Interlacing and Scaling Exponents for the Geodesic Watermelon in Last Passage Percolation

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Abstract: In a discrete planar last passage percolation (LPP), random values are assigned independently to each vertex in $\mathbb{Z}^2$, and each finite upright path in $\mathbb{Z}^2$ is ascribed the weight given by the sum of values attached to the vertices of the path. The weight of a collection of disjoint paths is the sum of its members’ weights. The notion of a geodesic, namely a path of maximum weight between two vertices, has a natural generalization concerning several disjoint paths. Indeed, a $k$-geodesic watermelon in $[1, n]^2 \cap \mathbb{Z}^2$ is a collection of $k$ disjoint upright paths contained in this square that has maximum weight among all such collections. While the weights of such collections are known to be important objects, the maximizing paths have remained largely unexplored beyond the $k = 1$ case. For exactly solvable models, such as exponential and geometric LPP, it is well known that for $k = 1$ the exponents that govern fluctuation in weight and transversal distance are $1/3$ and $2/3$; which is to say, the weight of the geodesic on the route $(1, 1) \rightarrow (n, n)$ typically fluctuates around a dominant linear growth of the form $\mu n$ by the order of $n^{1/3}$; and the maximum Euclidean distance of the geodesic from the diagonal typically has order $n^{2/3}$. Assuming a strong but local form of convexity and one-point moderate deviation estimates for the geodesic weight profile—which are available in all known exactly solvable models—we establish that, typically, the $k$-geodesic watermelon’s weight falls below $\mu nk$ by order $k^{5/3} n^{1/3}$, and its transversal fluctuation is of order $k^{1/3} n^{2/3}$. Our arguments crucially rely on, and develop, a remarkable deterministic interlacing property that the watermelons admit. Our methods also yield sharp rigidity estimates for naturally associated point processes. These bounds improve on estimates obtained by applying tools from the theory of determinantal point processes available in the integrable setting.

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1. Introduction and Main Results

In a discrete planar last passage percolation (LPP) model, each vertex in \( \mathbb{Z}^2 \) is independently ascribed a non-negative value sampled from a given law \( \nu \). An upright path in \( \mathbb{Z}^2 \) is a finite nearest-neighbour path on \( \mathbb{Z}^2 \) each of whose steps is upwards or to the right. To each such path a weight is ascribed by the LPP model, this being the sum of the values assigned to the vertices in \( \mathbb{Z}^2 \) visited by the path. For \( n \in \mathbb{N} \), let \( X_n \) denote the maximum weight assigned to an upright path that begins at the vertex \((1, 1)\) and that ends at the vertex \((n, n)\). Any upright path with these endpoints whose weight equals \( X_n \) is called a geodesic between \((1, 1)\) and \((n, n)\), and \( X_n \) is called the last passage time from \((1, 1)\) to \((n, n)\); these two concepts make sense for any pair of coordinate-wise ordered vertices.

Let the transversal fluctuation \( TF(n) \) denote the maximum anti-diagonal fluctuation of any geodesic between \((1, 1)\) and \((n, n)\). That is, \( TF(n) \) equals the minimum of positive real \( r \) such that every vertex of any geodesic between \((1, 1)\) and \((n, n)\) has Euclidean distance from the diagonal \( y = x \) at most \( r \). Planar LPP models are paradigmatic examples of models predicted to exhibit features of the Kardar-Parisi-Zhang (KPZ) universality class; in particular, the characteristic KPZ exponents of one-third and two-thirds for the weight and transversal fluctuations of geodesics.

A few planar LPP models—for which the law \( \nu \) has a special form—enjoy integrable properties that vastly facilitate their analysis. Two such models are geometric LPP with parameter \( p \in (0, 1) \), where \( \nu(k) = p^{k-1}(1 - p) \) for \( k \in \mathbb{N} \); and exponential LPP, where \( \nu \) is the exponential law. (By scaling properties of the exponential distribution, the rate of the exponential distribution is irrelevant, and we will consider exponentials of rate one.) For any non-negative \( \nu \), the existence of the limiting growth rate \( \mu \) for the weight maximum, specified as the almost sure limiting value of \( n^{-1}X_n \) as \( n \to \infty \), is a consequence of a straightforward superadditivity argument; the positivity of \( \mu \) holds as soon as \( \nu \) is not degenerate at 0, while the finiteness is guaranteed under mild moment assumptions. For geometric and exponential LPP, \( \mu \) can be explicitly evaluated. The integrable structure of these and a handful of other models has been crucial in proofs establishing the pair of KPZ scaling exponents.

The first model for which such a program was carried out was Poissonian last passage percolation on the plane, where the statistic of interest \( L_n \) is the maximum number of points on an upright path from \((0, 0)\) to \((n, n)\) in a rate one Poisson process on \( \mathbb{R}^2 \); conditionally on the total number of points in \([0, n]^2\), this is Ulam’s problem for the longest increasing subsequence in a uniform random permutation. In their seminal work
Fig. 1. Simulations of $k$-geodesic watermelons in Poissonian LPP; $k$ equals ten on the left and twenty on the right, with $n = 200$ in both. Grey are the points of the underlying Poisson process of intensity one on $[0, n]^2$; blue are the watermelon’s curves; and red is the diagonal $y = x$ around which the watermelon fluctuates on the scale $k^{1/3}n^{2/3}$.

[BDJ99], Baik, Deift, and Johansson showed that $n^{-1/3}(L_n - 2n)$ converges weakly to the GUE Tracy-Widom distribution, namely the high-$n$ limiting law of a scaled version of the largest eigenvalue of an $n \times n$ random matrix picked according to the Gaussian unitary ensemble. Shortly thereafter, Johansson [Joh00b] used this longitudinal fluctuation result to establish the transversal fluctuation exponent of two-thirds in this model, showing that with high probability the smallest strip around the diagonal containing any geodesic has width $n^{2/3+o(1)}$. Similar transversal fluctuation results have been proved for last passage percolation on $\mathbb{Z}^2$ with exponential and geometric passage times [BCS06]; and for the semi-discrete model of Brownian last passage percolation [Ham20].

The Baik-Deift-Johansson theorem was derived by noting that the random statistic $L_n$ has the distribution of the length of the top row of a Young tableau picked according to the Poissonized Plancherel measure of appropriate parameters. The latter observation was obtained via the Robinson-Schensted-Knuth correspondence in [Sch09,Knu70] and was first exploited in [LS77,VK77]. The correspondence extends to other rows of the tableau. Indeed, Greene [Gre82] established that the sum of the lengths of the first $k$ rows of the random Young tableau picked from the same measure has the distribution of the maximum number of Poisson points on $k$ upright paths from $(0, 0)$ to $(n, n)$ that are disjoint except at these shared endpoints. Baik, Deift and Johansson conjectured that the scaled lengths of the top $k$ rows converge jointly in distribution to the top $k$ points of a determinantal point process on $\mathbb{R}$ called the Airy point process. The conjecture was proved soon after, independently, by Borodin, Okounkov and Olshanski [BOO00]; Johansson [Joh01]; and Okounkov [Oko00]. By Greene’s theorem, then, the highest scaled weight of a set of $k$ disjoint upright paths in Poissonian LPP converges in law to the sum of the top $k$ points of the Airy point process. Other integrable LPP models exhibit a similar correspondence; we discuss it in the case of exponential LPP in Sect. 10.

We have recounted these fragments of KPZ history to advocate the conceptual importance of systems of $k$ upright disjoint paths with given endpoints that maximize collective weight. We call them $k$-geodesic watermelons and devote this article to a unified geometric treatment of them.
The parameter $k \in \mathbb{N}$ will be positive, with $k \leq n$. Define $X_n^k$ to be the maximum weight of any collection of $k$ disjoint upright paths contained in the square $[1, n]^2 \cap \mathbb{Z}^2$ which start in the bottom left corner (containing $(1, 1)$) and end in the top right corner (containing $(n, n)$); see Definition 2.1 ahead for a precise definition. Note that the collection attaining the maximum need not be unique, and that $X_n^1 = X_n$. The transversal fluctuation of any such collection of paths is the maximum Euclidean distance between a vertex in $\mathbb{Z}^2$ lying in the paths’ union and the diagonal $y = x$. We specify $\overline{TF}(n, k)$ to equal the maximum value of the transversal fluctuation over such sets of $k$ paths whose weight realises $X_n^k$, and $\underline{TF}(n, k)$ to be the minimum value of the transversal fluctuation over the same collection.

Our main result establishes the values of the exponents that govern weight and transversal fluctuations of $k$-geodesic watermelons in the context of geometric and exponential LPP. We note a simple heuristic to predict the exponents by considering the case that $k = n$, where the $k$-geodesic watermelon uses all the vertices in $[1, n]^2$: then by the strong law of large numbers, $X_n^k$ will be of order $\mathbb{E}(|\xi|) n^2$, with $\xi$ distributed according to $\nu$, and $\overline{TF}(n, k) = \overline{TF}(n) = \Theta(n)$. It is reasonable to believe that $X_n^k$ to first-order should be linear in $nk$, just as $X_n$ is to first-order linear in $n$. Then if we assume that the fluctuations are governed by exponents $\alpha$ and $\beta$ as $X_n^k = \mu nk - \Theta(k^\alpha n^{1/3})$ and $\overline{TF}(n) = \Theta(k^\beta n^{2/3})$, we obtain, from the $k = n$ case, the prediction that $\alpha = 5/3$ and $\beta = 1/3$.

The same predictions are obtained by considering the asymptotics of the Airy point process mentioned above. The $k^{th}$ point of this point process is asymptotically $-(3\pi/2)^{2/3}k^{2/3}$ [Sos00], which implies that the sum of the top $k$ points (to which $X_n^k$ corresponds in the limit of large $n$, as in the case of Poissonian LPP) is of order $-k^{5/3}$. From this, the prediction for the transversal fluctuation follows from the fact that paths with transversal fluctuations $x_n^{2/3}$ suffer a quadratic weight loss given by $x_n^{2/3}$ (see Theorem 3.3 ahead): if the weight loss of the extreme curves of the $k$-geodesic watermelon is of order $k^{-1} \cdot k^{5/3} n^{1/3} = k^{2/3} n^{1/3}$, it must be at distance $k^{1/3} n^{2/3}$ from the diagonal.

Our main result establishes these predictions with high probability, for all $k$ up to $n$.

**Theorem I.** Consider geometric LPP of given parameter $p \in (0, 1)$, or exponential LPP.

(1) Weight fluctuation: There exist positive constants $C_1$, $C_2$, $C$, and $c$ such that

$$\mathbb{P} \left( X_n^k - \mu nk \notin -k^{5/3} n^{1/3} \cdot (C_1, C_2) \right) \leq Ce^{-ck^2}$$

for $1 \leq k \leq n$, where we denote $a \cdot I = \{ ax : x \in I \}$ for $a \in \mathbb{R}$ and $I \subset \mathbb{R}$.

(2) Transversal fluctuation:

(a) There exist positive constants $M$, $C$, and $c$ such that, for $1 \leq k \leq n$,

$$\mathbb{P} \left( \overline{TF}(n, k) > Mk^{1/3} n^{2/3} \right) \leq Ce^{-ck}.$$

(b) A matching lower bound holds: there exist positive constants $C$, $c$, and $\delta$ such that, for $1 \leq k \leq n$,

$$\mathbb{P} \left( \underline{TF}(n, k) < \delta k^{1/3} n^{2/3} \right) \leq Ce^{-ck^2}.$$
Since the fundamental advances [BDJ99, Joh01], much analysis of integrable LPP models has been based on exact formulas for the point-to-point last passage time and for finite-dimensional distributions of the passage time profile from a point to certain special lines [Joh00a, BF08, BFS08]. These integrable techniques extend to the continuous scaling limits, such as the KPZ fixed point [MQR]. More recently, probabilistic and geometric technique in alliance with integrable input has been brought to bear on LPP problems. The Brownian Gibbs property, introduced in [CH14], is a resampling invariance enjoyed by the random ensemble of curves associated to Brownian LPP by the RSK correspondence. It has been used in [Ham20] to analyse the weight of the $k$-geodesic watermelon on the route from $(0,0)$ to $(n,n)$ in Brownian LPP, and in [CHH19] to gain strong control on LPP weight profiles and the Airy$_2$ process. The resampling property is also a central tool in the recent advance of [DOV], which, with the aid of [DV21], constructs the full scaling limit of Brownian LPP. Similar Gibbs properties in other models have been explored in works such as [CD18, CG20, Agg19, Wu19].

The present paper falls within the scope of a separate program of probabilistic and geometric inquiry into KPZ, focused on exponential and Poissonian LPP. By exploiting moderate deviation estimates from integrable probability and aspects of geodesic geometry, the slow bond problem was solved in the preprint [BSS14]. Developing this vein, [BSS19, HS20, BG21] and [Zha20] have offered information about coalescence structure of geodesics; their local fluctuations; and temporal correlation exponents. A combination of geometric and integrable methods, including the use of one-point estimates, has also been crucially used in [FO18, FO19] to establish universality of the GOE Tracy-Widom distribution in point-to-line LPP with general slope and time correlation exponents with generic initial conditions.

This paper pursues the preceding geometric and probabilistic program while adopting a novel geometric perspective: $k$-geodesic watermelons interlace, each with the next as the parameter $k \in \mathbb{N}$ rises; as we will explain in Sect. 2, this property is a tool that governs our ideas and the proofs of our results. The technique is robust. Although our main theorem addresses geometric and exponential LPP, its derivation makes very limited use of integrable inputs, holding sway under weak assumptions. Indeed, Theorem I follows directly from a more general result, Theorem II, valid under a rather natural set of assumptions that all known integrable models satisfy, which we state next.

1.1. Assumptions. Here we state a set of assumptions and our main result in its general form Theorem II. This form generalizes Theorem I because its hypotheses are the concerned assumptions, which we will show in Appendix A to be satisfied by exponential and geometric LPP.

We recall first that $\nu$ is the distribution of the vertex weights and has support contained in $[0, \infty)$. Consider next the limit shape defined by the map $[-1, 1] \to \mathbb{R} : x \mapsto \lim_{r \to \infty} r^{-1} E[X_{r(1-x), r(1+x)}, X_{r(1-x), r(1+x)}]$ is the last passage value from $(1, 1)$ to $(r(1-x), r(1+x))$. (Note then that $X_{r,r}$ coincides the existing usage $X_{r}$; we will write the latter in this special case.) Standard superadditivity arguments yield that the last limit in fact exists and that this map is concave. Recall also that $\mu = \lim_{r \to \infty} r^{-1} E[X_{r,r}]$ is this map evaluated at zero. Superadditivity also yields that $r^{-1} E[X_{r(1-x), r(1+x)}]$, for finite $r$, is at most the value of the limiting map at $x$. An important problem for general LPP models is to bound the non-random fluctuation given by the difference of these two quantities, for example, $\mu r - E[X_{r}]$ when $x = 0$, which is expected to be of fluctuation order $\sqrt{\text{Var}(X_{r})} = \Theta(r^{1/3})$. With this context, we state our assumptions. stop

(1) **Limit shape existence:** The distribution $\nu$ is such that $\mu < \infty$. 

(2) **Strong concavity of limit shape and non-random fluctuations:** There exist positive constants $\rho$, $G_1$, $G_2$, $g_1$, and $g_2$ such that, for large enough $r$ and $x \in [-\rho, \rho]$,

$$\mathbb{E}[X_r(1-x), r(1+x)] - \mu r \in [-G_1 x^2 r - g_1 r^{1/3}, -G_2 x^2 r - g_2 r^{1/3}].$$

(3) **Moderate and large deviation estimates, uniform in direction:**

(a) Fix any $\delta > 0$, and let $|x| \in [0, 1 - \delta]$. Then, there exist positive finite constants $c = c(\delta)$, $\theta_0 = \theta_0(\delta)$, and $r_0 = r_0(\delta)$ such that, for $r > r_0$ and $\theta > \theta_0$,

$$\mathbb{P}
\left(X_{r(1+x)}, r(1-x) - \mathbb{E}[X_{r(1+x)}, r(1-x)] > \theta r^{1/3}\right) \leq \exp(-c \min(\theta^{3/2}, \theta r^{1/3}))$$

and

$$\mathbb{P}
\left(X_{r(1+x)}, r(1-x) - \mathbb{E}[X_{r(1+x)}, r(1-x)] < -\theta r^{1/3}\right) \leq \exp(-c\theta^{3/2}).$$

(b) There exist convex functions $I_r : [0, \infty) \rightarrow \mathbb{R}$ for $r \in \mathbb{N}$ and a constant $c' > 0$ such that $I_r(\theta) \geq c' \min(\theta^{3/2}, \theta r^{1/3})$ for all $\theta \geq 0$, such that

$$\mathbb{P}(X_r - \mathbb{E}[X_r] > \theta r^{1/3}) \leq \exp(-I_r(\theta)).$$

We will call these, naturally enough, Assumptions 1, 2, and 3. They are expected to hold for a wide class of distributions $\nu$ and, in particular, are known to hold for the geometric and exponential cases. It is worth pointing out that a significant portion of our main result, Theorem II ahead, holds without Assumption 3b.

Assumption 2 encodes a non-trivial random fluctuation (the $\Theta(r^{1/3})$ translation) about a locally strongly concave limit shape (the $\mu r - \Theta(x^2 r)$ term). The assumption indicates that, even in the diagonal case, $X_r$ falls short in mean of the linear growth rate $\mu r$ by the order of typical fluctuation. This phenomenon is associated to the negativity of the mean of the GUE Tracy-Widom distribution. In regard to this assumption, we will say the endpoint of a path starting at $u \in \mathbb{Z}^2$ satisfies the “$\rho$-condition” if it lies in the interval joining $u + ((1 - \rho)r, (1 + \rho)r)$ and $u + ((1 + \rho)r, (1 - \rho)r)$.

Also note that Assumption 2 rules out trivial situations such as where the weights are almost surely constant—as then the limit shape would also be constant—in which case any set of $k$ paths with appropriate endpoints would be a $k$-geodesic watermelon, and we would not be able to make any non-trivial statement about transversal fluctuations.

It can be shown quite easily that Assumption 3a implies that $\nu$ has finite exponential moment, which in particular implies Assumption 1. Assumption 3a is itself expected to hold in a stronger form in general, with the lower tail bound with exponent 3 in place of the weaker 3/2 as we have assumed. Finally, Assumption 3b is a slightly stronger version of the upper tail bound of 3a when $x = 0$. The existence of the convex functions $I_r$ follows from the superadditivity of the sequence $\{X_r\}_{r \in \mathbb{N}}$ and is not actually an assumption, but the lower bound on $I_r$ is.

We are ready to state Theorem II. But first we remark that much recent progress in understanding the geometry of first passage percolation and other non-integrable models have been conditional results that hinged on similar assumptions on the limit shape and concentration estimates about the limit shape. This approach goes back to Newman’s work in the 90s (see e.g. [New95]) where geodesics and fluctuations in FPP were studied under curvature assumptions on the limit shape. More prominent recent examples include the work [Cha13] of Chatterjee where the KPZ relation between the weight and transversal fluctuation exponents were proved assuming in a strong form the existence of these exponents; see also [AD14]. The geometry of geodesics and bi-geodesics has been addressed under assumptions of strong convexity of the limit shape and concentration estimates about the limit shape. This approach goes back to Newman’s work in the 90s (see e.g. [New95]) where geodesics and fluctuations in FPP were studied under curvature assumptions on the limit shape. More prominent recent examples include the work [Cha13] of Chatterjee where the KPZ relation between the weight and transversal fluctuation exponents were proved assuming in a strong form the existence of these exponents; see also [AD14]. The geometry of geodesics and bi-geodesics has been addressed under assumptions of strong convexity of the limit shape.
shape [DH14] and moderate deviations around it [Ale20]. Similar results have also been obtained in exactly solvable cases where the essential integrable ingredients used were estimates analogous to the ones in the above assumptions. It is thus of much interest to extract the minimal set of assumptions under which one can establish sharp geometric results for LPP models.

**Theorem II.** Consider a last passage percolation model on $\mathbb{Z}^2$ that satisfies Assumptions 1, 2 and 3a.

1. **Weight fluctuation:** There exist positive constants $C_1, C_2, c_1$ and $c$ such that, for $1 \leq k \leq cn^{1/2}$,

$$\mathbb{P}\left(X_n^k - \mu n - k^{5/3}n^{1/3} \cdot (C_1, C_2)\right) \leq Ce^{-ck^2}.$$

2. **Transversal fluctuation:**
   - (a) There exist positive constants $M, C, c$ and $c_1$ such that, for $1 \leq k \leq cn^{1/2}$,
     $$\mathbb{P}\left(\text{TF}(n, k) > Mk^{1/2}n^{2/3}\right) \leq Ce^{-ck}.$$
   - (b) There exist positive constants $C, c, c_1$ and $\delta$ such that, for $1 \leq k \leq cn^{1/2}$,
     $$\mathbb{P}\left(\text{TF}(n, k) < \delta k^{1/2}n^{2/3}\right) \leq Ce^{-ck^2}.$$

If Assumption 3b also holds, then the above hold for $k \leq n$.

Note that Theorem II applies to two regimes of $k$ (either up to $n$ or order $n^{1/2}$) depending on whether Assumption 3b is in force or not. This will also be the case for many of the statements in this paper. The precise source of this is a technical statement ahead, Lemma 3.4.

**Remark 1.1.** An aspect of Theorem II(1), namely the bound $\mathbb{P}(X_n^k < \mu n - C_1k^{5/3}n^{1/3}) \leq e^{-ck^2}$, holds in the broader range $k \leq n$ without Assumption 3b: see Theorem 3.1.

A simple argument shows that the tail exponent for this probability is sharp; i.e.,

$$\mathbb{P}\left(X_n^k < \mu n - C_1k^{5/3}n^{1/3}\right) \geq e^{-ck^2}$$

for some constant $c > 0$. Indeed, the event that $X_n < \mu n - C_1k^{2/3}n^{1/3}$ implies that $X_n^k < \mu n - C_1(k^{2/3})n^{1/3}$ for any $k$, and $n$ correspondingly large enough. The preceding display follows from

$$\mathbb{P}\left(X_n < \mu n - C_1k^{2/3}n^{1/3}\right) = e^{-\Theta(k^2)},$$

a bound that is known, for example, in Exponential LPP: see [BGHK21, Theorem 2].

In fact, note that the above argument gives an upper bound $\mathbb{P}(X_n < \mu n - cn^{1/3}) \leq \exp(-c't^3)$, i.e, with exponent 3, though our assumption was with exponent 3/2. This sharp exponent can actually be obtained with an even weaker initial assumption, and this is done in the recent article [GH20], which makes use of the proof of the inequality $\mathbb{P}(X_n^k < \mu n - C_1k^{5/3}n^{1/3}) \leq \exp(-ck^2)$ given here.
Remark 1.2. A weaker form of Assumption 3a, namely that for some $\alpha < 1$, all $\delta > 0$, $|x| \in [0, 1 - \delta]$, and $\theta > \theta_0 = \theta_0(\delta)$,

$$\mathbb{P}\left(|X_{r(1+x), r(1-x)} - \mathbb{E}[X_{r(1+x), r(1-x)}]| > \theta r^{1/3}\right) \leq \exp(-c\theta^\alpha),$$

for some $c = c(\delta) > 0$, is enough to imply a variant of Theorem II where the tail probability exponents are suitable functions of $\alpha$; the $5/3$ and $1/3$ exponents for weight and transversal fluctuations are unchanged. It does not appear to us to be challenging to chase through our arguments to compute the forms of these upper bound tail exponents, although we do not do so.

Remark 1.3. Note that Theorems I and II (with Assumption 3b) are stated for $k \leq n$. In fact, it suffices to prove them for $k \leq c_1 n$ for any fixed constant $c_1 > 0$. This is since, for $c_1 n \leq k \leq n$, it holds that $k^{5/3} n^{1/3} = \Theta(n^2)$ and $k^{1/3} n^{2/3} = \Theta(n)$, along with the trivial observations that $X_n^k$ is at least 0 and at most the sum of $n^2$ i.i.d variables, while $\overline{\mathbb{F}}(n, k) \leq n$ and $\overline{\mathbb{F}}(n, k) \geq k$. In the remainder of the paper we will be concerned with the reduced regime $1 \leq k \leq c_1 n$ for a constant $c_1 > 0$ that will be specified later.

1.1.1. A non-determinantal setting: the point-to-line geodesic watermelon Though our main result Theorem II is stated for last passage percolation on $\mathbb{Z}^2$, our technique is robust, and an inviting prospect is to adapt our method to other integrable models such as the semi-discrete model of Brownian LPP or the continuum model of Poissonian LPP; the adaptations appear to be for the most part minor, but we have not pursued this direction carefully. However, all four examples—exponential and geometric LPP and these last two—are determinantal in the sense that there is an exact representation of the geodesic watermelon weight as the sum of the position of the top $k$ particles in a determinantal point process with explicit, albeit complex, formulas available for the distributions. We shall provide a more detailed discussion regarding the determinantal process connections for exponential and geometric LPP in Sect. 10. But to illustrate the power of our geometric methods we end by treating a particular “non-determinantal” setting: where although there exist explicit formulas for the one point distribution (the geodesic weight), the weight of the geodesic watermelon is not known to admit any connection to a determinantal process for $k > 1$.

Formally, consider point-to-line LPP with independent vertex weights such that Assumptions 1, 2, and 3 are satisfied. Let $\Upsilon_n^k$ denote a collection of $k$ disjoint paths contained in the triangular region

$$\{(x, y) \in \mathbb{Z}^2 : x \geq 1, y \geq 1, x + y \leq 2n\},$$

that maximizes the total weight among all such collection of $k$ disjoint paths. Let $Z_n^k$ denote the total weight of the paths in $\Upsilon_n^k$. Clearly $Z_n^1$ is the point-to-line last passage time from $(1, 1)$ to the line $x + y = 2n$. While it is known in exponential LPP that $Z_n^1$ has the same distribution as the largest eigenvalue of the Laguerre Orthogonal Ensemble (implicitly in the works [BR01b, Bai02, BR01a] with an explicit statement in [BGHK21]), as far as we are aware there does not exist any representation for $Z_n^k$ for $k > 1$ as a functional of a determinantal point process.

As before, we specify the transversal fluctuation of a collection of paths to be the maximum Euclidean distance between a vertex in $\mathbb{Z}^2$ lying in the paths’ union and the diagonal $y = x$. We specify $\overline{\mathbb{F}}^*(n, k)$ to equal the maximum value of the transversal
fluctuation over such sets of \( k \) paths whose weight realizes \( Z^k_{n^k} \) and \( \text{TF}^*(n, k) \) to be the minimum value of the transversal fluctuation over the same collection of sets.

The following is the analogue of Theorem I in this setting.

**Theorem III.** Consider a model, such as geometric or exponential LPP, that satisfies Assumptions 1, 2 and 3a. The statements in Theorem II that invoke Assumption 3a hold after the replacements \( X_k \rightarrow Z^k_{n^k} \), \( \text{TF}(n, k) \rightarrow \text{TF}^*(n, k) \) and \( \text{TF}(n, k) \rightarrow \text{TF}^*(n, k) \) are made. In particular, the results for point-to-line geodesic watermelons hold for \( k \leq c_1 n^{1/2} \) for an absolute constant \( c_1 > 0 \).

Note that we do not claim Theorem III for \( k \) up to \( \Theta(n) \); this is because we don’t have the analogue of Assumption 3b regarding the convexity of the exponent of the one-point tail probability for \( Z^k_{n^k} \). In fact, all our arguments for Theorem III use an estimate for the one-point tail probability of \( Z^k_{n^k} \) which is derived from only the assumed one-point information for \( X_n \) and the parabolic curvature of the limit shape, i.e., Assumptions 2 and 3a; see Proposition 3.6 ahead.

The next two sections develop an overview of the paper’s concepts and results. The vital phenomenon of interlacing of geodesic watermelons is surveyed in Sect. 2, with some of its main consequences being indicated. Section 3 offers an outline to the paper’s main proofs. We state some important technical results needed to obtain Theorem II; outline how these results are proved; and explain how they are used, alongside interlacing, to prove Theorem II. Section 3 ends with an indication of the structure of the later sections of the paper, which are devoted to giving the proofs.

2. Watermelon Interlacing

In this section, we will specify notation and important definitions concerning geodesic watermelons; state the important interlacing property that they enjoy; and state monotonicity and rigidity results for a natural point process associated to the weights of these watermelons. The results of this section do not rely on the assumptions introduced in Sect. 1.1, and, excepting Proposition 2.4, are deterministic.

2.1. The geodesic watermelon. Let \( \{ \xi_v : v \in \mathbb{Z}^2 \} \) denote a field of values. For now, we take these values to be deterministic non-negative reals. For \( a, b \in \mathbb{Z} \) with \( a \leq b \), we denote the integer interval \( \{ x \in \mathbb{Z} : a \leq x \leq b \} \) by \([a, b]\). We let \( \mathbb{N} = \{1, 2, \ldots\} \) denote the natural numbers (without zero).

Let two elements \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) of \( \mathbb{Z}^2 \) be such that \( u_1 \leq v_1 \) and \( u_2 \leq v_2 \). An upright path from \( u \) to \( v \) is a function \( \gamma : [a, b] \rightarrow \mathbb{Z}^2, \gamma_a = u \) and \( \gamma_b = v \), with each increment of \( \gamma \) equalling either \((0, 1)\) or \((1, 0)\); thus, \( a, b \in \mathbb{Z} \) satisfy \( b-a = (u_2-u_1)+(v_2-v_1) \). (We will sometimes omit ‘upright’: every path is upright.) The weight \( \ell(\gamma) \) of \( \gamma \) equals \( \sum \{ \xi_v : v \in \gamma \} \), where we have abused notation, in a way that we often will, by mistaking \( \gamma \) for its range. The weight of a disjoint collection of upright paths is the sum of the weights of the elements of the collection.

