Stationary Localized States Due to a Nonlinear Dimeric Impurity Embedded in a Perfect 1-D Chain

B. C. Gupta and K. Kundu

*Institute of physics, Bhubaneswar - 751 005, India*

**Abstract**

The formation of Stationary Localized states due to a nonlinear dimeric impurity embedded in a perfect 1-d chain is studied here using the appropriate Discrete Nonlinear Schrödinger Equation. Furthermore, the nonlinearity has the form, $\chi|C|^\sigma$ where $C$ is the complex amplitude. A proper ansatz for the Localized state is introduced in the appropriate Hamiltonian of the system to obtain the reduced effective Hamiltonian. The Hamiltonian contains a parameter, $\beta = \phi_1/\phi_0$ which is the ratio of stationary amplitudes at impurity sites. Relevant equations for Localized states are obtained from the fixed point of the reduced dynamical system. $|\beta| = 1$ is always a permissible solution. We also find solutions for which $|\beta| \neq 1$. Complete phase diagram in the $(\chi, \sigma)$ plane comprising of both cases is discussed. Several critical lines separating various regions are found. Maximum number of Localized states is found to be six. Furthermore, the phase diagram continuously extrapolates from one region to the other. The importance of our results in relation to solitonic solutions in a fully nonlinear system is discussed.

PACS numbers : 71.55.-i, 72.10.Fk
I. INTRODUCTION

Discrete Nonlinear Schrödinger equation (DNLSE) is a set of $n$ coupled differential equations.

$$i \frac{dC_m}{dt} = -\chi_m f_m(|C_m|)C_m + V_{m,m+1}C_{m+1} + V_{m,m-1}C_{m-1}$$

where $V_{m,m+1} = V^*_{m+1,m}$; and $m = 1, 2, 3, \ldots, n$. (1)

In eqn.(1) the nonlinearity appears through the functions $f_m(|C_m|)$ and $\chi_m$ is the nonlinearity parameter associated with the $m$-th grid point. Since, $\sum_m |C_m|^2$ is made unity by choosing appropriate initial conditions, $|C_m|^2$ can be interpreted as the probability of finding a particle at the $m$-th grid point. One way to derive this set of equations is to couple the vibration of masses at the lattice points of a lattice of $n$ sites to the motion of a quasi particle in the same lattice in the adiabatic approximation. The motion of the quasi particle is described, however, in the framework of a tight binding Hamiltonian (TBH). Same type of equation can also be obtained by nonlinear coupling of anharmonic oscillators through both positions and momenta of the oscillators. The set of equations, thus derived, is called the discrete self-trapping equation (DST). These equations also posses a constant of motion analogous to $\sum_m |C_m|^2$ in the DNLSE. In fact, both the DST and the DNLSE contain the same number of constants of motion. However, in general the analytical solutions of eqn.(1) are not known. Numerous works, both analytical as well as numerical, on the DNLSE and the DST have been reported [1–16].

One important feature of this type of nonlinear equations is that these can yield stationary localized (SL) states and soliton–like solutions. It has been shown that the presence of a nonlinear impurity can produce SL states in one dimension. The first study was made using Green’s function approach and authors considered $f(|C|) = |C|^2$ [17]. Later, one nonlinear impurity case has been generalized by taking $f(|C|) = |C|^\sigma$ and formation of SL states studied in one, two and three dimensions [18–20]. Furthermore, considering $f(|C|) = |C|^\sigma$, some discussions have been made about the formation of SL states in the
presence of two impurities in a 1-d chain \[18\]. In another development the Green function
approach and the ansatz approach were synthesized to find SL states self consistently. Two
types of nonlinear impurities, namely, \( f(|C|) = |C|^\sigma \) where \( \sigma \) is arbitrary and the rotational
nonlinear impurity, embedded in linear hosts like 1-d perfect chain and Caley tree have been
considered \[21\]. The purpose of considering Caley tree is to study the effect of connectivity
on the formation of SL states. In case of nonlinear dimer, \(|C_0|^2 = |C_1|^2 \) ( probability of the
particle to stay at the nonlinear sites are same ) is assumed. However, for \( \sigma = 2 \), it was found
that a \( \chi_{cr} = 8/3 \) exists above which an extra SL state appears. In this SL state \(|C_0|^2 \neq |C_1|^2 \).
It is, therefore, imperative to study if such a solution occurs for all \( \sigma \). This is the motivation
behind this paper. The formation of SL states is studied here starting from a Hamiltonian.
The fixed point of the Hamiltonian \[22\] which generates the appropriate DNLSE can also
produce the correct equations governing the formation of SL states. Although this approach
is simpler, it however needs the proper ansatz. We further note that the appropriate ansatz
has been obtained in our earlier analysis \[21\].

