KK–LIFTING PROBLEM FOR DIMENSION DROP INTERVAL ALGEBRAS

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Abstract. In this paper, we investigate KK-theory of (generalized) dimension drop interval algebras (with possibly different dimension drops at the endpoints), especially on the problem that which KK-class is representable by a $\ast$-homomorphism between two such C*-algebras (allowing tensor product with matrix for the codomain algebra). This lifting problem makes sense on its own in KK-theory, and also has application on the classification of C*-algebras which are inductive limits of these building blocks. It turns out that when the dimension drops at the two endpoints are different, there exist KK-elements which preserve the order structure defined by M. Dadarlat and T. A. Loring in [3] on the mod $p$ K-theory, but fails to be lifted to $\ast$-homomorphism. This is different from the equal dimension drops case as shown by S. Eliers in [6].

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1. INTRODUCTION

The Elliott program has been a very active research program in the area of operator algebra. A great number of simple C*-algebras have been classified by the standard Elliott invariant. For non-simple case, however, as soon as there is torsion in $K_1$, even if the algebra has real...
rank zero, the graded ordered $K$-group is no longer sufficient. This was shown first by G. Gong in [9] for the case of approximately homogeneous $C^*$-algebras (AH-algebras), and was shown by M. Dadarlat and T. A. Loring in [3] for the case of approximately (classical) dimension drop $C^*$-algebras considered by Elliott in [7]. (In other words, while the main theorem of [7] was correct as it dealt with circles, it was not correct in the setting of dimension drop algebras, except in the simple case — in which case, interestingly, the proof was also essentially correct.)

M. Dadarlat and T. A. Loring did prove an isomorphism theorem for real rank zero inductive limits of (classical) dimension drop algebras, using the total $K$-theory group (i.e., $K$-theory with coefficient) introduced by G. Gong in [9] (for the purpose of giving a counterexample), together with a new order structure which they defined in a completely general text, but they had to assume that an order isomorphism arose from a $KK$-element. For the case of bounded torsion in $K_1$-group, this assumption was avoided by S. Eilers in [6]; and in [4], M. Dadarlat and T. A. Loring showed that the $KK$-element existed in very great generality (the well-known universal multicoefficient theorem). This was extended by M. Dadarlat and G. Gong to include a classification of the AH-algebras considered by Gong in [9] (and by Elliott and Gong in [EG] in the simple case).

All this refers only to the original (classical) dimension drop interval algebras introduced by Elliott in [7]. The non-completeness of the order structure defined by Elliott on the $K^*$-group is essentially related to the $KK$-lifting problem in real rank zero setting. Namely, there exist $KK$-elements between the classical dimension drop algebras which preserve this order structure, but fail to be represented by a $\ast$-homomorphism between two such algebras. The new order structure on $K$-theory with coefficient defined by M. Dadarlat and T. A. Loring together with the so called Bockstein operations (see also [3]) can exactly solve this $KK$-lifting question.

Now, we work on the generalized dimension drop algebras, of course we need to investigate the $KK$-lifting problem between two such algebras. The main purpose of the present paper is to show that generalized dimension drop interval algebras, with different dimension drops at the two ends of the interval, raise a new problem. As we show, the existence theorem with the Dadarlat-Loring order structure on total $K$-theory fails at the level of building blocks.

**Theorem 1.1.** For generalized dimension drop algebras $A_m = I[m_0, m, m_1]$ and $B_n = I[m_0, n, m_1]$ with $(m_0, m_1) = 1$, there exists $KK$-elements
in $KK(A_m, B_n)$ (certain linear combinations of horizontal eigenvalue patterns with non symmetric coefficients), such that they preserve the Dadarlat-Loring order structure on total K-theory, but fail to be lifted to a $*$-homomorphism between $A_m$ and $B_n$.

To achieve this, we realize each KK-element on both ordered K-groups with coefficient and ordered K-homology groups of two such algebras. On one hand, we need to calculate the K-groups with coefficient for the generalized dimension drop algebras, the positive cone and all the related Bockstein operations. Moreover, we need to have complete analysis on the structure of induced elements of KK-elements on ordered K-groups with coefficient, from this analysis we can obtain the condition under which a KK-element can preserve the Dadarlat-Loring order structure. On the other hand, X. Jiang and H. Su studied these generalized dimension drop interval algebras in [10], they gave a criterion for KK-lifting problem between two such algebras. Their criterion is some positivity on the K-homology groups of these algebras. By our analysis, the different dimension drops at the two endpoints give us flexibility for KK-elements to preserve the Dadarlat-Loring order but do not satisfy Jiang-Su’s criterion for KK-lifting. Therefore, we succeed in finding the KK-elements in the main theorem above.

The paper is organized as follows. In section 2, some preliminaries are given about the generalized dimension drop algebras and the mod $p$ K-theory, including the Dadarlat-Loring order structure and all the Bockstein operations we need. In section 3, we calculate the K-theory with coefficient for generalized dimension drop algebras, including the positive cone, etc. In section 4, we make analysis on the behavior of KK-elements on K-theory with coefficient, and figure out the structure of morphisms on K-theory with coefficient. In section 5, we investigate the KK-lifting problem for generalized dimension drop algebras, and prove the main Theorem [11] namely, give examples of KK-elements which preserve the Dadarlat-Loring order but fail be lifted to a $*$-homomorphism.

2. Notation and preliminaries

Definition 2.1. A dimension drop algebra, denoted by $I[m_0, m, m_1]$, is a C*-algebra of the form:

$$I[m_0, m, m_1] = \{ f \in C([0, 1], M_m) : f(0) = a_0 \otimes \text{id}_{m_0}, f(1) = \text{id}_{m_1} \otimes a_1 \},$$

where $a_0$ and $a_1$ belong to $M_{m_0}(\mathbb{C})$ and $M_{m_1}(\mathbb{C})$ respectively.

Note that the classical dimension drop interval algebras, both the non-unital one $I_p = \{ f \in C([0, 1], M_p) : f(0) = 0, f(1) \in \mathbb{C} \}$ and
the unital one $\tilde{I}_p = \{ f \in C([0,1], M_p) : f(0), f(1) \in \mathbb{C} \}$ are included in the definition above. For the (generalized) dimension drop interval algebra, the two singular irreducible representations at the endpoints $V_0(f) = a_0$ and $V_1(f) = a_1$ are important for further analysis, these two representations exactly reflect the information of different dimension drops.

Moreover, there are four basic $\ast$-homomorphisms $\delta_0$, $\delta_1$, $id_{m,n}$, and $\overline{id}_{m,n}$ between such dimension drop interval algebras which play central roles later, we introduce them here:

$$\delta_i : I[m_0, m, m_1] \to I[m_0, n, m_1] \otimes M_{m_i}, \ i = 0, 1,$$

is defined by

$$\delta_i(f) = \begin{pmatrix} V_i(f) & V_i(f) & \ldots & V_i(f) \end{pmatrix},$$

we need $(m_0, m_1) = 1$ for this.

$$id_{m,n} : I[m_0, m, m_1] \to I[m_0, n, m_1] \otimes M_{m\frac{m}{(m,n)}}$$

is defined by

$$id_{m,n}(f)(t) = \begin{pmatrix} f(t) & f(t) & \ldots & f(t) \end{pmatrix},$$

where $f(t)$ repeats $\frac{n}{(m,n)}$ times.

$$\overline{id}_{m,n} : I[m_0, m, m_1] \to I[m_0, n, m_1] \otimes M_{m\frac{m}{(m,n),s}},$$

is defined by

$$\overline{id}_{m,n}(f)(t) = \begin{pmatrix} f(1-t) & f(1-t) & \ldots & f(1-t) \end{pmatrix},$$

where $s = \frac{m}{m_0m_1}$, we also need $(m_0, m_1) = 1$. Note that the non symmetric sizes appear above are also caused by the different dimension drops at the two endpoints.

In the next, we give a few basic preliminaries about K-theory with coefficient.

Given a natural number $p \geq 2$ (not necessarily a prime), denote $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$, the mod $p$ K-theory for C*-algebras was studied by [2], [13] and
Definition 2.2. For any C*-algebra $A$, and any natural number $p \geq 2$, the mod $p$ $K$-theory is defined as follows:

(i) $K_0(A;\mathbb{Z}_p) \cong KK(I_p, A),$

(ii) $K_0(A; \mathbb{Z}_p) \cong KK(\bar{I}_p, A).$

Similarly, the mod $p$ $K_1$-group can also be defined.

As we mentioned in the introduction, the crucial thing about mod $p$ $K$-theory is the order structure defined by M. Dadarlat and T. A. Loring on $K_0(A; G_p)$, and together with the Bockstein operations (see [3]).

Definition 2.3. (Dadarlat-Loring order)

$K_0^+(A; G_p) \triangleq \{([\varphi(1)], [\varphi|_{I_p}]) | \varphi \in Hom(\bar{I}_p, M_k(A)) \text{ for some integer } k\}.$

Lemma 2.4. (see [3]) There is a natural short exact sequence of groups:

$K_0(A) \xrightarrow{\times p} K_0(A) \xrightarrow{\mu_{A,p}} K_0(A; \mathbb{Z}_p) \xrightarrow{\nu_{A,p}} K_1(A) \xrightarrow{\times p} K_1(A),$

where $p \geq 2$, $\mu_{A,p}, \nu_{A,p}$ are the Bockstein operations defined by the Kasparov product with the element of $KK(I_p, \mathbb{C})$ given by the evaluation $\delta_1 : I_p \to \mathbb{C}$ and the element of $KK^1(\mathbb{C}, I_p)$ given by the inclusion $i : SM_p \to I_p$ respectively.