We will denote by \( X_{u \to v} \) the maximum weight of all upright paths from \( u \) to \( v \). Recall that, for \( r \in \mathbb{N} \), \( X_r \) is a shorthand for \( X_{r, r} \), itself shorthand for the last passage value \( X_{(1, 1) \to (r, r)} \) for the route from \((1, 1)\) to \((r, r)\).

Let \( k \in \mathbb{N} \) be positive. We may wish to specify the \( k \)-geodesic watermelon in the square \([1, n]^2\) as a maximum weight collection of \( k \) disjoint upright paths from \((1, 1)\) to \((n, n)\); but, naturally, we cannot, because such paths meet when they begin and end. The next definition succinctly deals with the need to unpick these points of contact.
Definition 2.1. Let \( n \in \mathbb{N} \) and \( k \in [1, n] \). A \( k \)-geodesic watermelon is a maximum weight collection of \( k \) disjoint upright paths in \([1, n]^{2}\) which has one curve moving from \((1, k - i + 1)\) to \((n, n - i + 1)\) for each \( i \in [1, k] \).

The nomenclature of watermelons is not new. It was introduced in the physics literature to denote certain ensembles of non-intersecting curves, such as non-intersecting Brownian bridges, whose curves bear a faint likeness to the stripes on the surface of watermelon fruit. These bridge systems arise, for example, when describing the weight of collections of disjoint paths in LPP models as the common endpoint of the collection varies. The name ‘geodesic watermelon’ distinguishes the denoted concept from the existing one, which we might call a weight watermelon. We will not allude to the latter, beyond mentioning that [JO20] treats weight watermelons related to the geometric RSK correspondence. We will sometimes write ‘\( k \)-melon’ as shorthand for ‘\( k \)-geodesic watermelon’.

The two quantities measuring transversal fluctuation of the \( k \)-geodesic watermelon, \( \overline{TF}(n, k) \) and \( \underline{TF}(n, k) \), were specified before Theorem I. They respectively equal the maximum and minimum, over the collection of \( k \)-geodesic watermelons, of the maximum Euclidean distance to the diagonal \( y = x \) of the vertices lying in some path belonging to that \( k \)-geodesic watermelon.

The set of vertices in a \( k \)-geodesic watermelon may not be unique; it is not in geometric LPP, for example. Further, for a given set of vertices which is a \( k \)-geodesic watermelon, the collection of constituent curves is not unique. It is easy to see that a sufficient condition for the geodesic watermelons to be almost surely unique is for \( \nu \) to have no atoms, as, for example, in the case of exponential LPP.

Theorem II concerns the vertex sets in geodesic watermelons considered simultaneously, rather than any particular watermelon. For this reason, we do not attempt to single out a melon when there are several distinct watermelon vertex sets. A subtlety regarding non-uniqueness and interlacing will however be addressed in the upcoming Sect. 2.2.

For any given set of vertices which form a geodesic watermelon, Definition 2.1 permits us to label the curves of the watermelon, which will aid us in stating the upcoming interlacing result. We wish to do this in a left-to-right manner, and to make this precise we next introduce a partial order on paths.

Definition 2.2. The time-range \( \mathcal{R}(\gamma) \) of an upright path \( \gamma \) is the interval of integers \( t \) such that \( x + y = t \) for some \((x, y) \in \gamma \). For \( t \in \mathcal{R}(\gamma) \), the point \((x, y) \) is unique, and we abuse notation by writing \( \gamma(t) = x - y \). (In contrast, subscripts were used to express \( \gamma \) as a function on an integer interval earlier in the section.)

Two upright paths \( \gamma \) and \( \phi \) satisfy \( \gamma \preceq \phi \) if the set \( \mathcal{R}(\gamma) \cap \mathcal{R}(\phi) \) is non-empty and every element \( t \) satisfies \( \gamma(t) \leq \phi(t) \). If each inequality is strict, we write \( \gamma \prec \phi \). Informally, we say that \( \gamma \) is to the left of \( \phi \).

A vector of upright paths is ordered if its component increase under \( \prec \), and weakly ordered if they do so under \( \preceq \). For a vector of upright paths \( \gamma \), note that if its components \( \gamma_i \) are disjoint and every pair has non-disjoint time ranges, then by planarity there is exactly one labelling of the paths which is ordered from left to right. In that case, we shall simply say the collection of paths is ordered.

With this definition, for any given \( k \)-geodesic watermelon \( \Gamma_n^k \), we record its elements as the components of the vector

\[
\Gamma_n^k = \left( \Gamma_{n,1}^{k,1}, \ldots, \Gamma_{n,k}^{k,k} \right),
\]
Each red curve remains within the region defined by adjacent blue curve boundaries. Observe that the red curves may overlap with the blue ones as shown between the top red and top blue curve; in fact, simulations suggest that a much more significant degree of overlap than depicted is typically present. Interlacing concerns consecutive values of $k$: the $k$- and $(k + 2)$-melons may cross, as the right figure illustrates for $k = 1$. The geodesic curve in green make uses of the overlapping purple portion of the 2- and 3-melons, while remaining in the envelope defined by the red curves. The middle blue curve cannot use the purple portion, and this causes the green curve to cross over the middle blue curve choosing the left-to-right order, so that $\Gamma_{n, \rightarrow}^{k, i} \prec \Gamma_{n, \rightarrow}^{k, i+1}$ for $i \in [1, k - 1]$; thus, $\Gamma_{n, \rightarrow}^{k, i}$ begins at $(1, k - i + 1)$ and ends at $(n, n - i + 1)$.

2.2. Interlacing. The basic property that drives our proofs is simple and intuitive. Consider exponential LPP, so that all $k$-geodesic watermelons are almost surely unique. The square $[1, n]^2$ is partitioned into two regions, NW and SE, by the geodesic $\Gamma_{n, \rightarrow}^1$, which lies in both. Then $\Gamma_{n, \rightarrow}^{2, 1}$ lies in NW and $\Gamma_{n, \rightarrow}^{2, 2}$ in SE. Three regions are specified by the boundaries $\Gamma_{n, \rightarrow}^{2, 1}$ and $\Gamma_{n, \rightarrow}^{2, 2}$ and the sides of the square, and these contain, one apiece, the three paths in the 3-geodesic watermelon: see Fig. 2 ahead. The partial order introduced in Definition 2.2 allows us to put interlacing on a precise footing.

**Definition 2.3.** A $k$-vector $\gamma$ and a $(k + 1)$-vector $\gamma'$ whose components are upright paths interlace when their components satisfy $\gamma'_i \preceq \gamma_i \preceq \gamma'_{i+1}$ for $i \in [1, k]$.

Figure 2 shows a pair of interlacing geodesic watermelons. We are ready to state our interlacing results, starting with the case where $\nu$ is continuous, so that we have a unique $k$-geodesic watermelon $\Gamma_{n, \rightarrow}^k$ for every $k \in [1, n]$.

**Proposition 2.4 (Geodesic watermelons interlace).** Suppose that the law $\nu$ is continuous. For $k \in [1, n - 1]$, the geodesic watermelons $\Gamma_{n, \rightarrow}^k$ and $\Gamma_{n, \rightarrow}^{k+1}$ interlace almost surely.

When the distribution $\nu$ has atoms, the set of vertices visited by a geodesic watermelon may be variable. Here is a deterministic interlacing result that is valid in this setting.

**Proposition 2.5.** For $n \in \mathbb{N}$, suppose that the values $\xi_v$ are non-negative for $v \in [1, n]^2$. Let $k \in [1, n]$, and let $\phi_{n, \rightarrow}^k$ be any $k$-geodesic watermelon. For $i \in [1, n]$, we may find an $i$-geodesic watermelon $\gamma_{n, \rightarrow}^i$ such that consecutive terms in the sequence $\{\gamma_{n, \rightarrow}^i : i \in [1, n]\}$
Fig. 3. A depiction of the argument for interlacing. Here the 2-geodesic watermelon $\gamma_2^n$ is blue, the 3-geodesic watermelon $\gamma_3^n$ is red, and they are assumed to be unique. Initially (first panel), they do not interlace, because the first two curves from the left of $\gamma_2^n$ cross those of $\gamma_3^n$. In the first step of the modification (second panel), we plan to switch the colours of the highlighted portions. However, doing so would result in blue curves intersecting, which is not allowed for the curves of geodesic watermelons. So we need to switch the colours of more curves to avoid this (third panel). The result of the modification is shown in the fourth panel. There is still overlap, but it is less than before (only one curve crosses now). However, we do not use this: instead, the weight maximization property of geodesic watermelons implies that the collections of blue curves and red curves in the fourth panel are respectively 2- and 3-geodesic watermelons, contradicting the uniqueness of $\gamma_2^n$ and $\gamma_3^n$.

interlace, with the union of the vertices in the curves in $\gamma_k^n$ coinciding with this union for $\phi_k^n$.

Here is the basic idea of the rather technical proofs of Propositions 2.4 and 2.5. Suppose the curves of the $k$-geodesic watermelons are unique for each $k \in [1, n]$ (a small random perturbation would ensure this), and that the $k$- and $(k+1)$-geodesic watermelons do not interlace. This means that two of the constituent curves must cross. Our argument will show that the crossing provides a global way of modifying the paths to obtain new $k$- and $(k+1)$-geodesic watermelons that defer from the earlier ones; this contradicts uniqueness. (The modified geodesic-watermelons will also in some sense be closer to interlacing, but we do not make use of this.) See Fig. 3.

Interlacing is hardly new. James Martin gave a proof of interlacing to the third author around 2009. The paper [FW98] proves a similar phenomenon for a last passage percolation model on $\mathbb{R}^2$ akin to Poissonian LPP; points are scattered in the plane, and upright paths consist of piecewise linear increasing functions. More precisely, [FW98] proves that the $k$-melon in a given set of points $A$ interlaces with the $(k+1)$-melon in the points defined by the skeleton of $A$, which is a sort of dual set of points to $A$. The skeleton of $A$ can look quite different from $A$, it does not appear that interlacing of $k$- and $(k+1)$-melons in the original set of points, i.e., the analogue of Proposition 2.4, is implied by this result.

2.3. Monotonicity of watermelon weight increments. A natural point process $\{Y_{n,i} : i \in [1, n]\}$ associated to the sequence $\{X_k^n : k \in [1, n]\}$ is obtained by stipulating that

$$X_k^n = \sum_{i=1}^{k} Y_{n,i}$$

for each $k \in [1, n]$. This process records increments $Y_{n,i} = X_i^n - X_{i-1}^n$, $i \in [2, n]$, in geodesic watermelon weight as the curve number rises, starting out at $Y_{n,1} = X_1^n$, the geodesic weight for the route from $(1, 1)$ to $(n, n)$. 
We mention that, in exponential LPP, the process \( \{Y_{n,i} : i \in [1,n]\} \) equals in law a list in decreasing order of the eigenvalues of an \( n \times n \)-matrix picked randomly according to the Laguerre Unitary Ensemble (LUE) \cite{AVMW13}. In particular, this random list decreases almost surely. This is essentially an instance of the deterministic fact that \( \{Y_{n,i} : i \in [1,n]\} \) is monotone for any given non-negative weights, which follows from the well-known Greene’s theorem \cite{Gre82} and the non-decreasing form of row lengths in Young tableaux. These results are stated for non-negative integer-valued weights; that the weights here are continuous can be handled by a limiting procedure, or by writing the Robinson-Schensted-Knuth correspondence in terms of the \( (\max,+) \) algebra, with respect to which Greene’s theorem generalizes \cite{BK01}.

Here, we are able to obtain the same result of deterministic monotonicity using purely geometric considerations, in contrast to the algebraic approaches just outlined. Indeed, arguments of the flavour of those that establish interlacing yield the following.

**Proposition 2.6** (Increment monotonicity: C’est magnifique, mais ce n’est pas Laguerre). For \( n \) a positive integer, suppose that the values \( \xi_v \) are non-negative for \( v \in [1,n]^2 \). Let \( k \in [2,n] \). Then \( Y_{n,k-1} \geq Y_{n,k} \). Suppose further that the values \( \{\xi_v : v \in [1,n]^2\} \) are independently picked according to a continuous law \( \nu \). Then the stated inequality is almost surely strict.

An analogous deterministic statement for semi-discrete LPP is proved by algebraic methods in \cite{DOV}.

### 3. Technical Ingredients and the Key Ideas in the Proofs

Here we describe some important technical tools, relying on the above geometric features, which form key ingredients for our proofs. We will also indicate roughly how these tools are proved. This done, we will outline how the tools will aid in proving Theorem II.

In this section, and indeed in the rest of this paper, Assumptions 1, 2, and 3a will be in force in all statements. When Assumption 3b is used in place of 3a, we will indicate this.

#### 3.1. The technical tools

There are three major technical tools, each a result of independent interest. The first two results give existence and non-existence of certain numbers of disjoint paths in \([1,n]^2\) with high probability, while the third one gives a quantitative result on paths with high transversal fluctuation.

**3.1.1. First tool: construction of disjoint paths of high weight** The key starting point is an explicit construction of \( m \in [1,k] \) disjoint paths achieving, with high probability, a cumulative weight \( \mu nm - O(mk^{2/3}n^{1/3}) \). By taking \( m = k \), this proves the weight lower bound aspect of Theorem II(1).

**Theorem 3.1.** There exist \( c, c_1, C_1 > 0 \), and \( k_0 \in \mathbb{N} \) such that, for \( k_0 \leq k \leq c_1 n \) and \( m \in [1,k] \), it is with probability at least \( 1 - e^{-ckm} \) that there exists a set of \( m \) disjoint paths \( \{\gamma_i : i \in [1,m]\} \) in the square \([1,n]^2\), with \( \gamma_i \) for each index \( i \) being a path from \((1,k-i+1)\) to \((n,n-i+1)\) satisfying \( \max_i \text{TF}(\gamma_i) \leq 2mk^{-2/3}n^{2/3} \), such that

\[
\sum_{i=1}^{m} \ell(\gamma_i) \geq \mu nm - C_1 mk^{2/3}n^{1/3}.
\]
The multi-scale construction of \( m \) disjoint curves with \( m = k = 5 \). In between each pair of consecutive dark lines is one scale of the construction. Observe that in a single scale, each of the \( k \) curves passes through a parallelogram of a particular size (as determined by the scale) and aspect ratio. A more detailed depiction of this part of the construction is provided in Fig. 9a.

The construction, in Sect. 8, is a multi-scale one where at each scale we consider geodesics constrained to lie in \( m \) disjoint parallelograms with a carefully chosen aspect ratio: see Fig. 4. The paths \( \gamma_i \) for \( i = 1, \ldots, m \) are now formed by concatenating these constrained geodesics at different scales. The weight lower bound in Theorem 3.1 can then be obtained by analyzing the typical weight of the constrained geodesics, and a concentration estimate implies that the weight of a concatenation of independent geodesics is typically close to its mean.

It is worth pointing out at this stage that while we will be using Theorem 3.1 and the implied lower bound in Theorem II(1) in many arguments appearing before we actually prove it in Sect. 8, its proof does not rely on any of the arguments where it is applied.

3.1.2. Second tool: several disjoint paths constrained in a strip typically have low weight

This tool asserts that the event that there exist \( k \) curves, each meeting a certain weight lower bound and packed within a thin strip around the diagonal, has very low probability. It will serve to help prove Theorem II(2b), the transversal fluctuation lower bound. Letting

\[
U_{n,w} := \{2 \leq x + y \leq 2n, |x - y| \leq w\},
\]

we show that if the width \( w \) equals \( k^{1/3-o(1)} n^{2/3} \), then it is unlikely that there are \( k \) disjoint paths in \( U_{n,w} \) each of which has weight at least \( \mu n - \Theta(k^{2/3} n^{1/3}) \).

**Theorem 3.2.** Given \( C_3 > 0 \), there exist \( \delta = \delta(C_3) > 0 \) and \( c_1 = c_1(C_3) > 0 \) such that, for \( k \in \mathbb{N} \) and \( k \leq c_1 n \), the following holds. Let \( \mathcal{E} = \mathcal{E}(n, k, \delta, C_3) \) denote the
event that there exist \( k \) disjoint paths \( \gamma_1, \gamma_2, \ldots, \gamma_k \) contained in \( U_{n, 8k^{1/3}n^{2/3}} \) such that \( \ell(\gamma_i) \geq \mu n - C_3k^{2/3}n^{1/3} \) for \( i = 1, 2, \ldots, k \). Then there exists an absolute constant \( c > 0 \) such that \( \mathbb{P}(\mathcal{E}) \leq e^{-ck^2} \).

This result will be proved in Sect. 4. How? If there are a large number of disjoint paths in a strip, then some of them must be restricted to some thin region. Our assumptions will imply that any path contained in a thin region has an exponentially-low-in-\( k \) probability of having weight shortfall from \( \mu n \) less than \( C_3k^{2/3}n^{1/3} \), and the van den Berg-Kesten (BK) inequality will complete the proof. Recall that the BK inequality bounds the probability of occurrence of a number of events disjointly by the product of their individual probabilities.

A similar argument has appeared in [BHS18] to show that the maximum number of disjoint geodesics across the shorter sides of an \( n \times n^{2/3} \) rectangle is uniformly tight with good tails. The argument in this paper is a refinement of this approach, requiring more careful entropy calculations to take care of the growing width of \( U_{n, 8k^{1/3}n^{2/3}} \) in \( k \).

3.1.3. Third tool: the weight shortfall of paths with high transversal fluctuations. This result will help to prove Theorem II(2a), the upper bound on geodesic watermelon transversal fluctuation. The tool asserts that any path from \((1, 1)\) to \((r, r)\) of transversal fluctuation greater than \( sr^{2/3} \) typically suffers a weight loss of order \( s^2r^{1/3} \). The result applies to any path and is not a bound on the probability of the maximal weight path having such a transversal fluctuation. The statement may be expected from known relations between transversal and longitudinal fluctuations (which are consequences of the curvature assumption in Assumption 2 and one-point fluctuation information in Assumption 3a). While similar results with suboptimal exponents have appeared (see [BSS14, BGZ21]), we obtain the following sharp result, adapting a multi-scale argument from [BSS14]. The proof appears in Sect. 6.

We introduce some notation for two line segments to state the result; we will be using this notation at several other places in the paper as well. Define, for \( r, t > 0 \)

\[
\mathbb{L}_{\text{left}} = \mathbb{L}_{\text{left}}(r, t) = \{(x, y) : x + y = 2, |y - x| \leq 2tr^{2/3}\} \quad \text{and} \quad \mathbb{L}_{\text{right}} = \mathbb{L}_{\text{right}}(r, t) = \{(x, y) : x + y = 2r, |y - x| \leq 2tr^{2/3}\};
\]

in words, \( \mathbb{L}_{\text{left}} \) and \( \mathbb{L}_{\text{right}} \) are the line segments of length \( 2tr^{2/3} \) respectively centred at \((1, 1)\) and \((r, r)\).

**Theorem 3.3.** Let \( X^{r,s,t} \) be the maximum weight over all paths \( \Gamma^{r,s,t} \) from \( \mathbb{L}_{\text{left}} \) to \( \mathbb{L}_{\text{right}} \) such that \( TF(\Gamma^{r,s,t}) > (s + t)r^{2/3} \), with \( t \leq s \). There exist absolute constants \( r_0, s_0, c > 0 \) and \( c_2 > 0 \) such that, for \( s > s_0 \) and \( r > r_0 \),

\[
\mathbb{P}\left(X^{r,s,t} - \mu r > -c_2s^2r^{1/3}\right) < e^{-cs^3}.
\]

We now show how the interlacing property and the above results will yield Theorem II.

3.2. Proof outline of theorem II. We already noted how Theorem 3.1 implies the weight lower bound of Theorem II(1). This is a collective weight lower bound, but for the remaining parts of Theorem II we will roughly need a lower bound on individual curve weights of order \( \mu n - \Theta(k^{2/3}n^{1/3}) \); note that this quantity is \( k^{-1} \) times the weight lower bound \( \mu nk - \Theta(k^{5/3}n^{1/3}) \) of Theorem II(1). This individual curve lower bound is provided in a slightly weaker (but sufficient) sense by an averaging argument, which is a simple but core ingredient of our proofs.
3.2.1. The averaging argument Let $X_n^{j,i}$ be the weight of the $i^{th}$ heaviest curve of a given $j$-geodesic watermelon. The averaging argument yields that there exists with high probability $j \in \llbracket k + 1, 2k \rrbracket$ such that, for all $i \leq j$,

$$X_n^{j,i} \geq X_n^{j,j} \geq \mu n - \Theta(k^{2/3}n^{1/3}),$$

and similarly for some $j \in \llbracket \lfloor k/2 \rfloor, k \rrbracket$. The argument is powered by a simple but very useful inequality: for $j \in \mathbb{N}$,

$$X_n^{j,j} \geq X_n^{j} - X_n^{j-1}.$$  \hspace{1cm} (6)

This is proved by considering the lowest weight curve and the remaining $(j - 1)$ curves of the $j$-melon separately. The weight of the latter is of course at most the weight of the $(j - 1)$-melon, and this gives the inequality. Notice that while $X_n^{j,j}$ is defined only once a $j$-geodesic watermelon has been specified, the lower bound in (6) is independent of the choice of $j$-geodesic watermelon. This allows us to prove that (5) holds for some $j \in \llbracket k + 1, 2k \rrbracket$ simultaneously for all choices of $j$-geodesic watermelon with high probability.

Deriving (5) from (6) is a matter of appealing to the lower bound on $X_n^{2k}$ from Theorem II(1) and the following crude upper bound on $X_n^{k}$, which is proved using one-point information (Assumption 3a) and the BK inequality (see Sect. 4); note the change in sign of the fluctuation term when compared with the weight upper bound of Theorem II(1).

**Lemma 3.4.** Under Assumption 3a, we may find $c > 0$ such that, if $t > 0$, there exists $k_0 = k_0(t)$ for which $n > n_0$ and $k_0 < k \leq t^{-3/4}n^{1/2}$ imply that

$$\mathbb{P} \left( X_n^k > \mu nk + tk^{5/3}n^{1/3} \right) \leq \exp \left( -ct^{3/2}k^2 \right).$$

If instead Assumption 3b is available, then the upper bound on $k$ may be taken to be $\min(1,t^{-3/2})n$.

The relaxation on the condition on $k$ we obtain under Assumption 3b is important to prove Theorem I with the full claimed range of $k$. Lemma 3.4 is the source of all future statements which give different conditions on $k$ under Assumption 3b. The proof of Lemma 3.4 using Assumption 3b is slightly different than with 3a, and is handled separately in Appendix B.3, while the proof with 3a appears in Sect. 4.

With the result of the averaging argument in hand, Theorem II(2) is a straightforward consequence of interlacing and Theorems 3.2 and 3.3, as we briefly describe next.

3.2.2. Bounding transversal fluctuations Equation (5), along with Theorem 3.2, implies the following transversal fluctuation lower bound for the random index $j$,

$$\mathbb{P} \left( \exists j \in \{ \lfloor k/2 \rfloor, \ldots, k \} \text{ such that } TF(n, j) > \delta k^{1/3}n^{2/3} \right) \geq 1 - e^{-ck^2}.$$  

This can then be upgraded to Theorem II(2b) by a simple application of interlacing. A similar argument with Theorem 3.3 implies that

$$\mathbb{P} \left( \exists j \in \{ k + 1, \ldots, 2k \} \text{ such that } \overline{TF}(n, j) < Ck^{1/3}n^{2/3} \right) \geq 1 - e^{-ck},$$

which when combined with interlacing gives Theorem II(2a).

What remains is the proof of the weight upper bound, Theorem II(1), which will use essentially all of the gathered ingredients.
3.2.3. Bounding above watermelon weight The proof has a few parts. Consider, as we did above, a random $k/2 \leq j \leq k$ that satisfies (5), so that for all $1 \leq i \leq j$, $X^j_i$ is not too light (for any choice of $j$-geodesic watermelon). By an application of Theorem 3.2 with $k$ replaced by $k/4$, it follows that with probability $1 - e^{-ck^2}$, at least $k/4$ paths in all $j$-geodesic watermelons exit a strip of width $\Theta(k^{1/3}n^{2/3})$. The interlacing property implies that the latter property holds for any fixed $k$-geodesic watermelon $\Gamma^k_n$ as well.

We also know that the cumulative weight of the heaviest $7k/8$ paths of $\Gamma^k_n$ cannot be too high, by the crude estimate Lemma 3.4 (with $k$ replaced by $7k/8$). This implies that on the event that the $k$-geodesic watermelon’s weight is high, the cumulative weight of the lightest $k/8$ paths of $\Gamma^k_n$ is high. Hence the heaviest path among the latter has unusually high weight which, along with monotonicity, implies the same lower bound for all the heaviest $7k/8$ paths of $\Gamma^k_n$.

Since we have already established that at least $k/4$ paths in the ensemble $\Gamma^k_n$ exit a strip around the diagonal of width $\Theta(k^{1/3}n^{2/3})$, this implies that at least $k/8$ disjoint paths both exit a strip of width $\Theta(k^{1/3}n^{2/3})$ and each have an unusually high weight. The probability that a single curve has both these properties is at most $\exp(-ck)$ by Theorem 3.3, and so the probability that $O(k)$ many disjoint curves have these properties is $\exp(-ck^2)$ by the BK inequality. We note that, without interlacing, the above would give a single curve with unusually high weight and large transversal fluctuation, which would yield the weaker probability bound of $\exp(-ck)$.

3.2.4. A few further comments in overview The proof strategies make it apparent that the arguments have minimal dependence on the precise nature of the model and indeed straightforward modifications will yield Theorem III.

The proofs critically rely on the averaging argument and watermelon interlacing. The latter is a central theme of the paper and is a consequence of non-local geometric aspects of the watermelons as will be clear from the proofs in Sect. 9. The monotonicity result, Proposition 2.6, will also be proved in Sect. 9, by similar geometric considerations. Interlacing is the principal means by which we have expressed this point of view, but in fact the stated monotonicity offers an alternative, but still geometric, route to proving Theorem II. In Sect. 10, we explore this approach and explain how it leads to certain sharp concentration bounds on each $Y_{n,k}$. We contrast this with weaker concentration bounds obtainable in the integrable setting of exponential LPP by straightforward determinantal point process techniques: see Proposition 10.1 and (44). It would be interesting to seek to sharpen Proposition 10.1 by more sophisticated determinantal point process technique.

3.3. Basic tools. To implement the ideas of the preceding sections, we will require certain basic estimates on geometrically relevant quantities. These include, for example, tail estimates for last passage values where the endpoints are allowed to vary over intervals whose length is of order $n^{2/3}$, or where one endpoint is allowed to vary over the line $x + y = 2n$; and tail and mean estimates for last passage values constrained to remain in certain parallelograms. In this section, we state the precise forms of these estimates.

Similar bounds have appeared in the literature, for example in the preprint [BSS14], but never under the general assumptions that we adopt. Our proofs follow similar strategies as in these previous works and are provided in Appendix B for completeness. The
statements are nonetheless technical and the reader might choose to skip this section at a first reading and only refer back to it as needed later in the paper.

As we prepare to give the statements, we adopt a convention that will prove convenient at several moments in the article: we will refer to measurements taken along the anti-diagonal as “width”, and measurements along the diagonal as “height”. For example, the parallelogram defined by \( \{(x, y) \in \mathbb{Z}^2 : |x - y| \leq 2\ell r^{2/3}, 2 \leq x + y \leq 2r\} \) will be said to have width \( 2\ell r^{2/3} \) and height \( r \). The implicit sense of the diagonal direction as vertical will find expression in usages such as ‘upper’ and ‘lower’ to refer to more or less advanced diagonal coordinates.

### 3.3.1. Interval-to-interval estimates

We start with estimates for the deviations of the weight when the endpoints are allowed to vary over intervals. For this we define some notation.

Extending the notation (3), let \( U = U_{r, \ell r^{2/3}, zr^{2/3}} \) be a parallelogram of height \( r \) with one pair of sides parallel to the anti-diagonal, each of width \( \ell r^{2/3} \), the lower side centred at \((1, 1)\) and the upper side at \((r - zr^{2/3}, r + zr^{2/3})\) for a \( z \) such that \(|z| \leq r^{1/3}\). Let \( A \) and \( B \) be the lower and upper sides of \( U \) and let \( S(U) = A \times B \). With \( G_2 \) and \( \rho \) as in Assumption 2, let

\[
\tilde{Z} := \sup_{(u, v) \in S(U)} \frac{X_{u \rightarrow v} - \mu r}{r^{1/3}} + \frac{G_2}{1 + 2\ell r^{3/2}} (z \wedge \rho r^{1/3})^2. 
\]

Although we have yet to employ the notation \( Z \) without a tilde, it will be used to denote a scaled form for weight of certain LPP paths. The tilde indicates a parabolic adjustment which compensates (and possibly over-compensates) for the curvature penalty incurred for off-diagonal movement. The parabolic form of this adjustment is indicated in Assumption 2. This form is guaranteed by Assumption 2 only up to anti-diagonal displacement of order \( z = \rho r^{1/3} \); concavity nonetheless guarantees that at least this loss occurs even beyond this point, and this accounts for the \( z \wedge \rho r^{1/3} \) term in the specification of \( \tilde{Z} \).

However, Assumption 3a only holds for paths whose endpoints are bounded away from the coordinate axes, and so for extreme anti-diagonal displacement we will need to work with a different object: define, for \( \delta > 0 \), \( \mathbb{L}_{\text{ext}}(\delta) = \{(r + zr^{2/3}, r - zr^{2/3}) : |z| \geq (1 - \delta)r^{1/3}\} \), which is an interval of points of extreme slope, and recall \( \mathbb{L}_{\text{left}} = \mathbb{L}_{\text{left}}(r, 1) = \{(x, y) \in \mathbb{Z}^2 : x + y = 2, |x - y| \leq 2r^{2/3}\} \) from (4). Finally define

\[
Z^{\text{ext}, \delta} = \sup_{u \in \mathbb{L}_{\text{left}}, v \in \mathbb{L}_{\text{ext}}(\delta)} \frac{X_{u \rightarrow v} - \mu r}{r^{1/3}},
\]

the interval-to-interval passage time between \( \mathbb{L}_{\text{low}} \) and \( \mathbb{L}_{\text{ext}}(\delta) \).

