On Pure and (approximate) Strong Equilibria of Facility Location Games

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Abstract

We study social cost losses in Facility Location games, where n selfish agents install facilities over a network and connect to them, so as to forward their local demand (expressed by a non-negative weight per agent). Agents using the same facility share fairly its installation cost, but every agent pays individually a (weighted) connection cost to the chosen location. We study the Price of Stability (PoS) of pure Nash equilibria and the Price of Anarchy of strong equilibria (SPoA), that generalize pure equilibria by being resilient to coalitional deviations. A special case of recently studied network design games, Facility Location merits separate study as a classic model with numerous applications and individual characteristics: our analysis for unweighted agents on metric networks reveals constant upper and lower bounds for the PoS, while an \( O(\ln n) \) upper bound implied by previous work is tight for non-metric networks. Strong equilibria do not always exist, even for the unweighted metric case. We show that \( e \)-approximate strong equilibria exist (\( e = 2.718... \)). The SPoA is generally upper bounded by \( O(\ln W) \) (\( W \) is the sum of agents’ weights), which becomes tight \( \Theta(\ln n) \) for unweighted agents. For the unweighted metric case we prove a constant upper bound. We point out several challenging open questions that arise.

1 Introduction

Modern computer and communications networks constitute an arena of economic interactions among multiple autonomous self-interested entities (network access providers, end-users, electronic commerce enterprises, content storage and distribution enterprises). The internet is perhaps the most massive and global field where economic networking interactions take place. The recently established study of network formation games [14](chapter 19) aims at understanding how the competitive (or coalitional) activity of multiple such selfish agents affects the network’s characteristics and its efficiency. In this paper we consider the setting of \( n \) Content Distribution Network enterprises interacting over a network. Enterprise \( i \) takes decisions on where to store replicas of digital content over the network, so as to satisfy access demand of local customers, situated at node \( u_i \). Demand is expressed by a non-negative weight \( w_i \) for enterprise \( i \). Every enterprise chooses strategically a location \( v \) on the network for installation of content replicas, so as to minimize its individual expenses for (a) storage/management of the content, and (b) weighted connection (bandwidth/delay) costs to the chosen location. We study a Facility Location game played among selfish agents (enterprises), that naturally models this situation. We use a fair cost allocation rule - known as Shapley cost-sharing [3] - for facility installation cost

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\( \beta_v \) is shared in a fair manner among agents that receive service from \( v \), so that agent \( i \) pays 
\[ w_i \frac{\beta_v}{W(v)}, \]
where \( W(v) \) denotes the total demand forwarded to \( v \). The weighted connection cost is given by 
\[ w_i d(u_i,v), \]
where \( d(u_i,v) \) denotes connection cost per unit of demand.

Our work focuses on bounding the social cost of cost-efficient and stable network infrastructures; 
stable networks will be represented by pure strategy Nash equilibria and strong equilibria of the 
corresponding Facility Location game. The social cost will be the sum of individual costs experienced by the agents-players of the game. Strong equilibria are an extension of pure Nash equilibria, 
discussed in \([2, 9]\), and essentially introduced by Aumann in \([5]\). Strong equilibria are resilient to 
coalitional pure deviations: a strategy profile is a strong equilibrium if no subset of agents can 
deviate (by jointly adopting a different pure strategy), so that all of its members are better off. 
Thus this notion captures the possibility of coalitional behavior among agents. Occurrence of such 
behavior is naturally expected in rapidly evolving modern markets, especially the ones involving 
the exploitation and exchange of digital goods and services.

We derive bounds on the Price of Stability (PoS) of pure equilibria, defined as the cost of 
the least expensive equilibrium relative to the socially optimum cost \([3]\). For strong equilibria we 
derive bounds on their (strong) Price of Anarchy (SPoA), i.e. the cost of the most expensive strong 
equilibrium relative to the socially optimum cost. Let us note that the Price of Anarchy of pure 
Nash equilibria was introduced in \([12]\) as the cost of the most expensive pure Nash equilibrium 
relative to the socially optimum cost, and measures essentially social cost losses incurred due to 
selfishness of agents, and lack of coordination among them. The PoS on the other hand measures 
social cost losses only due to selfishness: if all players coordinate (even by following some externally 
provided instructions, or by interacting on the basis of a common protocol) they may reach the 
least expensive equilibrium. The notion of strong equilibria inherently permits coordination among 
subsets of agents. Therefore we can intuitively expect the SPoA to almost match the PoS. Our 
results confirm this intuition.

The study of the price of stability in network design games with fair allocation of network 
link costs was initiated in \([3]\). One challenging problem remaining open since then is improving 
upon an upper bound of \( \text{PoS} = O(\log n) \) and a lower bound (recently shown in \([10]\)) of 
\( \frac{12}{7} \), for unweighted players on undirected networks. A series of recent works \([10, 6, 7, 9, 1]\) provides results 
(polylogarithmic upper bounds) with respect to the social cost of pure Nash and (approximate) 
strong equilibria in the network design game model of \([3]\), also for the case of weighted players. We 
review these results in section 2. We also explain how the Facility Location game that we study 
is a special case of the more general model introduced in \([3]\), that also includes “delay” costs in 
using network links (referred to as connection costs in the case of Facility Location), apart from 
fairly allocated “installation” costs. For the case of metric connection costs we were able to prove 
greatly improved and almost tight bounds for the PoS and the SPoA; this makes the metric Facility 
Location game and exceptional special case of the model introduced in \([3]\).

**Summary of Results** We analyze the PoS of the unweighted metric Facility Location game 
(section 4), and prove constant upper and lower bounds. Note that an \( O(\ln n) \) general upper 
bound implied by the work of \([3]\) is tight for non-metric networks (we discuss this in section 3). 
Our technique relies on direct analysis of social cost evolution during an iterative best response 
performed by the agents, until they reach equilibrium. We show in particular that given any 
initial configuration (including the social optimum), the social cost of the reached equilibrium 
is at most 2.36 times the initial social cost and at least 1.45 times this cost in the worst case. 
Metric Facility Location is the first case of the model proposed in \([3]\), in which an additional 
structural network property (triangle inequality) yields a constant almost tight Price of Stability.
Table 1: Summary: Facility Location Games with Fair Allocation of Facility Costs.

|                | Metric Unweighted | Non-metric Unweighted | Weighted |
|----------------|-------------------|-----------------------|----------|
| PNE | SE                | √ | e-apx            | e-apx | e-apx |
| PoS          | ∈ ([1.45, 2.36]) | Θ(ln n)               | O(ln W) |
| SPoA         | O(1)              | Θ(ln n)               | O(ln W) |

It is also interesting that incorporation of link delays (connection costs) makes the previously derived $O(H(n))$ upper bounds tight for the non-metric unweighted game. Subsequently we study strong equilibria (section 5). Although they do not always exist, we prove that $\alpha$-approximate strong equilibria exist (no subset deviation causes factor $\alpha$ improvement to all of its members), for $\alpha \geq e = 2.718\ldots$ in general networks and weighted agents. For the Price of Anarchy of approximate strong equilibria (SPoA) we show an $O(\ln W)$ general upper bound which becomes $\Theta(\ln n)$ for unweighted agents (section 6). For the metric unweighted case we prove a constant upper bound (section 6.1).

Except for the SPoA analysis in paragraph 6.1 (theorem 4), the rest of the described results have appeared in [11], in a shorter version. We note that the analysis of the SPoA of approximate strong equilibria for the metric unweighted case in paragraph 6.1 essentially generalizes the analysis of the corresponding result appearing in [11] (theorem 2); this result concerned the SPoA of exact strong equilibria only when they exist.