Lemma 2.5. For any $KK$-element $\alpha \in KK(A, B)$, where $A, B$ are two C*-algebras, then $\alpha$ induces the following commutative diagram:

$$
\begin{array}{ccc}
K_0(A) & \xrightarrow{\times p} & K_0(A) \\
\downarrow K_0(\alpha) & & \downarrow K_0(\alpha;\mathbb{Z}_p) \\
K_0(B) & \xrightarrow{\times p} & K_0(B)
\end{array}
\quad
\begin{array}{ccc}
K_0(A;\mathbb{Z}_p) & \xrightarrow{\nu_{A,p}} & K_1(A) \\
\downarrow K_0(\alpha;\mathbb{Z}_p) & & \downarrow K_1(\alpha) \\
K_0(B;\mathbb{Z}_p) & \xrightarrow{\nu_{B,p}} & K_1(B)
\end{array}
\quad
\begin{array}{ccc}
K_1(A) & \xrightarrow{\times p} & K_1(A) \\
\downarrow K_1(\alpha) & & \downarrow K_1(\alpha) \\
K_1(B) & \xrightarrow{\times p} & K_1(B)
\end{array}
$$

Proof. This follows from the associativity of the Kasparov product. □

We will denote by $Hom(K(A; p), K(B; p))$ the group of the triples $(x, \varphi, y)$ such that the diagram above commutes. For each $\alpha \in KK(A, B)$, the induced triple

$\Gamma(\alpha; p) = (K_0(\alpha), K_0(\alpha;\mathbb{Z}_p), K_1(\alpha))$
lives in $\text{Hom}(K(A; p), K(B; p))$.

3. K-Theory with Coefficient for Generalized Dimension Drop Interval Algebras

In previous section, we have four $\ast$-homomorphism between two generalized dimension drop interval algebras. We also have these homomorphisms from $\tilde{I}_p$ to $A_m = I[m_0, m, m_1]$ as follows: $\delta_0, \delta_1, id,$ and $\overline{id}$.

$\delta_i : \tilde{I}_p \to I[m_0, m, m_1], i = 0, 1,$ is defined by

$$\delta_i(f) = \begin{pmatrix} V_i(f) \\ V_i(f) \\ \ddots \\ V_i(f) \end{pmatrix}.$$  

$V_i(f)$ repeats $m$ times.

$id : \tilde{I}_p \to I[m_0, m, m_1] \otimes M_{p(m,p)}$ is defined by

$$id(f)(t) = \begin{pmatrix} f(t) \\ f(t) \\ \ddots \\ f(t) \end{pmatrix},$$  

where $f(t)$ repeats $\frac{m}{(m,p)}$ times.

$\overline{id} : \tilde{I}_p \to I[m_0, m, m_1] \otimes M_{p(m,p)}$ is defined by

$$\overline{id}(f)(t) = \begin{pmatrix} f(1-t) \\ f(1-t) \\ \ddots \\ f(1-t) \end{pmatrix},$$  

where $f(1-t)$ also repeats $\frac{m}{(m,p)}$ times.

Before the calculation of the K-theory with coefficient for generalized dimension drop interval algebras $A_m = I[m_0, m, m_1]$, we need the following lemma in [3].

**Lemma 3.1.** For the complex number $\mathbb{C}$, we have that $K_0(\mathbb{C}; G_p) = \mathbb{Z} \oplus \mathbb{Z}_p$, and $K_0^+(\mathbb{C}; G_p) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z}_p \mid a \geq b\}$. Moreover, we have the identification given by $[\delta_0] = (1, 0)$, and $[\delta_1] = (1, 1)$.
**Remark 3.2.** Note that we not only use this result, but also the way of identification, namely, for every $*$-homomorphism $\phi : \tilde{I}_p \to M_n(\mathbb{C})$, by using its irreducible representation, we assume $\phi$ can be decomposed into $c_0$ copies of $\delta_0$, $c_1$ copies of $\delta_1$, and $m$ copies of $\delta_t$ for $t \in (0, 1)$, then $[\phi] = (c_0 + c_1 + mp, c_1)$.

**Theorem 3.3.** For any natural number $p \geq 2$, we have that $K_0(A_m; G_p) = \{(b', \tilde{b}, c', \tilde{c}) \in (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus \mathbb{Z}_p) \mid \frac{m}{m_1} c' - \frac{m}{m_0} b' = 0, \frac{m}{m_1} c - \frac{m}{m_0} b \in p\mathbb{Z} \} \cong \mathbb{Z} \oplus Z(m, p)$, where $Z(m, p) = \{(b, \tilde{c}) \in \mathbb{Z}_p \oplus \mathbb{Z}_p \mid \frac{m}{m_1} c - \frac{m}{m_0} b \in p\mathbb{Z}\}$, and the isomorphism is given by $(a, \tilde{b}, \tilde{c}) \in \mathbb{Z} \oplus Z(m, p) \rightarrow (a \frac{m_0}{m_1}, \tilde{b}, a \frac{m_1}{m_0}, \tilde{c})$.

Then, $K_0'(A_m; G_p) = \{(a, \tilde{b}, \tilde{c}) \in \mathbb{Z} \oplus Z(m, p) \mid a \frac{m_0}{m_1} \geq b, a \frac{m_1}{m_0} \geq \tilde{c}\}$.

And, we have the following identifications:

$[\delta_0] = ((m_0, m_1), 0, 0), \ [\delta_1] = ((m_0, m_1), \tilde{m}_0, \tilde{m}_1), \ [\text{id}] = \frac{p}{(p, m)}((m_0, m_1), 0, \tilde{m}_1), \ [\tilde{\text{id}}] = \frac{p}{(p, m)}((m_0, m_1), \tilde{m}_0, 0)$.

If we further assume that $(m_0, m_1) = 1$ and $m | p$, then $[\delta_0], [\delta_1], \text{ and } [\text{id}]$ generate $K_0(A_m; G_p)$. Then, the Bockstein operations are given by

$\mu_{A_m; p} = \left(\frac{m_0}{m_1}\right), \text{ and } \nu_{A_m; p} = (-\frac{m}{mp_0}, \frac{m}{pm_1})$.

**Proof.** Consider the following short exact sequence

$$0 \to SM_m \to I[m_0, m, m_1] \to M_{m_0} \oplus M_{m_1} \to 0, \quad (*)$$

it induces the following six term exact sequence for KK-groups:

$$\begin{array}{cccccc}
\text{KK}(\tilde{I}_p, SM_m) & \longrightarrow & \text{KK}(\tilde{I}_p, I[m_0, m, m_1]) & \longrightarrow & \text{KK}(\tilde{I}_p, M_{m_0} \oplus M_{m_1}) \\
\downarrow \partial & & \downarrow \partial & & \\
\text{KK}^1(\tilde{I}_p, M_{m_0} \oplus M_{m_1}) & \longleftarrow & \text{KK}^1(\tilde{I}_p, I[m_0, m, m_1]) & \longleftarrow & \text{KK}^1(\tilde{I}_p, SM_m). \\
\end{array}$$

We know that $\text{KK}(\tilde{I}_p, SM_m) = K^1(\tilde{I}_p) = 0$, see for example ([10], Lemma 3.1), and

$\text{KK}(\tilde{I}_p, M_{m_0} \oplus M_{m_1}) = (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus \mathbb{Z}_p)$, $\text{KK}^1(\tilde{I}_p, SM_m) = \mathbb{Z} \oplus \mathbb{Z}_p$

by Lemma 3.1. Hence by exactness, we obtain that

$\text{KK}(\tilde{I}_p, I[m_0, m, m_1]) = \text{Ker} \partial$, $\text{KK}^1(\tilde{I}_p, I[m_0, m, m_1]) = (\mathbb{Z} \oplus \mathbb{Z}_p)/\text{Im} \partial$. 
To calculate $KK(\tilde{I}_p, I[m_0, m, m_1])$, it is enough to figure out the index map $\partial$ on the right hand of the diagram above.

To do this, we use the identification in Lemma 3.1, assume $(b', \bar{b}, c', \bar{c}) \in (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus (\mathbb{Z}_p))$, since the index map $\partial$ is induced by the extension $(*)$, one gets that

$$\partial(b', \bar{b}, c', \bar{c}) = \left( \frac{m}{m_1} c' - \frac{m}{m_0} b', \frac{m}{m_1} c - \frac{m}{m_0} b \right).$$

Hence,

$$KK(\tilde{I}_p, I[m_0, m, m_1]) = \{(b', \bar{b}, c', \bar{c}) \mid \frac{m}{m_1} c' - \frac{m}{m_0} b' = 0, \frac{m}{m_1} c - \frac{m}{m_0} b \in p\mathbb{Z}\}.$$  

Similarly, $KK^1(\tilde{I}_p, I[m_0, m, m_1])$ can also be calculated, but we don’t need it here.

