Monoidally Graded Manifolds

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Abstract

We give a generalization of the theory of \(\mathbb{Z}_2\)-graded manifolds to a theory of \(\mathcal{I}\)-graded manifolds, where \(\mathcal{I}\) is a commutative semi-ring with some additional properties. We prove Batchelor’s theorem in this generalized setting. To our knowledge, such a proof is still missing except for some special cases.

1 Introduction

The notion of a supermanifold appeared in the late 70s when mathematicians tried to understand the concept of supersymmetry proposed by physicists \(^1\). It extends the notion of a manifold \(M\) naturally by attaching Grassmann algebras locally to \(M\). The mysterious anticommutativity property of a fermionic field over \(M\) can be then interpreted in terms of the anticommutativity of Grassmann algebras. When multiplying two fermionic fields, one gets a bosonic field. This process can be tracked by assigning \(0 \in \mathbb{Z}_2\) to bosonic fields and \(1 \in \mathbb{Z}_2\) to fermionic fields. Hence, supermanifolds are also called \(\mathbb{Z}_2\)-graded manifolds. Though the grading in this case is merely used to distinguish commutative and anticommutative objects.

In 1982, Witten published a seminal paper relating Morse theory to supersymmetric quantum mechanics \(^2\). It was realized since then that there exists a very deep connection between supersymmetric theories in physics and cohomology theories in mathematics. To establish such a connection, one needs to update the language of \(\mathbb{Z}_2\)-graded manifolds to the language of \(\mathbb{Z}\)-graded (or graded) manifolds \(^3\). One major achievement in that direction is the AKSZ formalism of topological quantum field theories \(^5\), where the topological sigma models \(^6\) are reinterpreted in the language of so-called \(Q\)-manifolds\(^1\).

In cohomological field theories (or topological quantum field theories of Witten type), one can obtain useful invariants of smooth manifolds by studying observables \(\mathcal{O}(p)\) satisfying the following descent equations \(^6\)\(^7\)

\[Q(\mathcal{O}(p)) = d(\mathcal{O}(p-1)),\]  

for \(p \geq 1\) with \(Q \mathcal{O}(0) = 0\), where \(d\) is the de Rham differential. \(^1\)\(^1\) is equivalent to saying that \(\mathcal{O} := \sum_p \mathcal{O}(p)\) is closed in the total complex of some bicomplex with horizontal differential \(d\) and vertical differential \(Q\). Such a bicomplex can be obtained by applying a “change of coordinates” to

\(^1\)A \(Q\)-manifold is a graded manifold equipped with a vector field \(Q\) of degree 1 satisfying \(Q^2 = 0\).
the variational bicomplex of a fiber bundle \cite{8}. It is then also interesting to study \( \mathbb{Z} \times \mathbb{Z} \)-graded (or bigraded) manifolds.

In this paper, we follow the algebraic-geometric approaches in \cite{1, 9–13} to give a definition of \( \mathcal{I} \)-graded manifolds, where \( \mathcal{I} \) is an arbitrary commutative semi-ring with some additional properties. We also apply the techniques in \cite{10} to give a proof of Batchelor’s theorem, i.e., that every \( \mathcal{I} \)-graded manifold can be obtained from an \( \mathcal{I} \)-graded vector bundle. To our knowledge, such a proof is still missing except for some special cases \cite{14–16}.

## 2 Commutative Monoids and Parity Functions

Let \((\mathcal{I}, 0, +)\) be a commutative monoid. Let \(\mathbb{Z}_q\) denote the cyclic group of order \(q\).

**Definition 2.1.** A parity function is a (non-trivial) monoid homomorphism \(p : \mathcal{I} \to \mathbb{Z}_2\).

Not every \(\mathcal{I}\) has a non-trivial parity function. For example, there is no non-trivial homomorphism from \(\mathbb{Z}_q\) to \(\mathbb{Z}_2\) when \(q\) is odd. Let \(I_a\) denote \(p^{-1}(a)\) for \(a \in \mathbb{Z}_2\). We have \(I_a + I_b \subseteq I_{a+b}\). Recall that an element \(x\) in \(\mathcal{I}\) is called cancellative if \(x + y = x + z\) implies \(y = z\) for all \(y\) and \(z\) in \(\mathcal{I}\).

Suppose that there is a cancellative element in \(I_1\). It is easy to see that such an element induces an injective map from \(I_a\) to \(I_{a+1}\). It follows from the Cantor-Bernstein theorem that there exists a bijection between \(I_0\) and \(I_1\). A monoid is called cancellative if every element in it is cancellative.

We have shown that

**Proposition 2.1.** Let \(\mathcal{I}\) be an commutative cancellative monoid. If \(\mathcal{I}\) has a non-trivial parity function \(p\), then the submonoid \(I_0\) and its complement \(I_1\) have the same cardinality.

**Remark 2.1.** In the finite case, proposition 2.1 is no longer true if we drop the cancellative condition. For example, we can consider the commutative monoid defined by the following table. A non-trivial \(p\) is defined by setting \(p(0) = p(b) = 0\) and \(p(a) = 1\).

|     | 0 | a | b |
|-----|---|---|---|
| 0   | 0 | a | b |
| a   | a | b | a |
| b   | b | a | b |

Table 2.1: A commutative non-cancellative monoid of order 3.

The question now is, given an appropriate commutative cancellative monoid \(\mathcal{I}\), how can one construct a parity function for it? If \(\mathcal{I}\) is finite, it is not hard to show that \(\mathcal{I}\) is actually an abelian group. The fundamental theorem of finite abelian groups then tells us that \(\mathcal{I}\) is isomorphic to a direct product of cyclic groups of prime-power order. By Proposition 2.1, one of these cyclic groups must be \(\mathbb{Z}_{2^k}\), \(k \geq 1\). We can write \(\mathcal{I} = \mathbb{Z}_{2^k} \times \cdots\) and define \(p\) by sending \((x, \cdots) \in \mathcal{I}\) to \(a - 1\) (mod 2), where \(a\) is the order of \(x \in \mathbb{Z}_{2^k}\). If \(\mathcal{I}\) is infinite, the construction of \(p\) is hard, perhaps not possible in general. However, one can easily work out the case when \(\mathcal{I}\) is free. (\(\mathcal{I}\) is then cancellative, but not a group.) Let \(I_0\) be the submonoid of elements generated by even number of generators. Let \(I_1\) be the subset of elements generated by odd number of generators. Note that

\footnote{We say that \(I_0\) is the even part of \(\mathcal{I}\), and that \(I_1\) is the odd part of \(\mathcal{I}\). We also say that an element of \(I_0\) has parity \(a\) for \(a = 0, 1\).}
We obtain a parity function which sends elements in \( I_a \) to \( a \). As an example, let \( I \) be \( \mathbb{N} \), the monoid of natural numbers under addition. \( p \) is then defined by sending even numbers to 0 and odd numbers to 1.

Let \( K(I) \) denote the Grothendieck group of \( I \). Recall that it can be constructed as follows. Let \( \sim \) be the equivalence relation on \( I \times I \) defined by \( (a_1, a_2) \sim (b_1, b_2) \) if there exists a \( c \in I \) such that \( a_1 + b_2 + c = a_2 + b_1 + c \). The quotient \( K(I) = I \times I / \sim \) has a group structure by \( [(a_1, a_2)] + [(b_1, b_2)] = [(a_1 + b_1, a_2 + b_2)] \).

**Proposition 2.2.** Let \( p \) be a parity function for \( I \). The map
\[
p : K(I) \to \mathbb{Z}_2 \\
[(a_1, a_2)] \mapsto p(a_1) + p(a_2)
\]
is well-defined and gives a parity function for \( K(I) \).

