The subobject decomposition in enveloping tensor categories

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To the memory of Tonny A. Springer

Abstract. To every regular category $\mathcal{A}$ equipped with a degree function $\delta$ one can attach a pseudo-abelian tensor category $\mathcal{T}(\mathcal{A}, \delta)$. We show that the generating objects of $\mathcal{T}$ decompose canonically as a direct sum. In this paper we calculate morphisms, compositions of morphisms and tensor products of the summands. As a special case we recover the original construction of Deligne’s category $\text{Rep} S_t$.

1. Introduction

Deligne constructed in [Del07] the non-Tannakian tensor category $\text{Rep} S_t$ which can be interpreted as the category of representations of the symmetric group on $t$ letters where $t$ does not have to be a natural number. His construction is based on the fact that the space of morphisms between certain objects of $\text{Rep} S_n$, $n \in \mathbb{N}$, stabilize when $n$ goes to infinity.

In [Kno07], another construction of $\text{Rep} S_t$ was given which is not based on this stabilization property. Instead, $\text{Rep} S_t$ was obtained as a twisted version of the category of relations attached to $\mathcal{A} = \text{Set}^{\text{op}}$, where Set denoted the category of finite sets. This construction has the advantage that it readily generalizes to much more general setting. In [Kno07] a pseudo-abelian category tensor category $\mathcal{T}(\mathcal{A}, \delta)$ has been constructed from any regular category $\mathcal{A}$ which is equipped with a degree function $\delta$. Moreover, precise conditions on $\mathcal{A}$ and $\delta$ were established for $\mathcal{T}(\mathcal{A}, \delta)$ to be a semisimple (hence abelian) category. Taking for example for $\mathcal{A}$ the category of finite dimensional $\mathbb{F}_q$-vector spaces, this leads to a category $\text{Rep} \text{Gl}(V)$ where $V$ is an $\mathbb{F}_q$-vector space with any number $t \in \mathbb{C}$ elements.

Both constructions, Deligne’s and the authors, have in common that the tensor category is built up from generating objects and a description of the morphisms between them. The set of generating objects is not the same, though. Even though both sets are parameterized by natural numbers, Deligne’s objects have much smaller morphism spaces between them.

In this paper we elucidate how these two sets of generating objects are related to each other. More generally, we define the analogues of Deligne’s generators in the more general setting of an arbitrary regular category $\mathcal{A}$ and show how $\mathcal{T}(\mathcal{A}, \delta)$ can be constructed from them.

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More specifically, each object \( x \) of \( \mathcal{A} \) yields a generating object \([x]\) of \( \mathcal{T}(\mathcal{A}, \delta) \). The morphisms are linear combinations of all relations. It turns out that \([x]\) contains a specific direct summand \([x]^*\) and that \([x]\) is the direct sum of all \([y]^*\) where \( y \) runs through all subobjects of \( x \).

We show that these \([x]^*\) are precisely Deligne’s generators. For this, we exhibit bases for the morphism spaces \( \text{Hom}_T([x]^*, [y]^*) \) and calculate how composition is expressed in terms of these bases. Moreover, we decompose the tensor product \([x]^* \otimes [y]^*\) and calculate the tensor product of morphisms. The formulas obtained turn out to specialize exactly to formulas used by Deligne to define \( \text{Rep} \mathcal{S}_t \). This shows conclusively that our category \( \mathcal{T}(\text{Set}^{op}, \delta) \) is equivalent to \( \text{Rep} \mathcal{S}_t \). More precisely, we show:

1.1. Theorem. Let \( \mathcal{A} \) be a subobject finite, regular category, let \( \delta \) be a degree function on \( \mathcal{A} \) and \( \mathcal{T} := \mathcal{T}(\mathcal{A}, \delta) \). For all objects \( x \) and \( y \) of \( \mathcal{A} \) let \( R(x, y) \) be the set of subobjects of \( x \times y \) such that both projections \( r \to x \) and \( r \to y \) are surjective. Then for every object \( x \) of \( \mathcal{A} \) there is a direct summand \([x]^*\) of \([x]\) such that

\[
\bigoplus_{y \subseteq x} [y]^* \rightarrow [x] \tag{1.1}
\]

for all \( x \). Moreover, the objects \([x]^*\) have the following properties:

(a) The objects \([x]^*\) are natural with respect to isomorphism and generate \( \mathcal{T} \) as a pseudo-abelian category.

(b) For all \( x \) and \( y \), the morphism space \( \text{Hom}_T([x]^*, [y]^*) \) has two natural (with respect to isomorphisms) bases \( (r) \) and \( \{r\} \) where \( r \) runs through \( R(x, y) \).

(c) The composition of morphisms is computed with formulas (5.5) and (5.10) below for the \( (r) \)- and the \( \{r\} \)-basis, respectively.

(d) Each tensor product \([x]^* \otimes [y]^*\) is naturally a direct sum of the objects \([r]^*\) where \( r \) runs through \( R(x, y) \). The unital, associativity, and symmetry constraints are induced by those of the direct product \( x \times y \).

(e) The tensor product of morphisms is computed with formulas (5.26) and (5.32) below for the \( (r) \)- and the \( \{r\} \)-basis, respectively.

In particular, \( \mathcal{T} \) is uniquely determined by the properties (a) through (e) (with either of the morphisms \( (r) \) or \( \{r\} \)).

Summarizing: In the construction of \( \mathcal{T}(\mathcal{A}, \delta) \) with the \([x]\) as basic objects, a basis of \( \text{Hom}_T([x], [y]) \) is parameterized by the set of all subobjects of \( x \times y \). The formulas for composition and tensor product of morphisms are very simple. In contrast, a basis of the morphism space \( \text{Hom}_T([x]^*, [y]^*) \) for the \([x]^*\) is parameterized by subobjects of \( x \times y \) which map surjectively onto each factor. There are much fewer of them. On the other hand the formulas for composition and tensor product are much more involved.

Actually, the comparison of \([\text{Del07}]\) and \([\text{Kno07}]\) is a bit more complicated. For \( \text{Rep} \mathcal{S}_t \) the morphism space between \([x]\) and \([y]\) is spanned by all partitions of the disjoint union \( X \cup Y \). For \([x]^*\) and \([y]^*\) only those partitions are considered where both \( X \) and \( Y \) meet every part in at most one element. It follows that a partition of this kind corresponds to a *gluing* of \( X \) and \( Y \), i.e., subsets \( X_0 \subseteq X \) and \( Y_0 \subseteq Y \) together with a bijective map \( X_0 \rightarrow Y_0 \). This alternate description of the morphisms by way of gluings works for any so-called Mal’tsev category. Now Deligne gives just a multiplication formula for the \((r)\)-morphisms in terms of gluings which we generalize in the last section. Formula (5.10) for the \( \{r\} \)-morphisms seems to be new.
1.2. Remark. For the benefit of the reader we indicate in detail how the construction in [Del07] is a special case of ours. The base category $\mathcal{A}$ is chosen to be the opposite category $\text{Set}^{op}$ of the category of finite sets. Then we need first of all that $\mathcal{A}$ is regular, exact, subobject finite, and, in the last section, Mal’cev.

