On comparison of expert

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Abstract

A policy maker faces a sequence of unknown outcomes. At each stage two (self-proclaimed) experts provide probabilistic forecasts on the outcome in the next stage. A comparison test is a protocol for the policy maker to (eventually) decide which of the two experts is better informed. The protocol takes as input the sequence of pairs of forecasts and actual outcomes and (weakly) ranks the two experts.

We focus on anonymous and non-counterfactual comparison tests and propose two natural properties to which such a comparison test must adhere. We show that these determine the test in an essentially unique way. The resulting test is a function of the derivative of the induced pair of measures at the realized outcomes.

Keywords: forecasting, probability, testing

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1. Introduction

The literature on expert testing has, by and large, treated the question of whether a self-proclaimed expert can be identified as such, while also not allowing for charlatans to pass the test. A striking result due to Sandroni (2003) is that no such test exists without additional structural assumptions regarding the problem. The basic premise of this literature is the validity of the underlying question of whether a forecaster, or rather a probabilistic model, is correct or false. In a hypothetical world, where only one model exists and the tester can only entertain the services of a single expert, this may make sense. Even then, one might wonder what is the tester to do whenever she rejects the expert. Does she turn to another expert or to her own intuition? In any case she would probably, implicitly, utilize an alternative (possibly untested) model.

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This motivates us to seek an alternative approach to the issue of expert testing and that is a comparison of experts, which is the approach we pursue here. In this approach the tester is exposed to a few alternative models (forecasters) and a single realization of events. The tester then compares the alternative forecasters and decides which is the better informed one. Facing many (possibly conflicting) experts is commonplace in weather forecasting, financial forecasting, medical prognosis and more. Nevertheless, the design of comparison tests has been almost entirely ignored in the literature on expert testing. Two exceptions are Al-Najjar & Weinstein (2008) and Feinberg & Stewart (2008) which we discuss in the section on related literature.

The approach we take in this paper is axiomatic. After defining exactly what is meant by a comparison test we will turn to discuss some desirable properties for such tests. We then construct a test that complies with all the desired properties and show it is essentially unique. The setting we focus on is that of two experts and a test which (weakly) ranks the two and hence its range consists of three outcomes. It may either point at one of the two experts as being better informed or it may be indecisive. Let us discuss the properties that are central to our main results.

**Anonymity** - A test is anonymous if it does not depend on the identity of the agents but only on their forecasts.

**Error-free** - Let us assume that one of the experts has the correct model (namely, he would have passed a standard single expert test which has no type-1 errors). An error-free test will surely not point at the second expert as the superior one (albeit, it may provide a non-conclusive outcome).

**Reasonable** - Let us consider an event, $A$, that has positive probability according to the first expert but zero probability according to the second. Conditional on the occurrence of the event $A$, a reasonable test must assign positive probability to the first expert being better informed than the second.

The approach taken in this paper can be considered as a contribution to the hypothesis testing literature in statistics where a forecaster is associated with a hypothesis. In this context we propose a hypothesis test that complies with a set of fundamental properties which we refer to as axioms. In contrast, a central thrust for the hypothesis testing literature (for two hypotheses) is the pair of notions of significance level and power of a test. In that literature one hypothesis is considered as the null hypothesis while the other serves as an alternative. A test is designed to either reject the null hypothesis, in which case it accepts the alternative, or fail to reject it (a binary outcome). The significance level of a test is the probability of rejecting the null hypothesis whenever it is correct (type-1 error) while the power of the test is the probability of rejecting the null hypothesis assuming the alternative one is correct.
(the complement of a type-2 error).

In contrast with the aforementioned binary outcome that is prevalent in the hypothesis testing literature we allow, in addition, for an inconclusive outcome. Recall the celebrated Neyman-Pearson lemma which characterizes a test with the maximal power subject to an upper bound on the significance level. The possibility of an inconclusive outcome, in our framework, allows us to design a test where both type-1 and type-2 errors have zero probability[1].

Interestingly, the test proposed in the Neyman-Pearson lemma, similar to ours, also hinges on the likelihood ratio[2]. In our approach we, a priori, treat both hypotheses symmetrically. In the statistics literature, however, this is not the case and the null hypothesis is, in some sense, the status quo hypothesis. This asymmetry is manifested, for example, in the Neyman-Pearson lemma.

Note that in order to design a test that complies with a given significance level and a given power one must know the full specification of the two hypotheses. This is in contrast with our test which is universal, in the sense that it does not rely on the specifications of the two forecasts. Finally, let us comment that whereas hypothesis testing is primarily discussed in the context of a finite sample, typically from some IID distribution, our framework allows for sequences of forecasts that are dependent on past outcomes as well as past forecasts of the other expert.

1.1. Results

We construct a specific comparison test based on the derivative of two measures that are induced by the two forecasters. We prove that this test is anonymous, error-free and reasonable.

Two tests are essentially equal if their verdict is equal with probability one for any pair of forecasters[3]. The test we construct turns out to be unique modulo this equivalence relation. In other words, for any test that is not equivalent to ours and is anonymous and reasonable there exist two forecasters for which

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1Note that we abuse the statistical terminology. In statistics the notion of rejection is always used in the context of the null hypothesis. In our model we assume symmetry between the alternatives and so we discuss rejection also in the context of the alternative hypothesis. As a consequence, an error of type-1 is defined as the probability of accepting the alternative hypothesis whenever the null hypothesis is correct, and symmetrically, an error of type-2 is the probability of accepting the null hypothesis whenever the alternative one is correct.

2The test proposed in the Neyman-Pearson lemma rejects the null hypothesis whenever the likelihood ratio falls below some positive threshold.

3"with probability one" is meant with respect to the probability measure induced by either of the two forecasters. Please refer to Definition 5 for a more rigorous statement.
an error will be made (the probability of reversing the order) and hence that test cannot be error-free.

Finally, our constructed test perfectly identifies the correct forecaster whenever the two measures induced by the forecasters are mutually singular with respect to each other. Requiring the test to identify the correct expert when the measures are not mutually singular is shown to be impossible.

1.2. Related literature

Much of the literature on expert testing focuses on the single expert setting. This literature dates back to the seminal paper of Dawid (1982) who proposes the calibration test as a means to evaluate a forecaster (in particular a weather forecaster) and shows that a true expert will never fail this test. Foster & Vohra (1998) show how a charlatan, who has no knowledge of the weather, can produce forecasts which are always calibrated. The basic ingredient that allows the charlatan to fool the test is the use of random forecasts. Lehrer (2001) and Sandroni, Smorodinsky & Vohra (2003) extend this observation to a broader class of calibration-like tests. Finally, Sandroni (2003) shows that there exists no error-free test that is immune to such random charlatans (see also extensions of Sandroni’s result in Shmaya (2008) and Olszewski & Sandroni (2008)).

To circumvent the negative results various authors suggest to limit the set of models for which the test must be error-free (e.g., Al-Najjar, Sandroni, Smorodinsky & Weinstein (2010), and Pomatto (2016)), or to limit the computational power associated with the charlatan (e.g., Fortnow & Vohra (2009)) or to replace measure-theoretic implausibility with topological implausibility by resorting to the notion of category one sets (e.g., Dekel & Feinberg (2006)).

As previously mentioned, the comparison of experts has drawn little attention in the community studying expert testing, with two exceptions. Al-Najjar & Weinstein (2008) proposed a test based on the likelihood ratio for comparing two experts. They show that if one expert knows the true process whereas the other is uninformed, then one of the following must occur: either, the test correctly identifies the informed expert, or the forecasts made by the uninformed expert are close to those made by the informed one. It turns out that the test they propose is anonymous and reasonable but is not error-free (Subsection 6.1 Claim[1]). An asymptotic version of this likelihood ratio, however, will play a crucial role in our construction.

Feinberg & Stewart (2008) study an infinite horizon model of testing multiple experts using a cross-calibration test. In their test $N$ experts are tested simultaneously; each expert is tested according to a calibration restricted to dates where not only does the expert have a fixed forecast but the other
experts also have a fixed forecast, possibly with different values (a formal definition is given in Appendix B). They showed that a true expert is guaranteed to pass the cross-calibration no matter what strategies are employed by the other experts.

