Abstract

We introduce a two-dimensional sigma model on surfaces with boundary and target space a Jacobi manifold. The model yields a topological open string theory. In the Hamiltonian approach first class constraints are derived, which generate gauge invariance of the model under diffeomorphisms. By introducing a metric term, a non-topological sigma model is obtained, yielding a Polyakov action with metric and $B$-field, whose target space is a Jacobi manifold.
1 Introduction

Jacobi sigma models are here introduced as a natural generalisation of Poisson sigma models. The latter, first introduced in the context of two-dimensional gravity [1, 2], have been widely investigated in relation with symplectic groupoids, BF theory, branes and deformation quantisation [3–12]. They were also analysed from the point of view of holography and noncommutative geometry in [13]. In two dimensions these are topological field theories on a Riemannian surface \((\Sigma, g)\), with target space a Poisson manifold, \((M, \Pi)\) and a first order action

\[
S = \int_{\Sigma} \left[ \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij} \eta_i \wedge \eta_j \right]
\]

(1.1)

where the fields are given by the bundle maps \((X, \eta) : T\Sigma \to T^*M\),

\[
X : \Sigma \to M, \quad \eta \in \Omega^1(\Sigma, X^*T^*M)
\]

and \(\Pi\) is a Poisson structure, namely a skew-symmetric bi-vector satisfying Jacobi identity. When the latter is invertible, it is possible to eliminate the auxiliary field \(\eta\), and obtain a second order formulation with target the tangent space \(TM\). This yields a topological sigma model, the so-called A-model [14, 15], with only a \(B\)-field term, being \(B = \Pi^{-1}\) and \(dB = 0\). More details about the model are going to be discussed in next section,
but, what we would like to stress at the moment is that, precisely due to the properties of the Poisson bi-vector field $\Pi$, a number of interesting facts can be proven. First, under suitable assumptions, the space of solutions $\mathcal{C}$, quotiented with respect to symmetries, is a finite dimensional symplectic groupoid, generalising a well known result which holds for $M = \mathfrak{g}^*$, the dual of a given Lie algebra, where $\mathcal{C}/\text{Sym}$ is found to be diffeomorphic to $T^*G$ [3]; second, the path integral quantisation of the model furnishes a field-theoretical proof of Kontsevich star product quantisation of Poisson manifolds [4, 5]; moreover, the model is gauge invariant under space-time diffeomorphisms and the algebra of gauge parameters closes under Koszul bracket [3]; finally, if the Lagrangian in (1.1) is complemented with a dynamical term $\frac{1}{2}G^{ij}\eta_i \wedge \star \eta_j$, with $G$ a metric tensor on $M$, by integrating away the auxiliary field $\eta$ it is possible to get back the full Polyakov string action (see for example [16]). It is also possible to twist the Poisson structure by generalizing the Poisson sigma model with the introduction of a Wess-Zumino term [17].

A natural question for us is then, whether it is possible to relax the condition that $\Pi$ be Poisson, namely $[\Pi, \Pi]_S = 0$. An almost obvious generalisation, although not considered insofar in the literature\(^2\) is to consider a Jacobi structure, $(M, \Pi, E)$, with $\Pi$ a bi-vector field and $E \in \mathfrak{X}(M)$ a vector field on $M$ such that

$$[\Pi, \Pi]_S = 2E \wedge \Pi \quad \text{and} \quad [E, \Pi]_S = 0. \quad (1.2)$$

The goal is thus to build and study a two-dimensional sigma model with target space a Jacobi manifold. To this, we start from a theorem [19] which states that a Jacobi structure on $M$ always gives rise to a Poisson structure on $M \times \mathbb{R}$, say $P$, with the help of a kind of dilation vector field. A Poisson sigma model is then defined on $(M \times \mathbb{R}, P)$ whose dynamics may be reduced by means of a projection to the Jacobi manifold, $M \times \mathbb{R} \rightarrow M$. Thus we show that the projected dynamics can be obtained directly from an action on the Jacobi manifold, solely in terms of its defining structures.

The paper is organised as follows. In Section 2 we shortly review the Poisson sigma model, mainly following notations and conventions of [3]. In Section 3 Jacobi brackets and Jacobi manifolds are introduced and a Poisson sigma model on the extended manifold $M \times \mathbb{R}$ is defined. An action on the Jacobi manifold is thus proposed, which reproduces the projected dynamics. The model exhibits first class constraints, which generate gauge transformations. On using a consistent definition of Hamiltonian vector fields for Jacobi manifolds (see for example [20, 21]), we show that the latter can be associated with gauge transformations and verify that they close under Lie bracket, generating space-time diffeomorphisms. In section 4 we investigate the possibility of introducing a metric term, in analogy with what is done for the Poisson sigma model, so to obtain a model which is non-topological. We manage to integrate out the auxiliary fields and obtain a Polyakov action, with metric $g$ and $B$-field determined in terms of the defining structures.

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\(^1\)[, ]\(_S\) is the Schouten-Nijenhuis bracket defined on multivector fields.

\(^2\)While being in the process of submitting the manuscript we have been aware of a new submission on the archives [18] where the same idea is explored.
of the Jacobi bracket, \((\Lambda, E)\).

In order to better understand the novelties and peculiarities of the model, we build in Section 5 an explicit example with the group manifold of \(SU(2)\) as target space. We conclude with final remarks and perspectives.

## 2 Poisson sigma models

Let \((M, \Pi)\) be a Poisson manifold, where \(\Pi \in \Gamma(\wedge^2 TM)\) is a Poisson structure on the smooth \(m\)-dimensional manifold \(M\), and \(\Sigma\) a 2-dimensional orientable smooth manifold (possibly with boundary). The Poisson sigma model is defined by the fields \((X, \eta) : T\Sigma \to T^* M\), with \(X : \Sigma \to M\) and \(\eta \in \Omega^1(\Sigma, T^* M)\) a one-form on \(\Sigma\) with values in the one-forms over \(M\). To be rigorous, \(\eta\) should be further pull-backed to \(\Sigma\) by means of the pull-back map \(X^*\), which we omit from now in order not to burden the notation. The embedding of \(\Sigma\) in \(M\) is thus realised by the field \(X\), while \(\eta\) may be regarded as an auxiliary field.

