Critical exponents of the quark-gluon bags model with the critical endpoint

A. I. Ivanytskyi, K. A. Bugaev, A. S. Sorin, and G. M. Zinovjev

1Bogolyubov Institute for Theoretical Physics of the National Academy of Sciences of Ukraine, Metrologichna str. 144, Kiev-03680, Ukraine; and
2Bogoliubov Laboratory of Theoretical Physics, JINR, Joliot-Curie str. 6, 141980 Dubna, Russia

The critical indices \(\alpha', \beta, \gamma'\) and \(\delta\) of the Quark Gluon Bags with Surface Tension Model that has the critical endpoint are calculated and compared with the exponents of other models. These indices are expressed in terms of the most general parameters of the model. Despite the usual expectations the found critical indices do not depend on the Fisher exponent \(\tau\) and on the parameter \(z\) which relates the mean bag surface to its volume. The scaling relations for the obtained critical exponents are verified and it is demonstrated that for the standard definition of the index \(\alpha'\) the Fisher and the Griffiths scaling inequalities are not fulfilled in general case, whereas the Liberman scaling inequality is always obeyed. This is not surprising for the phase diagram with the asymmetric properties of pure phases, but the present model also provides us with the first and explicit example that the specially defined index \(\alpha'_c\) does not recover the scaling relations as well. Therefore, here we suggest the physically motivated definition of the index \(\alpha' = \alpha'_c\) and demonstrate that such a definition recovers the Fisher scaling inequality, while it is shown that the Griffiths inequality should be generalized for the phase diagram with the asymmetric properties. The critical exponents of several systems that belong to different universality classes are successfully described by the parameters of the present model and hence its equation of state can be used for a variety of practical applications.

Keywords: critical exponents, critical endpoint, deconfinement phase transition

I. INTRODUCTION

Scaling has been widely accepted as “a pillar of modern critical phenomena” [1]. The scaling hypothesis used in the study of critical phenomena was independently developed by the well-known scientists as Widom, Domb, Hunter, Kadanoff, Fisher, Patashinskii and Pokrovskii (see review [2] for the details). Further it was developed within the renormalization group approach [3, 4]. One of the most striking predictions of the scaling hypothesis is called the scaling laws. For instance, for the ordinary liquids these scaling laws relate the critical exponents \(\alpha, \beta, \gamma\) and \(\delta\) which describe the behavior of the specific heat capacity \(C \sim |t|^{-\alpha}\), here \(t = (T - T_{cep}) / T_{cep} < 0\) denotes a relative deviation of the temperature \(T\) from the critical one \(T_{cep}\), density differences of the liquid and gaseous phases \((\rho - \rho_g \sim |t|^\beta)\), isothermal compressibility \((K_T \sim |t|^{-\gamma'})\) and the shape of the critical isotherm which is given by the critical index \(\delta\) (for the formal definitions see below)

\[
\text{Fisher [5]} : \quad \alpha' + 2\beta + \gamma' \geq 2, \quad (1) \\
\text{Griffiths [6]} : \quad \alpha' + \beta(1 + \delta) \geq 2, \quad (2) \\
\text{Liberman [7]} : \quad \gamma' + \beta(1 - \delta) \geq 0. \quad (3)
\]

Similar equalities can be also introduced for magnetic systems [8] and for percolating systems [9]. The corresponding exponent inequalities for magnetic systems are often called Rushbrooke’s [10], Griffiths’ and Widom’s [11] inequalities, respectively.

The superscript prime in Eqs. (1–3) is necessary to introduce for the systems with the phase diagram for which the behavior of specific heat capacity and/or compressibility on the gaseous side of phase diagram, where \(t < 0\), differs from that one on the liquid side, i.e. for \(t > 0\). Although in many physical systems the scaling laws \(\frac{\alpha}{2} \geq 1\) are obeyed as equalities, it is customary to write them as inequalities since, as was proven by M. E. Fisher for liquids [5] [12], in the most general case they can be established as inequalities only. It was, however, found that in such exactly solvable models as the Fisher-Felderhof one-dimensional model [13], the statistical multifragmentation model (SMM) [13], the quark-gluon bags with surface tension model (QGBSTM1) with the tricritical endpoint [15] and its generalization [16], the Fisher and the Griffiths scaling inequalities (1) and (2), in which the index \(\alpha'\) is involved, may be broken. The corresponding proofs are given in [13] for the Fisher-Felderhof model, in [17] for the SMM and in [18] for the QGBSTM1 and its generalization [16]. In all these cases the phase diagram is rather asymmetric, i.e. the behavior of pure phases on the both sides of the phase equilibrium curve are different, and, as a result, the heat capacity of gaseous and liquid phases are quite different. To some extent this problem was resolved by M. E. Fisher in [13] by introduction of a specially defined index \(\alpha'_c\) which measures a divergence of the heat capacity difference of two phases at the critical endpoint (CEP). In this case the \(\alpha'_c\) index gives the maximal value among the \(\alpha'\) index of the gaseous and the liquid phases. The usage of such an index instead of the traditional exponent \(\alpha'\) allows one to formally recover the Fisher and the Griffiths inequalities in all exactly solvable models mentioned above, although neither the physical meaning of the index \(\alpha'_c\) nor its relation to the experimental procedure of the heat capacity measurement were ever justified. Therefore, we are faced to three principal questions: (I) Is it possible to justify the \(\alpha'_c\) definition? (II) What definition should be used instead of the \(\alpha'_c\) index, if it fails to recover the scaling...
inequalities 1 - 3? (III) What should we do with the scaling laws in the latter case?

In order to clarify these questions in the present work we calculate the critical exponents of the quark-gluon bags with surface tension model (QGBSTM2 hereafter) which has the CEP [19]. This phenomenological model is a novel development of the well known gas of bags model [20] which, however, contains entirely new mathematical mechanism of the CEP generation. Of course, the QGBSTM2 was developed for the CEP modeling without specifying its universality class and the present work is devoted to the finding of the critical exponents of this model and to the determination of its universality classes. Since our main subject of interest is to describe the endpoint properties of the deconfinement phase transition (PT) of quantum chromodynamics (QCD), we would like to pay a special attention to the QCD CEP properties. For this purpose we, in accord with the contemporary knowledge, suppose that the CEP of the deconfinement PT has the properties typical for 3-D Ising model in the case of 3 quark flavors degenerated QCD [21, 23], whereas for 2+1 quark flavors we assume that the QCD endpoint belongs to the universality class of the O(4) symmetric 3-dimensional spin model [24, 25]. In other words we would like to determine the QGBSTM2 parameters and fix them in order to reproduce the critical exponents of respective universality class.

In contrast to the comparable solvable models [13–16], including the QGBSTM1, the QGBSTM2 has entirely different structure of isobaric ensemble singularities describing the PT, and, as we shall show, for some values of parameters the Fisher and the Griffiths inequalities [1] and [2] of this model are not fulfilled for both \( \alpha \) and \( \alpha_s \) indices. Therefore, here we introduce a physically motivated definition of the supremum index \( \alpha'_s \) which is found from the linear combination of the specific heat capacities of pure phases taken with the nonsingular weights. On the examples of solvable models [13–16] we demonstrate that such a definition recovers the scaling inequalities in those case, when the traditional \( \alpha \) index fails, since \( \alpha'_s \geq \alpha'_s \). However, below we show that even with an improved definition of the index \( \alpha' = \alpha'_s \) for the QGBSTM2 critical exponents there are two regimes: the traditional scaling regime, when the scaling inequalities 1 - 3 are held as the equalities only, and the generalized scaling regime, when the Fisher [1] and the Liberman [3] inequalities are fulfilled, but the Griffiths inequality [2] is only obeyed in its generalized form which is suggested here. The performed thorough analysis of the scaling laws 1 - 5 and their generalization onto the case of the phase diagram with the asymmetric properties seem to be very important nowadays in a view of fast technological and computational progress which, respectively, allows one to study the substances and models with new and unusual thermodynamic properties.

The work is organized as follows. Section II is devoted to a brief discussion of the QGBSTM2 main ingredients. The model is analyzed in details in Section III. The QGBSTM2 critical exponents are calculated in that section also. The analysis of the scaling relations between the found critical exponents is given in Section IV. Conclusions and perspectives are discussed in Section V.

II. QUARK GLUON BAGS WITH SURFACE TENSION MODEL

The QGBSTM2 [15, 19] treats the quark-gluon plasma (QGP) bags and hadrons as relevant degrees of freedom. Similarly to the original statistical bootstrap model [26] the attraction between the degrees of freedom in this model is accounted via many sorts of the constituents, while the repulsion between them is introduced a la Van der Waals equation of state [15, 19, 20]. The phase structure of the QGBSTM2 is completely defined by the mass-volume spectrum that for a given temperature \( T \), baryonic chemical potential \( \mu \) is defined as

\[
F(z, T, \mu) = F_H(z, T, \mu) + u(T, \mu) I_\tau(\Delta z, \Sigma),
\]

This spectrum defines the isobaric partition [15, 19, 20]

\[
Z(z, T, \mu) = \frac{1}{z - F(z, T, \mu)},
\]

where \( z \) denotes the isobar variable.

