Stability analysis and control chaos for fractional 5D Maxwell-Bloch model

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Abstract. In this paper we investigate the dynamical behavior of fractional differential systems associated to 5D Maxwell-Bloch model in terms of fractional Caputo derivatives.

1 Introduction

The fractional calculus has been found to be an important tool in various fields, such as mathematics, physics, engineering, chemistry, biology, economics, chaotic dynamics, optimal control and other complex dynamical systems [1, 12, 4, 9, 11].

In this paper is used the Caputo definition of fractional derivatives. Let \( f \in C^\infty(\mathbb{R}) \) and \( \alpha \in \mathbb{R}, \alpha > 0 \). The \( \alpha \)-order Caputo differential operator \( D_\alpha f(t) \), is described by

\[
D_\alpha f(t) = J^{m-\alpha} f^{(m)}(t), \quad \alpha > 0,
\]

where \( f^{(m)}(t) \) represents the \( m \)-order derivative of the function \( f \), \( m \in \mathbb{N}^* \) is an integer such that \( m - 1 \leq \alpha \leq m \) and \( J^\beta \) is the \( \beta \)-order Riemann - Liouville integral operator \( [12] \), which is expressed by

\[
J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \quad \beta > 0,
\]

where \( \Gamma \) is the Euler Gamma function. If \( \alpha = 1 \), then \( D_1 f(t) = \frac{df}{dt} \).

In this paper we suppose that \( \alpha \in (0,1] \).

The paper is structured as follows. In Section 2 we recall some results concerning the study of stability for fractional systems. The problem of the existence and uniqueness of solution for the fractional 5D Maxwell-Bloch system (3.3) is analyzed in Section 3. Section 4 is devoted to studying of the stability of equilibrium states for fractional system (3.3). Also, the unstable equilibrium states of this system can be controlled via fractional stability theory. In Section 5, the numerical integration and numerical simulation for the controlled fractional 5D Maxwell-Bloch model (4.1) are given.

2 Preliminaries on fractional dynamical systems

We consider the following system of fractional differential equations on \( \mathbb{R}^n \):

\[
D_\alpha^i x^i(t) = f_i(x^1(t), x^2(t), \ldots, x^n(t)), \quad i = 1, n,
\]

where \( \alpha \in (0,1) \), \( f_i \in C^\infty(\mathbb{R}^n, \mathbb{R}) \), \( D_\alpha^i x^i(t) \) is the Caputo fractional derivative of order \( \alpha \) for \( i = 1, n \) and \( t \in [0, \tau) \) is the time.

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The fractional dynamical system (2.1) can be written as follows:

\[ D^\alpha_t x(t) = f(x(t)), \]  

(2.2)

where \( f(x(t)) = (f_1(x^1(t), \ldots, x^n(t)), f_2(x^1(t), \ldots, x^n(t)), \ldots, f_n(x^1(t), \ldots, x^n(t)))^T \) and \( D^\alpha_t x(t) = (D^\alpha_t x^1(t), \ldots, D^\alpha_t x^n(t))^T. \)

A point \( x_e = (x^1_e, x^2_e, \ldots, x^n_e) \in \mathbb{R}^n \) is said to be equilibrium state of the system (2.1), if \( D^\alpha_t x^i(t) = 0 \) for \( i = 1, n. \)

The equilibrium states of the fractional dynamical system (2.1) are determined by solving the set of equations: \( f_i(x^1(t), x^2(t), \ldots, x^n(t)) = 0, \quad i = 1, n. \)

The Jacobian matrix associated to system (2.1) is \( J(x) = (\frac{\partial f_i}{\partial x_j}), \quad i, j = 1, n. \)

The stability of the system (2.1) has been studied by Matignon in [10], where necessary and sufficient conditions have been established.

