The convergence Newton polygon of a \( p \)-adic differential equation I : Affinoid domains of the Berkovich affine line

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Abstract

We prove the finiteness of the radius of convergence function \( R^M \) of a differential module \( M \) over an affinoid domain \( X \) of the Berkovich affine line \( \mathbb{A}^1_{\mathbb{K}} \). This means that there exists a finite graph \( \Gamma(R^M) \subset X \), together with a canonical retraction \( \delta_{R^M} : X \to \Gamma(R^M) \), such that the function \( R^M : X \to \mathbb{R}_{>0} \) factorizes through \( \delta_{R^M} \). More generally, for each \( \xi \in X \), we define the convergence Newton polygon \( NP_{\text{conv}}(M, \xi) \) of \( M \), whose first slope is the logarithm of \( R^M(\xi) \), and the other slopes are the logarithms of the radii \( R^M_i(\xi) \) of convergence of all the Taylor solutions of \( M \) at \( \xi \). We prove the finiteness of all the slopes \( R^M_i(\xi) \) of \( NP_{\text{conv}}(M, \xi) \), and of its partial heights \( H^M_i(\xi) \), as functions on \( X \), together with their fundamental properties. Roughly speaking this result implies that there are only a finite number of numerical invariants that one can extract from the slopes of \( R^M_i \) and \( H^M_i(\xi) \) along the branches of \( X \). As a corollary we have their continuity.

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Introduction

In the ultrametric context the (one variable) radius of convergence function of a differential module \( M \) is an important invariant by isomorphisms. It is a function defined over a certain Berkovich space \( X \) and its slopes along the branches of \( X \) are numerical invariants (by isomorphism) of \( M \). If \( M \) is a differential module over \( \mathbb{K}((T)) \), where \( \mathbb{K} \) is trivially valued and of characteristic 0, then from the knowledge of the radius of convergence function of \( M \) (and of its submodules) one can recover the B.Malgrange irregularity of \( M \), the Poincaré-Katz rank of \( M \), and more generally the entire formal Newton polygon. The Radius of convergence function is also a major tool in the proof of the \( p \)-adic local monodromy theorem (cf. [And02], [Meb02], [Ked04]), and more recently of the Sabbah’s conjectures (cf. [Ked10a]). The radius of convergence function is today one of the most
important invariants of an ultrametric differential module. In this paper we prove its finiteness. Roughly speaking this implies that the numerical invariants that one can extract from the slopes of the radius of convergence function along the branches of $X$ are finite in number. We now explain what this means, and we give an idea of the proof.

Let $(K, |.|)$ be a complete valued ultrametric field of characteristic $0$. Let $X$ be a connected affinoid domain of the Berkovich affine line $A^1_K$ over $K$, and let $\mathcal{O}(X)$ be the $K$-affinoid algebra of its global sections. It is known that $X$ is always the quotient by $\text{Gal}(K_{\text{alg}}/K)$ of a standard set (cf. [Ber90, 4.2]). Roughly speaking $X$ is obtained from a closed disk $D^+(c_0, R_0)$ by removing a finite number of open disks (cf. section 1.1). In the whole paper a coordinate $T$ of $A^1_K$ is chosen. This determines the size of the disks as well as the choice of the derivation $d/dT$. The radius of convergence function will depend on this choice. For all $\xi \in X$ one has a canonical path $\lambda_\xi : [0, R_0] \to X$ with initial point $\xi$, and with end point the point $\xi_{0,R_0}$ of the Shilov boundary of $X$ defined by $D^+(c_0, R_0)$ (cf. section 1.3, [Ber90, 1.4.4]). A (closed) branch of $X$ is the image of such $\lambda_\xi$, in fact $X$ is the union of such branches and it has the structure of a so called polyhedron (cf. [Ber90, 4.1]). Moreover if $\Gamma$ is a finite union of closed branches, then the inclusion $\Gamma \subset X$ admits a canonical retraction $\delta_\Gamma : X \to \Gamma$ (cf. section 1.4), and $X$ is the topological projective limit of such retractions (cf. [BR10, Thm.2.20]). It is natural to ask whether a given function $\mathcal{R} : X \to \mathcal{T}$, where $\mathcal{T}$ is a set, factorizes through such a retraction $\delta_\Gamma$. The first point of this paper is to associate to $\mathcal{R}$ a canonical (possibly not finite) union of branches $\Gamma(\mathcal{R})$, called the constancy skeleton (or simply skeleton) of $\mathcal{R}$, on which $\mathcal{R}$ factorizes under convenient assumptions (cf. section 2). Roughly speaking $\Gamma(\mathcal{R})$ is the complement in $X$ of the union of all the disks on which $\mathcal{R}$ is constant. If $\Gamma(\mathcal{R})$ is a finite union of closed branches we say that $\mathcal{R}$ is finite or that it has a finite skeleton. Next we provide a sufficient set of conditions that guarantee that $\mathcal{R}$ is finite, continuous, and it factorizes through $\delta_{\Gamma(\mathcal{R})}$. In order to have an idea we list a rough version of them here (cf. section 2.3.1, Thm. 2.14 for a more accurate statement):

(C1) For all $\xi \in X$ one has $\rho_\mathcal{R}(\xi) > 0$.

(C2) $\mathcal{R}$ is piecewise linear, continuous, with a finite number of breaks on each closed branch of $X$.

(C3) There exists a finite union of closed branches $\Gamma$ such that if $D^-(t, \rho) \cap \Gamma = \emptyset$, then $\mathcal{R}$ is log-concave (hence decreasing by (C1)) on the branches inside $D^-(t, \rho)$.

(C4) The modulus of all possible non zero slopes of $\mathcal{R}$ at any point is lower bounded by a positive real number $\nu_\mathcal{R} > 0$, which is independent on the Berkovich point.

(C5) $\Gamma(\mathcal{R})$ is directionally finite at all its bifurcation points i.e. there are a finite number of branches of $\Gamma(\mathcal{R})$ passing through a bifurcation point $\xi$ of $\Gamma(\mathcal{R})$.

(C6) $\mathcal{R}$ is super-harmonic outside a finite set $\mathcal{E}(\mathcal{R}) \subset X$ (cf. Def. 2.10).

Among the functions satisfying these properties there are the functions of $\mathcal{O}(X)$, but also those of the type $\min(|f_1|^{-\alpha_1}, \ldots, |f_m|^{-\alpha_m})$, with $\alpha_i > 0$, and many others (compare with (4.2)). These properties are modeled on those satisfied by the partial height of the Newton polygon of a differential operator (cf. section 4.1). The rough idea of the proof is that the super-harmonicity implies that at each bifurcation point of $\Gamma(\mathcal{R})$ the function $\mathcal{R}$ has a break, while the assumption (C2) provides that there are a finite number of breaks, and hence a finite number of bifurcation points.

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1The case of $(K, |.|)$ trivially valued is allowed along the whole paper (up to section 6). Indeed the convergence polygon $NP^{\text{conv}}$ is invariant by scalar extension of $K$, so we can extend $K$ to a non trivially valued base field $\Omega/K$, apply the theorems over $\Omega$, and then re-descend to $K$ as explained in section 2.4. If $K$ is trivially valued the ring of formal power series $K[[T]]$ coincides with the ring of bounded analytic functions over $D^-(0, 1)$, its fraction field $K((T))$ coincides with the ring of bounded analytic functions over $\{|T| \in [0, 1]\}$.

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Let now \((M, \nabla)\) be a differential module over the differential ring \((\partial(X), \frac{d}{dt})\). Let \(Y' = G(T) \cdot Y, G \in M_p(\partial(X))\), be the differential equation associated to \(M\) in a basis. One is allowed to consider Taylor solutions of this equation and test their radius of convergence at each point of \(X(\Omega)\), for all complete valued field extension \(\Omega/K\). This fact permits to associate to any Berkovich point \(\xi \in X\) a radius of convergence by testing Taylors solutions at \(t_\xi := T(\xi) \in X(\mathcal{H}(\xi))\).\(^2\) Namely denote by \(Y(T, t_\xi)\) the Taylor solution of this equation around \(t_\xi\), with initial value \(Y(t_\xi, t_\xi) := \text{Id}.\) If \(Y(n) = G_n(T) \cdot Y\) is the \(n\)-th iterate of the equation, then \(Y(T, t_\xi) := \sum_{n \geq 0} G_n(t_\xi)^{T-t_\xi}/n!\). The minimum of the radii of convergence at \(t_\xi\) of the entries of \(Y(T, t_\xi)\) is given by \(\mathcal{R}^Y(\xi) := \liminf_n \frac{|G_n(t_\xi)|}{n!}^{-1/n}\). One obtains a function \(\mathcal{R}^Y : X \to \mathbb{R}_{>0}\) depending on the chosen basis of \(M\). In order to make this number invariant by base changes in \(M\) one sets
\[
\mathcal{R}^M(\xi) := \min \left( \liminf_n |G_n(\xi)/n!|^{-1/n}, \rho_{\xi, X} \right),
\]where \(\rho_{\xi, X}\) is the radius of the largest open disk centered at \(t_\xi \in X(\mathcal{H}(\xi))\) contained in \(X(\Omega)\). \(\mathcal{R}^M : X \to \mathbb{R}_{>0}\) is the radius of convergence function of \(M\). It represents the smallest radius of convergence of a Taylor solution of \(M\) around \(t_\xi\). We now refine this construction by taking in account the other radii. The vector space of germs of convergent solutions at \(t_\xi \in X(\mathcal{H}(\xi))\) is naturally filtered by the radius of convergence of its elements. We associate a polygon \(N P_{\mathcal{C}^{\mathcal{H}}}^{\mathcal{C}}(M, \xi)\) to this filtration, called convergence polygon of \(M\) at \(\xi\) (cf. section 4.3). Its first slope \(s^M_1(\xi) = h^M_1(\xi)\) is equal to \(\ln(\mathcal{R}^M(\xi))\). For \(i = 1, \ldots, r\) its \(i\)-th slope is given by \(s^M_i(\xi) := \ln(\mathcal{R}^M(\xi))\), where \(\mathcal{R}^M_i(\xi) \leq \rho_{\xi, X}\) is the radius of the largest open disk centered at \(t_\xi\) on which \(M\) admits at least \(r - i + 1\) linearly independent Taylor solutions, where \(r\) is the rank of \(M\). This defines univocally \(N P_{\mathcal{C}^{\mathcal{H}}}^{\mathcal{C}}(M, \xi)\) as the epigraph\(^3\) of the convex function \(h : [0, r) \to \mathbb{R}\) defined by the fact that \(h(0) = 0\), and that \(h(\xi)\) is linear on \([i - 1, i]\) with slope \(s^M_i(\xi)\). The values \(h^M_i(\xi) := h(i)\) are called the \(i\)-th partial heights. The main result of this paper (cf. Thm. 4.7) provides important properties on the behavior of \(N P_{\mathcal{C}^{\mathcal{H}}}^{\mathcal{C}}(M, \xi)\) as a function of \(\xi\). Namely we proves that the functions \(\mathcal{R}^Y, \mathcal{R}^M, s^M_i, h^M_i : X \to \mathbb{R}_{>0}\) are all finite functions i.e. they have a finite skeleton and factorize through it. As a consequence one has their continuity. We precise moreover a family of formal properties enjoyed by them as the piecewise linearity, convexity, super-harmonicity, integrality. Roughly speaking this result means that there are a finite number of numerical invariants of \(M\) that one can extract from the slopes of \(\mathcal{R}^Y, \mathcal{R}^M, s^M_i, h^M_i\) along the branches of \(X\), and that these functions are all definable in the sens of [LH10]. The proof is an induction on \(i = 1, \ldots, r\), and the first step consists in proving the finiteness of \(\exp(h^M_1) = \mathcal{R}^M\). The aforementioned criterion works for \(h^M_i = \mathcal{R}^M\) with respect to \(\Gamma\) equal to the skeleton of \(X\), and \(\mathcal{C}(\mathcal{R}^M)\) being the Shvaf boundary. For \(i \geq 2\) it holds for \(h^M_i\) with respect to \(\Gamma := \cup_{j=1}^{i-1} \Gamma(h^M_j)\), and \(\mathcal{C}(h^M_i)\) being a certain finite set depending on \(h^M_1, \ldots, h^M_{i-1}\). In order to prove the six properties (C1)–(C6) we compare the convergence polygon \(N P_{\mathcal{C}^{\mathcal{H}}}^{\mathcal{C}}(M, \xi)\) with two other polygons: the spectral Newton polygon \(N P^{sp}(M, \xi)\), and the spectral Newton polygon \(N P^{sp}(\mathcal{L}, \xi)\) of a differential operator \(\mathcal{L} := \left( \frac{d}{dt} \right)^r + \sum_{i=0}^{r-1} g_{r-i}(T) \left( \frac{d}{dt} \right)^i\) defining \(M\) in a convenient cyclic basis.

\(^2\)The original idea (and one of the most fruitful one) of considering “generic points” is due to Bernard Dwork. In his language a generic point for \(\xi\) is a \(t \in X(\Omega)\) satisfying \(\xi(f) = [f(t)]_\Omega\) for all \(f \in \partial(X)\) (cf. section 1.2), where \(\Omega\) is a large unspecified valued field extension of \(K\). Considering such generic points have been for long time a common practice (cf. [Dwo74], [Rob75],[CD94],...), and is still a “routine” by the specialists (cf. [CM02],[Meb02],...). As a matter of facts in the papers of Dwork and Robba \(X\) is always considered as the functor associating to \(\Omega\) the set \(X(\Omega)\) considered as a metric subspace of \([\Omega, \bullet]\). Indeed a very large \(\Omega\) (making all the points of \(X\) \(\Omega\)-rational) is often fixed once for all, in order to work with an individual metric space \(X(\Omega)\). Although unnecessary, in order to make the link between the two worlds it is convenient to systematically practice the “forma” of functor of points defined by \(X\). In fact a point \(t \in X(\Omega)\) provides a bounded character \(\partial(X) \to \Omega \to \partial(X)\), by \(T \mapsto t\). Eventually, by the description given in [Ber90, 1.2.2.ii]], this amounts to consider another description of \(X\) itself (cf. section 1.0.1). The reader knowing the language of Berkovich will not have any problem in recognizing the usual underlying objects of Berkovich theory.

\(^3\)i.e. the set of points of \(\mathbb{R}^2\) on or above the graph of \(h(\xi)\).
The slopes and the partial heights of $NP^{sp}(\mathcal{L}, \xi)$ are explicitly given in terms of the coefficients $g_i \in \mathcal{O}(X)$, and are hence finite. Now $NP^{sp}(M, \xi)$ is obtained from $NP^{conv}(M, \xi)$ “by truncation” of the large slopes (cf. section 3.2 and (4.10), see also [Ked10b, Notes of Ch.9, p.166]), while a classical result due to Young [You92] proves that the “small” slopes of these three polygons coincide (see [CM02, Thm.6.2] i.e. Prop.4.3 and Thm.5.1). In the non p-adic case, this is enough to control all the slopes since they are always “small”. In the p-adic case the “big” values of the slopes are reduced to the “small” values by using the Frobenius push-forward techniques as in [Ked10b] and [CD94].

We conclude this introduction by discussing the different definitions of the radii. The first slope of $NP^{sp}(M, \xi)$ is the logarithm of the so called spectral radius

$$R^{M,sp}(\xi) := \min(\liminf_n \xi(G_n/n!)^{-1/n}, r(\xi)) .$$

(0.2)

where $r(\xi)$ is the generic radius of $\xi$. The name of this function is due to the fact that $R^{M,sp}(\xi) = \omega/\|\nabla\|_{sp,\xi}$, where $\|\nabla\|_{sp,\xi}$ is the spectral norm\(^5\) of the connection of $M$ with respect to the norm $\xi$ (cf. (3.16)). $R^{M,sp}(\xi)$ is the spectral radius studied in the whole literature (cf. [Ked10b, Def.9.4.4], [CD94, Section 2.3], [CM02], . . . ). $R^{M,sp}$ is for certain reasons a better function than $R^{M}$ since it only depends on the restriction of $M$ to $\mathcal{H}(\xi)$, and it is hence invariant by restriction to a sub-affinoid. For this reason it is much more intrinsic than $R^{M}$. Expressing the radius in terms of spectral norm permits to make the theory much more algebraic and hence easier to generalize in more variables (cf. [Ked10a], [KX10], . . . ). Unfortunately the fact that $R^{M,sp}(\xi) = 0$ for all $K_{alg}$-rational point $\xi$ implies that $R^{M,sp}$ is not continuous, in fact if the valuation of $K$ is not trivial the set of $K_{alg}$-rational points is dense in $X$. Moreover its skeleton $\Gamma(R^{M,sp})$ is always equal to the whole space $X$, and $NP^{sp}(M, \xi)$ is not invariant under (non algebraic) scalar extensions of $K$. On the other hand $R^{M}$ is not stable by restriction to a sub-affinoid, mainly because of the presence of $\rho_{\xi,X}$. Hence it can not easily glued to give a function on a general curve. This has been recently done by F.Baldassarri [Bal10] where the definition of $R^{M}$ have been widely improved in order to make it much more intrinsic i.e. independent on the choice of the coordinate $T$. We notice that Baldassarri’s definition still preserves the dependence on a chosen skeleton, and it seems that a completely intrinsic definition does not exist. The definition of $R^{M}$ adopted here is due to F.Baldassarri itself and L.Di Vizio [BV07]. And the definitions of [Bal10] reduces to ours in this more elementary case (cf. section 8). As already mentioned a direct corollary of the finiteness is the continuity of $R^{M,sp}$, $R^{M}$, $s^{M}_{i}$, $l^{M}_{i} : X \to \mathbb{R}_{>0}$. For $i = 1$, i.e. for $R^{M} = \exp(h^{M})$, this is the one-variable case of [BV07], and a special case of [Bal10]. The proof of the continuity presented here is different in nature from those of [BV07], [Bal10], and [CD94]. All of them use a result of Dwork and Robba [DR80] providing an effective growth condition on the coefficients of the generic Taylor solution. Our proof do not use it since the continuity follows from the finiteness, and eventually from the continuity of the Newton polygon of $\mathcal{L}$.

Remark 0.1. Jérôme Poineau and Amaury Thuillier (independently) pointed out that if one works with over-convergent coefficients, the continuity of $R^{M}$ on $X$ (but also on curves) is a consequence of the super-harmonicity. The proof generalizes the fact that a concave function on an open interval is continuous, and is based on the explicit expression (0.1). This method seems to fail for $R^{M}_{i}$ since it can not be expressed globally on $X$ as in (0.1), but only outside $\bigcup_{j=1}^{i-1} \Gamma(R^{M}_{j})$ (cf. section 7).

Notes. A first proof of the harmonicity properties is due to P.Robba [Rob84] and [Rob85] for rank one differential equations with rational coefficients. He obtained the harmonicity of the radius

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\(^4\)This number is denoted by $\text{diam}(\xi)$ in [BR10, Section 1.4]. It is often called the radius of the point (cf. [Ber90, 4.2]). In the theory of differential equation $r(\xi)$ is the radius of the Dwork’s generic disk, this is why we call it generic radius.

\(^5\)The spectral norm $\|\nabla\|_{sp,\xi}$ is defined only if $\xi$ is a norm, in particular if $r(\xi) > 0$. We extend this definition to the points satisfying $r(\xi) = 0$ by setting $\|\nabla\|_{sp,\xi} := +\infty$. In this way $R^{M,sp}(\xi) = \omega/\|\nabla\|_{sp,\xi}$ for all $\xi \in X$. 

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function by expressing its slopes by means of the index (cf. [Rob84, Thm. 4.2, p.201]), and then deducing the harmonicity from the additivity of the indexes (cf. [Rob84, Prop.4.5, p.207]).

For differential modules over an annulus the super-harmonicity properties along the skeleton of the annulus are obtained in [Ked10b] that have been for us a constant source of inspiration. This paper is intended to generalize [Ked10b, Thm. 11.3.2] to the context of Berkovich spaces.

The existence of the skeleton of the radius of convergence function found his genesis in [Bal10] where F.Baldassarri announced a forthcoming paper with the proof of the finiteness of the individual function $\mathcal{R}^M$ in the more general framework of Berkovich curves.

A first proof of the finiteness of a function appears in [Ked10a, Section 5] in the case of a Berkovich closed unit disk over $K := k((z))$, where $k$ is a trivially valued field. The definitions of [Ked10a] are given ad hoc to deal with a closed disk and there are discrepancies with those of this paper, especially for the definition of the skeleton of a function (which is defined in [Ked10a] in term of the slopes). It turns out that the two definitions eventually coincide over a closed disk, and in fact certain techniques of this paper are not far from those of [Ked10a] and [Ked10b].

A proof of the finiteness of the radius function have been obtained by G.Christol [Chr11] for differential equations of rank one with polynomial coefficients. The proof uses an explicit formula for the radius function that we have contributed to realize (cf. the introduction of [chr]). The generalization of such a formula to rank one differential equation with arbitrary coefficients is the object of a forthcoming paper. This have been the starting point of the present paper.

The complete proof of the continuity and finiteness of the convergence Newton polygon over a quasi-smooth $K$-analytic Berkovich curve is obtained in the sequel of this paper [PP12].

Structure of the paper. In sections 1 we provide notations, and is section 2 we introduce the skeleton $\Gamma(\mathcal{R})$ together with the aforementioned finiteness criterion (cf. Thm. 2.14). In sections 3 and 4 we define all the polygons and we state the main result (cf. Thm. 4.7). In sections 5 and 6 we adapt to our context some results, and in section 7 we give the proof of 4.7.

1. Notation

All rings are commutative with unit element. $\mathbb{R}$ is the field of real numbers, and $\mathbb{R}_{\geq 0} := \{ r \in \mathbb{R} \mid r \geq 0 \}$. For all field $L$ we denote its algebraic closure by $L^{alg}$, by $\hat{L}$ its completion (if it has a meaning), by $L[T]$ the ring of polynomial with coefficients in $L$, and by $L(T)$ the fraction field of $L[T]$. In this paper $(K,|.|)$ will be a complete field of characteristic 0 with respect to an ultrametric absolute value $|.| : K \rightarrow \mathbb{R}_{\geq 0}$ i.e. verifying $|1| = 1$, $|a \cdot b| = |a||b|$, and $|a + b| \leq \text{max}(|a|,|b|)$ for all $a,b \in K$, and $|a| = 0$ if and only if $a = 0$. We denote by $|K| := \{ r \in \mathbb{R}_{\geq 0} \text{ such that } r = |t| \text{ with } t \in K \}$.

Define $E(K)$ as the category of complete valued ultrametric field $(\Omega,|.|_{\Omega})$ together with an isometric inclusion $e_{\Omega} : (K,|.|) \rightarrow (\Omega,|.|_{\Omega})$. A morphism $\Omega \rightarrow \Omega'$ in $E(K)$ is an isometric morphism of rings inducing the identity on $K$. The category $E(K)$ is filtering in the sense that for all $\Omega, \Omega' \in E(K)$ there exists $\Omega'' \in E(K)$ together with two morphisms $\Omega \subseteq \Omega''$ and $\Omega' \subseteq \Omega''$.

1.0.1 Berkovich spaces. An ultrametric Banach ring $(\mathfrak{A},|.|)$ is a ring $\mathfrak{A}$ together with a norm $|.| : \mathfrak{A} \rightarrow \mathbb{R}_{\geq 0}$ verifying for all $a,b \in \mathfrak{A}$ (i) $|a + b| \leq \text{max}(|a|,|b|)$, (ii) $|ab| \leq |a||b|$, (iii) $|1| = 1$, (iv) $|a| = 0$ if and only if $a = 0$. A bounded, multiplicative semi-norm of $\mathfrak{A}$ is a map $\xi : \mathfrak{A} \rightarrow \mathbb{R}_{\geq 0}$ satisfying (i) $\xi(a + b) \leq \text{max}(\xi(a),\xi(b))$, (ii) $\xi(ab) = \xi(a)\xi(b)$, (iii) $\xi(1) = 1$, (iv) $\xi(0) = 0$, (v) $\xi \leq C|.|$ for a constant $C > 0$. By definition the Berkovich space $\mathcal{M}(\mathfrak{A})$ of $\mathfrak{A}$ is space of all bounded multiplicative semi-norm on $\mathfrak{A}$, and its topology is the weakest one making

6 The principle is used in [Rob85, top of p.50] to construct the so called Robba’s exponentials.
continuous each map $\mathcal{M}(\mathcal{A}) \to \mathbb{R}_{\geq 0}$ of the form $\xi \mapsto \xi(a)$. With this topology $\mathcal{M}(\mathcal{A})$ is a non empty compact Hausdorff topological space (cf. [Ber90]). If $\xi \in \mathcal{M}(\mathcal{A})$, the kernel of $\xi$ is a prime ideal, and we denote by $\mathcal{H}(\xi)$ the completion of the fraction field of $\mathcal{A}/\text{Ker}(\xi)$. As in algebraic geometry if $Y := \mathcal{M}(\mathcal{A})$ the functor of points of $Y$ associates to a Banach ring $\mathcal{B}$ the set $\mathcal{Y}(\mathcal{B})$ of bounded ring homomorphisms $\mathcal{A} \to \mathcal{B}$. The data of an individual point $\xi \in Y$ is equivalent to the data of the character $\chi_{\xi} : \mathcal{A} \to \mathcal{H}(\xi)$ since $\xi$ can be recovered as the composite $\xi = |.|_{\mathcal{H}(\xi)} \circ \chi_{\xi}$.

Hence the restriction of the functor of points of $Y$ to the category $E(K)$ is enough to distinguish the points of $Y$. In the sequel we only consider $K$-algebras $\mathcal{A}$ topologically generated by $K$, by an element $T \in \mathcal{A}$, and by the elements of $\mathcal{A}$ that are fraction of polynomials in $K[T] \subset \mathcal{A}$. Hence, if $\Omega \in E(K)$, a bounded $K$-linear character $\mathcal{A} \to \mathcal{H}$ will be determined by the image of $T$. In this situation the equivalence relation among characters indicated in [Ber90, 1.2.2,(ii)] is equivalent to say that two $K$-linear bounded characters $\chi : \mathcal{A} \to \Omega$ and $\chi' : \mathcal{A} \to \Omega'$ are equivalent if and only if there exists a larger extension $\Omega'' \in E(K)$, and two morphisms $\Omega \to \Omega''$ and $\Omega' \to \Omega''$ in the category $E(K)$, such that $\chi = \chi'$ as characters with values in $\Omega''$.

1.0.2 Disks. Let $\Omega \in E(K), t \in \Omega, \rho \geq 0$. Consider the ring $\Omega\{\rho^{-1}(T - t)\}$ formed by power series $f(T) := \sum_{i \geq 0} a_i(T - t)^i$ such that $a_i \in \Omega$ for all $i \geq 0$, and $\lim_{i \to +\infty} |a_i|\rho^i = 0$. The setting $\xi_{t,\rho}(f) := \sup_{i \geq 0} |a_i|\rho^i$ is a multiplicative norm on $\Omega\{\rho^{-1}(T - t)\}$ that makes it a Banach $\Omega$-algebra, and the Berkovich space $D^+(t,\rho) := \mathcal{M}(\Omega\{\rho^{-1}(T - t)\})$ is called the closed disk, centered at $t$ with radius $\rho$. For all $\Omega' \subset E(\Omega)$ the $\Omega'$-valued points of $D^+(t,\rho)$ are given by $D^+_\Omega(t,\rho) := \{t' \in \Omega' \text{ such that } |t' - t| \leq \rho\}$. The ring $\Omega\{\rho^{-1}(T - t)\}$ is a principal ideal domain, whose ideals are generated by polynomials in $\Omega[T]$. Moreover $\Omega[T] \subset \Omega\{\rho^{-1}(T - t)\}$ is dense. If $t$ lies in $K$ we say that the disk is $K$-rational, in this case $\Omega\{\rho^{-1}(T - t)\} = K\{\rho^{-1}(T - t)\} \otimes_K \Omega$. The open disk $D^-(t,\rho)$ is the analytic space obtained by the union of $D^+(t,\rho')$ for all $\rho' < \rho$. Set $A_\Omega(t,\rho) := \cap_{\rho' < \rho} \Omega\{\rho'\}^{-1}(T - t)$. An element in this ring is a formal power series $\sum_{i \geq 0} a_i(T - t)^i$ with $a_i \in \Omega$ for all $i \geq 0$, satisfying $\lim_{i \to +\infty} |a_i|\rho^i = 0$ for all $\rho' < \rho$. The topology of $A_\Omega(t,\rho)$ is defined by the family of norms $\{\xi_{t,\rho'}\}_{\rho' < \rho}$ (cf. [Bou98, Ch.IX, par. 1.2, Def.3, p.139]). If $\Omega' \subset E(\Omega)$ one has $D^+_\Omega(t,\rho) := \{t' \in \Omega' \text{ such that } |t' - t| < \rho\}$. As above $\Omega[T] \subset A_\Omega(t,\rho)$ is dense.

1.0.3 Analytic elements, bounded analytic and analytic functions. Let $I \subset \mathbb{R}_{\geq 0}$ be an interval, $t \in \Omega$, and let $A(t, I) := \{[T - t] \in I\}$ be an annulus or disk. The ring $A_\Omega(t, I)$ of (Krasner) analytic elements on $A(t, I)$ is the completion, under the sup-norm on $A(t, I)$, of rational function in $K(T)$ without poles in $A(t, I)$ (cf. [CR94], [Chr12], [Ked10b, Def.8.5.1]). The ring $A_\Omega(t, I)$ of analytic functions on $A(t, I)$ is equal to $A_\Omega(t, R)$ if $I = [0, R]$ (cf. section 1.0.2), while if $0 \notin I$, then it is formed by power series $\sum_{i \in \mathbb{Z}} a_i (T - t)^i$, with $a_i \in \Omega$, satisfying $\lim_{i \to +\infty} |a_i|\rho^i = 0$ for all $\rho \in I$ (cf. [CR94], [Chr12], [Ked10b, Def.8.4.2]). The ring $B_\Omega(t, I)$ of Bounded analytic functions on $A(t, I)$ are analytic function satisfying $\sup_{i \geq 0} |a_i|\rho^i \leq C < +\infty$ for all $\rho \in I$, where $C$ is a convenient constant depending on the power series (cf. [CR94], [Chr12], [Ked10b, Def.8.1.5]).

1.1 Affinoid domains of the Berkovich affine line

$A_K^{\text{an}}$ is the analytic space obtained as $\cup_{\rho \geq 0} D^+(0,\rho)$. Since $K[T]$ is dense on each $K\{\rho^{-1}\}$, one proves that set theoretically $A_K^{\text{an}}$ is constituted by the multiplicative semi-norms on $K[T]$ extending the absolute value of $K$ (cf. [Ber90]). Such a multiplicative semi-norm is always given by

$$|f|_{t,\rho} := \xi_{t,\rho}(f) := \sup_{i \geq 0} \frac{|f(t)|_\Omega}{n!} \cdot \rho^i = \sup_{i \geq 0} \xi_{t}(\frac{f(t)}{n!}) \cdot \rho^i,$$ (1.1)

for all $f \in K[T]$, where $\Omega \in E(K), t \in \Omega,$ and $\rho \geq 0$. Here $\xi_t$ denotes $\xi_t(f) := |f(t)|_\Omega$ (cf. (1.6)). The choice of $t$ and $\rho$ is not unique, and in fact there is a canonical choice $\Omega := \mathcal{H}(\xi), t$ equal to

...
the image of $T$ in $\mathcal{H}(\xi)$, and $\rho = 0$. In section 1.3.2 we give more details. An affinoid domain $\text{X}_{K_{\text{an}}}^1$ is included in some closed disk $D^+(0, \rho)$, and is hence a Laurent domain of $D^+(0, \rho)$ (cf. [Ber90, 2.2.2]). If $K$ is algebraically closed such a rational domain is of the form

$$X = D^+(c_0, R_0) - \cup_{i=1}^d D^-(c_i, R_i),$$

(1.2)

where $c_0, \ldots, c_d \in K = \hat{K}_{\text{alg}}$ satisfy $|c_i - c_0| \leq R_0$ for all $i = 1, \ldots, d$, and $0 \leq R_1, \ldots, R_d \leq R_0$. The ring $\mathcal{O}(X)$ of global section of $X$ is the $K$-affinoid algebra $K\{p^{-1}T\}f, \frac{f}{g}\}$, where $p = R_0$, $f = T - c_0$, $q = (R_1, \ldots, R_d)$, $g = (T - c_1, \ldots, T - c_d)$ (cf. [Ber90, 2.2.2]). Denote by $\| \cdot \|_X$ its norm. By a Mittag-Leffler theorem [CR94, 5.3] the elements of $\mathcal{O}(X)$ can be uniquely write as $f = f_0 + f_1 + \cdots + f_d$, where $f_0 = \sum_{i \geq 0} a_{i,0}(T - c_0)^i \in K\{p^{-1}T\}f, \frac{f}{g}\}$, and, for all $j = 1, \ldots, d$, $f_j = \sum_{i \geq 0} a_{j,i}(T - c_i)^{j-i}$ with $a_{j,i} \in K$ and $\lim_{t \to 0} |a_{j,i}t^{j-i}| = 0$. The ring $\mathcal{O}(X)$ is a ring of rational fractions without poles in $K_{\text{alg}} = \{ t \in K_{\text{alg}} \text{ such that } |t - c_0| \leq R_0, \text{ and } |t' - c_i| \geq R_i \}$ is dense in $\mathcal{O}(X)$ with respect to $\| \cdot \|_X$.

