Snyder-type spacetimes, twisted Poincaré algebra and addition of momenta

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Abstract

We discuss a generalisation of the Snyder model compatible with undeformed Lorentz symmetries, which we describe in terms of a large class of deformations of the Heisenberg algebra. The corresponding deformed addition of momenta, the twist and the $R$-matrix are calculated to first order in the deformation parameters for all models. In the particular case of the Snyder realisation, an analytic formula for the twist is obtained.
1 Introduction

In his seminal paper [1], Snyder observed that, assuming a noncommutative structure of spacetime, and hence a deformation of the Heisenberg algebra, it is possible to define a discrete spacetime without breaking the Lorentz invariance. In this way the short-distance behavior of quantum field theory can be improved, possibly avoiding ultraviolet divergences.

More recently, noncommutative geometry has become an important field of research [2]. New models have been introduced, as for example the Moyal plane [3] and κ-Minkowski geometry [4], and the formalism of Hopf algebras has been applied to their study [5]. However, contrary to Snyder’s, these models either break or deform the action of the Lorentz group on spacetime.

It is therefore interesting to investigate the Snyder model from the point of view of noncommutative geometry. The Hopf algebra associated with the Snyder model has been studied in a series of papers [6, 7, 8], where the model has been generalised and the star product, coproduct and antipodes have been calculated using the method of realisations. A different approach was used in [9], where the Snyder model was considered in a geometrical perspective as a coset in momentum space, and results equivalent to those of refs. [6, 7] were obtained. More recently, in [10] a further generalisation was introduced and the construction of QFT on Snyder spacetime was undertaken.

However, some basic properties of the Hopf algebra formalism for Snyder spaces have not yet been investigated: for example the twist and the related $R$-matrix have not been explicitly calculated, although they have proven to be very useful tools, especially in the construction of a QFT. In particular, the knowledge of the $R$-matrix is useful for the definition of a twisted statistics in QFT. Actually, some difficulties arise because the coproduct in Snyder spaces is non-coassociative, so that the twist will not satisfy the cocycle condition for the Hopf algebra.

From a different point of view, phenomenological aspects of the Snyder model have been investigated in classical and quantum physics, especially in the nonrelativistic 3D limit [11, 12, 13]. The most interesting results in this context are the clarification of its lattice-like properties, leading to deformed uncertainty relations, and the study of the corrections induced on the energy spectrum of some simple physical systems.

In this paper, we extend previous investigations on the noncommutative geometry of the generalised Snyder models, by calculating the twist and the $R$-matrix to first order in the deformation parameter in the general case. We also obtain the expression of the twist for the so-called Snyder realisation, introduced in the original paper [1].

We note that our results could be rephrased using the formalism of Hopf algebroids [14, 15, 16, 17, 18, 19, 20, 21, 22], which is for some aspects more suitable for the description of the Snyder models than the usual one based on Hopf algebras, since it deals with the full phase space; however we leave this subject to future investigations.

2 Snyder space and its generalisation

Generalised Snyder spaces are a deformation of ordinary phase space, generated by noncommutative coordinates $\bar{x}_\mu$ and momenta $\bar{p}_\mu$ that span a deformed Heisenberg
algebra $\tilde{\mathcal{H}}(\tilde{x}, p)$,

$$[\tilde{x}_\mu, \tilde{x}_\nu] = i\beta M_{\mu\nu} \psi(\beta p^2), \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, \tilde{x}_\nu] = -i\varphi_{\mu\nu}(\beta p^2),$$

(1)

together with Lorentz generators $M_{\mu\nu}$ that satisfy the standard relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}),$$

$$[M_{\mu\nu}, p_\lambda] = i(\eta_{\mu\lambda}p_\nu - \eta_{\nu\lambda}p_\mu), \quad [M_{\mu\nu}, \tilde{x}_\lambda] = i(\eta_{\mu\lambda}\tilde{x}_\nu - \eta_{\nu\lambda}\tilde{x}_\mu),$$

(2)

where the functions $\psi(\beta p^2)$ and $\varphi_{\mu\nu}(\beta p^2)$ are constrained so that the Jacobi identities hold, $\beta$ is a constant of the order of $1/M_p^2$, and $\eta_{\mu\nu} = \text{diag} \ (-1, 1, 1, 1)$. The commutation relations (1)-(2) generalise those originally investigated in [1], that are recovered for $\psi = \text{const}$.

We recall that in its undeformed version, the Heisenberg algebra $\mathcal{H}(x, p)$ is generated by commutative coordinates $x_\mu$ and momenta $p_\mu$, satisfying

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}.$$  

(3)

The action of $x_\mu$ and $p_\mu$ on functions $f(x)$ belonging to the enveloping algebra $\mathcal{A}$ generated by the $x_\mu$ is defined as

$$x_\mu \triangleright f(x) = x_\mu f(x), \quad p_\mu \triangleright f(x) = -i\frac{\partial f(x)}{\partial x^\mu}.$$  

(4)

The noncommutative coordinates $\tilde{x}_\mu$ and the Lorentz generators $M_{\mu\nu}$ in (1)-(2) can be expressed in terms of commutative coordinates $x_\mu$ and momenta $p_\mu$ as

$$\tilde{x}_\mu = x_\mu \varphi_1(\beta p^2) + \beta x \cdot p p_\mu \varphi_2(\beta p^2) + \beta p_\mu \chi(\beta p^2),$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu.$$

(5)

(6)

Note that in the Snyder algebra the generators $M_{\mu\nu}$ are in principle independent of $x_\mu$ and $p_\mu$, thus the condition (6) characterises a particular representation of the model. We also remark that the function $\chi$ does not appear in the defining relations (1)-(2), but takes into account the ambiguities arising from the ordering of the operators $x_\mu$ and $p_\mu$ in equation (5).

