Local Nature of Coset Models

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Abstract

The local algebras of the maximal Coset model $C_{\text{max}}$ associated with a chiral conformal subtheory $\mathcal{A} \subset \mathcal{B}$ are shown to coincide with the local relative commutants of $\mathcal{A}$ in $\mathcal{B}$, provided $\mathcal{A}$ possesses a stress-energy tensor.

Making the same assumption, the adjoint action of the unique inner-implementing representation $U^\mathcal{A}$ associated with $\mathcal{A} \subset \mathcal{B}$ on the local observables in $\mathcal{B}$ is found to define net-endomorphisms of $\mathcal{B}$. This property is exploited for constructing from $\mathcal{B}$ a conformally covariant holographic image in $1 + 1$ dimensions which proves useful as a geometric picture for the joint inclusion $\mathcal{A} \vee C_{\text{max}} \subset \mathcal{B}$.

Immediate applications to the analysis of current subalgebras are given and the relation to normal canonical tensor product subfactors is clarified. A natural converse of Borchers’ theorem on half-sided translations is made accessible.

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1 Introduction

Structural and conceptual questions of quantum field theory are often addressed best within the framework of local quantum physics, where physics is described by assigning local algebras $\mathcal{B}(\mathcal{O})$ of observables to localisation regions $\mathcal{O}$ rather than in terms of quantum fields [Haa92]. In this picture it is natural to investigate the relative position of subnets $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ in the larger theory $\mathcal{B}$.

In $3 + 1$-dimensional spacetime this problem may be dealt with by means of the powerful reconstruction method of Doplicher and Roberts [DR90] and according to the results of Carpi and Conti [CC01] any subtheory satisfying certain assumptions is in a tensor product position in the larger theory. In lower dimensions, however, the situation is less restrictive and a lot of interesting examples are known.
In this article we study a class of inclusions of local quantum theories $\mathcal{A} \subset \mathcal{B}$, where $\mathcal{B}$ is given in its vacuum representation. When not all of the energy-content of $\mathcal{B}$ belongs to $\mathcal{A}$, there is space for other subtheories $\mathcal{C} \subset \mathcal{B}$ which commute with all of $\mathcal{A}$. We call such subtheories $\mathcal{C}$ Coset models (associated with $\mathcal{A} \subset \mathcal{B}$), and we want to derive typical features of these. By placing the problem into the setting of chiral conformal quantum field theory the large spacetime symmetry enables us to state and discuss the problems concerned clearly and rigorously.

Chiral conformal Coset models are studied for various reasons, in most cases connected to inclusions of chiral current algebras. These models exhibit a rich and yet tractable structure. One of the major achievements in this direction was the construction of the discrete series of Virasoro theories as Coset models by Goddard, Kent, and Olive [GKO86]. In the algebraic approach to quantum field theory much has been achieved for inclusions of local quantum theories generated by chiral current algebras and closely related structures [Lok94, Xu00a, Xu99, Xu01, Lon01, KL02].

Here, we want to broaden the perspective by using methods which do not make use of structures specifically connected to chiral current algebras, but which apply in a more general context: For chiral conformal quantum field theories the natural localisation regions are open, non-dense intervals $I$ in the circle. The local algebras $\mathcal{C}(I)$ of a Coset model associated with a subnet $\mathcal{A} \subset \mathcal{B}$ are contained in the local relative commutants $\mathcal{C}_I := \mathcal{A}(I)' \cap \mathcal{B}(I)$. Actually, if the local relative commutants fulfill isotony, ie the local relative commutants $\mathcal{C}_I$ increase with $I$, the $\mathcal{C}_I$ define a Coset model themselves which is obviously maximal.

Isotony for the $\mathcal{C}_I$ holds for chiral current subalgebras because of the strong additivity property of these models [LL97] (corollary IV.1.3.3): If $I_1, I_2$ arise from $I$ by removing a point in its interior, the local algebra $\mathcal{A}(I)$ is generated by its subalgebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$. This property is absent in many chiral conformal models [BSM90, Yng94] and, naturally, the question arises, under which circumstances the equality of $\mathcal{C}_I$ and $\mathcal{C}_{\text{max}}(I)$ can be proven. Our intention is to find more general conditions which secure this equality for various reasons and we refer to this task as the isotony problem.

In a recent work [Kos02] we constructed for any chiral conformal subtheory $\mathcal{A} \subset \mathcal{B}$ a globally $\mathcal{A}$-inner representation $U^A$ which implements the (global) chiral conformal transformations on $\mathcal{A}$. The result provides a factorisation of $U$, the implementation of chiral conformal symmetry in the vacuum representation of $\mathcal{B}$, into two commuting representations, $U^A$ and $U^{A'}$, which share with $U$ the properties of leaving the vacuum invariant and of positivity of energy. It was proved, by a simple argument, that the local operators in $\mathcal{B}$ which commute with $U^A$ form the maximal Coset model $\mathcal{C}_{\text{max}}$ associated with every particular inclusion $\mathcal{A} \subset \mathcal{B}$. $\mathcal{C}_{\text{max}}$ can be non-trivial, if we have $U^{A'} \neq 1$, ie if not all of the energy-content of $\mathcal{B}$ belongs to $\mathcal{A}$.

We would like to have a simple and applicable characterisation of local operators in $\mathcal{B}$ which belong to a Coset model associated with a subtheory $\mathcal{A}$, and we
want this characterisation to involve only local data according to the conviction that all observation is of finite extension and of finite duration. Of course, it is in principle possible to make this decision simply by taking all operators from $C_I$ and discarding all operators which do not commute with all operators belonging to an algebra $\mathcal{A}(J), J$ slightly enlarged. But chiral conformal quantum field theories usually behave well when $J$ tends to $I$, and we are led to the conjecture that the local algebras of the maximal COSET model and the corresponding local relative commutants should coincide in very general circumstances. As it stands for the moment, the maximal COSET model is determined by global data, the inner-implementing representation $U^A$, and establishing the equality $C_I = C_{\text{max}}(I)$ would prove that the COSET model is of a local nature, its local operators being singled out by a simple algebraic relation only involving local data associated with the very same localisation region.

For dealing with the isotony problem in our context\(^1\), we look at the action of $\text{Ad}_{U^A}$ on the local observables of $\mathcal{B}$. Because the construction of $U^A$ does not refer to the local structure of $\mathcal{A}$ at all, we need some information on the way this representation is generated by local observables. In chiral conformal field theory it is natural to assume that the inner implementing representation is generated by integrals of a stress-energy tensor affiliated with $\mathcal{A}$. This assumption does not imply strong additivity [BSM90] and concerning the models known today (at least to the author) is more general, since all strongly additive models contain a stress-energy tensor.

Because of the special features stress-energy tensors of chiral (and 1+1-dimensional) conformal field theory have according to the Lüscher-Mack theorem [FST89], solving the isotony problem proves possible, but the presence of a stress-energy tensor does not trivialise it at all. In fact, one is led to pinpoint the problem very much using arguments independent of the additional assumption, before the stress-energy tensor actually is needed to prove two crucial, but natural lemmas. Our discussion should, therefore, serve well as a setup for further generalisations.

Even for current subalgebras, which always contain a stress-energy tensor by the Sugawara construction, the action of a stress-energy tensor $\Theta^A$ of a current subalgebra on general currents in the larger current algebra $\mathcal{B}$ has not been studied as such, yet. Only in connection with the classification of conformal inclusions, ie the case that the stress-energy tensor $\Theta^\mathcal{B}$ coincides with that of $\mathcal{A}$ [SW86, AGOS7, BB87], this action has been object of research. The new perspective of analysing the action of $U^A$ on $\mathcal{B}$ (in this context: of $\Theta^A$ on $\mathcal{B}$) directly has led to a simple and natural characterisation of conformal inclusions by methods familiar in (axiomatic) quantum field theory [Kö83a].

\(^1\)Apparently, Carpi and Conti encountered the same problem while generalising their analysis [CC01] to general field algebras and solved it by methods quite different from the ones applied here [CC].
As mentioned above, the local nature of maximal Coset models associated with current subalgebras $\mathcal{A} \subset \mathcal{B}$ is clear because of the strong additivity property of $\mathcal{A}$. If the embedding $\mathcal{A} \vee \mathcal{C}_{\text{max}} \subset \mathcal{B}$ is known to be of finite index, then $\mathcal{C}_{\text{max}}$ inherits the strong additivity property from $\mathcal{A}$ and $\mathcal{B}$ by results of Longo [Lon01]. According to Xu [Xu00a], a large number of current algebra inclusions are known to satisfy this condition (cofinite inclusions $\mathcal{A} \subset \mathcal{B}$), but for a lot of others the situation has not been clarified, yet. If we now look at the embedding $\mathcal{C}_{\text{max}} \subset \mathcal{B}$ and consider the (“iterated”) Coset models associated with this inclusion, we arrive at another isotony problem. In case $\mathcal{C}_{\text{max}}$ is strongly additive as well, the local relative commutants of $\mathcal{C}_{\text{max}}$ and the local algebras of $\mathcal{C}_{\text{max}}$ form a pair of subnets which locally are their respective relative commutants. Inclusions of this type are of particular interest and Rehren called them a normal pair of subnets [Reh00].

Our analysis applies to the maximal Coset models associated with current subalgebras as these always contain the Coset stress-energy tensor $\Theta^B - \Theta^A$. This way we extend the finding on normal pairs for cofinite current subalgebras to all inclusions $\mathcal{A} \subset \mathcal{B}$ where both $\mathcal{B}$ and $\mathcal{A}$ contain a stress-energy tensor, independent of strong additivity or the index of the inclusion $\mathcal{A} \vee \mathcal{C}_{\text{max}} \subset \mathcal{B}$.

In the next section we first state our general assumptions and conventions and then discuss the “geometric impact” of $U^A$ on $\mathcal{B}$. Intuitively, we do not expect an observable of $\mathcal{B}$ to be more sensitive to the action of $\text{Ad}_{U^A}$ than to that of $\text{Ad}_{U^B}$: the generator of translations, $P$, is known to decompose into two commuting positive parts, $P = P^A + P^A'$, and regarding them as chiral analogues of Hamiltonians leads us to the expectation that $P^A$ should not transport observables of $\mathcal{B}$ “faster” than $P$ itself. A typical local observable $\mathcal{B}$ in $\mathcal{B}$ should exhibit a behaviour interpolating between invariance ($\mathcal{B}$ in $\mathcal{C}_{\text{max}}$) and covariance ($\mathcal{B}$ in $\mathcal{A}_{\text{max}}$).

For this behaviour to be ensured we have, as it turns out, only to show that scale transformations represented through $U^A$ respect the two fixed points of scale transformations, namely 0 and $\infty$, when acting on $\mathcal{B}$. We can prove this to be the case in presence of a stress-energy tensor and it seems natural in any case. The sub-geometrical transformation behaviour for translations, which we expect, then follows by results of Borchers [Bor97a, Bor97b] using the spectrum condition and modular theory. We collected, rearranged and reformulated results of Borchers and Wiesbrock in order to provide a natural converse of Borchers’ theorem on half-sided translations, which was not yet available in the literature. By extending the analysis to general conformal transformations we arrive at the notion of net-endomorphism property for the action of $U^A$ on $\mathcal{B}$.

In the third section we use the net-endomorphic action of $U^A$ to construct from the chiral conformal theory $\mathcal{B}$ a conformal net in 1+1 dimensions which contains the chiral algebras as time-zero algebras. The result satisfies all axioms of a 1+1-dimensional conformal quantum theory, except that its translations in spacelike directions to the right have positive spectrum rather than in future-like directions. While this prohibits interpreting the picture of chiral holography as
(completely) physically sensible, it gives a satisfactory geometric interpretation to the net-endomorphisms induced by $U^A$ and it provides a rather helpful geometrical framework of a quasi-theory in $1+1$ dimensions. The subnet $\mathcal{A} \subset \mathcal{B}$ and its coset models appear as subtheories of chiral observables and thus we make connection with results of REHREN [Reh00], which have interesting consequences for known examples.

In the closing section we provide our solution to the isotony problem (main theorem 12), i.e., we establish the local nature of the maximal coset model. We start by giving a new characterisation of $C_{\text{max}}$ making use of the particular structure of the group of chiral conformal transformations. And then, again, the presence of a stress-energy tensor for $\mathcal{A}$ is only needed in order to establish a rather natural, but crucial lemma on the representation of scale transformations through $U^A$. At the very end we discuss possible generalisations to models having no stress-energy tensor and to subtheories in other spacetimes. The appendix contains background on our additional assumption on the inner-implementing representation $U^A$, while we will use an abstract formulation of it in the main sections, and a simple, technical lemma on scale transformations as elements of the group of orientation preserving diffeomorphism on the circle, $\text{Diff}_+(S^1)$.

## 2 Net-endomorphism property

The fundamental object of this study is an inclusion of a chiral conformal theory, $\mathcal{A}$, in another chiral conformal theory, $\mathcal{B}$. The theory $\mathcal{B}$ shall be given in its vacuum representation, of which we summarise the general assumptions and some of its properties (cf. [GL96, GF93] and references therein), and we describe the embedding of $\mathcal{A}$ in $\mathcal{B}$ in this setting.

The localisation regions for chiral conformal theories are taken to be the proper intervals contained in the unit circle $S^1$, which is to be regarded as the conformal compactification of a (chiral) light-ray; the point $+1$ on $S^1$ corresponds to the point 0 on the light-ray and $-1 \in S^1$ corresponds to $\infty$. A proper interval $I$ is an open, connected subset of $S^1$ which has a causal (open) complement, $I' := \{S^1 \setminus I\}^\circ \neq \emptyset$. The inclusion of such a proper interval $I$ in the unit circle will be denoted as $I \subset S^1$.