**Lemma 1.1.** Let $\Omega \in E(K)$, $t \in \Omega$, $\rho \geq 0$. The semi-norm $\xi_{t,\rho} : K|T| \to \mathbb{R}_{\geq 0}$ extends by multiplicativity to $\mathcal{O}(X)$, and by continuity to $\mathcal{O}(X)$ if and only if $t \in D^+_\Omega(c_0, R_0)$ and $\rho \in I_t$, where

$$I_t := \begin{cases} [0, R_0] & \text{if } t \in X(\Omega) \\ [R_0, \rho] & \text{if } t \in D^+_\Omega(c_1, R_1) \end{cases}.$$  

(1.3)

If $\mathcal{O}(X)$ is not $\mathcal{O}(X)$, then $X \mathcal{O}(X) \mathcal{O}(X)$ is of the above form and $\mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X) = \mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X)$. By [Ber90, 1.3.6] one has $X \cong X \mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X)$, hence $\mathcal{O}(X)$ can be described as the completion of the ring $\mathcal{O}(X) = \mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X)$, $G$,

$$\mathcal{O}(X) = \mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X) = \mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X),$$

for all $\Omega \in E\mathcal{O}(X)$ one still has

$$X(\Omega) = \{ t \in \text{X such that } |t - c_0| \leq R_0, \text{ and } |t - c_i| \geq R_i \},$$

(1.4)

for some $c_0, \ldots, c_d \in K_{\text{alg}}$, $|c_i - c_0| \leq R_0$, and $0 < R_1, \ldots, R_d \leq R_0$, as above, with the additional condition that $D^+_\mathcal{O}(X)(c_0, R_0)$ is fixed by $G$, and the disks $\{D^+_\mathcal{O}(X)(c_i, R_i)\}_{i=1,\ldots,d}$ are permuted by $G$. We say that $X$ is $K$-rational if $c_0, \ldots, c_d \in K$. One has $\mathcal{O}(X) = \mathcal{O}(X) \mathcal{O}(X) \mathcal{O}(X)^G$, hence $\mathcal{O}(X)$ can be described as the completion of the ring $\mathcal{O}(X) = (\mathcal{O}(X) \mathcal{O}(X)^G)^G$. Each $f \in \mathcal{O}(X)$ can be seen as a function on $X(\Omega)$ for all $\Omega \in E(K(X))$, and one has

$$\|f\|_X = \max_{i=0,\ldots,d} \xi_{c_i, R_i}(f) = \sup_{\xi \in X} \{f(\xi)\} \sup_{\xi \in X} |f(t)|_\Omega.$$  

(1.5)

**1.1.1 Overconvergent functions on $X$.** For $\varepsilon > 0$ let $X_\varepsilon := D^+(c_0, R_0 + \varepsilon) - \cup_{i=1}^d D^-(c_i, R_i - \varepsilon)$ (over $K$ or $\hat{K}_{\text{alg}}$). We call overconvergent analytic functions on $X$ the ring $\mathcal{O}(X) := \cup_{\varepsilon > 0} \mathcal{O}(X_\varepsilon)$. Differential modules over $\mathcal{O}(X)$ are important especially because of the good properties of their de Rham cohomology. The material of this paper can be easily translate to the overconvergent case by replacing $X$ with an $X_\varepsilon$ with an unspecified choice of $\varepsilon > 0$ conveniently small.

**1.2 Dwork generic points**

For all $\Omega \in E(K)$ there is a natural map $i_\Omega : X(\Omega) \to X$ associating to $t \in X(\Omega)$ the semi-norm $f \mapsto |f(t)|_\Omega$. We will also use the following notation for the same semi-norm

$$i_\Omega(t)(f) = \xi(t,f) = \xi_{t,0}(f) = |f|_t = |f|_{t,0} = |f(t)|_\Omega = \lambda(t)(0)(f) = |f(\xi)|.$$  

(1.6)

If $j : \Omega' \to \Omega$ is a morphism in $E(K)$, then $i_\Omega \circ j = i_{\Omega'}$. For all semi-norm $\xi \in X$ there exists a $\Omega \in E(K)$ and a possibly not unique point $t \in X(\Omega)$ such that $|t|_\Omega = \xi$. i.e. $\xi = |t|_\Omega$ for all $f \in \mathcal{O}(X)$. Such a point $t$ is called a Dwork generic point for $\xi$. As mentioned in the introduction there is a canonical choice given by $\Omega := \mathcal{H}(\xi)$, and $t$ equal to the image $t_\xi$ of $T$ in $\mathcal{H}(\xi)$.
Lemma 1.2. Let $\xi \in X$. If $\Omega \in E(K)$ is algebraically closed and maximally complete, then $i^{-1}_\Omega(\xi)$ is an orbit under $\Gal^\cont(\Omega/K)$.

Proof. Assume $\xi_t = \xi_{t'}$. Let $K(t)$ and $K(t')$ be the completions of the sub-fields of $\Omega$ generated by $t$ and $t'$. Since the semi-norms $\xi_t = \xi_{t'}$ coincide on $K[T] \subset \mathcal{O}(X)$, then $K(t) \cong \mathcal{H}(\xi_t) = \mathcal{H}(\xi_{t'}) \cong K(t')$. Hence there exists a continuous isometric $K$-linear isomorphism $\sigma : K(t) \cong K(t')$ such that $\sigma(t) = t'$. By [DR77, Lemma 8.3] $\sigma$ extends to an isometric automorphism of $\Omega/K$. \hfill $\Box$

Recall that $\Gal^\cont(\Omega/K)$ acts isometrically on $\Omega$. In the sequel of this paper $\Omega/K$ will be conveniently chosen, and often replaced by a larger one, without further specifications.

Lemma 1.3. Let $\Omega \in E(K)$, $t \in D^+_t(c_0, R_0)$, $\rho \in I_t$ in order that $\xi_{t,\rho} \in X$ (cf. (1.3)). There exists a $\Omega' \in E(\Omega)$ and a Dwork generic point $t' \in X(\Omega')$ for $\xi_{t,\rho}$ satisfying $|t' - t| = \rho$.

Proof. Let $\tilde{\xi}_{t,\rho} \in X \hat{\otimes} \Omega$ be the lifting of $\xi_{t,\rho} \in X$ defined in the same way by (1.1). A Dwork generic point for $\tilde{\xi}_{t,\rho}$ is also a Dwork generic point for $\xi_{t,\rho}$. So we can assume $t \in K = \Omega$. In this case $T - t \in \mathcal{O}(X)$ hence $\rho = \xi_{t,\rho}(T - t) = \xi_{t'}(T - t) = |t' - t|$. \hfill $\Box$

Remark 1.4. For $t \notin \hat{K}^\alg$, and $\rho < r(\xi_t)$ (cf. (1.9)), then any point $t'$ satisfying $|t' - t| \leq \rho$ is a Dwork generic point of $\xi_{t,\rho} = \xi_{t,0}$.

1.3 Canonical paths on $X$ and generic radius of a point

Let $\Omega \in E(K)$ and $t \in D^+_t(c_0, R_0)$. The path $\lambda_t : I_t \rightarrow X$ (cf. (1.1)) associating to $\rho$ the Berkovich point $\lambda_t(\rho) := \xi_{t,\rho} = |.|_{t,\rho}$ is continuous. More precisely let $f \in \mathcal{O}(X)$, then the map $\rho \mapsto |f|_{t,\rho} : I_t \rightarrow \mathbb{R}_{>0}$ is continuous and enjoys the following properties:

1. (LA) One has a partition $I_t = \bigcup_{k=1}^{n_k} I_k$, and $\alpha_1, \ldots, \alpha_n \in |K|$, such that $|f|_{t,\rho} = \alpha_k \cdot \rho^{\alpha_k}$, for all $\rho \in I_k$;
2. (LC) Let $I \subseteq I_t$ be a subinterval. If the annulus $A(t, I) := \{|x - t| \in I\}$ is contained in $X \hat{\otimes} \Omega$, then $n_{k+1} \geq n_k$ for all $k$ such that $I_k \cap I \neq \emptyset$ and $I_{k+1} \cap I \neq \emptyset$;
3. (Z) With the notation just introduced in (LC) assume that $I_k \cap I, I_{k+1} \cap I \neq \emptyset$, then $f$ has exactly $n_{k+1} - n_k$ zeros $z \in K^\alg$ such that $|z - t| = \sup I_k = \inf I_{k+1} \in |K^\alg|$;
4. (M) Let $\Omega \in E(K)$, and let $a \in X(\Omega)$ be such that $|a - t|$ lies in the interior of $I_k$, and such that $D^-(a, |t - a|) \subseteq X$. Then one has $|f(a)|_{\Omega} = |f|_{a,|a-t|} = |f|_{t,|a-t|}$.

If $t \in X(\Omega)$, then $\xi_t \in X$ and $I_t = [0, R_0]$. Since $\xi_{t,\rho}$ is determined by continuity and multiplicativity by its restriction to $K[T]$, then by the last expression of (1.1) the path $\lambda_t$ only depends on $\xi_t \in X$ and not on the choice of $t$. If $\xi = \xi_t$ from now on we indicate it by $\lambda_{\xi} := \lambda_t$.

The conditions (LA) and (LC) are known as log-affinity and log-convexity respectively. Namely in the sequel the log-function $Lh$ attached to a function $h : I_t \rightarrow \mathbb{R}_{>0}$ will be defined by

$$Lh := \ln ch \circ \exp : \ln(I_t) \rightarrow \mathbb{R} \cup \{-\infty\}, \quad (1.7)$$

where if $0 \in I_t = [0, R_0]$ (i.e. if $t \in X(\Omega)$), then by definition $\ln(I_t) = [-\infty, \ln(R_0)]$. We say that $h$ has logarithmically a given property if $Lh$ has that property. Since $|f|_{t,\rho} \neq 0$ for all $\rho > 0$ if $t$ is often convenient to exclude the value $\rho = 0$, and consider $Lh$ as a function on $]-\infty, \ln(R_0)]$ with values in $\mathbb{R}$. If $t \in X(\Omega)$, and if $D^-(t, r) \subseteq X \hat{\otimes} \Omega$, then $\rho \mapsto |f|_{t,\rho}$ is log-increasing for all $\rho \leq r$, and

$$|f|_{t,\rho} = \sup_{\Omega \in E(K), t' \in D^-_{\Omega}(t,\rho)} |f(t')| \quad \rho \leq r. \quad (1.8)$$

More precisely there exists a particular $\Omega \in E(K)$ such that $|f|_{t,\rho} = \sup_{t' \in D^-_{\Omega}(t,\rho)} |f(t')|$.

\footnote{Note that $A(t, I)$ is an $\Omega$-rational analytic space. $A(t, I) \subseteq X \hat{\otimes} \Omega$ if and only if there is no holes of $X$ in $A(t, I)$.}

\footnote{Note that the zeros of $f$ are always algebraic, cf. [CR94].}
1.3.1 **Generic radius of a point.** We call generic radius of $\xi$ the number
\[
  r_K(\xi) := \max(\rho \in [0, R_0] \text{ such that } \lambda_\xi(\rho) = \lambda_\xi(0)).
\] (1.9)
We denote it by $r(\xi) := r_K(\xi)$ if no confusion is possible. The canonical path $\lambda_\xi$ is constant on $[0, r(\xi)]$, and it induces an homeomorphism of $[r(\xi), R_0]$ with its image in $X$.

**Lemma 1.5.** Let $\xi \in X$, and let $t \in X(\Omega)$ be a Dwork generic point for $t$. Assume that $K^{\text{alg}} \subset \Omega$. Then $r(\xi)$ equals the distance of $t$ from $K^{\text{alg}}$, i.e. $r(\xi) = \inf_{c \in K^{\text{alg}}} |t - c|$. 

**Proof.** Let $d_t := \inf_{c \in K^{\text{alg}}} |t - c|$. The zeros of any $f \in \partial(X)$ are algebraic, then by the properties of section 1.3, $|f(t')| = |f(t)|$ for all $t' \in D_\Omega(t, d_t)$. So by (1.8) one has $\lambda_\xi(d_t) = \lambda_\xi(0)$ and hence $d_t \leq r(\xi)$. To show $r(\xi) \leq d_t$ observe that $\lambda_\xi(r(\xi)) = \lambda_\xi(0)$, so by (1.8) any polynomial in $K[T]$ has no zeros in $D_\Omega(t, r(\xi))$ (cf. (Z) of section 1.3), and so $D_\Omega(t, r(\xi)) \cap K^{\text{alg}}$ is empty.

If $\Omega' \subset E(\Omega)$, each point in $D_\Omega(t, r(\xi))$ is a Dwork generic point for $\xi$. For this reason $D^-(t, r(\xi))$ is called a generic disk for $\xi$ (cf. section 1.3.3). We call $X^\text{gen}$ the subset of $X$ of points $\xi$ satisfying $r(\xi) > 0$. A point $\xi$ lies in $X - X^\text{gen}$ if and only if it admits a Dwork generic point in $X(K^{\text{alg}})$.

1.3.2 **Exact dependence of $\lambda_\xi(\rho)$ on the pair $(t, \rho)$.** Arguing as above one proves that if $t, t' \in D_\Omega(c_0, R_0)$ satisfy $\lambda_\xi(\rho) = \lambda_{\xi'}(\rho')$ for some $\rho \geq r(\xi)$, $\rho' \geq r(\xi')$, then $\rho = \rho'$, and $\lambda_\xi(\rho') = \lambda_{\xi'}(\rho')$ for all $\rho \geq \rho$. Moreover, up to enlarge $\Omega$, there exists an automorphism $\sigma \in \text{Gal}^\text{cont}(\Omega/K)$ such that $|\sigma(t') - t| \leq \rho$. Reciprocally if for some $\sigma \in \text{Gal}^\text{cont}(\Omega/K)$ one has $|\sigma(t') - t| \leq \rho$, then $\lambda_\xi(\rho') = \lambda_{\xi'}(\rho')$ for all $\rho \geq \rho$. With this description one proves the following

**Lemma 1.6.** One has $r(\xi_{t, \rho}) = \max(\rho, r(\xi_t))$, in particular if $t \in X(K^{\text{alg}})$, then $r(\xi_{t, \rho}) = \rho$.

1.3.3 **Image in the Berkovich space of an open disk or annulus.** Let $D^-(t, \rho)$ be an $\Omega$-rational open disk contained in $X(\overline{\Omega})$ (this amounts to ask $\rho \leq r_{t, X}$, cf. section 1.5). If $\rho \leq r(\xi)$ the image of $D^-(t, \rho)$ in $X$ by the canonical map $X(\overline{\Omega}) \to X$ is reduced to $\{\xi\}$. In this case we say that the open disk is generic. On the other hand if $\rho > r(\xi)$, then by Lemma 1.5, up to enlarge $\Omega$, there is a center of the disk in $K^{\text{alg}}$. So the image of $D^-(t, \rho)$ in $X$ is a genuine $K^{\text{alg}}$-rational disk. In both cases the image of $D^-(t, \rho)$ in $X$ equals $\cup_{\Omega' \subset E(\Omega)} D_{\Omega'}(t, r(\xi))$ (cf. (1.6)).

Let $S \subset X$ be a subset. We say that an open disk $D^-(t, \rho) \subset X(\overline{\Omega})$ is tangent to $S$ at $\xi \in S$ if $\xi = \xi_{t, \rho}$ and if either the disk is generic with $\rho = r(\xi)$ or, if it is not generic, the intersection of $S$ with the image of $D^-(t, \rho)$ in $X$ is empty. By section 1.3.2 all open disks tangent to $\xi$ have the same radius $\rho = r(\xi)$.

1.3.4 **Description of $X(\overline{K^{\text{alg}}}) \to X$.** Sections 1.3.3 and 1.3.2 provide a complete description of the map $X(\overline{K^{\text{alg}}}) \to X$. Indeed if $t \in X(K^{\text{alg}})$ the Galois group $G := \text{Gal}(K^{\text{alg}}/K)$ acts on the branch $\Lambda(\xi_t)$ as $\sigma(\lambda_\xi(\rho)) := \lambda_{\xi_t}(\rho) \circ \sigma = \lambda_{\xi_{\sigma(t)}(\rho)}$. On the other hand every semi-norm $\xi \in \overline{\xi}(\text{alg})$ is the infimum of a totally ordered family of semi-norms $\lambda_\xi(t) \in X(\overline{K^{\text{alg}}})$ (cf. [Ber90, 1.4.4]), and since $G$ preserves the diameters $\rho$, then it commutes with the infimum.

1.4 **Branches and saturated subsets.**

A closed branch $\Lambda(\xi)$ with starting point $\xi \in X$ is the image in $X$ of the interval $[0, R_0]$ by the canonical path $\lambda_\xi$. An branch is the union of a totally ordered family of closed branches by inclusions of subsets. The union of such branches is the whole space $X$ (cf. section 1.2). An open or closed segment of a branch $\Lambda(\xi)$ is the image $\lambda_\xi(I) \subset X$ of an open or closed interval $I$ of $\mathbb{R}$ which is contained in $[r(\xi), R_0]$. A saturated subset $\Gamma$ of $X$ is by definition an arbitrary union of branches
of $X$. The family of all branches of $X$ contained in $\Gamma$ is a partially ordered set by the inclusion of subsets of $X$. A maximal branch of $\Gamma$ is a maximal element of this family. Each point of $\Gamma$ is contained in a maximal branch of $\Gamma$ which is hence the union of its maximal branches. We say that $\Gamma$ is branch-closed if its maximal branches are all closed. Finally $\Gamma$ is called finite if it has a finite number of maximal branches. Each branch $\Lambda(\xi)$ always intersects all saturated subsets of $X$ because all branches have a common point $\lambda_\xi(R_0) = |\cdot|_{0_0,R_0}$. If $S \subseteq X$ is a subset, we denote by $\text{Sat}(S) = \bigcup_{\xi \in S} \Lambda(\xi)$ the smallest saturated subset containing $S$. We say that $\Gamma$ is $K$-rational if $\Gamma = \text{Sat}(S)$ with $S = \{\xi_{t,\rho}\}_{t \in I}$ and for all $i \in I$ one has $t_i \in X(K)$ ($I$ is possibly not finite).

**Definition 1.7.** Let $\Gamma$ be a saturated subset and let $|.| \in X$. We denote by

$$pr(\xi) := \inf(\rho \geq r(\xi) \text{ such that } \lambda_\xi(\rho) \in \Gamma). \quad (1.10)$$

Define $\delta r(\xi) := \lambda_\xi(\rho_T(\xi))$. The map $\delta r : X \rightarrow X$ is the identity on the smallest branch-closed saturated subset $\overline{\Gamma}$ containing $\Gamma$, and $\delta r(X) = \overline{\Gamma}$. If $\Gamma = \overline{\Gamma}$ we call $\delta r : X \rightarrow \Gamma$ the canonical retraction. If $\Gamma = \overline{\Gamma}$ is non empty, branch-closed, and finite, and if it is equipped with the quotient topology induced by the topology of $[0, R_0] \subseteq \mathbb{R}$ via the canonical paths $\lambda_\xi$, then $\delta r$ is continuous, and moreover $\Gamma$ is the topological quotient of $X$ by the map $\delta r : X \rightarrow \overline{\Gamma}$.

**1.5 The skeleton of $X$ and the function $\rho_{-X}$.**

The skeleton $\Gamma_X = \text{Sat}(S)$ of $X$ is the smallest finite branch-closed saturated subset containing the Shilov boundary $S = \{\xi_{c_i,R_i}\}_{c_i=0,...,\mu}$ (cf. (1.2)). $\Gamma_X$ is also the set of semi-norms of $X$ that are maximal with respect to the partial order given by $\xi \leq \xi'$ if and only if $\xi(f) \leq \xi'(f)$ for all $f \in \mathcal{O}(X)$. We denote by $\rho_{\xi,X} := \rho_{t,X}(\xi)$ (cf. Def. 1.7). If $t \in X(\Omega)$ is a Dwork generic point for $\xi$ one has

$$\rho_{\xi,X} = \rho_{t,X} := \min_{i=1,...,\mu} \left( |t - c_i|_{\Omega}, R_0 \right). \quad (1.11)$$

This expression represents $\rho_{\xi,X}$ as the radius $\rho$ of the largest open disk $D^-(t, \rho)$ contained in $X \otimes \Omega$. If $\xi \leq \xi'$, then $\rho_{\xi,X} = \rho_{\xi',X}$. In fact the inequality $\xi \leq \xi'$ applied to $T - c_i$ and $(T - c_i)^{-1}$ provides $|t - c_i| = |t' - c_i|$ in (1.11). For all $t \in X(\Omega)$, and all $\rho > 0$ one has $\rho_{t,\rho,X} = \max(\rho, \rho_{t,X})$.

**Remark 1.8.** For a closed disk this definition differs from [Ber90, 1.4] which gives $\Gamma_X = \emptyset$.

**1.6 Directions and directional finiteness.**

Let $X_{\text{int}} \subset X_{\text{gen}} \subset X$ be the set of Berkovich points $\xi$ of the form $\xi = \lambda_\xi'(\rho)$, with $\rho > r(\xi')$. In other words $X_{\text{int}} \subset X$ is formed by the points $\xi_{t,\rho}$ defined by a non generic open disk $D^-(t, \rho) \subset X$. These are the points of type (2) and (3) in the terminology of [Ber90, 1.4.4]. For $\xi \in X_{\text{int}}$ we denote by $\mathcal{B}(\xi)$ be the family of branches $\Lambda(\xi')$ admitting an open segment containing $\xi$ i.e. such that $\xi = \lambda_\xi(\overline{\rho})$ for some $\overline{\rho} > r(\xi')$. One defines an equivalence relation on $\mathcal{B}(\xi)$ as follows. We say that $\Lambda(\xi_1) \sim \Lambda(\xi_2)$ if and only if the two paths meet before meeting $\xi$ i.e. if and only if there exists $\rho < \overline{\rho}$ such that $\lambda_\xi(\rho) = \lambda_\xi(\rho)$. A direction through $\xi$ is an equivalence class in $\Delta(\xi) := \mathcal{B}(\xi)/\sim$. A representative branch $\Lambda(\xi_\delta)$ for a direction $\delta \in \Delta(\xi)$ is an arbitrary element of the class $\delta$. Let $\Gamma \subseteq X$ be a saturated subset, and let $\xi \in \Gamma \cap X_{\text{int}}$. We say that a direction $\delta \in \Delta(\xi)$ belongs to $\Gamma$ if there exists a representative branch $\Lambda(\xi_\delta)$ for $\delta$ contained in $\Gamma$. The set of directions belonging to $\Gamma$ will be denoted by $\Delta(\xi, \Gamma)$. If $\Delta(\xi, \Gamma)$ is finite, we say that $\Gamma$ is directionally finite at $\xi$.

**Remark 1.9.** The extra direction directed toward the infinity is not considered here as a direction.

**Remark 1.10.** Since $\overline{\rho} > r(\xi')$, then, by section 1.3.2, one can chose $\xi' = \xi_t$ with $t \in X(K^{\text{alg}})$. Hence if $\xi \in X_{\text{int}}$ and if there exists two distinct direction $\delta, \delta' \in \Delta(\xi)$, then $\xi = \xi_{t,\overline{\rho}} = \xi_{t',\overline{\rho}}$ with $t, t' \in X(K^{\text{alg}})$ and $\overline{\rho} = |t - t'| \in |K^{\text{alg}}| - \{0\}$. $\xi$ is then a point of type (2) in (cf. [Ber90, 1.4.4]). If
Let $\mathcal{T}$ be a set and let $\mathcal{R} : X \to \mathcal{T}$ be an arbitrary function. For all $\xi \in X$ consider the composite map $\mathcal{R}_\xi : X \widehat{\otimes} \mathcal{H}(\xi) \to X \to \mathcal{T}$, and define the constancy radius $\rho_\mathcal{R}(\xi)$ of $\mathcal{R}$ at $\xi$ as the maximum value of $\rho$ such that $\mathcal{R}_\xi$ is constant on the open disk $D^-(t_\xi, \rho) \subset X \widehat{\otimes} \mathcal{H}(\xi)$, where $t_\xi$ is the image of $T$ in $\mathcal{H}(\xi)$. Define the constancy skeleton, or simply the skeleton, $\Gamma(X, \mathcal{R}) \subseteq X$ of $\mathcal{R}$ as the set of points of $X$ of the form $\lambda_\xi(\rho_\mathcal{R}(\xi)) \in X$. We write $\Gamma(\mathcal{R})$ if no confusion is possible. Since $D^-(t_\xi, r(\xi))$ is contained in the inverse image of $\xi$, then from the definition one immediately has

$$r(\xi) \leq \rho_\mathcal{R}(\xi) \leq \rho_{\xi, X} \leq R_0 . \quad (2.1)$$

### 2. Constancy skeleton of a function on $X$.

#### 2.0.1 Functorial point of view. By composing with $i_\Omega$ (cf. (1.6)) we obtain for all $\Omega \in E(K)$ a map $\mathcal{R}_\Omega : X(\Omega) \to X \to \mathcal{T}$ which is obviously compatible with the inclusions of CVFE’s of $K$. In other words the family $\{\mathcal{R}_\Omega\}_{\Omega}$ is a natural transformation between the functor $X : \Omega \mapsto X(\Omega)$ and the constant functor $\Omega \mapsto T$. Let $\Omega \in E(K)$, $t \in X(\Omega)$ and $\rho > 0$. We say that $\mathcal{R}$ is constant on the open disk $D^-(t, \rho) \subset X \widehat{\otimes} \Omega$, if for all $\Omega' \in E(\Omega)$ the function $\mathcal{R}_{\Omega'} : \mathcal{D}_{\Omega'}^{-}(t, \rho) \to \mathcal{T}$ is constant. We define the constancy radius $\rho_\mathcal{R}(t)$ of $\mathcal{R}$ at $t$ as the radius of the largest open disk $D^-(t, \rho_\mathcal{R}(t))$ contained in $X \widehat{\otimes} \Omega$ (i.e. $\rho_\mathcal{R}(t) \leq \rho_{t, X}$) on which $\mathcal{R}$ is constant. If $\rho_{\mathcal{R}, \Omega'}(t) \leq \rho_{t, X}$ denotes the largest radius such that $\mathcal{R}_{\Omega'}$ is constant on the set $D_{\Omega'}^-(t, \rho_{\mathcal{R}, \Omega'}(t))$, then

$$\rho_\mathcal{R}(t) := \inf_{\Omega' \in E(\Omega)} \rho_{\mathcal{R}, \Omega'}(t) . \quad (2.2)$$

#### Lemma 2.1. This definition coincides with that of the above section : $\rho_\mathcal{R}(t) = \rho_\mathcal{R}(\xi_t)$. 

**Proof.** Firstly we prove the independence on $t$. Let $t, t'$ be two Dwork generic points for $\xi$. Up to enlarge $\Omega'$ one can assume that $t, t' \in X(\Omega')$ and that there exists $\sigma \in \text{Gal}^\text{cont}(\Omega'/K)$ such that $t' = \sigma(t)$. Since $\sigma$ is isometric one has $\sigma(D_{\Omega'}^-(t, \rho)) = D_{\Omega'}^-(\sigma(t), \rho)$. By construction $\mathcal{R}_{\Omega'}$ is constant on the orbit $i_{\Omega'}^1(\xi')$, for all $\xi' \in X$. So $\mathcal{R}_{\Omega'}$ is constant on $D_{\Omega'}^-(t, \rho)$ if and only if it is constant on $\sigma(D_{\Omega'}^-(t, \rho))$. This proves that $\rho_{\mathcal{R}, \Omega'}(t) = \rho_{\mathcal{R}, \Omega'}(t')$, and since this holds for all $\Omega' \in E(\Omega)$ large enough one also has $\rho_\mathcal{R}(t) = \rho_\mathcal{R}(t')$ by (2.2). Equality $\rho_{\mathcal{R}}(t) = \rho_\mathcal{R}(\xi_t)$ then follows from the fact that the image of a disk $D^-(t, \rho)$ in $X$ is equal to $\cup_{\Omega' \in E(\Omega)} i_{\Omega'}(\mathcal{D}_{\Omega'}^-(t, \rho))$ (cf. section 1.3.3). \qed

#### 2.0.2 Basic properties.

**Proposition 2.2.** $\Gamma(\mathcal{R})$ is a saturated subset satisfying moreover:

1. $\Gamma(\mathcal{R})$ is branch-closed (i.e. its maximal branches are closed);
2. $\Gamma_X \subseteq \Gamma(\mathcal{R})$ (i.e. $\Gamma(\mathcal{R})$ always contains the skeleton of $X$, cf. section 1.5);
3. $\xi \in \Gamma(\mathcal{R})$ if and only if $\rho_\mathcal{R}(\xi) = r(\xi)$;
4. $\lambda_\xi(\rho_\mathcal{R}(\xi)) \in \Gamma_X$ if and only if $\rho_\mathcal{R}(\xi) = \rho_{\xi, X}$;
5. $\rho_\mathcal{R}(\xi) = \rho_{\Gamma(\mathcal{R})}(\xi)$ for all $\xi \in X$ (cf. (1.10));
6. For all $\xi \in X$, and all $\rho \in [0, R_0]$ one has $\rho_\mathcal{R}(\lambda_\xi(\rho)) = \max(\rho, \rho_\mathcal{R}(\xi))$.
7. If $\mathcal{R}$ is constant on a non generic disk $D^-(t, \rho) \subset X$, $t \in X(\Omega)$, then $D^-(t, \rho) \cap \Gamma(\mathcal{R})$ is empty.

**Proof.** One can assume $K = \widehat{K}^\text{alg}$. Property vii) is evident. This implies iii) because if $r(\xi) < \rho_\mathcal{R}(\xi)$ then $\xi \notin \Gamma(\mathcal{R})$, since $\xi \in D^-(t_\xi, \rho_\mathcal{R}(\xi))$. Conversely if $r(\xi) = \rho_\mathcal{R}(\xi)$, then $\xi = \lambda_\xi(r(\xi)) = \lambda_\xi(\rho_\mathcal{R}(\xi)) \in \Gamma(\mathcal{R})$. By (2.1) iii) implies ii) since $\xi \in \Gamma_X$ and if only if $r(\xi) = \rho_{\xi, X}$. These properties imply that
\( \Gamma(\mathcal{R}) \) is saturated. In fact if \( \xi \in \Gamma_X \), then \( \Lambda(\xi) \subseteq \Gamma_X \subseteq \Gamma(\mathcal{R}) \) because \( \Gamma_X \) is saturated. On the other hand let \( \xi \in \Gamma(\mathcal{R}) - \Gamma_X \), and let \( \xi' := \lambda_\xi(\rho) \notin \Gamma_X \). Then one must have \( \rho_\mathcal{R}(\xi') = r(\xi') \) and hence \( \xi' \in \Gamma(\mathcal{R}) \), because otherwise \( \mathcal{R} \) would be constant on the non generic disk \( D^{-}(t_\xi', \rho_{\mathcal{R}}(\xi')) \) which contains \( \xi \in \Gamma(\mathcal{R}) \), contradicting vii). To prove i) let \( \Lambda \) be a maximal branch. If \( \Lambda \subseteq \Gamma_X \) there is nothing to prove since \( \Lambda = \Lambda(\xi_i, \rho_i) \). If \( \Lambda \not\subseteq \Gamma_X \), then we can express \( \Lambda \) as the disjoint union of a closed segment \( J \) with a segment \( I \) such that \( I \cap \Gamma_X \) is empty. We claim that the infimum semi-norm \( \xi := \inf_{\xi' \in I} \xi' \) belongs to \( \Gamma(\mathcal{R}) \) and hence to \( I \). In fact \( \inf_{\xi' \in I} \xi' = \inf_{r(\xi') < \rho < \rho_{\mathcal{R}}(\xi)} \lambda(\rho) \), and by (2.1) \( r(\xi') \leq \rho_{\mathcal{R}}(\xi) \). If by contrapositive \( r(\xi) < \rho_{\mathcal{R}}(\xi) \), then \( \mathcal{R} \) is constant on a non generic disk \( D^{-}(t_\xi, \rho) \) with \( \rho > r(\xi) \) close enough to \( r(\xi) \), and \( \xi' := \lambda(\rho) \) do not belongs to \( \Gamma(\mathcal{R}) \) by vii). So \( r(\xi) = \rho_{\mathcal{R}}(\xi) \) and \( \xi \in \Gamma(\mathcal{R}) \) by iii). v) and vi) are straightforward.

We denote by

\[
\delta_{\mathcal{R}} : X \to \Gamma(\mathcal{R})
\]

the canonical retraction defined by \( \delta_{\mathcal{R}}(\xi) := \lambda_\xi(\rho_{\mathcal{R}}(\xi)) = \delta_{\Gamma(\mathcal{R})}(\xi) \). The map \( \delta_{\mathcal{R}} \) is well defined since \( \Gamma(\mathcal{R}) \) is branch-closed. We say that \( \mathcal{R} \) is finite if \( \Gamma(\mathcal{R}) \) is a finite saturated subset. If \( \mathcal{R} \) is finite one proves easily that \( \delta_{\mathcal{R}} \) is a continuous map.

**Remark 2.3.** The correspondence \( \mathcal{R} \mapsto \delta_{\mathcal{R}} \) is idempotent (i.e. \( \delta_{\delta_{\mathcal{R}}} = \delta_{\mathcal{R}} \)). More precisely if \( \Gamma \subseteq X \) is a saturated subset, and if \( \mathcal{R} = \delta_{\Gamma} : X \to \Gamma \) is its retraction, then \( \delta_{\delta_{\mathcal{R}}} = \delta_{\Gamma(\mathcal{R})} \). Any branch-closed saturated subset \( \Gamma \) of \( X \) containing \( \Gamma_X \) is the skeleton of its retraction map \( \delta_{\Gamma} \) (i.e. \( \Gamma = \Gamma(\delta_{\Gamma}) \)).

**Remark 2.4.** Let \( \mathcal{R}_i : X \to \mathcal{T}_i \), \( i = 1, 2 \), and let \( g : \mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}_3 \) be any functions. If \( \mathcal{R}_3 := g \circ (\mathcal{R}_1 \times \mathcal{R}_2) \), then \( \Gamma(\mathcal{R}_3) \subseteq \Gamma(\mathcal{R}_1) \cup \Gamma(\mathcal{R}_2) \). Indeed clearly \( \rho_{\mathcal{R}_3}(\xi) > \min(\rho_{\mathcal{R}_1}(\xi), \rho_{\mathcal{R}_2}(\xi)) \), and \( \Gamma(\mathcal{R}_1) \cup \Gamma(\mathcal{R}_2) \) is saturated. This holds in particular for max(\( \mathcal{R}_1, \mathcal{R}_2 \)) or min(\( \mathcal{R}_1, \mathcal{R}_2 \)) if \( \mathcal{T}_i = \mathbb{R} \).

