Exact solution of a model of qubit decoherence due to telegraph noise

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We present a general and exact formalism for finding the evolution of a quantum system subject to external telegraph noise. The various qubit decoherence rates are determined by the eigenvalues of a transfer matrix. The formalism can be applied to a qubit subject to an arbitrary combination of dephasing and relaxational telegraph noise, in contrast to existing non-perturbative methods that treat only one or the other of these limits. As the applications: 1) We obtain the full qubit dynamics on time scales short compared with the environmental correlation times. In the strong coupling cases this reveals unexpected oscillations and induced magnetization components; 2) We find in strong coupling case strong violations of the widely used relation \(1/T_2 = 1/2T_1 + 1/T_\phi\), which is a result of perturbation theory; 3) We discuss the effects of bang-bang and spin-echo controls of the qubit dynamics in general settings of the telegraph noises. Finally, we discuss the extension of the method to the cases of many telegraph noise sources and multiple qubits. The method still works when white noise is also present.

I. INTRODUCTION

Decoherence of quantum systems is a fundamental issue with implications across all branches of physics. In this context, two-level quantum systems have served as a very useful paradigm. The subject of two-level systems in random time-dependent fields originated decades ago in the context of spin resonance [1]. Later experiments on macroscopic quantum coherence also focused attention on this problem [2]. The recent surge of interest in quantum computing and quantum control has rejuvenated the field, and much work has gone into solving various models of decoherence. All physical realizations of qubits are subject to external noise since all couple, however weakly, to the external environment. The most popular models are those that explicitly involve a bath whose degrees of freedom must be traced out[3]. In many cases, however, one can neglect the flow of quantum information from the bath to the system, and then it is sufficient to consider the qubit as being subject to random classical external fields. The conditions under which this assumption is valid have been considered in detail by Galperin et al. [4]. Here we merely note that when the coupling of the bath to its thermalizing external environment is very strong or on time scales longer than the characteristic microscopic times of the bath, we expect that even fully quantum system-bath models reduce to this case.

The most important type of noise in solid-state systems is telegraph noise: the qubit interacts with one or more random fluctuators in its neighborhood, and these fluctuators go back and forth between only two states [5]. Usually the qubit interacts with many such fluctuators. Depending on the distribution of fluctuator transition rates this may give rise to \(1/f\) or other types of noise. In this paper, however, we shall focus on the case of a single fluctuator, and will only mention the generalization to many fluctuators at the end. This special case is also extremely important in experimental practice [6].

Thus we investigate the following model: in a sequence of \(N\) time intervals each of length \(\Delta t\), the fluctuator producing the noise undergoes a sequence of states labelled by \(\left\{s_1, s_2, ..., s_N\right\}\), \(s_i = \pm 1\) as shown in Fig. 1, and so the hamiltonian within the \(i\)-th interval is given by

\[
H_{s_i} = -\frac{1}{2}(B_0 + s_i g) \cdot \vec{\sigma},
\]

where \(\sigma_x, \sigma_y, \sigma_z\) are the Pauli matrices acting on the qubit spin, \(B_0 = B_0 \hat{z}\) pointing in the \(z\)-direction for definiteness, and \(\pm g\) is the effective noise field arising from the coupling to the fluctuator. The average energy level separation of the qubit is \(B_0\) in units where \(\hbar = 1\) and \(g\mu_B = 1\). Our notation is that of a spin qubit but the model obviously applies to any two-level quantum system.

If \(g\) is in the \(\hat{z}\)-direction, then we refer to dephasing noise, and if \(g\) is in the \(x-y\) plane, then we have relaxational noise.

We characterize the fluctuator as follows. Within a small time interval \(\Delta t\) the fluctuator state \(s'\) stays unchanged, but it changes to \(s\) in going to the next inter-
val with the conditional probability $W_{ss'}$. The switching probabilities are parameterized as $W_{-+} = p + \delta$ and $W_{+-} = p - \delta$, and normalization requires that $W_{++} = 1 - p - \delta$ and $W_{--} = 1 - p + \delta$. To mimic telegraph noise we require $p = \gamma \Delta t \ll 1$ and $|\delta| = |\eta| \Delta t \ll 1$ with finite rates $\gamma$ and $\eta$. For a fluctuator in thermal equilibrium the stationary level population of the two fluctuator states is easily shown to be $p_s = (p - s \delta)/2p = (\gamma - s \eta)/2\gamma$ by a standard detailed-balance argument [7].

