Abstract. This article takes at heart Boolos’ intentions when presenting “The Hardest Logic Puzzle Ever.” We stress the fact that Boolos instructs us to solve the puzzle by asking three yes-no questions. In addition, it is a requirement that all the gods, including Random—whose behavior is determined by the outcome of flipping a coin inside its head—are always obliged to answer. Moreover, it is implied in Boolos’ solution that the meaning of “Da” and “Ja” is irrelevant to solve the puzzle, and, at the same time, that all of this would be in virtue of the irreducible fundamentality of “the Law of the Excluded Middle.” The purpose of this paper is, then, twofold. First, we prove that Boolos’ original puzzle cannot be solved, in an absolute deterministic way, in less than three yes-no questions. This does not mean that one could identify the gods’ identities in less than three questions, which chances, second, we compute here as well.

1. Introduction

The main aim of this paper is to emphasize the importance of what Boolos actually suggested as the proper way of addressing “The Hardest Logic Puzzle Ever,” whose origin he attributes to Raymond Smullyan (cf. [1]). Regarding this, we would like to start by reminding the reader that, according to Boolos, the significance of the puzzle is the following:

“There is a law of logic called “the law of excluded middle,” according to which either $X$ is true or not-$X$ is true, for any statement $X$ at all. (“The law of non-contradiction” asserts that statements $X$ and not-$X$ aren’t both true.) Mathematicians and philosophers have occasionally attacked the idea that excluded middle is a logically valid law. We can’t hope to settle the debate here, but can observe that our solution to puzzle 1 made essential use of excluded middle, exactly when we said “Whether the middle card is an ace or not…” It is clear from The Hardest Logic Puzzle Ever, and even more plainly from puzzle 1, that our ability to reason about alternative possibilities, even in everyday life, would be almost completely paralyzed were we to be denied the use of the law of excluded middle.” [1, p. 65]

What this paragraph tells us is, precisely, that if one wants to obtain a valid solution according to the rules posed by The Hardest Logic Puzzle Ever in the spirit and form professed by Boolos, one has to apply the Law of Excluded Middle. Otherwise, if one employs only other laws, primarily the Law of Non-Contradiction, some scenarios immediately follow that invalidate a solution according to Boolos’ guidelines: Say, one has to accept that gods’ heads...
might explode or that some gods might not be able to answer “Da” or “Ja” to certain questions.

One then should take at heart this conservative claim regarding the classical laws of logic to solve Boolos’ puzzle. It is plain obvious that we assumed that one has to first accept the feasibility of the Law of Excluded Middle so that there are admissible questions to pose to the gods. We want to remind that this law of logic tells us that in scenarios where there are three possibilities, $XYZ$, when something is identified as non-$X$, there is an open possibility that that same thing could be either $Y$ or $Z$. Keeping this in mind, let us think again about puzzle 1 in Boolos’ original article:

Puzzle 1: Noting their locations, I place two aces and a jack face down on a table, in a row; you do not see which card is placed where. Your problem is to point to one of the three cards and then ask me a single yes-no question, from the answer to which you can, with certainty, identify one of the three cards as an ace. If you have pointed to one of the aces, I will answer your question truthfully. However, if you have pointed to the jack, I will answer your question yes or no, completely at random. \[1\] p. 63

Besides the fact that it seems to be an election between two alternatives (Ace or Jack), Boolos points out that it is actually indifferent which is the identity of the card that was placed in the middle. This makes us believe that one should treat this as a case where one has to find out the identities of three different individuals/tokens (Ace1, Ace2 and Jack) if one wants to have an absolute answer to the puzzle. As we said before, this is due to the fact that when the ordinary reasoning that follows from the Law of Excluded Middle is applied, one can only find out whether the selected card in the first place is an Ace or the Jack, and from there one can then deduce the rest. Formally, one finds out either $J$ (which then gives options $A1$ and $A2$ as only follow-ups) or non-$J$ (which then gives $J$ and $A2$ as the only options).

We can think about this on a related case: From the fact that “I have purchased a car painted in a primary color and my new car is not blue,” it does not follow that the car is red, since it is a plausible possibility that the color could be yellow (if you are into this kind of flashy cars, of course).

Keeping all of this in mind, this article will reinforce Boolos’ \[1\] and Roberts’ \[3\] original intuitions regarding the laws of logic and our ordinary ways of reasoning. To do so we deepen a little bit more into some arguments that, we believe, are crucial for understanding the importance of Boolos’ purpose when formulating The Hardest Logic Puzzle Ever and his 3-question solution. All of this, combined, shall help us confirm our suggestion that it is not possible to have an absolute solution to the puzzle in less than three questions.

