Forcing Total $dr$-Power Domination Number of Graphs
Under Some Binary Operations

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Abstract. In this paper, the total $dr$-power domination number of graphs such as complete
bipartite graph, generalized fan and generalized wheel are obtained. The forcing total $dr$-power
domination number of graphs resulting from some binary operations such as join, corona and
lexicographic product of graphs were determined.

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product

1. Introduction

Let $G = (V, E)$ be a graph representing the electrical power system, where a vertex
represents an electrical node and an edge represents a transmission line joining two electrical
nodes. Some measurement devices must be placed at selected locations so that all the state
variables of the system can be measured in order to monitor the power system. A Phase
Measurement Unit (PMU) is a measurement device placed on a vertex and has the ability
to measure the state of the vertex and the edges connected to the vertex. The vertices and
edges that are measured by PMU’s are said to be observed. In this study, it is necessary
that each vertex with PMU is adjacent to another vertex with PMU also. But because of
the high cost value of a PMU, it is desirable to minimize their number while maintaining
the ability to monitor the entire power system.

The graphs considered in this paper are simple, connected, undirected and without
loops or multiple edges.

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The open neighborhood of $v$ in
$G$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is
the set $N[v] = N(v) \cup \{v\}$. For $X \subseteq V(G)$, the open neighborhood of $X$ is the set
$N(X) = \cup_{v \in X} N_G(v)$ and its closed neighborhood is the set $N[X] = N(X) \cup X$.

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A set $S \subseteq V(G)$ is a dominating set (resp. total dominating set) of $G$ if $N[S] = V(G)$ (resp. $N(S) = V(G)$). The domination number $\gamma(G)$ (resp. total domination number $\gamma_t(G)$) of $G$ is the minimum cardinality of a dominating set (resp. total dominating set). If $S$ is a dominating set (resp. a total dominating set) with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$), then we call $S$ a $\gamma$-set (resp. a $\gamma_t$-set) of $G$.

Let $G = (V,E)$ be a simple graph. Let $P \subseteq V(G)$. An edge $e = uv$ of $G$ is directly observed by $P$ if $u \in P$ or $v \in P$. A vertex $u$ of $G$ is directly observed if $u$ is incident to a directly observed edge. An edge $e' = xy$ is remotely observed by $P$ if $x,y \notin P$ and $x,y$ are directly observed vertices or at least one of $x$ and $y$ is incident to $k$ edges where $k - 1$ of these edges are directly observed by $P$. A non-directly observed vertex $u$ of $G$ which is incident to a remotely observed edge is called remotely observed vertex. Let $O^P_V(G)$ be the set of all directly and remotely observed vertices and $O^P_G(G)$ be the set of all directly and remotely observed edges. Then $P \subseteq V(G)$ is a $dr$-power dominating set (dr-pds) of $G$ if $O^P_V(G) = V(G)$ and $O^P_G(G) = E(G)$. The minimum cardinality of a $dr$-power dominating set is called the $dr$-power domination number of $G$ and is denoted by $\gamma_{pw}^*(G)$. A subset $P$ of $V(G)$ with cardinality $\gamma_{pw}^*(G)$ is called a $\gamma_{pw}^*$-set of $G$. A $dr$-power dominating set $D$ is said to be a total $dr$-power dominating set (tdr-pds) if the induced subgraph $(D)$ has no isolated vertex. The minimum cardinality of a total $dr$-power dominating set (tdr-pds) is called the total $dr$-power domination number of $G$ and is denoted by $\gamma_{tpw}^*(G)$. A subset $T$ of $V(G)$ with cardinality $\gamma_{tpw}^*(G)$ is called a $\gamma_{tpw}^*$-set of $G$. Moreover, there exists a connected graph $G$ such that $\gamma_{tpw}^*(G) \leq \gamma_t(G)$.