2 Related Work

Anshelevich et al. first studied the Price of Stability for network design games with general [4] and fair cost allocation [4]. In the latter case $n$ agents wish to connect node subsets over a network, by strategically selecting links to use. Every network link is associated to two cost components, an installation cost and a delay cost. Link installation cost is shared fairly among agents using the same link in [3], while every agent experiences a delay cost given by a polynomial function of the number of agents using the link. The authors showed that for unweighted agents these network design games belong to the class of potential games introduced by Monderer and Shapley in [13], hence they have pure strategy Nash equilibria. In particular, a potential function defined in [13] is associated to these games in the following manner: an arbitrarily initialized iterative best response procedure carried out by the players reaches a local minimum of the potential function, which corresponds to a pure strategy Nash equilibrium of the game. In [3] the authors developed elegant arguments using the potential function, so as to upper bound the PoS for unweighted agents. In case of polynomial delay costs of degree at most $k$ the PoS was shown to be at most $O((k+1) \ln n)$. For the case of directed networks this bound was shown to be tight (for $k=0$ and zero delays), whereas for undirected networks a lower bound of $\frac{4}{3}$ was given. This was recently improved to $\frac{12}{7}$ by Fiat et al. [10], who also showed an upper bound of $O(\log \log n)$ for PoS, when exactly one agent resides on each node of the network and all agents need to connect to a common sink node (single-sink case). For weighted agents a potential function proving existence of pure equilibria was developed for only 2 agents in [3]. Chen et al. [7] showed that pure strategy Nash equilibria do not always exist for weighted network design games with fair allocation of link installation costs. They

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1A result of 2.36-approximate strong equilibria that appeared in a previous version of this report was erroneous and has been removed.
developed a trade-off for the social cost of approximate equilibria versus the approximation factor, as a function of the maximum weight taken over all agents.

An extensive study of strong equilibria for network design games with fair cost allocation (but without delays) appeared recently by Albers in [1], for unweighted and weighted agents. Strong equilibria do not always exist, but she showed that $O(H(n))$ and $O(\ln W)$-approximate strong equilibria do exist ($W$ is the sum of the agents’ weights) for unweighted and weighted games respectively. She proved $O(H(n))$ and $O(\ln W)$ upper bounds for the SPoA of approximate strong equilibria which are tight for directed graphs. For undirected graphs she showed $\Omega(\sqrt{\log n})$ and $\Omega(\sqrt{\log W})$ lower bounds. Finally and most importantly, she showed an $\Omega(\frac{\log W}{\log \log W})$ lower bound for the PoS of weighted network design games. Strong equilibria in the context of (single-sink) unweighted network design games with fair cost sharing were first studied in [9]. The authors gave topological characterizations for the existence of strong equilibria on directed networks, and proved that $\text{SPoA} = \Theta(\log n)$.

As of the recent literature, the gap for the identification of the PoS of network design games with fair cost allocation remains open for unweighted agents, with the lower bound of $\frac{12}{7}$ being the best known so far, to the best of our knowledge. In effect, this means that the impact of an undirected network structure to the PoS is not well understood, since the upper bounding potential function arguments developed in [3] do not incorporate network structure. The Facility Location game is a special case of the model studied in [3], that is interesting on its own right: it finds numerous applications and exhibits intriguing characteristics. It embodies non-shareable delays explicitly and in a sense specializes single-sink network design considered in [9, 7]: we can simply augment the network with a node $t$ and set links $(v, t)$ to have fairly shareable cost, equal to the facility opening cost at $v$. The original network links have a delay cost only. Then every agent needs to choose at most two edges from the node it resides on, to $t$.

**Facility Location Games** An unweighted metric facility location game with uniform facility costs was studied in the context of selfish caching in [8]. The network model of that work is essentially equivalent to the one we discuss here, apart from the fact that fair cost-sharing of facility costs was not used: an agent could connect to a facility paid exclusively by another agent. Another difference is that facility opening cost was uniform across all nodes of the network, whereas we consider node-dependent costs. The PoA and PoS of this model were shown to be unbounded. The authors devised an extension of their game model, with payments exchanged among agents, in which the socially optimum configuration of the original game is rendered a pure strategy Nash equilibrium (hence $\text{PoS} = 1$ for the extended game). Vetta studied a class of games for competitive facility location [15], for which he proved existence of pure strategy Nash equilibria and an upper bound of 2 for the PoA. In this competitive setting enterprises open facilities at certain nodes and try to attract customers to connect to them. He also illustrated applicability of his model in the context of the $k$-median problem model. In [6] the authors considered single-sink network design with fair cost-sharing of all resources (network links) used by agents. They argued how this model can be viewed as a Facility Location model, but did not take connection costs (non-shareable delays) into account.

### 3 Definitions and Preliminaries

The network will be a complete graph $G(V, E)$, each edge $(u, v)$ of which is associated to a non-negative cost $d(u, v)$. We consider a set $A$ of $n$ agents. Each agent $i$ resides on a node $u_i \in V$, and is associated to a non-negative demand weight $w_i$. The strategy space of agent $i$ is $V$: $i$ chooses
Thus the socially optimum cost is $\beta v^2 n$ for facility costs equal to 1. Fig. 1 presents a non-metric lower bounding example for unweighted agents. Assume uniform of a single facility opened on $u$. $v$ indicates the impact of the network’s structure (triangle inequality) to the social cost of efficient networks however, the PoS is constant as we show in the following section. This result essentially says that the PoA of pure equilibria can be $\alpha n$. Every agent $i$ pays a fraction $\frac{w_i}{W(s)}$ of the facility installation cost at $s_i$. Denote by $F_s \subseteq V$ the set of facility locations specified under $s$. The social cost $c(s)$ is then:

$$
c(s) = \sum_i c_i(s) = \sum_i w_i d(u_i, s_i) + \sum_i \frac{w_i \beta s_i}{W(s)} = \sum_i w_i d(u_i, s_i) + \sum_{v \in F_s} \beta v
$$

We use $W(I)$ for the sum of weights of agents in set $I$. By $c_i(s)$ we denote the total cost of agents in $I$ under $s$. Furthermore, given a facility $v \in F_s$, we use $c_v(s)$ to denote $\sum_{s_i = v} c_i(s)$.

**Pure Nash Equilibria** The unweighted Facility Location game specializes network design games first studied in [3], and is a potential game [13]; it is associated to a potential function, that can be locally minimized by iterative best response performed by players, and the reached local minimum corresponds to a pure Nash equilibrium. In iterative best response we choose iteratively an arbitrary player $i$, and let him/her decide a strategy that minimizes its individual cost with respect to the current configuration $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ for the rest of the players. The process ends when no player $i$ can improve his/her individual cost under $s_{-i}$. Then $s$ is a pure Nash equilibrium.

A simple example shows that the PoA of pure equilibria can be $n$ in the worst case. Take a network of two nodes $u, v$. Opening a facility on either node has a cost $\beta$. We set $d(u, v) = \frac{(n-1)\beta}{n}$, and let $n$ agents reside on $u$. Then if every agent $i$ plays $s_i = v$, $i = 1 \ldots n$, the resulting configuration is a pure Nash equilibrium of cost $n\beta$, because every agent pays exactly $\beta$ and has no incentive to open a facility at node $u$ by paying the same cost $\beta$. The social optimum consists of a single facility opened on $u$ and all agents being locally serviced by it, at zero connection cost. Thus the socially optimum cost is $\beta$.

Potential function arguments developed in [3] yield an upper bound of $H(n)$ for the PoS, where $H(n)$ is the $n$-th harmonic number. This bound is tight for unweighted non-metric Facility Location. Fig. 1 presents a non-metric lower bounding example for unweighted agents. Assume uniform facility costs equal to 1. $2n$ unweighted agents reside on boldly drawn nodes, $n$ of them on $v_2$. The social optimum consists of facilities on $v_1$ and $v_2$, where $v_2$ serves only the $n$ agents residing on $v_2$. The rest are served by $v_1$. However they deviate to $v_2$ from $v_1$ one by one. It is then: $\text{PoS} \geq (2H(n) - H(2n))/(2 + n \epsilon) \geq (\ln n - \ln 2 - 1)/(2 + n \epsilon) = \Omega(\ln n)$. When it comes to metric networks however, the PoS is constant as we show in the following section. This result essentially indicates the impact of the network’s structure (triangle inequality) to the social cost of efficient pure Nash equilibria.

**Definition 1 (Strong Equilibria) [2, 5]** A strategy profile $s$ is a strong equilibrium if no subset of agents can deviate in coordination, so that each and every one of its members is better off. It is an $\alpha$-approximate strong equilibrium if no subset of agents can deviate in coordination, so that each and every one of its members is better off by a factor strictly more than $\alpha \geq 1$.

