ABELIAN 2-FORM GAUGE THEORY: SUPERFIELD FORMALISM

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Abstract: We derive the off-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations for all the fields of a free Abelian 2-form gauge theory by exploiting the geometrical superfield approach to BRST formalism. The above four (3 + 1)-dimensional (4D) theory is considered on a (4, 2)-dimensional supermanifold parameterized by the four even spacetime variables $x^\mu$ (with $\mu = 0, 1, 2, 3$) and a pair of odd Grassmannian variables $\theta$ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta \bar{\theta} + \bar{\theta} \theta = 0$). One of the salient features of our present investigation is that the above nilpotent (anti-)BRST symmetry transformations turn out to be absolutely anticommuting due to the presence of a Curci-Ferrari (CF) type of restriction. The latter condition emerges due to the application of our present superfield formalism. The actual CF condition, as is well-known, is the hallmark of a 4D non-Abelian 1-form gauge theory. We demonstrate that our present 4D Abelian 2-form gauge theory imbibes some of the key signatures of the 4D non-Abelian 1-form gauge theory. We briefly comment on the generalization of our superfield approach to the case of Abelian 3-form gauge theory in four (3 + 1)-dimensions of spacetime.

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1 Introduction

One of the most attractive and geometrically intuitive theoretical approaches, that provides a glimpse of the “physical” understanding of the mathematical properties associated with the nilpotent (anti-)BRST symmetries and their corresponding generators (i.e. conserved and nilpotent charges), is the superfield approach to BRST formalism (see, e.g., [1-8]).

In particular, the superfield approaches, proposed in [3-6], are such that the geometrical interpretations for (i) the nilpotent \( s_{(a)b}^2 = 0 \) (anti-)BRST symmetry transformations \( s_{(a)b} \) (and their corresponding nilpotent \( Q_{(a)b}^2 = 0 \) and conserved generators \( Q_{(a)b} \)), (ii) the nilpotency property \( s_{(a)b}^2 = 0, Q_{(a)b}^2 = 0 \) itself, and (iii) the anticommutativity property \( s_b s_{ab} + s_{ab} s_b = 0, Q_b Q_{ab} + Q_{ab} Q_b = 0 \), etc., become very transparent\(^1\). These results are the indispensable consequences of the superfield formulation developed in [3-6].

The above superfield approaches (especially the ones in [3-6]) have been exploited in the context of the gravitational theory and the (non-)Abelian 1-form \( A^{(1)} = dx^\mu A_\mu \) gauge theories where the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the above theories have been derived very accurately. The geometrical origin and interpretations for the nilpotent transformations (and their corresponding generators) have also been provided within the framework of the above superfield formulations.

The key role, in the application of the above approaches [1-8] to 1-form gauge theories, is played by the so-called horizontality condition where the super curvature 2-form (i.e. \( \tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + g\tilde{A}^{(1)} \wedge \tilde{A}^{(1)} \)) is equated to the ordinary curvature 2-form (i.e. \( F^{(2)} = dA^{(1)} + gA^{(1)} \wedge A^{(1)} \)). In the above, the symbol \( \tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}} \) (with \( d^2 = 0 \)) is the super exterior derivative and \( \tilde{A}^{(1)} \) stands for the super 1-form connection defined on the (4, 2)-dimensional supermanifold that is parametrized by the four spacetime variables \( x^\mu \) (with \( \mu = 0, 1, 2, 3 \)) and a pair of Grassmannian variables \( \theta \) and \( \bar{\theta} \) (with \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \)).

On the ordinary four \((3+1)\)-dimensional (4D) spacetime manifold (parametrized by the ordinary spacetime variable \( x^\mu \) alone), the ordinary exterior derivative \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) and the 1-form connection \( A^{(1)} = dx^\mu A_\mu \) define the ordinary 2-form \( F^{(2)} \).

In the expressions for the above (super) 2-forms, \( g \) is the coupling constant whose limiting case (i.e. \( g \to 0 \)) produces the horizontality condition for the 4D Abelian 1-form gauge theory. This horizontality condition has been referred to as the soul-flatness condition in [13] which amounts to setting equal to zero all the Grassmannian components of the (anti)symmetric curvature tensor that constitutes the super 2-form \( \tilde{F}^{(2)} \).

Recently, in a set of papers [14-24], the above superfield approaches [3-6] have been consistently extended so as to derive the nilpotent (anti-)BRST symmetry transformations that exist for the matter fields together with such a set of nilpotent symmetry transformations for the gauge and (anti-)ghost fields. The latter set of transformations, as pointed out earlier, are derived due to the application of the horizontality condition alone. We have christened the extended version [14-24] of the above superfield approaches [3-6] as the

\(^1\)We have chosen here the standard notations used in [9-24].
augmented superfield formalism. In this approach, in addition to the horizontality condition (that is applied on the gauge superfield), a few restrictions have been imposed on the matter as well as the gauge superfields of the supersymmetric gauge theory [14-24].

In our very recent works [21-24], we have been able to generalize the horizontality condition itself where a single restriction, on the superfields of the suitably chosen supermanifold, produces all the nilpotent (anti-)BRST symmetry transformations for all the fields of a given interacting (non-)Abelian 1-form gauge theory without spoiling the cute geometrical interpretations that emerge from the horizontality condition alone.

It would be very nice endeavouer to study the impact of the geometrical superfield approach [3-6,14-24] in the context of the (non-)Abelian 2-form gauge theories that have become very popular and pertinent in the realm of modern developments in the (super)string theories, related extended objects and supergravity theories (see, e.g. [25-27]). As a first modest step, we apply, in our present endeavour, the geometrical superfield formulation to the 4D free Abelian 2-form gauge theory and derive the off-shell nilpotent (anti-)BRST symmetry transformations for all the fields of the theory. In addition, we provide their geometrical interpretations in the language of the translational generators along the Grassmannian directions of the appropriately chosen supermanifolds.

There appear some novel features in the realm of the application of the above superfield approach to the Abelian 2-form gauge theory which do not crop up in the application of the very same approach to the 4D Abelian 1-form gauge theory. For instance, we obtain a CF type of restriction on the 4D bosonic local fields of the theory which enables us to obtain an absolutely anticommuting set of (anti-)BRST symmetry transformations. It is to be noted that this type of restriction happens to be a key signature of the non-Abelian 1-form gauge theory where the bosonic and fermionic (ghost) fields participate in the explicit form of the CF condition that ensures anticommutativity of the (anti-)BRST transformations [28].

The 4D free Abelian 2-form gauge theory, with its antisymmetric (i.e. $B_{\mu\nu} = -B_{\nu\mu}$) gauge potential $B_{\mu\nu}$, is interesting in its own right as it provides a dual description of the massless scalar fields [29,30]; appears in the supergravity multiplets [27] and excited states of the quantized (super)strings [25,26]; plays a crucial role in the existence of the noncommutative structure for string theory [31]; provides mass to the 4D Abelian 1-form ($A^{(1)} = dx^\mu A_\mu$) gauge field $A_\mu$ through a topological coupling (i.e. the celebrated $B \wedge F$ term) where the $U(1)$ gauge invariance and mass co-exist together without taking any recourse to the presence of the Higgs fields, etc.

Furthermore, in our earlier works [32-34], we have been able to show that the 4D Abelian 2-form gauge theory provides

(i) an interesting field theoretical model for the Hodge theory [32,33] because all the de Rham cohomological operators find their analogue(s) in the language of the conserved charges and the continuous symmetry transformations they generate,

(ii) a tractable model where the connection between the gauge symmetry and the translation subgroup of the Wigner’s little group turns out to be quite transparent [34], and
(iii) a gauge field theoretic model for the quasi-topological field theory [34].

Thus, it is important to know about this gauge potential and the corresponding gauge theory from various points of view. Our present endeavour is an attempt in that direction.

The purpose of the present paper is to study the geometrical structure behind the nilpotent (anti-)BRST symmetry transformations (and their corresponding generators) that are associated with the 4D free Abelian 2-form gauge theory in the framework of the superfield approach to BRST formalism. We exploit the power of the gauge (i.e. (anti-)BRST) invariant horizontality condition to derive the off-shell nilpotent (anti-)BRST symmetry transformations for all the basic fields of the appropriate Lagrangian densities (cf. (2.4), (2.5) below). The nilpotent transformations for the auxiliary fields are determined by the requirement of the absolute anticommutativity \(s_b s_{ab} + s_{ab} s_b = 0\) of the (anti-)BRST symmetry transformations \(s_{(a)b}\) (when they act on any field of the theory).

One of the key results of our present investigation is the derivation of the CF type restriction (cf. (3.12) below) within the framework of the superfield approach to BRST formalism. In fact, it is because of our present investigation that we were able to derive an absolutely anticommuting set of (anti-)BRST symmetry transformations in the case of 4D free Abelian 2-form gauge theory [35]. In this work, we were also able to demonstrate that the analogue of the CF restriction (cf. (2.8)) would always be required for the derivation of the above kind of anticommuting symmetry transformations in the context of higher p-form \((p \geq 2)\) Abelian gauge theories. It was also claimed that there was a deep connection between the restriction (2.8) and the concept of gerbes [35].

Our present investigation is interesting and essential on the following grounds.

First and foremost, to the best of our knowledge, the geometrical superfield approach to BRST formalism (especially proposed in [3-6,14-24]) has never been applied to the 2-form (and/or higher form) gauge theories.

Second, one of the key features of our present superfield approach is the derivation of the nilpotent (anti-)BRST symmetry transformations that always turn out to be absolutely anticommuting\(^\dagger\). As a result, it is important for us to apply the superfield formulation to the 4D Abelian 2-form gauge theory where the known (anti-)BRST symmetry transformations were not absolutely anticommuting in nature [10,32,33] (see, e.g. subsection 2.1 below).

Third, it is for the first time, that we are coming across a CF type of restriction in the context of an Abelian gauge theory for the proof of the anticommutativity of the (anti-)BRST transformations. The derivation of this restriction is a completely new result.

Finally, our present endeavour is our first modest step towards our main goal of applying the superfield approach to the 4D non-Abelian 2-form gauge theory, higher p-form \((p \geq 3)\) gauge theories as well as the gravitational theories.

