HILBERT SCHEMES OF DEGREE FOUR CURVES

SCOTT NOLLET AND ENRICO SCHLESINGER

Abstract. In this paper we determine the irreducible components of the Hilbert schemes $H_{4,g}$ of locally Cohen-Macaulay space curves of degree four and arbitrary arithmetic genus $g$. We show that these Hilbert schemes are connected, in spite of having $\sim \frac{g^2}{24}$ irreducible components. For $g \leq -3$ we exhibit a component that is disjoint from the component of extremal curves and use this to give a counterexample to a conjecture of Aït-Amrane and Perrin.

1. Introduction

Liaison theory has played a prominent role in the classification of algebraic curves in $\mathbb{P}^3_k$ since the pioneering work of Max Noether. It has only recently become clear that locally Cohen-Macaulay curves - locally Cohen-Macaulay schemes of pure dimension one - are the natural object of study [16], even if one is only interested in smooth connected curves. For example, in Gruson and Peskine’s classification of smooth irreducible curves of degree 8 and genus 5 [7], one family of curves is in the biliaison class of double lines of genus $-2$ (non-reduced curves) and another is in the biliaison class of the disjoint unions of a line and a twisted cubic (non-connected curves). In general, every biliaison class contains an essentially unique minimal curve, from which every other curve in the class maybe obtained by a rather explicit procedure - known to the expert as the Lazarsfeld-Rao property [4].

While every biliaison class contains smooth connected curves, minimal curves need only be locally Cohen-Macaulay. This explains the interest in the Hilbert schemes $H_{d,g}$ parametrizing locally Cohen-Macaulay curves in $\mathbb{P}^3$ of degree $d$ and arithmetic genus $g$.

Here are some results on these Hilbert schemes. In general, $H_{d,g}$ is non-empty precisely when $g = \frac{1}{2}d(d-1)(d-2)$ (plane curves) or $d > 1$ and $g \leq \frac{1}{2}(d-2)(d-3)$ [10, 3.3 and 3.4]. It is also known that $H_{d,g}$ is reducible for $d \geq 3$ and $g \leq \frac{1}{2}(d-3)(d-4)$ with two exceptions: $(d,g) = (3,0)$ and $(3,-1)$ [18]. More recently it has been shown that $H_{3,g}$ is connected (it has $\lceil \frac{d^2}{2} \rceil$ irreducible components) [13] and that $H_{d,g}$ is connected for $g > \binom{d-3}{2} - 2$ [24, 1, 23].

Each connectedness result above was obtained by specializing various families of curves to extremal curves as introduced by Martin-Deschamps and Perrin [17]: these are the curves $C$ which have the largest Rao function $h^1\mathcal{I}_C(n)$ with respect to $d$ and $g$. The extremal curves are geometrically characterized as the curves of degree $d$ that contain a planar subcurve of degree $d-1$ - unless $g = \binom{d-1}{2}$, when $C$ itself is planar, or $(d,g) \in \{(3,0),(4,1)\}$.
and their closure forms an irreducible component $E \subset H_{d,g}$ \cite{18}. The existence of a component of curves with the largest Rao function led Hartshorne to ask the following questions:

**Question 1:** Is $H_{d,g}$ connected for all $d$ and $g$?

**Question 2:** Does each component $B \subset H_{d,g}$ meet the extremal component $E$?

Hartshorne showed that various families of curves can be connected to extremal curves, for example smooth rational and elliptic curves, smooth curves of degree $d \geq g+3$, arithmetically Cohen-Macaulay curves and many others \cite{11}. Related to this is a conjecture of Aït-Amrane and Perrin stating that if $X$ is a family of curves whose cohomology does not exceed that of a family $X_0$ and the Rao module of the general curve in $X$ is a flat deformation of a subquotient of the Rao module of curves in $X_0$, then $X \cap X_0 \neq \emptyset$ (they have shown \cite{2} that semi-continuity alone is insufficient).

Now we specialize to curves of degree $d = 4$. The Hilbert scheme $H_{4,3}$ parametrizes plane curves and is smooth irreducible of dimension 17; $H_{4,1}$ is smooth irreducible of dimension 16 by \cite{3} and \cite{11}, 3.3 and 3.5] and its general curve is a complete intersection of two quadrics. The Hilbert scheme $H_{4,0}$ has two irreducible components, whose general members are respectively rational quartic curves and disjoint unions of a plane cubic and a line. Hartshorne first noticed that these two families can be connected; now there are several published proofs: \cite{19, 3.10], \cite{13}, 5.21 and 5.22], \cite{11}, 1.1]. One of our motivations is to extend the systematic study of these Hilbert schemes and complete the picture when $d = 4$.

Our first theorem is the classification of curves in $H_{4,g}$: we describe the irreducible components and give their dimensions. Our method is to first identify components whose general curve $C$ is very special in the sense that $C$ is contained in quadric surface ($h^0(I_C(2)) \geq 1$) or $C$ has large speciality ($h^1(O_C(-1)) \geq 2$). There are very few such components: if the general curve $C$ of a family lies on a quadric surface, then either $C$ is an extremal curve, a subextremal curve (these are the curves with largest Rao function among the non-extremal curves \cite{2}) or a double conic.

On the other hand, if the general curve $C$ of an irreducible component of $H_{4,g}$ has large speciality but does not lie on a quadric, we show (Proposition \cite{13}) that $C$ is either a thick 4-line or the union of a conic and a double line meeting at a point with multiplicity 2. A thick 4-line is a curve of degree 4 supported on a line $L$ and containing the first infinitesimal neighborhood of $L$ in $\mathbb{P}^3$.

Having disposed of these few very special components, it is relatively easy to list the other irreducible components. Their general member is either (a) a quasiprimitive (i.e. non-thick \cite{3}) 4-line, (b) the disjoint union of a line and a general curve of an irreducible component of $H_{3,g}$, or (c) the disjoint union of two double lines. Asymptotically the Hilbert scheme $H_{4,g}$ has $\sim \frac{g^2}{24}$ irreducible components, most of which are families of 4-lines (there are roughly $\frac{3g}{24}$ components which are not). For example, from our table (Theorem \cite{5.2}) we find
that $H_{4,-100}$ has 530 components (377 of these arise from 4-lines) while $H_{4,-1000}$ has 42755 components (of which 41252 arise from 4-lines).

Our second theorem states that $H_{4,g}$ is connected whenever it is not empty. For $g \leq -3$, the main novelty is the presence of a component $G_4$ of thick 4-lines that consists entirely of curves with generic embedding dimension three. We prove the connectedness theorem by showing that each irreducible component can be connected either to the extremal component $E$ or to $G_4$, and that the component of subextremal curves meets both $E$ and $G_4$. Specifically, the quasi-primitive 4-lines and the curves with large speciality may be deformed to thick 4-lines ($\S 2$ and $\S 4$). We show that a disjoint union of double lines specializes to a quasi-primitive 4-line on a double quadric surface in section 3 and section 5 is devoted to showing that families of unions of triple lines and reduced lines can be connected to the extremal component.

The component $G_4$ of thick 4-lines turns out to be rather interesting. Since these curves are scheme-theoretically (although not cohomologically) the most special, they cannot specialize to extremal curves, answering Question 2 in the negative. Since their Rao modules are flat deformations of subquotients of the Rao modules of extremal curves (Example 6.7), we obtain a counterexample to the conjecture of Aït-Amrane and Perrin above. Question 1 remains open.

Notation and conventions
We work over an algebraically closed field $k$ of arbitrary characteristic. A curve for us is a locally Cohen-Macaulay scheme over $k$ of pure dimension 1.

We will freely use the sentence “the family of curves of degree $d$ and genus $g$ with property $P$ is irreducible of dimension $m$” meaning there is a (unique) irreducible $m$-dimensional constructible subset $S$ of the Hilbert scheme $H_{d,g}$ whose closed points parametrize the curves of degree $d$ and genus $g$ with property $P$. Note that, since $S$ is constructible, the closure of $S$ in the Hilbert scheme is also an $m$-dimensional irreducible subset.

The symbol $L \cup_{nP} C$ denotes the schematic union of a line $L$ and a curve $C$, whose intersection is the divisor $nP$ on $L$.

2. Multiplicity four structures on lines
In this section we study locally Cohen-Macaulay curves in $\mathbb{P}^3$ which are supported on a line; we will simply call these d-lines, where $d$ is the degree of the curve.

We begin with the general theory of Banica and Forster [3, $\S$ 3]. Let $C$ be a locally Cohen-Macaulay curve on a smooth threefold $X$ with smooth support $Y$. Letting $Y^{(i)}$ be the subscheme of $X$ defined by $T_Y^i$ and $C_i$ be the subscheme of $X$ obtained by removing the embedded points from $C \cap Y^{(i)}$, we obtain the Cohen-Macaulay filtration for $Y \subset C$:

$$Y = C_1 \subset C_2 \subset \cdots \subset C_k = C$$
for some \( k \geq 1 \). The quotients \( L_j = \mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} \) are vector bundles on \( Y \) and the multiplicity is given by \( \mu(C) = 1 + \sum \text{rank} L_j \). The natural inclusions \( \mathcal{I}_{C_j} \subset \mathcal{I}_{C_{j+1}} \) induce generically surjective maps \( L_i \otimes L_j \to L_{i+j} \) and hence we obtain generic surjections \( L_i^j \to L_j \).

As in [3, §4], we say that \( C \) is thick if it contains \( Y^{(2)} \), i.e. \( C_2 = Y^{(2)} \). This is also equivalent to the condition that \( C \) have embedding dimension three at each point. In this case \( L_1 = \mathcal{I}_Y/\mathcal{I}_Y^2 \) is the conormal bundle of \( Y \) on \( X \). If further \( \mu(C) = 4 \), then \( \text{rank} \, L_2 = 1 \) and there is an exact sequence

\[
0 \to \mathcal{I}_C \xrightarrow{\mathcal{I}_Y^2} \mathcal{I}_Y^3 \to L_2 \to 0.
\]

If \( Y \) is a line in \( \mathbb{P}^3 \), we obtain the following.

**Proposition 2.1.** For \( g \leq 1 \) the set of thick 4-lines of genus \( g \) is parametrized by an irreducible closed subscheme of \( H_{4,g} \) of dimension \( 9 - 3g \).

**Proof.** The condition that \( C \) contain \( Y^{(2)} \) is clearly closed. If \( C \) is a thick 4-line with support \( Y \subset \mathbb{P}^3 \), then \( \mu(C) = 4, \frac{\mathcal{I}_Y^2}{\mathcal{I}_Y} \cong \mathcal{O}_Y(-2)^{\oplus 3} \) and \( L_2 \cong \mathcal{O}_Y(-g - 1) \) (because Pic \( Y \cong \mathbb{Z} \)).

It follows from the exact sequence above that giving such a curve \( C \) is the same as giving a surjective morphism \( \mathcal{O}_Y(-2)^{\oplus 3} \to \mathcal{O}_Y(-g - 1) \) modulo an automorphism of \( \mathcal{O}_Y(-g - 1) \).

