Abstract. We show that the page at which the Lee spectral sequence collapses gives a bound on the unknotting number, \( u(K) \). In particular, for knots with \( u(K) \leq 2 \), we show that the Lee spectral sequence must collapse at the \( E_2 \) page. An immediate corollary is that the Knight Move Conjecture is true when \( u(K) \leq 2 \).

1. Introduction

In [Kho00], Khovanov defined a bigraded knot invariant \( H_{Kh}(K) \) which categorifies the Jones polynomial. This invariant comes in the form of a homology theory based on a planar diagram for a knot and the Frobenius algebra \( \mathbb{Q}[X]/X^2 = 0 \).

There is a basic structural theory about Khovanov homology known as the Knight Move Conjecture, which can be stated as follows:

**Conjecture 1.1** (Knight Move Conjecture, [Kho00], [BN02]). The Khovanov homology of any knot \( K \) decomposes as a single ‘pawn move’ pair

\[
\mathbb{Q}\{0, n - 1\} \oplus \mathbb{Q}\{0, n + 1\}
\]

together with a set of knight move pairs

\[
\bigoplus_i \mathbb{Q}\{l_i, m_i\} \oplus \mathbb{Q}\{l_i + 1, m_i + 4\}
\]

where \( \mathbb{Q}\{i, j\} \) denotes a generator in bigrading \((i, j)\).

In [Lee05], Lee defined a deformation of Khovanov homology by changing the Frobenius algebra to \( \mathbb{Q}[X]/(X^2 = 1) \). The corresponding complex can be viewed as the original complex with additional differentials, resulting in a spectral sequence \( E_\infty(K) \) from Khovanov homology to Lee homology. Lee showed that this spectral sequence is an invariant of the knot \( K \), and that it converges to \( \mathbb{Q} \oplus \mathbb{Q} \). Rasmussen [Ras10] added to this result that \( E_\infty(K) = \mathbb{Q}\{0, s - 1\} \oplus \mathbb{Q}\{0, s + 1\} \), where \( s \) is Rasmussen’s slice invariant.

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The differential $d_n$ on $E_n(K)$ has bigrading $(1, 4n)$, so the Knight Move Conjecture is true whenever the Lee spectral sequence collapses at the $E_2$ page.

In this paper, we construct a lower bound for the unknotting number $u(K)$ using Lee’s homology theory, and we apply this bound to prove that the Lee spectral sequence must collapse at the $E_2$ page whenever $u(K) \leq 2$.

We technically use a lift of Lee’s complex obtained by setting each $X^2 = t$ as described in [Kho06]. The resulting homology $H_{\text{Lee}}(K)$ is a module over $\mathbb{Q}[t]/(X^2 = t)$, and it consists of two towers $\mathbb{Q}[t] \oplus \mathbb{Q}[t]$ and an $X$-torsion summand $T_X(H_{\text{Lee}}(K))$. Note that since $X^2 = t$, $X$-torsion and $t$-torsion are the same, i.e. $T_X(H_{\text{Lee}}(K)) = T_t(H_{\text{Lee}}(K))$. We define $u_X(K)$ to be the maximal order of $X$-torsion in $H_{\text{Lee}}(K)$.

**Theorem 1.2.** For any knot $K$, $u_X(K)$ gives a lower bound for the unknotting number of $K$.

We prove this by defining crossing change maps $f$ and $g$ as shown below such that on homology, both $f \circ g$ and $g \circ f$ are either equal to $2X$ or $-2X$. The diagrams $D_+$ and $D_-$ differ at a single crossing $c$, where $D_+$ has a positive crossing and $D_-$ has a negative crossing.

\[
\begin{array}{ccc}
C_{\text{Lee}}(D_+) & \xrightarrow{f} & C_{\text{Lee}}(D_-) \\
\downarrow{g} & & \downarrow{g} \\
C_{\text{Lee}}(D_-) & & C_{\text{Lee}}(D_+)
\end{array}
\]

Using similar chain maps, the first author gives a lower bound for the unknotting number from Bar-Natan homology [Ali].

Since $X^2 = t$, we have

\[ [u_X(K)/2] = u_t(K) \]

where $[x]$ is the ceiling of $x$. The variable $t$ keeps track of the Lee filtration, so if we add 1 to the maximal order of $t$-torsion in $H_{\text{Lee}}(K)$, the result is exactly the page at which the Lee spectral sequence collapses.

**Theorem 1.3.** If $K$ is a knot with $u(K) \leq 2$ and $K$ is not the unknot, then the Lee spectral sequence for $K$ collapses at the $E_2$ page.

