Collisional decoherence of a tracer particle moving in one dimension

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We study decoherence of the external degree of freedom of a tracer particle moving in a one dimensional dilute Boltzmann gas. We find that phase averaging is the dominant decoherence effect, rather than information exchange between tracer and gas particles. While a coherent superposition of two wave packets with different mean positions quickly turns into a mixed state, it is demonstrated that such superpositions of different momenta are robust to phase averaging, until the two wave packets acquire a different position due to the different velocity of each wave packet.

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I. INTRODUCTION

The transition from quantum mechanics to classical physics is one of the most debated problems in the history of modern physics. In particular, the question arises why one can not observe macroscopic objects in a superposition of spatial distinct locations, despite the fact that all objects are made up of microscopic particles which indeed can be observed in such position superposition states. Several conceptually very different solution to this problem were proposed, as e.g. the theory of spontaneous localization [1], which modifies the Schrödinger equation by adding an incoherent part. A less drastic approach is the theory of environmentally induced decoherence [2]. This assumes that the combined state of system and environment evolves according to Schrödinger’s equation, but if only the system density operator is observed, it seems as if the coupling to the environment destroys the quantum feature that a system can be in a superposition of several distinct states, a process known as decoherence.

During the last two decades, there has been increasing interest in the engineering of large quantum systems, e.g. for quantum information processing. A major limitation to these efforts is posed by their fragileness to decoherence. Therefore, a detailed understanding of different decoherence processes is no more just an academic problem, but necessary for future quantum technologies.

A paradigm of environmentally induced decoherence is collisional decoherence, where the system of interest is a tracer particle, possibly macroscopic in size, which experiences random collisions with particles of a thermal reservoir. The colliding particles can be molecules, much as in Brownian motion, but one could also think of photons of the cosmic background radiation. Several authors put forward increasingly complicated master equations for a tracer particle in a thermal gas, first for an infinitely heavy tracer particle [3–6], and later for a tracer particle with finite mass [7–10]. The latter were used to study collisional decoherence by applying quantum trajectory methods [11, 12]. However, the validity of the single collision calculations used in the derivations of the respective master equations for tracer particle with finite mass was recently questioned [13, 14], and a consensus is still missing.

The mechanisms by which an environment can destroy measurable superpositions can roughly be divided into two categories. First, the environmental state can get entangled with the system state. This effectively delocalizes the relative phase of any superposition state of the system into the combined state of system and environment. After tracing out the environmental degrees of freedom, this leads to a reductions of the coherences of the system. We say, the environment measures the system (see figure 1 (b)). In the second process, sometimes called phase averaging, the interaction with the environment changes the relative phase of the superposition state. If this phase change is random and different for each run of an experiment, then the coherence of the superposition state can no longer be observed (see figure 1 (c)). In some sense, phase averaging does not fundamentally destroy the coherences, but rather makes the relative phase...
One should mention that neither mechanisms completely solves the problem of the quantum-classical transition because the measurement problem remains. Nevertheless, both mechanisms can successfully describe the observed lack of coherences of macroscopic objects. However, it is the first decoherence mechanism which is mostly cited in connection with the quantum-classical transition, possibly because it appears to destroy coherences more fundamentally compared to phase averaging. This view is especially maintained within the topic of collisional decoherence.

Although a colliding gas particle certainly carries away some information about the state of the tracer particle, we show in this article that the decoherence due to this information exchange is negligible compared to phase averaging. The latter arises because a collision adds a relative phase to a spatial superposition state, which depends on the momentum of the colliding gas particle and is therefore random. This article hence aims at a fundamental change of the understanding of the collisional decoherence process.

Because we are interested in a general understanding of the collisional decoherence process, rather than in details depending on a particular interaction model, we use the simplest possible model. That is, we assume that the tracer particle as well as the gas particles only move in one dimension. Further, the gas is in a thermal state using Boltzmann statistics, and we apply the low density and high temperature limit. Within this limit, each collision is an independent event and we can neglect three particle collisions. The gas particles do not interact with each other (ideal gas), and their interaction potential with the tracer particle is of the hard core type, i.e. \( V(x-x_0) = \lim_{a \to \infty} a\delta(x-x_0) \) where the index \( g \) labels the gas particle.

