Semantics and Axiomatization for Stochastic Differential Dynamic Logic

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Abstract. Building on previous work by André Platzer, we present a formal language for Stochastic Differential Dynamic Logic, and define its semantics, axioms and inference rules. Compared to the previous effort, our account of the Stochastic Differential Dynamic Logic follows closer to and is more compatible with the traditional account of the regular Differential Dynamic Logic. We resolve an issue with the well-definedness of the original work’s semantics, while showing how to make the logic more expressive by incorporating nondeterministic choice, definite descriptions and differential terms. Definite descriptions necessitate using a three-valued truth semantics. We also give the first Uniform Substitution calculus for Stochastic Differential Dynamic Logic, making it more practical to implement in proof assistants.

Keywords: Stochastic reasoning · Dynamic logic · Proof calculus · Hybrid systems · Theorem proving

1 Introduction

It is well known that safety of complex hybrid systems, such as cyber-physical systems (whether autonomous or not), cannot be achieved with just simulation and testing \cite{6,8}. The space of possible behaviors is so big that testing and simulation cannot provide sufficient coverage. Achieving high confidence in correctness requires the ability to model the system mathematically and to prove its properties with an aid of an automated reasoning system. Moreover, cyber-physical systems operate in uncertain environments and even modeling such system is a nontrivial task. Thus, we need a system that is able to reason about properties that incorporate such uncertainties.

Differential Dynamic Logic (dDL) has proven a useful tool for certifying hybrid systems \cite{13,14}. However, it and its simplest probabilistic extensions can only reason about those systems whose continuous behavior is fully deterministic, but many hybrid systems are best modeled as stochastic processes. This may be because they are deployed in a setting where the underlying dynamics are stochastic, such as in processes interacting with physical materials with
stochastic properties, or with financial markets — or because they represent a controller acting under measurement uncertainty. Reasoning about such systems in a dDL style was formulated in Stochastic Differential Dynamic Logic \cite{10,11}.

Part of the reason that dDL has been successful in practice is the substantial amount of work done on its theory since it was first proposed. Notably, the adoption of universal substitution based reasoning \cite{13} allowed the move from KeYmaera to KeYmaera X \cite{4} with a much smaller Trusted Code Base, and the introduction of indefinite descriptions in dL \cite{2} allowed for reasoning about terms that may not always be defined but which are required in practice (such as square roots). In this work, we seek to develop the foundational theory required for a practical implementation of stochastic differential logic, by introducing a stochastic differential logic with definite descriptions in the uniform substitution style.

These are not the only differences between the logic presented here and the original sDL \cite{11,12}. We also allow for programs with true non-determinism, as this is important for reasoning about hybrid systems whose design is not fully specified.

Defining semantics for stochastic differential dynamic logic is a non-trivial task. We also found what we believe to be an error in the proof that the original sDL semantics are well-defined, in the sense that the semantics of formulas should always be measurable. To resolve this, our semantics differ from those of most dDL-style logics in that the “continuous” programs $x' = \theta \& H$ can not terminate non-deterministically at any point while $H$ is true, but instead only at pre-chosen stopping times.

Our semantics differs from those of $dL_i$ (dDL with definite descriptions \cite{2}) in another key way: we define formulas to be indeterminate in the case of program failure, and interpret program modalities to quantify over failures, while $dL_i$ ignores them. So we would evaluate $[x := 1 \cup \text{fail}]x = 1$ to be indeterminate, while the original $dL_i$ would evaluate it to true. By formulating a nondeterministic controller as a nondeterministic guarded choice operator which would fail when all guards are simultaneously false, we avoid the common challenge encountered by KeYmaera X users (particularly novices), where it is easy to accidentally state and prove a safety lemma that is vacuously true — that is, true not because the controller would always keep the system safe, but because the controller definition accidentally excludes the unsafe trajectories from consideration.

This paper is structured as follows. We first present the syntax of our SDL formulation, with a brief outline of the intended semantics in Section 2. We then present the full formal semantics in Section 3. We define the semantics of our validity judgment, and present some proof rules in Section 4. We present an axiomatization of our SDL in Section 5. We extend our validity judgment to statements about probabilities, and present proof rules and axioms for prob-.

\footnote{Namely, the proof of measurability for the semantics of $\langle \alpha \rangle f$ in \cite{12} Appendix A.2 uses the right-continuity of $[\alpha]_t$ to capture the value of $[f][[\alpha]_{t'}]$ for times $t'$ with converging sequences of rational nets. However it’s true only if $[f]$ is continuous on each path. This is not always the case.}
bilistic statements in Section \[6\] We then outline a uniform substitution calculus for SDL in Section \[7\] We conclude and discuss the next steps in Section \[8\] The proofs of correctness for our axioms with respect to the semantics are presented in Appendix \[A\] The paper assumes some familiarity with stochastic processes or stochastic differential equations; for relevant exposition please see \[11\].

2 Syntax

2.1 Terms

\[\theta, \kappa := c \mid x \mid \theta \ast \kappa \mid \theta + \kappa \mid f_d(\theta_1, \theta_2...\theta_d) \mid d_t(\theta) \mid d_{B,x}(\theta) \mid i\iota \phi_d\]

For a constant, \(x\) a variable, and \(f_d\) a function symbol of arity \(d\). In the definite description \(i\iota \phi_d\), \(d\) is a positive integer, \(i \in [1..d]\), and \(\phi_d\) is a formula with no program or formula symbols as subexpressions, and containing \(d\) special variable symbols \(\phi^{\alpha,\kappa}_{\iota i}\) for \(n \in [1..d]\), that are not used in any other context. Call the set of such special variables appearing in \(\phi_d\), \(\phi^{\alpha,\kappa}_{\iota i}\).

Variables come from a countable set \(V\) that is closed under sub-scripting by \(t\) or \(B, x\). The term \(d_t(\theta)\) is a differential that expresses the rate of change of \(\theta\) with respect to time. \(d_{B,x}\) is similar, but expresses the rate of change relative to a Brownian motion that is associated to variable \(x\).

