KNOT FLOER HOMOLOGY AND RATIONAL SURGERIES

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Abstract. Let $K$ be a rationally null-homologous knot in a three-manifold $Y$. We construct a version of knot Floer homology in this context, including a description of the Floer homology of a three-manifold obtained as Morse surgery on the knot $K$. As an application, we express the Heegaard Floer homology of rational surgeries on $Y$ along a null-homologous knot $K$ in terms of the filtered homotopy type of the knot invariant for $K$. This has applications to Dehn surgery problems for knots in $S^3$. In a different direction, we use the techniques developed here to calculate the Heegaard Floer homology of an arbitrary Seifert fibered three-manifold.

1. Introduction

Heegaard Floer homology [24] is an invariant for closed, oriented three-manifolds $Y$, taking the form of a collection of homology groups which are functorial under cobordisms. In [22] and [27], this invariant is extended to an invariant for null-homologous knots $K$. (Here, we say that a knot is null-homologous if its induced homology class in $H_1(Y;\mathbb{Z})$ is trivial. If the induced homology class of a knot $K \subset Y$ in $H_1(Y;\mathbb{Q})$ is trivial, we call the knot rationally null-homologous.) The knot invariant takes the form of a $\mathbb{Z} \oplus \mathbb{Z}$-filtration of the chain complex whose homology calculates the Heegaard Floer complex for $Y$. It is the filtered chain homotopy type of this filtered complex which depends on the particular knot $K$.

The knot filtration gives rise to collection of chain complexes $\{A^+_s(K)\}_{s \in \mathbb{Z}}$ and chain maps $\{v^+_s: A^+_s(K) \to B^+\}_{s \in \mathbb{Z}}$ and $\{h^+_s: A^+_s(K) \to B^+\}_{s \in \mathbb{Z}}$, where here $B^+ = CF^+(Y)$ is a chain complex whose homology is the Heegaard Floer homology $HF^+(Y)$. Indeed, the homology groups of the chain complex $A^+_s(K)$ represents the homology $HF^+$ of sufficiently large integer surgeries on $Y$ along $K$, in a sense which can be made precise (c.f. Theorem 4.4 of [22] and also [27]). (These complexes are defined in a more general setting in Section 3.)

Suppose that $K$ is a null-homologous knot in a three-manifold $Y$. Given a rational number $r$, let $Y_r(K)$ denote the three-manifold obtained by Dehn filling $Y - nd(K)$ with a solid torus with slope $r$ (with respect to the canonical Seifert framing of $K$). In the case where $r$ is an integer, the Heegaard Floer homology of $Y_r(K)$ can described in terms of the above-mentioned data coming from the knot filtration, according to [25].

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The primary aim of this article is to generalize this construction to the case of Morse surgery on a knot \( K \subset Y \) which is only rationally null-homologous. (By Morse surgery, we mean here Dehn surgery on a knot which can be realized as the boundary of a single two-handle addition to \([0, 1] \times Y\); in the case where \( K \) is null-homologous, this corresponds to Dehn surgery with an integral slope). This construction has new consequences even in the case of null-homologous knots in a three-manifold: since the result of Dehn surgery on a null-homologous knot \( K \subset Y \) can be viewed as Morse surgery on a knot in the connected sum of \( Y \) with a lens space, we obtain a description of the Heegaard Floer homology of \( Y_r(K) \) in terms of the original knot Floer homology of \( K \).

Rather than introducing the generalization of the knot package to knots which are only rationally null-homologous in this introduction, which will require some additional material in its statement (c.f. Sections 3, 5, and 6 below), we focus now in the description of the Floer homology of \( Y_r(K) \) when \( r \) is a rational number, and \( K \subset Y \) is null-homologous.

As a preliminary point, recall that the Heegaard Floer homology of \( Y \) admits a direct sum splitting indexed by the set of Spin\(^c\) structures over \( Y \), which in turn is an affine space for \( \mathbb{H}_2(Y; \mathbb{Z}) \). In particular, if \( K \subset Y \) is a knot in an integral homology sphere, then there is a splitting

\[
HF^+(Y_{p/q}(K)) \cong \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} HF^+(Y_{p/q}(K), i).
\]

Fix an integer \( i \), and consider the chain complexes

\[
A^+_i = \bigoplus_{s \in \mathbb{Z}} (s, A^+_{\lceil i/p \rceil}(K)) \quad \text{and} \quad B^+_i = \bigoplus_{s \in \mathbb{Z}} (s, B^+_s),
\]

where \( \lceil x \rceil \) denotes the greatest integer smaller than or equal to \( x \), and all \( B^+_{\lceil i/p \rceil} = CF^+(Y) \). We view the above chain homomorphisms \( v^+ \) and \( h^+ \) as maps

\[
v^+: (s, A^+_{\lceil i/p \rceil}(K)) \rightarrow (s, B^+) \quad \text{and} \quad h^+: (s, A^+_{\lceil i/p \rceil}(K)) \rightarrow (s - 1, B^+).
\]

Adding these up, we obtain a chain map

\[
D^+_{i,p/q}: A^+_i \rightarrow B^+_i;
\]

i.e.

\[
D^+_{i,p/q}\{(s, a_s)\}_{s \in \mathbb{Z}} = \{(s, b_s)\}_{s \in \mathbb{Z}},
\]

where here

\[
b_s = v^+_{\lceil i/p \rceil}(a_s) + h^+_{\lceil i/p(s-1) \rceil}(a_{s-1}).
\]

Let \( X^+_{i,p/q} \) denote the mapping cone of \( D^+_{i,p/q} \). Note that \( X^+_{i,p/q} \) depends on \( i \) only through its congruence class modulo \( p \). Note also that \( A^+_s \) and \( B^+_s \) are relatively \( \mathbb{Z} \)-graded, and the homomorphisms \( v^+_s \) and \( h^+_s \) respect this relative grading. The mapping
cone $X^+_i$ can be endowed with a relative grading, with the convention that $D^+_i,p/q$ drops the grading by one.

This mapping cone, whose ingredients are extracted from the knot filtration, captures Heegaard Floer homology of $p/q$ surgeries on $Y$ along $K$, according to the following:

**Theorem 1.1.** Let $K \subset Y$ be a null-homologous knot, and let $p, q$ be a pair of relatively prime integers. Then, for each $i \in \mathbb{Z}/p\mathbb{Z}$, there is a relatively graded isomorphism of groups

$$H_*(X^+_i,p/q) \cong HF^+(Y_{p/q}(K), i).$$

The proof of Theorem 1.1 is based on generalization of the knot filtration of [22] and [27] to the case of rationally null-homologous knots, together with a generalization of the integer surgeries description from [25] (where in fact Theorem 1.1 is proved in the case where $q = 1$). Note that in the more general case, the knot filtration is naturally a filtration by relative Spin$^c$ structures on the knot complement (rather than integers). Rational surgeries on $K \subset Y$ can be realized as Morse surgeries on a knot in the connected sum of $Y$ with a lens space. The resulting knot is gotten by forming the connected sum of $K$ with a model (homologically non-trivial) knot in the lens space. Theorem 1.1 is then realized as a combination of a straightforward calculation involving this model knot, combined with a Künneth principle for connected sums, followed by the general Morse surgeries description.

We turn now to various applications of Theorem 1.1.

1.1. **Applications to knots with $L$-space surgeries.** In Section 8, we give applications of Theorem 1.1 to knots in $S^3$ which admit $L$-space surgeries.

Recall that an $L$-space is a rational homology three-sphere $Y$ whose Floer homology $HF^+$ in each Spin$^c$ structure is isomorphic (as a relatively-graded $\mathbb{Z}[U]$-module) to $HF^+(S^3)$. This is equivalent to the condition that $\widehat{HF}(Y, s) \cong \mathbb{Z}$ for each $s \in \text{Spin}^c(Y)$. Recall also [14], [17] that if $Y$ is a rational homology three-sphere, then $\widehat{HF}(Y, s)$ is a $\mathbb{Q}$-graded group. Thus, the Heegaard Floer homology of an $L$-space is determined by the “correction term” function

$$d: \text{Spin}^c(Y) \longrightarrow \mathbb{Q}$$

which associates to each $s \in \text{Spin}^c(Y)$ the degree in which $\widehat{HF}(Y, s)$ is supported, compare also [6].

The set of $L$-spaces includes all lens spaces and indeed all three-manifolds with elliptic geometry, c.f. [16]; for more examples, see also [12]. Another interesting family is given by the branched double-covers of alternating knots in $S^3$, c.f. [20].

Let $K \subset S^3$ be a knot in the three-sphere. Write its symmetrized Alexander polynomial as

$$\Delta_K(T) = a_0 + \sum_{i>0} a_i (T^i + T^{-i}).$$
and let

\[(1)\quad t_i(K) = \sum_{j=1}^{\infty} ja_{|i|+j}.\]

Note that for any knot \(C\) in \(S^3\), there is a canonical affine map \(\mathbb{Z}/p\mathbb{Z} \cong \text{Spin}^c(S^3_{p/q}(C))\).

**Theorem 1.2.** Let \(K \subset S^3\) be a knot which admits an \(L\)-space surgery, for some \(r = \frac{p}{q} \in \mathbb{Q}\) with \(r \geq 0\). Then, for all integers \(i\) with \(|i| \leq \frac{p}{2q}\) we have that

\[(2)\quad d(S^3_{p/q}(K), i) - d(S^3_{p/q}(O), i) = -2t_{\lfloor \frac{|i|}{q} \rfloor}(K),\]

while for all \(|i| > \frac{p}{2q}\), we have that \(t_i(K) = 0\).

In the case where \(q = 1\), a version of the above theorem is established in [17] (c.f. Theorem 7.2 of [17]). Theorem 1.2 (in the case where \(q = 2\)) also gives the symmetry used in [15] to find an obstruction to a knot having unknotting number equal to one.

The following is a quick consequence of this result, together with the fact that knot Floer homology distinguishes the unknot (c.f. [21]) (though alternative proofs could be given which model the proof in [9] more closely):

**Corollary 1.3.** If \(S^3_{p/q}(K) \cong S^3_{p/q}(O)\) as oriented three-manifolds, then \(K = O\).

The above result, which was conjectured by Gordon in [8], was first established using Seiberg-Witten monopoles in [9]. Thanks to a theorem of Eliashberg and Etnyre (c.f. [4] and [5]), it is now possible to prove results of this type purely in the context of Heegaard Floer homology, see also [21].

### 1.2. On cosmetic surgeries.

Let \(Y\) be a closed, oriented three-manifold, and \(K \subset Y\) be a framed knot. Given a rational number \(r\), let \(Y_r(K)\) denote the three-manifold obtained by Dehn surgery along \(K\) with slope \(r\) (with respect to the initial framing). If there are two distinct rational numbers \(r\) and \(s\) with the property that \(Y_r(K)\) and \(Y_s(K)\) are homeomorphic (but the homeomorphism is not required to preserve the orientation inherited from \(Y\)), then the surgeries are called **cosmetic**. A pair of surgeries on \(K\) with \(r \neq s\) is called **truly cosmetic** if \(Y_r(K) \cong Y_s(K)\) as oriented manifolds.

Amphicheiral knots have cosmetic surgeries; specifically, if \(K\) is an amphicheiral knot, then \(S^3_{p/q}(K) \cong -S^3_{-p/q}(K)\). The unknot \(O\) admits infinitely many truly cosmetic surgeries: \(S^3_{p/q}(O) = S^3_{p/p+q}(O)\). Lackenby [10] has shown that under general conditions on a knot \(K \subset Y\), there are at most finitely many cosmetic surgeries, see also [2]. It is conjectured [2] that if \(Y_r(K) \cong Y_s(K)\), then \(Y - \text{nd}(K)\) admits an automorphism which carries the slope \(r\) to the slope \(s\).

The present state of Heegaard Floer homology – and specifically the surgery formulas given here – work best for excluding cosmetic surgeries on knots in \(S^3\). For example, we have the following result:
Theorem 1.4. If $K$ is a knot with Seifert genus equal to one, and $S^3_r(K) \cong S^3_s(K)$ with $r \neq s$, then $S^3_r(K)$ is an $L$-space.

The conclusion of the above theorem places severe restrictions on $K$. In particular, according to results of [16], it follows that $K$ must have the same knot Floer homology (and in particular the same Alexander polynomial) as the trefoil knot $T$, and thus according to Theorem 1.1, $S^3_r(K)$ and $S^3_s(T)$ have the same (graded) Floer homology groups. For particular integers $p$, the existence of a truly cosmetic surgery on such a knot $K$ with specified numerator $p$ can be ruled out by an explicit, finite search.

Theorem 1.5. Let $K \subset S^3$ and suppose that $S^3_r(K) \cong \pm S^3_s(K)$, then either $S^3_r(K)$ is an $L$-space or $r$ and $s$ have opposite signs.

For the above theorem, both possible conclusions can hold. The simplest example is the unknot which admits cosmetic surgeries with positive slopes. A more interesting example of cosmetic surgeries with positive slopes is provided by the trefoil knot $K$, which has the property that $S^3_9(K) \cong -S^3_{9/2}(K)$, c.f. [11]. Examples where $r$ and $s$ have opposite signs are given by amphicheiral knots.

Our methods can be refined to exclude cosmetic surgeries for certain numerators $p$. We study here the case where $p = 3$.

Theorem 1.6. Suppose that $K \subset S^3$ is a knot with the property that $S^3_{p/q}(K) \cong S^3_{p/q'}(K)$ as oriented manifolds. In the case where $p = 3$, we can conclude that $q = q'$.

1.3. Heegaard Floer homology of Seifert fibered spaces. We give some other applications of the general surgeries description along a rationally null-homologous knot. In Section 10, we use it to describe the Heegaard Floer homology of any Seifert fibered space whose first Betti number is even.

1.4. Organization. This paper is organized as follows. In Section 2, we review some of the topological preliminaries required by the knot filtrations, including the notion of relative Spin$^c$ structures for three-manifolds with torus boundary. In Section 3, we give the construction of the knot filtration. In the next two sections, we turn to some properties of the knot Floer homology which are rather straightforward adaptations of the corresponding results for null-homologous knots ([22], [27]): the relationship between knot Floer homology and “large” surgeries on a rationally null-homologous knot (Section 4), and the Künneth principle for connected sums of knots (Section 5). The first result is an ingredient in the Morse surgery formula from Section 6. In Section 7, we show how the Künneth principle, together with the Morse surgery formula, give the rational surgery formula described in this introduction. In Section 8, we turn to knots which admit $L$-space surgeries. In Section 9, we give the applications of the rational surgery formula to the problem of cosmetic surgeries on a knot in $S^3$. In Section 10, we turn to the Heegaard Floer homology groups of Seifert fibered spaces with even first Betti number.
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2. Preliminaries

We recall some of the background material and notation which will be required for the construction of the knot filtration. The bulk of this material is about the constructions relating (doubly-pointed) Heegaard diagrams for knots, and relative Spin\textsuperscript{c} structures for three-manifolds with boundary. We include also a few key properties of Heegaard Floer homology which will be used repeatedly throughout, see also [24].

2.1. Surgeries. Let $K \subset Y$ be a knot. the boundary of a tubular neighborhood of the knot $K$ is a torus $T$ equipped with a canonical choice of (isotopy class of) embedded curve $\mu$, a meridian for $K$. A longitude for $K$ is any embedded curve $\lambda$ in $T$ which meets a meridian transversally in a single point.

Given a homologically non-trivial, embedded curve $\gamma$ in $T$, we can form the three-manifold $Y_\gamma(K)$ which is gotten by attaching a solid torus to $Y - nd(K)$ so that $\gamma$ bounds a disk in the attached solid torus. We say that $Y_\gamma(K)$ is obtained from $Y$ by Dehn surgery along $K$. In the special case where $\gamma$ is a longitude for $K$, there is a canonical two-handle cobordism from $Y$ to $Y_\gamma(K)$, and we say that $Y_\gamma(K)$ is obtained from $Y$ by Morse surgery on $K$. A choice of longitude for $K$ is also called a framing of $K$.

Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with a framing $\lambda$. Since $K \in Y$ has finite order, there is a pair of integers $n$ and $d$ with minimal absolute value which satisfy the property that

$$d \cdot \lambda = n \cdot \mu \in H_1(Y - K; \mathbb{Z}),$$

In particular, it follows that the induced homology class $[K_\lambda] \in H_1(Y; \mathbb{Z})$ has order $|d|$. Here, and in the future, $K_\lambda$ denotes a copy of $K$ displaced into $Y - nd(K)$ using the framing $\lambda$.

2.2. Relative Spin\textsuperscript{c} structures for three-manifolds with boundary. Following Turaev [30], we say that two vector fields $v_1$ and $v_2$ on a closed, oriented three-manifold are homologous if they are homotopic in the complement of a ball in $Y$. The set of homology classes of vector fields can be given the structure of an affine space for $H^2(Y; \mathbb{Z})$, and indeed, it can be identified with the space of Spin\textsuperscript{c} structures over $Y$, Spin\textsuperscript{c}($Y$).

This construction can be readily generalized to the case of a three-manifold with torus boundary, c.f. Chapter I.4 of [30]. Specifically, on an oriented three-manifold $M$ with torus boundary, two vector fields $v_1$ and $v_2$ on $M$ which vanish nowhere and point outwards at $\partial M$ are said to be homologous if they are homotopic in the complement of a ball in the interior of $M$. The set of homology classes of nowhere vanishing vector fields can be naturally given the structure of an affine space for the relative cohomology $H^2(M, \partial M; \mathbb{Z})$. The homology classes of nowhere vanishing vector fields are called relative Spin\textsuperscript{c} structures on $M$, and are denoted Spin\textsuperscript{c}(M, $\partial M$).
Let $K \subset Y$ be a knot in a closed, oriented three-manifold, we can construct the three-manifold with torus boundary $M = Y - \text{nd}(K)$. In this case, we denote the relative Spin$^c$ structures over $M$ by $\text{Spin}^c(Y, K)$.

Orienting the core of the solid torus, we obtain a canonical isotopy class of nowhere vanishing vector field which points inward at the boundary, and which has a closed orbit which is the core of the solid torus (with its given orientation). More explicitly, the isotopy class of this vector field on $D^2 \times S^1$ is characterized by the property that in the interior of the solid torus, $w$ is everywhere transverse to the tangent planes to $D^2$.

Giving $K$ an orientation (and denoting this oriented knot by $\overline{K}$), we glue in this vector field $w$ to obtain a natural map

$$G_{Y, \overline{K}}: \text{Spin}^c(Y, K) \to \text{Spin}^c(Y)$$

which is equivariant with respect to the action by $H^2(Y, K; \mathbb{Z})$; i.e. letting

$$\iota: H^2(Y, K; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$$

be the natural map, we have for each $k \in H^2(Y, K; \mathbb{Z})$,

$$G_{Y, \overline{K}}(\xi + k) = G_{Y, \overline{K}}(\xi) + \iota(k).$$

As the notation suggests, $G_{Y, \overline{K}}$ depends on the orientation for $K$; indeed, for any $\xi \in \text{Spin}^c(Y, K)$,

$$(4) \quad G_{Y, \overline{K}}(\xi) - G_{Y, \overline{K}}(\xi) = \text{PD}[K].$$

Note that if $Y$ is a three-manifold, and $K \subset Y$ is a knot with meridian $\mu$, then the fibers of the map $G_{Y, \overline{K}}$ are the orbits of $\text{Spin}^c(Y, K)$ under the action by $\mathbb{Z} \cdot \text{PD}[\mu]$, where we think of $\text{PD}[\mu] \in H^2(Y, K; \mathbb{Z})$; i.e. $G_{Y, \overline{K}}$ realizes an identification

$$\text{Spin}^c(Y) \cong \frac{\text{Spin}^c(Y, K)}{\mathbb{Z} \cdot \text{PD}[\mu]}.$$

2.3. Doubly-pointed Heegaard diagrams. A doubly-pointed Heegaard diagram is a collection of data $(\Sigma, \alpha, \beta, w, z)$ where $\Sigma$ is an oriented surface of genus $g$, $\alpha = \{\alpha_1, ..., \alpha_g\}$ is a $g$-tuple of homologically linearly independent, pairwise disjoint, embedded curves in $\Sigma$ (a $g$-tuple of attaching circles), and $\beta = \{\beta_1, ..., \beta_g\}$ is another $g$-tuple of attaching circles, and $w$ and $z$ are two distinct pair of points in $\Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$.

A doubly-pointed Heegaard diagram gives rise to an oriented three-manifold $Y$ together with an oriented knot $\overline{K} \subset Y$. The orientation on $\Sigma$ induces an orientation on $U_\alpha$ (so that $\partial U_\alpha = \Sigma$), which can then be uniquely extended to an orientation over $Y$. The knot $K$ is obtained as a union of two arcs, $\eta_\alpha$ and $\eta_\beta$. The arc $\eta_\alpha$ is gotten by connecting $w$ and $z$ in $\Sigma - \alpha_1 - ... - \alpha_g$, and orienting it as a path from $w$ to $z$. This arc is then pushed into $U_\alpha$ so that only its boundary meets $\Sigma$ (at $w$ and $z$). The arc $\eta_\beta$ is obtained in an analogous manner, only reversing the roles of the circles in $\alpha$ and $\beta$. The oriented knot $\overline{K}$ is gotten by the difference $\eta_\alpha - \eta_\beta$. 
2.4. Relative Spin\(c\) structures associated to intersection points. Let \((\Sigma, \alpha, \beta, w, z)\) be a doubly-pointed Heegaard diagram. Fix \(x, y \in T_\alpha \cap T_\beta\). There are paths

\[ a: [0, 1] \rightarrow T_\alpha, \quad b: [0, 1] \rightarrow T_\beta \]

with \(\partial a = \partial b = x - y\). These paths can be viewed as arcs in \(\Sigma\) (supported inside the \(\alpha \cup \beta\)). The difference \(a - b\) is a closed one-cycle in \(\Sigma\) which is disjoint from \(w\) and \(z\). Indeed, since \(\Sigma - w - z\) is a subset of \(Y - K\), this one-cycle can be viewed as an element \(\xi(x, y) \in H_1(Y - K; \mathbb{Z})\).

