FORWARD EXPONENTIAL PERFORMANCES: PRICING AND OPTIMAL RISK SHARING

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Abstract. In a Markovian stochastic volatility model, we consider financial agents whose investment criteria are modelled by forward exponential performance processes. The problem of contingent claim indifference valuation is first addressed and a number of properties are proved and discussed. Special attention is given to the comparison between the forward exponential and the backward exponential utility indifference valuation. In addition, we construct the problem of optimal risk sharing in this forward setting and solve it when the agents' forward performance criteria are exponential.

Keywords: Forward performance criteria, stochastic utility, stochastic risk aversion, indifference price, optimal risk sharing, exponential utility, contingent claim pricing

Introduction

Contingent claim pricing in incomplete markets is one of the most challenging problems in mathematical finance. In incomplete markets, there exist contingent claims for which there is no dynamic self-financing portfolio that perfectly replicates their payoffs. A consequence of this is that the non-arbitrage arguments provide only an interval of prices consistent with the non-arbitrage
assumption. The answer to the question which is the “correct” price within this interval requires a model of agents’ risk preferences modelling and perhaps their endowments or/and their beliefs.

One of the most fruitful literature on financial agents’ risk preferences is the one on utility function. Based on the work of R. Merton [26] and [27], this theory suggests that an agent, who models her risk preferences through a utility function, is going to invest in the financial market with the aim to maximize the expectation of her utility function. The utility maximization has been extensively studied and developed in a variety of market models and utility functions (see for instance, [14], [20], [23] and [36] for an overview). If an agent’s investment criterion is the utility maximization, it is reasonable to assume that she evaluates each contingent claim by comparing the following two situations: maximization of the expected utility after buying (selling) the claim and maximization of the utility without any transaction on the claim. The price that makes these situations indifferent for the agent’s perspective is the so-called indifference price. This pricing mechanism was introduced in mathematical finance literature in [17] and then further developed by a number of authors (see among others [12], [25], [29] and [16] for an overview). We should highlight at this point that the indifference pricing mechanism is subjective, in the sense that a utility maximizer quotes prices at which she is willing to buy or sell a given contingent claim. However, there is no guarantee that these prices are the ones at which any kind of transaction actually takes place. Therefore, throughout this paper we prefer to call these prices values to emphasize their subjective nature.

One of the main flaws of the utility maximization (and of the induced indifference valuation) is the dependence on the time horizon at which the utility function stands. Although for investment goals and single claim pricing, fixing a certain investment/pricing time horizon may not be problematic, it creates consistency concerns. In particular, this theory does not provide a way to set another time horizon and the continuation of the investments to be consistent. Similarly, the valuing of continent claims with maturity later than the chosen time horizon can not be addressed with the available tools. This is because there is no forward shifting of a utility function. Even more inconvenient is that fixing a utility at some time in the future leaves no room for updating the utility function (and by extension the investment goals) until the terminal horizon. It looks like an agent is stuck with her utility function and a given subjective probability measure, no matter what happens to the market, her endowment or her beliefs.

The problem of time horizon dependence of investment choices has been recently studied by a number of authors ([6], [9], [15], [37] and [38]). A common concept of these works is that agents aim to maximize, instead of a utility function, a family of state-dependent utility functions in a time-consistent way. In this paper, we work on the notion of forward performance or forward utility, which has been introduced in the works of M. Musiela and T. Zariphopoulou [30] and [31] (see also [37] for an overview). In words, this concept suggests that in contrast to the backward utility
function maximization, the agents choose a family of state-dependent utility functions and their investment goal is to find the admissible trading strategy that keeps the expectation of this family at the same level (see Definition 1.1 for the exact definition and 33 for an extended discussion). If the family of utility functions is of certain exponential type, the forward performance is called exponential. Explicit formulas for the optimal strategy and the optimal wealth process under this type of forward performance criteria has been provided in 31 under a Markovian market model (see also 38 for some related discussion). More recently, G. Žitković in 39 establishes the characterization of the forward exponential performance process in a general semimartingale market model and as a special case in a diffusion stochastic volatility model, similar to the one we shall impose in the present paper. One of the important part of the characterization of the forward exponential utility functions is that the risk aversion becomes stochastic process instead of constant (as in the classic exponential utility function). Furthermore, for the stochastic risk aversion process, usually denoted by $\gamma_t$, it holds that the quantity $1/\gamma_t$, which can be thought as agent’s (stochastic) risk tolerance, is replicable.

The first aim of this paper (Section 2) is to contribute to the theory of exponential forward performance by investigating how agents value contingent claims under such investment criteria. Valuation in a forward manner has several differences in comparison with the backward valuation (which is induced by classic utility functions) both in financial and technical sense. In Section 2, we state and prove a number of properties of the forward indifference valuation and we point out the differences to the corresponding backward valuation. Namely, based on the characterization results in 39, we are able to prove that the dynamic version of the forward indifference valuation solves a certain type of backward stochastic differential equation (see Proposition 2.1). This equation is similar to the corresponding equation solved by the dynamic exponential utility indifference price (provided in 25), where the differences are limited to the referred martingale probability measure and to that the risk aversion becomes stochastic in the forward valuation. Using this stochastic differential equation, we also show exactly how the forward performance changes when the agent buys or sells a contingent claim. The next step in analyzing the forward exponential indifference valuation is its robust representation. As pricing functional in incomplete markets, the forward indifference value can be seen as a convex map, i.e. a convex risk measure in the sense of 11. Following the related literature on dynamic convex risk measure (see 7 and 22), we write the forward indifference value as a minimum (over martingale probability measures) of the expectation of the claim plus a penalty function, which incorporates the agent’s specific characteristics. This representation is useful for proving several properties of the indifference valuation such as continuity, differentiability with respect to the units of the claim and monotonicity with respect to risk aversion (see Propositions 2.2 and 2.3).
The second part of the paper (Section 3) is dedicated to the optimal risk sharing between two agents whose investment criteria are based on forward exponential performances. Optimal risk sharing problem is about two agents which negotiate the sharing of their endowments in such a way that the sum of their positions is optimized in terms of risk. This problem is well studied under several models from classic utility functions (see [4] and [5]) to convex risk measures (see among others [2], [3], [10] and [18]). All of these models are set in a backward fashion, that is the optimization criteria have a fixed time horizon, which in fact equals to the maturity of the agents’ endowments. In this paper, we initiate this problem in the forward setting and establish its solution in three different cases regarding the model parameters: (a) when both agents have constant risk aversions, (b) when agents have common but stochastic risk aversions and (c) when agents have different and stochastic risk aversions. In case (a), the problem is reduced to the backward setting. As it is pointed out in Remark 2.6, given a fixed time horizon, the forward exponential performance criterion with constant risk aversion can be considered as a simple utility function under a specific random endowment. By exploiting the robust representation of the indifference value, we get the exact form of the contracts that optimally share the agents’ random endowments and we bring out its similarity with the classic entropic risk measures case (studied among others in [3]). In case (b), a similar closed form solution of the optimal contract can be provided, where the impact of agents’ characteristics is clear (for example, the more different the agents’ beliefs about the probability measure are, the bigger the expected size of the optimal contract is). Finally, for the more general case of different and stochastic risk aversions we need to look at the time evolution of the inf-convolution risk measure induced by agents’ forward performance criteria. We first establish the necessary and sufficient conditions under which this measure can be seen as one induced by some other forward exponential performance. Then, we generalize the results of [3] in the forward setting and get the stochastic differential equation satisfied by the inf-convolution measure. This result enables us to derive the form of the optimal risk sharing contract, which also allows the comparison with the analogous backward valuation setting. The structure of the optimal risk sharing contract admits a mild generalization in the case of forward exponential performance. In particular, in the forward setting the optimal risk sharing consists of three terms, one that has to do with the sharing of the endowments, one that incorporates the difference of beliefs between agents and a replicable term (which can be ignored since it does not transfer any risk). The first term has the same form as in the case of entropic risk measures (however the risk aversion becomes stochastic); the second term, i.e. the sharing of agents’ difference of beliefs does not depend on agents’ random endowments, but does depend on agents’ risk aversion levels.

The market model used in this manuscript consists of one riskless asset and one risky asset, whose volatility is driven by a generalized Itô process (the same model has also been used in [35] and [39]). It should be pointed out that the majority of the results in this paper can be
generalized in a straightforward way to models with more risky assets and continuous martingales instead of Brownian motions. We choose to work in this simplified model in order to focus on the interpretation of the results and the explanation of how agents evaluate claims and share risks under forward looking investment criteria. Furthermore, this work deals with the exponential type of forward performance, since this type is more tractable and closed form solutions can be provided. It also helps the comparison with the backward case, where there are several well-known results regarding valuation and risk sharing issues. Finally, as mentioned and illustrated in [31], exponential forward performance is quite general and captures a variety of agents’ distinct characteristics.

1. Market Setting

1.1. Assets and admissible strategies. The market consists of a risky and a risk-free asset. The risk-free asset is used as a numéraire and its price process evolution is given by

$$dB_t = rB_t dt,$$

where $r > 0$ is a constant. The price process of the risky asset satisfies the following stochastic differential equation (SDE)

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW^1_t,$$  \hspace{1cm} (1)

where $S_0 > 0$ and process $Y$ solves the equation

$$dY_t = b(Y_t) dt + a(Y_t) \left( \rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t \right)$$  \hspace{1cm} (2)

Pair $(W^1_t, W^2_t)_{t \in [0, \infty)}$ is a 2-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is the augmented $\sigma$-algebra generated by $(W^1_t, W^2_t)_{t \in [0, \infty)}$. As usual, we assume that $\mu, \sigma, b$ and $a$ satisfy global Lipschitz and linear growth conditions (with $\sigma > 0$) and $\rho \in (-1, 1)$ is a constant.

We define the market price of risk process $(\lambda_t)_{t \in [0, \infty)}$, via

$$\lambda_t = \lambda(Y_t) = \frac{\mu(Y_t) - r}{\sigma(Y_t)}, \quad t \in [0, \infty).$$  \hspace{1cm} (3)

Throughout this paper, we impose the following technical assumption.

Assumption 1.1. For every $T > 0$, there exists $\varepsilon > 0$ such that

$$\mathbb{E} \left[ e^{(1/2 + \varepsilon) \int_0^T \lambda^2_u du} \right] < \infty.$$  

The set of admissible strategies is given by

$$\mathcal{A} = \{ \mathbb{F} \text{-progressively measurable } \pi : \mathbb{E} \left[ \int_0^t \sigma^2(Y_s) \pi^2_s ds \right] < \infty, \forall t > 0 \}.$$  

The discounted wealth process of an admissible strategy $\pi$ with initial capital $x$ is denoted by $X^{x,\pi}$ and satisfies the following SDE

$$dX^{x,\pi}_t = \sigma(Y_t)\pi_t(\lambda(Y_t) dt + dW^1_t)$$  \hspace{1cm} (4)
with \( X_0^{x,\pi} = x \) (when the initial wealth is zero, we simply write \( X^\pi \)). We also define the set \( A^\infty = \{ \pi \in A : X_t^\pi \in L^\infty(F_t), \forall t \geq 0 \} \).

