Chiral Symmetry Breaking in QED in a Magnetic Field at Finite Temperature

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Abstract

The catalysis of chiral symmetry breaking in the massless weakly coupled QED in a magnetic field at finite temperature is studied. The temperature of the symmetry restoration is estimated analytically as $T_c \approx m_{\text{dyn}}$, where $m_{\text{dyn}}$ is the dynamical mass of a fermion at zero temperature.

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Recently it has been shown that a constant magnetic field leads to a dynamical breaking of chiral symmetry at the weakest attractive interaction between fermions [1]. This effect is universal and is due to the dimensional reduction \( D \to D - 2 \) in the fermion pairing in a strong magnetic field when the dynamics of the lowest Landau level (LLL) plays the crucial role. As concrete models, the Nambu-Jona-Lasinio (NJL) model as well as QED in 2+1 and 3+1 dimensions were considered [1–9]. The effect of catalyzing the dynamical chiral symmetry breaking under the influence of a magnetic field was extended also to the case of external non-abelian chromomagnetic fields and finite temperatures [10–12], as well as to the supersymmetric NJL model [13], confirming the universality of the mechanism.

An important question of the chiral symmetry restoration in QED\( _4 \) at finite temperature was addressed in [14]. Lee, Leung, and Ng have obtained for the critical temperature \( T_c \simeq \frac{\alpha}{\pi} \sqrt{2\pi |eB|} \), where \( \alpha \) is the fine structure constant and \( B \) is the magnetic field strength. However, their \( T_c \) can be considered only as a rough upper estimate. In this paper we reconsider the problem and show that the correct estimate for the critical temperature is \( T_c \approx m_{dyn} \), where \( m_{dyn} \) is the dynamical mass of a fermion at zero temperature which in its turn is given by \( m_{dyn} \approx \sqrt{|eB|} \exp\left(-\sqrt{\pi/\alpha}\right) \) [6].

The Lagrangian density of massless QED\( _4 \) in a magnetic field is

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} [\bar{\psi}, i\gamma^\mu D_\mu \psi],
\]

(1)

where the covariant derivative \( D_\mu \) is

\[
D_\mu = \partial_\mu - ie(A^{ext}_\mu + A_\mu), \quad A^{ext}_\mu = \left(0, -\frac{B}{2} x_2, \frac{B}{2} x_1, 0\right).
\]

(2)

The Lagrangian density [1] is chiral invariant (we do not discuss the anomaly connected with the current \( j_{5\mu} \), in any case it is not manifested in quenched approximation dealt with in this paper). As is known, there is no spontaneous chiral symmetry breaking at \( B = 0 \) in the weak coupling phase of QED [13]. However, the magnetic field changes the situation drastically: at \( B \neq 0 \) the chiral symmetry is broken and there appears a gapless Nambu-Goldstone (NG) boson composed from a fermion and an antifermion. The dynamical mass
(energy gap) for a fermion can be defined by considering the Bethe-Salpeter (BS) equation for NG boson [8] or the Schwinger-Dyson (SD) equation for the dynamical mass function [9]. Both these approaches lead to the following equation for the dynamically generated fermion mass in the quenched approximation (taking into account the dominance of LLL at strong magnetic field) and in the Feynman gauge:

$$\Sigma(p_{||}) = \frac{\alpha^2}{2\pi^2 i} \int \frac{d^2k_{||}}{k_{||}^2 - \Sigma^2(k_{||})} \int_0^\infty \frac{dk_1^2 \exp(-k_1^2/2)}{(k_{||} - p_{||})^2 - k_1^2},$$

(3)

where $p_{||}$ is a two-dimensional momentum, $p_{||} = (p_0, p_3)$ (henceforth we will omit the subscript $||$ in $p$ and $k$), $\ell = 1/\sqrt{|eB|}$ is the magnetic length. In Euclidean region, the equation (3) with the replacement $\Sigma^2(k) \rightarrow \Sigma^2(0) \equiv m_{dyn}^2$ in the denominator

$$\Sigma(p) = \frac{\alpha}{2\pi^2} \int \frac{d^2k \Sigma(k)}{k^2 + m_{dyn}^2} \int_0^\infty \frac{dx \exp(-x\ell^2/2)}{(k - p)^2 + x},$$

