Numerical approximation of singular Forward-Backward SDEs

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Abstract

In this work, we study the numerical approximation of a class of singular fully coupled forward backward stochastic differential equations. These equations have a degenerate forward component and non-smooth terminal condition. They are used, for example, in the modeling of carbon market [9] and are linked to scalar conservation law perturbed by a diffusion. Classical FBSDEs methods fail to capture the correct entropy solution to the associated quasi-linear PDE. We introduce a splitting approach that circumvent this difficulty by treating differently the numerical approximation of the diffusion part and the non-linear transport part. Under the structural condition guaranteeing the well-posedness of the singular FBSDEs [8], we show that the splitting method is convergent with a rate 1/2. We implement the splitting scheme combining non-linear regression based on deep neural networks and conservative finite difference schemes. The numerical tests show very good results in possibly high dimensional framework.

Key words: singular FBSDEs, splitting scheme, non-linear regression.

MSC Classification (2020): 65C30, 65C20, 65M12, 60H35.

1 Introduction

In this work, we study the approximation of a class of singular fully coupled Forward Backward Stochastic Differential Equations (FBSDE). Let (Ω, F, P) be a stochastic basis supporting a d-dimensional Brownian motion W and T > 0 a terminal time. We denote by (Ft)t≥0 the filtration generated by the Brownian motion (augmented and completed).

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The singular FBSDE system, with solution \((P_t, E_t, Y_t, Z_t)_{0 \leq t \leq T}\), has the following form:

\[
\begin{align*}
&dP_t = b(P_t)dt + \sigma(P_t)dW_t \\
&dE_t = \mu(Y_t, P_t)dt \\
&dY_t = Z_t \cdot dW_t
\end{align*}
\]  

\text{(1.1)}

The function \(b : \mathbb{R}^d \to \mathbb{R}^d\), \(\sigma : \mathbb{R}^d \to \mathcal{M}_d\) where \(\mathcal{M}_d\) is the set of \(d \times d\) matrices on \(\mathbb{R}\), and \(\mu : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) are Lipschitz-continuous. These equations have been introduced in [9] as models for carbon emission market. They can model, more generally, cap-and-trade scheme used by government to limit the emission of certain pollutant. In these models, \(Y\) is the price of a pollution permit, \(E\) is the cumulative emission of the pollutant and \(P\) represents some state variables influencing the emission (demand, energy prices etc.). The coefficient \(\mu\) is naturally decreasing in the \(y\)-variable. The terminal condition is given by \(\phi(E_T, P_T)\), where \(\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) is a measurable function, non-decreasing in its \(E\)-variable and Lipschitz continuous in the \(P\)-variable. In its simplest form, it is given typically by:

\[
e \mapsto \phi(e) = 1_{(e > \Lambda)}, \ \Lambda > 0.
\]  

\text{(1.2)}

The constant \(\Lambda\) appears as a cap on emissions set by the regulator. The shape given in (1.2) translates the fact that a penalty (here set to one) is paid if the emission are above the regulatory cap at \(T\).

We observe that (1.1) has a forward one dimensional \(E\)-component of bounded variation and a backward component with an irregular terminal condition (1.2). This renders the mathematical analysis of the FBSDE system difficult. Nevertheless, the well-posedness and main features of (1.1) have been thoroughly studied in [8], see also Section 2.1 below. Notably, the authors of [8] prove existence and uniqueness of the solution to (1.1) but show at the same time that the terminal condition can only be attained in the following weak sense:

\[
1_{(\Lambda, +\infty)}(E_T) \leq Y_T \leq 1_{(\Lambda, +\infty)}(E_T),
\]  

\text{(1.3)}

using to simplify the presentation at this point the terminal function (1.2). Their study is based on the celebrated markovian representation of \(Y\) as

\[
Y_t = \mathcal{V}(t, P_t, E_t), \text{ for } t < T,
\]  

\text{(1.4)}

and the careful analysis of the property of \(\mathcal{V}\), where \(\mathcal{V}\), known as the decoupling fields, is solution to a quasilinear PDE. As mentioned in [8], the FBSDE system can be seen as a random perturbation of a scalar conservation law. The behavior at the terminal time is reminiscent of shocks appearing in conservation law. Let us note that the markovian

\footnote{To alleviate the notation, we assume that \(P\) and \(W\) have the same dimension and the coefficient functions of \(P\) are time-homogeneous. Note however that \(\sigma\) will not be assumed to be uniformly elliptic, which allows to consider a dimension of \(P\) as time and to embed the case of different dimension for \(P\) and \(W\) in our framework.}
representation breaks down at $T$ as indicated by (1.3). Moreover, the function $V$ is only locally Lipschitz-continuous on $[0, T)$:

$$|V(t, p, e) - V(t, p', e')| \leq c_1|p - p'| + \frac{1}{c_2(T - t)}|e - e'|,$$  

(1.5)

for some constants $c_1, c_2 > 0$, $(p, p', e, e') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$.

The application to carbon market is also a key motivation for our numerical study here: efficient numerical simulation of the price $Y$ would allow to calibrate properly the model to market data and validate its efficiency in practice. A first approach for the numerical approximation of $Y$ or $V$ would be to use PDE methods, and this is suggested in [26]. However, in the economic applications we have in mind, the dimensionality of the process $P$ prevents generally the use of these methods. In order to work on problems in moderate dimension, say 5 to 10, some probabilistic methods could be introduced. Probabilistic schemes have already been designed for FBSDEs and one could be tempted (as we were) to use the already known methods to tackle the numerical approximation of (1.1). In [3], the authors use a Picard Iteration method to decouple the FBSDE system and then obtain an approximation of $V$ by performing iteratively linear regression. Unfortunately, this method has only been shown to be convergent in the case of Lipschitz coefficient and for small coupling between the forward and backward part (or equivalently small time horizon), see [3] for details. Recently, machine learning methods have been considered for BSDEs approximation, especially for their applicability in very high-dimensional setting. In particular, [23] has analysed the deep BSDE solver introduced in [22] again in the setting of small coupling. In [17], a grid algorithm is introduced where the decoupling is obtained by a predictor: there, the time horizon or the coupling is arbitrary but the diffusion coefficient of the forward process must be uniformly elliptic. As observed, the FBSDE system under study is degenerated in the $E$-component and the terminal condition is not Lipschitz, so that none of the known methods for FBSDEs are proved to be convergent in the setting of (1.1). Moreover, the above methods fail, in practice, to approximate correctly the solution to (1.1). To empirically illustrate this fact, we consider the following toy model borrowed from [9]:

**Example 1.1 (Linear model).**

\[
\begin{align*}
\frac{dP_t}{t} &= \sigma dW_t \\
\frac{dE_t}{t} &= \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P^\ell_t - Y_t \right) dt \\
\frac{dY_t}{t} &= Z_t \cdot dW_t
\end{align*}
\]

with terminal function given by (1.2) and where $W$ is a $d$-dimensional Brownian motion and $\sigma > 0$.

By using a change of variable, this $d + 1$ dimensional model can be reduced to a one dimension model. Indeed, from [9] Proposition 6], there exists $\nu \in C^{1,2}([0, T), \mathbb{R})$,
solution to
\[ \hat{c}_t \nu - \nu \hat{c}_t \nu + \frac{\sigma^2 (T - t)^2}{2} \hat{c}_{\xi \xi} \nu = 0 \quad \text{and} \quad \nu(T, \xi) = \phi(\xi). \tag{1.9} \]

By essentially applying Ito's formula (see the proof of [9, Proposition 7] for details on proving (1.9)), one obtains that
\[ \mathcal{V}(t, e, \nu) = \nu(t, e + (T - t) \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} \xi \ell). \]

This observation allows us to use efficient methods to solve (1.9) numerically and to compare them to numerical solutions obtained by “classical” FBSDE scheme. In particular, we use a probabilistic method studied in [5] using interacting particle system, a class of mean field SDE, to obtain a “proxy” for \( \mathcal{V}(0, 0, e) \), see also [4, 28].

Going back to the approximation of (1.6)-(1.7)-(1.8) by “classical” FBSDEs methods, we first note that, in [11, Chapter 4], the authors report an application of the Bender-Zhang scheme [3]. The main issue is then that the Picard iteration does not converge to a single limit. Next, we have tested the Delarue-Menozzi scheme [17] and the deep FBSDE solver [23] for different value of \( \sigma \), the results are given in Figure 1.

Except maybe for the Delarue-Menozzi scheme in Figure 1c, the methods fail clearly to approximate the correct solution \( \mathcal{V}(0, 0, e) \). The problem comes from the nonlinear transport part of the equation in this degenerate setting. Indeed, the methods seem unable to recover the correct weak entropy solution. This is particularly clear on Figure 1a, where the level of noise is extremely small and the correct solution is almost the solution to the inviscid Burger’s equation. This leads us to introduce a new method to approximate the FBSDE system (1.1).

As already mentioned, in the socio-economic applications, the dimension of the \( P \)-variable is generally large. On the contrary, the \( E \)-variable is constrained to be of dimension one. We note also that approximating the dynamics of the \( P \)-variable corresponds to approximating simply a diffusion process, which can be easily done. To take into account these key differences in the two variables, we follow a splitting approach to compute numerically the solution \( \mathcal{V} \). On a discrete time grid, we iterate backward in time, a diffusion operator where the \( E \)-variable is fixed to capture the effect of the \( P \)-dynamics in (1.1), and a transport operator where the \( P \) variable is fixed to capture the effect of \( E \)-dynamics in (1.1). Our main theoretical result, see Theorem 2.2, proves...
that this scheme is convergent at a rate $\frac{1}{2}$ with respect to the time step. Our analysis is done under the minimal assumption used in [8] to obtain existence and uniqueness of the solution $V$. One of the main difficulty encountered is therefore due to the gradient explosion at the end of the time interval (1.5).

Then, we propose various implementations of the splitting scheme. They have however a common structure: given a discrete transport operator, the diffusion part is computed by means of probabilistic methods. The overall scheme is a then a sequence of (non linear) regression in the high dimensional space where lives the approximation of $V$ with respect to the $E$-variable. In our numerical experiments, we consider approximations of the transport operator by conservative finite difference methods (Lax-Friedrichs scheme or Upwind scheme), see e.g. [31]. As we do not always have access to a proxy for the tested models, we introduce an alternative implementation of the splitting scheme to validate our numerical results. It combines a particle approximation of the transport operator with a tree based regression for the diffusion operator. We validate empirically both approaches on Example 1.1 for which we have a one dimensional proxy at hand. We then test models with no equivalent one-dimensional PDE but whose $d + 1$ dimensional version can be reduced to 2-dimensional specification, see Section 3.3 for details. The tree-based algorithm is then used as a proxy as it is very efficient in low-dimension (we test dimension 4) for the models under consideration. When combining feedforward neural networks to compute the regression step and finite difference scheme for the transport step, we show that our splitting procedure can compute precisely and in reasonable amount of time the solutions of $10 + 1$ dimensional models.

