ARITHMETIC INFINITE FRIEZES FROM PUNCTURED DISCS

MANUELA TSCHABOLD

Abstract. We define the notion of infinite friezes as a variation of Conway-Coxeter frieze patterns and study their properties. We introduce operations on infinite friezes generating bigger and smaller infinite friezes. It turns out that triangulations of punctured discs give rise to infinite friezes having special properties. Furthermore, we work out a combinatorial interpretation of the entries of infinite friezes associated to triangulations of punctured discs via matchings for certain combinatorial objects, namely periodic triangulations of strips.

1. Introduction

Frieze patterns in mathematics were introduced and studied in [8] by Conway and Coxeter. The authors had the focus on patterns of \( n \) bi-infinite rows of positive integers bounded from above and below, respectively, by a row of 0’s, followed by a row of 1’s, whose entries satisfy a local rule. In this paper we refer to them as Conway-Coxeter friezes. There are several connections between Conway-Coxeter friezes and classical objects in mathematics. A well known correspondence between Conway-Coxeter friezes and triangulated polygons, first conjectured in [9] and proved in [8], is that every Conway-Coxeter friezes arises from the matching numbers of a triangulated polygon. By the work of Caldero and Chapoton in [7] Conway-Coxeter, friezes are closely related to Fomin-Zelevinsky’s cluster algebras of type \( A \). This fact serves as motivation for the work in this paper. Various versions of Conway-Coxeter friezes have been introduced and studied recently providing new information about cluster algebras, like friezes in [1], \( \text{SL}_2 \)-tilings in [4, 1], 2-frieze patterns in [14, 13], or see also [2, 5].

In this paper we generalize and extend the notion of Conway-Coxeter friezes and introduce similar patterns of positive integers without the condition of bounding rows at the bottom. We call them infinite friezes. It turns out that various properties known for Conway-Coxeter friezes can be adapted to infinite friezes. We shall present some of them. Unlike Conway-Coxeter friezes, infinite friezes are not necessarily periodic. The main result is the connection between periodic infinite friezes and triangulations of punctured discs. Moreover, for these particular periodic infinite friezes we are able to provide a combinatorial interpretation of the numbers occurring in them.

This paper is organized as follows. In Section 2, we introduce and study the main objects. We also focus on a special class of infinite friezes which are invariant under translation, called periodic infinite friezes. In Section 3, we define two algebraic operations on infinite friezes producing new infinite friezes, namely gluing and cutting. Moreover, we extend these operations to periodic infinite friezes. In Section 4, we explain how particular periodic infinite friezes arise from triangulations of punctured discs. More precisely, following the original idea for triangulations of polygons and [2], triangulations of punctured discs yield sequences of non-negative integers, called quiddity sequences.

We formulate and prove one of our main results, namely that these sequences provided from triangulations of punctured discs give rise to periodic infinite friezes. We also prove that the periodic infinite friezes obtained through this exhibit arithmetic progressions and are thus examples of so-called arithmetic friezes. In Section 5 we explain how triangulations of punctured discs correspond to periodic triangulations of a certain combinatorial structure we call the strip. A similar model was introduced by Holm and Jørgensen in [11]. They viewed triangulations of the strip as infinite simplicial complexes motivated by the cluster category introduced in [12]. Moreover, we show how periodic triangulations of the strip provide all entries in the arithmetic frieze associated to the corresponding triangulation of the disc by generalizing one of the main
results by Broline, Crowe and Issacs in [6]. We will use the terminology of matching numbers as used by Baur and Marsh in [2] to construct frieze pattern of type $D$. In the final Section 6 we will give an alternative way to describe the entries in a frieze pattern and see that periodic triangulations of strips also provide the diagonals in the associated periodic infinite frieze. We use a similar method of assigning labels to the vertices of a periodic triangulation of a strip as used by Conway and Coxeter in [8].

2. Periodic Infinite Friezes

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & \\
2 & 2 & 2 & 2 & 2 & \\
\vdots & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & \\
5 & 5 & 5 & 5 & 5 & \\
6 & 6 & 6 & 6 & 6 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

**Figure 2.1.** The basic infinite frieze $F_\ast$. 

**Definition 2.1.** An infinite frieze $F$ is an array $(m_{ij})_{i\geq 0, j\in \mathbb{Z}}$ of shifted infinite rows of positive integers bounded at the top by a row filled with 0’s, followed by a row of 1’s, i.e. $m_{0j} = 0$, $m_{1j} = 1$ for all $j \in \mathbb{Z}$, and $m_{ij} > 0$ else,

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & \\
\vdots & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & \\
5 & 5 & 5 & 5 & 5 & \\
6 & 6 & 6 & 6 & 6 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

such that the unimodular rule is satisfied, i.e. for every diamond in $F$ of the form

\[
m_{i-1,j+1} \\
m_{ij} \\
m_{i+1,j}
\]

the relation $m_{ij}m_{i,j+1} - m_{i-1,j+1}m_{i+1,j} = 1$ holds, where $i \geq 1$ and $j \in \mathbb{Z}$.

The easiest example of an infinite frieze we may think of is the basic infinite frieze $F_\ast = (m_{ij})_{i\geq 0, j\in \mathbb{Z}}$ with $m_{ij} = i$, for all $i \geq 0, j \in \mathbb{Z}$, as shown in Figure 2.1.

One can easily convince oneself that if an entry 1 appears in a non-trivial row of an infinite frieze, it is not possible that it has a 1 as a neighbour entry to the left, or right, respectively. Moreover, the product of the two neighbouring entries of the entry 1 is strictly bigger than 4.

**Definition 2.2.** For an infinite frieze $F = (m_{ij})_{i\geq 0, j\in \mathbb{Z}}$ its quiddity sequence $q_F$ is the infinite sequence $(a_j)_{j\in \mathbb{Z}}$ of positive integers given by the first non-trivial row of $F$. 
We use the following notions below. For a fixed $j \in \mathbb{Z}$ we denote by a \textit{se-diagonal} $f(a_j) = (f_i^j)_{i \geq 0}$ of an infinite frieze $F = (m_{ij})_{i,j \in \mathbb{Z}}$ an infinite sequence of entries in $F$ starting at the top of $F$ and running south-east through $a_j$, i.e. $f_i^j = m_{ij}$. We will usually drop the superscript $j$ as it will be clear from the context. Similarly, $\tilde{f}(a_j) = (\tilde{f}_i^j)_{i \geq 0}$ is a \textit{sw-diagonal} of $F$ through $a_j$ if $\tilde{f}_i^j = m_{i,j-i+2}$, where the \textit{sw} sign stands for south-west, the direction of the diagonal.

For our purpose we are mainly interested in a particular family of infinite friezes which stay invariant under horizontal translation, such as the ones in Figures 2.1 and 2.2.

**Definition 2.3.** An infinite frieze $(m_{ij})_{i,j \in \mathbb{Z}}$ is called \textit{n-periodic} and denoted by $F_n$ if there is a number $n \geq 1$ such that $m_{ij} = m_{i,j-n}$ for all $i \geq 0$ and $j \in \mathbb{Z}$. A fundamental region $D$ for $F_n$ is given by $n$ consecutive se-diagonals in $F_n$.

Note that every non-trivial row of an $n$-periodic infinite frieze is given by a repeating sequence of $n$ integers and therefore is determined by an $n$-tuple of positive integers, up to cyclic permutation. Note also that the entire periodic infinite frieze is covered by a fundamental region by successive copies in horizontal direction, thus a fundamental region contains all the information about the periodic infinite frieze.

**Definition 2.4.** Given an $n$-periodic infinite frieze $F_n$ with fundamental region $D = (m_{ij})_{i,j \in \mathbb{Z}, 1 \leq i \leq n}$, the \textit{quiddity sequence} $q_{F_n}$ of $F_n$ is an $n$-tuple $(a_1, a_2, \ldots, a_n)$, determined up to cyclic equivalence, where $a_j = m_{2j}$ for all $j \in \{1, 2, \ldots, n\}$.

**Remark 2.5.** (i) Clearly, two consecutive rows of an infinite frieze, except the first two rows, determine the rest of the infinite frieze, since we can fill the next row below and above these two by using the unimodular rule. It follows that an infinite frieze is determined by its quiddity sequence. In particular, if the quiddity sequence of an infinite frieze is periodic the whole infinite frieze is periodic.

(ii) On the other hand, given a se-diagonal of an infinite frieze the unimodular rule enables us to fill the next se-diagonal to the right (east), starting at the top. The analogous result is true for sw-diagonals. Hence a fundamental region of a periodic infinite frieze, and thus the periodic infinite frieze itself, is also determined as soon as one se-diagonal or one sw-diagonal, respectively,
is given. If an infinite frieze is not periodic, a \( \mathit{se} \)-diagonal and a \( \mathit{sw} \)-diagonal with a common entry different from zero are needed to determine the infinite frieze.

Motivated by the work of Conway and Coxeter in [8] the next lemma describes how the entries of an infinite frieze and its quiddity sequence depend on each other. Completely analogously we could have formulated the lemma for \( \mathit{sw} \)-diagonals.

**Lemma 2.6.** Let \( q_F = (a_j)_{j \in \mathbb{Z}} \) be the quiddity sequence of an infinite frieze \( F \) and \( f(a_k) = (f_i)_{i \geq 0} \) be the \( \mathit{se} \)-diagonal of \( F \) through \( a_k \) for some integer \( k \), i.e. \( f_0 = 0, f_1 = 1, f_2 = a_k, \ldots \) as shown in Figure 2.3. Then

\[
\begin{align*}
\begin{array}{cccccccc}
f_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\vdots & f_2 = a_k & a_{k+1} & a_{k+2} & a_{k+3} & \cdots & a_{k+i-1} & a_{k+i} & \cdots \\
f_3 & \ast & \ast & \ast & \ast & \cdots & \ast & \ast \\
f_4 & \ast & \cdots & \ast & \ast & \cdots & \ast \\
\vdots & f_5 & \ast & \cdots & \ast & \ast & \cdots & \ast \\
\vdots & f_i & \vdots & \cdots & \vdots & \ast & \cdots & \vdots \\
\vdots & f_{i+1} & \vdots & \cdots & \vdots & \ast & \cdots & \vdots \\
f_{i+2} & \\
\end{array}
\end{align*}
\]

**Figure 2.3.** An infinite frieze \( F \) with quiddity sequence \( q_F = (a_j)_{j \in \mathbb{Z}} \) and \( \mathit{se} \)-diagonal \( f(a_k) = (f_i)_{i \geq 0} \) through \( a_k \).

\[a) \ a_{k+i} = \frac{f_i + f_{i+2}}{f_{i+1}} \quad \text{for all } i \geq 0,\]

\[b) \ f_i = \det \begin{pmatrix} a_k & 1 & 0 \\ 1 & a_{k+1} & 1 \\ \vdots & \vdots & \vdots \end{pmatrix} =: \det A_{k+1-2}^k \quad \text{for all } i \geq 2.\]

**Proof.** Let \( q_F = (a_j)_{j \in \mathbb{Z}} \) be the quiddity sequence of an infinite frieze \( F \). We choose an arbitrary integer \( k \) and consider the \( \mathit{se} \)-diagonal \( f(a_k) = (f_i)_{i \geq 0} \) of \( F \) through \( a_k \).

**a)** Clearly, the equality is true for \( i = 0 \). Now, similar to the proof of the analogue relation valid for Conway-Coxeter-friezes for \( i \geq 1 \) we use that by the unimodular rule the entries in any two \( \mathit{se} \)-diagonals \( f, f' \) in \( F \) arranged as in the following figure

\[
\begin{array}{cccccccc}
f & f' \\
0 & 0 \\
1 & 1 \\
\vdots & \vdots \\
x_1' & x_1 \\
x_2' & x_2 \\
x_3' & x_3 \\
\end{array}
\]

We use that by the unimodular rule the entries in any two \( \mathit{se} \)-diagonals \( f, f' \) in \( F \) arranged as in the following figure...
where by definition $x_2$ and $x'_2$ are positive integers.

