On the Expanding Phase of a Singular Bounce and Intermediate Inflation: The Modified Gravity Description

V.K. Oikonomou

1Tomsk State Pedagogical University, 634061 Tomsk, Russia
2Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia

We demonstrate that the intermediate inflation scenario, is a singular inflation cosmology, with the singularity at the origin \( t = 0 \) being a pressure and energy density singularity and particularly a Type III singularity. Also, we show that the expanding phase of a singular bounce, can be identical to the intermediate inflation scenario, if the singular bounce has a Type III singularity at the origin. For the intermediate inflation scenario we examine the cosmological implications on the power spectrum in the context of various forms of modified gravity. Particularly we calculate the power spectrum in the context of \( F(R) \), \( F(G) \) Gauss-Bonnet gravity and also for \( F(T) \) gravity and we discuss the viability of each scenario by comparing the resulting spectral index with the latest observational data.

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I. INTRODUCTION

Finite-time singularities [1] are timelike singularities which frequently occur in modified gravity cosmologies [1–6], and there exist various types of singularities of this sort, with the most phenomenologically severe being the Big Rip. The most phenomenologically “soft” type of singularity is the Type IV [7–11], and the terminology “soft” refers to the fact that the Universe can smoothly pass through these singularities and the physical quantities are finite on these singularities. Type IV singularities only cause dynamical instabilities and these may cause exit from inflation in some inflationary models, see for example [10, 11]. Also in the context of bouncing cosmologies, it was demonstrated that it is possible for a Type IV singularity to occur at the bouncing point, so a singular bounce may occur [12, 13]. This singular bounce however is called singular, but the singularity is not of crushing type, and the effects of the singularity is dynamical instabilities, see Ref. [10, 11] for details.

On the other hand, it is possible that various inflationary scenarios may contain some sort of singularity during the inflationary era, or even at the end of the inflationary era [10, 11]. One quite interesting inflationary scenario is the intermediate inflation scenario [14–22], which in most cases it was realized in the context of scalar-tensor cosmology. The aim of this paper is three-fold: Firstly we will simply highlight the fact that the intermediate inflation scenario is a singular inflationary cosmology, since it contains a Type III, or pressure-energy density singularity at the origin \( t = 0 \). Secondly, we will demonstrate that the intermediate inflation scenario can be identical with the expanding phase of a singular bounce, with a Type III singularity occurring at the bouncing point. Thirdly, we will examine whether the Type III singular bounce, or equivalently the intermediate inflation scenario, can produce a nearly scale invariant power spectrum compatible with the observations, in the context of modified gravity. With regards to modified gravity, we shall be interested for the \( F(R) \) gravity, the \( F(G) \) gravity and the \( F(T) \) gravity descriptions. The calculation of the power spectrum is particularly easy, since these where performed for the Type IV singular bounce [12, 13], so the case at hand is a simple generalization.

This paper is organized as follows: In section II we briefly present some essential features of finite-time singularities, and in addition we demonstrate that the intermediate inflation scenario is a Type III singular inflation scenario, according to the classification of finite-time singularities. In addition, in section II, we show that the expanding phase of a Type III singular bounce can be identical with the intermediate inflation scenario, at least functionally. In section III, we investigate when the power spectrum of the intermediate inflation scenario in the context of \( F(R) \), \( F(G) \) and \( F(T) \) gravity, can be compatible with the observational data. Finally, the conclusions follow in the end of the paper.

*Electronic address: v.k.oikonomou1979@gmail.com
II. INTERMEDIATE INFLATION AS TYPE II SINGULAR INFLATION AND THE SINGULAR BOUNCE

Before we start discussing the singularity structure of the intermediate inflation scenario, we shall present some essential information with regards to finite-time singularities. These were firstly classified in Ref. [1], from which we shall use the terminology and notation. According to the classification in [1], there are four types of finite-time singularity, varying from crushing types to softer types, as follows:

- **Type I Singularity (“The so-called Big Rip Singularity”):** This is a crushing type singularity, and therefore all the physical quantities defined on a three dimensional spacelike hypersurface, which is determined by the time instance that the singularity occurs, are singular. Particularly, this is a timelike singularity which if it occurs at the time instance \( t = t_s \), then as \( t \to t_s \), the scale factor \( a(t) \), the total energy density \( \rho_{\text{eff}} \) and the total pressure \( p_{\text{eff}} \), strongly diverge, that is, \( a \to \infty, \rho_{\text{eff}} \to \infty \), and \( |p_{\text{eff}}| \to \infty \).