A sequence of the noise simulated the model is shown in Fig.1. We emphasize that telegraph noise has a finite correlation time, unlike white noise, for which exact methods are already available. Indeed, the telegraph sequence is a Poisson process and the noise spectrum is Lorentzian.

We wish to solve for the qubit density matrix $\rho$: given $\rho(t = 0)$, find $\rho$ at all later times. It has been shown that this decoherence process is exactly solvable by methods coming from the theory of stochastic differential equations [8]. The formal solution is given in terms of a complicated Laplace transform, which would be difficult to invert analytically and numerically unstable in most cases. In special cases the solution can be simplified and results have been presented when (1) $g_x = g_y = 0$: pure dephasing ($T_2$) noise [4, 9–11] and (2) $g_z = 0$: pure transverse or relaxation ($T_1$) noise [10, 11]. These special cases are important. However, there are clearly many situations in which both types of noise are present. For example, it has recently been shown that the character of the noise in flux qubits can be changed continuously from one type of noise to the other by changing the bias voltage [12].

The crossover region can only be described by the more general model, and the only available treatment is to use the phase-memory functional in the linear-coupling regime where $g \ll B_0$ [13]. In this paper, we show how to solve the general problem for a single qubit by a new algebraic method. This greatly simplifies the special cases and makes possible the presentation of results for all values of $(g_x, g_y, g_z)$. The solution is exact and does not make perturbative approximations. The main results are Eqs.(3)-(4) for the discrete-time formalism, Eqs.(6)-(7) for the continuous-time formalism, and the discussions thereafter.

The remainder of this paper is organized as follows. We develop the exact solution of the problem by the generalized transfer matrix method in section II, apply the theory to the case of bang-bang control in section III, and to the case of echo decay in section IV. We discuss in Sec.V possible generalization of the theory to the cases of combined white and telegraph noise, many fluctuators, and many qubits. Finally, Sec.VI is a summary of the work.

II. TRANSFER MATRIX ENSEMBLE AVERAGED OVER TELEGRAPH NOISE SEQUENCES

Our task is to solve for $\rho(t = N\Delta t)$ in the presence of a time-dependent Hamiltonian, averaged over all $2^N$ sequences with the appropriate probabilities. Formally

$$\rho(t) = \overline{U_{sN}U_{sN-1} \cdots U_{s1}\rho(0)U_{s1}^\dagger \cdots U_{s2-1}^\dagger U_{s1}^\dagger},$$

where $U_s = \exp(-iH_s\Delta t)$ is the evolution operator at noise level $s$ for one time interval and the overbar indicates the averaging over the sequences. However, it is much more convenient to parameterize $\rho$ by

$$\rho(t) = \frac{1}{2}I + \frac{1}{2}\mathbf{n}(t) \cdot \vec{\sigma}.$$  

The Bloch vector $\mathbf{n}(t)$ takes on only real values and satisfies $|\mathbf{n}(t)| \leq 1$ (the equality holds for a pure state). $I$ is the $2 \times 2$ unit matrix. Note that $\mathbf{n}(t)$ also gives the qubit magnetization as $\langle \vec{\sigma} \rangle = \text{Tr}(\rho \vec{\sigma}) = \mathbf{n}(t)$. The time development is given by

$$\mathbf{n}(t) \equiv T \mathbf{n}(0) = \overline{T_{sN}T_{sN-1} \cdots T_{s1} \mathbf{n}(0)},$$

which defines the ensemble averaged transfer matrix $T$, and the matrix $T_s$ is given by

$$T_s = \exp[i\Delta t (B_0L_z + s\vec{g} \cdot \vec{L})],$$

where $L_{x,y,z}$ are the usual generators of SO(3): $(L_i)_{jk} = i\epsilon_{ijk}$, where $\epsilon_{ijk}$ is the completely antisymmetric symbol. The $T_s$ are easily calculated explicitly for arbitrary $\vec{g}$. For noise that is uncorrelated between time intervals of equal length, the ensemble average in Eq.(2) can be performed within each interval, allowing exact solution of certain noise models [14]. For true telegraph noise the ensemble average has to be done in another way which we now describe.

