Null Controllability for Some Systems of Two Backward Stochastic Heat Equations with One Control Force

Hongheng Li¹ Qi Lü²

Abstract The authors establish the null controllability for some systems coupled by two backward stochastic heat equations. The desired controllability result is obtained by means of proving a suitable observability estimate for the dual system of the controlled system.

Keywords Backward stochastic heat equation, Null controllability, Observability estimate

2000 MR Subject Classification 93B05, 93B07

1 Introduction

Let \( T > 0, G \subset \mathbb{R}^n \) (\( n \in \mathbb{N} \)) be a given bounded domain with a \( C^4 \) boundary \( \Gamma \), and \( G_0 \) be a nonempty open subset of \( G \). Put

\[
Q \triangleq (0, T) \times G, \quad \Sigma \triangleq (0, T) \times \Gamma.
\]

Throughout this paper, we will use \( C \) to denote a generic positive constant depending only on \( G \) and \( G_0 \), which may change from line to line.

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \) be a complete filtered probability space on which a one dimensional standard Brownian motion \( \{B(t)\}_{t \geq 0} \) is defined, such that \( \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration generated by \( \{B(t)\}_{t \geq 0} \). Let \( H \) be a Banach space. Denote by \( L^2(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( X(\cdot) \) such that \( \mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty \), with the canonical norm; by \( L^\infty(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted bounded processes; by \( L^2(\Omega; C([0, T]; H)) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( X(\cdot) \) such that \( \mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty \), with the canonical norm.

This paper is devoted to the study of the null controllability for the following coupled backward stochastic heat equations:

\[
\begin{align*}
\begin{cases}
\frac{dy}{dt} = -\Delta y + (a_1 y + a_2 z + a_3 Y) dt + Y dB(t) & \text{in } Q, \\
\frac{dz}{dt} = -\Delta z + (b_1 y + b_2 z + b_3 Z + \chi_{G_0} f) dt + Z dB(t) & \text{in } Q, \\
y(0) = z(0) = 0 & \text{on } \Sigma, \\
y(T) = y_T, \quad z(T) = z_T & \text{in } G,
\end{cases}
\end{align*}
\]

1 Conference of Mathematics and Information Science, Leshan Normal University, Leshan 614000, Sichuan, China. E-mail: hongheng.li@163.com
2 School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 610054, China; Basque Center for Applied Mathematics, Mazarredo 14, 48009, Bilbao, Basque Country, Spain. E-mail: luqi59@163.com
* Project supported by the National Natural Science Foundation of China (No. 11110170), the Grant MTM2011-29306-C02-00 of the MICINN, Spain, the Project PI2010-04 of the Basque Government, the ERC Advanced Grant FP7-246775 NUMERIWAVES, the ESP Research Networking Programme OPT-PDE, Scientific Research Fund of Sichuan Provincial Education Department of China (No. 10ZC110) and the project Z1062 of Leshan Normal University of China.
where

\[
\begin{align*}
\{ a_i & \in L^2_T(0, T; L^\infty(G)), & i &= 1, 2, & a_3 & \in L^\infty_T(0, T; W^{1,\infty}(G)), \\
b_i & \in L^2_T(0, T; L^\infty(G)), & i &= 1, 2, & b_3 & \in L^2_T(0, T; W^{1,\infty}(G)),
\end{align*}
\] (1.2)

and \( \chi_{G_0} \) is the characteristic function of \( G_0 \). In the system (1.1), \( (y_T, z_T) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times L^2(G)) \) is a terminal state, \( (y, z) \) is a state variable and \( f \in L^2_T(0, T; L^2(G_0)) \) is a control variable. By duality analysis as in [1], we can establish the existence and uniqueness for the solutions to the system (1.1) in the class of

\[
(y, z; Y, Z) \in (L^2_T(\Omega; C([0, T]; L^2(G) \times L^2(G))) \cap L^2_T(0, T; H^1_0(G) \times H^1_0(G)))
\times L^2_T(0, T; L^2(G) \times L^2(G)).
\]

The null controllability of the system (1.1) is formulated as follows.

**Definition 1.1** The system (1.1) is said to be null controllable at a time \( T > 0 \) if for any given \( (y_T, z_T) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times L^2(G)) \), one can find a control \( f \in L^2_T(0, T; L^2(G_0)) \) such that the solution \( (y, z) \) to the system (1.1) satisfies \((y(0), z(0)) = (0, 0)\) in \( G, P\)-a.s.