The action functional is given by

\[
S(X, \eta) = \int_{\Sigma} \left[ \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j \right]
\]

with equations of motion:

\[
dX^i + \Pi^{ij}(X) \eta_j = 0, \quad (2.2)
\]

\[
d\eta_i + \frac{1}{2} \partial_i \Pi^{jk} \eta_j \wedge \eta_k = 0. \quad (2.3)
\]

Note that consistency of the e.o.m. requires that \(\Pi\) satisfies \([\Pi, \Pi]_S = 0\). \([\ , \ ]_S\) is the Schouten-Nijenhuis bracket, namely a skew-symmetric bilinear map \(\Lambda^p(M) \times \Lambda^q(M) \to \Lambda^{p+q-1}(M)\) given by

\[
[A_1 \wedge \cdots \wedge A_p, B_1 \wedge \cdots \wedge B_q]_S = \sum (-1)^{t+s} A_1 \wedge \cdots \hat{A}_s \cdots \wedge A_p \wedge [A_s, B_t] \wedge B_1 \wedge \cdots \hat{B}_t \cdots \wedge B_q
\]

(2.4)

where \(A_1, ..., A_p, B_1, ..., B_q\) are vector fields over \(M\) and \(\hat{A}\) indicates the omission of the vector field \(A\). Explicitly, we have

\[
0 = [\Pi, \Pi]_S^{ijk} = \Pi^{id} \partial_d \Pi^{jk} + \text{cycl}
\]

(2.5)

reproducing the Jacobi identity, which is true for a Poisson structure.

Note that if the worldsheet \(\Sigma\) has a boundary, the boundary conditions \(\eta(u)v = 0 \ \forall \ v \in T(\partial \Sigma)\), with \(u \in \partial \Sigma\), are chosen.

The sigma model action (2.1) contains a number of different interesting models. For example, the most natural one corresponds to the choice \(\Pi^{ij} = 0\), in which case one has simply an Abelian BF theory with action \(\int_{\Sigma} d^2u \, e^{\mu \nu} \eta_{\mu \nu} \partial_\nu X^i\), while an interesting nontrivial case has a linear Poisson structure on \(M\), \(\Pi^{ij} = f^{ij}_k X^k\). The latter leads to a non-Abelian
BF theory with action $S = \int_{\Sigma} d^2 u \left( \epsilon^{\mu\nu} \eta_{ij} X^i X^j + \frac{1}{2} \epsilon^{\mu\nu} f^{ij}_{k} X^k \eta_{\mu i} \eta_{\nu j} \right)$. In fact, in this case the Jacobi identity for $\Pi$ becomes a Jacobi identity for the structure constants of a Lie algebra $f^{ij}_{k}$. Another special case is the one with non-degenerate Poisson structure, which can be inverted to a symplectic form $\omega$ (which plays the role of $B$-field in the language of strings), leading to the so-called A-model, with action $S = \int \omega_{ij} dX^i \wedge dX^j$. It is also possible to show that 2-dimensional Yang-Mills, $R^2$-gravity theories and gauged WZW models can be obtained \cite{8, 22}.

We will now focus on the Hamiltonian approach. Let us choose locally a time coordinate $u^0 = t$ and denote with $u^1 = u$ the space coordinate, which can be taken to belong to a closed interval, $u \in [0, 1]$, if one wants to describe open strings. By denoting $\beta_i = \eta_{0i}$, $\zeta_i = \eta_{1i}$ and $\dot{X} = \partial_t X$, $X' = \partial_u X$, the first order Lagrangian can be written as

$$L(X, \zeta; \beta) = \int_{I} du \left[ -\zeta_i \dot{X}^i + \beta_i \left( X^i + \Pi^{ij}(X) \zeta_j \right) \right], \quad (2.6)$$

from which it is clear that $X$ and $\zeta$ are canonically conjugate variables, with Poisson brackets $\{\zeta_i(u), X^j(v)\} = \delta^i_j \delta(u - v)$ and all other brackets vanishing. Since $\beta$ has no conjugate variable, it has to be understood as a Lagrange multiplier imposing the constraints

$$X^i + \Pi^{ij}(X) \zeta_j = 0. \quad (2.7)$$

Therefore, the Hamiltonian

$$H_\beta = \int_{I} du \beta_i \left[ X^i + \Pi^{ij}(X) \zeta_j \right], \quad (2.8)$$

is a pure constraint and the space of solutions, say $C$, can be equivalently defined as the set of common zeroes of $H_\beta$. It is also possible to prove \cite{3} that these constraints are first class, namely they satisfy the following relations, provided that $\beta, \beta'$ vanish on the boundary:

$$\{H_\beta, H_{\beta'}\} = H_{[\beta, \beta']} \quad (2.9)$$

with

$$[\beta, \beta'] = d\left( \langle \beta, \Pi(\beta') \rangle \right) - \iota_{\Pi(\beta)} d\beta' + \iota_{\Pi(\beta')} d\beta \quad (2.10)$$

being the Koszul bracket of one-forms, which closes thanks to the Jacobi identity of $\Pi$. Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and one-forms at a point in $M$. Being the Hamiltonian of the model a pure constraint, the system is invariant under time-diffeomorphisms. The infinitesimal generators are the Hamiltonian vector fields associated with $H_\beta$,

$$\xi_\beta = \{H_\beta, \cdot\} = \dot{X}^i \frac{\partial}{\partial X^i} + \dot{\zeta}_i \frac{\partial}{\partial \zeta_i} \quad (2.11)$$

where $(\dot{X}^i, \dot{\zeta}_i)$ can be read from Eqs. (2.2)-(2.3) as:

$$\dot{X}^i = -\Pi^{ij} \beta_j, \quad (2.12)$$
\[
\dot{\zeta}_i = \partial_u \beta_i + \partial_i \Pi^{jk} \zeta_j \beta_k.
\] 

(2.13)

Moreover, by indicating with \( f(u) \partial_u \) a generic space diffeomorphism, it is immediate to check that this is the generator of an infinitesimal symmetry for the model, being it the Hamiltonian vector field associated with \( H_\beta \), for \( \beta_i = f(u) \zeta_i \). Thus, space diffeomorphisms are symmetries as well and the reduced phase space of the model can be defined as \( \mathcal{G} = \mathcal{C}/\text{Diff}(\Sigma) \). It can be proven [3, 4] that the latter is a finite-dimensional, closed subspace of phase space, of codimension 2dim(\( M \)), with a natural groupoid structure. Under certain conditions this is a symplectic groupoid integrating the Lie algebroid associated with the Poisson manifold [23].

Finally, the absence of a Hamiltonian implies that there is no dynamics and the model is topological in the bulk.