**Remark 2.2.** When \( I \) is cancellative, it can be seen as a submonoid of \( K(I) \) by the embedding
\[
i : I \to K(I) \\
a \mapsto [(a, 0)].
\]
For this reason, we sometimes simply write \( a - b \) to denote \( [(a, b)] \in K(I) \). The cancellative property is not necessary for the proof of Proposition 2.2. It guarantees the non-triviality of \( p' \), since \( p' \) restricted to \( I \) must coincide with \( p \).

**Proof.** Let \( (a_1, a_2) \) and \( (b_1, b_2) \) represent the same element of \( K(I) \), i.e., there exist some \( c \) such that \( a_1 + b_2 + c = a_2 + b_1 + c \). One then concludes that \( a_1 + b_2 \) and \( a_2 + b_1 \) must have the same parity. Note that, for \( a, b \in \mathbb{Z}_2 \), \( a = b \) if and only if \( a + b = 0 \). We have
\[
p'([(a_1, a_2)]) + p'([(b_1, b_2)]) = p(a_1 + b_2) + p(a_2 + b_1) = 0.
\]
Hence \( p'([a_1, a_2]) = p'([b_1, b_2]). \)

As an example, consider \( K(\mathbb{N}) = \mathbb{Z} \), the monoid of integers under addition. The parity function \( p' \) induced from the parity function \( p \) for \( \mathbb{N} \) again sends even numbers to 0 and odd numbers to 1.

### 3 Monoidally Graded Ringed Spaces

Let \( R \) be a commutative ring. Let \( I \) be a countable commutative cancellative monoid equipped with a parity function \( p \).

**Definition 3.1.** An \( I \)-graded \( R \)-module is an \( R \)-module \( V \) with a family of sub-modules \( \{ V_i \}_{i \in I} \) indexed by \( I \) such that \( V = \bigoplus_{i \in I} V_i \). \( v \in V \) is said to be homogeneous if \( v \in V_i \) for some \( i \in I \). We use \( d(v) \) to denote the degree of \( v \), \( d(v) = i \).

Given two \( I \)-graded \( R \)-modules \( V \) and \( W \), we make the direct sum \( V \oplus W \) and the tensor product \( V \otimes W \) into \( I \)-graded \( R \)-modules by setting
\[
V \oplus W = \bigoplus_{i \in I} (V_i \oplus W_i), \quad V \otimes W = \bigoplus_{k \in I} \left( \bigoplus_{i+j=k} V_i \otimes W_j \right).
\]
We can also make the space $\text{Hom}(V, W)$ of $R$-linear maps from $V$ to $W$ into a $K(\mathcal{I})$-graded $R$-module by setting

$$\text{Hom}(V, W) = \bigoplus_{\alpha \in K(\mathcal{I})} \text{Hom}(V, W)_\alpha, \quad \text{Hom}(V, W)_\alpha = \{ f \in \text{Hom}(V, W) | f(V_i) \subset W_j, [(j, i)] = \alpha \}.$$ 

A morphism from $V$ to $W$ is just an element of $\text{Hom}(V, W)_0$, i.e., an $R$-linear map of degree 0.

**Remark 3.1.** $\text{Hom}(V, W)$ is in general not $\mathcal{I}$-graded. This is because that we should assign degree “$j - i$” to a map $f$ which maps elements in $V_i$ to elements in $w \in W_j$. But the minus operation does make sense for a general monoid $\mathcal{I}$. So we have to work with $K(\mathcal{I})$, the group completion of $\mathcal{I}$. Note that $V^* = \text{Hom}(V, R)$, the dual of $V$, is in particular $K(\mathcal{I})$-graded. (The degree of elements in $V^*_i$ is $-i$.) Hence $V^* \otimes W$, which is isomorphic to $\text{Hom}(V, W)$, is $K(\mathcal{I})$-graded by assigning degree $j - i$ to elements in $V^*_i \otimes W_j$. Everything is consistent.

Now, suppose that $\mathcal{I}$ also has a commutative multiplicative structure which is compatible with the additive structure. That is, it is a commutative cancellative semi-ring. We write $ab$ as the multiplication of $a$ and $b$ in $\mathcal{I}$.

**Definition 3.2.** An $\mathcal{I}$-graded $R$-module $A$ is called an $\mathcal{I}$-graded $R$-algebra if $A$ is a unital associative $R$-algebra and if the multiplication $\mu : A \otimes A \to A$ is a morphism of $\mathcal{I}$-graded $R$-modules. We write $xy = \mu(x \otimes y)$ as the shorthand notation for multiplications of $A$. $A$ is said to be commutative if

$$xy - (-1)^{p(x)p(y)}yx = 0 \tag{3.1}$$

for all homogeneous $x, y \in A$, where we use $p(x)p(y)$ to denote $p(d(x)d(y)) \in \mathbb{Z}_2$.

**Remark 3.2.** Here we have to be careful about the sign factor appearing in the right hand side of (3.1). Although both of $\mathcal{I}$ and $\mathbb{Z}_2$ are semi-rings, $p$ is not necessarily a semi-ring homomorphism and we do not have $p(d(x)d(y)) = p(d(x))p(d(y))$ in general. To choose which as the sign factor is just a matter of convention.

Morphisms of $\mathcal{I}$-graded algebras are simply linear maps of degree 0 which preserves the algebraic structures. We use $\text{Comm-Alg}_I$ to denote the category of commutative $\mathcal{I}$-graded algebras.

**Definition 3.3.** The tensor algebra $T(V)$ is the $\mathcal{I}$-graded $R$-module $T(V) = \bigoplus_{n \in \mathbb{N}} V^\otimes_n$, together with the tensor product $\otimes$ as the canonical multiplication. The symmetric algebra $S(V)$ is the quotient algebra of $T(V)$ by the $\mathcal{I}$-graded two-sided ideal generated by

$$v \otimes w - (-1)^{p(v)p(w)}w \otimes v,$$

where $v, w \in V \subset T(V)$ are homogeneous.

**Remark 3.3.** $S(V)$ has a canonical $\mathbb{N}$-grading inherited from $T(V)$ which should not be confused with its $\mathcal{I}$-grading. We write $S(V) = \bigoplus_{n \in \mathbb{N}} S^n(V)$ to indicate that fact. Note that $S^0(V) = R$, but $S(V)_0$, the sub-space of homogeneous elements of degree 0, is in general larger than $R$.

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3The multiplicative structure on $\mathbb{Z}_2$ is inherited from the one on $\mathbb{Z}$.
S(V) is universal in the sense that, given a commutative \(I\)-graded \(R\)-algebra \(A\) and a morphism \(f : V \to A\). There exists a unique algebraic homomorphism \(\tilde{f} : S(V) \to A\) such that the following diagram commutes

\[
\begin{array}{ccc}
V & \xrightarrow{i} & S(V) \\
\downarrow{f} & & \downarrow{\tilde{f}} \\
A & = & A
\end{array}
\]

where \(i : V \to S(V)\) is the canonical embedding. Note that \(\tilde{f}\) preserves the \(I\)-grading, i.e., it is a morphism in \(\text{Comm-Alg}_I\). Choosing \(A\) to be \(R\) (viewed as an \(I\)-graded \(R\)-algebra whose components of non-zero degree are 0.) and \(f\) to be the zero map, we obtain an \(R\)-algebra homomorphism from \(S(V)\) to \(R\). We denote this map by \(\epsilon\). Note that \(\ker \epsilon = \bigoplus_{n>0} S^n(V)\).

Let \(k\) be a field and \(R\) be a commutative \(k\)-algebra. Let \(A\) be a commutative \(I\)-graded \(k\)-algebra.

**Definition 3.4.** A \(k\)-algebra epimorphism \(\epsilon : A \to R\) is called a body map of \(A\) if \(\ker \epsilon \supset I\), where \(I\) is the ideal in \(A\) generated by homogeneous elements of non-zero degree.

By definition, \(\epsilon\) preserves the \(I\)-grading of \(A\).

**Definition 3.5.** Let \(\epsilon\) be a body map of \(A\). \(A\) is said to be projected if the short exact sequence

\[
0 \to \ker \epsilon \to A \xrightarrow{\epsilon} R \to 0
\]

splits.