For the model at hand, we introduce the following notations for any time horizon \( T > 0 \).

\[
\mathcal{P}_T = \{ \mathbb{F}\text{-progressively measurable } \nu : \int_0^T \nu_u^2 du < \infty, \text{ a.s.} \} \tag{5}
\]

\[
\mathcal{P} = \bigcap_{T > 0} \mathcal{P}_T \tag{6}
\]

and

\[
\mathcal{N} = \{ (\beta, \nu) \in \mathcal{P} \times \mathcal{P} : Z^{\beta,\nu}_T \text{ is a true } \mathbb{P}\text{-martingale} \}, \tag{7}
\]

where \((Z^{\beta,\nu}_t)_{t \in [0,\infty)}\) is the solution of the equation

\[
dZ^{\beta,\nu}_u = -Z^{\beta,\nu}_u (\beta_u dW^1_u + \nu_u dW^2_u). \]

Note that under Assumption [1] \((\lambda, 0) \in \mathcal{N}\).

For every arbitrarily chosen time horizon \( T \) and every \((\beta, \nu) \in \mathcal{N}\), we define the probability measure \( Q^{\beta,\nu} \sim \mathbb{P}_{|F_T} \) by its R-N derivative

\[
dQ^{\beta,\nu} = Z^{\beta,\nu}_T. \]

We also define the set \( \mathcal{P}^\lambda \subseteq \mathcal{P} \), which contains all processes \( \nu \in \mathcal{P} \) such that \((\lambda, \nu) \in \mathcal{N}\) (similarly we define the set \( \mathcal{P}^\lambda_T \)).

A simple application of Girsanov Theorem implies that for every \( \nu \in \mathcal{P}^\lambda \) the discounted stock price \( S_t^{\mathcal{P},\mathcal{P}^\lambda} \) is a local-martingale under the measure \( Q^{\lambda,\nu} \). In fact, for the set of equivalent local-martingale measures \( \mathcal{M}^\mathcal{P} = \{ Q \sim \mathbb{P} : S_t^{\mathcal{P},\mathcal{P}^\lambda} \text{ is a } Q\text{-local martingale in } [0,T] \} \) it holds that:

\[
\mathcal{M}^\mathcal{P} = \{ Q^{\lambda,\nu} : \nu \in \mathcal{P}^\lambda_T \}
\]

(see [8] for the proof).

1.2. The forward exponential performance criteria. In this manuscript we assume that agent’s investment goals are modelled by so-called forward performance criteria (also called forward or stochastic utilities) introduced in [30] (see also [32]).

**Definition 1.1.** A map \( U : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) is called a forward performance process if:

(i) It is measurable with respect to the product of the optional \( \sigma \)-algebra on \( \Omega \times [0, \infty) \) and the Borel \( \sigma \)-algebra on \( \mathbb{R} \).

(ii) For fixed \( \omega \in \Omega \) and \( t \in [0, \infty) \), the mapping \( x \mapsto U_t(x) \) is strictly increasing and strictly concave.

(iii) \( \forall \pi \in A, \forall x \in \mathbb{R} \) and \( s \geq t \)

\[
\mathbb{E} \left[ U_s(X_s^{x,\pi}) | F_t \right] \leq U_t(X_t^{x,\pi}). \tag{8}
\]
There exists $\pi_\ast \in A$ such that
\[
\mathbb{E} \left[ U_s(X_{s^\pi}) | \mathcal{F}_t \right] = U_t(X_{t^\pi}),
\]
for every $t, s$ with $t \leq s$.

A forward performance is called exponential if there exist processes $(A_t)_{t \in [0, \infty)}$ and $(\gamma_t)_{t \in [0, \infty)}$ such that $\gamma_t > 0$ a.s. for every $t \geq 0$ and
\[
U_t(x) = -e^{-\gamma_t x + A_t},
\]
for $x \in \mathbb{R}$, and $t \geq 0$.

When agent’s risk preferences are modelled by a utility function $U(x)$, her investment criterion (up to some certain time horizon $T$) is the maximization of the expected utility function. That is for every $t \in [0, T]$, optimal trading strategy is defined by
\[
\sup_{\pi \in A} \mathbb{E}[U(x + X_{T}^{x, \pi}) | \mathcal{F}_t].
\]
The utility given by $U(x) = -e^{-\gamma x}$ is called exponential.

The forward performance process can be seen as a modification of the above utility maximization investment criterion, in the following sense: There is no terminal time horizon set and the choice of the utility is made at time 0, i.e. $U_0(x) = -e^\gamma_0 x + A_0$. For every futures time $t$, the utility is updated by replacing the risk aversion coefficient $\gamma_0$ with a stochastic one $\gamma_t$, and the term $A_0$ with $A_t$. Hence, we shall call the investment criterion (11) *backward* to emphasize its main difference with the forward performance criteria.

An analysis of the portfolio management problem with forward exponential criteria has been done in [31], where the authors provide explicit formulas for the optimal portfolio $\pi_\ast$ and the associated optimal wealth process $X^{x, \pi_\ast}$, for a variety of model parameters.

The following characterization of the forward exponential performance processes has been proven in [39].

**Theorem 1.1** (Žitković, 2009). Suppose that Assumption [LA] holds and let $U_t(x) = -e^{-\gamma_t x + A_t}$ be a forward exponential performance where $1_{\gamma_t} \in L^\infty(\mathcal{F}_t)$ for all $t \geq 0$. Then, there exist processes $(\vartheta_t)_{t \in [0, \infty)}$, $(\phi_t)_{t \in [0, \infty)}$ and $(p_t)_{t \in [0, \infty)}$ such that $\vartheta \in A^\infty$ and $1_{\gamma_t} = X_t^{1_{\gamma_0}, \vartheta}$, with $\gamma_0 \in \mathbb{R}_+$, $\phi \in \mathcal{P}$, $p \in A$ and
\[
A_t = A_0 + \frac{1}{2} \int_0^t (\lambda(Y_u) - \delta_u)^2 du + \gamma_t X_t^p - \frac{1}{2} \int_0^t \phi_u^2 du - \int_0^t \phi_u dW_u^2
\]
where $\delta_t = \gamma_t \vartheta_t \sigma(Y_t)$.

**Example 1.1.** A simple example of a forward exponential performance is the case where $\vartheta_t = \phi_t = 0$ for all $t \geq 0$. This corresponds to constant risk aversion coefficient ($\gamma$ is not a stochastic
process). If we further set \( p_t = -\frac{\lambda(Y_t)}{\sigma(Y_t)} \), for every time \( t \geq 0 \), we have that \( A_t = A_0 - \frac{1}{2} \int_0^t \lambda^2(Y_u)du - \int_0^t \lambda(Y_u)dW_u^1 \) (an example that has been used in [37]).

**Remark 1.1.** In view of the definition of forward exponential performance, we can assume without loss of generality that \( p_t = 0 \) for all \( t \geq 0 \). Similarly, we may ignore the initial term \( A_0 \).

Note also that the boundness assumption of \( 1/\gamma_t \) is not very restrictive in financial sense, since for a given time \( t \), \( 1/\gamma_t \) denotes agent’s the risk tolerance, which is normally a bounded quantity.

The characterization of forward exponential performances in Theorem 1.1 allows us to identify the exact elements of these performance criteria. More precisely, the decomposition of process \( A_t \) consists of two parts:

- The integral \( \int_0^t (\lambda(Y_u) - \delta_u)^2 du \), which reflects how the agent incorporates the market’s development in her investment criteria, in a way that this incorporation takes into account her stochastic risk tolerance level.
- The sum \( \frac{1}{2} \int_0^t \phi_u^2 du + \int_0^t \phi_u dW_u^2 \). This term does not depend on the level of risk aversion \( \gamma_t \) and it can be considered as the way the changes of the unhedgeable source of the market make the agent update her subjective probability measure \( \mathbb{P} \) (see also the related comment in [38]).

In what follows we will identify a forward exponential performance process by its **characterization pair** \( (\vartheta_t, \phi_t)_{t \in [0, \infty)} \), where \( \vartheta \in \mathbb{A}^\infty \) and \( \phi \in \mathbb{P} \) for all \( t \geq 0 \). The associate process \( A \) will be called the **characteristic process** of the forward exponential performance.

An important difference between the forward and the standard (backward) exponential investment criteria (defined in (11)) is that in the former the optimal strategy in (9) does depend on the initial wealth. However, this dependence is quiet clear (thanks to the replicability of the \( 1/\gamma_t \)).

**Proposition 1.1.** Let \( (\vartheta_t, \phi_t)_{t \in [0, \infty)} \) be the characterization pair of a forward exponential performance. If \( \pi^*(x) \) denotes the optimal trading strategy process in (9), with initial wealth \( X_0 = x \), then

\[
\pi^*(y) = \pi^*(x) + \gamma_0(y - x) \vartheta_t
\]

and

\[
X_t^{y, \pi^*(y)} - X_t^{x, \pi^*(x)} = \gamma_0 \frac{y - x}{\gamma_t}.
\]

for all \( x, y \in \mathbb{R} \) and \( t \geq 0 \).

**Proof.** The proof is a straightforward consequence of the facts that \( \frac{1}{\gamma_t} \) is replicable and that the backward exponential indifference valuation is independent on the initial wealth (see for instance [16]). \( \square \)

\*This is because \( -e^{-\gamma_t x + A_t} \) can be written as \( -e^{-\gamma_t x + \frac{1}{2} \int_0^t (\lambda(Y_u) - \delta_u)^2 du} Z_t^{\vartheta_t, \phi} \).
In what follows we assume that the agents’ initial wealth is equal to zero. For any nonzero initial wealth we may apply Proposition 1.1.

2. Valuation of Contingent Claims based on Forward Indifference

If an agent’s investment goals are determined by a forward exponential performance, it is reasonable to suppose that she uses indifference arguments in order to give values to contingent claims. The idea of indifference valuation was introduced in the finance literature in [17] and then developed and analyzed for a number of utility functions and market settings (see among others [16] and the references within). This (subjective) valuation concept compares two situations, the one where a contingent claim is bought or sold and another where there is no transaction on this claim.

