PRINCIPALLY SPECIALIZED CHARACTERS OF $\hat{sl}(m|1)$-MODULES

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ABSTRACT. In this paper, we calculate a series of principally specialized characters of the $\hat{sl}(m|1)$-modules of level 1. In particular, we show that the principally specialized characters of the basic modules $L(\Lambda_0)$ is expressed as an infinite product. In addition, we deduce the specialized character formula of “quasiparticle” type.

0. INTRODUCTION

Character formulas of basic representation of $\hat{sl}(m|n)$ are given in [KW2], based on their explicit construction in terms of bosonic and fermionic fields.

In this paper, we calculate the principally specialized characters of some $\hat{sl}(m|1)$-modules. In §1, we describe that the principally specialized characters of the basic $\hat{sl}(m|1)$-modules $L(\Lambda_0)$ is expressed as an infinite product. In §2, we deduce the specialized character formula of “quasiparticle” type.

We follow notation and terminologies from [KW2] without repeating their explanation.

1. SPECIALIZED CHARACTER FORMULA FOR SOME SERIES OF $\hat{sl}(m|1)$-MODULES

Throughout this paper, we assume that $m \geq 2$ and let $\alpha_0, \ldots, \alpha_m$ denote the set of simple roots for $\hat{sl}(m|1)$ where $\alpha_0$ and $\alpha_m$ are odd and $\alpha_i (i = 1, \ldots, m - 1)$ are even. Provided that all $s_i$ are positive integers, the sequence $s = (s_0, \ldots, s_m)$ defines a homomorphism $F_s : \mathbb{C}[e^{-\alpha_0}, \ldots, e^{-\alpha_m}] \rightarrow \mathbb{C}[q]$ by $F_s(e^{-\alpha_i}) = q^{s_i} (i = 0, \ldots, m)$, called the specialization of type $s$. In this paper, we consider the specialization of type $s = (1, \ldots, 1, 0)$ which makes sense for the characters of integrable representations, and we write simply $\mathcal{F}$ for this specialization $F_s$ when no confusion can arise.

Since the set of simple roots for the even part of $\hat{gl}(m|1)$ is

$$\hat{\Pi}' = \{\alpha'_0 = \alpha_0 + \alpha_m, \alpha_1, \ldots, \alpha_{m-1}\},$$

this specialization is the principal specialization with respect to the even part of $\hat{gl}(m|1)$; namely

$$\mathcal{F}(e^{-\alpha'_0}) = \mathcal{F}(e^{-\alpha_1}) = \cdots = \mathcal{F}(e^{-\alpha_{m-1}}) = q.$$

We recall the Fock space and its charge decomposition:

$$F = \oplus_{s \in \mathbb{Z}} F_s$$

from §3 of [KW2].

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Lemma 1.1. Let $m \geq 2, s \in \mathbb{Z}$. Then we have:

(a) $q^{\frac{sm}{2}} F(e^{-\Lambda_0} ch F_s) + q^{\frac{sm}{2}} F(e^{-\Lambda_0} ch F_{-s}) = \frac{2 \prod_{i=1}^{\infty} (1 + q^i)^2}{\varphi(q^m)^2}$.

(b) $F(e^{-\Lambda_0} ch F_s) = F(e^{-\Lambda_0} ch F_{m-1-s})$.

Here and further $\varphi(q) = \prod_{j=1}^{\infty} (1 - q^j)$.

Proof. In the case $\hat{\mathfrak{gl}}(m|1)$, the formula (3.15) in [KW2] gives the following:
\[
ch F = e^{\Lambda_0} \prod_{k=1}^{\infty} \frac{1 + ze^{-i-(k-\frac{1}{2})\delta} + z^{-1}e^{-i-(k-\frac{1}{2})\delta}}{(1 - ze^{-m+(k-\frac{1}{2})\delta})(1 - z^{-1}e^{-m+(k-\frac{1}{2})\delta})},
\]
where $z$ is the “charge” variable.

