FUNDAMENTAL GROUP OF $AF$-ALGEBRAS WITH FINITE DIMENSIONAL TRACE SPACE

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Abstract. We consider the realization of fundamental groups of $AF$-algebras in a certain class. We find the fundamental groups of $AF$-algebras with finite dimensional trace space which is not realizable as a fundamental group of von Neumann algebras.

1. Introduction

Let $\mathcal{M}$ be a factor of type $II_1$ with a normalized trace $\tau$. Murray and von Neumann introduced the fundamental group $F(\mathcal{M})$ of $\mathcal{M}$ in [7]. They showed that if $\mathcal{M}$ is hyperfinite, then $F(\mathcal{M}) = \mathbb{R}_+^\times$. Since then there has been many works on the computation of the fundamental groups. Voiculescu showed that $F(L(F_\infty))$ of the group factor $L(F_\infty)$ of the free group $F_\infty$ contains the positive rationals in [15] and Radulescu proved that $F(L(F_\infty)) = \mathbb{R}_+^\times$ in [14]. Connes [1] showed that $F(L(G))$ is a countable group if $G$ is an ICC group with property (T). Popa and Vaes showed that either countable subgroup of $\mathbb{R}_+^\times$ or any uncountable group belonging to a certain "large" class can be realized as the fundamental group of some factor of type $II_1$ in [12] and in [13].

Nawata and Watatani [10], [11] introduced the fundamental group of simple $C^*$-algebras with unique trace whose study is essentially based on the computation of Picard groups by Kodaka [4], [5], [6]. Nawata defined the fundamental group of non-unital $C^*$-algebras [8] and calculate the Picard group of some projectionless $C^*$-algebras with strict comparison by the fundamental groups [9]. We introduced the fundamental group of $C^*$-algebras with finite dimensional trace space in [2] whose study is essentially based on the computation of [10], [11]. Moreover, We introduced the fundamental group of finite von Neumann algebras with finite dimensional normal trace space in [3] and identified the realizable fundamental groups of them.

In this paper, we will consider the difference between fundamental groups of $C^*$-algebras and those of finite von Neumann algebras. We will consider the realization of fundamental groups of $AF$-algebras. Let $a, b$ be non-zero positive real numbers. We suppose that $a \neq b$. Then we can notice that there is no finite von Neumann algebra $\mathcal{M}$ with finite dimensional trace space such that $F(\mathcal{M}) = \{ \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} : n \in \mathbb{Z} \}$ by the form of fundamental group of von Neumann algebras.

If $a$ is algebraic number and $b$ is rational (or specific algebraic number), then there is no simple $AF$-algebra $\mathcal{A}$ in a certain class such that $F(\mathcal{A}) = \{ \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} : n \in \mathbb{Z} \}$.
However, we show this as one of the main theorems at last, if $a$ is transcendental, then there is a simple AF-algebra $A$ such that $F(A) = \{ \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} : n \in \mathbb{Z} \}.$

2. Realization of fundamental group of AF-algebra

We are interested in the existence of fundamental group. We will focus on the fundamental group of the specific AF-algebras. Let $A$ be an AF-algebra with $n$-dimensional trace space. Say $\{ \varphi_i \}_{i=1}^n = \partial, T(A)_1$. Let $g$ be an element of $K_0(A)$. Put $\Gamma(a) = (\varphi_1^*(g), \varphi_2^*(g), \ldots, \varphi_n^*(g))$. From now, we will consider AF-algebras which satisfies that $\Gamma$ is an injective unit preserving ordered abelian group homomorphism from $(K_0(A), K_0(A)^+, [1_A])$ to $(\mathbb{R}^n, \mathbb{R}^n_+, (1, 1, \ldots, 1))$. We call that $A$ has $K_0(A)$ embedding on $\mathbb{R}^n$ if $\Gamma$ is an injective unit preserving ordered abelian group homomorphism. Then we can see that $(K_0(A), K_0(A)^+, [1_A])$ is an ordered abelian subgroup of $(\mathbb{R}^n, \mathbb{R}^n_+, (1, 1, \ldots, 1))$. Conversely, we will conduct the condition that an ordered abelian subgroup of $(\mathbb{R}^n, \mathbb{R}^n_+, (1, 1, \ldots, 1))$ is considered as $(K_0(A), K_0(A)^+, [1_A])$ for some $A$ the dimension of trace space of which is $n$.

**Proposition 2.1.** Let $A$ be an AF-algebra with $n$-dimensional trace space. We suppose that $K_0(A)$ is embedding on $\mathbb{R}^n$. Let $D$ be an invertible positive diagonal matrix of $M_n(\mathbb{C})$ and let $U$ be a permutation unitary of $M_n(\mathbb{C})$. Then $\Gamma(K_0(A))DU = \Gamma(K_0(A))$ if and only if $DU \in F(A)$.

**Proof.** We consider that $K_0(A) \subset \mathbb{R}^n$ and that $\Gamma(K_0(A)) = K_0(A)$. We suppose that $K_0(A)DU = K_0(A)$. Let $p$ be a corresponding projection of $M_k(A)$ to $(1, 1, \ldots, 1)DU$. We define an isomorphism of ordered abelian group from $(K_0(A), K_0(A)^+, (1, 1, \ldots, 1))$ onto $(K_0(A), K_0(A)^+, (1, 1, \ldots, 1)DU)$ by $M_DUg = g(DU)$. Then an isomorphism $\Phi A \to pM_k(A)p$ is induced by $M_DU$ and the matrix representation of $T_{(k, p, \Phi)}$ is $DU$. Therefore $DU \in F(A)$. Converse is followed in [2].

