The Paneitz Curvature Problem on Lower Dimensional Spheres

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Abstract. In this paper we prescribe a fourth order conformal invariant (the Paneitz curvature) on the \(n\)-spheres, with \(n \in \{5, 6\}\). Using dynamical and topological methods involving the study of critical points at infinity of the associated variational problem, we prove some existence results.

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1 Introduction and the Main Results

In [24], Paneitz introduced a conformally fourth order operator defined on 4-manifolds. In [8], Branson generalized the definition to \(n\)-dimensional Riemannian manifolds, \(n \geq 5\). Given a smooth compact Riemannian \(n\)-manifold \((M,g)\), \(n \geq 5\), let \(P^n_g\) be the operator defined by

\[ P^n_g u = \Delta^2_g u - \text{div}_g (a_n S_g g + b_n \text{Ric}_g) du + \frac{n-4}{2} Q^n_g u, \]

where

\[ a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = \frac{-4}{n-2} \]

\[ Q^n_g = -\frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |\text{Ric}_g|^2 \]

and where \(S_g\) denotes the scalar curvature of \((M,g)\) and \(\text{Ric}_g\) denotes the Ricci curvature of \((M,g)\).

Such a \(Q^n\) is a fourth order invariant and we call it the Paneitz curvature. For more details about the properties of the Paneitz operator, see for example [8], [9], [10], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and the references therein.

If \(\bar{g} = u^{4/(n-4)} g\) is a metric conformal to \(g\), where \(u\) is a smooth positive function, then for all \(\varphi \in C^\infty(M)\) we have

\[ P^n_g (u \varphi) = u^{(n+4)/(n-4)} P^n_{\bar{g}} (\varphi). \]

Taking \(\varphi \equiv 1\), we then have

\[ P^n_g (u) = \frac{n - 4}{2} Q^n_{\bar{g}} u^{(n+4)/(n-4)} \quad (1.1) \]

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In view of equation (1.1), a natural question is whether it is possible to prescribe the Paneitz curvature, that is: given a function \( f : M \to \mathbb{R} \), does there exist a metric \( \tilde{g} \) conformally equivalent to \( g \) such that \( Q^n_{\tilde{g}} = f \)? According to equation (1.1), the problem is equivalent to finding a smooth positive solution of the following equation

\[
P^n_g(u) = \frac{n - 4}{2} f u^{(n+4)/(n-4)}, \quad u > 0 \quad \text{in } M.
\]

In this paper, we consider the case of standard sphere \( S^n \) endowed with its standard metric \( g_0 \), and in particular the cases \( n = 5 \) and \( n = 6 \). We are thus reduced to find a positive solution \( u \) of the problem

\[
P u := \Delta^2 u - c_n \Delta u + d_n u = K u^{n+4}, \quad u > 0 \quad \text{in } S^n,
\]

where \( c_n = \frac{1}{2}(n^2 - 2n - 4) \) and \( d_n = \frac{n-4}{16} n(n^2 - 4) \) and where \( K \) is a given \( C^3 \) function defined on \( S^n \).

More precisely, our aim is to give sufficient conditions on \( K \) such that equation (1.3) possesses a solution. It is easy to see that a necessary condition on \( K \) for solving equation (1.3) is that \( K \) has to be positive somewhere. In addition, there are topological obstructions of Kazdan-Warner type to solve (1.3) (see [17] and [27]) and so a natural question arises : under which conditions has to be positive somewhere. In addition, there are topological obstructions of Kazdan-Warner type to solve (1.3) (see [17] and [27]) and so a natural question arises : under which conditions on \( K \), (1.3) has a solution. Our aim is to handle such a question, using some topological and dynamical tools of the theory of critical points at infinity, see Bahri [1].

To state our main results, we need to introduce some notations. Throughout this paper \( K \) denotes a positive \( C^3 \) function on \( S^n \) \(( n = 5, 6 \) which has only nondegenerate critical points \( y_1, ..., y_N \) such that \( -\Delta K(y_i) \neq 0 \) for any \( i = 1, ..., N \). Each \( y_i \) is assumed to be of Morse index \( k_i \). For the sake of simplicity, we assume that

\[-\Delta K(y_i) > 0 \quad \text{for } 1 \leq i \leq l \quad \text{and} \quad -\Delta K(y_i) < 0 \quad \text{for } l + 1 \leq i \leq N.\]

For any \( s \in \{1, ..., l\} \) and for any \( s \)-tuple \( \tau_s = (i_1, ..., i_s) \in \{1, ..., l\}^s \) such that \( i_p \neq i_q \) for \( p \neq q \), we introduce a matrix \( M(\tau_s) = (m_{pq})_{1 \leq p, q \leq s} \) with

\[
m_{pp} = \frac{-\Delta K(y_{i_p})}{K(y_{i_p})^2}, \quad m_{pq} = -30 \frac{G(y_{i_p}, y_{i_q})}{(K(y_{i_p}) K(y_{i_q}))^4}, \quad \text{if } p \neq q,
\]

where \( G \) is a Green function for \( \mathcal{P} \) on \( S^6 \). It is given by \( G(x, y) = (1 - \cos d(x, y))^{-1} \).

Let \( Z \) be a pseudogradient of \( K \), of Morse-Smale type (that is, the intersections of the stable and the unstable manifolds of the critical points of \( K \) are transverse.)

Set

\[
X = \bigcup_{1 \leq i \leq l} W_s(y_i),
\]

where \( W_s(y_i) \) is the stable manifold of \( y_i \) for \( Z \).

Now, we are able to state our main results.

**Theorem 1.1** Let \( n = 5 \). Assume that the following two assumptions hold :

1. \( X \) is not contractible
2. \( W_s(y_i) \cap W_u(y_j) = \emptyset \) for any \( i \in \{1, ..., l\} \) and for any \( j \in \{l + 1, ..., m\} \).

Then (1.3) has a solution.
Theorem 1.2 Let $n = 6$ and assume that the following assumption holds:

(H) $\Delta K(y_i)\Delta K(y_j) < 900G(y_i, y_j)^2K(y_i)K(y_j)$ for any $i \neq j$ in $\{1, ..., l\}$. If

$$1 \neq \sum_{s=1}^{l} (-1)^{k_s},$$

where $k_s$ is the Morse index of $K$ at $y_s$, then (1.3) has a solution.

Remark 1.3 The assumption (H) implies that the least eigenvalue $\rho(y_i, y_j)$ of the matrix $M(y_i, y_j)$ defined by (1.4) is negative for any indices $i \neq j$ in $\{1, ..., l\}$.

Theorem 1.4 Let $n = 6$. Assume that the following two assumptions hold:

$(H_1)$ $X$ is not contractible

$(H_2)$ $\Delta K(y_i)\Delta K(y_j) < 900G(y_i, y_j)^2K(y_i)K(y_j)$ for any $i \neq j$ in $\{1, ..., l\}$. If

$W_s(y_i) \cap W_u(y_j) = \emptyset$ for any $i \in \{1, ..., l\}$ and for any $j \in \{l+1, ..., m\}$,

then (1.3) has a solution.

Our approach extends the topological and dynamical methods developed by Bahri [2], Bahri-Coron [3] and Ben Ayed et al [7], to the framework of such higher order equations. To perform such an extension, a fine analysis of the gradient flow of the Euler Lagrange Functional is needed. It turns out that such a gradient flow satisfies the Palais smale condition on its decreasing flow lines far from a finite number of isolated blow up. Then we construct a special pseudogradient near such "singularities” and perform a Morse reduction. Such a fine analysis of these "singularities” has its own interest, and plays a central role in the derivation of further existence results to be published in forthcoming papers.

Another main issue in our approach is to prove the positivity of the critical point obtained by our process. It is known that in the framework of such a higher order equation, such an issue is far from been trivial in general (see [19] for example), and the way we handle it here is very simple compared with the literature.

Besides the above results, we point out that our method enables us to reprove some existence results, obtained recently by Djadli, Malchiodi and Ould Ahmedou [19], namely:

Theorem 1.5 Assume that $n = 5$. If

$$\sum_{1 \leq i \leq l} (-1)^{k_i} \neq -1,$$

where $k_i$ is the Morse index of $K$ at $y_i$, then (1.3) has a solution.

Theorem 1.6 Let $n = 6$ and assume that for any $s \in \{1, ..., l\}$ and for any $\tau_s$, $M(\tau_s)$ is nondegenerate. If

$$1 \neq \sum_{s=1}^{l} \sum_{\tau_s = (i_1, ..., i_s)/\rho(\tau_s) > 0} (-1)^{7s-1-\sum_{j=1}^{s} k_{ij}},$$

then (1.3) has a solution, where $\rho(\tau_s)$ denotes the least eigenvalue of $M(\tau_s)$. 
We organize our paper as follows. In section 2, we set up the variational structure and recall some preliminaries. In section 3, we perform an expansion of the Euler functional associated to (1.3) and its gradient near critical points at infinity, then in section 4, we give the characterization of the critical points at infinity. In section 5, we provide the proofs of our results. The proofs require some technical results which, for the convenience of the reader, are given in the appendix.

2 Preliminary Tools

In this section we recall the functional setting and the variational problem and its main features. For \( K \equiv 1 \), the solutions of (1.3) are the family \( \tilde{\delta}_{(a,\lambda)} \) defined by

\[
\tilde{\delta}_{(a,\lambda)}(x) = \beta_n \frac{1}{2^n} \frac{\lambda^{\frac{n-4}{2}}}{\left(1 + \frac{\lambda^2-1}{2}(1 - \cos d(x,a))\right)^{\frac{n-4}{2}}},
\]

where \( a \in S^n \), \( \lambda > 0 \) and \( \beta_n \) is a positive constant. After performing a stereographic projection \( \pi \) through the point \(-a\) as pole, the function \( \tilde{\delta}_{(a,\lambda)} \) is transformed into

\[
\delta_{(0,\lambda)}(y) = \beta_n \frac{\lambda^{\frac{n-4}{2}}}{\left(1 + \lambda^2 |y|^2\right)^{\frac{n-4}{2}}},
\]

which is a solution of the problem

\[
\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{(see [22])}.
\]

The space \( H^2_2(S^n) \) is equipped with the norm:

\[
|| u ||^2 = \langle u, u \rangle = \int_{S^n} \mathcal{P} u \cdot u = \int_{S^n} |\Delta u|^2 + c_n \int_{S^n} |\nabla u|^2 + d_n \int_{S^n} u^2.
\]

We denote by \( \Sigma \) the unit sphere of \( H^2_2(S^n) \) and we set \( \Sigma^+ = \{ u \in \Sigma / u > 0 \} \).

We introduce the following functional defined on \( \Sigma \) by

\[
J(u) = \frac{1}{\left(\int_{S^n} K |u|^{\frac{2n}{n-4}}\right)^{\frac{n-4}{n}}} = \frac{|| u ||^2}{\left(\int_{S^n} K |u|^{\frac{2n}{n-4}}\right)^{\frac{n-4}{n}}}.
\]

The positive critical points of \( J \), up to a multiplicative constant, are solutions of (1.3). The Palais-Smale condition fails for \( J \) on \( \Sigma^+ \). This failure can be described using similar arguments as in [11], [23], [26].

**Proposition 2.1** Assume that \( J \) has no critical point in \( \Sigma^+ \) and let \( (u_k) \) be a sequence in \( \Sigma^+ \) such that \( J(u_k) \) is bounded and \( \nabla J(u_k) \) goes to 0. Then there exist an integer \( p \) and a sequence \( \varepsilon_k \) such that \( u_k \in V(p,\varepsilon_k) \), where \( V(p,\varepsilon) \) is defined by

\[
V(p,\varepsilon) = \{ u \in \Sigma / \exists a_1, ..., a_p \in S^n, \exists \lambda_1, ..., \lambda_p > \varepsilon^{-1}, \exists \alpha_1, ..., \alpha_p > 0 \text{ with} \}
\]

\[
|| u - \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} || < \varepsilon; \quad \langle J(u)^{\frac{n}{n-4}}, \sum_{i=1}^{p} \frac{8 \alpha_i^4}{\alpha_i^4 + K(a_i)} - 1 \rangle < \varepsilon \forall i, \quad \varepsilon_{ij} < \varepsilon \forall i \neq j \}.
\]
Here
\[
\varepsilon_{ij} = \left( \lambda_i \frac{\lambda_j}{\lambda_j} + \lambda_i \lambda_j (1 - \cos d(a_i, a_j)) \right) - \frac{n-4}{n}.
\]

The following result defines a parametrization of the set \( V(p, \varepsilon) \).

**Proposition 2.2** For any \( p \in \mathbb{N}^* \), there exists \( \varepsilon_p > 0 \) such that, if \( 0 < \varepsilon < \varepsilon_p \) and \( u \in V(p, \varepsilon) \), then the following minimization problem
\[
\min \left\{ \| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \|, \alpha_i > 0, \lambda_i > 0, a_i \in S^n \right\}
\]
has a unique solution \( (\alpha, a, \lambda) = (\alpha_1, ..., \alpha_p, a_1, ..., a_p, \lambda_1, ..., \lambda_p) \) (up to permutation). In particular, we can write \( u \in V(p, \varepsilon) \) as follows
\[
u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + v,
\]
where \( v \in H^2_2(S^n) \) such that, for any \( i = 1, ..., p \)
\[
(V_0) : \quad < v, \varphi_i > = 0 \quad \text{for} \quad \varphi \in \{ \delta_{(a_i, \lambda_i)}, \partial \delta_{(a_i, \lambda_i)}/\partial \lambda_i, \partial \delta_{(a_i, \lambda_i)}/\partial (a_i) \} \quad \forall \ j = 1, ..., n, \quad (2.1)
\]
for some system of coordinates \( (a_i)_1, ..., (a_i)_n \) on \( S^n \) near \( a_i \).

The proof of Proposition 2.2 is similar, up to minor modifications, to the corresponding statements in [4] and [2].

