Exact non-circular symmetric $N$-skyrmions in helical magnets without inversion symmetry

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Topological objects of intricate structures have been found in a wide range of systems, like cosmology to liquid crystal, DNA chains, superfluid $^3$He (Ref. 3), quantum Hall magnets, Bose-Einstein condensates etc. The topological spin texture in helical magnets, observed in recently, makes a new entry into this fascinating phenomenon. The unusual magnetic behaviour of helimagnet MnSi, noticed in recent years, prompted the suspicion that the magnetic states arising in such crystals are of topological nature. Experiments based on the topological Hall effect confirmed such topologically nontrivial states as the skyrmions, located on a plane perpendicular to the applied magnetic field. However, the available models close to MnSi, investigating the formation of skyrmion states, are based mostly on approximate or numerical methods. We present here a theoretical model for chiral magnets with competing exchange and Dzyaloshinskii-Moriya type interaction, which leads to an exact skyrmionic solution with integer topological charge $N$. Such topologically stable spin states with analytic solution on a two-dimensional plane show helical structures of partial order, without inversion and circular symmetry. These exact N-skyrmions, though represent higher excited states, correspond to the lowest energy stable configuration in each topological sector and are likely to appear in MnSi under suitable experimental conditions. The present exact topological solitons, with explicit non-circular symmetry could be applicable also to other fields, where skyrmions are observed, especially in natural systems with less symmetries.

Topological properties can be revealed through a mapping from a continuum space to a differentiable manifold. Therefore for describing topological objects one has to shift from the lattice to a continuum picture. In a magnetic model with the spin configuration varying slowly over the lattice spacing, one can approximate using a long wavelength description, the dimensional lattice to a continuum space $R^d$. At the same time the associated spin $S_j$ would go to a vector field $n^a(x)$, $a = 1, 2, 3$ of unit length $|n|^2 = 1$, at the classical limit, after a proper renormalisation, which might induce topological invariants of different nature, at different dimensions $d$, under suitable boundary conditions.

We intend to use this construction on a two-dimensional xy-plane perpendicular to the applied magnetic field, to simulate the result on MnSi found through the topological quantum Hall experiment. The physically motivated boundary condition demands that at large distances: $|x| = \rho \to \infty$, the spin field $n^a(x)$ should go to its vacuum solution, orienting itself to a fixed vector $n^a_\infty = \delta_{a3}$, along the applied field. This condition in turn introduces a nontrivial topology by identifying the infinities of the coordinate space to a single point, which compactifies the vector space $R^2$ to a sphere $S^2$ and defines thus a
mapping: \( S^2_x \to S^2_n \), linked to the nontrivial homotopy group \( \pi_2(S^2) = \mathbb{Z} \) (see Fig. 1). Each topological sector is labeled by an integer \( N \)-valued topological charge \( Q = \int d^2 x \Phi(x) \), which defined as the degree of this mapping, shows how many times such a skyrmionic field configuration \( \mathbf{n}(x) \) winds up or covers the target sphere, when the coordinate space is swept once. The corresponding charge density can be expressed through the spin vector field as

\[
\Phi(x) = \frac{1}{8\pi} (\mathbf{n}[\partial_x \mathbf{n} \times \partial_y \mathbf{n}]).
\]

Figure 1 | The induction of nontrivial topology. The two-dimensional coordinate space \( x \in \mathbb{R}^2 \), due to the boundary condition \( \lim_{\rho \to \infty} \mathbf{n} = \mathbf{n}_\infty = (0, 0, 1) \), compactifies to a 2-sphere \( S^2_x \), with its north-pole \( P \) identified to all points at the space-infinities. The unit vector field \( \mathbf{n}(x) \in S^2_n \), therefore describes a sphere to sphere mapping with the integer-valued topological charge \( Q = N \), defined as the degree or the winding number of this mapping.

Another important idea, that we use for constructing our magnetic model is to maintain a lower bound \( H \geq \text{const.}|Q| \) for energy \( H \) of the system through its topological charge \( Q \). Note that, such an energy bound would guarantee the stability of the finite energy solution in each topological sector, since the topological charge is conserved independent of the dynamics of the system and can not be changed by unwinding the spin configuration by smooth transformations. Moreover, when the energy reaches its lowest value saturating this bound: \( H = H_{\text{min}} = \text{const.}|Q| \), called Bogomolny limit\(^\text{24} \), some extraordinary thing happens in almost all known models \(^\text{17,18,21,23–25} \): it yields an exact solution to the field model, in some cases for all topological sectors.