The vacuum representation of $\mathcal{B}$ is given by a map from the set of proper intervals to v. NEUMANN algebras of bounded operators on a separable HILBERT space $\mathcal{H}$ satisfying isotony, i.e., for $I_1 \subset I_2 \subset S^1$ we have $\mathcal{B}(I_1) \subset \mathcal{B}(I_2)$, and locality, that is: If $I_1 \subset I_2'$, then $\mathcal{B}(I_1)$ is contained in $\mathcal{B}(I_2)'$, the commutant of $\mathcal{B}(I_2)$. It is required as well that there is a unitary, strongly continuous representation $U$ of the group of global, chiral, conformal transformations, $\text{PSL}(2, \mathbb{R})$, which satisfies the following: the generator of translations has positive spectrum (positivity of energy), $U$ implements the corresponding symmetry of $\mathcal{B}$, i.e., for $g \in \text{PSL}(2, \mathbb{R})$ the adjoint action of $U(g)$ on local algebras of $\mathcal{B}$ defines an isomorphism $\alpha_g$ from
any $\mathcal{B}(I)$ onto the corresponding $\mathcal{B}(gI)$, and, finally, $U$ has to contain the trivial representation exactly once. We choose a vector $\Omega$, the vacuum, of length 1 in the corresponding representation space. $\Omega$ has to be cyclic for $\mathcal{B}$, which, by the Reeh-Schlieder theorem, amounts to demanding $\mathcal{B}(I)\Omega$ to be dense in $\mathcal{H}$ for all $I \in S^1$.

A chiral conformal subtheory $\mathcal{A}$ embedded in $\mathcal{B}$, written as $\mathcal{A} \subset \mathcal{B}$, is given by a map from the set of proper intervals to local v.Neumann algebras, $S^1 \ni I \mapsto \mathcal{A}(I)$, with the following properties:

- **Inclusion:** $\mathcal{A}(I) \subset \mathcal{B}(I)$ for $I \in S^1$.
- **Isotony:** If $I_1 \subset I_2$, then $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.
- **Covariance:** For all $g \in \text{PSL}(2, \mathbb{R})$ and $I \in S^1$ we have: $\mathcal{A}(gI) = \alpha_g(\mathcal{A}(I))$.

These assumptions have a lot of interesting consequences of which we only name a few directly involved in this work. For instance, $\mathcal{B}$ has the Bisognano-Wichmann property, ie the modular data of the local algebra assigned to the upper half circle have a direct geometrical interpretation. If the action of scale transformations on the (chiral) light-ray, which we identify with $\mathbb{R}$, reads $D(t) : x \mapsto e^{t}x$, $x \in \mathbb{R}$, and the modular group of $\mathcal{B}(S^1_+)$ is given by $\Delta^{it}$, then we have $U(D(-2\pi t)) = \Delta^{it}$. Furthermore, the modular conjugation of $\mathcal{B}(S^1_+)$, denoted by $J$, implements the reflection $x \mapsto -x$. By covariance, this means in particular: the vacuum representation of $\mathcal{B}$ satisfies Haag duality (on the circle), namely we have $\mathcal{B}(I)' = \mathcal{B}(I)$, $I \in S^1$.

The local algebras $\mathcal{B}(I)$, $I \in S^1$, are continuous from the inside as well as from the outside, that is: $\mathcal{B}(I)$ coincides with the intersection of all local algebras assigned to proper intervals $J$ containing $I$ and is generated by all its local subalgebras (assigned to proper intervals $J$ with $J \subset I$), respectively. Continuity from the inside implies weak additivity, ie $\mathcal{B}(I)$ is generated by the subalgebras $\mathcal{B}(J_i)$ for each covering $\bigcup J_i = I$ \cite{F.J96}.

The vacuum representation of $\mathcal{A}$ is contained in the representation induced by the embedding $\mathcal{A} \subset \mathcal{B}$. In fact, the local inclusions $\mathcal{A}(I) \subset \mathcal{B}(I)$, $I \in S^1$, define a (quantum field theoretical) net of subfactors in the sense of LONGO and REHREN \cite{LR95}. By the Reeh-Schlieder theorem, the projection $e_{\mathcal{A}}$ onto the Hilbert space resulting from the closure of $\mathcal{A}(I)\Omega$, $I \in S^1$, does not depend on $I$ and, because the local subalgebras $\mathcal{A}(I) \subset \mathcal{B}(I)$ are modular covariant by the Bisognano-Wichmann property of $\mathcal{B}$ and conformal covariance of $\mathcal{A} \subset \mathcal{B}$, it follows \cite{Lak72, Jon33} that for every $I \in S^1$ we have: $\mathcal{A}(I) = \{e_{\mathcal{A}}\}' \cap \mathcal{B}(I)$. For a general summary on modular covariant subalgebras see eg \cite{Bor97a}.

We denote the v.Neumann algebra which is generated by all local algebras $\mathcal{A}(I)$, $I \in S^1$, by $\mathcal{A}$ as well (with a slight abuse of notation); this algebra contains all local observables of the theory $\mathcal{A}$ and all global observables associated with the subtheory $\mathcal{A} \subset \mathcal{B}$, that is all bounded operators which are weak limits of
local observables of the subtheory but which are not local themselves. By the Borchers–Sugawara construction \([\text{Kôs02]}\) there is a unique representation \(U^A\) of \(\text{PSL}(2, \mathbb{R})^-\), the universal covering group of \(\text{PSL}(2, \mathbb{R})\), which consists of global observables of \(A\) and implements conformal covariance on \(A\) by its adjoint action. Namely, the representation \(U\) factorises as

\[
U \circ p(g) = U^A(g)U^A'(g),
\]

where \(p\) denotes the covering projection from \(\text{PSL}(2, \mathbb{R})^-\) onto \(\text{PSL}(2, \mathbb{R})^-\) and \(U^A'\) is another representation of \(\text{PSL}(2, \mathbb{R})^-\) by unitaries in \(A'\).

In the following there will appear frequently the subgroups of translations,

\[
T(a)x = x + a, \quad x, a \in \mathbb{R},
\]

and of special conformal transformations,

\[
S(n)x = \frac{x}{1 + nx}, \quad x, n \in \mathbb{R}.
\]

Both groups are inverse-conjugate in \(\text{PSL}(2, \mathbb{R})\), ie \(T(a)\) is conjugate to \(S(-a)\), and the same holds true for their images in \(\text{PSL}(2, \mathbb{R})^-\), which we will denote by \(\tilde{T}\) and \(\tilde{S}\), respectively. Furthermore, we have the groups of rigid conformal rotations, denoted by \(R\) and \(\tilde{R}\), respectively, and of dilatations (scale transformations), \(D, \tilde{D}\). We adopt the physicists’ convention on the Lie algebra and use the same symbols for elements of the Lie algebra and their representatives as elements of an infinitesimal representation by (essentially) self-adjoint operators. We use parameters on the three subgroups mentioned so far which make the subgroup \(R\) of rotations naturally isomorphic to \(\mathbb{R}/2\pi\mathbb{Z}\) and yield the following relation between the generator of translations, \(P\), the generator of special conformal transformations, \(K\), and the generator of rigid conformal rotations, the conformal Hamiltonian \(L_0\): \(2L_0 = P - K\). \(P\) is a positive operator iff \(L_0\) is positive or, equivalently, iff \(-K\) is positive (eg \([\text{Kôs02}], \text{prop.1}\)\). \(U^A\) and \(U^A'\) both are of positive energy since \(U\) is. Furthermore, both representations leave the vacuum invariant \([\text{Kôs02}]\) (corollary. 6 and 7).

In the following we deduce, step by step, the sub-geometric character of the adjoint action of \(U^A\) (and of \(U^A'\)) on \(B\). The analysis relies on a single property of the dilatations in \(U^A\). The notion of net-endomorphisms arises naturally in the course of the argument and will be discussed at the end of this section. We, therefore, define:

**Definition 1** \(U^A\) is said to have the net-endomorphism property, if the adjoint action of \(U^A(\tilde{D}(t))\), \(t \in \mathbb{R}\), defines a group of automorphisms of \(B(S^1_+)\).

This property holds making the following

**Additional Assumption 1** There is a unitary, strongly continuous, projective representation \(\Upsilon^A\) of the universal covering group of orientation preserving diffeomorphisms of the circle, \(\text{Diff}_+(S^1)^-\), on \(\mathcal{H}\) such that:

- If a diffeomorphism \(\varphi \in \text{Diff}_+(S^1)\) is localised in \(I \in S^1\), ie \(\varphi \upharpoonright I' = \text{id} \upharpoonright I'\), it is represented by a local observable of \(A\), namely: \(\Upsilon^A(p^{-1}(\varphi)) \in A(I)\).

- \(\Upsilon^A(\tilde{D}(t))U^A(\tilde{D}(t))^* \in \mathbb{C}1\) for all \(t \in \mathbb{R}\).
Here, the covering projection from $\text{Diff}_+(S^1)$ onto $\text{Diff}_+(S^1)$ is denoted by $p$. Localised diffeomorphisms $\varphi$ are identified with their preimage $p^{-1}(\varphi)$ in the first sheet of the covering.

The Additional Assumption only enters through the lemmas 2 and 11, which we believe to hold true in a lot more general circumstances. It can be verified in presence of an integrable stress-energy tensor for $\mathcal{A}$ (see discussion in appendix). In this case the representations $\Upsilon^{\mathcal{A}} \upharpoonright \text{PSL}(2, \mathbb{R})$ and $U^A$ coincide, whereas we have only assumed that the respective generators agree up to a multiple of $\mathbb{I}$. At this point we want to stress: We do not assume $\mathcal{A}$ to be diffeomorphism covariant, i.e. the adjoint action of $\Upsilon^{\mathcal{A}}$ on $\mathcal{A}$ to implement a geometric, automorphic action of $\text{Diff}_+(S^1)$ on $\mathcal{A}$.

**Lemma 2** $U^A$ has the net-endomorphism property, if the Additional Assumption holds.

**Proof:** By lemma 13 there exist, for small $t \in \mathbb{R}$, diffeomorphisms $g_\varepsilon$, $g_\delta$ localised in arbitrarily small neighbourhoods of $-1$ and $1$, respectively, and diffeomorphisms $g_+$, $g_-$ localised in $S^1_+$ and $S^1_-$, respectively, such that we have: $D(t) = g_+g_-g_\delta g_\varepsilon$. If the closure of a proper interval $I$ is contained in $S^1_+$, we have with an appropriate choice of $g_\delta$, $g_\varepsilon$ by the Additional Assumption:

$$U^A(\tilde{D}(t))\mathcal{B}(I)U^A(\tilde{D}(t))^* = \Upsilon^{\mathcal{A}}(p^{-1}(g_+))\mathcal{B}(I)\Upsilon^{\mathcal{A}}(p^{-1}(g_-))^* \subset \mathcal{B}(S^1_+) \quad (1)$$

Because $\mathcal{B}(S^1_+)$ is continuous from the inside, we see that $\text{Ad}_{U^A(\tilde{D}(0))}$ induces an endomorphism of $\mathcal{B}(S^1_+)$. The same holds true for $U^A(\tilde{D}(-t))$ and, therefore, these endomorphisms are automorphisms.

□

The next step is to give a natural characterisation of one-parameter groups of unitary operators which define, by their adjoint action, endomorphism semigroups of a standard v.Neumann algebra. The following theorem is a mainly a new formulation of results by Borchers and Wiesbrock. Its present form is new and appears to be a natural converse of Borchers’ theorem on half-sided translations. The methods of proof are completely standard, but the result ought to be made available.

**Theorem 3** Assume $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ to be a v.Neumann algebra having a cyclic and separating vector $\Omega$ in the separable Hilbert space $\mathcal{H}$. $J, \Delta$ shall stand for the modular data of this pair. Let $V(t)$, $t \in \mathbb{R}$, be a strongly continuous one-parameter group. Then any two from \{a), b), c)\} imply the remaining two in the list below: d) yields a), b), c).

\begin{align*}
a) & \quad \text{i. } V(s) = e^{iHs}, H \geq 0, \\
\end{align*}

\footnote{Compare \cite{Dav96} for another characterisation of endomorphism semigroups related to Borchers’ theorem.}
\[ V(s)\mathcal{M}V(s)^* \subset \mathcal{M}, \ s \geq 0. \]

**b)**

i. \( V(s)\Omega = \Omega, \ s \in \mathbb{R} \),

ii. \( V(s)\mathcal{M}V(s)^* \subset \mathcal{M}, \ s \geq 0. \)

**c)**

i. \( \Delta^itV(s)\Delta^{-it} = V(e^{-2\pi t}s), \ JV(s)J = V(-s), \ t, s \in \mathbb{R} \),

ii. \( V(s)\mathcal{M}V(s)^* \subset \mathcal{M}, \ s \geq 0. \)

**d)**

i. \( V(s) = e^{iHs}, \ H \geq 0, \)

ii. \( \Delta^itV(s)\Delta^{-it} = V(e^{-2\pi t}s), \ t, s \in \mathbb{R} \),

iii. \( \langle m'_+\Omega, V(s)m_+\Omega \rangle \geq 0, \ s \geq 0, \ m_+ \in \mathcal{M}_+, \ m'_+ \in \mathcal{M}'_+. \)

\( \mathcal{M}_+ \) denotes the cone of positive elements in \( \mathcal{M} \), \( \mathcal{M}'_+ \) the cone of positive elements in its commutant \( \mathcal{M}' \).

