The nature of electromagnetic energy

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Abstract

The nature of electromagnetic energy for general charge and current distributions is analyzed. We compare several forms for the electromagnetic energy, and discuss under what conditions there will be electromagnetic energy within a specific volume. Our conclusion is that electromagnetic energy resides in charge and current densities, and there is no electromagnetic energy in any volume that does not contain electric charge or current.

1 Electromagnetic Energy of a Charge-Current Distribution

The rate at which energy is put into matter by an electric current in a volume, \( V \), is given by\(^1\)

\[
\frac{dU_{\text{Matter}}}{dt} = \int_V j \cdot E \, d^3r.
\]

By \( U_{\text{Matter}} \), we do not necessarily mean that any matter is in motion. For instance, in a wire with resistance, the electric field will increase the current density, \( j \), increasing the heat energy in the wire, but not producing any mechanical motion. Consequently, by conservation of energy, the rate of

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\(\text{2We are using Gaussian unit with c=1.}\)
change of electromagnetic energy is the negative of the rate of change of the energy in matter, so

$$\frac{dU_{EM}}{dt} = -\frac{dU_{Matter}}{dt} = -\int_V j \cdot E d^3r. \quad (2)$$

At this point, the usual procedure has been to use Maxwell’s equations to eliminate the current from Eq. (2). However, we will follow an alternate procedure, using the definition of the electric field in terms of the scalar and vector potentials:

$$E = -\nabla \phi - \partial_t A. \quad (3)$$

Then, Eq. (2) becomes

$$\frac{dU_{EM}}{dt} = \int_V j \cdot (\nabla \phi + \partial_t A) d^3r. \quad (4)$$

Integrating this equation with respect to time leads to

$$U_{\rho j} = \int_0^t dt \int_V j \cdot (\nabla \phi + \partial_t A) d^3r$$

$$= \int_0^t dt \int_V [\nabla \cdot (j \phi) - \phi(\nabla \cdot j) + j \cdot (\partial_t A)] d^3r$$

$$= \int_0^t dt \int dS \cdot (j \phi) + \int_0^t dt \int_V [\phi(\partial_t \rho) + j \cdot (\partial_t A)] d^3r$$

$$= \int_V \frac{1}{2} (\rho \phi + j \cdot A) d^3r + \int_0^t dt \int dS \cdot (j \phi). \quad (5)$$

In the above, we have used the continuity equation, and have assumed that $\phi$ and $\rho$, and $A$ and $j$, have the same time dependence, so

$$\phi \partial_t \rho = \rho \partial_t \phi = \frac{1}{2} \partial_t (\rho \phi) \quad \text{and} \quad j \cdot (\partial_t A) = A \cdot (\partial_t j) = \frac{1}{2} \partial_t (j \cdot A). \quad (6)$$

In this paper, we will consider the case where the current densities are either completely enclosed within the volume or completely outside of the volume (or are surface currents), so that the surface integral in Eq. (5) vanishes. Then, Eq. (5) reduces to

$$U_{\rho j} = \frac{1}{2} \int_V (\rho \phi + j \cdot A) d^3r. \quad (7)$$
From the volume integral in Eq. (7) for the electromagnetic energy, we can define an electromagnetic energy density
\[ u_{\rho j} = \frac{1}{2} (\rho \phi + j \cdot A), \]  
whose integral gives the electromagnetic energy within any volume of integration.

Equation (7) shows that the electromagnetic energy of a charge-current distribution resides in the charge-current distribution. This means that there is no electromagnetic energy in any volume that does not contain electric charge or current.\(^2\)

2 Energy of Electromagnetic Fields

The electromagnetic energy can be put in terms of the electromagnetic fields, \( E \) and \( B \), by using Maxwell’s equations, leading to
\[ U_{\rho j} = \frac{1}{2} \int_V (\rho \phi + j \cdot A) \, d^3r \]  
\[ = \frac{1}{8\pi} \int \left[ \rho \phi (\nabla \cdot E) + (\nabla \times B) \cdot A - A \cdot (\partial_t E) \right] \, d^3r \]
\[ = \frac{1}{8\pi} \int \left[ \nabla \cdot (E\phi) - E \cdot (\nabla \phi) + \nabla \cdot (B \times A) + B \cdot (\nabla \times A) - A \cdot (\partial_t E) \right] \, d^3r \]
\[ = \frac{1}{8\pi} \int (E \cdot E + B \cdot B) \, d^3r + \frac{1}{8\pi} \oint \mathbf{dS} \cdot \left[ \mathbf{E} \phi + \mathbf{B} \times \mathbf{A} + (\mathbf{E} \cdot \partial_t \mathbf{A} - \mathbf{A} \cdot \partial_t \mathbf{E}) \right]. \]  