2.2 Stochastic Hybrid Programs

\[\alpha, \beta := x_i := \theta \mid x_i := * \mid dx = bdt + \sigma dW \& H \mid \text{if } H \text{ then } \alpha \text{ else } \beta \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^\ast \mid \text{skip} \mid \text{fail} \mid \gamma\]

For \(H\) a formula containing no program, \(\gamma\) a program symbol, \(x\) is a vector of variables, and \(b, \sigma\) respectively a vector and a square matrix of terms of corresponding dimension. \(dx = bdt + \sigma dW \& H\) evolves the system according to the expressed stochastic differential equation for some non-deterministically chosen length of time. \(x_i := *\) draws a new value for \(x_i\) uniformly from \([0,1]\). \(\alpha \cup \beta\) represents the nondeterministic choice between \(\alpha\) and \(\beta\) (note: in some accounts of dDL, notation \(\alpha \mid \beta\) is used instead), \textit{alpha;beta}\) represents the sequential execution of \(\alpha\) and \(\beta\), \(\alpha^\ast\) represents an arbitrary (nondeterministic) number of repetitions of program \(\alpha\), \textit{skip} is an empty “no-op” program, and \textit{fail} is a “failure” program — once executed, it causes everything to become indeterminate.

2.3 Formulas

\[\phi, \psi ::= \theta \geq \kappa \mid \neg \phi \mid \phi \land \psi \mid p_d(\theta_1, \theta_2...\theta_d) \mid \langle \alpha \rangle \phi \mid \text{sure}(\phi)\]

\text{sure}(\phi) is true when \(\phi\) is true, and false otherwise (that is, if \(\phi\) is false or indeterminate). The program modality \(\langle \alpha \rangle \phi\) gives the maximum value of \(\phi\) that could be achieved after running \(\alpha\).

We will use \(\phi \lor \psi\) as syntactic sugar for \(\neg (\neg \phi \land \neg \psi)\) and \([\alpha] \phi\) for \(\neg (\langle \alpha \rangle \neg \phi)\). Also, \(\text{ind}(\phi)\) for \(\neg \text{sure}(\phi) \land \neg \text{sure}(\neg \phi)\). Then we define \(\phi_1 \rightarrow \phi_2\) as \(\neg \phi_1 \lor \phi_2 \lor (\text{ind}(\phi_1) \land \text{ind}(\phi_2))\), and \(\phi_1 \leftrightarrow \phi_2\) as \((\phi_1 \rightarrow \phi_2) \land \phi_2 \rightarrow \phi_1\).
3 Denotational Semantics

In order to account for partial definitions, we take $\mathbb{R}_\perp := \mathbb{R} \cup \perp$ with $\perp$ standing in for “undefined”, so that $\perp$ added to or multiplied by anything is $\perp$. We consider $\mathbb{R}_\perp$ to have the “extended topology” generated by the open sets of $\mathbb{R}$ and the singleton $\{ \perp \}$, and take the Borel sigma algebra on this topology. Similarly take $\mathcal{L} := \{ \oplus, \ominus, \otimes \}$ the truth values of Łukasiewicz logic [15], ordered $\ominus < \ominus \otimes \oplus$.

Define the negation operator as: $\oplus := \ominus, \ominus := \ominus, \otimes := \oplus$.

At any point in a program, only finitely many variables will have been used. Let $\mathbb{R}_\perp^\mathbf{Y}$ be the set of maps from $V$ to $\mathbb{R}_\perp$ that are $\perp$ in all but finitely many places. Alternatively, we may be at a point where the program has crashed, $\nabla$, or where it has gone past the bounds of its differential equation, $\Delta$. Call the set of valuations $\text{Val} := \mathbb{R}_\perp^\mathbf{Y} \cup \{ \nabla, \Delta \}$.

Then let a state $z$ be a random variable on $\text{Val}$: $\Omega \to \text{Val}$. Let the set of states be $\mathcal{Z}$.

As in [11], we will fix a canonical sample space $\Omega$ and a sigma algebra $\mathcal{F}$ on it. We endow it with a probability measure $\mathcal{P}$ and a family of IID uniform random variables $U : \Omega \to \mathbb{R}$. We give a semantics of non-deterministic choice as maps from adversarial choice sequences to outcomes; these choice sequences are binary streams that are measurable.

We give term semantics as $\mathbf{I} \triangleright \mathbf{x} \vdash [\theta] : \mathcal{I} \to \mathbb{R}_\perp^\mathbf{Y} \to \mathbb{R}_\perp$, written $\mathbf{I} \triangleright \mathbf{x} [\theta]$. Then we define the function $\mathbf{I} \triangleright \lambda \mathbf{x} [\theta] : \mathbb{R}_\perp^\mathbf{Y} \to \mathbb{R}_\perp$ as $\lambda \mathbf{x}. \mathbf{I} \triangleright \mathbf{x} [\theta]$.

This semantics is to be extended to $\mathcal{I} \triangleright \text{Val} \to \mathbb{R}_\perp$ by defining $\mathcal{I} \triangleright [\theta] (\omega) = \perp = \mathcal{I} \triangleright [\theta] (\omega) = \perp$ for all $\theta, \omega$. This further extends to $\mathcal{I} \triangleright \mathcal{Z} \to \Omega \to \mathbb{R}_\perp$ as $\mathcal{I} \triangleright [\theta] (\omega) := \mathcal{I} \triangleright [\theta] (\omega)$. 

