The Lax-Milgram Theorem.
A detailed proof to be formalized in Coq

François Clément, Vincent Martin
The Lax-Milgram Theorem.
A detailed proof to be formalized in Coq

François Clément∗, Vincent Martin†

Project-Team Serena

Research Report n° 8934 — July 2016 — 97 pages

Abstract: To guarantee the correction of numerical simulation programs implementing the finite element method, it is necessary to formalize the mathematical notions and results that allow to establish the soundness of the method. The Lax-Milgram theorem is one of those theoretical cornerstones: under some completeness and coercivity assumptions, it states existence and uniqueness of the solution to the weak formulation of boundary value problems. The purpose of this document is to provide the formal proof community with a very detailed pen-and-paper proof of the Lax-Milgram theorem.

Key-words: Lax-Milgram theorem, finite element method, detailed mathematical proof, formal proof in real analysis

This research was partly supported by GT ELFIC from Labex DigiCosme - Paris-Saclay.

∗ Équipe Serena. Francois.Clement@inria.fr.
† LMAC, UTC, BP 20529, FR-60205 Compiègne, France. Vincent.Martin@utc.fr
Le théorème de Lax-Milgram.
Une preuve détaillée en vue d’une formalisation en Coq

Résumé : Pour garantir la correction de programmes de simulation numérique implémentant la méthode des éléments finis, il est nécessaire de formaliser les notions et résultats mathématiques qui permettent d’établir la justesse de la méthode. Le théorème de Lax-Milgram est l’un de ces fondements théoriques : sous des hypothèses de complétude et de coercivité, il énonce l’existence et l’unicité de la solution de problèmes aux limites posés sous forme faible. L’objectif de ce document est de fournir à la communauté preuve formelle une preuve papier très détaillée du théorème de Lax-Milgram.

Mots-clés : théorème de Lax-Milgram, méthode des éléments finis, preuve mathématique détaillée, preuve formelle en analyse réelle
Contents

1 Introduction

2 State of the art
  2.1 Brézis ................................................. 4
  2.2 Ciarlet .................................................. 5
  2.3 Ern–Guermond ............................................. 5
  2.4 Quarteroni–Valli ......................................... 5

3 Statement and sketch of the proof

4 Detailed proof
  4.1 Supremum, infimum ....................................... 7
  4.2 Metric space ............................................... 11
    4.2.1 Topology of balls ................................... 11
    4.2.2 Completeness ......................................... 11
    4.2.3 Continuity ............................................. 15
    4.2.4 Fixed point theorem .................................. 16
  4.3 Vector space ............................................... 19
    4.3.1 Basic notions and notations ........................... 19
    4.3.2 Linear algebra ........................................ 20
  4.4 Normed vector space ...................................... 26
    4.4.1 Topology ............................................... 31
      4.4.1.1 Continuous linear map ............................ 31
      4.4.1.2 Bounded bilinear form ............................ 38
  4.5 Inner product space ...................................... 40
    4.5.1 Orthogonal projection ................................ 43
  4.6 Hilbert space .............................................. 49

5 Conclusions, perspectives

References

A Lists of statements
  List of Definitions .......................................... 59
  List of Lemmas ............................................... 60
  List of Theorems .............................................. 62

B Depends directly from...

C Is a direct dependency of...
1 Introduction

As stated and demonstrated in [2], formal proof tools are now mature to address the verification of scientific computing programs. The most thrilling aspect of the approach is that the round-off error due to the use of IEEE-754 floating-point arithmetic can be fully taken into account. One of the most important issue in terms of manpower is that all the mathematical notions and results that allow to establish the soundness of the implemented algorithm must be formalized.

Our present long term purpose is to formally prove programs using the Finite Element Method. The Lax–Milgram theorem is the key ingredient that establishes existence and uniqueness of the solution to the weak formulation and its discrete approximation. Of course, we will also have to deal with measure theory to formalize Sobolev spaces such as $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ on some reasonable domain $\Omega$, and establish that they are Hilbert spaces on which the Lax–Milgram theorem applies. As well as with the notion of distributions to set up the correct framework to deal with weak formulations. And finally with chapters of the interpolation theory to define the discrete finite element approximation spaces.

The purpose of this document is to provide the formal proof community with a very detailed pen-and-paper proof of the Lax–Milgram theorem. The most basic notions and results such as ordered field properties of $\mathbb{R}$ and properties of elementary functions over $\mathbb{R}$ are supposed known and are not detailed further. One of the key issues is to select in the literature the proof involving the most simple notions, and in particular not to justify the result by applying a more general statement.

Once a detailed proof of the Lax–Milgram theorem is written, the next step is to formalize all notions and results in the formal proof tool Coq. Then, it will be necessary to take care of the specificities of the classical logic commonly used in mathematics: where do we need the law of excluded middle? And to discuss decidability issues.

The paper is organized as follows. Different ways to prove variants of the Lax–Milgram theorem collected from the literature are first reviewed in Section 2. The chosen proof path is then sketched in Section 3 and fully detailed in Section 4. Lists of statements and direct dependencies are gathered in the appendix.

2 State of the art

We review the works of a few authors, mainly from the French school, that provide some details about statements similar to the Lax–Milgram theorem.

Of course, proofs provided in the literature are not comprehensive, and we have to cover a series of books to collect all the details necessary for a formalization in a formal proof tool such as Coq. Indeed, this is the main purpose of the present document. Usually, Lecture Notes in undergraduate mathematics are very helpful. We mainly used the books written by Gostiaux, since he was the teacher of one of us.

2.1 Brézis

In [3], the Lax–Milgram theorem is stated as Corollary V.8 (p. 84). Its proof is obtained from Theorem V.6 (Stampacchia, p. 83) and by means similar to the ones used in the proof of Corollary V.4 (p. 80) for the characterization of the projection onto a closed subspace.

The proof of the Stampacchia theorem for a bilinear form on a closed convex has four main arguments: the Riesz–Fréchet representation theorem (Theorem V.5 p. 81), the characterization of the projection onto a closed convex (Theorem V.2 p. 79), the fixed point theorem on a complete metric space (Theorem V.7 p. 83), and the continuity of the projection onto a closed convex (Proposition V.3 p. 80).

\footnote{\url{http://coq.inria.fr/}}
The proof of the fixed point theorem uses the notions of distance, completeness, and sequential continuity (e.g. see [7] Theorem 4.102 p. 115).

The proof of the Riesz–Fréchet representation theorem and the existence of the projection onto a closed convex share the possibility to use the notion of reflexive space through Proposition V.1 (p. 78) and Theorem III.29 (Milman–Pettis, p. 51). The latter states that uniformly convex Banach spaces are reflexive (i.e. isomorphic to their topological double dual), and its proof uses the more complex notions of weak and weak-* topologies. In this case, the existence of the projection onto a closed convex also needs the notions of compactness and lower semi-continuity through Corollary III.20 (p. 46), and the appeal to Hahn–Banach theorem which depends on Zorn’s lemma or the axiom of choice through Theorem III.7 and Corollary III.8 (p. 38).

Fortunately, [3] also provides more elementary proofs. The Riesz–Fréchet theorem only needs the closed kernel lemma (for continuous linear maps) and the already cited Corollary V.4. And the existence of the projection onto a closed convex can be obtained through elementary and geometrical arguments. Then, the uniqueness and the characterization of the projection onto a closed convex follows the parallelogram identity and Cauchy–Schwarz inequality.

Complements about the projection onto a closed convex subset can be found in [8] Lemmas 14.30 and 14.32 pp. 225–228. See also [12] p. 90 and [5] Theorem A.28 p. 467 for proofs of the Riesz–Fréchet representation theorem.

2.2 Ciarlet
In [4], the Lax–Milgram theorem is stated as Theorem 1.1.3 (pp. 8–10). The structure of the proof is similar to the one proposed in [3] for Stampacchia theorem, but simplified to the case of a subspace instead of a closed convex subset (e.g. see [8] Theorems 14.27 and 14.29 pp. 224–225).

2.3 Ern–Guermond
In [5], the Lax–Milgram theorem is stated as Lemma 2.2 (p. 83). The first proof is obtained as a consequence of the more general Banach–Nečas–Babuška theorem set on a Banach space (Theorem 2.6 p. 85). The proof is spread out in Section A.2 through Theorem A.43 (p. 472) for the characterization of bijective Banach operators, Lemma A.39 (p. 470) which is a consequence of the closed range theorem (Theorem A.34 p. 468, see also [12] pp. 205–208 and [3] p. 28) and of the open mapping theorem (Theorem A.35 p. 469).

A simpler alternative proof without the use of the Banach–Nečas–Babuška theorem is proposed in Exercise 2.11 (p. 107) through the closed range theorem and a density argument. For the latter, Corollary A.18 (p. 466) is a consequence of the Hahn–Banach theorem (Theorem A.16 p. 465, see also [11] Theorem 5.19 and [3] p. 7).

2.4 Quarteroni–Valli
In [10], the Lax–Milgram theorem is stated as Theorem 5.1.1 (p. 133). A variant, also known as Babuška–Lax–Milgram theorem, is stated for a bilinear form defined over two different Hilbert spaces (Theorem 5.1.2 p. 135). Their proofs are similar: they both use the Riesz–Fréchet representation theorem and the closed range theorem. Note that when the bilinear form is symmetric, the Riesz–Fréchet representation theorem and a minimization argument are sufficient to build the proof (Remark 5.1.1 p. 134).

3 Statement and sketch of the proof
Let $H$ be a real Hilbert space. Let $H'$ be its topological dual (i.e. the space of continuous linear forms on $H$). Let $H_h$ be a closed vector subspace of $H$ (in practice, $H_h$ is finite dimensional).
The Lax–Milgram theorem states existence and uniqueness of the solution to the following general problems:

\[
\begin{align*}
\text{find } u \in H & \text{ such that: } \forall v \in H, \quad a(u, v) = f(v); \\
\text{find } u_h \in H_h & \text{ such that: } \forall v_h \in H_h, \quad a(u_h, v_h) = f(v_h).
\end{align*}
\]

(1)

(2)

The main statement is the following:

**Lax–Milgram theorem.** Let \((H, (\cdot, \cdot)_H)\) be a real Hilbert space. Let \(\|\cdot\|_H\) be the associated norm. Let \(a\) be a bounded bilinear form on \(H\). Let \(f \in H'\) be a continuous linear form on \(H\). Assume that \(a\) is coercive with constant \(\alpha > 0\). Then, there exists a unique \(u \in H\) solution to Problem (1). Moreover,

\[\|u\|_H \leq \frac{1}{\alpha} \|f\|_{H'}.
\]

We choose simplicity and collect ingredients gathered in [3] and [4]. The main arguments are:

- Riesz–Fréchet representation theorem
  - closed kernel,
  - projection onto a closed subspace, and \(H = F \oplus F^\perp\)
    - projection onto a nonempty closed convex
      - notions of infimum, convexity and completeness,
      - parallelogram identity and Cauchy–Schwarz inequality,
- fixed point theorem on a complete metric space
  - notions of completeness and distance,
  - sequential continuity,
- continuity of the projection.
4 Detailed proof

To identify them at a glance, statements are given inside boxes colored accordingly to their nature:

- light gray for remarks
- light green is for definitions
- light blue for lemmas
- light red for theorems

Furthermore, inside the bodies of proof for lemmas and theorems, the most basic results are supposed known and are not detailed further; they are written in bold red. This includes:

- properties from propositional calculus;
- basic notions and results from set theory such as the complement of a subset, the composition of functions, injective and surjective functions;
- basic results from group theory;
- ordered field properties of \( \mathbb{R} \), ordered set properties of \( \mathbb{R} \);
- basic properties of the complete valued fields \( \mathbb{R} \) and \( \mathbb{C} \);
- definition and properties of basic functions over \( \mathbb{R} \) such the square, square root, and exponential functions, and the discriminant of a quadratic polynomial;
- basic properties of geometric series (sum of the first terms);

The last statement of this document is dedicated to the finite dimensional case. To prove Theorem 203 (Lax–Milgram–Céa), we use the closedness of finite dimensional subspaces, which is a direct consequence of the closedness of the sum of a closed subspace and a linear span. Such a result is of course valid in any normed space, but the general proof is based on the equivalence of norms in a finite dimensional space, and the latter needs more advanced results on continuity involving compactness. To avoid that, we propose a much simpler proof that is only valid in inner product spaces; which is fine here since we apply it on an Hilbert space.

Some facts about infima and suprema are first collected in Section 4.1, they will be useful to define the operator norm for continuous linear maps, and orthogonal projections in inner product spaces. Then, Section 4.2 is devoted to complements on metric spaces, it concludes with the fixed point theorem. Section 4.3 is for the general notion of vector spaces. Normed vector spaces are introduced in Section 4.4, with the continuous linear map equivalency theorem. Then, we add an inner product in Section 4.5 to define pre-Hilbert spaces and state a series of orthogonal projection theorems. Finally, Section 4.6 is dedicated to Hilbert spaces with the Riesz–Fréchet representation theorem, and variants of the Lax–Milgram theorem.

4.1 Supremum, infimum

**Remark 1.** From the completeness of the set of real numbers, every nonempty subset of \( \mathbb{R} \) that is bounded from above has a least upper bound in \( \mathbb{R} \). On the affinely extended real number system \( \overline{\mathbb{R}} \), every nonempty subset has a least upper bound that may be \( +\infty \) (and a greatest lower bound that may be \( -\infty \)). Thus, we have the following extended notions of supremum and infimum for a numerical function over a set.

**Definition 2 (supremum).** Let \( X \) be a set. Let \( f : X \to \mathbb{R} \) be a function. The extended number \( L \) is the *supremum of \( f \) over \( X \)*, and is denoted \( L = \sup(f(X)) \), iff it is the least upper bound of \( f(X) = \{ f(x) \mid x \in X \} \subset \mathbb{R} \):

\[
\forall x \in X, \quad f(x) \leq L; \quad (3)
\]

\[
\forall M \in \overline{\mathbb{R}}, \quad (\forall x \in X, \ f(x) \leq M) \implies L \leq M. \quad (4)
\]

RR n° 8934

RR n° 8934

Lemma 3 (finite supremum). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. Assume that there exists a finite upper bound for $f(X)$, i.e. there exists $M \in \mathbb{R}$ such that, for all $x \in X$, $f(x) \leq M$. Then, the supremum is finite and $L = \sup(f(X))$ iff (5) and
$$
\forall \varepsilon > 0, \exists x_{\varepsilon} \in X, \quad L - \varepsilon < f(x_{\varepsilon}).
$$

Proof. From hypothesis, and completeness of $\mathbb{R}$, $\sup(f(X))$ is finite. Let $L$ be a number. Assume that $L$ is an upper bound of $f(X)$, i.e. (3).

(4) implies (5). Assume that (4) holds. Let $\varepsilon > 0$. Suppose that for all $x \in X$, $f(x) \leq L - \varepsilon$. Then, $L - \varepsilon$ is an upper bound for $f(X)$. Thus, from hypothesis, we have $L \leq L - \varepsilon$, and from ordered field properties of $\mathbb{R}$, $\varepsilon \leq 0$, which is impossible. Hence, there exists $x \in X$, such that $L - \varepsilon < f(x)$.

(5) implies (4). Conversely, assume now that (5) holds. Let $M$ be an upper bound, i.e. for all $x \in X$, $f(x) \leq M$. Suppose that $M < L$. Let $\varepsilon = \frac{L - M}{2} > 0$. Then, from hypotheses, and ordered field properties of $\mathbb{R}$, there exists $x_{\varepsilon} \in X$ such that $f(x_{\varepsilon}) > L - \varepsilon = \frac{L + M}{2} > M$, which is impossible. Hence, we have $L \leq M$.

Therefore, (3) implies the equivalence between (4) and (5).

Lemma 4 (discrete lower accumulation). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. Let $L$ be a number. Then, (5) iff
$$
\forall n \in \mathbb{N}, \exists x_n \in X, \quad L - \frac{1}{n+1} < f(x_n).
$$

Proof. (5) implies (6). Assume that (5) holds. Let $n \in \mathbb{N}$. Let $\varepsilon = \frac{1}{n+1} > 0$. Then, from hypothesis, there exists $x_n = x_{\varepsilon} \in X$ such that
$$
L - \frac{1}{n+1} = L - \varepsilon < f(x_{\varepsilon}) = f(x_n).
$$

(6) implies (5). Conversely, assume now that (6) holds. Let $\varepsilon > 0$. From the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$ (e.g. $n = \lceil \frac{1}{\varepsilon} \rceil$). Then, from ordered field properties of $\mathbb{R}$, and hypothesis, we have $\varepsilon > \frac{1}{n+1}$, and there exists $x_{\varepsilon} = x_n \in X$ such that
$$
L - \varepsilon < L - \frac{1}{n+1} < f(x_n) = f(x_{\varepsilon}).
$$

Lemma 5 (supremum is positive scalar multiplicative). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. Let $\lambda \geq 0$ be a nonnegative number. Then, $\sup((\lambda f)(X)) = \lambda \sup(f(X))$ (with the convention that $0$ times $+\infty$ is $0$).

Proof. Let $L = \sup(f(X))$ and $M = \sup((\lambda f)(X))$ be extended numbers of $\overline{\mathbb{R}}$. Let $x \in X$.

From Definition 2 (supremum) $L$ is an upper bound for $f(X)$, and ordered set properties of $\overline{\mathbb{R}}$, we have
$$
(\lambda f)(x) = \lambda f(x) \leq \lambda L.
$$
Thus, $\lambda L$ is an upper bound of $(\lambda f)(X)$. Hence, from Definition 2 (supremum) $M$ is the least upper bound of $(\lambda f)(X)$, we have $M \leq \lambda L$.