There are a great many works on the controllability theory of deterministic heat equations and heat systems (see [2–7] and the references therein). However, things are quite different in the stochastic case. To our best knowledge, [8–11] are the only four published papers in which the null controllability for stochastic heat equations is studied. As far as we know, there is no published paper which is concerned with the null controllability of stochastic heat system.

Noting that we only act one control on the system (1.1), it is reasonable to expect that the action of \( z \) to \( y \) will be sufficiently effective. Hence we put the following condition on \( a_2 \).

**Condition 1.1** There exists a nonempty subdomain \( G_1 \subset G_0 \) and a constant \( \sigma > 0 \) such that \( a_2(x, t) \geq \sigma \) or \( a_2(x, t) \leq -\sigma \), a.e. \((x, t) \in G_1 \times (0, T), P\)-a.s.

In this paper, we prove the following result.

**Theorem 1.1** Let Condition 1.1 hold. For any terminal state \( (y_T, z_T) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times L^2(G)) \), we can find a control \( f \in L^2_T(0, T; L^2(G)) \) such that the solution to the system (1.1) with this control satisfies that \( (y(0), z(0)) = (0, 0) \) in \( G, P\)-a.s. Moreover, we have the following estimate for the control:

\[
|f|_{L^2_T(0, T; L^2(G))} \leq C\epsilon^{(T^{-4}+T)(1+p^2)}|y_T, z_T|_{L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times L^2(G))}
\] (1.3)

with

\[
p \triangleq \sum_{i=1}^2 (|a_i|_{L^\infty(0, T; L^\infty(G))} + |b_i|_{L^\infty(0, T; L^\infty(G))} + |a_3|_{L^\infty(0, T; W^{1,\infty}(G))} + |b_3|_{L^\infty(0, T; W^{1,\infty}(G))}).
\]

By means of the classical dual argument (see [11]), the null controllability of the system (1.1) can be reduced to the observability estimate for the following coupled forward stochastic heat equations:

\[
\begin{align*}
\begin{cases}
dw = \Delta w dt - (a_1 w + b_1 v) dt - a_3 w dB(t) & \text{in } Q, \\
dv = \Delta v dt - (a_2 w + b_2 v) dt - b_3 v dB(t) & \text{in } Q, \\
w = v = 0 & \text{on } \Sigma, \\
w(0) = w_0, \quad v(0) = v_0 & \text{in } G.
\end{cases}
\end{align*}
\] (1.4)
where \((w_0, z_0) \in L^2(\Omega, \mathcal{F}_0, P; L^2(G) \times L^2(G))\). We refer to [12] for the well-posedness of the system (1.4) under suitable assumptions in the class
\[(w, v) \in L^2_T(\Omega; C([0, T]; L^2(G) \times L^2(G))) \cap L^2_T(0, T; H^1_0(G) \times H^1_0(G)).\]
In order to prove Theorem 1.1, we only need to derive the following observability estimate for the system (1.4).

**Theorem 1.2** Let Condition 1.1 hold. Then any solution of the system (1.4) satisfies
\[|(w, v)|_{L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times L^2(G))} \leq C e^{C(T^{-4} + T)(1 + p^2)} |v|_{L^2_T(0, T; L^2(G_0))}. \tag{1.5}\]

The idea for the proof of Theorem 1.2 comes from the proof of an analogous result of Theorem 1.2 for deterministic heat systems (see [6]). We construct a functional \(A(t)\) (see Section 3 for the details) to connect the suitable norms of \(w\) and \(v\). The difference here is that we need to utilize Itô calculus for the computation. This will lead to some additional terms, compared with the deterministic case. How to treat these additional terms is the main difficulty which we need to overcome.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.2. At last, in Section 4, we prove Theorem 1.1.

## 2 Some Preliminaries

This section is dedicated to giving some preliminaries. To begin with, we introduce the following function.

Let \(G_2\) and \(G_3\) be two nonempty open subsets of \(G\) such that \(\overline{G_2} \subset G_1\) and \(\overline{G_3} \subset G_2\). From Lemma 5.1 in [11], we know that there is a \(\psi \in C^4(\overline{G})\) such that
\[
\begin{align*}
\psi &> 0 \quad \text{in} \; G, \\
\psi &\equiv 0 \quad \text{on} \; \partial G, \\
|\nabla \psi| &> 0 \quad \text{for all} \; x \in G \setminus G_3.
\end{align*}
\tag{2.1}
\]

Put
\[
\alpha(t, x) = \frac{e^{\lambda \psi(x)} - e^{2\lambda |\psi|_{L^\infty(G)}}}{t^2(T - t)^2}, \quad \varphi(t, x) = \frac{e^{\lambda \psi(x)}}{t^2(T - t)^2}. \tag{2.2}
\]

We have the following lemma for the observability estimate of backward stochastic heat equations.