**Proposition 2.2.** Let $M$ be an additive subgroup of $\mathbb{R}^n$. Let $\varepsilon > 0$ arbitrary. We suppose that there exist $\{ a_i^n \}_{i=1}^n \subset M$ satisfying following properties;

(i) $\{ a_i^n \}_{i=1}^n$ are $\mathbb{R}$-linearly independent.

(ii) $\| a_i^n \| < \varepsilon$, where $\| \cdot \|$ is an Euclidean norm of $\mathbb{R}^n$.

Then $M$ is dense in $\mathbb{R}^n$.

**Proof.** Let $M$ be an additive subgroup of $\mathbb{R}^n$ satisfying the hypothesis and let $\varepsilon > 0$. Then we can choose $\{ a_i^n \}_{i=1}^n$. By (i), $\{ a_i^n \}_{i=1}^n$ is a basis of $\mathbb{R}^n$. Let $a$ be in $\mathbb{R}^n$. Then $a = \sum_{i=1}^n r_i a_i^n$ for some $r_i$ in $\mathbb{R}$. Let $m_i$ be an integer satisfying $m_i \leq r_i \leq m_i + 1$. Then $\| a - \sum_{i=1}^n m_i a_i^n \| \leq \sum_{i=1}^n (r_i - m_i) \| a_i^n \| < \varepsilon$ by (ii). Therefore $M$ is dense in $\mathbb{R}$.

Let $M$ be an additive subgroup of $\mathbb{R}^n$. We define the positive cone of $M$ by $\{ (h_i)_{i=1}^n \in M : h_i > 0 \} \cup \{ (0, 0, \ldots, 0) \}$. We denote it by $M^+$. Then $(M, M^+)$ is an ordered abelian group. Moreover, we define the additive group homomorphism $\varphi_k$ from $M$ into $\mathbb{R}$ by $\varphi_k((h_i)_{i=1}^n) = h_k$. Then $\varphi_k$ are positive homomorphisms. Let $\{ g_n \}_{n=1}^\infty$ be a sequence of $M$. We call $\{ g_n \}_{n=1}^\infty$ increasing (decreasing) if $g_n \leq g_m$ ($g_n \geq g_m$) for any $n < m$. We denote by $S(M, M^+, (1, 1, \ldots, 1))$ the state space of $(M, M^+, (1, 1, \ldots, 1))$ and by $\partial S(M, M^+, (1, 1, \ldots, 1))$ the set of extremal points of $S(M, M^+, (1, 1, \ldots, 1))$. 


Proposition 2.3. Let \((M, M^+)\) be an ordered abelian subgroup of \((\mathbb{R}^n, (\mathbb{R}^n)^+)\).

We suppose that \(M\) is dense in \(\mathbb{R}^n\) and \((1, 1, \cdots, 1) \in M\).

Then \((M, M^+, (1, 1, \cdots, 1))\) is a dimension group and \(\partial_e S(M, M^+, (1, 1, \cdots, 1)) = \{ \varphi_k \}_{k=1}^n\).

Proof. Since \(M\) is dense in \(\mathbb{R}^n\), \(M\) satisfies Riesz property.

Then \((M, M^+, (1, 1, \cdots, 1))\) is a dimension group.

We will show that \(\partial_e S(M, M^+, (1, 1, \cdots, 1)) = \{ \varphi_k \}_{k=1}^n\).

Let \(\varphi\) be in \(S(M, M^+, (1, 1, \cdots, 1))\) and let \(g\) be in \(M\) arbitrary. First, we will show that \(\varphi(g_i)\) converges to \(\varphi(g)\) for any increasing sequence \(\{g_i\}_{i=1}^\infty\) which converges to \(g\) and \(g_i \neq g\) for any \(i\). By positivity of \(\varphi\), \(\{ \varphi(g_i) \}_{i=1}^n\) is an increasing converging sequence. Put \(\lim_{i \to \infty} \varphi(g_i) = \alpha\). Conversely, we suppose that \(\alpha \neq \varphi(g)\). Put \(\varepsilon = |\alpha - \varphi(g)| > 0\). Then \(0 < \frac{1}{n} < \frac{\varepsilon}{2}\) for some \(n \in \mathbb{N}\). Since \(g_i \to g\), there exists a natural number \(M_1\) such that \(0 \leq g - g_i \leq \frac{1}{n}(1, 1, \cdots, 1)\) for any \(i \geq M_1\).

Therefore \(0 \leq \varphi(g) - \varphi(g_i) \leq \frac{1}{n}\). Otherwise, \(\lim_{i \to \infty} \varphi(g_i) = \alpha\), so there exists a natural number \(M_2\) such that \(0 \leq \alpha - \varphi(g_i) \leq \frac{1}{n}\) for any \(i \geq M_2\). Therefore \(|\alpha - \varphi(g)| \leq \frac{1}{n}\). Hence \(0 < \varepsilon = |\alpha - \varphi(g)| \leq \frac{1}{n} < \frac{\varepsilon}{2}\). This leads to contradiction. As is the same with this previous proof, we can show the case of decreasing sequences.

