Quantitative two-scale stabilization on the Poisson space

*Joint work with R. Lachièze-Rey and G. Peccati.*

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Let $\eta$ be a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda(dx)$. The fluctuation of a generic functional $F$ is governed by some principles

- **Poincaré inequality**

\[
\text{Var}[F] \leq \int \mathbb{E}[|D_x F|^2] \lambda(dx).
\]

where $D_x F = F(\eta + \delta_x) - F(\eta)$ is the "add-one-cost".

- **Second-order Poincaré inequality**

\[
d_W(F, N) \lesssim \text{integrated moments of } D_{x,y}^2 F
\]

where $D^2 = DD$ is the iterated add-one-cost, cf. **Chatterjee** ('09), **Nourdin, Peccati et Reinert** ('09), **Last, Schulte et Peccati** ('16), **Schulte et Yukich** ('19) ...

- The add-one-cost controls the variance, the iterated add-one-cost gives gaussianity.
Fluctuation of Poisson functionals

▶ **Applications**: Spatial networks, coverage processes, tessellations etc. useful objects in telecommunication, topological/geometrical data analysis, machine learning...

▶ This talk is concerned with a *principle alternative to 2nd order Poincaré*. What happens if the iterated add-one-cost is not tractable?

▶ We address this problem with a two-scale stabilisation theory, which is a quantified version of the stabilisation theory of Penrose (’01), Penrose and Yukich (’01), Penrose (’05).

▶ This work is along the line of **Malliavin-Stein** methodology for normal approximation, combined with ideas from a quantitative CLT for the MST by Chatterjee and Sen (’17)
The iterated add-one-cost is not always tractable

Figure 1: Right: MST. Left: MST after adding a point to the origin.
Setting

- Let $\eta$ be a Poisson process with unit intensity on $\mathbb{R}^d$, identified with its support $\mathcal{P}$.
- For a Poisson functional $F = F(\eta)$ and $B \in \mathcal{B}(\mathbb{R}^d)$, define the add-one-cost
  
  $$D_x F(B) = F((\eta + \delta_x)|_B) - F(\eta|_B)$$

  and the two-scale discrepancy

  $$\psi := \sup_{x \in B} \mathbb{E}[|D_x F(B) - D_x F(A_x)|]$$

- The set $B$ represents the observation window growing to $\mathbb{R}^d$ and $A_x$ is a local window of $x$ with $\text{Leb}(A_x) \ll \text{Leb}(B)$.
- In practice, $B = B_n, A_x = B_{b_n}(x) \cap B$ with $b_n = o(n)$. In such case, the two-scale discrepancy is denoted by $\psi_n$. Define also

  $$\psi'_n = \sup_{x \in B(n-b_n)} \mathbb{E}[|D_x F(B_n) - D_x F(A_x)|].$$
Main (user friendly) result

**Theorem (Lachièze-Rey, Peccati and Y. ('20+))**

*Suppose that the following holds:*

- **there exists** $p > 4$ **and** $C < \infty$ **such that for all** $n \in \mathbb{N}$

\[
\sup_{x \in B_n} \mathbb{E}[|D_x F(B_n)|^p] + \mathbb{E}[|D_x F(A_x)|^p] \leq C^p,
\]

- **there exists** $c > 0$ **such that**

\[
\mathbb{V}ar[F(B_n)] \geq c \cdot \text{Leb}(B_n) = cn^d.
\]

*Then there exists** $c \in (0, \infty)$ **such that**

\[
\frac{1}{c} d_{\mathcal{W}} \left( \frac{F(B_n) - \mathbb{E}[F(B_n)]}{\sqrt{\mathbb{V}ar[F(B_n)]}}, N(0, 1) \right) \leq \begin{cases} 
\psi_n \frac{1}{2} (1 - \frac{4}{p}) + \left( \frac{b_n}{n} \right)^{d/2} \\
\psi_n' \frac{1}{2} (1 - \frac{4}{p}) + \left( \frac{b_n}{n} \right)^{1/2}
\end{cases}.
\]