(4)

has been analyzed in [7,9]. Particularly, as was shown in [7] (see Appendix C), in the case of weak coupling $\alpha$, the mass function $\Sigma(p)$ remains almost constant in the range of momenta $0 < p^2 \lesssim 1/\ell^2$ and decays like $1/p^2$ outside that region. To get an estimate for $m_{dyn}$ at $\alpha \ll 1$, we set the external momentum to be zero and notice that the main contribution of the integral is formed in the infrared region with $k^2 \lesssim 1/\ell^2$. The latter validates in its turn the substitution $\Sigma(k) \rightarrow \Sigma(0)$ in the integrand of (4), and finally we come to the following gap equation

$$\Sigma(0) \simeq \frac{\alpha}{2\pi^2} \Sigma(0) \int \frac{d^2k}{k^2 + m_{dyn}^2} \int_0^\infty \frac{dx \exp(-x\ell^2/2)}{k^2 + x},$$

(5)

i.e.

$$1 \simeq \frac{\alpha}{2\pi} \int_0^\infty \frac{dx \exp(-ax)}{x - 1} \log x, \quad a \equiv \frac{m_{dyn}^2 \ell^2}{2}.$$

(6)

The main contribution in (6) comes from the region $x \lesssim 1/a$, thus at $a \ll 1$ we get

$$1 \simeq \frac{\alpha}{4\pi} \log^2 \left(\frac{m_{dyn}^2 \ell^2}{2}\right),$$

(7)

$$m_{dyn} \simeq C \sqrt{|eB|} \exp \left[-\sqrt{\frac{\pi}{\alpha}}\right],$$

(8)
where $C$ is a constant of order one. The exponential factor displays the nonperturbative nature of this result. We note that the double logarithmic asymptotics of the electron self-energy in a magnetic field in the standard perturbation theory was obtained in earlier papers [16,17].

More accurate analysis which takes into account the momentum dependence of the mass function leads to the result [7]

$$m_{\text{dyn}} \simeq C\sqrt{|eB|} \exp \left[ -\frac{\pi}{2\sqrt{2}} \alpha \right].$$  

(9)

Notice that the ratio of the powers of this exponent and that in Eq.(8) is $\pi/2\sqrt{2} \simeq 1.1$, thus the approximation used above is rather reliable.

To study chiral symmetry breaking in an external field at nonzero temperatures we use the imaginary-time formalism [18]. Now the analogue of the equation (3) (with the replacement $\Sigma^2(\omega_n, k) \to m^2(T)$ in the denominator) reads

$$\Sigma(\omega_n', p) = \frac{\alpha}{\pi} T \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk \Sigma(\omega_n, k)}{\omega_n^2 + k^2 + m^2(T)} \int_0^{\infty} \frac{dx \exp(-x\ell^2/2)}{(\omega_n - \omega_n')^2 + (k - p)^2 + x},$$  

(10)

where $\omega_n = \pi T (2n + 1)$.

If we take now $n' = 0, p = 0$ in the left hand side of Eq.(10) and put $\Sigma(\omega_n, k) \approx \Sigma(0, 0) = \text{const}$ in the integrand, we come to the equation

$$1 = \frac{\alpha}{\pi} T \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{\omega_n^2 + k^2 + m^2(T)} \int_0^{\infty} \frac{dx \exp(-x\ell^2/2)}{(\omega_n - \omega_0)^2 + k^2 + x}.$$  

(11)

It is easy to check that the gap equation (11) coincides with the equation (58) in [14].

