Wong–Zakai approximations for quasilinear systems of Itô’s type stochastic differential equations driven by fBm with $H > \frac{1}{2}$

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Abstract

In a recent article Lanconelli and Scorolli (2021) extended to the multidimensional case a Wong-Zakai-type approximation for Itô stochastic differential equations proposed by Øksendal and Hu (1996). The aim of the current paper is to extend the latter result to system of stochastic differential equations of Itô type driven by fractional Brownian motion like those considered by Hu (2018).

The covariance structure of the fractional Brownian motion (fBm) precludes us from using the same approach as that used by Lanconelli and Scorolli and instead we employ a truncated Cameron-Martin expansion as the approximation for the fBm. We are naturally led to the investigation of a semilinear hyperbolic system of evolution equations in several space variables that we utilize for constructing a solution of the Wong–Zakai approximated systems. We show that the law of each element of the approximating sequence solves in the sense of distribution a Fokker–Planck equation and that the sequence converges to the solution of the Itô equation, as the number of terms in the expansion goes to infinite.

Key words and phrases: Wong Zakai approximation, fractional Brownian motion, Wick product, Fokker Planck, stochastic differential equations.

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1 Introduction and statements of the main results.

Let $T > 0$ be some arbitrary real number, $\{W_t\}_{t \in [0,T]}$ be a standard one dimensional Brownian motion and for $\epsilon > 0$ denote with $\{W^{\epsilon}_t\}_{t \in [0,T]}$ some smooth stochastic process converging to the latter as $\epsilon \to 0$.

Under suitable conditions on the coefficients $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ the solution $\{Y^{\epsilon}(t)\}_{t \in [0,T]}$ of the random ordinary differential equation

$$\frac{dY^{\epsilon}(t)}{dt} = b(t, Y^{\epsilon}(t)) + \sigma(t, Y^{\epsilon}(t)) \cdot \frac{dW^{\epsilon}(t)}{dt}$$

(1.1)

converges as $\epsilon \to 0$ to the strong solution of the Stratonovich stochastic differential equation (SDE for short)

$$dY(t) = b(t, Y(t))dt + \sigma(t, Y(t)) \circ dW_t,$$

(1.2)

or equivalently of the Itô SDE

$$dY(t) = \left[ b(t, Y(t)) + \frac{1}{2} \sigma(t, Y(t)) \partial_y \sigma(t, Y(t)) \right] dt + \sigma(t, Y(t))dW_t.$$ 

This is the statement of the famous Wong-Zakai theorem [1] whose multidimensional version can be found in [2].

In [3] Øksendal and Hu suggested how to modify equation (1.1) to get in the limit the Itô’s interpretation of (1.2); they considered the case with $\sigma(t, x) = \sigma(t)x$, where $\sigma : [0, T] \to \mathbb{R}$ is a deterministic function, and proved that the solution $\{X^{\pi}(t)\}_{t \in [0,T]}$ of the differential equation

$$\frac{dX^{\pi}(t)}{dt} = b(t, X^{\pi}(t)) + \sigma(t)X^{\pi}(t) \cdot \frac{dW^{\pi}(t)}{dt},$$

converges, as $\pi$ tends to zero, to the strong solution $\{X(t)\}_{t \in [0,T]}$ of the Itô SDE

$$dX(t) = b(t, X(t))dt + \sigma(t)X(t)dW(t).$$

(1.3)

Here, the symbol $X^{\pi}(t) \cdot \frac{dW^{\pi}(t)}{dt}$ stands for the Wick product between $X^{\pi}(t)$ and $\frac{dW^{\pi}(t)}{dt}$. (We postpone to the next section all the necessary mathematical details for the tools utilized in this introduction). In [4] Lanconelli and Scorolli presented a multidimensional version of this result in which the linear diffusion matrix diagonal and the $d$-dimensional Brownian motion is approximated by a piecewise interpolation also known as polygonal approximation; just as in [3] the product between diffusion coefficients and smoothed white noise is interpreted as a Wick product.

The aim of this article is to extend the aforementioned result to the case of a quasilinear system of SDEs driven by a fractional Brownian motion (fBm for short) with Hurst...
paramenter $H > 1/2$ as those considered by Hu in [5]. In order to do that we introduce
the following Cauchy problem
\begin{equation}
\begin{aligned}
\frac{dX^K(t)}{dt} &= b_i(t, X^K(t)) + \sigma_i(t)X^K(t) \diamond \frac{dB^K(t)}{dt}, \quad t \in [0, T] \\
X_i(0) &= c_i \in \mathbb{R}, \quad \text{for } i \in \{1, \ldots, d\},
\end{aligned}
\end{equation}
(1.4)
where $\{B^K(t)\}_{t \in [0, T]}$ stands for the truncated (up to the $K$-th term) Cameron-Martin expansion of the $d$-dimensional fractional Brownian motion $\{B(t)\}_{t \in [0, T]}$ (the reason for which we have to employ a different approximation from that used in [4] will be addressed latter on), the coefficients $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \to \mathbb{R}^d$ satisfy certain condition which will be specified in the following, while $c \in \mathbb{R}^d$ is a deterministic initial condition. In order to ease the notation we have omitted (and will do so for the rest of the article) the Hurst parameter $H > 1/2$.

We must interpret (1.4) as Wong-Zakai approximation of the stochastic Cauchy problem of the Itô type:
\begin{equation}
\begin{aligned}
dX_i(t) &= b_i(t, X(t))dt + \sigma_i(t)X_i(t)dB_i(t), \quad t \in [0, T] \\
X_i(0) &= c_i \in \mathbb{R}, \quad \text{for } i \in \{1, \ldots, d\}.
\end{aligned}
\end{equation}
(1.5)
It’s important to stress the fact that (1.4) is not a system of random ordinary differential equations, but rather an evolution equation involving an infinite dimensional gradient (see for instance [5, equation 1.5]).

All throughout this article we will assume that the coefficients $b$ and $\sigma$ posses enough regularity to ensure that the Cauchy problem (1.5) has a unique strong solution (e.g. [5]).

**Assumption 1.1.**

- The functions $b(t, x)$, $\partial_{x_1} b(t, x), \ldots, \partial_{x_d} b(t, x)$ are bounded and continuous;
- the functions $\sigma_1(t), \ldots, \sigma_d(t)$ are bounded and continuous.

We are now ready to state our main results; the first of which ensures the existence of a solution for the approximating equation.

**Theorem 1.2 (Existence).** Let Assumption [1.1] be in force. Then (1.4) has a mild solution in the sense of definition [2.2].

Our second theorem states that the law of the approximation solves a Fokker-Planck-like equation.

**Theorem 1.3 (Fokker-Planck equation).** The law
\[ \mu^K(t, A) := \mathbb{P}(\{\omega \in \Omega : X^K(t, \omega) \in A\}), \quad t \in [0, T], A \in \mathcal{B}(\mathbb{R}^d) \]
of the random vector $X^K(t)$ solves in the sense of distributions the Fokker-Planck equation
\[
\left( \partial_t + \sum_{i,j=1}^d \sum_{k=1}^K \sigma_i(t)\xi_k(t)x_i g^{(j)}_{ik}(t,x) \partial^2_{x_i x_j} + \sum_{i=1}^d b_i(t,x) \partial_{x_i} \right) u(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^d
\]
(1.6)
where $g^{(j)}_{ik} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function defined in (5.1).

