Energy and momentum of the Friedman and more general universes

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(Dated: October 15, 2018)

Abstract

Recently some authors concluded that the energy and momentum of the Friedman universes, flat and closed, are equal to zero locally and globally (flat universes) or only globally (closed universes). The similar conclusion was also done for more general only homogeneous universes (Kasner and Bianchi type I). Such conclusions originated from coordinate dependent calculations performed only in comoving Cartesian coordinates by using the so-called energy-momentum complexes. But it is known that the energy-momentum complexes can be reasonably use only in precisely defined asymptotically flat spacetimes (at null or at spatial infinity) to calculate global energy and momentum. In this paper we show, by using new coordinate independent expressions on energy and momentum that the Friedman and more general universes needn’t be energetic nonentity.

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I. INTRODUCTION

Recently many authors have calculated the energy and momentum of the Friedman universes and also more general, only spatially homogeneous universes, like Kasner, Bianchi type I and Bianchi type II universes [1].

The linear elements for these universes read [9]:

1. Spatially isotropic and homogeneous Friedman universes in the coordinates \((r, \chi, \vartheta, \varphi)\)

\[
ds^2 = dr^2 - R^2(t)[d\chi^2 + S^2(\chi)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)],
\]

where

\[
S(\chi) = \sin \chi \text{ if } k = 1 \text{ (closed universe)},
\]

\[
S(\chi) = \chi \text{ if } k = 0 \text{ (flat universe)},
\]

\[
S(\chi) = \sinh \chi \text{ if } k = -1 \text{ (open universe)}.
\]

2. Spatially flat and homogeneous vacuum Kasner’s universes in “Cartesian” coordinates \((t, x, y, z)\)

\[
ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2.
\]

The constants \(p_1, p_2, p_3\) satisfy the following constraints which follow from the vacuum Einstein equations

\[
p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1.
\]

If \(p_1 = p_2 = 0, \ p_3 = 1\), then one gets flat Minkowskian spacetime.

3. Spatially homogeneous Bianchi type I universes filled with stiff matter in “Cartesian” coordinates \((t, x, y, z)\)

\[
ds^2 = dt^2 - e^{2l} dx^2 - e^{2m} dy^2 - e^{2n} dz^2,
\]

where \(l = l(t), \ m = m(t), \ n = n(t)\).

4. Spatially homogeneous Bianchi type II universes in “Cartesian” coordinates \((t, x, y, z)\)

\[
ds^2 = dt^2 - D^2(t) dx^2 - H^2(t) dy^2 - [D^2(t) + x^2 H^2(t)] dz^2
- 2x H^2(t) dy dz.
\]
The above mentioned authors performed their calculations in special comoving coordinates called “Cartesian coordinates” despite that they used coordinate dependent double index energy-momentum complexes, matter and gravitation.

The authors have applied the six most frequently used energy-momentum complexes: Einstein canonical complex, Landau-Lifshitz complex, Bergmann-Thomson complex, Möller complex, Papapetrou complex and Weinberg energy-momentum complex. These all energy-momentum complexes are neither geometrical objects nor coordinate independent objects, e.g., they can vanish in some coordinates locally or globally and in other coordinates they can be different from zero. It results that the double index energy-momentum complexes and the gravitational energy-momentum pseudotensors determined by them have no physical meaning to a local analysis of a gravitational field, e.g., to study gravitational energy distribution. In fact, up to now, they were reasonably used only to calculate the global quantities for the very precisely defined asymptotically flat spacetimes (in spatial or in null direction).

The best one of the all possible double index energy-momentum complexes from physical and geometrical points of view is the Einstein canonical double index energy momentum complex $E K^i_k$ (For details see, e.g., [2]):

$$E K^i_k := \sqrt{|g|} (T^i_k + E t^i_k) = F_{[kl]} U^j_{i,k,l}. \quad (7)$$

Here $F_{[kl]} = (-) F_{[lk]}$ are Freud superpotentials, $T^i_k = T^{ki}$ mean the components of the symmetric energy-momentum tensor for matter, [10] and $E t^i_k$ are the components of the Einstein canonical energy-momentum pseudotensor for gravitational field $\Gamma^i_{kl} = \{^i_{kl}\}$.