The organization of the paper is as follows. In sec.II we describe the formalism part. In
section III(A) the phase diagram and energy diagram for the SL states satisfying \(|\beta| \neq 1\)
is presented. In sec.III(B) we discuss the full phase diagram of SL states considering both
the cases, namely, \(|\beta| = 1\) and \(|\beta| \neq 1\). In the last section we give a summary of our
investigation.

II. FORMALISM

The eqn.(1) describing the system of 1-d chain consisting of a nonlinear dimer impurity
of the kind \( \chi|C|^{\sigma} \) can be derived from the Hamiltonian given by

\[
H = \frac{1}{2} \sum_m (C_m^* C_{m+1} + C_m C_{m+1}^*) + \frac{\chi}{\sigma + 2} (|C_0|^{\sigma+2} + |C_1|^{\sigma+2}).
\]

(2)

A model derivation of the Hamiltonian is given in ref. \[21\]. Since we are interested in the
possible solutions for SL states, we assume
\[ C_m = \phi_m e^{x (-i E t)} \]

where \[ \phi_m = [\text{sgn}(E) \eta]^{m-1} \phi_1, \quad m \geq 1 \]

and \[ \phi_{-|m|} = [\text{sgn}(E) \eta]^{|m|} \phi_0, \quad m \leq 0 \] (3)

Eqn.(3) is the exact form of \( \phi_m \) in the presence of a dimeric nonlinear impurity and can be derived from Greens function analysis \[ 21 \]. Here \( 0 < \eta < 1 \) and is given by

\[ \eta = \frac{|E| - \sqrt{E^2 - 4}}{2}. \] (4)

It is also to be noted that for \( \chi = 0 \) all possible states of the resulting linear system lies in the band defined by \( |E| \leq 2 \). Since eqn.(3) defines a localized state it will appear either above the upper band edge or below the lower band edge of the linear system depending on the sign of \( \chi \). Introducing the \( \text{sgn}(E) \) or the signature of \( E \) in eqn.(3) we take care of that possibility. We further define \( \beta = \phi_1/\phi_0 \) if \( |\phi_1| \leq |\phi_0| \). Otherwise, we invert the definition of \( \beta \). Because of the symmetry in the system we shall get the same result. So, we restrict \( \beta \) in [-1,1]. Now from the normalization condition, \( \sum_m |C_m|^2 = 1 \), we get

\[ |\phi_0|^2 = \frac{1 - \eta^2}{1 + \beta^2}. \] (5)

Using eqn.(3), and eqn.(5) and the definition of \( \beta \) we get an effective Hamiltonian, \( H_{\text{eff}} \) where

\[ H_{\text{eff}} = \frac{\beta(1 - \eta^2)}{1 + \beta^2} + \text{sgn}(E) \eta + \frac{\chi}{\sigma + 2} (1 - \eta^2)^{\sigma/2 + 1} (1 + \beta^2)^{-(\sigma/2 + 1)} (1 + |\beta|^\sigma + 2). \] (6)