Second, we will show that \(\varphi\) is continuous on \(M\) by Euclidean norm of \(\mathbb{R}\) for any \(\varphi\) in \(S(M, M^+, (1, 1, \cdots, 1))\). Let \(g\) be in \(M\) and let \(h_i\) be a converging sequence to \(g\). Since \(M\) is dense in \(\mathbb{R}^n\), we can get some \(\{g_i\}_{i=1}^\infty\) which converges to \(g\) increasingly and \(g_i \neq g\) for any \(i\). Then \(\{2g - g_i\}_{i=1}^\infty\) is a deceasing sequence, converges to \(g\), and \(2g - g_i\) is not \(g\) for any \(i\). Let \(\varepsilon > 0\) arbitrary. Then there exists a natural number \(M_3\) such that \(0 \leq \varphi(g) - \varphi(g_i) < \varepsilon\) for any \(i \geq M_3\). Since \(h_i\) converges to \(g\), there exists a natural number \(M_4\) such that \(g_{M_4} \leq h_i \leq 2g - g_{M_4}\) for any \(i \geq M_4\). Then \(\varphi(g_{M_4}) \leq \varphi(h_i) \leq \varphi(2g - g_{M_4})\). Therefore \(\varphi(h_i) - \varphi(g_i) < \varepsilon\) for any \(i \geq M_4\). Since \(\varphi(h_i)\) converges to \(\varphi(g)\), \(\varphi\) is continuous on \(M\). Third, we will show that \(\{ \varphi(h_i) \}_{i=1}^\infty\) is a Cauchy sequence if \(\{h_i\}_{i=1}^\infty\) is so. Let \(\{g_i\}_{i=1}^\infty\) be a decreasing sequence in \(M\) satisfying that \(g_i \to (0, 0, \cdots, 0)\) and that \(g_i \neq (0, 0, \cdots, 0)\) for any \(i\). Then \(\{ -g_i \}_{i=1}^\infty\) is an increasing sequence, converges to \((0, 0, \cdots, 0)\), and \(-g_i\) is not \((0, 0, \cdots, 0)\) for any \(i\). Let \(\varepsilon > 0\) arbitrary. Then there exists a natural number \(M_5\) such that \(0 \leq \varphi(g_i) < \varepsilon\) for any \(i \geq M_5\). Since \(\{h_i\}_{i=1}^\infty\), there exists a natural number \(M_6\) such that \(-g_{M_6} \leq h_i - h_j \leq g_{M_6}\) for any \(i, j \geq M_6\). Then \(-\varphi(g_{M_6}) \leq \varphi(h_i) - \varphi(h_j) \leq \varphi(g_{M_6})\). Therefore \(\varphi(h_i) - \varphi(h_j) \leq \varepsilon\) for any \(i \geq M_6\). Hence \(\{ \varphi(h_i) \}_{i=1}^\infty\) is a Cauchy sequence if \(\{h_i\}_{i=1}^\infty\) is so. By the second result, the third result and by the density of \(M\), we can extend \(\varphi\) to be a continuous functional \(\hat{\varphi}\) on \(\mathbb{R}^n\). Since \(M\) is \(\mathbb{Z}\)-linear, then \(\hat{\varphi}\) is \(\mathbb{R}\)-linear. We denote by \(\hat{\varphi}_k\) the \(\mathbb{R}\)-linear continuous functional on \(\mathbb{R}^n\) such that \(\hat{\varphi}_k((x_i)_{i=1}^n) = x_k\). Then \(\{ \hat{\varphi}_k \}_{k=1}^n\) is a basis of \(\mathbb{R}\)-linear functionals on \(\mathbb{R}^n\). Therefore \(\hat{\varphi} = \sum_{k=1}^n r_k \hat{\varphi}_k\) for some \(r_k\) in \(\mathbb{R}^n\). Let \(\{e_k\}_{k=1}^n\) be a canonical basis of \(\mathbb{R}^n\). If we consider the decreasing sequence of \(M\) which converges to \(e_k\), we can see that \(r_k \geq 0\). Then \(\varphi = \sum_{k=1}^n r_k \hat{\varphi}_k\) for some \(r_k \geq 0\). At last, we will show that \(\{ \varphi_k \}_{k=1}^n = \partial_e S(M, M^+, (1, 1, \cdots, 1))\). Let \(\varphi_1\) and \(\varphi \in S(M, M^+, (1, 1, \cdots, 1))\). We suppose that \(0 \leq \varphi \leq \varphi_1\). Then \(0 \leq \hat{\varphi} \leq \hat{\varphi}_1\). Since \(\hat{\varphi}_1\) is in \(\partial_e S(M, M^+, (1, 1, \cdots, 1))\), then \(\hat{\varphi} = r\hat{\varphi}_1\) for some
0 \leq r \leq 1$. Therefore $\varphi = r \varphi_i$. Hence $\varphi_i \in \partial_* S(M, M^+, (1, 1, \ldots, 1))$. Converse is followed by the result that $\{ \varphi_i \}_{i=1}^n$ generates $S(M, M^+, (1, 1, \ldots, 1))$. Hence $\{ \varphi_k \}_{k=1}^n = \partial_* S(M, M^+, (1, 1, \ldots, 1))$. □

By 2.2 and 2.3 we can see this proposition.

**Proposition 2.4.** Let $M$ be an additive subgroup of $\mathbb{R}^n$ including $(1, 1, \ldots, 1)$. Put $M^+ = \{ (g_i)_{i=1}^n : g_i > 0 \} \cup \{ (0, 0, \ldots, 0) \}$. We suppose that for any $\varepsilon > 0$, there exist $\{ a_i^n \}_{i=1}^n \subset M$ satisfying following properties;

(i) $\{ a_i^n \}_{i=1}^n$ are $\mathbb{R}$-linearly independent.

(ii) $\| a_i^n \| < \varepsilon$, where $\| \cdot \|$ is an Euclidean norm of $\mathbb{R}^n$.

Then $(M, M^+, (1, 1, \ldots, 1))$ is a simple dimension group and

$\partial_* S(M, M^+, (1, 1, \ldots, 1)) = \{ \varphi_k \}_{k=1}^n$.

**Proof.** Simplicity is followed by $M^+$. □

We will think about the case $n = 2$. Let $A$ be an invertible matrix of $M_2(\mathbb{R})$. We call an additive subgroup $G$ of $\mathbb{R}^2$ including $(1, 1)$ $A$-invariant if $AG = G$.

Let $G$ be an additive subgroup of $\mathbb{R}^2$ including $(1, 1)$.

Put $I(G) = \{ A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in GL_2(\mathbb{R}) : AG = G, a > 0, b > 0 \}$.

