Uncertainty Quantification for the 12-lead ECG: 
a Lead Field Approach

Michael Multerer and Simone Pezzuto

Center for Computational Medicine in Cardiology, Institute of Computational Science, Università della Svizzera italiana, via G. Buffi 13, 6900 Lugano, Switzerland
{michael.multerer,simone.pezzuto}@usi.ch

Abstract. The standard electrocardiogram (ECG) is a point-wise evaluation of the body potential at certain given locations. These locations are subject to uncertainty and may vary from patient to patient or even for a single patient. In this work, we estimate the uncertainty in the ECG induced by uncertain electrode positions when the ECG is derived from the bidomain model. In order to avoid the high computational cost associated to the solution of the bidomain model in the entire torso, we propose a low-rank approach to solve the uncertainty quantification problem. More precisely, we exploit the sparsity of the ECG and the lead field theory to translate it into a set of deterministic, time-independent problems, whose solution is eventually used to evaluate expectation and covariance of the ECG. We assess the approach with numerical experiments in a simple geometry.

Keywords: Random electrode location · Uncertainty quantification · Lead field · Electrophysiology.

1 Introduction

The standard 12-lead ECG is a routinely acquired recording of the electric activity of the heart. It consists in 12 recordings of the electric potential on the chest. Each recording or lead is the potential difference between two electrodes or between an electrode and a reference potential. For instance, lead I is the potential difference between left and right arm, whereas lead V1 is obtained from the difference of an electrode on the chest (called again V1) and the Wilson Central Terminal, that is the average of limb electrode potentials.

In patient-specific modeling, the ECG could in principle be used to identify the parameters of sophisticated cardiac models, as advocated by several authors. However, the parameter identification procedure should be aware of uncertainties in the data. Such uncertainties may include the ECG electric potential, the electrodes location, the shape of the segmentation process of the anatomy, the electric conductivity, and thus on. From a computational perspective, uncertainty quantification may result in a very expensive task even when limiting the analysis to the forward problem. This is due to the
high non-linearity in the stochastic variables and the intrinsic cost in a single model evaluation.

In this work, we propose a computationally very efficient methodology to propagate the uncertainty in the electrode location to the simulated ECG. Making the assumption, that the electrode positions are very localized, we employ a low-rank approach to decouple the tensor product lead field problem for the correlation into a set of lead field problems for different right hand sides, see [11]. If the low-rank approach is not applicable, an alternative would be to resort to hierarchical matrix techniques, see [7]. The presented approach is heavily inspired by the lead field theory [14]. More specifically, we focus on estimating the expectation and the correlation function of the ECG when this is resulting from a bidomain simulation in the whole torso. Instead of solving the space-time problem, we recast it into a limited number of simpler time-independent problems by a low-rank approximation of the covariance function.

This paper is organized as follows: in Sec. 2 we review the bidomain theory for the ECG, the lead field approach and describe our method. In Sec. 3 we validate the approach on a simple geometry. We conclude in Sec. 4 with a brief discussion and outlook.

2 Methods

The ECG with bidomain theory. The electric potential $u_0(x,t)$ in the torso $\Omega_T \subset \mathbb{R}^d$, and consequently the ECG, can be modelled from the transmembrane potential $V_m(x,t)$ in the active myocardium $\Omega_H \subset \mathbb{R}^d$, cp. Figure 1 for an illustration, with the bidomain model [5] (see Fig. 1):

\[
\begin{align*}
-\nabla \cdot (G_b \nabla u_e) &= \nabla \cdot (G_i \nabla V_m), & \text{in } \Omega_H \times [0, \infty), \\
-\nabla \cdot (G_0 \nabla u_0) &= 0, & \text{in } \Omega_T \times [0, \infty), \\
-G_0 \nabla u_0 \cdot n &= 0, & \text{on } \Sigma \times [0, \infty), \\
u_e &= u_0, & \text{on } \Gamma \times [0, \infty), \\
-G_b \nabla u_e \cdot n + G_0 \nabla u_0 \cdot n &= G_i \nabla V_m \cdot n, & \text{on } \Gamma \times [0, \infty). 
\end{align*}
\]