**Proposition 3.5.** Suppose \( \delta > 0 \) is such that \(|z| < (1 - \delta)r^{1/3}\). Then there exist \( \theta_0 = \theta_0(\ell, \delta), c = c(\delta) > 0 \), and \( r_0 = r_0(\ell, \delta) \) such that, for \( r > r_0 \) and \( \theta > \theta_0 \),

\[
\mathbb{P} \left( \tilde{Z} > \theta \ell^{1/2} \right) < \exp \left( -c \min(\theta^{3/2}, \theta r^{1/3}) \right). \tag{7}
\]

Further, there exist \( \delta > 0 \) and \( c > 0 \) such that, for \( \theta > \theta_0 \) and \( r > r_0 \),

\[
\mathbb{P} \left( Z^{\text{ext}, \delta} > \theta \right) \leq \exp(-cr - c \min(\theta^{3/2}, \theta r^{1/3})). \tag{8}
\]
3.3.2. Point-to-line estimates
Here we bound the upper tail of point-to-line geodesic weights, using only the parabolic curvature of the limit shape (Assumption 2) and the point-to-point upper tail (Assumption 3a). We recall \( L_{\text{left}} = L_{\text{left}}(r, t) \) from (4).

**Proposition 3.6.** Let \( X \) be the maximum weight of all paths \( \Gamma \) from \( L_{\text{left}} \) to the line \( x + y = 2r \) whose endpoint does not lie on the line segment connecting \((r - (s + t))r^{2/3}, r + (s + t)r^{2/3}\) and \((r + (s + t))r^{2/3}, r - (s + t)r^{2/3}\). Then there exist constants \( c_1 > 0, c > 0, s_0, \theta_0, \) and \( r_0 \) such that, for \( r > r_0 \) and either (i) \( s = 0 \) and \( \theta \geq \theta_0 \), or (ii) \( s > s_0, 0 \leq t \leq s^2 \) and \( \theta = 0 \), we have that
\[
\mathbb{P}(X \geq \mu r + \theta r^{1/3} - c_1 s^2 r^{1/3}) \leq \exp\left(-c\left(\min(\theta^{3/2}, \theta r^{1/3}) + s^3\right)\right).
\]

3.3.3. Lower tail of constrained point-to-point
Next we come to a crucial ingredient of Theorem 3.1, an upper bound on the lower tail of the weight of the point-to-point geodesic constrained to not exit a given parallelogram, and a lower bound on the mean of the same. We obtain precisely the tail exponent of one that will be used in Theorem 3.1; the optimal exponent, however, is expected to be three, matching the optimal exponent of the point-to-point weight lower tail. Even given the optimal lower tail exponent of three as input via Assumption 3a, the argument we give for Proposition 3.7 would only yield the exponent 3/2 for the constrained lower tail.

With notation for parallelograms as above, let \( U = U_{r, \ell r^{2/3}, czr^{2/3}} \). Set \( u = (1, 1) \) and \( u' = (r - (z + h))r^{2/3}, r + (z + h)r^{2/3} \) where \(|h| \leq \ell/2\). Then define \( X_{u \to u'}^U \) to be the maximum weight over all paths from \( u \) to \( u' \) that are constrained not to exit \( U \). Recall \( G_1 \) from Assumption 2. We have the following lower tail and mean estimates for \( X_{u \to u'}^U \).

**Proposition 3.7.** (Lower tail and mean of constrained point-to-point). Let \( L_1, L_2 > 0 \) and \( K > 0 \) be fixed. Let \( z \) and \( \ell \) be such that \( |z| \leq K \) and \( L_1 \leq \ell \leq L_2 \). There exist positive constants \( r_0(K, L_1, L_2) \) and \( \theta_0(K, L_1, L_2) \), and an absolute constant \( c > 0 \), such that, for \( r > r_0 \) and \( \theta > \theta_0 \),
\[
\mathbb{P}(X_{u \to u'}^U \leq \mu r - \theta r^{1/3}) \leq \exp(-c\ell \theta).
\]

As a consequence, there exists \( C = C(K, L_1, L_2) \) such that, for \( r > r_0 \),
\[
\mathbb{E}[X_{u \to u'}^U] \leq \mu r - G_1 z^2 r^{1/3} - C r^{1/3}.
\]

We have kept the parabolic loss term in (10) to match with earlier expressions, but note that it may be absorbed into the term \(-C r^{1/3}\) for the ranges of \( z \) mentioned.

3.4. Organization.
In this section, we have signposted the location of several upcoming aspects. With some repetition, and for the sake of convenience, we now summarise the structure of the rest of the paper, which is principally devoted to proving Theorem II. There are four elements: lower and upper bounds on the transversal fluctuation of geodesic watermelons; and upper and lower bounds on the watermelons’ weight. The respective arguments are offered in Sects. 5, 6, 7 and 8. The first of the four elements is aided by the ‘thin strip means low weight’ Theorem 3.2. First, in Sect. 4, we will recall the BK inequality and use it to prove this theorem. The deterministic interlacing results assembled in Sect. 2 are important inputs, and, in Sect. 9, we prove them, so that the proof of Theorem II is completed in this section. Sect. 10 compares concentration bounds for
the weight increments $Y_{n,k}$ specified in (2) obtained by our geometric methods and by
determinantal point process techniques (in the case of exponential LPP); a proof of the
concentration bound via determinantal techniques is provided in the arXiv version of
this article. The proof of the point-to-line result, Theorem III, is provided in Sect. 11.

There are three appendices: Appendix 11 is devoted to proving that exponential
and geometric LPP satisfy the Assumptions; Appendix 11 to proving the three basic
tools, Propositions 3.5, 3.6 and 3.7, laid out in Sect. 3.3; and Appendix B.3 to proving
Lemma 3.4 under the stronger Assumption 3b. The proofs in Appendix B have been
deferred because they roughly mimic publicly available LPP arguments; we judge the
flow of the arguments at large to be aided by the few proof deferrals that we have
mentioned. The appendices render the article independent of LPP inputs beyond fairly
well-known assertions such as the expression of the cardinality of a determinantal point
process as a sum of independent Bernoulli random variables, and an estimate on the
mean of the eigenvalue count in a given interval for the Laguerre unitary ensemble.

4. Disjoint Paths and the van den Berg-Kesten Inequality

In this section, we prove Theorem 3.2, our assertion that several disjoint high-weight
paths may not typically coexist in a narrow strip. The proof is founded on the notion that
at least one among $k$ disjoint paths packed into the strip $U_{n, \delta k^{1/3} n^{2/3}}$ is forced to inhabit
a thin region of width $\delta k^{-2/3} n^{2/3}$. Here in outline are the principal steps that we will
follow to realize the proof.

1. We will argue that it is with probability decaying exponentially in $k$ that a path
   existing in a given thin region—of length $n$ and of width roughly $k^{-2/3} n^{2/3}$—has
   weight larger than $\mu n - \Theta(k^{2/3} n^{1/3})$.
2. The van den Berg-Kesten (BK) inequality bounds above the probability of events
   occurring in a certain disjoint fashion by the product of the probabilities of the
   events. This inequality permits us to conclude that the probability that there exist
   $k$ disjoint paths whose weight satisfies the above lower bound, each contained in a
   specified thin region, is exponentially small in $k^2$.
3. A union bound will then be taken indexed by a collection of $k$-tuples of thin regions
   whose elements capture all possible routes for the $k$ paths. The collection will be
   selected by means of a grid-based discretization; the cardinality of the collection
   will be small enough that the upper bound on probability for a given $k$-tuple will
   not be significantly undone by taking the union bound.

   To implement this three-step plan, we need to make precise the notion of ‘thin region’
in Step 1, and this in fact involves specifying the grid used in Step 3. This we do in a
first subsection. But first we remind the reader of our norm that statements by default
suppose Assumptions 1, 2 and 3a and that we indicate when Assumption 3b is needed
in place of 3a.

4.1. Specifying a grid that classifies paths running in a narrow strip. The grid will
be called $G$. It will comprise anti-diagonal planar intervals whose elements track the
progress of any path of transversal fluctuation at most $\delta k^{1/3} n^{2/3}$.

   Recall that width and height refer to anti-diagonal and diagonal length. Beyond the
   height $n$, path number $k$ and strip thinness parameter $\delta > 0$, the grid will be constructed
   by the use of two positive parameters $\eta$ and $\varepsilon$. We hope that the explanation we next
   offer alongside Fig. 5 will make their roles clear.
The strip of width $\delta k^{1/3} n^{2/3}$ and height $n$ will be divided into cells. Each cell will have width $\eta \epsilon n^{2/3}$ and height $\epsilon^{3/2} n$. The width is the product of $\eta$ and the two-thirds’ power of the height, so $\eta$ is a measure of the thinness of the cell, judged on the KPZ scale; we will choose this parameter to be a small constant, independently of $k$ and $n$. We will choose $\epsilon$ to have order $k^{-2/3}$, so that a union of intervals at consecutive heights may play the role of a ‘thin region’ indicated in the three-point summary. Each column of the grid contains $N$ cells, where $N = n/(\epsilon^{3/2} n) = \epsilon^{-3/2}$; and each row contains $M$ cells, where $M = 2\delta k^{1/3} n^{2/3}/(\eta \epsilon n^{2/3}) = 2\delta k^{1/3} \eta^{-1} \epsilon^{-1}$.

To begin the definition, let $\epsilon > 0$ and $\eta > 0$; we will specify the conditions on these values presently. We suppose $n \in \mathbb{N}$ to be large enough that $\epsilon \eta n^{2/3} \geq 1$, in order that cell width be at least the difference of $x$- or $y$-coordinates between consecutive elements in $\mathbb{Z}^2$ along an anti-diagonal line. Set $M, N \in \mathbb{N}$ according to

\[ N = \lceil \epsilon^{-3/2} \rceil \quad \text{and} \quad M = 2 \cdot \lceil \delta k^{1/3} \eta^{-1} \epsilon^{-1} \rceil. \tag{11} \]

The grid $G$ is a set of planar anti-diagonal intervals $\mathbb{L}_{i,j}$:

\[ G = \{ \mathbb{L}_{i,j} : i \in [0, N], \ j \in [0, M] \}, \]

where $\mathbb{L}_{i,j}$ connects the points

\[ (v_i - h_j, v_i + h_j) \quad \text{and} \quad (v_i - h_{j+1}, v_i + h_{j+1}), \]

with $v_i = \lfloor i \epsilon^{3/2} n \rfloor$ and $h_j = \lfloor (\delta k^{1/3} - j \eta \epsilon) n^{2/3} \rfloor$ for $i \in [0, N - 1]$; for $i = N$ we take $v_i = n$. Thus the last row of cells has greater height than do the first $N - 1$ rows, by a factor lying in $[1, 2)$, but this will not affect our arguments. Note that each element of the grid belongs to the rectangle $\{ |x - y| \leq 2\delta k^{1/3} n^{2/3}, \ 0 \leq x + y \leq 2n \}$. The grid is partitioned into subsets at given height by setting $G_i = \{ \mathbb{L}_{i,j} : j \in [0, M] \}$ for $i \in [0, N]$. To avoid cumbersome expressions, we employ, in the remainder of this section, a notational abuse by which $(x, y)$ for $x, y \in \mathbb{R}$ will actually denote the rounded integer lattice point $\lfloor (x, y) \rfloor$.

4.2. Bounding the maximum weight of paths between grid elements at consecutive heights. A grid journey is an $(N + 1)$-vector $\{ L_i : i \in [0, N] \}$ such that $L_i \in G_i$ for each index $i$. A grid journey plays in the implementation the conceptual role of a thin region in the outline. Indeed, for any grid journey $\mathcal{L}$, we write $X_{\mathcal{L}}$ for the supremum of the weights of paths contained in $[1, n]^2$ that intersect every one of the $N + 1$ components of $\mathcal{L}$. In Step 1, we will bound the upper tail of $X_{\mathcal{L}}$. To be ready to do so, we first gain control, in Lemma 4.1, on the mean of the weight of the heaviest path that begins in a given element of $G_i$ and that ends in a given element of $G_{i+1}$. Since the height discrepancy is $r = \epsilon^{3/2} n$, Assumption 2 indicates that the weight of the geodesic running between given endpoints in these intervals has leading term $\mu r$, with a negative correction whose order is $r^{1/3}$. When the endpoints are varied by at most $\eta r^{2/3}$, a Gaussian-order perturbation is transmitted to the maximum weight. This change of order $\eta^{1/2} r^{1/3}$ is less than the order of the negative mean provided that $\eta$ is small enough (and it is this consideration that determines our upper bound $\eta_0$ on the parameter $\eta$). This is how we will prove Lemma 4.1.

For $\eta > 0$, $r > 0$ and $|s| \leq r$, let $\mathbb{L}_{\text{low}}$ be the line segment joining $(-\eta r^{2/3}, \eta r^{2/3})$ and $(\eta r^{2/3}, -\eta r^{2/3})$ and let $\mathbb{L}_{\text{up}}$ be the line segment joining $(r - s - \eta r^{2/3}, r + s + \eta r^{2/3})$.
Fig. 5. The grid $G$. There are $N + 1$ row lines and $M + 1$ column lines, and the width and height of a single cell are $\eta \epsilon n^{2/3}$ and $\epsilon^{3/2} n$ respectively. The width of the grid as a whole is $28k^{1/3} n^{2/3}$. The role of the grid is to be a discretization of paths from the bottom side to the top. On the right, such a path is depicted in sky blue along with its corresponding discretization, which is the sequence of intervals, one on each grid line, which are coloured blue (or purple). Also depicted is a sequence of red intervals (when a red interval coincides with a blue interval, it is coloured purple). These red intervals satisfy the property that on each grid line, the red interval coincides with, or is to the right of, the blue interval on the same grid line. This is a property that is satisfied by the discretization of any path from bottom to top which is disjoint from and to the right of the sky blue path.

and $(r - s + \eta r^{2/3}, r + s - \eta r^{2/3})$. Define $Z$ by

$$Z = \sup \left\{ r^{-1/3} (X_{u \rightarrow v} - \mu r) : u \in \mathbb{L}_{\text{low}}, v \in \mathbb{L}_{\text{up}} \right\},$$

so that we seek control on the mean of $Z$.

**Lemma 4.1.** Under Assumption 2, there exist positive constants $c_1$, $r_0$ and $\eta_0$ such that, for $\eta < \eta_0$, $|s| \leq r$ and $r > r_0$, $\mathbb{E}[Z] \leq -2c_1$.

The basic strategy of the argument is to back away from $\mathbb{L}_{\text{low}}$ and $\mathbb{L}_{\text{up}}$ on either side and consider a point-to-point weight; such ideas have appeared in, for instance, [BSS14].

**Proof.** Let $m_1$ and $m_2$ be the points on $\mathbb{L}_{\text{low}}$ and $\mathbb{L}_{\text{up}}$ which are closest to the average of the ends of the respective intervals; we make this specification since these averages need not be lattice points.

The $\ell^1$ distance of these two points is $2r$. Let $u^* \in \mathbb{L}_{\text{low}}$ and $v^* \in \mathbb{L}_{\text{up}}$ be such that $X_{u^* \rightarrow v^*}$ equals $\mu r + Z r^{1/3}$. For $\eta > 0$, let $\phi_1 = m_1 - (\eta^{3/2} r, \eta^{3/2} r)$ and $\phi_2 = m_2 + (\eta^{3/2} r, \eta^{3/2} r)$. Then we have

$$X_{\phi_1 \rightarrow u^*} + X_{u^* \rightarrow v^*} + X_{v^* \rightarrow \phi_2} \leq X_{\phi_1 \rightarrow \phi_2}.$$

Note that the $\ell^1$ distance between $\phi_1$ and $\phi_2$ lies in $(2 + 4\eta^{3/2})r + [-2, 0]$. First we consider the case that this pair of points satisfies the $\rho$-condition in Assumption 2. We get that, for large $r$,

$$\mathbb{E}[X_{\phi_1 \rightarrow \phi_2}] \leq \mu (1 + 2\eta^{3/2})r - g_2 \left((1 + 2\eta^{3/2})r - 2\right)^{1/3} \leq \mu (1 + 2\eta^{3/2})r - g_2 r^{1/3}.$$
To get an upper bound on $\mathbb{E}[X_{u^* \rightarrow v^*}]$, we need a lower bound on $\mathbb{E}[X_{\phi_1 \rightarrow u^*}]$, which will also apply to $\mathbb{E}[X_{v^* \rightarrow \phi_2}]$ by symmetry. Note that $u^*$ is independent of the environment outside $U$, so $\mathbb{E}[X_{\phi_1 \rightarrow u^*}] \geq \inf_{\mu \in \mathbb{L}_{\text{low}}} \mathbb{E}[X_{\phi_1 \rightarrow u}]$. We bound this below using Assumption 2, and note that that expression is minimized if $u$ is either endpoint of $\mathbb{L}_{\text{low}}$. So we take $u = (-\eta r^{2/3}, \eta r^{2/3})$. This gives

$$\mathbb{E}[X_{\phi_1 \rightarrow u}] \geq \mu^3/2 r - \mu - G_1 \eta^{1/2} r^{1/3} - g_1 \eta^{1/2} r^{1/3},$$

for $r$ such that $\eta^{3/2} r > r_0$ for a large $r_0$. Thus, we get an upper bound on $\mathbb{E}[X_{u^* \rightarrow v^*}]$:

$$\mathbb{E}[X_{u^* \rightarrow v^*}] - \mu r \leq 2\mu - g_2 r^{1/3} + 2(G_1 + g_1) \eta^{1/2} r^{1/3} \leq -2c_1 r^{1/3}$$

for $c_1 = g_2/4, \eta$ small enough, and $r$ large enough. This completes the proof when the $\rho$-condition in Assumption 2 is satisfied.

If the $\rho$-condition is not satisfied, we have, using the monotonicity of the upper bound in Assumption 2 for all $x$ and a simple calculation,

$$\mathbb{E}[X_{u^* \rightarrow v^*}] \leq \mathbb{E}[X_{\phi_1 \rightarrow \phi_2}] \leq (\mu + 2\mu \eta^{3/2} - \rho^2 G_2)r.$$ 

In this case, the proof is completed by noting that, for $\eta$ sufficiently small, the coefficient of $r$ is strictly smaller than $\mu$. \hfill $\square$

The upper bound of $\eta_0$ from Lemma 4.1 will be imposed on $\eta$, in order that the mean of $Z$ be negative. A further upper bound on $\eta$ will later be needed; namely,

$$\eta c_1 C_3^{-1} < 1,$$

where $c_1$ is as in Lemma 4.1 and $C_3$ is as in Theorem 3.2. Let $\eta < \eta_0$ satisfy (12). The next three subsections in turn carry out the outlined three-step plan for proving Theorem 3.2.

4.3. Step one: bounding above the weight of the heaviest path on a given grid route. We rigorously take Step 1 by stating and proving the next result. The tail of $\exp(-\Theta(\varepsilon^{-3/2}))$ we obtain here is crucial, as this will become $\exp(-\Theta(k))$ when we later set $\varepsilon$ to be of order $k^{-2/3}$.

**Proposition 4.2.** Let $c_1$ be as in Lemma 4.1 and let $\eta > 0$ be fixed as after (12). There exist positive $c_3$ and $C$ such that, when $\varepsilon > 0$ and $n \in \mathbb{N}$ satisfy $n \geq C \varepsilon^{-3/2}$,

$$P \left( X_{\mathcal{L}} > \mu n - c_3 \varepsilon^{-3/2} \right) \leq \exp \left\{ -c_3 \varepsilon^{-3/2} \right\}.$$ 

**Proof.** Set $Z_i = \sup_{u \in \mathcal{L}_i, v \in \mathcal{L}_{i+1}} \varepsilon^{-1/2} n^{-1/3} (X_{u \rightarrow v} - \mu \varepsilon^{3/2} n)$ for $i \in [0, N - 1]$, where $\mathcal{L}_i$ is the anti-diagonal interval which is the union of the two line segments obtained by displacing $L_i$ by $(1, 0)$ and $(0, 1)$; this choice is made so that the $Z_i$ are determined by disjoint regions of the noise field and so are independent. Then $X_{\mathcal{L}} - \mu n \leq \varepsilon^{1/2} n^{1/3} \sum_{i=0}^{N-1} Z_i$. As noted, the collection $\{Z_i : [0, N - 1]\}$ is independent; and, by Lemma 4.1, the elements’ means lie uniformly to the left of zero. We will verify that these random variables have exponential upper tails and apply Bernstein’s inequality to $\sum Z_i$ in order to obtain the proposition.
If we set $C > r_0$ in the statement that we are seeking to prove, then a choice of $r = \varepsilon^{3/2}/n$ in Lemma 4.1 will satisfy the hypothesis that $r \geq r_0$. Applying this lemma with $\eta$ as specified, we find that

$$E[Z_i] \leq -2c_1$$

in view of translation invariance of the noise field $\{\xi_v : v \in \mathbb{Z}^2\}$. With $\eta$ now fixed, we claim that, for $\theta > \theta_0$,

$$\mathbb{P}(Z_i > \theta \eta^{1/2}) \leq \exp(-c\theta).$$

This is because $Z_i \leq \tilde{Z}_i$ (where $\tilde{Z}_i$ is defined analogously to $\tilde{Z}$ in Proposition 3.5) in the case that $L_i$ and $L_{i+1}$ form opposite sides of a parallelogram whose midpoint-to-midpoint anti-diagonal displacement is at most $(1-\delta)\varepsilon^{3/2}/2n$ for the $\delta > 0$ fixed in the second part of Proposition 3.5; while $Z_i \leq Z_{i}^{\text{ext},\delta}$ (again defined analogously to $Z^{\text{ext},\delta}$ in Proposition 3.5) in the other case. Applying Proposition 3.5 completes the claim. Note that we have again used that $r = \varepsilon^{3/2}/n > r_0$ by setting $C = \varepsilon^{3/2}/n > r_0$ in the statement of Proposition 4.2.

The above displayed bound establishes that $Z_i$ has exponential upper tails and so satisfies the conditions of Bernstein’s inequality (see equation 2.18 and Theorem 2.13 of [Wai19]). Applying the latter and since $N = \varepsilon^{-3/2}$ (see (11)), we get, for some $c_3 > 0$,

$$\mathbb{P}(X_{c_2} > \mu n - c_1 \varepsilon^{-1/3} n^{1/3}) \leq \mathbb{P}\left(\sum_{i=0}^{N-1} Z_i > -c_1 \varepsilon^{-3/2}\right) \leq \mathbb{P}\left(\sum_{i=0}^{N-1} (Z_i - E[Z_i]) > c_1 \varepsilon^{-3/2}\right) \leq \exp\left(-c_3 \varepsilon^{-3/2}\right).$$

\[\square\]

4.4. Step 2: the rarity of $k$ high weight paths in a narrow strip. The BK inequality is the principal tool enabling the second step. The inequality provides an upper bound on the probability of a number of events occurring disjointly in terms of the individual events’ probabilities. To state it we need a precise definition of disjointly occurring events; this is taken from [AGH18], which also proves the inequality in the setting of infinite spaces that we require.

**Definition 4.3.** For $d, n \in \mathbb{N}$, let $A_i \subseteq \mathbb{R}^d$ be Borel measurable for $i \in [1,n]$. For $\omega \in \mathbb{R}^d$ and $K \subseteq [1,d]$, define the cylinder set $\text{Cyl}(K, \omega) = \{\omega' : \omega'_i = \omega_i, i \in K\}$. Also define, for $A \subseteq \mathbb{R}^d$,

$$[A]_K := \{\omega : \text{Cyl}(K, \omega) \subseteq A\} \quad \text{and} \quad \bigcap_{i=1}^{n} A_i := \bigcup_{J_1, \ldots, J_n} \bigcap_{i=1}^{n} [A_i]_{J_i},$$

where the union is over all choices of disjoint subsets $J_1, \ldots, J_n$ of $\{1, \ldots, d\}$.

With the notation established, we may state the BK inequality.
Proposition 4.4 (BK inequality, Theorem 7 of [AGH18]). Fix $n \in \mathbb{N}$ and let $A_i \subseteq \mathbb{R}^d$ be Borel measurable for $i \in [1, n]$. Under any complete product probability measure $\nu$ on $\mathbb{R}^d$,

$$\nu \left( \bigcap_{i=1}^{n} A_i \right) \leq \prod_{i=1}^{n} \nu(A_i).$$

Note that we may take the $A_i$ to be the same event $A$ in this bound, in which case $\bigcap_{i=1}^{n} A_i$ is the event of the presence of $n$ disjoint instances of $A$. Many of our applications will take this form. The condition that $\nu$ is a complete probability measure is a technical one which does not affect our arguments, as we may assume that our vertex weight distribution $\nu$ is complete.

To state the formal realization of Step 2 of the outline, we define a $k$-disjoint grid journey to be a collection of grid journeys $\{L_m : m \in [1, k]\}$ such that there exist $k$ disjoint curves $\gamma_1, \ldots, \gamma_k$, ordered from left to right, with $\gamma_m$ intersecting each component of $L_m$ for each $m \in [1, k]$.

This can be expressed equivalently as the following constraint on the components of the $L_m$, which we label as $(L^m_1, \ldots, L^m_n)$; if $L^m_i = \mathbb{I}_{i, j_i, m}$, then $j_i, m \geq j_{i, m-1}$ for each $i \in [0, N], m \in [1, k]$. This means that, for every grid line of $G$ of slope $-1$, the interval chosen for $L_m$ coincides with or is to the right of that for $L_{m-1}$ for every $m \in [1, k]$. See the red and blue intervals depicted in Fig. 5.

We set $\varepsilon > 0$ for the rest of the proof of Theorem 3.2 to be

$$\varepsilon = \min(1, c_1 C_3^{-1}) \cdot k^{-2/3}, \quad (13)$$

where $c_1$ is as in Proposition 4.2. Thus with both $\eta$ and $\varepsilon$ fixed, the grid cells of $G$ have been fully specified.

With these definitions, we may state the next result, which obtains the $\exp(-ck^2)$ bound promised for Step 2 on the probability of there existing $k$ disjoint paths, each constrained to be in a thin region and each of which is not too light.

Proposition 4.5. Let $C_3 > 0$ be given and $\varepsilon$ as in (13). There exist positive finite $c_3$ and $C$ such that, for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$ satisfying $n > Ck$ and any $k$-disjoint grid journey $\{L_m : m \in [1, k]\}$,

$$\mathbb{P} \left( \bigcap_{m=1}^{k} \left( X_{L_m} > \mu n - C_3 k^{2/3} n^{1/3} \right) \right) \leq e^{-ck^2}.$$

Proof. In view of Proposition 4.2 and the value of $\varepsilon$, there exists $C < \infty$ such that, for $m \in [1, k]$ and $n > Ck$,

$$\mathbb{P} \left( X_{L_m} > \mu n - c_1 \varepsilon^{-1} n^{1/3} \right) \leq e^{-c_3 \varepsilon^{-3/2}}.$$

By (13), we may apply $\varepsilon < c_1 C_3^{-1} k^{-2/3}$ on the left and $\varepsilon < k^{-2/3}$ on the right. This yields

$$\mathbb{P} \left( X_{L_m} > \mu n - C_3 k^{2/3} n^{1/3} \right) \leq e^{-ck}.$$  \hspace{1cm} (14)

By the BK inequality Proposition 4.4 and (14), we obtain Proposition 4.5.  \hfill \square
4.5. Step 3: Bounding the number of $k$-disjoint grid journeys. Following Step 3, we plan a union bound over the collection of $k$-disjoint grid journeys $\{L_m : m \in \llbracket 1, k \rrbracket\}$. For this, we need a bound on the cardinality of this collection, which we record next.

**Lemma 4.6.** Let $\epsilon$ and $\eta$ be as set earlier. Then there exists $\delta = \delta(C_3, \eta) > 0$ such that the number of $k$-disjoint grid journeys is bounded by $e^{\frac{1}{2}c_3 k^2}$ for $k \geq 1$, with $c_3$ as in Proposition 4.5.

This lemma permits us to finish the proposed proof.

**Proof of Theorem 3.2.** We set $\delta$ to be as in Lemma 4.6 and $\epsilon$ as in Proposition 4.5. Let $E = E(n, k, \delta, C_3)$ be the event in the statement of Theorem 3.2. Note that $E \subseteq \bigcup (m=1^k) \{X_{L_m} > \mu n - C_3 k^{2/3} n^{1/3}\}$, where the union is over the collection of $k$-disjoint grid journeys $\{L_m : m \in \llbracket 1, k \rrbracket\}$. Lemma 4.6 says that the cardinality of this set is $\exp\left(\frac{1}{2}c_3 k^2\right)$, while Proposition 4.5 says that the probability of a single member of the union is at most $\exp(-c_3 k^2)$. Taking a union bound thus yields $P(E) < e^{-c_3 k^2 + \frac{1}{2}c_3 k^2} = e^{-\frac{1}{2}c_3 k^2}$, if $n \geq C \epsilon^{-3/2} = Ck$ and $k \in \mathbb{N}$. This proves Theorem 3.2 with $c = \frac{1}{2}c_3$. $\square$

One task remains.

**Proof of Lemma 4.6.** We first observe a bound on the number of ways to select the collection of intervals from the $i$th gridline $G_i$ that constitutes the union of the $i$th component of $L_m$ over $m \in \llbracket 1, k \rrbracket$ for a $k$-disjoint grid journey $\{L_m : m \in \llbracket 1, k \rrbracket\}$. Raising this bound to the power $N$ will then yield a bound on the cardinality of the set of $k$-disjoint grid journeys.