That $\text{Set}^{op}$ has these properties is most easily seen by observing that $\text{Set}^{op}$ is equivalent to the category $\text{Bool}$ of finite Boolean algebras. In fact, the functor $C \mapsto \text{Maps}(C, \mathbb{F}_2)$ is an equivalence. Now $\text{Bool}$ is the set of finite models of an algebraic theory (unital rings with all elements idempotent). Thus it is regular and exact by [Bor94, Thm. 3.5.4]. It is clearly subobject finite. The Mal’cev property follows from the fact that the algebraic theory contains a group operation (namely addition, see [BB04, Thm. 2.2.2, Ex. 2.2.5]).

The degree function on $\mathcal{A}$ is $\delta(C) = t^{\lvert C \rvert}$ where $t$ is a free variable and $\lvert C \rvert$ is the order of the finite set $C$. An epimorphism $e : x \rightarrow x'$ in $\mathcal{A}$ corresponds to an injective map $j : A' \hookrightarrow A$ of finite sets. Then our $\omega_e$ equals $P_A$ in the notation of [Del07, (2.10.2)]. This follows from [Kno07, Lemma 8.4 and 8.7 Example 1].

The objects $[x]^*$ correspond to the objects $[U]$ in [Del07, 2.12]. Our objects $[x]$ do not occur in [Del07] while, on the other side, Deligne’s objects $\{\lambda\}$ will be considered in a forthcoming paper. Our morphisms $(r)$ and $\{r\}$ correspond to $(C)$ and $\{C\}$ in [Del07, 2.12], respectively.

Finally the precise correspondence between the various Lemmas, Propositions and Theorems is (with [Del07] on top):

| \(\{r\}\) | \((s)(r)\) | \(\{s\}\{r\}\) | \(\{v\}^{\prime}\{u\}^{\prime}\) | \([x]^* \otimes [y]^*\) | assoc./com. | \((r) \otimes (r')\) | \(\{r\} \otimes \{r'\}\) |
|---|---|---|---|---|---|---|---|
| 2.2 | 2.10 | 2.11 | 2.4 | 2.7 | 2.8 |
| (5.2) | (5.5) | (5.10) | (6.5) | (5.16) | (5.20)/(5.21) | (5.26) | (5.32) |

Note that a more elaborate exposition of our construction in the case $\mathcal{A} = \text{Set}^{op}$ is contained in the paper [CO11] by Comes and Ostrik.

2. The category $\mathcal{T}(\mathcal{A}, \delta)$

The monoidal category $\mathcal{T}(\mathcal{A}, \delta)$ is build from two ingredients, a category $\mathcal{A}$ and a degree function $\delta$.

The category $\mathcal{A}$ has to be rich enough such that the usual relational calculus makes sense. This means roughly that it has images, pull-backs, and that images commute with pull-backs.

Let’s be more precise. For any fixed object $x$ of $\mathcal{A}$ the class of monomorphisms $m : y \hookrightarrow x$ carries a transitive relation by stipulating $m \leq m'$ if $m$ factors through $m'$. Two monomorphisms $m, m'$ are equivalent if both $m \leq m'$ and $m \geq m'$. Equivalence classes of monomorphisms are called subobjects of $x$. The collection of all subobjects will be denoted by $s(x)$. It is now partially ordered.

The image $\text{im}(f)$ of a morphism $f : x \rightarrow y$ is the minimal subobject $m$ of $y$ through which $f$ factors (if one exists). The image is also denoted by $f(x)$. If $f(x) = y$ then $f$ is by definition an extremal epimorphism (denoted $f : x \rightarrow y$). More generally, if the image of $f : x \rightarrow y$ is $m : u \hookrightarrow y$ then $f = me$ where $e : x \rightarrow u$ is an extremal epimorphism. Moreover, this epi-mono-factorization is unique up to unique isomorphism.
2.1. Remark. The monomorphism/extremal epimorphism language is a bit too clumsy for our needs. Henceforth, these morphisms will be called injective and surjective, respectively. This is justified by the fact that in most examples injective/surjective morphisms are just injective/surjective maps. This applies e.g. to the categories of sets, groups or vector spaces with one notable exception namely the category attached to Deligne’s original category $\text{Rep}(S_t)$. There $\mathcal{A} = \text{Set}^{\text{op}}$ is the opposite category of the category of finite sets. Then injective morphisms correspond to surjective maps and vice versa. Similarly, we write $u \subseteq v$ instead of $u \leq v$ for subobjects $u, v \in s(x)$ and $u \cap v$ for the fiber product (intersection) $u \times_x v$.

Now we formulate our main requirements on $\mathcal{A}$.

2.2. Definition. A category $\mathcal{A}$ will be called regular if the following conditions hold:

- **R0** For every object the collection of its subobjects is a set.
- **R1** Every morphism has an image.
- **R2** There is a terminal object (denoted by 1).
- **R3** For every commutative diagram

\[
\begin{array}{ccc}
  u & \to & y \\
  \downarrow & & \downarrow \\
  x & \to & z
\end{array}
\]

the pullback $x \times_z y$ exists.

- **R4** The pull-back of a surjective morphism by an arbitrary morphism exists and is surjective.

2.3. Remark. (1) Condition **R4** can be rephrased as: Let $f : x \to z$ and $g : y \to z$ be morphisms. Then

\[
(2.2) \quad f(x) \times_z y = f(x \times_z y)
\]

in the sense that one side exists if and only if the other does and in that case they are canonically isomorphic.

(2) The possibility for a pull-back not to exist is mainly included to accommodate the category of affine spaces over a field, since considering the empty set as an affine space creates more problems than it solves. For example the dimension formula does not hold for empty intersections. Nevertheless, one can form a new category $\mathcal{A}^{\emptyset}$ by adjoining an initial object $\emptyset$ to $\mathcal{A}$ which is then finitely complete. More precisely, all finite limits which do not exist in $\mathcal{A}$ exist in $\mathcal{A}^{\emptyset}$ and are equal to $\emptyset$.

(3) The axioms of regular categories are usually formulated in terms of regular epimorphisms. It is not difficult to show that all surjective morphisms are regular (even effective). Thus the definitions are equivalent.