The contents of our present paper are organized as follows.

In section 2, we discuss the bare essentials of the off-shell nilpotent and (i) anticommutativity property encodes the linear independence of the nilpotent (anti-)BRST symmetry transformations which emerge from a given “classical” local gauge symmetry transformation.

\(^\dagger\)
muting up to a U(1) vector gauge transformation, as well as (ii) absolutely anticommuting (anti-)BRST symmetry transformations for the 4D free Abelian 2-form gauge theory in the framework of Lagrangian formulation to set up the notations and conventions.

The latter (anti-)BRST symmetry transformations and the CF type restriction (cf. (2.8) below) are derived in section 3 by exploiting a gauge-invariant restriction on the super 2-form gauge connection that are defined on the (4, 2)-dimensional supermanifold.

Section 4 deals with the (anti-)BRST invariance of the appropriate Lagrangian densities of the present theory in the language of the superfield formalism.

Finally, in section 5, we summarize our key results, make some concluding remarks and point out a few future directions for further investigations.

Our Appendix A deals concisely with the generalization of our superfield approach to the Abelian 3-form gauge theory in four (3 + 1)-dimension of spacetime.

2 Preliminary: off-shell nilpotent (anti-)BRST symmetry transformations in Lagrangian formulation

Here we discuss briefly the off-shell nilpotent (anti-)BRST symmetry transformations for the 4D free Abelian 2-form gauge theory where (i) the transformations are anticommuting up to a U(1) vector gauge transformation, and (ii) the above transformations are absolutely anticommuting due to a specific restriction on the fields of the theory.

2.1 Non-anticommuting but off-shell nilpotent (anti-)BRST symmetry transformations

We begin with the following off-shell nilpotent (anti-)BRST invariant Lagrangian density of the 4D\(^8\) free Abelian 2-form gauge theory (see, e.g., [10,32,33]):

\[
\mathcal{L}_B = \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + B^\mu \left( \partial^\nu B_{\nu\mu} - \partial_\mu \phi \right) - \frac{1}{2} B^\mu B_\mu - \partial_\mu \beta \partial^\mu \beta \\
+ \left( \partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu \right) (\partial^\mu C^\nu) + \rho \left( \partial \cdot C + \lambda \right) + \left( \partial \cdot \bar{C} + \rho \right) \lambda, \tag{2.1}
\]

where the totally antisymmetric field strength tensor \( H_{\mu\nu\kappa} = \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu} \) is derived from the 3-form \( H(3) = dB(2) \equiv (1/3!)(dx^\mu \wedge dx^\nu \wedge dx^\kappa) H_{\mu\nu\kappa} \) that is constructed with the help of the nilpotent (\( d^2 = 0 \)) exterior derivative \( d = dx^\mu \partial_\mu \) and the 2-form connection \( B(2) = (1/2!)(dx^\mu \wedge dx^\nu) B_{\mu\nu} \). The latter defines the antisymmetric potential (i.e. the gauge field) \( B_{\mu\nu} \) of the present 4D free Abelian 2-form gauge theory. The bosonic auxiliary field \( B_\mu = -(\partial^\nu B_{\nu\mu} - \partial_\mu \phi) \) is the Nakanishi-Lautrup auxiliary field.

\(^8\)We adopt here the conventions such that the 4D flat Minkowskian metric is a diagonal metric with the signatures \((+1, -1, -1, -1)\). This implies that \((P \cdot Q) = P^\mu Q_\mu \equiv \eta_{\mu\nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i \) corresponds to the dot product between two non-null four-vectors \( P_\mu \) and \( Q_\mu \) where the Greek indices \( \mu, \nu, \kappa,... = 0, 1, 2, 3 \) stand for the spacetime directions on the 4D Minkowskian spacetime manifold and Latin indices \( i, j, k,... = 1, 2, 3 \) denote only the space directions on the above spacetime manifold.
It will be noted that there are a pair of fermionic (i.e. $\rho^2 = 0, \lambda^2 = 0, \rho \lambda + \lambda \rho = 0$) Lorentz scalar auxiliary ghost fields $\rho = -\frac{1}{2}(\partial \cdot \bar{C})$ and $\lambda = -\frac{1}{2}(\partial \cdot C)$ in the theory. Furthermore, there exists a set of fermionic ($C^2_\mu = 0, \bar{C}^2_\mu = 0, C_\mu \bar{C}_\nu + \bar{C}_\mu C_\mu = 0$, etc.) Lorentz vector (anti-)ghost fields ($\bar{C}_\mu C_\mu$ (with the ghost number $+1$) and a pair of bosonic (i.e. $\beta^2 \neq 0, \bar{\beta}^2 \neq 0, \beta \bar{\beta} = \bar{\beta} \beta$) Lorentz scalar (anti-)ghost fields ($\bar{\beta})\beta$ (with the ghost number $+2$) in the theory. These (anti-)ghost fields are required for the gauge-fixing term (i.e. $(1/2)(\partial^\nu B^\mu - \partial_\mu \phi)^2$) that is present in the Lagrangian density (2.1) of the present theory. The field $\phi$, that appears in the gauge-fixing term, is a massless $\Box \phi = 0$ scalar field where $\Box = \partial_\mu - \partial^\mu$. This field is required due to the stage-one reducibility in the theory.

The above Lagrangian density for the free Abelian 2-form gauge theory is endowed with the following off-shell nilpotent ($\tilde{s}^2_{(a)b} = 0$) but not absolutely anticommuting ($\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b \neq 0$) (anti-)BRST symmetry transformations $\tilde{s}_{(a)b}$, namely:

$$
\tilde{s}_b B_{\mu\nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad \tilde{s}_b C_\mu = -\partial_\mu \beta, \quad \tilde{s}_b \bar{C}_\mu = -B_\mu, \\
\tilde{s}_b \phi = \lambda, \quad \tilde{s}_b \beta = -\rho, \quad \tilde{s}_b \bar{\beta} = \rho, \quad \tilde{s}_b \bar{\beta} = -\lambda, \quad \tilde{s}_b \bar{\beta} = -\lambda, \quad \tilde{s}_b \bar{\beta} = \rho.
$$

(2.2)

$$
\tilde{s}_{ab} B_{\mu\nu} = -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad \tilde{s}_{ab} \bar{C}_\mu = +\partial_\mu \bar{\beta}, \quad \tilde{s}_{ab} C_\mu = +B_\mu, \\
\tilde{s}_{ab} \phi = \rho, \quad \tilde{s}_{ab} \beta = -\lambda, \quad \tilde{s}_{ab} \bar{\beta} = \lambda, \quad \tilde{s}_{ab} \bar{\beta} = -\lambda.
$$

(2.3)

It will be noted that:

(i) The above nilpotent (anti-)BRST symmetry transformations differ from the ones, given in our earlier works [32,33], by a sign factor. The above choice has been taken only for the algebraic convenience.

(ii) Under the above nilpotent symmetry transformations, the Lagrangian density transforms as: $\tilde{s}_b \mathcal{L}_B = -\partial_\mu [B^\mu \lambda + (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu - \rho \partial^\mu \beta]$ and $\tilde{s}_{ab} \mathcal{L}_B = -\partial_\mu [B^\mu \rho + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu - \lambda \partial^\mu \bar{\beta}]$. Thus, $\mathcal{L}_B$ is quasi-invariant under (2.2) and (2.3).

(iii) The anticommutativity property $\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b = 0$ is not precisely valid for the (anti-)ghost fields ($\bar{C}_\mu C_\mu$ because $\tilde{s}_b \tilde{s}_{ab} \bar{C}_\mu = -\partial_\mu \rho$ but $\tilde{s}_{ab} \tilde{s}_b \bar{C}_\mu = 0$. Furthermore, we have $\tilde{s}_b \tilde{s}_{ab} C_\mu = 0$ but $\tilde{s}_{ab} \tilde{s}_b C_\mu = +\partial_\mu \lambda$. Thus, the above (anti-)BRST transformations $\tilde{s}_{(a)b}$ in (2.2) and (2.3) are not absolutely anticommuting in nature.

(iv) The above anticommutativity property is valid up to an Abelian vector gauge transformation [i.e. $(\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b) \bar{C}_\mu = -\partial_\mu \rho, (\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b) \bar{C}_\mu = +\partial_\mu \lambda]$ for the fermionic vector (anti-)ghost fields ($\bar{C}_\mu C_\mu$ because they transform up to a total derivative term.

(v) The above observation is totally different from the anticommutativity property that is found for the 4D (non-)Abelian 1-form gauge theories. To be precise, the anticommutativity property is very much sacrosanct in the case of the 4D (non-)Abelian 1-form gauge theories where the (anti-)ghost fields are only fermionic in nature and they are found to be Lorentz scalars only. The vector (anti-)ghost fields do not exist in these theories.

(vi) The absolute anticommutativity, in the case of the non-Abelian 1-form gauge theory, emerges only due to the presence of the Curci-Ferrari restriction [28]. The superfield formulation, developed in [3], leads to the explicit derivation of the above restriction.
(vii) The superfield approaches [3-6, 14-24] can never be able to capture the nilpotent symmetry transformations (2.2) and (2.3) because the latter are not absolutely anticommuting in nature. The absolute anticommutativity, however, is a key consequence of [3-6,14-24].

