 + \Phi_q(\lambda, p, p + 1) |a_{p+1}| \right),
\]
this shows that the result is true for $n = 2$. Now for $n = 3$ we have
\[
|a_{p+2}| \leq \frac{[p]_q|b|Q_1}{q^p[2]_q\Phi_q(\lambda, p, p + 2)} \left( 1 + \frac{[p]_q|b|Q_1}{q^p[1]_q} \right),
\]
using (3.8), we obtain
\[
|a_{p+2}| \leq \frac{[p]_q|b|Q_1}{q^p[2]_q\Phi_q(\lambda, p, p + 2)} \left( 1 + \frac{[p]_q|b|Q_1}{q^p[1]_q} \right),
\]
which is true for $n = 3$. Let us assume that (3.7) is true for all $n \leq t$, that is
\[
|a_{t+p-1}| \leq \frac{[p]_q|b|Q_1}{q^p[t-1]_q\Phi_q(\lambda, p, t + p - 1)} \prod_{j=1}^{t-2} \left( 1 + \frac{[p]_q|b|Q_1}{q^p[j]_q} \right).
\]
Consider
\[
|a_{t+p}| \leq \frac{[p]_q|b|Q_1}{q^p[t]_q\Phi_q(\lambda, p, t + p)} \times \left\{ 1 + \Phi_q(\lambda, p, p + 1) |a_{p+1}| + \cdots + \Phi_q(\lambda, p, t + p - 1) |a_{t+p-1}| \right\}
\]
\[
\leq \frac{[p]_q|b|Q_1}{q^p[t]_q\Phi_q(\lambda, p, t + p)} \left( 1 + \frac{[p]_q|b|Q_1}{q^p} \right) \left( 1 + \frac{[p]_q|b|Q_1}{q^p[2]_q} \right) \left( 1 + \frac{[p]_q|b|Q_1}{q^p[3]_q} \right) \left( 1 + \frac{[p]_q|b|Q_1}{q^p[4]_q} \right) + \cdots + \frac{[p]_q|b|Q_1}{q^p[t-1]_q} \prod_{j=1}^{t-2} \left( 1 + \frac{[p]_q|b|Q_1}{q^p[j]_q} \right)
\]
\[
= \frac{[p]_q|b|Q_1}{q^p[t]_q\Phi_q(\lambda, p, t + p)} \prod_{j=1}^{t-1} \left( 1 + \frac{[p]_q|b|Q_1}{q^p[j]_q} \right).
\]
So, the result is true for $n = t + 1$. Also, we proved that the result true for all $n(n \geq 2)$ using mathematical induction. \hfill \Box

Taking $p = 1$ in Theorem 3.10, we obtain the following corollary.
Corollary 3.11. Let $f \in A$ be given by (1.1) (with $p = 1$). If $f \in ST_q(\lambda, k, b)$, then

$$|a_2| \leq \frac{[\lambda + 1]_q |b| Q_1}{q[2]_q},$$

and

$$|a_n| \leq \frac{|b| Q_1}{q[n - 1]_q \Phi_q(\lambda, 1, n)} \prod_{j=1}^{n-2} \left( 1 + \frac{|b| Q_1}{q[j]_q} \right) (n \geq 3).$$

Taking $b = 1 - \alpha$ ($0 \leq \alpha < 1$) and $p = 1$ in Theorem 3.10, we obtain the following consequence.

Corollary 3.12. Let $f \in A$ be given by (1.1) (with $p = 1$). If $f \in ST_q(\lambda, k, \alpha)$, then

$$|a_2| \leq \frac{P_1[\lambda + 1]_q}{q[2]_q},$$

and

$$|a_n| \leq \frac{[p]_q P_1}{q[n - 1]_q \Phi_q(\lambda, n)} \prod_{j=1}^{n-2} \left( 1 + \frac{P_1}{q[j]_q} \right) (n \geq 3),$$

where $P_1 = (1 - \alpha)Q_1$ and $Q_1$ is given by (2.3).

