An improved control technique for designing of composite non-linear feedback control in constrained time-delay systems

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Abstract
In this research, a novel delay-dependent composite non-linear feedback (CNF) control would be pointed for non-linear time-delay systems under actuator saturation. For this aim, first, the CNF controller would be formulated in the constrained time-delay systems. Then, a linear matrix inequality (LMI) approach is presented to obtain the controller parameters. Hence, the control synthesis of the constrained non-linear time-delay systems is transformed into an optimisation problem subjected to some LMIs. The suggested controller is simulated in some numerical examples like a chemical reactor with separator and recycle. The computed quantitative results will demonstrate the capability of the proposed CNF control approach compared to other best methods.

1 | INTRODUCTION

In the last decade, the composite non-linear feedback (CNF) control has been introduced to the set-point tracking of the linear time-invariant (LTI) system under actuator saturation [1]. The CNF control method enhances transient performance by using an additional non-linear damping term. Thus, the transient response improvement of the constrained systems would be the main properties of the CNF control scheme [2–4]. There exist three important terms in the CNF control law. They are referred to as the constant part, the linear and non-linear feedback terms. Normally, the CNF control would be synthesised as below:

(a) The constant part is computed to meet precisely the steady-state requirement.
(b) The linear feedback gains should be chosen to serve a small damping ratio and fast response satisfying the actuator limitations. For example, the closed-loop poles of the LTI systems are located in the left-half plane away from the origin and near to the imaginary axis.
(c) The non-linear feedback parameters are selected to increase the damping ratio of the overall system. Thus the damping term would be mainly activated near the equilibrium point.

The linear part of the CNF control operates on the unsaturated region of the actuator. Nevertheless, the damping term may have entirely proceeded on the boundaries of the constraints. Although, there exist sharp corners in the ideal saturation function, but it would be analytically proved that such non-linear term cannot destabilise the closed-loop system. Consequently, the induced overshoot by the linear part may be diminished nearby the set-point [5]. It is shown that the CNF control method can outperform by more than 30% as to similar attempts like the time-optimal control [1].

The CNF control has received much attention over the last two decades. For the first time, the CNF control has been evaluated via the state feedback scheme. Then, coupling with an observer block, it is formulated in the output feedback form [6, 7]. The CNF idea has been studied in the other control problems, as well as discrete-time systems [8, 9], multi-input multi-output (MIMO) systems [10], singular control systems [11–13], adaptive gain design [14], and synchronisation issues [15, 16].

The performance indexes may be considerably improved by cooperating the CNF technique with the well-known sliding-mode control (SMC). Thus, some versions of the SMC like discrete integral SMC [8], terminal SMC [10], and integral SMC [16–18] have been used to enhance the transient response.

In recent times, various CNF technique has been implemented in some practical control problems. These shortly include the chemical reactor [19, 20], high-speed positioning [2], robot arms [21], flight control [22], overhead crane servo system [23], path-following of actuated autonomous vehicle [24],...
servo-positioning [25], hard disk drive [9, 26], voltage source inverter [27, 28], planar motor [29] and inverted pendulum with magnetic ball suspension [27].

Traditionally, time-delay and controller saturation are two main sources of instability in the control problems. The closed-loop performance would be substantially degraded even if the stability is not changed. As a result, the non-linear systems would be very complicated regarding the constrained controller input. The command-filtering control laws have been suggested to approximate the derivative of the errors for some classes of the non-linear systems [30], as well as switched systems with backlash hysteresis [31], MIMO systems with input saturation [32], and full-state constrained systems [33].

The control design and tuning task may be accomplished via numerical and heuristic methods, as well as the Monte Carlo techniques [34], and iterative learning control [35]. But the analytic and systematic ways have been a concern in this study. The linear matrix inequality (LMI)-based control methods would be a robust framework for stability analysis and controller design. Thus, the closed-loop stability is guaranteed via the feasibility checking of an LMI. Then, a stabilising controller can be determined in the control systems through the solution of an LMI [36].

In many process applications (like chemical and thermal reactors, distillation columns, polymerisation, biological, Czochralski processes, and the others), a constant time-delay may emerge in the governed differential equations. As an example, a time-delay would be induced in a continuous stirred-tank reactor (CSTR)-separator due to the recycling action [37]. Hence, a class of the time-delay control systems is examined with the actuator saturation. The constrained non-linear time-delay system would have complex behaviour regarding the stability and transient response [38, 39]. The damping effect of CNF control can be useful in time-delay systems under control limitations [19, 20].

A sufficient stability statement can be derived from the Lyapunov stability theory. Hence, some CNF controllers have been proposed for non-linear time-delay systems [6, 20, 40]. Recently, various CNF controllers have been designed with the LMI representation in the non-linear control systems [10, 20, 41]. Lately, using a memory-state feedback CNF control, the stability analysis of the non-linear time-delay control systems is reported utilising the LMI [20]. Then, a systematic algorithm is addressed to the CNF control design and tuning in the time-delay systems [19]. This study seeks to address an improved CNF control system. However, the main contributions and differences of the proposed controller over the previous CNF control methods [19, 20] may be itemised as follows:

(a) A new delay-dependent criteria will be developed in the suggested CNF control scheme. For this goal, a delay-dependent Lyapunov functional is employed for the CNF control synthesis. Thus the time-delay bounds are directly incorporated in the controller design. As a consequence, it would be less-conservative in comparison to the best existing ones [19]. This is a remarkable feature of the methodology.

(b) The actuator limitations (i.e. non-symmetric upper- and lower-bound) are explicitly taken into account in the suggested approach. So, the linear part of the CNF control is designed to provide a small damping ratio while the actuator limitations are simultaneously handled. It would be a striking point of the method.

(c) A sufficient stability condition is addressed to the CNF controller design. Then some further suggestions are provided to reduce the conservatism. The minimisation problem is solved in an offline way. Thus the time-consuming and the complexities of LMI would not be a challenging issue. In addition, the numerical results declare that the method would be an improved version of the seminal work [19].

(d) The time-varying delays with decreasing property can obviously be handled by the presented method.

(e) Above all, the proposed procedure has some free and adjustable parameters. Clearly, they can be tuned by the controller designer to improve the closed-loop performance.

Nonetheless, the CNF control design will be done by means of an LMI based minimisation problem in non-linear time-delay systems under the control signal saturation. Thus, an optimisation problem with some LMI constraints will be found in the CNF controller design. Finally, two numerical examples, as well as a chemical reactor, are given to evaluate the superiority of the outlined control method in comparison to the other approaches. Moreover, the finding would discover the advantages of suggested CNF in comparison to the existing ones.

The rest of the study is organised as follows: Some useful lemmas are first introduced in Section 2. Then, in Section 3, the CNF control problem is formulated in the time-delay systems. The main results of the study are presented in Section 4. Finally, two numerical control examples are studied in Section 5. The concluding remarks and future works are drawn in the last section.

2 | MATHEMATICAL PRELIMINARIES

The operator $\| \cdot \|$ describes the 2-norm of a vector, and $I_n$ is an identity matrix. The star symbol $*$ denotes the symmetric property in a given square matrix. The set $\mathbb{R}$ is all real numbers, $\mathbb{R}^n$ is the set of all real vectors that contain $n$ elements, and $\mathbb{R}^{a \times n}$ is the set of all $a \times n$ real matrices. The following mathematical lemmas are useful to verify the main theorem of the study:

Lemma 1. Suppose $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ to be some well-defined time-domain functions. For any positive definite matrix $\mathbf{Z} \in \mathbb{R}^{a \times a}$ and a constant $\rho > -1$, the following inequality would hold [42]:

$$-2x^T(t) \int_{a(t)}^{b(t)} y(\tau) d\tau \leq \int_{a(t)}^{b(t)} y^T(\tau) \mathbf{Z} y(\tau) d\tau + 2\rho \int_{a(t)}^{b(t)} y^T(\tau) d\tau x(t)$$

$$+ (\rho + 1)^2 (b(t) - a(t))^2 x^T(t) \mathbf{Z}^{-1} x(t)$$

(1)
Lemma 2. Suppose \( x(t) \in \mathbb{R}^n \) and \( y(t) \in \mathbb{R}^n \) to be some well-defined functions in time-domain. Then the following inequality holds [43]:

\[
x^T(t)Hj(t) + y^T(t)H^Tj(t) \leq \rho y^T(t)Hj(t) + \frac{1}{\rho} x^T(t)jc(t)
\]

where \( H \in \mathbb{R}^{n \times n} \) is a given matrix, and \( \rho \) is a positive constant.