- **Type II Singularity (“The so-called Sudden Singularity”):** This is a pressure singularity, which firstly appeared in Refs. [23–30], and as it is obvious from the terminology pressure singularity, only the pressure diverges as \( t \to t_s \), that is \( |p_{\text{eff}}| \to \infty \), but both the scale factor and the total energy density are finite as \( t \to t_s \), that is \( a \to a_s, \rho_{\text{eff}} \to \rho_s \).

- **Type III Singularity:** This type of singularity is the second most severe after the Big Rip singularity, and in this case both the pressure and the energy density diverge as \( t \to t_s \), that is \( |p_{\text{eff}}| \to \infty \) and \( \rho_{\text{eff}} \to \infty \), but the scale factor is finite, \( a \to a_s \). We shall call this type of singularity the “energy-pressure” singularity.

- **Type IV Singularity:** This singularity is the most “harmless” singularity from a phenomenological point of view, since all the physical quantities are finite at \( t \to t_s \), that is, \( a \to a_s, \rho_{\text{eff}} \to \rho_s \) and \( |p_{\text{eff}}| \to |p_s| \). However, in this case the higher derivatives \( n \geq 2 \) of the Hubble rate diverge as \( t \to t_s \). These singularities cause dynamical instabilities on the inflationary dynamics [10, 11] and also to cosmological phenomenology at early-times, and these were extensively studied further in [31, 32].

Having the classification for the finite-time singularities, we now demonstrate that the intermediate inflation scenario has a Type III singularity at the origin \( t = 0 \). The intermediate inflation scale factor is [14, 15],

\[
a(t) = e^{At^n},
\]

with \( 0 < n < 1 \) and also \( A > 0 \), and the corresponding Hubble rate is,

\[
H(t) = Ant^{n-1}.
\]

The values of \( n \) determine the singularity structure of the intermediate inflation scenario, and specifically the singularity structure depending on the value of \( n \) is as follows,

- When \( n < 0 \), a Type I singularity (Big-Rip) occurs at \( t = 0 \).
- When \( 0 < n < 1 \), a Type III singularity occurs at \( t = 0 \).
- When \( 1 < n < 2 \), a Type II singularity occurs at \( t = 0 \).
- When \( n > 1 \), a Type IV singularity occurs at \( t = 0 \).

Since for the intermediate inflation scenario, \( n \) must be chosen as \( 0 < n < 1 \), then, from the classification above it is obvious that at \( t = 0 \) the intermediate inflation has a Type III singularity, which is an energy-pressure singularity.

The singular bounce cosmology [12, 13] has the same functional form for the scale factor and for the Hubble rate as in the intermediate inflation scenario, with the difference that the parameter \( n \) is restricted for physical consistency reasons as follows,

\[
n - 1 = \frac{2n + 1}{2m + 1}.
\]

The restricted form for \( n \) in Eq. (3) is necessary since we need the scale factor to be real for \( t < 0 \) and for \( t > 0 \), and also it is required that the Hubble rate is negative for \( t < 0 \) and positive for \( t > 0 \). Hence if \( n \) is chosen as in Eq. (3), the Hubble rate for \( t < 0 \) is negative and for \( t > 0 \) it is positive, which is the normal behavior expected for
In effect, if \( n \) is chosen as in Eq. (3), and also if \( 0 < n < 1 \), the bounce occurs, however it has a Type III singularity at the origin. Effectively, it differs from other bouncing cosmologies, since it has a singularity at the bouncing point \( t = 0 \), however our aim was to show that the expanding phase of this Type III singular bounce and of the intermediate inflation scenario are identical, at least functionally. It is conceivable however, that the primordial perturbations in the two scenarios will have different origin.

### III. INTERMEDIATE INFLATION FROM MODIFIED GRAVITY AND COMPARISON WITH OBSERVATIONS

The functional similarity of the intermediate inflation scenario with the singular bounce will enable us to provide analytic results on the power spectrum of primordial curvature perturbations, and the corresponding spectral index, by using the related literature. For example, the \( F(R) \) gravity description of the singular bounce was performed in Ref. [12], and the \( F(G) \) description of the singular bounce was performed in [13], so in these sections we share the results of these works and we shall apply these for the case of the intermediate inflation scenario.