Taking $b = 1 - \frac{\alpha}{[p]_q}$ ($0 \leq \alpha < [p]_q$) in Theorem 3.10, we obtain the following result.

Corollary 3.13. Let $f \in A(p)$ be given by (1.1). If $f \in ST_q(\lambda, p, k, \alpha)$, then

$$|a_{p+1}| \leq \frac{([p]_q - \alpha) Q_1}{q^p \Phi_q(\lambda, p, n + p - 1)},$$

and for all $n \geq 3$,

$$|a_{n+p-1}| \leq \frac{([p]_q - \alpha) Q_1}{q^p [n - 1]_q \Phi_q(\lambda, p, n + p - 1)} \prod_{j=1}^{n-2} \left( 1 + \frac{([p]_q - \alpha) Q_1}{q^p [j]_q} \right).$$

Putting $b = \left( 1 - \frac{\alpha}{[p]_q} \right) \cos \gamma e^{-i\gamma} \ (0 \leq \alpha < [p]_q, |\gamma| < \frac{\pi}{2})$ in Theorem 3.10, we obtain the following consequence.

Corollary 3.14. Let $f \in A(p)$ be given by (1.1). If $f \in ST_q(\lambda, p, k, \alpha)$, then

$$|a_{p+1}| \leq \frac{([p]_q - \alpha) \cos \gamma Q_1}{q^p \Phi_q(\lambda, p, p + 1)},$$

and for all $n \geq 3$,

$$|a_{n+p-1}| \leq \frac{([p]_q - \alpha) \cos \gamma Q_1}{q^p [n - 1]_q \Phi_q(\lambda, p, n + p - 1)} \prod_{j=1}^{n-2} \left( j + \frac{([p]_q - \alpha) \cos \gamma Q_1}{q^p [j]_q} \right).$$
Theorem 3.15. Let \( f \in \mathcal{ST}_q(\lambda, p, k, b) \). Then \( f(U) \) contains an open disc
\[
  r = \frac{q^p[p + 1]_q}{q^p(p + 1)[p + 1]_q + [p]_q |b|}
\]

Proof. Let \( w_0 \in \mathbb{C} \) and \( w_0 \neq 0 \) such that \( f(z) \neq w_0 \) for \( z \in U \). Then
\[
  f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z^p + \left(a_{p+1} + \frac{1}{w_0}\right) z^{p+1} + \cdots.
\]

Since \( f_1 \) is univalent, so
\[
  \left|a_{p+1} + \frac{1}{w_0}\right| \leq p + 1.
\]
Now using Theorem 3.10, we have
\[
  \left|\frac{1}{w_0}\right| \leq p + 1 + \frac{[p]_q |b| Q_1[\lambda + p]_q}{q^p[p + 1]_q},
\]
and hence we have
\[
  |w_0| \geq \frac{q^p[p + 1]_q}{q^p(p + 1)[p + 1]_q + [p]_q |b| Q_1[\lambda + p]_q}.
\]
This completes the proof of Theorem 3.15 \( \Box \)

Theorem 3.16. Let \( 0 \leq k < \infty \) be fixed and let \( f \in \mathcal{ST}_q(\lambda, p, k, b) \) with the form (1.1). Then for a complex \( \mu \), we have
\[
  |a_{p+2} - \mu a_{p+1}^2| \leq \frac{[p]_q |b| Q_1[\lambda + p, q]_2}{2[2]_q q^p[p + 1, q]_2} \max\{1, |2v - 1|\},
\]
where
\[
  v = \frac{1}{2} \left(1 - \frac{Q_2}{Q_1} - \frac{[p]_q b Q_1}{q^p} \left(1 - \frac{[2]_q[\lambda + p]_q[p + 2]_q}{[\lambda + p + 1]_q[p + 1]_q \mu}\right)\right),
\]
where \( Q_1 \) and \( Q_2 \) are given by (2.3) and (2.4), respectively. The result is sharp.

