Bound states with complex frequencies near the continuum on lossy periodic structures

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On a lossless periodic dielectric structure sandwiched between two homogeneous media, bound states in the continuum (BICs) with real frequencies and real Bloch wavevectors may exist, and they decay exponentially in the surrounding homogeneous media and do not couple with propagating plane waves with the same frequencies and wavevectors. The BICs are of significant current interest, because they give rise to high-

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resonances when the structure or the Bloch wavevector is slightly perturbed. In this paper, the effect of a small material loss on the BICs is analyzed by a perturbation method and illustrated by numerical results. It is shown that bound states with complex frequencies near the continuum appear, but they behave differently depending on whether the BIC is symmetry-protected or not. The Bloch wavevector of a bound state with a complex frequency can be real if the original BIC is symmetry-protected, and it is usually complex if the original BIC is not symmetry-protected. Our study improves the theoretical understanding on BICs and provides useful insight for their practical applications.

I. INTRODUCTION

There is currently a significant research interest on bound states in the continuum (BICs) in the photonics community [1, 2]. A BIC is a trapped or guided mode with a frequency in the radiation continuum [3]. Although a BIC has the same frequency (and wavevector when appropriate) as the radiative waves, it does not couple with these waves and does not leak power to infinity. Some early examples of BICs are trapped modes in waveguides with local distortions [4–6], where the radiative waves are propagating modes of the waveguide without the local distortion. BICs are also studied on uniform waveguides with lateral leaky structures [7–10]. In that case, a BIC is guided mode of the waveguide and the radiative waves are propagating waves on the lateral structure. A particularly simple structure supporting BICs is a slab waveguide with anisotropic core and substrate [11, 12]. Many recent works are concerned with BICs on periodic structures, including two-dimensional (2D) structures with one periodic direction [13–27], three-dimensional (3D) biperiodic structures [28–37], and 3D rotationally symmetric structures [38, 39]. In these studies, the periodic structures are sandwiched between or surrounded by homogeneous media, the BICs are guided Bloch modes above the lightline, and the radiative waves are propagating plane waves in the homogeneous media.

Importantly, a BIC can be regarded as a resonant mode (or resonant state) with an infinite quality factor (Q-factor). This implies that resonant modes with arbitrarily high Q-factors can be created by modifying the structure [40] or varying a physical parameter such as a component of the Bloch wavevector [11, 12]. Applications of optical BICs include lasing [40], sensing [41], filtering [42], switching [43], nonlinear optics [44, 45], etc. They are mostly related to the high-

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resonances, and are based on local fields enhanced by the resonances [50, 51], or special features in scattering, transmission or other spectra [52, 53]. The existence of a BIC implies that the associated scattering or diffraction problem loses uniqueness [13, 16]. The non-uniqueness may be explored for potential applications in optical switching and nonlinear optics [49].

For theoretical understanding and potential applications, it is crucial to study how a BIC is affected by a small perturbation of the structure or a physical parameter. A BIC can be robust (i.e. it continues to exist) to a special family of perturbations, but it usually turns to a resonant mode under a generic perturbation. Some BICs are symmetry-protected in the sense that they exhibit a symmetry mismatch with the corresponding radiative waves [4, 5, 11, 12]. These BICs are usually found on structures with certain symmetry, and they are robust to perturbations preserving the required symmetry [13, 16]. There are BICs that are not protected by symmetry in the usual sense (i.e. there is no symmetry mismatch), but depend crucially on symmetry for their existence and robustness [24, 25, 33]. If a BIC is turned to a resonant mode under a perturbation, it is important to find a relation between the Q-factor and the magnitude of the perturbation [40–42].

