Generalization of Kuramoto model in vector product form

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We generalize the Kuramoto model of coupled phase oscillators to 3-dimensional space using vector product. It is demonstrated that the structure of the equation of the generalized model is identical to that of the rigid body rotation, well-known in classical mechanics. From that structure, the numerical integration method for the 3-dimensional Kuramoto model is immediate, which successfully mimics the analytic and arguable solutions. We find the conventional numerical method is not suitable for simulating the 3-dimensional Kuramoto model, and this problem is resolved by the new method direct from the model equation derived this time in vector product form.

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I. INTRODUCTION

The Kuramoto model [1] is a widely-used mathematical model for studying synchronization behavior in populations of coupled oscillators [2–4]. The model consists of coupling terms describing the interactions among phase oscillators, and natural frequencies randomly distributed according to a distribution function. When the coupling strength surpasses the diversity of natural frequencies, collective synchronization emerges. Variations of the Kuramoto model have been proposed, including time delays [5], inertia effects [6, 7], and thermal noise [8]. Coupled oscillator models have been also used to study the swarming [9–14] and flocking [15, 16] behavior of natural and artificial systems.

One of the recent interests in the study of Kuramoto model is to increase the dimension of the phase variable or the embedding space. The pioneering work in this direction is the consensus model, where an individual’s opinion is represented by a vector [17]. Later, it was shown that the algebraic generalization can be done systematically [18] using the rotation group theory [19–21]. With these tools, the Kuramoto model in higher dimensions has been steadily studied [22–31]. These studies often illustrate the results in 3-dimensional figure because visualization in the higher dimensions is not practical. The geometric sense or intuition in 3-dimension plays the irreplaceable role in understanding the results, and therefore a solid description of the model in 3-dimension is important.

In this work, we derive a new and suitable numerical method for the integration of the generalized Kuramoto model in 3-dimension, which manifests that the validity of the conventional method should be limited to the case where natural frequency is not taken into account. For this, we first generalize the Kuramoto model to 3-dimension in a vector product form, whose structure exactly corresponds to that of the equation of the rigid body rotation, well-known in classical mechanics. The angular velocity of the rotation of the oscillator is explicitly written in the model equation, from which the numerical method is immediate. It is demonstrated that the new method successfully mimics the available analytic and arguable solutions while the conventional one does not. We illustrate and argue the artifact of the conventional numerical integration method.

This paper is organized as follows. In Sec. II, we present our extension of the Kuramoto model to 3-dimensional space. The new numerical method direct from the model equation is explained in Sec. III. In Sec. IV, we show numerical results that are consistent with the analytic and arguable solutions, and compare them to the artifacts produced by conventional numerical methods. This paper finishes with the brief summary and perspective in Sec. V.

II. KURAMOTO MODEL IN VECTOR PRODUCT FORM

The Kuramoto model is given by a differential equation for the dynamics of coupled phase oscillators:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i),$$

(1)

where $\theta_i \in [0, 2\pi)$ is the phase of oscillator $i$ and $\omega_i$ is its natural frequency. The $K > 0$ is the coupling constant and $N$ is the number of oscillators in the system.

We consider a circle of unit radius at the origin of the $x$-$y$ plane in 3-dimensional space. Then, in the Cartesian coordinate system, we introduce the position vector for oscillator $i$ of phase $\theta_i$ on the circle: $\mathbf{r}_i = (x_i, y_i, 0) = (\cos \theta_i, \sin \theta_i, 0)$. This way, the oscillator of phase $\theta_i$ can be also identified with a particle or an agent at position $\mathbf{r}_i$. Below, for convenience, we interchangeably use oscillator, particle, or agent. Similarly, for another oscillator $j$ on the unit circle, it reads that $\mathbf{r}_j = (\cos \theta_j, \sin \theta_j, 0)$. For
$r_i$ and $r_j$ written this way, one can observe that i) $\dot{r}_i = (-\sin \theta_i, \cos \theta_i, 0) \dot{\theta}_i$, ii) $w_i \times r_i = (-\sin \theta_i, \cos \theta_i, 0)\omega_i$ for $w_i = (0, 0, \omega_i)$ and vector product operator $\times$ [21], and iii) $(r_i \times r_j) \times r_i = (-\sin \theta_i, \cos \theta_i, 0)\sin(\theta_j - \theta_i)$.