Applying this result to $x'_1 = 0$, $x'_2 = 1$, $x'_3 = a_{k+1}$ the corresponding entries in $f$ are $x_1 = f_i$, $x_2 = f_{i+1}$, $x_3 = f_{i+2}$, see Figure 23 and we have the desired result

$$
\frac{f_i + f_{i+2}}{f_{i+1}} = \frac{0 + a_{k+i}}{1} = a_{k+i}.
$$

b) The claim follows immediately by induction on $i$ and with a).

\[ \square \]

Remark 2.7. In particular, for an $n$-periodic infinite frieze $F_n$ with quiddity sequence $q_{F_n} = (a_1, a_2, \ldots, a_n)$ the entries of the first non-trivial row $(a_j)_{j \in \mathbb{Z}}$ of $F_n$ satisfy $a_j = a_{j+ln}$, for all $j \in \{1, \ldots, n\}$ and every integer $l$. Let $f(a_k) = (f_i)_{i \geq 0}$ be the SE-diagonal of $F_n$ through $a_k$ for some $k \in \{1, \ldots, n\}$. Then Lemma 2.6 implies

$$a_{k+i} = \frac{f_{i+ln} + f_{i+2+ln}}{f_{i+1+ln}}$$

for all $i \in \{0, \ldots, n-1\}$ and every non-negative integer $l$. Moreover, the entries of $f$ can be expressed by $q_{F_n}$: for all $i \geq 2$, we have

$$f_i = \det \begin{pmatrix} d_1 & 1 & \cdots & \cdots & 0 \\ 1 & d_2 & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1 & d_{i-2} \\ 0 & 1 & 1 & \cdots & d_{i-1} \end{pmatrix},$$

where $d_j = a_{k+j-1} \mod n$ for all $j \in \{1, \ldots, i-1\}$.

Note that by Lemma 2.6(a) the entries in a SE-diagonal of an infinite frieze are proportional to the sum of its two neighbours. This result can be formulated as follows.

Corollary 2.8. Let $F$ be an infinite frieze with quiddity sequence $q_F = (a_j)_{j \in \mathbb{Z}}$ and $k$ be a fixed integer. Then the SW-diagonal $\tilde{f}(a_k) = (\tilde{f}_i)_{i \geq 0}$ through $a_k$ is given by the two neighbouring SW-diagonals $\tilde{f}(a_{k-1}) = (\tilde{f}_{i-1})_{i \geq 0}$ through $a_{k-1}$ and $\tilde{f}(a_{k+1}) = (\tilde{f}_{i+1})_{i \geq 0}$ through $a_{k+1}$ via

$$a_{k+1} \cdot \tilde{f}_i = \tilde{f}_{i-1} + \tilde{f}_{i+1},$$

for all $i \geq 1$. Moreover, if $a_{k-1} = 1$, then $\tilde{f}(a_k)$ is given by the “shifted sum” of $\tilde{f}(a_{k-1})$ and $\tilde{f}(a_{k+1})$.

Again, in the corollary we could have used SE-diagonals instead of SW-diagonals. Before studying operations on infinite friezes in the next section we have a short look at 1-periodic infinite friezes. It makes sense to call them complete infinite friezes since they arise from complete graphs, see Section 4.2. It is an easy induction argument using Corollary 2.8 to prove that complete infinite friezes are determined as follows.

Proposition 2.9. Let $a \geq 2$ be an integer and let $q = (a_j)_{j \in \mathbb{Z}}$ be the constant sequence with $a_j = a$ for all $j \in \mathbb{Z}$. Then $q$ is the quiddity sequence of a complete infinite frieze $F_n = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$. Moreover, the entries of $F_n$ are given by

$$m_{ij} = m_i = \sum_{k=0}^{\lfloor \frac{i-j}{a-1} \rfloor} (-1)^k \binom{i-1-k}{k} a^{i-1-k}$$

for $i \geq 1$ and $j \in \mathbb{Z}$. 
3. CUTTING AND GLUING INFINITE FRIEZES

In this section we point out two particular ways how an infinite frieze can be modified to obtain a new infinite frieze by extending the work of Conway and Coxeter in [8]. We also describe how these algebraic operations can be used on periodic infinite friezes.

3.1. Gluing. The first operation that we describe on infinite friezes produces a new infinite frieze starting from a smaller infinite frieze by inserting a pair of diagonals. Note that this is not the only possibility to define an operation on infinite friezes that enlarges them.

**Theorem 3.1.** Let \( F \) be an infinite frieze with quiddity sequence \( q_F = (a_j)_{j \in \mathbb{Z}} \) and \( k \) be a fixed integer. Then the sequence \( \hat{q}_F = (\hat{a}_j)_{j \in \mathbb{Z}} \) defined by

\[
\hat{a}_j = \begin{cases} 
  a_j, & \text{if } j \leq k - 1 \\
  a_k + 1, & \text{if } j = k \\
  1, & \text{if } j = k + 1 \\
  a_{k+1} + 1, & \text{if } j = k + 2 \\
  a_{j-1}, & \text{if } i \geq k + 3,
\end{cases}
\]

is the quiddity sequence of an infinite frieze \( \hat{F} \).

For an infinite frieze \( F \) with quiddity sequence \( q_F \) we call the operation that maps \( q_F \) to \( \hat{q}_F \) gluing above the pair \( a_k, a_{k+1} \), where \( \hat{q}_F \) is as in Theorem 3.1.

**Proof.** Let \( F = (m_{ij})_{i \geq 0, j \in \mathbb{Z}} \) be an infinite frieze with quiddity sequence \( q_F = (a_j)_{j \in \mathbb{Z}} \). We choose \( k \in \mathbb{Z} \) and consider the array \( \hat{F} = (\hat{m}_{ij})_{i \geq 0, j \in \mathbb{Z}} \) with \( \hat{m}_{0j} = 0, \hat{m}_{1j} = 1 \) and given by the entries of \( F \), for \( i \geq 2 \),

\[
\hat{m}_{ij} = \begin{cases} 
  m_{ij}, & \text{if } i + j \leq k + 1 \\
  m_{i-1,j} + m_{ij}, & \text{if } i + j = k + 2 \\
  m_{i-1,j}, & \text{if } i + j \geq k + 3 \text{ and } j \leq k + 1 \\
  m_{i-1,j} + m_{i-1,j}, & \text{if } j = k + 2 \\
  m_{i,j-1}, & \text{if } j \geq k + 3
\end{cases}
\]

as shown in Figure 3.1. We show that \( \hat{F} \) is an infinite frieze. Clearly, since \( F \) is an infinite frieze, we have \( \hat{m}_{ij} > 0 \) for all \( i \geq 2 \) and \( j \in \mathbb{Z} \). It remains to show that the unimodular rule is satisfied.
for every diamond in $\mathcal{F}$, i.e. $\hat{m}_{i,j} \hat{m}_{i,j+1} - \hat{m}_{i-1,j+1} \hat{m}_{i+1,j} = 1$ for all $i \geq 1$ and $j \in \mathbb{Z}$. As long none of the entries in the yellow diagonals involved, see Figure 3.1, this property is immediately inherited by $\tilde{\mathcal{F}}$. Otherwise, we get

$$\hat{m}_{i,k} + 2 \hat{m}_{i,k+3} = \hat{m}_{i-1,k+2} \hat{m}_{i,k+2} - \hat{m}_{i-1,k+2} \hat{m}_{i,k+3} = \hat{m}_{i,k+1} \hat{m}_{i+1,k+2} - \hat{m}_{i,k+1} \hat{m}_{i+1,k+3} = 1,$$

and similarly $\hat{m}_{i,k} + 1 \hat{m}_{i,k+2} - \hat{m}_{i-1,k+2} \hat{m}_{i,k+2} = 1, \hat{m}_{i-1,k+1} \hat{m}_{i,k+1} - \hat{m}_{i-1,k+1} \hat{m}_{i,k+2} = 1$. Hence $\tilde{\mathcal{F}}$ is the infinite frieze with quiddity sequence $\ldots a_{k-2} a_{k-1} a_k + 1 a_{k+1} + 1 a_{k+2} a_{k+3} \ldots$, as desired. \hfill \Box

**Remark 3.2.** In the proof of the theorem the effect of gluing into the initial infinite frieze is given explicitly: $\tilde{\mathcal{F}}$ is obtained from $\mathcal{F}$ by inserting simultaneously a pair of diagonals. In particular, a SE-diagonal and a SW-diagonal with common entry in the first row of $\tilde{\mathcal{F}}$ are inserted such that every entry of the new diagonals in $\tilde{\mathcal{F}}$ is given by the sum of the two closest entries in the neighbouring diagonals to the left and to the right, see Figure 3.1. We view $\tilde{\mathcal{F}}$ as being bigger than $\mathcal{F}$ and the symbol $\tilde{\cdot}$ denotes that we glued in a pair of diagonals.

Note that if we start with the basic frieze $\mathcal{F}_a$ given in Figure 2.1 then gluing serves as a tool to produce new infinite friezes from $\mathcal{F}_a$. The next result follows immediately.

**Corollary 3.3.** There exists infinitely many infinite friezes.

Clearly, as soon as we consider periodic infinite friezes we lose the periodicity after applying the operation of gluing once. To remedy this we define a slightly different operation that preserves periodicity of an infinite frieze using the current set up of gluing.

![Figure 3.2](image-url)  

**Figure 3.2.** The $(n+1)$-periodic infinite frieze $\tilde{\mathcal{F}}^n$ obtained by $n$-gluing above $a_k, a_{k+1}$ in an $n$-periodic infinite frieze $\mathcal{F}$ colored light grey.
Proposition 3.4. Given an $n$-periodic infinite frieze $F$ with quiddity sequence $q_F = (a_1 \ a_2 \ \cdots \ a_n)$ and let $k \in \{1, 2, \ldots, n\}$ be an integer. Then the $(n+1)$-tuple
\[
\hat{q}_F^n = \begin{cases} 
(a_1 + 2\ 1), & \text{if } n = 1 \\
(a_1 \ \cdots \ a_{k-1} \ a_k + 1 \ a_{k+1} + 1 \ a_{k+2} \ \cdots \ a_n), & \text{otherwise},
\end{cases}
\]
where indices are taken modulo $n$, leads to an infinite frieze $\hat{F}_n$ of period $n+1$.

For an $n$-periodic infinite frieze $F$ with quiddity sequence $q_F$, the operation mapping $q_F$ to the finite sequence $\hat{q}_F^n$, defined in Proposition 3.4, is called $n$-gluing above the pair $(a_k, a_{k+1})$.

Proof. Let $q_F = (a_1 \ a_2 \ \cdots \ a_n)$ be a quiddity sequence of an $n$-periodic frieze $F$. We choose $k \in \{1, 2, \ldots, n\}$. Then for every integer $l$ we glue above the pair $a_{k+ln}, a_{k+1+ln}$ as defined in Theorem 3.1 and obtain an infinite frieze, denoted by $\hat{F}_n$. By construction the first non-trivial row of $\hat{F}_n$ is $n+1$ periodic and determined, up to cyclic permutation, by the 2-tuple $(a_1+2\ 1)$, if $n = 1$, or by the $(n+1)$-tuple $(a_1 \ \cdots \ a_{k-1} \ a_k + 1 \ a_{k+1} + 1 \ a_{k+2} \ \cdots \ a_n)$ else. Hence the result. \[\Box\]

Note that the operation of $n$-gluing may provide an infinite frieze of period dividing $n+1$ but strictly smaller than $n+1$. Note also that $n$-gluing only depends on the pair $a_k, a_{k+1}$ of integers not on the choice of the quiddity sequence. Clearly, there are other operations on periodic infinite friezes producing new periodic infinite friezes. E.g., given a quiddity sequence $(a_1 \ a_2 \ \cdots \ a_n)$ of an $n$-periodic infinite frieze the sequence $(a_1 \ \cdots \ a_{k-1} \ a_k + 1 \ a_{k+1} + 1 \ a_{k+2} \ \cdots \ a_n)$ also leads to a periodic infinite frieze, namely of period $2n+1$, and so on.