Since $\alpha_0 = \delta - \epsilon_1 + \epsilon_{m+1}$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $(i = 1, \ldots, m)$, our principal specialization $F = F_s$ is written in terms of $e^{-\epsilon_i}$ as follows:
\[
F(e^{-\epsilon_i}) = q^{m+i}(i = 1, \ldots, m), \quad F(e^{-\epsilon_{m+1}}) = 1. \tag{1.2}
\]

Thus, we obtain
\[
F(e^{-\Lambda_0} ch F) = \prod_{k=1}^{\infty} \frac{1 + zq^{-m}q^{m(k-\frac{1}{2})} + z^{-1}q^{-m+i}q^{m(k-\frac{1}{2})}}{(1 - zq^{m(k-\frac{1}{2})})(1 - z^{-1}q^{m(k-\frac{1}{2})})}
= \prod_{k=1}^{\infty} \frac{1 + (zq^{-m}q^{k-\frac{1}{2}}) + (z^{-1}q^{-m+i}q^{k-\frac{1}{2}})}{(1 - zq^{m(k-\frac{1}{2})})(1 - z^{-1}q^{m(k-\frac{1}{2})})}
= \prod_{k=1}^{\infty} \frac{1 + (zq^{-1+m}q^{k-\frac{1}{2}}) + (z^{-1}q^{-m+i}q^{k-\frac{1}{2}})}{(1 - z^{-1}q^{m(k-\frac{1}{2})})(1 - (z^{-1}q^{m(k-\frac{1}{2})})}. \tag{1.3}
\]

In order to compute the coefficient of $z^s$, we use the Jacobi triple product identity:
\[
\prod_{n=1}^{\infty} (1 + zq^{n-\frac{1}{2}})(1 + z^{-1}q^{n-\frac{1}{2}}) = \frac{1}{\varphi(q)} \sum_{j \in \mathbb{Z}} z^j q^{\frac{j^2}{2}}, \tag{1.4}
\]
and also the following well-known identity (see (5.26) in [KP] and §5.8 in [KZ]):
\[
\prod_{k=1}^{\infty} (1 + zq^{k-\frac{1}{2}})^{-1} + (1 + z^{-1}q^{k-\frac{1}{2}})^{-1} = \frac{1}{\varphi(q)^2} \sum_{m \in \mathbb{Z}} (-1)^m z^m q^{\frac{m(m+1)}{2}} q^{-m+m+\frac{1}{2}}.
= \varphi(q)^{-2} \left( \sum_{m,k \geq 0} - \sum_{m,k < 0} \right) ((-1)^{m+k} z^m q^{\frac{m(m+1)}{2}} q^{-m+m+\frac{1}{2}k}). \tag{1.5}
\]

Replacing $z$ by $zq^{-m}$ in (1.4), $z$ by $-z^{-1}$ and $q$ by $q^m$ in (1.5), we rewrite the right side of (1.3) by
\[
F(e^{-\Lambda_0} ch F) = \frac{1}{\varphi(q)^2} \sum_{k \in \mathbb{Z}} \left( \sum_{a+p \geq 0} - \sum_{a+p < 0} \right) ((-1)^a z^{k-p} q^{\frac{k(k+1)}{2}} q^{-m+m+\frac{1}{2}a}) \quad (a+p \geq 0).
\]
and thus we have
\[ \mathcal{F}(e^{-\Lambda_0 c_h F_s}) = \frac{1}{\varphi(q)\varphi(q^m)^2}(\sum_{a,p\geq 0} - \sum_{a,p<0})(-1)^a q^{\frac{1}{2}(p+s)(p+s+1) - \frac{sm}{2} + \frac{m}{2}a(a+1)+map}. \] (1.6)

For convenience, we introduce the following functions:
\[ f_s(a,p) = (-1)^a q^{\frac{1}{2}(p+s)(p+s+1) - \frac{sm}{2} + \frac{m}{2}a(a+1)+map}, \]
\[ h_s = (\sum_{a,p\geq 0} - \sum_{a,p<0})f_s(a,p). \] (1.7)

