PRODUCT FORMULAS FOR CERTAIN SKEW TABLEAUX

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Abstract. The hook length formula gives a product formula for the number of standard Young tableaux of a partition shape. The number of standard Young tableaux of a skew shape does not always have a product formula. However, for some special skew shapes, there is a product formula. Recently, Morales, Pak and Panova joint with Krattenthaler conjectured a product formula for the number of standard Young tableaux of shape \( \lambda/\mu \) for \( \lambda = ((2a + c)^{c+a}, (a+c)^{a}) \) and \( \mu = (a+1, a^{a-1}, 1) \). They also conjectured a product formula for the number of standard Young tableaux of a certain skew shifted shape. In this paper we prove their conjectures using Selberg-type integrals. We also give a generalization of MacMahon’s box theorem and a product formula for the trace generating function for a certain skew shape, which is a generalization of a recent result of Morales, Pak and Panova.

1. Introduction

For a partition \( \lambda \) of \( n \), the number \( f_\lambda \) of standard Young tableaux of shape \( \lambda \) is given by the celebrated hook length formula due to Frame, Robins and Thrall \[3\]:

\[
f_\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h_\lambda(i,j)},
\]

where \( h_\lambda(i,j) \) is the hook length \( \lambda_i + \lambda'_j - i - j + 1 \). In general the number \( f_{\lambda/\mu} \) of standard Young tableaux of a skew shape does not have a product formula because it may have a large prime factor. However, in sporadic cases of skew shapes, some product formulas are known \[6, 8, 9\].

Recently, Naruse \[11\] found the following generalization of the hook length formula:

\[
f_{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h_\lambda(i,j)},
\]

where \( \mathcal{E}(\lambda/\mu) \) is the set of subsets \( D \subset \lambda/\mu \), called excited diagrams, satisfying certain conditions. Morales, Pak and Panova \[10\] found the following \( q \)-analog of Naruse’s hook length formula using semistandard Young tableaux:

\[
s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in \text{SSYT}(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{q^{\lambda'_j - i}}{1 - q^{h_\lambda(i,j)}}.
\]

The precise definitions of notations used in the introduction will be given in Section 2.

Using \(2\), Morales, Pak and Panova \[9\] found product formulas for the number of standard Young tableaux of certain skew shapes. In \[9\], joint with Krattenthaler, they conjectured the following product formula.

Conjecture 1.1. \[9\] Conjecture 5.17 Let \( \lambda = ((n + 2a)^{n+a}, (n + a)^a) \) and \( \mu = (a+1, a^{a-1}, 1) \). Then

\[
f_{\lambda/\mu} = |\lambda/\mu|! \frac{\Phi(a)^4 \Phi(n) \Phi(n + 4a) \Phi(2a)^2 \Phi(2n + 4a)}{\Phi(2a)^2 \Phi(2n + 4a)} \frac{a^2((n^2 + 4an + 2a^2)^2 - a^2)}{4a^2 - 1},
\]

where \( \Phi(n) = \prod_{i=1}^{n-1} i! \).

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In Section 3, we prove the following theorem, which is a generalization of Conjecture 1.1.

**Theorem 1.2.** Let \( \lambda = ((n + c + d)^{n+a}, (n + c)^d), \mu = (c + 1, c^a - 1, 1) \) and \( \rho = \lambda / \mu \). Then

\[
f_\rho = |\rho| \Phi(n) \Phi(a) \Phi(b) \Phi(c) \Phi(d) \Phi(n + a + b) \Phi(n + a + b + c + d)
\]

\[
\times \frac{(m - 1)\delta_{n+a}}{(a + b - 1)(a + b + 1)}.
\]

See Figure 1 for the Young diagram of the skew shape \( \rho \) used in Theorem 1.2. Note that Conjecture 1.1 is obtained as a special case \( a = b = c = d \) of Theorem 1.2. Our proof of Theorem 1.2 consists of several steps. First, we consider the generating function for reverse plane partitions of shape \( \rho \) and interpret it as a \( q \)-integral. Although it seems hopeless to evaluate the resulting \( q \)-integral because it has large irreducible factors, the \( q \to 1 \) limit becomes a Selberg-type integral which has a product formula. Using the well known connection between linear extensions and \( P \)-partitions of a poset, we obtain a product formula for \( f_\rho \), which is then shown to be equivalent to the formula in Theorem 1.2.

For integers \( n, a, b \geq 0 \) and \( m \geq 1 \), let \( V(n, a, b, m) \) denote the shifted skew shape \( \lambda / \mu \) for

\[
\lambda = ((n + a + b, n + a + b - 1, \ldots, b + 1) + (m - 1)\delta_{n+a})^*, \quad \mu = (\delta_{a+1})^*\]

where \( \nu^* \) denotes the shifted Young diagram of a strict partition \( \nu \). See Figure 2 for the Young diagram of \( V(n, a, b, m) \). Morales, Pak and Panova [9] also conjectured the following product formula.

**Figure 1.** The Young diagram of the skew shape \( \rho \) in Theorem 1.2.

**Figure 2.** The Young diagram of \( V(n, a, b, m) \).
**Conjecture 1.3.** [9, Conjecture 9.6] For \( \pi = V(n, a, b, m) \), the number \( g^\pi \) of standard Young tableaux of shape \( \pi \) is

\[
g^\pi = \frac{\mid \pi \mid!}{2^a} \cdot \frac{\Phi(n + 2a) \Phi(a)}{\Phi(2a) \Phi(n + a)} \cdot \sum_{n}^{2a} \frac{1}{\prod_{(i,j) \in D} h_{\lambda^*}(i,j)},
\]

where \( \sum (n) = \prod_{i=1}^{\lfloor n/2 \rfloor} (n - 2i)! \), \( \lambda \) is given in [3], \( D \) is the set of cells \((i, n + j)\) with \(1 \leq i \leq j \leq n\) and \( h_{\lambda^*}(i,j) \) is the shifted hook length.

In Section 4, we prove Conjecture 1.3 by a similar approach used in the proof of Theorem 1.2.

For integers \( n, a, b, c, d \geq 0 \) and \( m \geq 1 \), we let \( M(n, a, b, c, d, m) \) denote the skew shape \( \lambda/\nu' \), where \( \lambda = ((n + c + b)n + a) + (m - 1)\delta_{n+a} \), \( \nu = (d^{n+c}) + (m - 1)\delta_{n+c} \), see Figure 3.

In Section 5, we show that for \( \pi = M(n, a, b, c, d, 1) \), the generating function \( s_\pi(1, q, q^2, \ldots, q^N) \) for SSYTs of shape \( \pi \) with bounded entries also has a product formula.

**Theorem 1.4.** For \( \pi = M(n, a, b, c, d, 1) \) and an integer \( N \geq 0 \), we have

\[
s_\pi(1, q, q^2, \ldots, q^N) = q^\sum_{(i,j) \in \lambda/(c^a)} (\lambda'-i) \prod_{i=1}^{b} (q^{N-a+1+i}; q)_a \prod_{i=1}^{d} (q^{N+2-i}; q)_c \times \prod_{i=1}^{n} (q^{N-n-a-d+1+i}; q)_{n+a+b+c+d} \prod_{i=1}^{n} \prod_{j=1}^{a} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \prod_{(i,j) \in \lambda(0^c,c^a)} 1 - q^{h_{\lambda^*}(i,j)},
\]

where \( (0^n, c^a) \) is the set of cells in the \( a \times c \) rectangle, starting from the \((n+1)\)-st row. (In other words, \( (0^n, c^a) = (c^{n+a})/(c^a) \).)