**Proposition 2.1** ([10]) Let \( x_e \) be an equilibrium state of system (2.1) and \( J(x_e) \) be the Jacobian matrix \( J(x) \) evaluated at \( x_e. \)

(i) \( x_e \) is locally asymptotically stable, iff all eigenvalues \( \lambda(J(x_e)) \) of \( J(x_e) \) satisfy:

\[ |\arg(\lambda(J(x_e)))| > \frac{\alpha \pi}{2}. \]  

(2.3)

(ii) \( x_e \) is locally stable, iff either it is asymptotically stable, or the critical eigenvalues of \( J(x_e) \) which satisfy \( |\arg(\lambda(J(x_e)))| = \frac{\alpha \pi}{2} \) have geometric multiplicity one. \( \square \)

In the case when \( x_e \) is an unstable equilibrium state of the fractional system (2.2), we associate to (2.2) a new fractional system as follows.

The controlled fractional system associated to system (2.2) is described by:

\[ D^\alpha_t x(t) = f(x(t)) - k(x(t) - x_e), \]  

(2.4)

where \( k = \text{diag}(k_1, \ldots, k_n), \) \( k_i \geq 0, i = 1, n \) and \( x_e \) is an equilibrium state of (2.2).

If one selects the appropriate parameters \( k_i, i = 1, n \) which then make the eigenvalues of the linearized equation of (2.4) satisfy one of the conditions from Proposition 2.1, then the trajectories of (2.4) asymptotically approaches the unstable equilibrium state \( x_e \) in the sense that \( \lim_{t \to \infty} \|x(t) - x_e\| = 0, \) where \( \| \cdot \| \) is the Euclidean norm.

### 3 The fractional 5D Maxwell-Bloch model

In the physics of self-induced transparency for the most lasers and the most atoms the so called two level lossless model is an excellent approximation and is quite adequate for an understanding of the basic physics behind many coherent transient phenomena [2]. Self-induced transparency equations based upon this model are derived from the Maxwell-Schrödinger equations in the paper of Holm and Kovacic [7]. More precisely,
after averaging and neglecting non-resonant terms, the unperturbed Maxwell-Bloch dynamics in the rotating wave approximation (RWA) can be written on $\mathbb{C}^2 \times \mathbb{R}$ in the following form:

$$
\frac{du}{dt} = v, \quad \frac{dv}{dt} = uw, \quad \frac{dw}{dt} = \frac{1}{2}(\bar{u}v + u\bar{v}), \quad (3.1)
$$

where the superscript "$-$" denotes the complex conjugation. Physically speaking the complex scalar functions $u, v$ represent the self-consistent electric field and respectively the polarization of the laser-matter, the real scalar function $w$ describes the difference of its occupation numbers \[6, 8\].

Using the transformations $u = x^1 + ix^2$, $v = x^3 + ix^4$, $w = x^5$, the dynamical system (3.1) becomes:

$$
\dot{x}^1 = x^3, \quad \dot{x}^2 = x^4, \quad \dot{x}^3 = x^1x^5, \quad \dot{x}^4 = x^2x^5, \quad \dot{x}^5 = -x^1x^3 + x^2x^4, \quad (3.2)
$$

where $\dot{x}^i = \frac{x^i(t)}{dt}$ for $i = 1, 2, 5$. The phase space of (3.2) is $\mathbb{R}^5$.

The dynamical system (3.2) is called the five-dimensional Maxwell-Bloch equations or the 5D Maxwell-Bloch model.

In [6], Fordy and Holm discuss the phase space geometry of the solutions of the system (3.1) and show that it has three Hamiltonian structures. More recently, Birtea and Casu \[3\] solve the stability problem for the isolated equilibria of the system (3.2).

The fractional 5D Maxwell-Bloch model associated to 5D Maxwell-Bloch model (3.2) is defined by the following set of equations:

$$
\begin{align*}
D^\alpha_t x^1 &= x^3 \\
D^\alpha_t x^2 &= x^4 \\
D^\alpha_t x^3 &= x^1x^5, \\
D^\alpha_t x^4 &= x^2x^5 \\
D^\alpha_t x^5 &= -(x^1x^3 + x^2x^4)
\end{align*} \quad \alpha \in (0, 1). \quad (3.3)
$$