N.B. The choice of $b_n$ is done by optimizing the final bound.
THE CENTRAL LIMIT THEOREM FOR WEIGHTED MINIMAL SPANNING TREES ON RANDOM POINTS

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Let \( \{X_i, 1 \leq i < \infty\} \) be i.i.d. with uniform distribution on \([0, 1]^d\) and let \( M(X_1, \ldots, X_n; \alpha) = \min\{\sum_{e \in T'} |e|^{\alpha}; T' \text{ a spanning tree on } \{X_1, \ldots, X_n\}\} \). Then we show that for \( \alpha > 0 \),

\[
\frac{M(X_1, \ldots, X_n; \alpha) - EM(X_1, \ldots, X_n; \alpha)}{n^{(d-2\alpha)/2d}} \rightarrow N(0, \sigma_{\alpha, d}^2)
\]

in distribution for some \( \sigma_{\alpha, d}^2 > 0 \).
Strong stabilization: ∃ a.s. finite random variable $R_0$ such that

$$D_0 F(\mathcal{P} \cap B_{R_0}) = D_0 F((\mathcal{P} \cap B_{R_0}) \cup U)$$

for any finite $U \subset (B_{R_0})^c$.

Weak stabilization: for any $(E_n)$ with $\lim \inf E_n = \mathbb{R}^d$, we have

$$D_0 F(E_n) \rightarrow \delta_0(\infty) \text{ a.s.}$$

for some random variable $\delta_0(\infty)$.

**Theorem (Penrose and Yukich ('01))**

Assume i) uniform 4th-moment condition; ii) weak stabilization at 0. Then

$$\frac{\operatorname{Var}[F(B_n)]}{n^d} \rightarrow \sigma^2 \in [0, \infty) \quad \text{and} \quad \frac{F(B_n) - \mathbb{E}[F(B_n)]}{n^{d/2}} \xrightarrow{\mathcal{L}} N(0, \sigma^2).$$

If $\delta_0(\infty)$ is non-degenerate, then $\sigma^2 > 0$. 
Relation with our bounds

▶ Corollary of our bound:

\[
d_W \left( \frac{F(B_n) - \mathbb{E}[F(B_n)]}{\sqrt{\text{Var}[F(B_n)]^{1/2}}}, N(0, 1) \right)
\leq c \left[ \sup_{x \in B_n} \mathbb{P}[R_x \geq b_n] \frac{1}{2} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{4}{p} \right) + \left( \frac{b_n}{n} \right)^{d/2} \right],
\]

where \( R_x \) the radius of strong stabilization at \( x \).

▶ Assume \( F(\tau_x P \cap \tau_x B) = F(P \cap B) \) and weak stabilization \( \Rightarrow \)

\[
D_x F(E_n) \to \delta_x(\infty) \quad \text{a.s.}
\]

for any \( (E_n) \uparrow \mathbb{R}^d \). Therefore, the required condition

\[
\psi'_n = \sup_{x \in B_n - b_n} \mathbb{E}[|D_x F(B_n) - \delta_x(\infty) + \delta_x(\infty) - D_x F(A_x)|] \to 0
\]

is a uniform strengthening of weak stabilization. Note however that we do not require the existence of \( \delta_0(\infty) \).
Far reach of the Penrose-Yukich theory (thus ours)

Weights, subgraph counts, components counts of

- $k$-nearest neighbor graphs
- sphere of influence graphs
- Voronoi tessellations
- minimal spanning trees

**PY:** (Multivariate) Gaussian approximation holds if strong/weak stabilisation holds for the functional of interest.

**LrPY:** To obtain rates, if suffices to compute $\psi_n$ (or $\psi'_n$), or $P[R_x \geq b_n]$.

Not always easy, here is an open problem
The optimal travelling salesman tour on Poisson points is believed to be stabilizing (implying CLT if proved).
Applications (in our paper)

- **Online NNG (S):** Mark $\mathcal{P} \cap B_n$ with iid uniform $[0, 1]$ representing the arrival time, each point is attached to its nearest neighbour prior to its arrival. We obtain $n^{-c}$ for the rate of normal approximation of the weighted edge length.