Assign to any collection of $k$ intervals from $G_i$ the $M$-vector whose $j$th coordinate records the number of times $L_{ij}$ was picked. The $M$-vector assigned has components which sum to $k$, and the cardinality of the set of such vectors is $\binom{k+M-1}{k}$. Not all such vectors can be achieved by the specified map, as we are ignoring the constraint imposed by the selection of intervals in $G_{i-1}$, and so this binomial coefficient is an upper bound. This then gives an upper bound of

$$\left[\binom{k+M-1}{k}\right]^N$$

on the number of $k$-disjoint grid journeys. Now to bound this quantity we recall that $M = 2[\delta k^{1/3} \eta^{-1} \epsilon^{-1}]$. Given $\alpha > 0$, we set $\delta$ to be

$$\delta = 2^{-1} \eta (1 + c_1^{-1} C_3)^{-1} \alpha.$$ 

Then, since $\epsilon^{-1} < (1 + c_1^{-1} C_3) k^{2/3}$ from the statement of Proposition 4.5, we get $M \leq \alpha k$ from (11). We recall the well-known fact [Ash90, eq. (4.7.4)] that for $0 < \beta < 1/2$ and $n \geq 1$,

$$\binom{n}{\beta n} \leq \exp(n H(\beta)),$$

where $H(x) = -x \log x - (1 - x) \log(1 - x)$,
for \( x \in [0, 1] \), is the entropy function. We use this to bound (15):
\[
\left( \frac{k + M - 1}{k} \right)^N \leq \left( (1 + \alpha) \frac{k}{\alpha k} \right)^N \leq \exp \left[ (1 + \alpha) k N \cdot H \left( \frac{\alpha}{1 + \alpha} \right) \right].
\]
Using that \( N = \varepsilon^{-3/2} \) and \( \varepsilon = \min(1, c_1 C_3^{-1} k^{-2/3}) \), we set \( \alpha \) small enough that the exponent is smaller than \( \frac{1}{2} c_3 k^2 \), since \( H(x) \to 0 \) as \( x \to 0 \). This bound holds for all \( k \geq 1 \), yielding
\[
\left( \frac{k + M - 1}{k} \right)^N \leq \exp \left( \frac{1}{2} c_3 k^2 \right).
\]

4.6. A crude upper bound of the watermelon weight: proving Lemma 3.4. Theorem II(1) implies that \( X_k^n - \mu nk \) is typically at most a negative quantity of order \( k^{5/3} n^{1/3} \). We will first need a cruder upper bound, Lemma 3.4, where this order is asserted without any claim about the sign of the difference. We begin by restating this application of the BK inequality.

**Lemma 3.4.** Under Assumption 3a, we may find \( c > 0 \) such that, if \( t > 0 \), there exists \( k_0 = k_0(t) \) for which \( n > n_0 \) and \( k_0 < k \leq t^{-3/4} n^{1/2} \) imply that
\[
P \left( X_k^n > \mu nk + tk^{5/3} n^{1/3} \right) \leq \exp \left( -ct^{3/2} k^2 \right).
\]
If instead Assumption 3b is available, then the upper bound on \( k \) may be taken to be \( n \).

As mentioned in Sect. 3, we will prove here Lemma 3.4 under only Assumption 3a, which limits the range of \( k \) up to \( O(n^{1/2}) \). The stronger conclusion of the lemma under Assumption 3b has a slightly more complicated argument, and this is provided in Appendix C.

**Proof of Lemma 3.4 under Assumption 3a.** For any fixed \( k \)-geodesic watermelon \( \mathcal{W} \), let \( X_{n,\mathcal{W}}^{k,j} \) be the weight of its \( j \)th heaviest curve, imitating notation introduced in Sect. 2. Define the event \( A_j \) by
\[
A_j = \left\{ \exists \mathcal{W} : X_{n,\mathcal{W}}^{k,j} > \mu n + \frac{1}{4} t j^{-2/3} k^{4/3} n^{1/3} \text{ for all } i \in [1, j] \right\}
\]
\[
= \left\{ \exists \mathcal{W} : X_{n,\mathcal{W}}^{k,i} > \mu n + \frac{1}{4} t j^{-2/3} k^{4/3} n^{1/3} \right\},
\]
where the second equality is due to the monotonicity relation \( X_{n,\mathcal{W}}^{k,i} \geq X_{n,\mathcal{W}}^{k,i+1} \) for all \( i \in [1, k] \). We start by claiming that, for \( j \in [1, k] \),
\[
P \left( A_j \right) \leq \exp \left( -ct^{3/2} k^2 \right).
\]
This is because the probability on the left-hand side is bounded by the probability that there exist \( j \) disjoint curves, each with weight at least \( \mu n + \frac{1}{4} t j^{-2/3} k^{4/3} n^{1/3} \). This probability, by Assumption 3a and the BK inequality (Proposition 4.4), is bounded by
\[
\left( \exp(-ct^{3/2} k^2 j^{-1}) \right)^j = \exp \left( -ct^{3/2} k^2 j \right),
\]
for large enough $n$, if $tk^{2/3}$ is large enough. For this bound, we also need $\frac{1}{4}tj^{-2/3}k^{4/3}n^{1/3} \leq n$, which is provided by the assumed upper bound on $k$. We now claim that

$$\left\{ X_n^k > \mu nk + tk^{5/3}n^{1/3} \right\} \subseteq \bigcup_{j=1}^{k} A_j.$$  

Indeed, suppose that $A^c_j$ occurs for each $j \in \llbracket 1, k \rrbracket$. For each $j \in \llbracket 1, k \rrbracket$, by definition, on $A^c_j$, $X_{n,W}^{k,j} \leq \mu n + \frac{1}{4}tj^{-2/3}k^{4/3}n^{1/3}$ for all $k$-geodesic watermelons $W$. This then gives, for any $W$, that

$$X_n^k = \sum_{j=1}^{k} X_{n,W}^{k,j} \leq \mu nk + \frac{1}{4}tk^{4/3}n^{1/3} \sum_{j=1}^{k} j^{-2/3} \leq \mu nk + tk^{5/3}n^{1/3},$$

which completes the proof of the claim. The derivation of Lemma 3.4 is concluded by applying (16), using the union bound, and reducing $c$ in (16) to $c/2$ (which is possible for $k$ large enough depending on only $t$).

5. Not Too Thin: Bounding Below the Transversal Fluctuation

The aim of this section is to prove the lower bound on the transversal fluctuation exponent for the geodesic, i.e., Theorem II(2b). In view of Theorem 3.2, this would be immediate if we could show that, with high probability, $X_n^{k,k} \geq \mu n - Ck^{2/3}n^{1/3}$ for some $C > 0$. However, in the absence of such an estimate so far, we follow the strategy outlined in Section 3, relying on an averaging argument and geodesic watermelon interlacing. More precisely, the interlacing result Proposition 2.4 implies that it is sufficient to show the existence of some $j \in \llbracket 1, k \rrbracket$ such that no $j$-geodesic watermelon is contained in the strip $U_{n,\delta k^{1/3}n^{2/3}}$. To this end, we shall establish an averaged version of the lower bound of $X_n^{k,k}$, i.e., with high probability, that $X_n^{k,j} \geq \mu n - Ck^{2/3}n^{1/3}$ for some $j \in \llbracket \lfloor \frac{k}{2} \rfloor, k \rrbracket$. This, together with the above observation and Theorem 3.2, will complete the proof.

We note in the above discussion that $X_n^{j,j}$, unlike $X_n^j$, is only well-defined after specifying the $j$-geodesic watermelon—and this we have not done in the case where $\nu$ has atoms. However, using a uniform lower bound on $X_n^{j,j}$ which holds over all possible choices of the curves of the $j$-geodesic watermelon, we will be able to show that the averaging statement holds for $X_n^{j,j}$, defined for $j \in \llbracket 1, n \rrbracket$ as

$$X_n^{j,j} = \min_{W} X_{n,W}^{j,j},$$

where $X_{n,W}^{j,j}$ is the weight of the lightest curve of the $j$-geodesic watermelon $W$, and the minimum is over all $j$-geodesic watermelons.

We now state the averaged result.

**Lemma 5.1.** For $C_1$ as in Theorem II(1), and under Assumption 3a (respectively 3b), there exist positive constants $c_1$, $c > 0$, $C$ and $n_0$ such that, for $k \geq 1$ and $n \geq n_0$ for which $k \leq c_1 n^{1/2}$ (resp. $k \leq c_1 n$),

$$\mathbb{P} \left( \bigcup_{j=\lfloor \frac{k}{2} \rfloor}^{k} \left\{ X_n^{j,j} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \right) \geq 1 - Ce^{-ck^2}.$$
Here $k = 4$. The four curves shown are the curves of the $k$-melon, and the orange curve is the curve of lowest weight among the four, and so has weight $X_{kn}$. The remaining green curves form a collection of three disjoint curves, and hence have cumulative weight bounded above by $X_{kn}$. Thus $X_n^{k,k} + X_n^{k-1} \geq X_n^k$, and rearranging gives the inequality

$$X_{kn} \geq X^k_{n} - X_n^{k-1}.$$

**Proof.** Recall from (6) that, for every $j$, $X_n^{j,j} \geq X_n^j - X_n^{j-1}$. Indeed, any $j$-geodesic watermelon is comprised of a lightest curve and $j-1$ other curves that together weigh at most as much as the $(j-1)$-geodesic watermelon. See Fig. 6. We see then that, for any $k$,

$$\mathbb{P}\left( \bigcup_{j=\lceil k/2 \rceil}^k \left\{ X_n^{j,j} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \right) \geq \mathbb{P}\left( \bigcup_{j=\lceil k/2 \rceil}^k \left\{ X_n^j - X_n^{j-1} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \right).$$

We now observe that

$$\mathbb{P}\left( \bigcap_{j=\lceil k/2 \rceil}^k \left\{ X_n^j - X_n^{j-1} \leq \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \right) \leq \mathbb{P}\left( X_n^k - X_n^{\lceil k/2 \rceil-1} \leq \mu n (k + 1 - \lceil k/2 \rceil) - 2C_1 k^{5/3} n^{1/3} \right) \leq e^{-ck^2}.$$

From Lemma 3.4 with $t = 2^{5/3} C_1$, with the valid range depending on whether Assumption 3b is in force in that statement, we have that $\mathbb{P}\left( X_n^{\lceil k/2 \rceil-1} > \mu n ([k/2] - 1) + C_1 k^{5/3} n^{1/3} \right) \leq e^{-ck^2}$. Thus,

$$\mathbb{P}\left( X_n^k - X_n^{\lceil k/2 \rceil} \leq \mu n (k + 1 - \lceil k/2 \rceil) - 2C_1 k^{5/3} n^{1/3} \right) \leq e^{-ck^2}.$$
\[ P \left( X^k_n \leq \mu nk - C_1 k^{5/3} n^{1/3} \right) + e^{-ck^2}. \]

The weight lower bound of Theorem II(1) tells us that the first term is bounded by \( e^{-ck^2} \).

By the three preceding displays, we find that

\[ P \left( \bigcup_{j=\lfloor k/2 \rfloor}^k \{ X^{j,n}_n > \mu n - 4C_1 k^{2/3} n^{1/3} \} \right) \geq 1 - 2e^{-ck^2}, \]

so we are done.

We are now ready to complete the proof of Theorem II(2b).

**Proof of Theorem II(2b).** Under Assumption 3a (resp. 3b) and \( c_1 \) as in Lemma 5.1, fix \( k \) sufficiently large and \( n \) such that \( k \leq c_1 n^{1/2} \) (resp. \( k \leq c_1 n \)). Let \( C_1 \) be as in Theorem II(1). Let \( \delta = \delta(2^{2/3} \cdot 4C_1) \) be as in Theorem 3.2 and let \( \delta' = 2^{-1/3} \delta \). Let \( A_k \) denote the event that there exist \( \lfloor k/2 \rfloor \) disjoint paths contained in \( U_{n,\delta' k^{1/3} n^{2/3}} \), each of which has weight at least \( \mu n - 4C_1 k^{2/3} n^{1/3} \). Further, let \( B_k \) denote the event from Lemma 5.1:

\[ B_k := \left\{ \bigcup_{j=\lfloor k/2 \rfloor}^k \{ X^{j,n}_n > \mu n - 4C_1 k^{2/3} n^{1/3} \} \right\}. \]

Clearly, by Theorem 3.2 and Lemma 5.1, we have \( P(A_k \cap B_k) \geq 1 - e^{-ck^2} \) for some \( c > 0 \).

Observe next that on \( A_k^c \cap B_k \), there exists \( j \in \{ \lfloor k/2 \rfloor, \ldots, k \} \) such that all the curves of all \( j \)-geodesic watermelons have weight at least \( \mu n - 4C_1 k^{2/3} n^{1/3} \), and hence some of them must exit \( U_{n,\delta' k^{1/3} n^{2/3}} \). By the interlacing result Proposition 2.4, the same is true for all \( k \)-geodesic watermelons. This completes the proof of Theorem II(2b) with \( \delta \) there replaced by \( \delta' \).

6. Not Too Wide: Bounding Above the Transversal Fluctuation

Our objective in this section is to prove Theorem II(2a). We derive this result using Theorem 3.3, whose proof appears at the end of the section. The basic idea is same as in Sect. 5. To show that the \( k \)-geodesic watermelon has transversal fluctuation of order \( k^{1/3} n^{2/3} \) with large probability, we shall rely on the interlacing property and show that, with large probability, there exists \( j \in \lfloor k, 2k \rfloor \) such that the \( j \)-geodesic watermelon has transversal fluctuations at most of order \( k^{1/3} n^{2/3} \). Admitting Theorem 3.3 for now, we may prove Theorem II(2a).

**Proof of Theorem II(2a).** Let \( C_1 \) be as in the Lemma 5.1 and let

\[ B'_k = \bigcup_{j=k}^{2k} \{ X^{j,n}_n > \mu n - 7C_1 k^{2/3} n^{1/3} \}; \]

note that \( 4 \times 2^{2/3} < 7 \), so that Lemma 5.1 with \( 2k \) in place of \( k \) implies that \( P(B'_k) \geq 1 - e^{-ck^2} \).
Let \( A_k' = A_k'(C'') \) denote the event that there exists a path from \((1, 1)\) to \((n, n)\) that exits \( U_{n,C''k^{1/3}n^{2/3}} \) (recall the notation from (3)) and has weight at least \( \mu n - 7C_1 k^{2/3} n^{1/3} \). Now choose \( C'' > 0 \) (possible by invoking Theorem 3.3 with \( s \) a multiple of \( k^{1/3} \)) such that
\[
\mathbb{P}(A_k') \leq e^{-c'' k}
\]
for some \( c'' > 0 \), for all \( k \) sufficiently large and all \( n \) sufficiently large, depending on \( k \). Clearly, it now suffices to show that, on \( B_k' \cap (A_k')^c \), no \( k \)-geodesic watermelon exits \( U_{n,C''k^{1/3}n^{2/3}} \). By definition of \( B_k' \), there exists \( j \in [k, 2k] \) such that all paths of all \( j \)-geodesic watermelons have weight at least \( \mu n - 7C_1 k^{2/3} n^{1/3} \), and \( (A_k')^c \) ensures that all these paths are contained in \( U_{n,C''k^{1/3}n^{2/3}} \). The proof is completed by interlacing, invoking Proposition 2.4, with the parameter \( M \) in Theorem II(2a) being \( C'' \). \( \square \)

Now we turn to the proof of Theorem 3.3. We will follow the argument that yields Theorem 11.1 of [BSS14], but merely suppose Assumptions 2 and 3a. We first prove a companion result concerning paths that have large transversal fluctuation at the midpoint. The idea is that the weight of such a path is less than the sum of the weights of two point-to-line paths whose endpoints lie outside a central interval. We will make use of the upper tail estimate for such point-to-line weights, Proposition 3.6.

For an upright path \( \Gamma \) from \((1, 1)\) to \((r, r)\), and \( x \in [1, r] \), let \( \Gamma(x) \) denote the unique integer such that \((x - \Gamma(x), x + \Gamma(x))\) lies on \( \Gamma \). Recall the definitions of \( \mathbb{L}_{\text{left}} \) and \( \mathbb{L}_{\text{right}} \) (which depend on \( r \) and \( t \geq 1 \)) from (4).

**Proposition 6.1.** Let \( X_{\text{mid}}^{r,s,t} \) be the maximum weight of all paths \( \Gamma \) from \( \mathbb{L}_{\text{left}} \) to \( \mathbb{L}_{\text{right}} \) such that \( |\Gamma(r/2)| > (s + t)r^{2/3} \). Then there exist constants \( r_0, s_0, c > 0 \) and \( \tilde{c}_2 > 0 \) (all independent of \( t \)) such that for \( s > s_0 \), \( r > r_0 \) and \( 0 < t \leq s^2 \),
\[
\mathbb{P} \left( X_{\text{mid}}^{r,s,t} - \mu r > -\tilde{c}_2 s^2 r^{1/3} \right) < e^{-c s^3}.
\]

**Proof.** Let \( X_1 \) be the maximum weight of all paths \( \Gamma \) from \( \mathbb{L}_{\text{left}} \) to the line \( x + y = r \) whose endpoint is outside the interval joining \((r/2 - (s + t)r^{2/3}, r/2 + (s + t)r^{2/3})\) and \((r/2 + (s + t)r^{2/3}, r/2 - (s + t)r^{2/3})\). Let \( X_2 \) be the same for paths starting on the line \( x + y = r \) outside the mentioned interval and ending on \( \mathbb{L}_{\text{right}} \). Then
\[
\left\{ X_{\text{mid}}^{r,s,t} > \mu r - \tilde{c}_2 s^2 r^{1/3} \right\} \subseteq \bigcup_{i=1}^{2} \left\{ X_i > \mu r - 0.5\tilde{c}_2 s^2 r^{1/3} \right\}.
\]

From here, we apply Proposition 3.6 with \( \theta = 0 \), which yields
\[
\mathbb{P} \left( X_i > \mu r - 0.5\tilde{c}_2 s^2 r^{1/3} \right) \leq e^{-c s^3}
\]
for \( i \in \{1, 2\} \); a union bound now completes the proof of Proposition 6.1. \( \square \)

We now use a multi-scale argument to extend the result at the midpoint to transversal fluctuations at any point. Roughly, we will dyadically place points on the diagonal \( x = y \) between \((1, 1)\) and \((r, r)\). Suppose that some path has transversal fluctuation greater than \((s + t)r^{2/3}\) at some point while satisfying the weight lower bound. Then a consecutive pair of points in the dyadic division exists such that the intervening path is not too light and has a midpoint whose transversal fluctuation has order \((s + t)r^{2/3}\). The probability
of this circumstance is bounded by Proposition 6.1. Bounds on path weights for each side of this dyadic interval will finish the proof.

We remind the reader from the statement of Theorem 3.3 that \( \Gamma^{r,s,t} \) refers to the maximum weight over all paths \( \Gamma^{r,s,t} \) from \( \mathbb{L}_{\text{left}} \) to \( \mathbb{L}_{\text{right}} \) with transversal fluctuation greater than \( (s + t)r^{2/3} \) at some point. We now fix \( \Gamma^{r,s,t} \) to be the leftmost such path (which is uniquely defined by the weight-maximization property of the path) whose weight is \( X_{r,s,t} \).

**Proof of Theorem 3.3.** We may assume that \( s \leq r^{1/3} \), as otherwise the theorem is trivial. We may also simplify by assuming that \( \Gamma^{r,s,t}(x) > (s + t)r^{2/3} \) for some \( x \in [1, r] \) (the case where \( \Gamma^{r,s,t}(x) < -(s + t)r^{2/3} \) is symmetric). Let \( A \) be the event \( \{ X_{r,s,t} > \mu r - c_2 s^2 r^{1/3} \} \) for \( c_2 \) to be specified later. For \( j \geq 1 \), let \( S_j \) be the dyadic points, i.e.,

\[
S_j = \left\{ \ell 2^{-j} r : \ell \in [0, 2^j] \right\}.
\]

For \( j \geq 1 \), let \( T_j \) be the event defined by

\[
T_j = \left\{ \Gamma^{r,s,t}(x) < (s_j + t)r^{2/3} \quad \forall x \in S_j \right\},
\]

where

\[
s_j = \frac{s}{M} \prod_{i=1}^{j-1} \left( 1 + 2^{-i/3} \right) \quad \text{and} \quad M = 2 \cdot \prod_{i=1}^{\infty} \left( 1 + 2^{-i/3} \right) < \infty,
\]

so that \( s_j < \frac{1}{2}s \) for all \( j \). We fix \( j_0 \) so that the separation between points in \( S_{j_0} \) is \( \frac{s}{10} r^{2/3} \), i.e., \( j_0 \) is such that \( 2^{-j_0} r = \frac{s}{10} r^{2/3} \). The next lemma asserts that the defined dyadic breakup is fine enough to capture any path which has a high transversal fluctuation at some point.
Lemma 6.2. We have that \( \bigcap_{j=1}^{j_0} T_j = \emptyset \).

**Proof.** By definition of \( \Gamma^{r,s,t} \), there is a \( x \in [1, r] \) such that \( \Gamma^{r,s,t}(x) > (s+t)r^{2/3} \). Let \( \tilde{x} \) be the smallest point of \( S_{j_0} \) bigger than \( x \). Observe that since \( \Gamma^{r,s,t} \) is an upward path, its y-coordinate \( x + \Gamma^{r,s,t}(x) \) is increasing in \( x \), so that \( x + \Gamma^{r,s,t}(x) \leq \tilde{x} + \Gamma^{r,s,t}(\tilde{x}) \). From the definition of \( S_{j_0} \), we also have that \( \tilde{x} \leq x + \frac{s}{10}r^{2/3} \). This implies

\[
\Gamma^{r,s,t}(x) \leq (\tilde{x} - x) + \Gamma^{r,s,t}(\tilde{x}) \leq \left( \frac{s}{10} + s_{j_0} + t \right) r^{2/3},
\]

the last inequality holding on \( T_{j_0} \). However \( s_{j_0} + \frac{s}{10} + t < \frac{s}{2} + \frac{s}{10} + t < s + t \), a contradiction.

Let \( T_0 \) be the whole probability space. From Lemma 6.2 we have

\[
\mathbb{P}(A) = \mathbb{P}\left( \bigcup_{j=0}^{j_0} T_j^c \cap A \right) \leq \sum_{j=1}^{j_0} \mathbb{P}\left( T_{j-1} \cap T_j^c \cap A \right).
\]

We set \( r > r_0 \) and \( s > s_0 \), where \( r_0 \) and \( s_0 \) are obtained from Proposition 6.1. Then Proposition 6.1 establishes that \( \mathbb{P}(T_1^c \cap A) \leq e^{-cs^3} \), and so the proof of Theorem 3.3 is completed by the next lemma.

**Lemma 6.3.** Let \( r \) and \( s \) be as before. There exists \( c > 0 \) such that, for \( j \geq 2 \),

\[
\mathbb{P}\left( T_{j-1} \cap T_j^c \cap A \right) \leq 2^{-j} e^{-cs^3}.
\]

**Proof.** We split \( T_j^c = \bigcup_{h=0}^{2^j} T_{j,h}^c \) based on at which dyadic point \( h2^{-j}r \) of \( S_j \) we have \( \Gamma^{r,s,t}(h2^{-j}r) > (s_j + t)r^{2/3} \).

Let \( \mathbb{L}_1 \) be the line segment of length \( 2(s_{j-1} + t)r^{2/3} \) of slope \(-1\) centred at \((h - 1)2^{-j}r, (h - 1)2^{-j}r\), and let \( \mathbb{L}_2 \) be the same centred at \((h + 1)2^{-j}r, (h + 1)2^{-j}r\).

Now, on \( T_{j-1} \cap T_j^c \cap A \), there exists a path \( \Gamma_1 \) from \( \mathbb{L}_1 \) to \( \mathbb{L}_1 \), a path \( \Gamma_2 \) from \( \mathbb{L}_1 \) to \( \mathbb{L}_2 \), and a path \( \Gamma_3 \) from \( \mathbb{L}_2 \) to \( \mathbb{L}_2 \) with the following properties: (i) \( \Gamma_2(h2^{-j}r) > (s_j + t)r^{2/3} \) and (ii) \( \ell(\Gamma_1) + \ell(\Gamma_2) + \ell(\Gamma_3) > \mu r - c5^{2/3}r^{1/3} \).

We will first show that, due to condition (i), \( \Gamma_2 \) must suffer a large weight loss, via Proposition 6.1. Then we will show using Proposition 3.5 that \( \Gamma_1 \) and \( \Gamma_3 \) are unlikely to be able to make up this loss sufficiently well for (ii) to occur. The basic reason for the large weight loss of \( \Gamma_2 \) is that the loss from the transversal fluctuation is much amplified by its being defined on the scale \( 2^{-3(j-1)}r \) instead of \( r \).

We set the parameters for the application of Proposition 6.1. In order to avoid confusion, the parameter values will be distinguished by tildes. So set \( \tilde{r} = 2^{-3(j-1)}r \). As the endpoint of \( \Gamma_2 \) can lie anywhere on an interval of length \( 2(s_{j-1} + t)r^{2/3} \), set \( \tilde{r} \) such that \( \tilde{r}(\tilde{r})^{2/3} = (s_{j-1} + t)r^{2/3} \); i.e., \( \tilde{r} = (s_{j-1} + t)2^{j-3}r^{2/3} \). The path \( \Gamma_2 \) at its midpoint it must be at least \( (s_j + t)r^{2/3} \) away from the diagonal, so the minimum transversal fluctuation the path undergoes is \( (s_{j-1} + t)(s_{j-1} + t)2^{j-3}r^{2/3} \); (using (17)). Accordingly we set \( \tilde{s} = s_{j-1}2^{j-3}r^{2/3} \), so that \( \tilde{s}(\tilde{s})^{2/3} = (s_{j-1} - s_{j-1})2^{j-3}r^{2/3} \).

To apply Proposition 6.1 with these parameters, we require \( \tilde{s} \leq \left( \tilde{s}^2 \right)^{2/3} \). Since the hypotheses of Theorem 3.3 include that \( t \leq s \), and since \( s_j \in [s/M, s/2] \) for all \( j \), the requirement is implied if

\[
\frac{3s}{2} \cdot 2^{2(j-1)/3} \leq \frac{s^2}{M^2} \cdot 2^{2(j-1)/3},
\]
which clearly holds for all large enough $s$. So, making use of distributional translational invariance of the environment and applying Proposition 6.1, we obtain
\[ \mathbb{P}\left( \ell(\Gamma_2) > \mu \tilde{r} - \tilde{c}_2(\tilde{s})^2(\tilde{r})^{1/3} \right) \leq \exp\left( -c(\tilde{s})^3 \right); \]
i.e.,
\[ \mathbb{P}\left( \ell(\Gamma_2) > \mu \tilde{r} - \tilde{c}_2 s_{j-1}^2(2(j-1)/3 r)^{r^{1/3}} \right) \leq \exp\left( -c2^{(j-1)/3} s^3 \right). \]
This yields
\[ \mathbb{P}\left( \sum_{i=1}^{3} \ell(\Gamma_i) > \mu r - c_2 s^2 r^{2/3} \right) \leq \mathbb{P}\left( \ell(\Gamma_2) > \mu \tilde{r} - \tilde{c}_2 (2(j-1)/3 s_{j-1}^2 r)^{r^{1/3}} \right) \]
\[ + \mathbb{P}\left( \ell(\Gamma_1) + \ell(\Gamma_3) > \mu (r - \tilde{r}) + (\tilde{c}_2 2^{(j-1)/3} s_{j-1} - c_2 s^2) r^{1/3} \right). \]
We must address the second term. Set $c_2 = \tilde{c}_2 / 2 M^2$, so that $\tilde{c}_2 s_{j-1}^2 / 2 > c_2 s^2$. After dividing $\mathbb{L}_1$ and $\mathbb{L}_2$ into segments of length $r^{2/3}$, applying Proposition 3.5, and taking a union bound over the segments, we get that the second probability is bounded by $\exp\left( -c2^{(j-1)/2} s^3 \right)$.

For large enough $s$, these two quantities are bounded by $4^{-j} e^{-c s^3}$, and taking a union bound over the $2^j$ values of $h$, we obtain Lemma 6.3.

Theorem 3.3 readily implies that the transversal fluctuation of the geodesic $\Gamma_n$ from $(1, 1)$ to $(n, n)$ is of order $n^{2/3}$, with the optimal tail exponent of three.

**Corollary 6.4.** There exist constants $s_0$ and $n_0$ such that for $s > s_0$ and $n > n_0$,
\[ \mathbb{P}\left( \text{TF}(\Gamma_n) > s n^{2/3} \right) \leq e^{-c s^3}. \]

**Proof.** We have
\[ \mathbb{P}\left( \text{TF}(\Gamma_n) > s n^{2/3} \right) \leq \mathbb{P}\left( \text{TF}(\Gamma_n) > s n^{2/3}, X_n > \mu n - c_2 s^2 n^{1/3} \right) \]
\[ + \mathbb{P}\left( X_n \leq \mu n - c_2 s^2 n^{1/3} \right) \leq e^{-c s^3} + e^{-c s^3}, \]
the last inequality from Theorem 3.3 for the first term and Assumption 3a for the second. □

We end this section by recording a version of Theorem 3.3 for point-to-line paths which will be used in Sect. 11.

