Self-dual Vertex Operator Superalgebras with Shadows of large minimal weight

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Abstract

The shadow $V'$ of a self-dual vertex operator superalgebra $V$ is defined as the direct sum of the irreducible modules of its even vertex operator subalgebra $V(0)$ not contained in $V = V(0) \oplus V(1)$. We describe the self-dual "very nice" unitary rational vertex operator superalgebras $V$ of rank $c$ whose shadows have the largest possible minimal weights $c/8$ or $c/8 - 1$. The results are analogous to and imply the corresponding results for self-dual binary codes and lattices.

1 Introduction

There exist a deep not completely understood analogy between codes, lattices and vertex operator superalgebras, as studied for example in [God89, Höh95]. In this short note we establish a further example of this analogy. Elkies has recently been given a description of self-dual binary codes and lattices with long shadows [Elk95a, Elk95b]. We obtain such a description for vertex operator superalgebras (SVOAs) which form a mathematical algebraic definition of chiral algebras in conformal field theory. The miracle is, that we are working with different objects and different numbers and getting results of the same structure.² By using the constructions that attach a lattice to a code and a SVOA to a lattice the theorems of Elkies for codes and lattices follow from our theorems for SVOAs.

Let $V$ be a self-dual unitary "very nice" rational vertex operator superalgebra of rank $c \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ defined over the complex numbers. (A short definition is given at the end of this introduction. For more details see [Höh95], Chapter 1 and 2.) The even vertex operator subalgebra $V(0)$ of $V = V(0) \oplus V(1)$ consists of the vectors of integral weight and has besides $V(0)$ and $V(1)$ one (if $c \in \mathbb{Z} + \frac{1}{2}$) or two (if $c \in \mathbb{Z}$) more irreducible modules $V(2)$ (and $V(3)$). We call $V' = V(2)$ ($c \in \mathbb{Z} + \frac{1}{2}$) resp. $V' = V(2) \oplus V(3)$ ($c \in \mathbb{Z}$) the shadow of $V$. In the special case that $V$ is a vertex operator algebra (VOA), i.e. $V = V(0)$, we set $V' = V$.

In [Höh95], Chapter 5 we have given, in analogy to the known results for codes and lattices, a description of all extremal SVOAs, i.e. SVOAs $V$ with the largest possible minimal weight $\frac{c}{8}$ or $\frac{c}{8} - 1$. An in some sense dual problem is the classification of SVOAs with shadows of large minimal weight $h(V') \in \mathbb{Z} + \frac{c}{8}$, this is the smallest conformal weight of a nonzero

¹The reader will find that the structure of the results and proofs of this paper are exactly the same as in [Elk95b]. This is intended to emphasise the analogy.
The rank \( c \) of a self-dual “nice” rational SVOA is a integral or half-integral number (see \[H¨oh93\], Th. 2.2.2). For a self-dual SVOA \( V \) to be “very nice” the following additional properties for its even sub-VOA \( V_0 \) are assumed:

1. One has \( V = V_0 \) or the fusion algebra of \( V_0 \) is given by \( \mathbb{Z} [ \mathbb{Z}_2 \times \mathbb{Z}_2 ] \) if \( c \in 2 \mathbb{Z} \), \( \mathbb{Z} [ \mathbb{Z}_4 ] \) if \( c \in 2 \mathbb{Z} + 1 \) and the Ising model fusion rules if \( c \in \mathbb{Z} + \frac{1}{2} \).
2. The \( \text{SL}_2(\mathbb{Z}) = \langle S, T \rangle \)-representation on the genus one correlation functions is given by the diagonal matrix \( \bar{T} = e^{-2\pi i \frac{c}{24}} (1,-1,e^{2\pi i \frac{c}{24}},e^{2\pi i \frac{c}{24}}) \) (if \( c \) is integral, \( V \neq V_0 \)) or \( \bar{T} = e^{-2\pi i \frac{c}{24}} (1,-1) \) (if \( c \) is half-integral). For the values of \( \bar{S} \) compare \[H¨oh93\], (2.7).

