A rigidity result for extensions of braided tensor $C^*$-categories derived from compact matrix quantum groups

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Dedicated to Sergio Doplicher on the occasion of his seventieth birthday

Abstract

Let $G$ be a classical compact Lie group and $G_\mu$ the associated compact matrix quantum group deformed by a positive parameter $\mu$ (or $\mu \in \mathbb{R} \setminus \{0\}$ in the type $A$ case). It is well known that the category of unitary representations of $G_\mu$ is a braided tensor $C^*$-category. We show that any braided tensor $C^*$-functor $\rho : \text{Rep}(G_\mu) \to \mathcal{M}$ to another braided tensor $C^*$-category with irreducible tensor unit is full if $|\mu| \neq 1$. In particular, the functor of restriction $\text{Rep}(G_\mu) \to \text{Rep}(K)$ to a proper compact quantum subgroup $K$ cannot be made into a braided functor. Our result also shows that the Temperley–Lieb category $\mathcal{T}_{\pm d}$ for $d > 2$ cannot be embedded properly into a larger category with the same objects as a braided tensor $C^*$-subcategory.

1 Introduction

There are various strategies for studying the structure of or classifying semisimple rigid tensor categories. As an oversimplification, sometimes one focuses on the structure of simple objects. A basic result is Ocneanu’s rigidity, see [11] for a proof, asserting that there are finitely many $\mathbb{C}$-linear fusion categories with a prescribed fusion ring. Important results have been obtained for categories that may or may not have finitely many irreducibles, or a braiding, see [23, 26, 29, 30, 38, 39, 11], however this is an incomplete list.

Another approach, when there is a relevant subcategory with few arrows, is to try and construct the whole category $\mathcal{M}$ from a smaller subcategory $\mathcal{A}$. This subcategory is regarded as a symmetry of $\mathcal{M}$, and the classification problem becomes that of classifying extensions $\mathcal{M}$ with the given symmetry.

The first example, motivated by AQFT [9] (see also [15]), is that of permutation symmetry. For tensor $C^*$-categories with conjugates (and subobjects, direct sums and irreducible tensor unit) realizations of this symmetry are few, they are classified by a single integer parameter and a sign, the statistics phase. When all
the statistics phases are one, $M \simeq \text{Rep}(G)$ with its natural permutation symmetry, for a unique compact group $G$ \cite{10}. A related independent result is also well known \cite{8}.

The theory of subfactors \cite{19} or low dim AQFT \cite{12}, although differing, provide further remarkable instances of this general scheme. In the first case the Jones projections play a fundamental role, leading to the Temperley–Lieb symmetry; whereas the Virasoro algebra symmetry with central charge $< 1$ plays a major role in CFT on the circle. Deep classification results have been obtained in both these areas \cite{13, 17, 34, 2, 22}.

For the categories arising from low dimensional QFT or from certain quantum groups one has braid group symmetries. The main difference from the permutation symmetric case is the great variety of realizations in tensor categories.

In this paper we start with braided categories $\mathcal{A}$ arising from quantum groups where the braiding comes from an $R$–matrix. The matrix carries more information on the quantum group than in the group case, where most of the information gets lost, so that, for example, for any proper closed subgroup $K$ of a group $G$ the restriction functor $\text{Rep}(G) \rightarrow \text{Rep}(K)$ is a not full permutation symmetric tensor $\ast$–functor $\text{Rep}(G) \rightarrow \text{Rep}(K)$.

The $R$–matrix depends on the structural constants of the quantum group, making it an object intimately related to the quantum group and raising the question of whether the class of tensor categories with that specific braided symmetry has a rigid structure.

A recent result in \cite{6} for the categories arising from CFT on the circle seems to confirm a certain rigidity when constructing inclusions of unitarily braided, but not permutation symmetric, tensor categories.

More precisely, the authors, working with inclusions of conformal nets on the circle, where unitary braided symmetries arise, have shown that far fewer low values of the Jones index of the inclusion can occur than in subfactor situation. They exclude all non-integer index values $< 3 + \sqrt{3}$ (hence in particular all non-integer index values in the Jones discrete series) except one, $4 \cos^2 \pi/10$, which is realized \cite{35}. The integer values are known to be realized from inclusions derived by taking fixed points under group actions.

$\alpha$–induction relates the tensor category associated with a net $\mathcal{A}$ to that associated with an extension $\mathcal{B}$, see \cite{22} and references there. We also mention the work of \cite{27} for a categorical relation corresponding to inclusions of nets.

The aim of this paper is to derive a rigidity result for tensor categories with a braided symmetry derived from quantum groups of all Lie types but not at a root of unity. Note that in this case the $R$–matrix carries maximal information on the quantum group.

We shall work with tensor $C^*$–categories, $\mathcal{A}$ will be the category of unitary representations of any deformation $G_\mu$ of the classical compact Lie groups $G$ by a real parameter $\mu$.

It is known that the $R$–matrix for such quantum groups makes the associated representation category into a braided tensor $C^*$–category with conjugates, the analogues of duals. Except for the extreme values of $\mu$, the coinverse of $G_\mu$ is not involutive and the braiding not unitary and we show that every braided tensor $\ast$–
functor $\rho : \text{Rep}(G_\mu) \to M$ to a braided tensor $C^*$-category is full (see Theorem 3.1 for a more general statement). We derive this result from Theorem 5.4, showing an obstruction to construct extensions of braided tensor $C^*$-categories.

Note that, in our class of examples, $A$ has infinitely many irreducibles whose indices (or dimensions) can be arbitrarily large.

The category $M$ is not assumed to be embeddable into the category of Hilbert spaces. However, when it is, our result shows that the braiding of $\text{Rep}(G_\mu)$ does not extend to any $\text{Rep}K$, where $K$ is a proper quantum subgroup.

This corollary does not, to our knowledge, seem to have been noticed in the literature. It shows how the results are discontinuous in the classical limit $\mu = 1$.

Another application concerns Temperley–Lieb categories. It states that when this category is generated by an object of intrinsic dimension $d > 2$, (i.e. not in the discrete series of Jones), it cannot be embedded properly as a braided tensor $C^*$-subcategory into a larger such category with the same objects. We recall here the situation for subfactors $N \subset M$ with finite index where the associated $N$–$N$–bimodule tensor category contains a copy of the TL category as a (non-braided) tensor $C^*$-category.

The relation to our result can be seen by regarding the TL category $\mathcal{T}_{\pm d}$ as a full braided tensor subcategory of the representation category of $S_\mu U(2)$ for a deformation parameter determined by $|\mu + \mu^{-1}| = d$, and $\mu$ is either positive or negative corresponding to the minus or plus sign.

Our result relies on the theory of induction of [33]. We showed there that if a tensor $C^*$-category with conjugates $A$ admits a tensorial embedding $\tau$ into the category of Hilbert spaces, and hence, by Woronowicz duality, is a full tensor $C^*$-subcategory of the representation category of a compact quantum group $G_\tau$, then given an embedding of $\rho : A \to M$ of tensor $C^*$-categories, the full subcategory of $M$, whose objects are in the image of $\rho$, can be identified with a category of Hilbert bimodule representations of $G_\tau$ for a noncommutative $G_\tau$–ergodic $C^*$-algebra $(\mathcal{C},\alpha)$ intrinsically associated with $\rho$ and $\tau$. $(\mathcal{C},\alpha)$ is to be regarded as a virtual subgroup. For a fixed $\tau$, isomorphic pairs $(M,\rho), (M',\rho')$ yield conjugate ergodic actions.