**Proof:** Most of the implications were proved by Borchers and Wiesbrock, respectively: a) \( \land b) \implies c) \) \cite{Bor92} (cf \cite{Flo98}); b) \( \land c) \implies a) \) \cite{Wie92}; a) \( \land b) \implies c) \) \cite{Bor98}; a) \( \land b) \land c) \implies d) \) \cite{BR87, proposition 2.5.27}.

We prove the remaining statement, namely d) \( \implies a) \land b) \land c) \), by reduction to \cite{Bor97a, theorem 1.1} \footnote{Alternatively, one may use the same statement in \cite{Bor97b, theorem 2.5].}. As a first step we look at the domain of entire analytic vectors with respect to \( \Delta^{iz} \), which we denote by \( D_{\Delta}^{iz} \), and derive an analytic continuation of relation d)ii. as a quadratic form on \( D_{\Delta}^{iz} \). We define:

\[
F(z, w) := \langle \Delta^{iz}\psi, e^{ie^{2\pi wH}\Delta^{iz}\phi} \rangle .
\]

According to the spectrum condition on \( H \), \( F \) is analytic in \( w \) for \( 0 < \text{Im}(w) < \frac{1}{2} \), and this function is bounded and continuous for the closure of this region; the region itself shall be denoted by \( \mathbb{S} \). In fact, by Hartog’s theorem, \( F \) is analytic on \( \mathbb{C} \times \mathbb{S} \) as a function in two complex variables. We make full use of relation d)ii. by looking at another function \( G \), which agrees with \( F \) for \( 0 < \text{Im}(w) + \text{Im}(z) < \frac{1}{2} \):

\[
G(z, w) := \langle \psi, e^{ie^{2\pi (w+z)H}\phi} \rangle .
\]

Evaluating at \( w \in \mathbb{R} \) and \( z = \frac{i}{2} \) we get:

\[
\langle \Delta^{\frac{i}{2}}\psi, e^{ie^{2\pi wH}\Delta^{-\frac{i}{2}}\phi} \rangle = \langle \psi, e^{-e^{2\pi wH}\phi} \rangle . \tag{2}
\]

Both \( \psi, \phi \) are of the form \( \psi = \Delta^{-\frac{i}{2}}\psi', \phi = \Delta^{\frac{i}{2}}\phi', \psi', \phi' \in D_{\Delta} \). Since the set of such \( \psi', \phi' \) is dense in \( \mathcal{H} \), the equation above becomes an equation for bounded operators, which yields:

\[
e^{isH} = \Delta^{-\frac{i}{2}}e^{-sH}\Delta^{\frac{i}{2}}, \quad s \geq 0 . \tag{3}
\]
Next, we show invariance of $\Omega$ following arguments from the proof of \cite[lemma 2.3.c]{Bor98}: let $E$ be the projection onto the eigenvectors of $\Delta$ having eigenvalue 1. Multiplying the identity (3) from both sides by $E$ leads to:

$$E e^{isH} E = E e^{-sH} E, \quad s \geq 0.$$  

Here, the right hand side is a positive operator and thus we have as well:

$$(E e^{isH} E)^* = E e^{-isH} E = E e^{-sH} E = E e^{isH} E, \quad s \geq 0.$$  

According to a standard argument\footnote{Such an argument is given, for example, in \cite{Kost02} (proof of corollary 7) and uses the spectrum condition, the Phragmen-Lindelöf theorem, Schwarz’ reflection principle and Liouville’s theorem.}, this invariance with respect to conjugation yields: $E e^{isH} E = E e^{i0H} E = E$. Therefore, all vectors $\xi$ satisfying $\xi = E \xi$ are invariant under the action of $V$ and this means in particular: $V(s)\Omega = \Omega, \forall s \in \mathbb{R}$.

It now follows from \textbf{d)iii.} and \cite[proposition 2.5.28]{BR87} that $e^{-sH}, s \geq 0$, leaves the natural cone of $(M, \Omega)$ globally fixed. The other assumptions of \cite[theorem 1.1]{Bor97a} are the identities:

$$\Delta^it e^{-Hs} \Delta^{-it} = e^{-se^{-2\pi t}H}, \quad s \geq 0,$$

$$e^{-Hs}\Omega = \Omega, \quad s \geq 0.$$  

These relations are obvious by analytic continuation of results derived above. By \cite[theorem 1.1]{Bor97a} the adjoint action of $V(s), s \geq 0$, does indeed induce endomorphisms of $M$ and we have completed the proof.

\[ \square \]

The arguments in the proof of theorem 3 apply, with minor alterations, to translation groups with negative generator, as eg the special conformal transformations $U(S(n))$. While $J$ has the same action, $JU(S(n))J = U(S(-n))$, the scaling behaviour is opposite:

$$\Delta^it U(S(n)) \Delta^{-it} = U(S(e^{2\pi t} n)) .$$  

(4)

The negative spectrum together with the opposite scaling law shows that the condition characterising endomorphism semi-groups is just the same as in condition \textbf{d)iii.}. Since the arguments are completely analogous as for the case of positive spectrum and scaling law \textbf{c)i., d)ii.} we state the following corollary without proof:

**Corollary 4** The statements in theorem 3 still hold, if one replaces a)i., d)i. by $V(s) = e^{isK}, K \leq 0, and uses $\Delta^it V(s) \Delta^{-it} = V(e^{2\pi t} s), s, t \in \mathbb{R}$, instead of c)i., d)ii.$
Corollary 5  Assume $U^A$ to have the net-endomorphism property. Then the adjoint action of $U^A(\tilde{D}(\cdot))$ on $\mathcal{B}(S^1_\gamma)$ defines a group of automorphisms.

For $s \geq 0$ the adjoint action of $U^A(\tilde{T}(s))$ induces endomorphisms of $\mathcal{B}(S^1_\gamma)$ and the adjoint action of $U^A(\tilde{T}(-s))$ maps $\mathcal{B}(S^1_\gamma)$ into $\mathcal{B}(T(-s)S^1_\gamma)$. The corresponding statements hold true, if one replaces $\mathcal{A}$ by $\mathcal{A}'$ or $\tilde{T}(\cdot)$ by $\tilde{S}(\cdot)$.

Proof: The statement on $Ad_{U^A(\tilde{D}(\cdot))}$ follows from $U^A = U^\mathcal{A}pU^{\mathcal{A}^*}$ and covariance of $\mathcal{B}$. Using the factorisation of $U(T(s)) = U^A(\tilde{T}(s))U^{A'}(\tilde{T}(s))$, covariance and isotony of $\mathcal{B}$, the statement on $Ad_{U^A(\tilde{D}(\cdot))}$ and invariance of $\Omega$ with respect to $U^A$, we have the following inequality for all $t \in \mathbb{R}$, $s \geq 0$, $B_+ \in \mathcal{B}(S^1_\gamma)$, $B'_+ \in \mathcal{B}(S^1_\gamma)$:

$$0 \leq \langle U^A(\tilde{D}(t))^*B'_+U^{A'}(\tilde{D}(t))\Omega, U(T(s))U^A(\tilde{D}(t))^*B_+U^{A'}(\tilde{D}(t))\Omega \rangle = \langle B'_+\Omega, U^A(\tilde{T}(s))U^{A'}(\tilde{T}(e^ts))B_+\Omega \rangle .$$

In the limit $t \to -\infty$ strong continuity of $U^{A'}$ implies $\langle B'_+\Omega, U^A(\tilde{T}(s))B_+\Omega \rangle \geq 0$, which in turn yields the statement on $U^A(\tilde{T}(s))$, $s \geq 0$, by theorem 3. Following the same argument with $\mathcal{A}$ instead of $\mathcal{A}'$ and vice versa leads to the corresponding statement on $U^{A'}(\tilde{T}(s))$, $s \geq 0$. If one replaces in both statements $\tilde{T}(s)$ by $\tilde{S}(s)$, one may apply the argument as well, but using the limit $t \to \infty$ and corollary 4.

The remainder follows immediately from the following argument, which we indicate for the translations represented through $U^A$:

$$Ad_{U^A(\tilde{T}(-s))}B(S^1_\gamma) = Ad_{U(T(-s))}Ad_{U^{A'}(\tilde{T}(s))}B(S^1_\gamma) \subset B(T(-s)S^1_\gamma) .$$

The geometric impact of a general $U^A(\tilde{g})$, $\tilde{g} \in PSL(2,\mathbb{R})^\sim$, on an arbitrary local algebra $\mathcal{B}(I)$ is discussed easily. We may restrict our attention to group elements $\tilde{g}$ for which there is a single sheet of the covering projection $p$ containing both $\tilde{g}$ and the identity, as the following discussion indicates.

Every element $g$ in $PSL(2,\mathbb{R})$ is contained in (at least) one one-parameter group $^5 \text{Mos94 Mos97}$. We use the local identification of one-parameter subgroups in $PSL(2,\mathbb{R})$ and in $PSL(2,\mathbb{R})^\sim$, choose a parametrisation such that $\tilde{g} = \tilde{g}(1)$, $id = \tilde{g}(0)$, and we set $\gamma_{\tilde{g}}(I) := \bigcup_{r=0}^1 p(\tilde{g}(\tau))I$. For $\tilde{g}$ further away from the identity we set $\gamma_{\tilde{g}}(I) = S^1$ and take $\mathcal{B}(S^1)$ to be the algebra of all bounded operators on $\mathcal{H}$. Then we have:

Proposition 6  Assume $U^A$ to have the net-endomorphism property. Then we have for any $\tilde{g} \in PSL(2,\mathbb{R})^\sim$ and any $I \in S^1$: $Ad_{U^{A}(\tilde{g})}B(I) \subset B(\gamma_{\tilde{g}}(I))$, and $Ad_{U^{A'}(\tilde{g})}B(I) \subset B(\gamma_{\tilde{g}}(I))$.

\textsuperscript{5}I am indebted to D. Guido for providing the reference. In the particular case of $PSL(2,\mathbb{R})$ this fact may be checked directly (cf \textsuperscript{Kos03}).
Proof: Each proper interval $I$ in $S^1$ may be identified by the ordered pair consisting of its boundary points, $z_+$ and $z_-$. We define three one-parameter subgroups in $PSL(2,\mathbb{R})$ referring to each $I \in S^1$ with respect to a particular choice $h \in PSL(2,\mathbb{R})$ satisfying $hS^1_+ = I$: $D_I(.) = hD(.)h^{-1}$, $T_I(.) = hT(.)h^{-1}$, $S_I(.) = hS(.)h^{-1}$.

Each element $g$ in $PSL(2,\mathbb{R})$ is fixed, up to a dilatation $D_I(t)$, by its action on $\{z_+, z_-\}$. Under the action of elements $g(\tau)$, $\tau = 0, \ldots, 1$, interpolating in the one-parameter group associated with $g$ between the identity and $g = g(1)$, the orbits of $z_\pm$ are given by monotonous functions $z_+(\tau)$, $z_-(\tau)$. Demanding $s$, $n$, $t$ to depend continuously on $\tau$ and to take value 0 at $\tau = 0$, every $g(\tau)$ may be represented as $g(\tau) = S_{T_I(n(\tau))}I(n(\tau))T_I(s(\tau))D_I(t(\tau))$ or as $g(\tau) = T_{S_I(n(\tau))}I(s(\tau))S_I(n(\tau))D_I(t(\tau))$. We choose one form which works for all interpolating elements. By the requirements we have made it is ensured that the representation works (after obvious identifications) in $PSL(2,\mathbb{R})^\sim$ as well. Corollary 5 implies the claim of the proposition now.

□

This proves in particular: For every $I \in S^1$ there is a neighbourhood of the identity in $PSL(2,\mathbb{R})^\sim$ for which the action of $Ad_{U_A(.)}$ on $B(I)$ delivers local observables.

We have found $Ad_{U_A}$ to induce homomorphisms from local algebras of $B$ into algebras associated with an enlarged localisation region. This sub-geometrical action respects isotony, ie the net-structure. The adjoint action of $U$ induces the covariance isomorphisms of local algebras and one usually regards these as automorphisms of the net $B$. We consider, therefore, the term net-endomorphisms appropriate. The automorphic action of $Ad_{U_{A'(D(.))}}$ on $B(S^1_1)$ which we proved in lemma 2 does not, apparently, follow from the endomorphism property for the translation subgroups in corollary 3. This motivated definition 4 above.

In the next section we give a holographic interpretation of the net-endomorphism property. This shows that the results achieved so far are satisfactory and yield an interesting and useful new insight into structures associated with chiral conformal subnets and their Coset models.

3 Chiral holography

The mapping $(\tilde{g}, \tilde{h}) \mapsto U^A(\tilde{g})U^{A'}(\tilde{h})$ defines a representation $U^A \times U^{A'}$ of the group $PSL(2,\mathbb{R})^\sim \times PSL(2,\mathbb{R})^\sim$. This is, in fact, a representation of the conformal symmetry group of a local conformal quantum theory in 1+1 dimensions, which is isomorphic to $(PSL(2,\mathbb{R})^\sim \times PSL(2,\mathbb{R})^\sim)/\mathbb{Z}$. This factor group arises, if one identifies the simultaneous rigid conformal rotation by $2\pi$, namely $(\tilde{R}(2\pi), \tilde{R}(2\pi))$, with the trivial transformation. The last section taught us a lot about the sub-geometrical action of $U^A$, $U^{A'}$ on the local observables in $B$. So, it is natural to look for a relation between the geometrical character of this action
and structures in 1+1 dimensions.