The third and last theorem states that the approximation indeed converges to the strong solution of the Itô SDE;

**Theorem 1.4 (Convergence).** The mild solution $\{X^K(t)\}_{t \in [0,T]}$ converges as $K$ tends to infinite, to the unique strong solution $\{X(t)\}_{t \in [0,T]}$ of the Itô SDE (1.5). More precisely,
\[
\lim_{K \to \infty} \sum_{i=1}^d \mathbb{E} \left[ \left| X^K_i(t) - X_i(t) \right| \right] = 0, \quad \text{for all } t \in [0,T].
\]

## 2 Preliminaries

### 2.1 Elements on fractional Brownian motion.

In this section we will introduce the basic concepts that will be needed in order to prove our results. For further details the interested reader is referred to the excellent references [8][13][10].

Start by fixing $H \in (\frac{1}{2}, 1)$, and let $\Omega := C_0([0,T], \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions endowed with the topology of uniform convergence. There is a probability measure $P^H$ on $(\Omega, \mathcal{B}(\Omega))$, such that on $(\Omega, \mathcal{B}(\Omega), P^H)$ the coordinate process $B : \Omega \to \mathbb{R}^d$ defined as
\[
B(t, \omega) = \omega(t), \quad \omega \in \Omega
\]
is a $d$-dimensional fBm, i.e. a $d$-dimensional centered Gaussian stochastic process in which for each $i \in \{1, ..., d\}$ it holds that
\[
\mathbb{E} [B_i(t)B_i(s)] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad s, t \in [0,T].
\]

Let
\[
\phi(s,t) := H(2H-1)|s-t|^{2H-2}, \quad s, t \in [0,T],
\]
and define
\[
\mathcal{H}_\phi := \left\{ f : [0,T] \to \mathbb{R} : |f|_\phi^2 = \int_0^T \int_0^T f(s)f(t)\phi(s,t)dsdt < \infty \right\}.
\]
If $\mathcal{H}_\phi$ is equipped with the inner product

$$\langle f, g \rangle_\phi = \int_0^T \int_0^T f(s)g(t)\phi(s,t)dsdt,$$

then it becomes a separable Hilbert space, moreover we can see that $\mathcal{H}_\phi$ equals the closure of $L^2([0,T])$ with respect to the inner product $\langle \cdot, \cdot \rangle_\phi$. For $f \in \mathcal{H}_\phi$ we denote with $\Phi[f] : [0,T] \to \mathbb{R}$ the following continuous map

$$[0,T] \ni t \mapsto \int_0^T f(s)\phi(t,s)ds.$$

For a deterministic function $f \in \mathcal{H}_\phi$ we can define in the usual manner a fractional Wiener integral satisfying the following isometry property

$$\mathbb{E} \left[ \left( \int_0^T f(s)dB_i(s) \right)^2 \right] = |f|_\phi^2. \quad (2.2)$$

For $f \in \mathcal{H}_\phi$ and $i \in \{1,\ldots,d\}$ define the stochastic exponential of $f$ by

$$\mathcal{E}_i(f) := \exp \left\{ \int_0^T f(s)B_i(s) - \frac{1}{2}|f|_\phi^2 \right\},$$

then the linear span of $\{\mathcal{E}_i(f); f \in \mathcal{H}_\phi, i \in \{1,\ldots,d\}\}$ is dense in $L^2(\Omega)$.

### 2.2 Approximating equation.

The first step when constructing a Wong-Zakai approximation is to choose a sequence of smooth stochastic processes converging to the Brownian motion that drives the original equation.

**Remark 2.1.** In [4] we have employed the so called polygonal approximation; however the non-independence of the increments of the fBm precludes us from using this approach (see [4, Remark 3.2]).

Instead let’s assume that $\{e_k\}_{k \geq 1}$ is a complete orthonormal system (CONS) of the Hilbert space $\mathcal{H}_\phi$, then (e.g. [6, equation 3.21]) the $d$-dimensional fractional Brownian motion has the following Cameron-Martin expansion

$$B_i(t) = \sum_{k=1}^\infty \left[ \int_0^t \left( \int_0^T e_k(r)\phi(v,r)dr \right) dv \right] \int_0^T e_k(s)dB_i(s), \ t \in [0,T], \ \text{for} \ i \in \{1,\ldots,d\}.$$

From this expression it’s then straightforward to see that a natural approximation for the fractional white noise is given by

$$\frac{dB_i^K(t)}{dt} := \sum_{k=1}^K \left( \int_0^T e_k(r)\phi(t,r)dr \right) \int_0^T e_k(s)dB_i(s), \ t \in [0,T], \ \text{for} \ i \in \{1,\ldots,d\};$$
i.e. the time derivative of the truncated Cameron-Martin expansion. The convergence of this object to the “singular fractional white noise” as \( K \) goes to infinity must be understood in a space of generalized random variables (see [6] for further details).

Notice that due to (2.2) and the orthonormality of \( \{e_k\}_{k \geq 1} \) if we let \( Z_k^{(i)} := \int_0^T e_k(s)dB^i_H(s) \), then \( \left(Z_k^{(i)}\right)_{(k,i) \in \{1,\ldots,K\} \times \{1,\ldots,d\}} \) is a family of i.i.d. standard Gaussian random variables.

For the ease of notation let \( \xi_k(\cdot) = \Phi[e_k](\cdot) \) for all \( k \in \{1,\ldots,K\} \) and hence our approximation for the fractional white noise can be written as

\[
\frac{dB^K_i(t)}{dt} = \sum_{k=1}^K \xi_k(t)Z_k^{(i)}, \quad t \in [0,T], \quad \text{for } i \in \{1,\ldots,d\}.
\]

One of the key tools that will be employed in this article is the so called Wick product which can be defined for any couple of random variables \( X \) and \( Y \) belonging to \( L^p(\Omega) \) for some \( p > 1 \), (e.g. [7],[8],[9]). For our purposes it’s enough to consider just a few particular cases:

- if \( X \in L^p(\Omega) \) for some \( p > 1 \) and \( f \in H_\phi \) we set
  \[
  X \diamond \mathcal{E}_i(f) := T_{-\Phi[f]}X \cdot \mathcal{E}_i(f)
  \]
  where \( T_{-\Phi[f]} \) stands for the translation operator
  \[
  (T_{-\Phi[f]}X)(\omega) := X \left( \omega - \epsilon_i \int_0^\omega \Phi[f](r)dr \right).
  \]
  Here \( \{\epsilon_1, \ldots, \epsilon_d\} \) denotes the canonical basis of \( \mathbb{R}^d \). This is the fractional analog of the so called Gjessing lemma ([7] Theorem 2.10.6). From the latter we are able to see that the Wick product with a stochastic exponential preserves the monotonicity, i.e. :

  if \( X \leq Y \) then \( X \diamond \mathcal{E}_i(f) \leq Y \diamond \mathcal{E}_i(f) \).