The Hamiltonian consists of two variables namely \( \beta \) and \( \eta \) because \( \chi \) and \( \sigma \) are constants. The stationary localized states correspond to fixed points of the reduced dynamical system described by \( H_{\text{eff}} \). So to obtain the possible stationary localized states we need solving two coupled algebraic equations in \( \eta \) and \( \beta \) arising from setting \[ 11,22 \]

\[ \frac{\partial H_{\text{eff}}}{\partial \beta} = 0 \quad \text{and} \quad \frac{\partial H_{\text{eff}}}{\partial \eta} = 0. \] (7)

From the first relation of (7) we obtain
\[(\beta^2 - 1)[\frac{(1 + \beta^2)\sigma^2}{\beta \chi} - \frac{(1 - \eta^2)^{\sigma/2}(|\beta|^\sigma - 1)}{\beta^2 - 1}] = 0. \quad (8)\]

We note that the term in the parentheses is finite for \(|\beta| = 1\). So, \(|\beta| = 1\) is always a solution of eqn.(8). In the other equation, namely, \(\partial H_{\text{eff}}/\partial \eta = 0\), if we set \(|\beta| = 1\), we obtain

\[\frac{2\sigma^2}{|\chi|} = \eta_{\pm}(1 \mp \eta_{\pm})^{-1}(1 - \eta_{\pm}^2)^{\sigma/2}. \quad (9)\]

Here \(\eta_{\pm}\) refers to the symmetric and antisymmetric cases respectively. The phase diagram of SL states arising from eqn.(9) has been discussed in detail in ref. [21]. We further note that in our previous analysis for \(\sigma = 2\), we obtained another \(\chi_{\text{cr}} = \frac{8}{3}\) above which \(|\beta| \neq 1\). So, to explore the possibility of SL states in which \(|\beta| \neq 1\) for \(\sigma\) other than 2, we set the term in the parentheses in eqn.(8) to zero. This then defines a relation between \(\eta\) and \(\beta\).

Furthermore, if we introduce the expression for \((1 - \eta^2)\), thus derived in the other equation of eqn.(7) we obtain

\[\text{sgn}(E)\eta = \text{sgn}(\beta)\frac{|\beta|^{-\sigma/2} - |\beta|^\sigma/2}{|\beta|^{-\sigma/(2+1)} - |\beta|^\sigma/(2+1)}. \quad (10)\]

We note that \(\eta\) is a symmetric function of \(\beta\) and \(\frac{1}{\beta}\) and is always less than unity in magnitude irrespective of the magnitude of \(\beta\). Furthermore, when \(|\beta| \to 1\), we obtain \(\eta = \pm \text{sgn}(E)\frac{\sigma}{\sigma + 2}\).

At \(\sigma = 2\), \(\eta = \frac{1}{2}\) and this has been obtained in our earlier work [21]. Since \(\eta\) is by definition positive, if \(\beta < 0\), we must have \(E < -2\) and vice versa. Again from the expression of \((1 - \eta^2)\) which should always be positive, it can be shown that for \(\beta < 0\) will imply \(\chi < 0\) and vice versa. Hence, for \(\beta < 0\), we must have \(\chi < 0, E < -2\) and on the other hand for \(\beta > 0\), we need \(\chi > 0\) and \(E > 2\). Consequently, by choosing \(\chi\) positive we can restrict \(\beta\) in \([0,1]\). We also note that \(\beta \neq 1\) is not possible for the antisymmetric set and the analytical argument for this has been presented in ref. [21]. Now introducing eqn.(10) in the equation of \((1 - \eta^2)\) we obtain

\[\frac{1}{\chi^{2/\sigma}} = \frac{(\beta^{-1} - \beta)(\beta^{-\sigma/2} - \beta^{\sigma/2})\beta^{-\sigma/(2+1)} - \beta^{\sigma/(2+1)}}{(\beta^{-1} + \beta)(\beta^{-1} - \beta)^{2/\sigma}(\beta^{-\sigma/(2+1)} - \beta^{\sigma/(2+1)})^2} = F(\beta, \sigma) \quad (11)\]

We note that we have \(\beta \geq 0\). Furthermore, the right hand side of eqn.(11) is also a symmetric function of \(\beta\) and \(\beta^{-1}\) as like \(\eta\). For \(\sigma = 2\), we find, from eqn.(11), \(\chi_{\text{cr}} = \frac{8}{3}\) and
this is consistent with the result obtained from Green’s function analysis. So, to obtain the
phase diagram of SL states for $\beta \neq 1$, we analyze eqn.(11) and the results are discussed in
the next section.