For the model at hand, for a certain time horizon $T > 0$ we consider an $\mathcal{F}_T$-measurable payoff $C$. The (buyer) indifference value is the price $p$ that makes the agent indifferent between buying the claim at $p$ and not buying it at all. In our forward performance setting the buyer’s value of a payoff $C$ at any time $t \in [0, T]$, denoted by $v^b_t(C)$, is the $\mathcal{F}_t$-measurable solution of the following equation

$$U_t(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[ U_T \left( x - v^b_t(C) + \int_t^T \pi_s dS_s \right) \right| \mathcal{F}_t]$$

Due to replicability of $1/\gamma_t$, definition (15) can equivalently be written as

$$U_t \left( x + v^b_t(C) \right) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[ U_T \left( x + C + \int_t^T \pi_s dS_s \right) \right| \mathcal{F}_t]$$

(16)

**Remark 2.1.** By following the same arguments as the ones of Proposition 1.1 we get that the indifference value $v^b_t(C)$ does not depend on the initial wealth. (see also [24], [28], [29] and [38]).

Results on the value that solves equation (16) have been provided in [29] and in [28] for specific types of forward exponential performances in a stochastic factor model and in a binomial-type market model respectively. The use of forward exponential performance criteria for pricing contingent claims of American-type has been studied in [24] under the assumption of constant risk aversion.

One interesting question about these pricing mechanisms is how the forward exponential indifference valuation differs with the indifference valuation induced by exponential utilities. More precisely, we want to compare the solution $v^b_t(C)$ of (16), with the solution $pr^b_t(C)$ of the following equation

$$\sup_{\pi \in \mathcal{A}} \mathbb{E} \left[ -e^{-\gamma(x+pr^b_t(C)) + \int_t^T \pi_s dS_s} \right| \mathcal{F}_t] = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[ -e^{-\gamma(x+C + \int_t^T \pi_s dS_s)} \right| \mathcal{F}_t]$$

(17)

where $\gamma \in \mathbb{R}_+$ is the risk aversion coefficient. Throughout this paper the price process $pr^b_t(C)$ will be called backward exponential indifference value process.

Such a comparison has been studied in [28] and [30] (for constant risk aversion process). In the present section, we aim to extend the results on forward indifference valuation for general forward
exponential performance process and by doing so to highlight the special properties of the forward valuation regarding its comparison with the backward exponential valuation.

2.1. The BSDE representation of the indifference value. It has been proved in [25] that, under continuous filtration, the indifference value process under backward exponential utility satisfies a certain type of backward stochastic differential equation (BSDE). Adapted to our market model, the (buyer) indifference value of a contingent claim $C \in L^\infty(F_T)$ satisfies the following BSDE

$$C_t = C_0 + \frac{\gamma}{2} \int_0^t \zeta_u^2 du + \int_0^t \zeta_u dW_u^2 + X_t^\theta$$

(18)

and

$$C_T = C,$$

(19)

where $\gamma \in \mathbb{R}_+$ is the constant risk aversion coefficient, $\theta \in \mathcal{A}$ and $\left( \int_0^t \zeta_u dW_u^2 \right)_{t \in [0,T]}$ is a true martingale under the minimal entropy martingale measure. We call the triple $(C_t, \zeta_t, \theta_t)_{t \in [0,T]}$ a solution of BSDE (18) with terminal condition (19). Theorem 13 in [25] guarantees that for $C \in L^\infty(F_T)$ the solution is unique.

In the following proposition, it is established that the above representation has a nice extension in the case of the forward exponential performance.

Proposition 2.1. Let $(\vartheta_t, \phi_t)_{t \in [0,\infty)}$ be the characterization pair of a forward exponential performance and assume that there exists a constant $K_\gamma$ such that $\gamma_t \leq K_\gamma$ a.s. for every time $t \in [0,T]$ and that $\phi \in \mathcal{P}_\lambda^{\gamma}$. The forward exponential indifference (buyer) value process of a contingent claim $C \in L^\infty(F_T)$ is the unique solution, $(C_t)_{t \in [0,T]}$, of the following BSDE under the martingale measure $Q^{\lambda, \phi}$

$$C_t = C_0 + \frac{1}{2} \int_0^t \gamma_u \zeta_u^2 du + \int_0^t \zeta_u d\tilde{W}_u^2 + X_t^\theta$$

(20)

and

$$C_T = C,$$

(21)

for some processes $(\theta_t, \zeta_t)_{t \in [0,\infty)}$, such that

$$\mathbb{E}_{Q^{\lambda, \phi}} \left[ \int_0^T \theta_u^2 du \right] < \infty \quad \text{and} \quad \mathbb{E}_{Q^{\lambda, \phi}} \left[ \int_0^T \zeta_u^2 du \right] < \infty,$$

where $\tilde{W}_t^2 = W_t^2 + \int_0^t \phi_u du$.

Proof. The indifference valuation problem (16) becomes

$$e^{-\gamma t V^{(b)}(C) + A_t} = \inf_{\pi \in \mathcal{A}} \mathbb{E} \left[ e^{-\gamma T X_T^\pi - \gamma T C + A_T} \right]$$

(22)

where process $A$ is given by the characterization (12), with $A_0 = 0$.

Problem (22) leads to another forward exponential performance, where the risk aversion process remains $\gamma_t$ and the characteristic process is given by $\bar{A}_t = -\gamma_t V^{(b)}_t (C) + A_t$ for $t \in [0,T]$ and
\( A_t = -\gamma_t C + A_t \) for \( t \in (T, \infty) \). For this exponential forward performance, there exists an analogous characterization

\[
\tilde{A}_t = \tilde{A}_0 + \frac{1}{\gamma_t} \int_0^t (\lambda(Y_u) - \delta_u)^2 du + \gamma_t X^2_t - \frac{1}{2} \int_0^t z^2_u du - \int_0^t z_u dW^2_u,
\]

for some processes \( z \in \mathcal{P} \) and \( \tilde{p} \in \mathcal{A} \). Hence, for any \( t \in [0, T] \)

\[
v^{(b)}_t(C) = \frac{A_t - \tilde{A}_t}{\gamma_t} = \frac{1}{\gamma_t} \left( A_0 - \tilde{A}_0 + \gamma_t X^2_t - \frac{1}{2} \int_0^t (\phi_u^2 - z^2_u) du - \int_0^t (\phi_u - z_u) dW^2_u \right) = -\frac{\tilde{A}_0}{\gamma_0} + X^2_t - \frac{1}{\gamma_t} \left( \int_0^t (\phi_u^2 - z^2_u) du + \int_0^t (\phi_u - z_u) dW^2_u \right).
\]

Note that with the above notation \( v^{(b)}_0(C) = -\frac{\tilde{A}_0}{\gamma_0} \) and \( \tilde{W}^2 \) is a Brownian motion under the measure \( \mathbb{Q}^{\gamma,\phi} \) and strongly orthogonal to \( S_t \).

Hence, the indifference value process satisfies the following equation

\[
v^{(b)}_t(C) = v^{(b)}_0(C) + X^2_t + \frac{1}{\gamma_t} \left( \int_0^t (\phi_u - z_u)^2 du + \int_0^t (\phi_u - z_u) d\tilde{W}^2_u \right).
\]

Let \( \Lambda_t = \frac{1}{2} \int_0^t (\phi_u - z_u)^2 du + \int_0^t (\phi_u - z_u) d\tilde{W}^2_u \), for every \( t \in [0, \infty) \). A simple application of the Itô’s formula implies that

\[
\frac{\Lambda_t}{\gamma_t} = \frac{1}{2} \int_0^t (\phi_u - z_u)^2 du + \int_0^t \frac{(\phi_u - z_u)^2}{\gamma_u} d\tilde{W}^2_u + \int_0^t \Lambda_u \phi_u \sigma(Y_u) d\tilde{W}^1_u,
\]

where \( \tilde{W}^1_t = W^1_t + \int_0^t \lambda_u du \) and \( \theta_t = (\Lambda_t \phi_u + \tilde{p}_t) \sigma_t \in \mathcal{A} \). Also, the process \( \zeta = \frac{\phi - z}{\gamma} \) belongs in \( \mathcal{P}_T \), thanks to uniform boundness of \( \gamma_t \) for all \( t \in [0, T] \) and the fact that \( \phi, z \in \mathcal{P}_T \).

The fact that \( \int_0^T \theta^2_u du, \int_0^T \zeta^2_u du \in \mathbb{L}^1(\mathcal{F}_T, \mathbb{Q}^{\gamma,\phi}) \) follows from Lemma 2.1 below.

Finally, the uniqueness of the solution follows from Proposition 2.1 for \( \gamma = g \) and \( \phi = \psi \). \( \square \)

**Lemma 2.1.** Impose the same assumptions as in Proposition 2.1 and let \( (C_t, \zeta_t, \theta_t) \) be the solution of \( (20) \) and terminal condition \( (21) \) for some contingent claim \( C \in \mathbb{L}^\infty(\mathcal{F}_T) \). Then, there exists a constant \( K > 0 \) such that

\[
\sup_{\tau} \mathbb{E}_{\mathbb{Q}^{\gamma,\phi}} \left[ \int_\tau^T \sigma^2(Y_t) \theta^2_t dt \bigg| \mathcal{F}_\tau \right] + \sup_{\tau} \mathbb{E}_{\mathbb{Q}^{\gamma,\phi}} \left[ \int_\tau^T \zeta^2_t dt \bigg| \mathcal{F}_\tau \right] < K
\]

where the supremum is taken under any stopping time \( \tau \in [0, T] \).

**Proof.** We first apply the Itô’s formula for the process \( e^{-C_t} \) which implies that

\[
d(e^{-C_t}) = e^{-C_t} \left( \frac{\zeta^2_t}{2} + \frac{\theta^2_t \sigma^2(Y_t)}{2} \right) dt + e^{-C_t} \zeta_t d\tilde{W}^2_t + e^{-C_t} \theta_t \sigma(Y_t) d\tilde{W}^1_t
\]

where, \( \tilde{W}^1_t = W^1_t + \int_0^t \lambda_u du \). There is a sequence of stopping times \( \tau_n \), with \( \tau_n \nearrow T \) such that \( \int_{\tau_n}^{T \wedge \tau_n} \zeta_u d\tilde{W}^2_u \) and \( \int_{\tau_n}^{T \wedge \tau_n} \theta_u \sigma_u d\tilde{W}^1_u \) are \( \mathbb{Q}^{\gamma,\phi} \)-martingales. The boundness of \( C \) implies that for any
Remark 2.2. Note that the constant risk aversion of equation (20) solutions of \( \tau \), stopping time \( \tau \) ∈ [0, T] and her characterization process becomes

\[
\phi_t^C = \begin{cases} 
\phi_t - \gamma_t \xi_t, & t \in [0, T]; \\
\phi_t, & t \in (T, \infty).
\end{cases}
\]  

(24)

Hence, \( \mathcal{E}_{\mathbb{Q}^{\lambda, \phi}} \left[ \int_{\tau \wedge \tau_n} \zeta_t^2 dt \right| \mathcal{F}_{\tau \wedge \tau_n} \right] + \mathcal{E}_{\mathbb{Q}^{\lambda, \phi}} \left[ \int_{\tau \wedge \tau_n} \sigma^2(Y_t) \theta_t^2 dt \right| \mathcal{F}_{\tau \wedge \tau_n} \right] \leq 2e^{||C||_\infty} \). Letting \( n \to \infty \) completes the proof. \( \square \)

Corollary 2.1. Impose the same assumptions as in Proposition 2.1 and let \( (C_t, \xi_t, \theta_t) \) be the solution of (20) and terminal condition (21) for some contingent claim \( C \in \mathbb{L}^\infty(\mathcal{F}_T) \). If the agent sells the claim at price \( C_0 \), then the characterization pair of her forward exponential performance criteria becomes \( (\gamma_t, \phi_t^C, \rho_t) \in [0, \infty) \), where

\[
\phi_t^C = \begin{cases} 
\phi_t - \gamma_t \xi_t, & t \in [0, T]; \\
\phi_t, & t \in (T, \infty).
\end{cases}
\]  

(24)

and her characterization process becomes

\[
\bar{A}_t = -\frac{C_0}{\gamma_0} + \frac{1}{2} \int_0^t \left( \lambda(Y_u) - \delta_u \right) du + \frac{1}{2} \int_0^t (\phi_u^C)^2 du - \int_0^t \phi_u^C dW_u^2
\]

for some strategy \( \theta \).