**Corollary 1.4.** The Knight Move Conjecture is true for all knots $K$ with $u(K) \leq 2$. 
2. Background

In this section we will describe the Khovanov chain complex and the Lee deformation. We will use a notation that makes the module structure clear.

2.1. The Standard Khovanov Complex. Assume $L$ be a link in $S^3$ with diagram $D \subset \mathbb{R}^2$. Let $\mathcal{C} = \{c_1, c_2, ..., c_n\}$ denote the crossings in $D$, and viewing $D$ as a 4-valent graph, let $E = \{e_1, e_2, ..., e_m\}$ denote the edges of $D$. The edge ring is defined to be

$$R := \mathbb{Q}[X_1, X_2, ..., X_m]/\{X_2^1 = X_2^2 = ... = X_2^m = 0\}$$

with each variable $X_i$ corresponding to the edge $e_i$.

Each crossing $c_i$ can be resolved in two ways, the 0-resolution and the 1-resolution (see Figure 1). For each $v \in \{0, 1\}^n$, let $D_v$ denote the diagram obtained by replacing the crossing $c_i$ with the $v_i$-resolution. The diagram $D_v$ is a disjoint union of circles - denote the number of circles by $k_v$. The vector $v$ determines an equivalence relation on $E$, where $e_p \sim_v e_q$ if $e_p$ and $e_q$ lie on the same component of $D_v$.

![Figure 1](image.png)

The module $C_{Kh}(D_v)$ is defined to be a quotient of the ground ring:

$$C_{Kh}(D_v) := R/\{X_p = X_q \text{ if } e_p \sim_v e_q\}$$

we will denote this quotient by $R_v$.

There is a partial ordering on $\{0, 1\}^n$ obtained by setting $u \leq v$ if $u_i \leq v_i$ for all $i$. We will write $u < v$ if $u \leq v$ and they differ at a single crossing, i.e. there is some $i$ for which $u_i = 0$ and $v_i = 1$, and $u_j = v_j$ for all $j \neq i$. Corresponding to each edge of the cube, i.e. a pair $(u < v)$, there is an embedded cobordism in $\mathbb{R}^2 \times [0, 1]$ from $D_u$ to $D_v$ constructed by attaching a 1-handle near the crossing $c_i$ where $u_i < v_i$. This cobordism is always a pair of pants, either going from one circle to two circles (when $k_u = k_v - 1$) or from two circles to one circle (when $k_u = k_v + 1$). We call the former a merge cobordism and the latter a split cobordism.
For each vertex \( v \) of the cube, the quotient ring \( R_v \) is naturally isomorphic to \( \mathcal{A}^{\otimes k_v} \), where \( \mathcal{A} \) is the Frobenius algebra \( \mathbb{Q}[X]/(X^2 = 0) \). Recall that the multiplication and comultiplication maps of \( \mathcal{A} \) are given as:

\[
m : \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{A} \to \mathcal{A} : \begin{cases} 1 \mapsto 1, & X_1 \mapsto X \\ X_1X_2 \mapsto 0, & X_2 \mapsto X \end{cases}
\]

and

\[
\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{A} : \begin{cases} 1 \mapsto X_1 + X_2 \\ X \mapsto X_1X_2 \end{cases}
\]

The chain complex \( C_{Kh}(D) \) is defined to be the direct sum of the \( C_{Kh}(D_v) \) over all vertices in the cube:

\[
C_{Kh}(D) := \bigoplus_{v \in \{0,1\}^n} C_{Kh}(D_v)
\]

The differential decomposes over the edges of the cube. When \( u < v \) corresponds to a merge cobordism, define

\[
\delta_{u,v} : C_{Kh}(D_u) \to C_{Kh}(D_v)
\]

to be the Frobenius multiplication map, and when \( u < v \) corresponds to a split cobordism, define \( \delta_{u,v} \) to be the comultiplication map. In terms of the quotient rings \( R_u \) and \( R_v \), the map \( m \) is projection, while \( \Delta \) is multiplication by \( X_j + X_k \), where \( e_i, e_j, e_k, e_l \) are the edges at the corresponding crossing as in Figure 1. Note that \( X_j + X_k = X_i + X_l \).

