REMARK ON THE LOCAL NATURE OF METRIC MEAN DIMENSION

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Abstract. Metric mean dimension is a metric invariant of dynamical systems. It is a dynamical analogue of Minkowski dimension of metric spaces. We explain that old ideas of Bowen (1972) can be used for clarifying the local nature of metric mean dimension. We also explain the generalization to $\mathbb{R}^D$-actions and an illustrating example.

1. Introduction

1.1. Background. The purpose of this paper is to explain that old ideas of Bowen [Bow72] can be used in the context of mean dimension theory [Gro99, LW00, Lin99].

A pair $(X, T)$ is called a dynamical system if $X$ is a compact metrizable space and $T : X \to X$ is a homeomorphism. The mean dimension of $(X, T)$ (denoted by $\text{mdim}(X, T)$) is a topological invariant measuring the number of parameters per iterate for describing the orbits of $(X, T)$. It was first introduced by Gromov [Gro99].

One of the big difficulties in the study of mean dimension is that it is often very hard to prove an upper bound on mean dimension. So far, the most successful way for proving upper bounds on $\text{mdim}(X, T)$ is to use metric mean dimension. Metric mean dimension is a dynamical analogue of Minkowski dimension introduced by Lindenstrauss–Weiss [LW00]. It is not a topological invariant but a metric-dependent quantity.

Let $d$ be a metric (distance function) on $X$ compatible with the topology. For $\varepsilon > 0$ we denote by $S(X, \varepsilon)$ the entropy of $(X, T)$ at the scale $\varepsilon$. (The definition will be given in the next subsection.) We define the upper and lower metric mean dimensions of $(X, T, d)$ by

\begin{equation}
\underline{\text{mdim}}_M(X, T, d) := \limsup_{\varepsilon \to 0} \frac{S(X, \varepsilon)}{\log(1/\varepsilon)}, \quad \overline{\text{mdim}}_M(X, T, d) := \liminf_{\varepsilon \to 0} \frac{S(X, \varepsilon)}{\log(1/\varepsilon)}.
\end{equation}

They bounds the mean dimension $\text{mdim}(X, T)$ from above [LW00, Theorem 4.2]:

\begin{equation}
\text{mdim}(X, T) \leq \underline{\text{mdim}}_M(X, T, d) \leq \overline{\text{mdim}}_M(X, T, d).
\end{equation}
This is a dynamical analogue of the fact that Minkowski dimension bounds topological dimension. It is conjectured that there always exists a metric $d$ for which the equalities hold in (1.2).

The inequality (1.2) is a very useful result because it is often much easier to prove an upper bound on $\overline{\text{mdim}}_M(X, T, d)$ than to (directly) prove an upper bound on $\text{mdim}(X, T)$. This idea was first successfully used in [Tsu18a, Tsu18b] for proving tight upper bounds on mean dimension of certain dynamical systems coming from geometric analysis.

Why is it (relatively) easy to prove an upper bound on metric mean dimension? The main reason is that we can calculate metric mean dimension by using only some local information. (We will explain this more precisely in the next subsection.) On the contrary, mean dimension has a highly global nature. So far, we do not know any direct method to calculate it by using local information.

The purpose of this paper is to explain that we can understand the local nature of metric mean dimension more clearly by using ideas of Bowen [Bow72].

1.2. **Main result.** Let $(X, d)$ be a compact metric space. Let $\varepsilon > 0$ and $K$ a subset of $X$. A subset $S$ of $X$ is said to be an $\varepsilon$-spanning set of $K$ (or $(d, \varepsilon)$-spanning set of $K$ when we need to clarify what metric is used) if for every $x \in K$ there exists $y \in S$ satisfying $d(x, y) \leq \varepsilon$. We denote by $\#(K, d, \varepsilon)$ the minimum cardinality of such $S$.

Let $(X, T)$ be a dynamical system with a metric $d$ compatible with the topology. For a subset $\Omega$ of $\mathbb{Z}$, we define a metric $d_\Omega$ on $X$ by

$$d_\Omega(x, y) := \sup_{a \in \Omega} d(T^a x, T^a y).$$

If $\Omega$ is a finite subset, then $d_\Omega$ is also compatible with the given topology. However, if $\Omega$ is infinite (in particular, if $\Omega = \mathbb{Z}$), the metric $d_\Omega$ often defines a topology different from the originally given one. (It is often the case that $(X, d_\Omega)$ is noncompact if $\Omega$ is infinite.)

For a natural number $N$, we also denote

$$d_N(x, y) := d_{\{0, 1, 2, \ldots, N-1\}}(x, y) = \max_{0 \leq n < N} d(T^nx, T^ny).$$

For $\varepsilon > 0$ and a subset $K$ of $X$ we define

$$S(K, \varepsilon) := \limsup_{N \to \infty} \frac{\log \#(K, d_N, \varepsilon)}{N}.$$ 

This quantity is called the entropy of $K$ at the scale $\varepsilon$. We define the upper and lower metric mean dimensions of $(X, T, d)$ as in (1.1):

$$\overline{\text{mdim}}_M(X, T, d) := \limsup_{\varepsilon \to 0} \frac{S(X, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\text{mdim}}_M(X, T, d) := \liminf_{\varepsilon \to 0} \frac{S(X, \varepsilon)}{\log(1/\varepsilon)}.$$
Now we have defined metric mean dimension. It is not difficult to see that its definition has a local nature: If \( X = K_1 \cup K_2 \cup \cdots \cup K_n \) then for any \( \varepsilon > 0 \) and natural number \( N \)

\[
\#(X, d_N, \varepsilon) \leq \sum_{i=1}^{n} \#(K_i, d_N, \varepsilon).
\]

This means that we can easily decompose the problem of estimating \( \#(X, d_N, \varepsilon) \) into local pieces. Then

\[
S(X, \varepsilon) = \max_{1 \leq i \leq n} S(K_i, \varepsilon)
\]

and hence

\[
\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \frac{\max_{1 \leq i \leq n} S(K_i, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\text{mdim}}_M(X, T, d) = \liminf_{\varepsilon \to 0} \frac{\max_{1 \leq i \leq n} S(K_i, \varepsilon)}{\log(1/\varepsilon)}.
\]

So metric mean dimension can be calculated by local information (i.e. the entropy of each piece \( K_i \) at the scale \( \varepsilon \)). This is easy to see and not surprising. The main result below shows that indeed we can calculate metric mean dimension by using much more local quantity.

Let \( \delta > 0 \) and \( \Omega \subset \mathbb{Z} \). For \( x \in X \), we define \( B_\delta(x, d_\Omega) \) as the closed \( \delta \)-ball around \( x \) with respect to \( d_\Omega \):

\[
B_\delta(x, d_\Omega) := \{ y \in X \mid d_\Omega(x, y) \leq \delta \}.
\]

The following is our main result.

**Theorem 1.1.** For any positive number \( \delta \)

\[
\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \frac{\sup_{x \in X} S(B_\delta(x, d_\Omega), \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\text{mdim}}_M(X, T, d) = \liminf_{\varepsilon \to 0} \frac{\sup_{x \in X} S(B_\delta(x, d_\Omega), \varepsilon)}{\log(1/\varepsilon)}.
\]

This is surprising because the set

\[
B_\delta(x, d_\Omega) = \{ y \in X \mid d_\Omega(x, y) \leq \delta \} = \{ y \in X \mid d(T^n x, T^n y) \leq \delta \text{ for all integers } n \}
\]

is a very small subset. It is usually not a neighborhood of the point \( x \). For example, if \( (X, T) \) is the full-shift on \( \{0, 1\}^\mathbb{Z} \) with some metric \( d \) (or, more generally, if \( (X, T) \) is expansive), then the set \( B_\delta(x, d_\Omega) \) is just one-point \( \{ x \} \) for sufficiently small \( \delta \). Nevertheless, we can calculate metric mean dimension only by studying such small subsets.

We prove Theorem 1.1 in the next section. The proof is just a slight modification of the argument given by Bowen [Bow72]. Essentially, Theorem 1.1 is a mere simple corollary of [Bow72]. However Theorem 1.1 seems to be useful in the future study\(^1\) of mean dimension theory, so the author thinks that it is worth publishing.

\(^1\)What I have in my mind is the problem of estimating mean dimension of various geometric examples in [Gro99]. For such examples, probably we can study the sets \( B_\delta(x, d_\Omega) \) by some deformation theory techniques. See also §4.
Remark 1.2. Bowen [Bow72] considered the quantity

\[ h_{T,\text{homeo}}^*(\delta) := \sup_{x \in X} \left\{ \lim_{\varepsilon \to 0} S(B_\delta(x, d_Z), \varepsilon) \right\}. \]

A dynamical system \((X, T)\) is said to be \(h\)-expansive if \(h_{T,\text{homeo}}^*(\delta) = 0\) for some \(\delta > 0\).