Remark 2.2. Note that the constant risk aversion of equation (18) becomes stochastic in (20) in a mild manner. In addition, the minimal entropy martingale measure is replaced by the measure \( \mathbb{Q}^{\lambda, \phi} \).

2.2. The robust representation. As in the backward indifference valuation, the forward valuation can be considered as a dynamic (convex) risk measure in the sense of \( [7] \) (see also \( [38] \)). For a fixed time horizon \( T \), the map \(-v_t^b(\cdot) : L^\infty(\mathcal{F}_T) \mapsto L^\infty(\mathcal{F}_t)\) is convex, cash invariant and decreasing. The following theorem states its robust representation.

Theorem 2.1. Let \((\gamma_t, \phi_t)_{t \in [0, \infty)}\) be the characterization of a forward exponential performance and \( T > 0 \) some time horizon. The forward indifference (buyer) valuation process \( (v_t^b(\cdot))_{t \in [0, T]} \) defined in (10) for claims in \( L^\infty(\mathcal{F}_T) \) has the following representation

\[
v_t^b(C) = \text{essinf}_{\nu \in \mathcal{P}_T^\lambda} \{ \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}}[C|\mathcal{F}_t] + \alpha_t,T(\mathbb{Q}^{\lambda, \nu}) \}
\]  

(25)

for every \( C \in L^\infty(\mathcal{F}_T) \), where

\[
\alpha_t,T(\mathbb{Q}^{\lambda, \nu}) = \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[ \int_t^T \frac{(v_s - \phi_s)^2}{\gamma_s} \right| \mathcal{F}_t \]  

(26)
If we further assume that \( \exists K_\gamma > 0 \) such that \( \gamma_t < K_\gamma \) for all \( t \in [0, T] \) and that \( \phi \in \mathcal{P}_T^\lambda \), the infimum in (24) is attained by the process \( \hat{\nu}_t = \phi_t + \zeta_t \gamma_t \), for \( t \in [0, T] \), where \( \zeta \) is the corresponding part of the solution of BSDE (20).

Proof. We fix an arbitrary chosen contingent claim \( C \in \mathbb{L}^\infty(\mathcal{F}_T) \) and we follow the steps in the proof of Proposition 2.1. For this, we set \( \hat{A}_t = -\gamma_t \nu_t^{(b)}(C) + A_t \), which is the characterization process of the forward exponential utility after selling claim \( C \). Furthermore, from Theorem 4.4 in [39], we have that

\[
\frac{1}{\gamma_t} \left( \ln \left( \frac{1}{\gamma_t} \right) - 1 - \hat{A}_t \right) = \operatorname{essinf}_{Q \in \mathcal{M}_T^\lambda} \mathbb{E}_Q \left[ \frac{1}{\gamma_T} \left( \ln \left( \frac{1}{\gamma_T} \frac{Z_T^Q}{Z_t^Q} \right) - 1 - \hat{A}_T \right) \bigg| \mathcal{F}_t \right] \quad (27)
\]

which implies that

\[
\frac{1}{\gamma_t} \left( \ln \left( \frac{1}{\gamma_t} \right) - 1 - A_t \right) + \nu_t^{(b)}(C) = \operatorname{essinf}_{Q \in \mathcal{M}_T^\lambda} \mathbb{E}_Q \left[ \frac{1}{\gamma_T} \left( \ln \left( \frac{1}{\gamma_T} \frac{Z_T^Q}{Z_t^Q} \right) - 1 - A_T \right) + C \bigg| \mathcal{F}_t \right]. \quad (28)
\]

As simple rearrangement of the terms gives that indifference value process has the following dual representation

\[
\nu_t^{(b)}(C) = \operatorname{essinf}_{Q \in \mathcal{M}_T^\lambda} \{ \mathbb{E}_Q[C|\mathcal{F}_t] + H_t(Q, T) \} - H_t^{(0)}(T). \quad (29)
\]

where

\[
H_t(Q, T) = \mathbb{E} \left[ h \left( \frac{1}{\gamma_T} \frac{Z_T^Q}{Z_t^Q} - \frac{1}{\gamma_T} \frac{Z_T^Q}{Z_t^Q} A_T \right) \bigg| \mathcal{F}_t \right]
\]

with \( h(y) = y \log(y) - y \) for \( y > 0 \), and \( H_t^{(0)}(T) = \operatorname{essinf}_{Q \in \mathcal{M}_T^\lambda} H_t(Q, T) \).

We first verify that the martingale measure that minimizes the term \( H_t(Q, T) \) is the measure \( Q^{\lambda, \phi} \). Indeed,

\[
\operatorname{essinf}_{Q \in \mathcal{M}_T^\lambda} H_t(Q, T) = \operatorname{essinf}_{Q \in \mathcal{M}_T^\lambda} \mathbb{E} \left[ h \left( \frac{1}{\gamma_T} \frac{Z_T^Q}{Z_t^Q} - \frac{1}{\gamma_T} \frac{Z_T^Q}{Z_t^Q} A_T \right) \bigg| \mathcal{F}_t \right] = \operatorname{essinf}_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E} \left[ h \left( \frac{1}{\gamma_T} \frac{Z_T^\lambda}{Z_t^\lambda} - \frac{1}{\gamma_T} \frac{Z_T^\lambda}{Z_t^\lambda} \nu \bigg| \mathcal{F}_t \right] = \operatorname{essinf}_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E} \left[ h \left( \frac{1}{\gamma_T} \frac{Z_T^\lambda - \delta, \nu}{Z_t^\lambda - \delta, \nu} - \frac{1}{\gamma_T} \frac{Z_T^\lambda - \delta, \nu}{Z_t^\lambda - \delta, \nu} A_T \bigg| \mathcal{F}_t \right] = \frac{1}{\gamma_t} \operatorname{essinf}_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E} \left[ \log \left( \frac{1}{\gamma_t} \frac{Z_T^\lambda - \delta, \nu}{Z_t^\lambda - \delta, \nu} \right) - A_T \bigg| \mathcal{F}_t \right] - \frac{1}{\gamma_t}
\]

But

\[
\operatorname{essinf}_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E} \left[ \log \left( \frac{1}{\gamma_t} \frac{Z_T^\lambda - \delta, \nu}{Z_t^\lambda - \delta, \nu} \right) - A_T \bigg| \mathcal{F}_t \right] = \operatorname{essinf}_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E} \left[ \frac{-1}{2} \int_t^T (\nu_u^2 - \phi_u^2 + 2\phi_u \nu_u - 2\nu_u^2) du \bigg| \mathcal{F}_t \right]
\]
and hence

$$
\begin{align*}
\text{essinf}_{Q \in \mathcal{M}^T_T} H_t(Q, T) &= h \left( \frac{1}{\gamma_t} \right) - \frac{A_t}{\gamma_t} + \frac{1}{2\gamma_t} \text{essinf}_{\nu \in \mathcal{P}^\lambda_T} \left[ \int_t^T (\nu_u - \phi_u)^2 du \right] \mathbb{F}_t \\
&= h \left( \frac{1}{\gamma_t} \right) - \frac{A_t}{\gamma_t}
\end{align*}
$$

which also shows that the minimizer is the measure $Q^{\lambda, \phi}$. In addition, we have that the penalty function for every martingale measure $Q^{\lambda, \nu}$ is given by

$$
\alpha_{t,T}(Q^{\lambda, \nu}) = \frac{1}{\gamma_t} \mathbb{E}_{Q_{\lambda, \nu}} \left[ \log \left( \frac{1}{\gamma_t} \frac{Z^{\lambda, \nu}_u}{Z^{\lambda, \phi}_u} \right) - A_T \right] \mathbb{F}_t - h \left( \frac{1}{\gamma_t} \right) + \frac{A_t}{\gamma_t}
$$

Hence,

$$
v^{(b)}_t(C) \leq \mathbb{E}_{Q^{\lambda, \nu}}[C \mathbb{F}_t] + \alpha_{t,T}(Q^{\lambda, \nu})
$$

for every $\nu \in \mathcal{P}^\lambda_T$.

If we further assume that $\exists K_T > 0$ such that $\gamma_t < K_T$ for all $t \in [0, T]$ and that $\phi \in \mathcal{P}^\lambda_T$, we get from Proposition 2.1 that there is a triple $(v^{(b)}_t(C), \zeta_t, \theta_t)$ that solves the BSDE (20). Then, for every $\nu \in \mathcal{P}^\lambda_T$

$$
v^{(b)}_t(C) = \mathbb{E}_{Q^{\lambda, \nu}} \left[ C + \frac{1}{2} \int_t^T (2\zeta_u (\nu_u - \phi_u) - \gamma_u \zeta_u^2) du \right] \mathbb{F}_t
$$

It is left to observe (thanks to Lemma 2.1) that $\hat{\nu}_t = \phi_t + \zeta_t \gamma_t \in \mathcal{P}^\lambda_T$ and that $2\zeta_t (\hat{\nu}_t - \phi_t) - \gamma_t \zeta_t^2 = \frac{(\hat{\nu}_t - \phi_t)^2}{\gamma_t}$.

**Remark 2.3.** Note that it is due to the replicability of $1/\gamma_t$, that the penalty function can also be written as $\alpha_{t,T}(Q^{\lambda, \nu}) = \frac{1}{2} \mathbb{E}_{Q^{\lambda, \nu}} \left[ \frac{1}{\gamma_T} \int_t^T (\nu_u - \phi_u)^2 du \right] \mathbb{F}_t$.

**Remark 2.4.** It is clear that the martingale measure which minimizes the penalty process $\alpha_{t,T}$ is the measure $Q^{\lambda, \phi}$, which does not depend on the time horizon $T$. This means that the agent’s marginal utility valuation (the so-called Davis price) is the (conditional) expectation of the payoff under the same martingale measure regardless the maturity of the claim. This is in contrast with the backward exponential valuation, where the corresponding measure is the minimal entropy martingale measure, which depends on the time horizon the utility lives in.