Proof. Let \( f \in \mathcal{ST}_q(\lambda, p, k, b) \), then there exist a function \( w \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that
\[
  1 + \frac{1}{b} \left(\frac{z D_{q,p}(I_{q}^{k+p-1} f(z))}{I_{q}^{k+p-1} f(z)} - 1\right) = p_k(w(z)) \ (z \in U).
\]
(3.10)

Let \( h \in P \) be a function defined by
\[
  h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \ (z \in U).
\]
This gives
\[
  w(z) = \frac{c_1}{2} z + \frac{1}{2} \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots.
\]
and
\[ p_2(w(z)) = 1 + \frac{1}{2} c_1 Q_1 z + \frac{1}{2} \left( \frac{c_1^2 Q_2}{2} + (c_2 - \frac{c_1^2}{2}) Q_1 \right) z^2 + \ldots. \quad (3.11) \]

Using (3.11) in (3.10) along with (1.9), we obtain
\[ a_{p+1} = \frac{[p]_q b c_1 Q_1 [\lambda + 1]_q}{2 q [p + 1]_q}, \]

and
\[ a_{p+2} = \frac{[p]_q b [\lambda + p, q]_2}{2 [2]_q q [p + 1, q]_2} \left\{ \frac{c_1^2 Q_2}{4} + \frac{1}{2} (c_2 - \frac{c_1^2}{2}) Q_1 + \frac{[p]_q b Q_1^2 c_1^2}{4 q^p} \right\}. \]

For any complex number \( \mu \), we have
\[ a_{p+2} - \mu a_{p+1}^2 = \frac{[p]_q b [\lambda + p, q]_2}{2 [2]_q q [p + 1, q]_2} \left\{ \frac{c_1^2 Q_2}{4} + \frac{1}{2} (c_2 - \frac{c_1^2}{2}) Q_1 + \frac{[p]_q b Q_1^2 c_1^2}{4 q^p} \right\} - \frac{[p]_q^2 b^2 c_1^2 Q_1^2}{4 q^2} \left( [\lambda + 1]_q \right)^2 \mu. \quad (3.12) \]

Thus (3.12) can be written as
\[ a_{p+2} - \mu a_{p+1}^2 = \frac{[p]_q b Q_1 [\lambda + p, q]_2}{2 [2]_q q [p + 1, q]_2} \left\{ c_2 - \nu c_1^2 \right\}, \quad (3.13) \]

where
\[ \nu = \frac{1}{2} \left\{ 1 - \frac{Q_2}{Q_1} - \frac{[p]_q b Q_1}{q} \left( 1 - \frac{[\lambda + 1]_q [3]_q}{[\lambda + 2]_q} \mu \right) \right\}. \quad (3.14) \]

Now, taking absolute value and using Lemma 2.2, we obtain the required result. The sharpness of (3.9) follows from the sharpness of (2.5).

Putting \( p = 1 \) in Theorem 3.16, we obtain the following consequence.

**Corollary 3.17.** Let \( 0 \leq k < \infty \) be fixed and let \( f \in \mathcal{ST}_q(\lambda, k, b) \) with the form (1.1) (with \( p = 1 \)). Then for a complex parameter \( \mu \), we have
\[ |a_3 - \mu a_2^2| \leq \frac{|b| Q_1 [\lambda + 1, q]_2}{2 [2]_q q [2, q]_2} \max \{ 1, |2 \nu - 1| \}, \]

where
\[ \nu = \frac{1}{2} \left\{ 1 - \frac{Q_2}{Q_1} - \frac{b Q_1}{q} \left( 1 - \frac{[\lambda + 1]_q [3]_q}{[\lambda + 2]_q} \mu \right) \right\}, \]

where \( Q_1 \) and \( Q_2 \) are given by (2.3) and (2.4), respectively. The result is sharp.