(1)

Herein, $\Gamma = \partial \Omega_H \cap \partial \Omega_T$ is the heart-torso interface, $\Sigma = \partial \Omega_T \setminus \Gamma$ is the body surface, $u_e(x,t)$ is the extra-cellular potential in the heart, $G_b = G_i + G_e$ is
myocardium bulk conductivity, \( G_i \) and \( G_e \) are respectively intra- and extracellular conductivity, \( G_0 \) is the torso conductivity, and \( n \) is the outward normal for both \( \Gamma \) and \( \Sigma \). For sake of simplicity in the notation, we define

\[
G := \begin{cases} 
G_b & \text{in } \Omega_H, \\
G_0 & \text{in } \Omega_T,
\end{cases}
\quad u := \begin{cases} 
u_e & \text{in } \Omega_H, \\
u_0 & \text{in } \Omega_T,
\end{cases}
\]

and assume, without loss of generality, that \( u(\cdot, t) \in H^1(\Omega) \), where \( \Omega = \Omega_H \cup \Omega_T \).

In this case, the variational formulation for Eq. (1) is can be written according to

For every \( t \in \mathbb{R} \), find \( u(\cdot, t) \in H^1(\Omega) \) such that

\[
\int_{\Omega} G \nabla u(t) \cdot \nabla v \, dx = - \int_{\Omega_H} G_i \nabla V_m(t) \cdot \nabla v \, dx
\]

for all \( v \in H^1(\Omega) \). Well-posedness of the problem follows from standard application of the Riesz Theorem [8], given that \( \Omega_H, \Omega_T \) are Lipschitz domains and \( V_m(\cdot, t) \in H^1(\Omega) \). We remark that the formulation in Eq. (2) is equivalent to Eq. (1) when the restriction of the solution \( u|_{\Omega_i} \) belongs to \( H^2(\Omega_i) \), see e.g. [2,3] for a treatment of interface problems.

The ECG is a set of so-called leads, typically 12 in the standard ECG. Each lead reads as follows:

\[
V(t, \xi_1, \ldots, \xi_M) = \sum_{\ell=1}^M a_\ell u(\xi_\ell, t),
\]

where \( \Xi := \{\xi_i\}_{i=1}^M \) is the set of electrodes and \( a = [a_1, \ldots, a_M]^T \) is a zero-sum vector of weights defining the lead. For instance, a limb lead is the potential difference of 2 electrodes, whereas a precordial leads involves 4 electrodes (3 are used to build the Wilson Central Terminal, that is the reference potential). It is worth noting that Eq. (3) is valid only if \( u(\cdot, t) \in C^0(\Sigma) \), which is not true for \( u(\cdot, t) \in H^1(\Omega) \) and \( d \geq 2 \). For a rigorous discussion, see [5].

Quantities of interest. In this work, we are interested in computing statistics of \( V(t, \Xi(\omega)) \) when the electrode positions \( \Xi(\omega) := \{\xi_{m}(\omega)\}_{i=1}^M \) are not know exactly. Here, we denote by \( \xi_{\ell}(\omega) \) the random variable associated to the \( \ell \)-th electrode and assume that it’s distribution is given by the density \( \rho_\ell(x) \) with respect to the surface measure on \( \Sigma \). We assume that the collection \( \Xi(\omega) \) is independent. According to the definition in Eq. (3), the lead \( V(t, \Xi) \) is a random field as well, with expectation and correlation respectively reading as follows:

\[
\mathbb{E}[V](t) = \int_{\Sigma^M} V(t, X) \rho(X) d\sigma_X,
\]

\[
\text{Cor}[V](t, s) = \int_{\Sigma^M} V(t, X)V(s, X) \rho(X) d\sigma_X,
\]