- **Boolean model (S):** The number of connected components of the Boolean model

  $$O_u(\mathcal{P} \cap B_n) = \bigcup_{x \in \mathcal{P} \cap B_n} S_u(x).$$

  approaches normal with rate $n^{-c}$ in $d = 2$ and $\log(n)^{-c}$ in $d \geq 3$.

- **Minimal spanning tree (W):** The total weighted edge length of MST approaches normal distribution with the same rate as the percolation example. In both cases, $\psi'_n$ is bounded by the two arm events.

- **Excursion of heavy tail shot noise fields (W):** The intrinsic volumes of excursion sets $E_u = \{t \in B_n : X(t) \geq u\}$ of heavy tail shot noise field $X$ approaches normal with rate $n^{-c}$. 

Figure 2: $R_x := \inf\{r > u : \text{at most 1 arm in } B_r(x) \setminus B_u(x)\}$ where 1 arm means that the Boolean model contains a path connecting the boundary of two boxes.
\( \{R_x > b_n\} \subset \{\text{at least 2 arms at distance } b_n\} \).

### Phase transition of occupied and vacant regions

\[
 u_c := \inf \{ u : \mathbb{P}[0 \leftrightarrow \infty \text{ in } O_u] > 0 \} \in (0, \infty), \\
u^*_c := \sup \{ u : \mathbb{P}[0 \leftrightarrow \infty \text{ in } V_u] > 0 \} \in (0, \infty),
\]

and \( u_c = u^*_c \) in dimension 2 by Roy ('90), \( u_c < u^*_c \) in dimension \( d \geq 3 \) by Penrose ('96), Sarkar ('97).

- **Subcritical phase** \( u < u_c \)
  \[
  \mathbb{P}[R_x > b_n] \leq \mathbb{P}[\text{at least 1 arm at distance } b_n] \leq e^{-cb_n}.
  \]

- **Supercritical phase** \( u > u^*_c \)
  \[
  \mathbb{P}[R_x > b_n] \leq \mathbb{P}[\text{at least 1 vacant arm at distance } b_n] \leq e^{-cb_n}.
  \]

- **Critical phase** \( u \in [u_c, u^*_c] \)
  Two-arm event decays as \( b_n^{-c} \) in 2D and \( [\log(b_n)]^{-c} \) in \( d \geq 3 \) by a quantitative Burton-Keane argument of Chatterjee-Sen ('17).
Minimal spanning tree (W)

- Minimal spanning tree over a finite point set $\mathcal{U}$

$$\text{MST}(\mathcal{U}) = \text{Argmin} \left\{ \sum_{e \in \mathcal{T}} |e|, \mathcal{T} \text{ connected with } \forall(\mathcal{T}) = \mathcal{U} \right\}$$

- Functional of interest $M(B_n) \in \mathbb{R}^m$ given by

$$M(\varphi_i; B_n) := \sum_{e \in \text{MST}(\mathcal{P}|_{B_n})} \varphi_i(|e|), \quad 1 \leq i \leq m.$$ 

- Suppose $\varphi$ is given by $\varphi(x) = \psi(x) \mathbb{1}(x \leq r)$ for some non-decreasing function $\psi$ and some truncation level $r \in (0, \infty]$. If (and only if) $r = \infty$, suppose

$$\exists k \in \mathbb{N}, \quad \psi(x) \leq (1 + x)^k \text{ and } \int_0^\infty e^{-cu^d} d\psi(\sqrt{du}) < \infty.$$ 

- Examples: power-weighted edge length $\varphi(x) = x^\alpha$ or empirical process $\varphi(x) = \mathbb{1}(x \leq r)$. 
Theorem (LrPY ’20+)

Let $N = N(n)$ be a centered Gaussian vector with the same covariance matrix as

$$n^{-d/2}M(B_n).$$

Then, one has that

$$d_3(n^{-d/2}(M(B_n) - \mathbb{E}[M(B_n)]), N) \leq \begin{cases} cn^{-\theta} & \text{if } d = 2, \\ c \exp(-c \log \log(n)) & \text{if } d \geq 3, \end{cases}$$

for some $0 < \theta < 1$. The above bound continues to hold for the distances $d_2, d_c$, if $\text{Cov}[n^{-d/2}M(B_n)] \to \Sigma_\infty > 0$.