The next corollary describes the modification to the diagonals of the initial periodic infinite friezes caused by $n$-gluing, see Figure 3.2. To simplify argumentation we will use the following notation: for $j \in \{1, 2, \ldots, n+1\}\setminus\{k+2\}$, where $k+2$ is reduced modulo $n+1$, we set
\[
j' = \begin{cases} 
j, & k = n \text{ or } 1 \leq j < k + 2 \\
j - 1, & \text{else},
\end{cases}
\]
where we reduce $j'$ modulo $n$.

Corollary 3.5. Let $F$ be an $n$-periodic infinite frieze with quiddity sequence $q_F = (a_1 \ a_2 \ \cdots \ a_n)$. Let $\hat{F}_n$ be the $(n+1)$-periodic infinite frieze obtained from $F$ by $n$-gluing above the pair $a_k, a_{k+1}$, for some $k \in \{1, 2, \ldots, n\}$, with quiddity sequence $\hat{q}_F^n = (\hat{a}_1 \ \hat{a}_2 \ \cdots \ \hat{a}_{n+1})$. If $(\hat{f}_i)_{i \geq 0} = f(a_{j'}) = (\hat{f}_i')_{i \geq 0}$ and $(\hat{f}_i)_{i \geq 0} = \hat{f}(\hat{a}_j) = (\hat{f}_i)_{i \geq 0}$, then
\[
\hat{f}_{i+l(n+1)} = \begin{cases} 
f_{i+l(n+1)}, & 0 \leq i < k + 2 - j \\
f_{(i-1)+l(n+1)} + \hat{f}_{i+l(n+1)}, & i = k + 2 - j \\
\hat{f}_{(i-1)+l(n+1)}, & k + 2 - j < i \leq n
\end{cases}
\]
for all $l \geq 0$ and $j \neq k + 2 \pmod{n+1}$, where $k + 2 - j$ is reduced modulo $n+1$.

Before considering the reverse operation to gluing we illustrate Corollary 3.5 with an example. The example points out that $n$-gluing preserves more than the first two rows of the initial friezes. Diamond-shaped fragments of the old friezes appear in the new frieze.

Example 3.6. Let $n = 3$ and consider the basic infinite frieze $F_3$ given in Figure 2.4 with quiddity sequence $q_{F_3} = (2 \ 2 \ 2)$. Now we perform a 3-gluing above the pair $a_2, a_3$. This leads to the new quiddity sequence $\hat{q}_{F_3}^3 = (2 \ 3 \ 1 \ 3)$ determining a fundamental region of the infinite frieze $\hat{F}_3$ of period 4, see the figure below. The inserted pairs of diagonals are colored yellow. In particular, outside these diagonals $\hat{F}_3$ coincides with $F_3$. 
Example 3.6 already illustrates how the reverse operation to gluing will work.

3.2. Cutting. The second operation on infinite friezes we define produces a new infinite frieze starting from a bigger infinite frieze whenever an entry 1 appears in its quiddity sequence.

**Theorem 3.7.** Let \( \mathcal{F} \) be an infinite frieze with quiddity sequence \( q_\mathcal{F} = (a_j)_{je\mathbb{Z}} \) such that \( a_k = 1 \) for some integer \( k \). Then the sequence \( \tilde{q}_\mathcal{F} = (\tilde{a}_j)_{je\mathbb{Z}} \) with

\[
\tilde{a}_j = \begin{cases} 
a_j, & \text{if } j \leq k - 2, \\
a_{k-1} - 1, & \text{if } j = k - 1, \\
a_{k+1} - 1, & \text{if } j = k, \\
a_{j+1}, & \text{if } j \geq k + 1, 
\end{cases}
\]

is the quiddity sequence of an infinite frieze \( \tilde{\mathcal{F}} \).

We say that that \( \tilde{q}_\mathcal{F} = (\tilde{a}_j)_{je\mathbb{Z}} \) is obtained from \( q_\mathcal{F} = (a_j)_{je\mathbb{Z}} \) by **cutting above** \( a_k = 1 \), where \( \tilde{q}_\mathcal{F} = (\tilde{a}_j)_{je\mathbb{Z}} \) is as given in Theorem 3.7. The proof of the theorem is a straightforward consequence of the work we have done in the previous sections. In particular, Corollary 2.8 and Theorem 3.1.

**Remark 3.8.** In reverse to gluing, the effect caused on the initial infinite frieze by cutting is that a pair of diagonals is deleted and the remaining entries stay unchained. Namely, the se-diagonal through the right neighbouring entry of the entry 1 we cut above and the sw-diagonal through the left neighbouring entry of the same entry 1. Thus, we view \( \tilde{\mathcal{F}} \) as being smaller than \( \mathcal{F} \). The symbol \( \tilde{\sim} \) indicates that two diagonals are removed. Clearly, first gluing above a pair and then cutting above the new entry that occurred in the quiddity sequence after gluing yields the original infinite frieze.

Similarly, as in Section 3.1 the periodicity is lost if we consider periodic infinite friezes after cutting once. Thus, we define an operation on periodic infinite friezes that preserves periodicity. Recall that the two neighbours of an entry 1 in a given row are both strictly bigger than 1. Thus, if a periodic infinite frieze has an entry 1 in its quiddity sequence, the period has to be at least 2.

**Proposition 3.9.** Given a quiddity sequence \( q_\mathcal{F} = (a_1 a_2 \cdots a_n) \) of an \( n \)-periodic infinite frieze \( \mathcal{F} \) such that \( a_k = 1 \) for some \( k \in \{1, 2, \ldots, n\} \), the \( (n-1) \)-tuple

\[
\tilde{q}_n^{\mathcal{F}} = \begin{cases} 
(a_{k+1} - 2), & \text{if } n = 2 \\
(a_1 \cdots a_{k-2} a_{k-1} - 1 a_{k+1} - 1 a_{k+2} \cdots a_n), & \text{otherwise},
\end{cases}
\]

where we reduce indices modulo \( n \), yields an infinite frieze \( \tilde{\mathcal{F}}_n \) of period \( n-1 \).

We say that the quiddity sequence \( \tilde{q}_n^{\mathcal{F}} \) in Proposition 3.9 is obtained from \( q_\mathcal{F} \) by **\( n \)-cutting above** \( a_k \). The proof of Proposition 3.9 works quite similar as the one of Proposition 3.4. Note again that, strictly speaking, the operation of \( n \)-cutting leads to a periodic infinite frieze with period a divisor of \( n-1 \), possibly smaller than \( n-1 \).
4. Arithmetic Friezes

It is natural to ask which sequences yield a periodic infinite frieze. A partial answer is given in this section. We generalize the basic construction for obtaining a Conway-Coxeter frieze from triangulated polygons. Instead of triangulations of polygons we consider triangulations of punctured discs and associate sequences of positive integers to them thus generating periodic infinite friezes.

We provide a geometrical interpretation of the two operations $n$-gluing and $n$-cutting on periodic infinite friezes in terms of triangulations of punctured discs. Moreover, we prove a remarkable arithmetic property of these periodic infinite friezes.

**Figure 4.1.** A triangulation $\Pi$ of $S^1_n$ with quiddity sequence $q_\Pi = (1\ 4\ 1\ 2\ 6)$.

Let us briefly recall some basic notions of triangulated punctured discs, for more details on triangulations of bordered surfaces with marked points see [10]. For $n \geq 1$, the punctured disc $S^1_n$ is a closed disc with $n$ marked points on the boundary, numbered by $1, 2, \cdots n$ in clockwise order, and one marked point in the interior, namely the puncture labelled by 0. An arc in $S^1_n$ is a non-boundary, non-self-intersecting curve, connecting two marked points of $S^1_n$. In the sequel $S^1_n$ is always meant to be a punctured disc together with a fixed labelling and we only consider isotopy classes of arcs. An arc whose endpoints coincide is called a loop. We shall use the following notation for arcs of punctured discs: $0j$ indicates the bridging arc connecting the puncture with a marked point $j$ on the boundary. For two marked points $i, j$ on the boundary $ij$ denotes the arc isotopic to the boundary segment going clockwise from $i$ to $j$ by ignoring all marked points on the boundary other than $i$ and $j$. We use the following convention for marked points: if $i < 1$, $i \equiv n$, and if $i = n$, $i + 1$ equals 1.

Two arcs are said to be non-crossing if they have no point of intersection in the interior of $S^1_n$. A maximal collection of pairwise non-crossing arcs in $S^1_n$ is called a triangulation $\Pi$ of $S^1_n$. One can easily verify that every triangulation of $S^1_n$ consists of exactly $n$ arcs and cuts $S^1_n$ into $n$ disjoint regions, called triangles. Since the number of arcs in $S^1_n$ is finite, there exists only finitely many triangulations of $S^1_n$. When considering symmetries of triangulations, we always assume that the marked points on the boundary of $S_n$ are evenly distributed. For combinatorial reasons it is useful to consider triangulations of punctured discs up to rotation through $\frac{2\pi}{n}$ about the puncture. If two triangulations of $S^1_n$ are rotation-equivalent they are said to be of the same shape. Clearly, there are at most $n$ different triangulations of $S^1_n$ of the same shape, depending on the symmetries the triangulation has.

Note that every triangulation of $S^1_n$ contains at least one bridging arc. The special case where a triangulation consists entirely of bridging arcs is called star-triangulation and denoted by $\Pi_\ast$.

**Definition 4.1.** Let $\Pi$ be a triangulation of $S^1_n$. The quiddity sequence $q_\Pi$ of $\Pi$ is the finite sequence $(a_1\ a_2\ \cdots\ a_n)$ of positive integers, where $a_i$ is the number of connected components of $S^1_n\setminus \Pi \cap U$, for $U$ a small neighbourhood of $i$.

We will see that every $q_\Pi$ gives rise to a periodic infinite frieze (cf. Theorem 4.6). Hence it makes sense to call $q_\Pi$ a quiddity sequence. Figure 4.1 gives an example of a triangulation of $S^1_5$ together with its quiddity sequence.

Clearly, for two triangulations of $S^1_n$ with the same shape the quiddity sequences coincide up to cyclic permutation. Note also that the reflection at a diameter through a marked point on the
This completes the proof. □

Remark 4.5. It is noteworthy that every triangulation with \( r \) bridging arcs of \( S^1_n \) can be obtained from the star-triangulation on \( r \) arcs by gluing triangles successively.

We now come to one of the main results of the paper.

Theorem 4.6. Let \( \Pi \) be a triangulation of \( S^1_n \). Then the quiddity sequence \( q_\Pi = (a_1 a_2 \cdots a_n) \) of \( \Pi \) is a quiddity sequence of an infinite frieze \( \mathcal{F}_\Pi \) of period \( n \).

Proof. We prove the result by induction on \( n \). For \( n = 1 \), there is only the star-triangulation \( \Pi_* \) with quiddity sequence \( q_{\Pi_*} = (2) \) and this is a quiddity sequence \( q_{\mathcal{F}_*} \) for the basic infinite frieze \( \mathcal{F}_* \), cf. Figure 2.1, thus \( \mathcal{F}_{\Pi_*} = \mathcal{F}_* \).