**Theorem 6.5.** Let $\Gamma_{\text{line}}^{r,s}$ be the maximum weight path among all paths $\Gamma$ from $(1, 1)$ to the line $x + y = 2r$ with $\text{TF}(\Gamma) > s r^{2/3}$, and let $X_{\text{line}}^{r,s}$ be its weight. Under Assumptions 2 and 3a, there exist absolute constants $r_0$, $s_0$, $c > 0$ and $c_3 > 0$ such that, for $s_0 < s < r^{1/3}$ and $r > r_0$,
\[ \mathbb{P}\left( X_{\text{line}}^{r,s} - \mu r > -c_3 s^2 r^{1/3} \right) \leq e^{-c s^3}. \]

**Proof.** Let $x$ be such that $\Gamma_{\text{line}}^{r,s}$ has endpoint $(r - x, r + x)$. Let $A$ denote the event that $|x| < s r^{2/3} / 2$. On the event $A$, we have the claimed bound by applying Theorem 3.3 with $t = s / 2$. On $A^c$, the claim arises by applying Proposition 3.6 with $t = 0$, $\theta = 0$, and $s / 2$ in place of $s$ in its statement. □
7. Not Too Heavy: Bounding Above the Watermelon Weight

Our aim in this section is to complete the proof of the weight upper bound in Theorem II(1) and to provide the proof of Proposition 10.1. Recall that for the former our rough aim is to prove that there exist \( C_2 > 0 \) and \( c > 0 \) such that, for appropriate ranges of \( k \) and \( n \),

\[
\mathbb{P} \left( X_n^k > \mu n - C_2 k^{5/3} n^{1/3} \right) \leq e^{-ck^2}.
\]

We start by proving that with high probability at least \( k/4 \) curves of the \( k \)-geodesic watermelon must exit a strip \( U = U_{n, \frac{1}{2} \delta k^{1/3} n^{2/3}} \) around the diagonal of width \( \frac{1}{2} \delta k^{1/3} n^{2/3} \), with \( \delta \) as in Theorem 3.2. We will resort to proving an averaged version, as in Lemma 5.1; i.e., we will show that there exists \( j \in \llbracket [k/2], k \rrbracket \) such that \( k/4 \) curves of the \( j \)-melon exit \( U \). This will suffice by the interlacing guaranteed by Proposition 2.4.

For \( j \in \llbracket 1, n \rrbracket \) and \( \delta > 0 \), define \( E^j_n(\delta) \) to be the minimum of the number of curves of a \( j \)-geodesic watermelon \( \Gamma_n^j \) which exit \( U \), minimized over all \( j \)-geodesic watermelons \( \Gamma_n^j \).

**Lemma 7.1.** Let \( C_1 > 0 \) be as given in Theorem II(1), and let \( \delta = \delta(4C_1) \) be as in Theorem 3.2. There exists \( c_1 > 0 \) and \( c > 0 \) such that, under Assumption 3a (resp. 3b), for \( k \leq c_1 n^{1/2} \) (resp. \( k \leq c_1 n \)),

\[
\mathbb{P} \left( \bigcup_{j=[k/2]}^k \left\{ E^j_n(\delta) > \frac{k}{4} \right\} \right) \geq 1 - e^{-ck^2}.
\]

**Proof.** Recall from Sect. 5 the definition of \( X_n^{j,i} \) as the minimum weight of the lightest curve of a \( j \)-geodesic watermelon \( \mathcal{W} \) over all such \( \mathcal{W} \). We will bound below

\[
\mathbb{P} \left( \bigcup_{j=[k/2]}^k \left\{ X_n^{j,i} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \cap \left\{ E^j_n(\delta) > \frac{k}{4} \right\} \right),
\]

which clearly suffices. We have

\[
\bigcap_{j=[k/2]}^k \left\{ X_n^{j,i} \leq \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \cup \left\{ E^j_n(\delta) \leq \frac{k}{4} \right\} \leq \bigcup_{j=[k/2]}^k \left\{ E^j_n(\delta) \leq \frac{k}{4}, X_n^{j,i} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\}.
\]

Focus on the right-hand side of this display. Lemma 5.1 says that the probability of the first term in parentheses is bounded by \( e^{-ck^2} \). For the second term, note that, on each inner event, there exist \( 3k/4 \) curves contained in \( U_{n, \frac{1}{2} \delta k^{1/3} n^{2/3}} \) whose weight is at least \( \mu n - 4C_1 k^{2/3} n^{1/3} \). Theorem 3.2 then implies that the probability of the inner event is bounded by \( e^{-ck^2} \) for all \( j \); and a union bound, accompanied by a reduction in the value of \( c \), completes the proof. \( \square \)
Fig. 8. An illustration of how curves of $\Gamma_n^j$ exiting a strip force the same for curves of $\Gamma_n^k$ for $j = 3$ (red) and $k = 5$ (blue), where the two ensembles are elements of an interlacing family of geodesic watermelons. The underlying observation is that interlacing allows one to match each curve of $\Gamma_n^j$ with one curve of $\Gamma_n^k$ which has larger transversal fluctuation. Perhaps contrary to intuition, interlacing does not imply that a single curve of $\Gamma_n^j$ exiting a strip forces at least $k - j$ curves of $\Gamma_n^k$ to exit the same strip. This can also be seen in the figure, as there is only a single curve of $\Gamma_n^k$ which has a larger transversal fluctuation on the left than $\Gamma_n^j$ even though $k - j = 2$.

Proof of the weight upper bound in Theorem II(1). Let $B_1$ denote the event that for no $j \in \llbracket [k/2], k \rrbracket$ are there at least $k/4$ curves in every $j$-geodesic watermelon that exit the strip of width $\frac{1}{2} \delta k^{1/3} n^{2/3}$. By Lemma 7.1, there exists $c > 0$ such that

$$\mathbb{P}(B_1) \leq e^{-ck^2}.$$  \hfill (19)

By interlacing Proposition 2.5, we see that when $B_1^c$ occurs, at least $k/4$ of the curves of every $k$-geodesic watermelon must exit the strip of width $\frac{1}{2} \delta k^{1/3} n^{2/3}$: see Fig. 8.

Let $B_2$ denote the event that there exists an upright path $\Gamma$ from $(1, 1)$ to $(n, n)$ for which $\ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}$ and $\text{TF}(\Gamma) > \frac{1}{2} \delta k^{1/3} n^{2/3}$, for a value of $c' > 0$ to be specified next. Indeed, by Theorem 3.3, there exists $c' > 0$ and $c > 0$ such that

$$\mathbb{P}(B_2) \leq e^{-ck}.$$ \hfill (20)

Set $A = \{ X_n^k > \mu nk - \frac{1}{16} c' k^{5/3} n^{1/3} \}$ and

$$B_3 = \left\{ X_n^{[7k/8]} > \mu n \cdot [7k/8] + \frac{1}{16} c' k^{5/3} n^{1/3} \right\},$$

so that, by Lemma 3.4 under Assumption 3a (resp. 3b), for $k \leq c_1 n^{1/2}$ (resp. $k \leq c_1 n$),

$$\mathbb{P}(B_3) \leq e^{-ck^2}.$$ \hfill (21)

We now fix some $k$-geodesic watermelon $\Gamma_n^k$, and let $X_n^{k:j}$ be the weight of the $j^{th}$ heaviest curve of $\Gamma_n^k$ for each $j \in \llbracket 1, k \rrbracket$. The weight of the $[7k/8]$ heaviest curves of
Here, we prove Theorem 3.1. The concerned task is the development of one of the rated on the order of \( k \) occurs, \( \left(X_n^{[7k/8]} + \left(X_n^{[7k/8]+1} + \ldots + X_n^{k,k}\right) \right) \geq X_n > \mu nk - \frac{1}{16}c'k^{5/3}n^{1/3} \)
\[ \implies X_n^{[7k/8]+1} + \ldots + X_n^{k,k} > \frac{1}{8} \mu nk - \frac{1}{8}c'k^{5/3}n^{1/3}. \]

By the ordering of \( \{X_n^{k,i} : i \in [1,k]\} \), this then implies that \( X_n^{k,[7k/8]+1} > \mu n - c'k^{2/3}n^{1/3} \). Again by the ordering, this bound applies to \( X_n^{k,1}, \ldots, X_n^{k,[7k/8]} \) as well. This means we have \([7k/8]\) disjoint curves \( \Gamma \), each with \( \ell(\Gamma) > \mu n - c'k^{2/3}n^{1/3} \).

Thus, on \( A \cap B_1^c \cap B_3^c \) and by the pigeonhole principle, we must have at least \([k/8]\) disjoint curves \( \Gamma \), each satisfying \( \text{TF}(\Gamma) > \frac{1}{2}\delta k^{1/3}n^{2/3} \) and \( \ell(\Gamma) > \mu n - c'k^{2/3}n^{1/3} \).

By the BK inequality and (20), the probability of this occurrence is seen to be at most \( \exp(-ck \cdot k) = \exp(-ck^2) \). Noting the bounds (19) and (21) on \( \mathbb{P}(B_1) \) and \( \mathbb{P}(B_3) \), we complete the proof of the weight upper bound of Theorem II(1), by taking \( C_2 \) in its statement to be \( c'/16 \).

\[ \square \]

8. Not Too Light: Bounding Below the Watermelon Weight

Here, we prove Theorem 3.1. The concerned task is the development of one of the principal new tools offered in this paper: the construction of collections of disjoint paths in the square \([1,n]^2\) that achieve a certain weight with high probability.

In view of the form of the theorem, the task is to construct, for some \( c > 0, c_1 > 0, k \leq c_1n, \) and \( m \leq k, m \) disjoint paths \( \gamma_1, \ldots, \gamma_m \) such that \( \sum_{i=1}^m \ell(\gamma_i) \geq \mu mm - C_1mk^{2/3}n^{1/3} \) and max, \( \text{TF}(\gamma_i) \leq 2mk^{-2/3}n^{1/3} \), with probability at least \( 1 - \exp(-cmk) \).

8.1. The construction in outline. The construction leading to Theorem 3.1 may be explained in light of Theorem I(2) on the width of the \( k\)-geodesic watermelon having order \( k^{1/3}n^{2/3} \), even if the latter assertion is a consequence of the theorem rather than a means for deriving it. Indeed, that \( k \) watermelon curves coexist in a strip of width \( k^{1/3}n^{2/3} \) suggests that, at least around the mid-height \( n/2 \), adjacent curves will separated on the order of \( k^{-2/3}n^{2/3} \). We will demand this separation for the \( m \) curves in our construction. The curves will begin near \((1,1)\) and end near \((n,n)\) at unit-order distance, so we must guide them apart to become separated during their mid-lives.

We will index the life of paths in the square \([1,n]^2\) according to distance along the diagonal interval, indexed so that \([a,b]\) refers to the region between the lines \( x + y = 2a \) and \( x + y = 2b \). The diagonal interval \([1,n] \) that indexes the whole life of paths in the construction will be divided into five consecutive intervals called phases that carry the names take-off, climb, cruise, descent and landing. By the start of cruise, the middle phase, the sought separation has been obtained, and it will be maintained there. This separation is gained during take-off and climb, and it is undone in a symmetric way during descent and landing.

Take-off is a short but intense phase that takes the curves at unit-order separation on the tarmac to a consecutive separation of order \( k^{-1/3}n^{1/3} \) in a duration (or height) of order \( k^{2/3}n^{1/3} \). Climb is a longer and gentler phase, of duration roughly \( n/3 \), in which separation expands geometrically until it reaches the scale \( k^{-2/3}n^{2/3} \). Cruise is a stable phase of rough duration \( n/3 \).
The shortfall in weight of the $m$ paths in Theorem 3.1 relative to the linear term $\mu nm$ has order $mk^{2/3}n^{1/3}$. The weight shortfall in each phase is the difference in total weight contributed by the curve fragments in the phase and the linear term given by the product of $\mu m$ and the duration of the phase. The weight shortfall will be shown to have order $mk^{2/3}n^{1/3}$ for each of the five phases.

Take-off is a phase where gaining separation is the only aim. No attempt is made to ensure that the constructed curves have weight, and the trivial lower bound of zero on weight is applied. The weight shortfall is thus at most $\mu m \cdot \Theta(1)k^{2/3}n^{1/3}$.

The climb phase is where a rapid increase in separation is obtained, via a geometrically increasing separation across scales (which depend on $k$). A calculation is needed to bound above the shortfall by $mk^{2/3}n^{1/3}$ for the climb (and for the descent) phase. The calculation will estimate the loss incurred across the scales using the parabolic loss in weight of Assumption 2, relying on a precise choice of paths which do not incur too large a parabolic loss. The climb and descent phases are at the heart of the construction.

Cruise must maintain consecutive separation of order $k^{-2/3}n^{2/3}$ for a duration of roughly $n/3$. We construct paths that travel through an order of $k$ consecutive boxes of height $k^{-1}n$ and width $k^{-2/3}n^{2/3}$. As the KPZ one-third exponent for energy predicts, each passage of a path across a box incurs weight shortfall of order $k^{-1}n^{1/3}$. Cruise weight shortfall is thus of order $m \cdot k \cdot k^{-1/3}n^{1/3} = mk^{2/3}n^{1/3}$.

Now we turn to giving the details of each phase. As in Sect. 4, we adopt a rounding convention for coordinates which are not integers, but, in contrast, we will be ignoring the resulting terms of $\pm 1$. More precisely, all expressions for coordinates of points should be rounded down, but extra terms of $\pm 1$ which thus arise in non-coordinate quantities, such as expected values of weights, will be absorbed into constants without explicit mention.

8.2. The construction in detail.

8.2.1. Take-off. In this phase, the $m$ curves will travel from $(1, 1), \ldots, (1, m)$ to the line $x + y = 2k^{2/3}n^{1/3}$. Since we will make no non-trivial claim about the weight of take-off curves, we may choose these $m$ curves to be any disjoint upright paths, with the $i^{th}$ starting at $(1, i)$ and ending at

\[
\left( k^{2/3}n^{1/3} - \left( i - \frac{m}{2} \right) k^{-1/3}n^{1/3}, \ k^{2/3}n^{1/3} + \left( i - \frac{m}{2} \right) k^{-1/3}n^{1/3} \right). \tag{22}
\]

The next statement suffices to show that these $m$ disjoint paths exist; we omit the straightforward proof.

Lemma 8.1. Suppose given $m$ starting points \{(1, $i$) : $i \in \llbracket 1, m \rrbracket$\}; and $m$ ending points \{(xi, yi) : $i \in \llbracket 1, m \rrbracket$\} on the line $x + y = r$, for some $r \geq m + 1$, with $1 \leq x_1 < x_2 < \ldots < x_m \leq r$. Then there exist $m$ disjoint paths such that the $i^{th}$ connects $(1, i)$ to $(x_m - i+1, y_m - i+1)$.

The hypothesis of this lemma that $2k^{2/3}n^{1/3} \geq m + 1$ is satisfied when $n \geq 1$ because $m \leq k \leq n$.

8.2.2. Climb. This phase concerns the construction of curves as they pass through diagonal coordinates between $k^{2/3}n^{1/3}$ and a value $h$ that we will specify in (27). In Lemma 8.2, we will learn that $cn \leq h \leq n/2$; the climb phase thus has duration $\Theta(n)$, since $k$ is supposed to be at most a small constant multiple of $n$. 
During climb, the order of separation rises from \( k^{-1/3}n^{1/3} \) to \( k^{-2/3}n^{2/3} \). Separation will increase by a factor \( 2^{1/k} \) during each of several segments into which the phase will be divided. The number \( N \) of segments is chosen to satisfy
\[
2^{N/k} \in k^{-1/3}n^{1/3} \cdot [1, 2).
\tag{23}
\]
A depiction of one of the segments is shown in Fig. 9a.

In order that the constructed curves remain disjoint and incur a modest weight shortfall, we will insist that they pass through a system of disjoint parallelograms whose geometry respects KPZ scaling: the width of each parallelogram, and the anti-diagonal offset between its lower and upper sides, will have the order of the two-thirds power of the parallelogram’s height.

Segments \([2\ell_{j-1}, 2\ell_j]\) will be indexed by \( j \in \llbracket 1, N \rrbracket \), with
\[
\ell_j := \ell_{j-1} + 3^{-1}2^{3(j-1)/2}k^{-1/2}n^{1/2}, \tag{24}
\]
and \( \ell_0 = k^{2/3}n^{1/3} \). Climb begins where takeoff ends, at the diagonal coordinate \( 2\ell_0 \): see (22).

By level \( j \), we mean \( \{(x, y) \in \mathbb{Z}^2 : x + y = 2\ell_j\} \). The \( i \)th curve will intersect level \( j \) at a unique point \((\ell_j - p_{i,k}^{(j)} : \ell_j + p_{i,k}^{(j)})\), where \( p_{i,k}^{(j)} \) is inductively defined by
\[
p_{i,k}^{(j)} := p_{i,k}^{(j-1)} + \left( i - \frac{m}{2} \right) 2^{(j-1)/k} \left( 2^{1/k} - 1 \right) k^{-1/3}n^{1/3} \tag{25}
\]
from initial data \( p_{i,k}^{(0)} := (i - m/2)k^{-1/3}n^{1/3} \) that is chosen consistently with (22).

We indicated that separation would increase by a factor of \( 2^{1/k} \) during each segment in climb from an initial value of \( k^{-1/3}n^{1/3} \). This is what our definition ensures: the separation \( \text{sep}_{k}^{(j)} \) between positions on the \( j \)th level is given by

\[
\text{sep}_{k}^{(j)} := p_{i,k}^{(j)} - p_{i-1,k}^{(j)} = k^{-1/3}n^{1/3} + \sum_{r=0}^{j-1} 2^{r/k} (2^{1/k} - 1)k^{-1/3}n^{1/3} = 2^{j/k}k^{-1/3}n^{1/3},
\]

(26)

where the second equality is due to repeated applications of (25) and since \( \text{sep}_{k}^{(0)} = p_{i,k}^{(0)} - p_{i-1,k}^{(0)} = k^{-1/3}n^{1/3} \).

The height \( h \) that marks the end of the climb phase may now be set:

\[
h := \ell_{N}.
\]

(27)

8.2.3. Flight corridors We have indicated that the curves under construction will be forced to pass through certain disjoint parallelograms. The latter regions will be called flight corridors. We may consider a flight corridor delimited by levels \( j \) and \( j + 1 \) of width \( \text{sep}_{k}^{(j)} \).

Each curve will have a single flight corridor through which it passes between levels \( j \) and \( j + 1 \). The corridor corresponding to the \( i \)th curve will be denoted \( F^{(i,j)} \), and we define it next. Consider the planar line segment that runs from

\[
\left( \ell_{j} - p_{i,k}^{(j)} \right) + (1,0)
\]

to

\[
\left( \ell_{j+1} - p_{i,k}^{(j+1)} \right) + (1,0)
\]

(28)

The flight corridor \( F^{(i,j)} \) consists of those \( (x,y) \in \mathbb{Z}^{2} \) that are displaced from some element in the just indicated line segment by a vector \( (t, -t) \) for some \( t \) with absolute value at most \( 2^{-1} \text{sep}_{k}^{(j)} - 1 \). Thus the flight corridors associated to the \( i \)th curve for different values of \( i \) are disjoint. See Fig. 9. The addition of \( (1,0) \) in the first line of (28) is to ensure that the consecutive flight corridors of the same curve are disjoint.

The subpath of the \( i \)th curve from level \( j \) to \( j + 1 \) is defined to be a path of maximum weight between the two planar points just displayed that remains in the flight corridor \( F^{(i,j)} \). Its weight will be denoted by \( X^{(i,j)}_{n,k,2} \). Note that the level \( j \) subpath’s ending point in \( F^{(i,j)} \) and the level \( j + 1 \) subpath’s starting point in \( F^{(i,j+1)} \) are adjacent to each other, which allows us to concatenate them.

The height of \( F^{(i,j)} \) is \( \ell_{j+1} - \ell_{j} = 3^{-1}2^{3j/2}k^{-1/2}n^{1/2} \) and its width is \( \text{sep}_{k}^{(j)} = 2^{j/k}k^{-1/3}n^{1/3} \), so the flight corridor satisfies the relation height = \( \frac{1}{2} \cdot \text{width}^{3/2} \).

Next we will consider the anti-diagonal displacement (see Fig. 10) of the flight corridors. Recall that Proposition 3.7 says that, if \( z \) is of height \( z^{2/3} \), then the weight loss of the heaviest path constrained to connect the midpoints of the opposite sides of \( F^{(i,j)} \) is of order \( z^{2} \) of height \( z^{1/3} \). Thus to ensure a low enough weight loss, it will be important that the anti-diagonal displacement of \( F^{(i,j)} \) is at most of order \( (\text{height})^{2/3} \), i.e., that \( z \) is of constant order. We check that this holds next; indeed, it is for this relation to hold that we defined our separation to increase by
For these paths, this displacement between levels $j$ of simply a constant factor.

The anti-diagonal displacement of the flight corridor is highest for curves 1 and $m$. For these paths, this displacement between levels $j$ and $j + 1$ is

$$|p_{i,k}^{(j+1)} - p_{i,k}^{(j)}| = \frac{1}{2}m2^{j/k}(2^{1/k} - 1)k^{-1/3}n^{1/3} \leq Cm2^{j/k}k^{-4/3}n^{1/3}$$

in view of (25), with the inequality following from the bound $2^{1/k} - 1 \leq Ck^{-1}$ for some absolute constant $C$. As the height gain in $F^{(i,j)}$ is $3^{-1}2^{3j/2k}k^{-1/2}n^{1/2}$, and $m \leq k$, the anti-diagonal displacement for each corridors is indeed at most $C \cdot \text{(height)}^{2/3}$, the constant $C$ is uniform over $i, j, k, \text{and } m$.

We next verify that the height traversed in the climb phase is $\Theta(n)$, a fact which will be useful in verifying that the KPZ relations hold for the next phase, cruise.

**Lemma 8.2.** There exist positive $c_1$ and $c$ such that if $k < c_1n$, then $h = \ell_N$ satisfies $cn \leq h \leq n/2$ for large enough $n$.

**Proof.** This inference follows by noting from the definition (24) of $\ell_N$ that

$$h = \ell_0 + \sum_{j=0}^{N-1} 3^{-1}2^{3j/2k}k^{-1/2}n^{1/2} = k^{2/3}n^{1/3} + \frac{1}{3} \times \frac{2^{3N/2k} - 1}{2^{3/2k} - 1} \times k^{-1/2}n^{1/2}. \quad (29)$$

Now, since $2^{3/2k} - 1 = \Theta(k^{-1})$ and $2^{N/k} \in [1, 2) \cdot k^{-1/3}n^{1/3}$ from (23), the claim follows after simplification. \hfill $\square$

### 8.2.4. Cruise

Curves enter cruise at separation $k^{-2/3}n^{2/3}$, and must be maintained there for a duration of order $n$: see Fig. 9b. More precisely, and recalling that $h = \ell_N$, the $i^{th}$ curve enters cruise at

$$\left( h - \left( i - \frac{m}{2} \right) k^{-2/3}n^{2/3}, h + \left( i - \frac{m}{2} \right) k^{-2/3}n^{2/3} \right).$$

We set $\ell'_0 = h$, and $\ell'_j = \ell'_{j-1} + \frac{n-2h}{k} = h + (n-2h) \cdot \frac{i}{k}$ for $j \in [1, k]$. We will demand that the $i^{th}$ curve visits each point

$$\left( \ell'_j - \left( i - \frac{m}{2} \right) k^{-2/3}n^{2/3}, \ell'_j + \left( i - \frac{m}{2} \right) k^{-2/3}n^{2/3} \right). \quad (30)$$
Between consecutive points, this curve will be constrained to remain in a flight corridor that comprises those vertices in $\mathbb{Z}^2$ that are displaced from the planar line segment that interpolates the pair of points by a vector $(t, -t)$ for some $t$ with absolute value at most $2^{-1}k^{-2/3}n^{2/3}$, which will be made to be disjoint by a unit displacement of the bottom side, as in (28). Let $X^{(i,j)}_{n,k,3}$ be the weight of the $i$th curve in the $j$th corridor; namely, the maximum weight of paths that interpolate the endpoint pair and remain in the corridor.

We saw in Lemma 8.2 above that $h = \Theta(n)$. The new flight corridor thus satisfies the relation

$$\text{width} \in \left[ C^{-1} \cdot \text{height}^{2/3}, C \cdot \text{height}^{2/3} \right]$$

for a positive constant $C$. The corridor has no anti-diagonal displacement.

8.2.5. Descent and landing These two phases are specified symmetrically with climb and take-off.

8.3. Bounding below expected weight in the construction. We will argue, in two steps, that the constructed curves attain the sought weight $\mu nm - \Theta(mk^{2/3}n^{1/3})$ with high probability. In this section, we will show that, under Assumptions 2 and 3a, the expected total curve weight is at least $\mu nm - Cmk^{2/3}n^{1/3}$ for a constant $C > 0$. In the next, the desired bound on curve weight will be obtained by showing that this weight concentrates around its mean due to the independence of contributions from the various flight corridors.

We will control the contribution from a given flight corridor using the lower bound on the expectation of the constrained point-to-point weight from Proposition 3.7. That proposition will permit us to derive the next result, the conclusion of the present section.

**Proposition 8.3.** Let $X$ denote the sum of the weights of the $m$ curves in our construction. There exist $n_0 \in \mathbb{N}$, $C > 0$ and $c_1 > 0$ such that, for $n > n_0$, $k < c_1 n$ and $m \leq k$, $\mathbb{E}[X] \geq \mu nm - Cmk^{2/3}n^{1/3}$.

**Proof.** The values of $C$ and $c$ may change from line to line but they do not depend on $n$, $m$, or $k$.

We start with the weight of a curve fragment between two levels during climb. That is, we find a lower bound on $\mathbb{E}[X^{(i,j)}_{n,k,2}]$, where recall that $X^{(i,j)}_{n,k,2}$ the weight of the heaviest path that travels between the points (28) without exiting the flight corridor $F^{(i,j)}$.

In specifying the climb phase, we noted that the condition in Proposition 3.7 that the anti-diagonal displacement of the flight corridor is of order $r^{2/3}$ holds, where $r$ is the height of the flight corridor. Further, this holds uniformly over the $m$ flight corridors corresponding to the $m$ values of $i$ and the $N$ levels indexed by $j$.

Note that this implies that the parabolic weight loss term of $\mathbb{E}[X^{(i,j)}_{n,k,2}]$ in (10) of Proposition 3.7 is of order $r^{1/3}$ and so, by absorbing it into the term $-cr^{1/3}$, does not need to be explicitly calculated. Thus, using Proposition 3.7 with the settings $r = \ell_{j+1} - \ell_{j}$ and $zr^{2/3} = p^{(j+1)}_{i,k} - p^{(j)}_{i,k}$, we obtain

$$\mathbb{E}[X^{(i,j)}_{n,k,2}] \geq \mu (\ell_{j+1} - \ell_{j}) - C (\ell_{j+1} - \ell_{j})^{1/3}. \quad (31)$$

Here $C$ also includes the weight loss from the parabolic term in (10) of Proposition 3.7, since, as noted, $z$ is uniformly bounded.
Recall from (24) that \( \ell_{j+1} - \ell_j = 3^{j} 2^{j/2} k^{-1/2} n^{1/2} \). Simplifying the second term of (31) gives

\[
C \left( \ell_{j+1} - \ell_j \right)^{1/3} = C \cdot 2^{j/2} k^{-1/6} n^{1/6}.
\]

where we have absorbed the constant factor of \( 3^{1/3} \) into the value of \( C \). Combining the preceding two displays and summing over \( i \) gives a lower bound on the expected weight of all the curves between levels \( j \) and \( j + 1 \):

\[
\sum_{i=1}^{m} \mathbb{E}[X_{n,k,2}^{(i,j)}] \geq \frac{\mu}{3} \cdot 2^{j/2} k^{-1/2} n^{1/2} - C 2^{j/2} k^{-1/6} n^{1/6}.
\]

Summing from \( j = 0 \) to \( N - 1 \) gives a lower bound for the total weight of the climb phase of

\[
\sum_{j=0}^{N-1} \sum_{i=1}^{m} \mathbb{E}[X_{n,k,2}^{(i,j)}] \geq \frac{\mu m}{3} \cdot \sum_{j=0}^{N-1} 2^{j/2} k^{-1/2} n^{1/2} - C \cdot \sum_{j=0}^{N-1} 2^{j/2} k^{-1/6} n^{1/6}
\]

\[
= \mu (h - k^{2/3} n^{1/3}) \cdot m - C \cdot \frac{2^{N/2} k^{1/2} - 1}{21/2k - 1} \cdot mk^{-1/6} n^{1/6}
\]

\[
= \mu hm - C mk^{2/3} n^{1/3}, \tag{32}
\]

where we used that \( 2^{N/2} k = k^{-1/6} n^{1/6} \) from (23); that \( 2^{1/2k} - 1 \geq c k^{-1} \) for some \( c > 0 \); the expression of \( h \) from (29); and that \( \ell_0 = k^{2/3} n^{1/3} \).

As for cruise, an easy computation from Proposition 3.7 yields that

\[
\mathbb{E}[X_{n,k,3}^{(i,j)}] \geq \frac{\mu(n - 2h)}{k} - c k^{-1/3} (n - 2h)^{1/3} \geq \frac{\mu(n - 2h)}{k} - c k^{-1/3} n^{1/3}. \tag{33}
\]

We write \( X_{n,k,2}^{(i,j)} \) for the descent counterpart of the climb flight corridor weight maximum. The bound on mean valid for climb holds equally for descent. Combining climb, descent and cruise weight bounds with the zero bound for take-off and landing, the total weight \( X \) in our construction is seen to satisfy

\[
X \geq \sum_{i,j} \left( X_{n,k,2}^{(i,j)} + X_{n,k,2}^{(i,j)} + X_{n,k,3}^{(i,j)} \right) \tag{34}
\]

(where the indices that \( j \) is summed over differ for the terms corresponding to the climb phase and the cruise phase). Taking expectation, and applying (32) and (33), we obtain

\[
\mathbb{E} \left[ \sum_{i,j} \left( X_{n,k,2}^{(i,j)} + X_{n,k,2}^{(i,j)} + X_{n,k,3}^{(i,j)} \right) \right] \geq \mu nm - C mk^{2/3} n^{1/3},
\]

the conclusion of Proposition 8.3. □
8.4. One-sided concentration of the construction weight. We now know that the expected weight of the construction is correct. To prove Theorem 3.1, we argue that the weight is unlikely to fall much below its mean. We will stochastically dominate the summands in the expression for \( X \) from (34) with independent exponential random variables of varying parameters, and use the following concentration result from [Jan18] for sums of such variables. We denote by \( \text{Exp}(\lambda) \) the exponential distribution with rate \( \lambda \).