The initial value problem of fractional model (3.3) can be represented in the following matrix form:

$$
D^\alpha_t x(t) = Ax(t) + x^1(t)A_1x(t) + x^2(t)A_2x(t), \quad x(0) = x_0, \quad (3.4)
$$

where $0 < \alpha < 1$, $x(t) = (x^1(t), x^2(t), x^3(t), x^4(t), x^5(t))^T$, $t \in (0, \tau)$ and

$$
A = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

**Proposition 3.1** The initial value problem of the fractional 5D Maxwell-Bloch model (3.4) has a unique solution.
Proof. Let \( f(x(t)) = Ax(t) + x^1(t)A_1x(t) + x^2(t)A_2x(t) \). It is obviously continuous and bounded on \( D = \{ x \in \mathbb{R}^5 \mid x^1 \in [x_0^1 - \delta, x_0^1 + \delta], x^2 \in [x_0^2 - \delta, x_0^2 + \delta] \} \) for any \( \delta > 0 \). We have \( f(x(t)) - f(x_1(t)) = A(x(t) - x_1(t)) + y(t) + z(t) \), where \( y(t) = x^1(t)A_1x(t) - x^1_1(t)A_1x_1(t) \) and \( z(t) = x^2(t)A_2x(t) - x^2_1(t)A_2x_1(t) \). Then

(1) \( |f(x(t)) - f(x_1(t))| \leq \|A\| \cdot |x(t) - x_1(t)| + |y(t)| + |z(t)| \),

where \( \| \cdot \| \) and \( | \cdot | \) denote matrix norm and vector norm respectively.

It is easy to see that \( y(t) = (x^1(t) - x^1_1(t))A_1x(t) + x^1_1(t)A_1(x(t) - x_1(t)) \). Then

(2) \( |y(t)| \leq \|A_1\|((x(t) - x_1(t))A_1x(t)) + |x^1_1(t)| \cdot |x(t) - x_1(t)| \)

and using the inequality \( \|x^1(t) - x^1_1(t)\| \leq |x(t) - x_1(t)| \) one obtains

(3) \( |z(t)| \leq \|A_2\|((x(t)) + |x^2_1(t)|)|x(t) - x_1(t)| \).

Similarly, we prove that

(4) \( |f(x(t)) - f(x_1(t))| \leq (\|A\| + \|A_1\|(|x(t)| + |x^1_1(t)|)) + \|A_2\|(|x(t)| + |x^2_1(t)|))|x(t) - x_1(t)| \).

Replacing \( \|A\| = \|A_1\| = \|A_2\| = \sqrt{2} \), from the above we deduce that

(4) \( |f(x(t)) - f(x_1(t))| \leq L|x(t) - x_1(t)| \), where \( L = \sqrt{2}(1 + 4|x_0| + 2\delta) > 0 \).

The inequality (4) shows that \( f(x(t)) \) satisfies a Lipschitz condition. Using Theorems 1 and 2 in [3], it follows that (3.4) has a unique solution. \( \square \)

The equilibrium states of the fractional 5D Maxwell-Bloch model (3.3) are given as the union of the following two families:

\[ E_1 := \{ e^{m,n}_1 = (m, n, 0, 0, 0) \in \mathbb{R}^5 \mid m^2 + n^2 \neq 0 \}, \quad E_2 := \{ e^m_2 = (0, 0, 0, 0, m) \in \mathbb{R}^5 \mid m \in \mathbb{R} \}. \]

4 Stability study of fractional 5D Maxwell-Bloch model

We start with the study of stability of equilibrium states for the fractional system (3.3). Finally, we will discuss how to stabilize the unstable equilibrium states of the system (3.3) via fractional order derivative.

The Jacobian matrix of the system (3.3) is

\[ J(x) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
x^5 & 0 & 0 & 0 & x^1 \\
0 & x^5 & 0 & 0 & x^2 \\
-x^3 & -x^4 & -x^1 & -x^2 & 0
\end{pmatrix}. \]

Proposition 4.1 All equilibrium states of the fractional system (3.3) are unstable.
Proof. Case \( e_1^{mn} \in E_1 \). The characteristic polynomial of the matrix \( J(e_1^{mn}) \) is 
\[ p_{J(e_1^{mn})}(\lambda) = \det(J(e_1^{mn}) - \lambda I) = -\lambda^3(\lambda^2 + m^2 + n^2). \]
Then the characteristic roots of \( J(e_1^{mn}) \) are \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) and \( \lambda_{4,5} = \pm i\sqrt{\lambda^2 + m^2 + n^2} \). Since the eigenvalues of \( J(e_1^{mn}) \) are at least one positive, by Proposition 2.1, it follows that \( e_1^{mn} \) is unstable.

