§1. Introduction

In a series of works [Bo3-5], Borcherds developed a theory of modular forms over domains of type IV which admits an infinite product expansion. Such modular forms are said to be Borcherds’s product in this paper. Among all Borcherds’s products, Borcherds’s $\Phi$-function ([Bo4]) has an interesting geometric background; it is a modular form on the moduli space of Enriques surfaces characterizing the discriminant locus. In his construction, $\Phi$-function is obtained as the denominator function of one of the fake monster Lie superalgebras ([Bo2, §14]), although Enriques surface itself plays no role. After Borcherds, Jorgenson-Todorov ([J-T2,3]) and Harvey-Moore ([H-M]) discovered that the Ray-Singer analytic torsion ([R-S]) of an Enriques surface equipped with the normalized Ricci-flat Kähler metric coincides with Borcherds’s $\Phi$-function at its period point. The goal of this paper is to give a rigorous proof to their observation and generalize it to an interesting class of $K3$ surfaces studied by Nikulin ([Ni4]). Let us briefly recall these surfaces.

Let $(X, \iota)$ be a $K3$ surface with an anti-symplectic involution. Let $M$ be a 2-elementary hyperbolic lattice. The pair $(X, \iota)$ is said to be a 2-elementary $K3$ surface of type $M$ if the invariant sublattice of $H^2(X, \mathbb{Z})$ with respect to the action of $\iota$ is isometric to $M$. Since $X/\iota$ is an Enriques surface when $M \cong \text{II}_{1,9}(2)$, $K3$ surfaces of this class are a kind of generalizations of Enriques surfaces. (We denote by $I_{p,q}$ (resp. $\text{II}_{p,q}$) the odd (resp. even) unimodular lattice of signature $(p,q)$.)

By the Torelli theorem ([P-S-S]) and surjectivity of the period map ([To1]), the moduli space of 2-elementary $K3$ surfaces of type $M$ is isomorphic to an arithmetic quotient of an open subset of the symmetric bounded domain of type IV via the period map. The period domain is denoted by $\Omega_M$, the modular group by $\Gamma_M$, and the moduli space by $\mathcal{M}_M^0 = (\Omega_M \setminus \mathcal{D}_M)/\Gamma_M$. Here, $\mathcal{D}_M$ is the discriminant locus. Let $X^i$ be the fixed locus of $\iota$. Since only one of the irreducible components can have positive genus, $g(M)(\geq 0)$, unless $M \cong \text{II}_{1,9}(2)$ or $\text{II}_{1,1} \oplus E_8(-2)$, one can define a map $j_M^0 : \mathcal{M}_M^0 \to \mathcal{A}_g(M)$ to the Siegel modular variety by taking the period point of the Jacobi variety of $X^i$. (If $M \cong \text{II}_{1,1} \oplus E_8(-2)$, $j_M$ takes its value in $S^2(A_1)$, the second symmetric product of $A_1$.) Let $\mathcal{F}_g(M)$ be the sheaf of Siegel modular forms of weight 1 on $\mathcal{A}_g(M)$. We define a $\Gamma_M$-invariant sheaf on (an open subset of) $\Omega_M$ by $\lambda_M := j_M^* \mathcal{F}_g(M)$. A $\lambda_M^{\otimes q}$-valued modular form of weight $p$ (with a character of $\Gamma_M$) is said to be an automorphic form of weight $(p,q)$ in this paper whose Petersson norm is denoted by $\| \cdot \|$.

In this paper, we focus on the Ray-Singer analytic torsion regarded as a function on the moduli space of $K3$ surfaces. Since it coincides with (the Petersson norm...
of) the Jacobi $\Delta$-function for elliptic curves ([R-S]), it is natural to expect that it may yield an interesting modular form for $K3$ surfaces ([J-T1]). With small modifications, it is the case at least for 2-elementary $K3$ surfaces. We introduce the following function (see §K modifications, it is the case at least for 2-elementary $K3$ surfaces of type $M$;

$$(1.1) \quad \tau_M(X, \iota, \kappa) := \text{vol}(X, \kappa)^{14 - r(M)} \tau(X/\iota, \kappa) \sqrt{\text{vol}(X^\iota, \kappa|_{X^\iota})} \tau(X^\iota, \kappa|_{X^\iota}).$$

Here $\kappa$ is an $\iota$-invariant Ricci-flat Kähler metric of $X$, $r(M)$ is the rank of $M$, $\tau(X/\iota, \kappa)$ is the Ray-Singer analytic torsion of orbifold $(X/\iota, \kappa)$ (or equivalently, the equivariant Ray-Singer analytic torsion of $(X, \iota, \kappa)$) and $\tau(X^\iota, \kappa|_{X^\iota})$ is that of the fixed curve $(X^\iota, \kappa|_{X^\iota})$. Note that $\text{vol}(X^\iota, \kappa|_{X^\iota})$ and $\tau(X^\iota, \kappa|_{X^\iota})$ are defined multiplicatively with respect to the irreducible decomposition of divisors.

**Theorem 1.1 (Theorems 3.4, 7.2, 7.3).** Suppose that $r(M) \leq 17$.

1. $\tau_M$ is independent of a choice of Ricci-flat Kähler metrics and defines an invariant of a 2-elementary $K3$ surface of type $M$. It descents to a smooth $\Gamma_M$-invariant function on $\Omega^0_M := \Omega_M \setminus \mathcal{D}_M$.
2. There exists an automorphic form, $\Delta_M$, for $\Gamma_M$ of weight $(r(M) - 6, 4)$ such that $\tau_M = \|\Delta_M\|^{-1/4}$ and $\text{div}(\Delta_M) = \mathcal{D}_M$.
3. If $\delta$ is a root of $M^\perp$ and $(M \oplus \delta)$ is the smallest 2-elementary lattice containing $M$ and $\delta$, then $\Delta_{(M \oplus \delta)}$ coincides with the regularized restriction of $\Delta_M$ to $\Omega_{(M \oplus \delta)}$.

It may be worth mentioning that analogous statment is valid for theta divisors of dimension $\geq 1$ ([Yo2]). By (3), any $\Delta_M$ is the regularized restriction of such $\Delta_S$ that $S$ is root free. Therefore, it is important to know an explicit formula for them. By Nikulin’s classification, there are three such lattices up to isometry; $S \cong A_1, \Pi_{1,1}(2), \Pi_{1,9}(2)$.

**Theorem 1.2 (Theorems 10.1, 10.2, 9.3).**

1. $\Delta_{A_1}^{15} = \Delta_6^8/\Psi_{A_1}^4$ where $\Delta_6$ is the discriminant of plane sextic curves, and $\Psi_{A_1}^4$ is Borcherds’s product attached to the transcendental lattice and $\Theta_{E_7}(\tau)/\Delta(\tau)$. Here, $\Theta_{E_7}(\tau)$ is the vector valued theta series of $E_7$-lattice and $\Delta(\tau)$ is the Jacobi $\Delta$-function.
2. $\Delta_{\Pi_{1,1}(2)}^{17} = \Delta_{4,4}^9/\Psi_{\Pi_{1,1}(2)}$ where $\Delta_{4,4}$ is the discriminant of the linear system of bidegree $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\Psi_{\Pi_{1,1}(2)}$ is Borcherds’s product attached to the transcendental lattice and $\eta(\tau)^{-8}\eta(2\tau)^{-8}$. Here, $\eta(\tau)$ is the Dedekind $\eta$-function.
3. $\Delta_{\Pi_{1,9}(2)}$ coincides with the Borcherds $\Phi$-function associated to the fake monster Lie superalgebra of rank 10 up to a constant, and $\Delta_{\Pi_{1,9}(2)}^2$ is Borcherds’s product attached to $\mathcal{I}_{2,10}(2)$ and $\eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$.

For the precise meaning of Borcherds’s products used in this paper, see Theorem 8.1 and Definition 8.1. Concerning (3), another product formula can be found in [Bo5]. Jorgenson-Kramer ([J-K]) treat Borcherds’s $\Phi$-function from the view point of Green currents. Since Lefschetz fixed point formula in Alakelov theory is established by Kähler-Roessler recently ([K-R]), it will be possible to represent $\Delta_M$ by Green currents.

Among all the 2-elementary lattices, there are some interesting ones from the view point of Nikulin’s classification. As in the case of $\Pi_{1,9}(2)$, $\Delta_M$ for them is closely related to Borcherds’s theory ([Bo1-5]).
Theorem 1.3 (Corollary 6.1, Theorems 8.2, 8.3, 9.4, 9.6).

(1) If \( M \cong \Pi_{1,1} \oplus E_8(-2) \), one has \( \Delta_M^3 = C_M \Psi_{M\perp} \otimes j_M^*(\Delta_1 \Delta_2) \) where \( \Delta_i \) is the Jacobi \( \Delta \)-function in the \( i \)-th variable on \( A_i \times A_1 \), \( C_M \) is a constant, and \( \Psi_{M\perp} \) is the denominator function of a generalized Kac-Moody superalgebra. Moreover, \( \Psi_{M\perp} \) is Borcherds’ product attached to the transcendental lattice and \( \Theta_{A_{16}}(\tau)/\Delta(\tau) \). Here, \( \Theta_{A_{16}}(\tau) \) is the vector valued theta series of the Barnes-Wall lattice.

(2) If \( g(M) = 0 \), namely \( M\perp \cong I_{2,20} - r(M)(2) \) \((11 \leq r(M) \leq 17)\), \( \Delta_M \) is the denominator function of a generalized Kac-Moody superalgebra up to a constant. Moreover, \( \Delta_M^2 = \text{Borcherds’s product attached to the transcendental lattice and } \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8} \theta_{A_1}(\tau)^{r(M)-10} \text{ where } \theta_{A_1}(\tau) \text{ is the theta series of } A_1\text{-lattice.} \)

In view of Theorems 1.2, 1.3, it seems natural to conjecture that any \( \Delta_M \) in Theorem 1.1 is represented by Borcherds’ product and discriminant of curves. Compared to fake monster Lie algebras, generalized Kac-Moody superalgebras appearing in Theorem 1.3 are not well understood. We shall show in Theorem 11.1 and Corollary 11.1 that product of certain 10 theta functions on the domain of type \( I_{2,2} \) ([Ma]) admits Borcherds’ product which is analogous to Gritsenko-Nikulin’s product formula for Igusa’s modular form ([G-N1]).

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2.1 Moduli Space of K3 Surfaces. A compact complex surface $X$ is a K3 surface if and only if $q(X) = 0$ and $K_X \cong \mathcal{O}_X$. The cohomology lattice $H^2(X, \mathbb{Z})$ together with the intersection product is isometric to $L_{K3} = \mathbb{H}^1_{1,1} \oplus E_8(-1)^2$ where $E_8$ is the positive definite $E_8$-lattice. (For a lattice $M$, $M(k)$ denotes the lattice with $\langle , \rangle_{M(k)} = k \langle , \rangle_M$. $\Delta(M) = \{ m \in M; \langle m, m \rangle_M = -2 \}$ the root of $M$, and $M_\mathbb{R}$ (resp. $M_\mathbb{C}$) the vector space $M \otimes \mathbb{R}$ (resp. $M \otimes \mathbb{C}$). A sublattice $M' \subset M$ is said to be primitive if $M/M'$ is torsion free.) We denote by $\text{Pic}_X := H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$ the Picard lattice and by $\Delta(X)$ its root. By the Riemann-Roch theorem, $\Delta(X) = \Delta^+(X) \sqcup -\Delta^+(X)$ where $\Delta^+(X)$ consists of effective classes. The Kähler cone of $X$ is denoted by $C_X^+ := \{ \kappa \in H^{1,1}(X, \mathbb{R}); \langle \kappa, \delta \rangle > 0, \forall \delta \in \Delta^+(X) \}$ and its closure in $H^{1,1}(X, \mathbb{R})$ by $C_X$. Let $\phi_X : H^2(X, \mathbb{Z}) \to L_{K3}$ be an isometry. A pair $(X, \phi_X)$ is said to be a marked K3 surface. Two marked K3 surfaces $(X, \phi_X)$ and $(X', \phi_{X'})$ are isomorphic when there exists an isomorphism $f : X \to X'$ such that $\phi_X \circ f^* = \phi_{X'}$. There exist the fine moduli space of marked K3 surfaces. We briefly recall it ([P-S-S], [B-R], [B-P-V-V], [Be], [Mo], [Ni2,3] etc.). Define $\Omega$ by

\begin{equation}
\Omega := \{ [\eta] \in \mathbb{P}(L_{K3, \mathbb{C}}); \langle \eta, \eta \rangle_{L_{K3}} = 0, \langle \eta, \bar{\eta} \rangle_{L_{K3}} > 0 \}.
\end{equation}

For $\eta \in \Omega$, let $C(\eta) = \{ x \in L_{K3, \mathbb{R}}; \langle x, \eta \rangle_{L_{K3}} = 0, \langle x, x \rangle_{L_{K3}} > 0 \}$ be the positive cone over $\eta$ which consists of two connected components: $C(\eta) = C(\eta)^+ \sqcup C(\eta)^-$. Let $S(\eta) := L_{K3} \cap \eta^\perp$ be the Picard lattice over $\eta$, $\Delta(\eta)$ the root of $S(\eta)$, and $h_\delta := \{ x \in L_{K3, \mathbb{R}}; \langle x, \delta \rangle_{L_{K3}} = 0 \}$ the hyperplane orthogonal to $\delta$. Let $C(\eta) \setminus \bigcup_{\delta \in \Delta(\eta)} h_\delta = \bigsqcup P \subset \mathcal{C}$ be the decomposition into the connected components. $P$ is parametrized by the pair $(C(\eta)^+, \Delta^+_P(\eta))$ where $\Delta^+_P(\eta)$ is a partition $\Delta(\eta) = \Delta^+_P(\eta) \sqcup -\Delta^+_P(\eta)$ with the following property:

(P1) If $\delta_1, \ldots, \delta_k \in \Delta^+_P(\eta)$ and $\delta = \sum_i n_i \delta_i \ (n_i \geq 0)$, then $\delta \in \Delta^+_P(\eta)$.

Set $K\Omega := \{ (\eta, \kappa) \in \Omega \times L_{K3, \mathbb{R}}; \kappa \in C(\eta) \}$ and

\begin{equation}
K\Omega^0 := \{ (\eta, \kappa) \in K\Omega; \kappa \in C(\eta) \setminus \cup_{\delta \in \Delta(\eta)} h_\delta \}, \quad \tilde{\Omega} := K\Omega^0 / \sim.
\end{equation}

Here $(\eta, \kappa) \sim (\eta', \kappa')$ when $\eta = \eta'$ and $\kappa, \kappa' \in C_P(\eta)$ for some $P$. We denote by $p : K\Omega^0 \to \tilde{\Omega}$ the projection map. Let $\pi : \tilde{\Omega} \to \Omega$ be the projection to the first factor. It is immediate by above construction that (1) is a non-separated smooth analytic space, (2) $\pi$ is étale and surjective, (3) $\pi^{-1}(\eta) = \{ P \}$; the set of all pairs $(C(\eta)^+, \Delta(\eta)^+)$ where $\Delta(\eta)^+$ satisfies (P1).

For $\eta, \kappa \in K\Omega^0$, put $C(\eta)^g_\kappa$ for the connected component containing $\kappa$, and $\Delta(\eta)^g_\kappa := \{ \delta \in \Delta(\eta); \langle \eta, \delta \rangle > 0 \}$. Then, $[(\eta, \kappa)] \in \tilde{\Omega}$ corresponds to $(\eta, C(\eta)^g_\kappa, \Delta(\eta)^g_\kappa)$.

Let $\eta_X$ be a symplectic form; $H^0(X, K_X) = \mathbb{C} \eta_X$. Then, $\pi(X, \phi_X) := [\phi_X(\eta_X)] \in \mathbb{P}(L_{K3, \mathbb{C}})$ is the period of $(X, \phi_X)$ which lies on $\tilde{\Omega}$. Let $K_X$ be a Kähler metric of $X$ and identify it with its class in $H^2(X, \mathbb{R})$. The pair $(X, K_X)$ is said to be a polarized K3 surface and the triplet $(X, \phi_X, K_X)$ a marked polarized K3 surface. The polarized period point of $(X, \phi_X, K_X)$ is defined by $\pi(X, \phi_X, K_X) := ([\phi_X(\eta_X)], \phi_X(K_X)) \in K\Omega^0$ and the Burns-Rapoport period point by $[\pi(X, \phi_X, K_X)] \in \tilde{\Omega}$.

**Theorem 2.1.** There exists the universal family of marked K3 surfaces over $\tilde{\Omega}$; $P : \mathcal{X} \to \tilde{\Omega}$, such that the period map coincides with $\pi : \tilde{\Omega} \to \Omega$.

Let $M$ be a primitive hyperbolic sublattice of $L_{K3}$ with signature $(1, k)$. A marked K3 surface $(X, \phi_X)$ is marked (ample) $M$-polarized if $\phi_X(\text{Pic}_X) \supset M$ (and
As $M$ is hyperbolic, a marked $M$-polarized $K3$ surface is projective. Let $N := M^\perp = \{ l \in L_{K3}; \langle l, M \rangle = 0 \}$ be the orthogonal compliment of $M$ whose signature is $(2, 19 - k)$. By definition, the period point of a marked $M$-polarized $K3$ surface is contained in the following subset of $\Omega$;

$$\phi_X(C_X^\perp \cap M_R \neq \emptyset).$$

(2.3) $$\Omega_M := \{ [\eta] \in \mathbb{P}(N_C); \langle \eta, \eta \rangle_N = 0, \langle \eta, \eta \rangle_N > 0 \}.$$ 

Put $C(M) := \{ x \in M_R; \langle x, x \rangle_M > 0 \}$ or equivalently $C(M) = C(\eta) \cap M_R$. Define

(2.4) $$K\Omega_M := \{ ([\eta], \kappa) \in K\Omega; [\eta] \in \Omega_M, \kappa \in C(M) \}, \ K\Omega^0_M := K\Omega \cap K\Omega_M.$$ 

Set $\tilde{\Omega}_M^0 := K\Omega^0_M/\sim$ and $\tilde{\Omega}_M$ for the closure of $\tilde{\Omega}_M^0$ in $\tilde{\Omega}$.

**Lemma 2.1.** For any $\eta \in \Omega_M, M \cap C(M) \setminus \bigcup_{\delta \in \Delta(\eta) \setminus \Delta(N)} h_\delta \neq \emptyset$.

**Proof.** If $M_Q \cap C(M) \setminus \bigcup_{\delta \in \Delta(\eta) \setminus \Delta(N)} h_\delta = \emptyset$, then $C(M) \setminus \bigcup_{\delta \in \Delta(\eta) \setminus \Delta(N)} h_\delta = \emptyset$ and thus there exists $\delta \in \Delta(\eta) \setminus \Delta(N)$ such that $M_R \subset h_\delta$. This implies that $\delta \in M^\perp = N$ and contradicts the choice of $\delta$. \hfill $\square$

Let $H_l := \{ [\eta] \in \Omega; \langle \eta, l \rangle_{L_{K3}} = 0 \}$ be the hyperplane defined by $l \in L_{K3}$. We define the discriminant locus and an open subset by

(2.5) $$\mathcal{D}_M := \bigcup_{\delta \in \Delta(N)} H_\delta, \quad \Omega_M^0 := \Omega_M \setminus \mathcal{D}_M.$$ 

Let $\pi_M : \tilde{\Omega}_M^0 \to \Omega_0^0_M$ be the period map restricted to $\tilde{\Omega}_M^0$. By Theorem 2.1 and Lemma 2.1, we get the following. (See [Ni2,3] for the detail.)

**Theorem 2.2.** $\tilde{\Omega}_M^0$ is the fine moduli space of marked ample $M$-polarized $K3$ surfaces whose period map is $\pi_M$. The universal family $\mathcal{P}_M : \mathcal{X}_M^0 \to \tilde{\Omega}_M^0$ is obtained by putting $\mathcal{X}_M^0 := \mathcal{X}|_{\Omega_0^0_M}$ and $\mathcal{P}_M := \mathcal{P}|_{\mathcal{X}_M^0}$.

We are now interested in the boundary of $\tilde{\Omega}_M^0$ in $\tilde{\Omega}_M$ and the family of marked $M$-polarized $K3$ surfaces over it. For $\delta \in \Delta(N)$, put $H_0^\delta := H_\delta \setminus \bigcup_{d \in \Delta(N)} \{ \pm \delta \} H_d$.

**Lemma 2.2.** Let $\delta \in \Delta(N), \eta \in H_0^\delta$ and $\kappa \in M \cap C(M) \setminus \bigcup_{d \in \Delta(\eta) \setminus \Delta(N)} h_d$. Then,

$$\Delta(\eta) = \Delta^+_\kappa(\eta) \cup -\Delta^+_\kappa(\eta) \cup \{ \pm \delta \}$$

where $\Delta^+_\kappa(\eta) = \{ d \in \Delta(\eta); \langle \kappa, d \rangle > 0 \}$. We put

$$\Delta^+_\kappa(\eta) := \Delta^+_\kappa(\eta) \cup \{ \delta \} \quad \text{and} \quad \Delta^+_\kappa(\eta) := \Delta^+_\kappa(\eta) \cup \{ -\delta \}.$$ 

Then, $\Delta^+_\kappa(\eta)$ satisfies $\text{(P)}$.

**Proof.** Suppose that $d \in \Delta(\eta) \setminus \Delta(N)$. Then, either $\langle \kappa, d \rangle > 0$ or $\langle \kappa, d \rangle < 0$ because $\kappa \notin \bigcup_{d \in \Delta(\eta) \setminus \Delta(N)} h_d$. This implies $\Delta(\eta) \setminus \Delta(N) = -\Delta^+_\kappa(\eta) \cup \{ -\delta \}$. Suppose that $d \in \Delta(\eta) \cap \Delta(N)$ which means $d \in \Delta(N)$ and $\eta \in H_d$. By the choice of $\eta, d = \pm \delta$ which prove the first assertion. Since the Weyl group $W(\eta)$ acts properly discontinuously on $C(\eta)$, there exists a small neighborhood $K$ of $\kappa$ in $C(\eta)$ such that $s_d(K) \cap K \neq \emptyset$ for $d \in \Delta(\eta)$ means $d = \pm \delta$ where $s_d$ is the reflection in $d$. Thus there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0, \kappa + \epsilon \delta \in V(\eta) \setminus \bigcup_{d \in \Delta(\eta)} h_d$. Clearly, $\Delta^+_\kappa(\eta) \cup \{ -\delta \} = \{ d \in \Delta(\eta); \langle \kappa + \epsilon \delta, d \rangle > 0 \}$. Similarly, $\Delta^+_\kappa(\eta)$ is the partition associated to $\kappa - \epsilon \delta$. \hfill $\square$
Lemma 2.4. Let $C^1_δ$ (resp. $C^2_δ$) be the effective cycle on $X^1_0$ (resp. $X^2_0$) such that $φ^0_0(C^1_δ) = δ$ (resp. $φ^0_0(C^2_δ) = -δ$). Then, $C^1_δ$ (resp. $C^2_δ$) is an irreducible $2$-curve.

Proof. We only prove the case $i = 1$. Since $⟨κ, d⟩ > 0$ for any $d ∈ Δ^+_1(η_0)$ and $⟨κ, κ⟩ > 0$, $(φ^1_0)^{-1}(κ)$ is a pseudo-ample class ([Mo, §5]). As $⟨κ, δ⟩ = 0$, $C^1_δ$ consists of a chain of $-2$-curves by Mayer’s theorem ([Mo, §5]). Since $C^1_δ$ is effective, we can write $C^1_δ = ∑_j m_j E_j$ where $E_j$ is an irreducible $-2$-curve such that $⟨κ, φ^1_0(E_j)⟩ = 0$, and $m_j ∈ Z_+$. Suppose $C^1_δ$ is not irreducible. Then, $E_1 ⊈ C^1_δ$. This implies $φ^1_0(E_1) ∈ Δ(η_0) ∩ κ^−$. By Lemma 2.2 and effectiveness of $E_1$, $φ_0^1(E_1) = δ$. Thus, $C^1_δ = E_1$ and contradicts the assumption. □

Let $π_i : γ^*_i X → D$ be the pullback of the universal family by $γ_i$. Recall that a compact complex surface with at most rational double points is said to be a generalized $K3$ surface when its minimal resolution is a $K3$ surface.

Proposition 2.1. There exists a family of generalized $K3$ surface $π : Y → D$, a contraction morphism $b_i : γ^*_i X → Y$ which commutes with the projections, and a birational map $e : γ^*_i X → γ^*_2 X$ such that

1. $b_1$ (resp. $b_2$) is the blow-down of $C^1_δ$ (resp. $C^2_δ$) to point $o$,
2. $e : X^1_0 \setminus C^1_δ → X^2_0 \setminus C^2_δ$ is an isomorphism such that $b_1 = b_2 ∘ e$,
3. $e$ is the identity map on $X_t$ for any $t ≠ 0$,
4. On $X_{1,0}$ and $X_{2,0}$, $φ_1 ∘ e^* = w_δ ∘ φ_2$ where $w_δ$ is the reflection in $δ$.
5. $(Y, o)$ and $(Y_0, o)$ are nodes of dimension $3$ and $2$ respectively.

Proof. See [Mo, §3 Cor.2] and [Be, pp.143 Remarques]. □

Lemma 2.4. There exists an embedding $j : Y → P^N × D$ such that (1) $π = pr_2 ∘ j$
(2) $φ^1_i(c_1(b^1_i O_{p^N}(1))) = mκ$ for some $m ∈ Z_+$ and any fiber of $π_i$.

Proof. Let $L_t → γ^*_i X$ be the holomorphic line bundle such that $φ^1_i(c_1(L_i)) = κ$ for any fiber of $π_i$. There exists $m ∋ 1$ such that the linear system $|L_t|$ is very ample on $X^1_t$ for $t ≠ 0$, and is base point free and an embedding modulo $C_δ$: $Φ|_{m L_t} × π_i : γ^*_i X ↪ P^N × D$. By construction and Proposition 2.1, we get $Y = (Φ|_{m L_t} × π_i)(γ^*_i X)$ and $L^m = Φ^*|_{m L_t} O_{p^N}(1)$. □

2.2 2-Elementary K3 Surfaces and 2-Elementary Lattices. For a lattice $M$, we denote by $M^\vee$ its dual lattice relative to the quadratic form of $M$. $A(M) := M^\vee / M$ is called the discriminant group. A lattice $M$ is said to be 2-elementary when $A(M) ≅ (Z/2Z)^l(M)$ for some $l(M) ∈ Z_{≥0}$. We denote by $r(M)$ the rank of $M$. Let $δ(M)$ be the parity of the discriminant form ([Ni1,4]). By Nikulin, the triplet $(r, l, δ)$ determines the isometry class of 2-elementary lattices ([Ni1]).

Let $M$ be a primitive hyperbolic 2-elementary lattice of $L_{K3}$. $N$ is also 2-elementary and there exists a natural isomorphism between $A(M)$ and $A(N)$. Consider the sublattice $L' := M ⊕ N$ and the involution $I_M$ on $L'$;

\[(2.6)\]
\[I_M(x, y) = (x, -y) \quad (x ∈ M, y ∈ N).\]

As $I_M$ uniquely extends to an involution of $L_{K3}$, identify $I_M$ with the extended one. By construction, $M$ is the fixed part and $N$ is the anti-fixed part;

\[(2.7)\]
\[M = \{l ∈ L; I_M(l) = l\}, \quad N = \{l ∈ L; I_M(l) = -l\}.\]
$I_M$ induces an involution on $\tilde{\Omega}$ by $I(X, \phi_X, \kappa_X) = (X, I_M \circ \phi_X, \kappa_X)$ where $(X, \phi_X, \kappa_X)$ is a marked polarized $K3$ surface. Let $\Phi : R^2 P_* \mathbb{Z} \to L_{K3}$ be the global trivialization of the second cohomology group. (Thus $\Phi|_{(X, \phi_X)} = \phi_X$.) By the universality, there exists an involution $\iota_X : \mathcal{X} \to \mathcal{X}$ such that the following diagrams commute;

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\iota_X} & \mathcal{X} \\
\downarrow \phi & & \downarrow \phi \\
\tilde{\Omega} & \xrightarrow{I} & \tilde{\Omega}, \quad L_{K3} & \xrightarrow{I_M} & L_{K3}
\end{array}$$

(2.8)

By construction, $\tilde{\Omega}^0_M$ is contained in the fixed locus of $I$. Put $\iota_M := \iota_X|_{\mathcal{X}_M}$. By (2.8), $\iota_M$ is an involution over $X_M^0$ which induces an involution on each fiber $(X, \phi_X)$ such that $\phi_X \circ I_X^* = I_M \circ \phi_X$. In this way, associated to the 2-elementary primitive hyperbolic lattice $M$, any fiber of $X^0_M$ is a $K3$ surface with an anti-symplectic involution. Here, an involution $\iota : X \to X$ is anti-symplectic when $\iota^* \eta_X = - \eta_X$. For a pair $(X, \iota)$ of $K3$ surface and its anti-symplectic involution, put $H^2(X, \mathbb{Z}) = \{ l \in H^2(X, \mathbb{Z}); \iota_X^*(l) = \pm l \}$.

**Definition 2.1.** A pair $(X, \iota_X)$ of a $K3$ surface and an anti-symplectic involution is said to be a 2-elementary $K3$ surface of type $M$ if there exists a marking $\phi_X$ such that $\phi_X \circ I_X^* \circ \phi_X^{-1} = I_M$, or equivalently $\phi_X(H^2(X, \mathbb{Z})^\perp) = M$. Two 2-elementary $K3$ surfaces $(X, \iota)$, $(X', \iota')$ are isomorphic if there exists an isomorphism $f : X \to X'$ such that $f \circ \iota = \iota' \circ f$. Marking as above is said to be a marking of 2-elementary $K3$ surfaces of type $M$.

The following is clear by Theorem 2.2 and above argument. (See [Ni2-4].)

**Theorem 2.3.** $\tilde{\Omega}^0_M$ is the fine moduli space of marked 2-elementary $K3$ surface of type $M$ and $\mathcal{P}_M : (X^0_M, \iota_M) \to \tilde{\Omega}^0_M$ is the universal family.

Let $\Gamma(M)$ (resp. $\Gamma_M$) be the following subgroup of $O(L_{K3})$ (resp. $O(N)$);

$$\Gamma(M) := \{ g \in O(L), I_M \circ g = g \circ I_M \}, \quad \Gamma_M := \{ g|_N; g \in \Gamma(M) \}.$$  

Then, $\Gamma(M)$ acts on $\tilde{\Omega}^0_M$ and $\Gamma_M$ acts on $\Omega^0_M$ by $g \cdot [(\eta, \kappa)] := [(g\eta, g\kappa)]$ and $(g|_N)\eta = g\eta$ respectively. By the global Torelli theorem, two 2-elementary $K3$ surface of type $M$ are isomorphic if their period points in $\tilde{\Omega}^0_M$ lie on the same $\Gamma(M)$-orbit. The following theorem can be proved in the same manner as the proof of weak Torelli theorem and surjectivity theorem for the period map of Enriques surfaces ([Na, Corollary 4.14 and Theorem 7.1]).

**Theorem 2.4.** Via the period map $\pi_M$, $\tilde{\Omega}^0_M / \Gamma(M) = \Omega^0_M / \Gamma_M$.

Since $\Gamma_M$ is an arithmetic subgroup of $O(2, 20 - r(M))$, and $\Omega_M$ is two copies of the symmetric bounded domain of type IV, both $\Omega_M / \Gamma_M$ and $\tilde{\Omega}_M^0 / \Gamma_M$ are quasiprojective algebraic variety by Baily-Borel. Define modular varieties by

$$\mathcal{M}_M := \Omega_M / \Gamma_M, \quad \mathcal{M}_M^0 := \Omega^0_M / \Gamma_M = \tilde{\Omega}^0_M / \Gamma(M).$$

By Theorem 2.4, $\mathcal{M}_M^0$ is the moduli space of 2-elementary $K3$ surfaces of type $M$. To see what happens on the involution along the generic point of discriminant
locus, let \( \pi_i : \gamma_1^*X \to D \) be the same as in Proposition 2.1, and \( I, \iota_X \) as in (2.8). As \( I \circ \gamma_1 = \gamma_2 \), it follows from the universal property of \( \mathcal{P} : \mathcal{X} \to \tilde{\Omega} \) that \( \iota_X \) induces an isomorphism between \( (\pi_1, \gamma_1^*X, D) \) and \( (\pi_2, \gamma_2^*X, D) \). Namely, \( I \circ \pi_1 = \pi_2 \circ \iota_X \).

Thus, by Proposition 2.1, there exist two (rational) maps \( e \) and \( \iota_X \) between \( \gamma_1^*X \) and \( \gamma_2^*X \). Then, \( \iota_1 := e^{-1} \circ \iota_X : \gamma_1^*X \setminus C_1^1 \to \gamma_2^*X \setminus C_1^1 \) extends to a rational involution of \( \gamma_1^*X \) which commutes with the projection.

**Proposition 2.2.** There exists a holomorphic involution \( \iota_Y : Y \to Y \) which commutes with the projection \( \pi : Y \to U \) such that \( \iota_Y = b_1^{-1} \circ \iota_Y \circ b_1 \) on \( \gamma_1^*X \setminus C_1^1 \).

Namely, \( \pi_1 : \gamma_1^*X|_{D \setminus \{0\}} \to D \setminus \{0\} \) extends to the family \( \pi : Y \to D \) by contracting \( C_1^1 \). The central fiber \( Y_0 \) is a generalized K3 surface with one node \( o \) on which \( \iota_Y \) induces an anti-symplectic involution. Moreover, \( o \) is a fixed point of \( \iota_Y \).