We construct a map

\[ s_{w,z}: T_\alpha \cap T_\beta \rightarrow \text{Spin}^c(Y, K), \]

as follows (compare the analogous construction from [24]). A Heegaard diagram \((\Sigma, \alpha, \beta, w, z)\) can be realized by a self-indexing Morse function \(f: Y \rightarrow [0, 3]\), with a single index zero and three critical point (and \(g\) index one and two critical points), together with a Riemannian metric \(g\), for which \(\Sigma\) is the mid-level \(f^{-1}(3/2)\). \(\alpha_i\) is the locus of points flowing out of the \(i^{th}\) index one critical point (via gradient flow), and \(\beta_j\) is the locus of points flowing into the \(j^{th}\) index two critical point. Thus, \(x \in T_\alpha \cap T_\beta\) can be thought of as a \(g\)-tuple of gradient flow-lines \(\gamma_x\) containing all the index one and two critical points.

Now, the knot \(K\) is realized as a union of two flow-lines \(\gamma_w\) and \(\gamma_z\) which connect the index zero and index three critical points, meeting \(\Sigma\) in the points \(w\) and \(z\) respectively. The oriented knot \(\overline{K}\) is gotten by \(\gamma_z - \gamma_w\).

We construct a vector field representing the relative Spin\(c\) structure \(s_{w,z}(\xi)\) as follows. Modify the gradient vector field \(\nabla f\) in a neighborhood of the flows \(\gamma_x\) so that it has no zeros at any of the index one or two critical points. This modification involves a choice of nowhere vanishing vector field in a regular neighborhood of \(\gamma_x\), but it will follow easily from the construction that this choice will not affect the relative Spin\(c\) structure of the induced vector field. Next, we modify the vector field in a neighborhood of \(\gamma_w\) to obtain a new vector field \(v\) which has no zeros at either the index zero or three-critical points. In fact, this can be achieved so that the knot \(K\) is a closed orbit of the resulting vector field, whose orientation as a flow-line agrees with the orientation induced from \(\overline{K}\). This \(v\) modification involves a choice \(X\) of nowhere vanishing vector field on the neighborhood of \(\gamma_w\), with fixed behaviour on \(\overline{K}\). When calling attention to this choice, we write \(v = v(X)\). The resulting vector field \(v\) over \(Y\) is the vector field representing the Spin\(c\) structure \(s_w(x) \in \text{Spin}^c(Y)\) associated to the intersection point \(x\) and reference point \(w\), as described in [24] (and this fact is independent of the choice of \(X\)).

Our representative \(v\) has been constructed so that there is a neighborhood \(D_2 \times S^1\) of the closed flow-line \(\{0\} \times S^1 \cong \overline{K}\) (where here \(D_2\) is a disk of radius two centered at the origin) with the property that the disks \(D_2 \times \theta\) for any \(\theta \in S^1\) are transverse to the vector field \(v\). Consider a concentric disk \(D_1 \subset D_2\). We can continuously extend \(v_2 = v|_{Y - D_2 \times S^1}\) to a new vector field \(v_1 = v_1(x, X)\) over \(Y - D_1 \times S^1\), by using a vector
field over \((D_2 - D_1) \times S^1\) which is everywhere transverse to the annuli \((D_2 - D_1) \times \theta\), and which point towards the origin at \(\partial D_1 \times S^1\). Thus, the vector field \(v_1\) over \(Y - (D_1 \times S^1)\) inherits the vector field \(v\) which points outwards at the boundary. It is easy to see that the isotopy class of \(v_1\) is uniquely determined by the isotopy class of our initial vector field \(v\).

The induced relative Spin\(^c\) structure \(v_1\) over \(Y - \text{nd}(())K\) = \(Y - (D_1 \times S^1)\) depends, of course, on \(w, z, x\), and our choice \(X\), and we write it correspondingly as \(v_1(w, z, x, X)\). It is easy to see that

\[
G_{Y, K}(v_1(w, z, x, X)) = s_w(x).
\]

It is easy to see that the induced relative Spin\(^c\) structures over \(Y - \text{nd}(K)\) depends on the choice of \(X\) by

\[
v_1(w, z, x, X) - v_1(w, z, x, X') = a \cdot \text{PD}[\mu],
\]

where here \(a \in \mathbb{Z}\) is a universal constant (depending on only \(X\) and \(X'\); in fact, it is even independent of the ambient three-manifold \(Y\)). We choose \(X\) now to satisfy a normalization condition as follows.

Consider the unknot \(K \subset S^3\). An orientation \(\overline{K}\) specifies a canonical relative Spin\(^c\) structure \(\mathfrak{s}_0 \in \text{Spin}^c(S^3, \overline{K})\). This is the relative Spin\(^c\) structure represented by a vector field \(v\) over \(S^2 - \text{nd}(\overline{K}) \cong D^2 \times S^1\) which is everywhere transverse to the disks \(D^2\). The direction is specified by the condition that \(v\) can be represented by a vector field with closed orbits which have linking number one with our original knot \(K\). Our condition on \(X\) now is that for the standard genus one doubly-pointed diagram for the oriented unknot with a single intersection point \(x\), the relative Spin\(^c\) structure induced by \(v_1(w, z, x, X)\) is the canonical Spin\(^c\) structure for the oriented unknot.

With this choice for \(X\), it is easy now to see that the relative Spin\(^c\) structure underlying \(v_1(w, z, x, X)\) depends only on \(w, z, x\), inducing the required map

\[
\mathfrak{s}_{w, z}: T_\alpha \cap T_\beta \longrightarrow \text{Spin}^c(Y, K).
\]

We investigate its dependence on \(w, z, x\) in the following lemma. Continuing notation from [24], letting \(x, y \in T_\alpha \cap T_\beta\), let \(\pi_2(x, y)\) denote the space of homotopy classes of Whitney disks connecting \(x\) and \(y\), and for fixed

\[
p \in \Sigma - \alpha_1 - \ldots - \alpha_g - \beta_1 - \ldots - \beta_g
\]

and \(\phi \in \pi_2(x, y)\), let \(n_p(\phi)\) denote the intersection number of \(\phi\) with the submanifold \(\{p\} \times \text{Sym}^{g-1}(\Sigma) \subset \text{Sym}^g(\Sigma)\).

**Lemma 2.1.** Given \(x, y \in T_\alpha \cap T_\beta\), we have that

\[
(5) \quad \mathfrak{s}_{w, z}(y) - \mathfrak{s}_{w, z}(x) = \text{PD}[\epsilon(x, y)].
\]

In particular, if there is some \(\phi \in \pi_2(x, y)\), then

\[
(6) \quad \mathfrak{s}_{w, z}(x) - \mathfrak{s}_{w, z}(y) = (n_z(\phi) - n_w(\phi)) \cdot \text{PD}[\mu].
\]
Proof. We begin by establishing Equation (5). Vector fields representing $\mathfrak{g}_{w,z}(x)$ and $\mathfrak{g}_{w,z}(y)$ can be chosen so that they agree everywhere except in a regular neighborhood of $\xi(x, y)$. It follows that $\mathfrak{g}_{w,z}(x)$ and $\mathfrak{g}_{w,z}(y)$ differ by some multiple of the Poincaré dual of this curve (in the case where the curve is one). The fact that this multiple is one follows from the analogous property of $s_w(x)$ established in Lemma 2.19 of [24].

We turn to Equation (6). The homotopy class $\phi \in \pi_2(x, y)$ gives rise to a null-homotopy of $\xi(x, y)$ inside $Y$. This null-homotopy meets the knot $K$ with intersection number $n_z(\phi) - n_w(\phi)$. Thus, it can be modified to give a homology of $\xi(x, y)$ with $(n_z(\psi) - n_w(\psi)) \cdot \text{PD}[\mu]$ in $Y - K$. Equation (6) now follows from Equation (5). $\square$

2.5. Heegaard triples and relative Spin$^c$ structures. A Heegaard triple is a closed, oriented two-manifold $\Sigma$, equipped with three $g$-tuples of attaching circles, $(\Sigma, \alpha, \beta, \gamma)$, c.f. [24]. This gives rise to a four-manifold $X_{\alpha\beta\gamma}$ which has three boundary components $-Y_{\alpha\beta}, Y_{\beta\gamma}$, and $Y_{\alpha\gamma}$. (Of course, $Y_{\alpha\beta}$ here denotes the three-manifold described by the Heegaard diagram $(\Sigma, \alpha, \beta)$; $Y_{\alpha\gamma}$ and $Y_{\beta\gamma}$ are defined analogously.)

Suppose $K \subset Y$ is a framed knot. We can construct a Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$, where $Y_{\alpha\beta}$ represents $Y$, $Y_{\alpha\gamma}$ represents framed surgery along $K$, and $Y_{\beta\gamma}$ is a connected sum of $g - 1$ copies of $S^2 \times S^1$, and filling in $Y_{\beta\gamma}$ by a boundary connected sum of $g - 1$ copies of $B^3 \times S^1$, we obtain a four-manifold which is diffeomorphic to $W_\lambda(K)$. This is obtained by first writing down a Heegaard triple with the property that $K$ is supported entirely inside $U_\beta$, as the core of one of the handles in the handlebody, and $\beta_g$ represents its meridian, and realizing the framing $\lambda$ as a curve $\gamma_g$ which is disjoint from the $\beta_i$ for $i = 1, \ldots, g - 1$. We then let $\gamma_i$ for $i = 1, \ldots, g - 1$ be small, isotopic translates of the corresponding $\beta_i$. We would like to choose convenient reference points. These are points $w$ and $z$ are chosen so that there is an arc from $z$ to $w$ which crosses none of the $\alpha_i$ or $\gamma_i$ for $i = 1, \ldots, g$, or $\beta_j$ for $j = 1, \ldots, g - 1$, so that an arc from $z$ to $w$ crosses $\beta_g$ transversely once. An ordering on $w$ and $z$ is specified by an orientation on $K$.

We call such a doubly-pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, w, z)$ to be a doubly-pointed Heegaard triple subordinate to the framed, oriented knot $K \subset Y$ with framing $\lambda$.

By constructing the diagram carefully, we can arrange for there to be a unique intersection point $\Theta \in T_\beta \cap T_\gamma$ representing the generator $\Theta$ of the top-most non-trivial Floer homology group of $HF(\#^{g-1}(S^2 \times S^1))$. We denote this intersection point, and its corresponding homology class, by $\Theta$.

As in [24], we let $\pi_2(x, \Theta, y)$ denote the space of homotopy classes of Whitney triangles, i.e. continuous maps of the triangle into $\text{Sym}^g(\Sigma)$ which map the three edges to $T_\alpha$, $T_\beta$, and $T_\gamma$, and the vertices to the given points. In Section 8 of [24], it is shown that homotopy classes of triangles give rise to Spin$^c$ structures over $X_{\alpha\beta\gamma}$.

Proposition 2.2. Let $(\Sigma, \alpha, \beta, \gamma, w, z)$ be a doubly-pointed Heegaard triple subordinate to an oriented knot $K \subset Y$, equipped with framing $\lambda$. The map

$$\psi \in \pi_2(x, \Theta, y) \mapsto \mathfrak{g}_{w,z}(x) + (n_w(\psi) - n_z(\psi)) \mu$$
descends to give a well-defined map
\[ E_{Y,\lambda,\mathcal{F}} : \text{Spin}^c(W_\lambda(K)) \to \text{Spin}^c(Y, \overline{K}) \]
(i.e. which is independent of the choice of Heegaard triple). Moreover, letting \( \widetilde{F} \in H_2(W_\lambda(K), Y) \cong \mathbb{Z} \) denote a generator, we have that
\[ (7) \quad E_{Y,\lambda,\mathcal{F}}(s + \text{PD}[F_\lambda]) = E_{Y,\lambda,\mathcal{F}}(s) + \text{PD}[K_\lambda]. \]

**Proof.** Recall [24] that two triangles \( \psi \in \pi_2(x, \Theta, y) \) and \( \psi' \in \pi_2(x', \Theta, y') \) induce the same \( \text{Spin}^c \) structure on \( W_\lambda(K) \) is and only if there are \( \phi_1 \in \pi_2(x, x') \) (relative to the subspaces \( T_\alpha \) and \( T_\beta \)) and \( \phi_2 \in \pi_2(y, y') \) (relative to the subspaces \( T_\alpha \) and \( T_\gamma \)) with the property that
\[ \mathcal{D}(\psi) - \mathcal{D}(\psi') = \mathcal{D}(\phi_1) - \mathcal{D}(\phi_2) = n \cdot [\Sigma] \]
for some integer \( n \). (Here, as in [24], \( \mathcal{D}(\psi) \) denotes the two-chain in \( \Sigma \) induced by the homotopy class \( \psi \), whose multiplicity at some point \( p \) is the intersection number of \( \psi \) with \( p \times \text{Sym}^{g-1}(\Sigma) \).) But now \( n_w(\phi_2) = n_z(\phi_2) \) (since \( w \) and \( z \) lie in the same component of \( \Sigma - \alpha_1 - ... - \alpha_g - \gamma_1 - ... - \gamma_g \)); combining this with the fact that
\[ \bar{s}_{w,z}(x') = \bar{s}_{w,z}(x) + (n_w(\phi_1) - n_z(\phi_1)) \cdot \mu \]
(Lemma 2.1), it follows that in this case
\[ \bar{s}_{w,z}(x) + (n_w(\psi) - n_z(\psi))\mu = \bar{s}_{w,z}(x') + (n_w(\psi') - n_z(\psi'))\mu. \]

Independence of the Heegaard triple is a routine consequence that any two Heegaard triples can be connected by a sequence of stabilizations and handle slides among the \( \alpha_i \), \( \beta_j \) for \( j = 1, ..., g - 1 \), and the distinguished meridian \( \beta_g \) handlesliding over the other \( \beta_j \).

We now verify Equation (7). Suppose that we have \( \psi \in \pi_2(x, \Theta, y) \) and \( \psi' \in \pi_2(x', \Theta, y) \). The difference \( \mathcal{D}(\psi) - \mathcal{D}(\psi') \) gives a two-chain \( C \) in \( \Sigma \), which has no corners on \( \gamma_i \) for \( i = 1, ..., g \). The corners of \( C \) occur at \( x'_i \in x' \) and \( x_i \in x \). Indeed, the boundary of the two-chain consists of \( \xi(x, x') \) and some number of copies of the \( \gamma_i \). As such, it can be thought of as furnishing a homology
\[ \xi(x, x') = \ell \cdot \gamma_g + (n_w(\psi - \psi') - n_z(\psi - \psi'))\mu \]
in the knot complement (we are thinking of \( \lambda \) as a curve in the Heegaard surface; of course, it is identified with \( K_\lambda \) thought of as a curve in the knot complement). Indeed, \( \psi - \psi' \) corresponds to a surface-with-boundary \( Z \) in the four-manifold underlying the Heegaard triple, whose boundary is supported entirely \( Y \). The integer \( \ell \) corresponds to the multiplicity with which \( Z \) meets the cocore of the attaching two-handle. Thus, \( s_w(\psi) - s_w(\psi') = \ell \cdot \widetilde{F} \), and the equation follows. \qed
For a pointed Heegaard triple \((\Sigma, \alpha, \beta, \gamma, z)\), the group of two-dimensional homology classes is identified with the group of relations \(a + b + c = 0\) in \(H_1(\Sigma; \mathbb{Z})\) where \(a\) resp. \(b\) resp. \(c\) lies in the span of \(\{[\alpha_i]\}_{i=1}^g\) resp. \(\{[\beta_i]\}_{i=1}^g\) resp. \(\{[\gamma_i]\}_{i=1}^g\). These relations correspond to two-chains \(P\) in \(\Sigma\) with boundary a formal linear combination of the attaching circles, and with \(n_z(P) = 0\). Given a Whitney triangle \(\psi \in \pi_2(x, y, w)\), there is a formula for the evaluation \(\langle c_1(s_\psi)_e, [P]\rangle\), where \([P]\) denotes the second homology class corresponding to \(P\), in terms of the Heegaard triple, cf. Proposition 6.3 of [14]. To describe this, we need two notions, the Euler measure of the periodic domain and the dual spider number of the triangle with respect to the triply-periodic domain.

Let \(\Phi: F \to \Sigma\) be a representative for \(P\), where here \(\Phi\) is an immersion in a neighborhood of \(\partial F\). The line bundle \(\Phi^*(T\Sigma)\) has a canonical trivialization over \(\partial F\), since \(\Phi\) induces an isomorphism

\[
TF \cong \Phi^*(T\Sigma),
\]

and \(TF\) is canonically trivialized near \(\partial F\) (using the outward normal orientation on \(F\)). We define \(\tilde{\chi}(P)\) to be the Euler number of \(\Phi^*(T\Sigma)\), relative to this trivialization at \(\partial F\),

\[
\tilde{\chi}(P) = \langle c_1(\Phi^*(T\Sigma), \partial F), F \rangle.
\]

Note first that the orientation of \(\Sigma\), and the orientations on all the attaching circles \(\alpha, \beta, \gamma\) naturally induce “inward” normal vector fields to the attaching circles (i.e. if \(\gamma: S^1 \to \Sigma\) is a unit speed immersed curve, this inward normal vector is given by \(J \frac{\partial \gamma}{\partial t}\)). Let \(\alpha'_i, \beta'_i, \gamma'_i\) denote copies of the corresponding attaching circles \(\alpha_i, \beta_i, \gamma_i\), translated slightly in these normal directions. Let \(\alpha', \beta', \gamma'\) denote the corresponding \(g\)-tuples, and \(T'_\alpha, T'_\beta, T'_\gamma\) be the corresponding tori in \(\text{Sym}^g(\Sigma)\). By construction, then, \(u(e_\alpha)\) misses \(T'_\alpha, u(e_\beta)\) misses \(T'_\beta\), and \(u(e_\gamma)\) misses \(T'_\gamma\).

Let \(x \in \Delta\) be a point in the interior, chosen in general position, so that the \(g\)-tuple \(u(x)\) misses all of \(\alpha', \beta', \gamma'\). Choose three paths \(a, b, c\) from \(x\) to \(e_0, e_1, e_2\) respectively. The central point \(x\) and the three paths \(a, b, c\) is called a dual spider. We can think of the paths \(a, b, c\) as one-chains in \(\Sigma\). Recall that \(\partial P\) has three types of boundaries: the \(\alpha\), \(\beta\), and \(\gamma\) boundaries, which we denote \(\partial_\alpha P, \partial_\beta P, \partial_\gamma P\). Let \(\partial_\alpha P, \partial_\beta P, \partial_\gamma P\) respectively denote the one-chains obtained by translating the corresponding boundary components using the induced normal vector fields. The dual spider number of \(u\) and \(P\) is defined by

\[
\sigma(u, P) = n_{u(x)}(P) + \#(a \cap \partial_\alpha P) + \#(b \cap \partial_\beta P) + \#(c \cap \partial_\gamma P).
\]

According to Proposition 6.3 of [14],

\[(8) \quad \langle c_1(s_\psi(u)), H(P) \rangle = \tilde{\chi}(P) + \#(\partial P) - 2n_z(P) + 2\sigma(u, P),\]

where here \#(\partial P) represents the number of boundary components of \(P\), with multiplicity.
2.6. Filtered complexes. A \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered complex is a chain complex \( C \) whose underlying Abelian group decomposes as
\[
C = \bigoplus_{(i,j) \in \mathbb{Z} \oplus \mathbb{Z}} C\{i,j\},
\]
and whose boundary operator \( \partial \) carries \( C\{i_0, j_0\} \) into the subset
\[
C\{i \leq i_0 \text{ and } j \leq j_0\} = \bigoplus_{\{i', j'\} \mid i' \leq i \text{ and } j' \leq j} C\{i', j'\} \subset C.
\]

A map \( f: C \rightarrow C' \) between filtered complexes is called a filtered map if it carries \( C\{i_0, j_0\} \) into \( C'\{i \leq i_0 \text{ and } j \leq j_0\} \). Two filtered chain maps \( f_0, f_1: C \rightarrow C' \) are called filtered homotopic if there is a filtered map \( H: C \rightarrow C' \) with
\[
f_0 - f_1 = \partial' \circ H + H \circ \partial.
\]

Two filtered chain maps \( C \) and \( C' \) are called filtered chain homotopy equivalent if there are filtered chain maps \( f: C \rightarrow C' \) and \( g: C' \rightarrow C \) with the property that \( f \circ g \) and \( g \circ f \) are filtered homotopic to the corresponding identity maps.

A \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered \( \mathbb{Z}[U] \)-complex \( C \) is a filtered chain complex equipped with an endomorphism \( U: C \rightarrow C \) whose restriction to \( C\{i,j\} \) maps to \( C\{i-1, j-1\} \).

If \( C_1 \) and \( C_2 \) are \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered \( \mathbb{Z}[U] \)-chain complexes, then we can form their tensor product \( C_1 \otimes_{\mathbb{Z}[U]} C_2 \). This can be given a \( \mathbb{Z} \oplus \mathbb{Z} \)-filtration by
\[
(C \otimes C')\{i,j\} = \bigoplus_{\{(i_1,j_1),(i_2,j_2)\} \mid (i_1,j_1)+(i_2,j_2)=(i,j)} \frac{C\{i_1,j_1\} \otimes C\{i_2,j_2\}}{U_1 \cdot \xi_1 \otimes U_2 \cdot \xi_2}. \quad (C \otimes C')\{i,j\} = \bigoplus_{\{(i_1,j_1),(i_2,j_2)\} \mid (i_1,j_1)+(i_2,j_2)=(i,j)} \frac{C\{i_1,j_1\} \otimes C\{i_2,j_2\}}{U_1 \cdot \xi_1 \otimes U_2 \cdot \xi_2}.
\]

If \( C \) is a filtered complex, and \( (a, b) \in \mathbb{Z} \oplus \mathbb{Z} \), let \( C[(a, b)] \) denote the filtered complex whose underlying chain complex is isomorphic, but whose filtration is shifted by \( (a, b) \); i.e.
\[
C[(a, b)]\{i, j\} = C\{a + i, b + j\}.
\]

A relatively filtered map \( f: C \rightarrow C' \) is a chain map which respects the filtration on \( C \) and the filtration \( C'[(a, b)] \) for some \( (a, b) \in \mathbb{Z} \oplus \mathbb{Z} \).