Now, for \( n \geq 1 \), we assume that any triangulation of \( S^1_n \) yields an \( n \)-periodic infinite frieze. Let \( \Pi \) be a triangulation of \( S^1_n \) and \( q_\Pi = (a_1 a_2 \cdots a_{n+1}) \) its quiddity sequence. If \( \Pi = \Pi_* \), the claim follows as in the base case. Otherwise, if \( \Pi \neq \Pi_* \), there is a special marked point \( x \) of \( \Pi \) (Lemma 4.3). By Corollary 4.4, \( \Pi_{x} \) is a triangulation of \( S^1_{n+1} \) with quiddity sequence \( q_{\Pi_x} = (a_1 \cdots a_{n+1}) \) (or (2) if \( n = 1 \)). By induction \( q_{\Pi_{x}} \) is the quiddity sequence of an \( n \)-periodic infinite frieze \( \mathcal{F}_{\Pi_{x}} \). Now we \( n \)-glue above the pair \( a_{x-1}, a_{x+1} \) in \( \mathcal{F}_{\Pi_{x}} \) and by Proposition 4.3, this gives an infinite frieze \( \tilde{\mathcal{F}}_{\Pi_{x}} \) of period \( n+1 \) such that \( \tilde{q}_{\mathcal{F}_{\Pi_{x}}} = q_\Pi \). This completes the proof. □
Figure 4.2. The arithmetic frieze of period 5 associated to the triangulation of $S^1_5$ shown in Figure 4.1.

For the triangulation of $S^1_5$ given in Figure 4.1 the associated periodic infinite frieze is illustrated in Figure 4.2.

Note that from Theorem 4.6 it follows that cutting and gluing of triangles for triangulations of punctured discs provide a geometric interpretation via triangulations for the operations $n$-cutting and $n$-gluing defined on periodic infinite friezes in Section 3.

Corollary 4.7. Given a triangulation $\Pi \neq \Pi_0$ of $S^1_{n+1}$ with special marked point $x$, let $F_\Pi$ be the associated $(n+1)$-periodic infinite frieze. Then $F_{\Pi_0}$ equals $F_\Pi$.

Remark 4.8. We already observed that triangulations of $S^1_5$ with the same shape provide the same quiddity sequence, up to cyclic permutation, thus give rise to the same periodic infinite frieze. In general, the periodic infinite frieze associated to a triangulation of $S^1_5$ has period $n$. But, it might also have shorter periods: if a triangulation of $S^1_5$ has rotational symmetries, the shortest period of the associated infinite frieze is a factor of $n$, as pictured on the right in the figure below. Indeed, we can construct many triangulations of different punctured discs giving rise to the same periodic infinite frieze.

Hence the associated periodic infinite friezes are not uniquely determined. Moreover, let us point out that there are periodic infinite friezes which can not be given by a triangulation of a punctured disc. Examples for this fact are the complete infinite frieze with $a > 2$ or the periodic infinite frieze in Figure 2.2. They come from triangulated annuli (see Section 5 of [3]).

We now will see that periodic infinite friezes associated to triangulations of punctured discs satisfy a beautiful arithmetic property. For example, in Figure 4.2 the numbers marked respectively by red and green circles form a sequence with entries in a SE-diagonal given always by jumping 5 entries down. We claim that such a sequence is an increasing arithmetic progression. In the example, the indicated arithmetic progressions have respectively $d = 15$ for the red sequence, and $d = 9$ for the green sequence.
Definition 4.9. An infinite frieze $\mathcal{F} = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$ is arithmetic if there exists $d_{ij} > 0$ such that $m_{i+(k+1)n,j} - m_{i+kn,j} = d_{ij}$ for all $k \geq 0$.

Proposition 4.10. Every $n$-periodic infinite frieze $\mathcal{F}_n$ associated to a triangulation $\Pi$ of $S^1_n$ is arithmetic.

Proof. Clearly, if $\Pi = \Pi_n$ is the star-triangulation of $S^1_n$, the claim is true with $d_{ij} = n$ for all $i \geq 0, j \in \mathbb{Z}$. In particular the claim is true for $n = 1$. We proceed with the inductive step and assume the claim holds for every $n$-periodic infinite frieze $\mathcal{F}_n$. By Lemma 4.3 $\Pi$ contains a special marked point $x$ where we consider a triangulation $\Pi$ obtained from $\Pi$ by $n$-gluing above the appropriate entries. We denote the $n$-periodic infinite frieze $\mathcal{F}_n$ by $(m_{ij})_{i \geq 0, j \in \mathbb{Z}}$ and let $\mathcal{F}_n = (\hat{m}_{ij})_{i \geq 0, j \in \mathbb{Z}}$.

Since $\hat{a}_x = 1$ in $q\mathcal{F}_n$ we $n$-glue above $a_{x-1}, a_x$ in $\mathcal{F}_{n+1}$ to obtain $\mathcal{F}_n = (\hat{F}_{n+1})_{i \geq 0}$ for a fixed index $j_0 \in \{1, 2, \ldots, n+1\}$ let $\hat{f}(\hat{a}_{j_0}) = (\hat{f}_{i})_{i \geq 0}$ be the SE-diagonal through $\hat{a}_{j_0}$ in $\mathcal{F}_n$. Then there are two cases to consider. If $j_0 \neq x + 1$, then by Corollary 3.5 (for $k = x + 1$) we have for all $l \geq 0$

$$\hat{f}_{i+ln} = \begin{cases} f_{i+ln}, & 0 \leq i < x + 1 - j_0 \\ f_{i-1+ln} + f_{i+ln}, & i = x + 1 - j_0 \\ f_{i+ln}, & x + 1 - j_0 < i \leq n, \end{cases}$$

where $f(a_{j_0}) = (f_i)_{i \geq 0}$ is the SE-diagonal in $\mathcal{F}_{p+1}$ through $a_{j_0}$ and we reduce $x + 1 - j_0$ modulo $n + 1$. In the first case, i.e. if $0 \leq i < x + 1 - j_0$, by induction, this gives $\hat{f}_{i+(l+1)(n+1)} - \hat{f}_{i+l(n+1)} = \hat{f}_{i+l(n+1)} - \hat{f}_{i+ln} = d_{i,j_0}$ for all $l \geq 0$. In the second case, i.e. if $x + 1 - j_0 < i \leq n$, we get $\hat{f}_{i+l(n+1)} - \hat{f}_{i+l(n+1)} = f_{i+1} - f_{i+l(n+1)} = d_{i+1,j_0}$ (by induction) for all $l \geq 0$. If $i = x + 1 - j_0$, we have $\hat{f}_{i+(l+1)(n+1)} - \hat{f}_{i+l(n+1)} = f_{i-1+ln} + f_{i+ln} - f_{i-1+ln} = d_{i,j_0}$ for all $l \geq 0$, which is again a constant.

In the second case, if $j_0 = x + 1$, by Corollary 2.2 $\hat{f}(\hat{a}_{x+1})$ is entirely given by the shifted sum of its two neighbouring SE-diagonals. From the first case and since the sum of two such increasing arithmetic progressions is again an increasing arithmetic progression it follows that $\hat{f}(\hat{a}_{x+1})$ contains also $n + 1$ arithmetic progressions. We showed that a fundamental region of $\mathcal{F}_n$ satisfies the arithmetic property, hence $\mathcal{F}_n$ is arithmetic.

Note that every SE-diagonal of $\mathcal{F}_n$ can be split into $n$ increasing arithmetic progressions. Moreover, since a fundamental region for $\mathcal{F}_n$ is given by $n$ diagonals we have $n^2$ increasing arithmetic progressions overall that occur in $\mathcal{F}_n$. Note also that the work in the proof of Proposition 4.10 can be used to show that the operations $n$-gluing and $n$-cutting preserve the arithmetic property of periodic infinite friezes.

Lemma 4.11. If $\mathcal{F}$ is an $n$-periodic arithmetic infinite frieze, then the $(n+1)$-periodic infinite frieze $\hat{\mathcal{F}}$, respectively the $(n-1)$-periodic infinite frieze $\hat{\mathcal{F}}$ if defined, is arithmetic.

So far there are no known examples of non-periodic arithmetic infinite friezes. From now on an arithmetic frieze is assumed to be infinite and periodic.

5. Description via Matchings

In this section we shall focus on arithmetic friezes associated to triangulations of punctured discs. We present a combinatorial interpretation of the numbers in such an arithmetic friezes using matchings in the same way as in [2]. In order to do this, we introduce periodic triangulations of strips which we can interpret as triangulations of punctured discs. We will show that the number of matchings between vertices of a strip and triangles in a periodic triangulation of it are exactly the entries of the associated arithmetic frieze. Thus we receive an analogous result as the one by Broline, Crowe and Isaacs in [10] for Conway-Coxeter friezes and triangulated polygons.
Definition 5.1. For $n \geq 1$, the strip $\mathcal{U}_n$ in $\mathbb{R}^2$ is the Cartesian product of the real numbers and a closed interval with two disjoint countably infinite set of vertices on the upper, and on the lower boundary, respectively. The vertices on the upper boundary are labelled by $\{0^{(k)} | k \in \mathbb{Z}\}$ and the vertices on the lower boundary are labelled in groups of $n$ vertices by $\{1^{(k)}, \ldots, n^{(k)} | k \in \mathbb{Z}\}$. The vertices are arranged such that $0^{(k)}$ lies above the vertices $1^{(k)}, \ldots, n^{(k)}$ and $k$ increases to the right.

Note that the vertices on the lower boundary correspond to $\mathbb{Z}$ successive copies of the $n$ marked points on the boundary of the punctured disc $S^1_n$, whereas the vertices on the upper boundary correspond to $\mathbb{Z}$ copies of the puncture. Since we are mainly interested in combinatorics we assume that the vertices are evenly distributed. Throughout this paper we use the following convention for all boundary segments in between two vertices vanish.

A triangulation of the strip $\mathcal{U}_3$ is non-self-intersecting curve, up to isotopy, connecting two vertices of $\mathcal{U}_n$ such that

(A1) at least one vertex belongs to the lower boundary of $\mathcal{U}_n$,

(A2) the two vertices are neither equal nor neighbours,

(A3) if one vertex belongs to the upper boundary of $\mathcal{U}_n$, then the superscripts of the two vertices are equal.

An arc in $\mathcal{U}_n$ connecting two vertices on the lower boundary is called peripheral, it is called bridging otherwise.

Remark 5.3. Compared with the model of the strip used in [11] in this paper we exclude arcs connecting two vertices on the upper boundary. Moreover, we also assign a unique bridging arc to every vertex on the lower boundary, see (A3) in Definition 5.2. These additional conditions are motivated by the idea to identify the vertices on the upper boundary with a single vertex, where all boundary segments in between two vertices vanish.

We use the following notation for arcs of strips: the peripheral arc with vertex $i^{(k)}$ on the left and vertex $j^{(l)}$ on the right is denoted by $i^{(k)}j^{(l)}$, in this case either $k < l$, or $i < j - 2$ for $k = l$. Moreover, $0^{(k)}j^{(k)}$ denotes the unique bridging arc connecting the vertex $j^{(k)}$ on the lower boundary with the vertex $0^{(k)}$ on the upper boundary.

Definition 5.4. Two arcs in $\mathcal{U}_n$ are called non-crossing if they have no point of intersection in the interior of $\mathcal{U}_n$. A triangulation of $\mathcal{U}_n$ is a maximal collection $\mathcal{T}$ of pairwise non-crossing arcs in $\mathcal{U}_n$.

Note that $\mathcal{U}_n$ is partitioned by a triangulation into regions called triangles. These have three, or more sides. If we identify all vertices on the upper boundary, the segments on this boundary vanishes and all regions are 3-sided. Figure 5.1 shows part of a triangulation of $\mathcal{U}_3$. 