Let us define $G_{ss'}^{ss'}$ to be the $3 \times 3$ transfer matrix for an $N$-step qubit evolution that starts at the fluctuator state $s'$ and end up at the state $s$, but ensemble averaged over all intermediate fluctuator states. For $N = 1$ no intermediate intervals are involved, so that $G_{ss'}^{ss'} = G_{ss}^{ss} = W_{ss'}T_{s'}$. Here $T_{s'}$ is the transfer matrix in the starting state $s'$, and $W_{ss'}$ signifies the conditional probability for the change to $s$ immediately after the end intervals. By definition, we find that

$$G_{ss'}^{ss'} = \sum_{s''} \Gamma_{ss''} G_{s''-1}^{ss'},$$

which is already in the form of a matrix product, and by iteration we see that $G_N = \Gamma^N$, where we defined an operator

$$\Gamma = (1 - p - \delta \tau_3 + p \tau_1 - i\delta \tau_2) \times \exp(i\Delta t B_0 \cdot \vec{L} + i\Delta t \vec{g} \cdot \vec{L}\tau_3),$$  

(3)
The eigenvectors, and finally the transfer matrix $T$ also be easily obtained. We shall not go into these details.

$$T = \sum_{s,s'} G_{ss'}^{s'} = \langle x_f | \Gamma^N | i_f \rangle,$$

where $|x_f\rangle = \frac{1}{\sqrt{2}} \sum_{s=\pm} |s\rangle$ is formally one of the eigenstates of $\tau$, and $|i_f\rangle = \sqrt{2} \sum_{s=\pm} p_s |s\rangle$ encodes the initial stationary level distribution of the fluctuator. Note that the formal inner product is performed in the fluctuator level space, leaving a $3 \times 3$ matrix in the cases treated here.

In the limit of $\Delta t \to 0$,

$$\Gamma \to 1 + i\Delta t (B_0 L_z + g \cdot L \tau_3) - tU \rightarrow 1 + i\Delta t \gamma + \Delta t (\gamma \tau_1 - i\eta \tau_2 - \eta \tau_3) \sim \exp(-\Delta t P),$$

where we define

$$P = \gamma - iB_0 L_z - ig \cdot L \tau_3 - \gamma \tau_1 + i\eta \tau_2 + \eta \tau_3.$$ (6)

Thus in the continuum time limit $G = \Gamma^N = \exp(-tP)$ and so

$$T = \langle x_f | \exp(-tP) | i_f \rangle$$ (7)

for $t = N \Delta t$, and the problem reduces to the diagonalization of $P$ which can be cast into a $6 \times 6$ matrix. Assuming that $P$ is not defective (an assumption we have checked in the cases treated here), $T$ can be decomposed as

$$T = \sum_\lambda \langle x_f | \lambda \rangle \langle \lambda | i_f \rangle \exp(-\lambda t),$$ (8)

where $|\lambda\rangle$ and $\langle \lambda |$ are the right and left eigen vectors of $P$ with the eigenvalue $\lambda$, normalized such that $\langle \lambda | \lambda' \rangle = \delta_{\lambda \lambda'}$. (Notice again the partial inner products with $|i_f\rangle$ and $|x_f\rangle$.) Each eigenvalue corresponds to a relaxation time of the system. Typically, the shorter times correspond to transients and the longest two times correspond to $T_1$ and $T_2$ - this can be verified by examining $n(t)$ in detail.

Now let $\theta$ be the angle between $g$ and the $z$–axis. For $\theta = 0$, $g = g^z$, $L_z$ is conserved by $P$, so that we can use the quantum numbers $m = 0, \pm 1$ in place of $L_z$, and diagonalize the $2 \times 2$ matrix in the fluctuator spin space. The eigenvalues $\lambda$ are given by

$$\lambda = \gamma - iB_0 m \pm \sqrt{\gamma^2 - g^2 m^2 - 2i\eta m}, \quad m = 0, \pm 1.$$ (9)

The eigenvectors, and finally the transfer matrix $T$ can also be easily obtained. We shall not go into these details here. By inspection of $\text{Re}(\lambda)$, we see that the relaxation rate $1/T_1 = 0$ in the $n_x - y$ channel. The result agrees with the results in Ref. [9].