In the next section, the CNF control design would be formulated in the constrained time-delay systems.

3 | PROBLEM SETUP

Consider a non-linear time-delay system as the following form:

\[
\begin{align*}
\frac{d}{dt} x(t) &= A_0 x(t) + \sum_{i=1}^{N} A_i x(t-d_i) + B \hat{u}(t) \\
&+ f(x(t), x(t-d_1), ..., x(t-d_N)) \\
y(t) &= Cx(t), t \geq t_0
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R} \) is the constrained control input, and \( y(t) \in \mathbb{R} \) is the measured output of the non-linear system (Equation 3). The constant terms \( d_1, d_2, ..., d_N \) denote some unknown but bounded time-delays. Although the time-delays are assumed to be uncertain, there has been an upper-bound like \( \tilde{d} \) (i.e. \( d_i \leq \tilde{d}, i = 1, 2, ..., N \)). Moreover, \( A_0, A_1, A_2, ..., A_N \) are some given constant matrices. The time-delay control system (Equation 3) starts from \( x(t) = x_0 \), \( t_0 - \tilde{d} \leq t \leq t_0 \).

The saturation function \( \text{sat}(u) \) is defined as follows:

\[
\text{sat}(u) = \begin{cases} 
\hat{u}_{\text{min}} & u < \hat{u}_{\text{min}} \\
n & \hat{u}_{\text{min}} \leq u \leq \hat{u}_{\text{max}} \\
\hat{u}_{\text{max}} & u > \hat{u}_{\text{max}} 
\end{cases}
\]

Assumption 1. The following inequality holds for the non-linear function \( f(\eta(t)) \):

\[
\| f(\eta_1) - f(\eta_2) \| \leq \| M(\eta_1 - \eta_2) \|
\]

where \( M = \text{diag}\{M_0, M_1, M_2, ..., M_N\} \) and \( M_i \in \mathbb{R}^{n \times n}, i = 0, 1, 2, ..., N \) are some known matrices.

Generally, the Lipchitz assumption leads to a conservative condition. But the conservativeness may be reduced by appropriate selection of the matrices \( M_i \in \mathbb{R}^{n \times n}, i = 0, 1, 2, ..., N \).

In the non-linear time-delay system (Equation 3), it is interesting to determine an input signal \( u(t) \) so that the output signal \( y(t) \) tracks a constant reference \( r \) in the presence of the actuator saturation and some uncertain time-delays. The operating points (i.e. \( \bar{x} \) and \( \bar{u} \)) of the time-delay system (Equation 3) can be found as

\[
\begin{align*}
\begin{pmatrix} A_0 + \sum_{i=1}^{N} A_i \end{pmatrix} \bar{x} + B \bar{u} + f(\bar{\eta}) &= 0 \\
C \bar{x} &= r
\end{pmatrix}
\]

\[
\begin{align*}
u_L(t) &= K_0 x(t) + K_d (t - d_m) + u \\
u_N(t) &= \phi(x) B^T P_C (x(t) - \bar{x})
\end{align*}
\]

The non-linear term \( \phi(x) \) would be the damping property of the CNF controller. Usually, the non-linear function \( \phi(x) \) may be preferred in exponential form. But it would be selected in many different approaches. The simplest damping term may be chosen as the following scalar function:

\[
\phi(x) = -\beta e^{-\alpha_0|x|-\gamma/p}
\]

In Equation (9), the positive constants \( \beta, \alpha_1, \) and \( \alpha_0 \) may be treated as some tuning parameters. However, the scaling variable \( \alpha_0 \) would change with different tracking values produced by the set-point \( y_p \) as

\[
\alpha_0 = \begin{cases} 
\frac{1}{|y(0) - y_p|} & y(0) \neq y_p \\
1 & y(0) = y_p
\end{cases}
\]
It is clear that the non-linear term \( u_N(x) \) has the following properties:

(a) It tends to be zero at infinity (i.e. \( u_N(x) \to 0 \) as \( |y - y_{ref}| \to +\infty \)).

(b) It is zero around the equilibrium point (i.e. \( u_N(x) \to 0 \) as \( |y - y_{ref}| \to 0 \)).

By defining the deviated vector \( \xi(t) \overset{\text{def}}{=} x(t) - \bar{x} \) and the error signal \( e(t) = r - y(t) \), Equation (3) may be rewritten as

\[
\frac{d}{dt} \xi(t) = A_0 \xi(t) + \sum_{i=1}^{N} A_i \xi(t - d_i) + B \xi(t) + Bsat(u(t)) - B\bar{u} + F(\eta(t)) \tag{11}
\]

where \( F(\eta) \overset{\text{def}}{=} f(\eta) - f(\bar{\eta}) \). The non-linear function \( F(\eta) \) would also satisfy the following inequality:

\[
\|F(\eta(t))\| \leq \|M_0 \xi(t)\| + \|M_1 \xi(t) - d_1\| + \cdots + \|M_N \xi(t) - d_N\| \tag{12}
\]

As a result, by taking \( \bar{u} = (K_0 + K_d) \bar{x} + v \), Equation (5) can be modified as follows:

\[
\left\{ \begin{aligned}
A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \xi(t - d_i) + Bsat(u(t)) \\
C \bar{x} = r
\end{aligned} \right. \tag{13}
\]

Equation (13) has a unique solution, namely, \( \bar{x} \) and \( \bar{v} \), satisfying \( u_{min} \leq \bar{u} \leq u_{max} \). Hence, the equilibrium points would be directly determined from Equation (13).

Thus, the CNF control may be written as follows:

\[
u(t) = K_0 \xi(t) + K_d \xi(t - d_w) + \bar{v} + \phi(x(t)) B^T R \bar{\xi}(t) \tag{14}
\]

Let us define \( \omega(t) \) as follows:

\[
\omega (t) \overset{\text{def}}{=} sat \left( K_0 \xi(t) + K_d \xi(t - d_w) + u_N(t) + \bar{v} \right) - \left( K_0 \xi(t) + K_d \xi(t - d_w) + \bar{v} \right)
\]

Then, the closed-loop system would be found as

\[
\frac{d}{dt} \bar{x}(t) = (A_0 + BK_0) \bar{x}(t) + BK_d \bar{x}(t - d_w) + \sum_{i=1}^{N} A_i \bar{x}(t - d_i) + B \omega(t) + F(\eta(t)), t \geq t_0 \tag{15}
\]

An optimal CNF control can be designed by taking the following quadratic cost function:

\[
f_0 = \int_{t_0}^{+\infty} \left( (x(t) - \bar{x})^T Q (x(t) - \bar{x}) \right. + \left. (u_L(t) - \bar{u})^T R (u_L(t) - \bar{u}) \right) dt \tag{16}
\]

In the CNF control design, the damping term \( u_N(t) \) has been chosen to increase the damping ratio of the closed-loop system around the reference point. Hence, the non-linear term is ignored in the cost function in Equation (16). The cost function in Equation (16) can be compactly written as follows:

\[
f_0 = \int_{t_0}^{+\infty} \left( (\xi^T(t) Q \xi(t) + \xi^T(t) K_0^T R K_0 \xi(t) \right)
+ 2 \xi^T(t) K_0^T R K_d \xi(t - d_w) + \xi^T(t - d_w) K_d^T R K_d \xi(t - d_w) dt \tag{17}
\]

The controller parameters would be found by minimising the cost function in Equation (17). Hence, an LMI-based representation is derived for the CNF control design in the next section.