The splitting gives \(A\) an \(R\)-module structure depending on \(\epsilon\), with respect to which \(\epsilon\) becomes an \(R\)-algebra homomorphism. Conversely, \(A\) is projected if \(A\) has an \(R\)-module structure and \(\epsilon\) preserves that structure.

**Lemma 3.1.** Let \(V\) be an \(I\)-graded \(R\)-module with \(V_0 = 0\). Let \(\epsilon\) be an \(R\)-linear body map of \(S(V)\). Then \(\epsilon\) is unique.

**Proof.** In this case, \(S(V) = R \oplus I\) where \(I = \bigoplus_{n>0} S^n(V)\). Since \(I \subset \ker \epsilon\) and \(\epsilon\) is \(R\)-linear, the only possible choice of \(\epsilon\) is the canonical one. \(\Box\)

**Remark 3.4.** Let \(V\) be as in Lemma 3.1. Suppose \(A \cong S(V)\) as \(I\)-graded \(k\)-algebras. In particular, this implies that \(A\) admits a decomposition \(A = A' \oplus I\) where \(A' \cong R\) and \(I\) is the ideal generated by homogeneous elements of non-zero degree. Let \(\epsilon\) be a body map of \(A\). Since \(I \subset \ker \epsilon\), \(\epsilon\) is determined by \(\epsilon|_{A'}\). In other words, \(\epsilon\) is determined by a \(k\)-algebra endomorphism of \(R\).

More can be said if \(V\) is free.

**Lemma 3.2.** Let \(V\) be a free \(I\)-graded \(R\)-module with \(V_0 = 0\). Let \(\epsilon\) be an \(R\)-linear body map of \(S(V)\). (By Lemma 3.1, \(\epsilon\) is the canonical one.) Let \(I\) denote the kernel of \(\epsilon\). Then there exists an \(R\)-algebra isomorphism

\[
S(V) \cong S(I/I^2),
\]

where \(I^2\) is the square of the ideal \(I\).
Proof. Let \( \iota : V \hookrightarrow S(V) \) be the canonical embedding. Since \( I = \bigoplus_{n \geq 0} S^n(V) \), we have \( \iota(V) \subset I \), which yields another embedding \( V \hookrightarrow I/I^2 \hookrightarrow S(I/I^2) \), which induces the desired isomorphic map between \( S(V) \) and \( S(I/I^2) \). \( \square \)

**Definition 3.6.** The \( \mathcal{I} \)-graded algebra of formal power series on \( V \) is the \( R \)-module

\[
S(V) = \prod_{n \in \mathbb{N}} S^n(V)
\]
equipped with the canonical algebraic multiplication.

**Remark 3.5.** As is in the case of \( \mathcal{I} = \mathbb{Z}[4] \), it is actually crucial to work with \( \widetilde{S(V)} \) instead of \( S(V) \) when the even part of \( V \) is non-trivial. The former allows us to have a coordinate description of morphisms between "\( \mathcal{I} \)-graded domains", a notion of partition of unity for "\( \mathcal{I} \)-graded manifolds", and more.

Let \( I \) be the kernel of the canonical body map of \( S(V) \). One can equip \( S(V) \) with the so-called \( I \)-adic topology\(^4\). Moreover, one can consider the \( I \)-adic completion of \( S(V) \) which is defined as the inverse limit

\[
\widehat{S(V)}_I := \varprojlim S(V)/I^n
\]
of the inverse system \( ((S(V)/I^n)_{n \in \mathbb{N}}, (\pi_{m,n})_{n \leq m \in \mathbb{N}}) \), where \( \pi_{m,n} : S(V)/I^m \to S(V)/I^n \) is the canonical projection. Note that there is also a canonical projection \( S(V) \to S(V)/I^n \) for each \( n \in \mathbb{N} \). By the universal property of the inverse limit, one obtains a morphism

\[
\iota_I : S(V) \to \widehat{S(V)}_I
\]
with kernel being \( \bigcap_{n \geq 0} I^n = \{0\} \). On the other hand, it is easy to see that \( S(V)/I^n \cong \bigoplus_{n=0}^{n-1} S^i(V) \) for \( n \geq 1 \). It follows that there is a canonical isomorphism \( \widehat{S(V)}_I \cong \overline{S(V)} \) under which \( \iota_I \) coincides with the canonical inclusion \( S(V) \hookrightarrow S(V) \).

In fact, \( S(V) \) can be made into a metric space such that \( \overline{S(V)} \) is the completion of \( S(V) \) with respect to the metric structure \( [17] \). The metric-induced topology on \( S(V) \), with a slight abuse of notation, coincides with the \( I \)-adic topology on \( S(V) \), where \( I = \prod_{n \geq 0} S^n(V) \).

**Lemma 3.3.** Let \( A \) be a commutative \( \mathcal{I} \)-graded \( R \)-algebra. Let \( J \) be an ideal of \( A \) such that \( A \) is \( J \)-adic complete. \( S(V) \) is universal in the sense that, given a morphism \( f : V \to A \) such that \( f(V) \subset J \), there exists a unique (continuous) algebraic homomorphism \( \hat{f} : S(V) \to A \) such that the following diagram commutes

\[
\begin{array}{ccc}
V & \xleftarrow{\iota} & S(V) \\
\downarrow{f} & & \downarrow{j} \\
A & \xrightarrow{j} & A
\end{array}
\]

**Proof.** We already know that \( f \) induces a unique morphism \( f' : S(V) \to A \) such that \( f' \circ \iota = f \). By assumption, \( f' \) extends naturally to a morphism \( \hat{f} : S(V) \to \widehat{A}_J \cong A \).

\(^4\)To each point \( x \) of \( S(V) \) one assigns a collection of subsets \( B(x) = \{x + I^n\}_{x \in A, n \geq 0} \). The \( I \)-adic topology is then the unique topology on \( S(V) \) such that \( B(x) \) forms a neighborhood base of \( x \) for all \( x \).
Claim: \( \tilde{f} \) is continuous.

Proof: It suffices to show that \( \tilde{f}^{-1}(J^m) \) is a neighborhood of 0 for any \( m \in \mathbb{N} \). By assumption, \( I \subset \tilde{f}^{-1}(J) \). It follows that \( I^m \subset \tilde{f}^{-1}(J^m) \).

Since \( S(V) \) is dense in \( \overline{S(V)} \) and \( \tilde{f}|_{S(V)} = f' \), \( \tilde{f} \) is also unique.

Remark 3.6. Likewise, we have a canonical body map of \( \overline{S(V)} \) induced from the zero map \( V \to R \). Similar results like Lemma 3.1 and Lemma 3.2 also hold. For example, we have

\[
\overline{S(V)} \cong \overline{S(I/I^2)},
\]

where \( V \) and \( I \) are as in Lemma 3.2.

Lemma 3.4. Let \( \epsilon \) be the canonical body map of \( \overline{S(V)} \). Then for \( f \in \overline{S(V)} \), \( f \) is invertible if and only if \( \epsilon(f) \) is invertible.

Proof. “\( \Rightarrow \)”: Trivial.

“\( \Leftarrow \)”: Suppose \( \epsilon(f) = c \) where \( c \in R \) is invertible. We can write \( f = c + f' \) where \( f' \in \prod_{n \geq 1} S^n(V) \).

Note that \( (f')^k \in \prod_{n \geq k} S^n(V) \) for all \( k > 0 \). We can then set the inverse of \( f \) to be the formal sum

\[
f^{-1} := c^{-1} \sum_{k \in \mathbb{N}} (-1)^k (c^{-1} f')^k.
\]

(\( f^{-1} \) is well-defined because the formal sum restricted to each \( S^n(V) \) is a finite sum.)

Corollary 3.1. \( \overline{S(V)} \) is local if \( R \) is local.

