Renormalisation scale setting for D-mixing

Alexander Lenz\textsuperscript{a, b}, Maria Laura Piscopo\textsuperscript{a} and Christos Vlahos\textsuperscript{a}
\textsuperscript{a} IPPP, Department of Physics, University of Durham, DH1 3LE, UK
\textsuperscript{b} Physik Department, Universität Siegen, Walter-Flex-Str. 3, 57068 Siegen, Germany
email: alexander.josef.lenz@gmail.com, maria.l.piscopo@durham.ac.uk, christos.elahos@durham.ac.uk
(Dated: July 8, 2020)

A naive application of the heavy quark expansion (HQE) yields theory estimates for the decay rate of neutral $D$ mesons that are four orders of magnitude below the experimental determination. It is well known that this huge suppression results from severe GIM cancellations. We find that this mismatch can be solved by individually choosing the renormalisation scale of the different internal quark contributions. For $b$ and $c$ hadron lifetimes, as well as for the decay rate difference of neutral $B$ mesons the effect of our scale setting procedure lies within the previously quoted theory uncertainties, while we get enlarged theory uncertainties for the semileptonic CP asymmetries in the $B$ system.

INTRODUCTION

An improvement of our theoretical understanding of charm physics is crucial to make use of the huge amount of current and future experimental charm data obtained by LHCb\[11], BESIII\[2] and Belle II\[3]. The recent discovery of direct CP violation in the charm system by the LHCb collaboration\[4] is an example of this necessity. Briefly after the announcement of a non-vanishing measurement of $\Delta A_{CP} = A_{CP}(D^0 \rightarrow K^+ K^-) - A_{CP}(D^0 \rightarrow \pi^+ \pi^-)$ both theory papers arguing for a beyond standard model (BSM)\[5, 6] (partly based on the calculation of $\pi^0$) and a standard model (SM)\[8–11] origin of this measurement appeared (a summary of references investigating a previous claim for evidence of CP violation can be found in Ref.\[12]). Thus a decisive conclusion about the potential size of the SM contribution to $\Delta A_{CP}$ is mandatory to fully exploit the significant experimental progress in this field. A long-standing puzzle in this regard is the theoretical description of mixing of neutral $D$ mesons. Charm-mixing is by now experimentally well established and HFLAV\[13] finds as an average of \[13\]:

$$x = \frac{\Delta M_D}{\Gamma_{D^0}} = 0.39^{+0.11}_{-0.12}\%,$$

$$y = \frac{\Delta \Gamma_D}{2 \Gamma_{D^0}} = 0.651^{+0.063}_{-0.069}\%,$$

where $\Delta M_D$ is the mass difference of the two mass eigenstates of the neutral $D^0$ mesons and $\Delta \Gamma_D$ the corresponding decay rate difference. However, theory predictions for $x$ and $y$ cover a vast range of values - differing by several orders of magnitude, see e.g. the compilations of theory predictions in Refs.\[17, 43]. Future measurements will not only increase the precision of $x$ and $y$, but also give stronger bounds or even a measurement of the CP violation in mixing\[49] encoded e.g. in the phase $\phi_{12}$, which is currently constrained to be within $[-2.5^\circ, 1.8^\circ]$\[13]. A reliable range of potential SM values is pivotal to benefit from the coming experimental improvements.

HQE

The heavy quark expansion (HQE)\[50–56] (see Ref.\[57] for a recent overview) describes the total decay rate of heavy hadrons and the decay rate difference of heavy neutral mesons as an expansion in inverse powers of the heavy quark mass. In the case of $B_s$-mixing and $b$-hadron lifetimes the HQE predicts values\[57–63] which are in good agreement with the experimental ones\[13]:

| Decay Rate | HFLAV 2019 | HQE 2019 |
|------------|------------|----------|
| $\tau(B_s)$ | 0.994(4)   | 1.0007(25) |
| $\tau(B_d)$ | 1.076(4)   | 1.082$^{+0.022}_{-0.026}$ |
| $\tau(B_s) / \tau(B_d)$ | 0.969(6)   | 0.935(54)  |
| $\Delta \Gamma_{B_s}$ | 0.091(13)ps$^{-1}$ | 0.090(5)ps$^{-1}$ |