Strictly speaking, an optical BIC only exists on a lossless dielectric structure given by a real dielectric function...
(or tensor) $\epsilon$. Therefore, to study the robustness of a BIC, one assumes that the perturbation to $\epsilon$ is also real \textsuperscript{24}. Since actual materials are always lossy, it is important to analyze the effect of a small material loss to the BIC. For a symmetry-protected BIC, if the absorption profile keeps the symmetry of the structure, the BIC should remain as a bound state uncoupled with the radiative waves, but the frequency of the bound state becomes complex. In general, a BIC unprotected by symmetry is likely to become a resonant mode that radiates power to infinity, and also loses power to dissipation. But are there special class of BICs (unprotected by symmetry) that remain as bound states (without radiating power to infinity) when a small material loss is added to the structure? In this paper, we attempt to answer this question by considering propagating BICs on 2D periodic structures with reflection symmetries in both periodic and orthogonal directions. A propagating BIC on a 2D periodic structure is characterized by a real frequency and a real Bloch wavenumber. Using a perturbation method, we show that when a small absorption is added to a periodic structure with a propagating BIC, a bound state emerges, but both the frequency and the Bloch wavenumber of the bound state are complex. Our perturbation method is also applicable to symmetry-protected BICs on periodic structures. In that case, bound states with complex frequencies emerge, but the Bloch wavenumber remains real and fixed.

The rest of this paper is organized as follows. The mathematical formulation and the definitions of various modes are recalled in Sec. II. The main steps of the perturbation theory are presented in Sec. III. Numerical examples that validate the perturbation theory are shown in Sec. IV. The paper is concluded with a summary and a brief discussion in Sec. V.

### II. FORMULATION AND DEFINITIONS

To analyze the effect of a small material loss on the BICs, we consider a 2D periodic structure that is invariant in $z$, periodic in $y$ with period $L$, bounded in the $x$ direction by $|x| < D$ for some $D > 0$, and sandwiched by two homogeneous media given in $x > D$ and $x < -D$, respectively, where $\{x, y, z\}$ is a Cartesian coordinate system. Let $\epsilon = \epsilon(x, y)$ be the dielectric function describing the structure and the surrounding medium, then

$$\epsilon(x, y + L) = \epsilon(x, y) \quad \text{for all } (x, y),$$

and $\epsilon(x, y)$ is a positive constant for $x > D$ and $x < -D$, respectively. For simplicity, we assume the surrounding medium is vacuum, thus

$$\epsilon(x, y) = 1 \quad \text{for } |x| > D.$$  \hspace{1cm} (1)

For time-harmonic waves in the $E$ polarization, the $z$ component of the electric field, denoted by $u$, satisfies the following Helmholtz equation

$$\left[\partial_x^2 + \partial_y^2 + k^2\epsilon(x, y)\right] u = 0,$$  \hspace{1cm} (3)

where $k = \omega/c$ is the free space wavenumber, $\omega$ is the angular frequency, $c$ is the speed of light in vacuum, and the time dependence is assumed to be $e^{-i\omega t}$. For a real frequency, an incident plane wave given in the homogeneous media $(x > D$ or $x < -D$) induces reflected and transmitted waves. If $(\alpha, \beta)$ is the wavevector of a propagating plane wave, then $\alpha^2 + \beta^2 = k^2$, $\beta$ is real, and $|\beta| < k$ so that $\alpha = \sqrt{k^2 - \beta^2}$ is also real. Therefore, the radiation continuum for a fixed real $\beta$ is the frequency interval given by $\omega > c|\beta|$.

Without any incident field, electromagnetic waves may exist on the periodic structure as Bloch modes. For the 2D structure and the $E$ polarization, a Bloch mode is a solution of Eq. (3) given as

$$u(x, y) = e^{i\beta y} \phi(x, y),$$

where $\phi$ is periodic in $y$ with the same period $L$, $\beta$ is the Bloch wavenumber, and $u$ is outgoing or exponentially decaying as $|x| \to \infty$. The periodicity of $\phi$ implies that the real part of $\beta$ can be restricted by $|\text{Re}(\beta)| \leq \pi/L$. For $|x| > D$, the Bloch mode can be expanded in plane waves as

$$u(x, y) = \sum_{j=-\infty}^{\infty} c_j^\pm e^{i(\beta_j y \pm \alpha_j x)}, \quad x > D,$$

where $c_j^\pm$ are expansion coefficients,

$$\beta_j = \beta + \frac{2\pi j}{L}, \quad \alpha_j = \sqrt{k^2 - \beta_j^2}.$$  \hspace{1cm} (6)