Recall that MacMahon’s box theorem states that

\[
\sum_{T \in \text{RPP}(b^a)} q^{\max(T)} \leq c \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.
\]
By the simple connection between SSYTs and RPPs of any partition shape, (4) is equivalent to

\[ q^{|T|} = q^{b^a} \prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k+2}}. \]

Since the shape \( M(n, a, b, c, d, 1) \) is more general than \((b^a)\), Theorem 1.4 can be considered as a generalization of MacMahon’s box theorem.

Using [2], Morales, Pak and Panova [9] found a product formula for the generating function for semistandard Young tableaux of shape \( M(n, a, b, c, d, m) \).

**Theorem 1.5.** [9, Theorem 4.2] Let \( \lambda/\mu = M(n, a, b, c, d, m) \). Then

\[ \sum_{T \in \text{SSYT}(\lambda/\mu)} q^{|T|} = q^{\sum_{(i,j) \in \lambda/(\lambda, \mu)} (\lambda')_j - i} \prod_{i=1}^n \prod_{j=1}^a \prod_{k=1}^c \frac{1 - q^{m(i+j+k-1)}}{1 - q^{m(i+j+k+2)}} \prod_{(i,j) \in \lambda/(\lambda, \mu)} \frac{1}{1 - q^{h_{\lambda, (i,j)}}}. \]

In Section 2 we show the following trace generating function formula, which is a generalization of Theorem 1.5.

**Theorem 1.6.** Let \( \pi = M(n, a, b, c, d, m) \). Then

\[ \sum_{T \in \text{SSYT}(\pi)} x^{\text{tr}(T)} q^{|T|} = x^{na+\binom{n}{2}} q^{\sum_{(i,j) \in \lambda/(\lambda, \mu)} (\lambda')_j - i} \prod_{i=1}^n \prod_{j=1}^a \prod_{k=1}^c \frac{1 - q^{m(i+j+k-1)}}{1 - q^{m(i+j+k+2)}} \prod_{(i,j) \in \lambda/(\lambda, \mu)} \frac{1}{1 - x\chi(i,j) q^{h_{\lambda, (i,j)}}}, \]

where

\[ \chi(i,j) = \begin{cases} 1, & \text{if } (i,j) \in (n+c)^{n+a}, \\ 0, & \text{otherwise.} \end{cases} \]

2. Preliminaries

The following notations will be used throughout this paper:

\[(2n-1)!! = 1 \cdot 3 \cdots (2n-1), \quad (a;q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}), \]

\[\Phi(n) = \prod_{i=1}^{n-1} i!, \quad \mathfrak{Z}(n) = \prod_{i=1}^{[n/2]} (n-2i)!, \]

\[\Phi_q(n) = \prod_{i=1}^{n-1} (q;q)_i, \quad \mathfrak{Z}_q(n) = \prod_{i=1}^{[n/2]} (q;q)_{n-2i}. \]

A **partition** is a sequence \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \). Each \( \lambda_i > 0 \) is called a **part** of \( \lambda \). The **length** \( \ell(\lambda) \) of \( \lambda \) is the number of parts in \( \lambda \). We denote by \( \text{Par}_n \) the set of partitions with at most \( n \) parts. We will use the convention that \( \lambda_i = 0 \) for all \( i > \ell(\lambda) \). We define

\[ \delta_n = (n-1, n-2, \ldots, 1, 0) \in \text{Par}_n. \]

For a partition \( \lambda \), let

\[ n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i. \]

For \( \lambda \in \text{Par}_n \) and a sequence \( x = (x_1, \ldots, x_n) \) of variables, we define

\[ a_\lambda(x) = \text{det}(x_j^{\lambda_i+n-1})_{i,j=1}^n, \]

\[ e_\lambda(x) = \text{det}(x_j^{\lambda_i+n-1})_{i,j=1}^n = (-1)^{\binom{n}{2}} a_\lambda(x), \]

\[ \Delta(x) = a_{\delta_n}(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j), \]
generally, for any subset \( \lambda \) also define two partitions \( \lambda, \mu = (m_\lambda) \). The transpose \( M^\lambda \) of \( M_\lambda \) is constructed from the generating function of RPPs or SSYTs.

A row strict tableau (RST) is a tableau \( T \) of shape \( \pi \) such that the entries are weakly increasing in each row and strictly increasing in each column. An SYT, an SSYT, an RST and an RPP of shape \((4, 3, 1)\) from left to right are shown in Figure 5. For each element in \( \pi \), we denote by \( m_\pi \) the size of \( \pi \), denoted by \( |\pi| \), is the number of cells in \( \pi \).

A standard Young tableau (SYT) of shape \( \pi \) is a filling of the cells in \( \pi \) with \( 1, 2, \ldots, |\pi| \) such that the entries are increasing in each row and in each column. A semistandard Young tableau (SSYT) of shape \( \pi \) is a filling of the cells in \( \pi \) with nonnegative integers such that the entries are weakly increasing in each row and strictly increasing in each column. A row strict tableau (RST) of shape \( \pi \) is a filling of the cells in \( \pi \) with nonnegative integers such that the entries are strictly increasing in each row and weakly increasing in each column. A reverse plane partition (RPP) of shape \( \pi \) is a filling of the cells in \( \pi \) with nonnegative integers such that the entries which are weakly increasing in each row and in each column. See Figure 5 for examples of these objects. We denote by SYT(\( \pi \)), SSYT(\( \pi \)), RST(\( \pi \)) and RPP(\( \pi \)), respectively, the set of SYTs, SSYTs, RSTs and RPPs of shape \( \pi \). We will simply call an element in one of these sets a tableau of shape \( \pi \). For a tableau \( T \), we denote by \( |T| \) the size of the entries in \( T \), by \( \max(T) \) the largest entry in \( T \) and by \( \min(T) \) the smallest entry in \( T \).

The following well known lemma tells us that the number of standard Young tableaux can be computed from the generating function of RPPs or SSYTs.
Lemma 2.1. For any skew shape or shifted skew shape \( \pi \), we have

\[
f^\pi = \lim_{q \to 1} (q; q)_{|\pi|} \sum_{T \in \text{RPP}(\pi)} q^{|T|} = \lim_{q \to 1} (q; q)_{|\pi|} \sum_{T \in \text{SSYT}(\pi)} q^{|T|}.
\]

Proof. This can be proved using the \( (P, \omega) \)-partition theory. We follow the terminologies in [13 Section 3.15]. Let \( P \) be the poset whose elements are the cells \((i, j) \in \pi \) with relation \((i, j) \leq_P (i', j') \) if \( i \geq i' \) and \( j \geq j' \). Let \( \omega \) be any natural labeling of \( P \). Then \( \text{SYT}(\pi) \) is in bijection with the set \( \mathcal{L}(P, \omega) \) of linear extensions of \( P \), and the RPPs of shape \( \pi \) can be considered as the \( (P, \omega) \)-partitions. By [13 Theorem 3.15.7],

\[
\sum_{w \in \mathcal{L}(P, \omega)} q^{\maj(w)} = (q; q)_{|\pi|} \sum_{T \in \text{RPP}(\pi)} q^{|T|}.
\]