### 4 Unweighted Metric Facility Location: The Price of Stability

We analyze evolution of an equilibrium through iterative best response, initialized at configuration $s^*$. In fact, for any strategy profile $s$ that is reached through iterative best response initialized at $s^*$,
we will upper bound the ratio \( c(s)/c(s^*) \). Thus we do not require that \( s^* \) be the socially optimum configuration, nor that \( s \) is an equilibrium strategy profile. We only require that \( s \) is reached by iterative best response initialized at \( s^* \). Then taking \( s^* \) to be the socially optimum configuration and \( s \) to be the reached equilibrium will yield an upper bound for the Price of Stability, as a corollary. To facilitate clarity, we give a brief description of the analysis’ plan. Given any facility node \( v \in F_s^* \), denote by \( \mathcal{A}_s^*(v) = \{v_i^* = v\} \) the subset of agents connected to \( v \) in \( s^* \). Then \( c_v(s^*) \) will be the total cost incurred by \( \mathcal{A}_s^*(v) \) collectively. We will upper bound \( c(s)/c(s^*) \) as follows:

\[
\frac{c(s)}{c(s^*)} \leq \sum_{i \in F_{s^*}} \frac{\sum_{i \in \mathcal{A}_s^*(v)} c_i(s)}{\sum_{i \in \mathcal{A}_s^*(v)} c_i(s^*)} = \sum_{i \in F_{s^*}} \frac{\sum_{i \in \mathcal{A}_s^*(v)} c_i(s)}{\sum_{i \in \mathcal{A}_s^*(v)} c_i(s^*)} = \max_{i \in F_{s^*}} \sum_{i \in \mathcal{A}_s^*(v)} c_i(s) c_v(s^*)
\]

Define \( \Delta c(i) = c_i(s) - c_i(s^*) \) to be the increase in cost caused by agent \( i \) during iterative best response initialized at \( s^* \) and reaching \( s \). For any subset \( A' \subseteq A \) of agents we also use \( \Delta c(A') = \sum_{i \in A'} (c_i(s) - c_i(s^*)) \). To evaluate the upper bounding expression (1), we are going to use an upper bound on \( \Delta c(A_s^*(v)) = \sum_{i \in \mathcal{A}_s^*(v)} c_i(s) - c_i(s^*) \), valid for any \( v \in F_{s^*} \). This is the increase in social cost caused during iterative best response by agents connected to \( v \) in \( s^* \). Then for any \( v \in F_{s^*} \) we will maximize \( \frac{c_v(s^*)}{c_v(s^*)} \), by lower bounding \( c_v(s^*) \) for any \( v \in F_{s^*} \). In determining an upper bound on \( \Delta c(A_s^*(v)) \) we prove the following lemma, which charges any specific agent \( i \) a bounded amount of social cost increase during the execution of iterative best response.

**Lemma 1** Let \( \mathcal{A}_s^*(v) \) be the subset of agents that are connected to \( v \) in \( s^* \). For any \( i \in \mathcal{A}_s^*(v) \) define \( \mathcal{A}_s^i(v) \subseteq \mathcal{A}_s^*(v) \) to be the subset of agents that have not yet deviated from \( v \), exactly before the first deviation of \( i \). Then \( \Delta c(i) \leq \beta_{i_v}/|\mathcal{A}_s^i(v)| \).

**Proof.** For simplicity let \( |\mathcal{A}_s^i(v)| = k_i(v) \). Clearly \( i \in \mathcal{A}_s^i(v) \). Let us analyze contribution of \( i \) to social cost increase during its first deviation. By deviating, \( i \) reduces its individual cost from \( c_i \) to \( c_i' \), by joining another facility node \( v' \). Then:

\[
c_i = x_i(v) + \frac{\beta_{i_v}}{k_i(v)}, \quad c_i' = x_i(v') + \frac{\beta_{i_{v'}}}{\lambda_i(v')}
\]

\( x_i(v) \) and \( x_i(v') \) is the connection cost paid by \( i \) before and after its first deviation. \( \lambda_i(v') \) is the number of agents sharing facility cost at \( v' \), including \( i \). Since \( c_i(v') < c_i(v) \), it is \( x_i(v') - x_i(v) \leq \frac{\beta_{i_{v'}}}{k_i(v')} - \frac{\beta_{i_v}}{k_i(v)} \).

Let \( \Delta c_v(i) = c_i' - c_i \) be the difference caused to the social cost by this single deviation of \( i \). We examine the following cases:
1. \( k_i(v) > 1, \lambda_i(v') > 1 \): Then \( \Delta c_i(v) \leq \frac{\beta_{x_i(v')}}{k_i(v')} - \frac{\beta_{x_i(v')}}{k_i(v)} \).

2. \( k_i(v) = 1, \lambda_i(v') > 1 \): Then \( \Delta c_i(v) = -\beta_{x_i(v')} + x_i(v') - x_i(v) \leq -\frac{\beta_{x_i(v')}}{\lambda_i(v')} \).

3. \( k_i(v) > 1, \lambda_i(v') = 1 \): Then \( \Delta c_i(v) = \beta_{x_i(v')} + x_i(v') - x_i(v) \leq \frac{\beta_{x_i(v')}}{k_i(v')} \).

4. \( k_i(v) = 1, \lambda_i(v') = 1 \): Then \( \Delta c_i(v) = \beta_{x_i(v')} - \beta_{x_i(v')} + x_i(v') - x_i(v) \leq 0 \).

Clearly the above hold in general during execution of iterative best response, for any agent that performs a single deviation from a node \( v \) to a node \( v' \).

Now we implement a charging procedure along with iterative best response. Give all agents an initial label \( \ell(i) = i \), before executing iterative best response. The labels will be updated during iterative best response, so that increases caused by agent \( i \) will be charged to an appropriate agent \( \ell(i) \). Initialize \( \Delta c(\ell(i)) = 0 \) for all \( i \in A \). We will make use of the auxiliary variables \( k_i(v) \) and \( \lambda_i(v') \), as they were described before. At any time \( \lambda_i(v) \) is the number of agents (including \( i \) connected to \( v \), right after \( i \) joined \( v \)). We need to initialize \( \lambda_i(v) \) to a value for every \( v \in F_{s^*} \) and \( i \in A_{s^*}(v) \), to implement the charging scheme. Set \( \lambda_{\ell(i)} = \lambda_i \) to a distinct value from \( \{1, 2, \ldots, |A_{s^*}(v)|\} \). \( k_{\ell(i)}(v) \) will be always (during iterative best response) the number of agents connected to \( v \) exactly before deviation of \( i \) from \( v \) (before \( i \) leaves \( v \)). Charging is then implemented by relabeling deviating agents in the following manner.

1. If \( k_{\ell(i)}(v) = \lambda_{\ell(i)}(v) \) no relabeling is needed.

2. Otherwise there must be some \( j \neq i \) connected to \( v \) such that \( \lambda_{\ell(j)}(v) = k_{\ell(i)}(v) \). In this case exchange labels of \( i \) and \( j \).

Subsequently add the increase caused by deviation of \( i \) to \( \Delta c(\ell(i)) \). Finally, set \( \lambda_{\ell(i)}(v') \) equal to the number of agents connected to \( v' \) right after \( i \) has joined \( v' \).

By the previous definitions it follows that if \( k_{\ell(i)}(v) \neq \lambda_{\ell(i)}(v) \), then it is always \( k_{\ell(i)}(v) > \lambda_{\ell(i)}(v) \), i.e. \( i \) has joined \( v \) before some agent \( j \) with \( \lambda_{\ell(j)}(v) = k_{\ell(i)}(v) \), but leaves \( v \) “out of order”, i.e. before \( j \) leaves. By exchanging labels of \( i,j \) we add the increase caused by \( i \) to the agent that previously labeled \( j \). Possible increases in 1, 2, 3, 4, imply that any agent is charged by the end of iterative best response at most \( \frac{\beta_{x_i(v')}}{|A_{s^*}(v)|} \) for some \( i \). If we “guess” the exact order of first deviation of all agents, we can initialize \( \lambda_i(v) = k_i(v) \). Then each \( i \) itself will be charged at most \( \frac{\beta_{x_i(v')}}{|A_{s^*}(v)|} \).

In what follows we are going to upper bound the ratio \( \frac{\sum_{l \in A_{s^*}(v)} c_l(s)}{c_v(s^*)} \) for any \( v \in F_{s^*} \). If no agent \( i \in A_{s^*}(v) \) ever deviates from playing \( v \), then clearly it will be \( \frac{\sum_{l \in A_{s^*}(v)} c_l(s)}{c_v(s^*)} = 1 \). Our analysis focuses on two remaining cases: either \( (i) \) there is a non-empty set of agents \( A_s(v) \subset A_{s^*}(v) \) that never deviate from \( v \) during iterative best response or \( (ii) \) all agents \( i \in A_{s^*}(v) \) deviate from \( v \) during iterative best response. Case \( (i) \) is examined in proposition 1, whereas the analysis is concluded by analysis of \( (ii) \) in theorem 1.