More general, next proposition follows.

**Proposition 2.5.** Let $a$ be a positive algebraic number and let $b$ be positive rational numbers. We suppose that $a \neq b$, $a \neq 1$ and that $b \neq 1$. Put $\frac{p}{q}$ where $\gcd(p, q) = 1$ and where $p > 0, q > 0$. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Let $G$ be an $A$-invariant additive subgroup of $\mathbb{R}^2$ including $(1, 1)$. Then $I(G) \supseteq \{ A^n : n \in \mathbb{Z} \}$.

**Proof.** Since $b \neq 1$, $p \neq 1$ or $q \neq 1$. We suppose $q \neq 1$. Let $\psi(x) = \sum_{i=0}^n a_i x^i$ be a $\mathbb{Z}$-coefficient minimal polynomial of $a$. We will show that $\begin{bmatrix} 1 & 0 \\ 0 & q^n \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & q^{-n} \end{bmatrix}$ are in $I(G)$ for some $n$. Let $\alpha$ be an integer satisfying that there exist an integer $r$ relatively prime to $q$ and a natural number $m$ such that $\alpha \sum_{i=1}^n a_i p^i q^{n-i} = r q^m$. Then there exist a natural number $N > -m + n$ and an integer $s$ such that $sr + qN = 1$. Otherwise $p^{N+m-n}$ and $q^{N+m-n}$ are relatively prime. Then $xp^{N+m-n} + y q^{N+m-n} = s$. In the same way, $p^{m-n}$ and $q^{m-n}$ are relatively prime, so $x'p^{m-n} + y' q^{m-n} = -s$. Let $(g, h)$ be an element of $G$. Then $(g, h)(\alpha(x A^{N+m-n} + y)\psi(A) + I) = \left( g, \frac{1}{q^N h} \right)$ and $(g, h)(\alpha(x' A^{m-n} + y'I)\psi(A) + I) = \left( g, \frac{1}{q^N h} \right)$. Therefore $\begin{bmatrix} 1 & 0 \\ 0 & q^n \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & q^{-n} \end{bmatrix}$ are in $I(G)$. Hence $I(G) \supseteq \{ A^n : n \in \mathbb{Z} \}$. □

**Corollary 2.6.** Let $a$ be a positive algebraic number and let $b$ be positive rational numbers. We suppose that $a \neq b$, $a \neq 1$ and that $b \neq 1$. Put $\frac{p}{q}$ where
\( \gcd(p, q) = 1 \) and where \( p > 0, q > 0 \). Put \( A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \). There is no \( AF \)-algebras \( \mathcal{A} \) with 2-dimensional trace space such that \( \mathcal{A} \) is embedding on \( \mathbb{R}^2 \) and that \( F(A) = \{ A^n : n \in \mathbb{Z} \} \).

**Proof.** Contrarily, we suppose that \( F(A) = \{ A^n : n \in \mathbb{Z} \} \) for some \( AF \)-algebra \( \mathcal{A} \) with 2-dimensional trace space such that it satisfies (1). Then \( K_0(A) \) is \( A \)-invariant. By 2.5, \( F(A) \supset \{ A^n : n \in \mathbb{Z} \} \). This leads to contradiction. \( \Box \)

More generally, next proposition follows. Let \( R \) be an abelian ring. We denote by \( \mathbb{R}[s, t] \) the \( \mathbb{R} \)-coefficient polynomial ring generated by \( s, t \).

**Proposition 2.7.** Let \( a, b \) be non-zero positive algebraic numbers. We suppose that \( a \neq b \). Put \( A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \). If there exist \( p, q \in \mathbb{Z} \) such that \( p \neq 0, q \neq 0, 1 \), and that \( p \) and \( q \) are relatively prime and that \( \frac{p}{q} \in \mathbb{Z}[\beta, \beta^{-1}] \), then there is no \( AF \)-algebras \( \mathcal{A} \) with 2-dimensional trace space such that \( \mathcal{A} \) is embedding on \( \mathbb{R}^2 \) and that \( F(A) = \{ A^n : n \in \mathbb{Z} \} \).

**Proof.** We denote by \( \psi_1(t), \psi_2(t) \) the \( \mathbb{Z} \)-coefficient minimal polynomials of \( a, b \) respectively. Since \( a \neq b, \psi_1(t) \) and \( \psi_2(t) \) are relatively prime in the \( \mathbb{Q} \)-coefficient polynomial ring \( \mathbb{Q}[t] \). Then there exist \( \mathbb{Z} \)-coefficient polynomials \( a(t), b(t) \) and \( m \in \mathbb{Z} \setminus \{ 0 \} \) such that \( a(t)\psi_1(t) + b(t)\psi_2(t) = m \). Therefore \( a(b)\psi_1(b) = m \). We can get \( \alpha, r \in \mathbb{Z} \) satisfying that \( \alpha m = rq^n \) for some \( N \in \mathbb{N} \) and that \( r \) is relatively prime to \( q \). Then there exist \( s \in \mathbb{Z} \) and \( n \in \mathbb{N} \) such that \( sr+1 = q^n \). By hypothesis, we can get a \( \phi(t) \in P[t, t^{-1}] \) satisfying that \( \phi(b) = \frac{p}{q} \). Since \( \gcd(p, q) = 1, \gcd(p^n, q^n) = 1 \) and \( \gcd(p^{n+N}, q^{n+N}) = 1 \). Then there exist \( x, x', y, y' \in \mathbb{Z} \) such that \( xp^n + yq^n = s \) and that \( x'p^{n+N} + y'q^{n+N} = -s \). Let \( G \) be an \( A \)-invariant sub-additive group in \( \mathbb{R}^2 \). Let \( (g, h) \) be an element of \( G \). Then \( (g, h)(q^n\alpha(x'\phi(A)^n + y/I)\psi_1(A) + I) = (g, q^n h) \) and \( (g, h)(\alpha(x'\phi(A)^n + y/I)\psi_1(A) + I) = (g, q^n h) \). Therefore \( \begin{bmatrix} 1 & 0 \\ 0 & q^n \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 \\ 0 & q^n \end{bmatrix} \) are in \( I(G) \). Hence \( I(G) \supset \{ A^n : n \in \mathbb{Z} \} \). By 2.1 there is no \( AF \)-algebras \( \mathcal{A} \) with 2-dimensional trace space such that \( \mathcal{A} \) is embedding on \( \mathbb{R}^2 \) and that \( F(A) = \{ A^n : n \in \mathbb{Z} \} \). \( \Box \)