He studied several important consequences of this condition. In the study of entropy theory of [Bow72], the entropy of \(B_\delta(x, d_Z)\) is more or less a “remainder term”. On the other hand, Theorem 1.1 shows that \(S(B_\delta(x, d_Z))\) becomes a “main term” in the study of metric mean dimension. Therefore, interestingly, the viewpoints here and in [Bow72] are almost opposite although the proof of Theorem 1.1 is very close to [Bow72].

2. Proof of Theorem 1.1

Throughout this section, we assume that \((X, T)\) is a dynamical system with a metric \(d\). In this section, for two integers \(a \leq b\), we write \([a, b]\) as the set of integers \(x\) with \(a \leq x \leq b\). (This convention is used only in this section. In the later sections, \([a, b]\) means the set of real numbers \(x\) with \(a \leq x \leq b\).)

The next lemma is [Bow72, Lemma 2.1].

Lemma 2.1 ([Bow72]). Let \(\varepsilon > 0\) and \(F \subset X\). Let

\[ 0 = t_0 < t_1 < t_2 < \cdots < t_{r-1} < t_r = n \]

be a sequence of integers. Then

\[ \#(F, d_n, \varepsilon) \leq \prod_{i=0}^{r-1} \#(T^{t_i}F, d_{t_{i+1}-t_i}, \frac{\varepsilon}{2}). \]

Proof. Let \(E_i\) be a \((d_{t_{i+1}-t_i}, \frac{\varepsilon}{2})\)-spanning set of \(T^{t_i}(F)\) of minimum cardinality. For each \((x_0, x_1, \ldots, x_{r-1}) \in E_0 \times E_1 \times \cdots \times E_{r-1}\), consider

\[ V(x_0, x_1, \ldots, x_{r-1}) := \left\{ y \in F \middle| d_{t_{i+1}-t_i} (T^{t_i}y, x_i) \leq \frac{\varepsilon}{2} \text{ for all } 0 \leq i \leq r-1 \right\}. \]

These form a covering of \(F\). The diameter of \(V(x_0, \ldots, x_{r-1})\) with respect to \(d_n\) is less than or equal to \(\varepsilon\). We take one point from each non-empty \(V(x_0, x_1, \ldots, x_{r-1})\). They form a \((d_n, \varepsilon)\)-spanning set of \(F\).

The next proposition is a key result. This is a very small modification of [Bow72, Proposition 2.2].

Proposition 2.2. Let \(\varepsilon\) and \(\delta\) be positive numbers. Set

\[ a := \sup_{x \in X} S \left( B_\delta(x, d_Z), \frac{\varepsilon}{4} \right). \]

For any positive number \(\beta\) we have

\[ \sup_{x \in X} \#(B_\delta(x, d_n), d_n, \varepsilon) \leq C \cdot e^{(\alpha + \beta)n} \text{ for all integers } n, \]

where \(C = C(\varepsilon, \delta, \beta)\) is a positive number depending on \(\varepsilon, \delta, \beta\).
Proof. The proof is almost identical to [Bow72, Proposition 2.2]. But we provide a full proof for the completeness. For each \( y \in X \), since we know that
\[
\lim_{N \to \infty} \frac{1}{N} \log \# \left( B_\delta(y, d_Z), d_N, \frac{\varepsilon}{4} \right) \leq a + \alpha,
\]
there exist a natural number \( m(y) \) and a subset \( E(y) \subset X \) which is a \( (d_{m(y)}, \frac{\varepsilon}{4}) \)-spanning set of \( B_\delta(y, d_Z) \) with \( |E(y)| < e^{(a + \alpha)m(y)} \). We define an open set \( U(y) \) by
\[
U(y) := \left\{ x \in X \mid \exists z \in E(y) : d_{m(y)}(x, z) < \frac{\varepsilon}{2} \right\}.
\]
We have \( B_\delta(y, d_Z) \subset U(y) \) and
\[
(2.1) \quad \# \left( U(y), d_{m(y)}, \frac{\varepsilon}{2} \right) \leq |E(y)| < e^{(a + \alpha)m(y)}.
\]
We have
\[
B_\delta(y, d_Z) = \bigcap_{N=1}^\infty B_\delta \left( y, d_{[-N,N]} \right),
\]
and each \( B_\delta \left( y, d_{[-N,N]} \right) \) is closed. Since \( X \) is compact, there exists \( N(y) > 0 \) satisfying
\[
B_\delta \left( y, d_{[-N(y),N(y)]} \right) \subset U(y).
\]
If we take a sufficiently small \( \eta > 0 \) (depending on \( y \)) then we also have
\[
B_{\delta + \eta} \left( y, d_{[-N(y),N(y)]} \right) \subset U(y).
\]
Set \( V(y) := \{ x \in X \mid d_{[-N(y),N(y)]}(x, y) < \eta \} \). This is an open neighborhood of \( y \) and
\[
\forall v \in V(y) : B_\delta \left( v, d_{[-N(y),N(y)]} \right) \subset U(y).
\]
We choose \( y_1, \ldots, y_s \in X \) with \( X = V(y_1) \cup \cdots \cup V(y_s) \). Set
\[
N := \max \left( N(y_1), \ldots, N(y_s), m(y_1), \ldots, m(y_s) \right) + 1.
\]
Let \( n > 2N \) and \( x \in X \). For each \( t \in [N, n - N] \) we can choose \( y_t \) with \( T^t x \in V(y_t) \) and then
\[
T^t \left( B_\delta(x, d_n) \right) \subset B_\delta \left( T^t x, d_{[-N(y_i),N(y_i)]} \right) \subset U(y_i).
\]
From (2.1)
\[
\# \left( T^t \left( B_\delta(x, d_n) \right), d_{m(y_t)}, \frac{\varepsilon}{2} \right) \leq \# \left( U(y_t), d_{m(y_t)}, \frac{\varepsilon}{2} \right) < e^{(a + \alpha)m(y)}.
\]
We will inductively choose integers \( 0 = t_0 < t_1 < \cdots < t_r = n \) and points \( y_{i_1}, y_{i_2}, \ldots, y_{i_{r-1}} \) with \( T^{t_k} x \in V(y_{i_k}) \) \((1 \leq k \leq r - 1)\). First we set \( t_1 = N \) and choose \( y_{i_1} \) with \( T^N x \in V(y_{i_1}) \). Next, suppose we have defined \( t_1, \ldots, t_k(< n) \) and \( y_{i_1}, \ldots, y_{i_k} \). If \( t_k > n - N \) then we set \( r = k + 1 \) and \( t_r = n \). (And the induction process stops.) If \( t_k \leq n - N \) then we set \( t_{k+1} := t_k + m(y_{i_k})(< n) \) and choose a point \( y_{i_{k+1}} \) with \( T^{t_{k+1}} x \in V(y_{i_{k+1}}) \). This process eventually stops.
From the above construction, for \(1 \leq k \leq r-2\)
\[
\# \left( T^{t_k} (B_\delta(x, d_n)), d_{t_{k+1} - t_k}, \frac{\varepsilon}{2} \right) = \# \left( T^{t_k} (B_\delta(x, d_n)) , d_{m(y_k)}, \frac{\varepsilon}{2} \right) < e^{(a+\beta)m(y_k)} = e^{(a+\beta)(t_{k+1} - t_k)}.
\]
For \(k = 0\) or \(k = r-1\), we have \(t_{k+1} - t_k \leq N\) and hence
\[
\# \left( T^{t_k} (B_\delta(x, d_n)), d_{t_{k+1} - t_k}, \frac{\varepsilon}{2} \right) \leq \# \left( X, d_N, \frac{\varepsilon}{2} \right).
\]
Now we use Lemma 2.1 and get
\[
\# \left( B_\delta(x, d_n), d_n, \varepsilon \right) \leq \prod_{k=0}^{r-1} \left\{ \# \left( X, d_N, \frac{\varepsilon}{2} \right) \right\}^2 \cdot \prod_{k=1}^{r-2} e^{(a+\beta)(t_{k+1} - t_k)} \leq \left\{ \# \left( X, d_N, \frac{\varepsilon}{2} \right) \right\}^2 \cdot e^{(a+\beta)n}.
\]
We can regard the term \(\left\{ \# \left( X, d_N, \frac{\varepsilon}{2} \right) \right\}^2\) as a positive constant depending on \(\varepsilon, \delta, \beta\). This proves the statement. \(\Box\)

Now we are ready to prove the main result. We write the statement again.