Let us briefly review a single collision following reference [14]. The effect of a collision on the tracer particle depends on the state of the colliding gas particle. Therefore, we start this discussion with a convenient convex decomposition of the thermal density operator of a gas particle. A particular useful convex decomposition was given by Hornberger and Sipe [6] in terms of Gaussian minimum uncertainty wave packets \( |x_g, p_g⟩_g \) with

\[
⟨x'_g | x_g, p_g⟩_g = \frac{e^{-ix_g p_g/2\hbar}}{\sqrt{\sqrt{\pi}σ_g}} e^{ix'_g p_g/\hbar} e^{-(x_g-x'_g)^2/2σ^2},
\]

where \( x_g \) and \( p_g \) label the mean position and momentum of the wave packet, respectively, and \( σ_g \) labels the position uncertainty. It was shown that the density operator can be written as

\[
\hat{ρ}_g = \frac{2\pi\hbar}{L} μ_T(p_g)
\]

\[
= \int \frac{dx_g}{L} dp_g \ μ_{Tσ}(p_g) |x_g, p_g⟩_g ⟨x_g, p_g|.
\]

Here, \( μ_T(p_g) = e^{-p_g^2/(2m_g k_B T)} / \sqrt{2πm_g k_B T} \) is the Maxwell-Boltzmann distribution, \( L \) is a normalization length which is taken to infinity, and \( T_{σ} = T - \frac{m_g k_B σ^2}{2} \). The reason for a lower temperature in the Maxwell-Boltzmann distribution in Eq. 3 is that part of the thermal energy of the gas particle has been transferred to being a contribution to the momentum uncertainty of the states \(|x_g, p_g⟩_σ\).

With Eq. 3 at hand, we can assume that every gas particle is in a minimum uncertainty state with position uncertainty \( σ_g \), while the probability density for a particular combination of \( x_g \) and \( p_g \) is given by \( n_g μ_{Tσ}(p_g), \) where \( n_g \) is the particle density of the gas.

For a complete collision of the gas particle wave packet with the tracer particle wave packet, it is required that the velocity uncertainty of the gas particle state \(|x_g, p_g⟩_σ\) is small compared to the relative velocity of the two colliding particles. It was shown in [14] that this is the case (at least for most gas particles) if \( 2m_g k_B T σ^2 > \hbar^2 \), and therefore we will choose \( σ_g \) sufficiently large and approximate \( μ_{Tσ}(p_g) \) by \( μ_T(p_g) \) in the following. Furthermore, to avoid the discussion of three particle collisions, we require \( n_g σ_g \ll 1 \). Therefore, the position uncertainty has to satisfy

\[
\frac{\hbar}{\sqrt{m_g k_B T}} \ll σ_g \ll \frac{1}{n_g},
\]

which is generally possible in the high-temperature and low-density limit

\[
\frac{n_g \hbar}{\sqrt{m_g k_B T}} \ll 1.
\]

Note that this limit must also be satisfied to use Boltzmann statistics to describe an ideal gas.

It was further shown in [14], that, under the additional assumption of a slow (compared to the gas particles) tracer particle, the collision rate is

\[
R = n_g \sqrt{2k_B T / \sqrt{\pi m_g}}.
\]

It is well understood that if the tracer particle is initially (at time \( t = 0 \)) also in a minimum uncertainty state \(|x, p⟩_σ\), but with a position uncertainty \( σ \) related to the gas particle’s position uncertainty via their relative masses according to

\[
\frac{m_σ}{g} = m_g σ^2,
\]

then a collision results in the remarkable simple product state [14, 16]

\[
U_g(t) |x_g, p_g⟩_σ ⊗ U(t) |x, p⟩_σ.
\]

Here, \( t \) is some time after the collision, and \( U(t) \) is the free evolution operator of the single particle. The mean positions and momenta after the collision relate to the initial values according to

\[
x_g = \frac{2ε - (1 - α)x_g}{1 + α}, \quad p_g = \frac{2αp - (1 - α)p_g}{1 + α},
\]

\[
x = \frac{2αx_g + (1 - α)x}{1 + α}, \quad p = \frac{2p_g + (1 - α)p}{1 + α},
\]

where \( ε \) is some information about the state of the tracer particle, which depends on the state of the colliding gas particle. There-
Coherences of a quantum state are often described by the off-diagonals of the density matrix. This leads to the question: What basis should we use to examine decoherence? The two bases which first come to mind are the position basis and the momentum basis, and indeed, these are the bases usually used in the literature [3-5,11,12,15]. Nevertheless, they are not without problems.