From (7) there follow the local conservation laws for gravity and matter

$$E K^i_k \equiv 0. \quad (8)$$

Remark: The other double index energy-momentum complexes have structure which is very like to the structure of the complex $E K^i_k$ and they, of course, also satisfy suitable local conservation laws.

The conclusion of the authors which calculated the energy and momentum of the Friedman and more general universes by using double index energy-momentum complexes is the following: the energy and momentum of the closed Friedman universes are equal to zero globally, and in the case of the flat Friedman universes and their generalizations (Kasner, Bianchi type I, Bianchi type II universes) these quantities are equal to zero locally and globally.
One can have at least the following objections against the calculations of such a kind and against the above conclusion:

1. The authors despite that they used coordinate dependent expressions had performed their calculations only in Cartesian comoving coordinates. The results obtained in other comoving coordinates, e.g., in coordinates \((t, \chi, \vartheta, \varphi)\) or in coordinates \((t, r, \vartheta, \varphi)\) are dramatically different.

2. The local “energy-momentum distribution” as given by any energy-momentum complex has no physical sense but the authors try to give a physical sense of this distribution, e.g., they assert that the total energy density for flat Friedman universes, for Kasner and Bianchi type I universes, is null.

3. The conclusion leads us to Big-Bang which has no singularity in total energy density.

4. The global energy and momentum have physical meaning only when spacetime is asymptotically flat either in spatial or null direction and when these quantities can be measured. But this is not a case of the cosmological models. So, the problem of the global energy and global linear (or angular) momentum for Friedman, and for more general universes also, is not well-posed from the physical point of view because these universes are not asymptotically flat spacetimes, and, in consequence, their global quantities cannot be measurable. This problem can only have a mathematical sense.

Thus, one can doubt in validity of the conclusion that the energy and momentum of the Friedman, Kasner, Bianchi type I and Bianchi type II universes are equal to zero; especially that all these universes are energy-free.

By using double index energy-momentum complexes one should rather conclude that the energy and momentum of the Friedman, Kasner, Bianchi type I, and Bianchi type II universes explicite depend on the used coordinates and, therefore, they are undetermined not only locally but also globally. The last conclusion is very sensible because, as we mentioned beforehand, one cannot measure the global energy and global linear (or angular) momentum of the Friedman and any more general universe. One can do this only in the case of an isolated system when spacetime is asymptotically flat.
One cannot use the coordinate independent Pirani \([2]\) and Komar \([3]\) expressions in order to correctly prove (at least from the mathematical point of view) the statement that the energy of the Friedman, Kasner, Bianchi type I and Bianchi type II universes disappears, i.e., that these universes have zero net energy. It is because we have no translational timelike Killing vector field (descriptor of energy in Komar expression) in these universes, and the privileged normal congruence of the fundamental observers which exists in these universes is geodesic.\([11]\)

One also cannot use for this purpose the coordinate independent Katz-Bičak-Lynden (BKL) bimetric approach \([4]\) because the results obtained in this approach depend on the used background and on mapping of the spacetime under study onto this background.

Thus, the “academic” statement that the Friedman, Kasner, Bianchi type I and Bianchi type II universes have no energetic content is still not satisfactory proved. But by using Komar expression, one can correctly (from mathematical point of view) prove that the linear momentum for these universes disappears in a comoving coordinates.

In the following we will apply some new expressions on averaged relative energy and momentum to analyze of the energetic content of the Friedman, Kasner and Bianchi type I universes.\([12]\)

It is interesting that these new expressions lead us to \textit{positive-definite} results for the all these universes.

II. USING OF THE AVERAGED RELATIVE ENERGY-MOMENTUM TENSORS TO ANALYZE ENERGY AND MOMENTUM OF THE FRIEDMAN AND MORE GENERAL HOMOGENEOUS UNIVERSES

In the papers \([5]\) we have defined the canonical superenergy and angular supermomentum tensors, matter and gravitation, in general relativity (GR) and studied their properties and physical applications. In the case of the gravitational field these tensors gave us some substitutes of the non-existing gravitational energy-momentum and gravitational angular momentum tensors.