**Theorem 2.3** (= Theorem 1.1). For any positive number \(\delta\)
\[
\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \sup_{x \in X} \frac{\text{S} \left( B_\delta(x, d_\varepsilon), \varepsilon \right)}{\log(1/\varepsilon)},
\]
\[
\underline{\text{mdim}}_M(X, T, d) = \liminf_{\varepsilon \to 0} \sup_{x \in X} \frac{\text{S} \left( B_\delta(x, d_\varepsilon), \varepsilon \right)}{\log(1/\varepsilon)}.
\]

**Proof.** The proof is similar to the proof of [Bow72, Theorem 2.4]. Let \(\varepsilon\) be a positive number. Set \(a := \sup_{x \in X} S \left( B_\delta(x, dZ), \frac{\varepsilon}{4} \right)\).

Let \(n\) be a natural number and \(\beta\) a positive number. From Proposition 2.2
\[
\sup_{x \in X} \# \left( B_\delta(x, d_n), d_n, \varepsilon \right) \leq C \cdot e^{(a+\beta)n}.
\]
We choose points \(x_1, \ldots, x_{M_n}\) in \(X\) with
\[
X = \bigcup_{m=1}^{M_n} B_\delta(x_m, d_n), \quad M_n = \# \left( X, d_n, \delta \right).
\]
Then
\[
\#(X, d_n, \varepsilon) \leq \sum_{m=1}^{M_n} \# \left( B_\delta(x_m, d_n), d_n, \varepsilon \right) \leq CM_n \cdot e^{(a+\beta)n}.
\]
Namely
\[
\log \#(X, d_n, \varepsilon) \leq \log C + \log \#(X, d_n, \delta) + (a + \beta)n.
\]
Divide this by \( n \) and let \( n \to \infty \). Noting that \( C \) is independent of \( n \), we get

\[
S(X, \varepsilon) \leq S(X, \delta) + a + \beta.
\]

Let \( \beta \to 0 \):

\[
S(X, \varepsilon) \leq S(X, \delta) + a = S(X, \delta) + \sup_{x \in X} S\left( B_\delta(x, d_Z), \frac{\varepsilon}{4} \right).
\]

Divide this by \( \log(1/\varepsilon) \) and let \( \varepsilon \to 0 \). We get the statement. (Notice that \( S(X, \delta) \leq \log \#(X, d, \frac{\delta}{2}) < \infty \) by Lemma 2.1.) \( \square \)

3. Generalization to \( \mathbb{R}^D \)-actions

3.1. Statement of the result. In the preceding sections, a “dynamical system” means a compact metrizable space \( X \) with a homeomorphism \( T : X \to X \). In other words, it is a continuous action of the group \( \mathbb{Z} \) on a compact metrizable space. In order to broaden the applicability of Theorem 1.1, we would like to generalize it to more general group actions. This is essential for studying various geometric examples [Gro99, MT15, Tsu18b].

Here we consider actions of the group \( \mathbb{R}^D \) where \( D \) is a natural number. Probably this is the most basic case\(^2\) for geometric applications. \( \mathbb{R}^D \) has the standard topology.

A pair \( (X, T) \) is called an \( \mathbb{R}^D \)-action if \( X \) is a compact metrizable space and \( T : \mathbb{R}^D \times X \to X \) is a continuous action. The (topological) mean dimension of an \( \mathbb{R}^D \)-action \( (X, T) \) is denoted as \( \text{mdim}(X, T) \). (We do not provide its definition because the topological mean dimension is not the main object of this paper.)

Let \( (X, T) \) be an \( \mathbb{R}^D \)-action with a metric \( d \) on \( X \). For a subset \( \Omega \subset \mathbb{R}^D \) we define a metric \( d_\Omega \) on \( X \) by

\[
d_\Omega(x, y) := \sup_{a \in \Omega} d(T^a x, T^a y).
\]

For \( \varepsilon > 0 \) and \( K \subset X \) we define the entropy of \( K \) at the scale \( \varepsilon \) by

\[
S(K, \varepsilon) := \limsup_{L \to \infty} \frac{1}{LD} \log \#(K, d_{[0,L]^D}, \varepsilon).
\]

We define the upper and lower metric mean dimensions by

\[
\overline{\text{mdim}}_M(X, T, d) := \limsup_{\varepsilon \to 0} \frac{S(X, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\text{mdim}}_M(X, T, d) := \liminf_{\varepsilon \to 0} \frac{S(X, \varepsilon)}{\log(1/\varepsilon)}.
\]

They bound the mean dimension from above [LW00, Theorem 4.2]:

\[
\underline{\text{mdim}}_M(X, T) \leq \overline{\text{mdim}}_M(X, T, d) \leq \overline{\text{mdim}}_M(X, T, d).
\]

For \( x \in X, \delta > 0 \) and a subset \( \Omega \subset \mathbb{R}^D \), we set

\[
B_\delta(x, d_\Omega) := \{ y \in X | d_\Omega(x, y) \leq \delta \}.
\]

In particular

\[
B_\delta(x, d_{\mathbb{R}^D}) = \{ y \in X | d_{\mathbb{R}^D}(x, y) \leq \delta \} = \{ y \in X | \forall a \in \mathbb{R}^D : d(T^a x, T^a y) \leq \delta \}.
\]

\(^2\)The argument also works for \( \mathbb{Z}^D \)-actions.
The following is the main result of this section:

**Theorem 3.1.** Let \((X, T)\) be an \(\mathbb{R}^D\)-action with a metric \(d\). For any \(\delta > 0\)

\[
\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \frac{\sup_{x \in X} S(B_\delta(x, d_{\mathbb{R}^D}), \varepsilon)}{\log(1/\varepsilon)},
\]

\[
\underline{\text{mdim}}_M(X, T, d) = \liminf_{\varepsilon \to 0} \frac{\sup_{x \in X} S(B_\delta(x, d_{\mathbb{R}^D}), \varepsilon)}{\log(1/\varepsilon)}.
\]

We would like to mention a related result. In the study of the mean dimension of the system of Brody curves (see §4) in [Tsu18b], the author encountered a problem of how to formulate the local nature of metric mean dimension. At that time the author did not realize the paper of Bowen [Bow72]. In [Tsu18b, Lemma 2.5], the following lemma was proved.

**Lemma 3.2 ([Tsu18b]).** Let \((X, T)\) be an \(\mathbb{R}^D\)-action with a metric \(d\). For any \(\delta > 0\) and \(R \geq 0\),

\[
\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \left( \limsup_{L \to \infty} \frac{\sup_{x \in X} \log \#(B_\delta(x, d_{[-R,L+R]D}), d_{[0,L]D}, \varepsilon)}{L^D \cdot \log(1/\varepsilon)} \right).
\]

We also have a similar result for the lower metric mean dimension.

This is an important technical ingredient of [Tsu18b]. The basic philosophy behind this lemma is the same as in Theorem 3.1. It utilizes the local nature of metric mean dimension. We can reduce the global problem to a more local problem of studying the Bowen balls \(B_\delta(x, d_{[-R,L+R]D})\). This was very useful for studying the mean dimension of the system of Brody curves.

Lemma 3.2 easily follows from Theorem 3.1. So Theorem 3.1 is a stronger result. Moreover the statement of Theorem 3.1 is substantially simpler than Lemma 3.2. So the author thinks that it provides a “right” formulation of the local nature of metric mean dimension.

3.2. **Tiling argument.** Conceptually, the proof of Theorem 3.1 is the same as the proof of Theorem 1.1. But technically it is a bit more complicated. We need a kind of tiling argument originally due to Ornstein–Weiss [OS87]. This subsection is a preparation for it.

We consider the \(\ell^\infty\)-norm on \(\mathbb{R}^D\):

\[
| (x_1, \ldots, x_D) |_\infty := \max_{1 \leq n \leq D} |x_n|.
\]

We always think that \(\mathbb{R}^D\) is endowed with this norm (not the \(\ell^2\)-norm). In this section a “cube” \(\Lambda\) in \(\mathbb{R}^D\) means a set of the form

\[
\Lambda = u + [0, L]^D = \{ u + x \mid x \in [0, L]^D \},
\]
where \( u \in \mathbb{R}^D \) and \( L > 0 \). We set \( \ell(\Lambda) := L \). We also set
\[
3\Lambda := u + [-L, 2L]^D.
\]
We denote by \( \text{vol}(\cdot) \) the \( D \)-dimensional Lebesgue measure on \( \mathbb{R}^D \).

The following is a kind of (finite) Vitali covering lemma.