**Example 2.8.** Let \( c_1, c_2 \in \mathbb{Z} \). We suppose that \( c_1 > 1 \) and that \( \gcd(c_1, c_2) = 1 \). If \( b > 0 \) is a solution of the quadratic equation \( c_1 x^2 + c_2 x + c_1 = 0 \). Then \( b + \frac{1}{c_1} = -\frac{c_2}{c_1} \).

Let \( a \) be a non-zero positive algebraic number. Put \( A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \). By 2.7 there is no \( AF \)-algebras \( \mathcal{A} \) with 2-dimensional trace space such that \( \mathcal{A} \) is embedding on \( \mathbb{R}^2 \) and that \( F(A) = \{ A^n : n \in \mathbb{Z} \} \).

**Lemma 2.9.** Let \( \psi \) be a \( \mathbb{Z} \)-coefficient monic polynomial and let \( m \) be in \( \mathbb{Z}_{>1} \). Then there exist \( \varphi_1, \varphi_2 \in \mathbb{Z}[t, t^{-1}] \) and \( n \neq 0 \) in \( \mathbb{Z} \) such that \( m\varphi_1(t) + \psi(t)\varphi_2(t) + t^n = 1 \).

**Proof.** Since \( \psi \) is monic, \( \mathbb{Z}[t]/(m, \psi) \) is finite set. By considering \( \{ t^n \}_{n=1}^{\infty} \) in \( \mathbb{Z}[t]/(m, \psi) \), there exists \( n_1, n_2 \in \mathbb{N} \) such that \( n_1 \neq n_2 \) and that \( t^{n_1} - t^{n_2} \in \mathbb{Z} \).
$\mathbb{Z}[t]/(m, \psi)$. Therefore there exist $\varphi_1', \varphi_2' \in \mathbb{Z}[t]$ such that $m\varphi_1'(t) + \psi(t)\varphi_2'(t) + t^{n_1} = t^{n_2}$. Put $\varphi_1(t) = t^{-n_2}\varphi_1'(t)$, $\varphi_2(t) = t^{-n_2}\varphi_2'(t)$, and $n = n_1 - n_2$. Hence $m\varphi_1(t) + \psi(t)\varphi_2(t) + t^{n} = 1$. □

**Proposition 2.10.** Let $a, b$ be non-zero positive algebraic numbers. We suppose that $a \neq b$, $a \neq 1$, $b \neq 1$, minimal $\mathbb{Z}$-coefficient polynomials of $a$ and $b$ are different, and that the minimal $\mathbb{Z}$-coefficient polynomial of $b$ is monic. Put $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. There is no AF-algebras $A$ with 2-dimensional trace space such that $A$ is embedding on $\mathbb{R}^n$ and that $F(A) = \{ A^n : n \in \mathbb{Z} \}$.

**Proof.** Let $\psi_1$ and $\psi_2$ be minimal $\mathbb{Z}$-coefficient polynomials of $a$ and $b$ respectively. Since $\psi_1$s are minimal and different, $\psi_1$ and $\psi_2$ are relatively prime in $\mathbb{Q}[t]$. Then there exist $\varphi_1, \varphi_2 \in \mathbb{Z}[t]$ and $m \in \mathbb{Z}_{>0}$ such that $\varphi_1(t)\psi_1(t) + \varphi_2(t)\psi_2(t) = m$. If $m = 1$, $(A^n - 1)\psi_1(A)\psi_1(A) + I = \begin{bmatrix} 1 & 0 \\ 0 & b^n \end{bmatrix}$ for any $n \in \mathbb{Z}$. We suppose that $m > 1$. By previous lemma there exist $\varphi_1, \varphi_2 \in \mathbb{Z}[t, t^{-1}]$ and $n(\neq 0) \in \mathbb{Z}$ such that $m\varphi_1(t) + \psi(t)\varphi_2(t) + t^n = 1$. Then $-\varphi_1(A)\psi_1(A) + I = \begin{bmatrix} 1 & 0 \\ 0 & b^n \end{bmatrix}$ and $\varphi_1(A) = \varphi_2(A) + I = \begin{bmatrix} 1 & 0 \\ 0 & b^{-n} \end{bmatrix}$. Let $G$ be an $A$-invariant sub-additive group in $\mathbb{R}^2$. Let $(g, h)$ be an element of $G$. Then $\begin{bmatrix} 1 & 0 \\ 0 & b^n \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & b^{-n} \end{bmatrix}$ are in $I(G)$ for some $n \in \mathbb{N}$. Hence there is no AF-algebras $A$ with 2-dimensional trace space such that $A$ is embedding on $\mathbb{R}^n$ and that $F(A) = \{ A^n : n \in \mathbb{Z} \}$. □

**Proposition 2.11.** Let $\alpha$ be a non-zero positive transcendental number and let $\beta$ be a non-zero positive number. We suppose that $\beta \neq 1, \alpha, 1/\alpha$.

Put $G = \cup_{N \in \mathbb{N}}(\sum_{i=-N}^{N}(Q(\alpha_2^i, \beta_2^i) + Z(\alpha_2^{i+1}, \beta_2^{i+1})))).