### 3 Expansion of the Functional and its Gradient

In this section, we perform a useful expansion of the functional associated to (1.3) and its gradient near a critical point at infinity.

**Proposition 3.1** For \( \varepsilon \) small enough and \( u = \sum_{i=1}^{p} \alpha_i \delta_i + v \in V(p, \varepsilon) \), we have the following expansion
\[
J(u) = \left( \sum_{i=1}^{p} \frac{2n+4}{2n-4} K(a_i) \right) \left[ 1 - \frac{n-4}{n} c_2 \sum_{i=1}^{p} \frac{\alpha_i^{2n}}{2n-4} K(a_i)S_n \right] 4\Delta K(a_i) \lambda_i^2
\]
\[
+ \frac{c_1}{S_n} \sum_{i \neq j} \alpha_i \alpha_j \varepsilon_{ij} \left( \frac{1}{\sum_{k=1}^{p} \alpha_k^2} - \frac{2\alpha_i \alpha_j}{\sum_{j=1}^{p} \alpha_j^2} K(a_i) \right) - f(v) + \frac{1}{\sum_{i=1}^{p} \alpha_i^2 S_n} Q(v, v)
\]
\[
+ o \left( \sum_{i \neq j} \varepsilon_{ij} + \sum \frac{1}{\lambda_i^2} + \| v \|_{\min(\frac{2n}{n+3})} \right),
\]
where $c_1$ and $c_2$ are positive constants (defined in Lemmas 6.8 and 6.9), $S_n = \int_{R^n} \delta^{2n/(n-4)}$, 

$$Q(v, v) = \| v \|^2 - \frac{n + 4}{n - 4} \frac{\sum_{i=1}^{p} \alpha_i^2}{\sum_{i=1}^{p} \alpha_i} \int_{S^n} K(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i) \frac{n}{n-4} v^2$$

and

$$f(v) = \frac{2}{\sum_{j=1}^{p} \alpha_j^{\frac{2n}{n-4}} K(a_j) S_n} \int_{S^n} K(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i) \frac{n+4}{n-4} v.$$ 

(Here and in the sequel $\tilde{\delta}_i$ denotes $\tilde{\delta}_{(a_i, \lambda_i)}$).

**Remark 3.2** According to Proposition 3.1, we see that there is a difference between the three cases $n = 5$, $n = 6$ and the higher dimensions. In the case $n = 5$, the interaction between two masses dominates the self interaction, while for $n = 6$, there is a balance phenomenon, and for $n \geq 7$, the self interaction dominates the interaction between two masses.

**Proof of Proposition 3.1** Let us recall that

$$J(u) = \frac{\| u \|^2}{(\int_{S^n} K u^{2n/(n-4)})^{\frac{n}{n-4}}}.$$

Using Lemmas 6.7 and 6.8 in the Appendix, we have

$$\| u \|^2 = \sum_{i=1}^{p} \alpha_i^2 \| \tilde{\delta}_i \|^2 + \sum_{i \neq j} \alpha_i \alpha_j < \tilde{\delta}_i, \tilde{\delta}_j > + \| v \|^2$$

$$= \sum_{i=1}^{p} \alpha_i^2 S_n + \sum_{i \neq j} \alpha_i \alpha_j (c_1 \varepsilon_{ij} + o(\varepsilon_{ij})) + \| v \|^2$$

$$= (\sum_{i=1}^{p} \alpha_i^2 S_n) \left(1 + c_1 \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\sum_{k=1}^{p} \alpha_k^2 S_n} \varepsilon_{ij} + \frac{1}{\sum_{i=1}^{p} \alpha_i^2 S_n} \| v \|^2 + o(\sum_{i \neq j} \varepsilon_{ij}) \right).$$

Furthermore, we have

$$\int_{S^n} K(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + v)^{\frac{2n}{n-4}} = \int_{S^n} K(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i)^{\frac{2n}{n-4}} + \frac{2n}{n-4} \int_{S^n} K(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i)^{\frac{n+4}{n-4}} v$$

(3.1)

$$+ \frac{n(n+4)}{(n-4)^2} \int_{S^n} K(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i)^{\frac{8}{n-4}} v^2 + O\left(\int (\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i)^{\frac{12-n}{n-4}} \inf(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i), v) + \| v \|^2 \right)^\frac{2n}{n-4}.)$$

Since the Sobolev embedding of $H^2_2(S^n)$ in $L^{\frac{2n}{n-4}}$ is continuous, then there exists a constant $c$ such that

$$\int \| v \|^{(2n)/(n-4)} \leq c \| v \|^{(2n)/(n-4)}.$$
We also have
\[
\int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^{2n} = \sum_{i=1}^{p} \alpha_i^{2n} \int K \delta_i^{2n} + \frac{2n}{n-4} \sum_{i \neq j} \alpha_i \alpha_j \int K \delta_i \delta_j^{n+4} \delta_j^{n} + O \left( \sum_{i \neq j} \int \delta_i^{2n} \inf(\tilde{\delta}_i, \tilde{\delta}_j)^2 \right).
\]

For \( n \geq 8 \), we have \( 8/(n-4) \leq 2 \) and using Lemma 6.10 we find
\[
\int \frac{8}{n-4} \inf(\tilde{\delta}_i, \tilde{\delta}_j)^2 \leq \int (\delta_i \delta_j)^{n-4} = O(\varepsilon_{ij}^n \log \varepsilon_{ij}^{-1}).
\]

For \( n < 8 \), we have \( 8/(n-4) > 2 \) and using Lemma 6.10 we obtain
\[
\int \delta_i^{2(n-4)} \inf(\tilde{\delta}_i, \tilde{\delta}_j)^2 \leq \int \delta_i^{2(n-4)} \delta_j^2 \leq c \left( \int (\delta_i \delta_j)^{n-4} \right)^{2(n-4)} = O(\varepsilon_{ij}^2 (\log \varepsilon_{ij}^{-1})^{2(n-4)}).
\]

Using Lemmas 6.9, 6.10, (3.2) and (3.3), we derive that
\[
\int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^{2n} = \sum_{i=1}^{p} \alpha_i^{2n} \left( K(a_i) S_n + c_2 \frac{4 \Delta K(a_i)}{\lambda_i^3} + O \left( \frac{1}{\lambda_i^3} \right) \right)
\]
\[
+ \frac{2n}{n-4} \sum_{i \neq j} \alpha_i \alpha_j \left( c_1 K(a_i) \varepsilon_{ij} + o(\varepsilon_{ij} + \frac{1}{\lambda_i^3}) \right).
\]

Then (3.1) becomes
\[
\int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + v \right)^{2n} = \sum_{i=1}^{p} \alpha_i^{2n} \left( K(a_i) S_n + c_2 \frac{4 \Delta K(a_i)}{\lambda_i^3} \right)
\]
\[
+ \frac{2n}{n-4} \sum_{i \neq j} \alpha_i \alpha_j c_1 K(a_i) \varepsilon_{ij} + \frac{2n}{n-4} \int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^{n+4} v
\]
\[
+ \frac{n(n+4)}{(n-4)^2} \int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^{n} v^2 + O \left( \| v \|^{\min \left( \frac{n(n+4)}{2(n-4)}, 3 \right)} \right) + o \left( \sum_{i \neq j} \frac{1}{\lambda_i^2} + \varepsilon_{ij} \right).
\]

Thus our result follows. \( \square \)

As usual in this type of problem, we first deal with the \( v \)-part of \( u \). Let us introduce the following set
\[
E = \{ v/v \text{ satisfies } (V_0) \text{ and } \| v \| < \varepsilon \},
\]
where \( (V_0) \) is defined in (2.1).

**Proposition 3.3** For any \( u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon) \) given, there exists a unique \( \varpi = \varpi(a, \alpha, \lambda) \) which minimizes \( J(u+v) \) with respect to \( v \in E \). Moreover, we have the following estimate
\[
\| \varpi \| \leq c \| f \| \leq c \left( \sum_{i=1}^{p} \left| \frac{\nabla K(a_i)}{\lambda_i} \right| + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \min \left( \frac{1}{\lambda_i}, \frac{1}{\lambda_j}, \frac{n(n+4)}{2(n-4)} \right) \log \varepsilon_{ij}^{-1} \right) \min \left( \frac{n(n+4)}{2(n-4)}, 3 \right).
\]
Before we prove this result, we give the following proposition, whose proof is deferred to the Appendix

**Proposition 3.4** For any \( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon) \) given, \( Q(v, v) \) is a quadratic positive form in the space \( E \).

**Proof of Proposition 3.3** On one hand, using Proposition 3.4, we derive \( ||v|| < c ||f|| \), with \( c > 0 \). On the other hand, we have

\[
f(v) = 2 \left( \sum_{j=1}^{p} \alpha_j \tilde{n} \right) K(a_j) S_n^{-1} \int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^{\frac{n+4}{n-4}} v.
\]

Observe that

\[
\int_{S^n} K \left( \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \right)^{\frac{n+4}{n-4}} v = \sum_{i=1}^{p} \alpha_i \tilde{n} \int K \left( \tilde{\delta}_i \right)^{\frac{n+4}{n-4}} v + O \left( \sum_{i \neq j} \int \tilde{\delta}_i^{\frac{8}{n-4}} \inf(\tilde{\delta}_i, \tilde{\delta}_j) \left| v \right| \right)
\]

\[
= O \left( \sum_{i=1}^{p} \left| \nabla K(a_i) \right| \int \left| x - a_i \right| \tilde{\delta}_i^{\frac{4}{n-4}} \left| v \right| + \frac{||v||}{\lambda_i^2} + \sum_{i \neq j} \tilde{\delta}_i^{\frac{8}{n-4}} \inf(\tilde{\delta}_i, \tilde{\delta}_j) \left| v \right| \right)
\]

\[
\leq c ||v|| \left( \sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i} + 1 \right) + \sum_{i \neq j} \varepsilon_{ij} \min(1, \frac{4}{2(n-4)} \left( \log \varepsilon_{ij} \right)^{\frac{n+4}{n} \min(n, \frac{n+4}{2})} \left| v \right| \right).
\]

Thus the result follows. \( \square \)

**Proposition 3.5** For any \( u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon) \), we have the following expansion

\[
< \nabla J(u), \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} > = 2J(u) \left[ -c_1 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{n} c_2 \alpha_i^{\frac{n+4}{n-4}} J(u)^{\frac{n}{n-4}} \frac{4 \Delta K(a_i)}{\lambda_i^2} \right.
\]

\[
+ o \left( \sum_{i} \frac{1}{\lambda_i^2} + \sum_{j \neq k} \varepsilon_{kj} \right) \right] .
\]

**Proof.** We have

\[
\nabla J(u) = 2J(u) \left[ u - J(u) \frac{n}{n-4} \mathcal{P}^{-1}(Ku^{\frac{n+4}{n} - 1}) \right].
\]

Thus

\[
< \nabla J(u), \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} > = 2J(u) \left[ \sum_{j=1}^{p} \alpha_j < \tilde{\delta}_j, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} > - J(u)^{\frac{n}{n-4}} \int K \left( \sum_{j=1}^{p} \alpha_j \tilde{\delta}_j \right)^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \right].
\]

Observe that

\[
\int K \left( \sum_{j=1}^{p} \alpha_j \tilde{\delta}_j \right)^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} = \sum_{j=1}^{p} \alpha_j^{\frac{n+4}{n-4}} \int K \left( \tilde{\delta}_j \right)^{\frac{n+4}{n-4}} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} + \frac{n+4}{n-4} \sum_{i \neq j} \int K \left( \alpha_i \tilde{\delta}_j \right)^{\frac{n}{n-4}} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \left( \alpha_j \tilde{\delta}_j \right)
\]

\[
+ O \left( \left( \text{if } n \geq 8 \right) \sum_{k \neq j} \int \left( \tilde{\delta}_j \tilde{\delta}_k \right)^{\frac{n}{n-4}} + \left( \text{if } n < 8 \right) \sum_{k \neq j} \int \tilde{\delta}_j^{\frac{n}{n-4}} \tilde{\delta}_k^{\frac{n}{n-4}} \right). \tag{3.4}
\]
Thus using Lemmas 6.7, 6.9, 6.10, and the fact that $J(u)^{n/(n-4)}\alpha_i^{8/(n-4)}K(a_i) = 1 + o(1)$, for each $i$, the result follows.

**Proposition 3.6** For any $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta_i} \in V(p, \varepsilon)$, we have

$$<\nabla J(u), \frac{1}{\lambda_i} \partial a_i > = -2c_3J(u)^{\frac{2n-4}{n-4}} \frac{\nabla K(a_i)}{\lambda_i} + O\left(\frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}\right).$$

**Proof.** We have

$$<\nabla J(u), \frac{1}{\lambda_i} \partial a_i > = 2J(u)\left[\sum_{j=1}^{p} \alpha_j <\tilde{\delta_j}, \frac{1}{\lambda_i} \partial a_i > - J(u)^{\frac{n}{n-4}} \int K\left(\sum_{j=1}^{p} \alpha_j \tilde{\delta_j} \frac{1}{\lambda_i} \partial a_i \right) \right].$$

Furthermore we can obtain (3.4) but with $\frac{1}{\lambda_i} \partial \tilde{\delta_i}$ instead of $\lambda_i \frac{\partial \tilde{\delta_i}}{\partial \lambda_i}$. Thus using Lemmas 6.7 and 6.9, the result follows.

**4 Characterization of the Critical Points at Infinity**

This section is devoted to the characterization of the critical points at infinity for lower dimensions ($n = 5$ and $n = 6$). We recall that the critical points at infinity are the orbits of the flow that remain in $V(p, \varepsilon(s))$, where $\varepsilon(s)$ is a given function such that $\varepsilon(s)$ tends to zero when $s$ tends to $+\infty$ (see [1]).