Putting \( p = 1 \) and \( b = 1 - \alpha \) \( (0 \leq \alpha < 1) \) in Theorem 3.16, we get the following corollary.

**Corollary 3.18.** Suppose that the function \( f(z) \) given by (1.1) (with \( p = 1 \)) is in the class \( \mathcal{ST}_q(\lambda, k, \alpha) \). Then for a complex parameter \( \mu \), we have
\[ |a_3 - \mu a_2^2| \leq \frac{P_1 [\lambda + 1, q]_2}{2 q [2]_q [2, q]_2} \max \left\{ 1, \frac{P_2}{P_1} - \frac{P_1}{q} \left( 1 - \frac{[\lambda + 1]_q [3]_q}{[\lambda + 2]_q} \mu \right) \right\}, \quad (3.15) \]

where \( P_1 = (1 - \alpha) Q_1 \) and \( P_2 = (1 - \alpha) Q_2 \), \( Q_1 \) and \( Q_2 \) are given by (2.3) and (2.4), respectively. The result is sharp.
Putting $b = 1 - \frac{\alpha}{[p]q}$, $(0 \leq \alpha < [p]q)$ in Theorem 3.16, we get the following corollary.

**Corollary 3.19.** Let $0 \leq k < \infty$ be fixed and let $f \in ST_\varphi(\lambda, p, k, \alpha)$ with the form (1.1). Then for a complex parameter $\mu$, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{([p]q - \alpha)Q_1 \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right),$$

where $Q_1$ and $Q_2$ are given by (2.3) and (2.4), respectively. The result is sharp.

Putting $b = \left(1 - \frac{\alpha}{[p]q}\right)\cos \gamma e^{-i\gamma}$, $(0 \leq \alpha < [p]q, |\gamma| < \frac{\pi}{2})$ in Theorem 3.16, we get the following corollary.

**Corollary 3.20.** Let $0 \leq k < \infty$ be fixed and let $f \in ST_\varphi(\lambda, p, k, \alpha)$. Then for a complex parameter $\mu$, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{([p]q - \alpha)\cos \gamma Q_1 \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right).$$

The result is sharp.

**Theorem 3.21.** Let

$$\sigma_1 = \frac{([p]q b Q_2^2 + q\varphi(Q_2 - Q_1)) \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right),$$

$$\sigma_2 = \frac{([p]q b Q_2^2 + q\varphi(Q_2 - Q_1)) \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right),$$

$$\sigma_3 = \frac{([p]q b Q_2^2 + q\varphi(Q_2 - Q_1)) \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right).$$

If $f$ given by (1.1) belong to the class $ST_\varphi(\lambda, p, k, b)$ $(b > 0)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{([p]q b Q_1 \varphi)^2}{4[p]q[p+1]} \left(\frac{Q_2}{Q_1} + \frac{[p]q b Q_1 \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right)\right), \mu \leq \sigma_1, \\
\frac{([p]q b Q_1 \varphi)^2}{4[p]q[p+1]} \left(\frac{Q_2}{Q_1} + \frac{[p]q b Q_1 \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right)\right), \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{([p]q b Q_1 \varphi)^2}{4[p]q[p+1]} \left(\frac{Q_2}{Q_1} + \frac{[p]q b Q_1 \varphi}{4[p]q[p+1]} \left(1 - \frac{[2]q[p+1]q[p+1]}{[\lambda + p + 1]q[p+1]} \mu\right)\right), \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then
Further, if

