The Abelian Higgs Model
as an Ensemble of Vortex Loops

Dmitri Antonov *

Institute of Theoretical and Experimental Physics,
B. Cheremushkinskaya 25, RU-117 218 Moscow, Russia

Abstract

In the London limit of the Ginzburg-Landau theory (Abelian Higgs model), vortex dipoles (small vortex loops) are treated as a grand canonical ensemble in the dilute gas approximation. The summation over these objects with the most general rotation- and translation invariant measure of integration over their shapes leads to effective sine-Gordon theories of the dual fields. The representations of the partition functions of both grand canonical ensembles are derived in the form of the integrals over the vortex dipoles and the small vortex loops, respectively. By virtue of these representations, the bilocal correlator of the vortex dipoles (loops) is calculated in the low-energy limit.

It is further demonstrated that once the vortex dipoles (loops) are considered as such an ensemble rather than individual ones, the London limit of the Ginzburg-Landau theory (Abelian Higgs model) with external monopoles is equivalent up to the leading order in the inverse UV cutoff to the compact QED in the corresponding dimension with the charge of Cooper pairs changed due to the Debye screening.

*E-mail address: antonov@vxitep.itep.ru
†Address after October 1999: INFN-Sezione di Pisa, Università degli studi di Pisa, Dipartimento di Fisica, Via Buonarroti, 2 - Ed. B - 56127 Pisa, Italy.
1 Introduction

It is commonly argued that Abrikosov vortices [1] (Nielsen-Olesen strings [2]) in the dual Ginzburg-Landau theory (Abelian Higgs model) reveal the properties similar to those of strings in 3D- and 4D QCD, respectively. This observation is based on the so-called ’t Hooft-Mandelstam scenario of confinement [3], according to which confinement in QCD can be macroscopically thought of as the dual Meissner effect. Once being put forward, such a correspondence then enabled one to develop various phenomenological models of confinement, e.g., the so-called dual QCD approach [4]. However, an extremely strong support has been given to the ’t Hooft-Mandelstam scenario by the so-called Abelian projection method [5] (see e.g., Refs. [6, 7, 8, 9, 10, 11, 12, 13, 14] for recent developments and Ref. [15] for a review). The main outcome of this method is that under certain quite plausible assumptions (like the so-called Abelian dominance hypothesis [16, 18]), SU(2)-QCD can indeed be with a good accuracy viewed as the London limit of the dual Abelian Higgs model with external electrically charged particles. An independent support of this conclusion comes out from the evaluation of field correlators in the dual Abelian Higgs model and comparison of them with those in QCD, introduced within the so-called Stochastic Vacuum Model [17] and measured in the lattice experiments in Ref. [18]. This calculation has been performed in Refs. [19, 20, 21], and as a result a very good agreement between the two sets of correlators has been established [1]. All that encourages one to work further on investigating the confining properties of the Abelian Higgs model, which is a good and simple example of a model exhibiting the property of confinement. This is the main motivation for the present research.

In a variety of an existing literature on the Abelian Higgs model from the point of view of the Abelian projection method (see e.g., Refs. [22, 23, 21]), the interaction of the Nielsen-Olesen strings has not been studied. In this respect, string representations of various objects (like partition function or field correlators) obtained there are insensitive to the properties of the ensemble of strings as a whole and depend actually on a single string only [2]. Apart from that, there exist some publications where vortices in the 2D- and 3D Ginzburg-Landau theory have been treated as a grand canonical ensemble of dipoles [25, 26]. The results of these investigations are as follows. In the 2D case [25], the effective field theory, emerging after the summation over the vortex dipoles, is the sine-Gordon one with the action of the type

\[ S_{G-L}^{2D} = \int d^2 x \left[ (\partial \mu \chi)^2 + \bar{m}^2 \chi^2 - 2 \bar{\zeta} \cos \chi \right] . \]  

Here, \( \bar{m} \) stands for the gauge boson mass, and \( \bar{\zeta} \) is the so-called fugacity of dimension (mass)\(^2\), which is the statistical weight (Boltzmann factor) of a single vortex dipole. As far as the 3D case [26] is concerned, there the resulting effective field theory is again the Ginzburg-Landau type theory of magnetic Higgs field (albeit with an additional mass term of the dual vector field \( h \)), whose partition function has the form

\( ^1 \)A similar calculation of field correlators in the Abelian-projected \( SU(3) \)-gluodynamics has been performed in Ref. [12].

\( ^2 \)An obvious generalization of the string representation for the partition function of the theory with a global \( U(1) \)-symmetry to an ensemble of noninteracting strings has been performed in Ref. [24].

\( ^3 \)Throughout the present paper, we work in the Euclidean space-time.
\[ Z_{3D}^{\text{G-L}} = \int \mathcal{D}h \mathcal{D}\Phi \mathcal{D}\Phi^* \exp \left\{ - \int d^3x \left[ \frac{1}{4\eta^2} H_{\mu\nu}^2 + \frac{q^2}{2} h^2 + |(\partial_\mu + 2\pi i h_\mu) \Phi|^2 + m_H^2 |\Phi|^2 + \lambda |\Phi|^4 \right] \right\}. \]  

Here, \( H_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu \) is the field strength tensor of the dual field \( h \), \( \eta \) is the \( v.e.v. \) of the original Higgs field, which describes electric Cooper pairs, whose charge \( q \) is the double electron one. In order to arrive at Eq. (2), one should sum up over the grand canonical ensemble of the vortex dipoles, specifying the path-integral measure to be the one of the gas with a short-range repulsion as follows:

\[
Z_{3D}^{\text{G-L}} = \int \mathcal{D}h \exp \left\{ - \int d^3x \left[ \frac{1}{4\eta^2} H_{\mu\nu}^2 + \frac{q^2}{2} h^2 \right] \right\} \times 
\times \left\{ 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \left[ \prod_{n=1}^{N} \int_{0}^{\infty} \frac{ds_n}{s_n} e^{-m_H^2 s_n} \int_{x(0)=x(s_n)}^{x(s_n)} \mathcal{D}x \left( s_n' \right) \right] \right\} \times
\times \exp \left\{ \sum_{l=1}^{N} \int_{0}^{s_l} ds_l' \left( -\frac{1}{4} \mathbf{\dot{x}}^2 (s_l') + 2\pi i \mathbf{\dot{x}}_l (s_l') h_\mu (x (s_l')) \right) - \lambda \sum_{l,k=1}^{N} \int_{0}^{s_l} ds_l' \int_{0}^{s_k} ds_k'' \delta [x (s_l') - x (s_k'')] \right\}. \tag{3} \]