**Proposition 8.4** (Theorem 5.1 (i) of [Jan18]). Let \( W = \sum_{i=1}^{n} W_i \) where \( W_i \sim \text{Exp}(a_i) \) are independent. Define

\[
\nu := \mathbb{E} W = \sum_{i=1}^{n} \mathbb{E} W_i = \sum_{i=1}^{n} \frac{1}{a_i}, \quad a_* := \min_i a_i.
\]

Then for \( \lambda \geq 1 \),

\[
\mathbb{P} (W \geq \lambda \nu) \leq \lambda^{-1} e^{-a_* \nu (\lambda - 1 - \log \lambda)}.
\]

**Proof of Theorem 3.1.** Let \( K \geq 0 \) and \( 0 < L_1 \leq L_2 \). As before, let \( U = U_{r, \ell r^{2/3}, zr^{2/3}} \) be a parallelogram with width \( \ell r^{2/3} \), height \( r \), and anti-diagonal displacement \( zr^{2/3} \), where \( |z| \leq K \) and \( L_1 \leq \ell \leq L_2 \). Let \( X_r^U \) denote the weight of the heaviest midpoint-to-midpoint path that lies in \( U \). By Proposition 3.7, and \( \mathbb{E} [X_r^U] \leq \mathbb{E} [X_r] \leq \mu r \) (which is due to Assumption 2), there exists \( c_4 = c_4(L_1, L_2, K) > 0 \) such that, for \( r > r_0 = r_0(L_1, L_2, K) \) and \( \theta > \theta_0 = \theta_0(L_1, L_2, K) \),

\[
\mathbb{P} \left( X_r^U - \mathbb{E} [X_r^U] < -\theta r^{1/3} \right) \leq e^{-c_4 \theta};
\]

or

\[
\mathbb{P} \left( X_r^U - \mathbb{E} [X_r^U] < -\theta \right) \leq \exp(-c_4 \theta / r^{1/3}), \tag{35}
\]

the latter for \( \theta \) larger than \( \theta_0 r^{1/3} \) and \( r > r_0 \). Equation (35) says that we have the stochastic domination

\[
- \left( X_r^U - \mathbb{E} [X_r^U] \right) \leq_{\text{sd}} \text{Exp}\left( c_4 / r^{1/3} \right) + \theta_0 r^{1/3},
\]

where, in an abuse of notation, \( \text{Exp}(\lambda) \) denotes a random variable with the exponential distribution of rate \( \lambda \) (i.e., not the distribution itself), and \( X \leq_{\text{sd}} Y \) denotes that the distribution of \( X \) is stochastically dominated by that of \( Y \). Thus, in our construction, we have a coupling such that, almost surely,

\[
- \left( X_{n,k,2}^{(i,j)} - \mathbb{E} [X_{n,k,2}^{(i,j)}] \right) \leq W_2^{(i,j)} + \theta_0 k^{-1/6} n^{1/6}, \tag{36}
\]

\[
- \left( X_{n,k,2}^{(i,j)} - \mathbb{E} [X_{n,k,2}^{(i,j)}] \right) \leq W_2^{(i,j)} + \theta_0 k^{-1/6} n^{1/6}, \quad \text{and}
\]

\[
- \left( X_{n,k,3}^{(i,j)} - \mathbb{E} [X_{n,k,3}^{(i,j)}] \right) \leq W_3^{(i,j)} + \theta_0 k^{-1/3} n^{1/3}.
\]
Here the random variables on the right-hand side are independent and distributed as

\[ W_2^{(i,j)} \sim \text{Exp} \left( c_4 2^{-j/2k} k^{1/6} n^{-1/6} \right), \]
\[ \overline{W}_2^{(i,j)} \sim \text{Exp} \left( c_4 2^{-j/2k} k^{1/6} n^{-1/6} \right), \quad \text{and} \]
\[ W_3^{(i,j)} \sim \text{Exp} \left( c_4 k^{1/3} n^{-1/3} \right). \]

(37)

By our construction, \( K, L_1, \) and \( L_2 \) in Proposition 3.7 may be chosen independently of \( i \) and \( j \). Thus, whatever the dependence of \( c_4 \) and \( \theta_0 \) on \( i \) and \( j \), they are uniformly bounded away from 0 and \( \infty \) respectively, and so may be assumed to be constant. We set

\[ W := \sum_{i,j} \left( W_2^{(i,j)} + \overline{W}_2^{(i,j)} \right) + \sum_{i,j} W_3^{(i,j)}. \]

We wish to apply Proposition 8.4 to the sum \( W \) of independent exponential random variables, and so we must estimate Proposition 8.4’s parameters. We have

\[ \mathbb{E} \left[ \overline{W}_2^{(i,j)} \right] = \mathbb{E} \left[ W_2^{(i,j)} \right] = c_4^{-1} 2^{-j/2k} k^{-1/6} n^{1/6} \quad \text{and} \quad \mathbb{E} \left[ W_3^{(i,j)} \right] = c_4^{-1} k^{-1/3} n^{1/3}. \]

Summing the expressions of the last display over the indices and using that \( 2^{N/2k} = k^{-1/6} n^{1/6} \) from (23) and that \( 2^{1/k} - 1 = \Theta(k^{-1}) \) gives that the total mean \( \nu \) satisfies

\[ \nu := \mathbb{E}[W] \leq 8 c_4^{-1} m k^{2/3} n^{1/3}. \]

(38)

Noting that the coefficients of \( \theta_0 \) in (36) are the same as the mean of the corresponding exponential random variable up to a factor of \( c_4 \), similarly summing these coefficients shows the stochastic domination

\[ - (X - \mathbb{E}[X]) \leq W + 8 \theta_0 m k^{2/3} n^{1/3}. \]

Using that \( 2^{N/k} = k^{-1/3} n^{1/3} \) from (23), we see that the minimum \( a_* \) of the rates of the exponential random variables defined in (37) is given by

\[ a_* = \min(c_4 2^{-N/2k} k^{1/6} n^{-1/6}, c_4 k^{1/3} n^{-1/3}) = c_4 k^{1/3} n^{-1/3}. \]

(39)

We will prove something slightly stronger than required. Let \( S = \max(c_4^{-1}, \theta_0) \) and \( x \geq 1 \). By Proposition 8.3, with this result contributing the value of \( C \), and the stochastic domination (36),

\[ \mathbb{P} \left( X < \mu n m - (C + 20 Sx) m k^{2/3} n^{1/3} \right) \leq \mathbb{P} \left( -(X - \mathbb{E}[X]) > 20 S x m k^{2/3} n^{1/3} \right) \leq \mathbb{P} \left( W \geq 20 S x m k^{2/3} n^{1/3} - 8 \theta_0 m k^{2/3} n^{1/3} \right). \]

Since \( 20 S x - 8 \theta_0 \geq 12 S x \geq 12 c_4^{-1} x \), the latter quantity is bounded using Proposition 8.4, (38), and (39):

\[ \mathbb{P} \left( W > 12 c_4^{-1} x m k^{2/3} n^{1/3} \right) \leq \mathbb{P} \left( W > \frac{12}{8} \nu x \right) \leq \frac{2}{3} \exp \left( -a_* \nu \left( \frac{3}{2} x - 1 - \log \frac{3x}{2} \right) \right) \leq \exp \left( -c m k x \right). \]

(40)
Numerical evaluation of the exponent shows that we may take \( c = 3/4 \); taking \( x = 1 \) completes the proof.

In view of the paragraphs after (28) and (30), flight corridors during climb, descent and cruise lie within the strip around the diagonal of width \( 2mk^{-2/3}n^{2/3} \); thus, the transversal fluctuations of the constructed curves in these phases also satisfy this bound. By setting \( C_1 = C + 20S \), we complete the proof of Theorem 3.1. \( \square \)

Remark 8.5. Note that (40) shows that there exist positive constants \( c \) and \( x_0 \) such that, for \( x \geq x_0 \),

\[
\mathbb{P}
\left( X < \mu nm - (C + x)mk^{2/3}n^{1/3} \right) \leq \exp \left( -cmkx \right).
\]

9. Combinatorics of Watermelon Geometry

Here we prove two geometric results that we have invoked several times; namely, Propositions 2.4 and 2.6. The arguments are geometric and combinatorial, rather than probabilistic, and, we believe, of independent interest.

9.1. Proof of melon interlacing, Proposition 2.4. Instead of assuming that \( \nu \) is continuous as in Proposition 2.4, we will prove the following deterministic statement, whose hypotheses hold almost surely when \( \nu \) has no atoms. For a subset \( A \subseteq [1, n]^2 \) of vertices, we first define its weight by

\[
\ell(A) := \sum_{v \in A} \xi_v.
\]

**Proposition 9.1.** Suppose the vertex weights \( \{\xi_v : v \in [1, n]^2\} \) are non-negative and such that no two distinct subsets \( A, B \subseteq [1, n]^2 \) have equal weight. Then the consecutive terms in the unique sequence of geodesic watermelons \( \{\phi^k_n : j \in [1, n]\} \) interlace.

The sequence of geodesic watermelons is unique because the set of vertices of the \( k \)-geodesic watermelon is unique for every \( k \in [1, n] \) under the hypotheses of Proposition 9.1.

**Proof of Proposition 2.4.** This is immediate since when \( \nu \) has no atoms and has support contained in \([0, \infty)\), the hypotheses of Proposition 9.1 are almost surely satisfied. \( \square \)

After proving Proposition 9.1, we will upgrade it to Proposition 2.5, which handles the case that \( \nu \) has atoms, by a perturbation argument. We next introduce some notation to facilitate the proof of Proposition 9.1.

Recall the partial order \( \preceq \) introduced in Definition 2.2, where informally \( \gamma \preceq \gamma' \) means \( \gamma \) is to the left of \( \gamma' \). Let \( \phi^k_n \) be any fixed \( k \)-geodesic watermelon in the deterministic environment (we distinguish from the symbol \( \gamma^k_{\ell,j} \) which will play a different but related role in the proof). Then as in (1), for any \( k \), let \( \phi^k_{n,\gamma} \) be the curves of \( \phi^k_n \) ordered from left to right, i.e., \( \phi^k_{n,\gamma_1} < \phi^k_{n,\gamma_2} < \ldots < \phi^k_{n,\gamma_m} \). For a set of vertices \( A \), we call \( \{\gamma_1, \ldots, \gamma_m\} \) a path fragment decomposition of \( A \) if each \( \gamma_i \) is a connected component of \( A \) (under the connectivity structure of \( \mathbb{Z}^2 \)) as well as an upright path and \( A = \bigcup_{i=1}^m \gamma_i \); note that not all sets have a path fragment decomposition. We then call the \( \gamma_i \) path fragments.
For \( j \in [1, k-1] \), let \( R^k_j \) be the region between \( \phi_n^{k,j} \) and \( \phi_n^{k,j+1} \), defined by

\[
R^k_j = \{(x, y) \in [1, n]^2 : \phi_n^{k,j} \leq \gamma(x,y) \leq \phi_n^{k,j+1}\},
\]

where \( \gamma(x,y) \) is the singleton path at \((x, y)\). Recall that an order relation between two paths is well-defined only if their time ranges are not disjoint. Since the elements of \( \{\phi_n^{k,j} : j \in [1, k]\} \) do not share the same time range, for certain \((x, y)\), it may be that only one of the inequalities \( \phi_n^{k,j} \leq \gamma(x,y) \) and \( \gamma(x,y) \leq \phi_n^{k,j+1} \) is well-defined (regardless of whether that inequality is true or false). In these cases, the definition of \( R^k_j \) above will only require that inequality for which the corresponding time-ranges are not disjoint to be true, and will disregard the remaining inequality which is not well-defined. We also define \( R^0_k = \{(x, y) \in [1, n]^2 : \gamma(x,y) \leq \phi_n^{k,1}\} \) and \( R^k_k = \{(x, y) \in [1, n]^2 : \phi_n \leq \gamma(x,y)\} \). We note that, by the ordering of \( \{\phi_n^{k,j} : j \in [1, k]\} \),

\[
R^k_{j_1} \cap R^k_{j_2} = \emptyset \quad \text{if} \quad |j_1 - j_2| \geq 2.
\]

We also note that \( \phi_n^k \) interlacing with \( \phi_n^{k-1} \) is equivalent to \( \phi_n^{k,j} \subseteq R^k_{j-1} \) for each \( j \in [1, k] \). We say that \( \phi_n^{k,j} \) is in its home region if \( \phi_n^{k,j} \subseteq R^k_{j-1} \). We define the wilderness region \( W^k_j \) (for \( \phi_n^{k,j} \)) to be the complement of \( R^k_{j-1} \), i.e.,

\[
W^k_j := [1, n]^2 \setminus R^k_{j-1}.
\]

The idea of the proof of Proposition 9.1 is the following. Consider the case that \( \phi_n^k \) and \( \phi_n^{k-1} \) do not interlace, and look at the excursions of \( \phi_n^{k,j} \) into the respective wilderness regions \( W_j^k \) for each \( j \in [1, k] \). For each excursion, we identify a candidate path along the home region border, i.e., a subpath of \( \phi_n^{k-1,j-1} \) or \( \phi_n^{k,j} \), which has the same endpoints as the excursion. We then simultaneously swap all the excursions with the candidate paths. This results in two new collections of \( k \) and \( k-1 \) disjoint paths (which do not necessarily interlace) which cover the same set of vertices with the same multiplicities. The weight maximality property of \( \phi_n^k \) and \( \phi_n^{k-1} \) will then imply that the two new collections of paths are also \((k-1)\)- and \(k\)-geodesic watermelons, contradicting the uniqueness of the watermelons guaranteed by the hypothesis. See Fig. 11.

To implement this plan, we define for each \( j \in [1, k] \) the set \( \text{WE}^k_j \) of wilderness excursions to be the path fragment decomposition of \( \phi_n^{k,j} \cap W^k_j \). The first step is to identify the border candidate paths—subpaths of the paths of \( \phi_n^{k,j} \)—which will be swapped with the excursions into the wilderness—into elements of \( \text{WE}^k_j \).

**Lemma 9.2.** For every \( \gamma \in \text{WE}^k_j \), there exists a path \( C_j(\gamma) \), which we call a candidate path, such that \( C_j(\gamma) \subseteq \phi_n^{k-1,j-1} \) (if \( \gamma \) is to the left of \( \phi_n^{k-1,j-1} \)) or \( C_j(\gamma) \subseteq \phi_n^{k-1,j-1} \) (if \( \gamma \) is to the right of \( \phi_n^{k,j} \)). Further, \( (\phi_n^{k,j} \cup C_j(\gamma)) \setminus \gamma \) and \( (\phi_n^{k,j} \cup \gamma) \setminus C_j(\gamma) \) are upright paths, where \( i \in \{j - 1, j\} \) is chosen so that \( C_j(\gamma) \subseteq \phi_n^{k-1,j-1} \); that is, the sets obtained by swapping the wilderness excursion with the candidate path are upright paths.

**Proof.** Let \( \gamma \in \text{WE}^k_j \) be a given wilderness excursion, and let \( u \) be its startpoint and \( v \) be its endpoint. Note by planarity, the ordering of the paths \( \{\phi_n^{k-1,j} : j \in [1, k-1]\} \), and the adjacency of the starting and ending points of the curves of \( \phi_n^k \) and \( \phi_n^{k-1} \) that both \( u \) and \( v \) are adjacent to \( \phi_n^{k,j} \) or both are adjacent to \( \phi_n^{k-1,j-1} \).
Suppose both are adjacent to $\phi_{n,\to}^{k-1,j}$; the other case is argued in the same way. Then $\gamma$ must be to the right of $\phi_{n,\to}^{k-1,j}$. The points on $\phi_{n,\to}^{k-1,j}$ adjacent to $u$ and $v$ must also lie on $\phi_{n,\to}^{k-1,j}$; call these points $\overline{u}$ and $\overline{v}$. Then define $C_j(\gamma) \subseteq \phi_{n,\to}^{k-1,j}$ to be the portion of $\phi_{n,\to}^{k-1,j}$ lying between $u$ and $\overline{v}$, excluding these two points. It is easy to see that this path satisfies the claims made in Lemma 9.2.

Having identified which subsets of vertices to swap, we define the resulting new set of $k$ curves. For each $j \in [1, k]$, define

$$\gamma_{n,\to}^{k,j} = \phi_{n,\to}^{k,j} \cup \left( \bigcup_{\gamma \in \operatorname{WE}_j^{k} \cap C_j(\gamma) \setminus \bigcup_{\gamma \in \operatorname{WE}_j^{k}} \gamma \right).$$

This is an upright path by Lemma 9.2. We will define an analogous collection of $k - 1$ curves by considering the other half of the swap after the next lemma, which asserts that these just defined $k$ curves are in fact disjoint.

**Lemma 9.3.** $\{\gamma_{n,\to}^{k,j} : j \in [1, k]\}$ is a disjoint collection of $k$ curves.

**Proof.** Suppose to the contrary that, for some $j_1, j_2 \in [1, k]$ with $j_1 \neq j_2$, there is a vertex $v$ such that $v \in \gamma_{n,\to}^{k,j_1} \cap \gamma_{n,\to}^{k,j_2}$. Since $\phi_{n,\to}^{k,j_1} \cap \phi_{n,\to}^{k,j_2} = \emptyset$, there are two remaining cases:

1. there exist $\gamma_1 \in \operatorname{WE}_j^{k}$ and $\gamma_2 \in \operatorname{WE}_j^{k}$ such that $v \in C_{j_1}(\gamma_1) \cap C_{j_2}(\gamma_2)$; or
2. there exists $\gamma \in \operatorname{WE}_j^{k}$ such that $v \in C_{j_1}(\gamma) \cap \phi_{n,\to}^{k,j_2}$ and $v \notin \bigcup\{\gamma' : \gamma' \in \operatorname{WE}_j^{k}\}$ (without loss of generality).

We note immediately that Case 1 cannot occur as $C_{j_1}(\gamma_1)$ and $C_{j_2}(\gamma_2)$ are subsets of distinct curves of $\phi_{n,\to}^{k-1}$, which are disjoint. We next show that Case 2 leads to a contradiction.

There are again two cases, depending on whether $C_{j_1}(\gamma)$ is a subset of $\phi_{n,\to}^{k-1,j_1}$ or of $\phi_{n,\to}^{k-1,j_1}$. We handle the first as the argument for the second is similar. In this case, by planarity, $\gamma \subseteq \phi_{n,\to}^{k,j_1}$ is to the right of $\phi_{n,\to}^{k,j_1}$. Since $v \in C_{j_1}(\gamma) \cap \phi_{n,\to}^{k,j_2}$, it follows
that \( j_2 \leq j_1 - 1 \); for planarity and the ordering of \( \{ \phi_{n \to}^{k,j} : j \in [1, k] \} \) precludes \( j_2 \geq j_1 + 1 \). But this implies that \( v \in W_{j_2}^k = [1, n]^2 \setminus R_{j_2-1}^{k-1} \) since \( v \in \phi_{n \to}^{k-1,j_1} \subseteq R_{j_1}^{k-1} \) and \( R_{j_1}^{k-1} \cap R_{j_2-1}^{k-1} = \emptyset \) when \( j_2 \leq j_1 - 1 \) by (41).

We have arrived at a contradiction since this implies that \( v \in \cup \{ \gamma' : \gamma' \in \operatorname{WE}_j^k \} \).

We now define the new set of \( k-1 \) disjoint curves by doing the swap in the other way: for each \( j \in [1, k-1] \), define

\[
\gamma_{n \to}^{k-1,j} = \phi_{n \to}^{k-1,j} \cup \left( j+1 \right) \bigcup_{i=j}^{j+1} \bigcup_{\gamma \in \operatorname{WE}_i^k : \gamma \ni \phi_{n \to}^{k-1,j}} C_i(\gamma) \setminus \left( j+1 \right) \bigcup_{i=j}^{j+1} \bigcup_{\gamma \in \operatorname{WE}_i^k : \gamma \ni \phi_{n \to}^{k-1,j}} C_i(\gamma).
\]

From Lemma 9.2, we see that each \( \gamma_{n \to}^{k-1,j} \) is an upright path. We also need the analogue of Lemma 9.3, that \( \{ \gamma_{n \to}^{k-1,j} : j \in [1, k-1] \} \) is a collection of \( k-1 \) disjoint curves. The proof bears some similarities to that of Lemma 9.3 but we provide it here for completeness. For convenience, we first define

\[
A_j = \bigcup_{i=j}^{j+1} \bigcup_{\gamma \in \operatorname{WE}_i^k : \gamma \ni \phi_{n \to}^{k-1,j}} C_i(\gamma).
\]

**Lemma 9.4.** \( \{ \gamma_{n \to}^{k-1,j} : j \in [1, k-1] \} \) is a disjoint collection of \( k-1 \) curves.

**Proof.** Suppose to the contrary that, for some \( j_1, j_2 \in [1, k-1] \) with \( j_1 \neq j_2 \), there is a vertex \( v \) such that \( v \in \gamma_{n \to}^{k-1,j_1} \cap \gamma_{n \to}^{k-1,j_2} \). Since \( \phi_{n \to}^{k-1,j_1} \cap \phi_{n \to}^{k-1,j_2} = \emptyset \), there are two remaining cases:

1. \( v \in A_{j_1} \cap A_{j_2} \); or
2. \( v \in \phi_{n \to}^{k-1,j_1} \cap A_{j_2} \) and \( v \notin \bigcup \{ C_i(\gamma) : i \in [j_1, j_1 + 1], \gamma \in \operatorname{WE}_i^k, C_i(\gamma) \subseteq \phi_{n \to}^{k-1,j_1} \} \)
   (without loss of generality).

We immediately note that Case 1 cannot occur since any \( \gamma \in \cup_{i=1}^{j_1} \operatorname{WE}_i^k \) can belong to at most one of the \( A_j \), and distinct elements of \( \cup_{i=1}^{j_1} \operatorname{WE}_i^k \) are disjoint. We next show that Case 2 leads to a contradiction.

Since \( v \in A_{j_1} \), there are two possibilities: there exists \( \gamma \in \operatorname{WE}_i^k \) such that \( v \in \gamma \) and \( C_i(\gamma) \subseteq \phi_{n \to}^{k-1,j_2} \) for \( i = j_2 \) or \( j_2 + 1 \). We deal with the case that \( i = j_2 \); the argument for the other is similar.

That \( C_{j_2}(\gamma) \subseteq \phi_{n \to}^{k-1,j_2} \) implies that \( \gamma \) is to the right of \( \phi_{n \to}^{k-1,j_2} \) from Lemma 9.2. Thus \( v \in \phi_{n \to}^{k-1,j_2} \setminus \phi_{n \to}^{k-1,j_1} \) is to the right of \( \phi_{n \to}^{k-1,j_2} \), yielding that \( j_1 \geq j_2 + 1 \) since \( j_1 \neq j_2 \).

Now \( \phi_{n \to}^{k-1,j_1} \setminus \phi_{n \to}^{k-1,j_2} \) is to the right of \( \phi_{n \to}^{k-1,j_2} \), and so also of \( \gamma \). Since \( v \in \gamma \) lies on \( \phi_{n \to}^{k-1,j_1} \), which is the left boundary of \( R_{j_1}^{k-1} \), we see that \( \phi_{n \to}^{k,j_1+1} \) intersects \( R_{j_1}^{k-1} \) for some \( j \geq j_2 + 1 \), which for all such \( j \) is to the right of the home region \( R_{j_2}^{k-1} \) for \( \phi_{n \to}^{k,j_2+1} \). This implies that \( \phi_{n \to}^{k,j_1} \) also intersects its wilderness region, and that it is to the right of its home region, by the ordering of \( \{ \phi_{n \to}^{k,j} : j \in [1, k] \} \) by considering the anti-diagonal line through \( v \). But then by the definition of the candidate paths, there must exist \( \tilde{\gamma} \in \operatorname{WE}_{j_1}^k \) such that \( v \in C_{j_1}(\tilde{\gamma}) \), which is a contradiction.

Finally we record a sufficient condition, which we will shortly use, for the absence of wilderness fragments; note that this absence is equivalent to \( \phi_n^k \) and \( \phi_n^{k-1} \) interlacing.
Lemma 9.5. If \( \bigcup_{j=1}^{k} \text{WE}_j^k \subseteq \phi_n^{k-1} \), then \( \bigcup_{j=1}^{k} \text{WE}_j^k = \emptyset \).

The proof idea is to assume the contrary and to then consider the leftmost path \( \phi_{n,\to}^{k,j} \) of \( \phi_n^k \) which has a wilderness excursion. This excursion must be a subset of some path of \( \phi_n^{k-1} \). By considering an anti-diagonal line \( x + y = \ell \) through a vertex of this wilderness excursion, we see that each path of \( \phi_n^k \) to the right of \( \phi_n^{k,j} \), must have a wilderness excursion which intersects this anti-diagonal line and which must be a subset of a distinct path of \( \phi_n^{k-1} \). But this is a contradiction as there are not enough paths in \( \phi_n^{k-1} \) to accommodate all these paths of \( \phi_n^k \).

**Proof of Lemma 9.5.** Suppose \( \bigcup_{j=1}^{k} \text{WE}_j^k \neq \emptyset \). Then there is a minimum \( j_0 \) such that \( \phi_{n,\to}^{k,j_0} \not\subseteq R_{j_0-1}^{k-1} \). Let \( v \in \phi_{n,\to}^{k,j_0} \setminus R_{j_0-1}^{k-1} \). That \( \phi_{n,\to}^{k,j_0} \) is a path implies that \( \phi_{n,\to}^{k,j_0} \subseteq \phi_n^{k-1} \), and therefore there is an \( i_0 \geq 1 \) such that \( \phi_{n,\to}^{k,i_0} = \phi_{n,\to}^{k-1,j_0+i_0} \) on the unique anti-diagonal line \( x + y = \ell \) that passes through \( v \). The ordering of \( \{ \phi_{n,\to}^{k,j} : j \in [1,k] \} \) implies that, for each \( i \geq 1 \), \( \phi_{n,\to}^{k,i} \) equals a distinct curve of \( \phi_n^{k-1} \), again on the line \( x + y = \ell \), because the vertex of \( \phi_{n,\to}^{k,j+i_0} \) on this line cannot lie in \( R_{j_0+i_0-1}^{k-1} \). This is a contradiction as there are \( k-1-j_0-i_0 \) curves of \( \phi_n^{k-1} \) to the right of \( \phi_{n,\to}^{k-1,j+i_0} \) but \( k-j_0-1 \) to the right of \( \phi_{n,\to}^{k,j_0} \). \( \square \)

With the two new collections of \( k \) and \( k-1 \) disjoint curves, we may implement the last step of the outline of the proof of Proposition 9.1 by showing that \( \gamma_n^k \) and \( \gamma_n^{k-1} \) are new geodesic watermelons, contradicting the uniqueness of geodesic watermelons as implied by the hypothesis that distinct sets have distinct weights.

**Proof of Proposition 9.1.** We first note that \( \phi_n^k \) and \( \phi_n^{k-1} \) interlacing is equivalent to \( \bigcup_{j=1}^{k} \text{WE}_j^k = \emptyset \), and we will show the latter for \( k \in [1,n] \).

Fix \( k \in [1,n] \). The weight maximality property of \( \phi_n^k \) and \( \phi_n^{k-1} \), combined with Lemmas 9.3 and 9.4, implies that

\[
\ell(\gamma_n^k) \leq \ell(\phi_n^k) \quad \text{and} \quad \ell(\gamma_n^{k-1}) \leq \ell(\phi_n^{k-1}).
\]

But in the specification of \( \gamma_n^k \) and \( \gamma_n^{k-1} \) compared to \( \phi_n^k \) and \( \phi_n^{k-1} \), we have only swapped vertices. Thus we have

\[
\ell(\gamma_n^k) + \ell(\gamma_n^{k-1}) \leq \ell(\phi_n^k) + \ell(\phi_n^{k-1}).
\]

Combining the last display with (42) implies that

\[
\ell(\gamma_n^{k-1}) = \ell(\phi_n^{k-1}) \quad \text{and} \quad \ell(\gamma_n^k) = \ell(\phi_n^k).
\]

Then by the hypothesis that distinct sets have distinct weights, \( \gamma_n^k = \phi_n^k \) and \( \gamma_n^{k-1} = \phi_n^{k-1} \), where the equalities refer to the vertex sets. This can only happen if \( \bigcup_{j=1}^{k} \text{WE}_j^k = \emptyset \).

This is because \( \gamma_n^{k-1} = \phi_n^{k-1} \) implies that \( \bigcup_{j=1}^{k} \text{WE}_j^k \subseteq \phi_n^{k-1} \), which allows us to reach the conclusion of Lemma 9.5. This completes the proof of Proposition 9.1. \( \square \)

Next we turn to proving Proposition 2.5, which handles the case where \( v \) may have atoms. Its proof proceeds by introducing a deterministic perturbation which makes the given \( k \)-geodesic watermelon \( \phi_n^k \) the unique such melon; adding a second random perturbation to make all distinct subsets have distinct weights; invoking Proposition 9.1 to get interlacing; and showing that the \( k \)-geodesic watermelons in the perturbed environment are also \( k \)-geodesic watermelons in the original environment.
Proof of Proposition 2.5. Consider the perturbed environment obtained from $[1, n]^2$ as follows: each point $v$ in $\phi_n^k$ is deterministically given weight $\xi_v + \varepsilon/4n^2$, where $\xi_v$ is the original weight of $v$ and $\varepsilon > 0$ will be chosen later; the remaining points in $[1, n]^2$ have weight unchanged. We now further randomly perturb weights, by increasing the weight of each $v \in [1, n]^2$ by an independent random variable distributed uniformly on $[0, \varepsilon/8n^2]$. Then it is clear that, for every $\varepsilon > 0$, $\phi_n^k$ is the vertices of the unique $k$-geodesic watermelon in the perturbed environment and that every distinct subset of vertices in this environment almost surely has distinct weight.