- A consequence of the latter and the density of the stochastic exponentials is that if \( g \in H_\phi, \ F \in L^p(\Omega) \) and \( \langle D^{(i)}_\phi F, g \rangle \in L^p(\Omega) \) for some \( p > 1 \) then
  \[
  F \diamond \int_0^T g(s)dB^i(s) = F \int_0^T g(s)dB^i(s) - \langle D^{(i)}_\phi F, g \rangle
  \]
  where \( D^{(i)}_\phi \) denotes the \( \phi \)-derivative (e.g. [6],[10]) with respect to the \( i \)-th fBm and \( \langle \cdot, \cdot \rangle \) denotes the inner product of \( L^2([0,T]) \).

With all this in hand we are able to provide a solution concept for (1.4):
Definition 2.2. A $d$-dimensional stochastic process $\{X^K(t)\}_{t \in [0,T]}$ is said to be a mild solution of equation (1.4) if:

1. the function $t \mapsto X^K(t)$ is almost surely continuous;

2. for all $i \in \{1, \ldots, d\}$ and $t \in [0,T]$, the random variable $X^K_i(t)$ belongs to $L^p(\Omega)$ for some $p > 1$;

3. for all $i \in \{1, \ldots, d\}$, the identity

$$X^K_i(t) = c_i \mathcal{E}^K_i(0,t) + \int_0^t b_i(s, X^K(s)) \circ \mathcal{E}^K_i(s,t) ds, \quad t \in [0,T],$$

holds almost surely, where for any $t, r \in [0,T], r \leq t$, $\mathcal{E}^K_i(r,t)$ is a shorthand for $\mathcal{E}_i(\sigma^K_i(r,t))$ where $\sigma^K_i(r,t)$ denotes the orthogonal projection of $X_{[r,t]} \sigma_i(\cdot)$ on $\text{span}\{e_1, \ldots, e_K\} \subset \mathcal{H}_\phi$.

3 Proof of theorem 1.2

In order to prove the existence of a mild solution for (1.4) we will introduce a system of partial differential equations which is related to (1.4) by the following heuristic considerations.

Remark 3.1. Formally applying identity (2.5) we can rewrite (1.4) as

$$\begin{cases}
\frac{dX^K_i(t)}{dt} = b_i(t, X^K(t)) + \sigma_i(t) X_i(t) \cdot \left( \sum_{k=1}^K \xi_k(t) Z_k^{(i)} \right) - \sum_{k=1}^K \sigma_i(t) \xi_k(t) \left\langle D^{(i)}_\phi X_i(t), e_k \right\rangle, \\
t \in [0,T] \\
X_i(0) = c_i \in \mathbb{R}, \text{ for } i \in \{1, \ldots, d\}.
\end{cases}$$

If we now search for a solution of the form

$$X^K_i(t, \omega) = u_i(t, z(\omega)),$$

for $u_i : [0,T] \times \mathbb{R}^{K \times d} \to \mathbb{R}$ where we identify $\mathbb{R}^{K \times d}$ with the space of $(K \times d)$ and $z_{ki}(\omega) = Z_k^{(i)}(\omega)$ then by a simple application of the chain rule for the $\phi$-derivative we see that $u = (u_1, \ldots, u_d)$ has to solve the following semilinear hyperbolic system of partial differential equations

$$\begin{cases}
\partial_t u_i = b_i(t, u) + \sigma_i(t) \sum_{k=1}^K \xi_k(t) [x_{ki} u_i - \partial_{x_k} u_i], \\
(t, x) \in [0,T] \times \mathbb{R}^{K \times d}, \\
u_i(0, x) = c_i \in \mathbb{R}, \text{ for } i \in \{1, \ldots, d\}.
\end{cases}$$

Unfortunately to the best of our knowledge the latter does not satisfy the basic assumption of the main existence-uniqueness theorems present in the literature. For that reason
we will introduce the following auxiliary Cauchy problem

$$\begin{cases}
\partial_t v_i = b_i(t, v(t)) \exp \left\{ \frac{1}{2} \| x \|^2_F \right\} \exp \left\{ - \frac{1}{2} \| x \|^2_F \right\} - \sigma_i(t) \sum_{k=1}^K \xi_k(t) \partial_{x_i} v_i, \\
(t, x) \in [0, T] \times \mathbb{R}^{K \times d}, \\
v_i(0, x) = c_i \exp \left\{ - \frac{1}{2} \| x \|^2_F \right\}, \text{ for } i \in \{1, \ldots, d\},
\end{cases} \tag{3.2}$$

where \( \| \cdot \|_F \) denotes the Frobenious norm, i.e. \( \| x \|^2_F := \sum_{i=1}^d \sum_{k=1}^K |x_{ki}|^2 \). Our motivation for doing so will be clear in a moment.

A closer inspection would allow the reader to see that the latter is a \( d \)-dimensional semi-linear symmetric hyperbolic system of evolution equations in \((K \times d)\) spatial variables. The validity of assumption 1.1 implies the existence of a unique classical solution of (3.2) for any arbitrary time interval \([0, T]\) (see for instance [11], [12] and [13]).

If we let

$$\Sigma_{i,k}(r, t) := \int_r^t \sigma_i(s) \xi_k(s) ds = \langle \chi_{\{r, t\}} \sigma_i, e_k \rangle_\phi, \tag{3.3}$$

then we can write down a mild solution for (3.2) as

$$\begin{cases}
v_i(t, x) = c_i \exp \left\{ - \frac{1}{2} \| x - \Sigma^{(i)}(t) \|^2_F \right\} \\
+ \int_0^t b_i(s, v(s, x - \Sigma^{(i)}(s, t))) \exp \left\{ \frac{1}{2} \| x - \Sigma^{(i)}(s, t) \|^2_F \right\} \exp \left\{ - \frac{1}{2} \| x - \Sigma^{(i)}(s, t) \|^2_F \right\} ds,
\end{cases} \tag{3.4}$$

for \( t \in [0, T], x \in \mathbb{R}^{K \times d}, i \in \{1, \ldots, d\} \).

where for any pair \( r, t \in [0, T], t \geq r \), we denote with \( \Sigma^{(i)}(r, t) \) the \((K \times d)\)-matrix where the \( i \)-th column is given by \( [\Sigma_{i,1}(r, t), \ldots, \Sigma_{i,K}(r, t)]^T \) and all the remaining components are equal 0.

Now if we let \( u_i(t, x) := v_i(t, x) \exp \left\{ \frac{1}{2} \| x \|^2_F \right\} \) for all \( i \in \{1, \ldots, d\} \) a simple application of the chain rule shows that \( u \) solves (3.1), furthermore using (3.1) we have that the following mild representation holds

$$\begin{cases}
u_i(t, x) = c_i \exp \left\{ \sum_{k=1}^K \left[ x_{ik} \Sigma_{i,k}(0, t) - \frac{1}{2} |\Sigma_{i,k}(0, t)|^2 \right] \right\} \\
+ \int_0^t b_i(s, u(s, x - \Sigma^{(i)}(s, t))) \exp \left\{ \sum_{k=1}^K \left[ x_{ik} \Sigma_{i,k}(s, t) - \frac{1}{2} |\Sigma_{i,k}(s, t)|^2 \right] \right\} ds,
\end{cases} \tag{3.5}$$

for \( t \in [0, T], x \in \mathbb{R}^{K \times d}, i \in \{1, \ldots, d\} \).