Thus the set of thick 4-lines of genus \( g \) is parametrized by an open subset of a \( \mathbb{P}^{5 - 3g} \)-bundle over the Hilbert scheme of lines in \( \mathbb{P}^3 \). \( \square \)

If \( C \) has generic embedding dimension two it is said to be quasiprimitive. In this case \( \text{rank} L_1 = 1 \) and the generic surjections \( L_i^j \to L_j \) yield effective divisors \( D_j \) such that \( L_j \cong L_1^j(D_j) \); the multiplication maps show that \( D_i + D_j \leq D_{i+j} \). If \( C \) is a quasiprimitive 4-line in \( \mathbb{P}^3 \), then the Cohen-Macaulay filtration takes the form \( Y \subset D \subset W \subset C \) where \( Y \) is a line, \( D \) a double line and \( W \) a triple line. Setting \( a = \deg L_1, b = \deg D_2 \) and \( c = \deg D_3 \), we call the triple \((a, b, c)\) the type of \( C \). Note that \( a \geq -1 \) because the surjection \( \mathcal{I}_Y \to \mathcal{O}_Y(a) \) factors through \( \mathcal{I}_Y/\mathcal{I}_Y^2 \cong \mathcal{O}_Y(-1)^2 \). The isomorphisms \( \mathcal{I}_{Y,D} \cong \mathcal{O}_Y(a), \mathcal{I}_{D,W} \cong \mathcal{O}_Y(2a + b) \) and \( \mathcal{I}_{W,C} \cong \mathcal{O}_Y(3a + c) \) show that \( p_a(C) = -6a - b - c - 3 \).

If \( C \) has type \((a, b, c)\) with \( a = -1 \), then \( C \) necessarily lies in a double plane and hence is a flat limit of double conics [14, 8.1 and 8.2]. We are mainly interested in families that form irreducible components of the Hilbert scheme, so we will assume that \( a \geq 0 \) in the sequel. The proof of the following proposition is based on [3 §3.8].

**Proposition 2.2.** For a triple of integers \((a, b, c)\) satisfying \( a \geq 0 \) and \( 0 \leq b \leq c \), the set of quasiprimitive 4-lines of type \((a, b, c)\) is parametrized by an irreducible constructible subscheme of \( H_{4,g} \) of dimension \( 9a + 2b + 2c + 13 \).

**Proof.** The set of double lines of type \( a \) is parametrized by an open subscheme \( V_2 \) of a \( \mathbb{P}^{2a+3} \)-bundle over the Grassmannian of lines in \( \mathbb{P}^3 \) [19, 1.6]. Indeed, to give a double structure \( D \) of type \( a \) on a line \( Y \) is equivalent to give a surjective morphism \( \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y(a) \) modulo an automorphism of \( \mathcal{O}_Y(a) \).
Similarly, the set of quasiprimitive triple lines of type \((a, b)\) containing a double line \(D \in V_2\) is given by surjective morphisms \(I_D/I_D D \to O_Y(2a + b)\) modulo automorphisms of \(O_Y(2a + b)\). Since \(I_D/I_D D \cong O_Y(2a) \oplus O_Y(-a - 2)\) \([13, 2.3\text{ and } 2.6]\), it follows that the set of triple lines of type \((a, b)\) is parametrized by an open subscheme \(V_3\) of a \(\mathbb{P}^{3a+2b+3}\)-bundle over \(V_2\), hence is irreducible of dimension \(5a + 2b + 10\).

In the same way, 4-lines of type \((a, b, c)\) containing a fixed quasiprimitive 3-line \(W\) of type \((a, b)\) with support \(Y\) are given by surjections \(\nu_W = I_W/I_W W \to O_Y(3a + c)\) modulo automorphisms of \(O_Y(3a + c)\). Noting that \(I_D^3 \subset I_Y I_W\) on an open set, we see that the image of \(I_D^3\) in \(\nu_W\) is torsion, hence \(\phi\) factors through \(I_W/J\), where \(J = I_Y I_W + I_D^3\). We claim that (a) \(I_W/J \cong O_Y(3a + b) \oplus O_Y(-a - b - 2)\) and (b) the induced map \(I_D^3 \to O_Y(-a - b - 2)\) is zero. Claim (b) shows that each surjection \(\psi\) defines a quasiprimitive 4-line \(C\) (since \(O_Y(-a - b - 2) \to O_Y(3a + c)\) cannot be surjective for \(a \geq 0\)) and (a) shows that the set of surjections is parametrized by an open subscheme \(V_4\) of a \(\mathbb{P}^{4a+2c+3}\)-bundle over \(V_3\), hence is irreducible of dimension \(9a + 2b + 2c + 13\).

Noting the exact sequence
\[
0 \to \frac{I_Y I_D}{J} \to \frac{I_W}{J} \to \frac{I_W}{I_Y I_D} \to 0,
\]
we observe the following: The multiplication map \(I_{Y,D} \otimes O_Y I_{D,W} \to I_{Y,D}/J\) is an isomorphism, since it is surjective and \(I_{Y,D}/J\) has rank one (recall that \(J = I_Y I_W\) on an open set) and hence the kernel is zero. It follows that \(I_Y I_D/J \cong O_Y(3a + b)\). Further, the exact sequence
\[
0 \to \frac{I_W}{I_Y I_D} \to \frac{I_D}{I_Y I_D} \to \frac{O_Y(2a + b)}{0}
\]
shows that \(I_W/I_Y I_D \cong O_Y(-a - b - 2)\), since \(I_D/I_Y I_D \cong O_Y(2a) \oplus O_Y(-a - 2)\). Finally, the exact sequence \(\square\) splits because \(a, b \geq 0\), which proves part (a) of the claim. Part (b) is clear because \(I_D^3 \subset I_Y I_D\).

We now show that the irreducible family of quasiprimitive 4-lines of type \((a, b, c)\) contains thick 4-lines in its closure for \(a \geq 0\). This proposition will ultimately allow us to prove the connectedness of the Hilbert scheme \(H_{a,b}\).

**Proposition 2.3.** Let \((a, b, c)\) be a triple satisfying \(a \geq 0\) and \(c \geq b \geq 0\). Then there exists a flat family of curves \(C \subset \mathbb{P}^3_{\mathbb{A}^1}\) such that
1. the fibre \(C_t\) is a quasiprimitive 4-line of type \((a, b, c)\) for \(t \neq 0\)
2. the fibre \(C_0\) is a thick 4-line.

**Proof.** The outline of the proof is as follows. We fix a double structure \(Z\) of type \((a)\) on the line \(L\). We have seen that a quasiprimitive triple line of type \((a, b)\) containing \(Z\) is defined via a morphism \(I_Z \to O_L(2a + b)\). We construct a family of triple lines \(W_t\) specializing such a morphism to a morphism \(I_Z \to O_L(-a - 2) \oplus O_D\) where \(D\) is an effective divisor on \(L\). The general triple line \(W_t\) in the family is quasiprimitive of type \((a, b)\), while the special fibre \(W_0\) is the first infinitesimal neighborhood of \(L\) plus embedded points along \(D\).
Finally, we construct the desired family \( C_t \) by picking a morphism \( \mathcal{I}_W \to \mathcal{O}_L(3a + c) \) in such a way that the “extra line” in \( C_0 \) covers the embedded points of \( W_0 \), so that \( C_0 \) is locally Cohen-Macaulay. We fix coordinates so that \( \mathbb{P}^3 = \text{Proj}(k[x, y, z, w]) \). Let \( L_0 \) be the line of equations \( x = y = 0 \), and let \( Z_0 \) be the double structure on \( L_0 \) defined by the homogeneous ideal \((x^2, xy, y^2, xg - yf)\) where \( f = z^{a+1} \) and \( g = w^{a+1} \). Thus \( Z_0 \) is a double line of genus \(-a - 1\).

Now, over \( \mathbb{A}^1 = \text{Spec}(k[t]) \), we consider the trivial families \( L = L_0 \times \mathbb{A}^1 \) and \( Z = Z_0 \times \mathbb{A}^1 \). Let \( \mathcal{I}_Z \) denote the ideal sheaf of \( Z \) in \( \mathbb{P}^3_{\mathbb{A}^1} \). The epimorphism \( \pi : \mathcal{I}_Z \to \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_{L}(2a) \) mapping \( xg - yf \) to \((1, 0)\) and \( xy \) and \( y^2 \) to \((0, f^2), (0, fg), (0, g^2)\) induces an isomorphism \( \mathcal{I}_Z \otimes \mathcal{O}_L \cong \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_{L}(2a) \). The matrix

\[
M = \begin{bmatrix}
tw^b \\
z^{3a+b+2}
\end{bmatrix}
\]
defines a morphism

\[
\psi : \mathcal{O}_L(-a - b - 2) \to \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_{L}(2a),
\]
and we let \( \mathcal{G} = \text{Coker}(\psi) \). Note that \( \mathcal{G} \) is flat over \( \mathbb{A}^1 \) because \( \psi \) remains injective on the fibres over \( \mathbb{A}^1 \). Now let \( \chi : \mathcal{I}_Z \to \mathcal{G} \) be the surjective morphism

\[
\mathcal{I}_Z \xrightarrow{\pi} \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_{L}(2a) \to \mathcal{G}
\]
and define \( W \) by letting \( \mathcal{I}_W = \text{Ker}(\chi) \).

It is clear that \( W \) is a flat family of curves over \( \mathbb{A}^1 \). For \( t \neq 0 \), \( \mathcal{G}_t = \mathcal{G} \otimes k(t) \) is isomorphic to \( \mathcal{O}_{L_0}(2a + b) \), so that \( W_t \) is a quasiprimitive triple line of type \((a, b)\). On the other hand, \( \mathcal{G}_0 \) is isomorphic to \( \mathcal{O}_{L_0}(-a - 2) \oplus \mathcal{O}_D \), where \( D \) is the divisor in \( L_0 \) defined by the global section \( s = z^{3a+b+2} \) of \( \mathcal{O}_{L_0}(3a + b + 2) \). By construction \( W_0 \) contains \( L_0^{(2)} \), and \( \mathcal{I}_{L_0^{(2)}}, W_0 \cong \mathcal{O}_D \).