As already mentioned, the definition of a regular category is tailored to allow a calculus of relations. Recall that a relation between two objects $x$ and $y$ is a subobject $r$ of $x \times y$ or, equivalently a jointly injective pair of morphisms $r \to x, y$. If $s$ is another relation between $y$ and $z$ then their product $r \circ s$ is the image of $p_x \times p_z : r \times_y s \to x \times z$ with the stipulation that $r \circ s = \emptyset$ if $r \times_y s$ does not exist.
Axiom R3 is now instrumental to show that the product of relations is associative. This way one can define a new category Rel\(\mathcal{A}\) with the same objects as \(\mathcal{A}\) and with relations as morphisms. The identity morphism in Rel\(\mathcal{A}\) is given by the diagonal \(\Delta \subseteq x \times x\).

Given a relation \(r\) between \(x\) and \(y\) then the adjoint relation \(r^\vee \subseteq y \times x\) is obtained by switching factors. This way, Rel\(\mathcal{A}\) becomes a symmetric monoidal category with the tensor product being the Cartesian product \(x \times y\) and the terminal object \(1\) being the unital object. It is even rigid with every object being self-dual and

\[
(2.3) \quad \xymatrix{ 1 \ar[r]^-\Delta & x \times x \ar[l]_\Delta & 1}
\]

being the evaluation and coevaluation morphism, respectively.

Our ultimate goal is to produce tensor categories, i.e. rigid symmetric monoidal categories which are also abelian. It is only the last property which is lacking for Rel\(\mathcal{A}\). A first step in the right direction is to make it additive. This is very easy by considering linear combinations of relations as morphisms. But it turns out that the ensuing category is quite degenerate. It turns out that a twist of the construction makes things much better.

2.4. Definition. Let \(\mathcal{A}\) be a regular category and \(\mathbb{K}\) any base ring. A degree function on \(\mathcal{A}\) is a map which assigns every surjective morphism \(e : x \rightarrow y\) an element \(\delta(e) \in \mathbb{K}\) with the following properties:

- \(\textbf{D1}\) \(\delta(\text{id}_x) = 1\) for all \(x\).
- \(\textbf{D2}\) \(\delta(\overline{e}) = \delta(e)\) whenever \(\overline{e}\) is a pull-back of \(e\).
- \(\textbf{D3}\) \(\delta(e \overline{e}) = \delta(e) \delta(\overline{e})\) whenever \(e\) can be composed with \(\overline{e}\).

2.5. Remark. (1) If \(\mathcal{A}\) has no initial element we extend the degree function to \(\mathcal{A}^\circ\) by defining \(\delta(\emptyset \rightarrow \emptyset) := 1\). But beware: This does not define a degree function on \(\mathcal{A}^\circ\), since \(\textbf{D2}\) is violated for pull-backs along \(\emptyset \rightarrow x\).

(2) It is sometimes convenient to extend \(\delta\) to all morphisms \(f : x \rightarrow y\) by defining \(\delta(f) := \delta(x \rightarrow \text{im } f)\). Then \(\textbf{D2}\) holds unconditionally for all morphisms while \(\textbf{D3}\) is valid whenever \(e\) is injective or \(\overline{e}\) is surjective.

(3) If \(e\) and \(\overline{e}\) are isomorphic in an obvious sense then \(\delta(\overline{e}) = \delta(e)\). This is a consequence of \(\textbf{D2}\). In particular \(\delta(e) = 1\) for all isomorphisms \(e\).

(4) Every regular category \(\mathcal{A}\) has at least two degree function. First, the one with \(\delta(e) = 1\) for all \(e\). Secondly, the one with \(\delta(e) = 0\) for all non-isomorphisms \(e\).

(5) For most \(\mathcal{A}\) there aren’t any other degree functions. This holds for example for the category \(\text{Set}\) of finite sets.

(6) As a positive example let \(\mathcal{A}\) be an abelian category such that every object \(x\) has finite length \(\ell(x)\). Let \(t \in \mathbb{K}\) be arbitrary. Then \(\delta(e) := t^{\ell(\ker e)}\) is a non-trivial degree function.

(6) Another example is the category \(\text{Set}^{\text{op}}\). Then \(e : x \rightarrow y\) is represented by an injective map \(Y \rightarrow X\) and \(\delta(e) = t^{|X-Y|}\) is the degree function which gives rise to Deligne’s category \(\text{Rep} S_t\).

Now we can define an intermediate category \(\mathcal{T}^0(\mathcal{A}, \delta)\).
2.6. Definition. Let $\mathcal{A}$ be a regular category, $\mathbb{K}$ a field, and $\delta$ a $\mathbb{K}$-valued degree function on $\mathcal{A}$. Then the category $\mathcal{T}^0 = \mathcal{T}^0(\mathcal{A}, \delta)$ is defined as follows:

(a) The objects of $\mathcal{T}^0$ are those of $\mathcal{A}$. If an object $x$ of $\mathcal{A}$ is regarded as an object of $\mathcal{T}^0$ then we will denote it by $[x]$.
(b) A morphism from $[x]$ to $[y]$ is a formal $\mathbb{K}$-linear combination of relations between $x$ and $y$. The morphism corresponding to a relation $r$ is denoted by $\langle r \rangle$. We set $\langle \varnothing \rangle := 0$.
(c) The composition of $\mathcal{T}^0$-morphisms is defined as follows: let $r \mapsto x \times y$ and $s \mapsto y \times z$ be relations. Then

$$\langle s \rangle \langle r \rangle := \begin{cases} \delta(r \times_y s \Rightarrow s \circ r) \langle s \circ r \rangle, & \text{if } r \times_y s \text{ exists;} \\ 0, & \text{otherwise.} \end{cases}$$

(d) The tensor product of $[x]$ and $[y]$ is $[x] \otimes [y] = [x \times y]$. Let $\langle r \rangle : [x] \rightarrow [x']$ and $\langle s \rangle : [y] \rightarrow [y']$ be two morphisms given by relations $r$ and $s$. Then $\langle r \rangle \otimes \langle s \rangle : [x] \otimes [y] \rightarrow [x'] \otimes [y']$ is given by the relation $r \times s \Rightarrow (x \times x') \times (y \times y') \Rightarrow (x \times y) \times (x' \times y')$.

The unit object is $1 := [1]$ (with 1 being the terminal object of $\mathcal{A}$).