In terms of the realisation (5), the functions $\varphi_{\mu\nu}$ in (1) read

$$\varphi_{\mu\nu} = \eta_{\mu\nu}\varphi_1 + \beta p_\mu p_\nu \varphi_2,$$

(7)

while the Jacobi identities are satisfied if

$$\varphi = -2\varphi_1\varphi_1' + \varphi_1\varphi_2 - 2\beta p^2\varphi_1'\varphi_2,$$

(8)

where the prime denotes a derivative with respect to $\beta p^2$. In particular, the function $\varphi$ does not depend on the function $\chi$.

From (8) it follows that the coordinates $\tilde{x}_\mu$ are commutative for $\varphi_2 = \frac{2\varphi_1\varphi_1'}{\varphi_1 - 2\beta p^2\varphi_1'}$, and correspond to Snyder space for $\varphi_2 = \frac{1 + 2\varphi_1\varphi_1'}{\varphi_1 - 2\beta p^2\varphi_1'}$. In particular, the Snyder realisation (1)
is recovered for \( \varphi_1 = \varphi_2 = 1 \), and the Maggiore \([24]\) realisation for \( \varphi_1 = \sqrt{1-\beta p^2} \), \( \varphi_2 = 0 \) \([6, 7]\). There is also another interesting exact realisation of Snyder space for \( \psi = s = \text{const.} \), given by

\[
\hat{x}_\mu = x_\mu + \frac{\beta s}{4} K_\mu,
\]

which corresponds to \( \chi = 0 \), and where \( K_\mu = x_\mu p^2 - 2x \cdot p p_\mu \) are the generators of conformal transformations in momentum space, with \([K_\mu, K_\nu] = 0\). The algebra \([1]\) unifies commutative space, \( \psi = 0 \), and Snyder space, \( \psi = 1 \). Since the Lorentz transformations are not deformed, the Casimir operator of the algebra \((1)-(2)\) is \( C = p^2 \).

The Hopf algebra associated with these spaces can be investigated using the formalism introduced in refs. \([7,8,24,25,26]\) and generalised in \([28]\), to which we refer for more details. It turns out that the generalised addition of momenta \( k_\mu \) and \( q_\mu \) is given by \([7,8,27]\)

\[
k_\mu + q_\mu = D_\mu(k,q), \quad \text{with} \quad D_\mu(k,0) = k_\mu, \quad D_\mu(0,q) = q_\mu,
\]

where \( k,q \in M_{1,3} \). The function \( D_\mu(k,q) \) can be calculated in terms of \( \varphi_\mu \), as

\[
D_\mu(k,q) = P_\mu(\mathcal{K}^{-1}(k),q),
\]

where we have introduced the function \( \mathcal{K}_\mu(k) = P_\mu(k,0) \) and its inverse \( \mathcal{K}_\mu^{-1}(k) \), such that \( \mathcal{K}_\mu^{-1}(\mathcal{K}(k)) = k_\mu \). The function \( P_\mu(\lambda k,q) \) satisfies the differential equation

\[
\frac{dP_\mu(\lambda k,q)}{d\lambda} = k_\mu \varphi_\mu \left( P(\lambda k,q) \right), \quad \lambda \in \mathbb{R},
\]

with

\[
P_\mu(k,0) = \mathcal{K}_\mu(k), \quad P_\mu(0,q) = q_\mu.
\]

From \([12]\) and \([7]\) it follows that \( P_\mu(k,q) \) and hence \( D_\mu(k,q) \) do not depend on the function \( \chi \) in \([5]\).

It can be shown that \([24,25,26]\)

\[
e^{ik \cdot x} * e^{iq \cdot x} = e^{P(k,q) \cdot x + iQ(k,q)},
\]

where \( Q(k,q) \) satisfies the differential equation

\[
\frac{dQ(\lambda k,q)}{d\lambda} = k_\mu \varphi_\mu \left( P(\lambda k,q) \right),
\]

with \( Q(0,q) = 0 \) and \( \chi'' \equiv p^\mu \chi (\beta p^2) \).

Calculating the star product of two plane waves one then obtains

\[
e^{ik \cdot x} \ast e^{iq \cdot x} = e^{\mathcal{G}(k,q) \cdot x + i\mathcal{Q}(k,q)},
\]

with

\[
\mathcal{G}(k,q) = Q(\mathcal{K}^{-1}(k),q) - Q(\mathcal{K}^{-1}(k),0).
\]

Note that \( \mathcal{G} \) vanishes if \( \chi(k) = 0 \).
The algebra $\mathcal{A}$, generated by the coordinates $x_\mu$, can be extended to the algebra $\mathcal{U}$ generated by the $x_\mu$ and the $p_\mu$, symbolically indicated as $\mathcal{U} = \mathcal{A}\mathcal{T}$, where $\mathcal{T}$ is the algebra generated by the $p_\mu$ [14, 27]. The coproduct for the momenta $\Delta p_\mu$, is obtained from $D_\mu(k, q)$ as
\[ \Delta p_\mu = D_\mu(p \otimes 1, 1 \otimes p). \] (18)
Notice that the previous definitions imply that the addition of momenta and the coproduct do not depend on $\chi(\beta p^2)$.