Proof. Choose a non-unit \( f \in \overline{S(V)} \). Let \( c = \epsilon(f) \). By Lemma 3.4, \( c \) is a non-unit. Since \( R \) is local, \( 1 - c \) is invertible. \( 1 - f \) is then a unit by Lemma 3.4.

Recall that a ringed space \((X, \mathcal{O})\) is a topological space \(X\) with a sheaf of rings \(\mathcal{O}\) on \(X\).

Definition 3.7. An \( \mathcal{I} \)-graded ringed space is a ringed space \((X, \mathcal{O})\) such that

1. \( \mathcal{O}(U) \) is an \( \mathcal{I} \)-graded algebra for any open subset \(U\) of \(X\);

2. the restriction morphism \( \rho_{V,U} : \mathcal{O}(U) \to \mathcal{O}(V) \) is a morphism of \( \mathcal{I} \)-graded algebras.

A morphism between two \( \mathcal{I} \)-graded ringed spaces \((X_1, \mathcal{O}_1)\) and \((X_2, \mathcal{O}_2)\) is just a morphism \( \varphi = (\tilde{\varphi}, \varphi^*) \) between ringed spaces such that \( \varphi^* : \mathcal{O}_2(U) \to \mathcal{O}_1(\tilde{\varphi}^{-1}(U)) \) preserves the \( \mathcal{I} \)-grading for any open subset \(U\) of \(X_2\).

Let \((X, C)\) be a ringed space where \(C(U)\) are commutative rings. One can define \( \mathcal{I} \)-modules and commutative \( \mathcal{I} \)-graded \(C\)-algebras in a similar way. In particular, the structure sheaf \(\mathcal{O}\) of an \( \mathcal{I} \)-graded ringed space can be viewed as an \( \mathcal{I} \)-graded \(C\)-algebra if \(C\) is a sub-sheaf of \(\mathcal{O}\) such that \(C(U)\) are homogeneous sub-algebras of degree 0 of \(\mathcal{O}(U)\).

Definition 3.8. Let \( \mathcal{F} \) be an \( \mathcal{I} \)-graded \(C\)-module. The formal symmetric power \( \overline{S(\mathcal{F})} \) of \(\mathcal{F}\) is the sheafification of the presheaf

\[
U \to \overline{S(\mathcal{F}(U))},
\]

where \( \overline{S(\mathcal{F}(U))} \) is the \( \mathcal{I} \)-graded algebra of formal power series on the \(C(U)\)-module \(\mathcal{F}(U)\).

By definition, \( \overline{S(\mathcal{F})} \) is a commutative \( \mathcal{I} \)-graded \(C\)-algebra.
Lemma 3.5. Let \( A \) be a commutative \( I \)-graded \( C \)-algebra. Let \( B \) be a sub-sheaf of \( A \) such that \( A(U) \) is \( B(U) \)-adic complete for all open subsets \( U \). \( S(\mathcal{F}) \) is universal in the sense that, given a morphism of \( I \)-graded \( C \)-modules \( F : \mathcal{F} \to A \) such that \( F(\mathcal{F}(U)) \subset B(U) \) for all open subsets \( U \), there exists a unique morphism of \( I \)-graded \( C \)-algebras \( \tilde{F} : S(\mathcal{F}) \to A \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\iota} & S(\mathcal{F}) \\
\downarrow F & & \downarrow \tilde{F} \\
\tilde{F} & & \to A
\end{array}
\]

where \( \iota : \mathcal{F} \to S(\mathcal{F}) \) is the canonical monomorphism.

Proof. This follows directly from the universal property of sheafification\(^5\) and the universal property of \( S(\mathcal{F}(U)) \) stated in Lemma 3.3. \( \square \)

To end this section, we state the following lemma taken from [10].

Lemma 3.6. Let

\[ 0 \to G \to H \to F \to 0. \]  \hspace{1cm} (3.2)

be a short exact sequence of \( C \)-modules where \( F \) and \( G \) are locally free \( C \)-modules. Then the obstruction of the existence of a splitting of (3.2) can be represented as an element in the first sheaf cohomology group \( H^1(X, \text{Hom}(F, G)) \) of \( \text{Hom}(F, G) \).

4 Monoidally Graded Domains

Throughout this section, \( V \) is a real \( I \)-graded vector space with \( V_0 = 0 \). The dimension of the homogeneous sub-space \( V_i \) of \( V \) is \( m_i \). We also assume that only finitely many of \( m_i \) are non-zero.

Definition 4.1. Let \( U \) be a domain of \( \mathbb{R}^n \). An \( I \)-graded domain \( U \) of dimension \( n \mid (m_i)_{i \in \mathbb{Z}} \) is an \( I \)-graded ringed space \( (U, \mathcal{O}) \), where \( \mathcal{O} \) is the sheaf of \( S(\mathcal{V}) \)-valued smooth functions.

Remark 4.1. \( U \) is a locally ringed space by Corollary 3.1.

For example, a domain \( U \) with the sheaf \( C^\infty \) of smooth functions on \( U \) is an \( I \)-graded domain of dimension \( n \mid (0, \cdots) \), which is denoted again by \( U \) for simplicity.

Lemma 4.1. Let \( F : C^\infty \to C^\infty \) be an endomorphism of sheaves of commutative rings on \( U \). Then \( F \) must be the identity.

Proof. First, we show that \( F \) is actually an endomorphism of sheaves of unital \( \mathbb{R} \)-algebras on \( U \). It suffices to show that \( F \) restricted to any open subset of \( U \) sends a constant function to itself. We know this is true for \( \mathbb{Q} \)-valued constant functions. Now, if \( F \) sends a constant function \( f \) to a non-constant function \( g \), then one can find two rational number \( b_1 \) and \( b_2 \) such that \( g - b_1 \) and \( g - b_2 \) are non-invertible. But then the pre-images \( f - b_1 \) and \( f - b_2 \) are non-invertible, which implies that

\(^5\)That is, given a presheaf \( \mathcal{F} \), a sheaf \( \mathcal{G} \), and a presheaf morphism \( F : \mathcal{F} \to \mathcal{G} \), there exists a unique sheaf morphism \( \tilde{F} : \mathcal{F}^\sharp \to \mathcal{G} \) such that \( \tilde{F} \circ \iota = F \), where \( \mathcal{F}^\sharp \) is the sheafification of \( \mathcal{F} \) and \( \iota : \mathcal{F} \to \mathcal{F}^\sharp \) is the canonical morphism.
\( f \) is non-constant: a contradiction. To show that \( g \) actually equals \( f \), use the fact that the only field endomorphism of \( \mathbb{R} \) is the identity.

Let \( p \in U \). \( F \) induces a unital ring endomorphism \( F_p \) on the stalk \( C^\infty_p \). On the other hand, for any open neighborhood \( U_p \subset U \) of \( p \), the evaluation map
\[
\text{ev} : C^\infty(U_p) \to \mathbb{R} \\
f \mapsto f(p)
\]
induces a map \( \text{ev}_p : C^\infty_p \to \mathbb{R} \). For \( f_p \in C^\infty_p \), it is easy to see that \( f_p \) is invertible if and only if \( \text{ev}_p(f_p) \neq 0 \). Let \( c = \text{ev}_p(F_p(f_p)) \). \( f_p - c \) is non-invertible. Hence \( \text{ev}_p(f_p) = c \). In other words, for any open subset \( U' \) of \( U \), we have \( F_{U'}(f)(p) = f(p) \) for all \( f \in C^\infty(U') \) and all \( p \in U' \). This implies \( F = \text{id} \).

A morphism between \( \mathcal{I} \)-graded domains is just a morphism of \( \mathcal{I} \)-graded locally ringed spaces. Recall that we have the canonical body map \( \epsilon : C^\infty(U) \otimes S(V) \to C^\infty(U) \).

**Proposition 4.1.** There exists a unique monomorphism \( \varphi : U \to \mathcal{U} \) with \( \tilde{\varphi} = \text{id} \).