The discrete part of the mass-volume spectrum \( F_H \) in [4] is successfully used as the hadron resonance gas model to describe the experimental hadron multiplicities which allow one to recover the thermodynamic quantities of strongly interacting matter created in the heavy ion collisions, when this matter reaches the chemical freeze-out stage (an incomplete list of related works can be found in [29]). Here we consider the simplest parameterization of \( F_H \), since both the quantum statistics and the width of hadron resonances are important for the temperatures below 50 MeV and the baryonic chemical potentials larger than 940 MeV [30, 31]. Thus the spectrum \( F_H \) is parameterized as follows

\[
F_H(z, T, \mu) = \sum_{j=1}^{n} g_j e^{\delta z - v_j \phi(T, m_j)}.
\]

The continuous part of the spectrum [4] is chosen in the simplest form (compare it that one used in [16]), which can be cast as an integral \((V_0 \approx 1 \text{ fm}^3)\)

\[
I_\tau(\Delta z, \Sigma) = \int_{V_0}^{\infty} e^{-\Delta z v - \Sigma v^\tau},
\]

where the Fisher exponent \( \tau > 2 \) provides the convergence of the integral [7] for \( \Delta z = 0 \) and \( \Sigma = 0 \). Also here the notation \( \Delta z \equiv z - z_M(T, \mu) \) is introduced. In [6] the particle density of a hadron of mass \( m_j \), baryonic charge \( b_j \), eigenvolume \( v_j \) and degeneracy \( g_j \) is denoted
The functions \( u(T, \mu) \) and \( z_M(T, \mu) \) in (4) and (7) are the parameters of the present model which are smooth and finite together with all their first and second derivatives [15, 19]. The \( \tau \)-dependent exponentials in (6) and (7) describe the short range repulsion of the Van der Waals type [15, 19]. To parameterize the surface of a QGP bag in the continuous part of the spectrum (7) the parameter \( \kappa \) is introduced. Usually the constant \( \kappa \) is defined by the dimensionality \( d \) as \( \kappa = \frac{d-1}{2} \), but in what follows it is treated as a free parameter with the range of values \( 0 < \kappa < 1 \).

A few words should be added here about the hadronic surface tension. In principle, it can be included into the QGBSTM2 discrete spectrum [4]. The first and interesting results about the surface tension of hadrons which fit well into the QGBSTM2 framework can be found in [31]. They clearly demonstrate that the hadronic surface tension is rather small, although it changes the sign at the temperature about 150 MeV. However, in the present work we do not consider this element and set the hadronic surface tension to zero since its inclusion does not affect the expressions for the critical indices.

The new element of principal importance of the present model is the parameterization of the surface tension coefficient \( T \Sigma(T, \mu) \) (\( \Sigma(T, \mu) \) in (7) denotes the reduced surface tension coefficient) which in the vicinity of the phase equilibrium curve \( T = T_c(\mu) \) is defined as

\[
\Sigma^\pm(T, \mu) = \mp \frac{\kappa_0}{T} \left( T - T_c(\mu_{cep}) + \frac{dT_c}{d\mu}(\mu_{cep} - \mu) \right)^{\xi^\pm} \times \left| \frac{\Sigma_0(T, \mu)}{T_c(\mu)} \right|^{\xi^\pm},
\]

with the following values of constants \( \xi^\pm \geq 1, \xi^\pm > 0 \). Here \( \kappa_0 \) is chosen to be a positive constant, but the obtained results hold, if \( \kappa_0 > 0 \) is a smooth function of \( T \) and \( \mu \). As shown below it is also of crucial importance that the parameters \( \xi^\pm \) and \( \xi^\pm \) have different values below and above the phase coexistence curve \( T = T_c(\mu) \) which exists for \( \mu \geq \mu_{cep} \). It can be shown [19] that the necessary condition for the deconfinement PT existence with the CEP is that the QGBSTM2 surface tension coefficient changes the sign exactly at the phase equilibrium curve. In other words, the solution \( T_\Sigma(\mu) \) of the equation \( \Sigma(T, \mu) = 0 \) should coincide with the PT curve \( T_c(\mu) \) for \( \mu \geq \mu_{cep} \), i.e. \( T_\Sigma(\mu) = T_c(\mu) \) for \( \mu \geq \mu_{cep} \) (see Fig. 1 for details).

The important mathematical consequence of such a matching is that the discontinuity of the partial \( \mu \) and \( T \) derivatives of the reduced surface tension coefficient across the line \( T = T_\Sigma(\mu) \) provides the 1st order deconfinement PT existence [19], and hence in (9) one has \( \xi^+ \neq \xi^- \) and \( \xi^+ \neq \xi^- \) in general. The quantities introduced in (9) have the superscript \( +(-) \), if they are taken for \( T \) above (below) the curve \( T_\Sigma(\mu) \) in the whole \( \mu - T \) plane. For \( 0 \leq \mu < \mu_{cep} \) the nil line of the surface tension coefficient is located in such a way that the deconfinement PT degenerates into a cross-over since in this region \( \Sigma(\mu) < 0 \) (see [19] and Fig. 1 for more details).

Also note that the different slopes of the surface tension coefficient below and above its nil line \( T = T_\Sigma(\mu) \) are not unusual since this property is successfully used in such well known models as the Fisher droplet model (FDM) [33] and the SMM [14, 37], but, additionally, in the present model the reduced surface tension coefficient is negative (positive) for \( T \) above (below) the line \( T = T_\Sigma(\mu) \). As it is argued in [15, 19] there is nothing wrong or unphysical with the negative values of surface tension coefficient, since in the grand canonical ensemble the quantity \( T \Sigma v^\Sigma \) is the surface free energy \( f_{surf} = e_{surf} - T s_{surf} \) of the bag of mean volume \( v \), were \( e_{surf} \) and \( s_{surf} \) are the surface energy and entropy. Therefore, \( \Sigma < 0 \) means that the surface entropy contribution simply exceeds the surface energy part, i.e. \( T s_{surf} > e_{surf} \) and then \( f_{surf} < 0 \). It can be shown on the basis of exactly solvable model of surface deformations [35] that negative values of the surface free energy is a consequence of very large number of non-spherical configurations at high temperatures. To our best knowledge, the exactly solvable models of the liquid-gas PT with negative values of the surface tension coefficient provide us with the only physical reason preventing the condensation of small droplets into a liquid phase (an infinite droplet) at supercritical temperatures, and, thus, they naturally explain the existence of a cross-over both in QGP [15, 19] and, probably, in the ordinary liquids [39]. For the field-theoretical arguments in favor of the negative surface tension of quark gluon bags see [40, 41].

Another strong line of arguments in favor of the negative surface tension of quark gluon bags at high temperatures is provided by the recent analysis of the relation between the confining color string tension and the surface tension of QGP bag [42]. It clearly demonstrates that at the cross-over region the surface tension coefficient of large bags is unavoidably negative and, as shown in [42], this should lead to an appearance of surfaces with the fractal dimension. Note that the fractal surfaces are well known in the lattice QCD formulation [43, 44], but their principal role in the lattice entropy maximum formation of the confining tube (the so called ‘mysterious maximum’ [45]) and it leads to the formation of fractals. Such a power law may, in principle, naturally explain the appearance of the non-Boltzmann fluctuations in the high energy collisions experiments [34, 35] without appealing to the Tsallis statistics [36].
and their relation to the negative surface tension values of such tubes were revealed only recently [12].

The phases of the QGBSTM2 include the hadronic phase and the QGP which in the \( \mu - T \) plane are separated by the nil line of the surface tension coefficient and they can be distinguished by the sign of the surface tension coefficient (see Fig. 1). For a given \( \mu \) and temperature \( T \) above (below) the line \( T_{\Sigma}(\mu) \) the pressure is marked by the superscript \(+\)(\(-\)) and is defined by the equation

\[
p^\pm(T, \mu) = T \left[ F_H(z^\pm, T, \mu) + u(T, \mu)I_\tau(\Delta^\pm z, \Sigma^\pm) \right],
\]

where the following notations \( z^\pm \equiv \frac{\rho^\pm(T, \mu)}{\rho^0} \), \( \Delta^\pm z \equiv z^\pm(T, \mu) - z_M(T, \mu) \) are used. The expression \( p^\pm \) for pressures \( p^+ \) and \( p^- \) is determined, respectively, by the simple poles \( s^+ \) and \( s^- \) of the isobaric partition [5].

The mixed phase of these two phases corresponds to the vanishing value of the surface tension and in this respect it is similar to the CEP in ordinary liquids [8], but, in contrast to the ordinary liquids, the PT in the QGBSTM2 is not of the 2-nd order, but of the 1-st order everywhere at the phase diagram except for the CEP where it is, indeed, of the 2-nd order [19]. The corresponding pressure is given by the essential singularity of the isobaric partition [5]

\[
p^M(T, \mu) = Tz_M(T, \mu).
\]

This equation also demonstrates the meaning of the function \( Tz_M(T, \mu) \) which would give the QGP pressure in the absence of the surface tension.

As usual, the pressure of the stable phase is defined by the rightmost singularity of the isobar partition [5] [14] [19]. By construction the PT occurs at \( T = T_{\Sigma}(\mu) \) at which the simple pole singularities \( z^\pm(T, \mu) \) coincide with the essential singularity \( z_M(T, \mu) \). The colliding singularities automatically provide the fulfillment of the Gibbs criterion of phase equilibrium. The necessary condition for the PT existence is the following relation between the parameters of the model

\[
z_M(T, \mu) = F_H(z_M(T, \mu), T, \mu) + u(T, \mu)I_\tau(0, 0),
\]

at \( T = T_c(\mu) \equiv T_{\Sigma}(\mu) \).

Assuming that [12] is fulfilled we parameterize the shape of the phase coexistence curve \( T_c(\mu) \) in the vicinity of the CEP by the constant \( \xi^\tau > 0 \):

\[
T_{cep} - T_c(\mu) \sim (\mu - \mu_{cep})^{\xi^\tau}.
\]

The crucial importance of this new index for the QGBSTM1 was demonstrated recently [18]. Surprisingly this index was not considered in such well known models as the FDM [33] and the SMM [13] [67], although it is known that a related quantity \( K_c \) introduced in [16] plays a decisive role in the classification of the CEP stability types [17]. Below it will be shown that just this index determines the values of the exponent \( \alpha^\tau \).