Let $H = \{(b', c') \in \mathbb{Z} \oplus \mathbb{Z} \mid \frac{m}{m_1} c' - \frac{m}{m_0} b' = 0\}$, then $H \cong \mathbb{Z}$ via the map $(b', c') \mapsto \frac{c' + b'}{m_0 + m_1}(m_0, m_1)$, so $(\frac{m}{m_0}(m_0, m_1), \frac{m}{m_1}(m_0, m_1))$ corresponds to $1 \in \mathbb{Z}$, then the inverse of this map is $a \in \mathbb{Z} \mapsto (a \frac{m_0}{m_0(m_1)}, a \frac{m_1}{m_0(m_1)}).$

Therefore, by the identification od Lemma 3.1 we obtain the positive cone of $K_0(A_m; G_p)$:

$$K_0^+(A_m; G_p) = \{(a, \bar{b}, \bar{c}) \in \mathbb{Z} \oplus \mathbb{Z}(m, p) \mid a \frac{m_0}{m_0(m_1)} \geq b, a \frac{m_1}{m_0(m_1)} \geq c\}.$$  

To see the corresponding elements for the homomorphisms $\delta_0$, $\delta_1$, $id$, and $\overline{id}$, we just need to go through the identification of Lemma 3.1. This is straightforward, but easily confused, so we spell out these calculations here for the convenience of readers.

For $\delta_0 : \tilde{I}_p \rightarrow I[m_0, m, m_1]$, apply the quotient map $\pi : I[m_0, m, m_1] \rightarrow M_{m_0} \oplus M_{m_1}$, we get

$$\pi(\delta_0(f)) = \left( \begin{array}{ccc} V_0(f) & \cdots & V_0(f) \\ \cdots & \cdots & \cdots \\ V_0(f) & \cdots & V_0(f) \end{array} \right) \oplus \left( \begin{array}{ccc} V_0(f) & \cdots & V_0(f) \\ \cdots & \cdots & \cdots \\ V_0(f) & \cdots & V_0(f) \end{array} \right),$$

$V_0(f)$ repeats $m_0$ times in the first direct summand, and repeats $m_1$ times in the second direct summand. So by the identification in Lemma 3.1 this gives us the group element $(m_0, \bar{0}, m_1, \bar{0}) \in (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus (\mathbb{Z}_p))$; Similarly, $\delta_1$ gives us the group element $(m_0, \bar{m}_0, m_1, \bar{m}_1) \in (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus (\mathbb{Z}_p))$. For the map $id : \tilde{I}_p \rightarrow M_{(m_0, m_1)}(I[m_0, m, m_1])$, apply the
quotient map $\pi$, we get

$$
\pi(id(f)) = \begin{pmatrix}
V_0(f) & \cdots & V_0(f) \\
\vdots & & \vdots \\
V_1(f) & \cdots & V_1(f)
\end{pmatrix} + \begin{pmatrix}
\cdots & \cdots & \cdots \\
V_0(f) & \cdots & V_0(f) \\
\vdots & & \vdots \\
V_1(f) & \cdots & V_1(f)
\end{pmatrix},
$$

$V_0(f)$ repeats $\frac{pm_0}{(p,m)}$ times in the first direct summand, and $V_1(f)$ repeats $\frac{pm_1}{(p,m)}$ times in the second direct summand. Hence, it gives the element $\frac{p}{(p,m)}(m_0, \bar{0}, m_1, \bar{m}_1) \in (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus \mathbb{Z}_p)$. Similarly, the map $\underline{id}$ corresponds the element $\frac{p}{(p,m)}(m_0, 0, m_1, \bar{0}) \in (\mathbb{Z} \oplus \mathbb{Z}_p) \oplus (\mathbb{Z} \oplus \mathbb{Z}_p)$.

If we further assume that $(m_0, m_1) = 1$, and $m|p$, then

$$
[\delta_0] = (1, \bar{0}, 0), [\delta_1] = (1, \bar{m}_0, \bar{m}_1), [id] = \frac{p}{m}(1, \bar{0}, \bar{m}_1), [\underline{id}] = \frac{p}{m}(1, \bar{m}_0, 0).
$$

Moreover, $(\bar{m}_0, \bar{m}_1)$ is the smallest choice of non-negative pairs $(b, c)$ such that $\frac{m}{m_1}c - \frac{m}{m_0}b = 0$ (we have $(m_0, m_1) = 1$); by the assumption $m|p$, we know that $(0, \frac{p}{m}\bar{m}_1)$ is the smallest choice of non-negative integer pairs whose first coordinate is zero and such that $\frac{m}{m_1}c - \frac{m}{m_0}b = p$.

Then the linear combination of $(\bar{m}_0, \bar{m}_1)$, and $(0, \frac{p}{m}\bar{m}_1)$ with integer coefficients (could be negative numbers) will be the group $K_0(A_m, \mathbb{Z}/p\mathbb{Z})$. Therefore, for any element $\gamma = (x, \bar{b}, \bar{c}) \in K_0(A_m; G_p)$, there exist integers $l_1$ and $l_2$ (could be negative numbers), such that $\gamma = (x, l_1\bar{m}_0, l_2\bar{m}_1)$. Suppose

$$
\gamma = c_1[\delta_0] + c_2[\delta_1] + c_3[id], \quad (*)
$$

we can always set $c_2 = l_1$, $l_2 - l_1 = \frac{p}{m}c_3$, since $\gamma \in K_0(A_m; G_p)$, we have $\frac{m}{m_1}l_2m_1 - \frac{m}{m_0}l_1m_0 = pj$ for some integer $j$. Therefore, $c_3 = j$, then $c_1 = x - l_2$. So we can find integers $c_1, c_2, c_3$ which satisfy $(*)$.

In the next, we calculate the Bockstein operations $\mu_{A_m,p}$ and $\nu_{A_m,p}$. To do so, we need to go through the definition of these operations on the generators, actual compositions of maps on generators. Recalling that the generators of $K_0(A_m)$ is the matrix value projection $h(t)$ such that $h(0) = a_0 \otimes id_{m_0}$, $a_0 = 1 \otimes id_{m_0}$; and $h(1) = a_1 \otimes id_{m_1}$, $a_1 = 1 \otimes id_{m_1}$. Then the composition with the map $\delta_1 : I_p \to \mathbb{C}$ gives us the map $\delta_1(f)h(t)$ from $I_p$ to $A_m$. By our identification, this map corresponds the element $(\bar{m}_0, \bar{m}_1) \in \mathbb{Z}(m, p)$. So $\mu_{A_m,p} = \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}$.
For the Bockstein operation $\nu_{A_m;p}$, note that the generator of $K_1(A_m)$ $(K_1(I_p))$ is the matrix value function
\[
g(t) = \begin{pmatrix} e^{2\pi it} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},
\]
then the composition of the map $1 \to g(t)$ with $id$ gives the element $\overline{m/(p,m)} \in K_1(A_m)$, and with $\delta_i, i = 0,1$ gives zero (then the composition with $\overline{id}$ gives $-\overline{m/(p,m)}$). Hence $\nu_{A_m;p} = (-\frac{m}{pm_0}, \frac{m}{pm_1})$.

\[\square\]

Although $[\delta_0], [\delta_1]$ and $[id]$ can already generate the group $K_0(A_m; G_p)$, but it is not true that the positive cone is the linear span of these three elements with non-negative integer coefficients, for example, the element $[\overline{id}]$ cannot be written as a linear combination of these three with non-negative integer coefficients. For our purpose, the following stronger version of generators of the positive cone is needed.

Lemma 3.4. Given a generalized dimension drop algebra $A_m = I[m_0, m, m_1]$ with $(m_0, m_1) = 1$, and any positive integer $p$ with $m$ divides $p$, then any element of the Dadarlat-Loring positive cone of $K_0(A_m; G_p)$ can be written as a linear combination of $[\delta_0], [\delta_1], [id]$, and $[\overline{id}]$ with non-negative integer coefficients.

Proof. Recalling the calculation of the K-theory with coefficient, we know that: 
\[
K_0(A_m, \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \{(\bar{b}, \bar{c}) \in \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \mid \frac{m}{m_1}c - \frac{m}{m_0}b \in p\mathbb{Z}\},
\]
and
\[
[\delta_0] = (1, \bar{0}, \bar{0}), [\delta_1] = (1, \bar{m_0}, \bar{m_1}), [id] = \frac{p}{m}(1, \bar{0}, \bar{m_1}), [\overline{id}] = \frac{p}{m}(1, \bar{m_0}, \bar{0}).
\]
Since $(\bar{m_0}, \bar{m_1})$ is the smallest choice of non-negative pairs $(b, c)$ such that $\frac{m}{m_1}c - \frac{m}{m_0}b = 0$ (we have $(m_0, m_1) = 1$); by the assumption $m|p$, we know that $(0, \frac{p}{m}\bar{m_1})$ is the smallest choice of non-negative integer pairs whose first coordinate is zero and such that $\frac{m}{m_1}c - \frac{m}{m_0}b = p$, similarly, $(\frac{p}{m}\bar{m_0}, 0)$ is the smallest choice of non-negative integer pairs whose second coordinate is zero and such that $\frac{m}{m_1}c - \frac{m}{m_0}b = p$. Then
the linear combination of \((\bar{m}_0, \bar{m}_1)\), \((\frac{p}{m} \bar{m}_0, 0)\), and \((0, \frac{p}{m} \bar{m}_1)\) with non-negative integer coefficients will be the group \(K_0(A_m, \mathbb{Z}/p\mathbb{Z})\).