3.1 Definitions

Term Semantics We give term semantics as $[\theta] : \mathcal{I} \to \mathbb{R}_\perp^\mathbf{Y} \to \mathbb{R}_\perp$, written $\mathbf{I} \triangleright [\theta]$. Then we define the function $\mathbf{I} \triangleright \lambda \mathbf{x} [\theta] : \mathbb{R}_\perp^\mathbf{Y} \to \mathbb{R}_\perp$ as $\lambda \mathbf{x}. \mathbf{I} \triangleright \mathbf{x} [\theta]$.
For definite descriptions and differentials, we will first define candidate semantic functions \( \text{Can}(I, \theta) : \mathbb{R}_V^\perp \rightarrow \mathbb{R}_\perp \). If the candidates are not measurable we will discard them in favor of the constant \( \perp \) function.

\[
\text{Can}(I, \iota_i \phi_d)(x) := \begin{cases} 
\{ y(\diamond \phi_d, i) \} & \exists! y \in \mathbb{R}_V^\perp. x^{\diamond \phi_d} = y^{\diamond \phi_d}, \forall \omega.Iy[\phi_d](\omega) = \top \\
\perp & \text{else} 
\end{cases}
\]

Since \( \phi_d \) doesn’t contain programs or formula symbols, its semantics are constant across \( \Omega \)

\[
\text{Can}(I, d_t(\theta))(x) := \left( \sum_{x \in V} x \frac{\partial \hat{\theta}}{\partial x} + \frac{1}{2} \sum_{x, y \in V} \frac{\partial^2 \hat{\theta}}{\partial x \partial y} \left( \sum_{j \in V} x_{B,j} y_{B,j} \right) \right)(x)
\]

\[
\text{Can}(I, d_{B,x}(\theta))(x) := \left( \sum_{y \in V} \frac{\partial \hat{\theta}}{\partial y} y_{B,x} \right)(x)
\]

Where the last two definitions evaluate to \( \perp \) when derivatives are undefined.

Now:

\[
\text{Ix}[\iota_i \phi_d] := \begin{cases} 
\text{Can}(I, \iota_i \phi_d)(x) & \text{Can}(I, \iota_i \phi_d) \text{is measurable} \\
\perp & \text{else} 
\end{cases}
\]

\[
\text{Ix}[d_t \theta] := \begin{cases} 
\text{Can}(I, d_t \theta)(x) & \text{Can}(I, d_t \theta) \text{is measurable} \\
\perp & \text{else} 
\end{cases}
\]

\[
\text{Ix}[d_{B,x} \theta] := \begin{cases} 
\text{Can}(I, d_{B,x} \theta)(x) & \text{Can}(I, d_{B,x} \theta) \text{is measurable} \\
\perp & \text{else} 
\end{cases}
\]

**Program Semantics** \( [\alpha] : I \rightarrow \text{Val} \rightarrow \Omega \rightarrow \mathbb{C} \rightarrow \text{Val} \times \mathbb{C} \), written \( Iv[\alpha](\omega, C) \).

Then we can extend this to \( [\alpha] : I \rightarrow Z \rightarrow \Omega \rightarrow \mathbb{C} \rightarrow \text{Val} \times \mathbb{C} \) by setting \( Iz[\alpha](\omega, C) := I(z(\omega))[\alpha](\omega, C) \). For all \( I, \alpha, \omega, C, Iv[\alpha](\omega, C) := \nabla, C \) and \( I\Delta[\alpha](\omega, C) := \Delta, C \). Thus these cases are ignored below.

\[
Iv[x := \theta](\omega, C) := \begin{cases} 
\nabla, C & I\theta[\omega](\omega) = \perp \\
v[Iv[\theta](\omega)/x], C & \text{else} 
\end{cases}
\]

\[
Iv[x := v](\omega, C) := v[U(\omega)/v], C
\]

for \( U \) a never-before-used random variable.
\[ I^v[dx = bdt + sdW & H](\omega, C) := \]
\[ \text{let } f^t = v[x \mapsto v(x) + \int_0^t [b](\omega)dt + \int_0^t [s](\omega)dW(\omega) \]
\[ \text{or } \neg \text{ if that is undefined because } b \text{ or } s \text{ take on a value of } \bot \text{ in some location before } t, \text{ or if } I \int_0^t [H](\omega) \neq \oplus \text{ in:} \]
\[ \begin{cases} 
\neg & \exists t', t' \leq t \wedge f^{t'} = \neg , \text{tail}(\text{nat}(C), C) 
\end{cases} \]
\[ I^v[\text{if } H \text{ then } \alpha \text{ else } \beta](\omega, C) := \begin{cases} 
I^v[\alpha](\omega, C) & I^v[H](\omega) = \oplus \\
I^v[\beta](\omega, C) & I^v[H](\omega) = \ominus \\
\neg, C & I^v[H](\omega) = \ominus 
\end{cases} \]
\[ I^v[\alpha \cup \beta](\omega, C) := \begin{cases} 
I^v[\alpha](\omega, \text{tail}(C)) & \text{head}(C) = 0 \\
I^v[\beta](\omega, \text{tail}(C)) & \text{head}(C) = 1 
\end{cases} \]
\[ I^v[\alpha; \beta](\omega, C) := \text{let } (v_\alpha, C_\alpha) = I^v[\alpha](\omega, C) \text{ in } I^v[\alpha; \beta](\omega, C_\alpha) \]
\[ I^v[\alpha^*](\omega, C) := I^v[\alpha^{\text{nat}(C)}](\omega, \text{tail}(\text{nat}(C), C)) \]
\[ \text{where } \alpha^0 = \text{skip} \text{ and } \alpha^{k+1} = \alpha; \alpha^k. \]
\[ I^v[\gamma](\omega, C) := (I^v)(v, \omega, C) \]
\[ I^v[\text{skip}](\omega, C) := v, C \]
\[ I^v[\text{fail}](\omega, C) := \neg, C \]

**Formula Semantics** \[ \llbracket \phi \rrbracket : \mathcal{I} \rightarrow \text{Val} \rightarrow \Omega \rightarrow \mathcal{L}, \] written \( I^v[\phi](\omega). \) Similar to above we can extend this to \( \llbracket \phi \rrbracket : \mathcal{I} \rightarrow \mathcal{Z} \rightarrow \Omega \rightarrow \mathcal{L}, \) written \( I^z[\phi](\omega). \) For all \( \phi, \) we define \( I^v[\varnothing](\omega) := I^v[\phi](\omega) := \ominus. \) The behavior at all other values is defined below:

\[ I^v[\theta \geq \kappa](\omega) := \text{let } a = I^v[\theta](\omega), \ b = I^v[\kappa](\omega) \text{ in } \begin{cases} 
\oplus & a - b \geq 0 \\
\ominus & a - b < 0 \\
\ominus & a - b = \bot 
\end{cases} \]
\[ I^v[\neg \phi](\omega) := I^v[\neg \phi](\omega) \]
\[ I^v[\phi \land \kappa](\omega) := \max(I^v[\phi](\omega), I^v[\kappa](\omega)) \]
\[ I^v[p_d(\theta_1 \ldots \theta_d)](\omega) := \text{sup } I^v[p_d(I^v[\theta_1] \ldots I^v[\theta_d])](\omega) \]
\[ I^v[\langle \alpha \rangle \phi](\omega) := \text{let } v_C = \pi_1 I^v[\alpha](\omega, C) \text{ in } \sup_{C \text{ st } v_C \neq \Delta} I^v[V_C[\phi](\omega) \]
\[ I^v[\text{sure}(\phi)] := \begin{cases} 
\oplus & I^v[\phi](\omega) = \oplus \\
\ominus & \text{else} 
\end{cases} \]