**Remark 2.5.** Let \( X' \subseteq X \) a sub-affinoid, and \( \mathcal{R}' : X' \to \mathcal{T} \) be the restriction of \( \mathcal{R} : X \to \mathcal{T} \) to \( X' \). To avoid confusion we denote by \( \Gamma(X, \mathcal{R}) \subseteq X, \Gamma(X', \mathcal{R}') \subseteq X', \rho_{\mathcal{R}}(X, -), \rho_{\mathcal{R}'}(X', -) \) the respective skeletons and constancy radii. For all \( \xi' \in X' \) one clearly has \( \rho_{\mathcal{R}'}(X', \xi') = \min(\rho_{\mathcal{R}}(X, \xi'), \rho_{\mathcal{R}'}(X', \xi')) \), hence \( \Gamma(X', \mathcal{R}') = \left( \Gamma(X, \mathcal{R}) \cap X' \right) \cup \Gamma_X' \). So the finiteness of \( \mathcal{R} \) on \( X \) implies that of \( \mathcal{R}' \) on \( X' \).

### 2.0.3 Examples of skeletons.

i) Let \( \mathcal{R} = \text{Id}_X : X \to X \) be the identity, then \( \Gamma(\text{Id}_X) = \Gamma(r(-)) = X \) (cf. (1.9)).

ii) Let \( \mathcal{R} = 1 : X \to \{pt\} \) be a constant map, then \( \Gamma(1) = \Gamma(\rho_{-, X}) = \Gamma_X \) is the skeleton of \( X \).

iii) Let \( f_1, \ldots, f_n \in \mathcal{O}(X) \), let \( \alpha_1, \ldots, \alpha_n > 0 \), and let \( \mathcal{R}(\xi) := \min_i(||f_i(\xi)||^{-\alpha_i}) \). Then \( \Gamma(\mathcal{R}) = \text{Sat}((\{z_1, \ldots, z_r\}) \cup \Gamma_X, \{z_1, \ldots, z_r\} \subseteq X(\mathcal{R}^{\text{alg}}) \) is the union of all zeros of \( f_1, \ldots, f_n \).

iv) With the above notations if \( \mathcal{R}(\xi) := \max_i(||f_i(\xi)||^{-\alpha_i}) \), intended as a function with values in the set \( \mathcal{T} := \mathbb{R}_{>0} \cup \{\infty\} \), then one again has \( \Gamma(\mathcal{R}) = \text{Sat}((\{z_1, \ldots, z_r\}) \cup \Gamma_X, \{z_1, \ldots, z_r\} \subseteq X(\mathcal{R}^{\text{alg}}) \).

v) Assume now that \( \mathcal{R}(\xi) := \max_i ||f_i(\xi)||^{\alpha_i} \) (resp. \( \mathcal{R}(\xi) := \min_i ||f_i(\xi)||^{-\alpha_i} \) as a function with values in \( \mathcal{T} := \mathbb{R}_{>0} \cup \{\infty\} \)). In this case the explicit description of the skeleton \( \Gamma(\mathcal{R}) \) is more complicate, but one can easily deduce its finiteness from Remark 2.4.

### 2.1 Branch continuity and dag-skeleton.

We investigate now whether the function \( \mathcal{R} \) admits a factorization as \( \mathcal{R} = \mathcal{R}_{|\Gamma(\mathcal{R})} \circ \delta_{\mathcal{R}} \):

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{R}} & \mathcal{T} \\
\delta_{\mathcal{R}} \downarrow & & \downarrow \mathcal{R}_{|\Gamma(\mathcal{R})} \\
\Gamma(\mathcal{R}) & \xrightarrow{} & \Gamma(\mathcal{R})
\end{array}
\]
This is not automatically verified. In fact for a given $\xi \in X$ the restriction $R \circ \lambda_\xi : [0, R_0] \to T$ is constant for $\rho \in [0, \rho_R(\xi)]$, but one may have a different value at $\rho = \rho_R(\xi)$. We say that $R$ is \emph{branch continuous} if for all $\xi \in X$ one has $R(\lambda_\xi(\rho_R(\xi))) = \lim_{\rho \to \rho_R(\xi)} R(\lambda_\xi(\rho)) = R(\xi)$. A branch continuous map factorizes as $R = R\big|_{[0,R_0]} \circ \delta_R$ and is determined by its values on $\Gamma(R)$. A continuous function with values in a Hausdorff space $T$ is branch continuous. Conversely a finite and branch continuous function is continuous if and only if its restriction to $\Gamma(R)$ is continuous.

For some purposes this situation may be unsatisfactory since we wants to factorize all functions. For this we define the \emph{dag-skeleton} $\Gamma(R)^\dagger$ as follows. A \emph{germ of a direction} of $\delta \in \Delta(\xi)$ is an arbitrary \emph{unspecified} representative branch $\Lambda(\xi)$ for $\delta$ (cf. section 1.6). We define $\Gamma(R)^\dagger$ as the union of the skeleton $\Gamma(R)$ together with a \emph{germ of each directions} $\delta \in \Delta(\xi)$ of each point $\xi$ of $\Gamma(R)$. Any function $R$ factorizes through its dag-skeleton $\Gamma(R)^\dagger$. This situation will not occur in this paper since all the functions will be branch continuous. This idea can be better expressed in term of Huber spaces [Hub96], but this lies outside the scopes of this paper.

2.2 Boundary, bifurcation points, and smooth points.
We distinguish three kinds of points in $\Gamma(R)$:

i) The \emph{boundary} of $\Gamma(R)$ is by definition constituted by $\xi_{c_0,R_0}$, and by those points $\xi \in X$ such that $\Lambda(\xi)$ is a maximal branch of $\Gamma(R)$.

ii) A \emph{bifurcation point} of $\Gamma(R)$ is by definition a point $\xi \in \Gamma(R) \cap X_{\text{int}}$ for which there exists at least two distinct directions $\delta_1, \delta_2 \in \Delta(\xi)$ belonging to $\Gamma(R)$ (cf. section 1.6). The unique point which is possibly simultaneously a bifurcation and boundary point of $\Gamma(R)$ is $\xi_{c_0,R_0}$.

iii) We call \emph{smooth} point of $\Gamma(R)$ any other point of $\Gamma(R)$.

iv) Among smooth point there are those called \emph{punctured smooth}. These are the smooth points $\xi$ for which there exists at least a hole $D^-(c_i, R_i)$ of $X$ which is tangent to $\Gamma(R)$ at $\xi = \xi_{c_i,R_i}$ (cf. section 1.3.3). Punctured smooth points of $\Gamma(R)$ are the smooth points of $\Gamma(R)$ belonging to the Shilov boundary on $\Gamma_X$.

Remark 2.6. Functions $f \in \mathcal{O}(X)$ may not be log-convex along $\Gamma(R)$ in correspondence of a punctured smooth point (cf. property (LC) of section 1.3). Moreover if one works with over-convergent functions $\mathcal{O}^\dagger(X)$ on $X$, then punctured smooth points have to be considered as bifurcation points.

Remark 2.7. Bifurcation points are all of type (2) in the sense of [Ber90, 1.4.4] (cf. Remark 1.10).

Definition 2.8. We call critical point of $X$, denoted by $\mathcal{C}_X$ the points of the Shilov boundary of $X$ together with the bifurcation points of $\Gamma_X$. Explicitly one has $\mathcal{C}_X := \{\xi_{c_i,R_j}\}_{i=0,\ldots,\mu} \cup \{\xi_{c_i,|c_i-c_j|}\}_{i \neq j, i,j=1,\ldots,\mu}$ (cf. (1.2)). More generally if $\Gamma$ is any branch-closed saturated subset containing $\Gamma_X$ we call $\mathcal{C}(\Gamma)$ the union of $\mathcal{C}_X$ with the set of bifurcation and boundary points of $\Gamma$.

2.3 A Criterion for the finiteness of a real valued function $R$.
Let $R : X \to \mathbb{R}_{\geq 0}$ be any function. For all path $\lambda_\xi$ we indicate by
\[
R_\xi := R \circ \lambda_\xi : [0, R_0] \to \mathbb{R}_{\geq 0}
\] (2.5)
the composite map. If $t$ is a Dwork generic point for $\xi$ we also use the notation $R_\xi := R_\xi$. By definition $R_\xi$ is constant on $[0, r(\xi)]$. The log-function $L R_\xi : ]-\infty, \ln(R_0)] \to \mathbb{R}$ attached to $R_\xi$ is given by (cf. (1.7))
\[
L R_\xi(\tau) := \ln(R_\xi(\exp(\tau))) , \quad \tau \in ]-\infty, \ln(R_0)]
\] (2.6)
Let $\xi \in X_{\text{int}}$, and let $\Lambda(\xi_\delta) \in \mathcal{B}(\xi)$ be a representative branch for a direction $\delta \in \Delta(\xi)$ through $\xi$. The left slope of $L R_\xi$ at $\xi$ (if it exists) only depends on the direction $\delta$ defined by $\Lambda(\xi_\delta)$. Let now
\( \xi \in X - \{\xi_{c_0, R_0}\} \), the right slope of \( L^R \xi \) at \( \xi \) (if it exists) only depend on the branch \( \Lambda(\xi) \). We denote them by

\[
\partial_{+} R(\xi) := \lim_{\tau \to \tau_0^+} \frac{L^R(\xi)(\tau) - L^R(\xi)(\tau_0)}{\tau - \tau_0}, \quad \partial_{-} R(\xi) := \lim_{\tau \to \tau_0^-} \frac{L^R(\xi)(\tau) - L^R(\xi)(\tau_0)}{\tau - \tau_0}
\]

(2.7)

where \( \tau \in ]-\infty, \ln(R_0)\) is defined by the relation \( \xi = \lambda_{\xi}(\exp(\tau)) \), and \( \tau_0 := \ln(r(\xi)) \).

**Definition 2.9** (Flat directions). We say that a direction \( \delta \in \Delta(\xi) \) is flat for \( \mathcal{R} \) if \( \partial_{-} R(\xi) = 0 \). If all directions are flat for \( \mathcal{R} \), and if \( \partial_{+} R(\xi) = 0 \) too, we say that \( \mathcal{R} \) is flat at \( \xi \).

**2.3.1 Conditions and criterion.** Let as usual \( X = D^+(c_0, R_0) - \bigcup_{i=1}^\mu D^-(c_i, R_i) \). Let \( \mathcal{R} : X \to \mathbb{R}_{>0} \) be a function. Let \( \Gamma \) be a finite and branch-closed saturated subset containing \( \Gamma_X \). Consider the following conditions:

(C1) For all \( \xi \in X \) one has \( \rho(\mathcal{R}(\xi)) > 0 \). We say that \( \mathcal{R} \) is locally constant. By (2.1) this condition is automatically verified by all \( \xi \in X_{\text{gen}} \).

(C2) For all \( \xi \in X \) the function \( L^R(\xi) : ]-\infty, \ln(R_0)\) \( \to \mathbb{R}_{>0} \) is piecewise linear and continuous, with a finite number of breaks. Note that if \( \mathcal{R} \) verifies (C1) and (C2) then it is branch-continuous.

(C3) Let \( D^-(t, \rho) \subset X \) be a non-generic disk such that \( D^-(t, \rho) \cap \Gamma = \emptyset \). Then \( L^R(\xi) \) is concave on \([0, \ln(\rho)]\). Note that concavity implies continuity on \([0, \rho]\), hence the left and right slopes of \( L^R(\xi) \) along \( ]-\infty, \ln(\rho)[ \) exists and are finite.

(C4) The modulus of all possible non zero slopes of \( \mathcal{R} \) at any point is lower bounded by a positive real number \( \nu(\mathcal{R}) > 0 \), which is independent on the Berkovich point. Namely for all \( \xi \in X_{\text{int}} \) (resp. \( \xi \in X - \{\xi_{c_0, R_0}\} \)) and all \( \delta \in \Delta(\xi) \) one has \( \partial_{-} R(\delta)(\xi) \neq -\nu(\mathcal{R}), \nu(\mathcal{R}) = \{0\} \) (resp. \( \partial_{+} R(\xi) \neq \nu(\mathcal{R}), \nu(\mathcal{R}) = \{0\} \)).

(C5) \( \Gamma(\mathcal{R}) \) is directionally finite at all its bifurcation points i.e. \( \Delta(\xi, \Gamma(\mathcal{R})) \) is finite for all bifurcation point \( \xi \in \Gamma(\mathcal{R}) \). If (C5) holds we will say that \( \mathcal{R} \) is directionally finite.

(C6) There exists a finite set \( \mathcal{C}(\mathcal{R}) \subset X \) such that if \( \xi \in \Gamma(\mathcal{R}) - \mathcal{C}(\mathcal{R}) \) is a bifurcation point of \( \Gamma(\mathcal{R}) \) not in the Shilov boundary of \( X \), then \( \mathcal{R} \) is super-harmonic at \( \xi \) (cf. Def. 2.10 below).

**Definition 2.10.** Let \( \xi \in X_{\text{int}} \) not belonging to the Shilov boundary of \( X \). Assume that \( \partial_{+} R(\xi) \) and \( \partial_{-} R(\xi) \) exists for all \( \delta \in \Delta(\xi) \). We say that \( \mathcal{R} \) is super-harmonic (resp. sub-harmonic; harmonic) at \( \xi \) if the directions \( \delta \in \Delta(\xi) \) such that \( \partial_{-} R(\xi) = 0 \) are finite in number, and one has

\[
\partial_{+} R(\xi) \leq \sum_{\delta \in \Delta(\xi)} \partial_{-} R(\xi)
\]

(2.8)

(resp. one has \( \geq \); equality holds). We say that \( \mathcal{R} \) is super-harmonic (resp. sub-harmonic; harmonic) if it is super-harmonic at all point \( \xi \in X - \Gamma_X \). We call Laplacian of \( \mathcal{R} \) at \( \xi \) the negative number \( dd^c(\mathcal{R}, \xi) := \partial_{+} R(\xi) - \sum_{\delta \in \Delta(\xi, \Gamma(\mathcal{R}))} \partial_{-} R(\xi) \).

**Remark 2.11.** In the definition one has to exclude the points of the Shilov boundary of \( X \) because polynomials in \( K[T] \) are not sub-harmonic at such points, indeed some directions are removed.

**Remark 2.12.** Below we prove that if \( \mathcal{R} \) verifies (C1), (C2), (C3), (C5), then the sum (2.8) is automatically finite. In fact if \( \xi \notin \Gamma(\mathcal{R}) \), then the sum is trivially verified with \( 0 \leq \sum_{\delta \in \Delta(\xi)} 0 \). And if \( \xi \in \Gamma(\mathcal{R}) \), then in the sum one can replace \( \Delta(\xi) \) by the finite set \( \Delta(\xi, \Gamma(\mathcal{R})) \). Indeed by Prop. 2.17 the directions \( \delta \in \Delta(\xi) - \Delta(\xi, \Gamma(\mathcal{R})) \) are all flat for \( \mathcal{R} \) i.e. \( \partial_{-} R(\xi) = 0 \) (cf. Def. 2.9).

**Remark 2.13.** Definition 2.10 is less general with respect to the usual definition of super-harmonicity, as for example those in [BR10], [Thu05], [FJ04]. The general definition allows an infinite number direction of non zero slope and the finite sum of (2.8) is replaced by an infinite one.
Theorem 2.14. If $\mathcal{R} : X \to \mathbb{R}_{>0}$ satisfies the six conditions (C1)–(C6), then $\mathcal{R}$ is finite.

Proof. Since $\Gamma$ is finite we are reduced to prove that $\Gamma' := \Gamma(\mathcal{R}) \cup \Gamma$ is finite. Since $\Gamma(\mathcal{R})$ is directionally finite at its bifurcation points, it is enough to prove that there are a finite number of bifurcation points of $\Gamma'$. The points in $\mathcal{C} := \mathcal{C}(\mathcal{R}) \cup \mathcal{C}(\Gamma)$ are finite in number and we can neglect them (cf. Def. 2.8). Moreover up to replace $\Gamma$ by $\Gamma \cup \text{Sat}(\mathcal{C})$ we can assume $\mathcal{C}(\mathcal{R}) \subset \Gamma$. We distinguish the points of $\Gamma$ from those in $\Gamma'$. Each point $\xi \in \Gamma - \mathcal{C}$ is a smooth point of $\Gamma$ which is not punctured smooth. Hence $\xi \in \Gamma - \mathcal{C}$ is a bifurcation point of $\Gamma'$ if and only if there exists a direction $\delta$ through $\xi$ belonging to $\Gamma'$ but not to $\Gamma$ (i.e. $\delta \in \Delta(\xi, \Gamma') - \Delta(\xi, \Gamma)$). A representative branch $\Lambda(\xi)$ for $\delta$ defines a non generic open disk $D^{-}(t, \rho(t))$ which is tangent to $\Gamma$ at $\xi = \xi_{t,\rho(t)}$ and which intersects $\Gamma'$. The proof is then divided in two parts. Firstly we prove that there are finitely many bifurcation points of $\Gamma'$ belonging to $\Gamma - \mathcal{C}$ (cf. Proposition 2.18). This amounts to prove that there are finitely many disks of the above type. Secondly we prove that inside each disk tangent to $\Gamma$ there are finitely many bifurcation points of $\Gamma'$ (cf. Proposition 2.19).

Lemma 2.15 (Flat directions do not belong to the skeleton of $\mathcal{R}$). Let $\xi \in X$, let $t$ be a Dwork generic point for $\xi$ and let $\rho \leq \rho_{t}(t)$. Assume that $\mathcal{R}$ satisfies for all $\xi' \in D^{-}(t, \rho)$ the conditions (C1'): $\rho_{t}(\xi') > 0$ and (C3'): $L_{\mathcal{R}_{\xi'}}$ is concave on $] - \infty, \ln(\rho)[$. Then $L_{\mathcal{R}_{\xi}}$ is non constant along the segment $] - \infty, \ln(\rho)[$ if and only if $\rho_{t}(\xi) < \rho$, and in this case $L_{\mathcal{R}_{\xi}}$ has a break at $\tau := \ln(\rho_{t}(\xi))$.

Proof. If $L_{\mathcal{R}_{t}}$ is not constant on $] - \infty, \ln(\rho)[$, then $\mathcal{R}$ is non constant in $D^{-}(t, \rho)$, so $\rho_{t}(\xi) < \rho$. Conversely assume that $\rho_{t}(\xi) < \rho$ and, by contrapositive, that $L_{\mathcal{R}_{t}}$ is constant on $] - \infty, \tau'[$. Some $\tau < \tau' < \ln(\rho)$. Let $\rho' := \exp(\tau') > \rho_{t}(\xi)$. Since $\mathcal{R}$ is not constant on $D^{-}(t, \rho')$, there exists $\Omega' \in E(\Omega)$ and $t' \in D_{t'}^{-}(t, \rho')$ such that $\mathcal{R}(\Omega') \neq \mathcal{R}(t')$. We now consider $L_{\mathcal{R}_{t'}}$. For all $\rho \geq |t - t'|$ one has $\mathcal{R}_{t}(\rho) = \mathcal{R}_{t'}(\rho)$ because $\lambda_{t}(\rho) = \lambda'_{t'}(\rho)$. So $L_{\mathcal{R}_{t'}}$ is constant and equal to $L_{\mathcal{R}_{t}}$ along $|t - t'|, \rho'[ with value $\mathcal{R}(\xi')$. This contradicts the concavity of $L_{\mathcal{R}_{t'}}$ along $] - \infty, \ln(\rho)[$ because, by (C1'), $L_{\mathcal{R}_{t'}}$ is also constant in a neighborhood of $-\infty$ with value $\ln(\mathcal{R}(\xi')) \neq \ln(\mathcal{R}(\xi'))$.

Remark 2.16. A open disk $D^{-}(t, \rho)$ is generic if and only if $\rho \leq \rho_{t}(\xi)$. The left slope of $L_{\mathcal{R}_{t}}$ along $\lambda_{t}$ can be defined also at $\tau := \ln(\rho) \leq \ln(\rho_{t}(\xi))$, but it is always zero since $\mathcal{R}$ is constant on each generic disk (cf. (2.1)). In other words generic disks define flat directions.

Proposition 2.17. Assume that $\mathcal{R}$ satisfies (C1),(C2),(C3). Let $D^{-}(t, \rho)$ be a non generic disk which is tangent to $\Gamma$ at $\xi = \xi_{t,\rho}$. Let $\delta \in \Delta(\xi)$ be the direction defined by $D^{-}(t, \rho)$. Then $D^{-}(t, \rho)$ intersects $\Gamma'$ (or equivalently $\Gamma(\mathcal{R})$) if and only if $\delta$ is not a flat direction i.e. if $\partial_{-} \mathcal{R}_{t}(\delta) < 0$ (cf. Def. 2.9).

Proof. Conditions (C1), (C2) and (C3) imply that the slopes of $L_{\mathcal{R}_{t}}$ along $|0, \rho|$ are all negatives or equal to zero. Since $\rho_{t}(\xi) = \rho_{\mathcal{R}(t)}(\xi)$ (cf. section 2.0.2), then $D^{-}(t, \rho)$ intersects $\Gamma(\mathcal{R})$ if and only if $\rho_{t}(\xi) < \rho$. Lemma 2.15 implies $\partial_{-} \mathcal{R}_{t}(\delta) < 0$ because $L_{\mathcal{R}_{t}}$ has at least a break at $\rho_{t}(\xi) < \rho$.

Proposition 2.18. There are a finite number of bifurcation points of $\Gamma'$ belonging to $\Gamma$.

Proof. It is enough to prove that along each individual maximal branch $\Lambda(\xi)$ of $\Gamma$ there is a finite number of bifurcation points $\xi$ of $\Gamma(\mathcal{R})$. As observed the points of $\mathcal{C}$ are finite in number and we can neglect them. By Proposition 2.17 the super-harmonicity (C6) implies that $L_{\mathcal{R}_{\xi}} : [r(\xi), \mathcal{R}_{t}[ \to \mathbb{R}$ has a break at each bifurcation point of $\Gamma(\mathcal{R})$ belonging to $\Lambda(\xi) - \mathcal{C}$. By (C3) there are a finite number of breaks along $\Lambda(\xi)$, and hence a finite number of bifurcation points of $\Gamma(\mathcal{R})$.

Proposition 2.19. There is a finite number $N$ of bifurcation points of $\Gamma(\mathcal{R})$ inside a given (non generic) disk $D^{-}(t, \rho_{t}(t))$ which is tangent to $\Gamma$. Moreover, since $\mathcal{C}(\mathcal{R}) \subset \Gamma$, then $0 \leq N \leq \frac{\partial_{-} \mathcal{R}_{t}(\delta)}{2 \nu_{t}}$, where $\xi = \xi_{t,\rho_{t}(t)}$ and $\delta \in \Delta(\xi)$ is the direction defined by the disk $D^{-}(t, \rho_{t}(t))$. 


In particular the function \( L_R \) exists a compact sub-interval \( \Omega' \) for an unspecified \( \Omega' \) equal to \( L_R \) then \( \partial_- R(\xi') \leq \sum_{\delta \in \Delta(\xi', \Gamma(\Omega))} \partial_- \mathcal{R}_\theta(\xi') \leq -\nu_R \cdot N(\xi') \leq -2\nu_R < 0 \). (2.9)

In particular the function \( L^t \mathcal{R}_t \) has a break in correspondence to each bifurcation point of \( \Gamma(\Omega) \) in \( \Lambda(\xi_i) \). At each break point along \( \Lambda(\xi_i) \) one has (2.9) hence \( \partial_- \mathcal{R}_\delta(\xi) \leq -\nu_R \cdot N_t \), where \( N_t \) is the number of bifurcation points along \( \Lambda(\xi_i) \). If by contrapositive one has an infinite number of bifurcation points in \( D^-(t, \rho(t)) \), then for all integer \( n \) there exists a branch \( \Lambda(\xi_n) \) with \( t_n \) in \( D^-(t, \rho(t)) \) having at least \( n \) bifurcation points of \( \Gamma(\Omega) \). This forces \( \partial_- \mathcal{R}_\delta(\xi) \) to be less than or equal to \( -2\nu_R \cdot n \) for all \( n \), which is absurd because \( \partial_- \mathcal{R}_\delta(\xi) > -\infty \). So the number of bifurcation points of \( \Gamma(\Omega) \) inside \( D^-(t, \rho(t)) \) is finite. \( \square \)

This completes the proof of theorem 2.14. \( \square \)

**Definition 2.20.** Let \( A(t, I) := \{ |T - t| \in I \} \) be a possibly not closed annulus or disk (if \( 0 \in I \) one has a disk), and let \( \mathcal{R} : A(t, I) \to \mathbb{R} \) be a function. We say that \( \mathcal{R} \) is finite over \( A(t, I) \) if there exists a compact sub-interval \( J \subset I \) such that \( \Gamma(\mathcal{R}_{|J}) \) is finite over \( A(t, J) \), and for all compact \( J \subseteq J' \subseteq I \) one has \( \Gamma(\mathcal{R}_{|J'}) = \Gamma(\mathcal{R}_{|J}) \cup \Gamma_{A(t, J')} \) over \( A(t, J') \). In this case we define the skeleton \( \Gamma(\mathcal{R}) \) over \( A(t, I) \) as \( \Gamma(\mathcal{R}) := \Gamma(\mathcal{R}_{|J}) \cup \{ \xi_{0, \rho} \}_{\rho \in I} \).

**Corollary 2.21.** Let \( \mathcal{R} \) be a function defined on a possibly not closed annulus or disk \( A(t, I) := \{ |T - t| \in I \} \) (if \( 0 \in I \) one has a disk). Fix once for all two finite sets \( S, \mathcal{C}(\mathcal{R}) \subset X \), and set \( \Gamma = \{ \text{Sat}(S) \cup \{ \xi_{0, \rho} \}_{\rho \in I} \} \). If \( \mathcal{R} \) verifies the six properties (C1)–(C6) with respect to \( \Gamma \) and \( \mathcal{C}(\mathcal{R}) \) on each sub-annulus (resp. sub-disk) \( A(t, J) \), with \( J \) compact, \( J \subseteq I \) (resp. \( 0 \in J \) if \( 0 \in I \)). Assume moreover that \( \mathcal{R} \) has a finite number of breaks along \( \{ \xi_{0, \rho} \}_{\rho \in I} \). Then \( \Gamma(\mathcal{R}) \) is continuous, finite and it factorizes through \( \Gamma(\mathcal{R}) \). Moreover if \( A(t, I) \) is an open disk, if \( \Gamma \) and \( \mathcal{C}(\mathcal{R}) \) are both empty, and if \( s \) is the last slope of \( L^t \mathcal{R}_t \) as in Prop. 2.19, then the number \( N \) of bifurcation points of \( \Gamma(\mathcal{R}) \) verifies \( 0 \leq N \leq \frac{s}{2\nu_R} \).

**Proof.** By Thm. 2.14 the restriction of \( \mathcal{R} \) is finite on each closed sub-annulus \( \{ |T - t| \in J \} \subseteq \{ |T - t| \in I \} \). By super-harmonicity, each new branch of \( \Gamma(\mathcal{R}) \) generates a break along \( \{ \xi_{0, \rho} \}_{\rho \in I} \). \( \square \)

**Example 2.22.** 1. The function \( \xi \mapsto \rho_{\xi, X} \) verifies the six properties (C1)–(C6) with \( \Gamma = \Gamma_X \), and \( \mathcal{C}(\rho_{\cdot, X}) = \emptyset \). It is moreover super-harmonic in the sense of definition 2.10, and \( \Gamma(\rho_{\cdot, X}) = \Gamma_X \).

2. If \( \mathcal{R}_1, \ldots, \mathcal{R}_n \) are functions satisfying the six properties (C1)–(C6), then so does \( \min(\mathcal{R}_1, \ldots, \mathcal{R}_n) \).

3. Let \( f_1, \ldots, f_n \in \mathcal{O}(X) \) and \( \alpha_1, \ldots, \alpha_n > 0 \). Assume that each \( f_i \) has no zeros on \( \mathcal{O}(X) \). Then the function \( \mathcal{R}(\xi) := \min_i |f_i(\xi)|^{-\alpha_i} \) verifies (C1)–(C6), with \( \Gamma = \Gamma_X \), and \( \Gamma(\mathcal{R}) = \Gamma_X \). If \( K = K_{\text{alg}} \), then \( \mathcal{R} \) is also super-harmonic (cf. Def. 2.10) because so does each function \( \xi \mapsto |f_i(\xi)|^{-\alpha_i} \).

### 2.4 Permanence of (C1)–(C6) by descent of the ground field.

For \( \Omega \in E(K) \) let \( \Pr_K^\Omega : X \circ \Omega \to X \) be the canonical projection, and let \( \mathcal{R}' = \mathcal{R} \circ \Pr_K^\Omega \). For all \( \Omega' \in E(\Omega) \) and all \( t \in X(\Omega') \) one obviously has \( \rho_{\mathcal{R}(\Omega')}(t) = \rho_{\mathcal{R}(\Omega)}(t) \), because \( \rho_{\mathcal{R}(\Omega)}(t) \) only depend on \( X(\Omega') \) for an unspecified \( \Omega' \in E(K) \) (cf. (2.2)). This immediately gives \( \Gamma(\mathcal{R}) = \Pr_K^\Omega(\Gamma(\mathcal{R}')) \). The finiteness and the directionality finiteness (C5) of \( \Gamma(\mathcal{R}') \) implies then that of \( \Gamma(\mathcal{R}) \). Moreover the restrictions \( \mathcal{R}_t \) and \( \mathcal{R}'_t \) coincide as functions on \( [0, R_0] \) (cf. (2.5)). So the first four properties (C1)–(C4) hold for \( \mathcal{R}' \) if and only if they hold for \( \mathcal{R} \). Conversely the super-harmonicity of \( \mathcal{R}' \) over \( X' \) does not imply that of \( \mathcal{R} \). The problem arises at the bifurcation points of \( \Gamma(\mathcal{R}) \cap \Gamma, \) in particular at those of \( \Gamma_X \).

This is the reason for which one introduces property (C6) instead of the full super-harmonicity. In
sections 2.4.1 and 2.4.2 below we prove that if $\mathcal{R}'$ satisfies (C6) then so does $\mathcal{R}$. We first analyze the case $\Omega = \hat{K}^{\text{alg}}$ (cf. section 2.4.1) and then the general case (cf. section 2.4.2).

**Remark 2.23.** If $K = \hat{K}^{\text{alg}}$ the functions in $\mathcal{E}(X)$ are harmonic at each point $\xi \in \text{X}_{\text{int}} - \Gamma_X$. If $K \neq \hat{K}^{\text{alg}}$, then the functions in $\mathcal{E}(X)$ are not all super-harmonic nor all sub-harmonic. As a counterexample consider $p = 3$, $K = \mathbb{Q}_p$, $f(T) := (T - 1)^p / (T - 1)$, with roots $\alpha_1, \alpha_2$, and $X \otimes \hat{K}^{\text{alg}} = D^+(0, 1)$ with holes $D^-(\alpha_i, \varepsilon), 0 < \varepsilon < |\alpha_1 - \alpha_2|$. Then $f, f^{-1} \in \mathcal{E}(X)$, and the two paths $\lambda_{\alpha_1}$ and $\lambda_{\alpha_2}$ are identified in $X$ by $\text{Gal}(K^{\text{alg}}/K)$. So $\Gamma(f) = \Gamma(f^{-1}) = \Lambda(\xi_{\alpha_1, \varepsilon})$ in $X$. $f$ and $f^{-1}$ are both harmonic as functions on $X \otimes \hat{K}^{\text{alg}}$. While on $X$ one sees that $f$ (resp. $f^{-1}$) is log-convex (resp. log-concave) along $\lambda_{\alpha_1}$, hence sub-harmonic (resp. super-harmonic) at $\xi_{\alpha_1, |\alpha_1 - \alpha_2|}$.