**Proposition 4.1** Let $n = 5$, for $p \geq 2$, there exists a pseudogradient $W$ so that the following holds.

There is a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta_i} \in V(p, \varepsilon)$ so that

$$( -\nabla J(u + v), W + \frac{\partial \Pi}{\partial (\alpha_i, a_i, \lambda_i)}(W) ) \geq c \left( \sum_{i=1}^{p} \frac{\nabla K(a_i)}{\lambda_i} + \frac{1}{\lambda_i} + \sum_{i \neq i} \varepsilon_{ij} \right).$$

Furthermore, $|W|$ is bounded and the $\lambda_i$’s decrease along the flow lines.

**Proof.** We order the $\lambda_i$’s, for the sake of simplicity we can assume that: $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_p$. Let $I = \{i / \lambda_i \mid \nabla K(a_i) \geq 1\}$. Set

$$Z_1 = -\sum_{i=2}^{p} 2^i \alpha_i \lambda_i \frac{\partial \tilde{\delta_i}}{\partial \lambda_i}, \quad Z_2 = \frac{1}{\lambda_i} \partial a_i \nabla K(a_i).$$

Using Propositions 3.5 and 3.6, we derive that

$$< -\nabla J(u), Z_1 > \geq c \sum_{k \neq r} \varepsilon_{kr} + O\left(\sum_{i=2}^{p} \frac{1}{\lambda_i^2}\right) + o\left(\frac{1}{\lambda_i^2}\right).$$

(4.1)
Let \( \mu > 0 \) such that, for any critical point \( y \) of \( K \), if \( d(a, y) \leq 2\mu \) then \( |\Delta K(a)| > c > 0 \). Two cases may occur.

**Case 1** \( \lambda_2 \leq \lambda_1^2 \) or \( d(a_1, y) > \mu \) for any critical point \( y \).

In this case, we set \( W_1 = MZ_1 + Z_2 \) where \( M \) is a large constant. Observe that in the case where \( d(a_1, y) > \mu \), we can appear \( 1/\lambda_1 \) on the lower bound of (4.2) and therefore all the \( 1/\lambda_i \)'s. Combining (4.1) and (4.2), we derive

\[
< -\nabla J(u), W_1 > \geq c \sum_{i \in I} \frac{|\nabla K(a_i)|}{\lambda_i} + O\left( \sum_{k \neq r} \varepsilon_{kr} \right) + O\left( \sum_{i \in I} \frac{1}{\lambda_i^2} \right). \tag{4.2}
\]

In the other case, that is, \( \lambda_2 \leq \lambda_1^2 \), we can easily prove that \( \frac{1}{\lambda_i} = o(\varepsilon_{12}) \) (since we have \( (\lambda_1 \lambda_2)^{1/2}d(a_1, a_2) \leq c\lambda_1^{3/2} = o(\lambda_1^2) \) and \( (\lambda_2/\lambda_1)^{1/2} \leq \lambda_1^{1/2} = o(\lambda_1^2) \)). Therefore we can also obtain (4.3) in this case.

**Case 2** \( \lambda_2 \geq \lambda_1^2 \) and \( d(a_1, y) \leq 2\mu \) for a critical point \( y \).

We set \( Z_3 = \text{sign}(\Delta K(y)) \lambda_1 \frac{\partial \delta_i}{\partial \lambda_1} \), that is, we increase \( \lambda_1 \) if \( -\Delta K(y) > 0 \) otherwise we decrease it. We define \( W_2 = MZ_1 + Z_3 + mZ_2 \), where \( M \) is a large constant and \( m \) is a small constant. Observe that

\[
< -\nabla J(u), Z_3 > \geq \frac{c}{\lambda_1} + O\left( \sum_{j \neq 1} \varepsilon_{1j} \right)
\]

\[
< -\nabla J(u), W_2 > \geq cM \sum_{k \neq r} \varepsilon_{kr} + O\left( \frac{1}{\lambda_1^2} \right) + c \sum_{j \neq 1} \varepsilon_{1j} + m \sum_{i \in I} \frac{|\nabla K(a_i)|}{\lambda_i} + O\left( m \sum_{k \neq r} \varepsilon_{kr} \right) + O\left( \frac{m}{\lambda_1^2} \right) \geq c \left( \sum_{i \in I} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_1^2} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]

(since \( M \) is large and \( m \) is small). The pseudogradient \( W \) will be built as a convex combination of \( W_1 \) and \( W_2 \).

Arguing as in Appendix B of [7], we easily derive that

\[
< -\nabla J(u + \tau), W + \frac{\partial \psi}{\partial (\alpha_i, a_i, \lambda_i)}(W) > \geq < -\nabla J(u), W > + o\left( \sum_{i \in I} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_1^2} + \sum_{i \neq j} \varepsilon_{ij} \right) \tag{4.4}
\]

and therefore the proposition follows under (4.4).

\[ \Box \]

**Proposition 4.2** For \( n = 5 \), there exists a pseudogradient \( W \) so that the following holds. There is a constant \( c > 0 \) independent of \( u = \alpha \tilde{\delta}_{(\alpha, \lambda)} \in V(1, \varepsilon) \) such that

1) \( < \nabla J(u), W > \geq c\left( \frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^2} \right) \)
2) \(- < \nabla J(u + \varpi), W + \frac{\delta \varpi}{\delta (a, a', \lambda)}(W) > \geq c\left(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^2}\right)\)

3) \(W\) is bounded

4) the only region where \(\lambda\) increases along the flow lines of \(W\) is the region where \(a\) is near a critical point \(y\) of \(K\) with \(-\Delta K(y) > 0\).

**Proof.** Let \(\mu > 0\) such that, for any critical point \(y\) of \(K\), if \(d(x, y) \leq 2\mu\) then \(|\Delta K(x)| > c > 0\). Two cases may occur.

**Case 1** \(d(a, y) > \mu\) for any critical point \(y\). In this case we have \(|\nabla K(a)| > c > 0\). Set

\[ Z_1 = \frac{1}{\lambda} \frac{\partial \delta}{\partial a} \frac{\nabla K(a)}{|\nabla K(a)|} \]

From Proposition 3.6, we have

\[- < \nabla J(u), Z_1 > \geq c\left(\frac{|\nabla K(a)|}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right) \geq c\frac{|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^2}.\]

**Case 2** \(d(a, y) \leq 2\mu\) where \(y\) is a critical point of \(K\). Set

\[ Z_2 = \text{Sign}(-\Delta K(y))\lambda \frac{\partial \delta}{\partial \lambda} + m\varphi(\lambda |\nabla K(a)|) Z_1, \]

where \(m\) is a small constant and \(\varphi\) is a \(C^\infty\) function which satisfies \(\varphi(t) = 1\) if \(t \geq 2\) and \(\varphi(t) = 0\) if \(t \leq 1\). Using Propositions 3.5 and 3.6, we derive that

\[- < -\nabla J(u), Z_2 > \geq \frac{c}{\lambda^2} + cm\left(\frac{|\nabla K(a)|}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right) \geq \frac{c|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^2}.\]

Hence \(W\) will be built as a convex combination of \(Z_1\) and \(Z_2\). Thus the proof of claim 1) is completed. Claims 3) and 4) can be easily derived from the definition of \(W\). Regarding the estimate 2), it can be obtained, arguing as in [2] and [7], using Claim 1). \(\square\)

**Proposition 4.3** Let \(n = 5\). Assume that \(J\) has no critical point in \(\Sigma^+\). Then the only critical points at infinity of \(J\) correspond to \(\delta_{(y, \infty)}\), where \(y\) is a critical point of \(K\) with \(-\Delta K(y) > 0\). Moreover, such a critical point at infinity has a Morse index equal to \(5 - \text{index}(K, y)\).

**Proof.** Using Proposition 2.1, we derive that \(|\nabla J| \geq c\) outside of \(\cup_{p \geq 1} V(p, \varepsilon)\), where \(c\) is a positive constant which depends on \(\varepsilon\). From Proposition 4.1, we easily deduce the fact that there is no critical point at infinity in \(V(p, \varepsilon)\) for \(p \geq 2\). It only remains to see what happens in \(V(1, \varepsilon)\). From Proposition 4.2, we know that the only region where \(\lambda\) increases along the pseudogradient \(W\), defined in Proposition 4.2, is the region where \(a\) is near a critical point \(y\) of \(K\) with \(-\Delta K(y) > 0\). Arguing as in [2] and [7], we can easily deduce from Proposition 4.2 the following normal form:

If \(a\) is near a critical point \(y\) of \(K\) with \(-\Delta K(y) > 0\), we can find a change of variable \((a, \lambda) \rightarrow (\tilde{a}, \tilde{\lambda})\) such that

\[ J(\delta_{(a, \lambda)} + \tilde{v}) = \Psi(\tilde{a}, \tilde{\lambda}) := \frac{S_5^{1/5}}{K(\tilde{a})^{1/5}} \left(1 - \frac{(c - \eta)\Delta K(y)}{\lambda^2 K(y)}\right), \]
where \( c \) is a positive constant and \( \eta \) is a small positive constant.

This yields a split of variables \( a \) and \( \lambda \), thus it is easy to see that if \( \tilde{a} = y \), only \( \tilde{\lambda} \) can move. To decrease the functional \( J \), we have to increase \( \lambda \), thus we obtain a critical point at infinity only in this case and our result follows.

Next, we are going to study the case when \( n = 6 \). For this purpose, we divide the set \( V(p, \varepsilon) \) into five sets.

\[
V_1(p, \varepsilon, \eta) = \{ u/a_i \in B(y_j, \eta), j_i \neq j_k \text{ for } i \neq k \text{ and for } \tau = (j_1, \ldots, j_p), \rho(\tau) > 0 \}
\]

\[
V_2(p, \varepsilon, \eta) = \{ u/a_i \in B(y_j, \eta), j_i \neq j_k \text{ for } i \neq k, -\Delta K(y_j) > 0, \rho(j_1, \ldots, j_p) < 0 \}
\]

\[
V_3(p, \varepsilon, \eta) = \{ u/a_i \in B(y_j, \eta), j_i \neq j_k \text{ for } i \neq k, \exists j_1, \ldots, j_r \text{ s.t. } -\Delta K(y_j) < 0 \}
\]

\[
V_4(p, \varepsilon, \eta) = \{ u/a_i \in B(y_j, \eta), \exists i \neq k \text{ such that } j_i = j_k \}
\]

\[
V_5(p, \varepsilon, \eta) = \{ u/\exists i_1, \ldots, i_q \text{ such that } |a_{i_j} - y| > \eta/2 \text{ for all critical points } y \}
\]

where \( \eta \) is a positive constant such that \( \eta < (1/4) \inf_{i \neq j} d(y_i, y_j) \) and for each \( i \), if \( d(x, y_i) \leq \eta \) then we have \( |\Delta K(x)| > C > 0 \).

We then have the following crucial result.

**Proposition 4.4** Let \( n = 6 \), for \( p \geq 1 \), there exists a pseudogradient \( W \) so that the following holds.

There is a constant \( c > 0 \) independent of \( u = \sum_{i=1}^{p} \alpha_i \tilde{d}_i \in V(p, \varepsilon) \) so that

1. \( -\nabla J(u), W \geq c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i} \frac{\nabla K(a_i)}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right) \).

2. \( (\nabla J(u + \tau), W + \frac{\partial \tau}{\partial (\alpha_i, a_i, \lambda_i)}(W)) \geq c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i} \frac{\nabla K(a_i)}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right) \).

3. \( |W| \) is bounded. Furthermore, when \( u \in \bigcup_{j=3,4,5} V_j \), we have \( d\lambda_i(W) \leq 0 \). When \( u \in V_1 \cup V_2 \), \( |d\lambda_i(W)| \leq c\lambda_i \) for any \( i \). Moreover, the only case where the maximum of the \( \lambda_i \)'s is not bounded is when \( u \in V_1 \).

**Proof.** We start by proving Claim (i). By the assumption, for any critical point \( y \), we have \( \Delta K(y) \neq 0 \). Thus we can choose \( \eta > 0 \) such that for any \( x \in B(y, \eta) \), we have \( |\Delta K(x)| > c > 0 \).

We will define the pseudogradient depending on the sets \( V_i \) to which \( u \) belongs.

First, we consider the case of \( u = \sum_{i=1}^{p} \alpha_i \tilde{d}_i \in V_1(p, \varepsilon, \eta) \), we have for any \( i \neq j, |a_i - a_j| > c \) and therefore

\[
\varepsilon_{ij} = \frac{2}{\lambda_i \lambda_j (1 - \cos d(a_i, a_j)) (1 + o(1))} = \frac{2G(a_i, a_j)}{\lambda_i \lambda_j} (1 + o(1)),
\]

where \( G(a_i, a_j) = (1 - \cos d(a_i, a_j))^{-1} \) is the Green’s function of \( \mathcal{P} \). Thus

\[
\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\varepsilon_{ij} (1 + o(1)) = -\frac{2G(a_i, a_j)}{\lambda_i \lambda_j} (1 + o(1)).
\]

Observe that, since \( u \in V(p, \varepsilon) \), we have \( \alpha_i^4 J(u)^3 K(a_i) = 1 + o(1) \). Thus, Proposition 3.5 becomes

\[
< \nabla J(u), \alpha_i \lambda_i \frac{\partial \tilde{d}_i}{\partial \lambda_i} > = 2J(u)^3 \left[ c_2 \frac{4\Delta K(a_i)}{3 K(a_i)^2 \lambda_i^2} + \sum_{j \neq i} \frac{2c_1 G(a_i, a_j)}{(K(a_i)K(a_j))^{1/2} \lambda_i \lambda_j} + o\left( \sum_k \frac{1}{\lambda_k^2} \right) \right]
\]
We define $Z_1$ by $Z_1 = \sum_{i=1}^{p} \alpha_i \lambda_i (\partial \bar{\delta}_i)/(\partial \lambda_i)$. Thus, we derive

$$< -\nabla J(u), Z_1 > = c^T \Lambda M \Lambda + o(\sum_{k} \frac{1}{\lambda_k^2}) \geq c \sum_{k} \frac{1}{\lambda_k^2} \geq c \sum_{k} \frac{1}{\lambda_k^2} + c \sum_{i \neq j} \epsilon_{ij},$$

where $M$ is the matrix defined by (1.4) and $\Lambda =^T (1/\lambda_1, ..., 1/\lambda_p)$.