We start off with the \( F(R) \) gravity description, in vacuum, with the \( F(R) \) gravity action being,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R)
\]

and as it was shown in Ref. [12], the \( F(R) \) gravity which realizes the cosmology (11) during the early time era, has the following form,

\[
F(R) = R + a_2 R^2 + a_1,
\]

with the parameters \( a_1 \) and \( a_2 \) being positive and can be found in the Appendix of Ref. [12]. Following [12], the corresponding power spectrum \( \mathcal{P}_R \) at early times has the following form,

\[
\mathcal{P}_R \sim k^{3 + n - 4\mu - 2\mu(n-1)}/n-1.
\]

where the parameter \( \mu \) is \( \mu = \frac{1}{2}(2n-1) \). Obviously, the power spectrum is not scale invariant, but now we investigate whether the resulting spectral index can be compatible with the 2015 Planck data [33], for some value in the range \( 0 < n < 1 \), which corresponds to the intermediate inflation scenario. The spectral index of the power spectrum \( \mathcal{P}_R \), is defined as follows,

\[
n_s - 1 = \frac{d \ln \mathcal{P}_R}{d \ln k},
\]

and we easily find that, the spectral index of the power spectrum of Eq. (12), is equal to,

\[
n_s = 4 + \frac{n - 4\mu - 2\mu(n-1)}{n-1}.
\]

The 2015 Planck data [33], constraint the spectral index as follows,

\[
n_s = 0.9644 \pm 0.0049,
\]

and it can be shown that the spectral index of Eq. (5) cannot be compatible with the constraint (9), for any value of \( n \) in the range \( 0 < n < 1 \).

Now we turn our focus on the modified Gauss-Bonnet case, in which case, the vacuum \( F(G) \) gravity action has the following form,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (R + F(G)),
\]

with \( \kappa^2 = 1/M_{pl}^2 \), and also \( M_{pl} = 1.22 \times 10^{19} \text{GeV} \). As it was shown in Ref. [12], the \( F(G) \) gravity which realizes the singular bounce and in effect the intermediate inflation for \( 0 < n < 1 \), has the following form at early times,

\[
F(G) \sim C_2 G + B G^{\frac{4}{3(n-1)}}.
\]

\footnote{Note that for the number \((-1)^{1/3}\) has actually two complex branches and only one real negative, and we choose the negative branch.}
The resulting spectrum was found to be as follows,

$$P_R \sim k^{\frac{(2-2(n-1)+(n-1)^2)}{2n}},$$

(12)

with $\mu = -11/n$. Accordingly, the resulting spectral index is equal to,

$$n_s - 1 \equiv \frac{d\ln P_R}{d\ln k} \simeq 1 - \frac{11}{2n^2},$$

(13)

Then, it can be easily shown that the compatibility with Planck data \[33\] comes when $12.9 < n < 13.7$, and therefore the intermediate inflation inflation scenario in the context of modified Gauss-Bonnet gravity, is not compatible with the observational data.

The picture is entirely different when the intermediate inflation scenario is realized in the context of $F(T)$ gravity, see Ref. \[34\] for a review. We shall briefly present the calculation of the primordial curvature perturbations, for the intermediate inflation scenario in the context of $F(T)$ gravity. The details of this calculation will be presented elsewhere \[35\]. Studies on the evolution of primordial perturbations in $F(T)$ gravity were performed in Refs. \[36–41\], and we follow \[36\].

The perturbed metric in the longitudinal gauge is,

$$ds^2 = (1 + 2\Phi)dt^2 - a(t)^2(1 - 2\Psi)\sum_i dx_i^2,$$

(14)

so the scalar functions $\Phi$ and $\Psi$ practically quantify the scalar fluctuations of the metric. We can express the torsion scalar as a function of $\Phi$ and $\Psi$ in the following way,

$$\delta T = H(\dot{\Phi} + H\Phi),$$

(15)

with $H$ being the Hubble rate. By assuming an $F(T)$ gravity of the form $F(T) = T + f(T)$, the perturbation equations in $F(T)$ gravity are \[36\],