2.2 Absolutely anticommuting and nilpotent (anti-)BRST symmetry transformations

Let us begin with the modified versions \( \mathcal{L}_B^{(b)} \), \( \mathcal{L}_B^{(ab)} \) of the Lagrangian density (2.1) for the free 4D Abelian 2-form gauge theory [35]

\[
\mathcal{L}_B^{(b)} = \frac{1}{12} H^{\mu \nu \kappa} H_{\mu \nu \kappa} + B^\mu (\partial^\nu B_{\nu \mu}) + \frac{1}{2} (B \cdot B + B \cdot \tilde{B}) + \partial_\mu \tilde{\beta} \partial^\mu \beta \\
+ \left( \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu \right) (\partial^\mu C^\nu) + \left( \partial \cdot C - \lambda \right) \rho + \left( \partial \cdot \tilde{C} + \rho \right) \lambda \\
+ L^\mu \left( B_\mu - \tilde{B}_\mu - \partial_\mu \phi \right) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi,
\]

\[
\mathcal{L}_B^{(ab)} = \frac{1}{12} H^{\mu \nu \kappa} H_{\mu \nu \kappa} + B^\mu (\partial^\nu B_{\nu \mu}) + \frac{1}{2} (B \cdot B + B \cdot \tilde{B}) + \partial_\mu \tilde{\beta} \partial^\mu \beta \\
+ \left( \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu \right) (\partial^\mu C^\nu) + \left( \partial \cdot C - \lambda \right) \rho + \left( \partial \cdot \tilde{C} + \rho \right) \lambda \\
+ L^\mu \left( B_\mu - \tilde{B}_\mu - \partial_\mu \phi \right) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi,
\]

where an additional auxiliary vector field \( \tilde{B}_\mu \) and the Lagrange multiplier field \( L_\mu \) have been introduced. The above Lagrangian densities respect the off-shell nilpotent and absolutely anticommuting \( (s_b s_{ab} + s_{ab} s_b = 0) \) (anti-)BRST symmetry transformations \( s_{(a)b} \)

\[
s_b B_{\mu \nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = -\partial_\mu \beta, \quad s_b \tilde{C}_\mu = \tilde{B}_\mu, \quad s_b L_\mu = -\partial_\mu \lambda, \\
s_b \phi = \lambda, \quad s_b \tilde{\beta} = -\rho, \quad s_b \tilde{B}_\mu = -\tilde{\partial}_\mu \lambda, \quad s_b \left[ \rho, \lambda, \beta, B_\mu, H_{\mu \nu \kappa} \right] = 0,
\]

\[
s_{ab} B_{\mu \nu} = -(\partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu), \quad s_{ab} \tilde{C}_\mu = -\partial_\mu \tilde{\beta}, \quad s_{ab} C_\mu = +\tilde{B}_\mu, \quad s_{ab} L_\mu = -\partial_\mu \rho, \\
s_{ab} \phi = \rho, \quad s_{ab} \beta = -\lambda, \quad s_{ab} B_\mu = \partial_\mu \rho, \quad s_{ab} \left[ \rho, \lambda, \beta, \tilde{B}_\mu, H_{\mu \nu \kappa} \right] = 0.
\]

To be more precise, it can be checked that the above nilpotent transformations are absolutely anticommuting (i.e. \( \{ s_b, s_{ab} \} B_{\mu \nu} = 0 \)) if and only if the following constrained surface*, defined in terms of the 4D local fields, is satisfied, namely;

\[
B_\mu(x) - B_\mu(x) + \partial_\mu \phi(x) = 0.
\]

It is elementary to check that, for the rest of the fields of the theory, there is no need of the constrained equation (2.8) for the proof of the anticommutativity \( s_b s_{ab} + s_{ab} s_b = 0 \).

The noteworthy points, at this stage, are

(i) The Lagrangian densities (2.4) and (2.5) are equivalent on the constrained surface (2.8). This latter equation is the analogue of the Curci-Ferrari condition [28] that is invoked in the 4D non-Abelian 1-form gauge theory for the proof of the anticommutativity of the off-shell nilpotent (anti-)BRST symmetry transformations.

*This relation actually owes its origin to our present work (cf. equation (3.12) below).
(ii) The constrained condition (2.8) could be derived from (2.4) and (2.5) as an equation of motion with respect to the multiplier field $L_\mu$. Furthermore, it can be checked that the restriction (2.8) is an (anti-)BRST invariant quantity (i.e. $s_{(a)b}[\tilde{B}_\mu - B_\mu + \partial_\mu \phi] = 0$).

(iii) Under the absolutely anticommuting (anti-)BRST symmetry transformations, the Lagrangian densities (2.4) and (2.5) transform as:

$$s_b \mathcal{L}^{(b)} = -\partial_\mu [(\partial^\mu C^\nu - \partial^\nu C^\mu)B_\nu + \lambda B^\mu + \rho \partial_\mu \beta],$$

$$s_{ab} \mathcal{L}^{(ab)} = -\partial_\mu [(\partial^\mu \tilde{C}^\nu - \partial^\nu \tilde{C}^\mu)\tilde{B}_\nu - \rho \tilde{B}^\mu + \lambda \partial_\mu \beta].$$

(iv) Unlike the Curci-Ferrari condition of the 4D non-Abelian 1-form gauge theory [28] where the auxiliary fields and the (anti-)ghost fields are connected, the condition (2.8) invokes a relationship where the auxiliary fields and the derivative on the scalar field are taken into account. The latter relationship is deeply linked with the concept of gerbes [35].

(v) The above off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations are generated by the conserved charges $Q_{(a)b}$. For a generic field $\Omega(x)$, this statement can be succinctly expressed in the mathematical form, as

$$s_r \Omega(x) = -i [\Omega(x), Q_r]_{(\pm)}, \quad r = b, ab,$$

where the ($\pm$) signatures, as the subscripts on the above square brackets, correspond to the (anti)commutators for the generic field $\Omega(x)$, of the Lagrangian densities (2.4) and/or (2.5), being fermionic(bosonic) in nature.

3 Off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetries: superfield approach

To derive the off-shell nilpotent (anti-)BRST symmetry transformations (cf. (2.6),(2.7)) and the Curci-Ferrari type restriction$^\parallel$ (2.8), we begin with the superfields (that are the generalization of the basic 4D local fields of the Lagrangian densities (2.4) and (2.5)) on an appropriately chosen $(4, 2)$-dimensional supermanifold, parameterized by the superspace variable $Z^M = (x^\mu, \theta, \bar{\theta})$. These superfields can be expanded along the Grassmannian directions of the above supermanifold in terms of the basic fields of the Lagrangian densities (2.4)/(2.5) and some extra secondary fields as

$$\begin{align*}
\mathcal{B}_{\mu
u}(x, \theta, \bar{\theta}) &= B_{\mu\nu}(x) + \theta R_{\mu\nu}(x) + \bar{\theta} R_{\mu\nu}(x) + i \theta \bar{\theta} S_{\mu\nu}(x), \\
\tilde{\beta}(x, \theta, \bar{\theta}) &= \beta(x) + \theta f_1(x) + \bar{\theta} f_1(x) + i \theta \bar{\theta} b_1(x), \\
\tilde{\beta}(x, \theta, \bar{\theta}) &= \tilde{\beta}(x) + \theta f_2(x) + \bar{\theta} f_2(x) + i \theta \bar{\theta} b_2(x), \\
\tilde{\Phi}(x, \theta, \bar{\theta}) &= \tilde{\phi}(x) + \theta f_3(x) + \bar{\theta} f_3(x) + i \theta \bar{\theta} b_3(x), \\
\tilde{F}_\mu(x, \theta, \bar{\theta}) &= C_\mu(x) + \theta \tilde{B}^{(1)}_\mu(x) + \bar{\theta} \tilde{B}^{(1)}_\mu(x) + i \theta \bar{\theta} f^{(1)}(x), \\
\bar{F}_\mu(x, \theta, \bar{\theta}) &= \bar{C}_\mu(x) + \theta \tilde{B}^{(2)}_\mu(x) + \bar{\theta} \tilde{B}^{(2)}_\mu(x) + i \theta \bar{\theta} f^{(2)}(x).
\end{align*}$$

$^\parallel$It will be noted that the off-shell nilpotent and non-anticommuting transformations (2.2) and (2.3) cannot be derived by exploiting our present superfield formulation [3-6,14-24]. The absolute anticommutativity and nilpotency of the (anti-)BRST symmetry transformations are the key consequences of our geometrical superfield approach to BRST formalism.
In the limit \((\theta, \bar{\theta}) \to 0\), we retrieve the basic 4D fields of the Lagrangian densities (2.4)/(2.5) and the number of the fermionic (e.g. \(R_\mu, \bar{R}_\mu, f_1, f_2, \bar{f}_1, \bar{f}_2, f_3, \bar{f}_3, C_\mu, C_{\bar{\mu}}, f^{(1)}_\mu, f^{(2)}_\mu\)) and bosonic (e.g. \(B_\mu, S_\mu, b_1, \beta, b_2, \bar{\beta}, b_3, \phi, B^{(1)}_\mu, B^{(1)}_{\bar{\mu}}, B^{(2)}_\mu, B^{(2)}_{\bar{\mu}}\)) fields on the r.h.s. of the above expansion do match. Thus, the sanctity of the supersymmetry is maintained.

We have to exploit now the mathematical potential of the horizontality condition (i.e. \(\tilde{d}B^{(2)} = dB^{(2)}\))** to obtain the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations of (2.6) and (2.7). To this end in mind, we first of all, generalize the ordinary exterior derivative \(d = dx^\mu \partial_\mu\) as well as the 2-form \(B^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu)B_{\mu\nu}\) of the ordinary 4D spacetime manifold to their counterparts on the (4, 2)-dimensional supermanifold. These are

\[
d \rightarrow \tilde{d} = dZ^M \partial_{Z^M} \equiv dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \quad \partial_\mu \rightarrow \partial_{Z^M} = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}}),
\]

\[
B^{(2)} \rightarrow \tilde{B}^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu) \tilde{B}_{\mu\nu}(x, \theta, \bar{\theta})
\]

\[
+ (dx^\mu \wedge d\theta) \tilde{F}_\mu(x, \theta, \bar{\theta}) + (dx^\mu \wedge d\bar{\theta}) \tilde{F}_{\bar{\mu}}(x, \theta, \bar{\theta})
\]

\[
+ (d\theta \wedge d\bar{\theta}) \tilde{\beta}(x, \theta, \bar{\theta}) + (d\theta \wedge d\bar{\theta}) \tilde{\beta}(x, \theta, \bar{\theta})
\]

(3.2)

Taking the help of (3.1) and (3.2), it can be readily seen that the above definitions (on the (4, 2)-dimensional supermanifold) reduce to their counterparts (i.e. \(d, B^{(2)}\)) on the ordinary 4D spacetime manifold in the limit \((\theta, \bar{\theta}) \to 0\).