III. RESULTS AND DISCUSSIONS

(A) Phase diagram and energy diagram of SL states with $|\beta| \neq 1$

We first note that the number of possible SL states is the number of permissible solutions
of eqn.(11) for $\beta \in [0,1]$. So, we need to know the behavior of $F(\beta, \sigma)$ as a function of $\beta$
and $\sigma$. We see that for $\sigma = 2$, $F(\beta, \sigma)$ is monotonically increasing function of $\beta$ and this is
shown in fig.1. So, for $\sigma = 2$ we can have no solution (no SL state) or one solution (one SL
state) of eqn.(11) depending on the value of $\chi$. On the other hand for $\sigma = 5$, we find a local
maximum in $F(\beta, \sigma)$ and this is shown in fig.2. Therefore, for this value of $\sigma$ we can almost
have two solutions (two SL states) of eqn.(11). This clearly shows that there is a critical
value of $\sigma$ ($\sigma_{cr}$) above which $F(\beta, \sigma)$ develops a local maximum. This $\sigma_{cr}$ can be obtained
from the minimum value of $\sigma$ satisfying the relation, $\partial F(\beta, \sigma)/\partial \beta = 0$ for $\beta \in [0,1]$. This
is found graphically and $\sigma_{cr}$ comes out to be $\approx 2.33$. Therefore, for $\sigma < \sigma_{cr}$, since $F(\beta, \sigma)$
is monotonically increasing function of $\beta$, maximum value it takes for $\beta \in [0,1]$ is $F(1, \sigma)$.
Inasmuch as, the critical value of $\chi$ required to get a SL state for this case is

$$\chi_{cr} = \frac{2}{\sigma} \left[ \frac{(\sigma + 2)^2}{2(\sigma + 1)} \right]^{\sigma/2}, \quad (12)$$

it implies that for $\sigma \leq \sigma_{cr}$ we will have no SL states if $\chi < \chi_{cr}$ and only one SL state if
$\chi \geq \chi_{cr}$. On the other hand for $\sigma > \sigma_{cr}$ there will be two critical values of $\chi$ because $F(0, \sigma)$
$= 0, F(1, \sigma) \neq 0$ and there is a local maximum of $F(\beta, \sigma)$ at some $\beta_0 \in [0,1]$. Therefore, for
$\sigma > \sigma_{cr}$, lower critical value of $\chi$ ($\chi_{lcr}$) to get a SL state is given by $\chi_{lcr} = [1/F(\beta_0, \sigma)]^{\sigma/2}$.
The upper critical value of $\chi$ ($\chi_{ucr}$) separating the one and two SL states regions is same
as that given in eqn.(12). Therefore, for $\sigma > \sigma_{cr}$, we will have no SL states if $\chi < \chi_{lcr}$, one
SL state at $\chi = \chi_{lcr}$, two SL states for $\chi_{lcr} < \chi \geq \chi_{ucr}$ and again one SL state above $\chi_{ucr}$.
We further note that the upper critical line for $\sigma > \sigma_{cr}$ joins smoothly with critical line for $\sigma \leq \sigma_{cr}$. These lines are shown by solid and dotted lines respectively in fig.3. Salient features of the phase diagram of SL states in the $(\chi, \sigma)$ as shown in fig.3 are discussed below.