If \( D_u \) and \( D_v \) differ at crossing \( e_i \), define \( \epsilon_{u,v} = \sum_{j < i} u_j \). Then

\[
\delta = \sum_{u < v} (-1)^{\epsilon_{u,v}} \delta_{u,v}
\]

The Khovanov complex is bigraded, with a homological grading and a quantum grading. Up to an overall grading shift, the homological grading is just the height in the cube. Setting \( |v| = \sum_i v_i \), \( n_+ \) the number of positive crossings in \( D \), an \( n_- \) the number of negative crossings in \( D \), we have

\[
\text{gr}_h(R_v) = |v| - n_-
\]

For each vertex \( v \) of the cube, the quantum grading of \( 1 \in R_v \) is given by

\[
\text{gr}_q(1 \in R_v) = n_+ - 2n_- + |v| + k_v
\]

and each variable \( X_i \) has quantum grading \(-2\). With respect to the bigrading \((\text{gr}_h, \text{gr}_q)\), the differential \( \delta \) has bigrading \((1,0)\). The Khovanov homology \( H_{Kh}(D) \) is the homology of this complex.
2.2. The Lee Deformation. The Lee deformation on Khovanov homology comes from a small modification of the ring $R$. If we replace $R$ with the ring $R' := \mathbb{Q}[X_1, X_2, ..., X_k, t]/\{X_1^2 = X_2^2 = ... = X_l^2 = t\}$ and define everything as in the previous section, the result is a complex $C_{\text{Lee}}(D)$. The variable $t$ has homological grading 0 and quantum grading $-4$, so the complex is still bigraded. Note that $R'/ (t = 0) \cong R$, so $C_{\text{Lee}}(D)/ (t = 0) \cong C_{\text{Kh}}(D)$.

The edge maps are still given by projection for $m$ and multiplication by $X_j + X_k$ for $\Delta$. More precisely, for $u \lessdot v$ the edge homomorphism $\delta_{uv} : R_u \to R_v$ is given by

\[
1 \overset{m}{\mapsto} 1 \\
X_i \overset{m}{\mapsto} X_i \\
X_j \overset{m}{\mapsto} X_j \\
X_iX_j \overset{m}{\mapsto} t
\]

or

\[
1 \overset{\Delta}{\mapsto} X_j + X_k \\
X_i \overset{\Delta}{\mapsto} X_i\Delta + X_jX_k + t.
\]

As before, $e_i$, $e_j$, $e_k$ and $e_l$ are the edges at the corresponding crossing as in Figure 1. By a minor abuse of notation, we refer to this differential as $\delta$ as well.

The Lee homology is defined to be the homology of this complex,

\[
H_{\text{Lee}}(D) = \text{H}_*(C_{\text{Lee}}(D), \delta).
\]

Remark 2.1. The actual complex defined by Lee in \cite{Lee05} is given by $C_{\text{Lee}}(D)/ (t = 1)$. Setting $t = 1$ replaces the $q$-grading with a filtration, which induces the Lee spectral sequence. The number of page at which the Lee spectral sequence collapses is 1 more than the maximal degree of $t$-torsion in $H_{\text{Lee}}(D)$.

Theorem 2.2 (\cite{Lee05}). If $D$ is a diagram for a knot $K$, then ignoring gradings, $H_*(C_{\text{Lee}}(D))$ decomposes as

\[
H_{\text{Lee}}(D)) \cong \mathbb{Q}[t] \oplus \mathbb{Q}[t] \oplus T(H_{\text{Lee}}(D))
\]

where $T(C)$ is the $t$-torsion part of $C$.

In other words, the free part of $H_*(C_{\text{Lee}}(D))$ is isomorphic to the Lee homology of the unknot, and

\[
H_*(C_{\text{Lee}}(D)/ (t = 1)) \cong \mathbb{Q} \oplus \mathbb{Q}.
\]
3. The Crossing Change Map

Let \( D_+ \) and \( D_- \) be two diagrams that differ at a single crossing \( c \), with \( D_+ \) having a positive crossing and \( D_- \) having a negative crossing. Let \( e_i, e_j, e_k, e_l \) be the adjacent edges, as in Figure 2. In this section we will define chain maps

\[
\begin{align*}
  f : C_{Lee}(D_+) &\rightarrow C_{Lee}(D_-) \\
  g : C_{Lee}(D_-) &\rightarrow C_{Lee}(D_+) 
\end{align*}
\]

such that for any \( 1 \leq i \leq m \), both \( f \circ g \) and \( g \circ f \) are chain homotopy equivalent to multiplication by \( 2X_i \) or \(-2X_i\).