We show that, taking braided symmetries into account, there is an obstruction to constructing such non-trivial ergodic actions $(\mathcal{C},\alpha)$. More precisely, our main result states that the invariant $\kappa$ of $A$ must be a phase on irreducible spectral representations for the action $\alpha$ corresponding to $\rho : A \to M$ (Theorem 5.4). We are thus left to check that $\kappa(v)$ is never a phase for our quantum group $G_\mu$, unless $v$ is equivalent to the trivial representation (Theorem 3.1).

The invariant $\kappa$ appeared in its simplest form for categories derived from AQFT. It is just the statistics phase recalled above for permutation symmetric tensor $C^*$-categories [9]. In two spacetime dimensions the braid group replaces the permutation group. Unitarity of the braiding means that $\kappa$ is a phase on irreducible objects but it need not be a sign [12]. For general braided tensor $C^*$-categories, $\kappa$ has been discussed in [25].

A closely related invariant was independently introduced for certain braided tensor categories, the ribbon categories [36], and referred to as the twist, $\theta$. Its importance derives from the fact that ribbon categories associated with certain
quantum groups at roots of unity lead to the Reshetikhin–Turaev invariants of 3–manifolds and links in 3–manifolds.

Note that for quantum groups with unitary braided symmetries, \( \kappa \) is indeed always a phase, and our obstruction vanishes. In this case, braided extensions can be expected.

We describe the braided symmetries of \( \text{Rep}(S_\mu U(d)) \) got by twisting the universal \( R \)-matrix by \( d \)-th roots of unity. The invariant \( \kappa \) can be used to show that these braidings provide inequivalent braided tensor categories.

We finally mention that reconsidering the construction of these braidings, led us to compare explicitly the intrinsic characterizations of the quantum \( SU(d) \), when not at roots of unity, given by [23] and [31]. Whilst [23] starts from the fusion rules, [31] relies on the braiding. The twist \( \tau_C \) may be derived following the approach of [31], leading to a characterization of the tensor categories with fusion rules of type \( sl(d) \) in the spirit of [23], in terms of generators and relations analogous to [31], but now involving a \( d \)-th root of unity, see Prop. 7.2 and Cor. 7.3.

The paper is organized as follows. In Sect. 2 we recall the notion of a braided tensor \( C^* \)-category. We shall propose a variant weaker than the usual one, but sufficient for our main result, announced in Sect. 3. To this end, we recall the natural structure of \( \text{Rep}G_\mu \) as a braided tensor \( C^* \)-category. We shall in particular emphasize the case of \( S_\mu U(d) \) as it allows one to grasp the general idea of the proof quickly. In Sect. 4 we derive the main properties of the invariant \( \kappa \) of a braided tensor \( C^* \)-category with conjugates and compare it with the twist of a ribbon tensor category. In Sect. 5 we show that when the invariant \( \kappa \) of a braided tensor \( C^* \)-category \( A \) is not unitary it is an obstruction to construct braided extensions. In Sect. 6 we complete the proof by showing that \( \kappa(v) \) is never a phase for an irreducible representation \( v \) of the quantum groups \( G_\mu \), unless \( v \) is the trivial representation. In the last section we reconsider the results of [23] and [31].

2 Braided tensor \( C^* \)-categories

In this section we introduce various notions of braiding, from the weakest to the strongest. These will allow us to isolate that aspect to our main result.

The categories we shall consider will always be assumed to be strict, to be over the complex numbers, with irreducible tensor unit \( \iota \). We shall also assume existence of subobjects and direct sums, unless otherwise stated, but see the remarks at the beginning of Sect. 4. In this paper we shall mostly deal with tensor \( C^* \)-categories (or unitary categories).

The weakest notion requires that for any pair of objects \( u, v \) of a tensor \( C^* \)-category \( A \) there is an invertible intertwiner \( \sigma(u,v) \in (u \otimes v, v \otimes u) \) such that for arrows \( T \) belonging to spaces of the form \( (u, \iota) \),

\[
1_v \otimes T \circ \sigma(u,v) = T \otimes 1_v.
\] (2.1)

Note that the dual intertwiners \( \sigma_d(u,v) := \sigma(u,v)^{-1} \in (u \otimes v, v \otimes u) \) do not satisfy (2.1). If both \( \sigma \) and \( \sigma_d \) satisfy (2.1), \( \sigma \) will be referred to as a weak left...
braided symmetry. Notice that a weak left braided symmetry by itself, may not even be related to the braid group as we are not assuming properties (2.3) and (2.4) below. However, the relation to the braid group will follow automatically for the weak braided symmetries of interest in this paper (cf. the end of the section).

A stronger notion is that of a left braided symmetry. One requires the following relations for objects $u$, $u'$, $v$, and arrows $T \in (u, u')$,

$\sigma(u, u) = \sigma(u, u') = 1_u$, \hspace{1cm} (2.2)  

$\sigma(u \otimes u', v) = \sigma(u, v) \otimes 1_{u'} \circ \sigma(u') \otimes \sigma(u', v)$, \hspace{1cm} (2.3)  

$\sigma(u', v) \circ T \otimes 1_v = 1_v \otimes T \circ \sigma(u, v)$. \hspace{1cm} (2.4)  

Indeed, the notion of a weak left braided symmetry is just a special case of the naturality property (2.4) and (2.2) for $u$ or $u'$ the trivial object $i$.

A weak right braided symmetry is defined by the following equation for $T \in (u, i)$,

$1_v \otimes T = T \otimes 1_v \circ \sigma(v, u) = T \otimes 1_v \circ \sigma_d(v, u)$,

and right braided symmetry is defined replacing (2.3) and (2.4) by

$\sigma(u, v \otimes v') = 1_v \otimes \sigma(u, v') \circ \sigma(u, v) \otimes 1_{v'}$, \hspace{1cm} (2.3)'  

$\sigma(v, u') \circ 1_v \otimes T = T \otimes 1_v \circ \sigma(v, u)$. \hspace{1cm} (2.4)'  

Obviously, $\sigma$ is a (weak) right braided symmetry if and only if $\sigma^{-1}(v, u) := \sigma(v, u)^{-1}$ or $\sigma_{\ast}(u, v) := \sigma(v, u)^{\ast}$ are (weak) left braided symmetries.

We thus recover the usual notion of a braided symmetry considering a left and right braided symmetry. A (weak) half braided symmetry will just mean a (weak) left or right braided symmetry. Given (weak) half braided symmetry $\sigma$ in $\mathcal{A}$, the dual symmetry $\sigma_d(u, v)$ is another braided symmetry for $\mathcal{A}$ of the same type.

The representation categories of $S_\mu U(d)$, or, more generally, of $G_\mu$, where $G$ is a classical compact Lie group, or of $A_\mu(F)$ are well known examples of braided tensor $C^*$-categories. These will in fact be our main examples. They will be discussed in later sections.