Hence, for any element \(\gamma = (x, \bar{b}, \bar{c})\) in the positive cone of \(K_0(A_m, \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})\), there exist non-negative integers \(l_1\) and \(l_2\) such that \(\gamma = (x, l_1\bar{m}_0, l_2\bar{m}_1)\). In the next, we try to write \(\gamma\) as the linear combination of \([\delta_0]_{\mathbb{Z}}, [\delta_1]_{\mathbb{Z}}, [id]_{\mathbb{Z}}\), and \([\bar{id}]_{\mathbb{Z}}\) with positive integer coefficients.

Suppose \(\gamma = c_1[\delta_0] + c_2[\delta_1] + c_3[id] + c_4[\bar{id}]\), then one obtains that

\[
\begin{align*}
x &= c_1 + c_2 + c_3 \frac{p}{m} + c_4 \frac{p}{m}, \\
l_1 &= c_2 + c_4 \frac{p}{m}, \\
l_2 &= c_2 + c_3 \frac{p}{m}.
\end{align*}
\]

In case 1: assume that \(l_1 \geq l_2\), then \(\frac{p}{m}c_4 = \frac{p}{m}c_3 + l_1 - l_2\), let \(c_3 = 0\), one gets \(\frac{p}{m}c_4 = l_1 - l_2\). Since \(\frac{m}{m_1}l_2m_1 - \frac{m}{m_0}l_1m_0 \in p\mathbb{Z}\), i.e., \(l_2 - l_1 = \frac{p}{m}j\) for some integer \(j \leq 0\). Then \(c_4 = -j \geq 0\), therefore, \(c_2 = l_2 = l_1 + j\frac{p}{m}\).

This gives us \(c_1 = x - l_2 - \frac{p}{m}(-j) = x - l_2 + l_2 - l_1 = x - l_1\). Because \(x\) is in the positive cone, we know that \(x \geq l_1\), namely, \(c_1\) is non-negative.

In case 2: assume that \(l_2 \geq l_1\), then \(\frac{p}{m}c_3 = \frac{p}{m}c_4 + l_2 - l_1\), let \(c_4 = 0\), one gets \(\frac{p}{m}c_3 = l_2 - l_1\). Since \(\frac{m}{m_1}l_2m_1 - \frac{m}{m_0}l_1m_0 \in p\mathbb{Z}\), namely, \(l_2 - l_1 = \frac{p}{m}j\) for some integer \(j \geq 0\). Then \(c_3 = j \geq 0\), therefore, \(c_2 = l_1 = l_2 - \frac{p}{m}j\), this gives us \(c_1 = x - l_1 - \frac{p}{m}j = x - l_2\). Because \(\gamma\) is in the positive cone, we know that \(x \geq l_2\), namely, \(c_1\) is non-negative.

In all cases, we can always find non-negative integer solution for the system \((**)\) above. Then we are done. \(\square\)

4. Morphisms between K-theory with coefficient

In this section, we will look at the morphisms between K-theory with coefficient of two generalized dimension drop interval algebras, also preserving the Bockstein operations. Given \(A_m = \mathbb{I}[m_0, m, m_1]\), and \(B_n = \mathbb{I}[m_0, n, m_1]\), recall Lemma 2.3 and the notation there, we investigate the structure of \(\text{Hom}(K(A_m; p), K(B_n; p))\). For each KK-element \(\alpha \in KK(A_m, B_n)\), the induced triple

\[
\Gamma(\alpha; p) = (K_0(\alpha), K_0(\alpha; \mathbb{Z}_p), K_1(\alpha))
\]
lives in \( \text{Hom}(K(A_m;p), K(B_n;p)) \). Therefore, to analyze the structure of \( \text{Hom}(K(A_m;p), K(B_n;p)) \) is crucial in the following two senses: first, it is useful to figure out the condition under which a KK-element can preserve the Dadarlat-Loring order structure; second, more importantly, this analysis indicate the possible candidates of KK-elements which preserve the Dadarlat-Loring order structure, but fail to be representable by \( * \)-homomorphisms.

To analyze the structure of \( \text{Hom}(K(A_m;p), K(B_n;p)) \) means to give a general description of how an element looks like. A direct guess one could make is by direct calculation for a general element here. However, the equations set up from Lemma 2.5 for a general element in \( \text{Hom}(K(A_m;p), K(B_n;p)) \) is quite complicated, since we are working in mod \( p \) setting, to solve such equations is also difficult. Instead, we first give description of special elements of the form \( \Gamma = (0, \varphi, 0) \); and then construct concrete elements \((x, \phi, 0)\) and \((0, \psi, y)\) for given \( K_0 \)-multiplicity \( x \) and \( K_1 \)-multiplicity \( y \). Then we can have a description for general elements in \( \text{Hom}(K(A_m;p), K(B_n;p)) \). All these arguments become much easier for the classical dimension drop interval algebras due to the fact \( m_0 = m_1 = 1 \) (equivalently the equal dimension drop).

**Lemma 4.1.** Given \( A_m = I[m_0, m, m_1] \) and \( B_n = I[m_0, n, m_1] \), and any positive integer \( p \) with \( m | p \). For any induced triple \( \gamma = (x, \varphi, y) \), if \( \gamma = (0, \varphi, 0) \), then \( \varphi = d \left( \begin{array}{ccc} -m_1m_0 & m_0m_0 & m_0m_1 \\ -m_1m_1 & m_0m_0 & m_0m_1 \end{array} \right) \) for some integer \( d \) with \( 0 \leq d < \frac{m}{m_0m_1} \).

**Proof.** Recalling from Lemma 2.5 \( \gamma \) should fit the following commutative diagram:

\[
\begin{array}{c}
K_0(A_m) \xrightarrow{\mu_{A_m;p}} K_0(A_m, \mathbb{Z}_p) \xrightarrow{\nu_{A_m;p}} K_1(A_m) \\
\downarrow{\times x} \quad \downarrow{\varphi} \quad \downarrow{\times y} \\
K_0(B_n) \xrightarrow{\mu_{B_n;p}} K_0(B_n, \mathbb{Z}_p) \xrightarrow{\nu_{B_n;p}} K_1(B_n).
\end{array}
\]

Suppose \( \gamma = (0, \varphi, 0) \), then from the first (left) commuting square above, we get

\[
\varphi \circ \left( \begin{array}{c} m_0 \\ m_1 \end{array} \right) (1) = 0,
\]

and from the second (right) commuting square, we get

\[
\nu_{B_n;p} \circ \varphi \left( \frac{\bar{b}}{\bar{c}} \right) = 0 \quad (2)
\]
for any \((\bar{b}, \bar{c}) \in Z(m, p)\).

Assume \(\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}\), from (1), we get \(\varphi \left( \begin{pmatrix} \bar{m}_0 \\ \bar{m}_1 \end{pmatrix} \right) = 0\), namely,

\[
\begin{align*}
(4.1) & \quad \varphi_{11}m_0 + \varphi_{12}m_1 = pj \\
(4.2) & \quad \varphi_{21}m_0 + \varphi_{22}m_1 = pi.
\end{align*}
\]

Apply (2) on the element \(\left(0, \frac{p}{m}\bar{m}_1\right)\), we obtain

\[
\left(-\frac{n}{pm_0}, \frac{n}{pm_1}\right) \left(\begin{array}{c}
\varphi_{12} \frac{p}{m} \bar{m}_1 \\
\varphi_{22} \frac{1}{m} \bar{m}_1
\end{array}\right) = \varphi_{22} \frac{1}{m} \bar{m}_1 - \varphi_{12} \frac{n}{mm_1} \bar{m}_0 \in p\mathbb{Z}.
\]

Since \((m_0, m_1) = 1\), there exist \(\beta_0, \beta_1 \in \mathbb{Z}\), such that \(\beta_0m_0 + \beta_1m_1 = 1\). So we have \(pj\beta_0m_0 + pj\beta_1m_1 = pj\), this equation subtract (4.1), we get

\[
(\varphi_{11} - pj\beta_0)m_0 + (\varphi_{12} - pj\beta_1)m_1 = 0. \quad (3)
\]

Since \((m_0, m_1) = 1\), one has that \(\varphi_{11} - pj\beta_0 = d_1m_1\), and \(pj\beta_1 - \varphi_{12} = d_1m_0\) for some integer \(d_1\). Because \(\varphi\) is a morphism in the mod \(p\) setting, so we have \(\varphi_{11} = -d_1m_1, \varphi_{12} = d_1m_0\). Similarly, the the same argument applies on (4.2), we have that \(\varphi_{21} = -d_2m_1, \varphi_{22} = d_2m_0\).

Combine with (3), one gets \(\frac{d_2nm_0}{m} - \frac{d_1nm_1m_0}{mm_0} = pk\) for some integer \(k\). Then \(d_2nm_0 - d_1nm_1 = pkm\), use the same argument above with the pair \((\beta_0, \beta_1)\), we obtain that

\[
(d_2n - \beta_0pkm)m_0 - (d_1n + \beta_1pkm)m_1 = 0.
\]

Still note that \(\varphi\) is a morphism in the mod \(p\) setting, we have that \(d_2n = rm_1\) and \(d_1n = rm_0\) for some integer \(r\), from here, we get \(d_2m_0 = d_1m_1\). Therefore \(d_2 = dm_1\) and \(d_1 = dm_0\) for some integer \(d\).

