On the non-commutative Riemannian geometry of $GL_q(n)$

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Abstract

A recently proposed definition of a linear connection in non-commutative geometry, based on a generalized permutation, is used to construct linear connections on $GL_q(n)$. Restrictions on the generalized permutation arising from the stability of linear connections under involution are discussed. Candidates for generalized permutations on $GL_q(n)$ are found. It is shown that, for a given generalized permutation, there exists one and only one associated linear connection. Properties of the linear connection are discussed, in particular its bicovariance, torsion and commutative limit.

LPTHE-Orsay 95/51
q-alg/9507002
1 Introduction

Shortly after their discovery in the context of integrable models \[26, 16, 6, 11\], quantum groups were identified as interesting non-commutative generalizations of the algebra of functions on a Lie-group manifold \[30, 22, 10\]. Non-commutative differential calculi \[5\] have been proposed where the main constraint is the bi-covariance of the differential algebra \[31\]. In addition the $R$-matrix formulation \[10\] played a key role in further developments \[2, 21, 12, 25, 27, 28\].

The aim of this paper is to define linear connections and metrics on quantum groups. From the mathematical point of view, this is a step towards the understanding of which classical concepts can have a non-commutative generalization. From the physical point of view, it could be a first step towards the formulation of gravitational theories on quantum groups. Non-commutative manifolds in fact could represent a solution to the problem of short distance divergences of usual quantum field theories (See e.g. \[20, 18\]) and could also offer a more satisfactory description of space-time. In this respect, quantum groups are an interesting toy model where qualitative differences between the non-commutative ($q \neq 1$) and the non-deformed ($q = 1$) cases can be observed. In the context of the Dirac-operator-based differential calculus of Connes, an approach to the construction of such theories has been proposed using the Wodzicki residue of the Dirac operator \[13, 14\]. However, many interesting differential calculi, such as those on quantum groups \[31\] and spaces \[29\], are not defined by a Dirac operator. Here, as was proposed in \[17, 4, 15\], we follow the idea, which is suitable for all differential calculi, of a generalization to the non-commutative context of the usual commutative metrics and linear connections.

A general definition of linear connections, in the context of non-commutative geometry, has been recently proposed for the derivation-based differential calculus \[23, 8\] and other differential calculi \[23\] in which case the construction relies on a generalized permutation. In Section 2 we fix our notation concerning quantum groups. In Section 3 we briefly review the construction of \[23\] and add some restrictions on the generalized permutations which arise from the requirement that the set of covariant derivatives be stable under complex conjugation. Section 4 is devoted to the search for generalized permutations on $GL_q(n)$ which are restricted by the bicovariance condition. We find a two-parameter family of generalized permutations. In Section 5 we prove that for a given generalized permutation there exists only one linear connection. Properties of this linear connection are studied, in particular its bicovariance, torsion and curvature. Finally, we examine the commutative limit of our linear connections. We show that the limit of one class of these when $q \to 1$ corresponds to left and right invariant linear connections on $GL(n)$. We collect our conclusions in Section 6.
2 Quantum groups and their bicovariant differential calculi

The quantum group $\text{Fun}(GL_q(n))$ is a Hopf algebra $(A, \Delta, \epsilon, \kappa)$ generated, as an algebra, by the identity and $T^i_j, i, j = 1, \ldots n$. An exchange of the order of the generators, while maintaining the classical Poincaré series, is obtained by the RTT relation \cite{10}:

$$RT_1T_2 = T_1T_2R. \quad (2.1)$$

Here, $R$ is the $R$-matrix, which is an element of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ obeying the Yang-Baxter relation

$$\left((1 \otimes R) (R \otimes 1) (1 \otimes 1) (1 \otimes R) \right) (R \otimes 1) = (1 \otimes 1) (1 \otimes R) (R \otimes 1). \quad (2.2)$$

The $R$ matrix of $GL_q(n)$ is given by \cite{11}

$$R = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + \lambda \sum_{i < j} E_{ii} \otimes E_{jj}, \quad (2.3)$$

where $\lambda = q - q^{-1}$. It satisfies the Hecke condition

$$(R - q)(R + \frac{1}{q}) = 0. \quad (2.4)$$

The differential calculus on the quantum group is considerably restricted by the bicovariance condition \cite{31}. This means that there exist a right and left coaction of $A$ on $\Omega^1$, the space of 1-forms, such that

$$\Delta_L(a \ db) = \Delta(a) (1 \otimes d) \Delta(b), \quad (2.5)$$

$$\Delta_R(a \ db) = \Delta(a) (d \otimes 1) \Delta(b), \quad (2.6)$$

$$\left(1 \otimes \Delta_R\right) \Delta_L = \left(\Delta_L \otimes 1\right) \Delta_R. \quad (2.7)$$

Under some restrictions on $q$ and the assumption that $\Omega^1$ be generated as a left-module by $dT_j$, bicovariant differential calculi have been classified \cite{24} and shown to be obtained by the constructive method of Jurco \cite{12}.

For such differential calculi $\Omega^1$ is generated as a left (or right) module by left invariant 1-forms $\omega_j^i (\Delta_L(\omega_j^i) = 1 \otimes \omega_j^i)$:

$$\omega_j^i = \kappa(T_k^i) dT_j^k. \quad (2.8)$$

The differential algebra is entirely characterized by the commutation relations

$$\omega_j^i a = (1 \otimes f_j^k) \Delta(a)\omega_k^i, \quad (2.9)$$

where $f_j^k$ are linear functionnals representing the algebra $A$:

$$f_j^k(1) = \delta_j^k, \quad f_j^k(ab) = f_m^i(a)f_n^k(b).$$
They can be explicitly determined in terms of the $R$-matrix and some parameters [24]. Here we shall make the choice

$$f_{il}^{jk}(T_n^m) = (R^{-1})_{in}^{sk} (R^{-1})_{sl}^{jm}$$  \hspace{1cm} (2.10)

for the differential calculus. In the limit $q \to 1$, this differential calculus reduces to the usual one on $GL(n)$. It has been considered in Ref. [21, 28, 27, 32, 25]. In this case, the commutation relations are often written in the form

$$T_1 dT_2 = R dT_1 T_2 R.$$  \hspace{1cm} (2.11)