1. In fig.3 the region below the solid curve has no SL states. Every point of the blank region bound by the solid and the dotted curves represents only one SL state. The shaded region bound by the solid and the dotted curves has two possible SL states. (2) There are threshold values of $\chi$ and $\sigma$ below which no SL states appear. These values can be obtained from the relation $d\chi_{cr}/d\sigma = 0$ where $\chi_{cr}$ is given in eqn.(12). Thus we obtain $\chi_{th1} \approx 2.593$ and $\sigma_{th1} \approx 1.645$. This point is shown being surrounded by a small box in fig.3. There are also threshold values for both $\chi$ and $\sigma$ below which there is no possibility of getting two SL states. These threshold values are given by $\chi_{th2} = \chi_{lcr} \approx 2.866$ and $\sigma_{th2} = \sigma_{cr} = 2.33$ as discussed in earlier paragraph. This point is shown by a star in the figure.

Therefore, in the region bounded by $\chi \in [\chi_{th1}, \chi_{th2}]$ and $\sigma \in [\sigma_{th1}, \sigma_{th2}]$ we will have only one SL state. Consequently, for any $\chi$ between $\chi_{th1}$ and $\chi_{th2}$ there will be two critical values of $\sigma$, namely, $\sigma_1$ and $\sigma_2$. These points are shown in the figure for $\chi = 2.7 \in [\chi_{th1}, \chi_{th2}]$. For this value of $\chi \in [\chi_{th1}, \chi_{th2}]$ there will be no SL state below $\sigma_1$, one SL state between $\sigma_1$ and $\sigma_2$ and again there will be no SL states above $\sigma_2$. For a fixed value of $\chi > \chi_{th2}$ there will be three critical values of $\sigma$. For example, for $\chi = 4.25 > \chi_{th2}$, these three values of $\sigma$ are $0.585$, $3.29$ and $4.28$, respectively. These are denoted by $\sigma_3$, $\sigma_4$ and $\sigma_5$, respectively in the figure. Now for the fixed $\chi = 4.25$, there will be no SL states below $\sigma = 0.585$, one SL state for $0.585 < \sigma < 3.29$, two SL states for $3.29 < \sigma < 4.28$, one SL state at $\sigma = 4.28$ and again no SL state above $\sigma = 4.28$. We also notice that one SL state appears on the dotted line as well as on the solid line for $\sigma \leq \sigma_{cr}$ and these correspond to the case where $\beta = 1$. We further note that at $\sigma = 0$, the system contains a linear dimeric impurity with each site having the site-energy, $\chi$. So, to obtain a state fully localized on the dimer, but symmetric ($\beta = 1$) we need infinite magnitude for $\chi$. Precisely for this reason along the solid line for $\sigma \leq \sigma_{cr}$ in fig.3 $\chi \rightarrow \infty$ as $\sigma \rightarrow 0$. On the other hand, as $\sigma$ goes to infinity, the dimer site-energies will go to zero and a perfect system will be obtained. So, this cluster localized state will
vanish. This in turn implies that $\chi \to \infty$ as $\sigma \to 0$. So, along the critical line where $\beta = 1$, there will be another $\sigma_{cr}$, i.e., $\sigma_{th1}$ in our earlier discussion, for which $\chi_{cr}$ in eqn.(12) will assume the minimum value, i.e., $\chi_{th1}$. Similarly, the solid line for $\sigma > \sigma_{cr}$ represents one cluster localized SL state in which two impurity sites have different amplitude.