**2.4.1 Algebraic extensions.** Let $X' := X \otimes \hat{K}^{\text{alg}}$. By [Ber90, 1.3.6] the canonical map $\text{Pr}_K^{\text{alg}}$ identifies $X$ with $X'/\text{Gal}(K^{\text{alg}}/K)$. Let $\Gamma' := (\text{Pr}_K^{\text{alg}})^{-1}(\Gamma)$ and $\mathcal{E}'(\mathcal{R}') := (\text{Pr}_K^{\text{alg}})^{-1}(\mathcal{E}(\mathcal{R}))$. Assume that $\mathcal{R}'$ verifies the six properties (C1)–(C6) with respect to $\Gamma'$ and $\mathcal{E}'(\mathcal{R}')$. Let $\xi' \in X'$, and let $\xi := \text{Pr}_K^{\text{alg}}(\xi')$. If a set of directions $\delta_1, \ldots, \delta_{n_{\delta}} \in \Delta(\xi')$ form an orbit under $\text{Gal}(K^{\text{alg}}/K)$ and if $\delta$ is the corresponding direction of $\xi$, then as observed one has $\partial_+ \mathcal{R}'(\xi') = \partial_+ \mathcal{R}(\xi)$ and $\partial_- \mathcal{R}_{\delta}(\xi) = \partial_- \mathcal{R}_{\delta}(\xi')$ for all $i = 1, \ldots, n_{\delta}$. We call $n_{\delta}$ the multiplicity of $\delta$. Replacing each $\partial_- \mathcal{R}_{\delta}(\xi')$ by $\partial_- \mathcal{R}_{\delta}(\xi)$ in the formula (2.8) one sees that the contribution $\partial_- \mathcal{R}_{\delta}(\xi)$ of $\delta$ to the super-harmonicity of $\mathcal{R}$ at $\xi$ equals $\frac{1}{n_{\delta}} \cdot S'$, where $S' := \sum_{i=1}^{n_{\delta}} \partial_- \mathcal{R}_{\delta}(\xi')$ is the contribution of the orbit $\{\delta_i\}$ to the super-harmonicity of $\mathcal{R}'$ at $\xi'$. If $\delta_i \notin \Delta(\xi', \Gamma' \cap \Gamma(\mathcal{R}'))$, then $\partial_- \mathcal{R}_{\delta}(\xi') \leq 0$ for all $i$ (cf. Proposition 2.17) so that $S' \leq \partial_- \mathcal{R}_{\delta}(\xi)$. Hence if $\xi' \notin \Gamma(\mathcal{R}') \cap \Gamma'$ the super-harmonicity of $\mathcal{R}'$ at $\xi'$ implies that of $\mathcal{R}$ at $\xi$. The same holds if $\xi' \in \Gamma(\mathcal{R}') \cap \Gamma'$ is not a bifurcation point of $\Gamma(\mathcal{R}') \cap \Gamma'$ because in this case there are three kind of directions: $\Delta(\xi', \Gamma(\mathcal{R}') \cap \Gamma') = \{\delta\}$ is reduced to a single element, so $n_{\delta} = 1$ and its contribution to the super-harmonicity is the same over $K$ or $\hat{K}^{\text{alg}}$, the directions $\delta \in \Delta(\xi', \Gamma(\mathcal{R}'))$ satisfy $S' \leq \partial_- \mathcal{R}_{\delta}(\xi)$ as above, and finally the other directions do not contribute to the super-harmonicity. The problem arises if $\xi' \in \Gamma(\mathcal{R}') \cap \Gamma'$ is a bifurcation point of $\Gamma(\mathcal{R}') \cap \Gamma'$. In this case $\partial_- \mathcal{R}_{\delta}(\xi')$ can be positive for $\delta \in \Delta(\xi', \Gamma' \cap \Gamma(\mathcal{R}))$, and the super-harmonicity of $\mathcal{R}'$ at $\xi'$ does not imply the super-harmonicity of $\mathcal{R}$ at $\xi$ unless the multiplicity of each direction in $\Delta(\xi', \Gamma' \cap \Gamma(\mathcal{R}))$ is equal to one i.e. if $\Gamma(\mathcal{R}') \cap \Gamma' = \text{Sat}(\{\xi_{t_1, \rho_1}, \ldots, \xi_{t_n, \rho_n}\})$ with $t_1, \ldots, t_n \in X(K)$, in other words if $\Gamma'$ is $K$-rational (cf. section 1.4). This proves the following

**Proposition 2.24.** If $\mathcal{R}'$ verifies the five properties (C1)–(C5) then so does $\mathcal{R}$. If one choses $\mathcal{E}(\mathcal{R})$ in order that $\mathcal{E}'(\mathcal{R}')$ contains the bifurcation points of $\Gamma(\mathcal{R}') \cap \Gamma'$, then property (C6) descends from $\mathcal{R}'$ to $\mathcal{R}$. If moreover $\Gamma \cap \Gamma(\mathcal{R})$ is $K$-rational and if $\mathcal{R}'$ is super-harmonic, then so does $\mathcal{R}$ (Note that if $\Gamma$ is $K$-rational then so does $\Gamma \cap \Gamma(\mathcal{R}))$.

**2.4.2 Transcendental extensions.** Let $K = \hat{K}^{\text{alg}}$, let $\Omega \in E(K)$, and $X' := X \otimes_K \Omega$. Let $\xi' \in \Gamma(\mathcal{R}')$ and let $\xi \in \Gamma(\mathcal{R})$ be its image. Super-harmonicity concerns bifurcation points. They are all of type (2) in the sense of [Ber90, 1.4.4]. By Remark 1.10 the directions in $\Delta(\xi)$ and in $\Delta(\xi')$ all admit a representative branch of the type $\Lambda(\xi_i)$, with $t \in X(K) \subset X(\Omega)$. One has the following commutative diagram (cf. (1.6)) where both $i_K$ and $i_{\Omega}$ are injective maps:

$$
\begin{array}{ccc}
X(\Omega) & \xrightarrow{i_{\Omega}} & X' \\
\cup & & \downarrow \text{Pr}_K^{\text{alg}} \\
X(K) & \xrightarrow{i_K} & X.
\end{array}
$$

Each path $\lambda_{\xi} : [0, R_0] \to X$, with $t \in X(K) \subset X(\Omega)$, admits then a canonical lifting $\tilde{\lambda}_{\xi} : [0, R_0] \to X'$, where $\tilde{\xi} = i_{\Omega}(t) \in X'$. One hence has a canonical injective map $\Delta(\xi) \subseteq \Delta(\xi')$. Note that there
is no map $\Delta(\xi') \rightarrow \Delta(\xi)$ corresponding to $\Pr_{K}^{\Omega}$. In fact if a direction $\delta' \in \Delta(\xi')$ is defined by a disk $D^{-}(t', \rho)$ which is generic over $K$, but not over $\Omega$, then the image in $X$ of $D^{-}(t', \rho)$ is reduced to the point $\{\xi = \xi_{t', \rho}\}$, and there is no directions in $\Delta(\xi)$ corresponding to $\delta'$. In this case we say that $\delta'$ is *contracted* to $\xi$ by $\Pr_{K}^{\Omega}$. This is in fact the case for all directions $\delta' \in \Delta(\xi') - \Delta(\xi)$. Indeed $\Pr_{K}^{\Omega}$ identifies $\Gamma_{X'}$ with $\Gamma_{X}$ (which is explicitly given in both cases by Sat$(\{\xi_{ci,Ri}\}_{i=0,\ldots,m})$ cf. section 1.5) so one has the equality $\Delta(\xi', \Gamma_{X'}) = \Delta(\xi, \Gamma_{X})$. On the other hand if $\delta'$ does not belong to $\Gamma_{X'}$ then it is defined by a disk $D^{-}(t', \rho)$ which is non generic over $\Omega$. Clearly $D^{-}(t', \rho)$ is non generic over $K$ if and only if $\delta' \in \Delta(\xi)$. So each direction in $\Delta(\xi') - \Delta(\xi)$ is contracted to $\xi$. These directions are not in $\Gamma(\mathcal{R}')$ since defined by generic disks on which $\mathcal{R}'$ is constant.

**Proposition 2.25.** Let $\xi' \in \Gamma(\mathcal{R}')$, and let $\xi \in \Gamma(\mathcal{R})$ be its image in $X$. Then

$$\Delta(\xi, \Gamma(\mathcal{R})) = \Delta(\xi', \Gamma(\mathcal{R}')).$$

(2.11)

Hence the super-harmonicity is preserved by extending or descending the scalars from $K$ to $\Omega$. 

**Remark 2.26.** The radius of convergence function $\mathcal{R}^{M}$ will satisfy the six conditions (C1)–(C6) with respect to $\Gamma := \Gamma_{X}$ and $\mathcal{C}(\mathcal{R}) := \mathcal{C}_{X}$, and it will be also super-harmonic over any $\Omega \in E(K^{\text{alg}})$. If $X$ is $K$-rational, then $\mathcal{R}^{M}$ will be super-harmonic on $X$ (cf. Prop. 2.24)

### 3. Radius of convergence function of an ultrametric differential module

Thank to 2.4 one can assume $K = \widehat{K}^{\text{alg}}$. Nevertheless we do not apply systematically this assumption in order to point out the problems over $K$ (e.g. Lemma 3.4 and super-harmonicity in general).

#### 3.1 Preliminaries

A *differential ring* is a ring $A$ together with a derivation $d : A \rightarrow A$. A *differential module* over $(A, d)$ is a finite free $A$-module together with a linear map $\nabla : M \rightarrow M$, called the *connection* of $M$, satisfying $\nabla(am) = d(a)m + a\nabla(m)$ for all $a \in A$ and $m \in M$. The choice of a basis of $M$ provides an isomorphism of $A$ modules $M \cong A^{r}$, and the operator $\nabla$ is given in this basis by the rule

$$\nabla(a_{1}, \ldots, a_{r})^{t} = (d(a_{1}), \ldots, d(a_{r}))^{t} - G \cdot (a_{1}, \ldots, a_{r})^{t}$$

(3.1)

where $G \in M_{r}(A)$ is a matrix. Reciprocally the data of such a matrix defines a differential module structure on $A^{r}$ by the rule (3.1). A morphism between differential modules is an $A$-linear map $M \rightarrow N$ commuting with the connections. We denote by $d - \text{Mod}(A)$ the category of differential modules over $A$. Because we use Taylor solutions in the sequel the derivation will always be $d/dT$, and $A = \mathcal{O}(X)$, $\mathcal{H}(\xi)$, $\mathcal{A}_{\Omega}(t, \rho)$, ... If $A = \mathcal{O}(X)$ we will often restrict $M$ to a sub-affinoid $X' \subseteq X$ or to a subdisk $D^{-}(t, \rho) \subseteq X$. This means that we consider the scalar extension module $M \otimes_{\mathcal{O}(X)} \mathcal{O}(X')$ (resp. $M \otimes_{\mathcal{O}(X)} \mathcal{A}_{\Omega}(t, \rho)$) together with the connection $\nabla' := \nabla \otimes \text{Id} + \text{Id} \otimes d/dT$. A solution of $M$ with values in $\mathcal{A}_{\Omega}(t, \rho)$ is an element of the kernel of $\nabla'$ acting on $M \otimes_{\mathcal{O}(X)} \mathcal{A}_{\Omega}(t, \rho)$. If we chose a basis as above (cf. (3.1)), such a solution $\tilde{y} \in \mathcal{A}_{\Omega}(t, \rho)^{r}$ verifies $\tilde{y}' = G(T)\cdot \tilde{y}$. The Taylor solution of $M$ at $t$ is given by $Y(T, t) = \sum_{s \geq 0} G_{s}(T)(T-t)^{s}$, where $G_{s}$ is the matrix defined by $d^{s}(Y) = G_{s}(T)\cdot Y$. Namely $G_{0} := \text{Id}$, $G_{1} := G$, and recursively one has $G_{s+1} := d(G_{s}) + G_{s}\cdot G$ for all $s \geq 0$. The radius of convergence of this power series is given by

$$\mathcal{R}^{Y}_{\Omega}(t) := \liminf_{s} |G_{s}(t)/s|^{-1/s}_{\Omega}$$

(3.2)

where if $G_{s} = (g_{s,i,j})$, then $|G_{s}(t)| = \max_{i,j} |g_{s,i,j}(t)|$. It is well known that $\mathcal{R}^{Y}_{\Omega}(t) > 0$ for all $t \in X(\Omega)$ (cf. [DGS94, Appendix III]). $\mathcal{R}^{Y}_{\Omega}(t)$ only depend on the Berkovich point $\xi_{t} \in X$ defined by $t$, and so one has a well defined function $\mathcal{R}^{Y} : X \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$. 

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Definition 3.1. The radius of convergence of function of \( M \) is the function \( R^M : X \to ]0, R_0] \) associating to each \( \xi \in X \) the positive real number

\[
R^M(\xi) := \min( R^Y(\xi), \rho_{\xi,X} ) .
\]

(3.3)

The spectral radius of convergence function \( R^{M \hat{\otimes} K^{alg,sp}} : X \to ]0, R_0] \) of \( M \) is defined by

\[
R^{M \hat{\otimes} K^{alg,sp}}(\xi) := \min( R^Y(\xi), r(\xi) ) .
\]

(3.4)

The notation \( R^{M \hat{\otimes} K^{alg,sp}} \) will be justified in Remark 3.9. As for \( R^Y \) one always has \( R^M(\xi) > 0 \) (and also \( R^{M \hat{\otimes} K^{alg,sp}}(\xi) > 0 \) if \( r(\xi) > 0 \)). The matrices \( Y(T,t) \) and \( G_s \) depend on a basis of \( M \), as well the function \( R^Y \). On the other hand the presence of \( \rho_{\xi,X} \) and \( r(\xi) \) in the definitions of \( R^M \) and \( R^{M \hat{\otimes} K^{alg,sp}} \) respectively makes them invariant under base changes of \( M \). The Taylor solution matrix \( Y(T,t) \) lies then in \( GL_n(\mathbb{A}_\Omega(t, R^M(\xi_t))) \), and it verifies \( Y(T,t) = Y(T, t') \cdot Y(t', t) \) for all \( t' \) satisfying \( |t - t'| < R^M(\xi_t) \) (cf. [Chr12]). In particular the radius of convergence \( R^Y(\xi_t) \) of \( Y(T,t) \) is larger than or equal to the radius of convergence \( R^Y(\xi_t') \) of \( Y(T,t') \), by symmetry\(^{10}\) the two radii are actually equal \( R^Y(\xi_t) = R^Y(\xi_t') \). This fact together with (2.1) imply

\[
\max(R^M(\xi), r(\xi)) \leq \rho_{RM}(\xi) ,
\]

(3.5)

\[
\rho_{RY}(\xi) = \rho_{RM}(\xi) ,
\]

(3.6)

\[
\rho_{R^{M \hat{\otimes} K^{alg,sp}}}(\xi) = r(\xi) .
\]

(3.7)

From (3.6) one immediately has

\[
\Gamma(R^M) = \Gamma(R^Y)
\]

and from (3.7) one has \( \Gamma(R^{M \hat{\otimes} K^{alg,sp}}) = \Gamma(r) = X \) (here \( r : \xi \mapsto r(\xi) \) cf. Def. 1.10). So the important notion \( R^{M \hat{\otimes} K^{alg,sp}} \) is not finite. From (3.5) and (3.6) we deduce that if \( \rho_{RM}(\xi) = r(\xi) \) (i.e. if \( \xi \in \Gamma(R^M) \)), then

\[
R^M(\xi) = R^{M \hat{\otimes} K^{alg,sp}}(\xi) \leq r(\xi) ,
\]

(3.9)

and if the inequality is strict, then \( R^M(\xi) = R^{M \hat{\otimes} K^{alg,sp}}(\xi) = R^Y(\xi) \). From (3.5) and (2.1) one also obtains the following often useful expression of \( R^M \):

\[
R^M(\xi) = \min(R^Y(\xi), \rho_{RY}(\xi)) .
\]

(3.10)

Another important property of \( R^M \) and \( R^Y \) is the so called transfer theorem which affirms that they inverse the natural partial order of \( X \), that is : if \( \xi_1(f) \leq \xi_2(f) \) for all \( f \in \mathcal{O}(X) \), then

\[
R^Y(\xi_1) \geq R^Y(\xi_2) \quad \text{and} \quad R^M(\xi_1) \geq R^M(\xi_2) .
\]

(3.11)

For \( R^Y \) this immediately follows from (3.2). For \( R^M \) this follows from (3.3) since one has \( \rho_{\xi_1,X} = \rho_{\xi_2,X} \) (cf. section 1.5). The transfer theorem can be rephrased as follows. If \( 0 \leq \rho \leq \rho_{\xi,X} \), then \( R^M(\xi) \geq R^M(\lambda_\rho(\xi)) \) i.e. \( R^M \) is decreasing on \([0, \rho_{\xi,X}]\). Note that \( L^\infty R^M \) is moreover concave on \([\infty, \log(\rho_{\xi,X})] \), because by (3.2) it is lim inf of concave functions. More generally if \( I \) is an interval with interior \( \bar{I} \) and if the annulus \( \{ |T - \xi| \in \bar{I} \} \) is contained in \( X \), then \( R^M \) is log-concave on \( I \) by the same reason (cf. property (LC) of section 1.3).

Remark 3.2. The function \( R^Y \) and \( R^M \) are invariant by scalar extension of the ground field \( K \), hence all considerations of section 2.4 hold. More precisely let \( \tilde{\Omega} \in E(K), X' = X \otimes \tilde{\Omega}, M' := M \otimes \tilde{\Omega} \). Let \( R^{\tilde{\Omega}} \) be the function (3.2) on \( X' \) defined by \( M' \) in the basis \( e \otimes 1 \), where \( e \subset M \) is the basis that serves to define \( R^Y \). Then \( R^{\tilde{\Omega}} = Pr^\Omega \circ R^Y \) since the matrices \( \{ G_s \}_s \) of (3.2) are the same in

\(^9\)The spectral radius is often called generic radius by the authors, e.g. [Ked10b].

\(^{10}\)\( Y(t,t') \in GL_n(\Omega) \) is invertible with inverse \( Y(t',t) \).
both cases. This immediately gives $\mathcal{R}^M = \text{Pr}_K^\Omega \circ \mathcal{R}^M$ since $\rho_{-,X}$ is independent on $\Omega$. Conversely $\xi \mapsto r(\xi)$ depends on $K$ and so $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}$ is invariant if and only if $\Omega/K$ is algebraic.

3.2 Comparison between $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}$ and $\mathcal{R}^M$.

We now compare $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}$ and $\mathcal{R}^M$ along a branch. Definition 3.1 immediately gives $\mathcal{R}^{M \otimes K^{\text{alg,sp}}} = \min(r(\xi), \mathcal{R}^M(\xi))$, for all $\xi \in X$. Since $r(\lambda_\xi(\rho)) = \max(\rho, r(\xi))$ one finds

$$\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho) = \min(\max(r(\xi), \mathcal{R}^M(\xi))$$

for all $\rho \in [0, R_0]$, as in the following picture where $R := \mathcal{R}^M(\xi) > 0$:

$$\tau \mapsto \mathcal{R}^M(\tau)$$

In particular since $\rho \mapsto \mathcal{R}^M(\rho)$ is log-concave for $\rho \in [0, \rho_{\xi,X}]$ and $\rho \mapsto \max(r(\xi), \rho)$ is log-convex for all $\rho \in [0, R_0]$, then

i) For all $\rho \in [R, R_0]$ one has $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho) = \mathcal{R}^M(\rho)$.

ii) If $R \leq r(\xi)$, then $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho) = \mathcal{R}^M(\rho)$ for all $\rho \in [0, R_0]$.

iii) If $R > r(\xi)$, then $\forall \rho \in [0, R]$ one has $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho) = \max(\rho, r(\xi))$ and $\mathcal{R}^M(\rho) = \mathcal{R}^M(\xi) = R$.

Indeed for all $\rho \in [\rho_{\xi,X}, R_0]$ one has $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho) = \mathcal{R}^M(\rho)$ because $\lambda_\xi(\rho) \in \Gamma_X$ and hence $r(\lambda_\xi(\rho)) = \rho_{\lambda_\xi(\rho),X}$. So the above equalities have to be proved for $\rho < \rho_{\xi,X}$, and in this case they follow by the above convexity/concavity argument.

**Remark 3.3.** If $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho) = \mathcal{R}^M(\rho)$ then for all $\rho^\prime \in [\rho, R_0]$ one has $\mathcal{R}^{M \otimes K^{\text{alg,sp}}}(\rho^\prime) = \mathcal{R}^M(\rho^\prime)$.

3.3 Spectral radius and spectral norm of the connection.

We quickly recall some facts that are necessary for the correct understanding of this paper. Let $(F, |.|_F) \in E(K)$ and let $V$ be a finite dimensional vector space. A norm $|.|_V$ on $V$ compatible with $|.|_F$ is a map $|.|_V : V \to \mathbb{R}_{>0}$ such that (i) $|v|_V = 0$ if and only if $v = 0$; (ii) $|v - v'|_V \leq \max(|v|_V, |v'|_V)$ for all $v, v' \in V$; (iii) $|fv|_V = |f|_F \cdot |v|_V$ for all $f \in F, v \in V$. If $T : V \to V$ is a bounded $\mathbb{Z}$-linear operator, define $|T|_V := \sup_{v \neq 0} |T(v)|_V / |v|_V$ and $|T|_{\text{sp},|.|_F} := \lim_{s \to |.|_F}$ $|T^s|_V / |s|_V$. One proves that the limit exists, and that $|T|_{\text{sp},|.|_F}$ only depends on $|.|_F$ and not on the choice of $|.|_V$ compatible with $|.|_F$ (cf. [Ked10b, Def. 6.1.3]). Let $\omega := \lim_n |n|^{1/n}$. If the restriction of $|.|$ to $\mathbb{Q}$ is $p$-adic (resp. trivial), then $\omega = |p|^{-1/2}$ (resp. $\omega = 1$).

If $\xi \in X^{\text{gen}}$ the kernel of $\xi$ is zero, and $\xi$ is a norm on $\mathcal{O}(X)$. In this case $(\mathcal{H}(\xi), \xi) = (\mathcal{F}(X), \xi)$ is the completion of the fraction field $\mathcal{F}(X)$ of $\mathcal{O}(X)$ with respect to the norm $\xi$. The following lemma proves that the derivation $d/dT$ is continuous, and hence it extends by continuity to $\mathcal{H}(\xi)$.

**Lemma 3.4.** Let $\xi \in X^{\text{gen}}$ and let $V := F := \mathcal{H}(\xi)$. Then $|(d/dT)^n|_{\mathcal{H}(\xi)} \leq |n|^{-1}$ $r(\xi)^n$. Assume that
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\( K = \overline{K}^{\text{alg}} \) or, if \( K \neq \overline{K}^{\text{alg}} \), that \( \xi = \lambda_c(\rho) \) with \( c \in K \) and \( \rho > 0 \). Then

\[
\|(d/dT)^n|_{\mathcal{H}(\xi)} = \frac{|n!|}{r(\xi)^n} \quad \text{and} \quad \|d/dT\|_{Sp,\xi} = \frac{\omega}{r(\xi)}. \tag{3.14}
\]

Proof. If \( t \in X(\Omega) \) is a Dwork generic point for \( \xi \), then the Taylor expansion at \( t \) gives an injective map \( \mathcal{H}(\xi) \to A_0(t, r(\xi)). \) The image of \( f \in \mathcal{H}(\xi) \) is \( \sum_{i \geq 0} f^{(i)}(t)(T - t)^i/i! \) and \( \xi(f) = \xi_{t,0}(f) = \xi_{t,r(\xi)}(f) = \sup_{i \geq 0} |f^{(i)}(t)/i!| \cdot r(\xi)^i \). From this one easily has \( \|(d/dT)^n|_{\mathcal{H}(\xi)} \leq |n!|/r(\xi)^n \). Now we prove the converse inequality. Assume first that \( K = \overline{K}^{\text{alg}} \). For all \( c \in K \) one has \( |n!| = \|(d/dT)^n|_{\mathcal{H}(\xi)}(T - c)^n| \leq \|(d/dT)^n|_{\mathcal{H}(\xi)}|T - c|_n^n \). Hence \( \|(d/dT)^n|_{\mathcal{H}(\xi)} \geq |n!|/|t - c|_n^n \). Since this holds for all \( c \in K = \overline{K}^{\text{alg}} \) one finds \( \|(d/dT)^n|_{\mathcal{H}(\xi)} \geq |n!|/r(\xi)^n \), because \( r(\xi) = \inf_{c \in \overline{K}^{\text{alg}}} |t - c|_n \). If \( K \neq \overline{K}^{\text{alg}} \), but \( \xi = \xi_{c,\rho} \), with \( c \in K \), then \( r(\xi_{c,\rho}) = \rho \). The above computation for the individual polynomial \( (T - c)^n \) gives \( \|(d/dT)^n|_{\mathcal{H}(\xi)} \geq |n!|/|t - c|_n^n = |n!|/\rho^n = |n!|/r(\xi)^n \) (cf. Lemma 1.3).

Remark 3.5. The assumptions of Lemma 3.4 are possibly superfluous. The assumption \( K = \overline{K}^{\text{alg}} \) will not be relevant since \( R^M \) is insensitive to base change of the ground field \( K \) (cf. Remark 4.6).

Definition 3.6. Let \( \xi \in X^{\text{gen}} \), and let \( M = M_\xi \) be a differential module over \( (\mathcal{H}(\xi), d/dT) \). We set

\[
R^{M,sp}(\xi) := \omega \cdot \|\nabla\|_{Sp,\xi}^{-1}, \tag{3.15}
\]

where \( \|\nabla\|_{Sp,\xi} \) is the spectral norm of \( \nabla : M_\xi \to M_\xi \) with respect to the norm \( |.|_F = \xi \) on \( F := \mathcal{H}(\xi) \). One extends this definition to the whole \( X \) by setting \( R^{M,sp}(\xi) = 0 \) for all \( \xi \in X - X^{\text{gen}} \).

Remark 3.7. \( R^{M,sp}(\xi) \) coincides with the spectral radius \( R(V) \) studied in [Ked10b, Def.9.4.4] and [CD94, Section 2.3]. It is the radius studied in the whole literature (up to [BV07] and [Bal10]).

A direct computation gives (cf. [Ked10b, Lemma 6.2.5], [CD94, Prop.1.3])

\[
\|\nabla\|_{Sp,\xi} = \max\left(\limsup_s |G_s(\xi)|^{1/s}, \|d/dT\|_{Sp,\xi}\right), \tag{3.16}
\]

where \( G_s \) is the matrix of \( \nabla^s \) (cf. (3.2)).

Remark 3.8. Let \( \xi \in X^{\text{gen}} \). Assume, as in Lemma 3.4, that \( K = \overline{K}^{\text{alg}} \) or, if \( K \neq \overline{K}^{\text{alg}} \), that \( \xi = \xi_{c,\rho} \) with \( c \in K \), \( \rho > 0 \). Then \( \|d/dT\|_{Sp,\xi} = \omega / r(\xi) \) and \( R^{M,sp}(\xi) = \min(R^V(\xi), r(\xi)) \).

Remark 3.9. If \( K \neq \overline{K}^{\text{alg}} \) then \( R^{M,sp} = \min(R^V, r(\xi)) \). Both functions \( \xi \mapsto r(\xi) \) and \( \xi \mapsto R^V(\xi) \) are invariant by the action of \( G := \text{Gal}(K^{alg}/K) \) so \( R^{M,sp} \) defines a function on \( X = X^{gen}/G \). This was the function considered in sections 3.1 and 3.2 (cf. (3.4)).

4. Newton polygons.

Let \( r \geq 1 \) be a natural number. Let \( v : \{0, 1, \ldots, r\} \to \mathbb{R} \cup \{+\infty\} \), be any sequence \( i \mapsto v_i \) satisfying \( v_0 = 0 \). The Newton polygon \( NP(v) \subset \mathbb{R}^2 \) is the convex hull in \( \mathbb{R}^2 \) of the family of half-lines \( L_v := \{(x = i, y \geq v_i)\}_{i=0,\ldots,r} \) i.e. the intersection of all the upper half planes \( H_{a,b} := \{(x, y) \in \mathbb{R}^2 \text{ such that } y \geq ax + b\}, a, b \in \mathbb{R} \), containing \( L_v \). We call the \( i \)-th partial height of the polygon the value \( h_i := \min\{y \in \mathbb{R} \cup \{+\infty\} \text{ such that } (i, y) \in NP(v)\} \). If \( h : \{0, \ldots, r\} \to \mathbb{R} \cup \{+\infty\} \) denotes the function \( i \mapsto h_i \), then \( NP(v) = NP(h) \), and \( h \) is the smallest function with this property. We call slope sequence any increasing sequence \( s : \{1, \ldots, r\} \to \mathbb{R} \cup \{+\infty\} : s_1 \leq \ldots \leq s_r \). The slope sequence of \( NP(h) \) is defined by \( s_i := h_i - h_{i-1}, i = 1, \ldots, r \), where \( s_i = +\infty \) if \( h_i \) or \( h_{i-1} \) are equal to +\infty. The slope sequence of \( NP(h) \) determines the function \( h_i = s_1 + \cdots + s_i \), and hence \( NP(h) \).

Let : \( s_1 \leq \ldots \leq s_r \) be a slope sequence, the truncated slope sequence by the constant \( C \in \mathbb{R} \) is
by definition the sequence \( s_C := (s'_i)_{i=1,...,r} \), where \( s'_i := \min(s_i, C) \), for all \( i \). The corresponding polygons have the following shape:

\[
\begin{array}{c}
\includegraphics[width=0.8\textwidth]{polygons.png}
\end{array}
\]

As a matter of facts in the sequel we will deal only with truncated slope sequences by a convenient constant \( C \). Hence the \( i \)th slope \( s'_i \), as well as the \( i \)th partial height \( h'_i := s'_1 + \cdots + s'_i \) will never be equal to \(+\infty\). The explicit expression of \( h_i \) in terms of the \( v_i \) is \( h_i = \sup_{s \in \mathbb{R}} \left( s-i+\min_{j=0,...,r}(v_j-s \cdot j) \right) \).

In fact if \( y = sx + q_s \) is the line of slope \( s \) which is tangent to \( NP(v) \), then \( q_s = \min_{j=0,...,r}(v_j-s \cdot j) \), and \( h_i \) is the supremum of the values of those lines at \( x = i \).

**Example 4.1.** Let \( (F, | \cdot |_F) \) be a valued field and let \( P(T) := \sum_{i=0}^r a_{r-i} T^i \in F[T] \) be such that \( a_0 = 1 \). Let \( v_{P,i} := -\ln(|a_i|) \in \mathbb{R} \cup \{+\infty\} \). The Newton polygon of \( P(T) \) is by definition \( NP(v_P) \).

### 4.1 Spectral Newton polygon of a differential operator.

Let \( \xi \in X \). Let \( \mathcal{L} := \sum_{i=0}^r g_{r-i}(T) \cdot (d/dT)^i \), with \( g_0 = 1 \) and \( g_i \in \mathcal{O}(X) \). Let \( v_{\mathcal{L}} : i \mapsto -\ln(\omega^{-i} \cdot |g_i(\xi)|) \). The Newton polygon \( NP(L, \xi) := NP(v_{\mathcal{L}}) \) is called the *spectral Newton polygon* of \( \mathcal{L} \). Let \( s^{L,sp}(\xi) : s_1^{L,sp}(\xi) \leq \cdots \leq s_r^{L,sp}(\xi) \), be the slope sequence of \( NP(L, \xi) \). Then

\[
s_1^{L,sp}(\xi) = \left( \omega \cdot \min_{i=1,...,r} |g_i(\xi)|^{-1} \right)^{1/\lambda}.
\]

If \( \xi \in X_{\text{gen}} \) none of the values \( |g_i(\xi)| \) is equal to zero, so \( s_1^{L,sp}(\xi) \) and \( h_1^{L,sp}(\xi) := s_1^{L,sp}(\xi) + \cdots + s_r^{L,sp}(\xi) \) are all finite. To be consistent with the rest of the paper we set \( R_i^{L,sp}(\xi) := \exp(s_i^{L,sp}(\xi)) \) and \( H_i^{L,sp}(\xi) := \exp(h_i^{L,sp}(\xi)) \). Analogous definitions are given if \( g_i \in \mathcal{H}(\xi), \xi \in X_{\text{gen}} \).

**Proposition 4.2.** Assume that \( g_1, \ldots, g_r \in \mathcal{O}(X) \) have no zeros on \( X \), i.e. \( g_i, g_{i-1} \in \mathcal{O}(X) \) for all \( i = 1, \ldots, r \). Then:

i) For all \( i = 0, \ldots, r \) the function \( \xi \mapsto H_i^{L,sp}(\xi) \in \mathbb{R} \) verifies the six properties (C1)–(C6) with respect to \( \Gamma := \Gamma_X \) and \( \mathcal{C}(H_i^{L,sp}) := \mathcal{C}_X \), and is hence finite by Thm. 2.14. Moreover, if \( K = \overline{K_{\text{alg}}} \), then \( H_i^{L,sp} \) is super-harmonic in the sense of Definition 2.10;

ii) For all \( i = 0, \ldots, r \) one has \( \Gamma(h_i^{L,sp}) = \Gamma(h_i^{L,sp}) = \Gamma_X \);

iii) Assume that \( \xi \in X_{\text{int}} \), and that \( (i, h_i^{L,sp}(\xi)) \) is a vertex of \( NP(L, \xi) \) (i.e. \( i = r \) or \( s_i^{L,sp}(\xi) < s_{i+1}^{L,sp}(\xi) \)). Then:

(iii-a) There exists an open segment containing \( \xi \) of each branch in \( B(\xi) \) on which \( H_i^{L,sp} = \omega^i |g_i(\xi)|^{-1} \).

(iii-b) In this case for all \( \delta \in \Delta(\xi) \) the slopes \( \partial_- H_i^{L,sp}(\xi) \) and \( \partial_+ H_i^{L,sp}(\xi) \) are equal to those of \( \xi \mapsto |g_i(\xi)|^{-1} \) and lies hence in \( \mathbb{Z} \) by the property (Z) of section 1.3.

(iii-c) If moreover \( \xi \) is not belonging to the Shilov boundary of \( X \), and if \( K = \overline{K_{\text{alg}}} \), then \( H_i^{L,sp} \) is harmonic at \( \xi \) (i.e. (2.8) is an equality).

iv) From (iii-b) one deduces, by interpolation, that for all \( i = 1, \ldots, r \) the slopes of \( h_i^{L,sp} \) and \( s_i^{L,sp} = \sum_{\lambda=0}^i h_{i+1}^{L,sp} \) belong to \( \mathbb{Z} \cup \frac{1}{2} \mathbb{Z} \cup \cdots \cup \frac{1}{r} \mathbb{Z} \).

**Proof.** Since every \( g_i \) has no zeros on \( X \) the functions \( \xi \mapsto |g_i(\xi)| \) are constant on every maximal disk \( D^-(t, \rho_i, X) \). Hence ii) holds. The rest is straightforward (see for example [Ked10b, Thm.11.2.1]).
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Proposition 4.3 (Small radius). Let $\xi \in X^{\text{gen}}$, let $\mathcal{L} := \sum_{i=0}^{r} g_{r-i} \cdot (d/dT)^i$, $g_0 = 1$, $g_i \in \mathcal{H}(\xi)$, and let $(M, \nabla)$ be the corresponding differential module over $\mathcal{H}(\xi)$. Following [BGR84, 1.5.4] let

$$\|\mathcal{L}\|_{Sp, \xi} := \max_{1 \leq i \leq r} |g_i(\xi)|^\frac{1}{i}.\quad (4.3)$$

Then $\|\mathcal{L}\|_{Sp, \xi} > |d/dT|_{\mathcal{H}(\xi)}$ if and only if $\|\nabla\|_{Sp, \xi} > |d/dT|_{\mathcal{H}(\xi)}$. In this case one has

$$\|\nabla\|_{Sp, \xi} = \|\mathcal{L}\|_{Sp, \xi}.\quad (4.4)$$

Proof. Reproduce closely the proof of [CM02, Thm.6.2].