The time derivative term in Eq. (10) vanishes if there is no time dependence, or for electromagnetic radiation, where \( E \) and \( A \) each have the time dependence \( e^{-i\omega t} \). We consider that to be the case for the rest of this paper, which reduces Eq. (10) to
\[ U_{\rho j} = \frac{1}{8\pi} \int (E^2 + B^2) \, d^3r + \frac{1}{8\pi} \oint \mathbf{dS} \cdot (\mathbf{E} \phi + \mathbf{B} \times \mathbf{A}). \]  

If the surface integral vanishes, the electromagnetic energy can be put purely in terms of the electric and magnetic fields, with
\[ U_{EB} = \frac{1}{8\pi} \int (E^2 + B^2) \, d^3r. \]  

\(^2\)This conclusion holds even if the surface term in Eq. (5) doesn’t vanish.
Equation is (12) often used to give the electromagnetic energy within a volume, but that is only true if the surface integral in Eq. (10) vanishes. For instance, Jackson\[2\] derives Eq. (12) by ‘assuming’ that the surface integral, \(\oint\mathbf{dS} \cdot (E \phi + B \times A)\), in Eq. (11) can be in neglected\[3\]. Also, his derivation of Eq. (12) is for time independent fields, and then he ‘assumes’ it will also hold with time dependence\[4\]. So reference [2] agrees with us that Eq (12) only holds if those assumptions are made, and Eq. (10) would be needed if they were not made.

If the integral in Eq. (12) is over all space, and \(E\) and \(B\) decrease fast enough to make the surface integral vanish (This does not happen for radiation fields.), then Eq. (12) will give the same electromagnetic energy as Eq. (7). However, for volumes with finite surfaces, Eq. (10), including the surface integral, must be used to give the same result as Eq. (7) for the electromagnetic energy.

We note that, even if \((E^2 + B^2)\) does not equal zero within a volume that does not contain a charge or current distribution, the electromagnetic energy in that volume must vanish everywhere within the volume. This follows because we can enclose any point within the volume by an infinitesimal sphere. Then, Eq. (10) shows that the electromagnetic energy must vanish at that point, and so it must vanish anywhere within the volume.

From this discussion, we can conclude again that there is no electromagnetic energy in any volume that does not contain electric charge or current.

### 3 Poynting’s Theorem

In the above, we have derived properties of electromagnetic energy with an emphasis on the part played by the scalar and vector potentials. The usual textbook derivation uses Maxwell’s equations and some vector identities to replace \(\mathbf{j}\) in Eq. (2), leading to Poynting’s theorem in the form\[5\],

\[
-\int_V \mathbf{j} \cdot \mathbf{E} d^3r = \frac{1}{8\pi} \int_V \partial_t (E^2 + B^2) d^3r + \frac{1}{4\pi} \oint_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{B}).
\]

\[\text{See pages 166 and 213 of [2].}\]
\[\text{See page 259 of [2].}\]
\[\text{This is Eq. (6.107) of [2].}\]
Using conservation of energy, as discussed in Section 1, Eq. (13) can be written as

\[
\frac{dU_{EM}}{dt} = \frac{1}{8\pi} \int_V \partial_t (E^2 + B^2) d^3r + \frac{1}{4\pi} \oint_S \mathbf{dS} \cdot (\mathbf{E} \times \mathbf{B}). \tag{14}
\]

This equation can be interpreted as indicating that the rate of change in electromagnetic energy is given by the sum of a volume integral and a surface integral. The volume integral in Eq. (14) is the time derivative of the volume integral in Eq. (11), and we will show below that the surface integral is the time derivative of the surface integral in Eq. (11). At the same time, the surface integral represents the rate at which electromagnetic energy is entering the volume (if the surface integral is positive) or leaving the volume (if the surface integral is negative).