The canonical superenergy and angular supermomentum tensors were obtained pointwise as a result of some special averaging of the differences of the energy-momentum and angular momentum in normal coordinates \(\text{NC}(P)\). The role of the normal coordinates \(\text{NC}(P)\) is,
of course, auxilliary, only to extract tensorial quantities even from pseudotensorial ones.

The dimensions of the components of the canonical superenergy and angular supermomentum tensors can be written down as: \(\text{the dimensions of the components of an energy-momentum or angular momentum tensor (or pseudotensor)} \times m^{-2}\).

Recently, in the paper \cite{6}, we have proposed a new averaging of the energy-momentum and angular momentum differences in \(\textbf{NC}(P)\) which is very like to the averaging used in \cite{5} and which gives the averaged quantities with proper dimensionality of the energy-momentum and angular momentum densities.

Namely, we have proposed the following general definition of the averaged tensor (or pseudotensor) \(T_b^a(P)\)

\[
<T_b^a(P)> := \lim_{\varepsilon \to 0} \frac{\int_{\Omega} \left[ T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P) \right] d\Omega}{\varepsilon^2 / 2 \int_{\Omega} d\Omega}, \quad (9)
\]

where

\[
T_{(a)}^{(b)}(y) := T_{i}^{k}(y) e^{i}_{(a)}(y) e^{(b)}_{k}(y), \quad (10)
\]

\[
T_{(a)}^{(b)}(P) := T_{i}^{k}(P) e^{i}_{(a)}(P) e^{(b)}_{k}(P) = T_b^a(P) \quad (11)
\]

are the tetrad (or physical) components of a tensor or a pseudotensor \(T_i^k(y)\) which describes an energy-momentum distribution, \(y\) is the collection of normal coordinates \(\textbf{NC}(P)\) at a given point \(P\), \(e^i_{(a)}(y)\), \(e^{(b)}_k(y)\) denote an orthonormal tetrad field and its dual, respectively,

\[
e^i_{(a)}(P) = \delta^i_a, \quad e^{(a)}_k(P) = \delta^a_k, \quad e^{i}_{(a)}(y) e^{(b)}_{i}(y) = \delta^b_a, \quad (12)
\]

and they are parallelly propagated along geodesics through \(P\).

For a sufficiently small domain \(\Omega\) which surrounds \(P\) we required

\[
\int_{\Omega} y^i d\Omega = 0, \quad \int_{\Omega} y^i y^k d\Omega = \delta^{ik} M, \quad (13)
\]

where

\[
M = \int_{\Omega} (y^0)^2 d\Omega = \int_{\Omega} (y^1)^2 d\Omega = \int_{\Omega} (y^2)^2 d\Omega = \int_{\Omega} (y^3)^2 d\Omega, \quad (14)
\]

is a common value of the moments of inertia of the domain \(\Omega\) with respect to the subspaces \(y^i = 0, \quad (i = 0, 1, 2, 3)\).

We have choosen \(\Omega\) as a small analytic ball defined by

\[
(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2 = \varepsilon^2 L^2, \quad (15)
\]
which can be described in a covariant way in terms of the auxiliary positive-definite metric
\[ h^{ik} := 2v^i v^k - g^{ik}, \]
where \( v^i \) are the components of the four-velocity of an observer \( O \) at rest
at \( P \) (See, e.g., \[5\]). \( \varepsilon > 0 \) means an undimensional parameter, and \( L > 0 \) is a fundamental
length.

The mathematical trick with putting \( R = \varepsilon L \) was discovered by B. Mashhoon \[6\]. This
trick leads to proper dimensionality of the averaged quantities.

Since at \( P \) the tetrad and normal components are equal, from now on we will write the
components of any quantity at \( P \) without (tetrad) brackets, e.g., \( T^b_a(P) \) instead of \( T^{(b)}_{(a)}(P) \)
and so on.