**Proposition 1** Let \( s \) be a strategy profile reached by iterative best response initialized at strategy profile \( s^* \). For any facility \( v \in F_{s^*} \) define \( A_{s^*}(v) = \{i|s_i^* = v\} \) and let \( A_s(v) \subset A_{s^*}(v) \) the subset of agents that never deviated from \( v \), during iterative best response. If \( A_s(v) \neq \emptyset \), then \( \sum_{i \in A_{s^*}(v)} c_i(s) \leq 2.36 \cdot c_v(s^*) \).
Proof. As discussed previously, we will upper bound the ratio \( \frac{c_v(s^*)}{\Delta c(A_{s^*}(v))} \), by deriving an upper bound for \( \Delta c(A_{s^*}(v)) \) and a lower bound for \( c_v(s^*) \). By lemma 1 follows immediately:

\[
\Delta c(A_{s^*}(v)) \leq \beta_v |H(|A_s(v)|) - H(|A_s(v)|)|
\]

(2)

because agents in \( A_s(v) \) never deviated, whereas the rest caused a cost increase equal to \( \beta_v \) times a harmonic series term each. In what follows we show a lower bound on \( c_v(s^*) \).

Fix any facility \( v \in F_{s^*} \), and define \( I_{s^*}(v) = A_{s^*}(v) \setminus A_s(v) \). Without loss of generality assume such an order of agents, that agents \( i \in I_{s^*}(v) \) (and for every \( v \in F_{s^*} \) “best-respond” consecutively; e.g. assume that the algorithm scans the facilities in \( F_{s^*} \) in an arbitrary fixed order and for each facility, the agents connected to it in an arbitrary fixed order. We focus only on the first deviation of each agent \( i \in I_{s^*}(v) \) for some \( v \in F_{s^*} \). Let \( v' \) be the node that \( i \) deviates to. For convenience we use \( x_i^* = d(u_i, v) \) and let \( \delta x_i^* = d(u_i, v') - d(u_i, v) \). Let \( \lambda_i \) be the total number of agents connected to \( v' \) right after deviation of \( i \). The new cost of \( i \) right after its first deviation is: \( x_i^* + \delta x_i^* + \frac{\beta_v}{\lambda_i} \). For any other agent \( j \in A_{s^*}(v) \setminus \{i\} \) that, either deviates from \( v \) to some node \( v'' \) after the deviation of \( i \), or never deviated from \( v \) (in this case consider a trivial deviation with \( v'' = v \)), we have respectively:

\[
d(u_j, v) + \delta x_j^* + \frac{\beta_v}{\lambda_j} \leq d(u_j, v') + \frac{\beta_v}{\lambda_j}
\]

(3)

Substitute \( d(u_j, v') \) in (3) by triangle inequality: \( d(u_j, v') \leq d(u_j, v) + d(u_i, v) + d(u_i, v') \). Let \( k_i^* \) denote the number of agents connected to \( v \) right before deviation of \( i \) from \( v \). Also note that \( \delta x_i^* + \frac{\beta_v}{\lambda_i} \leq \frac{\beta_v}{k_i^*} \), because \( i \) decreases its individual cost by deviating. Thus we obtain:

\[
d(u_i, v) \geq \frac{1}{2} \left( \delta x_i^* + \frac{\beta_v}{\lambda_i} \right) \geq \frac{1}{2} \left( \delta x_i^* + \frac{\beta_v}{\lambda_i} - \frac{\beta_v}{k_i^*} \right)
\]

(4)

Because \( d(u_i, v) \geq 0 \), and because the latter has to hold for every pair of distinct agents \( i, j \in A_{s^*}(v) \), we deduce:

\[
d(u_i, v) \geq \frac{1}{2} \left[ \max_{i \in A_{s^*}(v) \setminus v} \left( \delta x_i^* + \frac{\beta_v}{\lambda_i} \right) - \frac{\beta_v}{k_i^*} \right]
\]

(5)

We use (5) for the connection cost of agents in \( I_{s^*}(v) \), and 0 for agents in \( A_s(v) = A_{s^*}(v) \setminus I_{s^*}(v) \); in essence for every \( i \in A_s(v) \) we simply set in (5) \( v'' = v \), \( k_i^* = |A_s(v)| \), and take \( \max_{i \in A_s(v) \setminus v} \left( \delta x_i^* + \frac{\beta_v}{\lambda_i} \right) = \frac{\beta_v}{|A_s(v)|} \). The cost \( c_v(s^*) \) is then:

\[
c_v(s^*) \geq \beta_v + \frac{\beta_v}{2} \sum_{k = |A_s(v)|}^{|A_{s^*}(v)|} \left( \frac{1}{|A_{s^*}(v)|} - \frac{1}{k} \right)
\]

\[
= \beta_v + \frac{\beta_v}{2} \left( \frac{|A_{s^*}(v)| - |A_s(v)|}{|A_s(v)|} - H(|A_{s^*}(v)|) + H(|A_s(v)|) \right)
\]

(6)

Using the bounds (6) and (2) we deduce the following upper bound:

\[
\left( \sum_{i \in A_{s^*}(v)} c_i(s) \right) / c_v(s^*) \leq \frac{1 + \frac{1}{2} \left( \frac{|A_{s^*}(v)| - |A_s(v)|}{|A_s(v)|} + H(|A_{s^*}(v)|) - H(|A_s(v)|) \right)}{1 + \frac{1}{2} \left( \frac{|A_{s^*}(v)| - |A_s(v)|}{|A_s(v)|} - H(|A_{s^*}(v)|) + H(|A_s(v)|) \right)}
\]

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the network. Facility opening costs are assumed least. We analyze a worst-case example, that makes the PoS for unweighted metric Facility Location at allocated facility costs is at most 2.36. Using (8) we end up with (7). This is technically equivalent to assuming that agents i with \( \ell_i \leq r \) never deviated. □

**Theorem 1** If \( s \) is a strategy profile for the unweighted Metric Facility Location game, reached by iterative best response initialized at a strategy profile \( s^* \), then \( c(s^*) \leq 2.36c(s) \).

**Proof.** We only need to complement the result of proposition 1 by considering the case where \( A_s(v) = \emptyset \). That is, \( I_{s^*}(v) = A_{s^*}(v) \), and all agents connected to \( v \) in \( s^* \) deviate during iterative best response. If for each \( i \in I_{s^*}(v) \) that deviates from \( v \) to \( v' \) (\( v' \) is not necessarily the same for every \( i \)) it is \( \delta x_i^* + \frac{\beta_i v_i}{\ell_i} \leq \frac{\beta_v}{|A_{s^*}(v)|} \), then:

\[
\sum_{i \in A_{s^*}(v)} c_i(s) = \sum_{i \in I_{s^*}(v)} \left( x_i^* + \delta x_i^* + \frac{\beta_i v_i}{\ell_i} \right) \leq \beta_v + \sum_{i \in A_{s^*}(v)} x_i^* = c_v(s^*)
\]

Thus, assume there is at least one agent \( j \) with \( \delta x_j^* + \frac{\beta_j v_j}{\ell_j} > \frac{\beta_v}{|A_{s^*}(v)|} \). In general, let \( r \) be the largest integer, so that \( \delta x_i^* + \frac{\beta_i v_i}{\ell_i} \leq \frac{\beta_v}{r} \) for all \( i \in I_{s^*}(v) = A_{s^*}(v) \). Using the same arguments that led to (5), and taking \( \max_{i} \left( x_i^* + \frac{\beta_i v_i}{\ell_i} \right) = \frac{\beta_v}{r} \), we deduce:

\[
x_i^* = d(u_i, v) \geq \frac{1}{2} \left( \frac{\beta_v}{r} - \frac{\beta_v}{k_i^*} \right), \text{ for } k_i^* \geq r \text{ and } x_i^* = d(u_i, v) \geq 0, \text{ for } k_i^* \leq r
\]

Using (8) we end up with a similar lower bound to (6) for \( 1 \leq r \leq |A_{s^*}(v)| \):

\[
c_v(s^*) \geq \beta_v + \frac{\beta_v}{2} \sum_{k=r}^{|A_{s^*}(v)|} \left( \frac{1}{k} - \frac{1}{k} \right) = \beta_v + \frac{\beta_v}{2} \left( \frac{|A_{s^*}(v)| - r}{r} - H(|A_{s^*}(v)|) + H(r) \right)
\]

To finish the proof, we note that by (8), deviation of agents with \( k_i^* \leq r \), yields a total cost increase (payed as new connection cost) of at most \( r \times \frac{\beta_v}{r} = \beta_v \), which is exactly equal to the cost saved by closing the facility at \( v \) (because all agents deviate). Thus we can simulate the situation with the case analyzed in proposition 1, by setting \( A_s(v) = \{ i \in A_{s^*}(v) | k_i^* \leq r \} \) and \( r = |A_s(v)| \), so as to end up with (7). This is technically equivalent to assuming that agents \( i \) with \( k_i^* \leq r \) never deviated. □

**Corollary 1** The Price of Stability for the unweighted Metric Facility Location game with fairly allocated facility costs is at most 2.36.