**Proof.** Put \( \iota_Y := b_1 \circ \iota_1 \circ b_1^{-1} \). As \( \iota_1 \) is a regular involution over \( \gamma_1^*X \setminus C_1^1 \), \( \iota_Y \) is an involution over \( Y \setminus \{o\} \) where \( o = b_1(C_1^1) \). By putting \( \iota_Y(o) = o \), \( \iota_Y \) extends to an involution over \( Y \). Since \((Y, o)\) is normal by Proposition 2.1, \( \iota_Y \) is regular. \( \square \)

For a 2-elementary K3 surface \((X, \iota)\) of type \( M \), let \( X' \) be the fixed locus; \( X' := \{x \in X; \iota(x) = x\} \) consisting of disjoint union of finitely many smooth curves. Nikulin determined the topological type of \( X' \) ([Ni4]).

**Theorem 2.5.** Let \( M \) be the lattice of Nikulin type \((r, l, \delta)\). Then,

\[
X' = \begin{cases} 
(1) & \emptyset, \quad (r, l, \delta) = (10, 10, 0) \\
(2) & C_1^{(1)} + C_2^{(1)}, \quad (r, l, \delta) = (10, 8, 0) \\
(3) & C^{(g(M))} + \sum_{i=1}^{k(M)} E_i, \quad (r, l, \delta) \neq (10, 10, 0), (10, 8, 0) 
\end{cases}
\]

where \( C^{(g)} \) is a smooth curve of genus \( g \), and \( E_i \) is a smooth \(-2\)-curve. In \( (3) \), \( g(M) = (22 - r - l)/2 \) and \( k(M) = (r - l)/2 \).

Let \( \pi : Y \to U \) and \( \iota_Y \) be the degenerating family of 2-elementary K3 surfaces of type \( M \) considered in Proposition 2.2. Set \( Y_0 \) for the central fiber. Since the embedding dimension of \((Y_0, o)\) is 3 and \( o \) is a fixed point of \( \iota_Y \), \((\iota_Y)_o \) induces an involution on \( C^3 \). As \( \iota_Y \) is anti-symplectic, \( \det((\iota_Y)_o) = -1 \) and \( (\iota_Y)_o \) is expressed by the diagonal matrix with eigenvalues as follows in a suitable coordinates;

\[
(2.11) \quad \text{Type } (0, 3) : \quad (-1, -1, -1), \quad \text{Type } (2, 1) : \quad (1, 1, -1).
\]

As \((Y, o)\) is a node, by a careful look at the morsification procedure, we get the following.

**Proposition 2.3.** Let \( Z \) be the fixed locus of \((Y, \iota_Y)\).

1. In case of type \((0, 3)\), \( o \) is an isolated point of \( Z \) and there exists a neighborhood \( V \) of \( o \) such that \( \pi(x, y, z, t) = t \) and \( \iota_Y(x, y, z, t) = (-x, -y, -z, t) \),
   \[
   (Y|_V, o) = \{(x, y, z, t); xy - z^2 - t^2 = 0\}.
   \]
2. In case of type \((2, 1)\), \( Z \) has a unique node at \( o \) and there exists a neighborhood \( V \) of \( o \) such that \( \pi(x, y, z, t) = t \) and \( \iota_Y(x, y, z, t) = (-x, -y, -z, t) \),
   \[
   (Y|_V, o) = \{(x, y, z, t); xy - z^2 - t^2 = 0\}.
   \]

For a 2-elementary primitive hyperbolic lattice \( M \) and \( \delta \in \Delta(N) \), set \( \langle M \oplus \delta \rangle \) for the smallest 2-elementary primitive hyperbolic lattice containing \( M \) and \( \delta \). It follows from definition that \( \Delta(\langle M \oplus \delta \rangle) = \delta^\perp \cap \Delta(N) \) and \( H_\delta \cap \Omega_M = \Omega_{\langle M \oplus \delta \rangle} \).
Lemma 2.5. For the family $\pi : \mathcal{Y} \to D$ in Proposition 2.2, $\iota^0_0 := b_1^{-1}(1_{\mathcal{Y}|X_0})b_1$ extends to an involution on the minimal resolution $X^1_0$, and $(X_0, \iota^0_0)$ becomes a 2-elementary $K3$ surface of type $(\mathcal{M} \oplus \delta)$.

Proof. By Proposition 2.3, it is clear that $\iota^0_0$ extends to an involution on $X^1_0$. Let $\phi_1$ and $\phi_2$ be the marking as in Proposition 2.1. By Proposition 2.1 (4) and Definition 2.1, we get $\phi_1^{-1}(\iota^0_0)^*\phi_1 = (\phi_1^{-1}\iota^0_0\phi_2) \circ (\phi_2^{-1}(e^{-1})^*\phi_1) = I_M \circ w_\delta$. Since $\delta \perp M$, we get $I_M \circ w_\delta = w_\delta \circ I_M$ and $I_M \circ w_\delta = I_{(M \oplus \delta)}$. □

Let $\mathcal{A}_g := \mathfrak{S}_g/Sp(2g; \mathbb{Z})$ be the Siegel modular variety. (When $g = 0$, $\mathcal{A}_g$ is a point.) We denote by $\mathcal{A}^*_g$ and $\mathcal{M}^*_M$ the Satake-Baily-Borel compactification. Let $C = \sum_{i=1}^l C_i$ be a disjoint union of smooth curves and $\text{Jac}(C_i)$ the Jacobi variety of $C_i$. Put $[\text{Jac}(C)] := ([\text{Jac}(C_1)], \ldots, [\text{Jac}(C_l)]) \in \mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_l}$, where $[\text{Jac}(C_i)]$ is the period of $\text{Jac}(C_i)$. By Theorems 2.4 and 2.5, we can define a morphism $j^0_M : \mathcal{M}^*_M \to \mathcal{A}^*_g(M)$ as follows.

Definition 2.2. $j^0_M : \mathcal{M}^*_M \ni [(X, \iota_X)] \longrightarrow [\text{Jac}(X^r)] \in \mathcal{A}^*_g(M)$. When $(r, l, \delta) = (10, 10, 0)$, $j^0_M$ is defined to be the constant map. When $(r, l, \delta) = (10, 8, 0)$, $j^0_M$ takes its value in $S^2(A_1)$, the second symmetric product of $A_1$.

Proposition 2.4. $j^0_M$ extends to a rational map $j_M : \mathcal{M}^*_M \dashrightarrow \mathcal{A}^*_g(M)$ if $r \leq 17$.

Proof. Since $\mathcal{M}^*_M$ and $\mathcal{A}^*_g(M)$ are projective algebraic varieties, it is enough to show that $j_M$ extends to a morphism from an open subset $V$ of $\mathcal{M}_M$ such that $\text{codim} \mathcal{M}_M \setminus V \geq 2$ because the boundary components of $\mathcal{M}_M$ have codimension $\geq 2$ when $r(M) \leq 17$. Put $V = \mathcal{M}_M^0 \cup \bigcup_{\delta \in \Delta(N)} H^0_\delta / \Gamma_M$. Take a point $\eta \in H^0_\delta$ and its small neighborhood $U$. By Proposition 2.2, we have a family of 2-elementary $K3$ surfaces of type $M$ whose fiber over $U \cap H_\delta$ is a generalized $K3$ surface with an anti-symplectic involution. Let $(X_{\eta}, \iota_{\eta})$ be the corresponding generalized $K3$ surface with involution over $\eta$. Let $X_{\eta}^0$ be the fixed locus and $X_{\eta}'$ its normalization. Define $j_M(X_{\eta}, \iota_{\eta}) := [\text{Jac}(X_{\eta}')] \in \mathcal{A}^*_g(M)$. Since $X_{\eta}^0$ has at most a node, it is well defined and gives an extension of $j_M$ to $U$. As $j_M$ is defined on $U \cap H_\delta$, its extension to $U$ is unique. This proves that $j^0_M$ extends to $V$. □

Proposition 2.5. For $\delta \in \Delta(N)$, $j_M|_{H^0_\delta} = j_{(M, \delta)}$.

Proof. Take a point $\eta_0 \in H^0_\delta$ and its small neighborhood $U$ in $\Omega_M$. Let $\tilde{U} \subset \tilde{\Omega}_M$ be a lift of $U$ such that $\tilde{\Omega}_M$ is dense in $\tilde{U}$. Let $\pi' : \mathcal{Y} \to \tilde{U}$ and $\pi' : \mathcal{Z} \to \tilde{U}$ be the family of 2-elementary $K3$ surfaces and its fixed locus. By the period map, identify $U$ with $\tilde{U}$. Then, over $U \cap H^0_\delta$, a fiber of $\mathcal{Y}$ is a generalized $K3$ surface with an anti-symplectic involution. Take $\tau \in U \cap H^0_\delta$. Let $(Y_\tau, \iota_{\tau})$ be the fiber over $\tau$ and $Z_\tau$ its fixed locus. Let $p : \tilde{Y}_\tau \to Y_\tau$ be the minimal resolution of $Y_\tau$. By Lemma 2.5, $\tilde{Y}_\tau := p^{-1} \circ \iota \circ \tilde{p}$ extends to an involution over $\tilde{Y}_\tau$. Set $\tilde{Z}_\tau$ for the fixed locus of $(\tilde{Y}_\tau, \tilde{\iota}_{\tau})$. By Proposition 2.3, $\tilde{Z}_\tau = Z_\tau + E$ when $(\mathcal{Y}, o)$ is of type $(0, 3)$, and $\tilde{Z}_\tau = \tilde{Z}_\tau$ is the normalization when $(\mathcal{Y}, o)$ is of type $(2, 1)$. By definition, we find that $j_{(M, \delta)} = j|_{H^0_\delta}$ on $U$. □
### §3. An Invariant of 2-Elementary K3 Surfaces

#### 3.1 Determinant Bundles and Quillen Metrics.

Let \((M, g)\) be a compact Kähler manifold on which acts a finite group \(G\) holomorphically and isometrically. Let \(\Box_{0,q}^G\) be the Laplacian restricted to the space of \(G\)-invariant \((0,q)\)-forms and \(\zeta_{0,q}^G(s)\) be the spectral zeta function of \(\Box_{0,q}^G\), which is regular at \(s = 0\). \(\zeta_{0,q}^G(s)\) is nothing but the spectral zeta function of the orbifold \((M/G, g)\).

**Definition 3.1.** The Ray-Singer analytic torsion of \((M/G, g)\) is defined by

\[
\tau(M/G, g) := \prod_{q \geq 0} (\det \Box_{0,q}^G)^{(-1)^q} q, \quad \det \Box_{0,q}^G := \exp \left( - \frac{d}{ds} \bigg|_{s=0} \zeta_{0,q}^G(s) \right).
\]

Let \(\pi : X \rightarrow S\) be a proper smooth morphism of Kähler manifolds on which acts the finite group \(G\) holomorphically. Assume that \(G\) preserves the fiber of \(\pi\). An equivariant determinant of cohomology is a line bundle on \(X/S\) tangent bundle sequel, we assume that a \(G\)-invariant Kähler metric \(g_t := g_{X/S}|_{X_t}\). Via the Hodge theory, the fiber of \(\pi\) is identified with the determinant of \(G\)-invariant harmonic forms which induces the \(L^2\)-metric on \(\lambda(X/S, G)\). We denoted it by \(\| \cdot \|_{L^2_{G}}\).

**Definition 3.2.** The Quillen metric of \(\lambda(X/S, G)\) relative to \(g_{X/S}\) is defined by \(\| \cdot \|_{Q}^2(t) := \tau(T/X, g_{X/S}); \| \cdot \|_{L^2_{G}}(t)\).

Let \(X_g = \{ x \in X; g(x) = x \}\) be the fixed locus of \(g\). Let \(R_{TX_g/S}\) the curvature form of \((TX_g/S, g_{X/S}|_{TX_g/S}), R_{N_{X_g/X}}\) the curvature form of the relative normal bundle \((N_{X_g/X}, g_{X/S}|_{N_{X_g/X}}); \exp(\theta_j)\) an eigenvalue of \(g_{N_{X_g/X}}\). Let \(T_d(A)\) and \(e(A)\) be the Todd and Euler genuses i.e., \(T_d(A) = \det(A(1 - e^{-A})^{-1})\) and \(e(A) = \det(A)\). The following theorems are due to Bismut ([Bi]) and Bismut-Gillet-Soulé ([B-G-S]). (Although detailed proof of Theorem 3.1 is not written in [B-G-S], it can be proved in the similar way as in the case \(G = \{1\}\). Details are left to the reader. We refer to Köhler-Roessler’s paper ([K-R]) for further generalizations.)

**Theorem 3.1.** The Chern form of \(\lambda(X/S, G)_{Q} := (\lambda(X/S, G), \| \cdot \|_{Q})\) is given by

\[
c_1(\lambda(X/S, G)_{Q}) = \frac{1}{|G|} \sum_{g \in G} \pi_*(T_d(T/X, g_{X/S}))(1,1).
\]

Here \(T_d(T/X, g_{X/S})\) is the \(g\)-Todd genus of \((TX/S, g_{X/S})\):

\[
T_d(T/X, g_{X/S}) = T_d \left( \frac{i}{2\pi} R_{TX_g/S} \right) \prod_{j=1}^q (T_d/e) \left( \frac{i}{2\pi} R_{N_{X_g/X}} + i \theta_j \right).
\]

**Theorem 3.2.** Let \(g_{X/S}\) and \(g'_{X/S}\) be \(G\)-invariant Kähler metrics on \(TX/S\). Let \(\| \cdot \|_{Q}\) and \(\| \cdot \|_{Q}'\) be the Quillen metrics of \(\lambda(X/S, G)\) relative to \(g_{X/S}\) and \(g'_{X/S}\). Then,

\[
\log \left( \frac{\| \cdot \|_{Q}'}{\| \cdot \|_{Q}} \right)^2 \geq \frac{1}{|G|} \sum_{g \in G} \pi_*(\overline{T_d(T/X, g_{X/S}, g'_{X/S}))}^{(0,0)}
\]
where $\widetilde{Td}(TX/S; g_{X/S}, g'_{X/S})$ is the Bott-Chern secondary class associated to the $g$-Todd form and $g_{X/S}, g'_{X/S}$.

### 3.2 An Invariant via Analytic Torsion

Let $(X, \iota)$ be a 2-elementary K3 surface with $\iota$-invariant Ricci-flat Kähler metric $\kappa$ and symplectic form $\eta_X$. The space of $L^2(0,q)$-forms splits into $L^2(X, \wedge^{0,q}) = \{ f \in L^2(X, \wedge^{0,q}); \iota^*f = \pm f \}$. Let $\Box_+^{0,q}$ be the Laplacian restricted to $L^2(X, \wedge^{0,q})$. By definition, the Ray-Singer analytic torsion of $(X/\iota, \kappa)$ is given by $\tau(X/\iota, \kappa) = \det \Box_+^{0,2}/\det \Box_0^{0,0}$ because $\zeta_+^{0,0}(s) - \zeta_+^{0,1}(s) + \zeta_+^{0,2}(s) = 0$ where $\zeta_+^{0,q}(s)$ is the zeta function of $\Box_+^{0,q}$. Since $\eta_X$ is a parallel section such that $\iota^*\eta_X = -\eta_X$, multiplication by $\eta_X$ induces an isometry between $(L^2(X, \wedge^{0,0}), \Delta_{-0})$ and $(L^2(X, \wedge^{0,2}), \Delta_{+2})$. Thus, we get the following.

**Lemma 3.1.** $\tau(X/\iota, \kappa) = \det \Box_+^{0,0}/\det \Box_0^{0,0}$.

Let $p : (X, \iota) \to S$ be a family of 2-elementary K3 surfaces of type $M$ and $p : Z \to S$ be the fixed locus. By Theorem 2.5, $Z$ decomposes into $Z = \sum_i C_i$ where $p : C_i \to S$ is a family of curves over $S$. Let $g_{X/S}$ be a family of Ricci-flat Kähler metrics such that $\iota^*g_{X/S} = g_{X/S}$. Set $g_{C_i/S} := g_{X/S}|_{C_i/S}$ for the induced metric on $TC_i/S$ and $g_{N_{C_i/X}}$ for the induced metric on the relative normal bundle $N_{C_i/X}$ which is defined by the following exact sequence;

(3.1) $0 \to TC_i/S \to TX/S|_{C_i} \to N_{C_i/X} \to 0$.

We define four smooth functions over $S$ as follows;

$$\tau(X/\iota, S, g_{X/S})(s) := \tau(X_s/\iota, g_s), \quad \tau(C_i/S, g_{C_i/S})(s) := \tau(C_i,s, g_{C_i,s}),$$

$$\text{vol}(X/\iota, S, g_{X/S})(s) := \text{vol}(X_s, g_s), \quad \text{vol}(C_i/S)(s) := \text{vol}(C_i,s, g_{C_i,s}).$$

By construction, the following is clear.

**Lemma 3.2.**

$$c_1(X, S, g_{X/S})|_{C_i/S} = c_1(C_i/S, g_{C_i/S}) + c_1(N_{C_i/S}, g_{N_{C_i/S}}).$$

**Proposition 3.1.**

$$[\text{Td}(X/S, g_{X/S})]^{(2,2)} = \sum_i \left\{ \frac{1}{8} c_1(X/S, g_{X/S}) c_1(C_i/S, g_{C_i/S}) - \frac{1}{12} c_1(C_i/S, g_{C_i/S})^2 \right\}.$$

**Proof.** Put $x = c_1(C_i/S, g_{C_i/S})$ and $y = c_1(X/S, g_{X/S}) - c_1(C_i/S, g_{C_i/S})$. By definition, $(\text{Td} + e)(y + \pi i) = (1 + \exp(-y))^{-1}$. Since

(3.3) $\text{Td}(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + O(x^3), \quad \frac{\text{Td}}{e}(y + \pi i) = \frac{1}{2} + \frac{y}{4} + O(y^3),$

we get the assertion by the definition of equivariant Todd form in Theorem 3.1 together with (3.3). □

Let $\eta_t$ be a symplectic form on $X_t$ depending holomorphically in $t \in S$ and $\kappa_t$ be a Ricci-flat Kähler metric on $X_t$ depending smoothly in $t$. Put $\|\eta_t\|^2 = \int_{X_t} \eta_t \wedge \eta_t$. [11]
Lemma 3.3.
\[ \kappa_t^2 = f(t) \eta_t \wedge \overline{\eta}_t, \quad f(t) = \frac{\text{vol}(X_t, \kappa_t)}{\|\eta_t\|^2}. \]

Proof. Since \( \kappa_t \) is Ricci-flat, there exists a constant \( C_t \) such that \( \kappa_t^2 = C_t \eta_t \wedge \overline{\eta}_t \). Integrating both hand sides, we get the assertion. \( \square \)

Let \( \Omega_M \) be the period domain of 2-elementary \( K3 \) surfaces of type \( M \). Let \( \omega_M \) be its Bergman metric and \( K_M(z, \overline{z}) \) its Bergman kernel function;

\begin{equation}
\omega_M(z) = \frac{i}{2\pi} \partial \overline{\partial} \log K_M(z, \overline{z}), \quad K_M(z, \overline{z}) = \frac{\langle z, \overline{z} \rangle_N}{|\langle z, l_M \rangle_N|^2},
\end{equation}

where \( l_M \in N_C \) is a vector such that \( H_{l_M} \) is the hyperplane at infinity of \( \mathbb{P}(N_C) \). Let \( \pi : S \to \Omega_M \) be the period map. Since \( \pi^* \omega_M = \frac{i}{2\pi} \partial \overline{\partial} \log \|\eta_t\|^2 \) by Schumacher’s formula ([Sch], [Ti], [To2]), we get the following by taking \( \frac{i}{2\pi} \partial \overline{\partial} \) of Lemma 3.3.

Proposition 3.2.
\[ c_1(X/S, g_{X/S}) = -p^* \left\{ \pi^* \omega_M + \frac{i}{2\pi} \partial \overline{\partial} \log \text{vol}(X/S, g_{X/S}) \right\}. \]

Proposition 3.3.
\[ \frac{1}{2} \left[ p_* Td(X/S, g_{X/S}) + p_* Td_1(X/S, g_{X/S}) \right]^{(1,1)} = \frac{r(M) - 6}{8} \left\{ -\pi^* \omega_M + \frac{i}{2\pi} \partial \overline{\partial} \log \text{vol}(X/S) \right\} - \frac{1}{2} \sum_i p_* Td(C_i/S, g_{C_i/S})^{(1,1)}. \]

Proof. Since \( \chi(X_t) = 24 \), it follows from Proposition 3.2 and the projection formula that
\begin{equation}
\frac{1}{2} Td(X/S, g_{X/S})^{(1,1)} = \frac{1}{24} \left[ p_* c_1(X/S, g_{X/S}) c_2(X/S, g_{X/S}) \right]^{(1,1)} = -\pi^* \omega_M + \frac{i}{2\pi} \partial \overline{\partial} \log \text{vol}(X/S).
\end{equation}

As \( \int_{C_i/t} c_1(C_{i,t}) = 1 - g(C_{i,t}) \), we get by Proposition 3.1 and the projection formula,
\begin{equation}
\frac{1}{2} Td_1(X/S, g_{X/S})^{(1,1)} = p_* \left[ \sum_i \frac{1}{8} c_1(C_i/S, g_{C_i/S}) p^* (-\pi^* \omega_M + \frac{i}{2\pi} \partial \overline{\partial} \log \text{vol}(X/S)) - \frac{1}{12} c_1(C_i/S, g_{C_i/S})^2 \right]
= \sum_i \frac{1}{8} \left\{ -\pi^* \omega_M + \frac{i}{2\pi} \partial \overline{\partial} \log \text{vol}(X/S) \right\} - \sum_i p_* Td(C_i/S, g_{C_i/S})^{(1,1)}.
\end{equation}

It follows from Theorem 2.4 that \( 1/2 + \sum_i (1 - g(C_i))/8 = (r(M) - 6)/8 \) for any \( M \) which, together with (3.5-6), yields the assertion. \( \square \)
Definition 3.3. For a 2-elementary K3 surface of type M with an $i$-invariant Ricci-flat Kähler metric $\kappa$, we define $\tau_M$ by

$$
\tau_M(X, \iota, \kappa) := \langle \kappa, \kappa \rangle^{\frac{14-r(M)}{8}} \tau(X/\iota, \kappa) \prod_i \langle C_i, \kappa | C_i \rangle^{\frac{i}{2}} \tau(C_i, \kappa | C_i)^{\frac{i}{2}}.
$$

Let $\tau_{M/S}$ be the function defined by $\tau_{M/S}(s) = \tau_M(X_s, \iota_s, g_{X_s})$. Let $j_{C_i/S} : S \ni s \to \text{Jac}(C_{i,s}) \in \mathcal{A}_g(C_i)$ be the period map. Let $\omega_{A_g}$ be the Bergman metric of $\mathcal{A}_g$ and $K_{A_g}(\tau)$ the Bergman kernel of $\mathcal{A}_g$;

$$
(3.7) \quad \omega_{A_g}(\tau) = \frac{i}{2\pi} \partial \bar{\partial} \log K_{A_g}(\tau), \quad K_{A_g}(\tau) = \det \text{Im} \tau.
$$

Theorem 3.3.

$$
\frac{i}{2\pi} \partial \bar{\partial} \log \tau_{M/S} \log \tau = -\frac{r(M)}{8} \omega_M - \frac{1}{2} \sum_i j_{C_i/S}^* \omega_{A_g(M)}.
$$

Proof. Consider the family $p : X \to S$. Put $G = \{1, \iota\}$. Since $H^q(X_s, \mathcal{O}_{X_s})^G = 0$ for $q > 0$, it follows that $\lambda(X/S, \iota) = \mathcal{O}_S 1_S$ where $1_S$ is the canonical section of $R^0_{G \bullet} \mathcal{O}_X$ such that $1_S(s) = 1$ in $H^0(X_s, \mathcal{O}_{X_s})^G$. Since the squared Quillen norm of $1_S$ is $\tau(X/\iota/S) \log(X/S)$, it follows from Theorem 3.1 and Proposition 3.3 that

$$
(3.8) \quad \frac{i}{2\pi} \partial \bar{\partial} \log \tau(X/\iota/S) \log(X/S) = -\frac{r(M)}{8} \omega_M - \frac{1}{2} \sum_i j_{C_i/S}^* \omega_{A_g(M)}.
$$

Fix a marking of the family of curves $p : C_i \to S$. Let $T(\text{Jac}(C_{i,s})) \in \mathfrak{S}_{g(C_i)}$ be the period matrix of $\text{Jac}(C_{i,s})$ relative to this marking. Let $T(C_i/S)$ be a function with values in $\mathfrak{S}_{g(C_i)}$ defined by $T(C_i/S)(s) = T(\text{Jac}(C_{i,s}))$. Theorem 3.1 applied to $p : C_i \to S$ yields

$$
(3.9) \quad \frac{i}{2\pi} \partial \bar{\partial} \log \{\tau(C_i/S) \log(C_i/S) \det \text{Im} T(J(C_i/S))\} = p_* \text{Td}(C_i/S, g_{C_i/S})^{(1,1)}.
$$

which, together with (3.8) and $\frac{i}{2\pi} \partial \bar{\partial} \log \det \text{Im} T(C_i/S) = j_* \omega_{A_g(C_i)}$, proves the assertion. \( \square \)

Theorem 3.4. $\tau_M(X, \iota, \kappa)$ does not depend on $\kappa$, and becomes an invariant of a 2-elementary $K3$ surface $(X, \iota)$.

Proof. Let $\kappa_0$ and $\kappa_1$ be two $\iota$-invariant Ricci-flat Kähler metrics of $(X, \iota)$, and set $T(X, \iota, \kappa) := \log T(X, \kappa)^{r(M)} / \log T_M(X, \iota, \kappa)$. Let $1_X \in H^0(X, \mathcal{O}_X) = \lambda(X, \iota)$, $1_{C_i} \in H^0(C_i, \mathcal{O}_{C_i})$ and $\sigma_i \in \det H^0(C_i, \Omega^1_{C_i})$ be the generators of each line. Let $\|\cdot\|_{Q, \kappa_j}$ be the Quillen metric relative to $\kappa_j$. Since the $L^2$-metric on $\det H^0(C_i, \Omega^1_{C_i})$ is independent of a choice of Kähler metric, it follows from Theorem 3.2 that

$$
(3.10) \quad \log \frac{T(X, \iota, \kappa_0)}{T(X, \iota, \kappa_1)} = \log \frac{\|1_X\|_{Q, \kappa_0}^2}{\|1_X\|_{Q, \kappa_1}^2} + \frac{1}{2} \sum \log \frac{\|1_{C_i} \otimes \sigma_i\|_{Q, \kappa_0}^2}{\|1_{C_i} \otimes \sigma_i\|_{Q, \kappa_1}^2}
$$

$$
= \frac{1}{2} \int_X \widetilde{Td}(X; \kappa_0, \kappa_1) + \frac{1}{2} \sum \int_{C_i} \widetilde{Td}_i(X; \kappa_0, \kappa_1) + \sum \int_{C_i} \widetilde{Td}(C_i; \kappa_0, \kappa_1).
$$
Since $\widetilde{Td}(X) = c_1(X)c_2(X)/24$, $\widetilde{Td}_i(X) = \sum_i c_1(X)c_2(C_i)/8 - c_1(C_i)^2/12$, and $\widetilde{Td}(C_i) = c_1(C_i)^2/12$, we get

$$\log \frac{T(X, \iota, \kappa_0)}{T(X, \iota, \kappa_1)} = \frac{1}{48} \int_X c_1c_2(X)(\kappa_0, \kappa_1) + \frac{1}{16} \sum_{C_i} c_1(X)c_1(C_i)(\kappa_0, \kappa_1). \tag{3.11}$$

By Yau’s theorem ([Ya]), there exists a family of Ricci-flat Kähler metrics $\kappa_t$ $(0 \leq t \leq 1)$ joining $\kappa_0$ and $\kappa_1$. Let $\eta$ be a fixed symplectic form on $X$. As in Lemma 3.3, put $f(t) = \text{vol}(X, \kappa_t)/\|\eta\|^2$. Let $L_t = \kappa_t^{-1}\hat{\kappa}_t$ be a section of $\text{End}(TX)$ such that $\kappa_t(L_tu, v) = \hat{\kappa}_t(u, v)$. By Lemma 3.3, we get $\text{Tr} \kappa_t^{-1}\hat{\kappa}_t = \partial_t \log f(t)$ which, together with the Bott-Chern formula ([B-C], [B-G-S, I, e]), yields

$$\int_X c_1c_2(X)(\kappa_0, \kappa_1) = \int_X \int_1^0 dt \frac{d}{d\epsilon} c_1(R_t + \epsilon\kappa_t^{-1}\hat{\kappa}_t)c_2(R_t + \epsilon\kappa_t^{-1}\hat{\kappa}_t)
= \int_1^0 dt \frac{d}{dt} \log f(t) \int_X c_2(R_t) = 24 \log \frac{\text{vol}(X, \kappa_0)}{\text{vol}(X, \kappa_1)}, \tag{3.12}$$

$$\int_{C_i} c_1(X)c_1(C_i)(\kappa_0, \kappa_1) = \int_{C_i} \int_1^0 dt \frac{d}{d\epsilon} c_1(R_t + \epsilon\kappa_t^{-1}\hat{\kappa}_t)c_1 \left( R(C_i) + \epsilon\frac{\hat{\kappa}_t|C_i}{\kappa_t|C_i} \right)
= \int_1^0 dt \frac{d}{dt} \log f(t) \int_{C_i} c_1(R_t(C_i)) = \chi(C_i) \log \frac{\text{vol}(X, \kappa_0)}{\text{vol}(X, \kappa_1)} \tag{3.13}$$

where $R_t$ is the curvature of $(X, \kappa_t)$ and $R_t(C_i)$ of $(C_i, \kappa_t|C_i)$. By (3.11-13), we get

$$\log \frac{T(X, \iota, \kappa_0)}{T(X, \iota, \kappa_1)} = \frac{r(M) - 6}{8} \log \frac{\text{vol}(X, \kappa_0)}{\text{vol}(X, \kappa_1)} \tag{3.14}$$

which, together with the definition of $T(X, \iota, \kappa)$, yields the assertion. □

Let $\tau_M$ be the function on $\tilde{\Omega}_M^0$, defined by $\tau_M(X, \phi, \iota) := \tau_M(X, \iota)$. By Theorems 3.3 and 3.4, $\tau_M$ is a smooth $\Gamma(M)$-invariant function on $\tilde{\Omega}_M^0$, and thus descends to a $\Gamma_M$-invariant smooth function (denoted by the same symbol $\tau_M$) on $\Omega_M^0$ by Theorem 2.4. Applying Theorem 3.3 to the universal family, we get the following.

**Theorem 3.5.** $\tau_M$ descends to a smooth $\Gamma_M$-invariant function on $\Omega_M^0$ and satisfies the following variational formula:

$$\frac{i}{2\pi} \bar{\partial} \partial \log \tau_M = - \frac{r(M) - 6}{8} \omega_M - \frac{1}{2} j_M^* \omega A_M(M).$$
§4. Degeneration of K3 Surfaces and Monge-Ampère Equation

4.1 Apriori Estimates for the Monge-Ampère Equation. Let \((X, \kappa)\) be a compact Kähler surface. Let \(F \in C^\infty(X)\) be a given function and \(\varphi \in C^\infty(X)\) satisfies the following complex Monge-Ampère equation;

\[
(\kappa + \frac{i}{2\pi} \partial \bar{\partial} \varphi)^2 = e^F \kappa^2, \quad \int_X \varphi \kappa^2 = 0, \quad \kappa + \frac{i}{2\pi} \partial \bar{\partial} \varphi > 0.
\]

Put \(\kappa' := \kappa + \frac{i}{2\pi} \partial \bar{\partial} \varphi\) for a new Kähler metric. Let \(\Delta\) (resp. \(\Delta'\)) be the Laplacian relative to \(\kappa\) (resp. \(\kappa'\)). Let \(R = (R_{ij\bar{k}\bar{l}})\) be the holomorphic bisectional curvature of \((X, \kappa)\). Let \(V\) be the volume of \((X, \kappa)\), \(\lambda > 0\) the first eigen value of \(\Delta\), and \(S\) the Sobolev constant of \((X, \kappa)\); \(\|f\|_{L^4} \leq S(\|df\|_{L^2} + \|f\|_{L^2})~(\forall f \in C^\infty(X))\). For \(f \in C^0(X)\), we denote by \(\|f\|_\infty\) the sup-norm.

Proposition 4.1.

1. If \(|F|_\infty \leq 1, |\varphi|_\infty \leq C(S, \lambda^{-1}, V)|F|_\infty\) where \(C(x, y, z)\) is bounded if all of \(x, y, z\) are bounded.
2. There exist \(C_i = C_i(|\Delta F|_\infty, |R|_\infty, |F|_\infty, |R|_\infty, |F|_\infty, S, \lambda^{-1}, V)\) \((i = 1, 2)\) such that \(C_i(x, y, z, w, s, t)\) is bounded from above and below if all of \(x, y, z, w, s, t\) are bounded, and the following inequality holds on \(X\); \(C_1 \kappa \leq \kappa' \leq C_2 \kappa\).