The unusual properties of the helical magnets like MnSi and especially their partial helical order created by the skyrmion spin states are believed to be due to the competing forces between the ferromagnetic exchange and an effective Dzyaloshinskii-Moriya (DM) type interaction. The ferromagnetic interaction tries to align the spins parallel to each other, while the DM-term with the broken inversion symmetry tends to orient them perpendicular to each other, settling finally to a partial helical order of topological origin. Moreover, the skyrmionic spin texture in MnSi, as observed in the recent topological Hall experiments, appears in the plane perpendicular to the applied magnetic field. Therefore to model such a system we construct our Hamiltonian \( H \) from a standard Heisenberg ferromagnet in two-dimensions: \( H_0 = -J \sum_{<ij>} \mathbf{S}_i \cdot \mathbf{S}_j \), where the spin \( \mathbf{S}_j \), located at site \( j = (i, j) \) interact with its nearest-neighbour at \( j' = (i, j \pm 1), (i \pm 1, j) \), in the 2-dimensional lattice. We include next the DM term: \( H_{DM} = J_1 \sum_{<ij,j'>} \mathbf{D}_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_{j'}) \), which in spite of its traditional form, bears some crucial differences in the structure of its Dzyaloshinskii (D)-vector \( \mathbf{D}_{ij} \), which would result to our field model allowing exact skyrmion solution. Firstly, we take the D-vector aligned along the applied field, which leaves nontrivial only its third-component \( D^3_{ij} \), breaking the rotational symmetry in the internal spin-space. Our next deciding assumption is the nonlinear nature of this vector, given through a spin-dependent structure:
Under this special geometric condition, the energy bound is exactly saturated attaining its lowest value at the solution manifold:

\( D^3_{1,j} = D^3(S_j^3, S_j^3) = f(S_j^3)(S_j^3 - S_j^3) \). Note that the second factor obeys the usual antisymmetric exchange of the D-vector, while the first factor, which we take in the explicit form \( f(S_j^3) = (1 - (S_j^3)^2)^{-1} \) manifestly breaks the space-inversion symmetry of the vector: \( D^3_{1,j} \neq D^3_{1,j} \), inducing the same property to the Hamiltonian of the system as usual through the DM interaction.

Our total Hamiltonian, which has the required broken inversion-symmetry and the competing spin interactions of opposite trends, should now be tested for its topological properties, for which we have to go for the continuum limit \( \mathbf{S}_{(i,j)} \rightarrow \mathbf{n}(x, y) \). For constructing the corresponding field model from the 2-dimensional lattice model, we have to take carefully the vanishing limit of the lattice constants in both the lattice directions \((i,j)\), which would result, for example, \( \mathbf{S}_{(i \pm 1,j)} - \mathbf{S}_{(i,j)} \rightarrow \pm \partial_x \mathbf{n}(x, y) \), \( \mathbf{S}_{(i,j \pm 1)} - \mathbf{S}_{(i,j)} \rightarrow \pm \partial_y \mathbf{n}(x, y) \) etc. This procedure reduces our lattice spin-model \( H = H_0 + H_{DM} \) to a field Hamiltonian with

\[
H_0 = J \int d^2 x (\nabla \mathbf{n})^2 \quad \text{and} \quad H_{DM} = J_1 \int d^2 x \ f(n^3) \nabla n^3 \cdot (n^1 \nabla n^2 - n^2 \nabla n^1).
\]

It is not difficult to see that, at this continuum limit the chirality of three non-coplanar spins \( \chi = \mathbf{S}_j \cdot [\mathbf{S}_y \times \mathbf{S}_y'] \) gets linked to the topological charge density \( \Phi(x) \), expressed through the \( \mathbf{n} \)-field as \( \mathbb{I} \).

It is important to note, that by tuning the coupling constants \( J \rightarrow J_1 \), our field Hamiltonian \( H \), as can be shown by using a simple school-level geometric inequality\(^{19}\), would acquire the crucial lower bound: \( H \geq 4\pi \sqrt{3}J|Q| \), through the topological charge \( Q \), the significance of which for the topological stability of the solutions is emphasized above. Focusing now on the conjecture, that the Bogomolny limit, at which the energy bound is saturated, should yield the exact finite-energy soliton solution, we look for the situation when this limit could be reached for nontrivial contributions from both parts of the Hamiltonian. We find that this is possible for a field configuration \( \mathbf{n}^* \), where the space-inversion as well as the circular symmetry is lost. More precisely, the spin texture \( \mathbf{n}^* = (\sin \beta \cos \gamma, \sin \beta \sin \gamma, \cos \beta) \), should be such that, the directions \( \nabla \beta \) and \( \nabla \gamma \), defining the helical order, would cross each other at an angle \( 60^\circ \), with three field vectors: \( \mathbf{a} = \nabla \beta \), \( \mathbf{b} = \sin \beta \nabla \gamma \) and \( \mathbf{c} = \mathbf{a} - \mathbf{b} \), forming an isosceles triangle. Under this special geometric condition, the energy bound is exactly saturated attaining its lowest value \( H_{min}[\mathbf{n}^*] = 4J\pi \sqrt{3}N \), in each sector with \( Q = \pm N \). Analysing this intriguing geometry\(^{19}\) we can extract the exact magnetic field solution as