This action of the surface integral in Eq. (14) can be used to define the Poynting vector,

\[
\mathbf{S_P} = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{B}). \tag{15}
\]

The Poynting vector has the significance that \( \mathbf{S_P} \cdot d\mathbf{S} \) gives the rate of transmission of electromagnetic energy through the infinitesimal surface \( d\mathbf{S} \).

If the volume in Eq. (14) is taken as a sphere with infinite radius, and \( (\mathbf{E} \times \mathbf{B}) \) approaches zero fast enough for the surface integral to vanish, Eq. (14) can be integrated over time to give

\[
U_{EB} = \frac{1}{8\pi} \int (E^2 + B^2) d^3r, \tag{16}
\]

as in Eq. (12). The discussion following Eq. (12) applies here too, and Eq. (16) would only hold if the surface integral in Eq. (14) is not included.

Although it is the time derivative of \( (E^2 + B^2) \), and not \( (E^2 + B^2) \), that appears in Eq. (14), it has often been suggested that Eq. (16) holds even when the surface integral does not vanish\[1, 2, 3\]. This is based on the assumption that the volume integral in Eq. (16) gives the electromagnetic energy in the volume, while the surface integral refers to energy that has left the volume, and need not be included. We show below that this is not the case.

Equation (14) can be modified by replacing the electric field in the surface integral by the scalar and vector potentials, resulting in

\[
\frac{dU_{EM}}{dt} = \frac{1}{8\pi} \int_V \partial_t (E^2 + B^2) d^3r - \frac{1}{4\pi} \oint_S \mathbf{dS} \cdot [(\nabla \phi + \partial_t \mathbf{A}) \times \mathbf{B}] 
\]
\[
\begin{align*}
&= \frac{1}{8\pi} \int \partial_t(E^2 + B^2)d^3r - \frac{1}{4\pi} \oint d\mathbf{S} \cdot [(\nabla \times (\phi \mathbf{B}) - \phi (\nabla \times \mathbf{B}) + (\partial_t \mathbf{A}) \times \mathbf{B})] \\
&= \frac{1}{8\pi} \int \partial_t(E^2 + B^2)d^3r - \frac{1}{4\pi} \nabla \cdot [\nabla \times (\phi \mathbf{B})] - \oint d\mathbf{S} \cdot (\phi \mathbf{j}) \\
&\quad + \frac{1}{4\pi} \oint d\mathbf{S} \cdot [((\phi \partial_t \mathbf{E}) - (\partial_t \mathbf{A}) \times \mathbf{B})] \\
&= \frac{1}{8\pi} \int \partial_t(E^2 + B^2)d^3r + \frac{1}{8\pi} \oint d\mathbf{S} \cdot \{\partial_t [\phi \mathbf{E} + (\mathbf{B} \times \mathbf{A})]\}. \quad (17)
\end{align*}
\]

In the derivation above, we have dropped the divergence of a curl, and assumed, as before, that the surface integral of \(\phi \mathbf{j}\) vanishes. We have also assumed, as in deriving Eq. (11), that any time dependence is given by \(e^{-i\omega t}\).

Now each term in Eq. (17) involves a time derivative, and it can be integrated over time to give

\[
U_{EM} = \frac{1}{8\pi} \int (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B})d^3r + \frac{1}{8\pi} \oint d\mathbf{S} \cdot [\mathbf{E} \phi + (\mathbf{B} \times \mathbf{A})], \quad (18)
\]

which agrees with Eq. (10). This shows again that a surface integral must be added to the volume integral of \((E^2 + B^2)\) to give the full amount of electromagnetic energy in the volume.

We see that Poynting’s theorem gives the same result for the electromagnetic energy as integrating charge and current densities if the time integration of Poynting’s theorem is done correctly. This means that Poynting’s theorem also leads to the conclusion that there is no electromagnetic energy in any volume that does not contain electric charge or current.

4 Electromagnetic Radiation

A number of questions can be raised about electromagnetic radiation. Doesn’t electromagnetic radiation carry energy as it travels through free space? How can light travel from the sun to the earth without having electromagnetic energy moving through space from the sun to the earth? How can an antenna receive radio signals if there is no electromagnetic energy between the broadcasting antenna and the receiving antenna? How can you account for radiation reaction if energy is not carried away by the radiation fields?