**Proof.** Existence is guaranteed by \( \epsilon \). Uniqueness follows from Remark 3.4 and Lemma 4.4.

We also have a canonical morphism for the other direction \( \mathcal{U} \to U \) induced by the canonical embedding \( \iota : C^\infty(U) \to C^\infty(U) \otimes S(V) \). Note that \( \epsilon \circ \iota = \text{id} \) on \( C^\infty(U) \).

**Proposition 4.2.** Let \( \varphi = (\tilde{\varphi}, \varphi^*) \) be a morphism from \( \mathcal{U}_1 = (U_1, \mathcal{O}_1) \) to \( \mathcal{U}_2 = (U_2, \mathcal{O}_2) \). The following diagram commutes.
\[
\begin{array}{ccc}
U_1 & \xrightarrow{\varphi} & U_2 \\
\uparrow & & \uparrow \\
U_1 & \xrightarrow{\tilde{\varphi}} & U_2
\end{array}
\]

**Proof.** Let \( U \) be an open subset of \( U_2 \). Let \( f \in \mathcal{O}_2(U) \). We need to show that
\[
\epsilon(\varphi^*(f)) = \epsilon(f) \circ \tilde{\varphi}.
\]
Suppose this does not hold. One can find a \( p \in \tilde{\varphi}^{-1}(U) \) such that \( \epsilon(\varphi^*(f))(p) = c \neq \epsilon(f)(\tilde{\varphi}(p)) \). Then there exists an open neighborhood \( U' \subset U \) of \( \tilde{\varphi}(p) \) such that \( \epsilon(f) - c \) is invertible. By Lemma 3.4, \( f - c \) is also invertible on \( U' \), which implies that \( \varphi^*(f - c) \) is invertible on \( \tilde{\varphi}^{-1}(U') \subset \tilde{\varphi}^{-1}(U) \), which contradicts the fact that \( \epsilon(\varphi^*(f - c)) \) is non-invertible on \( \tilde{\varphi}^{-1}(U') \).

**Definition 4.2.** A coordinate system of \( \mathcal{U} \) is a collection of functions \( (x^\mu, \theta_{i,a}) \) such that

1. \( x^\mu \) are elements of \( \mathcal{O}(U)_0 \) such that \( \epsilon(x^\mu) \) form a coordinate system of \( U \);
2. \( \theta_{i,a} \) are homogeneous elements of \( \mathcal{O}(U) \) of degree \( d(\theta_{i,a}) = i \), \( i \neq 0 \) and \( a = 1, \ldots, m_i \), which generate \( \mathcal{O}(U) \) as a \( C^\infty(U) \)-algebra.

\[\text{There will be no longer such a canonical morphism if we go the category of \( \mathcal{I} \)-graded manifolds.}\]
Suppose that $\mathcal{I}$ can be given a total order $\prec$. It follows that any function $f \in \mathcal{O}(U)$ can be written uniquely in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(x^\mu) \prod_{j \in \mathcal{J}} \theta_j^{\beta_j}, \quad (4.1)$$

where

- $\mathcal{J} \in \text{Pow}(\mathcal{I})$, $\beta = (\beta_j)_{j \in \mathcal{J}}$, $\beta_j = (\beta_{j,1}, \ldots, \beta_{j,m_j})$, $\beta_{j,k} \in \{0,1\}$ if $p(j) = 1$, $\beta_{j,k} \in \mathbb{N}$ if $p(j) = 0$;
- $\theta_j^{\beta_j} = \theta_{j,1}^{\beta_{j,1}} \cdots \theta_{j,m_j}^{\beta_{j,m_j}}$, the product $\prod_{j \in \mathcal{J}} \theta_j^{\beta_j}$ is arranged in a proper order such that $\theta_j^{\beta_j}$ is on the left of $\theta_{j'}^{\beta_{j'}}$ whenever $j < j'$;
- For a smooth function $g \in C^\infty(U)$, the notation $g(x^\mu)$ should be understood as

$$g(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{\mu} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} g(x^\mu)(x^1 - x^1)^{i_1} \cdots (x^n - x^n)^{i_n}. \quad (4.2)$$

Hence, $g(x^\mu)$ is an element in $\mathcal{O}(U)_0$ instead of $C^\infty(U)$.

The sum in (4.1) is well-defined because, by assumption, only finitely many of $m_j$ are non-zero.

**Remark 4.2.** One may wonder how we obtain (4.1). In fact, by definition, every function $f$ can be expressed in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(x^\mu) \prod_{j \in \mathcal{J}} \theta_j^{\beta_j}. \quad (4.1)$$

One can then define a map from $\mathcal{O}(U)$ to itself by sending $g(\epsilon(x^\mu))$ to $g(x^\mu)$. Now consider another map which sends $g(\epsilon(x^\mu))$ to $g^-(x^\mu)$, where

$$g^-(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{\mu} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} g(\epsilon(x^\mu))(\epsilon(x^1) - x^1)^{i_1} \cdots (\epsilon(x^n) - x^n)^{i_n}. \quad (4.2)$$

Using the binomial theorem, it is easy to see that the second map is the inverse of the first. In fact, the reader may notice that the map $g(\epsilon(x^\mu)) \mapsto g(x^\mu)$ is exactly the “Grassmann analytic continuation map” defined in [18].

**Corollary 4.1.** Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be as in Proposition 4.2. $\tilde{\varphi}$ is uniquely determined by $\varphi^*$.

**Proof.** Let $(x^\mu, \theta_{i,a})$ be a coordinate system of $U_2$. By Proposition 4.2 one has $\tilde{\varphi}^\mu = \epsilon(\varphi^* x^\mu)$, where $(\tilde{\varphi}^\mu)$ is $\tilde{\varphi}$ expressed in the coordinate system $(\epsilon(x^\mu))$ of $U_2$. \hfill $\blacksquare$

Let $\text{ev}$ be the evaluation map of $C^\infty(U)$ at $p \in U$. Let $s_p$ denote $\text{ev} \circ \epsilon$. Let $I_p$ denote the kernel of $s_p$. We follow [9] to prove the following lemmas.

**Lemma 4.2.** For any functions $f \in \mathcal{O}(U)$ and any integer $k \geq 0$, there is a polynomial $P_k$ in the coordinates $(x^\mu, \theta_{i,a})$ such that $f - P_k \in I_p^{k+1}$.
Proof. Use the classical Hadamard lemma and the decomposition (4.1). \( \square \)

**Lemma 4.3.** Let \( f \) and \( g \) be functions of \( \mathcal{O}(U) \), then \( f = g \) if and only if \( f - g \in I_p^k \) for all \( k \in \mathbb{N} \) and \( p \in U \). In other words, \( \bigcap_{p \in U} \bigcap_{k \in \mathbb{N}} I_p^k = \{ 0 \} \).

Proof. Let \( h = f - g \). Apply the decomposition (4.1) to \( h \), then by Lemma 4.2, \( h_{f,J,\beta} = 0 \) for all \( J \) and \( \beta \). Hence \( h = 0 \). \( \square \)

**Lemma 4.4.** Any morphism of \( \mathcal{I} \)-graded \( \mathbb{R} \)-algebras \( s : \mathcal{O}(U) \to \mathbb{R} \) must take the form \( s = s_p \).

Proof. Since we assume \( V_0 = 0 \), \( s \) can be reduced to a morphism \( C^\infty(U) \to \mathbb{R} \). Let \( x^\mu \) be a coordinate system of \( U \). Let \( f^\mu = x^\mu - s(x^\mu) \) and \( h = \sum_{\mu} (f^\mu)^2 \). Then \( s(h) = 0 \), which implies that \( h \) is non-invertible. In other words, there exists \( p \in U \) such that \( x^\mu(p) = s(x^\mu) \) for all \( \mu \).