where \( d\sigma_X = dx \cdots d\sigma_{XM} \) is the surface measure on \( \Sigma^M \), \( d\sigma_x \) the surface measure on \( \Sigma \), and the joint density is given by \( \rho(X) = \rho_1(x_1) \cdots \rho_M(x_M) \).
Lead field formulation. Typically, it is not convenient to compute the ECG from Eq. (1), because the ECG is only a very sparse evaluation of \( u(x, t) \). Moreover, in a patient-specific or personalization context, the ECG needs to be simulated several times with different instances of \( V_m(x, t) \), with no changes in the left hand side of Eq. (1). A convenient approach is based on the Green’s function, also known as lead field in the electrocardiographic literature \[14\]. In fact, it is possible to show that \( V(t, \xi) \) has the following representation \[5\]:

\[
V(t, \Xi) = \int_{\Omega^H} G_i(x) \nabla V_m(t) \cdot \nabla Z(\Xi) \, dx,
\]

where \( Z(\Xi) \) is the weak solution of the time-independent problem:

\[
\begin{cases}
-\nabla \cdot G \nabla Z = 0, & \text{in } \Omega, \\
- G \nabla Z \cdot n = \sum_{\ell=1}^M a_\ell \delta_{\xi_\ell}, & \text{on } \Sigma,
\end{cases}
\]

where \( \delta_{\xi_\ell} \) is the \((d-1)\)-dimensional Dirac delta centered at \( \xi_\ell \). Therefore, given that all measurement locations are fixed, Eq. (7) is only solved once, at the cost of a single time step of Eq. (1), and then used to compute \( V(t, \Xi) \) for any choice of \( V_m(x, t) \). To note, Eq. (6) is often adopted with approximated choices of \( Z(\Xi) \), e.g., assuming an isotropic and homogeneous torso via the boundary element method, or even simpler the fundamental solution of the Laplacian in \( \mathbb{R}^3 \). Finally, we remark that Eq. (7) is pure Neumann problem. Hence, the solution is only determined up to a constant. To fix this constant, we shall focus on solutions that vanish on average.

Expected ECG. In what follows, we exploit Eq. (6) to compute the the expectation and correlation of \( V \), according to Eq. (4) and (5). Substituting Eq. (6) into Eq. (4), we obtain by the linearity of the expectation that

\[
\mathbb{E}[V](t) = \int_{\Omega^H} G_i(x) \nabla V_m(t) \cdot \nabla \mathbb{E}[Z] \, dx.
\]

Again by linearity, the equation for the expected lead field \( \mathbb{E}[Z] \) follows from Eq. (7) and reads as follows:

\[
\begin{cases}
-\nabla \cdot G \nabla \mathbb{E}[Z] = 0, & \text{in } \Omega, \\
- G \nabla Z \cdot n = \sum_{\ell=1}^M a_\ell \rho_\ell, & \text{on } \Sigma,
\end{cases}
\]

where we recall that \( \rho_\ell \) is the probability density function of the random electrode location \( \xi_\ell(\omega) \). To show this, we observe that:

\[
\mathbb{E} \left[ \sum_{\ell=1}^M a_\ell \delta_{x_\ell} \right] = \sum_{\ell=1}^M a_\ell \int_{\Sigma^M} \delta_{x_\ell} \rho(X) \, d\sigma_X = \sum_{\ell=1}^M a_\ell \rho_\ell.
\]

Therefore, the cost of computing the average ECG is equivalent to that for solving for the point-wise ECG, i.e. one solution of Eq. (9) per lead. Moreover, the averaged Eq. (9) is more regular than the original problem with no singularity at the electrodes location. We remark however that the presence of the singularity is irrelevant for the computation of the ECG, since Eq. (6) considers the restriction of \( Z \) on \( \Omega^H \), clearly not in contact with the chest. However, the same approach
may be applied to simulate intracardiac or intramural signals as well, in which case special care is required.