2. THE HARDEST LOGIC PUZZLE EVER

George Boolos \[1\] p. 62 presents the Hardest Logic Puzzle Ever. The puzzle goes like this (boldface is ours):

Three gods $A$, $B$, and $C$ are called, in some order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities

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\[1\] We are referring here to the solutions offered by Rabern–Rabern \[2\] and Uzquiano \[4\], respectively, see also Wheeler–Barahona \[5\]. Indeed, as it will be apparent from our discussion in this paper, though insightful and interesting on their own right, their solutions do not apply to The Hardest Logic Puzzle Ever.
of $A$, $B$, and $C$ by asking three yes-no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for “yes” and “no” are “da” and “ja,” in some order. You do not know which word means which.

Boolos also gives the following guidelines (boldface is ours):

(a) It could be that some god gets asked more than one question (and hence that some god is not asked any questions at all).

(b) What the second question is, and to which god it is put, may depend on the answer to the first question. (And, of course, similarly for the third question).

(c) Whether Random speaks truly or not should be thought of as depending on the flip of a coin hidden in his brain: If the coin comes down heads, he speaks truly; if tails, falsely.

(d) Random will answer “Da” or “Ja” when asked any yes-no question.

We stress the fact that Boolos instructs us to solve the puzzle with three yes-no questions. It is also a requirement that all the gods, including Random, will always answer. Moreover, it is implied in Boolos’ solution that the meaning of “Da” and “Ja” is irrelevant to solve the puzzle (so, semantics does no determine our ontology), and, at the same time, that all of this would be in virtue of the irreducible fundamentality of “the Law of Excluded Middle.”

Roberts [3] reinforces Boolos’ instructions and solution. Indeed, he presents another three-question solution to the puzzle, where the gods always reply either “Da” or “Ja,” and the use of “the Law of Excluded Middle” is crucial. In addition, at the end of his paper he introduces two variations of the original puzzle that, to the best of our knowledge, have never been addressed.

The aim of this article is then twofold. First, we prove that Boolos’ original puzzle cannot be solved, in an absolute deterministic way, in less than three yes-no questions. Nevertheless, this does not mean that one could not be lucky enough to find the gods’ identities in less than three questions. It is for this reason that, second, we shall compute the chances to solve Boolos’ original puzzle with one and two yes-no questions, respectively, as well.

Before presenting our results, however, we need to make the following crucial considerations regarding Boolos’ instructions that will guide our analysis.

Boolos’ original formulation requires that the gods “will answer all questions in their own language”. In other words, the gods are obliged to answer. This in particular implies that not every yes-no question may be asked, as it comes out of the nature of the gods themselves. For example, the question

(Q) “Are you going to answer to this question with a word that means ‘no’ in your language?”

is a question that neither god True nor Random speaking truly can answer without violating their nature (for Random it would mean to reverse the outcome of the coin in its head).

However, it would be admissible as directed to god False or god Random speaking Falsely. This shows that there must be a subclass of the class of all yes-no questions that is admissible. Hence we introduce the following novel notion.

\[ \text{Rabern–Rabern [2] and Uzquiano [4] allow Random to “answer (or not) randomly,” namely, they make Random behave in a different way with respect to guideline (c) above. Indeed, Boolos specifies that the flipping of a coin will determine Random’s behavior prior to being addressed with a question.} \]
Definition A. A question $Q$ is *admissible* for god $A$ if and only if $A$ will answer either “Da” or “Ja” to $Q$.

It is important to observe that The Hardest Logic Puzzle Ever already has a class of permissible questions, namely, the class of *all* yes-no questions. Therefore, to keep Boolos’ spirit, we are forced to tailor the class of possible questions as they have to be dynamic in relation to the epistemic space. Notice as well that the set of admissible questions does not need to be static. In fact, Boolos remarks in guideline (b) that “What the second question is, and to which god it is put, may depend on the answer to the first question”. As our proof will show, either a static or a dynamic set of admissible questions will still yield the same outcome: three yes-no questions are needed to solve the puzzle with absolute certainty.

The previous differs strikingly with Rabern–Rabern [2], Uzquiano [4], see also Wheeler–Barahona [5]. They assume that any yes-no question can be asked and this opens the possibility of a third response: The impossibility to answer. For Rabern–Rabern this translates into the exploding-head scenario and for Uzquiano becomes the silent response, respectively.

This additional requirement is what allows them to solve the (by now, modified) puzzle in only two questions, reducing in this way Boolos’ requirements exclusively to the Law of Non-Contradiction. This is what actually shall trivialize Boolos’ puzzle, as we demonstrate here, and even Rabern–Rabern [2] have confirmed. As a matter of fact, all of this, again, dramatically deviates from Boolos’ original rules.