Case \( e_2^m \in E_2 \). The characteristic polynomial of the matrix \( J(e_2^m) \) is 
\[ p_{J(e_2^m)}(\lambda) = -\lambda(\lambda^2 - m)^2 \]
with characteristic roots \( \lambda_1 = 0, \lambda_{2,3} = \pm \sqrt{m}, \lambda_{4,5} = \mp \sqrt{m} \) for \( m > 0 \) and \( \lambda_1 = 0, \lambda_{2,3} = \pm i\sqrt{m}, \lambda_{4,5} = \mp i\sqrt{m} \) for \( m < 0 \). Applying now Proposition 2.1, it follows that \( e_2^m \) is unstable.

Similarly, it is easy to see that \( e_0 \) is unstable.

The controlled fractional 5D Maxwell-Bloch model associated to fractional 5D Maxwell-Bloch model (3.3) is defined by:

\[
\begin{align*}
D_t^\alpha x^1 &= x^3 - k_1(x^1 - x_e^1) \\
D_t^\alpha x^2 &= x^4 - k_2(x^2 - x_e^2) \\
D_t^\alpha x^3 &= x^1 x^5 - k_3(x^3 - x_e^3), \quad \alpha \in (0, 1), \\
D_t^\alpha x^4 &= x^2 x^5 - k_4(x^4 - x_e^4) \\
D_t^\alpha x^5 &= -(x^1 x^5 + x^2 x^4) - k_5(x^5 - x_e^5)
\end{align*}
\]

where \( x_e \) represents an arbitrary equilibrium state of (3.3) and \( k_i \in \mathbb{R}, i = 1, 5 \) are non-negative constants.

The parameters \( k_1, i = 1, 5 \) are feedback control gains which can make the eigenvalues of the linearized equation of the system (4.1) satisfy one of the conditions of Proposition 2.1 or one of the fractional Routh-Hurwitz conditions [1], then the trajectories of the system (4.1) asymptotically approaches the equilibrium state \( x_e \).

The Jacobian matrix of the controlled fractional system (4.1) is

\[
J(x, k) = \begin{pmatrix}
-k_1 & 0 & 1 & 0 & 0 \\
0 & -k_2 & 0 & 1 & 0 \\
x^5 & 0 & -k_3 & 0 & x^1 \\
0 & x^5 & 0 & -k_4 & x^2 \\
-x^3 & -x^4 & -x^1 & -x^2 & -k_5
\end{pmatrix}.
\]

Proposition 4.2 Let \( k_i > 0 \) for \( i = 1, 5 \). Then the equilibrium state \( e_2^m \in E_2 \) of the controlled fractional system (4.1) is locally asymptotically stable for all \( \alpha \in (0, 1] \), if one of the following conditions holds:

1. \(|k_1 - k_3| = |k_2 - k_4| \) and \( m = -\frac{1}{4}(k_1 - k_3)^2 \neq 0 \);
2. \( \max\{-\frac{1}{4}(k_1 - k_3)^2, -\frac{1}{4}(k_2 - k_4)^2\} < m < \min\{k_1 k_3, k_2 k_4\}; \)
3. \( -\frac{1}{4}(k_1 - k_3)^2 < m < \min\{-\frac{1}{4}(k_2 - k_4)^2, k_1 k_3\}; \)
4. \( -\frac{1}{4}(k_2 - k_4)^2 < m < \min\{-\frac{1}{4}(k_1 - k_3)^2, k_2 k_4\}; \)
5. \( m < \min\{-\frac{1}{4}(k_1 - k_3)^2, -\frac{1}{4}(k_2 - k_4)^2\}. \)
Proof. The Jacobian matrix of the system (4.1) at the point $e_2^m$ is

