ON THE RATIONALITY OF $SU(r, d)$

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INTRODUCTION

Let $C$ be a curve of genus $g$ over the complex numbers. For $0 < d < r$, let $SU = SU(r, r(g - 1) + d)$ be the moduli space of bundles with rank $r$ and fixed determinant of $\deg r(g - 1) + d$ over $C$. This paper shows that if $d$ divides $(r \pm 1)$, then $SU$ is rational (no matter what the value of $g$). In fact if the prime divisors of $\delta > 0$ are divisors of $r$, and $d$ divides $r - \delta > 0$ then $SU$ is rational for any genus. We conjecture that this holds for $r + \delta$ if $0 < \delta < r$.

Research on the rationality of $SU$ began shortly after the construction of the Moduli space. In 1964, Tyurin [9] proved the case of rank 2 bundles of odd degree. He went on to claim rationality for all ranks and degrees [10], but the paper contained fatal errors. Newstead corrected some errors, and in his unpublished thesis, he proved the case $d = 1$ for any $r$. He then went on to claim a proof for $(r, d) = 1$ [6]. That proof contained a trivial but fatal error. He concluded in his correction [6] that for any pair $(r, d)$ which are coprime the theorem holds for infinitely many genera. Which suggests, but does not prove, that the theorem is true. Ballico then noted that Newstead’s argument proved $SU$ was stably rational [1], where $X$ is stably rational with level $k$ if for all $n \geq k$, $X \times \mathbb{P}^n$ is rational. Unfortunately, Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer proved stable rationality does not imply rationality [2]. However, Ballico noted that stable rationality with small level and Newstead’s arguments would yield strong results. Aware of this, Boden and Yokogawa [3], used parabolic bundles to prove $SU(r, d)$ is stably rational.
of level \( \leq r - 1 \). This, and the argument of Newstead proved that if \((r, d) = 1\), then \(SU\) is rational provided either \((g, d) = 1\) or \((g, r - d) = 1\). And this result generalizes Newstead’s results.

Now our results continue the theme of widening the number of genera for which rationality is known. In particular, we find infinitely many pairs \((r, d)\) with \((r, d) = 1\) where \(SU(r, d)\) is rational.

Main Theorem. Let \(C\) be a smooth projective irreducible curve over \(\mathbb{C}\). Then \(SU(r, d')\) is rational if \(d' \equiv \pm d \pmod{r}\), \((r, d) = 1\) and if

A) \(d\) divides \((r \pm 1)\), or if

B) \(d\) divides \((r - \delta)\), where for all primes \(p\) such that \(p|\delta, p|r, \) and \(0 < \delta < r\).

Along the way we wind up proving a second, seemingly unrelated result:

Second Theorem. Let \(C\) be as above, and let \(\pi : \tilde{C} \to C\) be a cyclic étale cover of deg \(e\). For a general stable bundle \(E\) with \(\mu(\pi^*E) > g(\tilde{C})\), \(\pi^*(E)\) is generated by global sections.

Before we outline the proof, it should be noted that \(SU(r, d) \cong SU(r, rk \pm d)\) for any integer \(k\) by tensoring and/or dualizing.

Now we outline the proof giving a historic perspective. It begins with Newstead who observed that since a general bundle with rank \(r\) and deg \(r(g - 1) + d\) had \(h^0 = d\), there is a unique subbundle \(I = \oplus \mathcal{O}_C\) to a general \(E\). This gives an exact sequence

\[
0 \to I \to E \to F \to 0.
\]

It seems likely, and turns out to be true that for general \(E\), \(F\) is a stable bundle, and so a general \(E\) is given by an extension of a rank \(r - d = n\), deg \(r(g - 1) + d\) vector bundle \(F\) by \(I\).

Now we know \(SU = SU(r, r(g - 1) + d)\) has a map to \(SU' = SU(r - d, r(g - 1) + d)\) whose fiber over a closed point \([F]\) is \(\text{Ext}^1(F, I) \cong H^0(C, \omega_C \otimes F)^*\) modulo the action of \(\text{Aut}(I)\),
and this turns out to be a Grassmannian. But even if $SU' = SU(r - d, r(g - 1) + d)$ is rational, $SU$ need not be rational. Because unfortunately, if $X$ maps to a rational variety with rational fibers it is not necessarily true that $X$ is rational. But if $(r-d, r(g-1)+d) = 1$, then $SU'$ has a Poincare bundle $\mathcal{P}$ on $SU' \times C$. Tensoring by the pullback of $\omega_C$ then pushing down to $SU'$ and taking $d$ direct summands, we get a Zariski open set with a bundle parameterizing $\text{Ext}^1(F, I)$. Taking $\mathbb{G}$ the quotient by $\text{Aut}(I)$, we get a parameterization of an open set of $SU$ by a Grassmannian bundle over $SU'$. Since it is a bundle, it is trivial (on a perhaps smaller set) and so $SU$ is birational to $SU' \times \mathbb{G}$. If $SU'$ is rational by inductive hypothesis then $SU$ is rational. Unfortunately there is one problem. $r(g-1)+d$ and $r-d$ need not be relatively prime. And if they are not, there is no Poincare bundle.

To see that the above happens consider the following example. Set $r = 5$, $g = 6$ and $d = 2$. $5(6 - 1) + 2 = 27$ and is not relatively prime to $(5 - 2) = 3$. The same thing happens for $d = 5 - 2 = 3$. Now consider another example. Set $r = 7$, $g = 6$ and $d = 2$. $7(6 - 1) + 2 = 37$ which is relatively prime to $(7 - 2)$. But $SU' = SU(5, 5(6 - 1) + 2)$ which is the previous example. We can try $d = 7 - 2 = 5$ but $7(6 - 1) + 5 = 40$ is not relatively prime to $7 - 5 = 2$. The whole inductive argument breaks down unless $g$ is chosen carefully, or if $d = \pm 1$, in which case substituting $d = r - 1$ (By tensoring and perhaps dualizing) gives a rank 1 cokernel which is rational because it is a point and there is only one fiber.

Now Ballico notes that in the coprime-prime case we do not need the $SU'$ to be rational, but for some product of a rational variety with $SU'$ to be rational. If $SU'$ is stably rational with level less than or equal to the dimension of the Grassmannian, then $SU$ is rational. Unfortunately Ballico could not prove the moduli spaces were stably rational of small enough level; although he did prove them stably rational.

Boden and Yokogawa then proved rationality for most parabolic bundles. As a corollary they proved that bundles with $(\text{rank}, \text{deg}) = (r, d) = 1$ are stably rational with level $r - 1$. This implies $SU$ is stable if $(r(g - 1), r - d) = 1$ or $(r(g - 1), d) = 1$. And this simplifies
to \((g, d) = 1\) or \((g, r - d) = 1\).