This impressive result when the expansion parameter $\Lambda/m_b$ ($\Lambda$ denotes an hadronic scale of the order of $\Lambda^{QCD}$) suggests that one might still get reasonably well-behaving estimates moving to the charm system, where the expansion parameter increases by a factor of three. For the lifetime ratio $\tau(D^+)/\tau(D^0)$ both NLO-QCD corrections to the dimension-six contribution\[64] and values for the non-perturbative matrix elements of four quark operators\[58] are known - for all other charm hadrons this is not yet the case, thus, corresponding theory estimates have to be taken with care - and one finds indeed a nice agreement within the huge theory uncertainties:

$$\frac{\tau(D^+)}{\tau(D^0)}^{\text{HFLAV 2019}} = 2.536(19), \quad \frac{\tau(D^+)}{\tau(D^0)}^{\text{HQE 2019}} = 2.7^{+0.7}_{-0.8}. \quad (2)$$

Hence, it is quite surprising that a naive application of the HQE fails completely for $D$-mixing.

CHARM MIXING

Diagonalising the two dimensional mixing matrix of the $D^0$ and the $\bar{D}^0$ meson - containing the off-diagonal
Another method to determine the neutral mesons as well as the meson masses as well as the strong coupling, CKM elements from Ref. [74], non-perturbative matrix elements from Ref. [64], and the $D^0$ decay constant from Ref. [76].

GIM IN D-MIXING

In order to better understand the peculiarities of D-mixing we decompose $\Gamma_{12}$ according to the flavour of the internal quark pair. The three contributions are denoted $\Gamma^{ss}_{12}$, $\Gamma^{sd}_{12}$, and $\Gamma^{dd}_{12}$:

$$
\Gamma_{12} = - (\lambda^2_s \Gamma^{ss}_{12} + 2 \lambda_s \lambda_d \Gamma^{sd}_{12} + \lambda^2_d \Gamma^{dd}_{12}) + 2 \lambda_s \lambda_d \left( \Gamma^{sd}_{12} - \Gamma^{dd}_{12} \right) - \lambda^2_s \Gamma^{dd}_{12}.
$$

The CKM factor in the first term of Eq. (6) has by far the largest real part, while the second term has actually the largest imaginary part - it should thus be important for the determination of the potential size of CP violation in mixing. Since the relative imaginary part of $\lambda_s$ is much larger than that of $\lambda_d$, we suggest to keep all terms in $\lambda_d$. Eq. (6) shows very pronounced hierarchies:

$$
\begin{align*}
-\lambda^2_s &= -4.791 \cdot 10^{-2} + 3.094 \cdot 10^{-6} I, \\
+2\lambda_s \lambda_d &= +2.751 \cdot 10^{-5} + 6.121 \cdot 10^{-5} I, \\
-\lambda^2_d &= +1.560 \cdot 10^{-8} - 1.757 \cdot 10^{-8} I.
\end{align*}
$$

The CKM factor in the first term of Eq. (6) has by far the largest real part, while the second term has actually the largest imaginary part - it should thus be important for the determination of the potential size of CP violation in mixing. Since the relative imaginary part of $\lambda_s$ is much larger than that of $\lambda_d$, we suggest to keep all terms in $\lambda_d$. Moreover, extreme GIM cancellations affect the coefficients of the CKM elements in Eq. (6).

Expanding in the small mass parameter $\tilde{z} = m_{c}/m_{b}$ we find at LO-QCD (top line) and at NLO-QCD (lower line):

$$
\begin{align*}
\Gamma^{ss}_{12} &= \left\{ 1.62 - 2.34 \tilde{z} - 5.07 \tilde{z}^2 + \ldots, \\
1.42 - 4.30 \tilde{z} - 12.45 \tilde{z}^2 + \ldots, \\
-1.17 \tilde{z} - 2.53 \tilde{z}^2 + \ldots, \\
-2.15 \tilde{z} - 6.26 \tilde{z}^2 + \ldots, \\
-13.38 \tilde{z}^3 + \ldots, \\
0.07 \tilde{z}^2 - 29.72 \tilde{z}^3 + \ldots \right\},
\end{align*}
$$