Since \( f^\pi = |\text{SYT}(\pi)| = |\mathcal{L}(P, \omega)| \), by taking the \( q \to 1 \) limit, we obtain the first identity. The second identity can be proved similarly, see [12 Proposition 7.19.11]. \( \square \)

Let \( T \) be a tableau of shifted shape \( \lambda^* \) for a strict partition \( \lambda \) with \( \ell(\lambda) = \ell \). The reverse diagonal of \( T \) is the product \( \text{rdiag}(T) = (d_\ell, d_{\ell-1}, \ldots, d_1) \), where \( d_i \) is the entry in the cell \((i, i) \) of \( T \) for \( 1 \leq i \leq \ell \). For example, if \( T_1, T_2 \) and \( T_3 \) are the tableaux in Figure 6 from left to right, then \( \text{rdiag}(T_1) = (5, 1, 0) \), \( \text{rdiag}(T_2) = (4, 1, 0) \) and \( \text{rdiag}(T_3) = (2, 0, 0) \).

For a partition \( \mu = (\mu_1, \ldots, \mu_n) \), we denote \( q^\mu = (q^{\mu_1}, \ldots, q^{\mu_n}) \). The following theorem is a key ingredient in this paper.

Theorem 2.2. [7 Theorem 8.7] For \( \lambda, \mu \in \text{Par}_n \), we have

\[
\sum_{T \in \text{RPP}((\delta_{\ell+1} + \lambda)^*) \atop \text{rdiag}(T) = \mu} q^{|T|} = \prod_{j=1}^{n} (q; q)^{|\mu_j|} \sum_{T \in \text{SSYT}((\delta_{\ell+1} + \lambda)^*) \atop \text{rdiag}(T) = \mu} q^{|T|}\prod_{j=1}^{n} (q; q)^{|\lambda_j|} \pi_{|\lambda|+\delta_n} (q^{\mu_j}).
\]

Given an RPP, by adding \((i-1)\)'s to all the cells in the \( i \)-th column, we can get an RST, and similarly, by adding \((i-1)\)'s to the cells in the \( i \)-th row, we can get an SSYT. For example, the SSYT and the RST in Figure 6 are obtained in this way from the RPP on the right. Note that if \( T \in \text{RPP}((\delta_{\ell+1} + \lambda)^*) \) and \( \text{rdiag}(T) = \mu \), then the resulting SSYT or RST \( T' \) has \( \text{rdiag}(T') = \mu + \delta_n \). Applying this process to Theorem 2.2, we get the generating functions for SSYT and RST with fixed diagonal.

Corollary 2.3. For \( \lambda, \nu \in \text{Par}_n \), we have

\[
\sum_{T \in \text{SSYT}((\delta_{\ell+1} + \lambda)^*) \atop \text{rdiag}(T) = \nu} q^{|T|} = \prod_{j=1}^{n} (q; q)^{|\lambda_j|+n-j} \pi_{|\lambda|+\delta_n} (q^{\nu_j}),
\]

(6)

\[
\sum_{T \in \text{RST}((\delta_{\ell+1} + \lambda)^*) \atop \text{rdiag}(T) = \nu} q^{|T|} = \prod_{j=1}^{n} (q; q)^{|\lambda_j|+n-j} \pi_{|\lambda|+\delta_n} (q^{\nu_j}).
\]

(7)

For a partition \( \lambda \), the hook length of \((i, j) \in \lambda \) is defined by

\[
h_\lambda(i, j) = \lambda_i + \lambda_j' - i - j + 1.
\]
Now let $\lambda$ be a strict partition with $\ell(\lambda) = n$. Then $\lambda = \delta_{n+1} + \mu$ for some $\mu \in \operatorname{Par}_n$. The \textit{shifted hook length} of $(i, j) \in \lambda^*$ is defined by

$$h_{\lambda^*}(i, j) = \begin{cases} n + 1 + \mu_i, & \text{if } i = j, \\ \mu_i + \mu_j + 2(n + 1) - i - j, & \text{if } i < j \leq n, \\ h_{\mu}(i, j - n), & \text{if } j > n. \end{cases}$$

For an example, see Figure 7.

The $q$-integral of $f(x)$ over $[a, b]$ is defined by

$$\int_a^b f(x) \,dq_x = (1 - q) \sum_{i \geq 0} (f(bq^i) bq^i - f(aq^i) aq^i),$$

where $0 < q < 1$ and the sum is assumed to absolutely converge. We also define the multivariate $q$-integral

$$\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} f(x_1, \ldots, x_n) \,dq_{x_1} \cdots dq_{x_n} = \int_0^1 \int_0^{x_2} \cdots \int_0^{x_n-1} f(x_1, \ldots, x_n) \,dq_{x_1} \cdots dq_{x_n}.$$

The following lemma tells us how we make a change of variables in the $q$-integral.

\textbf{Lemma 2.4.} Let $f(x, q)$ be a function with two variables $x$ and $q$. For an integer $m \geq 0$, we have

$$\int_0^1 f(x^m, q^m) \,dq_x = \frac{1 - q}{1 - p} \int_0^1 f(x, p) \cdot x^{\frac{1 - m}{m}} \,dq_x,$$

where $p = q^m$.

\textbf{Proof.} By the definition of the $q$-integral, the left hand side is

$$\int_0^1 f(x^m, q^m) \,dq_x = (1 - q) \sum_{i \geq 0} f(q^{mi}, q^m) q^i$$

$$= (1 - q) \sum_{i \geq 0} f(p^i, p) p^{i/m}$$

$$= \frac{1 - q}{1 - p} \cdot (1 - p) \sum_{i \geq 0} f(p^i, p) p^{\frac{1 - m}{m}i} \cdot p^i,$$

which is equal to the right hand side. \hfill $\square$

The following lemma explains how we write the $q$-summation of certain functions in terms of the $q$-integral.

\textbf{Lemma 2.5.} \cite{4, Lemma 4.3} For a function $f(x_1, \ldots, x_n)$ satisfying $f(x_1, \ldots, x_n) = 0$ if $x_i = x_j$ for any $i \neq j$,

$$\sum_{\mu \in \operatorname{Par}_n} q^{\mu_{\delta_{n+1}}} f(q^{\mu_{\delta_{n+1}}}) = \frac{1}{(1 - q)^n} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} f(x_1, \ldots, x_n) \,dq_{x_1} \cdots dq_{x_n}.$$
Theorem 3.1. Let \( \lambda = ((n + b + c)^{n+a}, (n + c)^d) \), \( \mu = ((c + 1), c^{a-1}, 1) \) and \( \rho = \lambda / \mu \). Then

\[
f^\rho = |\rho|! \frac{\Phi(n) \Phi(a) \Phi(b) \Phi(c) \Phi(d) \Phi(n + a + c) \Phi(n + b + d) \Phi(n + a + b + c + d)}{\Phi(a + c) \Phi(b + d) \Phi(n + a + b) \Phi(n + c + d) \Phi(2n + a + b + c + d)}
\]
\[
\times \frac{ac(n(n + a + b + c + d)(n(n + a + b + c + d) + (a + c)(b + d)) + bd(a + c - 1)(a + c + 1))}{(a + c - 1)(a + c + 1)}.
\]