$G^+ = \{ (g, h) : g > 0, h > 0 \} \cup \{ (0, 0) \}$, and $u = (1, 1)$. Then $(G, G^+, u)$ is a simple dimension group satisfying that $\partial_n S(G, G^+, (u)) = \{ \varphi_1, \varphi_2 \}$.

**Proof.** Let $\varepsilon > 0$. Then there exist mon-zero rational numbers $q_1, q_2$ such that $||q_1q_2|| < \varepsilon$, $||q_2\alpha^2, q_2\beta^2|| < \varepsilon$. Moreover, $(q_1, q_1)$ and $(q_2\alpha^2, q_2\beta^2)$ are $\mathbb{R}$-linearly independent. Then $G$ is dense in $\mathbb{R}^2$. Obviously, $G$ is directed and unperforated, so $G$ is dimension group. Simplicity is also. □

We denote by $A$ the unital simple AF-algebra satisfying that $(K_0(A), K_0(A)^+, [1_A]) = (G, G^+, u)$.

**Proposition 2.12.** Let $A$ be as above. Then $F(A) = \{ \begin{bmatrix} \alpha^{2n} & 0 \\ 0 & \beta^{2n} \end{bmatrix} : n \in \mathbb{Z} \}$.

**Proof.** Since $\begin{bmatrix} \alpha^{2n} & 0 \\ 0 & \beta^{2n} \end{bmatrix} G = G$ and $\begin{bmatrix} \alpha^{2n} & 0 \\ 0 & \beta^{2n} \end{bmatrix} G^+ = G^+$, $\{ \begin{bmatrix} \alpha^{2n} & 0 \\ 0 & \beta^{2n} \end{bmatrix} : n \in \mathbb{Z} \} \subset F(A)$. Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in F(A)$. Then $a \in \varphi_1(G)$ and we can denote $a = \frac{1}{q_k} \sum_{i=0}^{m} q_i \alpha^i$ for some $q_i \in \mathbb{Q}$, $k \in \mathbb{N} \cup \{ 0 \}$. Since $a$ is invertible in $\varphi_1(G)$, $a = q_n \alpha^n$ for some $q \in \mathbb{Q}$, $n \in \mathbb{Z}$. Contravercially, we suppose that $n$ is odd. Let $l$ be a rational number satisfying that
If \( p \) and \( q \) are relatively prime in \( \mathbb{Z} \), then \( lq \) is even. Since the set of coefficients of \( \alpha^n \) is \( \mathbb{Z} \) because \( n \) is odd. It leads to contradiction. Therefore \( n \) is even. Since the set of coefficients of \( \alpha \) is \( \mathbb{Z} \), then that of \( \alpha^{n+1} \) is \( q\mathbb{Z} = \mathbb{Z} \). Then \( q = 1 \). Therefore \( a = \alpha^{2n} \) and \( b = \beta^{2n} \). We suppose that \( \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \in F(\mathcal{A}) \). Since \( \phi_2(1) = \phi_2(2) \), there exist \( n_1, n_2 \in \mathbb{Z} \) and polynomials \( p(x), q(x) \) with \( \mathbb{Q} \)-coefficient such that \( \phi(p)\alpha^{n_1} = 1, \phi(q)\alpha^{n_2} = \beta \), and that \( p(0) \neq 0, q(0) \neq 0 \). Therefore \( c = \alpha^{-n_1}\frac{1}{p(\alpha)} \) and \( \beta = \alpha^{n_2-n_1}\frac{q(\alpha)}{p(\alpha)} = \alpha^{n_2-n_1}\frac{q_1(\alpha)}{p_1(\alpha)} \) where \( p_1 \) and \( q_1 \) are relatively prime and where \( p_1(0) \neq 0, q_1(0) \neq 0 \). We suppose that \( \deg(p_1) > 0 \). Let \( k \in \mathbb{N} \) satisfying \( \deg(p_1^k(x)) > \deg(p(x)) \). Then \( \beta^k = \alpha^{k(n_2-n_1)}\frac{q_1^k(\alpha)}{p_1^k(\alpha)} \in \mathcal{C}\phi_1(2) \). Let \( g \) be an element of \( \mathcal{C}\phi_1(2) \). Then we can put \( \gamma = \alpha^{n_2} \frac{r_1(\alpha)}{r_2(\alpha)} \) for some \( n_3 \in \mathbb{N} \) and for some \( r_1(x), r_2(x) \in \mathbb{Q}[x] \), where \( r_1 \) and \( r_2 \) are relatively prime and where \( r_2(0) \neq 0 \). Since \( \deg(r_1(x)) \leq \deg(p) \), it leads to contradiction. Hence \( p_1(x) = p_0(x) \). We consider \( \frac{1}{\beta} \) and \( \frac{1}{\beta^k} \) for some \( k \), in the same with \( p_1(x) \), then \( q_1(x) = q_2(0) \). Therefore \( \beta = q_0\alpha^n \) for some \( q_0 \in \mathbb{Q} \), \( n \in \mathbb{Z} \). Since \( \phi_1(2) \) is generated by \( q_0\alpha^n, p(\alpha) = p(0) \). Moreover, \( c\alpha^n\mathbb{Z} \alpha = \frac{1}{p(0)}\mathbb{Z} \alpha^{n_0-n_1+1} \) and \( c\alpha^n\mathbb{Z}^{-1} = \frac{1}{p(0)}\mathbb{Z} \alpha^{n_0-n_1-1} \). Since the set of the coefficients of \( \alpha^{n_0-n_1+1} \) is \( q_0^{k_1} \mathbb{Z} \) for some \( k_1 \in \mathbb{Z} \) and that of \( \alpha^{n_0-n_1-1} \) is \( q_0^{k_2} \mathbb{Z} \) for some \( k_2 \in \mathbb{Z} \). Because \( \alpha^{n_0-n_1+1} \neq \alpha^{n_0-n_1-1}, k_1 \neq k_2 \). Then we can get the equation \( q_0^{k_1} = \frac{1}{p(0)} = q_0^{k_2} \) therefore \( p(0) = q_0 = 1 \). Then \( c = \alpha^{-n_1} \) and \( \phi_2 = \phi_1(2) \) or \( \phi_2(2) = \phi_1(2) \). Moreover, If \( |u| \geq 2 \), then \( \alpha^{n+1} \) is not in \( \phi_2(2) \). Therefore \( \beta = 1, \alpha, \frac{1}{\alpha} \). It leads to contradiction in hypothesis. Hence \( F(\mathcal{A}) = \{ \begin{bmatrix} \alpha^{2n} & 0 \\ 0 & \beta^{2n} \end{bmatrix} : n \in \mathbb{Z} \} \). \( \square \)