The rest of the paper is organised as follows. In Section 2 we first recall key properties of the theoretical solution. We then introduce the splitting approach and prove the convergence of the splitting scheme. In Section 3 we present a regression method for the splitting scheme at a theoretical level, which uses a grid approximation of the transport operator. We then introduce various implementation of the transport operator and a neural network approximation for the regression part. We finally present various numerical experiments to validate the efficiency of our method in practice.

**Notation.**

In the following we will use the following spaces

- For fixed $0 \leq a < b < +\infty$ and $I = [a, b]$ or $I = [a, b)$, $S^{2,k}(I)$ is the set of $\mathbb{R}^k$-valued càdlàg $\mathcal{F}_t$-adapted processes $Y$, s.t.

$$\|Y\|_{S^2}^2 := \mathbb{E}\left[\sup_{t \in I} |Y_t|^2\right] < \infty.$$  

Note that we may omit the dimension and the terminal date in the norm notation as this will be clear from the context. $S^{2,k}_c(I)$ is the subspace of process with continuous sample paths.

\footnote{French acronym for right continuous with left limits.}
For fixed $0 \leq a < b < +\infty$, and $I = [a, b]$, we denote by $\mathcal{H}^{2,k}(I)$ the set of $\mathbb{R}^k$-valued progressively measurable processes $Z$, such that
\[
\|Z\|^2_{\mathcal{H}^2} := \mathbb{E} \left[ \int_I |Z_t|^2 dt \right] < \infty.
\]
For $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, measurable and non-decreasing in its second variable, the functions $\varphi_-$ and $\varphi_+$ are the left and right continuous versions, respectively defined, for $(p, e) \in \mathbb{R}^d \times \mathbb{R}$, by,
\[
\varphi_-(p, e) = \sup_{e' < e} \varphi(p, e') \quad \text{and} \quad \varphi_+(p, e) = \inf_{e' > e} \varphi(p, e').
\]
Moreover, we denote by $\| \cdot \|_\infty$ the essential supremum:
\[
\|\varphi\|_\infty = \text{esssup}_{(p,e)\in\mathbb{R}^d \times \mathbb{R}} |\varphi(p, e)|.
\]

2. A splitting scheme

In this section, we introduce a theoretical splitting scheme to compute the solution of the singular FBSDEs. This scheme consists into iterating a "diffusion step" and a "transport step" on a discrete time grid $\pi := \{0 =: t_0 < \cdots < t_n < \cdots < t_N := T\}$, where $N$ is a positive integer. For latter use, we denote by $|\pi| := \max_{0 \leq n < N} (t_{n+1} - t_n)$.

Before defining the splitting scheme for the system (1.1), we recall some key theoretical properties of the solution obtained in [8], with slight extensions for the case of P-dependent terminal condition in [10]. The rest of the section is then dedicated to the proof of an upper bound for the convergence rate of the splitting scheme in terms of $|\pi|$. This is our main theoretical result, given in Theorem 2.2 Numerical implementations are presented in the next section.

2.1 Well-posedness and properties of singular FBSDEs

We first introduce two classes of functions, that will be useful in the sequel. The terminal condition function for (1.1) will belong to the first one.

Definition 2.1. Let $\mathcal{K}$ be the class of functions $\phi : \mathbb{R}^d \times \mathbb{R} \to [0, 1]$ such that $\phi$ is $L_\phi$-Lipschitz in the first variable for some $L_\phi > 0$ and non-decreasing in its second variable, namely
\[
|\phi(p, e) - \phi(p', e)| \leq L_\phi |p - p'| \quad \text{for all} \quad (p, p', e) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R},
\]
\[
\phi(p, e') \geq \phi(p, e) \quad \text{if} \quad e' \geq e.
\]
and moreover satisfying,

\[ \sup_e \phi(p, e) = 1 \quad \text{and} \quad \inf_e \phi(p, e) = 0 \quad \text{for all} \quad p \in \mathbb{R}^d. \quad (2.3) \]

Note that the bounds given in (2.3) are motivated by our main application, but up to a rescaling they can be arbitrary changed. We now introduce a class of admissible coefficient functions, for which the singular BSDE is well-posed, see Theorem 2.1 below. This class will be also useful to define the splitting scheme.

We consider three positive constants \(L, \ell_1\) and \(\ell_2\).

**Definition 2.2.** Let \(A\) be the class of functions \(B : \mathbb{R}^d \to \mathbb{R}^d, \Sigma : \mathbb{R}^d \to \mathcal{M}_d, F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) which are \(L\)-Lipschitz continuous functions. Moreover, \(F\) is strictly decreasing in \(y\) and satisfies, for all \(p \in \mathbb{R}^d\),

\[ \ell_1 |y - y'|^2 \leq (y - y')(F(y', p) - F(y, p)) \leq \ell_2 |y - y'|^2. \quad (2.4) \]

**Standing assumptions:** From now on, we assume that \((b, \sigma, \mu) \in \mathcal{A}\), recalling (1.1).

**Theorem 2.1** (Proposition 2.10 in [8], Proposition 3.2 in [10]). Let \(\tau > 0\), \((B, \Sigma, F) \in \mathcal{A}\) and \(\Phi \in \mathcal{K}\).

Given any initial condition \((t_0, p, e) \in [0, \tau) \times \mathbb{R}^d \times \mathbb{R}\), there exists a unique progressively measurable 4-tuple of processes \((P_{t_0}^{t_0, p, e}, E_{t_0}^{t_0, p, e}, Y_{t_0}^{t_0, p, e}, Z_{t_0}^{t_0, p, e})_{t_0 \leq t \leq \tau} \in \mathcal{S}^{2,1}_c([t_0, \tau]) \times \mathcal{S}^{2,1}_c([t_0, \tau]) \times \mathcal{H}^{2, d}([t_0, \tau])\) satisfying the dynamics

\[
\begin{align*}
    dP_{t_0}^{t_0, p, e} &= B(P_{t_0}^{t_0, p, e})dt + \Sigma(P_{t_0}^{t_0, p, e})dW_t, \quad P_{t_0}^{t_0, p, e} = p \in \mathbb{R}^d, \\
    dE_{t_0}^{t_0, p, e} &= F(P_{t_0}^{t_0, p, e}, Y_{t_0}^{t_0, p, e})dt, \quad E_{t_0}^{t_0, p, e} = e \in \mathbb{R}, \\
    dY_{t_0}^{t_0, p, e} &= Z_{t_0}^{t_0, p, e} \cdot dW_t,
\end{align*}
\]

and such that

\[ \mathbb{P} \left[ \Phi_-(P_{\tau}^{t_0, p, e}, E_{\tau}^{t_0, p, e}) \leq \lim_{t \uparrow \tau} Y_{t_0}^{t_0, p, e} \leq \Phi_+(P_{\tau}^{t_0, p, e}, E_{\tau}^{t_0, p, e}) \right] = 1. \quad (2.6) \]

The unique decoupling field defined by

\[ [0, \tau) \times \mathbb{R}^d \times \mathbb{R} \ni (t_0, p, e) \to w(t_0, p, e) = Y_{t_0}^{t_0, p, e} \in \mathbb{R} \]

is continuous and satisfies

1. For any \(t \in [0, \tau)\), the function \(w(t, \cdot, \cdot)\) is \(1/(l_1(\tau - t))\)-Lipschitz continuous with respect to \(e\),

2. For any \(t \in [0, \tau)\), the function \(w(t, \cdot, \cdot)\) is \(C\)-Lipschitz continuous with respect to \(p\), where \(C\) is a constant depending on \(L, \tau\) and \(L_\phi\) only.
3. Given \((p, e) \in \mathbb{R}^d \times \mathbb{R}\), for any family \((p_t, e_t)_{0 \leq t < \tau}\) converging to \((p, e)\) as \(t \uparrow \tau\), we have

\[
\Phi_-(p, e) \leq \liminf_{t \to \tau} w(t, p_t, e_t) \leq \limsup_{t \to \tau} w(t, p_t, e_t) \leq \Phi_+(p, e).
\]

(2.7)

4. For any \(t \in [0, \tau]\), the function \(w(t, \cdot, \cdot) \in \mathcal{K}\).

Using the previous result, we define the following operator associated to (1.1).

**Definition 2.3.** We define the operator \(\Theta\) by

\[
(0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \Theta_h(\psi) = v(0, \cdot) \in \mathcal{K}
\]

where \(v\) is the decoupling field given in Theorem 2.1 with parameters \(\tau = h, B = b, \Sigma = \mu\) and \(\Phi = \psi\).

We also deduce from Theorem 2.1 that \((\Theta_t)_{0 < t}\) is a semi-group of non-linear operators. In particular, we observe that \(V(0, \cdot) := \Theta_T(\phi) = \prod_{0 \leq n < N} \Theta_{t_{n+1} - t_n}(\phi)\), recall (1.4).

The following result arises from the proof of the previous Theorem, see [10].

**Corollary 2.1** (Approximation result). Let \(\tau > 0\), \((B, \Sigma, F) \in \mathcal{A}\) and \(\Phi \in \mathcal{K}\). Let \((\phi^k)_{k \geq 0}\) be a sequence of smooth functions belonging to \(\mathcal{K}\) and converging pointwise towards \(\phi\) as \(k\) goes to \(+\infty\). For \(\epsilon > 0\), consider then \(w^{\epsilon,k}\) the solution to:

\[
\partial_t u + F(p, u)\partial_p u + L_p u + \frac{1}{2} \epsilon^2 (\partial^2_{pe} u + \Delta_{pp} u) = 0 \quad \text{and} \quad u(\tau, \cdot) = \phi^k
\]

(2.9)

where \(\Delta_{pp}\) is the Laplacian with respect to \(p\), and \(L_p\) is the operator

\[
L_p(\varphi)(t, p, e) = \partial_p \varphi(t, p, e)B(p) + \frac{1}{2} \text{Tr} \left[ A(p) \partial_{pp}^2 \varphi \right](t, p, e),
\]

(2.10)

with \(\partial_p\) denotes the Jacobian with respect to \(p\), and \(A = \Sigma \Sigma^\top\), where \(\top\) is the transpose and \(\partial_{pp}^2\) is the matrix of second derivative operators. (For later use, we define \(L^\epsilon := L_p + \frac{1}{2}\epsilon^2 (\partial_{ee}^2 + \Delta_{pp})\).