4 MAIN RESULTS

The CNF control would be synthesised in the non-linear time-delay system in Equation (3). Hence, an LMI minimisation problem is developed to determine the controller parameters. Such a problem is solved in an offline way. Then, the control law is found via the solution of the optimisation problem subjected to some LMIs.

**Theorem 1.** Consider the constrained time-delay control system in Equation (3) with assumption 1. If there exist some \( y \) by \( y \) positive definite matrices \( X_0, X_d, Y_d, X_i, Y_i, i = 1, 2, \ldots, N \), the vectors \( G_0, G_d \) and the positive constants \( \gamma, \rho \), such that the following minimisation problem has a feasible solution:

\[
\min \gamma \\
\text{subject to}
\begin{bmatrix}
\xi_1 X_0 & X_0 A_1^T & \ldots & X_0 A_N^T & G_d^T B^T \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \frac{1}{2} X_1 & \ldots & 0 & 0 \\
\frac{1}{2} & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 1 & \frac{1}{d_w} X_d \\
\end{bmatrix} \geq 0 \tag{18}
\]

\[
\begin{bmatrix}
\frac{1}{1 + \epsilon_{x2}} X_0 \\
\rho \omega X_0 \\
* & (1 - \rho) X_0 \\
\end{bmatrix} \geq 0 \tag{19}
\]

\[
\begin{bmatrix}
\frac{1}{1 + \epsilon_{x2}} X_0 \\
\xi(t_0) \\
\end{bmatrix} \geq 0 \tag{20}
\]
where \( \Psi_{11} = (A_0 + \sum_{i=1}^{N} A_i) X_0 + B G_0 + (1 + \rho_0) B d + X_0 (A_0^T + \sum_{i=1}^{N} A_i^T) G_0^T B^T + (1 + \rho_0) G_j B^T + \sum_{i=1}^{N} \rho_i A_i X_0 + \sum_{i=1}^{N} \rho_i X_0 A_i^T + d_{m} (\rho_i + 1)^2 X_d + \sum_{i=1}^{N} (\rho_i + 1)^2 X_d + Y_d + \sum_{i=1}^{N} Y_d + \sum_{i=1}^{N} \nu_i + \rho_{d} I_{d} \) then, the asymptotic stability of the closed-loop system would be guaranteed by the CNF control law in Equation (6) with \( K_0 = C_d, \ X_d = C_d^{-1} \) and \( P_0 = \gamma X_0^{-1} \). Furthermore, the constant \( \gamma \) is the upper-bound of the cost function in Equation (16).

**Proof:** By substituting the following equations:

\[
\begin{align*}
\xi (t - d_m) &= \xi (t) - \int_{t-d_m}^{t} \dot{\xi} (\tau) d\tau \\
\xi (t - d_i) &= \xi (t) - \int_{t-d_i}^{t} \dot{\xi} (\tau) d\tau, \quad i = 1, 2, \ldots, N 
\end{align*}
\]

(22)

the closed-loop system in Equation (15) is modified as follows:

\[
\begin{align*}
\frac{d}{dt} \xi (t) &= \left( A_0 + B K_0 + B K_d + \sum_{i=1}^{N} A_i \right) \xi (t) \\
&- B K_d \int_{t-d_m}^{t} \dot{\xi} (\tau) d\tau - \sum_{i=1}^{N} A_i \int_{t-d_i}^{t} \dot{\xi} (\tau) d\tau \\
&+ B_{\omega} (t) + F (\eta (t)), \quad t \geq t_0 
\end{align*}
\]

(23)

The following Lyapunov–Krasovskii functional is taken:

\[
V (t) = \xi^T (t) P_0 \xi (t) + \sum_{i=1}^{N} \int_{t-d_i}^{t} \int_{0}^{+\infty} \xi^T (\sigma) A_i^T P_i A_i \xi (\sigma) d\sigma d\tau \\
+ \sum_{i=1}^{N} \int_{t-d_i}^{t} \int_{0}^{+\infty} \xi^T (\tau) S_i \xi (\tau) d\tau, \quad t \geq t_0 
\]

(24)

where \( P_0, P_i, S_i, \) \( S_i, \) and \( P_i, i = 1, 2, \ldots, N \) are some symmetric positive definite matrices.

The time-derivative of the Lyapunov function in Equation (24) is found as

\[
\frac{d}{dt} V (t) = \dot{\xi}^T (t) P_0 \dot{\xi} (t) + \xi^T (t) P_0 \dot{\xi} (t) \\
- \sum_{i=1}^{N} \int_{t-d_i}^{t} \xi^T (\tau) A_i^T P_i A_i \xi (\tau) d\tau + \sum_{i=1}^{N} \xi^T (t) S_i \xi (t) \\
- \sum_{i=1}^{N} \xi^T (t-d_i) S_i \xi (t-d_i)
\]
\[- \int_{t-d_n}^{t} \xi^T(\tau) K_d^T B^T P_d BK_d \dot{\xi}(\tau) \, d\tau + \xi^T(\tau) S_d \xi(\tau) \]
\[- \xi^T(t-d_n) S_d \xi(t-d_n) \] (25)

Then, along the closed-loop system in Equation (23), Equation (25) is obtained as follows:

\[
\frac{d}{dt} V(t) = \xi^T(t) \left( A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \right)^T P_0 \xi(t)
\]
\[+ P_0 \left( A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \right) + S_d + \sum_{i=1}^{N} S_i \]
\[= \xi^T(t) P_0 F(\eta(t)) - \sum_{i=1}^{N} \int_{t-d_i}^{t} \xi^T(\tau) A_i^T P_d A_i \dot{\xi}(\tau) \, d\tau \]
\[- \int_{t-d_n}^{t} \xi^T(\tau) K_d^T B^T P_d BK_d \dot{\xi}(\tau) \, d\tau \]
\[- 2\xi^T(t) P_d BK_d \int_{t-d_n}^{t} \dot{\xi}(\tau) \, d\tau - 2\xi^T(t) P_0 \sum_{i=1}^{N} A_i \]
\[\times \int_{t-d_i}^{t} \dot{\xi}(\tau) \, d\tau + 2\xi^T(t) P_0 B\omega(t) \] (26)

Applying Lemma 1 with \(\dot{x}(t) = P_0 \dot{\xi}(t)\) and \(\gamma(t) = BK_d \dot{\xi}(t)\), we have

\[-2\xi^T(t) P_0 BK_d \int_{t-d_n}^{t} \dot{\xi}(\tau) \, d\tau \leq d_n (\rho_0 + 1)^2 \xi^T(t) P_0 P_0^{-1} P_0 \xi(t) \]
\[+ 2\rho_0 \xi^T(t) P_0 BK_d \int_{t-d_n}^{t} \dot{\xi}(\tau) \, d\tau \]
\[\leq \int_{t-d_n}^{t} \xi^T(\tau) K_d^T B^T P_d BK_d \dot{\xi}(\tau) \, d\tau, \rho_0 > 0 \] (27)

In a similar way, applying Lemma 1 with \(\dot{x}(t) = R_0 \dot{\xi}(t)\) and \(\gamma(t) = A_i \dot{\xi}(t), i = 1, 2, \ldots, N\). Then,

\[-2\xi^T(t) R_0 \sum_{i=1}^{N} A_i \int_{t-d_i}^{t} \dot{\xi}(\tau) \, d\tau \leq (\rho_1 + 1)^2 d_i \xi^T(t) R_0 R_0^{-1} R_0 \xi(t) \]
\[+ \rho_1 2\xi^T(t) R_0 A_i \int_{t-d_i}^{t} \dot{\xi}(\tau) \, d\tau \]
\[+ \int_{t-d_i}^{t} \xi^T(\tau) A_i^T P_d A_i \dot{\xi}(\tau) \, d\tau, \rho_1 > 0 \] (28)

Using Lemma 2, we have

\[F^T(\eta(t)) R_0 \xi(t) + \xi^T(t) P_0 F(\eta(t)) \leq \bar{\rho}_F \xi^T(t) P_0^2 \xi(t) \]
\[+ \frac{1}{\bar{\rho}_F} F^T(\eta(t)) F(\eta(t)), \bar{\rho}_F > 0 \] (29)