If $\sigma$ satisfies (2.1) or is a weak half or half braided symmetry for a tensor $C^*$-category $\mathcal{A}$ and $\tau : \mathcal{A} \rightarrow \text{Hilb}$ is a tensor $\ast$-functor into the category of Hilbert spaces then the representation category of the compact quantum group $G^\tau$ associated to $\tau$ via Woronowicz duality has a braiding $\sigma_\tau$ of the same type defined by $\sigma_\tau(u, v) := \tau(\sigma(u, v))$, where $\hat{u}$ is the representation of $G^\tau$ corresponding to the object $u$ of $\mathcal{A}$.

It is easy to show that the category of Hilbert spaces has a unique braiding, even in the sense of (2.1), given by its unique permutation symmetry. If $(\mathcal{A}, \sigma)$ and $(\mathcal{M}, \varepsilon)$ are tensor $C^*$-categories with some type of braiding, a tensor $\ast$-functor $F : \mathcal{A} \rightarrow \mathcal{M}$ will be called braided if for any pair of objects $u, v$ in $\mathcal{A}$,

$F(\sigma(u, v)) = \varepsilon(F(u), F(v))$.

We shall consider cases where $\mathcal{A}$ has a stronger braided symmetry than $\mathcal{M}$. Note that, if for example properties (2.2), (2.3) hold for the braiding of $\mathcal{A}$ and $F$ is
a braided tensor *–functor to a weak half braided M, then those properties also hold for the braided symmetry of M for objects in the image of F. However, the naturality axioms (2.4) or (2.4)’ are not inherited by the full subcategory of M whose objects are in the image of F.

3 The rigidity result

Tensor categories derived from quantum groups arising from a deformation of classical Lie groups admit natural braided symmetries associated with the universal $R$–matrices of Drinfeld, see, e.g. [7, 24]. We work with compact matrix quantum groups. We start by describing briefly the corresponding braided symmetry for the type $A_{d-1}$ in this framework. After this, we recall how to construct braided tensor $C^*$–categories for the other Lie types, before illustrating our main result and its corollaries.

Woronowicz has introduced the compact quantum group $S_{\mu}U(d)$, for a nonzero $\mu \in [-1, 1]$, using his duality theorem [42]. It is the (maximal) compact quantum group whose representation category $\text{Rep}(S_{\mu}U(d))$ is generated, as an embedded tensor $C^*$–category, by the deformed determinant element

$$S = \sum_{p \in \mathbb{F}_d} (-\mu)^{V(p)} \psi_p(1) \otimes \cdots \otimes \psi_p(d),$$

where $\psi_i$ is an orthonormal basis of a $d$–dimensional Hilbert space.

For a nonzero complex parameter $q$, let $H_n(q)$ denote the Hecke algebra of type $A_{n-1}$, i.e. the quotient of the complex group algebra of the braid group $B_n$ with generators $g_1, \ldots g_{n-1}$ by the relations

$$g_i^2 = (1-q)g_i + q.$$  

For $q$ real or $|q| = 1$, $H_n(q)$ becomes a *–algebra with involution making the spectral idempotents of the $g_i$ into selfadjoint projections. However, there are nontrivial Hilbert space representations for $n$ arbitrarily large, only if $q > 0$ or $q = e^{\pm 2\pi i/\ell}$ [40]. In this paper we only consider the real case.

There is a well known remarkable representation of the Hecke algebra $H_\infty(q)$, with $q = \mu^2$, in $\text{Rep}(S_\mu U(d))$, due to Jimbo and Woronowicz [18, 42], and defined by

$$\eta(g_1)\psi_i \otimes \psi_j = \mu \psi_j \otimes \psi_i, \quad i < j,$$

$$\eta(g_1)\psi_i \otimes \psi_i = \psi_i \otimes \psi_i,$$

$$\eta(g_1)\psi_i \otimes \psi_j = \mu \psi_j \otimes \psi_i + (1-q)\psi_i \otimes \psi_j, \quad i > j.$$  

$\text{Rep}(S_\mu U(d))$ admits an intrinsic characterization [23], [31]. We follow the approach of [31] (based on the type of the braiding) and compare with that of [23] (based on the type of the fusion rules) in the last section.

$\text{Rep}(S_\mu U(d))$ is, up to equivalence, the unique tensor $C^*$–category $\mathcal{A}$ (with subobjects and direct sums) with a tensor *–functor from the braid category $\eta$:
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\( \mathbb{B} \rightarrow A \) factoring through representations of the complex Hecke algebras of type A, \( H_n(q) \) for \( q = \mu^2 \) and generated by an object \( u \) and an arrow \( S \in (i, u^d) \) satisfying

\[
S^* \circ S = d^*_q, \quad S \circ S^* = \eta(A_d), \tag{3.1}
\]

\[
S^* \otimes 1_u \circ 1_u \otimes S = (d - 1)!_q(-\mu)^{d-1}, \tag{3.2}
\]

\[
\eta(g_1 \ldots g_d) \circ S \otimes 1_u = \mu^{d-1} 1_u \otimes S, \tag{3.3}
\]

where \( n!_q \) is the usual quantum factorial and \( A_d \) is the antisymmetrized sum of the elements of the canonical basis of \( H_d(q) \), a scalar multiple of the analogue of the totally antisymmetric projection. The above relations are realized by the deformed determinant element \( S \) and the JW representation \( \eta \). They are easily verified for \( d = 2 \). (see [31] for details. Notice our \( g_i \) corresponds to \(-g_i\) there.)

**Remark** Note that for \( d \) odd, \( \text{Rep}(S_\mu U(d)) \) and \( \text{Rep}(S_{-\mu} U(d)) \) are canonically isomorphic.

It is well known that from Drinfeld’s theory of universal \( R \)-matrices, that the map \( \sigma_\omega : g_i \rightarrow \frac{\omega}{\mu} g_i \), where \( \omega \) is a complex \( d^{th} \) root of \( \mu \), makes \( \text{Rep}(S_\mu U(d)) \) into a braided tensor category. It is easy to check that \( \text{Rep}(S_\mu U(d)) \) actually becomes a braided tensor \( C^* \)-category in this way, cf. [31].

Working with a fixed deformation parameter leads to \( d \) inequivalent braided symmetries obtained by varying \( \omega \). This fact was first noted in [23], see also [39], Sect. 4.

Note that, for \( \mu > 0 \), the braided symmetry of \( \text{Rep}(S_\mu U(d)) \) corresponding to the positive \( d^{th} \) root \( \mu^{1/d} \) of \( \mu \) is a natural choice, as it reduces to the unique permutation symmetry of the category of Hilbert spaces for \( \mu = 1 \).

Deformation has been generalized to all classical compact Lie groups in the framework of compact matrix quantum groups [37][1][24]. The starting point was the dual picture of Drinfeld and Jimbo.

We start with a complex simple Lie algebra \( g \). Denote by \( (\alpha_i) \) a set of simple roots. Let \( A = \left( \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \right) \) be the Cartan matrix, where \( (\cdot, \cdot) \) is a symmetric invariant bilinear form on \( g \) such that \( (\alpha, \alpha) = 2 \) for a short root \( \alpha \). It follows that \( d_j := \frac{(\alpha_j, \alpha_j)}{2} \in \mathbb{N} \) for all \( j \).

For a complex deformation parameter \( \mu \) other than a root of unity, consider the quantized universal enveloping algebra \( U_\mu(g) \) as defined in Ch. 2, 7.1.1, [24].