Now suppose there exists an \( f \in C^\infty(U) \) such that \( s(f) \neq s_p(f) = f(p) \). Consider the function \( h' = h + (f - s(f))^2 \). Since \( h > 0 \) for all points of \( U/\{p\} \), we know \( h' > 0 \) on \( U \). But this contradicts the fact that \( s(h') = 0 \). Hence \( s \) must equal \( s_p \). \( \square \)

**Theorem 4.1.** Let \( \varphi = (\varphi, \varphi^*) \) be a morphism from \( \mathcal{U}_1 = (U_1, \mathcal{O}_1) \) to \( \mathcal{U}_2 = (U_2, \mathcal{O}_2) \). Let \( (x^\mu, \theta_{i,a}) \) be a coordinate system of \( U_2 \). Then \( \varphi^* \) is uniquely determined by the equations

\[
\varphi^* x^\mu = y^\mu, \quad \varphi^* \theta_{i,a} = \eta_{i,a},
\]

where \( y^\mu \in \mathcal{O}(U_1)_0, \eta_{i,a} \in \mathcal{O}(U_1) \), and \( (\epsilon(y^\mu))(p) \in U_2 \) for all \( p \in U_1 \).

Proof. Let \( f \in \mathcal{O}_2(U_2) \). By (4.1), to construct \( \varphi^* f \), we only need to define \( \varphi^* f_{J,\beta} \). But this is straightforward: one just replaces \( x^\mu \) with \( y^\mu \) and \( \theta_{i,a} \) with \( \eta_{i,a} \) in (4.2). By construction, we have \( \varphi^* 1 = 1, \varphi^* (f + g) = \varphi^* f + \varphi^* g \), and \( \varphi^* (fg) = \varphi^* f \varphi^* g \), hence \( \varphi^* \) is well-defined.

Now suppose there exists another \( \varphi'^* \) which equals \( \varphi^* \) on coordinates. Then they also equals on all polynomials of \( (x^\mu, \theta_{i,a}) \). By Lemma 4.2 and Lemma 4.3, \( \varphi'^* = \varphi^* \).

**Remark 4.3.** Theorem 4.1 can be seen as a generalization of the Global Chart Theorem in the \( \mathbb{Z}_2 \)-graded setting (see Theorem 4.2.5 in [11]).

**Corollary 4.2.** Let \( \varphi^* : \mathcal{O}_2(U_2) \to \mathcal{O}_1(U_1) \) be a ring homomorphism which preserves the \( \mathcal{I} \)-grading. Then there exists a unique morphism \( \varphi^* : \mathcal{U}_1 \to \mathcal{U}_2 \) such that \( \varphi^* = \varphi^* \).

Proof. First, one can easily show that \( \varphi^* \) is actually an \( \mathbb{R} \)-algebra homomorphism using arguments similar to those in Lemma 4.1. Choose a point \( p \in U_1 \), by Lemma 4.3, the morphism \( s_p \circ \varphi^* \) must take the form \( s_{p'} \) for some \( p' \in U_2 \). It follows that \( \varphi^* (I_p^\mu) \subset I_{p'}^\mu \). Let \( (x^\mu, \theta_{i,a}) \) be a coordinate system of \( U_2 \), we then have \( \varphi^* x^\mu = \varphi^* \epsilon(x^\mu)(p') \in I_{p'}^\mu \). Hence \( (\epsilon(\varphi^* x^\mu))(p) \in U_2 \) for all \( p \in U_1 \). Next, observe that a coordinate system of \( U_2 \) restricted to any open subset of it gives a coordinate system of that open subset. Now apply Theorem 4.1 and Corollary 4.1. \( \square \)

## 5 Monoidally Graded Manifolds

**Definition 5.1.** Let \( M \) be a \( n \)-dimensional manifold. An \( \mathcal{I} \)-graded manifold \( \mathcal{M} \) of dimension \( n \sum_{i \in \mathcal{I}} m_i \) is an \( \mathcal{I} \)-graded ringed space \( (M, \mathcal{O}_M) \) which is locally isomorphic to an \( \mathcal{I} \)-graded domain.
of dimension \( n|\{m_i\}_{i \in I} \). That is, for each \( x \in M \), there exist an open neighborhood \( U_x \) of \( x \), an \( I \)-graded domain \( \mathcal{U} \), and an isomorphism of locally ringed spaces

\[
\varphi = (\tilde{\varphi}, \varphi^*): (U_x, \mathcal{O}_M|_{U_x}) \to \mathcal{U}.
\]

\( \varphi \) is called a chart of \( M \) on \( U_x \).

\( M \) with the sheaf \( C^\infty \) of smooth functions on \( M \) is an \( I \)-graded manifold of dimension \( n|\{0, \cdots, m_i\} \), which is denoted again by \( M \) for simplicity. We call \( M \) together with a morphism \( \mathcal{O} \to C^\infty \) an underlying manifold of \( M \). Equivalently, an underlying manifold of \( M \) is a morphism \( \varphi : M \to M \) with \( \tilde{\varphi} = \text{id} \).

Let \( x \in M \). An open neighborhood \( U \) of \( x \) on which \( \mathcal{O}(U) \cong C^\infty(U) \otimes \mathcal{S}(V) \) is called a splitting neighborhood. Clearly, every chart is a splitting neighborhood, but not vice versa. The set of splitting neighborhoods form a base of the topology of \( M \). For a splitting \( U \), there exists sub-algebras \( C(U) \) and \( D(U) \) of \( \mathcal{O}(U) \) such that \( C(U) \cong C^\infty(U) \), \( D(U) \cong \mathcal{S}(V) \) and \( \mathcal{O}(U) = C(U) \otimes D(U) \). This induces an epimorphism

\[
\epsilon : \mathcal{O}(U) \to C^\infty(U)
\]

of graded commutative \( R \)-algebras, which is a body map of \( \mathcal{O}(U) \).

**Definition 5.2.** A local coordinate system of \( M \) is the data \((U, x^\mu, \theta_{i,a})\) where

1. \( U \) is a splitting neighborhood of \( M \);
2. \( x^1, \ldots, x^n \) are elements of \( C(U) \) such that \( \epsilon(x^1), \ldots, \epsilon(x^n) \) are local coordinate functions of \( M \) on \( U \);
3. \( \theta_{i,a} \) are homogeneous elements of \( D(U) \) of degree \( d(\theta_{i,a}) = i \), \( i \neq 0 \) and \( a = 1, \cdots, m_i \), which generate \( \mathcal{O}(U) \) as a \( C(U) \)-algebra.

**Remark 5.1.** By Theorem 4.11, every local coordinate system determines (non-canonically) a chart.

Now, let \( U \) be an arbitrary open subset of \( M \). We can choose a collection of charts \( \{U_\alpha\} \) such that \( U = \bigcup_\alpha U_\alpha \). For \( f \in \mathcal{O}(U) \), one can apply the restriction morphisms to \( f \) to get a sequence of sections \( f_\alpha \) in \( \mathcal{O}(U_\alpha) \). Now, apply \( \epsilon \) to each of them to get a sequence of smooth functions \( \tilde{f}_\alpha \) in \( C^\infty(U_\alpha) \). By Proposition 4.11, \( f_\alpha \) must be compatible with each other, hence can be glued together to get a smooth function \( f \) over \( U \). In this way, we construct a body map for every open subset of \( M \), which are compatible with restrictions. In other words, \( \epsilon \) can be seen as a sheaf morphism from \( \mathcal{O} \) to \( C^\infty \).

**Proposition 5.1.** There exists a unique monomorphism \( \varphi : M \to M \) with \( \tilde{\varphi} = \text{id} \).

**Proof.** Existence is guaranteed by \( \epsilon \). Uniqueness follows from Proposition 4.11.

**Proposition 5.2.** Let \( \varphi = (\tilde{\varphi}, \varphi^*) \) be a morphism from \( M = (M, \mathcal{O}_M) \) to \( N = (N, \mathcal{O}_N) \). The following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
\uparrow & & \uparrow \\
M & \xrightarrow{\tilde{\varphi}} & N
\end{array}
\]

\( ^7 \)We often refer to \( U_x \) as a chart too.
Proof. The proof is essentially the same as the one of Proposition 4.2.