If one uses e.g. the momentum basis, one would be interested in how long a superposition of the form $(|p_a\rangle + |p_b\rangle)/\sqrt{2}$ survives, or, equivalently, how fast the coherences $|p_a\rangle/\langle p_b|$ decrease. There is a major problem in this sort of question: A momentum eigenstate (or any state which is very localized in momentum) is itself a coherent superposition of widely spread position eigenstates, and one should expect that these position coherences of each individual momentum eigenstate do vanish on the same time scale, or even faster, as the momentum coherences of interest! The same reasoning applies to using the position basis.

Our reservations are clearly related to the concept of a pointer basis [2,17]. Pointer basis states should be fairly robust to decoherence. That is, if the system is prepared in one of these pointer states it will stay there for some time, whereas if the system is in a superposition of two pointer states, then the coherences typically decrease rapidly in time. These properties makes a pointer basis the basis of choice to study decoherence effects. Because a momentum eigenstate is highly unrobust to position decoherence, the momentum basis is not a good pointer basis. Similarly, a position eigenstate is unrobust to momentum decoherence, and therefore also not an appropriate pointer state.

To add some weight to our concern, we have a brief look at the decoherence rate found by [11] for an initial superposition $(|p\rangle + |−p\rangle)/\sqrt{2}$ of momentum states, i.e. their equation (67). Inserting the definitions (27, 31, 32) of [11] and assuming that the velocity of the tracer particle is small compared to an average gas particle, their equation (67) leads to the decoherence rate $D_p = \frac{\pi \sigma T}{\sqrt{2}}$, where we changed the notations according to ours, and $\sigma$ is a constant scattering cross section. This formula strongly opposes physical intuition because it predicts a decrease of the decoherence rate upon increasing the temperature, despite the fact that an increase of the temperature leads to more powerful and more frequent collisions.

Could we possibly single out a pointer basis from a measurement interpretation of a single collision? We showed in [14] that a colliding gas particle $|x_g,p_g\rangle_{sg}$ performs a smeared out measurement on the tracer particle in the basis $|x,p\rangle_{sg}$, indicating that these states could be used as pointer basis. We know from Eq. (5), that we have a choice in the width $\sigma_g$ of the gas particle states. Therefore, we can use minimum uncertainty states of any width $\sigma = \sqrt{m/m_g}\sigma_g$ as pointer basis, as long as $\sigma_g$ satisfies relation (1), which was required for the treatment of a single collision.

As is stated in e.g. [17], there are still many open questions about the emergence of a pointer basis from the coupling to an environment. Nevertheless, because of the heuristic reasoning in the previous paragraph, and the lack of sensible alternatives, we will indeed discuss decoherence by using superpositions of Gaussian states $|x_a,p_{a}\rangle_{sg} + |x_b,p_{b}\rangle_{sg}$, commonly referred to as cat states in the literature. We should note, that [12] superimposed several momentum eigenstates to produce a state simi-
lar to a cat state to study decoherence using quantum trajectory theory. This study was limited to the situation where the gas particle mass equals the tracer particle mass, and each collision leads to a complete loss of decoherence.

We will need the transformation of an initial (unnormalized) cat state \( |x_a, p_a\rangle_\sigma + |x_b, p_b\rangle_\sigma \) of the tracer particle upon a collision with a gas particle in the state \( |x_a, p_g\rangle_\sigma \). By using the linearity of quantum mechanics as well as Eqs. (8)-(10), and after tracing out the gas particle, we find for the density matrix of the tracer particle after a collision

\[
\hat{\rho}(t) = U(t)\left[ |\bar{x}_a, \bar{p}_a\rangle_\sigma \langle \bar{x}_a, \bar{p}_a| + \hat{c} e^{-i\hat{\varphi}} |\bar{x}_a, \bar{p}_a\rangle_\sigma \langle \bar{x}_b, \bar{p}_b| + \hat{c} e^{i\hat{\varphi}} |\bar{x}_b, \bar{p}_b\rangle_\sigma \langle \bar{x}_a, \bar{p}_a| + |\bar{x}_b, \bar{p}_b\rangle_\sigma \langle \bar{x}_b, \bar{p}_b| \right] U^\dagger(t). 
\]