Using the standard definitions for the entropy density \( s \) and the baryonic density \( \rho \) as \( T \) and \( \mu \) partial derivatives of the corresponding pressure one can explicitly write the Clapeyron-Clausius equation for pure phases \( \frac{\partial \mu}{\partial T} = -\frac{s^\pm - s_M}{\rho^\pm - \rho_M} \bigg|_{T=T_{cep}} \) with the help of [10]. However, in the present model there is an additional relation for the pressure of the mixed phase \( p^M \) [11], which by construction matches the pressures \( p^- \) and \( p^+ \) for the same value of the PT temperature. Therefore, one can establish two additional relations of the Clapeyron-Clausius type between the partial derivatives of the function \( p^M(T, \mu) \) and the partial derivatives of the pressure of each pure phase

\[
\frac{\partial \mu}{\partial T} = -\frac{s_M - s^\pm}{\rho_M - \rho^\pm} \bigg|_{T=T_{cep}} = -A_T \bigg|_{T=T_{cep}},
\]

for \( x \in \{T, \mu\} \) is used.

For further evaluation it is convenient to parameterize the behavior of the numerator and denominator in [14] at the CEP vicinity in the same way as it was suggested recently in [18]:

\[
A_T \bigg|_{T=T_{cep}} \sim (T_{cep} - T_c(\mu))^{\frac{1}{1+\frac{1}{\xi^\tau}} - 1},
\]

\[
A_\mu \bigg|_{T=T_{cep}} \sim (T_{cep} - T_c(\mu))^{\frac{1}{\xi^\tau}},
\]

where the finite values of the integral \( I_\tau(0, 0) \) and the functions \( F_H(z^\pm(T, \mu), T, \mu) \), \( u(T, \mu) \), \( z_M(T, \mu) \) along with their first derivatives for any finite values of \( T \) and \( \mu \) provide the validity of the condition \( \chi \geq \max(0, 1 - \frac{1}{\xi^\tau}) \).
Since the index $\chi$ unavoidably appears from an inspection of the Clapeyron-Clausius equation, which is a direct consequence of the Gibbs criterion, then it is quite general \[18\]. Moreover, the parameter $\chi$ played an important role in separating the different sets of solutions for the QGBSTM1 \[18\], but as it will be shown below, although such an index appears in the intermediate expressions for the analyzed singularities, surprisingly, it does not enter any equation for the critical exponents of the present model.

### III. THE STANDARD CRITICAL INDICES OF THE QGBSTM2

As usual, the standard set of the critical exponents $\alpha$, $\beta$ and $\gamma$ \[5\] \[8\] \[13\] describes the $T$-dependence of the system near the CEP:

- $C_\rho \sim |t|^{-\alpha'}$, for $t \leq 0$ and $\rho = \rho_{cep}$, \[18\]
- $\Delta \rho \sim |t|^\beta$, for $t \leq 0$, \[19\]
- $\Delta K_T \sim |t|^{-\gamma'}$, for $t < 0$, \[20\]

where $\Delta \rho \equiv (\rho^+ - \rho^-)_{t=T_c}$, defines the order parameter, $C_\rho \equiv \frac{T}{p}(\frac{\partial \rho}{\partial T})_\rho$ denotes the specific heat capacity at the critical density and $\Delta K_T \equiv (K_T^- - K_T^+)_T=T_c$ is the discontinuity in the isothermal compressibility $K_T \equiv \frac{1}{\rho}(\frac{\partial p}{\partial T})_T$ across the PT line, the variable $t$ is the reduced temperature $t \equiv \frac{T-T_{cep}}{T_{cep}}$. The critical isotherm shape is given by the index $\delta$ \[5\] \[8\]

$$p_{cep} - \tilde{p} \sim (\rho_{cep} - \tilde{\rho})^\delta \quad \text{for} \quad t = 0. \quad \tag{21}$$

Hereafter the tilde indicates that $T = T_{cep}$. Note that for the phase diagram shown in Fig. 1 one has $\tilde{p} = p^+$ and $\tilde{\rho} = \rho^+$.

The critical exponent $\alpha'$ describes the $T$-behavior of the specific heat capacity along the critical isochore $\rho = \rho_{cep}$ inside the mixture of the QGP and hadron phase. Similarly to \[18\] we do not use the Yang-Yang formula \[48\] to calculate $C_\rho$, since it leads to a less convenient representation. Therefore, some details of the heat capacity evaluation are given below. We have to note that there exist a few suggestions on how to possibly determine the specific heat capacity at the vicinity of the QCD CEP \[49\] \[50\] from the experimental data obtained in the heavy ion collisions and, hence, the question of correct definition of the $\alpha'$ index becomes of crucial importance for us.

As usual, the entropy density and the baryonic density of the mixed phase are defined via the corresponding values of pure phases and the volume fraction of hadronic phase $\lambda$ (the volume fraction of the QGP is, respectively,

$$\rho|_{T=T_c} = \lambda \rho^+|_{T=T_c} + (1 - \lambda) \rho^-|_{T=T_c}, \quad \tag{22}$$

$$s|_{T=T_c} = \lambda s^+|_{T=T_c} + (1 - \lambda) s^-|_{T=T_c}, \quad \tag{23}$$

Varying $\lambda$ from 0 to 1 one can describe all states inside the mixed phase. Fixing $\rho|_{T=T_c}$ to $\rho_{cep}$ in \[22\] one can first calculate the total $T$-derivative of the volume fraction $\lambda$ along the critical isochore $\rho = \rho_{cep}$ and then one can use \[23\] to determine the specific heat capacity at fixed baryonic density

$$C_\rho = \frac{T}{\rho_{cep}} \left[ \frac{ds^+}{dT} + \frac{d\mu_e}{dT} \frac{d\rho^+}{dT} \right. \right.$$ \[24\]

$$+ \left. \lambda \left( \frac{d}{dT}(s^- - s^+) + \frac{d\mu_e}{dT} \frac{d(\rho^- - \rho^+)}{dT} \right) \right]_{T=T_c}$$

$$= \frac{T}{\rho_{cep}} \left[ \frac{ds^+}{dT} + \frac{d\mu_e}{dT} \frac{d\rho^+}{dT} + (\rho^- - \rho_{cep}) \frac{d^2\mu_e}{dT^2} \right]_{T=T_c} \quad \tag{24}$$

where in the second step the Clapeyron-Clausius equation \[14\] for hadron phase and that one for the QGP together with the definition \[22\] for $\lambda$ were used. Using the Clapeyron-Clausius equation \[14\] for the QGP one can again rewrite \[24\] as

$$C_\rho = \frac{T}{\rho_{cep}} \left[ \frac{ds_M}{dT} + \frac{d\mu_e}{dT} \frac{d\mu_M}{dT} + \frac{d}{dT}(s^+ - s_M) \right. \right.$$ \[25\]

$$\left. + \frac{d\mu_e}{dT}(\rho^- - \rho_M) + (\rho^- - \rho_{cep}) \frac{d^2\mu_e}{dT^2} \right]_{T=T_c}$$

$$= \frac{T}{\rho_{cep}} \left[ (\rho_M - \rho_{cep}) \frac{d^2\mu_e}{dT^2} + \frac{ds_M}{dT} + \frac{d\mu_e}{dT} \frac{d\mu_M}{dT} \right]_{T=T_c} \quad \tag{25}$$

The obtained expression \[25\] for the specific heat capacity coincides with that one obtained for the QGBSTM1 \[18\] and, therefore, the $\alpha'$-index for the models with the CEP and with the triCEP are identical.

In this section we consider the divergent heat capacity at the CEP, i.e. $\alpha' > 0$, while the negative values of this index are analyzed in the subsequent section. Then the critical exponent $\alpha' > 0$ describes the temperature behavior of the most singular term in Eq. \[25\]. Expanding $\rho_M|_{T=T_c}$ into the series of $t$-powers and using the parameterization \[13\] of the PT curve one can show that the first term staying on the right hand side of \[25\] after the second equality behaves as $t^{min\{1, \frac{1}{\alpha' + 1}\}} \sim t^{\frac{2}{\alpha' + 1}}$. Since the entropy density and the baryonic density of QGP are, respectively, the $T$ and $\mu$ derivatives of the function $Tz_M(T, \mu)$, which is a regular function of its parameters together with its first and second derivatives, then the singularity in the second and third terms of \[25\] may appear from the square of the derivative $\frac{d\mu_e}{dT}$ and it has the form $\left( \frac{d\mu_e}{dT} \right)^2 \sim t^{\frac{2}{\alpha' + 1}}$. Accounting for these facts, one gets the critical exponent $\alpha'$ as

$$\alpha' = 2 - 2 \min\left( 1, \frac{1}{\xi T} \right). \quad \tag{26}$$

This equation shows that $\alpha' > 0$ for $\xi T > 1$ only, otherwise $\alpha' = 0$. As it was mentioned above, the critical exponent $\alpha'$ of the QGBSTM2 \[26\] exactly coincides with that one obtained for the QGBSTM1 \[18\] despite the fact
that the phase diagrams of these models have essential differences. Moreover, these models have entirely different ranges of the Fisher exponent \( \tau \): \( \tau > 2 \) within the QGBSTM2 while \( 1 < \tau \leq 2 \) within the QGBSTM1.