The horizontality condition is a gauge invariant restriction because \(dB^{(2)} = H^{(3)} \equiv (1/3!)(dx^\mu \wedge dx^\nu \wedge dx^\kappa)H_{\mu\nu\kappa}\) is an (anti-)BRST invariant quantity in the sense that \(s_{(\alpha)}H_{\mu\nu\kappa} = 0\). To see the consequences of the horizontality condition in its full blaze of glory, we have to compute explicitly the super 3-form \(\tilde{d}B^{(2)}\) and set all the Grassmannian components equal to zero. To this end in mind, we have the following explicit expression for \(\tilde{d}B^{(2)}\):

\[
\tilde{d}B^{(2)} = \frac{1}{2!}(dx^\kappa \wedge dx^\mu \wedge dx^\nu)(\partial_\kappa \tilde{B}_{\mu\nu}) + (d\theta \wedge d\theta \wedge d\theta)(\partial_\kappa \tilde{\beta}) + (d\theta \wedge d\bar{\theta} \wedge d\bar{\theta})(\partial_\kappa \tilde{\beta})
\]

\[
+ (d\theta \wedge d\bar{\theta} \wedge d\theta)[\partial_\kappa \tilde{F}_\mu + \partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu]
\]

\[
+ (dx^\mu \wedge d\theta \wedge d\theta)[\partial_\kappa \tilde{F}_\mu + \partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu]
\]

\[
+ (dx^\mu \wedge d\theta \wedge d\bar{\theta})[\partial_\kappa \tilde{F}_\mu + \partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu]
\]

\[
+ (dx^\mu \wedge d\theta \wedge d\bar{\theta})[\partial_\kappa \tilde{F}_\mu + \partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu]
\]

(3.3)

The first term is the above expression has to be equated with the r.h.s (i.e. \(dB^{(2)}\)). This
equality, in its full bloom, is as follows
\[
\frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\kappa) \left[ \partial_\mu \tilde{B}_{\nu\kappa}(x, \theta, \bar{\theta}) + \partial_\nu \tilde{B}_{\kappa\mu}(x, \theta, \bar{\theta}) + \partial_\kappa \tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}) \right] =
\frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\kappa) \left[ \partial_\mu B_{\nu\kappa}(x) + \partial_\nu B_{\kappa\mu}(x) + \partial_\kappa B_{\mu\nu}(x) \right].
\] (3.4)

It is clear that the l.h.s. of the above equation would have some coefficients of the Grassmannian variables \( \theta \) and \( \bar{\theta} \). These ought to be zero for the sanctity of the horizontality condition \( d\tilde{B}^{(2)} = dB^{(2)} \) because the r.h.s. of (3.4) does not contain any such kind of terms. To demonstrate this, we proceed, purposely step-by-step, so that all the key points of our computation could become clear.

Let us, first of all, set the coefficients of the 3-form differentials \( (d\theta \wedge d\theta \wedge d\theta) \) and \( (d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta}) \) equal to zero. These restrictions imply the following
\[
\begin{align*}
\partial_{\bar{\theta}} \tilde{\beta}(x, \theta, \bar{\theta}) &= 0 \Rightarrow \tilde{f}_2(x) = 0, \quad b_2(x) = 0, \\
\partial_{\bar{\theta}} \tilde{\beta}(x, \theta, \bar{\theta}) &= 0 \Rightarrow f_1(x) = 0, \quad b_1(x) = 0,
\end{align*}
\] (3.5)

which entail upon the above superfields to reduce to
\[
\tilde{\beta}(x, \theta, \bar{\theta}) \rightarrow \tilde{\beta}^{(r)}(x, \theta) = \beta(x) + \theta \bar{f}_1(x), \quad \tilde{\beta}(x, \theta, \bar{\theta}) \rightarrow \tilde{\beta}^{(r)}(x, \bar{\theta}) = \beta(x) + \bar{\theta} f_2(x).
\] (3.6)

We go a step further and set the coefficients of the differentials \( (d\theta \wedge d\bar{\theta} \wedge d\bar{\theta}) \) and \( (d\bar{\theta} \wedge d\theta \wedge d\theta) \) equal to zero. This condition leads to the following relationships
\[
\begin{align*}
\partial_{\theta} \tilde{\Phi}(x, \theta, \bar{\theta}) + \partial_{\bar{\theta}} \tilde{\beta}^{(r)}(x, \theta) &= 0 \Rightarrow b_3(x) = 0, \quad f_3(x) + \bar{f}_1(x) = 0, \\
\partial_{\bar{\theta}} \tilde{\Phi}(x, \theta, \bar{\theta}) + \partial_{\theta} \tilde{\beta}^{(r)}(x, \bar{\theta}) &= 0 \Rightarrow b_3(x) = 0, \quad \tilde{f}_3(x) + f_2(x) = 0.
\end{align*}
\] (3.7)

The above equation shows that the secondary fields of the superfields \( \tilde{\Phi}(x, \theta, \bar{\theta}) \), in the expansion (3.1), are connected with the secondary fields of (3.6).

The stage is set now to make a judicious choice so that the conditions in (3.5) and (3.7) could be satisfied. The following choices for the secondary fields, in terms of the auxiliary fermionic fields, satisfy the required conditions, namely;
\[
\begin{align*}
\tilde{f}_3(x) &= \rho(x) = -f_2(x), \quad f_3(x) = \lambda(x) = -\tilde{f}_1(x).
\end{align*}
\] (3.8)

These lead to the following expansions of the appropriate superfields
\[
\begin{align*}
\tilde{\beta}^{(r)}(x, \theta) &= \beta(x) + \theta (-\lambda(x)) \equiv \beta(x) + \theta (s_{ab}\beta(x)), \\
\tilde{\beta}^{(r)}(x, \bar{\theta}) &= \bar{\beta}(x) + \bar{\theta} (-\rho(x)) \equiv \bar{\beta}(x) + \bar{\theta} (s_{ab}\bar{\beta}(x)), \\
\tilde{\Phi}^{(r)}(x, \theta, \bar{\theta}) &= \phi(x) + \theta (\rho(x)) + \bar{\theta} (\lambda(x)) \equiv \phi(x) + \theta (s_{ab}\phi(x)) + \bar{\theta} (s_{ab}\phi(x)),
\end{align*}
\] (3.9)

where \( s_{(a)\bar{b}} \) are the off-shell nilpotent (anti-)BRST symmetry transformations quoted in (2.6) and (2.7). Thus, we have been able to derive the (anti-)BRST symmetry transformations associated with the local fields \( \beta(x) \), \( \bar{\beta}(x) \) and \( \phi(x) \) of the Lagrangian densities
(2.4) and (2.5) in the framework of the superfield approach to BRST formalism [3-6,14-24]. Furthermore, the above discussion (along with equation (2.9)) provides a glimpse of the mappings: 

\[ s_b \leftrightarrow \text{Lim}_{\theta \to 0}(\partial/\partial \bar{\theta}) \leftrightarrow Q_b, \quad s_{ab} \leftrightarrow \text{Lim}_{\theta \to 0}(\partial/\partial \theta) \leftrightarrow Q_{ab}. \]

It is worth emphasizing that one would have started with the explicit presence of the auxiliary fields \( B_\mu, \rho \) and \( \lambda \) in the expansion (3.1) itself as has been the case with the earlier works on superfield approach to BRST formalism in the context of (non-)Abelian 1-form gauge theories (see, e.g. [3-6] for details). However, just for the sake of generality, we have started out with an expansion of the superfields (cf. (3.1)) which looks quite general in nature. From the above equation (3.9), it is clear that \( s_{ab}\bar{\beta} = 0, s_b\beta = 0 \) and \( s_b s_{ab}\phi = 0 \) because these are the coefficients of \( \theta, \bar{\theta} \) and \( \theta \bar{\theta} \) in the superfield expansion.

Let us focus on the conditions \( \partial_\mu \bar{\beta}^{(r)} + \partial_\mu \bar{F}_\mu = 0 \) and \( \partial_\mu \bar{\beta}^{(r)} + \partial_\mu \bar{F}_\mu = 0 \). These requirements imply the following relationships:

\[ B^{(1)}_\mu = - \partial_\mu \beta, \quad f^{(1)}_\mu = +i \partial_\mu \lambda, \quad B^{(2)}_\mu = - \partial_\mu \bar{\beta}, \quad f^{(2)}_\mu = -i \partial_\mu \rho. \]  

(3.10)

The substitution of the above values in the expansions of the superfields \( \bar{F}_\mu \) and \( \bar{F}_\mu \) (cf. (3.1)), along with the identifications \( \bar{B}_\mu^{(1)} = \bar{B}_\mu \) and \( B^{(2)}_\mu = -B_\mu \), leads to the following version of their reduced form

\[ \bar{F}^{(r)}_\mu(x, \theta, \bar{\theta}) = C_\mu(x) + \bar{\theta} (\partial_\mu \beta(x)) + \theta (\partial_\mu \lambda(x)) \]

\[ \equiv C_\mu(x) + \theta (s_{ab} C_\mu(x)) + \bar{\theta} (s_b \partial_\mu \lambda(x)), \]

\[ \bar{F}^{(r)}_\mu(x, \theta, \bar{\theta}) = \bar{C}_\mu(x) + \theta (s_{ab} \bar{C}_\mu(x)) + \bar{\theta} (s_b \bar{C}_\mu(x)). \]  

(3.11)

It will be noted that

(i) the above transformations \( s_{(a)b} \) are from (2.6) and (2.7) that are absolutely anticommuting and off-shell nilpotent of order two,

(ii) the choices \( \bar{B}_\mu^{(1)} = \bar{B}_\mu \) and \( B^{(2)}_\mu = -B_\mu \), in terms of the auxiliary fields, is allowed within the framework of the superfield formulation, and

(iii) the above expansion is consistent with the mappings \( s_b \leftrightarrow \text{Lim}_{\theta \to 0}(\partial/\partial \bar{\theta}) \leftrightarrow Q_b, \quad s_{ab} \leftrightarrow \text{Lim}_{\theta \to 0}(\partial/\partial \theta) \leftrightarrow Q_{ab} \) which states that the charges \( Q_{(a)b} \) are like the translational generators along the Grassmanian directions of the (4, 2)-dimensional supermanifold.