2.7. Absolute gradings. Heegaard Floer homology is natural under cobordisms. Indeed, if \( W \) is a smooth, connected, oriented cobordism with \( \partial W = -Y_1 \cup Y_2 \) which is equipped with a \( \text{Spin}^c \) structure \( s \) whose restriction \( t_i = s|_{Y_i} \) for \( i = 1, 2 \) has torsion first Chern class, then there is an induced chain map
\[
f_{W,s}^+: \text{CF}^+(Y_1, t_1) \rightarrow \text{CF}^+(Y_2, t_2)
\]
which is homogeneous of degree
\[
\frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}
\]
(c.f. Theorem 7.1 of [14]).

Let \( T^+ \) denote the module \( \mathbb{Z}[U,U^{-1}]/U \cdot \mathbb{Z}[U] \). Recall [24] that \( HF^+(S^3) \cong T^+ \).

The \( \mathbb{Q} \)-grading on Floer homology is characterized by Equation (9), together with the normalization that \( HF_d^+(S^3) \) is trivial for all \( d < 0 \), non-trivial in degree \( d = 0 \).

Following our usual notational conventions, we write \( T^+_{(d)} \) for the module \( T^+ \), thought of as a graded \( \mathbb{Z}[U] \)-module, where multiplication by \( U \) lowers absolute degree by 2, and the non-zero homogeneous elements of lowest degree have degree \( d \). In this notation, then, \( HF^+(S^3) \cong T^+_{(0)} \).

2.8. **Approximating \( CF^+ \).** Following [25], it is useful to have the following approximation to \( CF^+(Y) \): fix an integer \( \delta \geq 0 \), let \( CF^\delta(Y) \subset CF^+(Y) \) denote the subcomplex which is annihilated by multiplication by \( U^{\delta+1} \), and let \( HF^\delta(Y) \) denote the homology of \( CF^\delta(Y) \). In particular, for \( \delta = 0 \), this construction gives \( \hat{HF}(Y) \). Note that for all \( \delta \geq 0 \), \( HF^\delta(Y) \) is a three-manifold invariant.
3. Construction of the knot filtration

The aim of this section is to construct the knot filtration: a filtration of $CF^\infty(Y)$ induced by a rationally null-homologous knot $K \subset Y$. Most of this discussion is a straightforward generalization of the corresponding constructions for null-homologous knots $K \subset Y$ described in [22] and [27].

Let $K \subset Y$ be a rationally null-homologous knot, and let $(\Sigma, \alpha, \beta, w, z)$ be a corresponding doubly-pointed Heegaard diagram.

For fixed $\xi \in \text{Spin}^c(Y, K)$, let $T(\xi) \subset (T_\alpha \cap T_\beta) \times \mathbb{Z} \times \mathbb{Z}$ be the subset of elements $[x, i, j]$ satisfying
\[ s_w(z)(x) + (i - j) \cdot \text{PD}[\mu] = \xi. \]

According to Lemma 2.1, if $[x, i, j] \in T(\xi)$, and $\phi \in \pi_2(x, y)$, then $[y, i - n_w(\phi), j - n_z(\phi)] \in T(\xi)$.

Let $CFK^\infty(\Sigma, \alpha, \beta, w, z, \xi)$ be the chain complex generated by $[x, i, j] \in T(\xi)$, endowed with the differential
\[
\partial^\infty[x, i, j] = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y)} \# \left( \frac{M(\phi)}{R} \right) \cdot [y, i - n_w(\phi), j - n_z(\phi)],
\]
where as usual, $M(\phi)$ is the moduli space of pseudo-holomorphic representatives of $\phi$, and $\mu(\phi)$ denotes its expected dimension.

The map $F: T(\xi) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$

given by $F[x, i, j] = (i, j)$ induces a $\mathbb{Z} \oplus \mathbb{Z}$ filtration on $CFK^\infty(\Sigma, \alpha, \beta, w, z)$.

**Theorem 3.1.** The filtered chain homotopy type of $CFK^\infty(\Sigma, \alpha, \beta, w, z, \xi)$ is an invariant of the underlying oriented knot $\overline{K} \subset Y$ and choice of relative Spin$^c$ structure $\xi \in \text{Spin}^c(Y, K)$.

**Proof.** This is a routine adaptation of the corresponding statement in Theorem 3.1 of [22].

We denote this complex by $CFK^\infty(Y, \overline{K}, \xi)$.

**Proposition 3.2.** Let $C_\xi = CFK^\infty(Y, \overline{K}, \xi)$. We have that
\[ C_{\xi + \text{PD}[\mu]} = C_\xi[(0, -1)]. \]

The short exact sequences of $\mathbb{Z}[U]$-chain complexes
\[
0 \rightarrow C_\xi\{i < 0 \} \rightarrow C_\xi \rightarrow C_\xi\{i \geq 0 \} \rightarrow 0
\]
and
\[
0 \rightarrow C_\xi\{i = 0 \} \rightarrow C_\xi\{i \geq 0 \} \xrightarrow{U} C_\xi\{i \geq 0 \} \rightarrow 0
\]
are isomorphic to the short exact sequences
\[ 0 \rightarrow CF^{-}(Y, s) \rightarrow CF^{\infty}(Y, s) \rightarrow CF^{+}(Y, s) \rightarrow 0, \]
and
\[ 0 \rightarrow \widehat{CF}(Y, s) \rightarrow CF^{+}(Y, s) \rightarrow U \rightarrow CF^{+}(Y, s) \rightarrow 0 \]
respectively, where here \( s = G_{Y, \overline{\mu}}(\xi). \) Similarly, the short exact sequences of \( \mathbb{Z}[U] \)-chain complexes
\[ 0 \rightarrow C_{\xi}\{ j < 0 \} \rightarrow C_{\xi} \rightarrow C_{\xi}\{ j \geq 0 \} \rightarrow 0 \]
and
\[ 0 \rightarrow C_{\xi}\{ j = 0 \} \rightarrow C_{\xi}\{ j \geq 0 \} \rightarrow U \rightarrow C_{\xi}\{ j \geq 0 \} \rightarrow 0 \]
are isomorphic to the short exact sequences
\[ 0 \rightarrow CF^{-}(Y, s') \rightarrow CF^{\infty}(Y, s') \rightarrow CF^{+}(Y, s') \rightarrow 0, \]
and
\[ 0 \rightarrow \widehat{CF}(Y, s') \rightarrow CF^{+}(Y, s') \rightarrow U \rightarrow CF^{+}(Y, s') \rightarrow 0, \]
where here \( s' = G_{Y, -\overline{\mu}}(\xi). \)

The set of relative Spin\(^c\) structures \( G_{Y, \overline{\mu}}^{-1}(s) \) inducing a fixed Spin\(^c\) structure over \( Y \) is a well-ordered set, under the rule that \( \xi_1 \leq \xi_2 \) if \( \xi_2 = \xi_1 + j \cdot PD[\mu] \) for some \( j \geq 0. \)

This ordering on relative Spin\(^c\) structures gives rise to an ordering of the generators of \( \widehat{CF}(Y) \) by \( \xi \mapsto \overline{s}_{w, z}(\xi). \) It is easy to see that this ordering gives rise to a filtration of the complex \( \widehat{CF}(Y) = \bigoplus_{x \in T_{\alpha} \cap T_{\beta}} \mathbb{Z}x \) endowed with the usual differential
\[ \partial x = \sum_{\{y \in T_{\alpha} \cap T_{\beta} \mid n_{w}(\phi) = 0, \mu(\phi) = 1\}} \# \left( \frac{M(\phi)}{\mathbb{R}} \right) \cdot y. \]

The homology of the associated graded object is the knot Floer homology
\[ \widehat{HFK}(Y, K) = \bigoplus_{\xi \in Spin^{c}(Y, \overline{\mu})} \widehat{HFK}(Y, K, \xi), \]
where \( \widehat{HFK}(Y, K, \xi) \) is generated by \( x \) with \( \overline{s}_{w, z}(x) = \xi, \) and whose differential counts holomorphic disks with \( n_{w}(\phi) = n_{z}(\phi) = 0. \)
3.1. **Relationship with knot Floer homology for null-homologous knots.** If \( K \subset Y \) is a null-homologous knot, then a choice of Seifert surface for \( Y \) gives an identification \( \text{Spin}^c(Y, K) \cong \mathbb{Z} \oplus \text{Spin}^c(Y) \). Thus, with this additional choice, we identify the \( \text{Spin}^c(Y) \)-filtration of \( \hat{C}F(Y, s) \) with a \( \mathbb{Z} \)-filtration. This identification is used, for example, in [22], where the knot filtration is described as a filtration by \( \mathbb{Z} \), rather than by relative \( \text{Spin}^c \) structures.

For example, if \( \overline{K} \) is a knot in \( S^3 \) and \( s \) is an integer, then \( \hat{HFK}(S^3, \overline{K}, \xi) \) in the present notation corresponds to \( \hat{HFK}(Y, \overline{K}, s) \) for \( s \in \mathbb{Z} \) in the notation from the introduction or [22], where here \( s \) and \( \xi \) are related by \( c_1(\xi) = 2s \text{PD}[\mu] \).
4. **Knot Floer homology and large surgeries**

We describe here the result of forming “large surgeries” on a rationally null-homologous, framed knot $K \subset Y$. This result generalizes a corresponding result in the null-homologous case, c.f. [22] and [27], and it is a special case of Theorem 6.1.

Let $Y$ be a knot given with framing $\lambda$. We can form a three-manifold $Y_{m\mu+\lambda}(K)$ obtained by filling the curve $m \cdot \mu + \lambda$ in $Y - \text{nd}(K)$. Let

$$W'_m(K): Y_{m\mu+\lambda}(K) \to Y$$

denote the two-handle cobordism obtained by turning around the two-handle cobordism from $-Y$ to $-Y_{m\mu+\lambda}(K)$. Note that

$$H_2(W'_m(K), Y; \mathbb{Z}) \cong \mathbb{Z},$$

and let $\hat{F} \subset W'_m(K)$ be a surface-with-boundary representing a generator. Clearly, PD[$\hat{F}$]|$_Y = \text{PD}[K] \in H^2(Y; \mathbb{Z})$. Note that for sufficiently large $m$, the self-intersection number of $\hat{F}$ is negative.

Let $\mathcal{K} \subset Y$ be an oriented knot in a three-manifold $Y$, whose induced homology class is trivial in rational homology. We fix also a framing $\lambda$ of $K$. For a fixed relative Spin$^c$ structure $\xi \in \text{Spin}^c(Y, K)$, let $C_\xi$ be the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex $CFK^\infty(Y, \mathcal{K}, \xi)$. There are two projection maps

$$(11) \quad C_\xi\{\max(i, j) \geq 0\} \to C_\xi\{i \geq 0\} \text{ and } C_\xi\{\max(i, j) \geq 0\} \to C_\xi\{j \geq 0\}.$$

Denoting

$$(12) \quad A^+_\xi(Y, \mathcal{K}) = C_\xi\{\max(i, j) \geq 0\} \text{ and } B^+_\xi(Y, \mathcal{K}) = CF^+(Y, G_{Y, \mathcal{K}}(\xi))$$

and using the identifications (from Proposition 3.2)

$$C_\xi\{i \geq 0\} \cong CF^+(Y, G_{Y, \mathcal{K}}(\xi)) \text{ and } C_\xi\{j \geq 0\} \cong CF^+(Y, G_{Y, -\mathcal{K}}(\xi)) \cong CF^+(Y, G_{Y, \mathcal{K}}(\xi + \text{PD}[K, \lambda])),$$

where here $K, \lambda$ is the push-off of $K$ inside $Y - K$ using the framing $\lambda$, we can view the canonical projection maps of Equation (11) as maps

$$(13) \quad u^+_\xi: A^+_\xi(Y, \mathcal{K}) \to B^+_\xi(Y, \mathcal{K}) \text{ and } h^+_\xi: A^+_\xi(Y, \mathcal{K}) \to B^+_{\xi + \text{PD}[K, \lambda]}(Y, \mathcal{K}).$$

**Theorem 4.1.** Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with a framing $\lambda$. Then, for all sufficiently large $m$, there is a map

$$\Xi: \text{Spin}^c(Y_{m, \mu+\lambda}(K)) \to \text{Spin}^c(Y, K)$$

with the property that for all $t \in \text{Spin}^c(Y_{m, \mu+\lambda}(K))$, the group $CF^+(Y_{m, \mu+\lambda}(K), t)$ is represented by the chain complex

$$A^+_{\Xi(t)} = C_\xi\{\max(i, j) \geq 0\},$$

in the sense that there are isomorphisms (of relatively $\mathbb{Z}$-graded $\mathbb{Z}[U]$-complexes)

$$\Psi^+_t : CF^+(Y_{m, \mu+\lambda}(K), t) \to A^+_{\Xi(t)}(Y, \mathcal{K}).$$
Furthermore, fix $t \in \text{Spin}^c(Y_{m\mu+\lambda}(K))$, and let $\xi = \Xi(t)$. There are Spin$^c$ structures $\mathfrak{r}, \eta \in W'_m(K)$ with $E_{Y,m\mu+\lambda}(\mathfrak{r}) = \xi$, and $\eta = \mathfrak{r} + \text{PD}[\hat{F}]$ with the property that the maps $v^+_{\xi}$ and $h^+_{\xi}$ correspond to the maps induced by the cobordism $W'_m(K)$ equipped with $\mathfrak{r}$ and $\eta$ respectively. More precisely, the following squares commute:

$$
\begin{array}{ccc}
CF^+(Y_{m\mu+\lambda}(K), t) & \xrightarrow{\Psi^+_{t,m}} & CF^+(Y, G_{Y,\overline{\mathfrak{r}}}(\xi)) \\
A^+_{\xi}(Y, K) & \xrightarrow{v^+_{\xi}} & B^+_{\xi}(Y, K)
\end{array}
$$

and

$$
\begin{array}{ccc}
CF^+(Y_{m\mu+\lambda}(K), t) & \xrightarrow{\Psi^+_{t,m}} & CF^+(Y, G_{Y,\overline{\mathfrak{r}}}(\xi)) \\
A^+_{\xi}(Y, K) & \xrightarrow{h^+_{\xi}} & B^+_{\xi+\text{PD}[\mathfrak{r}]}(Y, K).
\end{array}
$$

The following result is also easy consequences of the proof:

**Proposition 4.2.** Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with framing $\lambda$. For any $\delta > 0$, there is an integer $N$ with the property that for all $m \geq N$ and all $t \in \text{Spin}^c(Y_{m\mu+\lambda}(K))$, there are at most two Spin$^c$ structures in Spin$^c(W_{m\mu+\lambda}(K))$ with restriction to $Y_{m\mu+\lambda}(K)$ equal to $t$ for which the induced map

$$
F^\delta_{H_{m\mu+\lambda}(K),t}: HF^\delta(Y_{m\mu+\lambda}(K), t) \longrightarrow HF^\delta(Y)
$$

is non-trivial. These are the Spin$^c$ structures $\mathfrak{r}$ and $\eta$ associated to $t$ from Theorem 4.1 above.

We return to the proofs of Theorem 4.1 and Proposition 4.2 after some preliminary discussion and lemmas.

We work with a family of doubly-pointed Heegaard triples for the framed knot $\overline{\mathfrak{K}}$ ($\Sigma, \alpha, \beta, \gamma, w, z$), so that there are identifications $Y_{\alpha, \gamma} \cong Y_{m\mu+\lambda}(K)$, $Y_{\beta, \gamma} \cong \#_g(S^2 \times S^1)$, $Y_{\alpha, \beta} \cong Y$. We can give Heegaard triples for all of the $W'_m$ which differ only in $\gamma_g$, which winds along the meridian $\mu$. We call this region the “winding region” (c.f. Figure 1 for an illustration of a winding region; in this picture, the subscript for $\gamma_g$ is dropped).

**Definition 4.3.** An intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ is said to be supported in the winding region if its component along $\gamma_g$ lies in the winding region. Given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ (both supported in the winding region), we say that $\phi \in \pi_2(x, y)$ is supported in the if segment of $\partial(D(\phi))$ in $\gamma_g$ is a subset of the winding region. (Note that if $\phi \in \pi_2(x, y)$ is supported in the winding region, then so is any $\phi' \in \pi_2(x, y)$.) An equivalence class of intersection
point for \( Y_{m,\mu+\lambda} \) is said to be supported in the winding region if every intersection point in the equivalence class is supported in the winding region, and any two intersection points can be connected by Whitney disks supported in the winding region. Finally, \( \psi \in \pi_2(x,\Theta,y) \) is said to be a small triangle if the \( \gamma_g \) arc in \( \partial D(\psi) \) is supported in the winding region. Note that in this case, \( y \) is the “closest point” in \( \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) to \( x \).

**Definition 4.4.** The depth of a basepoint \( z \) in the winding region is the minimum absolute value of the algebraic intersection number of \( \gamma_g \) with any arc \( A \subset \Sigma - \alpha_1 - \ldots - \alpha_g \) which connects \( z \) with some basepoint \( z_0 \) outside the winding region with \( z \). Given an integer \( \epsilon > 0 \), we say that \( z \) is \( \epsilon \)-centered if its depth is less than or equal to \( (m - \epsilon)/2 \). Similarly, a choice of meridian \( \mu \) is called \( \epsilon \)-centered if each point \( z \in \mu \) is \( \epsilon \)-centered.

For a Heegaard diagram with an \( \epsilon \)-centered meridian, with basepoints \( w \) and \( z \) on either side of \( \mu \). Then, for each small triangle \( \psi \), we have that at least one of \( n_w(\psi) \) or \( n_z(\psi) = 0 \), and also

\[
\max(|n_w(\psi)|, |n_z(\psi)|) \leq \frac{m - \epsilon}{2}.
\]

This is an immediate consequence of the definitions.

**Lemma 4.5.** There is an integer \( \epsilon > 0 \) with the property that for all sufficiently large \( m \), each \( t \in \text{Spin}^c(Y_{m,\mu+\lambda}(K)) \) can be represented by an equivalence class of intersection points supported in the winding region and an \( \epsilon \)-centered choice of meridian.

**Proof.** If an equivalence class of intersection points in \( \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) is not supported in the winding region, we say it is bad. It is clear that for all large \( m \), the set of bad equivalence classes is bounded. We choose \( \epsilon \) so that \( 2\epsilon \) is greater than this number. Now move the basepoint as needed. \(\square\)

Given \( \xi \in \text{Spin}^c(Y,K) \), define

\[
\Psi_\xi : CF^\infty(Y_{m,\mu+\lambda}(K),t) \rightarrow CF^\infty(Y,K,\xi)
\]

**Figure 1. Illustration of the Heegaard triple** The integers denote (non-zero) local multiplicities of the “small triangle” connecting \( x \) and \( x' \). This picture is taking place in a cylindrical region in \( \Sigma \).
by
\[
\Psi^\infty_\xi[x, i] = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\psi \in \pi_2(x, \Theta, y) \mid E_{W_{m}(K)}(s_w(\psi)) = \xi} ([y, i - n_w(\psi), i - n_z(\psi)]),
\]
where here \(x\) represents \(s_w(\psi)|_{Y_{m\mu+\lambda}(K)}\). We claim that this is a chain map.

Given \(t \in \text{Spin}^c(Y_{m\mu+\lambda}(K))\), we realize \(t\) by an equivalence class of intersection points supported in the winding region. Define \(\xi = \Xi(t)\) by
\[
\xi = \mathfrak{g}_{w,z}(x) + (n_w(\psi) - n_z(\psi)) \cdot \mu,
\]
where \(\psi \in \pi_2(x', \Theta, x)\) is any small triangle connecting an intersection point \(x' \in T_\alpha \cap T_\gamma\) representing \(t\) with its “nearest point” \(x \in T_\alpha \cap T_\beta\). The restriction of \(\Psi^\infty_\xi\) to \(CF^+(Y_{m\mu+\lambda}(K), t)\) induces a chain map
\[
\Psi^+_\xi : CF^+(Y_{m\mu+\lambda}(K), t) \to A^+_\xi(Y, K).
\]

**Proof of Theorem 4.1.** The argument from [22] shows that \(\Psi^+\) induces an isomorphism. Post-composing \(\Psi^+\) with the projection \(C_t\{\max(i, j) \geq 0\} \to C\{i \geq 0\}\) (i.e. \(v^+\)), we obtain the map induced by the cobordism \(W'_m(K)\) equipped with the \(\text{Spin}^c\) structure \(\tau_s\); i.e. the first square in the statement of the theorem commutes. Commutativity of the second square follows similarly. \(\Box\)

**Lemma 4.6.** There is a constant \(c\) with the property that for all sufficiently large \(m\), for any of the \(\text{Spin}^c\) structures \(\{f(t), \eta(t)\} \in \text{Spin}^c(Y_{m\mu+\lambda}(K))\) appearing in Theorem 4.1, we have that
\[
-c \leq \langle c_1(f), [F] \rangle \leq 2m + c
\]
\[
-2m - c \leq \langle c_1(\eta), [F] \rangle \leq c.
\]

**Proof.** There is a constant \(C\) (depending on \(x \in T_\alpha \cap T_\beta\), but independent of \(m\)) with the property that for any small triangle \(\psi \in \pi_2(x', \Theta, x)\),
\[
\langle c_1(s_w(\psi)), [F] \rangle + [F] \cdot [F] = C + 2(n_w(\psi) - n_z(\psi)).
\]

This can be seen as follows. Consider the function
\[
f(x) = \langle c_1(s_w(\psi)), [F] \rangle - 2(n_w(\psi) - n_z(\psi)),
\]
where here \(\psi \in \pi_2(x', \Theta, x)\) is a small triangle. We claim that this is independent of the choice of \(x'\). This can be seen by varying \(x'\), and appealing to Equation (8). Similarly, to verify its independence of \(m\), it suffices to consider a fixed small triangle \(\psi \in \pi_2(x, \Theta, x')\), and observe that \(\langle c_1(s_w(\psi)), [F] \rangle\) increases by one as the number \(m\) is increased by one, and furthermore \([F] \cdot [F]\) also decreases by one. Note here that by “fixed small triangle”, we mean that \(x\) is fixed, as is \(n_w(\psi) - n_z(\psi)\), though, of course, \(\gamma_g\) is varied. This assertion is an easy consequence of Equation (8). More specifically, with the integer \(d\) chosen as in Equation (3), we see that \(d\cdot [F]\) can be represents a generator
of \(H_2(W'_m(K); \mathbb{Z})\). Specifically, let \(\mathcal{P}_m\) be a generator for the space of triply-periodic domains, and write \(\partial[\mathcal{P}_m] = A + B + C\), where \(A\), \(B\), and \(C\) are first homology classes in the spans of \(\alpha\), \(\beta\), and \(\gamma\) respectively. In view of Equation (3), \(\partial\mathcal{P}_m\) has \(C\) component given by \(d(m\mu + \lambda)\) and \(B\) component \(-\langle dm + n\rangle\mu\), modulo the other \(\beta_i\) for \(i < g\). Thus, \(\#(\partial\mathcal{P}_m) = dm + c\) for some constant \(c\). It is easy to see that the other quantities in Equation (8) are independent of \(m\).