To calculate the critical exponents \( \beta \) and \( \gamma' \) we need to know the behavior of the integrals \( I_{q}(0,0) \). As it was shown in [19] the necessary condition for the 1-st order deconfinement PT existence is that such integral should be finite for \( q = 1 \). This condition provides the range of values \( \tau > 2 \) for the Fisher exponent. The validity of this necessary condition can be demonstrated as follows. Indeed, using the definition of the baryonic density and Eqs. (10) for the phases on two sides of the PT curve one can find the baryonic density discontinuity across the deconfinement PT line [19]:

\[
\Delta \rho = -T \cdot \left( \frac{\partial \Sigma^+}{\partial \mu} - \frac{\partial \Sigma^-}{\partial \mu} \right) u_{I_{\tau=\kappa}}(0,0) \bigg|_{T=T_c} \sim \left( \frac{\partial \Sigma^-}{\partial \mu} - \frac{\partial \Sigma^+}{\partial \mu} \right) T=T_c,
\]

for \( \tau > 2 \) since in this case the integrals \( I_{\tau=\kappa}(0,0) \) and \( I_{\tau=1}(0,0) \) remain finite. Therefore, the temperature behavior of \( \Delta \rho \) is defined by the difference of the reduced surface tension coefficient partial derivatives calculated below and above the PT line. We would like to pay an attention to an analysis of the condition \( \Delta \rho \geq 0 \). Clearly, this condition simply means that the baryonic density increases during the deconfinement PT from hadrons to QGP, since the baryonic density of latter is higher.

Since the integral \( I_{\tau=\kappa}(0,0) \) and the denominator staying in the expression after the second equality sign in (27) is positive, then the condition \( \Delta \rho \geq 0 \) is provided by the inequality

\[
\frac{\partial \Sigma^+}{\partial \mu} < \frac{\partial \Sigma^-}{\partial \mu},
\]

which is the case for \( \zeta^+ = 1 \) only. Therefore, according to the parameterizations [9] and [13] one obtains

\[
\beta = \begin{cases} 
\beta^+, & \text{for } \zeta^- > \zeta^+ = 1, \\
\min(\beta^+, \beta^-), & \text{for } \zeta^- = \zeta^+ = 1, 
\end{cases}
\]

where the notation \( \beta^\pm = \zeta^\pm + \xi^\pm - \frac{1}{\zeta} \) is used. Since in accord with the adopted assumptions the densities of pure phases are finite at the CEP, it follows that the index \( \beta \) is non-negative.

To find the critical exponent \( \gamma' \) one has to calculate the isothermal compressibility \( K_T \equiv \frac{1}{\rho} \frac{\partial u}{\partial \mu} T \) for both pure phases. Using the baryonic density definition one can rewrite the isothermal compressibility as \( K_T = \frac{1}{\rho} \frac{\partial \rho}{\partial \mu} \). Therefore, using Eq. (10) at the CEP and keeping the most singular terms one finds

\[
\Delta K_T \sim -\left( \frac{\partial^2 \Sigma^-}{\partial \mu^2} - \frac{\partial^2 \Sigma^+}{\partial \mu^2} \right) T u_{I_{\tau=\kappa}}(0,0) \bigg|_{T=T_c} \sim -\left( \frac{\partial^2 \Sigma^-}{\partial \mu^2} - \frac{\partial^2 \Sigma^+}{\partial \mu^2} \right) T=T_c.
\]

Such a quantity, as suggested in [13], is rather convenient for the index \( \gamma' \) evaluation. The detailed derivation of the expression (29) can be found in the Appendix A. Indeed, differentiating Eq. (27) with the help of the parameterization [13] and recalling that its left hand side is just \( |t|^\beta \), one immediately finds

\[
\gamma' = \frac{1}{\xi T} - \beta.
\]

It is evident, that the direct way to calculate this index via the isothermal compressibilities \( K_T \) of pure phases would give us the same result for the present model. Thus, at the first glance it seems that the definition (29) suggested in [13] is rather appropriate for the case of the asymmetric phase diagram, when the \( \gamma' \) indices of two pure phases are different. We, however, will return to this question again in the subsequent section.

To calculate the critical exponent \( \delta \) one can use the definition of \( \Delta z \) at the critical isotherm and get

\[
\tilde{p} - p_{cep} = T_{exp} (\Delta z + z_M - z_{cep})
\]

where in the second step one has to expand \( z_M \) in powers of \( \Delta u \equiv u_{cep} - u_{cep} \) and keep the linear term. Similarly one can determine the deviation of the baryonic density at the critical isotherm from that at one of the CEP. Note that this critical isotherm is lying outside the mixed phase and necessarily it belongs to the high density phase. Then one finds

\[
\tilde{p} - p_{cep} = T_{exp} \left( \frac{\partial \Delta z}{\partial \mu} + \frac{\partial z_M}{\partial \mu} \right) \bigg|_{cep}.
\]

Analysis of the functions \( \Delta z \) and \( \frac{\partial \Delta z}{\partial \mu} \) near the CEP shows that it is convenient to substitute the expansion \( I_{\tau}(\Delta z, \Sigma) \approx I_{\tau}(0,0) - \Delta z I_{\tau-1}(\frac{\Delta z}{2}, \frac{\Sigma}{2}) - \Sigma I_{\tau-\kappa}(\frac{\Delta z}{2}, \frac{\Sigma}{2}) \) into Eq. (10) for the cross-over states in order to obtain

\[
\Delta z = \frac{\Delta \mu A_{\mu|cep} - \tilde{\Sigma} \tilde{u}_{I_{\tau=\kappa}}(\frac{\Delta z}{2}, \frac{\Sigma}{2})}{1 - \frac{\partial \rho}{\partial \mu} + \tilde{u}_{I_{\tau-1}}(\frac{\Delta z}{2}, \frac{\Sigma}{2})}.
\]

where it is sufficient to keep only the first order terms of the expansion and to use the fact that at the CEP \( \Delta z = 0 \). According to Eq. (31) the higher order terms of expansion could be neglected.

Using the same steps of derivation one can show that

\[
\frac{\partial \Delta z}{\partial \mu} = \frac{\Delta \mu \frac{\partial A_{\mu|cep}}{\partial \mu} - \tilde{\Sigma} \tilde{u}_{I_{\tau=\kappa}}(\frac{\Delta z}{2}, \frac{\Sigma}{2})}{1 - \frac{\partial \rho}{\partial \mu} + \tilde{u}_{I_{\tau-1}}(\frac{\Delta z}{2}, \frac{\Sigma}{2})}.
\]
The reduced surface tension coefficient partial derivative \( \frac{\partial \Sigma}{\partial \mu} \) vanishes at the CEP, whereas its second partial derivative \( \frac{\partial^2 \Sigma}{\partial \mu^2} \) diverges at this point. Therefore, from Eqs. \((31), (33), (32)\) and \((34)\) one finds that \( \Delta z \sim \Delta \mu \) and \( \frac{\Delta z}{\Delta \mu} \sim \frac{\partial \Sigma}{\partial \mu} \). Then using Eqs. \((9)\) and \((13)\) one concludes that

\[
\delta = \frac{1}{\xi_T^T \beta^+}. \tag{35}
\]

Note that the condition \( \delta > 1 \) requires \( \frac{1}{\xi_T} > \beta^+ \) which according to the result found for the index \( \gamma' \) \((30)\) is consistent with the constrain \( \gamma' > 0 \).

Despite some similarities the critical exponents of the QGBSTM2 are very different from the critical indices of the comparable cluster models such as the FDM \([33]\), the SMM \([17]\) and the QGBSTM1 \([18]\). Thus, surprisingly, the critical exponents of the present model do not depend on the Fisher topological exponent \( \tau \) and on the parameter \( \kappa \) which relates the mean surface of the QGP bag to its volume. It seems to be a unique feature of the QGBSTM2, since the critical exponents of the FDM, SMM and QGBSTM1 are expressed in terms these parameters. Furthermore, in contrast to the QGBSTM1 critical exponents \([18]\), the parameter \( \chi \) does not appear in any final expression for the critical indices of the QGBSTM2. This is another surprising fact since just this parameter switches the different regimes between the QGBSTM1 critical exponents \([18]\).

Having the explicit expressions for the QGBSTM2 critical exponents let us now discuss the question whether the obtained results are able to reproduce the indices of the 2-dimensional Ising model \([51]\), of the 3-dimensional Ising model \([52]\), of the simple liquids \([51]\) and of the O(4) symmetric 3-dimensional spin model \([53, 54]\) (see the Table I). It is an important question since, as it is expected, the universality class of the 3-D Ising model coincides with that one of the 3-flavor degenerated QCD, whereas the 2+1 flavor QCD falls into universality class of the O(4) spin model \([24, 25, 28, 55]\).

As one can see from the Table I \( \gamma' > 1 \) for all discussed systems. On the other hand, from the explicit expressions of the QGBSTM2 critical exponents one can see that within the present model \( \gamma' > 1 \) for \( \xi_T T < 1 \) only, which immediately leads to a conclusion that \( \alpha' = 0 \). Therefore, the QGBSTM2 with the traditional definition of index \( \alpha' \) is able to reproduce the critical exponents of the 2-dimensional Ising model only. Such a problem is not a new one. For example, the critical exponents of the SMM \([17]\) do not reproduce that ones of the simple liquids and of the 3-dimensional Ising model since \( \alpha'_{SMM} = 0 \). It is, however, believed that such a problem is related to the traditional definition of the critical exponent \( \alpha' \). In order to elucidate this fact let us study in detail the scaling relations for the indices of the present model.