Case $\lambda = 0$. Then, $\lambda f$ is the zero function (for all $x \in X$, $(\lambda f)(x) = \lambda f(x) = 0$). Thus, $(\lambda f)(X) = \{0\}$, and from Definition 2 (supremum) 0 is the least upper bound of $\{0\}$, $M = 0$. Hence, from ordered set properties of $\overline{\mathbb{R}}$, we have
$$
\lambda L = 0 \Leftrightarrow L = 0 \Leftrightarrow M = 0.
$$
Case $\lambda \neq 0$. Then, from hypothesis, $\lambda > 0$. From Definition 2 (supremum), $M$ is an upper bound for $(\lambda f)(X)$, field properties of $\mathbb{R}$, and ordered set properties of $\mathbb{R}$, we have

$$f(x) = \frac{1}{\lambda} \lambda f(x) = \frac{1}{\lambda} (\lambda f)(x) \leq \frac{M}{\lambda}.$$  

Thus, $\frac{M}{\lambda}$ is an upper bound for $f(X)$. Hence, from Definition 2 (supremum), $L$ is the least upper bound of $f(X)$, and ordered set properties of $\mathbb{R}$, we have $L \leq \frac{M}{\lambda}$, or equivalently $\lambda L \leq M$.

Therefore, $M = \lambda L$.

Remark 6. As a consequence, $\sup((\lambda f)(X))$ is finite iff $\lambda \sup(f(X))$ is finite.

Definition 7 (maximum). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. The supremum of $f$ over $X$ is called maximum of $f$ over $X$, and it is denoted $\max(f(X))$, if $f$ exists $y \in X$ such that $f(y) = \sup(f(X))$.

Lemma 8 (finite maximum). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. Let $y \in X$. Then,

$$f(y) = \max(f(X)) \iff \forall x \in X, \ f(x) \leq f(y). \tag{7}$$

Proof. “Left” implies “right”. Assume that $y$ realizes the maximum of $f$ over $X$. Let $x \in X$. Then, from hypothesis, Definition 2 (maximum), and Definition 2 (supremum) $f(y)$ is an upper bound of $f(X)$, we have $f(x) \leq f(y)$.

“Right” implies “left”. Conversely, assume now that $f(y)$ is an upper bound of $f(X)$. Let $\varepsilon > 0$. Let $x_{\varepsilon} = y \in X$. Then, from ordered field properties of $\mathbb{R}$, we have $f(y) - \varepsilon < f(y) = f(x_{\varepsilon})$. Hence, from Lemma 5 (finite supremum), and Definition 7 (maximum), we have $f(y) = \max(f(X))$.

Definition 9 (infimum). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. The extended number $l$ is the infimum of $f$ over $X$, and is denoted $l = \inf(f(X))$, if it is the greatest lower bound of $f(X) \subset \mathbb{R}$:

$$\forall x \in X, \ l \leq f(x); \tag{8}$$

$$\forall m \in \mathbb{R}, \ (\forall x \in X, \ m \leq f(x)) \implies m \leq l. \tag{9}$$

Lemma 10 (duality infimum-supremum). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. Then, $\inf(f(X)) = -\sup((-f)(X))$.

Proof. Let $L = \sup((-f)(X))$ be an extended number in $\mathbb{R}$. Let $l = -L$.

Let $x \in X$. From Definition 2 (supremum) $L$ is an upper bound of $(-f)(X)$, and ordered set properties of $\mathbb{R}$, we have $l = -L \leq f(x)$. Hence, $l$ is a lower bound of $f(X)$.

Let $m \in \mathbb{R}$ be a lower bound of $f(X)$. Let $x \in X$. Then, from ordered set properties of $\mathbb{R}$, we have $-f(x) \leq -m$. Thus, from Definition 2 (supremum) $L$ is the least upper bound of $(-f)(X)$, and ordered set properties of $\mathbb{R}$, we have $m \leq -L = l$. Hence, $l$ is the greatest lower bound of $f(X)$.

Therefore, from Definition 9 (infimum), $l = \inf(f(X))$.

Lemma 11 (finite infimum). Let $X$ be a set. Let $f : X \to \mathbb{R}$ be a function. Assume that there exists a finite lower bound for $f(X)$, i.e., there exists $m \in \mathbb{R}$ such that, for all $x \in X$, $m \leq f(x)$. Then, the infimum is finite and $l = \inf(f(X))$ if

$$\forall \varepsilon > 0, \exists x_{\varepsilon} \in X, \ f(x_{\varepsilon}) < l + \varepsilon. \tag{10}$$
Proof. Let \( x \in X \). Then, from hypothesis, and ordered field properties of \( \mathbb{R} \), we have \(-f(x) \leq -m\). Thus, from Lemma 3 (finite supremum), \( \sup((-f)(X)) \) is finite. Hence, from Lemma 10 (duality infimum-supremum), \( \inf(f(X)) = -\sup((-f)(X)) \) is finite too.

From Lemma 10 (duality infimum-supremum), Lemma 3 (finite supremum), and ordered field properties of \( \mathbb{R} \), we have

\[
\forall x \in X, \quad -f(x) \leq -l
\]

\[
\implies \forall x \in X, \quad f(x) \leq l
\]

\[
\implies \forall \epsilon > 0, \exists x_{\epsilon} \in X, \quad f(x_{\epsilon}) < l + \epsilon.
\]

\[
l = \inf(f(X)) \iff -l = \sup((-f)(X))
\]

\[
\implies \forall x \in X, \quad -f(x) \leq -l
\]

\[
\implies \forall \epsilon > 0, \exists x_{\epsilon} \in X, \quad -l - \epsilon < -f(x_{\epsilon})
\]

\[
\implies \forall \epsilon > 0, \exists x_{\epsilon} \in X, \quad f(x_{\epsilon}) < l + \epsilon.
\]

\[
\forall \epsilon > 0, \exists x_{\epsilon} \in X, \quad f(x_{\epsilon}) < l + \epsilon \iff \forall n \in \mathbb{N}, \exists x_n \in X, \quad f(x_n) < l + \frac{1}{n+1}.
\]

Lemma 12 (discrete upper accumulation). Let \( X \) be a set. Let \( f : X \to \mathbb{R} \) be a function. Let \( l \) be a number. Then,

\[
\forall \epsilon > 0, \exists x_{\epsilon} \in X, \quad f(x_{\epsilon}) < l + \epsilon \iff \forall n \in \mathbb{N}, \exists x_n \in X, \quad f(x_n) < l + \frac{1}{n+1}.
\]

Proof. Direct consequence of Lemma 4 (discrete lower accumulation) for \(-f\) and \( L = -l \), and ordered field properties of \( \mathbb{R} \).

Lemma 13 (finite infimum discrete). Let \( X \) be a set. Let \( f : X \to \mathbb{R} \) be a function. Assume that there exists a finite lower bound for \( f(X) \), i.e. there exists \( m \in \mathbb{R} \) such that, for all \( x \in X \), \( m \leq f(x) \). Then, the infimum is finite and \( l = \inf(f(X)) \) iff \( (8) \) and

\[
\forall n \in \mathbb{N}, \exists x_n \in X, \quad f(x_n) < l + \frac{1}{n+1}.
\]

Proof. Direct consequence of Lemma 11 (finite infimum), and Lemma 12 (discrete upper accumulation).

Definition 14 (minimum). Let \( X \) be a set. Let \( f : X \to \mathbb{R} \) be a function. The infimum of \( f \) over \( X \) is called minimum of \( f \) over \( X \) and it is denoted \( \min(f(X)) \), iff there exists \( y \in X \) such that \( f(y) = \inf(f(X)) \).

Lemma 15 (finite minimum). Let \( X \) be a set. Let \( f : X \to \mathbb{R} \) be a function. Let \( y \in X \).

Then,

\[
f(y) = \min(f(X)) \iff \forall x \in X, \quad f(y) \leq f(x).
\]

Proof. From Definition 14 (minimum), Lemma 10 (duality infimum-supremum), Lemma 8 (finite maximum), and ordered field properties of \( \mathbb{R} \), we have

\[
f(y) = \min(f(X)) \iff -f(y) = \max((-f)(X))
\]

\[
\iff \forall x \in X, \quad -f(x) \leq -f(y)
\]

\[
\iff \forall x \in X, \quad f(y) \leq f(x).
\]
4.2 Metric space

**Definition 16 (distance).** Let $X$ be a nonempty set. An application $d : X \times X \to \mathbb{R}$ is a *distance over $X$* iff it is nonnegative, symmetric, it separates points, and it satisfies the triangle inequality:

\[
\begin{align*}
\forall x, y \in X, \quad & d(x, y) \geq 0; \\
\forall x, y \in X, \quad & d(y, x) = d(x, y); \\
\forall x, y \in X, \quad & d(x, y) = 0 \iff x = y; \\
\forall x, y, z \in X, \quad & d(x, z) \leq d(x, y) + d(y, z).
\end{align*}
\] (14) (15) (16) (17)

**Definition 17 (metric space).** $(X, d)$ is a *metric space* iff $X$ is a nonempty set and $d$ is a distance over $X$.

**Lemma 18 (iterated triangle inequality).** Let $(X, d)$ be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points of $X$. Then,

\[
\forall n, p \in \mathbb{N}, \quad d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}).
\] (18)

**Proof.** Let $n \in \mathbb{N}$ be a natural number. For $p \in \mathbb{N}$, let $P(p)$ be the property

\[
d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}).
\]

**Induction:** $P(0)$. From Definition 16 (distance $d$ separates points), and ordered field properties of $\mathbb{R}$, $P(0)$ is obviously satisfied.

**Induction:** $P(p)$ implies $P(p + 1)$. Let $p \in \mathbb{N}$. Assume that $P(p)$ holds. Then, from Definition 16 (distance $d$ satisfies triangle inequality), we have $d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+p}) + d(x_{n+p}, x_{n+p+1})$. Hence, from hypothesis, and ordered field properties of $\mathbb{R}$, we have $P(p + 1)$.

Therefore, by induction on $p \in \mathbb{N}$, we have, for all $p \in \mathbb{N}$, $P(p)$. \qed

4.2.1 Topology of balls

**Definition 19 (closed ball).** Let $(X, d)$ be a metric space. Let $x \in X$ be a point. Let $r \geq 0$ be a nonnegative number. The *closed ball centered in $x$ of radius $r$*, denoted $B_d^c(x, r)$, is the subset of $X$ defined by

\[
B_d^c(x, r) = \{ y \in X \mid d(x, y) \leq r \}.
\] (19)

**Definition 20 (sphere).** Let $(X, d)$ be a metric space. Let $x \in X$ be a point. Let $r \geq 0$ be a nonnegative number. The *sphere centered in $x$ of radius $r$*, denoted $S_d(x, r)$, is the subset of $X$ defined by

\[
S_d(x, r) = \{ y \in X \mid d(x, y) = r \}.
\] (20)

**Definition 21 (open subset).** Let $(X, d)$ be a metric space. A subset $Y$ of $X$ is *open (for distance $d$)* iff

\[
\forall x \in Y, \exists r > 0, \quad B_d(x, r) \subset Y.
\] (21)

**Definition 22 (closed subset).** Let $(X, d)$ be a metric space. A subset $Y$ of $X$ is *closed (for distance $d$)* iff $X \setminus Y$ is open for distance $d$. 
Lemma 23 (equivalent definition of closed subset). Let \((X, d)\) be a metric space. A subset \(Y\) of \(X\) is closed (for distance \(d\)) if
\[
\forall x \in X \setminus Y, \exists r > 0, \quad B_d(x, r) \cap Y = \emptyset.
\] (22)

Proof. Direct consequence of Definition \ref{def:closed-subset} (closed subset), Definition \ref{def:open-subset} (open subset), and the definition of the complement from set theory.

Lemma 24 (singleton is closed). Let \((X, d)\) be a metric space. Let \(x \in X\) be a point. Then \(\{x\}\) is closed.

Proof. Let \(x' \in X\) be a point. Assume that \(x' \neq x\). Then, from Definition \ref{def:distance} (distance \(d\) separates points, contrapositive), and ordered field properties of \(\mathbb{R}\), \(\epsilon = \frac{1}{2} d(x', x)\) is positive. Hence, \(d(x', x) > \epsilon\) and \(B_d(x', \epsilon) \cap \{x\} = \emptyset\).

Therefore, from Lemma \ref{lem:equiv-def-closed-subset} (equivalent definition of closed subset), \(\{x\}\) is closed.

Definition 25 (closure). Let \((X, d)\) be a metric space. Let \(Y\) be a subset of \(X\). The closure of \(Y\), denoted \(\overline{Y}\), is the subset
\[
\overline{Y} = \{x \in X | \forall \epsilon > 0, B_d(x, \epsilon) \cap Y \neq \emptyset\}.
\] (23)

Definition 26 (convergent sequence). Let \((X, d)\) be a metric space. Let \(l \in X\). A sequence \((x_n)_{n \in \mathbb{N}}\) of \(X\) is convergent with limit \(l\) if
\[
\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \quad n \geq N \implies d(x_n, l) \leq \epsilon.
\] (24)

Lemma 27 (variant of point separation). Let \((X, d)\) be a metric space. Let \(x, x' \in X\) such that for all \(\epsilon > 0\), we have \(d(x, x') \leq \epsilon\). Then, \(x, x'\).

Proof. Assume that \(d(x, x') > 0\). Let \(\epsilon = \frac{d(x, x')}{2}\). Then, \(0 < d(x, x') \leq \epsilon = \frac{d(x, x')}{2}\). Hence, from ordered field properties of \(\mathbb{R}\) (with \(d(x, x') > 0\)), we have \(0 < 1 \leq \frac{1}{\epsilon}\), which is wrong. Thus, from Definition \ref{def:distance} (distance \(d\) is nonnegative), we have \(d(x, x') = 0\).

Therefore, from Definition \ref{def:distance} (distance \(d\) separates points), we have \(x = x'\).

Lemma 28 (limit is unique). Let \((X, d)\) be a metric space. Let \((x_n)_{n \in \mathbb{N}}\) be a convergent sequence of \(X\). Then, the limit of the sequence is unique. The limit is denoted \(\lim_{n \to +\infty} x_n\).

Proof. Let \(l, l' \in X\) be two limits of the sequence. Let \(\epsilon > 0\). Then, from ordered field properties of \(\mathbb{R}\), and Definition \ref{def:convergent-sequence} (convergent sequence) with \(\epsilon > 0\), let \(N, N' \in \mathbb{N}\) such that, for all \(n, n' \in \mathbb{N}, n \geq N\) and \(n' \geq N'\) implies \(d(x_n, l) \leq \frac{\epsilon}{2}\) and \(d(x_{n'}, l') \leq \frac{\epsilon}{2}\). Let \(M = \max(N, N')\). Let \(p \in \mathbb{N}\). Assume that \(p \geq M\). Then, from the definition of the maximum, we have \(d(x_p, l) \leq \frac{\epsilon}{2}\) and \(d(x_p, l') \leq \frac{\epsilon}{2}\). Hence, from Definition \ref{def:distance} (distance \(d\) is nonnegative, satisfies triangle inequality, and is symmetric), and ordered field properties of \(\mathbb{R}\), we have
\[
0 \leq d(l, l') \leq d(l, x_p) + d(x_p, l') = d(x_p, l) + d(x_p, l') \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Therefore, from Lemma \ref{lem:equiv-def-closed-subset} (variant of point separation), we have \(l = l'\).

Lemma 29 (closure is limit of sequences). Let \((X, d)\) be a metric space. Let \(Y\) be a nonempty subset of \(X\). Let \(a \in X\) be a point. Then,
\[
a \in \overline{Y} \iff \exists (a_n)_{n \in \mathbb{N}} \in Y^\mathbb{N}, \ a = \lim_{n \to \infty} a_n.
\] (25)
Proof. “Left” implies “right”. Assume that \( a \in \overline{Y} \). Let \( a_0 \) be a point of \( Y \). Let \( n \in \mathbb{N} \). Assume that \( n \geq 1 \). Then, \( \frac{1}{n} > 0 \), and from Definition 25 (closure), let \( a_n \) be in the nonempty intersection \( B_{\frac{1}{n}}(a_0) \cap Y \). From Definition 19 (closed ball), and Definition 16 (distance \( d \) is symmetric), we have \( a_n \in Y \) and \( d(a_n, a) \leq \frac{1}{n} \). Let \( \varepsilon > 0 \). Let \( N = \left\lceil \frac{1}{\varepsilon} \right\rceil \). Let \( n \in \mathbb{N} \). Assume that \( n \geq N \). Then, from ordered field properties of \( \mathbb{R} \), and the definition of ceiling function, we have

\[
d(a_n, a) \leq \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon.
\]

Hence, from Definition 26 (convergent sequence), the sequence \( (a_n)_{n \in \mathbb{N}} \) is convergent with limit \( a \).

“Right” implies “left”. Assume now that there exists a convergent sequence \( (a_n)_{n \in \mathbb{N}} \) in \( Y \) with limit \( a \). Let \( \varepsilon > 0 \). Then, from Definition 26 (convergent sequence), let \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \), \( n \geq N \) implies \( d(a_n, a) \leq \varepsilon \). Thus, \( a_N \) belongs to the ball \( B_{\frac{\varepsilon}{N}}(a) \). Hence, from Definition 25 (closure), \( a \) belongs to the closure \( \overline{Y} \).

Lemma 30 (closed equals closure). Let \((X, d)\) be a metric space. Let \( Y \) be a nonempty subset of \( X \). Then,

\[
Y \text{ is closed } \iff \ Y = \overline{Y}.
\]  

(26)

Proof. “Left” implies “right”. Assume that \( Y \) is closed. Then, from Definition 22 (closed subset), \( X \setminus Y \) is open. Let \( a \in \overline{Y} \). Then, from Definition 25 (closure), for all \( \varepsilon > 0 \), we have \( B_{\varepsilon}(a, \varepsilon) \cap Y \neq \emptyset \). Assume that \( a \notin Y \). Then, from the definition of the complement from set theory, and Lemma 23 (equivalent definition of closed subset), there exists \( \varepsilon > 0 \) such that \( B_{\varepsilon}(a, \varepsilon) \cap Y = \emptyset \). Which is impossible. Thus, \( a \) belongs to \( Y \). Hence, \( \overline{Y} \subset Y \). Moreover, from Definition 25 (closure), \( Y \) is obviously a subset of \( \overline{Y} \). Therefore, \( Y = \overline{Y} \).