Lemma 5.1. Let $O^1$ be the kernel of $\epsilon$. $O$ is $O^1$-adic complete. That is, for any open subset $U$, $O(U)$ is $O^1(U)$-adic complete.

Proof. Let $\hat{O}$ be the $O^1$-adic completion of $O$. There exists a canonical morphism $\iota : O \to \hat{O}$. Since $O$ is locally $O^1$-adic complete, the induced stalk morphism $\iota_p : O_p \to \hat{O}_p$ is an isomorphism for each $p \in M$. It follows that $O$ is $O^1$-adic complete.

Definition 5.3. An $I$-graded manifold $M$ is called projected if there exists a splitting of the short exact sequence of sheaves of rings

$$0 \to O^1 \to O \xrightarrow{\pi} C^\infty \to 0,$$

where $O^1$ is the kernel of $\epsilon$.

The structure sheaf $O$ of a projected manifold is a $C^\infty$-module.

Definition 5.4. A projected $I$-graded manifold $M$ is called split if there exists a splitting of the short exact sequence of $C^\infty$-modules

$$0 \to O^2 \to O^1 \xrightarrow{\pi} O^1/O^2 \to 0,$$

where $O^2$ is the square of $O^1$, $\pi$ is the canonical quotient map.

Let $O$ be the structure sheaf of a projected $I$-graded manifold. Let $F$ denote the sheaf $O^1/O^2$. $F$ is an $I$-graded $C^\infty$-module and we can define its formal symmetric power $S(F)$. By construction, the ringed space $M_S = (M, S(F))$ is also a projected $I$-graded manifold.

Lemma 5.2. Let $M = (M, O)$ be a projected $I$-graded manifold. $M$ is split if and only if $M \cong M_S$.

Proof. Let $\iota : F \to S(F)$ be the canonical monomorphism. $\iota$ splits the short exact sequence (5.2). Now if $M$ is split, one can find a monomorphism $F : F \to O$ of $C^\infty$-modules such that $F(F(U)) \subset O^1(U)$ for any open subset $U$. By Lemma 5.3 and Lemma 5.1 there exists a unique $C^\infty$-algebra morphism $\hat{F} : S(F) \to O$ such that $\hat{F} \circ \iota = F$. By Remark 5.4 $\hat{F}$ induces an isomorphism for each stalk. Hence $M \cong M_S$.

Lemma 5.3. Every projected $I$-graded manifold is split.

Proof. Due to the existence of a smooth partition of unity on $M$, $H^q(M, \text{Hom}(O^1/O^2, O^2))$ vanishes for $q \geq 1$. By Lemma 5.3 there is no obstruction of the existence of a splitting of (5.2).

Lemma 5.4. Every $I$-graded manifold is projected.

Proof. Let $O_{(i)} = O/O^{i+1}$. Let $\phi_{(0)} : C^\infty \to O_{(0)}$ be the identity. (By Proposition 5.1 there is a unique identification $O_{(0)} \cong C^\infty$.) One can construct by induction on $i$ mappings $\phi_{(i+1)} : C^\infty \to O_{(i+1)}$ such that $\pi_{i+1} \circ \phi_{(i+1)} = \phi_{(i)}$, where $\pi_{i+1} : O_{i+1} \to O_i$ is the canonical epimorphism. As is shown in [10], one can construct an element

$$\omega(\phi_{(i)} ) \in H^1(M, (T \otimes S^{i+1}(F))_0)$$

For each open subset $U$, one has $\hat{O}(U) = \varprojlim O(U)/O^n(U)$, where $O^n(U)$ is the $n$-th power of $O^1(U)$.
as the obstruction to the existence of $\phi_{(i+1)}$, where $T$ is the tangent sheaf of $M$. Due to the existence of a smooth partition of unity on $M$, $H^1(M, T \otimes S^{i+1}(\mathcal{F}))_0 = 0$ and $\omega(\phi_{(i)}) = 0$. It follows that there exists a unique morphism $\phi : C^\infty \to \lim \mathcal{O}(i)$ such that $\pi_i \circ \phi = \phi_{(i)}$, where $\pi_i : \lim \mathcal{O}(i) \to \mathcal{O}_i$ is the canonical epimorphism. By Lemma 5.1, $\phi$ can be seen as a morphism from $C^\infty$ to $\mathcal{O}$. Note that $\pi_0 = \epsilon$ and $\pi_0 \circ \phi = \phi_{(0)} = \text{id}$. $\phi$ splits (6.1).

**Corollary 5.1.** Every $\mathcal{I}$-graded manifold is split.

Let $V$ be a (finite dimensional) $\mathcal{I}$-graded vector space. An $\mathcal{I}$-graded vector bundle $\pi : E \to M$ is a vector bundle such that the local trivialization map $\varphi_U : \pi^{-1}(U) \to U \times V$ is a morphism of $\mathcal{I}$-graded vector spaces when restricted to $\pi^{-1}(x)$, $x \in U \subseteq M$. In other words, $E = \bigoplus_{k \in \mathcal{I}} E_k$ where $E_k$ are vector bundles whose fibers consist of elements of degree $k$. To any $\mathcal{I}$-graded vector bundle $E$ we can associate an $\mathcal{I}$-graded ringed space with the underlying topological space being $M$ and the structure sheaf being the sheaf of sections of $\bigoplus_{k \in \mathcal{I}} (E_k)^\ast$. (This is an $\mathcal{I}$-graded manifold in our sense if the fiber of $E$ does not contain elements of degree 0.) Corollary 5.1 can then be rephrased as

**Theorem 5.1.** Every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle.

### 6 Vector Fields and Tangent Sheaves

Throughout this section, every algebra is assumed to be real.

Let $R$ be a unital associative commutative $\mathcal{I}$-graded algebra. An $\mathcal{I}$-graded algebra $A$ over $R$ is defined to be an $\mathcal{I}$-graded algebra equipped with a left $R$-module structure such that $R_i A_j \subseteq A_{i+j}$, and

$$ r(ab) = (ra)b = (-1)^{p(r)p(a)}a(rb) $$

for $r \in R$ and $a, b \in A$. Recall that when we write $p(r)p(a)$, we mean $p(d(r)d(a))$. We also require that $1a = a$, where $1 \in R$ is the identity element.

**Definition 6.1.** An $\mathcal{I}$-graded Lie algebra over $R$ is an $\mathcal{I}$-graded algebra $L$ over $R$ whose multiplications (denoted by $[\cdot, \cdot]$) satisfy

$$ [a, b] = -(−1)^{p(a)p(b)}[b, a], $$

$$ [a, [b, c]] = [[a, b], c] + (−1)^{p(a)p(b)}[b, [a, c]], $$

for all $a, b, c \in L$.

The space of endomorphisms $\text{Hom}(A, A)$ (or $\text{gl}(A)$) of an $\mathcal{I}$-graded $R$-module $A$ is an associative $K(\mathcal{I})$-graded algebra over $R$. It can be also viewed as a $K(\mathcal{I})$-graded Lie algebra over $R$ by setting

$$ [f, g] = f \circ g - (−1)^{p(f)p(g)}g \circ f $$

for all $f, g \in \text{Hom}(A, A)$. In the case of $A$ being an $\mathcal{I}$-graded algebra, an endomorphism $D$ is said to be a derivation if

$$ D(ab) = D(a)b + (−1)^{p(D)p(a)}aD(b). $$

It is easy to check that derivations of $A$ form a $K(\mathcal{I})$-graded Lie subalgebra of $\text{gl}(A)$ over $R$. 

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Definition 6.2. Let \( M = (M, \mathcal{O}) \) be an \( I \)-graded manifold. A (local) vector field over \( M \) is a derivation of \( \mathcal{O}(U) \), where \( U \) is an open subset of \( M \).

