Relativistic quantum equation in nonstationary space-time geometry

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Abstract. An analytical solution for the 1-1 D relativistic quantum equation in dependent space-time geometry is given when the imaginary part of the particle mass is negligible. The solution is accomplished through covariant transformations of wave equation. The solutions have a modal structure and the modes are dynamics. The relativistic redshift turns out to be a manifestation of the space-time geometry motion. As illustrations, two space-time geometry motions corresponding to nonlinear mappings and the linear Lorentz transformations are discussed.

1. Introduction
This study follows up on previous works [1]-[2] proposing a new formulation of quantum theory in nonstationary space-time geometry paralleling the usual theory. In these previous papers the mass of particle turns out to be complex and affected by the space-time geometry motion. In view of their importance in the study of various quantum systems under the influence of nonstationary space-time geometry, exact solution of relativistic quantum equation is of a great interest. However, because of the interdependence of the space and time coordinates, the classical analytical methods cannot, in general, be applied. Therefore, the present work provides, for the 1-1D relativistic quantum equation, an analytical solution when the imaginary part of the mass is assumed negligible in relation to the real part. The solution is accomplished through covariant transformations of wave equation. We show that the solutions are dynamics and turn out to be the generalization of the classical ones. As illustrations, two space-time geometry motions corresponding to nonlinear mappings and the linear Lorentz transformations are discussed. The paper is organized as follows: Section 2 briefly summaries the analytical method used in this work. Section 3 outlines the position of the problem. Section 4 gives the analytical solution of the problem. Section 5 examines two cases of space-time geometry motion. Section 6 yields a conclusion.

2. Brief Summary of the used method
For more clarity of the work, we provide a brief summary of a covariant transformation of the wave equation [1]-[3]. The method consists in transforming the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial \tau^2} = \theta (\tau, x)$$

(1)

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for the field $\Psi(\tau, x)$ in the nonstationary space-time domain $R(\tau, x)$ into a wave equation of the covariant form, namely

$$\left[ \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \xi^2} \right] \varphi' (\tau - x) \varphi' (\tau + x) = \theta (\xi, \eta) \quad (2)$$

for the field $\Psi(\xi, \eta)$ in the fixed domain $S(\xi, \eta)$. $\varphi'$ is the first derivative of a transformation function $\varphi$ defined by the relationships between the coordinates in the two domains $S(\xi, \eta)$ and $R(\tau, x)$:

$$\xi + \eta = \varphi (\tau + x) \quad (3)$$
$$\xi - \eta = \varphi (\tau - x) \quad (4)$$

or

$$\eta = \frac{1}{2} [\varphi (\tau + x) - \varphi (\tau - x)] \quad (5)$$
$$\xi = \frac{1}{2} [\varphi (\tau + x) + \varphi (\tau - x)] \quad (6)$$

It should be noted that the real transformation functions

$$\xi = \xi (\tau, x) \quad (7)$$
$$\eta = \eta (\tau, \tau) \quad (8)$$

satisfy the following fundamental conditions

$$\frac{\partial \xi}{\partial \tau} = \frac{\partial \eta}{\partial x}, \quad \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial \tau} \quad (9)$$

In addition to the relations 9, the transformation functions 7 and 8 themselves satisfy wave equation

$$\frac{\partial^2 \xi}{\partial \tau^2} = \frac{\partial^2 \xi}{\partial x^2} \frac{\partial^2 \eta}{\partial \tau^2} = \frac{\partial^2 \eta}{\partial x^2} \quad (10)$$

Finally, we can say that the real function transformations, which leave the wave equation of the covariant form and map every point of a given nonstationary space-time geometry into a point of a static space-time geometry, are of the form 7-8 and satisfy the fundamental conditions 9.

It should be noted that $\varphi(\tau, x)$ and its inverse $F(\xi, \eta)$ are functions of a real variable not of a complex variable, since the conditions 9 play a similar role as the Cauchy-Riemann conditions in the transformation of the Laplace equation but are different.