“Right” implies “left”. Assume now that \( Y = \overline{Y} \). Let \( x \in X \setminus Y \). From the definition of the complement from set theory, and hypothesis, \( x \) does not belong to \( Y = \overline{Y} \). Thus, from Definition 25 (closure), there exists \( \varepsilon > 0 \) such that \( B_{\varepsilon}(x, \varepsilon) \cap Y = \emptyset \). Hence, from Lemma 23 (equivalent definition of closed subset), \( Y \) is closed.

Lemma 31 (closed is limit of sequences). Let \((X, d)\) be a metric space. Let \( Y \) be a nonempty subset of \( X \). Then,

\[
Y \text{ is closed } \iff \ \forall (a_n)_{n \in \mathbb{N}} \in Y^\mathbb{N}, \forall a \in X, \ a = \lim_{n \to \infty} a_n \implies a \in Y.
\]  

(27)

Proof. Direct consequence of Lemma 30 (closed equals closure), Definition 25 (closure), and Lemma 29 (closure is limit of sequences).

Definition 32 (stationary sequence). Let \((X, d)\) be a metric space. A sequence \((x_n)_{n \in \mathbb{N}}\) of \( X \)

is stationary iff

\[
\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \ n \geq N \implies x_n = x_N.
\]  

(28)

\( N \) is a rank from which the sequence is stationary and \( x_N \) is the stationary value.

Lemma 33 (stationary sequence is convergent). Let \((X, d)\) be a metric space. Let \((x_n)_{n \in \mathbb{N}}\)

be a stationary sequence of \( X \). Then, \((x_n)_{n \in \mathbb{N}}\) is convergent with the stationary value as limit.

Proof. From Definition 32 (stationary sequence), let \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N}, n \geq N \)

implies \( x_n = x_N \). Let \( \varepsilon > 0 \). Let \( n \in \mathbb{N} \). Assume that \( n \geq N \). Then, from Definition 16 (distance \( d \) separates points), we have \( d(x_n, x_N) = d(x_N, x_N) = 0 \leq \varepsilon \). Hence, from Definition 26 (convergent sequence), \((x_n)_{n \in \mathbb{N}}\) is convergent with limit \( x_N \).
4.2.2 Completeness

**Definition 34 (Cauchy sequence).** Let \( (X, d) \) be a metric space. Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of \( X \). \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence iff

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N}, \quad p \geq N \land q \geq N \quad \implies \quad d(x_p, x_q) \leq \varepsilon. \tag{29}
\]

**Lemma 35 (equivalent definition of Cauchy sequence).** Let \( (X, d) \) be a metric space. Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of \( X \). \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence iff

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, k \in \mathbb{N}, \quad p \geq N \quad \implies \quad d(x_p, x_{p+k}) \leq \varepsilon. \tag{30}
\]

**Proof.** \((29)\) implies \((30)\). Assume that \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence. Let \( \varepsilon > 0 \). From Definition 34 \((\text{Cauchy sequence})\), let \( N \in \mathbb{N} \) such that for all \( p, q \in \mathbb{N} \) and \( q \geq N \) implies \( d(x_p, x_q) \leq \varepsilon \). Let \( p, k \in \mathbb{N} \). Assume that \( p \geq N \). Then, we also have \( q = p + k \geq N \). Thus, \( d(x_p, x_{p+k}) = d(x_p, x_q) \leq \varepsilon \).

\((30)\) implies \((29)\). Conversely, assume now that \( (x_n)_{n \in \mathbb{N}} \) satisfies \((30)\). Let \( \varepsilon > 0 \). Then, let \( N \in \mathbb{N} \) such that for all \( p, k \in \mathbb{N}, p \geq N \) implies \( d(x_p, x_{p+k}) \leq \varepsilon \). Let \( p', q' \in \mathbb{N} \). Assume that \( p' \geq N \) and \( q' \geq N \). Then, we also have \( p = \min(p', q') \geq N \). Let \( k = \max(p', q') - p \geq 0 \). Then, we have \( d(x_p, x_{p+k}) \leq \varepsilon \). Assume that \( p' \leq q' \). Then, \( p = p' + k = q' \). Hence, we have \( d(x_p, x_q) = d(x_p, x_{p+k}) \leq \varepsilon \). Conversely, assume that \( p' > q' \). Then, \( p = q' \) and \( p + k = p' \). Hence, from Definition 16 \((\text{distance})\), we have \( d(x_p, x_q) = d(x_p, x_{p+k}) = d(x_p, x_{p+k}) \leq \varepsilon \).

**Lemma 36 (convergent sequence is Cauchy).** Let \( (X, d) \) be a metric space. Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of \( X \). Assume that \( (x_n)_{n \in \mathbb{N}} \) is a convergent sequence. Then, \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence.

**Proof.** Let \( \varepsilon > 0 \). From Lemma 28 \((\text{limit unique})\), let \( l = \lim_{n \to +\infty} x_n \in X \). From Definition 26 \((\text{convergent sequence})\), let \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N}, n \geq N \) implies \( d(x_n, l) \leq \frac{\varepsilon}{2} \). Let \( p, q \geq N \). Then, from Definition 16 \((\text{distance})\), \( d \) satisfies triangle inequality and is symmetric, and field properties of \( \mathbb{R} \) we have

\[
d(x_p, x_q) = d(x_p, l) + d(l, x_q) = d(x_p, l) + d(x_q, l) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, from Definition 34 \((\text{Cauchy sequence})\), \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence.

**Definition 37 (complete subset).** Let \( (X, d) \) be a metric space. A subset \( Y \) of \( X \) is complete for distance \( d \) iff all Cauchy sequences of \( X \) converge in \( Y \).

**Definition 38 (complete metric space).** Let \( X \) be a set. Let \( d \) be a distance over \( X \). \( (X, d) \) is a complete metric space iff \( (X, d) \) is a metric space and \( X \) is complete for distance \( d \).

**Lemma 39 (closed subset of complete is complete).** Let \( (X, d) \) be a complete metric space. Let \( Y \) be a closed subset of \( X \). Then, \( Y \) is complete.

**Proof.** Let \( (y_n)_{n \in \mathbb{N}} \) be a sequence in \( Y \). Assume that \( (y_n)_{n \in \mathbb{N}} \) is a Cauchy sequence. Since \( Y \) is a subset of \( X \), \( (y_n)_{n \in \mathbb{N}} \) is also a Cauchy sequence in \( X \). Then, from Definition 38 \((\text{complete metric space})\), \( X \) is complete, and Definition 37 \((\text{complete subset})\), the sequence \( (y_n)_{n \in \mathbb{N}} \) is convergent with limit \( y \in X \).

Moreover, from Lemma 29 \((\text{closure is limit of sequences})\), the limit \( y \) belongs to the closure \( \overline{Y} \). Hence, from Lemma 30 \((\text{closed equals closure})\), the limit \( y \) belongs to \( Y \).

Therefore, from Definition 37 \((\text{complete subset})\), \( Y \) is complete.
4.2.3 Continuity

**Remark 40.** The distance allows the definition of balls centered at each point of a metric space forming neighborhoods for these points. Hence, a metric space can be seen as a topological space.

**Remark 41.** The distance also allows the definition of entourages making metric spaces specific cases of uniform spaces. Let \((X,d)\) be a metric space. Then, the sets

\[ U_r = \{(x,x') \in X \times X \mid d(x,x') \leq r\} \]

for all nonnegative numbers \(r\) form a fundamental system of entourages for the standard uniform structure of \(X\). See Theorem 47 below.

**Definition 42 (continuity in a point).** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(x \in X\). Let \(f : X \to Y\) be a mapping. \(f\) is continuous in \(x\) iff

\[
\forall \varepsilon > 0, \exists \delta > 0, \forall x' \in X, \quad d_X(x,x') \leq \delta \implies d_Y(f(x),f(x')) \leq \varepsilon. \quad (31)
\]

**Definition 43 (pointwise continuity).** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(f : X \to Y\) be a mapping. \(f\) is (pointwise) continuous iff \(f\) is continuous in all points of \(X\).

**Lemma 44 (compatibility of limit with continuous functions).** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(f : X \to Y\) be a mapping. Assume that \(f\) is pointwise continuous. Then, for all sequence \((x_n)_{n \in \mathbb{N}}\) of \(X\), for all \(x \in X\), we have

\[
(x_n)_{n \in \mathbb{N}} \text{ is convergent with limit } x \implies (f(x_n))_{n \in \mathbb{N}} \text{ is convergent with limit } f(x). \quad (32)
\]

**Proof.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(X\). Let \(x \in X\). Assume that \((x_n)_{n \in \mathbb{N}}\) is convergent with limit \(x\). Let \(\varepsilon > 0\). Then, from Definition 42 (continuity in a point), there exists \(\alpha > 0\) such that,

\[
\forall n \in \mathbb{N}, \quad d_X(x_n,x) \leq \alpha \implies d_Y(f(x_n),f(x)) \leq \varepsilon.
\]

And from Definition 26 (convergent sequence) with \(\alpha > 0\), there exists \(N \in \mathbb{N}\) such that,

\[
\forall n \in \mathbb{N}, \quad n \geq N \implies d_X(x_n,x) \leq \alpha.
\]

Thus,

\[
\forall n \in \mathbb{N}, \quad n \geq N \implies d_Y(f(x_n),f(x)) \leq \varepsilon.
\]

Hence, from Definition 26 (convergent sequence), the sequence \((f(x_n))_{n \in \mathbb{N}}\) is convergent with limit \(f(x)\).

**Definition 45 (uniform continuity).** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(f : X \to Y\) be a mapping. \(f\) is uniformly continuous iff

\[
\forall \varepsilon > 0, \exists \delta > 0, \forall x,x' \in X, \quad d_X(x,x') \leq \delta \implies d_Y(f(x),f(x')) \leq \varepsilon. \quad (33)
\]

**Definition 46 (Lipschitz continuity).** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(f : X \to Y\) be a mapping. Let \(k \geq 0\) be a nonnegative number. \(f\) is \(k\)-Lipschitz continuous iff

\[
\forall x,x' \in X, \quad d_Y(f(x),f(x')) \leq k d_X(x,x'). \quad (34)
\]

Then, \(k\) is called Lipschitz constant of \(f\).

**Theorem 47 (equivalent definition of Lipschitz continuity).** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(f : X \to Y\) be a mapping. Let \(k \geq 0\) be a nonnegative number. \(f\) is \(k\)-Lipschitz continuous iff

\[
\forall x,x' \in X, \forall r \geq 0, \quad d_X(x,x') \leq r \implies d_Y(f(x),f(x')) \leq kr. \quad (35)
\]
Proof. “Left” implies “right”. Assume that \( f \) is \( k \)-Lipschitz continuous. Let \( x, x' \in X \). Let \( r \geq 0 \). Assume that \( d_X(x, x') \leq r \). Then, from Definition 46 (Lipschitz continuity), we have
\[
d_Y(f(x), f(x')) \leq k d_X(x, x') \leq kr.
\]

“Right” implies “left”. Conversely, assume now that \( f \) satisfies (35). Let \( x, x' \in X \). Let \( r = d_X(x, x') \). From Definition 16 (distance, \( d_X \) is nonnegative), \( r \) is also nonnegative. From ordered field properties of \( \mathbb{R} \), we have \( d_X(x, x') \leq r \). Hence, from hypothesis, we have
\[
d_Y(f(x), f(x')) \leq kr = k d_X(x, x').
\]
Therefore, from Definition 46 (Lipschitz continuity), \( f \) is \( k \)-Lipschitz continuous.

Definition 48 (contraction). Let \( (X, d) \) be a metric space. Let \( f : X \to Y \) be a mapping. Let \( k \geq 0 \) be a nonnegative number. \( f \) is a \( k \)-contraction iff \( f \) is \( k \)-Lipschitz continuous with \( k < 1 \).

Lemma 49 (uniformly continuous is continuous). Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. Let \( f : X \to Y \) be an uniformly continuous mapping. Then, \( f \) is continuous.

Proof. Direct consequence of Definition 45 (uniform continuity), Definition 43 (pointwise continuity), and Definition 42 (continuity in a point).

Lemma 50 (zero-Lipschitz continuous is constant). Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. Let \( f : X \to Y \) be a \( 0 \)-Lipschitz continuous mapping. Then, \( f \) is constant.

Proof. Let \( x, x' \in X \). Then, from Definition 46 (Lipschitz continuity), and Definition 16 (distance, \( d_Y \) is nonnegative and separates points), we have \( d_Y(f(x), f(x')) = 0 \) and \( f(x) = f(x') \).

Lemma 51 (Lipschitz continuous is uniformly continuous). Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. Let \( f : X \to Y \) be a Lipschitz continuous mapping. Then, \( f \) is uniformly continuous.

Proof. From Definition 46 (Lipschitz continuity), let \( k \geq 0 \) be the Lipschitz constant of \( f \). Let \( \varepsilon > 0 \) be a positive number.

Case \( k = 0 \). Then, from Lemma 50 (zero-Lipschitz continuous is constant), \( f \) is a constant function. Let \( \delta = 1 > 0 \). Let \( x, x' \in X \). Assume that \( d_X(x, x') < \delta \). Then, we have \( d_Y(f(x), f(x')) = 0 < \varepsilon \). Hence, from Definition 45 (uniform continuity), \( f \) is uniformly continuous.

Case \( k \neq 0 \). Then, \( k > 0 \). From ordered field properties of \( \mathbb{R} \), let \( \delta = \frac{\varepsilon}{k} > 0 \). Let \( x, x' \in X \). Assume that \( d_X(x, x') \leq \delta \). Then, from ordered field properties of \( \mathbb{R} \), we have
\[
d_Y(f(x), f(x')) \leq k d_X(x, x') \leq k \delta = \varepsilon.
\]
Hence, from Definition 45 (uniform continuity), \( f \) is uniformly continuous.

4.2.4 Fixed point theorem

Definition 52 (iterated function sequence). Let \( (X, d) \) be a metric space. Let \( f : X \to X \) be a mapping. An iterated function sequence associated with \( f \) is a sequence of \( X \) defined by
\[
x_0 \in X \quad \land \quad \forall n \in \mathbb{N}, \quad x_{n+1} = f(x_n).
\]
Then, from Definition 52 (Lipschitz continuity), let \( k \geq 0 \) be the Lipschitz constant of \( f \). From Definition 26 (convergent sequence), let \( a = \lim_{n \to +\infty} x_n \in X \) be the limit of the sequence.

**Case \( k = 0 \).** Then, from Lemma 50 (zero-Lipschitz continuous is constant), \( f \) is constant of value \( f(a) \). Thus, from Definition 32 (stationary sequence), the sequence \( (x_n)_{n \in \mathbb{N}} \) is stationary from rank 1. Hence, from Lemma 35 (stationary sequence is convergent), \( (x_n)_{n \in \mathbb{N}} \) is convergent with limit \( f(a) \).

**Case \( k \neq 0 \).** Then, from Definition 46 (Lipschitz continuity), we have \( k > 0 \). Let \( \varepsilon > 0 \). From Definition 26 (convergent sequence), let \( N \in \mathbb{N} \) such that, for all \( n \in \mathbb{N} \), \( n \geq N \) implies
\[
d(x_n, a) \leq \frac{\varepsilon}{k}.
\]
Let \( N' = N + 1 \). Let \( n \in \mathbb{N} \). Assume that \( n \geq N' \). Then, \( n - 1 \geq N \). Thus, from
Definition \[ \text{iterated function sequence} \], Definition \[ \text{Lipschitz continuity} \], and ordered field properties of \( \mathbb{R} \), we have
\[
d(x_n, f(a)) = d(f(x_{n-1}), f(a)) \leq k d(x_{n-1}, a) \leq \varepsilon.
\]
Hence, from Definition \[ \text{convergent sequence} \], the sequence \( (x_n)_{n \in \mathbb{N}} \) is convergent with limit \( f(a) \).
Therefore, in both cases, from Lemma \[ \text{limit is unique} \], \( f(a) = a \).

**Theorem 56 (fixed point).** Let \( (X,d) \) be a complete metric space. Let \( f : X \to X \) be a contraction. Then, there exists a unique fixed point \( a \in X \) such that \( f(a) = a \).
Moreover, all iterated function sequences associated with \( f \) are convergent with limit \( a \).

**Proof.** Uniqueness. Let \( a, a' \in X \) be two fixed points of \( f \). Then, from Definition \[ \text{contraction} \], and Definition \[ \text{Lipschitz continuity} \], we have \( d(a, a') = d(f(a), f(a')) \leq k d(a, a') \).
Thus, from ordered field properties of \( \mathbb{R} \), Definition \[ \text{contraction} \ (k < 1) \], and Definition \[ \text{distance} \ d \text{ is nonnegative} \), we have \( 0 \leq (1 - k) d(a, a') \leq 0 \). Therefore, from the zero-product property of \( \mathbb{R} \), Definition \[ \text{contraction} \ (k \neq 1) \], and Definition \[ \text{distance} \ d \text{ separates points} \), we have \( a = a' \).

Convergence of iterated function sequences and existence. Let \( x_0 \in X \). Let \( (x_n)_{n \in \mathbb{N}} \) be an iterated function sequence associated with \( f \). Let \( p, m \in \mathbb{N} \). Then, from Lemma \[ \text{iterated triangle inequality} \], Definition \[ \text{iterate Lipschitz continuous mapping} \], field properties of \( \mathbb{R} \), the formula for the sum of the first terms of a geometric series, and Definition \[ \text{contraction} \ (0 \leq k < 1) \), we have
\[
d(x_p, x_{p+m}) \leq \sum_{i=0}^{m-1} d(x_{p+i}, x_{p+i+1}) \\
\leq \left( \sum_{i=0}^{m-1} k^{p+i} \right) d(x_0, x_1) \\
= k^p \frac{1 - k^m}{1 - k} d(x_0, x_1) \\
\leq \frac{k^p}{1 - k} d(x_0, x_1).
\]

**Case** \( k = 0 \). Then, from Lemma \[ \text{zero-Lipschitz continuous is constant} \], \( f \) is constant.
Let \( a = f(x_0) \). Then, for all \( x \in X \), \( f(x) = a \). In particular, \( f(a) = a \), and for all \( n \in \mathbb{N} \), \( x_{n+1} = f(x_n) = a \). Thus, from Definition \[ \text{stationary sequence} \], the sequence \( (x_n)_{n \in \mathbb{N}} \) is stationary from rank 1 with stationary value \( a \).