The non-linear function \(F(\eta(t))\) is bounded as

\[\|F(\eta(t))\| \leq \|M_0 \xi(t)\| + \sum_{i=1}^{N} \|M_i \xi(t-d_i)\| \] (30)

Then

\[\|F(\eta(t))\|^2 \leq (N + 1) \|M_0 \xi(t)\|^2 \]
\[+ (N + 1) \sum_{i=1}^{N} \|M_i \xi(t-d_i)\|^2 \] (31)

Thus, the inequality in Equation (29) may be rewritten as

\[F^T(\eta(t)) R_0 \xi(t) + \xi^T(t) P_0 F(\eta(t)) \leq \bar{\rho}_F \xi^T(t) P_0^2 \xi(t) \]
\[+ \frac{1}{\bar{\rho}_F} (N + 1) \|M_0 \xi(t)\|^2 \]
\[+ \frac{1}{\bar{\rho}_F} (N + 1) \sum_{i=1}^{N} \|M_i \xi(t-d_i)\|^2, \bar{\rho}_F > 0 \] (32)

Therefore, the condition in Equation (26) would have an upper-bound as follows:

\[
\frac{d}{dt} V(t) \leq \xi^T(t) \left( A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \right)^T P_0 \xi(t)
\]
\[+ P_0 \left( A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \right) + S_d \]
\[+ \sum_{i=1}^{N} S_i + \bar{\rho}_F P_0^2 + \frac{1}{\bar{\rho}_F} (N + 1) M_0^T M_0 \]
\[
\xi(t) - \sum_{i=1}^{N} \xi^T(t - d_i) S_i(t - d_i) - \xi^T(t - d_m) S_d(t - d_m) \\
+ \frac{1}{\rho_F} (N + 1) \sum_{i=1}^{N} \xi^T(t - d_i) M_i^T M_i \xi(t - d_i) \\
+ d_m (\rho_0 + 1)^2 \xi^T(t) P_i P_i^T P_i \xi(t) \\
+ 2 \rho_0 \xi^T(t) P_i BK_d \int_{t - d_i}^{t} \xi(\tau) d\tau \\
+ \sum_{i=1}^{N} d_i (\rho_i + 1)^2 \xi^T(t) P_i P_i^T P_i \xi(t) \\
+ 2 \rho_0 \sum_{i=1}^{N} \xi^T(t) P_i A_i \int_{t - d_i}^{t} \xi(\tau) d\tau + 2 \xi^T(t) P_i B \omega(t)
\]  

(33)

It is interesting to satisfy the following inequality:

\[
\frac{d}{dt} V(t) \leq -\xi^T(t) \Omega \xi(t) - \left[ \xi^T(t) \xi(t - d_m) \right] [K_0 \ K_0] \xi(t) \\
\times R \left[ K_0 \ K_0 \right] \left[ \xi(t - d_m) \right]
\]

(34)

In the subsequent equations, it will be shown that the term \( \xi^T(t) P_i B \omega(t) \) is a negative value. As it is assumed, the linear term of the CNF controller would be designed such that it operates on the linear region (i.e., \( n_{min} \leq n(t) \leq n_{max} \)). Then \( n_{min} \leq u(t) - u_N(t) \leq n_{max} \). Recalling \( \omega(t) = u_N(t) - n(t) + \text{sat}(n(t)) \) and \( u_N(t) = \psi(\xi) B^T P_i \xi(t) \), three possible cases can be imagined as the following:

**Case I:** It is trivial that \( \omega(t) = u_N(t) \) in the unsaturated region \( n_{min} \leq n(t) \leq n_{max} \). Then,

\[
\xi^T(t) P_i B \omega(t) = \xi^T(t) P_i B \psi(\xi) B^T P_i \xi(t) \leq 0
\]

(35)

**Case II:** If \( n(t) > n_{max} \) then \( \omega(t) = n_{max} - n(t) + u_N(t) \geq 0 \). Hence, \( u_N(t) > n(t) - n_{max} > 0 \). It leads to \( B^T P_i \xi(t) < 0 \). Therefore,

\[
\xi^T(t) P_i B \omega(t) = \left( B^T P_i \xi(t) \right)^T \omega(t) < 0
\]

(36)

**Case III:** Similarly, if \( n(t) < n_{min} \) then \( \omega(t) = n_{min} - n(t) + u_N(t) < 0 \). Thus, \( u_N(t) < n(t) - n_{min} < 0 \). It leads to \( B^T P_i \xi(t) > 0 \). Therefore,

\[
\xi^T(t) P_i B \omega(t) = \left( B^T P_i \xi(t) \right)^T \omega(t) < 0
\]

(37)

Thus, the term \( \xi^T(t) P_i B \omega(t) \) is always negative in the proposed procedure.

Accordingly, the inequality in Equation (34) could be rewritten as follows:

\[
\xi^T(t) \left( A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \right) P_i \\
+ \rho_0 \left( A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i \right) + S_d \\
+ \sum_{i=1}^{N} S_i + \rho_F P_i^2 + \frac{1}{\rho_F} (N + 1) M_i^T M_i
\]

Thus, the term \( \xi^T(t) P_i B \omega(t) \) is always negative in the proposed procedure.
\[
\xi(t) - \sum_{i=1}^{N} \xi^T(t-d_i) S_d \xi(t-d) - \xi^T(t-d_m) S_d \xi(t-d_m) \\
+ \frac{1}{\bar{\rho}_F} (N+1) \sum_{i=1}^{N} \xi^T(t-d_i) M_i^T M_i \xi(t-d_i) \\
- 2\rho_0 \xi^T(t) P_0 B K_d \xi(t-d_m) - 2 \sum_{i=1}^{N} \rho_i \xi^T(t) P_i A_i \xi(t-d_i) \\
+ 2 \xi^T(t) P_0 B \omega(t) + \xi^T(t) Q_d \xi(t) + \xi^T(t) K_d^T R K_d \xi(t) \\
+ 2 \xi^T(t) K_d^T R K_d \xi(t-d_m) \\
+ \xi^T(t-d_m) K_d^T R K_d \xi(t-d_m) \leq 0
\]  
(39)

Then,
\[
\Theta(t) = [\xi^T(t) \xi^T(t-d_m) \xi^T(t-d)]^T \\
d \frac{d}{dt} V(t) \leq \Theta^T(t) \hat{\Psi}(t) + 2 \xi^T(t) P_0 B \omega(t)
\]  
(40)

and
\[
\Psi = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \cdots & \Psi_{1N} \\
* & \Psi_{22} & 0 & 0 & \cdots & 0 \\
* & * & \Psi_{33} & 0 & \cdots & 0 \\
* & * & * & \Psi_{44} & \cdots & 0 \\
* & * & * & * & \cdots & \Psi_{NN}
\end{bmatrix} \leq 0
\]  
(41)

where
\[
\Psi_{11} = \left(A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i\right)^T P_0 \\
+ P_0 \left(A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i\right) + Q + K_d^T R K_d \\
+ \rho_0 P_0 B K_d + \rho_0 K_d^T B^T P_0 + \sum_{i=1}^{N} \rho_i P_i A_i + \sum_{i=1}^{N} \rho_i A_i^T P_0 \\
+ \bar{\alpha}(\rho_0 + 1)^2 P_0 P_0^{-1} P_0 + \tilde{d} \sum_{i=1}^{N} (\rho_i + 1)^2 P_i P_i^{-1} P_0 + S_d \\
+ \sum_{i=1}^{N} S_i + \bar{\beta}_F P_0^2 + \frac{1}{\hat{\rho}_F} (N+1) M_i^T M_i
\]
\[
\Psi_{12} = \rho_0 P_0 B K_d + K_d^T R K_d, \quad \Psi_{13} = \rho_0 P_0 A_1, \quad \Psi_{14} = \rho_2 P_0 A_2, \\
\Psi_{1N} = \rho_N P_0 A_N, \quad \Psi_{22} = -S_d + K_d^T R K_d \\
\Psi_{1N} = \rho_N P_0 A_N, \quad \Psi_{22} = -S_d + K_d^T R K_d
\]

\[
\Psi_{33} = \frac{1}{\bar{\rho}_F} (N+1) M_i^T M_i - S_1, \quad \Psi_{44} = \frac{1}{\bar{\rho}_F} (N+1) M_i^T M_i - S_2, \\
\Psi_{NN} = \frac{1}{\bar{\rho}_F} (N+1) M_i^T M_i - S_N
\]