For the matter energy-momentum tensor \( mT^b_a(y) \) the averaging formula (9) gives
\[
\langle mT^b_a(P) \rangle = mS^b_a(P) \frac{L^2}{6},
\]
where
\[
mS^b_a(P) := \delta^{lm} \nabla_l \nabla_m \hat{T}^b_a
\]
is the canonical superenergy tensor of matter \[5\].

By introducing the four velocity \( \hat{v}^l = \delta^l_0, \quad v^i v_i = 1 \) of an observer \( O \) at rest at \( P \) and the
local metric \( \hat{g}^{ab} = \eta^{ab} \), where \( \eta^{ab} \) is the inverse Minkowski metric, one can write (17) in a
covariant way as
\[
mS^b_a(P; v^l) = (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \nabla_l \nabla_m \hat{T}^b_a.
\]
The sign \( \hat{=} \) means that an equality is valid only in some special coordinates.

For the gravitational field one gets the following tensor (if one uses the Einstein canonical
energy-momentum pseudotensor \( E^b_a(y) \) in the averaging process)
\[
\langle gE^b_a(P; v^l) \rangle = gS^b_a(P; v^l) \frac{L^2}{6},
\]
where the tensor \( gS^b_a(P; v^l) \) is the canonical superenergy tensor for the gravitational field
\[5\].

We have \[5\]
\[
gS^b_a(P; v^l) = \frac{2\alpha}{9} (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \left[ \hat{B}^b_{alm} + \hat{P}^b_{alm} \right] \\
- 1/2 \beta^b_a \hat{R}^{ijk}_m (\hat{R}_{ijkl} + \hat{R}_{ikjl}) + 2\beta^2 \hat{g}^b_a \hat{E}^l_{(g)(l)} \hat{E}^g_{(m)} \\
- 3\beta^2 \hat{E}^l_{a(l)} \hat{E}^b_{(m)} + 2\beta \hat{R}^b_{(ag)(l)} \hat{E}^g_{(m)} \right].
\]
Here $\alpha = \frac{1}{10\pi} = \frac{1}{2\pi}$ and $E_i^k := T_i^k - 1/2T^a_a$ is the modified energy-momentum tensor of matter. $B_{a\mu}^b$ mean the components of the Bel-Robinson tensor and $P_{a\mu}^{b\nu}$ are the components of a tensor which is very closely related to the Bel-Robinson tensor.[13]

In vacuum the tensor $<g t^b_a(P; v^j)>$ reduces to the simpler form

$$<g t^b_a(P; v^j)> = \frac{4\alpha}{27} \left( 2\dot{\nu}^i \dot{\nu}^m - \dot{g}^{lm} \right) \left[ \tilde{R}^{ik}(l) \tilde{R}_{aik}\dot{m} - 1/2 \delta^b_a \tilde{R}^{ikp}(l) \tilde{R}_{ikp}\dot{m} \right] L^2,$$

which is symmetric and the quadratic form $<g t_{ab}(P; v^j) > \dot{\nu}^a \dot{\nu}^b$ is positive-definite.

The averaged energy-momentum tensors $<m T^b_a(P; v^j)>$ and $<g t^b_a(P; v^j)>$ can be considered as the averaged tensors of the relative energy-momentum.

The averaged tensors

$$<g t^b_a(P; v^j)>, \quad <m T^b_a(P; v^j)>,$$

depend on the four-velocity $\ddot{v}$ of a fiducial observer $O$ which is at rest at the beginning $P$ of the normal coordinates $NC(P)$ used for averaging and on some fundamental length $L > 0$.

One can try to fix the fundamental length $L$ in a some way, e.g., with the help of the loop quantum gravity (LQG). Namely, one can take as $L$ the smallest length over which the classical model of the spacetime is admissable. But this is not necessary. One can effectively use the averaged relative energy-momentum tensors without fixing $L$ explicitly (See [6] for details).