This relation turns out to be a complete correspondence: We construct a 1+1-dimensional, local, conformal theory from the original chiral theory $\mathcal{B}$ applying the net-endomorphism property of $U^A$. In order to prove locality in 1+1 dimensions we are led to a particular choice of light-cone coordinates, by which the original local algebras $\mathcal{B}(I)$, $I \in S^1$, are included in the 1+1-dimensional picture as time zero algebras. This choice of coordinates yields an unphysical spectrum condition: translations in the right spacelike wedge have positive spectrum. Whereas this prohibits an interpretation of the new theory as a genuinely physical one, where we would have positivity of the spectrum in future-like directions, the construction does provide us with a useful geometrical picture for questions concerned with chiral subnets and their COSET models. For this reason we regard the result of our construction as a local, conformal quasi-theory in 1+1 dimensions.

If, on the opposite, one takes a (physical) conformal quantum theory in 1+1 dimensions and defines a chiral conformal net by restriction to time zero algebras, a similar phenomenon arises (cf [KLM01, Lon01]): the spectrum condition disappears altogether, but powerful tools of local quantum theory are available still, because the REEH-SCHLIEGER property survives. In our case there remains a spectrum condition from which one can still derive the REEH-SCHLIEGER property. In this sense we find a natural “converse” of the restriction process which justifies the term chiral holography for our construction.

The main result of this section will be proved by making contact with the analysis of BRUNETTI, GUIDO and LONGO [BGL93] who discussed conformal quantum field theories in general spacetime dimensions as local quantum theories on the conformal covering of the respective MINKOWSKI space as extensions of local nets living on MINKOWSKI space itself.

In 1+1 dimensions, MINKOWSKI space $\mathbb{M}$ is the Cartesian product of two chiral light-rays, which we take as light-cone coordinates of $\mathbb{M}$. One arrives at the (physical) conformal covering $\tilde{\mathbb{M}}$ of $\mathbb{M}$, if one compactifies both light-rays adding the points at infinity, takes the infinite, simply connected covering of the compactification $S^1 \times S^1$, which yields $\mathbb{R} \times \mathbb{R}$, and, finally, one identifies all points which are connected by the action of simultaneous rigid conformal rotations by $2\pi$. The result has the shape of a cylinder having infinite timelike extension: $\tilde{\mathbb{M}} = S^1 \times \mathbb{R}$. Without the final identification we would have spacelike separated copies of $\mathbb{M}$ in covering space, which we consider unphysical; conformally covariant quantum fields can be proven to live on this (physical) conformal covering of MINKOWSKI space, see [LM75].

Light-rays in $\tilde{\mathbb{M}}$ are infinitely extended, universal coverings of the compactified light-rays and serve well as light-cone coordinates. The localisation regions are 1+1-dimensional double cones given as Cartesian product of two intervals, $I \times J$, where $I$, $J$ are properly contained in a single copy of $S^1$ on the left and right
light-rays, respectively, in \( \tilde{M} \).

\( \text{PSL}(2, \mathbb{R})^{\sim} \) has an action on the infinite covering \( \mathbb{R} \) of \( S^1 \) which is transitive for the intervals properly contained in a single copy of \( S^1 \). We exclude the point of infinity from \( S^1 \) and choose a fixed interval \( I \) which is properly contained in the remainder. This interval is identified with its first (pre-)image in covering space and we choose for any pair of proper intervals \( J_{L,R} \) group elements \( \tilde{g}_{L,R} \in \text{PSL}(2, \mathbb{R})^{\sim} \) satisfying \( J_L = \tilde{g}_L I, \ J_R := \tilde{g}_R I \). Making use of this choice we define a set of (local) algebras indexed by \( 1+1 \)-dimensional double cones:

\[
\mathcal{B}^{1+1}(J_L \times J_R) := U^A(\tilde{g}_L)U^A(\tilde{g}_R)\mathcal{B}(I)U^A(\tilde{g}_L)^*U^A(\tilde{g}_R)^*.
\]

By covariance of \( \mathcal{B} \), the resulting algebra \( \mathcal{B}^{1+1}(J_L \times J_R) \) is uniquely determined by \( J_L \times J_R \).

Furthermore, we define a covering projection \( p \) from \( \mathbb{R} \) onto \( S^1 \) referring to the covering projection \( p : \text{PSL}(2, \mathbb{R})^{\sim} \to \text{PSL}(2, \mathbb{R}) \) such that we have: \( pJ_{L,R} := p(\tilde{g}_{L,R})I \). This definition enables us to state two identities for the algebras defined in equation (5):

\[
\mathcal{B}^{1+1}(J_L \times J_R) = U^A(\tilde{g}_L\tilde{g}_R^{-1})\mathcal{B}(pJ_R)U^A(\tilde{g}_L\tilde{g}_R^{-1})^* = U^A(\tilde{g}_R\tilde{g}_L^{-1})\mathcal{B}(pJ_L)U^A(\tilde{g}_R\tilde{g}_L^{-1})^*.
\]

Double cones \( J \times J \), which are centered at the time zero axis, are called \textit{time zero double cones} and we get for the corresponding \textit{time zero algebras} \( \mathcal{B}^{1+1}(J \times J) = \mathcal{B}(pJ) \). Thus, the local algebras of the original chiral conformal theory \( \mathcal{B} \) are included into the new quasi-theory \( \mathcal{B}^{1+1} \) as time zero algebras. Now we are prepared to state the main result of this section:

**Theorem 7** If \( \mathcal{A} \subset \mathcal{B} \) is an inclusion of chiral conformal theories and if the unique inner-implementing representation \( U^A \) associated with this inclusion has the net-endomorphism property, then equation (5) defines a set \( \mathcal{B}^{1+1} \) of local algebras assigned to double cones in \( 1+1 \)-dimensional conformal space time, \( \tilde{M} \), having all but one of the usual properties of a local, conformal, weakly additive quantum theory in \( 1+1 \) dimensions (see [BGL93]): the spectrum condition holds for translations in the right spacelike wedge.

**Proof:** Obviously, the set \( \mathcal{B}^{1+1} \) of local algebras is covariant with respect to the representation \( U^A \times U^A \). Because of the identity \( U^A(\hat{R}(2\pi))U^A(\hat{R}(2\pi)) = 1 \) the set \( \mathcal{B}^{1+1} \) is in fact labelled by the double cones in \( \tilde{M} \) and \( U^A \times U^A \) is a representation of the conformal group in \( 1+1 \) dimensions, namely the group \( (\text{PSL}(2, \mathbb{R})^{\sim} \times \text{PSL}(2, \mathbb{R})^{\sim})/\mathbb{Z} \). The spectrum condition for \( U^A \times U^A \) was proved in [Kôs02] (corollary 6).

The vacuum vector is invariant with respect to \( U^A \times U^A \) [Kôs02] (corollary 7) and it is a basis for the space of vectors with this property, because the space of \( U \)-invariant vectors is one dimensional. \( \Omega \) is cyclic for all local algebras in \( \mathcal{B}^{1+1} \) because of the REEH-SCHLIEDE property of \( \mathcal{B} \).
Isotony follows directly from the net-endomorphism property. An inclusion of 1+1-dimensional double cones $\tilde{g}_L I \times \tilde{g}_R I \subset \tilde{h}_L I \times \tilde{h}_R I$ contained in Minkowski space $\mathbb{M}$, yields the relations: $\tilde{h}_{L,R}^{-1} \tilde{g}_{L,R} I \subset I$. Applying proposition 6 we get: $Ad_{U A} (\tilde{h}_R^{-1} \tilde{g}_R) U A (\tilde{h}_L^{-1} \tilde{g}_L) B(I) \subset B(I)$. This is equivalent to $B^{1+1}(\tilde{g}_L I \times \tilde{g}_R I) \subset B^{1+1}(\tilde{h}_L I \times \tilde{h}_R I)$.

Locality for double cones in $\mathbb{M}$ is shown easily as well. We can reduce the discussion to the situation where there is a double cone $J_1 \times J_2$ spacelike to our basic time zero double cone $I \times I$ simply by applying an appropriate transformation. There is a time zero double cone $J \times J$ which contains $J_1 \times J_2$ and is spacelike to $I \times I$. Since we have shown isotony for $B^{1+1}$, locality for this set follows from locality of $B$.

Weak additivity may be proved as in the chiral case. By scale covariance the local algebras of $B^{1+1}$ are continuous from the inside as well as from the outside [LRT78]. Because we can restrict the discussion to time zero algebras and the argument of JÖRSS [Jö96] for the corresponding chiral situation may be extended directly, we have weak additivity for $B^{1+1}$.

The proof is complete, if one recognises that the proof of [BGL93] (proposition 1.9, on the unique extendibility of $B^{1+1}$ from $\mathbb{M}$ to all of $\tilde{M}$) only requires the prerequisites established so far. In particular, not the spectrum condition itself is needed, but only its consequence, the Reeh-Schlieder property.

In light of this theorem we obtain a straightforward interpretation of the subgeometrical action of $U^A$ on $B$. If we apply a chiral coordinate transformation $\tilde{g}_R$ to a time zero double cone $J \times J$ and if we test the localisation of the correspondingly transformed local algebra of $B^{1+1}$ only by looking at time zero algebras, then we find that the result commutes just with time zero algebras $B(K)$ assigned to proper intervals $K$ contained in the causal complement of $\gamma_{\tilde{g}_R} J$. The statement of proposition 6 follows from Haag duality of $B$.

The theorem has some direct applications to chiral subtheories and their Coset models: We have found that the maximal Coset model $\mathcal{C}_{\text{max}}$ associated with a subtheory $\mathcal{A} \subset B$ may be regarded as the chiral conformal theory of all right chiral observables in $B^{1+1}$ in the sense of Rehren [Reh00], ie the local observables of $B^{1+1}$ which are invariant under the action of transformations on the left-light-cone coordinate only.

The observables of $\mathcal{A}$ may be viewed as left chiral observables and the chiral conformal subnet $\mathcal{A}_{\text{max}} \subset B$ consisting of local observables invariant with respect to the action of $U^A$ (and hence covariant with respect to the action of $U^A$) is to be identified with the chiral theory of all left chiral observables in $B^{1+1}$.

Thus, we have identified $\mathcal{A}_{\text{max}}$ and $\mathcal{C}_{\text{max}}$ as fixed-points of a space-time symmetry acting on a suitably extended theory, namely $B^{1+1}$. In presence of the net-endomorphism property it is not necessary to extend the “classical” symmetry concept (see eg [Ara92]), if one wants to interpret the chiral subtheories.
$A_{\text{max}}$ and $C_{\text{max}}$ as fixed-points of a symmetry; all one has to do is to extend the theory $B$ to its holographic image. Generalisations of the symmetry concept are necessary for a large class of chiral conformal subtheories [LR95, Reh94].

Further remarks: Another interesting, direct consequence of theorem 7 is the following: The cyclic subspaces of $C_{\text{max}}$ and $A_{\text{max}}$, namely $C_{\text{max}}(I)\Omega$ and $A_{\text{max}}(I)\Omega$, coincide with the spaces of $U^A$- and $U^{A'}$-invariant vectors, respectively. By this, results from character arguments on inclusions of current algebras have a direct and rigorous meaning to the analysis of the respective inclusions of chiral conformal theories and Coset models.

Here one starts with an inclusion of current algebras, $A \subset B$, and looks at the decomposition of the vacuum representation of $B$ when restricted to the subtheory $A \vee C \subset B$, $C$ some Coset model associated with $A \subset B$. GODDARD, KENT, and OLIVE [GKOS86] constructed the minimal series of the Virasoro algebra, ie the quantum field theories generated by the stress-energy tensors having central charge less than 1, as Coset models associated with the inclusion of current algebras $SU(2)_{k+1} \subset SU(2)_1 \otimes SU(2)_k$, $k = 1, 2, \ldots$. Their decomposition formulae show the Coset stress-energy tensor to generate all of $C_{\text{max}}$ and $SU(2)_{k+1}$ to coincide with its maximal covariant extension, $A_{\text{max}}$. We call the chiral conformal quantum theories generated by a stress-energy tensor with central charge $c$ less than one "Vir$_c<1$ models".

KAC AND WAKIMOTO [KW88] gave a list of such decompositions for inclusions $A \vee C \subset B$, where $A \subset B$ is an inclusion of current algebras and $C$ is Vir$_{c<1}$-Coset model associated with this inclusion. Their list includes some examples in which $A_{\text{max}}$ and/or $C_{\text{max}}$ are non-trivial local extensions of $A$ and $C$, respectively.

The local extensions of all Vir$_{c<1}$ models have been classified completely by KAWAHIGASHI AND LONGO [KL02]. Most of the non-trivial ones are given by orbifolds: the local extension contains the Vir$_{c<1}$ model as fix-point subtheory with respect to a $\mathbb{Z}_2$ symmetry. Some of these are among the examples of [KW88]. Only four local extensions are of a different type. For two of these KAWAHIGASHI AND LONGO gave a rigorous interpretation as Coset models following suggestions of BÖCKENHAUER AND EVANS [BE99].