From (16) and the coproduct (18) one can then define the twist $\mathcal{F}$, such that $\Delta h = \mathcal{F}\Delta_0 h \mathcal{F}^{-1}$ for any $h \in \mathcal{U}$, as [14, 29, 30]
\[ \mathcal{F}^{-1} = \exp[i(1 \otimes x_\nu)(\Delta - \Delta_0)p_\alpha + i G(p \otimes 1, 1 \otimes p)] : , \] (19)
where $\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu$, and $:\;$ denotes normal ordering in which the coordinates $x_\nu$ stand on the left of the momenta $p_\mu$.

The star product $f \star g$ can be defined as
\[ (f \star g)(x) = m(\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g)), \quad f, g \in \mathcal{A}, \] (20)
with $m : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{H}$ the multiplication map of $\mathcal{A}$.

The relation (5) between $\bar{x}_\mu$ and $x_\mu$ can also be written in terms of the twist as
\[ \bar{x}_\mu = m(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)) = x_\alpha \varphi^\alpha_{\mu}(p) + \beta p_\mu \chi(p). \] (21)
It follows for consistency that
\[ \Delta p_\mu = \mathcal{F}(\Delta_0 p_\mu)\mathcal{F}^{-1}, \quad \Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu \] (22)
in accordance with (18).

The coproducts of momenta are found for special cases in [7]: for the Snyder realisation
\[ \Delta p_\mu = \frac{1}{1 - \beta p_\mu \otimes p^\mu} \left( p_\mu \otimes 1 - \frac{\beta}{1 + \sqrt{1 - \beta p^2}} p_\mu p_\alpha \otimes p^\alpha + \frac{1}{1 + \beta p^2} \otimes p_\mu \right), \] (23)
while for the Maggiore realisation
\[ \Delta p_\mu = p_\mu \otimes \sqrt{1 - \beta p^2} - \frac{\beta}{1 + \sqrt{1 - \beta p^2}} p_\mu p_\alpha \otimes p^\alpha + 1 \otimes p_\mu. \] (24)

The coproducts of the Lorentz generators are instead
\[ \Delta M_{\mu\nu} = \mathcal{F}(\Delta_0 M_{\mu\nu})\mathcal{F}^{-1}, \quad \Delta_0 M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}. \] (25)
Generally, because of the commutation relations (2), the coproduct of $M_{\mu\nu}$ will be trivial, i.e. $\Delta M_{\mu\nu} = \Delta_0 M_{\mu\nu}$ [7].

We recall that also the antipodes for Snyder space are trivial [7],
\[ S(p_\mu) = -p_\mu, \quad S(M_{\mu\nu}) = -M_{\mu\nu}, \] (26)
3 First order expansion

The study of the general form of the deformed Heisenberg algebra (1) is difficult, however one can study it perturbatively, by expanding the realisation (5) of the noncommutative coordinates in powers of $\beta$, namely,

$$\tilde{x}_\mu = x_\mu + \beta (s_1 x_\mu p^2 + s_2 x \cdot p p_\mu + c p_\mu) + O(\beta^2),$$

(27)

with parameters $s_1$, $s_2$, $c$, such that $\varphi_1(\beta p^2) = 1 + s_1 \beta p^2 + O(\beta^2)$, $\varphi_2(\beta p^2) = s_2 + O(\beta)$ and $\chi(\beta p^2) = c + O(\beta)$. Hence, the commutation relations do not depend on the parameter $c$ and to first order are given by

$$[\tilde{x}_\mu, \tilde{x}_\nu] = i\beta s M_{\mu\nu} + O(\beta^2), \quad [p_{\mu}, \tilde{x}_\nu] = -i [\eta_{\mu\nu}(1 + \beta s_1 p^2) + \beta s_2 p_\mu p_\nu] + O(\beta^2),$$

(28)

where $s = s_2 - 2s_1$.

The models of ref. [6, 7] are recovered for $s_2 = 1 + 2s_1$. Moreover, for $s_1 = 0$, $s_2 = 1$, eqs. (27), (28) reproduce the exact Snyder realisation, while for $s_1 = -\frac{1}{2}$, $s_2 = 0$ they give the first-order expansion of the Maggiore realisation. For $s_2 = 2s_1$, spacetime is commutative to first order in $\beta$, although the commutation relations are not canonical, while for $s_1 = -s/4$, $s_2 = s/2$, $c = 0$ one gets the exact realisation (9).

The first order expression for the function $P_\mu(k, q)$ is given by

$$P_\mu(k, q) = q_\mu + \int_0^1 d\lambda \left[ k_\mu + \beta \left( s_1 k_\mu (\lambda k + q)^2 + s_2 (\lambda k^2 + k \cdot q)(\lambda k_\mu + q_\mu) \right) \right] + O(\beta^2)$$

$$= k_\mu + q_\mu + \beta \left( s_1 q^2 + \left( s_1 + \frac{s_2}{3} \right) k \cdot q + s_2 \left( \frac{k^2}{2} \right) k_\mu + s_2 \left( k \cdot q + \frac{k^2}{2} \right) q_\mu \right) + O(\beta^2),$$

(29)

from where it follows that

$$K^{-1}_\mu(k) = k_\mu - \frac{\beta}{3} (s_1 + s_2) k^2 k_\mu + O(\beta^2).$$

(30)