Now pre- and post-multiply the matrix \(\hat{\Psi}\) by the symmetric matrix \(\Lambda\) as follows:
\[
\Lambda = \gamma^2 \text{ diag} \left( [P_0^{-1}, P_0^{-1}, P_0^{-1}, \ldots, P_0^{-1}] \right)
\]

Then, \(\hat{\Psi} = \Lambda \hat{\Psi} \Lambda\) is obtained as
\[
\hat{\Psi} = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \cdots & \Psi_{1N} \\
* & \Psi_{22} & 0 & 0 & \cdots & 0 \\
* & * & \Psi_{33} & 0 & \cdots & 0 \\
* & * & * & \Psi_{44} & \cdots & 0 \\
* & * & * & * & \cdots & \Psi_{NN}
\end{bmatrix} \leq 0
\]  
(42)

where
\[
\Psi_{11} = \gamma P_0^{-1} \left(A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i\right)^T \\
+ \gamma \left(A_0 + BK_0 + BK_d + \sum_{i=1}^{N} A_i\right) P_0^{-1} + \gamma P_0^{-1} Q P_0^{-1} \\
+ \gamma P_0^{-1} K_d^T R K_d P_0^{-1} + \gamma \rho_0 B K_d P_0^{-1} + \gamma \rho_0 P_0^{-1} K_d^T B^T \\
+ \gamma \sum_{i=1}^{N} \rho_i A_i P_0^{-1} + \gamma \sum_{i=1}^{N} \rho_i P_0^{-1} A_i + \gamma \bar{\alpha}(\rho_0 + 1)^2 P_0^{-1} \\
+ \gamma \tilde{d} \sum_{i=1}^{N} (\rho_i + 1)^2 P_i^{-1} + \gamma P_0^{-1} S_d P_0^{-1} + \gamma \sum_{i=1}^{N} P_0^{-1} S_i P_0^{-1} \\
+ \gamma \bar{\rho}_F \bar{L}_F + \gamma \frac{1}{\hat{\rho}_F} (N+1) P_0^{-1} M_i^T \sum_{i=1}^{N} M_i P_0^{-1}
\]
\[
\Psi_{12} = \rho_0 \gamma B K_d P_0^{-1} + \gamma P_0^{-1} K_d^T R K_d P_0^{-1}, \quad \Psi_{13} = \gamma P_0^{-1} A_1 P_0^{-1}, \\
\Psi_{1N} = \gamma P_0^{-1} A_N P_0^{-1}, \quad \Psi_{22} = -\gamma P_0^{-1} S_d P_0^{-1} + \gamma P_0^{-1} K_d^T R K_d P_0^{-1}, \quad \Psi_{33} = \gamma \frac{1}{\hat{\rho}_F} (N+1) P_0^{-1} M_i^T M_i P_0^{-1} - \gamma P_0^{-1} S_i P_0^{-1}
\]

Now, define some variables as follows:
\[
G_0 = \gamma K_d P_0^{-1}, \quad G_d = \gamma K_d P_0^{-1}, \quad X_0 = \gamma P_0^{-1}, \quad X_d = \gamma P_0^{-1}, \\
X_0 = \gamma P_0^{-1}, \quad X_d = \gamma P_0^{-1}, \quad Y_d = \gamma P_0^{-1}, \quad \rho_F = \gamma \bar{\rho}_F
\]
Then,

$$\begin{pmatrix}
\Psi_{11} & \rho_0 B G_d + \gamma^{-1} G_d^T R G_d & \rho_1 A_1 X_0 & \ldots & \rho_N A_N X_0 \\
* & -Y_d & 0 & \ldots & 0 \\
* & * & -Y_1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & * & -Y_N \\
\end{pmatrix} \leq 0 \quad (43)
$$

where

$$\Psi_{11} = (A_0 + \sum_{i=1}^{N} A_i) X_0 + B G_d + B G_d + X_0 (A_0^T + \sum_{i=1}^{N} A_i^T) + G_d^T B^T + G_d^T B^T + \gamma^{-1} X_0 Q X_0 + \gamma^{-1} G_d^T R G_d + \rho_0 B G_d + \rho_0 G_d^T B^T + \sum_{i=1}^{N} \rho_i A_i X_0 + \sum_{i=1}^{N} \rho_i X_0 A_i^T + \bar{d}_m (\rho_0 + 1)^2 X_d + \tilde{d} \sum_{i=1}^{N} (\rho_i + 1)^2 X_i + Y_d + \sum_{i=1}^{N} Y_i + \rho_F I_a + \frac{1}{\rho_F} (N + 1) X_0 M_N^T M_0 X_0.$$

The inequality in Equation (43) can be decomposed as the following:

$$\Psi = \begin{pmatrix}
\Psi_{11} & \rho_0 B G_d & \rho_1 A_1 X_0 & \ldots & \rho_N A_N X_0 \\
* & -Y_d & 0 & \ldots & 0 \\
* & * & -Y_1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & * & -Y_N \\
\end{pmatrix} + \begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & X_0 M_N^T & \ldots & 0 \\
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} \begin{pmatrix}
X_0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} + \begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_0 M_N^T & \ldots & X_0 M_N^T & \ldots & 0 \\
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} \begin{pmatrix}
X_0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} \leq 0 \quad (44)
$$

where

$$\Psi_{11} = (A_0 + \sum_{i=1}^{N} A_i) X_0 + B G_d + B G_d + X_0 (A_0^T + \sum_{i=1}^{N} A_i^T) + G_d^T B^T + G_d^T B^T + \rho_0 B G_d + \rho_0 G_d^T B^T + \sum_{i=1}^{N} \rho_i A_i X_0 + \sum_{i=1}^{N} \rho_i X_0 A_i^T + \bar{d}_m (\rho_0 + 1)^2 X_d + \tilde{d} \sum_{i=1}^{N} (\rho_i + 1)^2 X_i + Y_d + \sum_{i=1}^{N} Y_i + \rho_F I_a + \frac{1}{\rho_F} (N + 1) X_0 M_N^T M_0 X_0.$$

Applying the Schur’s complement lemma, the inequality in Equation (44) would be equivalent to the condition in Equation (21).
Applying the Schur’s complement lemma, the inequality in Equation (18) is equivalent to the following condition:
\[
\sum_{i=1}^{N} \bar{d} X_{0}^{-1} A_{i} X_{0}^{-1} A_{i} + \bar{d}_{u} G_{d}^{T} B_{d} X_{0}^{-1} B_{G_{d}} \leq \varepsilon_{1} X_{0}^{-1} \tag{49}
\]

It can be modified as
\[
\sum_{i=1}^{N} A_{i}^{T} X_{0}^{-1} A_{i} + \bar{d}_{u} X_{0}^{-1} G_{d}^{T} B_{d} X_{0}^{-1} B_{G_{d}} X_{0}^{-1} \leq \varepsilon_{1} X_{0}^{-1} \tag{50}
\]

Then the inequality in Equation (50) would be bounded as follows:
\[
V'(t_{0}) \leq \gamma \left(\varepsilon_{1} + \varepsilon_{2} \right) X_{0}^{-1} \xi(t_{0}) + \varepsilon_{1} \int_{t_{0}}^{+\infty} \xi^{T}(t) X_{0}^{-1} \xi(t) dt \tag{51}
\]

The following condition would hold when the closed-loop system is asymptotically stable:
\[
\int_{t_{0}}^{+\infty} \xi^{T}(t) X_{0}^{-1} \xi(t) dt = \xi^{T}(t_{0}) Y^{-1} \xi(t_{0}) \tag{52}
\]

for some matrix \( Y \). Therefore, there exists a positive constant \( \varepsilon_{2} \) such that the following inequality holds:
\[
\int_{t_{0}}^{+\infty} \xi^{T}(t) X_{0}^{-1} \xi(t) dt \leq \varepsilon_{2} \xi^{T}(t_{0}) X_{0}^{-1} \xi(t_{0}) \tag{53}
\]

Consequently, \( V'(t_{0}) \) has an upper-bound as follows:
\[
V'(t_{0}) \leq \gamma \left(1 + \varepsilon_{1} \varepsilon_{2} \right) \xi^{T}(t_{0}) X_{0}^{-1} \xi(t_{0}) \tag{54}
\]

Applying the Schur's complement lemma, the inequality in Equation (20) is equivalent to the following condition:
\[
(1 + \varepsilon_{1} \varepsilon_{2}) \xi^{T}(t_{0}) X_{0}^{-1} \xi(t_{0}) \leq 1 \tag{55}
\]

Then, \( V'(t) \leq V'(t_{0}) \leq \gamma \).