More property is needed: The tensor product theory defines on the direct sum of the irreducible \( V_0 \)-modules the structure of an \textit{intertwiner algebra} (see \[Hua95\], Th. 3.5) and this structure induces \textit{fusing} and \textit{braiding isomorphisms} between tensor products of intertwiner spaces (see \[Hua95\], Remark 3.3). In \[MSS98\], Appendix D 1 and E it was discussed that in the cases of the fusion algebra as above the fusing and braiding maps are well defined and depend only on the residue class \( c \) (mod 8). Since we don’t want to check that all the properties of the braiding and fusion maps as assumed in \[MSS98\] are satisfied, we simply assume that

3. the braiding and fusing is the one described in \[MSS98\].

The importance of the “braided fusion algebra” (this is the fusion algebra together with the braiding and fusion isomorphisms and the \( \text{SL}_2(\mathbb{Z}) \)-representation) of a VOA \( V \) is the fact that it describes completely the possible (S)VOA-extensions \( W \) of \( V \): Indeed, a vertex

\[
\sum_{n \in \frac{1}{2} \mathbb{Z} \geq 0} V_n = V(0) \oplus V(1), \quad V_0 = C \cdot 1,
\]
operator defining a (S)VOA-structure on the $V$-module $W$ is given by elements in the intertwiner spaces for $V$ and the Jacobi identity for $W$ is equivalent to the commutativity which can be expressed in terms of the braiding maps. Furthermore, the “braided fusion algebra” of $W$ can be described completely in terms of the “braided fusion algebra” of $V$ and the vertex operator of $W$ as elements inside.\footnote{The “braided fusion algebra” should be equivalent to a braided tensor category with extra properties, called “modular category” in [Tur94]. This means one can associate to a VOA a 3-dimensional topological quantum field theory, which should be thought as the VOA-analog of the discriminant form of an integral lattice.} As an application, one sees that the SVOA-structure of the $V_{(0)}$-module $V = V_{(0)} \oplus V_{(1)}$ is unique up to isomorphism.

## 2 Classification results

The subalgebra generated by the weight-$\frac{1}{2}$ part $V_{1/2}$ of a unitary SVOA $V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n$ is isomorphic to the tensor product $V_{\text{Fermi}}^\otimes k$ of $k = \dim V_{1/2}$ copies of the SVOA $V_{\text{Fermi}}$, the unique unitary SVOA with the smallest possible rank $\frac{1}{2}$. The graded vector space $V_{\text{Fermi}}^\otimes k$ is in fact an irreducible module for the infinite Clifford algebra generated by the coefficients of the vertex operators $Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ of elements $a \in V_{1/2}$.

**Theorem 1** For the minimal weight of the shadow of a self-dual unitary “very nice” rational vertex operator superalgebra $V$ of rank $c$ one has $h(V') \leq \frac{c}{8}$ with equality exactly if $V \cong V_{\text{Fermi}}^\otimes 2c$.

**Proof:** The character $\chi_M$ of a module $M$ of a vertex operator (super)algebra of rank $c$ is defined as the generating series

$$\chi_M = q^{-\frac{c}{24}} \sum_{n \in \mathbb{Q}} \dim M_n \cdot q^n$$

of the dimensions of its homogeneous pieces. Taking $q = e^{2\pi i \tau}$ the function $\chi_M(\tau)$ is for sufficient regular (S)VOAs holomorphic in the upper half-plane of the complex numbers [Zhu90]. For a SVOA as in the theorem we have proved in [Höh95] (Theorem 2.2.3):

The character $\chi_V$ is a weighted-homogeneous polynomial $P_V(x, y)$ of weight $c$ in the characters $x = \chi_{1/2} := \chi_{V_{\text{Fermi}}} = \sqrt{\frac{24}{\tau}}$ (weight 1/2) of the SVOA $V_{\text{Fermi}}$ and $y = \chi_8 := \chi_{V_{\text{Fermi}}} = \frac{\Theta_8}{\eta}$ (weight 8) of the lattice-VOA $V_{E_8}$.