Under the assumptions of Lemma 3.4 one has $|d/dT|_{\mathcal{H}(\xi)} = 1/r(\xi)$. So Proposition 4.3 can be rephrased as follows: $\|\mathcal{L}\|_{Sp, \xi} > r(\xi)^{-1}$ if and only if $\mathcal{R}^{M, sp}(\xi) < \omega \cdot r(\xi)$, and in this case

$$\mathcal{R}^{M, sp}(\xi) = \omega \cdot \|\mathcal{L}\|_{Sp, \xi} = \omega \cdot \min_{1 \leq i \leq r} |g_i(\xi)|^{-1/i} = \exp(s^L_{1, sp}(\xi)).\quad (4.5)$$

Proposition 4.4. Let $\alpha > 0$, be a constant and let $C(\xi) := \ln(\alpha \cdot r(\xi)) \in \{-\infty\} \cup \mathbb{R}$. Assume that $g_1, \ldots, g_r \in \mathcal{O}(X)$ have no zeros on $X$. Let $s'_{\xi}(\xi) : s'_{1}(\xi) \leq \cdots \leq s'_{r}(\xi)$ be the truncated sequence $s^L_{\xi}(\xi)_{|C(\xi)}$. Namely $s'_{i}(\xi) := \min(s^L_{i, sp}(\xi), C(\xi))$ for all $i = 1, \ldots, r$. In order to avoid to work with $-\infty$ let $R'_i := \exp(s'_{i})$, and let $H'_i(\xi) := \exp(h'_i(\xi))$ where as usual $h'_i(\xi) := s'_{i}(\xi) + \cdots + s'_{1}(\xi)$, and if $t \in X(K^{\text{alg}})$ we extend the above definition by $R'_i(\xi) := 0$ and $H'_i(\xi) = 0$. Then:

i) For all $i = 0, \ldots, r$ the function $\xi \mapsto H'_i(\xi) \in \mathbb{R}$ verifies $(C2),(C4)$, and $(C3)$ with $\Gamma = \Gamma_X$;

ii) $\Gamma(R'_i) = \Gamma(H'_i) = X$, because $R'_i(\xi) = 0$ for all $t \in X(K^{\text{alg}})$;

iii) Assume $\xi \in X_{\text{int}}$. If $(i, h'_i(\xi))$ is a vertex of the truncated Newton polygon at $\xi$ (i.e. $i = r$ or $s'_{i}(\xi) < s'_{i+1}(\xi)$) then for all $\delta \in \Delta(\xi)$ the slopes $\partial_{-}H'_i(\xi)$ and $\partial_{+}H'_i(\xi)$ lies in $\mathbb{Z}$. This implies, by interpolation, that for all $i = 1, \ldots, r$ the log-slopes of $H'_i$ always belong to $\mathbb{Z} \cup \frac{1}{2} \mathbb{Z} \cup \cdots \cup \frac{r}{2} \mathbb{Z}$.

iv) Assume that $\xi \in X_{\text{int}}$ does not belong to the Shilov boundary of $X$. Let $i_0 \in \{1, \ldots, r\}$ be the largest integer such that $s'_{i_0}(\xi) < C(\xi)$. If $K = K^{\text{alg}}$, then for all $i = 1, \ldots, i_0$, the function $H'_i$ is super-harmonic at $\xi$, and if moreover $(i, h'_i(\xi))$ is a vertex of the truncated Newton polygon as in iii), then $H'_i$ is harmonic at $\xi$.

Proof. Straightforward [Ked10b, Remark 11.2.4].

4.2 Spectral Newton polygon of a differential module.

By Lemma 3.4 if $\xi \in X^{\text{gen}}$, then $d/dT$ extends by continuity to $\mathcal{H}(\xi)$. Let $M = M_\xi$ be a differential module of rank $r$ over $(\mathcal{H}(\xi), d/dT)$. Let $0 = M_{\xi,0} \subset M_{\xi,1} \subset \cdots \subset M_{\xi,n} = M$ be a Jordan-Hölder sequence of $M$. This means that $N_{\xi,k} := M_{\xi,k}/M_{\xi,k-1}$ has no non trivial strict differential submodules for all $k$. Let $r_k$ be the rank of $N_{\xi,k}$, and let $R_k := \mathcal{R}^{N_{\xi,k}, sp}(\xi)$. Perform a permutation of the indexes in order to have $R_1 \leq \cdots \leq R_n$. Let $s^{M, sp}(\xi) : s^M_{1, sp}(\xi) \leq \cdots \leq s^M_{r, sp}(\xi)$ be the slope sequence obtained from $\ln(R_1) \leq \cdots \leq \ln(R_n)$ by counting $r_k$-times the slope $\ln(R_k)$:

$$s^{M, sp}(\xi) : \begin{array}{c}
\ln(R_1) = \cdots = \ln(R_1) \\
r_1\text{-times}
\ln(R_2) = \cdots = \ln(R_2) \\
r_2\text{-times}
\vdots
\ln(R_n) = \cdots = \ln(R_n) \\
r_n\text{-times}
\end{array}.\quad (4.6)$$

Set $h^M_{0, sp}(\xi) = 0$ and $h^M_{i, sp}(\xi) := s^M_{i, sp}(\xi) + \cdots + s^M_{1, sp}(\xi)$, for all $i = 1, \ldots, r$. The spectral Newton polygon $NP^{sp}(M, \xi)$ is by definition $NP(h^M_{sp}(\xi))$. We also set $\mathcal{R}^{M, sp}(\xi) := \exp(s^M_{1, sp}(\xi))$, and $H^M_{i, sp}(\xi) := \exp(h^M_{i, sp}(\xi))$. One has $\mathcal{R}^{M, sp}(\xi) = \mathcal{R}^{M, sp}(\xi)$. As for $\mathcal{R}^{M, sp}(\xi)$ (cf. Def. 3.6), we extend the definition of $\mathcal{R}^{M, sp}(\xi)$ to the whole $X$ by setting $\mathcal{R}^{M, sp}(\xi) = 0$, for all $\xi \in X - X^{\text{gen}}$. 

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4.3 Convergence Newton polygon of a differential module and main theorem

Let $M$ be a differential module over $\mathcal{O}(X)$ of rank $r$. Let $\xi \in X$ and let $t \in X(\Omega)$ be a Dwork generic point for $\xi$. For all $0 < R \leq \rho_{\xi,X}$ denote by $\text{Fil}^{\geq R} \text{Sol}(M,t,\Omega) \subset M \otimes \mathcal{A}_{\Omega}(t,R)$ the $\Omega$-vector space of solutions of $M$ with values in $\mathcal{A}_{\Omega}(t,R)$. If $R \leq R^M(\xi)$, then $\text{Sol}(M,t,\Omega) := \text{Fil}^{\geq R} \text{Sol}(M,t,\Omega)$ is independent on $R \leq R^M(\xi)$, and it has dimension $r$ over $\Omega$. The family $\{\text{Fil}^{\geq R} \text{Sol}(M,t,\Omega)\}_{0 < R \leq \rho_{\xi,X}}$ is a descending filtration of $\text{Sol}(M,t,\Omega)$ by $\Omega$-vector spaces.

The filtration is independent on the choice of $\Omega$ and $t$ in the following sense. A descent argument shows that if $t \in \Omega \subset \Omega'$, then $\text{Fil}^{\geq R} \text{Sol}(M,t,\Omega') = \text{Fil}^{\geq R} \text{Sol}(M,t,\Omega) \otimes \Omega'$ (cf. [Ked10b, Prop. 6.9.1]). If now $t' \in \Omega$ is another Dwork generic point for $\xi$, then, up to enlarge $\Omega$, one has $t' = \sigma(t)$ for some $\sigma \in \text{Gal}^{\text{cont}}(\Omega/K)$ (cf. Lemma 1.2). Since $\sigma$ is isometric, then $\sigma(D^-(t,R)) = D^-(t',R)$ for all $0 < R \leq \rho_{\xi,X}$. This provides an isomorphism of rings $\sum a_i(T-t)^i \mapsto \sum \sigma(a_i)(T-t')^i : \mathcal{A}_{\Omega}(t,R) \iso \mathcal{A}_{\Omega}(t',R)$ commuting with $d/dT$. Hence $\text{Fil}^{\geq R} \text{Sol}(M,t,\Omega)$ is identified by $\sigma$ to $\text{Fil}^{\geq R} \text{Sol}(M,t',\Omega)$.

For all $i = 1, \ldots, r$ define $R^M_i(\xi)$ as the largest value of $R \leq \rho_{\xi,X}$ such that $\dim_{\Omega} \text{Fil}^{\geq R} \text{Sol}(M,t,\Omega) \geq r - i + 1$. In other words $R^M_i(\xi)$ is the radius of the largest open disk centered at $t$ on which $M$ has at least $r - i + 1$ linearly independent solutions. For all $i = 1, \ldots, r$ set $s^M_i(\xi) := \ln(R^M_i(\xi))$ and $h^M_i(\xi) := s^M_1(\xi) + \cdots + s^M_i(\xi)$, $h^M_0(\xi) = 0$. By the above arguments $s^M_i(\xi)$ and $h^M_i(\xi)$ only depend on $M$ and $\xi$, and are independent on the choice of $t$ and $\Omega$. The convergence Newton polygon $NP^{\text{conv}}(M,\xi)$ is by definition $NP(h^M_\bullet(\xi))$. We also set $H^M_i(\xi) := \exp(h^M_i(\xi))$. One has (cf. (3.3))

$$R^M_i(\xi) = R^M(\xi).$$

For all $t,t' \in X(\Omega)$ satisfying $|t-t'| < R \leq \rho_{\xi,X}$ one has a canonical identification

$$\text{Fil}^{\geq R} \text{Sol}(M,t,\Omega) \iso \text{Fil}^{\geq R} \text{Sol}(M,t',\Omega)$$

induced by the canonical isomorphism $f(T) \mapsto \sum_{n \geq 0} f^{(n)}(t')(T-t')^n/n! : \mathcal{A}_{\Omega}(t,R) \iso \mathcal{A}_{\Omega}(t',R)$ which commutes with $d/dT$. So a solution in $\text{Fil}^{\geq R} \text{Sol}(M,t,\Omega)$ converges at all point $t' \in D^-(t,R)$ and belongs to $\text{Fil}^{\geq R} \text{Sol}(M,t',\Omega)$. This proves that for all $i = 1, \ldots, r$ one has

$$\max(R^M_i(\xi),r(\xi)) \leq \rho_{R^M_i(\xi)}.$$
iii) For all \( i = 1, \ldots, r \) the function \( H_i^M : X \to \mathbb{R}_{>0} \) satisfies (C1), (C2), (C4), (C5). Let \( \Gamma_0 := \Gamma_X \), and \( \Gamma_i := \bigcup_{j=1}^i \Gamma(R_j^M) \). Then \( H_i^M \) satisfies (C3) with respect to \( \Gamma := \Gamma_{i-1} \). More precisely \( H_i^M/\mathcal{R}_i^M \) is constant on each (non generic) disk tangent to \( \Gamma_{i-1} \), and \( \mathcal{R}_i^M \) enjoys on that disk all the properties of a genuine radius of convergence (cf. Prop. 7.5). In particular if \( \xi \notin \Gamma_{i-1} \), then \( \xi \in \Gamma(\mathcal{R}_i^M) \) if and only if \( \partial_- \mathcal{R}_i^M(\xi) < 0 \).

iv) [Concavity] Let \( t \in X(\Omega) \), \( \Omega \in E(K) \), and let \( I \subseteq [0, R_0] \) be a subinterval with interior \( \overset{0}{I} \). Then

a) If \( \rho_i,X \leq \inf(I) \), and if the open annulus \( \{|T-t| \in \overset{0}{I}\} \) is contained in \( X \overset{0}{\circ}\Omega \), then \( L H_i^M \) is concave on \( \ln(I) \), and equal to \( L H_i^{M, \bigcirc, \Omega, \text{sp}} \) on it.

b) Assume \( \sup(I) \leq \rho_i,X \). Then :

i. \( H_i^M \) is log-concave on each open subinterval \( J \subseteq \overset{0}{I} \) satisfying \( J \cap Q_t = \emptyset \), where \( Q_t := \{ \ln(\mathcal{R}_2^M(t)), \ln(\mathcal{R}_3^M(t)), \ldots, \ln(\mathcal{R}_i^M(t)) \} \). More generally if \( \rho \in Q_t \) and if for all \( j \) such that \( \rho = \mathcal{R}_j^M(t) \) the function \( \rho \to L \mathcal{R}_j^M(\rho) \) is concave at \( \rho \) (i.e. \( \partial_- \mathcal{R}_j^M(\xi_{\rho,i}) < 0 \)), then one can replace the set \( Q_t \) by \( Q_t - \{ \rho \} \).

ii. Let \( \rho \notin [0, \rho_i,X] \). If for all \( k = 1, \ldots, i \) one has \( \mathcal{R}_k^M(\xi_{\rho,i}) \neq \rho \), then the left and right log-slopes of \( H_i^M \) at \( \rho \) are less than or equal to 0 (i.e. \( H_i^M \) is log-decreasing at \( \rho \)).

v) [Weak super-harmonicity] Assume \( K = \widetilde{K}^{\text{alg}} \). Let \( \mathcal{C}_1 \) be the Shilov boundary of \( X \), and, inductively, define \( \mathcal{C}_i := \mathcal{C}_i \cup (\bigcup_{j=0}^{i-1} \mathcal{C}_j) \), where \( \mathcal{N} \) is the finite set of points \( \xi \in X \) satisfying (a) \( \xi \in \Gamma(\mathcal{R}_i^M) \cap \Gamma(H_i^M) \cap \Gamma_{i-1} \); (b) \( \mathcal{R}_i^M(\xi) = r(\xi) \); (c) \( \xi \) lies in the boundary of \( \Gamma(\mathcal{R}_i^M) \). Then for all \( i = 1, \ldots, r \) the function \( H_i^M \) is super-harmonic (at least) at all \( \xi \in X_{\text{int}} - \mathcal{C}_i \).

vi) [Weak harmonicity of the vertexes] Assume \( K = \widetilde{K}^{\text{alg}} \) and let \( \xi \in X_{\text{int}} \) be not in the Shilov boundary of \( X \). If \( \xi \notin \Gamma(H_i^M) \), then \( H_i^M \) is flat and hence harmonic at \( \xi \). Assume then \( \xi \in \Gamma(H_i^M) \). In this case if none of the values \( \{ \mathcal{R}_j^M(\xi) \}_{j=1, \ldots, i} \) is equal to \( r(\xi) \), and if \( (i, \mathcal{H}_i^M(\xi)) \) is a vertex of \( NP^{\text{conv}}(M, \xi) \) (i.e. \( i = r \), or \( s_i^1(\xi) < s_i^{1+1}(\xi) \)), then \( H_i^M \) is harmonic at \( \xi \).

**Corollary 4.8.** Let \( t \in K \), and \( A(t, I) := \{|T-t| \in I\} \) be a possibly not closed annulus or disk (if \( 0 \in I \) one has a disk). Let \( \mathcal{C} \) be one of the rings \( \mathcal{H}_K(t, I), \mathcal{B}_K(t, I), \mathcal{A}_K(t, I) \). (cf. Section 1.0.3). Let \( M \) be a differential module over \( \mathcal{C} \) of rank \( r \). Then Thm. 4.7 holds for \( M \) in the following cases:

i) if \( \mathcal{C} = \mathcal{H}_K(t, I); \)

ii) if \( \mathcal{C} \) is discretely valued, and \( \mathcal{C} = \mathcal{B}_K(t, I); \)

iii) if \( \mathcal{C} = \mathcal{B}_K(t, I) \) or \( \mathcal{C} = \mathcal{A}_K(t, I) \), and all \( \mathcal{R}_1^M, \ldots, \mathcal{R}_r^M \) (or equivalently all \( H_i^M \)) have a finite number of breaks along \( \{ \xi_{\rho,i} \}_{\rho \in I} \).

Moreover if \( \mathcal{C} = \mathcal{B}_K(t, I) \) or \( \mathcal{C} = \mathcal{A}_K(t, I) \), and if there exists \( i \leq r \) such that all \( \mathcal{R}_1^M, \ldots, \mathcal{R}_i^M \) have a finite number of breaks along \( \{ \xi_{\rho,i} \}_{\rho \in I} \), then \( \mathcal{R}_1^M, \ldots, \mathcal{R}_i^M \) (and hence \( H_1^M, \ldots, H_i^M \)) are finite.

The proof of Corollary 4.8 is placed at the end of section 7.

**Remark 4.9.** By v) of Thm. 4.7 one has the following statements.

i) If \( K = \widetilde{K}^{\text{alg}} \), then the function \( \mathcal{R}_1^M \) is super-harmonic on \( X \) (cf. Def. 2.10).

ii) If none of the values \( \{ \mathcal{R}_j^M(\xi) \}_{j=1, \ldots, i} \) is equal to \( r(\xi) \), then \( \partial_- H_i^M(\xi) = 0 \) for almost but a finite number of directions \( \delta \in \Delta(\xi) \), and \( H_i^M \) is super-harmonic at \( \xi \).

---

11See Definition 2.20 for the notion of finiteness over a possibly not open ring or annulus.

12If \( A(t, I) \) is a disk \( D \overset{0}{(t, \rho)} \), and if \( \lim_{\rho \to \rho_0^+} \mathcal{R}_j^M(\rho) = \rho \), then the same holds for all \( j \geq i \), and in this case \( \mathcal{R}_j^M \) has a finite number of breaks along \( \{ \xi_{\rho,i} \}_{\rho \in I} \) (cf. proof of Cor. 4.8).
iii) Let \( D^-(t, \rho) \) be a non generic disc. If \( D^-(t, \rho) \cap \Gamma_{i-1} = \emptyset \), then \( R_i^M \) and \( H_i^M \) differs by a constant on \( D^-(t, \rho) \), and \( \Gamma(R_i^M) \cap D^-(t, \rho) = \Gamma(H_i^M) \cap D^-(t, \rho) \). \( R_i^M \) and \( H_i^M \) enjoy the six properties on (C1)--(C5) with respect to \( \Gamma := \Gamma_{i-1} \) on that disk, in particular they are concave and decreasing on each branch \( \Lambda(\xi') \) for all \( \xi' \in D^-(t, \rho) \). If moreover \( K = \widehat{K}^{alg} \), then they are both super-harmonic at all \( \xi \in D^-(t, \rho) \).

iv) If \( M \) has rank \( r = 1 \), then the boundary points of \( \Gamma(R_i^M) \) not in the Shilov boundary of \( X \) are those \( \xi \in X - \Gamma_X \) satisfying \( R_i^M(\xi) = r(\xi) \) and \( \partial_i R_i^M(\xi) < 0 \) (cf. Cor. 7.9).

v) Assume \( K \) discretely valued. Since \( |a| = 1 \) for all \( a \neq 0 \) it follows from section 1.0.3 that \( K((T)) = A_K(0, I) = B_K(0, I) \), for all (open or closed) interval \( I \subseteq [0, 1] \). Moreover \( K[T, T^{-1}] = A_K(0, I) = B_K(0, I) \) for all \( I \) such that \( 1 \in I \) and \( 0 \notin I \). One has analogous interpretations for \( K[[T]] = A_K(0, R) = B_K(0, [0, R]) \), if \( R \leq 1 \), and \( K[T] = A_K(0, R) = B_K(0, [0, R]) \), if \( R > 1 \).

Corollary 4.8 holds if \( M \) is a differential module over \( K((T)) \), or \( K[T, T^{-1}] \). In this case \( \omega = 1 \), and the slopes \( \{ \partial_i s_{M,\rho}^i \} \) are directly related to the Formal Newton polygon of \( M \) [Ram78], [DMR07, p.97–107], [Rob80].

The remaining of the paper is devote to prove Theorem 4.7. For expository reasons below we give first a part of the proof (cf. section 4.3.1).

**Remark 4.10.** (C1) follows immediately by (4.9). One has \( \mathcal{R}_1^M \otimes \mathcal{K}^{alg,sp} = \mathcal{R}_1^M \) along the branches of \( \mathcal{G}(\mathcal{R}_i^M) \). In fact if \( \xi \in \mathcal{G}(\mathcal{R}_i^M) \) then \( r(\xi) = \rho R_M(\xi) \) and one has (3.9).  

4.3.1 Comparison between \( \mathcal{R}_i^M \) and \( \mathcal{R}_i^{M,sp} \). We now prove point i) of Theorem 4.7. Firstly we prove, for all \( \rho \in [0, R_0] \), the equality (4.10) : 

\[
\mathcal{R}_{i,t}^{M,\Omega,sp}(\rho) = \min(\mathcal{R}_{i,t}(\rho), \rho). 
\]

For \( \rho = 0 \) both functions are 0 and there is nothing to prove. Assume \( \rho > 0 \). Since \( \mathcal{R}_i^M \) is insensive by scalar extension of \( K \) we can assume \( \Omega = K \), \( t \in X(K) \). Let \( t_\rho \in X(\Omega_\rho) \) be a Dwork generic point for \( \xi_\rho \), and let \( D^-(t_\rho, \rho) \subset X \otimes \Omega_\rho \) be the generic disk with \( \rho = r(\xi_\rho) \) of Dwork that the restriction map \( \mathcal{G}(X) \to \mathcal{A}_{\Omega_\rho}(t_\rho, \rho) \) sending \( f \) into \( f_{|D^-(t_\rho, \rho)}(T) = \sum_{n \geq 0} f^{(n)}(t_\rho)(T - t_\rho)^n \) factorizes through an injective map \( \mathcal{G}(X) \to \mathcal{H}(\xi_\rho, \rho) \subset \mathcal{A}_{\Omega_\rho}(t_\rho, \rho) \). The truncated sequence \( \{ s_i^M(\xi_\rho, \rho) \} \) coincides with the slope sequence \( \{ s_i^{M,\Omega,sp}(\xi_\rho, \rho) \} \) of the restricted module \( M^{\otimes \mathcal{G}(X)} \otimes \mathcal{A}_{\Omega_\rho}(t_\rho, \rho) \) (cf. Def. 4.5). Both \( \{ s_i^{M,\Omega,sp}(\xi_\rho, \rho) \} \) and \( \{ s_i^{M,\mathcal{H}(\xi_\rho, \rho)} \} \) only depends on \( M_{\xi_\rho} := M^{\otimes \mathcal{G}(X)} \mathcal{H}(\xi_\rho, \rho) \), so we can assume \( M = M_{\xi_\rho} \). By Thm. 5.1 below we can assume \( M_{\xi_\rho} = M_{\xi_\rho}^R \). By definition of \( M_{\xi_\rho}^R \) one has \( \mathcal{R}_i^{M_{\xi_\rho}^R}(\xi_\rho, \rho) = R \) for all \( i = 1, \ldots, r \), and by (3.12) and (3.16) one has \( R = \mathcal{R}_i^{M_{\xi_\rho}^R}(\xi_\rho, \rho) \). By [Rob75] (cf. also [CM02], [Chr12]) \( M_{\xi_\rho}^R \) also admits a decomposition by the radii of convergence of its solutions around \( t_\rho \). So by (3.12) for all \( i = 2, \ldots, r \) one has \( \mathcal{R}_i^{M_{\xi_\rho}^R}(\xi_\rho, \rho) = R_{\xi_\rho}^{M_{\xi_\rho}^R}(\xi_\rho, \rho) \). Alternatively for all \( \rho \leq \rho \) let \( N := M_{\xi_\rho}^R \otimes \mathcal{H}(\xi_\rho, \rho) \mathcal{A}_{\Omega_\rho}(t_\rho, \rho) \). By Remark 3.8 for all \( i \) one has \( \mathcal{R}_{i,t}(\rho') = \min(\mathcal{R}(\rho), \rho') \) for all \( \rho' \leq \rho \). By contrapositive if there exists \( \rho' \) such that \( R < \rho < \mathcal{R}_{i,t}(\rho') \), then \( N \otimes \mathcal{A}_{\Omega_\rho}(t_\rho, \rho) \mathcal{H}(\xi_\rho, \rho) \) would have a trivial submodule in its Jordan-Hölder sequence, and hence \( \mathcal{R}_{i,t}(\rho) \) \( \geq \rho > R \) for some \( i \) which is absurd. This is also an old idea of Dwork [Dwo73] and Robba [Rob75]: the decomposition by the spectral radius is the decomposition by the radius at \( t_\rho \), see also [CD94], [CM02], [Chr12]. The proof presented here comes from [Ked10b, Thm. 11.9.2].
Lemma 4.11. Equality $\mathcal{R}_{i,t}^M(\rho) = \begin{cases} \mathcal{R}_{i,t}^M(t) & \text{if } \rho \in [0, \mathcal{R}_{i,t}^M(t)] \\ \mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho) & \text{if } \rho \in [\mathcal{R}_{i,t}^M(t), R_0] \end{cases}$ follows from (4.11). Moreover the function $\rho \mapsto \mathcal{R}_{i,t}^M(\rho)$ is continuous on $[0, R_0]$.

Proof. By (4.9), $\mathcal{R}_{i,t}^M$ is constant on the disk $D^-(t, \mathcal{R}_{i,t}^M(t))$ with value $R := \mathcal{R}_{i,t}^M(t)$. So $\mathcal{R}_{i,t}^M(\rho) = R$, for all $\rho \in [0, R]$. Again by (4.9), for all $\rho \in [\mathcal{R}_{i,t}^M(t), \rho_{t,X}]$ one has $\mathcal{R}_{i,t}^M(\rho) \leq \rho$. So, by (4.11), one has $\mathcal{R}_{i,t}^M(\rho) = \mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho)$ for all $\rho \in [R, \rho_{t,X}]$. If $\rho \in [\rho_{t,X}, R_0]$, then $\rho = \rho_{t,X}^\ast$ and $\mathcal{R}_{i,t}^M(\rho) \leq \rho$ by definition, and is hence equal to $\mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho)$ by (4.11). Now by Thm. 5.6 below, $\rho \mapsto \mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho)$ is continuous on $[0, R_0]$. Hence $\mathcal{R}_{i,t}^{M_{\Xi,sp}}(R) = R$ and $\rho \mapsto \mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho)$ is continuous too. In fact otherwise the condition $\mathcal{R}_{i,t}^{M_{\Xi,sp}}(R) < R$ implies $\mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho) < \rho$ in a neighborhood of $\rho = R$. Hence $\mathcal{R}_{i,t}^{M_{\Xi,sp}}(\rho) < \rho < R$ for $\rho < R$, contradicting (4.11). \(\square\)

5. Auxiliary results

5.1 Decomposition theorems

Theorem 5.1 (Decomposition over a point by the slopes of $NP_{\Xi}(M, \xi)$). Let $\xi \in X_{\text{int}}$, and let $L = \sum_{i=0}^r g_i \cdot (d/dT)^i$, $g_0 = 1$, $g_i \notin \mathcal{O}(\xi)$ be a differential operator defining the module $M_{\xi}$. Assume as in Lemma 3.4 that $K = K_{\text{alg}}$ or, if $K \neq K_{\text{alg}}$, that $\xi = \xi_{c,\rho}$, with $c \in K$ and $\rho > 0$. Then

i) $M_{\xi}$ decomposes into a direct sum $M_{\xi} = \bigoplus_{0<R<\rho(\xi)} M_{\xi}^R$, $\xi$ where $\mathcal{R}_{i,t}^{M_{\Xi}}(M_{\xi}^R) = R$ and the spectral Newton polygon $NP_{\Xi}(M_{\xi}^R, \xi)$ has a constant slope sequence of value $s_{i,t}^{M_{\Xi}}(\xi) = \ln(R)$, $\forall i$.

ii) Let $C_{\omega} := \ln(\omega \cdot r(\xi)) = \ln(\omega/ddT|_{\mathcal{O}(\xi)})$. Let $s_{i,t}^{M_{\Xi}}(\xi)$ (resp. $s_{i,t}^{M_{\Xi}}(\xi)$) be the slope sequence of $L$ (resp. $M$). Then $s_{i,t}^{M_{\Xi}}(\xi)|_{C_{\omega}} = s_{i,t}^{L_{\Xi}}(\xi)|_{C_{\omega}}$.

Proof. The original proof of i) is due Dwork [Dwo73] and Robba [Rob75, Rob80], in the case $\xi = \xi_{0,1}$. The generalization to a point of type $\xi_{0,\rho}$ with $\rho > 0$, can be found in [CM02]. One finds in [Ked10b, Thm.6.6.1 and 10.6.2] a proof which is closer to our context. Point ii) is [You92]. Translating by an element of $K$ the theorem holds for a point $\xi \in X_{\text{int}}$ satisfying the assumptions. \(\square\)

Remark 5.2. 1. The first assertion of Theorem 5.1 without the second one would be quite empty. In fact the dimensions of the terms of the Jordan Hölder sequence of Definition 4.2 are unknown.

2. The first part of the proof (cf. [Ked10b, Thm.6.6.1]) holds for all points $\xi \in X_{\text{gen}}$, the second part of the proof [Ked10b, Thm.10.6.2] requires Frobenius techniques and is stated only for points of type (2) or (3) i.e. in $X_{\text{int}}$. The result probably holds for all $\xi \in X_{\text{gen}}$.

Corollary 5.3. Assume $K = K_{\text{alg}}$, and let $\xi \in X_{\text{int}}$. Let $M$ be a differential module defined by an operator $L = \sum_{i=0}^r g_i \cdot (d/dT)^i$, $g_0 = 1$, with $g_i, g_i^{-1} \in \mathcal{O}(X)$. If $\mathcal{R}_{i,t}^{M_{\Xi}}(\xi) < \omega \cdot r(\xi)$, then for all $1 \leq i \leq i_0$ and for all direction $\delta \in \Delta(\xi)$ one has

\[
\partial_{-} H_{i,t}^{M_{\Xi}}(\xi) = \partial_{-} H_{i,t}^{L_{\Xi}}(\xi), \quad \partial_{+} H_{i,t}^{M_{\Xi}}(\xi) = \partial_{+} H_{i,t}^{L_{\Xi}}(\xi).
\]

Proof. Thm. 5.1 holds for all $\xi'$ in a small open segment containing $\xi$ of each branch through $\xi$. By Lemma 4.11 $\mathcal{R}_{i,t}^{M_{\Xi}}$ are continuous so the assumption $\mathcal{R}_{i,t}^{M_{\Xi}}(\xi) < \omega \cdot r(\xi)$ holds for $\xi'$ close to $\xi$. \(\square\)

Theorem 5.4 ([Ked10b, 12.4.1]). Let $t \in K$, $\rho > 0$. Let $M$ be a differential module over $A_K(t, \rho)$ of rank $r$. Assume that for some $i_0 \in \{1, \ldots, r-1\}$ there exists $\varepsilon > 0$ such that for all $\rho' \in [\rho - \varepsilon, \rho[, h_{i_0,t}^M(\rho')$ is constant and $s_{i,t}^{M_{\Xi}}(\rho') < s_{i_0+1,t}^{M_{\Xi}}(\rho')$. Then $M = M_1 \oplus M_2$, where:

i) $M_1$ has rank $i_0$ and for all $i = 1, \ldots, i_0$ one has $s_{i,t}^{M_{\Xi}}(\rho') = s_{i,t}^{M_{\Xi}}(\rho')$ for all $\rho' \in [\rho - \varepsilon, \rho[$.

ii) $M_2$ has rank $r - i_0$ and for all $i = i_0 + 1, \ldots, r$ one has $s_{i,t}^{M_{\Xi}}(\rho') = s_{i,t}^{M_{\Xi}}(\rho')$ for all $\rho' \in [\rho - \varepsilon, \rho[$. \(\square\)
Proposition 5.5. Let $M = M_1 \oplus M_2$ be a direct sum of differential modules over $\mathcal{O}(X)$ (resp. $\mathcal{A}_K(t, \rho)$) of ranks $r_1$ and $r_2$. Then for all $\xi \in X$ (resp. $\xi \in D^-(t, \rho)$) one has up to permutation\(^{13}\)
\[
\{ \mathcal{R}^M_1(\xi), \ldots, \mathcal{R}^M_{r_1+r_2}(\xi) \} = \{ \mathcal{R}^{M_1}_1(\xi), \ldots, \mathcal{R}^{M_1}_{r_1}(\xi) \} \cup \{ \mathcal{R}^{M_2}_1(\xi), \ldots, \mathcal{R}^{M_2}_{r_2}(\xi) \}.
\]

Proof. $Y_M = \left( \begin{array}{cc} Y_{M_1} & 0 \\ 0 & Y_{M_2} \end{array} \right)$, and $\text{Sol}(M, t, \Omega) = \text{Sol}(M_1, t, \Omega) \oplus \text{Sol}(M_2, t, \Omega)$. More precisely if $\vec{y} \in \text{Sol}(M, t, \Omega)$, the first $r_1$ (resp. the last $r_2$) entries of $\vec{y}$ forms a solution $\vec{y}_1 \in \text{Sol}(M_1, t, \Omega)$ (resp. $\vec{y}_2 \in \text{Sol}(M_2, t, \Omega)$). This proves that $\mathcal{R}(\vec{y}, t) = \min(\mathcal{R}(\vec{y}_1, t), \mathcal{R}(\vec{y}_2, t))$ because by definition the radius of $\vec{y}$ is the minimum of the radii of its entries. □

5.2 Behavior of the spectral Newton polygon along a branch

Theorem 5.6 ([Ked10b, Thm.11.3.2]). For simplicity assume $K = \overline{K}$\(^{\text{alg}}\). Let $M$ be a differential module over $\mathcal{O}(X)$. Let $t \in K$, $|t - c_0| < R_0$ be a $K$-rational point of $D^-((c_0, R_0)$ (cf. (1.2)). Then:

i) The functions $\rho \mapsto \mathcal{R}^{M, sp}_{\xi, t}(\rho)$ and $\rho \mapsto H^{M, sp}_{\xi, t}(\rho)$ verify properties (C2) and (C4) of section 2.3.1 along $I_t$ (cf. notation (1.3) and (2.5)). If moreover $I \subseteq I_t$ is an interval with interior $\bar{I}$ such that the annulus $\{|T-t| \in \bar{I}\}$ is contained in $\mathcal{X}$, then $\rho \mapsto H^{M, sp}_{\xi, t}(\rho)$ is log-concave on $I$.

ii) Let $\xi \in X_{\text{int}}$. If $(i, h^{M, sp}_{\xi, t}(\xi))$ is a vertex of $NP^{sp}(\mathcal{X}, \xi)$ (i.e. $i = r$ or $s^{M, sp}_{\xi}(\xi) < s^{M, sp}_{\xi+1}(\xi)$), then for all $\delta \in \Delta(\xi)$ one has $\partial_\delta H^{M, sp}_{\xi, t}(\xi) , \partial_+ H^{M, sp}_{\xi, t}(\xi) \in \mathbb{Z}$. This implies, by interpolation, that for all $i = 1, \ldots, r$ the log-slopes $\partial_\delta H^{M, sp}_{\xi, t}(\xi)$ and $\partial_+ H^{M, sp}_{\xi, t}(\xi)$ always belong to $\mathbb{Z} \cup \frac{1}{2} \mathbb{Z} \cup \cdots \cup \frac{r}{2} \mathbb{Z}$.

iii) Let $t \in X(\Omega)$ and let $\xi = \xi_{t, \overline{\xi}} \in X_{\text{int}}$ be a point such that there exists an open annulus $\{|T-t| \in [\overline{\xi} - \varepsilon, \overline{\xi} + \infty)\}$, with $\varepsilon > 0$, contained in $\mathcal{X}$. Let $i_0 \in \{1, \ldots, r\}$ be the largest integer such that $s_{i_0}^{M, sp}(\xi_{t, \overline{\xi}}) < \ln(\overline{\xi}) = \ln(r(\xi_{t, \overline{\xi}}))$. Then for all $i = 1, \ldots, i_0$, the function $H^{M, sp}_{\xi, t}(\xi)$ verifies $\partial_\delta H^{M, sp}_{\xi, t}(\xi_{t, \overline{\xi}}) = 0$ for almost but a finite number of directions $\delta \in \Delta(\xi_{t, \overline{\xi}})$, and it is super-harmonic at $\xi$.\(^{14}\) Moreover if $i \leq i_0$, and if $(i, h^{M, sp}_{\xi}(\xi))$ is a vertex of the spectral Newton polygon (i.e. $i = i_0$ or $s^{M, sp}_{\xi}(\xi) < s^{M, sp}_{\xi+1}(\xi)$), then $H^{M, sp}_{\xi, t}(\xi)$ is harmonic at $\xi$.

iv) If $\overline{\rho} < \rho_{t, \mathcal{X}}$ and if $\mathcal{R}^{M, sp}_{\xi, t}(\overline{\rho}) < \overline{\rho}$, then the left and right log-slopes of $\rho \mapsto H^{M, sp}_{\xi, t}(\rho)$ at $\overline{\rho}$ are less than or equal to 0 (the function $\rho \mapsto H^{M, sp}_{\xi, t}(\rho)$ is logarithmically non increasing at $\overline{\rho}$).