Let $S$ be a $\gamma_{tpw}^*$-set of a graph $G$. A subset $D$ of $S$ is said to be a forcing subset for $S$ if $S$ is the unique $\gamma_{tpw}^*$-set containing $D$. The forcing total $dr$-power domination number of $S$ is given by $f\gamma_{tpw}^*(S) = \min\{|D| : D$ is a forcing subset for $S\}$. The forcing total $dr$-power domination number of $G$ is given by

$$f\gamma_{tpw}^*(G) = \min\{f\gamma_{tpw}^*(S) : S$ is a $\gamma_{tpw}^*$-set of $G\}.$$

The join of two graphs $G$ and $H$, denoted by $G \circ H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The corona $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then forming the join $\langle \{v\} \rangle + H^v = v + H^v$, where $H^v$ is a copy of $H$, for each $v \in V(G)$.
The lexicographic product (composition) $G[H]$ of two graphs $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$, and $(u,u')(v,v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Amos [1] studied total domination. The relation between forcing and domination concepts was investigated by Chartrand et al. [5] and they defined “forcing domination number”. The following concepts: total $dr$-power domination [6], forcing domination number of graphs under some binary operations [7], forcing total domination number and forcing connected domination number under the lexicographic product of graphs [8], forcing independent domination number of a graph [4], and A-differential of graphs [3] was studied by Canoy, et al. The total $dr$-power domination number of some special graphs such as paths and cycles was studied by Armada [2].

2. Known Results

This section contains known results involving total $dr$-power domination, $dr$-power domination, total domination numbers of a graph $G$ that are very useful in proving the main results of this paper.

**Remark 2.1.** [6] For a graph $G$ without isolated vertices,

$$\gamma^*_t\gamma^*_p(G) \leq \gamma^*_t\gamma^*_p(G) \leq \gamma_t(G).$$

**Proposition 2.2.** [1] The total domination number of a cycle $C_n$ or a path $P_n$ on $n \geq 3$ vertices is given by

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2}, & n \equiv 0(\text{mod } 4), \\ \frac{n+2}{2}, & n \equiv 2(\text{mod } 4), \\ \frac{n+1}{2}, & \text{otherwise}. \end{cases}$$

**Theorem 2.3.** [6] Let $G$ and $H$ be any graphs. Then $P \subseteq V(G + H)$ is a total $dr$-power dominating set of $G + H$ if and only if it satisfies one of the following conditions:

(i) $P \subseteq V(G)$ and is a total dominating set, provided that $G$ is a graph with no isolated vertex;

(ii) $P \subseteq V(H)$ and is a total dominating set, provided that $H$ is a graph with no isolated vertex; or

(iii) $P = P_1 \cup P_2$, where $\emptyset \neq P_1 \subseteq V(G)$ and $\emptyset \neq P_2 \subseteq V(G)$.

**Corollary 2.4.** [6] Let $G$ and $H$ be any graphs. Then

$$\gamma^*_t\gamma^*_p(G + H) = 2.$$
Theorem 2.5. [6] Let $G$ be a nontrivial connected graph and $H$ be a graph with no isolated vertex. Then $P \subseteq V(G \circ H)$ is a total $dr$-power dominating set if and only if

$$P = A \cup \left( \bigcup_{v \in A} B_v \right) \cup \left( \bigcup_{u \notin A} D_u \right)$$

where $A \subseteq V(G)$, $B_v \subseteq V(H^v)$ for each $v \in A$ and $B_v \neq \emptyset$ for each $v \notin N_G(A)$, and $D_u \subseteq V(H^u)$ is a total dominating set of $H^u$ for each $u \notin A$.

Corollary 2.6. [6] Let $G$ be a nontrivial connected graph of order $m$ and $H$ be any graph with no isolated vertex. Then $\gamma_{t_{pw}}(G \circ H) = m$.

Theorem 2.7. [6] Let $G$ and $H$ be nontrivial connected graphs. Then $P = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, is a total $dr$-power dominating set of $G[H]$ if and only if $S$ is a dominating set of $G$, and $T_x$ is a total dominating set of $H$ for each $x \in S \setminus N(S)$.

Corollary 2.8. [6] Let $G$ and $H$ be nontrivial connected graphs. Then $P$ is a total $dr$-power dominating set of $G[H]$ if and only if $P$ is a total dominating set of $G[H]$. Moreover,

$$\gamma_{t_{pw}}^*(G[H]) = \gamma_t(G).$$

Theorem 2.9. [2] Let $G$ be a graph. Then

(i) $f\gamma_{t_{pw}}^*(G) = 0$ if and only if $G$ has a unique $\gamma_{t_{pw}}^*$-set.