Then the functions \(w^{\epsilon,k}\) are \(C^{1,2}\) (continuously differentiable in \(t\) and twice continuously differentiable in both \(p\) and \(e\)) and \(\lim_{\epsilon \to 0} \lim_{k \to \infty} w^{\epsilon,k} = w\) where the convergence is locally uniform in \([0, \tau) \times \mathbb{R}^d \times \mathbb{R}\). Moreover, for all \(k, \epsilon\), \(w^{k, \epsilon}(t, \cdot) \in \mathcal{K}\).

### 2.2 Scheme Definition

Let us first introduce the transport step where the diffusion part is frozen.

**Definition 2.4** (Transport step). We set

\[
(0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto T_h(\psi) = \tilde{v}(0, \cdot) \in \mathcal{K}
\]

where \(\tilde{v}\) is the decoupling field defined in Theorem 2.1 with parameters \(\tau = h, B = 0, \Sigma = 0, F = \mu\) and terminal condition \(\Phi = \psi\).
In the definition above, \( \tilde{v}(\cdot) \) is the unique entropy solution to
\[
\tilde{v}_t w + \tilde{v}_e (\mathfrak{M}(p, w)) = 0, \quad \text{where } \mathfrak{M}(p, y) = \int_0^y \mu(p, v) dv, \ 0 \leq y \leq 1, \tag{2.11}
\]
and \( \tilde{v}(h, \cdot) = \psi \). We will use this fact in the numerical section.

We now introduce the diffusion step, where conversely, the \( E \) - process is frozen to its initial value.

**Definition 2.5** (Diffusion step). We set
\[
(0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \mathcal{D}_h(\psi) = \tilde{v}(0, \cdot) \in \mathcal{K}
\]
where \( \tilde{v}(0, \cdot) \) is the decoupling in Theorem 2.1 with parameters \( \tau = h \), \( B = b \), \( \Sigma = \sigma \), \( F = 0 \) and terminal condition \( \Phi = \psi \).

Observe that, for \( t \in [0, h) \),
\[
\tilde{v}(t, p, e) = \mathbb{E}[\psi(P_{h}^{t, h}, e)] \quad \text{and} \quad \tilde{v}(t, \cdot) \in \mathcal{K} . \tag{2.12}
\]
We can now define the theoretical scheme on \( \pi \) by a backward induction.

**Definition 2.6** (Theoretical splitting scheme). We set
\[
(0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \mathcal{S}_h(\psi) := \mathcal{T}_h \circ \mathcal{D}_h(\psi) \in \mathcal{K}.
\]
For \( n \leq N \), we denote by \( u_{\pi}^n \) the solution of the following backward induction on \( \pi \):
- for \( n = N \), set \( u_{\pi}^N := \phi \),
- for \( n < N \), \( u_{\pi}^n = \mathcal{S}_{n+1-t_n}(u_{\pi}^{n+1}) \).

The \( (u_{\pi}^n)_{0 \leq n \leq N} \) stands for the approximation of the decoupling field \( \mathcal{V}(t, \cdot) \) for \( t \in \pi \).
Moreover, we observe, from the property of \( \mathcal{T} \) and \( \mathcal{D} \), that
\[
u_{\pi}^n \in \mathcal{K}, \ \text{for all } 0 \leq n \leq N. \tag{2.13}
\]

### 2.3 Convergence analysis

Our main theoretical result concerning the splitting is the following

**Theorem 2.2.** Under our standing assumptions, the following holds
\[
\int_{\mathbb{R}} |\mathcal{V}(0, p, e) - u_0^{\pi}(p, e)| de \leq C T(1 + |p|^2) \sqrt{|\pi|},
\]
for a positive constant \( C \).

The proof of the Theorem is postponed to the end of the section. It is classically based on the study of the scheme’s stability and its truncation error.
2.3.1 Truncation error

We need to compare, for \( \psi \in \mathcal{K}, \Theta_h(\psi) \) and \( \mathcal{S}_h(\psi), h > 0 \), to assess the truncation error. As already mentioned, the true solution \( \mathcal{V} \) has minimal locally Lipschitz regularity and it exhibits a gradient explosion in the \( E \)-variable near the terminal time \( T \). In the proof below, we thus need to consider smoothed version of the decoupling fields introduced in the definition of \( \Theta, T, \mathcal{D} \) and \( \mathcal{S} \).

First of all, for a given \( \psi \in \mathcal{K} \), we consider a smooth approximation sequence \( \psi^k \) as in Corollary 2.1. In particular, \( \nu^{k,\epsilon} \) is the smooth approximation of the decoupling field \( \nu(\Theta_h \psi) \) in Definition 2.3 and the associated FBSDEs, for \( 0 \leq t \leq h \),

\[
P^k_t = p + \int_0^t b(P^k_s)ds + \int_0^t \sigma(P^k_s)dW_s + \epsilon W^\prime_t, \tag{2.14}
\]

\[
E^{k,\epsilon}_t = e + \int_0^t \mu(Y^{k,\epsilon}_s, P^k_t)ds + \epsilon B_t, \tag{2.15}
\]

where \((W', B)\) is a Brownian motion independent from \( W \). Note that for the reader’s convenience, we omit the dependence upon the starting point \((0, p, e)\) in the FBSDEs notation. The convergence of \( \nu^{k,\epsilon} \) to \( \nu \) is given in Corollary 2.1.

We also need to consider a smooth version of \( \mathcal{S}_h(\psi) \), that we define now:

1. for \( 0 \leq t \leq h \), set:

\[
\bar{\nu}^{k,\epsilon}(t, p, e) = \mathbb{E}\left[ \psi^k(P^\prime_{h-t}, t, e) \right] \tag{2.16}
\]

2. then, \( \bar{\nu}^{k,\epsilon} \) is the decoupling of the following FBSDE, for all \( p \in \mathbb{R}^d, e \in \mathbb{R} \):

\[
d\bar{Y}^{k,\epsilon}_t = \bar{Z}^{k,\epsilon}_t dB_t, \tag{2.17}
\]

\[
d\bar{E}^{k,\epsilon}_t = \mu(Y^{k,\epsilon}_t, P^k_t)dt + \epsilon dB_t \tag{2.18}
\]

with terminal condition \( \bar{Y}^{k,\epsilon}_h = \bar{v}^{k,\epsilon}(0, p, \bar{E}^{k,\epsilon}_h) \) and initial condition \( \bar{E}^{k,\epsilon}_0 := e \).

Observe that the \( P \)-variable is frozen in the above definition and that \( \bar{Y}^{k,\epsilon}_t = \bar{v}^{k,\epsilon}(t, p, \bar{E}^{k,\epsilon}_t) \), for \( 0 \leq t \leq h \).

Before studying the truncation error, we give a strong local error control between the smooth approximations \( \nu^{k,\epsilon} \) and \( \bar{\nu}^{k,\epsilon} \). Note that this local error control in \( \sqrt{h} \) does not allow obtaining a converging global error control. We will however use it to obtain a better local control error in \( L1 \)-norm, see the proof of Proposition 2.1.

Lemma 2.1. Under our standing assumptions on \((\mu, b, \sigma)\), the following holds, for \( p \in \mathbb{R}^d, h > 0 \),

\[
\sup_{t \in [0, h], e \in \mathbb{R}} |v^{k,\epsilon}(t, p, e) - \bar{\nu}^{k,\epsilon}(t, p, e)| \leq C_{L, \psi}(1 + |p|) \sqrt{h}.
\]

Importantly, \( C_{L, \psi} \) does not depend on \( k \) nor \( \epsilon \), however it depends on the Lipschitz constant of \( \psi \) in the \( P \)-variable.
We set, for $v$, since $\dot{Y}$ is a martingale, recall (2.17),

\[ V_t^{k,\epsilon} = V_0^{k,\epsilon} - \int_0^t \left( \partial_t v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) + \mu(\tilde{Y}_s^{k,\epsilon}, P^*_s, \tilde{E}_s^{k,\epsilon}) + \mathcal{L}^\epsilon v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) \right) ds + M_t^{k,\epsilon}, \]

where $M_t^{k,\epsilon}$ is a square-integrable martingale.

From the PDE (2.9) satisfied by $V$, we get

\[ V_t^{k,\epsilon} = V_0^{k,\epsilon} - \int_0^t \left( \frac{\partial_y \mu(\tilde{Y}_s^{k,\epsilon}, P^*_s, \tilde{E}_s^{k,\epsilon})}{\partial y} - \lambda v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) + \lambda \delta P_s \right) ds + \int_0^t \frac{\partial_x \mu(\tilde{Y}_s^{k,\epsilon}, P^*_s, \tilde{E}_s^{k,\epsilon})}{\partial x} ds + M_t^{k,\epsilon}. \]

(2.19)

We set, for $0 \leq s \leq h$, $\delta P_s = P^*_s - p$ and

\[ c_s := \int_0^1 \frac{\partial_y \mu(\tilde{Y}_s^{k,\epsilon}, P^*_s, \tilde{E}_s^{k,\epsilon})}{\partial y} - \lambda v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) + \lambda \delta P_s \right) ds + \int_0^t \frac{\partial_x \mu(\tilde{Y}_s^{k,\epsilon}, P^*_s, \tilde{E}_s^{k,\epsilon})}{\partial x} ds + M_t^{k,\epsilon}. \]

(2.20)

From (2.4), we know that, for all $0 \leq s \leq h$,

\[ c_s \leq -\ell_1 < 0 \text{ and } |d_s| \leq L. \]

(2.22)

Then, (2.19) reads

\[ V_t^{k,\epsilon} = V_0^{k,\epsilon} - \int_0^t c_s v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) ds + \int_0^t d_s \delta P_s \partial_x v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) ds + M_t^{k,\epsilon}. \]

(2.23)

We set, for $0 \leq t \leq h$, $\mathcal{E}_t = e^{\int_0^t c_s \partial_x v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) ds}$ and, we have

\[ \mathcal{E}_0 = 1, \quad \text{ for all } 0 \leq t \leq h, \]

(2.24)

since $v^{k,\epsilon} \in K$, recall (2.22). We then compute

\[ \mathcal{E}_t V_t^{k,\epsilon} = V_0^{k,\epsilon} + \int_0^t d_s \delta P_s \partial_x v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) \mathcal{E}_s ds + N_t^{k,\epsilon} \]

(2.25)

where $N_t^{k,\epsilon}$ is a square-integrable martingale. In particular, we get

\[ |V_0^{k,\epsilon}| \leq \mathbb{E} \left[ |V_0^{k,\epsilon}| - \int_0^h \frac{|d_s|}{|c_s|} \delta P_s |c_s \partial_x v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) \mathcal{E}_s ds \right]. \]

(2.26)

recall (2.22). Observe that, for all $0 \leq s \leq h$,

\[ \dot{\mathcal{E}}_s := c_s \partial_x v^{k,\epsilon}(s, P^*_s, \tilde{E}_s^{k,\epsilon}) \mathcal{E}_s \]

(2.27)
where the dot denotes classically the time derivative. We thus deduce from (2.26)

\[ |V_0^{k,\epsilon}| \leq \mathbb{E}\left(|V_h^{k,\epsilon}| + C \sup_{s \in [0,h]} |\delta P_s| (E_0 - E_h) \right) \]
\[ \leq \mathbb{E}\left(|V_h^{k,\epsilon}| \right) + C(1 + |p|) \sqrt{h}. \quad (2.28) \]

Now, we observe that

\[ \mathbb{E}\left(|V_h^{k,\epsilon}| \right) = \mathbb{E}\left[|v^{k,\epsilon}(0, p, \tilde{E}_h^{k,\epsilon}) - \psi(P_h^{k,\epsilon}, \tilde{E}_h^{k,\epsilon})|\right] \]
\[ = \mathbb{E}\left[|\psi(P_h^{k,\epsilon}, \tilde{E}_h^{k,\epsilon}) - \psi(P_h^{k,\epsilon}, \tilde{E}_h^{k,\epsilon})|\right] \]
\[ \leq 2L_p \mathbb{E}[|\delta P_h|]. \quad (2.29) \]

We thus get

\[ \mathbb{E}\left(|V_h^{k,\epsilon}| \right) \leq C(1 + |p|) \sqrt{h}, \text{ which, combined with } (2.28), \text{ concludes the proof.} \]

We now turn to the main result for this part, which gives an upper bound to the error between \( S_h(\psi) \) and \( \Theta_h(\psi) \) that is effectively a control on the truncation error of the scheme.