After fixing the fundamental length $L$ one can determine univocally the averaged relative energy-momentum tensors along the world line of an observer $O$. In general one can unambiguously determine these tensors (after fixing $L$) in the whole spacetime or in some domain $\Omega$ if in the spacetime or in the domain $\Omega$ a geometrically distinguished timelike unit vector field $\ddot{v}$ exists. An example of such a kind of the spacetime is given by universes considered in this paper.

Let us apply the averaged relative energy-momentum tensors for gravitation $<g t_i^k(P; v^j)>$ and for matter $<m T_i^k(P; v^j)>$ to analyze the Friedman and more general universes (Kasner vacuum universes and Bianchi type I universes filled with stiff matter).

With this aim let us define

$$g\epsilon := <g t^b_a(P; v^j)> v^a v^b$$

—the averaged relative gravitational energy density,

$$m\epsilon := <m T^b_a(P; v^j)> v^a v^b$$

(24)
— the averaged relative matter energy density, and

\[ \epsilon := g \epsilon + m \epsilon \]  

(25)

— the averaged relative total energy density.

Here \( v^a \) are the components of the four-velocity of an observer \( O \) which is studying gravitational and matter fields.

If we take as the observers \( O \) the globally defined set of the fundamental observers,[14] then we can also define the global averaged total relative energy \( E \) for the considered universes:

\[ E := \int_{t=\text{const}} \epsilon \sqrt{|g|} d^3v, \]  

(26)

and, in analogous way, the global averaged relative energy for matter and for gravitation.

Here \( d^3v \) means the product of the differentials of the coordinates which parametrize slices \( t = \text{const} \), e.g., \( d^3v = dx dy dz \) in the Cartesian comoving coordinates \( (t, x, y, z) \).

After something tedious but very simple calculations we will obtain for Friedman universes:

1. \( g \epsilon, m \epsilon \) and, in consequence \( \epsilon \), are positive definite for the all Friedman universes.

2. \( \lim_{R \to 0} g \epsilon = \lim_{R \to 0} m \epsilon = \lim_{R \to 0} \epsilon = +\infty, \quad (k = 0, \pm 1). \)

   It follows from this that one can use the averaged relative energy densities to study the Big-Bang singularity.

3. \( \lim_{R \to \infty} g \epsilon = \lim_{R \to \infty} m \epsilon = \lim_{R \to \infty} \epsilon = 0, \quad (k = 0, -1). \)

4. The global averaged relative energies, gravitation, matter and total, are infinite \((+\infty)\) for flat and for open Friedman universes and they are finite and positive for closed Friedman universes.

For vacuum Kasner universes and for expanding Bianchi type I universes filled with stiff matter one obtains the following, coordinate independent results:

1. The averaged relative gravitational energy of a vacuum Kasner universe has positive-definite density and the same limits when \( t \to 0^+ \) or when \( t \to \infty \) as in the case of a flat Friedman universe. Also the global averaged relative energy is divergent to \(+\infty\).
2. For an expanding Bianchi type I universe filled with stiff matter the averaged relative
energy densities, gravitation and matter, are still positive definite and lead to divergent
to $+\infty$ global energies.

The other three invariant integrals which formally represent the components $P_{(\alpha)}$ $(\alpha = 1, 2, 3)$ of the global averaged relative linear momentum for Friedman and for more general,
only homogeneous, universes

$$P_{(\alpha)} := \int_{t=const} \{ \langle g^{i \ 0} \rangle + \langle m^{i \ 0} \rangle \} e^{i}_{(\alpha)} \sqrt{|g|} d^3v, \quad (\alpha = 1, 2, 3), \quad (27)$$

vanish trivially in a comoving coordinates because the integrands in these integrals (densities
of the averaged relative linear momentum components) identically vanish.

Here $e^{i}_{(\alpha)}$, $(\alpha = 1, 2, 3)$ mean the components of the three translational spatial Killing
vector fields (descriptors of the linear momentum) which exist in the Friedman universes
and in the more general, only homogeneous, universes (See, e.g., [8]).

We would like to emphasize that the integrals (26) and (27) do not depend on the used
coordinates. They depend only on a slice $t = const$.

The all above results are very sensible and satisfactory from the physical point of view.