**Lemma 3.3.** Let \( \Lambda_1, \ldots, \Lambda_N \) be cubes in \( \mathbb{R}^D \). There is a disjoint subfamily \( \{\Lambda_{n_1}, \ldots, \Lambda_{n_k}\} \) such that
\[
\Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_N \subset 3\Lambda_{n_1} \cup 3\Lambda_{n_2} \cup \cdots \cup 3\Lambda_{n_k}.
\]
In particular,
\[
\text{vol}(\Lambda_{n_1} \cup \cdots \cup \Lambda_{n_k}) \geq 3^{-D} \text{vol}(\Lambda_1 \cup \cdots \cup \Lambda_N).
\]

**Proof.** We use a greedy algorithm. Let \( \Lambda_{n_1} \) be one of the cubes in \( \{\Lambda_1, \ldots, \Lambda_N\} \) which have the maximum side length \( \ell(\Lambda_n) \). We discard all the cubes which have non-empty intersection with \( \Lambda_{n_1} \). Next let \( \Lambda_{n_2} \) be one of the remaining cubes which have the maximum side length. Then we discard all the cubes which have non-empty intersection with \( \Lambda_{n_2} \). We continue this process. It eventually stops and we get a disjoint family \( \{\Lambda_{n_1}, \ldots, \Lambda_{n_k}\} \).

Let \( \Lambda_m \) be an arbitrary cube in the initial family. If it is not chosen in the process, then there is \( \Lambda_{n_i} \) satisfying
\[
\Lambda_m \cap \Lambda_{n_i} \neq \emptyset, \quad \ell(\Lambda_m) \leq \ell(\Lambda_{n_i}).
\]
Then \( \Lambda_m \subset 3\Lambda_{n_i} \). \( \square \)

For \( r > 0 \) and a subset \( \Omega \subset \mathbb{R}^D \), we define the \( r \)-interior and \( r \)-boundary of \( \Omega \) by
\[
\text{int}(\Omega, r) := \{ x \in \Omega | x + [-r, r]^D \subset \Omega \},
\]
\[
\partial(\Omega, r) := \{ x \in \mathbb{R}^D | \exists y \in \Omega, \exists z \in \mathbb{R}^D \setminus \Omega : |x - y|_\infty \leq r, |x - z|_\infty \leq r \}.
\]
We also set
\[
B_r(\Omega) := \Omega \cup \partial(\Omega, r) = \{ x \in \mathbb{R}^D | \exists y \in \Omega : |x - y|_\infty \leq r \} = \text{int}(\Omega, r) \cup \partial(\Omega, r).
\]
For \( \Omega, \Omega' \subset \mathbb{R}^D \), we have
\[
\partial(\Omega \cup \Omega', r) \subset \partial(\Omega, r) \cup \partial(\Omega', r).
\]

Let \( \mathcal{C} = \{\Lambda_1, \ldots, \Lambda_N\} \) be a finite family of cubes in \( \mathbb{R}^D \). We set
\[
\ell_{\text{max}}(\mathcal{C}) := \max_{1 \leq n \leq N} \ell(\Lambda_n), \quad \ell_{\text{min}}(\mathcal{C}) := \min_{1 \leq n \leq N} \ell(\Lambda_n).
\]

The next proposition is a main technical ingredient of the proof of Theorem 3.1.

**Proposition 3.4.** For any \( \eta > 0 \) there exists a natural number \( K = K(\eta) > 0 \) satisfying the following statement. Let \( \Omega \subset \mathbb{R}^D \) be a bounded Borel subset, and let \( \mathcal{C}_1, \ldots, \mathcal{C}_K \) be finite families of cubes in \( \mathbb{R}^D \). We assume:

1. \( \ell_{\text{max}}(\mathcal{C}_1) \geq 1 \) and \( \ell_{\text{min}}(\mathcal{C}_{k+1}) \geq K \cdot \ell_{\text{max}}(\mathcal{C}_k) \) for all \( 1 \leq k \leq K - 1 \).
2. \( \text{vol}\{\partial(\Omega, \ell_{\text{max}}(\mathcal{C}_K))\} < \frac{4}{3} \cdot \text{vol}(\Omega) \).
(3) \( \Omega \subset \bigcup_{\Lambda \in \mathcal{C}_k} \Lambda \) for every \( 1 \leq k \leq K \).

Then there exists a disjoint subfamily \( \mathcal{A} \subset \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K \) satisfying

\[
\bigcup_{\Lambda \in \mathcal{A}} \Lambda \subset \Omega, \quad \text{vol} \left\{ B_1 \left( \Omega \setminus \bigcup_{\Lambda \in \mathcal{A}} \Lambda \right) \right\} < \eta \cdot \text{vol}(\Omega).
\]

This is a rather complicated statement. Before proving the lemma, we explain its meaning more intuitively. First, since \( K \) will be assumed very large, the number \( K \) of the given families \( \mathcal{C}_1, \ldots, \mathcal{C}_K \) is very large. The condition (1) means that that every cube in \( \mathcal{C}_{k+1} \) is much larger than cubes in \( \mathcal{C}_k \). (So we have \( K \) different scales.) The condition (2) means that the set \( \Omega \) is much larger than cubes in \( \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K \). The condition (3) is simple. The conclusion means that a large portion of \( \Omega \) can be covered by mutually disjoint cubes in \( \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K \).

**Proof.** We assume that \( K > 1 \) is sufficiently large so that

- \( K \cdot 3^{-D} \cdot \frac{2}{3} > 1 \).
- For \( r > 0 \) and a cube \( \Lambda \subset \mathbb{R}^D \), if \( \ell(\Lambda) \geq K r \) then \( \text{vol}(\partial(\Lambda, r)) < \frac{4}{3} \cdot \text{vol}(\Lambda) \).

We will inductively construct \( \mathcal{A}_k \subset \mathcal{C}_{K+1-k} \) for \( k = 1, 2, \ldots, K \) and finally define \( \mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_K \). We also set \( \Omega_k = \Omega \setminus \bigcup_{\Lambda \in \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k} \Lambda \). (Let \( \Omega_0 := \varnothing \).)

Suppose we have defined \( \mathcal{A}_1, \ldots, \mathcal{A}_k \). We are going to define \( \mathcal{A}_{k+1} \). (The initial step of the induction is the same; set \( k = 0 \) in the argument below.) Set

\[
\mathcal{C}_{K-k}' = \{ \Lambda \in \mathcal{C}_{K-k} \mid \Lambda \cap \text{int} (\Omega_k, \ell_{\max}(\mathcal{C}_{K-k})) \neq \emptyset \}.
\]

We have

\[
\text{int} (\Omega_k, \ell_{\max}(\mathcal{C}_{K-k})) \subset \bigcup_{\Lambda \in \mathcal{C}_{K-k}'} \Lambda \subset \Omega_k.
\]

In particular, every cube in \( \mathcal{C}_{K-k}' \) has no intersection with cubes in \( \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k \).

By Lemma 3.3 there exists a disjoint subfamily \( \mathcal{A}_{k+1} \subset \mathcal{C}_{K-k}' \) satisfying

\[
\text{vol} \left( \bigcup_{\Lambda \in \mathcal{A}_{k+1}} \Lambda \right) \geq 3^{-D} \cdot \text{vol} \left( \bigcup_{\Lambda \in \mathcal{C}_{K-k}'} \Lambda \right) \geq 3^{-D} \cdot \text{vol} \{ \text{int} (\Omega_k, \ell_{\max}(\mathcal{C}_{K-k})) \}.
\]

Now we have defined \( \mathcal{A}_k \) for all \( 1 \leq k \leq K \). We set \( \mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_K \). This is a disjoint family of cubes and \( \bigcup_{\Lambda \in \mathcal{A}} \Lambda \subset \Omega \).