In the use of these observations, one can write

$$\dot{r}_i = w_i \times r_i + \frac{K}{N} \sum_{j=1}^{N} (r_i \times r_j) \times r_i.$$ \hspace{1cm} (2)

This formula is straightforward after applying $(-\sin \theta_i, \cos \theta_i, 0)$ to both sides of Eq. (1). We then introduce $k \equiv K \sum_j r_j/N = K r_{CM}$, where $r_{CM}$ is the position center of the agents. For $k$, Eq. (2) is simply rewritten as

$$\dot{r}_i = (w_i - k \times r_i) \times r_i.$$ \hspace{1cm} (3)

The minus sign on the right hand side of Eq. (3) comes from $r_i \times r_j = -r_j \times r_i$.

Equation (3) is the spatial-coordinate representation of the standard Kuramoto model in Eq. (1) when subject to the restriction of $r_i = (\cos \theta_i, \sin \theta_i, 0)$ and $w_i = (0, 0, \omega_i)$. This restriction can be softened to give a generalization. Below, $r_i$ is allowed to reside on the unit sphere centered at the origin, and the angular velocity vector $w_i$ needs not to be parallel to z-direction. The differential equation in Eq. (3) with this softening is the generalized Kuramoto model we will study in this work.

Since the vector product $\times$ is a linear operation, one may consider a matrix $\Omega_{w_i}$ that satisfies $\Omega_{w_i} r_i = w_i \times r_i$. From the detail of the operation $\times$, it is given that

$$\Omega_{w_i} = \begin{pmatrix} 0 & -\omega_{i,z} & \omega_{i,y} \\ \omega_{i,z} & 0 & -\omega_{i,x} \\ -\omega_{i,y} & \omega_{i,x} & 0 \end{pmatrix}$$ \hspace{1cm} (4)

for $w_i = (\omega_{i,x}, \omega_{i,y}, \omega_{i,z})$. Additionally, by the identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ [21], Eq. (3) is rewritten as

$$\dot{r}_i = \Omega_{w_i} r_i + k - (k \cdot r_i)r_i.$$ \hspace{1cm} (5)

We remark that Eq. (5) is identical to the expression pioneered in Ref. [17] for $w_i = 0$ and also to what formulated in Ref. [18, 23] for general $w_i$. That is, Eqs. (3) and (5) are equivalent.

\section{III. NUMERICAL INTEGRATION AND ROTATION AXIS}

The conventional method for the numerical integration of differential equation Eq. (3) or (5) begins with to consider $r_i(t) + \Delta t \dot{r}_i(t)$ for small time-mesh $\Delta t$, and then normalization follows to make the agent stay on the unit sphere. Thus when the result is denoted with $r_i(t + \Delta t)$, it reads that

$$r_i(t + \Delta t) = \frac{r_i(t) + \Delta t \dot{r}_i(t)}{||r_i(t) + \Delta t \dot{r}_i(t)||}.$$ \hspace{1cm} (6)

This method, first proposed in [17], is supposed to yield the numerical results in various works such as [18, 22–25, 27, 29, 31] where the numerical technique is not explicitly explained. $r_i(t + \Delta t)$ appears to be valid as it approaches $r_i(t)$ as $\Delta t$ approaches zero. However, we note that this limit does not guarantee the validity of $r_i(t + \Delta t)$, as follows.

The differentiable motion on sphere is locally an arc with its own curvature. To know the local arc approximated by Eq. (6) for small $\Delta t$, one can consider the rotation from $r_i(t)$ to $r_i(t + \Delta t)$. Let $a_i(t, \Delta t)$ be the angle vector from $r_i(t)$ to $r_i(t + \Delta t)$. Then, it reads that $r_i(t) \times r_i(t + \Delta t) = a_i \sin(\|a_i\|)$, where $a_i$ is the short notation for $a_i(t, \Delta t)$ and $a_i$ is its direction vector. Since $|a_i(t, \Delta t)| \approx \Delta t$ for small $\Delta t$, from $\sin(\|a_i\|) = |a_i| + O(\|a_i\|^2)$, it follows that $a_i = (r_i(t) \times r_i(t + \Delta t))/\Delta t = O((\Delta t)^2)).$ Using rotation matrix $R(a_i)$, one may write $r_i(t + \Delta t) = R(a_i)r_i(t)$. Then, by substituting $r_i(t + \Delta t)$ in $a_i \equiv a_i(t, \Delta t)$ with Eq. (6), one can obtain