Hence, \(\varphi = d \left( \begin{pmatrix} -m_1m_0 & m_0m_0 \\ -m_1m_1 & m_0m_1 \end{pmatrix} \right) \).

Suppose that \(\varphi = d \left( \begin{pmatrix} -m_1m_0 & m_0m_0 \\ -m_1m_1 & m_0m_1 \end{pmatrix} \right) = 0\), then this happens if and only if \(\varphi \left( \begin{pmatrix} \bar{m}_0 \\ \bar{m}_1 \end{pmatrix} \right) = 0\), and \(\varphi \left( \begin{pmatrix} 0 \\ \frac{p}{m}\bar{m}_1 \end{pmatrix} \right) = 0\). The first equation is automatically satisfied since \(\varphi\) has this special form. For the second one, it means that \(\frac{dm_0m_0m_1}{m} \in p\mathbb{Z}\) and \(\frac{dm_0m_1}{m} \in p\mathbb{Z}\). This forces \(d\) to be a multiple of \(\frac{m}{m_0m_1}\). Hence we can always choose \(0 \leq d < \frac{m}{m_0m_1}\). \(\square\)
Lemma 4.2. Given $K_0$-multiplicity $x$ and $p$ with $m$ dividing $p$, there exists a morphism $\phi$ between $K_0(A_m, \mathbb{Z}_p)$ and $K_0(B_n, \mathbb{Z}_p)$, such that $(x, \phi, 0)$ is a triple in $\text{Hom}(K(A_m; p), K(B_n; p))$.

Proof. Suppose $\phi = \left( \begin{array}{cc} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{array} \right)$, as required, we need

\[(4.3) \quad \phi \left( \begin{array}{c} \bar{m}_0 \\ \bar{m}_1 \end{array} \right) = \left( \begin{array}{c} x\bar{m}_0 \\ x\bar{m}_1 \end{array} \right)\]

\[(4.4) \quad (-\frac{n}{pm_0}, \frac{n}{pm_1})\phi \left( \begin{array}{c} p \bar{0} \\ p \bar{1} \end{array} \right) = 0.\]

Expand these two equations, one get that

\[(4.5) \quad \phi_{11}m_0 + \phi_{12}m_1 \equiv xm_0 \mod p \]

\[(4.6) \quad \phi_{21}m_0 + \phi_{22}m_1 \equiv xm_1 \mod p,\]

and

\[(4.7) \quad \frac{n}{pm_1} \phi_{22} \frac{pm_1}{m} - \frac{n}{pm_0} \phi_{12} \frac{pm_1}{m} \equiv 0 \mod p \]

\[(4.8) \quad \frac{n}{pm_1} \phi_{21} \frac{pm_0}{m} - \frac{n}{pm_0} \phi_{11} \frac{pm_0}{m} \equiv 0 \mod p.\]

Namely,

\[(4.9) \quad \frac{nm_1}{m} \left( \frac{\phi_{22}}{m_1} - \frac{\phi_{12}}{m_0} \right) \equiv 0 \mod p \]

\[(4.10) \quad \frac{nm_0}{m} \left( \frac{\phi_{21}}{m_1} - \frac{\phi_{11}}{m_0} \right) \equiv 0 \mod p.\]

Since we have $(m_0, m_1) = 1$, there exists integer $\beta_0 \geq 0$, and $\beta_1 \leq 0$, such that $\beta_0 m_0 + \beta_1 m_1 = 1$. Then

\[(4.11) \quad xm_0 \beta_0 m_0 + xm_0 \beta_1 m_1 = xm_0 \]

\[(4.12) \quad xm_1 \beta_0 m_0 + xm_1 \beta_1 m_1 = xm_1.\]

Set $\phi_{11} = xm_0 \beta_0$, $\phi_{12} = xm_0 \beta_1$, $\phi_{21} = xm_1 \beta_0$, and $\phi_{22} = xm_1 \beta_1$, then with these data, (4.9) and (4.10) are satisfied. \qed

Lemma 4.3. Given $K_1$-multiplicity $y$ and $p$ with $m$ dividing $p$, there exists a morphism $\psi$ between $K_0(A_m, \mathbb{Z}_p)$ and $K_0(B_n, \mathbb{Z}_p)$, such that $(0, \psi, y)$ is a triple in $\text{Hom}(K(A_m; p), K(B_n; p))$. 
Proof. Assume $\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$, as required, we need

\[(4.13) \quad \psi \begin{pmatrix} \tilde{m}_0 \\ \tilde{m}_1 \end{pmatrix} = 0\]

\[(4.14) \quad \left(-\frac{n}{pm_0}, \frac{n}{pm_1}\right)\psi \begin{pmatrix} \frac{0}{m} \\ \frac{0}{m} \end{pmatrix} = y\left(-\frac{m}{pm_0}, \frac{m}{pm_1}\right)\begin{pmatrix} \frac{0}{m} \\ \frac{0}{m} \end{pmatrix}.\]

Expand these equations, we get

\[(4.15) \quad \psi_{11}m_0 + \psi_{12}m_1 \equiv 0 \pmod{p}\]

\[(4.16) \quad \psi_{21}m_0 + \psi_{22}m_1 \equiv 0 \pmod{p},\]

and

\[(4.17) \quad \frac{nm_1}{m} \left(\psi_{22} - \frac{\psi_{12}}{m_0}\right) \equiv y \pmod{p}\]

\[(4.18) \quad \frac{nm_0}{m} \left(\psi_{21} - \frac{\psi_{11}}{m_0}\right) \equiv -y \pmod{p}.\]

Since we have $(m_0, m_1) = 1$, there exists integer $\beta_0 \geq 0$ and $\beta_1 \leq 0$, such that $\beta_0 m_0 + \beta_1 m_1 = 1$. Set

$$
\psi = \begin{pmatrix} \frac{ym_0m_1\beta_1}{nm_0} & -\frac{ym_0m_1\beta_1}{nm_1} \\ -\frac{ym_0m_1\beta_0}{nm_0} & \frac{ym_0m_1\beta_0}{nm_1} \end{pmatrix},
$$

then $\psi$ satisfies the requirement. (Note that $\frac{ym}{n}$ is an integer, since $\frac{m}{m_0m_1}$ divided by $\frac{n}{m_0m_1}$, the quotient is $\frac{ym}{n}$.)

\[\square\]

**Theorem 4.4. (Structure of $\text{Hom}(K(A_m; p), K(B_n; p))$)** Any element $\Phi = (x, \rho, y)$ in $\text{Hom}(K(A_m; p), K(B_n; p))$ with $K_0$-multiplicity $x$ and $K_1$-multiplicity $y$ is of the following form:

$$
\Phi = (x, \sigma, y) + d(0, \begin{pmatrix} -m_1m_0 & m_0m_0 \\ -m_1m_1 & m_0m_1 \end{pmatrix}, 0). \tag{*}
$$

where $\sigma = \begin{pmatrix} \frac{xm_0\beta_0 + \frac{ym_0m_1\beta_1}{n}}{n} & \frac{xm_0\beta_1 - \frac{ym_0m_1\beta_0}{n}}{n} \\ \frac{xm_1\beta_0 - \frac{ym_0m_1\beta_0}{n}}{n} & \frac{xm_1\beta_1 + \frac{ym_0m_1\beta_0}{n}}{n} \end{pmatrix}$.\]
Suppose $\Phi = (x, \rho, y)$ is any element in $\text{Hom}(K(A_m; p), K(B_n; p))$ with $K_0$-multiplicity $x$ and $K_1$-multiplicity $y$, then by Lemma 4.1, we have $\Phi - \alpha = (0, \rho - \sigma, 0)$, so $\rho - \sigma = d \begin{pmatrix} -m_1m_0 & m_0m_0 \\ -m_1m_1 & m_0m_1 \end{pmatrix}$. So $\Phi$ has the general form:

$$\Phi = (x, \sigma, y) + d(0, \begin{pmatrix} -m_1m_0 & m_0m_0 \\ -m_1m_1 & m_0m_1 \end{pmatrix}, 0).$$  \hspace{1cm} (*)$$

In the next, we calculate the induced maps of the four homomorphisms mentioned in section 2. The following lemma is also basic in later use.

**Lemma 4.5.** For any integers $n, m$ and $p$ with $m$ dividing $p$, the four homomorphisms $\delta_0$, $\delta_1$, $\text{id}_{m,n}$, and $\overline{\text{id}}_{m,n}$ induce the following triples:

- $\Gamma(\delta_0; p) = (m_0, \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, 0)$,
- $\Gamma(\delta_1; p) = (m_1, \begin{pmatrix} 0 \\ m_1 \end{pmatrix}, 0)$,
- $\Gamma(\text{id}_{m,n}; p) = \left( \frac{m}{(m, n)}, \begin{pmatrix} m/n \\ 0 \end{pmatrix}, \frac{n}{(m, n)} \right)$,
- $\Gamma(\overline{\text{id}}_{m,n}; p) = \left( \frac{m}{(m, n)}, \frac{m}{m_0m_1}, \begin{pmatrix} 0 \\ \frac{m_0}{(m_0, m_1)} \end{pmatrix}, \frac{m_0}{(m_0, m_1)} \right)$, $- \frac{n}{(m, n), \frac{m}{m_0m_1}}$.