Now the author enters the picture. It would seem at first that Newstead’s original argument had reached its logical conclusion. But it could be improved by changing its very beginning. Newstead begins with the trivial bundle \(I\) of rank \(d\), because its embedding into a general \(E\) with \([E] \in SU\) is unique up to automorphisms of \(I\). But there are other bundles with this property. Namely any deg 0 twist of \(I\) or any direct sum of deg 0 twists of \(I\). We use the latter. Let \(\{L_1, \ldots L_e\}\) be a set of mutually non-isomorphic deg 0 line bundles and let \(U_i\) be a rank \(d\) vector space. Then set \(V_i = U_i \otimes L_i\) and \(W = \oplus V_i\). The bulk of our proof is to show that for any \(W\) with \(ed < r\) and a general \(E\) the sequence

\[
0 \to W \to E \to F \to 0
\]

has a stable bundle \(F\) as cokernel. And when rank \(W > \text{rank } E\) we have a surjection of \(W\) to \(E\) giving an exact sequence,

\[
0 \to K_{E,W} \to W \to E \to 0.
\]

For the plus case in part A, \(K_{E,W}\) is a fixed line bundle we call \(A\). And \(SU\) is isomorphic to \(H^0(C, A^* \otimes W)/\text{Aut}(W)\), and this turns out to be a product of Grassmannians.

Now for part B (which implies the minus case in part A). If \(ed = r - \delta\) where every prime factor of \(\delta\) is a prime factor of \(r\), then \((\delta, r(g - 1) + d) = 1\) since no divisor of \(r\) divides \(d\). That means \(SU' = SU(\delta, r(g - 1) + d)\) has a Poincare bundle. A general \(E\) is an extension of \(F \in SU'\) by \(W\) And since \(SU'\) has a Poincare bundle, it has a bundle whose fibers are \(\text{Ext}^1(F_t, W) \cong i^* H^0(C, \omega_C \otimes L_i^* \otimes F_t)^* \otimes U_i\) Taking an affine subset of \(SU'\) we can form a universal extension space. This gives us a map from the bundle over \(SU'\) to \(SU\) which is dominant and has fibers \(\text{Aut}(W)\). We take the quotient by the automorphism group and get a birational map from \(SU'\) cross a product of Grassmannians to \(SU\). Since the product of Grassmannians has dimension \(\geq r - 1\), \(SU\) is rational.
It would be nice to have similar results for when $ed = r + \delta$ but they are held up because we do not know when $K_{E,W}$ is stable. In the final section we make a natural conjecture and offer some evidence. We will also prove the Second Theorem.

I thank Johns Hopkins University and the Tata Institute for Fundamental Research for their hospitality. I would like to thank Shokurov who shared his expertise in birational geometry. I would also like to thank Mehta, Nitsure, Ramadas Ramanan and Srinivas for answering questions so numerous that I do not know who answered what. But I do know they were answered.

§1 Preliminaries

$C$ is a smooth projective irreducible curve of genus $g$ over $\mathbb{C}$

For a vector bundle $E$, $\mu(E) = \deg(E)/\text{rank}(E)$.

$E$ is stable (semistable) if $\mu(S) < \mu(E)$ (\leq) for all proper subbundles. If $E$ is not semistable, then it is called unstable. If $E$ is not stable, then it is called nonstable. If $E$ is semistable but not stable, then it is called strictly semistable.

As in the introduction $SU = SU(r, r(g - 1) + d)$ and $U = U(r, r(g - 1) + d)$, the moduli spaces of stable bundles of rank $r$ and $\deg r(g - 1) + d$ with fixed and unfixed determinants respectively. It is assumed that $0 < d < r$.

By abuse of notation we say $E \in SU$ (or $U$) if the isomorphism class of the vector bundle is in the moduli space.

We will use $E$ to denote a fixed bundle and $E_t$ to denote a general bundle in the moduli space.

$L$ will denote a single linebundle of $\deg = 0$.

We will consider sets of mutually non-isomorphic line bundles of $\deg 0$: $\{L_1, L_2, \ldots, L_e\}$. Our results will not depend on the the choice (except for the Second Theorem).

$U_i = \bigoplus k$ where $k$ is the ground field.

$V_i = U_i \otimes L_i$. 
\[ W_k = V_1 \oplus \cdots \oplus V_k. \]

\[ W = W_e. \]

We will use dimension counts to prove our results. These arguments are hampered because many bundles do not live on a moduli space, and even those that do may not have a universal or Poincare bundle. But we may (after Sundaram [8]) form very useful parameter spaces of bundles.

**Definition.** Let \( \text{Cl} \) be a class of vector bundles over \( C \). A parameter space \( T \) of \( \text{Cl} \) is a variety along with a bundle \( \mathcal{E} \) over \( T \times C \) such that any \( E \) in \( \text{Cl} \) is isomorphic to the restriction \( E_t \) of \( \mathcal{E} \) to some \( t \times C \).

**Remark 1.** It is not assumed that \( E_t \cong E_s \) implies \( s = t \). And even if it does, parameter spaces do not generally have the universal properties of a moduli space.

Stable vector bundles of rank \( r \) and degree \( d \) where \( (r, d) = 1 \) form a fine moduli space. In particular, there is a Poincare bundle \( \mathcal{P} \) over the moduli spaces \( SU \) and \( U \). In the non-coprime case, the variety is quasi-projective and has no Poincare bundle (even locally). However \( U \) and \( SU \) can be covered by Zariski open sets \( \{A_i\} \). The \( A_i \) in turn have etale covers \( p_i : T_i \to A_i \). There is a Poincare bundle \( \mathcal{P}_i \) over \( T_i \) such that the map given by the universal property of \( U \) and \( SU \) is \( p_i \). This gives a parameter space of stable bundles with rank \( r \) and degree \( d \) where \( (r, d) \neq 1 \).