### 3.2 Measurability

The following theorems are proven in Appendix A.
Theorem 1 (Measurability of Term Semantics). For any term $\phi$, any interpretation $I$, any state $z$, $Iz[\phi] : \Omega \rightarrow \mathbb{R}^\perp$ is a measurable function.

Theorem 2 (Measurability of Formula Semantics). For any formula $\phi$, any interpretation $I$, any state $z$, $Iz[\phi] : \Omega \rightarrow \mathcal{L}$ is a measurable function.

Theorem 3 (Measurability of Determinized Program Semantics). For any formula $\alpha$, any interpretation $I$, any state $z$, and any choice sequence $C$, $Iz[\alpha](C) : \Omega \rightarrow \text{Val}$ is a state (a measurable function).

4 Pathwise Reasoning

Definition 1 (Pathwise Validity). A formula $\phi$ is valid under a countable set of stop-times $T$ if $\forall I \in \mathcal{I} st. I_{\text{times}} = T, \forall v \in \mathbb{R}^\perp_1, \forall \omega \in \Omega. Iv[\phi] = \oplus$. Then we write $T \models \phi$. If $T$ has the property that $T' \supseteq T \rightarrow T' \models \phi$, we write $T \models^+ \phi$. $\phi$ is pathwise valid if $\exists T$ st. $T \models^+ \phi$.

Where a formula appears on a proof tree, it should be interpreted as the assertion that the formula is pathwise valid.

4.1 Proof Rules

Note that that if $\phi_1$, $\phi_2$ are valid, there exist $T_1$, $T_2$ st $T_1 \models^+ \phi_1$, $T_2 \models^+ \phi_2$, so $T_1 \cup T_2 \models^+ \phi_1$ and $T_1 \cup T_2 \models^+ \phi_2$.

The following proof rules are easily provable valid by inspection of the formula semantics, considering the consequent in the union of the two antecedant’s stop times.

\[
\text{AND-ELIM} \quad \frac{\phi_1 \\ \phi_2}{\phi_1 \land \phi_2} \\
\text{MP} \quad \frac{\phi_1 \rightarrow \phi_2}{\phi_2}
\]

As $\phi_1 \leftrightarrow \phi_2$ is valid if and only if they have the same semantics for every $I, v, \omega$, we have a syntactic substitution rule

\[
\text{IFF-SUB} \quad \frac{\phi_1 \leftrightarrow \phi_2}{\phi_3 \leftrightarrow \phi_3[\phi_1/\phi_2]}
\]

Note that one of the usual proof rules of dynamic logic, the rule “G” that says we may derive the validity of $[\alpha]\phi$ from that of $\phi$, is not sound in its usual form here, because $\alpha$ may be a failing program. Instead, let $\text{crash}(\alpha)$ be the formula $\text{ind}([\alpha]0 \geq 0)$, which has semantics of $\oplus$ if $\alpha$ may transition to $\triangledown$ and $\ominus$ otherwise.

\[
G \quad \frac{\phi}{\text{crash}(\alpha) \lor [\alpha]\phi}
\]

In addition, we will use a uniform substitution proof rule to instantiate axioms under some substitution $\sigma$. We will present and justify this rule in a later section.
5 Axioms

We present an axiom schema. Below, each $\theta, \phi, \alpha$ should be interpreted as a (term, formula, program) symbol of arity 0. Where the soundness proofs follow trivially from the defined semantics, we have omitted them (this is most cases).

5.1 Pathwise Axioms for Formulas

We start with the axioms that manipulate formulas.

**Id**
- $\phi \leftrightarrow \phi$

**Higher Order Equality**
- $(\phi_1 \leftrightarrow \phi_2) \leftrightarrow (\phi_2 \leftrightarrow \phi_1)$
- $(\phi_1 \leftrightarrow \phi_2) \leftrightarrow (\neg \phi_1 \leftrightarrow \neg \phi_2)$

**Conjunction**
- $(\phi_1 \land \phi_2) \leftrightarrow (\phi_2 \land \phi_1)$
- $\phi \leftrightarrow (\phi \land \phi)$
- $((\phi_1 \land \phi_2) \land \phi_3) \leftrightarrow (\phi_1 \land (\phi_2 \land \phi_3))$
- $\phi_1 \land \phi_2 \rightarrow \phi_1$

**Double Negation Elimination**
- $\neg \neg \phi \leftrightarrow \phi$

Note that double negation elimination is valid, but the law of the excluded middle is not, unless we are dealing with sure quantities.