Although both the free part of the world-line action standing in the exponent on the R.H.S. of Eq. (2) and the \( \delta \)-type interaction are quite natural, it looks desirable to perform the summation over the grand canonical ensemble of the vortex dipoles with the most general rotation- and translation invariant integration measure without specification of its concrete form. Clearly, such an approach would then enable one to perform also the summation over the grand canonical ensemble of the small vortex loops, built out of the Nielsen-Olesen strings in the Abelian Higgs model. Notice that the summation over the vortex loops seems to be difficult to perform by a direct generalization of Eq. (3), since the string world-sheet is a 2D object, rather than a 1D vortex line. It is our aim in the present paper to proceed with such a summation over the small vortex loops in the Abelian Higgs model. As far as the summation over the vortex dipoles in the Ginzburg-Landau theory is concerned, it turns out to be analogous to the summation over the grand canonical ensemble of magnetic loops emerging in the Abelian-projected \( SU(2) \)-gluodynamics \(^{[14]}\). In both cases of the Ginzburg-Landau theory and Abelian Higgs model, the effective field theories resulting from the summation occur to be of the sine-Gordon type \(^{[1]}\). They allow for the representations directly in terms of the integrals over the vortex dipoles (loops). As we shall see, besides the Biot-Savart type interaction of the objects under study, the effective actions, corresponding to these representations, contain a multivalued potential resulting from the cosine interaction. In the low-energy limit, the real branch of this potential takes a simple quadratic form, and the integration over the vortex dipoles (loops) becomes simply Gaussian. This will enable us to calculate exactly the bilocal correlation functions of the vortex dipoles and small vortex loops in this limit. After that, we shall investigate the effects brought about by such a way of summation over the grand canonical ensemble of the vortex dipoles (loops) to the Ginzburg-Landau theory (Abelian Higgs model) with external monopoles. In particular, it will be demonstrated that to the leading order in the inverse UV momentum cutoff, this method of summation leads to an equivalence of the so-extended Ginzburg-Landau theory (Abelian Higgs model) to the compact QED in the corresponding dimension.
The paper is organized as follows. In the next Section, we start our analysis with the Ginzburg-Landau theory and then apply the so-developed techniques in Section 3 to the Abelian Higgs model. The main results of the paper are summarized in Conclusions. Finally in three Appendices, some technical details of the performed calculations are outlined.

2 Grand Canonical Ensemble of the Vortex Dipoles in the Ginzburg-Landau Theory

Let us consider the partition function of the 3D Ginzburg-Landau theory in the London limit, i.e., the limit of infinitely heavy Higgs field,

\[ Z_{3D}^{G-L} = \int D A_\mu D \theta^{\text{sing}} D \theta^{\text{reg}} \exp \left\{ -\int d^3 x \left[ \frac{1}{4q^2} F^2_{\mu\nu} + \frac{\eta^2}{2} (\partial_\mu \theta - A_\mu)^2 \right] \right\} , \tag{4} \]

where from now on (both in the case of the Ginzburg-Landau theory and Abelian Higgs model) we adopt the notations of Refs. [24, 22, 8, 19, 23, 12, 21, 13]. Here, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength tensor of the vector potential \( A_\mu \), and \( \theta = \theta^{\text{sing}} + \theta^{\text{reg}} \) stands for the phase of the Higgs field, consisting of a singular and a regular parts. First of them describes a certain configuration of Abrikosov vortices (e.g., a single vortex) according to the equation

\[ \varepsilon_{\mu\nu\lambda} \partial_\nu \partial_\lambda \theta^{\text{sing}}(x) = 2\pi \delta_\mu(x). \tag{5} \]

Here, \( \delta_\mu(x) \equiv \oint dx_\mu(\tau) \delta(x - x(\tau)) \) is the \( \delta \)-function defined w.r.t. the vortex line parametrized by the vector \( x(\tau) \), \( 0 \leq \tau \leq 1 \). The vortex lines are closed, which is expressed by the equation \( \partial_\mu \delta_\mu = 0 \). Notice that Eq. (5) is nothing else but the local form of the Stokes theorem. As far as the usual regular part of the phase of the Higgs field is concerned, it describes fluctuations above this vortex background.

One can perform the so-called path-integral duality transformation (see the above mentioned Refs.), which eventually casts the partition function (4) into the form of the integrals over the dual magnetic field \( h \) and vortex lines. Referring the reader for details to the above cited papers, we present here the final result of this transformation, which has the form

\[ Z_{3D}^{G-L} = \int D x(\tau) D h D \varphi \exp \left\{ -\int d^3 x \left[ \frac{1}{4q^2} H^2_{\mu\nu} + \left( \frac{q}{\sqrt{2}} h_\mu + \partial_\mu \varphi \right)^2 - 2\pi i h_\mu \delta_\mu \right] \right\} . \tag{6} \]

The scalar field \( \varphi \) can be further eliminated by performing the gauge transformation, \( h_\mu \rightarrow h_\mu - \frac{\sqrt{2} q}{q} \partial_\mu \varphi \), after which the partition function takes the form

\[ Z_{3D}^{G-L} = \int D x(\tau) D h \exp \left\{ -\int d^3 x \left[ \frac{1}{4q^2} H^2_{\mu\nu} + \frac{q^2}{2} h^2 - 2\pi i h_\mu \delta_\mu \right] \right\} . \tag{6} \]

Notice that all the manipulations with Eq. (6), which will be performed below, should obviously be done at the saddle-point for the field \( h \), where due to the closeness of vortices, the second Proca equation, \( \partial_\mu h_\mu = 0 \), holds.

One can now sum up over the vortex dipoles in Eq. (6) in the sense of Eq. (3), which leads to Eq. (2). Our aim here is, however, to perform such a summation not specifying the measure to the
particular form of the R.H.S. of Eq. (3), but rather keeping it as the most general rotation- and translation invariant one. To start with, we shall consider partition function (6) as a contribution of $N$ vortex dipoles to their grand canonical ensemble. This can be done by replacing $\delta_\mu$ in Eq. (3) by the following expression describing the density of the gas of the vortex dipoles

$$\delta^{gas}_\mu(x) = \sum_{a=1}^{N} n_a \oint dz_a(\tau) \delta(x - x_a(\tau)),$$

(7)

Here, $n_a$’s stand for winding numbers, and we have decomposed the vector $x_a(\tau)$ as follows, $x_a(\tau) = y_a + z_a(\tau)$, where $y_a = \frac{1}{\tau} \int_0^\tau x_a(\tau)$ denotes the position of the $a$-th vortex dipole. In what follows, we shall restrict ourselves to the minimal values of winding numbers, $n_a = \pm 1$, which is just the essence of the dipole approximation. That is because the energy of a single vortex is known to be a quadratic function of the flux [1], due to which the existence of a dipole made out of two vortices of a unit flux is more energetically favorable than the existence of one vortex of the double flux. Besides that, we shall work in the approximation of a dilute gas of the vortex dipoles. According to this approximation, characteristic sizes of the vortex dipoles, $\frac{1}{\tau} \int_0^\tau \sqrt{\dot{z}_a^2}$, which we shall denote by $a$, are much smaller than characteristic distances $|y|$ between them, which we shall denote by $L$. In particular, this means that the vortex dipoles are short living objects.

