Instantons and the Large $N_c$ Limit of QCD

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We summarize our current understanding of instantons in the large $N_c$ limit of QCD. We also present some recent results from simulations of the instanton liquid in QCD for $N_c > 3$.

1 Historical Overview

Both instantons and the large $N_c$ expansion are important corner stones in our attempts to understand non-perturbative aspects of QCD. Instantons are related to topological aspects of QCD and play an important role in understanding the $U(1)_A$ anomaly. There is also a growing body of evidence that the mechanism for chiral symmetry breaking in QCD is intimately connected with instantons. The large $N_c$ expansion is based on the idea that $1/N_c$ can be used as expansion parameter in QCD. The main assumption is that QCD with many colors is a theory that exhibits confinement and chiral symmetry breaking, and that this theory is smoothly connected to the real world, in which $N_c$ is equal to three. There is an impressive amount of evidence, mainly of phenomenological nature, but also from the lattice, that this assumption is indeed correct.

It is commonly believed, however, that instantons and the large $N_c$ expansion are fundamentally incompatible with each other. This view is maybe best explained by providing a short overview of the literature on the subject. The instanton solution was discovered in 1975. In the following year ’t Hooft found the fermion zero mode in the background field of an instanton and explained the connection between instantons and the $U(1)_A$ anomaly. In 1978 Witten wrote a very influential paper entitled “Instantons, the Quark Model, and the $1/N$ expansion”. He noted that straightforward $N_c$ counting suggests that the $\eta'$ mass squared scales as $1/N_c$, whereas classical effects, such as instantons, scale as $\exp(-1/g^2) \sim \exp(-N_c)$. He explicitly stated that in trying to understand the $\eta'$ mass and other non-perturbative phenomena in QCD “it is necessary to choose between the large $N$ expansion and instantons”. In 1979 Witten and Veneziano derived a relation between the mass of the $\eta'$ and the
topological susceptibility in pure gauge theory,

\[ \frac{f^2}{2N_f} (m_{\eta}^2 + m_{\eta'}^2 - 2m_K^2) = \chi_{\text{top}}. \]  

(1)

Witten suggested that for all \( N_c \) instantons “melt” and cannot be distinguished from perturbative fluctuations. He also proposed that the topological susceptibility is of order \( O(1) \) in the large \( N_c \) limit, and that it is dominated by non-perturbative fluctuations other than instantons, presumably related to confinement. Indeed, Veneziano’s paper is entitled “\( U(1) \) without Instantons”.

Simple \( N_c \) counting implies that both the \( \eta' - \pi \) splitting and the \( \rho - \omega \) splitting are of order \( 1/N_c \). In the real world, of course, these two phenomena are very different in magnitude. This suggests that there is some physical effect in the \( \eta' \) channel that leads to unusually large \( 1/N_c \) corrections. This observation was discussed in more detail by Novikov et al. in a paper entitled “Are all hadrons alike?”. Novikov et al. observed that large corrections to the OZI rule appear many channels, not just pseudoscalar mesons, but also scalar mesons and glueballs. They also noticed that these are precisely the same channels in which direct instanton effects appear.

This observation led to the formulation of the instanton liquid model of the QCD vacuum. The instanton liquid model postulates that the \( N_c = 3 \) QCD vacuum is populated by localized, approximately dual or anti-self dual lumps, instantons. The density of instantons is approximately \( (N/V) \approx 1 \text{ fm}^{-4} \) while the size is \( \rho \approx 1/3 \text{ fm} \). These numbers reproduce the topological susceptibility in the pure gauge theory \( \chi_{\text{top}} \approx (200 \text{ MeV})^4 \) and the chiral condensate \( \langle \bar{\psi}\psi \rangle \approx -(230 \text{ MeV})^3 \). More detailed calculations show that the instanton liquid model successfully describes an impressive amount of data on hadronic correlation functions.

In the 1980’s researchers also began to study the topological structure of QCD on the lattice. It was found that the topological susceptibility in pure gauge QCD is \( \chi_{\text{top}} \approx (200 \text{ MeV})^4 \), as predicted by the Witten-Veneziano relation equ. (1). However, it was also observed that the topological susceptibility is very stable under cooling, and appears to be dominated by semi-classical configurations. Lattice simulations also appear to confirm the values of the key parameters of the instanton liquid, \( (N/V) \approx 1 \text{ fm}^{-4} \) and \( \rho \approx 1/3 \text{ fm} \).

