The Free Field Representation of SU(3) Conformal Field Theory

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Abstract

The free field representation structure as well as some four point correlation functions of the SU(3) conformal field theory are being considered.

1 Introduction

Because of the role the two dimensional conformal field theories play in various fields of theoretical physics, these theories have been under investigation for quite a long time. One of the most effective ways of solving conformal field theories, consists in reducing these theories to certain massless free field theories. The first models solved in this way were the minimal models, solved completely on the sphere. All the conformal blocks and correlation functions were represented in Feigin-Fuchs integrals and calculated explicitly. The operator algebra of the primary fields was derived then on the basis of Polyakov’s conformal bootstrap.

Another example (still not the last) of free field realization is the SU(2) conformal field theory. It was initiated by Wakimoto’s work, where the Fock space (or free field) representation of its chiral algebra for general \( k \) was constructed. The free field realization of the remaining ingredients of the theory, i.e. primary fields, screening operators and others, were given in. Afterwards the solution of the model was completed in, where the correlation functions and the operator algebra were constructed. There is an alternative approach to this model based on solving certain differential equations arising from the symmetries of the model. The results of these two solutions, of course, coincide.

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A common belief exists now that most of the conformal field theories admit representations in terms of free field theories, though the Hilbert spaces of the conformal field theories appear to be only subspaces of the total Fock space of the free field theories. A subsequent question arises whether the advances with the minimal and SU(2) models could be consistently generalized to the SU(N) or any other conformal field theory. An essential step was made recently towards this generalization. The Wakimoto constructions were developed for general Kac-Moody algebras and general forms for the primary fields and screening operators were obtained in terms of a Heizenberg algebra generators (free fields). Nevertheless the field-theoretical part of this generalization is, however, not transparent, yet. Particularly the integral representations of the conformal blocks are no more of Dotsenko-Fateev type. Moreover, their analytic behaviours in their poles and zeroes are unknown, which gives rise to formeadable difficulties in realizing the conformal bootstrap.

So, for these and other reasons it seems rather reasonable to start with the free field representation of the SU(3) conformal field theory on the sphere suspecting that it will contain all the principle features of such theories.

This is what this paper is dealing with. Anyhow, it succeeds only in performing the SU(2) parts of the SU(3) conformal field theory.

It is organized as follows. In section 2 the Wakimoto constructions, screening operators and the primary fields are being introduced following the notations and terminology of. A “Fock conjugated” primary operator is introduced as well, to build the conformal blocks and correlation functions.

In section 3 the bosonized structures of some SL(3) modules are reviewed and certain four point correlation functions are constructed and evaluated. The paper ends with some concluding remarks. The holomorphic part of the theory is being treated throughout the paper until stated otherwise.

2 Wakimoto constructions and primary fields

To construct the free field realization of the SU(3) conformal field theory, one firstly needs the bosonized form of the SU(3) currents:

\[ E_1(z) = - : a_{11}^* a_{11} a_{11}^* : - \nu \varphi_1^* a_{11}^* - (2 - \nu^2) a_{11}^* + a_{22} a_{12}^*; \]
\[ F_1(z) = a_{11} + a_{12} a_{22}^*; \]
\[ H_1(z) = 2 : a_{11} a_{11}^* : + : a_{12} a_{12}^* : - : a_{22} a_{22}^* : + \nu \varphi_1^*; \]
\[ E_2(z) = - : a_{22}^* a_{22} a_{22}^* : + : a_{22}^* a_{11} a_{11}^* : - : a_{22} a_{12} a_{12}^* : - \]
\[ - \nu \varphi_2^* a_{22}^* - (3 - \nu^2) a_{22}^* - a_{11} a_{12}^*; \]
\[ F_2(z) = a_{22}, \]
\[ H_2(z) = 2 : a_{22} a_{22}^* : + : a_{12} a_{12}^* : - : a_{11} a_{11}^* : + \nu \varphi_2^*. \]
which are the generating functions for the Chevally generators \( E_i(n), F_i(n), H_i(n) \)

\[
E_i(z) = \sum_n E_i(n) z^{-n-1} \quad ; \quad E_i(n) = \frac{1}{2i\pi} \oint dz (z - w)^n E_i(z);
\]

\[
F_i(z) = \sum_n F_i(n) z^{-n-1} \quad ; \quad F_i(n) = \frac{1}{2i\pi} \oint dz (z - w)^n F_i(z);
\]

\[
H_i(z) = \sum_n H_i(n) z^{-n-1} \quad ; \quad H_i(n) = \frac{1}{2i\pi} \oint dz (z - w)^n H_i(z).
\]

of the \( SL_\kappa(3) \) affine Kac - Moody algebra

\[
[E_i(n), F_j(m)] = \delta_{ij} H_j(n + m) + mk \delta_{m+n,0} \delta_{ij};
\]

\[
[H_i(n), E_j(m)] = C_{ij} E_j(n + m);
\]

\[
[H_i(n), F_j(m)] = -C_{ij} F_j(n + m);
\]

\[
[H_i(n), H_j(m)] = mk C_{ij} \delta_{n+m,0};
\]

\[
[E_i(n), E_j(m)] = N_{ij} E_{i+j}(n + m);
\]

\[
[F_i(n), F_j(m)] = -N_{ij}(n + m).
\]

Here and below \( N_{ij} = 1, \nu^2 = k + \tilde{h}, 1 \leq i, j \leq 2 \) and \( C_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1} \) where \( k \) is the central charge of the \( SL_\kappa(3) \) algebra , \( C_{ij} \) is the Cartan matrix of the Lie algebra \( SL(3) \) and \( \tilde{h} \) is its dual Coxeter number.

In (2.1) the \( SU(3) \) currents are expressed in terms of free fields ( bosonic ghosts ) \( a_{ij}(z), a_{ij}^*(z) \) and \( \varphi_i(z) (1 \leq i \leq j \leq 2) \), where stress on \( \varphi_i \) and dots on \( a_{ij}^* \) mean just derivatives of \( z \). They have the following two point functions and mode expansions

\[
< \varphi_i(z) \varphi_j(w) > = \frac{C_{ij}}{(z - w)^2};
\]

\[
< a_{ij}(z) a_{pq}^*(w) >= - < a_{ij}^*(z) a_{pq}(w) >= \frac{\delta_{ip} \delta_{jq}}{z - w};
\]

\[
\varphi_i(z) = \sum_n b_i(n) z^{-n-1}; \quad a_{ij} = \sum_n a_{ij}(n) z^{-n-1}; \quad a_{ij}^* = \sum_n a_{ij}^*(n) z^{-n-1}.
\]
The Fourier components of these expansions realize a Heisenberg algebra

\[ [a_{ij}(n), a_{pq}^*(m)] = \delta_{ip}\delta_{jq}\delta_{n+m,0}; \]  
\[ [b_i(n), b_j(m)] = nC_{ij}\delta_{n+m,0}; \]  
\[ [a, a] = [a^*, a^*] = [a, b] = [a^*, b] = 0, \quad \forall i, j, p, q, n, m. \]  

(2.5)

The normal ordering in the bosonization formulas (2.1) means the common prescription of keeping the annihilation operators right to the creation ones, whenever they are identified with respect to a vacuum state in a representation of the Heisenberg algebra (2.5).