![Figure 5.1. A triangulation of the strip $\mathcal{U}_3$.](image)
Figure 5.2. A 5-periodic triangulation of $\mathcal{U}_5$ with fundamental region $\mathcal{P}$ associated to the triangulation of $S^1_5$ given in Figure 4.1

Definition 5.5. A triangulation $\mathcal{T} = \mathcal{T}_n$ of $\mathcal{U}_n$ is called $n$-periodic if there exists an $(n+3)$-gon $\mathcal{P}$ in $\mathcal{T}_n$ such that $\mathcal{T}_n$ is covered by iteratively performing an appropriate translation of $\mathcal{P}$ in both horizontal directions. We say that $\mathcal{P}$ is a fundamental region for $\mathcal{T}_n$.

Note that we actually ignore the fact that covering the whole triangulation by $(n+3)$-gons leads to very small overlaps, as with every translation we place a bridging arc on a bridging arc.

While a fundamental region for a given periodic triangulation of a strip is not unique, they all give rise to the entire triangulation. For fixed $n$, the number of different fundamental regions equals, up to translation, the number of bridging arcs ending at a vertex on the upper boundary. Figure 5.2 shows an example of a 5-periodic triangulation of $\mathcal{U}_5$ with a fundamental region given by an octagon, where in this example the fundamental region is unique up to translation.

Figure 5.3. The star-triangulation $\mathcal{T}_n$ of $\mathcal{U}_n$ with a fundamental region $\mathcal{P}_0$.

We call the periodic triangulation consisting only of bridging arcs star-triangulation, denoted by $\mathcal{T}_n$. See Figure 5.3 where $\mathcal{P}_0$ is one of the $n$ possible choices for a fundamental region.

Remark 5.6. In general, if a triangulation of a strip has translational symmetry, bridging arcs occur repetitively in it. Since we restrict to fundamental regions for an $n$-periodic triangulation $\mathcal{T}_n$ of $\mathcal{U}_n$ given by $(n+3)$-gons it follows by definition that a fundamental region $\mathcal{P}$ for $\mathcal{T}_n$ looks as in the following figure

for some $i \in \{1, 2, \ldots, n\}$ and some integer $k$. In particular, for every integer $k$ there is a bridging arc at $0^{(k)}$. Moreover, the interior of $\mathcal{P}$ contains $n-1$ pairwise non-crossing arcs dividing $\mathcal{P}$ into $n$ triangles. One of these has four, and all the others have three distinct sides. Recall, our philosophy of viewing the vertices on the upper boundary to be a single vertex gives a suitable understanding of the quadrilateral in $\mathcal{P}$ as a triangle and explains why we ignore segments of the upper boundary.
We define the set \( \Phi^p_S \) of strips. First, from a triangulation \( \Pi \) of \( S \) the next goal is to give a bijection between triangulations of punctured discs and periodic triangulations.

5.2. From triangulations of punctured discs to periodic triangulations of strips. Our next goal is to give a bijection between triangulations of punctured discs and periodic triangulations of strips. First, from a triangulation \( \Pi \) of \( S^1 \) we construct an \( n \)-periodic triangulation \( T_n \) of \( U_n \) by associating a family of arcs in \( U_n \) to every arc of \( \Pi \).

This is realized as follows: for every integer \( k \) there is a natural embedding \( t_k \) of the set \( A(S^1_k) \) of arcs in \( S^1_k \) into the set \( A(U_n) \) of arcs in \( U_n \), given by

\[
t_k(ij) = \begin{cases} 
   i(k)j(k), & \text{if } i < j \\
   i(k)j(k+1), & \text{if } i \geq j.
\end{cases}
\]

We define the set \( \Phi(\Pi) \) of arcs in \( U_n \) by the union of the disjoint sets \( t_k(\Pi) \), \( k \in \mathbb{Z} \), i.e.

\[
\Phi(\Pi) = \bigcup_{k \in \mathbb{Z}} t_k(\Pi).
\]

Figure 5.4. A triangulation of \( S^1 \) with associated 5-periodic triangulation of \( U_5 \).
Proposition 5.10. Let $\Pi$ be a triangulation of $S_n^1$. Then $T_n = \Phi(\Pi)$ is an $n$-periodic triangulation of $U_n$.

Proof. Clearly, if $\Pi = \Pi_1$ is the star-triangulation of $S_n^1$, then the claim is true. We proceed by induction on $n$. If $n = 1$, there is only the star-triangulation. Now we assume the claim holds for $n \geq 1$. Let $\Pi$ be a triangulation of $S_{n+1}^1$ other than $\Pi_1$ with bridging arc $\partial j$, $j \in \{1, 2, \ldots, n + 1\}$, in $\Pi$. By Lemma 5.9, $\Pi$ contains a special marked point $x \in \{1, 2, \ldots, n + 1\} \setminus \{j\}$. Then, by Corollary 5.10 and induction, $\Pi_x$ is a triangulation of $S_n^1$ providing an $n$-periodic triangulation $T_n = \Phi(\Pi_x)$ of $U_n$. If $j \in \{1, 2, \ldots, x-1\}$, let $[j] = j$, and for $j \in \{x+1, \ldots, n+1\}$ let $[j] = j-1$. Then by construction, $T_n$ contains $0^{(k)} j^{(k)}$ for every $k \in \mathbb{Z}$. We choose the fundamental region for $T_n$ given by the $(n+3)$-gon $P'$ with vertices $0^{(0)}, 0^{(1)}, \ldots, n^{(1)}, n^{(0)}, (n-1)^{(k)}, \ldots, j^{(k)}$ containing $n-1$ pairwise non-crossing arcs. Clearly, since $x$ is a special marked point in $P'$ it follows, for every $k$, that there is no arc in $T_{n+1} = \Phi(\Pi)$ having $x^{(k)}$ as an endpoint. Then the union $P$ of all triangles in $P'$ together with the special triangle with vertices $x-1^{(0)}, x^{(0)}, x+1^{(0)}$ is an $(n+4)$-gon containing $n$ non-crossing arcs. Thus $P$ provides a fundamental region for $T_{n+1}$ and it follows that $T_{n+1}$ is an $(n+1)$-periodic triangulation of $U_{n+1}$. □

Note that triangulations of $S_n^1$ with the same shape provide the same $n$-periodic triangulation of $U_n$. Note also that if $x$ is a special marked point in $\Pi$, then $x^{(k)}$ is a special vertex in $\Phi(\Pi)$ for every integer $k$. Hence we have the following corollary.

Corollary 5.11. Let $T_n$ be an $n$-periodic triangulation of $U_n$ other than the star-triangulation with quiddity sequence $q_{T_n} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_n^{(0)})$. Then $a_i^{(0)} = 1$ for some $i \in \{1, 2, \ldots, n\}$ and $i^{(k)}$ is a special vertex with respect to $T_n$ for all $k \in \mathbb{Z}$.

5.3. From periodic triangulations of strips to triangulations of punctured discs. Let $T_n$ be an $n$-periodic triangulation of $U_n$ with fundamental region $P$. We associate a triangulation $\Pi = \Psi(T_n)$ of $S_n^1$ to $T_n$ as follows: there is natural projection $\pi$ from the set $A(U_n)$ of arcs in $U_n$ to the set $A(S_n^1)$ of arcs in $S_n^1$ given by

$$
\pi: A(U_n) \rightarrow A(S_n^1)
$$

$$
i^{(k)} j^{(l)} \mapsto ij
$$

(with $l \in \{k, k+1\}$ cf. Lemma 5.7). We define the set $\Pi$ of arcs in $S_n^1$ to be the image of $P$ under $\pi$, i.e.

$$
\Psi(T_n) = \bigcup_{i^{(k)} j^{(l)} \in T_n \cap P} \pi(i^{(k)} j^{(l)}) .
$$

One can easily convince oneself that the definition of $\Psi$ does not depend on the choice of a fundamental region. Note that $\pi$ sends bridging arcs to bridging arcs. Thus by Remark 5.6 $\Psi(T_n)$ contains at least one bridging arc. Note also that a special vertex in $T_n$ maps to a special marked point in $\Psi(T_n)$.

Clearly, the star-triangulation of $U_n$ leads to the star-triangulation of $S_n^1$. Another example for $n = 5$, Figure 5.2 shows the periodic triangulation of the strip whose image under $\Psi$ is the triangulation of the punctured disc given in Figure 4.1. The proof of the next result is straightforward.

Proposition 5.12. Let $T_n$ be a $n$-periodic triangulation of $U_n$. Then $\Pi = \Psi(T_n)$ is a triangulation of $S_n^1$. 

5.4. Bijection between triangulations of punctured discs and periodic triangulations of strips. In the sequel \( \Phi \) and \( \Psi \) denote the maps defined respectively in Subsections 5.2 and 5.3.

**Theorem 5.13.** The maps \( \Phi \) and \( \Psi \) are inverse bijections between triangulations of \( S_n^1 \) and \( n \)-periodic triangulations of \( U_n \).

**Proof.** We have to proof that \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \) are the identity on \( U_n \) and on \( S_n^1 \), respectively.

(i) Let \( T_n \) be an \( n \)-periodic triangulation of \( U_n \). We write \( \Pi = \Psi(T_n) \), \( T'_n = \Phi(\Pi) \) and show that \( T_n = T'_n \). First, we choose an arbitrary arc \( \gamma = i(k_j) \) in \( T_n \). We have \( \pi(\gamma) = ij \in \Pi \).

By Lemma 5.14 we know that \( \gamma = i(k_j) \) with \( i < j \), or \( \gamma = i(k_j) \) with \( i \geq j \). If \( i < j \), then \( \iota(k_{ij}) = i(k_j) \) in \( T'_n \), otherwise, if \( i \geq j \), then \( \iota(k_{ij}) = i(k_{j+1}) \) in \( T'_n \). In both cases \( \iota(k_{ij}) = \iota(k(\pi(\gamma))) \in T'_n \). Next, we show that \( T'_n \) is contained in \( T_n \). Let \( \gamma' = i(k_j) \) be an arbitrary arc in \( T'_n \). By Lemma 5.14 \( \gamma' = i(k_j) \) with \( i < j \), or \( \gamma' = i(k_j) \) with \( i \geq j \). Since \( T'_n \) is the image of \( T_n \), we have \( \gamma' = i(k_j) \) for some \( i, j \in \mathbb{Z} \), otherwise, if \( \gamma = i(k_j) \) in \( T_n \) with \( i < j \), or \( \gamma = i(k_j) \) in \( T_n \) with \( i \geq j \). Hence \( \gamma' \) is a translate of \( \gamma \). Since \( T_n \) is an \( n \)-periodic triangulation \( T_n \) contains all the translates of \( \gamma \) and we have \( \gamma' \in T_n \).

(ii) We consider a triangulation \( \Pi \) of \( S_n^1 \) and write \( T_n = \Phi(\Pi) \), \( \Pi' = \Psi(T_n) \). Let \( \gamma = ij \) be an arc in \( \Pi \). For some \( k \in \mathbb{Z} \) we have either \( \iota(k(\gamma)) = i(k_j) \) in \( T_n \) if \( i < j \), or \( \iota(k(\gamma)) = i(k_j) \) in \( T_n \) otherwise. In both cases we get \( \pi(\iota(k_{ij})) = ij \in \Pi' \), hence \( \gamma' \in \Pi' \). Since this is true for all non-crossing arcs in \( \Pi \) we have \( \Pi' = \Pi \) as desired. This establishes the bijection.

Before moving on to matchings, let us summarize some facts about periodic triangulations of strips implied by the bijection of Theorem 5.13 and our previous work on triangulations of punctured discs. Especially, how periodic triangulations of strips are linked to arithmetic friezes.

Exactly as for triangulations of punctured discs we can use the operations of cutting and gluing triangles for fundamental regions of periodic triangulations of strips. Here, we only allow adding and removing triangles at the lower boundary.