Within these two approximations, by substituting Eq. (7) into Eq. (6) one can proceed with the summation over the grand canonical ensemble of the vortex dipoles. This procedure essentially parallels a similar one of Ref. [14], and its details are outlined in Appendix A. As a result, we arrive at the following expression for the grand canonical partition function

$$Z^{3D}_{\text{grand}} = \int \mathcal{D}h \exp \left\{ -\int d^3x \left[ \frac{1}{4\eta^2} H_{\mu\nu}^2 + \frac{q^2}{2} h^2 - 2\zeta \cos \left( \frac{|h|}{\Lambda} \right) \right] \right\}.$$

(8)

Here, $\zeta \propto e^{-S_0}$ is the so-called fugacity, which has the dimension (mass)$^3$ with $S_0$ standing for the action of a single vortex dipole, and $\Lambda$ is the UV momentum cutoff. Thus, the summation over the grand canonical ensemble of the dipoles, built out of Abrikosov vortices, with the most general form of the measure of integration over their shapes yields in the dilute gas approximation the effective sine-Gordon theory (8). In particular, this way of treating the gas of the vortex dipoles leads to increasing the mass of the dual field $h$. Namely, expanding the cosine in Eq. (8) we get the square of the full mass, $M^2 = m^2 + m_D^2 = Q^2 \eta^2$, where $m = q\eta$ is the usual mass of the dual field (equal to the mass of the gauge boson), and $m_D = \frac{q}{2} \sqrt{2\zeta}$ is the additional contribution coming from the Debye screening. We have also introduced the full electric charge $Q = \sqrt{q^2 + \frac{2\zeta}{\Lambda^2}}$.

Our next aim is to derive the representation of the partition function (8) directly in the form of an integral over the vortex dipoles. This can be done by making use of the following equality (valid at the saddle-point of the field $h$, corresponding to the action standing in the exponent on the R.H.S. of Eq. (8)),

$$\exp \left\{ -\int d^3x \left[ \frac{1}{4\eta^2} H_{\mu\nu}^2 + \frac{q^2}{2} h^2 \right] \right\} =$$

$$\int \mathcal{D}j \exp \left\{ -\frac{\pi\eta^2}{2} \int d^3x d^3y j_\mu(x) \frac{e^{-m|y|}}{|x-y|} j_\mu(y) + 2\pi i \int d^3x h_\mu j_\mu \right\}.$$
Substituting it into Eq. (8), one can straightforwardly resolve the resulting saddle-point equation for the field $\mathbf{h}$, $\left. \frac{\partial}{\partial \mathbf{h}} \ln (|\mathbf{h}|) \right| = -i\frac{\pi \Lambda}{\zeta} j_\mu$, which yields the desired representation in terms of the vortex dipoles

$$Z_{\text{grand}}^{3D} = \int D\mathbf{j} \exp \left\{ -\left[ \frac{\pi \eta^2}{2} \int d^3x d^3y j_\mu(x) \frac{e^{-m|x-y|}}{|x-y|} j_\mu(y) + V[2\pi j] \right] \right\}. \quad (9)$$

Here, the complex-valued potential of the vortex dipoles reads (cf. Ref. [14])

$$V[j] = \sum_{n=\pm \infty} \int d^3x \left\{ \Lambda |j| \ln \left[ \frac{\Lambda}{2\zeta} |j| + \sqrt{1 + \left( \frac{\Lambda}{2\zeta} |j| \right)^2} + 2\pi i n \right] - 2\zeta \sqrt{1 + \left( \frac{\Lambda}{2\zeta} |j| \right)^2} \right\}. \quad (10)$$

The obtained representation (9) can now be applied to the calculation of correlators of the vortex dipoles. Indeed, it is possible to demonstrate that if we introduce into Eq. (6) with $\delta_\mu$ replaced by $\delta_\mu^{\text{gas}}$ a unity of the form

$$1 = \int D\mathbf{j} \delta \left( j_\mu - \delta_\mu^{\text{gas}} \right) = \int D\mathbf{j} Dl \exp \left\{ -2\pi i \int d^3x (j_\mu - \delta_\mu^{\text{gas}}) \right\}$$

and integrate out all the fields except $\mathbf{j}$, the result will coincide with Eq. (9). This is the reason why the correlators of $\mathbf{j}$’s are nothing else, but the correlators of the vortex dipoles. Such correlators can be calculated in the low-energy limit, i.e., when $\Lambda |j| \ll \zeta$ and one considers in Eq. (10) the stable minimum of the real branch of the potential by extracting from the whole sum the term with $n = 0$. In this case, the bilocal correlator of the vortex dipoles reads

$$\langle j_\mu(z) j_\nu(0) \rangle \equiv \frac{\int D\mathbf{j} j_\mu(z) j_\nu(0) \exp \left\{ -\left[ \frac{\pi \eta^2}{2} \int d^3x d^3y j_\mu(x) e^{-m|x-y|} j_\mu(y) + \int d^3x \left( -2\zeta + \frac{\pi \zeta |j|^2}{\zeta} \right) \right] \right\}}{\int D\mathbf{j} \exp \left\{ -\left[ \frac{\pi \eta^2}{2} \int d^3x d^3y j_\mu(x) e^{-m|x-y|} j_\mu(y) + \int d^3x \left( -2\zeta + \frac{\pi \zeta |j|^2}{\zeta} \right) \right] \right\}} = \delta_\mu \frac{m_D^2}{4\pi^2 \eta^2} \left[ \delta(z) - \frac{m_D^2}{4\pi} \frac{1}{|z|} e^{-M|z|} \right]. \quad (11)$$

This example illustrates how the vortex dipoles in the grand canonical ensemble are correlated to each other. In particular, we see that their correlator decreases exactly according to the Yukawa law with the screening provided by the full mass $M$.