We, however, take Boolos in his purest form. We prove that his original puzzle cannot deterministically be solved in less than three admissible yes-no questions. Our proof demonstrates that the necessity of a three-question solution to deterministically solve the puzzle is independent of what we actually ask, the meaning of “Da” and “Ja,” and the triviality of the distinction between truth-tellers and liars. In addition, even for Random behaving as in Boolos’ instructions (and not by “answering randomly”), a three-question solution is unavoidable (see footnote 2).

3. **THE THREE-QUESTION SOLUTION TO THE HARDEST LOGIC PUZZLE EVER**

As we mentioned before, we prove the following:

**Theorem B.** Boolos’ Hardest Logic Puzzle Ever cannot deterministically be solved in less than three admissible yes-no questions.

**Proof.** According to Boolos [1] and Roberts [3], the puzzle can deterministically be solved in three admissible yes-no questions. Therefore, the only thing we must prove is that there are no single-question or two-question deterministic solutions to the puzzle.

To restate the puzzle, the problem is to determine the identities of the three gods: $A$, $B$, and $C$, which are $T = \text{True}$, $F = \text{False}$, and $R = \text{Random}$, in some order. There are then 6 different possible identification scenarios, $S_1$–$S_6$:

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\(^3\)One additional undesirable (or, perhaps, convenient) consequence of the exploding-head option is that, more likely, the god whose head exploded will not be able to answer any other subsequent question. This might be a crucial difference with respect to the silent response, in which the god will still be available to answer further questions.
First, we shall prove that there is no a single-question deterministic solution to the puzzle. Without loss of generality, we address the first admissible yes-no question to god $A$. With independence of what we actually ask, $A$ will always answer either “Da” or “Ja.” Independently of what “Da” and “Ja” mean, the answer will not provide information enough to deterministically identify the three gods at the same time. The reason is given by the following Information Theory lemma from [5].

**Lemma C.** If a question has $N$ possible answers, these $N$ answers cannot distinguish $M > N$ different possibilities.

We have already specified that for our case there are 6 possible identification scenarios ($S_1$–$S_6$ above). Given that there are two possible answers, $M = 6$ and $N = 2$. Therefore, by Lemma C we cannot distinguish the identities of all three gods altogether.

Second, to provide a two-question solution to the problem in a deterministic way, in virtue of the same lemma, the first admissible yes-no question must reduce the remaining scenarios to no more than two. However, the latter cannot deterministically be guaranteed. The reason is that there is always a chance that $A$ is not-Random. In such a case, after the first question is answered, we are in a situation where $A$ could be either True or False, so scenarios $S_1$–$S_4$ still remain. Therefore, $M = 4 > 2 = N$. Notice that there is no admissible first question that would directly determine the identity of $A$, since there are $M = 3$ gods but still $N = 2$ possible answers to any admissible question.

Going back to the concept of admissible question, notice that the question (Q) we mentioned above is not an admissible question as a first question to ask because we do not know the identity of any of the gods yet. However, it may happen that, after the first question is answered, we gain some information that will allow (Q) to become an admissible question. In this regard, our set of admissible questions can be dynamic in relation with our interaction with the gods, and not just static or absolute. Remarkably enough though, no matter what we ask and how the set of admissible questions changes or not, we still need three yes-no questions to identify the gods with absolute certainty.

4. Computing the chances

We have demonstrated that Boolos’ puzzle cannot deterministically be solved in less than three questions. Now, we direct our attention to the second horn of our original concern, which is to compute the probabilities to identify the gods without asking any question, by asking one question and by asking two yes-no admissible questions, respectively.

**Theorem D.** Consider Boolos’ Hardest Logic Puzzle Ever.

(1) The event 

\[ \{ \text{solve The Hardest Logic Puzzle Ever without asking any questions} \} \]

occurs with probability $1/6$. 

|   | $A$ | $B$ | $C$ |
|---|-----|-----|-----|
| $S_1$ | $T$ | $F$ | $R$ |
| $S_2$ | $T$ | $R$ | $F$ |
| $S_3$ | $F$ | $T$ | $R$ |
| $S_4$ | $F$ | $R$ | $T$ |
| $S_5$ | $R$ | $T$ | $F$ |
| $S_6$ | $R$ | $F$ | $T$ |
The event \( \{ \text{solve The Hardest Logic Puzzle Ever by asking one question} \} \) occurs with probability \( \frac{1}{6} \).

The event \( \{ \text{solve The Hardest Logic Puzzle Ever by asking two questions} \} \) occurs with probability \( \frac{1}{3} \).

It may sound counterintuitive, but surprisingly enough, no matter what one actually asks, the probability to find all gods’ identities with only one single admissible yes-no question is the same that we have to find them without even asking any questions at all! Rolling a dice could save us some energy and effort.

**Proof of Theorem D.** For (1), it is clear that there is a probability of 1/6 of solving the puzzle without even asking a question. Indeed, we can simply roll a 6-faces uniform fair dice, observe the outcome \( X \), and then pick the corresponding scenario \( S_X \).

