ON THE GORENSTEIN AND $\mathfrak{F}$-COHOMOLOGICAL DIMENSIONS

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Abstract. We prove that for any discrete group $G$ with finite $\mathfrak{F}$-cohomological dimension, the Gorenstein cohomological dimension equals the $\mathfrak{F}$-cohomological dimension. This is achieved by constructing a long exact sequence of cohomological functors, analogous to that constructed by Avramov and Martsinkovsky in [3], containing the $\mathfrak{F}$-cohomology and complete $\mathfrak{F}$-cohomology. As a corollary we improve upon a theorem of Degrijse concerning subadditivity of the $\mathfrak{F}$-cohomological dimension under group extensions [13, Theorem B].

1. Introduction

Throughout, $G$ denotes a discrete group and $R$ a commutative ring.

Let $n_G$ denote the minimal dimension of a contractible proper $G$-CW-complex and $gd_G$ the minimal dimension of a model for $EG$, the classifying space for proper actions of $G$. Clearly $n_G \leq gd_G$ and Kropholler and Mislin have conjectured that if $n_G < \infty$ then $gd_G < \infty$ [11, Conjecture 43.1], they verified the conjecture for groups of type $FP\_\infty$ [25] and later Lück proved it for groups with a bound on the lengths of chains of finite subgroups [28].

The algebraic invariant best suited to the study of $gd_G$ is the Bredon cohomological dimension. Bredon cohomology was introduced in [8], and extended to infinite groups in [27]. We denote by $cd_G$ the Bredon cohomological dimension over $R$.

The $\mathfrak{F}$-cohomology was suggested by Nucinkis as an algebraic analog of $n_G$ [32], it is a special case of the relative homology of Mac Lane [29] and Eilenberg–Moore [15]. Let $\mathfrak{F}$ denote the family of finite subgroups of $G$ and let $\Delta$ denote the $\mathfrak{G}$-set $\bigsqcup_{H \in \mathfrak{F}} G/H$, we say that a module is $\mathfrak{F}$-projective if it is a direct summand of a module of the form $N \otimes_R R\Delta$ where $N$ is any $RG$-module. Short exact sequences are replaced with $\mathfrak{F}$-split short exact sequences—short exact sequences which split when restricted to any finite subgroup of $G$, or equivalently which split when tensored with $R\Delta$. The class of $\mathfrak{F}$-split short exact sequences is allowable in the sense of Mac Lane, and the projective modules with respect to these sequences are exactly the $\mathfrak{F}$-projectives. There are enough $\mathfrak{F}$-projectives and one can define a cohomology theory, denoted $\mathfrak{F}Ext^{\ast}_{RG}$:

$$\mathfrak{F}Ext^{\ast}_{RG}(M, N) = H^{\ast}\text{Hom}_{RG}(P\_\ast, N)$$

Where $P\_\ast$ is a $\mathfrak{F}$-split resolution of $M$ by $\mathfrak{F}$-projective modules. We define

$$\mathfrak{F}H^{\ast}(G, M) = \mathfrak{F}Ext^{\ast}_{RG}(R, M)$$

The $\mathfrak{F}$-cohomological dimension, denoted $\mathfrak{F}cd_G$, is the shortest length of a $\mathfrak{F}$-split $\mathfrak{F}$-projective resolution of $R$.

By a result of Bouc and Kropholler–Wall $\mathfrak{F}cd_G \leq n_G$ [7, 23] but it is unknown if $\mathfrak{F}cd_G < \infty$ implies $n_G < \infty$ or if there exist examples where the invariants differ.

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Nucinkis posed an algebraic version of the Kropholler–Mislin conjecture, asking if the finiteness of $\mathfrak{F} \text{cd} G$ and $\text{cd} G$ are equivalent \[33\].

A module is \textit{Gorenstein projective} if it is a cokernel in a strong complete resolution of $RG$-modules, these were first defined over an arbitrary ring by Enochs and Jenda \[17\]. We will give a full explanation of complete resolutions in Section 2.1. The Gorenstein projective dimension $\text{Gpd} M$ is the minimal length of a resolution of $M$ by Gorenstein projective modules. Equivalently, $\text{Gpd} M \leq n$ if and only if $M$ admits a complete resolution of coincidence index $n$ \[4\] p.864.

The \textit{Gorenstein cohomological dimension} of a group $G$, denoted $\mathfrak{G} \text{cd} G$, is the Gorenstein projective dimension of $R$. If $G$ is virtually torsion-free then $\text{Gcd} G = \text{vcd} G$ \[4\] Remark 2.9(1)]. Indeed the Gorenstein cohomology can be seen of as a generalisation of the virtual cohomological dimension. Bahlekeh, De mbegioti and Talelli have conjectured that $\mathfrak{G} \text{cd} G < \infty$ implies that $\text{cd} G < \infty$ \[4\] Conjecture 3.5.

By \[2\] Lemma 2.21], every permutation $RG$-module with finite stabilisers is Gorenstein projective, so combining with \[19\] Lemma 3.4] gives that $\text{Gcd} G \leq \mathfrak{G} \text{cd} G$.

In general we have the following chain of inequalities.

$$\mathfrak{G} \text{cd} G \leq \mathfrak{F} \text{cd} G \leq \text{cd} G$$

We prove the following:

\textbf{Theorem 3.11.} If $\mathfrak{F} \text{cd} G < \infty$ then $\mathfrak{G} \text{cd} G = \text{Gcd} G$.