$$(1 + f_T)\sum_i \dot{x}_i^2 \Psi - 3(1 + f_T)H\dot{\Psi} - 3(1 + f_T)H^2\Phi$$

$$+ 36f_{TT}H^3(\dot{\Psi} + H\Phi) = 4\pi G\delta \rho,$$

$$(1 + f_T - 12H^2f_{TT})(\dot{\Psi} + H\Phi) = 4\pi G\delta q,$$

$$(1 + f_T)(\Psi - \Phi) = 8\pi G\delta s,$$

$$+ 4\pi G\delta p,$$

(16)

where $f_T$, stands for $\partial_T f(T)$, and the derivatives $f_{TT}$ and $f_{TTT}$ are defined accordingly. Also the functions $\delta \rho$, $\delta p$, $\delta s$, $\delta q$, stand for the fluctuations of the total pressure, of the total energy density, of the anisotropic stress and of the fluid velocity respectively. If the matter fluids present are represented by a canonical scalar field with potential $V(\phi)$, we obtain the following equations,

$$\dot{\delta \rho} = \dot{\phi}(\delta \dot{\phi} - \dot{\phi}\delta \phi) + V_{,\phi}\delta \phi,$$

$$\dot{\delta q} = \dot{\phi}\delta \dot{\phi},$$

$$\dot{\delta s} = 0,$$

$$\dot{\delta p} = \dot{\phi}(\delta \dot{\phi} - \dot{\phi}\delta \phi) - V_{,\phi}\delta \phi,$$

(17)

As was shown in \[36\], due to the above relations we get $\Psi = \Phi$, and therefore, the scalar fluctuation $\delta \phi$ determines the gravitational potential $\Phi$, and thus we have a single degree of freedom. In effect, the evolution of scalar perturbations is determined by the following differential equation \[36\],

$$\ddot{\Phi}_k + \alpha \dot{\Phi}_k + \mu^2 \Phi_k + c_s^2 \frac{k^2}{a^2} \Phi_k = 0,$$

(18)
where $\Phi_k$ is the scalar Fourier mode of the potential $\Phi$, and moreover, the functions $\alpha$, $c_s^2$ and $\mu^2$ are the frictional term, the speed of sound parameter and the effective mass respectively, corresponding to the scalar potential $\Phi$. These functions are defined as follows,

$$
\alpha = 7H + \frac{2V_\phi}{\phi} - \frac{36H\dot{H}(f_{,TT} - 4H^2f_{,TTT})}{1 + f_{,T} - 12H^2f_{,TT}} ,
$$

$$
\mu^2 = 6H^2 + 2\dot{H} + \frac{2HV_\phi}{\phi} - \frac{36H^2\dot{H}(f_{,TT} - 4H^2f_{,TTT})}{1 + f_{,T} - 12H^2f_{,TT}} ,
$$

$$
c_s^2 = \frac{1 + f_{,T}}{1 + f_{,T} - 12H^2f_{,TT}} .
$$

The equation of motion for the canonical scalar field is,

$$
\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 ,
$$

and therefore the $f(T)$ gravity equations of motion can be written as follows,

$$
(a + f_{,T} - 12H^2f_{,TT})\ddot{H} = -4\pi G \dot{\phi}^2 .
$$

In effect, the equation that determines the evolution of the scalar perturbations is,

$$
\ddot{\Phi}_k + \left( H - \ddot{H} \right) \Phi_k + \left( 2\dot{H} - H\ddot{H} \right) \Phi_k + \frac{c_s^2 k^2}{a^2} \Phi_k = 0 .
$$

A gauge invariant physical quantity that quantifies any cosmological inhomogeneities, is the comoving curvature fluctuation, denoted as $\zeta$, which is,

$$
\zeta = \Phi - \frac{H}{H} \left( \dot{\Phi} + H\Phi \right) .
$$

We introduce the quantity $v$ defined as follows,

$$
v = z\zeta ,
$$

with $z$ standing for,

$$
z = a\sqrt{2}\epsilon ,
$$

and also $\epsilon$ is $\epsilon = -\frac{H}{H^2}$. Then the equation that determines the evolution of perturbations is,

$$
v''_k + \left( c_s^2 k^2 - \frac{z''}{z} \right) v_k = 0 ,
$$

with $c_s$ appearing in Eq. (19). The “prime” above indicates differentiation with respect to the conformal time, defined as follows,