Note that for $s_1 + s_2 = 0$, $K_\mu(k) = K^{-1}_\mu(k) = k_\mu$, to the first order in $\beta$. These results allow us to write down the generalised addition law of the momenta $k_\mu$ and $q_\mu$ to first order

$$(k \oplus q)_\mu = D_\mu(k, q) = k_\mu + q_\mu + \beta \left( s_2 k \cdot q q_\mu + s_1 q^2 k_\mu + \left( s_1 + \frac{s_2}{2} \right) k \cdot q \ k_\mu + \frac{s_2}{2} k^2 q_\mu \right) + O(\beta^2).$$

(31)

In particular, for the "conformal" case [9] with parameters $s_1 = -s/4$, $s_2 = s/2,

$$(k \oplus q)_\mu = k_\mu + q_\mu + \frac{\beta s}{4} \left( 2 k \cdot q q_\mu - q^2 k_\mu + k^2 q_\mu \right) + O(\beta^2).$$

(32)

It is also interesting to remark that for $s_2 = 2s_1 \neq 0$, $s = 0$, although spacetime is commutative up to the first order in $\beta$, the addition of momenta is still deformed, but it is now commutative

$$(k \oplus q)_\mu = (q \oplus k)_\mu = k_\mu + q_\mu.$$
The Lorentz transformations of momenta are not deformed, and denoting them by \( \Lambda(\xi, p) \), with \( \xi \) the rapidity parameter, the law of addition of momenta implies that

\[
\Lambda(\xi, k \oplus q) = \Lambda(\xi_1, k) \oplus \Lambda(\xi_2, q) \quad (34)
\]

is satisfied for \( \xi_1 = \xi_2 = \xi \). Hence there are no backreaction factors in the sense of ref. [31, 32]. This means that in composite systems the boosted momenta of the single particles are independent of the momenta of the other particles in the system.

The coproduct to the first order can be read from (31) and is given by

\[
\Delta p_\mu = \Delta_0 p_\mu + \beta \left[ s_1 p_\mu \otimes \rho^2 + s_2 p_\alpha \otimes \rho^\alpha \rho_\mu + \left( s_1 + \frac{s_2}{2} \right) p_\mu p_\alpha \otimes \rho^\alpha + \frac{s_2}{2} p^2 \otimes p_\mu \right] + O(\beta^3). \quad (35)
\]

The corresponding twist operator \( \mathcal{F}^{-1} \) is

\[
\mathcal{F}^{-1} = 1 \otimes 1 + i \beta \left[ s_1 D \otimes \rho^2 + \frac{s_2}{2} \rho^2 \otimes D + s_2 p_\alpha \otimes D p^\alpha + \left( s_1 + \frac{s_2}{2} \right) D p_\alpha \otimes \rho^\alpha \right] \quad (36)
\]

or equivalently, in terms of dilatation \( D = x \cdot p \) and momenta \( p_\alpha \)

\[
\mathcal{F}^{-1} = 1 \otimes 1 + i \beta \left[ s_1 D \otimes \rho^2 + \frac{s_2}{2} \rho^2 \otimes D + s_2 p_\alpha \otimes D p^\alpha + \left( s_1 + \frac{s_2}{2} \right) D p_\alpha \otimes \rho^\alpha \right] \quad (37)
\]

From this one can calculate the coproduct \( \Delta M_{\mu \nu} \), and the antipodes \( S(p_\mu) \) and \( S(M_{\mu \nu}) \) to the first order in \( \beta \). Using the twist (36), (37) to calculate the coproduct of \( p_\mu \) as in (22), one gets again (35), the same result as when using the function \( \mathcal{D} \), while using (25) to calculate the coproduct of \( M_{\mu \nu} \) gives \( \Delta M_{\mu \nu} = \Delta_0 M_{\mu \nu} + O(\beta^2) \), which is consistent with the general result \( \Delta M_{\mu \nu} = \Delta_0 M_{\mu \nu} \).

In general, the twist (19) will not satisfy the cocycle condition, the star product (16), (20) will be non-associative and the coproduct \( \Delta p_\mu \) in (18) will be non-coassociative. Let us verify these claims.

The cocycle condition is \( (\mathcal{F} \otimes 1)(\Delta_0 \otimes 1)\mathcal{F} = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F} \). The left hand side calculated for the twist \( \mathcal{F} \) (36) to the first order in \( \beta \) is

\[
1 \otimes 1 \otimes 1 + i \beta \left[ s_1 p_\alpha \otimes 1 \otimes x^\alpha p^2 + s_1 p_\alpha \otimes 1 \otimes x^\alpha \rho^2 + s_2 p_\alpha \otimes 1 \otimes x \cdot p \rho^\alpha \\
+ s_2 p_\alpha \otimes x \cdot p \rho^\alpha + \left( s_1 + \frac{s_2}{2} \right) p_\alpha p_\beta \otimes 1 \otimes x^\alpha p^\beta + \left( s_1 + \frac{s_2}{2} \right) 1 \otimes p_\alpha p_\beta \otimes x^\alpha p^\beta \\
+ \left( s_1 + \frac{s_2}{2} \right) 1 \otimes p_\alpha p_\beta \otimes x^\alpha p^\beta + \left( s_1 + \frac{s_2}{2} \right) p_\alpha \otimes p_\beta \otimes x^\alpha \rho^\beta + \frac{s_2}{2} p^2 \otimes 1 \otimes x \cdot p \\
+ \frac{s_2}{2} 1 \otimes p^2 \otimes x \cdot p + s_2 p_\alpha \otimes p^\alpha \otimes x \cdot p + s_1 p_\alpha \otimes p^2 \otimes x^\alpha + s_2 p_\alpha \otimes p^\alpha p_\beta \otimes x^\beta \\
+ \left( s_1 + \frac{s_2}{2} \right) p_\alpha p_\beta \otimes p^\alpha \otimes x^\beta + \frac{s_2}{2} p^2 \otimes p_\alpha \otimes x^\alpha \right] + O(\beta^3),
\]
while the right hand side is