(ii) $f\gamma_{t_{pw}}^*(G) = 1$ if and only if $G$ has at least two $\gamma_{t_{pw}}^*$-sets and there exists a vertex $v$ which is contained in exactly one $\gamma_{t_{pw}}^*$-set of $G$.

Corollary 2.10. [2] Let $G$ be a connected graph. Then

$$0 \leq f\gamma_{t_{pw}}^*(G) \leq \gamma_{t_{pw}}^*(G).$$

Theorem 2.11. [2] Let $G$ be a nontrivial graph. Then $f\gamma_{t_{pw}}^*(G) = \gamma_{t_{pw}}^*(G)$ if and only if for every $\gamma_{t_{pw}}^*$-set $P$ of $G$ and for each $v \in P$, there exists $u \in V(G) \setminus P$ such that $[P \setminus \{v\}] \cup \{u\}$ is a $\gamma_{t_{pw}}^*$-set of $G$. 
3. Forcing Total $dr$-Power Domination Number of the Join of Graphs

This section contains the total $dr$-power domination number of the complete bipartite graphs, generalized fan graphs, generalized wheel graphs, $P_n + P_m$, $P_n + C_m$ and $C_n + C_m$ and their forcing total $dr$-power domination numbers.

**Corollary 3.1.** Let $G$ and $H$ be any graphs. Then $R \subseteq V(G + H)$ is a $\gamma_{tpw}^*$-set of $G + H$ if and only if at least one of the following holds:

(i) $R$ is a $\gamma_t$-set of $G$ and $|R| = 2$,

(ii) $R$ is a $\gamma_t$-set of $H$ and $|R| = 2$,

(iii) $|R \cap V(G)| = 1$ and $|R \cap V(H)| = 1$.

**Theorem 3.2.** For any graphs $G$ and $H$,

\[
 f_{\gamma_{tpw}^*}(G + H) = \begin{cases} 
 0, & \text{if } G \text{ and } H \text{ are both trivial,} \\
 1, & \text{if } G \text{ is trivial and (i) } H \text{ has an isolated vertex or} \\
 & \text{(ii) } \gamma_t(H) > 2 \text{ or (iii) } \gamma_t(H) = 2 \text{ and there exists} \\
 & \text{a vertex in } H \text{ which is not in any } \gamma_t\text{-set of } H, \\
 & \text{or if } H \text{ is trivial and (i) } G \text{ has an isolated vertex or} \\
 & \text{(ii) } \gamma_t(G) > 2 \text{ or (iii) } \gamma_t(G) = 2 \text{ and there exists} \\
 & \text{a vertex in } H \text{ which is not in any } \gamma_t\text{-set of } G, \\
 2, & \text{otherwise.} 
\end{cases}
\]

**Proof.** Consider the following cases:

Case 1: $G$ and $H$ are both trivial graphs. 
\{x, y\} such that $x \in V(G)$ and $y \in V(H)$ is the only $\gamma_t$-set of $G + H$ by Corollary 3.1. Thus, $f_{\gamma_{tpw}^*}(G + H) = 0$ by Theorem 2.9(i).

Case 2: $G$ is trivial and (i) $H$ has an isolated vertex or (ii) $\gamma_t(H) > 2$ or (iii) $\gamma_t(H) = 2$ and $H$ contains a vertex which is not in any $\gamma_t$-set of $H$

Let $V(G) = \{x\}$. Suppose that $H$ has an isolated vertex, say $w$. By Corollary 3.1, $R_w = \{x, w\}$ is the only $\gamma_{tpw}^*$-set of $G + H$ containing $w$. If $\gamma_t(H) > 2$, then by Corollary 3.1, for each $u \in V(H)$, $R_u = \{x, u\}$ is the only $\gamma_{tpw}^*$-set of $G + H$ containing $u$. If $\gamma_t(H) = 2$ and there exists a vertex, say $v$, in $H$ which is not in any $\gamma_t$-set of $H$, then by Corollary 3.1, $R_v = \{x, v\}$ is the only $\gamma_{tpw}^*$-set of $G + H$ containing $v$. Thus, in any of the three cases, there always exists a vertex which is contained in exactly one $\gamma_{tpw}^*$-set of $G + H$.