**Proof.** Consider the upper-right half part \( \rho^* \) and the lower-left half part \( \rho' \) of \( \rho \) divided at the main diagonal with the dented square part filled as shown in Figure 8. In other words, \( \rho^* = \kappa(a, b, n) \) and \( \rho' = \kappa(c, d, n)' \), where

\[
\kappa(a, b, n) = (n + a + b, n + a + b - 1, \ldots, b + 1)^*/(\delta_{a+1})^*
\]

Define \( f(a, b, n, \mu, t) \) to be the generating function of RPP’s \( T \) of shape \( \kappa(n, a, b) \) such that \( \text{rdiag}(T) = \mu \) and the value in the top-left corner cell is at least \( t \) (hence, all the values are greater than or equal to \( t \)), i.e.,

\[
f(a, b, n, \mu, t) := \sum_{T \in \text{RPP}(\kappa(a, b, n))} q^{|T|} = q^{-t^2}/2\sum_{T' \in \text{RPP}((\delta_{a+1} + (bn+a))^*)} q^{|T'|}.
\]

The right hand side of the above equation is obtained by filling the skewed part \((\delta_{a+1})^*\) by \( t \)'s. By applying Theorem 2.2 we get

\[
f(a, b, n, \mu, t) = \sum_{T \in \text{RPP}(\rho^*)} q^{|T|} = \sum_{\mu \in \text{Par}_n} q^{-|\mu|} \sum_{T \in \text{RPP}(\rho')} q^{|T|} \sum_{\text{rdiag}(T) = \mu, \text{min}(T) = 0} q^{|T|}
\]

Hence,

\[
\sum_{T \in \text{RPP}(\rho^*)} q^{|T|} = \sum_{\mu \in \text{Par}_n} q^{-|\mu|} \sum_{T \in \text{RPP}(\rho')} q^{|T|} \sum_{\text{rdiag}(T) = \mu, \text{min}(T) = 0} q^{|T|}
\]

\[
= \sum_{\mu \in \text{Par}_n} q^{-|\mu|} (f(a, b, n, \mu, 0) - f(a, b, n, \mu, 1))(f(c, d, n, \mu, 0) - f(c, d, n, \mu, 1))
\]

\[
= \sum_{\mu \in \text{Par}_n} q^{-|\mu|} \prod_{i=1}^{n(a+b+d+1)} (q^{|\mu_1|} + \cdots + q^{|\mu_n|})^2 \prod_{i=1}^{n(a+b)} (q^{|\mu_1|} + \cdots + q^{|\mu_n|})^2 \prod_{i=1}^{n(a+b)} (q^{|\mu_1|} + \cdots + q^{|\mu_n|})^2
\]

\[
\times \left( \prod_{i=1}^{n} (q^{|\mu_i|} + q^{|\mu_i|+1}; q)_a - q^{|\mu_i|+n-i}; q)_a \right) \left( \prod_{i=1}^{n} (q^{|\mu_i|} + q^{|\mu_i|+1}; q)_c - q^{|\mu_i|+n+i}; q)_c \right)
\]

\[
= (1 - q^n) \prod_{j=1}^{a+b+d} (q^{|j|} + \cdots + q^{|j|})^2 \prod_{i=1}^{n} (q^{|x_i|}; q)_a - q^{|n+b|} \prod_{i=1}^{n} (x_i; q)_a \prod_{i=1}^{n} (x_i; q)_a
\]

\[
\times \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b+d} \Delta(X)^2 \left( \prod_{i=1}^{n} (qx_i; q)_a - q^{|n+b|} \prod_{i=1}^{n} (x_i; q)_a \right)
\]

\[
\times \left( \prod_{i=1}^{n} (qx_i; q)_c - q^{|n+d|} \prod_{i=1}^{n} (x_i; q)_c \right) d_q x_1 \cdots d_q x_n,
\]
Now we compute the integral part applying this limit computation, with the change of variables $x_i \mapsto 1 - x_i$. Then we have

$$
 f^{[\rho]} = \lim_{q \to 1} \frac{q^{B(a,b,n)+B(c,d,n)-(b+d+1)}}{(1-q)^n \prod_{j=1}^{n}(q; q)_{n+a+b-j} \prod_{j=1}^{n+c}(q; q)_{n+c+d-j}}
 \times \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b+d} \Xi(X)^2 \left( \prod_{i=1}^{n} (q x_i; q)_a - q^a(n+b) \prod_{i=1}^{n} (x_i; q)_a \right)
 \times \left( \prod_{i=1}^{n} (q x_i; q)_c - q^c(n+d) \prod_{i=1}^{n} (x_i; q)_c \right) dx_1 \cdots dx_n.
$$

Let us calculate the limits separately:

$$
 \lim_{q \to 1} \prod_{i=1}^{n} (q x_i; q)_a - q^a(n+b) \prod_{i=1}^{n} (x_i; q)_a
 = \lim_{q \to 1} \prod_{i=1}^{n} (1 - q^a x_i) - q^a(n+b) \prod_{i=1}^{n} (1 - x_i)
 = \prod_{i=1}^{n} (1 - x_i)^{n-1} \lim_{q \to 1} \prod_{i=1}^{n} (1 - q^a x_i) \sum_{j=1}^{n} \frac{-aq^{a-1} x_j}{1-q^a x_j} - a(b+n)q^a(n+b)^{-1} \prod_{i=1}^{n} (1 - x_i)
 = a^{n} (1 - x_i)^{n} \left( \sum_{j=1}^{n} \frac{x_j}{1 - x_j} + (n+b) \right).
$$

Now we compute the integral part applying this limit computation, with the change of variables $x_i \mapsto 1 - x_i$. Then we have

$$
 \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b+d} \Xi(X)^2 \lim_{q \to 1} \prod_{i=1}^{n} (q x_i; q)_a - q^a(n+b) \prod_{i=1}^{n} (x_i; q)_a
 \times \lim_{q \to 1} \prod_{i=1}^{n} (q x_i; q)_c - q^c(n+d) \prod_{i=1}^{n} (x_i; q)_c
dx_1 \cdots dx_n
 = ac \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{a+c}(1 - x_i)^{b+d} \Xi(X)^2
 \times \left( \sum_{j=1}^{n} \frac{1 - x_j}{x_j} + (n+b) \right) \left( \sum_{j=1}^{n} \frac{1 - x_j}{x_j} + (n+d) \right) dx_1 \cdots dx_n
 = ac \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{a+c}(1 - x_i)^{b+d} \Xi(X)^2
 \times \left( \prod_{i=1}^{n} x_i^{-1} s_{(1^{n-1})}(X) + b \right) \left( \prod_{i=1}^{n} x_i^{-1} s_{(1^{n-1})}(X) + d \right) dx_1 \cdots dx_n
 = ac \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} s_{(2^n-2,1,1)}(X) \prod_{i=1}^{n} x_i^{a+c-2}(1 - x_i)^{b+d} \Xi(X)^2 dx_1 \cdots dx_n
$$
Adding the above four results gives us

\[
\begin{align*}
\Phi(n+1) \Phi(n+a+c) \Phi(n+b+d) \Phi(n+a+b+c+d+1) \\
\Phi(a+c+1) \Phi(b+d) \Phi(2n+a+b+c+d)
\end{align*}
\]

We obtain the formula for \( f^{|\alpha|} \) by replacing the integration part in (8) by the above computation.
4. Enumeration of standard Young tableaux of shifted skew shape

In this section we prove Conjecture 1.3, which is restated below.