Sundaram constructs parameter spaces from the Jordan Holder filtration, and the Harder Narasimhan filtration. The strictly semistable bundles are parameterized by finitely many components with dimension \( \leq \dim M - (r - 1) \) The unstable bundles have infinitely many components. So this case needs explaining. First of all, for any Bundle \( E \) with rank \( r \) and degree \( d \), there is a unique filtration:

\[ \sum E : 0 = \mathcal{E}_1 \subset \mathcal{E}_2 \cdots \subset \mathcal{E}_k = E, \]

where \( E_i = \mathcal{E}_i/\mathcal{E}_{i-1} \) is semistable, and \( \mu_i = \mu(E_i) \) is strictly decreasing. We say that \( E \) has Harder Narasimhan (or HN) type \( \{(r_1, \mu_1), (r_2, \mu_2), \cdots (r_k, \mu_k)\}; r_1 + r_2 + \cdots + r_k = r, \) where
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$r_i = \text{rank}(E_i)$. We set $\mu_1 = \mu^{\text{max}}$; it is the upper bound on $\mu$ of a subbundle. Similarly, we set $\mu_k = \mu^{\text{min}}$; it is the lower bound on $\mu$ of a quotient bundle. If $\mu^{\text{max}} < 0$ then $h^0 = 0$ as $\mathcal{O}_C$ is not a subbundle. And by Serre Duality, if $\mu^{\text{min}}(E) > 2g - 2$, then $h^1 = 0$. Since the complex numbers are characteristic zero, $\mu^{\text{max}}(E \otimes F) = \mu^{\text{max}}(E) + \mu^{\text{max}}(F)$.

If $r$ is fixed and $\mu^{\text{max}}$ is bounded from above and $\mu^{\text{min}}$ is bounded from below, there are only finitely many HN types possible. This is also the case when deg is bounded from above and $\mu^{\text{min}}$ is bounded from below, or if $\mu^{\text{max}}$ is bounded from above and deg is bounded from below. The advantage is that then there are only finitely many components of the parameter space of unstable bundles, and each has dimension $\leq \dim U - (r - 1)$ or $\leq \dim SU - (r - 1)$.

§2 Some Propositions

Proposition 1. Let $0 < c \leq d$ and $c < r$ and $L$ be a line bundle of deg = 0. For a general $E_t \in U(r, r(g - 1) + d)$ with $d > 0$ and $d$ not necessarily $< r$, there is an exact sequence:

$$0 \to \oplus L \to E_t \to F_u \to 0$$

with $F_u$ stable.

Remark 2. This does not mean that all embeddings of $\oplus L$ have a stable vector bundle cokernel. But a general subbundle of a general bundle such a cokernel. Of course if $c = d$ there is only one such subbundle for general $E_t$ and the cokernel is unique.

Proof of Proposition 1. The first thing that can go wrong is that the cokernel is not a bundle. In some sense this is all that can go wrong outside of stability. If $c = 1$ this is obvious. If $c = 2$, the one dimensional family of (twisted) sections may span a subbundle of $E_t$ of rank 1. But going back to $c = 1$, that means $L$ factored. If that never happened,
there is no problem at \( c = 2 \) except maybe factoring through a subbundle of rank 2, and so on. So we can assume \( \mathcal{E} \otimes L \) is a subbundle.

So now the idea is to count dimensions of the space of subbundles in various \( E_t \) that our trivial bundles could factor through and the dimensions of the space of subbundles of the form \( \mathcal{E} \otimes L \). We know \( \mathcal{E} \otimes L \) embeds in every \( E_t \). We also know \( h^0(C, E_t) = d \) for general \( t \) and there are \( c(d - c) \) dimensions of embeddings. So the total dimension of subbundles of the form \( \mathcal{E} \otimes L \) is

\[
(*) \quad r^2(g - 1) + 1 + c(d - c),
\]

which we will show is larger than the dimension of the space of bundles factored through.

So suppose \( \mathcal{E} \otimes L \) factors through a bundle \( S \) and that we have a sequence;

\[
0 \to S \to E_t \to Q \to 0
\]

where \( Q \) is a vector bundle. \( S \) is a rank \( c \) bundle and an elementary transformation of \( \mathcal{E} \otimes L \). So at first glance the dimension of possible bundles \( S \) is \( c\delta \) with \( \delta = \deg S \), since \( S \) must be an elementary transform of \( \mathcal{E} \otimes L \). But there are \( c^2 - 1 \) projective automorphisms of \( \mathcal{E} \otimes L \). So if \( \delta \geq c \) the dimension is \( c\delta - c^2 + 1 \). If \( \delta < c \) the dimension is 0 and there are at least \( c^2 - c\delta - 1 \) dimension of automorphism of \( S \). The dimension of automorphisms must be subtracted from the dimension of extensions, so we shall abuse terminology and say the dimension of the space of bundles \( S \) with a map \( \mathcal{E} \otimes L \to S \) is \( c\delta - c^2 + 1 \).

We know that

\[
\deg Q = r(g - 1) + d - \delta = (r - c)(g - 1) + c(g - 1) + d - \delta.
\]

And since \( \mu_{\min}(Q) \geq \mu(E_t) > 0 \) is bounded from below, only finitely many HN types are possible for \( Q \). The dimension is bounded by the dimension for stable \( Q \), that is
\((r-c)^2(g-1)+1\). Since \(Q\) is a quotient of the stable bundle \(E_t\), \(\mu_{\text{min}}(Q) > \mu(E_t)\). Similarly \(\mu_{\text{max}}(S) < \mu(E_t)\) And hence \(\mu_{\text{min}}(S^*) > -\mu(E_t)\). All of which gives us \(\mu_{\text{min}}(\omega_S \otimes S^* \otimes Q) > 2g-2\) And hence, \(h^1(C, \omega_S \otimes S^* \otimes Q) = 0\).

Now we estimate and bound the dimension of the space of sections.

\[
h^0(C, \omega_C \otimes S^* \otimes Q) = 2c(r-c)(g-1) - \delta(r-c) + c^2(g-1) + cd - c\delta.
\]

Now we add the dimension of bundles \(S\) and \(Q\) then subtract 1 from \(\text{Ext}^1\) in order to get the projective dimension. It all comes out to:

\[
** \quad r^2(g-1) + 1 + cd - c^2 - \delta(r-c).
\]

The upshot is that * is bigger than **, so \(\oplus L\) is a subbundle with a bundle cokernel.

To see that the general cokernel is stable, consider yet another dimension count. We count extensions of the form

\[
0 \to \oplus L \to E_t \to F_u \to 0.
\]

Let \(F_u \in U\) be a parameter space of stable bundles of rank \(r-c\) and \(\text{deg} r(g-1) + d\). It has dimension \((r-c)^2(g-1)+1\). Calculating as above we see the dimension of extensions is \(r^2(g-1) + c(d-c)\). A parameter space of nonstable bundles would have dimension at least \(n-1\) lower; therefore, the general cokernel is stable. That proves Proposition 1.