By Theorem 2.9(ii), $f_{\gamma_{tpw}^*}(G + H) = 1$. Similarly, if $H$ is trivial and (i) $G$ has an isolated vertex or (ii) $\gamma_t(G) > 2$ or (iii) $\gamma_t(G) = 2$ and there exists a vertex in $G$ which is not in any $\gamma_t$-set of $G$, then $f_{\gamma_{tpw}^*}(G + H) = 1$. 


Case 3: $G$ is trivial, $\gamma_t(H) = 2$ and every vertex $v \in V(H)$ is contained in a $\gamma_t$-set of $H$

Let $V(G) = \{x\}$. By Corollary 3.1, for each $u \in V(H)$, $R_u = \{x, u\}$ is a $\gamma^*_t$-set of $G + H$. Also by assumption and Corollary 3.1, for all $v, w \in V(H)$ such that $v \neq w$, $R = \{v, w\}$ is a $\gamma_t$-set of $H$ and a $\gamma^*_t$-set of $G + H$. Clearly, no single element is contained in exactly one $\gamma^*_t$-set of $G + H$, that is, $f\gamma^*_t(G + H) \geq 2$. Consequently, by Corollary 2.10, $2 \leq f\gamma^*_t(G + H) \leq \gamma^*_t(G + H) = 2$. Therefore, $f\gamma^*_t(G + H) = 2$. Similarly, if $H$ is trivial, $\gamma_t(G) = 2$ and every vertex $v \in V(G)$ is contained in a $\gamma_t$-set of $G$, then $f\gamma^*_t(G + H) = 2$. 

Case 4: $G$ and $H$ are both nontrivial graphs.

By Corollary 3.1, for each $x \in V(G)$ and for each $u \in V(H)$, $R = \{x, u\}$ is a $\gamma^*_t$-set of $G + H$. Thus, for each $u \in R$, there exists $u_y \in V(G + H) \setminus R$ such that $[R \setminus \{u\}] \cup \{u_y\}$ is a $\gamma^*_t$-set of $G + H$. By Theorem 2.11, it follows that $f\gamma^*_t(G + H) = \gamma^*_t(G + H) = 2$. 

The next result is a direct consequence of Corollary 2.4 and Theorem 3.2.

**Corollary 3.3.** For any graph $H$, the total $dr$-power domination number of the join $K_1 + H$ is given by $\gamma^*_t(K_1 + H) = 2$ and its forcing total $dr$-power domination number is given by

$$
f\gamma^*_t(K_1 + H) = \begin{cases} 0, & \text{if } H \text{ is trivial}, \\ 1, & \text{if } H \text{ has an isolated vertex, or } \gamma_t(H) > 2, \text{ or } \\ \gamma_t(H) = 2 \text{ and there exists a vertex in } H \text{ which is not in any } \gamma_t\text{-set of } H, \\ 2, & \text{otherwise.} \end{cases}
$$

**Corollary 3.4.** The total $dr$-power domination number of the complete bipartite $K_{n,m} = \overline{K}_n + \overline{K}_m$ such that $n, m \geq 1$, is given by $\gamma^*_t(K_{n,m}) = 2$ and its forcing total $dr$-power domination number is given by

$$
f\gamma^*_t(K_{n,m}) = \begin{cases} 0, & n = 1 \text{ and } m = 1, \\ 1, & n = 1 \text{ and } m \geq 2 \text{ or } m = 1 \text{ and } n \geq 2, \\ 2, & n \geq 2 \text{ and } m \geq 2. \end{cases}
$$

**Proof.** By Corollary 2.4, $\gamma^*_t(K_{n,m}) = \gamma^*_t(\overline{K}_n + \overline{K}_m) = 2$. If $n = 1$ and $m = 1$, then $\overline{K}_1 = K_1$ is trivial, and so by Corollary 3.3, $f\gamma^*_t(K_{1,1}) = 0$. If $n = 1$ and $m \geq 2$, then $\overline{K}_m$ has an isolated vertex, and so by Corollary 3.3, $f\gamma^*_t(K_{1,m}) = 1$. Similarly, if $m = 1$ and $n \geq 2$, then $f\gamma^*_t(K_{n,1}) = 1$. If $n \geq 2$ and $m \geq 2$, then $\overline{K}_n$ and $\overline{K}_m$ are nontrivial graphs and so, by Theorem 3.2, $f\gamma^*_t(K_{n,m}) = 2$. 