**Corollary 5.14.** Let \( T_n \) be an \( n \)-periodic triangulation of \( U_n \) with quiddity sequence \( q_{T_n} = \{a_{n(1)}, a_{n(2)}, \ldots, a_{n(0)}\} \).

a) Assume that \( a_{x(0)} = 1 \) for some \( x \in \{1, 2, \ldots, n\} \). Let \( T_{n; x} \) denote the union of all triangles in \( T_n \) other than the special triangles at \( x(k) \), for all \( k \in \mathbb{Z} \). Then \( T_{n; x} \) is an \((n-1)\)-periodic triangulation of \( U_{n-1} \) with quiddity sequence

\[
q_{T_{n; x}} = \begin{cases} 
(2), & \text{if } n = 2 \\
(a_{1(0)} \cdots a_{(x-2)(0)} a_{(x-1)(0)} - 1 a_{(x+1)(0)} - 1 a_{(x+2)(0)} \cdots a_{n(0)}), & \text{otherwise}.
\end{cases}
\]

b) For some \( j \in \{1, 2, \ldots, n\} \) insert a vertex \( x(k) \) between \( j(k) \) and \( j(k+1) \) together with the arc \( j(k) \) for every \( k \in \mathbb{Z} \). Let \( T_{n; x} \) denote the union of all triangles in \( T_n \) with the triangles having vertices \( j(k), x(k) \) and \( j(k+1) \), for all \( k \in \mathbb{Z} \). Then \( T_{n; x} \) is an \((n+1)\)-periodic triangulation of \( U_{n+1} \) with quiddity sequence

\[
q_{T_{n; x}} = \begin{cases} 
(4, 1), & \text{if } n = 2 \\
(a_{1(0)} \cdots a_{(j-1)(0)} a_{j(0)} + 1 a_{(j+1)(0)} + 1 a_{(j+2)(0)} \cdots a_{n(0)}), & \text{otherwise}.
\end{cases}
\]

**Remark 5.15.** By Corollary 5.14, starting with a fundamental region for the star-triangulation of \( U_n \) and iteratively inserting triangles provides all periodic triangulations with \( n \) bridging arcs at a vertex on the upper boundary.

**Lemma 5.16.** Let \( T_n \) be an \( n \)-periodic triangulation of \( U_n \) with quiddity sequence \( q_{T_n} \) and let \( \Pi = \Psi(T_n) \) be the associated triangulation of \( S_n^1 \) with quiddity sequence \( q_{T_n} \). Then \( q_{T_n} \) equals \( q_{T_n} \). In particular, \( T_n \) provides an \( n \)-periodic infinite frieze \( F_{T_n} \) that is equal to \( F_{T_n} \).

**Corollary 5.17.** Given an \((n+1)\)-periodic triangulation \( T_{n+1} \neq T_n \) of \( U_{n+1} \) with special vertex \( x(0) \), let \( F_{T_{n+1}} \) be the associated \((n+1)\)-periodic infinite frieze. Then \( F_{T_{n+1}} \) equals \( F_{T_{n+1}} \).
Given a triangulation of a punctured disc, we will consider the associated periodic triangulation of the strip and the ways to allocate triangles in the periodic triangulation of the strip to sets of consecutive vertices on the lower boundary of the strip.

**Definition 5.18.** Let $\mathcal{T}_n$ be an $n$-periodic triangulation of $\mathcal{U}_n$ and let $I$ be a set of $s \geq 1$ consecutive vertices $p_1, p_2, \ldots, p_s$ on the lower boundary of $\mathcal{U}_n$.

(i) A *matching* between $I$ and $\mathcal{T}_n$ is an $s$-tuple $(\tau_1, \tau_2, \ldots, \tau_s)$ of pairwise distinct triangles of $\mathcal{T}_n$ such that $\tau_i$ is incident with $p_i$, for all $i \leq i \leq s$.

(ii) $\mathcal{M}_I, \mathcal{T}_n$ is the set of all matchings between $I$ and $\mathcal{T}_n$.

**Remark 5.19.** Clearly, the number of matchings between a set $I = \{ p \}$ containing one single vertex $p$ and an $n$-periodic triangulation $\mathcal{T}_n$ of $\mathcal{U}_n$ equals the number of triangles incident with $p$. It follows, by Lemma 5.10, that $|\mathcal{M}_{i^{(k)}, \mathcal{T}_n}| = a_{i^{(0)}} = a_i$ for the associated triangulation $\Pi = \Psi(\mathcal{T}_n)$ of $S^1$ with quiddity sequence $q_{n}| = (a_1, a_2, \ldots, a_n)$. In particular, if $i^{(k)}$ is special, $|\mathcal{M}_{i^{(k)}, \mathcal{T}_n}| = 1$.

We shall use the following notation. For two vertices $i^{(k)} \leq j^{(l)}$ on the lower boundary of $\mathcal{U}_n$, $[i^{(k)}, j^{(l)}]$ denotes the non-empty ordered set of consecutive vertices on the lower boundary of $\mathcal{U}_n$ starting with $i^{(k)}$ and increasing up to $j^{(l)}$. Otherwise if $i^{(k)} > j^{(l)}$, $[i^{(k)}, j^{(l)}]$ is not defined. Moreover, we set $s([i^{(k)}, j^{(l)}]) := |[i^{(k)}, j^{(l)}]|$ for the size of the interval. One can easily convince oneself that for two vertices $i^{(k)} \leq j^{(l)}$ with $s = s(i^{(k)}, j^{(l)})$ we have

\[ s = j - i + (l - k)n + 1 \quad \text{and} \quad j^{(l)} = (i + s - 1)^{(l)}. \]

Clearly, given an $n$-periodic triangulation $\mathcal{T}_n$ of $\mathcal{U}_n$ and two vertices $i^{(k)} \leq j^{(l)}$ on the lower boundary of $\mathcal{U}_n$, then $|\mathcal{M}_{i^{(k)}, j^{(l)}, \mathcal{T}_n}| = |\mathcal{M}_{i^{(k+m)}, j^{(l+m)}, \mathcal{T}_n}|$ for all $m \in \mathbb{Z}$. The next lemma gives a useful link between "smaller" and "bigger" periodic triangulations. For fixed $x \in \{1, 2, \ldots, n + 1\}$ we will use the following notation

\[ \tilde{i} = \begin{cases} \tilde{j}, & 1 \leq \tilde{i} \leq x, \\ i - 1, & x < \tilde{i} \leq n + 1, \end{cases} \quad \text{and} \quad \tilde{j} = \begin{cases} \tilde{i}, & 1 \leq \tilde{j} < x, \\ j - 1, & x \leq \tilde{j} \leq n + 1. \end{cases} \]

In particular, if $x = n + 1$, $\tilde{i} = i$, and if $x = 1$, $\tilde{j} = j - 1$.

**Lemma 5.20.** Let $\mathcal{T}_{n+1}$ be an $(n + 1)$-periodic triangulation of $\mathcal{U}_{n+1}$ with quiddity sequence $q_{n+1} = (a_1, a_2, \ldots, a_{n+1})$ and assume that $a_{x^{(0)}} = 1$ for some $x \in \{1, 2, \ldots, n + 1\}$. Let $\mathcal{T}_n = \mathcal{T}_{n+1}|_x$. Then for two vertices $i^{(k)} \leq j^{(l)}$ on the lower boundary of $\mathcal{U}_{n+1}$

\[ |\mathcal{M}_{i^{(k)}, j^{(l)}, \mathcal{T}_{n+1}}| = \begin{cases} |\mathcal{M}_{i^{(k)}, j^{(l)}, \mathcal{T}_n}|, & i \neq x + 1, j \neq x - 1, \\ |\mathcal{M}_{i^{(k)}, (j+1)^{(l)}, \mathcal{T}_n}| + |\mathcal{M}_{i^{(k)}, (j+1)^{(l)}, \mathcal{T}_n}|, & i \neq x + 1, j = x - 1 \\ |\mathcal{M}_{i^{(k)}, (j+1)^{(l)}, \mathcal{T}_n}| + |\mathcal{M}_{i^{(k)}, (j-1)^{(l)}, \mathcal{T}_n}|, & i = x + 1, j \neq x - 1 \\ |\mathcal{M}_{(i-1)^{(k)}, (j+1)^{(l)}, \mathcal{T}_n}| + |\mathcal{M}_{(i-1)^{(k)}, (j+1)^{(l)}, \mathcal{T}_n}|, & i = x + 1, j = x - 1 \\ |\mathcal{M}_{(i-1)^{(k)}, (j-1)^{(l)}, \mathcal{T}_n}| + |\mathcal{M}_{(i-1)^{(k)}, (j-1)^{(l)}, \mathcal{T}_n}|, & i = x + 1, j = x - 1. \end{cases} \]

The proof of Lemma 5.20 is a tedious but straightforward case-by-case study. We thus omit it.

**Theorem 5.21.** Let $\mathcal{T}_n$ be an $n$-periodic triangulation of $\mathcal{U}_n$ with associated arithmetic frieze $\mathcal{F}_{\mathcal{T}_n} = (m_{i^j})_{i,j \geq 0, j \in \mathbb{Z}}$. Then for $i \geq 2$ and $j \in \{1, 2, \ldots, n\}$

\[ m_{ij} = |\mathcal{M}_{i^{(0)}, (j+i-2)^{(0)}, \mathcal{T}_n}|. \]

**Proof.** We first prove the claim for star-triangulations. Let $\mathcal{T}_n = \mathcal{T}_n$ be the star-triangulation of $\mathcal{U}_n$ with associated basic infinite frieze $\mathcal{F}_n = (m_{ij})_{i,j \geq 0, j \in \mathbb{Z}}$, see Figures 5.3 and 5.4. Clearly, for two vertices $i^{(k)} \leq j^{(l)}$ on the lower boundary of $\mathcal{U}_n$ we have $|\mathcal{M}_{i^{(k)}, j^{(l)}, \mathcal{T}_n}| = |\mathcal{M}_{i^{(k)}, j^{(0)}, \mathcal{T}_n}| + 1$. Let $i \geq 2$ and $j \in \{1, 2, \ldots, n\}$, so $|\mathcal{M}_{i^{(0)}, (j+i-2)^{(0)}, \mathcal{T}_n}| = a_j + i - 2 = i = m_{it}$, for arbitrary $t \in \mathbb{Z}$. By setting $t = j$ the result follows for $\mathcal{T}_n$, in particular, for $n = 1$. 


Now we use induction on $n$ to prove the claim for the remaining periodic triangulations. If $i = 2$, Remark 5.19 gives the desired result. Let $i \geq 3$ and we assume that the result holds for any $n$-periodic triangulation of $\mathcal{U}_n$. We consider an $(n+1)$-periodic triangulation $\mathcal{T}_{n+1}$ of $\mathcal{U}_{n+1}$, $\mathcal{T}_{n+1} \neq \mathcal{T}_n$, with associated arithmetic frieze $\mathcal{F}_{\mathcal{T}_{n+1}} = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$. By Corollaries 5.11 and 5.12, $\mathcal{T}_{n+1}$ has special vertices, i.e. there exists $x \in \{1, 2, \ldots, n+1\}$ such that $z_{x,0} = 1$ in the quiddity sequence $(a_1, a_2, \ldots, a_{n+1})$ and $\mathcal{T}_n := \mathcal{T}_{n+1}\backslash x$ is an $n$-periodic triangulation of $\mathcal{U}_n$. W.l.o.g. $x = n + 1$. We write $\mathcal{F}_{\mathcal{T}_n} = (m'_{ij})_{i \geq 0, j \in \mathbb{Z}}$ for the arithmetic frieze associated to $\mathcal{T}_n$ with quiddity sequence $(a'_1, a'_2, \ldots, a'_n)$. Note that $a'_j = a_j$ for $j \notin \{1, n\}$, $a'_1 = a_1 - 1$, $a'_n = a_n - 1$ and we obtain $\mathcal{F}_{\mathcal{T}_{n+1}} = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$ by n-gluing above $a'_n, a'_1$ (Corollaries 5.11 and 5.12).