Proof. See [Ko, pp.298-299] for (1) and [Ya, pp.350-351, pp.359 l.2 2-28], [Ko, pp.300-302] for (2). □

Proposition 4.2. Let \(\varphi \in C^\infty(\mathbb{B}(2r))\) satisfies the Monge-Ampère equation;

\[
\det(g_{ij} + \frac{i}{2\pi} \varphi_{ij}) = e^\varphi \det(g_{ij}), \quad \varphi_{ij} = \partial^2 \varphi / \partial z_i \partial \bar{z}_j \text{ on the ball of radius } 2r \text{ in } \mathbb{C}^2,
\]

and suppose \(\lambda(\delta_{ij}) \leq (g_{ij} + \frac{i}{2\pi} \varphi_{ij}) \leq \Lambda(\delta_{ij})\) over \(\mathbb{B}(2r)\). Then, there exist constants \(\alpha = \alpha(\lambda, \Lambda, r) > 0\) and \(C = C(\lambda, \Lambda, r, |\partial \bar{\partial} \varphi|_{C^0(\mathbb{B}(2r))}, |F|_{C^2(\mathbb{B}(2r))}, |g_{ij}|_{C^2(\mathbb{B}(2r))}) \geq 0\) such that \(|\varphi|_{C^{2, \alpha}(\mathbb{B}(r))} \leq C\).

Proof. See [Si, Chap.2, §4]. □

4.2 Construction of Approximate Ricci-Flat Metrics. Let \(\pi : Y \to D\) be the degenerating family of K3 surfaces over the disc \(D\) considered in Proposition 2.1 whose fiber is denoted by \(Y_t\). By Proposition 2.1 (5), there exists a coordinate neighborhood \((V, o)\) in \(Y\) such that

\[
(V, o) \cong \{(x, y, z, t) \in \mathbb{B}(2); xy - z^2 - t^2 = 0\}, \quad \pi(x, y, z, t) = t
\]

where \(\mathbb{B}(r)\) is the ball of radius \(r\) centered at 0. Let \(L\) be a very ample line bundle over \(Y\) as in Lemma 2.4 and put \(L_t := L|_{Y_t}\). Since \(L\) is very ample, we may assume that \(Y\) is a closed subvariety of \(\mathbb{P}^N \times D\) with \(\pi = pr_2 \circ o\) and \(L = O_{\mathbb{P}^N}(1)|_Y\). Let \(H\) be a hyperplane of \(\mathbb{P}^N\) such that \(i(o) \not\subset H\). Let \(\sigma\) be the section of \(L\) such that \((\sigma)_0 = H \cap Y\). Let \(h_L\) be a Hermitian metric of \(L\) with the following properties;

(P2) there exist open subsets \(W' \subset W \subset V\) such that \(h_L(\sigma, \sigma) \equiv 1\) on \(W'\), and \(c_1(L, h_L) := -\frac{i}{2\pi} \partial \bar{\partial} \log h_L(\sigma, \sigma) = \omega_{FS}\) on \(X\setminus W\) where \(\omega_{FS}\) is the restriction of the Fubini-Study form of \(\mathbb{P}^N\).

Let \(\kappa_{Y_t}\) be the Ricci-flat Kähler metric on \(Y_t\) cohomologous to \(c_1(L_t)\). By Kobayashi-Todorov ([K-T]), \(\kappa_{Y_0}\) is a Ricci-flat Kähler metric on \(Y_0\) in the sense
of orbifold. Let \( \eta_{Y_t} \in H^0(Y_t, \Omega^2) \) be a symplectic form on \( Y_t \) depending holomorphically in \( t \in D \). By a suitable choice of \( \eta_{Y_t} \), we may assume the following:

\[
(4.3) \quad \kappa_{Y_t}^2 = h(t) \eta_{Y_t} \wedge \bar{\eta}_{Y_t}, \quad h(t) = \deg L/\|\eta_{Y_t}\|^2, \quad h(0) = 1.
\]

Fix an isomorphism \( j : (\mathbb{C}^2/ \pm 1 \cap \mathbb{B}(2), 0) \to (Y_0 \cap V, o); \)

\[
(4.4) \quad j(z_1, z_2) = (z_1^2, z_2, z_1 z_2) \in Y_0.
\]

We denote by \((r, \sigma)\) the polar coordinates of \( \mathbb{C}^2; r = \|z\| \) and \( \sigma = z/\|z\| \in S^3 \). By [K-T, Theorem 1], we get the following.

**Lemma 4.1.** There exist \( u_0 \in C^\infty(Y_0), v \in C^\omega(\mathbb{B}(2)) \) and \( c > 0 \) such that \( \kappa_0 = c_1(L_0, e^{-u_0} h_L) \) and \( j^* u_0(z) = c(\|z\|^2 + v(z)) \). Here \( v \) has the expansion:

\[
v(z) = \sum_{k=2} a_{2k}(\sigma) r^{2k} = \sum a_{I,J} z^I \bar{z}^J \quad \text{where} \quad a_{2k}(\sigma) = r^{-2k} \sum |\cdot|+|\cdot|=2k a_{I,J} z^I \bar{z}^J.
\]

For simplicity, we assume \( c = 1 \) in the sequel. (General case is easily obtained by small modifications of this case.) Let \( D(\delta) \) be the disc of radius \( \delta(\ll 1) \) and fix a smooth trivialization over \( D(\delta) \); \( I : (Y_0 \backslash W) \times D(\delta) \cong Y \backslash W \) such that \( i_t := I(\cdot, t) : Y_0 \backslash W \cong Y_t \backslash W \) is a diffeomorphism for any \( t \). Set

\[
(4.5) \quad u_t := (i_t^{-1})^* u_0 \in C^\infty(Y_t \backslash W), \quad h_t := e^{-u_t} h_L.
\]

Then, \( c_1(L_t, h_t) \) is a \((1,1)\)-form on \( Y_t \backslash W \) approximating \( \kappa_0 \).

**Proposition 4.3.** There exist \( \epsilon, C > 0 \) with the following properties. If \( \delta > 0 \) is chosen small enough, then for any \( |t| < \delta \), one has

(1) \( c_1(L_t, h_t) \geq \epsilon c_1(L_t, h_L) \) on \( Y_t \backslash W \),

(2) \( |\ast_t c_1(L_t, h_t)^2 - 1| \leq C |t| \) on \( Y_0 \backslash W \).

Here, \( \ast_t \) denotes the Hodge \(*\)-operator relative to \( c_1(L_t, h_L) = \omega_{FS} \).

**Proof.** By construction, there exist \( \alpha, \beta, \gamma \in C^\infty(D(\delta) \times (Y_0 \backslash W)) \) such that

\[
(4.6) \quad i_t^* c_1(L_t, h_t) \wedge c_1(L_t, h_L) = \alpha, \quad i_t^* c_1(L_t, h_t)^2 = \beta, \quad \ast_0 i_t^* c_1(L_t, h_t)^2 = \gamma.
\]

By definition, we get \( \alpha(z, 0) \geq 3 \epsilon_0, \beta(z, 0) \geq 9 \epsilon_0^2 \) and \( \gamma(z, 0) = 1 \). As \( \alpha, \beta, \gamma \) are continuous in \( t \), if \( |t| \ll 1 \), we get \( \alpha(t, z) \geq 2 \epsilon_0, \beta(t, z) \geq \epsilon_0^2 \) and \( |\gamma(t, z) - 1| \leq C |t| \), from which the assertion follows. \( \square \)

**Remark.** In the sequel of this section, we denote by \( C(>0) \) a constant independent of \( s \in [0, 1], t \in D \), and \( x \in Y_t \), though its value may change in each estimate.

Our next task is to construct an approximating family of Kähler metrics on \( W \). Put \( X := \{(x, y, z, t) \in \mathbb{C}^4; xy - z^2 - t^2 = 0\} \) and \( \pi(x, y, z, t) = t \). The following is due to Kronheimer ([Kr]).

**Proposition 4.4.** There exists \( q \in C^0(X) \cap C^\infty(X \backslash \{0\}) \) such that

(1) \( j^* q_0 = \partial \bar{\partial} \|z\|^2 \) where \( j : \mathbb{C}^2/ \pm 1 \to X_0 \) is the map defined by \( (4.4) \),

(2) \( \partial \bar{\partial} q_t \) is a Ricci-flat ALE metric on \( X_t \) where \( q_t := q|X_t, X_t := \pi^{-1}(t) \),

(3) \( q \) is homogeneous of order 1; \( q(sx, sy, sz, st) = |s| q(x, y, z, t) \).

**Remark.** A Riemannian manifold \((M, g)\) is ALE if there exists a compact subset \( K \) of \( M \), a finite group \( \Gamma \subset O(m, \mathbb{R}) \) and a diffeomorphism \( f : \mathbb{R}^m \backslash \mathbb{B}(R)/\Gamma \to M \backslash K \) such that \( f^* g = \delta + O(r^{-2-k}) \) for some \( k > 0 \) where \( \delta \) is the Euclidean metric.
By (4.3), we identify a neighborhood $V$ of $o$ in $\mathcal{Y}$ with a neighborhood around 0 of $\mathcal{X}$. Thus $Y_t \cap V = \mathcal{X}_t \cap \mathbb{B}(2)$ and we regard $q_t$ to be a function on $Y_t \cap V$. Put $S := \{ x \in \mathcal{X}; q(x) = 1 \}$ and $S_t := S \cap \mathcal{X}_t$ for the level set of $q$, and $S_{<\delta} := \bigcup_{|t|<\delta} S_t$ for the sublevel set. We consider $S_t$ to be a subset in $\mathbb{C}^3$. As $\pi : S_{<\delta} \to D(\delta)$ is of maximal rank when $\delta \ll 1$, we can construct a trivialization of $S_{<\delta}$ by integrating vector fields $\xi, \zeta$ on $S_{<\delta}$ such that $\pi_x \xi = \partial / \partial u$ and $\pi_x \zeta = \partial / \partial v$ where $t = u + i v$ is the coordinate of $D(\delta)$. 

**Lemma 4.2.** There exists a trivialization $\Psi : S_0 \times D(\delta) \to S_{<\delta}$ with the property that $\Psi(\cdot, t) : S_0 \to S_t$ is a diffeomorphism and that $\Psi(\cdot, 0)$ is the identity map on $S_0$. We put $\psi_t(\cdot) := \Psi(\cdot, t)$ and $\phi_t = \psi_t^{-1} : S_t \to S_0$ for the inverse map.

With this identification of $S_t$ with $S_0$, we can construct a good deformation of the Kähler potential by using the polar coordinates in Lemma 4.1. Namely, if $u_0 = q_0 + \sum_{k>2} a_{2k}(\sigma) q_0^k$ is the expansion in the polar coordinates, then

$$u_t = q_t + \sum_{k>2} (\phi_t^* a_{2k}) q_t^{2k}$$

will be a good approximation of $u_0$. Let us verify it in the sequel.

Set $Y_{t,a} := \{ x \in Y_t \cap V; q_t(x) \geq a \}$. We extend $\phi_t : Y_{t,\delta-1|t|} \to Y_{0,\delta-1|t|}$ and $\psi_t : Y_{0,\delta-1|t|} \to Y_{t,\delta-1|t|}$ by using the $\mathbb{R}_+$-action. Namely, we define

$$\phi_t(x) := q_t(x) \cdot q_{\psi_t(x)^{-1}t}(q_t(x)^{-1}x), \quad \psi_t(x) := q_0(x) \cdot q_{\phi_t(x)^{-1}t}(q_0(x)^{-1}x)$$

where $\lambda \cdot x = (\lambda x, \lambda y, \lambda z)$ if $x = (x, y, z) \in \mathbb{C}^3$. Put $\mu_\lambda(x) := \lambda \cdot x$. Let $J_\lambda$ be the complex structure of $Y_t$. Let $\bar{\partial}_{Y_t}$ be the $\bar{\partial}$-operator of $Y_t$. Via $\psi_t$, identify $\bar{\partial}_{Y_t}$ with the $\bar{\partial}$-operator of $(Y_{0,\delta-1|t|} \cap V, \psi_t^* J_t)$ and similarly $\bar{\partial}_{Y_t}$ with the $\bar{\partial}$-operator. We denoted them by $\bar{\partial}_t$ and $\bar{\partial}_t$ respectively;

$$\bar{\partial}_t := \psi_t^* \circ \bar{\partial}_{Y_t} \circ \phi_t^*, \quad \bar{\partial}_t := \psi_t^* \circ \bar{\partial}_{Y_t} \circ \phi_t^*, \quad \bar{\partial}_t \bar{\partial}_t = \psi_t^* \circ \bar{\partial}_{Y_t} \bar{\partial}_{Y_t} \circ \phi_t^*.$$

Take $f \in C^\infty(Y_0 \cap V)$ and identify it with an even function on $\mathbb{C}^2 \cap \mathbb{B}(2)$ via $j$. Let $(z_1, z_2)$ be the complex coordinates of $\mathbb{C}^2$, and $z_1 = x + i x_2, z_2 = x + i x_4$ be the real coordinates. For $z \in Y_{0,\delta-1|t|} \cap V$, we can write

$$\bar{\partial}_t \bar{\partial}_t f(z) = \sum a_{ij}^{kl}(t, z) \bar{\partial}_k f(z) dx_i \wedge dx_j + \sum b_{ij}^k(t, z) \partial_k f(z) dx_i \wedge dx_j$$

where $a_{ij}^{kl}$ and $b_{ij}^k$ restricted to $D(\delta) \times S_0$ are $C^\infty$-functions, and $\partial_i f = \partial f / \partial x_i$ etc.

**Lemma 4.3.** Putting $\|z\| = r$, one has the following in the polar coordinates:

$$\bar{\partial}_t \bar{\partial}_t f(z) = \sum \left\{ a_{ij}^{kl}(r^2 t, \sigma) \bar{\partial}_k f(z) + r^{-1} b_{ij}^k(r^{-2} t, \sigma) \partial_k f(z) \right\} dx_i \wedge dx_j.$$

**Proof.** As $\mu_\lambda j(z) = j(\sqrt{\lambda} z)$ by (4.5), the action of $\mathbb{R}_+$ on $Y_0 \cap V$ is expressed by $\mu_\lambda(z) = (\sqrt{\lambda} z_1, \sqrt{\lambda} z_2)$. Since $\mu_\lambda \circ \mu_\lambda = \mu_\lambda \circ \phi_t$ on $\mathbb{R}_+ \times Y_{t,\delta-1|t|}$ by definition, we get $\bar{\partial}_t \bar{\partial}_t f = \mu_\lambda^{-1} \circ \partial_t \partial_t \circ \phi_t^*$ which yields $a_{ij}^{kl}(\lambda^{-1} t, \lambda^{-1/2} z) = a_{ij}^{kl}(t, z)$ and $\lambda^{-1/2} b_{ij}^k(\lambda^{-1} t, \lambda^{-1/2} z) = b_{ij}^k(t, z)$. Putting $\lambda = r$, we get the assertion. \(\square\)
Let $\omega_t$ be the Kronheimer’s Ricci-flat ALE metric on $Y_t \cap V$;

$\omega_t := \partial_t \bar{\partial}_t q_t.$

As $\psi_t q_t = q_0$, we find that $\omega_t := \partial_t \bar{\partial}_t q_0$ is the Kähler form on $(Y_{0,\delta^{-1}|t|} \cap V, \psi_t^* J_t)$.

Taking $f(z) = ||z||^2$ in (4.10), it follows that, for any $z \in Y_{0,\delta^{-1}|t|} \cap V$,

$\omega_t(z) = \sum g_{ij}(r^{-2}t, \sigma) \, dx_i \wedge dx_j$

where $g_{ij}$ is a smooth function on $D(\delta) \times S_0$. Let $\rho$ be a cut-off function such that $\rho(s) \equiv 0$ for $s \leq 2\delta^{-1}$, $\rho(s) \equiv 1$ for $s \geq 4\delta^{-1}$, $0 \leq \rho'(s) \leq C_0\delta$ and $|\rho''(s)| \leq C_0\delta^2$.

Let $v \in C^\omega(B(1))$ be the error term of $u_0$ appeared in Lemma 4.1. Set

$v_t(x) := \rho_t(x) \phi^*_t v(x), \quad \rho_t(x) := \rho(|t|^{-1}q_t(x)), \quad \bar{v}_t := \rho(|t|^{-1}r^2) v = \psi_t^* v_t.$

Then, $v_t, \rho_t \in C^\infty(Y_t)$ and $\bar{v}_t \in C^\infty(Y_0)$. By (4.8), we get $\psi_t^* \partial_t \bar{\partial}_t v_t = \partial_t \bar{\partial}_t \bar{v}_t$.

**Lemma 4.4.** Let $*_t$ be the Hodge $*$-operator relative to $\omega_t$. There exist $B_i \in C^\infty(D(\delta) \times [0,1] \times S_0)$ $(i = 1, 2)$ such that, for any $z \in Y_{0,\delta^{-1}|t|} \cap V$,

$\quad *_t (\partial_t \bar{\partial}_t \bar{v}_t \wedge \omega_t)(z) = r^2 B_1(r^{-2}|t|, r^2, \sigma), \quad \quad *_t (\partial_t \bar{\partial}_t \bar{v}_t)^2(z) = r^4 B_2(r^{-2}|t|, r^2, \sigma).$

**Proof.** As $v \in C^\omega([0,1] \times S^3)$ by Lemma 4.1, it follows from (4.12) and

$\begin{align*}
\frac{r^{-1}}{\partial_t} \{ \rho(|t|^{-1}r^2) v \} &= 2|t|^{-1}r^{-1} x_i \rho'(|t|^{-1}r) v + r^{-1} \rho(|t|^{-1}r^2) \partial_t v, \\
\partial_t \{ \rho(|t|^{-1}r^2) v \} &= 4|t|^{-2} \rho''(|t|^{-1}r) x_i x_j v + 2|t|^{-1}r^2 \rho'(|t|^{-1}r^2) \delta_{ij} v \\
&+ 2|t|^{-1} \rho'(|t|^{-1}r^2) (x_j \partial_t v + x_i \partial_j v) + \rho(|t|^{-1}r^2) \partial_{ij} v
\end{align*}$

that there exists $A_i \in C^\infty(D(\delta) \times [0,1] \times S_0)$ such that $\omega_t^2(z) = A_0(r^{-2}|t|, r^2, \sigma) \, dV$, $\partial_t \bar{\partial}_t \bar{v}_t \wedge \omega_t(z) = r^2 A_1(r^{-2}|t|, r^2, \sigma) \, dV$ and $(\partial_t \bar{\partial}_t \bar{v}_t)^2(z) = r^4 A_2(r^{-2}|t|, r^2, \sigma) \, dV$ where $dV$ is the volume form of $(Y_0, \omega_0)$. Since $\omega_0^2 = dV$, we get $A_0(0, r, \sigma) \equiv 1$ for any $(r, \sigma) \in [0,1] \times S_0$. By the compactness of $[0,1] \times S_0$, there exists $C_0$ such that $0 < C_0 \leq A_0(s, r, \sigma) \leq C_0 < \infty$ for $|s| < \delta$ and $(r, \sigma) \in [0,1] \times S_0$. Since $*_t F = F/\omega_t^2$ for a 4-form $F$, we get $B_1 = A_1/A_0$ and $B_2 = A_2/A_0$. □

We define $\Omega_t$ and $G_{Y_t}$ as follows;

$\Omega_t := \omega_t + \partial_t \bar{\partial}_t v_t, \quad G_{Y_t} := \Omega_t^2/\omega_t^2.$

As before, set $\Omega_t := \psi_t^* \Omega_t = \omega_t + \partial_t \bar{\partial}_t v_t$ and $G_t := \psi_t^* G_{Y_t}$.

**Proposition 4.5.** If $|t| \ll 1$, $\Omega_t$ becomes a Kähler metric on $Y_t \cap V$.

**Proof.** When $q_t(x) \leq \delta^{-1}|t|$, $v_t \equiv 0$ by (4.13) and $\Omega_t > 0$ because $\Omega_t = \omega_t$ in this case. Thus, it is enough to show that $*_t \Omega_t \wedge \omega_t(z) > 0$ and $*_t \Omega_t^2(z) > 0$ for any $z \in Y_{0,\delta^{-1}|t|} \cap V$ if $|t| \ll 1$. By Lemma 4.4, we get

$\begin{align*}
*_t \Omega_t \wedge \omega_t(z) &= 1 + r^2 B_1(|t|/r^2, r^2, \sigma), \quad *_t \Omega_t^2 = 1 + 2r B_1(|t|/r^2, r^2, \sigma) + r^2 B_2(|t|/r^2, r^2, \sigma).
\end{align*}$
As $1 + r^2 B_1(0, r^2, \sigma) \geq C > 0$ for any $z \in B(1) = Y_0 \cap V$ because $\Omega_0$ is a Kähler metric, choosing $t$ small enough, the right hand side of the first formula of (4.16) is greater than $C/2$. Similarly, the rest inequality can be shown. □

Let $\tau_{Y_t}$ be the holomorphic family of symplectic forms on $Y_t \cap V$ such that $\tau_{Y_t} \wedge \bar{\tau}_{Y_t} = \omega_{Y_t}^2$. Under the identification of $Y_t \cap V$ with $\mathcal{X}_t \cap \mathbb{B}(2)$, we get $\omega_{Y_t} = g_{Y_t}(I_t, \cdot, \cdot) + \sqrt{-1}g_{Y_t}(K_t, \cdot, \cdot)$ where $g_{Y_t}$ is the underlying Riemannian metric and $(I_t, J_t, K_t)$ are complex structures defining the hyper-Kähler structure. Since $(V, \omega)$ is normal, there exists a holomorphic function $f_V \in \mathcal{O}(V)$ such that $\eta_Y|V = f_Y \cdot \tau_Y$ and $f_Y := f_V|Y_t$. As $f_Y$ has no zero for any $t \in B(\delta)$, $f_V$ has no zero on $V$. Thus there exists the lower bound on $V$; $|f_V(x)| \geq C > 0$.

By (4.3), (4.15) and the definition of $f_Y$, we get

$$|f_Y|^2 = (\eta_{Y_0} \wedge \bar{\eta}_{Y_0})/(\tau_{Y_0} \wedge \bar{\tau}_{Y_0}) = \Omega_{Y_0}^2/\omega_{Y_0}^2 = G_0.$$ (4.17)

Let $u_t$ be the function on $Y_t \setminus W$ as in (4.6) and $h_L$ be the Hermitian metric of $L$ with the property (P2) as before. Let $\chi \geq 0$ be a cut-off function on $V$ such that $\chi(x) \equiv 0$ for $x \in W$ and $\chi(x) \equiv 1$ for $x \in V \setminus Y$. We shall use the following (1,1)-form $\tilde{\kappa}_{Y_t}$ as a background metric in approximating $\kappa_{Y_t}$;

$$\tilde{\kappa}_{Y_t} := c_1(L_t, e^{-\theta_t} h_L) = \frac{i}{2\pi} \partial Y_t \bar{\partial} Y_t \{\theta_t - \log h_L(\sigma, \sigma)\}, \quad \theta_t := \chi u_t + (1 - \chi)(q_t + v_t).$$ (4.18)

**Proposition 4.6.** If $|t| \ll 1$, $\tilde{\kappa}_{Y_t}$ is a Kähler metric with the following estimate;

$$\left| \frac{\eta_{Y_t} \wedge \bar{\eta}_{Y_t}}{\tilde{\kappa}_{Y_t}^2} - 1 \right| \leq C |t|.$$ (4.19)

**Proof.** (1) Suppose $x = \psi_t(z) \in Y_t, \delta - |t| \cap V$. By definition, one has

$$\eta_{Y_t} \wedge \bar{\eta}_{Y_t}/\Omega_{Y_t}^2 = (\eta_{Y_t} \wedge \bar{\eta}_{Y_t}/\omega_{Y_t}^2) \cdot (\Omega_{Y_t}^2/\omega_{Y_t}^2)^{-1} = |f_{Y_t}|^2 G_{Y_t}^{-1}. $$ (4.18)

As $G_t(z) = 1 + r^2 B(\frac{|z|}{r^2}, r^2, \sigma)$ where $B(s, r^2, \sigma) := B_1(s, r^2, \sigma) + r^2 B_2(s, r^2, \sigma)$,

$$|G_t(z) - G_0(z)| \leq r^2 |B(s, r^{-2}|t|, \sigma) - B(s, 0, \sigma)| \leq |dB|_\infty |t| \leq C |t|.$$ (4.20)

Let $\| \cdot \|$ be the Euclidean norm of $\mathbb{C}^3$. Then, we get

$$\|f_V(\psi_t(z))\|^2 - |f_{Y_0}(z)|^2 \leq |f_V|_\infty |df_V|_\infty \|\psi_t(j(z)) - j(z)\| \leq C |t|.$$ (4.21)

which, together with (4.19-20), yields that

$$\left| \left| \psi_t^* \frac{\eta_{Y_t} \wedge \bar{\eta}_{Y_t}}{\Omega_{Y_t}^2} (z) - 1 \right| \right| \leq \left| \psi_t^* |f_V|^2 - G_t \right| \leq \left| \psi_t^* |f_V|^2 - |f_{Y_0}|^2 \right| + \left| G_t - G_0 \right| \leq \frac{C |t|}{|G_0| - C |t|}.$$ (4.22)

Since both $\Omega_{Y_0}$ and $\omega_{Y_0}$ are Kähler metrics on $Y_0 \cap V$, there exists $C > 0$ such that $|G_0| - C |t| \geq C$ on $Y_0 \cap V$, which, together with (4.22), yields the assertion.
(2) If \( q_t(x) \leq \delta^{-1}|t| \), it follows from (4.13) and (4.15) that \( \Omega_{Y_t}(x) = \omega_{Y_t}(x) \) and

\[
\eta_{Y_t} \wedge \bar{\eta}_{Y_t}(x)/\Omega_{Y_t}^2(x) = \eta_{Y_t} \wedge \bar{\eta}_{Y_t}(x)/\omega_{Y_t}^2(x) = |f_{Y_t}(x)|^2.
\]

By (4.17), we get \( f_{Y_t}(0) = 1 \) because \( \Omega_0(0) = \omega_0(0) \) by construction which, together with (4.23), yields

\[
\left| \frac{\eta_{Y_t} \wedge \bar{\eta}_{Y_t}}{\Omega_{Y_t}^2}(x) - 1 \right| = \left| |f_V(x)|^2 - |f_V(0)|^2 \right| \leq |f_V|_{\infty} \|df_V\|_{\infty} \|x\|.
\]

By Proposition 4.4 (3), one has \( C^{-1}(\|x\| + |t|) \leq q_t(x) \leq C(\|x\| + |t|) \) which, together with \( q_t(x) \leq \delta^{-1}|t| \), implies \( \|x\| \leq C|t| \) and yields the assertion.

(3) Consider \( Y_t \setminus W \). By construction (Lemma 4.1 and (4.15)), we get \( u_0 = g_0 + v_0 \) on \( V \). Therefore, \( \bar{\kappa}_{Y_0} = \kappa_{Y_0} \) and there is a smooth function \( w \) on \( V \setminus W \) such that \( u_t - (q_t + v_t)|_{V \setminus W} = t w|_{Y_t \cap (V \setminus W)} + t \bar{w}|_{Y_t \cap (V \setminus W)} \). Thus there exist (1,1)-forms \( \xi \) and \( \xi' \) on \( V \setminus W \) such that \( \bar{\kappa}_{Y_t}|_{V \setminus W} = c_1(L_t, h_t) + t \xi + t \xi' \) which, together with Proposition 4.4, implies that \( \bar{\kappa}_{Y_t} \) is positive on \( Y_t \cap (V \setminus W) \). As \( \bar{\kappa}_{Y_t} > 0 \) on \( (Y_t \setminus V) \cup (Y_t \cap W) \) by Propositions 4.4 and 4.5, it becomes a Kähler metric on \( Y_t \). Now, the desired estimate follows from Propositions 4.3 and 4.6. □

If \( |t| \ll 1 \), the following extension of \( \omega_{Y_t} \) becomes a Kähler metric on \( Y_t \);

\[
\omega_{Y_t} := c_1(L_t, e^{-w_t}h_L) = \frac{i}{2\pi} \partial_Y \bar{\partial}_{Y_t} \{ w_t - \log h_L(\sigma, \sigma) \}, \quad w_t := \chi u_t + (1 - \chi) q_t.
\]

By construction, \( \omega_{Y_t}|_{Y_t \cap W} = \partial_Y \bar{\partial}_{Y_t} q_t \). Let \( \tilde{\kappa}_t(s) \) \( (0 \leq s \leq 1) \) be the homotopy of Kähler metrics on \( Y_t \) joining \( \bar{\kappa}_{Y_t} \) and \( \omega_{Y_t} \) defined by

\[
\tilde{\kappa}_t(s) := s \bar{\kappa}_{Y_t} + (1 - s) \omega_{Y_t}.
\]

**Proposition 4.7.** Let \( \text{Ric}(\tilde{\kappa}_t(s)) = \partial_Y \bar{\partial}_{Y_t} \log(\tilde{\kappa}_t(s)^2/\eta_{Y_t} \wedge \bar{\eta}_{Y_t}) \) be the Ricci curvature of \( (Y_t, \tilde{\kappa}_t(s)) \). Then, one has \( |\text{Ric}(\tilde{\kappa}_t(s))|_{\infty} \leq C \).

**Proof.** As the assertion is clear for \( x \in Y_t \setminus W \), we suppose that \( x \in Y_t \cap W \). When \( \tilde{\kappa}_t(s) \leq \delta^{-1}|t| \), \( \tilde{\kappa}_t(s) = \omega_{Y_t} \) by construction, which is followed by the assertion as \( \omega_{Y_t} \) is Ricci-flat. When \( x = \psi_t(r, \sigma) \in Y_{t, \delta^{-1}} \cap W \), it follows from (4.19) that

\[
\psi_t^* \text{Ric}(\tilde{\kappa}_t(s)) = \partial_t \bar{\partial}_t \log \{ 1 + r^2 B'(s, r^{-2}|t|, r^2, \sigma) \}
\]

because \( \partial_Y \bar{\partial}_{Y_t} \log |f_Y|^2 = 0 \) where \( B'(s, u, r^2, \sigma) := 2s B_1(u, r^2, \sigma) + s^2 r^2 B_2(u, r^2, \sigma) \).

As \( |r^{-1} \partial_U (r^2 B')|_{\infty} + |\partial_{ij} (r^2 B')|_{\infty} \leq C \), we get the assertion. □

Let \( \tilde{R}_t(s) \) be the curvature of \( (Y_t, \tilde{\kappa}_t(s)) \).

**Proposition 4.8.** For any \( (t, x) \in D(\delta) \times Y_t \cap W \), one has \( |\tilde{R}_t(s)(x)| \leq C q_t(x)^{-1} \).

**Proof.** By (4.10-11) and (4.15), there exists \( g_{ij} \in C^\infty(D(\delta) \times [0, 1] \times S_0 \times [0, 1]) \) such that \( \psi_t^* \tilde{\omega}_t(s) = \sum g_{ij}(r^{-1}|t|, r^2, \sigma, s) dx_i dx_j \) and \( (g_{ij}) \) is an uniformly positive definite matrix. As \( |\partial^k g_{ij}(r^{-1}|t|, r^2, \sigma, s)| \leq C_k r^{-k} \), we get the assertion. □
Theorem 4.1. For any \((t, s) \in D(\delta) \times [0, 1]\), one has \(C^{-1}\bar{\kappa}_t(s) \leq \kappa_{Y_t} \leq C \bar{\kappa}_t(s)\).

Proof. Since \(\bar{\kappa}_t(s) = \omega_{Y_t} + s\partial_t, \bar{\omega}_{Y_t}, v_t\) on \(Y_t \cap V\), one has \(C^{-1}\omega_{Y_t} \leq \bar{\kappa}_t(s) \leq C\omega_{Y_t}\) by (4.16). Thus, it is enough to show the inequality when \(s = 0\). Consider the Monge-Ampère equation (4.1) with \(X = Y_t, \kappa = \bar{\kappa}_t, \phi = \phi_t\) and \(F = F_t = \log(h(t)\eta_{Y_t} \wedge \bar{\eta}_{Y_t} / \bar{\kappa}_t^2)\). Here \(h(t)\) is defined in (4.3). Since \(h(t) = 1 + O(|t|)\) as \(t \to 0\), it follows from Proposition 4.7 that \(|F_t|_\infty \leq C|t|\). Thus, by Propositions 4.7-8, all of \(|\Delta_t F_t|_\infty / |R_t|_\infty, |F_t|_\infty, |R_t|_\infty |F_t|_\infty, V_t\) are uniformly bounded in \(t \in D(\delta)\) where the subscript \(t\) means that these quantities are considered relative to \(\bar{\kappa}_t\).