Here we consider variation of the energy of SL states with $\chi$ for $\sigma < 2.33$ as well as $\sigma > 2.33$. The energy of the SL states can be calculated using eqn.(4). Fig.4 shows the energy of the SL states as a function of $\chi$ for $\sigma = 2$. It is clear from the figure that the SL state starts appearing at $\chi = 8/3$ and lies above the upper band edge of the perfect system. We see that the energy of the SL state increases almost linearly with $\chi$. That means as $\chi$ increases, the localization length of the state decreases and hence localization becomes stronger with the increase of $\chi$. In fig.5 we have plotted the energy of the SL states as a function of $\chi$ for a fixed value of $\sigma = 4 > 2.33$. Here also no state, two states and one state regions get reflected as in the fig.2. It is clear that for $\chi < 4.025$ there is no energy of SL state, at $\chi = 4.025$ there is one energy of SL state, for $4.025 < \chi \leq 6.489$ there are two energies of SL states for each $\chi$ and above $\chi = 6.489$ there is again one energy of SL state for each $\chi$. All possible states appear above the upper band edge. As $\chi$ increases from 4.025 to 6.489, energy of one of the states increases with $\chi$ and that of the other state decreases towards the band edge state. So, as $\chi$ increases from 4.025 to 6.489, one of the state localizes strongly while localization of the other state becomes weaker. After $\chi = 6.489$, one state disappears and energy of the other state continues to increase with the increase of $\chi$.

We now consider the variation of energy with $\sigma$ by first considering critical lines. Along the upper critical line (solid curve in fig.3), $\beta = 1$ and $\eta = \sigma/(\sigma + 2)$. This is true for both $\sigma > \sigma_{cr}$, and $\sigma \leq \sigma_{cr}$. So, as $\sigma \to \infty, \eta \to 1$. Therefore, the energy of the SL states along this line will go towards the upper band edge as $\sigma \to \infty$. SL states that appear above this critical line also have this property. On the other hand, along the lower critical line (dotted curve in fig.3) $\beta_0 \to 0$ as $\sigma \to \infty$ which can be shown from the movement of the maximum of $F(\beta, \sigma)$ as a function of $\sigma$. So, along this critical line, $\eta \to 0$ as $\sigma \to \infty$. Consequently $E \to \infty$. Therefore, the energy of one of the SL states in the shaded region of phase diagram
will increase as $\sigma$ increases. The energy of the other SL state will probably decrease. We now consider a specific example.

Fig. 6 shows how the energy of SL state or states vary with $\sigma$ for fixed value of $\chi$. The example that we choose is $\chi = 5 > \chi_{th2}$. We have plotted for a range of $\sigma$ between 2 and 6. Here $\sigma_{cr} = 3.58$. We see that for $\sigma < 3.58$ only one state appear above the upper band edge and the energy of the state decreases with increasing $\sigma$. At $\sigma = 3.58$ another state (shown by dotted line in the figure) appears with lower energy above the upper band edge of the perfect system. The energy of the new state increases with the increase of $\sigma$. The energy of the first state goes on decreasing as $\sigma$ increases and hence the localization of the state becomes weaker. But localization of the new state becomes stronger as $\sigma$ increases. Ultimately energies of both the states join at certain value of $\sigma$ (5.618) and then they disappear as expected (see fig. 3). The disappearance of the SL states can be understood from the fact that as $\sigma$ increases for a fixed $\chi$, the effective nonlinearity at the impurity sites decreases and hence the system approaches towards the perfect system.