The linear term of the CNF controller would be designed to operate in the non-saturated region. Thus,
\[
\| u_{L}(t) - \bar{u} \|^2 = [\xi^{T}(t), \xi^{T}(t - d_{u})] \begin{bmatrix} K_{0} & K_{d} \end{bmatrix}^{T} \begin{bmatrix} \xi(t) \\ \xi(t - d_{u}) \end{bmatrix} \leq \gamma^{2} \tag{56}
\]

Substituting the following identity:
\[
\begin{bmatrix} K_{0} & K_{d} \end{bmatrix}^{T} \begin{bmatrix} K_{0} & K_{d} \end{bmatrix} = \begin{bmatrix} X_{0}^{-1} & 0 \\ 0 & X_{0}^{-1} \end{bmatrix} \begin{bmatrix} G_{0} & G_{d} \end{bmatrix}^{T} \times \begin{bmatrix} G_{0} & G_{d} \end{bmatrix} \begin{bmatrix} X_{0}^{-1} & 0 \\ 0 & X_{0}^{-1} \end{bmatrix} \tag{57}
\]

The condition in Equation (56) could be verified if the following inequality holds:
\[
\begin{bmatrix} G_{0} & G_{d} \end{bmatrix}^{T} \begin{bmatrix} G_{0} & G_{d} \end{bmatrix} \leq \gamma^{2} \begin{bmatrix} \rho_{u} X_{0} & 0 \\ 0 & (1 - \rho_{u}) X_{0} \end{bmatrix} \tag{58}
\]

where \( 0 < \rho_{u} < 1 \). Then,
\[
\| u_{L}(t) - \bar{u} \|^2 \leq \gamma^{2} \rho_{u} \xi^{T}(t) X_{0}^{-1} \xi(t) + \gamma^{2} (1 - \rho_{u}) \xi^{T}(t - d_{u}) \times X_{0}^{-1} \xi(t - d_{u}) \leq \gamma^{2} \tag{59}
\]

Therefore, the inequality \( u_{min} \leq u_{L}(t) \leq u_{max} \) could be concluded if the condition \( \| u_{L}(t) - \bar{u} \| \leq \gamma_{u} \) holds with \( \gamma_{u} = \min \left\{ \bar{u} - u_{min}, u_{max} - \bar{u} \right\} \).
It completes the proof.

A particular case of Theorem 1 can be directly applied to the dynamical systems without time-delay.

**Corollary 1**: Consider the following dynamical system:
\[
\frac{dx}{dt} = Ax + B \text{sat}(u) + f(x) \tag{60}
\]

where the non-linear function \( f(x) \) satisfies the following condition:
\[
\| f(x) - f(\tilde{x}) \| \leq \| M (x - \tilde{x}) \| \tag{61}
\]

The CNF controller may be realised as follows:
\[
\hat{u} = \hat{u} + K (x - \hat{x}) + F \hat{X} P (x(t) - \hat{x}) \tag{62}
\]

The CNF control law in Equation (62), with \( K = G X^{-1} \) and \( P = \gamma X^{-1} \), could be found if there exist some matrices \( X, G \) and constants \( \rho_{F}, \gamma \) such that the following minimisation problem has a solution:
\[
\min \gamma \tag{63}
\]

subject to
\[
\begin{bmatrix} \gamma_{0} I & G \\ * & X \end{bmatrix} \geq 0 \tag{64}
\]

\[
\begin{bmatrix} X & \xi(t_{0}) \\ * & 1 \end{bmatrix} \geq 0 \tag{65}
\]

\[
\begin{bmatrix} AX + BG + X A^{T} + G^{T} B^{T} + \rho_{F} I \\ * \end{bmatrix} \begin{bmatrix} X M^{T} & X & G^{T} \\ * & * & -\gamma Q^{T} \\ * & * & * \end{bmatrix} \leq 0
\]
Proof: Similarly, the results are obtained by taking $V(x) = x^T P x$ as the Lyapunov function.

In the numerical examples, the addressed control strategy can be implemented via the following algorithm:

**Algorithm 1**: CNF control design for constrained time-delay systems.

**Step 1**: Select some constants $\rho, \epsilon, \epsilon_1, \epsilon_2, \rho_0, \rho_i$, $i = 1, 2, ..., N$.

**Step 2**: Solve the optimisation problem of Theorem 1 in an offline way.

**Step 3**: Find the decision variables $\gamma, \rho_F, G_0, G_d, X_0, X_d, Y_d, X_i, Y_i, i = 1, 2, ..., N$.

**Step 4**: Modify the controller parameters $R_i = \gamma X_0^{-1}$, $K_0 = G_0 X_0^{-1}$, and $K_d = G_d X_0^{-1}$.

**Remark 1**: A stabilising controller may be found for the time-delay systems in Equation (3) by ignoring the actuator saturation. Thus, the control law would be selected as follows:

$$u(t) = K_0 x(t) + K_d x(t - d_{sat}) + v$$

(66)

Then, asymptotic stability would be guaranteed by the control law in Equation (66) with $K_0 = G_0 X_0^{-1}$ and $K_d = G_d X_0^{-1}$.

**Remark 2**: In many real-process control, as well as the chemical reactor, distillation column, and the others, there exists a single time-delay in the governed differential equations (i.e. special case $N = 1$). Hence, the differential Equation (3) may be written as the following form:

$$\begin{cases}
\frac{d}{dt} x(t) = A_0 x(t) + A_1 x(t - d) + B_{sat}(u(t)) + f(x(t), x(t - d)) \\
y(t) = C x(t), t \geq t_0
\end{cases}$$

(68)

Thus, the study result can be reduced as follows:

If there exist some $n$ by $n$ positive definite matrices $X_0, X_d, Y_d, X_i, Y_i, i = 1, 2, ..., N$, the vectors $G_0, G_d$ and the positive constant $\rho_F$ such that the following minimisation problem has a solution:

$$\min \gamma$$ subject to

$$\begin{bmatrix}
\epsilon X_0 & X_0 A_1^T X_0 & \cdots & X_0 M_0^T \cr X_0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & X_0 M_N^T \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\end{bmatrix} \leq 0$$

(67)

$$\begin{bmatrix}
\epsilon_1 X_0 & X_0 A_1^T X_0 & \cdots & X_0 M_0^T \\
0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & X_0 M_N^T \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\end{bmatrix} \leq 0$$

(69)

$$\begin{bmatrix}
\epsilon_1 X_0 & X_0 A_1^T X_0 & \cdots & X_0 M_0^T \\
0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & X_0 M_N^T \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\end{bmatrix} \leq 0$$

(70)

$$\begin{bmatrix}
\epsilon_1 X_0 & X_0 A_1^T X_0 & \cdots & X_0 M_0^T \\
0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & X_0 M_N^T \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\end{bmatrix} \leq 0$$

(71)
state feedback term is not presented (i.e. \( \rho_0 \)). It is clear that Theorem 1 could also be used when the memory-state feedback control can improve the transient performance in the time-delay systems.