The square root in Eq. (6) is chosen such that either $\alpha_j$ has a positive imaginary part or $\alpha_j$ is real and positive. If the periodic structure is a lossless dielectric structure, i.e., $\epsilon$ is real, it can have guided Bloch modes (with real $\beta$ and real $\omega$) that decay exponentially as $|x| \to \infty$. Typically, the guided modes exist below the lightline (i.e. $k < |\beta|$) and form bands that are continuous in $\beta$. In this case, all $\alpha_j$ are pure imaginary with positive imaginary parts. A BIC is a special guided mode above the lightline (i.e. $k > |\beta|$). In that case, one or more $\alpha_j$ (including $\alpha_0 = \alpha$) are real and positive, but since the BIC decays exponentially as $|x| \to \infty$, the corresponding coefficients $c_j^\pm$ must vanish. Clearly, a BIC does not couple with propagating plane waves with wavevectors $(\pm \alpha, \beta)$. We call a BIC a standing wave if $\beta = 0$, and a propagating BIC if $\beta \neq 0$. Notice that the BICs only exist at special values of $\beta$.

On a 2D periodic structure, a resonant mode is a Bloch mode satisfying outgoing radiation conditions as $|x| \to \infty$. It is usually defined for a real $\beta$ and has a complex $\omega$. Since it radiates power to infinity, the amplitude of a resonant mode decays with time, and the $Q$-factor can be defined as $Q = -0.5\text{Re}(\omega)/|\text{Im}(\omega)|$. The resonant modes form bands, and on each band they depend continuously on $\beta$, except for special values of $\beta$ at which no power is radiated to infinity and $\omega$ becomes real. These special values of $\beta$ correspond exactly to the BICs. Therefore, a
BIC can be regarded as a special resonant mode with an infinite $Q$-factor. The resonant modes mostly exist above the lightline (defined using the real part of $k$), but their endpoints are actually located below the lightline [55].

Leaky modes on wave-guiding structures are usually defined for a real frequency [54]. For 2D periodic structures, a leaky mode is also a Bloch mode given in Eq. (4), it also satisfies outgoing radiation conditions as $|x| \to \infty$, but its Bloch wavenumber $\beta$ is complex. If the leaky mode propagates toward $y = +\infty$, then $\text{Im}(\beta)$ should be positive, such that as it propagates forward, it loses power and its amplitude decays with $y$. Leaky modes can exist below and above the lightline defined using the real part of $\beta$ [55].

For structures with a reflection symmetry in the periodic direction (i.e., the $y$ direction), the antisymmetric standing waves (ASWs) are well-known symmetry-protected BICs [13, 14]. Assuming the origin is chosen such that

$$\epsilon(x, y) = \epsilon(x, -y) \quad \text{for all } (x, y),$$

an ASW is a special BIC with $\beta = 0$ and an antisymmetric wave field, i.e., $u(x, -y) = -u(x, y)$. Due to the symmetry mismatch between the wave field and the plane waves with wavevectors $(\pm \alpha, \beta) = (\pm k, 0)$, the coefficients $c_0^+ \text{ and } c_0^-$ in Eq. (5) are automatically zero. The existence of ASWs can be rigorously proved [13, 16], and they are robust under symmetric perturbations of the structure [24].

Periodic structures can have BICs that are not obviously protected by any symmetry [15, 17, 13, 21, 23, 31–34, 38, 39]. In the 2D case, it is relatively easy to find nontrivial BICs on structures that are not only symmetric in $y$, but also symmetric in $x$, i.e.,

$$\epsilon(x, y) = \epsilon(x, -y) = \epsilon(-x, y) \quad \text{for all } (x, y).$$

If $k < 2\pi/L - |\beta|$, then all $\alpha_j$ for $j \neq 0$ are pure imaginary, thus, a BIC needs only to satisfy the condition $c_0^+ = 0$. The reflection symmetry in $x$ implies that the BIC (with a nonzero $\beta$ in general) is either even in $x$ or odd in $x$. Therefore, the coefficients $c_0^+$ and $c_0^-$ are related by $c_0^+ = \pm c_0^-$. The reflection symmetry in $y$ is also useful. It allows us to scale the BIC by a complex constant, such that

$$u(x, y) = u(x, -y) \quad \text{for all } (x, y),$$

where $u$ is the complex conjugate of $u$ [15, 24]. This is a case of the $\mathcal{PT}$-symmetry. It is easy to verify that the periodic function $\phi$ in Eq. (4) is also $\mathcal{PT}$-symmetric.