One of the remaining two is given as a maximal Coset model by chiral holography and the results of [KW88]: the vacuum representations of the maximal Coset models associated with the current algebra inclusions $SU(9)_2 \subset E(8)_2$ and $E(8)_3 \subset E(8)_2 \otimes E(8)_1$ both decompose upon restriction to the Vir$_{c=4}$ model into the direct sum of the vacuum representation and the representation with highest weight 8. Following KAWAHIGASHI AND LONGO there is only one local

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6For further details see [Kös03b] and the appendix.

7By [Reh01] (lemma 2.3). The proof of proposition 9 includes an alternative argument leading to this statement.

8KAWAHIGASHI AND LONGO gave an alternative argument on this point [KL02] (lemma 3.2, corollary 3.3).
extension with this decomposition, namely the extension \((A_{10}, E_6)\) according to the classification scheme [KL02], which thus is identified as the maximal Coset model associated with both current algebra inclusions.

By classification results on inclusions of current algebras giving rise to \(Vir_{c<1}\)-Coset models [EG87], the forth exceptional local extension, namely \((A_{28}, E_8)\) of \(Vir_{c=144}\), does not seem to be available by a Coset construction using current algebra inclusions. However, the local extension is known to exist by an abstract construction relying on the DHR-data of \(Vir_{c=144}\) [KL02] and thus appears to be a genuine achievement of local quantum physics.

As we have mentioned before, the holographic image \(B^{1+1}\) may not be interpreted as a physical model because of its peculiar spectrum condition. The picture changes in this aspect, if the net-endomorphism property takes a sharper form, namely if the transformed algebra, \(Ad_{U^A(\tilde{g})}B(I)\), commutes with all \(B(J)\), \(J\) a proper interval contained in \(I' \cap (p(\tilde{g})I)'\). After transfer into the holographic picture this property can easily be seen to be equivalent to timelike commutativity of \(B^{1+1}\). In this case we may interchange the role of space and time in the holographic picture and get a physically sensible conformal quantum theory in \(1+1\) dimensions.

It is not difficult to extend arguments of Longo [Lon01] on chiral subtheories to the \(1+1\)-dimensional inclusions \(A_{\text{max}} \vee C_{\text{max}} \subset B^{1+1}\). One can show that the representation of \(A_{\text{max}} \otimes C_{\text{max}}\) induced by the inclusion is unitarily equivalent to a localised representation \(\rho\). In case \(\rho\) has finite statistical dimension and a finite decomposition into tensor products \(\sigma_i \otimes \tau_j\) of irreducible localised representations \(\sigma_i\) of \(A_{\text{max}}\) and \(\tau_j\) of \(C_{\text{max}}\), respectively, this can be applied to the situation where \(B^{1+1}\) fulfills timelike commutativity in order to derive a necessary criterion for this particular property to hold.

One knows that the statistical phases of \(\sigma_i\) and \(\tau_j\) in a tensor product \(\sigma_i \otimes \tau_j\) occurring in \(\rho\) have to be conjugate to each other because of spacelike commutativity [Reh01] (corollary 3.2.). The argument covers the situation of timelike commutativity, where it forces the same statistical phases to coincide. The conformal spin-statistics theorem [GL96] tells us then that \(U^A(\tilde{R}(2\pi))\) has to have spectrum in \(\{\pm1\}\), i.e., the conformal highest weights associated with \(\sigma_i\) and \(\tau_j\) have to lie in \(\frac{1}{2}\mathbb{N}\).

This necessary condition excludes all inclusions of current algebras known to the author (except the ones that one can make up trivially). The result on the conformal highest weights is well known for (quasi-) primary fields in \(1+1\) dimensions commuting with themselves not only for spacelike, but also for timelike separation. The analyticity properties of their two-point function force both chiral scaling dimensions of such fields to be half integers.
4 Isotony problem

In this section we use the **Additional Assumption** in order to solve the isotony problem for the local relative commutants $C_I$ of an inclusion of chiral conformal theories, $A \subset B$. Once their isotony is proved, they are known to coincide with the local algebras of the maximal COSET model $C_{\text{max}}$ associated with $A \subset B$. This way, we reach the main goal of this paper: the maximal COSET model is found to be of a local nature, i.e., it is determined completely by local data.

As we mentioned in the introduction, the isotony problem requires a discussion using an argument suited for our specific scenario. We reduce the task by a purely group theoretical lemma first, which is a slightly extended version of a result of **Guido and Longo** [GL96]. In a side remark we use this lemma to characterise the subnets $A_{\text{max}}$ and $C_{\text{max}}$ and the vacuum subrepresentation of $A_{\text{max}} \vee C_{\text{max}}$. The argument is continued by a proposition providing necessary and sufficient conditions for isotony of local relative commutants to hold. Aside of being an intermediate step of our analysis, it illustrates the character of the isotony problem. The argument is completed by an application of the **Additional Assumption** and summarised in the main theorem of this work. In the remainder of this section we make some remarks on immediate applications of the theorem and relations to other works.

**Lemma 8** $\mathcal{H}$ a separable HILBERT space, $V$ a unitary, strongly continuous representation of $\text{PSL}(2, \mathbb{R})^\sim$ on $\mathcal{H}$. If $H \subset \text{PSL}(2, \mathbb{R})^\sim$ is a subgroup having closed, non-compact image in $\text{PSL}(2, \mathbb{R})$ under the action of the covering projection $p$, then each $V|_H$-invariant vector is in fact $V$-invariant. If $V$ is a representation of positive energy, then each vector which is invariant with respect to $V(\tilde{R}(\cdot))$ is $V$-invariant as well.

**Proof:** The proof of the claim is, up to trivial modifications, identical to the one indicated by **Guido and Longo** for [GL96], corollary B.2. For the reader’s convenience we include a sketch of the argument.

First, one recognises that it is completely sufficient to discuss the complement of the trivial subrepresentation, $V^\perp$, on the HILBERT subspace $\mathcal{H}^\perp$, which contains no vectors invariant with respect to the whole of $V$. We decompose $V^\perp$ into a direct integral of irreducible representations $V_x$.

We look at $V_x \otimes \overline{V}_x$, which can easily be seen not to contain the trivial representation, because $V_x$ is infinite dimensional (cf e.g. [Gri98]). Moreover, $V_x \otimes \overline{V}_x$ is a representation of $\text{PSL}(2, \mathbb{R})$. Now we are in the position to apply [Zim84] (theorem 2.2.20) and thus we have for any $\xi_x \in \mathcal{H}_x$:

$$\lim_{p(g) \to \infty} |\langle V_x(g)\xi_x, \xi_x \rangle|^2 = \lim_{p(g) \to \infty} \langle V_x(g) \otimes \overline{V}_x(g)\xi_x \otimes \overline{\xi}_x, \xi_x \otimes \overline{\xi}_x \rangle = 0 .$$

If we apply this to a $V_x|_H$-invariant vector $\psi_x$, we readily see: $\psi_x = 0$. Integrating over $x$ yields the first statement of the lemma.
The result on rigid conformal rotations may be deduced in the same manner. The irreducible representations $V_x$ are almost all of positive energy and the only irreducible representation of $PSL(2, \mathbb{R})$ having positive energy and containing a non-trivial $\tilde{R}(.)$-invariant vector is the trivial representation $[\text{Gri93}]$. □

The following result is partly known from $[\text{Reh00}]$ (lemma 2.3); we give an alternative proof here. Together with the other parts, this proposition may be viewed as a generalised version of $[\text{Xu00a}]$ (theorem 2.4), which is formulated for a particular class of chiral subnets.

**Proposition 9** Assume $U^A$ to have the net-endomorphism property and denote the projections onto the subspaces of $U^A$- and $U^A'$-invariant vectors by $E_A$ and $E_A'$, respectively. Then we have for the maximal $U^A$-covariant extension of $A$, given by $A_{max}(I) := \{U^A\}' \cap \mathcal{B}(I)$, and the maximal Coset model associated with $A \subset B$, given by $C_{max}(I) := \{U^A\}' \cap \mathcal{B}(I)$, for arbitrary $I \in S^1$:

$$A_{max}(I)\Omega = E_A'\mathcal{H}, \quad C_{max}(I)\Omega = E_A\mathcal{H}.$$  \hspace{1cm} (6)

For any Coset model $C$ associated with $A \subset B$ we have a unitary equivalence of chiral conformal theories: $A \vee C_{e_{AVC}} \cong A_{e_A} \otimes C_{e_C}$. $E_A\mathcal{H}$ has a direct interpretation as multiplicity space of the vacuum subrepresentation of $A \subset B$.

**Proof:** Concerning the proof of (6) we may restrict to $I = S^1_+$ (because of the Reeh-Schlieder theorem). By lemma 8 the spaces of vectors which are invariant with respect to translations are identical with $E_A\mathcal{H}$ and $E_A'\mathcal{H}$, respectively. Taking into account corollary 5 above the statement was proved by Borchers $[\text{Bor98}]$ (theorem 2.6.3).

Straightforward verification shows $A_{e_A} \otimes C_{e_C}$ to be a chiral conformal theory with the obvious definitions: its vacuum is given by $\Omega \otimes \Omega$, the representation implementing covariance is $Ue_A \otimes Ue_C(\cdot)$, its representation space is $e_A\mathcal{H} \otimes e_C\mathcal{H}$. The factoriality of the local algebras proves that $\Omega \otimes \Omega$ is (up to scalar multiples) unique $[\text{GL96}]$ (proposition 1.2), $[\text{Tak79}]$ (IV.5., corollary 5.11). One can establish uniqueness of $\Omega \otimes \Omega$ by group theoretic arguments as in lemma 8 as well.

We now look at the restrictions of $A \vee C_{e_{AVC}}$ and $A_{e_A} \otimes C_{e_C}$ to the chiral light-ray, $\mathbb{R}$. $\Omega$ is separating for $\bigcup_{I \in \mathbb{R}} A \vee C_{e_{AVC}}(I)$, the union of all local algebras assigned to compact intervals in $\mathbb{R}$. Thus, we are allowed to define a linear operator $W$ densely by:

$$WAC\Omega := A\Omega \otimes C\Omega, \quad A \in \mathcal{A}(I), C \in \mathcal{C}(I), I \in \mathbb{R}.$$  \hspace{1cm} (7)

The vacuum is a product state for $\bigcup_{I \in \mathbb{R}} A \vee C_{e_{AVC}}(I)$ (a corollary to Takeuchi’s theorem on modular covariant subalgebras $[\text{Tak72}]$). Hence, $W$ is bounded and extends by continuity to an isometry, as one may readily verify. Moreover, it is elementary to check that $WW^*$ and $W^*W$ commute with the respective restricted nets on $\mathbb{R}$, but these are irreducible. Hence, $W$ is a unitary operator.
Ad_W induces a unitary equivalence of the respective local algebras associated with every \( I \in \mathbb{R} \) by its definition and the separating property of the vacuum. Furthermore, \( W \) is readily shown to be covariant. If we denote the covariance automorphisms of \( \mathcal{A} e_A \otimes \mathcal{C} e_C \) by \( \alpha^\otimes \), we have for \( g I \in \mathbb{R}, I \in \mathbb{R} : \alpha^\otimes g \text{Ad}_W|_{\mathcal{A} e_A \otimes \mathcal{C} e_C(I)} = \text{Ad}_W \alpha g|_{\mathcal{A} e_A \otimes \mathcal{C} e_C(I)}. \) Using the Reeh-Schlieder property of the local algebras, one may reconstruct the representations \( U e_A \otimes U e_C(.) \) and \( U(.) e_{\mathcal{A} e_A \otimes \mathcal{C} e_C} \). Finally, we reconstruct the conformal models from their restrictions to the light-ray using conformal covariance.

In the following discussion \( A \) denotes local observables in \( \mathcal{A} \subset \mathcal{B} \) and \( \pi_0(A) \) its representative in the vacuum representation on \( \mathcal{E} A H =: \mathcal{H}_0 \). The implementation of conformal covariance in \( \pi_0 \) shall be written \( U_0 \). For every vacuum subrepresentation in \( \mathcal{A} \subset \mathcal{B} \) there is a partial isometry \( R : \mathcal{H} \rightarrow \mathcal{H}_0 \) satisfying \( RA = \pi_0(A) R \), for all local \( A \) in \( \mathcal{A} \subset \mathcal{B} \).

The projection \( e_R := R^* R \) commutes with all of \( \mathcal{A} \). \( RU^A(.) R^* \) is a unitary strongly continuous representation of \( PSL(2, \mathbb{R})^\sim \) which implements global conformal covariance in \( \pi_0 \), thus: \( RU^A(.) R^* = U_0(.) \). It follows directly that \( \Phi_\Omega := R^* \Omega \), the vacuum of the subrepresentation associated with \( R \), is invariant with respect to \( U^A \), ie \( \Phi_\Omega \in \mathcal{E} A H \). This completes the proof of the last statement. 

It is not clear in general that the representation \( \mathcal{A} \vee \mathcal{C}_{\text{max}} \subset \mathcal{B} \) of the tensor-product theory defined by the vacuum representation of a chiral subnet \( \mathcal{A} \subset \mathcal{B} \) and the vacuum representation of its maximal Coset model has a (spatial) tensor-product decomposition. This is known under certain conditions [KLM01]. We write \( \mathcal{A} \otimes \mathcal{C} \) for the vacuum representation of \( \mathcal{A} \vee \mathcal{C} \).

We now give a characterisation of isotony for the local relative commutants. The statement on the cyclic projections is non-trivial since, although the local relative commutants are manifestly covariant with respect to \( U \), the Reeh-Schlieder theorem does not apply due to the unclear status of isotony.