We can now take advantage of representation (25) and prove some properties of the valuation under forward exponential performance criteria, where we use the notation $v^{(b)}_t(C; \gamma)$ to emphasize (when needed) the dependence of the indifference valuation on the risk aversion process.

**Proposition 2.2.** Let $(\gamma_t, \phi_t)_{t \in [0, \infty)}$ be the characterization of a forward exponential performance and $T$ some time horizon. For every contingent claim $C \in L^\infty(\mathbb{F}_T)$ the following statements hold true.
(i) The forward (buyer) indifference value is decreasing with respect to risk aversion in the following sense: If $g_t$ is another risk aversion process with $\gamma_T \geq g_T$, a.s., then

$$v_t^{(b)}(C;\gamma) \leq v_t^{(b)}(C;g) \text{ a.s.,}$$

for every $t \leq T$.

(ii) Let $((\gamma_t(n))_{t \in [0,T]}_{n \in \mathbb{N}}$ be a sequence of risk aversion processes, such that $\gamma_T(n) \searrow \infty$ in $\mathbb{P}$, as $n \to \infty$. Then

$$v_t^{(b)}(C;\gamma(n)) \to \inf_{\nu \in \mathcal{P}_\lambda} \mathbb{E}_{Q_{\lambda,\nu}}[C|\mathcal{F}_t] \text{ a.s.,}$$

for any $t \in [0,T]$.

Also if $\gamma_T(n) \searrow 0$ in $\mathbb{P}$, as $n \to \infty$ and $\phi \in \mathcal{P}_\lambda^\mu$ then,

$$v_t^{(b)}(C;\gamma(n)) \to \mathbb{E}_{Q_{\lambda,\phi}}[C|\mathcal{F}_t] \text{ a.s.,}$$

for any $t \in [0,T]$.

(iii) For each risk aversion coefficient, the forward indifference valuation is time consistent in the sense that

$$v_t^{(b)}(v_s^{(b)}(C)) = v_t^{(b)}(C) \text{ a.s.,}$$

for any stopping times $\tau, s$ with $\tau \leq s$.

(iv) The indifference valuation is replication invariance, i.e.

$$v_t^{(b)}(C + X^\theta_T) = v_t^{(b)}(C) + X^\theta_t \text{ a.s.,}$$

for every $\theta \in \mathcal{A}$.

Proof. Part (i) follows directly from the robust representation of the forward indifference valuation given in [25]. Again from [25] and the monotone convergence theorem we get the first item of part (ii). For the limit of the indifference value when $\gamma_T(n) \searrow 0$, we use similar arguments as the ones in [25]. Thanks to the robust representation of the $v_t^{(b)}(C;\gamma(n))$, it is enough to show that

$$\lim_{n \to \infty} \inf v_t^{(b)}(C;\gamma(n)) \geq \mathbb{E}_{Q_{\lambda,\phi}}[C|\mathcal{F}_t].$$
We shall show the above inequality for \( t = 0 \), since the more general case is proved similarly. By Fenchel-Young inequality \( xp \geq \frac{p - e^{-ax}}{a} - \frac{p \ln(p)}{a} \), we get that for every \( \nu \in \mathcal{P} \)

\[
\mathbb{E}_{Q_{\lambda,\nu}}[C] = \mathbb{E}_{Q_{\lambda,\phi}}\left[ \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} \right] \geq \mathbb{E}_{Q_{\lambda,\phi}}\left[ \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} - \frac{1}{\gamma_T(n)} \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} \ln \left( \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} \right) \right]
\]

\[
= \mathbb{E}_{Q_{\lambda,\phi}}\left[ \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} - \frac{e^{-\gamma_T(n)} C}{\gamma_T(n)} \right] - \mathbb{E}_{Q_{\lambda,\nu}}\left[ \frac{1}{\gamma_T(n)} \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} \ln \left( \frac{Z_T^{0,\nu}}{Z_T^{0,\phi}} \right) \right]
\]

\[
= \frac{1}{\gamma_0(n)} - \frac{1}{\gamma_T(n)} \mathbb{E}_{Q_{\lambda,\phi}}\left[ \frac{e^{-\gamma_T(n)} C}{\gamma_T(n)} \right] - \frac{1}{2} \mathbb{E}_{Q_{\lambda,\nu}}\left[ \frac{1}{\gamma_T(n)} \int_0^T (\phi_s - \nu_s)^2 ds \right]
\]

Thus,

\[
\liminf_{n \to \infty} (b)_0(C; \gamma(n)) \geq \liminf_{n \to \infty} \mathbb{E}_{Q_{\lambda,\phi}}\left[ \frac{1 - e^{-\gamma_T(n)} C}{\gamma_T(n)} \right],
\]

which gives the desired result.

Part (iii) follows from the definition of the forward indifference valuation (16) and a simple application of the dynamic programming principle. Finally, part (iv) is a consequence of the definition of the indifference valuation. □

Remark 2.5. All items of Proposition 2.2 can be considered as extensions of the properties of the backward indifference value \( p_{r_t}^{(b)}(C) \), defined through exponential utility function in \( 17 \). For example, although the forward performance criterion has stochastic risk aversion, the monotonicity of the indifference value is preserved, something that is consistent with the financial intuition: the higher the risk aversion at the time of maturity is, the lower price the buyer is going to bid.

The main difference between forward and backward valuation is that the so-called marginal martingale probability measure, i.e. the measure that minimizes penalty function, is \( Q_{\lambda,\phi} \) in forward valuation and the minimal entropy martingale measure in the backward case. The important differences between these measures is that \( Q_{\lambda,\phi} \) does not depend on a time horizon.

Remark 2.6. A special case of the forward exponential performance is when the risk aversion is constant. Then, the problem of the indifference valuation has an immediate relation with the associated problem of the backward valuation. This is because, given a maturity \( T \) of a contingent claim, the forward utility function at \( T \) can be written as \( U_T(x) = -e^{-\gamma_T(x - \frac{A_T}{\gamma})} \), and the term \( -\frac{A_T}{\gamma} \) can be thought as an \( \mathcal{F}_T \)-measurable random endowment. Therefore, we may consider the forward performance indifference valuation as a backward (standard) indifference valuation under
this random endowment. A number of properties of this value, called conditional indifference price, has been provided in the Appendix of [1].

One further property of the indifference value that can be proved using the robust representation of the price is the following.

**Proposition 2.3.** Impose the same assumptions as Proposition 2.1 and let $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ be a contingent claim for some time horizon $T > 0$. Then, the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(a) = v_0^{(b)}(aC)$ is differentiable and

$$f'(a) = \mathbb{E}_{\mathbb{Q}_{\lambda,\phi}(aC)}[C]$$

where $\phi_t^{(aC)} = \phi_t - \gamma_t \zeta_t(aC)$ and $\zeta_t(aC)$ is the corresponding part of the solution of (20) for boundary condition $aC$.

**Proof.** We will show the result when $a = 0$. Thanks to item (ii) of Proposition 2.2, we have

$$\lim_{\epsilon \to 0} \frac{v_0^{(b)}(\epsilon C)}{\epsilon} = \lim_{\epsilon \to 0} \min_{\nu \in \mathbb{P}} \mathbb{E}_{\mathbb{Q}_{\lambda,\nu}}\left[ C + \frac{1}{2\epsilon\gamma_T} \int_0^T (\nu_u - \phi_u)^2 du \right] = \mathbb{E}_{\mathbb{Q}_{\lambda,\phi}}[C]$$

We also observe that $\lim_{\epsilon \to 0} \frac{v_0^{(b)}(-\epsilon C)}{\epsilon} = -\lim_{\epsilon \to 0} \frac{v_0^{(b)}(-\epsilon C)}{\epsilon} = \mathbb{E}_{\mathbb{Q}_{\lambda,\phi}}[C]$, which means that $f'(0) = \mathbb{E}_{\mathbb{Q}_{\lambda,\phi}}[C]$.

The more general case of $a \neq 0$ follows from the same arguments and Corollary 2.1. □

Note that in the case of backward valuation the situation is similar. Namely, the function $g(a) = pr_0^{(b)}(aC)$ is also differentiable and its derivative is given as $\mathbb{E}_{\mathbb{Q}(aC)}[C]$, where the $\mathbb{Q}(aC)$ is the martingale measure that minimizes the relative entropy with risk to the measure $\mathbb{P}(aC)$ defined by its R-N derivative $\frac{d\mathbb{P}(aC)}{d\mathbb{P}} = ce^{-\gamma aC}$, for the appropriate constant $c$ (see [19] for details on this result).

**Remark 2.7.** The derivative of the indifference valuation with respect to the units of a given claim can be used in the determination of the number of units that the agent is willing to sell/buy when the price of the contingent claim is given. In other words, it leads to the agent’s demand function on this claim in the same manner as in [1]. This differentiation result can be applied for a vector of claims in straightforward way.

The arguments of the proposition below follow similar lines as those in Proposition 14 of [25].

**Proposition 2.4.** We impose the assumptions of Proposition 2.1. If $C^n$ is a bounded sequence in $\mathbb{L}^\infty(\mathcal{F}_T)$ such that $C^n \to C$ in probability for some $C \in \mathbb{L}^\infty(\mathcal{F}_T)$, then

$$\sup_{0 \leq t \leq T} |v_t^{(b)}(C^n) - v_t^{(b)}(C)| \to 0$$

in probability.
Proof. Thanks to Proposition 2.1, the indifference values of $C$ and $C^n$ satisfy the following relations

$$C = v_t^{(b)}(C) + \frac{1}{2} \int_0^T \gamma_u \zeta_u^2 du + \int_0^T \zeta_u d\tilde{W}_u + \int_0^T \theta_u d\tilde{W}_u^1$$

$$C^n = v_t^{(b)}(C^n) + \frac{1}{2} \int_0^T \gamma_u (\zeta_u^n)^2 du + \int_0^T \zeta_u^n d\tilde{W}_u + \int_0^T \theta_u^n d\tilde{W}_u^1$$

for some processes $\zeta, \zeta^n, \theta$ and $\theta^n$. Hence,

$$v_t^{(b)}(C^n) - v_t^{(b)}(C) = C^n - C + \int_0^T (\theta_u - \theta_u^n) d\tilde{W}_u^1 + \frac{1}{2} \int_0^T \gamma_u (\zeta_u^2 - (\zeta_u^n)^2) du + \int_0^T (\zeta_u - \zeta_u^n) d\tilde{W}_u^2$$