\[
1 \otimes 1 \otimes 1 + i \beta \left( s_1 p_\alpha \otimes p^\beta \otimes \chi^\alpha + s_1 p_\alpha \otimes 1 \otimes \chi^\alpha p^\beta + 2 s_1 p_\alpha \otimes p_\beta \otimes \chi^\alpha \right) \\
+ s_2 p_\alpha \otimes p^\beta p_\beta \otimes \chi^\alpha + s_2 p_\alpha \otimes p_\beta \otimes \chi^\alpha p^\beta + s_2 p_\alpha \otimes 1 \otimes x \cdot p p^\alpha \\
+ s_2 p_\alpha \otimes p^\alpha \otimes x \cdot p + \left( s_1 + \frac{s_2}{2} \right) p_\alpha p_\beta \otimes p^\beta \otimes \chi^\alpha + \left( s_1 + \frac{s_2}{2} \right) p_\alpha \otimes p_\beta \otimes 1 \otimes \chi^\alpha p^\beta \\
+ \frac{s_2}{2} p^\alpha \otimes p_\alpha \otimes \chi^\alpha + \frac{s_2}{2} p^\alpha \otimes 1 \otimes x \cdot p + s_1 1 \otimes p_\alpha \otimes \chi^\alpha p^\alpha \\
+ s_2 1 \otimes p_\alpha \otimes x \cdot p p^\alpha + \left( s_1 + \frac{s_2}{2} \right) 1 \otimes p_\alpha p_\beta \otimes \chi^\alpha p^\beta + \frac{s_2}{2} 1 \otimes p^\alpha \otimes x \cdot p \right) + O(\beta^3).
\]

Generally, the cocycle condition is not satisfied even to the first order in $\beta$. It is satisfied only in the special case $s_2 = 2s_1$, which corresponds to commutative space. The coassociativity condition for the coproduct $\Delta$ \((18), (35)\) is \((\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta\). In general, also the coassociativity condition is not satisfied even to the first order in $\beta$, except in the special case $s_2 = 2s_1$.

The associativity condition for the star product \((16), (20), (31)\) is

\[
e^{ijk;\chi} \star \left( e^{i\lambda;j;\chi} \star e^{k\mu;\chi} \right) = \left( e^{i\lambda;j;\chi} \star e^{k\mu;\chi} \right) \star e^{ijk;\chi}. \tag{40}\]

The left hand side, calculated to the first order in $\beta$ is

\[
k_{1\mu} + k_{2\mu} + k_{3\mu} + \beta \left( k_{1\mu} \left[ s_1 (k_2^2 + k_3^2 + 2k_2 \cdot k_3) + \left( s_1 + \frac{s_2}{2} \right) (k_1 \cdot k_2 + k_1 \cdot k_3) \right] + \\
k_{2\mu} \left[ s_1 k_3^2 + \left( s_1 + \frac{s_2}{2} \right) k_2 \cdot k_3 + s_2 (k_1 \cdot k_2 + k_1 \cdot k_3) + \frac{s_2}{2} k_1^2 \right] + \\
k_{3\mu} \left[ 2s_2 k_2 \cdot k_3 + \frac{s_2}{2} k_2^2 + s_2 (k_1 \cdot k_2 + k_1 \cdot k_3) + \frac{s_2}{2} k_1^2 \right] \right) + O(\beta^2),
\]

while the right hand side reads

\[
k_{1\mu} + k_{2\mu} + k_{3\mu} + \beta \left( k_{1\mu} \left[ s_1 k_2^2 + \left( s_1 + \frac{s_2}{2} \right) k_1 \cdot k_2 + s_1 k_3^2 + \left( s_1 + \frac{s_2}{2} \right) (k_1 \cdot k_3 + k_2 \cdot k_3) \right] + \\
k_{2\mu} \left[ s_2 k_1 \cdot k_2 + \frac{s_2}{2} k_1^2 + s_1 k_3^2 + \left( s_1 + \frac{s_2}{2} \right) (k_1 \cdot k_3 + k_2 \cdot k_3) \right] + \\
k_{3\mu} \left[ s_2 (k_1 \cdot k_3 + k_2 \cdot k_3) + \frac{s_2}{2} (k_1^2 + k_2^2 + 2k_1 \cdot k_2) \right] \right) + O(\beta^2).
\]

It follows that the associativity condition for the star product is again not satisfied even to the first order in $\beta$, except in the special case $s_2 = 2s_1$. In this case, the star product is commutative and associative, but non-local.

Finally, in the commutative case $s_2 = 2s_1$, it is easily seen from \((35)\) that

\[
\tilde{\Delta} p_{\mu} \equiv \tau_0 \Delta p_{\mu} \tau_0 = \Delta p_{\mu}, \tag{43}\]

i.e. the coproduct is left-right symmetric, with the flip operator $\tau_0$ defined in the usual way as

\[
\tau_0 (A \otimes B) = B \otimes A. \tag{44}\]
The coproduct is cocommutative and coassociative.