**Case** \( x_1 = x_0 \). Then, from Lemma \[ \text{stationary iterated function sequence} \], the sequence \( (x_n)_{n \in \mathbb{N}} \) is stationary from rank 0 with stationary value \( x_0 = x_1 = f(x_0) = a \).
Hence, in both cases, from Lemma \[ \text{stationary sequence is convergent} \], the sequence \( (x_n)_{n \in \mathbb{N}} \) is convergent with limit \( a \).

**Case** \( k \neq 0 \) and \( x_1 \neq x_0 \). Then, from Definition \[ \text{contraction} \ (0 \leq k < 1) \], and Definition \[ \text{distance} \ d \text{ separates points, contrapositive} \), we have \( 0 < k < 1 \) and \( d(x_0, x_1) \neq 0 \). Then \( \varepsilon > 0 \).
Let
\[
\xi = \max \left( \frac{\ln \xi}{\ln k} \right) \geq 0, \quad N = [\xi] \in \mathbb{N}.
\]
Let \( p, m \in \mathbb{N} \). Assume that \( p \geq N \).
Then, from the definition of ceiling and max functions, we have \( p \geq \xi \geq \frac{\ln \xi}{\ln k} \).
Thus, from ordered field properties of \( \mathbb{R} \) (in \( k \) is negative), and increase of the exponential function, we have \( p \ln k \leq \ln \zeta \), hence \( k^p \leq \xi \), and finally
\[
\frac{k^p}{1 - k} d(x_0, x_1) \leq \varepsilon.
\]
Hence, from Lemma 55 \textit{equivalent definition of Cauchy sequence}, Definition 38 \textit{complete metric space}, and Definition 37 \textit{complete subset}, \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence that is convergent with limit \(a \in X\).

Therefore, in all cases, from Lemma 55 \textit{convergent iterated function sequence}, \(a\) is a fixed point of \(f\).

### 4.3 Vector space

\[\textbf{Remark 57.}\] Statements and proofs are presented in the case of vector spaces over the field of real numbers \(\mathbb{R}\), but most can be generalized with minor or no alteration to the case of vector spaces over the field of complex numbers \(\mathbb{C}\). When the same statement holds for both cases, the field is denoted \(K\). Note that in both cases, \(\mathbb{R} \subset K\).

#### 4.3.1 Basic notions and notations

\textbf{Definition 68 (vector space).} Let \(E\) be a set equipped with two vector operations: an addition \((+ : E \times E \to E)\) and a scalar multiplication \((\cdot : K \times E \to E)\). \((E,+,\cdot)\) is a \textit{vector space over field} \(K\), or simply \(E\) is a \textit{space}, iff \((E,+\cdot)\) is an abelian group with identity element 0\(_E\) (zero vector), and scalar multiplication is distributive wrt vector addition and field addition, compatible with field multiplication, and admits 1\(_K\) as identity element (simply denoted 1):

\[
\begin{align*}
\forall \lambda \in K, \forall u,v \in E, & \quad \lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v; \\
\forall \lambda, \mu \in K, \forall u \in E, & \quad (\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u; \\
\forall \lambda, \mu \in K, \forall u \in E, & \quad \lambda \cdot (\mu \cdot u) = (\lambda \mu) \cdot u; \\
\forall u \in E, & \quad 1 \cdot u = u. 
\end{align*}
\]

\[\textbf{Remark 59.}\] The \(\cdot\) infix sign in the scalar multiplication may be omitted.

\[\textbf{Remark 60.}\] Vector spaces over \(\mathbb{R}\) are called \textit{real spaces}, and vector spaces over \(\mathbb{C}\) are called \textit{complex spaces}.

\textbf{Definition 61 (set of mappings to space).} Let \(X\) be a set. Let \(E\) be a space. The \textit{set of mappings from} \(X\) \textit{to} \(E\) is denoted \(F(X,E)\).

\textbf{Definition 62 (linear map).} Let \((E,+_E,\cdot_E)\) and \((F,+_F,\cdot_F)\) be spaces. A mapping \(f : E \to F\) \textit{is a linear map from} \(E\) \textit{to} \(F\) \textit{iff} it preserves vector operations, i.e. iff it is additive and homogeneous of degree 1:

\[
\begin{align*}
\forall u,v \in E, & \quad f(u+_E v) = f(u) +_F f(v); \\
\forall \lambda \in K, \forall u \in E, & \quad f(\lambda \cdot_E u) = \lambda \cdot_F f(u).
\end{align*}
\]

\textbf{Definition 63 (set of linear maps).} Let \(E,F\) be spaces. The \textit{set of linear maps from} \(E\) \textit{to} \(F\) \textit{is denoted} \(L(E,F)\).

\textbf{Definition 64 (linear form).} Let \(E\) be a vector space over field \(K\). A \textit{linear form on} \(E\) \textit{is a linear map from} \(E\) \textit{to} \(K\).

\textbf{Definition 65 (bilinear map).} Let \((E,+_E,\cdot_E)\), \((F,+_F,\cdot_F)\) and \((G,+_G,\cdot_G)\) be spaces. A mapping \(\varphi : E \times F \to G\) \textit{is a bilinear map from} \(E \times F\) \textit{to} \(G\) \textit{iff} it is left additive, right additive, and left and right homogeneous of degree 1:

\[
\begin{align*}
\forall u,v \in E, \forall w \in F, & \quad \varphi(u+_E v, w) = \varphi(u,w) +_G \varphi(v,w); \\
\forall u \in E, \forall v,w \in F, & \quad \varphi(u, v+_F w) = \varphi(u,v) +_G \varphi(u,w); \\
\forall \lambda \in K, \forall u \in E, \forall v \in F, & \quad \varphi(\lambda \cdot_E u, v) = \lambda \cdot_G \varphi(u,v) = \varphi(u, \lambda \cdot_F v). 
\end{align*}
\]
Definition 66 (bilinear form). Let \( E \) be a space. A bilinear form on \( E \) is a bilinear map from \( E \times E \) to \( \mathbb{K} \).

Definition 67 (set of bilinear forms). Let \( E \) be a space. The set of bilinear forms on \( E \) is denoted \( \mathcal{L}_2(E) = \mathcal{L}(E \times E, \mathbb{K}) \).

4.3.2 Linear algebra

Lemma 68 (zero times yields zero). Let \((E, +, \cdot)\) be a space. Then,
\[
\forall u \in E, \quad 0 \cdot u = 0_E. \tag{48}
\]

Proof. Let \( u \in E \) be a vector. From Definition 58 (vector space, \((E, +)\) is an abelian group, scalar multiplication admits 1 as identity element and is distributive wrt field addition), and field properties of \( \mathbb{K} \), we have
\[
0 \cdot u = 0 \cdot u + u + (-u) = 0 \cdot u + 1 \cdot u + (-u) = (0 + 1) \cdot u + (-u) = u + (-u) = 0_E.
\]

Lemma 69 (minus times yields opposite vector). Let \((E, +, \cdot)\) be a space. Then,
\[
\forall \lambda \in \mathbb{K}, \forall u \in E, \quad (-\lambda) \cdot u = -(\lambda \cdot u). \tag{49}
\]

Proof. Let \( \lambda \in \mathbb{K} \) be a scalar. Let \( u \in E \) be a vector. From Definition 58 (vector space, scalar multiplication is distributive wrt field addition), field properties of \( \mathbb{K} \), and Lemma 68 (zero times yields zero), we have
\[
\lambda \cdot u + (-\lambda) \cdot u = (\lambda - \lambda) \cdot u = 0 \cdot u = 0_E.
\]

Therefore, from Definition 58 (vector space, \((E, +)\) is an abelian group), \((-\lambda) \cdot u\) is the opposite of \(\lambda \cdot u\).

Definition 70 (vector subtraction). Let \((E, +, \cdot)\) be a space. Vector subtraction, denoted by the infix operator \(-\), is defined by
\[
\forall u, v \in E, \quad u - v = u + (-v). \tag{50}
\]

Definition 71 (scalar division). Let \((E, +, \cdot)\) be a space. Scalar division, denoted by the infix operator \(\div\), is defined by
\[
\forall \lambda \in \mathbb{K}^*, \forall u \in E, \quad u \div \lambda = \frac{1}{\lambda} \cdot u. \tag{51}
\]

Lemma 72 (times zero yields zero). Let \((E, +, \cdot)\) be a space. Then,
\[
\forall \lambda \in \mathbb{K}, \quad \lambda \cdot 0_E = 0_E. \tag{52}
\]

Proof. Let \( \lambda \in \mathbb{K} \) be a scalar. From Definition 58 (vector space, \((E, +)\) is an abelian group and scalar multiplication is distributive wrt vector addition), and Definition 70 (vector subtraction), we have
\[
\lambda \cdot 0_E = \lambda \cdot (0_E - 0_E) = \lambda \cdot 0_E - \lambda \cdot 0_E = 0_E.
\]

Lemma 73 (zero-product property). Let \((E, +, \cdot)\) be a space. Then,
\[
\forall \lambda \in \mathbb{K}, \forall u \in E, \quad \lambda \cdot u = 0_E \iff \lambda = 0 \lor u = 0_E. \tag{53}
\]
Proof. Let \( \lambda \in \mathbb{K} \) be a scalar. Let \( u \in E \) be a vector.

“Left” implies “right”. Assume that \( \lambda = 0 \) or \( u = 0_E \). Then, from Lemma 68 (zero times yields zero), and Lemma 72 (times zero yields zero), we have \( \lambda \cdot u = 0_E \).

“Right” implies “left”. Assume that \( \lambda \cdot u = 0_E \) and \( \lambda \neq 0 \). Then, from Definition 58 (vector space scalar multiplication admits 1 as identity element and is compatible with field multiplication), field properties of \( \mathbb{K} \), and Lemma 72 (times zero yields zero), we have

\[
\lambda \cdot u = \frac{1}{\lambda} \cdot (\lambda \cdot u) = \frac{1}{\lambda} \cdot 1 \cdot 0_E = 0_E.
\]

Hence, since \( (P \land \neg Q \Rightarrow R) \Leftrightarrow (P \Rightarrow Q \lor R) \), \( \lambda \cdot u = 0_E \) implies \( \lambda = 0 \) or \( u = 0_E \).

**Definition 74 (subspace).** Let \((E, +, \cdot)\) be a space. Let \( F \subseteq E \) be a subset of \( E \). Let \( +_{|F} \) be the restrictions of \( + \) to \( F \times F \). Let \( \cdot_{|F} \) be the restrictions of \( \cdot \) to \( \mathbb{K} \times F \). \( F \) is a **vector subspace** of \( E \), or simply a **subspace** of \( E \), iff \((F, +_{|F}, \cdot_{|F})\) is a space.

**Remark 75.** In particular, a subspace is closed under restricted vector operations.

**Remark 76.** Usually, restrictions \( +_{|F} \) and \( \cdot_{|F} \) are still denoted \( + \) and \( \cdot \).

**Lemma 77 (trivial subspaces).** Let \( E \) be a space. Then, \( E \) and \( \{0_E\} \) are subspaces of \( E \).

**Proof.** \( E \) and \( \{0_E\} \) are trivially subspaces of \( E \). \( \{0_E\} \) is trivially a space. Therefore, from Definition 74 (subspace), \( E \) and \( \{0_E\} \) are subspaces of \( E \).

**Lemma 78 (closed under vector operations is subspace).** Let \( E \) be a space. Let \( F \) be a subset of \( E \). \( F \) is a subspace of \( E \) iff \( 0_E \in F \) and \( F \) is closed under vector addition and scalar multiplication:

\[
\forall u, v \in F, \quad u + v \in F; \tag{54}
\]

\[
\forall \lambda \in \mathbb{K}, \forall u \in F, \quad \lambda u \in F. \tag{55}
\]

**Proof.** “If”. Assume that \( F \) contains \( 0_E \) and is closed under vector addition and scalar multiplication. Then, \( F \) is closed under the restriction to \( F \) of vector operations. Let \( u, v \in F \) be vectors. Then, from Lemma 69 (minus times yields opposite vector) with \( \lambda = 1 \), \(-v = (-1)v \) belongs to \( F \), and \( u - v = u + (-v) \) also belongs to \( F \). Thus, from group theory, \((F, +_{|F})\) is a subgroup of \((E, +)\). Hence, from Definition 58 (vector space \((E, +)\) is an abelian group), and group theory, \((F, +_{|F})\) is also an abelian group and \( 0_F = 0_E \). Since \( F \) is a subset of \( E \), and \( E \) is a space, properties (39) to (42) are trivially satisfied over \( F \). Therefore, from Definition 74 (subspace), \( F \) is a subspace of \( E \).

“Only if”. Conversely, assume now that \( F \) is a subspace of \( E \). Then, from Definition 74 (subspace, \( F \) is a space), and Definition 58 (vector space \((F, +_{|F})\) is an abelian group), \( F \) contains \( 0_F = 0_E \) and \( F \) is closed under the restriction to \( F \) of vector operations. Therefore, \( F \) is closed under vector addition and scalar multiplication.

**Lemma 79 (closed under linear combination is subspace).** Let \( E \) be a space. Let \( F \) be a subset of \( E \). \( F \) is a subspace of \( E \) iff \( 0_E \in F \) and \( F \) is closed under linear combination:

\[
\forall \lambda, \mu \in \mathbb{K}, \forall u, v \in F, \quad \lambda u + \mu v \in F. \tag{56}
\]

**Proof.** “If”. Assume that \( F \) contains \( 0_E \) and is closed under linear combination. Let \( u, v \in F \) be vectors. Let \( \lambda \in \mathbb{K} \) be a scalar. Then, from Definition 58 (vector space scalar multiplication in \( E \) admits 1 as identity element), \( u + v = 1u + 1v \) belongs to \( F \), and from Lemma 73 (zero-product),
property), and Definition 58 (vector space) \((E,+)\) is an abelian group, \(\lambda u = \lambda u + 0 \cdot 0_E\) belongs to \(F\). Thus, \(F\) contains \(0_E\) and is closed under vector operations. Therefore, from Lemma 78 (closed under vector operations is subspace), \(F\) is a subspace of \(E\).

“Only if”. Conversely, assume now that \(F\) is a subspace of \(E\). Let \(\lambda, \mu \in \mathbb{K}\) be scalars. Let \(u, v \in F\) be vectors. Then, from Lemma 78 (closed under vector operations is subspace), \(F\) contains \(0_E\), \(F\) is closed under scalar multiplication, hence \(u' = \lambda u + v' = \mu v\) belong to \(F\), and \(F\) is closed by vector addition, hence \(u' + v' = \lambda u + \mu v\) belongs to \(F\). Therefore, \(F\) is closed under linear combination.

\[\begin{align*}
\text{Definition 80 (linear span).} & \text{ Let } E \text{ be a space. Let } u \in E \text{ be a vector. The linear span of } u, \\
& \text{denoted } \text{span}(\{u\}), \text{ is defined by } \\
& \text{span}(\{u\}) = \{ \lambda u \mid \lambda \in \mathbb{K} \}. \\
\end{align*}\]

\[\begin{align*}
\text{Definition 81 (sum of subspaces).} & \text{ Let } E \text{ be a space. Let } F, F' \text{ be subspaces of } E. \text{ The sum of } F \text{ and } F' \text{ is the subset of } E \text{ defined by } \\
& F + F' = \{ u + u' \mid u \in F, u' \in F' \}. \\
\end{align*}\]

\[\begin{align*}
\text{Definition 82 (finite dimensional subspace).} & \text{ Let } E \text{ be a space. Let } F \text{ be a subspace of } E. \text{ \(F\) is a finite dimensional subspace iff there exists } n \in \mathbb{N}, \text{ and } u_1, \ldots, u_n \in E \text{ such that } \\
& F = \text{span}(\{u_1, \ldots, u_n\}) = \text{span}(\{u_1\}) + \ldots + \text{span}(\{u_n\}) \\
& = \{ \lambda_1 u_1 + \ldots + \lambda_n u_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{K} \}. \\
\end{align*}\]

\[\begin{align*}
\text{Definition 83 (direct sum of subspaces).} & \text{ Let } E \text{ be a space. Let } F, F' \text{ be subspaces of } E. \text{ The sum } F + F' \text{ is called direct sum, and it is denoted } F \oplus F', \text{ iff all vectors of the sum admit a unique decomposition: } \\
& \forall u, v \in F, \forall u', v' \in F', \quad u + u' = v + v' \implies u = v \quad \land \quad u' = v'. \\
\end{align*}\]

\[\begin{align*}
\text{Lemma 84 (equivalent definitions of direct sum).} & \text{ Let } E \text{ be a space. Let } F, F' \text{ be subspaces of } E. \text{ The sum } F + F' \text{ is direct iff one of the following equivalent properties is satisfied: } \\
& F \cap F' = \{0_E\}; \\
& \forall u \in F, \forall u' \in F', \quad u + u' = 0_E \implies u = u' = 0_E. \\
\end{align*}\]

Proof. (60) implies (61). Assume that the sum \(F + F'\) is direct. Let \(v \in F \cap F'\) be a vector in the intersection. Then, from Definition 58 (vector space) \((E,+)\) is an abelian group, \(v\) admits two decompositions, \(v = v + 0_E = 0_E + v\). Thus, from Definition 83 (direct sum of subspaces), \(v = 0_E\).

(61) implies (62). Assume now that \(F \cap F' = \{0_E\}\). Let \(u \in F\) and \(u' \in F'\) be vectors. Assume that \(u + u' = 0_E\). Then, from Lemma 69 (minus times yields opposite vector with \(\lambda = 1\)), and Lemma 78 (closed under vector operations is subspace scalar multiplication), \(u = -u' = (-1)u'\) belongs to \(F'\) and \(u' = -u = (-1)u\) belongs to \(F\). Hence, \(u, u' \in F \cap F'\), and \(u = u' = 0_E\).