Here, we used \( \bar{x}_a \equiv \bar{x}(x_a, x_a) \) etc. given by Eq. (10), as well as

\[
\hat{c} = \exp\left[ -\frac{\alpha}{(1+\alpha)^2} \left( \frac{x_D^2}{\sigma^2} + \frac{\sigma^2}{\hbar^2} \right) \right], \quad \hat{\varphi} = \frac{2\alpha(x_{APD} - x_D p_A) + (1-\alpha)p_g x_D - \alpha(1-\alpha)x_D p_D}{(1+\alpha)^2\hbar},
\]

which is smaller than one because of a reduction of coherences due to a measurement performed by the colliding gas particle, and

\[
W_{\rho}(x', p') = \frac{1}{\pi\hbar} \exp\left[ -\frac{(x' - x_a)^2}{\sigma^2} \right] \exp\left[ -\frac{\sigma^2(p' - p_a)^2}{\hbar^2} \right] + \frac{1}{2\pi\hbar} \exp\left[ -\frac{(x' - x_b)^2}{\sigma^2} \right] \exp\left[ -\frac{\sigma^2(p' - p_b)^2}{\hbar^2} \right] + \frac{2c}{\pi\hbar} \exp\left[ -\frac{(x' - x_A)^2}{\sigma^2} \right] \exp\left[ -\frac{\sigma^2(p' - p_A)^2}{\hbar^2} \right] \times \cos\left[ \varphi + \frac{x_{APD} - p_A x_D}{2\hbar} + \frac{x_D p_A - p_D x_A'}{\hbar} \right].
\]

As is well known, the strength of the coherences, indicated by oscillatory behavior of the Wigner function, does not change due to the unitary free evolution. Therefore, in the following, we will mostly use the interaction picture. We can then use the general formula Eq. (16), with \( c = 1 \) and \( \varphi = 0 \), to plot the Wigner function of a state \( |\psi\rangle = |x_a, p_a\rangle_\sigma + |x_b, p_b\rangle_\sigma \) without a collision in figure 7(a).

According to Eqs. (12)-(14), a collision with a gas particle state \( |x_g, p_g\rangle_\sigma \) results in

\[
|x_{a/b}, p_{a/b}, x_{A/D}, p_{A/D}\rangle \rightarrow (\bar{x}_{a/b}, \bar{p}_{a/b}, \bar{x}_{A/D}, \bar{p}_{A/D}) \quad c = 1 \rightarrow \bar{c}, \quad \varphi = 0 \rightarrow \bar{\varphi},
\]

where \( \bar{x}_a = \bar{x}(x_a, x_a), etc \) are given by Eq. (10), and \( \bar{c} \) and \( \bar{\varphi} \) by Eq. (13) and Eq. (14), respectively. The Wigner function after the collision is plotted in figure 7(b).

It is quite astonishing, that, despite choosing a gas particle with only four percent of the mass of the tracer particle, and, despite using a superposition of very close Gaussian wave functions, almost all coherences are lost after a single collision. If we had separated the initial Gaussians only slightly more, or had chosen only a slightly heavier gas particle, the coherences would be not visible at all, because \( \bar{c} \) in Eq. (13) decreases exponentially with these parameters. This observation is independent of the initial momentum and position of the colliding gas particle, as well as whether the Gaussian wave functions are separated predominantly in position or momentum. It is therefore fair to say, that, unless the gas particle
resolved. The reason is that the measurement performed by the colliding gas particle is so imprecise, that it can not distinguish between the two Gaussian wave functions of the tracer particle.\(^{(22)}\)

The main effect of a collision with a light gas particle is a shift of the entire Wigner function in phase space. Of particular interest is that the oscillating part just shifts from \(\langle x_A, p_A \rangle \) to \(\langle \bar{x}_A, \bar{p}_A \rangle \), without acquiring an additional phase shift (i.e., the relative heights of the oscillating peaks do not change). This feature can be explained by looking at the argument of the cosine in Eq. (10). The sum of the first two terms does not change in a collision, because \((x_A p_D - p_A x_D)/(2\hbar)\) equals \((\bar{x}_A p_D - \bar{p}_A \bar{x}_D)/(2\hbar) + \bar{\phi}\), with \(\bar{\phi}\) taken from Eq. (14). The third and fourth term account for a phase shift corresponding to the shift of the Gaussian, i.e. \(x_A - x' \rightarrow \bar{x}_A - x'\) and \(p_A - p' \rightarrow \bar{p}_A - p'\).

Of course, if the gas particle state \(|x_a, p_a\rangle\) is taken from a thermal gas, we have to average over all gas particle momenta \(p_a\), weighted by the Maxwell-Boltzmann distribution \(\mu_T(p_a)\), as well as over all the gas particle positions \(x_g\) which can reach the tracer particle in a given time interval \((0, t)\). Because each possible combination of \(x_g\) and \(p_g\) results in a different shift of the Wigner function in phase space, it is clear that this procedure strongly suppresses the oscillations. This is the decoherence effect we referred to as ‘phase averaging’ in the introduction. It can suppress coherences quickly, even if the measurements which the gas particles perform are very weak.