We also define

$$Z_a = \sum_{i=1}^{p} \varphi(\lambda_i |\nabla K(a_i)|) \frac{1}{\lambda_i} \frac{\partial \bar{\delta}_i}{\partial a_i} |\nabla K(a_i)|$$

and

$$W_1 = CZ_1 + Z_a,$$

where $C$ is a large constant and where $\varphi$ is a $C^\infty$ function which satisfies $\varphi(t) = 0$ if $t \leq 1$ and $\varphi(t) = 1$ if $t \geq 2$. Using Proposition 3.6 and (4.6), we derive that

$$< -\nabla J(u), W_1 > \geq c \left( \sum_{i} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k} \frac{1}{\lambda_k^2} + \sum_{i \neq j} \epsilon_{ij} \right).$$

Secondly, we study the case of $u = \sum_{i=1}^{p} \alpha_i \bar{\delta}_i \in V_2(p, \varepsilon, \eta)$. Let $\rho$ be the least eigenvalue of $M$. Then, there exists an eigenvector $e$ associated to $\rho$ such that $|| e ||= 1$ with $e_i > 0$ for all $i$. Let $\gamma > 0$ such that for any $x \in B(e, \gamma) = \{ y \in S^{p-1} / || y - e || \leq \gamma \}$, we have $^T x M x < (1/2)\rho$. Two cases may occur.

**Case 1.** $| \Lambda |^{-1} \Lambda \in B(e, \gamma)$. In this case, we define $W_2 = -CZ_1 + Z_a$. As in (4.6) and (4.7), we derive that

$$< -\nabla J(u), W_2 > \geq c \left( \sum_{i} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k} \frac{1}{\lambda_k^2} + \sum_{i \neq j} \epsilon_{ij} \right).$$

**Case 2.** $| \Lambda |^{-1} \Lambda \notin B(e, \gamma)$. In this case, we define

$$Z_2 = -\sum_{i=1}^{p} | \Lambda | \alpha_i \lambda_i^2 \frac{\partial \bar{\delta}_i}{\partial \lambda_i} \left[ \frac{|\Lambda|}{|| y(0) ||} - \frac{y_i(0)}{|| y(0) ||^3} (y(0), |\Lambda| e - \Lambda) \right],$$

where $y(t) = (1 - t)\Lambda + t |\Lambda| e$. Define $\Lambda(t) = y(t)/ || y(t) ||$. Using Proposition 3.5, it is easy to derive that

$$< -\nabla J(u), Z_2 > = -c | \Lambda |^2 \frac{\partial}{\partial t} (^T \Lambda(t) M \Lambda(t)) + o(\sum_{k} \frac{1}{\lambda_k^2}).$$

Observe that

$$^T \Lambda(t) M \Lambda(t) = \rho + \frac{(1 - t)^2}{|| y(t) ||^2} (^T \Lambda M \Lambda - \rho || \Lambda ||^2).$$

Thus, we derive $(\partial)/(\partial t)(^T \Lambda(t) M \Lambda(t)) < -c$. Therefore for $W_2' = CZ_2 + Z_a$, we obtain

$$< -\nabla J(u), W_2' > \geq c \left( \sum_{i} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k} \frac{1}{\lambda_k^2} + \sum_{i \neq j} \epsilon_{ij} \right).$$
Now, we deal with the case of $u = \sum_{i=1}^{p} a_{i} \tilde{\chi}_{i} \in V_{3}(p, \varepsilon, \eta)$. Without loss of generality, we can assume that $1, \ldots, q$ are the indices which satisfy $-\Delta K(a_{i}) < 0$.

Set $I = \{i/\lambda_{i} \leq (1/10) \inf_{k=1,\ldots,q} \lambda_{k}\}$ and let $M_{I}$ be the matrix defined by the points $(a_{i})_{i \in I}$ (as in (1.4)) and $\rho_{I}$ be the least eigenvalue of $M_{I}$. Define

$$Z_{3} = -\sum_{i=1}^{q} \alpha_{i} \lambda_{i} (\partial \tilde{\chi}_{i})/ (\partial \lambda_{i})$$

Then, since $|a_{i} - a_{j}| > c$, using (4.5), we obtain

$$< -\nabla J(u), Z_{3} > \geq c \sum_{k=1}^{q} \left(\frac{1}{\lambda_{k}^{2}} + \sum_{i \neq k, i \in I} \frac{G(a_{i}, a_{k})}{\lambda_{i} \lambda_{k}} \right) \geq c \sum_{k \notin I} \left(\frac{1}{\lambda_{k}^{2}} + \sum_{i \neq k} \varepsilon_{ik}\right). \quad (4.9)$$

If $I \neq \emptyset$, then the lower bound becomes limited to those indices such that $k \notin I$. We have to add another vector field. If the matrix $M_{I}$ is positive definite, we define $Z_{3}' = Z_{1}(\sum_{i \in I} \alpha_{i} \tilde{\chi}_{i})$, that means the action of $Z_{1}$ but using only the indices of $I$. In the other case, that is, the matrix $M_{I}$ is not positive definite, we define $Z_{3}' = Z_{2}(\sum_{i \in I} \alpha_{i} \tilde{\chi}_{i})$. In both cases, we have

$$< -\nabla J(u), Z_{3}' > \geq c \sum_{k \in I} \left(\frac{1}{\lambda_{k}^{2}} + \sum_{i \neq k, i \in I} \varepsilon_{ik}\right) - c \sum_{k \in I, i \notin I} \varepsilon_{ik}. \quad (4.10)$$

Now, we define $W_{3} = C Z_{3} + Z_{3}' + m Z_{a}$ where $C$ is a large constant and $m$ is a small constant. Using (4.9), (4.10) and Proposition 3.6, we derive that

$$< -\nabla J(u), W_{3} > \geq c \left( \sum_{i} \frac{\nabla K(a_{i})}{\lambda_{i}} + \sum_{k} \frac{1}{\lambda_{k}^{2}} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

Next, we will study the case of $u = \sum_{i=1}^{p} a_{i} \tilde{\chi}_{i} \in V_{4}(p, \varepsilon, \eta)$. Let $B_{i} = \{j/a_{j} \in B(y_{i}, \eta)\}$. In this case, there is at least one $B_{i}$ which contains at least two indices. Without loss of generality, we can assume that $1, \ldots, q$ are the indices such that the set $B_{i}$ (1 \leq i \leq q) contains at least two indices. We will decrease the $\lambda_{i}$’s for $i \in B_{i}$ with different speed. For this purpose, let $\chi$ be a smooth cutoff function such that $\chi \geq 0$, $\chi = 0$ if $t \leq \gamma'$ and $\chi = 1$ if $t \geq 1$, where $\gamma'$ is a small constant. For $j \in B_{k}$, set $\overline{\chi}(\lambda_{j}) = \sum_{i \neq j, i \in B_{k}} \chi(\lambda_{j}/\lambda_{i})$. Define

$$Z_{4} = -\sum_{k=1}^{q} \sum_{j \in B_{k}} \alpha_{j} \overline{\chi}(\lambda_{j}) \lambda_{j} \frac{\partial \tilde{\chi}_{j}}{\partial \lambda_{j}}.$$ 

Using Proposition 3.5, we obtain

$$< -\nabla J(u), Z_{4} > = 2J(u) \sum_{k=1}^{q} \sum_{j \in B_{k}} \alpha_{j} \overline{\chi}(\lambda_{j}) \left[ -c_{1} \sum_{i \neq j} \alpha_{i} \lambda_{j} \frac{\partial \varepsilon_{ij}}{\partial \lambda_{j}} + c_{2} \frac{\alpha_{j}^{3} J(u)^{3} 4 \Delta K(a_{j})}{\lambda_{j}^{3}} \right. \right.$$

$$\left. + o(\sum_{r} \frac{1}{\lambda_{r}^{2}} + \sum_{i \neq r} \varepsilon_{ir}) \right].$$
For $j \in B_k$, with $k \leq q$, if $\chi(\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $\lambda_j^{-2} = o(\varepsilon_{ij})$ (for $\eta$ small enough).

Furthermore, for $j \in B_k$, if $i \notin B_k$ or $i \in B_k$ with $\lambda_i$ and $\lambda_j$ are of the same order, that is, $\gamma' < \lambda_i/\lambda_j < 1/\gamma'$, then we have $-\lambda_r(\partial \varepsilon_{ij})/(\partial \lambda_r) = \varepsilon_{ij}(1 + o(1))$, for $r = i, j$. In the case where $i \in B_k$ with (assuming that $\lambda_i < \lambda_j$) $\lambda_i/\lambda_j < \gamma'$, we have $\chi(\lambda_j) - \chi(\lambda_i) \geq 1$. Thus

$$-\chi(\lambda_j)\lambda_j(\partial \varepsilon_{ij})/(\partial \lambda_j) - \chi(\lambda_i)\lambda_i(\partial \varepsilon_{ij})/(\partial \lambda_i) \geq -\lambda_j(\partial \varepsilon_{ij})/(\partial \lambda_j) = \varepsilon_{ij}(1 + o(1)).$$

Thus, we derive that

$$< -\nabla J(u), Z_4 > \geq c \sum_{k=1}^{q} \sum_{j \in B_k, \chi(\lambda_j) \neq 0} (\frac{1}{\lambda_j} + \varepsilon_{ij}).$$

The lower bound does not contain all the indices. We need to add some terms. Let

$$\lambda_{i_0} = \inf \{\lambda_i, i = 1, ..., p\}.$$  

(4.12)

Now, we distinguish two subcases.

Subcase 1. There exists $j$ such that $\chi(\lambda_j) \neq 0$ and $\lambda_{i_0}/\lambda_j > \gamma'$, then we can appear on the lower bound $1/\lambda_{i_0}^2$ and therefore all the $1/\lambda_i^2$ and the $\varepsilon_{ik}$. Thus, we can define $W_4^1 = CZ_4 + Z_a$ where $C$ is a large constant.

Subcase 2. For each $j$, we have $\chi(\lambda_j) = 0$ or $\lambda_{i_0}/\lambda_j \leq \gamma'$. In this case, we define

$$D = (\{i/\chi(\lambda_i) = 0\} \cup (U_{k=1}^q B_k)^c) \cap \{i \in q. \lambda_i/\lambda_{i_0} < 1/\gamma'\}.$$ 

It is easy to see that $\{i/\chi(\lambda_i) = 0\}$ contains at most one index from each $B_j$ for $1 \leq j \leq q$ and therefore for $i, r \in D$ such that $i \neq r$ we have $a_i \in B(y_i, \eta)$ and $a_r \in B(y_r, \eta)$ with $j_i \neq j_r$. Let

$$u_1 = \sum_{i \in D} a_i \delta_i.$$ 

$u_1$ has to satisfy one of the three cases above, that is, $u_1 \in V_i(card(D), \varepsilon, \eta)$ for $i = 1, 2$ or 3. Thus, we can apply the associated vector field which we will denote $Z_4^1$ and we have the estimate

$$< -\nabla J(u), Z_4^1 > \geq c \sum_{i \in D} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i,j \in D} \varepsilon_{ij} \right) + O(\sum_{k \in D, r \notin D} \varepsilon_{kr}) + O(\sum_{i \notin D} \frac{1}{\lambda_i^2}).$$

Observe that for $k \in D$ and $r \notin D$, we have either $r \in B^q := \{i/\chi(\lambda_i) \neq 0\} \cap (U_{j=1}^q B_j)$ or $r \in (B^q)^c$. If $r \in B^q$, we have $\varepsilon_{kr}$ in the lower bound of (4.11). If $r \in (B^q)^c$, in this case since $r \notin D$ we have $\lambda_{i_0}/\lambda_r < \gamma'$. Furthermore, we can prove that $a_k$ and $a_r$ are not in the same set $B(y, \eta)$ and therefore $|a_k - a_r| > c$. Thus

$$\varepsilon_{kr} \leq \frac{c}{\lambda_k \lambda_r} \leq \frac{c \gamma'}{\lambda_k \lambda_{i_0}} = o(\varepsilon_{ki_0}).$$
(γ' small). Since \( i_0 \in D \) (\( i_0 \) is defined by (4.12)), then from \( 1/\lambda_{i_0}^2 \), we can appear on the lower bound all the \( 1/\lambda_i^2 \) and \( \varepsilon_i \) for \( i, r \in \{B^4\}^c \) (since for those indices we have \( |a_i - a_r| > c \)). Thus, we derive that
\[
< -\nabla J(u), Z_i' > = c \left( \sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i^2} + \sum_{i = 1}^p \frac{1}{\lambda_i^2} + \sum_{i,j \in \{B^4\}^c} \varepsilon_{ij} \right) + O(\sum_{k \in D, r \in B^4} \varepsilon_{kr}). \tag{4.13}
\]

Now, we define \( W_4 = CZ_4 + Z_4' + mZ_a \) where \( C \) is a large constant and \( m \) is a small constant. We obtain
\[
< -\nabla J(u), W_4 > = c \left( \sum_{i = 1}^p \frac{|\nabla K(a_i)|}{\lambda_i^2} + \sum_{i = 1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \right). \tag{4.14}
\]

The vector field \( W_4 \) defined in \( V_4(p, \varepsilon, \eta) \) will be a convex combination of \( W_4^1 \) and \( W_4^2 \).