**Lemma 2.1** (see [11, Theorem 5.1]) For any \(T > 0\), there exists a constant \(\lambda_0 = \lambda_0(G, G_2) > 0\) such that for all \(\lambda \geq \lambda_0\), one can find two constants \(C = C(\lambda) > 0\) and \(s_0 = s_0(\lambda) > 0\) so that for all \(p \in L^2_T(\Omega; C([0, T]; L^2(G)) \cap L^2_T(0, T; H^1_0(G)), \; f \in L^2_T(0, T; L^2(G))\) and \(g \in L^2_T(0, T; H^1(G))\) satisfying
\[
dp - \Delta p dt = f dt + g dB(t), \tag{2.3}
\]
and all \(s \geq s_1 = s_1(\lambda, T) \triangleq s_0(\lambda) \max(1, T^2)\), it holds that
\[
s^3 \lambda^4 \mathbb{E} \int_Q \varphi^3 e^{2s \alpha p^2} dx dt + s \lambda^2 \mathbb{E} \int_Q \varphi e^{2s \alpha |\nabla p|^2} dx dt
\leq C \left\{ \mathbb{E} \int_Q e^{2s \alpha f^2} dx dt + s^3 \lambda^4 \mathbb{E} \int_0^T \int_{G_2} \varphi^3 e^{2s \alpha p^2} dx dt + s \lambda^2 \mathbb{E} \int_Q \varphi e^{2s \alpha g^2} dx dt \right\}.
\]
\[ + \mathbb{E} \int_Q \varphi e^{2sa} \sum_{i=1}^n [(g_{x_i} + s^2 \alpha_{x_i}, g)^2 - (s \alpha_{x_i}^2 + s \alpha_{x_i,x_i})g^2] \, dx \, dt \]. \tag{2.4}

Applying Lemma 2.1 to the first and second equations in the system (1.4), respectively, we obtain
\[
s^3 \lambda^4 \mathbb{E} \int_Q \varphi^3 e^{2sa} w^2 \, dx \, dt + s \lambda^2 \mathbb{E} \int_Q \varphi^{2sa} |\nabla w|^2 \, dx \, dt
\leq C \left\{ \mathbb{E} \int_Q e^{2sa} (a_1 w + b_1 v)^2 \, dx \, dt + s^3 \lambda^4 \mathbb{E} \int_0^T \int_{G_2} \varphi^3 e^{2sa} w^2 \, dx \, dt + s \lambda^2 \mathbb{E} \int_Q \varphi^{2sa} (a_3 w)^2 \, dx \, dt \right. \\
+ \mathbb{E} \int_Q \varphi^{2sa} \sum_{i=1}^n [(a_3 w)_{x_i} + s^2 \alpha_{x_i} (a_3 w)^2] - (s \alpha_{x_i}^2 + s \alpha_{x_i,x_i}) (a_3 w)^2] \, dx \, dt \} \tag{2.5}
\]

and
\[
s^3 \lambda^4 \mathbb{E} \int_Q \varphi^3 e^{2sa} v^2 \, dx \, dt + s \lambda^2 \mathbb{E} \int_Q \varphi^{2sa} |\nabla v|^2 \, dx \, dt
\leq C \left\{ \mathbb{E} \int_Q e^{2sa} (a_1 w + b_1 v)^2 \, dx \, dt + s^3 \lambda^4 \mathbb{E} \int_0^T \int_{G_2} \varphi^3 e^{2sa} v^2 \, dx \, dt + s \lambda^2 \mathbb{E} \int_Q \varphi^{2sa} (a_3 w)^2 \, dx \, dt \right. \\
+ \mathbb{E} \int_Q \varphi^{2sa} \sum_{i=1}^n [(b_3 v)_{x_i} + s^2 \alpha_{x_i} (b_3 v)^2] - (s \alpha_{x_i}^2 + s \alpha_{x_i,x_i}) (b_3 v)^2] \, dx \, dt \} \tag{2.6}
\]

From the inequality (2.5) and the inequality (2.6), and choosing
\[ s \geq s_2 \triangleq \max\{ \beta^2, s_1 \}, \]
we get
\[ \mathbb{E} \int_Q \varphi^3 e^{2sa} (w^2 + v^2) \, dx \, dt \leq C \mathbb{E} \int_0^T \int_{G_2} \varphi^3 e^{2sa} (w^2 + v^2) \, dx \, dt. \tag{2.7} \]

Hence we obtain the following proposition.