Finally, we generate also direct sums and direct summands:

2.7. Definition. The category $\mathcal{T}(\mathcal{A}, \delta)$ is the pseudo-abelian closure of $\mathcal{T}^0(\mathcal{A}, \delta)$, i.e., its objects are of the form $pX$ where $X = \bigoplus_{\mu} \langle x_{\mu} \rangle \otimes \langle y_{\nu} \rangle$ is a formal finite direct sum of objects of $\mathcal{T}^0$ with

$$\text{Hom}_{\mathcal{T}}(X, Y) := \bigoplus_{\mu, \nu} \text{Hom}_{\mathcal{T}^0}([x_{\mu}], [y_{\nu}])$$

and $p \in \text{End}_{\mathcal{T}}(X)$ is idempotent with morphisms

$$\text{Hom}_{\mathcal{T}}(pX, qY) := q \text{Hom}_{\mathcal{T}}(X, Y)p.$$ 

In [Kno07] it was shown that the category $\mathcal{T}(\mathcal{A}, \delta)$ inherits from $\mathcal{A}$ the structure of a rigid symmetric monoidal category with tensor product $[x] \otimes [y] = [x \times y]$ and dual $(p[x])^\vee = p^\vee[x]$. In particular the associativity of the product of morphisms is ensured by the axioms of a degree function. Additionally, $\mathcal{T}(\mathcal{A}, \delta)$ is $\mathbb{K}$-linear and additive.

3. An alternate construction of $\mathcal{T}(\mathcal{A}, \delta)$

Let $\mathcal{A}$ be a regular category and $\delta$ a $\mathbb{K}$-valued degree function on $\mathcal{A}$. In this section we give a description of $\mathcal{T}(\mathcal{A}, \delta)$ or rather $\mathcal{T}^0(\mathcal{A}, \delta)$ in terms of generators and relations while bypassing the category of relations.

Every morphism $f : x \rightarrow y$ gives rise to two morphisms in $\mathcal{T}^0$ namely

$$[f] := \langle \Gamma_f \rangle : [x] \rightarrow [y] \text{ and } [f]^\vee = \langle \Gamma_f^\vee \rangle : [y] \rightarrow [x]$$

where $\Gamma_f \mapsto x \times y$ is the graph of $f$. Moreover for every relation

$$\xymatrix{ x \ar@{..>}[r]^r & y \ar@{..>}[l]^b}$$
we have
\[(3.3) \quad \langle r \rangle = [r \to y][r \to x]'.\]
So the objects \([x]\) and the morphisms \([f], [f]'\) generate \(\mathcal{T}(A, \delta)\) as a pseudo-abelian category. It is easy to verify that they satisfy the following relations:

**Rel1** \([f][g] = [fg]\) and \([g][f]' = [fg]'\)
for all composable morphisms \([f], [g]\),

**Rel2** \([a'][b'] = [b][a]'\)
for each Cartesian diagram \[
\begin{array}{c}
\begin{array}{ccc}
& x & \\
\searrow & & \searrow \\
& y & \\
\end{array}
\end{array}
\]

**Rel3** \([f][f]' = \delta(f) \text{id}_{[y]}\)
for each surjective morphism \([f] : x \to y\).

Conversely:

3.1. **Theorem.** Let \(\mathbb{K}\) be a ring, let \(A\) be a regular category, and let \(\delta\) be a \(\mathbb{K}\)-valued degree function on \(A\). Then \(\mathcal{T}(A, \delta)\) is the free pseudo-abelian category which is generated by the objects \([x]\) with \(x\) an object of \(A\) and morphisms \([f] : [x] \to [y], [f]' : [y] \to [x]\) for each morphism \([f] : x \to y\) which are subject to the relations **Rel1, Rel2 and Rel3**.

**Proof.** Let \(\tilde{T}\) be the free category defined by the generators and relations above and let \(F : \tilde{T} \to \mathcal{T}(A, \delta)\) be the obvious functor. It suffices to show that \(F\) is an isomorphism on \(\text{Hom}_{\tilde{T}}([x], [y])\) for all objects \(x, y\) of \(A\). Let \(F(x, y)\) be this map. Then \(F(x, y)\) is surjective because of (3.3). Thus, it suffices to show that \(\text{Hom}_{\tilde{T}}([x], [y])\) is linearly generated by all morphisms of the form (3.3).

For this, we first show that
\[(3.4) \quad \text{id}_x = [\text{id}_x]' = \text{id}_x'.\]
It follows from **Rel1** that both \([\text{id}_x]\) and \([\text{id}_x]'\) are idempotents. Because of \(\delta(\text{id}_x) = 1\) and **Rel3** we have \([\text{id}_x][\text{id}_x]' = \text{id}_x\). Hence \([\text{id}_x] = [\text{id}_x][\text{id}_x][\text{id}_x]' = [\text{id}_x][\text{id}_x]' = \text{id}_x'.\]

The space of morphisms is spanned by products \(\varphi = \varphi_1 \ldots \varphi_n\) where each \(\varphi_i\) is either of type \([f]\) or \([f]'\). Because of **Rel1** we may assume that no two adjacent morphisms are both of type \([f]\) or both of type \([f]'\). Because of **Rel2** we may also assume that there is no morphism of type \([f]'\) followed by a morphism of type \([f]\). Thus, we have the possibilities \(\varphi = \text{id}_x\) (case \(n = 0\)), \(\varphi = [f], \varphi = [f]',\) or \(\varphi = [g][f]'\). Because of (3.4), we may assume that there are two morphisms \([f] : u \to x, [g] : u \to y\) such that \(\varphi = [g][f]'\). Let \(r\) be the image of \(f \times g : u \to x \times y\) and \(h : u \to r\) the corresponding epimorphism. Consider the morphisms \(\bar{f} : r \to x\) and \(\bar{g} : r \to y\). Then \([g][f]' = [\bar{g}h][\bar{f}h]' = [\bar{g}][h][\bar{f}]' = \delta(h)[\bar{g}][\bar{f}]'.\]

4. **The subobject decomposition**

Let \(A\) be a subobject finite regular category and \(\delta\) a \(\mathbb{K}\)-valued degree function on \(A\).

Of particular interest are the morphisms \([i] : [y] \to [x]\) where \(i : y \to x\) is injective. In this case \(y \times_x y = y\) and therefore
\[(4.1) \quad [i]'[i] = \text{id}_{[y]}'.\]
It follows that \([\iota]\) is a split monomorphism and \([\iota]^\vee\) a split epimorphism. Let \(u\) be the image of \(\iota\), i.e., the subobject represented by \(\iota\). Then (4.1) implies that \(p_u := [\iota][\iota]^\vee \in \text{End}_{\mathcal{T}}([x])\) is an idempotent for which one easily checks that it depends only on \(u\). Thus, every \(u \in \mathcal{s}(x)\) gives rise to a direct summand \(p_u[x]\) of \([x]\) which is via \(\iota\) isomorphic to \([y]\).