We now claim that, over the coordinate ring \( R = k[t][x, y, z, w] \) of \( \mathbb{P}^3_{\mathbb{A}^1} \), the saturated ideal \( I_W \) of \( W \subset \mathbb{P}^3_{\mathbb{A}^1} \) has a free graded resolution

\[
0 \to F_3 \xrightarrow{M_3} F_2 \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 = R
\]
where

\[
F_1 = R(-3)^{\oplus 4} \oplus R(-a - 3)^{\oplus 2} \oplus R(-a - b - 2)
\]

\[
F_2 = R(-4)^{\oplus 3} \oplus R(-a - 4)^{\oplus 4} \oplus R(-a - b - 3)^{\oplus 2}
\]

\[
F_3 = R(-a - 5)^{\oplus 2} \oplus R(-a - b - 2)
\]
and the maps are defined by the matrices

\[
M_1 = \begin{bmatrix}
x^3 & x^2y & xy^2 & y^3 & x(xg - yf) & y(xg - yf) & x^2z^{a+b} + tw^b(xg - yf)
\end{bmatrix}
\]
ideal of \( F \) defines a map \( \phi \) contains the regular sequence \((P, I, J)\). Recalling that \( f^{x, y} \) contains the regular sequence \((a, b, c)\), we see from the proof of Proposition 2.2 above that \( C \) is a quasiprimitive 4-line of type \((a, b, c)\). Since \( \phi \) and the morphism \( I \), we see that \( J = I \). We thus obtain a commutative diagram:

\[
\begin{array}{cccccc}
\ldots & \longrightarrow & I_{C_0} & \longrightarrow & I_{W_0} & \longrightarrow \phi_0 \longrightarrow \mathcal{O}_{L_0}(3a + c) & \longrightarrow \ldots \\
\alpha & \downarrow & \downarrow & & \downarrow & \downarrow & \\
\ldots & \longrightarrow & I_{C_0}/I_{L_0} & \longrightarrow & I_{W_0}/I_{L_0} & \longrightarrow \mathcal{O}_{L_0}(3a + b + 2) & \longrightarrow \ldots
\end{array}
\]

\( \phi_0 \) is zero on \( I_{W_0} \), so it factors through \( I_{W_0}/I_{L_0} \). On the other hand, looking at the presentation of \( I_{W_0} \), we see that \( I_{W_0}/I_{L_0} \) is isomorphic to

\[
\mathcal{O}_{L_0}(-a - 3)^{\oplus 2} \oplus \mathcal{O}_{L_0}(-a - b - 2).
\]

Thus we have a commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & I_{C_0}/I_{L_0} & \longrightarrow & I_{W_0}/I_{L_0} & \longrightarrow \phi_0 \longrightarrow \mathcal{O}_{L_0}(3a + c) & \longrightarrow 0 \\
\downarrow & & \downarrow \alpha & & \downarrow & \\
0 & \longrightarrow & I_{C}/I_{L_0} & \longrightarrow & I_{W_0}/I_{L_0} & \longrightarrow \mathcal{O}_{L_0}(6a + b + c + 2) & \longrightarrow 0
\end{array}
\]
where, identifying \( \mathcal{I}_{W_{a}}/\mathcal{I}_{L_{0}}^{3} \) with \( \mathcal{O}_{L_{0}}(-a-3)^{\oplus 2} \oplus \mathcal{O}_{L_{0}}(-a-b-2) \) and \( \mathcal{I}_{L_{0}}^{2}/\mathcal{I}_{L_{0}}^{3} \) with \( \mathcal{O}_{L_{0}}(-2)^{\oplus 3} \), the morphisms are

\[
\alpha = \begin{bmatrix}
  w^{a+1} & 0 & z^{a+b} \\
  -z^{a+1} & w^{a+1} & 0 \\
  0 & -z^{a+1} & 0
\end{bmatrix}
\]

\[
\beta = \left[ z^{2a+2}w^{4a+b+c+2} z^{a+1}w^{5a+b+c+3} - z^{6a+b+c+4} w^{6a+b+c+4} - 2z^{5a+b+c+3}w^{a+1} \right]
\]

\[
\varphi_0 = \left[ z^{4a+c+3} z^{3a+c+2}w^{a+1} w^{4a+b+c+2} \right]
\]

Since \( \alpha \) is injective and

\[
\deg(\mathcal{I}_{L_{0}}^{2}/\mathcal{I}_{L_{0}}^{3}) - \deg(\mathcal{I}_{W_{a}}/\mathcal{I}_{L_{0}}^{3}) = 3a + b + 2,
\]

we see that \( C_{0} = \bar{C} \) is a thick 4-line, and this concludes the proof.

**Remark 2.4.** It is not known what kind of specializations might occur between quasiprimitive four-lines, however we will at least note a necessary condition. If a family of quasiprimitive four-lines of type \((a, b, c)\) specializes to a four-line of type \((a', b', c')\), then \( a' \leq a \). To see this, consider the deformation of the underlying double line \( Z \): the general such \( Z \) has genus \(-1-a\), hence the limit \( Z' \) consists of a double line of genus \(-1-a'\) and possibly some embedded points. Since the arithmetic genus is constant, we conclude that \( a' \leq a \). By the same reasoning, if a family of triple lines of type \((a, b)\) specializes to another of type \((a', b')\), then \( a' \leq a \). That this actually happens can be seen in [19, 3.6 and 3.10].

### 3. A deformation on the double quadric

The goal of this section is to show that families of disjoint unions of double lines contain certain families of 4-lines in their closure. This follows readily from our study of curves on double surfaces [22].

Let \( F \) be a smooth surface on a smooth projective threefold \( T \) and let \( X \subset T \) be the effective divisor \( 2F \). For a curve \( C \subset X \), let \( P \) be the curve part of the scheme-theoretic intersection \( C \cap F \). We may write

\[
\mathcal{I}_{C \cap F, F} = \mathcal{I}_{Z, F}(-P)
\]
where $Z$ is zero-dimensional. The inclusion $P \subset C \cap F$ generates the commutative diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{I}_R(-F) \\
\downarrow & & \downarrow f \\
0 & \to & \mathcal{I}_C \\
\downarrow & & \downarrow & \\
0 & \to & \mathcal{I}_{Z,F}(-P) \\
\end{array}
\]

which defines the residual curve $R$ to $C$ in $F$. Thus we obtain a triple $T(C) = \{Z, R, P\}$ in which $R \subset P$ are effective divisors on $F$. Using depth arguments and results on generalized divisors \cite{12}, as in \cite{13, 22}, one finds that $Z$ is a Gorenstein divisor on $R$, $\mathcal{L} \cong \mathcal{O}_R(Z - F)$ is a rank one reflexive $\mathcal{O}_R$-module, and $\sigma$ gives a section of $\mathcal{L}(F)$ that defines $Z$ as a generalized divisor on $R$. Note that the arithmetic genus of $C$ is given by the formula

\[
g(C) = g(P) + g(R) + \deg_R \mathcal{O}_R(F) - \deg(Z) - 1.
\]

In the following lemma we consider the case $X$ is a double quadric surface $2Q$ in $\mathbb{P}^3$, and describe the triple $T(C)$ for a general quasiprimitive 4-line $C$ of type $(0, b, c)$. This will allow us to conclude that $C$ is a specialization of a disjoint union of double lines, a fact we will later use in our description of the irreducible components of the Hilbert scheme (Theorem \ref{thm:3.1}) and in the proof of connectedness of the Hilbert scheme (Theorem \ref{thm:6.4}).

**Lemma 3.1.** Let $C \subset \mathbb{P}^3$ be a general quasiprimitive 4-line of type $(0, b, c)$ with Cohen-Macaulay filtration $L \subset D \subset W \subset C$. Then there exists a smooth quadric surface $Q$ such that

1. $D \subset Q$ and $C \subset 2Q$.
2. The triple associated to $C \subset 2Q$ has form $T(C) = \{Z, D, D\}$, where $Z$ consists of $c - b$ simple points and $b + 2$ double points, none of which are contained in $L$.

Further, $H^1(\mathcal{O}_D(Z + D - Q)) = 0$.

**Proof.** Existence of the smooth quadric in the first statement follows from \cite{14, 1.5}. Fixing homogeneous coordinates $x, y, z, w$ on $\mathbb{P}^3$ so that $x$ and $y$ generate the ideal of $L$, we may assume that $F = xw - yz$ is an equation for $Q$. Then $D$ is the divisor $2L$ on $Q$, and the isomorphism $\mathcal{I}_{L,D} \cong \mathcal{O}_L$ maps the global sections of $\mathcal{I}_{L,D}(1)$ defined by $x$ and $y$ to $z$ and $w$ respectively, where we consider $z$ and $w$ as elements of $H^0(L, \mathcal{O}_L(1))$.

As explained in Proposition \ref{prop:2.2} and \cite{14, 2.3}, $W$ corresponds to an epimorphism

$$
\psi : \mathcal{I}_D/\mathcal{I}_L \mathcal{I}_D \cong \mathcal{O}_L \oplus \mathcal{O}_L(-2) \xrightarrow{[p \ q]} \mathcal{O}_L(b).
$$
in which \( \psi \) maps the global section defined by \( F \) to \( q \in H^0(L, O_L(b + 2)) \). Thus under the isomorphism \( I_{D,W} \cong O_L(b) \), the section defined by \( F \) is mapped to \( q \). Since \( W \) is not contained in \( Q \) by assumption, we see that \( q \) is nonzero.

Clearly \( W \) is contained in \( 2Q \) and its associated triple has the form \((Z_W, L, D)\) for some zero dimensional subscheme \( Z_W \subset L \) of degree \( b + 2 \). In fact, if \( q \) is an equation for \( Z_W \), then the bottom row of diagram 3 becomes

\[
0 \to O_L(-2) \xrightarrow{q} O_L(b) \cong I_{D,W} \to O_{Z_W}(-D) \to 0
\]

with the identifications above. Putting this together with the analogous sequence for the triple \(\{Z, D, D\}\) associated to \( C \) we obtain the commutative diagram:

\[
\begin{array}{ccccccc}
0 & \to & I_{L,D}(-2) & \cong & O_L(-2) & \xrightarrow{\alpha} & I_{W,C} \cong O_L(c) & \to & I_{Z,W,Z}(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O_D(-2) & \to & I_{D,C} & \to & O_Z(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O_L(-2) & \to & I_{D,W} \cong O_L(b) & \to & O_{Z,W}(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & & & & & & 0 \\
\end{array}
\]

We need to describe the first horizontal arrow \( \alpha \) in the diagram. First note that \( \alpha \) factors through the inclusion \( I_{L,D}(-2) \cong O_L(-2) \xrightarrow{\beta} I_L I_D / J \cong O_L(b) \) where \( J = I_L I_W + I_Z^2 \) as in Proposition 2.2. The map \( \beta \) in turn is obtained from the above inclusion \( O_L(-2) \xrightarrow{q} O_L(b) \cong I_{D,W} \) upon tensoring with \( I_{L,D} \cong O_L \). Finally, \( I_L I_D / J \) is the \( O_L(b) \) summand in the decomposition \( I_W / J \cong O_L(b) \oplus O_L(-b - 2) \) of Proposition 2.2, and the isomorphism \( I_{W,C} \cong O_L(c) \) arises from the exact sequence

\[
0 \to I_C / J \to I_W / J \cong O_L(b) \oplus O_L(-b - 2) \xrightarrow{[f \, g]} O_L(c) \to 0.
\]

We conclude that \( \alpha \) is multiplication by the global section \( fq \in H^0(L, O_L(c + 2)) \), and therefore \( I_{Z,W,Z} \cong O_L(fq) \).

In the irreducible family of quasiprimitive 4-lines of type \((0, b, c)\), the set of curves \( C \) for which the form \( qf \) has simple zeros is open. Looking at

\[
0 \to I_{Z,W,Z} \cong O_L / (fq) \to O_Z \to O_{Z,W} \cong O_L / (q) \to 0
\]

we see that subscheme \( Z \) associated to a general such curve \( C \) consists of one double point for each zero of \( q \), and one simple point for each zero of \( f \); Moreover, \( qf = 0 \) defines the divisor \( Z \cap L \subset L \).