**Proposition 2.1** Let \((w, v)\) be a solution to the system (1.4). Then for each \(\lambda \geq \lambda_0\) and all \(s \geq s_2\), the inequality (2.7) holds.

### 3 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2.

**Proof of Theorem 1.2** From Condition 1.1, we know that \(a_2(x, t) \geq \sigma\) or \(a_2(x, t) \leq -\sigma\), a.e. \((x, t) \in G_1 \times (0, T), P\text{-a.s.}\) Without loss of generality, we assume that \(a_2(x, t) \leq -\sigma\), a.e. \((x, t) \in G_1 \times (0, T), P\text{-a.s.}\)

By the definition of \(\alpha\), we know
\[ s^3 \lambda^4 \mathbb{E} \int_Q \varphi^3 (w^2 + v^2) \, dx \, dt \leq C \mathbb{E} \int_0^T \int_{G_2} \varphi^3 e^{2sa} (w^2 + v^2) \, dx \, dt. \tag{3.1} \]

Let \(\xi \in C^\infty(\mathbb{R}^n)\) be a cut-off function satisfying
\[ \xi = 1 \text{ in } G_2, \quad \xi = 0 \text{ in } \mathbb{R}^n \setminus G_1, \quad 0 \leq \xi \leq 1 \text{ in } G_1. \tag{3.2} \]
Then we have

$$A(t) = \mathbb{E} \int_{G} \left( (e^{k\tau t} \eta^s w^2 + \beta_0 e^{2\tau t} \eta w v + \beta_1 e^{(k-2)\tau t} \eta v^2) dt \right).$$  (3.3)

Then we have

$$dA(t) = \mathbb{E} \int_{G} \left\{ k e^{k\tau t} \alpha \eta^s w^2 dt + e^{k\tau t} \eta^s w (\Delta w - a_1 w - b_1 v) dt + e^{k\tau t} \eta^s (a_2 w^2) dt + 2\tau e^{2\tau t} \alpha \eta w v dt + \beta_0 e^{2\tau t} \eta (v w + w dv + \eta w) dt + \beta_1 e^{(k-2)\tau t} \eta v^2 dt + \beta_2 e^{(k-2)\tau t} \eta^2 (v w + w dv) dt + \beta_3 e^{(k-2)\tau t} \eta^2 (v w + w dv) dt \right\} dx.$$  (3.4)

Noting that $(w, v)$ is the solution to the system (1.4), from the equality (3.4), we obtain

$$dA(t) = \mathbb{E} \int_{Q} \left\{ k e^{k\tau t} \alpha \eta^s w^2 dt + 2 e^{k\tau t} \eta^s w (\Delta w - a_1 w - b_1 v) dt + e^{k\tau t} \eta^s (a_2 w^2) dt + \beta_0 e^{2\tau t} \eta (v w + w dv) dt + \beta_1 e^{(k-2)\tau t} \eta v^2 dt + \beta_2 e^{(k-2)\tau t} \eta v^2 (\Delta w - a_1 w - b_1 v) dt + \beta_3 e^{(k-2)\tau t} \eta^2 (v w + w dv) dt \right\} dx.$$  (3.5)

Integrating the equality (3.5) in $[0, T]$, we get

$$0 = -\beta_0 \mathbb{E} \int_{Q} e^{2\tau t} \eta a_2 w^2 dt + \mathbb{E} \int_{Q} \left\{ k e^{k\tau t} \alpha \eta^s w^2 - 2 e^{k\tau t} \eta^s a_1 w^2 + e^{k\tau t} \eta^s (a_2 w^2) dt + \beta_0 e^{2\tau t} \eta (v w + w dv) dt + \beta_1 e^{(k-2)\tau t} \eta v^2 dt + \beta_2 e^{(k-2)\tau t} \eta v^2 (\Delta w - a_1 w - b_1 v) dt + \beta_3 e^{(k-2)\tau t} \eta^2 (v w + w dv) dt \right\} dx.$$  (3.6)

Denoting by $I_i$ ($i = 1, 2, 3, 4$) the last four terms on the right-hand side of the equality (3.6), we obtain

$$\beta_0 \mathbb{E} \int_{Q} a_2 e^{2\tau t} \eta w^2 dt = I_1 + I_2 + I_3 + I_4.$$  (3.7)

Now we are going to estimate $I_i$ ($i = 1, 2, 3, 4$).