Proof. The assumptions guarantee that one can assume $\Omega = K$, and $t = 0$ by a translation. In [Ked10b, Thm.11.3.2] these facts are proved for an annulus.\(^{15}\) The general statement is easily deduced as follows. Let $0 < \rho_1 < \rho_2 < \cdots < \rho_m < R_0$ be the values of $\rho$ for which there exists a center of a hole $c_i$ of $X$ with valuation $|c_i| = \rho$. Since the holes are finite in number and since the spectral Newton polygon of $M$ only depend on its restriction $M_{\xi, \rho}$ to $\mathcal{H}(\xi_{t, \rho})$ (cf. section 1.0.3), we can assume that $X$ equals an annulus $\{|T| \in [\rho_1, \rho_{i+1}]\}$ having possibly some holes placed at distance $\rho_i$ and $\rho_{i+1}$. In this situation let $\mathcal{O}_{an}(\rho_i, \rho_{i+1})$ be the ring of analytic elements on the open annulus $\{|T| \in [\rho_i, \rho_{i+1}]\}$. For all $\rho \in [\rho_i, \rho_{i+1}]$ the morphism $\mathcal{O}(X) \to \mathcal{H}(\xi_{t, \rho})$ factorizes through the natural restriction map $\mathcal{O}(X) \to \mathcal{O}_{an}(\rho_i, \rho_{i+1}) \to \mathcal{H}(\xi_{t, \rho})$. So we can assume $M$ to be a differential module over $\mathcal{O}_{an}(\rho_i, \rho_{i+1})$. The assertions are hence proved in [Ked10b, Thm.11.3.2].

Notes: the super-harmonicity is stated in [Ked10b] for a point of type (2) i.e. such that $\Delta(\xi)$ has at least two elements. If $\Delta(\xi)$ has a unique element, then the super-harmonicity is just the log-concavity which is the claim i). Directional finiteness is not mentioned in [Ked10b], but from

\(^{13}\)Here the equality is intended with multiplicities i.e. if a slope $s$ appears $n_s$-times in $NP^{\text{conv}}(M, \xi)$, then it appears $n_1 + n_2$-times in $NP^{\text{conv}}(M, \xi)$.

\(^{14}\)Because of Lemma 3.4 one only controls the slopes along $K$-rational directions. This is the reason of taking $K = \overline{K}$\(^{\text{alg}}\).

\(^{15}\)Some of these results were previously proved in [CD94, Th.2.3], and [Fon00].
the proof of [Ked10b, Thm. 11.3.2] it follows that the log-slopes \( \partial_- H^{\text{sp}}_{i,\delta}(\xi,\mathcal{P}) \) are those of a convenient differential polynomial with coefficients in the fraction field \( \mathcal{O}(X) \) of \( \mathcal{O}(X) \) so the directional finiteness (C5) holds applying Proposition 4.2, after possibly replacing \( M \) with its restriction to a sub-affinoid \( X' \) of \( X \) preserving the directions at \( \xi,\mathcal{P} \) (cf. Def. 5.8) and such that the coefficients of the polynomial have all no poles in \( X' \) (cf. section 5.4.1). Finally notice that the harmonicity statement of [Ked10b, 11.3.2,(c)] implicitly assumes \( K = K^{\text{alg}} \) because Lemma 3.4 is used to control the slopes along the branches \( \Lambda(t) \) defined by algebraic \( t \in K^{\text{alg}} \).

**Remark 5.7.** We will reproduce and generalize a part of Theorem 5.6 (cf. section 7).

### 5.3 Restriction to a sub-affinoid

Let \( X' \subseteq X \) be a sub-affinoid domain. The polygon \( NP^{\text{sp}}(M, \xi) \) only depends on the restricted module \( M_\xi = M \otimes \mathcal{O}(X) \), so it is invariant by restriction to \( X' \). Conversely the slopes of \( NP^{\text{conv}}(M, \xi) \) are upper bounded by \( \rho_{\xi,X} \), and so the convergence polygon is not stable by restriction to \( X' \).

**Definition 5.8.** A sub-affinoid \( X' \subseteq X \) preserves the directions at \( \xi \in X \) if \( \xi \in X' \), and if none of the directions in \( \Delta(\xi) \) nor the extra direction toward \(+\infty\) is suppressed when intersecting with \( X' \).

**Remark 5.9.** If \( X' \subseteq X \) preserves the directions at \( \xi \in X \), then \( \xi \in X_{\text{int}} \) (resp. \( \xi \) is not in the Shilov boundary of \( X \)) if and only if \( \xi \in X'_{\text{int}} \) (resp. \( \xi \) is not in the Shilov boundary of \( X' \)).

We preserve the conventions of Remark 2.5.\(^{16}\)

**Proposition 5.10.** Let \( X' \subseteq X \) be a sub-affinoid. Let \( M' := M \otimes \mathcal{O}(X') \). Then

i) For all \( i = 1, \ldots, r \) and all \( \xi' \in X' \) one has \( R^M_i(\xi') = \min(R^M_i(\xi'), \rho_{\xi,X'}) \), and

\[
\Gamma(X', R^M_i) = \left( \Gamma(X, R^M_i) \cap X' \right) \cup \Gamma_X'.
\]

ii) \( R^M_i \) is directional finite at \( \xi' \in X' \) (cf. (C5)) if and only if \( R^M_i(\xi') \) is directional finite at \( \xi' \).

iii) If \( \Gamma_X' \subseteq \Gamma(X, R^M_i) \), then for all \( \xi'' \in X' \) one has \( R^M_i(\xi'') = R^M_i(\xi'') \) and \( H^M_i(\xi'') = H^M_i(\xi''). \) If moreover \( \Gamma_X' \) preserves the directions at \( \xi' \in X' \), and if \( K = K^{\text{alg}} \), then \( H^M_i \) is super-harmonic (resp. harmonic) at \( \xi' \) if and only if so does \( H^M_i \) at \( \xi' \).

iv) Assume that \( X' \) preserves the directions at \( \xi' \in X' \), and that \( R^M_i(\xi') < \rho_{\xi,X'} \). Then for all \( j = 1, \ldots, i \) and all \( \delta \in \Delta(\xi') \) one has \( \partial_- R^M_{\delta}(\xi') = \partial_- R^M_{\delta}(\xi') \), and hence \( \partial_- H^M_{\delta}(\xi') = \partial_- H^M_{\delta}(\xi') \).

The same holds for the right slopes. Hence if \( K = K^{\text{alg}} \), then \( H^M_i \) is super-harmonic (resp. harmonic) at \( \xi' \) if and only if so does \( H^M_j \) at \( \xi' \).

**Proof.** Let \( t \in X(\Omega) \) be a Dwork generic points for \( \xi' \). Clearly \( \text{Sol}(M, t, \Omega) = \text{Sol}(M, t, \Omega) \), and if \( R \leq \rho_{\xi,X'} \), then \( \text{Fil}^R \text{Sol}(M, t, \Omega) = \text{Fil}^R \text{Sol}(M', t, \Omega) \). Moreover \( \text{Fil}^R \text{Sol}(M, t, \Omega) \subseteq \text{Fil}^R \text{Sol}(M', t, \Omega) \) for all \( \rho_{\xi,X'} \leq R \leq \rho_{\xi,X'} \). This proves that the convergent slope sequence of \( M' \) at \( \xi' \) equals that of \( M \) truncated by \( \rho_{\xi,X'} \). In other words \( R^M_i(\xi') = \min(R^M_i(\xi'), \rho_{\xi,X'}) \) for all \( \xi' \in X' \). From this expression together with (4.9) one sees that \( \rho_{R^M_i(\xi')} = \min(\rho_{R^M_i(\xi'), \rho_{\xi,X'}}) \). This implies (5.3), and hence ii). If \( \Gamma_X' \subseteq \Gamma(X, R^M_i) \), then \( R^M_i(\xi') \leq R^M_i(\xi') \) and \( \rho_{R^M_i(\xi')} \leq \rho_{R^M_i(\xi')} \) (cf. point v) of Prop.2.2), and \( R^M_i(\xi') = R^M_i(\xi') \) for all \( \xi' \in X' \). The same holds for all \( j < i \) because \( R^M_i(\xi') \leq R^M_i(\xi') \). This proves that \( H^M_i(\xi') = H^M_i(\xi') \). This implies the assertion about the (super-)harmonicity since the equality of the functions implies the equality of the slopes. To prove iv) observe that \( R^M_j(\xi') = \min(R^M_j(\xi'), \rho_{\xi,X'}) = R^M_j(\xi') \), \( R^M_i(\xi') < \rho_{\xi,X'} \), and that this remains true by continuity on an open segment containing \( \xi' \) of each direction \( \delta \in \Delta(\xi') \). \( \square \)

\(^{16}\)Note that \( R^M_i \) is not the restriction to \( R^M_i \) to \( X' \), so Remark 2.5 does not applies here.
5.4 Base change by a matrix in the fraction field \( \mathcal{F}(X) \) of \( \mathcal{O}(X) \)

Let \( \mathcal{F}(X) \) denotes the fraction field of \( \mathcal{O}(X) \). If \( H(T) \in \text{GL}_n(\mathcal{F}(X)) \) is a matrix, its entries and those of its inverse have a finite number of poles in \( X \), and these poles are algebraic over \( K \). Then there exists a sub-affinoid \( X' \subseteq X \) having conveniently small holes around the zeros and poles of \( H(T) \) and its inverse, in order that \( H \in \text{GL}_n(\mathcal{O}(X')) \). If \( \xi \in X_{\text{int}} \), then \( \xi \) can not be a zero of \( H(T) \), and \( X' \) can be chosen in order to preserve the directions at \( \xi \) (cf. Def. 5.8).

5.4.1 Reduction to a cyclic module. Let \( r := \text{rk}(M) \) be the rank of \( M \). By the cyclic vector theorem (cf. [Kat87]) one finds a cyclic basis of \( M \otimes \mathcal{O}(X) \mathcal{F}(X) \) in which \( M \) is represented by an operator \( L := \sum_{i=0}^{r} g_{-i}(T)(d/dT)^i \), with \( g_i \in \mathcal{F}(X) \) for all \( i \), and \( g_0 = 1 \). Then \( L \) represents simultaneously each \( M_\xi = M \otimes \mathcal{O}(X) \mathcal{H}(\xi) \) for all \( \xi \in X^{\text{gen}} \). If \( H(T) \in \mathcal{F}(X) \) is the base change matrix, one can chose \( X' \subseteq X \) as indicated in section 5.4. We further restrict \( X' \) in order that none of the \( g_i \) has poles nor zeros on it. By Proposition 5.10 the restriction of \( M \) to \( X' \) does not affect the finiteness. If moreover \( \Gamma_{X'} \subseteq \Gamma(\mathcal{R}_t^M) \), the super-harmonicity of \( H^M_t \) is also preserved.

6. Pull-back and push-forward by Frobenius

In this section \( K \) is assumed of mixed characteristic \( (0, p) \) with \( p > 0 \). We assume moreover \( \mu_p(K^{\text{alg}}) = \mu_p(K) \). If \( \alpha, \alpha' \in \mu_p(K) \) are two distinct \( p \)-th root of 1, then \( |\alpha - \alpha'| = \omega = |p|^{-1/p} \). Let \( T, \tilde{T} \) be two variables. The ring morphism \( \varphi^+: K[T] \to K[\tilde{T}] \) sending \( f(T) \) into \( f(\tilde{T}^p) \), defines a analytic map \( \varphi: A_K^{\text{an}} \to A_K^{\text{an}} \). If \( f \in A_K^{\text{an}}(\Omega) \) is a Dwork generic point for \( \xi \in A_K^{\text{an}} \), then \( \tilde{T}^p \) is a Dwork generic point for \( \varphi(\xi) \). Indeed for all \( f \in K[T] \) one has \( \varphi(\xi)(f) = \xi(f(\tilde{T}^p)) = |f(\tilde{T}^p)|_\Omega \).

Now we describe the image of a point of type \( \xi_{t, \rho} \). For all \( h > 0 \) and \( \rho, \rho' \geq 0 \) we set

\[
\phi(h, \rho) := \max(p^h, |p|^{p-1} \rho) = \begin{cases} 
\rho^h & \text{if } \rho > \omega^h \\
|p|^{p-1} \rho & \text{if } \rho < \omega^h 
\end{cases}; \tag{6.1}
\]

\[
\psi(h, \rho') := \min\left((\rho')^{1/p}, \frac{\rho'}{|p|^{p-1}}\right) = \begin{cases} 
(\rho')^{1/p} & \text{if } \rho' > \omega^p\rho' \\
\frac{\rho'}{|p|^{p-1}} & \text{if } \rho' \leq \omega^p \rho' 
\end{cases}. \tag{6.2}
\]

For \( h \) fixed \( \phi \) and \( \psi \) are increasing functions such that \( \phi(h, \psi(h, \rho')) = \rho' \) and \( \psi(h, \phi(h, \rho)) = \rho \). In the sequel of this section by convention one sets \( \rho' = \phi(h, \rho) \) and \( \rho = \psi(h, \rho') \).

**Proposition 6.1.** Let \( t \in \Omega, \rho > 0 \). Then \( \varphi(\xi_{t, \rho}) = \xi_{t^p, \phi(|t|^p\rho)} \), hence \( \varphi^{-1}(\xi_{t^p, \rho'}) = \{\xi_{at^p, \psi(|t|^p\rho')}\}_{a \rho = 1} \).

**Proof.** By density and by multiplicativity it is enough to prove that for all \( a \in K \) one has \( \varphi(\xi_{t, \rho})(T-a) = \xi_{t^p, \phi(|t|^p\rho)}(T-a) \). Write \( \varphi(\xi_{t, \rho})(T-a) = \xi_{t, \rho}(\tilde{T}^p-a) = \xi_{t, \rho}(\sum_{k=0}^{p-1} \tilde{T}^k t^{p-k} - a) = \max(|t^p-a|, |p|^{p-1} \rho, \rho^p) \), in fact the terms corresponding to \( k = 1, \ldots, p-1 \) form either a non decreasing or a non increasing sequence. On the other hand \( \xi_{t^p, \phi(|t|^p\rho)}(T-a) = \xi_{t^p, \phi(|t|^p\rho)}(T-t^p+a) = \max(\phi(|t|^p\rho), |t^p-a|) = \varphi(\xi_{t, \rho})(T-a) \).

**Remark 6.2.** If \( t \) is a Dwork generic point for \( \xi \in X^{1/p} \), then \( |t| = |\xi(\tilde{T})| \) is independent on \( t \).

**Remark 6.3.** If \( \rho = \psi(|t|^p\rho') \geq \omega|t| = |a - a'||t| \), then \( \xi_{a^t\psi(|t|^p\rho')} = \xi_{a^t, \psi(|t|^p\rho')} \) for all \( \alpha, \alpha' \in \mu_p(K) \). Hence \( \varphi^{-1}(\xi_{t^p, \rho'}) \) has a single element. Conversely if \( \rho \leq \omega|t| \), then \( \varphi^{-1}(\xi_{t^p, \rho'}) \) has \( p \) distinct elements.

**Proposition 6.4.** Let \( t \in \Omega \) and let \( \rho, \rho' \geq 0 \) be such that \( \rho = \psi(|t|^p\rho') \) and \( \rho' = \phi(|t|^p\rho) \). Let \( D^-(t, \rho) \) and \( D^-(t^p, \rho') \) be the open disks with algebras \( A_\Omega(t, \rho) \) and \( A_\Omega(t^p, \rho') \) respectively. Then:

1. One has the following equalities

\[
\varphi(D^-(t, \rho)) = D^-(t^p, \phi(|t|^p\rho)), \quad \varphi^{-1}(D^-(t^p, \rho')) = \bigcup_{a \rho = 1} D^-(at, \psi(|t|^p\rho')) \tag{6.3}
\]
Convergence Newton polygon I: the affine line

ii) For all $\alpha \in \mu_p(K)$ the corresponding morphism $\varphi_{\#}^{\#} : A_\Omega(t^p, \rho') \to A_\Omega(\alpha t, \psi(|t|, \rho'))$ is injective and isometric in the following sense. For all $f \in A_\Omega(t^p, \rho')$ and all $\eta < \rho'$ one has

$$|f|_{\eta, \eta} = |\varphi_{\#}^{\#}(f)|_{\alpha t, \psi(|t|, \eta)}.$$  \hfill (6.4)

iii) If $\rho' \leq \varpi|t|^p$, then for all $\alpha \in \mu_p(K)$, $\varphi_{\#}^{\#}$ is an isomorphism of rings (satisfying (6.4)).

iv) If $\varpi|t|^p < \rho'$, then $\varphi_{\rho}^{\#}$ is independent on $\alpha$. Moreover $\mu_p(K)$ acts on $A_\Omega(t, \rho')$ by $\alpha(f)(T) := f(\alpha T)$, and

$$\varphi_{\rho}^{\#}(A_\Omega(t^p, \rho')) = A_\Omega(t, \rho')^{\mu_p(K)}.$$  \hfill (6.5)

v) For all $\rho > 0$ we denote by $\psi_{\rho, \varphi} : A_\Omega(\alpha t, \rho) \to A_\Omega(t^p, \rho')$ the $\Omega$-linear map defined as $\psi_{\rho, \varphi} := \{(\varphi_{\#}^{\#})^{-1} if 0 < \rho \leq |t| \}$ if $|\omega||t| < \rho$. Then for all $\alpha \in \mu_p(K)$ the maps $\{\psi_{\rho, \varphi}\}_{\rho > 0}$ satisfy $\psi_{\rho, \varphi} \circ \varphi_{\rho}^{\#} = \mathrm{Id}_{A_\Omega(t^p, \rho')}$ for all $\rho > 0$. Moreover if $|\omega||t| \notin [\rho_1, \rho_2]$, and if $\rho_i := \varphi(|t|, \rho_i)$, $i = 1, 2$, then the following diagram is commutative where the horizontal maps are the restrictions:

$$\begin{array}{ccc}
A_\Omega(\alpha t, \rho_1) & \xrightarrow{\psi_{\rho_1, \rho_1}} & A_\Omega(\alpha t, \rho_2) \\
\varphi_{\rho_1} & \circ & \varphi_{\rho_2}
\end{array}$$  \hfill (6.6)

If $|\omega||t| \in [\rho_1, \rho_2]$, then the diagram does not commute.

Proof. To prove i) one needs to evaluate $|a^p - \varpi|^p$ for all $a \in D_{\Omega}^\cap(t, \rho)$, and arbitrary $\Omega' \in E(\Omega)$. The skeleton of $f(\tilde{T}) = \tilde{T}^p - t^p = \prod_{a^p \in 1}\tilde{T} - \alpha t$ is $\Gamma(f) = \operatorname{Sat}(\{|\alpha t|_{\varphi} = 1\})$. Hence for all $\alpha \in \Omega$ one has $|a^p - \varpi| = |\tilde{T}^p - t^p|_{\varphi} = |\tilde{T}^p - t^p|_{a^p = 1} = |\tilde{T} - \alpha t|$. In fact by Prop. 6.1 one has $|\tilde{T}^p - t^p|_{\varphi} = \varphi(|\omega|, \rho)$. Since $\varphi(0) = \varphi(|\omega|, \rho)$ is strictly increasing and since $|\alpha a^p| = |a^p - \varpi|$, then $|a^p - \varpi| = \varphi(|\omega|, \rho)$. This proves $\varphi(D_{\Omega}^\cap(t, \rho)) \subseteq D_{\Omega}^\cap(t, \rho)$. Conversely applying $\psi(\omega(-)|t|, -) to |a^p - \varpi| = \varphi(|\omega|, \rho)$.

Lemma 6.5. Let $t \neq 0$ and $\alpha \in \mu_p(K)$. For $k = 1, \ldots, p - 1$ the power series $T^{k/p} - \alpha = (T - \varpi)^{k/p} - \alpha = (T - \alpha)^{k/p} + \sum_{s \geq 1} \frac{1}{s} \left(T - \alpha\right)^{k/s} - \alpha^{k/s}$ has radius of convergence $\varpi|t|^p$ around $t^p$. $\square$

Proof. Since $|\left(T^{k/p}\right)| = |k/p|^s/|s|$, then $\lim \inf s |(k/s)^p - |s|^{-1/s} = |p| |t|^p \lim \inf s |s|^{-1/s} = \varpi|t|^p$.  \hfill (6.7)

We now define $(\varphi_{\rho, \varphi}^{\#})^{-1}$. By Lemma 6.5, $y := T^{1/p} - \alpha \in A_\Omega(t^p, \rho')$ because $\rho' \leq \varpi|t|^p$. Hence $(\varphi_{\rho, \varphi}^{\#})^{-1}(\sum_{i=0}^n a_i (\tilde{T} - \alpha t)^i) = \sum_{i=0}^n a_i g'$. This defines a map $\Omega[\tilde{T} - t] \to A_\Omega(t^p, \rho')$ satisfying (6.4), that extends by continuity to $A_\Omega(t, \rho)$ and coincides with $(\varphi_{\rho, \varphi}^{\#})^{-1}$. This proves iii). Assume $|\omega||t| < \rho$. Since $\Omega[\tilde{T}]^{P^p} = \Omega[T] = \varphi_{\#}^{\#}(\Omega[T])$, then by density one has iv) together with $\psi_{\rho, \varphi}^{\#} \circ \varphi_{\rho, \varphi}^{\#} = \mathrm{Id}_{A_\Omega(t^p, \rho')}$.

The commutativity of the diagram is evident. If $|\omega||t| \notin [\rho_1, \rho_2]$ it does not commute since for example $\psi_{\rho_1} \circ \psi_{\rho_2}(\tilde{T}) = 0 \neq \psi_{\rho_1}(\tilde{T})$, because $\psi_{\rho_1} = (\varphi_{\rho_1, \rho_1})^{-1}$.  \hfill \square
Remark 6.6. Corollary 6.7 below proves that (6.7) are equalities in the most part of cases. The inequality is due to the fact that $R_{f(T^p)}$ is the radius of the composite of $f$ and $T^p$ as power series, while $\psi(|t|, R_f(T))$ is the radius of their composite as functions. For instance, by Lemma 6.5, $T^{k/p} - t$ converges with exact radius $\omega^p|t|^p$, but its pull-back $\tilde{T}^k - t$ converges with infinite radius.

Corollary 6.7. With the above notations one has

i) If $R_f(T) \neq \omega^p|t|^p$, then $R_f(\tilde{T}^p) \neq \omega|t|$ and $R_f(\tilde{T}^p) = \psi(|t|, R_f(T))$. In particular this equality holds if $R_f(T) = 0$ (then $R_f(\tilde{T}^p) = 0$) or if $R_f(T) > +\infty$ (then $R_f(\tilde{T}^p) = +\infty$).

ii) If $R_f(\tilde{T}^p) \leq \omega|t|$, then $R_f(T) = \phi(|t|, R_f(\tilde{T}^p)) = |p||t|^{p-1}R_f(\tilde{T}^p)$.

iii) If $R_f(\tilde{T}^p) \geq \omega^p|t|^p$, then $R_f(T) \geq \omega^p|t|^p$.

Proof. One has $\phi(|t|, \omega|t|) = \omega^p|t|^p$ and $\psi(|t|, \omega^p|t|^p) = \omega|t|$. Assume that $R_f(T) \neq \omega^p|t|^p$ and, by contrapositive, that (6.7) is strict. Let $R_f(\tilde{T}^p) > \rho > \psi(|t|, R_f(T))$ be such that $\omega|t| \notin |\psi(|t|, R_f(T)), \rho|$. By diagram (6.6) $f(\tilde{T}^p) \in A_\Omega(\alpha t, \rho)$ and $f(T) = \psi^{\#}_{\alpha, \rho}(f(\tilde{T})) \in A_\Omega(t^p, \varphi(|t|, \rho))$. Now $\varphi(|t|, \rho) > \varphi(|t|, \psi(|t|, R_f(T))) = R_f(T)$ which is a contradiction. ii) and iii) follows from Prop. 6.4, iii).

6.0.3 Degree of the residual fields. The morphism $\varphi$ induces a $K$-linear isometric inclusion

$$\varphi^\#: \mathcal{H}(\varphi(\xi)) \rightarrow \mathcal{H}(\xi).$$ (6.8)

Proposition 6.8. Let $\xi = \xi_{c, \rho}, c \in \Omega, \rho \geq r(\xi_c)$. Then :

i) If $|\omega|c < \rho$, then $\mathcal{H}(\xi)/\mathcal{H}(\varphi(\xi))$ is an extension of degree $p$.

ii) Conversely if $c \in K$, $c \neq 0$ and if $\rho < |\omega|c$, then $\mathcal{H}(\xi) = \mathcal{H}(\varphi(\xi))$.

Proof. $K(\tilde{T})$ (resp. $K(T)$) is dense on $\mathcal{H}(\xi)$ (resp. $\mathcal{H}(\varphi(\xi))$). The map $\varphi^\#: K(T) \rightarrow K(\tilde{T})$, $T \rightarrow \tilde{T}^p$ is an extension of degree $p$. By density the degree of $\mathcal{H}(\xi)/\mathcal{H}(\varphi(\xi))$ is equal to 1 or $p$, and it is 1 if and only if $\tilde{T} = T^{1/p} \in \mathcal{H}(\varphi(\xi))$. One has $r(\xi_{c, \rho}) = \max(r(\xi_c), \rho) = \rho$, and $r(\varphi(\xi)) = r(\xi_{c, \rho}, \varphi(\xi_{c, \rho})) = \min(r(\xi_{c, \rho}), \varphi(|c|, \rho)) = \min(\varphi(|c|, r(\xi_c)), \varphi(|c|, \rho)) = \varphi(|c|, \rho)$. Let $\Omega \in E(K)$, $t = t_{c, \rho} \in \Omega$ (resp. $t^p \in \Omega$) be a Dwork generic point for $\xi_{c, \rho}$ (resp. $\xi_{c, \rho, \varphi(\xi_{c, \rho})}$). As in section 4.3.1 one has a diagram

$$\begin{array}{c}
\mathcal{H}(\xi) \rightarrow A_\Omega(t_{c, \rho}, \rho) \\
\mathcal{H}(\varphi(\xi)) \circ \leftarrow A_\Omega(t_{c, \rho}^p, \varphi(|c|, \rho)).
\end{array}$$ (6.9)

If $\rho > |\omega|c$, then $|t_{c, \rho}| = |\tilde{T} - c + c|_{c, \rho} = \max(|c|, \rho) \geq \rho > \max(|\omega|c, \omega p) = |t_{c, \rho}|$. And hence $\varphi(|c|, \rho) = \varphi(|t_{c, \rho}|, \rho) = \rho^p$. By contrapositive if $T^{1/p} \in \mathcal{H}(\varphi(\xi))$, then $T^{1/p} - t_{c, \rho} \in A_\Omega(t_{c, \rho}^p, \rho^p)$ which is absurd by Lemma 6.5. So $[\mathcal{H}(\xi) : \mathcal{H}(\varphi(\xi))] = p$. To prove ii) write $T^{1/p} - c = \lim_{k \rightarrow \infty} c \sum_{k=1}^{s} (1/k^p)(\frac{t_{c, \rho}^p}{|c|^p})^k$. This limit converges in $\mathcal{H}(\varphi(\xi))$ with respect to $\varphi(\xi) = \xi_{c, \rho, |\varphi(\xi_{c, \rho})|p^{-1}}$. Indeed $|(1/k^p)(\frac{t_{c, \rho}^p}{|c|^p})^k|_{|c|^{p^{-1}}|c|^p} = \frac{|c|^p}{|c|^p} = 1$ since $\rho < |\omega|c$. Hence $T^{1/p} \in \mathcal{H}(\varphi(\xi))$. □

6.1 Behavior of the radii of convergence under pull-back by Frobenius

Let $X \subset \mathbb{A}^{1,x}_K$ and $X^{1/p} := \varphi^{-1}(X)$. The image of the injective map $\varphi^\#: \mathcal{O}(X) \rightarrow \mathcal{O}(X^{1/p})$ is not stable under $d/dT$. One has $(d/dT_{pT^{1/p}}) \circ \varphi^# = \varphi^# \circ d/dT$, so one is induced to assume that $0 \notin X$ (hence $0 \notin X^{1/p}$) in sections 6.1 and 6.3. Then $pT^{1/p} - 1 \in \mathcal{O}(X^{1/p})$ and $\varphi^\#: (\mathcal{O}(X), d/dT) \rightarrow (\mathcal{O}(X^{1/p}), \frac{d/dT}{pT^{1/p}})$ commutes with the derivations. The pull-back by Frobenius $\varphi^*$ is the composite

$$\frac{d}{dT} \rightarrow \frac{d}{d\tilde{T}} - \text{Mod}(\mathcal{O}(X)) \rightarrow (\frac{d}{d\tilde{T}}_{pT^{1/p}}) - \text{Mod}(\mathcal{O}(X^{1/p})) \rightarrow \frac{d}{dT} - \text{Mod}(\mathcal{O}(X^{1/p}))$$ (6.10)
where the first functor is the usual scalar extension functor associating to \((M, \nabla)\) the \(\mathcal{O}(X^{1/p})\)-differential module \(\tilde{M} = M \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{1/p})\) together with \(\tilde{\nabla} := \nabla \otimes 1 + 1 \otimes (\frac{d/dt}{pT^{p-1}})\), which is a connection with respect to the derivation \(\frac{d/dt}{pT^{p-1}}\) of \(\mathcal{O}(X^{1/p})\). The second functor is an equivalence of categories that only changes the derivation; it sends \((\tilde{M}, \tilde{\nabla})\) into \((\tilde{M}, pT^{p-1}\tilde{\nabla})\) and it is the identity on the morphisms. Concretely if if \(\frac{d}{dt}(Y) = G(T)Y\) is the equation in a basis \(e\) of \(M\), then \(\frac{d}{dt}(Y) = pT^{p-1}G(T)p^{p-1}Y\) is that of \(\varphi^*(M) := M \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{1/p})\) in the basis \(e \otimes 1\). If \(Y(T)\) is a Taylor solution of \(Y' = GY\), then \(Y(\tilde{T})\) is the Taylor solution of \(\frac{d}{dt}(Y) = pT^{p-1}G(T)p^{p-1}Y\).

**Proposition 6.9.** Let \(M\) be a differential module over \(\mathcal{O}(X)\) of rank \(r\). Let \(t \in X^{1/p}(\Omega)\) be a Dwork generic point for \(\xi \in X^{1/p}\). Then by (6.7) one has \(R_i^\varphi^*M(\xi) \geqslant \psi(|t|, R_i^\varphi M(\xi))\). Moreover:

i) If for all \(i = 1, \ldots, i_0\) one has \(R_i^M(\varphi(\xi)) \neq \omega p^i |t|^p\), then \(R_i^\varphi^*M(\xi) = \psi(|t|, R_i^M(\varphi(\xi)))\).

ii) If \(R_i^\varphi^*M(\xi) \leqslant \omega |t|\), then \(R_i^M(\varphi(\xi)) = \phi(|t|, R_i^\varphi M(\xi)) = |p||t|^{p-1}R_i^\varphi^*M(\xi)\).