### 4.1 A Lower Bound on the Price of Stability

We analyze a worst-case example, that makes the PoS for unweighted metric Facility Location at least 1.45 asymptotically. Experimental evidence shows a lower bound > 1.77, which we believe is tight for PoS. This is because the analysis of theorem 1 embodies some losses due to bounding of harmonic numbers by logarithms.

Our construction appears in Fig. 2(a). Take \( 2n \) agents, \( n \) of them residing on a single node \( v \) of the network. Facility opening costs are assumed 1 everywhere. The social optimum \( s^* \) has 1 + \( \sqrt{n} \) facilities: \( v \) and \( v_1^*, l = 1 \ldots k \), where \( k = \sqrt{n} \). Let the \( n \) agents residing on \( v \) be serviced by \( v \) in \( s^* \), whereas the rest are equally partitioned to facilities \( v_l^*, l = 1 \ldots k \); every facility \( v_l^* \) services a
connection cost \( k \) up as in (6) yields:
\[
\sum \text{agents that deviate from } v \text{ to pay strictly less than } k \text{ to the facility at } v \text{ and will yield asymptotically the same result. In particular, by choosing to analyze any batch of } k \text{ agents that deviate from } v^* \text{ to } v, \text{ the PoS will be lower bounded as:}
\]
\[
\text{PoS} = \lim_{n \to \infty} \frac{1 + \sum_{i=1}^{k} \sum_{v \in A_{s^*}^r(v^*)} c_i(s)}{1 + \sum_{i=1}^{k} c_i(v^*)} \geq \lim_{n \to \infty} \frac{\sum_{i \in A_{s^*}^r(v^*)} c_i(s)}{(1/\sqrt{\pi} + c_{v^*}(s^*))}
\]  
(10)

For some constant \( p \in (0, 1) \) we will have \( r = \lceil (1 - p)k \rceil \) of the agents in \( A_{s^*}^v(v) \) increase their connection cost significantly by deviating, hence the social cost. For every \( i \in A_{s^*}^v(v) \) let \( x_i^* \) be the distance of \( i \) from \( v^* \), and \( \delta x_i^* \) the increase in connection cost after deviation. We determine \( x_i^* \) and \( \delta x_i^* \) by following best responses of agents in \( A_{s^*}^r(v^*) \). Let \( \lambda \geq n \) be the number of agents already connected to \( v \) before agents of \( A_{s^*}^r(v^*) \) deviate to \( v \) in order. In particular:

- For \( i = 1 \ldots r \) set \( \delta x_i^* = \frac{1}{k - r + 1} - \frac{1}{\sqrt{\lambda + 1}} - \varepsilon; \) a small decrease \( \varepsilon > 0 \) causes \( i \) to deviate to \( v \).

- For each of the remaining \( k - r \) agents set \( \delta x_i^* = \frac{1}{k - r + 1} \).

By similar arguments as for the derivation of (3),(4) and triangle inequality, we obtain \( x_i^* = d(u_i, v^*) = \max \{ 0, \frac{1}{2} (\max \delta x_i^* - \delta x_i^*) \} \), where \( \max \delta x_i^* = \frac{1}{k - r + 1} \). This yields \( x_i^* = 0 \) for \( i > r \) (the remaining \( k - r \) agents). Note that the \( k - r \) remaining agents will deviate to \( v \), because they prefer to pay strictly less than \( \frac{1}{k - r + 1} + \frac{1}{\lambda + 1} \), instead of a cost share \( \frac{1}{k - r} \) each, for the facility at \( v^* \). By deviation they “shut-down” the facility at \( v^* \) and decrease the social cost by 1, but pay a total connection cost \( \frac{k - r}{k - r + 1} \) for \( v \), which tends to 1 for large \( n \). Let us now derive the cost \( c_{v^*}(s^*) \). For convenience we use the notation \( \Delta H(p, q) \) to denote \( H(p) - H(q) \). For the first \( r \) agents, summing up as in (6) yields:
\[
c_{v^*}(s^*) = 1 + \frac{r}{2(k - r + 1)} - \Delta H(k, k - r) - \frac{r}{2k + r} \Delta H(\lambda + r, \lambda)
\]  
(11)

The cost of agents in \( A_{s^*}^r(v^*) \) after deviation is: \( \sum_{i \in A_{s^*}^r(v^*)} c_i(s) = c_{v^*}(s^*) - 1 + \frac{k - r}{k - r + 1} + \sum_{i=1}^{r} \delta x_i^* \).

By substituting \( \sum_{i=1}^{r} \delta x_i^* = \Delta H(k, k - r) - \Delta H(\lambda + r, \lambda) \), we obtain:
\[
\sum_{i \in A_s \cap \{v\}} c_i(s) = \frac{k-r}{k-r+1} + \frac{1}{2} \left( \frac{r}{k-r+1} + \Delta H(k, k-r) \right) - \frac{1}{2} \left( \frac{r}{\lambda+r} + \Delta H(\lambda+r, \lambda) \right)
\] (12)

We simplify (11) and (12) and substitute to (10) appropriately. For \(\Delta H(k, k-r)\) we use the following bounds:

\[
\gamma - 1 + \ln \frac{1}{p+1/k} \leq \Delta H(k, k-r) \leq 1 - \gamma + \ln \frac{1}{p+1/k}
\]

where \(\gamma > 0.5\) is the Euler constant. Also we use \(1-p^{-1/k} \leq \frac{r}{k-r+1} \leq 1-p\), because \((1-p)k - 1 \leq r = \lceil (1-p)k \rceil \leq (1-p)k + 1\). Then (10) becomes:

\[
\text{PoS} \geq \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n}} + \frac{1}{2} \left( \frac{r}{k-r+1} + \Delta H(k, k-r) \right) - \frac{1}{2} \Delta H(\lambda+r, \lambda)}{\frac{1}{2} \Delta H(k, k-r) + \frac{1}{2} \Delta H(\lambda+r, \lambda)} \Rightarrow
\] (13)

Notice that for large \(n\), \(\frac{r}{\lambda+r} \to 0\), because \(r = (\sqrt{n})\), and \(\lambda \geq n\). Furthermore it is \(\Delta H(\lambda+r, \lambda) \to 0\), because \(\Delta H(\lambda+r, \lambda) \leq \frac{1}{\lambda^2}\), with \(\lambda \geq n\) and \(r = O(\sqrt{n})\). Assuming that \(p\) is a constant, the limits of numerator and denominator exist, and we can use them to calculate the limit of the fraction. Given \(\gamma > 0.5\) the PoS lower bound is simplified to:

\[
\text{PoS} > \left( 1/4 + \frac{1}{2} \left( \frac{1}{p} - \ln p \right) \right) / \left( 3/4 + \frac{1}{2} \left( \frac{1}{p} + \ln p \right) \right)
\]

Numerical maximization over \(p \in (0,1)\) yields \(\text{PoS} > 1.45\), for \(p \approx 0.18\). We also searched computationally for \(r\) maximizing (13), for increasing \(n\). For \(r = 0.27\sqrt{n}\) we found the values appearing in the table 2(b), that indicate \(\text{PoS} > 1.77\). One can verify that any configuration other than \(s^*\) and \(s\) is more expensive, by definition of distances of agents from \(v\) and \(v_i^*\).

## 5 Approximate Strong Equilibria

Strong equilibria do not generally exist in the Facility Location game, even for unweighted agents on metric networks with uniform facility costs. We illustrate this by an example, over the network shown in Fig. 3. Consider the metric case in Figure 3. There are three unweighted agents \(i = 1..3\) situated on distinct nodes \(u_i\) of the depicted 6-node cycle. For any equilibrium strategy profile \(s\) there exist two agents \(i, j,\) with \(j = (i+1) \mod 3\) with \(c_i(s) \geq 1, c_j(s) \geq \frac{8}{9}\). By agreeing to open a facility on vertex \(v_i\), \(i\) and \(j\) would change their costs to \(c_i(s') = \frac{8}{9}\) and \(c_j(s') = \frac{7}{9}\) respectively, each lowering their cost by at least \(\frac{1}{9}\). Hence, this example does not have strong equilibria.