\textbf{Proposition 10} Assume the unique inner-implementing representation \( U^A \) associated with a chiral subnet \( \mathcal{A} \subset \mathcal{B} \) to have the net-endomorphism property. Referring to \( I \in \mathbb{S}^1 \), \( e^i_I \) shall denote the projection onto the Hilbert subspace which the local relative commutant \( \mathcal{C}_I = \mathcal{A}(I)' \cap \mathcal{B}(I) \) generates from the vacuum. The following are equivalent:

\begin{enumerate}
  \item For some pair \( I, K \) of intervals satisfying \( K \subset I \in \mathbb{S}^1 \) holds: \( e^i_K \subset e^i_I \).
  \item \( \mathcal{C}_{\mathbb{S}^1} \subset \{ U^A(\tilde{D}(t)), t \in \mathbb{R} \}' \).
  \item \( \mathcal{C}_{\text{max}}(I) = \{ U^A(\tilde{g}), \tilde{g} \in PSL(2, \mathbb{R})^\sim \}' \cap \mathcal{B}(I) = \mathcal{C}_I \), \( I \in \mathbb{S}^1 \).
\end{enumerate}

Remark: The statement on the cyclic projections is non-trivial since, although the local relative commutants are manifestly covariant with respect to \( U \), the
Reeh-Schlieder theorem does not apply due to the unclear status of isotony (cf eg [Bor68]).

**Proof:** The implications [c] ⇒ [a] \[⇒\] [b] are obvious. We start the proof proper with a discussion on [a] ⇒ [c] and here we look at the case \( I = S^1_+ \) (general case by covariance). We set \( e^-_K = e^+_K \). The inclusion \( e^-_K \subset e^+_K \) yields by the separating property of the vacuum and modular covariance of \( C_{S^1_+} \subset B(S^1_+) : C_K \subset C_{S^1_+} \). Thus, any \( g \in PSL(2, \mathbb{R}) \) satisfying \( gS^1_+ = K \) leads to an operator \( U(g) \) which leaves \( e^+_{\mathcal{H}} \) globally invariant. \( g \) has the form \( g = S(n)T(s)D(t) \), \( n, s \geq 0 \). \( g \) may be chosen such that \( t = 0 \).

By modular covariance \( J \), the modular conjugation of \( B(S^1_+) \), and \( e^+_c \) commute and, by covariance and the Bisognano-Wichmann property of \( B \), \( \text{Ad}_{JU(R(\pi))} \) induces an automorphism of \( C_{S^1_+} \), so \( e^+_c \) commutes with \( U(R(\pi)) \), too. The relations \( JT(s)J = T(-s), JS(n)J = S(-n) \) lead to \( U(S(-n))U(T(-s))e^+_{\mathcal{H}} \subset e^+_{\mathcal{H}} \). We assume \( n, s > 0 \) and define

\[
g(n, s) := S\left(-n \frac{ns + (1 + ns)^2}{2 + ns}\right) T\left(-s \frac{2 + ns}{ns + (1 + ns)^2}\right) \left(S(n)T(s)^2\right) .
\]

Applying scale covariance we arrive at: \( U(g(n, s))e^+_c \subset e^+_c \). The group element \( g(n, s) \) leaves the point 1 \( \in S^1 \) invariant and is not a pure scale transformation. This proves that all special conformal transformations leave \( e^+_c \) invariant. The same follows for the translations because of \( R(\pi)S(n)R(\pi) = T(-n) \), which proves \( [U(g), e^+_c] = 0 \) for all \( g \in PSL(2, \mathbb{R}) \) recognising that translations and special conformal transformations generate the whole group. For \( n = 0 \) or \( s = 0 \) the last part applies directly. This proves: \( e^+_K = e^+_c \) for all \( K \subset S^1 \). By modular covariance of the inclusions \( C_K \subset B(K) \) we have \( C_K = \{ e^+_K \}' \cap B(K) \) and this yields isotony for the local relative commutants. The remainder follows by maximality of \( C_{\text{max}} \).

Finally we discuss the implication [d] ⇒ [c]. If \( B \in B(S^1_+) \) commutes with \( U^A(\tilde{D}(t)), t \in \mathbb{R} \), then \( B\Omega \) is invariant under the action of all of \( U^A \) (lemma [k]). If \( \tilde{g} \) is sufficiently close to the identity, \( \text{Ad}_{U^A(\tilde{g})}(B) \) is a local operator (proposition [k]), and the separating property of the vacuum proves that \( B \) commutes with all of \( U^A \). Thereby, we arrive at \( C_{S^1_+} \subset C_{\text{max}}(S^1_+) \), provided the assumption in [d] holds. The other inclusion is trivial.

\( \square \)

If the dilatations \( U^A(\tilde{D}(t)), t \in \mathbb{R} \), induce automorphisms of \( B(S^1_+) \), the last part of the proof shows \( C_{\text{max}}(S^1_+) \) to be the fixed-point subalgebra with respect to this automorphism group. Covariance leads to a corresponding identification of every \( C_{\text{max}}(I), I \subset S^1 \). This may be regarded as an alternative “local” characterisation of \( C_{\text{max}} \), but since the automorphism groups are determined by global observables, namely non-trivial unitaries from \( U^A \), this is not satisfactory.

Only for the final step of our analysis we need to invoke the **Additional Assumption** once again:
Lemma 11 Assume the Additional Assumption to hold. Then we have: 
$$U^A(\tilde{D}(t)) \in \mathcal{A}(S^1_+) \lor \mathcal{A}(S^1_-), \ t \in \mathbb{R}, \text{ and } U^A \text{ has the net-endomorphism property.}$$

Proof: According to the Additional Assumption and lemma 13 there exist, for small, fixed $t$, diffeomorphisms $g_\epsilon, g_\delta$ localised in arbitrarily small neighbourhoods of $+1 \in S^1$ and $-1 \in S^1$, respectively, and diffeomorphisms $g_{\tau_1, \tau_2}^+, g_{\tau_1, \tau_2}^-$ which are localised in $S^1_+$ and $S^1_-$, respectively, and phases $\varphi(\tau_1, \tau_2)$ such that for $\tau_{1, 2} \in \mathbb{R}_+$:

$$U^A(\tilde{D}(t)) = \varphi(\tau_1, \tau_2) \Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^+)) \Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^-)) \cdot \text{Ad}_{U^A(\tilde{D}(\tau_1))}(\Upsilon^A(p^{-1}(g_\epsilon))) \text{Ad}_{U^A(\tilde{D}(-\tau_2))}(\Upsilon^A(p^{-1}(g_\delta))).$$

Following Roberts [Rob74, corollary 2.5], dilatation invariance of the vacuum and the shrinking supports ensure that the last two operators converge weakly to their vacuum expectation values in the limit $\tau_{1, 2} \to \infty$. We rewrite the equation above:

$$\text{Ad}_{U^A(\tilde{D}(\tau_1))}(\Upsilon^A(p^{-1}(g_\epsilon)))\text{Ad}_{U^A(\tilde{D}(-\tau_2))}(\Upsilon^A(p^{-1}(g_\delta)))U^A(\tilde{D}(t))^* = \varphi(\tau_1, \tau_2) \Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^+)) \Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^-))^*. \quad (8)$$

The operators to the right converge weakly by this equation in the limit $\tau_1, \tau_2 \to \infty$. For small $t$, $g_\epsilon$ and $g_\delta$ may be chosen close to the identity, $\omega(.)$ is continuous and normalised, which means that for $g_\epsilon, g_\delta \approx id$ we have $\omega(\Upsilon^A(p^{-1}(g_\epsilon))) \neq 0$, $\omega(\Upsilon^A(p^{-1}(g_\delta))) \neq 0$. This implies $U^A(\tilde{D}(t)) \in \mathcal{A}(S^1_+) \lor \mathcal{A}(S^1_-)$ for small and hence for all $t$.

Because $\Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^+))$ and $\Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^-))$ are unitary operators, the right-hand side of equation (8) converges, up to a phase, strongly against $U^A(\tilde{D}(t))$ for small $t$. This strong convergence proves that for $B \in \mathcal{B}(S^1_+)$ and small $t$ holds true in the weak topology:

$$U^A(\tilde{D}(t))B U^A(\tilde{D}(t))^* = \lim_{\tau_1, \tau_2 \to \infty} \text{Ad}_{\Upsilon^A(p^{-1}(g_{\tau_1, \tau_2}^+))}(B) \in \mathcal{B}(S^1_+). \quad (9)$$

This establishes the net-endomorphism property (definition 11).

The statement of this lemma holds trivially, if the global algebra $\mathcal{A}$ coincides with $\mathcal{A}(S^1_+) \lor \mathcal{A}(S^1_-)$. This is a desirable property (e.g. for the Connes’ fusion approach to superselection structure) and it holds in presence of strong additivity, but a proof of it relying on general properties of chiral conformal subtheories seems out of reach.

We summarise and state the main result of this work, which proves that the maximal Coset models are of a local nature:

Theorem 12 (main theorem) $\mathcal{A} \subset \mathcal{B}$ an inclusion of chiral conformal quantum theories and suppose the Additional Assumption to hold. Then the
unique inner-implementing representation \( U^A \) has the net-endomorphism property and for all \( I \in S^1 \) holds:

\[
\mathcal{C}_{\text{max}}(I) := \{U^A\}' \cap \mathcal{B}(I) = \mathcal{C}_I := \mathcal{A}(I)' \cap \mathcal{B}(I)
\]

**Proof:** The net-endomorphism property of \( U^A \) holds by lemma 2 and b) in proposition 10 is fulfilled because of lemma 11.

\[\Box\]

In the cases where both \( \mathcal{A} \) and \( \mathcal{B} \) possess an integrable stress-energy tensor, and hence \( \mathcal{C}_{\text{max}} \) alike, the main theorem means in particular: \( \mathcal{A}_{\text{max}}(I) \) and \( \mathcal{C}_{\text{max}}(I), I \in S^1 \) arbitrary, are their mutual relative commutants in \( \mathcal{B}(I) \). The local algebras \( \mathcal{A}_{\text{max}}(I) \) are factors which shows the relative commutant of \( \mathcal{A}_{\text{max}}(I) \lor \mathcal{C}_{\text{max}}(I) \) in \( \mathcal{B}(I) \) is \( \mathcal{C}1 \), ie this inclusion is irreducible.

The main theorem proves the conclusions of REHREN [Reh00] to hold true which rely on the generating property of nets of chiral observables, if the 1+1-dimensional theory contains a stress-energy tensor in the sense of the LÜSCHER-MACK theorem [FST89]. Since such a stress-energy tensor factorises into its independent chiral components, our analysis applies directly. The generating property introduced in [Reh00] resisted attempts of proof even in presence of a stress-energy tensor, unfortunately.

Further remarks\(^9\): Results of XU [Xu00a, Xu00b], LONGO [Lon01] show that the current algebras \( SU(n)_k, n,k \in \mathbb{N} \) and all \( \text{Vir}_{c<1} \) models\(^10\) are completely rational [KLM01], ie they have finitely many sectors, all with finite statistics, they are strongly additive and satisfy the split property. Finiteness of statistics shows that the decomposition formulae of KAC and WAKIMOTO of inclusions in the current algebras just mentioned yield examples of nets of normal, irreducible canonical tensor product subfactors (normal CTPS) in the sense of REHREN [Reh00]. For these inclusions the fact that \( \mathcal{A}_{\text{max}} \) and \( \mathcal{C}_{\text{max}} \) are locally their mutual relative commutants follows from the heredity of strong additivity for inclusions of finite index [Lon01]. Our result gives an independent proof relying on the presence of stress-energy tensors only and covers directly all current algebra inclusions.

REHREN has shown for normal CTPS that the sectors \( \rho_i^{A_{\text{max}}} \prec \rho \), \( \rho_j^{C_{\text{max}}} \prec \rho \) form sets which are closed under conjugation and (up to direct sums) fusion, that the coupling matrix \( Z_{ij} \) has to be a permutation matrix and that the coupling matrix induces an isomorphism of the fusions rules of \( \mathcal{A}_{\text{max}} \) and \( \mathcal{C}_{\text{max}} \) as far as only sub-endomorphisms of \( \rho \) are involved. In particular, the statistical dimensions of \( \rho_i^{A_{\text{max}}} \) and \( \rho_j^{C_{\text{max}}} \) have to coincide for \( Z_{ij} \neq 0 \Rightarrow Z_{ij} = 1 \). Thereby, the results of KAC and WAKIMOTO and similar decomposition formulae allow us to translate information on the superselection structure of \( \mathcal{A}_{\text{max}} \) into information on \( \mathcal{C}_{\text{max}} \) and vice versa.

\(^9\)Further details in [Kös03b] and in the appendix.

\(^{10}\)This list may easily extended, eg by looking at branching rules as eg in [KW88, KNS88] and through conformal inclusions.
Recently, Müger [Müg] succeeded in extending the results of Rehren: He proved that for normal CTPS $A_{\text{max}} \vee C_{\text{max}} \subset B$ the coupling matrix induces even an isomorphism of the respective DHR subcategories, if $B$ has trivial superselection structure, i.e., the vacuum representation is its only locally normal representation. There is one current algebra which has trivial superselection structure, namely $E(8)$.1.