(B) The full phase diagram of SL states

Taking into account all possible solutions for $\beta = 1$ as well as $\beta \neq 1$ we will have the complete phase diagram. This is shown in fig. 7. This figure shows several separated regions and the number of SL states range from zero to six. First we consider the region for $\sigma < 2$. There are four separated regions on the left side of $\sigma = 2$ line. The region I contains only one SL state and this comes from the symmetric set corresponding to $\beta = 1$. In region II, there are three SL states. Two of them are from the antisymmetric set corresponding to $\beta = 1$ and the other one is the contribution from the symmetric set. Region III has two SL states, one from the symmetric set with $\beta = 1$ and the other is for $\beta \neq 1$. The region marked by IV has four SL states. Two of them from the antisymmetric set, one from the symmetric set for $\beta = 1$ and the last one is from the case where $\beta \neq 1$. Along the $\sigma = 2$ line there are three critical values of $\chi$ and they are marked each by a star. These values are 1, 8/3 and 8 respectively and have been discussed in our earlier work [21]. Now we look at the regions on the right side of $\sigma = 2$ line. The region V contains two SL states and both are contributed
from the symmetric set. Region VI contains no SL state. The region VII contain two SL states and both of them are from the solutions when \( \beta \neq 1 \). The region VIII has altogether three SL states, two of them are contributed from the symmetric set for \( \beta = 1 \) and the other one comes from the solution for \( \beta \neq 1 \). Region IX contains five SL states, two of them come from the symmetric set, two from the antisymmetric set and the last one arises from the case where \( \beta \neq 1 \). Region X and the triangular region bounded by regions V, VI, VII and VII contain four SL states. Two of them arise from the symmetric set and the other two come form the solution for \( \beta \neq 1 \). There is a region containing the maximum number of SL states and that region occurs for large value of \( \chi \). The line separating region VIII, IX and the line separating the regions VII, X will cross each other at a larger value of \( \chi \) and will produce a region of six SL states. Each of the symmetric, the antisymmetric set and the solutions for \( \beta \neq 1 \) contributes two SL states in this region. It is to be noticed that there is continuity throughout the phase diagram as we go from one region to the other. We also find that if there are \( N \) (\( N \) is even) SL states in a region, energies of \( N/2 \) states increase and that of \( N/2 \) states decrease as \( \chi \) is increased. On the other hand for odd \( N \), energies of \( (N + 1)/2 \) states increase and that of \( (N - 1)/2 \) states decrease with increasing \( \chi \).

**IV. CONCLUSION:**

The formation of Stationary Localized states due to a dimeric impurity in a perfect 1-d linear system has been studied using the DNLSE. The nonlinearity considered has the form, \( \chi |C|^\sigma \) where \( \chi \) and \( \sigma \) are arbitrary. In our previous work we used Green’s function approach and the phase diagram for two particular situations was obtained. However from the previous analysis we were able to find the exact analytical structure of SL states. This enabled us to find the effective Hamiltonian, \( H_{\text{eff}} \) corresponding to the DNLSE considered here. This Hamiltonian is a function of two variables \( \eta \) and \( \beta \). The equations for SL states are obtained from the fixed points of the dynamical system described by \( H_{\text{eff}} \).

The total phase diagram of SL states then consists of two situations. In one case \( |\beta| = 1 \)
and in the other case $|\beta| \neq 1$. We also show that the signs of $\beta$ as well as $\chi$ determine the position of SL states. For example, for $\beta > 0$ and $\chi > 0$ which has been considered here, SL states appear above the upper band edge of the linear system. It is important to note that the phase diagram is quite rich in structure. It contains several $\sigma_{cr}$ and $\chi_{cr}$ to get different number of SL states. The maximum number of SL states we obtain is six. Since the analysis is complete, we conclude that the maximum number of possible SL states for the system considered here is six. It is well established that for one nonlinear impurity, we can get atmost two SL states. On the other hand, here we have six. Therefore it is worthwhile to find out the relation between the number of possible SL states and the number of impurities. Furthermore, the study of SL states from nonlinear clusters of similar type is important for understanding the formation of solitons in totally nonlinear system. For example, the present dimer problem is important for analyzing the formation of solitons peaking in between lattice sites. This aspect will be discussed elaborately elsewhere.
REFERENCES

[1] Eilbeck, P. S. Lomdahl and A. C. Scott, Physica D16 318 (1985)

[2] Davydov’s Soliton Revisited: Self trapping of vibrational energy in protein, Vol. 243 of NATO Advanced Study Institute, Series B: Physics, edited by Peter L. Christian and Alwyn C. Scott (Plenum, New York, 1991)

[3] T. Holstein, Ann. Phys. 8 325 (1959)

[4] A. S. Davydov and N. I. Kislikha, Phys. Status Solidi (B) 59 465 (1973)

[5] Yi Wan and C. M. Soukoulis, Phys. Rev. B 40 12264 (1989)