In the general case $s_2 \neq 2s_1$, the algebraic structure of the generalised Hopf algebra defined by the enveloping algebra of Poincaré-Weyl algebra, the coproduct $Δ$, the antipode $S$ and the counit $ε$ is under investigation.

The flip operator, $τ = Fτ_0F^{-1}$, $τ^2 = τ_0^2 = 1 \otimes 1$ is relevant in the discussion of the twisted statistics of particles in quantum field theory on noncommutative spaces [29] [30]. Note that

$$mF^{-1} = mτ_0F^{-1} = mF^{-1}τ = mF^{-1}τ_0$$  \hspace{1cm} (45)

where $F = τ_0Fτ_0$. Another important operator in this context is the $R$-matrix, which satisfies the relation $RΔp_μR^{-1} = ˜Δp_μ$ and $τ = R^{-1}τ_0 = τ_0R$. The $R$-matrix can be written as

$$R = ˜F^2 = 1 \otimes 1 + R_{cl} + O(β^2),$$  \hspace{1cm} (46)

and it should lead to a generalization of the triangularity condition. The classical $R$-matrix $R_{cl}$ is

$$R_{cl} = (x_α \otimes 1)(Δ_0)p^α - (1 \otimes x_α)(Δ_0)p^α,$$  \hspace{1cm} (47)

where $Δp_μ$ is given in (45). From (37) and (47), we can write

$$R_{cl} = β\left(s_1 - \frac{s_1}{2}\right)(D \otimes p^2 - p^2 \otimes D + (D \otimes 1 - 1 \otimes D)p_α \otimes p^α],$$  \hspace{1cm} (48)

or equivalently $R_{cl} = β\left(s_1 - \frac{s_1}{2}\right)(M_{αβ}p_β \otimes p_α - p_α \otimes M_{αβ}p_β)$. For $s_1 = 2$ it follows that $R_{cl} = 0$. For commutative spaces, for which (45) holds, $R_{cl}$ is given by

$$R_{cl} = (x_α \otimes 1 - 1 \otimes x_α)(Δ_0)p^α \in I_0.$$  \hspace{1cm} (49)

where $I_0$ is the right ideal of $U$ with the property $m(I_0 ⊗ (f \otimes g)) = 0$.

### 3.1 Hopf algebroid approach

The Hopf algebroid structure was introduced in [15] [16]. We point out that the twist $F^{-1}$ in [19] [30], as well as in the classical $R$-matrix $R_{cl}$ in [47] [48] are obtained in the Hopf algebroid approach [14] [17] [18], where the set of generators $x_α$ and $p_μ$ defines the basis of the Heisenberg Hopf algebroid [19] [20]. In the general case $s_2 \neq 2s_1$, the algebraic structure of the generalised Hopf algebroid is currently under investigation.

An important result is that the twist $F^{-1}$ in the case $G = 0$ (i.e. $χ(p) = 0$) is identical [21] [22] to

$$F^{-1} = e^{-ip_αx^α}e^{iK_1^{-1}(p)\otimes i^γ} = e^{-ip_αx^α}e^{iK_1^{-1}(p)\otimes i^γ},$$  \hspace{1cm} (50)

where $K_1(p)$ is defined after (11).

To the first order in $β$, $K^{-1}_1(p)$ is given in (30). For example, in the Snyder case, $s_1 = 0$, $s_2 = 1$ and

$$K_1^{-1}(p) = p_γ - \frac{β}{3}p^2p_γ + O(β^2).$$  \hspace{1cm} (51)
In the Maggiore realisation, \( s_1 = -\frac{1}{2}, \ s_2 = 0 \) and
\[
\mathcal{K}^{-1}_\gamma(p) = p_\gamma + \frac{\beta}{6} p^2 p_\gamma + O(\beta^2).
\] (52)

4 Twist for the Snyder realisation

In this section, we construct the exact twist operator for the Snyder space using the perturbative approach introduced in [14], by expanding (22) in powers of \( \beta \). We first consider the special case of the Snyder realisation \( \bar{\varphi}_1 = \varphi_2 = 1, \chi = 0 \), for which
\[
\tilde{x}_\mu = x_\mu + \beta \cdot p \ p_\mu.
\] (53)