Finally, we consider the case of \( u = \sum_{i = 1}^p \alpha_i \delta_i \in V_5(p, \varepsilon, \eta) \). We order the \( \lambda_i \)'s in an increasing order: \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p \). Let \( i_1 \) be such that for any \( i < i_1 \), we have \( |a_i - y_i| \leq \eta/2 \) where \( y_i \) is a critical point of \( K \) and \( |a_{i_1} - y| > \eta/2 \), for any critical point \( y \). Let us define
\[
u_1 = \sum_{i \leq i_1} \alpha_i \delta_i \]

Observe that \( u_1 \) has to satisfy one of the four cases above that is, \( u_1 \in V_i(i_1 - 1, \varepsilon, \eta) \), for \( i = 1, 2, 3 \) or 4. Thus, we can apply the associated vector field which we will denote \( Z_5 \) and we have the following estimate
\[
< -\nabla J(u), Z_5 > = c \sum_{i < i_1} \left( \frac{|\nabla K(a_i)|}{\lambda_i^2} + \frac{1}{\lambda_i^2} + \sum_{j < i_1} \varepsilon_{ij} + O(\sum_{j > i_1} \varepsilon_{ij}) \right)
\]
\[
+ o(\sum_{i \geq i_1} \frac{1}{\lambda_i^2} + \sum_{k \neq r} \varepsilon_{kr}).
\]

We also define
\[
Z_5' = \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_{i_1}} \frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} - C' \sum_{i \geq i_1} 2^i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i},
\]
where \( C' \) is a large constant. Using Propositions 3.5 and 3.6 and the fact that \( |\nabla K(a_{i_1})| > c \), we find
\[
< -\nabla J(u), Z_5 > \geq \frac{c}{\lambda_i} + O(\sum_{i \neq i_1} \varepsilon_i) + cC' \sum_{i \geq i_1} \left( \sum_{j \neq i} \varepsilon_{ij} + O(\frac{1}{\lambda_i^2}) + o(\sum_{k \neq r} \varepsilon_{kr} + \frac{1}{\lambda_k^2}) \right)
\]
\[
\geq \sum_{i \geq i_1} \left( \frac{c}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} + o(\sum_{k \neq r} \varepsilon_{kr} + \frac{1}{\lambda_k^2}) \right) \tag{4.15}
\]
(since \( C' \) is large). Define \( W_5 = Z_5 + CZ_5' \), where \( C \) is a large constant. We derive that
\[
< -\nabla J(u), W_5 > \geq c \sum_{i = 1}^p \left( \frac{|\nabla K(a_i)|}{\lambda_i^2} + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} \right). \tag{4.16}
\]
Now, we define the pseudogradient $W$ as a convex combination of $W_i$ for $i = 1, \ldots, 5$. The construction of $W$ is completed. It satisfies Claims (i) and (iii) of Proposition 4.5. Regarding (ii), it can be obtained, arguing as in [2] and [7], using the estimate (i).

**Proposition 4.5** Let $n = 6$. Assume that $J$ has no critical point in $\Sigma^+$. Then the only critical points at infinity of $J$ in $\Sigma^+$ correspond to

$$\sum_{j=1}^{p} K(y_{ij})^{-1/4} \delta(\tau_{ij}, \infty) \quad \text{with } \rho(\tau_p) > 0 \quad \text{and } \tau_p = (i_1, \ldots, i_p)$$

where $p \geq 1$ and $\rho(\tau_p)$ denotes the least eigenvalue of $M(\tau_p)$. Moreover, such a critical point at infinity has a Morse index equal to $(7p - 1 - \sum_{j=1}^{p} \text{index}(K, y_{ij}))$.

**Proof.** From Proposition 2.1, we derive that we just need to see what happens in $V(p, \varepsilon)$ for $p \geq 1$. From Proposition 4.4, we deduce that the only region where the $\lambda_i$’s are not bounded is when each $a_j$ is near critical point $y_{ij}$ with $i_j \neq i_k$ for $j \neq k$ and the matrix $M(\tau_p)$ is positive definite. In this region, arguing as in [2] and [7], we can find a change of variable

$$(a_1, \ldots, a_p, \lambda_1, \ldots, \lambda_p) \rightarrow (\bar{a}_1, \ldots, \bar{a}_p, \bar{\lambda}_1, \ldots, \bar{\lambda}_p) := (\bar{a}, \bar{\lambda})$$

such that

$$J(p \sum_{i=1}^{p} \alpha_i \delta(a_i, \lambda_i) + \bar{v}) = \Psi(\alpha_1, \ldots, \alpha_p, \bar{a}, \bar{\lambda}) := \frac{\sum \alpha_i^2 S_{\delta/3}(\bar{a}_i)}{\sum \alpha_i^2 K(\bar{a}_i)} \left(1 + (c - \eta) \Lambda M(\tau_p) \Lambda \right),$$

where $c$ is a positive constant, $\eta$ is a small positive constant and $\Lambda = (1/\bar{\lambda}_1, \ldots, 1/\bar{\lambda}_p)$. Thus, we conclude as in the proof of Proposition 4.3.

It remains to compute the Morse index of such a critical point at infinity. For this purpose, we observe that $M(\tau_p)$ is positive definite and the function $\Psi$ admits on the variables $\alpha_i$’s an absolute degenerate maximum with one dimensional nullity space. Then the Morse index of such a critical point at infinity is equal to $p - 1 + \sum_{j=1}^{p} (6 - \text{index}(K, y_{ij}))$. Thus our result follows.

\[\square\]

## 5 Proofs of Theorems

Let us start by proving the following result adapted from [7].

**Lemma 5.1** For $\eta > 0$ small enough, we define the following neighborhood of $\Sigma^+$

$$V_\eta(\Sigma^+) = \{ u \in \Sigma/J(u)^{2n-4 \over n-4} e^{2J(u)} | u^{-8/(n-4)}_{L^{2n/(n-4)}} < \eta \},$$

where $u^- = \max(0, -u)$. Then, for $n \geq 5$, $V_\eta(\Sigma^+)$ is invariant under the flow generated by $-\nabla J$.  

Proof. Suppose $u_0 \in V_g(\Sigma^+)$ and consider
\[
\begin{aligned}
\frac{du(s)}{ds} &= -\nabla J(u(s)) = -2J(u) \left( u - J(u)^{n/(n-4)} \mathcal{P}^{-1}(K|u|^{8/(n-4)}u) \right) \\
u(0) &= u_0
\end{aligned}
\]
Then
\[
e^{2\int_0^s J(u) u(s)} = u_0 + 2 \int_0^s e^{2\int_0^t J(u)} J(u) \frac{2n-4}{n-4} \mathcal{P}^{-1}(K|u|^{8/(n-4)}u) dt.
\]
Therefore
\[
u^-(s) \leq e^{-2\int_0^s J(u)} \left( u_0 - 2 \int_0^s e^{2\int_0^t J(u)} J(u) \frac{2n-4}{n-4} \mathcal{P}^{-1}(K(u^-)^{n+4}) dt \right) = e^{-2\int_0^s J(u)} f(s).
\]
Observe that, each solution of $\mathcal{P} v = g$ with $g \in L^{\frac{2n}{n+4}}$, using a regularity argument, has to satisfy $v \in H^2_2(S^n)$ and then $v \in L^{\frac{2n}{n+4}}$. Thus $f \in L^{\frac{2n}{n+4}}$. Setting
\[
F(s) = e^{-\frac{4n}{n-4} \int_0^s J(u)} |f| \frac{2n}{n-4} L^{\frac{2n}{n-4}}
\]
we have $|\nu^-(s)| \frac{2n}{n-4} L^{\frac{2n}{n-4}} \leq F(s)$.

Now, without loss of generality, we can assume that $u_0 \neq 0$ and we want to prove that $F$ is a decreasing function. Observe that
\[
F'(s) = -\frac{4n}{n-4} J(u) e^{-\frac{4n}{n-4} \int_0^s J(u)} \left| f \right| \frac{2n}{n-4} L^{\frac{2n}{n-4}} + e^{-\frac{4n}{n-4} \int_0^s J(u)} \frac{2n}{n-4} \int_{S^n} f' f \frac{n+4}{n-4}
\]
\[
\leq \frac{2n}{n-4} e^{-\frac{4n}{n-4} \int_0^s J(u)} \left[ -2J(u) |u_0^-| \frac{2n}{n-4} + \int_{S^n} f' f \frac{n+4}{n-4} \right].
\]
Notice that
\[
|\int_{S^n} f' f \frac{n+4}{n-4}| \leq c \int_{S^n} |f'| |u_0^-\frac{n+4}{n-4} + c \int_{S^n} |f'| \left( \int_0^s |f'(t)| \right)^\frac{n+4}{n-4}.
\]
But, we have
\[
\int_{S^n} (u_0^-)^\frac{n+4}{n-4} \left( e^{2\int_0^s J(u)} J(u)^\frac{2n-4}{n-4} \mathcal{P}^{-1}(K(u^-)^{n+4}) \right) \leq C J(u)^\frac{2n-4}{n-4} e^{2\int_0^s J(u)} |u_0^-\frac{n+4}{n-4} u^- (s)|^{\frac{n+4}{n-4}}.
\]
and we also have
\[
\int_{S^n} |f'(s)| \left( \int_0^s |f'(t)| \right)^\frac{n+4}{n-4} \leq c s^\frac{n}{n+4} \int_{S^n} |f'(s)| \int_0^s |f'(t)| \frac{n+4}{n-4}
\]
\[\leq c s^\frac{n}{n+4} \left( e^{2sJ(u_0)} J(u_0)^\frac{2n-4}{n-4} \right) \frac{2n}{n-4} \int_0^s |u^-(t)|^{\frac{n+4}{n-4}} \int_0^s |u^-(t)|^{\frac{n+4}{n-4}}.
\]
Hence, if \(|u^-(s)| \leq 5|u_0|\), for \(0 \leq s \leq 1\), and using the fact that \(u_0 \in V_\eta(\Sigma^+), \) that is, 
\[ J(u_0) \frac{2n-4}{n-4} e^{2J(u_0)} |u_0|^{\frac{n-4}{n-4}} < \eta, \] 
and \(\eta\) is small enough, then \(F'(s) \leq 0, \) for \(0 \leq s \leq 1\). Therefore 
\[ J(u(s)) \frac{2n-4}{n-4} e^{2J(u(s))} |u(s)|^{\frac{n-4}{n-4}} < \eta, \] 
and our result follows. 

Proof of Theorem 1.1 Arguing by contradiction, we assume that \(J\) has no critical point in \(V_\eta(\Sigma^+), \) where 
\[ V_\eta(\Sigma^+) = \{ u \in \Sigma/e^{2J(u)} J(u)^6 \mid u^- |_{L^4} < \eta \}, \] 
\(\eta\) is a small positive constant and \(u^-\) denotes the negative part of \(u\), that is, \(u^- = \max(0, -u).\)

It follows from Proposition 4.3 that the only critical points at infinity of \(J\) in \(V_\eta(\Sigma^+)\) correspond to \(\delta_{(y, +\infty)}\), where \(y\) is a critical point of \(K\) with \(-\Delta K(y) > 0\). It follows that \(V_\eta(\Sigma^+)\) retracts by deformation on \(X_\infty = \cup_{y_t} -\Delta K(y_t) > 0 W_u(y_t)\), where \(W_u(y_t)\) is the unstable manifold at infinity of such a critical point at infinity. Using Assumption (A2) and Proposition 4.2, we see that \(X_\infty\) can be parametrized by \(X \times [A, +\infty[\), where \(A\) is a large positive constant.

In addition, we have \(X_\infty\) is contractible in \(V_\eta(\Sigma^+)\) and \(\Sigma^+\) retracts by deformation on \(X_\infty\), therefore \(X_\infty\) is contractible leading to the contractibility of \(X\), which is in contradiction with the assumption (A1) of our theorem. Thus there exists a critical point of \(J\) in \(V_\eta(\Sigma^+).\)

Now, it remains to prove that such a critical point is a positive function. Let us define the function \(w^-\) by the solution of the following problem

\[ \mathcal{P}w^- = -K(x)(u^-)^{\frac{n+4}{n-4}} \text{ on } S^n \]

Since \(K(x)(u^-)^{\frac{n+4}{n-4}} \in L^{\frac{2n}{n+4}}\), we see \(w^- \in H_2^2\). Furthermore, we have \(w^- \leq 0\). Thus we derive

\[ \int_{S^n} \mathcal{P}w^- . w^- = ||w^-||^2 = \int_{S^n} -K(x)(u^-)^{\frac{n+4}{n-4}} w^- \leq C ||w^-||_{L^{2n/(n-4)}} ||u^-||_{L^{2n/(n-4)}}^{(n+4)/(n-4)} \]

Thus, either \(|w^-|=0\) and therefore \(u^- = 0\), or \(|w^-| \neq 0\) and we derive

\[ ||w^-|| \leq C ||u^-||_{L^{2n/(n-4)}}^{(n+4)/(n-4)}. \quad (5.1) \]

Furthermore, on one hand we have

\[ \int_{S^n} u . \mathcal{P}w^- = \int_{S^n} uK(u^-)^{\frac{n+4}{n-4}} \geq c_K \int_{S^n} (u^-)^{\frac{2n}{n-4}} \geq c_K ||u^-||_{L^{\frac{2n}{n-4}}}^{\frac{2n}{n-4}} \quad (5.2) \]

(since \(K\) is bounded from below by a positive constant), and on the other hand, we have

\[ \int u . \mathcal{P}w^- = \int w^- K \mid u \mid^{\frac{1}{n-4}} u \leq \int_{u \leq 0} w^- K(u^-)^{\frac{n+4}{n-4}} \leq \int_{S^n} w^- K(u^-)^{\frac{n+4}{n-4}} = \int_{S^n} w^- \mathcal{P}w^- = ||w^-||^2. \quad (5.3) \]
Using (5.1), (5.2) and (5.3), we obtain

\[ c_K \left| u^- \right|_{L^{2n/(n-4)}}^{2n/(n-4)} \leq \left\| u^- \right\|^{2} \leq C \left| u^- \right|_{L^{2n/(n-4)}}^{2(n+4)/(n-4)}. \]