### Table I: The critical indices of the 2-dimensional Ising model \([51]\), of the simple liquids \([51]\), of the 3-dimensional Ising model \([52]\) and of the O(4) symmetric spin model \([53, 54]\).

| \( \alpha' \) | \( \beta \) | \( \gamma' \) | \( \delta \) |
|----------------|----------------|----------------|----------------|
| 0              | 0.335(15)      | 1.25(5)        | 15             |
| 1              | 0.3265(1)      | 1.2373(2)      | 4.53(3)        |
| 0.1096(5)      | 0.10(1)        | 1.44(4)        | 4.7893(8)      |
| 0.10(1)        | 0.3265(1)      | 1.44(4)        | 4.82(5)        |

### IV. THE SCALING RELATIONS OF THE QGBSTM2

Now we return to the discussion of the scaling inequalities \([1, 2, 3]\) of the present model. This inequalities are well established analytically \([5, 6]\) and experimentally \([8, 51, 56]\). As one can see from the Table II these inequalities are exactly established only for an exact solution of the 2-dimensional Ising model while for other systems shown there such inequalities are established within the error bars, which in some cases are not very small. From time to time in the literature there appear the models \([13, 17, 18]\) and even the experimental works (we mean the text-book example \([57]\)) discussed in the quoted edition of \([8]\) in which the problems related to the Fisher \([1]\) and the Griffiths \([2]\) inequalities are reported. It is remarkable that the reported problems are always related to the inequalities in which the exponent \( \alpha' \) is involved. The situation is somewhat mysterious since the formal conditions of the well known Fisher theorem \([5]\) proving the validity of these inequalities for liquids are fulfilled by the models \([13, 17, 18]\), but there is the range of parameters for which either one of the relations \((1)\) and \((2)\) or both of them are not obeyed. Therefore, it is interesting to verify the scaling relations for the indices of the QGBSTM2 which, as we discussed in the preceding sections, demonstrate the remarkable difference with that ones of the models \([13, 17, 18]\).

The explicit expressions for the critical exponents found above allow us to directly examine the scaling inequalities. Despite the usual expectations, the Fisher and Griffiths inequalities are not always obeyed, whereas the Liberman inequality is fulfilled for any values of the model parameters. Let’s first demonstrate the validity of the Liberman inequality \([3]\). Using the explicit expressions for the indices \( \beta, \gamma' \) and \( \delta \), i.e. Eqs. \((28), (30)\) and \((35)\), one obtains

\[
\gamma' + \beta(1 - \delta) = \frac{1}{\xi_T} \left( 1 - \frac{\beta}{\xi_T} \right)
\]

\[
= \begin{cases} 
0, & \text{for } \zeta^- > \zeta^+ = 1 \\
\frac{1}{\xi_T} \left( 1 - \frac{\min(\beta^+, \beta^-)}{\xi_T} \right) \geq 0, & \text{for } \zeta^- = \zeta^+ = 1
\end{cases}
\]

where an obvious inequality \( \min(\beta^+, \beta^-) \leq \beta^+ \) is accounted for. From the Liberman scaling law \([3]\) one im-
Since \( \rho \) equation one can find that of two pure phases. Then from the Clapeyron-Clausius idea behind such a suggestion is to get the most singular dependence of the specific heat capacity difference of two may be broken for the QGBSTM2, if 2 max(1, \( \frac{1}{\xi_T} \)) for the left hand side of the Griffiths inequality (2) and sequence of steps min(1, \( \frac{1}{\xi_T} \)) we employed the expression for index \( \alpha' \). Note that in the evaluation of the second equality above we follow the suggestion of [13] and replace the index \( \alpha' \) by \( \alpha'' \), where the latter describes the temperature dependence of the specific heat capacity of two phases at the CEP, i.e. \( \Delta C = (C^+_{\rho^+} - C^+_{\rho^-})_{T= T_c} \). The idea behind such a suggestion is to get the most singular term from the difference of the specific heats capacity of two pure phases. Then from the Clapeyron-Clausius equation one can find that

\[
\Delta C = T_c \rho^+ - \rho^- \rho^\prime \frac{d \rho}{dT} \frac{d \mu_c}{d T} \rho_M + s_M + \frac{d \mu_c}{d T} \rho_M
\]

\[
= T_c \frac{d \mu_c}{d T} \ln \frac{\rho^+}{\rho^-}.
\]

Since \( \rho^+ - \rho^- \sim t^\beta \) in the vicinity of the CEP, then using [13] and the fact that the function \( z_M(T, \mu) \) together with all its derivatives up to the second one are finite, we conclude that the first term on the right hand side of (30) behaves as \( t^{\alpha'_s \min(0, \frac{1}{\xi_T} - 2) + \beta} \) and \( \Delta C \). Similarly one finds that the second term on the right hand side of (39) behaves as \( t^{\alpha'_s + 2 \beta + \gamma'} \). This analysis shows that the first of the two discussed terms is the leading one and, hence, we have

\[
\alpha'_s = \max \left( 2, \frac{1}{\xi_T} \right) - \beta - \frac{1}{\xi_T}. \tag{40}
\]

Note that \( \alpha'_s \geq 0 \) for \( \frac{1}{\xi_T} < 2 - \beta \) only. Using the \( \alpha'_s \) expression (40) one can get

\[
\alpha'_s + 2 \beta + \gamma' = \max \left( 2, \frac{1}{\xi_T} \right) \geq 2, \tag{41}
\]

which is in a complete agreement with the expectation of [13]. However, such a replacement does not recover the Griffiths inequality. Indeed, in this case one finds

\[
\alpha'_s + \beta(1 + \delta) = 2 - \min \left( 2, \frac{1}{\xi_T} \right) + \frac{\beta}{\xi_T}. \tag{42}
\]

Obviously, the right hand side of (42) is less than 2 for \( \beta = \beta^+ < \beta^+ \min(1, 2\xi^T) \). Thus, the QGBSTM2 gives the first and explicit example that the definition of the \( \alpha'_s \) index does not recover the Griffiths scaling inequality, although it, indeed, redeems the Fisher scaling law. The reason that the \( \alpha'_s \) definition worked well in all preceding models, but fails for this one, is that within the models [13, 17, 18] the PT corresponds to the change of the leading singularity from the simple pole of the isobaric partition partition to its essential singularity, whereas in the QGBSTM2 the leading singularity above and below PT is the simple pole. Then for the models [13, 17, 18] the divergent terms in the difference \( \Delta C = (C^+_{\mu^+} - C^-_{\mu^-})_{T= T_c} \) did not cancel each other because they have rather different analytical behavior, while in the QGBSTM2 the most divergent terms in the specific heat capacity of two phases coincide with each other and they simply cancel each other in the expression for \( \Delta C \). Therefore, the QGBSTM2 provides us with a direct evidence that such a simple and attractive definition of the \( \alpha'_s \) index [13] which, so far, was designed for the asymmetric phase diagrams and worked well for the models [13, 17, 18], should be replaced now by a proper one.

Keeping in mind these facts we introduce a new definition for the index \( \alpha' \) which is based on the behavior of the linear combination of specific heat capacity of two phases \( C_{tot} = \hat{K}^+ C^+ + \hat{K}^- C^- \) with the positive and nonvanishing coefficients \( K^\pm > 0 \) which in general may depend on \( T \) or \( \mu \). The inequality \( \frac{K^+}{K^-} > 0 \) guarantees that no term in \( C_{tot} \) is missing or cancelled. On the other hand the condition \( K^\pm |_{T= T_c} > 0 \) provides that the index \( \alpha'_s \) has the maximal value among that one of pure phases. Note that this condition leads to a difference with the traditional definition of the index \( \alpha'_s \) since the coefficients \( K^+ |_{T= T_c} \) and \( K^- |_{T= T_c} \) are not related to

\[
\begin{array}{|c|c|c|c|}
\hline
 & \text{Ising model} & \text{Simple liquids} & \text{O(4) model} \\
\hline
\alpha' + 2 \beta + \gamma' & 2 & 1.99996(7) & 2.0200(55) \pm 2.01(8) \\
\alpha' + \beta(\delta + 1) & 2 & 2.000412(5000) & 1.9425(55) \pm 2.02(9) \\
\gamma' + (1 - \delta) & 0 & -0.000052(2000) & 0.0775(212) \pm 0.01(6) \\
\hline
\end{array}
\]

TABLE II: Scaling relations between the critical exponents taken from the Table I. The uncertainties were calculated from their values given in the Table I using the error determination method for indirect measurements [58].
the fractions of pure phases. Such a property allows us
to avoid the situation, when the curve at which the spe-
cific heat capacity is found does not match the boundary
with one of the phases which was the case for the models
analyzed in [13–17].

The index \( \alpha'_c \) definition has a clear physical inter-
pretation: when approaching the CEP the density fluc-
tuations get so strong that the specific heat capacity of
both phases contribute into the measurable value and,
hence, the largest term determines the critical exponent
\( \alpha'_c \). Then the coefficients \( K^+|_{T=\Theta} \) and \( K^-|_{T=\Theta} \) define
the weight of corresponding pure phase into the specific
heat capacity of their mixture. Obviously, the maximiza-
tion of the \( \alpha' \)-value is not well suited for the pure phases
with asymmetric properties, but also it should increase
the magnitude of the left hand side of the Fisher and
Griffiths inequalities and, thus, it should weaken the nec-
necessary conditions to obey them.