** Proposition 2.** Suppose that for a general bundle \(E_t \in U\) there is a natural sequence

\[
0 \to W_k \to E_t \to F_u \to 0
\]

where \(F_u\) is a stable vector bundle for general \(u\). If \(\oplus L\) (with \(\text{deg} L = 0\)) embeds in a general \(E_t\), \(L \notin W_k\), and \(dk + c < r\), then for (perhaps more general \(E_t\)) there is an exact sequence:

\[
0 \to W_k \oplus L \to E_t \to F' \to 0
\]
with $F'_v$ a stable vector bundle and $\oplus L$ a general subbundle of $E_t$. Hence, by setting $c = d$ and considering only $E_t$ with $h^0(C, E_t \otimes L_i^*) = d$, and with $ed < r$, we get for general $t$, a unique sequence:

$$0 \to W_e \to E_t \to F_u \to 0$$

with $F_u$ a stable bundle for general $t$.

**Proof of Proposition 2.** First we start with $F_u = F$ fixed. Let $\oplus L$ be a subbundle of $F$. Then for a general subbundle Proposition 1 applies and the cokernel is stable. Now we have to pullback the subbundle of $F$ to a subbundle of $E_t$.

For any given embedding $\oplus L \xrightarrow{\alpha} F$, there is a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W_k & \longrightarrow & E_{t\alpha} & \longrightarrow & \oplus L & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_k & \longrightarrow & E_t & \longrightarrow & F & \longrightarrow & 0 \\
\end{array}
$$

$E_{t\alpha}$ is base extension of $E_t$ over $F$ by $\alpha$. So $\alpha$ factors through $E_t$ iff it factors through $E_{t\alpha}$, which means the top sequence splits. To interpret this, consider the surjective induced map:

$$H^0(C, \omega \otimes W_k^* \otimes F)^* \rightarrow H^0(C, \omega \otimes W_k^* \otimes (\oplus L))^*.$$

The top sequence splits if the element corresponding to the bottom sequence maps to 0. It is easily verified by a dimension count that there are extensions which cause the upper sequence to split, and many of them. For the kernel is the dimension of the bottom extensions of $F$ to $W_k$ minus the dimension of top extensions $\oplus L$ to $W_k$.

We can choose $E_t$ so it has $h^1(C, E_t \otimes L_i^*) = 0$ for $1 \leq i \leq k$ and $h^1(C, E_t \otimes L^*) = 0$. This forces a choice of $F$ but we can choose $E_t$ so that $F$ is stable. Also note, the general extension of $F$ by $W_k$ will be stable and have the above vanishing of $h^1$. Now we count the dimension of pairs, $(\oplus L \subset E_t)$ denoting a specific subbundle $\oplus L$. It’s dimension is the
dimension of the bottom extensions plus \( c(d - c) \). Now this maps to the set of subbundles of \( F \). So we subtract the dimension of the kernel, which means we subtract the dimension of the bottom extensions and add the number of top extensions. This gives us a total of \( c(d - c) + kcd(g - 1) \).

Now we count the dimension of pairs \( (\oplus L \subset F) \). Since a general \( E_t \) has \( h^1(C, E_t \otimes L^*) = 0 \), so does \( F \) have \( h^1(C, F \otimes L^*) = 0 \). Since \( h^0(C, F \otimes L^*) = r(g - 1) + d - (r - dk)(g - 1) \), we get dimension of rank \( c \) subbundles given by, \( cdk(g - 1) + c(d - c) \). So the map is dominant and a general bundle \( c \oplus L \subset E_t \) has a stable cokernel (provided our bundle is an extension of \( F \)).

The preceding argument has two faults. The first is that we never constructed the object whose dimensions we counted. The second is that it handles \( F_u \) one \( u \) at a time, treating the family of \( F_u \) as a set (with no algebraic structure).

For the first, let \( V = \text{Ext}^1(F, W_k) \). Now there is a sequence over \( V \times C \):

\[
0 \to W_k \to E \to F \to 0.
\]

Given a point \( v \in V \) the sequence restricts over \( v \times C \) to:

\[
0 \to W_k \to E_t \to F \to 0,
\]

where the extension corresponds to \( v \in V \). See [5] [7] and [4]. If we tensor by \( L^* \) and push down \( \pi : V \times C \to V \), we get a sequence

\[
0 \to \pi_* E' \to \pi_* F'.
\]

When restricted to \( v \times C \) we get:

\[
0 \to H^0(C, E_t) \to H^0(C, F).
\]

A rank \( c \) subspace in \( H^0(C, E_t) \) corresponds to a unique subbundle \( \oplus L \subset E_t \). And its image in \( H^0(C, F) \) corresponds to the subbundle’s image in \( F \). So we take the Grassmannian
bundles and note that the second Grassmannian bundle maps from $V \times Gr$ to $Gr$. So a Grassmannian bundle over $V$ dominates a Grassmannian.

The previous dimension count now holds. And our dimension count has shown a birational map between subbundles of general $E_t$ and those of $F$. We have a general subbundle, $\bigoplus L \subset F$ has a stable cokernel by Proposition 1. And that implies a general subbundle $W_k \bigoplus L \subset E_t$ has a stable cokernel provided $E_t$ has cokernel $F$ corresponding to $W_k \subset E_t$.

We need to extend this result to a general cokernel $F_u$. So we need a general construction of a universal extension of two families of vector bundles. We have $W_k$ which does not really vary, and $F_u$ which does. So we have two families $T$ and $S$. $T$ is a point, and $S$ is some cover of an open subset of $SU$ the moduli of stable bundles of some fixed rank and degree. First we need $H^1(C, F_u \otimes L^*) = 0$. This is true for general $F_u$. Then we need to take an open set and an etale cover to get a Poincare bundle. We then take $S$ a Zariski open affine subspace of the etale cover. This will still dominate. We now satisfy the criteria of [Lemma 2.4, 7], because an affine space has no higher cohomology. So by that lemma, there is a universal extension with parameter $\mathbb{P}(V)$ which is a projective bundle over $U$ whose fiber at $u$ is $H^1(C, \text{Hom}(F_u, W_k))$. It can be formed by taking the pullback of $\omega_C \otimes W^*$, tensoring by the Poincare bundle $P$ and pushing down onto $U$ and dualizing. Now we can take Grassmannian bundles and globalize the proof, showing that for a general subbundle $\bigoplus L \subset E_t$ of a general bundle $E_t$ the cokernel is a stable bundle. □

**Proposition 3.** For a general bundle $E_t \in U$, a line bundle $L_{k+1}$ of deg 0 with $L_{k+1} \not\subset W$, a general $\bigoplus L_{k+1} \subset E_t$ with $c \leq d$, and $W_k \subset E_t$, the natural map

$$W_k \bigoplus L_{k+1} \to E_t$$

is surjective provided $dk + c \geq r + 1$.