Choosing $k > 2$, $r \in \left( \frac{1}{2}, 2 \right)$, by the definition of $\alpha$, we know that there exists an $s > 0$ such that for all $s \geq s_3$, it holds that

$$\left| e^{(k-2)\tau t} \alpha \right|_{L^\infty(Q)} \leq 1, \quad \left| e^{(k-2)\tau t} \alpha \right|_{L^\infty(Q)} \leq 1, \quad \left| \tau \phi e^{(k-2)\tau t} \alpha \right|_{L^\infty(Q)} \leq 1.$$  (3.8)

By virtue of the first and second inequalities in (3.8), we know

$$I_1 = \mathbb{E} \int_{Q} \left( k e^{k\tau t} \alpha \eta^s w^2 - 2 e^{k\tau t} \eta^s a_1 w^2 + e^{k\tau t} \eta^s a_2 w^2 \right) dx dt$$

$$= \mathbb{E} \int_{Q} e^{2\tau t} (k e^{(k-2)\tau t} \alpha \eta^s w^2 - 2 e^{(k-2)\tau t} \eta^s a_1 w^2 + e^{(k-2)\tau t} \eta^s a_2 w^2) dx dt.$$
\[ \leq C(p + p^2 + 1)E \int_Q e^{2\tau \alpha \eta}w^2dxdt. \] (3.9)

As the estimate of \( I_1 \), by the third, fourth and fifth inequalities in (3.8), one can easily obtain

\[
I_3 = \mathbb{E} \int_Q (\beta_1 t \epsilon^{(r \alpha)} \eta^\beta b_1^2 - \beta_0 e^{2\tau \alpha} \eta b_1^2 - 2\beta_1 e^{(l \alpha)} \eta^\beta b_2 v^2 + \beta_1 e^{(l \alpha)} \eta^\beta b_3^2)dxdt
\]

\[
= \mathbb{E} \int_Q e^{\tau \alpha} \eta^\beta v^2(\beta_1 t \epsilon^{(l-r \alpha)} \alpha \eta^\beta - \beta_0 e^{(l-r \alpha)} \eta b_1 - 2\beta_1 e^{(l-r \alpha)} \eta^\beta b_2 + \beta_1 e^{(l-r \alpha)} \eta^\beta b_3^2)dxdt
\]

\[
\leq C[(p + \beta_1)(p + p^2)]E \int_Q e^{\tau \alpha} \eta^\beta v^2dxdt + \frac{1}{4} \mathbb{E} \int_Q e^{2\tau \alpha} \eta w^2dxdt. \] (3.10)

Now we estimate \( I_2 \). By Cauchy-Schwartz inequality, utilizing the sixth, seventh, eighth and ninth inequalities in (3.8), we have

\[
I_2 = -\mathbb{E} \int_Q [2e^{k \tau} \eta^\beta b_1 w^2 - 2\beta_0 e^{2\tau \alpha} \alpha \eta w^2 + \beta_0 e^{2\tau \alpha} \eta(a_1 + b_2 - a_3 b_3)w^2
\]

\[+ 2\beta_1 e^{(l \alpha)} \eta^\beta a_2 w^2]dxdt.
\]

Recalling \( l > 1 + \frac{r}{2} \) and noticing \( 1 + \frac{r}{2} > r \), we obtain

\[
I_2 \leq C[(p + \beta_1)(p + p^2)]E \int_Q e^{\tau \alpha} \eta^\beta v^2dxdt + \frac{1}{4} \mathbb{E} \int_Q e^{2\tau \alpha} \eta w^2dxdt. \] (3.12)

At last, we estimate \( I_4 \).

\[
I_4 = \mathbb{E} \int_Q (2e^{k \tau} \eta^\beta w \Delta w + \beta_0 e^{2\tau \alpha} \eta(v \Delta w + w \Delta v) + 2\beta_1 e^{(l \alpha)} \eta^\beta v \Delta v)dxdt
\]

\[= \mathbb{E} \int_Q e^{k \tau} \eta^\beta \Delta w^2dxdt - 2\beta_0 \mathbb{E} \int_Q e^{2\tau \alpha} \eta \Delta w^2dxdt
\]

\[+ \beta_0 \mathbb{E} \int_Q e^{2\tau \alpha} \eta \Delta w^2dxdt - 2\beta_0 \mathbb{E} \int_Q e^{2\tau \alpha} \eta \Delta w \cdot \nabla v dxdt
\]

\[+ \beta_1 \mathbb{E} \int_Q e^{(l \alpha)} \eta^\beta \Delta v^2dxdt - 2\beta_1 \mathbb{E} \int_Q e^{(l \alpha)} \eta^\beta \Delta v dxdt. \] (3.13)