In addition, they prove that in the presence of an informed expert, the subset of data-generating processes under which an ignorant expert (a charlatan) will pass the cross-calibration test with positive probability, is topologically “small”. The cross calibration test naturally induces a comparison test for two experts: If one expert passes while the other does not then he is the better informed one, while in all other cases the test is inconclusive.4 This induced comparison test turns out to be anonymous and error-free but not reasonable (for further details see Claim 2 in Subsection 6.2).

Echenique & Shmaya (2008) study a setting where a decision maker (DM) has some initial belief about the evolution of a system and takes actions to maximize her payoff. The DM is offered an alternative hypothesis and the paper provides a scheme for choosing between the two hypotheses (a ‘test’) with a guarantee on the payoffs. In particular, whenever the scheme suggests to adopt the alternative hypothesis, the resulting payoffs do not diminish in comparison with the hypothetical payoff were that hypothesis rejected. In addition, their test is shown to accept the initial belief whenever it is true. Their test, once again, is based on the likelihood ratio but is obviously asymmetric and is not error-free.

Pomatto (2016) poses a question that can be interpreted as one about multiple expert testing. The paper characterizes classes of hypotheses (‘paradigms’ in his jargon) for which there exists a test that will pass the true hypothesis while rejecting any other hypothesis in the class as well as any convex combination thereof. The latter requirement is quite strong as it consequently means that any pair of hypotheses is not testable, in contrast with our results.

Finally, the likelihood ratio, central to our result, appears in the context of many statistical tests. Whereas our work derives a test based on the likelihood ratio as an essentially unique test that conforms with some fundamental properties, many papers and scholars in statistics consider the likelihood ratio as axiomatic. This is captured in Edwards (1972)’s well-cited Likelihood Axiom: “Within the framework of a statistical model, all the information which the data provide concerning the relative merits of two hypotheses is contained in the likelihood ratio of those hypotheses on the data, and the likelihood ratio

4In fact, any single expert test induces a comparison test as follows. Run the test for each of the two experts simultaneously and whenever one passes and the other one fails rank them accordingly. Otherwise, the test is inconclusive.
is to be interpreted as the degree to which the data support the one hypothesis against the other”.

1.3. Finite or infinite test?

A long-standing debate in the literature on expert testing is whether a test should be finite. A test is finite if its decision is made in some finite time. In contrast, an infinite test may require the infinite sequence of forecasts and realizations prior to making a verdict. The argument for considering finite tests is that infinite tests are impractical.

Although we sympathize with the argument that infinite tests are impractical we do think they have academic merit. The construction of well-behaved infinite, possibly impractical, tests would eventually shed light on their finite counterpart. Thus, if the technical analysis underlying the understanding of infinite tests is more tractable than that of finite tests, then the study of infinite tests should be the port of embarkation for this research endeavor. This is what motivates our approach in this paper.\(^5\)

Furthermore, in expert testing we should allow experts to calibrate their model given the data. Pushing the design of tests towards finite tests may result in tests that give a verdict before these models are refined and calibrated. Consider the classical example of an IID process. A forecaster who is aware that indeed the process is such may need time (and data) to calibrate the model and to learn its parameter. Initial forecasts may be wrong, yet those made after a calibration phase become more accurate and long-run predictions are spot-on. To capture the importance of such a preliminary calibration test and patience in model (expert) selection we introduce the following notion:\(^6\)

**Tail test** - A tail test is one which depends only on forecasts made eventually, after the calibration phase. Whereas much of the literature emphasizes tests that provide their verdict at some finite outcome, we take the opposite approach for some of our results and consider comparison tests that are based on a long-run performance. It turns out that the test proposed here, which is anonymous, error-free and reasonable, is also a tail test. It is also unique in a very strong sense—the error of any alternative tail test, which is also anonymous and reasonable, can be made arbitrarily close to one.

\(^5\)In a companion paper (Kavaler & Smorodinsky (2019)) we use some of the machinery developed here to formulate a ‘well-behaved’ finite test.

\(^6\)In a way the recent success of ‘deep learning’ based on enormous data sets (paralleling our interest in long-run observations) testifies to the importance of patience in model (expert) selection and the benefit of looking at many data points.
2. Model

At the beginning of each period \( t = 1, 2, \ldots \) an outcome \( \omega_t \), drawn randomly by Nature from the set \( \Omega = \{0, 1\} \), is realized. Before \( \omega_t \) is realized, two self-proclaimed experts (sometimes referred to as forecasters) simultaneously announce their forecast in the form of a probability distribution over \( \Omega \).

We assume that both forecasters observe all past outcomes and all previous pairs of forecasts. For any (infinite) realization, \( \omega := \{ \omega_1, \omega_2, \ldots \} \in \Omega^\infty \), we denote by \( \omega^t := \{ \omega_1, \omega_2, \ldots, \omega_t \} \) its prefix of length \( t \) (sometimes referred to as the partial history of outcomes up to period \( t \)), and set \( \omega^0 := \emptyset \).

We will abuse notation and use \( \omega^t \) to denote the cylinder set \( \{ \hat{\omega} \in \Omega^\infty | \hat{\omega}^t = \omega^t \} \). In other words, \( \omega^t \) will also denote the set of realizations which share a common prefix of length \( t \). For any \( t \) we denote by \( g_t \) the \( \sigma \)-algebra on \( \Omega^\infty \) generated by the cylinder sets \( \omega^t \) and let \( g_\infty := \sigma(\bigcup_{t=0}^{\infty} g_t) \) denote the smallest \( \sigma \)-algebra which consists of all cylinders (also known as the Borel \( \sigma \)-algebra). Let \( \Delta(\Omega^\infty) \) be the set of all probability measures defined over the measurable space \( (\Omega^\infty, g_\infty) \).

At each stage, two forecasts (elements in \( \Delta(\Omega) \)) are provided by two experts. Let \( (\Omega \times \Delta(\Omega) \times \Delta(\Omega))^t \) be the set of all sequences composed of outcomes and pairs of forecasts made up to time \( t \) and let \( \bigcup_{t \geq 0} (\Omega \times \Delta(\Omega) \times \Delta(\Omega))^t \) be the set of all such finite sequences.

A (pure) forecasting strategy, \( f \), is a function that maps finite histories to a probability distribution over \( \Omega \). Formally, \( f : \bigcup_{t \geq 0} (\Omega \times \Delta(\Omega) \times \Delta(\Omega))^t \rightarrow \Delta(\Omega) \). Note that each forecast provided by one expert may depend, inter alia, on those provided by the other expert in previous stages. Let \( F \) denote the set of all forecasting strategies.

A probability measure \( P \in \Delta(\Omega^\infty) \) naturally induces a (set of) corresponding forecasting strategy, denoted \( f_P \), that satisfies for any \( \omega \in \Omega^\infty \) and \( t > 0 \) such that \( P(\omega^t) > 0 \),

\[
f_P(\omega^t, \ldots, \omega^t)[\omega_{t+1}] = P(\omega_{t+1} | \omega^t).
\]

Thus, the forecasting strategy \( f_P \) derives its forecasts from the original measure \( P \) via Bayes rule. Note that this does not restrict the forecast of \( f_P \) over cylinders, \( \omega^t \), for which \( P(\omega^t) = 0 \).

In the other direction, we abuse notation and given an ordered pair of forecasting strategies, \( f := (f_0, f_1) \in F \times F \) (henceforth \( f \)), let \( h \) be a function that maps each triplet \( (\omega, f_0, f_1) \) to its uniquely induced play path: \( h(\omega, f_0, f_1) \in (\Omega \times \Delta(\Omega) \times \Delta(\Omega))^\infty \). Additionally, for any \( n \geq 0 \), the prefix (of length

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7 For expository reasons we restrict attention to a binary set \( \Omega = \{0, 1\} \). The results extend to any finite set.

8 Hereinafter we will often abuse notation and use \( P \) instead of \( f_P \).
n) and the suffix (starting at n) of \( h(\omega, f_0, f_1) \) are denoted by \( h^n(\omega, f_0, f_1) \) and \( h_n(\omega, f_0, f_1) \), respectively. Whenever \((\omega, f_0, f_1)\) is clear from the context we abuse notation and denote these by \( h \), \( h^n \), and \( h_n \), respectively.