**Theorem 4.1.** For $\pi = V(n, a, b, m)$, the number $g^{\pi}$ of standard Young tableaux of shape $\pi$ is

$$g^{\pi} = \frac{\pi!}{2^a} \cdot \frac{\Phi(n + 2a) \Phi(a) \Phi(n + a + 1)}{\Phi(2a) \Phi(n + a + 1)} \cdot \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h_{\lambda^*}(i,j)};$$

where $\mathfrak{S}(n) = \prod_{i=1}^{[n/2]} (n - 2i)!$, $\lambda = ((n + a + b, n + a + b - 1, \ldots, b + 1) + (m - 1)\delta_{n+a})^*$ and $D$ is the set of cells $(i, n + j)$ with $1 \leq i \leq j \leq n$.

**Proof.** Firstly, by computing the shifted hook lengths of the cells in $\lambda \setminus D$ explicitly, we can rewrite the conjectured formula for $g^{\pi}$ as

$$g^{\pi} = \frac{\pi!}{2^a} \cdot \frac{\Phi(n + 2a) \Phi(a) \Phi(n + a + 1)}{\Phi(2a) \Phi(n + a + 1)} \cdot \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h_{\lambda^*}(i,j)};$$

To utilize the generating function formula for the semistandard Young tableaux of shifted shapes with fixed diagonals given in Corollary 2.3, we fill the top row of the skew part $(\delta_{n+a})^*$ by 0’s, the second row by 1’s, and so on. If we say we fix the diagonal cells of $\lambda$ by $\nu := (\mu_1 + n - 1, \ldots, \mu_{n-1} + 1, \mu_n, a - 1, \ldots, 1)$ for some $\mu \in \text{Par}_n$, then we have

$$\sum_{T \in \text{SSYT}(\lambda)} q^{|T|} = \sum_{T \in \text{SSYT}(\pi)} q^{|T|} \cdot \sum_{\text{rdiag}(T) = \nu \atop \text{Par}_n} q^{|T|}$$

Summing this up over all partitions with at most $n$ parts gives

$$\sum_{T \in \text{SSYT}(\pi)} q^{|T|} = \sum_{T \in \text{SSYT}(\pi)} q^{|T|} \cdot \sum_{\text{rdiag}(T) = \nu \atop \text{Par}_n} q^{|T|}$$

$$= \frac{q^{(-\pi)}}{(1 - q)^n} \prod_{i=1}^{n+\pi} (q; q)_{b+m(n+a-j)} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \mathfrak{u}_{(b+n)+m\delta_{n+a}}(x_1, \ldots, x_n, q, \ldots, q^n) \, dq.$$ 

To simplify the integrand, we note the following:

$$a_{(b^n)+\lambda}(x_1, \ldots, x_n) = \det(x_i^{b+\lambda}) = \prod_{i=1}^{n} x_i^{b} \cdot \det(x_i^{\lambda}) = \prod_{i=1}^{n} x_i^{b} \cdot a_{\lambda}(x_1, \ldots, x_n)$$

and

$$a_{m\lambda}(x_1, \ldots, x_n) = \det(x_i^{m\lambda}) = a_{\lambda}(x_1, \ldots, x_n).$$

Thus we can rewrite the integrand as

$$\mathfrak{u}_{(b+n)+m\delta_{n+a}}(x_1, \ldots, x_n, q, \ldots, q^n) = q^{h(\pi)} \prod_{i=1}^{n} x_i^{b} \cdot \Delta(x_1^{m}, \ldots, x_n^{m}, q, \ldots, q^n).$$

So far we have

$$\sum_{T \in \text{SSYT}(\pi)} q^{|T|} = \frac{q^{h(\pi)} - (-\pi)}{(1 - q)^n} \prod_{i=1}^{n+\pi} (q; q)_{b+m(n+a-j)} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b} \cdot \Delta(x_1^{m}, \ldots, x_n^{m}, q, \ldots, q^n) \, dq.$$
We apply Lemma 2.4 to make the change of variables $x_i^n \mapsto x_i$, and by letting $p = q^m$ we get
\[
\sum_{T \in \text{SSYT}(\pi)} q^{|T|} = \frac{q^{b(\pi) - (\pi)}}{(1 - p)^n \prod_{i=1}^{n+a} (q; q)_{b+m(n+a-j)}} \times \int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b - \frac{m+1}{m}} \Delta(x_1, \ldots, x_n, p^{a-1}, \ldots, p^{0}) dp x_1 \cdots dp x_n.
\]

Note that
\[
\Delta(x_1, \ldots, x_n, p^{a-1}, \ldots, p^{0}) = p^{(\pi)^+ + n(\pi)} \prod_{i=1}^{a-1} (p; p) \cdot \prod_{i=1}^{n} (p^{1-a} x_i; p)_a \cdot \Delta(x_1, \ldots, x_n),
\]
and so we have
\[
\sum_{T \in \text{SSYT}(\pi)} q^{|T|} = \frac{q^{b(\pi) - (\pi)} + n(\pi)}{(1 - q^m)^n \prod_{i=1}^{n+a} (q; q)_{b+m(n+a-j)}} \times \int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b - \frac{m+1}{m}} (p^{1-a} x_i; p)_a \cdot \Delta(x_1, \ldots, x_n) dp x_1 \cdots dp x_n.
\]

Given this, by Lemma 2.1 we can get the number of standard Young tableaux of shape $\pi$ by
\[
g^\pi = \lim_{q \to 1} \left( (q; q)_{|\pi|} \sum_{T \in \text{SSYT}(\pi)} q^{|T|} \right)
= \frac{|\pi|! \cdot n(\pi)}{m^n \prod_{j=0}^{n+a-1} (b + m j)!} \int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b - \frac{m+1}{m}} (1 - x_i)^a \cdot \Delta(x_1, \ldots, x_n) dx_1 \cdots dx_n.
\]

Notice that the integral is a special case of the well-known Selberg integral:
\[
\int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{b - \frac{m+1}{m}} (1 - x_i)^a \cdot \Delta(x_1, \ldots, x_n) dx_1 \cdots dx_n
= \frac{1}{n!} \prod_{j=1}^{n} \frac{\Gamma(b + j + \frac{1}{2}) \Gamma(a + 1 + \frac{1}{2} j) \Gamma(1 + \frac{1}{2} j)}{\Gamma(\frac{a+1}{m} + a + 1 + \frac{1}{2} (n + j - 2) \Gamma(1 + \frac{1}{2} j))}
\]
\[
= \begin{cases} 
\frac{m^{N+2N(N+a)} 2^N \Phi(2N + 2a) \Gamma(2N) \Gamma(2a) \Gamma(2 N + 2a)}{m^{N+2(N+a)} \Phi(2N + 2a + 1) \Gamma(2N + 2 a + 1) \Gamma(2N + 2 a + 1)}, & \text{if } n = 2N, \\
\frac{m^{N+2(N+a)} 2^N \Phi(2N + 2a) \Gamma(2N) \Gamma(2a) \Gamma(2 N + 2a)}{m^{N+2(N+a)} \Phi(2N + 2a + 1) \Gamma(2N + 2 a + 1) \Gamma(2N + 2 a + 1)}, & \text{if } n = 2N + 1,
\end{cases}
\]
where (14) is obtained by explicitly computing the gamma function values. By replacing the integral part in $g^\pi$ by (14), we get
\[
g^\pi = \frac{|\pi|! n^{\left(\frac{a}{m}\right)} \Phi(a) \Phi(n + 2a) \Gamma(2a) \Gamma(n + 2a)}{\Phi(2a) \Gamma(n + 2a)} \cdot P(a, b, n, m),
\]
where
\[
P(a, b, n, m)
= \begin{cases} 
\prod_{j=0}^{2N+a-1} (b + mj)! \prod_{i=0}^{N-1} (b + 1 + mi) \prod_{j=1}^{2N} \prod_{i=0}^{N+a-1} (2b + 1 + (2i + j)m), & \text{if } n = 2N, \\
\prod_{j=0}^{2N+a} (b + mj)! \prod_{j=1}^{2N+a} \prod_{i=0}^{N+a-1} (2b + 1 + (2i + 1) m), & \text{if } n = 2N + 1.
\end{cases}
\]
By comparing this formula to (13), to verify Morales, Pak and Panova conjecture, we only need to prove