We don’t know if $\mathfrak{F} \text{cd} G < \infty$ implies $\mathfrak{G} \text{cd} G < \infty$, although if $\text{Gcd} G = 0$ or 1 then $\text{Gcd} G = \mathfrak{F} \text{cd} G = \text{cd} G$ \[2\] Proposition 2.19 \[4\] Theorem 3.6]. Additionally if $G$ is in Kropholler’s class $H$ $\mathfrak{F} \text{cd} G$ and has a bound on the orders of its finite subgroups then $\mathfrak{F} \text{cd} G = \text{Gcd} G$ (see Example 5.12).

Generalising a construction of Avramov–Martsinkovsky, Asadollahi–Bahlekeh–Salarian showed that if $\text{Gcd} G < \infty$ then there is a long exact sequence of cohomology functors relating the group cohomology, the complete cohomology and the Gorenstein cohomology \[3\] \[2\]. Our result follows from constructing a similar long exact sequence relating the $\mathfrak{F}$-cohomology, the complete $\mathfrak{F}$-cohomology (defined in Section 2.3) and a new cohomology theory we call the $\mathfrak{F}G$-cohomology defined in Section 3. These two long exact sequences fit into a commutative diagram, see Proposition 3.9. It appears that the requirement in Theorem 3.11 that $\mathfrak{F} \text{cd} G < \infty$ will be difficult to circumvent since this new long exact sequence cannot be constructed for all groups.

In Section 4 we use that the Gorenstein cohomological dimension is subadditive to improve upon a result of Degrijse on the behaviour of $\mathfrak{F} \text{cd}$ under group extensions \[13\] Theorem B]. Degrijse phrased his result in terms of Bredon cohomological dimension of $G$ with coefficients restricted to cohomological Mackey functors, but this invariant is equal to $\mathfrak{F} \text{cd} G$ \[34\] Theorem 6.2].

\textbf{Corollary 4.2.} Given a short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

if $\mathfrak{F} \text{cd} G < \infty$ then $\mathfrak{F} \text{cd} G \leq \mathfrak{F} \text{cd} N + \mathfrak{F} \text{cd} Q$.

In Section 5 we use the Avramov–Martsinkovsky long exact sequence to prove the following.

\textbf{Proposition 5.4.} If $\text{Gcd} G < \infty$ and $\text{cd} Q G < \infty$ then $\text{cd} Q G \leq \text{Gcd} G$.

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2. Preliminaries

2.1. Complete Resolutions and Complete Cohomology. A weak complete resolution of a module $M$ is an acyclic resolution $T_*$ of projective modules which coincides with an ordinary projective resolution $P_*$ of $M$ in sufficiently high degree. The degree in which the two coincide is called the coincidence index. A weak complete resolution is called a strong complete resolution if $\text{Hom}_{RG}(T_*, Q)$ is acyclic for every projective module $Q$. We avoid the term “complete resolution” since some authors use it to refer to a weak complete resolution and others to a strong complete resolution.

**Proposition 2.1.** [2] Proposition 2.8] A group $G$ admits a strong complete resolution if and only if $\text{Gcd} G < \infty$

The advantage of strong complete resolutions is that given strong complete resolutions $T_*$ and $S_*$ of module $M$ and $N$, any module homomorphism $M \to N$ lifts to a morphism of strong complete resolutions $T_* \to S_*$. [10 Lemma 2.4]. Thus they can be used to define a cohomology theory: given a strong complete resolution $T_*$ of $M$ we define

$$\hat{\text{Ext}}_{RG}^*(M, -) \cong H^* \text{Hom}_{RG}(T_*, -)$$

We also set $\hat{H}^*(G, -) = \hat{\text{Ext}}^*_{RG}(R, -)$. This coincides with the complete cohomology of Mislin [30], Vogel [21], and Benson–Carlson [5] (see [10 Theorem 1.2] for a proof). Recall that the complete cohomology is itself a generalisation of the Farrell–Tate Cohomology, defined only for groups with finite virtual cohomological dimension [3, §X].

Even weak complete resolutions do not always exist, for example a free Abelian group of infinite rank cannot admit a weak complete resolution [31, Corollary 2.10]. It is conjectured by Dembegioti and Talelli that a $ZG$-module admits a weak complete resolution if and only if it admits a strong complete resolution [14, Conjecture B].

2.2. $\mathfrak{F}$-Cohomology. This section contains two technical lemmas we will need later.

If $M$ is any $RG$-module and $F_i = R\Delta^i$ is the standard $\mathfrak{F}$-split resolution of $R$ [33, p.342], then $F_* \otimes_R M$ is an $\mathfrak{F}$-split $\mathfrak{F}$-projective resolution of $M$. Thus we’ve shown:

**Lemma 2.2.** $\mathfrak{F}$-split $\mathfrak{F}$-projective resolutions exist for all $RG$-modules $M$.

There is also a version of the Horseshoe Lemma, proved as in [18, Lemma 8.2.1].