(32)

where $\tilde{W}_t^1 = W_t^1 + \int_0^t \lambda_u du$ and $\tilde{W}_t^2 = W_t^2 + \int_0^t \phi_u du$. We then define for each $n \in \mathbb{N}$, the sequence of processes $\nu_t(n) = -\frac{1}{2} \gamma_t (\zeta_t + \zeta_t^n)$ and by Lemma 2.1 $\nu(n) \in \mathcal{P}_T$ for each $n$. Under the probability measure $Q(n)$ defined through its R-N derivative

$$\frac{dQ(n)}{dQ} = \mathcal{E}(\nu_t(n))$$

we have that

$$v_t^{(b)}(C^n) - v_t^{(b)}(C) = \mathbb{E}_{Q(n)}[C^n - C | \mathcal{F}_t].$$

(33)

The next step is to observe that the process $\frac{1}{2} \int_0^t \gamma_u (\psi_u + \psi_u(n)) d\tilde{W}_u^2$ is a $BMO(\mathbb{Q}^\lambda, \phi)$ martingale for $t \in [0, T]$. Indeed, thanks to the Lemma 2.1 we have that

$$\left\| \frac{1}{2} \int_0^t \gamma_u (\psi_u + \psi_u(n)) d\tilde{W}_u^2 \right\|_{BMO} \leq \frac{1}{4} \sup_{\tau} \left\| \mathbb{E}_{Q} \left[ \int_\tau^T \gamma_u^2 (\psi_u + \psi_u(n))^2 du \right] \right\|_{L^\infty} + \frac{K^2}{4} \sup_{\tau} \left\| \mathbb{E}_{Q} \left[ \int_\tau^T (\psi_u)^2 du \right] \right\|_{L^\infty} < \infty$$

where supremum is taken under all stopping times $\tau$ in $[0, T]$. Theorem 3.1 of [21] guarantees the existence of two constants $p > 1$ and $c > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}_{Q} \left[ e^{-\frac{t}{2}} \int_0^t \nu(n)^2 du - p \int_0^t \nu(n) d\tilde{W}_u^2 \right] < c$$

for each $n \in \mathbb{N}$. Then, (33) follows by applying the Hölder’s and the Doob’s maximal inequalities.

\[ \square \]

3. Optimal Risk Sharing

In the present section, we consider two financial agents whose investment criteria are modelled by forward exponential performances and we address the problem of optimal risk sharing. We denote the characterization pairs of agents’ performance criteria by $(\theta_t, \phi_t)_{t \in [0, \infty)}$ and $(\theta_t, \psi_t)_{t \in [0, \infty)}$, with the risk aversion processes $\gamma$, $g$ defined by the equations $\gamma = X^{\frac{1}{\theta}}$ and $g = X^{\frac{1}{\theta}}$. Also, $U_i^j$ stands for the corresponding agent’s forward utility at time $t$ and $A^i$ is the associated characterization process for each agent, $i = 1, 2$. In addition, we assume that each agent has some initial (non-replicable) endowment in her portfolio, denoted by $\mathcal{E}_i \in \mathbb{L}^\infty(\mathcal{F}_T)$, for some time horizon $T > 0$. The sum $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ is the so-called aggregated random endowment.
The problem of optimal risk sharing (as formed in the mathematical finance literature in [3]) is finding a contract $C^*$ and a price $p^*$ which solve the following problem

$$\arg\max_{C,p} \sup_{\pi \in A} E \left[ U_1^1(\mathcal{E}_1 + \int_0^T \pi_s dS_u + C - p) \right]$$

Given that

$$\sup_{\pi \in A} E \left[ U_2^2(\mathcal{E}_2 + \int_0^T \pi_s dS_u) \right] \leq \sup_{\pi \in A} E \left[ U_2^2(\mathcal{E}_2 + \int_0^T \pi_s dS_u - C + p) \right]$$

From the definition of the indifference valuation (16) and its replication invariance property (part (iv) of Proposition 2.2) we get the following more tractable equivalent problem

$$\arg\max_C \{v_0^1(\mathcal{E}_1 + C) + v_0^2(\mathcal{E}_2 - C)\}$$ \hspace{1cm} (34)

where, $v_0^i(\cdot)$ denotes the (buyer) indifference valuation of the corresponding agent at time $t = 0$, where $i = 1, 2$.

**Definition 3.1.** We say that agents are in Pareto optimal situation if the solution set of problem (34) is the replicable claims or equivalently if

$$v_0^1(\mathcal{E}_1 + C) + v_0^2(\mathcal{E}_2 - C) < v_0^1(\mathcal{E}_1) + v_0^2(\mathcal{E}_2).$$ \hspace{1cm} (35)

for every $C$ which is not replicable.

The inequality (35) introduces the so-called inf-convolution measure

$$\rho(\mathcal{E}) = \inf_C \{\rho^1(\mathcal{E} - C) + \rho^2(C)\}$$ \hspace{1cm} (36)

where $\rho^i(C) = -v_0^i(C)$ for $i = 1, 2$ is the convex risk measure induced by the associated forward exponential performance criteria. Note that $\rho(\cdot)$ maps payoffs in $L^\infty(\mathcal{F}_T)$ to $\mathbb{R} \cup \{-\infty\}$, where the time horizon $T$ is the maturity of the agents’ endowments.

**Assumption 3.1.** There exists a constant $K > 0$ such that $\gamma_t, g_t < K$ for all $t \in [0, T]$, a.s. and $\phi, \psi \in \mathcal{P}^\lambda$.

A first result, the proof of which is based on Theorem 2.1, is that the inf-convolution measure of two forward exponential performance processes is not a risk measure that is induced by another forward exponential performance criterion (something which is in contrast with the inf-convolution measure induced by exponential utility functions).

**Proposition 3.1.** Let Assumption 3.1 hold true. The inf-convolution risk measure $\rho(\cdot)$ defined in (36) is induced by a forward exponential performance process if and only if $\phi_t = \psi_t$ for all $t \in [0, T]$,
a.s. In this case, the characterization of the forward exponential performance is \((\theta_t + \vartheta_t, \phi_t)_{t\in[0,T]}\), i.e. the risk aversion process is given by

\[
\Gamma_t = \frac{\gamma_t g_t}{\gamma_t + g_t}
\]

and the characterization process by

\[
A_t = \frac{1}{2} \int_0^t (\lambda(Y_u) - \delta_u)^2 \, du - \frac{1}{2} \int_0^t \phi_u^2 \, du - \int_0^t \phi_u \, dW_u^2.
\]

where \(\delta_t = \Gamma_t (\theta_t + \vartheta_t) \sigma(Y_t)\).

**Proof.** Theorem 3.6 in [3] states that the penalty function of the inf-convolution measure is the sum of the penalty function of the involved risk measures. In our setting, the penalty function of \(\rho(\cdot)\) can be written as

\[
\alpha_{t,T}(\mathbb{Q}^{L}) = \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{L}} \left[ \int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} + \frac{(\nu_s - \psi_s)^2}{g_s} \, ds \right] \mathcal{F}_t
\]

for every \(t \in [0,T]\).

Assume that \(\rho(\cdot)\) is induced by a forward exponential performance. A necessary condition for this is the existence of a risk aversion process \(\Gamma\) and a process \(\tilde{\phi}\) such that

\[
\mathbb{E}_{\mathbb{Q}^{L}} \left[ \int_t^T \frac{(\nu_s - \tilde{\phi}_s)^2}{\Gamma_s} \, ds \right] \mathcal{F}_t = \mathbb{E}_{\mathbb{Q}^{L}} \left[ \int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} + \frac{(\nu_s - \psi_s)^2}{g_s} \, ds \right] \mathcal{F}_t
\]

for every \(t \in [0,T]\) and for every process \(\nu \in \mathcal{P}^\lambda_T\). Setting \(\nu = \tilde{\phi}\) and taking into account the positivity of \(\gamma\) and \(g\), we get that \(\phi_t = \psi_t\) for all \(t \in [0,T]\), a.s. But \(\frac{2+g}{\gamma g^2}\) is replicable and bounded, therefore (see also Remark 2.3)

\[
\mathbb{E}_{\mathbb{Q}^{L}} \left[ \frac{1}{\gamma T} \int_t^T (\nu_s - \tilde{\phi}_s)^2 \, ds \right] \mathcal{F}_t = \mathbb{E}_{\mathbb{Q}^{L}} \left[ \frac{\gamma_T + g_T}{\gamma_T g_T} \int_t^T (\nu_s - \phi_s)^2 \, ds \right] \mathcal{F}_t
\]

for every \(t \in [0,T]\) and for every process \(\nu \in \mathcal{P}^\lambda\), which first implies that \(\tilde{\phi} = \phi\) and then \(\Gamma_t = \frac{\gamma_T g_T}{\gamma_T + g_T}\), \(\forall t \in [0,T]\).

For the inverse part, we assume that \(\phi_t = \psi_t\) for all \(t \in [0,T]\), a.s. Equation (37) implies that

\[
\alpha_{t,T}(\mathbb{Q}^{L}) = \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{L}} \left[ \int_t^T \frac{\gamma_s + g_s}{\gamma_s g_s} (\nu_s - \phi_s)^2 \, ds \right] \mathcal{F}_t
\]

for every \(t\) and \(\nu\). Letting \(\Gamma_t = \frac{\gamma_T g_T}{\gamma_T + g_T}\) completes the proof. \(\square\)

**Remark 3.1.** Proposition 3.1 states that only in the case where agents adapt their subjective probability measure up to the maturity of their endowments in the same manner, the representative agent can be considered as behaving under forward exponential performance criteria. In other word, when \(\phi = \psi\) Theorem 2.3 in [3] has a direct extension in the case of stochastic risk aversion.
3.1. **The special case of replicable endowments.** A special case is when agents do not have any endowment in their initial portfolios or when both endowments are replicable, that is \( \exists \pi_1, \pi_2 \in A \) and \( c^1, c^2 \in \mathbb{R} \) such that \( E_i = X_{T_i}^{c^1, \pi_1} \), for \( i = 1, 2 \).

**Proposition 3.2.** Let Assumption \( \ref{assumption1} \) hold true and assume that \( E_1 \) and \( E_2 \) are replicable. Then, agents are in Pareto optimal situation if and only if \( \phi_t = \psi_t \) for every \( t \in [0, \infty) \) a.s.

**Proof.** The result follows from the dual representation of the indifference valuation \( \ref{dual_representation} \) and the Theorem 3.6 of \cite{3}, which states that Pareto optimality is equivalent to the equality of the agents’ marginal martingale measures, that is \( Q^\lambda \phi = Q^\lambda \psi \).

Proposition 3.2 states that agents are willing to trade some non-replicable claims if and only if the way they adapt their subjective probability measure is not the same at all times. Note that this statement is independent on the agents’ risk aversion processes \( \gamma \) and \( g \). Another way to interpret this result is that if agents do not include in their utilities the unhedgeable part of the market (i.e. when \( \phi = \psi = 0 \)), they are unwilling to make any non-replicable transaction, no matter how their risk aversion processes differ to each other. This can be seen as a generalization of the corresponding result in the case of backward exponential utility (see Proposition 3.8 in \cite{1}).