\[ P(a, b, n, m) = \frac{\prod_{i=1}^{n} \prod_{j=0}^{i-1} (2(b+1) + (n+i+j-1)m)}{2^n \prod_{i=0}^{n-a-1} (b+mi)! \prod_{i=0}^{n-a-1} (b+1+mi) \prod_{j=0}^{n-a-1} (2(b+1) + (i+j)m)} \text{.} \]

It is not very hard to check that they are two different ways of expressing multiplications of the same set of factors. This finishes the proof.

\[ \square \]

5. Generalized MacMahon’s box theorem using q-integrals

In this section, we prove Theorem 1.4, which is restated as follows.

**Theorem 5.1.** For \( \pi = M(a, b, c, d, 1) \) and an integer \( N \geq 0 \), we have

\[
s_{\pi}(1, q, q^2, \ldots, q^N) = q^{\sum_{(i,j) \in \lambda/(\mu+\nu)} (\lambda_{ji} - 1)} \prod_{i=1}^{n} (q^{N-a+a+1}; q)_{a} \prod_{i=1}^{d} (q^{N+b-b+1}; q)_{c} 
\times \prod_{i=1}^{n} (q^{N-a-a-d+1+1}; q)_{n+a+b+c+a} \prod_{i=1}^{n} \prod_{j=1}^{a} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{\lambda(i,j)}} \text{.}
\]

**Proof.** Recall from Corollary 2.3 that

\[
\sum_{T \in \text{SSYT}(\delta_{n+1}+\lambda)^*} q^{|T|} = \frac{q^{[\mu+\delta_n]}_{\lambda+\delta_n}(q^{\mu+\delta_n})}{\prod_{j=1}^{n} (q; q)_{\lambda_j+n-j}} \text{.}
\]

We first consider the upper right half part of \( \pi \), divided at the main diagonal, that is, \( \pi^{rh} := (n+a+b, n+a+b-1, \ldots, b+1)^*/(\delta_{a+1})^* \). To utilize (15), we fill the skew part \((\delta_{a+1})^* \) by 0’s in the first row, 1’s in the second row, and so on, and by \((a-1)\) in the \(a\)-th row. Similarly, we attach \((\delta_{b+1})^* \) below the last row and fill it with \((N+1)\) in the first row, \((N+2)\) in the second row, and so on, and with \((N+b)\) in the \(b\)-th row. Then

\[
\sum_{T \in \text{SSYT}(\pi^{rh}) \atop \text{rdiag}(T) = \mu+\delta_n \atop \text{min}(T) \geq 0, \text{max}(T) \leq N} q^{|T|} = q^{(-b+1) - (b+1) - N(b+1)} \sum_{T \in \text{SSYT}(\delta_{n+a+b+1}^*) \atop \text{rdiag}(T) = (N+b, \ldots, N+1, \mu_1+n-1, \ldots, \mu_n, a, \ldots, 1, 0)} q^{|T|} = \frac{q^{(b+1) - (b+1) - N(b+1) + [\mu+\nu] + (a_2)}}{\prod_{j=1}^{1} (q; q)_j} \Delta(q^{N+b}, \ldots, q^{N+1}, q^{\mu+\delta_n}, q^{a-1}, \ldots, q, 1) \text{.}
\]

On the other hand, to deal with the lower-left half of \( \pi \) using \( \pi^{ll} \) in Corollary 2.3, we reflect the lower-left half of \( \pi \) along the main diagonal and denote that part by \( \pi^{ll} = (n+c+d, n+c+d-1, \ldots, d+1)^*/(\delta_{c+1})^* \). Note that since we have reflected the diagram along the main diagonal, the fillings satisfying the conditions of semistandard Young tableaux becomes row strict tableaux. Keeping this in mind, again attach \((\delta_{c+1})^* \) in front and \((\delta_{d+1})^* \) below the last row, and fill the cells by \((c-1)\) in the first column, \((c+1)\) in the second column, and so on, and by \((-1)\) in the \(c\)-th column of the \((\delta_{c+1})^* \) part attached in front, and by \((N-d+1)\) in the first column of the \((\delta_{d+1})^* \) part, by \((N-d+2)\) in the second column, and so on, and by \(N\)’s in the last column. Then we have

\[
\sum_{T \in \text{RST}(\pi^{ll}) \atop \text{rdiag}(T) = \mu+\delta_n \atop \text{min}(T) \geq 0, \text{max}(T) \leq N} q^{|T|} = q^{(-c+1) + (d+1) - N(d+1)} \sum_{T \in \text{RST}(\delta_{n+c+d+1}^*) \atop \text{rdiag}(T) = (N, N-1, \ldots, N-d+1, \mu_1+n-1, \ldots, \mu_n-1, \ldots, -c)} q^{|T|} \text{.}
\]
Hence,

\[
\sum_{T \in \text{SSYT}(\pi)} q^{|T|} = \sum_{\mu \in \text{Par}_n} q^{-|\mu|-1} \sum_{T \in \text{SSYT}(\pi^\mu)} q^{|T|} \sum_{T \in \text{SSYT}(\pi^\mu)} q^{|T|} \sum_{T \in \text{SSYT}(\pi^\mu)} q^{|T|}
\]

\[
= q^{-\binom{a}{3}+\binom{b}{3}+\binom{c}{3}+N} \prod_{j=1}^{n+c+d+1} (q; q)_j
\]

\[
\prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j
\]

\[
\prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j
\]

Note that

\[
\Delta(q^{N+b}, q^{N+c}, x_1, \ldots, x_n, q^{a-1}x_1, \ldots, q^{a-1}x_n, q^{-c})
\]

\[
= q^{-\binom{a}{3}+\binom{b}{3}+\binom{c}{3}+N} \sum_{a=1}^{b} \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i \prod_{i=1}^{n} (q; q)_i
\]

Applying the above computations gives

\[
\sum_{T \in \text{SSYT}(\pi)} q^{|T|} = q^{-\binom{a}{3}+\binom{b}{3}+\binom{c}{3}+N} \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j
\]

\[
= q^{-\binom{a}{3}+\binom{b}{3}+\binom{c}{3}+N} \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j \prod_{j=1}^{n+c+d+1} (q; q)_j
\]