More recently lattice simulations have started to focus on the structure of low-lying eigenvectors of the Dirac operator and the mechanism of chiral symmetry breaking. The instanton model predicts that the lowest eigenstates of the Dirac operator, which dominate chiral symmetry breaking, are linear combinations of localized, approximately chiral states associated with the fermionic zero modes of individual instantons and anti-instantons. This picture has
been confirmed by lattice calculations, although there is some controversy concerning the question whether the size of the chiral lumps is in agreement with the instanton prediction.

2 Instantons and the OZI rule

In this section we would like to remind the reader how instantons lead to large violations of the OZI rule. ’t Hooft explained that the effect of an instanton on fermionic correlation functions can be summarized in terms of an effective interaction

$$L = G \left[ (\bar{\psi}\tau_a\psi)^2 - (\bar{\psi}\psi)^2 - (\bar{\psi}i\gamma_5\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right],$$

where $G$ is related to the tunneling amplitude and $\tau$ is an isospin matrix. On the single instanton level we can directly read off the interaction in the channel characterized by the current $\bar{\psi}\Gamma\psi$. The interaction is attractive in the pion $(\bar{\psi}i\gamma_5\tau\psi)$ and sigma $(\bar{\psi}\psi)$ channels, and repulsive in the eta prime $(\bar{\psi}i\gamma_5\psi)$ and $a_0$ $(\bar{\psi}\tau\psi)$ channels. We note that the OZI violating interaction the $\eta'$ and $\sigma$ channel is not suppressed with respect to the OZI allowed contribution in the $\pi$ and $a_0$ sector.

It is worth repeating why this is so. The instanton interaction corresponds to the contribution of fermionic zero modes to the quark propagator. Since there is exactly one zero mode for every flavor, there are no diagonal $(\bar{u}u)(\bar{u}u)$ or $(\bar{d}d)(\bar{d}d)$ interactions. Second, since the fermion zero modes for quarks and anti-quarks have opposite chirality, the interaction is also off-diagonal in the basis spanned by right and left-handed fermions $q_R, q_L$. As a consequence the sign of the interaction flips in going from the scalar to the pseudoscalar channel, and in going from $I = 0$ to $I = 1$ states. Also, to leading order there is no instanton contribution to the interaction in vector channels.

In order to be more quantitative we consider correlation functions involving the scalar and pseudoscalar currents introduced above. The correlators are defined by

$$\Pi_\pi(x, y) = \langle \text{Tr} (S(x, y)i\gamma_5S(y, x)i\gamma_5) \rangle,$$

$$\Pi_\delta(x, y) = \langle \text{Tr} (S(x, y)S(y, x)) \rangle,$$

$$\Pi_\eta'(x, y) = \langle \text{Tr} (S(x, y)i\gamma_5S(y, x)i\gamma_5) - 2\langle \text{Tr} (i\gamma_5S(x, x)) \text{Tr} (i\gamma_5S(y, y)) \rangle \rangle,$$

$$\Pi_\sigma(x, y) = \langle \text{Tr} (S(x, y)S(y, x)) - 2\langle \text{Tr} (S(x, x)) \text{Tr} (S(y, y)) \rangle \rangle,$$

where $S(x, y)$ is the fermion propagator and $\langle \rangle$ denotes the average over all gauge configurations. The OZI-violating difference between the $\pi, \eta'$ and $\sigma, a_0$
Figure 1: Correlation functions in the pion, delta ($a_0$), and rho meson channels. The correlators are normalized to free field behavior ($\Pi_0(x) \sim x^{-6}$).

Figure 2: OZI violating disconnected correlation functions in the pseudoscalar ($\eta' - \pi$), scalar ($\sigma - a_0$), and vector ($\omega - \rho$) channel. The correlators are normalized as in Fig.1.
channels is determined by “disconnected” (or double hairpin) contributions to the correlation functions.

In Figs. 1 and 2 we show results for the correlation functions obtained in unquenched instanton simulations of the instanton liquid. We observe that the correlation function in the OZI violating $\eta' - \pi$ and $\sigma - a_0$ channels is as large as the correlation function in the OZI allowed $\pi$ channel whereas the OZI violating $\omega - \rho$ channel is much smaller. This pattern was also found in a lattice calculation by Isgur and Thacker. It would be interesting to perform similar calculations in other channels in which large violations of the OZI rule might be present. Examples are the mixing of scalar glueballs with scalar mesons or multi-pion states, the $\Delta I = 1/2$ rule, and OZI suppressed decays of heavy quark bound states.