The next restriction \( \partial_\mu \Phi^{(r)} + \partial_\mu \bar{F}^{(r)}_\mu + \partial_\mu \bar{F}^{(r)}_\mu = 0 \) implies the following relationship

\[ \bar{B}_\mu(x) - B_\mu(x) + \partial_\mu \phi(x) = 0, \]  

(3.12)

where the expansions from (3.9) and (3.11) have been inserted into the above restriction. The above condition is the Curci-Ferrari type restriction (cf. (2.8)) that has been invoked in the proof of the anticommutativity of the (anti-)BRST symmetry transformations (2.6) and (2.7). Furthermore, it can be noted that \( \partial_\theta [\partial_\mu \Phi^{(r)} + \partial_\mu \bar{F}^{(r)}_\mu + \partial_\mu \bar{F}^{(r)}_\mu] = 0, \partial_\bar{\theta} [\partial_\mu \Phi^{(r)} + \partial_\mu \bar{F}^{(r)}_\mu + \partial_\mu \bar{F}^{(r)}_\mu] = 0 \). This observation (in view of \( s_b \leftrightarrow \partial_\theta \) and \( s_{ab} \leftrightarrow \partial_\bar{\theta} \)), ultimately, implies that the above condition (3.12) is an (anti-)BRST invariant relationship (i.e. \( s_{(a)b}[\bar{B}_\mu(x) - \)]
superfield approach to BRST formalism which provides the basis for the existence and (anti-)BRST invariance of the restriction (2.8). The geometrical origin of (2.8), in the language of gerbes, has already been discussed in our earlier work [35].

We are now well prepared to concentrate on the restrictions \( \partial_\theta \bar{B}_{\mu \nu} + \partial_\nu \bar{F}^{(r)} - \partial_\mu \bar{F}^{(r)} = 0 \) and \( \partial_\theta \bar{B}_{\mu \nu} + \partial_\mu \bar{F}^{(r)} - \partial_\nu \bar{F}^{(r)} = 0 \). The insertion of the super expansions in (3.11) and (3.1) leads to the following relationships

\[
\begin{align*}
R_{\mu \nu} &= - (\partial_\mu C_\nu - \partial_\nu C_\mu), \\
S_{\mu \nu} &= - i(\partial_\mu B_\nu - \partial_\nu B_\mu) \equiv - i(\partial_\mu B_\nu - \partial_\nu B_\mu).
\end{align*}
\]

(3.13)

It is clear that the last entry in the above equation is automatically satisfied due to the relationship given in (3.12). The next restriction is the final restriction which enables us to compare the l.h.s. and r.h.s. of the horizontality condition as given in (3.4). It is clear that the following relationships would emerge from this (cf. (3.4)) equality:

\[
\begin{align*}
\partial_\mu R_{\kappa \mu} + \partial_\nu R_{\kappa \mu} + \partial_\kappa R_{\mu \nu} &= 0, \\
\partial_\mu S_{\kappa \nu} + \partial_\nu S_{\kappa \mu} + \partial_\kappa S_{\mu \nu} &= 0.
\end{align*}
\]

(3.14)

These conditions are readily satisfied by the values obtained for the expressions of the secondary fields \( R_{\mu \nu}, \bar{R}_{\mu \nu} \) and \( S_{\mu \nu} \) in terms of the basic and auxiliary fields (cf. (3.13)). As a consequence, we note that the super curvature tensor \( \bar{H}^{(h)}(\mu \nu \kappa) = \partial_\mu \bar{B}^{(h)}_{\nu \kappa} + \partial_\kappa \bar{B}^{(h)}_{\mu \nu} + \partial_\mu \bar{B}^{(h)}_{\nu \kappa} \equiv H^{(h)}(\mu \nu \kappa) \) remains independent of the Grassmannian variables \( \theta \) and \( \bar{\theta} \) (because \( \bar{B}^{(h)}_{\mu \nu}(x, \theta, \bar{\theta}) = B_{\mu \nu}(x) - \theta(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) - \bar{\theta}(\partial_\nu \bar{C}_\nu - \partial_\kappa \bar{C}_\kappa) + \theta \bar{\theta} (\partial_\mu B_\nu - \partial_\nu B_\mu) \)) where the superscript \( (h) \) on \( \bar{H}^{(h)}(\mu \nu \kappa) \) denotes that the super curvature tensor has been obtained after the application of the horizontality condition due to which \( \bar{B}^{(h)}_{\mu \nu}(x, \theta, \bar{\theta}) \to \bar{B}^{(h)}_{\mu \nu}(x, \theta, \bar{\theta}) \).

The substitution of all the above values of the secondary fields, in terms of the auxiliary and basic fields, leads to the following expansion for (3.1), namely;

\[
\begin{align*}
\bar{B}^{(h)}_{\mu \nu}(x, \theta, \bar{\theta}) &= B_{\mu \nu}(x) + \theta (s_{ab} B_{\mu \nu}(x)) + \bar{\theta} (s_{ab} B_{\mu \nu}(x)) + \theta \bar{\theta} (s_{ab} \bar{B}_{\mu \nu}(x)), \\
\bar{\beta}^{(h)}(x, \theta, \bar{\theta}) &= \bar{\beta}(x) + \theta (s_{ab} \bar{\beta}(x)) + \bar{\theta} (s_{ab} \bar{\beta}(x)) + \theta \bar{\theta} (s_{ab} \bar{\beta}(x)), \\
\bar{\phi}^{(h)}(x, \theta, \bar{\theta}) &= \phi(x) + \theta (s_{ab} \phi(x)) + \bar{\theta} (s_{ab} \phi(x)) + \theta \bar{\theta} (s_{ab} \phi(x)), \\
\bar{F}^{(h)}_{\mu}(x, \theta, \bar{\theta}) &= C_\mu(x) + \theta (s_{ab} C_\mu(x)) + \bar{\theta} (s_{ab} C_\mu(x)) + \theta \bar{\theta} (s_{ab} \bar{C}_\mu(x)),
\end{align*}
\]

(3.15)

Here (i) the off-shell nilpotent transformations \( s_{(ab)} \) of equations (2.6) and (2.7) have been taken into account for the above expansions, and (ii) the superscript \( (h) \) on the above superfields denotes the expansion of the superfields after the application of the horizontality condition. Furthermore, it will be noted that, in the above expansion, we have taken into account \( s_{0 \beta} = 0, s_{ab} \bar{\beta} = 0, s_{ab} \phi \equiv s_{ab} \bar{\phi} = 0 \). Finally, the geometrical interpretations for the off-shell nilpotent (anti-)BRST symmetry transformations and their corresponding
charges emerge from the following relationships (cf. (2.9))

\[
\begin{align*}
\lim_{\theta \to 0} \frac{\partial}{\partial \theta} \tilde{\Omega}^{(h)}(x, \theta, \bar{\theta}) &= s_{ab} \Omega(x) \equiv -i[\Omega(x), Q_{ab}](\pm), \\
\lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\Omega}^{(h)}(x, \theta, \bar{\theta}) &= s_{b} \Omega(x) \equiv -i[\Omega(x), Q_{b}](\pm),
\end{align*}
\]

where \(\Omega(x)\) is the generic local field of the Lagrangian densities (2.4)/(2.5) and \(\tilde{\Omega}^{(h)}(x, \theta, \bar{\theta})\) is the corresponding superfield defined on the (4, 2)-dimensional supermanifold (cf. (3.15)).

The above expression implies that the off-shell nilpotent (anti-)BRST symmetry transformations \(s_{(a)b}\) and their corresponding generators \(Q_{(a)b}\) geometrically correspond to the translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold. To be more specific, the BRST symmetry transformation corresponds to the translation of the particular superfield along the \(\bar{\theta}\)-direction of the supermanifold when there is no translation of the same superfield along the \(\theta\)-direction of the supermanifold (i.e. \(\theta \to 0\)). This geometrical operation on the specific superfield generates the BRST symmetry transformation for a specific field in the language of the translational generator (i.e. \(\lim_{\theta \to 0} (\partial / \partial \theta)\)) on the above (4, 2)-dimensional supermanifold.

It will be noted that the (anti-)BRST symmetry transformations for the auxiliary fields \((B_{\mu}, \bar{B}_{\mu})\) and the Lagrange multiplier field \((L_{\mu})\) are derived from the requirement of the absolute anticommutativity \([s_{b}s_{ab} + s_{ab}s_{b}] \Omega = 0\) of the (anti-)BRST symmetry transformations \(s_{(a)b}\) for a generic field \(\Omega\). For instance, it can be seen, from (2.6) and (2.7), that \((s_{b}s_{ab} + s_{ab}s_{b}) C_{\mu} = 0\) and \((s_{b}s_{ab} + s_{ab}s_{b}) \bar{C}_{\mu} = 0\) yield the nilpotent transformations \(s_{b}B_{\mu} = -\partial_{\mu} \lambda\) and \(s_{ab}B_{\mu} = \partial_{\mu} \rho\), respectively. Similarly, the nilpotent transformations for the multiplier field \(L_{\mu}\) are found from the equations of motion \(L_{\mu} = \bar{B}_{\mu}\) and \(L_{\mu} = -B_{\mu}\) that emerge from the Lagrangian densities (2.4) and (2.5), respectively. It is elementary now to note that \(s_{b}L_{\mu} = -\partial_{\mu} \lambda\) and \(s_{ab}L_{\mu} = -\partial_{\mu} \rho\). Thus, ultimately, we obtain all the nilpotent (anti-)BRST symmetry transformations for all the fields of the theory.

4  (Anti-)BRST invariance: superfield approach

The nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations (2.6) and (2.7) leave the Lagrangian densities (2.4) and (2.5) quasi-invariant because the latter transform to the total spacetime derivatives. This observation can be captured within the framework of our present superfield approach to BRST formalism. To this end in mind, let us, first of all, note that the following relationship [35] is true, namely;

\[
\begin{align*}
s_{b} s_{ab} \left[ 2\beta \bar{\beta} + \bar{C}_{\mu} C^{\mu} - \frac{1}{4} B^{\mu \nu} B_{\mu \nu} \right] &= B^{\mu} (\partial^{\nu} B_{\nu \mu}) + B \cdot \bar{B} + \partial_{\mu} \beta \partial^{\mu} \beta \\
+ (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} \bar{C}_{\mu}) (\partial^{\mu} C^{\nu}) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda.
\end{align*}
\]
In the above, using (2.8)/(3.12), we can re-express

\[ B \cdot \bar{B} = \frac{1}{2} \left( B \cdot B + \bar{B} \cdot \bar{B} \right) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi, \]  

(4.2)
to obtain the gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian densities (2.4) and (2.5). This shows that, modulo some total spacetime derivatives terms, the gauge-fixing and Faddeev-Popov ghost terms are actually the (anti-)BRST exact terms.