**Proof of Proposition 3.** To begin with we show that the map is at least generically surjective for a general $E_t$. So let $i$ be the largest integer such that $di \leq r - 1$. Then for a general
$E_t$ and a general $\oplus L_{i+1} \subset E_t$, with $c' = r - 1 - d_1$, we get a subbundle whose quotient is a line bundle, by Proposition 2. Now another subbundle $L_{i+1} \subset E_t$ which does not factor through $W_i \oplus L$ maps to the line bundle quotient generically. We can choose another $L_{i+2}$, set $c' + 2 = c$, and the map $W_k \oplus L_{k+1}$ generically surjects onto $E_t$.

Now we have to show surjectivity for a general $E_t$. Once again, this will be done by dimension count. First we count the number of pairs $W_k \oplus L_{k+1} \subset E_t$. The dimension of the $E_t$ is $r^2(g-1) + 1 = (r^2 - 1)(g-1) + g$. The dimension of the subbundle (for given $E$) is the dimension of the Grassmannian $(c, d) = cd - c^2$. So our final dimension is:

$$(* *) \quad (r^2 - 1)(g-1) + g + cd - c^2.$$ 

Now suppose the map in the proposition does not surject onto $E_t$. It then surjects onto some bundle $Q$ and $E_t$ is an elementary transformation of $Q$. So this means

$$\text{deg } Q = \text{deg } E_t - \delta = r(g-1) + d - \delta$$

To calculate the dimension of the family of $Q$’s consider the sequence:

$$0 \to K \to W_k \oplus L_{k+1} \to Q \to 0,$$

where $K$ is a line bundle, with $\text{deg } K = r(g-1) + d - \delta$. Assume for now $h^1(C, K^* L_i) = 0$ for all $L_i$ and $K$. An informal dimension count gives:

$$(r + 1)(r - 1)(g-1) + g + (r + 1)d - (r + 1)\delta.$$ 

This follows from there being $r+1$ subline bundles in $W_k \oplus L_{k+1}$. All of them have $\text{deg } = 0$, and $K^*$ has $\text{deg } = r(g-1) + d - \delta$, and therefore, $h^0(C, K^* \otimes L_i) = (r - 1)(g-1) + d - \delta$. The $g$ is the dimension of the possible $K$. Now we need to account for the automorphisms which come to $kd^2 - c^2$. Now note that $(r + 1)d = (kd + c)d = (kd^2 + cd)$. And hence,
\[(r + 1)d - kd^2 - c^2 = cd - c^2.\] Finally we have to add \(r\delta\) the dimension of elementary transformations. The total dimension is:

\[\text{(***)} \quad (r^2 - 1)(g - 1) + g + cd - c^2 - \delta.\]

Since (*** \geq (****) with equality iff \(\delta = 0\), we can conclude surjectivity except for two problems. The first: what object have we calculated the dimension of? The second: what if \(K^* \otimes L_i\) is special?

For the first problem we can form a universal Hom space, for maps from \(K\) to \(W_k \oplus L_{k+1}\). The quotient comes for free. In general the Hom space is constructed as follows. Suppose we have families of bundles \(\mathcal{E}\) over \(S \times C\), and \(\mathcal{F}\) over \(T \times C\). Now consider \(\mathcal{E}^* \otimes \mathcal{F}\) over \(S \times T \times C\), and push down from \(C\), then pull it back up and call it \(U\). If \(h^0(C, E^* \otimes F)\) is constant, then restricting to a point \((E \in S, F \in T)\) the bundle is \(H^0(C, E^* \otimes F)\). And over a point \(v \in H^0(C, E^* \otimes F)\) we get a morphism \(E \to F\) over \(C\). If \(h^0(C, E^* \otimes F)\) is not constant we can take a stratification and check the dimension of each piece.

For the second problem we have \(\deg(K^*) = \alpha\) with \(1 \leq \alpha \leq 2g - 2\). So \(g \geq 2\). We will do the cases \(\alpha = 1\) and \(2\) separately. Then we will do the case \(\alpha \geq 3\). For \(\alpha = 1\), \(d\) must be \(1\) and \(k \geq 2\). For fixed \(L_1\) and \(L_2\) distinct there are only finitely many line bundles \(K\) of \(\deg = 0\) so that \(h^0(C, K^* \otimes L_i) \neq 0\) for \(i = 1\) and \(2\). Furthermore we must have \(d = 1\). The morphisms from \(K\) to \(L_i\) are unique up to automorphisms of \(W\) and hence there are only finitely many \(Q\), one for each \(K\). Now the dimension of elementary translations is \(r\) times the difference of degree or

\[r(r(g - 1) + d - 1) < ***.\]

Now we assume \(\alpha = 2\) and \(h^0(C, K^* \otimes L_i) = 1\) for all \(i\), and hence \(d = 1\). The dimension of \(K\) is \(\leq \alpha\). As above each \(K\) gives rise to a unique \(Q\). So the dimension of \(Q\) is \(\leq \alpha\) and adding the dimension of elementary transformations we get

\[\alpha + r(r(g - 1) + d - \alpha) < ***.\]
ON THE RATIONALITY OF $SU(r, d)$

Now we by reordering $i$ if necessary, we can assume $h^0(C, K^* \otimes L_i) \geq 2$ and $\alpha = 2$. $h^0(C, K^* \otimes L_i) = 1$ for $i \neq 1$. We have two case. Case i) $d=2$ and hence $k=1$. Case ii) $d=1$. For case i, the dimension of $K$ is zero. Furthermore, the dimension of $Q$ is zero because there is only one fixed map from $K$ to $W$ up to automorphisms of $W$. Finally, the dimension of elementary transformations is

$$r(r(g-1)+d-2) < ***.$$

For case ii, dimension $K$ is again zero, but there is a one dimensional family of maps from $K$ to $L_1$ up to automorphisms of $W$, and hence, there is a one dimensional family of $Q$. We add the dimension of elementary transformations to get

$$1 + r(r(g-1)+d-2) < ***.$$

Now assume $3 \leq \alpha \leq 2g - 2$. By Clifford’s Theorem, $h^0(C, K^* \otimes L_i) \leq \frac{\alpha}{2} + 1$. By Marten’s Theorem the dimension of $K$ with $h^0(C, K^* \otimes L_i) \leq \frac{\alpha}{2} + 1 - a$ equals $2a$ for $a$ non-negative. Now assume all $L_i$ have $h^0(C, K^* \otimes L_i) \leq \frac{\alpha}{2} + 1 - a$. The dimension of $K$ is $2a$. The dimension of $Q$ is $\leq kd(\frac{\alpha}{2} + 1 - a - d) + c(\frac{\alpha}{2} + 1 - a - c) + 2a$ or:

$$(kd + c)(\frac{\alpha}{2} + 1 - a - d) + cd - c^2 + 2a.$$

This is clearly maximal if $a = 0$, as $kd + c = r + 1 > 2$. Now adding the dimension of the elementary transformations and rearranging the terms we get:

$$r^2(g-1) + cd - c^2 + \frac{\alpha}{2}(r + 1) - r\alpha - d + r + 1.$$
To show this is less than *** it suffices to show:

\[
\frac{\alpha}{2} (r + 1) - r\alpha - d + r + 1 < 1
\]

\[
\frac{-(r - 1)}{2} \alpha - d + r + 1 < 1
\]

\[
-\frac{3}{2} r + \frac{3}{2} - d + r + 1 < 1
\]

\[
\frac{-r}{2} - d + \frac{5}{2} \leq \frac{1}{2}
\]

\[\square\]