**Remark 3.2.** This equation is the analog of [4] equation 3.4, where instead of shifting the \( i \)-th component of the vector of spatial variables we shift the \( i \)-th column of the matrix of spatial variables.
At this point we define the candidate solution \( \{ X^K(t) \}_{t \in [0,T]} \) as
\[
X^K_i(t, \omega) = u_i(t, z(\omega)), \ t \in [0,T], \omega \in \Omega, \text{ for } i \in \{1, \ldots, d\}
\] (3.6)
where again \( z(\omega) \) is the \( K \times d \)-matrix in which the \((k, i)\)-th component is given by \( Z^{(i)}_k(\omega) \).

Next we must verify that \( \{ X^K(t) \}_{t \in [0,T]} \) is indeed a mild solution of the system (1.4), i.e. that satisfies the conditions imposed by definition 2.2.

The almost surely continuity of the path is given by the continuity of \([0,T] \ni t \mapsto \exp \sum_{k=1}^{K} \left[ x_{ik} \sum_{i,k} \left( r, t \right) - \frac{1}{2} \left| \sum_{i,k} \left( r, t \right) \right|^2 \right] \bigg|_{x_{ki} = Z^{(i)}_k(\omega)} \]
\[
= \exp \left\{ \int_0^T \sum_{k=1}^{K} \left( \int_r^t \sigma_i(s) \xi_k(s) ds \right) e_k(q) dB^H_i(q) - \frac{1}{2} \sum_{k=1}^{K} \left( \int_r^t \sigma_i(s) \xi_k(s) ds \right)^2 \right\}
\]
\[
= \exp \left\{ \int_0^T \sum_{k=1}^{K} \left( \chi_{[r,t]} \sigma_i, \xi_k, e_k(q) dB^H_i(q) - \frac{1}{2} \sum_{k=1}^{K} \left| \chi_{[r,t]} \sigma_i, e_k(q) \right|^2 \right) \right\}
\]
\[
= \exp \left\{ \int_0^T \sigma_i^K(r, t; q) dB^H_i(q) - \frac{1}{2} \left| \sigma_i^K(r, t; \cdot) \right|^2 \right\} =: \xi^K_i(r, t), \tag{3.7}
\]
and using assumption 1.1 we have that
\[
|X^K_i(t)| \leq |c_i| \left\{ \int_0^T \sigma_i^K(0, t; q) dB^H_i(q) \right\} + M \int_0^t \exp \left\{ \int_0^T \sigma_i^K(s, t; q) dB^H_i(q) \right\} ds,
\]
where \( M > 0 \) is a constant such that \( |b(t, x)| \leq M \). Taking the \( L^p(\Omega) \)-norm on both sides above and using the triangular inequality we obtain
\[
\|X^K_i(t)\|_p \leq |c_i| \left\{ \int_0^T \sigma_i^K(0, t; q) dB^H_i(q) \right\} + M \int_0^t \left\{ \int_0^T \sigma_i^K(s, t; q) dB^H_i(q) \right\} ds
\]
\[
\leq |c_i| \left\{ \frac{p}{2} \left| \sigma_i^K(0, t; \cdot) \right|^2 \right\} + M \int_0^t \exp \left\{ \frac{p}{2} \left| \sigma_i^K(s, t; \cdot) \right|^2 \right\} ds
\]
\[
\leq |c_i| \left\{ \frac{p}{2} \left| \sigma_i^K(0, t; \cdot) \right|^2 \right\} + M t \sup_{s \in [0,t]} \exp \left\{ \frac{p}{2} \left| \sigma_i^K(s, t; \cdot) \right|^2 \right\} < \infty,
\]
where we used the fact that the fractional Wiener integral of \( \sigma_i^K(r, t; \cdot) \) is a Gaussian random variable. This proves the membership of \( X^K_i(t) \) to \( L^p(\Omega) \), \( p \geq 1 \) for all \( i \in \{1, 2, \ldots, d\} \) and \( t \in [0,T] \)
Last thing we need to do is to prove that the process $X^K(t)$ satisfies the representation (2.6). First we notice that for any $l \in \{1, \ldots, K\}$ and $i \in \{1, \ldots, d\}$

$$Z^{(i)}_l - \Sigma_{i,l}(s, t) = Z^{(i)}_l - \sum_{k=1}^{K} \langle \chi_{[s,t]} \sigma_i, e_k \rangle \phi \int_0^T \int_0^T e_k(r) e_i(q) \phi(r, q) drdq = T_{-\Phi[\sigma^K_i(s,t)]} Z^{(i)}_l$$

where $T_{-\Phi[\sigma^K_i(s,t)]}$ is a shorthand for $T_{-\Phi[\sigma^K_i(s,t)]}$. Then it follows from (3.5) and (3.7) that

$$X^K_i(t) = c_i E_i^K(0, t) + \int_0^T T_{-\Phi[\sigma^K_i(s,t)]} b_i(s, X^K(s)) \cdot E^K_i(s, t) ds.$$ 

Using identity (2.3) we have that for all $t \in [0, T]$ and $i \in \{1, \ldots, d\}$ the following holds a.s.

$$X^K_i(t) = c_i E^K_i(0, t) + \int_0^t b_i(s, X^K(s)) \cdot E^K_i(s, t) ds;$$

completing the proof.

4 Proof of theorem 1.4

The aim of this section is to show that the mild solution of

$$\begin{cases}
\frac{dX^K_i(t)}{dt} = b_i(t, X^K(t)) + \sigma_i(t) X^K_i(t) \odot dB^K_i(t), & t \in [0, T] \\
X_i(0) = c_i \in \mathbb{R}, & \text{for } i \in \{1, \ldots, d\};
\end{cases}$$

converges in $L^1(\Omega)$ to the unique strong solution of

$$\begin{cases}
dX_i(t) = b_i(t, X(t))dt + \sigma_i(t) X_i(t)dB_i(t), & t \in [0, T] \\
X_i(0) = c_i \in \mathbb{R}, & \text{for } i \in \{1, \ldots, d\}.
\end{cases}$$

The first thing we must do is to rewrite the solution of the latter stochastic Cauchy problem in a way that resembles identity (2.6). To that aim we will mimic a reduction method proposed by [3] which is based on techniques from the Wick calculus and fractional White noise (see also [7, theorem 3.6.1]). We stress the fact that the following steps are somewhat formal in our setting but can be performed rigorously under some more technical considerations (see for instance [6] or [14]).