In a study on branching rules associated with conformal inclusions in exceptional current algebras, Kac and Niculescu Sanielevici [KNS88] provided some decomposition formulae which yield examples of this structure. Particularly interesting is the embedding $SU(2)_{16} \vee SU(3)_{6} \subset E(8)$. If we regard $SU(2)_{16}$ as chiral conformal subtheory $A \subset E(8)$ and $SU(3)_{6}$ as associated Coset model, $C$, then both $A_{\text{max}}$ and $C_{\text{max}}$ are non-trivial extensions and the localised representation connected with $A_{\text{max}} \vee C_{\text{max}} \subset E(8)$ is found to be a “diagonal” sum of six tensor products. The latter are known to be inequivalent by the result of Rehren and Müger’s results show, that the respective DHR categories associated with the endomorphisms of $A_{\text{max}}$ and $C_{\text{max}}$, respectively, involved in $\rho$ are isomorphic.

5 Discussion

Information on the way in which the Borchers-Sugawara representation $U^A$ associated with a chiral conformal subtheory $A \subset B$ is generated by local observables led to further knowledge on $U^A$. This, in turn, was exploited for proving the maximal Coset model, $C_{\text{max}}$, associated with $A \subset B$ to be of a local nature, more specifically to coincide with the respective local relative commutant. This way, we provided a solution of the isotony problem for a large class of chiral subsystems $A \subset B$ (theorem 12).

All that turned out to be necessary were two special features of the implementers of dilatations (lemmas 2 and 11). The first one leads to an understanding how $U^A$ acts on general local observables in $B$ geometrically which we summarised as net-endomorphism property (proposition 6, definition 1). We found this property is in complete correspondence with the geometry of a 1+1-dimensional conformal quantum theory and derived all properties one can ask from a 1+1-dimensional holographic image of $B$ (theorem 7). The derivation of the net-endomorphism property relied mainly on a result on the interplay of modular theory and positivity of energy; we found it worth while to summarise and reformulate the facts known today in a natural converse of Borchers’ theorem on half-sided translations (theorem 3).

Our solution of the isotony problem made use of specific structures of the chiral conformal group, $PSL(2, \mathbb{R})^\sim$, and integrable positive energy representations of the group of orientation preserving diffeomorphisms of the circle, $Diff_+(S^1)$. The results are satisfactory in many respects:
The Borchers-Sugawara construction of $U^A$ is completely general, yet completely independent of local information and we have shown that additional input is needed only for deriving two natural lemmas (lemmas 2 and 11).

Our Additional Assumption, the presence of an integrable stress-energy tensor, is satisfied for a large class of well investigated examples, the inclusions of current algebras (cf appendix). The main results exhibit the natural objects of studies on Coset models to be the maximal Coset model, $C_{\text{max}}$ and the maximal covariant extension, $A_{\text{max}}$, and opened the gate for a direct incorporation of results which have been compiled in research in representation theory of affine Lie algebras and string theory. In particular, we made accessible examples of normal canonical tensor product subfactors [Reh00] in which $A_{\text{max}}$ and $C_{\text{max}}$ are both non-trivial local extensions.

Yet, there are chiral conformal models which do not possess a stress-energy tensor [Kös03b], so our analysis asks for a more general approach. The most general concept relating covariance and local observables is given in terms of local implementers constructed via the universal localisation map as an application of the split property, known as the quantum Noether theorem [BDL86]. Especially in connection with chiral conformal models this concept proved applicable: CARPI [Car99] reconstructed the stress-energy tensor of certain models via point like limits of local implementers by methods which were introduced and applied in the context of general chiral conformal quantum theory by FREDENHAGEN AND JÖRß [FJ96] with remarkable success.

Approaching the problem from this angle appears to be promising. First, we have reduced it to a question on the way a particular set of global observables, namely to the dilatation group $U^A(\tilde{D}(.)$, is generated by local observables of $A \subset B$. The dilatations proved natural and very useful to look at in connection with the isotony problem. Secondly, the models to look at first, the conformally covariant derivatives of the $U(1)$ current, obey canonical commutation relations and are well known in many respects (see eg [Yng94, GLW98]). Analytic problems connected with nuclearity and the split property have been addressed successfully for free fields [BW86, DDFL87], and the task looks interesting and difficult enough.

As we already mentioned, our analysis does not directly extend to subsystems in other spacetimes. Conformally invariant theories in higher dimensions might be accessible by the more general applicability of the Borchers-Sugawara construction [Kös02] and the presence of spacetime symmetry groups leaving compact localisation regions globally fixed (cf eg [BGL93]), so an analysis based on local implementers might work here as well. For other local quantum theories the isotony problem has been solved by methods less direct than ours, but very general ones [CC01, CC]. Thus, the quest for the heart of this problem still awaits further investigation.
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Appendix

The first part of this appendix discusses the background on our additional assumption. Since we only need to give a summary of (mostly) well known results, the discussion will be brief; further details may be found in [Kö03]. The second part contains a simple lemma on the position of dilatations (scale transformations) in $\text{Diff}_+(S^1)^\sim$.

Chiral current algebras provide a large class of interesting models. They constitute chiral conformal quantum field theories defined by fields, the currents, for which the commutator is linear in the fields. The current algebras which we are interested in are labelled by reductive, compact Lie algebras, which we call the respective colour algebra of the model. The structure constants of the colour algebra determine the current algebra up a central extension, which is labelled for any simple ideal in the colour algebra by a positive integer, called the level.

The current algebras associated with abelian and with simple, non-abelian colour algebras of type $A(n), B(n), C(n), D(n), G(2)$ can be constructed at level 1 as quark models [BH71], i.e. by combining Wick squares of free fermions (reviews: [GOS86, FST89]). The corresponding models of higher level, $k$, are constructed by tensoring the level 1 current algebra $k$ times with itself and extracting the vacuum representation of the level $k$ current (sub-)algebra from this tensor product representation. We denote the current algebra of type $A(n)$ at level $k$ as $A(n)_k$ and extend this notation to all other simple, non-abelian current algebras mutatis mutandis.

These models fulfill manifestly Wightman’s axioms. The conformal Hamiltonian yields an energy grading on them in terms of modes. In the mode picture, the current algebra associated with a simple, non-abelian colour algebra is given by commutation relations of the following form:

$$[j^a_m, j^b_n] = if^{abc}j^c_{m+n} + kg^{ab}m\delta_{m,-n}.$$  

The indices $a, b, c$ refer to a basis of the colour Lie algebra, $f^{abc}$ are its structure constants, $g^{ab}$ its Cartan metric (in a natural normalisation). Unitarity reads in terms of the modes $j^a_n$: $j^a_n = j^a_{-n}$. Using these commutation relations and positivity of energy it is possible to establish linear $H$ bounds by arguments as in [BSM90]. This yields the following properties of the fields: smeared currents which are symmetric and localised in a proper interval are essentially self-adjoint on the Wightman domain and
generate local von Neumann algebras which satisfy the Haag-Kastler axioms of local quantum theory \cite{HK64, Haa92}. For current algebras with an abelian colour algebra these axioms were established by Buchholz and Schulz-Mirbach \cite{BSM90}.

For the cases $E_6$, $E_7$, $E_8$, $F_4$ there are means to construct the current algebras in the mode picture at all levels, and Wightman’s axioms appear implicitly in the literature for these cases (see \cite{Kac90, GW84, TL97}). Hence, the Haag-Kastler axioms may be proved as indicated above. Another way of establishing current algebras as local quantum theories is to look at the exponentiated, positive-energy representations of loop groups stemming from the modes\footnote{This approach relies entirely on the structure and representation theory of loop groups and their Lie algebras and works for the integration of quark models as well. Mentioning the integration through linear $H$ bounds seems worth while since this is closer to the general approach for the transition from a local quantum theory in terms of quantum fields to the corresponding formulation in terms of local algebras of bounded operators.} \cite{GF93, TL97, Was98}.

In conformally covariant Wightman quantum field theories in one (chiral) or $1 + 1$ dimensions the theorem of Lüscher and Mack \cite{FST89} determines the commutation relations of a symmetric Wightman field implementing the infinitesimal conformal spacetime transformations on fields of the quantum field theory, the stress energy tensor (SET), up to a numerical constant, which is determined by the two-point function of the SET. In $1 + 1$ dimensions the SET is found to factorise into two (independent) chiral components. The chiral SETs form, by their commutation relations, an infinitesimal, positive energy representation of $\text{Diff}_+(S^1)$. In terms of its modes, $L_n$, the SET defines a Virasoro algebra with the numerical constant, the central charge $c$, determining the central extension:

$$[L_m, L_n] = (m - n)L_{m+n} + (m - 1)m(m + 1)\frac{c}{12}\delta_{m-n}.$$ 

Unitarity of this representation manifests itself in the relations $L_n = L_{-n}^\dagger$. Either by establishing linear $H$ bounds \cite{BSM90} or by integrating the Virasoro algebra \cite{GW85, Lok94} the SET can be shown to define a conformally covariant, local quantum theory.

In both formulations of current algebras the (Segal-) Sugawara construction (cf \cite{PS86} \S 9.4, \cite{FST89}) yields a SET which is a quadratic function of the currents, either in terms of the modes or as a Wick square. In fact, these models prove to be diffeomorphism invariant.

The embedding of one colour Lie algebra, $\mathfrak{h}$, into another one, $\mathfrak{g}$, yields an inclusion of current algebras and the respective chiral conformal quantum theories. By the (Segal-) Sugawara construction both current algebras contain a SET. Due to complete reducibility results \cite{KW88, Kac90} the respective current algebra associated with $\mathfrak{h} \subset \mathfrak{g}$ and the Sugawara-Virasoro algebras are known to be
represented as direct sums of irreducible highest-weight representations tensored by trivial representations on multiplicity spaces.

This ensures the integrability of the infinitesimal representations and even more: the cocycles of the respective representations are found to be completely determined by the infinitesimal central extensions, i.e., the cocycles for all irreducible subrepresentations agree and the group laws are fulfilled in the direct sum representations up to phases. Due to technical problems connected with the infinite dimension of the groups/Lie algebras involved this is not obvious at all, but it has been established by Toledano-Laredo [TL99] on the grounds indicated here.

This is of special importance for the SET of the current algebra associated with \( \mathfrak{g} \) embedded in the current algebra of \( \mathfrak{g} \). It is straightforward to prove that the modes \( f_{+1,0}^{BC} \) agree with the respective linear combinations of the generators of \( U^A \), where \( \mathcal{A} \subset \mathcal{B} \) is taken to be the corresponding inclusion of current algebras as local quantum theories. This identity follows by integrability of both infinitesimal representations (e.g. [Frö77]) and uniqueness of \( U^A \) [Kös02].

Our additional assumption is thus shown to hold in this class of examples: \( \Upsilon^A \), the integrated representation of \( Diff_+ (S^1)^{\sim} \) generated by the Sugawara SET of \( \mathcal{A} \), is a projective, unitary, strongly continuous, of positive energy and its restriction to \( PSL(2, \mathbb{R})^{\sim} \) agrees with \( U^A \). The statement of the additional assumption possibly covers a more general set of models, but due to technical difficulties connected with the infinite dimension of \( Diff_+ (S^1)^{\sim} \) one has to be content with discussing integrable representations.

We come now to a simple, technical lemma on the position of scale transformations in \( Diff_+ (S^1)^{\sim} \). If \( I_1 \) and \( I_2 \) are neighbouring intervals, the completed union which consists of \( I_1 \cup I_2 \) and the common boundary point will be denoted \( I_1 \bar{\cup} I_2 \); the result is assumed to be a proper interval in \( S^1 \).

**Lemma 13** For a fixed scale transformation \( D(t) \neq id, t \) small, there exist diffeomorphisms \( g_\delta, g_\epsilon \in Diff_+ (S^1) \) which are localised in arbitrarily small neighbourhoods of \( +1 \) and \( -1 \), respectively, and which agree with \( D(t) \) close to \( +1 \) and \( -1 \), respectively, such that, by defining

\[
\begin{align*}
g_\delta^{\tau_1} &:= D(\tau_1)g_\delta D(\tau_1)^{-1}, & g_\epsilon^{\tau_2} &:= D(\tau_2)^{-1}g_\epsilon D(\tau_2),
\end{align*}
\]

we have for all \( \tau_{1,2} \in \mathbb{R}_+ \):

\[
D(t) = g_\delta^{\tau_1}g_\epsilon^{\tau_2}g_\delta^{\tau_1}g_\epsilon^{\tau_2}.
\]

Here, the diffeomorphisms \( g_\delta^{\tau_1}, g_\epsilon^{\tau_2} \) are uniquely specified by their being localised in the upper and lower half circle, respectively. After a local identification of \( Diff_+ (S^1) \) with a sheet of \( Diff_+ (S^1)^{\sim} \) containing the identity, equation (10) still holds for the respective images in \( Diff_+ (S^1)^{\sim} \).
Proof: If $I_1$ and $I_2$ are neighbouring intervals, the completed union which consists of $I_1 \cup I_2$ and the common boundary point will be denoted $I_1 \bigcup I_2$.