As \(\bar{\kappa}_{Y_t}\) is quasi-isometric to \(\omega_{Y_t}\) on \(Y_t \cap V\), the Sobolev constant \(S_t\) is uniformly bounded. As \(\lambda_t\), the first eigenvalue of the Laplacian of \((Y_t, \bar{\kappa}_{Y_t})\) is continuous in \(t\) ([Yo1, Theorem 5.1]), we also get an uniform bound \(\lambda_t^{-1} < C\) because \(\lambda_0 > 0\). The assertion follows from Proposition 4.1 (2) together with these bounds. \(\square\)

Consider the following Monge-Ampère equation on \(Y_t\):

\[
(\bar{\kappa}_t + \frac{i}{2\pi} \partial \bar{\partial} \phi_t(s))^2 = e^{s F_t + a_t(s)} \bar{\kappa}_t^2, \quad \int_{Y_t} \phi_t(s) \bar{\kappa}_t^2 = 0
\]

where \(F_t = \log(h(t)\eta_{Y_t} \wedge \bar{\eta}_{Y_t} / \bar{\kappa}_t^2)\) and \(a_t(s) = \log(\deg L / \int_{Y_t} e^{s F_t} \bar{\kappa}_t^2)\). Set \(\kappa_t(s)\) for the smooth homotopy of Kähler metrics joining \(\bar{\kappa}_t\) and \(\kappa_{Y_t}\):

\[
\kappa_t(s) := \bar{\kappa}_t + \frac{i}{2\pi} \partial \bar{\partial} \phi_t(s) > 0, \quad \kappa_t(0) = \bar{\kappa}_t, \quad \kappa_t(1) = \kappa_{Y_t}.
\]

Theorem 4.2. For any \(t \in D, s \in [0, 1]\) and \(x \in Y_t\), one has

\[
|F_t|_\infty + |a_t(s)| \leq C|t|, \quad C^{-1} \bar{\kappa}_t(x) \leq \kappa_t(s)(x) \leq C \bar{\kappa}_t(x), \quad |\text{Ric}(\kappa_t(s))|_\infty \leq C.
\]

Proof. The first inequality follows from Proposition 4.8, the third one from Proposition 4.7 and (4.28). The second one is similarly proved as Theorem 4.1. \(\square\)

Let \(R_t(s)\) be the curvature of \((Y_t, \kappa_t(s))\) and \(\nabla_t(s)\) its covariant derivative.

Theorem 4.3. On \(Y_t \cap W\), one has \(|\nabla_t(s)^k R_t(s)(x)| \leq C q_t(x)^{-1 - \frac{k}{4}}\).

Proof. For \(p \in S_s\), let \(B(p, r)\) be the metric ball of radius \(r\) centered at \(p\) relative to the metric \(\omega_{Y_s}\), and \(w = (w_1, w_2)\) its holomorphic normal coordinates. Fix \(\epsilon > 0\) small enough so that \((B(p, \epsilon), w)\) becomes a coordinate neighborhood for any \(p \in S_s (s \in D(\delta))\). Take \(x_0 \in Y_{t, \delta - |t|} \cap W\). Put \(y_0 := q_t(x_0)^{-1} x_0 \in S_{q_t(x_0)^{-1} t}\) and \(B = B(y_0, \epsilon)\). For \(y \in B\), put \(x := q_t(x_0) y \in Y_t\) which makes \((B, w)\) to be a coordinate neighborhood of \(Y_t\) at \(x_0\). As \(y = y_0 + w, v(z) = O(||z||^4)\) and \(q_t(x_0)^{-1}(q_t(x) + v_t(x)) = q_t(x_0)^{-1} y + q_t(x_0)^{-1} v_t(x_0) \phi_t(x_0)^{-1} y\), we get \(|q_t(x_0)^{-1}(q_t + v_t)|_{C^k(B)} \leq C(k, ||v||_{C^k})\). Thus, by Propositions 4.5 and 4.8, if we write \(q_t(x_0)^{-1} \bar{\kappa}_{Y_t} = \sum_{i,j} \tilde{g}_{ij}(t, w) dw_i \wedge dw_j\) on \(B\), we get \(\lambda I \leq (\tilde{g}_{ij}) \leq \Lambda I\) and \(|\tilde{g}_{ij}|_{C^k(B)} \leq C_k\) where \(\lambda, \Lambda, C_k > 0\) are independent of \(x_0\) and \(t\). Putting \(\bar{\kappa}_t := q_t(x_0)^{-1} \bar{\kappa}_{Y_t}\) and \(\tilde{\phi}_t(s) := q_t(x_0)^{-1} \tilde{\phi}_t(s)\), the Monge-Ampère equation (4.28) becomes as follows on \(B\):

\[
(\frac{i}{2\pi} \partial \bar{\partial} \tilde{\phi}_t(s))^2 = e^{s F_t + a_t(s)} \det(\tilde{g}_{ij}) \partial \bar{\partial} \tilde{\phi}_t(s) = \frac{\partial^2 \tilde{\phi}_t(s)}{\partial w_i \partial w_j}\]

21
§5. Reduction to the ALE Instanton

5.1 Estimates of Anomaly. Let us consider the situation in §4.2 and keep notations there. In this section, we assume that \( \pi : \mathcal{Y} \to D \) is a degenerating family of 2-elementary K3 surfaces with involution \( \iota \) and \( L \) is \( \iota \)-invariant. Thus, \( \kappa_{Y_t} \) is also \( \iota \)-invariant by the uniqueness of Ricci-flat Kähler metric. As \( o \) is a fixed point of \( \iota \), according to the type in (2.11), we fix the local coordinates \( (x, y, z, t) \) as in Proposition 2.3. Relative to this local coordinates, we can construct \( \tilde{\kappa}_Y \) as in (4.18) and \( \omega_Y \) as in (4.25). By taking the average of the action of \( \iota \) if necessary, we may assume that both \( \tilde{\kappa}_Y \) and \( \omega_Y \) are \( \iota \)-invariant, and so is \( \tilde{\kappa}_t(s) \).

Let \( \widetilde{\text{Td}}(Y_t; \kappa_{Y_t}, \tilde{\kappa}_{X_t}) \) be the Bott-Chern secondary class associated to the Todd genus and \( \kappa_{Y_t}, \tilde{\kappa}_{X_t} \). By definition ([B-C], [B-G-S, I (e)]), we get

\[
\widetilde{\text{Td}}(Y_t; \kappa_{Y_t}, \tilde{\kappa}_{X_t})^{(2,2)} = \frac{1}{24} \int_1^0 \left( \text{Tr} \kappa_t(s)^{-1} \tilde{\kappa}_t(s) \right) c_2(R_t(s)) \, ds \\
+ \frac{1}{24} \int_1^0 c_1(R_t(s)) \cdot \left. \frac{d}{|s|} \right|_{|s|=0} c_2 \left( R_t(s) + \epsilon \frac{\tilde{\kappa}_t(s)}{\kappa_t(s)} \right) ds
\]

(5.1)

where \( \tilde{\kappa}_t(s) = \partial_s \kappa_t(s) \). As in §4.2, we denote by \( C \) a constant independent of \( s \in [0, 1] \), \( t \in D \) and \( x \in Y_t \), though its value may change in each estimate. For a norm \(| \cdot |_\kappa \), we mean that it uses the metric \( \kappa \) in measuring.

**Lemma 5.1.** For any \( t \in D \setminus \{0\} \), one has

\[
\left| \int_{Y_t} \widetilde{\text{Td}}(Y_t; \kappa_{Y_t}, \tilde{\kappa}_{X_t})^{(2,2)} \right| \leq C \sup_{s \in [0, 1]} \left| \kappa_t(s)^{-1} \tilde{\kappa}_t(s) \right|_{\infty, \kappa_{X_t}}.
\]

**Proof.** In the proof, every norms and volumes are those relative to \( \kappa_t(s) \). By (5.1),

\[
\left| \int_{Y_t} \widetilde{\text{Td}}(Y_t; \kappa_{Y_t}, \tilde{\kappa}_{Y_t})^{(2,2)} \right| \leq \sup_{s \in [0, 1]} |\kappa_t(s)^{-1} \tilde{\kappa}_t(s)| \int_0^1 ds \int_{Y_t} |R_t(s)|^2 \, dv_t(s).
\]

(5.2)

Let \( \tau_t(s) \) the scalar curvature of \( (Y_t, \kappa_t(s)) \). As is well known (cf. [G]),

\[
24 = \chi(Y_t) = \int_{Y_t} \frac{1}{32\pi^2} \left( |R_t(s)|^2 - 4|\text{Ric}(\kappa_t(s))|^2 + \tau_t(s)^2 \right) \, dv_t(s).
\]

(5.3)
Let $\theta_1, \theta_2$ be a local unitary frame and write $\text{Ric}(\kappa_t(s)) = i(\rho_1 \theta_1 \bar{\theta}_1 + \rho_2 \theta_2 \bar{\theta}_2)$ ($\rho_1, \rho_2 \in \mathbb{R}$). Then, $\tau_t(s) = \rho_1 + \rho_2$ and $\text{Ric}(\kappa_t(s))^2 = 2\rho_1 \rho_2 \, d\nu_t(s)$. Thus,

\[(5.4)\]
$$\int_{Y_t} \tau_t(s)^2 \, d\nu_t(s) = \int_{Y_t} |\text{Ric}(\kappa_t(s))|^2 \, d\nu_t(s) + \int_{X_t} c_1(Y_t)^2 = \int_{Y_t} |\text{Ric}(\kappa_t(s))|^2 \, d\nu_t(s)$$

because $c_1(Y_t)$ is cohomologous to zero which, together with (5.3), yields

\[(5.5)\]
$$\int_{Y_t} |R_t(s)|^2 \, d\nu_t(s) = 32\pi^2 \left(24 + 3 \int_{Y_t} |\text{Ric}(\kappa_t(s))|^2 \, d\nu_t(s)\right).$$

The assertion follows from (5.2) and (5.5) together with Theorem 4.2. $\square$

**Lemma 5.2.** For any $t \in D\setminus\{0\}$, one has

$$\left|\int_{Y_t} \tilde{T}_d(Y_t; \kappa_t, \rho_t)\right| \leq C.$$ 

**Proof.** Let $\Delta_t(s)$ be the Laplacian of $(Y_t, \kappa_t(s))$. Differentiating (4.28) by $s$, we get

\[(5.6)\]
$$\Delta_t(s) \dot{\varphi}_t(s) = F_t + \dot{a}_t(s), \quad \int_{Y_t} \dot{\varphi}_t(s) \kappa_t(0)^2 = 0.$$ 

Multiplying $|\dot{\varphi}_t(s)|^{p-2} \dot{\varphi}_t(s)$ to the both hand sides, the integration by parts yields

\[(5.7)\]
$$\int_{X_t} |d\dot{\varphi}_t(s)|^2 \, d\nu_t(s) \leq \frac{p^2}{4(p-1)} \int_{X_t} |s \, F_t + a_t(s)| |\dot{\varphi}_t(s)|^{p-1} \, d\nu_t(s)$$

for any $p \geq 2$. By Theorem 4.2, (5.7) is also valid after changing the metric from $\kappa_t(s)$ to $\kappa_t(0)$. Applying Moser’s iteration argument ([Ko, pp.298-299]), we get

\[(5.8)\]
$$|\dot{\varphi}_t(s)|_\infty \leq C \|\dot{\varphi}_t(s)\|_{L^2, \kappa_t(s)}$$

where $C$ depends only on the constants in Theorem 4.2, the Sobolev constant of $(Y_t, \kappa_t(0))$ and $|s \, F_t + a_t(s)|_\infty$. By Theorems 4.1 and 4.2, $C$ is uniformly bounded because so is the Sobolev constant of $(Y_t, \kappa_t(0))$. As in the proof of Theorem 4.1, we find that $\lambda_t$, the first eigenvalue of $\Delta_t(0)$, is also uniformly bounded away from zero; $\lambda_t \geq \lambda > 0$ which, together with (5.7), $p = 2$ and Theorem 4.2, yields

\[(5.9)\]
$$\|\dot{\varphi}_t(s)\|_{L^2}^2 \leq \lambda^{-1} \|d\dot{\varphi}_t(s)\|_{L^2}^2 \leq C \lambda^{-1} |F_t + \dot{a}_t(s)|_{L^2} \|\dot{\varphi}_t(s)\|_{L^2} \leq C |t| \|\varphi_t(s)\|_{L^2}$$

where norms are those relative to $\kappa_t(0)$. Comparing (5.8) and (5.9), we get

\[(5.10)\]
$$|\dot{\varphi}_t(s)|_\infty \leq C |t|.$$ 

Take $x \in Y_t \cap W$, and let $(B, w)$ be the coordinate neighborhood centered at $x$ as in the proof of Theorem 4.3. Put $\tilde{\kappa}_t(s) := q_t(x)^{-1} \kappa_t(s)$ for the rescaled metric on $B$. Let $\tilde{\Delta}_t(s) = q_t(x) \Delta_t(s)$ be the Laplacian of $(X_t, \tilde{\kappa}_t(s))$. Then, (5.6) becomes

\[(5.11)\]
$$\tilde{\Delta}_t(s) \dot{\varphi}_t(s) = q_t(x) (F_t + \dot{a}_t(s)).$$
By Proposition 4.7, Theorem 4.2 and (5.10), the Schauder estimate ([G-T, Theorem 6.2]) applied to (5.11) yields (one can apply it because the geometry of $(B(0, r_0), \tilde{\kappa}_t(s))$ is bounded as shown in the proof of Theorem 4.3.)

\begin{equation}
|\tilde{\phi}_t(s)|_{C^{2+\alpha}(B(0, r_0))} \leq C(|\tilde{\phi}_t(s)|_{\infty} + |q_t(x) (F_t + \partial_t(s))|_{C^{\alpha}(B(0, 2r_0))}) \\
\leq C(|t| + q_t(x)) \leq C q_t(x).
\end{equation}

As $\kappa_t(s)$ converges smoothly in $t$ to a Kähler metric on $Y_0$ outside of $W$ and thus $|\tilde{\phi}_t(s)|_{C^2(Y_1 \setminus W)} \leq C$, it follows from (5.12) that, for any $x \in Y_t \cap W$,

\begin{equation}
|\partial_{Y_t} \tilde{\partial}_{Y_t} \tilde{\phi}_t(s)(x)|_{\kappa_t(s)} \leq C q_t(x).
\end{equation}

Rewriting (5.13) in terms of $\kappa_t(s)$, we get the assertion by Lemma 5.1 because

\begin{equation}
|\kappa_t(s)^{-1} \kappa_t(s)(x)|_{\kappa_t(s)} = |\partial_{Y_t} \tilde{\partial}_{Y_t} \tilde{\phi}_t(s)(x)|_{\kappa_t(s)} \leq C q_t(x)^{-1} q_t(x) \leq C.
\end{equation}

\textbf{Lemma 5.3.} For any $t \in D \setminus \{0\}$, one has

\[ \left| \int_{Y_t} \tilde{Td}(Y_t; \tilde{\kappa}_Y, \omega_Y) \right| \leq C. \]

\textbf{Proof.} Since (5.2) and (5.5) are also valid for $\tilde{\kappa}_t(s)$, we get

\begin{equation}
\left| \int_{Y_t} \tilde{Td}(Y_t; \tilde{\kappa}_Y, \omega_Y) \right| \leq C \sup_{s \in [0,1]} |\tilde{\kappa}_t(s)^{-1} \kappa_t(s)|_{\infty}
\end{equation}

As $\partial_s \tilde{\kappa}_t(s) = \tilde{\kappa}_Y - \omega_Y$ by (4.26), we get $\sup_{s \in [0,1]} |\tilde{\kappa}_t(s)^{-1} \kappa_t(s)|_{\infty} \leq C$ by Theorem 4.1 which, together with (5.15), implies the assertion. \quad \Box

\textbf{Proposition 5.1.} For any $t \in D \setminus \{0\}$, one has

\[ \left| \int_{Y_t} \tilde{Td}(Y_t; \kappa_Y, \omega_Y) \right| \leq C. \]

\textbf{Proof.} Clear by Lemmas 5.2 and 5.3 because the Bott-Chern secondary class does not depend on a path joining two metrics ([B-C], [B-G-S I (e)]). \quad \Box

Let $\pi : Z \to D$ be the family of fixed curves of $t$; $Z = \{ x \in Y; \ T(x) = x \}$. Let $\kappa_{Z_t}(s)$ be the restriction of $\kappa_t(s)$ to $Z_t$; $\kappa_{Z_t}(s) = \kappa_t(s)|_{Z_t}$. By definition,

\begin{equation}
\int_{Z_t} c_1(Y_t) c_1(Z_t)(\kappa_Y, \tilde{\kappa}_Y) \\
= \int_1^0 ds \int_{Z_t} (\operatorname{Tr} \kappa_t(s)^{-1} \kappa_t(s)) c_1(Z_t, \kappa_{Z_t}(s)) + \kappa_{Z_t}(s)^{-1} \kappa_{Z_t}(s) c_1(Y_t, \kappa_t(s)).
\end{equation}
Lemma 5.4. For any $t \in D \setminus \{0\}$, one has
\[ \left| \int_{Z_t} c_1(Y_t) c_1(Z_t)(\kappa_{Y_t}, \tilde{\kappa}_{Y_t}) \right| \leq C. \]

Proof. When $o = \text{Sing} \ Y \notin Z$, the assertion is obvious. Thus, we assume that $o \in Z_0$. By Proposition 2.5, $Z_t$ is a degenerating family of curves such that $Z_0$ has only one node at $o$. By (5.16), it is enough to show the followings;

(1) $|\text{Tr} \, \kappa_t(s)^{-1} \tilde{\kappa}_t(s)| \leq C |t|$, (2) $|c_1(Z_t, \kappa_{Z_t}(s))|_{\infty, \kappa_{Z_t}} \leq C |t|^{-1}$,
(3) $|\kappa_{Z_t}(s)^{-1} \tilde{\kappa}_{Z_t}(s)| \leq C$, (4) $|c_1(X_t, \kappa_t(s))|_{\infty, \kappa_t(s)} \leq C$.

As $\text{Tr} \, \kappa_t(s)^{-1} \tilde{\kappa}_t(s) = F_t + \dot{a}_t(s) = O(|t|)$ by Theorem 4.2, we get (1). (4) is proved in Theorem 4.2. (3) follows from (5.14) because, for any $v \in T Z_t$,
\[ (5.17) \quad -|\kappa_t(s)^{-1} \tilde{\kappa}_t(s)|_{\infty} \leq \kappa_t(s)(v, v)^{-1} \partial_s \kappa_t(s)(v, v) \leq |\kappa_t(s)^{-1} \tilde{\kappa}_t(s)|_{\infty}. \]

As $\kappa_t(s)$ is $\iota$-invariant, $R_t(s)$ restricted to $Z_t$ splits as follows ([B-G-V, §6.3 (6.1)]);
\[ (5.18) \quad R_t(s)|_{Z_t} = R(Z_t, \omega_t(s)) \oplus R(N_{Z_t/Y_t}) \]
where $R(N_{Z_t/Y_t})$ is the curvature of the normal bundle of $Z_t$ relative to the induced metric, and the splitting is orthogonal. Therefore, by Theorem 4.3, we get
\[ (5.19) \quad |c_1(Z_t, \kappa_{Z_t}(s))|_{\infty} = |R(Z_t, \kappa_{Z_t}(s))|_{\infty} \leq |R_t(s)|_{Z_t} \leq |R_t(s)|_{\kappa_t(s)} \leq C |t|^{-1}. \quad \square \]

Lemma 5.5. For any $t \in D \setminus \{0\}$, one has
\[ \left| \int_{Z_t} c_1(Y_t) c_1(Z_t)(\tilde{\kappa}_{Y_t}, \omega_{Y_t}) \right| \leq C. \]

Proof. In the same way as the proof of Lemma 5.4, we may assume that $o \in Z_0$. Put $\tilde{\kappa}_{Z_t}(s) := \tilde{\kappa}_t(s)|_{Z_t}$. By (5.16), it is enough to prove the following; for any $x \in Y_t \cap W$,

(1) $|\text{Tr} \, \tilde{\kappa}_t(s)^{-1} \tilde{\kappa}_t(s)(x) \leq C q_t(x)$, (2) $|c_1(Z_t, \tilde{\kappa}_{Z_t}(s))(x) \leq C q_t(x)^{-1}$,
(3) $|\tilde{\kappa}_{Z_t}(s)^{-1} \tilde{\kappa}_{Z_t}(s)|(x) \leq C$, (4) $|c_1(Y_t, \tilde{\kappa}_t(s))(x) \leq C$
where length is measured relative to $\tilde{\kappa}_{Z_t}$ and $\tilde{\kappa}_{Y_t}$. As $\tilde{\kappa}_t(s)^{-1} \tilde{\kappa}_t(s) = \tilde{\kappa}_{Y_t} - \omega_{Y_t}$, we get (3) by Theorem 4.1 and (1) by Lemma 4.4. (4) follows from Proposition 4.7. Since $\tilde{\kappa}_t(s)$ is $\iota$-invariant, by the same argument as the proof of (2) in Lemma 5.2, we get (2) by Proposition 4.8. \quad \square

Proposition 5.2. For any $t \in D \setminus \{0\}$, one has
\[ \left| \int_{Z_t} c_1(Y_t) c_1(Z_t)(\kappa_{Y_t}, \omega_{Y_t}) \right| \leq C. \]

Proof. Clear by Lemmas 5.4 and 5.5 \quad \square

Define $\tilde{\tau}_M(Y_t, \iota_t, \omega_{Y_t})$ in the same manner as Definition 3.3. By Propositions 5.1 and 5.2, we get the following.
Theorem 5.1. For any \( t \in D \setminus \{0\} \), one has
\[
| \log \tau_M(Y_t, \iota t) - \log \tilde{\tau}_M(Y_t, \iota t, \omega_{Y_t}) | \leq C.
\]

5.2 Singularity of Type \((0,3)\) and Asymptotics of \( \tau_M \). In this subsection, we assume that \((Y_0, \iota, o)\) is a singularity of type \((0,3)\) in the sense of \((2.11)\). Put \( \tau_M(t) := \tau_M(Y_t/\iota t, \omega_{Y_t}) \).

Theorem 5.2. As \( t \to 0 \), one has the following asymptotic formula:
\[
\log \tau_M(t) = -\frac{1}{8} \log |t|^2 + o(\log |t|).
\]

Proof. Let \( \Delta_t \) be the Laplacian of \((Y_t, \omega_{Y_t})\). As \( \omega_{Y_t} \) is \( \iota \)-invariant, \( \Delta_t \) splits into \( \Delta_t^\pm \) as in Lemma 3.1. Let \( K_t^\pm(s, x, y) \) be the heat kernel of \( \Delta_t^\pm \) and \( K_t(s, x, y) \) that of \( \Delta_t \). Since \( K_t^\pm(s, x, y) = K_t(s, x, y)/2 \pm K_t(s, x, y)/2 \) and thus
\[
(5.20) \quad \text{Tr} e^{-s\Delta_t^+} - \text{Tr} e^{-s\Delta_t^-} = \int_{X_t} K_t(s, x, \iota t x) \, dv_{X_t},
\]
it follows from [B-G-V, Theorem 6.11] that there exists \( a_t(z, t) \in C^\infty(Z_t) \) such that
\[
(5.21) \quad \text{Tr} e^{-s\Delta_t^+} - \text{Tr} e^{-s\Delta_t^-} \sim \int_{Z_t} \left( \frac{a_1(z, t)}{s} + a_0(z, t) \right) \, dv_{Z_t} + O(s) \quad (s \to 0).
\]

Put \( a_t(t) = \int_{Z_t} a_t(z, t) \, dv_{Z_t} \). From (5.20-21) and Lemma 3.1, it follows that
\[
(5.22) \quad \tau(Y_t/\iota t, \omega_{Y_t}) = \log \det \Delta_t^- - \log \det \Delta_t^+
\]
\[
= \int_0^1 \frac{ds}{s} \left( \text{Tr} e^{-s\Delta_t^+} - \text{Tr} e^{-s\Delta_t^-} - \frac{a_{-1}(t)}{s} - a_0(t) \right)
+ \int_1^\infty \frac{ds}{s} \left( \text{Tr} e^{-s\Delta_t^+} - \text{Tr} e^{-s\Delta_t^-} - 1 \right) + a_{-1}(t) - \Gamma'(1) a_0(t) + \Gamma'(1)
\]
\[
= \int_0^1 \frac{ds}{s} \left\{ \int_{Y_t} K_t(s, x, \iota x) \, dv_{Y_t} - \int_{Z_t} \left( \frac{a_1(z, t)}{s} + a_0(z, t) \right) \, dv_{Z_t} \right\}
+ \int_1^\infty \frac{ds}{s} \left\{ \int_{Y_t} K_t(s, x, \iota x) \, dv_{Y_t} - 1 \right\} + a_{-1}(t) - \Gamma'(1) a_0(t) + \Gamma'(1).
\]

Let \((X, \omega_X)\) be the ALE instanton (see (6.1) below). By (4.25), we have
\((Y_t \cap W, \omega_{Y_t}) \cong (X \cap B(|t|^{-1/2}), |t| \omega_X)\). Let \( K(s, x, y) \) be the heat kernel of \((X, \omega_X)\). As \( \omega_{Y_t} \) is Ricci-flat on \( Y_t \cap W \), it follows from [C-L-Y, §3, 4] that there exist constants \( C, \gamma > 0 \) such that
\[
(5.23) \quad 0 < K_t(s, x, y) \leq C s^{-\varepsilon} e^{-\frac{-\varepsilon d(x, y)}{s}}, \quad |dxK_t(s, x, y)| \leq C s^{-\frac{\gamma}{2}} e^{-\frac{-\varepsilon d(x, y)}{s}}
\]
for any \( t \in D \setminus \{0\} \), \( s \in (0, 1] \) and \( x, y \in Y_t \cap W \), and one has the same estimates for \( K(s, x, y) \) for any \( s > 0 \) and \( x, y \in X \). By Duhamel’s principle (cf. [B-B, pp.63-67]) together with (5.23), there exist \( c, C > 0 \) such that
\[
(5.24) \quad \left| \int_{Y_t \cap W} K_t(s, x, \iota x) \, dv_{Y_t} - \int_{X \cap B(|t|^{-1/2})} K(|t|^{-1}s, x, \iota x) \, dv_X \right| \leq C e^{-\frac{\varepsilon}{2}}
\]
for any \( t \in D\backslash\{0\} \) and \( s \in (0, 1] \). Put \( J(T) := \int_0^T s^{-1} ds \int_{B(\sqrt{T})} K(s, x, ix) dv_X \). Since \( W \cap Z_t = \emptyset \), there appears no contribution from \( Z_t \) to the divergence of \( \tau(Y_t/\lambda \omega_t) \) which, together with (5.24), yields

\[
\int_0^1 \frac{ds}{s} \int_{Y_t \cap W} K_t(s, x, ix) dv_{Y_t} = J(|t|^{-1}) + O \left( \int_0^1 \frac{ds}{s} e^{-s} \right) = J(|t|^{-1}) + O(1).
\]

As \( \omega_{Y_t} \) is a smooth family of metrics outside of \( Y_t \cap W \), one has

\[
\int_0^1 \frac{ds}{s} \left( \int_{Y_t \backslash W} K_t(s, x, ix) dv_{Y_t} - \int_{Z_t} \left( \frac{a_1(z, t)}{s} + a_0(z, t) \right) dv_{Z_t} \right) \leq C.
\]

Let \( \lambda_k(t) \) be the \( k \)-th eigenvalue of \( \Delta_t \). Since the Sobolev constant of \( (Y_t, \omega_{Y_t}) \) is uniformly bounded away from zero, it follows from [C-L] that there exists \( N \geq 0 \) such that, for any \( t \in D \) and \( k \geq N \), one has \( \lambda_k(t) \geq C k^{1/2} \). By [Yo1, Theorem 5.1], \( \lambda_k(t) \) is a continuous function on \( D \) for any \( k \geq 0 \) and \( \lambda_1(0) > 0 \). Thus, we get the bound \( \lambda_k(t) \geq C k^{1/2} \) and \( 0 \leq \text{Tr} e^{-s \Delta_t} - 1 \leq C e^{-\lambda_1(0)s/2} \) for \( s \geq 1 \), which yields

\[
\int_1^\infty \frac{ds}{s} \left| \int_{Y_t} K(s, x, ix) dv_{Y_t} - 1 \right| = \int_1^\infty \frac{ds}{s} \left| \text{Tr} e^{-s \Delta_t^+} - \text{Tr} e^{-s \Delta_t^-} - 1 \right|
\leq \int_1^\infty \frac{ds}{s} (\text{Tr} e^{-s \Delta_t} - 1) \leq C.
\]

Let \( K(s, x, y; \lambda \omega_X) \) be the heat kernel and \( B(r; \lambda \omega_X) \) the metric ball of radius \( r \) of \( (X, \lambda \omega_X) \). Let \( ds_{C^2}^2 \) be the Euclidean metric of \( C^2 \) and \( K_{C^2/\pm 1}(s, x, y) \) be the heat kernel of \( (C^2/\pm 1, ds_{C^2}^2) \). Since \( (X, \lambda \omega_X, t, o) \) converges to \( (C^2/\pm 1, ds_{C^2}^2, t', 0) \) as \( \lambda \to 0 \) where \( t'(z_1, z_2) = (\sqrt{-1}z_1, \sqrt{-1}z_2) \), we get

\[
\lim_{T \to \infty} \frac{J(T)}{\log T} = \lim_{T \to \infty} \int_0^1 d\sigma \int_{B(T^{1/2}, T^{-\sigma} \omega_X)} K(1, x, ix; T^{-\sigma} \omega_X) dv_{T^{-\sigma} \omega_X} = \int_0^1 d\sigma \int_{C^2/\pm 1} K_{C^2/\pm 1}(1, x, l'x) dv_{C^2} = \frac{1}{2} \int_{C^2} \left( \frac{e^{-|l'x|}}{(4\pi s)^2} + \frac{e^{-|lx|}}{(4\pi s')^2} \right) dv_{C^2} = \frac{1}{4}.
\]

As \( \log \tau_M(t) = J(|t|^{-1}) + O(1) \quad (t \to 0) \) by (5.22), (5.26-27) and (1.1), the assertion follows from (5.28) because \( \pi : Z \to D \) is a smooth morphism and thus \( \log \tau(Z_t, \omega_{Z_t}) \) is bounded on \( D \).  \( \square \)
§6. Singularity of $\tau_M$ along the Discriminant Locus

6.1 Heat Kernel on the ALE Instanton. Consider the following affine quadric with involution and the fixed locus;

$$X := \{(z_1, z_2, z_3) \in \mathbb{C}^3; z_1 z_2 - z_3^2 = 1\}, \quad \iota(z_1, z_2, z_3) = (z_1, z_2, -z_3),$$

$$X^\iota = Z := \{(z_1, z_2) \in \mathbb{C}^2; z_1 z_2 = 1\}.$$

Let $\omega_X = \partial_X \bar{\partial}_X q$ be the Kronheimer’s ALE metric on $X$ ([Kr]) as in §4.2, and put $\omega_Z := \omega_X|_Z$ for its restriction to $Z$. Fix $o \in X$ and put $r(x) := \text{dist}(o, x)$. Let $B(\rho)$ be the metric ball of radius $\rho$ centered at $o$. By Kronheimer [Kr], there exist $c, C_\alpha > 0$ with the following properties; (1) For any $y \in X$, the injectivity radius at $y$ is greater than $j_y = c(1 + r(y))$. (2) On the metric ball $B(y, j_y)$ with the normal coordinates $x = (x_1, \cdots, x_4)$, the metric tensor $g(x) = \sum g_{ij}(x) dx_i dx_j$ satisfies

$$\sup_{x \in B(y, j_y)} |\partial^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha (1 + r(y))^{-|\alpha|}.$$

For simplicity, we assume $c = 2$ by considering $(2c)^{-1} \omega_X$ if necessary. Let $K(t, x, y)$ be the heat kernel of $(X, \omega_X)$ and define its parametrix ([B-G-V, Theorem 2.26]);

$$p(t, x, y) := (4\pi t)^{-m} e^{-\frac{d(x, y)^2}{4t}} \{u_0(x, y) + t u_1(x, y)\}$$

where $u_i(\cdot, y) \in C^\infty(B(y, j_y))$. (In [B-G-V], $\Phi_i$ is used instead of $u_i$.)