**Proof.** The map \(\text{id} \otimes \varphi^* : M \otimes A_Q(t^p, \rho^p) \rightarrow M \otimes A_Q(t, \psi(|t|, \rho^p))\) induces an \(\Omega\)-linear isomorphism \(\text{Sol}(M, t^p, \Omega) \xrightarrow{\sim} \text{Sol}(\varphi^*M, t, \Omega)\). By Proposition 6.4 one has \(R_\xi^M(\varphi(\xi)) \leqslant \rho^\varphi(\xi, X)\). Identifying \(M \otimes A_Q(t^p, \rho^p) \xrightarrow{\sim} A_Q(t^p, \rho^p)\) with a solution \(\tilde{y}(T) \in A_Q(t^p, \rho^p)\) sent into \(\tilde{y}(\tilde{T}) \in A_Q(t, \psi(|t|, \rho^p))\). Corollary 6.7 gives \(\text{id} \otimes \varphi^*(\text{Fil} \geqslant p^i \text{Sol}(M, t^p, \Omega)) \subseteq \text{Fil} \geqslant p^i \psi(|t|, \rho^p) \text{Sol}(\varphi^*M, t, \Omega)\). The only “pathology” that can happens is that a solution \(\tilde{y}(T)\) with radius strictly larger than \(\omega |t| = \psi(|t|, \omega p^i |t|^p)\). This increase the dimension of \(\text{Fil} \geqslant p^i \psi(|t|, \rho^p) \text{Sol}(\varphi^*M, t, \Omega)\) with respect to that of \(\text{Fil} \geqslant p^i \text{Sol}(M, t^p, \Omega)\). For \(p^i \leqslant \omega |t|^p\) the dimensions are equal, so i) and ii) hold. For \(p^i > \omega |t|^p\) the assumption i) implies that there is no “pathology”, so all the radii are transformed by the rule \(\psi(|t|, -)\), and the dimensions are equal.

**Corollary 6.10.** Assume \(\xi \in X_{\text{int}}\) of the form \(\xi_{t, \rho}\) with \(t \in K\) and \(\rho > 0\). Then Proposition 6.9 holds replacing \(R_i^M(\xi)\) and \(R_i^\varphi^*M(\xi)\) by \(R_i^{1, \text{sp}} M(\xi)\) and \(R_i^{1, \text{sp}} \varphi^*M(\xi)\) respectively.

**Proof.** The claim follows by truncation from point i) of Theorem 4.7 (cf. section 4.3.1).

### 6.2 Antecedent by Frobenius.

**Proposition 6.11.** Let \(t \in \Omega\) be a Dwork generic point for \(\xi \in (\mathbb{A}_K^{1,\text{an}})_{\text{gen}}\). Let \(\tilde{M}\) be a \(\mathcal{H}(\xi)\)-differential module satisfying \(R_i^{1, \text{sp}}(\xi) > \omega |t| = \omega \xi(\tilde{T})\). Then there exists a unique \(\mathcal{H}(\varphi(\xi))\)-differential module \(M\) satisfying both conditions \(\varphi^*(M) \cong \tilde{M}\) and \(R_i^{1, \text{sp}}(\varphi(\xi)) = R_i^{1, \text{sp}}(\xi)^p\).

**Proof.** The proof is the same as [Ked10b, Thm.10.4.2].

### 6.3 Definition of the push-forward by Frobenius

If \(\varphi^* : \mathcal{H}(\varphi(\xi)) \xrightarrow{\sim} \mathcal{H}(\xi)\), then the Frobenius pull-back \(\varphi^*\) is an equivalence, and the push-forward \(\varphi_*\) is by definition its quasi inverse. We then consider point i) of Prop. 6.8:

**Hypothesis 6.12.** In this section we assume \(\xi = \xi_{c, p}\) with \(c \in \Omega\), \(\rho \geq r(\xi_c)\) and \(\rho > \omega |c|\).

**Remark 6.13.** Under 6.12 one has \(\varphi(\xi) = \varphi(\xi_{c, p}) = \xi_{c^p, p^p}\). If one needs to perform \(n\)-times the Frobenius push-forward, then one has to assume \(p > (\omega |c|)^{1/p^n}\) to have \([\mathcal{H}(\varphi(\xi)) : \mathcal{H}(\varphi^n(\xi))] = p^n\).

---

\(^{17}\)Namely if \(\nabla : M \rightarrow \tilde{M}\) is a connection with respect to \(\frac{d/dT}{pT^{p-1}}\), then \(pT^{p-1}\nabla : M \rightarrow \tilde{M}\) is a connection with respect to \(d/d\tilde{T}\), and one sees that an \(\mathcal{O}(X^{1/p})\)-linear morphism commutes with \(\nabla\)s if and only if it commutes with \(pT^{p-1}\nabla\)s.
The Frobenius push-forward functor \( \varphi_* \) is the composite functor

\[
\frac{d}{dT} - \Mod(\mathcal{H}(\xi)) \xrightarrow{\sim} \left( \frac{d/dT}{pT^{p-1}} \right) - \Mod(\mathcal{H}(\xi)) \xrightarrow{\sim} \frac{d}{dT} - \Mod(\mathcal{H}(\varphi(\xi)))
\]  
(6.11)

where the first equivalence is the inverse of the change of derivation functor (6.10) and it associates to the \((\mathcal{H}(\xi), d/dT)\)-differential module \((\widetilde{M}, \tilde{\nabla})\) the \((\mathcal{H}(\xi), \frac{d/dT}{pT^{p-1}})\)-differential module \((\widetilde{M}, \frac{\tilde{\nabla}}{pT^{p-1}})\). The second is the scalar restriction functor associating to \((\widetilde{M}, \frac{\tilde{\nabla}}{pT^{p-1}})\) the \((\mathcal{H}(\varphi(\xi)), d/dT)\)-differential module \((\tilde{M}, \tilde{\nabla})\) itself viewed as a \(\mathcal{H}(\varphi(\xi))\)-module via \(\varphi^\#\). We denote by \((\varphi_*(\tilde{M}), \varphi_*(\tilde{\nabla}))\) the differential module over \(\mathcal{H}(\varphi(\xi))\) so obtained.

### 6.3.1 Matrix of \(\varphi_*(\tilde{\nabla})\).

One has a direct sum decomposition \(\mathcal{H}(\xi) = \bigoplus_{k=0}^{p-1} \varphi^\#(\mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^k\), so that each \(g(\tilde{T}) \in \mathcal{H}(\xi)\) can be uniquely written as \(g(\tilde{T}) = \sum_{k=0}^{p-1} g_k(T)p^k\). Since \(\frac{d}{dT} = \frac{1}{pT^{p-1}} \frac{d}{dT}\) stabilizes globally each factor and \(\frac{1}{pT^{p-1}} \frac{d}{dT}(g_k(T)p^k) = g'_k(T)p^k\). For all \(g(\tilde{T}) \in \mathcal{H}(\xi)\) we define \(\varphi_*(g)(\tilde{T}) \in M_{p \times p}(\mathcal{H}(\varphi(\xi)))\) to be the matrix of the multiplication in \(\mathcal{H}(\xi)\) by \(g(\tilde{T})/(pT^{p-1})\), with respect to the basis \(1, \tilde{T}, \ldots, \tilde{T}^{p-1}\) over \(\mathcal{H}(\varphi(\xi))\). One has

\[
\varphi_*(g)(\tilde{T}) = \left(\begin{array}{cccc}
g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & Tg_0(T) \\
g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & Tg_1(T) \\
g_1(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & Tg_2(T) \\
g_{p-2}(T) & g_{p-1}(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & Tg_3(T) \\
g_{p-3}(T) & g_{p-2}(T) & g_{p-1}(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots \\
g_{p-2}(T) & g_{p-3}(T) & g_{p-2}(T) & g_{p-1}(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & \cdots & \cdots \\
g_{p-3}(T) & g_{p-2}(T) & g_{p-3}(T) & g_{p-2}(T) & g_{p-1}(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & \cdots \\
g_{p-4}(T) & g_{p-3}(T) & g_{p-4}(T) & g_{p-3}(T) & g_{p-2}(T) & g_{p-1}(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g_1(T) & g_0(T) & g_1(T) & g_0(T) & g_1(T) & g_0(T) & g_1(T) & g_0(T) & g_1(T) \\
g_0(T) & g_{p-1}(T) & g_0(T) & g_1(T) & g_0(T) & g_1(T) & g_0(T) & g_1(T) & g_0(T) \\
\end{array}\right)
\]
(6.12)

Notice that the terms over the diagonal are multiplied by \(T\). Let \((\tilde{M}, \tilde{\nabla})\) be a differential module over \(\mathcal{H}(\xi)\). Fix a \(H(\xi)\)-linear isomorphism \(\mathcal{H}(\xi) \cong \tilde{M}\) (i.e. a basis of \(\tilde{M}\)), and let \(\frac{d}{dT} - G(\tilde{T})\) be the map \(\tilde{\nabla}\) in this basis. Writing \(\mathcal{H}(\xi)^r = (\bigoplus_{k=0}^{p-1} \varphi^\#(\mathcal{H}(\varphi(\xi))) \cdot \mathcal{T}^k)^r\) one sees that if \(G(\tilde{T}) = (g_{i,j}(\tilde{T}))_{i,j=1,...,r} \in M_{r \times r}(\mathcal{H}(\xi))\), then the matrix of \(\varphi_*(\tilde{\nabla})\) is given by the block matrix

\[
\varphi_*(G)(\tilde{T}) = : (\varphi_*(g_{i,j})(\tilde{T}))_{i,j=1,...,r} \in M_{pr \times pr}(\mathcal{H}(\varphi(\xi))).
\]
(6.13)

### 6.4 The rank one modules \(\mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^{k/p}\).

We preserve the assumption 6.12. For all \(k = 0, \ldots, p-1\) the function \(y_k := T^{k/p}\) verifies \(\frac{d}{dT}(y_k) = \frac{k}{pT^{k/p}} y_k\). Let \(\mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^{k/p}\) be the corresponding differential module over \((\mathcal{H}(\varphi(\xi)), \frac{d}{dT})\). Since \([\mathcal{H}(\xi) : \mathcal{H}(\varphi(\xi))] = p\) one has a decomposition of \(\mathcal{H}(\varphi(\xi))\)-differential modules \(\varphi_*(\mathcal{H}(\xi)) = \bigoplus_{k=0}^{p-1} \mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^{k/p}\). In the sequel we often identify \(\tilde{T}\) with \(T^{1/p}\) and \(\mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^{k/p}\) with \(\varphi^\#(\mathcal{H}(\varphi(\xi))) \cdot \tilde{T}^k \subseteq \mathcal{H}(\xi)\). For all \(\mathcal{H}(\varphi(\xi))\)-differential module \(M\) we set \(M[k] := M \otimes_{\mathcal{H}(\varphi(\xi))} \mathcal{H}(\varphi(\xi)) \cdot T^{k/p}\).

**Lemma 6.14.** [Ked10b, Lemma 10.3.6]. One has the following properties:

i) For all \(\mathcal{H}(\xi)\)-differential module \(\tilde{M}\) and all \(k = 0, \ldots, p-1\) one has a canonical isomorphism of \(\mathcal{H}(\varphi(\xi))\)-differential modules \(\mu_k : \varphi_*(\tilde{M})[k] \cong \varphi_*(\tilde{M})\) defined by \(m \otimes h(T)T^{k/p} \mapsto h(T^p)\tilde{T}^km\).

ii) An \(\mathcal{H}(\varphi(\xi))\)-differential sub-module \(N \subseteq \varphi_*(\tilde{M})\) is the push-forward of a submodule of \(\tilde{M}\) if and only if \(N\) viewed as a subgroup of \(\tilde{M}\) is stable under the multiplication by scalars of \(\mathcal{H}(\xi)\).

This happens if and only if for all \(k = 0, \ldots, p-1\) one has \(\mu_k[N[k]] \subseteq N\).

iii) For all differential module \(M\) over \(\mathcal{H}(\varphi(\xi))\) one has \(\varphi_*(\varphi_* M) \cong \bigoplus_{k=0}^{p-1} M[k]\).

iv) For all differential module \(\tilde{M}\) over \(\mathcal{H}(\xi)\) one has \(\varphi^* \varphi_* \tilde{M} \cong \tilde{M}^\otimes p\).

**Proposition 6.15.** Let \(t\) be a Dwork generic point for \(\xi\). For all \(k = 1, \ldots, p-1\) one has \(\mathcal{R} \mathcal{H}(\varphi(\xi)) T^{k/p}(\varphi(\xi)) = \omega^p|t|^p\).

**Proof.** The solution of \(y_k'(T) = \frac{k}{p}y_k(T)\) around \(t^p\) is \(T^{k/p}\). Then apply Lemma 6.5. \(\square\)
6.5 Behavior of the radius by Frobenius push-forward

We preserve the assumption 6.12. Let \( t \) be a Dwork generic point for \( \xi = \xi_{c,p} \).

**Proposition 6.16.** Let \( \tilde{M} \) be a differential module over \( \mathcal{H}(\xi) \). Assume that \( \mathcal{R}_{i_{1}}^{\tilde{M},sp}(\xi) \leq \omega |t| < \mathcal{R}_{i_{1}+1}^{\tilde{M},sp}(\xi) \). Then, up to a permutation, the list of the spectral radii of \( \varphi_{*} \tilde{M} \) is given by

\[
\bigcup_{i \geq i_{1}} \left\{ p^{i} |t|^{p-1} \mathcal{R}_{i}^{\tilde{M},sp}(\xi), \ldots, p^{i} |t|^{p-1} \mathcal{R}_{p}^{\tilde{M},sp}(\xi) \right\} \bigcup_{i > i_{1}} \left\{ \mathcal{R}_{i}^{\tilde{M},sp}(\xi), \omega^{p} |t|^{p}, \ldots, \omega^{p} |t|^{p} \right\}.
\]

(6.14)

**Proof.** The proof comes from [Ked10b, Thm. 10.5.1] with slide modifications. We reproduce it for the convenience of the reader. We can assume that \( M \) has no non trivial sub-objects, so \( \mathcal{R}_{1}^{M,sp}(\xi) = \cdots = \mathcal{R}_{r}^{M,sp}(\xi) = R \). Assume \( R > \omega |t| \). Let \( M \) be such that \( \tilde{M} = \varphi_{*}(M) \) (cf. Prop.6.11). Then \( M \) is simple, since a sub-object \( N \subset M \) produces a sub-object \( \varphi^{*}N \) of \( M \). Hence \( \mathcal{R}_{1}^{M,sp}(\varphi(\xi)) = \cdots = \mathcal{R}_{r}^{M,sp}(\varphi(\xi)) = R^{p} \). By Lemma 6.14 one has \( \varphi_{*} \tilde{M} = \varphi_{*}\varphi^{*}M = \varphi_{*}^{p} \mathcal{R}_{k}[M] \).

Each \( \mathcal{R}_{k}[M] = \varphi_{*} \tilde{M} \mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^{k/p} \) is irreducible since \( \mathcal{H}(\varphi(\xi)) \cdot \mathcal{T}^{k/p} \) has rank one. Since \( \mathcal{R}_{1}^{M,sp}(\varphi(\xi)) \cdot \mathcal{T}^{k/p}(\varphi(\xi)) = \left\{ \omega^{p} |t|^{p} \right\} \) if \( k \neq 0 \), and since \( \omega^{p} |t|^{p} < R^{p} \), then \( \mathcal{R}_{1}^{M,sp,sp}(\varphi(\xi)) = \left\{ \omega^{p} |t|^{p} \right\} \) if \( k \neq 0 \), because the radius of a tensor product is the minimum of the radii if they are different (cf. [Ked10b, Lemma 9.4.6]). This proves the result. Assume now that \( R \leq \omega |t| \). Let \( R^{'} := \max \left( \frac{p^{i} |t|^{p-1} R}{\omega}, |t| \right) \). One has a commutative diagram compatible with the derivations like (6.9) with \( A_{\Omega}(t_{c,p}, p) \) (resp. \( A_{\Omega}(t_{c,p}, \varphi([c, p])) \)) replaced by \( A_{\Omega}(t, R) \) (resp. \( A_{\Omega}(t^{p}, R^{p}) \)). The restriction \( \mathcal{H}(\varphi(\xi)) \to \mathcal{H}(\varphi(\xi)) \to \mathcal{H}(\varphi(\xi)) \to A_{\Omega}(t, R) \) \( \to A_{\Omega}(t^{p}, R^{p}) \) \( \to A_{\Omega}(t^{p}, R^{p}) \) (cf. Prop. 6.4, iii). Hence \( \varphi_{*} \tilde{M} \otimes \mathcal{H}(\varphi(\xi)) A_{\Omega}(t^{p}, R^{p}) = \varphi_{*} \varphi_{*} \tilde{M} \otimes \mathcal{H}(\varphi(\xi)) A_{\Omega}(t^{p}, R^{p}) = \mathcal{M} \otimes \mathcal{H}(\varphi(\xi)) A_{\Omega}(t^{p}, R^{p}) \) (cf. Lemma 6.14). By assumption \( \tilde{M} \) is trivialized by \( A_{\Omega}(t, R) \) and hence by \( A_{\Omega}(t^{p}, R^{p}) \), so \( \mathcal{R}_{\varphi^{*} M,sp}(\varphi(\xi)) \geq R^{p} \). If now \( 0 \neq N \subset \varphi_{*} \tilde{M} \), then \( \mathcal{R}_{N,sp}(\varphi(\xi)) \geq \mathcal{R}_{\varphi^{*} M,sp}(\varphi(\xi)) \geq R^{p} \). We claim that this is an equality for all \( N \). Since \( \varphi^{*} N \subset \varphi^{*} \varphi_{*} \tilde{M} = \mathcal{M} \), each simple sub-quotient of \( \varphi^{*} N \) appears among those of \( M \), and its radius is \( R \). So \( \mathcal{R}_{\varphi^{*} N,sp}(\xi) = R \leq |t| \) and by Proposition 6.9 one has \( \mathcal{R}_{\varphi^{*} N,sp}(\varphi(\xi)) = R^{p} \).

6.6 Stability of the directional finiteness and harmonicity by Frobenius push-forward

An affinoid \( X \) is called a *pseudo-annulus* if \( \Gamma_{X} \) has, at most, a unique bifurcation point \( \xi \), and if \( \Gamma_{X} \) has no punctured smooth points other than possibly \( \xi \). A pseudo-annulus is obtained from an annulus \( X' := \{|T - t| \in [R_{1}, R_{0}]\} \), \( t \in K^{alg} \), by removing a finite number of disks \( D_{i}(c_{i}, R_{i}) \), \( c_{i} \in K^{alg} \), in order that the bifurcation point is \( \xi = \xi_{0,\overline{\tau}} \) with \( \overline{\tau} := \{ |c_{i} - t| = |c_{j} - t| \} \) for all \( i, j = 2, \ldots, \mu \). We now assume \( t = 0 \) and \( \xi = \xi_{0,\overline{\tau}} \). Let \( X^{p} := \varphi(X) \). One defines as above the push-forward functor

\[
\varphi_{*} : \frac{d}{dT} - \text{Mod}(\theta(X)) \overset{\sim}{\longrightarrow} \frac{d}{dT} - \text{Mod}(\theta(X^{p})) \overset{\sim}{\longrightarrow} \frac{d}{dT} - \text{Mod}(\theta(X^{p})) .
\]

(6.15)

If \( \xi' \in X \) is verifies the assumption 6.12, then (6.15) is compatible with (6.11) at \( \xi' \). By Proposition 6.1 the map \( \varphi : X \to X^{p} \) identifies the directions through \( \xi \) and \( \varphi(\xi) \):

\[
\varphi : \Delta(\xi) \overset{\sim}{\longrightarrow} \Delta(\varphi(\xi)) , \quad \xi_{c,p} \mapsto \xi_{c',p'} ,
\]

(6.16)

For \( \rho \) close to \( \overline{\tau} \), \( \xi_{c,p} \) verifies the assumption 6.12, and \( [\mathcal{H}^{1}\xi_{c,p} : \mathcal{H}(\xi_{c',p'})] = p \) by Prop. 6.8.

**Proposition 6.17.** Let \( X \) be a pseudo-annulus and \( \xi := \xi_{0,\overline{\tau}} \) as above. Let \( \tilde{M} \) be a differential module over \( \theta(X) \). Let \( i_{0} \) (resp. \( i_{1} \)) be the largest value of \( i \) such that \( \mathcal{R}_{i_{1}}^{\tilde{M}}(\xi) < \omega(\xi) = \overline{\rho} \) (resp. \( \mathcal{R}_{i_{1}}^{\tilde{M}}(\xi) \leq \omega(\xi) = \omega(\overline{\rho}) \)). For \( i \leq i_{0} \) we define \( \phi(i) := \left\{ \begin{array}{ll} p_{i} & \text{if } 1 \leq i_{1} < i_{0}, \\ \overline{p}_{r} & \text{if } i_{1} < i \leq i_{0} \end{array} \right. \)

\footnote{It is understood that if \( i_{1} = i_{0} \), then \( \phi(i) = p_{i} \) and \( d_{i} = i \) for all \( i \in \{1, \ldots, i_{0}\} \).}
\[ f_i(\overline{T}) := (p_{\overline{T}}^{(p-1)})^{d_i} \in \mathcal{O}(X). \]
Let \( i \in \{1, \ldots, i_0\} \), then for all \( c \in X(\Omega) \), \( |c| \leq p \), for all \( \rho \) close enough to \( p \), one has

\[ H^{\hat{M}}_i(\xi_{c,\rho}) = H^{\varphi^{\hat{M}}}_{\phi(i)}(\xi_{c,\varphi^{\hat{M}}})^{1/p}/|f_i(\xi_{c,\rho})|. \]  

(6.17)

Hence if \( \delta \in \Delta(\xi) \) is the direction defined by \( \Lambda(\xi_c) \), and if \( \delta' \in \Delta(\varphi(\xi)) \) is the corresponding direction defined by \( \Lambda(\xi_{\varphi}) \), then \( \partial_+ H^{\hat{M}}_i(\xi_{0,\overline{p}}) = \partial_+ H^{\varphi^{\hat{M}}}_{\phi(i)}(\xi_{0,\overline{p}}') - d_i \), and

\[ \partial_- H^{\hat{M}}_{i,\delta}(\xi_{0,\overline{p}}) = \begin{cases} 
\partial_- H^{\varphi^{\hat{M}}}_{\phi(i),\delta}(\xi_{0,\overline{p}}') - d_i & \text{if } \delta \text{ is defined by } \Lambda(\xi_c) \text{ with } c = 0 \\
\partial_- H^{\varphi^{\hat{M}}}_{\phi(i),\delta}(\xi_{0,\overline{p}}') & \text{if otherwise}
\end{cases} \]  

(6.18)

Here the log-slopes \( \partial_+ H^{\hat{M}}_{i,\delta}(\xi_{0,\overline{p}}) \) and \( \partial_- H^{\hat{M}}_{i,\delta}(\xi_{0,\overline{p}}) \) are computed with respect to the variable \( \tau = \ln(\rho) \) (cf. (2.7)), while the slopes \( \partial_+ H^{\varphi^{\hat{M}}}_{\phi(i)}(\xi_{0,\overline{p}}') \) and \( \partial_- H^{\varphi^{\hat{M}}}_{\phi(i),\delta}(\xi_{0,\overline{p}}') \) are computed with respect to the variable \( \tau' := \ln(p^{\overline{p}}) \). Indeed the natural variable on \( \mathcal{O}(X^{\overline{p}}) \) is \( T = T^{\overline{p}} \). Moreover:

i) There are a finite number of directions \( \delta \in \Delta(\xi) \) such that \( \partial_- H^{\hat{M}}_{i,\delta}(\xi) \neq 0 \) if and only if the same is true for the directions \( \delta' \in \Delta(\varphi(\xi)) \) such that \( \partial_- H^{\varphi^{\hat{M}}}_{\phi(i),\delta}(\varphi(\xi)) \neq 0 \).

ii) \( (i, H^{M}_i(\xi)) \) is a vertex of \( NP^{\text{conv}}(\hat{M}, \xi) \) (i.e. \( i = r \) or \( s^M_i(\xi) < s^M_{i+1}(\xi) \) if and only if \( \phi(i), H^{\varphi^M_{\phi(i)}}(\varphi(\xi)) \) is a vertex of \( NP^{\text{conv}}(\varphi(\hat{M}), \varphi(\xi)) \) (i.e. \( \phi(i) = pr \) or \( s^M_{\phi(i)}(\varphi(\xi)) < s^M_{\phi(i)+1}(\varphi(\xi)) \)).

iii) If \( K = K^{\text{alg}} \), and if \( \xi \in X_{\text{int}} \) is not in the Shilov boundary of \( X \), then \( f_i \) is harmonic at \( \xi \) with slopes in \( Z \). Hence \( H^{\hat{M}}_i \) is super-harmonic (resp. harmonic, has slopes in \( Z \)) at \( \xi := \xi_{0,\overline{p}} \) if and only if \( H^{\varphi^{\hat{M}}}_{\phi(i)} \) is super-harmonic (resp. harmonic, has slopes in \( Z \)) at \( \varphi(\xi) = \xi_{0,\overline{p}} \).

The same holds for the spectral polygon since for \( i \leq i_0 \) one has \( H^{M}_i(\xi) = H^{M,sp}_i(\xi) \), and for all \( j \leq \phi(i_0) \) one has \( H^j_i = H^j_i,sp(\xi) \).

Proof. Write \( s^M_{i_1}(\xi) \leq \cdots \leq s^M_{i_0}(\xi) \leq \omega|t| < s^M_{i_0+1}(\xi) \leq \cdots \leq s^M_{i_0+1}(\xi) < \ln(\overline{p}) \leq s^M_{i_0+1}(\xi) \leq \cdots \leq s^M_\xi(\xi) \). For all \( i = 1, \ldots, i_0 \) one has \( \mathcal{R}^M_i = \mathcal{R}^M_{i,sp} \) along a conveniently small open segment containing \( \xi = \xi_{0,\overline{p}} \) of each branch through \( \xi \). By Prop. 6.16 for all \( \xi' \) belonging to such segments one has

\[ s^{\varphi^{\hat{M}}}_{\phi(i)}(\varphi(\xi')) : \]

\[ \leq \ln(|\omega||t'||^p|) < ps^M_{i_0+1}(\xi') \leq \cdots \leq ps^M_{i_0}(\xi') < p\ln(\overline{p}) \leq \cdots 
\]

(6.19)

where \( t' \) is a Dwork generic point for \( \xi' \). Hence \( h^{\varphi^{\hat{M}}}_{\phi(i)}(\varphi(\xi')) = p \cdot h^{\hat{M}}_i(\xi') + p \cdot i \cdot \ln(|\omega||t'||^p|) \) for all \( i \leq i_1 \). And if \( i_1 < i \leq i_0 \), then

\[ h^{\varphi^{\hat{M}}}_{\phi(i)+1}(\varphi(\xi')) = h^{\varphi^{\hat{M}}}_{\phi(i)}(\varphi(\xi')) + (p-1)(r-i_1)\ln(\omega^p|t'||^p) + ps^M_{i_0+1}(\xi') + \cdots + ps^M_{i_0}(\xi') \]

(6.22)

\[ = p \cdot h^{\hat{M}}_i(\xi') + p \cdot i_1 \cdot \ln(|\omega||t'||^p|) + (p-1)(r-i_1)\ln(\omega^p|t'||^p) \]

(6.23)

\[ = p \cdot h^{\hat{M}}_i(\xi') + p \cdot r \cdot \ln(|\omega||t'||^p|) .
\]

(6.24)

This proves (6.17). This gives \( H^{\hat{M}}_{i,\delta}(\rho) = H^{\varphi^{\hat{M}}}_{\phi(i),\delta}(\rho^{p})^{1/p}/|f_i(\xi_{c,\rho})| \), for all \( \rho \to \overline{p} \). So (6.18) holds. □
7. Proof of the main Theorem 4.7

7.1 Structure of the proof

Remark 7.1. Let $\mathcal{R}^M_i : X \to \mathbb{R}^i$ be defined by $\mathcal{R}^M_i(\xi) := (R^M_i(\xi), \ldots, R^M_i(\xi))$. Defines analogously $H^M_i, s^M_i, h^M_i$. Clearly $p_{R^M_i}(\xi) = \min_{j=1, \ldots, i} p_{R^M_i}(\xi)$, so that $\Gamma(R^M_i) = \bigcup_{j=1, \ldots, i} \Gamma(R^M_i(\xi))$. Hence the finiteness of $\mathcal{R}^M_i$ is equivalent to the finiteness of all $\mathcal{R}^M_i$. The same holds for $H^M_i, s^M_i, h^M_i$. Of course $\mathcal{R}^M_i$ and $H^M_i$ are the exponential of $s^M_i$ and $h^M_i$ respectively, so we are reduced to prove the finiteness of these latter. The functions $s^M_i$ and $h^M_i$ are related by the bijective map $h^M_i(\xi) = U \cdot s^M_i(\xi)$, where $U \in GL_\nu(\mathbb{Z})$ is the matrix $U = (u_{ij})$ with $u_{ij} = 1$ if $i \geq j$ and $u_{ij} = 0$ otherwise. This proves that

$$\Gamma_i := \Gamma(R^M_i) = \Gamma(H^M_i) = \Gamma(h^M_i) = \Gamma(s^M_i) \quad \text{for all } i = 1, \ldots, r. \quad (7.1)$$

The aim is to apply Theorem 2.14 to the function $H^M_i$ with respect to $\Gamma := \Gamma_{i-1}$, and a convenient finite set $\mathcal{C}_i \subseteq \Gamma_{i-1}$. The proof is an induction on $i$. The first step is Proposition 7.8 proving the claims for $H^M_1 = R^M_1$ with respect to $\Gamma_0 := \Gamma_X$, and $\mathcal{C}_1$ equal to the Shilov boundary.

- It easy to prove (C1),(C2),(C4), this is done in section 7.2.

- The proof of (C3) is done in section 7.4. For $\mathcal{R}^M_1 = H^M_1$, (C3) is the concavity of the radius outside $\Gamma_0 = \Gamma_X$, and coincides with transfer (3.11). Now the behavior of $H^M_i$ along a branch is not concave since it is “perturbed” by the variation of the other $H^M_j$, with $j < i$. The idea is then to study the locus $\Gamma_{i-1}$ outside which the first $i-1$ radii are all constants. By Remark 7.1 the finiteness of all the $\Gamma_i$ is equivalent to that of all the $\Gamma(H^M_i)$. So from now on we prove the finiteness of the $\Gamma_i = \Gamma_{i-1} \cup \Gamma(H^M_i)$. Proposition 7.5 proves that $H^M_i$ behaves as a genuine radius (of a direct factor of $M$) outside $\Gamma_{i-1}$, in particular $H^M_3$ is concave outside $\Gamma_{i-1}$, and (C3) can be considered as a sort of “relative transfer” satisfied by $H^M_i$ with respect to $\Gamma_{i-1}$.

- It remains to prove the direction finiteness (C5) and the super-harmonicity (C6) of $H^M_i$. They are both local properties at a point $\xi \in \Gamma(H^M_i)$, and we can assume that $M$ is always cyclic by section 5.4.1. For this we distinguish 2 cases: $\mathcal{R}^M_i(\xi) < r(\xi)$ and $\mathcal{R}^M_i(\xi) \geq r(\xi)$.

* If $\mathcal{R}^M_i(\xi) < r(\xi)$ one applies Frobenius push-forward to make $\mathcal{R}^M_i, \ldots, \mathcal{R}^M_i$ small, and apply Proposition 4.4 (via Thm. 5.1 iii)). This proves that $H^M_i$ is super-harmonic at $\xi$, and that it has a finite number of non zero slopes $\partial H^M_i(\xi)$ (cf. section 7.3). To prove that $\Gamma_i$ is directionally finite at $\xi$, we can forget the directions belonging to $\Gamma_{i-1}$ because this last is finite by induction. Since $H^M_i$ is concave outside $\Gamma_{i-1}$ (by relative transfer (C3)) the other directions belong to $\Gamma_i$ (which coincides with $\Gamma(H^M_i)$ outside $\Gamma_{i-1}$) if and only if $\partial H^M_i(\xi) < 0$, so $\Gamma_i$ is directionally finite at $\xi$ (i.e. (C6)). This is explained in Prop.7.10.

* If $\mathcal{R}^M_i(\xi) \geq r(\xi)$, Lemma 7.7 essentially guarantee that $\Gamma_i = \Gamma_{i-1}$ around $\xi$, or, if $\xi \notin \Gamma_{i-1}$, $\xi$ is an end point of $\Gamma_i$. So in this case the direction finiteness (C5) is easy. The super-harmonicity of $H^M_i$ is more delicate, and it holds only outside a particular finite set $\mathcal{C}_i$ contained in $\Gamma_{i-1}$ as prescribed by Thm. 2.14. This is explained in Proposition 7.11.

7.2 Proof of Theorem 4.7 up to the finiteness and super-harmonicity.

The functions $\mathcal{R}^M_i$ and $H^M_i$ are insensitive to scalar extension of the ground field $K$, so one can always assume that the branch $\Lambda(t)$ satisfies $t \in X(K)$. So from (4.10) the function $\mathcal{R}^M_i$ acquires immediately all the properties of $\mathcal{R}^M_{i+1} sp$ along a branch. In fact, similarly to the picture (3.13), these two functions differs from an individual slope which is in both cases equal to 1 or 0. The slopes of $H^M_i$ differs from those of $H^M_{i+1} sp$ by an integer, hence ii) and iv) of Theorem 4.7 are a straightforward consequence of Thm. 5.6, and also the fact that $\mathcal{R}^M_i$ and $H^M_i$ verify (C2) and (C4).

Remark 7.2. For $i = 2, \ldots, r$ the functions $\mathcal{R}^M_{i,t}$ and $\mathcal{R}^M_{i,sp}$ are possibly not concave nor monotone.
in } = \infty \rho_{t,X} [\cdot]. The picture of } \mathcal{R}_{i,\xi}^M \text{ will present then a great difference with respect to that of } \mathcal{R}_{i}^M \text{ (cf. (3.13)). If } i = 1, \text{ then } \mathcal{R}_{1}^M \text{ satisfies } (C3) \text{ with } \Gamma := \Gamma_X \text{ (cf. section 3.1).}

**Remark 7.3.** A part of v) and vi) of Thm. 4.7 is automatic from Thm. 5.6 if } \xi \in X_{\text{int}} \text{ admits, as a neighborhood, an open annulus. However we are going to reproduce entirely the general proof.}

We are now reduced to prove the claims of Theorem 4.7 concerning the directional finiteness (cf. (C5)), the finiteness, the super-harmonicity, and property (C3) (cf. points iii) v), vi)).