\[\square\]
Corollary 3.5. The total $dr$-power domination number of the generalized fan $F_{n,m} = \overline{K}_n + P_m$, where $n \geq 1$ and $m \geq 2$, is given by $\gamma^*_p(F_{n,m}) = 2$ and its forcing total $dr$-power domination number is given by

$$f_{\gamma^*_p}(F_{n,m}) = \begin{cases} 1, & n \geq 1 \text{ and } m \geq 4, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_p(F_{n,m}) = \gamma^*_p(\overline{K}_n + P_m) = 2$. Let $P_n = [u_1, u_2, \ldots, u_n]$. If $n = 1$ and $m = 4$, then by Proposition 2.2, $\gamma(P_4) = 2$ and $P_4$ has exactly one $\gamma_t$-set which is $\{u_2, u_3\}$, that is, $u_1$ is a vertex not in a $\gamma_t$-set of $P_4$. By Corollary 3.3, $\gamma^*_p(F_{1,4}) = 1$. If $n = 1$ and $m > 4$, then by Proposition 2.2, $\gamma(P_m) > 2$. By Corollary 3.3, $\gamma^*_p(F_{1,m}) = 1$. If $n = 1$ and $m < 4$, then by Proposition 2.2, $\gamma(P_2) = \gamma(P_3) = 2$ and so, $\{u_1, u_2\}$ is the $\gamma_t$-set of $P_2$ while $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are $\gamma_t$-sets of $P_3$. Clearly, for $m = 2, 3$, every vertex $u_i \in V(P_m)$ is contained in a $\gamma_t$-set of $P_m$.

Corollary 3.6. The total $dr$-power domination number of the generalized wheel $W_{n,m} = \overline{K}_n + C_m$, where $n \geq 1$ and $m \geq 3$, is given by $\gamma^*_p(W_{n,m}) = 2$ and its forcing total $dr$-power domination number is given by

$$f_{\gamma^*_p}(W_{n,m}) = \begin{cases} 1, & n \geq 1 \text{ and } m \geq 5, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_p(W_{n,m}) = \gamma^*_p(\overline{K}_n + C_m) = 2$. Let $C_n = [u_n, u_1, u_2, \ldots, u_n]$. If $n = 1$ and $m \geq 5$, then by Proposition 2.2, $\gamma(C_m) > 2$. By Corollary 3.3, $\gamma^*_p(W_{1,m}) = 1$. If $n = 1$ and $m < 5$, then by Proposition 2.2, $\gamma(C_3) = \gamma(C_4) = 2$ and so, $\{u_1, u_2\}, \{u_2, u_3\}$ and $\{u_3, u_4\}$ are the $\gamma_t$-sets of $C_3$ while $\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}$ and $\{u_4, u_1\}$ are $\gamma_t$-sets of $C_4$. Clearly, for $m = 3, 4$, every vertex $u_i \in V(C_m)$ is contained in a $\gamma_t$-set of $C_m$. By Corollary 3.3, $\gamma^*_p(W_{1,m}) = 2$. If $n \geq 2$ and $m \geq 3$, then $\overline{K}_n$ and $C_m$ are nontrivial graphs. By Corollary 3.3, $f_{\gamma^*_p}(W_{n,m}) = 2$. \hfill $\square$