Observe that \( 2n/(n-4) < 2(n+4)/(n-4) \). Thus, either \( u^- = 0 \) or \( \left| u^- \right|_{L^{2n/(n-4)}} \geq C \) and this case cannot occur since by the definition of the neighborhood of \( \Sigma^+ \) we have this norm is small. This completes the proof of our result. \( \square \)

**Proof of Theorem 1.2** By Proposition 4.5 and assumption \((H)\), we derive that the only critical points at infinity of \( J \) in \( V_{\eta}(\Sigma^+) \) correspond to \( \tilde{\delta}_{(y,\infty)} \), where \( y \) is a critical point of \( K \) with \( -\Delta K(y) > 0 \). We order the critical values of \( K \): \( K(y_{i_1}) \geq K(y_{i_2}) \geq \ldots \geq K(y_{i_l}) \) (those critical points \( y_{i_j} \) satisfy \( -\Delta K(y_{i_j}) > 0 \)). Let \( c_r = (S_0)^{4/6}(K(y_{i_r}))^{-1/6} \) be the critical value at infinity. For the sake of simplicity, we can assume that \( c_r \)'s are different. Then, we have

\[ b_1 < \min_{\Sigma^+} J = c_1 < b_2 < c_2 < b_3 < c_3 < \ldots < b_l < c_l < b_{l+1}. \]

Recall that we already built in Proposition 4.4 a vector field \( W \) defined in \( V(p, \varepsilon) \) for \( p \geq 1 \), \( \varepsilon \) will be chosen so that \( V(p, \varepsilon) \subset V_{\eta}(\Sigma^+) \). Outside \( \bigcup_{p \geq 1} V(p, \varepsilon/2) \), we will use \( -\nabla J \) and our global vector field \( Z \) will be built using a convex combination of \( W \) and \(-\nabla J\). Now, according to Proposition 4.5, there is no critical value above the level \( b_{l+1} \). Let \( J_c = \{ u \in V_{\eta}(\Sigma^+) \mid J(u) < c \} \). Using the vector field \( Z \), we have \( J_{b_{l+1}} \) retracts by deformation on \( J_{b_1} \cup W_u(y_{i_r})_{\infty} \), where \( W_u(y_{i_r})_{\infty} \) is the unstable manifold at infinity (see sections 7 and 8 of [6]). Then, denoting by \( \chi \) the Euler-Poincaré characteristic, we have

\[ \chi(J_{b_{l+1}}) = \chi(J_{b_1}) + (-1)^{6-k_r}, \]

where \( k_r = \text{index}(K, y_{i_r}) \). It is easy to see that \( \chi(J_{b_1}) = \chi(\emptyset) = 0 \) and \( \chi(V_{\eta}(\Sigma^+)) = 1 \). Therefore

\[ 1 = \sum_{r=1}^{l} (-1)^{6-k_r} = \sum_{r=1}^{l} (-1)^{k_r}. \]

If (5.4) is violated, \( J \) has a critical point in \( V_{\eta}(\Sigma^+) \). Arguing as in the proof of Theorem 1.1, we conclude that this critical point is a positive function and hence our theorem follows. \( \square \)

**Proof of Theorem 1.4** As in the proof of Theorem 1.2, we derive that the only critical points at infinity of \( J \) in \( V_{\eta}(\Sigma^+) \) correspond to \( \tilde{\delta}_{(y,\infty)} \), where \( y \) is a critical point of \( K \) with \( -\Delta K(y) > 0 \). Thus, the sequel of the proof of our theorem is exactly the same as in the proof of Theorem 1.1, so we will omit it. \( \square \)

**Proof of Theorem 1.5** Arguing by contradiction, we assume that \( J \) has no critical point in \( \Sigma^+ \). Using Proposition 4.3, the only critical points at infinity correspond to \( \tilde{\delta}(y, \infty) \), where \( y \) is a critical point of \( K \) with \( -\Delta K(y) > 0 \). Such a critical point at infinity has a Morse index equal to \( (5 - \text{index}(K, y)) \). Using the same arguments as in the proof of Theorem 1.2, the result follows. \( \square \)
Proof of Theorem 1.6  The proof is the same as the proof of Theorem 1.2. But here, the critical points at infinity correspond to

$$\sum_{r=1}^{p} K(y_{i_r})^{-1/4} \tilde{\delta}(y_{i_r}, \infty)$$

with $\rho(\tau_p) > 0$ and $\tau_p = (y_{i_1}, \ldots, y_{i_s})$),

where $p \geq 1$ and $\rho(\tau_p)$ denote the least eigenvalue of $M(\tau_p)$. Such a critical point at infinity has an index equal to $\sum_{r=1}^{p} (6 - k_{i_r}) + (p - 1) = 7p - 1 - \sum_{r=1}^{p} \text{index}(K, y_{i_r})$. Using the same argument as the proof of Theorem 1.2, the result follows. □

6 Appendix

6.1 The Coercivity of the Quadratic Form

In this appendix, we give the proof of Proposition 3.4, adapted from [1].

Proposition 6.1  For any $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon)$ given, $Q(v, v)$ is a quadratic positive form in the space $E = \{v/ v \text{ satisfies } (V_0)\}.

Proof.  Using a stereographic projection, we need to prove the proposition on $\mathbb{R}^n$ with the bilaplacian.

Let us define the sets, for $i = 1, \ldots, p$

$$\Omega_i = \{x \in \mathbb{R}^n/ |x - x_i| < \frac{1}{8\lambda_i} \min \varepsilon_1^{-1} \text{ and } |x - x_j| > \frac{1}{8\lambda_j} \min \varepsilon_1^{-1} \text{ for } \lambda_j \text{ s.t. } \lambda_j \geq \lambda_i\}$$

By construction $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Now, we define

$$H = \{u \in L^{\infty}(\mathbb{R}^n)/\Delta u \in L^2(\mathbb{R}^n)\},$$

$H$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\int_{\mathbb{R}^n} |\Delta u|^2$.

For $\varphi$ belongs to $H$, we introduce the projection $Q_i$ by: $\varphi_i = Q_i \varphi$ satisfies

$$\Delta^2 \varphi_i = \Delta^2 \varphi \text{ in } \Omega_i, \quad \Delta \varphi_i = \varphi_i = 0 \text{ on } \partial \Omega_i.$$

Let us define also $E_i^- = \text{span} <\delta_i, (\partial \delta_i)/(\partial \lambda_i), (\partial \delta_i)/(\partial a_i)>$ and $E_i^+ = (E_i^-)^\perp$ (the orthogonal being taken in the sense of the scalar product $\int \Delta \psi \Delta \varphi$).

Next, we will use the following lemmas which we will prove in the end.

Lemma 6.2  If, for $i \neq j$, $\varepsilon_1$'s are small enough, then

$$\int_{\mathbb{R}^n} \frac{1}{\delta_{1}^{n+4}} \frac{n+4}{n-4} \frac{n+4}{n-4} \int_{\mathbb{R}^n} |\Delta \varphi_i|^{2} |^2 \geq \frac{c}{c} \int_{\mathbb{R}^n} |\Delta \varphi_i|^{2} \frac{1}{2}.$$

Lemma 6.3  There exists $\alpha_1 > 0$ s.t. for any $\varphi \in E_i^+$, we have

$$\int_{\mathbb{R}^n} |\Delta \varphi |^2 \frac{n+4}{n-4} \int_{\mathbb{R}^n} \delta_{1}^{n-4} \varphi^2 \geq \alpha_1 \int_{\mathbb{R}^n} |\Delta \varphi |^2.$$
Lemma 6.4 For \( v \in H \) satisfying (\( V_0 \)) and \( v_i = Q_iv \), we write \( v_i = v_i^- + v_i^+ \), where \( v_i^- \in E_i^- \) and \( v_i^+ \in E_i^+ \). Then, we have
\[
\int_{\Omega_i} |\Delta v_i^-|^2 = o(\int_{R^n} |\Delta v|^2).
\]
Using those Lemmas, we are able to give the proof of the above proposition. Indeed:
Let \( v \) satisfy (\( V_0 \)), we denote \( v_i = Q_i v \) for each \( i = 1, ..., p \). We can assume that \( v_i \) is defined on \( R^n \) by taking \( v_i = 0 \) on \( \Omega_i^c \). We split \( v_i \) into two parts: \( v_i = v_i^- + v_i^+ \) where \( v_i^- \in E_i^- \) and \( v_i^+ \in E_i^+ \). Since the sets \( \Omega_i \)'s are disjoint, we derive
\[
\sum_{i=1}^p \int_{\Omega_i} |\Delta v_i|^2 \leq \sum_{i=1}^p \int_{\Omega_i} |\Delta v|^2
\]
Thus
\[
Q(v, v) = \int_{R^n} |\Delta v|^2 - \frac{n+4}{n-4} \sum_{i=1}^p \int_{R^n} \delta_i^{n+4} v^2
\]
\[
= \int_{(\cup \Omega_i)^c} |\Delta v|^2 + \sum_{i=1}^p \left( \int_{\Omega_i} |\Delta v|^2 - \int_{\Omega_i} |\Delta v_i|^2 \right)
\]
\[
+ \sum_{i=1}^p \left( \int_{\Omega_i} |\Delta v_i|^2 - \frac{n+4}{n-4} \int_{R^n} \delta_i^{8/(n-4)} v_i^2 \right) - \frac{n+4}{n-4} \sum_{i=1}^p \int_{R^n} \delta_i^{n-4}(v^2 - v_i^2) \tag{6.1}
\]
Observe that, using Lemmas 6.3 and 6.4, we have
\[
\int_{R^n} |\Delta v_i|^2 - \frac{n+4}{n-4} \int_{R^n} \delta_i^{n-4} v_i^2 = \int_{R^n} |\Delta v_i^+|^2 + \int_{R^n} |\Delta v_i^-|^2 - \frac{n+4}{n-4} \int_{R^n} \delta_i^{n-4}(v_i^+)^2
\]
\[
- \frac{n+4}{n-4} \int_{R^n} \delta_i^{n-4}((v_i^-)^2 + 2v_i^+v_i^-) \geq \alpha_1 \int_{R^n} |\Delta v_i^+|^2 + o(\int_{R^n} |\Delta v|^2)
\]
\[
\geq \frac{\alpha_1}{2} \int_{\Omega_i} |\Delta v_i|^2.
\]
We also have, using Lemma 6.2,
\[
\int_{R^n} \delta_i^{n-4}(v^2 - v_i^2) = \left[ \int |v + v_i| \frac{2n}{2n-4} \right]^{\frac{n+4}{n-4}} \left[ \int \delta_i^{n+4} |v - v_i| \right]^{\frac{n-4}{n+4}} \left[ \int |v - v_i| \frac{2n}{2n} \right]^{\frac{2n}{2n(n+4)}}
\]
\[
= o(\int |\Delta v|^2).
\]
Thus, (6.1) becomes
\[
Q(v, v) \geq \int_{(\cup \Omega_i)^c} |\Delta v|^2 + \sum_{i=1}^p \left[ \int_{\Omega_i} |\Delta v|^2 - \int_{\Omega_i} |\Delta v_i|^2 \right] + \frac{\alpha_1}{2} \sum_{i=1}^p \int_{\Omega_i} |\Delta v_i|^2
\]
\[
+ o(\int_{R^n} |\Delta v|^2) \geq \int_{(\cup \Omega_i)^c} |\Delta v|^2 + \frac{\alpha_1}{2} \sum_{i=1}^p \int_{\Omega_i} |\Delta v|^2 + o(\int_{R^n} |\Delta v|^2)
\]
\[
\geq \alpha_0 \int_{R^n} |\Delta v|^2 . \tag{6.3}
\]
Thus the proof of Proposition 6.1 is completed under Lemmas 6.2, 6.3 and 6.4.

**Proof of Lemma 6.3** Observe that the family of functions \( \delta_{(a, \lambda)} \) are the solutions of the Yamabe problem on \( \mathbb{R}^n \), that is, the functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^n} | \Delta u |^2 - \frac{n-4}{2n} \int_{\mathbb{R}^n} | u |^{2n/4} 
\]

has only the family of functions \( \delta_{(a, \lambda)} \) as critical points. Those critical points are degenerated and of index 1. The nullity space is of dimension \( n+1 \) and it is generated by the derivative of \( \delta_{(a, \lambda)} \) with respect to \( \lambda \) and \( a \). Furthermore, the set of negativity is generated by the function \( \delta := \delta_{(a, \lambda)} \). Let

\[
F = \text{span}\{ \delta_i, \frac{\partial \delta}{\partial \lambda}, \frac{\partial \delta}{\partial (a)_i}, i = 1, ..., n \}.
\]

Thus, on the orthogonal of \( F \), the second derivation of the functional \( I \) on the point \( \delta \) is positive definite. Therefore

\[
\exists \, \alpha_1 > 0 \text{ s.t. } \forall v \in F^\perp \text{ we have } \| v \|^2 - \frac{n+4}{n-4} \int_{S^n} \delta_i^{n/4} v^2 \geq \alpha_1 \| v \|^2. \tag{6.4}
\]

**Proof of Lemma 6.4** We have

\[
v_i^- = a \delta_i + b \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + \sum_{j=1}^n c_j \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j}. \]

Multiply \( v_i^- \) by \( \delta_i^{(n+4)/(n-4)} \) and integrate on \( \mathbb{R}^n \), using Lemma 6.2, we derive that

\[
a \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} v_i^- = \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} (v_i^- - v) = O \left( \sum \varepsilon_{ij}^{1/2} \left( \int_{\mathbb{R}^n} | \Delta v |^2 \right)^{1/2} \right).
\]

In the same way, we have

\[
b \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} | \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} |^2 = \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v_i^- = \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} (v_i^- - v) = O \left( \int_{\mathbb{R}^n} \delta_i^{n+4} | v_i - v | \right) = O \left( \sum \varepsilon_{ij}^{1/2} \left( \int_{\mathbb{R}^n} | \Delta v |^2 \right)^{1/2} \right)
\]

\[
c_j \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} | \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} |^2 = \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v_i^- = \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{4}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (v_i^- - v) = O \left( \int_{\mathbb{R}^n} \delta_i^{n+4} | v_i - v | \right) = O \left( \sum \varepsilon_{ij}^{1/2} \left( \int_{\mathbb{R}^n} | \Delta v |^2 \right)^{1/2} \right).
\]

Using the fact that \( \delta_i, \lambda_i(\partial \delta_i)/(\partial \lambda_i) \) and \( \lambda_i^{-1}(\partial \delta_i)/(\partial (a_i)_j) \) have a constant norme, the lemma follows.

