CONCAVITY AND CONVEXITY OF THE GROUND STATE ENERGY

HERBERT KOCH

Abstract. This note proves convexity resp. concavity of the ground state energy of one dimensional Schrödinger operators as a function of an endpoint of the interval for convex resp. concave potentials.

1. Main result and context

Let $I = (a, b) \subset \mathbb{R}$ an open interval, $V \in C(a, b)$ be a convex or concave potential with $\liminf_{t \to -\infty} V = \infty$ if $a = -\infty$. Consider for $t \in (a, b]$ the energy

$$E_t(u) = \int_a^t u_x^2 + Vu^2 \, dx.$$ 

There is a unique positive minimizer $u \in H_0^1(a, t)$ under the constraint $\|u\|_{L^2(a, t)} = 1$. It satisfies the Euler-Lagrange equation

$$-u_{xx} + Vu = \lambda(t) u$$

on $(a, t)$ with boundary conditions $u(a) = u(t) = 0$ (and obvious modifications if $a = -\infty$). Here $\lambda(t)$ is the Lagrangian multiplier, and $\lambda(t) = E_t(u)$. The map $t \to \lambda(t)$ is the main object of interest.

**Theorem 1.** The map $(a, b] \ni t \to \lambda(t)$ is twice differentiable, strictly decreasing and $\lim_{t \to a} \lambda(t) = \infty$. The map $t \to \lambda(t)$ is convex if $V$ is convex, strictly convex if $V$ is convex and not affine, concave if $V$ is concave and strictly concave if $V$ is concave and not affine.

The convexity part follows from a much stronger celebrated result by Brascamp and Lieb [3]. It is related to a weaker statement in Friedland and Hayman [5] with a computer based proof there. These statements found considerable interest and use in the context of monotonicity formulas beginning with the seminal work of Alt, Caffarelli and Friedman [1]. Caffarelli and Kenig [4] prove a related monotonicity formula using the results by Brascamp-Lieb [3]. They attribute an analytic proof to Beckner, Kenig and Pipher [2] which the author has never seen. To the best knowledge of the author the the concavity statements are new.

This note has its origin in a seminar of free boundary problems at Bonn. It is a pleasure to acknowledge that it would not exist without my coorganizer Wenhui Shi.

2. A short elementary proof

**Proof.** Monotonicity and $\lim_{t \to a} \lambda(t) = \infty$ are an immediate consequence of the definition. We consider the equation (1) on the interval $(a, t)$ and denote by $u(x) = u(x, t)$ the unique $L^2$ normalized non negative ground state with ground
state energy $\lambda = \lambda(t)$. Differentiability with respect to $x$ and $t$ is an elementary property of ordinary differential equations. We argue at a formal level and do not check existence of integrals resp. derivatives below, which follows from standard arguments. We differentiate the equation with respect to $t$, denote the derivative of with respect to $t$ by $\dot{u}$ and obtain
\begin{equation}
-\frac{d^2 \dot{u}}{dx^2} + V \dot{u} - \lambda \dot{u} = \dot{\lambda} u \tag{2}
\end{equation}
with boundary conditions $\dot{u}(a) = 0$ and $\dot{u}(t) = -u_x(t)$. We multiply (2) by $u$ and integrate. Since $\|u\|_{L^2} = 1$ we obtain
\begin{equation}
\dot{\lambda} = \dot{u}(t)u_x(t) = -u_x^2(t) \tag{3}
\end{equation}
Due to the normalization $\dot{u}$ is orthogonal to $u$, i.e. $\int_a^t u \dot{u} \, dx = 0$. The quotient $w = \frac{\dot{u}}{u}$ satisfies
\begin{equation}
\frac{u_x}{u}w_{xx} + \frac{u_x^2}{u^2}w - \frac{u_x^2}{u^2}w = \dot{\lambda} < 0.
\end{equation}
In particular $w$ has no non positive local minimum. Since $w \to \infty$ as $x \to t$ there can be at most one sign change. Since $\dot{u}$ is orthogonal to $u$ there is exactly one sign change of $\dot{u}$, lets say at $t_0 < t$. We multiplying (1) by $u_x$ and integrate to get
\begin{equation}
\dot{\lambda} = -u_x(t)^2 = \frac{1}{2} \int_a^t V' u^2 \, dx.
\end{equation}
Using the orthogonality $\int u \dot{u} \, dx = 0$ we obtain an partly implicit formula for the second derivative of $\lambda$ with respect to $t$,
\begin{equation}
\ddot{\lambda} = \int_a^t V' u \, dx = \int_a^t (V'(x) - V'(t_0))wu^2 \, dx.
\end{equation}
Thus $t \to \lambda$ is convex if $V$ is convex, it satisfies $\ddot{\lambda} > 0$ if $V$ is convex and not affine (i.e. $V'$ is not constant), it is concave if $V$ is concave and $\ddot{\lambda} < 0$ if $V$ is concave and not affine.

\begin{thebibliography}{9}
\bibitem{1} Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman. Variational problems with two phases and their free boundaries. \textit{Trans. Amer. Math. Soc.}, 282(2):431–461, 1984.
\bibitem{2} W. Beckner, C.E. Kenig, and J. Pipher. A convexity property for gaussian measures. 1998.
\bibitem{3} Herm Jan Brascamp and Elliott H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. \textit{J. Functional Analysis}, 22(4):366–389, 1976.
\bibitem{4} Luis A. Caffarelli and Carlos E. Kenig. Gradient estimates for variable coefficient parabolic equations and singular perturbation problems. \textit{Amer. J. Math.}, 120(2):391–439, 1998.
\bibitem{5} S. Friedland and W. K. Hayman. Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions. \textit{Comment. Math. Helv.}, 51(2):133–161, 1976.
\end{thebibliography}