Let us close this preliminary discussion with a note regarding the collision rate. In the limit of a small mass ratio \(\alpha\), the tracer particle will be very localized compared to the gas particles, in both, position and velocity, as is evident from Eq. (7). Therefore, we do not have to use the full rate operator formalism developed in \(^{(14)}\), but we can simply assume that the tracer particle is reasonably localized somewhere near the origin, and a gas particle collides with the tracer particle during a time interval \((0, t)\) exactly if

\[
0 < -x_g m_g / p_g < t
\]

is satisfied.

### III. COLLISIONAL DECOHERENCE FOR LIGHT GAS PARTICLES

In this section, we discuss the more interesting situation, when a collision only partially destroys the coherences, which is the case if the colliding gas particle is extremely light (mass ratios much smaller than one percent, as we will see). Then, we need a more quantitative measure of the decoherence process, which we will develop below.

Figure 4 shows how the Wigner function of a superposition state changes due to a collision with a light gas particle. The coherences after the collision are still well

\[
\text{FIG. 3: (color online) The Wigner function of the initial coherent superposition } |\psi\rangle = |x_a, p_a\rangle_\sigma + |x_b, p_b\rangle_\sigma \text{ (a), and of the state resulting from a collision with } |x_g, p_g\rangle_\sigma \text{ (b). Parameters are: } x_a = 15, x_b = 0, p_a = 0, p_b = 1.5, \sigma = 4, m = 1, \hbar = 1, x_g = 100, p_g = -1, \text{ and } \alpha = 0.04.\]

\[
\text{FIG. 4: (color online) As in figure 3 but with different parameters for the gas particle: } p = -0.2, x_g = 500, \text{ and } \alpha = 0.002.\]

is much lighter than the tracer particle, the decoherence rate equals the collision rate.

This result becomes even more pronounced, if we average over different initial gas particle positions and momenta, and we will study this effect in the following section for extremely light gas particles, for which the decoherence per collision due to information exchange of the colliding particles, \((1 - c)\), will be small.

#### A. Position decoherence

To study position decoherence, we consider an initial tracer particle state \(|x_a, p_a\rangle_\sigma + |x_b, p_b\rangle_\sigma\). An example of the corresponding Wigner function is plotted in figure 3(a). The Wigner function after a collision with a gas particle state \(|x_g, p_g\rangle_\sigma\) is obtained following section II, but to study the decoherence due to a thermal gas, we average over different initial positions \(x_g\) and momenta \(p_g\) of the gas particle.

The Wigner function \(W_{\rho_1}\) after a collision, averaged over 200 pairs \((x_g, p_g)\) taken from an appropriate distribution (see below), is shown in figure 3(c), (e), and (g),
FIG. 5: (color online) (a): The Wigner function for the tracer particle before a collision. (c), (e), (g): The average Wigner function after one collision with a gas particle at temperature $T = 0.2$, $0.5$, and $1.5$, respectively. (d), (f), (h): The change of the Wigner function due to a collision at temperature $T = 0.2$, $0.5$, and $1.5$, respectively. (b): The relative change of the Wigner function at the origin due to a collision. This serves as a quantitative measure of the ‘decoherence per collision’. The solid line is the first order expansion in temperature, Eq. (27). Parameters are: $x_d = 20$, $x_h = -20$, $p_a = 0$, $p_b = 0$, $\sigma = 4$, $m = 1$, $k_B = 1$, $t = 20$, and $\alpha = 0.0001$.

For the respective temperatures $T = 0.2$, $0.5$, and $1.5$, the corresponding changes of the Wigner function from the initial one (figure 5(a)) are shown figure 5(d), (f), and (g).

As a quantitative measure of coherence, we use the height of the maximum peak of the oscillations. In particular, the relative change of the maximum peak serves as ‘decoherence per collision’, and is plotted over the temperature $T$ in figure 5(b).

In the following, we derive an expression for the ‘decoherence per collision’. In the limit of a light gas particle we can use $\tilde{\epsilon} \approx 1$, and concentrate on the cosine of the Wigner function Eq. (16)

$$\cos \left[ \varphi - \frac{p x_D}{2 \hbar} + x_D \frac{p - p'}{\hbar} \right],$$

where we used $p_D = 0$ and $p_A = p$ for our choice of the initial cat state.