**Correlation of the ECG.** The natural continuation of the above argument yields the correlation for the ECG according to

$$\text{Cor}[V](t, s) = \int_{\Sigma^2} (G_i \nabla \otimes G_i \nabla) V_m(t) V_m(s) : (\nabla \otimes \nabla) \text{Cor}[Z] \, d\sigma_x d\sigma_y,$$

where the tensor product is \([u \otimes v]_{ij} = u_i(x) v_j(y)\) and the inner product between tensors is \(A : B = \sum_{ij} [A]_{ij} [B]_{ij}\). The problem for the correlation \(\text{Cor}[Z]\), obtained as above from Eq. (7), reads as follows:

\[
\begin{align*}
\langle \nabla \cdot G \nabla \otimes \nabla \cdot G \nabla \rangle \text{Cor}[Z] &= 0, \quad \text{in } \Omega \times \Omega, \\
\langle \mathbf{n} \cdot G \nabla \otimes \nabla \cdot G \nabla \rangle \text{Cor}[Z] &= 0, \quad \text{on } \Sigma \times \Omega, \\
\langle \nabla \cdot G \nabla \otimes \mathbf{n} \cdot G \nabla \rangle \text{Cor}[Z] &= 0, \quad \text{on } \Omega \times \Sigma, \\
\langle \mathbf{n} \cdot G \nabla \otimes \mathbf{n} \cdot G \nabla \rangle \text{Cor}[Z] &= R, \quad \text{on } \Sigma \times \Sigma, \\
\end{align*}
\]

where the correlation \(R(x, x')\) of the Neumann data in Eq. (7) is

\[
R(x, x') = \text{Cor} \left[ \sum_{\ell=1}^M a_\ell \delta_{x, \ell}, \sum_{\ell'=1}^M a_{\ell'} \delta_{x', \ell'} \right] = \sum_{\ell=1}^M a_\ell^2 \rho_\ell(x) \delta_{x}(x') + \sum_{\ell \neq \ell'} a_\ell a_{\ell'} \rho_\ell(x) \rho_{\ell'}(x').
\]

As the computation of \(\text{Cor}[Z]\) requires the solution of a tensor product boundary value problem, it is computationally rather expensive. In what follows, we will exploit the particular structure of \(R(x, y)\) to significantly reduce the computational cost and implementation effort.

**Discretization.** The variational formulation of the averaged lead field problem Eq. (8) resemble Eq. (2) with a different right hand side. With \(Y = H^1(\Omega)\), the problem is:

Find \(E[Z] \in Y\): \(\int_\Omega G \nabla E[Z] \cdot \nabla v \, dx = \int_\Sigma \sum_{i=1}^M a_i \rho_i v \, dx, \quad \text{for all } v \in Y.\)

The Galerkin approximation in the space \(Y_h \subset Y\), with \(Y_h = \text{span}\{\phi_k\}_{k=1}^{N_h}\), reads as follows:

\([K]_{k\ell} = \int_\Omega G \nabla \phi_k \cdot \nabla \phi_\ell \, dx, \quad [g]_k = \int_\Sigma \sum_{i=1}^M a_i \rho_i(x) \phi_k(x) \, dx.\)
Concerning the correlation problem in Eq. (10), the variational formulation is as follows:

Find \( \text{Cor}[Z] \in Y \times Y \) such that

\[
\int_{\Omega^2} (G \nabla \otimes G \nabla) \text{Cor}[Z] : (\nabla \otimes \nabla) v \, dx \, dx' = \int_{\Sigma^2} R v \, dx \, dx'
\]

for all \( v \in Y \times Y \). The corresponding Galerkin formulation on \( Y_h \times Y_h \) is:

\[
(K \otimes K)[Z] = R,
\]

where \( \text{Cor}[Z] \approx \sum_{k,\ell} [Z]_{k\ell} \phi_k \otimes \phi_\ell \) and

\[
[R]_{pq} = \int_{\Sigma^2} R \phi_p \phi_q \, dx \, dx' = \sum_{\ell=1}^M a_\ell^2 \int_{\Sigma} \rho_\ell \phi_p \phi_q \, dx + \sum_{\ell \neq \ell'} a_\ell a_{\ell'} \left( \int_{\Sigma} \rho_\ell \phi_p \, dx \right) \left( \int_{\Sigma} \rho_{\ell'} \phi_q \, dx \right).
\]