The horizontality condition leads to the explicit expression of the 2-form gauge super-field, in terms of the basic and auxiliary fields, as

\[ \tilde{B}^{(h)}_{\mu \nu}(x, \theta, \bar{\theta}) = B_{\mu \nu}(x) - \theta \left( \partial_\mu \bar{C}_\nu - \partial_\nu C_\mu \right) - \bar{\theta} \left( \partial_\mu C_\nu - \partial_\nu \bar{C}_\mu \right) + \theta \bar{\theta} \left( \partial_\mu B_\nu - \partial_\nu B_\mu \right), \]  

(4.3)
where the last term can also be expressed as \( \theta \bar{\theta} \left( \partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu \right) \) (cf. (3.13)). Using the explicit expansions in (3.9), (3.11) and (4.3), it can be seen that the Lagrangian densities (2.4) and (2.5) can be expressed, in terms of the superfields, as

\[ \tilde{L}^{(b,ab)}_B = \frac{1}{12} \tilde{H}^{(h)}_{\mu \nu \kappa} + \frac{\partial}{\partial \theta} \left[ 2 \tilde{\beta}^{(r)} \right] - \frac{1}{4} \tilde{B}^{(h)}_{\mu \nu} \tilde{B}^{(h)\mu \nu}, \]  

(4.4)
where \( \tilde{H}^{(h)}_{\mu \nu \kappa} = \partial_\mu B^{(h)}_{\nu \kappa} + \partial_\nu B^{(h)}_{\mu \kappa} + \partial_\kappa B^{(h)}_{\mu \nu} \) is the curvature super tensor derived after the application of the horizontality condition. It is straightforward to note that this super tensor is independent of the Grassmannian variables. As a consequence, the kinetic energy term for the 2-form gauge field is an (anti-)BRST invariant quantity (i.e. \( s_{(ab)h} H_{\mu \nu \kappa} = 0 \)).

Taking the help of discussions after equation (3.12) and exploiting the nilpotency property of the translational generators (i.e. \( \partial_\theta^2 = 0, \partial_{\bar{\theta}}^2 = 0 \)), it is evident that the following relationship is sacrosanct, namely;

\[ \operatorname{Lim}_{\theta \to 0} \frac{\partial}{\partial \theta} \tilde{L}^{(b,ab)}_B = 0 \iff s_b \tilde{L}^{(b)}_B = 0, \]  

\[ \operatorname{Lim}_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{L}^{(b,ab)}_B = 0 \iff s_{ab} \tilde{L}^{(ab)}_B = 0. \]  

(4.5)
Thus, we conclude that the Grassmannian independence of the super Lagrangian density, expressed in terms of the (4, 2)-dimensional superfields (derived after the application of the horizontality condition), provides a clear-cut proof for the (anti-)BRST invariance of the 4D Lagrangian densities (2.4) and (2.5). In other words, if the operation of the partial derivatives w.r.t. the Grassmannian variables, on the appropriate (4, 2)-dimensional super Lagrangian density, turns out to be zero, the corresponding 4D Lagrangian density of a given gauge theory would respect the nilpotent (anti-)BRST symmetry invariance. This conclusion is in complete agreement with our recent works on 1-form gauge theories [36-39].

We wrap up this section with the assertion that the superfield approach to BRST formalism does simplify the understanding of the (anti-)BRST invariance in a given theory.
5 Conclusions

In our present endeavour, we have concentrated on the application of the geometrical superfield approach to BRST formalism to derive the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for all the fields of the Lagrangian densities (cf. (2.4),(2.5)) of a 4D free Abelian 2-form gauge theory. To the best of our knowledge, this is for the first time that the idea of the geometrical superfield approach to BRST formalism (especially proposed in [3-6, 14-24] for the 4D (non-)Abelian 1-form gauge theories) has been generalized to the case of the free 4D Abelian 2-form gauge theory. The above geometrical superfield approach, we firmly believe, can be extended so as to derive the (anti-)BRST symmetry transformations in the case of the higher p-form (p ≥ 3) gauge theories which have become important in the context of the (super)string theories.

Our present study illustrates that there is no existence of an absolutely anticommuting set of on-shell nilpotent (anti-)BRST symmetry transformations for the 4D free Abelian 2-form gauge theory. This feature of our present Abelian gauge theory is exactly same as that of the 4D non-Abelian 1-form gauge theory where the on-shell nilpotent and anticommuting (anti-)BRST symmetry transformations do not exist together [9-12]. The off-shell nilpotent and absolutely anticommuting (anti-)BRST transformations exist for both the above theories due to the CF type restrictions which emerge from a pair of coupled Lagrangian densities. Thus, our present free Abelian 2-form theory does imbibe some of the key features of the non-Abelian 1-form gauge theory.

It will be noted that, in our very recent work [40], we have been able to show the existence of the on-shell and off-shell nilpotent BRST symmetry transformations for a specific Lagrangian density for the 4D free Abelian 2-form gauge theory. In a similar fashion, for another specific Lagrangian density of the above 2-form gauge theory, the on-shell and off-shell nilpotent anti-BRST symmetry transformations have also been shown to exist. However, the on-shell nilpotent (anti-)BRST symmetry transformations do not exist together for a single Lagrangian density of the above 2-form gauge theory.

We note that there is a great deal of difference between the 4D Abelian 1-form gauge theory (that is endowed with the off-shell as well as on-shell nilpotent and anticommuting (anti-)BRST symmetries) and the 4D free Abelian 2-form gauge theory. The latter theory has deep connection with the geometrical objects called gerbes [35] (due to the restriction (2.8)/(3.12) which is not the case with the 4D Abelian 1-form gauge theory where there is no need of any CF type restriction). In fact, for the Abelian 1-form theory, the (anti-)BRST symmetry transformations are found to be automatically anticommuting.

One can encapsulate the geometrical interpretations (see, e.g., [14-24] for details) of specific quantities, connected with the BRST formalism, in the language of the following
The above (geometrically intuitive) mappings are possible only in the super field approach to BRST formalism proposed in [3-6,14-24]. This is not the case, however, with the mathematical superfield approach to BRST formalism proposed in [41,42].

It is clear from the above equation (cf. (5.1)) that the BRST and anti-BRST charges have their own identity and they play completely independent roles. In fact, they correspond to the translational generators along the independent Grassmannian directions $\theta$ and $\bar{\theta}$ of the $(4, 2)$-dimensional supermanifold on which the present Abelian 2-form gauge theory is considered. A clear-cut proof of the above assertion has been corroborated in our earlier work [40] where it has been demonstrated that the BRST and anti-BRST charges lead to completely distinct and independent constraint conditions. These conditions emerge from the physicality criteria $Q(a)_{b}|_{phys} = 0$ [40]. The latter is exploited in the context of discussions connected with the constraint structure of the 4D Abelian 2-form gauge theory.

We touch upon another very decisive feature of the present superfield approach to BRST formalism. This formulation always ensures the nilpotency and an absolute anticommutativity of the (anti-)BRST symmetry transformations as is evident from the last two entries in the above equation (5.1). In fact, it is due to the above key points that we obtain the CF type restriction (cf. (3.12)) from the superfield approach to BRST formalism which ensures the anticommutativity of the (anti-)BRST symmetry transformations $s(a)b$. The validity of $s_b s_{ab} + s_{ab} s_b = 0$ turns out explicitly from the super expansion (3.15) because it can be noted that, for all the superfields $\tilde{\Omega}(h)$ (derived after the application of the horizontality condition), the operator equation $\left[(\partial/\partial\theta)(\partial/\partial\bar{\theta}) + (\partial/\partial\bar{\theta})(\partial/\partial\theta)\right] \tilde{\Omega}(h)(x, \theta, \bar{\theta}) = 0$ is always true. Thus, one of the key results of our present investigation is the derivation of the absolutely anticommuting (anti-)BRST symmetry transformations.

It will be noted that the horizontality condition $\tilde{d}B^{(2)} = dB^{(2)}$ is a gauge (i.e. nilpotent (anti-)BRST) invariant restriction on an appropriately chosen super 2-form gauge connection $\tilde{B}^{(2)}$ that is defined on a suitably selected supermanifold. This is due to the fact that the curvature tensor $H_{\mu\nu\kappa}$, that constitutes the 3-form $H^{(3)} = dB^{(2)}$, remains invariant under the (anti-)BRST symmetry transformations (i.e. $s_{(a)b}H_{\mu\nu\kappa} = 0$). As commented earlier after equation (2.6), the key reasons behind the emergence of the nilpotent (anti-)BRST symmetry transformations together, within the framework of the superfield formulation, is encoded (i) physically in the observation that $s_{(a)b}H_{\mu\nu\kappa} = 0$, and (ii) mathematically in the nilpotency $d^2 = 0$ of the super exterior derivative ($\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}$).
It has been shown in our earlier works [32-34] that, in addition to the above nilpotent (anti-)BRST symmetry transformations, there exist nilpotent (anti-)co-BRST symmetry transformations for the Abelian 2-form gauge theory. However, these transformations have been found to be anticommuting only up to a U(1) vector gauge transformation. In our very recent works [43,44], we have derived the absolutely anticommuting set of (anti-)BRST as well as (anti-)co-BRST symmetry transformations and shown that the Abelian 2-form gauge theory is a field theoretic model for the Hodge theory. It would be very interesting to extend our present work and exploit the superfield approach to derive the above absolutely anticommuting (anti-)co-BRST symmetry transformations for the theory.