**Claim 3.5.** There exists \( 0 \leq k < K \) satisfying

\[
(3.2) \quad \text{vol} \{ \text{int} (\Omega_k, \ell_{\max}(\mathcal{C}_{K-k})) \} < \frac{\eta}{3} \cdot \text{vol}(\Omega).
\]

**Proof.** Suppose this is false. Then for all \( 0 \leq k < K \)

\[
\text{vol} \left( \bigcup_{\Lambda \in \mathcal{A}_{k+1}} \Lambda \right) \geq 3^{-D} \cdot \text{vol} \{ \text{int} (\Omega_k, \ell_{\max}(\mathcal{C}_{K-k})) \} \geq 3^{-D} \cdot \frac{\eta}{3} \cdot \text{vol}(\Omega).
\]
Then
\[
\text{vol}(\Omega_K) \leq \text{vol}(\Omega) - K \cdot 3^{-D} \cdot \frac{\eta}{3} \cdot \text{vol}(\Omega) = \left(1 - K \cdot 3^{-D} \cdot \frac{\eta}{3}\right)\text{vol}(\Omega) < 0
\]
because we assumed \(K \cdot 3^{-D} \cdot \frac{\eta}{3} > 1\). This is a contradiction. \(\square\)

Let \(0 \leq k < K\) be the integer satisfying (3.2). Set \(r = \ell_{\max}(C_{K-k}) \geq 1\). We have
\[
\partial(\Omega_k, r) \subset \partial(\Omega, r) \cup \bigcup_{\Lambda \in \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k} \partial(\Lambda, r).
\]
From the choice of \(K\) in the beginning, for \(\Lambda \in \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k\)
\[
\text{vol}(\partial(\Lambda, r)) < \frac{\eta}{3} \cdot \text{vol}(\Lambda).
\]
Then\[
\text{vol}(\partial(\Omega_k, r)) \leq \text{vol}(\partial(\Omega, r)) + \sum_{\Lambda \in \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k} \frac{\eta}{3} \cdot \text{vol}(\Lambda) < \frac{\eta}{3} \text{vol}(\Omega) + \frac{\eta}{3} \text{vol}(\Omega) = \frac{2\eta}{3} \cdot \text{vol}(\Omega).
\]
Then we estimate the volume of \(B_r(\Omega_k) = \text{int}(\Omega_k, r) \cup \partial(\Omega_k, r)\) by
\[
\text{vol}(B_r(\Omega_k)) < \frac{\eta}{3} \cdot \text{vol}(\Omega) + \frac{2\eta}{3} \cdot \text{vol}(\Omega) = \eta \cdot \text{vol}(\Omega).
\]
Since \(B_1(\Omega_K) \subset B_r(\Omega_k)\), this proves the statement. \(\square\)

3.3. Proof of Theorem 3.1. Throughout this subsection we assume that \((X, T)\) is an \(\mathbb{R}^D\)-action with a metric \(d\).

**Lemma 3.6.** Let \(\varepsilon > 0\) and \(F \subset X\). Let \(\Omega, \Omega_1, \ldots, \Omega_N \subset \mathbb{R}^D\) be bounded subsets. If \(\Omega \subset \Omega_1 \cup \cdots \cup \Omega_N\) then
\[
\#(F, d_\Omega, \varepsilon) \leq \prod_{n=1}^N \#\left(F, d_{\Omega_n}, \frac{\varepsilon}{2}\right).
\]

**Proof.** Let \(E_n \subset X\) be a \((d_{\Omega_n}, \frac{\varepsilon}{2})\)-spanning set of \(F\). For each \((x_1, \ldots, x_N)\) in \(E_1 \times \cdots \times E_N\), we set
\[
V(x_1, \ldots, x_N) := \left\{ y \in X \mid \forall 1 \leq n \leq N : d_{\Omega_n}(y, x_n) \leq \frac{\varepsilon}{2}\right\}.
\]
The diameter of this set with respect to \(d_\Omega\) is less than or equal to \(\varepsilon\). The sets \(V(x_1, \ldots, x_N)\), \((x_1, \ldots, x_N) \in E_1 \times \cdots \times E_N\), cover \(F\). We pick a point from each (non-empty) \(V(x_1, \ldots, x_N)\). Then we get a \((d_\Omega, \varepsilon)\)-spanning set of \(F\). \(\square\)

**Lemma 3.7.** For any \(\varepsilon > 0\) and any bounded Borel subset \(\Omega \subset \mathbb{R}^D\)
\[
\#(X, d_\Omega, \varepsilon) \leq \left(\#(X, d_{[0,1]^D}, \frac{\varepsilon}{2})\right)^{\text{vol}(B_1(\Omega))}.
\]
Proof. Let $A$ be the set of $u \in \mathbb{Z}^D$ satisfying $(u + [0, 1]^D) \cap \Omega \neq \emptyset$. Then

$$\Omega \subset \bigcup_{u \in A} (u + [0, 1]^D) \subset B_1(\Omega).$$

Then $|A| \leq \text{vol} (B_1(\Omega))$ and (by Lemma 3.6)

$$\# (X, d_{\Omega}, \varepsilon) \leq \prod_{u \in A} \# (X, d_{u + [0, 1]^D}, \varepsilon/2).$$

We have

$$\# (X, d_{u + [0, 1]^D}, \varepsilon/2) = \# (X, d_{[0, 1]^D}, \varepsilon/2).$$

This proves the statement. □

Proposition 3.8. Let $\varepsilon, \delta, \beta$ be positive numbers. There exists a positive number $N = N(\varepsilon, \delta, \beta) > 0$ satisfying the following statement. Set

$$a := \sup_{x \in X} \mathcal{S} \left( B_\delta(x, d_{[0, 1]^D}), \frac{\varepsilon}{4} \right).$$

Then

$$\sup_{x \in X} \# \left( B_\delta \left( x, d_{[-N,L+N]^D}, d_{[0,L]^D}, \varepsilon \right) \right) < e^{(a+\beta)L^D}$$

for all sufficiently large $L > 1$.

Proof. We choose $\eta > 0$ so small that

$$(3.3) \quad \left( \# \left( X, d_{[0,1]^D}, \frac{\varepsilon}{4} \right) \right)^\eta < e^{\frac{\beta}{2}}.$$ 

Let $K = K(\eta) > 1$ be the positive number for this $\eta$ introduced in Proposition 3.4.

For $k = 1, 2, \ldots, K$, we will inductively construct a finite subset $Y_k \subset X$ and, for each $y \in Y_k$, positive numbers $L_k(y), M_k(y)$ and open neighborhoods $U_k(y), V_k(y)$ of $y$ satisfying the following.

- $L_1(y) > 1$ and $L_k(y) > K \max_{z \in Y_{k-1}} L_{k-1}(z)$ for $k \geq 2$.
- $\# \left( U_k(y), d_{[0,L_k(y)]^D}, \frac{\varepsilon}{2} \right) < \exp \left( (a + \frac{\beta}{2}) L_k(y)^D \right)$.
- For every $v \in V_k(y)$, $B_\delta \left( v, d_{[-M_k(y),M_k(y)]^D} \right) \subset U_k(y)$.
- $X = \bigcup_{y \in Y_k} V_k(y)$ for every $1 \leq k \leq K$.

Suppose we have defined the data for the $(k-1)$-th step. We are going to construct the data for the $k$-th step. (The initial step, $k = 1$, can be treated in the same way.) Let $y \in X$ be an arbitrary point. Since we have

$$\mathcal{S} \left( B_\delta \left( y, d_{[0,1]^D} \right), \frac{\varepsilon}{4} \right) \leq a < a + \frac{\beta}{2},$$

we can choose $L_k(y) > K \max_{z \in Y_k} L_{k-1}(z)$ (when $k = 1$, we assume $L_1(y) > 1$) satisfying

$$\frac{1}{L_k(y)^D} \log \# \left( B_\delta \left( y, d_{[0,L_k(y)]^D} \right), \frac{\varepsilon}{4} \right) < a + \frac{\beta}{2}.$$
Then there is $E_k(y) \subset X$ which is a $(d_{[0,L_k(y)]^D}, \frac{\varepsilon}{2})$-spanning set of $B_\delta (y, d_{\mathbb{R}^D})$ with

$$|E_k(y)| < \exp \left( \left( a + \frac{\beta}{2} \right) |L_k(y)|^D \right).$$

We define

$$U_k(y) := \left\{ x \in X \mid \exists z \in E_k(y) : d_{[0,L_k(y)]^D}(x, z) < \frac{\varepsilon}{2} \right\}.$$

This is an open set containing $B_\delta (y, d_{\mathbb{R}^D})$ with

$$\# \left( U_k(y), d_{[0,L_k(y)]^D}, \frac{\varepsilon}{2} \right) \leq |E_k(y)| < \exp \left( \left( a + \frac{\beta}{2} \right) |L_k(y)|^D \right).$$

Since

$$B_\delta (y, d_{\mathbb{R}^D}) = \bigcap_{M=1}^{\infty} B_\delta (y, d_{[-M,M]^D}),$$

there exists $M_k(y) > 0$ satisfying

$$B_\delta \left( y, d_{[-M_k(y),M_k(y)]^D} \right) \subset U_k(y).$$

If $\delta' > \delta$ is sufficiently close to $\delta$ then we also have

$$B_{\delta'} \left( y, d_{[-M_k(y),M_k(y)]^D} \right) \subset U_k(y).$$

Set

$$V_k(y) := \left\{ x \in X \mid d_{[-M_k(y),M_k(y)]^D}(x, y) < \delta' - \delta \right\}.$$

Then for every $v \in V_k(y)$ we have $B_{\delta} \left( v, d_{[-M_k(y),M_k(y)]^D} \right) \subset U_k(y)$. $V_k(y)$ is an open neighborhood of $y$. So we choose a finite set $Y_k \subset X$ such that $V_k(y), y \in Y_k$, cover $X$. We have finished the construction of the $k$-th step. So the induction works.