As the number of degrees of freedom for the correlation problem is \( N_h^2 \), it may easily become computationally prohibitive. However, assuming that the densities \( \rho_\ell, \ell = 1, \ldots, M \) are strongly localized, the right hand side in (12) may be represented by a low-rank approximation according to

\[
R \approx \sum_k r_k \otimes r_k, \quad r_k \in \mathbb{R}^{N_h},
\]

with \( K \ll N_h \). In this case, we also expect a low-rank solution, that is

\[
Z \approx \sum_k \zeta_k \otimes \zeta_k, \quad \zeta_k \in \mathbb{R}^{N_h},
\]

with \( K \ll N_h \) and \( \zeta_k \in \mathbb{R}^{N_h} \). Then, due to the tensor product structure of (12), there simply holds

\[
K \zeta_k = r_k, \quad k = 1, \ldots, K.
\]

In practice, we compute the low-rank approximation by a diagonally pivoted, truncated Cholesky decomposition, see [10].

Finally, for the computation of statistics of the ECG, we insert the computed Galerkin approximations into Eq. (4) and Eq. (5) and obtain

\[
\mathbb{E}[V](t) \approx \mathbf{V}(t) \cdot \mathbf{z}, \quad \text{Cor}[V](t, s) \approx \sum_{k,\ell=1}^K (\mathbf{V}(t) \cdot \zeta_k) (\mathbf{V}(s) \cdot \zeta_\ell),
\]

where

\[
[V(t)]_j = \int_{\Omega_h} G_i \nabla V_{\alpha}(t) \cdot \phi_j \, dx.
\]

3 Numerical Validation

In this Section we consider an idealized torso-heart domain in \( \mathbb{R}^2 \) as depicted in Fig. 1. The conductivity tensors are defined as follows:

\[
G_0 = \sigma_{\text{torso}} \mathbf{I}, \quad G_i = \sigma_{i,\ell} \mathbf{I} + (\sigma_{i,\ell} - \sigma_{i,t}) \mathbf{f} \otimes \mathbf{f}, \quad G_e = \sigma_{e,\ell} \mathbf{I} + (\sigma_{e,\ell} - \sigma_{e,t}) \mathbf{f} \otimes \mathbf{f},
\]
with \( f \) being a unit-vector field representing the cardiac fibers, oriented circularly in \( \Omega_H \). For the transmembrane potential, we consider the function \( V_m(x, t) = \tanh(x_1 - t) \).

For this example we only consider 2 random electrodes, \( \xi_1(\omega) \) and \( \xi_2(\omega) \), which are uniformly distributed with mean \( \xi_1 \) and \( \xi_2 \). The support of the distribution is the same for both and equal to \( 2\delta \). The lead is defined with \( a_1 = 1 \) and \( a_2 = -1 \). In particular, we have:

\[
R(x, x') = (\rho_1(x) + \rho_2(x))\delta_\pi(x') - \rho_1(x')\rho_2(x) - \rho_2(x)\rho_1(x').
\]

The computational domain is approximated by a triangular mesh, and parameters are set as in \[14\]. In the Galerkin formulation, we considered linear finite elements, with a total of 46363 degrees of freedom. The implementation is done with the Python backend of FEniCS 2019.1.0.

In the experiment we compare the proposed method based on the lead field against the solution of the bidomain model. In Fig. 2 we show the computed ECG, along with average and confidence interval, and an average lead field in the torso. The number of modes in the low-rank representation of the correlation was 74, and linearly scales with the mesh size.

4 Discussion and Conclusions

In this work, we have solved the problem of quantifying the uncertainty in the ECG when electrode positions are uncertain. Our method recasts the problem in a fully deterministic setting by using the lead field theory and a low-rank approximation for the correlation. The computational advantage is significant over the standard forward simulation of the bidomain model, and it is even more advantageous in the setting of parameter estimation and the inverse problem of electrocardiography. The formulation can be generalized to address other relevant problems, such as quantifying the uncertainty in the ECG due to, e.g., uncertain transmembrane potential or torso-heart segmentation, leading to more robust simulations.
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