3 Jacobi sigma models

In order to formulate a consistent sigma model with target configuration space a Jacobi manifold \( M \), we will first briefly review the main definitions of Jacobi brackets and Jacobi manifold (see for example [20, 24–27]), hence we will build a Poisson sigma model on the extended Poisson manifold \( M \times \mathbb{R} \), according to a Poissonization procedure of the Jacobi structure. We will thus project the obtained dynamics on the underlying Jacobi manifold and finally propose a consistent model, directly defined on the Jacobi manifold, whose dynamics coincides with the projected one.

3.1 Jacobi brackets and Jacobi manifold

Jacobi brackets are defined by means of a a bi-differential operator acting on the algebra of functions on a smooth manifold \( M \), as

\[
\{ f, g \}_J = \Lambda(df, dg) + f(Eg) - g(Ef),
\]

(3.1)

where \( \Lambda \) is a bivector field and \( E \) is a vector field (called Reeb vector field) on the manifold \( M \), satisfying

\[
[\Lambda, \Lambda]_S = 2E \wedge \Lambda, \quad [\Lambda, E]_S = \mathcal{L}_E \Lambda = 0.
\]

As can be easily understood from the defining equation, Jacobi brackets are skew-symmetric and satisfy Jacobi identity just like Poisson brackets, but in general a Jacobi structure does not satisfy Leibniz rule, which is instead replaced by the condition

\[
\{ f, gh \}_J = \{ f, g \}_J h + g \{ f, h \}_J + gh(Ef).
\]

(3.3)

In other words, the Jacobi bracket endows the algebra of functions \( \mathcal{F}(M) \) with the structure of a Lie algebra, but, unlike the Poisson bracket, it is not a derivation of the point-wise
product among functions. Clearly, Jacobi brackets are a generalisation of Poisson brackets since the latter can be obtained from the former if the Reeb vector field is vanishing, $E = 0$.

Analogously to the Poisson framework, a Hamiltonian vector field can be associated with a function $f \in \mathcal{F}(M)$, according to the following definition (see for example [20]):

$$\xi_f = \Lambda(df, \cdot) + fE.$$  \hspace{1cm} (3.4)

The map $f \rightarrow \xi_f$ is homomorphism of Lie algebras, it being $[\xi_f, \xi_g] = \xi_{\{f,g\}_J}$, where the bracket $[\cdot, \cdot]$ is the standard Lie bracket of vector fields.

Examples of Jacobi manifolds are locally conformal symplectic manifolds and contact manifolds. The former ones are even-dimensional manifolds endowed with a two-form $\omega$ and an open covering of charts $\{U_i\}$ such that locally the restriction $\omega|_{U_i} = e^{a_i} \Omega_i$, with $\Omega_i$ symplectic form on the chart $U_i$ and $a_i$ smooth functions on the local chart. Locally, they have then a Poisson structure $\{\cdot, \cdot\}_i$ but globally $e^{-a_i}\{e^{a_i}f, e^{a_i}g\}$ is a Jacobi bracket. More explicitly, and being this more useful for applications, one can define a locally conformal symplectic manifold by a pair $(\omega, \alpha)$, with $\omega$ a two-form with rank equal to the dimension of the manifold and $\alpha$ a one-form, such that

$$d\alpha = 0, \quad d\omega + \alpha \wedge \omega = 0.$$  \hspace{1cm} (3.5)

The Jacobi structure $(\Lambda, E)$ is thus defined as the unique bi-vector field and the unique vector field which satisfy:

$$\iota_E \omega = -\alpha, \quad \iota_{\Lambda(\gamma)} = -\gamma \quad \forall \gamma \in T^* M.$$  \hspace{1cm} (3.6)

Contact manifolds are instead odd-dimensional manifolds which are endowed with a one-form called contact form (or contact structure), i.e. a one-form satisfying $\vartheta \wedge (d\vartheta)^n \neq 0$ everywhere, where $2n + 1$ is the dimension of the manifold. This means that a one-form $\vartheta$ is a contact structure on an odd-dimensional manifold if $\vartheta \wedge (d\vartheta)^n$ is a volume form. Obviously, contact forms are defined up to multiplication by a never vanishing function. On a contact manifold one can define a Lie algebra structure on the space of functions as

$$\{f, g\} \vartheta \wedge (d\vartheta)^n = (n - 1)df \wedge dg \wedge \vartheta \wedge (d\vartheta)^{n-1} + (fdg - gdf) \wedge (d\vartheta)^n,$$  \hspace{1cm} (3.7)

which is local by construction and satisfies Jacobi identity. It is possible to show that this is actually a Jacobi bracket by defining $\Lambda$ and $E$ as follows:

$$\iota_E \vartheta \wedge (d\vartheta)^n = (d\vartheta)^n$$  \hspace{1cm} (3.8)

$$\iota_{\Lambda} \vartheta \wedge (d\vartheta)^n = n\vartheta \wedge (d\vartheta)^{n-1}.$$
These relations trivially imply that
\[ \iota_E \vartheta = 1, \quad \iota_E d\vartheta = 0, \quad (3.9) \]
as well as
\[ \iota_\Lambda \vartheta = 0, \quad \iota_\Lambda d\vartheta = 1. \quad (3.10) \]
An interesting property of a contact manifold is that its Poissonization is actually a
Symplectification, as will be further commented in next section.

Interesting examples of contact manifolds are three-dimensional semi-simple Lie groups,
where one of the basis left- or right-invariant one-forms can be chosen as a contact structure.
Especially interesting to us is the group \( SU(2) \), whose associated sigma models
have been widely studied. Besides being simple and fairly well behaved in many respects,
\( SU(2) \) is the prototypical example of a Poisson-Lie group. It has been investigated in
relation with Poisson sigma models in [7, 28]. Moreover, Poisson-Lie duality of the \( SU(2) \)
Principal Chiral model, with and without Wess-Zumino term, has been considered by the
authors in [29–31]. Therefore, we are interested in the possibility of generalising previous
results obtained in [29–31] to Jacobi sigma models on \( SU(2) \) and we will exhibit a
preliminary analysis in Section 5.

3.1.1 Homogeneous Poisson structure on \( M \times \mathbb{R} \) from Jacobi structure

The starting point for the subsequent analysis is provided by the following theorem [19]:

**Theorem 3.1.** \( J(f, g) = \Lambda(df, dg) + f(Eg) - g(Ef) \) defines a Jacobi structure on the
manifold \( M \) iff the bivector \( P \) defined as
\[
P \equiv \frac{1}{t} \Lambda + \frac{\partial}{\partial t} \wedge E, \quad t \in \mathbb{R}_+ \quad (3.11)
\]
is a Poisson structure on \( M \times \mathbb{R}_+ \).