- Two vertices $x, y \in \mathcal{P}$ form an edge of MST if and only if $x$ and $y$ belong to different component of $O_{\frac{|x-y|}{2}}(\mathcal{P})$.

- In $d = 2$, consider $(\log(n))^a$ Boolean models with random radius and relate $\psi'_n$ to the 2-arm estimates.
Proof of the general bound (i) Stein’s bound (’72, ’86)

- **Stein’s lemma**

  \[ \mathbb{E}[f'(N)] = \mathbb{E}[Nf(N)]. \]

  if and only if \( N \sim N(0, 1) \).

- **Heuristic:** \( F \approx N \) if and only if

  \[ \mathbb{E}[f'(F)] \approx \mathbb{E}[Ff(F)]. \]

- **Stein’s equation**

  \[ f'(x) - xf(x) = h(x) - \mathbb{E}[h(N)] \]

  with \( h \in \text{Lip}_1 \). Evaluate the expectation wrt \( \mathbb{P} \circ F^{-1} \), then take sup over \( h \) gives

  \[ d_W(F, N) := \sup_{h \in \text{Lip}_1} |\mathbb{E}h(F) - \mathbb{E}h(N)| \]

  \[ \leq \sup_{\|g'\|, \|g''\| \leq 1} |\mathbb{E}[Fg(F)] - \mathbb{E}[g'(F)]|. \]
For $F = F(B)$ with $\mathbb{E}[F] = 0$, $\mathbb{E}[F^2] = 1$, we integrate by parts

$$\mathbb{E}[Fg(F)] = \mathbb{E}\left[ \int_B D_x(g(F)) (-D_x L^{-1} F) \, dx \right]$$

$$\approx \mathbb{E}\left[ g'(F) \int_B D_x F (-D_x L^{-1} F) \, dx \right]$$

where $L^{-1}$ involves thinning and (independent) superposition.

Proof of IBP by (birth and death) semigroup interpolation: in 1 dimension, $(\Omega, \mathcal{F}, P) = (\mathbb{N}_0, \text{Po}(1))$,

$$P_t f(k) = \mathbb{E}[f(\text{Bin}(k, e^{-t}) + \text{Po}(1 - e^{-t}))]$$

and

$$L f(k) = 1(f(k + 1) - f(k)) - k(f(k) - f(k - 1)).$$

satisfying $-\mathbb{E}[fLg] = \mathbb{E}[DfDg]$ with $Df(k) = f(k + 1) - f(k)$. 
Thus, interpolation and $-\mathbb{E}[FLG] = \mathbb{E}[\langle DF, DG \rangle]$ gives

$$
\mathbb{E}[Fg(F)] = \mathbb{E}[(P_0 F - P_\infty F)g(F)] \\
= -\int_0^\infty \mathbb{E}[(LP_tF)g(F)]dt \\
= \int_0^\infty \mathbb{E} \left[ \int_B D_x(g(F))D_xP_tF dx \right] dt \\
= \mathbb{E} \left[ \int_B D_x(g(F))(-D_xL^{-1}F) dx \right]
$$

by setting

$$-L^{-1} := \int_0^\infty P_t dt$$

Combining Stein’s bound, integration by parts, and Cauchy-Schwarz

$$d_W(F, N) \lesssim \text{Var} \left[ \int_B D_x F (-D_xL^{-1}F) dx \right]^{1/2}$$

$$= \left( \int \int_{B^2} \text{Cov}[D_xFD_xL^{-1}F, D_yFD_yL^{-1}F] dxdy \right)^{1/2}.$$
Proof of the general bound (iii) two-scale stabilization

- **When** $x$ and $y$ are close i.e. $A_x \cap A_y \neq \emptyset$, bound the covariance by

  \[
  \mathbb{E}[|D_x L^{-1} F(B)|^p] \leq \mathbb{E}[|D_x F(B)|^p] \leq C,
  \]

  yielding a term $(\frac{b_n}{n})^{d/2}$.