It was observed before that QCD corrections lower the GIM suppression by one power of $\tilde{z}$. The peculiarity of Eq. (6) is that the CKM dominant factor $\lambda^2_c$ multiplies the extremely GIM suppressed term given in Eq. (12), the CKM suppressed factor $\lambda_s \lambda_d$ multiplies the GIM suppressed term given in Eq. (11) and the very CKM suppressed factor $\lambda^2_d$ multiplies $\Gamma^{dd}_{12}$, where no GIM suppression is present. Thus the three contributions in...
Eq. (6) have actually a similar size:
\[
\Gamma_{12} = (2.08 \cdot 10^{-7} - 1.34 \cdot 10^{-11} I) \ (1\text{st term})
- (3.74 \cdot 10^{-7} + 8.31 \cdot 10^{-7} I) \ (2\text{nd term})
+ (2.22 \cdot 10^{-8} - 2.5 \cdot 10^{-8} I) \ (3\text{rd term}). \tag{13}
\]
It is also clear that a sizeable phase in \( D \)-mixing can only arise, if the slightly GIM suppressed term is enhanced. Different solutions have been suggested in order to explain the mismatch between the HQE prediction and experimental determination. i) Higher orders in the HQE could be less affected by GIM suppression - first estimates of the dimension nine contribution to \( D \)-mixing show indeed such an enhancement, but not on a scale to reproduce the experimental number. For a final conclusion about this possibility a full determination of dimension nine and twelve would be necessary. ii) Large violations of quark-hadron duality are excluded by the many successful tests of the HQE as stated above. In Ref. [8] it was shown that violations as small as 20 per cent could be sufficient to explain the experimental value of \( D \) mixing. iii) The HQE is not applicable and we have to rely on different methods, like summing over the exclusive decays channels contributing to the decay rate difference, see e.g. Refs. [8,8].