Such a Poisson structure may be seen to be homogeneous, i.e. if the vector field
\( Z = t \frac{\partial}{\partial t} \) is introduced, then it is easy to show that \( P \) in (3.11) satisfies \( \mathcal{L}_Z P = -P \), being
the first term in (3.11) homogeneous of degree \(-1\) with respect to \( t \).

On performing the change of variables \( t = e^\tau \), the Poisson structure gets defined on
\( M \times \mathbb{R} \) as follows:
\[
P = e^{-\tau} \left( \Lambda + \frac{\partial}{\partial \tau} \wedge E \right), \quad (3.12)
\]
with \( Z = \frac{\partial}{\partial \tau} \), where \( \mathbb{R} \) denotes the \( \tau \)-axis. This redefinition will be particularly useful
for simplifying forthcoming computations. We will also consider the immersion \( j : M \hookrightarrow M \times \mathbb{R} \) through the identification of \( M \) with \( M \times \{0\} \).

The association of a Poisson structure on an extended manifold with a Jacobi structure
on the original manifold is usually referred to as Poissonization.
As it was already mentioned in the previous section, an interesting property of a contact manifold is that its Poissonization is actually a symplectic manifold, hence one could refer to it as a symplectification. Indeed, if $M$ is a contact manifold, one can define a non-degenerate closed 2-form $\omega$ on $M \times \mathbb{R}$ by using the contact form $\theta$: $\omega = d(e^\tau \pi^* \theta) = e^\tau (d\tau \land \pi^* \theta + d\pi^* \theta)$, where $\pi : M \times \mathbb{R} \to M$ is the projection map. This is obviously closed, and since $\tau \in \mathbb{R}$ is a zero-form, $\omega$ is a well-defined two-form. Because of the properties of $\theta$, it is possible to prove that $\omega$ is also non-degenerate, so it is a legitimate symplectic form and makes $(M \times \mathbb{R}, \omega)$ into a symplectic manifold.

### 3.2 Poisson sigma model on $M \times \mathbb{R}$

Let us consider an $m$-dimensional Jacobi manifold $(M, \Lambda, E)$ and a Poisson sigma model having the Poisson manifold $(M \times \mathbb{R}, P)$ as target space, with Poisson structure $P = e^{-X_0} \left( \Lambda + \frac{\partial}{\partial X_0} \land E \right)$. The field configurations in this case are maps $X^i = (X^i, X^0) : \Sigma \to M \times \mathbb{R}$ and $\eta \in \Omega^1(\Sigma, T^*(M \times \mathbb{R}))$, with $\eta_I = (\eta_i, \eta_0)$, where the capital indices $I, J = 0, \ldots, m$ are related to the Poisson manifold $M \times \mathbb{R}$, while $i, j = 1, \ldots, m$ are related to the Jacobi manifold $M$. The Poisson bi-vector field can be written explicitly in a coordinate basis as

$$P^{IJ} = e^{-X_0} \left( \Lambda^{ij} \begin{pmatrix} -E^1 \\ \vdots \\ -E^m \end{pmatrix} \right), \quad (3.13)$$

with $P = P^{IJ} \partial_I \land \partial_J$ and $E = E^i \partial_i$ (note that the Reeb vector field has only non-zero components on $M$).

The decomposition of the equations of motion, (2.2) and (2.3), then results in the following equations:

$$dX^i + e^{-X_0} \left( \Lambda^{ij} \eta_j - E^i \eta_0 \right) = 0, \quad (3.14)$$
$$dX^0 + e^{-X_0} E^i \eta_i = 0, \quad (3.15)$$
$$d\eta_i + \frac{1}{2} e^{-X_0} \partial_i \Lambda^{jk} \eta_j \land \eta_k + e^{-X_0} \partial_i E^j \eta_0 \land \eta_j = 0, \quad (3.16)$$
$$d\eta_0 - \frac{1}{2} e^{-X_0} \Lambda^{jk} \eta_j \land \eta_k - e^{-X_0} E^j \eta_0 \land \eta_j = 0. \quad (3.17)$$

Let us now project the dynamics to $M$ via projection map $\pi : M \times \mathbb{R} \to M$, namely by
considering $X_0 = \text{const}$. We find (by choosing $X_0 = 0$ for simplicity)

$$
\begin{align*}
  &dX^i + \Lambda^{ij}\eta_j - E^i\eta_0 = 0, \\
  &E^i\eta_i = 0, \\
  &d\eta_i + \frac{1}{2} \partial_i \Lambda^{jk}\eta_j \wedge \eta_k + \partial_i E^j\eta_0 \wedge \eta_j = 0, \\
  &d\eta_0 - \frac{1}{2} \Lambda^{jk}\eta_j \wedge \eta_k = 0.
\end{align*}
$$

(3.18)

One can notice that equation $E^i\eta_i = 0$ is purely algebraic, i.e. it is a constraint.

In next section we will show that it is possible to derive the projected dynamics (3.18) from an action principle, directly defined on the Jacobi manifold, in a consistent manner. We will thus analyse the space of solutions, the algebra of constraints and the gauge invariance of the model.

### 3.3 Action principle on the Jacobi manifold

Let $(M, \Lambda, E)$ be a Jacobi manifold, with $\Lambda \in \Gamma(\wedge^2 TM)$ and $E \in \Gamma(TM)$ satisfying

$$
[\Lambda, \Lambda]_S = 2E \wedge \Lambda, \quad [\Lambda, E]_S = \mathcal{L}_E \Lambda = 0.
$$

(3.19)

**Proposition 3.1.** The action functional

$$
S(X, \eta, \lambda) = \int_\Sigma \left[ \eta_i \wedge dX^i + \frac{1}{2} \Lambda^{ij}(X)\eta_i \wedge \eta_j - E^i(X)\eta_i \wedge \lambda \right]
$$

(3.20)

with field configurations $(X, \eta, \lambda)$, $X : \Sigma \to M$, $\eta \in \Omega^1(\Sigma, T^* M)$, $\lambda \in \Omega^1(\Sigma, \mathcal{F}(M))$, and boundary condition $\eta(v) = 0, v \in T(\partial \Sigma)$, defines a sigma model on the Jacobi manifold $M$, whose dynamics reproduces Eqs. (3.18).

Notice in particular the need for the auxiliary field $\lambda$ which is a one-form on $\Sigma$ but a function on $M$, to take into account the contribution of the Reeb vector field.