$$r_i(t + \Delta t) = R \left( (r_i(t) \times r_i(t)) \Delta t (1 + O((\Delta t)^2)) \right) r_i(t),$$ \hspace{1cm} (7)

where the higher-order correction comes from the expansion of $\|r_i(t + \Delta t)\|$ for small $\Delta t$. Equation (7) demonstrates $r_i(t + \Delta t)$ lies on the great-circle tangent to $r_i(t)$ because $(r_i(t) \times r_i(t)) \cdot r_i = 0$. We remark that the position vector $r_i(t + \Delta t)$ is always located on the great circle, no matter how small $\Delta t$ is.

Figure 1 shows two circles tangent to a tangential vector at a point on sphere. The black arrow represents the tangential vector. The green circle is the great circle that is tangent to the arrow. The purple circle that turns around from the north pole is also tangent to the arrow. When the black arrow represents $r_i(t)$, the green great circle provides the place on which $r_i(t + \Delta t)$ of Eq. (6) should lie (this is the message of Eq. (7)). Here, we may question if there can be another tangential circle along which a difference equation for position update is written in the condition that its difference rate approaches $r_i(t)$ in the $\Delta t \to 0$ limit. The purple circle show there is no reason to adhere to the green great circle. Note, when a position update is considered along the purple circle with displacement proportional to $\Delta t$, the difference rate can be tuned to approach $r_i(t)$ in the $\Delta t \to 0$ limit.

Actually, there is an infinite number of circles that are tangent to the black arrow though only two are illustrated in Fig. 1. This highlights that the choice of a specific circle from the infinite possibilities, like Eq. (6), can result in a numerical artifact that persists regardless how small $\Delta t$ is. Instead, the recognition of correct circle should be operated by the model equation itself, Eqs. (3) or (5), where every information about the motion of the Kuramoto model is described or encoded. The edge of Eq.(3) over Eq. (5) is that the former explicitly displays the most important information about the motion on sphere, the axis of rotation at any instant.

According to the physics of the rotation of rigid body [32], the parenthesis part of Eq. (3), $w_i - k(t) \times r_i(t)$,
the latter is along 

When this velocity is considered constant over a small $\Delta t$, where $I$ is the identity matrix. The concept of $\Omega_n$, already well-known as an infinitesimal transformation in traditional literature, had led to the study of Lie algebra [33]. The infinitesimal transformation of the conventional numerical method Eq. (6) is $\Omega_{r_i \times r_i}$ as shown in Eq. (7) while that for the new method Eq. (8) is $\Omega_{w_i \times k \times r_i}$. Since $\Omega_{r_i \times r_i} \neq \Omega_{w_i \times k \times r_i}$ and this is $\Delta t$-independent, the numerical integration methods in Eqs. (6) and (8) are distinct even for arbitrarily small $\Delta t$. Thus, $r_i^h(t+\Delta t)$ [Eq (6)] is not a suitable method for the numerical integration of Eq. (5) (or, equivalently, (3)). Note this model equation is the destination of the new method $r_i^h(t+\Delta t)$ [Eq (8)] in the $\Delta t \to 0$ limit.

Although the idea behind $r_i^h(t+\Delta t)$ seems to be valid, we have argued that it is actually not. The Lie algebra attributed to rotation axis is finer than the idea for $r_i^h(t+\Delta t)$. We were motivated to discern $\Omega_{w_i \times k \times r_i}$ of $r_i^h(t+\Delta t)$ from $\Omega_{r_i \times r_i}$ for $r_i^h(t+\Delta t)$ by the observation that there are infinitely many circles tangent to each others at a point on sphere. $r_i^h(t+\Delta t)$ appears to be suggested in disregard of the ambiguity caused by such infinite number of possibilities.