Finally, we show that \( H^1(O_D(Z + D - Q)) = 0 \). Consider \( W = Z \cap L \) and the residual scheme \( Y \) to \( W \) in \( Z \). From the description above, \( \deg W = c + 2 \) and \( \deg Y = b + 2 \). Since \( D^2 = 0 \) on \( Q \), the sequence relating \( Y \) and \( W \) to \( Z \) takes the form

\[
0 \to I_{Y,L} \to I_{Z,D} \to I_{W,L} \to 0
\]
and applying $\mathcal{H}om_{\mathcal{O}_D}(-, \mathcal{O}_D)$ yields the exact sequence

$$0 \to \mathcal{O}_L(W) \to \mathcal{O}_D(Z) \to \mathcal{O}_L(Y) \to 0$$

(one checks locally that $\mathcal{E}xt^1_{\mathcal{O}_D}(\mathcal{O}_L, \mathcal{O}_D) = 0$ and $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{O}_L(a), \mathcal{O}_D) \cong \mathcal{O}_L(-a)$ by [8, III,6.7]). Tensoring by $\mathcal{O}_D(-Q)$ and taking the long exact cohomology sequence now gives the desired vanishing.

**Proposition 3.2.** Let $C$ be a general quasiprimitive 4-line of type $(0, b, c)$ lying on a double quadric surface $2Q$ as in Lemma 3.1. Then there is a flat family of curves $X_t, t \in \mathbb{P}^1$ on $2Q$ with $X_0 = C$ and general member a disjoint union of double lines of arithmetic genera $-1 - b$ and $-1 - c$.

**Proof.** Since the double line $D$ underlying $C$ is linearly equivalent on $Q$ to the disjoint union of two lines, our claim is a consequence of Lemma 3.1 and of the following theorem from [22]

**Theorem 3.3.** Let $C \subset X$ be a curve with triple $T(C) = \{Z, R, P\}$ such that $H^1(\mathcal{O}_R(Z + P - F)) = 0$. Then $C$ is a specialization of curves $C'$ with triples $\{Z', R', P'\}$ for which $R'$ is general in its divisor class on $F$.

4. **Curves of degree 4 with large speciality**

In this section we study curves which have large speciality. We express the speciality of a curve $C$ by its spectrum which can be defined as the non-negative function

$$h_C(n) = \Delta^2 h^0(\mathcal{O}_C(n)) = h^0(\mathcal{O}_C(n)) - 2h^0(\mathcal{O}_C(n - 1)) + h^0(\mathcal{O}_C(n - 2)),$$

which we represent by the tuple of integers with exponents $\{n \cdot h_C(n)\}$. The curves with the largest speciality are the extremal curves, which form an irreducible component $E \subset H_{4,0}$ of dimension $15 - 2g$ [18, Theorem 2.5 and Theorem 3.7]. Extremal curves are defined as those curves achieving upper bounds [17] on the Rao function $h^1(I_C(n))$, but for $d = 4$ and $g \leq 0$ they may also be characterized as (a) nonplanar curves containing a plane cubic curve, or (b) curves with spectrum $\{g\} \cup \{0, 1, 2\}$ [20, 2.2].

Similarly, there are sharp upper bounds on the Rao function for non-extremal curves [20]. The curves achieving these bounds are called subextremal and have spectrum $\{g + 1, 0, 1^2\}$, although they are not characterized by this fact [20, 2.15]. The curves with the speciality of a subextremal curve are characterized as follows.

**Lemma 4.1.** Let $C$ be a curve of degree 4 and genus $g \leq -2$. Then $C$ has spectrum $\{g + 1, 0, 1^2\}$ if and only if $C$ contains a subcurve $T$ of degree 3 and genus 0.

**Proof.** If $C$ contains a curve $T \in H_{3,0}$, then the principal spectrum spectrum $\{0, 1^2\}$ of $T$ is contained in that of $C$ [25, §3] and the remaining element $g + 1$ is determined by the genus of $C$. 

Conversely, suppose that $C$ has spectrum \{ $g+1, 0, 1^2$ \}. Then $h^0 \mathcal{O}_C(g+1) = 1$ and choosing $0 \neq \alpha \in H^0 \mathcal{O}_C(g+1)$ gives a map $\mathcal{O}_C \to \mathcal{O}_C(g+1)$ with image $\mathcal{O}_D$ for some closed subscheme $D \subset C$. The local depth of $D$ is one because $\mathcal{O}_D \subset \mathcal{O}_C(g+1)$, hence $D$ is a locally Cohen-Macaulay curve. The inclusion above also shows that $h^0 \mathcal{O}_D(1) \leq 2$: for $g < -2$ this is because $h^0 \mathcal{O}_C(g+2) = 2$. If $g = -2$, then $h^0 \mathcal{O}_C = 3$ and the inclusion $H^0(\mathcal{O}_D(1)) \subset H^0(\mathcal{O}_C)$ is strict, as otherwise we obtain a surjection $\mathcal{O}_D(1) \to \mathcal{O}_C$, which is absurd. It follows that $D = L$ is a line. This yields an exact sequence

$$0 \to \mathcal{O}_L(-g-1) \to \mathcal{O}_C \to \mathcal{O}_T \to 0$$

for a closed subscheme $T \subset C$ of degree 3 and genus 0.

If $T$ is not purely one-dimensional, then the purely one-dimensional part $P \subset T$ is planar because $g(P) > 0$ and $\deg P = 3$ \cite{14, 3.1}, but this is not possible because $C$ is not extremal. Thus $T$ is locally Cohen-Macaulay, and this finishes the proof.

**Proposition 4.2.** Let $C$ be a curve of degree 4 and genus $g \leq -1$ having spectrum \{ $-g+1, 0, 1^2$ \}. Then there is a line $L$ such that the Rao module $M_C = H^1_\bullet(\mathcal{I}_C)$ is a graded module over the coordinate ring $S_L$ with resolution

$$0 \to S_L(-j) \oplus S_L(j-5+g) \xrightarrow{\alpha} S_L(-2)^{\oplus 3} \to S_L(-g-1) \to M_C \to 0$$

for some integer $2 \leq j \leq n(g) = \left\lfloor \frac{5-g}{2} \right\rfloor$. The cohomology of $C$ is determined by $j$ and we denote the corresponding family of curves by $H_j$.

**Proof.** We first treat the case $g \leq -2$. In this case $C$ contains a degree 3 and genus 0 curve $T$ by \cite{14}. As every curve of degree 3 and genus 0, $T$ is arithmetically Cohen-Macaulay with total ideal $I_T$ generated by three quadrics \cite{10, 3.5}. Furthermore, by the proof of \cite{14}, there is a line $L$ such that $\mathcal{I}_{T,C} \cong \mathcal{O}_L(-g-1)$. Factoring the surjection $\mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3} \to I_T \to \mathcal{O}_L(-g-1)$ through $\mathcal{O}_L(-2)^{\oplus 3}$ and writing the kernel of the induced map as $\mathcal{O}_L(-j) \oplus \mathcal{O}_L(j-5+g)$ for some integer $j \in \left[2, \frac{5+g}{2} \right]$, we obtain the resolution \cite{14}.

Now assume that $g = -1$. The spectrum shows that $C$ is neither ACM (because $h_C(0) > 1$) nor extremal, hence $h^1 \mathcal{I}_C \leq 1$, $h^1 \mathcal{I}_C(1) \leq 2$ and $h^1 \mathcal{I}_C(2) \leq 1$ \cite{20, Theorem 2.11}. The first two of these are equalities in view of the Euler characteristics (since $h^3 \mathcal{I}_C = h^3 \mathcal{I}_C(1) = 0$) and $h^2 \mathcal{I}_C(1) = 0$. If $h^1 \mathcal{I}_C(2) = 1$, then $C$ is subextremal by definition and resolution \cite{14} for $j = 2$.

If $h^1 \mathcal{I}_C(2) = 0$, then $\mathcal{I}_C$ is 3-regular (hence $h^1 \mathcal{I}_C(n) = 0$ for $n \geq 2$) and $h^0 \mathcal{I}_C(2) = h^2 \mathcal{I}_C(2) = 0$. Since $h^1 \mathcal{I}_C(n)$ is increasing or zero for $n \leq 0$, we see that $h^1 \mathcal{I}_C(n) = 0$ for $n < 0$. In particular, the Rao module $M_C$ has a generator $m$ in degree 0. If $m$ does not generate $M_C$ as a module over the homogeneous coordinate ring $S = k[x, y, z, w]$ of $\mathbb{P}^3$, then $m$ is annihilated by 3 independent linear forms, which implies that $C$ lies on a quadric by \cite{24, 3.4.5}, a contradiction. Thus $m$ generates the Rao module and $M_C \cong S/(x, y, z^2, zw, w^2)$ after a change of coordinates, so that $M_C$ has resolution \cite{14} for $I_L = (x, y)$ and $j = 3$. □

Finally, we describe the families $H_j$ and how they fit together in the Hilbert scheme.
Proposition 4.3. For fixed \( g \leq -1 \), let \( H_j \) denote the family of curves \( C \in H_{4,g} \) with spectrum \( \{g + 1, 0, 1^3\} \) and Rao module \( M_C \) having resolution \([\text{I}].\) Let \( G_4 \) denote the family of thick 4-lines. Then

1. The family \( H_2 \) is irreducible of dimension \( 13 - 2g \) (resp. \( 16 \) if \( g = -1 \)). It is the family of subextremal curves and meets \( G_4 \).
2. The family \( H_3 \) is irreducible of dimension \( 13 - 2g \) (resp. \( 16 \) if \( g = -1 \)). It meets \( G_4 \) in general and \( G_4 \subset H_3 \) if \( g \geq -2 \).
3. Suppose \( g \leq -3 \). Then \( H_j \subset G_4 \) for \( 3 < j \leq n(g) \), and \( G_4 = \overline{H_{n(g)}} \) is irreducible of dimension \( 9 - 3g \).

Proof. We consider the last statement first: suppose \( g \leq -3 \) and let \( C \in H_j \) for \( 3 < j \leq n(g) \). The sequence \([\text{I}].\) shows that \( h^1I_C(3) = -g - 3 \), hence \( h^0I_C(3) = 4 \). If \( L \) and \( T \) are as in the proof of \([\text{II}].\) above, then \( \dim(I_L I_T) \leq 4 \) because \( I_L I_T \subset I_C \) and so \( I_T = I_T^2 \) by Lemma \([\text{II}].\) below. It follows that \( C \) is a thick 4-line supported on \( L \), hence \( H_j \subset G_4 \). As we saw in Proposition \([\text{II}].\) \( G_4 \) is irreducible and the thick 4-lines supported on \( L \) are parametrized by the open subset

\[
U \subset \text{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(I_T, \mathcal{O}_L(-g - 1)) \cong \text{Hom}_{\mathcal{O}_L}(\mathcal{O}_L(-2)^3, \mathcal{O}_L(-g - 1))
\]

corresponding to surjective maps. For \( j = 2 \) and 3, we can use the specific surjection given by \((w^{1-g}, w^{3-g-j} z^{j-2}, z^{1-g})\) to see that \( G_4 \) meets \( H_j \). Since \( I_C \) is determined by its image in \( I_T/I_L I_T \cong S_L(-2)^3 \) and hence by the image of the first map in sequence \([\text{I}].\) above, we find by counting dimensions that \( H_j \) is irreducible of dimension \( 5 + 2j - 2g \) (except if \( 2j = 5 - g \), when the dimension is \( 4 + 2j - 2g \)). For \( j = n(g) = \lfloor \frac{5 - g}{2} \rfloor \), the closure of \( H_{n(g)} \) is irreducible of dimension \( 9 - 3g \), hence is equal to \( G_4 \).