**Lemma 2.3** (Horseshoe Lemma). If

$$0 \to A \to B \to C \to 0$$

is an $\mathfrak{F}$-split short exact sequence and $P_*$ and $Q_*$ are $\mathfrak{F}$-split $\mathfrak{F}$-projective resolutions of $A$ and $C$ respectively then there is an $\mathfrak{F}$-split $\mathfrak{F}$-projective resolution $S_*$ of $B$ such that $S_i = P_i \oplus Q_i$ and there is an $\mathfrak{F}$-split short exact sequence of augmented complexes

$$0 \to \hat{P}_* \to \hat{S}_* \to \hat{Q}_* \to 0$$

2.3. Complete $\mathfrak{F}$-cohomology. Nucinkis constructs a complete $\mathfrak{F}$-cohomology in [32], we give a brief outline here. An $\mathfrak{F}$-complete resolution $T_*$ of $M$ is an acyclic $\mathfrak{F}$-split complex of $\mathfrak{F}$-projectives which coincides with an $\mathfrak{F}$-split $\mathfrak{F}$-projective resolution of $M$ in high enough dimensions. An $\mathfrak{F}$-strong $\mathfrak{F}$-complete resolution $T_*$ has $\text{Hom}_{RG}(T_*, Q)$ exact for all $\mathfrak{F}$-projectives $Q$. Given such a $T_*$ we define

$$\hat{\text{Ext}}_{RG}^*(M, -) = H^* \text{Hom}_{RG}(T_*, -)$$
\[ \hat{\mathfrak{H}}^*(G, -) = \hat{\mathfrak{E}}\text{xt}_{RG}^*(R, -) \]

Nucinkis also describes a Mislin style construction and a Benson–Carlson construction of complete $\mathfrak{H}$-cohomology defined for all groups, proves they are equivalent, and proves that whenever there exists an $\mathfrak{H}$-complete resolution they agree with the definition above.

2.4. Gorenstein Cohomology. The Gorenstein cohomology is, like the $\mathfrak{H}$-cohomology, a special case of the relative homology of Mac Lane \[29\] and Eilenberg–Moore \[15\].

Recall that a module is Gorenstein projective if it is a cokernel in a strong complete resolution. An acyclic complex $C_*$ of Gorenstein projective modules is G-proper if $\text{Hom}_{RG}(Q, C_*)$ is exact for every Gorenstein projective $Q$. The class of G-proper short exact sequences is allowable in the sense of Mac Lane \[29, \S IX.4\]. The projectives objects with respect to G-proper short exact sequences are exactly the Gorenstein projectives. For $M$ and $N$ any $RG$-modules, we define

\[ \text{GExt}_{RG}^*(M, N) = H^* \text{Hom}_{RG}(P_*, N) \]
\[ \text{G}\mathfrak{H}^*(G, N) = \text{GExt}_{RG}^*(R, N) \]

Where $P_*$ is a G-proper resolution of $M$ by Gorenstein projectives.

The usual method of producing a “Gorenstein projective dimension” of a module $M$ in this setting would be to look at the shortest length of a G-proper resolution of $M$ by Gorenstein projectives. A priori this could be larger than the Gorenstein projective dimension defined in the introduction, where the G-proper condition is not required. Fortunately there is the following theorem of Holm:

**Theorem 2.4.** \[22, Theorem 2.10\] If $M$ has finite Gorenstein projective dimension then $M$ admits a $G$-proper Gorenstein projective resolution of length $\text{Gpd} M$.

Generalising an argument of Avramov and Martsinkovsky in \[3\], Asadollahi Bahlekeh and Salarian construct a long exact sequence:

**Theorem 2.5** (Avramov–Martsinkovsky long exact sequence). \[2, Theorem 3.11\] For a group $G$ with $\text{Gcd} G < \infty$, there is a long exact sequence of cohomology functors

\[ 0 \to \text{G}H^1(G, -) \to H^1(G, -) \to \cdots \]
\[ \cdots \to \text{G}H^n(G, -) \to H^n(G, -) \to \hat{H}^n(G, -) \to G\text{H}^{n+1}(G, -) \to \cdots \]

The construction relies on the complete cohomology being calculable via a complete resolution, hence the requirement that $\text{Gcd} G < \infty$.

We will need the following lemma later:

**Lemma 2.6.** Any G-proper resolution of $R$ is $\mathfrak{H}$-split.

**Proof.** If $P_*$ is a G-proper resolution of $R$ then since $R[G/H]$ is a Gorenstein projective \[2, Lemma 2.21\],

\[ \text{Hom}_{RG}(R[G/H], P_*) \cong \text{Hom}_{RG}(R, P_*) \cong P^H_* \]

is exact, thus $P_*$ is $\mathfrak{H}$-split \[54, Remark 5.5, Lemma 5.11\]. \qed

3. $\mathfrak{H}_G$-COHOMOLOGY

3.1. Construction. We define another special case of relative homology, which we call the $\mathfrak{H}_G$-cohomology. It enables us to build an Avramov–Martsinkovsky long exact sequence of homological functors containing $\hat{\mathfrak{H}}^*$ and $\hat{\mathfrak{H}}^*$.

We define an $\mathfrak{H}_G$-projective to be the cokernel in a $\mathfrak{H}$-complete $\mathfrak{H}$-strong resolution and say a complex $C_*$ of $RG$-modules is $\mathfrak{H}_G$-proper if $\text{Hom}_{RG}(Q, C_*)$ is exact for any $\mathfrak{H}_G$-projective $Q$. The $\mathfrak{H}_G$-proper short exact sequences form an allowable class in the sense of Mac Lane, whose projective objects are the $\mathfrak{H}_G$-projectives — to
check the class of $\mathcal{G}_G$-proper short exact sequences is allowable we need only check that given a $\mathcal{G}_G$-proper short exact sequence, any isomorphic short exact sequence is $\mathcal{G}_G$-proper and that for any $RG$-module $A$ the short exact sequences 

$$0 \to A \xrightarrow{id} A \to 0 \to 0$$

and

$$0 \to 0 \to A \xrightarrow{id} A \to 0$$

are $\mathcal{G}_G$-proper.