Let’s find out the index \( \alpha'_c \). Making the same steps as
in deriving Eq. (39) we obtain

\[
C_{tot} = T \left[ K^+ \left( \frac{d \xi}{d T} \right) + K^- \left( \frac{d \xi}{d T} \right) \right] \bigg|_{T=\Theta} = 0
\]

From this formula one can draw an important conclusion
that negative values of the index \( \alpha'_c \) are possible, if and
only if each full \( T \)-derivative on the right hand side of
Eq. (43) including \( d \mu |_{\Theta} \) and \( d \mu |_{T=\Theta} \) vanish at the CEP.
The necessary conditions for \( \alpha'_c < 0 \) are \( d \mu |_{\Theta} = 0 \) and

\[
\left. \frac{d s_M}{d \xi} \right|_{T=\Theta} = \frac{d s_M}{d \xi} \bigg|_{T=\Theta} \sim |t|^\omega, \quad (44)
\]

where \( \omega > 0 \) for \( \alpha'_c < 0 \) and \( \omega = 0 \) for \( \alpha'_c \geq 0 \). The
geometrical meaning of Eq. (44) is that the function
\( s_M(T, \mu_{\Theta}) \) has a kink point at the critical temperature
for \( \omega > 0 \). The case \( \omega = 0 \) does not add anything new
to the above evaluation of the indices \( \alpha' \) and \( \alpha'_c \). Us-
ing the parameter \( \omega \) one can show that the first term on
the right hand side of Eq. (40) behaves as \( t^{\min(\omega, \frac{1}{\xi'^c} - 1)} \).

Since the coefficients \( K^+ \left|_{T=\Theta} \right. \) are positive, then each
term \( K^+ \left|_{T=\Theta} \right. \) behaves as \( t^{\min(\omega, \frac{1}{\xi'^c} - 1)} \).

Similarly the exponents \( \alpha' \) and \( \alpha'_c \) can be found for the
nonzero values of \( \omega \)

\[
\alpha'_c = 2 - 2 \min \left( \frac{1}{\xi'^c}, \frac{1}{\xi'^c} + \frac{1 + \xi'^c}{2} \right), \quad (46)
\]

Comparing Eqs. (45) and (47), one can easily see that
\( \alpha'_c > \alpha'_c \) for the same value of \( \omega \). Moreover, for the case
\( \xi'^c > \omega > 2 \) one finds that \( \alpha'_c = \alpha' + \beta = -\omega \), i.e. the
index \( \alpha'_c \) can be essentially larger then \( \alpha'_c \) since \( \beta > 0 \).
Therefore, it is evident that in the models [13–17] in which the \( \alpha'_c \) index recovers the scaling inequalities
(1)–(3) the \( \alpha'_c \) index should recover them too, since by
the construction it is not smaller than \( \alpha'_c \), but for other
models the properties of this index should be studied.

Let us demonstrate explicitly that the proposed defi-
nition of the critical exponent \( \alpha'_c \) is more adequate with
respect to the QGBSTM2 scaling inequalities. From the
expression (40) for the index \( \gamma' \) one gets \( \frac{1}{\xi'^c} = \gamma' + \beta \).
Substituting this result into (45) one finds

\[
\alpha'_c = \begin{cases} -\omega, & \text{for } \omega \leq \gamma' + 2\beta - 2, \\ 2 - 2\beta - \gamma', & \text{for } \omega \geq \gamma' + 2\beta - 2 = -\alpha'_c, \end{cases} \quad (48)
\]

and, hence, \( \alpha'_c > 2 - 2\beta - \gamma' \) which is just the Fisher
inequality written for the \( \alpha'_c \) exponent. Note that in con-
trast to the \( \alpha'_c \) definition the Fisher inequality for the
\( \alpha'_c \) definition is not fulfilled anymore. For instance, if
\( \alpha'_c = -\omega \), then from Eqs. (47) and (48) it follows that for
the same value of the index \( \omega \) the relation \( \alpha'_c + 3\beta + \gamma' \geq 2 \)
holds which, obviously, differs from the Fisher ineq-
uality. The considered example clearly demonstrates that
for \( \omega < 0 \) the most general definition of the index \( \alpha'_c \)
leads to the difficulties with the Fisher scaling relation
and, hence, it has to be replaced by the \( \alpha'_c \) definition.

Since the Liberman inequality (36) does not depend on
the \( \alpha' \) definition, it holds, but here we would like to
pay an attention to the case when it turns into an
equality. From Eq. (39) one concludes that this occurs
for \( \beta = \beta^+ \), which applied to the relations for the indices
\( \delta \) and \( \gamma \) yields a very important relation

\[
\gamma' + \beta = \beta \delta = \frac{1}{\xi'^c}, \quad (49)
\]

that should hold for any model in which the scaling re-
lation (3) is obeyed as equality, for instance, for \( O(2) \),
\( O(3) \) and \( O(4) \) spin models. For such models the expon-
ent \( \xi'^c = \frac{2m}{y} \) is the ratio of the so-called magnetic \( y_\chi \)
and thermal \( y_t \) exponents \( [59, 59, 60] \) and it defines the
curvature radius of the phase diagram in \( \mu - T \) plane
[59]. If, however, the scaling law (3) is obeyed as an
inequality, the relation (49) reads as an inequality too, i.e.
\( \beta \delta \xi'^c < 1 \).
For the Griffiths inequality \(2\), the direct substitution of the expressions \(28\), \(30\), \(35\) and \(45\) yields
\[
\alpha_c' + \beta(1 + \delta) =
\begin{cases}
-\omega + \beta(1 + \delta), & \text{for } \omega \leq \beta + \frac{1}{\xi T} - 2, \\
2 - \frac{1}{\xi T} (1 - \frac{\beta}{\xi T}), & \text{for } \omega \geq \beta + \frac{1}{\xi T} - 2.
\end{cases}
\tag{50}
\]
Note that the right hand side of Eq. \(50\) can be smaller than 2 in two cases. Indeed, if the index \(\omega\) obeys the inequalities \(\beta(1 + \delta) < \omega < \beta + \frac{1}{\xi T} - 2\), then the Griffiths scaling law \(2\) is broken down. The other possibility for the break down of this scaling law corresponds to the lower equality in \(50\), i.e. for
\[
\alpha_c' + \beta(1 + \delta) = 2 - \delta(\beta^+ - \beta),
\tag{51}
\]
that occurs for \(\omega \geq \beta + \frac{1}{\xi T} - 2\) and \(\beta < \beta^+\). Thus, from these examples one might conclude that even the improved definition \(45\) of the index \(\alpha_c'\) which perfectly works for the Fisher inequality does not save the Griffiths scaling law. We, however, would like to stress that these examples clearly show us that the problem of formulating the scaling inequalities for the asymmetric phase diagrams is much deeper and cannot be resolved by the redefinition of the indices. It also requires the modification of traditional scaling relations. Indeed, using the Fisher inequality with the \(\alpha_c'\) exponent and the explicit expressions for the indices \(\gamma\) and \(\delta\) we obtain another inequality
\[
\alpha_c' + \beta + \beta^+ \delta \geq 2,
\tag{52}
\]
which is an analog of the Griffiths inequality for the QGBSTM2 exponents.

Let us show that the scaling law \(52\) cannot be broken within the QGBSTM2. For this purpose we assume that \(\alpha_c' + \beta + \beta^+ \delta = \alpha_c' + \beta + \frac{1}{\xi T} < 2\). Then, comparing the latter assumption with the upper equality in \(50\), we arrive at the contradiction, since in this case \(\omega = -\alpha_c' \leq \beta + \frac{1}{\xi T} - 2\), but our assumption is equivalent to the different inequality, i.e. \(\omega = -\alpha_c' > \beta + \frac{1}{\xi T} - 2\). Analogously, from the lower expression in \(50\) it follows that \(\alpha_c' + \beta + \beta^+ \delta = 2\) and we again obtain the contradiction with the original assumption which, as we proved, is the false one.

From Eq. \(50\) one can deduce that Eqs. \(40\) and \(51\) hold as equalities for \(\omega \geq \beta + \frac{1}{\xi T} - 2\) and \(\beta = \beta^+\), and, hence, the Fisher scaling relation \(1\) becomes an equality too. Thus, in this case all scaling laws \(1\)–\(3\) are fulfilled as equalities and this is the traditional scaling regime. In all other cases defined by Eq. \(50\), i.e. for \(\beta(1 + \delta) < \omega < \beta + \frac{1}{\xi T} - 2\) or for \(\omega \geq \beta + \frac{1}{\xi T} - 2\) and \(\beta < \beta^+\), the Fisher and Liberman scaling laws are obeyed as inequalities, but the Griffiths one in its usual form \(2\) is broken down and, hence, we suggest to use the inequality \(52\) instead of \(2\). Indeed, the latter, as we proved above, is valid in more general case than the original Griffiths one and also it seems to be more natural, since, in contrast to \(2\), for \(\beta^+ > \beta = \beta^-\) the high density phase exponent \(\delta\) in \(52\) is multiplied by other high density phase exponent \(\beta^+\). Since in this regime the generalized inequality \(52\) is valid, we call it the generalized scaling regime.

Also, we have to stress that the break down of the traditional Griffiths inequality \(2\) does not require some special conditions. Indeed, it is easy to see that, if the Fisher relation \(1\) is fulfilled as equality and the Liberman one \(3\) is obeyed as a strict inequality, then the traditional Griffiths inequality \(2\) would be broken, which is not the case for the inequality \(52\). Note that for the QGBSTM2 exponents the Liberman inequality can be also generalized similarly to the Griffiths inequality \(52\). Indeed, rewriting the first equality in the expression \(36\) one finds
\[
\gamma' + \beta - \beta^+ \delta = 0,
\tag{53}
\]
which is stronger than the usual Liberman scaling inequality, since it is possible that \(\beta^+ > \beta\). Here it is appropriate to discuss the definition \(20\) of the critical index \(\gamma'\) suggested in \(13\). Similarly to the definition of the \(\alpha_c'\) index \(13\) such a definition is simple and it may provide one with the maximal value of the compressibility exponent of two pure phases. However, it has the very same defect as the \(\alpha_c'\) definition, namely, it does not guarantee that for some models with the asymmetric phase diagrams the leading terms of two compressibilities may simply cancel each other as we found such a possibility for the QGBSTM2 \(\alpha_c'\) index and this would lead to the problems with the scaling relations. Therefore, in order to avoid these problems for such cases we suggest to employ the definition of the \(\gamma'\) exponent which is similar to the \(\alpha_c'\) definition \(13\) in which the heat capacities \(C_\pm\) are replaced by the corresponding thermal compressibilities \(K_T^\pm\)
\[
k^+ K_T^+ + k^- K_T^- \sim |t|\gamma',
\tag{54}
\]
with the positive and nonvanishing coefficients \(k^+ > 0\). The interpretation of such coefficients is similar to that one introduced for the \(\alpha_c'\) index.