As is well known, the full symmetry algebra of the SU(N) conformal field theories is a semiprod- uct of the \( \hat{SL}_k(N) \) Kac-Moody and Virasoro algebras. The Virasoro algebra

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \]  

(2.6)

in these theories is realized by the Fourier components \( L_n \) of the stress-energy tensor \( T(z) \), which is constructed as a normal ordered inner product of the corresponding currents:

\[ T(z) = \frac{1}{2\nu^2} \left[ \sum_{i,j=1,2} (C^{-1}_{ij} : H_iH_j : + 2\delta_{ij} : E_iF_j :) \right], \]  

(2.7)

where

\[ T(z) = \sum_n \frac{L_n}{z^{n+2}}, \quad L_n = \frac{1}{2\pi i} \oint dz (z - w)^{n+1}T(z). \]

The central charge \( c \) of the obtained Virasoro algebra can be calculated from the two point function of the stress-energy tensor (2.7), using the operator product expansions between the SU(3) currents

\[ E_i(z)F_j(w) = \frac{\delta_{ij}k}{(z - w)^2} + \frac{\delta_{ij}}{z - w}H_j(w) + R.T.; \]
\[ H_i(z)E_j(w) = \frac{C_{ij}}{z - w}E_j(w) + R.T.; \]  

(2.8)
\[ H_i(z)F_j(w) = -\frac{C_{ij}}{z - w}F_j(w) + R.T.; \]
\[ H_i(z)H_j(w) = \frac{C_{ij}k}{(z - w)^2} + R.T., \]

where R.T. means terms which are regular as \( z \rightarrow w \). It yields to

\[ \langle T(z)T(w) \rangle = \frac{4k}{k + 3} \frac{1}{(z - w)^4} = \frac{c/2}{(z - w)^4}, \]  

(2.9)
i.e.
\[ c = \frac{8k}{k+3} = 8 - 24\alpha_0^2, \alpha_0^2 = (k+3)^{-1}. \] (2.10)

Easy to obtain the bosonized form of the stress-energy tensor \( T(z) \), using the corresponding expressions for the currents (2.1) and the two point functions (2.4):

\[ T(z) = \sum_{i,j=1,2} C_{ij}^{-1}(1/2) \varphi_i' \varphi_j' + \frac{1}{\nu} \varphi_j'' + \sum_{1 \leq i \leq j \leq 2} a_i^* a_{ij} :. \] (2.11)

To pass to the field-theoretical content of the free field representation one needs the bosonized forms of the primary fields\(^1\) in the SU(3) conformal field theory. They were shown in\(^{11}\) to have the following general forms

\[ \Phi(z) = \prod_{1 \leq i \leq j \leq \text{rank} g} a_{ij}^* V(\beta, z), \] (2.12)

where \( V(\beta, z) \) is a vertex operator\(^{11}\) and \( d_{ij} \) are positive integer numbers. For the case of the SU(3) conformal field theory they admit the following form

\[ \Phi(z) = a_{11}^* d_{11} a_{12}^* d_{12} a_{22}^* d_{22} \exp(\beta_1 \varphi_1 + \beta_2 \varphi_2). \] (2.13)

In conformal field theories the primary fields serve as lowest weight vectors of Verma modules over the Virasoro algebra, by their definition\(^1\):

\[ T(z)\Phi(w) = \frac{\Delta}{(z-w)^2}\Phi(w) + \frac{1}{z-w} \partial_w \Phi(w) + R.T. \] (2.14)

\[ L_0 = \Phi(w) = \Delta \Phi(w), \] (2.15)

\[ L_n = 0 \quad \text{if} \quad n \geq 0; \]

\[ L_n \Phi(w) = \text{other states of the module} \quad \text{if} \quad n \leq 0. \]

To make the primary fields (2.12) serve as the lowest weight vectors for the SL(3) algebra modules also, they must be put to satisfy

\[ F_i(z)\Phi_{L.W.}(w) = R.T., \quad i = 1, 2. \] (2.16)

These conditions bring the primary fields (2.13) to the form of a "bare" vertex operator

\[ \Phi_{L.W.}(z) = V(\beta, z) = \exp(\vec{\beta}, \vec{\varphi}), \] (2.17)

where two dimensional vectors \( \vec{\beta} = (\beta_1, \beta_2) \) and \( \vec{\varphi} = (\varphi_1, \varphi_2) \) are introduced for the sake of having compact expressions. The operator product expansions of the primary field (2.17) with the generators of the Cartan subalgebra are as follows

\[ H_i(z)\Phi_{L.W.}(w) = \frac{\nu(\vec{\alpha}_i, \vec{\beta})}{(z-w)} \Phi_{L.W.}(w) + R.T.. \] (2.18)
Here \( \alpha_1, \alpha_2 \) are the simple roots of the SL(3) Lie algebra, realized by the null components of the \( \hat{SL}_k(3) \) generators:

\[
[E_i(0), F_j(0)] = \delta_{ij} H_j(0), \quad [H_i(0), F_j(0)] = -C_{ij} F_j(0),
\]

\[
[H_i(0), E_j(0)] = C_{ij} E_j(0), \quad [H_i(0), H_j(0)] = 0.
\]

All the remaining states of the \( \hat{SL}_k(3) \) modules are available, by acting on the lowest weight vector with the generators \( E_i(z) \):

\[
E_i^{\Lambda_{i}+1}(z) \Phi_{L.W.}^{(\lambda, \mu)}(w) = \frac{\Lambda_i + \nu(\vec{\alpha}_i, \vec{\beta})}{(z-w)} a_i^{\Lambda_i} \Phi_{L.W.}^{(\lambda, \mu)}(w) + R.T.
\]

Easy to verify, that putting \( \Lambda_i = -\nu(\vec{\alpha}_i, \vec{\beta}) \), finite dimensional irreducible representations \( D^{(\lambda, \mu)} \) of the SL(3) algebra are obtained with the dimensions

\[
\dim[D^{(\lambda, \mu)}] = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2),
\]

where \( \Lambda_1 \) is denoted by \( \lambda \) and \( \Lambda_2 \) is denoted by \( \mu \). This allows one to assign indices \( (\lambda, \mu) \)

\[
\begin{align*}
\lambda &= \Lambda_1 = -\nu(\vec{\alpha}_1, \vec{\beta}) = -\nu \sum_{j=1,2} C_{1j} \beta_j, \\
\mu &= \Lambda_2 = -\nu(\vec{\alpha}_2, \vec{\beta}) = -\nu \sum_{j=1,2} C_{2j} \beta_j.
\end{align*}
\]

to the lowest weight vector (2.17), indicating its belonging to a certain finite dimensional representation \( D^{(\lambda, \mu)} \) with the weight vector \( \vec{\Lambda} = (\lambda, \mu) \). Subsequently the conformal charges \( \beta_1 \) and \( \beta_2 \) get quantized in "quantum numbers" of the SL(3) algebra

\[
\beta_i = -\alpha_0 \sum_{j=1,2} C_{ij}^{-1} \Lambda_j.
\]