The space of 2-forms is constructed as the image of $\Omega^1 \otimes_A \Omega^1$ under the “multiplication” map $\pi$:

$$\pi : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$$  \hspace{1cm} (2.12)

$$\pi = 1 - \Lambda,$$  \hspace{1cm} (2.13)

where $\Lambda$ is a bimodule automorphism, obeying the Yang-Baxter equation, which generalizes the permutation map of the commutative case. Let $\eta^i_j$ be right invariant 1-forms:

$$\eta^i_j = T^i_l \omega^l_k \kappa(T^k_j).$$  \hspace{1cm} (2.14)

Then $\Lambda$ is determined by [31]

$$\Lambda(\omega^j_i \otimes \eta^k_l) = \eta^i_l \otimes \omega^j_k.$$  \hspace{1cm} (2.15)

When applied to $\omega^j_i \otimes \omega^k_l$, one can show that

$$\Lambda(\omega^j_i \otimes \omega^k_l) = \Lambda_{jk}^{i} \omega^{i,p}_{n,q} \omega^{m}_{p} \otimes \omega^{q}_{n},$$  \hspace{1cm} (2.16)

$$\Lambda^{i,j}_{k} \omega^{i,p}_{n,q} = f_{j,p}^{i,n}(\kappa(T^k_m)T^l_n).$$  \hspace{1cm} (2.17)

When applied to $dT_1 \otimes dT_2$, the map $\Lambda$ yields

$$\Lambda(dT_1 \otimes dT_2) = R dT_1 \otimes dT_2 R^{-1}.$$  \hspace{1cm} (2.18)

The Hecke relation for the $R$ matrix [24], combined with the previous equation, yields the following characteristic equation for $\Lambda$:

$$(\Lambda - 1)(\Lambda + q^2)(\Lambda + q^{-2}) = 0.$$  \hspace{1cm} (2.19)

Higher order forms can be constructed in a similar way using the map $\Lambda$ [31]. The exterior derivative is defined with the help of the right and left-invariant 1-form $\theta$:

$$\theta = -\frac{q^{2n+1}}{\lambda} \sum_i q^{-2i} \omega^i_i,$$  \hspace{1cm} (2.20)
by
\[ d\omega = [\theta, \omega], \quad (2.21) \]
where \([,]\) is the graded commutator and the product is in \(\Omega\).

For real values of \(q\) or for \(|q| = 1\), an involution may be defined on \(GL_q(n)\) reducing it respectively to \(U_q(n)\) or to \(GL_q(n, \mathbb{R})\). Setting the \(q\)-determinant equal to one gives rise to \(SU_q(n)\) and \(SL_q(n, \mathbb{R})\) \[10\]. The bicovariant differential calculus on these reductions is characterized either by a larger set of 1-forms than the classical case \[3\] or by a modified Leibniz rule \[4\].

3 Linear connections in non-commutative geometry

In this section we collect the main definitions and results concerning the general construction of linear connections as proposed in \[23\]. We add some new restrictions on the generalized permutation by imposing the stability of the set of covariant derivatives under complex conjugation. In the following \(\mathcal{A}\) is a unital associative algebra over \(\mathbb{C}\) equipped with the differential calculus \((\Omega, d)\).

Definition 3.1: Let \(\pi\) be the multiplication in \(\Omega\). A generalized permutation, \(\sigma\), is a bimodule automorphism of \(\Omega^1 \otimes_A \Omega^1\) satisfying
\[ \pi \circ \sigma = -\pi. \quad (3.1) \]

A generalized flip, \(\tau\), is defined as a generalized permutation satisfying \(\tau^2 = 1\).

Remarks:
1- Note that \(\sigma = -1\) is a generalized flip.
2- When the algebra \(\mathcal{A}\) is the algebra of \(C^\infty\) functions on a manifold \(M\), the permutation
\[ \tau(\omega \otimes \omega') = \omega' \otimes \omega \quad (3.2) \]
is a generalized flip.
3- When \(\Omega^2\) is realized as a subspace of \(\Omega^1 \otimes \Omega^1\) with an imbedding \(i\) verifying \(\pi \circ i = 1_{\Omega^2}\) then
\[ 1 = 2i \circ \pi \quad (3.3) \]
is a generalized flip. The generalized flip of the derivation-based differential calculus proposed in Ref.\[8, 23\] is of this form, as are the generalized flips of Ref.\[19, 15\].
4- If \(\sigma\) is a generalized permutation then so is \(\sigma^{-1}\) as well as \(\sigma^{2n+1}\) for an arbitrary integer \(n\).
5- If \(\sigma\) and \(\sigma'\) are two generalized permutations then so is \(\mu(\sigma + 1) + \mu'(\sigma' + 1) - 1\). The \(\sigma + 1\) form a linear space.
Definition 3.2: A linear connection associated to a generalized permutation, \( \sigma \), is a linear map, \( \nabla^\sigma \), from \( \Omega^1 \) to \( \Omega^1 \otimes_A \Omega^1 \) satisfying the two Leibniz rules

\[
\begin{align*}
\nabla^\sigma(a\omega) &= da \otimes \omega + a \nabla^\sigma \omega, \quad (3.4) \\
\nabla^\sigma(\omega a) &= \sigma(\omega \otimes da) + \nabla^\sigma \omega \ a, \quad (3.5)
\end{align*}
\]

for any \( a \in A \) and any \( \omega \in \Omega^1 \).

Remarks:

1- When the algebra \( A \) is the commutative algebra of smooth functions on a manifold the only possible linear connections are those associated to the permutation (3.2).

2- If \( \sigma \) and \( \sigma' \) are two generalized permutations then \( \nabla^\sigma - \nabla^\sigma' \) is a left-module homomorphism.

3- If \( \nabla \) and \( \nabla' \) are two linear connections associated to the same generalized permutation, then their difference is a bimodule homomorphism.