Let

$$E_i(0, t) := \exp \left\{ - \int_0^t \sigma_i(s)dB_i(s) - \frac{1}{2} \chi_{[0,t]} |\sigma_i|^2 \right\},$$
and
\[ \mathcal{E}_i(0,t) := \exp \left\{ \int_0^t \sigma_i(s)dB_i(s) - \frac{1}{2} \chi_{[0,t]} \sigma_i^2 \right\}. \]

Using equation (3.41) of we can formally write (1.5) as
\[
\begin{cases}
\frac{dX_i(t)}{dt} = b_i(t, X(t)) + \sigma_i(t)X_i(t) \cdot \frac{dB_i(t)}{dt}, \ t \in [0, T] \\
X_i(0) = c_i, \ \text{for} \ i \in \{1, \ldots, d\},
\end{cases}
\]

were we must bare in mind that the time derivative of the fBm is not well defined as a random variable, so in order to make sense of the expression above we must interpret it as a differential equation in some space of generalized random variables (e.g. [13][6]). Then we can Wick-multiply both sides of the equality above by \( E_i(0, t) \) which gives, after rearranging
\[
\frac{dX_i(t)}{dt} \cdot E_i(0, t) - \sigma_i(t)X_i(t) \cdot \frac{dB_i(t)}{dt} \cdot E_i(0, t) = b_i(t, X(t)) \cdot E_i(0, t).
\]

By means of the identity
\[
\frac{dE_i(0,t)}{dt} = \sigma_i(t)E_i(0,t) \cdot \frac{dB_i(t)}{dt}.
\]
and the Leibniz rule for the Wick product we obtain
\[
\frac{dX_i(t)}{dt} = b_i(t, X(t)) \cdot E_i(0, t), \tag{4.1}
\]
where
\[
\mathcal{X}_i(t) := X_i(t) \cdot E_i(0, t).
\]

It follows that
\[
\mathcal{X}_i(t) = X_i(t) \cdot E_i(0, t) = c_i + \int_0^t b_i(s, X(s)) \cdot E_i(0, s)ds
\]
or which is equivalent
\[
X_i(t) = c_iE_i(0, t) + \int_0^t b_i(s, X(s)) \cdot E_i(s, t)ds
\]
where we used the identity
\[
E_i(0, t) \cdot E_i(0, t) = 1, \ a.s. \ for \ all \ t \in [0, T].
\]

We conclude that the unique strong solution of (1.5) satisfies the following integral equation
\[
X_i(t) = c_iE_i(0, t) + \int_0^t b_i(s, X(s)) \cdot E_i(s, t)ds \tag{4.2}
\]
for all \( t \in [0, T] \) and \( i \in \{1, \ldots, d\} \).
Remark 4.1. Under assumption (1.4) it follows that for any $t \in [0, T]$ the strong solution of (1.5) belongs to $L^p(\Omega)$ for any $p \geq 1$.

Now we are ready to prove the convergence:

$$|X_i(t) - X_i^K(t)| \leq |c_i||\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)|$$

$$+ \int_0^t |b_i(s, X(s)) \cdot \mathcal{E}_i(s, t) - b_i(s, X^K(s)) \cdot \mathcal{E}_i^K(s, t)|ds$$

$$= |c_i||\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)|$$

$$+ \int_0^t |b_i(s, X(s)) \cdot \mathcal{E}_i(s, t) - b_i(s, X(s)) \cdot \mathcal{E}_i^K(s, t)|ds$$

$$+ b_i(s, X(s)) \cdot \mathcal{E}_i^K(s, t) - b_i(s, X^K(s)) \cdot \mathcal{E}_i^K(s, t)|ds$$

$$\leq |c_i||\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)|$$

$$+ \int_0^t |b_i(s, X(s)) \cdot (\mathcal{E}_i(s, t) - \mathcal{E}_i^K(s, t))|ds$$

$$+ \int_0^t |b_i(s, X(s)) - b_i(s, X^K(s))| \cdot \mathcal{E}_i^K(s, t)ds.$$

Using the Lipschitz continuity of $b_i$ and the fact that the Wick product with a stochastic exponential preserves the monotonicity we have that

$$|X_i(t) - X_i^K(t)| \leq |c_i||\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)| + \int_0^t |b_i(s, X(s)) \cdot (\mathcal{E}_i(s, t) - \mathcal{E}_i^K(s, t))|ds$$

$$+ L \int_0^t \sum_{j=1}^d |X_j(s) - X_j^K(s)| \cdot \mathcal{E}_i^K(s, t)ds$$

where $L$ is a positive constant such that for all $t \in [0, T]$ it holds $|b_i(t, X) - b_i(t, Y)| \leq L|X - Y|_1$; here $| \cdot |_1$ denotes the $\ell^1$ norm. Now we take expectation yielding

$$\mathbb{E} \left[|X_i(t) - X_i^K(t)|\right] \leq |c_i|\mathbb{E} \left[|\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)|\right] + \int_0^t \mathbb{E} \left[|b_i(s, X(s)) \cdot (\mathcal{E}_i(s, t) - \mathcal{E}_i^K(s, t))|\right]ds$$

$$+ L \int_0^t \sum_{j=1}^d \mathbb{E} \left[|X_j(s) - X_j^K(s)|\right]ds.$$

The previous inequality is valid for all $i = 1, \ldots, d$ and $t \in [0, T]$; therefore, summing over $i$ and setting

$$X^K(t) := \sum_{i=1}^d \mathbb{E} \left[|X_i(t) - X_i^K(t)|\right]$$
we obtain
\[ X^K(t) \leq \sum_{i=1}^{d} |c_i| \mathbb{E} \left[ |\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)| \right] + \sum_{i=1}^{d} \int_{0}^{t} \mathbb{E} \left[ |b_i(s, X(s)) \circ (\mathcal{E}_i(s, t) - \mathcal{E}_i^K(s, t))| \right] ds \\
+ Ld \int_{0}^{t} X^K(s) ds
\]
\[ = M^K(t) + Ld \int_{0}^{t} X^K(s) ds, \]
where
\[ M^K(t) := \sum_{i=1}^{d} |c_i| \mathbb{E} \left[ |\mathcal{E}_i(0, t) - \mathcal{E}_i^K(0, t)| \right] + \sum_{i=1}^{d} \int_{0}^{t} \mathbb{E} \left[ |b_i(s, X(s)) \circ (\mathcal{E}_i(s, t) - \mathcal{E}_i^K(s, t))| \right] ds. \]

According to Gronwall’s inequality the previous estimate yields
\[ X^K(t) \leq M^K(t) + Ld \int_{0}^{t} M^K(s)e^{Lt} ds; \quad (4.3) \]
and hence the proof will be complete if we show that $M^K(t)$ is bounded for all $t \in [0, T]$ and it holds that
\[ \lim_{K \to \infty} M^K(t) = 0, \quad \text{for all } t \in [0, T]; \]
this will allow us to use dominated convergence for the Lebesgue integral appearing in $\text{(4.3)}$ and conclude that
\[ \lim_{K \to \infty} X^K(t) = 0. \]

In order to prove the boundedness we write
\[ M^K(t) \leq \sum_{i=1}^{d} |c_i| \left( \mathbb{E} \left[ |\mathcal{E}_i(0, t)| \right] + \mathbb{E} \left[ |\mathcal{E}_i^K(0, t)| \right] \right) \\
+ \int_{0}^{t} \mathbb{E} \left[ |b_i(s, X(s)) \circ \mathcal{E}_i(s, t) - b_i(s, X(s)) \circ \mathcal{E}_i^K(s, t)| \right] ds \\
\leq 2 \sum_{i=1}^{d} |c_i| + \sum_{i=1}^{d} \int_{0}^{t} \mathbb{E} \left[ |b_i(s, X(s)) \circ \mathcal{E}_i(s, t)| \right] + \mathbb{E} \left[ |b_i(s, X(s)) \circ \mathcal{E}_i^K(s, t)| \right] ds \\
\leq 2 \sum_{i=1}^{d} |c_i| + 2dMt; \]
the boundedness also follows from the continuity of $t \mapsto M^K(t)$ and the compactness of $[0, T]$. 