§3 The Main Theorem

Now we are prepared to prove the Main Theorem.

Proof of Main Theorem part A. Suppose \(ed = r - 1\). By Proposition 1 and repeated use of Proposition 2, for a general \(E_t \in SU\) we have a sequence:

\[
0 \to W \to E_t \to A \to 0
\]

for a fixed line bundle \(A\). Now the family of \(E_t\) is given by elements of \(\text{Ext}^1(W, A)\) modulo the automorphism of \(W\). \(\text{Ext}^1\) is given by \(H^0(C, \omega_C \otimes W^* \otimes L)^*\). This decomposes into a direct sum of \(H^0(C, \omega_C \otimes V_i^* \otimes A)^*\), which in turn is isomorphic to a direct sum of \(H^0(C, \omega_C \otimes A \otimes L_i)^* \otimes U_i^*\). Concentrating on one \(i\) we can see that the quotient by the automorphisms of \(U_i\) gives a Grassmannian. So the whole thing is just a product of Grassmannians.

Now for \(ed = r + 1\), Proposition 3 shows that for a general \(E_t \in SU\), we have an exact sequence

\[
0 \to A \to W \to E_t \to 0,
\]

for some fixed bundle \(A\). So the family is parameterized by \(\text{Hom}(A, W)\) modulo the automorphisms on \(W\). We can decompose and get a direct sum of \(H^0(C, L^* \otimes L_i) \otimes U_i\)
modulo the automorphisms of $W$ — but again, this is just a product of Grassmannians. And we are done. □

To prove part B we need a result of Boden and Yokogawa.

**Definition.** A variety $X$ is stably rational of level $k$ if $X \times \mathbb{P}^k$ is rational.

**Theorem 3 (Boden and Yokogawa [3]).** If $(r, d) = 1$ then $SU(r, d)$ is stably rational of level $r - 1$

Now we prove part B.

**Proof of Main Theorem part B.** As in the argument for part A, using Propositions 1 and 2 repeatedly we get a sequence for general $E_t$,

$$0 \to W \to E_t \to F_u \to 0,$$

where $F_u$ is stable. This means a general $E_t$ is an extension of a stable bundle $F_u$ by $W$. So for a general $F_u$ and a general extension by $W$ we get $E_t \in SU$. The map to $SU$ is not one to one, but its fiber is just the automorphisms of $W$ since for general $E_t$, $W$ is a unique subbundle.

So now we need to construct the map more carefully, then take care of the fiber. First note that by the hypothesis the rank and degree of $F_u$ (which is $(\delta, r(g - 1) + d)$) are coprime because all factors of $\delta$ divide $r$ and not $d$. So now let $SU' = SU(\delta, r(g - 1) + d)$. $SU'$ is a fine moduli space with a Poincare bundle $\mathcal{P}$ over $SU' \times C$. We also have projections $p_1$ and $p_2$ onto $SU'$ and $C$ respectively. Now we will form a bundle parameterizing extensions of $F_u \in SU'$ by $W$. Consider

$$p_2^*p_2^*(p_1^*(W^* \otimes \omega_C) \otimes \mathcal{P})^*.$$

The general fiber is $H^0(C, W^* \otimes \omega_C \otimes F_u)^*$ which is canonically $\text{Ext}^1(F_u, W)$. It is also canonically $\oplus H^0(C, L_i^* \otimes \omega_C \otimes F_u)^* \otimes U_i^*$. By taking an affine open subset on $SU'$, we can
assume the bundle is trivial, and that we have a universal extension space parameterized by the total space of the bundle, which by restricting to a smaller affine open subspace of $SU'$ is trivial and hence has no higher cohomology which satisfies the conditions of [Lemma 2.4, 7]. And therefore this gives a dominant map from the bundle over $SU'$ to $SU$ which has fiber isomorphic to $\text{Aut}(W)$. But clearly the action of $\text{Aut}(W)$ on the bundle over $SU$ gives a product of Grassmannians cross $SU'$. The dimension of the product of Grassmannians is the rank of the bundle minus $ed^2$. For the rank of the bundle we have

$$h^0(\omega_C \otimes L_i^* \otimes F_u) = \delta(g - 1) + (r - \delta)(g - 1) + d + 2(g - 1)\delta - \delta(g - 1).$$

This is $(r+\delta)(g-1)+d$. Now to get the rank of the full bundle we multiply by $e$ the number of $L_i$, and $d$ the rank of $U_i$. But $ed = r-\delta$, so we get $(r+\delta)(r-\delta)(g-1)+ed^2$. Now $\text{Aut}(W)$ has dimension $ed^2$. So the product of Grassmannians has dimension $(r+\delta)(r-\delta)(g-1) \geq r-1$. Now by Boden Yokogawa $SU'$ cross the product of Grassmannians is rational. And therefore so is $SU$. □

**Remark 3.** The condition that $ed = r - \delta$ where all prime divisors of $\delta$ are divisors of $r$ is necessary. Suppose to the contrary that a prime $p$ divides $\delta$ but $p$ does not divide $r$, then we can solve the equation $r(g - 1) + d \equiv 0 \pmod{p}$. The rank and degree would then not be relatively prime for those $g$ which solve the equation.

**Remark 4.** Solutions for $g$ that satisfy the above remark are the only $g$ where rationality can fail, because otherwise $(r(g - 1) + d, \delta) = 1$ and the proof goes through. For a given $g$, it is enough if $(r(g - 1) + d, \delta) = 1$ for one possible $\delta$. In the case of $r = 13$ and $d = 5$ or $d = 8 = (13 - 5)$ (the first case where $(r,d) = 1$ but the hypothesis of the Main Theorem does not hold) we get three equations. First the Boden Yokogawa equations $(g,5) = 1$ and $(g,8) = 1$. The Chinese Remainder Theorem gives $g \equiv 0 \pmod{10}$. Now we let $e = 2$ so $\delta = 13 - 2(5) = 3$ and we get $13(g - 1) + 5 \equiv 0 \pmod{3}$. The final equation is then

$$g \equiv 20 \pmod{30}.$$
We shall have more to say about this in the next section, after making a conjecture.