Let us now see what will be the consequences of accounting for interaction of the vortex dipoles in the grand canonical ensemble if we introduce into the Ginzburg-Landau theory external monopoles. To start with, we shall introduce the monopoles into the Ginzburg-Landau theory with noninteracting Abrikosov vortices. This can be done by replacing the field strength tensor $F_{\mu \nu}$ in Eq. (3) by $F_{\mu \nu} + F_{\mu \nu}^M$, where the monopole field strength tensor $F_{\mu \nu}^M$ obeys the equation $\frac{1}{2}e_{\mu \nu \lambda} \partial_\lambda F_{\nu \lambda}^M = 2\pi \rho$ with $\rho$ standing for the density of monopoles. The path-integral duality transformation of the so-extended theory (4) has been performed in Ref. [13] and effectively results to adding to the Lagrangian standing in the exponent on the R.H.S. of Eq. (3) the term

$$\frac{i}{2} \hbar \frac{\partial}{\partial x_\mu} \int d^3y \frac{\rho(y)}{|x-y|}. \quad (12)$$

We are now in the position to investigate what will be the effect of summation over the grand canonical ensemble of interacting vortex dipoles to the Ginzburg-Landau theory with external
monopoles. If we restrict ourselves to the leading term in the \( \frac{1}{\Lambda} \)-expansion of Eq. (8), \( \text{i.e., keep in the expansion of the cosine only the term quadratic in } h, \) than the integration over \( h \) is Gaussian, and the result has the form

\[
Z_{\text{grand}}^{3D}[\rho] \xrightarrow{\Lambda \to \infty} \exp \left[ -\frac{\pi}{2Q^2} \int d^3xd^3y \frac{\rho(x)\rho(y)}{|x-y|} \right]. \tag{13}
\]

The details of a derivation of Eq. (13) are presented in Appendix B. This equation means that in the physical limit, when the UV momentum cutoff infinitely increases, the partition function of the Ginzburg-Landau theory with external monopoles, where the vortex dipoles are summed up in the sense of the grand canonical ensemble, is equivalent to the statistical weight of 3D compact QED with the charge \( q \) of Cooper pairs replaced by the full one \( Q \).

Finally, it is worth noting that accounting for all the terms in the \( \frac{1}{\Lambda} \)-expansion of Eq. (8), rather than only for the leading one, leads simply to the following substitution in Eq. (9)

\[
V[2\pi j] \rightarrow V \left[ 2\pi j + \frac{1}{2} \nabla x \int d^3y \frac{\rho(y)}{|x-y|} \right].
\]

3 Grand Canonical Ensemble of the Small Vortex Loops in the Abelian Higgs Model

In the present Section, we shall extend the analysis of the grand canonical ensemble of the vortex dipoles to the 4D case of the small vortex loops, built out of Nielsen-Olesen strings, in the London limit of the Abelian Higgs model. The partition function under study is given by Eq. (4) with the replacement \( d^3x \rightarrow d^4x \). Notice that in what follows, we shall mark the 4D quantities (\( \text{e.g., charge of Cooper pairs, v.e.v. of the Higgs field, etc.} \)) by prime in order to distinguish them from the corresponding 3D ones. Equation (5) goes over into

\[
\varepsilon_{\mu\nu\lambda\rho} \partial_\lambda \partial_\rho \theta^{\text{sing.}}(x) = 2\pi \Sigma_{\mu\nu}(x),
\]

where \( \Sigma_{\mu\nu}(x) \equiv \int d\sigma_{\mu\nu}(x(\xi)) \delta(x-x(\xi)) \) is the so-called vorticity tensor current defined at the string world-sheet \( \Sigma \), parametrized by the vector \( x_\mu(\xi) \). Here, \( \xi = (\xi^1, \xi^2) \in [0, 1] \times [0, 1] \) denotes the 2D coordinate, and \( d\sigma_{\mu\nu}(x(\xi)) \) stands for the infinitesimal world-sheet element. Due to the closeness of strings, the vorticity tensor current is conserved, \( \text{i.e., } \partial_\mu \Sigma_{\mu\nu} = 0 \).

The path-integral duality transformation of the Abelian Higgs model accounting for noninteracting Nielsen-Olesen strings has been performed in the above mentioned papers, and the result reads

\[
Z_{\text{AHM}} =
\]

\[
= \int \mathcal{D}x_\mu(\xi) \mathcal{D}h_{\mu\nu} \mathcal{D}B_\mu \exp \left\{ -\int d^4x \left[ \frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \left( \frac{q'}{2} h_{\mu\nu} + \partial_\mu B_\nu - \partial_\nu B_\mu \right)^2 - i\pi h_{\mu\nu} \Sigma_{\mu\nu} \right] \right\}.
\]

Here, \( H_{\mu\nu\lambda} \equiv \partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} + \partial_\lambda h_{\mu\nu} \) stands for the field strength tensor of the massive antisymmetric tensor field \( h_{\mu\nu} \), usually referred to as the Kalb-Ramond field [27]. Analogously to the
In order to proceed from the individual string to the gas of the small vortex loops, one should analogously to the 3D case substitute for $\Sigma_{\mu\nu}$ in Eq. (14) the following expression

$$\Sigma_{\mu\nu}^{\text{gas}}(x) = \sum_{a=1}^{N} n_a \int d\sigma_{\mu\nu}(x^a(\xi)) \delta(x - x^a(\xi)).$$

After that, the summation over the grand canonical ensemble of the vortex loops is straightforward. Referring the reader for the details to Appendix C, we shall present here the result of this procedure, which has the form

$$Z_{\text{grand}}^{4D} = \int Dh_{\mu\nu} \exp \left\{ - \int d^4 x \left[ \frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{q'^2}{4} h_{\mu\nu}^2 - 2\zeta' \cos \left( \frac{|h_{\mu\nu}|}{\Lambda'} \right) \right] \right\} .$$

Here, $|h_{\mu\nu}| \equiv \sqrt{h_{\mu\nu}^2}$, $\Lambda'$ is a new UV momentum cutoff, and the fugacity $\zeta'$ (Boltzmann factor of a single vortex loop) has now the dimension (mass)$^4$. The square of the full mass of the field $h_{\mu\nu}$ following from Eq. (14) reads $M^2 = m'^2 + m_D^2 \equiv Q'^2 \eta^2$. Here, $m' = q' \eta'$ is the usual Higgs contribution, $m_D = \frac{2\eta' \sqrt{\zeta'}}{\Lambda'^2}$ is the Debye contribution, and $Q' = \sqrt{q'^2 + \frac{4\zeta'}{\Lambda'^2}}$ is the full electric charge.