Let us consider (2). Without loss of generality, we address the only question we have available to god \( A \). There are two possibilities. Either we addressed our question to Random (with probability 1/3, and we are in scenarios \( S_5-S_6 \)) or we addressed our question to a god that is not Random (with probability 2/3, and we are in scenarios \( S_1-S_4 \)). Notice that after this has happened, we have no questions left. In the first case, where \( A \) is Random, there is a fifty/fifty chance to fall under either scenario \( S_5 \) or scenario \( S_6 \). Hence, since we started with a probability of 1/3 of the question being addressed to Random, and after the first admissible yes-no question we are left with a 1/2 probability of deciding the correct scenario, the total probability is then \( (1/3) \times (1/2) = 1/6 \). In the second case, that of \( A \) being non-Random with probability 2/3, there is a 1/4 probability that one and only one of scenarios \( S_1-S_4 \) occurs. Therefore, the total probability to find the identities of each god in one question in this second case is \( (2/3) \times (1/4) = 1/6 \).

Finally, let us show (3), namely, that there is a probability of 1/3 to determine the gods’ identities in only two admissible yes-no questions. Independently of the first question, that without loss of generality we address to \( A \), and its answer, \( A \) is either Random with probability 1/3 or not-Random with probability 2/3. In the first case, we are again in scenarios \( S_5 \) or \( S_6 \). We can use the second question to further decide in an absolute, deterministic way, that is, with probability 1, whether we are actually situated in either scenario \( S_5 \) or scenario \( S_6 \). Indeed, ask god \( B \) the question *Does da mean yes iff Rome is in Italy?* God \( B \) will answer Da if and only if \( B \) is True, and will answer Ja if and only if \( B \) is False. Therefore, the total probability for this case is \( (1/3) \times 1 = 1/3 \). In the second case, we fall under scenarios \( S_1-S_4 \), where we know that \( A \) is not random. By using the same question as before, we identify \( A \) as either True or False with probability 1. This means that we can narrow down the space of epistemic possibilities to either \( S_1-S_2 \) or \( S_3-S_4 \). Say we are situated in the space \( S_1-S_2 \). Then, there is a probability of 1/2 of having either \( S_1 \) or \( S_2 \) as the correct identification scenario. (The same actually holds for the other space). Hence, the total probability is \( (2/3) \times 1 \times (1/2) = 1/3 \). And with this our computations are done. □

Our proof of Theorem D(3) is consistent with the original strategy of Boolos: “Your first move is to find a god who you can be certain is not Random” [1]. After that you ask to the not-Random god the question *Does Da mean yes iff Rome is in Italy?* to decide if you are talking to god True or god False.

It is a remarkable fact that, independently of having a static or dynamic set of admissible questions, the event \{identify the gods in 3 questions\} occurs with probability 1, while...
identify the gods in less than 3 questions} has probability strictly less than 1. Notice also that once the event \{identify the gods in 2 questions\} has occurred (and this would happen 1 out of 3 times, as Theorem D(3) showed), the gods have definitely been identified, with absolute certainty.

One can also think about the chances of whether a yes-no question is admissible or not. For example, say we consider question \(Q\) as a first question. Flipping the coin means that Random is speaking truly with probability \(1/2\). Then \(Q\) is not an admissible question not only for god True, but also for god Random. So the probability that \(Q\) is an admissible first question when posed to any of the gods is \((1/2) \times (1/3) = 1/6\). On the other hand, Random speaks falsely with probability \(1/2\). In this case \(Q\) is not admissible only for god True. Thus the probability that \(Q\) is admissible in this case is \((1/2) \times (2/3) = 1/3\).

5. Conclusion

The proper understanding of Boolos’ original purpose when proposing The Hardest Logic Puzzle Ever is crucial, as we have demonstrated in our paper, to offer not only a proper solution to the puzzle but also to consider the natural issue of admissibility of the questions posed to the gods. To address the puzzle properly, one has to put close attention to Boolos’ requirements, specifically to the claim that the Law of Excluded Middle is a better explanatory source for certain ways of reasoning than the Law of Non-Contradiction. This is the main difference that, as the present article evidences, one can find in other solutions to variations of the puzzle, and it is the reason why they fail to be proper answers to Boolos’ original formulation of the puzzle. By rescuing the spirit and form of Boolos’ Hardest Logic Puzzle Ever, we have reinforced and done justice to such a claim.

We still believe that Boolos’ Hardest Logic Puzzle Ever is indeed the hardest logic puzzle ever, as one must be clever enough to find three convenient yes-no questions that the gods will answer in order to solve it (and not just two questions, as in other modified versions). As Victor Hugo once wrote: “An invasion of armies can be resisted, an invasion of ideas cannot be resisted.”

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