Applying Proposition 9.1 gives a sequence of interlacing geodesic watermelons $\{\gamma_n^j : j \in [1, n]\}$ (which is $\varepsilon$-dependent) in the perturbed environment, and the uniqueness at level $k$ implies that the set of vertices of $\gamma_n^k$ is the same as that of $\phi_n^k$.

We set $\varepsilon$ to be the minimum gap, over $j \in [1, n]$, between the $j$-geodesic melon’s weight and the weight of the best collection of $j$ disjoint paths which is not a $j$-geodesic melon, both in the unperturbed environment. It is clear that $\varepsilon > 0$.

We now argue that $\{\gamma_n^j : j \in [1, n]\}$, which is a sequence of geodesic watermelons in the perturbed environment, must also be a sequence of geodesic watermelons in the original environment. If we let $\tilde{\ell}(\cdot)$ denote the perturbed length, then it is easy to see that, for any subset of vertices $A \subseteq [1, n]^2$,

$$\tilde{\ell}(A) - \frac{3\varepsilon}{8} \leq \ell(A) \leq \tilde{\ell}(A). \tag{43}$$

Let $\gamma_n^j$ be a $j$-geodesic melon in the unperturbed environment and suppose by way of contradiction that for some $j \in [1, n]$, $\gamma_n^j$ is not a $j$-geodesic melon in the unperturbed environment. Then observing $\tilde{\ell}(\gamma_n^j) < \tilde{\ell}(\gamma_n^j)$ and applying (43) with $A = \gamma_n^j$ for the first inequality and $A = \gamma_n^j$ for the final inequality yields

$$\ell(\gamma_n^j) \leq \tilde{\ell}(\gamma_n^j) < \tilde{\ell}(\gamma_n^j) \leq \ell(\gamma_n^j) + \frac{3\varepsilon}{8}.$$

But the definition of $\varepsilon$ implies that this inequality does not hold. So $\gamma_n^j$ must be a $j$-geodesic melon in the original environment for every $j \in [1, n]$. \qed

9.2. Proving Proposition 2.6. We must prove that the sequence $Y_{n,k} = X_n^k - X_n^{k-1}$ is non-decreasing in $k > 1$, for any given set of non-negative weights. Note that $Y_{n,k} \leq Y_{n,k-1}$ is equivalent to $X_n^k + X_n^{k-2} \leq 2X_n^{k-1}$. To prove the latter, we will decompose the $2(k-1)$ curves of $\gamma_n^k$ and $\gamma_n^{k-2}$ into two sets of $k-1$ disjoint curves and then invoke the maximality of the weight $X_n^{k-1}$. This is easy when $k = 3$, as Figure 12 illustrates.

The case of general $k$ will be proved by induction. We will establish a stronger inductive hypothesis, Lemma 9.6. We remind the reader of the abuse of notation we have adopted till now where a path is sometimes identified with the set of vertices in its range. It serves the purpose of stating and proving Lemma 9.6 to adopt a further abuse, where we will sometimes regard a path as a multiset of vertices in its range. A multiset in this instance is an $\mathbb{N}$-valued map on $[1, n]^2$, i.e, a subset of $[1, n]^2$ where the elements may have multiplicities. We use the symbol $\equiv$ to denote equality of two multisets. So, for example, it will hold for two paths $\gamma_1$ and $\gamma_2$ that $(\gamma_1 \cup \gamma_2) \setminus \gamma_1 \equiv \gamma_2$, even if the two paths have portions of overlap.
Lemma 9.6. Suppose $\gamma_1 \leq \ldots \leq \gamma_k$ and $\gamma'_1 \leq \ldots \leq \gamma'_{k-2}$ are such that all $2k-2$ curves have the same time-range (recall Definition 2.2) and no vertex is counted with multiplicity strictly greater than two in $\bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma'_i$. Then there exist two collections $A_1$ and $A_2$ of $k-1$ disjoint curves each such that the curves have the same time range as before and

$$A_1 \cup A_2 \equiv \bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma'_i,$$

with $\gamma_1 \in A_1$ and $\gamma_k \in A_2$.

Here is how we will prove Lemma 9.6: regard the union of the paths as a subset of $[1, n]^2$; peel off the left boundary and label it as one path in $A_1$; and peel off the right boundary and label it as a path in $A_2$. Now peel off the right boundary of the remaining set of vertices and label it as a path in $A_1$, and do likewise with the left boundary and $A_2$. Continue iteratively in this fashion, alternately assigning the left boundary to $A_1$ and $A_2$, till all the vertices have been depleted. This will yield the desired decomposition: see Figure 12 to see that this is how the orange curves have been assigned to $A_1$ and the green curves to $A_2$.

To render a proof from this idea, we will need to define the max and min of two curves $\gamma$ and $\gamma'$ in order to specify precisely what we called peeling off the right and left boundaries. In each excursion between $\gamma$ and $\gamma'$, we will retain the right path fragment in the max, the left fragment in the min, and the overlapping portion in both. If the curves are disjoint, the right path is the max. In the notation introduced in Definition 2.2, for $t \in \mathcal{R}(\gamma) \cap \mathcal{R}(\gamma')$,

$$\max(\gamma, \gamma')(t) = \max(\gamma(t), \gamma'(t))$$

$$\min(\gamma, \gamma')(t) = \min(\gamma(t), \gamma'(t)),$$

where the max and min on the right are with respect to the normal $\leq$ ordering. Note that $\min(\gamma, \gamma') \leq \gamma, \gamma' \leq \max(\gamma, \gamma')$. We also record without proof the following simple statement.

Lemma 9.7. Let $\gamma_1$, $\gamma'_1$, $\gamma_2$, and $\gamma'_2$ be upright paths with $\mathcal{R}(\gamma_1) = \mathcal{R}(\gamma_2)$, $\mathcal{R}(\gamma'_1) = \mathcal{R}(\gamma'_2)$, and $\gamma_i \leq \gamma'_i$ for $i = 1, 2$. Then

Fig. 12. The base case of Lemma 9.6. In the first panel are the $\gamma$ curves in blue and $\gamma'$ curve in red (with overlap in purple), which in our application of Lemma 9.6 will be the 3-melon and 1-melon. In the remaining two panels are highlighted how these four paths can be decomposed into a pair of two disjoint paths while respecting multiplicities. The rightmost orange path in the second panel is $\max(\gamma'_1, \gamma'_2)$. 
\begin{enumerate}[\hspace{1.5em}(I)]
\item $\gamma_1 \cup \gamma_2 \equiv \max(\gamma_1, \gamma_2) \cup \min(\gamma_1, \gamma_2)$;
\item $\max(\gamma_1, \gamma_2) \leq \max(\gamma_1', \gamma_2')$ and $\min(\gamma_1, \gamma_2) \leq \min(\gamma_1', \gamma_2')$; and
\item $\max(\gamma_1, \gamma_2)$ and $\min(\gamma_1, \gamma_2)$ are upright paths.
\end{enumerate}

**Proof of Lemma 9.6.** The proof is by induction on $k$. We first verify the base case of $k = 3$; see Figure 12. We take $A_1 = \{\gamma_1, \max(\gamma_1', \gamma_2)\}$ (highlighted orange in the second panel of Figure 12). We take $A_2 = \{\min(\gamma_1', \gamma_2), \gamma_3\}$ (highlighted green in the third panel there). We now show that $A_1$ is a pair of disjoint curves; the argument of the same for $A_2$ is similar. We observe that $\gamma_1 \leq \gamma_2 \leq \max(\gamma_1', \gamma_2)$ and $\gamma_1 \leq \gamma_1' \leq \max(\gamma_1', \gamma_2)$. Thus any vertex in both $\gamma_1$ and $\max(\gamma_1', \gamma_2)$ must be a member of $\gamma_1$, $\gamma_2$, and $\gamma_1'$, contradicting the multiplicity assumption. This completes the case $k = 3$.

Now we move to the inductive step. The idea is similar to the base case’s. Consider the set

$$B \equiv \bigcup_{i=2}^{k-1} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma_i';$$

i.e., remove the left and right boundaries $\gamma_1$ and $\gamma_k$ from the original collection. We claim that this set satisfies the inductive hypotheses for $k - 1$. To see this, define $\gamma_i = \min(\gamma_{i+1}, \gamma_i')$ for $i \in \left[1, k-2\right]$ and $\gamma_{k-1} = \max(\gamma_{k-1}, \gamma_{k-2}')$, and define $\gamma_i' = \max(\gamma_{i+1}, \gamma_i')$ for $i \in \left[1, k-3\right]$. Lemma 9.7 asserts that each of these is an upright path, and that their union is $B$. It is also immediate from Lemma 9.7 that the ordering relations of $\gamma_i$ and $\gamma_i'$ are satisfied, and the multiplicity condition is inherited from the original $\gamma_i$ and $\gamma_i'$ curves. Thus $B$ satisfies the inductive hypotheses for $k - 1$.

We therefore obtain two sets, $A_1'$ and $A_2'$, each with $k - 2$ disjoint curves, such that

$$\tilde{A}_1 \cup \tilde{A}_2 \equiv \bigcup_{i=2}^{k-1} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma_i',$$

$\min(\gamma_2, \gamma_1') = \tilde{\gamma}_1 \in \tilde{A}_1$, and $\max(\gamma_{k-1}, \gamma_{k-2}') = \tilde{\gamma}_{k-1} \in \tilde{A}_2$. We now form $A_1$ and $A_2$ by defining

$$A_1 = \tilde{A}_2 \cup \gamma_1 \quad \text{and} \quad A_2 = \tilde{A}_1 \cup \gamma_k.$$

All we have to verify is that $\gamma_1$ is disjoint from the curves in $\tilde{A}_2$; and likewise for $\gamma_k$ and $\tilde{A}_1$. We will show the first as the argument for the second is similar.

Suppose there is $\gamma \in \tilde{A}_2$ such that there exists $v \in \gamma \cap \gamma_1$. Notice that $\gamma_1 \leq \min(\gamma_2, \gamma_1') \leq \gamma$.

Thus if $v$ is an element of the left-hand and right-hand paths in this display, then it must also be an element of the middle path. But then $v$ must have multiplicity at least three in $\bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma_i'$ as the three paths in the last display belong to three disjoint collections of paths, namely $\bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma_i' \setminus (\tilde{A}_1 \cup \tilde{A}_2)$, $\tilde{A}_1$, and $\tilde{A}_2$ respectively. This is a contradiction which completes the induction step and the proof of Lemma 9.6.

$\square$
Remark 9.8. In fact, the hypotheses that \( \{ \gamma_i : i \in [1, k] \} \) and \( \{ \gamma'_i : i \in [1, k-2] \} \) are ordered, \( \gamma_1 \leq \gamma'_1 \), and \( \gamma_{k-2} \leq \gamma_k \) are all not needed for the conclusion of Lemma 9.6 to hold; but if dropping those hypotheses, we have to require the left boundary of the collection to lie in \( A_1 \) and the right boundary in \( A_2 \), instead of \( \gamma_1 \) and \( \gamma_k \). Without these hypotheses, the set \( B \) in the proof above is taken to be the original collection with the left boundary and right boundary curves removed; that the remaining curves can be decomposed as needed to apply the induction hypothesis is proved by a separate induction argument. As the argument becomes more involved and we do not require this generality for our application, we have chosen to not make this more general statement.

Now we use Lemma 9.6 to complete the proof of Proposition 2.6.

Proof of Proposition 2.6. That \( Y_{n,1} \geq Y_{n,2} \) is equivalent to \( 2X^1_n \geq X^2_n \), which is obvious from the definition of the two quantities.

To prove \( Y_{n,k-1} \geq Y_{n,k} \) for \( k \geq 3 \), as we noted, it is sufficient to prove \( X^{k-2}_n + X^k_n \leq 2X^{k-1}_n \). Let \( \gamma^k_n = \{ \gamma^k_{n,j} \ : j \in [1, k] \} \) and \( \gamma^{k-2}_n = \{ \gamma^{k-2}_{n,j} \ : j \in [1, k-2] \} \) be \( k \)- and \( (k-2) \)-geodesic watermelons. We extend them by horizontal line segments so that \( \gamma_i \) is extended to begin at \((i - k + 1, k - i + 1)\) and end at \((n + i - 1, n - i + 1)\) for \( i \in [1, k] \), and \( \gamma'_i \) is extended to begin at \((i - k + 3, k - i - 1)\) and end at \((n + i - 1, n - i + 1)\) for \( i \in [1, k-2] \); this is done so that all the paths have the same time-range.

We apply Lemma 9.6 to these curves; the hypothesis on the ordering of the curves and the multiplicity assumption is satisfied by the disjointness of the curves in geodesic watermelons and an invocation of interlacing (Proposition 9.1). Then we obtain two collections \( A_1 \) and \( A_2 \) of \( k - 1 \) disjoint curves. We intersect these curves with \([1, n]^2\) to get \( A'_1 \) and \( A'_2 \), which are still two collections of \( k - 1 \) disjoint curves and whose union is the same as the union of \( \gamma^k_n \) and \( \gamma^{k-2}_n \), regarded as multisets. The maximum weight property of \((k - 1)\)-geodesic watermelons completes the proof by implying that

\[
X^{k-2}_n + X^k_n = \ell(A'_1) + \ell(A'_2) \leq 2X^{k-1}_n.
\]

\( \square \)

10. Connections to Determinantal Point Processes and Eigenvalue Rigidity

We begin this section by showing how the just proved deterministic monotonicity of \( \{ Y_{n,i} : i \in [1, n] \} \) can be used, somewhat surprisingly, to quickly obtain a concentration of measure result for the same point process. We expect the tail exponent of two for \( k \) to be optimal for a deviation of order \(-\Theta(k^{2/3} n^{1/3})\), i.e., that there is a matching lower bound of the form \(\exp(-\Theta(k^2))\) for deviations of order \(k^{2/3} n^{1/3}\).

Proposition 10.1. Suppose that Assumptions 1, 2, and 3a (resp. 3b) hold. There exist positive constants \( C_1, C_2 \) and \( c_1 \), such that, for some \( c > 0 \), and for \( 1 \leq k \leq c_1 n^{1/2} \) (resp. \( 1 \leq k \leq c_1 n \)),

\[
P\left(Y_{n,k} - \mu n \neq -k^{2/3} n^{1/3} \cdot (C_1, C_2)\right) \leq e^{-ck^2}.
\]

Proof. By Proposition 2.6, \( \{ Y_{n,i} : i \in [1, n] \} \) are ordered. Thus, taking \( C_1 \) as in Theorem II(1),

\[
P\left(Y_{n,k} < \mu n - 4C_1 k^{2/3} n^{1/3}\right) \leq P\left(\sum_{i=k}^{2k-1} Y_{n,i} < \mu n k - 4C_1 k^{5/3} n^{1/3}\right).
\]
The latter quantity is bounded by
\[
\mathbb{P}
\left(\sum_{i=1}^{2k-1} Y_{n,i} < \mu n (2k - 1) - 2^{5/3} C_1 k^{5/3} n^{1/3}\right) \\
+ \mathbb{P}
\left(\sum_{i=1}^{k-1} Y_{n,i} > \mu n (k - 1) + (4 - 2^{5/3}) C_1 k^{5/3} n^{1/3}\right).
\]

Note that \(\sum_{i=1}^{2k-1} Y_{n,i} = X_n^{2k-1}\) and \(\sum_{i=1}^{k-1} Y_{n,i} = X_n^{k-1}\). Thus the preceding display is equal to
\[
\mathbb{P}
\left(X_n^{2k-1} < \mu n (2k - 1) - C_1 (2k)^{5/3} n^{1/3}\right) \\
+ \mathbb{P}
\left(X_n^{k-1} > \mu n (k - 1) + (4 - 2^{5/3}) C_1 k^{5/3} n^{1/3}\right)
\]
the first term is bounded by \(e^{-ck^2}\) by the construction in Theorem 3.1, and the second term by the crude upper bound in Lemma 3.4 (noting that \(4 - 2^{5/3} > 0\)). Both these inferences hold for ranges of \(k\) and \(n\) depending on whether Assumption 3a or 3b is in force, as in the statement of Proposition 10.1.

Now we turn to the other bound. Since the sequence \(\{Y_{n,i} : i \in \llbracket 1, n \rrbracket\}\) is ordered,
\[
\mathbb{P}
\left(Y_{n,k} > \mu n - C_2 k^{2/3} n^{1/3}\right) \leq \mathbb{P}
\left(\sum_{i=1}^{k} Y_{n,i} > \mu n k - C_2 k^{5/3} n^{1/3}\right) \\
= \mathbb{P}
\left(X_n^k > \mu n k - C_2 k^{5/3} n^{1/3}\right) \leq e^{-ck^2}
\]
where Theorem II(1) was used in the last line, with \(C_2\) as in the latter, and where \(k\) and \(n\) are again supposed to belong to certain ranges depending on whether Assumption 3a or 3b is in force.

Integrable methods also yield a lot of information about the process \(\{Y_{n,i} : i \in \llbracket 1, n \rrbracket\}\). For instance, in exponential LPP, this process has the distribution of \(\lambda_1 \geq \cdots \geq \lambda_n\), the eigenvalues of the LUE [AVMW13], i.e., the Hermitian matrix \(X^*X\) where \(X\) is an \(n \times n\) matrix of i.i.d. standard complex Gaussian entries. This can be used to show that the edge, given by \(\{Y_{n,i} : i \in \llbracket 1, k \rrbracket\}\) with \(k\) fixed, converges weakly in the limit of high \(n\) to the \(k\) uppermost elements in the Airy point process, after appropriate centering and scaling. Straightforward determinantal techniques prove the following weaker counterpart to Proposition 10.1:
\[
\mathbb{P}
\left(\lambda_k \notin (4n - C_5 k^{2/3} n^{1/3}, 4n - C_6 k^{2/3} n^{1/3})\right) \leq e^{-ck^2}.
\]
(44)

This is proved in the version of this article on arXiv. The basic argument for (44) uses the representation of the number of points of a determinantal point process in a given interval as the sum of independent Bernoulli random variables [HKPV09, Theorem 4.5.3] and then applies standard concentration inequalities for such sums. However, we also need a sharp estimate on the mean of this sum of Bernoulli random variables, which is available in the exponential LPP case from the literature on the LUE. The corresponding estimate does not seem to be available for geometric LPP, where the relevant determinantal process is the less-studied Meixner ensemble.
10.1. Another route to Theorems I and II using the lower bound of $Y_{n,k}$. The proof of Theorem II that we have presented entails obtaining a high probability lower bound on $X^{j,j}_n$ for a random $j$ by averaging (5), and applying interlacing to relate $j$ and $k$. Since $X^{k,k}_n \geq Y_{n,k}$ by (6), there is another route to Theorems I and II, which uses the lower bound of $Y_{n,k}$ provided by Proposition 10.1 and (44), allowing the direct application of Theorems 3.2 and 3.3 to prove Theorem II(2), bypassing averaging and interlacing. The upper bound on $X^{k,k}_n$ follows the argument given in Section 7, but again invokes the lower bound on $X^{k,k}_n$ directly.

Note that the lower bound on $Y_{n,k}$ in Proposition 10.1 relies only on the construction in Theorem 3.1, the upper bound in Lemma 3.4, and the key monotonicity property in Proposition 2.6.

For exponential LPP, a more direct approach that yields weaker bounds invokes (44) to obtain the required lower bound on $Y_{n,k}$ and proceeds as in the preceding paragraph. The monotonicity result in this case is an immediate consequence of the correspondence with LUE [AVMW13].

11. Point-to-Line LPP

In this section, we prove Theorem III. We start by noting that the proof of interlacing for $\Gamma^k_n$ applies verbatim to $\Upsilon^k_n$.

Proof of Theorem III. We will first bound the weight fluctuations, proving the analogue of Theorem II(1), and then provide the arguments to verify the versions of Theorem II(2) concerning transversal fluctuations.

Weight fluctuations: The lower bound follows immediately from the weight lower bound of Theorem II(1) as the point-to-point $k$-geodesic watermelon’s weight is stochastically dominated by the point-to-line $k$-geodesic watermelon’s weight. So we need to only prove the upper bound.

In fact, the proofs of all the ingredient lemmas and propositions that went into the proof of the weight upper bound of Theorem II(1) apply verbatim to the point-to-line watermelon by replacing $X^k_n$ with $Z^k_n$ and $X^{k,k}_n$ with $Z^{k,j}_n$. The only ingredient for which this is not true is Theorem 3.3, which must be replaced by its point-to-line version Theorem 6.5.

We will now follow the steps of the proof of the weight upper bound of Theorem II(1). We first observe the analogue of Lemma 3.4 for the point-to-line watermelon weight:

$$
\mathbb{P}\left(Z^k_n > \mu nk + tk^{5/3}n^{1/3}\right) \leq e^{-ck^{3/2}k^2},
$$

for $k < t^{-3/4}n^{1/2}$. We have this inequality because the proof of Lemma 3.4 requires only the one-point input (here for the point-to-line single geodesic weight, which is provided from Assumption 3a by Proposition 3.6 with $t = s = 0$) and the BK inequality. With (45) and the weight lower bound of Theorem II(1), we obtain the analogue of Lemma 5.1, which holds for large enough $n$, some $C < \infty$, and $k < C^{-3/4}n^{1/2}$:

$$
\mathbb{P}\left(\bigcup_{j=\left\lfloor \frac{k}{2} \right\rfloor}^{k} \left\{ Z^{j,j}_n > \mu n - Ck^{2/3}n^{1/3}\right\} \right) \geq 1 - e^{-ck^2}.
$$
Let $C_3 > 0$. Similarly to Lemma 7.1’s statement, let $E_n^{j,*}(\delta)$ denote the minimum number of curves of $\gamma_n^j$, over all $j$-geodesic point-to-line watermelons which exit the strip $U_n, \frac{1}{2}\delta k^{1/3}n^{2/3}$, with $\delta$ as given in Theorem 3.2. (Note that Theorem 3.2 applies equally well without any modification to its statement in this situation.) Thus we obtain the analogue of Lemma 7.1:

$$
\mathbb{P}\left( \bigcup_{j=[k/2]}^{k} \left\{ E_n^{j,*}(\delta) > \frac{k}{4} \right\} \right) \geq 1 - e^{-ck^2}. \tag{47}
$$

At this point the proof of the weight upper bound of Theorem II(1) applies essentially verbatim, but we reproduce it here for the reader’s benefit.

Let $B_1$ be the complement of the event whose probability is bounded below in (47), so $\mathbb{P}(B_1) \leq e^{-ck^2}$. On $B_1^c$, by interlacing, at least $k/4$ of the curves of any $k$-geodesic point-to-line watermelon $\gamma_n^k$ must exit the strip of width $\frac{1}{2}\delta k^{1/3}n^{2/3}$. By Theorem 6.5, there is a $c’$ such that, if $B_2$ is defined as

$$
B_2 = \left\{ \exists \Gamma : \ell(\Gamma) > \mu n - c’k^{2/3}n^{1/3}, \text{TF}(\Gamma) > \frac{1}{2}\delta k^{1/3}n^{2/3} \right\},
$$

where $\Gamma$ is an upright path from $(1, 1)$ to the line $x + y = 2n$, then

$$
\mathbb{P}(B_2) \leq e^{-ck}. \tag{48}
$$

Consider the events $A = \{ Z_n^k > \mu nk - \frac{1}{16}c’k^{5/3}n^{1/3} \}$ and $B_3 = \{ Z_n^{[k/8]} > \mu n[7k/8] + \frac{1}{16}c’k^{5/3}n^{1/3} \}$. By (45), $\mathbb{P}(B_3) \leq e^{-ck^2}$. Fix some $k$-geodesic point-to-line watermelon $\gamma_n^k$, and let $Z_n^{k,j}$ be the weight of its $j$th heaviest curve for each $j \in [1, k]$. Now on $A \cap B_3^c$ we have

$$
Z_n^{[k/8]} + \left( Z_n^{k, [k/8]+1} + \ldots + Z_n^{k,k} \right) \geq Z_n^k > \mu nk - \frac{1}{16}c’k^{5/3}n^{1/3} \implies Z_n^{[k/8]+1} + \ldots + Z_n^{k,k} > \frac{1}{8}\mu nk - \frac{1}{8}c’k^{5/3}n^{1/3}.
$$

By definition, the $Z_n^{k,j}$ are ordered; thus, we learn that $Z_n^{k, [k/8]+1} > \mu nk - c’k^{2/3}n^{1/3}$. Again by the ordering, this bound applies to $Z_n^{k,1}, \ldots, Z_n^{k,[k/8]}$ as well. This means that we have $[k/8]$ disjoint curves $\Gamma$, each with $\ell(\Gamma) > \mu n - c’k^{2/3}n^{1/3}$. Thus, on $A \cap B_1^c \cap B_3^c$, we must have at least $[k/8]$ disjoint curves $\Gamma$, each satisfying $\text{TF}(\Gamma) > \frac{1}{2}\delta k^{1/3}n^{2/3}$ and $\ell(\Gamma) > \mu n - c’k^{2/3}n^{1/3}$. By the BK inequality Proposition 4.4 and the bound on $\mathbb{P}(B_2)$ in (48), the probability of this occurring is bounded by $\exp(-ck \cdot k) = \exp(-ck^2)$. Noting the bounds on $\mathbb{P}(B_1)$ and $\mathbb{P}(B_3)$ completes the proof.

Transversal fluctuations: We first prove the exponent lower bound, analogous of Theorem II(2b); namely, that there exist $C < \infty$, $c > 0$ and $\delta > 0$ such that, for $k > k_0$ and $k < C^{-3/4}n^{1/2}$,

$$
\mathbb{P}\left( \text{TF}^*(n, k) < \delta k^{1/3}n^{2/3} \right) \leq e^{-ck^2}. \tag{49}
$$

The proof proceeds in exactly the same way as that of Theorem II(2b). Let $C$ be as in (46). Let $\delta = \delta’(2^{2/3}C)$ be as in Theorem 3.2 and let $\delta’ = 2^{-1/3}\delta$. Let $A_k$ denote the
event that there exist \([k/2]\) disjoint paths contained in \(U_{n,8’k^{1/3}n^{2/3}}\), each of which has weight at least \(\mu n - Ck^{2/3}n^{1/3}\). Further, let \(B_k\) denote the event from (46):

\[
B_k := \bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{ Z_n^{j,j} > \mu n - Ck^{2/3}n^{1/3} \right\}.
\]

Clearly, by Theorem 3.2 and (46), we have \(\mathbb{P}(A_k^c \cap B_k) \geq 1 - e^{-ck^2}\) for some \(c > 0\).

Observe next that, on \(A_k^c \cap B_k\), there exists \(j \in \{\frac{k}{2}, \ldots, k\}\) such that each of the curves of any \(j\)-geodesic point-to-line watermelon \(\Upsilon_n^j\) has weight at least \(\mu n - Ck^{2/3}n^{1/3}\), and hence some of them must exit \(U_{n,8’k^{1/3}n^{2/3}}\). By the interlacing result for point-to-line watermelons, the same must be true for any \(k\)-geodesic point-to-line watermelon. This completes the proof of (49) with \(\delta\) in the statement replaced by \(\delta’\).

Now we turn to the exponent upper bound, analogous to Theorem II(2a)); namely, that there exist \(C'' < \infty, C < \infty, c > 0\) and \(k_0\) such that, for \(k > k_0\) and \(k < C^{-1}n^{1/2},
\[
\mathbb{P}\left( \bar{T}_F(n, k) > C''k^{1/3}n^{2/3} \right) \leq e^{-ck},
\]
which immediately implies the upper exponent bound. This proof proceeds exactly as does Theorem II(2a)’s.

Let \(C'\) be as in (46) with \(2k\) in place of \(k\), and \(B_k'\) denote the large probability event from there:

\[
B_k' := \bigcup_{j=k}^{2k} \left\{ X_n^{j,j} > \mu n - C’k^{2/3}n^{1/3} \right\}.
\]

Let \(A_k' = A_k'(C'')\) denote the event that there exists a path from (1, 1) to the line \(x + y = 2n\) that exits \(U_{n,C''k^{1/3}n^{2/3}}\) and has weight at least \(\mu n - C’k^{2/3}n^{1/3}\). Now choose \(C'' > 0\) (possible by Theorem 6.5) such that \(\mathbb{P}(A_k') \leq e^{-ck^2}\), for some \(c'' > 0\) for all \(k\) and all \(n\) sufficiently large. Clearly now it suffices to show that, on \(B_k' \cap (A_k')^c\), no \(k\)-geodesic point-to-line watermelon exits \(U_{n,C''k^{1/3}n^{2/3}}\). By the definition of \(B_k'\), there exists \(j \in [k, 2k]\) such that all paths of all \(j\)-geodesic point-to-line watermelons have weight at least \(\mu n - C’k^{2/3}n^{1/3}\), and \(A_k'\) ensures that all these paths are contained in \(U_{n,C''k^{1/3}n^{2/3}}\). The proof of (50) is completed by invoking the interlacing of geodesic point-to-line watermelons. \(\square\)

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Appendix A. Exponential & Geometric LPP Satisfy the Assumptions

In this appendix, we cite the results which show that exponential and geometric LPP satisfy Assumptions 1, 2, and 3.

We start with the foundational result on the Tracy-Widom fluctuations of the maximal path weight, due to Johansson [Joh00a], and continue with moderate deviations estimates and expectation asymptotics for the maximal path weight.