By virtue of integration by parts, we get

\[
\mathbb{E} \int_Q e^{k \tau} \eta^\beta \Delta w^2dxdt
\]

\[= \mathbb{E} \int_Q \Delta(e^{k \tau} \eta^\beta)w^2dxdt
\]

\[= \mathbb{E} \int_Q e^{k \tau} w^2\left(\kappa^2 \epsilon^2 \Delta \alpha \eta^\beta + k \tau \Delta \alpha \eta^\beta + \frac{4}{3} |\nabla \eta|^2 \eta^{-\frac{5}{2}}
\]

\[+ \frac{4}{3} \eta^\beta \Delta \eta + \frac{8}{3} k \tau \eta^\beta \nabla \alpha \cdot \nabla \eta \right) . \] (3.14)
Null Controllability of BSHSs

It is easy to check that
\[ \eta^{-\frac{3}{2}} \nabla \eta = 6 \nabla \xi \in L^\infty(Q), \quad \eta^{-\frac{3}{2}} \Delta \eta = 30 |\nabla \xi|^2 + 6 \xi \Delta \xi \in L^\infty(Q). \] (3.15)

Recalling \( k > 2 \), by means of the last two inequalities in (3.8), we can obtain
\[ \mathbb{E} \int_Q e^{k \tau_0} \eta^{\frac{4}{5}} \Delta u^2 \, dx \, dt \leq C \mathbb{E} \int_Q e^{2 \tau_0} \eta u^2 \, dx \, dt. \] (3.16)

By the similar argument to obtain the inequality (3.12) and the inequality (3.16), we can show
\[ \beta_0 \mathbb{E} \int_Q e^{2 \tau_0} \eta \Delta (w v) \, dx \, dt \leq \frac{1}{4} \mathbb{E} \int_Q e^{2 \tau_0} \eta w^2 \, dx \, dt + C \beta_0^2 \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt \] (3.17)
and
\[ \beta_1 \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} \Delta u^2 \, dx \, dt \leq C \beta_1 \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt. \] (3.18)

Then, it follows from (3.13)–(3.18) that
\[ I_4 \leq C \mathbb{E} \int_Q e^{2 \tau_0} \eta w^2 \, dx \, dt + C(\beta_0^2 + \beta_1) \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt \]
\[ - 2 \mathbb{E} \int_Q e^{k \tau_0} \eta^{\frac{4}{5}} |\nabla w|^2 \, dx \, dt - 2 \beta_0 \mathbb{E} \int_Q e^{2 \tau_0} \eta \nabla w \cdot \nabla v \, dx \, dt \]
\[ - 2 \beta_1 \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} |\nabla v|^2 \, dx \, dt. \] (3.19)

Let \( k + l < 4 \) and \( \beta_1 > \frac{\beta_0^2}{4} \). Then we know
\[ - 2 \mathbb{E} \int_Q e^{k \tau_0} \eta^{\frac{4}{5}} |\nabla w|^2 \, dx \, dt - 2 \beta_0 \mathbb{E} \int_Q e^{2 \tau_0} \eta \nabla w \cdot \nabla v \, dx \, dt \]
\[ - 2 \beta_1 \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} |\nabla v|^2 \, dx \, dt \leq 0. \] (3.20)

Therefore, we find
\[ I_4 \leq C \mathbb{E} \int_Q e^{2 \tau_0} \eta w^2 \, dx \, dt + C(\beta_0^2 + \beta_1) \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt. \] (3.21)

From (3.9)–(3.10), (3.12) and (3.21), we see
\[ \beta_0 \mathbb{E} \int_Q a_2 e^{2 \tau_0} \eta w^2 \, dx \, dt \leq C(1 + p^2)(\beta_0^2 + \beta_1^2) \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt \]
\[ + C(p^2 + 1) \int_Q e^{2 \tau_0} \eta w^2 \, dx \, dt. \] (3.22)
Hence, by setting \( \beta_0 = 2C(1 + p^2) \), we obtain
\[ \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt \leq C(1 + p^{10}) \mathbb{E} \int_Q e^{\tau_0} \eta^{\frac{4}{5}} v^2 \, dx \, dt. \] (3.23)
Taking into account Proposition 2.1, the inequality (3.1) and the inequality (3.23), for \( \lambda \geq \lambda_0 \) and \( s \geq \max\{s_2, s_3\} \), we deduce

\[
E \int_Q \varphi^3 e^{2s\alpha} (w^2 + v^2) dx dt \leq C(1 + p^{10}) E \int_0^T \int_{G_0} e^{2s\alpha} v^2 dx dt. \tag{3.24}
\]