IV. COMPARISON OF THE TWO METHODS

In this section, it will be demonstrated that $r_i^h(t+\Delta t)$ successfully mimics the known analytic or arguable solutions while $r_i^h(t+\Delta t)$ does not. The artifact of $r_i^h(t+\Delta t)$ is also analyzed. Before this, we first mention a special case where $r_i^h(t+\Delta t)$ can replace $r_i^h(t+\Delta t)$.

A. $w_i = 0$: unique matching case

When $w_i = 0$, Eqs. (4) and (5) gives $r_i \times \dot{r}_i = r_i \times k$. Thus, in this case, $\Omega_{r_i \times r_i} = \Omega_{w_i \times k \times r_i}$, holds. This explains why the numerical result of [17], where the method of Eq. (6) was suggested, is still valid: note $w_i = 0$ for all $i$ therein. Figure 2 is drawn with the data obtained by $r_i^h(t+\Delta t)$ and $r_i^b(t+\Delta t)$. It shows that the difference between the two methods is already negligible for $\Delta t = 0.01$. By the positive coupling without natural frequency, all agents are expected to converge to a single point, and the result is so. The convergence is similar to the ferromagnetic ordering found in spin systems [35] like the classical Heisenberg model in an infinite dimension. It can be interpreted as an extension of the relationship between the standard Kuramoto model with identical natural frequencies and the mean-field XY model.

One can directly find the condition of $w_i$, for which $r_i \times \dot{r}_i = w_i \times k \times r_i$ holds, as follows. When Eq. (3) is plugged in the left-hand side, it reads that $r_i \cdot w_i = 0$. Since $r_i = r_i(t)$ is the dynamic variable that can be arbitrarily assigned, initially, on the sphere, $r_i \cdot w_i = 0$ for all $i$.
is possible only when \( \mathbf{w}_i = 0 \). This way, the two numerical methods become identical when natural frequency is not taken into account or vanishing. However, the equality between the infinitesimal transformations, \( \Omega_{x \times r_i} = \Omega_{x \times -C} = \), is generally not the case. Now we will examine the impact of the natural frequency.

### B. \( N = 1 \) & \( \mathbf{w} \neq 0 \): single agent mismatch

The simplest case with natural frequency is when \( N = 1 \) for constant \( \mathbf{w} \neq 0 \). In this case, from Eq. (3), the equation of motion reads

\[
\dot{\mathbf{r}} = \mathbf{w} \times \mathbf{r} = \Omega_{\mathbf{w}} \mathbf{r},
\]

where the subscript \( i \) is omitted for simplicity, and \( C \)-involved term disappears because \( C \propto R \) for \( N = 1 \). With no loss of generality, we use \( \mathbf{w} = (0, 0, w) \) to write the solution as

\[
\mathbf{r}(t) = (\rho_0 \cos(wt + \phi_0), \rho_0 \sin(wt + \phi_0), z_0),
\]

where \( \rho_0 = \sqrt{x_0^2 + y_0^2} \) and \( \phi_0 = \arctan(y_0/x_0) \) for initial \( \mathbf{r}(0) = (x_0, y_0, z_0) \) with \( ||\mathbf{r}(0)|| = 1 \) (the case of \( z_0 = 1 \) that gives no motion is excluded).

Because \( \mathbf{w} \) is constant, the numerical solution to Eq. (9) by Eq. (8) is \( \mathbf{r}^{n}(n\Delta t) = R^{n}(\mathbf{w}\Delta t)\mathbf{r}(0) = (\rho_0 \cos(nw\Delta t + \phi_0), \rho_0 \sin(nw\Delta t + \phi_0), z_0) \) for \( n = 1, 2, \ldots \). Then, all data points generated by \( R^{n}(n\Delta t) \) lie on the solution circle in Eq. (10), and the occupation by the data points becomes denser for smaller \( \Delta t \). As a result, Eq. (8) provides a proper numerical integration method for the analytic solution Eq. (10). However, the method in Eq. (6) does not, as follows.