**Excluded Middle of Sureness**
- $\text{sure}(\phi) \lor \neg \text{sure}(\phi)$

**Sureness**
- $\text{sure}(\phi) \leftrightarrow \text{sure}(\text{sure}(\phi))$
- $\text{sure}(\phi) \rightarrow \phi$
- $\text{sure}(\phi) \rightarrow (\phi_2 \rightarrow \phi_1 \land \phi_2)$
- $(\neg \text{sure}(\phi) \leftrightarrow \neg \phi \lor \text{ind}(\phi))$
- $\text{sure}(\phi_1 \land \phi_2) \leftrightarrow \text{sure}(\phi_1) \land \text{sure}(\phi_2)$

5.2 Pathwise Axioms for Terms

All formulas that are valid in the theory of real closed fields are valid here, so we can take as axioms any axiomatization of real closed fields. Additionally, we have the following axioms for differentiating terms:
\[ d_c = 0 \]
\[ d_{B,i} c = 0 \]
\[ d_t x = x_i \]
\[ d_B x = x_{B,i} \]
\[ d_t (f + g) = d_t f + d_t g \]
\[ d_{B,i} (f + g) = d_{B,i} f + d_{B,i} g \]
\[ d_t (f * g) = g * d_t f + f * d_t g + \frac{1}{2} \sum_i d_{B,i} f * d_{B,i} g \]
\[ d_{B,i} (f * g) = g * d_w f + f * d_w g \]

Finally, we have an axiom for substituting equal terms:
\[ (\theta_1 = \theta_2) \rightarrow (p_1(\theta_1) \Leftrightarrow p_1(\theta_2)) \]

### 5.3 Pathwise Axioms for Programs

**Distributivity**
- \( (\langle \alpha \rangle (\phi_1 \lor \phi_2) \Leftrightarrow \langle \alpha \rangle \phi_1 \lor \langle \alpha \rangle \phi_2) \)

**Skip and Fail**
- \( \langle \text{skip}; \alpha \rangle \phi \Leftrightarrow \langle \alpha \rangle \phi \)
- \( \langle \alpha \rangle \phi \Leftrightarrow \langle \alpha; \text{skip} \rangle \phi \)
- \( \langle \text{skip} \rangle \phi \Leftrightarrow \phi \)
- \( \text{ind}(\langle \text{fail}; \alpha \rangle \phi) \)
- \( \text{ind}(\langle \text{fail} \rangle \phi) \)

**Nondeterministic Choice**
- \( \langle \alpha \cup \beta \rangle \phi \Leftrightarrow \langle \beta \cup \alpha \rangle \phi \)
- \( \langle \alpha \cup \beta \rangle \phi \Leftrightarrow \langle \alpha \rangle \phi \lor \langle \beta \rangle \phi \)
- \( \langle \alpha \rangle \phi \rightarrow \langle \alpha \cup \beta \rangle \phi \)

**Conditionals**
- \( \langle \text{if} \ H \ \text{then} \ \alpha \ \text{else} \ \beta \rangle \phi \Leftrightarrow (H \land \langle \alpha \rangle \phi) \lor (\neg H \land \langle \beta \rangle \phi) \)

**Iteration**
- \( \langle \alpha^* \rangle \phi \Leftrightarrow \phi \lor \langle \alpha; \alpha^* \rangle \phi \)
- \( [\alpha^*]\text{sure}(\phi \rightarrow [\alpha]\phi) \rightarrow \text{sure}(\phi \rightarrow [\alpha^*]\phi) \)

**Composition**
- \( \langle \alpha; \beta \rangle \phi \Leftrightarrow \langle \beta \rangle \phi \)

This one necessitates some justification:

**Lemma 1 (No Look-Ahead Consumption).** There exists a natural number \( n_C \) such that \( \pi_1(Iv[\alpha](\omega, C)) = \text{tail}^{n_C} C \). Furthermore for any \( C' \) that agrees with \( C \) in the first \( n_C \) places, \( \pi_1(Iv[\alpha](\omega, C)) = \pi_1(Iv[\alpha](\omega, C')) \)
Assignment

Proof. By structural induction on programs.

Lemma 2 (Compositionality of Supremum Semantics).

\[ I v \langle \alpha; \beta \rangle \phi(\omega) = I v \langle \alpha \rangle \phi(\omega) \]

Proof. \[ I v \langle \alpha; \beta \rangle \phi(\omega) = \text{let } v_1 = \pi_1(I v [\alpha](\omega, C_1)) \text{ in } \max_{C_1} I v_1 [\beta] \phi(\omega) = \text{let } v_1 = \pi_1(I v \langle \alpha \rangle(\omega, C_1)) \text{ in } \max_{C_1} I v_1 [\beta] \phi(\omega) \]

Differential Axioms

These axioms correspond to the differential axioms of ODEs. Stochastic behavior to ensure it is exactly axiom DI of [13]. The first two correspond to DW and DC, which say that in the absence of crashes, we may move constraints into postconditions and established postconditions into constraints. The next two correspond to DE, and say that after an sde is run, it has performed assignments on the differentials of all involved variables. The fifth axiom uses the condition that it have no stochastic behavior to ensure it is exactly axiom DI of [13] - reducing to the case of ODEs.

1. crash(dx = bd + σdW & H) ∨ [dx = bd + σdW & H] H
2. sure([dx = bd + σdW & H] H) \[dx = \begin{cases} bd + \sigma dW & H \text{ or } H \\ H \end{cases}\]
3. [dx = bd + σdW & H] \[d_t x_i = \theta_1 \rightarrow d_t \theta_1 \geq \theta_2 \]
4. \[dx = bd + \sigma dW & H] \[\sigma = 0 \text{ or } d_t \theta_1 \geq d_t \theta_2 \rightarrow ([dx = bd + \sigma dW & H] \theta_1 \geq \theta_2) \]

6 Reasoning in Distribution

We have been careful to establish that our semantics is measurable. Now we can reason about real arithmetic extended with terms of the form \( P(\phi) \). As before a set of stop times models a formula ♠ of this language, \( T \models ♠ \) when for any \( I \) st \( I_{\text{times}} = t, z \) a state such that \( P(z = \Delta) = P(z = \vee) = 0 \), the ♠ is a true formula of arithmetic under the substitution \( P(\phi) \rightarrow P(I z[\phi](\omega) = \oplus) \). Again we say that a set of stop times validates ♠, \( T \models ♠ \) when \( T' \supseteq T \rightarrow T' \models ♠ \), and ♠ is valid if \( \exists T \) that validates it.