As noted earlier, a collision does not change the sum $\varphi - px_D/(2\hbar)$. Further, we use $1 + \alpha \approx 1$ in the limit of light gas particles, which leads to $\bar{x}_D \approx x_D$ and $\bar{p} \approx p + 2p_g$. Therefore, we find the oscillating term of the Wigner function $W_{p_g}$ after a collision by averaging over

$$\cos \left[ \varphi - \frac{p x_D}{2 \hbar} + x_D \frac{p + 2p_g - p'}{\hbar} \right]$$

In particular, we use the reduction of the maximum of the oscillations as a quantitative measure of decoherence. Because this maximum before a collision is at $p'$ given by $\varphi - px_D/(2\hbar) + x_D(p - p')/\hbar = 0$, we substitute this into Eq. (20), and the coherences after one collision are then obtained by averaging over

$$\cos \left( \frac{2x_D p_g}{\hbar} \right).$$

We see that the relative phase $2p_g x_D/\hbar$ added by the collision depends solely on the momentum of the colliding gas particle, and we will need the momentum probability distribution of the colliding gas particles to perform the averaging over Eq. (21).

According to Eq. (3), the probability density of finding a gas particle in the state $x_g, p_g)_{\sigma_g}$ is given by $n_g \mu_T(p_g)$, where $n_g$ is the particle density of the gas. Knowing that a gas particle of momentum $p_g$ collides with the tracer particle exactly if the position $x_g$ satisfies Eq. (18), we can write down the normalized momentum probability distribution of a colliding gas particle

$$C(p_g) = \frac{|p_g|}{2m_g k_B T} \exp \left( -\frac{p_g^2}{2m_g k_B T} \right).$$

Note that this distribution does not depend on the length of the considered time interval $(0, t)$. This will lead to a ‘decoherence per collision’ (and therefore to a decoherence rate) which is independent of the considered time interval, as should be expected in a Markovian process.

We finally find for the coherences after one collision with a thermal gas particle

$$\langle \cos \left( \frac{2x_D p_g}{\hbar} \right) \rangle_{C(p_g)} = \int_0^\infty dp_g \frac{p_g}{m_g k_B T} \exp \left( -\frac{p_g^2}{2m_g k_B T} \right) \cos \left( \frac{2x_D p_g}{\hbar} \right).$$

(23)
\begin{equation}
= \frac{2x_D \sqrt{2m_g k_BT}}{\hbar} \int_0^\infty du \, e^{-u^2} \sin\left(\frac{2x_D \sqrt{2m_g k_BT}}{\hbar} u\right),
\end{equation}

where we used integration by parts. We deduce for the ‘decoherence per collision’ of position superposition states

\begin{equation}
\frac{\text{Decoherence}}{\text{Collision}} = \frac{2x_D \sqrt{2m_g k_BT}}{\hbar} \int_0^\infty du \, e^{-u^2} \sin\left(\frac{2x_D \sqrt{2m_g k_BT}}{\hbar} u\right),
\end{equation}

or for the decoherence rate

\begin{equation}
D_x = \frac{4x_D m_g k_BT}{\sqrt{\pi} \hbar} \int_0^\infty du \, e^{-u^2} \sin\left(\frac{2x_D \sqrt{2m_g k_BT}}{\hbar} u\right).
\end{equation}

The decocherence rate \(D_x\) agrees up to some constant with the one found by [3] for three dimensional collisions.

The ‘decoherence per collision’ Eq. (25) is plotted over temperature in figure 6 (a) for the same parameters as in figure 5 (b), but for higher temperatures. It might come as a surprise, that the decoherence rate exceeds the collision rate for \(x_D \gtrsim \sqrt{\pi} \Lambda\), where we used integration by parts. The reason is that in this regime, the Wigner function after a collision shows oscillations, which are out of phase with the oscillations of the initial Wigner function, as shown in figure 6 (b). Therefore, if we write down the actual (interaction picture) Wigner function after some small time \(t\) as

\begin{equation}
W_{\rho(t)} = (1 - Rt)W_{\rho_0} + RtW_{\rho_1},
\end{equation}

where \(R\) is the collision rate Eq. (6), and \(\rho_0\) and \(\rho_1\) are the density operators corresponding to no collision and to one collision, respectively, then the oscillations in \(W_{\rho_0}\) and \(W_{\rho_1}\) interfere destructively. These out-of-phase coherences in turn can be understood by noting that the momentum distribution of the colliding particles, Eq. (22), is not peaked at \(p_g = 0\), but rather at \(p_g = \pm \sqrt{2m_g k_BT}\).