ALTERNATIVE SCALE SETTING

In \( \Gamma_{12} \) the two renormalisation scales \( \mu_1 \) and \( \mu_2 \) are arising, see Fig. 1. The dependence on \( \mu_1 \) in the \( \Delta C = 1 \) Wilson coefficients of the effective Hamiltonian cancels, up to terms of higher order, the corresponding dependence of the radiative corrections to the diagrams, Fig. 1(a). Similarly the dependence on \( \mu_2 \) arises from loop-corrections to the HQE diagrams, Fig. 1(b) and cancels the corresponding dependence of the matrix elements of the \( \Delta C = 2 \) four quark operators. We will not discuss the \( \mu_2 \)-dependence any further since this cancellation is very effective. For the \( \mu_1 \)-dependence, in the \( B_s \) system the cancellation is numerically only weakly realised when moving from LO-QCD to NLO-QCD, see Refs. [87,88]. This indicates the importance of higher order corrections and first steps in that direction show indeed large NNLO-QCD effects [87,88]. In the \( D \) system a reduction of the \( \mu_1 \)-dependence, when moving from LO-QCD to NLO-QCD, is present in the individual contributions \( \Gamma_{12}^{ss,sd,dd} \) but not in \( \Gamma_{12} \), see Fig. 2 which seems to be again a consequence of the severe GIM cancellations. Making the scale dependence explicit we can write:
\[
\Gamma_{12} = \sum_{q_1, q_2 = ss, sd, dd} \Gamma_{12}^{q_1 q_2} (\mu_1^{q_1 q_2}, \mu_2^{q_1 q_2}) \langle Q \rangle (\mu_2^{q_1 q_2}) \ \frac{1}{m_{\pi}^2} + \ldots \tag{14}
\]
In general different internal quark pairs contribute to different decay channels of the \( D^0 (\bar{D}^0) \) meson e.g. \( s \bar{s} \) to a \( K^+ K^- \) final state and \( s d \) to a \( \pi^+ \pi^- \) final state. For each of these different observables the choice of the renormalisation scales is a priori arbitrary, nevertheless one typically fixes \( \mu_1^{ss} = \mu_1^{sd} = \mu_1^{dd} = \mu \) which is then chosen to be equal to the mass of the decaying heavy quark, i.e. \( \mu = m_Q \) for \( Q \) quark decays, to minimize terms of the form \( \alpha_s(\mu) \ln(\mu^2/m_Q^2) \). Uncertainties due to unknown higher order corrections are estimated varying \( \mu \) between \( m_Q/2 \) and \( 2m_Q \) - in the case of the charm quark we fix the lower bound to 1 GeV in order to still ensure reliable perturbative results. Here we propose two different ways to treat the renormalisation scale \( \mu_1^{q_1 q_2} \), both will reduce the mismatch between the HQE prediction and the experimental determination of \( D \)-mixing, while leaving the other HQE predictions unchanged: i) \( \mu_1^{ss}, \mu_1^{sd} \) and \( \mu_1^{dd} \) are set to the common scale \( m_c \) but varied independently between 1 GeV and 2\( m_c \). ii) \( \mu_1^{ss}, \mu_1^{sd} \) and \( \mu_1^{dd} \) are set to different scales according to the size of the available phase space. In particular we will evaluate \( \Gamma_3^{ss} \) at the scale \( \mu_1^{ss} = \mu - 2 \epsilon \), \( \Gamma_3^{sd} \) at the scale \( \mu_1^{sd} = \mu - \epsilon \) and \( \Gamma_3^{dd} \) at the scale \( \mu_1^{dd} = \mu \), where \( \epsilon \) is related to the kinematics of the decays. If \( \epsilon \) is not too large, then both methods will yield results for the individual \( \Gamma_3^{ss}, \Gamma_3^{sd} \) and \( \Gamma_3^{dd} \) which lie within the usually quoted theory uncertainties obtained following the prescription stated above, but they will clearly affect in a sizeable way the severe GIM cancellations in Eqs. (11) and (12). The first method gives a considerably enhanced range of values for \( \Omega \):
\[
\Omega \in [4.6 \cdot 10^{-5}, 1.3], \tag{15}
\]
which nicely covers also the experimental determination of the decay rate difference. Scanning independently over \( \mu_1^{ss}, \mu_1^{sd} \) and \( \mu_1^{dd} \) in 11 equidistant steps we find that out of the 1331 points only 14 give a value of \( \Omega < 0.001 \), while 984 give a value of \( \Omega > 0.1 \). The very small HQE prediction seems thus to be an artefact of fixing the scales \( \mu_1^{ss}, \mu_1^{sd} \) and \( \mu_1^{dd} \) to be the same. The range of values shown in Eq. (15) is similar even if we use the pole scheme for the quark masses, lattice results instead of the HQET results or a different \( \Delta C = 2 \) operator basis. In all these cases the value \( \Omega \geq 1 \) can be obtained. For \( \alpha \) we get in general results ranging from \(-\pi \) to \( \pi \). A closer look
Within the SM we get the weakly GIM suppressed contribution in But the semi-leptonic CP asymmetries are governed by only get a shift within the usually quoted theory range. \(\epsilon\) values of enhancement up to the experimental value is possible for \(\Omega\) would be affected in this scenario. Again an enhancement up to the experimental value is possible for values of \(\epsilon \approx 0.2\) GeV. Finally we have to test the effect of our alternative scale setting procedure on all the other HQE predictions. For the lifetimes (i.e. \(\tau(D^+)/\tau(D^0)\) as well as \(b\) hadron lifetimes) and the decay rate difference \(\Delta \Gamma_s\) no GIM-like cancellations arise and we can only get a shift within the usually quoted theory range. But the semi-leptonic CP asymmetries are governed by the weakly GIM suppressed contribution in \(B_s\)-mixing. Within the SM we get

\[
\text{Re}\left(\frac{\Gamma_{12}^{s}}{M_{12}^{s}}\right)^{\text{SM}} = -\frac{\Delta \Gamma_q}{\Delta M_q} = \left\{\begin{array}{l}
-\left(49.9 \pm 6.7\right) \cdot 10^{-4} \quad q = s \\
-\left(49.7 \pm 6.8\right) \cdot 10^{-4} \quad q = d
\end{array}\right.
\]

\[
\text{Im}\left(\frac{\Gamma_{12}^{s}}{M_{12}^{s}}\right)^{\text{SM}} = a_s^{q d} = \left\{\begin{array}{l}
+\left(2.2 \pm 0.2\right) \cdot 10^{-5} \quad q = s \\
-\left(5.0 \pm 0.4\right) \cdot 10^{-4} \quad q = d
\end{array}\right.
\]