Before giving the proof of Lemma 6.2, we need the following lemma.
Lemma 6.5 Let $w \in H$ and $h$ satisfies
\[ \Delta^2 h = 0 \text{ on } B_\lambda, \Delta h = \Delta w \text{ and } h = w \text{ in } \partial B_\lambda, \]
where $B_\lambda = \{ x / |x| < \lambda \}$. We have
\[ \int_{B_\lambda} \delta^{(n+4)/(n-4)} \frac{|h|}{\lambda^{(n-4)/2}} \leq \frac{c}{\lambda^{(n-4)/2}} \left( \int_{\mathbb{R}^n} |\Delta w|^2 \right)^{1/2}. \]

Proof. First, observe that we have, for $w \in H$, if a function $u$ satisfies
\[ \Delta^2 u = 0 \text{ on } B_1, \Delta u = \Delta w \text{ and } u = w \text{ in } \partial B_1, \]
thus
\[ \int_{\partial B_1} |\Delta u| + \int_{\partial B_1} |u| \leq C \left( \int_{\mathbb{R}^n} |\Delta w|^2 \right)^{1/2}. \]

Notice that we can assume that $w \geq 0$ and $\Delta w \geq 0$. Indeed:

Remark 6.6 If the function $w$ does not satisfy "$-\Delta w \geq 0$" and "$w \geq 0$", we can introduce the function $w'$ defined by
\[ w' \in H^1(\mathbb{R}^n) \text{ and } -\Delta w' = |\Delta w| \text{ on } \mathbb{R}^n. \]
Thus $w' \in H$ and it satisfies "$-\Delta w' \geq 0$" and "$w' \geq 0$" on $\mathbb{R}^n$ and it easy to see that
\[ w' - w \geq 0, \quad |w - Pw| \leq w' - Pw' \quad \text{and} \quad \int_{\mathbb{R}^n} |\Delta w'|^2 = \int_{\mathbb{R}^n} |\Delta w|^2, \]
where $P$ is the projection operator on any subset.

Hence if the lemma holds with $w \geq 0$ and $\Delta w \geq 0$, it will hold for all $w$. If we assume $w$ and $-\Delta w$ to be positive, the function $h$ will also be positive. Then
\[ \int_{B_\lambda} \delta^{\frac{n+4}{n-4}} h = \int_{B_\lambda} \Delta^2 (\delta - \theta) h = \int_{\partial B_\lambda} \frac{\partial}{\partial n} (\Delta (\delta - \theta)) h + \int_{\partial B_\lambda} \frac{\partial}{\partial n} (\delta - \theta) \Delta h \]
where $\theta$ satisfies
\[ \Delta^2 \theta = 0 \text{ on } B_\lambda, \Delta \theta = \Delta \delta \text{ and } \theta = \delta \text{ in } \partial B_\lambda \]
It is easy to see that $\theta$ is equal to
\[ \theta = \frac{1}{2n} c_\lambda (|x|^2 - \lambda^2) + \frac{c_0}{(1 + \lambda^2)(n-4)/2}, \]
where $c_0$ is defined in the definition of $\delta$ and $c_\lambda$ is equal to
\[ c_\lambda = \Delta \delta |_{\partial B_\lambda} = \frac{(n-4)c_0}{(1 + \lambda^2)^{n/2}} (-n - 2\lambda^2) \sim \frac{-c}{\lambda^{n-2}}. \]
(for $\lambda$ large). Therefore, we have

$$
\frac{\partial}{\partial n}(\delta - \theta) \mid_{\partial B_{\lambda}} = -\frac{(n - 4)c_0\lambda}{(1 + \lambda^2)(n - 2)/2} - \frac{\lambda}{n}c_{\lambda} = -\frac{(n - 2)(n - 4)c_0\lambda^3}{n(1 + \lambda^2)n^2/2} \sim -\frac{c}{\lambda^{n-3}} \tag{6.8}
$$

and

$$
\frac{\partial}{\partial n}(\Delta(\delta - \theta)) \mid_{\partial B_{\lambda}} = \frac{c_0(n - 2)(n - 4)\lambda}{(1 + \lambda^2)^{n+2}/2}(n + 2 + 2\lambda^2) \sim \frac{c}{\lambda^{n-1}} \tag{6.9}
$$

(for $\lambda$ large). Using (6.8) and (6.9), (6.7) becomes

$$
\int_{\partial B_{\lambda}} \delta_{n+2} h \leq \frac{c}{\lambda^{n-1}} \int_{\partial B_{\lambda}} h + \frac{c}{\lambda^{n-3}} \int_{\partial B_{\lambda}} -\Delta h \tag{6.10}
$$

Let $\overline{h}(x) = \lambda^{(n-4)/2}h(\lambda x)$ and $\overline{w}(x) = \lambda^{(n-4)/2}w(\lambda x)$. The function $\overline{h}$ satisfies (6.5) with $\overline{w}$ instead of $w$. Thus, it satisfies (6.6) with $\overline{w}$. Observe that

$$
\int_{\partial B_{1}} \overline{h} = \frac{1}{\lambda^{n+2}} \int_{\partial B_{\lambda}} h , \quad \int_{\partial B_{1}} \Delta \overline{h} = \frac{1}{\lambda^{n+2}} \int_{\partial B_{\lambda}} \Delta h , \quad \int_{R^n} |\Delta \overline{w}|^2 = \int_{R^n} |\Delta w|^2 . \tag{6.11}
$$

Thus, using (6.8), (6.9), (6.10) and (6.11), the lemma follows. \hfill \square

**Proof of Lemma 6.2** First, we assume that we have only two masses. Take $i = 1$ in the Lemma. We can make a translation and a dilatation so that $\lambda_1 = 1$ and $\overline{\lambda}_1 = 0$. Let

$$
\overline{\varphi} = \lambda_1^{-\frac{n-4}{2}} \varphi(\lambda_1 x + a_1) \quad , \quad \overline{\delta}_j = \lambda_1^{-\frac{n-4}{2}} \delta_j(\lambda_1 x + a_1)
$$

Notice that

$$
\overline{\varphi}_{12} = \varepsilon_{12} , \quad \overline{\lambda}_1 = 1 , \quad \overline{\lambda}_2 = \frac{\lambda_2}{\lambda_1} , \quad \overline{\alpha}_1 = 0 , \quad \overline{\alpha}_2 = \lambda_1(a_2 - a_1)
$$

Assume first that $\lambda_1 \geq \lambda_2$, hence $\overline{\lambda}_2 \leq 1$. Then

$$
\overline{\Omega}_1 = \{x / |x| < (8\varepsilon_{12}^{-1/2})^{-1}\}
$$

Let $\overline{\varphi}_1 = \overline{\varphi} - \overline{\overline{h}}$ with $\Delta^2\overline{\overline{h}} = 0$ in $\overline{\Omega}_1$, $\Delta \overline{\overline{h}} = \Delta \overline{\varphi}$ and $\overline{\overline{h}} = \overline{\varphi}$ on $\partial\overline{\Omega}_1$, we have

$$
\int_{R^n} \delta_{1}^\frac{n+4}{4} | \varphi - \varphi_1 | \leq \int_{\overline{\Omega}_1} + \int_{|x| \geq (8\varepsilon_{12}^{-1/2})^{-1}} \leq c\varepsilon_{12}^{1/2}(\int_{R^n} |\Delta \overline{\varphi}|^2)^{1/2} + \varepsilon_{12}^{\frac{n+4}{4}(n-4)}(\int_{R^n} |\Delta \varphi|^2)^{1/2}
$$

$$
\leq c(\varepsilon_{12}^{1/2} + \varepsilon_{12}^{\frac{n+4}{4}(n-4)})(\int_{R^n} |\Delta \varphi|^2)^{1/2} \tag{6.12}
$$

(using Lemma 6.5 and Holder’s inequality.) Observe that

$$
\int_{R^n} \delta_{1}^\frac{n+4}{4} | \varphi - \varphi_1 | = \int_{R^n} \delta_{1}^\frac{n+4}{4} | \varphi - \varphi_1 | .
$$
Thus, the proof is completed in this case (the case where $\lambda_2 \leq \lambda_1$). We will now see the other case i.e. $\lambda_1 \leq \lambda_2$. Thus

$$\Omega_1 = \{x/ |x| < (8\varepsilon_1^{1/(n-4)})^{-1} \text{ and } |x - \vec{\sigma}_2| > (8\lambda_2^{1/(n-4)})^{-1} \}.$$  

Let

$$\tilde{\Omega} = \{x/ |x| < (8\varepsilon_1^{1/(n-4)})^{-1} \} \quad \text{and} \quad \tilde{W} = \{x/ |x - \vec{\sigma}_2| > (8\lambda_2^{1/(n-4)})^{-1} \}.$$  

Observe that we have $\partial\overline{\Omega} = \partial\tilde{\Omega} \cup \partial\tilde{W}$. We define $\tilde{\varphi}_1$ to be the projection of $\varphi$ on $\tilde{\Omega}$ and $\tilde{\psi}_1$ to be the projection of $\varphi$ on $\tilde{W}$.

In the following, we will assume that $-\Delta \varphi > 0$ and $\varphi \geq 0$. The general case can be deduced by Remark 6.6.

Hence we derive

$$| \varphi - \tilde{\varphi}_1 | \leq (\varphi - \tilde{\varphi}_1) + (\varphi - \tilde{\psi}_1) \quad \text{in} \quad R^n$$

(6.13)

and thus

$$\int_{R^n} \sigma_{n+4}^{n+4} | \varphi - \tilde{\varphi}_1 | \leq \int_{\overline{\Omega}} \sigma_{n+4}^{n+4} (\varphi - \tilde{\varphi}_1) + \int_{\tilde{W}} \sigma_{n+4}^{n+4} (\varphi - \tilde{\psi}_1) + \int_{\Omega} \sigma_{n+4}^{n+4} \varphi + \int_{\tilde{W}^c} \sigma_{n+4}^{n+4} \varphi. \quad (6.14)$$

As in (6.12), using Holder’s inequality, we have

$$\int_{\Omega} \sigma_{n+4}^{n+4} \varphi \leq c \varepsilon_1^{\frac{n+4}{2n}} (\int_{R^n} |\Delta \varphi|^2)^{1/2}.$$  

We estimate now $\int_{\tilde{W}^c} \sigma_{n+4}^{n+4} \varphi$. As in [1], we prove that

$$\overline{\sigma}_1 = \frac{c_0}{1 + x^2} \leq c_0 \frac{2x_2^{n+4} \lambda_2^{n+4}}{(1 + \lambda_2^2 |x - \vec{\sigma}_2|^2)^{n+4}} \quad \text{if} \quad \lambda_2 |x - \vec{\sigma}_2| \leq \frac{1}{8\varepsilon_1^{n+4}}. \quad (6.15)$$

We then have

$$\int_{\tilde{W}^c} \sigma_{n+4}^{n+4} \varphi \leq c \left( \int_{\tilde{W}^c} \sigma_{n+4}^{n+4} \right)^n \int_{R^n} |\Delta \varphi|^2 \left( \int_{\tilde{W}^c} \sigma_{n+4}^{n+4} \right)^\frac{n+4}{2n} \left( \int_{R^n} |\Delta \varphi|^2 \right)^\frac{1}{2}$$

$$\leq c \varepsilon_1^{\frac{n+4}{2n}} \left( \int_{R^n} |\Delta \varphi|^2 \right)^\frac{1}{2} \leq c \varepsilon_1^{\frac{n+4}{2n}} \left( \int_{R^n} |\Delta \varphi|^2 \right)^\frac{1}{2}. \quad (6.16)$$

Using Lemma 6.5, and as in (6.12), we have

$$\int_{\Omega} \sigma_{n+4}^{n+4} (\varphi - \tilde{\varphi}_1) \leq c \varepsilon_1^{\frac{1}{2}} \left( \int_{R^n} |\Delta \varphi|^2 \right)^{\frac{1}{2}}. \quad (6.17)$$

It remains to estimate $\int_{\tilde{W}} \sigma_{n+4}^{n+4} (\varphi - \tilde{\psi}_1)$(6.18)

$$\int_{\tilde{W}} \sigma_{n+4}^{n+4} (\varphi - \tilde{\psi}_1) = \int_{\tilde{W}} \Delta \tilde{\varphi}_1 (\varphi - \tilde{\psi}_1) = \int_{\partial\tilde{W}} \frac{\partial}{\partial n} (\tilde{\sigma}_1 - \tilde{\varphi}_1) \Delta \varphi + \int_{\partial\tilde{W}} \frac{\partial}{\partial n} (\Delta (\tilde{\varphi}_1 - \tilde{\varphi}_1)) \varphi$$

$$\leq \sup \left| \frac{\partial}{\partial n} (\tilde{\sigma}_1 - \tilde{\varphi}_1) \right| \int_{\partial\tilde{W}} \Delta \varphi + \sup \left| \frac{\partial}{\partial n} (\Delta (\tilde{\varphi}_1 - \tilde{\varphi}_1)) \right| \int_{\partial\tilde{W}} \varphi,$$
where $\overline{\theta}_1$ is the projection of $\overline{\delta}_1$ on $\overline{W}$. Now, we need to estimate the normal derivatives which appear in (6.18). For this effect, let us introduce the Green's function $G_{\overline{W}}$ which satisfies

$$
\Delta^2 G_{\overline{W}}(x, \cdot) = \delta_x, \quad \text{in } \overline{W}, \quad \Delta G_{\overline{W}} = G_{\overline{W}} = 0 \text{ on } \partial \overline{W}.
$$