The category \( \mathcal{C}(g, \mu) \) of finite dimensional representations of \( U_\mu(g) \) admitting a weight decomposition has the structure of a braided ribbon tensor category [24], see also [24].

If \( \mu \in (0, 1) \), \( U_\mu(g) \) becomes a Hopf \(^*\)-algebra with real form \( E^*_i = F_i, K^*_i = K_i \). The representation space of every object of \( \mathcal{C}(g, \mu) \) with positive weights on the \( K_i \)'s admits a natural Hilbert space structure making it into a \(^*\)-representation. One gets in this way a tensor \( C^* \)-category with conjugates and braided symmetry embedded into the category of Hilbert spaces. A compact quantum group \( G_\mu \) may then be defined via Woronowicz duality, see [37][1][24].

The category \( \text{Rep}(G_\mu) \) of unitary finite dimensional representations of \( G_\mu \) will be regarded as a braided tensor \( C^* \)-category as described above. In the particular
case of type $A$, we allow negative parameters for $S_{\mu}U(d)$ and the braided symmetry may be any of the $\sigma_w$'s.

The aim of this paper is to prove the following rigidity result (a consequence of Theorem 5.4.)

3.1. Theorem For $\mu \neq 1$ (and also $\mu \neq -1$ in the type $A$ case), every braided tensor $^*$-functor $\rho : \operatorname{Rep}(G_{\mu}) \to \mathcal{M}$ into a weak half braided tensor $C^*$-category is full.

We derive the following consequences.

3.2. Corollary For $\mu \neq 1$ (and also $\mu \neq -1$ in the type $A$ case) the braided symmetry of $\operatorname{Rep}(G_{\mu})$ does not make the representation category of any proper compact quantum subgroup into a weak half braided tensor $C^*$-category.

Proof If $K$ is a proper compact quantum subgroup then the tensor $^*$-functor $\operatorname{Rep}(G_{\mu}) \to \operatorname{Rep}(K)$ of restriction is not full, hence the image of the braided symmetry of $\operatorname{Rep}(G_{\mu})$ can not be a braided symmetry for $\operatorname{Rep}(K)$.

The next corollary is an application to the Temperley–Lieb category $T_{\pm d}$. Recall that for $d > 0$, $T_{\pm d}$ may be defined as the universal tensor $^*$-category with objects $\mathbb{N}_0$ whose arrows are generated by a single arrow $R \in (0, 2)$ satisfying $R^* \otimes 1_1 \circ 1_1 \otimes R = \pm 1_1$ and $R^* \circ R = d$. The following assertions are well known [14]. The units and generating objects of these categories are irreducible and the spaces of arrows are finite dimensional. The categories are simple except at roots of unity, $d = 2 \cos \frac{\pi}{n}$, when they have a single non-zero proper ideal. They are tensor $C^*$-categories when $d \geq 2$ and at roots of unity their quotients by the unique non-zero proper ideal are tensor $C^*$-categories. The projection $e = \frac{R \circ R}{d}$ defines a Hecke algebra representation $\eta(g_1) := (1 + q)e - q$ where $q = \mu^2$ and $-1 < \mu < 1$ is determined by $|\mu + \frac{1}{\mu}| = d$ and $\mu > 0$ for $T_{-d}$ and $\mu < 0$ for $T_d$. It is also known that $T_{\pm d}$ is a full braided tensor $C^*$-subcategory of $\operatorname{Rep}(S_{\mu}U(2))$, see [32] with previous results in [4] for embedded categories. It is equivalent to $\operatorname{Rep}(S_{\mu}U(2))$ after completion under subobjects and direct sums. We may thus apply Theorem 3.1.

3.3. Corollary The category $T_{\pm d}$ generated by a single selfadjoint object of dimension $d > 2$ regarded as a braided tensor $C^*$-category can not be embedded properly as a weak half braided tensor $C^*$-subcategory with the same objects.

4 Invariant $\kappa$ and the twist of ribbon categories

In this section we recall the main properties of the invariant $\kappa$ of a braided tensor $C^*$-category with conjugates [25] and we compare it with the twist $\theta$ of ribbon categories [36].

If $v$ is an irreducible object of $A$ and $R \in (\iota, \overline{\iota} \otimes v)$, $\overline{R} \in (\iota, v \otimes \overline{\iota})$ a solution of the conjugate equations for $v$ then $\sigma(v, \overline{\iota}) \circ \overline{R}$ must be a nonzero scalar multiple of $R$ whenever $\sigma(v, \overline{\iota}) \in (v \otimes \overline{\iota}, \overline{\iota} \otimes v)$ is an invertible intertwiner, as the space $(\iota, \overline{\iota} \otimes v)$
is one dimensional. This scalar in general depends on the choice of the conjugate equations. However, the following proposition shows that it is independent if we choose a standard solution of the conjugate equations, in the sense of [25]. For irreducible objects \( v \) solutions are standard if and only if they are normalized, \( \|R\| = \|\overline{R}\| \).

4.1. *Proposition* If \( \sigma \) is a right braided symmetry for \( A \), then for any irreducible object \( v \) the scalar \( \kappa_r(v) \) defined by

\[
\sigma(v, \overline{v}) \circ \overline{R} = \kappa_r(v) R
\]

does not depend on the choice of standard solutions of the conjugate equations for \( v \).

If \( \sigma \) is a braided symmetry, \( \kappa_r \) is a class function, i.e. \( \kappa_r(v) = \kappa_r(u) \) if \( v \) and \( u \) are unitarily equivalent.

*Remark* Note that, unlike in the case of tensor categories, we do not need to make a choice \( v \rightarrow R_v \) to define \( v \rightarrow \kappa_r(v) \).

We may extend \( \kappa_r \) to reducible objects \( v \) of \( A \) by the equation

\[
\sigma(v, \overline{v}) \circ \overline{R} = \kappa_r(v) R
\]

where we again use standard solutions. Note that \( \kappa_r(v) \in (v, v) \). Similarly, for left braided symmetries we may define \( \kappa_l(v) \).

For a braided symmetry, \( \kappa_r(v) = \kappa_l(\overline{v}) =: \kappa(v) \). In this case \( \kappa(v) = \sum \kappa(v_n) E_n \) where \( v_n \) are the irreducible components of \( v \) and \( E_n \) minimal central projections in \( (v, v) \) with \( \sum_n E_n = 1_v \). It follows that \( v \rightarrow \kappa(v) \) is central,

\[
\kappa(v) \circ T = T \circ \kappa(u), \quad T \in (u, v),
\]

see [25].

4.2. *Proposition* If \( \sigma \) is a right braided symmetry, then for any object \( v \) of \( A \)

\[
\kappa_r^{-1}(v) = \kappa_r^\sigma(v)^{-1}, \quad \kappa_l(v) = \kappa_r(v), \quad \kappa_r^\sigma(v) = \kappa_r^\sigma(v)^{-1}.
\]

*Proof* We may assume \( v \) irreducible. The first relation is obvious and the third follows from the first two. Now \( \sigma(v, \overline{v})^* \circ R_v = \kappa_r^\sigma(v)^* R_v \), so

\[
\kappa_r^\sigma(v) \|R_v\|^2 = \overline{R_v}^* \circ \sigma(v, \overline{v})^* \circ R_v = (\sigma(v, \overline{v}) \circ \overline{R_v})^* \circ R_v = \overline{\kappa_r(v)} \|R_v\|^2,
\]

completing the proof.