Lemma 6.1. For any $y \in X$ and $(x, t) \in B(\bar{y}, j_y/2) \times [0, 1 + |y|^2]$, one has

$$|K(t, x, y) - p(t, x, y)| \leq C(c, C_\alpha) (1 + r(y)^2)^{-2} e^{-\frac{\gamma d(x, y)^2}{t}}.$$

Proof. Let $B(r)$ be the ball of radius $r < 1$ in $\mathbb{R}^4$. Let $(M, g)$ be a Riemannian 4-manifold such that $B(1)$ is embedded into $M$ and $g = \sum g_{ij} dx_i dx_j$ on $B(1)$. Suppose that $|\partial^k (g_{ij}(x - \delta_{ij})| \leq A_k$ for any $k \geq 0$ and $x \in B(1)$. Let $L(t, x, y)$ be the heat kernel of $(M, g)$. By [C-L-Y, §3.4], there exists $r, C, \gamma > 0$ depending only on $A_k$ ($k \leq 6$) and $r$ such that, for any $(x, y) \in B(r) \times B(r)$ and $t \in (0, 1]$,

$$0 < L(t, x, y) \leq C t^{-2} e^{-\frac{\gamma d(x, y)^2}{4t}}, \quad |d_x L(t, x, y)| \leq C t^{-\frac{3}{2}} e^{-\frac{\gamma d(x, y)^2}{2t}}.$$

Let $L'(t, x, y)$ be the Dirichlet heat kernel of $(B(1), g)$ which satisfies the same estimates (6.4). By Duhamel’s principle, for any $(x, y) \in B(r) \times B(r)$ and $t \in (0, 1]$,

$$|L(t, x, y) - L'(t, x, y)| \leq C e^{-\frac{ct}{t}}$$

where $c_0, C > 0$ depend only on $C_k$. Since $C^k$-norm of $g_{ij}$ is bounded by $A_k$ ($k = 6$ is enough), it follows from the usual asymptotic expansion of the heat kernel that there exit a constant $C, \gamma > 0$ depending only on $A_k$ ($k \leq 6$) and $r$ such that, for any $(x, y) \in B(r) \times B(r)$ and $t \in (0, 1]$,

$$|L'(t, x, y) - p L'(t, x, y)| \leq C e^{-\frac{\gamma d(x, y)^2}{t}}.$$
where \( p_L(t,x,y) \) is defined by the formula (6.3) for \( L' \) and \( d(x,y) \) is the distance relative to the metric \( g \). Comparing (6.5) and (6.6), we find

\[
|L(t,x,y) - p_L(t,x,y)| \leq C e^{-\frac{\gamma d(x,y)^2}{t}}.
\]

For any \( y \in X \), consider the rescaled space \((X, j_y^{-2}\omega_X)\) and its heat kernel \( L(t,x,y) \). As \( j_y^4 K(j_y^2 t, x, y) = L(t, x, y) \) and \( j_y^4 p(j_y^2 t, x, y) = p(t, x, y) \), it follows from (6.7) that

\[
|K(j_y^2 t, x, y) - p(j_y^2 t, x, y)| \leq C (1 + r(y))^{-4} \exp \left( -\frac{\gamma d(x,y;\omega_X)^2}{j_y^2 t} \right), \quad (t \in (0,1]).
\]

The assertion follows from (6.8) by putting \( s = j_y^2 t \). □

Let \( N_{Z/X} \) be the normal bundle of \( Z \) in \( X \) and \( v \) its fiber coordinate. Set

\[
\Omega := \{(z, v) \in N_{Z/X}; |v| \leq r(z) + 1\}, \quad \Omega(a) := \{(z, v) \in \Omega; r(\exp_z(v)) \leq a\}.
\]

Via the exponential map, identify \( \Omega \) with a tubular neighborhood of \( Z \) in \( X \). Let \( dv_Z \) be the volume form of \((Z, \omega_Z)\). Then, the volume form of \((X, \omega_X)\) can be written on \( \Omega \) as follows; \( dv_X(x) = J(z,v) dv dv_Z(z) \) where \( x = \exp_z(v) \). Set

\[
q_i(z,v) := J(z,v) u_i(x,\iota(x)), \quad q_i^{(k)}(z,v) := \frac{1}{k!} \partial^k |t=0 q_i(z, tv).
\]

In the sequel, \( C \) denotes a constant which depends only on \( C_\alpha (|\alpha| \leq 6), c \) and \( \gamma \). By definition, we get \( q_0(z,0) = 1 \). By (6.2), the following is clear.

**Lemma 6.2.** For any \((z,v) \in \Omega\), one has the following estimates;

1. \( |q_0(z,v) - q_0(z,0)| \leq C |v|(1 + r)^{-1} \)
2. \( |\partial_\alpha q_0(z,v)| \leq C (1 + r)^{-2} \)
3. \( |q_0(z,v) - q_0^{(1)}(z,v) - q_0^{(2)}(z,v)| \leq C |v|^3 (1 + r)^{-3} \)
4. \( |q_1(z,v) - q_1(z,0)| \leq C |v|(1 + r)^{-3} \)

In (2), \( |\alpha| > 0 \) when \( i = 0 \) and \( |\alpha| \geq 0 \) when \( i = 1 \).

Let \( dx dy = \sqrt{-1} dv dv_Z \) be the volume form on the fiber of \( N_{Z/X} \). Set

\[
I_1(s,z) := \int_{|v| \leq r(z) + 1} s^{-2} e^{-|v|^2/s} \left( q_0(z, v) - q_0(z) - q_0^{(1)}(z, v) - q_0^{(2)}(z, v) \right) dx dy.
\]

**Lemma 6.3.** For any \( T > 1 \), one has

\[
\left| \int_0^T \frac{ds}{s} \int_{r \leq \sqrt{T}} I_1(s,z) dv_Z \right| \leq C (\log T + 1).
\]

**Proof.** Suppose \( r \geq \sqrt{s} \). It follows from Lemma 6.2 (3) that

\[
|I_1(s,z)| \leq \int_{|v| \leq r + 1} C e^{-\frac{|v|^2}{s^2}} \frac{|v|^3}{(1 + r)^3} \leq C \int_{r \leq r + 1} e^{-\frac{t}{\rho^2}} \frac{\rho^4 dp}{s^2 (1 + r)^3} \leq C \frac{s^{\frac{1}{2}}}{(1 + r)^{\frac{3}{2}}}
\]

which yields

\[
\left| \int_1^T \frac{ds}{s} \int_{\sqrt{r} \leq r \leq \sqrt{T}} I_1(z,s) dv_Z \right| \leq C \int_1^T \frac{ds}{s} \int_{\sqrt{r}}^{\sqrt{T}} s^{\frac{1}{2}} (1 + r)^{-3} r dr \leq C \log T.
\]
Suppose $r \leq \sqrt{s}$. It follows from Lemma 6.2 (1) that
\[
(6.12) \quad \left| \int_{|v| \leq r+1} e^{-|v|^2/s} (q_0(z, v) - q_0(z)) \frac{dx \, dy}{s^2} \right| \leq C \int_{|v| \leq r+1} e^{-|v|^2/s} \frac{|v| \, dx \, dy}{s^2(1+r)} \leq C \frac{(1+r)^2}{s^2}.
\]
Similarly, using Lemma 6.2 (2), we get
\[
(6.13) \quad \left| \int_{|v| \leq r+1} e^{-|v|^2/s} (q_0^2(z, v)) \frac{dx \, dy}{s^2} \right| \leq C \int_{|v| \leq r+1} e^{-|v|^2/s} \frac{|v|^2 \, dx \, dy}{s^2(1+r)^2} \leq C \frac{(1+r)^2}{s^2}.
\]
Since \( \int_{|v| \leq r+1} e^{-|v|^2/s} v^k \, dx \, dy = 0 \) \((k > 0)\), we get
\[
(6.14) \quad \int_{|v| \leq r+1} s^{-2} e^{-|v|^2/s} q_0^1(z, v) \, dx \, dy = 0
\]
which, together with (6.12-13), yields
\[
(6.15) \quad \left| \int_1^T \frac{ds}{s} \int_{r \leq \sqrt{s}} I_1(z, s) \, dv \right| \leq C \int_1^T \frac{ds}{s} s^{-2} \int_{r \leq \sqrt{s}} (1+r)^2 \, dr \leq C \log T.
\]
When \( s \leq 1, 1+r(z) \geq s \) for any \( z \in Z \). Thus, by (6.10), we get
\[
(6.16) \quad \left| \int_0^1 \frac{ds}{s} \int_{r \leq \sqrt{T}} I_1(z, s) \, dv \right| \leq C \int_0^1 \frac{ds}{s} \sqrt{s} \int_{r \leq \sqrt{T}} (1+r)^{-3} \, dr \leq C.
\]
The assertion follows from (6.11) and (6.15-16). \( \square \)

Set \( I_2(s, z) := \int_{|v| \leq r(z) + 1} s^{-1} e^{-|v|^2/s} (q_1(z, v) - q_1(z, 0)) \, dx \, dy \).

**Lemma 6.4.** For any \( T > 1 \), one has
\[
\left| \int_0^T \frac{ds}{s} \int_{r \leq \sqrt{T}} I_2(s, z) \, dv \right| \leq C (\log T + 1).
\]

**Proof.** Suppose \( r \geq \sqrt{s} \). It follows from Lemma 6.2 (4) that
\[
(6.17) \quad |I_2(s, z)| \leq C \int_{|v| \leq r+1} e^{-|v|^2/s} \frac{|v| \, dx \, dy}{s (1+r)^3} \leq C \int_{\rho \leq r+1} e^{-|v|^2/\rho^2} \frac{\rho \, dp}{s (1+r)^3} \leq C \frac{\sqrt{s}}{(1+r)^3}
\]
which yields
\[
(6.18) \quad \left| \int_1^T \frac{ds}{s} \int_{\sqrt{r} \leq r \leq \sqrt{T}} I_2(s, z) \, dv \right| \leq C \int_1^T \frac{ds}{s} s^{\frac{1}{2}} \int_{\sqrt{r}}^{\infty} (1+r)^{-3} \, dr \leq C \log T.
\]
Suppose \( r \leq \sqrt{s} \). It follows from the same lemma that
\[
(6.19) \quad |I_2(s, z)| \leq C(1+r)^{-3} \int_{\rho \leq r+1} s^{-1} e^{-|v|^2/\rho^2} \, dv \rho \leq C s^{-\frac{1}{2}}(1+r)^{-1}
\]
which yields

\[
(6.20) \quad \left| \int_1^T \frac{ds}{s} \int_{r \leq \sqrt{s}} I_2(s,z) dv_Z \right| \leq C \int_1^T \frac{ds}{s} s^{-\frac{1}{2}} \int_0^{\sqrt{s}} (1 + r)^{-1} r dr \leq C \log T.
\]

When \( s \leq 1 \), it follows from (6.17) that

\[
(6.21) \quad \left| \int_0^1 \frac{ds}{s} \int_{r \leq \sqrt{T}} I_2(s,z) dv_Z \right| \leq C \int_0^1 \frac{ds}{s} \frac{s}{2} \int_0^\infty (1 + r)^{-1} r dr \leq C.
\]

The assertion follows from (6.17), (6.19) and (6.21). \( \square \)

**Lemma 6.5.** For any \( T > 1 \), one has

\[
\left| \int_0^T \frac{ds}{s} \int_{\Omega(\sqrt{T}) \setminus \Omega(\sqrt{s})} \{ K(s,x,tx) - p(s,x,tx) \} dv_X \right| \leq C \log T + 1.
\]

**Proof.** Put \( E(s,x) := K(s,x,tx) - p(s,x,tx) \), \( I_3(s) = \int_{\Omega(\sqrt{T}) \setminus \Omega(\sqrt{s})} E(s,x) dv_X \), and \( I_4(s) = \int_{\Omega(\sqrt{s})} E(s,x) dv_X \). Suppose \( 1 \leq s \leq T \). By Lemma 6.1, we get

\[
|I_3(s)| \leq C \int_{\Omega(\sqrt{T}) \setminus \Omega(\sqrt{s})} (1 + r^2)^{-2} e^{-\frac{|v|^2}{s}} dv_Z dx dy
\]

\[
\leq C \int_{\sqrt{s}}^{\sqrt{T}} \frac{r dr}{(1 + r^2)^2} \int_{|v| \leq r} e^{-\frac{|v|^2}{s}} dx dy \leq C \int_{\sqrt{s}}^{\sqrt{T}} \frac{dr}{r^3} \leq C.
\]

As \((X,\omega_X)\) is Ricci-flat, we get a bound ([L-Y, Theorem 3.2]); \( K(t,x,y) \leq C t^{-2} \) for any \( t > 0 \) and \( x,y \in X \). Then, we have

\[
|I_4(s)| \leq C \int_{r \leq \sqrt{s}} dv_Z \int_{|v| \leq r} \left( \frac{1}{s^2} + \frac{1}{s(1 + r^2)} \right) dx dy
\]

\[
\leq C \frac{1}{s^2} \int_{r \leq \sqrt{s}} r^2 dv_Z + C \int_{r \leq \sqrt{s}} dv_Z \leq C \frac{1}{s^2} \int_{r \leq \sqrt{s}} r^3 dr + C \frac{1}{s} \int_{r \leq \sqrt{s}} r dr \leq C.
\]

Suppose that \( s \leq 1 \). By Lemma 6.1, we get

\[
(6.23) \quad |I_3(s) + I_4(s)| \leq C \int_{\Omega(\sqrt{T}) \setminus \Omega(\sqrt{s})} e^{-\frac{|v|^2}{s}} dv_X \leq C s \int_{\sqrt{s}}^{\sqrt{T}} \frac{r dr}{(1 + r^2)^2} \int_{C} e^{-\frac{|v|^2}{s}} \frac{dx dy}{s} \leq C s.
\]

Together with (6.23-25), we get

\[
(6.25) \quad \left| \int_0^T \frac{ds}{s} (I_3(s) + I_4(s)) \right| \leq C \int_1^T \frac{ds}{s} + C \int_0^1 \frac{ds}{s} s \leq C (\log T + 1). \quad \square
\]
Lemma 6.6. Put \( q_{0;11}(z,0) = \partial_s \partial_t |_{v=0} q_0(z,v) \). For any \( T > 1 \), one has

\[
\int_{B(\sqrt{T})} K(s, x, lx) dv_X - \frac{1}{16\pi s} \int_{r \leq \sqrt{T}} dv_Z - \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) dv_Z
\]

\[
= \int_{B(\sqrt{T}) \setminus \Omega(\sqrt{T})} K(s, x, lx) dv_X + I_3(s) + I_4(s) - \frac{1}{16\pi s} \int_{r \leq \sqrt{T}} e^{-\frac{r}{s}} dv_Z
\]

\[
- \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) e^{-\frac{r^2}{s}} (1 + \frac{r^2}{s}) dv_Z + \int_{r \leq \sqrt{T}} (I_1(s, z) + I_2(s, z)) dv_Z.
\]

Proof. By the definition of \( p(s, x, y) \), \( J(z, v) \) and \( q_i^{(k)}(z, v) \), we get

\[
(4\pi s)^2 p(s, x, lx) J(z, v) = e^{-\frac{|v|^2}{s}} \left\{ q_0(z,0) + q_1^{(1)}(z, v) + q_0^{(2)}(z, v) \right\} + s e^{-\frac{|v|^2}{s}} q_1(z,0)
\]

\[
+ e^{-\frac{|v|^2}{s}} \left\{ q_0(z, v) - q_0(z,0) - q_1^{(1)}(z, v) - q_0^{(2)}(z, v) \right\} + s e^{-\frac{|v|^2}{s}} \left\{ q_1(z, v) - q_1(z,0) \right\}
\]

which, together with \( q_0(z,0) = 1 \), \( q_1(z,0) = 0 \) (Ric(\( \omega_X \)) = 0), (6.14) and the definition of \( I_1 \) and \( I_2 \), yields

\[
\int_{\Omega(\sqrt{T})} p(s, x, lx) dv_X - \int_{r \leq \sqrt{T}} (I_1(s, z) + I_2(s, z)) dv_Z
\]

\[
= \int_{\Omega(\sqrt{T})} \left\{ (4\pi s)^2 - e^{-\frac{|v|^2}{s}} + (4\pi s)^{-2} e^{-\frac{|v|^2}{s}} q_{0;11}(z,0) \right\} dv_X
\]

\[
= \int_{|v| \leq r+1} e^{-\frac{r^2}{s}} dv_Z + \int_{r \leq \sqrt{T}} q_{0;11}(z,0) dv_Z \int_{|v| \leq r+1} e^{-\frac{r^2}{s}} dv_X
\]

\[
- \frac{1}{16\pi s} \int_{|v| \leq r+1} dv_Z - \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) e^{-\frac{(r+1)^2}{s}} (1 + \frac{(r+1)^2}{s}) dv_Z.
\]

Therefore, we get

\[
\int_{\Omega(\sqrt{T})} p(s, x, lx) dv_X - \frac{1}{16\pi s} \int_{r \leq \sqrt{T}} dv_Z - \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) dv_Z
\]

\[
= - \frac{1}{16\pi s} \int_{r \leq \sqrt{T}} e^{-\frac{(r+1)^2}{s}} dv_Z - \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) e^{-\frac{(r+1)^2}{s}} (1 + \frac{(r+1)^2}{s}) dv_Z
\]

\[
+ \int_{r \leq \sqrt{T}} (I_1(s, z) + I_2(s, z)) dv_Z
\]

which, together with the definition of \( I_3, I_4 \) and the following, yields the assertion;

\[
\int_{B(\sqrt{T})} K(s, x, lx) dv_X - \frac{1}{16\pi s} \int_{r \leq \sqrt{T}} dv_Z - \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) dv_Z
\]

\[
= \int_{B(\sqrt{T}) \setminus \Omega(\sqrt{T})} K(s, x, lx) dv_X + \int_{\Omega(\sqrt{T})} \{ K(s, x, lx) - p(s, x, lx) \} dv_X
\]

\[
+ \int_{\Omega(\sqrt{T})} p(s, x, lx) dv_X - \frac{1}{16\pi s} \int_{r \leq \sqrt{T}} dv_Z - \frac{1}{16\pi} \int_{r \leq \sqrt{T}} q_{0;11}(z,0) dv_Z. \quad \Box
\]
Lemma 6.7. For any $s \in (0,1]$, one has

$$\left| \int_{B(\sqrt{T})} K(s,x,tx) dv_X - \frac{1}{s} \int_{r \leq \sqrt{T}} \frac{dv_Z}{16\pi} - \int_{r \leq \sqrt{T}} q_{0;1\bar{1}}(z,0) \frac{dv_Z}{16\pi} \right| \leq C \sqrt{s}.$$  

Proof. Put $I_5(s) := \int_{B(\sqrt{T}) \setminus \Omega(\sqrt{T})} K(s,x,tx) dv_X$, $I_6(s) := \int_{r \leq \sqrt{T}} s^{-1} e^{-\frac{r^2}{4s}} dv_Z$, and $I_7(s) := \int_{r \leq \sqrt{T}} q_{0;1\bar{1}}(z,0) e^{-\frac{r^2}{4s}} (1 + \frac{r^2}{s}) dv_Z$. For any $x \in B(\sqrt{T}) \setminus \Omega(\sqrt{T})$, we get $2d(x,tx) \geq r(x) + 1$ which, together with (6.23), yields that, for any $s > 0$,

$$|I_5(s)| \leq C \int_X s^{-2} e^{-\frac{r^2}{4s}} dv_X \leq C e^{-\frac{r}{2}}, \quad |I_6(s)| \leq C \int_Z s^{-1} e^{-\frac{r^2}{4s}} dv_Z \leq C e^{-\frac{r}{2}},$$

$$|I_7(s)| \leq C \int_Z e^{-\frac{(r+1)^2}{4s}} \left( 1 + \frac{(1+r)^2}{s} \right) \frac{dv_Z}{(1+r)^2} \leq C \int_0^\infty e^{-\frac{(1+r)^2+1}{4s}} r dr \leq C e^{-\frac{r}{2}}.$$  

By (6.10), (6.17) and (6.24), we get

$$\int_{r \leq \sqrt{T}} |I_1(s,z)| + |I_2(s,z)| \leq C \sqrt{s} \quad \text{and} \quad |I_3(s) + I_4(s)| \leq C s \quad \text{for any} \ s \in (0,1]$$  

which, together with (6.30) and Lemma 6.6, yields the assertion. □

Proposition 6.1. For any $T > 1$, one has

$$\left| \int_0^T \frac{ds}{s} \left\{ \int_{B(\sqrt{T})} K(s,x,tx) dv_X - \int_{r \leq \sqrt{T}} \left( \frac{dv_Z}{16\pi s} + q_{0;1\bar{1}}(z,0) \frac{dv_Z}{16\pi} \right) \right\} \right| \leq C (\log T + 1).$$  

Proof. By Lemmas 6.3-7 together with (6.30), we get

$$|I_5(s)| + |I_6(s)| + |I_7(s)| \leq C (\log T + 1).$$  

6.2 Singularity of Type $(2,1)$ and Asymptotics of $\tau_M$. Let us consider the same situation as in §5.2. Here, we assume that $(Y_0, t, o)$ is of type $(2,1)$ in the sense of (2.11). By Proposition 2.3, $Z_0$ has only one node at $o$. In the sequel, we use the same notations as in §5.2.

Lemma 6.8. There exists a function $I(T)$ defined for $T > 1$ such that, as $t \to 0$,

$$\log \tau(Y_t/\iota_t, \omega_Y) = I(|t|^{-1}) + O(1), \quad |I(T)| \leq C (\log T + 1).$$  

Proof. Let $K_t(s,x,y)$ be the heat kernel of $(Y_t, \omega_Y)$ and

$$K_t(s,x,x_t x) \sim (a_1(x,t) s^{-1} + a_0(x,t) + O(s)) \delta_{Z_t}(x) \quad (s \to 0)$$  

its pointwise asymptotic expansion ([B-G-V, Theorem 6.11]) where $\delta_{Z_t}$ is the Dirac $\delta$-function supported along $Z_t$ and $a_i(z,t)$ is a smooth function on $Z_t$. By Lemma 3.1, we get in the same manner as (5.25)

$$\log \tau(Y_t/\iota_t, \omega_Y) = \int_0^1 \frac{ds}{s} \left\{ \int_{Y_t} K_t(s,x,x_t x) dv_Y - \int_{Z_t} \left( \frac{a_1(z,t)}{s} + a_0(z,t) \right) dv_{Z_t} \right\}$$

$$+ \int_1^\infty \frac{ds}{s} \left\{ \int_{Y_t} K_t(s,x,x_t x) dv_Y - 1 \right\} + a_{-1}(t) - \Gamma'(1)(a_0(t) - 1)$$

where $\Gamma'(1)$ is the derivative of the Gamma function.
where \( a_i(t) := \int_{Y_t} a_i(x,t) \, dv_{Y_t} \). By the same argument as in the proof of Theorem 5.4 using (5.23) and Duhamel’s principle, we get (5.24). Put

\[
I(T) := \int_0^T \frac{ds}{s} \left\{ \int_{B(\sqrt{T})} K(s,x,\omega) \, dv_X - \frac{1}{s} \int_{r \leq \sqrt{T}} \frac{dv_Z}{16\pi} - \int_{r \leq \sqrt{T}} q_{0;11}(z,0) \frac{dv_Z}{16\pi} \right\}.
\]

By Lemma 6.7 and (5.24), the integrand of \( ds/s \) in \( I(T) \) should coincide with the asymptotic expansion (6.33) on \( W \cap Y_t \) and we get \( \int_{Y_t \cap W} a_0(z,t) \, dv_{Z_t} = \int_{r \leq |t|^{-\frac{1}{2}}} dv_{Z_t}/16\pi \) and \( \int_{Y_t \cap W} a_1(z,t) \, dv_{Z_t} = \int_{r \leq |t|^{-\frac{1}{2}}} q_{0;11}(z,0) \, dv_{Z_t}/16\pi \) which, together with (6.33), (5.24) and the definition of \( I(T) \), yields

\[
\int_0^1 \frac{ds}{s} \int_{Y_t \cap W} \left\{ K_1(s,x,\omega) - \frac{a_1(x,t)}{s} - a_0(x,t) \right\} \, dv_{Y_t} = I(|t|^{-1}) + O \left( \int_0^1 \frac{ds}{s} e^{-\frac{r}{s}} \right).
\]

In the same way as §5.2, we get (5.27) which, together with (6.33-34) and Proposition 6.1, yields the assertion. □

**Theorem 6.1.** As \( t \to 0 \), one has the following asymptotic formula:

\[
\log \tau_M(Y_t, \omega_t) = -\frac{1}{8} \log |t| + O(1).
\]

**Proof.** Let \( \omega_{Z_t} := \omega_{Y_t}|_{Z_t} \) be the induced metric on \( Z_t \). In the similar way as [B-B, Théorème 6.2], there exists an universal constant \( \beta \in \mathbb{R} \) such that one has

\[
\log \tau(Z_t, \omega_{Z_t}) = \beta \log |t| + O(\log \log |t|^{-1}) \quad \text{as} \quad t \to 0,
\]

which, together with Lemma 6.8 and Theorem 5.1, yields

\[
\log \tau_M(Y_t, \omega_t) = \bar{I}(|t|^{-1}) + O(\log \log |t|^{-1}), \quad |\bar{I}(T)| \leq C (\log T + 1)
\]

where \( \bar{I}(T) = I(T) + \beta \log T \). Let \( \eta(t) \) be a relative canonical form such that \( \eta(t) \neq 0 \). Let \( \omega_1(t), \ldots, \omega_q(t) \) be a basis of \( \pi_* \Omega^1_{Y/D} \). We may assume that \( \omega_1(0) \) has at most logarithmic poles at \( o \) and \( \omega_2(0), \ldots, \omega_q(0) \) are holomorphic as \( Z_0 \) has only one node. By Theorem 3.5, one has the following on \( D \setminus \{0\} \):

\[
\frac{i}{2\pi} \bar{\partial} \bar{\partial} \left\{ \log \tau_M(Y_t, \omega_t) + \frac{r(M) - 6}{8} \log \|\eta(t)\|^2 + \frac{1}{2} \log \det \left( \int_{Z_t} \omega_i(t) \wedge \bar{\omega}_j(t) \right) \right\} = 0.
\]

Since \( \log \|\eta(t)\|^2 = O(1) \) and \( \log \det \left( \int_{Z_t} \omega_i(t) \wedge \bar{\omega}_j(t) \right) = O(\log \log |t|^{-1}) \) as \( t \to 0 \), by the same argument as [B-B, Proposition 10.1] together with (6.35-36), there exists \( \alpha \in \mathbb{R} \) such that

\[
\log \tau_M(Y_t, \omega_t) = \alpha \log |t|^2 + O(1) \quad (t \to 0).
\]

By (6.35), \( \alpha = \lim_{T \to \infty} \bar{I}(T)/\log T \) is an invariant of the instanton \( (X, \omega_X) \).

To determine \( \alpha \), let us compute an example. Take \( M = \Pi_{1,1} \oplus E_8(-2) \). Let \( \Phi \) be Borcherds’s \( \Phi \)-function of weight 12 over \( \Omega \), the Hermitian domain of type IV associated to the lattice \( \Pi_{1,1} \oplus \Pi_{1,1} \oplus \Lambda_{24}(-1) \), with zero divisor \( D \) (the discriminant locus) where \( \Lambda_{24} \) is the 24-dimensional Leech lattice. (For \( \Phi \), see [B2,3]). Put
Ψ_{M^⊥} := \Phi|_{\Omega_M}$ for the restriction. As the orthogonal compliment of $E_8(2)$ in $\Lambda_{24}$ is the 16-dimensional Barnes-Wall lattice $\Lambda_{16}$ ([B2], [C-S]) which is free from roots, $\Psi_M$ is a nonzero holomorphic modular form of weight 12. Let $d \in \Delta(\Pi_{1,1} \oplus \Pi_{1,1} \oplus \Lambda_{24}(-1))$, and $\pi(d) \in \Lambda_{16}^\vee(-1)$ be the orthogonal projection. Then, $H_d \cap \Omega_M \neq \emptyset$ if and only if $-2 < \langle \pi(d), \pi(d) \rangle \leq 0$. Computing $\theta_{\Lambda_{16}^\vee}(\tau)$ (cf. (8.19)), we find that there is no norm 1 or 3 element in $\Lambda_{16}^\vee$, and thus $\pi(d) = 0$. Namely, the zero divisor of $\Psi_{M^⊥}$ coincides with $D_M$. Since $j_M$ takes its value in $S^2(A_1)$ by Theorem 2.5 and $j_M^*(\Delta_1 \Delta_2)$ vanishes of order 2 along $D_M$ (because the quotient map $\Omega_M \to M$ blanches of order 2 along $D_M$) where $\Delta_i$ is the Jacobi $\Delta$-function in the $i$-th variable, $\Delta' := \Psi_M \otimes j_M^*(\Delta_1 \Delta_2)$ vanishes of order 3 along $D_M$. As $D_M$ is the divisor of type (2, 1) by Theorem 2.5, it follows from Theorem 3.3 and (3.37) together with [B-B, Proposition 10.1] that

$$i \frac{\partial}{2\pi i} \partial \log \left[ \tau_M \parallel \Delta' \parallel \tau \right] = -\left( \alpha + \frac{1}{8} \right) \delta_{D_M}$$

outside of subvarieties of codimension $\geq 2$ where $\delta_{D_M}$ is the current $\int_{D_M}$. Since $\tau_M$ and $\parallel \Delta' \parallel$ are $\Gamma_M$-invariant, (6.38) can be regarded as an equation of currents over $M$ by Hartogus’s extension theorem. From the residue theorem, it follows that $\alpha = -1/8$.

Corollary 6.1. If $\Lambda = \Pi_{1,1} \oplus E_8(-2)$, there exists a constant $C_\Lambda \neq 0$ such that $\tau_\Lambda = \parallel \Delta_\Lambda \parallel^{-1/4}$ where $\Delta_\Lambda = C_\Lambda \Phi|_{\Omega_\Lambda} \otimes j_M^*(\Delta_1 \Delta_2)$ and $\Phi$ is the denominator function of the fake monster Lie algebra.

§7. Identification of $\tau_M$ with an Automorphic Form

Let $A_g$ be the Siegel modular variety and $A_g^*$ be the Satake compactification. Let $\hat{j}_M : \Omega_M \dashrightarrow A_g(M)$ be the rational map as in Proposition 2.4, and $\hat{\Omega}_M = \Gamma \times j_M(\Omega_M)$ be the closure of the graph of $j_M$ in $\Omega_M \times A_g(M)$. Let $p_1 : \hat{\Omega}_M \to \Omega_M$ and $p_2 : \hat{\Omega}_M \to A_g$ be the morphisms induced by the projections. We regard $j_M^* \omega_{A_g}$ (which is originally defined on $\Omega_M^0$) as a current on $\Omega_M^0 \cup D_M$ by putting $j_M^* \omega_{A_g} = p_1^* p_2^* \omega_{A_g}$ where $D_M^0 := \bigcup_{\delta \in \Delta(N)} H_\delta^0$ ($H_\delta^0 = H_\delta \setminus \bigcup_{d \neq \pm \delta} H_d$).

Theorem 7.1. One has the following equation of currents on $\Omega_M^0 \cup D_M^0$:

$$i \frac{\partial}{2\pi i} \partial \log \tau_M = \frac{1}{8} \delta_{D_M} - \frac{r(M) - 6}{8} \omega_M - \frac{1}{2} j_M^* \omega_{A_g(M)}.$$

Proof. Theorems 3.5, 5.2, 6.1, together with Bismut-Bost’s extension argument ([B-B, Proposition 10.2]), yields the assertion. □

Let $F_g^0 (g = g(M))$ be the sheaf of Siegel modular forms of weight 1 over $A_g$. Set $\lambda_M := i_* \mathcal{O}_{\Omega_M}(j_M^* F_g^0)$ where $i : \Omega_M^0 \cup D_M^0 \hookrightarrow \Omega_M$ is the inclusion. It is an invertible sheaf on $\Omega_M$ because $j_M^0 : \Omega_M^0 \cup D_M^0 \to A_g$ is regular and is defined out side of subvarieties of codimension 2. Since $j_M^0$ is independ of markings, $\lambda_M$ is invariant under $\Gamma_M$. Namely, it is a $\Gamma_M$-module. Let $\chi : \Gamma_M \to \mathbb{C}^*$ be a character.
**Definition 7.1.** \( f \in H^0(\Omega_M, \lambda_M^{\otimes q}) \) is said to be an automorphic form of weight \((p, q)\) with character \(\chi\) if
\[
f(\gamma \cdot z) = \chi(\gamma) j(\gamma, z)^p \gamma^* f(z)
\]
for any \(z \in \Omega_M\) and \(\gamma \in \Gamma_M\) where \(j(\gamma, z) := (\gamma \cdot z, l_M)/(z, l_M)\) is an automorphic factor. The Petersson norm, \(\|f\|\), is defined by
\[
\|f(z)\|^2 := K_M(z, \bar{z})^p \det \text{Im}(j_M(z))^q |f(z)|^2.
\]

Here, \(K_M(z, \bar{z})\) is the Bergman kernel and \(l_M \in N_C\) is the same vector as in (3.4).

In the sequel, we often omit character. Thus, an automorphic form is rigorously speaking an automorphic form with some character. Since \(\log j_M\|\psi\|\) becomes a locally integrable function on \(\Omega_M\) for any meromorphic Siegel modular form \(\psi\), the curvature current of \((\lambda_M, \|\cdot\|)\) \((\|\cdot\|\) is the Petersson norm) can be defined in the usual manner, and coincides with \(c_1(\lambda_M, \|\cdot\|) = j_M^* \omega_{A_g}\) on \(\Omega_M^0 \cup D_M^0\).

**Theorem 7.2.** Suppose \(r(M) \leq 17\). Then, there exists a modular form \(\Delta_M\) of weight \((r(M) - 6, 4)\) such that \(\tau_M = \|\Delta_M\|^{-1/4}\) and \(\text{div}(\Delta_M) = D_M\).

**Proof.** Take a non-zero meromorphic modular form \(\phi\) of weight \((r - 6, 4)\) such that \(D_M\) is not contained in the zero and polar locus of \(\phi\). Put \(F := \tau^8_M \|\phi\|^2\). By Theorem 7.1 and Hartog's theorem, one has \(\frac{\partial}{\partial z} \text{log } F = \delta_{D_M} - \delta_{\text{div}(\phi)}\) on \(\Omega_M\).

Therefore, \(\partial \text{log } F\) is a \(\Gamma_M\)-invariant meromorphic 1-form on \(\Omega_M\) with at most logarithmic poles, and thus \(G(y) := \exp (\int_y^y \partial \text{log } F)\) \((\ast\) is a reference point in \(\Omega_M\)\) is a meromorphic function on \(\Omega_M\) such that \(\text{div}(G) = D_M - \text{div}(\phi)\). Let \(\gamma \in \Gamma_M\).

Let \([\gamma]\) be a simple closed real curve in \(M_M = \Omega_M/\Gamma_M\) corresponding to a path joining \(y\) and \(\gamma \cdot y\). As \(\Omega_M\) is diffeomorphic to the cell, the homotopy class of \([\gamma]\) does not depend on a choice of \(y\). Thus, \(\chi(\gamma) = \exp (\int_{[\gamma]} \partial \text{log } F) = G(\gamma \cdot y)/G(y)\) is independent of \(y\) and becomes a character of \(\Gamma_M\). Since \(\Gamma_M/\Gamma_M, \Gamma_M\) is finite by Kazhdan's theorem ([Kz]), \(\chi\) takes its values in \(S^1 = U(\mathbb{C})\). Therefore, \(\log |G|^2\) is a \(\Gamma_M\)-invariant pluriharmonic function whose divisor is the same as \(\log F\).