Existence of pure equilibria for weighted agents is also an open issue. However, we were able to reduce to a constant the logarithmic approximation factor \(\alpha\) known for general network design \([1, 7]\). In fact, our result is even more general, as it concerns strong equilibria. We make use of the following remark.
We prove that dam follows from Lemma 2. Let I. For every
Theorem 2
max
W e derive an approximation factor as an upper bound of dam
max
Because
caused by consecutive deviations of coalitions. The game in stance has an
Lemma 2
W e derive an approximation factor that eliminates cycles.
Remark 1 If an instance of the Facility Location game does not have strong equilibria, then there
is at least one cycle of deviations of particular coalitions that results in a circular sequence of
configurations \((s^i)_{j=1}^k\) with \(s^1 = s^k\).
Given such a sequence \((s^i)_{j=1}^k\), we denote the coalition that deviates from \(s^1\) to form \(s^{i+1}\) by \(I_j\). Such a deviation causes a cost decrease of agents in \(I_j\) and possibly a cost increase of agents in \(A \setminus I_j\). Recall that \(A\) is the set of all agents. We define two quantities, the weighted improvement imp\(r(I_j)\) for agents in \(I_j\) and the weighted damage dam\(r(I_j)\) caused by agents in \(I_j\) respectively:

\[
\text{impr}(I_j) = \prod_{i \in I_j} \left( \frac{c_i(s^1)}{c_i(s^{i+1})} \right)^{w_i} \quad \text{dam}(I_j) = \prod_{i \in A \setminus I_j} \left( \frac{c_i(s^{i+1})}{c_i(s^1)} \right)^{w_i}
\]

We derive an approximation factor that eliminates cycles.

Lemma 2 Let \((s^i)_{j=1}^k\) with \(s^1 = s^k\) be a cycle of configurations in a Facility Location game instance, caused by consecutive deviations of coalitions. The game instance has an \(\alpha\)-approximate strong equilibrium if for all such sequences \(\alpha \geq \text{dam}_{\text{max}}((s^i)_{j=1}^k)\), where \(\text{dam}_{\text{max}}((s^i)_{j=1}^k) = \max_{j=1...k-1} \text{dam}(I_j)^{1/W(I_j)}\).

Proof. If there is no \(\alpha\)-approximate strong equilibrium we know that there is at least one cycle \((s^i)_{j=1}^k\) such that \(\forall j \in \{1, \ldots, k-1\} \forall i \in I_j : \frac{c_i(s^j)}{c_i(s^{j+1})} > \alpha\).
Because \(s^1 = s^k\) we have that \(\prod_{j=1}^{k-1} \frac{c_i(s^j)}{c_i(s^{j+1})} = 1\) for every agent \(i\). Then:

\[
1 = \prod_{i=1}^{n} \left( \prod_{j=1}^{k-1} \frac{c_i(s^j)}{c_i(s^{j+1})} \right)^{w_i} = \prod_{i=1}^{k-1} \frac{\text{impr}(I_j)}{\text{dam}(I_j)} > \prod_{i=1}^{k-1} \frac{\alpha W(I_j)}{W(I_j)}
\]

It follows that \(\text{dam}_{\text{max}}((s^i)_{j=1}^k) > \alpha\). Hence the lemma follows by contradiction.

We derive an approximation factor as an upper bound of \(\text{dam}_{\text{max}}((s^i)_{j=1}^k)\) for any cycle.

Theorem 2 For every \(\alpha \geq e\) there exist \(\alpha\)-approximate strong equilibria in the Facility Location game with fairly allocated facility costs, even for weighted agents and general networks.

Proof. We prove that \(\text{dam}_{\text{max}}((s^i)_{j=1}^k) < e\) for every cycle \((s^i)_{j=1}^k\) of configurations and the result follows from Lemma 2. Let \(I_j(v)\) be the set of agents going to \(v\) in \(s^j\), but not in \(s^{j+1}\), and \(A_j(v)\) be the set of agents going to \(v\) in both \(s^j\) and \(s^{j+1}\). Note that \(I_j = \bigcup_{v \in V} I_j(v)\), therefore:
The Strong Price of Anarchy

We derive next an upper bound on the SPoA of $\alpha$-approximate strong equilibria, for the general Facility Location game.

**Theorem 3** For any constant $\alpha \geq \epsilon$, the Price of Anarchy of $\alpha$-approximate strong equilibria for the Facility Location game with fairly allocated facility costs, is upper bounded tightly by $O(\ln n)$ for unweighted and by $O(\ln W)$ for weighted agents, where $W$ is the sum of weights.

**Proof.** Let $s$ and $s^*$ be the most expensive strong equilibrium and the socially optimum configuration respectively. For any facility node $v \in F_{s^*}$ let $A^*_v(v)$ and $I^*_v(v)$ be respectively the subsets of agents connected to $v$ in $s^*$, and connected to $v$ in $s^*$ but not in $s$. We will upper bound the SPoA by $\max_{v \in F_{s^*}} \left( \sum_{i \in A^*_v(v)} c_i(s^*) \right) / c_v(s^*)$. Because $s$ is an $\alpha$-approximate strong equilibrium, for every subset of $I^*_v(v)$ there is at least one agent $i$, that is not willing to deviate to $v$ in coordination with the rest agents of the subset. Let $I^*_v(v) = \{1, \ldots, |I^*_v(v)|\}$ and $I^*_v(v) = \{1, \ldots, i\}$, where $i$ is not willing to deviate in coordination with $I^*_v(v) \setminus \{i\}$. Then:

$$c_i(s^*) \leq \alpha \left( w_i d(u_i, v) + \frac{w_i \beta_v}{W(I^*_v(v)) + W_v(v)} \right) \leq \alpha \left( w_i d(u_i, v) + \frac{w_i \beta_v}{W(I^*_v(v))} \right) \Rightarrow$$

$$\text{dam}_{\max}(s^*_j) = \max_j \left( \prod_{v \in V} \left( \prod_{i \in A^*_v(v)} \frac{c_i(s^*_j)}{c_i(s^*_i)} \right)^{w_i} \frac{W(I^*_v(v))}{W(I^*_v(v))} \right)^{\frac{1}{w_v}}$$

Hence, we need only consider what happens at the worst case node. For an agent $i$ in $A^*_v(v)$ we get that:

$$\frac{c_i(s^*_j)}{c_i(s^*_i)} = \frac{w_i \left( d(u_i, v) + \frac{\beta_v}{W(s^*_j+1(v))} \right)}{w_i \left( d(u_i, v) + \frac{\beta_v}{W(I^*_v(v)) + W(A^*_v(v))} \right)} \leq 1 + \frac{W(I^*_v(v))}{W(A^*_v(v))}$$

It follows that:

$$\text{dam}_{\max}(s^*_j) \leq \max_j \left( 1 + \frac{W(I^*_v(v))}{W(A^*_v(v))} \right)^{\frac{W(A^*_v(v))}{W(I^*_v(v))}} < \lim_{r \to \infty} \left( 1 + \frac{1}{r} \right)^r = \epsilon$$

Approximate strong equilibria are also approximate pure Nash equilibria, thus:

**Corollary 2** The Facility Location game with fairly allocated facility costs and weighted agents has $\alpha$-approximate pure strategy Nash equilibria for every $\alpha \geq \epsilon$. 


\[
\sum_{i \in I_s^*(v)} c_i(s) = \sum_{i \in I_s^*(v)} c_i(s) \leq \alpha \left( \sum_{i \in I_s^*(v)} w_i d(u_i, v) + \beta \sum_{i=1}^{|I_s^*(v)|} \frac{w_i}{W(I_s^*(v))} \right) 
\]

\[
\sum_{i \in I_s^*(v)} c_i(s) \leq \alpha \left( \sum_{i=1}^{|I_s^*(v)|} \frac{w_i}{W(I_s^*(v))} \right) \sum_{i \in I_s^*(v)} w_i d(u_i, v) + \beta \frac{|I_s^*(v)|}{W(I_s^*(v))} 
\]

\[
\sum_{i \in I_s^*(v)} c_i(s) \leq \alpha \left( \sum_{i=1}^{|I_s^*(v)|} \frac{w_i}{W(I_s^*(v))} \right) c_{I_s^*(v)}(s^*) 
\]