§4 Conjectures

Inspired by the Main Theorem, it seems natural to prove that $SU$ is rational if $ed = r + \delta$ where $\delta$ has the property that any prime divisor of $\delta$ is a prime divisor of $r$, and $0 < d < r$.

We attempt a proof.

We begin by constructing the universal hom space over $SU$ and $\{W\}$. It might have more than one irreducible component, corresponding to bundles $E_t$ where $h^0(C, W^* \otimes E_t)$ is large because $h^1$ is large. But for a general bundle we have the above $h^1 = 0$. So we get one irreducible component which dominates $SU$. Now given a morphism $W \rightarrow E$. We get a morphism $K_{E,W} \rightarrow W$. We would like $K_{E,W}$ to be stable. And given $SU' = SU(\delta, -r(g - 1) - d)$, there is a dominant map from $SU$ to $SU'$. This would imply that $\delta < r$ since if it is bigger than $r$ the map $SU$ to $SU'$ cannot dominate for dimension reasons. And if $\delta = r$ we get birationally isomorphic spaces $SU$ and $SU'$.

But how do we know $K_{E,W}$ is stable for a general $E$ when $0 < \delta < r$? The problem is we do not! So for now, suppose that $K = K_{E,W}$ is stable. For a general $K$ we have $h^1(C, K^* \otimes W) = 0$. So we calculate the dimension of the irreducible component of the universal Hom space containing $\{K|h^1(C, K^* \otimes W) = 0\}$. First we count the dimension of $\text{Hom}(K, W)$.

$$h^0(C, K^* \otimes W) = \delta ed\left(\frac{r(g - 1) + d}{\delta} - (g - 1)\right)$$

$$= ed(r - \delta)(g - 1) + ed^2$$

$$= (r^2 - \delta^2)(g - 1) + ed^2$$

Adding $\dim U' = \dim U(\delta, -r(g - 1) - d) = (\delta^2(g - 1))$, we get $r^2(g - 1) + ed^2$. This is the dimension of $U$ plus the dimension of Aut $W$. So we get a dominant map from the
irreducible component of the Universal Hom space dominating $SU'$ to $SU$ and the general fiber is $\text{Aut}(W)$.

Since $(\delta, -r(g-1)-d) = 1$ because all prime divisors of $\delta$ divide $r$, we can then restrict to an open set of $SU'$ and construct the Universal Hom bundle by taking the pullback of the pushdown of the Poincare bundle $P$ tensored by the pullback of $W^*$. This bundle is locally trivial and has fibers $H^0(C, E \otimes W^*)$ which equals the direct sum $\bigoplus H^0(C, E \otimes L_i^*) \otimes U_i$. Now the action of Aut$(W)$ gives $SU'$ cross a product of Grassmannians. Now $SU$ is rational because of Boden Yokogawa and the fact that the dimension of the product of Grassmannians is $\geq r-1$.

**Conjecture 1.** Let $C$ be a smooth projective irreducible curve over $\mathbb{C}$. If $(r, d) = 1$ and $d$ divides $r + \delta$ where $0 < \delta < r$ and every prime divisor of $\delta$ divides $r$, then $SU = SU(r, r(g-1)+d)$ is rational.

As noted in the above plausibility argument, Conjecture 1 would follow from the stability of the bundle $K_{E,W}$ and the vanishing of $h^1(C, K_{E,W}^* \otimes W)$

But it is actually enough to show the vanishing of one $h^1(C, K_{E,W}^* \otimes W)$ provided $h^1(C, W^* \otimes E) = 0$. This is because we would then get a family of $K_{E,W}$ for which $h^1(C, K_{E,W}^* \otimes W) = 0$ generically. The dimension of that component of the Universal Hom is, by the calculations preceding Conjecture 1, the dimension of the family $+(r^2 - \delta^2)(g-1) + ed^2$. Now this adds up to $r^2(g-1) + 1 + ed^2$ (the dimension of Hom$(W, E)$) if and only if the dimension of the family is $\delta^2(g-1)+1$. But deg $K_{W,E}$ is fixed, and $\mu_{\text{max}} < 0$ because it is a subbundle of $W$. This means there are only finitely many HN types. But the dimension of a parameter space for unstable bundles is less than $\delta^2(g-1)+1$. Hence, $K_{W,E}$ is generically stable. So it would suffice to prove the following conjecture in order to prove Conjecture 1.

**Conjecture 2.** Let $C$ be as above. There is an $E \in SU$ such that $h^1(C, W^* \otimes E) = 0$, $E$
is spanned by $W$, and $h^1(C, K_{W,E}^* \otimes W) = 0$.

We can prove Conjecture 2 if $e = 1$ or 2.

**Proposition 4.** Conjecture 2 holds if $e = 1$ or 2.

To prove this we need a Lemma belonging to folklore. By an elementary transformation of a bundle $E$ we mean the kernel, $E_p$, of a sequence.

$$0 \rightarrow E_p \rightarrow E \rightarrow \mathcal{O}_p \rightarrow 0,$$

where $p$ is a reduced point on $C$. This corresponds to blowing up a point $p \in \mathbb{P}(E)$ and taking the pushdown of the fiber. Corresponding to a point on $\mathbb{P}(E)$ there is a $r-2$ plane in $\mathbb{P}(E_p)$ and this corresponds to a point in $\mathbb{P}(E_p^*)$. If we blow up this point, blow down the fiber, and then dualize we get $E$ back. This is known as an inverse elementary transformation.

**Lemma 1.** Let $C$ be as above. For a general bundle $E \in U(r, d)$ or $E \in SU(r, d)$ a general elementary transformation is stable and so is a general inverse elementary transformation.

**Proof of Lemma 1.** Suppose to the contrary that the general $E_p$ and hence all $E_p$ are unstable. $E_p$ has fixed degree $(d-1)$ and $\mu_{max}(E_p) < \frac{d}{r}$ because $E_p$ is a subbundle of the stable bundle $E$. This means there are finitely many HN types.