Because the lowest weight vector (2.17) behaves analogous to a vector \( |\alpha> \) of a bosonic Fock space \( F_{\alpha} \) with respect to the momentum \( p_{i}\) \(^{14}\),

\[
p_i |\alpha> = \alpha_i |\alpha> \quad [p_i, \Phi_{L.W.}^{(\lambda, \mu)}(z)] = (\vec{\alpha}_i, \vec{\beta}) \Phi_{L.W.}^{(\lambda, \mu)}(z);
\]

an identification is possible,

\[
\Phi_{L.W.}^{(\lambda, \mu)}(0) |0> = |\vec{\beta}>, \quad p_i |\vec{\beta}> = (\vec{\alpha}_i, \vec{\beta}) |\vec{\beta}>;
\]

which allows one to refer to the above constructed modules as to "charged" bosonic Fock spaces \( F_{\vec{\beta}, \lambda} \) with conformal charge \( \vec{\beta} = (\beta_1, \beta_2)^9 \). These Fock spaces act as representations of the Heisenberg algebra

\[
[b_i(n), b_j(m)] = nC_{ij} \delta_{n+m,0}
\]

with central elements \( b_i(0) \), identified with the \( p_i \) components of the momentum \( \vec{p} \):

\[
[b_i(0), \Phi_{L.W.}^{(\lambda, \mu)}(z)] = (\vec{\alpha}_i, \vec{\beta}) \Phi_{L.W.}^{(\lambda, \mu)}(z).
\]

\(^{1}\)Recall the mode expansion of a boson field \( \phi_i(z) \), and put \( [p_i, a_j] = C_{ij} \).
In consistency with the interpretations above, the conformal blocks in free field represented conformal field theories arise as expectation values in these charged Fock spaces $F_{\vec{\beta},k}$. So, to actually construct the conformal blocks, one needs the notion of the "Fock conjugated" lowest weight vector, to act as the conjugate vacuum state $|\lambda,0\rangle$. On the other hand, the appropriately chosen conjugate vector allows one to avoid the redundant contour integrations in the conformal blocks. This vector will be built with the same SL(3) quantum numbers from a "Fock conjugated" primary field, represented in terms of the contravariant Heisenberg algebra

\[ [\tau b_i(n), \tau b_j(m)] = -nC_{\hat{h}_{-i}, \hat{h}_{-j}}\delta_{m+n,0}, \]  
\[ [\tau a_{ij}(n), \tau a_{pq}^*(m)] = -\delta_{ip}\delta_{jq}\delta_{m+n,0}, \]  

as far as the conjugation attaches only the Fock space background of the representation constructed above, with $C_{\hat{h}_{-i}, \hat{h}_{-j}} = \tau C_{ij} = C_{ij}$ and $\hat{h} = 3$ - being the dual Coxeter number of SU(3).

One can think then of the following expression for the general form of the conjugate operator

\[ \tilde{\Phi}^{(\lambda,\mu)}_{L.W.}(z) = (\tau a_1^*)^q(\tau a_2^*)^t(\tau a_{22}^*)^s exp[\tilde{\beta}_i \varphi_{\hat{h}_{-i}}]. \]  

(2.29)

Notice now that the same commutation relations (2.28) are satisfied if one makes the following substitutions in the Heisenberg algebra (2.5)

\[ \begin{cases} 
    b_i(n) \leftrightarrow -b_{\hat{h}_{-i}}(-n), \\
    a_{ij}(n) \leftrightarrow a_{ij}^*(-n).
\end{cases} \]  

(2.30)

The isomorphism between these two sets of Heisenberg generators (2.28) and (2.30), allows one to put

\[ \begin{cases} 
    \tau b_i(n) \equiv -b_i(-n), \\
    \tau a_{ij}(n) \equiv a_{ij}^*(-n).
\end{cases} \]  

(2.31)

So one arrives then with the following form of the conjugate operator

\[ \tilde{\Phi}^{(\lambda,\mu)}_{L.W.}(z) = a_1^qa_1^t a_{22}^s exp(\sum_{i=1,2} \tilde{\beta}_i \varphi_{\hat{h}_{-i}}). \]  

(2.32)

The parameters $q, t, s$, and $\tilde{\beta}_i$ are yet undefined. To do this one must subject the conjugate operator (2.32) to the following conditions. Firstly it must act as a lowest weight vector

\[ F_i(z)\tilde{\Phi}^{(\lambda,\mu)}_{L.W.}(w) = R.T., \]  

(2.33)

which implies $s = 0$.

Second, its SL(3) quantum numbers must coincide with those of the vector (2.17):

\[ H_i(z)\tilde{\Phi}^{(\lambda,\mu)}_{L.W.}(w) = H_i(z)\Phi^{(\lambda,\mu)}_{L.W.}(w); \quad i = 1, 2. \]  

(2.34)
The conditions (2.34) leave one with the following system of equations on the parameters $q$, $t$, $\tilde{\beta}_1, \tilde{\beta}_2$:

$$
\begin{aligned}
-2q - t + \nu(2\tilde{\beta}_2 - \tilde{\beta}_1) &= -\lambda, \\
-q - t + \nu(2\tilde{\beta}_1 - \tilde{\beta}_2) &= -\mu.
\end{aligned}
$$

(2.35)

Third, because the $\tilde{\Phi}_{L.W.}^{(\lambda,\mu)}(z)$ is thought to be a primary field, all the coefficients at the singularities higher than the first order in the operator product expansions $E_i(z)\tilde{\Phi}_{L.W.}^{(\lambda,\mu)}(w)$ must be put to be zero:

$$
\begin{aligned}
-q(q-1) - q(2 - \nu^2) + q\nu(2\tilde{\beta}_2 - \tilde{\beta}_1) &= 0, \\
-t(t-1) - qt + \nu(\tilde{\beta}_1 + \tilde{\beta}_2) - t(3 - \nu^2) &= 0.
\end{aligned}
$$

(2.36)

Solving the obtained equations on the parameters $q$, $t$, $\tilde{\beta}_1, \tilde{\beta}_2$ (2.35) and (2.36), four possible sets of these parameters are found. The last subjection to these parameters

$$
\Delta^{(\lambda,\mu)} = \tilde{\Delta}^{(\lambda,\mu)}
$$

(2.37)

leaves one with two of the four solutions:

$$
q = 0, t = 0, \tilde{\beta}_1 = \beta_2, \tilde{\beta}_2 = \beta_1
$$

(2.38)

and

$$
q = 0, t = -\eta + \nu(\tilde{\beta}_1 + \tilde{\beta}_2), \tilde{\beta}_1 = \eta\alpha_0 - \beta_1, \tilde{\beta}_2 = \eta\alpha_0 - \beta_2,
$$

(2.39)

where $\eta = -k - 1$.