$$J(e_2^m,k) = \begin{pmatrix} -k_1 & 0 & 1 & 0 & 0 \\ 0 & -k_2 & 0 & 1 & 0 \\ m & 0 & -k_3 & 0 & 0 \\ 0 & m & 0 & -k_4 & 0 \\ 0 & 0 & 0 & 0 & -k_5 \end{pmatrix},$$

whose characteristic polynomial $p_{J(e_2^m,k)}(\lambda) = \det(J(e_2^m,k) - \lambda I)$ is

$$p_{J(e_2^m,k)}(\lambda) = -(\lambda + k_5)[\lambda^2 + (k_1 + k_3)\lambda + k_1k_3 - m][\lambda^2 + (k_2 + k_4)\lambda + k_2k_4 - m].$$

Its characteristic roots are $\lambda_1 = -k_5$, $\lambda_{2,3} = -(k_1 + k_3)\pm\sqrt{(k_1 - k_3)^2 + 4m} / 2$, $\lambda_{4,5} = -(k_2 + k_4)\pm\sqrt{(k_2 - k_4)^2 + 4m} / 2$. We denote:

$\Delta_1 = (k_1 - k_3)^2 + 4m$, $\Delta_2 = (k_2 - k_4)^2 + 4m$, $u = \frac{1}{2}(k_1 - k_3)^2$, $v = \frac{1}{2}(k_2 - k_4)^2$.

(1) Let $\Delta_1 = \Delta_2 = 0$. Then $|k_1 - k_3| = |k_2 - k_4|$ and $m = \frac{1}{4}(k_1 - k_3)^2 \neq 0$. The eigenvalues $\lambda_1 = -k_5$, $\lambda_{2,3} = -(k_1 + k_3)$, $\lambda_{4,5} = -(k_2 + k_4)$ are all negative. Then $|\arg(\lambda_1)| = \pi > \frac{\pi}{2}\alpha$ for any $\alpha \in (0, 1)$ and so $e_2^m$ is asymptotically stable.

(2) Suppose that $\Delta_1 > 0$ and $\Delta_2 > 0$. Then $m > u$ and $m > v$. We have $\lambda_5 < 0$. The eigenvalues $\lambda_i, i = 1, 4$ are all negative iff $u < m < k_1k_3$ and $v < m < k_2k_4$. Hence, for $\max\{u, v\} < m < \min\{k_1k_3, k_2k_4\}$ it implies that $e_2^m$ is asymptotically stable for $\alpha \in (0, 1)$.

(3) - (4) We suppose now that $\Delta_1 < 0$ and $\Delta_2 > 0$. It follows $m < u$ and $m > v$. In this case the eigenvalues $\lambda_{4,5}$ are negative iff $v < m < k_2k_4$. For $\Delta_1 < 0$, we have $\lambda_{2,3} = -(k_1 + k_3)\pm i\sqrt{-\Delta_1} / 2$. Since $\text{Re}(\lambda_{2,3}) = -\frac{1}{2}(k_1 + k_3) < 0$ we have $|\arg(\lambda_{2,3})| = \pi > \frac{\alpha \pi}{2}$ for all $0 < \alpha < 1$. Applying now Proposition 2.1 (i), we can conclude that $e_2^m$ is asymptotically stable if $v < m < \min\{u, k_2k_4\}$ and $\alpha \in (0, 1]$.

Similarly, we discuss the case $\Delta_1 > 0$ and $\Delta_2 < 0$.

(5) Finally, we suppose $\Delta_1 < 0$ and $\Delta_2 < 0$. It follows $m < u$ and $m < v$. We have $\lambda_{2,3} = -(k_1 + k_3)\pm i\sqrt{-\Delta_1} / 2$, $\lambda_{4,5} = -(k_2 + k_4)\pm i\sqrt{-\Delta_2} / 2$. Since $\text{Re}(\lambda_i) < 0$ for $i = 2, 5$, we have $|\arg(\lambda_i)| = \pi > \frac{\alpha \pi}{2}$ for all $0 < \alpha < 1$. By Proposition 2.1(i), $e_2^m$ is asymptotically stable iff $m < \min\{u, v\}$ and $\alpha \in (0, 1]$.