After the thorough discussion of the QGBSTM2 critical exponents it is not surprising that for the traditional scaling regime the present model with the \(\alpha_c'\) and \(\omega\) definitions is able to reproduce the critical indices of the 2-dimensional Ising model, of the simple liquids, of the 3-dimensional Ising model and of the O(4) symmetric spin model for which, as one can see from the Table II, the inequalities \(1\)–\(3\) are well established. The list of the corresponding QGBSTM2 parameters is given in the Table III. However, an existence of the generalized scaling regime for the QGBSTM2 with the most favorable definition of the \(\alpha_c'\) index for which the Griffiths inequality is broken down is rather surprising, since it is widely accepted that the validity of the Fisher \(1\) and the Griffiths \(2\) relations does not require any additional assumptions except for the conditions of the Fisher theorem \(5\)–\(8\)–\(50\).
|                  | 2D Ising Model | 3D Ising Model | Simple Liquids | O(4) Model |
|------------------|----------------|----------------|----------------|------------|
| $\xi^+$          | 0.6010(1)      | 0.631(2)       | 0.55(1)        |            |
| $\xi^-$          | 1.0112(2)      | 0.93(23)       | 1.0112(2)      |            |
| $\omega^+$       | 1.0112(2)      | 0.93(23)       | 1.0112(2)      |            |
| $\omega^-$       | 1.20(4)        |                |                |            |
| $\zeta^-$        | 0.9903(2)      | 1.20(4)        | > 0            |            |
| $\kappa^-$       | 0.19(6)        |                |                |            |
| $\xi^+$          | 1.0112(2)      | 0.93(23)       | 1.0112(2)      |            |
| $\xi^-$          | 1.0112(2)      | 0.93(23)       | 1.0112(2)      |            |
| $\omega^+$       | 1.0112(2)      | 0.93(23)       | 1.0112(2)      |            |
| $\omega^-$       | 1.20(4)        |                |                |            |
| $\zeta^-$        | 0.9903(2)      | 1.20(4)        | > 0            |            |
| $\kappa^-$       | 0.19(6)        |                |                |            |

TABLE III: The QGBSTM2 parameters which describe the corresponding exponents given in the Table I. The values of the parameter $\omega$ extracted from the expressions for $\alpha'$, $\alpha'_s$ and $\alpha'_c$ are marked with $^*$, $^{**}$ and $^{***}$, respectively. The symbol $\emptyset$ means that it is impossible to find the value of the corresponding parameter which allows us to describe the critical exponents shown in the Table I.

Now the QGBSTM2 provides us with an explicit example that this may not be the case for the Griffiths inequality $^2$ and the latter has to be replaced by $^52$.

It is worth to note that from the physical point of view the successful description of the critical exponents belonging to the different universality classes by a single model not only shows that the QGBSTM2 is very general, but it also evidences for the fact that the physical clusters or bags employed in this model are, indeed, the relevant degrees of freedom at the CEP for all analyzed systems, despite the very different physics exhibiting by the original Hamiltonians of the corresponding spin models. On the one hand this conclusion completes the finding $^61$ that the FDM $^33$ correctly describes the distribution of large clusters of the 2- and 3-dimensional Ising model in the wide range of temperatures, including the CEP. On the other hand the QGBSTM2 generalizes the results of the noninteracting cluster models $^62, 63$ which quite successfully describe the CEP properties of real liquids. Since the QGBSTM1 $^19$ is also able to reproduce the critical exponents $^18$ of the same universality classes as the QGBSTM2, we may hope that these models can be effectively used to describe the QCD endpoint properties and they can help to experimentally distinguish the CEP case from the tricritical endpoint case.

V. CONCLUSIONS

The practical necessity to describe the thermodynamic functions of the QGP with the (tri)critical endpoint which has the required properties simulated the development of a variety of the exactly solvable statistical models $^15, 16, 19, 64–68$. Since it is not exactly know whether the QCD phase diagram endpoint is critical or tricritical it was necessary to developed the exactly solvable models for both cases. However, in contrast to the tricritical endpoint case which was worked out in many versions $^15, 16, 19, 64–68$, a formulation of a realistic exactly solvable model with the CEP required many years and took many additional efforts, because neither the physical mechanism of its generation nor the mathematical properties of such a model were known before its development in $^19$. The main result of $^19$ clearly demonstrates that the traditional framework of liquid-gas PT models $^2, 13, 33, 37$ has to be essentially modified in order to degenerate the 1st order deconfinement PT into a CEP and into a cross-over. Furthermore, on the one hand the liquid-gas PT models require the vanishing of surface tension coefficient at the CEP, and on the other hand the positive values of the confining tube entropy unavoidably demand for the negative surface tension coefficient at the cross-over region $^42, 69$. The natural conclusion that follows from these findings is that at the cross-over region there should exist the curve of nil values of the surface tension coefficient which must pass through the endpoint of the 1st order deconfinement PT. If the nil line of the surface tension coefficient does cross the deconfinement PT curve at the endpoint only, then, as shown in $^15$, such an endpoint of the deconfinement PT is the tricritical one and this is the case of QGBSTM1. If, however, the nil curve of the surface tension coefficient matches the deconfinement PT curve as described above (see Fig. 1 for details), then the model has the CEP and it corresponds to the QGBSTM2 $^19$.

The present paper is devoted to the analysis of the critical exponents of the QGBSTM2 which are necessary to be studied before applying the present model to describe either the lattice QCD thermodynamics or various experimental data. Such an analysis allowed us to find some general restrictions on the model parameters and figure out the important relations between them. Also it is found that the QGBSTM2 exponents essentially differ from that ones of the FDM, SMM and QGBSTM1 despite many similarities between these models. Thus, in the FDM and the SMM the critical exponents can be expressed in terms of a few major input parameters like the Fisher topological exponent $\tau$ and the exponent $\alpha$ which is usually related to a dimension, whereas the QGBSTM1 critical exponents also depend on two recently introduced parameters $\xi^T$ and $\chi$. In addition to all these parameters the QGBSTM2 depends on the input exponents $\xi^\pm$, $\zeta^\pm$ and $\omega$, but in this work it is found that, in contrast to the FDM, SMM and QGBSTM1, the QGBSTM2 exponents do not depend on the input exponents $\alpha$, $\chi$ and $\tau$, although here we for the first time found that the present model formulation is valid for $\tau > 3$ only.

Here we also showed that neither with the traditionally defined index $\alpha'_c$ of $^18$ nor with the specially introduced one $\alpha'_s$ $^13$ it is possible to fulfill the Fisher and Griffiths scaling inequalities, while the Liberman one is always obeyed. Such a situation is known for the models $^13, 16$, but in the case of present model the index $\alpha'_c$ does not in general allow one to recover the scaling inequalities $^1$ and $^2$ as well. However, in the present work we proved that for a physically motivated definition of the index $\alpha'_c$ which corresponds to the maximal value...
of the index $\alpha'$ of two pure phases, the QGBSTM2 exponents recover the Fisher scaling inequality (1), while the Griffiths inequality (2) in its traditional form is fulfilled only in a case, when all scaling laws (1)-(3) become the equalities. It is also shown that for this traditional scaling regime the QGBSTM2 exponents reproduce the values of critical indices of the 2-dimensional Ising model, of the simple liquids, of the 3-dimensional Ising model and of the O(4) symmetric spin model and this evidences for the fact that the bags (or physical clusters) used in the QGBSTM2 are the relevant degrees of freedom at the CEP for all studied systems despite quite a different physics of their phases. The present analysis demonstrates that the negative values of the $\alpha'$ index can be achieved with the help of newly introduced exponent $\omega$ in (11), which, as we explicitly demonstrated, plays a decisive role in the recovering of scaling inequalities.

Besides the traditional scaling regime, we found that the QGBSTM2 exponents have the generalized scaling regime, for which the Griffiths inequality should be generalized to Eq. (52). Also here we showed that for the QGBSTM2 critical indices the Liberman inequality can be similarly generalized to a stronger relation (53). It is clear that a similar results should be obtained for the systems whose surface tension at the vicinity of the CEP can be reduced to the form of Eq. (9). The case of the asymmetric phase diagrams with other surface tension parameterization should be analyzed both experimentally and theoretically. The example of the present model suggests that for the asymmetric phase diagrams the Fisher scaling inequality can be recovered by the proper definition of the index $\alpha'$, while for such phase diagrams the Griffiths scaling inequality in its usual form can be broken down. This can occur for the linear temperature dependency of the surface tension coefficient in both pure phases since $\xi^+ = \xi^- = 1$ which is not too restrictive as it is demonstrated by the FDM and by the exactly solvable model of surface deformations [33]. The other important constraint on such systems is that their surface tension coefficient should change its sign while crossing the PT line. Clearly, such a property is rather unusual for the ordinary liquids and is more resembling the transition from a mixture of gases into a strongly interacting plasma, which, as one can argue [13,40,42], is the case for the QGP.