The preceding definition of the linear connection has the advantage of allowing an extension to the tensor product over \( A \) of several copies of \( \Omega^1 \). This is formulated in:

Proposition 3.3: A linear connection associated to a generalized permutation \( \sigma \) admits a unique extension as a linear map from \( \Omega^1 \otimes_A \ldots \otimes_A \Omega^1 \) to \( \Omega^1 \otimes_A \ldots \otimes_A \Omega^1 \) of the form

\[
\nabla^\sigma(\omega \otimes \omega') = \nabla^\sigma(\omega) \otimes \omega' + \sigma_s(\omega \otimes \nabla^\sigma \omega'), \quad (3.6)
\]

for any \( \omega \in \Omega^1 \) and any \( \omega' \in \Omega^1 \otimes_A \ldots \otimes_A \Omega^1 \), with \( \sigma_s \) an automorphism of \( \Omega^1 \otimes_A \ldots \otimes_A \Omega^1 \). The unique \( \sigma_s \) is given by

\[
\sigma_s = \sigma \otimes 1 \otimes \ldots \otimes 1. \quad (3.7)
\]

Proof. The proof can be carried out by induction. For \( s = 2 \) an identification of \( \nabla^\sigma(\omega f \otimes \omega') \) with \( \nabla^\sigma(\omega \otimes f \omega') \), where \( f \) is an arbitrary element of \( A \) and \( \omega \) and \( \omega' \) are 1-forms, gives \( \sigma_2 = \sigma \otimes 1 \); so the proposition is true for \( s = 2 \). Suppose it is true to order \( s - 1 \) and let \( \omega' \) be an element of the tensor product of \( s - 1 \) copies of \( \Omega^1 \) then, by the induction hypothesis,

\[
\nabla^\sigma f \omega' = df \otimes \omega' + f \nabla^\sigma \omega'. \quad (3.8)
\]

The identification of \( \nabla^\sigma(\omega \otimes f \omega') \) with \( \nabla^\sigma(\omega f \otimes \omega') \) where \( \omega \) is an element of \( \Omega^1 \) completes the Proof. ♣
Suppose that $\mathcal{A}$ is an algebra over $\mathbb{C}$ equipped with an involution $\ast$. Then $\Omega^1$ carries a natural involution defined by $(bda)\ast = (da\ast)b\ast$. The involution on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ is not a priori determined. In fact, if we define the anti-homomorphism $\alpha$ by

$$\alpha(\omega \otimes \omega') = \omega'^\ast \otimes \omega^\ast,$$

and if $\phi$ is an automorphism of $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ such that $(\phi \circ \alpha)^2 = 1$, then $\phi \circ \alpha$ defines an involution on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$. We would like to define an involution on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ which in the commutative limit reduces to $(\omega \otimes \omega')^\ast = \tau(\omega'^\ast \otimes \omega^\ast)$, where $\tau$ is the usual permutation operator, and which allows us to define the complex conjugate of a linear connection, as in the commutative case, by

$$\nabla^\sigma \omega = \left(\nabla^\sigma (\omega^\ast)\right)^\ast.$$

The requirement that $\nabla^\sigma$ be a linear connection imposes constraints on the involution on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ and on the generalized permutation, $\sigma$:

**Proposition 3.4:** Suppose that $\mathcal{A}$ is equipped with an involution $\ast$, then the following assertions are equivalent

1- The map $\nabla^\sigma$ defined in (3.10) is a linear connection.
2- the generalized permutation, $\sigma$, verifies

$$(\sigma \circ \alpha)^2 = 1$$

and the involution on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ is given by

$$(\omega \otimes \omega')^\ast = \sigma(\omega'^\ast \otimes \omega^\ast).$$

**Proof.** $2 \Rightarrow 1$ is a direct calculation. We prove $1 \Rightarrow 2$. Calculate, with the aid of Equation (3.10), $\nabla^\sigma (\omega a)$:

$$\nabla^\sigma (\omega a) = \left(\nabla^\sigma (a^\ast \omega^\ast)\right)^\ast$$

$$= (da^\ast \otimes \omega^\ast)^\ast + (\nabla^\sigma \omega^\ast)^\ast a$$

$$= (da^\ast \otimes \omega^\ast)^\ast + (\nabla^\sigma \omega)^\ast a.$$  

If the map $\nabla^\sigma$ is a covariant derivative then there exists a generalized permutation, $\phi$, such that

$$\nabla^\sigma (\omega a) = \phi(\omega \otimes da) + \nabla \omega a.$$  

Comparing the two equations (3.13) and (3.14) we obtain

$$(da^\ast \otimes \omega^\ast)^\ast = \phi(\omega \otimes da).$$
This equation is valid for arbitrary \( a \) and \( \omega \) so the involution in \( \Omega^1 \otimes_A \Omega^1 \) verifies:

\[
\left( \omega' \otimes \omega \right)^* = \phi(\omega^* \otimes \omega^*). \tag{3.16}
\]

The involution property, \(* = 1\), gives \((\phi \circ \alpha)^2 = 1\). It remains to prove that \( \phi = \sigma \). In order to do this, calculate, using Equation (3.10), \( \nabla'(a\omega): \)

\[
\nabla'(a\omega) = a\left( \nabla'(\omega^*) \right)^* + \left( \sigma(\omega^* \otimes da^*) \right)^*.
\tag{3.17}
\]

Since \( \nabla' \) is a linear connection we have

\[
\nabla'(a\omega) = a\nabla' \omega + da \otimes \omega.
\tag{3.18}
\]

Comparing equation (3.17) and (3.18) we get

\[
da \otimes \omega = \left( \sigma(\omega^* \otimes da^*) \right)^*.
\tag{3.19}
\]

This equation is valid for arbitrary \( a \) and \( \omega \), so we have

\[
\omega' \otimes \omega = \left( \sigma(\omega^* \otimes \omega^*) \right)^*.
\tag{3.20}
\]

Comparing equations (3.16) and (3.20) leads to the equality of \( \phi \) and \( \sigma \). ♣

Definition 3.5: For a given involution \(*\) on \( \Omega^1 \otimes_A \Omega^1 \), a generalized permutation \( \sigma \) is defined to be real if it satisfies the following property:

\[
\sigma \circ * = * \circ \sigma \tag{3.21}
\]
on \( \Omega^1 \otimes_A \Omega^1 \).