We don’t know if the class of $\mathcal{G}_G$-projectives is precovering (see [18, §8]), so we don’t know if there always exists an $\mathcal{G}_G$-proper $\mathcal{G}_G$-projective resolution. However if $A$ and $B$ admit $\mathcal{G}_G$-proper $\mathcal{G}_G$-resolutions $P_*$ and $Q_*$ respectively then any map $A \to B$ induces a map of resolutions $P_* \to Q_*$ which is unique up to chain homotopy equivalence [29, IX.4.3] and we have a slightly weaker form of the Horseshoe Lemma, the proof of which is as in [18, 8.2.1]:

Lemma 3.1 (Horseshoe Lemma). Suppose

$$0 \to A \to B \to C \to 0$$

is a $\mathcal{G}_G$-proper short exact sequence of $RG$-modules and both $A$ and $C$ admit $\mathcal{G}_G$-proper $\mathcal{G}_G$-projective resolutions $P_*$ and $Q_*$ then there is an $\mathcal{G}_G$-proper resolution $S_*$ of $B$ such that $S_1 = P_1 \oplus Q_1$ and there is an $\mathcal{G}_G$-proper short exact sequence of augmented complexes

$$0 \to \tilde{P}_* \to \tilde{S}_* \to \tilde{Q}_* \to 0$$

For any module $M$ which admits an $\mathcal{G}_G$-proper resolution $P_*$ by $\mathcal{G}_G$-projectives we define

$$\mathcal{G}_G Ext^*_R (M, N) = H^* \text{Hom}_{RG} (P_*, N)$$

We define also

$$\mathcal{G}_G H^* (G, -) = \mathcal{G}_G Ext^*_R (R, -)$$

The next lemma follows from Lemma [18] see [18, 8.2.3].

Lemma 3.2. Suppose

$$0 \to A \to B \to C \to 0$$

is a $\mathcal{G}_G$-proper short exact sequence of $RG$-modules and both $A$ and $C$ admit $\mathcal{G}_G$-proper $\mathcal{G}_G$-projective resolutions, then there is an $\mathcal{G}_G Ext^*_R (-, M)$ long exact sequence for any $RG$-module $M$.

For any $RG$-module $M$ the $\mathcal{G}_G$ projective dimension of $G$ denoted $\mathcal{G}_G pd M$ is the minimal length of an $\mathcal{G}_G$-proper resolution of $M$ by $\mathcal{G}_G$-projectives. We set $\mathcal{G}_G cd G = \mathcal{G}_G pd R$. Note that these finiteness conditions will not be defined unless $R$ admits an $\mathcal{G}_G$-proper resolution by $\mathcal{G}_G$-projectives.

One could think of $\mathcal{G}_G$-cohomology as the “Gorenstein cohomology relative $\mathcal{G}$”.

3.2. Technical Results. We need a couple of results for the $\mathcal{G}_G$-cohomology whose analogs are well known for Gorenstein cohomology [22].

We say an $RG$-module $M$ admits a right resolution by $\mathcal{G}$-projectives if there exists an exact chain complex

$$0 \to M \to T_{-1} \to T_{-2} \to \cdots$$

where the $T_i$ are $\mathcal{G}$-projectives, $\mathcal{G}$-strong right resolutions and $\mathcal{G}$-split right resolutions are defined as for any chain complex.

Lemma 3.3. An $RG$-module $M$ is $\mathcal{G}_G$-projective if and only if $M$ satisfies

\begin{itemize}
  \item[(\ast)] $\mathcal{G} Ext^1_{RG}(M, Q) \cong 0$ for all $\mathcal{G}$-projective $Q$
\end{itemize}

for all $i \geq 1$ and $M$ admits a right $\mathcal{G}$-strong $\mathcal{G}$-split resolution by $\mathcal{G}$-projectives.
Proof. If $M$ is the cokernel of a $\mathcal{F}$-strong $\mathcal{F}$-complete resolution $T_*$ then for all $i \geq 1$ and any $\mathcal{F}$-projective $Q$,
\[
\mathfrak{F}\text{Ext}^i_{\mathcal{R}G}(M, Q) \cong H^i \text{Hom}_{\mathcal{R}G}(T^+_i, Q)
\]
Where $T^+_i$ denotes the resolution $T^+_i = T_i$ if $i \geq 0$ and $T^+_i = 0$ for $i < 0$. Then $\mathfrak{F}$ follows because $T_i$ is $\mathcal{F}$-strong.

Conversely given $\mathfrak{F}$ and an $\mathcal{F}$-strong right resolution $T^-_*$ then let $T^+_i$ be the standard $\mathcal{F}$-split resolution for $M$ (Lemma [2,2], $\mathfrak{F}$) ensures that $T^+_i$ is $\mathfrak{F}$-strong and splicing together $T^+_i$ and $T^-_i$ gives the required resolution. \hfill \Box

Lemma 3.4. If $\mathfrak{F}pd N < \infty$ and $M$ is $\mathfrak{F}G$-projective then $\mathfrak{F}\text{Ext}^i_{\mathcal{R}G}(M, N) = 0$ for all $i \geq 1$.

Proof. Let $P_* \rightarrow \rightarrow N$ be a $\mathfrak{F}$-split $\mathfrak{F}$-projective resolution then by a standard dimension shifting argument
\[
\mathfrak{F}\text{Ext}^i(M, N) \cong \mathfrak{F}\text{Ext}^{i+j}(M, K_j)
\]
where $K_j$ is the $j$th syzygy of $P_*$. Since $K_j$ is projective for $j \geq n$ the result follows from Lemma [3,3]. \hfill \Box

Proposition 3.5. Let $A$ be any $\mathcal{R}G$-module and $P_* \rightarrow \rightarrow A$ a length $n$ $\mathfrak{F}$-split resolution of $A$ with $P_i$ $\mathfrak{F}$-projective for $i \geq 1$, then $P_*$ is $\mathfrak{F}G$-proper.