The other case for which compact explicit expressions can be given is $\theta = \pi/2$ (so that $g$ is perpendicular to $B_0$) and at the same time the switching rate imbalance $\eta = 0$. For simplicity let us assume that $g$ is in $x$–direction and $B_0$ is still in the $z$-direction. Upon a rotation in the level space, we have $U \tau_1 U^\dagger = \tau_3$ and $U \tau_3 U^\dagger = -\tau_1$, where $U$ is the SU(2) rotation about the $y$-axis by 90 degrees (in the level spin space). Under this transformation

$$U P U^\dagger = \gamma - \gamma \tau_3 - iB_0 L_z + ig L_z \tau_1.$$ (10)

This matrix is the same as that of a Hamiltonian for the coupling of a spin-1 particle with angular momentum $L$ and a spin-1/2 particle with angular momentum $S = \tau/2$. Inspection reveals that the operator only mixes states whose $z$-components of $L + S$ differ by 2. (This is seen by writing $L_x = (L_+ + L_-)/2$ and $S_x = \tau_1/2 = (S^+ + S^-)/2$). As the result, the Hilbert space is divided into two invariant subspaces, spanned respectively by the two sets of states

$$\{|-1,-1/2), |1,-1/2), |0,1/2\rangle\},$$

$$\{|-1,1/2), |0,-1/2), |1,1/2\rangle\}.$$ (11)

Here in each basis state the first index refers to $L_z$, and the second one to that of $S_z = \tau_3/2$. The diagonalization can be carried out separately in the two subspaces, and only $3 \times 3$ matrices are involved. The $T$-matrix is formally given by $T = \langle x_f | \exp(-tU P U^\dagger) | z_f \rangle$, where $|z_f\rangle = (1,0)^T$. Here we used the facts that $|i_f\rangle = |x_f\rangle$ for $\eta = 0$ and that $U |x_f\rangle = |z_f\rangle$. Without going into details we mention that the eigenvalues of $U P U^\dagger$ (and thus of $P$) satisfy one of the two equations below:

$$\lambda^3 + 2\gamma \lambda^2 + (B_0^2 + g^2) \lambda + 2B_0^2 \gamma = 0,$$

$$\lambda^3 + 4\gamma \lambda^2 + (B_0^2 + g^2 + 4\gamma^2) \lambda + 2g^2 \gamma = 0.$$ (12)

The result can also be obtained from the stochastic differential equation approach [10].

For general $\theta$, the analysis (in terms of phase memory function) in the literature is limited so far to the so-called linear-coupling regime $g \ll B_0$ [13]. Our model is however exact for any coupling strength. All that we have to do is to diagonalize a $6 \times 6$ matrix to obtain the full qubit dynamics. In Fig.2 we plot the evolution of the qubit vector $n(t)$ in the rotating frame defined by $B_0$. The parameters are $B_0 = 1, \gamma = 0.1, g = 0.3, \eta = 0$ and $\theta = \pi/4$. We see from Fig.2(a) that starting from $n(0) = \hat{x}$, $n_x$ decays with oscillations even in the rotating frame. On the other hand the z-component is induced in the intermediate stage and actually decays more slowly than the x-component. We checked that this feature is visible in the strong coupling regime ($\gamma < g$), but is much weaker in the weak coupling regime ($\gamma > g$), and is completely absent in the case of $\gamma \gg g$, the limit of white
is available from Redfield theory, which applies only to the intermediate stages, in agreement with the behavior of decays slower than the x- and y-components induced in relations. In Fig.2(b) the qubit starts from noise. This signifies the unique role of long time correlation of the rates is used to improve clarity. The perturbative formula breaks down completely for strong coupling. FIG. 2: (Color online) The evolution of the qubit vector \( \mathbf{n} = (n_x, n_y, n_z) \) in the rotating frame, starting from (a) \( \mathbf{n}(0) = \hat{z} \), and (b) \( \mathbf{n}(0) = \hat{z} \). The parameters are \( B_0 = 1, \gamma = 0.1, g = 0.3, \eta = 0, \) and \( \theta = \pi/4 \). The time \( t \) is in units of \( 1/B_0 \). The same legend for the curves is used in (a) and (b).

One should be aware that none of this qubit dynamics is available from Redfield theory, which applies only to the regime \( \gamma t \gg 1 \).

We can decide precisely the asymptotic decay rates in the different channels by matching \( n_{x,y,z}(t) \) with the envelope curves \( \exp[-\text{Re}(\lambda) t] \), where \( \lambda \) are the numerical eigenvalues of \( P \). The resulting decay rates \( 1/T_1 \) (in the \( n_z \)-channel) and \( 1/T_2 \) (in the \( n_{x,y} \)-channel) as a function of \( \theta \) are plotted in Fig.3. First consider the case of \( \eta = 0 \) (solid lines). Fig.3(a) is in the weak-coupling regime, where we see a smooth change and a crossing of the two rates as \( \theta \) increases. This is similar to the case of uncorrelated noise[15]. The reason is that for \( g < \gamma \), in a time scale determined by \( 1/g \) many switches occur and as such the noise is essentially uncorrelated beyond a time scale of \( 1/g \).