The representation of the partition function (16) in terms of the vortex loops can be obtained by virtue of the following equality valid at the saddle-point of Eq. (14) (with $\Sigma_{\mu\nu} \rightarrow \Sigma_{\mu\nu}^{\text{gas}}$), at which $\partial_\mu h_{\mu\nu} = 0$,

$$\exp \left\{ - \int d^4 x \left[ \frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{q'^2}{4} h_{\mu\nu}^2 \right] \right\} =$$

$$= \int D\!S_{\mu\nu} \exp \left\{ - \left[ \frac{q' \eta'^3}{4} \int d^4 x d^4 y S_{\mu\nu}(x) K_1( m'|x-y|) \frac{S_{\mu\nu}(y)}{|x-y|} + i\pi \int d^4 x h_{\mu\nu} S_{\mu\nu} \right] \right\} ,$$

where $K_1$ stands for the modified Bessel function. Substituting this equality into Eq. (16), we can integrate the field $h_{\mu\nu}$ out, which yields the desired representation for the partition function (16),

$$Z_{\text{grand}}^{4D} = \int D\!S_{\mu\nu} \exp \left\{ - \left[ \frac{q' \eta'^3}{4} \int d^4 x d^4 y S_{\mu\nu}(x) K_1( m'|x-y|) \frac{S_{\mu\nu}(y)}{|x-y|} + V[\pi \Lambda' S_{\mu\nu}] \right] \right\} ,$$

where the effective potential $V$ is given by Eq. (14) with $d^3 x \rightarrow d^4 x$.

Similarly to the 3D case, correlation functions of $S_{\mu\nu}$’s, calculated by virtue of the partition function (17), are nothing else, but the correlation functions of the small vortex loops in the gas. This can be seen by mentioning that if we insert into Eq. (14) with $\Sigma_{\mu\nu} \rightarrow \Sigma_{\mu\nu}^{\text{gas}}$ the following unity

$$1 = \int D\!S_{\mu\nu} \delta \left( S_{\mu\nu} - \Sigma_{\mu\nu}^{\text{gas}} \right) = \int D\!S_{\mu\nu} Dl_{\mu\nu} \exp \left[ -i\pi \int d^4 x l_{\mu\nu} \left( S_{\mu\nu} - \Sigma_{\mu\nu}^{\text{gas}} \right) \right]$$
and integrate out all the fields except \( S_{\mu\nu} \), the result will coincide with Eq. (17). Such correlation functions of the small vortex loops can be most easily calculated in the low-energy limit, \( \Lambda^2 |S_{\mu\nu}| \ll \zeta' \), by considering the stable minimum of the real branch of the potential (10), where it takes a simple parabolic form. In particular, the bilocal correlation function reads (cf. Eq. (11))

\[
\langle S_{\alpha\beta}(z) S_{\lambda\rho}(0) \rangle \equiv 
\]

\[
\equiv \frac{\int D S_{\mu\nu} S_{\alpha\beta}(z) S_{\lambda\rho}(0) \exp \left\{ -\frac{g'2}{4} \int d^4x d^4y S_{\mu\nu}(x) \frac{K_1(m'|x-y|)}{|x-y|} S_{\mu\nu}(y) + \frac{\pi^2\Lambda^4}{4\zeta} \int d^4x S^2_{\mu\nu} \right\}}{\int D S_{\mu\nu} \exp \left\{ -\frac{g'2}{4} \int d^4x d^4y S_{\mu\nu}(x) \frac{K_1(m'|x-y|)}{|x-y|} S_{\mu\nu}(y) + \frac{\pi^2\Lambda^4}{4\zeta} \int d^4x S^2_{\mu\nu} \right\}}
\]

\[
= (\delta_{\alpha\lambda}\delta_{\beta\rho} - \delta_{\alpha\rho}\delta_{\beta\lambda}) \frac{m_D^2}{4\pi^2\eta^2} \left[ \delta(z) - \frac{m_D^2 M' K_1(M'|z|)}{4\pi^2 |z|} \right].
\]

Thus, we see that if Nielsen-Olesen strings are considered not as individual ones, but as a gas of the small vortex loops, the interaction between which has the Yukawa form, the bilocal correlator of the vortex loops in the low-energy limit has also the Yukawa behaviour (albeit screened not by the mass of the gauge boson, but by the full mass \( M' \)).

Let us now investigate grand canonical ensemble of the vortex loops in the presence of external monopoles. To start with, we first clarify what will be the modifications of the Abelian Higgs model with Nielsen-Olesen strings treated as individual ones due to external monopoles. The starting partition function has the form

\[
Z_{\text{AHM}} [j] = \int \mathcal{D}A_\mu \mathcal{D}\theta^{\text{sing}} \mathcal{D}\theta^{\text{reg}} \exp \left\{ -\int d^4x \left[ \frac{1}{4q'^2} (F_{\mu\nu} + F_{\mu\nu}' M')^2 + \frac{\eta^2}{2} (\partial_\mu \theta - A_\mu)^2 \right]\right\}. \tag{18}
\]

Here, the monopole field strength tensor \( F_{\mu\nu}' M' \) obeys the equation \( \partial_\mu \tilde{F}_{\mu\nu}' M' = j_\nu \), where \( \tilde{F}_{\mu\nu}' M' \equiv \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}' M' \), according to which

\[
F_{\mu\nu}' M'(x) = -\frac{1}{4\pi^2} \varepsilon_{\mu\nu\lambda\rho} \frac{\partial}{\partial x_\lambda} \int d^4y \frac{j_\rho(y)}{(x-y)^2}. \tag{19}
\]

with \( j_\mu \) standing for the (conserved) monopole current. The path-integral duality transformation of the partition function (18) has been performed in Ref. [21] and leads to Eq. (14) with the substitution \( \Sigma_{\mu\nu} \to \hat{\Sigma}_{\mu\nu} \equiv \Sigma_{\mu\nu} + \frac{1}{2\pi} \tilde{F}_{\mu\nu}' M' \).