Remark 4. Sometimes, a memory-state feedback control can improve the transient performance in the time-delay systems. It is clear that Theorem 1 could also be used when the memory-state feedback term is not presented (i.e. \( d_m = 0 \)) in the control law in Equation (6).

Remark 5. The LMIs in Equations (18) to (21) may seem hard to be feasible due to the restriction of the parameter. Some reasonable guidelines to reduce the conservatism may be summarised as follows:

(a) The matrices \( M_i \in \mathbb{R}^{d \times d} \) for \( i = 0, 1, 2, \ldots, N \) are fitted appropriately.
(b) The bounds of the uncertain terms \( \bar{d}_i \) for \( i = 0, 1, 2, \ldots, N \) and \( \bar{d}_m \) are chosen tightly.
(c) The weight matrices \( Q \) and \( R \) are tuned using the optimisation method.
(d) A reasonable control constraints \( u_{\min} \) and \( u_{\max} \) are chosen for the considered system. The operating point \( \bar{u} \) is placed near to the centre line rather than the saturation corners. Hence, the region \( u_{\min} < \bar{u} < u_{\max} \) is freely satisfied as well.
(e) The free parameters \( \rho, \rho_0, \rho_0, \rho_i, i = 1, 2, \ldots, N \) are adjusted using the trial and error.

Remark 6. In case, the time-delays \( d_1, d_2, \ldots, d_N \) and \( d_m \) are time-varying. The following expression would be added to

\[
\begin{bmatrix}
\Psi_{11} & \rho_0 G_d & \rho_1 A_1 X_0 & \cdots & \rho_N A_N X_0 & X_0 M_0^T & 0 & \cdots & 0 & X_0 Q^T & 0 & 0 & C_0^T & G_d^T & 0 \\
* & -Y_d & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & G_d^T & 0 & 0 \\
* & * & -Y_1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -Y_N & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \frac{\rho_0}{N+1} I_d & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \frac{\rho_0}{N+1} I_d & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \frac{\rho_0}{N+1} I_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \frac{\rho_0}{N+1} I_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \frac{\rho_0}{N+1} I_d & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \gamma I_d & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \gamma I_d & 0 & 0 & 0 & 0 \\
\end{bmatrix} \leq 0
\]

where \( \Psi_{11} = (A_0 + A_1) X_0 + G_d X_0 + (1 + \rho_0) G_d X_0 + (1 + \rho_0) G_d X_0 \) and \( A_{II} = (A_0 + A_1) X_0 + G_d X_0 + (1 + \rho_0) G_d X_0 + (1 + \rho_0) G_d X_0 + \bar{d}_m \) and \( \rho_0 \) is chosen.
Equation (25):
\[
\sum_{j=1}^{N} \int_{t-d_j}^{+\infty} \xi_j^T (\tau) A_j^T P_j A_j \xi_j (\tau) \, d\tau + \sum_{j=1}^{N} \int_{t-d_j}^{+\infty} \xi_j^T (\tau) K_j^T B_j^T P_j B_j K_j \xi_j (\tau) \, d\tau + \int_{t-d_m}^{+\infty} \xi_j^T (\tau - d_m) S_j \xi_j (\tau - d_m) \, d\tau
\]

\eqref{eq:25}

The matrices \(P_j, S_j, \xi_j\) and \(P_j, i = 1, 2, \ldots, N\) are positive definite. Then, Theorem 1 could still valid if the time-delays have the decreasing property.

5 \ NUMERICAL SIMULATIONS

Example 1: Consider the following non-linear time-delay system:
\[
y(t) + \dot{y}(t) + y(t) + 0.5y(t - 0.5) + y(t - 0.5)
\]
\[
+ y(t) \sin \left( \frac{y(t)}{4 + y^2(t)} \right) = u(t)
\]

\eqref{eq:75}

Equation \(\eqref{eq:75}\) maybe in the form of Equation \(\eqref{eq:3}\) by taking the state vector \(x = [x_1 \ x_2]^T\) with \(x_1 = y\) and \(x_2 = \dot{y}\). Then,
\[
\begin{aligned}
\dot{x}_1 &= \begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix} x(t) + \begin{bmatrix}
0 & 0 \\
-1 & -0.5
\end{bmatrix} x(t - 0.5) \\
&\quad - \begin{bmatrix}
\frac{x_1 \sin(x_1)}{4 + x_1^2} \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u(t)
\end{aligned}
\]

\eqref{eq:76}

In this example, first, the output tracks the constant reference \(y_p = 4\). Then, the results are used to evaluate the closed-loop performance via the time-varying signal \(y_p(t) = 4 + 0.2 \sin(t)\).

The equilibrium points could be computed as \(\bar{x}_1 = y_p = 4\), \(\bar{x}_2 = 0\), and \(\bar{u} = 4y_p + \frac{y_p \sin(y_p)}{4 + y_p^2} = 7.8486\).

It can be checked that the inequality \(|\frac{5x_1 \sin(x_1) - (4 + x_1^2) \sin(4)}{5(4 + x_1^2)}| \leq 0.2|x_1 - 4|\) holds for \(x_1 \in \mathbb{R}\). Then, the non-linear function meets assumption 1 with \(M_0 = \text{diag}(0.2, 0)\). In the cost function in Equation \(\eqref{eq:16}\), the weight matrices are selected as \(R = 1\) and \(Q = \text{diag}(0.08, 0)\). The damping parameters are \(\beta = 100\) and \(\alpha = 0.1\). The time-delay system in Equation \(\eqref{eq:76}\) starts from \(x_0 = [3 \ 0]^T\). The non-symmetric control limitations are considered as \(u_{\text{min}} = 6.5\) and \(u_{\text{max}} = 9\). The proposed control law in Equation \(\eqref{eq:6}\) with \(d_m = 0.1\) s is implemented in the time-delay system in Equation \(\eqref{eq:76}\). In order to assess the control technique, the outcomes are compared with the existing control methods \[19, 20\]. Similar to Equation \(\eqref{eq:16}\), the following cost function is computed as the performance index:
\[
J_0 = \int_0^{+\infty} \left( (y(t) - y_p)^2 + R(u(t) - \bar{u})^2 \right) \, dt.
\]

\eqref{eq:77}

The quantitative results, as well as the cost function \(J_0\) and settling-time, are computed in Table 1.

The states of the time-delay system in Equation \(\eqref{eq:76}\) are shown in Figures 2 and 3. The applied control input and its constraints are observed in Figure 4. The actuator constraints are significantly activated during the simulation.

As anticipated, the transient performance improvement is the main achievement of the CNF controller. The settling-time is notably enhanced by more than 58% compared to the best similar method \[19\]. Therefore, the transient response would be improved by the proposed method when the reference signal is constant.

| Criterion | Proposed CNF control | Existing method [19] | 2-Term CNF control [20] |
|-----------|----------------------|----------------------|------------------------|
| \(J_0\)   | 31.7672              | 32.8642              | 33.3257                |
| Settling-time (s) | 1.7890              | 4.2580              | 5.3990                  |

Table 1: Comparison of the performance index

FIGURE 2 The first state of example 1 with constant reference

FIGURE 3 The second state of example 1 with constant reference
Next, considering the above conditions, proposition 1 is applied in example 1 with taking a time-varying reference $y_{sp}(t) = 4 + 0.2 \sin(t)$. Although the suggested procedure is exclusively concentrated on the constant set-point regulation, but a bounded error would be induced in the presence of the time-varying reference. Thus, the ultimate boundedness is guaranteed rather than asymptotic stability. More detail on this topic can be found in [7].

Considering the sine set-point, the states of the system in Equation (76) are plotted in Figures 5 and 6. However, a bounded tracking error emerges in the simulation as expected. The applied control effort and control limitations are depicted in Figure 7. Accordingly, the numerical indexes and simulation results can declare the benefits of the suggested CNF controller over the previous methods in a typical time-delay system under actuator saturation.