Clearly \(p_u = \langle r \rangle\) where \(r\) is the relation

\[
\begin{array}{ccc}
\iota & \downarrow & \iota \\
 y & \downarrow & x \\
 x & \downarrow & x
\end{array}
\]

This implies that the collection of all idempotents \(p_u, u \in \mathcal{s}(x)\), is linearly independent. One also easily checks that

\[
(4.3) \quad p_u p_v = p_w \quad \text{for all } u, v \in \mathcal{s}(x).
\]

In particular, the idempotents \(p_u\) commute with each other. Thus, they can be expressed by primitive idempotents \(p^*_u\).

More precisely, let \(\mathbb{K}[\mathcal{s}(x)]\) be the Möbius algebra of \(\mathcal{s}(x)\), i.e., the \(K\)-vector space with basis \(\mathcal{s}(x)\) and product induced by intersection. Then the formula

\[
(4.4) \quad p_v = \sum_{u \subseteq y} p^*_u \quad \text{for all } v \in \mathcal{s}(x).
\]

defines recursively a new basis \((p^*_y)_{y \in \mathcal{s}(x)}\) satisfying

\[
(4.5) \quad p_u^* p_v^* = \delta_{u,v} p_u^* \quad \text{and} \quad p_u^* p_v = \begin{cases} 
 p_u^* & \text{if } u \subseteq v \\
 0 & \text{otherwise}
\end{cases}.
\]

(see e.g. [Sta12, Thm. 3.9.1]). Conversely, one has

\[
(4.6) \quad p_v^* := \sum_{u \subseteq v} \mu(u, v) p_u
\]

where \(\mu(u, v) \in \mathbb{Z}\) is the Möbius function of the lattice \(\mathcal{s}(x)\). Plugging in (4.4) into (4.6) and vice versa one obtains the relations

\[
(4.7) \quad \sum_{u \subseteq v \subseteq w} \mu(v, w) = \delta_{u,w} = \sum_{u \subseteq v \subseteq w} \mu(u, v).
\]

Thus each subobject \(u \subseteq x\) gives rise to a direct summand \(p^*_u[x]\) of \([x]\). We abbreviate \([x]^* := p^*_u[x]\). Then \(p^*_u[x] = p^*_u p_u[x] \cong p^*_u[u] = [u]^*\). Thus we get the subobject decomposition of \([x]\)

\[
(4.8) \quad [x] = \sum_{u \subseteq x} p^*_u[x] = \sum_{u \subseteq x} [u]^*.
\]

Observe that because the \(p_u\) are linearly independent, each summand \([u]^*\) is non-zero.

Next, we describe the functorial properties of the subobject decomposition.

\textbf{4.1. Lemma.} Let \(f : x \to y\) be a morphism and \(z \subseteq y\). Then

\[
(4.9) \quad p_z[f] = [f] p_{f^{-1}(z)}.
\]
Proof. Let \( \iota : z \rightarrow y \) be the inclusion. Then the two diagrams

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
\tau 
\ar[r] 
& x 
\ar[r]_f 
& y 
\ar[r] 
& \tau \ar[l]_\iota \\
\ar[u]_{f^{-1}(z)} \end{array}
&
\begin{array}{c}
\xymatrix{
\tau 
\ar[r] 
& x 
\ar[r]_f 
& y 
\ar[r] 
& \tau \ar[l]_\iota \\
\ar[u]_{f^{-1}(z)} \end{array}
\end{array}
\end{array}
\]

represent the left and right hand side of (4.9). We conclude with \( \overline{f} = f\tau \).

From this we derive

4.2. Lemma. Let \( f : x \rightarrow y \) be a morphism and \( u \subseteq x, z \subseteq y \) subobjects. Then

\[
(4.11) \quad p_z^*[f]p_u^* = \begin{cases} [f]p_u^* & \text{if } f(u) = z \\ 0 & \text{otherwise.} \end{cases}
\]

and

\[
(4.12) \quad p_z^*[f] = \sum_{u \subseteq x \atop f(u) = z} [f]p_u^* \quad \text{and} \quad [f]^\vee p_z^* = \sum_{u \subseteq x \atop f(u) = z} p_u^*[f]^\vee.
\]

Proof. First, observe that the right hand formulas are just the adjoints of the ones on the left hand side.

For \( u \subseteq x \) multiply both sides of (4.9) by \( p_u^* \). Then

\[
(4.13) \quad p_z^*[f]p_u^* = \begin{cases} [f]p_u^* & \text{if } f(u) \subseteq z \\ 0 & \text{otherwise.} \end{cases}
\]

From this we get

\[
(4.14) \quad p_z^*[f]p_u^* = \sum_{u \subseteq x \atop f(u) = z} \mu(v, z)p_v^*[f]p_u^* = \sum_{u \subseteq x \atop f(u) \subseteq v \subseteq z} \mu(v, z)[f]p_u^* = \begin{cases} [f]p_u^* & \text{if } f(u) = z, \\ 0 & \text{otherwise.} \end{cases}
\]

Then (4.12) follows:

\[
(4.15) \quad p_z^*[f] = \sum_{u \subseteq x} p_z^*[f]p_u^* = \sum_{u \subseteq x \atop f(u) = z} [f]p_u^*.
\]

4.3. Corollary. Let \( f : x \rightarrow y \) be a morphism and \( u \subseteq x, v \subseteq y \) subobjects. Then \( [f] \) maps \( [u]^* \) into \( [f(u)]^* \) and \( [f]^\vee \) maps \( [v]^* \) into \( \bigoplus_{u \subseteq x \atop f(u) = v} [u]^* \).

Proof. By (4.11) we have \((1 - p_{f(u)}^*)[f]p_u^* = 0\) which means that \([u]^*\) is mapped into \([f(u)]^*\).

For the next formula, we need a certain numerical invariant attached to a surjective morphism. Since \( \mathcal{A} \) is subobject finite the set \( s(x) \) is a finite poset (even a lattice, possibly without a minimum). Let \( \mu(y, z) \) be its Möbius function and let \( e : x \rightarrow y \) be surjective. Then we define

\[
(4.16) \quad \omega_e := \sum_{u \subseteq x \atop e(u) = y} \mu(u, x)\delta(u \rightarrow y) \in \mathbb{K}.
\]
4.4. Lemma. Let $e : x \to y$ be a surjective morphism. Then
\[(4.17) \quad [e] p^*_x [e]^\vee = \omega_e p^*_y.\]

Proof. Let $\iota : u \to x$ be a subobject. Then \((4.12)\) implies that $p^*_y[e]p_u = 0$ unless $e(u) = y$. Using $p_u = [\iota][\iota]^\vee$ we get
\[
[e] p^*_x [e]^\vee \overset{(4.11)}{=} p^*_y [e] p^*_x [e]^\vee \overset{(4.6)}{=} \sum_{u \subseteq x} \mu(u, x) p^*_y [e] p_u [e]^\vee = \\
\overset{(4.18)}{=} \sum_{u \subseteq x \atop e(u) = y} \mu(u, x) p^*_y [e] [e]^\vee \overset{\text{Relation}}{=} \sum_{u \subseteq x \atop e(u) = y} \mu(u, x) \delta(u \to y) p^*_y \overset{(4.16)}{=} \omega_e p^*_y. \quad \square
\]

5. The subobject construction of $\mathcal{T}(\mathcal{A}, \delta)$

Let $\mathcal{A}$ be a subobject finite regular category and $\delta$ a $\mathbb{K}$-valued degree function. In this section we show how to build up $\mathcal{T}$ from its generators $[x]^*$ thereby proving Theorem 1.1.