We will show that for every $j \in \{1, 2, \ldots, n+1\}$, the se-diagonal $\tilde{f} = \tilde{f}(a_j) = (\tilde{f}_i)_{i \geq 0} = (m_{ij})_{i \geq 0}$ in $\mathcal{F}_{\mathcal{T}_{n+1}}$ through $a_j$ has as entries the matching numbers of the intervals starting at $j^{(0)}$, i.e. that
\[ m_{ij} = |\mathcal{M}_{j^{(0)}, (j+i-2)^{(0)}}, \mathcal{T}_{n+1}| \]
holds for $i \geq 3$. Note that $s = s(j^{(0)}, (j+i-2)^{(0)}) = i - 1$. We will distinguish the cases $j \neq x + 1$ and $j = x + 1$ in (i) and (ii), respectively.

(i) Let $2 \leq j \leq n+1$. Corollary 5.14 tells, that the diagonals in $\mathcal{F}_{\mathcal{T}_{n+1}}$ through $a_j$ equals the diagonals in $\mathcal{F}_{\mathcal{T}_n}$ through $a'_j$ up to every $(n+1)$st entry, where the subscript $j$ of $a'_j$ is reduced modulo $n$. We obtain for $t \in \{1, 2, \ldots, n+1\}$ and $l \geq 0$
\[ m_{t+l(n+1),j} = \begin{cases} m'_{t+ln,j}, & 1 \leq t \leq n + 1 - j \\ m'_{(t-1)+ln,j} + m'_{t+ln,j}, & t = n + 2 - j \\ m'_{(t-1)+ln,j}, & n + 3 - j \leq t \leq n + 1. \end{cases} \]

Let $t \geq l(n+1) \geq 3$.
- First we consider $1 \leq t \leq n + 1 - j$. By induction $m'_{t+ln,j} = |\mathcal{M}_{j^{(0)}, (j+t-2)^{(0)}}, \mathcal{T}_n|$ where the interval has endpoints $j^{(0)}, (j+t-2+ln)^{(0)} = (j+t-2)^{(0)}$ and its size is $t - 1 + ln$ (formula (c)). To see that $|\mathcal{M}_{j^{(0)}, (j+t-2)^{(0)}}, \mathcal{T}_{n+1}| = |\mathcal{M}_{j^{(0)}, (j+t-2)^{(0)}}, \mathcal{T}_{n+1}|$, we use Lemma 5.20 observing that the first case applies.
- For $n + 1 - j + 2 \leq t \leq n + 1$ the proof works similarly.
- Let $t = n + 2 - j$. By induction $m'_{(t-1)+ln,j} = |\mathcal{M}_{j^{(0)}, (n-1)^{(0)}}, \mathcal{T}_n|$ and $m'_{t+ln,j} = |\mathcal{M}_{j^{(0)}, (n)^{(0)}}, \mathcal{T}_n|$ (again using formula (c) to determine the endpoints of the intervals). We are now in the second case of Lemma 5.20 and thus we see that
\[ m_{n+2-j+l(n+1),j} = |\mathcal{M}_{j^{(0)}, (n-1)^{(0)}}, \mathcal{T}_n| + |\mathcal{M}_{j^{(0)}, n^{(0)}}, \mathcal{T}_n| = |\mathcal{M}_{j^{(0)}, n^{(0)}}, \mathcal{T}_{n+1}|, \]
as claimed, since by formula (c) the endpoint of the interval is $(j + n + 2 - j + l(n+1) - 2)^{(0)} = (n + l(n+1))^{(0)} = n^{(0)}$.

(ii) Let $j = 1$. We use Corollaries 2.8 and 5.5 to obtain
\[ m_{t+l(n+1),1} = \begin{cases} m'_{(t-1)+ln,2} + m'_{t+ln,1}, & 1 \leq t \leq n \\ m_{n+1-ln,2} + m_{n+ln,1} + m_{0+(l+1)n,2} + m'_{1+(l+1)n,1}, & t = n + 1. \end{cases} \]

We then use similar arguments as in (i) to prove the claim. \hfill \Box

To illustrate Theorem 5.21 we give an example.

**Example 5.22.** We consider the 5-periodic triangulation of $\mathcal{U}_5$ with associated triangulation of $S_5^1$ given in Figure 7.4. Then the associated arithmetic frieze $\mathcal{F}_5 = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$ has quiddity sequence $q_{\mathcal{F}_5} = (4 \ 3 \ 3 \ 1 \ 2)$ and the fundamental region $\mathcal{D} = (m_{ij})_{i \geq 0, 1 \leq j \leq 5}$ looks as follows
Using matchings between sets of $s \geq 1$ consecutive vertices, starting at $j^{(0)}$, and $T_5$ we obtain the matching numbers $\mathcal{M}_{\{(j^{(0)}, (j+s-1)^{(0)}), T_5\}}$ given in the table below. For fixed $j$, this provides exactly the non-trivial entries in the diagonals of $F_5$ whereas we get the non-trivial rows of $F_5$ if $s$ is fixed.

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| 1   | 4 | 11| 29| 18| 7 | 10| 23|
| 2   | 3 | 8 | 5 | 2 | 3 | 7 | 18|
| 3   | 3 | 2 | 1 | 2 | 5 | 13| 8 |
| 4   | 1 | 1 | 3 | 8 | 21| 13| 5 |
| 5   | 2 | 7 | 19| 50| 31| 12| 17|

6. Alternative Description

In this section we give an alternative description of the entries in arithmetic friezes associated to triangulated punctured discs. This will provide, in a simple way, all diagonals in an arithmetic frieze. Conway and Coxeter in (32) of [8] allocated non-negative numbers to the vertices of triangulated polygons and these numbers appear in the diagonals of the associated Conway-Coxeter frieze. Adapting their strategy we now assign numbers to the vertices of periodic triangulations of strips in an iterative way.

![Diagram of a periodic triangulation]

**Figure 6.1.** The labels with starting vertex $2^{(0)}$ attached to the 5-periodic triangulation of $U_5$ in Figure 5.2 with associated arithmetic frieze given in Figure 4.2.

Let $T_n$ be an $n$-periodic triangulation of $U_n$. We consider a fixed vertex $j^{(l)}$ on the lower boundary of $U_n$ and attach labels $n_{j^{(l)}}(i^{(k)}) \in \mathbb{Z}_{\geq 0}$ to every vertex $i^{(k)}$ of $U_n$ as follows: we first
set $n_{j^{(i)}}(j^{(i)}) = 0$, and $n_{j^{(i)}}(i^{(k)}) = 1$ whenever the vertex $i^{(k)}$ is joint with $j^{(i)}$ by a boundary segment or an arc in $T_n$. Note that $i^{(k)}$ may lie on the upper boundary. As soon as a label is given for a vertex $0^{(k)}$ on the upper boundary of $U_n$ we label all the remaining vertices on the upper boundary by the same number, i.e. we then set

$$n_{j^{(i)}}(0^{(k')}) := n_{j^{(i)}}(0^{(k)}) \text{ for all } k' \neq k.$$ 

Once all neighbours of $j^{(i)}$ (through arcs or boundary segments) have obtained their label, we iteratively define the labels for the remaining vertices on the lower boundary. Whenever there is a triangle in $T_n$ given by three vertices $i^{(k)}, j^{(k)}$ and $0^{(k')}$ on the lower boundary such that two of its vertices already have a label, e.g. $i^{(k)}$ and $j^{(k')}$, we take their sum for the label of the remaining vertex:

$$n_{j^{(i)}}(0^{(k')}) := n_{j^{(i)}}(i^{(k)}) + n_{j^{(i)}}(j^{(k')}).$$

If there is no such a triangle in $T_n$ left, we consider triangles in $T_n$ of one of the following types

\begin{align*}
\cdots & j^{(k)} & \cdots & j^{(k)} & \cdots \\
\cdots & i^{(k)} & \cdots & i^{(k)} & \cdots
\end{align*}

where two (or three) vertices already have a label. For the triangle on the left we use the same rule as before, and take the sum of the two given labels for the remaining label. In the triangle on the right we consider $0^{(k)}$ and $0^{(k+1)}$ to be a single vertex with label $n_{j^{(i)}}(0^{(k)})$ and also take the sum of the two given labels as the remaining label.

Note that either $0^{(k)}$ and $0^{(k+1)}$ already have the same label or they will get the same label now. As mentioned before once a vertex on the upper boundary has a label, all vertices on the upper boundary obtain the same label.

Finally, we continue the labelling for the remaining vertices on the lower boundary as done before: if the labels are given for two vertices in a triangle of $T_n$, we take their sum for the remaining one taking care that whenever a triangle is a four-sided region, we only take one of the labels on the upper boundary for the sum.

Figure 6.1 gives an example of this labelling for the 5-periodic triangulation of $U_5$ shown in Figure 5.2 with starting vertex $2^{(0)}$. We get the two sequences

$$\begin{align*}
\left(n_{2^{(0)}}(i^{(k)})\right)_{i^{(k)}>2^{(0)}} &= (0 \, 1 \, 1 \, 1 \, 5 \, 4 \, 1 \, 1 \, 7 \, 3 \, \cdots) \quad \text{and} \quad \left(n_{2^{(0)}}(i^{(k)})\right)_{i^{(k)}<2^{(0)}} = (0 \, 1 \, 1 \, 5 \, 9 \, 4 \, 7 \, 3 \, \cdots).
\end{align*}$$

Comparing these numbers with those in the associated arithmetic frieze, see Figure 4.2, we observe that the first sequence is the SE-diagonal through $a_3 = 1$, and the second sequence gives the SW-diagonal through $a_1 = 1$.

**Remark 6.1.** Clearly, if a special vertex $x^{(k)}$ with respect to $n$-periodic triangulation $T_n$ of $U_n$ is not equal to the starting vertex $j^{(i)}$, we have

$$n_{j^{(i)}}(x^{(k)}) = n_{j^{(i)}}((x - 1)^{(k)}) + n_{j^{(i)}}((x + 1)^{(k)}).$$

The next lemma gives a useful relation if the starting vertex is special.

**Lemma 6.2.** Let $T_n$ be an $n$-periodic triangulation of $U_n$, $T_n \neq T_n$. Let $x^{(i)}$ be a special vertex with respect to $T_n$. Then for any vertex $i^{(k)} \neq x^{(i)}$ on the lower boundary of $U_n$

$$n_{x^{(i)}}(i^{(k)}) = n_{(x-1)^{(i)}}(i^{(k)}) + n_{(x+1)^{(i)}}(i^{(k)}).$$
Proof. We proof the result by induction on \( n \). If \( n = 1 \), there is only the star-triangulation. For \( n = 2 \), up to translation there is only one 2-periodic triangulation \( \mathcal{T}_2 \neq \mathcal{T}_s \) of \( \mathcal{U}_2 \), see figure below.

W.l.o.g. we may assume that \( 2^{(0)} \) is special and show the the claim for \( x^{(i)} = 2^{(0)} \). One easily checks that \( n_{2(0)}(1^{(k)}) = |2k - 1| \) and \( n_{2(0)}(2^{(k)}) = |4k| \) for all \( k \in \mathbb{Z} \). Moreover, we have \( n_{1(0)}(1^{(k)}) = |k| \) and \( n_{1(0)}(2^{(k)}) = |2k + 1| \) for all \( k \in \mathbb{Z} \). Finally, \( n_{1(1)}(1^{(k)}) = |k - 1| \) and \( n_{1(1)}(2^{(k)}) = |2k - 1| \) for all \( k \in \mathbb{Z} \). Hence, for \( k \in \mathbb{Z} \), we have \( n_{1(0)}(1^{(k)}) + n_{1(1)}(1^{(k)}) = |2k - 1| = n_{2(0)}(1^{(k)}) \), and if \( k \neq 0 \), \( n_{1(0)}(2^{(k)}) + n_{1(1)}(2^{(k)}) = |4k| = n_{2(0)}(2^{(k)}) \) as desired.