FIG. 3: (Color online) The relaxation rate \( 1/T_1 \) and dephasing rate \( 1/T_2 \) (in units of \( B_0 \)) as functions of the angle \( \theta \) between the static field \( B_0 \) and the noise field \( g \). \( 1/T_2 \) (triangle points) is \( 1/T_2 \) calculated from the perturbative formula in Eq. 11. The weak coupling case (a) has the parameters \( \gamma = 0.5 \) and \( g = 0.1 \). Here \( \eta = 0 \) (0.1) for solid (dashed) lines, while the strong coupling case (b) uses \( \gamma = 0.1 \) and \( g = 0.3 \). Here \( \eta = 0 \) (0.05) for solid (dashed) lines. Rescaling of the rates is used to improve clarity. The perturbative formula breaks down completely for strong coupling.

noise. This signifies the unique role of long time correlations. In Fig.2(b) the qubit starts from \( \mathbf{n} = \hat{z} \) and decays slower than the x- and y-components induced in the intermediate stages, in agreement with the behavior in Fig.2(a).

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\[
1/T_2 = 1/T_1 + 1/T_1^* \quad \text{(11)}
\]

with

\[
1/T_1 = \cos^2 \theta S(0)/2, \quad \text{(12)}
\]

\[
1/T_1^* = \sin^2 \theta S(\Omega)/2. \quad \text{(13)}
\]

Here \( 1/T_3 \) is the dephasing rate as if in an effective z-direction random field with amplitude \( B_0 \cos \theta \) alone, \( S(\omega) \) is the power spectrum of telegraph noise and \( \Omega = B_0 \). In view of the wide use of this formula, it is important to check how well it holds for general coupling strengths. The comparison is made in Fig.3 for \( \eta = 0 \) only, where \( 1/T_2 \) calculated according to the above formula is denoted by \( 1/T_2^* \) (triangles) and thus can be compared to the actual \( 1/T_2 \). We see that in the weak (strong) coupling case of Fig.3(a) [Fig.3(b)] it agrees perfectly with (deviates considerably from) our exact result.
(solid lines). The formula cannot be used when the coupling of the source to the qubit is large compared to the inverse correlation time of the noise. This illustrates the importance of our exact results in the strong coupling cases.

III. DYNAMICAL DECOUPLING BY BANG-BANG CONTROL

The formalism can be easily adapted to echo decay measurements and the bang-bang control protocol where control pulses are applied. In such processes the transfer matrix can be formally written as

\[ T = \langle f | \exp\left[ -\int_0^t dt' P + i \sum_i \phi(t_i) \cdot \mathbf{L} \right] | i_f \rangle, \]

where \( \hat{T} \) time-orders the operators, encoding the instantaneous rotations with vectors \( \{ \phi(t_i) \} \) caused by pulses at time \( \{ t_i \} \). These manipulations are important for comparison with experiments such as that in Ref.

Let us consider an open loop quantum control (or bang-bang control), in which a sequences of \( \pi_x \) or \( \pi_y \) pulses with fixed intervals are applied to reduce the decoherence due to slowly low frequency noise. In the ideal case, we assume that each pulse is of zero width in time. The problem of ideal dynamical decoupling by bang-bang control has been exactly solved in the special cases of pure dephasing noise (\( \theta = 0 \)) and pure transverse noise (\( \theta = \pi/2 \)) [11]. With our method we can get the exact solution at arbitrary working points.

In some cases \( \pi_x \) pulses would cause the anti-Zeno effect in experiment[17], so that we consider \( \pi_y \) pulses. After a \( \pi_y \) pulse is applied, the Bloch vector is rotated by \( \pi \) about the \( y \)-axis: \( \mathbf{n} \rightarrow \exp(i\pi L_y) \mathbf{n} \). For a periodic sequence of \( \pi_y \) pulses, the transfer matrix is explicitly given by

\[ T(N\tau) = \langle f | \exp(-P\tau) \exp(i\pi L_y)^N | i_f \rangle, \]

where \( \tau \) is the interval between two adjacent pulses, \( N \) is the number of pulses applied. By diagonalizing the operator (again a \( 6 \times 6 \) matrix) \( \exp(-P\tau) \exp(i\pi L_y) \), we get the eigenvalues \( \lambda_1=\ldots,6 \) and the candidate decay rates \( \Gamma_i = -\ln|\lambda_i|/\tau \). In principle only one of the six \( \Gamma \)’s control the long time asymptotic behavior in a specific channel. This rate can be decided precisely, as we used in Sec.II, by matching \( n_{x,y}(t) \) with the envelope curve \( \exp(-\Gamma_i t) \).