III. PERTURBATION THEORY

Suppose there is a BIC on a lossless dielectric structure described by a real dielectric function $\epsilon$, it is clearly important to find out what happens to the BIC, if the structure is slightly perturbed [24, 40]. In general, a real perturbation of the dielectric function $\epsilon$ will destroy the BIC and produce a resonant mode with a complex frequency, but special perturbations may preserve the BIC [24]. The case for symmetry-protected BICs is well understood. If the dielectric function $\tilde{\epsilon}$ of the perturbed structure is still real and symmetric, the BIC is preserved and will have a slightly different real frequency. The case for BICs without symmetry protection (i.e., there is no symmetry mismatch between the BIC and the radiating waves) is more complicated. For some cases, it is still possible to identify special perturbations that preserve the BIC. For example, a propagating BIC on a periodic structure with reflection symmetry in both $x$ and $y$ directions is preserved if the perturbation is also symmetric in $x$ and $y$ [24]. The perturbed structure will have a BIC with a slightly different frequency and a slightly different Bloch wavenumber.

Here, we investigate the effect of a small material loss on the BICs. The dielectric function of a lossy periodic structure is written as

$$\tilde{\epsilon}(x, y) = \epsilon(x, y) + \delta F(x, y),$$

where $\delta$ is real, $F$ is a complex function with $\text{Im}(F) \geq 0$ and $\max[\text{Im}(F)] = 1$, and $\delta$ is a small positive number. We study the problem by a perturbation method, assuming the lossless structure given by $\epsilon(x, y)$ has a BIC $u = \phi \exp(i\beta y)$ with frequency $\omega$, and the lossy structure has a Bloch mode $\tilde{u} = \phi \exp(i\beta y)$ with frequency $\tilde{\omega}$. Due to the material loss, it is clearly impossible to have a real $\tilde{\omega}$ and a real $\beta$, if the Bloch mode $\tilde{u}$ is required to decay exponentially or radiate power outward as $|x| \to \infty$. Moreover, if $\tilde{u}$ is allowed to radiate power to infinity, we can specify a real $\tilde{\omega}$ around $\beta$ and find a Bloch mode with a complex $\tilde{\omega}$, or specify a real $\tilde{\omega}$ around $\omega$ and calculate a solution with a complex $\tilde{\beta}$. These solutions correspond to resonant and leaky modes, respectively, and they are not the topic of this paper. Since the BIC decays exponentially as $|x| \to \infty$, we look for a perturbed mode $\tilde{u}$ that also decays exponentially as $|x| \to \infty$. More precisely, we assume the frequency and Bloch wavenumber of the original BIC satisfy

$$|\beta| < k < \frac{2\pi}{L} - |\beta|,$$

and look for a nearly bound state $\tilde{u}$ such that the coefficients $c_0^\pm$ for $\tilde{u}$ (defined as in Eq. (4)) vanish. For that purpose, we need to determine both $\tilde{\omega}$ and $\tilde{\beta}$.

Let us consider a lossless periodic structure with a real dielectric function $\epsilon$ satisfying Eqs. (11), (24) and (8), and a BIC with frequency $\omega$ and a Bloch wavenumber $\beta$ satisfying Eq. (11), and assume the BIC is scaled to satisfy the $\mathcal{PT}$-symmetric condition [9]. If $\tilde{u}$ is a bound state (close to the BIC) on the lossy structure with a dielectric function $\tilde{\epsilon}$ given in Eq. (10), then $\tilde{u}$ satisfies Helmholtz equation (3) with $\epsilon$ and $k$ replaced by $\tilde{\epsilon}$ and $\tilde{k} = \tilde{\omega}/c$, respectively. To find $\tilde{u}$, we use a perturbation method by
expanding \( \tilde{\phi}, \tilde{k} \) and \( \tilde{\beta} \) in power series of \( \delta \): 