We fix

(3.4) \hspace{1cm} N > \max\{M_k(y) \mid 1 \leq k \leq K, y \in Y_k\}.

We assume that $L > 1$ is any sufficiently large number such that the cube $\Omega := [0,L]^D$ satisfies

$$\text{vol} \left( \partial \left( \Omega, \max_{y \in Y_k} L_k(y) \right) \right) < \frac{\eta}{3} \cdot \text{vol}(\Omega).$$

We are going to prove

$$\sup_{x \in X} \# \left( B_\delta \left( x, d_{[-N,L+N]^D}, d_{[0,L]^D} \right) \right) < e^{(a+\beta)L^D}.$$

Take any $x \in X$. For each $1 \leq k \leq K$ and $t \in \mathbb{Z}^D \cap \Omega$, we pick $y \in Y_k$ with $T^t x \in V_k(y)$. Set $\Lambda_{k,t} := t + [0,L_k(y)]^D$. From the choice of $N$ in (3.4),

$$T^t \left( B_\delta \left( x, d_{[-N,L+N]^D} \right) \right) \subset B_\delta \left( T^t x, d_{[-M_k(y),M_k(y)]^D} \right) \subset U_k(y).$$

Hence

(3.5) \hspace{1cm} \# \left( B_\delta \left( x, d_{[-N,L+N]^D}, d_{\Lambda_{k,t}}, \frac{\varepsilon}{2} \right) \right) \leq \# \left( U_k(y), d_{[0,L_k(y)]^D}, \frac{\varepsilon}{2} \right) < e^{(a+\frac{\beta}{2})\text{vol}(\Lambda_{k,t})}.\]
Let $C_k$ be the set of cubes $\Lambda_{k,t}$ ($t \in \mathbb{Z}^D \cap \Omega$). (Notice that this depends on $x \in X$.) The cubes in $C_k$ cover $\Omega = [0, L]^D$.

Now we apply Proposition 3.4 to $\Omega$ and $C_1, \ldots, C_K$. Then there is a disjoint subfamily $\mathcal{A} \subset C_1 \cup \cdots \cup C_K$ such that (set $\Omega' := \Omega \setminus \bigcup_{\Lambda \in \mathcal{A}} \Lambda$)
\[
\bigcup_{\Lambda \in \mathcal{A}} \Lambda \subset \Omega, \quad \text{vol } (B_1(\Omega')) < \eta \cdot \text{vol } (\Omega).
\]

From Lemma 3.7 and the choice of $\eta$ in (3.3)
\[
\# \left( B_\delta \left( x, d_{[0,1]^D}, \frac{\varepsilon}{4} \right) \right) \leq \left( \# \left( X, d_{[0,1]^D}, \frac{\varepsilon}{4} \right) \right)^\text{vol } (B_1(\Omega')) \leq \left( \# \left( X, d_{[0,1]^D}, \frac{\varepsilon}{4} \right) \right)^{\eta \cdot \text{vol } (\Omega)} \leq e^{\frac{\beta}{2} \text{vol } (\Omega)}.
\]

On the other hand, for each $\Lambda \in \mathcal{A}$, from (3.5)
\[
\# \left( B_\delta \left( x, d_{[-N,L+N]^D}, \frac{\varepsilon}{2} \right) \right) < e^{(a + \frac{\beta}{2}) \text{vol } (\Lambda)}.
\]

From Lemma 3.6
\[
\# \left( B_\delta \left( x, d_{[-N,L+N]^D}, d_{[0,L]^D}, \varepsilon \right) \right) \leq \# \left( B_\delta \left( x, d_{[-N,L+N]^D}, d_{\varepsilon/2} \right) \right) \cdot \prod_{\Lambda \in \mathcal{A}} \# \left( B_\delta \left( x, d_{[-N,L+N]^D}, d_{\varepsilon/2} \right) \right) \leq e^{\frac{\beta}{2} \text{vol } (\Omega)} \cdot \prod_{\Lambda \in \mathcal{A}} e^{(a + \frac{\beta}{2}) \text{vol } (\Lambda)} \leq e^{(a + \beta) \text{vol } (\Omega)} = e^{(a + \beta) L^D}.
\]

This holds for any $x \in X$. So we have proved the statement of the proposition. □

Now we are ready to prove Theorem 3.1. We write the statement again.

**Theorem 3.9** (= Theorem 3.1). For any $\delta > 0$
\[
\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \sup_{x \in X} \frac{S(B_\delta(x, d_{R^D}), \varepsilon)}{\log(1/\varepsilon)},
\]
\[
\underline{\text{mdim}}_M(X, T, d) = \liminf_{\varepsilon \to 0} \sup_{x \in X} \frac{S(B_\delta(x, d_{\mathbb{R}^D}), \varepsilon)}{\log(1/\varepsilon)}.
\]

**Proof.** Let $\varepsilon$ and $\beta$ be any positive numbers. Let $N(\varepsilon, \delta, \beta)$ be the positive number given in Proposition 3.8. Set
\[
a := \sup_{x \in X} S \left( B_\delta (x, d_{\mathbb{R}^D}), \frac{\varepsilon}{4} \right).
\]
For any \( L > 0 \), there are \( x_1, \ldots, x_M \in X \) such that
\[
X = \bigcup_{m=1}^{M} B_\delta \left( x_m, d_{[-N,L+N]^D} \right), \quad M = \# \left( X, d_{[-N,L+N]^D} \delta \right) = \# \left( X, d_{[0,L+2N]^D} \delta \right).
\]
Then
\[
\# \left( X, d_{[0,L]^D} \varepsilon \right) \leq \sum_{m=1}^{M} \# \left( B_\delta \left( x_m, d_{[-N,L+N]^D} \right), d_{[0,L]^D} \varepsilon \right)
\leq M \cdot \sup_{x \in X} \# \left( B_\delta \left( x, d_{[-N,L+N]^D} \right), d_{[0,L]^D} \varepsilon \right)
< M \cdot e^{(a+\beta)L^D}
\]
for all sufficiently large \( L \) by Proposition 3.8. So, for large \( L \)
\[
\log \# \left( X, d_{[0,L]^D} \varepsilon \right) \leq \log \# \left( X, d_{[0,L+2N]^D} \delta \right) + (a + \beta)L^D.
\]
Divide this by \( L^D \) and let \( L \to \infty \).

Let \( \beta \to 0 \):
\[
S(X, \varepsilon) \leq S(X, \delta) + a + \beta.
\]

Divide this by \( \log(1/\varepsilon) \) and let \( \varepsilon \to 0 \). Notice that \( S(X, \delta) \leq \log \# \left( X, d_{[0,1]^D} \delta \right) < \infty \) by Lemma 3.7. We get the conclusion. \( \square \)

### 4. Example: Brody curves

In this section we explain how to use Theorem 3.1 by an example. Our example is the \( \mathbb{C} \)-action on the space of Brody curves (Lipschitz holomorphic curves). We revisit a result proved in [Tsu18b].

#### 4.1. \( \mathbb{C} \)-action on the space of Brody curves.

Let \( z = x + y\sqrt{-1} \) be the standard coordinate of the complex plane \( \mathbb{C} \). We consider the complex projective space \( \mathbb{C}P^N \) with the Fubini–Study metric. For a holomorphic map \( f : \mathbb{C} \to \mathbb{C}P^N \), we denote by \(|df|(z) \geq 0 \) the local Lipschitz constant at \( z \in \mathbb{C} \). Explicitly, for \( f = [f_0 : f_1 : \cdots : f_N] \)
\[
|df|^2(z) = \frac{1}{4\pi} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \left( |f_0|^2 + |f_1|^2 + \cdots + |f_N|^2 \right).
\]
For \( \lambda > 0 \) we define \( \mathcal{M}_\lambda(\mathbb{C}P^N) \) as the space of holomorphic maps \( f : \mathbb{C} \to \mathbb{C}P^N \) satisfying \(|df| \leq \lambda \) all over the plane\(^3\). This is endowed with the compact-open topology and becomes a compact metrizable space. The group \( \mathbb{C} = \mathbb{R}^2 \) continuously acts on it by
\[
T : \mathbb{C} \times \mathcal{M}_\lambda(\mathbb{C}P^N) \to \mathcal{M}_\lambda(\mathbb{C}P^N), \quad (a, f(z)) \mapsto f(z + a).
\]

\(^3\)A Lipschitz holomorphic map from \( \mathbb{C} \) is called a Brody curve. Brody [Bro78] found its importance in the study of Kobayashi hyperbolicity.
We would like to study the mean dimension of $(\mathcal{M}(\mathbb{C}P^N), T)$. It is easy to see that
\[ \text{mdim} \left( \mathcal{M}_\lambda(\mathbb{C}P^N), T \right) = \lambda^2 \cdot \text{mdim} \left( \mathcal{M}_1(\mathbb{C}P^N), T \right). \]

So it is enough to study the case of $\lambda = 1$. (Nevertheless, it is useful to consider other $\mathcal{M}_\lambda(\mathbb{C}P^N)$ even for the study of $\mathcal{M}_1(\mathbb{C}P^N)$. See Proposition 4.2 below.)