Now observe that a single forecasting strategy need not induce a measure as its output may also depend on what another expert forecasts. However, an ordered pair of forecasting strategies \( f \), does induce a pair of probability measures, denoted \( P^f_0(\cdot) \), \( P^f_1(\cdot) \), over \( \Omega^\infty \). By Kolomogorov's extension theorem, it is enough to define these probabilities over the cylinder sets of the form \( \omega^t \), denoted for an arbitrary \( \omega \in \Omega^\infty \), \( t > 0 \) and \( i \in \{0, 1\} \):

\[
P^f_i(\omega^t) = \prod_{n=1}^{t} f_i(h^{n-1})[\omega_n]. \tag{1}
\]

### 2.1. Comparison test

A comparison test is a measurable function whose input is a pair of two forecasting strategies and a realization, and whose output is a (weak) order over the two experts. Formally, \( T : \Omega^\infty \times F \times F \rightarrow \{0, \frac{1}{2}, 1\} \) where \( T = i \neq \frac{1}{2} \) implies that expert \( i \) is claimed as better informed, while \( T = \frac{1}{2} \) implies the test is inconclusive (this cannot be avoided, for example, when both experts' forecasts always agree).

A comparison test should, a priori, treat both experts similarly. This is captured by the following notion of anonymity of a test.

**Definition 1.** A test \( T \) is **anonymous** if for all \( \omega \in \Omega^\infty \) and \( f_0, f_1 \in F \),

\[
T(\omega, f_0, f_1) = 1 - T(\omega, f_1, f_0).
\]

In other words, the expert chosen by \( T \) does not depend on the expert’s identity (0 or 1). Note that whenever \( f_0 = f_1 \), an anonymous test \( T \) must be inconclusive and always output 0.5.

We follow the lion’s share of the literature on single expert testing and, furthermore, require that the outcome of the comparison test depends only on predictions made along the realized play path. Formally,

**Definition 2.** A test \( T \) is **non-counterfactual** if there exists a function \( \hat{T} : (\Omega \times \Delta(\Omega) \times \Delta(\Omega))^\infty \rightarrow \{0, \frac{1}{2}, 1\} \) such that \( T = \hat{T} \circ h \).
Hereafter we restrict attention to anonymous and non-counterfactual tests.

2.2. Desired properties

We now turn to formally define two desired properties for a comparison test followed by the motivation. We will later argue that these induce an essentially single comparison test.

The first property requires that whenever one of the experts has the correct model (namely, he would have passed a standard single expert test which has no type-1 errors) the test will surely not point at the second expert as the superior one (albeit, it may provide a non-conclusive outcome).

For any test, \( T \), and an ordered pair of forecasting strategies, \( f \), we denote by \( \{ T(\cdot, f) = k \} \) the set of realizations for which the test outputs \( k \).

**Definition 3.** A test \( T \) is error-free if for all \( f \) and \( i \in \{0, 1\} \), \( P^f_{1-i}(\{ T(\cdot, f) = i \}) = 0 \).

In other words, whenever one expert knows the probability distribution governing the realizations of Nature, the test must not identify the other expert as the true expert. In the jargon of hypothesis testing, Definition 3 implies that an error-free test must eliminate errors of type-1 and, symmetrically, type-2.

One trivial example of an error-free test is the test that constantly outputs \( \frac{1}{2} \). Note that it is also anonymous and non-counterfactual. We shall later propose a non-trivial error-free test. Unfortunately that test will also be indecisive at times but not always. In fact, it turns out that error-free tests must be indecisive whenever the experts induce a pair of measures that are mutually absolutely continuous. Formally,

**Proposition 1.** Let \( f \) be such that \( P^f_1 \ll P^f_0 \). If \( T \) is error-free then \( P^f_0(\{ T(\cdot, f) = 0 \}) < 1 \).

Thus, expert 0, from his own perspective, cannot be confident that the test will identify him as better informed. Combine this with the definition of an error-free test to conclude that from the expert’s point of view that test must, at times, be inconclusive:

**Corollary 1.** Let \( f \) be such that \( P^f_1 \ll P^f_0 \). If \( T \) is error-free then \( P^f_0(\{ T(\cdot, f) = \frac{1}{2} \}) > 0 \).

We now turn to the proof of Proposition 1.

**Proof.** Assume that

\[
P^f_0(\{ T(\cdot, f) = 0 \}) = 1.
\] (2)
Since $P_f^1 \ll P_f^0$ it follows from (2) that

$$P_f^0(\{T(\cdot, f) = 0\}^c) = 0 \implies P_f^1(\{T(\cdot, f) = 0\}^c) = 0.$$ 

Therefore

$$P_f^1(\{T(\cdot, f) = 0\}) = 1,$$

which by the anonymity of $T$ contradicts the assumption that $T$ is error-free.$^9$

The next property of a comparison test asserts that for any set of realizations assigned zero probability by one forecaster and positive probability by the other forecaster, there must be some subset of realizations for which that other forecaster is deemed superior. Formally,

**Definition 4.** A test $T$ is *reasonable* if for all $f$ and $i \in \{0, 1\}$, and for all measurable set $A$,

$$P_f^i(A) > 0 \text{ and } P_{1-i}^i(A) = 0 \implies P_f^i(A \cap \{T(\cdot, f) = i\}) > 0. \quad (3)$$

It should be emphasized that reasonableness and error-free are not related notions. To see why error-free does not imply reasonableness, just consider the constant error-free test $T \equiv \frac{1}{2}$. An example of a reasonable test that is not error-free is deferred to the end of Subsection 4.

### 3. The derivative test

We now turn to our construction of a non-counterfactual, anonymous, error-free and reasonable comparison test. Before doing so, some preliminaries are required.

Given an ordered pair of forecasting strategies, $f$, a realization of Nature, $\omega \in \Omega^{\infty}$, we define the likelihood ratio between the two forecasters at time $t$ as,

$$D_{f_0}^t f_1(\omega) := \prod_{n=1}^{t} \frac{f_1(h^{n-1})[\omega_n]}{f_0(h^{n-1})[\omega_n]}.$$ 

Define the following limit functions:

$$\overline{D}_{f_0} f_1(\omega) := \begin{cases} \limsup_{t \to \infty} D_{f_0}^t f_1(\omega), & f_0(h^{n-1})[\omega_n] > 0 \text{ for all } n \geq 1 \\ +\infty, & f_0(h^{n-1})[\omega_n] = 0 \text{ for some } n. \end{cases}$$

$^9$In the context of hypothesis testing, Corollary 1 implies that an error-free test will not have a power of one whenever the null hypothesis is absolutely continuous w.r.t to the alternative one.
\[ D_{f_0}f_1(\omega) := \begin{cases} 
\liminf_{t \to \infty} D_{f_0}f_1(\omega), & f_0(h^{n-1})[\omega_n] > 0 \text{ for all } n \geq 1 \\
+\infty, & f_0(h^{n-1})[\omega_n] = 0 \text{ for some } n. 
\end{cases} \]

Whenever the two limits coincide and take a finite value, we refer to this value as the derivative of the forecasting strategy \( f_1 \) with respect to the forecasting strategy \( f_0 \) at \( \omega \). Formally, if \( D_{f_0}f_1(\omega) = D_{f_0}f_1(\omega) < \infty \), let \( D_{f_0}f_1(\omega) = \overline{D}_{f_0}f_1(\omega) \) be the derivative of \( f_1 \) with respect to \( f_0 \) at \( \omega \). We are now ready to define the derivative test, denoted \( \mathcal{D} \), a non-counterfactual and anonymous test which we will show is error-free and reasonable:

\[ \mathcal{D}(\omega, f_0, f_1) = \begin{cases} 
1, & D_{f_0}f_1(\omega) = 0 \\
0.5, & \text{other} \\
0, & D_{f_0}f_1(\omega) = 0. 
\end{cases} \tag{4} \]

Expert \( i \) is indicated as the true forecaster at \( \omega \) whenever the derivative of \( f_1 - i \) with respect to \( f_i \) exists and equals 0. Intuitively, this happens when the probability assigned by expert \( i \) to the actual realization is infinitely larger than that assigned by expert \( 1 - i \).