We start with the description of morphisms:

5.1. Definition. Let $R(x, y) \subseteq s(x \times y)$ denote the set of relations $r \subseteq x \times y$ such that both projections $a : r \to x$ and $b : r \to y$ are surjective. For every $r \in R(x, y)$ one can define two morphisms $[x]^* \to [y]^*$ namely
\[
\begin{align*}
(r) & := p^*_y [b] p^*_x [a]^\vee p^*_x \overset{(4.11)}{=} [b] p^*_x [a]^\vee \\
\{r\} & := p^*_x (r) p^*_x = p^*_y [b] [a]^\vee p^*_x
\end{align*}
\]

These two sets of morphisms are related as follows:

5.2. Lemma.
\[(5.2) \quad \{r\} = \sum_{s \in R(x, y) \atop x \subseteq r} (s) \quad \text{and} \quad (r) = \sum_{s \in R(x, y) \atop x \subseteq r} \mu(s, r) \{s\}.
\]

Here $\mu$ is the Möbius function of $s(x \times y)$.

Proof. The first formula follows from
\[
\begin{align*}
\{r\} = p^*_y [b] [a]^\vee p^*_x \overset{(4.4)}{=} \sum_{s \subseteq r} p^*_y [b] p^*_s [a]^\vee p^*_x \overset{(4.11)}{=} \sum_{s \subseteq r \atop x \to x, y} p^*_y [b] p^*_s [a]^\vee p^*_x = \sum_{s \subseteq R \atop x \to x, y} (s).
\end{align*}
\]

The second formula now follows from Möbius inversion. \quad \square

Now we have:

5.3. Proposition. Each of the two sets of morphisms $(r)$ and $\{r\}$ with $r \in R(x, y)$ forms a basis of $\text{Hom}_{\mathcal{T}}([x]^*, [y]^*)$.

Proof. Because of $(5.2)$ it suffices to prove the assertion for $\{r\}$. Let $\overline{R} := s(x \times y)$ the set of all relations, $\overline{R} := R(x, y)$, and $R' := \overline{R} \setminus R$ the complement. By definition we have
\[
(5.4) \quad \text{Hom}([x]^*, [y]^*) = p^*_x \text{Hom}([x], [y]) p^*_x \subseteq \text{Hom}([x], [y]) = \mathbb{K}[[\overline{R}]].
\]

It follows from $(4.12)$ that $p^*_y (R') p^*_x = 0$. Moreover, one checks easily that $p^*_x (r) p^*_x \in \langle r \rangle + \mathbb{K}[[\langle R' \rangle]]$ for all $r \in R$. So $\langle R \rangle \to \langle \overline{R} \rangle \to \langle R' \rangle \overset{\sim}{\to} p^*_x (\overline{R}) p^*_x$ implies the assertion. \quad \square
Next we calculate the composition of basis elements. We do that for the \((r)\)-basis first.

### 5.4. Lemma
Let \(r \in R(x, y)\) and \(s \in R(y, z)\). Then

\[
(s)(r) = \sum_{u \in R(r,s)} \omega_{u \to y} (\pi).
\]

Here, \(\pi\) is the image of \(u\) in \(x \times z\).

**Proof.** With \(t := x \times y s\) we get the diagram

\[
\begin{array}{ccc}
 a'' & \rightarrow & b'' \\
 \downarrow & & \downarrow \\
 a & \rightarrow & b \\
 \downarrow & & \downarrow \\
 x & \rightarrow & y \\
 \end{array}
\]

Now we calculate

\[
(s)(r) = [b'] p^*_s [a'] \uparrow [b] p^*_r [a] \overset{\text{Re}12}{=} [b'] p^*_s [a''] \uparrow [a'] \uparrow [b'] p^*_r [a] \overset{\text{Re}12}{=} \sum_{u \in R(r,s)} [b'][b'] p^*_u [a''][a] \uparrow (5.8)
\]

\[
= \sum_{u \in R(r,s)} \omega_f[\bar{b}] p^*_u [f] \uparrow [a] (4.17) = \sum_{u \in R(r,s)} \omega_f[\bar{b}] p^*_u (\pi) = \sum_{u \in R(r,s)} \omega_f (\pi)
\]

with \(\pi\) the image of \(u\) in \(x \times z\):

\[
\begin{array}{ccc}
 a'' & \rightarrow & b'' \\
 \downarrow & & \downarrow \\
 a & \rightarrow & b \\
 \downarrow & & \downarrow \\
 x & \rightarrow & y \\
 \end{array}
\]

The same for \(\{r\}\):

### 5.5. Lemma
Let \(r \in R(x, y)\) and \(s \in R(y, z)\). For a subobject \(y' \subseteq y\) put, by abuse of notation,

\[
r \times_{y'} s := r \times_y y' \times_y s \quad \text{and} \quad r \circ_{y'} s := \text{im}(r \times_{y'} s \to x \times z).
\]

Then

\[
\{s\} \{r\} = \sum_{y' \subseteq y \atop r \times_{y'} y \to x \atop y' \times_{y'} y \to z} \mu(y', y) \delta(r \times_{y'} s \to r \circ_{y'} s)\{r \circ_{y'} s\}.
\]

**Proof.** By definition

\[
\{s\} \{r\} = p^*_z[b'][a'] \uparrow p^*_y [b][a] \uparrow p^*_x = \sum_{y' \subseteq y \atop 11} \mu(y', y) p^*_z[b'][a'] \uparrow p^*_y [b][a] \uparrow p^*_x
\]

\[
\square
\]
Taking preimages of \( y' \) yields the diagram

\[
\begin{array}{ccc}
\rarr{x} & \rarr{r \times_y y'} & \rarr{r \times_y y' } \\
\rarr{f} & \rarr{z} & \rarr{y' \times_y y' } \\
\rarr{\alpha} & \rarr{\beta} & \rarr{y' \times_y y' }
\end{array}
\]

(5.12)

Thus

\[
p_{x}^{*} [b'] [a']^{\vee} p_{y'} [b] [a]^{\vee} p_{z}^{*} = p_{x}^{*} [b] [f] [f]^{\vee} [a]^{\vee} p_{z}^{*} = \delta(f) p_{x}^{*} [b] [a]^{\vee} p_{z}^{*}.
\]

(5.13)

The last expression equals \( \delta(f) \{ r \circ_y s \} \) if both \( \overline{\alpha} \) and \( \overline{\beta} \) are surjective and is zero otherwise. \( \square \)

5.6. Remark. Using equations (5.2) and (5.5) one can express \( \{ s \} \{ r \} \) also in terms of the \( (r) \)-basis resulting in

\[
\{ s \} \{ r \} = \sum_{u \subseteq r \times y, s \subseteq u \times z} \omega_{u \rightarrow \overline{\pi}} (\overline{\pi})
\]

(5.14)

where \( \overline{\pi} \) denotes the image of \( u \) in \( x \times z \). For exact Mal’tsev categories we will show the more stringent formula (6.5) below.