Above all, when Eq. (6) is used, the first step location \( \mathbf{r}^0(\Delta t) = (\mathbf{r}(0) + \Delta t\dot{\mathbf{r}}(0))/||\mathbf{r}(0) + \Delta t\dot{\mathbf{r}}(0)|| \) is not on the solution circle in Eq. (10), unless \( z_0 = 0 \). This is because i) the \( z \)-component of \( \mathbf{r}(0) + \Delta t\dot{\mathbf{r}}(0) \) is \( z_0 \) and ii) \( ||\mathbf{r}(0) + \Delta t\dot{\mathbf{r}}(0)|| > 1 \). The first comes from the fact that \( \dot{\mathbf{r}}(0) = (0, 0, w) \times \mathbf{r}(0) \) has no \( z \)-component, and the second comes from that \( \dot{\mathbf{r}}(0) \) of unit length is perpendicular to \( \mathbf{r}(0) \). It is thus follows that \( z_0/(1 + \delta_0) < z_0 \) for \( \delta_0 \propto \Delta t^2 > 0 \) (\( z_0 > 0 \) is assumed with no loss of generality). The decrease of \( z \)-coordinate value is repeated to show \( z_0 > z_1 > z_2 > \ldots \) with \( z_{i+1} \equiv z_i/(1 + \delta_i) \) for \( \delta_i \propto \Delta t^2 > 0 \). Interestingly, the decrease at each step \( \Delta_i \equiv z_i - z_{i+1} \) follows \( \Delta_i \propto z_i \). Then, the updated data points approach to the equator at \( z = 0 \) for small \( \Delta t \), as demonstrated in Fig. 3 with the numerical data generated by Eq. (6). The limit cycle rotating along the equator is always the long-term state of the numerical method Eq. (6) independent of initial conditions when \( N = 1 \) and \( \mathbf{w} \neq 0 \).

### C. \( \mathbf{w}_i = \mathbf{w} \neq 0 \): many agents mismatch I

For the system with \( \mathbf{w}_i = \mathbf{w} \) for all \( i \), using Eq. (8), we observed that all agents converge to a rotating point of angular velocity \( \mathbf{w} \) as time increases. We use \( \mathbf{w} = (0, 0, 1) \) with no loss of generality. The results are illustrated in the upper hemisphere of Fig. 4. All agents gather together before long; the data points at \( t = 10 \) already almost look like a single point. The deviation of all agents’ locations is much smaller than the point size. Thereafter, \( \mathbf{r}_i \approx \mathbf{r}_{CM} \) so that \( \mathbf{k} \times \mathbf{r}_i \approx 0 \). When these approximations are applied to Eq. (3), it follows that \( \mathbf{r}_{CM} \approx \mathbf{w} \times \mathbf{r}_{CM} \) whose structure is similar to that of Eq. (9). Then, \( \mathbf{r}_{CM} \) readily gives the limit cycle like Eq. (10). The circle on a latitude in the upper hemisphere of Fig. 4 is its numerical realization.

Analytically, if viewed in the rotating frame of angular velocity \( \mathbf{w} \), the behavior of all particles/agents is simply a convergence to a fixed point. It should looks like Fig. 2. A compact mathematical proof of this behavior is provided in [23]. The steady state is a limiting circle whose latitude is determined by the initial conditions. Obviously, the latitude of the limiting circle, which corresponds to the \( z \)-coordinate of the fixed point in the rotating frame, needs not to be zero. However, this natural feature is not the case when Eq. (6) is employed.
Another notable artifact of Eq. (6) is that its steady state is always the great circle at equator for the same setting of \( \mathbf{w} \) as above. In the subsection IV B, we examined the destination of a single agent with natural frequency vector parallel to \( z \)-axis is the equator at \( z = 0 \) when Eq. (6) is used for numerical integration. Thus if there are more agents with the same natural frequency, it is natural to expect that i) all trajectories will approach the equator at \( z = 0 \) because this is the destination of each agent and ii) all agents will collapse to one point by the positive coupling \( K \). In the lower hemisphere of Fig. 4, the grey arrows by Eq. (6), representing the motion of positions’ center, is the numerical observation that corresponds with the arguments above. Regardless of the initial condition, the collective steady state is the rotation of collapsed points along the great circle at \( z = 0 \), perpendicular to the common natural frequency vector. This is the artifact by the conventional method in Eq. (6).