3.2. **The case of constant risk aversions.** In the simplified case where the agents’ risk aversion are constant, we can explicitly solve the sharing problem \( \ref{eq:sharing_problem} \). This is in fact because under constant risk aversion, the contingent claim valuation problem can be written as conditional indifference valuation under classical exponential utility (see Remark \( \ref{remark:constant_risk} \)).

**Proposition 3.3.** Impose Assumption \( \ref{assumption1} \) and suppose in addition that agents’ risk aversions are constants, i.e. \( \gamma, g \in \mathbb{R}_+ \). Any claim of the form

\[
\frac{1}{\gamma + g} \left( \int_0^T \frac{\psi^2 - \phi^2}{2} dt + \int_0^T (\psi_t - \phi_t) dW_t^2 \right) + \frac{gE_2 - \gamma E_1}{\gamma + g} + X_{T_i}^{c, \pi} \tag{38}
\]

for some constant \( c \) and admissible strategy \( \pi \in A \), solves the optimal sharing problem \( \ref{eq:sharing_problem} \).

**Proof.** By robust representation \( \ref{eq:robust_representation} \), we get that for every claim \( C \) it holds that

\[
v_0^1(E_1 + C) + v_0^2(E_2 - C) \leq \inf_{\nu \in P^\lambda_{T_i}} E_{Q^\lambda, \nu} \left[ E + \frac{1}{2\gamma} \int_0^T (\nu_t - \phi_t)^2 ds + \frac{1}{2g} \int_0^T (\nu_t - \psi_t)^2 ds \right] \tag{39}
\]

For very claim \( C^* \) of the form \( \ref{eq:claim_form} \), we have that

\[
v_0^1(E_1 + C^*) = \inf_{\nu \in P^\lambda_{T_i}} E_{Q^\lambda, \nu} \left[ \frac{g}{\gamma + g} \left( E + \frac{1}{2\gamma} \int_0^T \left( \frac{(\nu_t - \phi_s)^2}{\gamma} + \frac{\psi^2 - \phi^2}{\gamma + g} - \frac{2(\psi_t - \phi_t)}{\gamma + g} \right) ds \right) + c \right]
\]

\[
= \frac{g}{\gamma + g} \inf_{\nu \in P^\lambda_{T_i}} E_{Q^\lambda, \nu} \left[ E + \frac{1}{2\gamma} \int_0^T (\nu_t - \phi_t)^2 ds + \frac{1}{2g} \int_0^T (\nu_t - \psi_t)^2 ds \right] + c
\]
Similarly, we get that
\[
v_0^2(\mathcal{E}_2 - C^*) = \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ \frac{\gamma}{\gamma + g} \mathcal{E} + \frac{1}{2} \int_0^T \left( \frac{(\nu_s - \psi_s)^2}{g} - \frac{\psi_s^2 - \phi_s^2}{\gamma + g} + \frac{2(\psi_s - \phi_s)}{\gamma + g} \right) ds \right] - c
\]
\[
= \frac{\gamma}{\gamma + g} \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ \mathcal{E} + \frac{1}{2} \int_0^T (\nu_s - \phi_s)^2 ds + \frac{1}{2g} \int_0^T (\nu_s - \psi_s)^2 ds \right] - c
\]
Therefore,
\[
v_0^1(\mathcal{E}_1 + C^*) + v_0^2(\mathcal{E}_2 - C^*) = \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ \mathcal{E} + \frac{1}{2} \int_0^T (\nu_s - \phi_s)^2 ds + \frac{1}{2g} \int_0^T (\nu_s - \psi_s)^2 ds \right]
\]
which together with (39) completes the proof. □

In other words, the optimal risk sharing part consists of three elements: the optimal sharing of the agents’ random endowments which is exactly the same as in the backward exponential utility case (see [2]), the sharing of the agents’ perspectives about the probability measure (in the way they are incorporated on the agents’ forward performances) and a replicable part (which can essentially be ignored since it does not transfer any risk).

If there are no endowments, the agents share their differences of beliefs regarding the evolution of the probability measure through the contract with payoff
\[
\frac{1}{\gamma + g} \left( \int_0^T \frac{\psi_s - \phi_s}{2} dt + \int_0^T (\psi_t - \phi_t) dW_t^2 \right).
\]
Note that the expectation of this payoff increases (in absolute terms) as the differences of the processes \(\phi\) and \(\psi\) increases (this means that intense difference in beliefs implies high volume of transaction).

3.3. The case of stochastic risk aversions. In the case where the agents’ risk aversion coefficients are stochastic, the optimal sharing problem is more involved, since the methods used in the backward exponential utility case can not be applied. Recall that problem (34) is equivalent to finding a claim \(C^*\) that maximizes the sum
\[
v_0^1(\mathcal{E}_1 + C) + v_0^2(\mathcal{E}_2 - C)
\]
or equivalently
\[
v_0^{(1)}(\mathcal{E}_1 + C) + v_0^{(2)}(\mathcal{E}_2 - C)
\]
\[
= \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ \mathcal{E}_1 + C + \frac{1}{2} \int_0^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \right] + \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ \mathcal{E}_2 - C + \frac{1}{2} \int_0^T \frac{(\nu_s - \psi_s)^2}{g_s} ds \right]
\]
\[
= \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ \mathcal{E} + C + \frac{1}{2} \int_0^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \right] + \inf_{\nu \in \mathcal{P}_T^\lambda} \mathbb{E}_{Q,\lambda,\nu} \left[ -C + \frac{1}{2} \int_0^T \frac{(\nu_s - \psi_s)^2}{g_s} ds \right]
\]
In the special case where the agents have the same stochastic risk aversion process, the optimal risk sharing problem can be solved explicitly.
Proposition 2.1. where $\pi$ contingent claim which gives the existence and an implicit form of the optimal risk sharing contract. can not be applied. However, we are able to construct the dynamics of the inf-convolution measure, $E$ applying the same calculations to $E$ we first observe that for every such $C^*$ and every $\nu \in P_T, E_{Q,\nu} \left[ E_1 + C^* + \frac{1}{2} \int_0^T (\nu_s - \phi_s)^2 \right]$ equals to

$$E_{Q,\nu} \left[ \frac{E}{2} + \frac{1}{2\gamma T} \left( \int_0^T (\nu_s - \phi_s)^2 + \psi_s^2 - \phi_s^2 \right) ds + \int_0^T (\psi_s - \phi_s) dW_s^2 \right]$$

Applying the same calculations to $E_{Q,\nu} \left[ E_2 - C^* + \frac{1}{2} \int_0^T (\nu_s - \psi_s)^2 \right]$, we get that

$$v_0^{(1)} (E_1 + C^*) + v_0^{(2)} (E_2 - C^*) = 2 \inf_{\nu \in P_T} E_{Q,\nu} \left[ \frac{E}{2} + \frac{1}{4\gamma T} \left( \int_0^T (\nu_s - \phi_s)^2 + (\nu_s - \psi_s)^2 \right) ds \right]$$

which is equivalent to $41$. □

If the agents have the same risk aversion process the sharing of the endowments is exactly the same as the corresponding situation of entropic risk measures (i.e., when agents have the same risk aversion coefficient). This implies that after the transaction both agents have the same random endowment, $E/2$. Note also the similarity of the other terms of the optimal contract with those of the case analyzed in subsection 3.2.

For the more general case of different and stochastic risk aversion processes the above arguments cannot be applied. However, we are able to construct the dynamics of the inf-convolution measure, which gives the existence and an implicit form of the optimal risk sharing contract.

We first note that under Assumption 3.1. the agents’ dynamic indifference values for any bounded contingent claim $C$ solve the following BSDE’s under the minimal martingale measure $Q^{\lambda,0}$ (see Proposition 2.1).

$$- d\rho^1_t (C) = dX^\pi_t + \frac{1}{2} (\gamma t (\zeta^1_t)^2 - 2\zeta^1_t \phi_t) dt - \zeta^1_t dW_t^2, \quad \rho^1_t (C) = -C$$

$$- d\rho^2_t (C) = dX^\pi_t + \frac{1}{2} (g(t) (\zeta^2_t)^2 - 2\zeta^2_t \psi_t) dt - \zeta^2_t dW_t^2, \quad \rho^2_t (C) = -C$$

where $\pi^1, \pi^2 \in A^\infty$ and $\zeta^1, \zeta^2 \in P_T$. 

**Proposition 3.4.** Impose Assumption 3.1. and assume that $\gamma_t = g_t$ for every $t \in [0, T]$. Then, any claim of the form

$$\frac{1}{2} \int_0^T \frac{\psi_t - \phi_t}{\gamma_t} dt + \int_0^T \frac{\psi_t - \phi_t}{\gamma_t} dW_t^2 + \frac{E_2 - E_1}{2} + X_t^{c,\pi}$$

for some constant $c$ and strategy $\pi \in A$, solves the optimal sharing problem $34$. 

**Proof.** It is enough to show that for every claim $C^*$ of the form $40$

$$v_0^{(1)} (E_1 + C^*) + v_0^{(2)} (E_2 - C^*) = \inf_{\nu \in P_T} E_{Q,\nu} \left[ E_1 + C^* + \frac{1}{2} \int_0^T (\nu_s - \phi_s)^2 ds \right]$$

We first observe that for every such $C^*$ and every $\nu \in P_T, E_{Q,\nu} \left[ E_1 + C^* + \frac{1}{2} \int_0^T (\nu_s - \phi_s)^2 ds \right]$ equals to

$$E_{Q,\nu} \left[ \frac{E}{2} + \frac{1}{2\gamma T} \left( \int_0^T (\nu_s - \phi_s)^2 + \psi_s^2 - \phi_s^2 \right) ds + \int_0^T (\psi_s - \phi_s) dW_s^2 \right]$$

Applying the same calculations to $E_{Q,\nu} \left[ E_2 - C^* + \frac{1}{2} \int_0^T (\nu_s - \psi_s)^2 ds \right]$, we get that

$$v_0^{(1)} (E_1 + C^*) + v_0^{(2)} (E_2 - C^*) = 2 \inf_{\nu \in P_T} E_{Q,\nu} \left[ \frac{E}{2} + \frac{1}{4\gamma T} \left( \int_0^T (\nu_s - \phi_s)^2 + (\nu_s - \psi_s)^2 \right) ds \right]$$

which is equivalent to $11$. □
Adapting the argument lines of Section 3 of [2], we consider a claim \( C \in \mathbb{L}^\infty(F_T) \) and introduce the BSDE

\[
-dF_t = f(t, \zeta_t)dt - \zeta_tdW_t^2 + dX^\theta_t, \quad F_T = -C
\]  

(44)

where for every \( t \in [0, T] \)

\[
f(t, \zeta_t) = \frac{1}{2} \frac{\gamma_t g_t}{\gamma_t + g_t} (\zeta_t^2 + (\psi_t - \phi_t)^2) - \frac{\zeta_t (\gamma_t \psi_t + g_t \phi_t) + (\psi_t - \phi_t)^2}{\gamma_t + g_t}
\]

(45)

The solution of (44) is given by a triple \( (F_t(C), \zeta_t, \theta_t) \).