It is obvious that \( \mathcal{D} \) is non-counterfactual and anonymous. We turn to prove that \( \mathcal{D} \) is also error-free and reasonable. To do so, we will need the following two technical observations regarding derivatives of forecasting strategies:

**Lemma 1.** Fix \( 0 < \alpha < \infty \) and let \( A \subset \Omega^\infty \) be a measurable set. Then
\[ a) \ A \subset \{ \omega \mid D_{f_0}f_1(\omega) \leq \alpha \} \implies P_{f_0}^i(A) \leq \alpha P_{f_0}^i(A). \]
\[ b) \ A \subset \{ \omega \mid D_{f_0}f_1(\omega) \geq \alpha \} \implies P_{f_0}^i(A) \geq \alpha P_{f_0}^i(A). \]

**Lemma 2.** For all \( f \), \( D_{f_0}f_1 \) exists and is finite \( P_{f_0}^i \) - a.e.

The proof of Lemmas 1 and 2 are relegated to Appendix A.

3.1. \( \mathcal{D} \) is error-free and reasonable

Now that we have established the existence and the finiteness of the test \( \mathcal{D} \), let us prove it complies with the two central properties for comparison tests:

**Theorem 1.** The derivative test, \( \mathcal{D} \), is a reasonable and error-free test.
Proof. **Part 1 - $\mathcal{D}$ is reasonable:** Let $A$ be a measurable set and assume (w.l.o.g) that

$$P_0^f(A) > 0 \text{ and } P_1^f(A) = 0. \tag{5}$$

For $a > 0$ let us denote $R_a := A \cap \{\omega | 0 < a \leq D_{f_0} f_1(\omega) < \infty\}$. Note that if $P_0^f(R_a) > 0$ then applying part b of Lemma 1 yields

$$P_1^f(R_a) \geq aP_0^f(R_a) > 0$$

which contradicts (5). Therefore,

$$P_0^f(A \cap \{\omega | 0 < D_{f_0} f_1(\omega) < \infty\}) = P_0^f \left( \bigcup_{0 < a} R_a \right) \leq \sum_{0 < a} P_0^f(R_a) = 0. \tag{6}$$

Since, by Lemma 2, $D_{f_0} f_1$ exists and is finite $P_0^f - a.e.$, we conclude that

$$P_0^f(A \cap \{\omega | D_{f_0} f_1(\omega) = 0\}^c) = 0.$$ 

Hence,

$$0 < P_0^f(A) = P_0^f(A \cap \{\omega | D_{f_0} f_1(\omega) = 0\}) = P_0^f(A \cap \{\mathcal{D}(\cdot, f) = 0\}), \tag{6}$$

where the right-most equality follows from (4). Inequality (6) implies that the test $\mathcal{D}$ is reasonable.

**Part 2 - $\mathcal{D}$ is error-free:** Note (w.l.o.g) that

$$\{\mathcal{D}(\cdot, f) = 1\} = \{\omega | \lim_{t \to \infty} D_t^1 f_0(\omega) = 0 \text{ and } f_1(h^{n-1})[\omega_n] > 0 \text{ for all } n \geq 1\}$$

$$\subset \{\omega | \lim_{t \to \infty} D_t^1 f_1(\omega) = \infty\} \cup \{\omega | f_0(h^{n-1})[\omega_n] = 0 \text{ for some } n\}$$

$$\subset \{\omega | D_{f_0} f_1(\omega) = D_{f_0} f_1(\omega) = \infty\}. \tag{7}$$

By Lemma 2, $D_{f_0} f_1$ is finite $P_0^f - a.e.$; thus

$$P_0^f(\{\mathcal{D}(\cdot, f) = 1\}) \leq P_0^f(\{\omega | D_{f_0} f_1(\omega) = D_{f_0} f_1(\omega) = \infty\}) = 0,$$

and $\mathcal{D}$ is error-free. \hfill \Box
Remark 1. The test $\mathcal{D}$ and its key properties can be usefully viewed as an implication of the Lebesgue decomposition (Billingsley (1995), Section 31). A standard decomposition usually involves a decomposition of one measure with respect to another into a singular part and an absolutely continuous part. Here, it is applied in both directions in such a way that allows some flexibility on how measure-zero sets are handled. Given a pair of forecasting strategies $f$, we decompose the set $\Omega^\infty$ into three sets: $\{\mathcal{D}(\cdot, f) = 1\}$ which corresponds to expert 1’s induced measure $P_f^1$, $\{\mathcal{D}(\cdot, f) = 0\}$ which corresponds to expert 0’s induced measure $P_f^0$, and $\{\mathcal{D}(\cdot, f) = \frac{1}{2}\}$ where the measures are mutually absolutely continuous. The outcome of the test is found accordingly.

3.2. The uniqueness of $\mathcal{D}$

Although there may be other error-free and reasonable comparison tests they are essentially equivalent to the derivative test. To capture this idea we introduce the following equivalence relation over tests:

Definition 5. We say that the test $T$ is equivalent to the test $\hat{T}$ with respect to the pair of forecasters $f$, denoted $T \sim_{f} \hat{T}$, if and only if for all $i \in \{0, 1\}$,

$$P_f^i(\{\omega | T(\omega, f_0, f_1) \neq \hat{T}(\omega, f_0, f_1)\}) = 0.$$ 

$T$ is equivalent to the test $\hat{T}$, denoted $T \sim \hat{T}$, if and only if $T$ is equivalent to the test $\hat{T}$ with respect to any pair of forecasters.

Proposition 2. The relation $\sim$ is an equivalence relation over the set of all comparison tests.

The proof of Proposition 2 is relegated to Appendix A. To establish the theorem about the essential uniqueness of the derivative test we will consider an arbitrary anonymous, non-counterfactual, reasonable test, $T$, that is not equivalent to $\mathcal{D}$. We will then argue that $T$ cannot be error-free. We will do so by constructing a pair of forecasting strategies for which the error-free condition fails.

Theorem 2. Let $T$ be an anonymous, non-counterfactual, reasonable test. If $T \not\sim \mathcal{D}$ then $T$ is not error-free.

Proof. Assume by contradiction that $T$ is error-free. Let $f$ be such that $T \not\sim \mathcal{D}$, then (w.l.o.g.) $\exists k, l(\neq k) \in \{0, \frac{1}{2}, 1\}$ such that

10In fact we show a much stronger result; Theorem 2 asserts that $T$ admits an error with respect to any pair of forecasting strategies for which $T$ is not equivalent to $\mathcal{D}$. 
\[ P_0^f(\{T(\cdot, f) = 1\} \cap \{\mathcal{D}(\cdot, f) = k\}) > 0. \]

In addition, by Part 2 of Theorem 1, \( \mathcal{D} \) is error-free; therefore

\[ P_0^f(\{T(\cdot, f) = 1\}) = P_0^f(\{\mathcal{D}(\cdot, f) = 1\}) = 0 \]

and consequently,

\[ P_0^f(A_1 := \{T(\cdot, f) = 0\} \cap \{\mathcal{D}(\cdot, f) = \frac{1}{2}\}) > 0 \text{ or } P_0^f(A_2 := \{T(\cdot, f) = \frac{1}{2}\} \cap \{\mathcal{D}(\cdot, f) = 0\}) > 0. \]

Case 1: \( P_0^f(A_1) > 0 \). By Part 1 of Theorem 1, \( \mathcal{D} \) is reasonable; thus

\[ P_1^f(A_1) = 0 \implies P_0^f(A_1 \cap \{\mathcal{D}(\cdot, f) = 0\}) > 0 \]

which leads to a contradiction, since \( \{\mathcal{D}(\cdot, f) = 0\}, \{\mathcal{D}(\cdot, f) = \frac{1}{2}\} \) are disjoint. Thus

\[ P_1^f(\{T(\cdot, f) = 0\}) > 0 \]

which contradicts the assumption that \( T \) is error-free.

Case 2: \( P_0^f(A_2) > 0 \). By the assumption, \( T \) is a reasonable test where, by Part 2 of Theorem 1, \( \mathcal{D} \) is error-free; therefore the contradiction

\[ P_1^f(\{\mathcal{D}(\cdot, f) = 0\}) > 0 \]

follows analogously from Case 1.