**Proposition 2.1** (truncation error). Under our standing assumptions on the coefficients \((\mu, b, \sigma)\), the following holds, for \( \psi \in \mathcal{K} \):

\[ \int |S_h(\psi)(p,e) - \Theta_h(\psi)(p,e)| \, de \leq C_{L_{\psi}} (1 + |p|^3) h^{\frac{3}{2}}, \quad (2.32) \]

for \( p \in \mathbb{R}^d, \, h > 0 \).

**Proof.** 1. We first consider the regularised version of the decoupling fields, as introduced in (2.16)-(2.17). Let \( V_{t}^{r,k,\epsilon} = \tilde{Y}_{t}^{r,k,\epsilon} - \epsilon V^{r,k}(t, P_t^{r,\epsilon}, \tilde{E}_t^{r,\epsilon}), \) for \( t \leq h \), with \( (P_t^{r,\epsilon}, \tilde{E}_t^{r,\epsilon}) = (p, e) \). We first observe that by definition (2.16) and the fact that \( P^r \) and \( B \) (and thus subsequently \( \tilde{E}^{r,k,\epsilon} \)) are independent,

\[ \mathbb{E}\left[V_{t}^{r,k,\epsilon} | B \right] = \tilde{v}(0, p, \tilde{E}_t^{k,\epsilon}) - \mathbb{E}\left[\psi(P_{t}^{\epsilon}, \tilde{E}^{k,\epsilon}_{t}) | B \right] = 0. \quad (2.33) \]

We also observe that, for \( 0 \leq t \leq h \),

\[ |V_{t}^{r,k,\epsilon}| \leq |\tilde{v}^{r,k}(t, p, \tilde{E}_t^{k,\epsilon}) - v^{r,k}(t, p, \tilde{E}_t^{k,\epsilon})| + |v^{r,k}(t, p, \tilde{E}_t^{k,\epsilon}) - v^{r,k}(t, P_t^{r,\epsilon}, \tilde{E}_t^{r,\epsilon})| \]
\[ \leq C(1 + |p|) \sqrt{h} + C |P_t^{r,\epsilon} - p|, \]

where for the last inequality we used Lemma 2.1 and the uniform Lipschitz continuity of \( v^{r,k} \) (Note that \( C \) depends upon the Lipschitz constant of \( \psi \)). This leads to

\[ \sup_{t \in [0,h]} \mathbb{E}\left[\text{esssup}_{e}|V_{t}^{r,k,\epsilon}| \right] \leq C(1 + |p|) \sqrt{h}. \quad (2.34) \]
2. Let us consider the tangent process $\partial E_t^{k,\epsilon}$ given by

$$
\partial E_t^{k,\epsilon} = 1 + \int_0^t \partial_y (\tilde{Y}_s^{k,\epsilon}, p) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds
$$

(2.35)

$$
= \partial_y (\tilde{Y}_s^{k,\epsilon}, p) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) ds.
$$

(2.36)

And we observe that $0 \leq \partial E_t^{k,\epsilon} \leq 1$, for all $0 \leq t \leq h$.

In order to bound the error $\int |V_0^{k,\epsilon,\epsilon}| de$, we will study the dynamics of $t \mapsto \left| \mathbb{E} \left[ V_t^{e,k,\epsilon} \partial E_t^{k,\epsilon} \right] \right| de$.

Using (2.19), we compute

$$
V_t^{e,k,\epsilon} \partial E_t^{k,\epsilon} = V_0^{e,k,\epsilon} + \int_0^t V_s^{e,k,\epsilon} \partial_y (\tilde{Y}_s^{k,\epsilon}, p) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds + N_t^{k,\epsilon}
$$

(2.37)

$$
= \int_0^t \left( \mu (\tilde{Y}_s^{k,\epsilon}, p) - \mu (v^{k,\epsilon} (s, P_s^\epsilon, \tilde{E}_s^{k,\epsilon}), P_s^\epsilon) \right) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds
$$

(2.38)

where $N_t^{k,\epsilon}$ is a square-integrable martingale. Taking expectation on both sides of the above equality, we get

$$
|V_0^{e,k,\epsilon}| \leq \mathbb{E} \left[ \int_0^h \left( \mu (\tilde{Y}_s^{k,\epsilon}, p) - \mu (v^{k,\epsilon} (s, P_s^\epsilon, \tilde{E}_s^{k,\epsilon}), P_s^\epsilon) \right) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds \right]
$$

(2.39)

$$
+ \mathbb{E} \left[ \int_0^h V_s^{e,k,\epsilon} \partial_y (\tilde{Y}_s^{k,\epsilon}, p) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds \right].
$$

(2.40)

Recall (2.33). Since $\partial E_t^{k,\epsilon}$, $\partial E_t^{k,\epsilon}$ and $-\partial_y (\epsilon)$ are non-negative, we deduce

$$
|V_0^{e,k,\epsilon}| \leq \mathbb{E} \left[ \int_0^h \left| \mu (\tilde{Y}_s^{k,\epsilon}, p) - \mu (v^{k,\epsilon} (s, P_s^\epsilon, \tilde{E}_s^{k,\epsilon}), P_s^\epsilon) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds \right] \right]
$$

(2.41)

$$
- \mathbb{E} \left[ \int_0^h V_s^{e,k,\epsilon} \partial_y (\tilde{Y}_s^{k,\epsilon}, p) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds \right].
$$

(2.42)

Integrating the previous inequality, we get

$$
\int |V_0^{e,k,\epsilon}| de \leq \mathbb{E} \left[ \int_0^h \left| \mu (\tilde{Y}_s^{k,\epsilon}, p) - \mu (v^{k,\epsilon} (s, P_s^\epsilon, \tilde{E}_s^{k,\epsilon}), P_s^\epsilon) \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) \partial E_t^{k,\epsilon} ds \right] \right]: = A_1
$$

(2.43)

$$
- \mathbb{E} \left[ \int_0^h V_s^{e,k,\epsilon} \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) ds \right].
$$

We know study the term $A_2$ above: Since, $\tilde{v}^{e,k}$ is bounded and $\mu$ is Lipschitz continuous, we have

$$
| \int \partial E_t^{k,\epsilon} (s, p, \tilde{E}_s^{k,\epsilon}) ds | \leq C (1 + |p|)
$$

(2.44)
and then

\[ A_2 \leq C(1 + |p|) \mathbb{E}\left[ \int_0^h \text{esssup}_e |V^e_{s,k,\epsilon}| \, ds \right] \]

\[ \leq C(1 + |p|^2) h^{\frac{3}{2}} , \]

recalling (2.34).

We now compute an upper bound for \( A_1 \). Since \( \mu \) is Lipschitz-continuous, we have

\[ A_1 \leq C \mathbb{E}\left[ \int_0^h \left( |V^e_{s,k,\epsilon}| + |P^e_s - p| \right) \hat{c} \left[ v^{k,\epsilon}(s, P^e_s, \tilde{E}^{k,\epsilon}_s) \right] \, ds \, de \right] \]

\[ \leq \mathbb{E}\left[ \int_0^h (\text{esssup}_e |V^e_{s,k,\epsilon}| + |P^e_s - p|) \hat{c} \left[ v^{k,\epsilon}(s, P^e_s, \tilde{E}^{k,\epsilon}_s) \right] \, de \right] \]

Since \( v^{k,\epsilon} \) is (uniformly) bounded and using (2.34), we obtain

\[ A_1 \leq C(1 + |p|^2) h^{\frac{3}{2}} . \tag{2.45} \]

Combining the estimate for \( A_1 \) and \( A_2 \), we conclude:

\[ \int |v^{k,\epsilon}(0, p, e) - v^{k,\epsilon}(0, p, e)| \, de \leq C(1 + |p|^2) h^{\frac{3}{2}} . \tag{2.46} \]

Then passing to the limits in \( k, \epsilon \) and using the dominated convergence theorem conclude the proof.

\[ \square \]

### 2.3.2 Scheme stability

We now study the scheme’s stability by introducing a perturbed version of the scheme given in Definition 2.6.

**Definition 2.7** (Perturbed scheme). For \( n \leq N \), let \( \eta_n : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) be measurable functions satisfying

\[ \int |\eta_n(p, e)| \, de \leq c(1 + |p|^{\kappa}) \text{ for some } \kappa \geq 1, \epsilon > 0, \tag{2.47} \]

where \( \kappa, \epsilon \) do not depend on \( n \). We denote by \( (u'_n)_{0 \leq n \leq N} \) the solution of the following backward induction:

- for \( n = N \), set \( u'_N := \phi + \eta_N \),
- for \( n < N \), \( u'_n = S_{t_{n+1} - t_n}(u'_{n+1}) + \eta_n \).