Now, if one wants to find an involution \(*\) on \( \Omega^1 \otimes_A \Omega^1 \) such that \( \nabla' \) is a linear connection, then one should take, according to Proposition 3.4, (3.12) as a definition of \(*\). The condition for this to be possible is \((\sigma \circ \alpha)^2 = 1\). If one further demands that \( \sigma \) be real then one has to use the following:

Proposition 3.6: Suppose that the generalized permutation, \( \sigma \), verifies Equation (3.14) and that the involution on \( \Omega^1 \otimes_A \Omega^1 \) is given by \( \sigma \circ \alpha \) then \( \sigma \) is real iff it is a generalized flip.

Proof. The reality condition reads

\[
\sigma \circ \sigma \circ \alpha = \sigma \circ \alpha \circ \sigma.
\tag{3.22}
\]

Since \( \sigma \) is an automorphism, this equation leads to

\[
\sigma \circ \alpha = \alpha \circ \sigma.
\tag{3.23}
\]

This relation in \((\sigma \circ \alpha)^2 = 1\) gives \( \sigma^2 = 1\). ♣
The definition of the complex conjugate of a linear connection allows the following:

**Definition 3.7:** A real linear connection associated to a generalized permutation \( \sigma \) is defined by \( \nabla' = \nabla^\sigma \).

**Remark:**
The involution on \( \Omega^1 \otimes A \Omega^1 \) defined above induces an involution on \( \Omega^2 \) by \( (\omega \wedge \omega')^* = \pi((\omega \otimes \omega')^*) = -\omega^* \wedge \omega^* \). This is due to the property (3.1).

**Definition 3.8:** The torsion \( T \) of a linear connection \( \nabla^\sigma \), is defined as the linear map from \( \Omega^1 \) to \( \Omega^2 \) given by

\[
T = d - \pi \circ \nabla^\sigma. \quad (3.24)
\]

**Proposition 3.9:** The torsion map is a bimodule homomorphism.
**Proof.** It is an immediate consequence of the condition (3.1). ♣

**Definition 3.10:** The curvature \( R \) of a linear connection \( \nabla^\sigma \) is defined as the linear map from \( \Omega^1 \) to \( \Omega^2 \otimes_A \Omega^1 \) given by

\[
R = ((T \otimes 1) + (\pi \otimes 1) \nabla^\sigma) \nabla^\sigma \quad (3.25)
\]

**Proposition 3.11:** The curvature is a left-module homomorphism.
**Proof.** A straightforward calculation. ♣

**Definition 3.12:** A metric \( g \) is defined as an element of \( \Omega^1 \otimes_A \Omega^1 \) satisfying

\[
\pi(g) = 0. \quad (3.26)
\]

If \( \Omega^1 \otimes_A \Omega^1 \) is equipped with an involution, a real metric is defined by \( g^* = g \).

The definition of a non-degenerate metric requires some more structure on the algebra \( A \). This structure must guarantee that the dimension of \( \Omega^1 \) as a left module be well defined. For example if \( A \) is a Hopf algebra then it is well known that this is so (See e.g. [31]). If it exists, let \( \omega^a, \ a = 1, \ldots N \), be a free basis of \( \Omega^1 \) as a left module then a metric can be written uniquely in the form

\[
g = g_{ab} \omega^a \otimes \omega^b, \quad (3.27)
\]

with \( g_{ab} \in A \). We will call a metric nondegenerate if the matrix whose elements are \( g_{ab} \) is invertible.

**Definition 3.13:** A metric \( g \) and a linear connection \( \nabla^\sigma \) are said to be compatible if the condition \( \nabla^\sigma g = 0 \) is satisfied.
4 Determination of $\sigma$ on $GL_q(n)$

In addition to the previous requirements on $\sigma$, it is natural, in the context of quantum groups, to add the requirement of bicovariance:

**Definition 4.1:** A generalized permutation, $\sigma$, is called bicovariant iff:

\[
(1 \otimes \sigma)\Delta_L = \Delta_L \sigma, \\
(\sigma \otimes 1)\Delta_R = \Delta_R \sigma.
\] (4.1)

Following the $R$-matrix technique, that is the determination of all unknown maps from the $R$-matrix and $q$, we will determine the candidates for the map $\sigma$ in terms of $R$. We recall from (2.13) and (3.1) that the generalized permutation $\sigma$ is an automorphism of $\Omega^1 \otimes_A \Omega^1$ verifying

\[
(1 - \Lambda) \circ (\sigma + 1) = 0,
\] (4.2)

the bicovariance requirements (4.1), and when $A$ is equipped with an involution, that is for real $q$ and for $|q| = 1$, the involution property (3.11).

In order to find candidates for $\sigma$, we shall prove the following Proposition, which, in its first part, is a generalization of Proposition 3.1 of [31]:

**Proposition 4.2:** Let $\alpha_{ij}$, $i, j = 0, 1$, be complex numbers.

1. There exists a unique bimodule homomorphism, $\Phi$, of $\Omega^1 \otimes_A \Omega^1$ such that

\[
\Phi(dT_1 \otimes dT_2) = \sum_{i,j} \alpha_{ij} R^i dT_1 \otimes dT_2 R^j.
\] (4.3)

Moreover,

2. The map $\Phi$ is bicovariant.

3. The map $\Phi$ is a generalized permutation iff

\[
\alpha_{01} - \alpha_{10} = 0, \\
\alpha_{00} + \lambda \alpha_{10} - \alpha_{11} = -1,
\] (4.4)

where, we recall, $\lambda = q - q^{-1}$.