**Proof.** To prove this lemma, we need to check the induced map of these four homomorphisms on generators of $K$-theoretic groups. We spell out the details for the convenience of readers.

For $\delta_0 : A_m \to B_n \otimes M_{m_0}$, recalling the generator of $K_0(A_m)$ is the projection $h(t)$ with $h(0) = 1 \otimes \text{id}_{m_0} \otimes \text{id}_{\frac{m}{m_0}}$, and $h(1) = 1 \otimes \text{id}_{m_1} \otimes \text{id}_{\frac{m}{m_1}}$. Apply $\delta_0$ on $h(t)$, we get that $\delta_0(h)(0) = 1 \otimes \text{id}_{m_0} \otimes \text{id}_{m_0} \otimes \text{id}_{\frac{m}{m_0}}$, and $\delta_0(h)(1) = 1 \otimes \text{id}_{m_0} \otimes \text{id}_{m_1} \otimes \text{id}_{\frac{m}{m_1}}$. So the image is $m_0$ times the generator of $K_0(B_n)$. So the $K_0$-multiplicity of $\Gamma(\delta_0; p)$ is $m_0$. Similarly, the $K_0$-multiplicity of $\Gamma(\delta_1; p)$ is $m_1$. Moreover, it is obvious that the $K_1$-multiplicities of both $\Gamma(\delta_0; p)$ and $\Gamma(\delta_1; p)$ are zero.
For the part of $\Gamma(\delta_0; p)$ on $K_0(A_m, \mathbb{Z}_p)$, note that the restriction of $id : \tilde{I}_p \to A_m$ and $\delta_1 : \tilde{I}_p \to A_m$ are the generators of $K_0(A_m, \mathbb{Z}_p)$. Apply $\delta_0$ on them, by our identification, we get

$$\Gamma(\delta_0; p) \left( \begin{array}{c} \tilde{p} \\ \tilde{m} \end{array} \right) = \left( \begin{array}{c} \tilde{0} \\ \tilde{0} \end{array} \right)$$

$$\Gamma(\delta_0; p) \left( \begin{array}{c} \tilde{m} \\ \tilde{m} \end{array} \right) = \left( \begin{array}{c} \tilde{m} \\ \tilde{m} \end{array} \right)$$

which indicates that the middle part of $\Gamma(\delta_0; p)$ is $\left( \begin{array}{c} m_0 \\ m_1 \end{array} \right)$. Similarly, we can get $\Gamma(\delta_1; p)$.

For $id_{m,n} : A_m \to B_n \otimes M_{m,n}$, it is easily seen that the $K_0$-multiplicity of $\Gamma(id_{m,n}; p)$ is $\frac{m}{(m,n)}$. But for the $K_0$-multiplicity of $\Gamma(id_{m,n})$, we need to be careful (the different dimension drops cause non symmetric size compare with $id_{m,n}$): the $K_0$-multiplicity of $\Gamma(id_{m,n})$ is $\frac{m}{((m,n), \frac{m}{m_0 m_1})}$.

The $K_1$-multiplicity of $\Gamma(id_{m,n}; p)$ is $\frac{n}{(m,n)}$: The $K_1$-multiplicity of $\Gamma(id_{m,n})$ is $-\frac{n}{((m,n), \frac{m}{m_0 m_1})}$.

For the middle part of $\Gamma(id_{m,n}; p)$ and $\Gamma(id_{m,n}; p)$, we also go through the compositions on generators, and the identification in Theorem 3.3, it is straightforward but tedious, then we get the formulas.

\[ \square \]

**Proposition 4.6.** $KK(A_m, B_n) \cong \mathbb{Z} \oplus \mathbb{Z}_{(m,n)} \oplus \mathbb{Z}_{\frac{m}{m_0 m_1}}$, and $\delta_0, \delta_1$ and $id_{m,n}$ generate $KK(A_m, B_n)$.

**Proof.** By UCT, it is easily seen that $KK(A_m, B_n) \cong \mathbb{Z} \oplus \mathbb{Z}_{(m,n)} \oplus \mathbb{Z}_m$, moreover, $\mathbb{Z} = \text{Hom}(K_0(A_m), K_0(B_n))$, $\mathbb{Z}_{(m,n)} = \text{Hom}(K_1(A_m), K_1(B_n))$, and $\text{Ext}(K_1(A_m), K_0(B_n)) = \mathbb{Z}_{\frac{m}{m_0 m_1}}$.

By Lemma 1.5, $\mathbb{Z}$ is generated by $\delta_0$, and $\mathbb{Z}_{(m,n)}$ is generated by $id_{m,n} - \frac{m}{(m,n)}(\beta_0 \delta_0 + \beta_1 \delta_1)$, where $\beta_0, \beta_1$ are integers such that $\beta_0 m_0 + \beta_1 m_1 = 1$. On the other hand, the non zero KK-element $m_1 \delta_0 - m_0 \delta_1$ induces trivial map on K-groups, hence it gives an nontrivial extension. Since $m_1 \delta_0 - m_0 \delta_1$ has order $\frac{m}{m_0 m_1}$, then we are done. \[ \square \]
Theorem 4.7. Given positive integers \( n, m \) and \( p \) with \( n, m \mid p \), then the canonical map

\[ \Gamma : KK(A_m, B_n) \to \text{Hom}(K(A_m; p), K(B_n; p)) \]

is an isomorphism.

Proof. Given any element \( \Phi = (x, \tau, y) \) in \( \text{Hom}(K(A_m; p), K(B_n; p)) \), by UCT, we know that there is a KK-element \( \alpha \), such that \( \Gamma(\alpha; p) = (x, \eta, y) \) in \( \text{Hom}(K(A_m; p), K(B_n; p)) \). By Lemma 4.1,

\[ \Phi - \Gamma(\alpha; p) = (0, \tau - \eta, 0) = d(0, \begin{pmatrix} -m_1 m_0 & m_0 m_0 \\ -m_1 m_1 & m_0 m_1 \end{pmatrix}, 0) \]

for some integer \( d \) with \( 0 \leq d < \frac{m}{m_0 m_1} \). By Lemma 4.5, we have that

\[ d(0, \begin{pmatrix} -m_1 m_0 & m_0 m_0 \\ -m_1 m_1 & m_0 m_1 \end{pmatrix}, 0) = d m_0 \Gamma(\delta_1; p) - d m_1 \Gamma(\delta_0; p) \]

Hence,

\[ \Phi = \Gamma(\alpha; p) + d m_0 \Gamma(\delta_1; p) - d m_1 \Gamma(\delta_0; p) \]

So, \( \Gamma \) is surjective.

By Proposition 4.6, we know that \( KK(A_m, B_n) \) has \( (m, n) \frac{m}{m_0 m_1} \) torsion elements; on the other hand, \( \text{Hom}(K(A_m; p), K(B_n; p)) \) has at least \( (m, n) \frac{m}{m_0 m_1} \) torsion elements by Lemma 4.1 (we have the assumption \( m \mid p \) there). Since any surjective morphism from \( \mathbb{Z} \) to \( \mathbb{Z} \) is automatically injective, so the map \( \Gamma \) on torsion part must be injective (since we have shown \( \Gamma \) is surjective above). Therefore \( \Gamma \) is an isomorphism.

\[ \square \]

Now, we can have a better understanding about the structure of \( \text{Hom}(K(A_m; p), K(B_n; p)) \), the torsion elements

\[ d(0, \begin{pmatrix} -m_1 m_0 & m_0 m_0 \\ -m_1 m_1 & m_0 m_1 \end{pmatrix}, 0) \]

are exactly \( d m_0 \Gamma(\delta_1; p) - d m_1 \Gamma(\delta_0; p) \) by Lemma 4.5. Hence, for any triple \( \alpha = (x, \omega, 0) \) with \( K_1 \)-multiplicity zero, by Theorem 4.4, we know it is of the following form:

\[ \alpha = (x, \begin{pmatrix} x m_0 \beta_0 & x m_0 \beta_1 \\ x m_1 \beta_0 & x m_1 \beta_1 \end{pmatrix}, 0) + d(0, \begin{pmatrix} -m_1 m_0 & m_0 m_0 \\ -m_1 m_1 & m_0 m_1 \end{pmatrix}, 0). \]

Where \( \beta_0 m_0 + \beta_1 m_1 = 1, \beta \geq 0, \beta_1 \leq 0 \). By Lemma 4.5 again, we get

\[ \alpha = (\beta_0 x - d m_1) \Gamma(\delta_0; p) + (\beta_1 x + d m_0) \Gamma(\delta_1; p) - d m_1 \Gamma(\delta_0; p) \]

\[ = (\beta_0 x - d m_1) \Gamma(\delta_0; p) + (\beta_1 x + d m_0) \Gamma(\delta_1; p). \quad (\dagger) \]
For a general triple $\Phi = (x, \sigma, y)$ in Theorem 4.4, first, there is an integer $k$, such that $y = k \frac{n}{(m, n)}$, then $\frac{ny}{n} = k \frac{m}{(m, n)}$. Therefore,

$$k\Gamma(id_{m,n}; p) = \left(\frac{ny}{n}, \begin{pmatrix} \frac{ny}{n} & 0 \\ 0 & \frac{ny}{n} \end{pmatrix}, y \right).$$

Then it is straightforward to verify that

$$\Phi - k\Gamma(id_{m,n}; p) = ((x - \frac{ny}{n})\beta_0 - dm_1)\Gamma(\delta_0; p) + ((x - \frac{ny}{n})\beta_1 + dm_0)\Gamma(\delta_1; p).$$

Then

$$\Phi = k\Gamma(id_{m,n}; p) + ((x - \frac{ny}{n})\beta_0 - dm_1)\Gamma(\delta_0; p) + ((x - \frac{ny}{n})\beta_1 + dm_0)\Gamma(\delta_1; p).$$

So in K-theory with coefficient picture, KK-group is also generated by $\delta_0, \delta_1$ and $id_{m,n}$.