To prove this lemma, it is sufficient to prove the following two equations:
(a’) \[ q^{sm}h_s + q^{-sm}h_{-s} = 2\varphi(q)\prod_{i=1}^\infty(1 + q^i)^2. \]
(b’) \[ h_s = h_{m-1-s}. \]

First we shall prove (a’).
Since \( f_s(-a, -p - 1) = q^{-sm}f_{-s}(a, p) \), we have
\[ \sum_{a,p<0} f_s(a,p) = \sum_{a,p<0} f_s(-a, -p - 1) = q^{-sm}\sum_{a,p<0} f_{-s}(a,p). \]
Hence we obtain
\[ h_s = \sum_{a,p<0} f_s(a,p) + \sum_{p\geq 0} f_s(0,p) - \sum_{a,p<0} f_s(a,p) = q^{-sm}\sum_{a,p<0} f_{-s}(a,p) + \sum_{p\geq 0} f_s(0,p) - \sum_{a,p<0} f_s(a,p). \]

We then have
\[ q^{\frac{1}{2}sm}h_s = q^{-\frac{1}{2}sm}\sum_{a,p<0} f_{-s}(a,p) + q^{\frac{1}{2}sm}\sum_{p\geq 0} f_s(0,p) - q^{\frac{1}{2}sm}\sum_{a,p<0} f_s(a,p). \] (1.8)
A straightforward computation replacing \( s \) by \(-s\) yields
\[ q^{-\frac{1}{2}sm}h_{-s} = q^{\frac{1}{2}sm}\sum_{a,p<0} f_s(a,p) + q^{-\frac{1}{2}sm}\sum_{p\geq 0} f_{-s}(0,p) - q^{-\frac{1}{2}sm}\sum_{a,p<0} f_s(a,p). \] (1.9)

By (1.8) and (1.9), we obtain
\[ q^{\frac{1}{2}sm}h_s + q^{-\frac{1}{2}sm}h_{-s} = 2\sum_{p\geq 0} q^{\frac{1}{2}p(p+1)}. \] (1.10)

It is known (see e.g. §5 in [KW1]) that the right side of (1.10) has the following product expansion:
\[ \sum_{p\geq 0} q^{\frac{1}{2}p(p+1)} = \prod_{k\geq 1} \frac{1 - q^{2k}}{1 - q^{2k-1}} = \varphi(q)\prod_{i=1}^\infty(1 + q^i)^2, \]
and hence (a’) follows.
Next we shall prove (b’).
Since \( f_s(-a - 1, 0) = -f_s(a, 0) \), we have
\[
\sum_{a \geq 0} f_s(a, 0) = \sum_{a < 0} f_s(-a - 1, 0) = -\sum_{a < 0} f_s(a, 0).
\]
Hence we obtain
\[
h_s = \sum_{a \geq 0} f_s(a, p) + \sum_{a < 0} f_s(a, 0) - \sum_{a, p < 0} f_s(a, p)
= \sum_{a \geq 0} f_s(a, p) - \sum_{a < 0} f_s(a, 0) - \sum_{a, p < 0} f_s(a, p)
= (\sum_{a \geq 0} - \sum_{a < 0}) f_s(a, p).
\]
Replacing \( a \) by \(-a - 1\) and \( p \) by \(-p\), we have
\[
h_s = (\sum_{a, p < 0} - \sum_{a, p \geq 0}) f_s(-a - 1, -p)
= (\sum_{a, p < 0} - \sum_{a, p \geq 0}) (-1)^{a+1} q^{-(p+s)(-p+s+1)-\frac{a+1}{2}a+1+m(a+1)p}
= (\sum_{a, p \geq 0} - \sum_{a, p < 0}) (-1)^{a} q^{\frac{1}{2}(p-s)(p-s-1)-\frac{a+1}{2}a+1+m(a+1)p}.
\]
(1.11)
Replacing \( s \) by \( m - 1 - s \) in (1.11) and comparing it with (1.7), we get (b’).

Using Lemma [14] inductively, we have the following.