These questions have straightforward answers:
The propagation of electromagnetic waves is by means of the Poynting vector,

\[ \mathbf{S}_p = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{B}) \],

(19)

behaving like a current for the flow of electromagnetic energy. Just like an electric current, which carries electric charge through regions where there is no charge, the Poynting vector current carries electromagnetic energy through regions where there is no electromagnetic energy.

A simple example of this (apart from the electric current in a wire) is what happens if a blob of charge is placed in the middle of a conducting object. The blob retains its shape, but vanishes exponentially in time, while the charge appears on the outer surface of the conductor. At no time is there any charge in the conductor between the original blob and the outer surface of the conductor. It looks like magic, but that’s how a current can carry charge (or electromagnetic energy) without charge or energy ever appearing between the source and the receiver.

When light travels from the sun to a solar panel on earth, no light is actually seen until it hits the solar panel, or is reflected from dust in the atmosphere. This is like a quantum mechanical wave function, which governs the propagation of a wave that is not detected until it strikes a detector, where it is identified as a particle. There is no ‘particle’ between the source and the detector, which is why we can’t say that the particle went through one slit or the other.

The absorption of electromagnetic energy by a detector or antenna is determined by the equation,

\[ \frac{d^2W}{dtd\Omega} = r^2(\hat{\mathbf{r}} \cdot \mathbf{S}_p) \],

(20)

which again uses the Poynting vector as an electromagnetic current, with no mention of electromagnetic energy.

Similarly, calculations of radiation reaction are generally based on calculating the radiated power,

\[ dP = \mathbf{S}_p \cdot dA \],

(21)

using the Poynting vector as an electromagnetic current without any direct consideration of electromagnetic energy. The question of what happens after six

See, for instance, Section 6.9.3 of [3].
the emission as the radiation propagates has no effect on radiation reaction\footnote{See, for instance, Section 11.2.2 of \cite{1}.}.

5 Conclusion

The electromagnetic energy of a charge-current distribution within any volume is given by the integral

\[ U_{\rho j} = \frac{1}{2} \int_V (\rho \phi + \mathbf{j} \cdot \mathbf{A}) d^3r. \] (22)

The electromagnetic energy within a volume can also be given in terms of the electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) by the volume and surface integrals

\[ U_{EB} = U_{\rho j} = \frac{1}{8\pi} \int_V (E^2 + B^2) d^3r + \frac{1}{8\pi} \oint_S \mathbf{dS} \cdot (\mathbf{E}\phi + \mathbf{B} \times \mathbf{A}). \] (23)

Only if the surface integral in Eq. (23) vanishes, can the electromagnetic energy within the volume be given by just the volume integral.

Equation (22) shows that electromagnetic energy is contained in charge and current densities. There is no electromagnetic energy in any volume that does not contain electric charge or current. Consequences of this, as it applies to electromagnetic radiation, are discussed in Section 4. It should not be surprising that electromagnetic energy resides in matter because the only time it is ever detected or seen is when electromagnetic radiation excites matter.

The fact that we have found that the location of electromagnetic energy is within charge and current distributions in the form \( (\rho \phi + \mathbf{j} \cdot \mathbf{A}) \) resolves the long-standing question of whether the electromagnetic potentials or the \( \mathbf{E} \) and \( \mathbf{B} \) fields are the fundamental objects of electromagnetism\footnote{See, for instance, Section 2.4.4 of \cite{1}.} Although it had long been assumed that \( \mathbf{E} \) and \( \mathbf{B} \) were the fundamental fields, with the potentials \( \phi \) and \( \mathbf{A} \) being mathematical supplements, it has recently been emphasized by Sebens\footnote{See, for instance, Section 4.4 of \cite{1}} that the potentials could be of fundamental importance.

Also, the understanding of classical electromagnetism as a gauge theory derived from quantum mechanics can only be understood in terms of the
In fact, the electromagnetic fields $E$ and $B$ almost never appear in quantum electrodynamics (QED). In this paper, we have shown that it is the potentials that contain electromagnetic energy, and should be considered fundamental, with the fields $E$ and $B$ being convenient shorthand to simplify equations.

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**References**

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$^9$See Section 6.2 of [3].