In this case, $\Phi$ obeys the characteristic equation

\[
(\Phi + 1)(\Phi - \lambda_1)(\Phi - \lambda_2) = 0,
\] (4.5)

\[
\lambda_1 = -1 + \alpha_{10}(q + q^{-1}) + \alpha_{11}(1 + q^2),
\] (4.6)

\[
\lambda_2 = -1 - \alpha_{10}(q + q^{-1}) + \alpha_{11}(1 + q^{-2}).
\] (4.7)

**Proof.** An element $\nu$ of $\Omega^1 \otimes_A \Omega^1$ can be written in a unique way as

\[
\nu = \sum a_{ki}^j d^k T_i \otimes d^j T_j = Tr (a (dT_1 \otimes dT_2)) ,
\] (4.8)
where \( a \in M_n(A) \otimes M_n(A) \). This is a consequence of the fact that the \( dT \) generate \( \Omega^1 \) as a left-module. The action of \( \Phi \) on \( \nu \) is defined by

\[
\Phi(\nu) = \text{Tr}(a \alpha_{ij} R_i dT_1 \otimes dT_2 R^j).
\]  
(4.9)

It clearly satisfies (4.3). It remains to check that \( \Phi \) defined in this way is a bimodule homomorphism. The left-module homomorphism property is assured by construction. To check the right-module homomorphism property it suffices to verify that

\[
\Phi(dT_1 \otimes dT_2 T_3) = \Phi(dT_1 \otimes dT_2) T_3. 
\]  
(4.10)

This is so because the \( T \) generate the algebra. The left-hand side of Equation (4.10) can be written, after successive use of Equation (2.11), as

\[
\Phi(dT_1 \otimes R_{12}^{-1} T_2 dT_3 R_{23}^{-1}) = R_{23}^{-1} R_{12}^{-1} \Phi(T_1 dT_2 \otimes dT_3) R_{12}^{-1} R_{23}^{-1}, 
\]  
(4.11)

here the subscripts of the \( R \)-matrix denote the two spaces on which it acts. Next, we use the left-module property to write the right-hand side of Equation (4.11) as

\[
\alpha_{ij} R_{12}^{-1} R_{23}^{-1} R_{i2} dT_1 \otimes dT_2 dT_3 R_{i2} R_{12}^{-1} R_{23}^{-1}. 
\]  
(4.12)

The right-hand side of Equation (4.10) can be written as

\[
\alpha_{ij} R_{i2}^{-1} dT_1 \otimes dT_2 R_{i2} R_{12}^{-1} R_{23}^{-1} R_{i2}. 
\]  
(4.13)

The commutation relations (2.11) allow us to write this term as

\[
\alpha_{ij} R_{i2}^{-1} dT_1 \otimes R_{23}^{-1} T_2 dT_3 R_{23}^{-1} R_{i2} = \alpha_{ij} R_{i2}^{-1} R_{23}^{-1} R_{12}^{-1} R_{23}^{-1} R_{12}^{-1} R_{i2} R_{23}^{-1} R_{12}^{-1} R_{23}^{-1}. 
\]  
(4.14)

As a consequence of the Yang-Baxter equation we have

\[
R_{12}^{-1} R_{23}^{-1} R_{12}^{-1} R_{23}^{-1} = R_{23}^{-1} R_{12}^{-1} R_{23}^{-1},
\]

\[
R_{i2}^{-1} R_{23}^{-1} R_{i2}^{-1} R_{23}^{-1} = R_{12}^{-1} R_{23}^{-1} R_{i2}^{-1}. 
\]  
(4.15)

The right hand sides of equations (4.12) and (4.14) are thus equal. This proves the first point of the Proposition.

In order to prove the bicovariance of \( \Phi \), it suffices to prove that

\[
\Delta_L \Phi(dT_1 \otimes dT_2) = (1 \otimes \Phi) \Delta_L(dT_1 \otimes dT_2), 
\]  
(4.16)

\[
\Delta_R \Phi(dT_1 \otimes dT_2) = (\Phi \otimes 1) \Delta_R(dT_1 \otimes dT_2). 
\]  
(4.17)

This is due to the fact that \( dT_1 \otimes dT_2 \) generate \( \Omega^1 \otimes \mathcal{A} \Omega^1 \) as a left module. Using Equation (4.3) and

\[
\Delta_L(dT_1 \otimes dT_2) = T_1 T_2 \otimes dT_1 \otimes dT_2, 
\]  
(4.18)

\[
\Delta_R(dT_1 \otimes dT_2) = dT_1 \otimes dT_2 \otimes T_1 T_2, 
\]  
(4.19)

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equations (4.16) and (4.17) can be written as
\[ \alpha_{ij} R^i T_1 T_2 \otimes dT_1 \otimes dT_2 R^j = \alpha_{ij} T_1 T_2 \otimes R^i dT_1 \otimes dT_2 R^j, \quad (4.20) \]
\[ \alpha_{ij} R^i dT_1 \otimes dT_2 \otimes T_1 T_2 R^j = \alpha_{ij} R^i dT_1 \otimes dT_2 R^j \otimes T_1 T_2. \quad (4.21) \]

These equations are true due to the commutation relations (2.1). This proves point 2 of the Proposition.

Point 3 is a straightforward calculation using equations (2.18) and the Hecke condition (2.4).

Proposition 4.2 gives us a two-parameter family of bicovariant generalized permutations. We turn to examine some of their properties. First, note that the maps \( \Phi \) have the same eigenspaces even though their eigenvalues might be different. In fact, if we introduce the projectors
\[
\begin{align*}
\Pi_1(dT_1 \otimes dT_2) &= \hat{P}_q dT_1 \otimes dT_2 \hat{P}_q, \\
\Pi_2(dT_1 \otimes dT_2) &= \hat{P}_{-q^{-1}} dT_1 \otimes dT_2 \hat{P}_{-q^{-1}}, \\
\Pi_3(dT_1 \otimes dT_2) &= \hat{P}_{-q^{-1}} dT_1 \otimes dT_2 \hat{P}_q, \\
\Pi_4(dT_1 \otimes dT_2) &= \hat{P}_q dT_1 \otimes dT_2 \hat{P}_{-q^{-1}},
\end{align*}
\]

(4.22)

with
\[
\hat{P}_q = \frac{R + q^{-1}}{q + q^{-1}}, \quad \hat{P}_{-q^{-1}} = \frac{q - R}{q + q^{-1}},
\]