$$
\tau = \int dt \frac{1}{a} .
$$

The first Friedmann equation in $f(T)$ gravity is,

$$
H^2 = -\frac{f(T(t))}{6} - 2f_{,T}H^2 ,
$$

and due to the fact that $T = -6H^2$, for the intermediate inflation case we have,

$$
T = -6A^2n^2t^{2n-2} .
$$

By inverting the above and using Eq. (28), we easily obtain the approximate $f(T)$ gravity realizing the intermediate inflation scenario, which is,

$$
f(T) = c_1 T^{\frac{4n}{2}} - \frac{T}{2 \left( 1 - \frac{4n}{2} \right)} .
$$
where $c_1$ is an arbitrary integration constant. In effect, the total $F(T)$ gravity is $F(T) = T + f(T)$. Due to the fact that we are interested in early times, the exponential $e^{A t^n}$ is approximated as $e^{A t^n} \sim 1$, and via the relation \cite{27} we can see that the conformal time can be identified with the cosmic time. Therefore, we have for the intermediate inflation scenario,

$$\begin{align*}
    z(t) &= \sqrt{\frac{2(1 - n)t^{-n}}{An}}, \\
    c_s^2(t) &= \frac{A c_1 n \frac{d}{dt}(An - 2) (-A^2 n^2 t^{2n-2}) \frac{\Delta n}{t^2}}{2(An - 1)S(t)} - \frac{12 A^2 n^2 (An - 1)t^{2n-2}}{2(An - 1)S(t)},
\end{align*}$$

with $S(t)$ being,

$$S(t) = \left( c_1 t^{\frac{An}{2}} (An - 2) (-A^2 n^2 t^{2n-2}) \frac{\Delta n}{t^2} - 6A^2 n^2 t^{2n-2} \right).$$

Hence, at leading order, the equation which gives the evolution of perturbations becomes,

$$v_k''(t) + \left( k^2 - \left( \frac{n}{2t^2} \right)^n \right) v_k(t) = 0,$$

which can be solved to yield at leading order \cite{35},

$$v_k(t) = \frac{C_2 \sqrt{t} \left( 2^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right) (kt)^{-\frac{n-1}{2}} \right)}{\pi}.$$

The power spectrum is defined as follows,

$$\mathcal{P}_\zeta = \frac{k^3}{2\pi^2} \left| \frac{v_k(t)}{z(t)} \right|^2 \bigg|_{k = aH},$$

so by using the previous relations, it can be shown that \cite{35},

$$\mathcal{P}_\zeta \simeq \frac{(n-1)^{n-1} A^{1-n} \Gamma^{\frac{n-1}{2}} k^{n+3}}{4\pi^2 (1-n)}.$$

Therefore, the corresponding spectral index is,

$$n_s = \frac{1}{n-1} + 4.$$

The 2015 Planck data indicate that the spectral index has to be in the interval $n_s = \{0.9595, 0.9693\}$, and it can be shown that in order for this to occur, the parameter $n$ in Eq. \cite{37} has to be chosen in the interval $n = \{0.67, 0.6711\}$. Hence compatibility with the Planck can be achieved for the intermediate inflation scenario, in the context of $F(T)$ gravity, by appropriately choosing the parameters of the theory.

### IV. CONCLUSIONS

In this work we showed that the intermediate inflation scenario is a singular type of inflation, with the singularity occurring at the origin being a Type III singularity. The intermediate inflation scenario can be identical with the expanding phase of a Type III singular bounce cosmology. By using the existing results in the literature, we calculated the power spectrum of primordial curvature perturbations for the intermediate inflation scenario, in the context of vacuum $F(R)$, vacuum $F(G)$ and vacuum $F(T)$ gravity. As we demonstrated, only the $F(T)$ gravity realizations yielded results compatible with the 2015 observational data.

Our results show an important feature of modified gravities in general. Particularly, it might occur that a modified gravity description might be viable or not, in the context of some theory, so this indicates that not all modified gravity descriptions yield equivalent results. To our opinion, the theory which is actually closer to physical reality will have the characteristic of being compatible with most of the observations, and also it will be at the same time, relatively simple, but ingeniously constructed. Here we aimed to show that many different descriptions of modified gravity are not equivalent.
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