Here $\Theta_2 \text{ resp. } \Theta_8$ denote the theta series of the lattices $\mathbb{Z}$ resp. $E_8$, the root lattice of the Lie group $E_8$, and $\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function. The functions $\chi_{1/2}$ resp. $\chi_8$ are modular functions (with character) for the theta group $\Gamma_2 = \langle S, T^2 \rangle$ respectively for the full modular group $\text{PSL}(2, \mathbb{Z}) = \langle S, T \rangle$, where the matrices $S$ and $T$ correspond to the modular transformations $\tau \mapsto -\frac{1}{\tau}$ and $\tau \mapsto \tau + 1$, i.e. $\chi_V$ is a modular function for $\Gamma_2$.

The modular curve $\mathbb{H}/\Gamma_2$ has two cusps represented by $\infty$ and 1. Mainly by definition the character $\chi_V'$ of the shadow $V'$ is given by the expansion of the modular function $\chi_V$ in the cusp 1 (see [Höh95], Def. 2.2.4):

$$\chi_{V'}(\tau) = \alpha e^{2\pi i \tau} \chi_V\left(\frac{1}{\tau} + 1\right),$$

with $\alpha = 1$ if $c \in \mathbb{Z}$ and $\alpha = \frac{1}{\sqrt{2}}$ if $c \in \mathbb{Z} + \frac{1}{2}$. Since

$$\chi_{V'}^{\otimes k} = \frac{1}{\sqrt{2}} e^{\frac{2\pi i}{k}} \chi_{V_{\text{Fermi}}} \left(\frac{1}{\tau} + 1\right) = q^{\frac{1}{24}} (1 + q + q^2 + 2 q^3 + \cdots)$$

### 3
and
\[ e^{\frac{\pi i}{h}} \chi_8(-\frac{1}{h} + 1) = \chi_8(\tau) = q^{-\frac{1}{3}}(1 + 248q + 4124q^2 + \cdots) \]  
(3)

it follows that the character of the shadow can be written as
\[ \chi_V' = \alpha P_V(\sqrt{2} \chi_V^{Fermi}'; \chi_8) = \dim V_{h(V')} \cdot q^{-\frac{1}{2}h(V')} + \cdots. \]
(4)

From this and the expansions (2) and (3) we conclude that for the coefficients of the polynomial
\[ P_V(x, y) = \sum_{i=0}^{[\frac{s}{8}]} A_i \cdot x^{2c-16i}y^i \]
one gets
\[ A_k = 0 \quad \text{for} \quad k > \frac{c}{8} - h(V'). \]
(5)

Since the series \( \chi_V = P_V(\chi_{1/2}, \chi_8) \) starts with \( q^{-\frac{\pi i}{2}} \) we get the relation \( \sum_{i=0}^{[\frac{s}{8}]} A_i = 1 \). The assumption \( h(V') \geq \frac{c}{8} \) implies \( P_V(x, y) = x^{2c} \) and one has \( h(V') = \frac{c}{8} \). Finally \( \chi_V = P_V(\chi_{1/2}, \chi_8) = \chi_{2c/8} \) gives \( \dim V_{1/2} = 2c \) and the SVOA \( V \) is completely generated by \( V_{1/2} \), i.e. \( V \cong V_{Fermi}^{2c/8} \).

Similar as in the case of codes and lattices every unitary SVOA \( V \) can be decomposed as a tensor product \( W \otimes V_{Fermi}^{\otimes r} \), where \( V_{Fermi}^{\otimes r} \) is the SVOA generated by the elements of conformal weight \( \frac{1}{2} \) in \( V \) and \( W \) is a SVOA with no nonzero elements of weight \( \frac{1}{2} \). If \( V \) is self-dual, so is \( W \) (see [Höh97], Th. 2.2.8). For the shadow of a tensor product we have the following result:

**Lemma 1** Let \( V \) and \( W \) self-dual unitary “very nice” rational SVOAs. Then \( V \otimes W \) is “very nice” and for the shadows one has \( (V \otimes W)' = mV' \otimes W' \) with \( m = 2 \) if the ranks of \( V \) and \( W \) are both half-integral and \( m = 1 \) else.