\[
\tilde{\phi} = \phi + \delta \phi_1 + \delta^2 \phi_2 + \cdots \tag{12}
\]

\[
\tilde{\beta} = \beta + \delta \beta_1 + \delta^2 \beta_2 + \cdots \tag{13}
\]

\[
\tilde{k} = k + \delta k_1 + \delta^2 k_2 + \cdots \tag{14}
\]

Inserting the above into the governing equation of \( \tilde{\phi} \) (derived from the Helmholtz equation for \( \tilde{u} \), see Appendix), and considering the different powers of \( \delta \), we obtain \( \mathcal{L} \phi = 0 \) for 

\[
\mathcal{L} = \partial_x^2 + \partial_y^2 + 2i\beta \partial_y + k^2 \epsilon - \beta^2, \tag{15}
\]

and 

\[
\mathcal{L} \phi_j = B_1 \beta_j + B_2 k_j - C_j, \tag{16}
\]

for \( j \geq 1 \), where 

\[
B_1 = 2\beta \phi - 2i \partial_y \phi, \quad B_2 = -2k \epsilon \phi, \quad C_1 = k^2 F \phi,
\]

and \( C_j \) for \( j \geq 2 \) are given in Appendix.

The operator \( \mathcal{L} \) is singular, since \( \mathcal{L} \phi = 0 \) has a nonzero solution corresponds to the BIC. Equation (16) is a Helmholtz equation with a source term in the right hand side. It does not have a solution, unless the right hand side is orthogonal with the BIC, that is

\[
\int_{\Omega} \mathcal{L} (B_1 \beta_j + B_2 k_j - C_j) \, dr = 0, \tag{17}
\]

where \( r = (x, y) \) and \( \Omega \) is the domain for one period of the structure, given by

\[
\Omega = \{(x, y) \mid -\infty < x < \infty, -L/2 < y < L/2\}.
\]

Since \( \tilde{u} \) is required to have \( c_{0,0}^\pm = 0 \) (defined as in Eq. (12)), \( \phi_j \) must decay exponentially as \( |x| \to \infty \), then the right hand side of Eq. (16) must be orthogonal to a diffraction solution \( \phi \) for the corresponding frequency \( \omega \) and wavenumber \( \beta \). That is,

\[
\int_{\Omega} \mathcal{L} (B_1 \beta_j + B_2 k_j - C_j) \, dr = 0. \tag{18}
\]

More details on \( \phi \) are given in Appendix. In particular, \( \phi \) is chosen to be even (or odd) in \( x \) if the BIC is even (or odd) in \( x \), and it is scaled to satisfy the \( \mathcal{PT} \)-symmetry condition (9). Equations (17) and (18) give rise to

\[
\begin{bmatrix} \beta_j \\ k_j \end{bmatrix} = \begin{bmatrix} b_{1j} \\ b_{2j} \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{19}
\]

where

\[
\begin{align*}
a_{11} &= \int_{\Omega} \phi B_1 \, dr, \\
a_{12} &= \int_{\Omega} \phi B_2 \, dr, \\
a_{21} &= \int_{\Omega} \phi^* B_1 \, dr, \\
a_{22} &= \int_{\Omega} \phi^* B_2 \, dr, \\
b_{1j} &= \int_{\Omega} \phi C_j \, dr, \\
b_{2j} &= \int_{\Omega} \phi^* C_j \, dr.
\end{align*}
\]