The following proof rules contain assertions about both the validity of formulas in this arithmetic language, as well as in the language of sDL formulas. Clearly we can take all the axioms of real arithmetic here, as well the following axioms and proof rules whose soundness is self-evident:

- \( P(\phi) \geq 0 \)
\[ \cdot P(\neg \phi) \leq 1 - P(\phi) \]
\[ \cdot P(\phi_1 \lor \phi_2) \geq P(\phi_1) \]
\[ \cdot P(\text{sure}(\phi)) = P(\phi) \]

\[
\text{VALID-PROB} \quad \frac{\phi}{P(\phi) = 1}
\]

\[
\text{DISJOINT-PROB} \quad \frac{(\text{sure}(\phi_1) \rightarrow \neg \text{sure}(\phi_2)) \land (\text{sure}(\phi_2) \rightarrow \neg \text{sure}(\phi_1))}{P(\phi_1 \lor \phi_2) = P(\phi_1) + P(\phi_2)}
\]

The hardest reasoning principals are those that deal with the program modality. The following axiom is valid because every time we randomize a variable, we do so independently.

\[ \cdot P(\langle x = \ast \rangle \phi) = \int_0^1 P(< x = s > \phi) \, ds. \]

Note that integrals aren’t a part of our language, so what we really mean is that the left hand side = c for some constant with the side condition that c = the right hand side.

In particular we can derive from this: 0 \leq c \leq 1 \rightarrow P(\langle x = \ast; \text{if } x < c \text{ then } \alpha \text{ else } \beta \rangle \phi) = c \ast P(\langle \alpha \rangle \phi) + (1 - c) \ast P(\langle \beta \rangle \phi)

6.1 Inequality Axiom for Stochastic Differential Equations

Theorem 7 of [11] gives a soundness proof of an axiom for reasoning about stochastic differential equations, drawing heavily on [9, Section 7.2]. We present the same axiom here, modified to fit our syntax (in particular, note that the extra-syntactic construction L of [11] is handled by our \( d_t \) syntax). We omit the soundness proof, as it is essentially unchanged from [11].

The following is sound under the following conditions: \( \forall I, \omega, \hat{\theta} \theta \) has compact real support, and \( b, \sigma \) are real and Lipschitz when restricted to \( \{v \in \text{Val} | I v[H] = \oplus\} \).

\[
(\langle ' \rangle) \quad \frac{(\langle \alpha \rangle) H \rightarrow (\theta \leq \lambda p) \quad H \rightarrow (\theta \geq 0) \quad H \rightarrow (d_t \theta \leq 0)}{P(\langle \alpha \rangle \langle dx = bd t + \sigma dW \& H \rangle \theta \geq \lambda) \leq p}
\]

7 Uniform Substitution

In order to make use of our axioms, we need to be able to instantiate them into specific formulas we want to reason about by substituting in concrete terms for function, program, and predicate symbols. A substitution \( \sigma \) maps each function symbol \( f_d \) to some term \( \phi \) containing \( d \) special 0-ary function symbols \( \cdot_{0}^{f_d} \) for \( i \in 1...d \), no other 0-ary function symbols, and no variables. Similarly it maps \( p_d \) to a formula \( \phi \) containing \( d \) special 0-ary function symbols \( \cdot_{0}^{p_d} \) for \( i \in 1...d \), no other 0-ary function symbols, and no variables. It maps \( \gamma \) to some program \( \alpha \) We specify a substitution by a list of mappings, with all unlisted symbols mapped to themselves (in the case of function or predicate symbols, themselves
applied to their appropriate \( \bullet \) symbols). Define the signature of a substitution \( \Sigma(\sigma) := \{ \text{symbols that } \sigma \text{ does not map to themselves} \} \).

\( \sigma \) acts on terms, programs, and formulas (written \( \sigma \tau \) for \( \tau \) a term, program, or formula) by simultaneously and recursively applying all its mappings to produce a new term, program, or formula respectively, as follows:

**Terms**

\[
\begin{align*}
\sigma(c) & \quad \quad c \\
\sigma(x) & \quad \quad x \\
\sigma(\theta \star \kappa) & \quad \quad [\sigma\theta] \star [\sigma\kappa] \\
\sigma(\theta + \kappa) & \quad \quad [\sigma\theta] + [\sigma\kappa] \\
\sigma(d_i(\theta)) & \quad \quad d_i([\sigma\theta]) \\
\sigma(d_{B,x}(\theta)) & \quad \quad d_{B,x}([\sigma\theta]) \\
\forall i \in (1...d) \bullet_0^i f_{i, \kappa} & \quad \quad [\sigma\theta_i] (\sigma f_d) \\
\sigma(\iota \phi_d) & \quad \quad \iota_i[\sigma\phi_d] \\
\end{align*}
\]

**Programs**

\[
\begin{align*}
\sigma(x_i := \theta) & \quad \quad x_i := [\sigma\theta] \\
\sigma(x_i := *) & \quad \quad x_i := * \\
\sigma(dx = b dt + \sigma dW \& H) & \quad \quad dx = [\sigma b] dt + [\sigma\sigma] dW \& [\sigma H] \\
\sigma(\text{if } H \text{ then } \alpha \text{ else } \beta) & \quad \quad \text{if } [\sigma H] \text{ then } [\sigma\alpha] \text{ else } [\sigma\beta] \\
\sigma(\alpha \star) & \quad \quad [\sigma\alpha]^* \\
\sigma(\gamma) & \quad \quad \sigma(\gamma) \\
\sigma(\text{fail}) & \quad \quad \text{fail} \\
\sigma(\text{skip}) & \quad \quad \text{skip} \\
\end{align*}
\]

**Formulas**

\[
\begin{align*}
\sigma(\theta \geq \kappa) & \quad \quad [\sigma\theta] \geq [\sigma\kappa] \\
\sigma(\neg \phi) & \quad \quad \neg[\sigma\phi] \\
\sigma(\phi \land \psi) & \quad \quad [\sigma\phi] \land [\sigma\psi] \\
\forall i \in (1...d) \bullet_0^i f_{i, \kappa} & \quad \quad [\sigma\theta_i] (\sigma f_d) \\
\sigma(\text{sure}(\phi)) & \quad \quad \text{sure}(\sigma[\sigma\phi]) \\
\sigma(\iota \phi_d) & \quad \quad \iota_i[\sigma\phi_d] \\
\end{align*}
\]

\( \sigma \) also acts on formulas in our probability meta-language; let \( \sigma(\bullet) \) be \( \bullet \) with \( P([\sigma\phi]) \) substituted for \( P(\phi) \).