3 Instantons at Large $N_c$

In the previous section we argued that the OZI violating direct instanton contribution in the $\eta'$ channel is not suppressed compared to the instanton contribution in the $\pi$ channel. But what is the $N_c$ dependence of these effects? Witten argued that instanton effects are proportional to $\exp(-8\pi^2/g^2)$ and therefore scale as $\exp(-N_c)$. But this is only true for instantons at the cutoff scale. In perturbation theory, the tunneling rate is of the form

$$dn(\rho) \sim \rho^{-5}(\rho\Lambda)^b d\rho,$$

where $b = 11N_c/3 - 2N_f/3$ is the first coefficient of the beta function. This implies that small instantons $\rho < \Lambda^{-1}$ are suppressed as $N_c \to \infty$, but large instantons are not.

This observation is not very useful, because instantons of size $\Lambda^{-1}$ have action $S \sim 1$ and are clearly not semi-classical objects. We can be somewhat more quantitative, however. The one-loop result contains additional $N_c$ dependent factors that are related to the collective coordinate measure. Exponentiating everything we can write

$$\frac{dN}{d\rho} \sim \exp[N_cF(\rho)],$$

with $F(\rho) = 2 - s(\rho) + 2\log(s(\rho)) + \ldots$ and $s(\rho) \equiv S(\rho)/N_c = (8\pi^2)/(N_cg^2(\rho))$. The function $F(\rho)$ has a non-trivial zero for $s^* \sim 5$ so the density of instantons of size $\rho = \rho^* \sim 0.2\Lambda^{-1}$ fm is fixed as $N_c \to \infty$. This is a much more optimistic conclusion, because it implies that instantons with action $S = N_cs^* \sim 5N_c \gg 1$ can survive in the large $N_c$ limit. The precise value of $\rho^*$ and $s^*$ depends on the renormalization scheme and higher order corrections. A fixed point in
the instanton size distribution was indeed observed in the lattice calculation of Lucini and Teper, see also Fig. 3. However, these authors also find that the density of instantons of size $\rho > \rho^*$ keeps growing as $N_c \to \infty$, and that as a result there is no well defined average instanton size.

4 Supersymmetric Theories

While the fate of instantons in the large $N_c$ limit of QCD is still mysterious some progress has been made in understanding instantons in the large $N_c$ limit of supersymmetric gauge theories. The most impressive accomplishment is Maldacena’s discovery of an explicit master field in the case of $N = 4$ supersymmetric QCD at large $N_c$ in the limit of large ’t Hooft coupling $\lambda = g^2 N_c$. The master field is given by supergravity in an $AdS_5 \times S_5$ background together with a set of rules that relate gauge theory to supergravity observables. It is very interesting to study how instantons appear in this correspondence.

On the supergravity side gauge theory instantons appear as D-instantons and are characterized by a point on $AdS_5 \times S_5$. This integration over this point corresponds to the collective coordinate integration on the gauge theory side. The $AdS_5$ part is related to the integration over $d^4 x/\rho^5$ while the $S_5$ arises from the fermion zero modes.

The case of multi-instanton configurations is even more interesting. Naively one would think that a $k$-instanton contribution involves an integration over $(AdS_5 \times S_5)^k$. In the large $N_c$ limit one finds, however, that the saddle point configuration is given by $k$ instantons in commuting $SU(2)$ subgroups of $SU(N_c)$ all with the same size and located at the same point. As a result, the collective coordinate measure contains only one copy of $AdS_5 \times S_5$. The multi-instanton configuration is bound by fermion exchanges.

Progress was also made in understanding $N = 2$ SUSY QCD. Seiberg and Witten determined the low energy effective action of this theory. In the semi-classical limit, the result can be expressed as the perturbative one-loop contribution plus an infinite series on $k$-instanton corrections, and, remarkably, nothing else. This would seem to imply that the large $N_c$ limit of this theory is rather boring. Instantons are suppressed as $\exp(-N_c)$, monopoles have masses $O(N_c)$, and one is left with only the perturbative part.

This is not correct, however. The methods of Seiberg and Witten were generalized to arbitrary $N_c$ by Klemm et al. This result was analyzed in more detail by Douglas and Shenker. The moduli space of the theory is characterized by $N_c - 1$ Higgs expectation values. At most points on the moduli space the large $N_c$ limit is indeed trivial but Douglas and Shenker identified a special form of the large $N_c$ limit in which instantons and monopoles survive.