A system of such agents with identical natural frequencies was studied in [31], and numerical data were presented. However, since the quantity in interest therein is invariant to \( \mathbf{w} \), the numerical data were obtained for \( \mathbf{w} = 0 \). Our finding of Eq. (8) implies, if the \( \mathbf{w} \neq 0 \) case was numerically tested with Eq. (6) or with one of its refinements that are still in disregard of \( \Omega_{\mathbf{w}_i - \mathbf{k} \times \mathbf{r}_i} \), the result would become different. Interestingly, we found in [23] such numerical trajectory similar to the one drawn in the upper hemisphere of Fig. 4. A simple way to obtain such data without using \( \Omega_{\mathbf{w}_i - \mathbf{k} \times \mathbf{r}_i} \) is following; first, prepare the trajectory data of all agents using Eq. (6) with \( \mathbf{w}_i = 0 \) for all \( i \), and then rotate the data using \( \mathbf{w}_i = \mathbf{w} \) in the consideration of the time label of each data point.

FIG. 3. (Color Online) Approaching to the great circle by the conventional method Eq. (6) when \( N = 1 \) and \( \mathbf{w} \neq 0 \). The great circle in this figure is the equator that is perpendicular to the used \( \mathbf{w} = (0, 0, 2.2) \). Each arrow is the displacement during \( 2n \leq t \leq 2n + 0.15 \) for \( n = 0, 1, 2, \ldots \). Though not shown, there is one round trip between adjacent arrows (we did not display that because such details make visualization rather confusing). As the time-mesh, \( \Delta t = 0.01 \) is used.

FIG. 4. (Color Online) [Upper hemisphere] Formation of the limit cycle of collapsed agents in a system of size \( N = 10 \) by the method in Eq. (8) when \( \mathbf{w}_i = \mathbf{w} = (0, 0, 1) \) for all \( i \). The solid circles with different colors represent the agents at the specified time, and the black bold arrow line is the displacement of the position center \( r_{CM} \) per unit time. Initially, the agents are randomly distributed on the surface. The sequence of the black bold line shows that the limit cycle is formed around \( t = 5 \), and this is consistent with the analytic understanding in the text. We use \( \Delta t = 0.01 \) and \( K = 1 \).

[Lower hemisphere] Result by the conventional method in Eq. (6) for the same numerical setting used above. For the visualization and comparison, the data are displayed on the lower hemisphere by using \( -z \) instead of \( z \). The four grey bold arrows are the displacement of \( r_{CM} \) per unit time during \( 0 \leq t \leq 4 \). Each short grey arrow is the displacement of \( r_{CM} \) during \( n \leq t \leq n + 0.15 \) for \( n = 4, 5, \ldots \). The final destination is the great circle parallel to \( z \)-axis. This observation on such great circle as the long-term state of Eq. (6) is independent of the initial conditions (see text for its reason). Note the limit cycle by Eq. (8) is not necessarily a great circle as illustrated in the upper hemisphere.
FIG. 5. (Color Online) Spiral to the fixed point by non-identical parallel $w_i$s. The vertical value of the main plot is the numerical obtained distance between the position center $(x, y, z)$ at time $t$ of the agents in the system and the rotation axis; $z$-axis is chosen as the direction of the rotation, i.e., $w_i = (0, 0, \omega_i)$. As shown, $\sqrt{x^2 + y^2}$ stays around 1 even up to $t \approx 10^4$ (the horizontal black line). However, it decays thereafter following $\rho \sim t^{-c}$ for a positive $c$ (the declining purple line), where $c \approx 1$ by fitting the data. This asymptotic form is used in the inset to draw the spiral to the fixed point at the pole (the purple curve in the upper hemisphere). The black curve in the inset shows the position center up to $t \approx 10^4$. We use $\Delta t = 0.01$, $K = 32$, and $N = 10$. Initially, the agents are randomly distributed. The $\omega_i$ are chosen from the uniform distribution with unit mean and the width 0.2.