Thus for any function $u$ we have

$$
u(y) = \int_{\overline{W}} G_{\overline{W}} \Delta^2 u + \int_{\partial \overline{W}} \frac{\partial}{\partial \nu} G_{\overline{W}} \Delta u + \int_{\partial \overline{W}} \frac{\partial}{\partial \nu} (\Delta G_{\overline{W}}) u.
$$

Observe that $\overline{\bigtriangledown}_1 - \overline{\nu}_1$ satisfies

$$
\Delta^2 (\overline{\bigtriangledown}_1 - \overline{\nu}_1) = \overline{\bigtriangledown}^{n+4}_1 \text{ in } \overline{W}, \quad \Delta (\overline{\bigtriangledown}_1 - \overline{\nu}_1) = \overline{\bigtriangledown}_1 - \overline{\nu}_1 = 0 \text{ on } \partial \overline{W}.
$$

Thus, we derive

$$
(\overline{\bigtriangledown}_1 - \overline{\nu}_1)(y) = \int_{\overline{W}} G_{\overline{W}}(x, y) \overline{\bigtriangledown}^{n+4}_1(x, \cdot) \big| \frac{\partial}{\partial \nu} (\overline{\bigtriangledown}_1 - \overline{\nu}_1)(y) \big| = \int_{\overline{W}} \frac{\partial}{\partial \nu} G_{\overline{W}}(x, y) \overline{\bigtriangledown}^{n+4}_1(x). \quad (6.19)
$$

But we have

$$
G_{B^c(0,1)}(x, y) = \frac{1}{|x|^{n-4}} G_{B(0,1)}(\frac{x}{|x|}, \frac{y}{|y|}),
$$

$$
G_{\overline{W}}(x, y) = \frac{1}{r^{n-4}} G_{B^c(0,1)}(\frac{x}{r}, \frac{y}{r}) = \frac{1}{|x|^{n-4}} G_{B(0,1)}(\frac{rx}{|x|^2}, \frac{ry}{|y|^2}).
$$

Let $y \in \partial \overline{W}$ and let $\pi_y$ be the half space which contains $B(0,1)$ and satisfies $y \in \partial \pi_y$, then we have

$$
G_{B(0,1)}(x, y) \leq G_{\pi_y}(x, y) \quad \text{and} \quad \big| \frac{\partial}{\partial \nu} G_{B(0,1)}(x, y) \big| \leq \big| \frac{\partial}{\partial \nu} G_{\pi_y}(x, y) \big| \leq \frac{c}{|x-y|^{n-3}}
$$

and therefore, since $y \in \partial \overline{W}$,

$$
\big| \frac{\partial}{\partial \nu} G_{\overline{W}}(x, y) \big| \leq \frac{c}{r|x-y|^{n-3}} \leq \frac{c|x|}{r|x-y|^{n-3}} \leq \frac{c}{r|x-y|^{n-3}} \quad \text{and}
$$

$$
\big| \frac{\partial}{\partial \nu} (\overline{\bigtriangledown}_1 - \overline{\nu}_1)(y) \big| \leq \int_{R^n} \frac{c\delta^{n+4}_1}{r|x-y|^{n-3}} + c \int_{R^n} \frac{\delta^{n+4}_1(x-y)}{|x|^{n-3}}
$$

$$
\leq \frac{c}{r} \delta(y) + \frac{4c}{(1+r^2)^{1/2}} \int_{|x|^2 \leq (1+r^2)/4} \frac{\delta^{n+4}_1(x-y)}{|x|^{n-4}} + c \int_{4|x|^2 \leq (1+r^2)} \frac{\delta^{n+4}_1(y)}{|x|^{n-3}}
$$

$$
\leq \frac{c}{r} \delta(y) + c\delta(y) \frac{n+4}{(1+r^2)^{3/2}} \leq \frac{c}{r(1+r^2)^{(n-4)/2}}.
$$
For the second term, we introduce the Green’s function $\tilde{G}_\Omega$ for $-\Delta$, i.e. $\tilde{G}_\Omega$ satisfies

$$-\Delta \tilde{G}_\Omega(x,\cdot) = \delta_x \text{ in } \tilde{\Omega}, \quad \tilde{G}_\Omega = 0 \text{ on } \partial\tilde{\Omega}.$$ 

By the same argument we prove that

$$\left| \frac{\partial}{\partial y} \tilde{G}_\Omega \right| \leq \frac{c}{r|x-y|^n} + \frac{c}{|x-y|^n}.$$ 

Arguing as above, for $g = \Delta(\delta_1 - \bar{\theta}_1)$, we have

$$\left| \frac{\partial g(y)}{\partial y} \right| = \left| \frac{\partial}{\partial y} (\Delta(\delta_1 - \bar{\theta}_1))(y) \right| \leq \int_{\bar{\Omega}} \frac{c\delta_1^{n+4}}{n} + \int_{\bar{\Omega}} \frac{\delta(x-y)^{n+4}}{|x|^n}$$

$$\leq \frac{c}{r} g(y) + \frac{c}{(1+r^2)^{n/2}} g(y) + \delta(y)^{n+4}(1+r^2)^{1/2} \leq \frac{\delta(y)^{n+4}}{r(1+r^2)^{n/2}}.$$ 

Thus (6.18) becomes

$$\int_{\tilde{\Omega}} \frac{\delta_1^{n+4}}{|x-y|^n} |(\varphi - \bar{\varphi}_1)| \leq \frac{c}{r(1+r^2)^{n/2}} (\int_{\partial\tilde{\Omega}} \Delta \varphi + \frac{1}{r^2} \int_{\partial\tilde{\Omega}} \varphi).$$

Using (6.6) and (6.11), we derive

$$\int_{\tilde{\Omega}} \frac{\delta_1^{n+4}}{|x-y|^n} |(\varphi - \bar{\varphi}_1)| \leq \frac{c}{r(1+r^2)^{n/2}} \int_{\tilde{\Omega}} |\Delta \varphi|^2 \leq \frac{cR^{n+2}}{(1+r^2)^{n/2}} |\varphi|^2.$$ 

Recall that $r = (8\lambda_2 \varepsilon_{12}^{1/(n-4)})^{-1}$. If $|\tilde{x}_2| < 1$ then $\varepsilon_{12} \sim (\lambda_2^{(4-n)/2} \leq \lambda_2^{(4-n)/4} \leq C \varepsilon_{12}^{1/2}$. In the other case, that is, $|\tilde{x}_2| \geq 1$, we have $\varepsilon_{12} \sim (\lambda_2^{2/|\tilde{x}_2|^{(n-4)/2}}$ and therefore $r^{(n-4)/2} \leq (\lambda_2/|\tilde{x}_2|^{(n-4)/2}) \leq \varepsilon_{12}^{1/2}$. Thus, in all cases we obtain

$$\int_{\tilde{\Omega}} \frac{\delta_1^{n+4}}{|x-y|^n} |(\varphi - \bar{\varphi}_1)| \leq \varepsilon_{12}^{1/2} |\varphi|^2.$$ 

This completes the proof in the case where we are dealing with two points.

In the general case, one introduces the sets, assuming $\lambda_1 = 1$, $a_1 = O$ and $\varphi \geq 0$

$$W_i = \{x \in R^n \mid x | < \varepsilon_{1i}^{-1/(n-4)}, \quad |x - a_i| > \lambda_i^{-1}\varepsilon_{1i}^{-1/(n-4)} \forall \lambda_i > 1\}$$

Then $\partial \Omega_1 \subseteq \partial W_i$. Let $\varphi_1$ be the projection of $\varphi$ on $\Omega_1$ and $\tilde{\varphi}_i$ be the projection of $\varphi$ on $W_i$. Then the above arguments, in the case of two points, imply

$$\int_{W_i} \frac{\delta_1^{n+4}}{|x-y|^n} |(\varphi - \tilde{\varphi}_i)| \leq c\varepsilon_{1i}^{1/2} (\int_{R^n} |\Delta \varphi|^2)^{1/2} \quad (6.20)$$

$$\int_{W_i} \frac{\delta_1^{n+4}}{|x-y|^n} \varphi \leq c\varepsilon_{1i}^{1/2} (\int_{R^n} |\Delta \varphi|^2)^{1/2} \quad (6.21)$$

From (6.20) and (6.21) the general case follows. \qed
6.2 Some estimates

In this subsection, we collect some technical estimates of the different integral quantities which occur in the paper. The proof of these estimates are similar to their analogous for Laplacian in [1] and [25].

Lemma 6.7 Let $a \in S^n$ and $\lambda > 0$ large enough. Using the stereographic projection $\pi_a$ the function $\delta_{(a, \lambda)}$ will be transformed to $\delta_{(0, \lambda)}$ (see [5]). Furthermore, we have

$$\int_{S^n} \tilde{L}_a \tilde{\delta} = \int_{S^n} \tilde{\delta} \frac{2a}{\nu} = \int_{R^n} \delta \frac{2a}{\nu} = S_n, \quad <\tilde{\delta}, \lambda \frac{\partial \delta}{\partial \lambda}> = 0, \quad <\tilde{\delta}, 1 \frac{\partial \delta}{\partial a}> = 0.$$

Lemma 6.8 For $a_1, a_2 \in S^n, \lambda_1, \lambda_2 > 0$ large enough, let $b \in S^n$ such that $d(a_1, b) = d(a_2, b)$. Using the stereographic projection $\pi_b$, the function $\delta_{(a_i, \lambda_i)}$ will be transformed to $\delta_{(\tilde{a}_i, \lambda_i)}$ with

$$\tilde{a}_i = \frac{(\lambda_i^2 - 1) \text{Proj}_{R^n} a_i}{2 + (\lambda_i^2 - 1)(1 - \cos (\theta_0))}, \quad \tilde{\lambda}_i = \frac{2 + (\lambda_i^2 - 1)(1 - \cos (\theta_0))}{2 \lambda_i}, \quad \theta_0 = \pi - d(a_i, b)$$

(see [5]). Furthermore, we have for $i \neq j$,

$$\int_{S^n} \tilde{L}_{\tilde{a}_i} \tilde{\delta}_{\tilde{j}} = \int_{S^n} \tilde{\delta}_{\tilde{i}} \tilde{\delta}_{\tilde{j}} = \int_{R^n} \delta_{\tilde{i}} \delta_{\tilde{j}} = c_1 \tilde{\varepsilon}_{ij} + o(\tilde{\varepsilon}_{ij}) = c_1 \varepsilon_{ij} + o(\varepsilon_{ij})$$

where

$$\varepsilon_{ij} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_i \lambda_j}{2} (1 - \cos d(a_i, a_j)) \right)^{-\frac{n+4}{2}} \quad \text{and} \quad \tilde{\varepsilon}_{ij} = \left( \frac{\tilde{\lambda}_i}{\tilde{\lambda}_j} + \frac{\tilde{\lambda}_i \tilde{\lambda}_j}{2} (1 - \cos \tilde{d}(a_i, a_j)) \right)^{-\frac{n+4}{2}}$$

and $c_1 = \beta^{2n/(n-4)} \int_{R^n} (1 + |x|^2)^{-(n+4)/2}$. If $n = 6$, $c_1 = \frac{\beta^6}{24} w_5$, $\beta_n$ is defined in the definition of $\delta$ and $w_5$ is the volume of the five dimensional sphere.

Lemma 6.9 We have the following estimates.

$$\int_{S^n} K \delta^{2n-4} = \int_{R^n} K \delta^{2n-4} = K(a)S_n + c_2 \frac{4\Delta K(a)}{\lambda^2} + O\left( \frac{1}{\lambda^3} \right)$$

$$\int_{S^n} K \delta^{n+4} \frac{\partial \delta}{\partial \lambda} = -\frac{n-4}{n} c_2 \frac{4\Delta K(a)}{\lambda^2} + O\left( \frac{1}{\lambda^3} \right)$$

$$\int_{S^n} K \delta^{n+4} \frac{\partial \delta}{\partial a} = c_3 \frac{\nabla K(a)}{\lambda} + O\left( \frac{1}{\lambda^2} \right),$$

where $c_2 = \frac{1}{2n} \int_{R^n} |x|^2 \delta^{2n/(n-4)}_{(0,1)}$. If $n = 6$, $c_2 = \frac{\beta^6}{48} w_5$. 

Lemma 6.10 For \( i \neq j \), we have the following estimates

\[
\int_{S^n} K^{\frac{n+4}{2(n-4)}} v^i \delta_j = \int_{\mathbb{R}^n} \tilde{K}^{\frac{n+4}{n-4}} v^i \delta_j = c_1 K(a_i) \varepsilon_{ij} + o(\varepsilon_{ij} + \frac{1}{\lambda_i^2}) \\
\int_{S^n} (\delta_i \delta_j)^{\frac{n}{n-4}} = O(\varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1}) \\
< \delta_j, \lambda_i \frac{\partial \varepsilon_i}{\partial \lambda_i} > = c_1 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij}) \\
\int_{S^n} K^{\frac{n+4}{2(n-4)}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = c_1 K(a_j) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij} + \frac{1}{\lambda_j^2}) \\
\int_{S^n} K^{\frac{n+4}{2(n-4)}} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j = c_1 K(a_i) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij} + \frac{1}{\lambda_i^2}).
\]

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