In Figs.4 we plot the normalized decay rates \( T_1/T_{1d} \) (in the \( n_x \)-channel) and \( T_2/T_{2d} \) (in the \( n_{x,y} \)-channel) as functions of the interval \( \tau \) between pulses. \( 1/T_1 \) and \( 1/T_2 \) are the rates in the absence of the pulses. A normalized rate of 1 therefore corresponds to no suppression of decoherence by the pulses. We set \( B_0 = 1, \eta = 0, \theta = \pi/4, \) and \( \gamma = 0.1 \). In addition, \( g = 0.03 \) and \( g = 3 \) in Fig.4(a) and (b), respectively. We observe that as the interval \( \tau \) between \( \pi_y \) pulses decreases, the decay rates of both \( n_x \)-channel and \( n_{x,y} \)-channel decrease. The reduction is significant as soon as \( \tau \approx 1/g \). Moreover, in Fig.4(b) where \( \theta = \pi/4 \) we observe oscillatory behavior of the relative rates similar to the case of \( \theta = \pi/2 \) studied elsewhere.[11]

IV. SPIN-ECHO DECAY

In a spin-echo process, a \( \pi_x \) pulse is applied half-way between two \( \pi_x/2 \) pulses. The first \( \pi_x/2 \) pulse rotate the initial qubit state \( n_z \) to \( n_y \), the second \( \pi_x/2 \) rotate \( n_y \) back to \( n_z \) right before the measurement. The \( \pi_x \) pulse reverses the \( n_y \)-component, reducing the line-broadening associated with low frequency noise in the final measurement. The transfer matrix in the combined process is easily shown to be given by

\[ T(t) = \langle f | \exp(i\pi L_x/2) \exp(-tP/2) \exp(i\pi L_x) \times \exp(-tP/2) \exp(i\pi L_x/2) | i_f \rangle, \]

where \( t \) is the time interval between the two \( \pi_x/2 \) pulses.

In Fig.5 we plot the time dependence of the echo signal for the parameters \( g = 0.8 \) (strong coupling case), \( \gamma = 0.1 \), and (a) \( \theta = 0 \), (b) \( \theta = \pi/4 \) and (c) \( \theta = \pi/2 \). We can see that the echo signal shows steps. The steps in the special case of \( \theta = 0 \) have been observed experimentally,[18] as has been pointed out in Ref.[4]. Here we point out that steps should occur for all \( \theta \), and the period of the steps depends sensitively on \( \theta \). Indeed,
θ behavior for any values of \((\text{weak coupling case})\). In this case no steps are ob-
cantly. This indicates that a many-fluctuator model (plus

can be extended to the case of combined noise sources: a white-noise field \(h\) and a two-level fluctuator. We assume that in a time
sequence \(h\) does not change within an interval \(\Delta t\), but
is uncorrelated between different intervals. This enables
us to do the ensemble average over \(h\) within each time
interval, where we assume \((h_i h_j) = (\nu_i / \Delta t) \delta_{ij}\), with \(i =
x, y, z\). The effect of \(h\) enters \(\Gamma\) as

\[
\Gamma = (1 - p - \delta \tau_3 + p \tau_1 - i \delta \tau_2) \times 
\exp[i \Delta t(B_0 \cdot h \cdot L + i \Delta t g \cdot L \tau_3)].
\tag{16}
\]

Expanding the exponential function (up to the second
order of \(h\)) and performing the ensemble average over \(h\),
and finally taking the limit \(\Delta t \to 0\) (recalling that
\(p = \gamma \Delta t\) and \(\delta = \eta \Delta t\)) we again obtain \(\Gamma = \exp(-\Delta t P)\)
with a modified operator

\[
P = \gamma + \sum \nu_i L_i^2 / 2 - iB_0 L_z
-i g \cdot L \tau_3 - \gamma \tau_1 + i \eta \tau_2 + \eta \tau_3.
\tag{17}
\]

We assume that the contributions from higher or-
der order moments of \(h\) are of higher orders in \(\Delta t\) and can be
ignored. As such the result is independent of the concrete
form of the distribution function for \(h\).