(62) implies (60). Assume finally that \(0_E\) admits a unique decomposition. Let \(u, v \in F\) and \(u', v' \in F'\) be vectors. Assume that \(u + u' = v + v'\). Then, from Definition 58 (vector space) \((E,+)\) is an abelian group, and Definition 70 (vector subtraction), we have \((u - v) + (u' - v') = 0_E\). Thus, from hypothesis, we have \(u - v = u - v = 0_E\), and from Definition 58 (vector space) \((E,+)\) is an abelian group, we have \(u = v\) and \(u' = v'\).

Therefore, all three properties are equivalent.

\[\begin{align*}
\text{Lemma 85 (direct sum with linear span).} & \text{ Let } E \text{ be a space. Let } F \text{ be a subspace of } E. \text{ Let } u \in E \text{ be a vector. Assume that } u \not\in F. \text{ Then, the sum } F + \text{span}(\{u\}) \text{ is direct.}
\end{align*}\]
Proof. From Lemma 78 (closed under vector operations is subspace), we have \(0_E \in F\) and \(\text{span}(\{u\})\) are subspace). Assume that \(v \neq 0_E\). Then, from Definition 80 (linear span), \(\lambda \in \mathbb{K}\) such that \(v = \lambda u\). Thus, from hypothesis, and Lemma 73 (zero-product property (contrapositive)), we have \(\lambda \neq 0_E\). Hence, from field properties of \(\mathbb{K}\), and Lemma 81 (closed under vector operations is subspace), we have \(\frac{1}{\lambda} v = u \in F\). Which is impossible by hypothesis.

Therefore, from Lemma 84 (equivalent definitions of direct sum), the sum \(F + \text{span}(\{u\})\) is direct.

Definition 86 (product vector operations). Let \((E, +_E, \cdot_E)\) and \((F, +_F, \cdot_F)\) be spaces. The product vector operations induced on \(E \times F\) are the mappings \(+_{E \times F} : (E \times F) \times (E \times F) \to E \times F\) and \(\cdot_{E \times F} : \mathbb{K} \times (E \times F) \to E \times F\) defined by

\[
\forall (u, v), (u', v') \in E \times F, \quad (u, v) +_{E \times F} (u', v') = (u +_E u', v +_F v');
\]

\[
\forall \lambda \in \mathbb{K}, \forall (u, v) \in E \times F, \quad \lambda \cdot_{E \times F} (u, v) = (\lambda \cdot_E u, \lambda \cdot_F v).
\]

Lemma 87 (product is space). Let \((E, +_E, \cdot_E)\) and \((F, +_F, \cdot_F)\) be spaces. Let \(+_{E \times F}\) and \(\cdot_{E \times F}\) be the product vector operations induced on \(E \times F\). Then, \((E \times F, +_{E \times F}, \cdot_{E \times F})\) is a space.

Proof. From group theory, \((E \times F, +_{E \times F})\) is an abelian group with identity element \(0_{E \times F} = (0_E, 0_F)\). Distributivity of the product scalar multiplication wrt product vector addition and field addition, compatibility of the product scalar multiplication with field multiplication, and 1 is the identity element for the product scalar multiplication are direct consequences of Definition 58 (vector space), and Definition 86 (product vector operations).

Therefore, from Definition 58 (vector space), \((E \times F, +_{E \times F}, \cdot_{E \times F})\) is a space.

Definition 88 (inherited vector operations). Let \(X\) be a set. Let \((E, +_E, \cdot_E)\) be a space. The vector operations inherited on \(\mathcal{F}(X, E)\) are the mappings \(+_{\mathcal{F}(X, E)} : \mathcal{F}(X, E) \times \mathcal{F}(X, E) \to \mathcal{F}(X, E)\) and \(\cdot_{\mathcal{F}(X, E)} : \mathbb{K} \times \mathcal{F}(X, E) \to \mathcal{F}(X, E)\) defined by

\[
\forall f, g \in \mathcal{F}(X, E), \forall x \in X, \quad (f +_{\mathcal{F}(X, E)} g)(x) = f(x) +_E g(x);
\]

\[
\forall \lambda \in \mathbb{K}, \forall f \in \mathcal{F}(X, E), \forall x \in X, \quad (\lambda \cdot_{\mathcal{F}(X, E)} f)(x) = \lambda \cdot_E f(x).
\]

Remark 89. Usually, inherited vector operations are denoted the same way as the vector operations of the target space.

Lemma 90 (space of mappings to a space). Let \(X\) be a set. Let \((E, +_E, \cdot_E)\) be a space. Let \(+_{\mathcal{F}(X, E)}\) and \(\cdot_{\mathcal{F}(X, E)}\) be the vector operations inherited on \(\mathcal{F}(X, E)\). Then, \((\mathcal{F}(X, E), +_{\mathcal{F}(X, E)}, \cdot_{\mathcal{F}(X, E)})\) is a space.

Proof. From Definition 61 (set of mappings to space), and group theory, \((\mathcal{F}(X, E), +_{\mathcal{F}(X, E)})\) is an abelian group with identity element

\[
0_{\mathcal{F}(X, E)} = (x \mapsto 0_E).
\]

Distributivity of the inherited scalar multiplication wrt inherited vector addition and field addition, compatibility of the inherited scalar multiplication with field multiplication, and 1 is the identity element for the inherited scalar multiplication are direct consequences of Definition 58 (vector space), and Definition 88 (inherited vector operations).

Therefore, from Definition 58 (vector space), \((\mathcal{F}(X, E), +_{\mathcal{F}(X, E)}, \cdot_{\mathcal{F}(X, E)})\) is a space.

Lemma 91 (linear map preserves zero). Let \(E\) and \(F\) be spaces. Let \(f\) be a linear map from \(E\) to \(F\). Then, \(f(0_E) = 0_F\).
Proof. From Lemma \[68\] (zero times yields zero), and Definition \[62\] linear map \( f \) is homogeneous of degree 1), we have
\[
f(0_E) = f(0_K \cdot 0_E) = 0_K \cdot F f(0_E) = 0_F.
\]

**Lemma 92** (linear map preserves linear combinations). Let \((E,+_E, \cdot_E)\) and \((F,+_F, \cdot_F)\) be spaces. Let \( f : E \to F \) be a mapping from \( E \) to \( F \). Then, \( f \) is a linear map from \( E \) to \( F \) iff it preserves linear combinations:
\[
\forall \lambda, \mu \in K, \forall u, v \in E, \quad f(\lambda \cdot_E u +_E \mu \cdot_E v) = \lambda \cdot_F f(u) +_F \mu \cdot_F f(v).
\]

**Proof.** “If”. Assume that \((67)\) holds. Let \( u, v \in E \) be vectors. Then, from Definition \[58\] vector space scalar multiplications in \( E \) and \( F \) admit 1 as identity element, we have
\[
f(u +_E v) = f(1 \cdot_E u +_E 1 \cdot_E v) = 1 \cdot_F f(u) +_F 1 \cdot_F f(v) = f(u) +_F f(v).
\]
Hence, \( f \) is additive. Let \( \lambda \in K \) be a scalar. Let \( u \) be a vector. Then, from Lemma \[68\] zero times yields zero in \( E \) and \( F \), and Definition \[58\] vector space \((E,+_E)\) and \((F,+_F)\) are abelian groups, we have
\[
f(\lambda \cdot_E u) = f(\lambda \cdot_E u +_E 0 \cdot_E 0_E) = \lambda \cdot_F f(u) +_F 0 \cdot_F f(0_E) = \lambda \cdot_F f(u).
\]
Hence, \( f \) is homogeneous of degree 1. Therefore, from Definition \[62\] linear map, \( f \) is a linear map from \( E \) to \( F \).

“Only if”. Conversely, assume now that \( f \) is a linear map from \( E \) to \( F \). Let \( \lambda, \mu \in K \) be scalars. Let \( u, v \in E \) be vectors. Then, from Definition \[62\] linear map \( f \) is additive, \( f(\lambda \cdot_E u +_E \mu \cdot_E v) = f(\lambda \cdot_E u) +_F f(\mu \cdot_E v) \), and \( f \) is homogeneous of degree 1) \( f(\lambda \cdot_E u) = \lambda \cdot_F f(u) \) and \( f(\mu \cdot_E v) = \mu \cdot_F f(v) \). Hence, we have
\[
f(\lambda \cdot_E u +_E \mu \cdot_E v) = \lambda \cdot_F f(u) +_F \mu \cdot_F f(v).
\]

**Lemma 93** (space of linear maps). Let \( E \) and \( (F,+_F, \cdot_F) \) be spaces. \(+_F(E,F)\) and \( \cdot_F(E,F)\) be the vector operations inherited on \( F(E,F) \). Then, \( L(E,F) \) is a subspace of \( (F(E,F),+_F(E,F), \cdot_F(E,F)) \).

**Proof.** From Definition \[63\] set of linear maps, and Lemma \[90\] space of mappings to a space, \( L(E,F) \) is a subset of \( F(E,F) \). From Lemma \[72\] zero times yields zero, and Definition \[58\] vector space \((F,+_F)\) is an abelian group, \( 0_{F(E,F)} \) trivially preserves vector operations. Hence, from Definition \[62\] linear map, \( 0_{F(E,F)} \) is a linear map from \( E \) to \( F \). From Definition \[88\] inherited vector operations, Definition \[62\] linear map, and Definition \[58\] vector space \((E,+_E)\) and \((F,+_F)\) are abelian groups and scalar multiplications in \( E \) and \( F \) are compatible with field multiplication), \( L(E,F) \) is trivially closed under linear combination.

Therefore, from Lemma \[73\] closed under linear combination is subspace, \( L(E,F) \) is a subspace of \( F(E,F) \).

**Definition 94** (identity map). Let \( E \) be a space. The identity map on \( E \) is the mapping \( \text{Id}_E : E \to E \) defined by
\[
\forall u \in E, \quad \text{Id}_E(u) = u.
\]

**Lemma 95** (identity map is linear map). Let \( E \) be a space. Then, the identity map \( \text{Id}_E \) is a linear map.

**Proof.** Direct consequence of Definition \[94\] identity map, and Definition \[62\] linear map.
Lemma 96 (composition of linear maps is bilinear). Let $E, F, G$ be spaces. Then, the composition of functions is a bilinear map from $L(E, F) \times L(F, G)$ to $L(E, G)$.

Proof. From Lemma 93 [space of linear maps], $L(E, F)$, $L(F, G)$ and $L(E, G)$ are spaces, and Lemma 87 [product is space], $L(E, F) \times L(F, G)$ is a space.

Let $f \in L(E, F)$ and $g \in L(F, G)$ be linear maps. Let $\lambda, \mu \in \mathbb{K}$ be scalars. Let $u, v \in E$ be vectors. Then, from the definition of composition of functions, and Lemma 92 [linear map preserves linear combinations], for $f$ and $g$, we have

$$(g \circ f)(\lambda u + \mu v) = g(f(\lambda u + \mu v)) = g(\lambda f(u) + \mu f(v)) = \lambda g(f(u)) + \mu g(f(v)) = \lambda(g \circ f)(u) + \mu(g \circ f)(v).$$

Hence, $g \circ f$ belongs to $L(E, G)$, and composition is a mapping from space $L(E, F) \times L(F, G)$ to space $L(E, G)$.

From the definition of composition of functions, and Definition 88 [inherited vector operations], composition of linear maps is trivially left additive and left homogeneous of degree 1. From the definition of composition of functions, Definition 88 [inherited vector operations], and Definition 62 [linear map], left argument “$g$” is additive and homogeneous of degree 1, composition of linear maps is trivially right additive and right homogeneous of degree 1.

Therefore, from Definition 65 [bilinear map], composition of linear maps is a bilinear map from $L(E, F) \times L(F, G)$ to $L(E, G)$. \qed

Definition 97 (isomorphism). Let $E$ and $F$ be spaces. An isomorphism from $E$ onto $F$ is a linear map from $E$ to $F$ that is bijective.

Definition 98 (kernel). Let $E$ and $F$ be spaces. Let $f \in L(E, F)$ be a linear map from $E$ to $F$. The kernel of $f$ (or null space of $f$), denoted $\ker(f)$, is the subset of $E$ defined by

$$\ker(f) = \{ u \in E \mid f(u) = 0_F \}. \quad (69)$$

Lemma 99 (kernel is subspace). Let $E$ and $F$ be spaces. Let $f \in L(E, F)$ be a linear map from $E$ to $F$. Then, $\ker(f)$ is a subspace of $E$.

Proof. From Lemma 91 [linear map preserves zero], and Definition 98 [kernel], $0_E$ belongs to $\ker(f)$. Let $\lambda, \mu \in \mathbb{K}$ be scalars. Let $u, v \in \ker(f)$ be vectors in the kernel. Then, from Lemma 92 [linear map preserves linear combinations], Definition 98 [kernel], and Definition 58 [times zero yields zero], we have

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) = \lambda 0_F + \mu 0_F = 0_F.$$

Hence, from Definition 98 [kernel], $\lambda u + \mu v$ belongs to $\ker(f)$.

Therefore, from Lemma 79 [closed under linear combination is subspace], $\ker(f)$ is a subspace of $E$. \qed

Lemma 100 (injective linear map has zero kernel). Let $E$ and $F$ be spaces. Let $f \in L(E, F)$ be a linear map from $E$ to $F$. Then, $f$ is injective iff $\ker(f) = \{ 0_E \}$.

Proof. “If”. Assume that $\ker(f) = \{ 0_E \}$. Let $u, v \in E$ be vectors. Assume that $f(u) = f(v)$. Then, from Definition 70 [vector subtraction], Definition 32 [linear map], and Definition 58 [vector space $(F, +)$ is an abelian group], we have $f(u - v) = 0_F$, hence $u - v$ belongs to $\ker(f)$. Thus $u - v = 0_E$, and from Definition 70 [vector subtraction], and Definition 58 [vector space $(E, +)$ is an abelian group], we have $u = v$. Therefore, from the definition of injectivity, $f$ is injective.

“Only if”. Conversely, assume now that $f$ is injective. Let $u \in \ker(f)$ be a vector in the kernel. Then, from Definition 98 [kernel], we have $f(u) = 0_F$. Thus, from Lemma 91 [linear map preserves zero], we have $f(u) = f(0_E)$. Therefore, from the definition of injectivity, $u = 0_E$. \qed

RR n° 8934
Lemma 101 (K is space). The commutative field K equipped with its addition and multiplication is a space.

Proof. Direct consequence of the commutative field structure.

4.4 Normed vector space

Definition 102 (norm). Let E be a space. An application \( \| \cdot \| : E \rightarrow \mathbb{R} \) is a norm over E iff it separates points (or it is definite), it is absolutely homogeneous of degree 1, and it satisfies the triangle inequality:

\[
\forall u \in E, \quad \|u\| = 0 \implies u = 0_E; \\
\forall \lambda \in K, \forall u \in E, \quad \|\lambda u\| = |\lambda| \|u\|; \\
\forall u, v \in E, \quad \|u + v\| \leq \|u\| + \|v\|. \tag{72}
\]

Remark 103. The absolute value over field K is a function \(|\cdot| : K \rightarrow \mathbb{R}\) that is nonnegative, definite, multiplicative, and satisfies the triangle inequality. It is the modulus for the field of complex numbers.

Definition 104 (normed vector space). \((E, \|\cdot\|)\) is a normed vector space, or simply a normed space, iff E is a space and \(\|\cdot\|\) is a norm over E.

Lemma 105 (K is normed space). The commutative field K equipped with its absolute value is a normed space.

Proof. Direct consequence of Definition 104 (normed vector space), Definition 102 (norm), and properties of the absolute value over K (see Remark 103).

Lemma 106 (norm preserves zero). Let \((E, \|\cdot\|)\) be a normed space. Then, \(\|0_E\| = 0\).

Proof. From Definition 104 (normed vector space, E is a space), and Definition 58 (vector space, \(0_E \) belongs to \(E\)). From Lemma 68 (zero times yields zero), Definition 102 (norm \(\|\cdot\|\) is absolutely homogeneous of degree 1), definition of the absolute value, and field properties of \(\mathbb{R}\), we have

\[
\|0_E\| = \|0_K \cdot 0_E\| = |0_K| \|0_E\| = 0 \|0_E\| = 0. \tag*{□}
\]

Lemma 107 (norm is nonnegative). Let \((E, \|\cdot\|)\) be a normed space. Then, \(\|\cdot\|\) is nonnegative.

Proof. From Definition 104 (normed vector space, E is a space). Let \(u \in E\) be a vector. Then, from Definition 102 (norm \(\|\cdot\|\) is absolutely homogeneous of degree 1 and satisfies triangle inequality), Definition 58 (vector space, \((E, +)\) is an abelian group), and ordered field properties of \(\mathbb{R}\), we have we \(\|-u\| = \|u\|\) and

\[
\|u\| = \frac{1}{2}(\|u\| + \|-u\|) \geq \frac{1}{2} \|u - u\| = \frac{1}{2} \|0_E\| = 0. \tag*{□}
\]

Lemma 108 (normalization by nonzero). Let \((E, \|\cdot\|)\) be a normed space. Then,

\[
\forall \lambda \in K, \forall u \in E, \quad u \neq 0 \implies \left\| \frac{u}{\|u\|} \right\| = |\lambda|. \tag{73}
\]

Inria
Proof. Direct consequence of Definition 109 (distance associated with norm), Definition 102 (norm $\| \cdot \|$ is definite and absolutely homogeneous of degree 1), Lemma 107 (norm is nonnegative), and field properties of $\mathbb{R}$.

**Definition 109 (distance associated with norm).** Let $(E, \| \cdot \|)$ be a normed space. The distance associated with norm $\| \cdot \|$ is the mapping $d : E \times E \to \mathbb{R}$ defined by

$$\forall u, v \in E, \quad d(u, v) = \| u - v \|.$$  \hfill (74)

**Remark 110.** The mapping $d$ will be proved below to be a distance; hence its name.

**Lemma 111 (norm gives distance).** Let $(E, \| \cdot \|)$ be a normed space. Let $d$ be the distance associated with norm $\| \cdot \|$. Then, $(E, d)$ is a metric space.