Thus, we have seen that for any small triangle \(\psi \in \pi_2(\mathcal{X}', \Theta, x)\),

\[
\langle c_1(s_w(\psi)), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] - 2(n_w(\psi) - n_z(\psi))
\]

depends only on \(x \in T_\alpha \cap T_\beta\). Since there are only finitely many intersection points \(T_\alpha \cap T_\beta\), it follows that there is some constant \(C'\) with the property that

\[
|\langle c_1(s_w(\psi)), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] - 2(n_w(\psi) - n_z(\psi))| \leq C'.
\]

By restricting to \(\epsilon\)-centered base points for some integer \(\epsilon\) independent of \(m\) (which we can do in light of Lemma 4.5), we can arrange for \(2|n_w(\psi) - n_z(\psi)| \leq m - \epsilon\) for all small triangles (c.f. Equation (14)). The result now follows, bearing in mind that \(\mathfrak{r} = s_w(\psi)\), and \(\eta = s_z(\psi) = \mathfrak{r}(\psi) - PD[\widehat{F}]\).

\[\square\]

**Proof of Proposition 4.2.** We choose \(m\) large enough that \(W'_m(K)\) has negative-definite intersection form. Given \(t \in \text{Spin}^c(Y)\), let \(\mathcal{S}(t) \subseteq \text{Spin}^c(W'_m(K))\) denote the set of Spin\(^c\) structures whose restriction to \(Y\) is \(t\). This set \(\mathcal{S}(t)\) has the form \(\{s_0 + n \cdot PD[\widehat{F}]\}_{n \in \mathbb{Z}}\) for some fixed \(s_0 \in \text{Spin}^c(W'_m(K))\). The function \(n \mapsto c_1(s_0 + n \cdot PD[\widehat{F}])^2\) is a quadratic function of \(n\) which is bounded above.

Choosing \(m\) larger than the constant \(c\) from Lemma 4.6, it follows easily that \(c_1(\eta)^2 \geq c_1(\eta + PD[\widehat{F}])^2\), while \(c_1(\mathfrak{r})^2 \geq c_1(\mathfrak{r} - PD[\widehat{F}])^2\). Since \(\eta = \mathfrak{r} + PD[\widehat{F}]\), it follows readily that at least one of \(c_1(\eta)^2\) or \(c_1(\mathfrak{r})^2\) is a maximum of \(c_1(s)^2\) for \(s \in \mathcal{S}(t)\).

As in the proof of Lemma 4.6 (cf. Inequality (16)), there is a constant \(B\) independent of \(m\) with the property that if \(\langle c_1(\mathfrak{r}), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] \geq B\), then for any corresponding small triangle \(\psi\) representing \(\mathfrak{r}\), we have \(n_w(\psi) > 0\). Moreover, in this case, \(h^+\), which corresponds to the Spin\(^c\) structure \(\eta\), induces an isomorphism, while \(\eta\) maximizes \(c_1(s)^2\) among all \(s \in \mathcal{S}(t)\). In the same way, when \(\langle c_1(\mathfrak{r}), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] \leq -B\), then \(\mathfrak{r}\) induces an isomorphism, and it maximizes \(c_1(s)^2\) among all \(s \in \mathcal{S}(t)\).

In either case, in view of Equation (9), we see that the degree of any element of \(\widetilde{HF}(Y_{m\mu + \lambda}(K), t)\) lies within a bounded distance (independent from \(m\) and \(t\)) from \(-c_1(s)^2/4\), where \(s \in \mathcal{S}(t)\) minimizes \(c_1(s)^2\). The claimed result follows.

**Corollary 4.7.** Let \(K \subset Y\) be a null-homologous knot in an integer homology three-sphere, and fix an integer \(\delta \geq 0\). There is a constant \(C\) with the property that for
all sufficiently large $m$ and any $t \in \text{Spin}^c(Y_{m\mu+\lambda}(K))$, there is a chain complex for $CF^\delta(Y_{m\mu+\lambda}(K), t)$ with the property that if $CF^\delta_d(Y_{m\mu+\lambda}(K), t)$ is non-trivial, then

$$|d - \frac{m}{4}| \leq C.$$ 

**Proof.** It is easy to see that for fixed $t \in \text{Spin}^c(Y)$, if we consider $S(t)$, the element $s_0$ which maximizes $c_1(s)^2$ has $c_1(s_0)^2 = -m + c$, where here $c$ is some constant (independent of $m$). Thus, according to Equation (9), the map $f^\delta_{W, s_0}$ carries an element of degree $d$ to an element of $d - \frac{m}{4} + c$. Now, according to Theorem 4.1, we have a chain complex representing $CF^\delta(Y_{m\mu+\lambda}(K), t)$ (for any choice of $t$) whose breadth is constant, independent of $m$ and the choice of $t$. According to the proof of Proposition 4.2, $f^\delta_{W, s_0}$, the map of degree $d - \frac{m}{4} + c$, carries some element of this complex non-trivially to $CF^\delta(Y)$. 

\[\square\]
5. K"unneth principle

Let $K_1 \subset Y_1$ and $K_2 \subset Y_2$ be a pair of oriented three-manifolds equipped with oriented knots. Then, we can form the connected sum to obtain an oriented knot $K_1 \# K_2 \subset Y_1 \# Y_2$. Indeed, given $\xi_i \in \text{Spin}^c(Y_i, K_i)$, we can form their connected sum $\xi_1 \# \xi_2$. This induces a gluing map

$$\text{Spin}^c(Y_1, K_1) \times \text{Spin}^c(Y_2, K_2) \longrightarrow \text{Spin}^c(Y_1 \# Y_2, K_1 \# K_2),$$

written $\xi_1, \xi_2 \mapsto \xi_1 \# \xi_2$, which is equivariant with respect to the natural map

$$H^2(Y_1, K_1; \mathbb{Z}) \oplus H^2(Y_2, K_2; \mathbb{Z}) \longrightarrow H^2(Y_1 \# Y_2, K_1 \# K_2; \mathbb{Z}).$$

More explicitly, we can think of $\xi_i$ ($i = 1, 2$) as specified by a non-vanishing vector field which we can think of as a closed orbit. We realize the connected sum as attaching a one-handle to $Y_1 \coprod Y_2$ along a pair of three-balls $B_i$ supported on $K_i$. We can assume that on each sphere $S_i = \partial B_i$, there are exactly two points where the vector field is normal to $S_i$, the two points where $S_i$ meets $K_i$: the “in-going” and “out-going points”.

Here, $p \in S_i$ is “in-going” if $\xi_i$ points into $Y_i - B_i$. We can then match the in-going point on $Y_1 - B_1$ with the out-going point on $Y_2 - B_2$ and vice versa, to construct a nowhere vanishing vector field on $Y_1 \# Y_2$. This new vector field has $K_1 \# K_2$ as a closed orbit, and has prescribed from in the connected sum region. This vector field gives rise to a relative Spin$^c$ structure Spin$^c(Y_1 \# Y_2, K_1 \# K_2)$, inducing the gluing map above.

The gluing map can be described in terms of Heegaard diagrams as follows. Let $(\Sigma_1, \alpha_1, \beta_1, w_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, w_2, z_2)$ be the doubly-pointed Heegaard diagrams compatible with the oriented knots $K_1 \subset Y_1$ and $K_2 \subset Y_2$. Consider the oriented two-manifold $\Sigma_3$ obtained as the connected sum of $\Sigma_1$ with $\Sigma_2$, identifying neighborhoods of $z_1$ with $w_2$. Then, the doubly-pointed Heegaard diagram $(\Sigma_3, \alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2, w_1, z_2)$ is compatible with $K_1 \# K_2 \subset Y_1 \# Y_2$. Now, given $x_1 \in T_{\alpha_1} \cap T_{\beta_1}$ and $x_2 \in T_{\alpha_2} \cap T_{\beta_2}$, we can think of $x_1 \times x_2$ as an intersection point $T_{\alpha_1 \cup \alpha_2} \cap T_{\beta_1 \cup \beta_2}$. Then,

$$\mathfrak{g}_{w_1, z_2}(x_1 \times x_2) = \mathfrak{g}_{w_1, z_1}(x_1) \# \mathfrak{g}_{w_2, z_2}(x_2).$$

The following is a straightforward generalization of the connected sum principle for knot homology in the case of null-homologous knots, c.f. [22], [27]. A proof can be found, for example, in Theorem 7.1 of [22]; we omit it here.

**Theorem 5.1.** Fix $\xi_i \in \text{Spin}^c(Y_i, K_i)$ for $i = 1, 2$. There is a filtered chain homotopy equivalence

$$CFK^\infty(Y_1, K_1, \xi_1) \otimes_{\mathbb{Z}[U]} CFK^\infty(Y_2, K_2, \xi_2) \longrightarrow CFK^\infty(Y_1 \# Y_2, K_1 \# K_2, \xi_1 \# \xi_2).$$

**Definition 5.2.** A $U$-knot is a knot in a three-manifold $Y$ with the property that the induced filtration of $CF^\infty(Y, s)$ is trivial. More precisely, given any $\xi \in \text{Spin}^c(Y, K)$, $CFK^\infty(Y, K, \xi)$ is chain homotopy equivalent via a relatively $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy to the chain complex $R$ which is a free, rank one $\mathbb{Z}[U, U^{-1}]$ module with the trivial differential.
The unknot in $S^3$ is a $U$-knot. Indeed, the results of [21] can be interpreted as saying that the only $U$-knot in $S^3$ is the unknot. Other $U$-knots will be described in Lemma 7.1.

**Corollary 5.3.** If $K_2 \subset Y_2$ is a $U$-knot, then for each $\xi_1 \in \text{Spin}^c(Y_1, \overline{K}_1)$ and $s_2 \in \text{Spin}^c(Y_2)$, there is some $\xi_2 \in \text{Spin}^c(Y_2)$ representing $s_2$, with the property that

$$CFK^\infty(Y_1, \overline{K}_1, \xi_1) \cong CFK^\infty(Y_1 \# Y_2, \overline{K}_1 \# \overline{K}_2, \xi_1 \# \xi_2)$$

as $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complexes.

**Proof.** Let $\xi \in \text{Spin}^c(Y_2)$ be any relative Spin$^c$ structure representing $s_2$. Then, $CFK^\infty(Y_2, \overline{K}_2, \xi)$ is quasi-isomorphic to the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex $\mathbb{Z}[U, U^{-1}]$ which contains a non-trivial element in filtration $(0, n)$ for some $n$. Letting

$$\xi_2 = \xi + n \text{PD}[\mu]$$

(where $\mu$ of course is a meridian for $K_2$), we see that $CFK^\infty(Y_2, \overline{K}_2, \xi_2)$ is quasi-isomorphic to $\mathbb{Z}[U, U^{-1}]$ with a generator in filtration level $(0, 0)$ (c.f. Proposition 3.2). For this choice of $\xi_2$, the corollary is an immediate consequence of Theorem 5.1. □
6. The Morse surgery formula

We describe the Morse surgery formula, which expresses the Heegaard Floer homologies of Morse surgery along a rationally null-homologous knot in terms of its knot Floer homology.

Given $s \in \text{Spin}^c(Y_{\lambda}(K))$, let
\[
A^+_{\xi}(Y, K) = \bigoplus_{\xi \in \text{Spin}^c(Y_{\lambda}(K), \pi_1(Y_{\lambda}(K))) | G_{Y_{\lambda}(K), \pi_1}(\xi) = s} A^+_{\xi}(Y, K),
\]
\[
B^+_{\xi}(Y, K) = \bigoplus_{\xi \in \text{Spin}^c(Y_{\lambda}(K), \pi_1(Y_{\lambda}(K))) | G_{Y_{\lambda}(K), \pi_1}(\xi) = s} B^+_{\xi}(Y, K),
\]
where here $A^+_{\xi}(Y, K)$ and $B^+_{\xi}(Y, K)$ are defined as in Equation (12).

In the above definition, we view $K$ also as a knot in $Y_{\lambda}(K)$. An orientation on $K \subset Y$ naturally induces also an orientation on the induced knot in $Y_{\lambda}(K)$ in a natural way: an orientation on $K$ corresponds to an orientation on a meridian for $K$, which can be thought of as supported in $Y - nd(K)$, which in turn can be thought of as a subset of $Y_{\lambda}(K)$, where it in turn induces an orientation of the induced knot in $Y_{\lambda}(K)$.

Theorem 6.1. Let $K \subset Y$ be a knot in a rational homology three-sphere, and let $\lambda$ be a framing on $K$ with the property that $Y_{\lambda}(K)$ is also a rational homology three-sphere. The Heegaard Floer homology $HF^+(Y_{\lambda}(K), s)$ is calculated by the homology of the mapping cone of the chain map
\[
D^+_s: A^+_{\xi}(Y, K) \longrightarrow B^+_{\xi}(Y, K)
\]
defined by
\[
D^+_s \left( \{ a_{\xi} \}_{\xi \in \text{Spin}^c(Y_{\lambda}(K), \pi_1(Y_{\lambda}(K)))} \right) = \{ b_{\xi} \}_{\xi \in \text{Spin}^c(Y_{\lambda}(K), \pi_1(Y_{\lambda}(K)))},
\]
where
\[
b_{\xi} = h^+_\xi(a_{\xi} - PD[K_{\lambda}]) - v^+_\xi(a_{\xi}).
\]

The proof is modeled on the proof of the main result from [25]. Specifically, it is based on two key ingredients: the relationship between the knot Floer homology of $K \subset Y$ and the Heegaard Floer homology of three-manifolds obtained as sufficiently large surgeries on $K$ from Section 4, and also an exact exact triangle which relates, for all integers $m$, the Heegaard Floer homologies of $Y_{\lambda}(K)$, $Y_{m, \mu+\lambda}(K)$, and $Y$ (the latter taken with twisted coefficients).

We have not strived for maximal generality in Theorem 6.1. As a technical tool, we make heavy use of the rational grading on $HF^+$ which is defined on the Heegaard Floer homology of any three-manifold equipped with a Spin$^c$ structure whose first Chern class is torsion, cf. Section 7 of [14]. Thus, Theorem 6.1 holds – and indeed the proof we give below applies – whenever we consider Spin$^c$ structures over $Y_{\lambda}(K)$ whose first Chern class is torsion.
6.1. **A surgery exact sequence.** The following is essentially a restatement of Theorem 3.1 of [25]:

**Theorem 6.2.** Let $Y$ be a closed, oriented three-manifold, and $K \subset Y$ be a knot with framing $\lambda$. Then, for all positive integers $m$, there is a long exact sequence

$$
\ldots \longrightarrow HF^+(Y_\lambda(K)) \xrightarrow{F_1^+} HF^+(Y_{m\mu+\lambda}(K)) \xrightarrow{F_2^+} \bigoplus^m HF^+(Y) \xrightarrow{F_3^+} \ldots
$$

where here $\mu$ denotes the meridian of $K$.

We describe the maps in the above theorem below, after which we make a few comments on the proof.

We place a basepoint $p$ on $\beta_\gamma$, and consider twisted homology with coefficients in $\mathbb{Z}/m\mathbb{Z}$; i.e. write $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] = \mathbb{Z}[T]/(T^m-1)$, and consider the chain complex $CF^+(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$ endowed with the differential

$$
\partial^+[x, i] = \sum_{y \in \pi_2^1 \cap \pi_2^2} \sum_{\phi \in \pi_2^1(x, y) \mid \mu(\phi)=1} \# \left( \frac{M(\phi)}{R} \right) \cdot T^{m_p(\phi)} \cdot [y, i - n_z(\phi)]
$$

where as usual here $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i \geq 0$, $\pi_2^1(x, y)$ denotes the space of homotopy classes of Whitney disks connecting $x$ and $y$, $\mu(\phi)$ denotes the Maslov index of $\phi$, and terms in the above equation for which $i - n_z(\phi) < 0$ are to be dropped. Moreover, $m_p(\phi)$ denotes the multiplicity of the basepoint $p$ in the boundary of $\phi$; i.e. $p$ determines a codimension one submanifold $\beta_1 \times \ldots \times \beta_{g-1} \times \{p\} \subset \mathbb{T}_\beta$, and $m_p(\phi)$ denotes the intersection number with the restriction of the boundary of $\phi$ with this subset. We denote the complex by $CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$. (In the terminology of [24], this is the chain complex for $Y$ with twisted coefficients in $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$, where it is denoted $\underline{CF}^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$, however, as in [25], we drop the underline here in the interest of notational simplicity.) Recall that there is an isomorphism of chain complexes of modules over $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$,

$$
\theta: CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]) \cong CF^+(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}],
$$

where here the right-hand-side is endowed with the differential which is the original differential on $CF^+(Y)$ tensored with the identity map on $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$. There is a corresponding identification

$$
HF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]) \cong HF^+(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \cong \bigoplus^m HF^+(Y).
$$

The quantity $m_p(\phi)$ has an expression more in the spirit of the constructions from Section 3. Fix two basepoints, $w$ and $z$ on either side of $\beta_\gamma$, so that there is an arc in $\Sigma$ connecting them, but disjoint from all the attaching circles except for $\beta_\gamma$, which it meets transversally in the single intersection point $p$. Then,

$$
m_p(\phi) = nw(\phi) - nz(\phi).
$$
We now define the chain maps \( f_1^+ \), \( f_2^+ \), and \( f_3^+ \) inducing maps on homology \( H_1^+ \), \( H_2^+ \), and \( H_3^+ \) appearing in Theorem 6.1.

The map \( f_1^+ \) is defined by counting pseudo-holomorphic triangles between \( \mathbb{T}_\alpha \), \( \mathbb{T}_\gamma \), and \( \mathbb{T}_\delta \). More precisely, note that the Heegaard triple \((\Sigma, \alpha, \gamma, \delta, z)\) determines a four-manifold \( X_{\alpha,\gamma,\delta} \) with three boundary components
\begin{equation}
Y_{\alpha,\gamma} \cong Y_\alpha(K), \quad Y_{\alpha,\delta} \cong Y_{m,\mu+\lambda}(K), \quad \text{and} \quad Y_{\gamma,\delta} \cong \#^g(S^2 \times S^1)\#L(m, 1).
\end{equation}

\textbf{Definition 6.3.} Let \( \ell \in \text{Spin}^c(\#^g(S^2 \times S^1)\#L(m, 1)) \) denote the canonical \text{Spin}^c structure, i.e. this is the one which extends over the tubular neighborhood \( N \) of a sphere with self-intersection number \( m \) (after attaching a two-handle and a collection of \( g-1 \) three-handles) in such a way that its first Chern class evaluates as \( \pm m \) on the two-sphere generator of the two-dimensional homology of \( N \).

Let \( \Theta_{\gamma,\delta} \) denote the Floer homology class corresponding to the generator (over \( L^*H_1(Y_{\gamma,\delta}) \otimes \mathbb{Z}[U] \)) of
\[ \text{HF}_{\leq 0}(Y_{\gamma,\delta}, \ell) \cong L^*H^1(Y_{\gamma,\delta}) \otimes \mathbb{Z}[U] \]
in its canonical \text{Spin}^c structure \( \ell \). (As usual, we arrange for the homology class \( \Theta_{\gamma,\delta} \) to be represented by a single intersection point in \( \mathbb{T}_\gamma \cap \mathbb{T}_\delta \), which we also denote by \( \Theta_{\gamma,\delta} \).)

We then define
\begin{equation}
f_1^+([x, i]) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\{\psi \in \pi_2(x, \Theta_{\gamma,\delta}, y) \mid \mu(\psi) = 0\}} \#M(\psi) \cdot [y, i - n_z(\psi)].
\end{equation}

Similarly, we define \( f_2^+: CF^+(Y_{m,\mu+\lambda}(K)) \rightarrow CF^+(Y; \mathbb{Z}/m\mathbb{Z}) \) by
\begin{equation}
f_2^+([y, i]) = \sum_{w \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\psi \in \pi_2(y, \Theta_{\beta,\delta}, w) \mid \mu(\psi) = 0\}} \#M(\psi) \cdot [w, i - n_z(\psi)] \cdot T^{m_p(\psi)},
\end{equation}
where \( m_p(\psi) \) is the natural extension of \( m_p \) to triangles. (In particular, in the case where we consider doubly-pointed Heegaard diagrams, \( m_p(\psi) = n_w(\psi) - n_z(\psi) \) as in Equation (18)).