If \( j = 2 \), then \( H_j \) consists of subextremal curves, which are those obtained from extremal curves of degree 2 and genus \( g' = g - 1 \) by a height one biliaison on a quadric surface \([\text{II}].\) 2.11 and 2.14. Let then \( \gamma, \rho \) (resp. \( \gamma', \rho' \)) be the gamma and Rao functions for the extremal curves of degree 2 and genus \( g - 1 \) (resp. subextremal curves of degree 4 and genus \( g \)). Letting \( B_{\gamma, \rho, 2, 1} \) denote the universal biliaison scheme of Martin-Deschamps and Perrin \([\text{II}].\) VII \( \S 4 \), we have smooth irreducible projections

\[
\begin{align*}
B_{\gamma, \rho, 2, 1} \xrightarrow{q_3} & H_{\gamma', \rho'} \\
q_1 \downarrow & E \xrightarrow{} H_{\gamma, \rho}
\end{align*}
\]

to the spaces \( H_{\gamma, \rho} \) (resp. \( H_{\gamma', \rho'} \)) of curves with constant cohomology. The family \( E = H_{\gamma', \rho'} \) of extremal curves is irreducible of dimension \( 7 - 2g \) \([\text{I}].\) 2.5 and using \([\text{II}].\) VII, 4.8] we compute that the fibre dimension of \( q_1 \) is 8 (resp. 9 if \( g = -1 \)) and the fibre dimension of \( q_2 \) is 2, hence the family \( H_{\gamma, \rho} \) of subextremal curves is irreducible of dimension \( 13 - 2g \) (resp. 16 if \( g = -1 \)).

For \( j = 3 \), we take an indirect approach. Consider the family of arithmetically Cohen-Macaulay curves \( D \) with resolution of the form

\[
0 \rightarrow \mathcal{O}(2g - 1) \oplus \mathcal{O}(g - 2) \rightarrow \mathcal{O}(g - 1)^3 \rightarrow I_D \rightarrow 0.
\]
This family is irreducible of dimension $2\left(-\frac{g+3}{3}\right) + \left(-\frac{g+2}{2}\right) + 2$ (resp. $12$ if $g = -1$) by \([3]\) and the general member is smooth and irreducible (the numerical character has no gaps).

Let $D$ be a general such curve. Then a general (disjoint) union $D \cup L$ with a line $L$ lies on an integral surface of degree $-g + 2$; To see this, use the linear systems $\mathbb{P}H^0\mathcal{I}_D(-g + 1)$ and $\mathbb{P}H^0\mathcal{I}_L(1)$ to obtain a map $\tau : \mathbb{P}^3 - D - L \to \mathbb{P}^2 \times \mathbb{P}^1$. Since the fibres of $\tau (= S \cap H : S \in H^0\mathcal{I}_D(-g + 1), H \in H^0\mathcal{I}_L(1))$ are generally of dimension one, the image of $\tau$ has dimension two. Composing with the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$, we apply Jouanolou’s Bertini theorem \([13, 6.10]\) to see that the general surface of degree $-g + 2$ containing $D \cup L$ is irreducible. Furthermore, the resolution for $\mathcal{I}_D$ shows that $H^1\mathcal{O}_{D \cup L}((-g+2) - 4 - (g+1)) \neq 0$ and we find that \([16, III, 2.7(b)]\) $D \cup L$ can be bilinked on a surface of degree $-g + 2$ with height $g + 1$ to a curve $C$, which lies in $H_3$ by direct calculation.

Let $\gamma, \rho$ (resp. $\gamma’, \rho’$) be the gamma and Rao functions for curves in $H_3$ (resp. $D \cup L$). Letting $\mathcal{B}_{\gamma, \rho, -g+2, -g-1}$ be the universal biliaison scheme \([16, VII \S 4]\), we obtain smooth irreducible projections
\[
\mathcal{B}_{\gamma, \rho, -g+2, -g-1} \xrightarrow{q_3} H_{\gamma’, \rho’} \\
q_1 \downarrow \\
H_3 = H_{\gamma, \rho}
\]

From the last paragraph, the image of $q_2$ is dense in the irreducible component consisting of the closure of the family of disjoint unions $D \cup L$ considered above. Using the resolutions given, we compute the dimension of the fibres of $q_1$ and $q_2$ via \([16, VII 4.8]\) and conclude that $H_3$ is irreducible of dimension $13 - 2g$ (resp. $16$ if $g = -1$).

**Remark 4.4.** One can check by a dimension count that the general members of the families $H_2$ and $H_3$ are described as follows.

1. For $g = -1$, the general member of $H_2$ is a disjoint union of conics. For $g \leq -2$, the general member of $H_3$ is the union of a double line $Z$ of genus $g - 2$ and two disjoint lines $L_1$ and $L_2$, each meeting $Z$ in a scheme of length 2.
2. For $g = -1$, the general member of $H_3$ is a disjoint union of a line and a twisted cubic curve. For $g \leq -2$, the general member of $H_3$ is the union of a double line of genus $g - 1$ and a smooth conic meeting in a scheme of length 2.

The following lemma and its proof are well known:

**Lemma 4.5.** Let $L \subset \mathbb{P}^N$ be a linear subvariety of codimension two and let $I$ be the homogeneous ideal generated by a subspace $V \subset H^0(\mathbb{P}^N, \mathcal{O}(d))$ of dimension $r$. Then the image $W$ under the multiplication map $V \otimes H^0(\mathbb{P}^N, \mathcal{I}_L(1)) \to H^0(\mathbb{P}^N, \mathcal{O}(d+1))$ satisfies $\dim(W) \geq r + 1$ with equality if and only $I = I_L^{-1}f$ for some form $f$ of degree $d - r - 1$.

**Proof.** Let $S = \text{Sym} H^0(\mathbb{P}^N, \mathcal{I}_L(1)) \cong k[x, y]$ be the symmetric algebra and set $\mathbb{P}^1 = \text{Proj}(S)$. Sheafifying the natural map $V \otimes_k S \to \bigoplus_n H^0(\mathbb{P}^N, \mathcal{O}(n))$ of free graded $S$-modules over $\mathbb{P}^1$ and letting $\mathcal{F}$ denote the image, we obtain an exact sequence $0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}^r \to \mathcal{F} \to 0$ of
with \( s \leq r - 1 \) and \( a_i > 0 \), hence \( h^0(\mathcal{E}(1)) \leq r - 1 \). Since \( H^0(\mathcal{E}(1)) \) is the kernel of the surjection \( V \otimes S_1 \to W \), we see that \( \dim(W) \geq r + 1 \) with equality if and only if \( s = r - 1 \) and \( a_i = 1 \) for \( 1 \leq i \leq r - 1 \), which is equivalent to saying that \( \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(r - 1) \).

\[ \square \]

## 5. Triple lines union a line

In this section we are interested in families of curves \( C \) that are unions of a quasi-primitive triple line \( W \) of type \((a, b)\) and a reduced line \( L \), when \( W \cap L \) is non-empty. Note that we have

\[ g(C) = g(W) + g(L) + \text{length}(W \cap L) - 1 = -3a - 3 - b(W) + \text{length}(W \cap L). \]

In what follows, we will fix \( g \) and \( a \) with \( a \geq 0 \). If \( Z \subset W \) denotes the underlying double line of genus \(-1 - a\), we have four different families of such curves \( C \) in \( H_{4,g} \):

- \( F_1 = \{ W \cup L : \text{length}(W \cap L) = 3, b(W) = -3a - g \} \)
- \( F_2 = \{ W \cup L : \text{length}(W \cap L) = 2, \text{length}(Z \cap L) = 2, b(W) = -3a - g - 1 \} \)
- \( F_3 = \{ W \cup L : \text{length}(W \cap L) = 2, \text{length}(Z \cap L) = 1, b(W) = -3a - g - 1 \} \)
- \( F_4 = \{ W \cup L : \text{length}(W \cap L) = 1, b(W) = -3a - g - 2 \} \)

The main results of this section are that families \( F_1 \) and \( F_2 \) are irreducible (Proposition 5.2) and that the other two families lie in their closures (Proposition 5.4).

Our arguments hinge on the following correspondence: Fix a double line \( Z \) of type \( a \geq 0 \) with support \( Y \) and let \( b \geq a \) be an integer. Let \( S \supset Z \) be a surface of degree \( a + b + 2 \) with equation \( h \) which does not contain the first infinitesimal neighborhood \( Y^{(2)} \) of \( Y \). Removing the possible embedded points from the scheme cut out by \( I = (I_Y, I_Z, h) \) yields a quasi-primitive 3-line \( W \supset Z \). This defines \( \Phi \):

\[ \mathcal{S} = \{ h \in H^0(I_Z(a + b + 2)) : h \notin I_Y^2 \} \xrightarrow{\Phi} \{ \text{Quasiprimitive 3-lines } W \supset Z \} \]

**Lemma 5.1.** Let \( \Phi \) be the map above. Then

1. The image of \( \Phi \) is the set of quasiprimitive 3-lines \( W \supset Z \) of type \((a, b')\) with \( b' \leq b \).
2. For \( h \) as above, set \( I = (I_Y, I_Z, h) \). Then the following are equivalent.
   (a) \( W = \Phi(h) \) has type \((a, b)\).
   (b) The scheme defined by \( I \) is locally Cohen-Macaulay.
   (c) \( h \) is irreducible modulo \( I_Y I_Z \).
   If any of these conditions hold, then \( I = I_W \).

**Proof.** For \( h \in (I_Z)_{a+b+2} : h \notin I_Y^2 \), let \( W \) be the purely one-dimensional part of the scheme \( V \) defined by the ideal \( I = (I_Y, I_Z, h) \). Since \( W \) is quasi-primitive and contains \( Z \), \( W \) has type \((a, b')\) for some \( b' \geq a \) and the total ideal may be written \( I_W = (I_Y I_Z, h') \) with \( h' \) of
degree $a + b' + 2$ by [19, 2.3]. The inclusions $(I_Y I_Z, h) \subset I_Y \subset I_W$ now show that $b' \leq b$. On the other hand, if $W$ is a quasi-primitive 3-line of type $(a, b')$ with $b' \leq b$, then writing $I_W = (I_Y I_Z, h')$ as above and choosing a hypersurface $F$ of degree $b - b'$ with equation $f$ meeting $Z$ properly, we see that $\Phi(fh') = W$.

For the equivalences in statement 2, let $C$ be the scheme defined by $I = (I_Y I_Z, h)$ so that $W = \Phi(h)$ is obtained from $C$ by removing possible embedded points.