Then instantiating axioms is done via the following rules, with the admissibility side-condition to be specified later:

\[
\begin{align*}
\text{US} \quad \frac{\phi}{\sigma[\phi]} \quad \sigma \text{ is } \phi\text{-admissible} \\
\text{USP} \quad \frac{\bullet}{\sigma(\bullet)} \quad \sigma \text{ is } \phi\text{-admissible for all } P(\phi) \text{ appearing in } \bullet
\end{align*}
\]

### 7.1 Read and Write Variables

For a value \( v \in \mathbb{R}_V \), and \( V' \subset V \), let \( v^{V'} \) be its projection onto the subspace without \( V' \). Let \( v^x \) be shorthand for \( v^{(x)} \). Let \( v(x) \) be the projection onto only \( x \).
Definition 2.

\[ \text{RV}(\theta) := \{ x \in V | \exists I, v_1, v_2. v_1^I = v_2^I, I v_1[\theta] \neq I v_2[\theta] \} \]

\[ \text{RV}(\alpha) := \{ x \in V | \exists I, v_1, v_2, v_1^I = v_2^I, I v_1[\alpha](\omega, C) = v_1' \} \]

\[ \text{RV}(\phi) := \{ x \in V | \exists I, v_1, v_2, v_1^I = v_2^I, I v_1[\phi](\omega) = v_2 \} \]

\[ \text{WV}(\alpha) := \{ x \in V | \exists I, v_1, v_2, v_1^I = v_2^I, I v_1[\alpha](\omega) = v_2, v_1(x) \neq v_2(x) \} \]

\[ \text{WV}(\phi) := \bigcup_{s \in \Sigma} \text{RV}(s) \cap \text{WV}(\sigma[s]) \]

Definition 3. The signature \( \Sigma \) of a term, program, or formula is set of function, predicate, program symbols it contains.

7.2 Admissibility Condition

Along the lines of \cite{13}, define the read variables introduced by \( \sigma \) to a program or formula \( e \),

\[ \text{RV}(\sigma, e) := \bigcup_{s \in \Sigma} \text{RV}(s) \cap \text{WV}(\sigma[s]) \]

\( \sigma \) is defined admissible for \( e \) when for any subexpression \( e' \) of \( e \),

\[ \text{RV}(\sigma, e) \cap \text{WV}(\sigma[e']) = \emptyset \]

Syntactic Approximations

Note that along the lines of the original US paper \cite{13}, we can syntactically compute over-approximations of the sets of read and write variables. Thus in some cases we can prove that substitutions are admissible solely syntactically.

7.3 Soundness

We follow the strategy of the original US paper with modifications to fit our semantics.

Adjoints Substitutions

For \( q \in \mathbb{R}^d_+ \), let \( I^q_{\sigma, f_d} \) be the interpretation that is the same as \( I \), but with \( \bullet^0_{\sigma, f_d, i} \) mapped to \( q(i) \) for all \( i \), and let \( I^q_{\sigma, p_d} \) be the interpretation that is the same as \( I \) but with \( \bullet^0_{\sigma, p_d, i} \) mapped to \( q(i) \).

Definition 4. Substitution Adjoint (Taken from paper)

The adjoint to substitution \( \sigma \) is the operation that maps \( I, v \) to the adjoint interpretation \( \sigma(I, v) \).

\[ \sigma(I, v)(f_d) = \lambda q : \mathbb{R}^d_+, I^q_{\sigma, f_d} v[\sigma f_d] \]

\[ \sigma(I, v)(p_d) = \lambda \omega : \mathbb{R}^d_+, I^q_{\sigma, p_d} v[\sigma p_d](\omega) \]

\[ \sigma(I, v)(\gamma) = \lambda v', \omega', C'. I v'[\sigma[\gamma]](\omega', C') \]

\[ \sigma(I, v)_{\text{times}} = I_{\text{times}} \]

Note that \( \sigma(I, v)(\gamma) \) does not depend on \( v \). The following corollary follows directly from the above definitions.
Corollary 1. (Equal Adjoint Interpretations) If for some $e$ a term, program, or formula symbol $v_1 = v_2$ on $\text{RV}(\sigma, e)$, then $\sigma(I, v_1)(e) = \sigma(I, v_2)(e)$. Therefore for any program or formula $e$, if $v_1 = v_2$ on $\text{RV}(\sigma, e)$, $e$ has the same semantics under the interpretations $\sigma(I, v_2)$ and $\sigma(I, v_1)$.

The following Lemmas are proven in Appendix A.

Lemma 3. Uniform substitution for terms For all $I, v$,

$$Iv[\sigma\theta] = \sigma(I, v)v[\theta]$$

Lemma 4. Uniform substitution for programs When $\sigma$ is admissible for $\alpha$, uniform substitution and its adjoint interpretation have the same semantics for all $I, v, \omega$:

$$Iv[\sigma\alpha](\omega, C) = \sigma(I, v)v[\alpha](\omega, C)$$

Lemma 5. Uniform substitution for formulas When $\sigma$ is admissible for $\phi$, The uniform substitution $\sigma$ and its adjoint interpretation have the same semantics for all $I, v, \omega$:

$$Iv[\sigma\phi](\omega) = \sigma(I, v)v[\phi](\omega)$$

US, USP US is thus sound: $T \models \phi$ if and only if for any $I$ with times a superset of $T$, for all $v, \omega$, $Iv[\phi] = \oplus$. If $\sigma$ is admissible for $\phi$ then by theorem 5 for any $I$ with times a superset of $T$, $Iv[\sigma\phi](\omega) = \sigma(I, v)v[\phi](\omega) = \oplus$ as $\sigma(I, v)$ has the same times as $I$.

We obtain the soundness of USP similarly.

8 Conclusion

We have given a logic for reasoning about stochastic hybrid programs, with a measurable semantics. To make it suitable for implementation in a practical proof-assistant, we have extended it with definite descriptions and differentials, and given it a proof calculus in a uniform substitution style.