**Proposition 1.2.** Let \( m \geq 2 \) and \( k \in \mathbb{Z}_+ \). Then we have
\[
\mathcal{F}(e^{-\Lambda_0} chF_{k(1)+m-1}) = \mathcal{F}(e^{-\Lambda_0} chF_{k(m-1)})
= \frac{q^m}{2} \left( \sum_{|j| \leq k} (-1)^{k-j} x^{k^2-j^2} \right) \prod_{i=1}^{\infty} \frac{(1 + q^i)^2}{\varphi(q^m)^2}.
\]

**Proof.** By Lemma [14], we get the recurrence formula
\[
\mathcal{F}(e^{-\Lambda_0} chF_{s+m-1}) = q^m \left( \frac{2 \prod_{i=1}^{\infty} (1 + q^i)^2}{\varphi(q^m)^2} - q^m \mathcal{F}(e^{-\Lambda_0} chF_s) \right).
\]
(1.12)
We now set for \( k \in \mathbb{Z}, s = k(m-1), x = q^{m(m-1)} \) and
\[
H_k = \mathcal{F}(e^{-\Lambda_0} chF_{k(m-1)}) \left( \prod_{i=1}^{\infty} \frac{(1 + q^i)^2}{\varphi(q^m)^2} \right)^{-1}.
\]
For the proof of this proposition, it is sufficient to show that
\[
H_{k+1} = x^k \left( \sum_{|j| \leq k} (-1)^{k-j} x^{k^2-j^2} \right), \quad k \in \mathbb{Z}_+.
\]
(1.13)
We shall show (1.13) by induction on $k$. Using these notation, (1.12) is rewritten as

$$H_{k+1} = x^k(2 - x^k H_k).$$

Assume that (1.13) is true for $k - 1$. Then we obtain

$$H_{k+1} = x^k \left\{ 2 - x^k \cdot x^{k-1} \left( \sum_{|j| \leq k-1} (-1)^{k-1-j} x^{(k-1)^2-j^2} \right) \right\}$$

$$= x^k \left( 2 + \sum_{|j| \leq k-1} (-1)^{k-j} x^{k^2-j^2} \right)$$

$$= x^k \left( \sum_{|j| \leq k} (-1)^{k-j} x^{k^2-j^2} \right),$$

proving the proposition. \qed

We consider principally specialized characters of $\hat{\mathfrak{sl}}(m|1)$-modules $L(\Lambda(s))$ where $\Lambda(s)$ are defined in Remark 3.2 of [KW2], given by

$$\Lambda(s) = \begin{cases} 
\Lambda_s & \text{if } 0 \leq s \leq m \\
-(s-m)\Lambda_0 + (1+s-m)\Lambda_m & \text{if } s \geq m \\
(1-s)\Lambda_0 + s\Lambda_m + s\delta & \text{if } s \leq 0.
\end{cases} \quad (1.14)$$

**Theorem 1.3.** Let $m \geq 2$. Then we have the following:

(a) For $\Lambda = \Lambda_0$ or $\Lambda_{m-1}$, we have

$$\mathcal{F}(e^{-\Lambda}chL(\Lambda)) = \prod_{i=1}^{\infty} \frac{(1 + q^i)^2}{\varphi(q^m)}. \quad (1.15)$$

(b) For $\Lambda = \{k(m-1) + 1\}\Lambda_0 - k(m-1)\Lambda_m$ ($k \in \mathbb{Z}$), we have

$$\mathcal{F}(e^{-\Lambda}chL(\Lambda)) = \left( \sum_{|j| \leq |k|} (-1)^{k-j} q^{(k^2-j^2)\frac{m(m-1)}{2}} \right) \prod_{i=1}^{\infty} \frac{(1 + q^i)^2}{\varphi(q^m)}. \quad (1.16)$$