Recalling the definitions of \( \alpha \) and \( \phi \) (see (2.2)), we have

\[
E \int_Q \varphi^3 e^{2s\alpha} (w^2 + v^2) dx dt \geq \min_{x \in G} \left[ \varphi^3 \left( x, \frac{T}{2} \right) e^{2s\alpha(x, \frac{T}{2})} \right] E \int_0^T \int_G (w^2 + v^2) dx dt. \tag{3.25}
\]

and

\[
E \int_0^T \int_{G_0} e^{2s\alpha} v^2 dx dt \leq \max_{(x,t) \in G} (e^{2s\alpha(x,t)}) E \int_0^T \int_{G_0} v^2 dx dt. \tag{3.26}
\]

From (3.24)–(3.26), we obtain

\[
E \int_{\frac{T}{4}}^T \int_G (w^2 + v^2) dx dt \leq C e^{CT^{-4}(1+p^2)} E \int_0^T \int_{G_0} v^2 dx dt. \tag{3.27}
\]

Noting \( d(w^2 + v^2) = 2wdw + (dw)^2 + 2vdv + (dv)^2 \), and applying the usual energy estimate to the system (1.4), it is easy to see that, for any \( 0 \leq t_1 \leq t_2 \leq T \), it holds

\[
E \int_{G_0} [w^2(t_2) + v^2(t_2)] dx - E \int_{G_0} [w^2(t_1) + v^2(t_1)] dx \\
= E \int_{t_1}^{t_2} \int_G [2wdw + (dw)^2 + 2vdv + (dv)^2] dx dt \\
= E \int_{t_1}^{t_2} \int_G [2w(\Delta w - a_1w - b_1v) + (a_3w)^2 + 2v(\Delta v - a_2w - b_2v) + (b_3v)^2] dx dt \\
\leq (1 + p^2) E \int_{t_1}^{t_2} \int_G (w^2 + v^2) dx dt. \tag{3.28}
\]

Hence, in terms of Gronwall inequality, it follows

\[
E \int_{G_0} [w^2(t_2) + v^2(t_2)] dx \leq e^{C(T(1+p^2))} E \int_{G_0} [w^2(t_1) + v^2(t_1)] dx. \tag{3.29}
\]

By the inequality (3.27) and the inequality (3.29), we conclude that the solution \((w, v)\) to the system (1.4) satisfies the inequality (1.5).

4 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1 which is our controllability result. The proof is almost the standard dual argument. However, for the sake of completeness, we still give it here.

**Proof of Theorem 1.1** For any \((y_T, z_T) \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; \mathcal{L}^c(\mathcal{G}) \times \mathcal{L}^c(\mathcal{G}))\), we need to find a control \( f \in L^2_2(0, T; L^2(G_0)) \) such that the solution to the system (1.1) satisfies \((y(0), z(0)) = (0, 0)\) in \( G \), \( P\)-a.s. We use the duality argument.

We introduce the following linear subspace of \( L^2_2(0, T; L^2(G_0)) \times L^2_2(0, T; L^2(G)) \):

\[
X \triangleq \{ (v|_{[0,T] \times G_0 \times \Omega} | (w, v) \) solves the system (1.4) with some
\]
\[(w_0, v_0) \in L^2(\Omega, \mathcal{F}_0; L^\infty(\mathcal{G}) \times L^\infty(\mathcal{G}))\],

and define a linear functional on \(X\) as follows:

\[
L(v|[0,T] \times G_0 \times \Omega) = \mathbb{E} \int_G (y_T w(T) + z_T v(T)) dx.
\]

By means of the observability estimate (see Theorem 1.2), we know

\[
|L(v|[0,T] \times G_0 \times \Omega)| \leq \left( \mathbb{E} \int_G (|y_T|^2 + |z_T|^2) dx \right)^{\frac{1}{2}} \left( \mathbb{E} \int_G (|w(T)|^2 + |v(T)|^2) dx \right)^{\frac{1}{2}}
\]

\[
\leq C e^{C(T^{-1} + p^2) + T(1 + p^2)} \left( \mathbb{E} \int_G (|y_T|^2 + |z_T|^2) dx \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^T \int_{G_0} |v|^2 dz dt \right)^{\frac{1}{2}}.
\]

Thus, \(L\) is a bounded linear functional on \(X\) such that the norm of \(l\) is bounded by

\[
C e^{C(T^{-1} + p^2) + T(1 + p^2)} \left( \mathbb{E} \int_G (|y_T|^2 + |z_T|^2) dx \right)^{\frac{1}{2}}.
\]