To define \( f_3^+ \), we proceed as follows. Fix \( \psi \in \pi_2(\Theta_{\gamma,\beta}, \Theta_{\beta,\delta}, \Theta_{\gamma,\delta}) \). The congruence class \( c \) of \( m_p(\psi) \) modulo \( m \) is independent of the choice of \( \psi \). The map
\[ f_3^+: CF^+(Y; \mathbb{Z}/m\mathbb{Z}) \rightarrow CF^+(Y_{\lambda}(K)) \]
is given by the formula
\begin{equation}
f_3^+(T^s \cdot [x, i]) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\{\psi \in \pi_2(x, \Theta_{\beta,\gamma}, y) \mid \mu(\psi) = 0, s+m_p(\psi) \equiv c \pmod{m}\}} \#M(\psi) \cdot [y, i - n_z(\psi)].
\end{equation}

\textbf{Proof of Theorem 6.2.} Let \( H_1^+ \) and \( H_2^+ \) denote the null-homotopies of \( f_2^+ \circ f_1^+ \) and \( f_3^+ \circ f_2^+ \) respectively. With this notation, the proof of Theorem 3.1 applies to show that
the chain maps
\[ \phi^+: CF^+(Y_\lambda(K)) \rightarrow M(f_2^+) \quad \text{and} \quad \psi^+: M(f_2^+) \rightarrow CF^+(Y_\lambda(K)) \]
defined by
\[ \phi^+(\xi) = (f_1^+(\xi), H_1^+(\xi)). \]
and
\[ \psi^+(x, y) = H_2^+(x) + f_3^+(y) \]
respectively are quasi-isomorphisms, and hence establishing Theorem 6.2 as stated above. \( \square \)

6.2. The analogue of Theorem 6.1 for the case of \( CF^\delta \).

Given an integer \( \delta \geq 0 \), let \( CF^\delta \) denote the approximation of \( CF^+ \) described in Subsection 2.8. We state and prove analogue of Theorem 6.1 for this group, following the pattern of proof from [25].

Given \( \delta \geq 0 \), let
\[ A^\delta_\xi(Y, K) = C_\xi \{ 0 \leq \max(i, j) \leq \delta \} \quad \text{and} \quad B^\delta_\xi(Y, K) = CF^\delta(Y, G_{Y\lambda}(\xi)). \]

Given \( s \in \text{Spin}^c(Y_\lambda(K)) \), let
\[ A^\delta_\xi(Y, K) = \bigoplus_{\{ \xi \in \text{Spin}^c(Y_\lambda(K), K) \mid G_{Y\lambda(K), \pi}(\xi) = s \}} A^\delta_\xi(Y, K) \]
\[ B^\delta_\xi(Y, K) = \bigoplus_{\{ \xi \in \text{Spin}^c(Y_\lambda(K), K) \mid G_{Y\lambda(K), \pi}(\xi) = s \}} B^\delta_\xi(Y, K). \]

Consider the map
\[ D^\delta_s: A^\delta_\xi(Y, K) \rightarrow B^\delta_\xi(Y, K) \]
defined by
\[ D^\delta_s \left( \{ a_\xi \}_{\xi \in G_{Y\lambda(K), \pi}^{-1}(s)} \right) = \{ b_\xi \}_{\xi \in G_{Y\lambda(K), \pi}^{-1}(s)}; \]
where
\[ b_\xi = h^\delta_{\xi - \text{PD}[K_\lambda]}(a_\xi_{-\text{PD}[K_\lambda]}) - v^\delta_\xi(a_\xi). \]

Our aim in the present subsection is to prove the following analogue of Theorem 6.1 for \( CF^\delta \).

Theorem 6.4. The group \( CF^\delta(Y_\lambda(K), s) \) is quasi-isomorphic to the mapping cone \( X^\delta_s(\lambda) \) of
\[ D^\delta_s: A^\delta_\xi(Y, K) \rightarrow B^\delta_\xi(Y, K). \]
We deduce Theorem 6.4 from a combination of Theorems 6.2 and 4.1. Theorem 6.1 will follow from Theorem 6.4, together with some further observations about gradings, as explained in Subsection 6.3.

We turn our attention now towards proving Theorem 6.4. We assume that both $d$ and $n$ have the same sign, and hence without additional loss of generality that both are positive, returning to the case where their signs are opposite in Subsection 6.4.

It will be useful to have the following lemma. In the statement, “sufficiently large” is meant with respect to the ordering on $\text{Spin}^c(Y,K)$ with fixed filling $t$ over $Y$ described earlier. Specifically, recall that $\xi_1 \leq \xi_2$ if $\xi_2 = \xi_1 + j \cdot \text{PD}[\mu]$ for some $j \geq 0$. In particular, we say that a condition $P$ holds for all all sufficiently large relative $\text{Spin}^c$ structures $\text{Spin}^c(Y,K)$ if there is a finite collection $\Xi \subset \text{Spin}^c(Y,K)$ with the properties that each $t \in \text{Spin}^c(Y)$ can be represented by some $\eta \in \text{Spin}^c(Y,K)$ with $\eta \in \Xi$, and also for any $\xi \in \text{Spin}^c(Y,K)$ which has the property that $\xi \geq \eta$ for some $\eta \in \Xi$ then $P$ holds for $\xi$.

Lemma 6.5. If $\xi$ is sufficiently large, then
\[ v^\delta_\xi : A^\delta_\xi(Y,K) \longrightarrow B^\delta_\xi(Y,K) \]
is an isomorphism, while
\[ h^\delta_\xi : A^\delta_\xi(Y,K) \longrightarrow B^\delta_\xi+\text{PD}[K\lambda](Y,K) \]
is trivial; moreover, if $\xi$ is sufficiently small, then $h^\delta_\xi$ is an isomorphism, while $v^\delta_\xi$ is trivial.

Proof. Since there are finitely many $x \in T_\alpha \cap T_\beta$, there is a maximal $\xi_0$ among all $\sum_{x \in T_\alpha \cap T_\beta} x \cdot [i,j]$ with $x \in T_\alpha \cap T_\beta$. Clearly, for any $\xi \geq \xi_0$, there are no generators $[x,i,j]$ of $CFK^\infty(Y,K,\xi)$ with $i < 0$ and $j \geq 0$; thus, in fact, $A^+_\xi(Y,K) = C\xi\{i \geq 0\}$; i.e. $v^+_\xi$ is the identity map, and the corresponding statement for $v^\delta_\xi$ follows at once.

The other assertions follow from similar reasoning.

Proof of Theorem 6.4 when $d$ and $n$ have the same sign. Choose $m = nk$, where $n$ is as in Equation (3). It is easy to see that $\text{PD}[K\lambda]$ has order $dk + 1$ in $H^2(Y_{m\cdot\mu+\lambda})$ whereas $\text{PD}[K\lambda]$ has order $nk$ in the quotient group $H^2(Y, Y - K; \mathbb{Z})/m\text{PD}[\mu]$.

Theorem 6.2 expresses $CF^\delta(Y_{\lambda}(K))$ as the mapping cone of a map from $CF^\delta(Y_{m\cdot\mu+\lambda}(K))$ to $CF^\delta(Y, \mathbb{Z}/m\mathbb{Z})$. We can think of $CF^\delta(Y, \mathbb{Z}/m\mathbb{Z})$ more invariantly, as a sum
\[ \bigoplus_{[\xi] \in \text{Spin}^c(Y,K)/m\cdot\mu} ([\xi], CF^+(Y, G_{Y,K}(\xi))), \]
where, as the notation suggests, the index set consists of $m \cdot \mu$-orbits in $\text{Spin}^c(Y,K)$ (each of which induces the same $\text{Spin}^c$ structure over $Y$, of course). The identification
is induced by the map
\[ T^i \otimes \xi \mapsto (\mathfrak{g}_{m,2}(x) + i \cdot \text{PD}[\mu], x). \]

Now, the map
\[ f_3^\delta: CF^\delta(Y_{m\mu+\lambda}(K)) \longrightarrow CF^\delta(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]) \]
can be written as
\[ a \mapsto \sum_{s \in \text{Spin}^c(W'_m(K))} ([E_{Y,\overline{K}}(\delta)], f_3^\delta(s)), \]
where here \( W'_m(K) \) denotes the natural cobordism from \( Y_{m\mu+\lambda}(K) \), and \( E_{Y,\overline{K}} \) is the map from \( \text{Spin}^c \) structures over \( W'_m(K) \) to relative \( \text{Spin}^c(Y, \overline{K}) \) as in Proposition 2.2.

Choosing \( k \) sufficiently large, (where \( m = nk \)) according to Theorem 4.1, we have that the summand \( CF^\delta(Y_{m\mu+\lambda}, t) \subset CF^\delta(Y_{m\mu+\lambda}) \) is identified with \( A^\delta_0(Y, \overline{K}) \) for \( \xi = \Xi(t) \). In fact, there are only two homotopically non-trivial components of \( f_2^\delta |_{CF^\delta(Y_{m\mu+\lambda}(K), t)} \), according to Proposition 4.2: these are the components belonging to \( \mathfrak{r} \) and \( \mathfrak{n} \), whose corresponding maps are identified with \( v^\delta_\xi \) and \( h^\delta_\xi \) respectively. In view of Equation (7),
\[ E_{Y,\overline{K}}(\mathfrak{r} + \text{PD}[\mathcal{F}]) = E_{Y,\overline{K}}(\mathfrak{r}) + m\mu + \lambda; \]
and hence, if \( \Xi(t) = \xi \) and \( \Xi(t + \text{PD}[K_\lambda]) = \xi + \text{PD}(K_\lambda) \), then the ranges of \( v^\delta_\xi \) and \( h^\delta_\xi \) coincide.

In effect, we have shown that \( CF^\delta(Y_\xi(K)) \) is quasi-isomorphic to the mapping cone of a map which splits according to \( K_\lambda \) orbits in \( \text{Spin}^c(Y, K) \) or equivalently, \( \text{Spin}^c \) structures over \( Y_\lambda(K) \). In each such orbit, the map has the form
\[ (f_2^\delta)': \bigoplus_{s \in \mathbb{Z}/(dk+1)\mathbb{Z}} A^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, \overline{K}) \longrightarrow \bigoplus_{s \in \mathbb{Z}/dk\mathbb{Z}} B^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, \overline{K}), \]
defined by adding
\[ v^\delta_\xi: A^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, \overline{K}) \longrightarrow B^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, \overline{K}) \]
and
\[ h^\delta_\xi: A^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, \overline{K}) \longrightarrow B^\delta_{\xi+(s+1) \cdot \text{PD}[K_\lambda]}(Y, \overline{K}). \]
According to Lemma 6.5 and our hypothesis on the sign of \( d \), we see that for any \( \xi \), if \( s \) is sufficiently large, then letting \( \xi' = \xi + s \cdot \text{PD}[K_\lambda] \), we have that \( h^\delta_\xi \) is null-homotopic, and \( v^\delta_\xi \) is an isomorphism. It follows from this (together with the analogous statement for \( s \) sufficiently small) that the mapping cone of \( (f_2^\delta)' \) is identified with the mapping cone of
\[ (f_2^\delta)''': \bigoplus_{s \in \mathbb{Z}} A^\delta_{\xi+(sn) \cdot \text{PD}[\mu]}(Y, \overline{K}) \longrightarrow \bigoplus_{s \in \mathbb{Z}} B^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, \overline{K}), \]
gotten by adding all the maps \( v^\delta_\xi \) and \( h^\delta_\xi \). This establishes the theorem. \( \square \)

The case where \( d \) and \( n \) have opposite signs is handled in Subsection 6.4.
6.3. Gradings, and the proof of Theorem 6.1 (in the case where $d$ and $n$ have the same sign). In the proof of Theorem 6.4, for each $\delta \geq 0$, we establish quasi-isomorphisms

$$\phi^\delta: CF^\delta(Y_\lambda(K), s) \rightarrow X^\delta_\phi(\lambda).$$

We wish to conclude that

$$HF^+(Y_\lambda(K), s) \cong H_*(X^\delta_\phi(\lambda)).$$

To this end, we must establish that $X^\delta_\phi(\lambda)$ is a relative $\mathbb{Z}$-graded complex, which can be given an absolute grading so that the map $\phi^\delta$ is homogeneous of degree zero.

Clearly, the groups $A^\delta_\xi(Y, K)$ and $B^\delta_\xi(Y, K)$ are relatively $\mathbb{Z}$-graded, and the maps $v^+_\xi$ and $h^+_\xi$ are all relatively $\mathbb{Z}$-graded maps. It is now a formal consequence of the shape of

$$D^\delta_\phi: A^\delta_\phi(Y, K) \rightarrow B^\delta_\phi(Y, K)$$

that $A^\delta_\phi(Y, K)$ and $B^\delta_\phi(Y, K)$ can be given absolute $\mathbb{Z}$-gradings so that $D^\delta_\phi$ drops this grading by one. This naturally endows the mapping cone $X^\delta_\phi(\lambda)$ with an absolute $\mathbb{Z}$-grading. In fact, this grading is uniquely determined up to an overall shift.

In the proof of Theorem 6.2, we considered maps

$$f^\delta_1: CF^\delta(Y_\lambda(K)) \rightarrow CF^\delta(Y_{nk, \mu+\lambda}(K))$$

and

$$f^\delta_2: CF^\delta(Y_{nk, \mu+\lambda}(K)) \rightarrow CF^\delta(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]).$$

Fix $\xi \in \text{Spin}^c(Y, K)$. For $s \in \mathbb{Z}$, consider the projection

$$\Pi^A_\lambda: CF^+(Y_{nk, \mu+\lambda}(K)) \rightarrow CF^+(Y_{nk, \mu+\lambda}(K), G_{Y_{nk, \mu+\lambda}(K), K}(\xi + s \cdot \text{PD}[K_\lambda])).$$

We prove the following analogue of Proposition 4.6 of [25].

**Proposition 6.6.** Fix an absolute lift of the relative $\mathbb{Z}$-grading on $CF^+(Y_\lambda(K), G_{Y_\lambda(K), K}(\xi))$, and an integer $\delta \geq 0$. Then, there is a constant $b$ so that, for all sufficiently large $k$, there are absolute lifts of the relative $\mathbb{Z}$-gradings on both

$$A^\delta_{\xi+s \cdot \text{PD}[K_\lambda]}(Y, K) \cong CF^\delta(Y_{nk, \mu+\lambda}(K), t) \subset CF^\delta(Y_{nk, \mu+\lambda}(K))$$

and $CF^\delta(Y, G_Y(K)(\xi + s \cdot \text{PD}[K_\lambda]))$ for all $|s| \leq b$, with the property that $\Pi^A_\lambda \circ f^\delta_1$ and also the restriction of $f^\delta_2$ to

$$([s], CF^\delta(Y, G_Y(K)(\xi + s \cdot \text{PD}[K_\lambda]))) \subset CF^\delta(Y, \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$$

have degree zero.

We return to a proof of Proposition 6.6 after introducing a lemma.

Recall (c.f. Equation (19)) that the map $f^+_1$ was defined by counting holomorphic triangles for a Heegaard triple describing a four-manifold which we denoted $X_{\alpha\gamma\delta}$, whose boundaries consist of $Y_\lambda(K), Y_{(nk)\mu+\lambda}(K)$ and $\#^{g-1}(S^2 \times S^1)\# L(nk, 1)$. We denote this four-manifold here by $X(k)$, to call attention to its dependence on $k$. 
The following lemma is an adaptation of the proof of Lemma 4.7 from [25].

**Lemma 6.7.** Fix a constant $C_0$. Then, for all sufficiently large $k$, the following statement holds. Each Spin$^c$ structure over $Y_{nk\mu+\lambda}(K)$ has at most one extension $s$ over $X(k)$ whose restriction to $\#^{g-1}(S^2 \times S^1)\# L(nk, 1)$ is the canonical Spin$^c$ structure (in the sense of Definition 6.3, and for which

\[ C_0 \leq c_1(s)^2 + nk. \]

**Proof.** Observe that $H_2(X(k); \mathbb{Z}) \cong \mathbb{Z}$ is generated by a surface $\Sigma$ with

\[ \Sigma^2 = -nk d^2 (dk + 1). \]

This can be seen, for example, by observing that the group of triply-periodic domains in $X(k)$ is generated by a relation of the form

\[-(dk + 1)d \cdot \lambda + d \cdot (nk \cdot \mu + \lambda) + a + b,\]

where $a$ is a sum of curves among the $\alpha_i$ with $i = 1, \ldots, g$, and $b$ a sum of curves among the $\beta_j$ with $j = 1, \ldots, g - 1$. The intersection number of the first two curves – which gives the self-intersection number of the homology class corresponding to the triply-periodic domain – is $-nk d^2 (dk + 1)$.

Now, for any other Spin$^c$ structure over $X(k)$ which interpolates between the same two Spin$^c$ structures on $Y_\lambda(K)$ and $Y_{nk\mu+\lambda}(K)$ (and whose restriction to the remaining boundary boundary component $\#^{g-1}(S^2 \times S^1)\# L(nk, 1)$ is the canonical Spin$^c$ structure) has the form $s + j \text{PD} [\Sigma]$, for some integer $j \neq 0$; thus,

\[ c_1(s + j \cdot \text{PD} [\Sigma])^2 - c_1(s)^2 = 4(j^2 \Sigma \cdot \Sigma + j \langle c_1(s), [\Sigma] \rangle) \]

\[ \leq -4j^2 nk d^2 (dk + 1) \left( 1 - \frac{|\alpha|}{j} \right) \]

\[ \leq -2nk d^2 (dk + 1). \]

Of course, if $k$ is sufficiently large, then Inequality (25) is violated. \qed

**Proof of Proposition 6.6.** Fix an absolute lift of the relative $\mathbb{Z}$-grading on $CF^+(Y_\lambda(K), s)$, and fix some $\delta \geq 0$. Now, $f_1^\delta$ decomposes as a sum of homogeneous terms, indexed by those $s \in \text{Spin}^c(X(k))$ whose restriction to the boundary component $\#^{g-1}(S^2 \times S^1)\# L(nk, 1)$ is the canonical Spin$^c$ structure. By a suitable adaptation of Equation (9), we see that each term is homogeneous of degree $(c_1(s)^2 + nk)/4$ (in this application, note that $X(k)$ has three, rather than two, boundary components, and we are considering the pairing with a fixed homology class on the third term). It follows readily from Lemma 6.7 that there is at most one such Spin$^c$ structure which can induce a non-trivial map from $CF^\delta(Y_\lambda(K), G_{Y_\lambda(K)}(\xi))$ to $CF^\delta(Y_{nk\mu+\lambda}(K))$.

Thus, we can grade $CF^\delta(Y_{nk\mu+\lambda}(K))$ so that each component of $f_1^\delta$ has degree zero.
Similarly, we can endow \([s, CF^\delta(Y, \xi + s \cdot PD[K_\lambda])] \subset CF^\delta(Y, \mathbb{Z}/m\mathbb{Z})\) with the grading for which
\[
v^\delta_s : A^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K) \cong CF^\delta(Y_{nk+\mu+\lambda}(K), \xi + s \cdot PD[K_\lambda]) \
\rightarrow ([s], CF^\delta(Y, \xi + s \cdot PD[K_\lambda]))
\]
has degree zero. We must check that this is compatible with the grading for which
\[
h^\delta_s : A^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K) \cong CF^\delta(Y_{nk+\mu+\lambda}(K), \xi + s \cdot PD[K_\lambda]) \
\rightarrow ([s], CF^\delta(Y, \xi + (s+1) \cdot PD[K_\lambda]))
\]
has degree zero.

To this end, it suffices to prove the following. Fix \(K \subset Y\) with framing \(\lambda\). Fix also \(\xi \in \text{Spin}\mathbb{c}(Y, K)\) and \(t \in \mathbb{Z}\). Then, for sufficiently large \(\delta\), there is a homogeneous element
\[
(26) \quad a = \{a_s\}_{-b \leq s \leq b} \in \bigoplus_{-b \leq s \leq b} H_*(A^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K))
\]
with \(a_t, a_{t+1} \neq 0\), and \(H_*(D^\delta_s(a)) = 0\), i.e. where here \(H_*(D^\delta_s(a))\) denotes the map on homology induced by
\[
D^\delta_s : \bigoplus_{-b \leq s \leq b} A^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K) \
\rightarrow \bigoplus_{-b+1 \leq s \leq b} B^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K).
\]

To see this, we proceed as follows. Abbreviate \(\bigoplus_{-b \leq s \leq b} A^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K)\) by \(A^\delta\) (or \(A^\delta(b)\), when we wish to refer to its dependence on \(b\)). We have also \(A^{\infty}\) and \(A^+\) which are obtained analogously. Similarly, write \(B^\delta\) for \(\bigoplus_{-b+1 \leq s \leq b} B^\delta_{\xi+s\cdot PD[K_\lambda]}(Y, K)\).

We have maps \(D^\delta : A^\delta \rightarrow B^\delta, D^+ : A^+ \rightarrow B^+\) and \(D^\infty : A^{\infty} \rightarrow B^\infty\). Note that
\(H_*(M(D^\infty : A^{\infty} \rightarrow B^{\infty})) \cong \mathbb{Z}[U, U^{-1}]\). Moreover, there is a map
\[
\Pi^\infty : M(f^{\infty} : A^{\infty} \rightarrow B^{\infty}) \rightarrow B^{\infty}
\]
induced by \(\Pi^\infty(\{a_s\}_{-b \leq s \leq b}) = \{v^\delta_s(a_s)\}_{-b+1 \leq s \leq b}\). (Here, \(f^{\infty}\) is the map obtained by adding the \(v^\infty\) and \(h^\infty\) to give a map from \(A^{\infty}(b)\) to \(B^{\infty}(b)\).) It is not difficult to see that for a generator
\[
\alpha^{\infty} \in H_*(M(D^{\infty} : A^{\infty} \rightarrow B^{\infty}))
\]
each component of \(\Pi^\infty(\alpha^{\infty})\) is non-trivial. We can also arrange for \(a^{\infty}\) to be a homogeneous element. By multiplying the generator through by \(U^{-i}\) if necessary, we can arrange for each component of \(\Pi^\infty(\{a_s\}_{-b \leq s \leq b})\) to inject into \(H_*(B^{\infty}_{\xi+s\cdot PD[K_\lambda]}(Y, K))\).

Thus, we obtain a homogeneous element \(\alpha^{\infty} \in H_*(M(D^+ : A^+ \rightarrow B^+))\) (the image of \(\alpha^{\infty}\) under the natural map), whose image \(\Pi^+(\alpha^{\infty})\) has non-trivial components in \(H_*(B^{\infty}_{\xi+s\cdot PD[K_\lambda]}(Y, K))\) (for all \(-b + 1 \leq s \leq b\)). Now, for sufficiently large \(\delta\), \(U^\delta(\alpha^{\infty}) = 0\). Thus, we can find a homogeneous element \(\delta \in H_*(M(f^\delta : A^\delta \rightarrow B^\delta))\) with the property that \(D^\delta(a^\delta) = 0\), but \(\Pi^\delta(\delta)\) has non-trivial image in each component \(H_*(B^{\delta}_{\xi+s\cdot PD[K_\lambda]}(Y, K))\).