4. Tail tests

In the introduction, we state our intention to study tests in which decisions are made for the distant future. In this section we take this a step further and consider tests which not only enable decisions to be made for the distant future, but also **only** for the distant future.

The motivation for this is that a tester must allow the two forecasters (some time) to accumulate data so they can calibrate their model. A forecaster may have a very good parametric model in mind but can only calibrate the values of the parameters by observing enough data. A test that allows for such an initial calibration stage is called a **tail test**. Formally,
Definition 6. The pair of triplets, \((\omega, f_0, f_1), (\tilde{\omega}, \tilde{f}_0, \tilde{f}_1)\) \(\in \Omega^\infty \times F \times F\), eventually coincide if there exists \(n > 1\) such that for all \(1 \leq t \leq n - 1\), \(i \in \{0, 1\}\),

\[
h_n = \tilde{h}_n \quad \text{and} \quad f_i(h_t^{-1})(\omega_1) > 0, \tilde{f}_i(h_t^{-1})(\tilde{\omega}_1) > 0
\]

(7)

(where \(\tilde{h}_n := h_n(\tilde{\omega}, \tilde{f}_0, \tilde{f}_1)\), \(\tilde{h}_t := h_t^{-1}(\tilde{\omega}, \tilde{f}_0, \tilde{f}_1)\)).

In words, the two play paths agree from some time on whenever the prefix has mutually positive probability.

Definition 7. \(T\) is a tail test if whenever a pair of triplets, \((\omega, f_0, f_1), (\tilde{\omega}, \tilde{f}_0, \tilde{f}_1)\) \(\in \Omega^\infty \times F \times F\), eventually coincide, then \(T(\omega, f_0, f_1) = T(\tilde{\omega}, \tilde{f}_0, \tilde{f}_1)\).

In layman’s terms, a tail test ignores the prefix of the sequence and makes the comparison between the two experts based on the suffix of forecasts and realizations.

It turns out that the derivative test also conforms with the tail property:

Theorem 3. \(D\) is a tail test.

Proof. Let \((\omega, f_0, f_1), (\omega', f'_0, f'_1)\) \(\in \Omega^\infty \times F \times F\) be a pair of triplets that eventually coincide for some \(n > 1\). Let \((\omega'', f''_0, f''_1)\) \(\in \Omega^\infty \times F \times F\) be a triplet that satisfies

\[
h_1(\omega'', f''_0, f''_1) = h_n(\omega, f_0, f_1).
\]

(8)

Since, by the right part of (7), \(D_{f_0}f_1(\omega) > 0\) for all \(1 \leq t < n - 1\), it follows from (8) that \[1\]

\[
0 = D_{f_0}f_1(\omega'') \iff 0 = D_{f_0}^{-1}(\omega') \cdot D_{f_0}f''_1(\omega'') = D_{f_0}f_1(\omega) \iff D(\omega, f_0, f_1) = 0.
\]

Additionally, by the same consideration we have

\[
1 = D(\omega'', f''_0, f''_1) \iff 0 = D_{f_0}f''_1(\omega'') \iff D(\omega, f_0, f_1) = 1,
\]

and therefore

\[1\]

Note that \(D_{f_0}f''_1(\omega'') = 0\) if and only if \(D_{f_0}f_1(\omega'')\) exists and equals 0.
\[ \mathcal{D}(\omega'', f''_0, f''_1) = \mathcal{D}(\omega, f_0, f_1). \]

Similarly, we show that \( \mathcal{D}(\omega'', f''_0, f''_1) = \mathcal{D}(\omega', f'_0, f'_1) \) by replacing \( h_n(\omega, f_0, f_1) \) with \( h_n(\omega', f'_0, f'_1) \) in (8) and this concludes the proof.

To establish that \( \mathcal{D} \) is unique among all reasonable error-free tests, we showed that for an arbitrary non-equivalent yet reasonable test there must be some error. What we have shown is that there is a pair of experts where one expert will assign a positive probability to the test pointing at the other expert as more informative. That probability, the error probability, although positive is possibly very small. It turns out that if we restrict the discussion to tail tests, the uniqueness of \( \mathcal{D} \) comes in a stronger form, as the error probability can be made arbitrarily close to one. In other words, for an arbitrary non-equivalent reasonable tail test and \( 0 < \varepsilon < 1 \), there exists a pair of experts for which one expert assigns a probability of \( 1 - \varepsilon \) to the other expert being deemed more informative.

Before we state this theorem we require the following lemma.

**Lemma 3.** If \( T \) is reasonable then for all \( f \) and \( i \in \{0, 1\}, k \neq i \), and for all measurable set \( A \),

\[ P^i_f(A \cap \{T(\cdot, f) = k\}) > 0 \implies P^i_{1-i}(A \cap \{T(\cdot, f) = k\}) > 0. \]

**Proof.** Let \( A \) be a measurable set and (w.l.o.g) assume by contradiction that

\[ P^i_f(A \cap \{T(\cdot, f) = k\}) > 0 \text{ and } P^i_{1-i}(A \cap \{T(\cdot, f) = k\}) = 0 \]

for some \( k \in \{0, 1\} \). \( T \) is reasonable; thus (3) yields \( P^i_f(A \cap \{T(\cdot, f) = k\} \cap \{T(\cdot, f) = 1\}) > 0 \) which contradicts the fact that \( \{T(\cdot, f) = k\}, \{T(\cdot, f) = 1\} \) are disjoint sets. \( \Box \)

Now we turn to establish a strong version of the uniqueness of \( \mathcal{D} \):

**Theorem 4.** Let \( T \) be an anonymous, non-counterfactual, reasonable tail test. If \( T \not\sim \mathcal{D} \) then for all \( 0 < \varepsilon < 1 \) there exists \( \tilde{f} := (\tilde{f}_0, \tilde{f}_1) \) such that

\[ P^i_0(\{T(\cdot, \tilde{f}) = 1\}) > 1 - \varepsilon \text{ or } P^i_1(\{T(\cdot, \tilde{f}) = 0\}) > 1 - \varepsilon. \]
Proof. By Theorem (w.l.o.g.) there exists a pair \( f := (f_0, f_1) \) such that \( P^f_1(\{T(\cdot, f) = 0\}) > 0 \). In addition, since \( \{T(\cdot, f) = 0\} \) is \( g_\infty \) measurable we can apply the Levy upwards theorem (Williams (1991), Theorem 14.2.) to obtain

\[
\lim_{t \to \infty} P^f_1(\{T(\cdot, f) = 0\} \mid g_t) = \lim_{t \to \infty} E^{g_t}\left[1_{\{T(\cdot, f) = 0\}} \mid g_\infty\right] = 1_{\{T(\cdot, f) = 0\}}, P^f_1 - a.s.
\]

Therefore, there exists \( B^f \subset \{T(\cdot, f) = 0\} \) with \( P^f_1(B^f) = P^f_1(\{T(\cdot, f) = 0\}) \) such that for all \( \omega \in B^f \) and for all \( t \geq 1 \),

\[
\lim_{t \to \infty} P^f_1(\{\omega^t = 0\} \mid \omega^t) = 1 \text{ and } f_1(h^{t-1}(\omega, f_0, f_1)) f_t > 0.
\]

(9)

Let \( 0 < \epsilon < 1 \). Fix \( \tilde{\omega} \in B^f \) and observe that from (9) there exists \( n = n(\epsilon, \tilde{\omega}, f) > 1 \) such that for all \( t \geq n - 1 \),

\[
P^f_1(\{T(\cdot, f) = 0\} \cap \omega^t) > (1 - \epsilon)P^f_1(\omega^t) > 0.
\]

Thus, applying Lemma yields \( P^f_0(\{T(\cdot, f) = 0\} \cap \omega^{n-1}) > 0 \) and consequently, \( f_0(h^{t-1}(\omega, f_0, f_1)) f_t > 0 \) for all \( 1 \leq t \leq n - 1 \), is inferred from (1). Now, modify \( f \) to be the forecasting strategy \( \hat{f} \) which one-step-ahead conditionals satisfy

\[
\hat{f}_i(\omega^{t-1}, \cdot, \cdot)[\omega_t] = \begin{cases} 1, & \omega^t = \omega^t, t < n \\ \hat{f}_i(\omega^{t-1}, \cdot, \cdot)[\omega_t], & \text{other} \\ 0, & \omega^t \neq \omega^t, t < n. \end{cases}
\]