**Proof.** Prop. 3.1 is proven by direct derivation of the equations of motion. It is a straightforward calculation to get

$$
\begin{align*}
  &dX^i + \Lambda^{ij}\eta_j - E^i\lambda = 0, \\
  &d\eta_i + \frac{1}{2} \partial_i \Lambda^{jk}\eta_j \wedge \eta_k + \partial_i E^j\eta_0 \wedge \eta_j = 0, \\
  &E^i\eta_i = 0,
\end{align*}
$$

(3.21) (3.22) (3.23)

which are exactly the first three equations of (3.18), obtained from the reduction to $M$ of the Poisson sigma model on the extended manifold, provided that we identify $\eta_0$ with the pull-back of $\lambda$: $\eta_0 = \pi^*\lambda$.  

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The forth apparently missing equation in (3.18) is retrieved by a consistency require-
ment. On applying the exterior derivative to Eq. (3.21), one is left with
\[ \Lambda^{ij} d\eta_j - E^i d\lambda = 0. \] (3.24)
By substituting Eq. (3.22) into Eq. (3.24), using the constraint represented by Eq. (3.23)
and the definition of the Schouten bracket \([\Lambda, \Lambda] = 2\Lambda^{ij} \partial_j \Lambda^{kl} \partial_l \partial_k \wedge \partial_t,\) we find
\[ \frac{1}{4} \eta_j \wedge \eta_k \left( [\Lambda, \Lambda]_S \right)^{ijk} - E^i d\lambda = 0. \] (3.25)
Since the target space is a Jacobi manifold, we have \([\Lambda, \Lambda]_S = 2E^i \Lambda,\) from which
\[ d\lambda = \frac{1}{2} \Lambda^{ij} \eta_i \wedge \eta_j \] (3.26)
is obtained.

3.3.1 Hamiltonian description, constraints and gauge transformations

For the Hamiltonian formulation we follow the same approach as for the Poisson sigma
model. We choose \(\Sigma\) with the topology of \(\mathbb{R} \times \mathcal{I},\) with \(\mathcal{I} = [0,1],\) and pose \(\beta_i = \eta_{lt},\)
\(\zeta_i = \eta_{lu}, \ X^i = \partial_t X^i, \ X' = \partial_u X^i,\) so that the Lagrangian of the action (3.20) becomes
\[ L(X, \zeta; \beta; \lambda) = \int \! du \left[ -\dot{X}^i \zeta_i + \beta_i \left( X'^i + \Lambda^{ij} \zeta_j - E^i \lambda_u \right) + \lambda_t \left( E^i \zeta_i \right) \right], \] (3.27)
where \(\lambda_t\) and \(\lambda_u\) are the components of the one-form \(\lambda = \lambda_t dt + \lambda_u du.\) Explicitly, the
equations of motion read
\[ \dot{X}^i = -\Lambda^{ij} \beta_j + E^i \lambda_t \]
\[ \dot{\zeta}_i = \beta'_i + \partial_i \Lambda^{jk} \beta_j \zeta_k + \partial_u E^j \zeta_j \lambda_t. \] (3.28)
It is evident that \(X\) and \(\zeta\) are canonically conjugated variables, i.e.
\[ \{\zeta_i(u), X^j(v)\} = \delta^i_j \delta(u - v) \] (3.29)
with all other brackets vanishing, while \(\beta\) and \(\lambda_t\) are Lagrange multipliers imposing the
constraints
\[ C_1^i: = X'^i + \Lambda^{ij} \zeta_j - E^i \lambda_u = 0 \]
\[ C_2: = \zeta_i E^i = 0. \] (3.30)
Note that both the time components of \(\eta\) and \(\lambda\) have the role of Lagrangian multipliers
while the space components are possibly dynamical fields. As already specified, \(\lambda_t, \beta_i,\) are
smooth functions on $M$. The Hamiltonian function can be computed, yielding

$$H_{\beta,\lambda} = \int_I du \left[ \beta_i \left( X^i + \Lambda^{ij} \zeta_j - E^j \lambda_u \right) + \lambda_t \left( \zeta_i E^i \right) \right] = \int_I du \left( \beta_i \mathcal{C}_i^1 + \lambda_0 \mathcal{C}_2 \right)$$

(3.31)

namely, is itself a pure constraint, it being a combination of secondary constraints, $\mathcal{C}_1^1, \mathcal{C}_2$. However, the Lagrange multipliers $\beta$ and $\lambda_t$ are not really independent: by using the condition $d\lambda = \frac{1}{2} \Lambda \eta \wedge \eta$ we obtain the following relations:

$$\beta_i = \partial_i \lambda_t,$$  \hspace{1cm} (3.32)

$$\lambda_u = 0.$$ \hspace{1cm} (3.33)

Therefore, we can use Eq. (3.32) to rewrite the Hamiltonian in terms of $\lambda_t$ only:

$$H_\lambda = \int_I du \left[ \partial_i \lambda_t \left( X^i + \Lambda^{ij} \zeta_j \right) + \lambda_t \left( \zeta_i E^i \right) \right].$$

(3.34)

It is straightforward to show that $\mathcal{C}_1^1, \mathcal{C}_2$ are first class constraints, i.e.

$$\{ \mathcal{C}_1(u), \mathcal{C}_2(v) \} = 0,$$ \hspace{1cm} (3.35)

and

$$\{ \mathcal{C}_1^i(u), H_\lambda \} = 0, \quad \{ \mathcal{C}_2(u), H_\lambda \} = 0.$$ \hspace{1cm} (3.36)

In fact, let

$$\delta_{\xi_{H_\lambda}} X^i := \{ H_\lambda, X^i \} = -\Lambda^{ij} \partial_j \lambda_t + E^i \lambda_t$$ \hspace{1cm} (3.37)

$$\delta_{\xi_{H_\lambda}} \zeta_i := \{ H_\lambda, \zeta_i \} = (\partial_i \lambda_t)' + \partial_i \Lambda^{jk} \partial_j \lambda_t \zeta_k + \partial_i E^j \zeta_j \lambda_t$$ \hspace{1cm} (3.38)

denote the components of the Hamiltonian vector field $\xi_{H_\lambda}$. We have

$$\delta_{\xi_{H_\lambda}} \mathcal{C}_1^i = \delta_{\xi_{H_\lambda}} \left( X^i + \Lambda^{ij} \zeta_j \right) = \frac{1}{2} [\Lambda, \Lambda]_{ij} \partial_j \lambda_t \zeta_k + (\mathcal{L}_{E^j} \Lambda)^{ij} \partial_j \lambda_t \zeta_j + E_i (\lambda_t)' = 0$$ \hspace{1cm} (3.39)

it being $[\Lambda, \Lambda]_S = 2 E \wedge \Lambda$, $\mathcal{L}_{E^j} \Lambda = 0$ and $(\lambda_t)' = \partial_j \lambda_t X^j = -\Lambda^{jk} \partial_j \lambda_t \zeta_k$. The same arguments can be used to check that

$$\delta_{\xi_{H_\lambda}} \mathcal{C}_2 = \delta_{\xi_{H_\lambda}} (\zeta_i E^i) = (\mathcal{L}_{E^j} \Lambda)^{ij} \partial_j \lambda_t \zeta_j = 0$$ \hspace{1cm} (3.40)

Therefore, analogously to the Poisson sigma model, the reduced phase space $\mathcal{G}$ is represented by $\mathcal{C}$ modulo the symmetries (3.37)-(3.38), with $\mathcal{C}$ the space of solutions of the constraint equations, as before.