By Hahn-Banach theorem, \(L\) can be extended to be a bounded linear functional with the same norm on \(L^2_{\mathcal{F}}(0,T; L^2(G_0))\). For simplicity, we use the same notation for this extension. Now, the Riesz representation theorem allows us to find a random field \(f \in L^2_{\mathcal{F}}(0,T; L^2(G_0))\) such that

\[
\mathbb{E} \int_G (y_T w(T) + z_T v(T)) dx = \mathbb{E} \int_0^T \int_{G_0} f v dx dt
\]

and

\[
|f|_{L^2_{\mathcal{F}}(0,T; L^2(G))} \leq C e^{C(T^{-1} + p^2) + T(1 + p^2)} |(y_T, z_T)|_{L^2(\Omega, \mathcal{F}_0; L^2(G) \times L^2(\mathcal{G}))}. \tag{4.2}
\]

We claim that this random field \(f\) is exactly the control we need. In fact, by means of Itô formula, we know that

\[
d(yw) = ydw + wd y + dydw \tag{4.3}
\]

and

\[
d(zv) = zdv + vd z + dzdv, \tag{4.4}
\]

where \((y, z)\) is the solution to the system (1.1) and \((w, v)\) is the solution to the system (1.4). From (4.3), we obtain

\[
\mathbb{E} \int_G y_T w(T) dx - \mathbb{E} \int_G y(0) w_0 dx
\]

\[
= \mathbb{E} \int_Q (ydw + wd y + dydw) dx
\]

\[
= \mathbb{E} \int_Q y(\Delta w - a_1 w - b_1 v) dx dt + \mathbb{E} \int_Q w(-\Delta y + a_1 y + a_2 z + a_3 Y) dx dt
\]

\[
+ \mathbb{E} \int_Q Y(-a_3 w) dx dt
\]

\[
= \mathbb{E} \int_Q (a_2 wz - b_1 vy) dx dt. \tag{4.5}
\]
From (4.4), we know
\[
\begin{align*}
E \int_G z_T v(T) dx &- E \int_G z(0)v_0 dx \\
&= E \int_Q (zd v + vd z + dz dv) dx \\
&= E \int_Q z(\Delta v - a_2 w - b_2 v) dx dt + E \int_Q v(-\Delta z + b_1 y + b_2 z + b_3 Z + \chi_{G_0} f) dx dt \\
&\quad + E \int_Q Z(-b_3 v) dx dt \\
&= E \int_Q (b_1 v y - a_2 w z + \chi_{G_0} f v) dx dt. \tag{4.6}
\end{align*}
\]
Combining the equalities (4.1) and (4.5)–(4.6), we find
\[
E \int_G y(0) w_0 dx + E \int_G z(0) v_0 dx = 0.
\]
Since \((w_0, v_0)\) can be chosen arbitrarily, this implies that \((y(0), z(0)) = 0\) in \(G, P\text{-a.s.}\)

References

[1] Zhou, X., A duality analysis on stochastic partial differential equations, *J. Funct. Anal.*, 103, 1992, 275–293.
[2] Barbu, V., Controllability of parabolic and Navier-Stokes equations, *Sci. Math. Jpn.*, 56, 2002, 143–211.
[3] Doubova, A., Fernández-Cara, E., González-Burgos, M. and et al., On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, *SIAM J. Control Optim.*, 41, 2002, 798–819.
[4] Fernández-Cara, E. and Guerrero, S., Global Carleman inequalities for parabolic systems and applications to controllability, *SIAM J. Control Optim.*, 45, 2006, 1390–1446.
[5] Fursikov, A. V. and Imanuvilov, O. Y., Controllability of evolution equations, Lecture Notes Series, 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
[6] Guerrero, S., Null controllability of some systems of two parabolic equations with one control force, *SIAM J. Control Optim.*, 46, 2007, 379–394.
[7] Zuazua, E., Controllability and observability of partial differential equations: some results and open problems, *Handbook of Differential Equations: Evolutionary Differential Equations*, vol. 3, Elsevier Science, 2006, 527–621.
[8] Barbu, V., Răşcanu, A. and Tessitore, G., Carleman estimate and controllability of linear stochastic heat equations, *Appl. Math. Optim.*, 47, 2003, 97–120.
[9] Lü, Q., Some results on the controllability of forward stochastic heat equations with control on the drift, *J. Funct. Anal.*, 260, 2011, 832–851.
[10] Sirbu, M. and Tessitore, G., Null controllability of an infinite dimensional SDE with state- and control-dependent noise, *Systems Control Lett.*, 44, 2001, 385–394.
[11] Tang, S. and Zhang, X., Null controllability for forward and backward stochastic parabolic equations, *SIAM J. Control Optim.*, 48, 2009, 2191–2216.
[12] Da Prato, G. and Zabczyk, J., Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.