Note that the factor \((qx_i/q^{N+b+1}, q)^{b+d}\) in the integrand becomes zero for the values \(q^{N-b+1} \leq x_i \leq q^{N+b}\). Hence we can change the lower end of the integral by \(q^{N+b+1}\). Then the integral is a special case of the \(q\)-Selberg integral which was conjectured by Askey [11] and proved by Habsieger [3], Kadell [5] and Evans [2] :
\[
(16) \quad = (-1)^3 q^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{\Gamma_q(\alpha + i) \Gamma_q(\beta + i) \Gamma_q(\beta + i - 1) \left( \frac{q}{2} \right)^{\beta + i} \left( \frac{q}{2} \right)^{\alpha + i} (ab)^{i+1}}{\Gamma_q(\alpha + \beta + n + i - 1) (a - b)}.
\]

By specializing \(a = q^{N+b+1}, \ b = q^n, \alpha = b + d + 1, \beta = a + c + 1\), we get

\[
\int_{q^{N+b+1} \leq x_1 \leq \cdots \leq x_n \leq q^n} \sum_{i=1}^{n} (x_1/q^n; q)_{a+c} (q x_i/q^{N+b+1}; q)_{b+d} dx_1 \cdots dx_n
\]

\[
= \sum_{i=1}^{n} (1 - q)^n (q^{N-a-d+1+i}; q)_{n+a+b+c+d} \Phi_q(n) \Phi_q(n + a + c) \Phi_q(n + b + d) \Phi_q(n + a + b + c + d) \Phi_q(a + c) \Phi_q(b + d) \Phi_q(2n + a + b + c + d).
\]

By putting the result of evaluating the Selberg-type integral and simplifying the \(q\)-power, we get

\[
\sum_{T \in \text{SYT}(\lambda)} q^{[T]} = \sum_{T \in \text{SYT}(\lambda)} \prod_{i=1}^{n} (q^{N-a-d+1+i}; q)_{n+a+b+c+d} \Phi_q(n) \Phi_q(n + a + c) \Phi_q(n + b + d) \Phi_q(n + a + b + c + d) \Phi_q(a + c) \Phi_q(b + d) \Phi_q(n + c + d) \Phi_q(2n + a + b + c + d).
\]

Note that

\[
a \left( \binom{n + d}{2} \right) + b \left( \binom{n + a}{2} \right) + c \left( \binom{n + a + d}{2} \right) = \sum_{(i,j) \in \lambda / (c^n)} (X'_j - i).
\]

Also noting that

\[
\frac{\Phi_q(a) \Phi_q(c) \Phi_q(n) \Phi_q(n + a + c)}{\Phi_q(a + c) \Phi_q(n + a) \Phi_q(n + c + d) \Phi_q(2n + a + b + c + d)} = \prod_{i=1}^{n} \prod_{j=1}^{a} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}
\]

and considering the hook lengths of the cells in \(\lambda \setminus (0^n, c^n)\), we can rewrite the result as

\[
s_x(1, q, q^2, \ldots, q^n) = \sum_{T \in \text{SYT}(\lambda)} q^{[T]} \prod_{i=1}^{n} (q^{N-a-d+1+i}; q)_{n+a+b+c+d} \Phi_q(n) \Phi_q(n + a + c) \Phi_q(n + b + d) \Phi_q(n + a + b + c + d) \Phi_q(a + c) \Phi_q(b + d) \Phi_q(n + c + d) \Phi_q(2n + a + b + c + d).
\]

6. Skew trace generating function

In this section, we prove Theorem 1.6, which is restated as follows.
Theorem 6.1. Let $\pi = M(n, a, b, c, d, m)$. Then

$$
\sum_{T \in \text{SSYT}(\pi)} x^{\text{tr}(T)} q^{|T|} = x^{na + \binom{a}{2}} q^{\sum_{(i,j) \in \lambda_1(\pi)} (\lambda_i - 1)} \prod_{i=1}^{n} \prod_{j=1}^{a} \prod_{k=1}^{c} \frac{1-q^{m(i+j+k-1)}}{1-q^{m(i+j+k-2)}} \prod_{(i,j) \in \lambda_1((0^n,c^a))} \frac{1}{1-x^\chi(i,j) q^{b(i,j)^*}}
$$

where

$$
\chi(i,j) = \begin{cases} 1, & \text{if } (i,j) \in ((n+c)^{n+a}), \\ 0, & \text{otherwise}. \end{cases}
$$

Proof. Since both sides are power series in $x$ and $q$, is sufficient to show for $x = q^t$, where $t$ is an arbitrary integer. The idea is that we divide $\pi$ in two parts along the diagonal, compute the generating functions of the upper-right half and the lower-left half separately, and lastly combine them together.

The upper-right half is $V(n, a, b, m)$ (see Figure 2). We denote this upper-half by $\pi^u$ and the lower-half by $\pi^l$. Note that $\pi^d = (V(n, c, d, m))^t$.

We consider $\pi^u$ first. To utilize the generating function formula for the semistandard Young tableaux of shifted shapes with fixed diagonals given in Corollary 2.3, we fill the top row of the skewed part $\delta_{n+1}$ by 0's, the second row by 1's, and so on, and the $a$-th row by $a - 1$. If we say we fixed the diagonal cells by $\nu = (\mu_1 + n - 1, \ldots, \mu_{n-1}, \mu_n, a - 1)$, then we have

$$
\sum_{T \in \text{SSYT}((\pi^u)^{n+a}) \atop \text{rdiag}(T) = \nu} q^{|T| + t \cdot \text{tr}(T)} = q^{\binom{n+1}{2}} \sum_{T \in \text{SSYT}((\pi^u)^{n+a}) \atop \text{rdiag}(T) = \nu} q^{|T| + t \cdot \text{tr}(T)}
$$

For the lower-left half, we consider the transpose of $\pi^d$, i.e., $(\pi^d)^t = V(n, c, d, m)$. To satisfy the inequality condition of semistandard Young tableaux, after combined with the upper-right half, the fillings in this part should be row strict tableaux. Let us recall the generating function for the row strict tableaux with fixed diagonal:

$$(17) \sum_{T \in \text{RST}((\delta_n)^c) \atop \text{rdiag}(T) = \mu + \delta_n} q^{|T|} = \frac{\prod_{j=1}^{n} (q; q)^{a_j+n-j}}{\prod_{j=1}^{n} (q; q)^{\rho_j+n-j}} \sigma_\rho + \delta_n(q^{\mu+\delta_n}).
$$

To utilize the generating function for the row strict tableaux with fixed diagonal, we fill the skewed part $\delta_{n+c+1}$ by $-b$ in the first column, $(-b+1)$'s in the second, and so on, and by $(-1)$'s in the last column. Then, in our setting, $\rho$ part in (17) is $(d^{n+c}) + (m-1)\delta_{n+c}$. We can compute

$$
\begin{align*}
    b(\rho) &= (n+c)\left(\frac{d}{2}\right) + (m-1)^2\left(\frac{n+c+1}{3}\right) + d(m-1)\left(\frac{n+c}{2}\right) - \left(\frac{m}{2}\right)\left(\frac{n+c}{2}\right), \\
    b(\rho) &= d\left(\frac{n+c}{2}\right) + (m-1)\left(\frac{n+c}{3}\right), \\
    (c+n)|\rho| &= (n+c)\left(d(n+c) + (m-1)\left(\frac{n'+c'}{2}\right)\right).
\end{align*}
$$