Corollary 3.7. The total $dr$-power domination number of the join $P_n + P_m$, where $n \geq 1$ and $m \geq 1$, is given by $\gamma^*_p(P_n + P_m) = 2$ and its forcing total $dr$-power domination number is given by

$$f_{\gamma^*_p}(P_n + P_m) = \begin{cases} 0, & n = 1 \text{ and } m = 1, \\ 1, & \text{either } n = 1 \text{ and } m \geq 4, \text{ or } m = 1 \text{ and } n \geq 4, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_p(P_n + P_m) = 2$. If $n = 1$ and $m = 1$, then $P_1 = K_1$ is a trivial graph and so, by Corollary 3.3, $\gamma^*_p(P_1 + P_1) = 0$. If $n = 1$ and $m \geq 4$, then $f_{\gamma^*_p}(P_1 + P_n) = f_{\gamma^*_p}(P_{1,m}) = 1$ by Corollary 3.5. Similarly, if $m = 1$ and $n \geq 4$, $f_{\gamma^*_p}(P_n + P_1) = 1$. If $n = 1$ and either $m = 2$ or $m = 3$, then by Corollary 3.5,
The total $dr$-power domination number of the join $P_n + C_m$, where $n \geq 1$ and $m \geq 3$, is given by $\gamma^*_{tpw}(P_n + C_m) = 2$ and its forcing total $dr$-power domination number is given by

$$f\gamma^*_{tpw}(P_n + C_m) = \begin{cases} 1, & n = 1 \text{ and } m \geq 5 \\ 2, & \text{otherwise.} \end{cases}$$

**Proof.** By Corollary 2.4, $\gamma^*_{tpw}(P_n + C_m) = 2$. If $n = 1$ and $m \geq 5$, then by Corollary 3.6, $f\gamma^*_{tpw}(P_1 + C_m) = f\gamma^*_ {tpw}(W_1,m) = 1$. If $n = 1$ and $m < 5$, then by Corollary 3.6, $f\gamma^*_ {tpw}(P_1 + C_m) = f\gamma^*_ {tpw}(W_1,m) = 2$. If $n \geq 2$ and $m \geq 2$, then $P_n$ and $C_m$ are nontrivial graphs. By Corollary 3.3, $f\gamma^*_ {tpw}(P_n + C_m) = 2$.

**Corollary 3.9.** The total $dr$-power domination number of the join $C_n + C_m$, where $n \geq 3$ and $m \geq 3$, and its forcing total $dr$-power domination number is given by

$$f\gamma^*_ {tpw}(C_n + C_m) = \gamma^*_ {tpw}(C_n + C_m) = 2.$$

**Proof.** By Corollary 2.4, $\gamma^*_ {tpw}(C_n + C_m) = 2$. Note that the cycles $C_n$ and $C_m$ are nontrivial graphs. By Theorem 3.2, $f\gamma^*_ {tpw}(C_n + C_m) = 2$.

## 4. Forcing Total $dr$-Power Domination Number of the Corona of Graphs

This section contains the forcing total $dr$-power domination number of the coronas $G \circ H$ and $K_m \circ H$ such that $G$ is a nontrivial connected graph, $K_m$ is a complete graph and $H$ is any graph.

**Theorem 4.1.** Let $G$ be a nontrivial connected graph and let $H$ be any graph. Then $R \subseteq V(G \circ H)$ is a $\gamma^*_ {tpw}$-set of $G \circ H$ if and only if $R = V(G)$. In particular,

$$f\gamma^*_ {tpw}(G \circ H) = 0.$$

**Proof.** Suppose that $R \subseteq V(G \circ H)$ is a $\gamma^*_ {tpw}$-set of $G \circ H$. Note that $V(G)$ is a $\gamma^*_ {tpw}$-set of $G \circ H$ by Corollary 2.6. Suppose that $R \neq V(G)$ and let $A = R \cap V(G)$, that is, $|A| < |V(G)|$. Since $R$ is a $\gamma^*_ {tpw}$-set of $G \circ H$, $R = A \cup \bigcup_{v \in A} B_v \cup \bigcup_{u \notin A} D_u$ as described in Theorem 2.5 where $|B_v| = 0$ for each $v \in A$ and $|D_u| = \gamma_t(H)$ for each $u \notin A$ or $u \in V(G) \setminus A$. Hence,
\(|R| = |A| + \gamma_t(H)(|V(G)| - |A|) \\
\geq |A| + 2|V(G)| - 2|A| \quad \text{since } \gamma_t(H) \geq 2 \\
\geq 2|V(G)| - |A| \quad \text{since } |V(G)| > |A| \\
\geq |V(G)| = m \quad \text{since } \gamma_t(H) \geq 2 \\
> |A| + 2|V(G)| - 2|A| \quad \text{since } |V(G)| > |A| \\
> 2|V(G)| = m \quad \text{by Corollary 2.6.} \\
\]

This is a contradiction since |R| = m by Corollary 2.6. Thus, R = V(G). The converse is clear. In particular, since V(G) is the unique \(\gamma_t^*\)-set of \(G \circ H\), by Theorem 2.9(i), \(f\gamma_t^*(G \circ H) = 0\). 