It is now straightforward to see what will be the consequences of the summation over the grand canonical ensemble of the small vortex loops to the Abelian Higgs model with external monopoles. Namely, this summation leads to the following expression for the partition function:

\[
Z_{\text{grand}}^{4D} [j] = \int \mathcal{D}h_{\mu\nu} \exp \left\{ -\int d^4x \left[ \frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{q'^2}{4} h_{\mu\nu}^2 - 2\zeta' \cos \left( \frac{|h_{\mu\nu}|}{\Lambda^2} \right) - \frac{i}{2} h_{\mu\nu} \tilde{F}_{\mu\nu}' M' \right]\right\}. \tag{20}
\]

In particular, in the physical limit \( \Lambda' \to \infty \), keeping in Eq. (20) only the leading term in the \( \frac{1}{N} \)-expansion, we obtain

\[
Z_{\text{grand}}^{4D} [j] \overset{\Lambda' \to \infty}{\sim} \exp \left[ -\frac{1}{8\pi^2 Q^2} \int d^4x d^4y j_\mu(x) \frac{1}{(x-y)^2} j_\mu(y) \right]. \tag{21}
\]
The derivation of Eq. (21) is similar to that of Eq. (13). The only technical detail necessary for it is the integration over the Kalb-Ramond field, which can be found in Refs. [28, 21].

Thus we see that similarly to the 3D case, in this physical limit, the partition function of the grand canonical ensemble of the small vortex loops extended by external monopoles is equivalent to the statistical weight of the 4D compact QED with the electric charge $q'$ replaced by the full one $Q'$.

Finally, accounting for all the orders in the $1/\Lambda'$-expansion in Eq. (20) is equivalent to the following substitution in Eq. (17):

$$V[\pi \Lambda' S_{\mu\nu}] \rightarrow V\left[\Lambda' \left(\pi S_{\mu\nu} - \frac{1}{2} \tilde{F}^{M'}_{\mu\nu}\right)\right].$$

This means that external monopoles do not affect the Yukawa-type interaction of the vortex loops, but enter only their effective potential.

4 Conclusions

In the present paper, we have investigated grand canonical ensembles of the vortex dipoles and small vortex loops in the London limit of the Ginzburg-Landau theory and Abelian Higgs model, respectively. In the approximation where these objects form a dilute gas, the summation over them with the most general rotation- and translation invariant measure of integration over their shapes has been performed. The resulting effective theories turned out to have the form of the sine-Gordon theories due to the additional cosine interaction term of the dual field, emerging from the above mentioned summation. In the physical limit of the large UV momentum cutoff, one can keep in the expansion of this cosine interaction in powers of the inverse cutoff only the first term. This yields a certain (positive) correction to the mass of the dual field due to the Debye screening.

After that, we have casted the obtained sine-Gordon theories, corresponding to the Ginzburg-Landau theory and Abelian Higgs model, into the forms of the representations in terms of the vortex dipoles and small vortex loops, respectively. The resulting effective actions turned out to have similar forms and consist of the Biot-Savart Yukawa type interaction of the objects under study and a certain multivalued effective potential. In the low-energy limit, this potential takes a simple quadratic form, and the effective actions become Gaussian. This enables one to calculate correlation functions of the vortex dipoles (loops) in this limit, and as an example, we have calculated the bilocal correlators. Those turned out to have the Yukawa form, where the screening is governed by the full mass of the dual field, resulting both from the Higgs and Debye effects.

Then, we have addressed the problem of what will be the consequences of our approach to treatment the vortex dipoles (loops) as a grand canonical ensemble to the theories under study extended by external monopoles. In this way, we have demonstrated that to the leading order in the expansion in powers of the inverse momentum cutoff, summation over the grand canonical ensemble results to the complete equivalence of these theories to the compact QED in the corresponding dimension with the charge of Cooper pairs replaced by the full one (which accounts also for the Debye screening). This result differs from that, which one gets by considering Abrikosov vortices (Nielsen-Olesen strings) as individual ones. In that case, the equivalence to compact QED holds only in the limit of vanishing gauge boson mass. As far as the effects brought about by external monopoles to all the orders of the expansion in the inverse powers of the UV momentum cutoff are concerned, those result into certain shifts of the arguments of the effective potentials, but do not affect the Biot-Savart interaction.
It now looks attractive to investigate the grand canonical ensembles of the vortex dipoles (loops) emerging in the effective Abelian-projected theories, corresponding to the original $SU(N)$, $N > 2$, Yang-Mills theories, since the dual Abelian Higgs model can be under some assumptions considered as such an effective theory for the case $N = 2$. Recently, some progress in this direction has been achieved in Ref. [30].

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Appendix A. Summation over the Grand Canonical Ensemble of the Vortex Dipoles
In this Appendix, we shall present some details of a derivation of Eq. (8) of the main text. Substituting Eq. (7) into Eq. (6), one can perform the summation over the grand canonical ensemble of the vortex dipoles as follows:

$$1 + \sum_{N=1}^{\infty} \frac{\zeta^N}{N!} \prod_{i=1}^{N} \int d^3y^i \int Dz^i \mu \left[ z^i \right] \sum_{n_a = \pm 1} \exp \left\{ 2\pi i \sum_{a=1}^{N} n_a \oint dz^a h_{\mu} (x^a) \right\} =$$

$$= 1 + \sum_{N=1}^{\infty} \frac{(2\zeta)^N}{N!} \left\{ \int d^3y \int Dz \mu [z] \cos \left( 2\pi \oint dz h_{\mu}(x) \right) \right\}^N.$$  

Here, we have introduced the so-called fugacity $\zeta$, which is proportional to the statistical weight of a single vortex dipole and has the dimension (mass)$^3$. Next, $\mu [z]$ stands for a certain rotation- and translation invariant measure of integration over the shapes of dipoles. Let us now employ for the grand canonical ensemble of the vortex dipoles under consideration the dilute gas approximation described after Eq. (7). One can then expand $h_{\mu}(x)$ up to the first order in $a/L$ as follows:

$$h_{\mu}(x) = h_{\mu}(y) + L^{-1} z_{\mu} n_{\nu} h_{\mu}(y) + O \left( \left( \frac{a}{L} \right)^2 \right), \quad (A.1)$$

where $n_{\nu} = y_{\nu}/|y|$, and we have substituted $n_{\nu}/L$ for the derivative $\partial/\partial y_{\nu}$. By making use of this expansion, we obtain

$$\int Dz \mu [z] \cos \left( 2\pi \oint dz h_{\mu}(x) \right) \simeq \int Dz \mu [z] \cos \left( \frac{2\pi}{L} n_{\nu} h_{\mu}(y) P_{\mu\nu} [z] \right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \frac{2\pi}{L} \right)^{2n} n_{\nu_1} h_{\mu_1}(y) \cdots n_{\nu_{2n}} h_{\mu_{2n}}(y) \int Dz \mu [z] P_{\mu_1\nu_1} [z] \cdots P_{\mu_{2n}\nu_{2n}} [z] =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \frac{2\pi a^2}{L} \right)^{2n} n_{\nu_1} h_{\mu_1}(y) \cdots n_{\nu_{2n}} h_{\mu_{2n}}(y) \times$$