To repeat: given \(t\), we obtain \(\delta\) so that \(H_*(M(f^\delta : A^\delta(b) \rightarrow B^\delta(b)))\) has a homology class of the required shape as in Equation (26). Here \(b\) is any cut-off, chosen
sufficiently large (note that $H_*(M(f^\delta_b))$ stabilizes when $b$ is sufficiently large). Given this $\delta$, we then find an appropriately large $k$ so that $CF^\delta(Y_{nk,\mu+\lambda}(K),t)$ is for each $t \in \text{Spin}^c(Y_{nk,\mu+\lambda}(K))$ is represented by $A^\delta_{\xi+s,PD[K\lambda]}(Y,K)$ for some $s$ (Theorem 4.1). Thus, we have found a homology class in $H_*(M(f^\delta))$ of the required shape, establishing the compatibility of the various gradings.

With the above proposition in place, the proof of Theorem 6.1 follows from Theorem 6.4 by following the pattern from [25]. Specifically, according to Proposition 6.6, an absolute grading on $CF^\delta(Y_\lambda(K))$ induces absolute gradings on both

$$\bigoplus_{-b \leq s \leq b} A^\delta_{\xi+s,PD[K\lambda]}$$

and

$$\bigoplus_{-b+n \leq s \leq b} ([s],CF^\delta(Y,G_{Y,\overline{K}}(\xi + s \cdot PD[K\lambda])) \subset CF^\delta(Y;\mathbb{Z}[\mathbb{Z}/nk\mathbb{Z}]),$$

so that $f^\delta_1$ and $f^\delta_2$ both are graded maps with degree zero. Let

$$\Pi^B_s : CF^\delta(Y;\mathbb{Z}[\mathbb{Z}/nk\mathbb{Z}]) \rightarrow CF^\delta(Y,G_{Y,\overline{K}}(\xi + s \cdot PD[K\lambda]));$$

denote the projection onto the summand $([s],CF^\delta(Y,\xi + s \cdot PD[K\lambda]))$.

**Lemma 6.8.** With respect to the above gradings, given $\delta \geq 0$, we have that for any sufficiently large $k$ and $|s| \leq b$, the map $\Pi^B_s \circ H^1_1$ is homogeneous of degree $+1$.

**Proof.** This follows exactly the proof of Lemma 4.8 of [25].

**Proof of Theorem 6.1, in the case where $d$ and $n$ have the same sign.** This now follows exactly as in [25]. Specifically, Theorem 6.4 provides, for any $\delta > 0$, a quasi-isomorphism between $CF^\delta(Y_\lambda(K),s)$ and $M(D^\delta_s)$.

There is an integer $b$ with the property that for all $s \geq b$, $v^+_s$ and $h^+_s$ are isomorphisms. Thus, we can truncate the mapping cone at this level to obtain a quasi-isomorphic complex. According to Proposition 6.6, the truncated mapping cone inherits a grading, and according to Lemma 6.8, the quasi-isomorphism from Theorem 6.4 respects these gradings. In effect, we have shown that for any $\delta \geq 0$, there is a graded isomorphism of $HF^\delta(Y_\lambda(K))$ with $HF^\delta(M(D^\delta_s))$. It is now a formal consequence (see Lemma 2.7 of [25] for details) that $HF^+(Y_\lambda(K)) \cong HF^+(M(D^\delta_s))$, as well.

**6.4. The signs of $d$ and $n$ are opposite.** In the case where the signs of $d$ and $n$ in Equation (3) do not coincide, we write instead

$$d \cdot \mu = -n \cdot \mu \in H_1(Y - L;\mathbb{Z})$$

where in this new notation now, both $d$ and $n$ are positive. Letting $m = nk$, we see that in this case, $PD[K\lambda]$ has order $dk - 1$ in $H^2(Y_{m,\mu+\lambda})$. 


Theorem 6.4 is obtained as before, with minor notational changes. Specifically, applying Theorem 6.2 as before, we have a description of $HF^\lambda(Y_\lambda(K), \mathfrak{s})$ as the homology of a mapping cone of a map from $CF^\lambda(Y_{m, \kappa+\lambda}(K))$ to $CF^\lambda(Y, \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$, which decomposes as

$$(f_2^\delta)': \bigoplus_{s \in \mathbb{Z}/(dk-1)\mathbb{Z}} A_\lambda^{\delta+s \cdot \text{PD}[K\lambda]}(Y, K) \longrightarrow \bigoplus_{s \in \mathbb{Z}/dk\mathbb{Z}} B_\lambda^{\delta+s \cdot \text{PD}[K\lambda]}(Y, K),$$

defined by adding

$$v_\delta^s: A_\lambda^{\delta+s \cdot \text{PD}[K\lambda]}(Y, K) \longrightarrow B_\lambda^{\delta+s \cdot \text{PD}[K\lambda]}(Y, K)$$

and

$$h_\delta^s: A_\lambda^{\delta+(s+1) \cdot \text{PD}[K\lambda]}(Y, K) \longrightarrow B_\lambda^{\delta+(s+1) \cdot \text{PD}[K\lambda]}(Y, K).$$

Combining Lemma 6.5 with the positivity of $d$ and $n$ in Equation (27), we see that for any $\xi$, if $s$ is sufficiently large and $\xi' = \xi + s \cdot \text{PD}[K\lambda]$, we have that $h_\xi^s$ is null-homotopic, and $v_\xi^s$ is an isomorphism. It follows from this (together with the analogous statement for $s$ sufficiently small) that the mapping cone of $(f_2^\delta)'$ is identified with the mapping cone of

$$(f_2^\delta'')': \bigoplus_{s \in \mathbb{Z}} A_\lambda^{\delta+(sn) \cdot \text{PD}[\mu]}(Y, K) \longrightarrow \bigoplus_{s \in \mathbb{Z}} B_\lambda^{\delta+s \cdot \text{PD}[K\lambda]}(Y, K),$$

gotten by adding all the maps $v_\delta^s$ and $h_\delta^s$. This establishes the Theorem 6.4 in the present case.

For fixed $\delta \geq 0$ and $k$ sufficiently large, $f_2^k$ remains a homogeneous map as before. However, in the present case, we study $f_3^\delta$, the map associated to the cobordism $W_\lambda$ from $Y$ to $Y_\lambda$, in place of $f_1^\delta$. To this end, we have the following:

**Lemma 6.9.** Fix an integer $\delta \geq 0$ and constant $C_0$. For all sufficiently large $k$, the following condition holds. For each $\mathfrak{s}_0 \in \text{Spin}^c(W_\lambda(K))$, there is at most one Spin$^c$ structure of the form $\mathfrak{s} = \mathfrak{s}_0 + jk\text{PD}[\hat{F}]$ with $j \in \mathbb{Z}$ for which

$$C_0 \leq c_1(\mathfrak{s})^2 + nk.$$

Here, $\hat{F}$ denotes a generator of $H^2(W_\lambda(K), Y)$. 

**Proof.** This follows from the fact that $W_\lambda$ is a negative-definite cobordism. \(\square\)

The relevance of the lemma is the following: $f_3^\delta(\xi \otimes T^i)$ is a sum of the maps associated to Spin$^c$ structures $\mathfrak{s}$ over the cobordism $W_\lambda(K)$ which differ by addition of $nk\text{PD}[\mu] = dk\text{PD}[\lambda]$, under the identification $H^2(Y, K; \mathbb{Z}) \cong H^2(W_\lambda(K))$. By Equation (7), the latter in turn correspond to $dk\mathbb{Z} \cdot \hat{F}$ orbits.

It follows from Lemma 6.9 that if $k$ is sufficiently large, then given $t \in \text{Spin}^c(Y)$ and $i \in \mathbb{Z}/nk\mathbb{Z}$, there is a unique Spin$^c$ structure which contributes non-trivially to
for all $\xi \in HF^\delta(Y, t)$. Specifically, according to Equation (9), the non-triviality of the map places a lower bound (independent of $k$) on the square of the first Chern class of any such Spin$^c$ structure (c.f. Corollary 4.7).

With these remarks in place, it now follows quickly that the map

$$\psi^\delta: M(f_2^\delta) \rightarrow CF^\delta(Y_l(K))$$

given by $\psi^\delta(x, y) = H_2^\delta(x) + f_3^\delta(y)$ is a relatively graded map.

The proof of Theorem 6.1 is then completed as before.
7. The proof of Theorem 1.1

If $K \subset Y$ is a null-homologous knot, then $Y_{p/q}(K)$ can be realized by surgery with coefficient $a$ inside the knot $K \# O_{q/r} \subset Y \# L(q,r)$, where $a$ is the greatest integer smaller than or equal to $p/q$, $a = \lfloor \frac{p}{q} \rfloor$, and

$$\frac{p}{q} = a + \frac{r}{q}$$

and $O_{q/r} \subset L(q,r)$ is the knot which is obtained by viewing one component of the Hopf link as a knot inside the lens space $L(q,r)$, thought of as $q/r$ surgery on the other component of the Hopf link, c.f. Figure 2.

In view of these remarks, Theorem 1.1 is proved by combining a model calculation of the knot Floer homology of $O_{q/r}$, the Künneth principle for connected sums, together with the surgery formula of Theorem 6.1.

Of course, $L(q,r) - O_{q/r}$ is a solid torus, and consequently, there is an affine identification $\text{Spin}^c(L(q,r), O_{q/r}) \cong \mathbb{Z}$.

Lemma 7.1. The knot $O_{q/r} \subset L(q,r)$ is a $U$-knot. Indeed, there is an affine identification $\phi$ fitting into a commutative diagram

$$\begin{align*}
\mathbb{Z} & \xrightarrow{\phi} \text{Spin}^c(L(q,r), O_{q/r}) \\
\downarrow & \quad \downarrow \text{G}_{L(q,r), O_{q/r}} \\
\mathbb{Z}/q\mathbb{Z} & \cong \text{Spin}^c(L(q,r))
\end{align*}$$

(where the left vertical arrow is the natural quotient map) with the property that

$$\overline{HFK}(L(q,r), O_{q/r}, \phi(i)) \cong \begin{cases} 
\mathbb{Z} & \text{if } 0 \leq i \leq q - 1 \\
0 & \text{otherwise.}
\end{cases}$$

Figure 2. The knot $O_{q/r}$. Thinking of $K$ as a knot in the lens space obtained by performing $q/r$ on the other component of this link, we obtain the knot $O_{q/r}$.
Proof. Consider the standard genus one Heegaard decomposition of the lens space \( L(q, r) \), where \( \alpha \) is a curve of slope \( 0/1 \) and \( \beta \) is a curve of slope \( q/r \). Placing two basepoints \( w \) and \( z \), separated by an arc which is disjoint from \( \alpha \) and meets \( \beta \) exactly once, we obtain a doubly-pointed Heegaard diagram for \( O_{q/r} \subset L(q, r) \). Of course \( \alpha \cap \beta \) consists of exactly \( q \) points \( \{x_0, \ldots, x_{q-1}\} \), each one representing a different Spin\(^c\) structure over \( L(q, r) \). See Figure 3 for an illustration.

Now,

\[
(28) \quad H_1(L(q, r) - O_{q/r}) \cong \frac{H_1(T^2 - w - z)}{\mathbb{Z} \cdot \alpha + \mathbb{Z} \cdot \beta} \cong \frac{H_1(T^2)}{\mathbb{Z} \cdot \alpha} \cong \mathbb{Z},
\]

which is generated by the homology class represented by a curve \( \gamma \) in \( T^2 \) with slope \( 1/0 \). We specify an ordering on the \( \{x_i\}_{i=0}^{q-1} \) as follows. Let \( m \) be the intersection point of \( \beta \) with the arc connecting \( w \) and \( z \). Our ordering is specified by the conditions that the arc in \( \beta \) from \( x_i \) to \( x_{i+1} \) is disjoint from all the other \( x_k \) and \( m \). Clearly, with this ordering, \( \epsilon(x_i, x_{i+1}) = \gamma \). Let \( \phi(i) \) correspond to \( \mathfrak{a}_{w,z}(x_i) \). The lemma follows. \( \square \)

Suppose that \( K \) is a null-homologous knot in an integral homology three-sphere \( Y \). In this case, \( H^2(Y, K) \cong \mathbb{Z} \), and also \( H^2(Y \# L(q, r), K \# O_{q/r}) \cong \mathbb{Z} \).

Lemma 7.2. Suppose that \( K \subset Y \) is a null-homologous knot in an integer homology three-sphere. Under the connected sum

\[
(Y, K) \# (L(q, r), O_{q/r}) \to (Y \# L(q, r), K \# O_{q/r}),
\]

Figure 3. Heegaard diagram for \( O_{3/2} \). The intersection points \( x_0, x_1, x_2 \) are ordered so that \( \epsilon(x_0, x_1) = \epsilon(x_1, x_2) \) is a generator of the homology of \( L(3, 2) - O_{3/2} \).
the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\
\cong & & \cong \\
H^2(Y, K) \oplus H^2(L(q, r), O_{q/r}) & \xrightarrow{} & H^2(Y \# L(q, r), K \# O_{q/r}),
\end{array}
\]

where \( f(x, y) = qx + y \). Moreover, under this correspondence, if \( K_\lambda \) is the push-off of \( K \) with respect to the integral framing \( a \), then \( PD[K_\lambda] \) represents the element \( p \in \mathbb{Z} \cong H^2(Y \# L(q, r), K \# O_{q/r}) \), where \( p/q = a + q/r \).

Proof. Let \( m \) and \( \ell \) denote the meridian and the longitude of \( O_{q/r} \). We have an isomorphism \( H^2(L(q, r), O_{q/r}) \cong \mathbb{Z} \) under which \( m \) and \( \ell \) are mapped to \( r \) and \( q \) respectively. Now in \( Y \# L(q, r) - K \# O_{q/r} \), clearly the meridian of \( K \) is homologous to a longitude for \( O_{q/r} \), and hence it is mapped to \( q \) under the isomorphism. Moreover, the push-off of \( K_\lambda \) is homologous \( m + a \ell \), which in turn is mapped to \( r + aq = p \), under this above map. \( \square \)

Proof of Theorem 1.1. Fix an identification \( \text{Spin}^c(Y \# L(q, r), K \# O_{q/r}) \cong \mathbb{Z} \), and correspondingly think of \( A_+^a(Y \# L(q, r), K \# O_{q/r}) \) as indexed by \( s \in \mathbb{Z} \). Since \( O_{q/r} \) is a \( U \)-knot (c.f. Lemma 7.1), the Künneth principle for connected sums, in the form of Corollary 5.3 applies to show that

\[
A_+^a(Y \# L(q, r), K \# O_{q/r}) \simeq A_+^g(s)(Y)
\]

for some \( g: \mathbb{Z} \rightarrow \mathbb{Z} \) (where here \( \simeq \) means filtered quasi-isomorphic). Indeed, according to Lemma 7.2, the formula \( g(s) \) satisfies \( s = qg(s) + j \) where, according to Lemma 7.1, \( 0 \leq j < q \); i.e.

\[
g(s) = \left\lfloor \frac{s}{q} \right\rfloor.
\]

Recall also that \( PD[K_\lambda] \) represents \( p \) times a generator of \( H^2(Y \# L(q, r), K \# O_{q/r}) \).

With these remarks in place, Theorem 1.1 is obtained as a direct application of Theorem 6.1. \( \square \)

7.1. The case of \( \widehat{HF} \). Some of the algebra is simpler when one considers \( HF^+ \), rather than \( \widehat{HF} \). To this end, we let

\[
\widehat{D}_{i,p/q}: \widehat{A}_i \rightarrow \widehat{B}_i;
\]

i.e.

\[
\widehat{D}_{i,p/q}\{a_{\left\lfloor \frac{i}{q} \right\rfloor}\}_{s \in \mathbb{Z}} = \{b_{\left\lfloor \frac{i}{q} \right\rfloor}\}_{s \in \mathbb{Z}},
\]

where here

\[
b_{\left\lfloor \frac{i}{q} \right\rfloor} = \hat{v}_{\left\lfloor \frac{i}{q} \right\rfloor}(a_{\left\lfloor \frac{i}{q} \right\rfloor}) + \hat{h}_{\left\lfloor \frac{i}{q} \right\rfloor - 1}(a_{\left\lfloor \frac{i}{q} \right\rfloor - 1}).
\]

The proof of Theorem 1.1 adapts readily to this context to give the following:
Theorem 7.3. Let \( K \subset Y \) be a null-homologous knot. There is a relatively graded isomorphism of groups
\[
H_e(\mathcal{X}_{i,p/q}) \cong \widetilde{HF}(Y_{p/q}(K), i)
\]
for each \( i \in \mathbb{Z}/p\mathbb{Z} \).

7.2. Absolute gradings. In fact, Theorem 1.1 (and indeed Theorem 6.1) actually determines \( HF^+(Y_{p/q}(K), i) \) as an absolutely graded group.

An absolute grading on \( \mathcal{X}^+_{i,p/q}(K) \) compatible with the relative grading is specified by fixing an absolute grading on \( B^+_{\left\lfloor i + ps \right\rfloor} \) (for any \( s \)), thought of as a subcomplex of \( \mathcal{X}^+_{i,p/q}(K) \) (note that it is independent of the choice of \( K \)). This absolute grading in turn is in turn determined by a grading on its homology \( H_*(B^+_{\left\lfloor i + ps \right\rfloor}) \cong T^+ \). Finally, that datum is fixed by the requirement \( H_*(\mathcal{X}^+_{i,p/q}(O)) \cong T^+ \) in such a way that its bottom-most non-trivial element is supported in dimension \( d(S^3_{p/q}(O), i) = d(L(p,q), i) \).

This assertion follows easily from the proof of Theorem 1.1. Specifically, as in Subsection 6.3, the composite of
\[
f^\delta_1 : CF^\delta(Y_\lambda(K)) \longrightarrow CF^\delta(Y_{nk,\mu+\lambda}(K))
\]
with the projection onto \( CF^\delta(Y_{nk,\mu+\lambda}(K), t) \subset CF^\delta(Y_{nk,\mu+\lambda}(K)) \) is a homogeneous map (provided that \( n \) is sufficiently large). Moreover, its degree depends on the intersection form of the cobordism \( X(k) \) and the first Chern class of \( t \). Moreover, the proof of Theorem 6.4 shows that this map is also identified with the projection \( \mathcal{X}^\delta_s(\lambda) \longrightarrow A^\delta_s(Y, \overline{K}) \). Since the intersection form of \( X(k) \) does not depend on the particular knot \( \overline{K} \), the claim follows.
8. Knots which admit \( L \)-space surgeries

Suppose that \( K \subset S^3 \) is a knot which admits an \( L \)-space surgery with positive slope \( r \). Examples of such knots include all torus knots, and also knots from Berge’s list, c.f. [1]. Moreover, an alternating knot \( \tilde{K} \) with unknotting number equal to one gives rise to another knot \( C \subset S^3 \) which admits an \( L \)-space surgery. This new knot \( C \) is obtained by performing the unknotting operation, but connecting the two strands which were crossed by an arc \( \gamma \); \( C \) then is the branched double-cover \( \gamma \), compare [15].

Theorem 1.2 says that for such a knot, the Alexander polynomial is determined from the correction terms. In this application, we find it convenient to use the group \( HF^+ \); to do this, recall the characterization of \( L \)-spaces in terms of this group [16]: a rational homology three-sphere \( Y \) is an \( L \)-space if and only if for each \( t \in \text{Spin}^c(Y) \),

\[
HF^+(Y, t) \cong \mathcal{T}^+.
\]

The correction terms \( d(Y, t) \) of an \( L \)-space is the minimal grading of any homogeneous element of \( HF^+(Y, t) \).

The symmetrized Alexander polynomial \( \Delta_K(T) \) plays a role, since it is the Euler characteristic of the knot Floer homology; i.e.

\[
\sum_s \chi(HFK_s(S^3, K, s)) \cdot T^s = \Delta_K(T),
\]

c.f. Equation (1) of [22] or [27].

**Proof of Theorem 1.2.** It is easy to see that both maps \( v_s^+: H_s(A^+_s) \to H_s(B^+) \) and \( h_s^+: H_s(A^+_s) \to H_s(B^+) \) are surjective, since \( B^+ \cong HF^+(S^3) \), and both maps are isomorphisms in all sufficiently large degrees.

Fix \( i \in \mathbb{Z}/p\mathbb{Z} \), and let \( s \in \mathbb{Z} \) be an arbitrary representative. Split \( A^+_s = A^+_s \oplus J \) where \( J = \bigoplus_{i \neq s} A^+_i \cap J \). From the above remarks, it is clear that the restriction of \( D^+_{i,p/q} \) to \( H_s(J) \) surjects onto \( H_s(B^+_i) \). It follows at once that there is a surjection

\[
\varphi: \text{Ker} \left( H_s(D^+_{i,p/q}): H_s(A^+_i) \to H_s(B^+_i) \right) \to H_s(A^+_s).
\]

Moreover, it follows from this surjectivity, combined with Theorem 1.1, that

\[
\text{Ker} \left( H_s(D^+_{i,p/q}): H_s(A^+_i) \to H_s(B^+_i) \right) \cong HF^+(S^3_{p/q}(K), i) \cong \mathcal{T}^+.
\]

The surjectivity of \( \varphi \), together with the fact that \( H_s(A^+_s) \cong H_s(B^+) \) in all sufficiently large degree combine to show that \( H_s(A^+_s) \otimes \mathbb{Q} \cong \mathcal{T}^+ \otimes \mathbb{Q} \). Applying the same argument, only taking coefficients in \( \mathbb{Z}/p\mathbb{Z} \) for arbitrary \( p \) shows that \( H_s(A^+_s) \cong \mathcal{T}^+ \).