(a) $\Rightarrow$ (b) If $W$ has type $(a, b)$, then by [19, 2.3] the total ideal for $W$ takes the form $I_W = (I_Y I_Z, h')$ with $\deg h' = a + b + 2 = \deg h$. The inclusions $I \subset I_C \subset I_W$ show that all three ideals are equal, so $C = W$ is locally Cohen-Macaulay.

(b) $\Rightarrow$ (c) Suppose that $h = h't$ modulo $I_Y I_Z$. If $T$ is the surface with equation $t$, then we may assume $T$ meets $Y$ properly (since if both $t \in I_Y$ and $h' \in I_Y$, then $h \in I_F^2$, contrary to hypothesis). In this case $(I_Y I_Z, t)$ defines a scheme of length $4 \deg T$ (because $I_Y I_Z$ defines a locally Cohen-Macaulay 4-line) and if $C$ is locally Cohen-Macaulay, then $(I, t)$ defines a scheme of length $3 \deg T$; since $(I_Y I_Z, t) = (I, t)$, we must conclude that $C$ is not locally Cohen-Macaulay.

(c) $\Rightarrow$ (a) If $W$ has type $(a, b')$ for $b' < b$, then $I_W = (I_Y I_Z, h')$ with $\deg h' = a + b' + 2$. The inclusion $I \subset I_W$ shows that there exists $t$ of degree $b - b'$ such that $h = h't$ modulo $I_Y I_Z$.

Proposition 5.2. With the notation above, in $H_{4, 9}$ we have

1. $F_1$ is irreducible of dimension $11 - 2g - a$ if $0 \leq a \leq -\frac{g}{3}$ and empty if $a > -\frac{g}{3}$.

2. $F_3$ is irreducible of dimension $10 - 2g - a$ if $0 \leq a < -\frac{g - 1}{3}$ and empty if $a \geq \frac{-g - 1}{3}$.

Proof. We first prove statement 1, then indicate the changes to obtain statement 2. Let $H \subset H_{3, -a}$ be the family of unions $Z \cup_{2p} L$. By [19, 3.2(a)], $H$ is irreducible of dimension $9 + 2a$. For $b = -3a - g$ we interpret $H^0 \mathcal{O}_{\mathbb{P}^3}(a + b + 2)$ as an affine scheme. Pulling back the universal family over $H$ we obtain a diagram

$$
\mathcal{Z} \cup \mathcal{L} \subset \mathbb{P}^3 \times H \times H^0(\mathcal{O}_{\mathbb{P}^3}(a + b + 2)) \quad \downarrow
\quad H \times H^0(\mathcal{O}_{\mathbb{P}^3}(a + b + 2)).
$$

Consider the closed subset

$$
V = \{(Z \cup L, h) \in H \times H^0(\mathcal{O}_{\mathbb{P}^3}(a + b + 2)) : h \in I_Z \cap (I_Y^3, I_L)\}
$$

with first projection $V \xrightarrow{p_1} H$. The fibres of $p_1$ are vector subspaces of dimension $(a+b+5) - 3a - 2b - 7$. Indeed, after a change of coordinates we may write $I_Z = (x^2, xy, y^2, xy - yf)$ ([19, 1.4(c)]) and $I_L = (x, z)$, when the fibre is identified with the kernel $K$ of the composite map

$$
H^0(I_Z(a + b + 2)) \hookrightarrow (x, z, y^2)^{a+b+2} \to ((x, z, y^2)/(x, z, y^3))_{a+b+2} \cong k
$$

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$$
H^0(I_Z(a + b + 2)) \hookrightarrow (x, z, y^2)^{a+b+2} \to ((x, z, y^2)/(x, z, y^3))_{a+b+2} \cong k
$$
(the inclusion has the correct target because $Z$ meets $L$ in the double point $2P$). Since $I_Z$ is $(a+2)$-regular, $h^0(I_Z(a+b+2)) = \chi I_Z(a+b+2) = \binom{a+b+5}{3} - 3a - 2b - 6$ and $p_1 : V \to H$ is an affine bundle with fibres of dimension $\binom{a+b+5}{3} - 3a - 2b - 7$. In particular, $V$ is irreducible.

Consider the open subset $U = \{(Z \cup L, h) \in V : h \notin I_Y^2\}$. The correspondence of Lemma 7.1 shows that elements of $U$ determine unions $W \cup L$. If $\mathcal{S} \subset \mathbb{P}^3 \times U$ is the family of surfaces with equation $h$, we obtain flat families

$$\mathcal{Y}, Z \cup \mathcal{L}, \mathcal{S} \subset \mathbb{P}^3 \times U \xrightarrow{\mathcal{Y}} U$$

where $\mathcal{Y}$ is the support of $Z$. The subscheme $\mathcal{W} \subset \mathbb{P}^3 \times U$ defined by the ideal sheaf $I_W = I_Y I_Z + I_S$ is also flat over $U$. To see this, observe that the fibre of the sheaf $\mathcal{F} = I_W = I_Y I_Z + I_S/I_Y I_Z$ over $u = (Z \cup L, h)$ is isomorphic to $\mathcal{O}_Y(-a-b-2)$: Indeed, $\mathcal{F}_u$ is a quotient of $\mathcal{O}_Y(-a-b-2)$ via the generator $h$, and is a torsion free $\mathcal{O}_Y$-module because of the inclusion $\mathcal{F}_u \subset I_Z/I_Y I_Z$. It follows that the fibres of $\mathcal{W}$ have constant Hilbert polynomial and the family is flat by [8, III, 9.9].

Finally, let $U' \subset U$ be the open set for which the fibres of $\mathcal{W}$ are locally Cohen-Macaulay, taken with the induced reduced scheme structure. This is precisely the set for which the fibres of $\mathcal{W}$ are quasiprimitive 3-lines of type $(a,b)$ by Lemma 7.1. The curves used in the proof of Corollary 3.3 show that $U'$ is non-empty; the double line $Z$ with total ideal $I_Z = (x^2, xy, y^2, xz^{a+1} - yw^{a+1})$ has the line $L = \{x = w = 0\}$ as a double tangent and $h = z^b(xz^{a+1} - yw^{a+1}) - x^2w^{a+b} = 0$ satisfies the conditions above. The definition of $V$ above makes it clear that the fibres of $\mathcal{W}$ meet the lines $L$ in triple points, so the family $\mathcal{W} \cup \mathcal{L}$ is also flat over $U'$. The universal property of the Hilbert scheme gives a map $U' \to H_{4, -3a-b}$ whose image is precisely the family $F_1$ of unions $W \cup_{3P} L$. In particular, $F_1$ is irreducible.

The structure of the map $U' \to H$ shows that $U'$ has dimension $\binom{a+b+5}{3} - a - 2b + 2$. On the other hand, if $W$ is a triple line arising in the construction above, then $I_W$ is $(a+b+2)$-regular (see [19, 2.4]), hence $\dim H^0(I_W(a+b+2)) = \binom{a+b+5}{3} - 6a - 4b - 9$. Subtracting this redundancy shows that the family has dimension $5a + 2b + 11 = 11 - 2g - a$.

The proof of statement 2 goes through via the same outline. The main differences are as follows. The family $H \subset H_{3, -a-1}$ is now the family of unions $Z \cup_P L$, which is irreducible of dimension $10 + 2a$ by [19, 3.2(b)]. In the definition 7 of $\mathcal{V}$, $I_Y^3$ is replaced by $I_Y^2$ and in the map 8 the exponents of $y$ should be reduced by one. To see that $U'$ is nonempty, we can use the same triple line $W$ as in the proof above, but instead use the line $L$ given by $\{x = z = 0\}$. The remaining modifications are clear.

**Remark 5.3.** Since the family of triple lines $W$ is irreducible of dimension $10 + 5a + 2b$ [19, 2.6], we expect that the natural map $\{W \cup_{3P} L\} \xrightarrow{\mathcal{F}} \{W\}$ which forgets the line $L$ has generically one dimensional fibres. However, there are triple lines $W$ for which the fibre
$F^{-1}(W)$ has larger dimension. For example, the triple lines constructed in characteristic $p > 0$ by Hartshorne [19, 2.3] have a two-dimensional family of triple tangent lines.

**Proposition 5.4.** With the notation above, we have

1. $F_2 \subset F_1$ if $0 \leq a < \frac{-g-1}{3}$ and is otherwise empty.
2. $F_4 \subset F_3$ if $0 \leq a < \frac{-g-1}{3}$ and is otherwise empty.

**Proof.** Let $W_0 \cup_{2P} L$ be a curve in the family $F_2$, so that the underlying double line $Z \subset W_0$ satisfies $\text{length}(Z \cap L) = 2$. If $W_0$ has type $(a, -3a - g - 1)$ and support $Y$, we may write $I_Y = (x, y), I_Z = ((x, y)^2, xg - yf), I_L = (x, z)$ and $I_{W_0} = (I_Y I_Z, h_0)$ in suitable coordinates [19, 2.3]. If $K \subset H^0(I_Z(-2a - g + 2))$ is the vector subspace considered in the proof above, then $zh_0 \in K$. Fixing a member $(Z \cup L, h) \in U'$ as above, the deformation $(1 - t)zh_0 + th$ gives a map $\mathbb{A}^1 \rightarrow K$, which yields $\psi^{-1}(U') \rightarrow H_{4g}$. This extends to a map $\overline{\psi} : T = \psi^{-1}(U') \cup \{0\} \rightarrow \text{Hilb}^x_4$ into the full Hilbert scheme: by construction, it’s clear that the limit curve $\overline{\psi}(0)$ contains $W_0 \cup L$. Since this curve has genus $g$, it is equal to $\overline{\psi}(0)$, completing the proof. The limit of the triple lines $W_t$ is the triple line $W_0$ along with an embedded point, which is conveniently covered up by the line $L$. Statement 2 is similar. □

**Corollary 5.5.** The closure of family $F_{1,a}$ in $H_{4g}$ contains extremal curves.

**Proof.** Following [19, 3.6], the family of ideals $I_t$ below give a deformation from a triple line $W$ of type $(a, b)$ to an extremal curve of the same arithmetic genus.

$$I_t = ((x, y)^3, (x, y)(xz^{a+1} - tyw^{a+1}), z^b t^2(xz^{a+1} - tyw^{a+1}) - x^2 w^{a+b})$$

We simply observe that the line $L = \{x = w = 0\}$ is triple tangent to the triple line $W_t$ defined by $I_t$ for all $t \neq 0$ and that this same line is a triple tangent to the limit extremal curve having ideal

$$I_0 = (x^2, xy, y^3, xz^{3a+b+3} - y^2 w^{3a+b+2}).$$

☐

**Remark 5.6.** The closure of the family $F_3$ above forms an irreducible component of the Hilbert scheme (Theorem 6.2) with one exception. When $a = 0$ and $g \leq -2$, a curve $C_0 = W_0 \cup_{2P} L \in F_1$ is a flat limit of curves $C_t = Z_t \cup_{2P} L \cup L_t$, where $Z_t$ is a double line and $L_t$ is a line disjoint from $L$. This is not surprising in view of [19, Proposition 3.3], which says that $W$ is the limit of unions $Z_t \cup L_t$.