5. KK-lifting problem for generalized dimension drop interval algebras

In this section, we start to investigate the KK-lifting problem for generalized dimension drop interval algebras. Set $KK^+(A_m, B_n)$ to be following set:

$$\{ \kappa \in KK(A_m, B_n) \mid \kappa = [\varphi] \text{ for some } \varphi \in Hom(A_m, M_k(B_n)) \}.$$

Then we try to find some conditions under which a KK-element can lies in this set; moreover, try to relate these conditions to proper invariants of C*-algebras such that it is applicable to the classification program.

In [10], X. Jiang and H.Su investigated these kind building blocks, they already had a criterion for KK-lifting in terms of K-homology. We first recall the necessary preliminaries here.

**Lemma 5.1.** (Lemma 3.1 in [10]) Given $A_m = I[m_0, m, m_1]$, then its K-homology group is generated by the two irreducible representations $V_0, V_1$ up to the following relation:

$$\frac{m}{m_0}[V_0] = \frac{m}{m_1}[V_1].$$

**Remark 5.2.** For $\delta_0, \delta_1 \in KK(A_m, B_n)$, it is straightforward to see that

$$\delta_0([V_0^A]) = m_0[V_0^A], \delta_0([V_1^B]) = m_1[V_0^A]$$

$$\delta_1([V_0^A]) = m_0[V_1^A], \delta_1([V_1^B]) = m_1[V_1^A]$$

By this lemma, the relation between $\delta_0$ and $\delta_1$ is $\frac{m}{m_0}\delta_0 = \frac{m}{m_1}\delta_1$. 

Moreover, they defined an order structure on the K-homology groups:

\[ K^0_+ (A_m) \triangleq \{ \rho \in K^0(A_m) \mid \rho \text{ is a finite dimensional representation of } A_m \} . \]

Then they proved the following criterion for KK-lifting.

**Theorem 5.3.** (Theorem 3.7 in [10]) Given \( \alpha \in KK(A_m, B_n) \), then \( \alpha \) can be lifted to a \( * \)-homomorphism if and only if \( \alpha^* \) is positive from \( K^0(B_n) \) to \( K^0(A_m) \), where \( \alpha^* \) is the Kasparov product with K-homology groups.

**Remark 5.4.**

1. Moreover, Jiang and Su proved that \( KK(A_m, B_n) \cong Hom(K^0(B_n), K^0(A_m)) \) under the Kasparov product. Hence, the isomorphism is an ordered isomorphism.

2. We change a little bit the original statement of Jiang and Su’s theorem, since we don’t require the homomorphism to be unital, see the remark after Jiang and Su’s Theorem 3.7 in [10].

On the other hand, the Dadarlat-Loring order on K-theory with coefficient gave a lifting criterion for KK-elements between classical dimension drop interval algebras (see Proposition 3.2 in [6]), and succeeded as an invariant for the classification of real rank zero limits of these classical ones (see [3], [1], [6]). So it is natural to ask for the corresponding KK-lifting criterion for generalized dimension drop interval algebras in Dadarlat-Loring’s picture. Let us try to do this.

For simplicity, we first look at the KK-elements with zero \( K_1 \)-multiplicity, realize them on \( Hom(K(A_m; p), K(B_n; p)) \), they all have the form as in (\( \dagger \)), then by Theorem 4.7, these KK-elements are of the form

\[ \alpha = (\beta_0 x - dm_1)\delta_0 + (\beta_1 x + dm_0)\delta_1, 0 \leq d < \frac{m}{m_0m_1} . \] (**)

To relate the lifting of these elements to K-theory with coefficient, we first have the following proposition.

**Proposition 5.5.** Given any KK-element \( \alpha \in KK(A_m, B_n) \) with \( K_1 \)-multiplicity zero as in (**) with \( 0 \leq d < \frac{m}{m_0m_1m_0} \) or \( 0 \leq d < \frac{m}{m_0m_1m_1} \), given any \( p \) with \( m|p \), if \( \Gamma(\alpha; p) \) preserves the Dadarlat-Loring order, then \( \beta_0 x - dm_1 \geq 0 \).

**Proof.** We know that

\[ \Gamma(\alpha; p) = (x, \left( \begin{array}{cc} xm_0\beta_0 & xm_0\beta_1 \\ xm_1\beta_0 & xm_1\beta_1 \end{array} \right), 0) + d(0, \left( \begin{array}{cc} -m_1m_0 & m_0m_0 \\ -m_1m_1 & m_0m_1 \end{array} \right), 0). \]

Since we always assume \( \beta_0 \geq 0, \beta_1 \leq 0 \), if \( \alpha \) preserves the Dadarlat-Loring order, then \( x \geq 0 \). So \( -x\beta_1\Gamma(\delta_1; p) \) preserves the Dadarlat-Loring order, so \( \Gamma(\alpha; p) - x\beta_1\Gamma(\delta_1; p) \) also preserves this order.
Note that
\[-x\beta_1 \Gamma(\delta_1; p) = (-x\beta_1 m_1, \begin{pmatrix} 0 & -x\beta_1 m_0 \\ 0 & -x\beta_1 m_1 \end{pmatrix}, 0)\].

Hence,
\[\Gamma(\alpha; p) - x\beta_1 \Gamma(\delta_1; p) = (x(1 - \beta_1 m_1), \begin{pmatrix} x m_0 \beta_0 - dm_1 m_0 & dm_0 m_1 \\ x m_1 \beta_0 - dm_1 m_1 & dm_0 m_1 \end{pmatrix}, 0)\]
\[= (x\beta_0 m_0, \begin{pmatrix} (x\beta_0 - dm_1) m_0 & dm_0 m_0 \\ (x\beta_0 - dm_1) m_1 & dm_0 m_1 \end{pmatrix}, 0)\]

Because this is positive, then the image of \((\frac{p}{m}, \frac{m}{m}, \frac{m}{m}) \in K_0(A_m; G_p)\) under this triple is in the positive cone of \(K_0(B_n; G_p)\). This means that
\[
\frac{px\beta_0 m_0}{m} m_0 \geq \frac{pm_1}{m} dm_0 m_0, \text{ and } \frac{px\beta_0 m_0}{m} m_1 \geq \frac{pm_1}{m} dm_0 m_1.
\]

Since \(0 \leq d < \frac{m}{m_0 m_1 m_0}\) or \(0 \leq d < \frac{m}{m_0 m_1 m_0}\), then we have \(\frac{pm_1}{m} dm_0 m_0 < p \) or \(\frac{pm_1}{m} dm_0 m_1 < p\). Therefore, we have either \(\frac{px\beta_0 m_0}{m} m_0 \geq \frac{pm_1}{m} dm_0 m_0\) or \(\frac{px\beta_0 m_0}{m} m_1 \geq \frac{pm_1}{m} dm_0 m_1\), in any case, we get \(\beta_0 x - dm_1 \geq 0\). \(\square\)

**Remark 5.6.** For the classical dimension drop interval algebras, we can choose \(\beta_0 = 1, \beta_1 = 0\), then (**) becomes
\[\alpha = (x - d)\delta_0 + d\delta_1.\]

Hence \(\alpha\) is positive in Dadarlat-Loring’s sense if and only if it can be lifted to a homomorphism. This proposition is then exactly Lemma 3.1 in [6]. But now we could have \(\beta_1 \leq 0\), this indicates that the Dadarlat-Loring order may fail to guarantee the lifting of \(\alpha\).

**Proposition 5.7.** Given any KK-element \(\alpha \in KK(A_m; B_n)\) with \(K_1\)-multiplicity zero as in (**), if the \(K_0\)-multiplicity \(x \geq m\), then \(\alpha\) can be lifted to a homomorphism between the algebras.