The first one coincides with the lowest weight vector (2.17). So one can refer to the second set, as to the parameters of the conjugate operator, for it satisfies all the subjected conditions (2.33) - (2.37). To rewrite (2.39) in a more compact way, it’s convinient to introduce the following two vectors $\vec{\alpha}_0 = (\alpha_0, \alpha_0)$ and $\vec{\Theta} = \vec{\alpha}_1 + \vec{\alpha}_2$. Then

$$
t = \eta + (\vec{\Theta}, \vec{\Lambda}), \quad \vec{\beta} = \eta\vec{\alpha}_0 - \vec{\beta}
$$

(2.40)

$$
\tilde{\beta}_i = \eta\alpha_0 + \alpha_0 \sum_{j=1,2} C^{-1}_{\tilde{h}_i, \tilde{h}-j} \Lambda_j.
$$

Finally for the conjugate lowest weight vector, the following free field representation is constructed

$$
\tilde{\Phi}_{L.W.}^{(\lambda,\mu)}(z) = a_0^{\eta + (\vec{\Theta}, \vec{\Lambda})} \exp\left(\sum_{i=1,2} \tilde{\beta}_i \varphi_{\tilde{h}_i} \right).
$$

(2.41)

The existence of such a lowest weight vector is interpreted in charged Fock space terms, as a manifestation of an isomorphism between the dual Fock space $F_{\tilde{\beta},k}$ and the Fock space $F_{\eta\alpha_0 - \beta, k}$, if one puts

$$
\tau \tilde{b}_i(0) \equiv \eta(\alpha_0)_i - b_{\tilde{h}_i}(0),
$$

(2.42)
Another important ingredient of the theory is obtained, when one puts \( \vec{\lambda} = (0,0) \) and \( \vec{\beta} = (0,0) \):

\[
\tilde{\Phi}^{(\lambda,\mu)}_{L,W}(z) \rightarrow \tilde{1}(z) = a_{12}^n \exp[\sum_{i=1,2} \eta(a_0) i \varphi_{\tilde{h}-i}].
\]

It is called the conjugate identity operator, because it commutes with the current algebra (2.1) and has a null conformal dimension, like the identity operator \( \Phi^{(0,0)}_{L,W}(z) = 1 \).

The existence of such an operator is a consequence of the conformal charge asymmetric structure of the Fock space expectation values.

The last necessary objects for constructing the conformal blocks in a free field represented theory, are the so called screening operators. They have the following form in the case of the SU(3) theory:

\[
\begin{align*}
J_1^+(z) &= a_{11}^n \exp(\alpha_+ \varphi_1); \\
J_2^+(z) &= (a_{22} + a_{11}^* a_{12}) a_{11}^n \exp(\alpha_+ \varphi_2).
\end{align*}
\]

where \( n_- = -(k+3) \), \( n_+ = 1 \), \( \alpha_- = -\alpha^{-1} = \alpha_0 = (\sqrt{k+3})^{-1} \).

Actually, for the purposes pursued in this paper, two of them \( J_1^+ \) and \( J_2^+ \) will do quite well.

The screening operators have peculiar operator product expansions with the current algebra generators (2.1). The only non-vanishing operator product expansions (i.e. expansions with singularities) have the following forms of total derivatives

\[
\begin{align*}
E_i(z) J_+^+(w) &= \delta_{ij} (k+3) \partial_w \left( \frac{\exp(\alpha_0 \varphi_j)}{z-w} \right), \quad i = 1, 2 \\
E_{i+j}(z) J_+^+(w) &= (i-j) (k+3) \partial_w \left( \frac{a_{ij}^* \exp(\alpha_0 \varphi_i)}{z-w} \right).
\end{align*}
\]

which allows one to use them under the contour integrals effectively, while building the conformal blocks.

It’s worth noting as well, that this screening operators have their own, independent of the conformal blocks, interpretations in algebraic contents\(^{11,14}\). And the possibility of using them effectively in conformal blocks originates from the BRST - cohomological structure of the free field representations of the conformal field theories\(^{9,10,11,14}\).

So much for the preliminaries in dealing with the correlation functions and conformal blocks in the SU(3) conformal theory represented in free fields.

3 SL(3) modules and correlation functions

The prescription of building nonvanishing \( N \) - point functions in conformal field theories\(^{2,3}\) requires the conformal charges \( \vec{\beta} \) to satisfy certain constraints, called the neutrality conditions. In the case of the SU(3) conformal field theory they have the following form

\[
\sum_{a=1}^{N-1} \vec{\beta}_a + \vec{\beta}_N = \eta \vec{a}_0,
\]

\[
(i - j) (k+3) \partial_w \left( \frac{a_{ij}^* \exp(\alpha_0 \varphi_i)}{z-w} \right).
\]

\[
\sum_{a=1}^{N-1} \vec{\beta}_a + \vec{\beta}_N = \eta \vec{a}_0,
\]

\[
(3.1)
\]

\[
(2.45)
\]

\[
(2.44)
\]

\[
(2.43)
\]
if one needs to calculate an $N$ - point function of the type

$$\langle \Phi^{(\lambda_1, \mu_1)}(z_1, \bar{z}_1) \cdots \Phi^{(\lambda_{N-1}, \mu_{N-1})}(z_{N-1}, z_{N-1}) \tilde{\Phi}^{(\lambda_N, \mu_N)}(z_N, \bar{z}_N) \rangle. \quad (3.2)$$

To take into consideration the screening currents (2.45), one modifies the equations (3.1)$^2$: \[ \sum_{a=1}^{N-1} \vec{\beta}_a + \vec{\beta}_N + \vec{n}_a = \eta \vec{\alpha}_0, \quad (3.3) \]

which leads in its turn, to the following values for the numbers of screening currents needed

$$n_i = \sum_{j=1,2} \sum_{a=1}^{N-1} [C_{ij}^{-1} \Lambda_{ij}^a - C_{ij}^{-1} \Lambda_{ij}^N]. \quad (3.4)$$

Here $\vec{n} = (n_1, n_2)$ is a formal vector, $n_1$ being the number of screening currents $J_i^+$ and $n_2$ - the number of $J_2^+$. $^3$

For the $(a, a^*)$ - part of the $N$ - point function (3.2) the charge conservation conditions are subjected in the following way

$$\begin{cases} N(a_{12}) + n_2 - N(a_{12}^*) = \eta \\ N(a_{22}) + n_2 - N(a_{22}^*) = 0 \\ N(a_{11}) + n_1 - n_2 - N(a_{11}^*) = 0 \end{cases} \quad (3.5)$$

where $N(a)$ is the numbers of $a$ - fields of certain type in the correlator (3.2). These conditions are equivalent to the claim for the $N$ - point function (3.2) to be SL(3) invariant, i.e. to have the SL(3) charges conserved, because the SL(3) part of the representation of the symmetry algebra of the theory is spanned by the field operators $a_{ij}(z)$ and $a_{ij}^*(z)$. The conserved SL(3) charges are the third component of isospin $T_z$ and the hypercharge $Y$. They are connected with the Cartan basis generators $H_1$ and $H_2$ in the following way

$$\begin{cases} T_z = \frac{1}{2} H_2 \\ Y = \frac{2}{3} H_1 + \frac{1}{2} H_2 \end{cases} \quad (3.6)$$

In these quantities equations (3.5) are equivalent to more transparent equations

$$\begin{cases} \sum_{a=1}^{N} T_{za} = 0 \\ \sum_{a=1}^{n} Y_{za} = 0 \end{cases} \quad (3.7)$$

which are very convinient when correlation functions are being considered. $^4$ As a matter of fact all the information for having nonvanishing correlation functions is encoded in the equations (3.3) and (3.7).

$^2$\(\vec{n}\) can be interpreted as the weight vector of a $(\vec{n}_1, \vec{n}_2)$ representation of the SL$_q$(3).