3. Position of the problem
The relativistic quantum equation of a particle of a mass $m$ evolving in fixed space-time geometry is given by the well known Klein-Gordon equation. However, when considering nonstationary space-time geometry, the evolution of the same particle is described by the following equation[1]:

$$\frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \xi^2} \varphi' (\tau - x) \varphi' (\tau + x) = \theta (\xi, \eta) \quad (2)$$
\[ \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial \tau^2} - \frac{m^2 c^2}{\hbar^2 k} (k \varphi'^2 - i \varphi'') \Psi = 0 \] 

(11)

or

\[ \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial \tau^2} - M^2 c^2 \bar{\hbar}^2 \Psi = 0 \]

(12)

with

\[ M^2 = x^2 \frac{k \varphi'^2 - i \varphi''}{k} \]

(13)

where \( \varphi' \) and \( \varphi'' \) are respectively the first and second derivative of \( \varphi(\tau - x) \) and \( \tau = ct \). It should be noted that \( \varphi \) is an identity when the space-time geometry becomes static, so \( \varphi' = 1 \) and \( \varphi'' = 0 \).

Equation 11 is identical in appearance with the classical Klein-Gordon equation (pseudo-covariance). The mass \( M \) is complex and depends on the space-time geometry motion. We are interested in this study by the case of a stable particle that is when the imaginary part of the mass is negligible. Thus, equation 11 can be expressed as follows:

\[ \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial \tau^2} - \frac{m^2 c^2 \varphi'^2}{\hbar^2} \Psi = 0 \]

(14)

with the relativistic energy-momentum relation:

\[ E_{ns}^2 = P_{ns}^2 + m^2 \varphi'^2 c^4 \]

(15)

\( E_{ns} \) and \( P_{ns} \) are respectively the energy and momentum in nonstationary space-time geometry.

We note that energy at the rest \( (P_{ns} = 0) \) is \( E_{ns} = m \varphi'' c^2 \) and when the space-time geometry becomes static \( (\varphi' = 1) \), we find again the Einstein relation. Relation 15 means that equation 14 allows positive and negative energy as solutions like in classical case. However, because of the interdependence of the space and time coordinates, the classical analytical methods cannot be applied for determining these solutions. Actually, more general methods [4] - [10] are used, which are mathematically rather involved. In our case, we use the covariant transformation of wave equation cited in section 2. Therefore, the relativistic quantum equation 14 in nonstationary space-time geometry \( R(\tau, x) \) is transformed into modified Klein-Gordon equation in corresponding static space-time geometry \( S(\xi, \eta) \) and writes

\[ \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \tau^2} - \frac{m^2 \varphi'^2}{\hbar^2} F(\xi + \eta) \Psi = 0 \]

(16)

where \( F(\xi \pm \eta) \) is the first derivative of the inverse transformation function \( F(\xi \pm \eta) \) of \( \varphi(\tau \pm x) \).
Next, let be

\[ \frac{F_r(\xi + \eta)}{F_r(\xi - \eta)} = G(\xi, \eta) \]  \hspace{1cm} (17)

Finally, equation 14 is transformed in \( S(\xi, \eta) \) into the following equation of the covariant form:

\[ \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \tau^2} - \frac{m^2 c^2}{\hbar^2} G(\xi, \eta) \Psi = 0 \]  \hspace{1cm} (18)

4. Analytical solution

For determining the solutions of equation 14 in \( R(\tau, x) \), first of all, we have to solve eq. 18. However, equation 18 will not separate unless \( G(\xi, \eta) \) turns out to be the sum of a function of \( \xi \) alone and a function of \( \eta \) alone. In other words, \( G \) has to be in the following form:

\[ G(\xi, \eta) = G_1(\xi) + G_2(\eta) \]  \hspace{1cm} (19)

In this condition, equation 18 becomes of separable form and writes:

\[ \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \tau^2} - \frac{m^2 c^2}{\hbar^2} [G_1(\xi) + G_2(\eta)] \Psi = 0 \]  \hspace{1cm} (20)

So, we seek \( \Psi \) of the form:

\[ \Psi(\xi, \eta) = L(\xi) K(\eta) \]  \hspace{1cm} (21)

Therefore, we obtain two differential equations which are:

\[ \frac{\partial^2 L}{\partial \xi^2} + [\mu^2 + \frac{m^2 c^2}{\hbar^2} G_1]L = 0 \]  \hspace{1cm} (22)

\[ \frac{\partial^2 K}{\partial \eta^2} + [\mu^2 - \frac{m^2 c^2}{\hbar^2} G_2]K = 0 \]  \hspace{1cm} (23)