**Proof:** Only the property “very nice” has not been completely shown in [Höh97], especially it remains to prove that the even sub-VOA
\[ (V \otimes W)_{(0)} = V_{(0)} \otimes W_{(0)} \oplus V_{(1)} \otimes W_{(1)} \]
has the right “braided fusion algebra” and gives the right \( SL_2(Z) \)-representation. Let \( c, d \in \frac{1}{2}Z \) be the ranks of \( V \) and \( W \). As discussed at the end of the introduction, the “braided fusion algebras” of \( V_{(0)} \) and \( W_{(0)} \) are the same as for \( (V_{Fermi}^{\otimes 2c})_{(0)} \) and \( (V_{Fermi}^{\otimes 2d})_{(0)} \); the “braided fusion algebra” of \( V_{(0)} \otimes W_{(0)} \oplus V_{(1)} \otimes W_{(1)} \) is uniquely determined since \( V \otimes W \) is unique as \( V_{(0)} \otimes W_{(0)} \)-module. So the structure of the “braided fusion algebras” is the same as for \( (V_{Fermi}^{\otimes 2c}) \otimes (V_{Fermi}^{\otimes 2d})_{(0)} = (V_{Fermi}^{\otimes 2(c+d)})_{(0)} \), but \( V_{Fermi}^{\otimes 2(c+d)} \) is known to be “very nice”. In particular one gets the stated relations between the shadows and the structure of the \( SL_2(Z) \)-representation follows also.

Applying the lemma to \( V = W \otimes V_{Fermi}^{\otimes r} \) we get \( h(V') = h(W') + \frac{1}{16} \), i.e. the difference \( \frac{1}{8} \text{rk}(W') - h(W') \) doesn’t change if we tensorize \( W \) with copies of \( V_{Fermi} \) and so we can restrict us in the following theorem to self-dual SVOAs \( V \) with \( \text{dim} V_{1/2} = 0 \).

**Theorem 2** Let \( V \) be a self-dual unitary “very nice” rational SVOA \( V \) of rank \( c \) without nonzero elements of weight \( \frac{1}{2} \). Then one has the following:

i) The dimension of the weight 1 part \( V_1 \) is at least \( 2c(23\frac{1}{2} - c) \).

ii) The equality holds if and only if the minimal weight of the shadow \( V' \) is equal to \( \frac{c}{8} - 1 \).
iii) In this case the number of linear independent vectors in the shadow of weight $\frac{c}{8} - 1$ is $2^{|c| - 11}c$.

Proof: As in the proof of Theorem 1 we use the the polynomial $P_V(x, y)$. Assume first that the minimal weight of $V'$ is greater or equal then $\frac{c}{8} - 1$. Equation (3) shows that $P_V(x, y)$ is a linear combination of $x^{2c}$ and $x^{2c-16}y$. From the expansions

$$\chi_{1/2} = q^{-\frac{c}{8}}(1 + q^{\frac{c}{8}} + q^2 + q^2 + \cdots)$$

and (3) we find

$$\chi_V = x\chi_{1/2} - \frac{c}{8} x^{2c-16} \chi_{1/2}^{16} - \chi_8) + \frac{\dim V_1 - 2c(23^{\frac{1}{2}} - c)}{162} x^{2c-32} \chi_{1/2}^{16} - \chi_8)^2. \tag{6}$$

Equations (3), (3) and (4) give us then for the shadow

$$\dim V'_{\frac{c}{8}-1} = \alpha 2^{c-24} (\dim V_1 - 2c(23^{\frac{1}{2}} - c)).$$