Proof. The case $n = 0$ is obvious. If $n = 1$ then for any $\mathfrak{F}G$-projective $Q$, there is a long exact sequence
\[
0 \rightarrow \text{Hom}_{\mathcal{R}G}(Q, P_1) \rightarrow \text{Hom}_{\mathcal{R}G}(Q, P_0) \rightarrow \text{Hom}_{\mathcal{R}G}(Q, A) \rightarrow \mathfrak{F}\text{Ext}^1_{\mathcal{R}G}(Q, P_1) \rightarrow \cdots
\]
But $\mathfrak{F}\text{Ext}^1_{\mathcal{R}G}(Q, P_1) = 0$ by Lemma [3,3]

Assume $n \geq 2$ and let $K_*\rightarrow$ be the syzygies of $P_*$, then there is an $\mathfrak{F}$-split resolution
\[
0 \rightarrow P_n \rightarrow \cdots \rightarrow P_{i+1} \rightarrow K_i \rightarrow 0
\]
so $\mathfrak{F}pd K_i < \infty$ for all $i \geq 0$. Thus every short exact sequence
\[
0 \rightarrow K_i \rightarrow P_i \rightarrow K_{i-1}
\]
is $\mathfrak{F}G$-proper by Lemma [3,4] so $P_*$ is $\mathfrak{F}G$-proper. \hfill \Box

Lemma 3.6 (Comparison Lemma). Let $A$ and $B$ be two $\mathcal{R}G$-modules with $\mathfrak{F}$-strong $\mathfrak{F}$-split right resolutions by $\mathfrak{F}$-projectives called $S^*$ and $T^*$ respectively, then any map $f : A \rightarrow B$ lifts to a map $f_*$ of complexes as shown below:
\[
\begin{array}{cccccc}
0 & \rightarrow & A & \rightarrow & S^1 & \rightarrow & S^2 & \rightarrow & \cdots \\
& & f & \downarrow & f_1 & \downarrow & f_2 & \\
0 & \rightarrow & B & \rightarrow & T^1 & \rightarrow & T^2 & \rightarrow & \cdots
\end{array}
\]
The map of complexes is unique up to chain homotopy and if $f$ is $\mathfrak{F}$-split then so is $f_*$. \hfill \Box

Proof. The Lemma without the $\mathfrak{F}$-splitting comes from dualising [18, p.169], see also [22, Proposition 1.8].

Assume $f$ is $\mathfrak{F}$-split and consider the map of complexes restricted to $R\mathcal{H}$ for some finite subgroup $H$ of $G$. Let $i^T_*$ and $i^S_*$ denote the splittings of the top and
We may choose $\theta$ coinciding with an $F$ generality we may also assume that $F$ is $\mathcal{F}$-split. 

$\cdots$ ... $S_{i-1}$ $S_i$ ... $\cdots$ 

$\cdots$ ... $T_{i-1}$ $T_i$ ... $\cdots$ 

Let $s_i = \partial_{i-1}^S \circ s_{i-1} \circ \iota_{T_{i-1}}^T$. Then

$$f_i \circ s_i = f_i \circ \partial_{i-1}^S \circ s_{i-1} \circ \iota_{T_{i-1}}^T = \partial_{i-1}^T \circ f_{i-1} \circ s_{i-1} \circ \iota_{T_{i-1}}^T = \partial_{i-1}^T \circ \iota_{T_{i-1}}^T = \text{id}_{T_i}$$

Where the second equality is the commutativity condition coming from the fact that $f_*$ is a chain map.

3.3. An Avramov–Martsinkovsky Long Exact Sequence in $\mathcal{F}$-cohomology.

Theorem 3.7. Given an $\mathcal{F}$-strong $\mathcal{F}$-complete resolution of $R$ there is a long exact sequence

$$0 \longrightarrow \mathcal{F}H^1(G, -) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{F}H^{n-1}(G, -) \longrightarrow \mathcal{F}G H^n(G, -) \longrightarrow \mathcal{F}H^n(G, -) \longrightarrow \cdots$$

Proof. We follow the proof in [2, §3]. Consider an $\mathcal{F}$-strong $\mathcal{F}$-complete resolution $T_*$ coinciding with an $\mathcal{F}$-projective $\mathcal{F}$-split resolution $P_*$ in sufficiently high dimension. We may choose $\theta_* : T_* \longrightarrow P_*$ to be $\mathcal{F}$-split by Lemma 3.6 and without loss of generality we may also assume that $\theta_i$ is surjective for all $i$.

Truncating at position 0 and adding cokernels gives the bottom two rows of the diagram below, the row above is the row of kernels. Note that the map $A \rightarrow R$ is necessarily surjective since the maps $T_0 \rightarrow P_0$ and $P_0 \rightarrow R$ are surjective.

$$\cdots \longrightarrow 0 \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_0 \longrightarrow K \longrightarrow 0$$

$$\cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_0 \longrightarrow A \longrightarrow 0$$