$^3$Here the same notations for the generators $T_z$ and $Y$ in (3.6) and their eigenvalues in (3.7) are used, which can’t cause misunderstanding in what follows.
For a four point correlation function one gets from the equations (3.4):

\[
\begin{align*}
3n_1 &= \sum_{a=1}^{3}(2\lambda_a + \mu_a) - (2\mu_4 + \lambda_4) \\
3n_2 &= \sum_{a=1}^{3}(2\mu_a + \lambda_a) - (2\lambda_4 + \mu_4)
\end{align*}
\] (3.8)

The consideration of these equations shows that most of the four point correlation functions (3.2) with insertions of both screening currents \(J_1^+\) and \(J_2^+\) possess conformal block functions, the structure of zeroes and poles of which are yet unclear and they don’t admit reductions to the Dotsenko - Fateev type hypergeometric functions (at least naïvely). The main problem is, that the functional equation which is satisfied by the normalization integrals of the conformal blocks (see Appendix A in [3]) turn out to be ”less informative” in this case. Mainly, due to the fact of non-commutativity (in a quantum group sense) of the screening currents \(J_1^+\) and \(J_2^+\). Nevertheless, it seems that the Dotsenko - Fateev procedure for calculating the normalization integrals (though perhaps being not the only possible way) must be applicable to all the generic cases of conformal field theories with proper modifications, supported by the quantum group structure of the space of conformal blocks. In this paper the consideration of four point correlation functions is restricted to the ones, reducible to Dotsenko - Fateev type integrals. These are mainly the four point correlation functions which have only one type of screening currents inserted. They represent, in a certain sense, the SU(2) parts of the SU(3) conformal field theory.

For such kind of four point correlation functions the neutrality conditions (3.8) can be rewritten in the following ways:

\[
\vec{n} = (n_1, 0)
\]

\[
\begin{align*}
\sum_{a=1}^{3}\mu_a &= \lambda_4 - n_1 \\
\sum_{a=1}^{3}\lambda_a &= \mu_4 + 2n_1
\end{align*}
\] (3.9)

\[
\vec{n} = (0, n_2)
\]

\[
\begin{align*}
\sum_{a=1}^{3}\lambda_a &= \mu_4 - n_2 \\
\sum_{a=1}^{3}\mu_a &= \lambda_4 + 2n_2
\end{align*}
\] (3.10)

A sort of duality in \((\lambda, \mu)\) and \((n_1, n_2)\) present in (3.9) and (3.10) is worth noting.

The most convenient choice of the states from SL(3) multiplets, which will stand in four point correlators satisfying the SL(3) charge conservation (3.7), appears to be the following one

\[
\langle \Phi_{H.W.}^{(\lambda_1, \mu_1)}(z_1, \bar{z}_1)\Phi_{H.W.}^{(\lambda_2, \mu_2)}(z_2, \bar{z}_2)\Phi_{T_3^3 Y_3^3}^{(\lambda_3, \mu_3)}(z_3, \bar{z}_3)\tilde{\Phi}_{L.W.}^{(\lambda_4, \mu_4)}(z_4, \bar{z}_4) \rangle.
\] (3.11)

Here H.W (L.W.) means the highest (lowest) weight vector of a particular SL(3) module \((\lambda, \mu)\). The choice of \((T_3^3, Y_3^3)\) is fixed then by solving consistently the equations (3.9) and (3.10) with (3.7). The solutions yield to the following choices for \((T_3^3, Y_3^3)\) in correspondence with (3.9) and (3.10)

\[
\vec{n} = (n_1, 0)
\]
\[ \begin{cases} T^3_z = \frac{\lambda_3}{2} - n_1 = T^3_{z,H.W.} - n_1 \\ Y^3 = \frac{1}{3}(2\mu_3 + \lambda_3) = Y^3_{H.W.} \end{cases} \] (3.12)

\[ \bar{n} = (0, n_2) \]

\[ \begin{cases} T^3_z = \frac{\lambda_3}{2} + \frac{n_2}{2} = T^3_{z,H.W.} + \frac{n_2}{2} \\ Y^3 = \frac{1}{3}(2\mu_3 + \lambda_3) - n_2 = Y^3_{H.W.} - n_2 \end{cases} \] (3.13)

In the equations (3.12) and (3.13) the relations between the ”physical” and ”Cartan” quantum numbers are used for the H.W and L.W. states of a representation \((\lambda, \mu)\):

\[ \text{H.W. s.} = (T_z, Y) = \left( -\frac{1}{2}\lambda, -\frac{1}{3}(2\mu + \lambda) \right) \]

\[ \text{L.W. s.} = (T_z, Y) = \left( \frac{1}{2}\mu, -\frac{1}{3}(2\lambda + \mu) \right) \] (3.14)

To be more concrete in evaluating the four point correlation functions (3.11) and for the sake of simplicity as well, let’s consider the cases when only representations of the \((\lambda, 0)\) and/or \((0, \mu)\) types are inserted where possible.

For such insertions the neutrality conditions (3.9) and (3.10) together with (3.12) and (3.13) lead to the following conformal blocks for the four point correlation function\(^4\) (3.11):

\[ I^{(n_1, 0)}(z_1, z_2, z_3, z_4) \sim \int_{s_1} dv_1 \int_{s_2} dv_2 \ldots \int_{s_{n_1}} dv_{n_1} \times \]

\[ \times \langle \Phi^{(\lambda, 0)}_{H.W.}(z_1) \Phi^{(n_1, 0)}_{H.W.}(z_2) \Phi^{(n_1, 0)}_{T^3_z, Y^3}(z_3) \tilde{\Phi}^{(n_1, \lambda)}_{L.W.}(z_4) \times \]

\[ \times J^+_1(v_1) J^+_1(v_2) \ldots J^+_1(v_{n_1}) \rangle \] (3.15)

\[ I^{(0, n_2)}(z_1, z_2, z_3, z_4) \sim \int_{c_1} du_1 \int_{c_2} du_2 \ldots \int_{c_{n_2}} du_{n_2} \times \]

\[ \times \langle \Phi^{(0, \mu)}_{H.W.}(z_1) \Phi^{(0, n_2)}_{H.W.}(z_2) \Phi^{(0, n_2)}_{T^3_z, Y^3}(z_3) \tilde{\Phi}^{(\mu, n_2)}_{L.W.}(z_4) \times \]

\[ \times J^+_2(u_1) J^+_2(u_2) \ldots J^+_2(u_{n_2}) \rangle \] (3.16)

Let’s now have a look on how the modules of \((\lambda, 0)\) and \((0, \mu)\) types are constructed.

\(^4\)Here and below the conventional notations for the conformal blocks and other ingredients of the theory set up in\(^2,^3\) are used.
The root diagram of the SL(3) algebra (2.19), Fig.1, induces the following weight diagrams for the representations \((\lambda, 0)\) and \((0, \mu)\), Fig.2 and Fig.3.