Example 4.1 By choosing the parameters $k_i, i = 1, 5$ that satisfy one condition from Proposition 4.2, then the trajectories of the controlled fractional model are driven to the unstable equilibrium point $e_2^m$. The parameters are selected as: $k_1 = k_3 = \frac{1}{4}$, $k_2 = \frac{3}{2}$, $k_4 = \frac{3}{2}$, $k_5 > 0$. For $m = -\frac{1}{8}$ we have $-\frac{25}{144} < m < \min\{0, 1\}$. It follows that the stability condition (4) of Proposition 4.2 is achieved. This implies that, the
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trajectories of the controlled fractional system (4.1) converge to \( e_2 = (0, 0, 0, 0, -\frac{1}{9}) \) for any \( \alpha \in (0, 1] \). In this case we have \( \Delta_1 = -\frac{1}{2} \) and \( \Delta_2 = \frac{7}{36} \) for any \( \alpha \in (0, 1] \). The eigenvalues are \( \lambda_1 < 0, \lambda_{2,3} = -\frac{1}{4} \pm i\frac{\sqrt{2}}{4} \) and \( \lambda_{4,5} = -\frac{13}{12} \pm i\frac{\sqrt{2}}{12} < 0. \) \( \square \)

**Proposition 4.3** The equilibrium state \( e_0 \) of the controlled fractional system (4.1) is locally asymptotically unstable for \( k_i > 0, i = 1, 5 \) and \( \alpha \in (0, 1) \).

**Proof.** The characteristic polynomial of the Jacobian matrix \( J(e_0, k) \) is

\[
p_{J(e_0,k)}(\lambda) = -\prod_{i=1}^{5}(\lambda + k_i) \text{ with characteristic roots } \lambda_i = -k_i \text{ for } i = 1, 5.
\]

Since \( \arg(\lambda_i) = \pi > \frac{\alpha \pi}{2} \) for \( i = 1, 5 \), by Proposition 2.1(i) it follows that \( e_0 \) is locally asymptotically stable. \( \square \)

Let us study the problem of stabilizing of the fractional system (3.3) at the equilibrium state \( e_1^{mn} \in E_1 \).

The Jacobian matrix of the system (4.1) at the point \( e_1^{mn} \) is

\[
J(e_1^{mn}, k) = \begin{pmatrix}
-k_1 & 0 & 1 & 0 & 0 \\
0 & -k_2 & 0 & 1 & 0 \\
0 & 0 & -k_3 & 0 & m \\
0 & 0 & 0 & -k_4 & n \\
0 & 0 & -m & -n & -k_5
\end{pmatrix}.
\]

Its characteristic polynomial is

\[
p_{J(e_1^{mn}, k)}(\lambda) = -(\lambda + k_1)(\lambda + k_2)P(\lambda) \text{ with}
\]

\[
P(\lambda) = \lambda^2 + a_1 \lambda^2 + a_2 \lambda + a_3, \quad \text{where}
\]

\[
\begin{align*}
a_1 & = k_3 + k_4 + k_5 \\
a_2 & = k_3 k_4 + k_3 k_5 + k_4 k_5 + m^2 + n^2 \\
a_3 & = k_3 k_4 k_5 + k_3 n^2 + k_4 m^2
\end{align*} \quad \text{(4.3)}
\]

The eigenvalues of the characteristic equation are \( \lambda_1 = -k_1, \lambda_2 = -k_2 \) and the roots \( \lambda_{3,4,5} \) of the equation \( P(\lambda) = 0 \). In this case we apply the fractional Routh-Hurwitz conditions corresponding to polynomial \( P(\lambda) \). The discriminant \( D(P) \) of the polynomial \( P(\lambda) \) is

\[
D(P) = 18a_1 a_2 a_3 + a_1^2 a_2^2 - 4a_3 a_1^3 - 4a_2^3 - 27a_3^2. \quad \text{(4.4)}
\]

Because of the complexity of \( D(P) \), we only consider the following two situations:

(i) \( k_i > 0 \) for \( i = 1, 5 \); (ii) \( k_1 > 0, k_2 > 0, k_3 = k_4 = b > 0, k_5 = 0 \).