The result will follow from (14), because \( I_s^*(v) \subseteq A_s^*(v) \), and agents in \( A_s^*(v) \setminus I_s^*(v) \) play \( v \) in \( s \) and \( s^* \). We analyze the SPoA ratio. For \( w_i = 1 \) it is \( W(I_s^1(v)) = |I_s^1(v)| = 1 \), which yields \( \text{SPoA} = O(\alpha \mathcal{H}(n)) \). For weighted agents set \( W(I_s^0(v)) = 0 \) and use \( x \geq 1 + \ln x \) for \( x \in (0, 1) \) on (14):

\[
\alpha \sum_{i=1}^{|I_s^*(v)|} \left( 1 - \frac{W(I_s^{i-1}(v))}{W(I_s^i(v))} \right) \leq -\alpha \sum_{i=1}^{|I_s^*(v)|} \ln \frac{W(I_s^{i-1}(v))}{W(I_s^i(v))} = \alpha(1 + \ln W) 
\]

\[ \square \]

**Unweighted Lower Bound** We refer to Fig. 4 for a tight (non-metric) lower bounding example in case of unweighted agents. Facility opening costs are 1. It is clear that all agents being connected to node \( v_{eq} \) is the most expensive \( \alpha \)-approximate equilibrium, of social cost \( \alpha \mathcal{H}(n) \): no agent coalition has incentive to deviate to \( v_{opt} \), which is practically the node that all agents reside on, and results in a social cost of virtually 1, for \( \epsilon \to 0 \) (e.g. \( \epsilon = n^{-2} \)).

### 6.1 Unweighted Agents on Metric Networks

We prove the following for the SPoA of the unweighted metric Facility Location game:

**Theorem 4** The Price of Anarchy of \( \epsilon \)-approximate strong equilibria in the unweighted metric Facility Location game with fairly allocated facility costs is upper bounded by a constant.
The proof of theorem 4 consists of several partial results that we will synthesize. Let $s$ denote any $\alpha$-approximate strong equilibrium and $s^*$ the socially optimum configuration. We will upper bound the \textsc{SPoA} again by $\max_{v \in F_{s^*}} \left( \sum_{i \in A_{s^*}(v)} c_i(s) / c_i(s^*) \right)$. For any facility node $v \in F_{s^*}$, define $A_s(v) \subseteq A_{s^*}(v)$ to be the subset of those agents that are connected to $v$ both in $s^*$ and $s$. Define $I_s(v) = A_{s^*}(v) \setminus A_s(v)$ to be the subset of agents that are connected to $v$ in $s^*$ but not in $s$. See fig. 5 for an illustration of the definitions. To simplify notation we use $d(u_i,v) = x_i^*$, for $i \in A_{s^*}(v)$. At first we consider the following simple case:

**Lemma 3** Let $I_{s^*}(v)$ be a subset of misconnected agents under $\alpha$-approximate strong equilibrium profile $s$. Let $R$ denote agents connected to $v$ under $s$, with $R \subseteq A \setminus A_{s^*}(v)$. If under $s \setminus R$ no agent of $I_{s^*}(v)$ has incentive to deviate to $v$ in coordination with $I_{s^*}(v)$, then $\sum_{i \in A_{s^*}(v)} c_i(s) \leq (1 + \alpha) c_v(s^*)$.

**Proof.** If under $s \setminus R$ no agent $i \in I_{s^*}(v)$ has incentive to deviate in coordination with $I_{s^*}(v) \setminus \{i\}$ to $v$, then for every $i \in I_{s^*}(v)$ it is $c_i(s) \leq \alpha c_i(s^*)$, because $s$ is an $\alpha$-approximate strong equilibrium. Then:

$$
\sum_{i \in A_{s^*}(v)} c_i(s) = \sum_{i \in A_s(v)} c_i(s) + \sum_{i \in I_{s^*}(v)} c_i(s) \leq \beta_v + \sum_{i \in A_s(v)} x_i^s + \alpha \left( \beta_v + \sum_{i \in I_{s^*}(v)} x_i^{s^*} \right) \\
\leq (1 + \alpha) \left( \beta_v + \sum_{i \in A_{s^*}(v)} x_i^{s^*} \right)
$$

The latter equals $(1 + \alpha)c_v(s^*)$. \hfill $\Box$

For the rest of the analysis we treat the complementary case of that described in the previous lemma. Assume there exists at least one agent $i \in I_{s^*}(v)$ willing to deviate to $v$ in coordination with the coalition $I_{s^*}(v)$ under $s \setminus R$. Define a minimal disagreeing subset $I_0^s(v) \subseteq I_{s^*}(v)$ to be a minimal subset of misconnected agents containing an agent $i$ that would actually deviate to $v$ with $I_0^s(v)$, under $s \setminus R$. Then also define $I_{s^*}(v) = I_{s^*}(v) \setminus I_0^s(v)$. Fix $i \in I_0^s(v)$ to be from now on the agent (or one of them if there are many) that would deviate to $v$ in coordination with $I_0^s(v)$. We call $i$ the \textit{unstable} agent of the minimal disagreeing subset $I_0^s(v)$. By definition, the following holds for an \textit{unstable} agent $i$ of the minimal disagreeing subset:

![Figure 5: The situation examined in the proof of theorem 4.](image-url)
\[ c_t(s) > \alpha \left( x_t^* + \frac{\beta_v}{|I_s^0(v)| + |A_s(v)|} \right) \]  

(15)

The rest of the analysis consists of bounds for agents in \( I_s^0 \) and \( I_s^* \) separately. In particular, lemmas 4 and 5 describe upper bounds for these sets respectively.

**Lemma 4** Let \( I_s^* \) be a subset of misconnected agents under an \( \alpha \)-approximate strong equilibrium profile \( s \). Let \( I_{s_r}^0 \) be a minimal disagreeing subset of \( I_s^* \) and \( i \in I_{s_r}^0 \) an unstable agent. Then:

\[ \sum_{t \in I_{s_r}^0(v)} c_t(s) \leq 2\alpha \left( \beta_v + \sum_{t \in I_{s_r}^0(v)} x_t^* \right) \]  

(16)

**Proof.** By minimality of \( I_{s_r}^0(v) \), every agent \( l \in I_{s_r}^0(v) \) is not willing to deviate to \( v \) under \( s_{-R} \) with a coalition of size \( |I_{s_r}^0(v)| - 1 \). Thus for every \( l \in I_{s_r}^0(v) \):

\[ c_t(s) \leq \alpha \left( x_t^* + \frac{\beta_v}{|I_{s_r}^0(v)| + |A_s(v)| - 1} \right) \]

Summing over \( I_{s_r}^0(v) \) yields:

\[ \sum_{t \in I_{s_r}^0(v)} c_t(s) \leq \alpha \left( \sum_{t \in I_{s_r}^0(v)} x_t^* + \frac{|I_{s_r}^0(v)| \beta_v}{|I_{s_r}^0(v)| + |A_s(v)| - 1} \right) \]

which is upper bounded by at most as in (16).

\( \square \)

**Lemma 5** Let \( I_s^* \) be a subset of misconnected agents under an \( \alpha \)-approximate strong equilibrium profile \( s \). Let \( I_{s_r}^0 \) be a minimal disagreeing subset of \( I_s^* \) and \( I_{s_r}^0 = I_{s_r}^0 \setminus I_{s_r}^0(v) \). Then:

\[ \sum_{j \in I_{s_r}^0(v)} c_j(s) \leq \alpha \left( \sum_{j \in I_{s_r}^0(v)} x_j^* + \beta_v \left( H(|A_{s_r}(v)|) - H(|I_{s_r}^0(v)|) \right) \right) \]  

(17)

**Proof.** Without loss of generality name agents \( j \in I_{s_r}(v) \) by distinct indices \( 1, \ldots, |I_{s_r}(v)| \) and define a series of supersets of \( I_{s_r}^0(v) \), as follows: \( I_{s_r}^0(v) = I_{s_r}^{j-1}(v) \cup \{j\} \). Because \( s \) is an \( \alpha \)-approximate strong equilibrium, every set \( I_{s_r}^0(v) \) contains an agent that is not willing to deviate to \( v \) in coordination with \( I_{s_r}^0(v) \). This agent is found either in \( I_{s_r}^0(v) \setminus \{i\} \) or in \( I_{s_r}^0(v) \setminus I_{s_r}^0(v) \). We can assume without any loss of generality that for subset \( I_{s_r}^0(v) \) this agent is \( j \); otherwise we only need to exchange \( j \) with some agent from \( I_{s_r}^0(v) \setminus \{i\} \). One easily verifies that, by definition of a minimal disagreeing subset, such an exchange will not affect any of our previous results up to now. Then we have:

\[ c_j(s) \leq \alpha \left( x_j^* + \frac{\beta_v}{|I_{s_r}^0(v)| + |A_s(v)| + |R|} \right), \quad j = 1, \ldots, |I_{s_r}(v)| \in I_{s_r}^0(v) \]  

(18)

We omit \(|R|\) and sum the inequality over \( j \in I_{s_r}(v) \). The result follows.