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow R \longrightarrow 0$$

We make some observations about the diagram: Firstly since the module $A$ is the cokernel of a $\mathcal{F}$-strong $\mathcal{F}$-complete resolution, $A$ is $\mathcal{F}G$ projective. Secondly in degree $i \geq 0$ the columns are $\mathcal{F}$-split and the $P_i$ are $\mathcal{F}$-projective, thus the $K_i$ are $\mathcal{F}$-projective for all $i \geq 0$. Thirdly the far right vertical short exact sequence is $\mathcal{F}$-split since the degree 0 column and the rows are $\mathcal{F}$-split. Finally the top row is exact and $\mathcal{F}$-split since the other two rows are.
Apply the functor $\text{Hom}_{RG}(-, M)$ for an arbitrary $RG$-module $M$ and take homology. This gives a long exact sequence

$$\cdots \longrightarrow \mathfrak{H} H^i(G, M) \longrightarrow \mathfrak{H} H^i(G, M) \longrightarrow H^i\text{Hom}_{RG}(K, M) \longrightarrow \cdots$$

We can simplify the right hand term:

$$H^i\text{Hom}_{RG}(K, M) \cong \mathfrak{H} G\text{Ext}_G^i(K, M)$$

$$\cong \mathfrak{H} H^{i+1}(G, M)$$

Where the first isomorphism is because, by Proposition 3.5, the top row is $\mathfrak{H}G$-proper. For the second isomorphism note that the short exact sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow R \longrightarrow 0$$

is $\mathfrak{H}G$-proper by Proposition 3.5 so

$$0 \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_0 \longrightarrow A \longrightarrow R \longrightarrow 0$$

is an $\mathfrak{H}G$-proper $\mathfrak{H}G$-projective resolution of $R$. Thus the second isomorphism follows from the short exact sequence and Lemma 3.2

**Corollary 3.8.** If $R$ admits an $\mathfrak{H}$-strong $\mathfrak{H}$-complete resolution then $\mathfrak{H}G\text{cd}G < \infty$.

**Proof.** In the proof of the theorem we assumed an $\mathfrak{H}$-strong $\mathfrak{H}$-complete resolution of $R$ and built a finite length $\mathfrak{H}G$-proper resolution of $R$ by $\mathfrak{H}G$-projectives. □

**Proposition 3.9.** If the Avramov–Martsinkovsky long exact sequence and the long exact sequence of Theorem 3.7 both exist, there is a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathfrak{H} H^{n-1} & \longrightarrow & \mathfrak{H} \text{G} H^n & \longrightarrow & \mathfrak{H} H^n & \longrightarrow & \mathfrak{H} \text{G} H^{n+1} & \longrightarrow & \cdots \\
\bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \\
\cdots & \longrightarrow & \mathfrak{H} H \mathfrak{H} C^n & \longrightarrow & \mathfrak{H} \text{G} H^n & \longrightarrow & \mathfrak{H} H^n & \longrightarrow & \mathfrak{H} \text{G} H^{n+1} & \longrightarrow & \cdots \\
\end{array}$$

Where for conciseness we have written $H^n$ for $H^n(G, -)$ etc.

**Proof.** The construction of the Avramov–Martsinkovsky long exact sequence is analogous to the proof of Theorem 2.5, we give a quick sketch below as we will need the notation. Take a strong complete resolution $T'_* \rightarrow K'_*$ of $R$ coinciding with a projective resolution $P'_*$ in high dimensions and let $A'$ be the zeroth cokernel of $T'_*$. Thus $A'$ is Gorenstein projective. Again, the map $T'_* \rightarrow P'_*$ is assumed surjective and the kernel $K'_*$ is a projective resolution of $K'$, the kernel of the map $A' \longrightarrow R$. Applying $\text{Hom}_{RG}(-, M)$, for some $RG$-module $M$, to the short exact sequence of complexes

$$0 \longrightarrow K_* \longrightarrow T_* \longrightarrow P_* \longrightarrow 0$$

gives the Avramov–Martsinkovsky long exact sequence.

Let $T_*, P_*, K_*, K$ and $A$ be as defined in the proof of Theorem 2.5. There is a commutative diagram of chain complexes

$$\begin{array}{cccc}
0 & \longrightarrow & K_* & \longrightarrow & T_* & \longrightarrow & P_* & \longrightarrow & 0 \\
\bigg\uparrow & & \bigg\uparrow & & \bigg\uparrow & & \bigg\uparrow & & \\
0 & \longrightarrow & K'_* & \longrightarrow & T'_* & \longrightarrow & P'_* & \longrightarrow & 0 \\
\end{array}$$

Where the maps $\beta$ exists by the comparison theorem for projective resolutions and $\gamma$ exists by the comparison theorem for strong complete resolutions [10] Lemma 2.4. The map $\alpha$ is the induced map on the kernels. Applying $\text{Hom}_{RG}(-, M)$ for some $RG$-module $M$, and taking homology, the maps $\alpha$, $\beta$ and $\gamma$ induce the maps $\alpha_*$, $\beta_*$ and $\gamma_*$. 
Finally we construct the map $\eta_{\ast} : \mathcal{G}H^n(G, -) \to \mathfrak{f}H^n(G, -)$. Let $B_{\ast}$ be a $G$-proper Gorenstein projective resolution and recall $P_{\ast}$ is an $\mathfrak{f}$-split resolution by $\mathfrak{f}$-projectives. Then $B_{\ast}$ is $\mathfrak{f}$-split (Lemma 2.3) so there is a chain map $P_{\ast} \to B_{\ast}$ inducing $\eta_{\ast}$ on cohomology.

Commutativity is obvious for the diagram with the maps $\eta_{\ast}$ removed, leaving us with two relations to prove. Let $\varepsilon_n^G : \mathcal{G}H^n(G, -) \to H^n(G, -)$ denote the map from the commutative diagram. This is the map induced by comparison of a resolution of Gorenstein projectives and ordinary projectives $[2, 3.2, 3.11]$. We get $\beta_{\ast} \circ \eta_{\ast} = \varepsilon_n^G$, since all the maps are induced by comparison of resolutions, and such maps are unique up to chain homotopy equivalence.