For $f \in \mathcal{M}_\lambda(\mathbb{C}P^N)$, we define its **energy density** by
\[ \rho(f) := \lim_{L \to \infty} \left( \frac{1}{L^2} \sup_{a \in \mathbb{C}} \int_{a+[0,L]^2} |df|^2 dxdy \right). \]

We define $\rho_\lambda(\mathbb{C}P^N)$ as the supremum of $\rho(f)$ over $f \in \mathcal{M}_\lambda(\mathbb{C}P^N)$. We have
\[ \rho_\lambda(\mathbb{C}P^N) = \lambda^2 \cdot \rho_1(\mathbb{C}P^N). \]

In [MT15] we proved the lower bound
\[ \text{mdim} \left( \mathcal{M}_1(\mathbb{C}P^N), T \right) \geq 2(N+1)\rho_1(\mathbb{C}P^N). \]

On the other hand, it was proved in [Tsu18b] that
\[ \text{mdim} \left( \mathcal{M}_1(\mathbb{C}P^N), T \right) \leq 2(N+1)\rho_1(\mathbb{C}P^N). \]

So we get the formula
\[ \text{mdim} \left( \mathcal{M}_1(\mathbb{C}P^N), T \right) = 2(N+1)\rho_1(\mathbb{C}P^N). \]

The purpose here is to explain the proof of the upper bound $\text{mdim} \left( \mathcal{M}_1(\mathbb{C}P^N), T \right) \leq 2(N+1)\rho_1(\mathbb{C}P^N)$. The proof uses metric mean dimension.

For $f, g \in \mathcal{M}_\lambda(\mathbb{C}P^N)$ we define
\[ d(f, g) := \max_{z \in [0,1]^2} \mathbf{d}_{\text{FS}} \left( f(z), g(z) \right). \]

Here $\mathbf{d}_{\text{FS}}(\cdot, \cdot)$ is the Fubini–Study metric. This $d(f, g)$ becomes a metric on $\mathcal{M}_\lambda(\mathbb{C}P^N)$ by the unique continuation principle. Notice that we have
\[ d_\mathbb{C}(f, g) = \sup_{z \in \mathbb{C}} \mathbf{d}_{\text{FS}} \left( f(z), g(z) \right). \]

We would like to prove the upper bound on the upper metric mean dimension
\[ (4.1) \quad \overline{\text{mdim}}_M \left( \mathcal{M}_1(\mathbb{C}P^N), T, d \right) \leq 2(N+1)\rho_1(\mathbb{C}P^N). \]

Since metric mean dimension bounds mean dimension (see (3.1)), it follows from (4.1) that
\[ \text{mdim} \left( \mathcal{M}_1(\mathbb{C}P^N), T \right) \leq 2(N+1)\rho_1(\mathbb{C}P^N). \]

So the problem is how to prove (4.1). The proof will be given in §4.3.

In [Tsu18b] the upper bound (4.1) was proved by using Lemma 3.2 (a weaker version of Theorem 3.1). Here we will prove (4.1) by using Theorem 3.1. The basic structures of the two proofs are the same. But the use of Theorem 3.1 makes the argument clean.
By Theorem 3.1

\[ \overline{\text{mdim}}_M (\mathcal{M}_1(\mathbb{C}P^N), T, d) = \limsup_{\varepsilon \to 0} \frac{\sup_{f \in \mathcal{M}_1(\mathbb{C}P^N)} S(B_\delta (f, d_C), \varepsilon)}{\log(1/\varepsilon)}. \]

So we need to study \( B_\delta (f, d_C) = \{ g \in \mathcal{M}_1(\mathbb{C}P^N) \mid d_C(f, g) \leq \delta \} \).

We sometimes need to consider balls in \( \mathcal{M}_\lambda(\mathbb{C}P^N) \) for \( \lambda \neq 1 \). So we introduce a new notation for clarifying the value of the parameter \( \lambda \): For \( \delta > 0 \) and \( f \in \mathcal{M}_\lambda(\mathbb{C}P^N) \) we denote by \( B_\delta (f, d_C)_\lambda \) the \( \delta \)-ball around \( f \) in \( \mathcal{M}_\lambda(\mathbb{C}P^N) \) with respect to \( d_C \). Namely

\[ B_\delta (f, d_C)_\lambda := \left\{ g \in \mathcal{M}_\lambda(\mathbb{C}P^N) \left| \sup_{z \in \mathbb{C}} d_{FS}(f(z), g(z)) \leq \delta \right. \right\}. \]

In this notation,

\[ \overline{\text{mdim}}_M (\mathcal{M}_\lambda(\mathbb{C}P^N), T, d) = \limsup_{\varepsilon \to 0} \frac{\sup_{f \in \mathcal{M}_\lambda(\mathbb{C}P^N)} S(B_\delta (f, d_C)_\lambda, \varepsilon)}{\log(1/\varepsilon)}. \]

### 4.2. Nondegeneracy and key propositions.

The following definition is very important. For \( a \in \mathbb{C} \) and \( R > 0 \) we denote \( D_R(a) = \{ z \in \mathbb{C} \mid |z - a| \leq R \} \).

**Definition 4.1.** Let \( f : \mathbb{C} \to \mathbb{C}P^N \) be a holomorphic curve. For \( R > 0 \) it is said to be \( R \)-nondegenerate if for all \( a \in \mathbb{C} \)

\[ \max_{z \in D_R(a)} |df|(z) \geq \frac{1}{R}. \]

The basic idea behind this definition is as follows. For \( f \in \mathcal{M}_\lambda(\mathbb{C}P^N) \) we would like to study \( B_\delta (f, d_C)_\lambda \) by a deformation theory technique. As is usual in deformation theory, the description of a small deformation of \( f \) becomes simpler if a transversality condition is satisfied at the curve \( f \). The above definition provides the transversality condition we need in our deformation theory\(^4\). A trivial example which is not \( R \)-nondegenerate is a constant curve (i.e. a holomorphic curve whose image is one-point). Constant curves are the most “singular” object for our deformation theory. The condition (4.2) means that \( f \) is not close to a constant curve over the disk \( D_R(a) \). So an \( R \)-nondegenerate curve is not close to a constant curve over any \( R \)-disk\(^5\). For example, nonconstant elliptic functions are \( R \)-nondegenerate for some \( R > 0 \).

The next two propositions are slightly simpler versions of \([Tsu18b, \text{Proposition 3.2, Proposition 3.3}]\). They are key results for the proof of (4.1).

\(^4\) The condition in Definition 4.1 involves the parameter \( R \). So it is a kind of quantitative transversality condition.

\(^5\) Yosida \([Yos34]\) defined that a meromorphic function \( f \in \mathcal{M}_\lambda(\mathbb{C}P^1) \) is of first category if the closure of the \( \mathbb{C} \)-orbit of \( f \) does not contain a constant function. Definition 4.1 is a quantitative version of this old idea.
Proposition 4.2. There exist positive numbers $\delta_1$ and $C_1$ satisfying the following statement. For any $\lambda > 1$ there exists $R_1 = R_1(\lambda) > 0$ such that for any $f \in \mathcal{M}_1(\mathbb{C}P^N)$ we can construct a map

$$\Phi : B_{\delta_1} (f, d_{\mathcal{C}})_1 \to \mathcal{M}_\lambda(\mathbb{C}P^N)$$

satisfying

1. $\Phi(f)$ is $R_1$-nondegenerate.
2. For any $g, h \in B_{\delta_1} (f, d_{\mathcal{C}})_1$ and $z \in \mathbb{C}$
   $$d_{FS}(\Phi(g)(z), \Phi(h)(z)) \leq C_1 \cdot d_{FS}(g(z), h(z)),
   \quad d_{FS}(g(z), h(z)) \leq C_1 \cdot \max_{|w-z| \leq 3} d_{FS}(\Phi(g)(w), \Phi(h)(w)).$$

This is a kind of “resolution of singularity”. If we choose an arbitrary $f \in \mathcal{M}_1(\mathbb{C}P^N)$, it might be a degenerate curve. However we can replace it by a nondegenerate one $\Phi(f)$. The proof is an application of surgery (gluing). Given $f \in \mathcal{M}_1(\mathbb{C}P^N)$, we look for its “degenerate region” (i.e. the region where the norm $|df|$ is uniformly small). We glue sufficiently many rational curves to $f$ over the degenerate region. Then the resulting curve becomes nondegenerate. The condition (2) means that this surgery procedure does not destroy the metric structure.