**Proof.** To prove (a), we let $k = 0$ in Proposition 2. Then, since $chF_s = \varphi(e^{-\delta})^{-1}chL(\Lambda(s))$ and $\Lambda(0) = \Lambda_0$, we get

$$\mathcal{F}(e^{-\Lambda_0}chL(\Lambda_0)) = \prod_{i=1}^{\infty} \frac{(1 + q^i)^2}{\varphi(q^m)}.$$  

Since $\Lambda_{(m-1)} = \Lambda_{m-1} = \Lambda_0 - \alpha_m$, we have

$$\mathcal{F}(e^{-\Lambda_{m-1}}) = \mathcal{F}(e^{-\Lambda_0})\mathcal{F}(e^{\alpha_m}) = \mathcal{F}(e^{-\Lambda_0}),$$

and hence

$$\mathcal{F}(e^{-\Lambda_{m-1}}chL(\Lambda_{m-1})) = \mathcal{F}(e^{-\Lambda_0}chL(\Lambda_{(m-1)}))$$

$$= \mathcal{F}(e^{-\Lambda_0}chF_{m-1})\varphi(q^m)$$

$$= \prod_{i=1}^{\infty} \frac{(1 + q^i)^2}{\varphi(q^m)}.$$
which is (a).

To prove (b), we let \( \Lambda = \{ k(m-1) + 1 \} \Lambda_0 - k(m-1)\Lambda_m \) for \( k \in \mathbb{Z} \). First we consider in the case \( k \in \mathbb{Z}_+ \), since \( \Lambda_m = \Lambda_0 - \frac{1}{m-1} \alpha_1 - \cdots - \frac{m}{m-1} \alpha_m \) and by (1.14), it follows that

\[
\mathcal{F}(e^{-\Lambda(k(m-1))}) = \mathcal{F}(e^{-\Lambda}) q^{-km(m-1)} = \mathcal{F}(e^{-\Lambda_0}) q^{-\frac{km(m-1)}{2}},
\]

and hence we obtain

\[
\mathcal{F}(e^{-\Lambda chL(\Lambda)}) = \mathcal{F}(e^{-\Lambda(k(m-1))} chL(\Lambda(k(m-1))))
\]

\[
= q^{-\frac{km(m-1)}{2}} \mathcal{F}(e^{-\Lambda_0} chL(\Lambda(k(m-1))))
\]

\[
= \left( \sum_{|j| \leq k} (-1)^{k-j} q^{(k^2-j^2)\frac{m(m-1)}{2}} \right) \prod_{i=1}^{\infty} \frac{1 + q^i}{\phi(q^m)}.
\]

In the case \( k < 0 \), the same discussion as above yields

\[
\mathcal{F}(e^{-\Lambda((k+1)(m-1))}) = \mathcal{F}(e^{-\Lambda}) = \mathcal{F}(e^{-\Lambda_0}) q^{-\frac{km(m-1)}{2}},
\]

and hence we obtain

\[
\mathcal{F}(e^{-\Lambda chL(\Lambda)}) = \mathcal{F}(e^{-\Lambda((k+1)(m-1))} chL(\Lambda((-k+1)(m-1))))
\]

\[
= q^{-\frac{km(m-1)}{2}} \mathcal{F}(e^{-\Lambda_0} chL(\Lambda((-k+1)(m-1))))
\]

\[
= \left( \sum_{|j| \leq -k} (-1)^{k-j} q^{(k^2-j^2)\frac{m(m-1)}{2}} \right) \prod_{i=1}^{\infty} \frac{1 + q^i}{\phi(q^m)}
\]

which is (b). \( \square \)

**Remark 1.4.** This is an additional remark to Theorem 1.3.

We consider an asymptotic behavior of principally specialized characters \( \mathcal{F}(e^{-\Lambda_0 chL(\Lambda_0)}) \).

We shall write \( f(\tau) \sim g(\tau) \) if \( \lim_{\tau \downarrow 0} f(\tau)/g(\tau) = 1 \), where \( \tau \downarrow 0 \) means that \( \tau = iT \) \( (T > 0) \) and \( T \to 0 \).