To see this, let $Z$ be the underlying double line and $Y = \text{Supp} W$. First we use [19, Proposition 2.6] to write $I_W = ((x, y)^3, xq, yq, h = pq - ax^2 - bxy - cy^2)$ where $q = xz - yw$ may be taken to be the equation of a smooth quadric surface $Q$ by [19, Remark 1.5] and $I_L = (x,l)$ for some linear form $l$. Note that $L$ is not tangent to $Q$ at $P$ because $L$ meets the underlying double line $Z$ in a reduced point.
If \( L_t = \{ x + wt = y + zt = 0 \} \) for \( t \in \mathbb{A}^1 \), the family \( D_t = L_t \cup Y \cup P L \) gives a flat family of extremal curves whose limit is \( D_0 = Z \cup L \). In considering the total family \( D \subset \mathbb{P}^3 \times \mathbb{A}^1 \), we see by Grauert’s theorem that \( \pi_* (\mathcal{I}_D (-g + 1)) \) is locally free on \( \mathbb{A}^1 \) (the extremal curves have constant cohomology) and hence globally free. Since \( \mathcal{I}_{D_1}(3) \) is generated by global sections and \( D_1 \) is reduced of embedding dimension \( \leq 2 \), \( D_1 \) lies on smooth surfaces of degrees \( \geq 3 \). In particular, we can find a section \( s_t \) (yielding a corresponding surface \( S_t \)) such that \( S_1 \) is a smooth surface containing \( D_1 \) and \( s_0 = lh \).

Now consider the family \( C_t = S_t \cap (L \cup Y^{(2)} \cup L_t) \). For general \( t \neq 0 \), \( C_t \) is the disjoint union of \( L_t \) and \( Z_t \cup P L \), where \( Z_t \) is a double line of genus \( g \). Let \( D_0 \) be the flat limit in the (full) Hilbert scheme. It is clear that \( D_0 = L \cup W_0 + \) possible embedded points, with \( W_0 \) a triple line. Since the limit of \( L_t \cup Y \) is \( Z \), it is clear that \( Z \subset W_0 \) and hence \( W_0 \) has type \((0, b')\) for some \( b' \geq 0 \). Since \( lh \in I_{W_0} \) and the plane \( \{l = 0\} \) meets \( W_0 \) properly, we have that \( h \in I_{W_0} \) and it follows (from [19, Proposition 2.6]) that \( W_0 \subset W \). Since these are Cohen-Macaulay triple lines, we conclude that \( W_0 = W \) and hence \( D_0 = L \cup W \) (there are no embedded points because \( p_a(C_t) = p_a(L \cup W) = g \)).

6. The Hilbert Schemes \( H_{4,g} \)

In this section we prove the main results of the paper. The first of these is Theorem 6.2 that describes the irreducible components of the Hilbert schemes \( H_{4,g} \). The second is the fact 6.4 that \( H_{4,g} \) is connected. The cases when \( g \geq 0 \) are well known and described in the introduction. We begin with the case \( g = -1 \), since it has a somewhat different statement due to the existence of more reduced curves.

**Proposition 6.1.** The Hilbert scheme \( H_{4,-1} \) is connected and has 3 irreducible components:

1. The 17-dimensional family of extremal curves.
2. The 16-dimensional closure of the family of subextremal curves. The general member is the disjoint union of two conics.
3. The 16-dimensional closure of the family whose general member is a disjoint union of a twisted cubic and a line.

**Proof.** If a curve \( C \in H_{4,-1} \) is not extremal, then its spectrum is necessarily \( \{0^2, 1^2\} \), in which case \( C \in H_2 \) or \( H_3 \) by Proposition 4.3. The families \( H_2 \) and \( H_3 \) have general members as described in Remark 4.4 and meet because both contain thick 4-lines. Finally, \( H_2 \) meets the family of extremal curves by [21] or [14].

In the following theorem the letter \( L \) denotes a line and the symbol \( \cup \) the disjoint union of two curves. \( L \cup_n P C \) denotes the schematic union of a line \( L \) and a curve \( C \), whose intersection is the divisor \( nP \) on \( L \).

**Theorem 6.2.** The irreducible components of the Hilbert schemes \( H_{4,g} \) for \( g \leq -2 \) are those listed in the following table
| Label  | General Curve                                                                 | Dimension | Restrictions |
|--------|------------------------------------------------------------------------------|-----------|--------------|
| $G_1$  | $D \cup Z$                                                                   | $15 - 2g$ | none         |
|        | $D$ smooth conic, $\deg(Z) = 2$                                              |           |              |
|        | $g(Z) = g - 3$, $\text{length}(D \cap Z) = 4$                               |           |              |
| $G_2$  | $L_1 \cup_{2P} Z \cup_{2Q} L_2$                                             | $13 - 2g$ | none         |
|        | $L_1 \cap L_2 = \emptyset$                                                   |           |              |
|        | $\deg(Z) = 2$, $g(Z) = g - 2$                                                |           |              |
| $G_3$  | $D \cup_{2P} Z$                                                              | $13 - 2g$ | none         |
|        | $D$ smooth conic                                                              |           |              |
|        | $\deg(Z) = 2$, $g(Z) = g - 1$                                                |           |              |
| $G_4$  | general thick 4-line                                                          | $9 - 3g$  | $g \leq -3$  |
| $G_5$  | double conic                                                                  | $13 - 2g$ | none         |
| $G_6$  | $Z \cup_{2P} L_1 \cup L_2$                                                   | $11 - 2g$ | $g \leq -3$  |
|        | $\deg(Z) = 2$, $g(Z) = g$                                                    |           |              |
| $G_{7,a}$ | $W \cup_{3P} L$                                                               | $11 - 2g - a$ | $g \leq -3$  |
|        | $W$ quasiprimitive 3-line                                                    |           | $0 < a \leq \frac{-g}{3}$ |
|        | type of $W = (a,-3a-g)$                                                       |           |              |
| $G_{8,a}$ | $W \cup_{2P} L$                                                               | $10 - 2g - a$ | $g \leq -6$  |
|        | $W$ quasiprimitive 3-line                                                    |           | $0 < a \leq \frac{-g-1}{3}$ |
|        | type of $W = (a,-1-3a-g)$                                                    |           |              |
| $G_{9,a}$ | $W \cup L$                                                                    | $8 - 2g - a$ | $g \leq -6$  |
|        | $W$ quasiprimitive 3-line                                                    |           | $0 < a \leq \frac{-g-3}{3}$ |
|        | type of $W = (a,-3-3a-g)$                                                    |           |              |
| $G_{10,m}$ | $D_1 \cup D_2$                                                                | $d(-m) + d(g+m+1)$ | $0 \leq m \leq \frac{-g-1}{2}$ |
|        | $\deg(D_1) = 2$, $g(D_1) = -m$                                               |           |              |
|        | $\deg(D_2) = 2$, $g(D_2) = g + m + 1$                                        |           |              |
| $G_{11,a,b}$ | Quasiprimitive 4-line                                                        | $7 - 2g - 3a = \frac{-9}{2}$ | $0 \leq b \leq \frac{-6a-g-3}{2}$ |
|        | type $(a, b, -6a - b - g - 3)$                                                |           |              |

where $d(-m) = \dim H_{2,-m}$ equals $5 + 2m$ if $m > 1$, and $8$ if $m = 0$ or $m = 1$.

Proof. In the table $G_j$ denotes the closure in the Hilbert scheme $H_{4,g}$ of the set of curves described in the corresponding row. The outline of the proof is as follows. First we show the families listed in the table are irreducible of the stated dimension. Then we show there is no inclusion relation among them. Finally, we prove every curve of degree 4 and genus $g \leq -2$...
The limit of the underlying family of double lines \( Z \) again that \( L \) possible because the limit double line meets \( G \) a contradiction. Similarly \( L \) in a double point.

\( G \) has genus \( \geq -1 \), hence \( a \leq 0 \) contradicting the restriction imposed. 

If \( G_{7,a'} \subset G_{9,a} \), then in considering the underlying double line as in Remark 2.4 we see that \( a' \leq a \), which in turn implies that \( \dim G_{7,a'} = 11 - 2g - a' > 8 - 2g - a = \dim G_{9,a} \), a contradiction. Similarly \( G_{7,a'} \not\subset G_{9,a} \).

It remains to show that neither \( G_{7,a'} \) and \( G_{8,a} \) contains the other. There can be no containment \( G_{7,a'} \subset G_{8,a} \), because then \( a' \leq a \) by Remark 2.4 and hence \( \dim G_{8,a} = 10 - 2g - a < 11 - 2g - a' = \dim G_{7,a} \), a contradiction. Now suppose that \( G_{8,a'} \subset G_{7,a} \). Remark 2.4 tells us again that \( a' \leq a \), and since \( \dim G_{8,a'} < \dim G_{7,a} \) we conclude that \( a' = a \). In particular, the limit of the underlying family of double lines \( Z \) has no embedded points. This is not possible because the limit double line meets \( L \) in one point while the general member meets \( L \) in a double point.
Finally, $G_{11,a,b}$ cannot be contained in any of the families $G_j$ with $j \leq 10$ by semicontinuity: indeed, since $a > 0$, every 4-line $C$ in $G_{11,a,b}$ satisfies $h^1\mathcal{O}_C(-2) = 1$, while for any other curve $D \in H_{4,g}$ we have $h^1\mathcal{O}_D(-2) \geq 2$. On the other hand, there are no containments among the families $G_{11,a,b}$: if $G_{11,a,b} \subset G_{11,a',b'}$, then by Remark 2.4 we would have $a \leq a'$, while

$$7 - 2g - 3a' = \dim G_{11,a',b'} > \dim G_{11,a,b} = 7 - 2g - 3a$$

would yield $a' < a$, a contradiction.

To finish the proof, we still have to show our families cover the Hilbert scheme. Let $C \in H_{4,g}$ have support $B = C_{red}$.

**Case 1**: $\deg B = 4$

Here $C = B$ is reduced, and all reduced curves of degree 4 satisfy $g \geq -1$ with the following two exceptions: either (a) $C$ is the disjoint union of a conic (possibly degenerate) and two lines, when $g = -2$ and $C \in G_{10,0}$ or (b) $C$ is the disjoint union of four lines, $g = -3$ and $C \in G_{10,1}$.

**Case 2**: $\deg B = 3$

In this case $C = Z \cup D$, where $Z$ is a double line with support $L$ and $D$ is a reduced curve of degree 2. In particular, $B = L \cup D$.

First suppose that $D$ is planar and let $l = \text{length}(D \cap L)$. If $l = 0$, then $C \in G_{10,0}$. If $l = 1$, then $g(B) = 0$ and $C$ belongs to one of the families $G_2$, $G_3$ or $G_4$ by Proposition 4.3. If $l = 2$, then $B$ is planar and hence $C$ is extremal by $[20, 2.2]$.