![Figure 2.](image)

Let \( D_0^+ \) be the 0-resolution of \( D_+ \) at \( c \) and \( D_1^+ \) the 1-resolution \( D_+ \) at \( c \), and define \( D_0^- \), \( D_1^- \) analogously. Note that \( D_0^+ \) and \( D_1^- \) are the same diagram, as are \( D_1^+ \) and \( D_0^- \). We can write

\[
\begin{align*}
  C_{Lee}(D_+) &= C_{Lee}(D_0^+) \xrightarrow{\delta_+} C_{Lee}(D_1^+) \\
  C_{Lee}(D_-) &= C_{Lee}(D_0^-) \xrightarrow{\delta_-} C_{Lee}(D_1^-) 
\end{align*}
\]

where \( \delta_+ \), \( \delta_- \) are the edge maps for the respective complexes corresponding to the crossing \( c \). As modules (ignoring the differentials), we have \( C_{Lee}(D_+) = D_0^+ \oplus D_1^+ \) and \( C_{Lee}(D_-) = D_0^- \oplus D_1^- \).

For \( a \) in \( C_{Lee}(D_+) \), write \( a = (a^0, a^1) \). In order to pin down the signs on \( C_{Lee} \), we need to choose an ordering of the crossings. For simplicity, take \( c \) to be the last crossing. We define \( f : C_{Lee}(D_+) \rightarrow C_{Lee}(D_-) \) by

\[
f(a^0, a^1) = ((X_j - X_k)a^1, a^0)
\]

Similarly, we define \( g : C_{Lee}(D_-) \rightarrow C_{Lee}(D_+) \) by
Lemma 3.1. The maps $f$ and $g$ are chain maps.

Proof. We will prove it for $f$ (The proof for $g$ is completely analogous). The map by 1 doesn’t interact with the edge maps $\delta_+$ or $\delta_-$, and since we chose $c$ to be the last crossing, the negative signs are the same in $C_{Lee}(D^1_+)$ and in $C_{Lee}(D^1_-)$ in a manner to make the map commute with the respective differentials. Thus, it is a chain map.

Similarly, the negative signs in $C_{Lee}(D^1_+)$ and in $C_{Lee}(D^0_-)$ make the multiplication by $X_j - X_k$ anticommute with all differentials within these two
complexes. Thus, to see that the multiplication by $X_j - X_k$ is a chain map, we must show $(X_j - X_k)\delta_+ = \delta_-(X_j - X_k) = 0$.

The map $\delta_+$ is either a merge or a split, depending on the vertex in $C_{\text{Lee}}(D^0_+)$.
For the vertices where it is a merge, $e_j$ and $e_k$ will lie on the same circle in the corresponding vertex in $C_{\text{Lee}}(D^1_+)$, so $X_j = X_k$ at these vertices and $(X_j - X_k)\delta_+ = 0$. For the vertices where it is a split, $\delta_+$ is multiplication by $X_j + X_k$, so $(X_j - X_k)\delta_+ = X_j^2 - X_k^2 = t - t = 0$.

The argument that $\delta_-(X_j - X_k) = 0$ is similar. If we choose a vertex where $\delta_-$ is a merge map, then we will have $X_j = X_k$ on $C_{\text{Lee}}(D^0_-)$. But if $\delta_-$ is a split map, then $X_j - X_k$ is already equal to zero on $C_{\text{Lee}}(D^0_+)$. □

**Lemma 3.2.** For any $a$ in $C_{\text{Lee}}(D_+)$ and any $b$ in $C_{\text{Lee}}(D_-)$, $(g \circ f)(a) = (X_j - X_k)a$ and $(f \circ g)(b) = (X_j - X_k)b$.

This is clear from the definitions of $f$ and $g$. The lemma becomes interesting with the following result of Hedden and Ni.

**Lemma 3.3 ([HN13]).** Let $D$ be a planar diagram for a link $L$. If edges $e_j$ and $e_k$ are diagonal from one another at a crossing $c$ (positive or negative), then there is a chain homotopy $H : C_{\text{Lee}}(D) \to C_{\text{Lee}}(D)$ satisfying

$$\delta H + H\delta = X_j + X_k$$

Although their proof is for the Khovanov complex, the same argument applies for the Lee complex. For completeness, we will repeat it here.

**Proof.** Suppose the crossing $c$ is positive. Then the complex can be written

$$C_{\text{Lee}}(D^0_+) \xrightarrow{\delta_+} C_{\text{Lee}}(D^1_+)$$

where $D^0_+$ and $D^1_+$ are the 0- and 1-resolution at $c$.