[6] Yu. S. Kivshar, Phys. Lett A 173 (1993); Phys. Rev. Lett. 70 3055 (1993)

[7] M. I. Molina and G. P. Tsironis, Phys. Rev. Lett. 73 464 (1994)

[8] V. M. Kenkre and D. K. Campbell, Phys. Rev. B 34 4959 (1986)

[9] V. M. Kenkre, G. P. Tsironis, Phys. Rev. B 35 1473 (1987)

[10] V. M. Kenkre, G. P. Tsironis and D. K. Campbell, Nonlinearity in Condensed Matter, ed. A. R. Bishop, D. K. Campbell, P. Kumar and S. E. Trullinger, (Springer-Verlag 1987)

[11] A. B. Aceves, C. De Angelis, T. Peschel, R. Muschall, F. Lederer, S. Trillo and S. Wabnitz, Phys. Rev. E 53 1172 (1996)

[12] V. M. Kenkre and M. Kus, Phys. Rev. B 46 13792 (1992)

[13] M. I. Molina and G. P. Tsironis, Physica D 65 267 (1993)

[14] D. H. Dunlap, V. M. Kenkre and P. Reineker, Phys. Rev. B 47 14842 (1992)

[15] P. K. Datta and K. Kundu, Phys. Rev B 53 1 (1996)

[16] M. Johansson and R. Riklund, Phys. Rev. B 49 6587 (1994)
[17] M. I. Molina and G. P. Tsironis, Phys. Rev. B 47 15330 (1993).

[18] G. P. Tsironis, M. I. Molina and D. Hinnig, Phys Rev. E 50 2365 (1994)

[19] Y. Y. Yiu, K. M. Ng and P. M. Hui, Phys. Lett. A 200 325 (1995)

[20] Y. Y. Yiu, K. M. Ng and P. M. Hui, Preprint

[21] B. C. Gupta and K. Kundu, Phys. Rev. B (in print)

[22] Boris Malomed and Michel I. Weinstein, Phys. Lett A 220 91 (1996)
FIGURES

FIG. 1. $F(\beta, \sigma)$ and $1/\chi$ is plotted as a function of $\beta$ in the range between 0 and 1 for a fixed value of $\sigma = 2$.

FIG. 2. $F(\beta, \sigma)$ is plotted as a function of $\beta$ in the range between 0 and 1 for a fixed value of $\sigma = 5$. $\beta_0$ is the position of maximum of $F(\beta, \sigma)$

FIG. 3. The phase diagram of SL states for $\beta \neq 1$ is shown. Here $\sigma_1 = 1.645$, $\sigma_2 = 2.33$, $\sigma_3 = 0.585$, $\sigma_4 = 3.29$ and $\sigma_5 = 4.28$. The point surrounded by small box and the point marked by star has coordinates $(2.593, 1.645)$ and $(2.866, 2.33)$ respectively.

FIG. 4. Variation of energy of SL state as a function of $\chi$ for $\sigma = 2$ is shown. Vertical line touches the $\chi$ axis at $8/3$ and is drawn to show the critical value of $\chi$ to get SL state.

FIG. 5. Variation of energy of SL states as a function of $\chi$ for a fixed $\sigma = 4$ is shown. Vertical lines are drawn to show the critical values of $\chi$ separating no SL state, two SL states and one SL state regions. The dotted and the dashed vertical lines touch the $\chi$ axis at 4.025 and 6.489 respectively.

FIG. 6. This shows the variation of energy of SL states as a function of $\sigma$ for a fixed value of $\chi = 5$. Vertical lines are drawn to show the transition points. The small and long vertical lines touch the $\sigma$ axis at 3.58 and 5.618 respectively.

FIG. 7. This shows the full (both the case, namely, $\beta = 1$ and $\beta \neq 1$ is taken into account) phase diagram in the $(\chi, \sigma)$ plane. There are several marked regions containing different number of SL states.