Since $\dim V'_{\frac{c}{8}-1}$ is greater or equal than 24, this implies $\dim V_1 \geq 2c(23^{\frac{1}{2}} - c)$ and we have proved one direction of ii). For the converse we can use the the polynomial $P_V(x, y)$ and (3) to obtain

$$\chi_V/\alpha = (\sqrt{2} \chi'_{2c-16})^{2c-16} - \frac{c}{8} (\sqrt{2} \chi'_{2c-16})^{2c-16} \left(\frac{\sqrt{2} \chi'_{2c-16}}{162} \chi_{1/2}^{16} - \chi_8) \right)$$

$$= q^{-\frac{c}{8}} (\frac{c}{8} \cdot 2^{c-8} + q^2 + \cdots),$$

so indeed $\dim V'_{\frac{c}{8}-1} = 2^{|c| - 11} \cdot c$ which had to be shown.

To find all the SVOAs $V$ as in Theorem 1 with $h(V') = \frac{c}{8} - 1$ one needs the table of all self-dual SVOAs $V$ of rank $c$ smaller than 24. By the methods introduced in [HHS93], Chapter 3 one can compute the table and this work is in preparation [HHS98]. For $c < 16$ every $V$ with $\dim V_{1/2} = 0$ satisfies $h(V') = \frac{c}{8} - 1$ and the list of this SVOAs (making some additional assumptions) together with the characters of $V$ and $V'$ were already given in [HHS93] (Theorem 3.2.4 and Table 5.3); confirming the the values for $\dim V_1$ and $\dim V'_1$ as stated in Theorem 1. Using also [HHS93] we get the following list:

| $c$ | 8 | 12 | 14 | 15 | 15$\frac{1}{2}$ | 16 | 17 | 17$\frac{1}{2}$ | 18 | 18$\frac{1}{2}$ |
|-----|---|----|----|----|-------------|----|----|-------------|----|-------------|
| $\dim V_1$ | 248 | 276 | 266 | 255 | 248 | 240 | 221 | 210 | 198 | 185 |
| $V_1$ | $E_8$ | $D_{12}$ | $E_7^2$ | $A_{15}$ | $E_{8,2}$ | $D_3^2$ | $A_{11}E_6$ | $C_{10}$ | $D_6^2$ | $E_{7,2}F_4$ |

| $c$ | 19 | 19$\frac{1}{2}$ | 20 | 20$\frac{1}{2}$ | 21 | 21$\frac{1}{2}$ | 22 | 22$\frac{1}{2}$ | 23 | 23$\frac{1}{2}$ |
|-----|----|------|----|------|----|------|----|------|----|------|
| $\dim V_1$ | 171 | 156 | 140 | 123 | 105 | 86 | 66 | 45 | 23 | 0 |
| $V_1$ | $A_7^2D_5$ | $D_{8,2}B_4$ | $D_4^2$ | $A_{9,2}A_4$ | $A_3^2$ | $D_{4,2}C_3^2$ | $A_1^{22}$ | $A_{11}^{15}$ | $U_1^{23}$ | $0 \subset VB^2$ |

The first row gives the rank, the second the dimension of $V_1$ and the last row contains one example of a SVOA meeting the bound of Theorem 1, i) labelled by the sub-VOA $V_1$.
generated by $V_1$, this is the VOA associated to an integrable highest weight representation of the affine Lie algebra coming from the reductive Lie algebra $V_1$. The first index of a component denotes the rank of the simple Lie subalgebra, the second (if it exist) the level of the highest weight representation. This index is omitted if the level is 1.

As it is the case for lattices and codes the following holds for SVOAs: If there exist for a $c$ in the range $16 \leq c < 24$ a SVOA of rank $c$ with $\dim V_{1/2} = 0$ at all, then there exist such a SVOA meeting the bound $\dim V_1 = 2c(23^{1/2} - c)$. The first SVOA $V_{E_8}$ in the list is even a VOA. Especially interesting is the last one. The SVOA $VB^3$ of rank $23^{1/2}$ is the shorter moonshine module or Baby Monster SVOA and was constructed in [Höh95], Chapter 4. On $VB^3$ acts the group $2 \times B$ by automorphisms, where $B$ is the Baby Monster, the second largest sporadic simple group. It is conjectured that $VB^3$ is the unique self-dual unitary “very nice” rational SVOA with $\dim V_{1/2} = \dim V_1 = 0$.