Observe that, by construction, for all \( \omega \in \{T(\cdot, f) = 0\} \cap \omega^{n-1} \) we obtain \( h_n(\omega, f_0, f_1) = h_n(\omega, \hat{f}_0, \hat{f}_1) \), and in addition to that, for all \( 1 \leq t \leq n - 1 \), \( i \in \{0, 1\} \),

\[
f_i(h^{t-1}(\omega, f_0, f_1)) f_t > 0, \hat{f}_i(h^{t-1}(\omega, \hat{f}_0, \hat{f}_1)) f_t > 0.
\]

Note that the corresponding forecasting strategy \( \hat{f} \) determines the one-step-ahead forecasts up to time \( n \) only through the history of outcomes and does not depend on the full histories.
Hence, \((\omega, f_0, f_1), (\omega, \hat{f}_0, \hat{f}_1)\) eventually coincide by \((7)\) and since \(T\) is a tail test it follows that \(T(\omega, \hat{f}_0, \hat{f}_1) = T(\omega, f_0, f_1) = 0\) yielding \(\omega \in \{T(\cdot, \hat{f}) = 0\} \cap \bar{\omega}^{n-1}\). As a result,

\[
P^i_1(\{T(\cdot, f) = 0\})
\]

\[
= P^i_1(\{T(\cdot, f) = 0\}) \geq P^i_1(\{T(\cdot, f) = 0\}) \bar{\omega}^{n-1} = P^i_1(\{T(\cdot, f) = 0\}) \bar{\omega}^{n-1} > 1 - \epsilon,
\]

and therefore completes the proof.

Unfortunately, as the next example shows, this strong version of uniqueness cannot be established without resorting to tail tests. The same example also serves to demonstrate that a reasonable test is not necessarily error-free.

**Example 1.** Assuming that from day two onward, along a realization \(\hat{\omega} := (1, 1, 1, \ldots)\), two forecasting strategies are shown to have similar predictions, according to an IID distribution with parameter 1, where on day one, one expert assigns 1 to the outcome 1 whereas the other expert assigns half. Let \(\vec{h}, \overleftarrow{h}\) denote the corresponding uniquely induced play paths and consider the following test:

\[
T(\omega, f_0, f_1) = \begin{cases} 
\mathcal{D}(\omega, f_0, f_1), & \text{other} \\
0, & h = \overleftarrow{h} \\
1, & h = \vec{h}.
\end{cases}
\]

Note, for every triplet \((\omega, f_0, f_1)\), whose induced play path coincides with \(\vec{h}\) or \(\overleftarrow{h}\), there exists \(i \in \{0, 1\}\) such that

\[
P^i_1(\{T(\cdot, f) = 1 - i\}) = P^i_1(\{\hat{\omega}\}) = \frac{1}{2} < 1
\]

where the most-left equality holds, since \(\mathcal{D}\) is error-free. Moreover, since \(P^i_1(\{\hat{\omega}\}) > 0\) for all \(i \in \{0, 1\}\) and \(\mathcal{D}\) is a reasonable test, it follows that \(T\) is reasonable even as it admits a bounded error by \((10)\). The fact that \(T\) is not a tail test follows directly from the anonymity of \(T\) along \(\vec{h}, \overleftarrow{h}\).

5. Ideal Tests

Recall that an error-free test eliminates the occurrences in which the less-informed expert is pointed out. A stronger and more appealing property is to point out the better-informed expert. Informally,
we would like to consider tests that have the following property: \( P^f_i(\{T(\cdot, f) = i\}) = 1 \) whenever \( f_0 \neq f_1 \). However, there could be pairs of forecasters that are not equal but induce the same probability distribution.

**Definition 8.** A test \( T \) is ideal with respect to \( W \subseteq F \) if for all \( f \in W \times W \) and \( i \in \{0, 1\} \) such that \( p^f_i \neq p^f_{1-i} \),

\[
P^f_i(\{T(\cdot, f) = i\}) = 1.
\]

It is called ideal if it is ideal with respect to \( F \).

In other words, whenever expert \( i \) knows the actual data generating process and expert \( 1-i \) does not, an ideal test will surely identify the informed expert. In addition, it is a straightforward corollary of Proposition 1 that there exists no ideal test with respect to a set of forecasting strategies whenever one induced measure is absolutely continuous with respect to the other.

It is a common notion that two measures \( P, Q \) are mutually singular with respect to each other, denoted \( P \perp Q \), if there exists a set \( A \) such that \( P(A) = Q(A^c) = 1 \).

**Definition 9.** Two forecasting strategies, \( f_0, f_1 \in F \), are said to be mutually singular with respect to each other, if \( P^f_0 \perp P^f_1 \). A set \( W \subseteq F \) is pairwise mutually singular if for any pair \( f \in W \times W \) such that \( P^f_0 \neq P^f_1 \), \( f_0, f_1 \) are mutually singular with respect to each other.

In other words, two forecasting strategies are mutually singular with respect to each other if their corresponding induced measures are mutually singular with respect to each other. The next lemma asserts that a reasonable test is able to perfectly distinguish between ‘far’ measures which are induced from forecasting strategies which are said to be mutually singular with respect to each other.

**Lemma 4.** Let \( f_0, f_1 \in F \) be mutually singular with respect to each other. If \( T \) is reasonable then for all \( i \in \{0, 1\} \),

\[
P^f_i(\{T(\cdot, f) = i\}) = 1.
\]

The proof of Lemma 4 is relegated to Appendix A. It should be noted that Lemma 4 holds even for \( T \) which is not error-free.

The next theorem provides a necessary and sufficient condition for the existence of an ideal test over sets.
Theorem 5. There exists a non-counterfactual anonymous ideal test with respect to \( W \) if and only if \( W \) is pairwise mutually singular.

Proof. \( \Leftarrow \) From Lemma 4 and Part 1 of Theorem 1 we conclude that \( \mathcal{D} \) is an ideal test with respect to \( W \).

\( \implies \) Let \( T \) be a non-counterfactual anonymous ideal test with respect to a set \( W \). Let \( f \in W \times W \) be such that \( P^f_0 \neq P^f_1 \) and observe that since \( \{T(\cdot,f) = 0\}, \{T(\cdot,f) = 1\} \) are disjoint and \( T \) is ideal, we obtain

\[
1 = P^f_1(\{T(\cdot,f) = i\}) = P^f_{1-i}(\{T(\cdot,f) = i\})
\]

for all \( i \in \{0,1\} \), yielding \( W \) that is pairwise mutually singular.

We conclude the section with an example of a test over a domain of mutually singular forecasts:

Example 2. Let

\[
W_{IID} \times W_{IID} := \{ f \mid \forall i \in \{0,1\} \exists a_i \in [0,1] \text{ s.t. } \forall \omega \in \Omega^\infty, f_i(\omega^t,\cdot,\cdot)[1] \equiv a_i \}.
\]

For \( \omega \in \Omega^\infty \) denote the average realization by

\[
a_\omega := \lim_{t \to \infty} \left( \frac{\sum_{n=1}^{t} 1_{\omega_n = 1}}{t} \right)
\]

(whenever the limit exists) and consider the following comparable test

\[
T(\omega,f_0,f_1) = \begin{cases} 
1, & f_1(h^0)[1] = a_\omega \neq f_0(h^0)[1] \\
0.5, & \text{other} \\
0, & f_0(h^0)[1] = a_\omega \neq f_1(h^0)[1].
\end{cases}
\]

Obviously, \( T \) is well-defined, anonymous and non-counterfactual. Showing that \( T \) is ideal with respect to \( W_{IID} \) is a mere application of the law of large numbers.

6. Existing tests

It is natural to inquire whether comparison tests previously proposed comply with the properties we introduced. We turn to discuss the tests proposed in Al-Najjar & Weinstein (2008) and Feinberg &
It turns out that neither of these tests satisfies the full axiomatic system which was introduced in Subsections 2.1 and 2.2, and hence does not belong to the equivalence class represented by $\mathcal{D}$.