Let $p_{\rho}(c, d, n, m) := n(\rho') - n(\rho) + (n+c)|\rho| + \binom{n+c+1}{3}$. Then we have

$$
\sum_{T \in \text{SSYT}((\pi^d)^{n+c}) \atop \text{rdiag}(T) = \mu + \delta_n} q^{|T| + t \cdot \text{tr}(T)} = q^{\binom{n+c}{2} + t \binom{n+c}{2}} \sum_{T \in \text{SSYT}((\pi^d)^{n+c}) \atop \text{rdiag}(T) = \mu + \delta_n} q^{|T| + t \cdot \text{tr}(T)}
$$

$$
= \frac{\prod_{j=1}^{n+c} (q; q)^{a_j+n-c-j}}{\prod_{j=1}^{n+c} (q; q)^{\rho_j+n-c-j}} \sigma_\rho + \delta_n(q^{\mu+\delta_n}).
$$

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Hence the trace generating function for the semistandard Young tableaux of shape $\pi$ would be

$$\sum_{T \in \text{SSYT}(\pi)} q^{T|\hbar - \text{tr}(T)}$$

$$= \sum_{\mu \in \text{Par}_n} q^{-(t+1)|u + \delta_n|} \sum_{T \in \text{SSYT}(\pi^u)} q^{T|\hbar - \text{tr}(T)} \sum_{T \in \text{SSYT}(\pi^d)} q^{T|\hbar - \text{tr}(T)}$$

$$= \frac{1 - q^n}{1 - q} \prod_{j=1}^{n+a-1} (q; q)_{b+m(n+a-j)} \prod_{j=1}^{n+c} (q; q)_{d+m(n+c-j)}$$

$$\times \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{\ell} \cdot \bar{\pi}(b^{n+c}) \cdot (x_1^a, \ldots, x_n^a, q^{c-1}, \ldots, q^{0})$$

Note that we have already computed

$$\bar{\pi}(b^{n+c}) \cdot (x_1^a, \ldots, x_n^a, q^{c-1}, \ldots, q^{0})$$

$$= q^{m(b+n)m} \prod_{i=1}^{n} (q^{m(i-1)} \cdot x_i^{m})_a \cdot \Delta(x_1^m, \ldots, x_n^m),$$

and similarly, we can compute

$$\bar{\pi}(d^{n+c}) \cdot (x_1^a, \ldots, x_n^a, q^{c-1}, \ldots, q^{0})$$

$$= q^{-d\left(\frac{c-1}{2}\right) - 2m\left(\frac{c+1}{2}\right) - m\left(\frac{c+1}{2}\right)} \prod_{i=1}^{c-1} (q^{m(i-1)} \cdot x_i^{m})_a \cdot \Delta(x_1^m, \ldots, x_n^m).$$

Combining all these gives

$$\sum_{T \in \text{SSYT}(\pi)} q^{T|\hbar - \text{tr}(T)}$$

$$= q^{\frac{b+mn}{3} - \left(d+mn\right)\left(\frac{c+1}{2}\right) + \left(m-1\right)\left(\frac{c+1}{2}\right) + \left(1-2m\right)\left(\frac{c+1}{2}\right) + p_n(c,d,n,m)}$$

$$\times \frac{\Phi_{q^n}(d) \Phi_{q^n}(d)}{1 - q^n} \prod_{j=0}^{n+c-1} (q; q)_{d+mj}$$

$$\times \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^{n} x_i^{\ell + b + d} \cdot (q^{m(i-1)} \cdot x_i^{m})_a \cdot \Delta(x_1^m, \ldots, x_n^m)^2 d_q x_1 \cdots d_q x_n.$$
\[ q^{\alpha - (\frac{n}{2})} \prod_{i=1}^{n} \frac{\Gamma_{q}(\alpha - 1 + i) \Gamma_{q}(\beta - 1 + i) \Gamma_{q}(i) \cdot q^{(\alpha - 1 + i)} \cdot b^{\alpha + 2i - 2}}{\Gamma_{q}(\alpha - 1 + \beta - 1 + n + i)} = q^{(\alpha - 1)\left(\frac{n}{2}\right) + \frac{1}{n}(n-1)(2n-1)} \cdot b^{n \alpha + n(n-1)} (1 - q)^{n} \prod_{i=1}^{n} \frac{(q; q)_{\beta - i + 1}}{(q^{\alpha - 1 + i}; q)_{\alpha + \beta - 1}}. \]

If we let
\[ \alpha = t + b + d + 1 - m, \]
\[ \beta = a + c + 1, \]
\[ b = p^a = q^{a \cdot n} \]
to evaluate (*), then we obtain
\[ \int_{0 \leq x_1 \leq \ldots \leq x_n} \prod_{i=1}^{n} x_i^{t+b+d+1-m} (p^{-a} x_i; p)_{a+c} \cdot \Delta(x_1, \ldots, x_n)^2 d_p x_1 \cdots d_p x_n \]
\[ = q^{(t+b+d+1)(a + c + n) + 2m((\alpha) + a(n))} (1 - q^m)^{n_m} \Phi_{q^m}(a + c + n) \Phi_{q^m}(a + c + n) \Phi_{q^m}(a + c + n) \Phi_{q^m}(a + c + n) \Phi_{q^m}(a + c + n) \]
Applying the above result of integration gives
\[ \sum_{T \in SSYT(\pi)} q^{|T| + t - tr(T)} \]
\[ = q^{(**) t \left( a \cdot n + \left( \frac{n}{2} \right) \right) + (b + mn) \left( \frac{a}{2} \right) - (d + mn) \left( \frac{c + 1}{2} \right) + (m - 1) \left( \frac{a}{3} \right) + (1 - 2m) \left( \frac{c + 1}{3} \right) + (b + d + 1) \left( a \cdot n + \left( \frac{n}{2} \right) \right) + 2m \left( \left( \frac{n}{3} \right) + a \left( \frac{n}{2} \right) \right) + p_c(d, c, d, n, m).} \]

Note that
\[ (**) = t \left( a \cdot n + \left( \frac{n}{2} \right) \right) = \sum_{(i,j) \in \lambda / (c)^a} (\lambda'_j - i), \]
where \( \lambda = ((n + b + c)^{n+a}) + ((m - 1) \delta_{a+n} \cup \theta') \), \( \theta = (d^{n+c}) + (m - 1) \delta_{a+c} \). Also note that
\[ \Phi_{q^m}(a) \Phi_{q^m}(c) \Phi_{q^m}(n) \Phi_{q^m}(n + a + c) \]
\[ = \prod_{i=1}^{n} \prod_{j=1}^{c} \frac{1 - q^{m(i+j+k-1)}}{1 - q^{m(i+j+k-2)}} \prod_{(i,j) \in \lambda / (c)^a} \frac{1}{1 - q^d \chi(i,j) + h(i,j)}, \]
where
\[ \chi(i,j) = \begin{cases} 1, & \text{if } (i,j) \in ((n + c)^{n+a}), \\ 0, & \text{otherwise}. \end{cases} \]
Lastly, we replace \( q^t \) by \( x \).

\[ \square \]

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