**Proposition 2.13.** Let \( \alpha \) be a non-zero positive transcendental number. Put \( G = \cup_{N \in \mathbb{N}}(\sum_{n=-N}^{N}(\mathbb{Q}(\alpha^{2i}, \alpha^{2i-1}) + \mathbb{Z}(\alpha^{2i+1}, \alpha^{2i-1}) + \mathbb{Q}(\sqrt{2})(\alpha^{2i+1}, 0)) \). \( G^+ = \{(g,h) : g > 0, h > 0 \} \cup \{(0,0)\} \), and \( u = (1,1) \). Then \( (G, G^+, u) \) is a simple dimension group satisfying that \( \partial_S(G, G^+, (u)) = \{ \phi_1, \phi_2 \} \).

**Proof.** Same with [2.11] \( \square \)

We denote by \( \mathcal{A} \) the unital simple AF-algebra satisfying that \( (K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_\mathcal{A}]) = (G, G^+, u) \).  

**Lemma 2.14.** Let \( \alpha \) be a transcendental number and let \( p(x), q(x), r(x) \) be in \( \mathbb{Q}[x] \). If \( p(\alpha)(\sqrt{2}) + q(\alpha)(\sqrt{2}) + r(\alpha) = 0 \), then \( p(x) = q(x) = r(x) = 0 \).

**Proof.** It is sufficient to show in the case that \( p(x), q(x), r(x) \) are \( \mathbb{Z} \)-coefficient and relatively prime in \( \mathbb{Q}[x] \). Multiply \( -p(\alpha)r(\alpha) + q^2(\alpha)((\sqrt{2}) + (q(\alpha)r(\alpha) + 2p^2(\alpha)(\sqrt{2}) + (r^2(\alpha) - 2p(\alpha)q(\alpha)) \). Then we can get \( 4p^3(\alpha) + 2q^3(\alpha) + r^3(\alpha) - 6p(\alpha)q(\alpha)r(\alpha) = 0 \). Therefore \( r(x) = 2r_1(x) \) for some \( r_1(x) \in \mathbb{Z}[x] \). Substitute this in \( 4p^3(\alpha) + 2q^3(\alpha) + r^3(\alpha) - 6p(\alpha)q(\alpha)r(\alpha) = 0 \), then we can get \( 2p^3(\alpha) + \)
Let \( a \) be as above. Then \( F(A) = \{ \left[ \begin{array}{cc} \alpha^{2n} & 0 \\ 0 & \frac{1}{\alpha^{2n}} \end{array} \right] : n \in \mathbb{Z} \} \).

**Proposition 2.15.** Let \( A \) be as above. Then \( F(A) = \{ \left[ \begin{array}{cc} \alpha^{2n} & 0 \\ 0 & \frac{1}{\alpha^{2n}} \end{array} \right] : n \in \mathbb{Z} \} \).

**Proof.** Since 
\[
\left[ \begin{array}{cc} \alpha^{2n} & 0 \\ 0 & \frac{1}{\alpha^{2n}} \end{array} \right] G = G
\]
and 
\[
\left[ \begin{array}{cc} 0 & 1 \\ 1 & \frac{1}{\alpha^{2n}} \end{array} \right] G^+ = G^+, \quad \left[ \begin{array}{cc} 0 & \alpha^{2n} \\ \frac{1}{\alpha^{2n}} & 0 \end{array} \right] : n \in \mathbb{Z} \} ⊂ F(A). \]
Let \( \left[ \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] \in F(A) \). As is the same with \( 0 \), we can put \( a = (1 + l\sqrt{2})\alpha^{-2n} \) for some \( l \in \mathbb{Z} \). Then we can put \( a = (1 + l\sqrt{2})\alpha^{-2n} \) for some \( l \in \mathbb{Z} \). Contrarily we suppose that \( l \neq 0 \). For any \( n \in \mathbb{Z} \), the coefficients of \( \alpha^n \) includes \( \sqrt{2} \) in \( a\varphi_1(G) \). It leads to contradiction to \( a\varphi_1(G) = \varphi_1(G) \). Then \( l = 0 \) and \( a = \alpha^{-2n} \). We suppose that 
\[
\left[ \begin{array}{cc} 0 & c \\ d & 0 \end{array} \right] \in F(A). \]
Then \( c\varphi_1(G) = \varphi_2(G) \) and \( \varphi_2(G) = d\varphi_2(G) \). Therefore \( c \in (\varphi_1(G) \setminus \{0\})^{-1} ∩ \varphi_2(G) \) and \( d \in \varphi_1(G) ∩ (\varphi_2(G) \setminus \{0\})^{-1} \). Since \( \varphi_1(G) \supset \varphi_2(G) \), then we will consider that \( \varphi_1(G) ∩ (\varphi_1(G) \setminus \{0\})^{-1} \). Let \( c \in \varphi_1(G) \setminus (\varphi_1(G) \setminus \{0\})^{-1} \). Then \( c \in \varphi_1(G) \), then we can put 
\[
c = \frac{1}{\alpha^N}((\sqrt{2}p(\alpha) + q(\alpha)) \) for some \( p(x), q(x) \in \mathbb{Q}[x] \) and \( N \in \mathbb{Z}_{\geq 0} \). Then 
\[
\frac{1}{c} = \frac{\alpha^N((\sqrt{2}p(\alpha) + q(\alpha))}{2p(\alpha) + q^2(\alpha)} \]
and \( \frac{1}{c} \in \varphi_1(G) \), then 
\[
\frac{1}{c} = \frac{1}{\alpha^N}((\sqrt{2}p(\alpha) + q(\alpha)) \) for some \( p_1(x), q_1(x) \in \mathbb{Q}[x] \) and \( N \in \mathbb{Z}_{\geq 0} \). By \( p(x) = 0 \). Then 
\[
\frac{1}{c} = \frac{q(\alpha)}{\alpha^N} \setminus \frac{1}{c} = \frac{\alpha^N}{q(\alpha)} \]
and \( q(\alpha) \neq 0 \). Since \( c = \alpha^n \) for some \( n \in \mathbb{Z} \). But \( c\varphi_1(G) \neq \varphi_2(G) \). Then 
\[
F(A) \subset \{ \left[ \begin{array}{cc} \alpha^{2n} & 0 \\ 0 & \frac{1}{\alpha^{2n}} \end{array} \right] : n \in \mathbb{Z} \} \). Hence 
\[
F(A) = \{ \left[ \begin{array}{cc} \alpha^{2n} & 0 \\ 0 & 1 \end{array} \right] : n \in \mathbb{Z} \} \). □