Since the Satake-Baily-Borel boundary of \(M_M\) has codim \(\geq 2\) when \(r(M) \leq 17\), Hartog's theorem implies that there exists a constant \(C \neq 0\) such that \(F = C^2 |G|^2\).

Thus, we get \(\tau^8_M = C^2 |G|^2 \|\phi\|^2\), and \(\Delta_M := CG \cdot \phi\) is the desired form. \(\square\)

Let \(\delta \in \Delta(N)\). If \((M \oplus \delta)\) denotes the smallest 2-elementary lattice generated by \(M, \delta\), then \(\Omega_{(M \oplus \delta)}\) can naturally be identified with \(H_\delta\). As \(\Delta_M\) is a section of \(\Lambda_M^\otimes 4\) which vanishes of order one along \(H_\delta\), it follows from Proposition 2.5 that \(\Delta_M \langle \cdot, l_M \rangle/(\cdot, l_M)\) restricted to \(H_\delta^0\) is a section of \(\Lambda_{M \oplus \delta}^\otimes 4\). Note that if \(g((M \oplus \delta)) = g(M) - 1, A_{g-1}\) is considered to be one of the boundary components of \(A_g\), and the restriction map \(S : \mathcal{F}_g^\otimes 4 \rightarrow \mathcal{F}_g^{\otimes 4}_{g-1}\) coincides with the Siegel operator ([F]).

**Theorem 7.3.** Under the identification \(\Omega_{(M \oplus \delta)} = H_\delta\), one has
\[
\Delta_{(M \oplus \delta)}(y) = C_{(M \oplus \delta)} \lim_{z \to y} \langle \cdot, l_M \rangle \Delta_M(z) = C(M, \delta) \langle \cdot, l_M \rangle \Delta_M |_{H_\delta}(y).
\]

Here \(C_{(M \oplus \delta)}\) is a nonzero constant.

**Proof.** Put \(\delta := \delta_0\). We separate the proof into two cases.

**Case (1)** Assume \(\langle \delta_0, \delta_1 \rangle = -1\) for some \(\delta_1 \in \Delta(N)\). With this condition, \(M \oplus \mathbb{Z}\delta_0\)
is primitive in $L_K$ and $\langle M + \delta_0 \rangle = M + \mathbb{Z}\delta_0$ which implies $g(\langle M + \delta_0 \rangle) = g(M) - 1$. By the condition, $\mathbb{Z}\delta_0 + \mathbb{Z}\delta_1 = A_2(-1)$ where $A_2$ is the $A_2$-root lattice whose roots are $\pm\delta_0, \pm\delta_1, \pm\delta_2$ where $\delta_2 := \delta_0 - \delta_1$.

**Step 1.** Take $\omega_0 \in (H_{\delta_0} \cap H_{\delta_1})^0 := H_{\delta_0} \cap H_{\delta_1} \setminus \cup_{d \neq \pm \delta_1} H_d$. Choose $\kappa_0 \in C(M)$ such that $\langle \kappa_0, d \rangle \neq 0$ for any $d \in \Delta(M)$. By the surjectivity of the period map, there exists a marked $K$ surface $(X_0, \phi)$ with nef and big line bundle $L_0$ such that $\pi(X_0, \phi) = \omega_0$ and $\phi(c_1(L_0)) = \kappa_0$. Choosing $0 < \epsilon \ll 1$, we may suppose that there exists a Kähler class whose image by $\phi = \kappa := \kappa_0 - \epsilon\delta_0$. Let $C_{\delta_i}$ be the cycle corresponding to $\delta_i$. As $\langle \kappa, \delta_i \rangle > 0$, any $C_{\delta_i}$ is effective. We choose $\epsilon$ so small that $C_{\delta_1}, C_{\delta_2}$ are irreducible $-2$-curves and $C_{\delta_i} = C_{\delta_1} \cup C_{\delta_2}$.

Let $U$ be a small neighborhood of $(X_0, \phi)$ in $\tilde{\Omega}_M$, and $(X, \phi) \to U$ be the universal family. By construction, there exists a relatively nef and big line bundle $L \to X$ which restricted to $X_0$ is $L_0$. Choosing $m \gg 1$, let $\Phi_{|mL|}: X \to \mathbb{P}^N$ be the morphism associated to the complete linear system $|mL|$, and $Y := \Phi_{|mL|}(X) \to U$ be the image. As $\kappa_0$ does not intersect $\delta_1$, the cycle $C_{\delta_1}$ corresponding to $\delta_1$ is an irreducible $-2$-curve in $X_t$ if $t \in H_{\delta_1}^0$ (cf. Lemma 2.3). By above construction, $C_{\delta_0}$ splits into two components if $t \in (H_{\delta_0} \cap H_{\delta_1})^0$: $C_{\delta_0} = C_{\delta_1} \cup C_{\delta_2}$. By Mayer’s theorem, $\Phi_{|mL|}: X_t \to Y_t$ is the minimal resolution whose exceptional locus is $C_{\delta_1}$ if $t \in H_{\delta_1}^0 (i = 0, 1, 2)$, $C_{\delta_0} = C_{\delta_1} \cup C_{\delta_2}$ if $t \in (H_{\delta_0} \cap H_{\delta_1})^0$, and empty if outside of the discriminant locus. Put $o \in Y_0$ for the image of $\tilde{C}_{\delta_0}$. As $\mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2$ is the $A_2$-root system, $(Y_0, o)$ is a $K$3 surface with an $A_2$-singularity. Set

$$M(A_2) := \{ \alpha \in \mathbb{C}^3; \sum \alpha_i = 0 \}, \quad Z := \{(x, \alpha) \in \mathbb{C}^3 \times M(A_2); x_1 x_2 - \prod (x_3 + \alpha_i) = 0\}.$$  

Consider the deformation of $A_2$-singularity $(Z, 0) \to (M(A_2), 0)$ on which acts $S_3 = W(A_2)$ by the permutation of coordinates $\alpha$. Then, $(Z/S_3, 0) \to (M(A_2)/S_3, 0)$ is the semiumiversal deformation of $A_2$-singularity. The discriminant locus of $Z \to M(A_2)$ is $D = D_0 \cup D_1 \cup D_2$ where $D_i = \{ \alpha \in M(A_2); \alpha_k - \alpha_j = 0 \}$ (if $i, j, k \in \{0, 1, 2\}$).

By the versality, there exist maps $F: (Y, o) \to (Z, 0)$ and $f: (U, 0) \to (M(A_2), 0)$ which commute with the projections such that $(Y, o) = F^*(Z, 0)$ and $f|H_0 = D_i$.

Let $\phi' = I_M \circ \phi$ (cf. (2.6)) be another marking, and $(X', \phi') \to U'$ be the universal family such that $U' = I \circ U$ (cf. (2.8)). Let $L' \to X'$ be the relatively nef and big line bundle such that $\phi'(c_1(L'_p)) = \kappa_0$. By the similar construction as before, we get a family $Y' \to U'$ such that $X' \to Y'$ is the simultaneous resolution. Since $\phi^{-1}(\kappa_0)$ and $(\phi')^{-1}(\kappa_0)$ are weak polarizations ([Mo, pp.318]) of $(X_t, \phi)$ and $(X'_t, \phi')$ respectively, we can define the weakly polarized period map ([Mo, pp.318]) $U \to V$ and $U' \to V$ by sending $[(\omega_1), \kappa_t]$ to $[(\omega_1), \kappa_0]$ where $[\omega_1]$ is the period and $\kappa_t$ is the Kähler class. Note that the weak polarized period domain in our situation is $\Omega_M \times C(M)^+$ and $V$ is its subset. As $U \to V$ (resp. $U' \to V$) is an isomorphism, we may regard $Y$ and $Y'$ are families over $V$. Since there exists the universal marked family of generalized $K$3 surfaces over $\Omega_M \times C(M)^+$ ([Mo, pp.321]), we get an identification $e: Y' \cong Y$. Let $p: X \to Y$ and $p': X' \to Y'$ be the simultaneous resolution, and $\iota_M: X \to X'$ be the isomorphism as in (2.8). Then, $\iota := e \circ p' \circ \iota_M \circ p^{-1}$ is a rational automorphism over $Y$. By the weakly polarized global Torelli theorem ([Mo, pp.319]), $\iota_t$ is an anti-symplectic involution on $Y_t$ for any $t \in V$ and $\iota$ is holomorphic everywhere. By appropriate normalizations, we may suppose $(Y, o) = (Z, 0)$ and $e(x_0, x_1, x_2) = (x_1, x_0, x_2)$ in (7.1).

**Step 2.** Let $C \to V$ be the family of fixed curves. By Theorem 2.5, we get the decomposition $C = C^{(g)} + \sum E_i$ where $C^{(g)}_t$ is a smooth irreducible curve of genus

37
\( g = g(M) \) for generic \( t \in V \) and \( E_i \) is a family of smooth rational curves. As \( C_0^{(g)} \) is a fixed locus, \( \text{Sing } C_0^{(g)} = \text{Sing } Y_0 = o \). Moreover it is a Cartier divisor because 
\[
(C_0^{(g)}, o) = (Z_0, 0) \cap \{ x_0 - x_1 = 0 \} = \{ (x_2, x_3); x_2^2 - x_3^3 = 0 \}. \]
As \( C_0^{(g)} \cdot C_0^{(g)} = 2(g - 1) \geq 0, C_0^{(g)} \) is a nef and effective divisor. Since \( (Y_0, o) \) is an \( A_2 \)-singularity, \( o \) can not belong to the fixed part of the linear system \( |C_0^{(g)}| \) by Saint-Donat’s theorem. Therefore, we can pick up \( f_1, \cdots, f_g \), nonconstant meromorphic functions on \( Y \), such that \( \text{div}(f_i) = P_i - C^{(g)} \) with \( o \not\in P_g \) but \( \text{Sing } Y_i \in P_i \) for \( 1 \leq i < g \). Let \( \omega_{Y/V} \) be a nowhere vanishing relative 2-form on \( Y \). Let \( \varphi_i(t) \) be the 1-form on \( C_t^{(g)} \) defined by \( \varphi_i(t) := \text{Res}_C^{(g)} f_i \omega_{Y/V} \). As \( \text{Sing } Y_i \) is contained in the zero locus of \( f_j \) for \( j < g, \varphi_j(t) \) \( (j < g) \) becomes a holomorphic 1-form on the normalization of \( C_t^{(g)} \) when \( t \in H_{\delta_0} \). Thus, if \( j < g, \)
\[
(7.2) \quad \int_{C_t^{(g)}} \varphi_i(t) \wedge \overline{\varphi_j(t)} \rightarrow \int_{C_0^{(g)}} \varphi_i(0) \wedge \overline{\varphi_j(0)} < \infty \quad (t \to H_{\delta_0}).
\]
Let \( A_1, B_1, \cdots, A_g, B_g \) be the symplectic basis of \( H^1(C_t^{(g)}, \mathbb{Z}) \). As \( (C_0^{(g)}, 0) \) is the cusp, we may assume that \( A_g \) and \( B_g \) are vanishing cycles, and all other cycles converges to the symplectic basis of the normalization of \( C_0^{(g)} \). Since we have fixed the symplectic basis, \( j_M \) takes its value in \( \mathfrak{g} \),
\[
(7.3) \quad \tau(j_M(t)) = \left( \int_{A_i} \varphi_j(t) \right)^{-1} \left( \int_{B_i} \varphi_j(t) \right) \in \mathfrak{g}.
\]
As the bidegree \((1, 1)\) part of the diagonal in \( C_t^{(g)} \times C_t^{(g)} \) is homologous to \( \sum A_i^{(1)} B_i^{(2)} - B_i^{(1)} A_i^{(2)} \) in \( H^2(C_t^{(g)} \times C_t^{(g)}; \mathbb{Z}) \), it follows from (7.2) that
\[
(7.4) \quad \int_{C_t^{(g)}} \varphi_g(t) \overline{\varphi_g(t)} = \text{Im} \tau_{gg}(j_M(t)) \left| \int_{A_g} \varphi_g(t) \right|^2 + O(1).
\]
Let \( E_\beta := \{(x_2, x_3); x_2^2 = (x_3 + 1)(x_3 + \beta)(x_3 - 1 - \beta)\} \) be an elliptic curve and \( \gamma_1, \gamma_2 \) be the symplectic basis such that \( \gamma_1 \) converges to the cycle \( |x_3 + 1| = \epsilon \) as \( \beta \to 0 \). Since \( o \not\in P_g \), multiplying a constant if necessary, we may assume \( f_g = 1/\{x_2^2 - \prod(x_3 + \alpha_i)\} + O(1) \) on a neighborhood of \( o \). When \( t \to H_{\delta_0}, \alpha(t) \to D_0 \) and \( \alpha_2(t) \to \alpha_1(t) \). Putting \( \beta = \alpha_1/\alpha_2 \), we may suppose \( A_g \) is identified with \( \gamma_1 \) and get
\[
(7.5) \quad \int_{A_g} \varphi_g(t) = \int_{A_1} \frac{dz}{\sqrt{(z + \alpha_0)(z + \alpha_1)(z + \alpha_2)}} + O(1)
\]
\[
= \alpha_2^{-\frac{1}{2}} \int_{\gamma_1} \frac{dz}{\sqrt{(z + 1)(z + \beta)(z - 1 - \beta)}} + O(1) = \frac{2\pi}{\sqrt{3}} \alpha_2^{-\frac{1}{2}} + O(1).
\]
Let \( W \) be a small neighborhood of \( \omega_0 \in (H_{\delta_0} \cap H_{\delta_1})^0 \) in \( H_{\delta_0}^0 \). Since
\[
(7.6) \quad \left| \frac{\det \left( \int_{C^{(g)}} \varphi_i \varphi_j \right)}{\int_{C^{(g)}} \varphi_g \overline{\varphi_g}} \right|_W = \det \left( \int_{C^{(g)}} \varphi_i \varphi_j \right)_{i,j<g} \bigg|_W
\]

38
by (7.2-5), it follows that
\[
\frac{i}{2\pi} \bar{\partial} \left[ \log \left( \frac{\det \left( \int_C \varphi_i \overline{\varphi_j} \right)}{j_M^* \text{Im} \tau_{gg}} \right) \right]_W
\]
\[
= \frac{i}{2\pi} \bar{\partial} \left[ \log \left( \left| \int_{A^g} \varphi_g \right|^2 \right) \right]_W + \frac{i}{2\pi} \bar{\partial} \left[ \log \det \left( \int_C \varphi_i \overline{\varphi_j} \right)_{i,j<g} \right]_W
\]
\[
= \frac{1}{2} \delta_{H_{\delta_0} \cap H_{\delta_1} \cap W} + j_{(M \oplus \delta_0)}^* \omega_{\delta_0} \omega_{\delta_0} \omega_{\delta_0} |W|.
\]

Since \( H_d \cap H_{\delta_0} \cap W \neq \emptyset \) (\( d \in \Delta(N) \)) iff \( d = \pm \delta_0, \pm \delta_1, \pm \delta_2 \), it follows from Theorem 7.1 that
\[
\frac{i}{2\pi} \bar{\partial} \log \left[ \tau_M \det \left( \int_C \varphi_i \overline{\varphi_j} \right)^\frac{1}{2} \left( \frac{|\langle w, \delta_0 \rangle|^2}{\langle w, \overline{w} \rangle} \right)^\frac{1}{2} \right]_W
\]
\[
= \frac{1}{8} \sum_{d \in \Delta(N) \setminus \{ \pm \delta_0 \} / \pm 1} \delta_{H_d \cap H_{\delta_0} \cap W} - \frac{r(M) - 5}{8} \omega(M \oplus \delta_0)
\]
\[
= \frac{1}{4} \delta_{H_{\delta_0} \cap H_{\delta_1} \cap W} - \frac{r(M \oplus \delta_0) - 6}{8} \omega(M \oplus \delta_0)
\]
which, together with (7.7), yields
\[
\frac{i}{2\pi} \bar{\partial} \log \left[ \tau_M (j_M^* \text{Im} \tau_{gg})^\frac{1}{2} \left( \frac{|\langle w, \delta_0 \rangle|^2}{\langle w, \overline{w} \rangle} \right)^\frac{1}{2} \right]_W
\]
\[
= - \frac{r(M \oplus \delta_0) - 6}{8} \omega(M \oplus \delta_0) - \frac{1}{2} j_M^* \omega_{\delta_0} \omega_{\delta_0} \omega_{\delta_0} |W|.
\]

**Step 3.** Let \( \delta_3 \in \Delta(N) \) such that \( \langle \delta_3, \delta_0 \rangle = 0 \). Take \( \omega_0 \in (H_{\delta_0} \cap H_{\delta_3})^0 \) and choose \( \kappa_0 \in C_M \) as in Step 1. Similarly as before, there exists a marked K3 surface \( (X_0, \phi) \) with nef and big line bundle \( L_0 \) such that \( \pi(X_0, \phi) = \omega_0 \) and \( \phi(c_1(L_0)) = \kappa_0 \). Let \( C_{\delta_i} \) be the cycle corresponding to \( \delta_i \). Then, \( C_{\delta_0} \) and \( C_{\delta_4} \) are mutually disjoint irreducible -2-curves. Let \( U \) be a small neighborhood of \( (X_0, \phi) \) in \( \Omega_M \), \( (X, \phi) \to U \) the universal family, and \( L \to X \) the relatively nef and big line bundle whose restriction to \( X_0 \) is \( L_0 \). Using \( |mL| \), set \( Y := \Phi |mL| (X) \to U \) for \( m \gg 1 \). As \( \langle \kappa_0, \delta_i \rangle = 0 \), the cycle \( C_{\delta_i} \) corresponding to \( \delta_i \) is an irreducible -2-curve in \( X_t \) if \( t \in H_{\delta_i}^0 \). Thus, \( \Phi |mL| : X_t \to Y_t \) is the minimal resolution whose exceptional locus is \( C_{\delta_i} \) if \( t \in H_{\delta_i}^0 \) (\( i = 0, 3 \)), \( C_{\delta_0} \cup C_{\delta_3} \) if \( t \in (H_{\delta_0} \cap H_{\delta_3})^0 \), and empty if outside of the discriminant locus. Put \( o_i \) for the image of \( C_{\delta_i} \). By construction, \( (Y_0, o_0, o_3) \) is a K3 surface with two \( A_1 \)-singularities. Let \( \iota : Y \to Y \) be the anti-symplectic involution constructed as before, and \( C^{(g)} \to U \) be the family of fixed curves of maximal genus. By construction, \( C^{(g)} \to U \) is f.s.o. in the sense of [B-B] (though the total space admits two nodes here). Since \( C^{(g)}_t \) has a node for \( t \in H_{\delta_0}^0 \) (because \( g(M \oplus \delta_0) = g(M) - 1 \)), we find \( o_0 \in C^{(g)}_0 \). Let \( \varphi_1(t), \ldots, \varphi_g(t) \) be a basis of relative 1-forms for \( C \to U \) such that \( \varphi_g(t) \) has a logarithmic pole at \( \text{Sing} C^{(g)}_t \) if \( t \in H_{\delta_0} \) and that \( \varphi_1(t), \ldots, \varphi_{g-2}(t) \) are regular on the normalization of \( C^{(g)}_t \) for
any \( t \in U \). If \( o_3 \in C_0^{(g)} \), we may suppose that \( \varphi_{g-1}(0) \) has a logarithmic pole at \( o_3 \) and \( \varphi_{g-1}(t) \) is regular on the normalization of \( C_0^{(g)} \) for \( t \in H_0^{(g)} \). If \( o_3 \notin C_0^{(g)} \), we may assume that \( \varphi_{g-1}(t) \) is regular on the normalization of \( C_t^{(g)} \) for any \( t \in U \). Let \( W \) be a small neighborhood of \( \omega_0 \in (H_0 \cap H_{\delta_3})^0 \) in \( H_\delta \). From \([B-B, \text{Proposition 13.3}]\) together with Proposition 2.5, it follows that

\[
\frac{i}{2\pi} \partial \bar{\partial} \left[ \log \left( \frac{\det \left( \int_{\Omega} \varphi_i \varphi_j \right)}{j_M^{*} \text{Im} \tau_{gg}} \right) \bigg|_W \right] = \frac{i}{2\pi} \partial \bar{\partial} \left[ \log \left( \frac{\det \left( \int_{\Omega} \varphi_i \varphi_j \right)}{j_M^{*} \text{Im} \tau_{gg}} \right) \bigg|_W \right] = j_{(M+\delta_0)}^{*} \omega \varphi_{g-1} \bigg|_W.
\]

Since \( H_d \cap H_{\delta_3} \cap W \neq \emptyset \) \((d \in \Delta(N)) \) iff \( d = \pm \delta_0, \pm \delta_3 \), comparing (7.8) and (7.10), we get

\[
\frac{i}{2\pi} \partial \bar{\partial} \log \left[ \tau_M \left( j_M^{*} \text{Im} \tau_{gg} \right) \right] \left( \left\| \langle \omega, \omega, \rangle \right\| \left\| \langle \omega, \omega, \rangle \right\| \right) \bigg|_W = \frac{1}{8} \delta_{H \cap H_{\delta_3} \cap W} - \frac{r((M \oplus \delta_0) - 6)}{8} \omega_{(M \oplus \delta_0)} - \frac{1}{2} j_{(M+\delta_0)}^{*} \omega \varphi_{g-1} \bigg|_W.
\]

**Step 4.** In view of the proof of Theorem 7.2, we may write \( \Delta_M = \psi \otimes j_M^{*} E \) where \( \psi \) is a meromorphic modular form over \( \Omega_M \) of weight \( g(r-6) \) and \( E \) is an Eisenstein series of weight \( 4g \) \((|F|)\) whose divisor does not contain the boundary component \( A_g \setminus A_g \). For \( \tau = (\tau_{ij})_{i,j \leq g} \in \Gamma_g \), write \( \tau' = (\tau_{ij})_{i,j \leq g} \in \Gamma_{g-1} \). Let \( S : H^0(A_g, F_g^k) \rightarrow H^0(A_{g-1}, F_{g-1}^k) \) be the Siegel operator. Since \( j_M(H_{\delta_0}) \subset A_{g-1} \), it follows from Proposition 2.5 that

\[
\left( \frac{\langle \cdot, l_M \rangle}{\langle \cdot, \delta_0 \rangle} \Delta_M \right)^g \bigg|_{H_{\delta_0}} = \left( \frac{\langle \cdot, l_M \rangle}{\langle \cdot, \delta_0 \rangle} \right)^g \psi \bigg|_{H_{\delta_0}} \otimes j_{(M+\delta_0)}^{*} S(E).
\]

From the definition of Petersson norm, it follows that

\[
\| \frac{\langle \cdot, l_M \rangle}{\langle \cdot, \delta_0 \rangle} \Delta_M \|^g \bigg|_{H_{\delta_0}} \bigg( w \bigg) = K_M^{-g(r-5)} \left( \frac{\langle \cdot, l_M \rangle}{\langle \cdot, \delta_0 \rangle} \right)^g \bigg|_{z=w} 2 j_M^{*}(\text{det Im} \tau')^{4g} |S(E)|^2
\]

\[
= \| \Delta_M \|^g j_{(M+\delta_0)}^{*} \text{Im} \tau_{gg} \|^{-4g} \left( \left| \frac{\langle \cdot, \delta_0 \rangle}{\langle \cdot, \delta_0 \rangle} \right|^{g} \right) \bigg|_{z=w}
\]

\[
= \tau_M^{-8g} j_{(M+\delta_0)}^{*} \text{Im} \tau_{gg} \|^{-4g} \left( \left| \frac{\langle \cdot, \delta_0 \rangle}{\langle \cdot, \delta_0 \rangle} \right|^{g} \right) \bigg|_{z=w}
\]

where \( K_M \) is the Bergman kernel as in (3.4). Comparing (7.9) and (7.11) with (7.13), \( \| \Delta_M \cdot \frac{\langle \cdot, l_M \rangle}{\langle \cdot, \delta_0 \rangle} \|_{H_{\delta_0}} \|^2 \) satisfies the same \( \partial \bar{\partial} \)-equation as \( \| \Delta_{(M+\delta_0)} \|^2 \) over \( \Omega_{(M+\delta_0)}^0 \cup D_{(M+\delta_0)}^0 \). In view of the proof of Theorem 7.2, solution of the \( \partial \bar{\partial} \)-equation in Theorem 7.1 is unique up to constant. Thus, we get the assertion.

**Case (2)** Suppose that there is no \( d \in \Delta(N) \) such that \( |\langle d, \delta_0 \rangle| = 1 \). Thus, for any \( d \in \Delta(N) \), \( H_d \cap H_{\delta_0} \neq \emptyset \) iff \( \langle d, \delta \rangle = 0 \). Now, we can prove the assertion in the same way as Step 3 and 4 in Case (1), and details are left to the reader. \( \square \)
8.1 Borcherds’s Product. For a lattice $M$ of signature $(2, b^-)$, let $\{e_{\gamma}\}_{\gamma \in M^+/M}$ be the standard unitary basis of the group ring $\mathbb{C}[M^+/M]$. We denote by $\rho_M$ the Weil representation of the metaplectic group $Mp_2(\mathbb{Z})$.

\begin{equation}
\rho_M(T)e_{\gamma} = e^{\pi i(\gamma,T)e_{\gamma}}, \quad \rho_M(S)e_{\gamma} = \frac{\sqrt{b^-}}{\sqrt{|M^+/M|}} \sum_{\delta \in M^+/M} e^{2\pi i(\gamma,\delta)}e_{\delta}
\end{equation}

where $T = ((11), 1)$, $S = ((0^{-1}), 1)$ are the generators of $Mp_2(\mathbb{Z})$. In this section, we suppose $M = \Pi_{1,1}(N) \oplus K$ ($N = 1, 2$ and $K$ is a hyperbolic 2-elementary lattice.) Let $F(\tau) = \sum_{\gamma \in M^+/M} e_{\gamma} \sum_{k \in \mathbb{Q}} c_{\gamma}(k) q^k$ be a nearly holomorphic modular form of type $\rho_M$ with weight $1 - b^-/2$ which has the integral Fourier coefficients at the cusp. We denote by $\Psi_M(z, F)$ Borcherds’s product attached to $F(\tau)$ ([Bo5]);

\begin{equation}
-\log \|\Psi_M(z, F)\|^4 = \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} \Theta_M(\tau, z) F(\tau) y \, dx \, dy / y^2 + c_0(0)(\Gamma'(1) + 2 \log \sqrt{2\pi}).
\end{equation}

$\Psi_M(z, F)$ is a function on the Grassmannian $G(2, b^-)$ which is isomorphic to the tube domain $K_{\mathbb{R}} + \sqrt{-1}C(K^+)$. The following theorem is due to Borcherds.

**Theorem 8.1 ([Bo5, Theorem 13.3]).**

1. $\Psi_M(z, F)$ is an automorphic form on $G(2, b^-)$ for some arithmetic subgroup of $O(M)$ of weight $c_0(0)/2$.
2. The zeros or poles of $\Psi_M(z, F)$ lies on the divisor $\lambda^\perp$ ($\lambda \in M, \lambda^2 < 0$) of order $\sum_{0 < x \in \mathbb{R}, x \lambda \in M^\vee} c_{x\lambda}(x^2\lambda^2/2)$.
3. $\Psi_M(z, F)$ admits the following holomorphic infinite product expansion near the cusp and $z \in K_{\mathbb{R}} + \sqrt{-1}W$;

\begin{equation}
\Psi_M(z, F) = e^{2\pi i(\rho(K, W, F_K), z)} \prod_{\lambda \in K^\vee \cap W^\vee} \prod_{n \in \mathbb{Z}/N\mathbb{Z}} (1 - e^{2\pi i(\lambda, z+n/N)})^{c_{\lambda+n/f}/N(\lambda^2/2)}
\end{equation}

where $\rho(K, W, F_K)$ is the Weyl vector, $W$ is a Weyl chamber, $W^\vee$ the dual cone of $W$, and $f, f'$ are the generators of $\Pi_{1,1}(N)$ such that $f \cdot f = f' \cdot f' = 0, f \cdot f' = N$.

8.2 2-Elementary $K3$ Surfaces with $(g, \delta) = (0, 1)$. By Nikulin’s table ([Ni4]), all the primitive 2-elementary hyperbolic lattices in $L_{K3}$ with $(g, \delta) = (0, 1)$ are isometric to one of the following $S_k$ ($1 \leq k \leq 9$).

Let $\delta_0, \delta_1, \ldots$ be the basis of $I_{1,9}(2)$ such that $h^2 = 2, h \cdot \delta_i = 0, \delta_i \cdot \delta_j = -2\delta_{ij}$. Then, $I_{1,8}(2) = A_1 \oplus A_1(-1)^{\oplus 8} = \mathbb{Z} h \oplus \mathbb{Z} \delta_1 \oplus \cdots \oplus \mathbb{Z} \delta_8 = I_{1,9}(2) \cap \delta_0^\perp$. Put

\begin{equation}
\rho := (3h - \delta_0 - \cdots - \delta_8)/2 \in I_{1,9}(2)^\vee, \quad \kappa := 3h - (\delta_1 + \cdots + \delta_8).
\end{equation}

$\Lambda$ is a sublattice of $I_{1,9}(2)^\vee$ defined by $\Lambda = \mathbb{Z} \rho \oplus \mathbb{Z} \delta_0 \oplus \rho^\perp \cap \delta_0^\perp \cap I_{1,9}(2)$. Note that $\kappa^\perp \cap I_{1,8}(2) \cong E_8(-2)$ (cf. [Man]). As $\delta_0^\perp \cap \Lambda = \mathbb{Z} h \oplus \mathbb{Z} \delta_1 \oplus \cdots \oplus \mathbb{Z} \delta_8 = I_{1,8}(2)$, above $\delta_0, \cdots, \delta_8$ satisfy

\begin{equation}
\delta_0^\perp \cap \cdots \cap \delta_{k-1}^\perp \cap \Lambda = \mathbb{Z} h \oplus \mathbb{Z} \delta_k \oplus \cdots \oplus \mathbb{Z} \delta_8 = I_{1,9-k}(2), \quad \delta_i \cdot \rho = 1.
\end{equation}
Define 2-elementary lattices $\Lambda_k, S_k, T_k$ ($1 \leq k \leq 9$) by

\begin{align}
\Lambda_k := \delta_0^+ \cap \cdots \cap \delta_{k-1}^+ = I_{1,9-k}(2), \quad T_k := \Pi_{1,1}(2) \oplus \Lambda_k \cong I_{2,10-k}(2), \quad S_k := T_k^+. 
\end{align}

Here, the first orthogonal compliment is considered in $\Lambda_k$, and the last one in $L_{K3}$. Then, $(r(S_k), l(S_k), \delta(S_k)) = (10 + k, 12 - k, 1)$. Let $\pi_k : \Lambda \to \Lambda_k'$ be the orthogonal projection. As is well known (cf. [Man]), the Weyl vector of $\Lambda_k$ as that of $\Lambda$, we denote by $A_m(\Lambda_k')$.

Then, (8.10) $A_m(\Lambda_k')$ via the following map;

\begin{align}
A_m(\Lambda_k') := \rho_k := \pi_k(\rho) = (3h - \delta_k - \cdots - \delta_8)/2.
\end{align}

We denote by $\text{Am}(\Lambda_k, \mathbb{R})$ the Weyl chamber containing $\rho_k$ and by $\text{NE}(\Lambda_k, \mathbb{R})$ the dual cone of $\text{Am}(\Lambda_k, \mathbb{R})$ i.e., $\text{NE}(\Lambda_k, \mathbb{R}) = \{ r \in \Lambda_k; \langle r, \text{Am}(\Lambda_k, \mathbb{R}) \rangle > 0 \}$.