(4.23)

then the generalized permutation \( \Phi \) can be written as
\[ \sigma_{\lambda_1, \lambda_2} = \lambda_1 \Pi_1 + \lambda_2 \Pi_2 - \Pi_3 - \Pi_4, \]

(4.24)

and the expression for \( \Lambda \) is
\[ \Lambda = \Pi_1 + \Pi_2 - q^2 \Pi_3 - q^{-2} \Pi_4. \]

(4.25)

In the commutative limit \( \Pi_1 + \Pi_2 \) tends to the projector onto symmetric elements of \( \Omega^1 \otimes \Lambda \Omega^1 \) and \( \Pi_3 + \Pi_4 \) to the projector onto antisymmetric elements. The multiplication map \( \pi \) may be expressed in terms of these projections as
\[ \pi = (1 + q^2) \Pi_3 + (1 + q^{-2}) \Pi_4. \]

(4.26)

So \( \Omega^2 \) can be identified with the projection of \( \Omega^1 \otimes \Lambda \Omega^1 \)
\[ \Omega^2 = (\Pi_3 + \Pi_4) \Omega^1 \otimes \Lambda \Omega^1. \]

(4.27)

An imbedding \( i \) of \( \Omega^2 \) in \( \Omega^1 \otimes \Lambda \Omega^1 \) verifying \( \pi \circ i = 1_{\Omega^2} \) exists and is given by
\[ i = \frac{1}{1 + q^2} \Pi_3 + \frac{1}{1 + q^{-2}} \Pi_4. \]

(4.28)
With the aid of this imbedding we obtain the expression (3.3) for \( \sigma \):

\[
\sigma_\Lambda = 1 - 2i \circ \pi = -1 + 2 (\Pi_1 + \Pi_2). \tag{4.29}
\]

Note that this \( \sigma \) verifies \( \sigma^2 = 1 \); it is equal to \(-1\) on \( \Omega^2 \) and to \(+1\) on \((\Pi_1 + \Pi_2)(\Omega^1 \otimes_A \Omega^1)\). It corresponds to \( \lambda_1 = \lambda_2 = 1 \) in equation (4.24).

Another simple solution to equations (4.4), which in addition obeys the Yang-Baxter equation, is given by \( \sigma_{q^{-2},q^2} \),

\[
\sigma_R(dT_1 \otimes dT_2) = R^{-1}dT_1 \otimes dT_2R^{-1}. \tag{4.30}
\]

This \( \sigma \) is to be compared with the \( \sigma \) found in Ref. [7] for the quantum plane. Indeed, it could be obtained in the same way from the differential calculus (See Lemma 5.13).

We turn now to consider involutions for \(|q| = 1\). Then one can consider the involution \((T^*_j)_i = T^*_i_j \) on \( GL_q(n) \). This involution is compatible with the relations on the algebra because, for \(|q| = 1\) and \( R \) given by (2.3), one has

\[
\overline{R}^{ij}_{kl} = (R^{-1})^{ji}_{lk} \tag{4.31}
\]

where \( \overline{R} \) is the complex conjugate of \( R \). In this case, the quantum group is \( GL_q(n, \mathbb{R}) \).

**Proposition 4.3:** Let \(|q| = 1\). A generalized permutation, \( \sigma_{\lambda_1, \lambda_2} \), defines an involution iff

\[
|\lambda_1| = |\lambda_2| = 1. \tag{4.32}
\]

**Proof.** For \(|q| = 1\) relations \[4.31\] imply that

\[
\alpha \circ \Pi_i \circ \alpha = \Pi_i, \quad i = 1, 2, 3, 4. \tag{4.33}
\]

So, the condition \[3.11\] reads

\[
|\lambda_1|^2\Pi_1 + |\lambda_2|^2\Pi_2 + \Pi_3 + \Pi_4 = 1, \tag{4.34}
\]

which completes the Proof. ♦

**Remark:**

The previously defined \( \sigma_\Lambda \) and \( \sigma_R \) satisfy equations (3.11).

## 5 Linear connections on \( GL_q(n) \)

In this section we determine linear connections on the quantum group \( GL_q(n) \) and study their properties.
Proposition 5.1: Let $\sigma$ be any generalized permutation. The map $\nabla^\sigma_0$ defined by
\[
\nabla^\sigma_0 : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1 \\
\nabla^\sigma_0(\omega) = \theta \otimes \omega - \sigma(\omega \otimes \theta).
\] (5.1)
is a linear connection associated to $\sigma$.

Proof. Calculate first
\[
\nabla^\sigma_0(a\omega) = ([\theta, a] + a\theta) \otimes \omega - \sigma(a\omega \otimes \theta)
\] (5.2)
and then use the expression of the exterior derivative and the bimodule property to obtain
\[
\nabla^\sigma_0(a\omega) = da \otimes \omega + a\nabla^\sigma_0\omega.
\] (5.3)
Similarly, calculate
\[
\nabla^\sigma_0(\omega a) = \theta \otimes \omega a + \sigma(\omega \otimes ([\theta, a] - \theta a)),
\] (5.4)
\[
= \sigma(\omega \otimes da) + (\nabla^\sigma_0\omega)a.
\]
This completes the Proof. ♣

Remarks:
1-The linear connection $\nabla^\sigma_0$ can be defined on any differential calculus where the exterior derivative is a graded commutator. See [19] for another example.
2-For $\sigma = -1$ the resulting covariant derivative $\nabla^\sigma_0$ is $i \circ d$, where $i$ is the embedding of $\Omega^2$ into $\Omega^1 \otimes_A \Omega^1$, by equation (2.21).

Proposition 5.2: The extension of $\nabla^\sigma_0$ to the tensor product of $s$ copies of $\Omega^1$ is given by
\[
\nabla^\sigma_0^s \nu = \theta \otimes \nu + \sigma_s(\nu \otimes \theta), \ \forall \nu \in \Omega^1 \otimes_A \ldots \Omega^1
\] (5.5)

Proof. A direct application of Proposition 3.3. ♣

Proposition 5.3: There are no non-vanishing bimodule homomorphisms from $\Omega^1$ to $\Omega^1 \otimes_A \Omega^1$.