Let us prove the following

**Theorem 3.2.**
(i) For each $\lambda \in \Omega^1(\Sigma, \mathcal{F}(M))$ there exists a Hamiltonian vector field $\xi_{H,\lambda}$ associated with $H_\lambda$ such that its projection onto $M$, $\xi_\lambda$, is the Hamiltonian vector field associated to $\lambda_t$ through Jacobi bracket. $\xi_\lambda$ are infinitesimal generators of space-time diffeomorphisms.

(ii) The Jacobi sigma model is gauge-invariant under space-time diffeomorphisms, which close under Jacobi bracket.

(iii) The map $H_\lambda \rightarrow \xi_\lambda$ is a Lie-algebra homomorphism

$$\{ H_\lambda, H_\beta \} = H_{\{ \lambda_t, \beta_t \}}.$$  

where $\{,\}$ is the canonical Poisson bracket on $T^*PM$, $\{,\}_J$ is the Jacobi bracket on $PM$ (with $PM \ni X$ the configuration space).

Proof. According to the definition (3.4), with any smooth function on $M$, it is possible to associate a vector field through the Jacobi structure, which, because of its properties has the right to be called Hamiltonian, represented by

$$\xi_f = (\Lambda^i \partial_i f + f E^i) \frac{\partial}{\partial X^i}. \tag{3.42}$$

Let us apply the definition to $\lambda_t$. We find

$$\xi_{\lambda_t} = (\Lambda^i \partial_i \lambda_t + \lambda_t E^i) \frac{\partial}{\partial X^i}. \tag{3.43}$$

From the equations of motion we have $\Lambda^i \partial_j \lambda_t + \lambda_t E^i = \dot{X}^i$, which implies that

$$\xi_{\lambda_t} = \dot{X}^i \frac{\partial}{\partial X^i}. \tag{3.44}$$

The latter is the projection onto $M$ of the Hamiltonian vector field associated with the Hamiltonian (3.34) through the canonical Poisson bracket defined on $T^*\Sigma$, (3.29)

$$\xi_H = \dot{X}^i \frac{\partial}{\partial X^i} + \dot{\xi}_i \frac{\partial}{\partial \xi_i} \tag{3.45}$$

to wit

$$\xi_{\lambda_t} = \pi^* \xi_H. \tag{3.46}$$

But the Hamiltonian is a pure constraint, therefore the model is invariant under time diffeomorphisms generated by $\xi_{\lambda_t}$. It is also invariant under space diffeomorphisms $f(u)\partial/\partial u$, provided $f(u)$ is chosen appropriately, as it has been done for the Poisson sigma model. These infinitesimal symmetries are gauge transformations, being the Hamiltonian first class. Moreover, by definition

$$[\xi_{\lambda_t}, \xi_{\lambda}]=\xi_{\{\lambda_t, \lambda\}_J}. \tag{3.47}$$
Interestingly, we can reverse the argument given above to derive the relation between the Lagrange multipliers $\beta$ and $\lambda$, by the request that the Hamiltonian does generate gauge transformations, as it is for example the case for Yang-Mills theories. To this, let us consider the Hamiltonian

$$H_{\beta,\lambda_t} = \int_j du \left[ \beta_i C'_i(u) + \lambda_t C_2(u) \right]$$

(3.48)

where we suppose $\beta$ and $\lambda$ to be independent. The Hamiltonian vector field associated with it through the canonical brackets (3.29) reads

$$\xi_{\beta,\lambda_t} = \dot{X}^i(\beta, \lambda_t) \frac{\partial}{\partial X^i} + \dot{\lambda}(\beta, \lambda_t) \frac{\partial}{\partial \lambda}.$$  

(3.49)

In order for it to generate gauge transformations of the model, given the projection map $\pi: T^* M \to M$, the projected vector field $\pi^* \xi_{\beta,\lambda_t} = \dot{X}^i(\beta, \lambda_t) \frac{\partial}{\partial X^i}$ has to be a diffeomorphism on $M$. This imposes that $\beta_t$ be equal to $\partial_t \lambda_t$, namely, $\pi^* \xi_{\beta,\lambda_t}$ has to be the Hamiltonian vector field associated with $\lambda_t \in \mathcal{F}(M)$ via the Jacobi bracket.

Finally, let us prove the last statement of Prop. 3.2 by direct calculation. We have

$$\{H_{\lambda}, H_{\bar{\lambda}}\} = \int du du' \mathcal{L}_{\pi^* \xi_{\lambda}} H_{\bar{\lambda}} = \int du du' \left[ C'_i \mathcal{L}_{\xi_{\lambda}} (\partial_t \bar{\lambda}_t) + C_2 \mathcal{L}_{\xi_{\lambda}} \bar{\lambda}_t \right]$$

$$= \int du du' \left[ C'_i \partial_t \{\lambda_t, \bar{\lambda}_t\} J + C_2 \{\lambda_t, \bar{\lambda}_t\} J \right] = H\{\lambda_t, \bar{\lambda}_t\} J$$

(3.50)

which is what we wanted to prove. \qed

To summarise the results, the reduced phase space of the model is $G = C/\text{Diff}(\Sigma)$, where $C$ indicates the space of solutions of Eqs. (3.28) while $\text{Diff}(\Sigma)$ is the gauge group of diffeomorphisms generated by the Hamiltonian, which are in turn associated with the Jacobi structure through Eqs. (3.43), (3.3.1).