The central element \( \kappa \) is in fact an invariant for braided tensor \( C^* \)–categories. More precisely, the invariant \( \kappa_r \) is preserved under a full tensor \( * \)–functor of right braided tensor \( C^* \)–categories with conjugates.

We next give an alternative definition of \( \kappa_l(v) \), in the spirit of the relation between statistics parameter, dimension and statistics phase in AQFT [15].
4.3. Proposition If \( \sigma \) is a left braided symmetry of \( \mathcal{A} \), then for any object \( v \) and any standard solution of the conjugate equations,

\[
\kappa_l(v)^{-1} = R_v^* \otimes 1_v \circ 1_v \otimes \sigma(v, v) \circ R_v \otimes 1_v,
\]

\[
\kappa_l(v) = R_v^* \otimes 1_v \circ 1_v \otimes \sigma(v, v)^{-1} \circ R_v \otimes 1_v.
\]

The same relations hold for a right braided symmetry with \( \kappa_r(v) \) in place of \( \kappa_l(v) \).

Proof

\[
R_v^* \otimes 1_v \circ 1_v \otimes \sigma(v, v) \circ R_v \otimes 1_v = R_v^* \otimes 1_v \circ 1_v \otimes \sigma(v, v)^{-1} \circ 1_v \circ \sigma(v \otimes v, v) \circ R_v \otimes 1_v = (\sigma_d(\tau, v) \circ R_v)^* \otimes 1_v \circ 1_v \otimes R_v = R_v^* \otimes 1_v \circ \kappa_l(v)^{-1} \otimes R_v = \kappa_l(v)^{-1}.
\]

The second relation follows applying the first to \( \sigma_d \). For the remaining statements it suffices to replace \( \sigma \) with \( \sigma_+ \) and \( \sigma_- \).

The following known result exhibits the behaviour of \( \kappa \) under tensor products, see \cite{25}.

4.4. Proposition If \( \sigma \) is a braided symmetry of \( \mathcal{A} \), then for any pair of objects \( u, z \),

\[
\kappa(u \otimes z) = (\sigma(z, u) \circ \sigma(u, z))^{-1} \circ \kappa(u) \otimes \kappa(z).
\]

The following convention suggested by categories of endomorphisms of an algebra helps to simplify notation. The symbol \( T \) denoting an arrow in \( (w, z) \) will also be used for \( T \otimes 1_v \in (w \otimes u, z \otimes u) \). The meaning of \( T \) will be clear from the context. Whereas \( 1_v \otimes T \) will be denoted by \( \rho(T) \). The previous proposition easily yields the following formula.

4.5. Corollary For any object \( u \in \mathcal{A} \) and any integer \( n \),

\[
\kappa(u^{\otimes n}) = \Sigma_{n-1}^{-1} \circ \cdots \circ \Sigma_1^{-1} \circ \kappa(u)^{\otimes n},
\]

where

\[
\Sigma_k := \rho^{k-1}(\sigma) \circ \rho^{k-2}(\sigma) \circ \cdots \circ \rho^2(\sigma) \circ \rho(\sigma)
\]

and \( \sigma = \sigma(u, u) \).

Remark As \( \kappa(u) \) is a scalar, when \( u \) is irreducible, \( \kappa(v) \) is an eigenvalue of \( \Sigma_{n-1}^{-1} \circ \cdots \circ \Sigma_1^{-1} \kappa(u)^{\otimes n} \), if \( v \) is an irreducible summand of \( u^{\otimes n} \).

We next compare the invariant \( \kappa \) of a braided tensor \( C^* \)-category with the twist \( \theta \) of a ribbon category. Recall that in a \( \mathbb{C} \)-linear tensor category (always assumed abelian and semisimple, although not necessarily strict) a right dual \( u^* \) for any object \( u \) of the category, is defined by two arrows \( e \in (v^* \otimes v, i) \) and \( d \in (u, v \otimes v^*) \) such that, up to canonical associativity isomorphisms, omitted here for simplicity,

\[
1_v \otimes e \circ d \otimes 1_v = 1_v,
\]
As for conjugates, a right dual is unique up to a unique invertible $T \in (v_1^*, v_2^*)$ such that $e_1 = e_2 \circ T \otimes 1_v, d_2 = 1_v \otimes d_1$. A left dual $^*v$ of $v$ is similarly defined by arrows $e' \in (v \otimes ^* v, i), d' \in (i, ^* v \otimes v)$. In a tensor $^*$–category with conjugates, a conjugate $\overline{v}$ of $v$ is always a right and left dual, $\overline{v} = v^*$.

A right rigid tensor category is a tensor category with a specified choice of right dual ($v^*, e_v, d_v$) for every object $v$. A ribbon category is a right rigid braided tensor category with a choice of isomorphisms $\theta_v \in (v, v)$, called twists, natural in $v$ and satisfying
\[
\theta_{v \otimes w} = \sigma(w, v) \circ \sigma(v, w) \circ \theta_v \otimes \theta_w,
\]
\[
\theta_{v^*} = (\theta_v)^c,
\]
\[
\theta_1 = 1_v.
\]

In a ribbon category we also have an associated left duality ($v^* = v^*, e_v^*, d_v^*$) defined as in equation (3.5) of Ch. XIV in [20]. The associated contravariant functor coincides with that induced by right duality. As is well known, we may then define a scalar valued trace $\text{tr}_v$ as in Def. XIV.4.1 in [20], analogous to a left inverse in a tensor $C^*$–category [25]. The twist can be computed from the braided symmetry and trace, $(\theta_v)^{-1} = \text{tr}_v \otimes 1_v (\sigma_1^v(v, v))$. From this, it is easy to show that if $N$ is a ribbon category and $\mathcal{M}$ a tensor $C^*$–category with conjugates embedded in $\mathcal{N}$ as a full tensor subcategory then for any irreducible object $v$ of $\mathcal{M}$, the trace arising from the ribbon structure may be chosen positive and $\theta_v = \kappa_r(v)^{-1}$, cf. Ch. XIV in [20].

5 An obstruction to extending braided tensor $C^*$–categories

Throughout this section, $\mathcal{A}$ and $\mathcal{M}$ are tensor $C^*$–categories with conjugates and irreducible tensor units and $\rho : \mathcal{A} \to \mathcal{M}$ is a tensor $^*$–functor.

We start by assuming that $\mathcal{A}$ and $\mathcal{M}$ have braidings, $\sigma$ and $\varepsilon$, in the sense of (2.1) and that $\rho$ is a braided tensor functor. We shall refer $\mathcal{M}$ as a braided extension of $\mathcal{A}$.

Our main assumption is that $\mathcal{A}$ admits an embedding into the category of Hilbert spaces, and we fix a tensor $^*$–functor $\tau : \mathcal{A} \to \text{Hilb}$. (We shall not assume that $\tau$ is braided as this would imply that $\sigma$ is a permutation symmetry.) Hence, by Woronowicz duality, $\tau$ determines a compact quantum group $G^\tau$ with a braided representation category.