In the above conditions we have \( a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 - a_3 > 0 \).

**Proposition 4.4** Let \( e_1^{mn} \in E_1 \) the equilibrium state of the system (4.1).

(i) Let \( k_i > 0 \) for \( i = 1, 5 \).

(1) if \( D(P) > 0 \), then \( e_1^{mn} \) is locally asymptotically stable for \( \alpha \in (0, 1) \);

(2) if \( D(P) < 0 \), then \( e_1^{mn} \) is locally asymptotically stable for \( \alpha \in (0, \frac{4}{5}) \).
(ii) Let \( k_1 > 0, k_2 > 0, k_3 = k_4 = k_5 = 0 \).

1. if \( b > 2\sqrt{m^2 + n^2} \), then \( e_1^{mn} \) is locally asymptotically stable for \( \alpha \in (0, 1) \);
2. if \( 0 < b < 2\sqrt{m^2 + n^2} \), then \( e_1^{mn} \) is locally asymptotically stable for \( \alpha \in (0, \frac{2}{3}) \).

**Proof.**

(i)(1) From hypothesis we have \( \lambda_1 < 0 \) and \( \lambda_2 < 0, a_1 > 0, a_2 > 0 \) and \( a_1 a_2 > a_3 \). When \( D(P) > 0 \), the assertion (i) of fractional Routh-Hurwitz conditions ([4], p. 704) is satisfied. But Routh-Hurwitz conditions are the necessary and sufficient conditions for the fulfillment of Proposition 2.1(i). Then \( e_1^{mn} \) is asymptotically stable for any \( \alpha \in (0, 1) \).

(ii) We have \( \lambda_1 < 0 \) and \( \lambda_2 < 0, a_1 > 0, a_2 > 0 \) and \( a_3 > 0 \). When \( D(P) < 0 \), the assertion (ii) of fractional Routh-Hurwitz conditions ([4], p. 704) is satisfied. As above, we deduce that \( e_1^{mn} \) is asymptotically stable for any \( \alpha \in (0, \frac{2}{3}) \).

(ii) For \( k_3 = k_4 = b > 0, k_5 = 0 \), we have \( a_1 = 2b > 0, a_2 = b^2 + m^2 + n^2 > 0, a_3 = b(m^2 + n^2) \) and \( D(P) = (m^2 + n^2)b^2 - 4(m^2 + n^2) \). Using the same manner as in demonstration of assertions (i) we prove that (ii)(1) and (ii)(2) hold. \( \square \)

**Example 4.2** By choosing the parameters \( k_i, i = 1, 5 \) that satisfy one condition from Propositions 4.4, then the trajectories of the controlled fractional model are driven to the unstable equilibrium point \( e_1^{mn} \). If we select the parameters as follows: \( k_1 > 0, k_2 > 0, k_3 = k_4 = 0.5, k_5 = 0 \) and \( m^2 + n^2 = 0.25 \), then \( a_1 = 1, a_2 = 0.5, a_3 = 0.125 \). Since \( D(P) = -\frac{3}{64} < 0 \) it follows that the stability condition (2) of Proposition 4.4 (ii) is achieved. This implies that, the trajectories of the system (4.1) converge to \( e_1^{mn} = (m,n,0,0,0) \) when \( m^2 + n^2 = 0.25 \) and \( \alpha \in (0, 1) \). The eigenvalues are \( \lambda_3 = -0.5, \lambda_{4,5} = -0.25 \pm 0.433i \). For example, substituting \( k_1 = k_2 = 1.2, k_3 = k_4 = 0.5, k_5 = 0 \) and \( \alpha = 0.65 \) in (4.1) we obtains that the controlled fractional system is asymptotically stable at \( e_1 = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, 0, 0, 0\right) \). \( \square \)