8.1 Future Work

We have maintained the measurability of our semantics by adopting an all-or-nothing approach with respect to definite descriptions and differentials, where if the resulting semantics isn’t measurable we throw it out. It would be interesting to see if we could handle terms more delicately by only demanding that their semantics be restricted to the contexts of the formulas they appear in. Additionally, sufficient conditions for measurability of partial derivatives are given in [7], and it may be fruitful to incorporate them into our semantics.

We’ve presented just one proof rule for stochastic differential equations, based on prior work from [11] which leverages Doob’s Martingale inequality. There is a rich literature on concentration inequalities for martingales [15] as well as
inequalities on the solutions of SDEs, such as \cite{3}. We would like to derive sound proof rules based on these methods, and we believe that the differential terms of our languages will allow us to do so in a way that minimizes the need for semantic side-conditions.

Appendix A Proofs

Proof (Theorem 1). We proceed by structural induction. It is enough that the function $\hat{I}\theta$ be measurable; the intended result then follows from the structure of the $\sigma$-algebra on Val, and composing with $z$ as a measurable function.

The semantics of constants and variables satisfy this immediately. Term symbols, differentials, and definite descriptions do so by definition. Addition and multiplication follow from the inductive hypothesis.

Proof (Theorems 2, 3). We prove these theorems simultaneously by induction on the structure of programs and formulas.

Formulas: The measurability of inequalities follows from Theorem 1\cite{2}. The cases of negations and conjunctions, and sureness follow from the inductive hypothesis. The interpreted formula symbols are measurable by definition.

For $\langle \alpha \rangle \phi$, we have

$$\text{let } z_C(\omega) = \pi_1 I z(\omega) \|[\alpha]\| (\omega, C) \text{ in } \sup_{C \text{ st. } z_C(\omega) \neq \Delta} I z_C(\omega) \|[\phi]\| (\omega).$$

By \cite{3} and IH, each $z_C(\omega)$ must be measurable, so by IH so is $I z_C(\omega) \|[\phi]\| (\omega)$. The semantics here are then the pointwise supremum of countably many measurable functions, which is measurable \cite{7}.

Programs: Under a particular choice sequence, $\alpha \cup \beta$ and $\alpha^*$ behave as other programs, so we don’t need to consider these cases. $\gamma$ preserves measurability by definition. $\text{skip}$ acts as the identity transform on states, and $\text{fail}$ outputs a constant function, so these trivially preserve measurability. That $\alpha; \beta$ preserves measurability follows directly from the IH. Random variables and Ito integrals are measurable by definition. The semantics of assignments and conditionals can be rewritten as compositions of the semantics of terms, formulas, and programs with projections, and thus preserve measurability by the appropriate use of IH and \cite{1}.

Proof (Proof of Lemmas 3, 4, 5). We place the following well-founded partial order on substitutions: $\sigma_1 \sqsubseteq \sigma_2$ if $\Sigma(\sigma_1) \subset \text{sym}(\sigma_2)$ or if every element of $\Sigma(\sigma_1)$ is of the form $\sigma^d_\tau^{a,i}$ for $\tau_d$ some symbol $f_d$ or $p_d$ in $\text{sym}(\sigma_2)$.

The unique least substitution is then the identity substitution. Observe that from definition 4, the lemmas always hold when $\sigma$ is the identity substitution.

We proceed by mutual structural induction on terms, programs, and formulas, and simultaneously on the ordering on substitutions. Note that semantics where
\[v = \nabla \text{ don't change under reinterpretation, so below we just consider the cases when } v \in \mathbb{R}^V.\]

Let us start by considering terms:

- The semantics of constants and variables don’t change under different interpretations, and don’t substitutions don’t change them.
- The cases of additions, multiplications, and derivatives follow immediately by IH.
- As formulas appearing in definite descriptions contain no programs, they have no write variables, so \(\sigma\) is admissible for them and this case also follows by IH.

\[\text{let } v \in (1...d) \cdot f_{\alpha, i} \mapsto [\theta_\alpha] (v f_{\alpha, i}) (\omega)\]

\[\text{by definition. Then by cor. 1, the above expression}\]

Now consider programs:

- Differential equations follow similarly.
- The semantics of randomization, skip, fail don’t change under interpretation.
- \(\text{let } (v_\alpha, C_\alpha) = \text{IH} (iv\{\sigma\alpha\}, iv\{\sigma\alpha\}) (\omega, C_\alpha)\)

By definition, \(v = v_\alpha\) on \(\text{WV}(\sigma\alpha)\), which \(\supseteq \text{RV}(\sigma, \alpha; \beta)\) by admissibility, hence \(\supseteq \text{RV}(\sigma, \beta)\) by definition. Then by cor. 1, the above also holds.

- Conditional, union, star follow similarly from IH. Star requires a nested induction on \(n\) in \(\alpha^n\).

And finally formulas:

- For inequalities, this follows from Theorem 3 and IH.
- Since the write variables of \(\phi\) are a superset of the write variables of any of its subexpressions, \(\sigma\) must be admissible for each of its subexpressions. Then we can apply IH for the cases of conjunction, negation, and sureness.

\[\text{let } (v_\alpha, C_\alpha) = \text{IH} (iv\{\sigma\alpha\}, iv\{\sigma\alpha\}) (\omega, C_\alpha)\]

- By definition, \(v = v_\alpha\) on \(\text{WV}(\sigma\alpha)\), which \(\supseteq \text{RV}(\sigma, \alpha; \beta)\) by admissibility, hence \(\supseteq \text{RV}(\sigma, \beta)\) by definition. Then by cor. 1, the above also holds.

- Conditional, union, star follow similarly from IH. Star requires a nested induction on \(n\) in \(\alpha^n\).

And finally formulas:

- For inequalities, this follows from Theorem 3 and IH.
= σ(I, v)p_d(I_v[[σθ]]) .... I_v[[σθ]])(ω).
I^H = σ(I, v)p_d(σ(I, v)v[θ1] ... σ(I, v)v[θd]))(ω) = σ(I, v)][p_d(θ1 ... θd)](ω)
• I_v[[σ(α)φ]](ω) = I_v[[σα]]σφ](ω)

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