Consider the ergodic $C^*$–action $(\mathbb{C}, \alpha)$ of $G^\tau$ associated with the pair $(\rho, \tau)$, see [32]. For each object $u$ of $\mathcal{A}$, $\mathcal{H}_u$ is the Hilbert bimodule constructed in [33], in fact just depending on $\rho_u$ of $\mathcal{M}$.

We identify $\mathcal{H}_u$ with $\tau_u \otimes \mathbb{C}$ as right Hilbert bimodules. The canonical unitaries $S_u$ make the left module structures explicit, see Prop. 8.6 in [33],
\[
< \psi \otimes I, e^c(\phi) \cdot \psi' \otimes I > = (\rho(R^*_u) \circ 1_{\rho_u} \otimes T \otimes 1_{\rho_u}) \otimes (j_u \psi \otimes \phi \otimes \psi'),
\]
for every irreducible spectral representation \( v \) of the ergodic action of \( G^* \) on \( \mathcal{C} \) and every linear intertwiner \( c^v : \tau_v \to \mathcal{C} \) between \( v \) and \( \alpha \) i.e. of the form \( c^v(\phi) = T \otimes \phi \), \( T \in (\rho_v, \iota) \).

The next lemma shows that if \( \mathcal{A} \) and \( \mathcal{M} \) have a braiding in the weak sense of relation (2.1) and if \( \rho \) is a braided functor, the left bimodule structure is completely determined by the representation theory of \( G^* \) and the ergodic action.

**5.1. Lemma** If \( \sigma \) and \( \varepsilon \) satisfy (2.1) for tensor \( C^* \)-categories \( \mathcal{A} \) and \( \mathcal{M} \) and if \( \rho : \mathcal{A} \to \mathcal{M} \) is a braided tensor *-functor then the left module structure on \( \mathfrak{K}_u \) under the canonical identification with \( \tau_u \otimes \mathcal{C} \) is given by

\[
< \psi \otimes I, c^v(\phi) \cdot \psi' \otimes I > = c^v(\tau(R_u^* \otimes 1_v \otimes 1_\Pi \otimes \sigma(v, u))j_u \psi \otimes \phi \otimes \psi').
\]

**Proof** Writing, as above, \( c^v(\phi) = T \otimes \phi \), with \( \phi \in \tau_v \) and \( T \in (\rho_v, \iota) \),

\[
< \psi \otimes I, c^v(\phi) \cdot \psi' \otimes I >= (\rho(R_u^*) \circ 1_\Pi \otimes 1_{\rho_u} \otimes (j_u \psi \otimes \phi) \otimes \psi') =
\]

\[
(\rho(R_u^*) \circ 1_\Pi \otimes 1_{\rho_u} \otimes (1_{\rho_v} \otimes T \circ \varepsilon(\rho_v, \rho_u))) \otimes (j_u \psi \otimes \phi) \otimes \psi') =
\]

\[
(\rho(R_u^*) \circ 1_\Pi \otimes 1_{\rho_u} \otimes T \circ 1_\Pi \otimes \rho \sigma(v, u)) \otimes (j_u \psi \otimes \phi) \otimes \psi') =
\]

\[
T \circ \rho(R_u^*) \circ 1_v \circ 1_{\Pi} \otimes \sigma(v, u)) \otimes (j_u \psi \otimes \phi) \otimes \psi') =
\]

\[
T \otimes (\tau(R_u^* \otimes 1_v \circ 1_{\Pi} \otimes \sigma(v, u))j_u \psi \otimes \phi) \otimes \psi'))
\]

\[
c^v(\tau(R_u^* \otimes 1_v \circ 1_{\Pi} \otimes \sigma(v, u))j_u \psi \otimes \phi \otimes \psi').
\]

From Lemma 5.1, we derive a first property that spectral representations for ergodic actions arising from braided functors \( \rho : \mathcal{A} \to \mathcal{M} \) have to satisfy.

**5.2. Corollary** Let \( \sigma \) and \( \varepsilon \) be weak left braided symmetries of \( \mathcal{A} \) and \( \mathcal{M} \) respectively and \( \rho : \mathcal{A} \to \mathcal{M} \) a braided tensor *-functor. For every irreducible object \( v \) of \( \mathcal{A} \) such that \( (i, \rho_v) \neq \{0\} \) and for every object \( u \in \mathcal{A} \), and every solution \( R_u \) of the conjugate equations for \( u \),

\[
R_u^* \otimes 1_v \circ 1_{\Pi} \otimes \sigma(v, u) = R_u^* \otimes 1_v \circ 1_{\Pi} \otimes \sigma_d(v, u).
\]

**Proof** It suffices to apply Lemma 5.1 to \( \sigma \) and \( \sigma_d \), recalling that entries of spectral multiplets of an ergodic action corresponding to irreducible representations are linearly independent and that \( \tau \) is faithful on arrows being defined on a tensor \( C^* \)-category with conjugates.

**Remark** When \( \sigma \) and \( \varepsilon \) are weak right braided symmetries, (5.1) is replaced by \( R_u^* \otimes 1_v \circ 1_{\Pi} \otimes \sigma_{-1}(v, u) = R_u^* \otimes 1_v \circ 1_{\Pi} \otimes \sigma_*(v, u). \)
Note that (5.1) is automatically satisfied if \( \sigma \) is unitary. However, if it is non-unitary, the corollary provides an obstruction to constructing non-trivial braided extensions \((M, \rho)\) of \(A\).

We start with a given braided extension \((M, \rho)\) and draw conclusions about the eigenvalues \(\kappa_l(v)\) corresponding to irreducible objects \(v\) of \(A\) with \((\iota, \rho_v)\) non-trivial.

**5.3. Corollary** Let \(\sigma\) be a left (right) braided symmetry of \(A\), \(\varepsilon\) a weak left (right) braided symmetry of \(M\) and \(\rho\) a braided tensor \(*\)-functor. For any irreducible object \(v\) of \(A\) such that \((\iota, \rho_v) \neq \{0\}, \kappa_l(v) (\kappa_r(v))\) is a phase.

**Proof** Replacing \(\sigma\) and \(\varepsilon\) by \(\sigma_{-1}\) and \(\varepsilon_{-1}\) if necessary, we may assume, by Prop. 4.2, that \(\sigma\) and \(\varepsilon\) are left braided symmetries. We claim that for any irreducible \(v\) with \((\iota, \rho_v) \neq 0\) and any solution \(\overline{R}_v\) of the conjugate equations for \(v\), \(\sigma_*(v, \overline{\sigma}) \circ \overline{R}_v = \sigma_{-1}(v, \overline{\sigma}) \circ \overline{R}_v\). This would show that, choosing standard solutions, \(\kappa_r^*(v) (\kappa_l^*(v)) = \kappa_l^{\sigma_{-1}}(v) (\kappa_r^{\sigma_{-1}}(v))\) and the first two relations in Prop. 4.2 imply \(|\kappa_l(v)| = 1\). Now applying the previous proposition to \(\varepsilon\) and \(\sigma\), being left and hence weak left braided symmetries, it follows from (5.1) with \(v\) and \(\overline{\sigma}\) in place of \(u\) and \(v\) respectively, that

\[
R^*_v \otimes 1_{\overline{\sigma}} = R^*_v \otimes 1_{\overline{\sigma}} \circ \sigma_d(\overline{\sigma}, v) \sigma(\overline{\sigma}, v)^{-1}.
\]

Composing on the right by \(1_{\overline{\sigma}} \otimes \overline{R}_v\) gives

\[
1_{\overline{\sigma}} = R^*_v \otimes 1_{\overline{\sigma}} \circ (\sigma_d(\overline{\sigma}, v) \circ \sigma(\overline{\sigma}, v)^{-1} \circ \overline{R}_v).
\]

On the other hand \(\sigma_d(\overline{\sigma}, v) \circ \sigma(\overline{\sigma}, v)^{-1} \circ \overline{R}_v\) must be a scalar multiple of \(\overline{R}_v\) as \(v\) is irreducible. Our equation then shows that the scalar equals 1, so \(\sigma(\overline{\sigma}, v)^* \circ \overline{R}_v = \sigma(\overline{\sigma}, v)^{-1} \circ \overline{R}_v\).