Recall that $X_{n,\lfloor nh \rfloor}$ is the last passage value from $(1, 1)$ to $(n, \lfloor nh \rfloor)$.

**Theorem A.1.** Suppose that the vertex weight law in LPP is exponential of mean one, and fix $\psi > 1$. As $n \to \infty$, uniformly over $h \in [\psi^{-1}, \psi]$, 

$$
\frac{X_{n,\lfloor nh \rfloor} - (1 + \sqrt{h})^2 n}{h^{-1/6}(1 + \sqrt{h})^{4/3} n^{1/3}} \overset{d}{\to} F_{TW}.
$$

Suppose instead that the weight law is geometric with parameter $p \in (0, 1)$. Then, uniformly over $h \in [\psi^{-1}, \psi]$, as $n \to \infty$, 

$$
\frac{X_{n,\lfloor nh \rfloor} - \omega(h, p)n}{\sigma(h, p)n^{1/3}} \overset{d}{\to} F_{TW},
$$

where 

$$
\omega(h, p) = \frac{(1 + \sqrt{hp})^2}{1 - p} \quad \text{and} \quad \sigma(h, p) = \frac{p^{1/6}h^{-1/6}}{1 - p}((\sqrt{h} + \sqrt{p})^{2/3}(1 + \sqrt{hp})^{2/3}.
$$

Here $\overset{d}{\to}$ denotes convergence in distribution, $F_{TW}$ is the GUE Tracy-Widom distribution, and uniformly is meant in the sense of uniform (over $h$) convergence of distribution functions.

**Proof.** These results appear for fixed $h$ in [Joh00a], the first as Theorem 1.6 there and the second as Theorem 1.2.

In fact, the proofs also establish the uniformity of the convergence for $h \in [\psi^{-1}, \psi]$, though unfortunately this is not explicitly stated. For the aid of the reader, we give a few brief pointers to how this uniformity can be seen to hold in [Joh00a]. Let $F_{n,h}$ be the distribution function of $X_{n,\lfloor nh \rfloor}$. It is shown in [Joh00a] that $F_{n,h}$ converges pointwise to $F_{TW}$ for fixed $h$ (in [Joh00a], our $h$ is denoted by $\gamma$).

In the geometric case, $F_{n,h}$ is explicitly written down in terms of a determinantal formula involving the Meixner polynomials; in the exponential case the same is done in terms of the Laguerre polynomials. The argument for the exponential case is the same as for the geometric case given ahead, with Laguerre polynomials replacing Meixner polynomials.

Lemma 3.1 of [Joh00a] asserts that if certain estimates are satisfied by determinantal formulas of this form, then they converge pointwise to $F_{TW}$. That $F_{n,h}$ satisfies these estimates in the geometric case is then shown in [Joh00a, Sect. 5] by making use of asymptotics and estimates for Meixner polynomials. If these estimates hold uniformly over $h \in [\psi^{-1}, \psi]$, then the convergence claimed in [Joh00a, Lemma 3.1] holds uniformly.

That the estimates hold uniformly is a consequence of the fact that all the bounds in [Joh00a, Sect. 5] are continuous functions of $h$, and $[\psi^{-1}, \psi]$ is a compact set. The dependence on $h$ in many of these bounds is not written explicitly, but it is straightforward to check that their dependence on $h$ is continuous. \qed
The next three statements establish that exponential and geometric LPP satisfy Assumptions 2 and 3a.

**Theorem A.2** (Moderate deviation estimate). Consider exponential LPP. Fix $\psi > 1$ and let $h \in [\psi^{-1}, \psi]$. There exist $t_0 = t_0(\psi), n_0 = n_0(\psi)$ and $c = c(\psi) > 0$ such that, for $n > n_0$ and $t > t_0$,
\[
\mathbb{P}\left( X_{n,[hn]} - (1 + \sqrt{h})^2 n > tn^{1/3} \right) \leq \exp\left( -c \min(t^{3/2}, tn^{1/3}) \right).
\]
For the lower tail, for $t > t_0$,
\[
\mathbb{P}\left( X_{n,[hn]} - (1 + \sqrt{h})^2 < -tn^{1/3} \right) \leq \exp\left( -ct^3 \right).
\]
Similarly, in geometric LPP of parameter $p \in (0, 1)$, the above two displays hold with $\omega(h, p)$ (as in Theorem A.1) in place of $(1 + \sqrt{h})^2$.

**Proof.** For exponential LPP, the lower tail for $t > t_0$ and the upper tail for $t_0 < t < n^{2/3}$ are provided by Theorem 2 of [LR10]. For the remaining case of $t \geq n^{2/3}$ in the upper tail, see [Joh00a].

For geometric LPP, the upper tail bound is proved in [Joh00a] (combining Corollary 2.4 and equation (2.22) there), while the lower tail bound is implied by [BDM+01, Theorem 1.1]. □

**Theorem A.3** (Expected point-to-point weight). Fix $\psi > 1$ and let $h \in [\psi^{-1}, \psi]$. There exist $c_1 = c_1(\psi) > 0$, $c_2 = c_2(\psi) > 0$, and $n_0 = n_0(\psi)$ such that, for $n > n_0$, in exponential LPP,
\[
\mathbb{E}[X_{n,[hn]}] - (1 + \sqrt{h})^2 n \left/ h^{-1/6}(1 + \sqrt{h})^{4/3} n^{1/3} \right. \in (-c_1, -c_2),
\]
while for geometric LPP of parameter $p \in (0, 1)$,
\[
\frac{\mathbb{E}[X_{n,[hn]}] - \omega(h, p)n}{\sigma(h, p)n^{1/3}} \in (-c_1, -c_2),
\]
with $\omega(h, p)$ and $\sigma(h, p)$ as in Theorem A.1.

**Proof.** This follows from Theorem A.1 and Theorem A.2, and that the GUE Tracy-Widom distribution has negative mean. Note that we need the uniform convergence of the distribution functions in Theorem A.1 in order to have $n_0$ depend on only $\psi$ and not $h$. □

Assumption 2 can be verified in exponential and geometric LPP by replacing $n$ by $n - xn^{2/3}$ and setting $h$ to be $(n + xn^{2/3})/(n - xn^{2/3})$ in Theorem A.3.

We now note why the GUE Tracy-Widom distribution has negative mean, by pulling together known results to which Ivan Corwin has drawn our attention.

**Lemma A.4** (Negative mean of GUE Tracy-Widom). Let $X_{TW}$ be distributed according to the GUE Tracy-Widom distribution. Then we have $\mathbb{E}[X_{TW}] < 0$.

**Proof.** Remark 1.15 and equation 1.29 of [QR19] show that $X_{BR}$ strictly stochastically dominates $4^{1/3}X_{GUE}$, where the law of $X_{BR}$ is the Baik-Rains distribution (the cumulative distribution function of $X_{BR}$ is labeled $F_{stat}$ in [QR19, Equation 1.29]). Now [BR00, Proposition 2.1] asserts that $\mathbb{E}[X_{BR}] = 0$, completing the proof. □
Appendix B. Proofs of Basic Tools

This appendix contains some consequences of Assumptions 1, 2, and 3a that were used in the main text.

- Section B.1 proves Proposition 3.5, the upper tail bound on interval-to-interval weights;
- Section B.2 proves Proposition 3.6, concerning upper tails for point-to-line weights; and
- Section B.3 proves Proposition 3.7, concerning the lower tail and mean of the constrained point-to-point weights.

B.1. Interval-to-interval estimates.

Proof of Proposition 3.5. We start with the bound (7) on $\tilde{Z}$, and we treat only the case that $|z| \leq \rho r^{1/3}$; when $|z| > \rho r^{1/3}$, our argument works by setting $z = \rho r^{1/3}$ and using the concavity of the limit shape posited in Assumption 2.

By considering the event that $\sup_{(u,v) \in S(U)} X_{u \rightarrow v}$ is large and two events defined in terms of the environment outside of $U$, we find a point-to-point path which has large length. To define these events, first define points $\phi_{\text{low}}$ and $\phi_{\text{up}}$ on either side of $A$ and $B$:

$\phi_{\text{low}} := \left(-\ell^{3/2}r, -\ell^{3/2}r\right)$

$\phi_{\text{up}} := \left((1+\ell^{3/2})r - zr^{2/3}, (1+\ell^{3/2})r + zr^{2/3}\right)$.

Let $u^*$ and $v^*$ be points on $A$ and $B$ where the suprema in the definition of $\tilde{Z}$ are attained, and set

$E_{\text{low}} = \left\{ X_{\phi_{\text{low}} \rightarrow u^*-0(1)} > \mu \ell^{3/2}r - \frac{\theta \ell^{1/2}}{3} r^{1/3} \right\}$ and

$E_{\text{up}} = \left\{ X_{v^*+0(1) \rightarrow \phi_{\text{up}}} > \mu \ell^{3/2}r - \frac{\theta \ell^{1/2}}{3} r^{1/3} \right\}$.

Using Assumption 2 to bound the expectation in going from the second to the third line of the following, we find that

$$\mathbb{P}\left( \sup_{(u,v) \in S(U)} X_{u \rightarrow v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1 + 2 \ell^{3/2}} + \theta \ell^{1/2} r^{1/3}, E_{\text{low}}, E_{\text{up}} \right)$$

$$\leq \mathbb{P}\left( X_{\phi_{\text{low}} \rightarrow \phi_{\text{up}}} \geq \mu (1 + 2 \ell^{3/2})r - G_2 \frac{z^2 r^{1/3}}{1 + 2 \ell^{3/2}} + \frac{\theta \ell^{1/2}}{3} r^{1/3} \right)$$

$$\leq \mathbb{P}\left( X_{\phi_{\text{low}} \rightarrow \phi_{\text{up}}} \geq \mathbb{E}\left[ X_{\phi_{\text{low}} \rightarrow \phi_{\text{up}}} \right] + \frac{\theta \ell^{1/2}}{3} r^{1/3} \right) \leq \exp\left( -c \min(\theta^{3/2}, \theta r^{1/3}) \right),$$

for $c > 0$ independent of $\ell$. We used Assumption 2 for the former inequality of the final line, and Assumption 3a and the stationarity of the random environment for the latter.

Let us denote conditioning on the environment $U$ by the notation $\mathbb{P}(\cdot | U)$. By this, we mean we condition on the weights of vertices interior to $U$ as well as those on the lower and upper sides $A$ and $B$. Then we see
\[ P \left( \sup_{(u,v) \in S(U)} X_{u \to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1 + 2 \ell^{3/2}} + \theta \ell^{1/2} r^{1/3}, E_{\text{low}}, E_{\text{up}} \mid U \right) \]

\[ = P \left( \sup_{(u,v) \in S(U)} X_{u \to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1 + 2 \ell^{3/2}} + \theta \ell^{1/2} r^{1/3} \mid U \right) \cdot P \left( E_{\text{low}} \mid U \right) \cdot P \left( E_{\text{up}} \mid U \right). \]

So with (51), all we need is a lower bound on \( P \left( E_{\text{low}} \mid U \right) \) and \( P \left( E_{\text{up}} \mid U \right) \). This is straightforward using independence of the environment between \( U \) and the regions above and below it:

\[ P \left( E_{\text{lower}}^c \mid U \right) \leq \sup_{u \in A} P \left( X_{\phi_{\text{low}} \to u-(0,1)} \leq \mu \ell^{3/2} r - \frac{\theta \ell^{1/2}}{3} r^{1/3} \right) \leq \frac{1}{2} \]

for large enough \( \theta \) and \( r \) (depending on \( \ell \)), using Assumption 3a. A similar upper bound holds for \( P \left( E_{\text{upper}}^c \mid U \right) \). Together this gives

\[ P \left( \sup_{(u,v) \in S(U)} X_{u \to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1 + 2 \ell^{3/2}} + \theta \ell^{1/2} r^{1/3} \mid U \right) \]

\[ \geq \frac{1}{4} \cdot P \left( \sup_{(u,v) \in S(U)} X_{u \to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1 + 2 \ell^{3/2}} + \theta \ell^{1/2} r^{1/3} \right), \]

and taking expectation on both sides, combined with (51), gives the bound (7) of Proposition 3.5.

We now treat the bound (8) on \( Z^{\text{ext}, \delta} \). Observe that

\[ \{ Z^{\text{ext}, \delta} > \theta \} \subseteq \bigcup_{u \in L_{\text{low}}} \left( \left\{ X_{u \to (\delta r, 2r)} > \mu r + \theta r^{1/3} \right\} \cup \left\{ X_{u \to (2r, \delta r)} > \mu r + \theta r^{1/3} \right\} \right). \]

We bound the probability of \( \bigcup_{u \in L_{\text{low}}} \{ X_{u \to (\delta r, 2r)} > \mu r + \theta r^{1/3} \} \); the full bound then follows by a symmetric argument and a union bound.

The point \((\delta r, 2r)\) can be written as \((\tilde{r} - \tilde{z}, \tilde{r} + \tilde{z})\) with \( \tilde{r} = (1 + \delta/2) r \) and \( \tilde{z} = (1 - \delta/2) r \). By the definitions, we have that \( \tilde{r}/\tilde{r} \geq \rho \) if \( \delta \) is sufficiently small. So by the concavity guaranteed by Assumption 2 and by making \( \delta \) small enough, we see that

\[ E \left[ X_{u \to (\delta r, 2r)} \right] \leq \mu \tilde{r} - G_2 \rho^2 \tilde{r} \leq \mu r + \frac{\mu \delta}{2} r - G_2 \rho^2 \left( 1 + \frac{\delta}{2} \right) r \leq (\mu - \eta) r \]

for some \( \eta > 0 \). With such a value of \( \delta \) fixed, note that \((\delta r, 2r)\) is such that we may apply Assumption 3a; we find by so doing that

\[ P \left( X_{u \to (\delta r, 2r)} \geq \mu r + \theta r^{1/3} \right) \leq P \left( X_{u \to (\delta r, 2r)} - E \left[ X_{u \to (\delta r, 2r)} \right] \geq \eta r + \theta r^{1/3} \right) \]

\[ \leq \exp \left( -c \epsilon - c \min \left( \theta^{3/2}, \theta^{1/3} \right) \right). \]

Taking a union bound over the \( O(r^{2/3}) \) many \( u \) in \( L_{\text{low}} \) and reducing the value of \( c \) to absorb this factor into the exponent complete the proof of Proposition 3.5. \( \square \)
B.2. Point-to-line estimates.

Proof of Proposition 3.6. Note that we may assume $s \leq r^{1/3}$. Let $\rho$ and $G_1$ be as in Assumption 2. Let $\Gamma$ be the leftmost path with endpoint not on the line segment connecting $(r - (s + t)r^{2/3}, r + (s + t)r^{2/3})$ and $(r + (s + t)r^{2/3}, r - (s + t)r^{2/3})$ whose weight is $X$; this is well-defined by the weight maximization property of $\Gamma$. Let $p_{\text{left}}$ be the coordinates of the starting point of $\Gamma$ on $\mathbb{L}_{\text{left}}$ (recall from (4)), and $p_{\text{right}}$ be the coordinates of the endpoint of $\Gamma$ on the line $x + y = 2r$. Let $y_j = (s + t + j)r^{2/3}$, $\mathbb{L}_j$ be the line segment joining the points 

$$(r - y_j, r + y_j) \quad \text{and} \quad (r - y_{j+1}, r + y_{j+1}),$$

and let $A_j$ be the event that $p_{\text{right}} \in \mathbb{L}_j$ and $(p_{\text{left}}, p_{\text{right}})$ are such that the slope of the line connecting them is not extreme enough to apply the second part of Proposition 3.5, i.e., it holds that $|(p_{\text{left}} - p_{\text{right}})_y|/(p_{\text{left}} + p_{\text{right}})_x \leq 1 - \delta/2$, for $j = 0, \ldots, r^{1/3}$, with $\delta$ as in that proposition’s second part. Finally, let $A$ be the event that $(p_{\text{left}}, p_{\text{right}})$ satisfy $|(p_{\text{left}} - p_{\text{right}})_y|/(p_{\text{left}} + p_{\text{right}})_x > 1 - \delta/2$. Then clearly the whole probability space equals 

$$\bigcup_{j=0}^{r^{1/3}} A_j \cup A.$$

Thus we have

$$\mathbb{P}(X - \mu r > \theta r^{1/3} - 0.5c_1s^2r^{1/3}) \leq \sum_{j=0}^{r^{1/3}} \mathbb{P}(X > \mu r + \theta r^{1/3} - 0.5c_1s^2r^{1/3}, A_j)$$

$$+ \mathbb{P}(X > \mu r + \theta r^{1/3} - 0.5c_1s^2r^{1/3}, A).$$

(52)

We will bound the two terms using the next two lemmas.

Lemma B.1. In the notation of Proposition 3.6 and under Assumptions 2 and 3a, there exist $c > 0$, $c_1 > 0$, and $r_0$ such that, for $r > r_0$ and $j = 0, 1, \ldots, r^{1/3}$, 

$$\mathbb{P}(X > \mu r + \theta r^{1/3} - 0.5c_1s^2r^{1/3}, A_j) \leq \exp \left(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3 + j^3)\right).$$

Lemma B.2. In the notation of Proposition 3.6 and under Assumptions 2 and 3a, there exist constants $c > 0$, $c_1 > 0$, $\theta_0$, $s_0$, and $r_0$ such that, for $r > r_0$, $s > s_0$, and $\theta_0 < \theta < r^{2/3}$, 

$$\mathbb{P}(X > \mu r + \theta r^{1/3} + \theta r^{1/3} - 0.5c_1s^2r^{1/3}, A) \leq \exp \left(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3)\right).$$

Before proving these lemmas, we show how the proof of Proposition 3.6 is completed using them. Lemma B.1 says that each individual summand in the first term of (52) is bounded by $\exp \left(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3 + j^3)\right)$, while Lemma B.2 says that the second term is bounded by $\exp \left(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3)\right)$. So we have, by summing over $j$, 

$$\mathbb{P}(X > \mu r + \theta r^{1/3} - 0.5c_1s^2r^{1/3}) \leq C \exp \left(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3)\right)$$
for some $C < \infty$. Reducing the value of $c$ completes the proof. \hfill \Box

In seeking to prove Lemmas B.1 and B.2, we wish to show that when the endpoint of a particular geodesic is sufficiently extreme, it suffers a weight loss with high probability. Lemma B.1 addresses the case that the slope between the points is bounded away from 0 and $\infty$, while Lemma B.2 addresses when the slope between the points is extreme.

**Proof of Lemma B.1.** We fix $j$. We divide $\mathbb{L}_{\text{left}}$ into segments of size at most $r^{2/3}$ each, indexed by $i$ as $\mathbb{L}_{\text{left}}^i$. Thus there are at most $t$ segments.

In the notation of Proposition 3.5, we have that $z$ is bounded uniformly away from $r$, and so we may bound $X$ on $A_j$ by using Assumption 2 and Proposition 3.5. We set $c_1 = G_2 \rho^2/3$ with $G_2$ as in Assumption 2. Then,

$$
\mathbb{P}\left( X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A_j \right) \leq \mathbb{P}\left( \sup_{y \in \mathbb{L}_{\text{left}}^i, \ w \in \mathbb{L}_j} X_{y \to w} > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3} \right)
$$

$$
\leq \sum_{i=1}^t \mathbb{P}\left( \sup_{y \in \mathbb{L}_{\text{left}}^i, \ w \in \mathbb{L}_j} X_{y \to w} > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3} \right).
$$

Now note that $0.5c_1 s^2 < \frac{1}{6} G_2 \left( (s + j) \wedge \rho r^{1/3} \right)^2$ by our choice of $c_1$ and since $s \leq r^{1/3}$. Thus each summand in the previous display is in turn bounded by

$$
\mathbb{P}\left( \sup_{y \in \mathbb{L}_{\text{left}}^i, \ w \in \mathbb{L}_j} \left( X_{y \to w} - \mu r + \frac{1}{3} G_2 \left( (s + j) \wedge \rho r^{1/3} \right)^2 r^{1/3} \right) \right)
$$

$$
\leq \theta r^{1/3} + \frac{1}{6} G_2 \left( (s + j) \wedge \rho r^{1/3} \right)^2 r^{1/3}
$$

$$
\leq \exp\left( -c(\min(\theta^{3/2}, \rho r^{1/3}) + s^3 + j^3) \right).
$$

The last inequality is via the first part of Proposition 3.5, and holds for both (i) $s = 0$ and $\theta$ sufficiently large, as well as for (ii) $\theta = 0$ and $s > s_0$ (with the earlier mentioned assumption that $s \leq r^{1/3}$, which also allows us to put $s^3 + j^3$ in the exponent without a $\rho$ term). Using that $t \leq s^2$ and reducing the value of $c$ complete the proof. \hfill \Box

**Remark B.3.** Note that we only use the upper bound on $t$ at the very last step, and a weaker bound such as $t \leq s^\alpha$ for some $\alpha \geq 2$ in Proposition 6.1 would also work (we require $\alpha \geq 2$ in applications of the proposition, for example to prove Theorem 3.3).

**Proof of Lemma B.2.** Here the endpoints are not bounded uniformly away from the coordinate axes, and so we will make use of the second part of Proposition 3.5, which yields

$$
\mathbb{P}\left( X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A \right) \leq \exp\left( -c\left(r + \min(\theta^{3/2}, \rho r^{1/3})\right) \right)
$$

$$
\leq \exp\left( -0.5c\left(r + \min(\theta^{3/2}, \rho r^{1/3}) + s^3\right) \right).
$$

The last inequality is from the fact that $s \leq r^{1/3}$ (as otherwise the statement is trivial) and $\theta < r^{2/3}$, and again holds for both choices of parameters (i) and (ii) in Proposition 3.6. \hfill \Box
B.3. Lower tail and mean of constrained point-to-point.

Proof of Proposition 3.7. We first derive the lower tail bound (9). Fix \( J = \theta^{1/2} / \ell \) and define \( u_j = (J^{-1} \cdot j(r - z - hr^{2/3}), J^{-1} \cdot j(r + z + hr^{2/3})) \) for \( j = 0, \ldots, J \). By the stationarity of the random field and the union bound, we have

\[
\mathbb{P}\left( X_{u \to u'}^U \leq \mu r - \theta r^{1/3} \right) \leq J \cdot \mathbb{P}\left( X_{u \to u_1}^U \leq \frac{1}{J} \mu r - \frac{\theta}{J} r^{1/3} \right).
\]

We also have

\[
\mathbb{P}\left( X_{u \to u_1}^U \leq \frac{1}{J} \mu r - \frac{\theta}{J} r^{1/3} \right) \leq \mathbb{P}\left( X_{u \to u_1} \leq \mu r - \frac{\theta}{J^{2/3}} \left( \frac{r}{J} \right)^{1/3} \right) + \mathbb{P}\left( X_{u \to u_1} > \mu r - \frac{\theta}{J^{2/3}} \left( \frac{r}{J} \right)^{1/3}, \text{ TF}(\Gamma_{u, u_1}) \right) > \ell J^{2/3} \left( \frac{r}{J} \right)^{2/3} \leq \exp\left(-c\theta^{3/2} / J\right) + \exp\left(-c\ell J^2\right) \leq 2 \exp(-c\ell\theta),
\]

for sufficiently large \( \theta \) depending on \( K \). For the second-to-last inequality we have used Assumption 3a for the first term and Theorem 3.3 for the second. The parameters (marked here with tildes to avoid confusion) for the application of Theorem 3.3 are as follows: \( \tilde{r} = r / J, \tilde{\ell} = K J^{2/3} / J = K / J^{1/3} \), and \( \tilde{\theta} = \theta^{1/2} / J^{1/3} \). It is easy to check that the conditions of Theorem 3.3 are met with these parameter choices for sufficiently large \( \theta \).

For Theorem 3.3 to apply, we also need \( \ell J^{2/3} \geq \tilde{\theta} = \theta^{1/2} / J^{1/3} \), which is satisfied with equality by our choice of \( J \). This completes the proof of the lower tail estimate.

For the lower bound (10) on \( \mathbb{E}[X_{u \to u'}^U] \), we have

\[
\mathbb{E}[X_{u \to u'}^U] = \mathbb{E}[X_{u \to u'}] - \mathbb{E}[X_{u \to u'} - X_{u \to u'}^U] \geq \mu r - G_1 z^2 r^{1/3} - g_1 r^{1/3} - \mathbb{E}[X_{u \to u'} - X_{u \to u'}^U],
\]

using the lower bound for the first term from Assumption 2. Note that the second term is the expectation of a positive random variable. We claim that the expectation is bounded above by \( Cr^{1/3} \) for some \( C < \infty \). This follows from the tail probability formula for expectation:

\[
\mathbb{E}[X_{u \to u'} - X_{u \to u'}^U] = \int_0^\infty \mathbb{P}\left( X_{u \to u'} - X_{u \to u'}^U > tr^{1/3} \right) dt.
\]

Bounding the integrand by \( \mathbb{P}(X_{u \to u'} > \mu r + 0.5tr^{1/3}) + \mathbb{P}(X_{u \to u'}^U < \mu r - 0.5tr^{1/3}) \) and using the bounds from Assumption 3a and the tail bound from above to show that this expression is integrable, with the bound on the integral being independent of \( r \), complete the proof.
Appendix C. Proof of a Crude Upper Bound Under a Convexity Assumption

In this appendix, we complete the proof of Lemma 3.4 under the upper tail convexity hypothesis, Assumption 3b.

Proof of Lemma 3.4 under Assumption 3b. As in the proof of Lemma 3.4, for any fixed \( k \)-geodesic watermelon \( \mathcal{W} \), let \( X_{n,\mathcal{W}}^k \) be the weight of its \( j^{th} \) heaviest curve. For \( J \subseteq [1, k] \) and \( \delta = (\delta_1, \ldots, \delta_k) \) such that \( \delta_j \geq 0 \) for each \( j \in J \) and \( \delta_j = 0 \) for \( j \in [1, k] \setminus J \), define

\[
A_{\delta, J} = \{ \exists \mathcal{W} : X_{n,\mathcal{W}}^k > \mu n + \delta_j n^{1/3} \text{ for all } j \in J \}.
\]

We claim that if \( \sum_{j \in J} \delta_j = \lceil tk^{5/3} \rceil \), then, for an absolute constant \( c > 0 \) and for \( k \leq t^{-3/2}n \),

\[
P(A_{\delta, J}) \leq \exp\left(-ct^{3/2}k^2\right). \tag{53}
\]

This follows from the BK inequality and Assumption 3b, which gives that

\[
P(A_{\delta, J}) \leq P(\exists \text{ disjoint paths } \{\gamma_j : j \in J\} \text{ with } \ell(\gamma_j) \geq \mu n + \delta_j n^{1/3} \text{ for all } j \in J)
\leq \exp \left(-\sum_{j \in J} I_n(\delta_j)\right).
\]

Since \( I_n \) is convex and \( \sum_{j \in J} \delta_j = \lceil tk^{5/3} \rceil \), we get

\[
\sum_{j \in J} I_n(\delta_j) \geq k I_n\left(\frac{1}{k} \sum_{j \in J} \delta_j\right) = k I_n\left(k^{-1} \lceil tk^{5/3} \rceil\right) \geq c' \min(\frac{1}{2}k^2, tk^{5/3}n^{1/3}) \geq c't^{3/2}k^2
\]

as \( I_n(x) \geq c' \min(xn^{1/3}, x^{3/2}) \) for all \( x \geq 0 \) from Assumption 3b, and \( k \in [1, \lfloor \min(1, t^{-3/2}n) \rfloor] \). Let \( D \) be the set of \( \delta \) defined by

\[
D = \left\{ \delta : \sum_{j=1}^k \delta_j = \lceil tk^{5/3} \rceil, \delta_j \in \mathbb{N} \cup \{0\} \ \forall j \in [1, k] \right\}.
\]

Now we observe \( P\left(X_n^k > \mu nk + tk^{5/3}n^{1/3}\right) = P(\exists \mathcal{W} : \sum_{j=1}^k (X_{n,\mathcal{W}}^k - \mu n)n^{-1/3} > tk^{5/3}) \). By considering the ceiling of each summand in the event of this last probability, we obtain that the last expression is bounded by

\[
P\left(\exists \mathcal{W} : \sum_{j=1}^k \lceil (X_{n,\mathcal{W}}^k - \mu n)n^{-1/3} \rceil > tk^{5/3} \right)
\leq P\left(\bigcup_{\delta \in D} \left\{ \exists \mathcal{W} \text{ and } J \subseteq [1, k] : \lceil (X_{n,\mathcal{W}}^k - \mu n)n^{-1/3} \rceil > \delta_j \ \forall j \in J \right\} \right)
\leq 2^k \cdot |D| \cdot \exp\left(-ct^{3/2}k^2\right).
\]
where the first inequality is seen by considering $J$ to be the set of indices $j$ such that $\lceil (X^{k,j}_{n,W} - \mu n)n^{-1/3} \rceil > 0$ and taking $\delta_j$ to be an integer at most the latter quantity such that $\sum_{j \in J} \delta_j = \lceil tk^{5/3} \rceil$ for $j \in J$, and $\delta_j = 0$ for $j \notin [1,k] \setminus J$. The second inequality follows by the union bound; the bound (53) on $\mathbb{P}(A_{\delta,j})$; and the cardinality of the number of subsets of $[1,k]$. The cardinality of $D$ is the number of non-negative solutions to $x_1 + \ldots + x_k = tk^{5/3}$, which is $\binom{k^{5/3}+k-1}{k-1} \leq \binom{2(tk^{5/3}/k)}{k}$. Using the trivial bound $\binom{n}{k} \leq n^k$ and simplifying, we obtain that $|D| \leq \exp\left(\frac{1}{2}ct^{3/2}/k^2\right)$ for $k$ larger than some absolute $k_0$ depending on $t$, so the proof is completed by reducing the value of $c$. $\square$

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