$$\times \frac{1}{(2n-1)!!} \left[ \hat{1}_{\mu_1\nu_1,\mu_2\nu_2} \cdots \hat{1}_{\mu_{2n-1}\nu_{2n-1},\mu_{2n}\nu_{2n}} + \text{permutations} \right].$$
Here, \( P_{\mu\nu}[z] \equiv \oint dz^\mu dz^\nu \) is the so-called tensor area, and \( \hat{I}_{\mu\nu,\lambda\rho} = \frac{1}{2} (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \). Notice that the sum in square brackets on the R.H.S. of the last equality contains \((2n-1)!!\) terms, which is the reason for extracting explicitly this normalization factor. The general form of the tensor structure standing in these brackets is due to the rotation- and translation invariance of the measure \( \mu[z] \).

In the further contraction of indices, it is worth noting that within the dilute gas approximation, \( n_{\mu} h_{\mu}(y) = 0 \). Indeed, as we have already seen, this approximation allows for the substitution \( n_{\mu} \rightarrow L \frac{\partial}{\partial y} \). On the other hand, the divergency of the field \( h \) vanishes according to the second Proca equation, valid at the saddle point of the field \( h \) corresponding to Eq. (8). This completes the proof of our statement.

Finally, introducing the UV momentum cutoff as \( \Lambda = \frac{L}{\sqrt{2\pi a^2}} (\gg a^{-1}) \), we obtain

\[
\int Dz^\mu[z] \cos \left( 2\pi \int dz^\mu h_{\mu}(x) \right) \simeq \cos \left( \frac{|h(y)|}{\Lambda} \right),
\]

which leads to Eq. (8).

**Appendix B. Details of a Derivation of Eq. (13)**

In the present Appendix, some details of a derivation of Eq. (13) will be presented. Let us start with the following expression:

\[
J \equiv \int Dh \exp \left\{ -\int d^3x \left[ \frac{1}{4\eta^2} H_{\mu\nu}^2 + \frac{Q^2}{2} h^2 + \frac{i}{2} h_{\mu} \frac{\partial}{\partial x_{\mu}} \int d^3y \frac{\rho(y)}{|x-y|} \right] \right\},
\]

which (up to an inessential constant factor, which can be referred to the integration measure) is the leading term in the \( \frac{1}{a} \)-expansion of the partition function (8) including external monopole part (12). Gaussian integration over the field \( h \) yields

\[
J = \exp \left\{ -\frac{\eta^2}{2} \int d^3x d^3y \frac{e^{-M|x-y|}}{4\pi|x-y|} \left[ \frac{1}{4} \frac{\partial^2}{\partial x_{\mu} \partial y_{\mu}} \int d^3z \frac{\rho(z)}{|x-z|} \int d^3u \frac{\rho(u)}{|y-u|} + \frac{4\pi^2}{M^2} \rho(x)\rho(y) \right] \right\},
\]

where the normalization factor was assumed to be included into the measure \( Dh \).

Let us consider the first term in square brackets on the R.H.S. of Eq. (B.1) together with the factor in front of these brackets. By performing the partial integration, one can cast the derivative \( \partial/\partial y_{\mu} \) to the factor in front of the brackets, after which due to the translation invariance of this factor it can be replaced by \(-\partial/\partial x_{\mu}\). This derivative can be casted back by doing one more partial integration, which yields for this group of terms

\[
\frac{1}{4} \int d^3x d^3y \frac{e^{-M|x-y|}}{4\pi|x-y|} \frac{\partial^2}{\partial x_{\mu} \partial y_{\mu}} \int d^3z \frac{\rho(z)}{|x-z|} \int d^3u \frac{\rho(u)}{|y-u|} =
\]

\[
= -\frac{1}{4} \int d^3x d^3y \frac{e^{-M|x-y|}}{4\pi|x-y|} \Delta x \int d^3z \frac{\rho(z)}{|x-z|} \int d^3u \frac{\rho(u)}{|y-u|} = \pi \int d^3x d^3yd^3u \rho(x) \frac{e^{-M|x-y|}}{4\pi|x-y|} \frac{\rho(u)}{|y-u|}.
\]

Substituting this expression into Eq. (B.1), we obtain
\[ J = \exp \left\{ -\frac{\pi \eta^2}{2} \int d^3x d^3y \frac{e^{-M|x-y|}}{4\pi |x-y|} \left[ \int d^3u \frac{\rho(x)\rho(u)}{|y-u|} + \frac{4\pi}{M^2} \rho(x)\rho(y) \right] \right\}. \]  

(B.2)

The integral

\[ I \equiv \int d^3y \frac{e^{-M|x-y|}}{|x-y||y-u|} = \int d^3y \frac{e^{-M|y|}}{|y||w-y|}, \]  

(B.3)

where \( w \equiv x - u \), can be calculated by the two alternative ways. One of them is to divide the integration region over \( |y| \) into two parts, \([0,|w|],[|w|,+\infty)\) and expand \( \frac{1}{|w-y|} \) in Legendre polynomials \( P_n \)'s on both of them. Then, the integration over the azimuthal angle singles out from the whole series only the zeroth term, \( \frac{1}{-1} P_n(\cos \theta) d \cos \theta = 2\delta_{n0} \). After that, the integration over \( |y| \) at both intervals is straightforward and yields for the integral (B.3) the following result:

\[ I = \frac{4\pi}{M^2|w|} \left( 1 - e^{-M|w|} \right). \]  

(B.4)

Another way to calculate the integral (B.3) is to use the relations

\[ \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\eta}}{p^2 + M^2} = \frac{1}{4\pi} \frac{e^{-M|y|}}{|y|} \quad \text{and} \quad \int \frac{d^3q}{(2\pi)^3} \frac{e^{iq(w-y)}}{q^2} = \frac{1}{4\pi|w-y|}, \]

by virtue of which we have

\[ I = 16\pi^2 \int d^3y \frac{d^3p d^3q}{(2\pi)^6} \frac{e^{ip\eta+iq(w-y)}}{(p^2 + M^2)q^2} = 16\pi^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{ipw}}{(p^2 + M^2)} = \]

\[ = 16\pi^2 \int \frac{d^3p}{(2\pi)^3} \int_0^{+\infty} d\alpha \int_0^{+\infty} d\beta e^{ipw-\alpha p^2-\beta(p^2+M^2)} = 2\sqrt{\pi} \int_0^{+\infty} d\alpha \int_0^{+\infty} d\beta \frac{e^{-BM^2 - \frac{w^2}{4(\alpha + \beta)^2}}}{(\alpha + \beta)^2}. \]