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Thus it suffices to prove that
\[
\lim_{K \to \infty} \mathcal{M}^K(t) = 0, \quad \text{for all } t \in [0, T].
\]

Using the fact that \( \mathcal{E}_i^K(0, t) \) converges in \( L^p(\Omega), p \geq 1 \) to \( \mathcal{E}_i(0, t) \) (see Appendix A) it follows that
\[
\lim_{K \to \infty} \mathcal{M}^K(t) = \lim_{K \to \infty} \sum_{i=1}^d c_i \mathbb{E} \left[ \left| \mathcal{E}_i^K(0, t) - \mathcal{E}_i(0, t) \right| \right] + \lim_{K \to \infty} \sum_{i=1}^d \int_0^t \mathbb{E} \left[ \left| b_i(s, X(s)) \circ (\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)) \right| \right] ds
\]
\[
= \sum_{i=1}^d \lim_{K \to \infty} \int_0^t \mathbb{E} \left[ \left| b_i(s, X(s)) \circ (\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)) \right| \right] ds.
\]

We now prove that we can take the last limit inside the integral; first of all, note that the integrand is bounded: in fact,
\[
\mathbb{E} \left[ \left| b_i(s, X(s)) \circ (\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)) \right| \right] = \mathbb{E} \left[ \left| b_i(s, X(s)) \circ \mathcal{E}_i^K(s, t) - b_i(s, X(s)) \circ \mathcal{E}_i(s, t) \right| \right]
\]
\[
\leq \mathbb{E} \left[ \left| b_i(s, X(s)) \circ \mathcal{E}_i^K(s, t) \right| \right] + \mathbb{E} \left[ \left| b_i(s, X(s)) \circ \mathcal{E}_i(s, t) \right| \right]
\]
\[
= \mathbb{E} \left[ \left| b_i(s, X(s)) \circ \mathcal{E}_i^K(s, t) \right| \right] + \mathbb{E} \left[ \left| b_i(s, X(s)) \right| \cdot \mathbb{E} \left[ \left| \mathcal{E}_i(s, t) \right| \right] \right]
\]
\[
\leq 2M.
\]

We proceed by proving that
\[
\lim_{K \to \infty} \mathbb{E} \left[ \left| b_i(s, X(s)) \circ (\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)) \right| \right] = 0.
\]

Let us rewrite the expected value as follows:
\[
\mathbb{E} \left[ \left| b_i(s, X(s)) \circ (\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)) \right| \right] = \mathbb{E} \left[ \left| b_i(s, X(s)) \circ \mathcal{E}_i^K(s, t) - b_i(s, X(s)) \circ \mathcal{E}_i(s, t) \right| \right]
\]
Using (2.3) we get rid of the Wick product and write
\[
= \mathbb{E} \left[ \left| T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} b_i(s, X(s)) | \mathcal{E}_i^K(s, t) - \mathbf{T}_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} b_i(s, X(s)) | \mathcal{E}_i(s, t) \right| \right].
\]
Adding and subtracting \( T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} b_i(s, X(s)) \mathcal{E}_i^K(s, t) \) inside the absolute value and then using the triangular inequality yields
\[
\leq \mathbb{E} \left[ \left| T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} b_i(s, X(s)) \mathcal{E}_i^K(s, t) - T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} b_i(s, X(s)) \mathcal{E}_i^K(s, t) \right| \right]
\]
\[
= \mathbb{E} \left[ \left| b_i(s, T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} X(s)) - b_i(s, T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} X(s)) \right| \mathcal{E}_i^K(s, t) \right]
\]
\[
\leq L \mathbb{E} \left[ \left| T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} X(s) - T_{\mathbf{\Phi}_{(\mathbf{\sigma}, t)}} X(s) \right| \mathcal{E}_i^K(s, t) \right]
\]
\[
+ M \mathbb{E} \left[ \left| \mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t) \right| \right].
\]
Hence,
\[
\lim_{K \to \infty} E \left[ |b_i(s, X(s)) \diamond (\mathcal{E}^K_i(s, t) - \mathcal{E}_i(s, t))| \right] \leq \lim_{K \to \infty} L E \left[ |T_{-\Phi[\sigma^K_i(s, t)]} X(s) - T_{-\Phi[\sigma_i(s, t)]} X(s) \mathcal{E}^K_i(s, t)| \right] \\
+ \lim_{K \to \infty} M E \left[ |\mathcal{E}^K_i(s, t) - \mathcal{E}_i(s, t)| \right].
\]

The second term above converges to zero by the discussion on Appendix A.

At this point we will need the following lemma;

**Lemma 4.2.** Let \( Y \in L^q(\Omega) \) for some \( q \in (0, \infty) \) and let \( \{f_n\}_{n \geq 1} \) be a sequence converging to \( f \) in \( H_\phi \), then it holds that
\[
\lim_{n \to \infty} T_{\Phi f_n} Y = T_{\Phi f} Y, \quad \text{in } L^p(\Omega) \text{ for all } 0 < p < q < \infty.
\]

**Proof.** For simplicity we will consider the case of random variable \( Y \) depending only on a one dimensional fBm that can be seen as one of the components of our \( d \)-dimensional fBm, the general case does not present further difficulties. Notice that
\[
T_{\Phi f_n} B_t(\omega) = B_t^H + \int_0^t \int_0^T f_n(s) \phi(s, r) ds dr
\]
\[
= B_t + \int_0^T \int_0^T \chi_{[0,t]}(r) f_n(s) \phi(s, r) ds dr
\]
\[
= B_t + \langle f_n, \chi_{[0,t]} \rangle_\phi,
\]
at this point we use the fact that convergence in norm implies the weak convergence, and hence if \( f_n \) converges in \( H_\phi \) to \( f \) as \( n \to \infty \) we have that \( \langle f_n, \chi_{[0,t]} \rangle_\phi \) converges to \( \langle f, \chi_{[0,t]} \rangle_\phi \).