**Proposition 2.2** (\( L^1 \)-stability). Under our standing assumptions, the following holds true for \( (\eta_n) \), satisfying (2.47), perturbation of the scheme given in Definition 2.6:

\[ \max_{0 \leq n \leq N} \mathbb{E}\left[ \int |u'_n - u_n'|(K^{0,\mu}_n, e) \, de \right] \leq \sum_{n=0}^N \mathbb{E}\left[ \int |\eta_n|(K^{0,\mu}_n, e) \, de \right] . \tag{2.48} \]
Proof. 1.a In the proof, we denote \( \delta u_n^\pi = u_n - u_n' \), for all \( n \leq N \).
We observe that
\[
\int |\delta u_N|(p, e)de \leq \int |\eta_N|(p, e)de \leq \epsilon(1 + |p|^\kappa), \tag{2.49}
\]
where we used \ref{2.47} for the last inequality. We have then
\[
\mathbb{E}\left[ \int |\delta u_N|(P^{0,p}_T, e)de \right] \leq \mathbb{E}\left[ \int |\eta_N|(P^{0,p}_T, e)de \right] < \infty. \tag{2.50}
\]
We used the fact that for any \( q > 0 \),
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |P_t^{0,p}|^q \right] \leq C_q(1 + |p|^q). \tag{2.51}
\]
1.b Assume (induction hypothesis)
\[
|\delta u_{n+1}|(p, e)de \leq K_{n+1}(1 + |p|^\kappa). \tag{2.52}
\]
for some positive \( K_{n+1} < +\infty \). (Note that \( K_N = \epsilon \) from \ref{2.49}). Denoting \( \bar{u}_{n+1} = \mathcal{D}(t_{n+1} - t_n)(u_{n+1}^\pi) \) and \( \bar{u}'_{n+1} = \mathcal{D}(t_{n+1} - t_n)(u'_{n+1}) \), we have
\[
|u_n^\pi - u'_n| \leq |\mathcal{T}(t_{n+1} - t_n)(\bar{u}_{n+1}) - \mathcal{T}(t_{n+1} - t_n)(\bar{u}'_{n+1})| + |\eta_n|. \tag{2.53}
\]
From Lemma 3.6 in \cite{10} applied to \( \mathcal{T} \), we obtain
\[
\int |\mathcal{T}(t_{n+1} - t_n)(\bar{u}_n) - \mathcal{T}(t_{n+1} - t_n)(\bar{u}'_n)|(p, e)de \leq \int |\bar{u}_{n+1} - \bar{u}'_{n+1}|(p, e)de \tag{2.54}
\]
Moreover,
\[
\bar{u}_{n+1} - \bar{u}'_{n+1} = \mathbb{E}\left[ u_{n+1}^\pi(P_{t_{n+1} - t_n}^{0,p}, e) - u'_{n+1}(P_{t_{n+1} - t_n}^{0,p}, e) \right] \tag{2.55}
\]
which leads to
\[
\int |\delta u_n|(p, e)de \leq \int \mathbb{E}\left[ |\delta u_{n+1}|(P_{t_{n+1} - t_n}^{0,p}, e)de \right] + \int |\eta_n|(p, e)de. \tag{2.56}
\]
From the induction hypothesis, we know that
\[
\mathbb{E}\left[ |\delta u_{n+1}|(P_{t_{n+1} - t_n}^{0,p}, e)de \right] \leq K_{n+1}(1 + \mathbb{E}\left[ |P_{t_{n+1} - t_n}^{0,p}|^\kappa \right])
\]
leading to \( \mathbb{E}\left[ |\delta u_{n+1}|(P_{t_{n+1} - t_n}^{0,p}, e)de \right] \leq K_{n+1}C_\kappa(1 + |p|^\kappa) \), where we used \ref{2.51}. Using \ref{2.47}, we then obtain
\[
\int |\delta u_n|(p, e)de \leq K_n(1 + |p|^\kappa) \text{ with } K_n = K_{n+1}C_\kappa + \epsilon,
\]
proving the induction hypothesis (2.52) for the next step. Moreover, this shows that
\[ E \int |\delta u_{n+1}|(P_{tn+1}^{0,p}, e) de ] < +\infty \] and
\[ E \int |\delta u_n|(P_{tn}^{0,p}, e) de ] < +\infty. \] Thus, we deduce from
(2.56),
\[ E \int |\delta u_n|(P_{tn}^{0,p}, e) de ] \leq E \int |\delta u_{n+1}|(P_{tn+1}^{0,p}, e) de ] + E \int |\eta_n|(P_{tn}^{0,p}, e) de ] (2.57) \]

2. From step 1.a and 1.b above, we deduce that (2.57) holds for all \( n < N \). Iterating
the inequality on \( n \) concludes the proof.

2.3.3 Proof of Theorem 2.2

We classically write the true solution given by the decoupling field as a perturbed
splitting scheme, for a perturbation \( \zeta_n(\cdot) = \Theta_{tn+1-tn}(\mathcal{V}(tn+1, \cdot)) - \mathcal{S}_{tn+1-tn}(\mathcal{V}(tn+1, \cdot)) \). We observe that, indeed, for all \( n \leq N \),
\[ \mathcal{V}(tn, \cdot) = \mathcal{S}_{tn+1-tn}(\mathcal{V}(tn+1, \cdot)) + \zeta_n(\cdot) . (2.58) \]

From Proposition 2.1, we know that \( \zeta_n \) satisfies (2.47) with \( \kappa = 2 \), recall (2.32), and then
\[ E \int |\zeta_n|(P_{tn}, e) de ] \leq C(1 + |p|^2)(tn+1 - tn)^{\frac{3}{2}}. (2.59) \]

Using Proposition 2.2, we obtain that for \( n = 0 \) in particular,
\[ \int |\mathcal{V}(0, p, e) - u_0^e(p, e)| de ] \leq CT(1 + |p|^2)\sqrt{\pi] . \]

3 Numerical schemes

The possible difference in the dimension between the \( E \)-variable and the \( P \)-variable leads
us to treat these variables very differently in the numerical procedure. The convergence
result obtained in the previous section indicates that it is indeed reasonable to use a
splitting scheme. We then work toward a fully implementable scheme building on this
approach.

3.1 A regression method for the splitting scheme

We present here a – still theoretical – discrete-time scheme which combines a finite
difference approximation of the transport operator and a probabilistic approximation of
the diffusion operator. In the next section, we discuss various possible implementations.
We first suppose that the approximation of the transport operator is given as follows. Let $J$ be a positive integer and $\mathcal{E} = \{e_j\}_{1 \leq j \leq J}$ a discrete grid of $\mathbb{R}$. We denote by $T_h^\mathcal{E}$ an approximation of the operator $T_h$ on $\mathcal{E}$. Namely,

$$\mathbb{R}^d \times \mathbb{R}^J \ni (p, \theta) \mapsto T_h^\mathcal{E}(p, \theta) \in \mathbb{R}^d.$$  

This means that for each $p \in \mathbb{R}^d$, $T_h^\mathcal{E}(p, \cdot)$ is an approximation on the grid $\mathcal{E}$ of the corresponding equation (2.11) on $[0, h]$. We assume moreover that it satisfies, for some $q \geq 1$ and $q' \geq 1$,

$$|T_h^\mathcal{E}(p, \theta)| \leq C(1 + |p|^q + |\theta|^{q'}) .$$  

The terminal condition $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is simply approximated on $\mathcal{E}$ by $\theta^j = \psi(p, e_j)$, for all $1 \leq j \leq J$ and $p \in \mathbb{R}^d$.

Given this approximate transport operator, we now introduce a probabilistic approximation of $\mathcal{V}(0, p, \cdot)$ on $\mathcal{E}$. To this end, let us consider the Euler scheme associated to $P$ on $\pi$, namely, for $n \geq 0$,

$$\hat{P}^\pi_{t_{n+1}} = \hat{P}^\pi_{t_n} + b(\hat{P}^\pi_{t_n})(t_{n+1} - t_n) + \sigma(\hat{P}^\pi_{t_n}) \Delta \hat{W}_n \quad \text{and} \quad \hat{P}^\pi_0 = p .$$  

Here, $\Delta \hat{W}_n$, $0 \leq n \leq N-1$ are independent random variables that stands for an approximation of the law of $W(t_{n+1} - W_n) \quad 0 \leq n \leq N-1$ and we assume that their moments verify $\mathbb{E}[\Delta \hat{W}_n^\rho] \leq C_\rho |t_{n+1} - t_n|^\frac{\rho}{2}$, $\rho \geq 1$. It is well known from the Lipschitz continuity assumption on $b$ and $\sigma$ that, for any $\rho \geq 1$,

$$\mathbb{E}\left[\sup_{t \in \pi} |\hat{P}^\pi_t|^{\rho}\right] \leq C_\rho (1 + |p|^{\rho'}) .$$  

We now define a discrete time process $(\Gamma_n)_{0 \leq n \leq N}$ valued in $\mathbb{R}^J$ as follows.

**Definition 3.1.** $(\Gamma_n)_{0 \leq n \leq N}$ is solution to the following backward scheme:

1. For $n = N$, $\Gamma^j_N = \phi(\hat{P}^\pi_{t_N}, e_j)$ for $1 \leq j \leq J$.

2. For $n < N$, compute

$$\Gamma^j_n = T_h^\mathcal{E}(\hat{P}^\pi_{t_n}, \mathbb{E} \Gamma^j_{n+1} \hat{P}^\pi_{t_n}) .$$  

For later use, we define the auxiliary process $(\hat{\Gamma}_n)$ by

$$\hat{\Gamma}^j_n = \mathbb{E} \Gamma^j_{n+1} \hat{P}^\pi_{t_n} \quad \text{for all} \ 1 \leq j \leq J.$$  

We also importantly observe that, due to the Markovian property of $\hat{P}^\pi$ on $\pi$, $(\Gamma_n)$ satisfies

$$\Gamma_n := \gamma_n(\hat{P}^\pi_{t_n}) , \quad 0 \leq n \leq N ,$$  

where the functions $\gamma_n : \mathbb{R}^d \to \mathbb{R}^J$, are given by
Definition 3.2.  
1. For \( n = N, \gamma^j_N(p) = \phi(p, e_j), 1 \leq j \leq J, p \in \mathbb{R}^d \). 

2. Then, compute for \( n < N, p \in \mathbb{R}^d \),
\[
\bar{\gamma}^j_n(p) = \mathbb{E} \left[ \gamma^j_{n+1} \left( p + b(p) h + \sigma(p) \Delta \hat{W}_n \right) \right] \quad \text{for all } 1 \leq j \leq J, 
\]
\[
\gamma^j_n(p) = T^\xi_h(p, \bar{\gamma}_n(p)).
\]

With the above definitions, we have that \( \Gamma_0 = \gamma_0(P_0) \) which stands for an approximation of \( \mathcal{V}(0, P_0, \cdot) \) on the grid \( \mathcal{E} \).

To obtain the wellposedness of the previous definitions, we check that the conditional expectations at each step of the scheme are well defined. This follows from a direct backward induction using (3.2) and (3.4).