Having established that \( H_s(A^+_s) \cong \mathcal{T}^+ \), and that \( v_s^+ \) and \( h_s^+ \) are isomorphisms in all sufficiently large degrees, it follows that \( v_s^+ \) is modeled on multiplication by \( U^{V_s} \), and \( h_s^+ \) is multiplication by \( U^{H_s} \), where all \( V_s, H_s \geq 0 \).
The condition that $\text{Ker} \left( H_\ast \left( D^+_{i,p/q} : H_\ast(A_i^+) \to H_\ast(B_i^+) \right) \right) \cong T^+$ ensures readily that in each $i \in \mathbb{Z}/p\mathbb{Z}$, there can be at most one integer $s \in \mathbb{Z}$ for which both $V_{\lfloor \frac{i+ps}{q} \rfloor}$ and $H_{\lfloor \frac{i+ps}{q} \rfloor} \geq 0$. Our aim now is to determine which value of $s$ has this property.

Let $m(s) = \min(V_s, H_s)$. We claim that

$$m(s_1) \leq m(s_2) \text{ if } s_1 \leq s_2 \leq 0 \text{ or } s_1 \geq s_2 \geq 0. \quad (29)$$

To this end, we have an exact sequence

$$0 \to C\{i < 0 \text{ and } j \geq s\} \to C\{\max(i, j - s)\} \to C\{i \geq 0\} \to 0.$$  

Since $\chi(\widehat{HFK}(K, s)) = a_s$, it follows at once that for $s \geq 0$,

$$\chi(C\{i < 0 \text{ and } j \geq s\}) = t_s(K),$$

which in turn is the same as $V_s$. Similarly, $\chi(C\{i \geq 0 \text{ and } j < s\}) = t_s(K) + 2s$, which is $H_s$. In particular, this (together with a symmetric argument for $s \leq 0$) shows that

$$V_s \leq H_s \text{ for } s \geq 0 \quad \text{and} \quad V_s \geq H_s \text{ for } s \leq 0. \quad (30)$$

Incidentally, we have just established that $t_s = m(s) \geq 0.$

Note that there is a natural quotient map $A_s^+ \to A_{s+1}^+$, and indeed, the projection of $A_s^+ \to B_s^+$ factors through this quotient. It follows at once that $\{V_s\}_{s \in \mathbb{Z}}$ is a non-increasing sequence in $s$. Dually, $\{H_s\}_{s \in \mathbb{Z}}$ is a non-decreasing sequence in $s$. Combining this with Equation (30), Equation (29) follows.

Having established that for each $i \in \mathbb{Z}/p\mathbb{Z}$, there is exactly one non-zero integer among the $\{m(\lfloor \frac{i+ps}{q} \rfloor)\}_{s \in \mathbb{Z}}$. Equation (29) together with the fact that $m(s) = m(-s)$ (an easy consequence of Equation (31)) shows that $m(\lfloor \frac{i}{q} \rfloor) \neq 0$ implies that

$$\lfloor \frac{i}{q} \rfloor \leq \lfloor \frac{p}{2q} \rfloor$$

for all $s \in \mathbb{Z}$.

Finally, note that the bottom-most generator of $A_{\lfloor \frac{i}{q} \rfloor}^+$ is an element whose degree is $2m(\lfloor \frac{i}{q} \rfloor)$ less than the corresponding generator for the unknot. In view of the remarks from Subsection 7.2, the theorem now follows.

\[\square\]

**Lemma 8.1.** Let $K \subset S^3$ be a knot. The map $v_s^+: A_s^+ \to B^+$ is an isomorphism on homology for all $s \geq 0$ if and only if $\widehat{HFK}(K, s) = 0$ for all $s > 0$.

**Proof.** It is easy to see by descending induction on $s$ that $v_s^+$ is an isomorphism on homology for all $s \geq d$ if and only if $\widehat{HFK}(K, s) = 0$ for all $s > d$.  \[\square\]
Proof of Corollary 1.3. Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(O)$. Then, for some permutation $\sigma: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$, we have that $HF^+(S^3_{p/q}(O), i) \cong HF^+(S^3_{p/q}(K), \sigma(i))$ for all $i$. It follows from the proof of Theorem 1.2 (Equation (31)) that $t_i(K) \geq 0$ for all $i$. Summing Equation 2 over all $i$ and using the hypothesis that $S^3_{p/q}(K) \cong S^3_{p/q}(O)$, we conclude that $t_i(K) \equiv 0$. It follows that $V_s = 0$ for $s \geq 0$, i.e. $v^+_s: H_*(A^+_s) \rightarrow HF^+(S^3)$ is an isomorphism for all $s \geq 0$. According to Lemma 8.1, $\hat{HFK}(K, s) = 0$ for all $s \neq 0$, and hence, according to [21], $K$ is the unknot. \hfill $\Box$
9. On cosmetic surgeries

Under favorable circumstances, the existence of an orientation-preserving homeomorphism \( \hat{HF}(S^3_r(K)) \cong \hat{HF}(S^3_s(K)) \) for distinct \( r \) and \( s \) forces the knot Floer homology of \( K \) to agree with that for the unknot, and hence for the knot to be unknotted. This is not always the case, though. For example, the knot \( K = 9_{44} \) has the property that \( HF^+(S^3_{+1}(K)) \cong HF^+(S^3_{-1}(K)) \), although \( S^3_{+1}(K) \not\cong S^3_{-1}(K) \). Specifically, according to Theorem 6.1 of [26], the knot \( 9_{44} \) has

\[ \hat{HFK}(K, i) = \mathbb{Z}[a_i] \]

where the subscript indicates the degree in which the summand is supported, and \( a_i \) is the \( i \)th coefficient of the Alexander polynomial

\[ \Delta_K(T) = \sum_{i \in \mathbb{Z}} a_i \cdot T^i = T^{-2} - 4T^{-1} + 7 - 4T + T^2. \]

It is now a straightforward application of the surgery formula, either in the form given in the present paper or from [25], that

\[ HF^+(S^3_{+1}(K)) \cong \tau^+(0) \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2(-1) \cong HF^+(S^3_{-1}(K)). \]

On the other hand, Walter Neumann [13] informs us that the manifolds \( S^3_{+1}(K) \) and \( S^3_{-1}(K) \) can be distinguished (using the computer program Snap [3], see also [31]) by their hyperbolic volume; one has hyperbolic volume roughly 5.52, the other has hyperbolic volume roughly 5.27.

Our results here can be proved by restricting attention to Floer homology with coefficients in any field \( F \), which we suppress from the notation. For concreteness, we restrict to \( F = \mathbb{Z}/2\mathbb{Z} \), so that e.g. \( \hat{HF}(S^3) \) denotes the homology \( H_*(\mathcal{CF}(S^3) \otimes \mathbb{Z}) \).

**Definition 9.1.** Define \( \nu(K) \) by

\[ \nu(K) = \min\{s \in \mathbb{Z} | \hat{\alpha}_s : \hat{A}_s \rightarrow \hat{CF}(S^3) \text{ induces a non-trivial map in homology} \} \]

**Lemma 9.2.** If \( K \subset S^3 \) and \( -K \) denotes its reflection, then either \( \nu(K) \) or \( \nu(-K) \) is non-negative.

**Proof.** Let \( \{\mathcal{F}_s\}_{s \in \mathbb{Z}} \) be the knot filtration of \( \hat{CF}(S^3) \) (i.e. in the notation of Section 3 \( \mathcal{F}_s = C_{\xi_0} \{j < s\} \), where \( \xi_0 \in \text{Spin}^c(S^3, K) \) is the relative Spin\(^c \) structure with trivial first Chern class) and let

\[ \tau(K) = \min\{s \in \mathbb{Z} | H_*(\mathcal{F}_s) \rightarrow \hat{HF}(S^3) \text{ is non-trivial} \}. \]

Recall that \( \tau(K) = -\tau(-K) \) and also that \( \nu(K) = \tau(K) \) or \( \tau(K) + 1 \) (c.f. Lemma 3.3 of [18] and Proposition 3.1 from the same reference respectively; or alternatively, see [27]).
The integer \( \nu(K) \) dictates the maps on homology induced by all the \( \hat{h}_s \) and \( \hat{v}_s \), according to the following two lemmas.

**Lemma 9.3.** For all \( s \geq \nu(K) \), \( \hat{v}_s \) induces a non-trivial map in homology.

**Proof.** This follows at once from the fact that the image of \( \hat{v}_s \) is \( F_s \), and \( F_s \subseteq F_t \) if \( s \leq t \).

**Lemma 9.4.** If \( \nu(K) \geq 0 \), then for all \( s > 0 \), \((\hat{h}_s)^* = 0\).

**Proof.** The image of \( \hat{h}_s : \hat{A}_s \to \hat{B} \) is identified with \( F_{-s} \subseteq \hat{B} \), thus, if \( s > 0 \) and \( \tau(K) \geq 0 \), then the map on homology induced by \( \hat{h}_s \) is trivial.

**Proposition 9.5.** Let \( K \subset S^3 \) be a knot, and fix a pair of relatively prime integers \( p \) and \( q \). Then

\[
(32) \quad \text{rk} \hat{HF}(S^3_{p/q}(K)) = \left| p \right| + 2 \max(0, (2\nu(K) - 1)|q| - |p|) + |q| \left( \sum_s \left( \text{rk}H_*(\hat{A}_s) - 1 \right) \right)
\]

**Proof.** Recall that if \( K \subset S^3 \), and \(-K\) denotes its reflection, then \( S^3_{p/q}(K) \cong -S^3_{p/q'}(-K) \). Thus, in view of Lemma 9.2 and Equation (33), we can assume without loss of generality that \( \nu(K) \geq 0 \). Also, of course, we can assume that \( q \geq 0 \). Using Theorem 7.3 to express \( \hat{HF}(S^3_{p/q}(K)) \) in terms of \( H_*(\hat{X}_{p/q}) \), calculating the rank of \( H_*(\hat{X}_{p/q}) \) is now straightforward application of Lemmas 9.3 and 9.4. In fact, the cokernel \( \hat{D} \) has rank \( \max(0, (2\nu(K) - 1)q - p) \), while its kernel has rank

\[
P + \max(0, (2\nu(K) - 1)q - p) + q \left( \sum_s \left( \text{rk}H_*(\hat{A}_s) - 1 \right) \right).
\]

(The above proposition holds even in the case where \( p \) and \( q \) are not relatively prime, with the understanding that if \( (p, q) = a \), then \( \hat{HF}(S^3_{p/q}(K)) \) denotes the direct sum of \( a \) many copies of \( \hat{HF}(S^3_{p'/q'}(K)) \), where here \( p' = p/a, q' = q/a \).

The following result is analogous to Lemma 8.1.

**Proposition 9.6.** Let \( K \) be a knot with \( \nu(K) = 0 \) and \( \text{rk}H_*(\hat{A}_s(K)) = 1 \) for all \( s \in \mathbb{Z} \). Then, \( K \) is the unknot. More generally,

\[
g(K) = \max(\nu(K), \{ s \in \mathbb{Z} | \text{rk}H_*(\hat{A}_{s-1}) > 1 \}).
\]
Proof. Recall [21] that
\[ g(K) = \max \{ t \in \mathbb{Z} | \widehat{HFK}(K, t) \neq 0 \}. \]
This readily implies that the map \( \widehat{v}_s : \widehat{A}_s \to \widehat{CF}(S^3) \) induces an isomorphism in homology for all \( s \geq g(K) \): \( \widehat{A}_s \) has a subcomplex \( C\{i < 0, j = s\} \), the kernel of \( \widehat{v}_s \), which is easily seen to be acyclic (as it is filtered by subcomplexes whose associated graded has homology isomorphic to \( \bigoplus_{t > s} \widehat{HFK}(K, t) \)), and moreover its image in \( C\{i = 0\} \) consists of the subcomplex \( C\{i = 0, j \leq s\} \), whose quotient is acyclic.

Thus, we have proved that
\[ g(K) \geq \max(\nu(K), \{ s \in \mathbb{Z} | \text{rk} H_*(\widehat{A}_s) > 1 \}). \]

Now consider
\[
\begin{array}{ccc}
0 & \longrightarrow & C\{i < 0, j \geq g - 1\} \\
& & \longrightarrow C\{\max(i, j - g + 1) \geq 0\} \\
& & \xrightarrow{v_{g-1}^+} C\{i \geq 0\} \\
& & \longrightarrow 0,
\end{array}
\]
where the middle term here is, of course \( A_{g-1}^+ \). Since \( HFK(K, t) = H_*(C\{(0, t)\}) = 0 \) for all \( t > g \), it follows from the natural filtration that \( H_*(C\{i < 0, j \geq g - 1\}) \cong H_*(\{-1, g - 1\}) \cong H_*(C\{0, g\}) = \widehat{HFK}(K, g) \neq 0 \). It follows at once that \( \widehat{v}_{g-1} \) is not an isomorphism. It follows at once that either the map is not surjective, in which case \( \nu(K) = g - 1 \), or it has kernel, in which case \( \text{rk} H_*(\widehat{A}_{g-1}) > 1 \).

Before stating the next result, we recall that the total rank of \( \widehat{HF}(Y) \) is independent of the orientation of \( Y \):
\[ \text{rk} \widehat{HF}(-Y) = \text{rk} \widehat{HF}(Y). \]
(Indeed, in Proposition 2.5 of [23], it is shown that \( \widehat{HF}^*(\pm Y) \cong \widehat{HF}^*(Y) \).)

**Theorem 9.7.** Let \( K \subset S^3 \) be a knot, and suppose that \( r \) and \( s \) are distinct rational numbers with the property that \( S^3_{\pm p/q}(K) \cong \pm S^3_{\pm p/q}(K) \). Then, either \( S^3_{\pm p/q}(K) \) is an L-space, or \( r \) and \( s \) have opposite signs.

**Proof.** Since \( H_1(S^3_{\pm p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \), we can fix \( p \) throughout. Now, according to Proposition 9.5, for fixed \( p \), and positive integral \( q \), \( \text{rk} \widehat{HF}(S^3_{\pm p/q}(K)) \) is a monotone non-decreasing function of \( q \). In fact, the function is strictly monotone except possibly for sufficiently small \( q \), for which the rank is \( p \). But this ensures that \( S^3_{\pm p/q}(K) \) is an L-space. The same remarks hold for the function \( \text{rk} \widehat{HF}(S^3_{-p/q}(K)) \) for fixed \( p \) and positive, integral \( q \). Since the total rank of \( \widehat{HF}(Y) \) is an invariant of the underlying (unoriented) three-manifold, c.f. Equation (33), the result holds. \( \square \)
Proposition 9.8. If $K$ is a non-trivial knot with $\nu(K) = 0$, then if there are rational numbers $r, s \in \mathbb{Q}$ with $r \neq s$ and $S^3_r(K) \cong S^3_s(K)$, then $r = \pm s$.

Proof. According to Proposition 9.5, if $rk \widehat{HF}(S^3_{p/q}(K)) = rk \widehat{HF}(S^3_{p/q'}(K))$ for $q' \neq \pm q$, then $rk \hat{v}_s(\widehat{A}_s) = 1$ for all $s$. Now, in view of Proposition 9.6, $K$ is the unknot, contrary to our assumption. Thus, it follows that for a cosmetic surgery on $K$, $r = -s$.

Theorem 9.9. Let $K \subset S^3$ be a knot with Seifert genus equal to one. Then if $S^3_r(K) \cong S^3_s(K)$ as oriented manifolds, then either $S^3_r(K)$ is an L-space or $r = s$.

Proof. In view of Theorem 9.7, we can assume that $r > 0$ and $s < 0$.

As in the proof of Proposition 9.6, it is clear that

$$\widehat{v}_s : H_*(\widehat{A}_s(K)) \longrightarrow \widehat{HF}(S^3)$$

is an isomorphism for all $s > 0$. Also, $\nu(K) \leq 1$.

We exclude the possibility that $\nu(K) = 1$. For $S^3_r(K)$, we have that in any given Spin$^c$ structure, $\widehat{HF}(S^3_s(K), t)$ is described by

$$H_*(\widehat{A}_0(K))^m \oplus F^{m-1}.$$ 

Since this group is also described as the Floer homology of $S^3_s(K)$ with negative $s$, it has the form

$$H_*(\widehat{A}_0(K))^n \oplus F^{n+1}.$$ 

Since $rk \hat{v}_s(\widehat{A}_0(K)) \neq 0$ (since its Euler characteristic is 1), the equality of these two ranks forces at once that $m = n + 1$ and $rk \hat{H}_s(\widehat{A}_0(K)) = 1$. But it is easy to see that a relatively graded isomorphism

$$H_*(\widehat{A}_0(K))^{n+1} \oplus F^{n}(-1) \cong H_*(\widehat{A}_0(K))^n[1] \oplus F^{n+1}(0)$$

cannot possibly hold.

In the case where $\nu(K) = 0$, we force a relatively graded isomorphism

$$K^n \oplus F_{(0)} \cong K[1]^m \oplus F_{(0)},$$

where here $K$ is the kernel of the map on homology

$$(\widehat{v}_0 \oplus \widehat{h}_0)_* : H_*(\widehat{A}_0) \longrightarrow F \oplus F.$$ 

Such a relatively graded isomorphism can hold only if the rank of $K$ is zero. In turn, this forces $rk \hat{H}_s(\widehat{A}_0) = 1$. From Proposition 9.6, it follows now that $K$ is a trivial knot, contradicting our hypothesis.
By Proposition 9.5 (or alternatively [16]), a genus one knot with $L$-space surgeries is easily seen to have the knot Floer homology groups of the trefoil $T$. In particular, if $S^3_2(K) \cong S^3_3(K)$, then it follows that $HF^+(S^3_2(T)) \cong HF^+(S^3_3(T))$ as $\mathbb{Q}$-graded groups. Of course, $HF^+(S^3_2(T))$ can be calculated explicitly (for example using Theorem 1.2), and the authors know of no pair of distinct rational numbers $r$ and $s$ for which $HF^+(S^3_r(T)) \cong HF^+(S^3_s(T))$.

In a different direction, the algebra can be used in some cases to exclude cosmetic surgeries with a fixed numerator $p$ (i.e. first homology of the surgered manifold). We content ourselves here with a discussion of the case where $p = 3$.

**Theorem 9.10.** Suppose that $K$ is a non-trivial knot. Then, if $r, s \in \mathbb{Q}$ both with numerators having absolute value 3, and with $r \neq s$, we have that $S^3_r(K) \not\cong S^3_s(K)$ as oriented manifolds.

**Proof.** By Lemma 9.2, we can arrange that $\nu(K) \geq 0$. The possibility that $\nu(K) \geq 3$ is excluded by counting ranks. According to Proposition 9.5, we have that

$$\widehat{HF}(S^3_{-3/q}(K)) = 3 + q \cdot \left(2(2\nu(K)) - 1\right) + \sum_s \left(rk_{\mathbb{H}}(\hat{A}_s) - 1\right).$$

When $\nu(K) \geq 3$, then for all $q$, we have that

$$6 \leq (2\nu - 1)q,$$

and hence

$$HF(S^3_{3/q}(K)) = -3 + q \cdot \left(2(2\nu(K)) - 1\right) + \sum_s \left(rk_{\mathbb{H}}(\hat{A}_s) - 1\right).$$

It follows that

$$\text{rk}\widehat{HF}(S^3_{3/q}(K)) \leq \text{rk}\widehat{HF}(S^3_{-3/q}(K)) < \text{rk}\widehat{HF}(S^3_{3/q+1}(K)),$$

the latter inequality following from the fact that

$$6 < 2(2\nu(K)) - 1 + \sum_s \left(rk\mathbb{H}(\hat{A}_s) - 1\right).$$

It follows that three-manifolds $\{S^3_{3/q}(K)\}_{q \in \mathbb{Z}}$ are all distinct.

Next, we turn our attention to excluding $\nu(K) = 2$. In this case, the above argument can fail if Inequality (35) fails. (Note that when $\nu(K) = 2$ and $q = 1$, Inequality (34) fails, but it is still the case that $\text{rk}\widehat{HF}(S^3_2(K)) < \text{rk}\widehat{HF}(S^3_{3/q}(K))$.) Thus, it remains to exclude the possibility $S^3_{-3/q}(K) \cong S^3_{3/q+1}(K)$. Now, $\widehat{HF}(S^3_{-3/q}(K)) \cong \widehat{HF}(S^3_{3/q+1}(K))$ forces equality, rather than the inequality of Equation (35). In particular, this forces $\text{rk}\mathbb{H}(\hat{A}_s) = 1$ for all $s$ and, since $\chi(\hat{A}_s) = 1$, each $\mathbb{H}(\hat{A}_s)$ is supported in even degree.
The condition that $\nu(K) = 2$ and $\text{rk} H_*(\hat{A}) = 1$ forces both $H_*(\hat{A}_0)$ and $H_*(\hat{A}_1)$ to be supported in negative degrees (c.f. [16]). On the one hand, $H_*(S^3_{-3/(q+1)}(K), 0)$ is non-trivial in degree zero, while the even degree part of $H_*(S^3_{3/q}(K), 0)$ is carried by elements from $H_*(\hat{A}_0)$ and $H_*(\hat{A}_1)$, which in turn is supported in negative degrees. Observe that our use of absolute degree in this case is justified by the fact that $L(3, 2) \cong -L(3, 1)$.

We turn our attention now to the case where $\nu(K) = 1$. Let

$$C = \sum_s \left( \text{rk} H_*(\hat{A}_s) - 1 \right).$$

Inequality (35) holds (and hence leads to no cosmetic surgeries, as above) except if $C = 0, 2, 4$. (Note that $C$ is even.) In the case where $C = 2$, it follows from symmetry that $\text{rk} H_*(\hat{A}_0) = 3$, and $\text{rk} H_*(\hat{A}_s) = 1$ for all $s \neq 0$. Thus, in the cases where $C = 0$ or 2,

Proposition 9.6 ensures that $K$ is a knot with Seifert genus equal to one. According to Theorem 9.9, $S^3_2(K)$ must be an $L$-space. But if $K$ is a knot with Seifert genus one, some positive surgery on $K$ gives an $L$-space, then by considering Proposition 9.5, we can conclude $S^3_3(K) q = 1$ or 2. But the $\mathbb{Q}/\mathbb{Z}$-valued linking form ensures that $S^3_3(K) \not\cong S^3_3(K)$.