This implies that
\[
E[|T_{\Phi f} B_t - T_{\Phi f} B_t|^p] = |\langle f_n, \chi_{[0,t]} \rangle_\phi - \langle f, \chi_{[0,t]} \rangle_\phi|^p \to 0, \text{ as } n \to \infty.
\]

Furthermore notice that this holds for any random variable in the *Gaussian Hilbert space* (e.g. [9])
\[
\mathcal{G}_\phi := \left\{ \int_0^T g(s) dB_s; \ g \in H_\phi \right\}.
\]

At this point if \( Y \in L^q(\Omega) \) we have that for any \( \epsilon > 0 \) there’s a polynomial random variable \( P \) (which is a polynomial in some random variables in \( \mathcal{G}_\phi \)) such that \( \|Y - P\|_q < \epsilon \) (the existence of such a random variable is guaranteed by [9 Theorem 3.2] together with [9 Theorem 2.11]). By the triangle inequality we have that for \( 0 < p < q \)
\[
\|T_{\Phi f_n} X - T_{\Phi f} X\|_p \leq \|T_{\Phi f_n} X - T_{\Phi f_n} P\|_p + \|T_{\Phi f_n} P - T_{\Phi f} P\|_p + \|T_{\Phi f} P - T_{\Phi f} Y\|_p
\]
Now using the fractional Girsanov’s theorem we have
\[
\|T_\Phi f P - T_\Phi Y\|_p = \mathbb{E}[|P - Y|^p \mathcal{E}(f)]^{1/p} \\
\leq \mathbb{E}[|P - Y|^{pp_1}]^{1/(pp_1)} \mathbb{E} \left[ \mathcal{E}(f)^{p_2/p} \right]^{1/(pp_2)} \\
= \|P - Y\|_q \|\mathcal{E}(f)^{1/p^2}\|_r
\]
where \( q := p_1 p, r := p_2 p \) and \( \frac{1}{p_1} + \frac{1}{p_2} = 1 \). Same happens with \( |T_\Phi f_n P - T_\Phi f_n Y|_p \). At this point we notice that
\[
\|\mathcal{E}(f_n)^{1/p^2}\|_r \leq \mathbb{E} \left[ \exp \left\{ \frac{r}{2p^2} \int_0^T f_n(s) dB_H^s \right\} \right]^{1/r} \\
\leq \sup_n \exp \left\{ \frac{r}{2p^2} |f_n|^2 \right\} =: C.
\]
It follows that
\[
\|T_\Phi f_n X - T_\Phi X\|_p \leq 2C \|P - Y\|_q + \|T_\Phi f_n P - T_\Phi f P\|_p \\
\leq (2C + 1) \epsilon
\]
provided \( n \) is large enough; since \( \epsilon \) was arbitrary the proof is complete.

From remark 4.1, lemma 4.2 and the fact that
\[
\lim_{K \to \infty} \mathbb{E}^{K}_{i}(s, t) = \mathbb{E}_{i}(s, t), \quad \text{in } L^p(\Omega) \text{ for all } p \geq 1
\]
it follows that
\[
\lim_{K \to \infty} \mathbb{E} \left[ |T_{-\Phi \sigma_i^K(s, t)} X(s) - T_{-\Phi \sigma_i(s, t)} X(s)| \mathcal{E}_i^K(s, t) \right] = 0,
\]
completing the proof.

5 Proof theorem 1.3

Let \( \varphi \in C^2_0([0, T] \times \mathbb{R}^d) \), i.e. a two times continuously differentiable function on \([0, T] \times \mathbb{R}^d\) with compact support, and in order to ease the notation we set \( z := x|_{x_{ki} = z_k^{(i)}} \).

Then by [3.6] we have
\[
0 = \varphi(T, X^K(T)) - \varphi(0, c) \\
= \int_0^T \left[ \partial_t \varphi(r, u(r, z)) + \sum_{i=1}^d \partial_i \varphi(r, u(r, z)) \partial_t u_i(r, z) \right] dr \\
= \int_0^T \partial_t \varphi(r, u(r, z)) dr \\
+ \sum_{i=1}^d \int_0^T \partial_i \varphi(r, u(r, z)) \left( b_i(t, u(r, z)) + \sigma_i(t) \sum_{k=1}^K \xi_k(t) [x_{ki} u_i(r, z) - \partial x_{ki} u_i(r, z)] \right) dr \\
= \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D},
\]

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At this point we have that
\[ z \]
where

\[
\mathcal{A} = \int_0^T \partial_t \varphi(r, u(r, z))dr,
\]

\[
\mathcal{B} = \int_0^T \nabla \varphi(r, u(r, z)) \cdot b(s, u(r, z))dr,
\]

\[
\mathcal{C} = \sum_{i=1}^d \sum_{k=1}^K \int_0^T \partial_i \varphi(r, u(r, z)) \sigma_i(r) \xi_k(r)[x_{ki}u_i(r, z)]dr,
\]

\[
\mathcal{D} = -\sum_{i=1}^d \sum_{k=1}^K \int_0^T \partial_i \varphi(r, u(r, z)) \sigma_i(r) \xi_k(r) \partial_{ik}u_i(r, z)dr,
\]

where \( \cdot \) denotes the inner product in \( \mathbb{R}^d \). Taking expectation to the first and last term above we obtain

\[
0 = \mathbb{E}[\mathcal{A}] + \mathbb{E}[\mathcal{B}] + \mathbb{E}[\mathcal{C}] + \mathbb{E}[\mathcal{D}].
\]

Now using the fact that \( z \) is a standard Gaussian matrix where the components are mutually independent, we have

\[
\mathbb{E}[\mathcal{C}] = \sum_{i=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \xi_k(r) \int_{\mathbb{R}^{K \times d}} \partial_i \varphi(r, u(r, x))u_i(r, x)x_{ki}(2\pi)^{-K \times d/2} e^{-\frac{1}{2} \|x\|_2^2} dx dr,
\]

integration by parts yields

\[
= \sum_{i=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \xi_k(r) \int_{\mathbb{R}^{K \times d}} \partial_i \varphi(r, u(r, x))u_i(r, x)(2\pi)^{-K \times d/2} \partial_{x_{ki}} e^{-\frac{1}{2} \|x\|_2^2} dx dr
\]

\[
= \sum_{i,j=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \xi_k(r) \int_{\mathbb{R}^{K \times d}} \partial_j \partial_i \varphi(r, u(r, x)) \partial_{x_{ki}} u_j(r, x)u_i(r, x)(2\pi)^{-K \times d/2} \partial_{x_{ki}} e^{-\frac{1}{2} \|x\|_2^2} dx dr
\]

\[
+ \sum_{i=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \xi_k(r) \int_{\mathbb{R}^{K \times d}} \partial_i \varphi(r, u(r, x)) \partial_{x_{ki}} u_i(r, x)(2\pi)^{-K \times d/2} \partial_{x_{ki}} e^{-\frac{1}{2} \|x\|_2^2} dx dr
\]

\[
= \mathbb{E} \left[ \sum_{i,j=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \xi_k(r) \partial_j \partial_i \varphi(r, u(r, z)) \partial_{x_{ki}} u_j(r, z)u_i(r, z)dr \right]
\]

\[
+ \mathbb{E} \left[ \sum_{i=1}^d \sum_{k=1}^K \int_0^T \partial_i \varphi(r, u(r, z)) \sigma_i(r) \xi_k(r) \partial_{x_{ki}} u_i(r, z)dr \right]
\]

and now notice that the last term above equals \(-\mathbb{E}[\mathcal{D}]\).