\( \square \)

The following lemma will provide a lower bound for \( \sum_{j \in I_{s_r}(v)} x_j^* \) appearing in (17). Note that lemma 4 provides a concrete upper bound for the cost (under strategy profile \( s \)) of agents in
Ix*(v), as a function of their connection cost in the socially optimum configuration s* and the corresponding facility cost \( \beta_v \). This is not the case with lemma 5 for agents in \( I_x(v) \); the upper bound (17) involves socially optimum connection cost plus \( \beta_v \) multiplied by additional terms of harmonic numbers. Thus we need to determine how low the socially optimum connection cost can be, in order to derive an upper bounding ratio for the SPoA.

**Lemma 6** Let \( I_x(\nu) \) be a subset of misconnected agents under \( \alpha \)-approximate strong equilibrium profile s. Let \( I_0(\nu) \) be a minimal disagreeing subset of \( I_x(\nu) \) and \( I_{x^*}(\nu) = I_x(\nu) \setminus I_0(\nu) \). Then:

\[
\sum_{j \in I_x(\nu)} x_j^* \geq \frac{\beta_v}{1 + \alpha} \left( \left| A_s(\nu) \right| - \left| \alpha r \right| \right) - \alpha \left( H(|A_s(\nu)|) - H(|\alpha r|) \right), \quad r = |I_0(\nu)| + |A_s(\nu)|
\]  

**Proof.** Let i be the fixed unstable agent of \( I_0(\nu) \). Note that, under strategy profile s, i does not have an incentive to join facility node \( s_j \) for any \( j \in I_x(\nu) \). Thus if \( j \) pays for \( s_j \) a share of \( \frac{\beta_j}{\lambda_j} \) (that is, \( s_j \) serves \( \lambda_j \) agents in total in s):

\[
c_i(s) \leq \alpha \left( d(u_i, s_j) + \frac{\beta_j}{1 + \lambda_j} \right) \leq \alpha \left( d(u_i, v) + d(u_j, v) + d(u_j, s_j) + \frac{\beta_j}{\lambda_j} \right)
\]

This inequality derives by usage of (18) for \( c_i(s) \), and by safely omitting \( |R| \). Using (20) and the lower bound for \( c_i(s) \) given in (15), we can solve for \( x_j^* \). By the definition of \( I_{x^*}(\nu) \) in lemma 5, it is \( |I_{x^*}(\nu)| = |I_0(\nu)| + j \), \( j = 1, \ldots, |I_x(\nu)| \). Then we obtain:

\[
x_j^* \geq \max \left\{ 0, \frac{\beta_v}{1 + \alpha} \left( \frac{1}{|I_0(\nu)| + |A_s(\nu)|} - \frac{\alpha}{j + |I_0(\nu)| + |A_s(\nu)|} \right) \right\}, \quad j = 1, \ldots, |I_x(\nu)|
\]

Finally we sum up the latter bound over all \( j \). Notice that \( x_j^* \) becomes non-negative only when \( j + |I_0(\nu)| + |A_s(\nu)| \geq \alpha(|I_0(\nu)| + |A_s(\nu)|) \). Since \( j + |I_0(\nu)| + |A_s(\nu)| \) is an integral value, it turns out that \( x_j^* \) becomes non-negative for those values of \( j \) for which it is \( j + |I_0(\nu)| + |A_s(\nu)| \geq \lceil \alpha(|I_0(\nu)| + |A_s(\nu)|) \rceil \). Then, by setting \( r = |I_0(\nu)| + |A_s(\nu)| \), and by summing up over all \( j \) we obtain the specified lower bound.

**Proof of Theorem 4** Now we put everything together. A lower bound on \( c_\nu(s^*) \) is:

\[
c_\nu(s^*) \geq \beta_\nu + \sum_{j \in I_x(\nu)} x_j^* + \sum_{l \in I_0(\nu)} x_l^* + \sum_{i \in A_\nu} x_i^*
\]

Accordingly, we obtain the following upper bound on \( \sum_{i \in A_\nu} c_i(s) \) by (16):

\[
\sum_{i \in A_\nu} c_i(s) \leq \sum_{l \in I_0(\nu)} c_l(s) + \sum_{j \in I_x(\nu)} c_j(s) + \sum_{i \in A_\nu} c_i(s) 
\leq 2\alpha \left( \sum_{l \in I_0(\nu)} x_l^* + \beta_\nu \right) + \sum_{j \in I_x(\nu)} c_j(s) + \sum_{i \in A_\nu} c_i(s)
\]

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Using the latter bounds for $c_v(s^*)$ and $\sum_{l \in A_{s^*}(v)} c_i(s)$, we deduce:

$$\text{SPoA} \leq 1 + \left(2\alpha \beta_v + 2\alpha \sum_{l \in p_{s^*}(v)} x_l^s + \sum_{j \in I_{s^*}(v)} c_j(s) \right) \left(\beta_v + \sum_{l \in p_{s^*}(v)} x_l^s + \sum_{j \in I_{s^*}(v)} x_j^s \right)$$

$$\leq 1 + 2\alpha \times \frac{\beta_v + \sum_{j \in I_{s^*}(v)} x_j^s + \beta_v \left(H(|A_{s^*}(v)|) - H(r)\right)}{\beta_v + \sum_{j \in I_{s^*}(v)} x_j^s}$$

$$= 1 + 2\alpha + 2\alpha \times \frac{\beta_v \left(H(|A_{s^*}(v)|) - H(r)\right)}{\beta_v + \sum_{j \in I_{s^*}(v)} x_j^s}$$

(21)

The latter inequality emerged firstly by substitution from (17), and secondly by removal of $\sum_{l \in p_{s^*}(v)} x_l^s$ from numerator and denominator. We maximize (21) by taking the lower bound of the denominator, given in (19) and obtain:

$$\text{SPoA} \leq 1 + 2\alpha + 2\alpha - \frac{\text{H}(|A_{s^*}(v)|) - \text{H}(r)}{1 + \frac{\text{H}(|A_{s^*}(v)|) - \text{H}(r)}{\text{H}(|A_{s^*}(v)|) - \text{H}(\lceil\alpha r\rceil)}}$$

(22)

Because $\lceil\alpha r\rceil \leq (\alpha + 1)r$, and by using logarithmic bounds for the harmonic numbers:

$$\text{SPoA} \leq 1 + 2\alpha + 2\alpha - \frac{1 - \gamma + \ln \frac{|A_{s^*}(v)|}{r}}{1 + \frac{1 - \gamma + \ln \frac{|A_{s^*}(v)|}{r} - \alpha(\gamma + \ln \alpha - 1)}{\ln \frac{|A_{s^*}(v)|}{r} + \alpha(\gamma + \ln \alpha - 1)}}$$

By substituting 0.5 for $\gamma > 0.5$ and $\alpha = e$ we can maximize numerically the resulting upper bounding function of $y = \frac{|A_{s^*}(v)|}{r}$ to at most a constant. Note that, when strong equilibria exist, it is $\alpha = 1$. Substituting so in (22) yields an upper bounding expression similar to the one of PoS (7), up to constant multiplicative and additive terms.

7 Open Problems

Further investigation of the weighted game is mostly challenging: it is not known whether this game possesses pure equilibria. Designing a counter-example seems quite demanding, as the game specializes in some sense the single-sink weighted network design game studied in [7]. For this case pure equilibria were shown not to exist generally. Extending our analysis of the PoS and SPoA to the weighted metric case appears to be also non-trivial. Finally, derivation of good lower bounds for the PoS and the SPoA of the (non-metric) weighted case is an interesting aspect of research: lower bounding techniques developed in [1] do not readily apply for Facility Location.

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