### 7.3 Super-harmonicity if } \mathcal{R}_{i}^M(\xi) < r(\xi).

**Proposition 7.4.** Let } \xi \in X_{\text{int}}. \text{ If } \mathcal{R}_{i}^M(\xi) < r(\xi) \text{ then for all } j = 1, \ldots, i \text{ one has } \partial_- H_{j,\delta}^M(\xi) \neq 0 \text{ only for a finite number of directions } \delta \in \Delta(\xi). \text{ If } K = \widehat{K}_{\text{alg}}, \text{ and if } \xi \in X_{\text{int}} \text{ is not in the Shilov boundary of } X, \text{ then } H_{j}^M \text{ is super-harmonic at } \xi. \text{ If moreover } (i, h_{i}^M(\xi)) \text{ is a vertex of } NP^\text{conv}(M, \xi) \text{ (i.e. } i = r \text{ or } s_{i}^M(\xi) < s_{i+1}^M(\xi)), \text{ then } H_{i}^M(\xi) \text{ is harmonic at } \xi.

**Proof.** One has } \mathcal{R}_{i}^M(\xi) < r(\xi) \leq \rho_{\xi,X'}, \text{ for all } X' \subseteq X. \text{ Hence by Prop. } 5.10 \text{ all the assertions are local at } \xi. \text{ So we can assume that } X \text{ is a pseudo-annulus with bifurcation or punctured smooth point } \xi \text{ (cf. Section 6.6), and that } M \text{ is cyclic defined by an operator } \mathcal{L} = \sum_{i=0}^{r} g_i \partial_t (d/dT)^i, \text{ if } g_0 = 1, g_i, g_i^{-1} \in \mathcal{O}(X) \text{ for all } i \text{ (cf. section 5.4.1). By (4.11), one has } \mathcal{R}_{i}^M(\xi) = \mathcal{R}_{i}^{M,\text{sp}}(\xi) < r(\xi). \text{ This equality is preserved by continuity in an open segment containing } \xi \text{ of each direction } \delta \in \Delta(\xi). \text{ We distinguish two cases: } \mathcal{R}_{i}^{M,\text{sp}}(\xi) < \omega \cdot r(\xi) \text{ and } \omega \cdot r(\xi) \leq \mathcal{R}_{i}^{M,\text{sp}}(\xi) < r(\xi). \text{ In the first case by Prop. 4.3 (cf. also Prop. 4.2 and 4.4, Thm. 5.1, Cor. 5.3) for all } \delta \in \Delta(\xi) \text{ and all } j = 1, \ldots, i \text{ one has}

\[ \partial_- \mathcal{R}_{j,\delta}^M(\xi) = \partial_- \mathcal{R}_{j,\delta}^{M,\text{sp}}(\xi) = \partial_- \mathcal{R}_{j,\delta}^{C,\text{sp}}(\xi), \quad \partial_- H_{j,\delta}^M(\xi) = \partial_- H_{j,\delta}^{M,\text{sp}}(\xi) = \partial_- H_{j,\delta}^{C,\text{sp}}(\xi). \quad (7.2) \]

The same equalities hold for the right slopes. Since } H_{j}^{C,\text{sp}} \text{ is directional finite (cf. Prop. 4.2), then } \partial_- H_{j,\delta}^M(\xi) = 0 \text{ up to a finite number of directions. If } \xi \in X_{\text{int}} \text{ is not in the Shilov boundary of } X, \text{ then } H_{j}^{C,\text{sp}} \text{ is super-harmonic at } \xi, \text{ or harmonic if } (i, h_{i}^M(\xi)) \text{ is a vertex, and so does } H_{j}^M. \text{ If the absolute value of } K \text{ extends the trivial valuation of } \mathbb{Z}, \text{ then } \omega = 1 \text{ and the proof is completed. If the absolute value of } K \text{ is } p\text{-adic, then } \omega < 1. \text{ In this case assume that } \omega r(\xi) \leq \mathcal{R}_{i}^M(\xi) < r(\xi). \text{ Up to a translation we can assume that } X \text{ is obtained form an annulus } \{ T \in [R_1, R_0] \} \text{ by removing some disks, and that } \xi = \xi_0 , \pi \text{ as in section 6.6. Then by applying several times the Frobenius push-forward we reduce the value of the radii in order to reproduce the above computations. Namely, with the notation of Proposition 6.17, let } h > 0 \text{ be the smallest integer such that } \mathcal{R}_{\phi^h(i)}^{(\varphi,\lambda)M,\text{sp}}(\varphi^h(\xi)) < \omega \cdot r(\varphi^h(\xi_0, \pi)) = \omega \varphi^h, \text{ where } \varphi^h \text{ and } \varphi^h \text{ denotes the } h\text{-times iterated of } \varphi \text{ and } \varphi \text{ respectively. Then by Prop. 6.17 the slopes of } H_{i}^M \text{ at } \xi \text{ are equal to those of } H_{\phi^h(i)}^{(\varphi,\lambda)M,\text{sp}} \text{ at } \varphi^h(\xi), \text{ up to the left and right slopes along } A(\xi_0). \text{ And } H_{i}^M \text{ is super-harmonic at } \xi \text{ if and only if so does } H_{\phi^h(i)}^{(\varphi,\lambda)M,\text{sp}} \text{ at } \varphi^h(\xi). \text{ Up to restrict } X^h \text{ we can assume that } (\varphi,\lambda)M \text{ is cyclic represented by an operator } \mathcal{L}^{(h)} \text{ with invertible coefficients in } \mathcal{O}(X^h). \text{ Moreover, by the choice of } h, \text{ we are now in the domain of applicability of Thm. 5.1, ii), and Cor. 5.3. Hence the slopes of } H_{\phi^h(i)}^{(\varphi,\lambda)M,\text{sp}} \text{ at } \varphi^h(\xi) \text{ are those of } \mathcal{L}^{(h)}, \text{ and by Prop. 4.2 } H_{\phi^h(i)}^{(\varphi,\lambda)M,\text{sp}} \text{, and hence } H_{\phi^h(i)}^{(\varphi,\lambda)M,\text{sp}}, \text{ are super-harmonic at } \varphi^h(\xi). \text{ By Prop. 6.17 } (i, h_{i}^M(\xi)) \text{ is a vertex if and only if } (\phi^h(i), h_{\phi^h(i)}^{(\varphi,\lambda)M}(\varphi^h(\xi))) \text{ is a vertex, this implies the last assertion.} \]

### 7.4 Property (C3) for } H_{i}^M

Let } \Gamma_0 := \Gamma_X \text{ and } \Gamma_{i} := \bigcup_{j=1}^{i} \Gamma(\mathcal{R}_{j}^M). \text{ Let } D^{-}(t, \rho) \subset X \text{ be a non generic disk on which } \mathcal{R}_{1}^M, \ldots, \mathcal{R}_{i-1}^M \text{ are constant i.e. } \overline{D^{-}(t, \rho)} \cap \Gamma_{i-1} = \emptyset. \text{ Let } b_0 := 1, \text{ and if } i \geq 1 \text{ let } b_i := \prod_{j=1}^{i-1} \mathcal{R}_{j}^M(\xi_t). \text{ Then } H_{i}^M = b_{i-1} \cdot \mathcal{R}_{i}^M \text{ over } D^{-}(t, \rho). \text{ Both functions then have the same properties on } D^{-}(t, \rho).
Proposition 7.5. If $K = \widehat{K}_{\text{alg}}$, then $R_i^M$ is either constant on $D^-(t, \rho)$ or there exists a direct sum decomposition $M \otimes_{\mathbb{C}(X)} A_K(t, \rho) = M_1 \oplus M_2$ such that $R_i^M(\xi') = R_i^{M_1}(\xi')$ for all $\xi' \in D^-(t, \rho)$. In particular $R_i^M$ and $H_i^M$ verify (C3) with respect to $\Gamma := \Gamma_{i-1}$, and they both enjoy all the properties of a genuine radius of convergence outside $\Gamma_{i-1}$. In particular $\Gamma(R_i^M) \cap D^-(t, \rho) \neq \emptyset$ if and only if $\partial_- R_i^{M, \delta}(\xi, t, \rho) < 0$ where $\delta \in \Delta(\xi, t, \rho)$ is the direction defined by the disk.

Proof. Since the disk is non generic we can assume $t \in X(K)$. The slopes of $R_i^M$ on the disk coincides with those of $H_i^M = b_{i-1} R_i^M$. Hence, by section 7.2, $R_i^M$ verifies point iv) of Theorem 4.7 over $D^-(t, \rho)$. Assume that $R_i^M$ is non constant on $D^-(t, \rho)$. Then $R_i^M(\rho') > R_i^{M_1}(\rho')$, for all $\rho' \in [R_i^{M_1}(t), \rho]$. Otherwise point iv) of Theorem 4.7 is contradicted. By Theorem 5.4 there exists a direct sum decomposition $M \otimes_{\mathbb{C}(X)} A_K(t, \rho) = M_1 \oplus M_2$ of $A_K(t, \rho)$-differential modules such that for all $\rho' \in [R_i^{M_1}(t), \rho]$ one has $R_{i, t}^{M_1}(\rho') = R_{i, t}^{M_2}(\rho')$, for $k = 1, \ldots, r - i + 1 = \text{rank}(M_1)$, and $R_{i, t}^{M_2}(\rho') = R_{i, t}^{M_1}(\rho')$, for $j = 1, \ldots, i - 1 = \text{rank}(M_2)$. By Lemma 7.6 below the radius $R_i^M$ is equal to $R_i^{M_1}$ on the whole disk, and hence it enjoys its properties. Note that by (4.9) the non constancy of $R_i^M$ on $D^-(t, \rho)$ implies that $R_i^M(\xi') < \rho$ for all $\xi' \in D^-(t, \rho)$, and hence $R_i^M = R_i^{M_0} A_K(t, \rho)$ on $D^-(t, \rho)$ (cf. Def. 4.5). So $\Gamma(R_i^{M_0} A_K(t, \rho)) = \Gamma(R_i^M) \cap D^-(t, \rho)$. Now from Lemma 2.15 this intersection is not empty if and only if $\partial_- R_i^{M, \delta}(\xi, t, \rho) < 0$.

Lemma 7.6. For all $\xi' \in D^-(t, \rho)$ one has $R_i^{M_1}(\xi') = R_i^M(\xi')$.

Proof. By Prop. 5.5 the convergence radii of $M$ at $\xi'$ are the union (with multiplicities) of those of $M_1$ and of $M_2$. So it is enough to prove that for all $\xi' \in D^-(t, \rho)$ one has $R_{i-1}^{M_1}(\xi') < R_i^{M_1}(\xi')$. Indeed this implies by Prop. 5.5 that $R_j^M(\xi') = R_j^{M_1}(\xi')$ for all $j = 1, \ldots, i - 1$, and $R_i^{M_1}(\xi') = R_i^{M_2}(\xi')$ for all $k = 1, \ldots, r - i + 1$. Since $N^p_{\text{conv}}$ is insensitive to scalar extensions of $K$, one can assume that $\xi' = \xi'$ with $t' \in X(K)$. By changing the center one can assume that $t = t'$. Now $R_i^{M_1}$ is log-concave with negative log-slopes along $[0, \rho]$. Then for all $\rho'$ close enough to $\rho$ one has $R_i^{M_1}(\xi') = R_i^{M_1}(\xi') = R_i^{M_1}(\rho') = R_i^{M_1}(\rho') > R_i^{M_1}(\xi') = R_i^{M_1}(\xi')$ as desired.

7.5 Finiteness and super-harmonicity (end of proof)

Lemma 7.7. If $R_i^M(\xi) \geq r(\xi)$, then either $\xi \notin \Gamma(R_i^M)$ or

i) If $\xi \in \Gamma(R_i^M) - \Gamma_{i-1}$, then $\xi$ is a boundary point of $\Gamma(R_i^M)$;

ii) If $\xi \in \Gamma_{i-1} \cap \Gamma(R_i^M)$, then $\Delta(\xi, \Gamma(R_i^M)) \subseteq \Delta(\xi, \Gamma_{i-1})$.

Proof. It is enough to prove that $R_i^M$ is constant on each non generic disk $D^-(t, \rho)$ tangent to $\xi = \xi_{i, \rho}$ such that $D^-(t, \rho) \cap \Gamma_{i-1} = \emptyset$. Up to enlarge $K$ we can assume $K = \widehat{K}_{\text{alg}}$, and $t \in X(K)$ in order that $\rho = r(\xi)$. By Prop. 7.5 the function $R_i^M$ enjoys concavity properties on $D^-(t, \rho)$. So $R_i^M(\xi) \geq R_i^M(\xi_{i, \rho}) = R_i^M(\xi) \geq r(\xi) = \rho$. Hence by (4.9) $R_i^M$ and $H_i^M$ are constant on $D^-(t, \rho)$.

Proposition 7.8. $R_i^M$ is continuous, $\Gamma(R_i^M)$ is finite and factorizes through it. If $K = \widehat{K}_{\text{alg}}$, then $R_i^M$ is super-harmonic on $X$. Moreover $R_i^M$ satisfies the claim vi) of Theorem 4.7.

Proof. We can assume $K = \widehat{K}_{\text{alg}}$. In order to apply Thm. 2.14 it remains to prove directional finiteness (C5) and super-harmonicity. By Lemma 7.7 if $R_i^M(\xi) \geq r(\xi)$, then $\Delta(\xi, R_i^M)$ is finite in all the cases. The super-harmonicity in these cases is proved as follows. If $\xi \notin \Gamma(R_i^M)$ there is nothing to prove. If $\xi \in \Gamma(R_i^M) - \Gamma_X$ is a boundary point of $\Gamma(R_i^M)$, the super-harmonicity coincides with the concavity (cf. Prop. 2.15). If $\xi \in \Gamma_X$, then $\Delta(\xi, R_i^M) = \Delta(\xi, \Gamma_X)$, and all the directions through $\xi$, but those of $\Gamma_X$, are flat. Then along a branch $\Lambda(\xi_{i, \rho}, R_i)$ of $\Gamma_X$ the function $\rho \mapsto R_{\xi_i, i, \rho}(\rho)$ is bounded by $\rho \mapsto \rho \xi_{i, \rho}, X = \rho$ and it is equal to it at the value $\rho = \overline{\rho}$ for which
\( \xi = \lambda_{\xi_i, R_i}(\mathcal{P}) \). Then \( \partial_\xi \mathcal{R}_M(\xi) \leq 1 \) and \( \partial_{-\xi} \mathcal{R}_M(\xi) \geq 1 \) for all direction \( \delta \) corresponding to a branch of \( \Gamma_X \). Assume now \( \mathcal{R}_M(\xi) < r(\xi) \). Apply Proposition 7.4 to prove the super-harmonicity of \( \mathcal{R}_M = H_i^M \) at \( \xi \), and that \( \partial_{-\xi} \mathcal{R}_M(\xi) \neq 0 \) only for a finite number of directions \( \delta \in \Delta(\xi) \). By Proposition 7.5 \( \delta \in \Delta(\xi, \Gamma(\mathcal{R}_M)) - \Delta(\xi, \Gamma_X) \) if and only if \( \partial_{-\xi} \mathcal{R}_M(\xi) < 0 \). This proves the directional finiteness (C5). Finally assume that \( (1, h_i^M(\xi)) \) is a vertex (i.e. \( r = 1 \) or \( s_i^M(\xi) < s_{i+1}^M(\xi) \)) and \( \mathcal{R}_M(\xi) \neq r(\xi) \). If \( \mathcal{R}_M(\xi) > r(\xi) \), then by (3.5) \( \mathcal{R}_M \) is constant on the non generic open disk \( \mathcal{D}^*(i, \mathcal{R}_M(\xi)) \) that contains \( \xi \), and there is nothing to prove. If \( \mathcal{R}_M(\xi) < r(\xi) \) the harmonicity follows by Prop. 7.4.

**Corollary 7.9.** Assume the \( M \) is of rank \( r = 1 \), and that \( \xi \notin \Gamma_X \). Then \( \xi \) is a boundary point of \( \Gamma(\mathcal{R}_M) \) if and only if \( \mathcal{R}_M(\xi) = r(\xi) \) and \( \partial_{-\xi} \mathcal{R}_M(\xi) < 0 \).

**Proof.** If \( \mathcal{R}_M(\xi) = r(\xi) \) and \( \partial_{-\xi} \mathcal{R}_M(\xi) < 0 \), by Lemma 2.15, \( \mathcal{R}_M(\xi) = r(\xi) \), hence \( \xi = \mathcal{C}_M(\xi) \in \Gamma(\mathcal{R}_M) \). Now \( \xi \) lies in the boundary of \( \Gamma(\mathcal{R}_M) \) by Lemma 7.7. Reciprocally by Lemma 2.15 a boundary point \( \xi \) of \( \Gamma(\mathcal{R}_M) \) not in \( \Gamma_X \) verifies \( \partial_{-\xi} \mathcal{R}_M(\xi) = 0 \) for all \( \delta \in \Delta(\xi) \), and \( \partial_{-\xi} \mathcal{R}_M(\xi) < 0 \). So \( \mathcal{R}_M \) is not harmonic at \( \xi \), and if by contrapositive \( \mathcal{R}_M(\xi) \neq r(\xi) \) this contradicts point vi) of Thm. 4.7 (cf. Prop. 7.8), so one must have \( \mathcal{R}_M(\xi) = r(\xi) \).

**Proposition 7.10.** If \( H_i^M, H_{i-1}^M \) are finite, then \( H_i^M \) is directionally finite.

**Proof.** By assumption \( \Gamma_{i-1} \) is finite. If \( \xi \notin \Gamma_{i-1} \), then, \( \mathcal{R}_i^M, \mathcal{R}_{i-1}^M \) being constant outside \( \Gamma_{i-1} \), the function \( H_i^M \) is directionally finite at \( \xi \) if and only if so does \( \mathcal{R}_i^M \). By Prop. 7.5 the \( \mathcal{R}_i^M \) acquires the properties of a genuine radius outside \( \Gamma_{i-1} \), so it is finite and super-harmonic on each non generic disk \( \mathcal{D}^*(i, \rho) \) tangent to \( \Gamma_{i-1} \). Let \( \xi \in \Gamma_{i-1} \). If \( \mathcal{R}_i^M(\xi) \geq r(\xi) \) then one applies Lemma 7.7 to prove the directional finiteness of \( H_i^M \) since \( \Gamma(H_i^M) - \Gamma_{i-1} = \Gamma(\mathcal{R}_i^M) - \Gamma_{i-1} \) (cf. Remark 7.1). If \( \mathcal{R}_i^M(\xi) < r(\xi) \), by Prop. 7.5 one has \( \partial_{-\xi} H_i^M(\xi) < 0 \) for all \( \delta \in \Delta(\xi, \Gamma(H_i^M)) - \Delta(\xi, \Gamma_{i-1}) \). Now by Prop. 7.4 there are a finite number of directions \( \delta \in \Delta(\xi) \) such that \( \partial_{-\xi} H_i^M(\xi) \neq 0 \). Hence \( \Delta(\xi, \Gamma(H_i^M)) \) is finite.

**Proposition 7.11.** If \( H_i^M, H_{i-1}^M \) satisfy Theorem 4.7, then so does \( H_i^M \).

**Proof.** It is enough to prove that \( H_i^M \) verifies claims v) and vi) of Theorem 4.7. This guarantee that \( H_i^M \) fulfill the assumptions (C1)–(C6) of Thm. 2.14 with respect to \( \Gamma := \Gamma_{i-1} \) and \( \mathcal{C}(H_i^M) := \mathcal{C}_i \).

For this we assume \( K = \mathcal{K}_a\mathcal{C}_i \). We distinguish three cases: \( \mathcal{R}_M^i(\xi) < r(\xi) \), \( \mathcal{R}_M^i(\xi) = r(\xi) \), and \( \mathcal{R}_M^i(\xi) > r(\xi) \). Assume first that \( \mathcal{R}_M^i(\xi) < r(\xi) \). By Prop. 7.4 \( H_i^M \) is super-harmonic at \( \xi \) and it also enjoys property vi) of Thm. 4.7. Assume now that \( \mathcal{R}_M^i(\xi) > r(\xi) \). By (4.9) \( \mathcal{R}_M^i \) is constant on the disk non generic \( \mathcal{D}^*(i, \mathcal{R}_M^i(\xi)) \) that contains \( \xi \), and hence \( H_i^M = H_{i-1}^M \cdot \mathcal{R}_i^M(\xi) \) is super-harmonic at \( \xi \) and only if so does \( H_{i-1}^M \). This happens if \( \xi \notin \mathcal{C}_i \subseteq \mathcal{C}_i \) so v) of Thm. 4.7 is fulfilled. We now check vi). Assume that \( r(\xi) \notin \{\mathcal{R}_M^i(\xi)\}_{j=1,...,i} \) and that \( (i, h_i^M(\xi)) \) is a vertex. Let \( i_0 \) be the largest value of \( j \) such that \( \mathcal{R}_j^i(\xi) < r(\xi) \), or \( i_0 = 0 \) if \( \mathcal{R}_M^i(\xi) > r(\xi) \). By Prop. 7.4 \( H_i^M \) is harmonic at \( \xi \). Since all the functions \( \mathcal{R}_{i_0+1}^M, ..., \mathcal{R}_i^M \) are flat at \( \xi \), then \( H_i^M = H_{i_0}^M \cdot \prod_{j=i_0+1}^i \mathcal{R}_j^M \) is harmonic at \( \xi \). This concludes the case \( \mathcal{R}_M^i(\xi) > r(\xi) \). Assume now that \( \mathcal{R}_M^i(\xi) = \xi \). In this case we only have to check property v) of Thm. 4.7. We analyze the possible cases. If \( \xi \notin \Gamma(H_i^M) \), then \( H_i^M \) is flat at \( \xi \), and hence harmonic. If \( \xi \notin \Gamma_{i-1} \), then \( H_{i-1}^M \) is flat, and \( \mathcal{R}_i^M \) enjoys the properties of a radius (cf. Prop. 7.5) and it is super harmonic at \( \xi \). So \( H_i^M = H_{i-1}^M \cdot \mathcal{R}_i^M \) is super-harmonic at \( \xi \).

If \( \xi \notin \Gamma(H_i^M) \), then \( \mathcal{R}_i^M \) is flat at \( \xi \), and by induction \( H_{i-1}^M \) is super-harmonic outside \( \mathcal{C}_{i-1} \subseteq \mathcal{C}_i \). So if \( \xi \notin \mathcal{C}_i \), then \( H_i^M = H_{i-1}^M \cdot \mathcal{R}_i^M \) is super-harmonic at \( \xi \). Assume then that \( \mathcal{R}_i^M(\xi) = r(\xi) \), and that \( \xi \in \Gamma(H_i^M) \cap \Gamma_{i-1} \). We have to prove that \( H_i^M \) is super-harmonic outside \( \mathcal{C}_{i-1} \) and of the boundary of \( \Gamma(\mathcal{R}_i^M) \). Since \( \xi \notin \mathcal{C}_{i-1} \), then \( H_{i-1}^M \) is super-harmonic at \( \xi \), so it is enough to prove that \( \mathcal{R}_i^M \) is super-harmonic at \( \xi \). By Lemma 7.7 we know that \( \Delta(\xi, \Gamma(\mathcal{R}_i^M)) \subseteq \Delta(\xi, \Gamma_{i-1}) \).

Since \( \xi \) is not in the boundary of \( \Gamma(\mathcal{R}_i^M) \), then \( \Delta(\xi, \Gamma(\mathcal{R}_i^M)) \) is not empty. We now prove that for
all $\delta \in \Delta(\xi, \Gamma(\mathcal{R}_i^M))$ one has

$$\partial_+ \mathcal{R}_i^M(\xi) \geq 1 \geq \partial_- \mathcal{R}_i^M(\xi). \quad (7.3)$$

And hence that $\mathcal{R}_i^M$ is super-harmonic at $\xi$. Let $\Delta(\xi_t), t \in K$, be a representative branch of a direction $\delta \in \Delta(\xi, \Gamma(\mathcal{R}_i^M))$, and let $\xi = \xi_{t, \rho}$. If $\delta \in \Delta(\xi, \Gamma_X)$, then by definition $\mathcal{R}_i^M(\xi') \leq \rho_{\xi', X} = r(\xi')$ for all $\xi' \in \Gamma_X$, so that $\mathcal{R}_{i,t}(\rho) \leq \rho = r(\xi_{t, \rho})$ around $\rho$, and $\mathcal{R}_i^M(\xi) = \mathcal{R}_{i,t}^M(\rho) = \rho = r(\xi)$. This implies (7.3). Assume now that $\delta \in \Delta(\xi, \Gamma(\mathcal{R}_i^M)) - \Delta(\xi, \Gamma_X)$. Then one must have $\mathcal{R}_i^M(\xi) < \rho$ otherwise $\mathcal{R}_i^M$ is constant on $D^-(\rho, \xi)$ and $\delta \notin \Delta(\xi, \Gamma(\mathcal{R}_i^M))$. By (4.10) for all $\rho \geq \mathcal{R}_i^M(\xi)$ one has $\mathcal{R}_{i,t}(\rho) = \mathcal{R}_{i,t}^M(\rho) \leq \rho = r(\xi_{t, \rho})$, and as above $\mathcal{R}_i^M(\xi) = \mathcal{R}_{i,t}^M(\rho) = \rho = r(\xi)$. This implies (7.3). □

The above properties imply the finiteness of each $H_i^M$ (and hence of $\mathcal{R}_i^M$, $s_i^M$, $h_i^M$) by applying inductively Thm. 2.14 to $\mathcal{R} := H_i^M$, $\Gamma := \Gamma_{i-1}$, and $\mathcal{C}(\mathcal{R}) := \mathcal{C}_i$. This ends the proof of Thm. 4.7.

Proof of Corollary 4.8. By translation we can assume $t = 0$. Let $\mathcal{O}$ be equal to one of $\mathcal{H}_K(0, I), \mathcal{B}_K(0, I), \mathcal{A}_K(0, I)$. Almost all assertions can be proved from Thm. 4.7 by restriction to a sub-annulus (resp. sub-disk) $\{|T| \in J\}$, with $J$ compact (resp. $0 \in J$). The unique assertion that remains to prove are the global finiteness of each $H_i^M$, and the fact that along the branch $\lambda_0(0) := \{\lambda_{0, \rho}\}_{\rho \in I}$ each $H_i^M$ has a finite number of breaks. By Cor. 2.21, these two assertions are in fact equivalent, and this proves iii). This also proves the last assertion. Namely assume that, for $i \leq r$, all $H_i^M, \ldots, H_1^M$ have a finite number of breaks along $\lambda_0(I)$. Then $\mathcal{R}_i^M = H_i^M$ is finite by Cor. 2.21, and an induction on $k \leq i$ shows that $\Gamma(H_i^M)$ is contained in $A(0, J_k) \cup \lambda_0(I)$ for some compact $J_k$. Indeed each new branch generates by super-harmonicity a break along $\lambda_0(I)$. We now prove i) and ii). If $\mathcal{O} = \mathcal{H}_K(0, I)$ or if $\mathcal{O} = B_K(0, I)$ with $K$ discretely valued, then $\mathcal{R}_i^M$ and $H_i^M$ always have a finite number of breaks along $\lambda_0(I)$ by [Ked10b, Thm. 11.3.2, Remark 11.3.4]. The proof ends here, but for the convenience of the reader we now show why this is true. We assume that $I$ is open, since otherwise the result is Thm. 4.7. For all $i$ one has $\mathcal{R}_i^M = \mathcal{R}_i^{M, sp} \lambda_0(I)$. Let $L_i := \lim_{\rho \to s^+} \mathcal{R}_i^{M, sp} \lambda_0(\rho)$, where $s := \sup(I)$. The limit exists in $[0, s]$ since $H_i^{M, sp}$ is concave along $I$, for all $i$. If $L_1 = s$, then the sequence of slopes of $\mathcal{R}_i^{M, sp}$ is decreasing (by concavity), contained in $Z \cup 1/2Z \cup \ldots \cup 1/2Z$, and lower bounded by 1 (by definition of spectral radius). So the sequence of slopes is constant for $\rho \to s^-$: there is a “last slope” (this argument is due to Christol-Mebkhout [CM02]). So $\mathcal{R}_1^M$ is log-linear outside some compact $J_1 \subseteq I$, and hence finite by Cor. 2.21. Since $H_2^M$ is log-concave, then $\mathcal{R}_2^M$ is concave outside $J_1$, so the same argument proves that $\mathcal{R}_2^M$ is log-linear outside some $J_2 \supseteq J_1$, and hence finite by Cor. 2.21. By induction all $\mathcal{R}_i^M$ are finite. Assume now that $L_1 < s$. Let $i_0$ be the larger value of $i$ such that $L_i < s$. We perform Frobenius push-forward to have $L_{i_0} < s$. By continuity there exists $\varepsilon > 0$ such that $\mathcal{R}_i^M(\xi_{0, \rho}) < \omega \rho$ for all $\rho \in [s-\varepsilon, s]$. To prove that the number of breaks of $\mathcal{R}_1^M$ is finite on $[s-\varepsilon, s]$ one has to perform a global push-forward over $[s-\varepsilon, s]$ in order to control simultaneously all the $\{\mathcal{R}_i^M(\xi_{0, \rho})\}_{\rho \in [s-\varepsilon, s]}$. So one argue as in section 6.3 replacing $\mathcal{H}(\xi)$ by $\mathcal{H}_K([s-\varepsilon, s])$ or $\mathcal{B}_K([s-\varepsilon, s])$. One has the same results as in section 6.5. One sees from (6.12), that if the coefficients of $G(T)$ are bounded (resp. analytic elements), then so does $\varphi_*(G)(T)$. Indeed the sequence $\{a_i\}_i$ of the Taylor coefficients of the entries $h_{i,j}(T) = \sum_{i \in \mathbb{Z}} a_it^i$ of $\varphi_*(G)(T)$ are obtained as sub-sequences of those of $G(\bar{T})$. Now we perform a base change in the fraction field of $\mathcal{O}$, and possibly restrict the annulus, to find a differential operator that again has bounded coefficients (resp. analytic elements). Now bounded functions (resp. analytic elements) have a finite number of zeros, so the Newton polygon of the operator has a finite number of slopes. This proves that $\mathcal{R}_1^M, \ldots, \mathcal{R}_n^M$ have a finite number of breaks along $\lambda_0(I)$, and hence that they are finite by Cor. 2.21. For $i \geq i_0 + 1$ one proceeds inductively using the above argument of Christol-Mebkhout, to prove the finiteness of $\mathcal{R}_i^M$. □

19This condition is called solvability in [CM02].
20Bounded function have a finite number of zeros if and only if $K$ has a discrete valuation [Chr12].
8. S-skeleton

In [Bal10] and in [PP12] one considers a slight modified definition of \( i \)-th radii depending on a given skeleton \( \Gamma \) as follows. Let \( \mathfrak{f} \subset X_{\text{int}} \) be a finite subset and let \( \Gamma = \text{Sat}(\mathfrak{f}) \) be a finite and branch-closed saturated subset of the affinoid domain \( X \subseteq A_{K,\text{an}}^1 \). The union of all bifurcation, punctured smooth, and boundary points of \( \Gamma \) constitute the so called (weak) triangulation \( S \) of \( X \) (cf. [PP12]). The data of \( \Gamma \cup \Gamma_X \), or equivalently of \( S \), defines univocally a covering of \( X \) by (possibly not \( K \)-rational) closed annuli and closed disks such that

i) the union of their skeletons equals \( \Gamma \cup \Gamma_X \),

ii) the union of the boundary points of their skeletons is \( S \).

Imitating section 2 define the \( S \)-constancy radius of an arbitrary function \( R : X \to T \) as

\[
\rho_{R:S}(\xi) := \min(\rho_R(\xi), \rho_T(\xi)),
\]

and the \( S \)-skeleton \( \Gamma(R; S) \) of \( R \) as the image of the map \( \delta_{R:S} : X \to X \) defined by \( \delta_{R:S}(\xi) := \lambda_\xi(\rho_{R:S}(\xi)) \). From (8.1) one immediately has \( \Gamma(R; S) = \Gamma(R) \cup \Gamma \). So the finiteness of \( \Gamma(R) \) is equivalent to that of \( \Gamma(R; S) \).

Now let \( M \) be a differential module over \( \mathcal{O}(X) \) of rank \( r \). Define \( R_{S,i}^M(\xi) \) as the larger value of \( \rho \leq \rho_T(\xi) \) such that there exists \( \Omega \in E(K) \), and a Dwork generic point \( t \) for \( \xi \), in order that \( M \) has at least \( r - i + 1 \) linearly independent solution with values in \( A_\Omega(t, \rho) \). One sees that \( R_{S,i}^M(\xi) := \min(R_{i,S}^M(\xi), \rho_T(\xi)) \), so that by Remarks 2.4 and 2.3 one has\(^{21}\) \( \rho_{R_{S,i}^M}(\xi) \geq \min(\rho_T \cup \Gamma_X(\xi), \rho_{R_{i}^M}(\xi)) \) and \( \Gamma(R_{S,i}^M) \subseteq \Gamma(R_{i}^M) \cup \Gamma \). Its finiteness and branch continuity are clear by Thm. 4.7, and hence its continuity. On the other hand its \( S \)-skeleton is \( \Gamma(R_{S,i}^M, S) = \Gamma(R_{i}^M, S) = \Gamma(R_{i}^M) \cup \Gamma \). Both \( R_{i}^M \) and \( R_{S,i}^M \) are continuous and factorizes through \( \Gamma(R_{i}^M, S) \).

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\(^{21}\) More precisely by (4.9) and since \( \rho_T(\lambda_\xi(\rho)) = \max(\rho_T(\xi), \rho) \), one has \( \rho_{R_{S,i}^M}(\xi) = \left\{ \begin{array}{ll} \rho_{R_{i}^M}(\xi) & \text{if } R_{i}^M(\xi) \leq \rho_T(\xi) \\ \rho_T(\xi) & \text{if } R_{i}^M(\xi) > \rho_T(\xi) \end{array} \right. \).
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