The final commutativity relation, that $\eta_{\ast} \circ \alpha_{\ast} = \varepsilon_n^G$, is the most difficult to show. Here $\varepsilon_n^G : \mathfrak{f}G\mathcal{G}H^n(G, -) \to \mathfrak{f}H^n(G, -)$ denotes the map from the commutative diagram, it is induced by comparison of resolutions.

Here is a commutative diagram showing the resolutions involved:

$$
\begin{array}{cccccc}
0 & \to & K & \to & A & \to & R & \to & 0 \\
0 & \to & K' & \to & A' & \to & R & \to & 0 \\
0 & \to & K' & \to & A' & \to & R & \to & 0
\end{array}
$$

Let $L_{\ast}$ be the chain complex defined by $L_i = K_{i-1}$ for all $i \geq 1$ and $L_0 = A$, with boundary map at $i = 1$ the composition of the maps $K_0 \to K$ and $K \to A$. Thus $L_{\ast}$ is acyclic except at degree zero where $H_0L_{\ast} = R$. Similarly let $L'_{\ast}$ denote chain complex with $L'_i = K'_{i-1}$ for all $i \geq 1$ and $L'_0 = A'$ augmented by $A'$, so $L'_{\ast}$ is acyclic except at degree zero where $H_0L'_{\ast} = R$. Note that $L_{\ast}$ is an $\mathfrak{f}G$-proper resolution of $R$ by Proposition [2.3] and $L'_{\ast}$ is a $G$-proper resolution of $R$ by the Gorenstein cohomology version of the same proposition.

Recall that the maps $\varepsilon_n^G$ and $\eta_{\ast}$ are induced by comparison of resolutions: $\varepsilon_n^G$ is induced by a map $P_{\ast} \to L_{\ast}$ and $\eta_{\ast}$ is induced by a map $P_{\ast} \to L'_{\ast}$. The map $\mathfrak{f}G\text{Ext}_{RG}^i(K, -) \to \text{GExt}_{RG}^i(K', -)$ is induced by $\alpha : K'_i \to K_i$. Thus the map $\alpha_{\ast} : \mathfrak{f}G\mathcal{G}H^n(G, -) \to \mathcal{G}H^n(G, -)$ is induced by $L'_{\ast} \to L_{\ast}$. The diagram below is the one we must show commutes.

$$
\begin{array}{ccc}
\mathfrak{f}G\mathcal{G}H^n(G, -) & \cong & \mathcal{G}H^n(G, -) \\
\downarrow^{\alpha_{\ast}} & & \downarrow^{\eta_{\ast}} \\
\mathfrak{f}H^n(G, -) & \simeq & \mathcal{G}H^n(G, -) \\
\mathfrak{f}G\mathcal{G}H^n(G, -) & \cong & \mathcal{G}H^n(G, -)
\end{array}
$$

Since the composition $P_{\ast}$ to $L'_i$ to $L_{\ast}$ is a map of resolutions from $P_{\ast}$ to $L_{\ast}$, and such maps are unique up to chain homotopy equivalence, this completes the proof. □

**Corollary 3.10.** Given an $\mathfrak{f}$-strong $\mathfrak{f}$-complete resolution of $R$, $\text{Gcd } G = n < \infty$ implies $\mathfrak{f}H^i(G, -)$ injects into $\mathfrak{f}H^i(G, -)$ for all $i \geq n + 1$. 

Proof. \( \text{Gcd} \ G < \infty \) implies the Avramov–Martinskovsky long exact sequence exists (Theorem 2.5). Consider the the commutative diagram of Proposition 3.9. The map 
\[ \phi : \mathfrak{F}H^i(G, -) \rightarrow \mathfrak{F}H^i(G, -) \]
factors as \( \eta_i \circ \alpha_i = 0 \), so since \( \mathfrak{G}H^i(G, -) = 0 \) for all \( i \geq n + 1 \), \( \mathfrak{F}H^i(G, -) \) injects into \( \hat{\mathfrak{F}}H^i(G, -) \) for all \( i \geq n + 1 \).

\[ \square \]

**Theorem 3.11.** If \( \mathfrak{cd} G < \infty \) then \( \mathfrak{cd} G = \text{Gcd} \ G \).

**Proof.** We know already that \( \text{Gcd} \ G \leq \mathfrak{cd} G \) (see Section 1). If \( \mathfrak{cd} G < \infty \) then it is trivially true that \( \mathfrak{F} \) admits an \( \mathfrak{F} \)-strong \( \mathfrak{F} \)-complete resolution, thus \( \mathfrak{F}H^i(G, -) \) injects into \( \hat{\mathfrak{F}}H^i(G, -) \) for all \( i \geq \text{Gcd} \ G + 1 \), but \( \hat{\mathfrak{F}}H^i(G, -) \) is always zero since \( \mathfrak{cd} G < \infty \) \[24, 4.1(i)\].

\[ \square \]

**Example 3.12.** Let \( R = \mathbb{Z} \) for this example. Kropholler introduced the class \( H_\mathfrak{F} \) of hierarchically decomposable groups in \[24\] as the smallest class of groups such that if there exists a finite dimensional contractible \( G \)-CW complex with stabilisers in \( H_\mathfrak{F} \) then \( G \in H_\mathfrak{F} \). Let \( H_\mathfrak{F}_b \) denote the subclass of \( H_\mathfrak{F} \) containing groups with a bound on the orders of their finite subgroups.