Depending on how large \( d \), the dimension of the \( P \)-variable, is, we may choose various probabilistic schemes to compute (3.6). This has been thoroughly studied in the context of BSDEs approximation and various methods have been suggested: linear regression \([18, 19, 20]\), quantization methods \([2, 1, 32]\), cubature methods \([15, 16, 12]\) or Malliavin calculus approach \([6, 14]\). In the next section, we present a non-linear regression method used e.g. in \([27]\).

3.2 Implementation using non linear regression

We now turn to an implementation that can work in a high dimensional setting for \( P \). To perform the regression step in Definition 3.1, we will use Neural Networks representation of the value function. This will be coupled with conservative finite difference approximation of the transport operator that we first recall.

3.2.1 Conservative Finite Difference approximation of transport equation

We shall now consider conservative methods for the transport operator associated to the backward equation (2.11).

Recall that, for a given positive integer \( J \), \( \mathcal{E} = (e_j)_{1 \leq j \leq J} \) is a uniform grid of \( \mathbb{R} \) where we set \( \delta := e_{j+1} - e_j \). We also introduce \( \mathbb{R} = \{ r_0 = 0 < \cdots < r_k < \cdots < r_K = h \} \) a uniform grid for a given positive integer \( K \), and we set \( \delta := h/K \).

We first consider the Lax-Friedrichs approximation to the backward transport equation (2.11) and define \( T^\text{LF}_{h} : \mathbb{R}^d \times \mathbb{R}^J \rightarrow \mathbb{R}^J \) the approximation of the associated operator \( T_h \). It is defined as follows, see e.g. [31 Chapter 12].

**Definition 3.3** (Lax-Friedrichs). For a given \( p \in \mathbb{R}^d \) and \( \theta \in \mathbb{R}^J \), let \( (V^j_k)_{1 \leq k \leq K, 1 \leq j \leq J} \) denotes the approximation at the point \((r_k, e_j)\) The steps to compute \( V \) are:

- at time \( r_K = h \): set \( V^K_j = \theta_j, 1 \leq j \leq J \),
- for \( 0 \leq k < K \): set \( V^1_k = V^{k+1}_1 \), \( V^K_j = V^{k+1}_J \) and compute, for \( 1 < j < J \) :
\[
V^j_k = \frac{1}{2} (V^{k+1}_{j+1} + V^{k+1}_{j-1}) + \frac{\delta}{2\delta} \left( \mathcal{M}(p, V^{k+1}_{j+1}) - \mathcal{M}(p, V^{k+1}_{j-1}) \right).
\]
Then, set $T_{\mathcal{E},\mathbb{R},h}^{LF}(p, \theta) := V^0$.

When the function $\mu$ has constant sign, a more satisfactory method to use is the upwind method, as it is less diffusive. Since in the application to carbon markets given in Example 3.2 and 3.1 below, this will be the case, we consider the upwind method for $\mu \geq 0$. We thus now define $T_{\mathcal{E},\mathbb{R},h}^{U} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ the approximation of the associated operator $T_h$ as follows, see again e.g. [31].

**Definition 3.4** (Upwind for $\mu \geq 0$). For a given $p \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^J$ let $(V_j^k)_{1 \leq j \leq J, 1 \leq k \leq K}$ denotes the approximation at the point $(r_k, e_j)$ The steps to compute $V$ are:

1. at time $r_k = h$: set $V_j^K = \theta_j$, $1 \leq j \leq J$,
2. for $0 \leq k < K$: set $V_j^k = V_j^{k+1}$ and compute, for $1 \leq j \leq J$:

$$ V_j^k = V_j^{k+1} + \frac{\delta}{\delta} \left( \mathfrak{M}(p, V_j^{k+1}) - \mathfrak{M}(p, V_j^k) \right). \quad (3.11) $$

Then, set $T_{\mathcal{E},\mathbb{R},h}^{U}(p, \theta) := V^0$.

3.2.2 Non-linear regression and implemented scheme

We first mention that for this part the Euler scheme [33] is computed using real Brownian increment, namely $\Delta W_n = (W_{n+1} - W_n), 0 \leq n \leq N - 1$. We have seen in the last section two possible implementations of the transport operator $T_h$ on the spatial grid $\mathcal{E}$, that we shall denote for this part simply by $T_h^{\mathcal{E}}$. The last point to precise is the computation of the conditional expectation part of the scheme in Definition 3.1, where at each time step the quantities $\gamma_n(\hat{P}^n_{t_n}) = \mathbb{E}\left[ \Gamma_{n+1}|\hat{P}^n_{t_n} \right]$ has to be estimated, recall Definition 3.2. In order to do so, we will use deep learning as it was demonstrated to be very efficient for high dimensional system, already in the setting of FBSDEs, see e.g. [24, 27].

The functions $(\gamma_n)$ will be optimally approximated by a feedforward neural network. We denote by $N \mathcal{N}_{d_0,d_1,L,m}$ the set of neural nets, which are functions $\Phi(\cdot; \Theta) : \mathbb{R}^{d_0} \rightarrow \mathbb{R}^{d_1}$, parametrised by $\Theta$ and with the following characteristics: the input dimension is $d_0$, the output dimension is $d_1$, $L + 1$ is the number of layers, $m = (m_l)_{0 \leq l \leq L}$ where $m_l$ is the number of neurons on each layer, $l = 0, \ldots, L$: by default, $m_0 = d$ and $m_L = d_1$. The neural network has thus $L - 1$ hidden layers. We refer, to e.g. [27] Section 2] for a detailed description of feedforward neural network. The number of total parameters is $N_{L,m} = \sum_{l=0}^{L-1} m_l(1 + m_{l+1})$, and thus $\Theta \in \mathbb{R}^{N_{L,m}}$.

Given $T_h^{\mathcal{E}} = T_{\mathcal{E},\mathbb{R},h}^{U}$ or $T_h^{\mathcal{E}} = T_{\mathcal{E},\mathbb{R},h}^{LF}$, the scheme to compute $(\hat{\gamma}_n, \bar{\gamma}_n)$ approximation of $(\gamma_n, \bar{\gamma}_n)$ in Definition 3.2 is given as follows.

**Definition 3.5.**

1. At $t_N = T$, $\hat{\gamma}_j^N(p) = \hat{\gamma}_N^j(p) = \phi(p, e_j)$, $1 \leq j \leq J$, $p \in \mathbb{R}^d$.

2. For $n = N - 1, \ldots, 1$: given a simulation of $P^n_{t_n}$, optimize

$$ \hat{\mathcal{L}}_n(\Theta) = \mathbb{E}\left[ T_h^{\mathcal{E}}(\hat{P}^n_{t_{n+1}}, \hat{\gamma}_{n+1}(\hat{P}^n_{t_{n+1}})) - \left( y_n(\hat{P}^n_{t_n}, \Theta) + z_n(\hat{P}^n_{t_n}, \Theta)(W_{n+1} - W_n) \right) \right]^2 \quad (3.12) $$
where \((\mathcal{Y}_n(\cdot, \Theta), \mathcal{Z}_n(\cdot, \Theta)) \in \mathcal{N}_{d,d+1} \times \mathcal{L}_m,\) so that

\[
\Theta^*_n = \arg\min_{\Theta \in \mathcal{R}^N_m} \hat{L}_n(\Theta) \quad \text{and then} \quad \hat{\gamma}_n(\cdot) := \mathcal{Y}_n(\cdot, \Theta^*_n).
\]

3. At the initial time \(t_0 = 0,\) compute \(\hat{\gamma}_0(\hat{P}^*_0) = \mathbb{E}\left[ T^\varepsilon_H(\hat{P}^*_0; \hat{\gamma}_1(\hat{P}^*_1)) \right].\)

The function \(\hat{\gamma}_0(\cdot)\) stands for the numerical approximation of \(\mathcal{V}(0, P_0, \cdot).\)

Remark 3.1. (i) The loss minimisation in (3.12) is done using a Stochastic Gradient Descent algorithm: we use Adam Optimizer \([30]\) provided in the Keras API \([13]\). The good approximation of the function \(\hat{\gamma}_n\) is guaranteed by universal approximation theorem for neural networks \([25]\) and is quite efficient in practice, as demonstrated by our numerical examples below.

(ii) Definition 3.3 should be compared with the scheme DBDP1 in \([27, \text{Section 3}]\). In this paper, the authors compute a non-linear conditional expectation (related to BSDEs) at each step: Here, we only compute a conditional expectation and use \(\mathcal{Z}\) as a control variate. In particular, differently to DBDP1, we have to apply \(T^\varepsilon_H\) in \(\hat{L}_n(\cdot)\) at each step. Note also that one can not apply directly DBDP1 to the singular FBSDE (1.1) under study as it is a fully-coupled FBSDE.

(iii) A key point is to ensure the stability of the finite difference scheme for the transport equation, namely that the CFL condition is satisfied. For example, for the Lax-Friedrichs scheme given in Definition 3.3, one has to enforce, for each time \(t_n:\)

\[
\sup_{1 \leq k \leq K, 1 \leq j \leq J} \left| \mu(V^j_k, \hat{P}^*_n) \frac{\delta}{\delta} \right| < 1,
\]

see e.g. \([31, \text{Chapter 13}]\). In practice, we choose \(B,\) such that

\[
\sup_{y \in [0,1], p \in [-B,B]^d} \left| \mu(y, p) \frac{\delta}{\delta} \right| < 1.
\]

The constant \(B\) depends obviously on the parameters \(\delta, \bar{\delta}\) and should be large enough. Then, in the simulation, \(\hat{P}^*_n\) is projected on \([-B,B]^d\). We also ensure that \(0 \leq V^j_k \leq 1\) by truncating \(\hat{\gamma}_n^j\) if necessary and relying on the scheme monotony.

3.3 Numerical experiments

In this section, we present the results of our numerical experiments that show that the splitting scheme is efficient in practice to approximate \(\mathcal{V}(\cdot).\) The method presented in the previous section, will be tested on two complementary models to Example 1.1. The first one reads as follows.
Example 3.1 (BM with positive emission).

\[ dP_t^\ell = \sigma dW_t^\ell \text{ and } dE_t = \mu \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^d P_t^\ell, Y_t \right) dt \] (3.13)

with \( \mu(p, y) = 1 + \frac{1}{1+y} - y \) and \( \phi(p, e) = 1_{\{e \geq 0\}} \).

The above model will have non-negative \( \mu \) which is more realistic if one has in mind application to carbon market. A critic could be however that it is driven by a Brownian Motion and that it will not suffer any discrete time error. We then introduce a multiplicative model as follows.

Example 3.2 (Multiplicative model).

\[ dP_t^\ell = \mu P_t^\ell dt + \sigma P_t^\ell dW_t^\ell, \ P_0^\ell = 1, \ \text{and} \ dE_t = \tilde{\mu}(P_t, Y_t) dt \] (3.14)

with \( \tilde{\mu}(p, y) = \left( \prod_{\ell=1}^d p^\ell \right)^{\frac{1}{d}} e^{-\theta y}, \text{for some } \theta > 0 \) and \( \phi(p, e) = 1_{\{e \geq 0\}} \).