6
5 The Instanton Liquid at Large $N_c$

In this section we wish to study the question whether it is possible to construct a large $N_c$ instanton ensemble in QCD that is consistent with standard large $N_c$ counting. Equ. (8) shows that instantons of action $S \sim N_c \gg 1$ and size $\rho \sim N_c^0$ can survive in the large $N_c$ limit. It is quite natural to assume that the total density of these objects would scale as $(N/V) \sim N_c$ because instantons are essentially $SU(2)$ configurations and the number of mutually commuting $SU(2)$ subgroups of $SU(N_c)$ grows as $N_c$. This means that we can have $O(N_c)$ instantons and the instanton liquid remains dilute. Using the trace anomaly relation

$$\langle T_{\mu\nu} \rangle = -\frac{b}{32\pi^2} \langle g^2 C^a_{\mu\nu} F^a_{\mu\nu} \rangle$$

(9)

with $b = 11N_c/2 - 2N_f/2$ we also see that the instanton contribution to the vacuum energy scales as $\epsilon \sim N_c^2$. This is identical to the scaling behavior of the perturbative part.

In order to construct a consistent ensemble of instantons we have to make an assumption concerning the fate of large instantons. In the following we shall assume that there is a classical $O(S_0) = O(N_c)$ core in the instanton interaction for instantons that overlap in group space. This core excludes configurations with large or strongly overlapping instantons that are not semi-classical. The parameters of the core were fitted to reproduce the phenomenological values of the instanton size and density for $N_c = 3$. Instanton size distributions
obtained from numerical simulations of the instanton liquid for $N_c = 3, \ldots, 6$ are shown in Fig. 3. In these simulations we have assumed that $(N/V) \sim N_c$, but this assumption can be verified by computing the free energy of the instanton liquid as a function of $(N/V)$. We note that the distributions show the fixed point discussed above and observed in the lattice simulations of Lucini and Teper.

We have also computed the quark condensate and the topological susceptibility in pure gauge theory, see Fig. 4. We observe that the quark condensate scales as $\langle \bar{q}q \rangle \sim N_c$. This scaling can be understood using random matrix arguments. The quark condensate is approximately given by

$$\langle \bar{q}q \rangle = -\frac{1}{\pi \rho} \left( \frac{3N_c N}{2 V} \right)^{1/2}.$$  \hspace{1cm} (10)

The total number of states in the zero-mode zone scales as $N_{zmz} \sim (N/V) \sim N_c$. This contributes a factor $N_c^{1/2}$ to the scaling behavior of $\langle \bar{q}q \rangle$. The second factor of $N_c^{1/2}$ arises from the fact that the quark condensate is inversely proportional to the average matrix element $|T_{IA}|$ of the Dirac operator between fermion zero modes. Since instanton zero modes live in $SU(2)$ subgroups $|T_{IA}|^2$ scales as $1/N_c$. Similar arguments can be used to show that the pion decay constant scales as $f_{\pi}^2 \sim N_c$.

For a dilute gas of instantons we would expect that the topological sus-
ceptibility in pure gauge theory scales as $\chi_{\text{top}} = \langle Q_{\text{top}}^2 \rangle / V \simeq (N/V) \sim N_c$, contrary to the standard large $N_c$ assumption $\chi_{\text{top}} \sim N^0$. Further support for this assumption was recently provided by Witten. We have to keep in mind, however, that the result $\chi_{\text{top}} \simeq (N/V)$ is based on the idea that topological charge fluctuations are Poissonian. Since the classical interaction between instantons also grows with $N_c$, this is not necessarily the case. Indeed, simulations carried out for $N_c = 3, \ldots, 7$ seem to indicate that the topological susceptibility remains finite in the large $N_c$ limit, see Fig. 4.

6 Instead of Conclusions

We have argued that instantons provide a very successful explanation of the pattern of OZI violation observed in QCD for $N_c = 3$ colors. The fate of instantons in the large $N_c$ limit remains unclear. It may well be that instantons become large and overlap strongly, and the semi-classical description breaks down completely. However, it is also possible that the semi-classical picture survives. This would seem to require large cancellations and fine tuning, but for reasons that we do not understand the parameters of the instanton liquid in QCD with $N_c = 3$ colors are already remarkably close to the critical point where a smooth large $N_c$ limit is possible.

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