Having found the ‘decoherence per collision’ due to phase averaging, we can draw a quantitative comparison with the ‘decoherence per collision’ due to information exchange, which is \(1 - \hat{c} \approx \Delta x_D^2 / \sigma^2\). Because of the first inequality of (4), this decoherence effect is indeed negligible if the density and temperature of the gas are such, that it can be considered an ideal Boltzmann gas. This is also true in the regime discussed in section 1 because phase averaging is sufficient to remove any coherences in a single collision.

We note that it is often stated in the literature [3, 15] that, if the separation \(x_D\) of two interfering wave packets is larger than the thermal wave length \(\Lambda = h / \sqrt{2 \pi \sqrt{m_g k_BT}}\) of the gas, then a colliding gas particle can distinguish between the two interfering wave packets, therefore removing their coherences. In finding that the decoherence rate is about the collision rate if \(x_D \gtrsim \sqrt{\pi} \Lambda\), we confirm the latter part of this statement, but we also show that the loss of coherence is by no means related to a measurement performed by the gas particle, but due to classical phase averaging resulting from the randomness of the momentum transfer.

### B. Momentum decoherence

In this subsection, we will show that the decoherence of momentum superposition states is not a direct process. Instead, two coherent wave packets with momentum separation \(p_D\) will, after some time, acquire a position sep-
out to be negligible in a high temperature and low density gas. Any direct momentum decoherence will turn out to be negligible in a high temperature and low density gas.

We consider the initial tracer particle state \( |x, p_o\rangle + |x, p_0\rangle \), whose Wigner function is plotted in figure 7 (a). Again, the main source of decoherence will be phase averaging. Contrary to the previous subsection, where the relative phase of the two Gaussian wave packets after a collision depended on the initial momentum of the colliding gas particle, for momentum decoherence this phase depends on the initial position of the colliding gas particle \([23]\). As the variation of the initial position of a colliding gas particle is increased, the more ‘decoherence per collision’ we will find. For a given gas particle momentum \( p_g \), the gas particle position can be anywhere within the interval \((-p_g t/m_g, 0)\). Therefore, the ‘decoherence per collision’ will not only depend on the temperature \( T \) for the distribution of \( p_g \), but also on the considered time interval. In figure 7 (c) - (h), the effects of a collision on the Wigner function is shown for different temperatures and time intervals. The dependence of the ‘decoherence per collision’ for low temperatures and short times turns out to be linear in temperature and quadratic in time, as shown in figure 7 (b). As a result, it is not possible to define a time independent decoherence rate for momentum superpositions. We will provide a physical interpretation at the end of this subsection, and first give a mathematical explanation of these results.

For this purpose, we consider again the oscillating cosine within the Wigner function Eq. (10)

\[
\cos \left[ \varphi + \frac{x p_D}{2\hbar} - \frac{p_D}{2} \frac{x - x'}{\hbar} \right],
\]

at its maximum \( \varphi + xp_D/(2\hbar) - p_D(x - x')/\hbar = 0 \). As discussed before, a collision does not change the sum \( \varphi + xp_D/(2\hbar) \) in the cosine, and we only have to consider the change of \( \varphi \) to \( x + 2ax \) in the last term of the cosine. Therefore, we find the ‘decoherence per collision’ by averaging over \( \cos(2ax p_D/\hbar) \). Here, we need the normalized probability distribution \( \tilde{C}(x_g) \) of the initial position of the colliding gas particle, which is obtained from the probability density \( n_g \mu T(p_g) \) by integration over all \( p_g \), for which a gas particle with position \( x_g \) can reach the tracer particle (i.e. \( -x_g \gtrless p_g t/m_g \gtrless \pm \infty \), where the upper sign is for positive \( x_g \))

\[
\tilde{C}(x_g) \propto \frac{n_g}{\sqrt{2\pi m_g k_B T}} \int_{|x_g|/m_g/t}^{\infty} dp_g \exp \left( -\frac{p_g^2}{2m_g k_B T} \right).
\]