Put $M := T_k = \Pi_{1,1}(2) \oplus I_{9-k}(2)$. Vectors in $M$ are denoted by $(m, n, \lambda)$ ($m, n, \lambda \in \mathbb{Z}$, $\lambda \in I_{9-k}(2)$) whose norm is $4mn - \langle \lambda, \lambda \rangle$. The period domain $\Omega_{S_k}$ is isomorphic to the tube domain $\Lambda_k, \mathbb{R} + \sqrt{-1}C(\Lambda_k)$ via the following map;

\begin{align}
\Lambda_k, \mathbb{R} + \sqrt{-1}C(\Lambda_k) \ni \varepsilon \mapsto (1/2, -(\varepsilon, \varepsilon)/2, \varepsilon) \in \Omega_{S_k}^+.
\end{align}

We define $e_0, e_1, v_0, v_1, v_2, v_3 \in \mathbb{C}[M^\vee / M]$ by

\begin{align}
e_0 := e(0,0,0), \quad e_1 := e(0,0,\rho_k), \quad v_i := \sum_{2(\delta, \delta) \equiv i \mod 4} e_\delta.
\end{align}

Put $q = e^{2\pi i \tau}$. Let $\theta_{A_1+\delta/2}(\tau) := \sum_{\eta \in \mathbb{Z}} q^{(k+\delta/2)^2} \ (\delta \in \{0, 1\})$ be the theta function of $A_1$-lattice. Define $f_0(\tau), f_1(\tau)$, and $\{c_k(0)\}_{l \in \mathbb{Z}}, \{c_k(1)\}_{l \in \mathbb{Z}+1/4}$ by

\begin{align}
f_0(\tau) := \frac{\eta(2\tau)^8 \theta_{A_1}(\tau)^k}{\eta(\tau)^8 \eta(4\tau)^8} = \sum_{l \in \mathbb{Z}} c_k(0)(l) q^l = q^{-1} + 8 + 2k + O(q),
\end{align}

(8.10)

\begin{align}
f_1(\tau) := -16 \frac{\eta(4\tau)^8 \theta_{A_1+1/2}(\tau)^k}{\eta(2\tau)^{16}} = \sum_{l \in 1/4 + \mathbb{Z}} 2c_k(1)(l) q^l.
\end{align}

Then, $f_0(\tau)$ is a modular form of weight $(k-8)/2$ for $\Gamma(4)$ with the same character as that of $\theta_{A_1}(\tau)$. Define $g_i(\tau)$ ($i \in \mathbb{Z}/4\mathbb{Z}$) by

\begin{align}
g_i(\tau) = \sum_{l \equiv i \mod 4} c_k(0)(l) q^{l/4}, \quad \sum_{i \in \mathbb{Z}/4\mathbb{Z}} g_i(\tau) = \frac{\eta(\tau/2)^8 \theta_{A_1}(\tau/4)^k}{\eta(\tau)^8 \eta(\tau/4)^{8}} = f_0(\tau/4),
\end{align}

and a modular form of type $\rho_M$ by

\begin{align}
F_k(\tau) := f_0(\tau) e_0 + f_1(\tau) e_1 + \sum_{i \in \mathbb{Z}/4\mathbb{Z}} g_i(\tau) v_i.
\end{align}

**Theorem 8.2.** If $k < 8$, one has $\Delta_{S_k}(z)^2 = C_k \Psi_{1,2,10-k}(2)(z, F_k)$ and the following infinite product expansion;

\begin{align}
\Delta_{S_k}(z) = C_k e^{2\pi i \rho_k(z)} \prod_{\delta \in \{0,1\}} \prod_{\tau \in \Pi^+(A_k)} (1 - e^{2\pi i (r,z)^2}) c_{k, \delta}(r \cdot r/2)
\end{align}

where $\Pi^+(A_k) := (\delta \rho_k + A_k) \cap \text{NE}(\Lambda_k, \mathbb{R})$.

Let $M\Gamma(4) = \{(a,b)_4, \sqrt{c}\tau + d \in Mp_2(\mathbb{Z}); c \equiv 0 \mod 4\}$ be a subgroup of $Mp_2(\mathbb{Z})$. Put $Z := S^2$. 

\[42\]
Lemma 8.1. There exists a character $\chi$ of $M\Gamma_0(4)$ such that $\rho_M(g)e_0 = \chi(g)^8 - k e_0$.

Proof. From (8.1), it follows that $\ker \rho_M \setminus M\Gamma_0(4) = \{ Z^a T^b : 0 \leq a, b \leq 3 \}$ and $\rho_M(g) e_\gamma = i^{8 - k} e_{\gamma}^b$ if $g \in \ker \rho_M \cdot Z^a T^b$. Putting $\chi(g) = i^a$, we get the assertion. □

For $\gamma = (\gamma_{0,1}, \sqrt{\alpha c + d}) \in M\rho_2(\mathbb{Z})$, put $j(\gamma, \tau) := \sqrt{\alpha c + d}$. For $g \in M\rho_2(\mathbb{Z})$, we set $f|_g(\tau) := f(g \cdot \tau) j(g, \tau)^{-1}$ where $l$ is the weight of $f$. The following construction of modular forms of type $\rho_M$ from a scalar valued modular form of higher level, is due to Borcherds ([Bo6,7]). (It is a special case of his construction.)

Proposition 8.1 ([Bo6,7]). Let $\phi(\tau)$ be a modular form for $\Gamma_0(4)$ with character $\chi^{8 - k}$. Then, $\hat{\phi}(\tau) := \sum_{g \in \Gamma_0(4) \setminus \Gamma(1)} f|_g(\tau) \rho_M(g^{-1}) e_0$ is well defined and becomes a modular form of type $\rho_M$.

Definition 8.1. If the level of $M$ is 4 and $F = \hat{\phi}$ for some modular form for $\Gamma_0(4)$ in Theorem 8.1, we write $\Psi_M(z, \phi)$ in stead of $\Psi_M(z, \hat{\phi})$. $\Psi_M(z, \phi)$ is said to be Borcherds’s product attached to $M$ and $\phi$.

Lemma 8.2. $F_k(\tau)$ is a modular form of weight $(k - 8)/2$ of type $\rho_M$.

Proof. Pick up the representatives $\Gamma_0(4) \setminus \Gamma(1) = \{ 1, S, ST, ST^2, ST^3, V \}$. Here we put $V := S^{-1} T^2 S$. From (8.1), it follows that

$$
(8.13) \quad \rho_M((ST^l)^{-1}) e_0 = i^{k - 2} 2^{-i - 2} \sum_{j=0}^{3} i^{-j} v_j, \quad \rho_M(V^{-1}) e_0 = e_1.
$$

Take $\phi(\tau) = f_0(\tau)$ in Proposition 8.1. As $\theta_{A_1}(\tau)$ is a modular form of weight 1/2 with character $\chi^{-1}$ for $\Gamma_0(4)$ (cf. [Bo5, Theorem 4.1]), $f_0(\tau)$ is a modular form of weight $(k - 8)/2$ for $\Gamma_0(4)$ with character $\chi^{8 - k}$. (Note that $\chi^8 \equiv 1$.) Since $\rho_M((ST)^{-1}) = 2^{(8 - k)/2} \bar{f}_0((\tau + l)/4)$, comparing with (8.13), we get

$$
(8.14) \quad \sum_{l=0}^{3} f_0|_{ST^l}(\tau) \cdot \rho_M((ST^l)^{-1}) e_0 = \sum_{i=0}^{3} g_i(\tau) v_i.
$$

Similarly, since $f_0|_V(\tau) = f_1(\tau)$, comparing with (8.13),

$$
(8.15) \quad f_0|_V(\tau) \cdot \rho_M(V^{-1}) e_0 = f_1(\tau) e_1.
$$

Thus, $F_k(\tau) = \tilde{f}_0(\tau)$. As the weight of $f_0(\tau)$ is $(k - 8)/2$, so is $F_k(\tau)$. □

Proof of Theorem 8.2. Since it follows from (8.9-12) that

$$
(8.16) \quad F_k(\tau) = (q^{-1} + 8 + 2k + O(q)) e_0 + (8 + 2k + O(q)) v_0 + O(q^{1/4}) v_1 + O(q^{1/2}) v_2 + (q^{-1/4} + O(q^{3/4})) v_3 + O(1) e_1,
$$

$\Psi_{T_k}(z, f_0) := \Psi_{T_k}(z, F_k)$ is a modular form for some arithmetic subgroup of $O(T_k)$ of weight $8 + 2k$. Note that $v_0$ contains $e_0$ with multiplicity one. By Theorem 8.1 (2), the zero of $\Psi_{T_k}(z, f_0)$ consists of all the hyperplanes perpendicular to the root of $T_k$ with multiplicity 2. Thus, $\Delta_{S_k}(z)^2$ and $\Psi_{T_k}(z, f_0)$ have the same zero and weight which prove the first assertion. By [Bo5, Theorem 10.4], the Weyl vector of
Ψ(z, f₀) is 2ρk. Let f₀(τ/4) = ∑l∈Z c(l) q^l be the Fourier expansion at the cusp. Using c(⟨r, r⟩/2) = c₀,k(⟨2r, 2r⟩/2) for any r ∈ Λ'_k, it follows from Theorem 8.1 (3) that

\[
(8.16) \quad \Psi_{T_k}(z, f₀) = e^{4πi⟨ρ_k, z⟩} \prod_{r ∈ Λ_k ∩ NE(Λ_k, r)} (1 - e^{2πi⟨r, z⟩}c₀,k(r^2/2)) \times \prod_{r ∈ (ρ_k + Λ_k) ∩ NE(Λ_k, r)} (1 - e^{2πi⟨r, z⟩}2c₁,k(r^2/2))
\times \prod_{r ∈ Λ'_k ∩ NE(Λ_k, r)} (1 - e^{2πi⟨r, z⟩}c(r^2/2)(1 + e^{2πi⟨r, z⟩})c(r^2/2))
\]

\[
= e^{4πi⟨ρ_k, z⟩} \prod_{r ∈ Π^+_0(Λ_k)} (1 - e^{2πi⟨r, z⟩}2c₀,k(r^2/2)) \prod_{r ∈ Π^+(Λ_k)} (1 - e^{2πi⟨r, z⟩}2c₁,k(r^2/2))
\]

\[
= \left[ e^{2πi⟨ρ_k, z⟩} \prod_{δ ∈ \{0, 1\}} \prod_{r ∈ Π^+_0(Λ_k)} (1 - e^{2πi⟨r, z⟩}cδ,k(r^2/2)) \right]^2. \quad \Box
\]

8.3 Nikulin’s K3 Surfaces. For Λ = Π₁,₁ ⊕ E₈(−2), a Λ-2-elementary K3 surface is one of the exceptional type ((r, l, δ) = (10, 8, 0)) in Theorem 2.5 discovered by Nikulin ([Ni4]). It is an elliptic K3 surface whose fixed locus consists of two smooth fibers. Let Ψ_{Λ⁺} := Φ|_{Ω_{Λ}} be the same automorphic form as in Corollary 6.1.

Put θ_{Λ₁₆+δ}(τ) = \sum_{λ+δ ∈ Λ₁₆} q^{λ^2/2} for the theta series of Λ₁₆ where δ ∈ Λ′₁₆/Λ₁₆.

By [Bo2], there exists an involution I on Λ₂₄ (the Leech lattice) whose fixed lattice is E₈(2) and anti-fixed lattice is Λ₁₆. As Λ₂₄ is self-dual, there exists a canonical identification Λ′₁₆/Λ₁₆ ≅ E₈(2)'/E₈(2) = Λ'/Λ via which we identify the generators of the standard basis of the group rings C[Λ₁₆/Λ₁₆] and that of C[Λ'/Λ] (cf. [Bo5, §4]). Under this identification, ρ_{Λ₁₆} = ρ_{Λ} because δ(Λ₁₆) = δ(Λ) = 0. Let Θ_{Λ₁₆}(τ) = ∑δ∈Λ′₁₆/Λ₁₆ θ_{Λ₁₆+δ}(τ) eδ be the theta series of Λ₁₆. Let \{cδ(k)\} be the Fourier coefficients of the following modular form;

\[
(8.17) \quad Θ_{Λ₁₆}(τ)/Δ(τ) = \sum_{δ ∈ Λ′₁₆/Λ₁₆} eδ θ_{Λ₁₆+δ}(τ)/Δ(τ) = \sum_{δ ∈ Λ′₁₆/Λ₁₆} \sum_{k ∈ Z} cδ(k) q^{k/2} eδ.
\]

Theorem 8.3. Ψ_{Λ⁺}(z) = Ψ_{Λ@Π₁,₁}(z, Θ_{Λ₁₆}/Δ).

Proof. From [C-S, Chap.4] (note that q = e^{πiτ} in [C-S] although q = e^{2πiτ} here), it follows that θ_{Λ₁₆}(τ) = 1 + O(q^2), 1/Δ(τ) = q^{-1} + 24 + O(q), and

\[
θ_{Λ₁₆}(τ) = e^{2πi⟨θ, τ⟩} / Δ(τ) = 2^{24} \cdot 2^{-9} \left[ \{θ_3(τ)^2 - θ_2(τ)^2\}^8 + \{θ_3(τ)^2 + θ_2(τ)^2\}^8 + θ_3(τ)^8θ_2(τ)^8 \right]
\]

\[
+ 30\{θ₃(τ)^2 - θ₂(τ)^2\}^4 \cdot \{θ₃(τ)^2 + θ₂(τ)^2\}^4)
\]

\[
= 2^{-5}(1 + 16(1 + 2q^{1/4})) + (1 + 2q^{1/4}) + (4q^{1/4})^4(1 + 4q^{1/2})^4
\]

\[
+ 30(1 + 16)(1 + 2q^{1/4}) + O(q) = 1 + O(q).
\]

In particular, δ ≠ 0 implies that θ_{Λ₁₆+δ}(τ) = O(q) and thus

\[
θ_{Λ₁₆}(τ)/Δ(τ) = (q^{-1} + 24 + O(q)) e₀ + \sum_{δ≠0} O(1) eδ.
\]
From Theorem 8.1, it follows that $\Psi_{A \otimes \Pi_{k,1}}(z, F)$ is a holomorphic modular form of weight 12 whose zero divisor coincides with the discriminant. Comparing weight and zeros, we get the assertion. \hfill\Box

§9. GKM Superalgebras Arizing from $\Delta_M$

9.1 Generalized Kac-Moody Superalgebras and K3 Surfaces. Following [G-N2], let us recall generalized Kac-Moody superalgebras (GKM superalgebra for short) associated to an algebraic K3 surface. (For the general theory of GKM (super)algebras, see [Bo1,2] and [G-N1-3].)

Let $X$ be an algebraic K3 surface and $S := \text{Pic}_X$ its Picard lattice. Let $\text{Exc}(S)$ be the set of all -2-curves in $X$. Let $C(S) = \{ v \in S_R; \angle v, v > 0 \}$ be the light cone of $S$ and $C^+(S)$ be the connected component containing the ample class. Let $\text{Am}(S_R)$ be the ample cone: $\text{Am}(S_R) := \{ l \in C^+(S); \angle l, \delta > 0, \forall \delta \in \text{Exc}(S) \}$. Put $\overline{\text{Am}}(S) := \text{Am}(S_R) \cap S^v$ where closure is considered in $C^+(S)$. Let $W(S)$ be the Weyl group of $S$. Then, $\Delta(S) = W(S)(\text{Exc}(S))$ and $\overline{\text{Am}}(S_R)$ is the fundamental domain for the action of $W(S)$ on $C^+(S)$.

To define a Lie superalgebra associated to $X$, we need the set of simple roots. Let us put $s\Delta^r := \text{Exc}(S)$ for the set of all simple real roots. Let $s\Delta^i_0$ (resp. $s\Delta^i_1$) be a sequences of elements in $\overline{\text{Am}}(S)$ such that any $a \in \overline{\text{Am}}(S)$ can appear in $s\Delta^i_m$ finitely many times;

$$ (9.1) \quad s\Delta^i_m = \{ m(a)z; a \in \overline{\text{Am}}(S), \angle a, a > 0 \} \cup \{ \tau(a)z; a \in \overline{\text{Am}}(S), \angle a, a = 0 \}. $$

Here $m(a)z$, $\tau(a)z \in \mathbb{Z}$, and $m(a)z$ (resp. $\tau(a)z$) implies that $a$ appears $m(a)z$ (resp. $\tau(a)z$) times in $s\Delta^i_m$. An element of $s\Delta^i_0$ (resp. $s\Delta^i_1$) is called a simple even (resp. odd) imaginary root. Put $s\Delta^i := s\Delta^i_0 \cup s\Delta^i_1$ for the set of all simple imaginary roots, and $s\Delta := s\Delta^r \cup s\Delta^i$ for the set of all simple roots. Let us write $s\Delta = \{ h_{ij} \}; i, j \in I \}$ be the Gramm matrix. Namely, $a_{ij} = \langle h_i, h_j \rangle$. As $A$ satisfies the conditions of generalized Cartan matrix ([Bo1]):

(1) $i \neq j$ implies $a_{ij} \geq 0$, (2) $h_i \in s\Delta^re \Rightarrow a_{ii} = -2$ and $a_{ij} \in \mathbb{Z}$,

we get a GKM superalgebra $g(S, s\Delta^i) := g'(A)$. (See [Bo1,2], [G-N1,2] for details.)

Let $\Pi^+ := \{ a \in \Delta; a \notin 0 \} \subset \text{NE}(S) := \text{NE}(S_R) \cap S^v$ be the set of positive roots where $\text{NE}(S_R)$ is the dual cone (Mori cone) to the ample cone $\text{Am}(S_R)$. Let $g_\alpha$ be the root space attached to $\alpha \in \Pi^+ \cup \{ 0 \} \cup -\Pi^+$. Then, $g(S, s\Delta^i)$ admits root space decomposition: $g(S, s\Delta^i) = (\oplus_{\alpha \in \Pi^+} g_\alpha) \oplus g_0 \oplus (\oplus_{\alpha \in \Pi^-} g_{-\alpha})$, $g_0 = S_R$. According to the decomposition into even and odd part, we get $g_\alpha = g_{\alpha,0} \oplus g_{\alpha,1}$. Multiplicity of $\alpha \in \Pi^+$ is defined by $\text{mult}(\alpha) := \dim g_{\alpha,0} - \dim g_{\alpha,1} \in \mathbb{Z}$.

In view of Borcherds's works, it is the denominator formula that connects GKM superalgebras and automorphic forms. We recall it when $S$ admits a Weyl vector.

**Definition 9.1.** $\rho \in S_Q$ is a Weyl vector if $\langle \rho, \delta \rangle = 1$ for all $\delta \in \text{Exc}(S)$.

For $a \in s\Delta^i$, define $m(a) \in \mathbb{Z}$ by (1) $m(a) = m(a)_0 - m(a)_1$ if $\langle a, a \rangle > 0$, and (2) the following formal series if $\langle a, a \rangle = 0$ and $a$ is primitive,

$$ (9.2) \quad \prod_{n=1}^{\infty} (1 - q^n)^{\tau(na)} = 1 - \sum_{k=1}^{\infty} m(ka)q^k \quad (\tau(na) := \tau(na)_0 - \tau(na)_1). $$

45
Theorem 9.1 ([Bo1,2] (cf. [G-N1,2])). Suppose that $S$ has a Weyl vector $\rho$.

(1) For a GKM superalgebra $\mathfrak{g}(S, s\Delta^{im})$, one has the following identity:

$$
\Phi_{\mathfrak{g}(S, s\Delta^{im})}(z) : = \sum_{w \in W(S)} \det(w) \{ e^{2\pi i (w(\rho), z)} - \sum_{r \in s\Delta^{im}} m(r) e^{2\pi i (w(\rho + r), z)} \} 
= e^{2\pi i (\rho, z)} \prod_{\alpha \in \Pi^+} (1 - e^{2\pi i (\alpha, z)})^{\text{mult}(\alpha)}.
$$

The formal series $\Phi_{\mathfrak{g}(S, s\Delta^{im})}(z)$ is said to be the denominator function.

(2) Let $\Psi(y)$ be a formal series with the following integral Fourier expansion:

$$
\Psi(z) = \sum_{w \in W(S)} \det(w) \{ e^{2\pi i (w(\rho), z)} - \sum_{r \in \text{Am}(S)} m(r) e^{2\pi i (w(\rho + r), z)} \}.
$$

Define $\tau(nr) \in \mathbb{Z}$ by (8.2) for primitive norm zero $r \in \text{Am}(S)$ and $n \in \mathbb{N}$. Let $\mathfrak{g}(S, s\Delta^{im})$ be the GKM superalgebra whose simple imaginary roots are

$$
s\Delta^{im}_0 = \{ m(r) r; m(r) > 0, \langle r, r \rangle > 0 \} \cup \{ \tau(r) r; \tau(r) > 0, \langle r, r \rangle = 0 \},
$$

$$
s\Delta^{im}_1 = \{ -m(r) r; m(r) < 0, \langle r, r \rangle > 0 \} \cup \{ -\tau(r) r; \tau(r) < 0, \langle r, r \rangle = 0 \}.
$$

Then, $\Phi_{\mathfrak{g}(S, s\Delta^{im})}(z) = \Psi(z)$.

Among all the GKM superalgebras associated to $K3$ surfaces, one of the fake monster Lie algebras constructed by Borcherds is the most beautiful and interesting.

Let $\Lambda, S$ and $T$ be the 2-elementary lattices defined by $\Lambda := \Pi_{1,1} \oplus E_8(-2)$, $T = \Pi_{1,1}(2) \oplus \Lambda$, $S = T^\perp = \Pi_{1,1}(2) \oplus E_8(-2)$. A 2-elementary $K3$ surface of type $S$ is the universal cover of an Enriques surface and $T$ is the transcendental lattice. The period domain $\Omega_S$ is realized as the tube domain $\Lambda_\mathbb{R} + \sqrt{-1}C(\Lambda)$ as before (cf. (8.7)). Set $\rho := (0, 1, 0)$, $\rho' := (1, 0, 0)$. Then, $\rho$ is a Weyl vector of $\Lambda$.

Theorem 9.2 ([Bo2,4]). There exists a GKM superalgebra $\mathfrak{g}(\Lambda, s\Delta^{im})$ (one of the fake monster Lie algebras) whose denominator function, $\Phi$, is the automorphic form over $\Omega_S$ of weight 4 with zero divisor $D_S$. For $\Im y \gg 0$, one has the following:

$$
\Phi(z) = \sum_{w \in W(\Lambda)} \det(w) e^{2\pi i (\rho, w(z))} \prod_{n > 0} (1 - e^{2\pi i n (\rho, w(z))})(-1)^n 8
= e^{2\pi i (\rho, z)} \prod_{r \in \Pi^+} (1 - e^{2\pi i (r, z)})(-1)^{r(\rho - \rho')}c(r^2/2)
$$

where $\Pi^+ = \mathbb{N}\rho \cup \{ r \in \Lambda; \rho \cdot r > 0 \}$ and $\sum_{n \geq 1} c(n) q^n = \eta(\tau)^{-8}(2\tau)^8\eta(4\tau)^{-8}$.

Theorem 9.3. There exists a constant $C_S \neq 0$ such that $\Delta_S = C_S \Phi$. $\Phi(z)^2$ is Borcherds’s product attached to the modular form $F_0$ of (8.12).

Proof. Comparing the weight and zero of $\Delta_S$ and $\Phi$, we get the first assertion. Regarding $I_{1,9}(2)$ as a sublattice of $\Lambda$, we can prove the second assertion in the same manner as Theorem 8.1. □

Remark. We remark that $I_{2,10}(2)$ (not the transcendental lattices $\Pi_{1,1} \oplus I_{1,9}(-2)$) is used in (3) to construct Borcherds’s product. In [A], [Kn], Allcock and Kondo uses the similar relation between $\Lambda$ and $I_{1,9}$ to study the moduli space of Enriques surfaces.

9.2 GKM Superalgebras Arising from $\Delta_M$. We keep the notations in §8.1, and study two classes of 2-elementary $K3$ surfaces as before.
**Theorem 9.4.** There exists a GKM superalgebra attached to \( \Lambda_k \) whose denominator function is (up to a constant) \( \Delta_{S_k}(z) \).

**Proof.** In view of Theorem 9.1 (2), it is enough to show that \( \Delta_{S_k}(z) \) admits the following integral Fourier expansion at the cusp;

\[
\Delta_{S_k}(z) = C_{S_k} \sum_{w \in W(\Lambda_k)} \det(w) \{ e^{2\pi i (w(\rho),z)} - \sum_{r \in \text{Am}(\Lambda_k)} m(r) e^{2\pi i (w(\rho + r),z)} \}.
\]

By Theorem 8.2, \( \Delta_{S_k}(z) \) has the following Fourier expansion at the cusp;

\[
\Delta_{S_k}(z) = C_{S_k} \sum_{r \in \Lambda_k^+ \cap C(\Lambda_k)} n(r) e^{2\pi i (r,z)}
\]

where \( n(r) \in \mathbb{Z} \). By (8.4), (8.5), Theorems 7.3 and 9.2, one has

\[
\Delta_{S_k}(w(z)) = \lim_{v \to z} \left( \frac{i}{2\pi} \right)^k \frac{\Phi(w(v))}{\prod_{i=0}^{k-1} \langle w(v), \delta_i \rangle} = \det(w) \Delta_{S_k}(z) \quad (w \in W(\Lambda_k))
\]

because \( w(\delta_i) = \delta_i \) for \( i \leq k - 1 \). Namely, \( n(w(r)) = \det(w) n(r) \) for any \( r \in \Lambda_k^+ \) and \( w \in W(\Lambda_k) \). Together with the same argument as [G-N1, Theorem 2.3 (a)], one has \( r - \rho_k \in \text{Am}(\Lambda_k) \) if \( n(r) \neq 0 \). Putting \( m(r) := -n(r + \rho_k) \), (9.4) becomes

\[
\Delta_{S_k}(z) = C_{S_k} \sum_{w \in W(\Lambda_k)} \det(w) \sum_{r \in \text{Am}(\Lambda_k)} -m(r) e^{2\pi i (w(\rho + r),z)}.
\]

By Theorem 8.2, \( -m(0) = 1 \) which, together with (9.6), yields the assertion. \( \square \)

To study the case of 2-elementary K3 surface of type \( \Lambda \), Borcherds’s \( \Phi \)-function of rank 26 is crucial. Let \( \Lambda_{24} \) be the Leech lattice, and put \( L := \Pi_{1,1} \oplus \Lambda_{24}(-1) \). Let \( \rho = (0,1,0) \) be a Weyl vector of \( L \).

**Theorem 9.5 ([Bo2,3]).** The denominator function \( \Phi \) of the fake monster Lie algebra is the automorphic form over \( \Omega_{11,1} \oplus L \) of weight 12 with only simple zero along the discriminant. The denominator formula becomes

\[
\Phi(z) = \sum_{w \in W(L)} \sum_{n > 0} \det(w) \tau(n) e^{2\pi i (w(\rho),z)} = e^{2\pi i (\rho,z)} \prod_{r \in \Pi^+} (1 - e^{2\pi i (r,z)}) p_{24}(1 - r^2/2)
\]

where \( \tau(n) \) is the Ramanujan \( \tau \)-function, \( p_{24}(n) \) the number of partitions of \( n \) into 24 colors, and \( \Pi^+ = \mathbb{N} \rho \cup \{ r \in \Lambda; \rho \cdot r > 0 \} \).

**Theorem 9.6.** \( \Psi_{\Lambda^\perp} \) is the denominator function of a GKM superalgebra.

**Proof.** By Theorem 8.3, \( \Psi_{\Lambda^\perp}(z) \) has the similar Fourier expansion as (9.4). Since \( \Psi_{\Lambda^\perp} = \Phi|_{\Omega_{11,1} \oplus \Lambda} \), it follows from Theorem 9.5 that \( \Psi_{\Lambda^\perp}(w(z)) = \det(w) \Psi_{\Lambda^\perp}(z) \).

Thus, we get the assertion in the same manner as Theorem 9.4. \( \square \)
§10. An Explicit Formula for $\Delta_{A_1}$ and $\Delta_{II_{1,1}(2)}$

We determine an explicit formula for $\Delta_{A_1}$ and $\Delta_{II_{1,1}(2)}$ in this section.

10.1 An Explicit Formula for $\Delta_{A_1}$. Let $(X, i)$ be a 2-elementary $K3$ surface of type $A_1$. Then, $X/\iota = \mathbb{P}^2$ and the fixed curve $X^i$ is a smooth plane sextic curve where the quotient map $X \to \mathbb{P}^2$ is the morphism associated to the complete linear system of ample line bundle of degree 2. As $X^i \subset \mathbb{P}^2$ has an ambiguity of $PGL(3, \mathbb{C})$ (arizing from a choice of 3 sections), there exists a morphism $i : M^0_{A_1} \supset [(X, i)] \to [X^i] \in H^6_{sm}/PGL(3, \mathbb{C})$ where $H^6_{sm} = \mathbb{P}(Sym^6 \mathbb{C}^3)^\vee \setminus D_6$ is the set of all smooth sextic curves in $\mathbb{P}^2$, and $D_6$ is the discriminant locus of universal plane sextic curves $\pi : C_6 \to H_6 = \mathbb{P}(Sym^6 \mathbb{C}^3)^\vee$. Note that $\xi = (\xi_i) \in \mathbb{P}(Sym^6 \mathbb{C}^3)^\vee$ corresponds to the curve $C_\xi = \{x \in \mathbb{P}^2; \sum_{|I|=6} \xi_I x^I = 0\}$. Conversely, if $\xi, \xi' \in H^6_{sm}$ are in the same orbit of $PGL(3, \mathbb{C})$, $C_\xi$ and $C_{\xi'}$ are projectively equivalent and thus the double covers of $\mathbb{P}^2$ blanching along $C_\xi$ and $C_{\xi'}$ are isomorphic. In this way, one verify $i : M^0_{A_1} \cong H^6_{sm}/PGL(3, \mathbb{C})$ (Sha).

We denote by $\lambda(C_6/H^6_{sm})$ the determinant of cohomology in the sense of §3.1. Let $Jac : H^6_{sm} \ni \xi \to [Jac(C_\xi)] \in A_{10}$ be the Torelli map. As $j_{A_1} = Jac \circ i$ and $(Jac)^* H_{10} = \lambda(C_6/H^6_{sm})^{GL(3, \mathbb{C})}$, we find $\lambda_{A_1} = i^* \lambda(C_6/H^6_{sm})^{GL(3, \mathbb{C})}$ where $\lambda(C_6/H^6_{sm})^{GL(3, \mathbb{C})}$ is the sheaf of $GL(3, \mathbb{C})$-invariant sections of $\lambda(C_6/H^6_{sm})$. Moreover, this identification is an isometry if $\lambda_{A_1}$ is equipped with the Petersson metric and if $i^* \lambda(C_6/H^6_{sm})^{GL(3, \mathbb{C})}$ with the $L^2$-metric. Our first task is to construct a section of $\lambda_{A_1}^{\otimes 15}$ arizing from the discriminant of plane sextics. In the sequel, we put $F(x; \xi) = \sum_{|I|=6} \xi_I x^I$. By the Poincaré residue sequence, we get the following.

Lemma 10.1. If we denote by $x, y, z$ the homogeneous coordinates of $\mathbb{P}^2$, then

$$H^0(C_\xi, \Omega^1_C) = \bigoplus_{i+j+k=3} \mathbb{C} \operatorname{Res}_{C_\xi} x^i y^j z^k (x dy \wedge dz - y dx \wedge dz + z dx \wedge dy) / F(x, y, z; \xi).$$

We define a local section of $\lambda(C_6/H^6_{sm})$ by

$$\omega(\xi) := \bigwedge_{i+j+k=3} \operatorname{Res}_{C_\xi} x^i y^j z^k (x dy \wedge dz - y dx \wedge dz + z dx \wedge dy) / F(x, y, z; \xi) \in \det H^0(C_\xi, \Omega^1).$$

To be precise, if $p : (Sym^6 \mathbb{C}^3)^\vee \setminus \{0\} \to \mathbb{P}(Sym^6 \mathbb{C}^3)^\vee$ is the natural projection, then $\omega$ is a section of $p^* \lambda(C_6/H^6_{sm})$ over $(Sym^6 \mathbb{C}^3)^\vee$.

Lemma 10.2. For any $g \in GL(3, \mathbb{C})$, $g^* \omega = det(g)^{20} \omega$ where $GL(3, \mathbb{C})$ acts on $(Sym^6 \mathbb{C}^3)^\vee$ via the induced representation.

Proof. From the definition of representation of $GL(3, \mathbb{C})$ on $(Sym^6 \mathbb{C}^3)^\vee$, it follows that $F(g \cdot (x, y, z); g \cdot \xi) = F(x, y, z; \xi)$. Since $GL(3, \mathbb{C})$ acts on $Sym^3 \mathbb{C}^3$ by $det(\cdot)^{10}$, we get $g \cdot \Lambda_{i+j+k=3} x^i y^j z^k = det(g)^{10} \Lambda_{i+j+k=3} x^i y^j z^k$. By computation, we get $g^* (x dy \wedge dz - y dx \wedge dz + z dx \wedge dz) = det(g)(x dy \wedge dz - y dx \wedge dz + z dx \wedge dz)$. Together with all of these, we get the assertion: $g^* (\omega(g \cdot \xi)) = det(g)^{20} \omega(\xi)$.

Lemma 10.3. There exists a homogeneous polynomial $D_6(\xi)$ of degree 75 in the $\xi$-variable such that the discriminant locus of the plane sextics is $\operatorname{div}(D_6)$. Moreover, for any $g \in GL(3, \mathbb{C})$, $D_6(g \cdot \xi) = det(g)^{-150} D_6(\xi)$. 

48
Proof. Let \( v_6 : \mathbb{P}(\mathbb{C}^3) \hookrightarrow \mathbb{P}(\text{Sym}^6 \mathbb{C}^3) \) be the Veronese embedding. Put \( X := v_6(\mathbb{P}(\mathbb{C}^3)) \) and \( X^\vee \subset \mathbb{P}(\text{Sym}^6 \mathbb{C}^3)^\vee \) for the projective dual variety of \((X, \mathcal{O}_{\mathbb{P}(\text{Sym}^6 \mathbb{C}^3)}(1))\). By Katz’s formula ([Kt. Cor. 5.6]), \( X^\vee \) is a projective hypersurface of degree

\[
\text{deg} X^\vee = \frac{c(\mathbb{P}(\mathbb{C}^3))}{(1 + v_6^*c_1(\mathbb{P}(\text{Sym}^6 \mathbb{C}^3)))^2} = \frac{1 + 3H + 3H^2}{(1 + 6H)^2} = 75
\]

where \( H = c_1(\mathbb{P}(\mathbb{C}^3)) \) is the hyperplane section of \( \mathbb{P}(\mathbb{C}^3) \). Take \( D_6(\xi) \) as a defining equation of \( X^\vee \). As \( X^\vee \) is invariant under the action of \( \text{GL}(3, \mathbb{C}) \), there exists \( l \in \mathbb{Z} \) such that \( D_6(g \cdot \xi) = \text{det}(g)^l D_6(\xi) \). Putting \( g = \lambda I_3 \) (\( \lambda \in \mathbb{C}^\times \)), we find \( l = -150 \). □

Proposition 10.1. \( \Delta_6^2(\xi) := D_6(\xi)^2 \cdot \omega(\xi)^{\otimes 15} \) is a \( \text{GL}(3, \mathbb{C}) \)-invariant section of \( p^* \lambda(C_6/H_6^{sm}) \). In particular, we may regard \( \Delta_6^2 \in H^0(M_{A_1}^{0}, \lambda_{A_1}^{\otimes 15}) \).