**Lemma 3.1.** If we impose Assumption 3.1 and assume that \( \int_0^T (\psi_u - \phi_u)^2 du \in \mathbb{L}^\infty(F_T) \), the BSDE (44) admits a unique solution \( (F_t(C), \zeta_t, \theta_t) \) for every \( C \in \mathbb{L}^\infty(F_T) \). In addition, \( F_t(C) \) is for any time \( t \in [0, T] \) a convex risk measure.

**Proof.** Let us first define the process \( k_t = \frac{\gamma_t u + g_t \phi_t}{\gamma_t + g_t} \), for \( t \in [0, T] \), where \( k \in \mathcal{P}^n \). We then note that under the martingale measure \( Q^{\lambda,k} \) the BSDE (44) is written as

\[
-dF_t = \frac{1}{2} (\Gamma_t \hat{z}_t^2 dt - 2z_t d\hat{W}_t^2) + dX^\theta_t + dL_t, \quad F_T = -C
\]

(46)

where \( \Gamma_t = \frac{\gamma_t u + g_t \phi_t}{\gamma_t + g_t}, \hat{W}_t^2 = W_t^2 + \int_0^t k_u du \) is a standard Brownian Motion under \( Q^{\lambda,k} \), orthogonal to \( W^1 \) and \( L_t = \int_0^t \left( \frac{\Gamma_t (\psi_u - \phi_u)^2}{\gamma_u + g_u} - \frac{(\psi_u - \phi_u)^2}{\gamma_u + g_u} \right) du \). We then observe that the BDSE

\[
-d\hat{F}_t = \frac{1}{2} (\Gamma_t \hat{z}_t^2 dt - 2z_t d\hat{W}_t^2) + dX^\theta_t \quad \hat{F}_T = -C + L_T
\]

(47)

admits a solution \( (\hat{F}_t, \hat{z}_t, \hat{\theta}_t) \), which is in fact the dynamic risk measure \( \hat{\rho}_t(C - L_T) \) induced by the exponential performance criteria with characterization pair \( (\theta_t + \vartheta_t, k_t) \). This means that \( (\hat{F}_t - L_t, \hat{z}_t, \hat{\theta}_t) \) is a solution of (46).

The uniqueness of the solution follows by Proposition A.1.

Finally, the fact that \( F_t(C) \) is a dynamic convex risk measure is a consequence of Proposition A.1 and the convexity of \( f(t, \zeta_t) \) with respect to \( \zeta_t \) (see also Proposition 5.1 of [13] and Proposition 5.1 of [44]). □

The following theorem solves the optimal risk sharing problem under forward exponential performance criteria.

**Theorem 3.1.** Assume that \( \int_0^T (\psi_u - \phi_u)^2 du \in \mathbb{L}^\infty(F_T) \) and let \( (\rho^{1,2}_t(\mathcal{E}), \zeta_t, \theta^{1,2}_t) \) be the solution of (44). Then, for every \( t \in [0, T] \)

\[
\rho^{1,2}_t(\mathcal{E}) = \inf_C \{\rho_t^1(\mathcal{E} - C) + \rho_t^2(C)\}
\]

and the optimal risk sharing claims are of the form

\[
C^* = \mathcal{E}_2 - \int_0^T \left( \frac{g_t}{2} \left( \frac{\gamma_t \zeta_t + (\psi_t - \phi_t)}{\gamma_t + g_t} \right)^2 + \frac{\gamma_t \zeta_t + (\psi_t - \phi_t)}{\gamma_t + g_t} \right) dt - \int_0^T \frac{\gamma_t \zeta_t + (\psi_t - \phi_t)}{\gamma_t + g_t} dW_t^2 + X_T^c, \theta
\]

(48)
for $c \in \mathbb{R}$ and $\theta \in \mathcal{A}^\infty$.

*Proof.* After simple calculations we get that for every processes $z$ and $y$ and for every $t \in [0, T]$

$$f(t, z_t) \leq \frac{1}{2} \left( \gamma_t(z_t - y_t)^2 - 2(z_t - y_t)\phi_t \right) + \frac{1}{2} \left( g_t y_t^2 - 2y_t \psi_t \right).$$ (49)

Let $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ be an arbitrarily chosen claim and $(\rho_t^1(\mathcal{E} - C), \zeta_t^1, \pi_t^1)$ and $(\rho_t^2(C), \zeta_t^2, \pi_t^2)$ be the solutions of the BSDE’s \([42]\) and \([43]\) with boundary conditions $\mathcal{E} - C$ and $-C$ respectively. This implies that if we set $\tilde{\pi}_t = \pi_t^1 + \pi_t^2$ and $\tilde{\zeta}_t = \zeta_t^1 + \zeta_t^2$, the triple $(\rho_t^1(\mathcal{E} - C) + \rho_t^2(C), \tilde{\zeta}_t, \tilde{\pi}_t)$ is a solution of the following BSDE

$$-dC_t = \frac{1}{2} \left( \gamma_t(z_t - \zeta_t^2)^2 - 2(z_t - \zeta_t^2)\phi_t \right) + \frac{1}{2} \left( g_t(\zeta_t^2)^2 - 2\zeta_t^2 \psi_t \right) dt - z_t dW_t^2 + dX_t^p, \quad C_T = -\mathcal{E}$$ (50)

for $\zeta^2$ given. Then by Proposition A.1 and inequality (49), we get that $\rho_t^{1,2}(\mathcal{E}) \leq \rho_t^1(\mathcal{E} - C) + \rho_t^2(C)$ for every claim $C$ and every time $t \in [0, T]$. Also for process $\zeta$

$$f_t(\zeta_t) = \frac{1}{2} \left( \gamma_t(\zeta_t - \hat{\zeta}_t)^2 - 2(\zeta_t - \hat{\zeta}_t)\phi_t \right) + \frac{1}{2} \left( g_t \hat{\zeta}_t^2 - 2\hat{\zeta}_t \psi_t \right)$$

and for every $t \in [0, T]$, where $\hat{\zeta}_t = \frac{\zeta_t + \psi_t - \phi_t}{\gamma_t + \varphi_t} \in \mathcal{P}_T$.

The next step is to observe that for the process

$$\hat{C}_t = -\int_t^T \left( \frac{g_s z_s^2}{2} - \hat{z}_s \psi_s \right) ds + \int_0^t \hat{z}_s dW_s^2$$

the triple $(\hat{C}_t, \hat{z}_t, 0)$ is the unique solution of \([43]\) with boundary condition $\hat{C} = \hat{C}_T$, i.e., $\hat{C}_t = \rho_t^{(2)}(\hat{C})$.

We also have that if $(\rho_t^1(\mathcal{E} - \hat{C}), \hat{\zeta}_t, \hat{\pi}_t)$ is the solutions of \([42]\) with boundary condition $\mathcal{E} - C$, then $(\rho_t^1(\mathcal{E} - \hat{C}) + \rho_t^2(\hat{C}), \hat{\zeta}_t, \hat{\pi}_t)$ is the solution of \([44]\) with boundary condition $-\mathcal{E}$. Thanks to Lemma 3.1 we have that $\rho_t^1(\mathcal{E} - \hat{C}) + \rho_t^2(\hat{C}) = \rho_t^{1,2}(\mathcal{E})$, which in turn means that $C^* = \mathcal{E}_2 - \hat{C}$ is an optimal risk sharing contract. The fact that we can add/subtract any replicable claim on the optimal risk sharing contract is a consequence of the replication invariance property of the indifference valuation (see Proposition 2.2 item (iv)).

**Remark 3.2.** We note that when $\phi_t = \psi_t$ for every $t \in [0, T]$, equation \([44]\) becomes similar to \([42]\) and \([43]\), where the risk aversion process is given by $\Gamma_1$, i.e. the solution of \([44]\), which is the inf-convolution measure, is a dynamic risk measures that is induced by an agent with exponential forward performance criteria with characteristic pair $(\Gamma_1, \phi_t)$ (see also Proposition 3.1).

**APPENDIX A.**

**Proposition A.1.** Impose Assumption 3.1 and assume that $(C_t, z_t, \theta_t)$ and $(C_t', z_t', \theta_t')$ are solutions of the following BSDE’s

$$dC_t = f(t, z_t) dt + \theta_t dW_t^1 + z_t dW_t^2, \quad C_T = C$$

$$dC_t' = f'(t, z_t') dt + \theta'_t dW_t^1 + z_t' dW_t^2, \quad C_T' = C'$$
where \(C, C' \in L^\infty(F_T)\), with \(C \leq C'\) a.s., \((W^1_t, W^2_t)\) is a two dimensional Brownian Motion, \(f : \Omega \times [0, T] \times A \rightarrow \mathbb{R}\) is given by
\[
f(t, z_t) = \frac{\Gamma_t}{2} z_t^2 - z_t \gamma_t \psi_t + g_t \phi_t + (\psi_t - \phi_t)^2 \left( \frac{\Gamma_t}{2} - \frac{1}{\gamma_t + g_t} \right)
\]
and \(f' : \Omega \times [0, T] \times A \rightarrow \mathbb{R}\) is a smooth random function for which
\[
f(t, z_t) \leq f'(t, z_t), \text{ a.s.} \quad (51)
\]
Then, \(C_t \leq C'_t\), a.s. for every \(t \in [0, T]\).

Proof.
\[
C_t - C'_t - (C_0 - C'_0) = \int_0^t (f(s, z'_t) - f'(s, z'_s)) ds + \int_0^t (z_t - z'_t) dW^2_s
\]
\[
+ \int_0^t (f(s, z_s) - f(s, z'_s)) ds + \int_0^t (\theta_t - \theta'_t) dW^1_s
\]
We then observe that
\[
f(t, z_t) - f(t, z'_t) = \frac{\Gamma_t}{2} (z^2_t - (z'_t)^2) - (z_t - z'_t) \frac{\gamma_t \psi_t + g_t \phi_t}{\gamma_t + g_t}
\]
\[
= (z_t - z'_t) K_t
\]
where \(K_t = \frac{\Gamma_t}{2} (z_t + z'_t) - \frac{\gamma_t \psi_t + g_t \phi_t}{\gamma_t + g_t}\). Now \(K \in P^\lambda\) by Lemma 2.1 and the uniform boundness of \(\gamma\) and \(g\). Hence, \(Q^{\lambda,K}\) is a martingale measure and therefore \(C_t - C'_t - (C_0 - C'_0) - \int_0^t (f(s, z'_s) - f'(s, z'_s)) ds\) is a true \(Q^{\lambda,K}\)-martingale. Taking expectation under \(Q^{\lambda,K}\) and exploiting the assumed inequality \([51]\) gives the intended inequality of the solutions. \(\Box\)

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