Proof. We will use the following Lemma proved in [10, 25, 32]

Lemma 5.4: Let $c$ be the $q$-determinant of $T$,
\[
c = \text{det}_q T = \sum_p (-q)^{l(p)}T_{p(1)}^1T_{p(2)}^2 \ldots T_{p(n)}^n,
\] (5.6)
where the sum is over all permutations on $n$ elements and $l(p)$ is the number of transpositions in the permutation $p$. Then $c$ is in the center of $A$ and verifies $\omega c = q^{-2}c\omega$ for all $\omega$ in $\Omega^1$.  

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An immediate consequence of the preceding Lemma is the

**Corollary 5.5:** \( \nu c = q^{-4}c\nu, \forall \nu \in \Omega^1 \otimes_A \Omega^1. \)

We are now in position to prove the Proposition. Let \( \phi \) be a bimodule homomorphism from \( \Omega^1 \) to \( \Omega^1 \otimes_A \Omega^1 \) and let \( \omega \in \Omega^1 \). By the homomorphism property and the Lemma 5.4, we get
\[
\phi(\omega)c = q^{-2}c\phi(\omega).
\] (5.7)

On the other hand, since \( \phi(\omega) \in \Omega^1 \otimes_A \Omega^1 \), by Corollary 5.5 we obtain
\[
\phi(\omega)c = q^{-4}c\phi(\omega).
\] (5.8)

Comparing these two equations we prove the Proposition. ♣

As a direct consequence of the preceding and of the third remark following Definition 3.2 we obtain

**Theorem 5.6:** For any generalized permutation \( \sigma \) on \( GL_q(n) \), there exists one and only one associated linear connection, given by (5.1).

We now turn to the study of some of the properties of the linear connection \( \nabla_0^\sigma \).

**Proposition 5.7:** For any generalized permutation \( \sigma \), the linear connection \( \nabla_0^\sigma \) has vanishing torsion.

*Proof.* Calculate \( \pi \circ \nabla_0^\sigma \omega = \theta \wedge \omega + \omega \wedge \theta = d\omega \),
\[
\pi \circ \nabla_0^\sigma \omega = \theta \wedge \omega + \omega \wedge \theta = d\omega,
\] (5.9)

where we have used the property (3.1). The proof of the Proposition follows from (3.24). ♣

**Proposition 5.8:** For any generalized permutation \( \sigma \), the linear connection \( \nabla_0^\sigma \) has the expression
\[
\nabla_0^\sigma \omega^a = (\Lambda - \sigma)\omega^a \otimes \theta
\] (5.10)
\[
= \left((\lambda_1 - 1)\Pi_1 + (\lambda_2 - 1)\Pi_2 + (q^2 - 1)\Pi_3 + (q^{-2} - 1)\Pi_4\right)\omega^a \otimes \theta,
\]
on the left invariant 1-forms \( \omega^a \), and
\[
\nabla_0^\sigma \eta^a = (\Lambda^{-1} - \sigma)\eta^a \otimes \theta
\] (5.11)
\[
= \left((\lambda_1 - 1)\Pi_1 + (\lambda_2 - 1)\Pi_2 + (q^{-2} - 1)\Pi_3 + (q^2 - 1)\Pi_4\right)\eta^a \otimes \theta,
\]
on the right-invariant 1-forms \( \eta^a \).

*Proof.* This is an immediate consequence of the definition of \( \Lambda \), the right invariance of \( \theta \) and Equation (4.24). ♣
Definition 5.9: A bicovariant linear connection, $\nabla$, is defined by the properties
\[ (1 \otimes \nabla)\Delta_L = \Delta_L \nabla \quad (\text{left covariance}), \quad (5.12) \]
\[ (\nabla \otimes 1)\Delta_R = \Delta_R \nabla \quad (\text{right covariance}). \quad (5.13) \]

Proposition 5.10: The linear connections associated to the generalized permutations $\sigma_{\lambda_1,\lambda_2}$ of formula (4.24) are bicovariant.

Proof. First, one sees that $\Lambda$ and $\sigma_{\lambda_1,\lambda_2}$ are bicovariant. Then, using formula (5.10) and the left invariance of $\omega^a$ one sees that formula (5.12) is true when applied to $\omega^a$. Now, the 1-forms $\omega^a$ form a basis of the left module $\Omega^1$. Then, formula (3.4) and the previous result show that the associated linear connection is left invariant.

For the right invariance, one has to consider the right invariant 1-forms $\eta^a$, which constitute a basis of $\Omega^1$ as a right module and formulas (5.11), (3.5).

The following Proposition allows one to calculate explicitly the covariant derivative associated to a generalized permutation given by equation (4.24):

Proposition 5.11: Define $\nu$, $\gamma$ and $\beta$ by
\[ \nu = q + q^{-1}, \quad \gamma = \frac{\lambda_1 - \lambda_2}{q^2 - q^2}, \quad \beta = \frac{\lambda_1 q^2 - \lambda_2 q^{-2}}{q^2 - q^2}; \quad (5.14) \]
the linear connection associated to the generalized permutation, $\sigma_{\lambda_1,\lambda_2}$, acts on left-invariant 1-forms as follows:
\[ \nabla_{0}^{\sigma_{\lambda_1,\lambda_2}}\omega^i_j = -\frac{1}{\omega^2}(1 - \gamma - \beta) \ \omega^i_k \wedge \omega^k_j - \gamma \ \omega^i_k \otimes \omega^k_j + \frac{1}{2}(1 - \gamma + \beta) (\omega^i_j \otimes \theta + \theta \otimes \omega^i_j) + \frac{\lambda_2}{2\omega^2}(1 - \gamma - \beta) (\omega^i_j \otimes \theta - \theta \otimes \omega^i_j). \quad (5.15) \]