We are not aware of any explicit solution for these models but they have the property that any \( d + 1 > 2 \) dimensional model can be recast as a 2-dimensional model (one dimension for the \( P \)-variable, one dimension for the \( E \)-variable). This will be used for numerical validation of the non-linear regression scheme used for the multidimensional models by introducing an alternative scheme efficient in low dimension (see next Section). However, one should notice that there is no simple equivalent one-dimensional PDE available for Examples 3.2 or 3.13 as it is the case for Example 1.1 recall (1.9).

3.3.1 An alternative scheme

To validate empirically the results obtained with the non-linear regression scheme, we could use a PDE method in low dimension. However, we chose to use here another method based on the splitting scheme that will combine a particle method with tree-like regression. This method will be efficiently implemented on the Examples 3.1, 3.2 and 1.1 for two main reasons: We work in moderate dimension and the process \( P \) can be expressed as a function of the underlying Brownian motion, namely \( P_t = \mathcal{P}(t, W_t) \).

Here, contrarily to the previous section, \( \Delta \hat{W}_n := \hat{W}_{t_{n+1}} - \hat{W}_{t_n}, 0 \leq n \leq N-1 \) stands for discrete approximation of the Brownian increments \( (W_{t_{n+1}} - W_{t_n})_{0 \leq n \leq N-1} \). We also assume that the time grid \( \pi \) is equidistant and thus \( |\pi| = \frac{T}{N} =: \delta \). One could then use, for all \( 1 \leq \ell \leq d \), \( \mathbb{P}(\hat{W}_{t_n}^\ell = \sqrt{\delta}) = \mathbb{P}(\hat{W}_{t_n}^\ell = -\sqrt{\delta}) = \frac{1}{2} \) in (3.3) but this requires \( 2d \) points in total for the approximation. We use instead the cubature formula introduced in [21] Section A.2 which requires only 2d points. Denoting \( (\ell^\ell)_{1 \leq \ell \leq d} \) the canonical basis of \( \mathbb{R}^d \), we set, for \( 1 \leq i \leq I = 2d \), \( \mathbb{P}(\hat{W}_{t_n} = \omega^i_b) = \frac{1}{2d} \) and \( \omega^i_b = -\sqrt{\delta} \ell^\ell \), if \( i = 2\ell \) or \( \omega^i_b = \sqrt{\delta} \ell^\ell \) if \( i = 2\ell - 1 \). At a given point \((t_n, \hat{P}^\pi_{t_n} = \mathcal{P}(t_n, \hat{W}_{t_n}, e), \) the approximation of \( D_b(\psi) \), recall Definition 2.3 reads then simply

\[ \mathbb{E}\left[ \psi(\hat{P}^\pi_{t_{n+1}}, e) | \hat{P}^\pi_{t_n} \right] = \frac{1}{2d} \sum_{i=1}^I \psi(\mathcal{P}(t_{n+1}, \hat{W}_{t_{n+1}} + \omega^i_b), e). \] (3.15)
For $0 \leq n \leq N$, we denote by $\mathcal{S}_n$, the discrete support of the random variable $\hat{W}_n$. We observe that $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ and for $x \in \mathcal{S}_n$, $x + \Delta \hat{W}_n \in \mathcal{S}_{n+1}$. Thus, when computing (3.15), there is no need for an interpolation step, if $\psi(\cdot, e)$ is known on $\mathcal{S}_{n+1}$. We will obviously exploit this fact and compute recursively, backward in time, the approximation of $\mathcal{V}$ on the discrete sets $(\mathcal{S}_n)_{0 \leq n \leq N}$. The full approximation of the diffusion operator $D_{(t_{n+1} - t_n)}$ that acts at time $t_n$, will be given after discussing the discrete version of the operator $\mathcal{T}$, as it will be then more easily justified.

Let us thus now introduce a discrete version of the operator $\mathcal{T}$, recall Definition 2.4, that will compute an approximation to (2.11) written in forward form: We shall use the celebrated Sticky Particle Dynamics (SPD) [7] see also [29, Section 1.1]. The SPD is particularly simple to implement in our case, since, due to the monotonicity assumption on $(\mu, \psi)$, there is no particle colliding! For $M \geq 1$, let $D_M = \{ e = (e_1, \ldots, e_m, \ldots) \in \mathbb{R}^M \mid e_1 \leq \cdots \leq e_m \leq \cdots \leq e_M \}$. The discrete version of $\mathcal{T}$ will act on empirical CDF or equivalently on empirical distribution $\frac{1}{M} \sum_{m=1}^{M} \delta_{e_m}$ ($\delta_e$ is the Dirac mass at $e$). Generally, $\psi(p, \cdot)$, which is a CDF for each $p \in \mathbb{R}^d$, would need to be approximated in an optimal way on $D_M$. We observe here that the terminal condition $\phi$ to Examples 3.1, 3.2 and 1.1, is simply represented by $e = (0, \ldots, 0)$. The iterative algorithm allows us to restrict our study to terminal condition $\psi$, such that $\psi(p, \cdot) = H(\frac{1}{M} \sum_{m=1}^{M} \delta_{e_m})$ for some $e \in D_M$, where $H$ is the Heaviside function and $*$ the convolution operator. The approximation of $\mathcal{T}$ is then given by

$$\mathbb{R}^d \times D_M \ni (p, e) \mapsto \mathcal{T}^M_h(p, e) = (E_{p,m}^M)_{1 \leq m \leq M} \in D_M,$$

where $(E_{p,m}^t)_{1 \leq m \leq M}$ is a set (of positions) of particles computed as follows. Given the initial position $e \in D_M$ (representing $\psi$) and velocities $(\bar{F}_m(p))_{1 \leq m \leq M}$ set to $\bar{F}_m(p) = \int_{(m-1)/M}^{m/M} \mu(p, y)dy$, we consider $M$ particles $(E_{p,m}^t)_{1 \leq m \leq M}$, whose positions at time $t \in [0, h]$ are simply given by

$$E_{p,m}^t = e_m + \bar{F}_m(p)t.$$

We observe that $(E_{p,m}^t)_{1 \leq m \leq M} \in D_M$, for all $t \in [0, h]$, as $-\mu$ is non-decreasing.

We are now ready to define the approximation of the diffusion operator $D_{t_{n+1} - t_n}$, denoted $\mathcal{D}_n^M$: it will take into account that $\mathcal{T}^M_h$ acts at the level of particles. Introduce, to ease the presentation, $\mathcal{P}_{n+1} = \{ p = \Psi(t_{n+1}, w), w \in \mathcal{S}_{n+1} \}$, which is simply the (discrete) support of $\hat{P}_{n+1}$. Assume that

$$\mathcal{S}_{n+1} \ni w \mapsto \Psi(w) = e^w \in D_M$$

is given such that for $p \in \mathcal{P}_{n+1}$, $p = \Psi(t_{n+1}, w)$, we have $\psi(p, \cdot) = H \ast (\frac{1}{M} \sum_{m=1}^{M} \delta_{e_m})$. 

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Then, for \( w \in \mathcal{S}_n \), setting \( w_{in+1} \) in (3.15) reads
\[
\bar{v}_n(w, e) := \mathbb{E} \left[ \psi(P_{n+1}^e) \right| P_{tn}^\pi = \Psi(t_n, w) \right] = \frac{1}{2d} \sum_{i=1}^I \sum_{m=1}^M \delta_{w_{im+1}}(e),
\]
\[
= H_\ast \left( \frac{1}{2dM} \sum_{i=1}^I \sum_{m=1}^M \delta_{w_{im+1}}(e) \right).
\]
This means that, the function \( e \mapsto \bar{v}(w, e) \) is an empirical CDF and is determined by the particles \( \mathcal{E} = \bigcup_{i=1}^I \{ e^{w_{im+1}} \} \). There is no need to keep \( 2dM \) particles at step \( n \), when the function \( \psi \) at step \( n+1 \) is given by \( M \) particles (for each \( p \in \mathcal{P}_{n+1} \)). To reduce the number of particles, we first sort the cloud of particles \( \mathcal{E} \) to obtain \( \bar{e}_n \), then we consider \( \bar{\gamma}_n \) and then \( \gamma_n \).

**Definition 3.6 (Alternative scheme).**

1. At \( n = N \): Set \( \gamma_N := (0, \cdots, 0) \) whose empirical CDF is \( \phi \). Then define, \( \gamma_N \) by
\[
\mathcal{S}_N \ni w \mapsto \gamma_N(w) = e_N.
\]

2. For \( n < N \): Given \( \gamma_{n+1} : \mathcal{S}_{n+1} \rightarrow D_M \), define \( \bar{\gamma}_n : \mathcal{S}_n \rightarrow D_M \) by \( \bar{\gamma}_n = D_\pi^M(\gamma_{n+1}) \) and then \( \gamma_n \) by
\[
\mathcal{S}_n \ni w \mapsto \gamma_n(w) = T_h^M(\Psi(t_n, w), \bar{\gamma}_n(w)) \in D_M.
\]

The approximation of \( \mathcal{V}(0, 0, \cdot) \) is then given by \( H_\ast \gamma_0 \).

### 3.3.2 Numerical results

In this section, we report the findings of the numerical tests we performed on the models given in Examples 3.1, 3.2 and 1.1 using the non-linear regression and splitting method of Definition 3.1 and the alternative scheme, presented in Definition 3.6.

Concerning the non-linear regression, we use a common structure in all our experiments for the feedforward neural networks used in (3.12) to represent \( (Y, Z) \), namely:
- The output layer is of dimension \( (J + 1) \times d \), where \( J \) is size of the \( E \)-variable grid;
- Two intermediate layers of dimension \( \kappa J \times d + 10 \) (\( \kappa J \) is fixed to 20 below);
- An input layer of dimension \( d \).

As already mentioned, the training is done using the Adam optimiser using 100 minibatches with size 50 and batch normalization. We check validation loss every 30 iterations with the validation batch of size 500. The learning rate is initially fixed at \( \eta = 0.001 \).
We first observe that our schemes are able to reproduce the proxy for the true solution of Example 1.1 as reported in Figure 2. This has to be compared with the results of Figure 1 for the “classical” FBSDEs methods. Let us emphasize that the non-linear regression scheme (denoted NN&LF) is tested in dimension $d = 10$ and the alternative scheme (denoted BT&SPD) in dimension $d = 4$. Since the Lax-Friedrichs scheme presents a diffusive phenomenon, we increase the space discretization steps to overcome this effect when $\sigma$ decreases. In the numerical computations, we choose respectively $J = 500, 1000, 1500$ for $\sigma = 1.0, 0.3, 0.01$ to obtain satisfactory approximations.