6.1. The likelihood ratio test

Al-Najjar and Weinstein (2008) introduced the following test:

$$L(\omega, f_0, f_1) = \begin{cases} 
1, & \liminf_{t \to \infty} D_t^{f_0} f_1(\omega) > 1 \\
0.5, & \text{other} \\
0, & \limsup_{t \to \infty} D_t^{f_0} f_1(\omega) < 1.
\end{cases}$$

In other words, a likelihood ratio of one suggests that both experts are likely equal and so the test cannot determine which is better. Similarly, the same conclusion holds whenever the likelihood ratio oscillates infinitely often below and above one. Otherwise, if the likelihood ratio is eventually greater than one (smaller than one) then expert 1 (expert 0) is deemed superior. Note that this test differs from $\mathcal{D}$ whenever the likelihood ratio is high but finite. In our case, the test does not prefer any expert, whereas the test $L$ does. It turns out that this test does not satisfy all the properties we introduce:

Claim 1. $L$ is reasonable and is not error-free.

Proof. Let $f_1$ be a forecasting strategy which deterministically predicts $\hat{\omega}$. Let $0 < \varepsilon < 1$ and let $f_0$ be the forecasting strategy which predicts $(1 - \varepsilon)$ at day one and meets $f_1$ from day two onward regardless of any past history. Note that if $P_0^f$ is the true measure, then $L(\hat{\omega}, f_0, f_1) = \frac{1}{1-\varepsilon} > 1$ yielding $P_0^f(\{L(\cdot, f) = 1\}) \geq 1 - \varepsilon$. As a result, since $\varepsilon$ is taken arbitrarily, not only is $L$ not error-free but it admits an arbitrarily large error. The fact that $L$ is reasonable follows directly from Part 1 of Theorem 1.

6.2. The cross-calibration test

The cross-calibration test introduced in Feinberg and Stewart (2008) checks the empirical frequencies of the realization conditional on each profile of forecasts that occurs infinitely often (please refer to Appendix B for a formal definition). The test outputs a binary verdict (pass/fail) for each of the experts separately, but does not rank them; nevertheless, it induces a natural comparison test, $T_{\text{cross}}$, defined as follows: $T_{\text{cross}} = \frac{1}{2}$ if and only if both experts either pass or fail the cross-calibration test whereas $T_{\text{cross}} = i$ if and only if expert $i$ passes the cross-calibration test and expert $1 - i$ fails.

Electronic copy available at: https://ssrn.com/abstract=3288814
Claim 2. $T_{\text{cross}}$ is error-free and is not reasonable.

Proof. Let $f_0, f_1$ be forecasting strategies which deterministically predict $\omega := (0, 1, 1, \ldots), \omega$, respectively, and observe that since both $f_0$ and $f_1$ pass the cross-calibration test on $h(\omega, f_0, f_1)$ it follows that $T_{\text{cross}}(\omega, f_0, f_1) = \frac{1}{2}$ yielding

$$1 = P^f_0(\{\omega\}) \leq P^f_0(\{T_{\text{cross}}(\cdot, f) = \frac{1}{2}\}). \quad (11)$$

However, $f_0, f_1$ are mutually singular with respect to each other; so if $T_{\text{cross}}$ was a reasonable test then, by Lemma 4 it would satisfy

$$P^f_0(\{T_{\text{cross}}(\cdot, f) = 0\}) = 1$$

which contradicts (11) and therefore $T_{\text{cross}}$ is not reasonable. The fact that $T_{\text{cross}}$ is error-free follows immediately from Dawid (1982) and hence omitted.

One could suspect that the counterexample used in the proof of Claim 2 builds on the fact that both experts use some Dirac measure and so assign zero probability to any finite history that disagrees with that measure. Thus, a counterexample where both forecasters assign a positive probability to any finite history is provided in Appendix B.

7. Summary

We study tests that compare two (self-proclaimed) experts in light of some infinite sequence of forecasts and outcomes, where the goal of the test is to spot the better informed one. We propose some natural properties for such tests and construct the unique test (up to an equivalence class) that complies with these properties. In Kavaler & Smorodinsky (2019) we propose a framework where a comparison test provides a verdict in finite time. We adapt the four properties to the new setting and similarly propose a unique test for that environment. Some natural directions for future research are to extend our results to settings with more than two experts and to study alternative sets of properties.

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References

Al-Najjar, N., Sandroni, A., Smorodinsky, R., & Weinstein, J. (2010). Testing theories with learnable and predictive representations. *Journal of Economic Theory, 145*(6), 2203–2217.

Al-Najjar, N., & Weinstein, J. (2008). Comparative testing of experts. *Econometrica, 76*(3), 541–559.

Billingsley, P. (1995). *Probability and Measure*. (3rd ed.). New York.

Dawid, P. (1982). The well-calibrated bayesian. *Journal of the American Statistical Association, 77*, 605–613.

Dekel, E., & Feinberg, Y. (2006). Non-bayesian testing of a stochastic prediction. *Review of Economic Studies, 73*, 893–936.

Echenique, F., & Shmaya, E. (2008). You won’t harm me if you fool me. *Mimeo, .*

Edwards, A. (1972). *Likelihood*. Cambridge University Press.

Feinberg, Y., & Stewart, C. (2008). Testing multiple forecasters. *Econometrica, 76*, 561–582.

Fortnow, L., & Vohra, R. (2009). The complexity of forecast testing. *Econometrica, 77*, 93–105.

Foster, D., & Vohra, R. (1998). Asymptotic calibration. *Biometrika, 85*, 379–390.

Kavaler, I., & Smorodinsky, R. (2019). A cardinal comparison of experts. *Mimeo, .*

Lehrer, E. (2001). Any inspection is manipulable. *Econometrica, 69*, 1333–1347.

Olszewski, W., & Sandroni, A. (2008). Manipulability of future-independent tests. *Econometrica, 76*, 1437–1466.

Pomatto, L. (2016). Testable forecasts. *Caltech. Mimeo, .*

Sandroni, A. (2003). The reproducible properties of correct forecasts. *International Journal of Game Theory, 32*, 151–159.
APPENDIX

Appendix A. Missing proofs

Lemma 5. Let $B := \{ B_i \}_{i \in \mathbb{N}}$ be an arbitrary sequence of cylinders and set $B := \bigcup_{i \in \mathbb{N}} B_i$. Then, there exists an index set $J \subseteq \mathbb{N}$ such that $\{ B_j \}_{j \in J}$ are pairwise disjoint, and $B = \bigcup_{j \in J} B_j$.

Proof. A cylinder is called maximal in $B$ if it is not a subset of any other cylinders in $B$. Any cylinder in $B$ is contained in some maximal cylinder in $B$. Let $J \subseteq \mathbb{N}$ be such that $\{ B_j \}_{j \in J}$ is the set of all distinct maximal cylinders. Since any two distinct maximal cylinders are disjoint it follows that $B = \bigcup_{j \in J} B_j$. $\square$

Proof of Lemma 1. (a) Let $A$ be a measurable set which satisfies the left side of (a) and let $U \subset \Omega^\infty$ be any open set such that $A \subset U$. Fix $\epsilon > 0$, then for all $a \in A, N > 0$ there exists $t = t_{(a,N,\epsilon)} > N$ such that

$$D_{f_0}^t f_1(a) = \frac{\prod_{n=1}^t f_1(h^{n-1}(a,f_0,f_1))[a_n]}{\prod_{n=1}^t f_0(h^{n-1}(a,f_0,f_1))[a_n]} = \frac{P_1^t(a^t)}{P_0^t(a^t)} \leq (\alpha + \epsilon). \quad (A.1)$$

Consider the following set of cylinders

$$\mathcal{B} := \{ a^t \subset U \mid a \in A, \ t > 0, \ P_1^t(a^t) \leq (\alpha + \epsilon)P_0^t(a^t) \}. $$

Note, it follows from (A.1) that $\mathcal{B}$ is not empty where $\sup \{ t \mid a^t \in \mathcal{B} \} = \infty$. By Lemma 5 we are provided with an index set $J$ and a collection of pairwise disjoint sets $\{ B_j \in \mathcal{B} \}_{j \in J}$ such that

$$B := \bigcup_{B \in \mathcal{B}} B = \bigcup_{j \in J} B_j \quad (A.2)$$

yielding that \( A \subseteq B \) and \( B_j \in \mathcal{B} \). Hence,

\[
\begin{align*}
P_1^f(A) & \leq P_1^f(B) = P_1^f \left( \bigcup_{j \in J} B_j \right) \\
& \leq \sum_{j \in J} P_1^f(B_j) \leq (a + \epsilon) \sum_{j \in J} P_0^f(B_j) \\
& \leq (a + \epsilon) P_0^f(U),
\end{align*}
\]

where the most-right inequality holds since \( U \supset B_j \)'s are disjoint.