When $\nu(K) = 1$, we are left with the remaining case that $C = 4$. Once again, since $\chi(\hat{A}_s) = 1$, there are two ways in which the total rank $C$ can in principle distribute over the $H_*(\hat{A}_s)$: either $H_*(\hat{A}_0)$ has rank 5 (and $H_*(\hat{A}_s)$ has rank one for all other $s$) or there is exactly one positive $s$ with $\text{rk} H_*(\hat{A}_s) = H_*(\hat{A}_{-s}) = 3$ and for all $t$ with $|t| \neq |s|$, $\text{rk} H_*(\hat{A}_t) = 1$. In the first case, Proposition 9.6 again ensures that $g(K) = 1$, and hence we can apply Theorem 9.9.

We exclude $S^3_{3/q}(K) \cong S^3_{3/q+1}(K)$ as follows. It is easy to see that in this case, in the model for the spin structure has the form

$$\widehat{HF}(S^3_{-3/q}(K), 0) \cong H_*(\hat{A}_0)^n[1] \oplus \mathbb{F}_0 \oplus G'$$

while

$$\widehat{HF}(S^3_{3/q+1}(K), 0) \cong H_*(\hat{A}_0)^{m+1} \oplus \mathbb{F}_{-1} \oplus G.$$  

Here, $G$ consists of some even number of copies of $M$, the kernel of

$$\widehat{(v)}_* : H_*(\hat{A}_s) \longrightarrow \mathbb{Z},$$

each of which is shifted up some even amount; while $G'$ consists of direct sum of some even number copies of $M$, each of which is shifted by an odd degree. Except in the case where $n = 0$, which coincides with the case where $q = 2$, which we need not consider, these shifts in degree for $G'$ are downward.
Now, when \( q \) is even, then
\[
n = 2 \left( \left\lfloor \frac{q-4}{3} \right\rfloor + 1 \right)
\]
and
\[
m = n - 1,
\]
while if \( q \) is odd then
\[
n = 2 \left( \left\lfloor \frac{q}{3} \right\rfloor + 1 \right)
\]
and
\[
m = n + 1.
\]
In either case, we have that \( n \not\equiv m \pmod{2} \). It follows from the assumption that 
\[
\widehat{HF}(S^3_{3/q}(K),0) \cong \widehat{HF}(S^3_{-3/q+1}(K),0)
\]
readily that \( H_*(\widehat{A}_0) \cong \mathbb{F}(-2) \). Thus, we have that
\[
\mathbb{F}_n(-1) \oplus \mathbb{F}_{n+1}(0) \oplus G' \cong \mathbb{F}_m(-1) \oplus \mathbb{F}_{m+1}(-2) \oplus G.
\]
Assuming that \( n \geq 0 \), we have the maximal degree of any element of \( G' \) is smaller than the maximal degree of any element of \( G \). It follows at once that \( H_*(\widehat{A}_s) \) consists of \( \mathbb{Z} \) in degree 0, and also the minimal degree of any element of \( G' \) is \(-1\). In fact, it follows that the rank of the degree zero part of \( G \) is \( n + 1 \), while the rank of the degree \(-1\) part of \( G \) is \( m + 1 \). However, since \( G \) and \( G' \) both have even rank in each degree, this contradicts the above observation that \( n \) and \( m \) have different parity.

Finally, we turn attention to the case where \( \nu(K) = 0 \). Since \( K \) is a non-trivial knot, Proposition 9.8 ensures that the only possibility is that 
\[
S^3_{3/q}(K) \cong S^3_{-3/q}(K).
\]
But this is easily excluded by the \( \mathbb{Q}/\mathbb{Z} \)-valued linking form. \(\square\)

9.1. Proofs of theorems in Subsection 1.2. Theorem 1.4 is a restatement of Theorem 9.9 proved above. Theorem 1.5 is stated and proved as Theorem 9.7 above. Theorem 1.6 is a restatement of Theorem 9.10 stated and proved above.
10. Seifert fibered spaces

Methods from this paper lead to the calculation of the Heegaard Floer homology groups of a large class of Seifert fibered spaces. The primary ingredients here are the calculation of the knot Floer homology of the “Borromean knot” (c.f. Section 9 of [22]), the knot $O_{q,r}$ considered in Section 7, and Theorem 6.1. Our aim here is to state and prove these results.

Let $h: \mathbb{Z} \to \mathbb{Z}$ be a function with the property that
$$
\lim_{s \to \pm\infty} h(s) = +\infty.
$$

We describe here a natural $\mathbb{Z}[[U]]$-module associated to $X$. It is interesting to compare the following construction with a construction of Némethi [12], see also [19].

A well at height $n \geq 0$ is a pair of integers $(i,j)$ with $i < j - 2$ and the property that $h(k) \leq n$ for all $i \leq k \leq j$, while $h(i) > n$ and $h(j) > n$. Let $M_n(h)$ denote the free Abelian group generated by $W_n(h)$.

If $x \in W_n(h)$ and $y \in W_{n-1}(h)$, we write $x > y$ if $x = (i,j)$ and $y = (i',j')$ with $i \leq i' < j' \leq j$. Define $U: W_n(h) \to W_{n-1}(h)$ by the formula
$$
U \cdot x = \sum_{\{y \in W_{n-1}(h) | x > y\}} y,
$$
and let
$$
\mathbb{HF}^+(h) = \bigoplus_{n \in \mathbb{Z}} W_n(h)
$$
be the induced module over $\mathbb{Z}[U]$. Indeed, we can view this as a graded $\mathbb{Z}[U]$ module by the grading which sends $W_n(h)$ to $2n$.

In the language of Némethi, the set of wells forms a root, and $\mathbb{HF}^+(h)$ is the associated $\mathbb{Z}[U]$-module.

Let $Y$ be a Seifert fibered space over a genus $g$ orbifold with Seifert invariants $(a,r_1/q_1,\ldots,r_n/q_n)$ over a genus $g$ base, c.f. [29], [28], [7]. The orbifold degree is the quantity
$$
\deg(Y) = a + \sum_i r_i / q_i.
$$
Recall that $b_1(Y)$ is even if and only if $\deg(Y) \neq 0$. By reversing the orientation on $Y$, we can arrange for $\deg(Y) > 0$.

There is a presentation of the first homology of $Y$ as

(36) \hspace{1cm} H_1(Y; \mathbb{Z}) \cong \frac{H_1(\Sigma; \mathbb{Z}) \oplus \mathbb{Z}m_0 \oplus \mathbb{Z}m_1 \oplus \ldots \oplus \mathbb{Z}m_n}{(a \cdot m_0 + \sum_{i=1}^n r_i \cdot m_i = 0, r_i \cdot m_0 - q_i \cdot m_i = 0, i = 1, \ldots, n)}

(c.f. below for more specifics).
Theorem 10.1. Let $Y$ be a Seifert fibered space over a genus $g$ orbifold with positive degree, and Seifert invariants $(a, r_1/q_1, ..., r_n/q_n)$. There is an affine identification $\text{Spin}^c(Y) \cong H_1(Y; \mathbb{Z})$ with the following properties.

- The $HF^+(Y, s)$ is non-trivial only to those Spin$^c$ structures which are supported in the span of $m_0, ..., m_n$ (in the notation of Equation (36)).
- Let $s$ be some Spin$^c$ structure over $Y$, and let $\xi_0 \cdot m_0 + \xi_1 \cdot m_1 + ... + \xi_n \cdot m_n$ be a representative with $0 \leq \xi_i \leq q_i$. For integers $-g \leq t \leq g$, let $\delta_t : \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by
  \[ \delta_t(s) = (-1)^{s+1} t + \left( \xi_0 + a \cdot s + \sum_{i=1}^{n} \left\lfloor \frac{\xi_i + r_i \cdot s}{q_i} \right\rfloor \right), \]

and $h_t : \mathbb{Z} \rightarrow \mathbb{Z}$ be the function
  \[ h_t(s) = \begin{cases} \sum_{i=0}^{s-1} \delta_t(i) & \text{if } s \geq 0 \\ -\sum_{i=s}^{-1} \delta_t(i) & \text{if } s < 0. \end{cases} \]

Then, there is a relatively $\mathbb{Z}$-graded isomorphism of $\mathbb{Z}[U]$-modules:
\begin{equation}
HF^+(Y, s) \cong \bigoplus_{-g \leq t \leq g} \Lambda^{g+t} H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} HF^+(h_t).
\end{equation}

In the above statement, the tensor product is to be taken in the graded sense: $\Lambda^{g+t} H_1(\Sigma; \mathbb{Z})$ is supported in grading $t$. Note also that the right-hand-side is graded only in the relative sense, corresponding to our choice of $s$.

Note that in the case where $g = 0$, this recaptures (as a relatively $\mathbb{Z}$-graded group) the description of $HF^+$ for rational homology Seifert fibered spaces given by Némethi in terms of his “computational sequences”. We prove Theorem 10.1 in Subsection 10.2 below, after giving some sample calculations.

Although we have described here the Floer homology only as a relatively graded group, the absolute grading can be obtained by comparing the summand corresponding to $t = 0$ with the calculation of genus zero Seifert fibered spaces from [19], see also [12].

10.1. Sample calculations. We begin with some generalities. Suppose $h : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function with $\lim_{s \rightarrow \pm \infty} h(s) = +\infty$. Let $\delta(s) = h(s) - h(s-1)$. Clearly, the rank of $\text{Ker} U \subset \mathbb{H}F^+(h)$ agrees with the number of pairs of integers $(i, j)$ with $i < j$, $\delta(i) < 0$, $\delta(j) > 0$ and $\delta(k) = 0$ for all $i < k < j$ (i.e. these are the local minima of $h$).

In the case of Theorem 10.1, explicitly finding these local minima is a straightforward matter: in the statement of the theorem, $\delta_t(s)$ differs from a linear function of $s$ by a periodic function whose period is the least common multiple of 2 (when $t \neq 0$) and the integers $\{q_i\}_{i=1}^{n}$.

We use Theorem 10.1 to calculate the Seifert fibered space over a genus one base, and Seifert invariants $(-1, \frac{1}{2}, \frac{2}{3})$. Note that $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^2$, and there is a unique Spin$^c$ structure with non-trivial $HF^+$. 
To calculate it, we proceed as follows. Observe that
\[ \delta_t(s + 6) = 1 + \delta_t(s). \]
Moreover, the sequence \( \{\delta_0(s)\}_{s=0}^5 \) is
\[ \{0, -1, 0, 0, 0, 0\} \]
Set \( t = 0 \). The sequence of integers \( \{h_0(i)\} \) clearly has a unique local minimum. It follows that the corresponding summand of \( HF^+(Y) \) is isomorphic to
\[ H_1(T^2) \otimes \mathbb{Z} \cong (T^+)^2. \]
In our subsequence descriptions, we will fix an absolute grading lifting the relative grading, with the additional convention that this corresponding summand of \( HF^+(Y) \) is isomorphic to \((T^+_0)^2\). Indeed, this convention corresponds to the naturally induced absolute grading on \( HF^+(Y) \), as can be seen by comparing against the case where \( g = 0 \), and observing that the Seifert fibered space with corresponding invariants is \( S^3 \).
When \( t = -1 \), all of the minima of the sequence \( \{h_{-1}(i)\}_{i \in \mathbb{Z}} \) clearly occur for \(-5 \leq i \leq 12\), where \( \{\delta_{-1}(s)\}_{s=-5}^{12} \) takes the form
\[ \{-3, 0, -2, 0, -2, 1, -2, 1, -1, 1, -1, 2, -1, 2, 0, 2, 0, 3\}. \]
We plot the corresponding function \( \{h_{-1}(s)\}_{s=-5}^{12} \) in Figure 4. It is clear from this description that the corresponding summand of \( HF^+(Y) \) takes the form
\[ T_{(-1)}^+ \oplus \mathbb{Z}_{(-1)}^2 \oplus \mathbb{Z}_{(1)}^2, \]
under the normalization convention established in the previous paragraph.
In the case where \( t = 1 \), we have that \( \{\delta_1(i)\}_{i=-12}^5 \) takes the form
\[ \{-1, -2, 0, -2, 0, -1, 0, -1, 1, -1, 1, 0, 1, 0, 2, 0, 2, 1\}, \]
which clearly contains all the minima of \( \{h_{-1}(i)\} \) for \( i \in \mathbb{Z} \). It follows that the corresponding summand of \( HF^+(Y) \) is of the form \( T_{(-1)}^+ \oplus \mathbb{Z}_{(-1)} \).

\[ \text{Figure 4. Height function for the Seifert fibered space with invariants (-1,1/2,2/3) at } t = -1. \]
Putting this together, if $Y$ denotes the Seifert fibered space over a genus one base with Seifert invariants $(-1, 1/2, 2/3)$, then its Heegaard Floer homology can be described as a graded $\mathbb{Z}[U]$ module by

$$HF^+(Y) \cong \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}^3_{(-1)} \oplus (\mathcal{T}^+_{(0)})^2 \oplus (\mathcal{T}^+_{(-1)})^2.$$ 

10.2. Proof of Theorem 10.1. It is useful to have a mild generalization of the rational surgeries formula.

**Definition 10.2.** Let $K \subset Y$ be a null-homologous knot in a three-manifold $Y$. Fix an integer $a$ and an $n$-tuple of rational numbers $\{\frac{q_i}{r_i}\}_{i=1}^n$. Consider an $n$-tuple of unknotted circles $O_i$ each of which links $K$ once, and which are pairwise mutually unlinked. Let $Y(K, a, \{\frac{q_i}{r_i}\}_{i=1}^n)$ denote the three-manifold obtained as $a$-surgery on $K$, followed by $-\frac{q_i}{r_i}$ surgery on each $O_i$. This three-manifold $Y(K, a, \{\frac{q_i}{r_i}\}_{i=1}^n)$ is said to be obtained as a generalized rational surgery on $K \subset Y$, with Seifert invariants $(a, \{\frac{q_i}{r_i}\}_{i=1}^n)$.

For fixed $K \subset Y$, let

$$Y' = Y \# (\#_{i=1}^n L(q_i, r_i)) \quad \text{and} \quad K' = K \# (\#_{i=1}^n O_{q_i/r_i})$$

in the notation of Section 7. Of course, $Y(K, a, \{\frac{q_i}{r_i}\}_{i=1}^n)$ can be thought of as the three-manifold gotten by $a$-surgery on $K' \subset Y'$.

![A schematic illustration of generalized rational surgery](image)

**Figure 5.** A schematic illustration of generalized rational surgery. We take here $n = 3$. $K$ represents some initial knot, and $\{\frac{q_i}{r_i}\}_{i=1}^3$ represent surgery instructions on the unknots, while $K$ is framed with framing $a$. The lightly drawn circles represent generators of the homology of the complement of the dark link (they are meridians).
As an example, if we start with the unknot $K \subset S^3$ and form the three-manifold $S^3(K, a, \{q_i/r_i\}_{i=1}^n)$, we obtain the Seifert fibered space whose base has genus zero, $n$ singular fibers, and Seifert invariants $(a, \{q_i/r_i\}_{i=1}^n)$, with the standard conventions. More generally, if we start with the Borromean knot with presentation $\langle g \rangle$, over a genus $g$ base orbifold with Seifert invariants $(a, \{q_i/r_i\}_{i=1}^n)$, we obtain the Seifert space whose base has genus zero, and $n$ singular fibers, and Seifert invariants $(a, \{q_i/r_i\}_{i=1}^n)$, with the standard conventions.

Consider the map

$$\beta: H_1(Y', K'; \mathbb{Z}) \rightarrow H_1(Y - K; \mathbb{Z})$$

defined by

$$\beta(\xi_0 + \xi_1 \cdot g_1 + \ldots + \xi_n \cdot g_n) = \xi_0 + \left( \sum_{i=1}^{n} \frac{\xi_i}{q_i} \right).$$

Letting $N = a \cdot m_0 + \sum_{i=1}^{n} r_i \cdot g_i$ be the push-off of $K$, we have that

$$H_1(Y(K, a, \{q_i/r_i\}_{i=1}^n); \mathbb{Z}) \cong H_1(Y' - K')/\mathbb{Z} \cdot N.$$

Given $E \in H_1(Y - K)$, define

$$A_\beta^{+}[E] = \bigoplus_{s \in \mathbb{Z}} (s, A_{\beta(E+s \cdot N)}^{+}(Y, K)) \quad \text{and} \quad B_{E}^{+} = \bigoplus_{s \in \mathbb{Z}} (s, B_{E}^{+}(Y^s, \mathcal{N}(Y))),$$

and

$$D_{E}^{+}: A_{E}^{+} \rightarrow B_{E}^{+}$$

with

$$D_{E}^{+}(a_s)_{s \in \mathbb{Z}} = \{b_s\}_{s \in \mathbb{Z}}$$

where $a_s \in A_{\deg(E+s \cdot N)}^{+}$ and $b_s \in B_{\deg(E+s \cdot N)}^{+}$ by

$$b_s = h_{E}^{+}(a_{s-1}) + v_{E}^{+}(a_s).$$
Proposition 10.3. Let $K \subset Y$ be a null-homologous knot, and fix an $E \in \text{Spin}^c(Y' - K')$. Suppose moreover that $b_1(Y(K,a,\{q_i/r_i\})) = b_1(Y)$. There is a map $f : H_1(Y' - K') \rightarrow \text{Spin}^c(Y(K,a,\{q_i/r_i\}))$ with the property that for each $E \in Y' - K'$ for which $c_1(f(E))$ is a torsion class, we have that $HF^+(Y(K,a,\{q_i/r_i\}_{i=1}^n), f(E))$ is identified with the homology of the mapping cone $D^+_E : A^+_E \rightarrow B^+_E$.

**Proof.** Like Theorem 1.1, this follows from a direct application of Theorems 6.1, 5.1, and the calculation of the invariant for $O_{q/r} \subset L(q,r)$ of Lemma 7.1.

Theorem 10.1 follows quickly from Proposition 10.3, together with the calculation of the knot Floer homology of the Borromean knot (c.f. Section 9 of [22]), which we summarize here:

**Lemma 10.4.** Let $K \subset \#^g(S^2 \times S^1)$ be the Borromean knot. The Spin$^c$ structure over $\#^g(S^2 \times S^1)$ with trivial first Chern class is the only one whose induced knot filtration consists of non-zero groups. For that chain complex $C$, we have a splitting

$$C \cong \bigoplus_t \Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+,$$

where this splitting corresponds to the various summands $C\{i - j = t\}$. In particular, there are identifications

$$A^+_s(\#^g(S^2 \times S^1), K) \cong \bigoplus_t \Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+,$$

$$C\{i \geq 0\}(\#^g(S^2 \times S^1), K) \cong \bigoplus_t \Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+.$$

Moreover, the following squares commute:

$$\begin{array}{ccc}
\Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+ & \xrightarrow{U^*_{\text{max}(0,t-s)}} & \Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+ \\
\downarrow & & \downarrow \\
A^+_s & \xrightarrow{v^+} & B^+ \\
\end{array}$$

$$\begin{array}{ccc}
\Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+ & \xrightarrow{U^*_{\text{max}(0,s-t)}} & \Lambda^{t+g}H_1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{T}^+ \\
\downarrow & & \downarrow \\
A^+_s & \xrightarrow{h^+} & B^+ \\
\end{array}$$

where here all the vertical maps are induced by inclusions in the identification of Equation (38).
Proof. Most of the above statements are a direct result of Proposition 9.2 of [22], which, together with the Künneth principle, gives that
\[ C\{i, j\} = U^{-i} \otimes A^{g-i+j}H^1(\Sigma; \mathbb{Z}), \]
with no differentials. The second square involves the identification between \( C\{i \geq 0\} \) and \( C\{j \geq 0\} \), which is sends the subset \( C\{i - j = t\} \) to \( C\{i - j = -t\} \), as explained in Proposition 5.2 of [25]. \( \square \)

**Lemma 10.5.** Given a map \( \delta: \mathbb{Z} \rightarrow \mathbb{Z} \), consider the chain map
\[ D_\delta: \bigoplus_s A_s \longrightarrow \bigoplus_s B_s, \]
where all \( A_s \cong T^+ \cong B_s \), defined by
\[ D_\delta(\{a_s\}_{s \in \mathbb{Z}}) = \{b_s\}_{s \in \mathbb{Z}} \]
where
\[ b_s = U^{\max(-\delta(s), 0)}a_s + U^{\max(\delta(s+1), 0)}a_{s+1}. \]
Letting
\[ h(s) = \sum_{i=0}^s \delta(i), \]
we have that the homology of the mapping cone of \( D_\delta \) is \( \mathbb{H}^{F+}(h) \), provided that
\[ \lim_{s \to \pm \infty} h(s) = \pm \infty. \]

**Proof.** The map \( D_\delta \) is surjective. It remains to identify its kernel. Given \((i, j) \in W_n(h)\), consider the element \( \{a_s\}_{s \in \mathbb{Z}} \) defined by the property that
\[ a_s = \begin{cases} 
(-1)^{k}U^{h(k)-n} & \text{if } i < k < j \\
0 & \text{otherwise.} 
\end{cases} \]
By linearity, we can extend this to a homomorphism \( W_n(h) \longrightarrow \text{Ker}D_\delta \). It is straightforward to verify that this extends to an isomorphism \( \mathbb{H}^{F+}(h) \longrightarrow \text{Ker}D_\delta \). \( \square \)

**Proof of Theorem 10.1.** According to the adjunction inequality (c.f. Theorem 8.1 of [23]), since the Thurston norm of \( Y \) is trivial, it follows that \( HF^+(Y, s) \) is trivial for all Spin\(^c\) structures with non-torsion \( c_1(s) \). According to the combination of Proposition 10.3 and Lemma 10.4, given \( E \), the Floer homology \( HF^+(Y, [E]) \) splits as
\[ HF^+(Y, [E]) \cong \bigoplus_{t \in \mathbb{Z}} A^{g+t}H^1(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} X(t), \]
where here \( X(t) \) is the homology of a chain complex satisfying the hypotheses of Lemma 10.5, for the function \( \delta_t: \mathbb{Z} \rightarrow \mathbb{Z} \) as in the statement of the theorem. The
theorem now follows from the calculation in Lemma 10.5. Note that Equation (39) holds, since the orbifold has positive degree. □
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