Figure 2: Model of Example 1.1: Comparison of the two methods Neural Nets & Lax-Friedrichs (NN&LF) with $d = 10$ and the alternative scheme (BT&SPD) with $d = 4$. The Proxy solution is given by the same particle method used in Figure 1 on the one-dimensional PDE (1.9). Lax-Friedrichs scheme implemented with discretization of space $J = 1500, 1000, 500$, for $\sigma = 0.01, 0.3, 1$ respectively and number of time step $K = 30$. The number of time step for the splitting is $N = 64$. For BT&SPD, the number of particles is $M = 3500$ and the number of time steps $N = 20$.

Figure 3: Example 3.1 in dimension $d = 10$: Comparison of two methods Neural nets & Upwind scheme and solution obtained using the alternative scheme on equivalent 4-dimensional model. The Upwind scheme used discretization of space $J = 100, 300, 400$ respectively for $\sigma = 1.0, 0.3, 0.01$ and number of time step $K = 20$. The number of time step for the splitting is $N = 32$. For BT&SPD, the number of particles is $M = 3500$, and the number of time steps $N = 20$.

Next, we tested our scheme on the models of Example 3.1 and Example 3.2. The results are reported on the graphs in Figure 3 and 4 respectively. Since the function
\( \mu \) is always positive in these two examples, we can use an Upwind scheme. Unlike the Lax-Friedrichs scheme, the Upwind scheme is less diffusive, and we can lower the number of space discretization. In our example, we take \( J = 100, 300, 400 \) respectively for \( \sigma = 1.0, 0.3, 0.01 \). We are not aware of an exact solution for this model, so we compare both the non-linear regression scheme (NN\&U) for \( d = 10 \) and the alternative scheme (BT\&SPD) on an equivalent four dimensional model.

![Graphs showing comparison of methods](image)

**Figure 4:** Example 3.2 in dimension \( d = 10 \): Comparison of two methods Neural nets \& Upwind scheme and solution obtained using the alternative scheme on equivalent 4-dimensional model (BT\&SPD). The Upwind scheme used discretization of space \( J = 100, 400, 500 \) respectively for \( \sigma = 1, 0.3, 0.01 \) and number of time step \( K = 20 \). The number of time step for the splitting is \( N = 32 \). For BT\&SPD, the number of particles is \( M = 3500 \), and the number of time steps \( N = 20 \).

As we pointed out before, Lax-Friedrichs scheme is more diffusive than Upwind scheme: this is illustrated on Figure 5, by considering the case where \( \sigma = 0.01 \), and taking \( J = 400 \) only for the LF space discretisation. On this graph and the computations below, the ‘Proxy’ to the true solution is obtained by running an equivalent one-dimensional model using the alternative scheme (BT\&SPD) with parameters: number of particles \( M = 3500 \), number of time step \( N = 20 \). Table 1 presents the error obtained by comparing the non-linear regression scheme to this proxy, the computational times is also given.\(^3\) The \( L^1 \)-error is the error used in the theoretical part, but one can see that the \( L\infty \) error behaves also very well. The computational times can still be reduced on our examples by diminishing the batch size but this would certainly not generalise to other models more challenging for the training of the neural networks.

\(^3\)Intel Core i5-8265U, 16.0 GB RAM.
Finally, we want to empirically estimate the convergence rate of the error introduced by the splitting. We consider the model \( \sigma = 0.3 \) and for which there is no discrete-time simulation error (as the forward process is a Brownian Motion). We consider a set of number of time steps \( N := \{4, 8, 16, 32, 64, 128\} \), and compute the \( L^1 \) and \( L^\infty \) error by NN & Upwind method (with \( K = 20, J = 400 \)). The proxy solution is always given by alternative scheme in one dimensional equivalent model, to achieve a better precision. The empirical convergence rate with respect to the number of time step is close to one, which is slightly better than the upper bound obtained in Theorem \ref{thm:2.2}.

Finally, in Table \ref{table:2}, we report the computational time and the error associated to different dimensions \( d = 1, 5, 10 \) in Example \ref{ex:3.2} where \( \sigma = 0.3 \) with NNs & Upwind scheme \((K = 20, J = 400)\) and with fixed number of time step for the splitting \( N = 32 \). Per our specification, the computational time does not increase exponentially and, importantly, neither the empirical error. This behavior is expected from the non-linear regression using neural networks.
Figure 6: Convergence rate on $N$ for model Example 3.2 with parameters $d = 10, \sigma = 0.3$ and $K = 20, J = 400$.

| Dimension | $d = 1$ | $d = 5$ | $d = 10$ |
|-----------|---------|---------|----------|
| Time      | 673s    | 1077s   | 1488s    |
| $L^1$ Error | 0.0431  | 0.0594  | 0.0756  |
| $L^\infty$ Error | 0.0867 | 0.1041 | 0.1253 |

Table 2: Computational cost in example 3.2 for different dimension $d$ (for the $P$-variable).

References

[1] V. Bally and G. Pagès. Error analysis of the optimal quantization algorithm for obstacle problems. *Stochastic Processes and their Applications*, 106(1):1 – 40, 2003.

[2] V. Bally and G. Pagès. A quantization algorithm for solving multidimensional discrete-time optimal stopping problems. *Bernoulli*, 9(6):1003–1049, 2003.

[3] Christian Bender and Jianfeng Zhang. Time discretization and markovian iteration for coupled fbsdes. *The Annals of Applied Probability*, 18(1):143–177, 2008.

[4] Mireille Bossy and Denis Talay. Convergence rate for the approximation of the limit law of weakly interacting particles: application to the burgers equation. *The Annals of Applied Probability*, 6(3):818–861, 1996.

[5] Mireille Bossy and Denis Talay. A stochastic particle method for the mckean-vlasov and the burgers equation. *Mathematics of computation*, 66(217):157–192, 1997.

[6] B Bouchard and T. Touzi. Discrete-time approximation and monte-carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111(2):175 – 206, 2004.

[7] Yann Brenier and Emmanuel Grenier. Sticky particles and scalar conservation laws. *SIAM journal on numerical analysis*, 35(6):2317–2328, 1998.

[8] René Carmona and François Delarue. Singular FBSDEs and scalar conservation laws driven by diffusion processes. *Probability Theory and Related Fields*, 157(1-2):333–388, 2013.
[9] René Carmona, François Delarue, Gilles-Edouard Espinosa, and Nizar Touzi. Singular forward–backward stochastic differential equations and emissions derivatives. *The Annals of Applied Probability*, 23(3):1086–1128, 2013.

[10] Jean-François Chassagneux, Hinesh Chotai, and Dan Crisan. Modelling multi-period carbon markets using singular forward backward sdes. *arXiv preprint arXiv:2008.09044*, 2020.

[11] Jean-François Chassagneux, Hinesh Chotai, and Mirabelle Muûls. *A Forward-Backward SDEs Approach to Pricing in Carbon Markets*. SpringerBriefs in Mathematics of Planet Earth, Springer, 2017.

[12] Jean-Francois Chassagneux and Camilo Garcia Trillos. Cubature method to solve bsdes: Error expansion and complexity control. *Mathematics of Computation*, 89(324):1895–1932, 2020.

[13] Francois Chollet et al. Keras, 2015.

[14] Dan Crisan, K. Manolarakis, and Touzi Nizar. On the monte carlo simulation of bsdes: An improvement on the malliavin weights. *Stochastic Processes and their Applications*, 120(7):1133 – 1158, 2010.

[15] Dan Crisan and Konstantinos Manolarakis. Solving backward stochastic differential equations using the cubature method: application to nonlinear pricing. *SIAM Journal on Financial Mathematics*, 3(1):534–571, 2012.

[16] Dan Crisan and Konstantinos Manolarakis. Second order discretization of backward sdes and simulation with the cubature method. *The Annals of Applied Probability*, 24(2):652–678, 2014.

[17] François Delarue and Stéphane Menozzi. A forward-backward stochastic algorithm for quasi-linear PDEs. *The Annals of Applied Probability*, pages 140–184, 2006.

[18] Emmanuel Gobet, Jean-Philippe Lemor, and Xavier Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *The Annals of Applied Probability*, 15(3):2172–2202, 2005.

[19] Emmanuel Gobet, José G López-Salas, Plamen Turkedjiev, and Carlos Vázquez. Stratified regression monte-carlo scheme for semilinear pdes and bsdes with large scale parallelization on gpus. *SIAM Journal on Scientific Computing*, 38(6):C652–C677, 2016.

[20] Emmanuel Gobet and Plamen Turkedjiev. Approximation of backward stochastic differential equations using malliavin weights and least-squares regression. *Bernoulli*, 22(1):530–562, 2016.

[21] Lajos Gergely Gyurkó and Terry J Lyons. Efficient and practical implementations of cubature on wiener space. In *Stochastic analysis 2010*, pages 73–111. Springer, 2011.
[22] J. Han, A. Jentzen, and W. E. Solving high-dimensional partial differential equations using deep learning. Proceedings of the National Academy of Sciences, 115(34):8505–8510, 2018.

[23] Jiequan Han and Jihao Long. Convergence of the deep bsde method for coupled fbsdes. Probability, Uncertainty and Quantitative Risk, 5(1):1–33, 2020.

[24] Jiequn Han, Arnulf Jentzen, and E Weinan. Solving high-dimensional partial differential equations using deep learning. Proceedings of the National Academy of Sciences, 115(34):8505–8510, 2018.

[25] Kurt Hornik, Maxwell Stinchcombe, and Halbert White. Multilayer feedforward networks are universal approximators. Neural networks, 2(5):359–366, 1989.

[26] Sam Howison and Daniel Schwarz. Risk-neutral pricing of financial instruments in emission markets: a structural approach. SIAM Journal on Financial Mathematics, 3(1):709–739, 2012.

[27] Côme Huré, Huyen Pham, and Xavier Warin. Deep backward schemes for high-dimensional nonlinear pdes. Mathematics of Computation, 89(324):1547–1579, 2020.

[28] Benjamin Jourdain. Probabilistic characteristics method for a one-dimensional inviscid scalar conservation law. The Annals of Applied Probability, 12(1):334–360, 2002.

[29] Benjamin Jourdain and Julien Reygner. Optimal convergence rate of the multitype sticky particle approximation of one-dimensional diagonal hyperbolic systems with monotonic initial data. Discrete & Continuous Dynamical Systems-A, 36(9):4963.

[30] Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

[31] Randall J LeVeque. Numerical methods for conservation laws, volume 132. Springer, 1992.

[32] G. Pagès and A. Sagna. Improved error bounds for quantization based numerical schemes for bsde and nonlinear filtering. Stochastic Processes and their Applications, 128(3):847 – 883, 2018.