Local vector fields over \( M \) actually form a \((K(I))\)-graded sheaf on \( M \). To prove this, we need the partition of unity lemma in the \( I \)-graded setting.

Lemma 6.1. Let \( f \in \mathcal{O}(M) \) such that \( \epsilon(f)(x) \neq 0 \) for all \( x \in M \). \( f \) is invertible.

Proof. Choose an open cover \( \{U_\alpha\} \) of charts of \( M \). Let \( f_\alpha \) denote \( \rho_{U_\alpha,M}(f) \). Each \( f_\alpha \) is invertible by Lemma 6.2. Let \( f_\alpha^{-1} \) denote the inverse of \( f_\alpha \). By uniqueness of the inverse, \( f_\alpha^{-1} \) are compatible with each other, hence can be glued to give a section \( f^{-1} \in \mathcal{O}(M) \), which is the inverse of \( f \).

Lemma 6.2. Let \( \{U_\alpha\} \) be an open cover of \( M \). There exists a locally finite refinement \( \{V_\beta\} \) of \( \{U_\alpha\} \) and a family of functions \( \{l_\beta \in \mathcal{O}(M)\} \) such that

1. \( \text{supp } l_\beta \subset V_\beta \) is compact and \( \epsilon(l_\beta) \geq 0 \) for all \( \beta \);
2. \( \sum_\beta l_\beta = 1 \).

Proof. First, find a partition of unity \( \{\tilde{l}_\beta\} \) of \( M \) subordinate to \( \{V_\beta\} \). Choose \( l'_\beta \in \mathcal{O}(V_\beta) \) such that \( \epsilon(l'_\beta) = \tilde{l}_\beta \). Since \( \tilde{l}_\beta \) are invertible, we can then set \( l_\beta \) to be \( (\sum_\beta l'_\beta)^{-1} l'_\beta \).

Using Lemma 6.2, it is not hard to prove the following lemma.

Lemma 6.3. Let \( U \) and \( V \) be open in \( M \) such that \( V \subset U \). Let \( D \) be a derivation of \( \mathcal{O}(U) \). Then there exists a unique derivation \( D' \) of \( \mathcal{O}(V) \) such that \( D'(\rho_{V,U}(f)) = \rho_{V,U}(D(f)) \) for all \( f \in \mathcal{O}(U) \).

We skip the proof of Lemma 6.3 since it is essentially the same as the one in the \( \mathbb{Z}_2 \)-graded setting [9]. Note that Lemma 6.3 implies that local vector fields over \( M \) form a presheaf \( \mathfrak{X} \) on \( M \).

Proposition 6.1. \( \mathfrak{X} \) is a sheaf on \( M \).

Remark 6.1. \( \mathfrak{X} \) is called the tangent sheaf of \( M \).

Proof. Let \( U \) be an open subset of \( M \) with an open cover \( \{U_\alpha\} \). Let \( D_\alpha \in \mathfrak{X}(U_\alpha) \) be compatible with each other. We obtain a \( D \in \mathfrak{X}(U) \) by setting \( D(f) \) to be unique function obtained by gluing \( D_\alpha(f_\alpha) \), where \( f \in \mathcal{O}(U) \) and \( f_\alpha = \rho_{U_\alpha,U}(f) \). \( D(f) \) is well defined because \( \rho_{U_\alpha \cap U_\beta, U_\alpha}(D_\alpha(f_\alpha)) = \rho_{U_\alpha \cap U_\beta, U_\beta}(D_\beta(f_\beta)) \).

We are now ready to give the following definition.

Definition 6.3. A \( QK \)-manifold is a bigraded manifold equipped with three vector fields \( Q \), \( K \) and \( d \) of degree \((0, 1)\), \((1, -1)\) and \((1, 0)\), respectively, satisfying the following relations

\[
Q^2 = 0, \quad QK + KQ = d, \quad Kd + dK = 0.
\]

\( QK \)-manifolds can be used to study the descent equations [11] in a cohomological field theory. The physical observables \( \mathcal{O}^{(p)} \) can be interpreted as functions over a \( QK \)-manifold. Fix a \( Q \)-closed function \( \mathcal{O}^{(0)} \), a \( K \)-sequence is defined by setting \( \mathcal{O}^{(p)} = \frac{1}{p!} K^p \mathcal{O}^{(0)} \). A sequence \( \{\mathcal{O}^{(p)}\}_{p=0}^n \) is called an exact sequence if there exists another sequence \( \{\mathcal{P}^{(p)}\}_{p=0}^n \) such that \( \mathcal{O}^{(p)} = \mathcal{Q} \mathcal{P}^{(p)} + d \mathcal{P}^{(p-1)} \) for \( p \geq 1 \) and \( \mathcal{O}^{(0)} = \mathcal{Q} \mathcal{P}^{(0)} \). In [8], the author proved that

Theorem 6.1. Every solution to [11] is a \( K \)-sequence up to an exact sequence.
7 Conclusions

In this paper, we have given a definition of \( \mathcal{I} \)-graded manifolds, where \( \mathcal{I} \) is a countable cancellative commutative semi-ring. Such a definition unifies different objects such as supermanifolds, graded manifolds and colored supermanifolds. We have proved the existence and uniqueness of an underlying manifold of an \( \mathcal{I} \)-graded manifold. Furthermore, we have also proved Batchelor’s theorem in this generalized setting, namely that every \( \mathcal{I} \)-graded manifold can be obtained from an \( \mathcal{I} \)-graded vector bundle. At the end of this paper, we have discussed a special class of bigraded manifolds, the \( \mathcal{QK} \)-manifolds, and their applications to cohomological field theories.

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References

[1] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Differential Geometrical Methods in Mathematical Physics, 1977, pp. 177–306.
[2] E. Witten, *Supersymmetry and Morse theory*, Journal of Differential Geometry 17 (1982), no. 4, 661–692.
[3] A. S. Cattaneo and F. Schätz, *Introduction to supergeometry*, Reviews in Mathematical Physics 23 (2011), no. 06, 669-690.
[4] M. Fairon, *Introduction to graded geometry*, European Journal of Mathematics 3 (2017), no. 2, 208–222.
[5] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, International Journal of Modern Physics A 12 (1997), no. 07, 1405–1429.
[6] E. Witten, *Topological sigma models*, Communications in Mathematical Physics 118 (1988), no. 3, 411–449.
[7] ____, *Topological quantum field theory*, Communications in Mathematical Physics 117 (1988), no. 3, 353–386.
[8] S. Jiang, *Mathematical structures of cohomological field theories* (2022), available at arxiv:2202.12425
[9] D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Mathematical Surveys 35 (1980), no. 1, 1–64.
[10] Y. I. Manin, *Gauge Field Theory and Complex Geometry*, Vol. 289, Springer Berlin, Heidelberg, 1997.
[11] C. Carmeli, L. Caston, and R. Fioresi, *Mathematical Foundations of Supersymmetry*, Vol. 15, European Mathematical Society, 2011.
[12] C. Bartocci, U. Bruzzo, and D. H. Ruipérez, *The Geometry of Supermanifolds*, Springer Dordrecht, 2012.
[13] E. Keßler, *Supergeometry, Super Riemann Surfaces and the Superconformal Action Functional*, Vol. 2230, Springer Cham, 2019.
[14] M. Batchelor, *The structure of supermanifolds*, Transactions of the American Mathematical Society 253 (1979), 329–338.
[15] T. Covolo, J. Grabowski, and N. Poncin, *Splitting theorem for \( \mathbb{Z}_2 \)-supermanifolds*, Journal of Geometry and Physics 110 (2016), 393–401.
[16] A. Kotov and V. Salnikov, *The category of \( \mathbb{Z} \)-graded manifolds: what happens if you do not stay positive* (2021), available at arxiv:2108.13498
[17] B. Singh, *Basic Commutative Algebra*, World Scientific, 2011.
[18] A. Rogers, *Supermanifolds: Theory and Applications*, World Scientific, 2007.