5 **Numerical integration of the fractional system (4.1)**

Consider the fractional differential equations

\[
\begin{align*}
D_\alpha^\alpha x^i(t) &= F_i(x^1(t), x^2(t), x^3(t), x^4(t), x^5(t)), \quad t \in (0, \tau), \quad \alpha \in (0, 1) \\
x(0) &= (x^1_0, x^2_0, x^3_0, x^4_0, x^5_0)
\end{align*}
\]

where \( F_1(t) = x^3(t) - k_1(x^1(t) - x^1_0), \quad F_2(t) = x^4(t) - k_2(x^2(t) - x^2_0), \quad F_3(t) = x^1(t)x^5(t) - k_3(x^3(t) - x^3_0), \quad F_4(t) = x^2(t)x^5(t) - k_4(x^4(t) - x^4_0), \quad F_5(t) = -(x^1(t)x^3(t) + x^2(t)x^4(t)) - k_5(x^5(t) - x^5_0). \)

Since the function \( F(t) = (F_1(t), F_2(t), F_3(t), F_4(t), F_5(t)) \) is continuous, the initial value problem (5.1) is equivalent to the nonlinear Volterra integral equation ([5], which is given as follows:
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\[ x^i(t) = x_0^i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_i(x^1(s), x^2(s), x^3(s), x^4(s), x^5(s)) ds, \quad i = 1, 5. \] (5.2)

Diethelm et al. have given a predictor-corrector scheme \cite{Diethelm2003}, based on the Adams-Bashforth-Moulton algorithm to integrate the equation (5.2). We apply this scheme to the controlled fractional system (5.1). For this, let \( h = \frac{\tau}{N}, t_n = nh \) for \( n = 0, 1, \ldots, N \). We use the following notations:

\[ x^i[n] = x^i(nh), \quad x^i_p[n] = x^i_p(nh), \quad F^i[n] = F_i(x[n]), \quad F^i_p[n] = F_i(x_p[n]) \] for \( i = 1, 5 \).

The controlled fractional system (5.1) can be discretized as follows:

\[
\begin{align*}
x^i[n+1] &= x^i_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n a[j, n+1] F^i[j] + F^i_p[n+1], \\
x^i_p[n+1] &= x^i_p_0 + \frac{h^\alpha}{\alpha \Gamma(\alpha)} \sum_{j=0}^n b[j, n+1] F^i[j],
\end{align*}
\] (5.3)

where \( i = 1, 5 \) and:

\[
\begin{align*}
a[0, n+1] &= n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \\
a[j, n+1] &= (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, \quad j = 1, n, \\
b[j, n+1] &= (n+1-j)^\alpha - (n-j)^\alpha, \quad j = 0, n.
\end{align*}
\] (5.4)

The above scheme given by the relations (5.3) and (5.4) is called the **Moulton-Adams algorithm for controlled fractional system** (5.1) (see for details \cite{Diethelm2003}).

The error estimate for the algorithm described by (5.3) and (5.4) is

\[
\max_{0 \leq j \leq N} \{ x^i[j] - x^i_p[j] \} = O(h^{\alpha+1}).
\]

Applying the algorithm (5.3) – (5.4), the fractional system (5.1) is numerically integrated for \( \alpha = 0.65, \ k_1 = k_2 = 1.2, k_3 = k_4 = 0.5, k_5 = 0 \) and \( x_e = (\frac{\sqrt{3}}{4}, \frac{1}{4}, 0, 0, 0) \) (see Example 4.2). For this, we consider \( h = 0.01, \epsilon = 0.01, N = 500, t = 502 \) and the initial conditions \( x^1(0) = \epsilon + \frac{\sqrt{3}}{4}, \ x^2(0) = \epsilon + \frac{1}{4}, x^3(0) = x^4(0) = x^5(0) = \epsilon. \)

Using the software Maple 11, the orbits \((n, x^i(n)), i = 1, 5\) of system (5.1) are represented in the figures Fig. 1-5.
The numerical simulations show the validity of the theoretical analysis.

Conclusions. The dynamics of the fractional 5D Maxwell-Bloch model (3.3) was investigated in this paper. The analysis of the stability of equilibrium states for the controlled fractional 5D Maxwell-Bloch model (4.1) was studied. Finally, the numerical integration and numerical simulation for the fractional system (4.1) are given.

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