### III. CONCLUSION

The new, coordinate independent expressions on the averaged relative energy-momentum indicate that the Friedman, Kasner and Bianchi type I universes are not energetic nonentity: all they have positive-definite averaged relative energy densities.

The result of such a kind is very satisfactory from the physical point of view. Much more satisfactory than the strange and coordinate dependent results which one obtains when uses the double index energy-momentum complexes and pseudotensors.

We are planning to use in a near future the averaged relative energy-momentum tensors,
and also the averaged tensors of the relative angular momentum, [15] to analyze more general homogeneous universes than universes considered in this paper.
APPENDIX A: THE AVERAGED RELATIVE ENERGY DENSITIES FOR GRAVITY AND FOR MATTER IN KASNER AND IN BIANCHI TYPE I UNIVERSES

Here we give the final expression for $g\epsilon$ in Kasner vacuum universes and $g\epsilon$ and $m\epsilon$ in Bianchi type I universes filled with stiff matter. The corresponding expressions for Friedman universes can be easily found from the results presented in the old our papers [5] in which we have studied superenergy of the Friedman universes and from the formulas (16) and (19) of this paper which connect the averaged relative energy-momentum tensors with the canonical superenergy tensors.

1. $g\epsilon$ for gravity in a Kasner universe:

\[
g\epsilon = \frac{2\alpha}{9t^4} (p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2) L^2 + \frac{2\alpha}{27t^4} \left[ p_1^2 (p_1 - 1)^2 + p_2^2 (p_2 - 1)^2 + p_3^2 (p_3 - 1)^2 \right] L^2. \tag{A1}
\]

2. $g\epsilon$ and $m\epsilon$ in an expanding Bianchi type I universe filled with stiff matter:

\[
g\epsilon = \frac{2\alpha}{27} \left\{ \left[ (\dot{l})^2 + \ddot{l} \right]^2 + \left[ (\dot{m})^2 + \ddot{m} \right]^2 + \left[ (\dot{n})^2 + \ddot{n} \right]^2 \right. \\
+ 2(\dot{l}\dot{m} + \ddot{l}\ddot{m} + \dot{m}\ddot{m}) + \dot{m}^2 \dot{n}^2 + \dot{n}^2 \dot{m}^2 + \dot{l}^2 \dot{n}^2 \right\} L^2 \\
+ \frac{20\alpha}{27} (\dddot{l} + \dddot{n} + \dddot{m}) L^2, \tag{A2}
\]

\[
m\epsilon = \frac{\alpha}{3} \left[ \dddot{l} (\dot{m} + \dot{n}) + \dddot{m} (\dot{l} + \dot{n}) + \dddot{n} (\dot{l} + \dot{m}) + 2(\dddot{l} + \dddot{n} + \dddot{m}) \right] L^2 \\
+ \frac{4\alpha}{3} (\dot{l} \dot{m} + \dot{l} \dot{n} + \dot{m} \dot{n}) L^2 \\
- \frac{\alpha}{3} \left[ \dddot{l} (\dot{m} + \dot{n}) + \dddot{m} (\dot{l} + \dot{n}) + \dddot{n} (\dot{l} + \dot{m}) \right] (\dot{l} + \dot{m} + \dot{n}) L^2. \tag{A3}
\]

It is immediately seen that the averaged relative gravitational energy densities are in the both above cases positive definite.

With the help of the Einstein equations one can easily prove that the averaged relative energy density for matter in an expanding Bianchi type I universe is also positive-definite.

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[9] We will use geometrized units in which \( G = c = 1 \).

[10] This tensor is source in the Einstein equations.

[11] Pirani expression on energy only can be applied in a spacetime having a privileged normal
and timelike congruence. But for a geodesic congruence Pirani expression fails giving trivially
zero.

[12] More general universes were not investigated yet.

[13] \( P^b_{alm} \) has almost the same analytic form as \( B^b_{alm} \) and the same symmetries.

[14] For these observers \( v^a = \delta_o^a \) in a comoving coordinates.

[15] One can introduce such tensors in analogy to the averaged relative energy-momentum tensors.
See [6] for details.