D. $w_i \propto w_j$: many agents mismatch II

The next case is that all $w_i$s are not identical but parallel with each other, i.e., $w_i \propto w_j$ for all $i$ and $j$ (we still simply use $w_i = (0, 0, \omega_i)$). In this case, one may imagine a frustrating situation where the spreading of the agents along latitudinal direction by the different $w_i$s contradicts the convergence by positive coupling $K$; the spreading prevents the convergence, and vice versa. We note there is no such frustration along longitudinal direction. Thus, the agents readily form an arc on a latitude as the longitudinal attraction by positive $K$ lasts without intervention. Interestingly, there are two specific locations on the sphere, where the frustration does not happen.

The locations in interest are the poles, where the rotation axis (this is unique for all parallel $w_i$s) penetrates the surface of the sphere. The reason why the poles are special is that rotation at pole does not result in any actual motion. Thus, the conflict between spreading and converging disappears when all particles/agents move towards one of the poles. If this occurs, the system ultimately reaches a fixed point solution at the pole.

Figure 5, drawn with data generated by Eq. (8), shows that the trajectory of the position center $r_{\text{CM}}$ forms a spiral to the fixed point at a pole. In the early stage, a limit cycle seems to appear, but this arrives at the fixed point at a pole in the end. Although a cycle lasts longer for larger $K$ and smaller $N$, it finally disappears in the numerical tests. The data obtained for $K = 32$ and $N = 10$ are shown in Fig. 5. In the numerical simulations, we have not observed the other kind of long-term state except the fixed point at a pole for non-identical parallel $w_i$s.

A possible explanation why $r_{\text{CM}}$ heads to the pole is following. We first note that the spreading by the different $w_i$s occurs in the lateral direction only. Since there is no disturbance in converging along longitudinal direction, the agents are expected to form a cluster of arc on a latitude after initial duration. The positive coupling causes each agent in a pair to attract the other along the geodesic direction between them. The attraction on agent $i$ by agent $j$ is applied following $(r_i \times r_j) \times r_i$ (see Eq. (2)), which is tangent to the geodesic curve between $r_i$ and $r_j$. Here, we remark that the geodesic curve between any two points of the arc is on the northern side compared to the arc itself (this is explained now for the upper hemisphere). See the diagram in Fig. 6. As a result, the longitudinal component of any attraction always points towards the north pole.

Hence, all particles/agents are biased to the pole. The bias continues as long as the agents form an arc with lateral spreading by different $w_i$s on a latitude. The bias weakens as the arc approaches the pole. It vanishes when all agents converges the pole to form the fixed point at the pole. If this process happens below the equator, the final destination becomes the south pole.

The numerical result becomes different when the con-
v. summary & outlook

A natural interest in the study of the high-dimensional Kuramoto models is the dimension-dependent properties. The remarkable finding on the continuous/discontinuous synchronization transition depending dimension is one of the representative results [29]. In the present work, we have investigated how the rotation axis matters, which cannot be an interest for the standard Kuramoto model in spatial 2-dimension. The validity of a difference equation for the numerical integration of the given differential equation is a basic issue and, probably, one of the most important practical problems when no analytic solution is available except a few trivial cases.

We first have explained the standard Kuramoto model can be understood with vector product in 3-dimensional space. This naturally leads to a generalization of the model for the dynamics of the 3-dimensional spherical coordinate of the agent on the unit sphere or, equivalently, for that of the 2-dimensional spherical phase. It is revealed that the conventional method for the numerical integration of the Kuramoto model in 3-dimension [Eq. (6)] is valid when the natural frequency is not taken into account. A new numerical method [Eq. (8)] that is also valid in the presence of natural frequencies are written as a direct application of the model equation [Eq. (3)] we have derived this time.

For the comparison of the numerical integration methods in Eqs. (6) and (8), we have checked the reproducibility of the analytic or arguable solutions available for the rather simple systems. It is illustrated that Eq. (8) can mimic such solutions while Eq. (8) yields artifacts far from mimicking. Although Eq. (8) is expected to be most suitable in numerical integration, its validity is limited to 3-dimension. This is because vector product that explicitly reveals the rotation character is defined in 3-dimension only. Thus, a generalization of Eq. (8) to the higher dimensions remains an interesting problem, and this is now under investigation. Still, in 3-dimension, the comparison of Eqs. (6) and (8) for the more general setup of natural frequency vectors are also necessary.

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