**Proof.** By Remark 5.2 to determine whether \(\alpha \in (**)\) can be lifted, it is equivalent to count the numbers of \(\delta_0\) and \(\delta_1\). If \(x \geq m\), then \(\beta_0 x - dm_1 \geq 0\), and \(\beta_1 x + dm_0 \leq 0\). Assume that \(\beta_0 x - dm_1 = \frac{m}{m_0} j_0 + r_0, 0 \leq r_0 < \frac{m}{m_0}\), and \(|\beta_1 x + dm_0| = \frac{m}{m_1} j_1 + r_1, 0 \leq r_1 < \frac{m}{m_1}\). Then
\[
\alpha = \frac{m}{m_0} j_0 \delta_0 + r_0 \delta_0 - (\frac{m}{m_1} j_1 \delta_1 + r_1 \delta_1) = ((\frac{m}{m_0} - \frac{m}{m_1} j_1) - r_1) \delta_1 + r_0 \delta_0
\]
While
\[ j_0 - j_1 = \frac{\beta_0 x - dm_1 - r_0}{m_0} - \frac{|\beta_1 x + dm_0| - r_1}{m_1} \]
\[ = \frac{x - (r_0 m_0 - r_1 m_1)}{m} \]
Since \( x \geq m \), we have that \( j_0 - j_1 \geq 1 \). So \( \alpha \) can be lifted. \( \square \)

To make situation easier, we assume \( d = 0 \), we try to investigate the exact conditions which imply \( \alpha \) in (***) preserves the Dadarlat-Loring order. We have the following proposition.

**Proposition 5.8.** Given \( \alpha = \beta_0 x \delta_0 + \beta_1 x \delta_1 \), let \( R \) be the remainder of \( \beta_0 m_0 m_0 x \) divided by \( m \), and \( S \) be the remainder of \( \beta_0 m_0 m_1 x \) divided by \( m \). Then \( \Gamma(\alpha; p) \) preserves the Dadarlat-Loring order structure if and only if \( x = 0 \) or
\[
(5.1) \quad \beta_0 m_0 m_0 x \geq m, \beta_0 m_0 m_1 x \geq m
\]
\[
(5.2) \quad m_0 x \geq R, m_1 x \geq S.
\]

**Proof.** To determine whether \( \Gamma(\alpha; p) \) preserves the Dadarlat-Loring order, by Lemma 3.4, we only need to work on the generators of positive cone:
\[
[\delta_0] = (1, 0, 0), [\delta_1] = (1, \bar{m}_0, \bar{m}_1)
\]
\[
[id] = (\frac{p}{m}, 0, \frac{p}{m} \bar{m}_1), [id] = (\frac{p}{m}, \frac{p}{m} \bar{m}_0, 0)
\]

1. The image of \([\delta_0]\) is automatically positive.
2. For the image of \([\delta_1]\),
\[
\Gamma(\alpha; p)([\delta_1]) = (x, \begin{pmatrix} \beta_0 m_0 x & \beta_1 m_0 x \\ \beta_0 m_1 x & \beta_1 m_1 x \end{pmatrix} \begin{pmatrix} \bar{m}_0 \\ \bar{m}_1 \end{pmatrix}, 0)
\]
\[= (x, \begin{pmatrix} \bar{m}_0 x \\ \bar{m}_1 x \end{pmatrix}, 0), \]
This is also always positive.
3. The image of \([id]\) is
\[
\Gamma(\alpha; p)([id]) = (x \frac{p}{m}, \begin{pmatrix} \beta_0 m_0 x & \beta_1 m_0 x \\ \beta_0 m_1 x & \beta_1 m_1 x \end{pmatrix} \begin{pmatrix} \frac{p}{m} \bar{m}_0 \\ 0 \end{pmatrix}, 0)
\]
\[= (x \frac{p}{m}, \begin{pmatrix} \bar{m}_0 x \beta_0 m_0 p \\ \bar{m}_0 x \beta_0 m_1 m \end{pmatrix}, 0). \]
Positivity means that
\[ x \frac{p}{m} m_0 \geq \bar{m}_0 x \beta_0 m_0 \frac{p}{m}, \quad x \frac{p}{m} m_1 \geq \bar{m}_0 x \beta_0 m_1 \frac{p}{m}. \]

These force first that
\[ \beta_0 m_0 m_0 x \geq m, \quad \beta_0 m_0 m_1 x \geq m. \]

Moreover, write
\[ \beta_0 m_0 m_0 x = \ll \beta_0 m_0 m_0 x + r, 0 \leq r < 1 \]
\[ \beta_0 m_0 m_1 x = \ll \beta_0 m_0 m_1 x + s, 0 \leq s < 1 \]

, then positivity is equivalent to
\[ \bar{m} p \leq \frac{p m_0 x}{m}, \quad sp \leq \frac{p m_1 x}{m}, \]

which is
\[ R = rm \leq m_0 x \quad S = sm \leq m_1 x. \]

4. The image of \([id]\) is
\[ \Gamma(\alpha; p)([id]) = (x \frac{p}{m}, \begin{pmatrix} \beta_0 m_0 x & \beta_1 m_0 x \\ \beta_0 m_1 x & \beta_1 m_1 x \end{pmatrix} \begin{pmatrix} 0 \\ \frac{p}{m} \end{pmatrix}, 0) \]
\[ = (x \frac{p}{m}, \begin{pmatrix} \bar{m}_1 x \beta_1 m_0 \frac{p}{m} \\ \bar{m}_1 x \beta_1 m_1 \frac{p}{m} \end{pmatrix} ) 0). \]

Positivity means that
\[ x \frac{p}{m} m_0 \geq \bar{m}_1 x \beta_1 m_0 \frac{p}{m}, \quad x \frac{p}{m} m_1 \geq \bar{m}_1 x \beta_1 m_1 \frac{p}{m}. \]

Note that \( \beta_1 \leq 0 \), we need a little more work. Since \( \beta_1 m_1 m_0 \frac{p}{m} + (\beta_0 m_0 - 1) m_0 \frac{p}{m} = 0 \), so
\[ \bar{m}_1 x \beta_1 m_0 \frac{p}{m} = (-\beta_0 m_0 + 1) \frac{p}{m} x \bar{m}_0, \]
\[ = \bar{m}_0 \frac{p}{m} x - \bar{m}_0 \beta_0 m_0 \frac{p}{m} \]
\[ = \bar{m}_0 \frac{p}{m} x - \bar{m} p \]
\[ = m_0 \frac{p}{m} x - r p \]
The condition $R \leq m_0 x$ is equivalent to $m_0 \frac{p}{m} x - rp \geq 0$. Hence, $x \frac{p}{m} m_0 \geq \bar{m}_1 x \beta_1 m_0 \frac{p}{m}$. Similarly, with the condition $S \leq m_1 x$ we have $x \frac{p}{m} m_1 \geq \bar{m}_1 x \beta_1 m_1 \frac{p}{m}$. \hfill \Box

Remark 5.9. In the classical dimension drop algebra case, the conditions above becomes $x \geq m$, and this is enough to guarantee a lifting. So for general case, the non-positive number $\beta_1$ caused by different dimension drops at two endpoints really gives us the possibility for counterexamples.

Now, we are able to prove the main Theorem 1.1, namely, give examples of KK-elements which preserve the Dadarlat-Loring order structure, and fail to be lifted to a $\ast$-homomorphism.

Proof of Theorem 1.1. Let $m_0 = 2, m_1 = 3, m = 12$, then we take $\beta_0 = 2, \beta_1 = -1$, we want some KK-elements $\alpha = 2x \delta_0 - x \delta_1$, such that $\alpha$ preserves the Dadarlat-Loring order structure but fail to be lifted to a $\ast$-homomorphism. By Proposition 5.8, we need the $K_0$-multiplicity $x$ satisfies the following inequalities:

\begin{align}
(5.3) & \quad 8x \geq 12, 12x \geq 12 \\
(5.4) & \quad 2x \geq R, 3x \geq S
\end{align}

From (5.3), we get $x \geq 2$, take $x = 2$, then (5.2) is satisfied, then $\alpha = 4\delta_0 - 2\delta_1$ can not be lifted to a homomorphism by Jiang and Su’s criterion. By Proposition 5.7, such examples only exist for $x < m$, and under our algorithm, we can actually determine all the possible examples. If $x = 5$, we get $10\delta_0 - 5\delta_1 = 4\delta_0 - \delta_1$, which also fits our purpose. In fact, $x = 3$ and $x = 5$ are all the possibilities. \hfill \Box

Remark 5.10. 1. Let us summarize the KK-lifting story since Elliott’s paper [7], he defined an order structure which is good for KK-lifting of circle algebras, then S. Eilers found counterexamples for classical dimension drop algebras, i.e., $\delta_0 - \delta_1$ (with symmetric coefficients not exceed the generic size), then Dadarlat and Loring defined an order structure on the K-theory with coefficient, which can kill the counterexample $\delta_0 - \delta_1$. Now, we find other counterexamples for generalized dimension drop algebras, namely, certain linear combination of $\delta_0, \delta_1$ with non symmetric sizes, a natural question is do we have a new order structure to exclude these ones. This could be answered in another paper.
2. In the present paper, we show the existence theorem fails at building block level (as for generalized dimension drop interval algebras), but we didn’t investigate it for limit algebras. For simple limits, as it is well known, the $K_0$-multiplicities of any partial map would be arbitrarily large, then no trouble. For real rank zero case, we can still get a classification by the Dadarlat-Loring order structure, because real rank zero condition can more or less control the dynamical behavior of connecting maps. This is done in a forthcoming paper. However, the most general limits are different story.

3. Jiang and Su’s criterion for KK-lifting is useful for more general C*-algebras on interval with dimension drops, e.g. splitting interval algebras with dimension drops in [11].

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