It is now suitable to introduce new integration variables as \( \alpha = bt, \beta = b(1-t); b \in [0,+\infty), \ t \in [0,1] \), which yields

\[ I = 2\sqrt{\pi} \int_0^{+\infty} \frac{db}{\sqrt{b}} \frac{1}{b} \int_0^{+\infty} dt e^{-\frac{w^2}{2b} - bM^2(1-t)} = \frac{2\sqrt{\pi}}{M^2} \int_0^{+\infty} \frac{db}{b^2} e^{-\frac{w^2}{2b}} \left( 1 - e^{-bM^2} \right). \]

The remaining integral is already straightforward to evaluate by changing the variable \( b \rightarrow \frac{1}{b} \) and making use of the formula

\[ \int_0^{+\infty} \frac{db}{\sqrt{b}} e^{-Ab - \frac{w^2}{2b}} = \sqrt{\frac{\pi}{A}} e^{-2\sqrt{AB}}, \]

valid for positive \( A \) and \( B \). The result coincides with Eq. (B.4).

Finally, substituting Eq. (B.4) into Eq. (B.2) and recalling that \( M = Q\eta \), we arrive at Eq. (13) of the main text.
Appendix C. Summation over the Grand Canonical Ensemble of the Small Vortex Loops

In the present Appendix, we shall outline some steps of a derivation of Eq. (16). Let us first consider the infinitesimal world-sheet element of the $a$-th vortex loop, which has the form

$$d\sigma_{\mu\nu}(x^a(\xi)) = \varepsilon^{\alpha\beta}(\partial_\alpha x^a_\mu(\xi))(\partial_\beta x^a_\nu(\xi))d^2\xi,$$

where $\alpha, \beta = 1, 2$. Analogously to the 3D case, it is reasonable to introduce the center-of-mass coordinate (position) of the world-sheet $y^a_\mu \equiv \int d^2\xi x^a_\mu(\xi)$. The full world-sheet coordinate can be respectively decomposed as follows $x^a_\mu(\xi) = y^a_\mu + z^a_\mu(\xi)$ with the vector $z^a_\mu(\xi)$ describing the shape of the $a$-th vortex loop world-sheet. Then, substituting into Eq. (14) instead of $\Sigma_{\mu\nu}$ the vorticity tensor current of the vortex loop gas (15), we can perform the summation over the grand canonical ensemble of the vortex loops as follows:

$$1 + \sum_{N=1}^\infty \frac{\zeta N!}{N!} \left( \prod_{l=1}^N \int d^4 y^l \int Dz^l_\rho(\xi)\mu^l [z^l] \right) \sum_{n_a=\pm 1} \exp \left\{ i\pi \sum_{a=1}^N n_a \int d\sigma_{\mu\nu}(z^a(\xi))h_{\mu\nu}(x^a(\xi)) \right\} =$$

$$= 1 + \sum_{N=1}^\infty \frac{(2\zeta')^N}{N!} \left\{ \int d^4 y \int Dz_\rho(\xi)\mu'[z] \cos \left( \pi \int d\sigma_{\mu\nu}(z(\xi))h_{\mu\nu}(x(\xi)) \right) \right\}^N.$$

Here, $\mu'$ is a certain rotation- and translation invariant measure of integration over shapes of the world-sheets of the vortex loops. Employing now the dilute gas approximation, we can expand $h_{\mu\nu}$ up to the first order in $a'/L'$ (cf. Eq. (A.1)), where $a'$ stands for the typical value of $|z^a(\xi)|$'s (vortex loops sizes), which are much smaller than the typical value $L'$ of $|y^a|$'s (distances between vortex loops) $^4$. This yields

$$\int Dz_\rho(\xi)\mu'[z] \cos \left( \pi \int d\sigma_{\mu\nu}(z(\xi))h_{\mu\nu}(x(\xi)) \right) \simeq \int Dz_\rho(\xi)\mu'[z] \cos \left( \frac{\pi}{L'} n_\lambda h_{\mu\nu}(y)P'_{\mu\nu,\lambda}[z] \right) =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \left( \frac{\pi}{L} \right)^{2n} n_\lambda h_{\mu_1\nu_1}(y) \cdots n_\lambda h_{\mu_{2n}\nu_{2n}}(y) \int Dz_\rho(\xi)\mu'[z] P'_{\mu_1\nu_1,\lambda_1}[z] \cdots P'_{\mu_{2n}\nu_{2n},\lambda_{2n}}[z], \quad (C.1)$$

where $n_\lambda \equiv y_\lambda/|y|$ and $P'_{\mu\nu,\lambda}[z] \equiv \int d\sigma_{\mu\nu}(z(\xi))z_\lambda(\xi)$. Due to the rotation- and translation invariance of the measure $\mu'[z]$, the last average has the form

$$\int Dz_\rho(\xi)\mu'[z] P'_{\mu_1\nu_1,\lambda_1}[z] \cdots P'_{\mu_{2n}\nu_{2n},\lambda_{2n}}[z] =$$

$$= \left( \frac{a'^3}{(2n-1)!} \right)^{2n} \left[ \hat{1}_{\mu_1\nu_1} \hat{1}_{\mu_2\nu_2} \delta_{\lambda_1\lambda_2} \cdots \hat{1}_{\mu_{2n-1}\nu_{2n-1}} \hat{1}_{\mu_{2n}\nu_{2n}} \delta_{\lambda_{2n-1}\lambda_{2n}} + \text{permutations} \right].$$

Substituting this expression into Eq. (C.1), we finally obtain

$^4$Similarly to the 3D case, the approximation $a' \ll L'$ means that the vortex loops are short living objects.
\[
\int \mathcal{D}z_\rho(\xi)\mu'[z] \cos\left( \pi \int d\sigma_{\mu\nu}(z(\xi))h_{\mu\nu}(x(\xi)) \right) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \frac{\pi a'^3}{L'} \right)^{2n} |h_{\mu\nu}(y)|^{2n} = \cos \left( \frac{|h_{\mu\nu}(y)|}{\Lambda'^2} \right),
\]
where we have introduced a new UV momentum cutoff \( \Lambda' \equiv \sqrt{\frac{L'}{\pi a'^4}} \), which is much larger than \( 1/a' \). This yields the desired Eq. (16).
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