To generalize our present idea to the non-Abelian 2-form gauge theory is another promising direction. Yet another direction, that could be pursued for the application of the superfield approach to BRST formalism, is in the context of interesting field theoretical models proposed in [45,46] which also involve the 2-form gauge field. These are some of the issues that are presently being investigated and our results would be reported elsewhere [47].

Appendix A

Our present superfield formalism can be applied to any arbitrary Abelian p-form gauge theory in any arbitrary D-dimension of spacetime. To corroborate this assertion, we apply our method first to the 4D Abelian 3-form \( B^{(3)} = \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\eta)B_{\mu\nu\eta} \) gauge theory described by a totally antisymmetric potential \( B_{\mu\nu\eta} \). The generalization of the above ordinary 4D 3-form onto the \((4, 2)\)-dimensional supermanifold \((\text{with } Z^M = (x^\mu, \theta, \bar{\theta}))\) is

\[
B^{(3)} \rightarrow \tilde{B}^{(3)} = \frac{1}{3!} (dZ^M \wedge dZ^N \wedge dZ^K) \tilde{B}_{M N K} \equiv \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\eta) \tilde{B}_{\mu\nu\eta} + \frac{1}{2!} (dx^\mu \wedge dx^\nu \wedge d\bar{\theta}) \tilde{B}_{\mu\nu\bar{\theta}} + \frac{1}{3!} (d\theta \wedge d\theta \wedge d\bar{\theta}) \tilde{B}_{\theta\theta\bar{\theta}} \tag{A.1}
\]

We make the identifications: \( \tilde{B}_{\mu\nu\eta} = \tilde{B}_{\mu\nu\eta}(x, \theta, \bar{\theta}), \tilde{B}_{\mu\nu\bar{\theta}} = \tilde{F}_{\mu\nu}(x, \theta, \bar{\theta}), \tilde{B}_{\mu\nu\theta} = \tilde{F}_{\mu\nu}(x, \theta, \bar{\theta}), \tilde{B}_{\mu\theta\bar{\theta}} = \tilde{\Phi}_{\mu}(x, \theta, \bar{\theta}) \) and \( \tilde{B}_{\theta\theta\bar{\theta}} = \tilde{\Phi}_{\mu}(x, \theta, \bar{\theta}) \) as the generalization of the 4D local fields \( (B_{\mu\nu\eta}, \tilde{C}_{\mu\nu}, C_{\mu\nu}, \phi_{\mu}, \tilde{C}_{2}, \tilde{C}_{1}, C_{1}, \beta_{\mu}, \tilde{\beta}_{\mu}) \) of the Abelian 3-form gauge theory onto the corresponding superfields defined on the \((4, 2)\)-dimensional supermanifold.

Our present Abelian 3-form gauge theory has to be considered on this \((4, 2)\)-dimensional supermanifold in the framework of the superfield approach to BRST formalism so that we can derive the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for all the fields of the theory by exploiting the theoretical arsenal of horizontality condition. Towards this goal in mind, we shall quote the main results emerging from the application of the horizontality condition and shall avoid the algebraic details.
The super expansion of the above superfields along the Grassmannian directions of the supermanifold can be expressed as follows

\[
\begin{align*}
\tilde{B}_{\mu \nu \eta}(x, \theta, \bar{\theta}) &= B_{\mu \nu \eta}(x) + \theta \tilde{R}_{\mu \nu \eta}(x) + \bar{\theta} R_{\mu \nu \eta}(x) + i \theta \bar{\theta} S_{\mu \nu \eta}(x), \\
\tilde{\beta}_\mu(x, \theta, \bar{\theta}) &= \beta_\mu(x) + \theta \tilde{f}^{(1)}_\mu(x) + \bar{\theta} f^{(1)}_\mu(x) + i \theta \bar{\theta} b_\mu(x), \\
\bar{\tilde{\beta}}_\mu(x, \theta, \bar{\theta}) &= \bar{\beta}_\mu(x) + \theta \tilde{f}^{(2)}_\mu(x) + \bar{\theta} f^{(2)}_\mu(x) + i \theta \bar{\theta} b_\mu(x), \\
\Phi_\mu(x, \theta, \bar{\theta}) &= \phi_\mu(x) + \theta \tilde{f}^{(3)}_\mu(x) + \bar{\theta} f^{(3)}_\mu(x) + i \theta \bar{\theta} b^{(3)}_\mu(x), \\
\tilde{F}_{\mu \nu}(x, \theta, \bar{\theta}) &= C_{\mu \nu}(x) + \theta \tilde{B}^{(1)}_{\mu \nu}(x) + \bar{\theta} B^{(1)}_{\mu \nu}(x) + i \theta \bar{\theta} s_{\mu \nu}(x), \\
\tilde{\tilde{F}}_{\mu \nu}(x, \theta, \bar{\theta}) &= \tilde{C}_{\mu \nu}(x) + \theta \tilde{B}^{(2)}_{\mu \nu}(x) + \bar{\theta} B^{(2)}_{\mu \nu}(x) + i \theta \bar{\theta} \tilde{s}_{\mu \nu}(x), \\
\tilde{F}_1(x, \theta, \bar{\theta}) &= C_1(x) + \theta \tilde{b}^{(1)}_1(x) + \bar{\theta} b^{(1)}_1(x) + i \theta \bar{\theta} s_1(x), \\
\tilde{\tilde{F}}_1(x, \theta, \bar{\theta}) &= \tilde{C}_1(x) + \theta \tilde{b}^{(2)}_1(x) + \bar{\theta} b^{(2)}_1(x) + i \theta \bar{\theta} \tilde{s}_1(x), \\
\tilde{F}_2(x, \theta, \bar{\theta}) &= C_2(x) + \theta \tilde{b}^{(1)}_2(x) + \bar{\theta} b^{(1)}_2(x) + i \theta \bar{\theta} s_2(x), \\
\tilde{\tilde{F}}_2(x, \theta, \bar{\theta}) &= \tilde{C}_2(x) + \theta \tilde{b}^{(2)}_2(x) + \bar{\theta} b^{(2)}_2(x) + i \theta \bar{\theta} \tilde{s}_2(x),
\end{align*}
\]  

(A.2)

where \(B_{\mu \nu \eta}\) is the gauge field, \(\phi_\mu\) is a vector bosonic field, \((\tilde{C}_{\mu \nu})C_{\mu \nu}\) are the fermionic antisymmetric (anti-)ghost fields, \((\tilde{\beta}_\mu)\beta_\mu\) are the bosonic vector (anti-)ghost fields, \((\tilde{C}_2)C_2\) and \((\tilde{C}_1)C_1\) are a pair of fermionic (anti-)ghost Lorentz scalar fields. These fields are the basic (primary) fields of the 4D Abelian 3-form gauge theory within the framework of the BRST formalism. Rest of the fields, in the above expansion, are the secondary fields that have to be expressed in terms of the primary (basic) fields.

By exploiting the horizontality condition \(\tilde{d}\tilde{B}^{(3)} = dB^{(3)}\)\(^\dagger\), where \(\tilde{d}\) and \(\tilde{B}^{(3)}\) are defined in (3.2) and (A.1), one would be able to achieve the above goal as well as one would be able to derive the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations of the theory. The algebraic computations are a bit tedious but straightforward like that of our Sec. 3. The results, emerging from the above horizontality condition by setting all the Grassmannian components of the supercurvature tensor equal to zero, are

\[
\begin{align*}
b^{(1)}_2 &= 0, \quad s_2 = 0, \quad \tilde{b}^{(2)}_2 = 0, \quad \tilde{s}_2 = 0, \quad \tilde{s}_1 = 0, \quad s_1 = 0, \quad b^{(2)}_2 + \tilde{b}^{(2)}_1 &= 0, \\
b^{(1)}_2 + \tilde{b}^{(1)}_2 &= 0, \quad \tilde{b}^{(1)}_2 = 0, \quad \tilde{f}^{(2)}_\mu = \partial_\mu \tilde{C}_2, \quad f^{(1)}_\mu = \partial_\mu C_2, \quad b_\mu = i\partial_\mu \tilde{b}^{(2)}_1, \\
\tilde{b}_\mu &= -i\partial_\mu b^{(2)}_2, \quad b^{(3)}_\mu = -i\partial_\mu \tilde{b}^{(1)}_2, \quad B^{(1)}_{\mu \nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \quad f^{(2)}_\mu = \partial_\mu \tilde{C}_2, \\
\tilde{B}^{(2)}_{\mu \nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \quad s_{\mu \nu} = i(\partial_\mu f^{(1)}_\nu - \partial_\nu f^{(1)}_\mu) = -i(\partial_\mu f^{(3)}_\nu - \partial_\nu f^{(3)}_\mu), \quad \tilde{s}_{\mu \nu} = -i(\partial_\mu \tilde{f}^{(2)}_\nu - \partial_\nu \tilde{f}^{(2)}_\mu) \equiv -i(\partial_\mu \tilde{f}^{(2)}_\nu - \partial_\nu \til{f}^{(2)}_\mu), \\
\tilde{R}_{\mu \nu \eta} &= \partial_\mu \tilde{C}_{\nu \eta} + \partial_\nu \tilde{C}_{\eta \mu} + \partial_\eta \tilde{C}_{\mu \nu}, \quad \tilde{R}_{\mu \nu \eta} = \partial_\mu \tilde{C}_{\nu \eta} + \partial_\nu \tilde{C}_{\eta \mu} + \partial_\eta \tilde{C}_{\mu \nu}, \\
S_{\mu \nu \eta} &= -i(\partial_\mu B^{(2)}_{\nu \eta} + \partial_\nu B^{(2)}_{\eta \mu} + \partial_\eta B^{(2)}_{\mu \nu}) \equiv i(\partial_\mu B^{(2)}_{\nu \eta} + \partial_\nu B^{(2)}_{\eta \mu} + \partial_\eta B^{(2)}_{\mu \nu}).
\end{align*}
\]  

(A.3)

In addition to the above results, we obtain the following Curci-Ferrari type restrictions

\[
f^{(2)}_\mu + \tilde{f}^{(3)}_\mu = \partial_\mu \tilde{C}_1, \quad \tilde{f}^{(1)}_\mu + f^{(3)}_\mu = \partial_\mu C_1, \quad \tilde{B}^{(1)}_{\mu \nu} + B^{(2)}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu,
\]  

(A.4)

which ensure the consistency of the three equivalences shown in (A.3). It will be noted that the above restrictions emerge from setting the specific coefficients of the 4-form differentials

\[^\dagger\text{Note that } dB^{(3)} = \frac{1}{4}(dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau) H_{\mu \nu \sigma \tau} \text{ where } H_{\mu \nu \sigma \tau} = \partial_\mu B_{\nu \sigma \tau} + \partial_\nu B_{\sigma \tau \mu} + \partial_\sigma B_{\tau \mu \nu} + \partial_\tau B_{\mu \nu \sigma} \text{ is a totally antisymmetric curvature tensor that would be useful for the kinetic term of the gauge field.}\]
[e.g. \((dx^μ \wedge dθ \wedge dθ \wedge d\bar{θ})\), \((dx^μ \wedge dθ \wedge d\bar{θ} \wedge d\bar{θ})\), \((dx^{\mu+i} \wedge dx^{\mu+i} \wedge dx^3 \wedge dx^5)\)] of the l.h.s. of the horizontality condition \(\bar{d}B^{(3)} = dB^{(3)}\). Finally, it is worth pointing out that the coefficients of the differentials \((dx^{μ+i} \wedge dx^{ν} \wedge dx^3 \wedge dx^5)\) from the l.h.s. and r.h.s. of the condition \(\bar{d}B^{(3)} = dB^{(3)}\) match due to the precise form of \(R_{μνη}, \bar{R}_{μνη}\) and \(S_{μνη}\), quoted in (A.3).