Proposition 5.3 and Lemma 5.4 or Lemma 5.5 describe \( \mathcal{T} (\mathcal{A}, \delta) \) as an additive category. Now we turn to its monoidal structure.

5.7. Lemma. Let \( x \) and \( y \) be objects of \( \mathcal{A} \). Then

\[
p_{x}^{*} \otimes p_{y}^{*} = \sum_{r \in R(x, y)} p_{r}^{*} \in \text{End}([x \times y])
\]

(5.15)

In particular, there is a canonical isomorphism

\[
[x]^{\ast} \otimes [y]^{\ast} \cong \bigoplus_{r \in R(x, y)} [r]^{\ast}
\]

(5.16)

\[\text{Proof.} \ \text{The idempotent} \ P := p_{x}^{*} \otimes p_{y}^{*} \text{is contained in the algebra spanned by the projections} \ p_{u \times v} \ \text{with} \ u \subseteq x \ \text{and} \ v \subseteq y. \ \text{So it must be a sum} \ P = \sum_{r \in R_0} p_{r}^{*} \ \text{of minimal idempotents where} \ R_0 \ \text{is a certain subset of} \ s(x \times y). \ \text{Let} \ u \subseteq x \ \text{and} \ v \subseteq y \ \text{with} \ u \neq x \ \text{or} \ v \neq y. \ \text{Then} \ p_{u} \otimes p_{v} = p_{u \times v} \ \text{implies}
\]

\[
0 = p_{u}^{\ast} p_{x}^{\ast} \otimes p_{v} p_{y}^{\ast} = \sum_{r \in R_0} p_{u \times y}^{\ast} p_{r}^{\ast} = \sum_{r \in R_0} p_{r}^{\ast}.
\]

(5.17)

So the right hand sum is empty which means that \( R_0 \subseteq R(x, y) \). Because \( [x]^{\ast} \) is self-dual and because of \( \dim_{K} \text{Hom}_{\mathcal{T}} (1, [x]^{\ast}) = 1 \) we have

\[
|R(x, y)| = \dim_{K} \text{Hom}([x]^{\ast}, [y]^{\ast}) = \dim_{K} \text{Hom}(1, [x]^{\ast} \otimes [y]^{\ast}) = \\
\sum_{r \in R_0} \dim_{K} \text{Hom}_{\mathcal{T}} (1, [r]^{\ast}) = |R_0|.
\]

(5.18)

Thus \( R_0 = R(x, y) \). \( \square \)
5.8. Remark. Formula (5.16) generalizes readily to tensor products with more than two factors. More precisely, let \( x_1, \ldots, x_n \) be objects of \( \mathcal{A} \). Then

\[
(5.19) \quad [x_1]^* \otimes \cdots \otimes [x_n]^* \cong \bigoplus_r [r]^*
\]

where \( r \) runs through all subobjects of \( x_1 \times \cdots \times x_n \) such that the projections \( r \to x_i \) are surjective for \( i = 1, \ldots, n \). In particular, the associativity morphism \((x \times y) \times z \cong x \times (y \times z)\) yields the associativity constraint

\[
(5.20) \quad ([x]^* \otimes [y]^*) \otimes [z]^* \cong [x]^* \otimes ([y]^* \otimes [z]^*).
\]

Likewise, the canonical isomorphism \( x \times y \cong y \times x \) yields the commutativity constraint

\[
(5.21) \quad [x]^* \otimes [y]^* \cong [y]^* \otimes [x]^*.
\]

Next we investigate the tensor product as a functor. For this we need to determine for any morphisms \( \varphi : [x]^* \to [y]^* \) and \( \psi : [x']^* \to [y']^* \) the matrix coefficients of

\[
(5.22) \quad \bigoplus_{u \in R(x,x')} [u]^* = [x]^* \otimes [x']^* \overset{\varphi \otimes \psi}{\longrightarrow} [y]^* \otimes [y']^* = \bigoplus_{v \in R(y,y')} [v]^*.
\]

Clearly, it suffices to do this for a basis and we start with the \((r)-basis\).

5.9. Lemma. Let

\[
(5.23) \quad \begin{array}{ccc}
    & r & \\
\xymatrix{a & x & y \ar[rr]^-b & & x' & y'} \end{array}
\quad \text{and} \quad \begin{array}{ccc}
    & r' & \\
\xymatrix{a' & x & y' \ar[rr]^-b' & & x' & y'} \end{array}
\]

be elements of \( R(x,y) \) and \( R(x', y') \), respectively. For each \( w \in R(r, r') \) define

\[
(5.24) \quad r_w := (a \times a')(w) \subseteq x \times x', \quad r'_w := (b \times b')(w) \subseteq y \times y'
\]

and the morphism

\[
(5.25) \quad \tau_w : [x]^* \otimes [x']^* \to [r_w]^* \overset{(w)}{\longrightarrow} [r'_w]^* \cong [y]^* \otimes [y']^*
\]

where \((w)\) is induced by the inclusion \( w \hookrightarrow r_w \times r'_w \). Then

\[
(5.26) \quad (r) \otimes (r') = \sum_{w \in R(r, r')} \tau_w.
\]

Proof. The morphism \( w \to r_w \times r'_w \) is in fact injective since \( w \hookrightarrow r \times r' \hookrightarrow x \times y \times x' \times y' \) and therefore also \( w \to r_w \times r'_w \to x \times x' \times y \times y' \) is injective. Below is a diagram of the setup. All four quadrilaterals commute.

\[
(5.27)
\]
Let \( \overline{a} := a \times a' \) and \( \overline{b} := b \times b' \). Then
\[
(r) \otimes (r') = [b] p_r^* [a] \vee [b'] p_{r'}^* [a'] \vee = [b \times b'] (p_r^* \otimes p_{r'}^*) [a \times a'] \vee \quad (5.15)
\]
(5.28)
\[
= \sum_{u \in R(r,r')} [b] p_{ru}^* [\overline{a}] \vee = \sum_{u \in R(r,r')} p_{ru}^* [\overline{b}] p_{ru}^* [\overline{a}] \vee p_{ru}^* = \sum_{u \in R(r,r')} \tau_u. \quad \square
\]

Now the same for the \( \{r\} \)-basis.