Since \( \phi \) and \( \varphi \) satisfy the \( \mathcal{PT} \)-symmetry condition (9), and \( \epsilon \) satisfies Eq. (5), it is easy to show that all entries of matrix \( A \) are real. If \( A \) is invertible, we can solve \( \beta_j \) and \( k_j \) from the above system, and then solve \( \phi_j \) from Eq. (16). The two conditions (17) and (18) ensure that Eq. (16) has a solution that decays exponentially as \( |x| \to \infty \). Moreover, \( \phi_j \) has the same parity (even or odd in \( x \)) as the BIC. For the lossy structure, the function \( F \) in Eq. (16) is assumed to be pure imaginary. If we assume that \( F \) is also symmetric in both \( x \) and \( y \), i.e., \( F \) satisfies Eq. (8), then \( b_{11} \) and \( b_{22} \) are also pure imaginary. This means that \( \beta_1 \) and \( k_1 \) are pure imaginary and nonzero in general. Therefore, both \( \tilde{\omega} \) and \( \tilde{\beta} \) of the bound state \( \tilde{u} \) are complex in general.

The ASWs are important special cases with \( \beta = 0 \). The field profile \( \phi \) of an ASW is odd in \( y \). The diffraction solution \( \varphi \) is even in \( y \). To be consistent with the above perturbation theory, we can scale \( \phi \) as a pure imaginary function, and scale \( \varphi \) as a real function, then both satisfy Eq. (9). Due to these properties, it is easy to show that \( a_{21} = 0 \) and \( b_{21} = 0 \). It is also clear that \( a_{12} \neq 0 \). If \( a_{21} \neq 0 \), we conclude that \( \beta_1 = 0 \) and \( k_1 = b_{11}/a_{12} \). Similar to the general case above, Eq. (16) can be solved, and the solution \( \phi_j \) is also odd in \( y \). This implies that \( b_{2j} = 0 \) and \( \beta_j = 0 \) for all \( j \geq 1 \). Furthermore, for the ASWs, the reflection symmetry in \( x \) is not important. We can obtain the same conclusion if \( \epsilon \) and \( F \) have only a reflection symmetry in \( y \), that is, they satisfy Eq. (17), instead of Eq. (8).

The perturbation theory is valid only when the matrix \( A \) is invertible or \( a_{21} \neq 0 \) if the BIC is an ASW. The matrix \( A \) can indeed be singular, for example, when the BIC is a symmetric standing wave, i.e., \( \beta = 0 \) and \( \phi \) is even in \( y \). A similar restriction is present in the theory on the robustness of the BICs (24). However, for a generic BIC, the matrix \( A \) is typically invertible.

IV. NUMERICAL EXAMPLES

In this section, we present some numerical results to validate the perturbation theory. The periodic structure concerned is an array of parallel, identical and infinitely long circular cylinders with radius \( a \) and dielectric constant \( \epsilon_c \) surrounded by vacuum. The cylinders are parallel to the \( z \) axis and their centers are located on the \( y \) axis. The period in the \( y \) direction, i.e. the center-to-center distance between two nearby cylinders, is \( L \). The structure is chosen for its simplicity. In fact, a semi-analytic computational method based on cylindrical and plane wave expansions is available to calculate Bloch modes of the structure to high accuracy (20).

Antisymmetric standing waves on periodic arrays of circular cylinders have been thoroughly investigated before (14, 19, 20). On an array of cylinders with \( a = 0.3L \) and \( \epsilon_c = 11.6 \), there is an ASW with normalized frequency \( kL/(2\pi) = 0.41122783 \). Now we assume the cylinders are lossy, with a dielectric constant \( \epsilon_c = 11.6 + i\delta \).
for \( \delta > 0 \). According to the perturbation theory, when \( \delta \) is small, there should be a bound state with a complex frequency (and a zero Bloch wavenumber) near the original ASW. We calculate the bound state for a few different values of \( \delta \) and list the numerical results for \( \tilde{k} \) in Table I below. The field pattern of the bound state for \( \delta = 0.1 \) is shown in Fig. 1. As we know from Sec. III, the bound state near the ASW is also anti-symmetric (odd in \( y \)) and also decays exponentially as \( x \to \pm \infty \). These properties are noticeable in Fig. 1. The perturbation theory gives us the analytic formula \( k_1 = b_{11}/a_{12} \) for the first order term in the power series of \( \tilde{k} \), where \( b_{11} \) and \( a_{12} \) are related to the ASW, and functions \( \epsilon \) and \( F \). Since all these are known, we evaluate \( k_1 \) by this formula and obtain \( k_1L \approx -0.1068i \). From the numerical solutions of \( \tilde{k} \) for different values of \( \delta \), we can also estimate \( k_1 = \partial \tilde{k}/\partial \delta |_{\delta=0} \) by a difference formula. The numerical result is \( k_1L \approx -0.1069i \). The agreement with the analytic formula is excellent.

