Mode analysis method for the computation of guided wave dispersion in metal sheet

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Abstract. Mode analysis method based on finite element theory is proposed to plot dispersion curves in metal sheet. This method is implemented by adopting COMSOL software to numerically solve and plot the dispersion curves. Compared with results obtained by semi-analytical finite element method, the calculating results are basically consistent and the modal caused by mode blending is obtained, which proves the mode analysis method is effective. Finally, the wave structure characteristics of various models are analysed, this provides a basis for the application of ultrasonic guided wave testing.

1. Introduction

Ultrasonic guided wave detection is regarded as an essential technique of non-destructive testing. Due to the characteristics of fast speed, long propagation distance and convenient automatic detection, it is widely used in defect or quality inspection of industrial bars, pipelines, railway rails, plate materials and composite materials. In view of different detection objects, it is necessary to select the appropriate guided wave mode and excitation frequency to achieve the detection effect, which often depends on the analysis of the guided wave dispersion characteristics [1]. Therefore, accurately grasping the dispersion relationship of ultrasonic guided waves in the structure is a prerequisite and a key step for nondestructive testing [2, 3].

In recent years, the dispersion characteristics of sheet metal have aroused researchers’ closed attention. PAOLO BOCCHINI [4] developed the GUWGUI software based on the semi-analytical finite element method by adopting Matlab language, which can be used to plot the dispersion curve of ultrasonic guided waves in the plate structure, but the theoretical derivation and calculation process of this method is complicated. Zhang Yan [5] used the dichotomy method to analyze the dispersion curves of Lamb wave in composite laminates in Matlab environment, but did not involve solving the SH wave dispersion problems. Wu Bin [6] used the finite element software ANSYS to solve the characteristic frequency of the slab structure dispersion. However, the post-processing needs to compile the Matlab program for the modal and wave number identification, which was time consuming. Comsol software just overcomes the complicated features of the steps because of the solution steps of mode analysis, this can be used for the analysis of solid mechanical field, acoustic field and electromagnetic field, and easily obtain wave number values. In addition, the mode analysis method is generally used for electromagnetic field analysis. Considering electromagnetic wave
propagation has a certain similarity with ultrasonic guided wave propagation [7, 8], this paper proposes mode analysis method based finite element theory to calculate the dispersion relations in metal sheet through the characteristic equation of the relationship between frequency and wave number. The wave structure characteristics can be obtained after the post-processing. It provides a simple and rapid calculation method for the frequency and modal selection when employing ultrasonic guided wave testing.

2. Theoretical basis
Waveguide is assumed to be a homogeneous, isotropic linear elastomer material with uniform cross-sectional shape. Consider a micro-body in the waveguide. Its displacement, stress, and strain can be respectively expressed as \( u = [u_x, u_y, u_z]^T \), \( \sigma = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}]^T \) and \( \varepsilon = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}]^T \). According to the generalized Hooke’s law, stress-strain relations satisfy:

\[
\sigma = C \varepsilon
\]

In the formula, \( C \) is an elastic matrix that can be expressed as:

\[
C = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda \\
\lambda & \lambda + 2\mu & \lambda \\
\lambda & \lambda & \lambda + 2\mu
\end{bmatrix}
\]

In the formula (2), the Lame constants are calculated according to the following formulas (3) and (4):

\[
\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}
\]

\[
\mu = \frac{E}{2(1 + \nu)}
\]

Where \( E \) is the elastic modulus of the waveguide, \( \nu \) is its Poisson's ratio. In addition to length expansion in the micro-element, the vibration displacements generate due to shear. Therefore, the strain in the Cartesian coordinate system needs to satisfy the geometric equations:

\[
\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{1}{2} \left( \frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x} \right), \varepsilon_{yz} = \frac{\partial u_y}{\partial z}, \varepsilon_{zz} = \frac{1}{2} \left( \frac{\partial u_z}{\partial z} + \frac{\partial u_x}{\partial x} \right), \varepsilon_{xy} = \frac{\partial u_x}{\partial y}, \varepsilon_{xz} = \frac{\partial u_x}{\partial z}
\]