Proof. The first assertion follows from Lemmas 10.2 and 10.3, and the second from \( H^0(H_6^{sm}, \lambda(C_6/H_6^{sm})^{\otimes 15})^{GL(3, \mathbb{C})} = H^0(M_{A_1}^{0}, \lambda_{A_1}^{\otimes 15}) \). □

Let \( \delta_1, \delta_2 \in \Delta(\Pi_{12,18} \oplus A_1(-1)) \) such that \( \delta_1/2 \in (\Pi_{12,18} \oplus A_1(-1))^\vee \) and \( \delta_2/2 \notin (\Pi_{12,18} \oplus A_1(-1))^\vee \). Since \( \Gamma_{A_1} = O(\Pi_{12,18} \oplus A_1(-1)) \), it follows from a theorem of Nikulin ([N1, Proposition 1.15.1]) that \( \Delta(\Pi_{12,18} \oplus A_1(-1))/\Gamma_{A_1} = \{ \delta_1, \delta_2 \} \). In particular, \( \mathcal{D}_{A_1}/\Gamma_{A_1} \) consists of 2 irreducible components: \( \mathcal{D}_{A_1}/\Gamma_{A_1} = H_{\delta_1} + H_{\delta_2} \).

Lemma 10.4. \( \Delta_6^2 \) vanishes of order 2 along \( H_{\delta_2} \).

Proof. Since \( \langle A_1 \oplus \delta_2 \rangle = A_1 \oplus A_1(-1) \), 2-elementary \( K3 \) surfaces over \( H_{\delta_2} \) are those of type \( A_1 \oplus A_1(-1) \). Take a generic 2-elementary \( K3 \) surface \((X, \iota)\) of this type. Then, \( \text{Pic}(X/\iota) = \mathbb{I}_{1,1} \) and \( X/\iota \) is a blow-up of \( \mathbb{P}^2 \) at one point. This implies that the complete linear system associated to the nef and big line bundle of degree 2 maps \( X \to \mathbb{P}^2 \) and \( X^\iota \) to a nodal sextic curve with one singular point. Conversely, given such a nodal sextic curve, the minimal resolution of the double cover of \( \mathbb{P}^2 \) blanching along it is a 2-elementary \( K3 \) surface of type \( A_1 \oplus A_1(-1) \). This extends the isomorphism \( i : M_{A_1}^{0} \cup H_6^0 \to H_6^{\text{nod}}/\text{PGL}(3, \mathbb{C}) \) and \( \lambda_{A_1} = i^* \lambda(C_6/H_6^{\text{nod}})^{GL(3, \mathbb{C})} \) where \( H_6^{\text{nod}} \) is the set of all plane sextic curves with at most one node. By Proposition 10.1, we find \( \Delta_6^2(\xi) = H_6^{\text{sm}}(\lambda(C_6/H_6^{\text{nod}})^{\otimes 15})^{\text{GL}(3, \mathbb{C})} = H^0(M_{A_1}^{0} \cup H_6^0, \lambda_{A_1}^{\otimes 15}) \). To compute the vanishing order of \( \Delta_6^2 \), take a generic point \( p \in H_{\delta_2} \) and its small neighborhood \( U \). As \( \omega \) is regarded to be a nowhere vanishing section of \( \lambda_{A_1} \) on \( U \), the assertion follows from Proposition 10.1 because \( H_{\delta_2} = \text{div}(D_6) \). □

Lemma 10.5. \( \Delta_6^2 \) has an algebraic singularity along \( H_{\delta_1} \).

Proof. By Shah’s result ([Sha]), \( H_{\delta_1} \) contracts to a point in \( H_6^{ss}/\text{PGL}(3, \mathbb{C}) \) corresponding to a smooth conic of multiplicity 3 where \( H_6^{ss} \) is the set of all semi-stable sextic curves. If \( \{ C_{\xi} \}_{|t| < 1} \) is a degenerating family of sextic curves such that \( C_{\xi} \) is a smooth triple conic, then \( \| i^* \Delta_6^2(\xi) \| \sim |t|^a (\log |t|)^b \) \((t \to 0)\) with some \( a, b \in \mathbb{R} \). If \((U, s)\) is a holomorphic disc which meets transversally to \( H_{\delta_1} \) at \( s = 0 \), then

\[
\| i^* \Delta_6^2(s) \| \sim |s|^l (\log |s|)^m \quad (s \to 0)
\]

with some \( l, m \in \mathbb{R} \) because \( s \) and \( t \) are algebraically related. Since \( \langle A_1, \delta_1 \rangle = \mathbb{I}_{1,1} \) and \( g(\mathbb{I}_{1,1}) = 10 = g(A_1) \), there exists a section, \( \sigma \), of \( \lambda_{A_1}^{\otimes 15} \) over \( U \) such that

49
\[ \|\sigma(s)\| \sim 1 \ (s \to 0) \] because \( j_{A_1}(U) \) is away from the boundary component of the Siegel modular variety \( A_{10} \). Since \( i^* \Delta_0^2(s)/\sigma(s) \) is a holomorphic function on \( U \setminus \{0\} \), it must be meromorphic on \( U \) because of (10.3). In particular, \( m = 0 \) and \( \Delta_0^2 \) has zero (or pole) of order \( l \in \mathbb{Z} \) along \( H_\delta \). \( \square \)

Let \( E_1, E_2 \) be divisors on \( \Omega_{A_1} \) defined by \( E_1 = \sum_{\delta \sim \delta_1} H_\delta \) and \( E_2 = \sum_{\delta \sim \delta_2} H_\delta \) where \( \delta \) runs over \( \Delta(\Pi_{2,18} \oplus A_1(-1)) \) and \( \delta \sim \delta' \) if \( \delta = \gamma \cdot \delta' \) for some \( \gamma \in \Gamma_{A_1} \). Identify \( \Delta_0^2 \) to be a \( \Gamma_{A_1} \)-invariant section of \( \lambda_{A_1}^{\otimes 15} \) over \( \Omega_{A_1} \), namely \( \Delta_0^2 \in H^0(\Omega_{A_1}, \lambda_{A_1}^{\otimes 15})_{\Gamma_{A_1}} \). Since the quotient map \( \Omega_{A_1} \to \mathcal{M}_{A_1} \) blanches of order 2 along \( \mathcal{D}_{A_1} = E_1 + E_2 \), we get the following from Lemmas 10.4 and 10.5.

**Lemma 10.6.** \( \Delta_0^2 \) vanishes of order \( 2l \) \((l \in \mathbb{Z}) \) along \( E_1 \) and of order 4 along \( E_2 \).

**Proposition 10.2.** \( \Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta) \) is a modular form of weight 75 with zero divisor \( 57E_1 + E_2 \) where \( \Theta_{E_7}(\tau) \) is the theta series of the \( E_7 \)-lattice.

**Proof.** Since \( \Theta_{E_7}(\tau)/\Delta(\tau) \) is a modular form of type \( \rho_{\Pi_{2,18} \oplus A_1(-1)} \) of weight \(-17/2\), we can construct Borcherds’s product \( \Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta) \) by Theorem 8.1. Since \( \theta_{E_7}(\tau) = 1 + 126q + O(q^2) \), \( \theta_{E_7+1/2}(\tau) = 56q^{3/4} + 576q^{7/4} + O(q^2) \) and \( 1/\Delta(\tau) = q^{-1} + 24 + O(q) \) (cf. [C-S]), we get

\[
(10.4) \quad \Theta_{E_7}(\tau)/\Delta(\tau) = (q^{-1} + 150 + O(q)) e_0 + (56q^{-1/4} + O(q^{3/4})) e_1.
\]

By Theorem 8.1, \( \Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta) \) has weight \( c_0(0)/2 = 75 \) whose zero divisor is \( \sum_{\delta \in \Delta(\Pi_{2,18} \oplus A_1(-1))} H_\delta + 56 \sum_{\delta \in \Delta(\Pi_{2,18} \oplus A_1(-1))} H_\delta / 2 \in (\Pi_{2,18} \oplus A_1(-1))^{\vee} \) \( H_\delta = 57E_1 + E_2 \). \( \square \)

**Theorem 10.1.** \( \Delta_{A_1}^{15}(z) = C_{A_1} \Delta_0^5(z)/\Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta) \).

**Proof.** Put \( F(z) := \Delta_0^5(z)/[\Delta_{A_1}^{15}(z) \cdot \Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta)] \). Since \( \Delta_{A_1} \) is an automorphic form of weight \((-5,4)\), \( \Delta_0^2 \) of \((0,15)\) and \( \Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta) \) of \((75,0)\) by Theorem 7.2, Propositions 10.1 and 10.2, \( F \) is an automorphic form of weight \((0,0)\). Namely, taking a higher power of \( F \) if necessary, \( F \) is a \( \Gamma_{A_1} \)-invariant meromorphic function on \( \Omega_{A_1} \), thus a meromorphic function on the modular variety \( \mathcal{M}_{A_1} \). From Theorem 7.2, Lemma 10.6 and Proposition 10.2, it follows that \( \text{div}(F) = 8(l-9)E_1 \). Since \( F \) extends to a meromorphic function on the Satake-Baily-Borel compactification of \( \mathcal{M}_{A_1} \), the residue theorem implies \( l = 9 \) and \( F \) must be a (nonzero) constant. \( \square \)

**Remark.** It seems that it is in [B-K-P-S-B] that \( \Psi_{\Pi_{2,18} \oplus A_1(-1)}(z, \Theta_{E_7}/\Delta) \) first appeared. From their construction, its Weyl vector is the projection of that of \( \Pi_{1,1} \oplus E_8(-1)^{\oplus 3} \) to \( \Pi_{1,1} \oplus E_8(-1)^{\oplus 2} \oplus A_1(-1) \). They used this modular form to show that the moduli space of K3 surfaces of degree 2 is quasi-affine. See [B-K-P-S-B, Theorem 1.3 and Example 2.1] for the details.

**10.2 An Explicit Formula for \( \Delta_{\Pi_{1,1}(2)} \).** Let \( (X, \iota) \) be a 2-elementary K3 surface of type \( \Pi_{1,1}(2) \). (\( X, \iota \)) is said to be *generic* if \( E \cdot \iota(E) > 0 \) for any \(-2\)-curve \( E \). (Equivalently, \( E + \iota(E) \) lies in the positive cone: \( (E + \iota(E))^2 \geq 0 \).) Let \( e, f \) be a basis of \( \Pi_{1,1}(2) \) such that \( e^2 = f^2 = 0 \) and \( e \cdot f = 2 \). Let \( \phi \) be a marking of \( (X, \iota) \). If it is generic, we may suppose that \( C_e := \phi^{-1}(e) \) and \( C_f := \phi^{-1}(f) \) are nef divisors because \( 2C_e \cdot E = C_e \cdot (E + \iota(E)) \geq 0 \) for any \(-2\)-curve \( E \). Thus, \( X \) has two elliptic fibrations associated to the linear systems \( |C_e| \) and \( |C_f| \). Since \( C_e + C_f \)
is ample, the linear system \( |C_ε + C_f| \) induces a finite surjective morphism \( Φ : X → \mathbb{P}^1 × \mathbb{P}^1(⊂ \mathbb{P}^3) \), because \( H^0(X, φ^{-1}(e + f)) = H^0(X, φ^{-1}(e)) ⊗ H^0(X, φ^{-1}(f)) \). By the Lefschetz formula, these spaces consist of \( i \)-invariant sections. Thus \( Φ \) induces a map: \( Φ/ι : X/ι → \mathbb{P}^1 × \mathbb{P}^1 \). As it induces an isomorphism of the Picard lattice, it must be an isomorphism. Thus, any generic 2-elementary \( K3 \) surfaces of type \( Π_{1,1}(2) \) is realized as a double cover of \( \mathbb{P}^1 × \mathbb{P}^1 \) blanching along a smooth curve of bidegree \( (4, 4) \). By this description, we can construct the discriminant of fixed curves as that of curves of bidegree \( (4, 4) \) on \( \mathbb{P}^1 × \mathbb{P}^1 \).

Let \( H_{4,4} = \mathbb{P}(\text{Sym}^4 \mathbb{C}^2 ⊗ \text{Sym}^4 \mathbb{C}^2)^{\vee} \) be the set of all curves of bidegree \( (4, 4) \) on \( \mathbb{P}^1 × \mathbb{P}^1 \). As before, \( ξ = (ξ_{i,j,k,l})_{i+j=4,k+l=4} ∈ H_{4,4} \) represents the curve defined by \( C_ξ := \{(x, y), (z, w)\} ∈ \mathbb{P}^1 × \mathbb{P}^1; \sum ξ_{i,j,k,l}x^iy^jzw^l = 0\). Let \( π : C_{4,4} → H_{4,4} \) be the universal family i.e. \( π^{-1}(ξ) = C_{ξ,ι} \), on which acts the group of projective transformations \( PG := P(GL(2, \mathbb{C}) × GL(2, \mathbb{C})) \). Then, above description implies an isomorphism \( i : \mathcal{M}^{00}_{Π_{1,1}(2)} \cong [(X, ι)] → [X'] ∈ H_{4,4}^{sm}/PG \). As before, \( \mathcal{M}^{00}_{Π_{1,1}(2)} \) is the set of all smooth curves in \( H_{4,4} \) and \( \mathcal{M}^{00}_{Π_{1,1,1}(2)} \) is the set of all isomorphism classes of generic 2-elementary \( K3 \) surfaces of type \( Π_{1,1}(2) \). Let \( λ(C_{4,4}/H_{4,4}) \) be the determinant of cohomology on which acts \( G := GL(2, \mathbb{C}) × GL(2, \mathbb{C}) \). Note that the action of \( G \) is not effective. As before, we get an identification \( λ_{Π_{1,1;1}(2)} = i^∗λ(C_{4,4}/H_{4,4}^{nod})G \) on \( \mathcal{M}^{00}_{Π_{1,1,1}(2)} \) where \( H_{4,4}^{nod} \) is the set of all curve of bidegree \( (4, 4) \) with at most one node. Put \( F(x, y, z, w; ξ) := \sum_{i+j=k+l=4} ξ_{i,j,k,l}x^iy^jz^kw^l \). We regard \( (x, y) \) and \( (z, w) \) as the homogeneous coordinates of \( \mathbb{P}^1 \). By the Poincaré residue sequence, we get the following.

**Lemma 10.7.**

\[
H^0(C_ξ, Ω^1_{C_ξ}) = \bigoplus_{i+j=k+l=2} \mathbb{C} \text{Res}_{C_ξ} \frac{x^iy^jz^kw^l (x dy - y dx) ∧ (z dw - w dz)}{F(x, y, z, w; ξ)}.
\]

As before, let us define a local section of \( λ(C_{4,4}/H_{4,4}^{nod}) \) by

\[
ω(ξ) := \bigwedge_{i+j=k+l=2} \text{Res}_{C_ξ} \frac{x^iy^jz^kw^l (x dy - y dx) ∧ (z dw - w dz)}{F(x, y, z, w; ξ)} ∈ \det H^0(C_ξ, Ω^1).
\]

**Lemma 10.8.** For any \((g, h) ∈ G, (g, h)^∗ω = (\det(g) \det(h))^{18} ω\).

**Proof.** Since the proof is similar to that of Lemma 10.2, we leave it to the reader. □

**Lemma 10.9.** There exists a homogeneous polynomial \( D_{4,4}(ξ) \) of degree 68 in the \( ξ \)-variable such that the discriminant locus of \( π : C_{4,4} → H_{4,4} \) coincides with \( \text{div}(D_{4,4}) \). Moreover, for any \((g, h) ∈ G, D_{4,4}((g, h)−1 ξ) = (\det(g) \det(h))^{−136} D_{4,4}(ξ)\).

**Proof.** Let \( Q = \mathbb{P}^1 × \mathbb{P}^1(⊂ \mathbb{P}^3) \) be a hyperquadric. Since the line bundle of bidegree \((4, 4)\) is \( -2K_Q \) (\( K_Q \) is the canonical bundle), let us consider the Veronese embedding \( v_{4,4} : Q → \mathbb{P}(H^0(Q, -2K_Q))^\vee \). Let \( X := v_{4,4}(Q) \) be the image and \( X^\vee \) be the projective dual variety of \((X, O_P(H^0(Q, -2K_Q))^\vee(1))\). By Katz’s formula ([Kt, Cor. 5.6]), \( X^\vee \) is a projective hypersurface of degree

\[
\text{deg} X^\vee = \int_Q \frac{c(Q)}{(1 + v_{4,4}c_1(\mathbb{P}(H^0(Q, -2K_Q))^\vee))^2} = \int_Q \frac{1 + 2H + 2H^2}{(1 + 4H)^2} = 68
\]
where $H = \mathcal{O}_{D^3}(1)$. Let $D_{4,4}(\xi)$ be a defining equation of $X^\vee$. Since $X^\vee$ is stable under the action of $G$, there exists $l \in \mathbb{Z}$ such that $D_{4,4}((g, h) \cdot \xi) = (\det(g) \det(h))^l D_{4,4}(\xi)$. Putting $g = \lambda I_2$, $h = I_2$, we find $l = -136$. □

**Proposition 10.3.** $\Delta_{4,4}^G(\xi) := D_{4,4}(\xi)^9 \cdot \omega(\xi)^{\otimes 2}$ becomes a $G$-invariant section of $\lambda(C_{4,4}/H_{4,4}^{\text{red}})$. In particular, we may regard $\Delta_{4,4}(\xi)^9 \in H^0(M_{00}^{00}, \lambda_{11,11}^{\otimes 2})$.

**Proof.** The assertion follows from Lemmas 10.8 and 10.9. □

**Lemma 10.10.** The discriminant locus $H := D_{11,11}^2(\Gamma_{11,11}(2))$ is an irreducible divisor of $\mathcal{M}_{11,11}(2)$. $\Delta_{4,4}$ vanishes of order 9 along $H$.

**Proof.** The first assertion follows from Nikulin’s theory ([Ni, Proposition 1.15.1]), and the second from Proposition 10.3. □

Let us discuss special 2-elementary $K3$ surfaces of type $\Pi_{1,1}(2)$. A 2-elementary $K3$ surface of type $\Pi_{1,1}(2)$, $(X, \iota)$, is said to be special if it has a $-2$-curve $E$ such that $E$ and $\iota(E)$ is disjoint. (This definition is analogous to that of Enriques surfaces ([B-P-V-V, Theorem 18.2], [Na, Remark 4.6]).) Let $\phi$ be a marking of $(X, \iota)$ and $I := \phi \circ \iota^* \phi^{-1}$ be the involution induced by $\iota$. Put $\delta := \phi(E) \in \Delta(L_{K3})$. Then, $\langle \delta, I(\delta) \rangle = 0$ by assumption. Let $T := (\Pi_{1,1}(2))^{\perp} = \Pi_{1,1}(2) \oplus I_{1,1}$ be the transcendental lattice. Let $p : L_{K3} \to T^\vee$ be the orthogonal projection. Set $d := p(\delta) = (\delta - I(\delta))/2 \in T^\vee$. Then, $d^2 = -1$ and $d \equiv (e + f)/2 \mod T$ where $\{e, f\}$ is the basis of $\Pi_{1,1}(2)$ as before. Conversely, if $d' \in T^\vee$ is a vector such that $(d')^2 = -1$ and $d' \equiv (e + f)/2 \mod T$, one can easily verify that there exists $\delta' \in \Delta(L_{K3})$ such that $\langle \delta', I(\delta') \rangle = 0$ and $d' = p(\delta')$.

**Lemma 10.11.** $(X, \iota)$ is a special 2-elementary $K3$ surface of type $\Pi_{1,1}(2)$ iff its period lies on the divisor $D'_{11,11} := \sum_{d \in T^\vee, d^2 = -1} H_d$.

**Proof.** Suppose that $(X, \iota)$ is special. Let $\omega$ be a canonical form of $(X, \iota)$. Let $E$ be a $-2$-curve as above. Since $\langle \omega, E \rangle = \langle \phi(\omega), d \rangle = 0$, we get $\phi(\omega) \in H_d$. Conversely, let $(X, \iota, \phi)$ be a marked 2-elementary $K3$ surface of type $\Pi_{1,1}(2)$ whose period lies on $H_d \subset D'$. Let $\delta \in \Delta(L_{K3})$ such that $\langle \delta, I(\delta) \rangle = 0, p(\delta) = d$. Let $E = \phi^{-1}(\delta)$ be an effective divisor and $E = \sum m_i C_i + \sum n_j E_j$ be the irreducible decomposition where $C_i^2 \geq 0$, $E_j^2 = -2$ and $m_i, n_j > 0$. By assumption, $\langle E, \iota(E) \rangle = 0$. Thus, $0 \geq \sum n_j E_j, \sum n_j \iota(E_j))$. Suppose that there is no $E_j$ such that $\langle E_j, \iota(E_j) \rangle = 0$. Since there is no $\iota$-invariant $-2$-curve and $\Gamma_1^2 \geq -2$ for any irreducible curve, we get $\langle E_j, \iota(E_j) \rangle \geq 2$. Note that $\langle E, \iota(E) \rangle = (E + \iota(E))^2/2 - 2 \equiv 0 \mod 2$. Since

$$0 \geq \sum n_j E_j, \sum n_j \iota(E_j) \geq \sum_{E_j = \iota(E_j)} \langle n_i E_i + n_j E_j, n_i \iota(E_i) + n_j \iota(E_j) \rangle$$

(10.7)

$$= \sum_{E_j = \iota(E_i)} \langle n_i E_i + n_j E_j, n_i E_j + n_j E_i \rangle$$

$$\geq \sum_{E_j = \iota(E_i)} 2(n_i - n_j)^2 \geq 0,$$

we can rewrite $\sum n_i E_i = \sum n_k (E_k + \iota(E_k))$. As $(E_k + \iota(E_k))^2 \geq 0$, it is impossible that $E^2 = -2$. Thus, $\langle E_j, \iota(E_j) \rangle = 0$ for some $E_j$ and we get the assertion. □
Lemma 10.12. \( \Delta_{4,4}^0 \) has an algebraic singularity along \( H' := D'_{11,1}(2)/\Gamma_{11,1}(2) \).

Proof. Since special 2-elementary \( K \)3 surfaces of type \( \Pi_{1,1}(2) \) are realized as a double cover of singular quadric surface like special Enriques surfaces (cf. [B-P-V-V, Theorem 18.2]), we can prove the assertion analogously to Lemma 10.5. \( \square \)

Let us regard \( \Delta_{4,4}^0 \) as a \( \Gamma_{11,1}(2) \)-invariant section of \( \lambda_{11,1}^{(2)} \) over \( \Omega_{11,1}(2) \).

Lemma 10.13. \( \text{div} (\Delta_{4,4}^0) = 18 D_{11,1}(2) + l D'_{11,1}(2) \) for some \( l \in \mathbb{Z} \).

Proof. Since the projection map \( \Omega_{11,1}(2) \to \mathcal{M}_{11,1}(2) \) ramifies of order 2 along the discriminant locus, we get the assertion by Lemmas 10.10, 10.11, and 10.12. \( \square \)

Let \( \epsilon_{00}, \epsilon_{01}, \epsilon_{10}, \epsilon_{11} \) be the basis of \( \mathbb{C}[T^\nu/T] = \mathbb{C}[\Pi_{1,1}(2)^\nu/\Pi_{1,1}(2)] \) such that \( \epsilon_{\alpha \beta} \) corresponds to \( (\alpha e + \beta f)/2 \in \Pi_{1,1}(2)^\nu \). Let \( f(\tau) := \eta(\tau)^{-8}\eta(2\tau)^{-8} \) be a modular form for \( \Gamma_0(2) \). Modifying Proposition 8.1 for \( \Gamma_0(2) \), we can verify that

\[
(10.8) \quad \tilde{f}(\tau) := f(\tau) \epsilon_{00} + 8\{f(\tau/2) + f((\tau + 1)/2)(\epsilon_{00} + \epsilon_{01} + \epsilon_{10}) + 8\{f(\tau/2) - f((\tau + 1)/2)\epsilon_{11}
\]

is a modular form of type \( \rho_T \) of weight \(-8\).

Proposition 10.4. \( \Psi_T (z, f) := \Psi_T (z, \tilde{f}) \) is a modular form of weight 68 with zero divisor \( D_{11,1}(2) + 16 D'_{11,1}(2) \).

Proof. Since \( \eta(\tau)^{-8}\eta(2\tau)^{-8} = q^{-1} + 8 + O(q) \), we get

\[
(10.9) \quad \tilde{f}(\tau) = (q^{-1} + 13) + O(q)) \epsilon_{00} + O(1)(\epsilon_{01} + \epsilon_{10}) + (16q^{-1/2} + O(q^{1/2}))\epsilon_{11},
\]

which, together with Theorem 8.1, yields the assertion. \( \square \)

Theorem 10.2. \( \Delta_{11,1}^{17} = C_{11,1}(2) \Delta_{4,4}^0(z)/[\Psi_{11,1}(2) \oplus \Pi_{1,1}(2)](z, \eta(\tau)^{-8}\eta(2\tau)^{-8}) \).

Proof. Put \( F(z) := \Delta_{4,4}^0(z)/[\Delta_{11,1}(2) \Delta_{11,1}^{17} \Psi_{11,1}(2) \oplus \Pi_{1,1}(2)](z, \eta(\tau)^{-8}\eta(2\tau)^{-8}) \). Since \( \Delta_{11,1}(2) \) is an automorphic form of weight \((-4, 4)\), \( \Delta_{11,1}^{17} \) of \((0, 68)\), and \( \Psi_T(z, f) \) of \((68, 0)\), \( F(z) \) is a \( \Gamma_{11,1}(2) \)-invariant meromorphic function on \( \Omega_{11,1}(2) \). From Theorem 7.2, Lemma 10.3 and Proposition 10.4, it follows that \( \text{div}(F) = (l - 16)D'_{11,1}(2) \). Thus, \( F \) descends to a meromorphic function on \( \mathcal{M}_{11,1}(2) \) with divisor \((l - 16)H'/2 \). By the residue theorem, we get \( l = 16 \) and \( F \) must be a nonzero constant. \( \square \)

Remark. As in the case of Borcherds’s \( \Phi \)-function, we can show

\[
\Psi_{11,1}(2) \oplus \Pi_{1,1}(2)\eta(\tau)^{-8}\eta(2\tau)^{-8})^2 = \Psi_{12,18}(2)\eta(\tau)^{-16}\eta(2\tau)^{16}\eta(4\tau)^{16}
\]

if we regard \( \Pi_{1,1}(2) = \{m \cdot h + n \cdot \delta \in I_{1,1}; m \equiv n \bmod 2 \} \) and \( \Pi(2) \oplus I_{1,17} \) as a sublattice of \( I_{2,18} \). Here, \( h^2 = 1, \delta^2 = -1, \) and \( h \cdot \delta = 0 \) is the basis of \( I_{1,1} \). It is interesting that \( \eta(\tau)^{-16}\eta(2\tau)^{16}\eta(4\tau)^{16} \) is the square of the modular form used to get Borcherds’s \( \Phi \)-function.
§11. A Theta Product and Borcherds’s Product

The Nikulin type of the lattice $S_5$ in (8.5) is $(r, l, \delta) = (16, 6, 1)$ and a 2-elementary $K3$ surface of type $S_5$ is the minimal resolution of the double cover of $\mathbb{P}^2$ branching along generic 6 lines. Therefore, the moduli space of 2-elementary $K3$ surfaces of type $S_5$ is isomorphic to the configuration space $\mathbb{X}(3, 6)$ of six points in $\mathbb{P}^2$, and the period map induces a morphism $Prd: \mathbb{X}(3, 6) \to \mathcal{M}_{S_5}$. In [Ma], Matsumoto described the inverse of the period map by using the theta function. Let $\mathbb{H}_2$ be the domain defined by $\mathbb{H}_2 := \{ W \in M(2, 2; \mathbb{C}) ; (W - W^*)/2i > 0 \}$ where $W^* = i^* W$. Identification of $\mathbb{H}_2$ and $\Omega_{S_5} \cong \Lambda_5 + \sqrt{-1}C_{\Lambda_5}$ is given by

\[
\mathbb{H}_2 \ni y = \left( \frac{y_0 + y_1}{y_0 + y_1 + y_2 - iy_3}, \frac{y_0 + y_1 + y_2 + iy_3}{y_0 + y_2} \right) \to \left( 1 : -\det y : y_0 : y_1 : y_2 : y_3 \right) \in \Omega_{S_5}
\]

under which $\Gamma_M(1 + i)$ (an arithmetic subgroup of $\text{Aut}(\mathbb{H}_2)$) is identified with $\Gamma_{S_5}$ by [Ma, Proposition 1.5.1]. Note that the quadratic form attached to $\Lambda_5$ is given by $q(y) = 4\det(y)$. On $\mathbb{H}_2$ exist theta functions, ten of which are represented by

\[
\Theta_{a, b}(W) = \sum_{m \in \mathbb{Z}[\sqrt{-1}]} \exp \pi i \{(m + a)^* W(m + a) + 2\text{Re} b^* m\}
\]

where $a, b \in \{0, (1 + i)/2\}^2$ with $a^* b \in \mathbb{Z}$. Any one of these ten theta functions is said to be an even theta function in this paper.

**Theorem 11.1.** Via the identification of $\Omega_{S_5}$ with $\mathbb{H}_2$, $2^{12} \Delta_{S_5} = \prod_{\text{even}} \Theta_{a, b}$.

**Proof.** By [Ma, 1.4, Proposition 1.5.1 and Lemma 2.3.1], $\text{div}(\Theta_{a, b})$ is the $\Gamma_{S_5}$-orbit of $H_{\alpha(a, b)} + H_{\alpha'(a, b)}$ where $\alpha(a, b), \alpha'(a, b) \in \Delta(T_5)$ are roots defined in [Ma, 1.4]. (By [Ma, 2.3], $(a, b)$ corresponds uniquely to a partition of $\{1, \ldots, 6\}$ into $\{i, j, k\} \cup \{l, m, n\}$. Then, $\alpha(a, b) := \alpha(i, j, k)$ and $\alpha'(a, b) := \alpha(l, m, n)$ in [Ma, 1.4].) In particular, on $\Omega_{S_5}$, one has

\[
\text{div}(\prod_{\text{even}} \Theta_{a, b}) = \sum H_{\alpha(a, b)} + H_{\alpha'(a, b)}.
\]

By [Ma, Proposition 3.1.1], $(\prod \Theta_{a, b})^2$ becomes a modular form of weight 20 relative to $\Gamma_M(1 + i)$. Consider the function $\Delta_{S_5}^2/(\prod \Theta_{a, b})^2$ (or its higher power if necessary) which descents to a meromorphic function on $\mathcal{M}_{S_5}$ by the automorphic property. Compared with Theorem 7.2, it has no pole and thus is a constant. Namely, there exists a constant $C$ such that $C \Delta_{S_5} = \prod \Theta_{a, b}$. Comparing the first non-zero Fourier coefficient ($[\text{G-N1}, (1.7)]$), we get $C = 2^{12}$. \(\square\)

Put $h = \left( \frac{1}{1 - i}, \frac{1}{1 + i}, 1 \right)$, $\delta_6 = \left( \frac{1}{1 - i}, \frac{1}{1 + i}, 0 \right)$, $\delta_7 = \left( 0, \frac{1}{1 - i}, 1 \right)$, $\delta_8 = \left( 0, \frac{i}{1 + i}, 0 \right)$, and $\delta_8' = h - \delta_7 - \delta_8$, $\delta_8' = h - \delta_6 - \delta_8$, $\delta_8' = h - \delta_6 - \delta_7$. Then, the set of simple roots of $\Lambda_5$ is $\{\delta_6, \delta_7, \delta_8, \delta_6', \delta_8'\}$, and the Weyl vector is $2\rho_5 = \sum_{6 \leq \delta_6} \delta_i + \sum_{6 \leq \delta_8} \delta_i'$. Put $q_i = e^{2\pi i \omega_i}$ $(1 = 0, 1, 2, 3)$, $y = y_0 h + y_1 \delta_6 + y_2 \delta_7 + y_3 \delta_8$ and $r = r_0 h + r_1 \delta_6 + r_2 \delta_7 + r_3 \delta_8 \in \mathbb{H}_2$. Comparing Theorem 8.1 and 11.1, we get the following.

**Corollary 11.1.**

\[
2^{-12} \prod_{(a, b) \text{ even}} \Theta_{a, b}(y) = q_0^3 q_1^{-1} q_2^{-1} q_3^{-1} \prod_{\epsilon \in \{0, 1\}} \prod_{r \in \Pi_{\epsilon}^+(\Lambda_5)} (1 - q_0^\alpha q_1^{-r_1} q_2^{-r_2} q_3^{-r_3}) c_{5, \epsilon}(2 \det r)
\]

where $\Pi_{\epsilon}^+(\Lambda_5) = (\sum_{6 \leq \delta_6} (\mathbb{Z} + \frac{\epsilon}{2}) \geq 0 \delta_i + \sum_{6 \leq \delta_8} (\mathbb{Z} + \frac{\epsilon}{2}) \geq 0 \delta_i') \setminus \{0\}$, and $\{c_{5, \epsilon}(m)\}$ is the same as in Theorem 8.2.
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