Also we believe that for the asymmetric phase diagrams the question of the proper definition of the critical exponents and the scaling relations between them should be thoroughly investigated both experimentally and theoretically. In particular, in many works the index $\alpha'$ is difficult to determine directly and, hence, it is usually found from the hyperscaling relation [8,58] or even from one of the scaling equalities (1) or (2), while, as it is suggested by the above analysis, in some cases one can get the unexpected results for the traditional scaling inequalities.

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VI. APPENDICES

A. The isothermal compressibility

In this appendix we give the details of the isothermal compressibility evaluation. As usual, one can simplify the isothermal compressibility as $K_T = \frac{1}{\rho} \frac{\partial \rho}{\partial T} \bigg|_T = \frac{1}{\rho} \frac{\partial \rho}{\partial \mu}$. An explicit expression for the baryonic density of pure phases is as follows

$$\rho^\pm = T \left[ \frac{\partial z_M}{\partial \mu} + \frac{A_\mu - \frac{\partial \Sigma^\pm}{\partial \mu} u_{I_T - \kappa}}{1 - \frac{\partial F_H}{\partial \mu} + u_{I_{T - 1}}} \right], \quad (55)$$

from which one directly obtains its $\mu$ derivative as

$$\frac{\partial \rho^\pm}{\partial \mu} = T \left[ \frac{\partial^2 z_M}{\partial \mu^2} + f_1 + f_2 \right], \quad (56)$$

where the following notations are introduced

$$f_1 = \frac{1}{1 - \frac{\partial F_H}{\partial \mu} + u_{I_{T - 1}}} \left[ \frac{\partial A_\mu}{\partial \mu} - \frac{\partial^2 \Sigma^\pm}{\partial \mu^2} u_{I_T - \kappa} - \frac{\partial \Sigma^\pm}{\partial \mu} \frac{\partial \mu}{\partial \mu} \right] \times I_{T - \kappa} + \frac{\alpha \Sigma^\pm}{\partial \mu} \left( \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - \kappa} - \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - 2 \kappa} \right), \quad (57)$$

and

$$f_2 = \frac{A_\mu - \frac{\partial \Sigma^\pm}{\partial \mu} u_{I_T - \kappa}}{1 - \frac{\partial F_H}{\partial \mu} + u_{I_{T - 1}}}^2 \left[ \frac{\partial^2 F_H}{\partial s \partial \mu} - \frac{\partial \mu}{\partial \mu} I_{T - 1} + u \left( \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - 2} + \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - 1 - \kappa} \right) \right]. \quad (58)$$

All the regular terms in the expressions (56), (57) and (58) can be neglected, since the isothermal compressibility behavior near the CEP is defined by the singular ones. Hence, keeping only the singular terms we get

$$T^{-1} \frac{\partial \rho^\pm}{\partial \mu} \approx \frac{\frac{\partial \Sigma^\pm}{\partial \mu} u \left( \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - \kappa} - \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - 2 \kappa} \right)}{1 - \frac{\partial F_H}{\partial \mu} + u_{I_{T - 1}}} + \left( A_\mu - \frac{\partial \Sigma^\pm}{\partial \mu} u_{I_T - \kappa} \right) \left( \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - 2} + \frac{\partial \Sigma^\pm}{\partial \mu} I_{T - 1 - \kappa} \right) \left( 1 - \frac{\partial F_H}{\partial \mu} + u_{I_{T - 1}} \right)^2 - \frac{\frac{\partial \Sigma^\pm}{\partial \mu} u_{I_T - \kappa}}{1 - \frac{\partial F_H}{\partial \mu} + u_{I_{T - 1}}}. \quad (59)$$

In order to calculate the critical exponent $\gamma'$ one, according to the suggestion of [13], has to evaluate the
difference of the isothermal compressibilities across the PT line $\Delta K_T = \left[ \frac{1}{\rho^2} \frac{\partial \rho^+}{\partial \mu} - \frac{1}{\rho^2} \frac{\partial \rho^-}{\partial \mu} \right]_{T=T_c}$. Requiring that such a quantity remains finite everywhere at the phase coexistence curve, except for the CEP, one has to demand that all the integrals $I_{\tau-q}$ in Eq. (59) are finite for $T = T_c$, including $T = T_{\text{exp}}$. The analysis shows that it is sufficient to require that all such integrals, including $I_{\tau-2}$, converge for $\tau > 3$ at the PT line, where $\Delta z = 0$ and $\Sigma^\pm = 0$. Then, the only possible singularity at the CEP in the expression for $\frac{\partial \rho^\pm}{\partial \mu}$ is provided by the term which is proportional to $\frac{\partial^2 \Sigma^\pm}{\partial \mu^2}$. Then for the isothermal compressibility one finds

$$K_T^\pm \simeq -(\rho^\pm)^{-2} \frac{\partial^2 \Sigma^\pm}{\partial \mu^2} \cdot \frac{T u I_{\tau-\alpha}}{1 - \frac{\partial F}{\partial s} + u I_{\tau-1}}. \quad (60)$$

Just this expression is used to evaluate the right hand side of Eq. (29).

### B. Volume fraction of pure phases

In order to establish the relation between the indices $\beta$ and $\chi$ it is convenient to analyze the behavior of the pure phase volume fractions at the critical isochore near the CEP. The QGBSTM2 critical isochore completely lies inside the mixed phase. The critical baryonic density can be described by the volume fraction of hadrons $\lambda$ (the QGP volume fraction is $1 - \lambda$, respectively) as

$$\rho_{\text{exp}} = \lambda \rho^-|_{T=T_c} + (1 - \lambda) \rho^+|_{T=T_c}, \quad (61)$$

where $\rho^-$ and $\rho^+$ are, respectively, the baryonic densities of hadronic phase and QGP. From the previous equation one finds

$$\lambda = \frac{\rho^+ - \rho_{\text{exp}}}{\rho^+ - \rho^-} \quad \text{and} \quad 1 - \lambda = \frac{\rho_{\text{exp}} - \rho^-}{\rho^+ - \rho^-}. \quad (62)$$

Using an evident relation $\rho_{\text{exp}} = \rho M|_{\text{exp}}$ and the definition of the baryonic density one obtains

$$(\rho^\pm - \rho_{\text{exp}})|_{T=T_c} = (T_c - T_{\text{exp}}) \left[ \frac{\partial z M}{\partial \mu} \right]_{\text{exp}}$$

$$+ T_c \left[ A_\mu \frac{\partial \Sigma^\pm}{\partial \mu} u I_{\tau-\alpha} + \frac{\partial z M}{\partial \mu} - \frac{\partial z M}{\partial \mu} \right]_{\text{exp} \, I_{\tau-1}}. \quad (63)$$

Applying the parameterization defined by Eqs. (9), (13) and (17) to the right hand side of (63), one can easily demonstrate that either $(\rho^\pm - \rho_{\text{exp}})|_{T=T_c} \sim T_{\text{min}}(\chi, \frac{1}{\xi^T})$ for $\chi^+ = 1$ or $(\rho^\pm - \rho_{\text{exp}})|_{T=T_c} \sim T_{\text{min}}(\chi, \frac{1}{\xi^T})$ for $\chi^+ > 1$. Note that for this proof it is necessary to expand the difference $\frac{\partial z M}{\partial \mu} - \frac{\partial z M}{\partial \mu} |_{\text{exp}}$ up to linear terms of $\mu - \mu_{\text{exp}}$ and $T_c - T_{\text{exp}}$. Finally, using the definition of the index $\beta$, i.e. $(\rho^+ - \rho^-)|_{T=T_c} \sim T^\beta$, and the feature of the present model $\chi^+ = 1$ we find the following behavior of the volume fractions

$$\lambda \sim \min\left(\chi, \frac{1}{\xi^T}\right) - \beta, \quad (64)$$

$$1 - \lambda \sim \begin{cases} \min(\chi, \frac{1}{\xi^T}) - \beta, & \text{for } \chi^+ = 1; \\ \min(\chi, \frac{1}{\xi^T}) - \beta, & \text{for } \chi^+ > 1. \end{cases} \quad (65)$$

in the vicinity of the CEP. Combining these results with the fact that both $\lambda$ and $(1 - \lambda)$ are finite at the CEP, we obtain a very important consequences that $\min(\chi, \frac{1}{\xi^T}) - \beta \geq 0$, and either $\min(\chi, \frac{1}{\xi^T}) - \beta \geq 0$ for $\chi^+ = 1$ or $\min(\chi, \frac{1}{\xi^T}) - \beta \geq 0$ for $\chi^+ > 1$. An explicit expression for the index $\beta$ (28) allows us to rewrite all these conditions as

$$\beta \leq \min\left(\chi, 1, \frac{1}{\xi^T}\right), \quad (66)$$

which holds always. From the inequality above one immediately deduces that $\beta \leq \chi$. These results are important for finding the index $\alpha_\nu$. Indeed, recalling the expression (28) one can write

$$\min\left[ \min\left(\chi, \frac{1}{\xi^T}\right), \min\left(\chi, 1, \frac{1}{\xi^T}\right) \right] = \beta, \quad (67)$$

for the case $\chi^+ = 1$ and

$$\min\left[ \min\left(\chi, \frac{1}{\xi^T}\right), \min\left(\chi, 1, \frac{1}{\xi^T}\right) \right] = \beta, \quad (68)$$

for the case $\chi^+ > 1$. From the above equations one can immediately deduce that $\max((\rho^+ - \rho M), (\rho^- - \rho M)) \sim T^\beta$ at the vicinity of the CEP, since $\beta \geq 0$. Thus, here we showed that the term staying inside the square brackets on the right hand side of Eq. (43), indeed, behaves as $t^{\frac{1}{\chi^+} + \beta - 2}$.

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