In general, the states in these modules can be represented in the following way

\[
\Phi_{i,j}^{(\lambda,0)}(z) =: E_i^\lambda E_j^{\lambda-j} V^{(\lambda,0)} : (z) \tag{3.17}
\]

\[
\Phi_{i,j}^{(0,\mu)}(z) =: E_i^\mu E_j^{\mu-j} V^{(0,\mu)} : (z) \tag{3.18}
\]

with the corresponding weights

\[
H_1(z)\Phi_{i,j}^{(\lambda,0)}(w) = \frac{-(2j - \lambda + i)}{z - w} \Phi_{i,j}^{(\lambda,0)}(w) + R.T.
\]

\[
H_2(z)\Phi_{i,j}^{(\lambda,0)}(w) = \frac{2i - \lambda + j}{z - w} \Phi_{i,j}^{(\lambda,0)}(w) + R.T. \tag{3.19}
\]

\[
H_1(z)\Phi_{i,j}^{(0,\mu)}(w) = \frac{\mu - i - j}{z - w} \Phi_{i,j}^{(0,\mu)}(w) + R.T.
\]

\[
H_2(z)\Phi_{i,j}^{(0,\mu)}(w) = \frac{2i - j}{z - w} \Phi_{i,j}^{(0,\mu)}(w) + R.T. \tag{3.20}
\]

Or equivalently

\[
\begin{cases}
T_z = \frac{1}{7}(-\lambda + j + 2i) \\
Y = \frac{1}{3}(\lambda - 3j)
\end{cases} \tag{3.21}
\]

in \((\lambda, 0)\) and

\[
\begin{cases}
T_Z = -\frac{1}{7}(j - 2i) \\
Y = \frac{1}{3}(2\mu - 3j)
\end{cases} \tag{3.22}
\]

in \((0, \mu)\).

Naively one needs the explicit bosonized forms of the r.h.s. of (3.17) and (3.18) to evaluate the conformal blocks (3.15) and (3.16).

In fact it turns quite sufficient having the bosonized states from some of the simple modules like \(\lambda = 1, 2, 3\) and \(\mu = 1, 2, 3\) to generalize then the results for the conformal blocks of the form (3.15) and (3.16) containing arbitrary \(\lambda\) and \(\mu\).

In the Fig.4, the corresponding weight diagrams are represented for the modules \((1, 0)\), \((2, 0)\), \((3, 0)\) and for their duals on the Fig.5. Missing the rather tedious calculations let’s have just the bosonized forms of the states in the modules of Fig.4.
\( \Phi_{L.W.}^{(1,0)} = \Phi_{0,1}^{(1,0)} = V^{(1,0)} \)

\( \Phi_{0,0}^{(1,0)} = a_{11}^* V^{(1,0)} \)

\( \Phi_{H.W.}^{(1,0)} = \Phi_{1,0}^{(1,0)} = (a_{12}^* - a_{22}^* a_{11}^*) V^{(1,0)} \)

\( \Phi_{L.W.}^{(2,0)} = \Phi_{0,2}^{(2,0)} = V^{(2,0)} \quad \Phi_{1,0}^{(2,0)} = (a_{22}^* a_{11}^2 - a_{12}^* a_{11}^*) V^{(2,0)} \)

\( \Phi_{0,1}^{(2,0)} = a_{11}^* V^{(2,0)} \quad \Phi_{1,1}^{(2,0)} = (a_{12}^* - a_{11}^* a_{22}^*) V^{(2,0)} \quad (3.24) \)

\( \Phi_{0,0}^{(2,0)} = a_{11}^2 V^{(2,0)} \quad \Phi_{H.W.}^{(2,0)} = \Phi_{2,0}^{(2,0)} = (a_{22}^* a_{11}^2 - 2a_{12}^* a_{11}^* a_{22}^* + a_{12}^* a_{22}^2) V^{(2,0)} \)

\( \Phi_{L.W.}^{(3,0)} = \Phi_{0,3}^{(3,0)} = V^{(3,0)} \quad \Phi_{1,0}^{(3,0)} = (a_{22}^* a_{11}^3 - a_{12}^* a_{11}^2) V^{(3,0)} \)

\( \Phi_{0,2}^{(3,0)} = a_{11}^* V^{(3,0)} \quad \Phi_{2,0}^{(3,0)} = (a_{22}^* a_{11}^3 - 2a_{22}^* a_{12}^* a_{11}^2 + a_{12}^* a_{11}^2) V^{(3,0)} \)

\( \Phi_{0,1}^{(3,0)} = a_{11}^2 V^{(3,0)} \quad \Phi_{1,1}^{(3,0)} = (a_{11}^2 a_{22}^* - a_{12}^* a_{11}^*) V^{(3,0)} \)

\( \Phi_{0,0}^{(3,0)} = a_{11}^3 V^{(3,0)} \quad \Phi_{1,2}^{(3,0)} = (a_{11}^2 a_{22}^* - a_{12}^*) V^{(3,0)} \quad (3.25) \)

\( \Phi_{H.W.}^{(3,0)} = \Phi_{3,0}^{(3,0)} = (a_{11}^3 a_{22}^* - a_{12}^* a_{11}^* a_{22}^* + 3a_{11}^* a_{22}^* a_{12}^2 - a_{12}^* a_{12}^3) V^{(3,0)} \)

\( \Phi_{2,1}^{(3,0)} = (6a_{11}^* a_{22}^* a_{12}^* - 3a_{12}^* a_{11}^* a_{22}^2 V^{(3,0)} \)

For the states in the dual modules \((0,1), (0,2), (0,3)\), Fig. 5., one arrives with:

\( \Phi_{L.W.}^{(0,1)} = \Phi_{0,1}^{(0,1)} = V^{(0,1)} \)

\( \Phi_{1,1}^{(0,1)} = a_{22}^* V^{(0,1)} \quad (3.26) \)

\( \Phi_{H.W.}^{(0,1)} = \Phi_{0,0}^{(0,1)} = a_{12}^* V^{(0,1)} \)

\( \Phi_{L.W.}^{(0,2)} = \Phi_{0,2}^{(0,2)} = V^{(0,2)} \quad \Phi_{1,0}^{(0,2)} = a_{12}^* a_{22}^* V^{(0,2)} \)

\( \Phi_{1,1}^{(0,2)} = a_{22}^* V^{(0,2)} \quad \Phi_{0,1}^{(0,2)} = a_{12}^* V^{(0,2)} \quad (3.27) \)

\( \Phi_{2,2}^{(0,2)} = a_{22}^* V^{(0,2)} \quad \Phi_{H.W.}^{(0,2)} = \Phi_{0,0}^{(0,2)} = a_{12}^* V^{(0,2)} \)
\[
\Phi_{L,W}^{(0,3)} = \Phi_{0,3}^{(0,3)} = V^{(0,3)} \\
\Phi_{1,3}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{2,3}^{(0,3)} = a_{22}^* V^{(0,3)} \\
\Phi_{3,3}^{(0,3)} = a_{22}^* V^{(0,3)} \\
\Phi_{0,2}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{1,2}^{(0,3)} = a_{22}^* a_{12}^* V^{(0,3)} \\
\Phi_{1,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{H,W}^{(0,3)} = \Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{1,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{H,W}^{(0,3)} = \Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{1,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{H,W}^{(0,3)} = \Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{1,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{H,W}^{(0,3)} = \Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{1,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{H,W}^{(0,3)} = \Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{1,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,1}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \\
\Phi_{H,W}^{(0,3)} = \Phi_{0,0}^{(0,3)} = a_{12}^* V^{(0,3)} \]

Now let’s examine the states appearing in the conformal blocks (3.15) and (3.16) from the modules \((\lambda, 0)\) and \((0, \mu)\). As was mentioned in the sect. 2, for the sake of simplicity it’s convenient to to keep the ”Fock conjugated” representations \((\lambda, \mu)\) represented in the conformal blocks by the lowest weight states (2.41). Anyhow, there’s no obstacle in constructing ”Fock conjugated” modules \((\lambda, 0)\) and \((0, \mu)\), acting explicitly in the same way as in (3.17) and (3.18). All the states in these ”Fock conjugated” modules have the same corresponding weights (3.19) and (3.20). The only difference is that in the case of the ”Fock conjugated” modules null states occur in the places where numerical zeroes while building the modules \((\lambda, 0)\) and \((0, \mu)\) appear (e.g. \(E_1^{(\lambda,0)}(z)\Phi_{1,0}^{(\lambda,0)}(w) = \text{null state}\)).