- **When** they are far apart i.e. $A_x \cap A_y = \emptyset$, we replace everything by its local version

  \[
  \text{Cov}[D_x F(A_x)D_x L^{-1} F(A_x), D_y F(A_y)D_y L^{-1} F(A_y)]
  \]

  with 4 error terms like

  \[
  \text{Cov}[(D_x F(B) - D_x F(A_x))D_x L^{-1} F, D_y F D_y L^{-1} F].
  \]

- By independence of Poisson points over non-overlapping regions, (1) = 0, we bound (2) by

  \[
  \mathbb{E}[|D_x F(B) - D_x F(A_x)||D_x L^{-1} F D_y F D_y L^{-1} F|].
  \]

- Applying Hölder’s inequality and bounding the moments

  \[
  \mathbb{E}[|D_x L^{-1} F(A_x)|^p] \leq \mathbb{E}[|D_x F(A_x)|^p] \leq C
  \]

  leads to the two-scale discrepancy $\psi_n$, ending the proof.
Two-scale bounds of the type

\[(\psi_n)^{1/2}(1 - \frac{4}{p}) + \left(\frac{b_n}{n}\right)^{d/2}\]

holds for

- **Kolmogorov distance**

  \[d_K(F, N) = \sup_{x \in \mathbb{R}} |\mathbb{P}[F \leq x] - \mathbb{P}[N \leq x]|,\]

- probability metrics for **multivariate normal approximation**, including smooth ones \(d_2, d_3\) (generalizing \(d_W\)), and the non-smooth convex distance (generalizing \(d_K\))

  \[d_c(F, N_{\Sigma}) = \sup_{E \text{ convex}} |\mathbb{P}[F \in E] - \mathbb{P}[N_{\Sigma} \in E]|\]

possibly subject to stronger moment conditions \((p > 6)\).
Behind the scenes: a new Kolmogorov bound

**Theorem (LrPY ’20+)**

Let \( \hat{F} = (F - \mathbb{E}[F]) / \sigma \).

\[
d_K \left( \hat{F}, N \right) \leq \left| 1 - \frac{\text{Var}[F]}{\sigma^2} \right| + \frac{1}{\sigma^2} \mathbb{E} \left[ |\text{Var}[F] - \langle DF, -DL^{-1}F \rangle| \right] \\
+ \frac{2}{\sigma^2} \mathbb{E} \left[ |\delta(DF|DL^{-1}F)| \right],
\]

where \( \delta \) is the Kabanov-Skorohod integral.

- Starting point of the two-scale bound in \( d_K \).
- Two redundant terms in *Schulte (’16)* and *Eichelsbacher and Thäle (’14)* are removed.
- A good place to start if the 4th-moment assumption is not verified.
Final remarks

- Our theorem gives almost optimal rates $\log(n)^c n^{-d/2}$ in the case of exponential stabilization $\mathbb{P}[R(x) > t] \leq ce^{-c't}$.
- The second-order Poincaré estimates of Last, Peccati and Schulte (’16), Lachièze-Rey, Schulte and Yukich (’19) and Schulte and Yukich (’19) is concerned with

$$\mathbb{P}[D_{x,y}^2 F \neq 0],$$

yielding Berry-Esseen bounds $n^{-d/2}$ for exponential stabilization.
- The upshot of our theorem is that we do not require knowledge on the iterated add-one-cost operators, which can be very hard to access quantitatively for not necessarily exponentially stabilizing functionals such as critical percolation models.
Thanks!