The next result follows directly from Corollary 3.3 and Theorem 4.1. Note that \(K_1 \circ H = K_1 + H\).

**Corollary 4.2.** Let \(K_m\) be a complete graph of order \(m \geq 1\) and let \(H\) be any graph. Then

\[
f\gamma_t^*(K_m \circ H) = \\
\begin{cases} 
0, & \text{if either } m = 1 \text{ and } H \text{ is trivial, or } m > 1, \\
1, & \text{if } m = 1 \text{ and either (i) } H \text{ has an isolated vertex, or } \\
& (ii) \gamma_t(H) > 2, \text{ or (iii) } \gamma_t(H) = 2 \text{ and there exists a vertex in } H \text{ which is not in any } \gamma_t\text{-set of } H, \\
2, & \text{if } m = 1, \gamma_t(H) = 2 \text{ and every vertex in } H \\
& \text{is contained in any } \gamma_t\text{-set of } H. 
\end{cases}
\]

5. Forcing Total \(dr\)-Power Domination Number of the Lexicographic Product of Graphs

This section contains the forcing total \(dr\)-power domination number of the graphs \(G[H]\), \(P_n[H]\) and \(C_n[H]\) where \(G\) and \(H\) are nontrivial connected graphs.

**Theorem 5.1.** Let \(G\) and \(H\) be nontrivial connected graphs. Then

\[f\gamma_t^*(G[H]) = \gamma_t(G).\]

**Proof.** Note that \(\gamma_t^*(G[H]) = \gamma_t(G)\) by Corollary 2.8. Now, suppose that \(P = \bigcup_{u \in S} ((u) \times T_u)\), where \(S\) is a \(\gamma_t\)-set of \(G\) and \(T_u \subseteq V(H)\). Consequently, \(|P| = |S| = \gamma_t(G)\). By Theorem 2.7 and Corollary 2.8, \(P\) is a \(\gamma_t^*\)-set of \(G[H]\). Suppose that \(f\gamma_t^*(G[H]) = f\gamma_t^*(P)\). Moreover, suppose that \(P\) has a forcing subset \(R\) with \(|R| < |P|\), that is, \(P = R \cup N\), where \(N = \{(u, v) \in P : (u, v) \notin R\}\). Pick \((u, v) \in N\). Note that there always exists a vertex \((u, w) \in V(G[H])\) such that \(w \neq v\) and \(|P \setminus \{(u, v)\}| \cup \{(u, w)\} = Q\) is a \(\gamma_t^*\)-set of \(G[H]\) since all adjacent vertices \((r, s)\)
of \((u, v)\) with \(u \neq r\) are adjacent vertices of \((u, w)\) also. Thus, \(Q = R \cup M\), where 
\[M = [N \setminus \{(u, v)\}] \cup \{(u, w)\},\]
\[O^2_Q(G[H]) = V(G[H]) \quad \text{and} \quad O^2_Q(G[H]) = E(G[H]),\]
that is, \(Q\) is a \(\gamma^*_t\)-set containing \(R\), a contradiction. Thus, \(|R| = |P|\) and \(P\) is the only forcing subset for \(P\). Therefore, \(f_{\gamma^*_t}(G[H]) = |P| = \gamma_t(G)\). \(\square\)

The next result follows from Theorem 5.1 and Corollary 2.2.

**Corollary 5.2.** Let \(H\) be a nontrivial connected graph and \(n \geq 3\). Then
\[f_{\gamma^*_t}(P_n[H]) = f_{\gamma^*_t}(C_n[H]) = \begin{cases} \frac{n}{2}, & n \equiv 0(\text{mod } 4), \\ \frac{n+2}{2}, & n \equiv 2(\text{mod } 4), \\ \frac{n+1}{2}, & \text{otherwise}. \end{cases}\]

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