A periodic array of circular cylinders can support propagating BICs. We assume the cylinders have a radius \( a = 0.35L \) and dielectric constant \( \epsilon_r = 11.56 + i \delta \). For \( \delta = 0 \), the periodic array has one odd-in-\( x \) propagating BIC with \( kL/(2\pi) = 0.6702 \) and \( \beta/(2\pi) = 0.2483 \), and one even-in-\( x \) propagating BIC with \( kL/(2\pi) = 0.4854 \) and \( \beta L/(2\pi) = 0.0780 \). For a small \( \delta > 0 \), these two BICs are turned to bound states with a complex frequency and a complex Bloch wavenumber. We calculate the bound states for a few different values of \( \delta \). The numerical results for \( \tilde{k} \) and \( \tilde{\beta} \) of the bound state near the odd-in-\( x \) BIC are listed in Table II. The field pattern of the bound state for \( \delta = 0.1 \) is shown in Fig. 2. The perturbation theory allows us to solve \( \beta_j \) and \( k_j \) from the \( 2 \times 2 \) linear system Eq. (19). The first order terms \( \beta_1 \) and \( k_1 \) are particularly easy to evaluate, since they are only related to the BIC, the diffraction solution \( \varphi \), and the given functions \( \epsilon \) and \( F \). Using the odd-in-\( x \) solution \( \varphi \), we obtain \( \beta_1L \approx -0.062i \) and \( k_1L \approx -0.173i \). Since \( k_1 \) and \( \beta_1 \) are partial derivatives of \( \tilde{k} \) and \( \tilde{\beta} \) with respect to \( \delta \) and evaluated at \( \delta = 0 \), we can estimate \( k_1 \) and \( \beta_1 \) from the numerical solutions in Table II by some difference formulae. The estimated values are \( \beta_1L \approx -0.0620i \) and \( k_1L \approx -0.1733i \) and they agree very well with the exact values by the perturbation theory.

For the same periodic array, we calculate the bound state near the even-in-\( x \) propagating BIC. The complex \( \tilde{k} \) and \( \tilde{\beta} \) of the bound state are listed in Table III. The field pattern of the bound state for \( \delta = 0.1 \) is shown in Fig. 1. As we know from Sec. III,
given in Table III are exactly the same.

V. CONCLUSION

An optical BIC usually exists on an open lossless di-electric structure. It has a real frequency, and also a real Bloch wavevector if the structure is periodic. We study how BICs are affected when a small material loss is added to the structure. The result on symmetry-protected BICs is fully expected. For 2D periodic structures with a reflection symmetry, a symmetry-protected BIC becomes a bound state with a complex frequency, if the absorption profile preserves the relevant symmetry. The new bound state still has a symmetry mismatch with the radiative waves, and does not radiate power to infinity. The imaginary part of the frequency is solely the consequence of material loss, and the real part of the frequency remains in the radiation continuum.

The case for BICs without the usual symmetry-protection is more complicated. In general, adding a small material loss to the structure may open up a radiation channel and turn a BIC to a resonant mode that radiates power to infinity. Many BICs are propagating modes with a real and nonzero Bloch wavevector on periodic structures with certain symmetry, and their frequencies are within the interval where only one radiation channel exists. These propagating BICs are not protected by symmetry in the sense of a symmetry mismatch, but they depend on the symmetry for their existence and robustness [24]. We consider 2D periodic structures with reflection symmetry in both periodic and perpendicular directions, and study the effect of a small material loss that preserves the reflection symmetry. We show that a generic propagating BIC is turned to a bound state with a complex frequency and a complex Bloch wavenumber. The real part of the complex frequency still lies in the radiation continuum. The imaginary part accounts for both radiation and absorption losses. The complex Bloch wavenumber implies that the bound state no longer has a uniform magnitude along the periodic direction. In general, it is impossible to have a bound state with a real Bloch wavenumber near the original propagating BIC. All modes with a real Bloch wavenumber at and near that of the BIC are resonant modes.