3 Relations to codes and lattices

The analogs of Theorem 1 and 2 and of the above table for self-dual codes and lattices as given in [Elk95a, Elk95b] can easily be deduced from the results for SVOAs. Since the reduction from codes to lattices was described in [Elk95b] we show only the (analog) reduction from SVOAs to lattices. For the notation see also [Elk95b].

Corollary 3 Let $L$ be a self-dual lattice in $\mathbb{R}^n$. Then the analogs of Theorem 1 and 2 hold for $L$:

(1) The shortest characteristic vector of $L$ has norm $n$ if and only if $L \cong \mathbb{Z}^n$.

(2) Assume that $L$ has no vectors of norm 1. Then:

i) $L$ has at least $2n(23 - n)$ vectors of norm 2.

ii) Equality holds if and only if $L$ has no characteristic vectors of norm smaller then $n - 8$.

iii) In that case the number of characteristic vectors of length exactly $n - 8$ is $2^{n-11}n$.

Proof: To every integral lattice $L$ of rank $n$ one can construct a SVOA $V_L$ of rank $c = n$ whose irreducible modules $M_\lambda$ are indexed be the cosets $\lambda \in L^*/L$ of the dual lattice modulo the lattice. The character of $M_\lambda$ is determined by the theta series of $L + \lambda$:

$$\chi_{M_\lambda} = \left( \sum_{x \in L+\lambda} q^{\frac{1}{2}(x,x)} \right) \cdot \left( q^{1/24} \prod_{l=1}^{\infty} (1 - q^l) \right)^{-n}. \quad (7)$$

For self-dual lattices one has $V_L' = M_{L'}$ where $L' = L^* \setminus L$ is the shadow of $L$ and (7) shows that $h(V_L')$ is equal to $\frac{1}{8}$ times the minimal norm of a characteristic vector of $L$. Since $V_\mathbb{Z} \cong V_{\text{Fermi}}^{\otimes 2}$ and $V_K \otimes V_L \cong V_K \otimes V_L$ for two integral lattices $K$ and $L$ part (1) follows from Theorem 1.

For (2) we note first, that $\dim (V_L)_{1/2} = 0$ if and only if $L$ has no vectors of norm 1. From (7) we get that $\dim (V_L)_1$ is equal to $n$ plus the numbers of vectors in $L$ of norm 2 and $\dim (V_L')_{-1}$ is the number of characteristic vectors of $L$ of norm $(n - 8)$. Together with the identity $2n(23^{1/2} - n) = 2n(23 - n) + n$ part (2) of the corollary follows. \qed
A SVOA $V$ can only be come from a lattice by the above construction if its rank is integral and the affine Kac-Moody sub-VOA $\tilde{V}_1$ is a tensor product of level-1 representations of affine Kac-Moody algebras corresponding to the “simply laced” Lie algebras $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$. (One has $(\tilde{V}_L)_1 = V_{L_0}$, where $L_0$ is the root sublattice of $L$.) In fact every SVOA of Theorem 3 with $h(V') = \frac{c}{8} - 1$ and the above properties comes from a unique lattice, thus the corresponding list for lattices can be read off from the list of such SVOAs.

Remarks: One can define Kleinian codes as a natural fourth step before binary codes, lattices and vertex operator algebras and then again the analogous theorems hold [Höh96]. The analog of Theorem 1 for lattices was used in [Elk95a] as the final step in the proof of Donaldson famous theorem about positive definite intersection forms of closed differentiable simply connected 4-manifolds with the help of the Seiberg-Witten invariants. Theorem 1 relates quantum field theory in another way to the new proof of this theorem.

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