\[
|a_{p+2} - \mu a_{p+1}^2| + \frac{q^p \left( [p + 1]_q \right)^2 [\lambda + p, q]}{[2]_q [p]_q b Q_1 \left( [\lambda + p]_q \right)^2 [p + 1, q]_2} \times \left\{ 1 - \frac{Q_2}{Q_1} - \frac{b Q_1}{q^p} \left( 1 - \frac{[2]_q [\lambda + p]_q [p + 2]_q}{[\lambda + p + 1]_q [p + 1]_q} \mu \right) \right\} |a_{p+1}|^2 \leq \frac{[p]_q b Q_1 [\lambda + p, q]_2}{q^p [2]_q [p + 1, q]_2}.
\]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_{p+2} - \mu a_{p+1}^2| + \frac{q^p \left( [p + 1]_q \right)^2 [\lambda + p, q]}{[2]_q b Q_1 \left( [\lambda + p]_q \right)^2 [p + 1, q]_2} \times \left\{ 1 + \frac{Q_2}{Q_1} + \frac{[p]_q b Q_1}{q^p} \left( 1 - \frac{[2]_q [\lambda + p]_q [p + 2]_q}{[\lambda + p + 1]_q [p + 1]_q} \mu \right) \right\} |a_{p+1}|^2 \leq \frac{[p]_q b Q_1 [\lambda + p, q]_2}{q^p [2]_q [p + 1, q]_2}.
\]

The result is sharp.

**Proof.** Applying Lemma 2.3 to (3.12) and (3.13), we can obtain our results asserted by Theorem 3.21.

Putting \( p = 1 \) in Theorem 3.21, we obtain the following corollary.

**Corollary 3.22.** Let

\[
\begin{align*}
\varsigma_1 &= \frac{[b Q_1^2 + q (Q_2 - Q_1)] [\lambda + 2]_q}{b Q_1^2 [\lambda + 1]_q [3]_q}, \\
\varsigma_2 &= \frac{[b Q_1^2 + q (Q_2 + Q_1)] [\lambda + 2]_q}{b Q_1^2 [\lambda + 1]_q [3]_q}, \\
\varsigma_3 &= \frac{[b Q_1^2 + q Q_2] [\lambda + 2]_q}{b Q_1^2 [\lambda + 1]_q [3]_q}.
\end{align*}
\]

If \( f \) given by (1.1) (with \( p = 1 \)) belong to the class \( \mathcal{ST}_q(\lambda, k, b) \) with \( b > 0 \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b Q_1 [\lambda + 1]_q}{q [2]_q [2]_q} \left\{ \frac{Q_2}{Q_1} + \frac{b Q_1}{q} \left( 1 - \frac{[\lambda + 1]_q [3]_q}{[\lambda + 2]_q} \mu \right) \right\}, & \mu \leq \varsigma_1, \\
\frac{b Q_1 [\lambda + 1]_q}{q [2]_q [2]_q}, & \varsigma_1 \leq \mu \leq \varsigma_2, \\
\frac{b Q_1 [\lambda + 1]_q}{q [2]_q [2]_q} \left\{ \frac{Q_2}{Q_1} + \frac{b Q_1}{q} \left( 1 - \frac{[\lambda + 1]_q [3]_q}{[\lambda + 2]_q} \mu \right) \right\}, & \mu \geq \varsigma_2.
\end{cases}
\]

Further, if \( \varsigma_1 \leq \mu \leq \varsigma_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{q [2]_q [\lambda + 1, q]_2}{b Q_1 \left( [\lambda + 1]_q \right)^2 [2, q]_2} \times \left\{ 1 - \frac{Q_2}{Q_1} - \frac{b Q_1}{q} \left( 1 - \frac{[\lambda + 1]_q [3]_q}{[\lambda + 2]_q} \mu \right) \right\} |a_2|^2 \leq \frac{b Q_1 [\lambda + 1, q]_2}{q [2]_q [2, q]_2}.
\]
If \( \varsigma_3 \leq \mu \leq \varsigma_2 \), then

\[
|a_{p+2} - \mu a_{p+1}^2| + \frac{q[2]_q[\lambda + 1, q]_2}{qbQ_1([\lambda + 1]_q^2)[2, q]_2} \times \left( 1 + \frac{Q_2}{Q_1} + \frac{bQ_1}{q} \left( 1 - \frac{[\lambda + 1]_q[3]_q}{[\lambda + 2]_q} \mu \right) \right) |a_2|^2 \leq \frac{bQ_1[\lambda + 1, q]_2}{q[2]_q[2, q]_2}.
\]

The result is sharp.