### 4 Metric extension and Polyakov action

We will show in this section that, just like in the Poisson sigma model case [16], the topological model considered so far can be generalised into a non-topological model by introducing a dynamical term containing the metric of the worldsheet (via the Hodge star operator on $\Sigma$) and a metric tensor $G$ for the target space:

$$S(X, \eta, \lambda) = \int_\Sigma \left[ \eta_i \wedge dX^i + \frac{1}{2} \Lambda^{ij}(X) \eta_i \wedge \eta_j - E^i(X) \eta_i \wedge \lambda + \frac{1}{2} (G^{-1})^{ij}(X) \eta_i \wedge *\eta_j \right].$$  

(4.1)

We are now concerned with the integration of the auxiliary fields ($\eta$ and $\lambda$) to obtain a Polyakov action for the embedding maps $X$. To do this, we first write the new equations
of motion following by the introduction of the new metric term:
\[
dX^i + \Lambda^{ij} \eta_j - E^i \lambda + (G^{-1})^{ij} \eta_j = 0,
\]

\[
d\eta_i + \frac{1}{2} \partial_i \Lambda^{jk} \eta_j \wedge \eta_k - \partial_i E^j \eta_j \wedge \lambda + \frac{1}{2} \partial_i (G^{-1})^{jk} \eta_j \wedge \star \eta_k = 0,
\]

\[
E^i \eta_i = 0.
\]

(4.2)  
(4.3)  
(4.4)

Thanks to the new term and the fact that the metric tensor \( G \) is naturally non-degenerate, the equation for \( \eta \) can be extracted from Eq. (4.2):
\[
\star \eta_j = -G_{ij} \left( dX^i + \Lambda^{ik} \eta_k - E^i \lambda \right).
\]

(4.5)

On applying again the Hodge star operator (we choose the metric signature \((1, -1)\) for \( \Sigma \), so in this case \( \star^2 = 1 \)) and substituting back the expression (4.5) for \( \star \eta \) we have
\[
\eta_p = -(M^{-1})^j_p G_{ij} \left( \star dX^i - \Lambda^{ik} G_{\ell k} dX^\ell + \Lambda^{ik} G_{\ell k} E^\ell \lambda - E^i \star \lambda \right),
\]

(4.6)

where we defined the matrix \( M^p_j = \delta^p_j - G_{ji} \Lambda^{ik} G_{\ell k} \Lambda^{lp} \), which is symmetric and assumed to be non-degenerate, without any assumption on the non-degeneracy of the \( \Lambda \) bivector.

Remarkably, by substituting the expression for \( \star \eta \) into the term \( \frac{1}{2} (G^{-1})^{ij} \eta_i \wedge \star \eta_j \), the action acquires the simple form
\[
S = \frac{1}{2} \int_\Sigma \eta_i \wedge dX^i,
\]

(4.7)

where we also used the fact that on-shell \( E^i \eta_i = 0 \). Replacing the explicit expression for \( \eta \), Eq. (4.6), in the action we obtain
\[
S(X, \lambda) = \int_\Sigma \left[ \frac{1}{2} (M^{-1})^p_j G_{jp} dX^i \wedge \star dX^j - \frac{1}{2} (M^{-1})^p_j G_{\ell p} \Lambda^{\ell k} G_{jk} dX^i \wedge dX^j - \frac{1}{2} (M^{-1})^p_j G_{\ell p} \Lambda^{\ell k} G_{mk} E^m \lambda \wedge dX^i + \frac{1}{2} (M^{-1})^p_j G_{\ell p} E^\ell \star \lambda \wedge dX^i \right].
\]

(4.8)

There is still \( \lambda \) to be integrated out. This can be achieved by recognising \( \int_\Sigma \star \lambda \wedge dX \) as the scalar product on the space of 1-forms so that \( \int_\Sigma \star \lambda \wedge dX = - \int \lambda \wedge \star dX \). Thus the last two terms in Eq. (4.8) are proportional to \( \lambda \). The latter acting as a Lagrange multiplier, imposes the constraint
\[
(M^{-1})^{\ell \ell} \left( \Lambda^{\ell k} G_{mk} E^m dX^i + E^\ell \star dX^i \right) = 0
\]

(4.9)

where we used the metric tensor \( G \) to lower and raise the target space indices. This means that on-shell the term proportional to \( \lambda \) vanishes and what remains is the second order
action

\[ S = \int_{\Sigma} \left[ g_{ij} dX^i \wedge \ast dX^j + B_{ij} dX^i \wedge dX^j \right] \]  \hspace{1cm} (4.10)

with metric and $B$-field given by:

\[ g_{ij} = G_{jp} (M^{-1})^p_i, \quad B_{ij} = G_{ik} (M^{-1})^p_j G_{p\ell} \Lambda^{\ell k}. \]  \hspace{1cm} (4.11)

Eq. (4.10) represents a Polyakov string action with target space a Jacobi manifold, with the Jacobi structures hidden in the metric and $B$-field. Note that the Reeb vector field $E$ plays no role in the definition of $g$ and $B$ but it is present in the constraint (4.9).

5 An example: $SU(2)$

In this section we consider the group manifold of $SU(2)$ as target space. It provides an example of a contact manifold where the contact structure can be taken to be one of the left-invariant basis one forms of the group, say $\theta^i$ so that $\ell^{-1} d\ell = \theta^i e_i \in \Omega^1(SU(2), \mathfrak{su}(2))$, is the Maurer-Cartan left-invariant one form on the group, with $\ell \in SU(2), e_i$ the Lie algebra generators and choose, to be definite, $\vartheta = \theta^3$ as contact structure for $SU(2)$ (right invariant one-forms could be used equivalently). It is easily checked that it satisfies the conditions (3.8)-(3.10). Indeed, the Maurer-Cartan equation $d\theta^i = \frac{1}{2} \epsilon^k_{ij} \theta^i \wedge \theta^j$ leads to

\[ d\vartheta = \theta^1 \wedge \theta^2, \]  \hspace{1cm} (5.1)

so that $\vartheta \wedge d\vartheta = \theta^1 \wedge \theta^2 \wedge \theta^3 = \text{Vol}_{S^3}$.

From Eqs. (3.8) we get for the Jacobi structure

\[ \Lambda = Y_1 \wedge Y_2, \quad E = Y_3, \]  \hspace{1cm} (5.2)

where $Y_i$, with $i \in \{1, 2, 3\}$ are the left-invariant vector fields on $SU(2)$.