Performing the \(\epsilon\) analysis we find:

| \(\epsilon\) (GeV) | \(\Gamma_{12}^{s}/M_{12}^{s}\) | \(\Gamma_{12}^{s}/M_{12}^{d}\) |
|-----------------|-----------------|-----------------|
| 0.0             | -0.00499 + 0.0000221 | -0.00397 - 0.003501 |
| 0.2             | -0.00494 + 0.0000237 | -0.00492 - 0.000531 |
| 0.5             | -0.00484 + 0.0000266 | -0.00482 - 0.000591 |
| 1.0             | -0.00447 + 0.0000377 | -0.00448 - 0.000844 |
| 1.5             | -0.00287 + 0.0000911 | -0.00309 - 0.00211 |

We see, that for \(\epsilon\) values of up to 1 GeV the predictions for the real part lie with the usually quoted theory uncertainties (indicated in blue). The predictions for the semi-leptonic asymmetries can, however, be increased by almost 100% compared to the usually quoted values.

**CONCLUSIONS**

Our main finding is that the range of the HQE uncertainty for \(y\) is much larger than previously thought and it covers the experimental value if we modify the usually adopted scale setting. For a full solution of the \(D\)-mixing puzzle we nevertheless suggest a more precise estimate of higher order corrections in the HQE, as well as a completion of the NNLO-QCD corrections to the leading term. In our alternative scale setting procedure we find that a small contribution to CPV in mixing stemming from the decay rate can be up to one per mille within in the SM, which agrees with estimates made in Refs. [89, 90]. For a prediction of CP violation in mixing in addition the contribution coming from \(M_{12}\) has to be determined. This might be done in future via the help of dispersion relations, see e.g. Refs. [83, 90, 91]. We would like to note that our suggested procedure is still respecting the GIM mechanism, because for vanishing internal strange quark mass, also the parameter \(\epsilon\) will be zero. Finally this scale setting does not affect quantities like \(\tau(D^+)/\tau(D^0)\), \(b\) hadron lifetimes and \(\Delta \Gamma_s\) outside the range of their quoted theoretical errors, but it affects the semi-leptonic CP asymmetries and we get enhanced SM ranges:

\[
a_{sl}^q \in [-9.2; -4.6] \cdot 10^{-4}, \quad a_{sl}^s \in [2.0; 4.0] \cdot 10^{-5}. \quad (17)
\]

**ACKNOWLEDGEMENTS**

We thank Vladimir Braun, Marco Gersabeck, Thomas Rauh, Alexey Petrov and Aleksey Rusov for helpful discussions.