Putting \( p = 1 \) and \( b = 1 - \alpha \) (0 \( \leq \alpha < 1 \)) in Theorem 3.21, we obtain the following corollary.

**Corollary 3.23.** Let

\[
\begin{align*}
\vartheta_1 &= \frac{P_1^2 + q(P_2 - P_1)}{P_1^2[\lambda + 1]_q} [\lambda + 2]_q, \\
\vartheta_2 &= \frac{P_1^2 + q(P_2 + P_1)}{P_1^2[\lambda + 1]_q} [\lambda + 2]_q, \\
\vartheta_3 &= \frac{P_1^2 + qP_2}{P_1^2[\lambda + 1]_q} [\lambda + 2]_q.
\end{align*}
\]

If \( f \) given by (1.1) (with \( p = 1 \)) belong to the class \( ST_q(\lambda, k, \alpha) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{P_1[1+1]_q}{[2]_q[2, q]_2} \left( \frac{P_2}{P_1} + \frac{P_1}{q} \left( 1 - \frac{[\lambda + 1]_q[3]_q}{[\lambda + 2]_q} \mu \right) \right), & \mu \leq \vartheta_1, \\
\vartheta_1 \leq \mu \leq \vartheta_2, & \frac{P_1[1+1]_q}{[2]_q[2, q]_2}, \\
-\frac{P_1[1+1]_q}{[2]_q[2, q]_2} \left( \frac{P_2}{P_1} + \frac{P_1}{q} \left( 1 - \frac{[\lambda + 1]_q[3]_q}{[\lambda + 2]_q} \mu \right) \right), & \mu \geq \vartheta_2.
\end{cases}
\]

Further, if \( \vartheta_1 \leq \mu \leq \vartheta_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{q[2]_q[\lambda + 1, q]_2}{P_1 ([\lambda + 1]_q^2)[2, q]_2} \times \left( 1 - \frac{P_2}{P_1} - \frac{P_1}{q} \left( 1 - \frac{[\lambda + 1]_q[3]_q}{[\lambda + 2]_q} \mu \right) \right) |a_2|^2 \leq \frac{P_1[\lambda + 1, q]_2}{q[2]_q[2, q]_2}.
\]

If \( \vartheta_3 \leq \mu \leq \vartheta_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{q[2]_q[\lambda + 1, q]_2}{qP_1 ([\lambda + 1]_q^2)[2, q]_2} \times \left( 1 + \frac{P_2}{P_1} + \frac{P_1}{q} \left( 1 - \frac{[\lambda + 1]_q[3]_q}{[\lambda + 2]_q} \mu \right) \right) |a_2|^2 \leq \frac{P_1[\lambda + 1, q]_2}{q[2]_q[2, q]_2}.
\]

The result is sharp.
Putting $b = \left(1 - \frac{\alpha}{[p]_q}\right)$ ($0 \leq \alpha < [p]_q$) in Theorem 3.21, we obtain the following corollary.

**Corollary 3.24.** Let

$$
e_1 = \frac{\left([p]_q - \alpha \right) Q_1^2 + q^p (Q_2 - Q_1) \left[\lambda + P + 1\right]_q [P + 1]_q}{2_q \left([p]_q - \alpha \right) Q_1^2 [\lambda + P]_q [P + 2]_q},$$

$$
e_2 = \frac{\left([p]_q - \alpha \right) Q_1^2 + q^p (Q_2 + Q_1) \left[\lambda + P + 1\right]_q [P + 1]_q}{2_q \left([p]_q - \alpha \right) Q_1^2 [\lambda + P]_q [P + 2]_q},$$

$$
e_3 = \frac{\left([p]_q - \alpha \right) Q_1^2 + q^p Q_2 \left[\lambda + P + 1\right]_q [P + 1]_q}{2_q \left([p]_q - \alpha \right) Q_1^2 [\lambda + P]_q [P + 2]_q}.$$