The performance of devices based on BICs may be limited by a number of practical issues, such as material losses, fabrication errors, finite sizes (as true BICs only exist on infinite structures), and loss of periodicity (for periodic structures). The impact of material loss to resonant effects (Q-factors and field enhancement) around a BIC has been analyzed by Yoon et al. [51]. Our study on bound states with complex frequencies reveals a major difference between BICs protected or unprotected by symmetry. It is worthwhile to investigate the consequences of this difference on the resonant effects.

APPENDIX

The perturbation theory follows our earlier work [24]. Some key steps are listed below. More details can be found in [24]. The governing equation for $\phi$ is

$$\left[\partial_x^2 + \partial_y^2 + k^2 - \beta^2 + 2i\beta\partial_y\right]\tilde{\phi} = 0.$$  

In the right hand side of Eq. (16), $C_j$ (for $j \geq 2$) are given by

$$C_j = \left[\sum_{l=1}^{j-1} (k_l k_{j-l} + \beta_l \beta_{j-l}) + F \sum_{l=0}^{j-1} k_l k_{j-l} - i \sum_{n=1}^{j-1} \sum_{l=0}^{j-1} (k_l k_{n-l} + \beta_l \beta_{n-l}) + F \sum_{l=0}^{j-1} k_l k_{n-l} - i \sum_{n=1}^{j-1} \beta_n \partial_y \phi_{j-n}\right] \phi_{j-n} + 2i \sum_{n=1}^{j-1} \beta_n \partial_y \phi_{j-n}$$

where $\beta_0 = \beta$ and $k_0 = k$.

Equation (17) is derived by multiplying $\overline{\phi}$ to both sides of Eq. (16), integrating on $\Omega$, and showing that

$$\int_\Omega \phi \partial_* \phi dx = 0.$$  

If the BIC is even in $x$, we consider a diffraction problem with two incident plane waves: $\exp[i(\beta y + \alpha x)]$ for $x < -D$ and $\exp[i(\beta y - \alpha x)]$ for $x > D$. Since the structure is symmetric in $x$, the diffraction solution, denoted as $\tilde{\nu}_e$, is also even in $x$. Moreover, $\tilde{\nu}_e$ contains outgoing plane waves of identical amplitudes as $x \to \pm \infty$, that is

$$\tilde{\nu}_e(x, y) \sim e^{i(\beta y \pm \alpha x)} + S_\epsilon e^{i(\beta y \mp \alpha x)}, \quad x \to \pm \infty.$$  

Since $\epsilon$ is real, energy is conserved, thus $|S_\epsilon| = 1$. If $S_\epsilon = \exp(2i\theta_e)$ for a real $\theta_e$, we define $\nu_e = \tilde{\nu}_e \exp(-i\theta_e)$, then $\nu_e$ is even in $x$ and satisfies the $\mathcal{PT}$-symmetry condition, Eq. (9). The function $\varphi$ is given by $\varphi = \nu_e \exp(-i\beta y)$. If the BIC is odd in $x$, we similarly define a diffraction solution $\nu_o$ which is odd in $x$ and satisfies Eq. (9), and let $\varphi = \nu_o \exp(-i\beta y)$.

Equation (18) is derived by multiplying $\overline{\varphi}$ to both sides of Eq. (16) and integrating on the rectangle $\Omega_h$ for $|x| < h$,
and $|y| < L/2$. Assuming $\phi_j \to 0$ as $|x| \to \infty$ exponentially and taking the limit for $h \to \infty$, the left hand side gives

$$\int_{\Omega} \nabla \phi_j dx = 0.$$  

The right hand side must also vanish. This leads to Eq. (13). On the other hand, if $\beta$ and $k_j$ satisfy Eq. (19), then Eq. (16) (for $\phi_j$) has a solution and that solution must decay exponentially as $|x| \to \infty$.

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