The \( \mathbb{Z} \mathbb{G} \)-module \( B(G, \mathbb{Z}) \) of bounded functions from \( G \) to \( \mathbb{Z} \) was first studied in \[23\]. Kropholler and Mislin proved that if \( G \in H_\mathfrak{F}_b \) then \( \mathfrak{F}G \leq \mathfrak{cd} G \) \[25\].

Since Gorenstein cohomological dimension is subadditive under extensions \[4, \text{Remark 2.9(2)}\], an application of Theorem 3.11 removes the condition on the orders of finite subgroups:

**Corollary 4.2.** Given a short exact sequence of groups

\[ 1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1 \]

such that every finite index overgroup of \( N \) in \( G \) has a bound on the orders of the finite subgroups not contained in \( N \). If \( \mathfrak{cd} G < \infty \) then \( \mathfrak{cd} G \leq \mathfrak{cd} N + \mathfrak{cd} Q \).

**Remark 4.3.** Even in the case that \( \mathfrak{cd} Q < \infty \) and \( N \) is finite it is unknown if \( \mathfrak{cd} G < \infty \). However, if it fails in such a case then it necessarily fails when \( N \) is a cyclic group of order \( p \) \[34, \text{Lemma 6.10}\].
5. Rational Cohomological Dimension

For this section, let $R = \mathbb{Z}$. Gandini has shown that for groups in $\mathbf{H}_3$, $\text{cd}_Q G \leq \text{Gcd } G$ [14, Remark 4.14] and this is the only result we are aware of relating $\text{cd}_Q G$ and $\text{Gcd } G$. In Proposition 5.3 we show that $\text{cd}_Q G \leq \text{Gcd } G$ for all groups with $\text{cd}_Q G < \infty$. Recall there are examples of torsion-free groups with $\text{cd}_Q G < \text{cd}_Z G$ [12, Example 8.5.8] and $\text{Gcd } G = \text{cd}_Z G$ whenever $\text{cd}_Z G < \infty$ [2, Corollary 2.9], so we cannot hope for equality of $\text{cd}_Q G$ and $\text{Gcd } G$ in general.

Question 5.1. Are there groups $G$ with $\text{Gcd } G < \infty$ but $\text{cd}_Q G = \infty$?

Lemma 5.2. For any group $G$, $\text{silp } Q G \leq \text{silp } Z G$.

Proof. By [16, Theorem 4.4], $\text{silp } Q G = \text{silp } Q G$ and $\text{silp } Z G = \text{silp } Z G$. Combining with [20, Lemma 6.4] that $\text{silp } Q G \leq \text{silp } Z G$ gives the result. \hfill \Box

Lemma 5.3. If $\text{Gcd } G < \infty$ then for any $\mathbb{Q}$-$G$-module $M$ there is a natural isomorphism

$$\hat{H}^*(G, M) \otimes \mathbb{Q} \cong \hat{\text{Ext}}^*_Q (\mathbb{Q}, M)$$

Proof. Let $T_\ast$ be a strong complete resolution of $\mathbb{Z}$ by $\mathbb{Z}$-$G$-modules, then $T_\ast \otimes \mathbb{Q}$ is a strong complete resolution of $\mathbb{Q}$ by $\mathbb{Q}$-$G$-modules. By an obvious generalisation of [31, Lemma 2.2], if $\text{silp } Q G \leq \infty$ then any complete $\mathbb{Q}$-$G$-module resolution is a strong complete $\mathbb{Q}$-$G$-module resolution, so since $\text{silp } Q G < \text{silp } Z G < \infty$, $T_\ast \otimes \mathbb{Q}$ is a strong complete resolution. This gives a chain of isomorphisms for any $\mathbb{Q}$-$G$-module $M$:

$$\hat{H}^*(G, M) \otimes \mathbb{Q} \cong H^* \text{Hom}_{\mathbb{Z}G}(T_\ast, M) \otimes \mathbb{Q}$$

$$\cong H^* \text{Hom}_{\mathbb{Q}G}(T_\ast \otimes \mathbb{Q}, M)$$

$$\cong \hat{\text{Ext}}^*_Q (\mathbb{Q}, M)$$

\hfill \Box

Proposition 5.4. If $\text{cd}_Q G < \infty$ then $\text{cd}_Q G \leq \text{Gcd } G$.

Proof. There is nothing to show if $\text{Gcd } G = \infty$ so assume that $\text{Gcd } G < \infty$. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, tensoring the Avramov–Martsinkovsky long exact sequence with $\mathbb{Q}$ preserves exactness. Combining this with Lemma 5.3 and the well known fact that for any $\mathbb{Q}$-$G$-module $M$ there is a natural isomorphism [6, p.2]

$$H^*(G, M) \otimes \mathbb{Q} \cong \text{Ext}^*_Q (\mathbb{Q}, M)$$

gives the long exact sequence

$$\cdots \longrightarrow GH^i(G, M) \otimes \mathbb{Q} \longrightarrow \text{Ext}^i_Q (\mathbb{Q}, M) \longrightarrow \hat{\text{Ext}}^i_{\mathbb{Q}G} (\mathbb{Q}, M) \longrightarrow \cdots$$

Since $\text{cd}_Q G < \infty$, we have that $\hat{\text{Ext}}^i_{\mathbb{Q}G} (\mathbb{Q}, M) = 0$ [24, 4.1(i)]. Thus there is an isomorphism for all $i$,

$$G H^i(G, M) \otimes \mathbb{Q} \cong \text{Ext}^i_Q (\mathbb{Q}, M)$$

and the result follows. \hfill \Box

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