**5.4. Theorem** Let \((A, \sigma)\) be a tensor \(C^*\)-category with conjugates and a left (right) braided symmetry. Assume that \(A\) admits a tensor \(*\)-embedding \(\tau\) into the category of Hilbert spaces. If \(\kappa(v)\) is not a phase whenever \(v\) is an irreducible of \(A\) not equivalent to \(\iota\), then every braided tensor \(*\)-functor \(\rho : (A, \sigma) \to (M, \varepsilon)\) into a weak left (right) braided tensor \(C^*\)-category \((M, \varepsilon)\) is full.

**Proof** The arrow space \((\iota, \rho_v)\) of \(M\) can be identified with the spectral space of \(\hat{\rho}\) for the action of \(G^\tau\) on \(\mathcal{C}\). We may obviously replace \(M\) by the category whose objects are those of \(A\) and where the space of arrows from \(u\) to \(v\) is now \((\rho_u, \rho_v)\) defining \(\rho\) and the algebraic structure in the obvious way. By the main result of [33], Theorem 7.7, there is then a full and faithful \(*\)-functor \(\lambda\) from \(M\) to the category of bimodule \(G^\tau\)-representations, taking \(u\) to the \(G^\tau\)-bimodule \(\tau_u \otimes \mathcal{C}\) and \(\rho(T)\) to \(\tau(T) \otimes I\). By the previous corollary, \((\iota, \rho_v) = \{0\}\) for every irreducible \(v \neq \iota\), hence \(\mathcal{C} = \mathbb{C}\). Now regarding \(\lambda\) as taking values in the category of \(G^\tau\)-representations, \(\lambda \rho = \tau\) and is full, thus \(\rho\) is full.

6 Proof of Theorem 3.1

By Theorem 5.4 it suffices to show that \(\kappa(v)\) is not a phase whenever \(v\) is an irreducible unitary representation of \(G_\mu\) not equivalent to \(\iota\).
We want to compute the invariant $\kappa$ for the braided $C^*$-tensor category $(\text{Rep}S_{\mu}U(d), \sigma_{\omega})$ and begin with its value on the fundamental representation $u$, writing $\sigma$ for $\sigma_{\omega}$ for brevity.

Recall that $\overline{\pi}$ may be realized as the subobject of $u^{\otimes d-1}$ defined by $\eta(E_{d-1})$. We have the relations $g_i S = -q S$, with $q = \mu^2$ for $i = 1, \ldots, d-1$ (see e.g. Lemma 5.6 in [31]), hence $\sigma_i S = -\omega \mu S$.

By Theorem 5.5 in [31], a standard solution of the conjugate equations for the fundamental representation $u$ is given by $R = \lambda S$, $\overline{\pi} = (-1)^{d-1} R$, with $\lambda$ a suitable positive scalar. Hence

$$\sigma(u, \overline{\pi}) \circ R = \lambda \sigma(u, \overline{\pi}) \circ 1_u \otimes E_{d-1} \circ S =$$

$$(-1)^{d-1} \lambda E_{d-1} \otimes 1_u \circ \sigma(u, u^{\otimes d-1}) \circ S = (-1)^{d-1} (-\omega \mu)^{d-1} \lambda E_{d-1} \otimes 1_u \circ S =$$

$$(\omega \mu)^{d-1} R,$$

so that

$$\kappa(u) = (\omega \mu)^{d-1}.$$

**Remark** A similar computation shows that $\kappa(\overline{\pi}) = (\omega \mu)^{d-1}$ as well.

Note that these are not phases, unless $\mu = \pm 1$. As we shall see more precisely later, these eigenvalues can be well understood in terms of the representation theory of the quantum group.

From the above computation of $\kappa(u)$, and the remark following Corollary 4.5, we arrive at the following result expressed in terms of the original representation $\eta$ of the Hecke algebra $H_n(\mu^2)$

6.1. **Proposition**

$$\kappa(u^{\otimes n}) = (\frac{\omega}{\mu})^{-n(n-1)} (\omega \mu)^{n(d-1)} \eta(\lambda_{-1} \cdots G_{-1}^{1}),$$

where $G_k = g_k g_{k-1} \cdots g^2 \cdots g_{k-1} g_k$.

We shall use this formula to reduce the problem to the case $\mu > 0$ and a specified root.

6.2. **Theorem** Suppose that, for some $\mu > 0$ and a specified $d^{th}$-root of $\omega$ of $\mu$, $\kappa(v)$ is never a phase when $v$ is an irreducible object of $\text{Rep}(S_{\mu}U(d))$ not equivalent to $u$ then the same property holds if $\mu$ and $\omega$ are replaced independently by $-\mu$ and any other root $\omega'$.

**Proof** If $v$ is an irreducible summand of some tensor power $u^{\otimes n}$, $\kappa(v)$ appears as the eigenvalue of $\kappa(u^{\otimes n})$ corresponding to the central support of $v$ in $(u^{\otimes n}, u^{\otimes n}) = \eta(\lambda_{-1} H_n(q))$. On the other hand the representations $\eta_{\mu}$ and $\eta_{-\mu}$ have the same kernel $I_n$ in each $H_n(q)$ (cf. [23] and also [31]), hence this explicit dependence on the Hecke algebra $H_n(\mu^2)$ shows that the relevant central support does not change if one replaces $\mu$ by $-\mu$. This central support agrees with that of an irreducible summand $v'$ in $u^{\otimes n}$ where $u'$ is the fundamental representation of $S_{-\mu}U(q)$. The spectrum of $\eta_{\mu}(G_{-1}^{1} \cdots G_{-1}^{1})$ is just the spectrum of the image of $G_{-1}^{1} \cdots G_{-1}^{1}$ in
\( H_n(q)/I_n \) under the quotient map, and hence does not change if \( \mu \) is replaced by \(-\mu\). It follows that \( \kappa(v) \) for \( \mu \) and \( \sigma_\omega \) can differ from \( \kappa(v') \) for \( \mu \) or \(-\mu\) and \( \sigma_\omega \) only by a phase.

**Remark** The kernel of \( \eta_\mu \) is known and may be described in terms of Young diagrams. Hence the spectral analysis of the element \( G_{n-1}^{-1} \cdots G_1^{-1} \) of the Hecke algebra \( H_n(q) \) determines the invariant \( \kappa \) on all irreducibles contained in \( v \otimes u \), cf. [14]. However, we shall not pursue this approach, but rather look for a more general argument giving results for deformations of classical compact Lie groups of Lie types other than \( A \).