Proof. First we note that $\sigma_{\lambda_1,\lambda_2}$ of (4.24), can be written as
\[ \sigma_{\lambda_1,\lambda_2} = (\lambda_1 + 1)\Pi_1 + (\lambda_2 + 1)\Pi_2 - 1, \quad (5.16) \]
so that $\nabla \omega^i_j$ can be expressed as
\[ \nabla \omega^i_j = \theta \otimes \omega^i_j + \omega^i_j \otimes \theta - \left[(\lambda_1 + 1)\Pi_1 + (\lambda_2 + 1)\Pi_2\right] \omega^i_j \otimes \theta. \quad (5.17) \]

It remains to calculate the term in the brackets of (5.17). We will do so by calculating it for two different values of the couple $(\lambda_1, \lambda_2)$ with the aid of the following two Lemmatae.
Lemma 5.12: The covariant derivative associated to $\sigma_\Lambda$ acts on left-invariant 1-forms as follows:

\[
\nabla_0^{\sigma_\Lambda} \omega_j = -\frac{2}{\nu^2} \omega_k \wedge \omega_j^k - \frac{\lambda^2}{\nu^2} (\theta \otimes \omega_j^i - \omega_j^i \otimes \theta).
\] (5.18)

Proof. The Proof is a straightforward calculation exploiting the fact that $\sigma_\Lambda$ can be expressed in terms of $\Lambda$ as

\[
\sigma_\Lambda = -1 + 2 \frac{(\Lambda + q^2)(\Lambda + q^{-2})}{\nu^2},
\] (5.19)

as well as the equation

\[
d\omega_j^i = (1 - \Lambda) \theta \otimes \omega_j^i = -\omega_k^i \wedge \omega_j^k,
\] (5.20)

which allows to eliminate $\Lambda(\theta \otimes \omega_j^i)$ in $\nabla_0^{\sigma_\Lambda} \omega_j^i$. ♣

Lemma 5.13: The covariant derivative associated to $\sigma_R$ is determined by

\[
\nabla_{\sigma_R} dT^i_j = 0.
\] (5.21)

Proof. Calculate the covariant derivative associated to $\sigma_R$ of the two sides of equation (2.11). ♣

The Proof of the Proposition is completed after expressing $\nabla^{\sigma_{\lambda_1, \lambda_2}} \omega_j^i$ in terms of $\nabla_{\sigma_\Lambda} \omega_j^i$ and $\nabla_{\sigma_R} \omega_j^i$ as

\[
\nabla^{\sigma_{\lambda_1, \lambda_2}} \omega_j^i = \frac{1}{2} (1 - \gamma + \beta)(\theta \otimes \omega_j^i + \omega_j^i \otimes \theta)
+ \frac{1}{2} (1 - \gamma - \beta) \nabla^{\sigma_\Lambda} \omega_j^i + \gamma \nabla^{\sigma_R} \omega_j^i.
\] (5.22)

This equation is obtained after the evaluation of $\Pi_k \omega_j^i \otimes \theta, k = 1, 2$ in terms of $\nabla^{\sigma_\Lambda} \omega_j^i$ and $\nabla^{\sigma_R} \omega_j^i$. ♣

Finally, we consider the limit of the linear connections determined above when $q \to 1$. In this limit the differential calculus tends to the usual commutative differential calculus. The 1-form $\theta$ has a singular limit but $\lambda \theta$ tends to the right and left invariant 1-form $\alpha$ on $GL(n)$. First of all, a necessary condition for the limit to be non-singular is that the generalized permutation tend to the flip operator that is $\lambda_1 \to 1$ and $\lambda_2 \to 1$. A more precise statement, giving a necessary and sufficient condition for the limit to be non-singular is the following:

Proposition 5.14: Let

\[
\mu_i = \frac{\lambda_i - 1}{\lambda}, \quad i = 1, 2,
\] (5.23)
the linear connection associated to \(\sigma_{\lambda_1,\lambda_2}\) admits a non-singular limit iff \(\mu_1\) and \(\mu_2\) have finite limits \(\mu_i|_{q=1}\) when \(q\) tends to 1. The linear connection, in the limit, is determined by

\[
\nabla \omega^i_j = -\frac{1}{2} (1 - \gamma_0) \omega^i_k \wedge \omega^k_j - \gamma_0 \omega^i_k \otimes \omega^k_j,
\]

\[\quad - \frac{\mu_0}{2} (\alpha \otimes \omega^i_j + \omega^i_j \otimes \alpha), \]

where

\[
\gamma_0 = \frac{\mu_2|_{q=1} - \mu_1|_{q=1}}{2}, \quad \mu_0 = \frac{\mu_2|_{q=1} + \mu_1|_{q=1}}{2}. \tag{5.24}
\]

**Proof.** A direct application of Proposition 5.11. ♠

**Remark:** When \(\mu_1\) and \(\mu_2\) tend to 0, which is the case of \(\sigma_{\Lambda}\), \(\gamma_0\) and \(\mu_0\) vanish and the limiting linear connection is given by

\[
\nabla \omega^i_j = -\frac{1}{2} \omega^i_k \wedge \omega^k_j. \tag{5.26}
\]

### 6 Conclusion

The main result of this paper is the existence and uniqueness, for generic \(q\), of the linear connection associated to a given generalized permutation. This connection is bicovariant and torsion-free. This is in contrast to the commutative case \((q = 1)\) where there are an infinite number of linear connections not necessarily bicovariant and torsion-free and where the generalized permutation is constrained to be the flip operator. It is also in contrast to the cases with \(q\) a root of unity where Proposition (5.3) is not in general valid. The arbitrariness in the deformed case lies merely in the generalized permutation for which we have found a two parameter family (equation 4.24). These parameters may be arbitrary functions of \(q\) and are constrained by the involution property (Proposition 4.3). The commutative limit is non-singular for a class of such functions which tend to the identity when \(q \to 1\). The commutative limit of the linear connection is a subset of right and left invariant linear connections on \(GL(n)\).

We have used the differential calculus 2.10 to obtain our results. Had we used another differential calculus with the usual commutative limit the qualitative aspects of our conclusions, in particular the uniqueness of the linear connection associated to a given generalized permutation, are expected to remain the same.

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