To establish a propagation model of the ultrasonic guided wave in the structure, a simple harmonic of the linear perturbation can be applied in the z direction, and the finite element is discretized, so that a node with a displacement vector expression in the waveguide can be given by:

\[
u(x, y, z, t) = U_j N_j e^{i\omega t - ik_z z}
\]

where \( k_z \) is the wave number, \( \omega \) is the angular frequency, \( N_j \) is shape function of the j-th node, \( U_j \) is the displacement amplitude vector of the j-th node. According to the formulas (1)-(5), the elastic potential energy of the waveguide structure is:
According to the kinetic energy theorem, the expression of the kinetic energy of the waveguide structure can be written as:

\[ W_k = \int \int C \varepsilon \varepsilon^T dV = \frac{1}{2} \int C \varepsilon^2 dV = \frac{1}{2} \int \varepsilon^T C \varepsilon dV \]  

(7)

where \( \dot{u} \) is the derivative of displacement.

For a waveguide dielectric unit, the Lagrangian amount can be expressed as:

\[ L = W_D - W_E \]

(8)

For small time-varying variations, according to the Hamiltonian principle, we have:

\[ \delta L = \int_0^T \delta (W_D - W_E) dt = 0 \]

(10)

Substituting (7) and (8) into (10), we obtain the formula (11), which is weak form of the finite element solution:

\[ \int_0^T \left[ \int \delta (u^r) \rho \dot{u} dV + \int \delta (e^r) C \varepsilon dV \right] dt = 0 \]

(11)

From Eq. (5) and Eq. (6), strain can be defined as:

\[ \varepsilon(x, y, z, t) = (ik, k, k)N[J_{min}^{m-1} e^{i(\sigma - i\omega)}] \]

(12)

In Eq. (12), \( k_1, k_2 \) are the matrix with parameters. \( m \) is the order of the solution, and the wavelength is \( L, c \) is the wave velocity. Substituting Eq. (6) and Eq. (12) into Eq. (11), then becomes:

\[ (K_1 + ik_2 K_2 + k_2^2 K_3 - \omega^2 M)U = 0 \]

(13)

where \( K_1, K_2, K_3, M \) are all coefficient matrices, \( U \) is total displacement matrices. We can set \( \xi = ik \), then Eq. (13) is rewritten as:

\[ (K_1 + \xi K_2 - \xi^2 K_3 - \omega^2 M)U = 0 \]

(14)

Eq. (14) is a problem of finding generalized eigenvalues. By giving a certain value and setting the required mode order simultaneously, the eigenvalue solver in the commercial finite element software can be used to find a set of values, each of which corresponds to a mode.

3. Finite element model

At first, the geometric model is established in Comsol. Since the calculation scale of the 3D model is large, the plate is simplified into a two-dimensional section to reduce the number of degrees of freedom. An aluminum plate with a thickness of 5 mm and a width of 10 mm is taken as an example. Table 1 displays the aluminum material parameter, where \( \rho \) is density, \( E \) is Young's modulus and \( \nu \) is Poisson's ratio.
Then, the solid mechanics field is added and the solution step of the mode analysis is selected. Consider this model in the ideal case. There is no external force with a free boundary condition [6]. In order to simulate infinitely wide plate, periodic boundary condition is employed. To ensure the quality of the mesh, quadrilateral meshes separates the region. The resulting finite element model is shown in Figure 1. The frequency is taken as the scanning parameter, in the case of high frequency condition, there are more modes, causing the larger calculated amount. Thus, this sets a range from 0 to 800kHz and the step size is 10 kHz. The $\omega$ can be defined by:

$$\omega = 2\pi f$$

Lastly, set the required analysis mode to 30 in mode analysis step of Comsol where the reference speed value can be set as 300m/s. In this way, software can search for the wave number values of all orders corresponding to the current frequency.

### Table 1. Material parameters of aluminum

| $\rho$ (kg / m$^3$) | $E$ (GPa) | $\nu$ |
|---------------------|-----------|-------|
| 2700                | 72        | 0.33  |

![Fig. 1 2D model](image)

#### 4. Drawing of dispersion curve

After applying the method mentioned above, the complex solution and real solution of wave number are obtained. When the dispersion curves prepare to plot, only the real solutions are required. Therefore, the restricted condition that real part value of the solution is much larger than the imaginary part value is set to remove the complex solution of the wave number. The phase velocity and group velocity can be directly calculated based on the relationship between wave number and angular frequency:

$$c_p = \frac{\omega}{k} = \frac{2\pi f}{k}$$

Furthermore, the group speed can be expressed by:

$$c_g = \frac{d\omega}{dk} = \frac{2\pi df}{dk} = \frac{2\pi f}{k}$$
For equation (17), the solving method is to give two sets of frequency values, so that two sets of wave number values can be calculated correspondingly, then making the difference quotient, the group velocity values are obtained eventually.