It is known (see e.g. (13.13.5) in [K3]) that the asymptotic behavior of the Dedekind \( \eta \)-function

\[
\eta(\tau) = q^{\frac{1}{24}} \varphi(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = e^{2\pi i \tau}
\]

is given by

\[
\eta(\tau) \sim \tau \downarrow 0 \left( -i \tau \right)^{-\frac{1}{24}} e^{-\frac{\pi i}{12}},
\]

and so

\[
\varphi(q) = q^{-\frac{1}{24}} \eta(\tau) \sim \tau \downarrow 0 \left( -i \tau \right)^{-\frac{1}{24}} e^{-\frac{\pi i}{12}}.
\]
From this assertion, we have
\[ \prod_{n=1}^{\infty} (1 + q^n) = \frac{\varphi(q^2)}{\varphi(q)} \sim \frac{1}{\sqrt{2}} e^{\frac{\pi i}{24\tau}} \]
and hence
\[ F(e^{-\Lambda_0} chL(\Lambda_0)) \sim \frac{1}{\sqrt{2}} e^{\frac{\pi i}{24\tau}} \]
By applying the Tauberian theorem (see [I] and Proposition 4.22 in [KP]), we have
\[ a_n \sim \frac{1}{8\sqrt{3}} (m + 1)^{\frac{1}{2}} e^{\frac{\pi}{3} \frac{m+1}{m}} \]
where \( F(e^{-\Lambda_0} chL(\Lambda_0)) = \sum_{n=0}^{\infty} a_n q^n \).

Next, we consider for principally specialized characters of some series of level 1 integrable modules. By Theorem 1.3, we have for \( \Lambda = \Lambda_{m-1} \) or \( \{ k(m-1) + 1 \} \Lambda_0 - k(m-1)\Lambda_m \ (k \in \mathbb{Z}) \),
\[ F(e^{-\Lambda} chL(\Lambda)) \sim F(e^{-\Lambda_0} chL(\Lambda_0)) \sim \frac{\sqrt{m}}{2} (-i\tau)^{\frac{1}{2}} e^{\frac{2\pi i}{12m}}. \quad (1.17) \]
The formula (4.12) in [KW2] is the asymptotics of the basic specialization (the specialization of type \((1, 0, \ldots, 0)\)) of \( sl(m|1) \)-module \( L(\Lambda(s)) \). The formula which is obtained by replaced \( \tau \) by \( m\tau \) in (4.12) of [KW2] coincides with (1.17).

2. Specialized character formula of “quasiparticle” type

Proposition 2.1. For \( m \geq 2 \) and \( s \in \mathbb{Z} \), we have
\[ F(e^{-\Lambda_0} chF_s) = q^{-\frac{as}{2}} \sum_{a, b, c, d \in \mathbb{Z}_+ \atop a-b-c-d=s} \frac{q^{a(a+1)}q^{b(b-1)}q^{c(m)}}{(q)_a (q)_b (q^m)_c (q^m)_d} . \]

Here and further \( (q)_n = \prod_{i=1}^{n} (1-q^i) \).

Proof. This formula is shown just by the same argument as that in the proof of the formula (3.14) in [KW2], by making use of a basis of \( F_s \):
\[
\begin{align*}
&\left( \psi^{(1)}_{-\left( j_1, \frac{1}{2} \right)} \cdots \psi^{(1)}_{-\left( j_{a-1}, \frac{1}{2} \right)} \right) \cdots \left( \psi^{(m)}_{-\left( j_{m-1}, \frac{1}{2} \right)} \cdots \psi^{(m)}_{-\left( j_{m, a_m - 1}, \frac{1}{2} \right)} \right) \\
&\times \left( \psi^{(1)*}_{-(\frac{j_1}{2})} \cdots \psi^{(1)*}_{-(\frac{j_{a-1}}{2})} \right) \cdots \left( \psi^{(m)*}_{-(\frac{j_{m, a_m - 1}}{2})} \cdots \psi^{(m)*}_{-(\frac{j_{m, a_m - 1}}{2})} \right) \\
&\times \left( \varphi^{(1)}_{-(k_1, \frac{1}{2})} \cdots \varphi^{(1)}_{-(k_{c-1}, \frac{1}{2})} \right) \left( \varphi^{(1)*}_{-(k_1, \frac{1}{2})} \cdots \varphi^{(1)*}_{-(k_{c-1}, \frac{1}{2})} \right) [0] 
\end{align*}
\]
and
\[ a - b + c - d = s \]
where \( a = \sum_i a_i, b = \sum_i b_i \).