Now, we assume the claim holds for every \( n \)-periodic triangulation \( \mathcal{T}_n \neq \mathcal{T}_s \) of \( \mathcal{U}_n \). Let \( \mathcal{T}_{n+1} \neq \mathcal{T}_s \) be an \((n+1)\)-periodic triangulation of \( \mathcal{U}_{n+1} \) with special vertex \( y^{(0)} \). By Corollary 5.14, \( \mathcal{T}_n = \mathcal{T}_{n+1} \setminus y \) is an \( n \)-periodic triangulation of \( \mathcal{U}_n \). For a vertex \( i^{(k)} \) on the lower boundary of \( \mathcal{U}_{n+1} \) with \( i \neq y \) let

\[
[i] = \begin{cases} 
    i, & 1 \leq i < y \\
    i-1, & y < i \leq n+1.
\end{cases}
\]

and \([i]^{(i)}\) denotes the corresponding vertex on the lower boundary of \( \mathcal{U}_n \). In particular, if \( y = n+1 \), \( [j] = j \) for all \( j \in \{1, 2, \ldots, n\} \). We consider two cases.

(i) Assume, up to translates, there are at least two special vertices for \( \mathcal{T}_{n+1} \). For any special vertex \( x^{(i)} \neq y^{(0)} \) of \( \mathcal{T}_{n+1} \), the vertex \([x]^{(i)}\) is special for \( \mathcal{T}_n \). Let \( i^{(k)} \neq x^{(i)} \) be a vertex on the lower boundary of \( \mathcal{U}_{n+1} \).

- If \( i \neq y \). We consider the vertex \([i]^{(k)} \neq [x]^{(i)}\) on the lower boundary of \( \mathcal{U}_n \). Then by induction we have

\[
n_{[x]^{(i)}}([i]^{(k)}) = n_{[x-1]^{(i)}}([i]^{(k)}) + n_{[x+1]^{(i)}}([i]^{(k)}) \quad (x - 1 \neq y, x + 1 \neq y).
\]

Clearly, since both \( x \) and \( i \) are different from \( y \), we have \( n_{x^{(i)}}(i^{(k)}) = n_{[x]^{(i)}}([i]^{(k)}) \), and similarly

\[
n_{(x-1)^{(i)}}(i^{(k)}) = n_{[x-1]^{(i)}}([i]^{(k)}) \quad n_{(x+1)^{(i)}}(i^{(k)}) = n_{[x+1]^{(i)}}([i]^{(k)}).
\]

Hence the claim follows.

- If \( i = y \), by assumption we have \( x \neq y \) and Remark 6.1 tells us that

\[
n_{x^{(i)}}(y^{(k)}) = n_{x^{(i)}}((y - 1)^{(k)}) + n_{x^{(i)}}((y + 1)^{(k)}).
\]

With the result above for \( i \neq y \) it follows that

\[
n_{x^{(i)}}(y^{(k)}) = n_{(x-1)^{(i)}}((y - 1)^{(k)}) + n_{(x+1)^{(i)}}((y - 1)^{(k)}) + n_{(x-1)^{(i)}}((y + 1)^{(k)}) + n_{(x+1)^{(i)}}((y + 1)^{(k)}).
\]

Clearly, \((x - 1)^{(i)} \neq y^{(k)} \neq (x + 1)^{(i)}\), and again by Remark \ref{rem:6.1} we have

\[
n_{x^{(i)}}(y^{(k)}) = n_{(x-1)^{(i)}}(y^{(k)}) + n_{(x+1)^{(i)}}(y^{(k)}).
\]

(ii) If \( y^{(0)} \) is the only special vertex for \( \mathcal{T}_{n+1} \), up to translates, it follows that \( \mathcal{T}_{n+1} \setminus y \) is the star-triangulation \( \mathcal{T}_s \) of \( \mathcal{U}_n \). In this case the proof works similar as in the induction step and we leave it to the reader to check the details. \qed
The next result shows that this labelling algorithm provides all entries occurring in an arithmetic frieze $\mathcal{F}_n$ associated to a given $n$-periodic triangulation of $\mathcal{U}_n$, and hence to every triangulation of $S^1_n$.

**Theorem 6.3.** Let $\mathcal{T}_n$ be an $n$-periodic triangulation of $\mathcal{U}_n$ with associated arithmetic frieze $\mathcal{F}_{\mathcal{T}_n} = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$. Then for the fundamental region $\mathcal{D} = (m_{ij})_{i \geq 0, 1 \leq j \leq n}$

$$m_{ij} = n_{(j-1)0} \left( (j - 1 + i)^{(0)} \right).$$

**Proof.** First we consider the star-triangulation $\mathcal{T}_n$ of $\mathcal{U}_n$ with basic infinite frieze $\mathcal{F}_\ast = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$, $m_{ij} = i$ for all $j \in \mathbb{Z}$. Let $j^{(0)}$ be a fixed vertex on the lower boundary of $\mathcal{U}_n$. Then we have $n_{j^{(0)}} \left( (j - 1)^{(0)} \right) = n_{j^{(0)}} \left( (j + 1)^{(0)} \right) = 1$ and $n_{j^{(0)}} \left( 0^{(k)} \right) = 1$ for all $k \in \mathbb{Z}$.

Clearly, for $i \geq 1$ we get $n_{j^{(0)}} \left( (j + i)^{(0)} \right) = n_{j^{(0)}} \left( (j + i - 1)^{(0)} \right) + n_{j^{(0)}} \left( 0^{(k)} \right)$ and inductively $n_{j^{(0)}} \left( (j + i)^{(0)} \right) = i = m_{ij}$ for arbitrary $t \in \mathbb{Z}$. For $t = j + 1$ the desired result follows. Hence the claim holds for star-triangulations, in particular, for triangulations of $\mathcal{T}_1$.

We proceed by induction on $n$ and assume the claim is true for $n \geq 1$. Let $\mathcal{T}_{n+1} \neq \mathcal{T}_n$ be an $(n+1)$-periodic triangulation of $\mathcal{U}_{n+1}$ with associated arithmetic frieze $\mathcal{F}_{\mathcal{T}_{n+1}}$. By Corollaries 5.11 and 5.13 we know that $\mathcal{T}_{n+1}$ contains a special vertex $x^{(0)}$ and $\mathcal{T}_n := \mathcal{T}_{n+1,x}$ is an $n$-periodic triangulation of $\mathcal{U}_n$. Let $\mathcal{F}_{\mathcal{T}_n} = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$ denote the arithmetic frieze associated to $\mathcal{T}_n$ and recall that $\mathcal{F}_{\mathcal{T}_{n+1}} = \mathcal{F}_{\mathcal{T}_n} = (m_{ij})_{i \geq 0, j \in \mathbb{Z}}$, where we $n$-glue above $a_{x-1}, a_x$ in $\mathcal{F}_{\mathcal{T}_n}$ (Corollary 5.17).

As in the proof of Lemma 6.2, we set $[i] = i$ for any $1 \leq i < x$, and $[i] = i - 1$ for any $x < i \leq n + 1$. For a vertex $i^{(k)}$ of $\mathcal{U}_{n+1}$, with $i \neq x$, $[i]^{(k)}$ denotes the corresponding vertices of $\mathcal{U}_n$. In order to avoid confusion we denote the labels for $\mathcal{T}_{n+1}$ and $\mathcal{T}_n$ respectively by $\check{n}_{x}(-)$ and $n_{x}(-)$.

By definition, we have $n_{j^{(0)}} \left( (j - 1)^{(0)} \right) = 0 = \check{m}_{0,j}$ and $n_{j^{(0)}} \left( j^{(0)} \right) = 1 = \check{m}_{1,j}$ for every $j \in \{1, 2, \ldots n\}$. Now, we take $\check{m}_{j+1(i+1),j}$ in $\mathcal{F}_{\mathcal{T}_{n+1}}$ with $i \in \{1, 2, \ldots, n + 1\}$, $t \in \mathbb{Z}_{\geq 0}$ and assume that $i + t(n + 1) \geq 2$. We consider the three cases $1 \leq j < x + 1$, $x + 1 < j \leq n + 1$ and finally $j = x + 1$.

(i) Let $1 \leq j < x + 1$.
- By Corollary 5.3, if $1 \leq i < x + 1 - j \leq x$ (we have $1 \leq i < x , 1 \leq j - 1 + i < x$), we get
  
  $$\check{m}_{i+1(i+1),j} = \begin{cases} m_{i+1(i+1),j} & \text{if } j = 1 \\ n_{j+1(i+1),j} \left( (j - 1 + i)^{(0)} \right), & \text{else} \end{cases}$$

  By definition, we have $n_{j^{(0)}} \left( (j - 1)^{(0)} \right) = 0 = \check{m}_{0,j}$ and $n_{j^{(0)}} \left( j^{(0)} \right) = 1 = \check{m}_{1,j}$ for every $j \in \{1, 2, \ldots n\}$. Now, we take $\check{m}_{j+1(i+1),j}$ in $\mathcal{F}_{\mathcal{T}_{n+1}}$ with $i \in \{1, 2, \ldots, n + 1\}$, $t \in \mathbb{Z}_{\geq 0}$ and assume that $i + t(n + 1) \geq 2$. We consider the three cases $1 \leq j < x + 1$, $x + 1 < j \leq n + 1$ and finally $j = x + 1$.

(ii) Let $x + 1 < j \leq n + 1$.
- For $x + 1 - j < i \leq n + 1$, we have $x \leq j + i - 2 \leq x - 1 + n$ and if $j = 1$, $x \leq i - 1 \leq n$. By Corollary 5.3 we get
  
  $$\check{m}_{j+1(i+1),j} = \begin{cases} m_{j+1(i+1),j} & \text{if } j = 1 \\ n_{j+1(i+1),j} \left( (j + i - 2)^{(0)} \right), & \text{else} \end{cases}$$

(iii) Let $j = x + 1$.
- By Corollary 5.3, if $1 \leq i < x + 1 - j \leq x$ (we have $1 \leq i < x , 1 \leq j - 1 + i < x$), we get
  
  $$\check{m}_{j+1(i+1),j} = \begin{cases} m_{j+1(i+1),j} & \text{if } j = 1 \\ n_{j+1(i+1),j} \left( (j + i - 2)^{(0)} \right), & \text{else} \end{cases}$$
- If \( i = x + 1 - j \), we have \( j + i - 1 = x \) and by Corollary 3.5 together with the work done so far, we get
\[
\hat{m}_{i+1+(n+1),j} = \hat{m}_{i-1+(n+1),j} + \hat{m}_{i+1+(n+1),j} = \hat{n}_{(j-1)\circ1}(x - 1)^{(i)} + n_{(j-1)\circ1}(x + 1)^{(i)}.
\]
The latter is \( n_{(j-1)\circ1}(x)^{(i)} \) as desired (Remark 6.1).

(ii) If \( x + 1 < j < n + 1 \). The proof is similar as in the previous case by replacing \( m_{-j} \) with \( m_{-j} \).

(iii) Let \( j = x + 1 \) \((j - 1 = x)\). By Corollary 2.8 and the two previous cases, for \( i \geq 2 \), we have
\[
\hat{m}_{ij} = \hat{m}_{i+1,j-1} + \hat{m}_{i-1,j+1} = n_{i-2\circ1}(j + i - 1)^{(0)} + n_{j\circ1}(j + i - 1)^{(0)} = n_{(j-1)\circ1}(j + i - 1)^{(0)},
\]
where the last equality follows by Lemma 6.2.

This completes the proof. \(\square\)

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