We define $H : C_{\text{Lee}}(D) \to C_{\text{Lee}}(D)$ to be equal to 0 on $C_{\text{Lee}}(D^0_+)$ summand and $\delta_-$ on $C_{\text{Lee}}(D^1_+)$ summand, i.e. the edge map that would have appeared if $c$ were a negative crossing. Our complex with the total differential $\delta + H$ now looks like

$$C_{\text{Lee}}(D^0_+) \xrightarrow{\delta_+} C_{\text{Lee}}(D^1_+) \xrightarrow{\delta_-} C_{\text{Lee}}(D^0_+)$$
Since $\delta_-$ anticommutes with all edge maps except $\delta_+, \delta H + H\delta = \delta_+\delta_- + \delta_-\delta_+$, which by inspection is equal to $X_j + X_k$.

The negative crossing argument is similar. In this case, the chain homotopy $H$ is defined using $\delta_+$ instead of $\delta_-$. □

Putting the previous two lemmas together, we get the following corollary.

**Corollary 3.4.** For any $1 \leq i \leq m$, any $a$ in $H_{Lee}(D_+)$ and any $b$ in $H_{Lee}(D_-)$, we have $(g_* \circ f_*)(a) = \pm 2X_i a$ and $(f_* \circ g_*)(b) = \pm 2X_i b$, where the sign depends on $i$.

4. **Unknotting Number Bounds and the Knight Move Conjecture**

Suppose $C$ is a $\mathbb{Q}[X]$-module. Recall that the $X$-torsion in $C$, which we will denote by $T_X(C)$, is given by

$$T_X(C) = \{a \in C : X^n a = 0 \text{ for some } n \in \mathbb{N}\}$$

We define $u_X(C)$ to be the maximum order of a torsion element in $C$.

**Lemma 4.1.** Let $D_+$ and $D_-$ be two knot diagrams which differ at a single crossing $c$. Then

$$|u_X(H_{Lee}(D_+)) - u_X(H_{Lee}(D_-))| \leq 1$$

where $X$ refers to $X_i$ for some $i$.

Note that up to sign, multiplication by any $X_i$ is the same on the Lee homology, so $u_{X_i}(H_{Lee})$ does not depend on the choice of $i$.

**Proof.** Let $a \in T_{X_i}(H_{Lee}(D_+))$, and let $\text{ord}_{X_i}(a)$ denote the order of $a$ with respect to $X_i$. Then

$$\text{ord}_{X_i}(a) \geq \text{ord}_{X_i}(f_*(a)) \geq \text{ord}_{X_i}(g_*(f_*(a)))$$

Since $g_*(f_*(a)) = \pm 2X_1 a$ and we’re working over $\mathbb{Q}$, we get $\text{ord}_{X_i}(g_*(f_*(a))) = \text{max}(\text{ord}_{X_i}(a) - 1, 0)$. This gives

$$\text{ord}_{X_i}(a) - 1 \leq \text{ord}_{X_i}(f_*(a))$$

so $u_{X_i}(H_{Lee}(D_+)) - u_{X_i}(H_{Lee}(D_-)) \leq 1$. The reverse inequality is obtained by starting with $b$ in $T_{X_1}(H_{Lee}(D_-))$ and applying $f_* \circ g_*$. □

**Theorem 4.2.** For any knot $K$, $u_X(K)$ gives a lower bound for the unknotting number of $K$. 

Proof. This follows immediately from the previous lemma together with the observation that $H_{\text{Lee}}(\text{Unknot}) = \mathbb{Q}[t] \oplus \mathbb{Q}[t]$, so $u_X(\text{Unknot}) = 0$. □

To translate this result back to the Lee spectral sequence, we use the fact that $X^2 = t$. It follows that

$$\left\lceil u_X(K)/2 \right\rceil = u_t(K)$$

where $\left\lceil x \right\rceil$ is the ceiling of $x$. The Lee spectral sequence collapses at the $E_2$ page if and only if $u_t(K) = 1$.

**Theorem 4.3.** If $K$ is a knot with $u(K) \leq 2$ and $K$ is not the unknot, then the Lee spectral sequence for $K$ collapses at the $E_2$ page.

**Proof.** By the previous theorem, $u_X(K) \leq 2$, so $u_t(K) \leq 1$. Since Khovanov homology detects the unknot, we know that $u_t(K) \neq 0$. The theorem follows. □

**Corollary 4.4.** The Knight Move Conjecture is true for all knots $K$ with $u(K) \leq 2$.

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