[1] R. Aaij et al. (LHCb), (2018), arXiv:1808.08865
[2] M. Ablikim et al., Chin. Phys. C44 (2020), 10.1088/1674-1137/44/4/040001; arXiv:1912.05093 [hep-ex]
[3] W. Altmannshofer et al. (Belle-II), PTEP 2019, 123C01 (2019); Erratum: PTEP2020,no.2,029201(2020], arXiv:1808.10567 [hep-ex]
[4] R. Aaij et al. (LHCb), Phys. Rev. Lett. 122, 211803 (2019); arXiv:1903.08726 [hep-ex]
[5] M. Chaia, A. Lenz, A. V. Rusov, and J. Scholtz, JHEP 07, 161 (2019); arXiv:1903.10490 [hep-ph]
[6] A. Dery and Y. Nir, JHEP 12, 104 (2019); arXiv:1909.11242 [hep-ph]
[7] A. Khodjamirian and A. A. Petrov, Phys. Lett. B 774, 235 (2017); arXiv:1706.07780 [hep-ph]
[66] T. Jubb, M. Kirk, A. Lenz, and G. Télaltmatzi-Xolocotzi, Nucl. Phys. B 915, 431 (2017), arXiv:1603.07770 [hep-ph].
[67] M. Beneke, G. Buchalla, and I. Dunietz, Phys. Rev. D 54, 4419 (1996) [Erratum: Phys.Rev.D 83, 119902 (2011)], arXiv:hep-ph/9605259.
[68] M. Beneke, G. Buchalla, C. Greub, A. Lenz, and U. Nierste, Phys. Lett. B 459, 631 (1999) arXiv:hep-ph/9808385.
[69] A. Dighe, T. Hurth, C. Kim, and T. Yoshikawa, Nucl. Phys. B 624, 377 (2002) arXiv:hep-ph/0109088.
[70] M. Beneke, G. Buchalla, A. Lenz, and U. Nierste, Phys. Lett. B 576, 173 (2003) arXiv:hep-ph/0307344.
[71] M. Ciuchini, E. Franco, V. Lubicz, F. Mescia, and C. Tarantino, JHEP 08, 031 (2003) arXiv:hep-ph/0308029.
[72] A. Lenz and U. Nierste, JHEP 06, 072 (2007) arXiv:hep-ph/0612167.
[73] A. Bazavov et al., Phys. Rev. D97, 034513 (2018) arXiv:1706.04022 [hep-lat].
[74] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).
[75] J. Charles, A. Hocker, H. Lacker, S. Laplace, F. Le Diberder, J. Malcles, J. Ocariz, M. Pivk, and L. Roos (CKMfitter Group), Eur. Phys. J. C 41, 1 (2005), arXiv:hep-ph/0406164.
[76] S. Aoki et al. (Flavour Lattice Averaging Group), Eur. Phys. J. C 80, 113 (2020) arXiv:1902.08191 [hep-lat].
[77] S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D2, 1285 (1970).
[78] E. Golowich and A. A. Petrov, Phys. Lett. B 625, 53 (2005) arXiv:hep-ph/0506185.
[79] M. Bobrowski, A. Lenz, J. Riedl, and J. Rohrwild, JHEP 03, 009 (2010), arXiv:1002.4794 [hep-ph].
[80] H. Georgi, Phys. Lett. B 297, 353 (1992), arXiv:hep-ph/9209291.
[81] T. Ohl, G. Ricciardi, and E. H. Simmons, Nucl. Phys. B 403, 605 (1993) arXiv:hep-ph/9301212.
[82] I. I. Bigi and N. G. Uraltsev, Nucl. Phys. B 592, 92 (2001) arXiv:hep-ph/0005089.
[83] M. Bobrowski, A. Lenz, and T. Rauh, in Proceedings, 5th International Workshop on Charm Physics (Charm 2012): Honolulu, Hawaii, USA, May 14-17, 2012 (2012) arXiv:1208.6438 [hep-ph].
[84] A. F. Falk, Y. Grossman, Z. Ligeti, and A. A. Petrov, Phys. Rev. D 65, 054034 (2002) arXiv:hep-ph/0110317.
[85] H.-Y. Cheng and C.-W. Chiang, Phys. Rev. D 81, 114020 (2010) arXiv:1005.1106 [hep-ph].
[86] H.-Y. Jiang, F.-S. Yu, Q. Qin, H.-n. Li, and C.-D. L, Chin. Phys. C 42, 063101 (2018) arXiv:1705.07335 [hep-ph].
[87] H. Asatrian, A. Hovhannisyan, U. Nierste, and A. Yeghiazaryan, JHEP 10, 191 (2017) arXiv:1709.02160 [hep-ph].
[88] H. M. Asatrian, H. H. Asatryan, A. Hovhannisyan, U. Nierste, S. Tumasyan, and A. Yeghiazaryan, (2020) arXiv:2006.13227 [hep-ph].
[89] A. L. Kagan and L. Silvestrini, (2020), arXiv:2001.07207 [hep-ph].
[90] H.-N. Li, H. Umeeda, F. Xu, and F.-S. Yu, (2020), arXiv:2001.04079 [hep-ph].
[91] A. F. Falk, Y. Grossman, Z. Ligeti, Y. Nir, and A. A. Petrov, Phys. Rev. D 69, 114021 (2004) arXiv:hep-ph/0402204.