\[
\sin \beta = 2(\rho_0)\frac{N\alpha^2}{N\alpha} e^{\frac{\rho_0 N}{\sqrt{3}}} ((\rho_0)^N e^\frac{\rho_0 N}{\sqrt{3}} + (\rho_0)^N e^\frac{\rho_0 N}{\sqrt{3}}) \quad \text{and} \quad \gamma = \pm N\alpha,
\]

in polar coordinates \((\rho, \alpha)\) with a constant scaling parameter \( \rho_0 \). This exact solution, which can be checked by direct insertion, obeys the required boundary condition \( \mathbf{n}(\rho \rightarrow \infty) = \mathbf{n}_\infty = (0, 0, 1) \), and yields the topological charge \( Q = \pm N \). The distribution of the charge density \( \Phi(\rho, \alpha) \), which coincides also with the energy density at the solution manifold \( \mathbf{n}^* \) is shown in Fig. 2, together with the magnetization \( \sin \beta(\rho, \alpha) \) and the spin component \( n^2 = \cos \beta(\rho, \alpha) \), calculated for the exact result \( \mathbb{E} \), specific to the first excited state with \( Q = 1 \). The figure shows that at the skyrmion axis \( \rho = 0 \), the charge density becomes maximum, vanishing gradually with the distance, while the magnetisation and the spin component exhibit a toroidal structure without circular and space-inversion symmetry. Their maximum/minimum is attained at a distance \( \rho_m = \rho_0 e^{\frac{\alpha}{\sqrt{3}}} \), varying with \( \alpha \), which describes an intriguing form of the torus, with its radius spreading out with increasing polar angle. As revealed in quantum Hall experiments\(^{7,8}\), the peculiarly large Hall conductance is created by an effective magnetic field proportional to the topological charge density. Therefore the detailed and exact form of the charge density together with the magnetization and the spin field component, found here based on the exact skyrmion solution, should serve as a guiding resource for the future precision experiment.
Figure 2 | Exact skyrmion state with $Q = 1$.  

**a.** The charge density $\Phi(\rho, \alpha) = \sin^2 \beta(\rho, \alpha)/\rho^2$, which coincides also with the energy density for the exact skyrmion.  

**b.** The magnetization $|\mathbf{m}| = (|n_1|^2 + |n_2|^2)^{1/2} = \sin \beta(\rho, \alpha)$, and  

**c.** Spin component $n_3 = \cos \beta(\rho, \alpha)$, showing toroidal structure with explicitly broken inversion and circular symmetry.

The graphical result based on the numerical calculation on a model for $MnSi$, proposed in Ref 13, shows that the skyrmion lattice derived as a spontaneous ground state solution, preserves the circular as well as the space-inversion symmetry, when projected on two-dimensions. However, our exact result based on our magnetic model with the broken space-inversion and the spin rotational symmetry, induced by the DM interaction, gives a different picture. It shows, as depicted in Fig. 3, that the partial helical order of the spin texture on the $xy$-plane manifestly breaks both the circular and the inversion symmetry. The future experiments capable of identifying the finer details of the topological spin texture should be able to settle this issue. At the theoretical level, it is significant to note also that, the exact topological soliton without circular symmetry, presented here, is a nontrivial generalisation over the well known solution of the nonlinear $\sigma$-model$^{17,18,21}$, exhibiting this symmetry. In fact all other known exact solutions of the field models$^{23-25}$, obtained at the Bogomolny limit, are basically spherical symmetric, except only the present one.

Another valuable information, e.g. the detail spin structure in higher topological sectors with arbitrary integer charge $Q = N \geq 1$, can be found easily from our exact solution (3). Although the solution with higher topological charge was investigated by using a combination of analytic and numerical methods in connection with the Hall skyrmions$^5$ and recently for the bilayer graphene$^{26}$, no such study, as far as we
know, is undertaken for the magnetic models. Therefore the present result, designed specifically for the magnets like MnSi, and more significantly having purely analytic result with manifestly broken inversion symmetry, is novel in many respect. Such an exact skyrmion with higher topological charge $Q = 3$, is shown in Fig. 3b. These stable states in higher sectors with more intricate texture, as we see in the figure, would be challenging to detect in topological Hall experiments.

The topological soliton with non-circular symmetry, presented here, is exceptional as an exact field theoretic solution and would be interesting for its detection and applications in other systems, lacking such a symmetry.

Figure 3 | Skyrmion spin texture with partial helical order, described by the exact solution of the spin vector field $n(x)$. a , with the topological charge $Q = 1$ and b, $Q = 3$. Both the figures demonstrate manifestly broken space-inversion and the circular symmetry on the $xy$-plane, with a distinct difference in their helical pattern, describing different degrees of mapping in these two cases.

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