The theory can also be generalized to many indepen-
dent fluctuators described by the parameters \(\{g_n, \gamma_n, \eta_n\}\)
\((n = 1, ..., N)\). We assume that after each small time in-
terval at most one of the fluctuators could switch (joint
switches occur at higher orders of \(\Delta t\)). Since the fluctu-
ators are independent, it is straightforward to see that the
desired \(P\) operator is now given by

\[
P = \sum_n \left(\gamma_n - i g_n \cdot L \tau_3 - \gamma_n \tau_1 + i \eta_n \tau_2 + \eta_n \tau_3\right)
- iB_0 L_z,
\tag{18}
\]

where \(\tau_n\) denotes the \(i\)-th Pauli matrix operating on the
\(n\)-th fluctuator spin. This is a system with a spin-1
coupled to many independent spin-1/2’s. The problem
is still solvable algebraically provided that \([L_z, P] = 0\).
In general, \(P\) is a \((3 \times 2^N) \times (3 \times 2^N)\) matrix. The
final \(3 \times 3\) T-matrix according to which the Bloch vector
evolves is formally given by \(T = \langle x_f | \exp(-tP) | i_f \rangle\) with
\(|x_f\rangle = \Pi_n |x_{f_n}\rangle\) and \(|i_f\rangle = \Pi_n |i_{f_n}\rangle\), where \(|x_{f_n}\rangle\) and \(|i_{f_n}\rangle\)
describe the states of the \(n\)-th fluctuator.

Finally the method can be generalized to the case of
many qubits. Let \(D\) be the dimension of the Hilbert space
of the qubits. Then there are \(D^2 - 1\) Hermitian generators
\(X_i\) of the transformation group \(SU(D)\) instead of the
three Pauli matrices used in the single qubit case where

![FIG. 5: The time dependence of echo signals for \(g = 0.8,\n\gamma = 0.1\) and (a) \(\theta = 0\) (pure dephasing), (b) \(\theta = \pi/4\) and
(c) \(\theta = \pi/2\).](image)

![FIG. 6: The time dependence of echo signal for \(g = 0.08\) and
\(\gamma = 0.1\). (a) gives the case \(\theta = 0\), (b) is the case \(\theta = \pi/4\) and
(c) is the case \(\theta = \pi/2\), which corresponds to the experiment
in Ref. [12].](image)
\( D = 2 \). The \( D \times D \) density matrix may be expanded as a real linear combination of the \( X_i \) as in Eq.(2):

\[
\rho(t) = I/D + \sum_{i=1}^{D^2-1} a_i(t) X_i.
\]

The \((D^2 - 1) \times (D^2 - 1)\) matrix \( T_2 \) that evolves the vector \( a(t) \) forward in time when the joint state \( s \) of the \( N \) two-level fluctuators are given is now easily computed, and the steps in deriving the transfer matrix are formally identical to that in Sec.II. We see that the corresponding operator \( P \) will be a \((2^N(D^2 - 1)) \times (2^N(D^2 - 1))\) matrix, and its eigenvalues determine the \((D^2 - 1)\) relaxation times in the system, one corresponding to each possible observable. The rapidly increasing dimension of the superoperator space will begin to give difficulties for numerical calculations even at moderate values of \( N \) and \( D \).

VI. SUMMARY

We have developed an exact transfer matrix method to solve the problem of qubit decoherence caused by one fluctuator, and have applied the theory to free qubit decay, bang-bang control and spin-echo decay. We have reproduced all known exact results and have shown that the perturbative limits are correct. The method is relatively straightforward to apply, since it is completely algebraic, and the qubit decoherence rates are determined by the eigenvalues of a transfer matrix.

The formalism is also more powerful than previous exact methods in that it can be applied to a qubit subject to an arbitrary combination of dephasing and relaxational telegraph noise. The typical way of combining the two types of noise is according to the perturbative formula

\[ 1/T = 1/2T_1 + 1/T_\phi. \]

We have shown that this formula breaks down when the coupling is strong. The algebraic method can get qubit dynamics on time scales short compared with the environmental correlation times, which is often of experimental interest.

The method can be generalized to the case of many noise sources and multiple qubits, though the size of the matrices grows rapidly.

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