By replacing \( \frac{1}{\alpha} \) to \( \alpha \) in \( 2.14 \) and \( 2.15 \), we can show the next proposition.

**Proposition 2.16.** Let \( \alpha \) be an transcendal number. Then there exists an \( AF \)-algebra such that \( F(A) = \{ \left[ \begin{array}{cc} \alpha^{2n} & 0 \\ 0 & \alpha^{-2n} \end{array} \right] : n \in \mathbb{Z} \} \).

**Proposition 2.17.** Let \( \alpha \) be a non-zero positive transcendal number. Then there exists a unital \( AF \)-algebra such that 
\[
F(A) = \{ \left[ \begin{array}{cc} \alpha^n & 0 \\ 0 & 1 \end{array} \right] : n \in \mathbb{Z} \} \).
Proof. Let β be a non-zero positive transcendental number which is not including in the algebraic closure of $\mathbb{Q}[\alpha]$. Put $G = \{ \sum_{i=-N}^N a_i \alpha^i, l_1 + l_2 \beta \} : a_i, l_1, l_2 \in \mathbb{Z} \}$. Then $I(G) = \{ \begin{bmatrix} \alpha^n & 0 \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \}$. □

Theorem 2.18. Let α be a non-zero positive transcendental number and let $\beta > 0$. Then there exists a unital AF-algebra such that $F(A) = \{ \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} : n \in \mathbb{Z} \}$. Proof. If $\beta \neq 1$, then replace α to $\alpha^*$ in 2.12, 2.15, and 2.16. If $\beta = 1$, by 2.17. □

References

[1] A. Connes, A factor of type $\text{II}_1$ with countable fundamental group, J. Operator Theory 4, 1 (1980), 151–153.
[2] T. Kawahara, Fundamental group of $C^*$-algebras with finite dimensional trace space, J. Operator Theory 77 (2017), no. 1, 149-170.
[3] Fundamental group of finite von Neumann algebra with finite dimensional normal trace space, arxiv1608.06611 (preprint).
[4] K. Kodaka, Full projections, equivalence bimodules and automorphisms of stable algebras of unital $C^*$-algebras, J. Operator Theory, 37 (1997), 357-369.
[5] K. Kodaka, Picard groups of irrational rotation $C^*$-algebras, J. London Math. Soc. (2) 56 (1997), 179-188.
[6] K. Kodaka, Projections inducing automorphisms of stable UHF-algebras, Glasg. Math. J. 41 (1999), no. 3, 345–354.
[7] F. Murray and J. von Neumann, On rings of operators IV, Ann. Math. 44 (1943), 716–808.
[8] N. Nawata, Fundamental group of simple $C^*$-algebras with unique trace $\text{III}$, Canad. J. Math. 64 (2012), 573–587.
[9] N. Nawata, Picard groups of certain stably projectionless $C^*$-algebras, J. London Math. Soc. (2) 88 (2013), 161–180.
[10] N. Nawata, Y. Watatani, Fundamental group of simple $C^*$-algebras with unique trace, Adv. Math. 225, (2010), 307–318.
[11] N. Nawata, Y. Watatani, Fundamental group of simple $C^*$-algebras with unique trace $\text{II}$, Journal of Functional Analysis, 260, (2011), 428–435.
[12] S. Popa, string rigidity of $\text{II}_1$ factors arising from malleable actions of $w$-rigid groups, I, Invent. Math., 165, (2006), 369–408.
[13] S. Popa, S. Vaes, Actions of $\mathbb{F}_\infty$ whose $\text{II}_1$ factors and orbit equivalence relations have prescribed fundamental group, J. Amer. Math. Soc. 23 (2010), 383-403.
[14] F. Radulescu, The fundamental group of the von Neumann algebra of a free group with infinitely many generators is $\mathbb{R}_+ \setminus 0$, J. Amer. Math. Soc. 5, 3, (1992), 517–532.
[15] D. Voiculescu, Circular and semicircular systems and free product factors, in: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, in: Progr. Math., 92, Birkhauser, Boston, (1990), 45–60.

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