To this end, we use the dual picture of Drinfeld and Jimbo. See Sect. 3 for notation.

Let \( \rho \in h^* \) be the element defined by \((\alpha_i, 2\rho) = (\alpha_i, \alpha_i)\). Let \( \lambda \) be a dominant integral weight and \( v_\lambda \) the irreducible representation of \( \mathfrak{u}_\mu(g) \) with highest weight \( \lambda \). The twist \( \theta_{v_\lambda} \) is known to act on the space of \( v_\lambda \) as scalar multiplication by \( \mu^{- (\lambda, \lambda + 2\rho)} \) see Lemma 7.3.2 in [24].

Taking the comparison between the algebraic and analytic approach to the twist into account, see Prop. 5.1, we are reduced to showing that \((\lambda, \lambda + 2\rho)\) is strictly positive for \( \lambda \neq 0 \).

In particular, in the type \( A_{d-1} \) case that table in Humphrey’s book gives,

\[
(\lambda_i, \lambda_i + 2\rho) = \frac{2}{d}(d - i)(1 + \cdots + i - 1) + \frac{3}{d}i(d - i) + \frac{2}{d}i(1 + \cdots + d - i - 1) = \frac{1}{d}(d - i)(i - 1)i + \frac{3}{d}i(d - i) + \frac{1}{d}i(d - i - 1)(d - i) = \frac{1}{d^2}i(d - i)(d + 1),
\]

for the fundamental weights. For \( i = 1 \) and \( i = d - 1 \), this reduces to our previous computation of \( \kappa(u) \) and \( \kappa(\overline{u}) \).

**Remark** The computation of \( \kappa(u) \) and \( \kappa(\overline{u}) \) at the beginning of the section depended on relations (3.1)–(3.3). The above proof gives an independent derivation.
7 On the characterization of \( \text{Rep}(S_\mu U(d)) \)

An intrinsic characterization of the quantum deformation of the spatial linear group was first given in [23], and based on the analysis of the fusion rules. An independent approach was proposed in [31] for \( S_\mu U(d) \), where the starting point was the braided symmetry. The aim of this section is to make the relation between the two approaches explicit, when not at roots of unity.

To any semisimple rigid tensor category \( \mathcal{C} \) with Grothendieck semiring isomorphic to that of \( sl(d) \) (briefly, an \( sl(d) \)-category), Kazhdan and Wenzl associate an invariant \( \tau_{\mathcal{C}} \), the twist of the category. They start with a suitable idempotent \( a \in (X^2, X^2) \), \( X \) the fundamental object of \( \mathcal{C} \), and find a nonzero complex number \( q \) such that \( qa - (I - a) \) satisfies the Hecke algebra relations for the parameter \( q \). This defines representations \( \eta_{KW} : H_n(q) \to (X^n, X^n) \). If \( \nu \in (\iota, X^d) \) and \( p \in (X^d, \iota) \) are chosen so that \( p \circ \nu = 1 \) then the twist is defined by

\[
\tau_{\mathcal{C}} := p \otimes 1 \circ \eta_{KW}(g_d \cdots g_1) \circ 1 \otimes \nu.
\]

It is asserted in Prop. 5.1, [23] that \( \tau_{\mathcal{C}} \) is a \( d^1 \)-root of unity. Clearly, \( \text{Rep}(S_\mu U(d)) \) is an \( sl(d) \)-category. However, comparing with (3.3) leads to \( \tau = \mu^{d^{-1}} \). The origin of the inconsistency is the claim on page 135 of [23] that \( \nu \otimes \nu \) is an eigenvector of \( \eta_{KW}(X^d, X^d) \) with eigenvalue 1, whilst, for \( S_\mu U(d) \), iterating relations (3.3) \( d \) times gives, \( \eta(u^d, w^d) \nu \otimes \nu = \mu^{d(d-1)} \nu \otimes \nu \).

This value is related to Drinfeld’s theory of universal \( R \)-matrices. In fact, it is well known that universal \( R \)-matrices give rise to ribbon tensor categories. As a consequence, in the type \( A \) case, the generator of the Hecke algebra needs to be multiplied by a suitable scalar to ensure the naturality of the braiding. This scalar is well known, see Lemma 3.2.1 in [41], see also [5], [43]. (We would warn the reader that this scalar is often computed incorrectly in the literature.)

The well known characterization of the quasiequivalence class of a non-faithful Hecke algebra representation in a rigid tensor category \( \mathcal{C} \) [23, 42, 31] leads to

\[
\tau_{\mathcal{C}} = w \mu^{d^{-1}} \text{ where } \mu \text{ is a complex square root of } q \text{ and } w \text{ is a } d^{16} \text{ root of unity.}
\]

This expression easily leads to a presentation of \( sl(d) \)-categories analogous to [31]. To make the comparison more immediate, we shall assume that \( \mathcal{C} \) has a \( * \)-involution making it into a tensor \( * \)-category. Analogous relations may be derived in the general case. We omit the proof.

7.2. Proposition If \( \mathcal{C} \) is a tensor \( * \)-category of \( sl(d) \) type, \( q \) is derived from \( \mathcal{C} \) as in [23], \( \nu \in (\iota, X^d) \) satisfies \( \nu^* \circ \nu = 1 \), then \( \nu \circ \nu^* = \eta(E_d) \) and

\[
\nu^* \otimes 1_X \circ 1_X \otimes \nu = w \frac{(-\mu)^{d^{-1}}}{[d]_q}, \tag{7.2}
\]

\[
\eta(g_1 \cdots g_d) \nu \otimes 1_X = w^{d^{-1}} 1_X \otimes \nu, \tag{7.3}
\]

where \( \mu^2 = q \), \( \tau_{\mathcal{C}} = w \mu^{d^{-1}}, [d]_q = 1 + q + \cdots + q^{d-1} \), and \( \eta = \eta_{KW} \).

Remark If \( \mathcal{C} \) is a tensor \( C^* \)-category, there are Hilbert space representations for \( H_\infty(q) \). Hence \( q \) is either a root of unity or \( q > 0 \) by Wenzl’s result [40].
We characterize $\text{Rep}(S_\mu U(d))$ among $sl(d)$–categories in the spirit of [23].

7.3. Corollary Let $\mathcal{C}$ be a tensor $C^*$–category of $sl(d)$–type with associated parameter $q$. Then,

a) if $\tau_\mathcal{C} > 0$ then $\mathcal{C}$ is tensor $^*$–equivalent to $\text{Rep}(S_{\sqrt{q}}U(d))$,

b) if $\tau_\mathcal{C} < 0$ and $d$ is even then $\mathcal{C}$ is tensor $^*$–equivalent to $\text{Rep}(S_{-\sqrt{q}}U(d))$.

Remark Note that for $\mu > 0$, and $d$ even, $\text{Rep}(S_{-\mu}U(d))$ is a twist of $\text{Rep}(S_\mu U(d))$ by $w = -1$.

Acknowledgements. C.P. would like to thank C. De Concini, Y. Kawahigashi, R. Longo and M. Müger for discussions. We would like to thank the referee for pointing out Lemma 3.2.1 in [41] and [5] and for giving suggestions on how to shorten the presentation of the paper.

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