Since the above inequalities hold for any open set \( U \) which contains \( A \) and

\[
P_0^f(A) = \inf_{U-\text{open} : A \subseteq U} \{ P_0^f(U) \},
\]

it follows that for all \( \epsilon > 0 \),

\[
P_1^f(A) \leq (a + \epsilon) P_0^f(A)
\]

which completes the proof of Case (a). The proof of Case (b) is analogous and hence omitted. \( \square \)

We now turn to show that the derivative of one measure with respect to another exists and is finite almost surely.

**Proof of Lemma 2**

Let \( I := \{ \omega \mid \overline{D}_f f_1(\omega) = +\infty \} \). Therefore, for all \( \alpha > 0 \),

\[
I \subset \{ \omega \mid \overline{D}_f f_1(\omega) \geq \alpha \}
\]

and it follows from part b of Lemma 1 that \( P_0^f(I) \leq \frac{1}{\alpha} P_1^f(I) \). Now let \( \alpha \to \infty \) to obtain

\[
P_0^f(I) = 0, \tag{A.3}
\]

and consequently \( \overline{D}_f f_1 \) is finite \( P_0^f - a.e. \). For the second part let

\[
R(a, b) := \{ \omega \mid D_f f_1(\omega) < a < b < \overline{D}_f f_1(\omega) < \infty \}.
\]

Note that

\[
R(a, b) \subset \{ \omega \mid D_f f_1(\omega) \leq a \}
\]

as well as

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\[ R(a, b) \subset \{ \omega | \overline{D}_{f_0}f_1(\omega) \geq b \} \]

where applying Lemma 1 in both directions gives

\[ bP_0^f(R(a, b)) \leq P_0^f(R(a, b)) \leq aP_0^f(R(a, b)) \]

Hence, for all \(0 < a < b\),

\[ P_0^f(R(a, b)) = 0 \]

(A.4)

where from (A.3) and (A.4) we obtain

\[ P_0^f(\{ \omega | D_{f_0}f_1(\omega) < \overline{D}_{f_0}f_1(\omega) < \infty \}) = P_0^f(\bigcup_{0 < a < b} R(a, b)) \leq \sum_{a,b \in \mathbb{Q}} P_0^f(R(a, b)) = 0. \]

Therefore, \(D_{f_0}f_1\) exists \(P_0^f\) - a.e. ∎

**Proof of Proposition 2**

Let \(T, T_1, T_2 \in \mathbb{T}, f \in F, \) and \(i \in \{0, 1\} \).

**Reflexivity:**

\[ P_i^f(\{ \omega | T(\omega, f_0, f_1) \neq T(\omega, f_0, f_1) \}) = 0 \implies T \sim T. \]

**Symmetry:**

\[ P_i^f(\{ \omega | T_1(\omega, f_0, f_1) \neq T_2(\omega, f_0, f_1) \}) = 0 \iff P_i^f(\{ \omega | T_2(\omega, f_0, f_1) \neq T_1(\omega, f_0, f_1) \}) = 0; \]

hence, \(T_1 \sim T_2 \iff T_2 \sim T_1. \)

**Transitivity:** Assume that \(T_1 \sim T,\) and \(T \sim T_2;\) hence

\[ T_1 \sim T \implies P_i^f(\{ \omega | T_1(\omega, f_0, f_1) \neq T(\omega, f_0, f_1) \}) = 1, \]

as well as

\[ T \sim T_2 \implies P_i^f(\{ \omega | T(\omega, f_0, f_1) \neq T_2(\omega, f_0, f_1) \}) = 1. \]

Thus

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Proof of Lemma 4. W.l.o.g. let $A$ be such that: $P^f_0(A) = 1, P^f_1(A) = 0$. $T$ is reasonable, therefore $P^f_0(A \cap \{T(\cdot, f) = k\}) > 0$ from (3). Let $k \in \{\frac{1}{2}, 1\}$ and assume that

$$P^f_0(A \cap \{T(\cdot, f) = k\}) \neq 0.$$  

Lemma 3 yields

$$P^f_1(A \cap \{T(\cdot, f) = k\}) > 0$$

which contradicts the assumption that $P^f_1(A) = 0$. Hence, $P^f_0(A \cap \{T(\cdot, f) = k\}) = 0$. As a result,

$$P^f_0(A \cap \{T(\cdot, f) = 0\}) = P^f_0(A) = 1$$

and therefore $P^f_0(\{T(\cdot, f) = 0\}) = 1$.

Appendix B. The cross-calibration test

We now restate the cross-calibration test as suggested by Feinberg & Stewart (2008). Fix a positive integer $N > 4$ and divide the interval $[0, 1]$ into $N$ equal closed subintervals $I_1, ..., I_N$, so that $I_j = [\frac{j-1}{N}, \frac{j}{N}], \ 1 \leq j \leq N$. All results in their paper hold when $[0, 1]$ is replaced with the set of distributions over any finite set and the intervals $I_j$ are replaced with a cover of the set of distributions by sufficiently small closed convex subsets. At the beginning of each period $t = 1, 2, ..., \ all forecasters (or experts) $i \in \{0, ..., M-1\}$ simultaneously announce predictions $I^i_t \in \{I_1, ..., I_N\}$, which are interpreted as probabilities with which the outcome 1 will occur in that period. We assume that forecasters observe both the realized outcome and the predictions of the other forecasters at the end of each period.

The cross-calibration test is defined over sequences $(\omega_t, I^0_t, ..., I^{M-1}_t)_{t=1}^\infty$, which specify, for each period $t$, the outcome $\omega_t \in \Omega$, together with the prediction intervals announced by each of the $M$ forecasters. Given any such sequence and any $M$ - tuple $l = (I^0, ..., I^{M-1}) \in \{I_1, ..., I_N\}_t^M$, define $\xi^l_t = \ldots$
For every $l$ satisfying $\lim_{t \to \infty} \nu_t^l = \infty$. 

In the case of a single forecaster, the cross-calibration test reduces to the classic calibration test, which checks the frequency of outcomes conditional on each forecast that is made infinitely often. With multiple forecasters, the cross-calibration test checks the empirical frequencies of the realization conditional on each profile of forecasts that occurs infinitely often. Note that if an expert is cross-calibrated, he will also be calibrated.

Claim 2 demonstrated why $T_{cross}$ is not reasonable, and so does not satisfy the set of axioms we study. In that example, both forecasters used a Dirac measure. We now turn to a slightly more elaborate example that demonstrates the same thing; yet the forecasters assign a positive probability to any finite history.

**Example 3.** Set $N > 4$, $M = 2$. Let $f_0$ be a convex combination of two forecasting strategies. With probability 0.5 it deterministically predicts $\omega$ and with the remaining probability it is an IID sequence of fair coin flips. On the other hand, $f_1$ forecasts 1 in period $t$ with probability $1 - \frac{1}{(t+2)}$, independent of past outcomes.

Then, conditional on the realization of $\omega$, both experts repeatedly announce the interval $I_N$ from some finite time onward. Consequently, over the profile $l = (1, I_N, I_N)$, equation (A.1) holds for all $i$ and therefore both experts pass the cross-calibration test over $\omega$ yielding that

$$T_{cross}(\omega, f_0, f_1) = \frac{1}{2}. \quad (A.2)$$

However, by construction, $p^I_0(\omega) = \frac{1}{2}$ and $p^I_1(\omega) = 0$, and yet, if $T_{cross}$ would be a reasonable test, then $0 < p^{\omega}_0(\omega) = p^{\omega}_0(\{T_{cross}(\cdot, f) = 0\} \cap \{\omega\})$ which contradicts equality (A.2).