The distribution is normalized either by integration over \( x_g \), or directly by dividing by the collision probability \( Rt \). After substituting \( u = p_g/\sqrt{2m_g k_B T} \) we find

\[
\tilde{C}(x_g) = \frac{\sqrt{m_g}}{t \sqrt{2\pi k_B T}} \int_{\sqrt{|x_g|/m_g t}}^{\infty} du \, e^{-u^2},
\]

and therefore

\[
\left\langle \cos \left( \frac{2\alpha p_D x_g}{\hbar} \right) \right\rangle \tilde{C}(x_g) = \frac{\sqrt{m_g}}{t \sqrt{2\pi k_B T}} \int_{-\infty}^{\infty} dx_g \cos \left( \frac{2\alpha p_D x_g}{\hbar} \right) \tilde{C}(x_g) \int_{\sqrt{|x_g|/m_g t}}^{\infty} du \, e^{-u^2}
\]
Because this function represents the coherences after one collision, we have to subtract it from unity to obtain the ‘decoherence per collision’

$$\frac{\text{Decoherence}}{\text{Collision}} = 1 - \frac{mh}{t\sqrt{2m_gk_BT}p_D} \int_0^\infty du e^{-u^2} \sin \left( \frac{2t\sqrt{2m_gk_BT}p_D}{mh} u \right).$$  

(32)

The solid lines in figure [7] (b) are taken from Eq. (33), and agree well with the data (dots) obtained from the numerical Wigner functions directly. The approximation in Eq. (34) is valid if the ‘decoherence per collision’ is small.

Similar to position decoherence, we see from Eq. (34) that momentum decoherence due to information exchange \((1 - \varepsilon \approx \alpha\sigma^2p_D^2/h^2)\) is negligible (unless for temperatures and times so small, that relation (11) is violated). Hence, we established that also momentum decoherence is due to phase averaging.

At first, the increase of the decoherence rate with the considered time interval seems to be at odds with the uniformity in time in the following sense: If we split a time interval \((0, t)\) into sub intervals \((0, t/N), (t/N, 2t/N), \ldots, (t-t/N, t)\), the decoherence rate of the entire interval should be the averaged decoherence rate of all the sub intervals. If we now assume that the decoherence rate for each sub interval is the same (“uniformity in time”), we would be lead to the conclusion that the decoherence rate for the interval \((0, t)\) equals the decoherence rate for the subinterval \((0, t/N)\), clearly contradicting Eq. (33) and Eq. (34).

In the above argument, we made the following conceptual error: by assuming the same decoherence rate for each sub interval, we implied the same initial cat state at the beginning of each sub interval. But instead, by the position decoherence according to Eq. (33) with \(x_D(t') = p_Dt'/m\). Indeed, substituting this time dependent position separation into Eq. (25) and averaging over all times \(t' \in (0, t)\), one exactly recovers Eqs. (33) and (34).

In other words, the decoherence which an initial cat state \([x, p_a] + [x, p_b]\) experiences during a time interval \((0, t)\) is perfectly explained by position decoherence of the evolving state \(U(t')([x, p_a] + [x, p_b])\). This leads us to the physical interpretation that momentum decoherence is not a direct process, but results indirectly from position decoherence due to position separation which the tracer particle acquires over time.

IV. CONCLUSIONS FROM THE STUDY OF DECOHERENCE

We showed previously [14] that in one dimensional collisional decoherence, the quantity measured by a colliding gas particle does not only depend on physical parameters like density and temperature, but also on the choice of decomposition Eq. (3) of the density operator of a thermal gas particle. It is therefore reassuring to find in this article, that measurement effects as a source of decoherence are negligible in the high temperature and low density limit, where Eq. (3) is valid. The reason is that measurement effects are small compared to phase averaging effects, which arise from a random relative phase added to a superposition state during the collision process.

We further arrive at a neat interpretation of the decoherence process of a superposition of two Gaussians wave packets. The decoherence due to a collision depends on the position separation of the two Gaussian wave packets at the time of the collision. In contrast, there is no direct decoherence due to the momentum separation \(p_D\) of the two coherent wave packets. Instead, over time, the momentum separation changes the position separation according to \(x_D(t) = x_D + tp_D/m\). This leads to an indirect influence of \(p_D\) on the decoherence rate, which, if there is no initial position separation, is described by Eq. (33).

Further work is required to see whether this drastic change in the understanding of the collisional decoherence process also applies to three dimensional systems.

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[21] The time evolved Wigner function is obtained by replacing $x'$ by $(x' - p't/m)$ in Eq. (16).
[22] This can also be seen from the Kraus operators which represent the transformation of the tracer particle state due to a collision. The Kraus operators are derived in [14] and consist of the Glauber displacement operator with a small correction representing a weak phase space measurement.
[23] This can be spelled out in a more intuitive way by saying that the acquired relative phase depends on the actual time of collision $t' \in (0, t)$, multiplied by the momentum of the colliding gas particle.
[24] This is easier shown by recovering Eq. (32) from Eq. (23).