The coproduct of the momenta is given by (23). We expand it with respect to the deformation parameter \( \beta \) as \( \Delta p_\mu = \sum_{k=0}^\infty \Delta_k p_\mu \), with \( \Delta_k p_\mu \propto \beta^k \)
\[
\Delta p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu + \beta \left( \frac{1}{2} p_\mu p_\alpha \otimes p^\alpha + p_\alpha \otimes p^\alpha p_\mu + \frac{1}{2} p^2 \otimes p_\mu \right)
+ \beta^2 \left( \frac{1}{2} p_\mu p_\alpha p_\beta \otimes p^\alpha p^\beta + p_\alpha p_\beta \otimes p^\alpha p^\beta p_\mu + \frac{1}{8} p_\mu p_\alpha p^2 \otimes p^\alpha - \frac{1}{8} p^4 \otimes p_\mu \right)
+ \frac{1}{2} p_\mu p^2 \otimes p^\alpha p_\mu + \frac{1}{2} p_\mu p_\alpha p^2 \otimes p^\alpha p^\beta + \frac{1}{16} p_\mu p_\alpha p^4 \otimes p^\alpha + \frac{1}{8} p_\mu p_\alpha p^4 \otimes p_\mu + \frac{1}{8} p_\mu p^4 \otimes p^\alpha p_\mu
+ \frac{1}{2} p_\mu p^2 \otimes p^\alpha p_\mu + O(\beta^4)
\] (54)
and we look for the twist operator in the form
\[
\mathcal{F} = e^{\hbar \gamma_1 f_1 + \hbar \gamma_2 f_2 + \ldots},
\] (55)
where \( f_k \propto \beta^k \). From (22) we obtain the equations satisfied by the \( f_k \) order by order,
\[
[f_1, \Delta_0 p_\mu] = \Delta_1 p_\mu,
\] (56)
\[
[f_2, \Delta_0 p_\mu] = \Delta_2 p_\mu - \frac{1}{2} \left( [f_1, [f_1, \Delta_0 p_\mu]] \right),
\] (57)
\[
[f_3, \Delta_0 p_\mu] = \Delta_3 p_\mu - \frac{1}{2} \left( [f_1, [f_2, \Delta_0 p_\mu]] + [f_2, [f_1, \Delta_0 p_\mu]] \right)
- \frac{1}{3} \left( [f_1, [f_1, [f_1, \Delta_0 p_\mu]]] \right).
\] (58)
and so on. To calculate \( f_1 \) we write down the ansatz
\[
f_1 = \beta (\alpha_1 p^2 \otimes x \cdot p \ + \alpha_2 p_\alpha p_\beta \otimes x^\alpha p^\beta \ + \alpha_3 p_\alpha \otimes x \cdot p \ p^\alpha \ + \alpha_4 p_\alpha \otimes x^\alpha p^2)
\]
and insert it into (56) to determine the unknown coefficients $\alpha_i$. The resulting expression for $f_1$ is

$$f_1 = -i\beta \left( \frac{1}{2} p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right).$$

Inserting this and the ansatz

$$f_2 = \beta^2 (\alpha_1 p^4 \otimes x \cdot p + \alpha_2 p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + \alpha_3 p_\alpha p^2 \otimes x \cdot p p^\alpha + \alpha_4 p_\alpha p^2 \otimes x^\alpha p^\beta + \alpha_5 p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma),$$

into (57), we find

$$f_2 = i \frac{\beta^2}{2} \left( \frac{1}{2} p^4 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right).$$

An analogous procedure to third order gives

$$f_3 = -i \frac{\beta^3}{3} \left( \frac{1}{2} p^6 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta p^4 \otimes x^\alpha p^\beta + p_\alpha p^4 \otimes x \cdot p p^\alpha \right).$$

From the results for $f_1$, $f_2$, $f_3$, ..., we conjecture that the twist $F$ can be written as

$$F = \exp \left( -i \left( \frac{1}{2} p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right) \ln \left( \frac{1 + \beta p^2}{p^2} \otimes 1 \right) \right).$$

One can check that (62) gives the correct twist for the Snyder space by calculating

$$m(F^{-1} (\otimes 1)(x_\mu \otimes 1)) = x_\mu + \beta x \cdot p p_\mu.$$

An independent verification is to start from (19). We get

$$F^{-1} = : \exp \left[ \frac{i}{1 - \beta p_\alpha \otimes p^\alpha} \left( \frac{\beta \sqrt{1 + \beta p^2}}{1 + \sqrt{1 + \beta p^2}} p^\alpha \otimes x_\mu p_\nu + \left( \sqrt{1 + \beta p^2} - 1 \right) \otimes x \cdot p \right. \right.$$

$$\left. + \beta p_\alpha \otimes x \cdot p p^\alpha \right) : ,$$

which expanded up to second order gives

$$F^{-1} = 1 \otimes 1 + i\beta \left( \frac{1}{2} p^4 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right)$$

$$- \frac{ig^2}{2} \left( \frac{1}{4} p^4 \otimes x \cdot p - \frac{1}{4} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta - p_\alpha p^2 \otimes x \cdot p p^\alpha - p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma \right)$$

$$- 2p_\alpha p_\beta \otimes x \cdot p p^\alpha \otimes x \cdot p p^\beta$$

$$+ p_\alpha p_\beta \otimes x_\gamma x \cdot p p^\gamma + \frac{1}{2} p_\alpha p_\beta p_\gamma \otimes x^\alpha x^\beta p^\gamma + p_\alpha p_\beta p_\gamma \otimes x^\alpha x \cdot p p^\beta p^\gamma$$

$$+ p_\alpha p_\beta \otimes x_\gamma x \cdot p p^\gamma p^\beta + O(\beta^3).$$

(65)
The expression in eq. (65) agrees exactly with what one would get from (59) and (60) using the fact that \( F^{-1} = 1 \otimes 1 - f_1 \otimes f_2 + \frac{1}{2} f_1^2 + O(\beta^3) \).