In order to define the fields $X^i, \dot{X}^i, X^h$ in a chart independent way, we resort to the group valued map $g : \Sigma \to SU(2)$, and the pull-back map $g^* : \Omega^1(SU(2)) \to \Omega^1(\Sigma)$, so to get

\[ g^*(g^{-1} dg) = (g^{-1} \partial_t g) dt + (g^{-1} \partial_u g) du \]  \hspace{1cm} (5.3)

which is a one-form on $\Sigma$, valued in the Lie algebra of $SU(2)$. We shall omit the pull-back from now on, but it will be always understood, unless otherwise stated. The action (3.20) may thus be written as follows

\[ S(g, \eta, \lambda) = \int_{\Sigma} \left[ \eta_i \wedge (g^{-1} dg)^i + \frac{1}{2} \epsilon^{3ij} \eta_i \wedge \eta_j - \eta_3 \wedge \lambda \right] \]  \hspace{1cm} (5.4)

where $g^{-1} dg = (g^{-1} dg)^i e_i$ is Lie algebra valued, while $\eta = \eta_i e^{i*}$ is valued in the dual of

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the Lie algebra and \( e^{i*}(e_j) = \delta^i_j \).

The equations of motion acquire the form

\[
(g^{-1}dg)^i + \epsilon^{3ij} \eta_j - \delta^{3i} \lambda = 0 \tag{5.5}
\]

and

\[
d\eta_i = 0, \quad \eta_3 = 0. \tag{5.6}
\]

Differently from Poisson sigma models, despite the fact that \( \Lambda \) is degenerate, the field \( \eta \) can be integrated out from the action according to the following procedure, which is valid for any contact manifold. By using Eqs. (3.9)-(3.10) we can contract the equation of motion in Eq. (3.21) with \( \vartheta = \vartheta_i \theta^i \) to obtain

\[
\langle \vartheta, g^{-1}dg \rangle - \lambda \langle \partial, \eta \rangle = \langle \vartheta_i(g^{-1}dg)^i - \lambda = 0, \tag{5.7}
\]

with \( \langle \cdot, \cdot \rangle \) the natural pairing between \( T^*M \) and \( TM \), so that \( \lambda = \vartheta_i(g^{-1}dg)^i \) can be integrated out. To integrate the fields \( \eta \) we can contract again Eq. (3.21) with \( d\vartheta \), and using again Eqs. (3.9)-(3.10) we obtain

\[
- (d\vartheta)_{ij}(g^{-1}dg)^j + \eta_i = 0, \tag{5.8}
\]

so that \( \eta_i = (d\vartheta)_{ij}(g^{-1}dg)^j \) can be integrated out as well. Substituting the expressions for \( \lambda \) and \( \eta \) in the action in Eq. (3.20), we obtain the second order action

\[
S_2 = -\frac{1}{2} \int_\Sigma \langle d\vartheta, (g^{-1}dg) \wedge (g^{-1}dg) \rangle, \tag{5.9}
\]

which has the same form as an A-model. In particular, \( d\vartheta \) has the role of a \( B \)-field, and in this case it is closed. In particular, by writing \( d\vartheta \) explicitly and by further introducing the notation

\[
(g^{-1}\partial_t g) = A^i e_i, \quad (g^{-1}\partial_u g) = J^i e_i \tag{5.10}
\]

with \( (A^i, J^i) \) the currents of the sigma model, we have

\[
S_2 = -\frac{1}{2} \int_\Sigma \epsilon_{3ij}(g^{-1}dg)^i \wedge (g^{-1}dg)^j = \int_\Sigma d^2 u \epsilon_{3ij} A^i J^j. \tag{5.11}
\]

It is also interesting to specialise the Polyakov action obtained in Eq. (4.10) to the \( SU(2) \) target manifold by introducing the natural Cartan-Killing metric on the latter: \( G_{ij} = \delta_{ij} \). By using \( G_{ij} = \delta_{ij} \) and \( \Lambda^{ij} = \epsilon^{3ij} \), the background metric and \( B \)-field are then obtained as

\[
g_{ij} = (h^{-1})_{ij}, \quad B_{ij} = -\frac{1}{2} \epsilon_{3ij}, \tag{5.12}
\]
so to have
\[
S = \int \Sigma \left[ (h^{-1})_i^j (g^{-1}dg)^i \wedge \star (g^{-1}dg)^j - \frac{1}{2} \epsilon_{3ij} (g^{-1}dg)^i \wedge (g^{-1}dg)^j \right],
\]
complemented with the constraint
\[
(g^{-1}dg)^3 = 0,
\]
which follows from Eq. (4.9).

The metric \( h \) is defined as \( h^{ij} = \delta^{ij} + \epsilon^{i3} \delta_{pq} \epsilon^{j3} \). It is interesting to note that this metric \( h \) was obtained from the authors in previous works on Poisson-Lie symmetry of sigma models [29–33] as the restriction to the \( \mathfrak{sb}(2, \mathbb{C}) \) algebra of a kind of generalised metric of \( SL(2, \mathbb{C}) \). The Lie group \( SB(2, \mathbb{C}) \) plays the role of the Poisson-Lie dual partner of \( SU(2) \) in the Drinfeld double decomposition of \( SL(2, \mathbb{C}) \). Therefore, it would be interesting to further explore the reasons behind the presence of this particular metric, which is not clear yet, especially in view of future applications in Poisson-Lie T-duality.

6 Conclusions and Outlook

In this paper we have constructed a Jacobi sigma model, which is a two-dimensional sigma model with target space a Jacobi manifold, as a natural generalisation of Poisson sigma models. In particular, we started from the concept of Poissonization of a Jacobi manifold, which consists in the construction of a homogeneous Poisson structure on the extended manifold \( M \times \mathbb{R} \) from a Jacobi structure on \( M \). We projected the dynamics of this extended Poisson sigma model on the Jacobi manifold \( M \) and then formulated a new sigma model action having \( M \) as target space which reproduces the projected dynamics. This is outlined in the following summary diagram

\[
\begin{array}{ccc}
(M, \Lambda, E) \xrightarrow{\text{Poissonization}} (M \times \mathbb{R}, P) \\
\pi \\
(M, \Lambda, E) \xrightarrow{\text{M-projected model}} 
\end{array}
\]

We have analysed the Hamiltonian formulation of the model, which exhibits first class constraints generating gauge transformations. In particular, we have shown that by using the definition of Hamiltonian vector field associated with a Jacobi structure, these vector fields can be associated with gauge transformations generating space-time diffeomorphisms, and the model is topological.
We also investigated the possibility to include a metric term in the action, resulting in a non-topological sigma model in which the auxiliary fields can be integrated out to give a Polyakov action, where the background metric and $B$-field are related to the defining structures of the target Jacobi manifold.

In particular, we analysed $SU(2)$ as an example of contact target manifold in view of its relation with Poisson-Lie symmetry and T-duality. The action that is obtained is of the A-model type.

Issues such as quantisation, integrability and T-duality of the Jacobi model represent interesting directions of research, some of which are presently under investigation.

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