Now consider the parameter space of a general $E_p$ with it’s universal bundle $\mathcal{E}$. Take $X = \mathbb{P}(\mathcal{E})$. A point on $X$ is an elementary transformation of a bundle $F^*$ which when dualized gives a bundle $E \in U(r, d)$. If the map to $U(r, d)$ does not dominate then the Lemma is proved. So assume the map dominates. The dimension of $X$ is the dimension of the parameter space $+r - 1$. Which equals (by Sundaram) $r^2(g-1) - r + 1$ for the parameter space $+r - 1 = r^2(g - 1)$. That means the fiber for a general bundle $E \in U(r, d)$ consists of finitely many points. But those points represent all the elementary transformations of $E$ which are not stable (because every such transformation has an
inverse). The conclusion is a general elementary transformation of a stable bundle is stable. This proves the Lemma. □

Proof of Proposition 4. In the case \( e = 1 \) there is only one linebundle \( L_1 \). Twisting by \( L_1^* \) we can assume the bundle is \( \mathcal{O}_C \) and \( W = H^0(C, E) \otimes \mathcal{O}_C \). We will now construct a bundle for \( d = 2r \) then work our way down in degree. Let \( A \) be a general line bundle of deg \( g + 1 \). \( A \) is non special and spanned. So let \( E \) be the direct sum of \( r \) copies of \( A \). \( K^*_{E,W} = E \) and so it is nonspecial. The only problem is that the original bundle \( E \) is not stable. We can choose a one parameter family of stable bundles \( E_t \) degenerating to \( E_0 = E \). A general \( E_t \) will be stable and spanned and so will have a bundle \( K_{E_t,W} \). So taking a possibly smaller family we parameterize a family of \( K^*_{E_t,W} \) with a general bundle nonspecial. This finishes the case \( d = 2r \).

To get the case \( d = 2r - 1 \) we use elementary transformations. We can assume that a general \( E_t \in M_r = U(r, rg + r) \) has \( K^*_{E,W} \) non special. Furthermore, a general elementary transform of \( E_t \) is stable by Lemma 1. And for \( F \in M_{r-1} = U(r, rg + r - 1) \), a general inverse transformation is stable by Lemma 1, and a general bundle \( F \) is spanned. Putting these together shows that there is a pair \( E \in M_r \) and \( E_p \in M_{r-1} \). Where \( E \) is spanned with \( K^*_{E,W} \) nonspecial and \( E_p \) an elementary transformation of \( E \) with \( E_p \) spanned. We have an exact sequence

\[ 0 \to E_p \to E \to \mathcal{O}_p \to 0. \]

There is a second exact sequence

\[ 0 \to H^0(C, E_p) \otimes \mathcal{O}_C \to H^0(C, E) \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0. \]

The second sequence surjects onto the first and the kernel is

\[ 0 \to K_{E_p,W_p} \to K_{E,W} \to \mathcal{O}_C(-p) \to 0. \]

\( W_p \) is just \( H^0(C, E_p) \otimes \mathcal{O}_C \). And by dualizing the last sequence and taking cohomology we see that \( K^*_{E_p,W_p} \) is nonspecial. Repeating this argument proves the case \( e = 1 \).
For the case $e = 2$ there are line bundles $L_1$ and $L_2$ of deg 0. Now do the case where $d = r$. Choose a general line bundle $A$ with $\deg(A) = g$ so that $A \otimes L_i^*$ is not special for $i = 1$ or 2. And choose $A$ such that the divisors $|A \otimes L_1|$ and $|A \otimes L_2^*|$ have no points in common. Now let $E$ be the direct sum of $r$ copies of $A$. It is easily verified that $h^1(C, K_{E,W}^* \otimes L_i) = 0$ for $i = 1$ or 2. This proves the case $d = r$. The remaining cases are proved as above using elementary transformations. □

Remark 5. The case $e = 1$ has no impact on Conjecture 1. The case $e = 2$ seems to but does not. For suppose that $2d = r + \delta$ where every prime factor of $\delta$ divides $r$. Then $2(r - d) = r - \delta$ where $\delta$ has the above properties. So the result is already known because $SU(r, r(g - 1) + d) \cong SU(r, r(g - 1) + r - d)$.

However the case $e = 1$ may have some use. Throughout this paper we assumed the choice of bundles $\{L_1, \ldots, L_e\}$ was of arbitrary distinct bundles. But we never needed that fact (outside of distinctness). So consider $L_1$ such that $L_1^{\otimes i}$ is trivial iff $i$ is a multiple of $e$.

Now set $L_i = L_1^{\otimes i}$. The set of $L_i$ corresponds to a cyclic etale cover $\tilde{C}$ of $C$. The $L_i$ pull back to be trivial, so $\tilde{W}$ the pullback of $W$ is trivial. On $\tilde{C}$, a general $K_{E,W}^*$ is nonspecial. So we expect (but have not proven!) that for general $\tilde{E}$ a pullback of $E$ on $C$, $K_{E,W}^*$ is nonspecial. Proving this would prove Conjecture 2 and hence Conjecture 1. Because then $K_{E,W} \otimes L_i$ would be nonspecial for all $i$.

To prove this last conjecture it is required to prove that a general $\tilde{E}$ is spanned. That much we can do. But unfortunately we work over $C$ and not $\tilde{C}$. A new proof of the Second Theorem may lead to a proof of the above.

Now we prove the Second Theorem.

Proof of Second Theorem. A cyclic cover of degree $e$ (where $p$ does not divide $e$) corresponds to a subgroup of $e$ torsion points of the Jacobian. Or in other words a set:

$$\{L, L^{\otimes 2} \ldots L^{\otimes (e-1)}, L^{\otimes e}\},$$
with $L^\otimes i = \mathcal{O}_C$ iff $i = e$. Setting the above set of line bundles equal to $L_1, L_2, \ldots L_e$, and noting that the sections of $\pi^*E$ are the pullback of $W$, we then get the Second Theorem from Proposition 3. □

**Remark 6.** Assuming the conjecture on the stability of $K_{E,W}$ there are more equations (as in Remark 4) that $g$ must solve. We have $r = 13$ and $d = 5$ or 8. The case $e = 2$ gives nothing new (as in Remark 5). But for $e = 3$ we have $3 \times 8 = 24 = 13 + 11$ which gives the equation $13(g - 1) + 8 \equiv 0 \pmod{11}$ or

$$g \equiv 8 \pmod{11}.$$

But $e = 3$ and $d = 5$ gives nothing new. However, for $e = 4$ and $d = 5$ we get $20 = 13 + 7$ so $13(g - 1) + 5 \equiv 0 \pmod{7}$. This reduces to

$$g \equiv 6 \pmod{7}.$$

Taking these equations, the previous equations, and using the Chinese Remainder Theorem we get

$$g \equiv 2, 120 \pmod{2, 310}.$$

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