Proof. From Definition 109 (distance associated with norm), Lemma 107 (norm is nonnegative), Definition 70 (vector subtraction), and Definition 102 (norm $\| \cdot \|$ is absolutely homogeneous of degree 1 with $\lambda = -1$, definite and satisfies triangle inequality), $d$ is nonnegative and symmetric, separates points, and satisfies the triangle inequality. Thus, from Definition 10 (distance), $d$ is a distance over $E$. Therefore, from Definition 17 (metric space), $(E, d)$ is a metric space.

**Lemma 112 (linear span is closed).** Let $(E, \| \cdot \|)$ be a normed space. Let $d$ be the distance associated with norm $\| \cdot \|$. Let $u \in E$ be a vector. Then, span($\{u\}$) is closed for distance $d$.

Proof. Let $\lambda, \lambda' \in \mathbb{K}$ be scalars. From Definition 109 (distance associated with norm), Definition 58 (vector space), scalar multiplication is distributive wrt field addition, Definition 70 (vector subtraction), and Definition 102 (norm $\| \cdot \|$ is absolutely homogeneous of degree 1), we have

$$d(\lambda u, \lambda' u) = \| \lambda u - \lambda' u \| = |\lambda - \lambda'| \| u \|. \quad (75)$$

**Case $u = 0_E$.** Direct consequence of Definition 80 (linear span), and Lemma 24 (singleton is closed span($\{u\}$) = $\{0_E\}$).

**Case $u \neq 0_E$.** Then, from Definition 102 (norm $\| \cdot \|$ is definite, contrapositive), and Lemma 107 (norm is nonnegative), we have $\| u \| > 0$. Let $(\lambda_n u)_{n \in \mathbb{N}}$ be a sequence in span($\{u\}$). Assume that this sequence is convergent. Let $\varepsilon > 0$.

From Lemma 36 (convergent sequence is Cauchy), ordered field properties of $\mathbb{R}$ (with $\| u \| > 0$), Definition 34 (Cauchy sequence $(\lambda_n u)_{n \in \mathbb{N}}$ is a Cauchy sequence with $\varepsilon \| u \| > 0$), and Equation (75), there exists $N \in \mathbb{N}$ such that for all $p, q \in \mathbb{N}, p, q \geq N$ implies

$$|\lambda_p - \lambda_q| = d(\lambda_p u, \lambda_q u) \leq \frac{\varepsilon \| u \|}{\| u \|} = \varepsilon.$$

Hence, from Definition 34 (Cauchy sequence), Definition 37 (complete subset $(\lambda_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ complete), Lemma 28 (limit is unique) $(\lambda_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ (complete), let $\lambda \in \mathbb{K}$ be the limit $\lim_{n \to +\infty} \lambda_n$.

Then, from ordered field properties of $\mathbb{R}$ (with $\| u \| > 0$), Definition 26 (convergent sequence $(\lambda_n u)_{n \in \mathbb{N}}$ is convergent with limit $\lambda$, with $\varepsilon \| u \| > 0$), and Equation (75), there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$ implies

$$d(\lambda_n u, \lambda u) = |\lambda_n - \lambda| \| u \| \leq \frac{\varepsilon \| u \|}{\| u \|} = \varepsilon.$$

Hence, from Definition 26 (convergent sequence) $(\lambda_n u)_{n \in \mathbb{N}}$ has limit $\lambda u \in \text{span}(\{u\})$. Therefore, from Lemma 31 (closed is limit of sequences), span($\{u\}$) is closed (for distance $d$).
Definition 113 (closed unit ball). Let \((E, \|\cdot\|)\) be a normed space. Let \(d\) be the distance associated with norm \(\|\cdot\|\). The closed unit ball of \(E\) is \(B_d^1(0_E, 1)\) in the metric space \((E, d)\).

Lemma 114 (equivalent definition of closed unit ball). Let \((E, \|\cdot\|)\) be a normed space. Let \(B_d^1\) be the closed unit ball in \(E\). Then, \(B_d^1 = \{u \in E \mid \|u\| \leq 1\}\).

Proof. Direct consequence of Definition 113 (closed unit ball), Definition 109 (distance associated with norm), Lemma 111 (norm gives distance), and Definition 19 (closed ball).

Definition 115 (unit sphere). Let \((E, \|\cdot\|)\) be a normed space. Let \(d\) be the distance associated with norm \(\|\cdot\|\). The unit sphere of \(E\) is \(S_d(0_E, 1)\) in the metric space \((E, d)\).

Lemma 116 (equivalent definition of unit sphere). Let \((E, \|\cdot\|)\) be a normed space. Let \(S_1\) be the unit sphere in \(E\). Then, \(S_1 = \{u \in E \mid \|u\| = 1\}\).

Proof. Direct consequence of Definition 115 (unit sphere), Definition 109 (distance associated with norm), Lemma 111 (norm gives distance), and Definition 20 (sphere).

Lemma 117 (zero on unit sphere is zero). Let \((E, \|\cdot\|)\) be a normed space. Let \(S_1\) be the unit sphere in \(E\). Let \(F\) be a space. Let \(f \in \mathcal{L}(E,F)\) be a linear map from \(E\) to \(F\). Then, \(f = 0_{\mathcal{L}(E,F)}\) iff \(f\) is zero on \(S_1\).

Proof. “If”. Assume that \(f\) is zero on the unit sphere. Let \(u \in E\) be a vector.

Case \(u = 0_F\). Then, from Lemma 91 (linear map preserves zero), \(f(u) = f(0_E) = 0_F\).

Case \(u \neq 0_F\). Then, from Lemma 108 (normalization by nonzero) with \(\lambda = 1\), and Lemma 116 (equivalent definition of unit sphere), \(\xi = \frac{u}{\|u\|}\) belongs to \(S_1\). Thus, from Definition 58 (vector space), scalar multiplication is compatible with field multiplication, and field properties of \(\mathbb{R}\), we have \(u = \|u\| \cdot \xi\). Hence, from Definition 63 (linear map), homogeneity of degree 1, hypothesis, and Lemma 72 (times zero yields zero), we have \(f(u) = f(\|u\| \cdot \xi) = \|u\| f(\xi) = \|u\| 0_{F} = 0_{F}\).

Therefore, in both cases, \(f = 0_{\mathcal{L}(E,F)}\).

“Only if”. Conversely, assume now that \(f = 0_{\mathcal{L}(E,F)}\). Then, from Lemma 116 (equivalent definition of unit sphere), \(S_1\) is a subset of \(E\). Then, \(f\) is also zero on the unit sphere.

Lemma 118 (reverse triangle inequality). Let \((E, \|\cdot\|)\) be a normed space. Then,

\[
\forall u, v \in E, \quad \|u - v\| \leq \|u\| - \|v\| \leq \|u - v\|. \tag{76}
\]

Proof. Let \(u, v \in E\) be vectors. Then, from Definition 104 (normed vector space), \(\|\cdot\|\) is a norm, and Definition 102 (norm) \(\|\cdot\|\) satisfies triangle inequality, we have \(\|u\| \leq \|u - v\| + \|v\|\). Hence, from ordered field properties of \(\mathbb{R}\), we have \(\|u\| - \|v\| \leq \|u - v\|\). Thus, from Definition 102 (norm) \(\|\cdot\|\) is absolutely homogeneous of degree 1 with \(\lambda = -1\), we have

\[
\|v\| - \|u\| \leq ||v - u|| = \|u - v\|.
\]

Therefore, from properties of the absolute value in \(\mathbb{R}\), we have \(\|u\| - \|v\| \leq \|u - v\|\).

Lemma 119 (norm is one-Lipschitz continuous). Let \((E, \|\cdot\|)\) be a normed space. Let \(d\) be the distance associated with norm \(\|\cdot\|\). Then, \(\|\cdot\|\) is 1-Lipschitz continuous from \((E, d)\) to \((\mathbb{R}, \cdot|\cdot|)\).

Proof. Direct consequence of Lemma 118 (reverse triangle inequality), Definition 109 (distance associated with norm), and Definition 40 (Lipschitz continuity) with \(k = 1\).
Lemma 120 (norm is uniformly continuous). Let \((E, \| \cdot \|)\) be a normed space. Let \(d\) be the distance associated with norm \(\| \cdot \|\). Then, \(\| \cdot \|\) is uniformly continuous from \((E, d)\) to \((\mathbb{R}, | \cdot |)\).

**Proof.** Direct consequence of Lemma 119 (norm is one-Lipschitz continuous), and Definition 51 (Lipschitz continuous is uniform continuous).

Lemma 121 (norm is continuous). Let \((E, \| \cdot \|)\) be a normed space. Let \(d\) be the distance associated with norm \(\| \cdot \|\). Then, \(\| \cdot \|\) is continuous from \((E, d)\) to \((\mathbb{R}, | \cdot |)\).

**Proof.** Direct consequence of Lemma 120 (norm is uniformly continuous), and Lemma 49 (uniform continuous is continuous).

Definition 122 (linear isometry). Let \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\) be normed spaces. Let \(f \in \mathcal{L}(E, F)\) be a linear map from \(E\) to \(F\). \(f\) is a linear isometry from \(E\) to \(F\) if it preserves the norm:

\[
\forall u \in E, \quad \| f(u) \|_F = \| u \|_E.
\]  

(77)

Lemma 123 (identity map is linear isometry). Let \((E, \| \cdot \|)\) be a normed space. Then, the identity map \(1_E\) is a linear isometry.

**Proof.** Direct consequence of Lemma 95 (identity map is linear map), Definition 94 (identity map), and Definition 122 (linear isometry).

Definition 124 (product norm). Let \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\) be normed spaces. The product norm induced over \(E \times F\) is the mapping \(\|(\cdot, \cdot)\|_{E \times F} : E \times F \to \mathbb{R}\) defined by

\[
\forall (u, v) \in E \times F, \quad \| (u, v) \|_{E \times F} = \| u \|_E + \| v \|_F.
\]  

(78)

Remark 125. The mapping \(\|(\cdot, \cdot)\|_{E \times F}\) will be proved below to be a norm; hence its name and notation.

Remark 126. The norm \(\|(\cdot, \cdot)\|_{E \times F}\) is the \(L^1\)-like norm over the product \(E \times F\). \(L^p\)-like norms for \(p \geq 1\) and \(p = +\infty\) are also possible; they are all equivalent norms.

Lemma 127 (product norm is normed space). Let \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\) be normed spaces. Let \(\|(\cdot, \cdot)\|_{E \times F}\) be the product norm induced over \(E \times F\). Then, \((E \times F, \|(\cdot, \cdot)\|_{E \times F})\) is a normed space.

**Proof.** From Lemma 87 (product is space), \(E \times F\), equipped with product vector operations of Definition 86 (product vector operations), is a space.

Let \((u, v) \in E \times F\) be vectors. Assume that \(\|(u, v)\|_{E \times F} = 0\). Then, from Definition 124 (product norm), we have \(\|u\|_E + \|v\|_F = 0\). And, from Lemma 107 (norm is nonnegative) for \(\| \cdot \|_E\) and \(\| \cdot \|_F\), and ordered field properties of \(\mathbb{R}\), we have \(\|u\|_E = \|v\|_F = 0\). Thus, from Definition 102 (norm), \(\| \cdot \|_E\) and \(\| \cdot \|_F\) are definite, and Lemma 87 (product is space), \(0_{E \times F} = (0_E, 0_F)\), we have \((u, v) = 0_{E \times F}\). Hence, \(\|(\cdot, \cdot)\|_{E \times F}\) is definite.

Let \(\lambda \in \mathbb{K}\) be a scalar. Let \((u, v) \in E \times F\) be vectors. Then, from Definition 86 (product vector operations), scalar multiplication, Definition 124 (product norm), Definition 102 (norm \(\| \cdot \|_E\) and \(\| \cdot \|_F\) are absolutely homogeneous of degree 1), and field properties of \(\mathbb{R}\), we have

\[
\|\lambda(u, v)\|_{E \times F} = \| (\lambda u, \lambda v) \|_{E \times F} = \| \lambda u \|_E + \| \lambda v \|_F
\]

\[
= |\lambda| \| u \|_E + |\lambda| \| v \|_F = |\lambda| (\| u \|_E + \| v \|_F) = |\lambda| \|(u, v)\|_{E \times F}.
\]

Hence, \(\|(\cdot, \cdot)\|_{E \times F}\) is absolutely homogeneous of degree 1.
Let \((u, v), (u', v') \in E \times F\) be vectors. Then, from Definition \ref{product vector operations} (vector addition), Definition \ref{product norm} (product norm), Definition \ref{norm} (norm) \(\|\cdot\|_E\) and \(\|\cdot\|_F\) satisfy triangle inequality, and field properties of \(\mathbb{K}\), we have

\[
\| (u, v) + (u', v') \|_{E \times F} = \|(u + u', v + v')\|_{E \times F} = \|u + u'|_E + \|v + v'|_F \\
\leq (\|u\|_E + \|u'\|_E) + (\|v\|_F + \|v'\|_F) = (\|u\|_E + \|v\|_F) + (\|u'|_E + \|v'|_F) \\
= \|(u, v)\|_{E \times F} + \|(u', v')\|_{E \times F}.
\]

Hence, \(\|(\cdot, \cdot)\|_{E \times F}\) satisfies triangle inequality.

Therefore, from Definition \ref{norm} (norm), \(\|(\cdot, \cdot)\|_{E \times F}\) is a norm over \(E \times F\), hence, from Definition \ref{normed vector space} (normed vector space), \((E \times F, \|(\cdot, \cdot)\|_{E \times F})\) is a normed space.

\begin{lemma} \textbf{(vector addition is continuous).} \label{vector addition is continuous} \textit{Let} \((E, \|\cdot\|_E)\) \textit{be a normed space}. \textit{From Lemma} \ref{product norm} \textit{(product norm)}, \textit{let} \(\|(\cdot, \cdot)\|_{E \times F}\) \textit{be the product norm induced over} \(E \times E\). \textit{Let} \(d_E\) \textit{and} \(d_{E \times E}\) \textit{be the distances associated with norms} \(\|\cdot\|_E\) \textit{and} \(\|(\cdot, \cdot)\|_{E \times E}\). \textit{Then}, \(\|(\cdot, \cdot)\|_{E \times E}\) \textit{satisfies triangle inequality}. \textit{Therefore}, \(\|(\cdot, \cdot)\|_{E \times E}\) \textit{is a norm over} \(E \times F\), \textit{hence}, \textit{from Definition} \ref{normed vector space} \textit{(normed vector space)}, \(\|(\cdot, \cdot)\|_{E \times F}\) \textit{is a normed space}. \qedhere
\end{lemma}

\begin{proof}
Let \(u, v, u', v' \in E\). From Definition \ref{distance associated with norm} (distance associated with norm), Definition \ref{norm} (norm), \((E, +)\) \textit{is an abelian group}, Definition \ref{norm} (norm) \(\|\cdot\|_E\) \textit{satisfies triangle inequality}, Definition \ref{product norm} (product norm), and Definition \ref{product vector operations} (product vector operations), we have

\[
d_E(u + v, u' + v') = \|(u + v) - (u' + v')\|_E = \|u - u' + v - v'\|_E \\
\leq \|u - u'|_E + \|v - v'|_E = \|(u - u', v - v')\|_{E \times E} = \|(u, v) - (u', v')\|_{E \times E} \\
= d_{E \times E}((u, v), (u', v')).
\]

Let \(u, v \in E\) and \(\varepsilon > 0\). Set \(\delta = \varepsilon\), then for all \(u', v' \in E\), we have

\[
d_{E \times E}((u, v), (u', v')) \leq \delta \implies d_E(u + v, u' + v') \leq \delta = \varepsilon.
\]

Therefore, from Definition \ref{continuity in a point} (continuity in a point), and Definition \ref{pointwise continuity} (pointwise continuity), the vector addition is (pointwise) continuous from \((E \times E, d_{E \times E})\) to \((E, d_E)\). \qedhere
\end{proof}

\begin{lemma} \textbf{(scalar multiplication is continuous).} \label{scalar multiplication is continuous} \textit{Let} \((E, \|\cdot\|_E)\) \textit{be a normed space}. \textit{Let} \(d\) \textit{be the distance associated with norm} \(\|\cdot\|\). \textit{Let} \(\lambda \in \mathbb{K}\) \textit{be a scalar}. \textit{Then}, \textit{the scalar multiplication by} \(\lambda\) \textit{is continuous from} \((E, d)\) \textit{to itself}. \qedhere
\end{lemma}

\begin{proof}
\textbf{Case} \(\lambda = 0_E\). \textit{Let} \(u \in E\) \textit{and} \(\varepsilon > 0\). \textit{Set} \(\delta = 1\). \textit{Then}, \(\forall u' \in E\), \textit{from Lemma} \ref{zero times yields zero} (zero times yields zero), \textit{and Definition} \ref{distance} (distance \(d\) separates points), we have

\[
d(u, u') \leq \delta = 1 \implies d(\lambda u, \lambda u') = d(0_E, 0_E) = 0 \leq \varepsilon.
\]

Therefore, from Definition \ref{continuity in a point} (continuity in a point), and Definition \ref{pointwise continuity} (pointwise continuity), the scalar multiplication by \(0_E\) is (pointwise) continuous from \((E, d_E)\) to \((E, d)\).