The solutions of equations 22 and 23 need the expression of \( G(\xi, \eta) \) which itself depends on the function transformation. However, since the 2th derivative \( \varphi'' \) is assumed negligible, so the first derivative \( \varphi' \) can be assumed as a constant. As a consequence, according to 1 and 12, equation 14 is transformed in fixed space-time geometry \( S(\xi, \eta) \) into Klein-Gordon equation and writes:

\[ \frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \xi^2} - \frac{m^2 c^2}{\hbar^2} \Psi = 0 \]  \hspace{1cm} (24)

The solutions of 24 for positive energy are the plane wave type of the form:
\[ \psi(\xi, \eta) = \psi_0 \exp -ik(\xi - \eta) \]  

(25)

Transforming back to the original variables \( \tau, x \) in \( R(\tau, x) \) by means of 4, we obtain the solution of equation 14 in non stationary space-time geometry. This solution writes as follows:

\[ \psi(\tau, \eta) = \psi_0 \exp -ik\varphi(\tau - x) \]  

(26)

We ascertain that solutions are, in general, dynamics and of functional plane wave type. In practical, we are interesting by particular solutions corresponding to real situations. This case will be treated at the next subsection.

4.1. Particular solutions
Consider the 1-1D relativistic quantum equation 14 defined on the following nonstationary space-time geometry such as \( x = a(\tau) \) expanding or shrinking the \( x \) domain:

\[ 0 \leq x \leq a(\tau) \]  

(27)

according to 5, the corresponding fixed domain is given by

\[ 0 \leq \eta \leq \eta_0 \]  

(28)

with

\[ \eta_0 = \frac{1}{2}[\varphi(\tau + a(\tau)) - \varphi(\tau - a(\tau))] \]  

(29)

The boundary conditions are needed and they vary with the respective applications. For instance, for the following boundary conditions in \( R(\tau, x) \)

\[ \Psi(\tau, 0) = 0, \Psi(\tau, a(\tau)) = 0, \]  

(30)

the corresponding boundary conditions in \( S(\xi, \eta) \) are

\[ \Psi(\xi, 0) = 0, \Psi(\xi, \eta_0) = 0, \]  

(31)

Thus, the general solution of equation 24 is given by:

\[ \Psi(\xi, \eta) = A_n \sum_{1}^{\infty} \sin(k_n\eta) \cos(\varphi_n\xi + \alpha_n) \]  

(32)

with

\[ k_n = \frac{n\pi}{\eta_0} \]  

(33)
\[\bar{\omega}_n = \left( \frac{m^2 c^2 + n^2 \pi^2}{\eta_0^2} \right)^{\frac{1}{2}} \] (34)

Finally, transforming back to the variables \(\tau, x\) in \(R(\tau, x)\) by means of the function transformation \(\varphi\), we deduce the general solution of equation 14 in \(R(\tau, x)\). Indeed, the substitution of \(\eta, \xi\) and \(\eta_0\) in equations 32-34 by their respective expression 5, 6 and 29, we obtain

\[\Psi(\tau, x) = A_n \sum_{n=1}^{\infty} \left\{ \sin k_n \frac{1}{2} [\varphi(\tau + x) - \varphi(\tau - x)] \times \cos \left( \varpi_n \frac{1}{2} [\varphi(\tau + x) + \varphi(\tau - x)] + \alpha_n \right) \right\} \] (35)

Remarks

• 'Expression 35 shows that the modes are expressed in terms of functional series and are dynamics.'

• 'The boundary conditions 30 are satisfied by 35.'

• 'When the space-time geometry is fixed, the function transformation \(\varphi\) becomes identity, so that we find again the classical solutions'

• 'We remind that the pulsation,'

\[\omega_n = \bar{\omega}_n C \] (36)

5. Illustrations

5.1. Nonlinear transformations

For a linear motion \(a(\tau) = a_0 + \beta \tau\), expanding or shrinking the \(x\) domain, the corresponding real function transformations are [3], [11]:

\[\eta = \frac{1}{2} \ln \left[ \frac{a_0 + \beta(\tau + x)}{a_0 + \beta(\tau - x)} \right] \] (37)

\[\xi = \frac{1}{2} \ln \left[ \frac{a_0 + \beta(\tau + x)}{a_0 + \beta(\tau - x)} \right] a_0^2 \] (38)

Thus, according to 3 and 4, the transformation \(\varphi\) and its first derivative are respectively given by:

\[\varphi(\tau \pm x) = \ln \left[ 1 + \frac{\beta}{a_0} (\tau \pm x) \right] \] (39)

\[\varphi'(\tau \pm x) = \frac{\beta}{\beta (\tau \pm x) + a_0} \] (40)
and 29 gives

\[ \eta_0 = \frac{1}{2} \ln \frac{1 + \beta}{1 - \beta} \]  