To evaluate the sum in (11), we transform it into the integral in complex plane $\omega$:

$$T \sum_{n=-\infty}^{\infty} \frac{1}{(\omega_n^2 + a^2)(|\omega_n - \omega_0|^2 + b^2)} = \frac{1}{2\pi i} \int_C \frac{d\omega}{[1 + e^{-\omega\beta}] [\omega^2 - a^2](|\omega - i\omega_0|^2 + b^2)},$$  

(12)

where $\beta \equiv 1/T$ and the contour $C$ runs as usually around the poles of the function $(1 + e^{-\omega\beta})^{-1}$. By deforming the contour, we can represent the integral in (12) as the sum over four residues at $\omega = \pm a$ and $\omega = i\omega_0 \pm b$. Thus, we obtain:
which is just what one obtains from Eq. (5) after performing the integration over \(x\), where we also switched to dimensionless variables Eq. (17) comes from the region \(0 < x < \pi T\), the magnetic field, \(T\)

In the limit \(T \to 0\), Eq. (13) reduces to the following one

\[
1 = \frac{\alpha}{\pi} \int_0^\infty \int_0^\infty \frac{dk dx \exp[-x t^2/2]}{\sqrt{k^2 + x^2} + m^2(T)\sqrt{k^2 + m^2(T) + m^2(T) + m^2(T)}}
\]

which is just what one obtains from Eq. (5) after performing the integration over \(k_4 = -ik_0\).

The equation for the critical temperature is obtained from (15) putting \(m(T_c) = 0\):

\[
1 = \frac{\alpha}{\pi} \int_0^\infty \int_0^\infty \frac{dk dx e^{-2x(\pi T_c \ell)^2}}{1/4 + x^2 + \frac{1}{k^2} + \frac{1}{k^2 + x} + \frac{1}{k^2 + x}} \left\{ \frac{1/4 + x}{k^2} \tanh (\pi k) + \frac{1/4 - x}{k^2 + x} \coth (\pi \sqrt{k^2 + x}) \right\},
\]

where we also switched to dimensionless variables \(x \to (2\pi T_c)^2 x\) and \(k \to 2\pi T_c k\).

By assuming smallness of the critical temperature in comparison with the scale put by the magnetic field, \(T_c \ell \ll 1\), we see that the double logarithmic in field contribution in Eq. (17) comes from the region \(0 < x \lesssim 1/2(\pi T_c \ell)^2\), \(1/\pi \lesssim k < \infty\). Simple estimate gives:

\[
1 \simeq \frac{\alpha}{\pi} \int_0^1 dx \int_{1/\pi}^\infty \frac{dk}{1/4 + x^2 + k^2} \left[ \frac{1/4 + x}{k^2} + \frac{1/4 - x}{k^2 + x} \right]
\]

\[
\sum_{n=0}^\infty \frac{1}{(n+a)(n+b)(n+c)(n+d)} = -\frac{\psi(a)}{(b-a)(c-a)(d-a)} - \frac{\psi(b)}{(a-b)(c-b)(d-b)} - \frac{\psi(c)}{(a-c)(b-c)(d-c)} - \frac{\psi(d)}{(a-d)(b-d)(c-d)},
\]

where \(\psi(x) = d \log \Gamma(x)/dx\).
\[
\approx \frac{\alpha}{\pi} \int_0^{1/2(\pi T_c \ell)^2} dx \left[ \frac{1}{2(1/4 + x)} \log \left( 1 + (1/4 + x)^2 \pi^2 \right) + \frac{1/4 - x}{(1/4 + x)(1/4 - x)} \right]
\times \log \frac{(1/4 + x + |1/4 - x|)\sqrt{1/\pi^2 + (1/4 + x)^2}}{(1/4 + x)\sqrt{1/\pi^2 + x} + |1/4 - x|/\pi}
\approx \frac{\alpha}{4\pi} \log^2 \left[ \frac{1}{2(\pi T_c \ell)^2} \right].
\]

Thus, for the critical temperature, we obtain the estimate:

\[
T_c \approx \sqrt{|eB|} \exp \left[ -\frac{\pi}{\alpha} \right] \approx m_{dyn}(T = 0),
\]

where \(m_{dyn}\) is given by (8). The relationship \(T_c \approx m_{dyn}\) between the critical temperature and the zero temperature fermion mass was obtained also in NJL model in (2+1)- and (3+1)-dimensions \([4,11]\).