If $f$ given by (1.1) belong to the class $ST_q(\lambda, P, k, b)$ with $b > 0$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{\left([p]_q - \alpha \right) Q_1 [\lambda + p, q]_2}{q^p [2]_q [p + 1, q]_2} \left(\frac{Q_2}{Q_1} + \frac{([p]_q - \alpha) Q_1}{q} \left(1 - \frac{[2]_q \lambda [\lambda + p]_q \mu_1}{[\lambda + p + 1]_q [p + 1]_q}\right)\right), & \mu \leq \epsilon_1, \\
\frac{([p]_q - \alpha) Q_1 [\lambda + p, q]_2}{q^p [2]_q [p + 1, q]_2} \left(\frac{Q_2}{Q_1} + \frac{([p]_q - \alpha) Q_1}{q} \left(1 - \frac{[2]_q \lambda [\lambda + p]_q \mu_1}{[\lambda + p + 1]_q [p + 1]_q}\right)\right), & \epsilon_1 \leq \mu \leq \epsilon_2, \\
\frac{([p]_q - \alpha) Q_1 [\lambda + p, q]_2}{q^p [2]_q [p + 1, q]_2} \left(\frac{Q_2}{Q_1} + \frac{([p]_q - \alpha) Q_1}{q} \left(1 - \frac{[2]_q \lambda [\lambda + p]_q \mu_1}{[\lambda + p + 1]_q [p + 1]_q}\right)\right), & \mu \geq \epsilon_2. 
\end{cases}$$

Further, if $\epsilon_1 \leq \mu \leq \epsilon_3$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{q^p ([p + 1]_q)^2 [\lambda + p, q]_2}{2_q ([p]_q - \alpha) Q_1 (\lambda + p)_q [p + 1, q]_2} \times \left(1 - \frac{Q_2}{Q_1} - \frac{([p]_q - \alpha) Q_1}{q} \left(1 - \frac{[2]_q \lambda [\lambda + p]_q \mu_1}{[\lambda + p + 1]_q [p + 1]_q}\right)\right) |a_{p+1}|^2 \leq \left([p]_q - \alpha \right) Q_1 [\lambda + p, q]_2.$$

If $\epsilon_3 \leq \mu \leq \epsilon_2$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{q^p ([p + 1]_q)^2 [\lambda + p, q]_2}{2_q ([p]_q - \alpha) (\lambda + p)_q [p + 1, q]_2} \times \left(1 + \frac{Q_2}{Q_1} + \frac{([p]_q - \alpha) Q_1}{q} \left(1 - \frac{[2]_q \lambda [\lambda + p]_q \mu_1}{[\lambda + p + 1]_q [p + 1]_q}\right)\right) |a_{p+1}|^2 \leq \frac{\left([p]_q - \alpha \right) Q_1 [\lambda + p, q]_2}{q^p [2]_q [p + 1, q]_2}.$$

The result is sharp.
4. Conclusions

Studies of the coefficient problems including the Fekete-Szegö problems continue to motivate researchers in Geometric Function Theory of Complex Analysis. In our present investigation, we have introduced and studied a new class $ST_q(\lambda, p, k, b)$ of analytic functions associated with $q$-analogue of $p$-valent Noor integral operator in the open unit disc $U$. For functions in this class, we have derived the coefficient estimates of the coefficients $|a_{p+1}|$ and $|a_{n+p+1}|$ for $n \geq 3$, and Fekete-Szegö functional problems for functions belonging to this new class. Several of new results are shown to follow upon specializing the parameters involved in our main results.

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Conflict of interest

The authors declare that they have no competing interests.

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