Proposition 4.3. For any $R > 0$ and $0 < \varepsilon < 1$ there exist positive numbers $\delta_2 = \delta_2(R)$, $C_2 = C_2(R)$ and $C_3 = C_3(\varepsilon)$ satisfying the following statement. Let $f \in \mathcal{M}_2(\mathbb{C}P^N)$ be an $R$-nondegenerate curve, and let $\Lambda \subset \mathbb{C}$ be a square of side length $L \geq 1$. Then

$$\# (B_{\delta_2} (f, d_{\mathcal{C}})_2, d_{\Lambda}, \varepsilon) \leq \left(\frac{C_2}{\varepsilon}\right)^{2(N+1)} \int_{\mathbb{C}} |df|^2 dxdy + C_3 \cdot L.$$

This is proved by deformation theory. Given a nondegenerate curve $f$, we describe the ball $B_{\delta_2} (f, d_{\mathcal{C}})_2$ by a deformation theory technique and get the above estimate.

Here we do not provide the detailed proofs of the above two propositions. They are the same as the proofs of [Tsu18b, Proposition 3.2, Proposition 3.3]. (Indeed the above propositions are slightly simpler than [Tsu18b, Proposition 3.2, Proposition 3.3] because the argument of [Tsu18b] is based on Lemma 3.2 whereas the argument here uses much simpler Theorem 3.1.) It is not our purpose here to explain the proofs of the above two propositions. Our purpose is to explain how to use Theorem 3.1.

4.3. Proof of the upper bound. Now we start to prove the upper bound (4.1):

$$\text{mdim}_M \left(\mathcal{M}_1(\mathbb{C}P^N), T, d\right) \leq 2(N + 1) \rho_1(\mathbb{C}P^N).$$

By Theorem 3.1, for any $\delta > 0$

$$\text{mdim}_M \left(\mathcal{M}_1(\mathbb{C}P^N), T, d\right) = \lim_{\varepsilon \to 0} \sup \frac{\sup_{f \in \mathcal{M}_1(\mathbb{C}P^N)} S(B_{\delta} (f, d_{\mathcal{C}})_1, \varepsilon)}{\log(1/\varepsilon)}.$$
Recall that
\[ S(B_{\delta}(f, d_{C})_1, \varepsilon) = \limsup_{L \to \infty} \log \#(B_{\delta}(f, d_{C})_1, d_{[0,L]^2}, \varepsilon). \]

Let \( \delta_1 \) and \( C_1 \) be positive constants introduced in Proposition 4.2. Take an arbitrary \( 1 < \lambda < 2 \). Let \( R_1 = R_1(\lambda) \) be the positive constant introduced in Proposition 4.2 for this \( \lambda \). And, let \( \delta_2 = \delta_2(R_1) \) be the positive constant introduced in Proposition 4.3 for this \( R_1 \). Set
\[ \delta = \min \left( \delta_1, \frac{\delta_2}{C_1} \right). \]

Take an arbitrary \( f \in \mathcal{M}_1(\mathbb{C}P^N) \). Applying Proposition 4.2 to \( f \), there is a map \( \Phi: B_{\delta}(f, d_{C})_1 \to \mathcal{M}_\lambda(\mathbb{C}P^N) \) such that
- \( \Phi(f) \) is \( R_1 \)-nondegenerate.
- For any \( g, h \in B_{\delta}(f, d_{C})_1 \) and \( z \in \mathbb{C} \)
\[
\mathbf{d}_{FS}(\Phi(g)(z), \Phi(h)(z)) \leq C_1 \cdot \mathbf{d}_{FS}(g(z), h(z)),
\]
\[
\mathbf{d}_{FS}(g(z), h(z)) \leq C_1 \cdot \max_{|w-z| \leq 3} \mathbf{d}_{FS}(\Phi(g)(w), \Phi(h)(w)).
\]
From (4.3) and \( \delta \leq \frac{\delta_2}{C_1} \), for \( g \in B_{\delta}(f, d_{C})_1 \)
\[
\mathbf{d}_{FS}(\Phi(f)(z), \Phi(g)(z)) \leq C_1 \delta \leq \delta_2.
\]
Hence
\[ \Phi(B_{\delta}(f, d_{C})_1) \subset B_{\delta_2}(\Phi(f), d_{C})_\lambda \subset B_{\delta_2}(\Phi(f), d_{C})_2. \]
From (4.4), for any \( g, h \in B_{\delta}(f, d_{C})_1 \) and \( L > 0 \)
\[
d_{[0,L]^2}(g, h) \leq C_1 \cdot d_{[-3,L+3]^2}(\Phi(g), \Phi(h)).
\]
Hence for any \( 0 < \varepsilon < 1 \)
\[
\#(B_{\delta}(f, d_{C})_1, d_{[0,L]^2}, \varepsilon) \leq \#(\Phi(B_{\delta}(f, d_{C})_1), d_{[-3,L+3]^2}, \varepsilon)
\]
\[
\leq \#(B_{\delta_2}(\Phi(f), d_{C})_2, d_{[-3,L+3]^2}, \varepsilon).
\]
We apply Proposition 4.3 to the curve \( \Phi(f) \) and the square \([-3, L+3]^2 \) of side length \( L + 6 \). Then
\[
\#(B_{\delta_2}(\Phi(f), d_{C})_2, d_{[-3,L+3]^2}, \frac{\varepsilon}{2C_1}) \leq \left( \frac{2C_1C_2}{\varepsilon} \right)^{2(N+1)} \int_{[-3,L+3]^2} |d\Phi(f)|^2 \, dx dy + C_3 \cdot (L + 6).
\]
Here \( C_2 = C_2(R_1) \) and \( C_3 = C_3 \left( \frac{\varepsilon}{2C_1} \right) \) are the positive constants introduced in Proposition 4.3. Therefore
\[
\log \#(B_{\delta}(f, d_{C})_1, d_{[0,L]^2}, \varepsilon)
\]
\[
\leq \log \left( \frac{2C_1C_2}{\varepsilon} \right) \left\{ 2(N+1) \int_{[-3,L+3]^2} |d\Phi(f)|^2 \, dx dy + C_3 \cdot (L + 6) \right\}.
\]
Divide this by $L^2$ and let $L \to \infty$. Then

$$S(B_\delta(f,d_C)_1,\varepsilon) \leq \log \left( \frac{2C_1 C_2}{\varepsilon} \right) \{2(N+1)\rho(\Phi(f))\}.$$ 

Noting $\Phi(f) \in \mathcal{M}_\lambda(\mathbb{C}P^N)$, we get

$$\sup_{f \in \mathcal{M}_1(\mathbb{C}P^N)} S(B_\delta(f,d_C)_1,\varepsilon) \leq 2(N+1)\rho(\mathbb{C}P^N) \cdot \log \left( \frac{2C_1 C_2}{\varepsilon} \right).$$

Divide this by $\log(1/\varepsilon)$ and let $\varepsilon \to 0$. Then (noting $C_1 C_2$ is independent of $\varepsilon$)

$$\overline{\text{mdim}}_M(\mathcal{M}_1(\mathbb{C}P^N),T,d) = \limsup_{\varepsilon \to 0} \frac{\sup_{f \in \mathcal{M}_1(\mathbb{C}P^N)} S(B_\delta(f,d_C)_1,\varepsilon)}{\log(1/\varepsilon)} \leq 2(N+1)\rho(\mathbb{C}P^N).$$

Recall that $1 < \lambda < 2$ is arbitrary and $\rho(\mathbb{C}P^N) = \lambda^2 \rho(\mathbb{C}P^N)$. So, letting $\lambda \to 1$, we get the result

$$\overline{\text{mdim}}_M(\mathcal{M}_1(\mathbb{C}P^N),T,d) \leq 2(N+1)\rho_1(\mathbb{C}P^N).$$

The author recommends interested readers to compare the argument in this subsection with the one given in [Tsu18b, pp.947-949]. Their basic structures are the same, but probably the argument here is a bit clearer. This is because here we use Theorem 3.1 whereas [Tsu18b] used Lemma 3.2.

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