In passing, let us just briefly note that, the photon thermal mass, which is of the order of \(\sqrt{\alpha T}\) \([19]\), cannot change our result for the critical temperature. As is easy to check, the only effect of taking it into account will be the shift in \(x\) for a constant of the order of \(\alpha\) in the integrand of (18). However, such a shift is absolutely irrelevant for our estimate (19).

In conclusion, we notice that the main result of this paper is the analytic estimate for the temperature [given by Eqs.(19) and (8)] of the chiral symmetry restoration in the weakly interacting QED in a background magnetic field. In words, the critical temperature is proportional to the value of the fermion dynamical mass at zero temperature. Although such a result looks natural from the physical point of view, we felt necessity to clarify this point by means of a rigorous analytical calculation. The reason was the following: there appeared a rough estimate for the critical temperature in \([14]\) (such that \(T_c \gg m_{dyn}(T = 0)\)). This upper estimate was enough indeed for the purpose of that paper, i.e. for proving that the catalysis of chiral symmetry breaking is not important in dynamics of the electroweak phase transition. However, making use of that rough estimate in other problems may not be appropriate at all.

Finally, we would like to emphasize that in this paper as well as in all previous publications \([3,8,14]\) only was the weakly coupled QED \((\alpha \ll 1)\) considered. It would also be interesting to clarify the influence of an external magnetic field on the chiral symmetry
breaking in strongly coupled QED (\(\alpha > \alpha_c\)) [15] where the chiral symmetry is broken even in the absence of a magnetic field.

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REFERENCES

[1] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. Lett. 73, 3499 (1994).

[2] K.G. Klimenko, Z. Phys. C 54, 323 (1992).

[3] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Phys. Lett. B 349, 477 (1995).

[4] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Phys. Rev. D 52, 4718 (1995).

[5] A.V. Shpagin, *Dynamical mass generation in (2+1)-dimensional electrodynamics in external magnetic field*, hep-ph/9611412 (submitted to Physics of Atomic Nuclei).

[6] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Phys. Rev. D 52, 4747 (1995).

[7] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Nucl. Phys. B 462, 249 (1996).

[8] C.N. Leung, Y.J. Ng, and A.W. Ackley, Phys. Rev. D 54, 4181 (1996).

[9] D.K. Hong, Y. Kim and S.-J. Sin, Phys. Rev. D 54, 7879 (1996).

[10] I.A. Shovkovy and V.M. Turkowski, Phys. Lett. B 367, 213 (1996).

[11] D. Ebert and V.Ch. Zhukovsky, *Chiral phase transitions in strong chromomagnetic fields at finite temperature and dimensional reduction*, hep-ph/9701323.

[12] I.A. Shushpanov and A.V. Smilga, *Quark condensate in a magnetic field*, preprint ITEP-TH-6/97, hep-ph/9703201.

[13] V.Elias, D.G.C.McKeon, V.A.Miransky, and I.A.Shovkovy, Phys. Rev. D 54, 7884 (1996).

[14] D.-S. Lee, C. N. Leung, and Y. J. Ng, *Chiral Symmetry breaking in a uniform external magnetic field*, hep-th/9701172 (to appear in Phys.Rev D, May issue, 1997).

[15] P.I. Fomin, V.P. Gusynin, V.A. Miransky, and Yu.A. Sitenko, Riv. Nuovo Cim. 6, N5 (1983).
[16] B. Jancovici, Phys. Rev. 187, 2275 (1969).

[17] Yu.M. Loskutov and V.V. Skobelev, Theor. Mat. Fiz. 48, 44 (1981).

[18] A.A. Abrikosov, L.P. Gorkov and I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Nauka, Moscow, 1962).

[19] H.A. Weldon, Phys. Rev. D 26, 1394 (1982).