At this point we have that

\[
0 = \mathbb{E}[\mathcal{A}] + \mathbb{E}[\mathcal{B}] + \mathbb{E} \left[ \sum_{i,j=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \xi_k(r) \partial_j \partial_i \varphi(r, u(r, z)) \partial_{x_{ki}} u_j(r, z)u_i(r, z)dr \right].
\]
Using Tower’s property yields

\[
0 = \mathbb{E}[A] + \mathbb{E}[B] + \mathbb{E}\left[ \sum_{i,j=1}^{d} \sum_{k=1}^{K} \int_0^T \sigma_i(r) \xi_k(r) \partial_j \partial_i \varphi(r, u(r, z)) u_i(r, z) \mathbb{E}[\partial_{x_k} u_j(r, z) | \mathcal{G}_i(r)] dr \right]
\]

\[
= \mathbb{E}[A] + \mathbb{E}[B] + \mathbb{E}\left[ \sum_{i,j=1}^{d} \sum_{k=1}^{K} \int_0^T \sigma_i(r) \xi_k(r) \partial_j \partial_i \varphi(r, u(r, z)) u_i(r, z) g_{ki}^{(j)}(r, u_i(r, z)) dr \right]
\]

where \( \mathcal{G}_i(r) \) is the sigma algebra generated by the random variable \( u_i(r, z) \), and the function \( g_{ki}^{(j)} : [0, T] \times \mathbb{R}^{K \times d} \) is a measurable function, whose existence is guaranteed by the Doob’s lemma chosen to satisfy

\[
g_{ki}^{(j)}(r, u_i(r, z)) = \mathbb{E}[\partial_{x_k} u_j(r, z) | \mathcal{G}_i(r)]. \tag{5.1}
\]

Putting everything together and using (3.6) we obtain

\[
0 = \mathbb{E}\left[ \int_0^T \partial_t \varphi(r, X^K(r)) dr + \int_0^T \nabla \varphi(r, X^K(r)) \cdot b(s, X^K(r)) dr \right. \\
+ \left. \sum_{i,j=1}^{d} \sum_{k=1}^{K} \int_0^T \sigma_i(r) \xi_k(r) \partial_j \partial_i \varphi(r, X^K(r)) X^k_i(r) g_{ki}^{(j)}(r, X^k_i(r)) dr \right]
\]

Observe that the last member above contains expectations of functions of the random vector \( X^K(r) \), for \( r \in [0, T] \); therefore, writing the law of this random vector as

\[
\mu^K(r, A) := \mathbb{P}(\{\omega \in \Omega : X^K(r, \omega) \in A\}), \quad r \in [0, T], A \in \mathcal{B}(\mathbb{R}^d)
\]

\[
0 = \int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \varphi(r, x) + \nabla \varphi(r, x) \cdot b(s, x) \right. \\
+ \left. \sum_{i,j=1}^{d} \sum_{k=1}^{K} \int_0^T \sigma_i(r) \xi_k(r) \partial_j \partial_i \varphi(r, x) x_i g_{ki}^{(j)}(r, x_i) dr \right] d\mu^K(r, x) dr.
\]

The last equalities hold for any test function \( \varphi \in C^2_0([0, T] \times \mathbb{R}^d) \) and this completes the proof.
Appendix A

Fix \( s, t \in [0, T] \) and without loss of generality assume that \( t \geq s \), then using the basic inequality \(|e^X - e^Y| \leq |e^X + e^Y| \cdot |X - Y|\) it holds that

\[
|\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)| \leq |\mathcal{E}_i^K(s, t) + \mathcal{E}_i(s, t) - \mathcal{E}_i(s, t)|
\]

\[
\times \int_0^T \sigma_i^K(t, s; q) dB_i(q) - \frac{1}{2} |\sigma_i^K(t, s; \cdot)|_\phi^2 - \int_0^t \sigma_i(s) dB_i(s) + \frac{1}{2} |\chi_{[0, t]} \sigma_i|_\phi^2
\]

\[
\leq |\mathcal{E}_i^K(s, t) + \mathcal{E}_i(s, t)|
\]

\[
\times \left( \left| \left( \int_0^T [\sigma_i^K(t, s; q) - \sigma_i(q)] dB_i(q) \right) + \frac{1}{2} |\sigma_i^K(t, s; \cdot)|_\phi^2 - |\chi_{[s, t]} \sigma_i|_\phi^2 \right| \right).
\]

Now let’s write

\[
|\sigma_i^K(t, s; \cdot)|_\phi^2 - |\chi_{[s, t]} \sigma_i|_\phi^2 \leq |(\sigma_i^K(t, s; \cdot) - |\chi_{[s, t]} \sigma_i|_\phi) - |\chi_{[s, t]} \sigma_i|_\phi|
\]

\[
\leq S^2 T^{2Hp} |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi
\]

where we used the triangular inequality and \( S \) is a constant such that \( |\sigma_i(t)| \leq S \) for all \( t \in [0, T] \).

At this point we raise both sides to the \( p \geq 1 \) and take expectation yielding

\[
\mathbb{E} \left[ |\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)|^p \right] \leq 2^{p-1} \mathbb{E} \left[ |\mathcal{E}_i^K(s, t) + \mathcal{E}_i(s, t)|^p \left| I(\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi) \right|^p \right.
\]

\[
+ S^{2p} T^{2Hp} |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi|^p \]

where \( I(\cdot) \) denotes the fractional Wiener integral. Using Hölder’s inequality where \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \) we have

\[
\leq 2^{p-1} \left\{ \mathbb{E} \left[ |\mathcal{E}_i^K(s, t) + \mathcal{E}_i(s, t)|^p \right] |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi|^p \right\}
\]

\[
+ S^{2p} T^{2Hp} \mathbb{E} \left[ |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi|^p \right]
\]

\[
\leq 2^{p-1} \left\{ \left( 2p e^{p(p_1-1)/2} |\chi_{[s, t]} \sigma_i|_\phi^2 \right) 2^{p/2} \Gamma(p_2 + 1) / \sqrt{\pi} |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi|^p \right\}
\]

\[
+ S^{2p} T^{2Hp} \left( 2p e^{p(p_1-1)/2} |\chi_{[s, t]} \sigma_i|_\phi^2 \right) |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi|^p \right\}
\]

and hence

\[
\mathbb{E} \left[ |\mathcal{E}_i^K(s, t) - \mathcal{E}_i(s, t)|^p \right] \leq C |\sigma_i^K(t, s) - |\chi_{[s, t]} \sigma_i|_\phi|^p \rightarrow 0,
\]

as \( K \rightarrow \infty \) where

\[
C = 2^{3p/2-1} e^{p(p_1-1)/2} |\chi_{[s, t]} \sigma_i|_\phi^2 \Gamma(p_2 + 1) / \sqrt{\pi} + 2^{2p-1} S^{2p} T^{2Hp} e^{p(p_1-1)/2} |\chi_{[s, t]} \sigma_i|_\phi^2.
\]
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