In order to verify whether the guided wave dispersion curves drawn by the mode analysis method are correct, the curves are compared with the result obtained by the semi-analytical finite element (SAFE) method. As shown in Figure 2. Figure 3, the solid point represents the solution result of the mode analysis method, and the hollow point represents the solution result of the semi-analytical finite element method.

From figure 2 to figure 3, the dispersion curves of various modes were plotted by mode analysis method, including the symmetric mode of the Lamb wave (S_0 mode, S_1 mode), the anti-symmetric mode (A_0 mode, A_1 mode), the SH wave mode (SH_0 mode, SH_1 mode, SH_2 mode) and the structural mode (the curve is not coincident). There is no dispersion phenomenon in the SH_0 mode, because the phase velocity and the group velocity remain almost unchanged with increasing excitation frequency. So the SH_0 mode is more suitable for the detection of metal sheet. When the excitation frequency is within 400 kHz, there is dispersion in the S_0 and A_0 modes of Lamb wave modes. In this case, less modes are beneficial to decrease the difficulty for signal processing.

Besides, the structural modal is obtained by mode analysis method. There are 7 modes in this modal, and phase velocity of each mode decreases with increase of the excitation frequency. As the number of mode increases, the phenomenon of scattering is significant. The reason for this is that different modes are affected by the boundary and structure size and the blending modes are formed [6]. Therefore, the blending modes should be avoided as much as possible to reduce the complexity for signal processing. Dispersion curves of the Lamb wave and the SH wave mode drawn by the two methods are slightly different, which is caused by the difference of the cutoff frequency. Generally speaking, the results obtained by the SAFE method are agree with the numerical computation, verifying the validity and comprehensiveness of the mode analysis method.

5. Analysis of wave structure feature

In order to represent the structural characteristics of the guided wave, taking the displacement distribution of all modes at a frequency of 300 kHz as an example, a vertical cut line displayed in figure 1 is chosen in rectangular waveguide, and the displacement values of the discrete node on the cut line are outputted after post-processing. Each node contains three directional displacements that denoted as u_x, u_y and u_z, respectively, those are normalized based on the maximum value of the absolute value [3]. As shown in figure 4 to figure 7, the solid line in the figure represents u_x, the dotted line represents u_y, and the solid point represents u_z.

It can be seen from Figure 4 that obviously, of all displacements in different directions, u_x are the largest, and the displacements in x and y direction are almost zero, indicating that the guided wave of S_0 mode propagates in a good direction, so the length or width of the material should be detected by S_0
of Lamb wave. As shown in Figure 5, the structural modes have displacements in all directions, indicating that the propagation direction is uncertainty. If the structural mode had been generated in the process of guided wave testing, the more difficulties of the signal processing would have been made, so this modal should be avoided as much as possible. It can be seen from Figure 6 that there exist displacements in the direction of x and z at SH wave, and the displacements in y direction are nearly to 0. In Figure 7, displacements near the surface of the plate are decreased, and the displacements in y and z direction are consistent. It is concluded that the ultrasonic guided wave of A0 mode propagates in oblique incidence, reflected via the upper and lower interfaces in waveguide. Thus, it can be used to detect the thickness of the metal sheet.

6. Conclusion

In this paper, a mode analysis method is introduced to solve the dispersion relation of ultrasonic guided waves in metal sheet. This method based on the relationship between frequency and eigenvalue derived by wave equation. According to the method, Comsol software is adopted to calculate the dispersion by giving frequency, the order and the reference value of mode searching. After post-processing, the displacement field on the section line is analyzed. It provides a reference for the ultrasonic guided wave detection frequency and modal selection. The dispersion relations solved by the semi-analytical finite element method is compared. The results show a well agreement with dispersion relations obtained by semi-analytical finite element method. Besides, the dispersion curves of structural modes are obtained, which proves a comprehensive feature of this method.
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