As a further check, let us calculate the coproduct \( \Delta p_\mu = F \Delta_0 p_\mu F^{-1} \) with twist \( F \) given in (62)

\[
F \Delta_0 p_\mu F^{-1} = p_\mu \otimes 1 + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\beta^{n-k}}{k!} A_{n,k}(p^{2(n-k)} \otimes 1) \text{ad}_{\beta}^k (1 \otimes p_\mu),
\]

where

\[
A_{n,k} = \sum_{r_1, \ldots, r_k = n} \frac{1}{r_1 r_2 \cdots r_k}
\]

and

\[
\text{ad}_{\beta}^k (1 \otimes p_\mu) = \beta^k \left[ \sum_{l=0}^{k} c_{k-l,l}(p_\mu (p^2)^{k-l} \otimes 1)(p_\mu \otimes p^2)^l + \sum_{l=0}^{k} d_{k-l,l} ((p^2)^{k-l} \otimes p_\mu)(p_\mu \otimes p^2)^l \right].
\]

The coefficients \( c_{k-l,l} \) and \( d_{k-l,l} \) satisfy the following recursive relations

\[
c_{k-l+1,l} = l c_{k-l,l} + (l-1) c_{k-l+1,l-1} + \frac{1}{2} d_{k-l+1,l-1}
\]

\[
d_{k-l+1,l} = (l + \frac{1}{2}) d_{k-l,l} + l d_{k-l+1,l-1}
\]

with \( c_{0,0} = 0 \) and \( d_{0,0} = 1 \). Particularly, the coefficients \( c_{k,0}, c_{k-1,1} \) and \( d_{k,0} \) are

\[
c_{k,0} = 0, \quad c_{k-1,1} = 1 - \frac{1}{2^k}, \quad d_{k,0} = \frac{1}{2^k}.
\]

Using this result for \( d_{k,0} \), we sum the terms of the form \((p^{2(n-k)} \otimes 1)(p^2 \otimes p_\mu) = p^{2n} \otimes p_\mu\)

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\beta^{n-k}}{k!} A_{n,k} \frac{1}{2^k} p^{2n} \otimes p_\mu = e^{\frac{1}{4} \ln(1 + \beta p^2)} \otimes p_\mu = \sqrt{1 + \beta p^2} \otimes p_\mu.
\]

which agrees with the corresponding term in \( \Delta p_\mu \) (23). Proceeding similarly, using the result for \( c_{k-1,1} \) in (71), we get \( p_\mu \frac{1 + \beta p^2 - \sqrt{1 + \beta p^2}}{p^2} p_\mu \otimes p^2 \), in accordance with the corresponding term in \( \Delta p_\mu \) (23). The complete inductive proof for \( \Delta p_\mu \) (23) and \( F \) (62) will be given elsewhere.

As a consistency check, using the twist (62) to calculate the coproduct of \( M_\mu \), we can also verify that the coproduct of the Lorentz generators is undeformed to all orders i.e.,

\[
\Delta M_{\mu \nu} = \Delta_0 M_{\mu \nu}.
\]

Note that the twist corresponding to the Snyder realization can be written in terms of the dilatation \( D = \chi \cdot p \) and of \( p^2 \), in a form which slightly differs from eq. (62) but is equivalent to it.
5 Twist for the Maggiore realisation

The same procedure can be performed for the Maggiore realisation (24). The coproduct, when expanded up to the third order, takes the following form

\[ \Delta p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu - \frac{\beta}{2} \left( p_\mu p_\alpha \otimes p^\alpha + p_\mu \otimes p^2 \right) \]

\[ -\frac{\beta^2}{8} \left( p_\mu \otimes p^4 + p_\mu p_\alpha p^2 \otimes p^\alpha \right) - \frac{\beta^3}{16} \left( p_\mu \otimes p^6 + p_\mu p_\alpha p^4 \otimes p^\alpha \right) + O(\beta^4) \]  

Using the same procedure as in the previous section, we find

\[ f_1 = \frac{i\beta}{2} \left( p_\alpha \otimes x^\alpha p^2 + p_\alpha p_\beta \otimes x^\alpha p^\beta \right), \]  

\[ f_2 = \frac{i\beta^2}{8} \left( p_\alpha \otimes x^\alpha p^4 + p_\alpha p_\beta \otimes x^\alpha p^\beta + 2 p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma \right. \]

\[ + 2 p_\alpha p_\beta \otimes x^\alpha p^\beta p^2 + 2 p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^2 \]  

\[ f_3 = \frac{i\beta^3}{8} \left( \frac{1}{2} p_\alpha \otimes x^\alpha p^6 + \frac{4}{3} p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma + \frac{3}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta p^4 \right. \]

\[ + \frac{7}{12} p_\alpha p_\beta \otimes x^\alpha p^4 + \frac{5}{12} p_\alpha p_\beta \otimes x^\alpha p^4 + \frac{7}{3} p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^2 \]  

\[ + \frac{5}{7} p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^2 + \frac{4}{7} p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\delta + 2 p_\alpha p_\beta p_\gamma p_\delta \otimes x^\alpha p^\beta p^\delta \). \]

In this case, we were not able to obtain a closed form for the twist. However, the perturbative result, when used to calculate the coproduct of \( M_{\mu\nu} \), gives again the primitive coproduct.

6 Conclusions

In this paper we have investigated the most general realisations of the Snyder model compatible with undeformed Lorentz invariance, and have calculated the twist and the R-matrix for the generic case, at leading order in the deformation parameters. In particular, in the specific case of the Snyder realisation we were able to obtain an analytic expression for the twist.

Our results can be rephrased using the formalism of Hopf algebroids [14, 15, 16, 17, 18, 19, 20, 21, 22], that is for some aspects more suitable for the description of the Snyder models than the usual one based on Hopf algebras, since it deals with the full phase space. We leave however this subject to future investigations.

The results obtained in this paper may be important for the construction of a complete QFT on Snyder spaces. Some basic attempts in this direction have been put forward in refs. [7, 9, 10].
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