Choose a set \{ $I^0_\pm, \iota = +, -, \delta, \varepsilon$ \} of proper, disjoint intervals such that $I^0_\pm \subset S^1_\pm$, $+1 \in I^0_\delta$, $-1 \in I^0_\varepsilon$, the $I^0_i$ are separated by proper intervals $I_a, \ldots, I_d$ and a covering of $S$ by proper intervals $I^1_i$ is defined through:

$$I^1_+ := I_a \cup I^0_+ \cup I_b, \quad I^1_- := I_c \cup I^0_- \cup I_d, \quad I^1_\delta := I_a \cup I^0_\delta \cup I_d, \quad I^1_\varepsilon := I_c \cup I^0_\varepsilon \cup I_b.$$ 

For fixed $t$, one can choose these intervals such that $D(t)$ satisfies $D(t) I^0_\iota \subset I^1_\iota$. Since $D(t) S^1_\pm \subset S^1_\pm$, we may choose $g_\delta$, $g_\varepsilon$ close to $id$ such that $g_\delta$ agrees with $D(t)$ on $I^0_\delta$ and with $id$ on $I^1_\varepsilon$ and $g_\varepsilon$ agrees with $D(t)$ on $I^0_\varepsilon$ and with $id$ on $I^1_\varepsilon$. Referring to this choice we set:

$$g_\pm \upharpoonright I^1_\pm := D(t)g_\delta^{-1}g_\varepsilon^{-1} \upharpoonright S^1_\pm, \quad g_\pm \upharpoonright S^1_\mp := id \upharpoonright S^1_\mp.$$ 

Then we have $D(t) = g_+ g_- g_\delta g_\varepsilon$. We may now apply the definitions in the lemma to this choice and recognise the results to satisfy equally well the assumptions of the construction just given.

For a neighbourhood of the identity the covering projection $p : Diff_+ (S^1)^\sim \to Diff_+ (S^1)$ is a homeomorphism. If we apply $p^{-1}$ to $D(t)$, $g_\delta$, $g_\varepsilon$, $g_+$, $g_-$, we have $p^{-1}(D(t)) = p^{-1}(g_+) p^{-1}(g_-) p^{-1}(g_\delta) p^{-1}(g_\varepsilon)$. For small $\tau_1$, $\tau_2$ the equality (10) holds with the corresponding replacements, and the same is true for all $\tau_1, \tau_2 \in \mathbb{R}_+$ by continuity: denoting the covering projection from $\mathbb{R}$ onto $S^1$ by $p$, all the group elements involved belong to the identity component of the subgroup of $Diff_+ (S^1)^\sim$ which stabilises $p^{-1}(+1)$ and $p^{-1}(-1)$, ie we never leave the first sheet of the covering.

$\square$

References

[AGO87] R.C. Arcuri, J.F. Gomes, and D.I. Olive. *Conformal subalgebras and symmetric spaces*. Nuclear Phys. B285 (1987) 327.

[Ara92] H. Araki. *Symmetries in theory of local observables and the choice of the net of local algebras*. Rev. Math. Phys. SI 1 (Special Issue) (1992) 1–14.

[BB87] F.A. Bais and P.G. Bouwknegt. *A classification of subgroup truncations of the bosonic strings*. Nuclear Phys. B279 (1987) 561.

[BDL86] D. Buchholz, S. Doplicher, and R. Longo. *On Noether’s theorem in quantum field theory*. Ann. Physics 170 (1986) 1–17.

[BE99] J. Bockenhauer and D.E. Evans. *Modular invariants, graphs and $\alpha$-induction for nets of subfactors II*. Comm. Math. Phys. 200 (1999) 57–103.
[BG87] P. Bowcock and P. Goddard. *Virasoro algebras with central charge $c < 1$.* Nuclear Phys. **B285** [FS19] (1987) 651–670.

[BGL93] R. Brunetti, D. Guido, and R. Longo. *Modular structure and duality in conformal quantum field theory.* Comm. Math. Phys. **156** (1993) 201–219.

[BH71] K. Bardakci and M. B. Halpern. *New dual quark models.* Phys. Rev. **D3** (1971) 2493.

[Bor68] H.-J. Borchers. *On the converse of the Reeh-Schlieder theorem.* Comm. Math. Phys. **10** (1968) 269–273.

[Bor92] H.-J. Borchers. *The CPT theorem in two-dimensional theories of local observables.* Comm. Math. Phys. **143** (1992) 315–332.

[Bor97a] Hans-Jürgen Borchers. *Half-sided modular inclusions and structure analysis in quantum field theory.* In S. Doplicher, R. Longo, J.E. Roberts, and L. Zsido, eds., *Operator Algebras and Quantum Field Theory.* International Press, Cambridge, MA, 1997 pages 589–608.

[Bor97b] H.-J. Borchers. *On the lattice of subalgebras associated with the principle of half-sided modular inclusion.* Lett. Math. Phys. **40** (1997) 371–390.

[Bor98] H.-J. Borchers. *Half-sided translations and the type of von Neumann algebras.* Lett. Math. Phys. **44** (1998) 283–290.

[BR87] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics I. C*- and W*-Algebras, Symmetry Groups, Decomposition of States.* Springer, New York, 2nd edition, 1987.

[BSM90] D. Buchholz and H. Schulz-Mirbach. *Haag duality in conformal quantum field theory.* Rev. Math. Phys. **2** (1990) 105–125.

[BW86] D. Buchholz and E.H. Wichmann. *Causal independence and the energy-level density of states in local quantum field theory.* Comm. Math. Phys. **106** (1986) 321–344.

[Car99] S. Carpi. *Quantum Noether’s theorem and conformal field theory: Study of some models.* Rev. Math. Phys. **11** (1999) 519–532.

[CC] S. Carpi and R. Conti. *(In preparation).*

[CC01] S. Carpi and R. Conti. *Classification of subsystems for local nets with trivial superselection structure.* Comm. Math. Phys. **217** (2001) 89–106.
[Dav96] D.R. Davidson. *Endomorphism semigroups and lightlike translations*. Lett. Math. Phys. 38 (1996) 77–90.

[DDFL87] C. D’Antoni, S. Doplicher, K. Fredenhagen, and R. Longo. *Convergence of local charges and continuity properties of $W^*$-inclusions*. Comm. Math. Phys. 110 (1987) 325–348.

[DR90] S. Doplicher and J.E. Roberts. *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*. Comm. Math. Phys. 131 (1990) 51–107.

[FJ96] K. Fredenhagen and M. Jörß. *Conformal Haag-Kastler nets, point-like localized fields, and the existence of operator product expansions*. Comm. Math. Phys. 176 (1996) 541–554.

[Flo98] M. Florig. *On Borchers’ theorem*. Lett. Math. Phys. 46 (1998) 289–293.

[Frö77] J. Fröhlich. *Application of commutator theorems to the integration of representations of Lie algebras and commutation relations*. Comm. Math. Phys. 54 (1977) 135–150.

[FST89] P. Furlan, G.M. Sotkov, and I.T. Todorov. *Two-dimensional conformal quantum field theory*. Riv. Nuovo Cim. 12 (1989) 1–203.

[GF93] F. Gabbiani and J. Fröhlich. *Operator algebras and conformal field theory*. Comm. Math. Phys. 155 (1993) 569–640.

[GKO86] P. Goddard, A. Kent, and D. Olive. *Unitary representations of the Virasoro and super-Virasoro algebras*. Comm. Math. Phys. 103 (1986) 105–119.

[GL96] D. Guido and R. Longo. *The conformal spin and statistics theorem*. Comm. Math. Phys. 181 (1996) 11–35.

[GLW98] D. Guido, R. Longo, and H.-W. Wiesbrock. *Extensions of conformal nets and superselection sectors*. Comm. Math. Phys. 192 (1998) 217–244.

[GO86] P. Goddard and D. Olive. *Kac-Moody and Virasoro algebras in relation to quantum physics*. Internat. J. Modern Phys. A 1 (1986) 303–414.

[Gri93] D.R. Grigore. *The projective unitary irreducible representations of the Poincaré group in $(1 + 2)$-dimensions*. J. Math. Phys. 34 (1993) 4127–4189.
[GW84] R. Goodman and N.R. Wallach. *Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle.* J. Reine Angew. Math. **347** (1984) 69–222.

[GW85] R. Goodman and N.R. Wallach. *Projective unitary positive-energy representations of Diff(S¹).* J. Funct. Anal. **63** (1985) 299–321.

[Haa92] R. Haag. *Local Quantum Physics: Fields, Particles, Algebras.* Springer, Berlin, Heidelberg, 1992.

[HK64] R. Haag and D. Kastler. *An algebraic approach to quantum field theory.* J. Math. Phys. **5** (1964) 848–861.

[Jon83] V.F.R. Jones. *Index for subfactors.* Invent. Math. **72** (1983) 1–25.

[Jör96] M. Jörß. *Conformal Quantum Field Theory: From Haag-Kastler Nets to Wightman Fields.* Ph.D. thesis, Universität Hamburg, 1996. DESY 96-136, KEK 96-10-219.

[Kac90] V. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, 3rd edition, 1990.

[KL02] Y. Kawahigashi and R. Longo. *Classification of local conformal nets. Case c < 1.* [math-ph/0201015](http://arxiv.org/abs/math-ph/0201015) 2002.

[KLM01] Y. Kawahigashi, R. Longo, and M. Müger. *Multi-interval subfactors and modularity of representations in conformal field theory.* Comm. Math. Phys. **219** (2001) 631–669.

[KNS88] V.G. Kač and M. Niculescu Sanielevici. *Decompositions of representations of exceptional affine algebras with respect to conformal subalgebras.* Phys. Rev. **D 37** (1988) 2231–2237.

[Kös02] S. Köster. *Conformal transformations as observables.* Lett. Math. Phys. **61** (2002) 187–198.

[Kös03a] S. Köster. *Conformal covariance subalgebras,* 2003. [hep-th/0303201](http://arxiv.org/abs/hep-th/0303201).

[Kös03b] S. Köster. *Structure of Coset Models.* Ph.D. thesis, Georg-August-Universität Göttingen, 2003.

[KW88] V. Kac and M. Wakimoto. *Modular and conformal invariance constraints in representation theory of affine algebras.* Adv. Math. **70** (1988) 156–236.

[LM75] M. Lüscher and G. Mack. *Global conformal invariance in quantum field theory.* Comm. Math. Phys. **41** (1975) 203–234.
[Lok94] T. Loke. *Operator Algebras and Conformal Field Theory of the Discrete Series Representations of Diff(S^1)*. Ph.D. thesis, University of Cambridge, 1994.

[Lon01] R. Longo. *Conformal subnets and intermediate subfactors*. Comm. Math. Phys. 237 (2003) 7-30.

[LR95] R. Longo and K.-H. Rehren. *Nets of subfactors*. Rev. Math. Phys. 7 (1995) 567–597.

[LRT78] P. Leyland, J. Roberts, and D. Testard. *Duality for quantum free fields*, 1978. CNRS Marseille 78/P.1016, KEK 79-1-157.

[Mos94] M. Moskowitz. *On the surjectivity of the exponential map for certain Lie groups*. Annali di Matematica pura ed applicata (serie IV.) 166 (1994) 129–143.

[Mos97] M. Moskowitz. *Correction and addenda to: On the surjectivity of the exponential map for certain Lie groups*. Annali di Matematica pura ed applicata (serie IV.) 173 (1997) 351–358.

[Müg] M. Müger. (private communication).

[PS86] A. Pressley and G. Segal. *Loop Groups*. Clarendon Press, Oxford, 1986.

[Reh94] K.-H. Rehren. *Subfactors and coset models*. In H.-D. Doebner, Y.K. Dobrev, and A.G. Ushveridze, eds., *Generalized Symmetries in Physics*. World Scientific, Singapore, 1994 pages 338–356.

[Reh00] K.-H. Rehren. *Chiral observables and modular invariants*. Comm. Math. Phys. 208 (2000) 689–712.

[Reh01] K.-H. Rehren. *Locality and modular invariance in 2d conformal qft*. In Roberto Longo, ed., *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*, volume 30 of *Fields Institute Communications*. American Mathematical Society, Providence, RI, 2001 pages 341–354. Siena 2000 Proceedings.

[Rob74] J.E. Roberts. *Some applications of dilatation invariance to structural questions in the theory of local observables*. Comm. Math. Phys. 37 (1974) 273–286.

[SW86] A.N. Schellekens and N.P. Warner. *Conformal subalgebras of Kac-Moody algebras*. Phys. Rev. D 34 (1986) 3092–3096.
[Tak72] M. Takesaki. Conditional expectations in von Neumann algebras. J. Funct. Analysis 9 (1972) 306–321.

[Tak79] M. Takesaki. Theory of Operator Algebras I. Springer Verlag, New York, 1979.

[TL97] V. Toledano Laredo. Fusion of Positive Energy Representations of $LSpin_{2n}$. Ph.D. thesis, University of Cambridge, 1997.

[TL99] V. Toledano Laredo. Integrating unitary representations of infinite-dimensional Lie groups. J. Funct. Anal. 161 (1999) 478–508.

[Was98] A. Wassermann. Operator algebras and conformal field theory III. Fusion of positive energy representations of LSU(N) using bounded operators. Invent. Math. 133 (1998) 467–538.

[Wie92] H.-W. Wiesbrock. A comment on a recent work of Borchers. Lett. Math. Phys. 25 (1992) 157–159.

[Xu99] F. Xu. Algebraic coset conformal field theories II. Publ. Res. Inst. Math. Sci. 35 (1999) 795–824.

[Xu00a] F. Xu. Algebraic coset conformal field theories. Comm. Math. Phys. 211 (2000) 1–43.

[Xu00b] F. Xu. Jones-Wassermann subfactors for disconnected intervals. Commun. Contemp. Math. 2 (2000) 307–347.

[Xu01] F. Xu. On a conjecture of Kac-Wakimoto. Publ. Res. Inst. Math. Sci. 37 (2001) 165–190.

[Yng94] J. Yngvason. A note on essential duality. Lett. Math. Phys. 31 (1994) 127–141.

[Zim84] R.J. Zimmer. Ergodic Theory and Semisimple Groups, volume 81 of Monographs in Mathematics. Birkhäuser, Boston, Basel, Stuttgart, 1984.