The substitution of the results of (A.3) into (A.2) leads to the following super expansion of the superfields (after the application of the horizontality condition) in the language of the nilpotent and absolutely (anti-)BRST symmetry transformations

\[
\begin{align*}
\bar{B}^{(h)}_{μνη}(x, θ, \bar{θ}) &= B_{μνη}(x) + θ \left( s_{ab}B_{μνη}(x) \right) + \bar{θ} \left( s_bB_{μνη}(x) \right) + θ \bar{θ} \left( s_{ab}B_{μνη}(x) \right), \\
\bar{β}^{(h)}_μ(x, θ, \bar{θ}) &= β_μ(x) + θ \left( s_{ab}β_μ(x) \right) + \bar{θ} \left( s_bβ_μ(x) \right) + θ \bar{θ} \left( s_{ab}β_μ(x) \right), \\
\bar{β}^{(h)}_β(x, θ, \bar{θ}) &= \bar{β}_β(x) + θ \left( s_{ab}\bar{β}_β(x) \right) + \bar{θ} \left( s_b\bar{β}_β(x) \right) + θ \bar{θ} \left( s_{ab}\bar{β}_β(x) \right), \\
\bar{Φ}^{(h)}_μ(x, θ, \bar{θ}) &= φ_μ(x) + θ \left( s_{ab}φ_μ(x) \right) + \bar{θ} \left( s_bφ_μ(x) \right) + θ \bar{θ} \left( s_{ab}φ_μ(x) \right), \\
\bar{F}^{(h)}_{μν}(x, θ, \bar{θ}) &= C_{μν}(x) + θ \left( s_{ab}C_{μν}(x) \right) + \bar{θ} \left( s_bC_{μν}(x) \right) + θ \bar{θ} \left( s_{ab}C_{μν}(x) \right), \\
\bar{F}^{(h)}_μ(x, θ, \bar{θ}) &= C_μ(x) + θ \left( s_{ab}C_μ(x) \right) + \bar{θ} \left( s_bC_μ(x) \right) + θ \bar{θ} \left( s_{ab}C_μ(x) \right), \\
\bar{F}^{(h)}_η(x, θ, \bar{θ}) &= C_η(x) + θ \left( s_{ab}C_η(x) \right) + \bar{θ} \left( s_bC_η(x) \right) + θ \bar{θ} \left( s_{ab}C_η(x) \right), \\
\bar{F}^{(h)}_1(x, θ, \bar{θ}) &= C_1(x) + θ \left( s_{ab}C_1(x) \right) + \bar{θ} \left( s_bC_1(x) \right) + θ \bar{θ} \left( s_{ab}C_1(x) \right), \\
\bar{F}^{(h)}_2(x, θ, \bar{θ}) &= C_2(x) + θ \left( s_{ab}C_2(x) \right) + \bar{θ} \left( s_bC_2(x) \right) + θ \bar{θ} \left( s_{ab}C_2(x) \right), \\
\bar{F}^{(h)}_3(x, θ, \bar{θ}) &= C_3(x) + θ \left( s_{ab}C_3(x) \right) + \bar{θ} \left( s_bC_3(x) \right) + θ \bar{θ} \left( s_{ab}C_3(x) \right),
\end{align*}
\]

where the off-shell nilpotent and absolutely anticommutating (anti-)BRST symmetry transformations \(s_{(a)b}\), with the identifications \(b^{(2)}_1 = B_1, b^{(2)}_2 = B_2, \bar{b}^{(1)}_1 = \bar{B}, f^{(1)}_μ = \bar{F}_μ, f^{(2)}_μ = F_μ, f^{(3)}_μ = f_μ, \bar{f}^{(3)}_μ = \bar{f}_μ\), for the Abelian 3-form gauge theory are

\[
\begin{align*}
&\begin{align*}
s_bB_{μνη} &= \partial_μC_{νη} + \partial_νC_{μη} + \partial_ηC_{μν}, & s_bC_{μν} &= \partial_μβ_ν - \partial_νβ_μ, & s_bC_{μν} &= B^{(2)}_μ, \\
s_bβ_μ &= \partial_μC_2, & s_bC_2 &= 0, & s_bB^{(2)}_μ &= 0, & s_bC_1 &= -\bar{B}, & s_bB &= 0, & s_bB_2 &= 0, \\
s_bC_1 &= B_1, & s_bB_1 &= 0, & s_bC_2 &= B_2, & s_bβ_μ &= F_μ, & s_bF_μ &= 0, & s_bφ_μ &= f_μ, \\
s_bf_μ &= 0, & s_bF_μ &= -\partial_μB, & s_b\bar{f}_μ &= \partial_μB_1, & s_bB^{(1)}_μ &= \partial_μf_ν - \partial_νf_μ, \\
s_{ab}B_{μνη} &= \partial_μC_{νη} + \partial_νC_{μη} + \partial_ηC_{μν}, & s_{ab}C_{μν} &= \partial_μ\bar{β}_ν - \partial_ν\bar{β}_μ, & s_{ab}C_{μν} &= B^{(1)}_μ, \\
s_{ab}\bar{β}_μ &= \partial_μC_2, & s_{ab}C_2 &= 0, & s_{ab}B^{(1)}_μ &= 0, & s_{ab}C_1 &= -\bar{B}_1, & s_{ab}B_1 &= 0, & s_{ab}B_2 &= 0, \\
s_{ab}\bar{C}_1 &= -\bar{B}_2, & s_{ab}B &= \bar{B}, & s_{ab}\bar{C}_2 &= B, & s_{ab}\bar{β}_μ &= F_μ, & s_{ab}F_μ &= 0, & s_{ab}φ_μ &= \bar{f}_μ, \\
s_{ab}\bar{f}_μ &= 0, & s_{ab}\bar{F}_μ &= -\partial_μB_2, & s_{ab}f_μ &= -\partial_μB_1, & s_{ab}B^{(2)}_μ &= \partial_μ\bar{f}_ν - \partial_ν\bar{f}_μ.
\end{align*}
\end{align*}
\]

It is elementary to check that the above (anti-)BRST symmetry transformations are off-shell nilpotent of order two (i.e. \(s^{2}_{(a)b} = 0\)).

Furthermore, it can be seen that the anticommutativity property (i.e. \(s_b s_{ab} \neq s_{ab} s_b = 0\)) on the following basic fields of the Abelian 3-form gauge theory

\[
\{s_b, s_{ab}\} B_{μνη} = 0, \quad \{s_b, s_{ab}\} C_{μν} = 0, \quad \{s_b, s_{ab}\} \bar{C}_{μν} = 0,
\]

is true only when the Curci-Ferrari type restrictions (A.4) (i.e. \(B^{(2)}_{μν} + \bar{B}^{(1)}_{μν} = \partial_μφ_ν - \partial_νφ_μ, f_μ + \bar{F}_μ = \partial_μC_1, \bar{f}_μ + F_μ = \partial_μ\bar{C}_1\) are satisfied. The property of the anticommutativity

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of the (anti-)BRST symmetry transformations is trivially obeyed in the case of rest of the fields of our present 4D Abelian 3-form gauge theory.

Finally, one can write down the (anti-)BRST invariant Lagrangian density for the above Abelian 3-form gauge theory as (see, e.g. [35,40] for details)

\[
\mathcal{L}_B^{(3)} = \frac{1}{48} H^{\mu\nu\xi} H_{\mu\nu\xi} + s_b s_{ab} \left( \frac{1}{2} \bar{C}_1 C_1 - C_2 C_2 + \frac{1}{2} \bar{C}_{\mu\nu} C^{\mu\nu} + \bar{\beta}_\mu \beta_\mu + \frac{1}{2} \phi^\mu \phi_\mu - \frac{1}{6} B_{\mu\nu\xi} B^{\mu\nu\xi} \right).\]  

(A.9)

It will be noted that (i) all the individual terms in the big round bracket have the mass dimension two and the ghost number equal to zero, (ii) the ghost number of the ghost fields can be computed from the transformations (A.6) and (A.7), and (iii) the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations (A.7) and (A.6) have been taken into account in writing of the gauge-fixing and Faddeev-Popov ghost terms.

Our approach is quite general and can be applied to any arbitrary Abelian \( p \)-form gauge theory in any arbitrary \( D \)-dimension of spacetime. All one has to do is generalize (A.1) to super \( p \)-form connection that is defined on a (D, 2)-dimensional supermanifold. This will automatically provide the clue about the number of basic (primary) fields of the Abelian \( p \)-form gauge theory. The application of the celebrated horizontality condition would be able to produce the desired nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations of the theory which can be exploited, in turn, to produce the (anti-)BRST invariant Lagrangian density of the theory (see, e.g. (A.9)).

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