5.10. Lemma. Let
\[
(5.29)
\]
be elements of \( R(x,y) \) and \( R(x',y') \), respectively. For each \( u \in R(x,x') \) and \( v \in R(y,y') \) define
\[
(5.30) \quad w_{u,v} := u \times_{x \times x'} (r \times r') \times_{y \times y'} v
\]
(see diagram (5.33) below) and the morphism
\[
(5.31) \quad \sigma_{u,v} : [x]^* \otimes [x']^* \to [u]^* \{w_{u,v}\}^* \to [v]^* \lor [y]^* \otimes [y']^*
\]
Then
\[
(5.32) \quad \{r\} \otimes \{r'\} = \sum_{u \in R(x,x')} \sum_{v \in R(y,y')} \sigma_{u,v}.
\]

Proof. First we claim that \( w_{u,v} \to u \times v \) is injective. For this, let \( \overline{w} \) be the image of \( w_{u,v} \) in \( u \times v \). Then \( w_{u,v} \to x \times x' \times y \times y' \) would factor through \( \overline{w} \). Then the same holds for \( w_{u,v} \to r \times r' \) since \( r \times r' \to x \times y \times x' \times y' \) is injective. So also the injective morphism \( w_{u,v} \to u \times r \times r' \times v \) factors through \( \overline{w} \) which shows the claim \( w_{u,v} = \overline{w} \).

With \( \overline{a} := a \times a' \) and \( \overline{b} := b \times b' \) we get
\[
(5.34) \quad \{r\} \otimes \{r'\} = (p_{y}^* \otimes p_{y}^*) (p_{x}^* \otimes p_{x}^*) \quad (5.15)
\]
\[
= \sum_{u \in R(x,x')} \sum_{v \in R(y,y')} p_{v}^* [\overline{b}] p_{u}^* [\overline{a}] \vee p_{u}^* = \sum_{u \in R(x,x')} \sigma_{u,v}. \quad \square
\]

6. A formula for exact Mal’tsev categories

As explained in the introduction, formula (5.10) for the product \( \{s\} \{r\} \) does not appear in [Del07]. Instead another identity is proven ([Del07, 2.11]) which we consider now in our framework. For that we have to assume that \( \mathcal{A} \) is exact and Mal’tsev.
6.1. Definition. A regular category \( A \) is Mal’tsev if for every object \( x \) any subobject \( r \subseteq x \times x \) containing the diagonal is an equivalence relation. The category is exact if for every equivalence relation \( r \subseteq x \times x \) there is a surjective morphism \( x \twoheadrightarrow y \) with \( r = x \times_y x \).

Examples.

The quotient object \( y \) of \( x \) is uniquely determined by \( r \). Therefore, in an exact Mal’tsev category there is a duality between subobjects of \( x \times x \) containing the diagonal and quotient objects of \( x \). This can be generalized to relations between different objects \( x \) and \( y \). For this we define \( R_q(x, y) \) to be the set of isomorphism classes of diagrams

\[
\begin{array}{cccc}
& a & \rightrightarrows & b \\
\downarrow & & & \downarrow \\
& y & \rightrightarrows & y
\end{array}
\]

It is clear that for every \( u \in R_q(x, y) \) the fiber product \( x \times_u y \) is in \( R(x, y) \).

6.2. Lemma ([CKP93, Thm. 5.5]). Let \( x \) and \( y \) be objects of an exact Mal’tsev category \( A \). Then the map

\[
(6.2) \quad R_q(x, y) \rightarrow R(x, y) : u \mapsto r = x \times_u y
\]

is bijective, the inverse being the push-out \( r \mapsto u = x \sqcup_r y \).

Now for every \( u \in R_q(x, y) \) we define

\[
(6.3) \quad \{u\}' := p^*_{\beta}[\alpha][a]p^*_{x} \mathrel{\text{Rel}_2} \{x \times_u y\}.
\]

There is an order relation on \( R_q(x, y) \) by defining \( u \leq v \) when \( x, y \twoheadrightarrow v \) factors through \( x, y \twoheadrightarrow u \). This way, \( x, y \twoheadrightarrow 1 \) becomes the maximal element of \( R_q(x, y) \). Moreover, (6.2) is order preserving. In particular, for any \( u \in R_q(x, y) \) the interval \( [u, 1] \) can be identified with the set \( q(u) \) of quotient objects and therefore forms a lattice. Therefore it carries a Möbius function.

and (5.2) becomes

\[
(6.4) \quad \{u\}' = \sum_{t \in R_q(x, y)} (x \times_t y) \quad \text{and} \quad (r) = \sum_{u \in R_q(x, y)} \mu_q(u, x \sqcup_r y) \{u\}'.
\]

Then the following multiplication formula generalizes [Del07, 2.11]:

6.3. Lemma. Let \( A \) be an exact Mal’tsev category. Let \( u \in R_q(x, y) \), \( v \in R_q(y, z) \). Let \( \overline{y} := \text{im}(y \twoheadrightarrow u \times v) \) and \( w := u \sqcup_y v \) (such that \( u, v, \overline{y}, w \) forms a pull-back diagram, see (6.6) below). Then

\[
(6.5) \quad \{v\}' \{u\}' = \omega_{\overline{y} \rightarrow y} \sum_{t \in R_q(u, w)} \mu_q(t, w) \{t\}'.
\]
Proof. Below we write \( v \{t\}'_w \) or \( z \{t\}'_x \) to indicate whether we consider \( \{t\}' \) as a morphism \( [u]^* \to [v]^* \) or \( [x]^* \to [z]^* \). Then we have:

\[
\{v\}'\{u\}' = p_x^*[d]^* [c] = p_x^*[d]^* \left[ \prod_{j \in f} \prod_{k \in f} [g] \right] [\{a\}] [p_x]^* \quad (6.6)
\]

\[
= \omega_f p_x^*[d]^* \left[ \prod_{j \in f} \prod_{k \in f} [g] \right] [\{a\}] [p_x]^* \quad (6.7)
\]

\[
= \omega_f p_x^*[d]^* \left( \sum_{t \in F(q, u, v)} \mu_q(t, w) \right) \{t\}'_w \quad (6.8)
\]

\[
= \omega_f \sum_{t \in F(q, u, v)} \mu_q(t, w) \quad (6.9)
\]

\[
= \omega_f \sum_{t \in F(q, u, v)} \mu_q(t, w) \quad (6.10)
\]

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