(41)

with \( \tau = ct \) and \( \beta = c/v \), \( v \) is the velocity motion increasing or decreasing the given domain. Next, since the first derivative is not a constant, we have to solve equations 22 and 23. However, since the velocity \( v \) is assumed to be very small in relation to \( c \) (\( v \ll c \) or \( \beta \ll 1 \)), the following approximations hold extremely well:

\[ \varphi (\tau \pm x) = \frac{\beta}{a_0} (\tau \pm x), \varphi' (\tau \pm x) = \frac{\beta}{a_0}, \eta = \frac{\beta}{a_0} x, \xi = \frac{\beta}{a_0} \tau, \]  

(42)

Hence,

\[ k_n = \frac{n \pi}{\beta}, \varpi_n = \left( \frac{m^2 c^2}{\hbar^2} + \frac{n^2 \pi^2}{\beta^2} \right)^{\frac{1}{2}} \]  

(43)

Therefore, the general solution 35 writes as follows:

\[ \Psi (\tau, x) = A_n \sum_{1}^{\infty} \sin \left( \frac{n \pi}{a_0} x \right) \cos \left( \frac{\beta}{a_0} \varpi_n \tau + \alpha_n \right) \]  

(44)

or

\[ \Psi (\tau, x) = A_n \sum_{1}^{\infty} \sin (K_n x) \cos (\Omega_n \tau + \alpha_n) \]  

(45)

with

\[ K_n = \frac{n \pi}{a_0} = k_n \varphi' \]  

(46)

\[ \Omega_n = \frac{\beta}{a_0} \omega_n = \omega_n \varphi' \]  

(47)

We note a general Doppler shift which, according to the dispersion relation, affects also the mass \( m \).

### 5.2. Linear Lorentz transformations

Consider a 1-D confined region moving with the same velocity \( v \) expanding or shrinking the space-time geometry. This means that locally the geometry of the considered region is assumed static. This region constitutes, with its corresponding transformed region in the fixed space-time geometry, inertial frames. Therefore, the two regions are related by the linear Lorentz transformations. The linear Lorentz transformations are:
\[ \xi = \gamma (\tau - \beta x) \quad (48) \]

\[ \eta = \gamma (x - \beta \tau) \quad (49) \]

With \( \beta = v/c \) and \( \gamma = (1 - \beta^{-1})^{\frac{1}{2}} \). The Lorentz transformations satisfy the fundamental conditions 9 and are solutions of wave equation 10. From 48 and 49, we deduce the transformation \( \phi \) and its first derivative \( \phi' \):

\[ \xi \pm \eta = \phi (\tau \pm x) = \left( \frac{1 \mp \beta}{1 \pm \beta} \right)^{\frac{1}{2}} (\tau \pm x) \quad (50) \]

\[ \phi' (\tau \pm x) = \left( \frac{1 \mp \beta}{1 \pm \beta} \right)^{\frac{1}{2}} (\tau \pm x) \quad (51) \]

As we have seen, Klein-Gordon equation admits, in fixed space-time geometry, solutions of the type 25. So by transforming back to the \((\tau, x)\) variables, we find the solution of equation 14. It writes:

\[ \Psi (\tau, x) = \Psi_0 \exp \left[ -ik \left( \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} (\tau - x) \right] \quad (52) \]

or

\[ \Psi (\tau, x) = \Psi_0 \exp [-iK_{ns} (\tau - x)] \quad (53) \]

with

\[ K_{ns} = k \left( \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} = k\phi' \quad (54) \]

\[ \Omega_{ns} = \omega \left( \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} = \omega \phi' \quad (55) \]

We ascertain that we obtain the usual solution with the relativistic redshift which affects all the observables even the mass.
6. Conclusion
The relativistic quantum equation in nonstationary space-time geometry has been solved analytically when the imaginary part of the particle mass is assumed negligible. This study leads to the following results:

- The solutions express in terms of functional modes.
- The modes are, in general, dynamics.
- All the physical characteristics of the solutions are affected by a factor representing the redshift which is a manifestation of the space-time geometry rather than an effect due to object motions in space-time. The factor depends on the motion expanding or shrinking the space-time geometry.
- We ascertain that the mass of the particle is also affected by the redshift.
- The usual relativistic Dopplers effect turns out to be only a particular case corresponding specifically to Lorentz transformations.

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