Since
\[
\text{weight}(\psi_k^{(i)}) = \epsilon_i + k\delta, \quad \text{weight}(\psi_k^{(i)*}) = -\epsilon_i + k\delta, \\
\text{weight}(\varphi_k^{(1)}) = \epsilon_{m+1} + k\delta, \quad \text{weight}(\varphi_k^{(1)*}) = -\epsilon_{m+1} + k\delta,
\]
and (1.2), we have
\[
\mathcal{F}(e^{\text{weight}(\psi_k^{(i)})}) = q^{i + m(j - \frac{1}{2})}, \quad \mathcal{F}(e^{\text{weight}(\psi_k^{(i)*})}) = q^{-i + m(j + \frac{1}{2})}, \\
\mathcal{F}(e^{\text{weight}(\varphi_k^{(1)})}) = q^{m(j - \frac{1}{2})}, \quad \mathcal{F}(e^{\text{weight}(\varphi_k^{(1)*})}) = q^{m(j + \frac{1}{2})},
\]
for \( i = 1, \ldots, m \) and \( j \in \mathbb{N} \).

Hence we have
\[
\mathcal{F}(e^{-\Lambda_0 ch F_s}) = \sum_{a,b,c,d \in \mathbb{Z}_+} \left( \sum_{0 < i_1 < \cdots < i_a} q^{(i_1 - \frac{m}{2}) + \cdots + (i_a - \frac{m}{2})} \right) \left( \sum_{0 < j_1 < \cdots < j_b} q^{(j_1 + \frac{m}{2} - 1) + \cdots + (j_b + \frac{m}{2} - 1)} \right) \\
\times \left( \sum_{0 < k_1 \leq \cdots \leq k_c} q^{(mk_1 - \frac{m}{2}) + \cdots + (mk_c - \frac{m}{2})} \right) \left( \sum_{0 < l_1 \leq \cdots \leq l_d} q^{(ml_1 - \frac{m}{2}) + \cdots + (ml_d - \frac{m}{2})} \right) \\
= q^{-\frac{sm}{2}} \sum_{a,b,c,d \in \mathbb{Z}_+} \left( \sum_{0 < i_1 < \cdots < i_a} q^{i_1 + \cdots + i_a} \right) \left( q^{-b} \sum_{0 < j_1 < \cdots < j_b} q^{j_1 + \cdots + j_b} \right) \\
\times \left( \sum_{0 < k_1 \leq \cdots \leq k_c} q^{m(k_1 + \cdots + k_c)} \right) \left( q^{-dm} \sum_{0 < l_1 \leq \cdots \leq l_d} q^{m(l_1 + \cdots + l_d)} \right) \\
= q^{-\frac{sm}{2}} \sum_{a,b,c,d \in \mathbb{Z}_+} \frac{q^{\frac{1}{2}a(a+1)}}{(q)_a} \frac{q^{\frac{1}{2}b(b-1)}}{(q)_b} \frac{q^{cm}}{(q^m)_c} \frac{1}{(q^m)_d},
\]
proving the proposition.

By Theorem 1.3 and Proposition 2.1, we also have shown

**Corollary 2.2.** For \( m \geq 2 \), the following formula holds
\[
\prod_{i=1}^{\infty} \left( 1 + q^i \right)^2 \varphi(q^m)^2 = \sum_{a,b,c,d \in \mathbb{Z}_+} \frac{q^{a(a+1)} q^{b(b-1)} q^{cm}}{(q)_a (q)_b (q^m)_c (q^m)_d}. 
\]
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