The states standing at the point \((z_3, \bar{z}_3)\) from the modules \((n_1, 0)\) and \((0, n_2)\) in (3.15) and (3.16) have the following bosonized forms. From the equations (3.12) and (3.13) one learns that these are the states with \((-n_1/2, Y_{H,W})\) from \((n_1, 0)\) in the case of (3.12) and \((n_2/2, Y_{H,W}; -n_2)\) from \((0, n_2)\) in the case of (3.13). Taking into consideration (3.21) and (3.22) one arrives with the following states, \((j = 0, i = 0)\) for the case of \((n_1, 0)\) and \((j = n_2, i = n_2)\) for the case of \((0, n_2)\).

Examining the explicit bosonized forms for the states in the modules \((\lambda, 0)\) (eqs. 3.23 - 3.25) and \((0, \mu)\) (eqs. 3.26 - 3.28) as well as the Fig.4. and Fig.5. one can notice that in general the states \(\Phi_{0,0}^{(\lambda,0)}(z) = \Phi_{0,0}^{(0,\mu)}(z)\) are \(\lambda\) order \((\mu\) order\) monomials of the fields \(a_{12}^*(z)\) \((a_{22}^*(z)\). This leads to another statement, that the only terms from the \((a, a^*)\) part of the highest weight states of \((\lambda, 0)\) modules having nonzero contributions in the correlator (3.15) are the ones containing the fields \(a_{12}^*(z)\) in the corresponding \(\lambda\)th power. In the case of the correlator (3.16) this observation turns to be obvious, as far as the highest weight states in the modules \((0, \mu)\) are just \(\mu\)th order monomials of \(a_{12}^*(z)\).

Concluding these preliminary observations let’s write down the exact forms of the holomorphic conformal blocks (3.15) and (3.16) for general values of \(n_1 = p\) and \(n_2 = q\) in the projective group gauge \(z_1 = 0; z_2 = z; z_3 = 1; z_4 \to \infty\) as usual:

\[
I^{(p,0)}(z) = \int_{s_1} dv_1 \int_{s_2} dv_2 \ldots \int_{s_p} dv_p \times
\]

\[
\times (V^{(\lambda,0)}(0)V^{(p,0)}(z)V^{(p,0)}(1)\tilde{V}^{(p,\lambda)}(\infty)V_1^+(v_1)V_1^+(v_2)\ldots V_1^+(v_p))
\]
\[ \times \langle a_{12}^\lambda (0) a_{12}^p (z) a_{11}^p (1) a_{12}^{q+p+\lambda} (\infty) a_{11} (v_1) a_{11} (v_2) \ldots a_{11} (v_n) \rangle \] (3.29)

\[ I^{(0,q)} (z) = \int_{c_1} du_1 \int_{c_2} du_2 \ldots \int_{c_q} du_q \times \]
\[ \times \langle V^{(0,\mu)} (0) V^{(0,q)} (z) V^{(0,q)} (1) \tilde{V}^{(\mu,q)} (\infty) V_2^+(u_1) V_2^+(u_2) \ldots V_2^+(u_q) \rangle \]
\[ \times \langle a_{12}^\mu (0) a_{12}^p (z) a_{22}^p (1) a_{12}^{q+p+\mu} (\infty) a_{22} (u_1) a_{22} (u_2) \ldots a_{22} (u_q) \rangle \] (3.30)

Evaluating the constructed conformal blocks (3.29) and (3.30) with the help of the corresponding two point functions (2.4), one gets finally the following expressions of them:

\[ I^{(p,0)} (z) = z^{\alpha_{12}} (1 - z)^{\alpha_{23}} \int_{s_1} dv_1 \int_{s_2} dv_2 \ldots \int_{s_p} dv_p \times \]
\[ \times \prod_{i=1}^p v_i^a (1 - v_i)^{b-1} (z - v_i)^c \prod_{i<j} (v_i - v_j)^{2p} \] (3.31)

\[ I^{(0,q)} (z) = z^{\alpha'_{12}} (1 - z)^{\alpha'_{23}} \int_{c_1} du_1 \int_{c_2} du_2 \ldots \int_{c_q} du_q \times \]
\[ \times \prod_{i=1}^q u_i^{a'} (1 - u_i)^{b'-1} (z - u_i)^{c'} \prod_{i<j} (u_i - u_j)^{2p} \] (3.32)

Here:

\[ \alpha_{12} = \beta^{(\lambda,0)} C \tilde{\beta}^{(p,0)} ; \quad \alpha'_{12} = \beta^{(0,\mu)} C \tilde{\beta}^{(0,q)} \]
\[ \alpha_{23} = \beta^{(p,0)} C \tilde{\beta}^{(p,0)} ; \quad \alpha'_{23} = \beta^{(0,q)} C \tilde{\beta}^{(0,q)} \]
\[ a = \beta^{(\lambda,0)} C \bar{e}_1 \quad ; \quad a' = \beta^{(0,\mu)} C \bar{e}_2 \]
\[ b = c = \beta^{(p,0)} C \bar{e}_1 \quad ; \quad b' = c' = \beta^{(0,q)} C \bar{e}_2 \] (3.33)
\[ \vec{e}_1 = (\alpha_0, 0) \]
\[ \vec{e}_2 = (0, \alpha_0) \]
\[ 2\rho = \vec{e}_1^T C \vec{e}_2 = 2\alpha_0^2 \]

The integrals (3.31) and (3.32) are obviously of Dotsenko - Fateev type, the technique of handling which as well as their structures of zeroes and poles are worked out in\textsuperscript{2,3}. Due to the commutativity of the fields \( a_{11}^*(z) \), \( a_{22}^*(z) \) and \( a_{12}^*(z) \) the integrals (3.31) and (3.32) appear to be simpler than in the case of the SU(2) conformal field theory (comp.\textsuperscript{15}). In references\textsuperscript{2,3} a convinient basis of linearly independant conformal blocks was performed identifying the basis elements by a system of independent contour configurations, which assigns the conformal blocks a "monodromy" index \( k \) in the following way:

\begin{align*}
I_k^{(p,0)}(a, b, c; \rho; z) &= \int_1^\infty dv_1 \cdots \int_1^{v_{p-k-1}} dv_{p-k} \int_0^z dz_{p-k+1} \cdots \int_0^{v_p-1} dv_p \times \\
&\quad \times \prod_{i=1}^p v_i^a \prod_{i=1}^p (v_i - 1)^{b-1} (v_i - z)^c \prod_{i=p-k+1} v_i^a (1 - v_i)^{b-1} (z - v_i)^c \\
&\quad \times \prod_{i<j} (v_i - v_j)^{2\rho} \quad (3.34)
\end{align*}

\begin{align*}
I_k^{(0,q)}(a', b', c'; \rho; z) &= \int_1^\infty du_1 \cdots \int_1^{u_{q-k-1}} du_{q-k} \int_0^z du_{q-k+1} \cdots \int_0^{u_q-1} du_p \times \\
&\quad \times \prod_{i=1}^q u_i^{a'} \prod_{i=1}^{q-1} (u_i - 1)^{b'-1} (u_i - z)^c' \prod_{i=q-k+1} u_i^{a'} (1 - u_i)^{b'-1} (z - u_i)^c' \\
&\quad \times \prod_{i<j} (u_i - u_j)^{2\rho} \quad (3.35)
\end{align*}