\textbf{Case} \(\lambda \neq 0_E\). \textit{Let} \(u, u' \in E\). \textit{From Definition} \ref{distance associated with norm} (distance associated with norm), Definition \ref{norm} (norm) \(\|\cdot\|\) \textit{is absolutely homogeneous of degree 1}, we have

\[
d(\lambda u, \lambda u') = \|\lambda u - \lambda u'\| = |\lambda| \|u - u'\| = |\lambda| d(u, u')
\]

Let \(u \in E\) \textit{and} \(\varepsilon > 0\). \textit{From properties of the absolute value over} \(\mathbb{K}\), \(|\lambda| \neq 0\) \textit{and we can set} \(\delta = \frac{\varepsilon}{|\lambda|}\). \textit{Then} \(\forall u' \in E\), \textit{we have}

\[
d(u, u') \leq \delta \implies d(\lambda u, \lambda u') \leq |\lambda| \delta = \varepsilon.
\]

Therefore, from Definition \ref{continuity in a point} (continuity in a point), and Definition \ref{pointwise continuity} (pointwise continuity), the scalar multiplication is (pointwise) continuous from \((E, d)\) to \((E, d)\). \qedhere
\end{proof}
4.4.1 Topology

Remark 130. Since a distance can be defined from a norm, normed spaces can be seen as metric spaces, hence as topological spaces too. Therefore, the important notions of continuous linear map and of closed subspace.

Remark 131. There exists a purely algebraic notion of dual of a space $E$: the space of linear forms over $E$, usually denoted $E^* = \mathcal{L}(E, \mathbb{K})$. We focus here on the notion of topological dual of a normed space $E$: the space of continuous linear forms over $E$, usually denoted $E' = \mathcal{L}_c(E, \mathbb{K})$.

Remark 132. When $W$ is a subset of the set $X$, and $f$ a mapping from $X$ to $Y$, the notation $f(W)$ denotes the subset of $Y$ made of the images of elements of $W$. Applied to a norm on a vector space, when $X$ is a subset of a normed space $(E, \|\cdot\|)$, the notation $\|X\|$ denotes the subset of $\mathbb{R}$ of values taken by norm $\|\cdot\|$ on vectors of $X$:

$$\|X\| = \{\|u\| \mid u \in X\}.$$

4.4.1.1 Continuous linear map

Lemma 133 (norm of image of unit vector). Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. Let $S_1$ be the unit sphere in $E$. Let $f \in \mathcal{L}(E,F)$ be a linear map from $E$ to $F$. Let $u \in E$ be a vector. Assume that $u \neq 0_E$. Then, \[ \frac{u}{\|u\|_E} \] belongs to $S_1$ and

$$\|f \left( \frac{u}{\|u\|_E} \right) \|_F = \frac{\|f(u)\|_F}{\|u\|_E}. \quad (79)$$

Proof. From Definition 102 (norm $\|\cdot\|_F$ is definite, contrapositive), we have $\|u\|_F \neq 0$. Thus, from Lemma 107 (norm is nonnegative), and \underline{field properties of $\mathbb{R}$}, $\frac{1}{\|u\|_E} \geq 0$. Let $\xi = \frac{u}{\|u\|_E}$. Then, from Lemma 108 (normalization by nonzero with $\lambda = 1$), we have $\|\xi\|_E = 1$. Hence, $\xi$ belongs to $S_1$.

From Definition 62 (linear map homogeneity of degree 1), Definition 102 (norm $\|\cdot\|_F$ is absolutely homogeneous of degree 1), and Lemma 107 (norm is nonnegative), we have

$$\|f(\xi)\|_F = \left\| f \left( \frac{u}{\|u\|_E} \right) \right\|_F = \left\| \frac{f(u)}{\|u\|_E} \right\|_F = \frac{\|f(u)\|_F}{\|u\|_E}. \tag{79}$$

Lemma 134 (norm of image of unit sphere). Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. Let $S_1$ be the unit sphere in $E$. Let $f \in \mathcal{L}(E,F)$ be a linear map from $E$ to $F$. Then,

$$\|f(S_1)\|_F = \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \mid u \in E, u \neq 0_E \right\}. \quad (80)$$

Proof. From Definition 102 (norm $\|\cdot\|_F$ is definite, contrapositive), and \underline{field properties of $\mathbb{R}$}, let $g : E \to \mathbb{R}$ be the mapping defined by $g(0_E) = 0$, and for all $u \in E \setminus \{0_E\}$, $g(u) = \frac{\|f(u)\|_F}{\|u\|_E}$.

Let $\xi \in S_1$ be a unit vector. From Lemma 116 (equivalent definition of unit sphere), $\|\xi\|_E = 1$. Then, from Lemma 106 (norm preserves zero contrapositive), $\xi \neq 0_E$. Thus, from \underline{field properties of $\mathbb{R}$}, we have

$$\|f(\xi)\|_F = \frac{\|f(\xi)\|_F}{\|\xi\|_E} = g(\xi) \in g(E \setminus \{0_E\}).$$

Hence, $\|f(S_1)\|_F \subseteq g(E \setminus \{0_E\})$.
Let \( u \in E \) be a vector. Assume that \( u \neq 0_E \). Then, from Lemma 133 (norm of image of unit vector), \( \xi = \frac{u}{\|u\|_E} \) belongs to \( S_1 \) and
\[
g(u) = \frac{\|f(u)\|_F}{\|u\|_E} = \|f(\xi)\|_F \in f(S_1).
\]
Hence, \( g(E\setminus\{0_E\}) \subset \|f(S_1)\|_F \).
Therefore, \( \|f(S_1)\|_F = g(E\setminus\{0_E\}) \).

**Definition 135 (operator norm).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \( f \in \mathcal{L}(E, F) \) be a linear map from \( E \) to \( F \). The operator norm on \( \mathcal{L}(E, F) \) induced by norms on \( E \) and \( F \) is the mapping \( N_{E,F} : \mathcal{L}(E, F) \to \mathbb{R} \) defined by
\[
N_{E,F}(f) = \sup \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \mid u \in E, u \neq 0_E \right\}.
\]
(81)

**Remark 136.** When restricted to continuous linear maps, the mapping \( N_{E,F} \) will be proved below to be a norm; hence its name.

**Lemma 137 (equivalent definition of operator norm).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \( S_1 \) be the unit sphere in \( E \). Let \( f \in \mathcal{L}(E, F) \) be a linear map from \( E \) to \( F \). Then,
\[
N_{E,F}(f) = \sup(\|f(S_1)\|_F).
\]
(82)

**Proof.** Direct consequence of Definition 135 (operator norm), and Lemma 134 (norm of image of unit sphere).

**Lemma 138 (operator norm is nonnegative).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Then, \( N_{E,F} \) is nonnegative.

**Proof.** Let \( f \in \mathcal{L}(E, F) \) be a linear map from \( E \) to \( F \). Then, from Lemma 137 (equivalent definition of operator norm), we have \( N_{E,F}(f) = \sup(\|f(S_1)\|_F) \). Let \( \xi \in S_1 \) be a unit vector. Then, from Lemma 107 (norm is nonnegative), \( \|f(\xi)\|_F \) is nonnegative. Therefore, from Definition 2 (supremum) \( N_{E,F}(f) \) is an upper bound for \( \|f(S_1)\|_F \), \( N_{E,F}(f) \) is nonnegative too.

**Definition 139 (bounded linear map).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. A linear map \( f \) from \( E \) to \( F \) is bounded iff
\[
\exists C \geq 0, \forall u \in E, \quad \|f(u)\|_F \leq C \|u\|_E.
\]
(83)
Then, \( C \) is called continuity constant of \( f \).

**Definition 140 (linear map bounded on unit ball).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \( B_1^E \) be the closed unit ball in \( E \). A linear map \( f \) from \( E \) to \( F \) is bounded on the closed unit ball iff there exists an upper bound for \( \|f(B_1^E)\|_F \), i.e.
\[
\exists C \geq 0, \forall \xi \in B_1^E, \quad \|f(\xi)\|_F \leq C.
\]
(84)

**Definition 141 (linear map bounded on unit sphere).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \( S_1 \) be the unit sphere in \( E \). A linear map \( f \) from \( E \) to \( F \) is bounded on the unit sphere iff there exists an upper bound for \( \|f(S_1)\|_F \), i.e.
\[
\exists C \geq 0, \forall \xi \in S_1, \quad \|f(\xi)\|_F \leq C.
\]
(85)
Theorem 142 (continuous linear map). Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \(f \in L(E, F)\) be a linear map from \(E\) to \(F\). Then, the following propositions are equivalent:

1. \(f\) is continuous in \(0_E\);
2. \(f\) is continuous;
3. \(f\) is uniformly continuous;
4. \(f\) is Lipschitz continuous;
5. \(f\) is bounded;
6. \(N_{E,F}(f)\) is finite;
7. \(f\) is bounded on the unit sphere.
8. \(f\) is bounded on the closed unit ball.

Proof. Let \(S_1\) be the unit sphere in \(E\). Let \(B_1^E\) be the closed unit ball in \(E\).

5 implies 4. Assume that \(f\) is bounded. From Definition 139 (bounded linear map), let \(C \geq 0\) such that, for all \(u \in E\), we have \(\|f(u)\|_F \leq C \|u\|_E\). Let \(u, v \in E\) be vectors. Then, from Definition 70 (vector subtraction), Definition 62 (linear map), and hypothesis, we have

\[
\|f(u) - f(v)\|_F = \|f(u - v)\|_F \leq C \|u - v\|_E.
\]

Hence, from Definition 46 (Lipschitz continuity), \(f\) is \(C\)-Lipschitz continuous.

4 implies 3. Assume that \(f\) is Lipschitz continuous. Then, from Lemma 51 (Lipschitz continuous is uniform continuous), \(f\) is uniformly continuous.

3 implies 2. Assume that \(f\) is uniformly continuous. Then, from Lemma 49 (uniform continuous is continuous), \(f\) is (pointwise) continuous.

2 implies 1. Assume that \(f\) is (pointwise) continuous. Then, from Definition 43 (pointwise continuity), \(f\) is continuous in \(0_E\).

1 implies 8. Assume now that \(f\) is continuous in \(0_E\). Let \(\varepsilon = 1 > 0\). Then, from Definition 12 (continuity in a point), and Lemma 91 (linear map preserves zero), let \(\delta > 0\) such that, for all \(u \in E\), \(\|u - 0_E\|_E = \|u\|_E \leq \delta\) implies \(\|f(u) - f(0_E)\|_F = \|f(u)\|_F \leq 1\). Let \(C = \frac{1}{\delta} > 0 > 0\). Let \(\xi \in B_1^E\) be a vector in the unit ball. From Lemma 114 (equivalent definition of closed unit ball), \(\|\xi\|_E \leq 1\). Then, from Definition 102 (norm \(\|\cdot\|_F\) is absolutely homogeneous of degree 1), and ordered field properties of \(\mathbb{R}\), we have \(\|\xi\|_E \leq \delta \|\xi\|_E < \delta\). Thus, from Definition 102 (norm \(\|\cdot\|_F\) is absolutely homogeneous of degree 1), Definition 62 (linear map homogeneity of degree 1), ordered field properties of \(\mathbb{R}\), and hypothesis, we have \(\|f(\xi)\|_F = \frac{1}{\delta} \|f(\delta \xi)\|_F \leq \frac{1}{\delta} = C\).

Hence, from Definition 140 (linear map bounded on unit ball), \(f\) is bounded on the unit ball.

8 implies 7. Assume now that \(f\) is bounded on the unit ball. From Definition 140 (linear map bounded on unit ball), and Lemma 114 (equivalent definition of closed unit ball), let \(C \geq 0\) such that for all \(\xi \in B_1^E\), \(\|f(\xi)\|_F \leq C\). Let \(\xi \in S_1\) be a unit vector. Then, from Lemma 116 (equivalent definition of unit sphere), and Lemma 114 (equivalent definition of closed unit ball), we also have \(\xi \in B_1^E\). Thus, from hypothesis, \(\|f(\xi)\|_F \leq C\). Hence, from Definition 141 (linear map bounded on unit sphere with same constant \(C\)), \(f\) is bounded on the unit sphere.

7 implies 6. Assume then that \(f\) is bounded on the unit sphere. Then, from Definition 141 (linear map bounded on unit sphere), there exists a finite upper bound \(C \geq 0\) for \(\|f(S_1)\|_F\). Hence, from Lemma 3 (finite supremum), \(\sup(\|f(S_1)\|_F)\) is finite, and from Lemma 137 (equivalent definition of operator norm), \(N_{E,F}(f)\) is finite.
6 implies 5. Assume finally that \( N_{E,F}(f) \) is finite. Let \( C = N_{E,F}(f) \). Then, from Lemma 138 (operator norm is nonnegative), \( C \) is nonnegative. Let \( u \in E \) be a vector.

Case \( u = 0_E \). Then, from Lemma 101 (linear map preserves zero), \( f(u) = f(0_E) = 0_F \). Hence, from Lemma 106 (norm preserves zero for \( \|\cdot\|_F \) and \( \|\cdot\|_F \)), and ordered field properties of \( \mathbb{R} \), we have

\[
\|f(u)\|_F = 0 \leq 0 = C \|u\|_E.
\]

Case \( u \neq 0_E \). Then, from Definition 102 (norm \( \|\cdot\|_F \) is definite, contrapositive), \( \|u\|_F \neq 0 \). Thus, from field properties of \( \mathbb{R} \), Definition 135 (operator norm), and Definition 2 (supremum) \( N_{E,F}(f) \) is an upper bound for \( \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \mid u \in E, u \neq 0_E \right\} \), we have

\[
\|f(u)\|_F = \frac{\|f(u)\|_F}{\|u\|_E} \|u\|_E \leq N_{E,F}(f) \|u\|_E = C \|u\|_E.
\]

Hence, from Definition 139 (bounded linear maps), \( f \) is bounded.

Therefore, we have \( 5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 8 \Rightarrow 7 \Rightarrow 6 \Rightarrow 5 \) hence all properties are equivalent.

\[\square\]

**Definition 143 (set of continuous linear maps).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. The set of continuous linear maps from \( E \) to \( F \) is denoted \( \mathcal{L}_c(E, F) \).

**Lemma 144 (finite operator norm is continuous).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \( f \in \mathcal{L}(E, F) \) be a linear map from \( E \) to \( F \). Then, \( f \) belongs to \( \mathcal{L}_c(E, F) \) (i.e. \( f \) is continuous) if and only if \( N_{E,F}(f) \) is finite.

Moreover, let \( B_1^E \) and \( S_1 \) be the closed unit ball and the unit sphere in \( E \), and let \( C \geq 0 \), then we have the following equivalences:

\[
N_{E,F}(f) \leq C \iff C \text{ is an upper bound for } \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \mid u \in E, u \neq 0_E \right\} \quad (86)
\]

- \( C \) is a continuity constant for \( f \)
- \( C \) is an upper bound for \( \|f(B_1^E)\|_F \)
- \( C \) is an upper bound for \( \|f(S_1)\|_F \).

**Proof.** Direct consequences of Definition 143 (set of continuous linear maps), Theorem 142 (continuous linear map), Definition 2 (supremum) \( N_{E,F}(f) \) is the least upper bound of \( \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \mid u \in E, u \neq 0_E \right\} \), Definition 139 (bounded linear maps), Definition 140 (linear map bounded on unit ball), and Definition 141 (linear map bounded on unit sphere).

**Lemma 145 (linear isometry is continuous).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \( f \in \mathcal{L}(E, F) \) be a linear map from \( E \) to \( F \). Assume that \( f \) is a linear isometry from \( E \) to \( F \). Then, \( f \) belongs to \( \mathcal{L}_c(E, F) \) (i.e. \( f \) is continuous).

**Proof.** Let \( S_1 \) be the unit sphere in \( E \). Let \( \xi \in S_1 \) be a unit vector. Then, from Definition 122 (linear isometry), we have \( \|f(\xi)\|_F = \|\xi\|_E = 1 \leq 1 \). Hence, from Lemma 144 (finite operator norm is continuous) \( 1 \) is an upper bound for \( \|f(S_1)\|_F \), \( f \) belongs to \( \mathcal{L}_c(E, F) \).

**Lemma 146 (identity map is continuous).** Let \((E, \|\cdot\|_E)\) be a normed space. Then, the identity map \( \text{Id}_E \) belongs to \( \mathcal{L}_c(E, E) \) (i.e. \( \text{Id}_E \) is continuous).

**Proof.** Direct consequence of Lemma 123 (identity map is linear isometry), and Lemma 145 (linear isometry is continuous).

Inria
**Theorem 147 (normed space of continuous linear maps).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Let \(\|\cdot\|_{F,E}\) be the restriction of \(N_{E,F}\) to continuous linear maps. Then, \((\mathcal{L}_c(E,F), \|\cdot\|_{F,E})\) is a normed space.

*Proof.* Let \(S_1\) be the unit sphere in \(E\). From Definition 143 (set of continuous linear maps), \(\mathcal{L}_c(E,F)\) is obviously a subset of \(\mathcal{L}(E,F)\).

Let \(f \in \mathcal{L}_c(E,F)\) be a continuous linear map from \(E\) to \(F\). Then, from Lemma 144 (finite operator norm is continuous), \(\|f\|_{F,E}\) is finite. Hence, \(\|\cdot\|_{F,E}\) is a mapping from \(\mathcal{L}_c(E,F)\) to \(\mathbb{R}\).

From Definition 144 (linear map bounded on unit sphere, with \(C = 0\)), and Lemma 144 (finite operator norm is continuous upper bound for \(\|\cdot\|_{F,E}(S_1)\)), \(0 \leq \|f\|_{F,E}\) belongs to \(\mathcal{L}_c(E,F)\).

Let \(f \in \mathcal{L}_c(E,F)\) be a continuous linear map from \(E\) to \(F\). Assume that \(\|f\|_{F,E} = 0\). Let \(\xi \in S_1\) be a unit vector. Then, from Lemma 107 (norm is nonnegative), Lemma 137 (equivalent definition of operator norm), and Definition 2 (supremum \(\|\cdot\|_{F,E}\) is an upper bound for \(\|f(S_1)\|_F\)), we have

\[
0 \leq \|f(\xi)\|_F \leq \|f\|_{F,E} = 0.
\]

Thus, \(\|f(\xi)\|_F = 0\), and from Definition 102 (norm \(\|\cdot\|_E\) is definite), \(f(\xi) = 0_F\). Hence, from Lemma 117 (zero on unit sphere is zero), \(f = 0_{\mathcal{L}(E,F)}\), and \(\|\cdot\|_{F,E}\) is definite.