The other possibility is that $D = L_1 \cup L_2$ is a disjoint union of lines. If $D$ does not meet $Z$, then $C \in G_{10,1}$. If $Z$ meets $L_1$ but not $L_2$, then $Z \cup L_1$ is a specialization of a double line meeting $L_1$ in a double point by $[14, 3.2]$, so $C$ lies in $G_6$. If $Z$ meets both $L_1$ and $L_2$, then $g(B) = 0$ and $C$ again belongs to one of the families $G_2$, $G_3$ or $G_4$ by Proposition 4.3.

**Case 3**: $\deg B = 2$

If $B$ is a smooth conic, then $C$ belongs to $G_5$, so we may assume that $B = L \cup L'$ is a union of two lines. If $C$ is a union of two double lines, then either (a) the lines are disjoint and $C \in G_{10,m}$ for $m = -\max\{g(Z_1), g(Z_2)\}$ or (b) the lines meet and hence $C$ is contained in a double plane; in this case $C \in G_5$ by $[14, 8.1$ and 8.2]$

The remaining possibility is that $C = W \cup L$, where $L$ is a line and $W$ is a triple line which is quasi-primitive because $g(W) \leq -1$ (the only thick triple line has genus 0). If $W$ has type $(a,b)$, let $Z$ be the underlying double line and set $l = \text{length}(W \cap L)$.

If $l = 3$, then necessarily $\text{length}(Z \cap L) = 2$. If $a = -1$, then $Z \cup L$ is planar and $C$ is extremal. If $a = 0$, then $Z \cup L \in H_{3,0}$ and $C$ belongs to one of the families $G_2$, $G_3$ or $G_4$ by Proposition 4.3. If $a > 0$, then $C$ belongs to $G_{7,a}$.

Suppose $l = 2$. If $\text{length}(Z \cap L) = 2$ and $a = -1$ or 0, we argue as in the case $l = 3$. If $\text{length}(Z \cap L) = 2$ and $a > 0$, then $C \in G_{7,a}$ by Proposition 5.4. Thus we may assume
$Z \cap L = P$ a reduced point. If $a = -1$, then $Z \cup L \in H_{3,0}$ and $C \in G_2 \cup G_3$ by Proposition 4.3. If $a = 0$, then $C \in G_6$ by Remark 5.6. If $a > 0$, then $C \in G_{8,a}$.

Suppose $l = 1$. If $a > 0$, then $C \in G_8$ by Proposition 5.4. If $a = 0$, then $C \in G_6$ by Proposition 5.4 and Remark 5.6. If $a = -1$, then $Z \cup L \in H_{3,0}$ and $C \in G_2 \cup G_3$ by Proposition 4.3.

If $l = 0$, then $C = W \cup L$ with $W$ quasi-primitive of type $(a, -3 - 3a - g)$. If $a = -1$, then $W$ is extremal and $C \in G_6$ by [19, 3.2]. If $a = 0$, then $C \in G_{5,0}$ by [19, 3.3], while if $a > 0$, then $C \in G_{9,a}$.

**Case 4:** $\deg B = 1$

If $C$ is thick, then $C \in G_4$. If $C$ is a quasiprimitive 4-line, then $C$ has type $(a, b, c)$ for some integers $a \geq -1$ and $c \geq b \geq 0$. If $a = -1$, then the underlying double line $Z$ is planar and $C$ lies in a double plane, hence $C \in G_5$ by [14, 8.1 and 8.2]. If $a = 0$, then $C \in G_{10,m}$ for $m = -1 - b$ by Proposition 3.2. Finally, if $a > 0$, then $C \in G_{11,a,b}$.

**Example 6.3.** As the restrictions in the statement imply, some of these components do not show up if the genus $g$ is not small enough. For example, $H_{4,-2}$ has only 5 irreducible components:

1. The 19-dimensional family of extremal curves $G_1$, whose general member is the union of a conic and a double line of genus $-5$ meeting in a scheme of length 4.
2. The 17-dimensional family $G_2$, whose general member, the union $L_1 \cup_{2P} Z \cup_{2Q} L_2$ of lines $L_i$ and a double line $Z$ of genus $-4$, is a subextremal curve.
3. The 17-dimensional family $G_3$ whose general member is the union of a double line of genus $-3$ and a conic, meeting in a double point.
4. The 17-dimensional family $G_5$ whose general member is a double conic.
5. The 16-dimensional family $G_{10,0}$ of curves whose general member is the disjoint union of a conic and two lines.

We can now prove $H_{4,g}$ is connected:

**Theorem 6.4.** The Hilbert scheme $H_{4,g}$ is connected whenever nonempty.

*Proof.* We may assume $g \leq -2$, and it suffices to show that all the irreducible components can be connected to the component $G_1$ of extremal curves. The families $G_2$ and $G_3$ meet the family $G_4$ of thick 4-lines by Proposition 4.3, and $G_2$ meets $G_1$ by [1], [2], or [21]. In particular, thick 4-lines belong to the connected component of extremal curves, and it is enough to show that all other components can be connected to either $G_1$ or $G_4$. $G_5$ meets $G_1$ by [14, 5.1 and 8.2]. That $G_6$ meets $G_1$ follows immediately from [14, 2.1] and [19]. $G_7,a$ meets $G_1$ by Corollary 5.3 and $G_{8,a}$ meets $G_1$ by Proposition 5.4 and [11, 2.5]. One can connect $G_{9,a}$ to $G_1$ by applying [19, 3.8] and [11, 2.1]. The families $G_{11,a,b}$ meet $G_4$ by Proposition 2.3. Applying Proposition 3.2, we see $G_{10,m}$ contains $G_{11,0,m-1}$ for $m > 0$, and
hence meets $G_4$ as well. Lastly, we consider $G_{10,0}$. By definition this family contains curves $C = Z \cup L_1 \cup P \ L_2$ where $Z$ has degree two and $L_i$ are meeting lines. Since $Z$ is extremal, $C$ specializes to an extremal curve $E \in G_1$ by \cite[2.1 and 2.5]{[1]}.

As an application of our results, we can now give a counterexample to a conjecture of Aït-Amrane and Perrin \cite{[2]}. The conjecture regards the following question, which has been a recurring theme of this paper:

**Question 6.5.** Let $X$ and $X_0$ be two irreducible families of curves in $H_{d,g}$ having constant cohomology. Under what conditions do we have a nonempty intersection $\overline{X} \cap X_0 \neq \emptyset$ in $H_{d,g}$?

We have been lucky in that whenever we suspected the existence of such a deformation, we could actually prove it. In general, this question is difficult. A first necessary condition is provided by semicontinuity \cite[III, §12]{[3]}: If $\tau^i(n) = h^i(\mathbb{P}^3, I_C(n))$ for $C \in X$ and $\tau^i_0(n) = h^i(\mathbb{P}^3, I_{C_0}(n))$ for $C_0 \in X_0$, then whenever $\overline{X} \cap X_0 \neq \emptyset$ we must have $\tau^i(n) \leq \tau^i_0(n)$ for all $i$ and $n$ (we write $\tau \leq \tau_0$ for short). This condition is not sufficient, even when $X$ (resp. $X_0$) is an irreducible component of the Hilbert scheme $H_\tau$ (resp. $H_{\tau_0}$) of curves with fixed cohomology. This has been shown by a recent example of Aït-Amrane and Perrin \cite{[2]}.

A more subtle necessary condition is afforded by the Rao modules of the curves. Let $A$ be a discrete valuation ring with fraction field $K$ and residue field $k$ and let $C \subset \mathbb{P}^3_A$ be a family of locally Cohen-Macaulay curves over $A$. Then the Rao module $M_{C_K}$ of the generic curve is a flat deformation of a subquotient $M$ of the Rao module $M_{C_K}$ of the special curve (\cite[Proposition 5.9]{[13]}, \cite[§4.2.2]{[4]}, \cite[Proposition 13]{[4]}). This means that there are submodules $M_1 \subset J \subset M_{C_K}$ such that $M = J/M_1$ and $M$ is a flat deformation of $M_{C_K}$. Moreover,

\begin{equation}
\dim(M_1)_n = h^0I_{C_0}(n) - h^0I_{C_K}(n) \quad \text{and} \quad \dim(M_{C_K}/J)_n = h^2I_{C_0}(n) - h^2I_{C_K}(n)
\end{equation}

In view of this result, the following conjecture of Aït-Amrane and Perrin is natural:

**Conjecture 6.6** (\cite{[2]}, Conjecture 14). Let $X$ and $X_0$ be irreducible components of $H_\tau$ and $H_{\tau_0}$ respectively. Suppose that

1. $\tau \leq \tau_0$
2. The Rao module of the generic curve $C_\xi$ of $X$ is a flat deformation of a subquotient of the Rao module of a curve $C_0$ in $X_0$ and that the numerical conditions \cite{[4]} hold.

Then $\overline{X} \cap X_0 \neq \emptyset$ in $H_{d,g}$.

As it turns out, this is no longer true, as we note in the following example.

**Example 6.7.** For $g \leq -3$, let $X \subset H_{d,g}$ denote the irreducible family $H_{g(g)}$ from Proposition \cite[13]{[14]}. The closure $\overline{X}$ is precisely the family of thick 4-lines. Let $X_0 = G_1$ denote irreducible family consisting of extremal curves. We claim that $\overline{X} \cap X_0 = \emptyset$. Indeed, an extremal curve cannot specialize to a thick 4-line because this would violate semicontinuity, while a thick 4-line has everywhere embedding dimension 3, and so cannot specialize to an
extremal curve that has generic embedding dimension 2 \cite{8}. On the other hand, we will now show that the conditions of the conjecture hold.

First we compare the Rao modules. Let $C$ be a general thick 4-line with support $L$ and set $S = S_L$. Proposition \cite{4} shows that $M_C \cong S/(a,b,c)(-g-1)$ where $a, b, c$ are general forms of degree $-g+1$ in $S$. Choose a linear form $l \in S$ so that $(a,l)$ is a regular sequence and a form $f \in (b,c)$ of degree $-g+2$ so that $(a,f)$ is a regular sequence. We consider the extremal Koszul module $M = S/(a,lf)$. Since the multiplication $S/(a) \to S/(a,f)$ is injective and the image of the submodule $(f)$ is $(lf)$, we see that the submodule $J = lM$ is isomorphic to $S/(a,f)$. Since $f \in (b,c)$ by choice, $S/(a,b,c)$ is a quotient of $J$ by $M_1 = (lb,lc)$. If $E$ is an extremal curve corresponding to $M$, then $\deg E = 4$, $p_a(E) = g$ and we have just shown that $M_C$ is a subquotient of $M_E$. It is clear that $\dim (M/J)_n = 1$ for $g \leq n \leq 0$ and zero otherwise; this is seen to be precisely $h^2 I_E(n) - h^2 I_C(n)$ in comparing the spectra of these curves. Finally, since the Euler characteristics of $I_E$ and $I_C$ are the same, the exact sequences relating the Rao modules shows that $\dim (M_1)_n = h^0 I_E(n) - h^0 I_C(n)$. In particular, the semicontinuity conditions are immediate.

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Department of Mathematics, Texas Christian University, Fort Worth, TX 76129, USA
E-mail address: s.nollet@tcu.edu

Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
E-mail address: enrsch@mate.polimi.it