The overall factors \( z^{\alpha_{12}}(1 - z)^{\alpha_{23}} \) are suppressed as far as they are common to all the integrals and can be restored in the final expressions. Afterwards the consideration of the conformal blocks (3.34) and (3.35) will be restricted to one of them, particularly to the first one, keeping in mind that the analogous simulations are appliable to the second one, when \( a, b, c \), and \( \alpha \)'s are substituted by the stressed ones and \( p \) by \( q \). The choice of the basis (3.34) in the space of the
conformal block functions corresponds to the expansion of the four point function over the s-channel partial waves\(^2,3,15\).

The basis functions \(I^{(p,0)}_{k}(a, b, c; \rho; z)\) can be represented in the following way:

\[
I^{(p,0)}_{k}(a, b, c; \rho; z) = N^{(p,0)}_{k}(a, b, c; \rho; z) f_{k}(z) = N^{(p,0)}_{k} z^{\gamma_{k}} f_{k}(z)
\]

(3.36)

where \(f_{k}(z)\) is analytic at \(z = 0\), \(f_{k}(0) = 1\) and \(\gamma_{k} = (k - 1)(1 + a + c + (k - 2)\rho)\). \(N^{(p,0)}_{k}(a, b, c; \rho; z)\) is the so-called normalization integral, having the following form (\(^3\), Appendix A):

\[
N^{(p,0)}_{k} = \int_{0}^{1} dt_{1} \int_{0}^{t_{1}} dt_{2} \ldots \int_{0}^{t_{p-k-1}} dt_{p-k} \int_{0}^{1} ds_{1} \int_{0}^{s_{1}} ds_{2} \ldots \int_{0}^{s_{k-1}} ds_{k} \times
\]

\[
\times \prod_{i=1}^{p-k} (1 - a - c - b - 2(p - 2)\rho)(1 - t_{i})^{b-1} \prod_{i<j} (t_{i} - t_{j})^{2\rho} \times
\]

\[
\times \prod_{i=1}^{k} s_{i}^{a}(1 - s_{i})^{c} \prod_{i<j} (s_{i} - s_{j})^{2\rho} =
\]

\[
= \prod_{i=1}^{p-k} \frac{\Gamma(ip)}{\Gamma(\rho)} \prod_{i=0}^{p-k-1} \frac{\Gamma(-a - c - b - 2(p - 2)\rho + ip)\Gamma(b + ip)}{\Gamma(-a - c - 2\rho(k - 1) - ip)} \times
\]

\[
\times \prod_{i=1}^{k} \frac{\Gamma(ip)}{\Gamma(\rho)} \prod_{i=0}^{k-1} \frac{\Gamma(1 + a + ip)\Gamma(1 + c + ip)}{\Gamma(2 + a + c + (k - 2 + i)\rho)}
\]

(3.37)

Here the notations \(v_{i} = t_{i}\), if \(1 \leq i \leq p - k\) and \(v_{i} = s_{i}\), if \(p - k + 1 \leq i \leq p\) are found to be suitable. The same representation holds for the conformal blocks \(I^{(0,q)}_{k}(z)\) if one changes \(a, b, c\) to the stressed ones as well as \(p\) to \(q\).

To construct monodromy invariant physical correlation functions from the obtained conformal blocks, one must sew the holomorphic conformal blocks with the corresponding antiholomorphic ones, by a monodromy invariant metric, summing over the indice \(k\).

Due to the chosen above basis \(I^{(p,0)}_{k}(z)\) ( \(I^{(0,a)}_{k}(z)\) ) as well as to the fact that the \((a, a^{*})\) part of the conformal blocks doesn’t effect its monodromy properties (as far as \(\Delta(a) = 1\) and \(\Delta(a^{*}) = 0\)), the diagonal metric needed is the one constructed for the minimal models, i.e. the \(\chi_{k}\). It can be derived for the cases \(I^{(p,0)}_{k}(z)\) and \(I^{(0,q)}_{k}(z)\) in the same way as in\(^2,3\). The result coincides with the one for the minimal models up to an overall \(k\) independant factor and appears to have the following form:
The same formula holds for the case of \((0, q)\) conformal blocks, changing \(a, b, c\) to \(a', b', c'\) and \(p\) to \(q\) in (3.38).

Finally, using all the calculations above and restoring the antiholomorphic parts of the theory, one obtains the following expression for the four point correlation function (3.11) with the certain choices made in (3.12) and (3.13):

\[
\langle \Phi_{H.W.}(\lambda_1, \mu_1) \Phi_{H.W.}(z_1, \bar{z}_1) \Phi_{T^2_{\lambda_3, Y_3}}(z_3, \bar{z}_3) \Phi_{L.W.}(\lambda_4, \mu_4) \rangle = |z|^{2\alpha_{12}} |1-z|^{2\alpha_{23}} \prod_{s<t} |z_s - z_t|^{-2\alpha_{st}} G(z, \bar{z}).
\]
4 Concluding remarks

The free filed representation was reviewed and some four point correlation functions were constructed in this paper on the SU(3) conformal field theory. The four point correlators considered were the ones with conformal blocks having only one type of screening currents inserted. The latter were shown to have Dotsenko-Fateev type structures and perform the SU(2) subtheories of the SU(3) conformal field theory.

Anyhow, it must be noted as well that there are indeed other conformal blocks in the SU(3) conformal field theory containing the both $J_1^+$ and $J_2^+$ currents and nevertheless reducible to the Dotsenko-Fateev type integrals with normalization integrals different in that they are multiplied by a polynomial of $a$, $b$, $c$, $a'$, $b'$, $c'$, and $\rho$. But the consideration of such conformal blocks (i.e. reducible ones) does not provide a complete basis for calculating the structure constants of the complete operator algebra in the way, performed in\textsuperscript{4,15}.

Moreover, it seems most likely that to create such a basis one must be able to make use of the third screening current $J_3^+$ (see e.g.\textsuperscript{14,16,17}) without which the construction of an arbitrary representation of the $SL_q(3)$ would be impossible. Such a possibility, in its turn, could provide, in a certain sense, an evidence of completeness of the operator algebra structure constants.

The usage of the third screening current $J_3^+$ in constructing conformal blocks still needs to be investigated.

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Figure captions

Fig.1.: The root diagram of SL(3).

Fig.2.: The weight diagram of the (λ, 0) representations.

Fig.3.: The weight diagram of the (0, μ) representation.

Fig.4.: The (1,0), (2,0) and (3,0) representations.

Fig.5.: The (0,1), (0,2) and (0,3) representations.