**Example 2:** Consider CSTR with the jacket cooling and separator, as illustrated in Figure 8.

A first-order irreversible exothermic reaction $A \rightarrow B$ is taking place in the CSTR. A time-delay $d$ is induced regarding the recycling action. The differential equation governed on such process can be found as the following form [44, 45]:

$$
\begin{align*}
\dot{x}_1(t) &= \frac{F}{V_r} \left( \Delta C_{a0} + (1 - \Delta) x_1(t - d) - x_1(t) \right) \\
&\quad - K_0 x_1(t) e^{-\frac{E_{\text{R}_{20}}}{RT_0}} \\
\dot{x}_2(t) &= \frac{F}{V_r} \left( \Delta T_0 + (1 - \Delta) x_2(t - d) - x_2(t) \right) \\
&\quad - \frac{U_{\text{A}_{j}}}{\rho_{j} C_{j} V_{j}} \left( x_2(t) - x_3(t) \right) - \frac{\lambda_{j} K_0}{\rho_{j} C_{j}} x_1(t) e^{-\frac{E_{\text{R}_{20}}}{RT_0}} \\
\dot{x}_3(t) &= \frac{U_{\text{A}_{j}}}{\rho_{j} C_{j} V_{j}} \left( x_2(t) - x_3(t) \right) + \frac{T_{\text{in}} - x_3(t)}{V_j} u(t)
\end{align*}
$$

In Equation (78), $x_1$ is the reactant concentration (kmol/m$^3$), $x_2$ is the reactor temperature (K), $x_3$ is the jacket temperature
TABLE 2 The parameters of the CSTR

| Parameter | Description | Unit | Value |
|-----------|-------------|------|-------|
| $K_0$     | Pre-exponential factor | sec | $20.75 \times 10^6$ |
| $E$       | Activation energy | J/kmol | $69.71 \times 10^6$ |
| $R$       | Universal gas constant | J/kmol-K | 8314 |
| $F$       | Flow-rate of feed and product | m$^3$/s | $4.377 \times 10^{-3}$ |
| $\rho$    | Density of product stream | kg/m$^3$ | 801 |
| $C_{d0}$  | Concentration of reactant $A$ in feed | kmol/m$^3$ | 8.01 |
| $V_r$     | Volumetric holdup of liquid in reactor | m$^3$ | 102 |
| $T_0$     | Temperature of feed | K | 294 |
| $C_p$     | Heat capacity of product | J/kg-K | 3137 |
| $\lambda$ | Heat of reaction | J/kg-kmol | $-69.71 \times 10^6$ |
| $U$       | Overall heat transfer coefficient | W/K-m$^2$ | 851 |
| $A_j$     | Jacket heat transfer area | m$^2$ | 101 |
| $V_j$     | Jacket volume | m$^3$ | 1000 |
| $\rho_j$  | Density of coolant | kg/m$^3$ | 1000 |
| $C_j$     | Heat capacity of coolant | J/kg-K | 4183 |
| $T_{cin}$ | Supply temperature of cooling medium | K | 294 |

Equation (78) may be modified as follows:

$$\dot{x} = \left[ \begin{array}{ccc} - \frac{F}{V_r} & 0 & 0 \\ 0 & - \frac{F}{V_r} & \frac{UA_j}{\rho C_p V_r} \\ 0 & \frac{UA_j}{\rho C_p V_r} & - \frac{UA_j}{\rho C_j V_j} \end{array} \right] x(t) + \left[ \begin{array}{c} \frac{E}{V_r} (1 - \Delta) \\ 0 \\ 0 \end{array} \right] u(t) + \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \mu(t) + \left[ \begin{array}{c} \frac{F}{V_r} \Delta C_{d0} - K_0 X_1(t) e^{- \frac{E_{Rc}}{Rc}} \\ 0 \\ 0 \end{array} \right] \left( \begin{array}{c} T_{cin} - X_1(t) \\ T_{cin} - X_2(t) \end{array} \right)$$

(79)

The inequality in Equation (4) may be rewritten as

$$K_0 \sqrt{1 + \frac{\lambda^2}{\rho^2 C_p^2}} e^{- \frac{E_{Rc}}{Rc}} \left| X_1(t) - X_2(t) \right| \leq \left\| M_0 (x - \tilde{x}) \right\|$$

(80)

where $M_0 = \text{diag}(M^{11}_0, M^{22}_0, 0)$. Thus assumption 1 is locally met as the following:

$$M^{11}_0 = K_0 \sqrt{1 + \frac{\lambda^2}{\rho^2 C_p^2}} e^{- \frac{E_{Rc}}{Rc}} = 0.0010$$

$$M^{22}_0 = K_0 \sqrt{1 + \frac{\lambda^2}{\rho^2 C_p^2}} e^{- \frac{E_{Rc}}{Rc}} = 0.0004$$

It is evident that Equation (79) would be in the form of Equation (3). Hence, the results of the study can be directly applied to the dynamical system in Equation (79). In such a chemical process, the circulation coefficient is defined as $\Delta = \frac{F - F_{rec}}{F}$. Hence, $0 \leq \Delta \leq 1$ where $\Delta = 1$ means no recycle and $\Delta = 0$ means total recycle.

In the presented control problem, it is interesting that the reactor concentration $x_3$ tracks a fixed reference point $T_{ref} = 310$ K only by manipulating the jacket cooling. Then, the equilibrium point of Equation (78) can be computed as follows:

$$\tilde{x}_1 = \frac{F \Delta C_{d0}}{F \Delta + V_r K_0 e^{- \frac{E_{Rc}}{Rc}}} = 3.8443,$$

$$\tilde{x}_2 = T_{ref} = 310,$$

$$\tilde{x}_3 = \left( \frac{\rho C_p F \Delta}{UA_j} + 1 \right) \tilde{x}_2 - \frac{\rho C_p F \Delta T_0}{UA_j} + \frac{\lambda K_0 V_r}{UA_j} \frac{F \Delta C_{d0} e^{- \frac{E_{Rc}}{Rc}}}{F \Delta + V_r K_0 e^{- \frac{E_{Rc}}{Rc}}} = 299.8075,$$

$$\tilde{u} = \frac{UA_j}{\rho C_j T_{cin} - \tilde{x}_3} = 0.0361.$$
The quantitative results, as well as the cost function $J_0$ and the transient performances of the output $x_2(t)$, are computed and compared with the existing method, as seen in Table 3.

The transient response improvement is the main feature of the CNF control. In contrast to seminal strategy [19], both of the indexes, reported in Table 3, are appreciably enhanced by more than 20%.

The regulation problem is fulfilled in the chemical process in Equation (78) in spite of the time-delay and the control limitations. Table 3 reveals that the performance indexes are ultimately enhanced by the suggested CNF control, whereas the control constraints have been activated at the beginning. The most striking results to emerge from the data verify that the presented method is a more effective control approach in applying to the chemical reactor with separator and recycle. Accordingly, the proposed CNF control scheme can handle the actuator saturation and induced time-delay in the typical CSTR.

6 | CONCLUSION AND FUTURE DIRECTION

In this study, a delay-dependent criterion is discussed to the CNF control design in the time-delay systems subjected to the actuator saturation. For achieving this goal, by considering the actuator upper and lower limitations, an LMI-based approach is developed to formulate and find the CNF controller parameters in the constrained dynamical systems. Thus, the control synthesis issue is deliberately translated to an optimisation problem subjected to some LMIs. The suggested controller is numerically implemented and tested in two control examples (a typical non-linear time-delay system and a CSTR with separation and recycle). In comparison with the best similar approaches [19], the simulation results and computed transient performances (i.e. settling-time and overshoot) confirm that remarkable progress has been made by the proposed method.

The findings suggest the following directions for future research: The idea of the study may be improved or extended to other classes of the non-linear control systems. The efficient delay-dependent Lyapunov–Krasovskii functional can be incorporated to obtain a less-conservative sufficient stability condition. The other practical limitations, as well as the actuator non-linearity, rate saturation, and dead zone, may be carried out with a similar procedure. It is recommended to include the mentioned points in the stability analysis and control synthesis.

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