Let \(\lambda \in \mathbb{K}\) be a scalar. Let \(f, g \in \mathcal{L}_c(E,F)\) be a continuous linear map from \(E\) to \(F\). Let \(\xi \in S_1\) be a unit vector. Then, from Definition 88 (inherited vector operations, scalar multiplication), and Definition 102 (norm \(\|\cdot\|_F\) is absolutely homogeneous of degree 1), we have

\[
\|(\lambda f)(\xi)\|_F = \|\lambda f(\xi)\|_F = |\lambda| \|f(\xi)\|_F.
\]

Thus, from Lemma 137 (equivalent definition of operator norm), Lemma 5 (supremum is positive scalar multiplication), and nonnegativity of absolute value, we have

\[
N_{E,F}(\lambda f) = \sup\{\|(\lambda f)(\xi)\|_F\} = |\lambda| \sup\{\|f(\xi)\|_F\} = |\lambda| \|f\|_{F,E}.
\]

Then, from Lemma 144 (finite operator norm is continuous \(N_{E,F}(\lambda f)\) is finite), \(\lambda f\) belongs to \(\mathcal{L}_c(E,F)\). Hence, \(\|\cdot\|_{F,E}\) is absolutely homogeneous of degree 1, and \(\mathcal{L}_c(E,F)\) is closed under scalar multiplication.

Let \(f, g \in \mathcal{L}_c(E,F)\) be continuous linear maps from \(E\) to \(F\). Let \(\xi \in S_1\) be a unit vector. Then, from Definition 88 (inherited vector operations vector addition), Definition 102 (norm \(\|\cdot\|_F\) satisfies triangle inequality), Lemma 137 (equivalent definition of operator norm), Definition 2 (supremum \(\|\cdot\|_{F,E}\) resp. \(\|\cdot\|_{F,F}\) is an upper bounds for \(\|f(S_1)\|_F\) resp. \(\|g(S_1)\|_F\)), and field properties of \(\mathbb{R}\), we have

\[
\|(f + g)(\xi)\|_F \leq \|f(\xi)\|_F + \|g(\xi)\|_F \leq \|f\|_{F,E} + \|g\|_{F,E}.
\]

Thus, from Lemma 144 (finite operator norm is continuous), \(\|f\|_{F,E} + \|g\|_{F,E}\) is a finite upper bound for \(\|(f + g)(S_1)\|_F\), \(f + g\) belongs to \(\mathcal{L}_c(E,F)\) and

\[
\|f + g\|_{F,E} \leq \|f\|_{F,E} + \|g\|_{F,E}.
\]

Hence, \(\mathcal{L}_c(E,F)\) is closed under vector addition and \(\|\cdot\|_{F,E}\) satisfies triangle inequality.

Therefore, from Lemma 78 (closed under vector operations is subspace), Definition 102 (norm), and Definition 104 (normed vector space), \(\mathcal{L}_c(E,F)\) is a subspace of \(\mathcal{L}(E,F)\), \(\|\cdot\|_{F,E}\) is a norm over \(\mathcal{L}(E,F)\), and \((\mathcal{L}_c(E,F), \|\cdot\|_{F,E})\) is a normed space.

\[\square\]

**Lemma 148 (operator norm estimation).** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces. Then,

\[
\forall f \in \mathcal{L}_c(E,F), \forall u \in E, \quad \|f(u)\|_F \leq \|f\|_{F,E} \|u\|_E.
\]
Proof. Let \( f \in \mathcal{L}_c(E,F) \) be a continuous linear map from \( E \) to \( F \). Then, from Theorem \( \ref{thm:finite-operator-norm} \) (normed space of continuous linear maps), \( \|f\|_{F,E} \) is finite. Let \( u \in E \) be a vector.

Case \( u = 0_E \). Then, from Lemma \( \ref{lem:linear-map-preserves-zero} \) (linear map preserves zero), Lemma \( \ref{lem:norm-preserves-zero} \) (norm preserves zero for \( \|\cdot\|_E \) and \( \|\cdot\|_F \)), and ordered field properties of \( \mathbb{R} \), we have

\[
\|f(u)\|_F = \|f(0_E)\|_F = \|0_F\|_F = 0 \leq \|f\|_{F,E} = \|f\|_{F,E} \|u\|_E.
\]

Case \( u \neq 0_E \). Then, from Definition \( \ref{def:operator-norm} \) (operator norm), and Definition \( \ref{def:supremum} \) (supremum of \( \|f\|_{F,E} \)), we have \( \|f(u)\|_E \leq \|f\|_{F,E} \). From Definition \( \ref{def:norm} \) (norm is definite, contrapositive), and Lemma \( \ref{lem:norm-is-nonnegative} \) (norm is nonnegative for \( \|\cdot\|_E \)), \( \|u\|_E > 0 \).

Hence, from ordered field properties of \( \mathbb{R} \), \( \|f(u)\|_F \leq \|f\|_{F,E} \|u\|_E \).

\begin{lemma}[continuous linear maps have closed kernel] \label{lem:continuous-linear-maps-have-closed-kernel}
Let \( (E,\|\cdot\|_E) \) and \( (F,\|\cdot\|_F) \) be normed spaces. Then, \( f \in \mathcal{L}_c(E,F) \) is a continuous linear map from \( E \) to \( F \). Then, \( \ker(f) \) is closed in \( E \).
\end{lemma}

Proof. Direct consequence of Definition \( \ref{def:kernel} \) (kernel), Lemma \( \ref{lem:linear-map-preserves-zero} \) (linear map preserves zero), \( \ker(f) = f^{-1}(\{0_F\}) \), Lemma \( \ref{lem:singleton-is-closed} \) (singleton is closed), \( \{0_F\} \) is closed, and preimages of closed subsets by continuous mappings are closed.

\begin{lemma}[compatibility of composition with continuity] \label{lem:compatibility-composition-continuity}
Let \( (E,\|\cdot\|_E) \), \( (F,\|\cdot\|_F) \) and \( (G,\|\cdot\|_G) \) be normed spaces. Then,

\[
\forall f \in \mathcal{L}_c(E,F), \forall g \in \mathcal{L}_c(F,G), \quad g \circ f \in \mathcal{L}_c(E,G) \quad \text{and} \quad \|g \circ f\|_{G,E} \leq \|g\|_{G,F} \|f\|_{F,E}. \tag{88}
\]

Proof. Let \( S_1 \) be the unit sphere in \( E \). Let \( f \in \mathcal{L}_c(E,F) \) and \( g \in \mathcal{L}_c(F,G) \) be continuous linear maps. Then, from Lemma \( \ref{lem:composition-linear-maps-bilinear} \) (composition of linear maps is bilinear), \( g \circ f \) belongs to \( \mathcal{L}(E,F) \). Let \( \xi \in S_1 \) be a unit vector. Then, from the definition of composition of functions, Lemma \( \ref{lem:operator-norm-estimation} \) (operator norm estimation for \( g \) and \( f \)), Lemma \( \ref{lem:equivalent-definition-unit-sphere} \) (equivalent definition of unit sphere \( \|\xi\|_E = 1 \)), and field properties of \( \mathbb{R} \), we have

\[
\|(g \circ f)(\xi)\|_F = \|g(f(\xi))\|_F \leq \|g\|_{G,F} \|f(\xi)\|_F \leq \|g\|_{G,F} \|f\|_{F,E} \|\xi\|_E = \|g\|_{G,F} \|f\|_{F,E}. 
\]

Hence, from Lemma \( \ref{lem:finite-operator-norm-is-continuous} \) (finite operator norm is continuous), \( \|g \circ f\|_{G,E} \|f\|_{F,E} \) is a finite upper bound for \( \|(g \circ f)(S_1)\|_E \), \( g \circ f \) belongs to \( \mathcal{L}_c(E,F) \) and

\[
\|g \circ f\|_{G,E} \leq \|g\|_{G,F} \|f\|_{F,E}. 
\]

\begin{lemma}[complete normed space of continuous linear maps] \label{lem:complete-normed-space-linear-maps}
Let \( (E,\|\cdot\|_E) \) and \( (F,\|\cdot\|_F) \) be normed spaces. If \( (F,\|\cdot\|_F) \) is complete, then the normed space \( \mathcal{L}_c(E,F) \) is also complete (i.e. they are both Banach spaces).
\end{lemma}

Proof. Case \( E = \{0_E\} \). Then, from Lemma \( \ref{lem:linear-map-preserves-zero} \) (linear map preserves zero), \( \mathcal{L}_c(E,F) \) is also the singleton \( \{0_{\mathcal{L}_c(E,F)}\} \). From Definition \( \ref{def:complete-space} \) (complete metric space), and Lemma \( \ref{def:stationary-sequance-is-convergent} \) (stationary sequence is convergent), singletons are trivially complete metric spaces since they possess only one sequence which is constant, hence stationary, hence convergent. Therefore, \( \mathcal{L}_c(E,F) \) is complete.

Case \( E \neq \{0_E\} \).

Pointwise limit. Let \( d_{F,E} \) be the distance associated with norm \( \|\cdot\|_{F,E} \). From Lemma \( \ref{lem:norm-gives-distance} \) (norm gives distance), \( \mathcal{L}_c(E,F),d_{F,E} \) is a metric space. Let \( (f_n)_{n \in \mathbb{N}} \) be a Cauchy sequence in \( (\mathcal{L}_c(E,F),d_{F,E}) \). Then, from Definition \( \ref{def:cauchy-sequence} \) (Cauchy sequence), Definition \( \ref{def:distance-associated} \) (distance associated to a metric).
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

with norm. Theorem 147 (norm of continuous linear maps), Definition 136 (operator norm), and Lemma 144 (finite operator norm is continuous). \( \|f - f_0\|_{F,E} \) is lower than or equal to const. \( \forall \varepsilon > 0 \), \( \exists N \in \mathbb{N}, \forall p, q \in N, \ p, q \geq N \) \( \implies \forall u \in E, \ \|f_p(u) - f_q(u)\|_F \leq \varepsilon \|u\|_E \). (89)

Let \( u \in E \). Case \( u \not= 0_E \). Let \( \varepsilon' > 0 \). From Definition 102 (norm of \( \|\cdot\|_F \) is definite, contrapositive), \( \|u\|_E \not= 0 \) and from Equation (89) with \( \varepsilon = \varepsilon' \|u\|_E \), we have

\[ \exists N \in \mathbb{N}, \forall p, q \in N, \ p, q \geq N \implies \|f_p(u) - f_q(u)\|_F \leq \varepsilon'. \]

Thus, from Definition 34 (Cauchy sequence \( (f_n(u))_{n \in \mathbb{N}} \) is a Cauchy sequence), and Definition 37 (complete subset \( F \) is complete), let \( f(u) = \lim_{n \to +\infty} f_n(u) \) be the limit in \( F \).

Case \( u = 0_E \). Since from Lemma 91 (linear map preserves zero), we have for all \( n \in \mathbb{N} \), \( f_n(0_E) = 0_F \), let \( f(0_E) = 0_F = \lim_{n \to +\infty} f_n(0_E) \).

Linearity. Let \( u, v \in E \) and \( \lambda, \mu \in \mathbb{K} \). From Definition 62 (linear map for all \( n \in \mathbb{N} \), \( f_n \) is a linear map), Lemma 128 (vector addition is continuous), Lemma 129 (scalar multiplication is continuous), and Lemma 44 (compatibility of limit with continuous functions), we have

\[ f(\lambda u + \mu v) = \lim_{n \to +\infty} f_n(\lambda u + \mu v) = \lim_{n \to +\infty} (\lambda f_n(u) + \mu f_n(v)) = \lim_{n \to +\infty} (\lambda f_n(u)) + \lim_{n \to +\infty} (\mu f_n(v)) = \lambda \lim_{n \to +\infty} f_n(u) + \mu \lim_{n \to +\infty} f_n(v) = \lambda f(u) + \mu f(v). \]

Hence, from Lemma 92 (linear map preserves linear combinations), \( f \) belongs to \( \mathcal{L}(E,F) \).

Continuity. In Equation (89), we consider a fixed \( u \in E \) and we take the limit when \( q \) goes to \( +\infty \). Thus, from Lemma 121 (norm is continuous), Lemma 44 (compatibility of limit with continuous functions), and Definition 88 (inherited vector operations on \( \mathcal{L}(E,F) \)), we have

\[ \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in N, \ p \geq N \implies \forall u \in E, \ (\|f_p - f\|_F) \leq \varepsilon \|u\|_E. \] (90)

Hence, from Definition 139 (bounded linear map \( f_p - f \) is bounded), Theorem 142 (continuous linear map \( f_p - f \) is continuous), Theorem 147 (norm of continuous linear maps \( \mathcal{L}_c(E,F) \) is a space), and Definition 58 (vector space \( \mathcal{L}_c(E,F) \) is an abelian group), we have \( f = f_p - (f_p - f) \) belongs to \( \mathcal{L}_c(E,F) \).

Limit for \( \|\cdot\|_{F,E} \): From Lemma 144 (finite operator norm is continuous) \( \varepsilon \) is a continuity constant for \( f_p - f \), Equation (90) becomes

\[ \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in N, \ p \geq N \implies \|f_p - f\|_{F,E} \leq \varepsilon. \]

Hence, from Definition 109 (distance associated with norm \( \|\cdot\|_{F,E} \)), and Definition 26 (convergent sequence), the sequence \( (f_n)_{n \in \mathbb{N}} \) is convergent in \( \mathcal{L}_c(E,F) \) for the distance \( d_{F,E} \).

Therefore, from Definition 37 (complete subset), the normed space \( \mathcal{L}_c(E,F) \) is complete.

**Definition 152 (topological dual).** Let \( (E,\|\cdot\|_E) \) be a normed space. The set of continuous linear forms on \( E \), denoted \( E' = \mathcal{L}_c(E,\mathbb{K}) \), is called the topological dual of \( E \).

**Definition 153 (dual norm).** Let \( (E,\|\cdot\|_E) \) be a normed space. The dual norm associated with \( \|\cdot\|_E \), denoted \( \|\cdot\|_{E'} \), is the operator norm \( \|\cdot\|_{E,F} \) on \( E' = \mathcal{L}_c(E,\mathbb{K}) \) induced by norms \( \|\cdot\|_E \) and \( |\cdot| \) (absolute value over \( \mathbb{K} \)).

**Lemma 154 (topological dual is complete normed space).** Let \( (E,\|\cdot\|_E) \) be a normed space. Let \( E' \) be the topological dual of \( E \). Let \( \|\cdot\|_{E'} \) be the associated dual norm. Then, \( (E',\|\cdot\|_{E'}) \) is a complete normed space.

RR n° 8934
Proof. Direct consequence of Definition 153 [dual norm], Theorem 147 [normed space of continuous linear maps], Lemma 151 [complete normed space of continuous linear maps], Lemma 105 [\(K\) is normed space], and the completeness of \(K\).

**Definition 155 (bra-ket notation).** Let \((E, \| \cdot \|_E)\) be a normed space. A continuous linear form \(\varphi \in E^\ast\) is a bra, denoted \(\langle \varphi \rangle\). A vector \(u \in E\) is a ket, denoted \(|u\rangle\). In bra-ket notation (or Dirac notation, or duality pairing), the application \(\varphi\) is denoted \(\langle \varphi | u \rangle_{E'\times E}\).

**Lemma 156 (bra-ket is bilinear map).** Let \((E, \| \cdot \|_E)\) be a normed space. Then, \(\langle \cdot | \cdot \rangle_{E'\times E}\) is a continuous linear map from \(E' \times E\) to \(\mathbb{K}\).

**Definition 157 (bounded bilinear form).** Let \((E, \| \cdot \|_E)\) be a normed space. A bilinear form \(\varphi \in \mathcal{L}_2(E)\) is bounded iff

\[
\exists C \geq 0, \forall u, v \in E, \quad |\varphi(u, v)| \leq C \|u\|_E \|v\|_E. \quad (91)
\]

Then, \(C\) is called the continuity constant of \(\varphi\).

**Lemma 158 (representation for bounded bilinear form).** Let \((E, \| \cdot \|_E)\) be a normed space. Let \(S_1\) be the unit sphere of \(E\). Let \(\varphi \in \mathcal{L}_2(E)\) be a bilinear form on \(E\). Assume that \(\varphi\) is bounded. Then, there exists a unique continuous linear map \(A \in \mathcal{L}_c(E, E')\) such that

\[
\forall u, v \in E, \quad \varphi(u, v) = \langle A(u)|v \rangle_{E'\times E} = A(u)(v). \quad (92)
\]

Moreover, for all \(C\) continuity constant of \(\varphi\), we have

\[
\|A\|_{E'\times E} \leq C. \quad (93)
\]

**Proof.** From Definition 67 [set of bilinear forms], and Definition 66 [bilinear form], \(\varphi\) is a bilinear map.

**Existence.** Let \(u \in E\) be a vector. Let \(A_u : E \to \mathbb{R}\) be the mapping defined by

\[
\forall v \in E, \quad A_u(v) = \varphi(u, v).
\]

Let \(\lambda, \lambda' \in \mathbb{R}\) be scalars. Let \(v, v' \in E\) be vectors. Then, from Definition 65 [bilinear map], \(\varphi\) is right linear, we have

\[
A_u(\lambda v + \lambda' v') = \varphi(u, \lambda v + \lambda' v') = \lambda \varphi(u, v) + \lambda' \varphi(u, v') = \lambda A_u(v) + \lambda' A_u(v').
\]

F. Clément, & V. Martin

Inria
Hence, from Lemma 92 (linear map preserves linear combinations), and Definition 94 (linear form), $A_u$ is a linear form on $E$.

Let $v \in E$ be a vector. From Definition 157 (bounded bilinear form for $\varphi$), let $C \geq 0$ such that, for all $u', v' \in E$, we have $|\varphi(u', v')| \leq C \|u'\|_E \|v'\|_E$. Let $C_u = C \|u\|_E$. Then, from Lemma 107 (norm is nonnegative), and ordered field properties of $\mathbb{R}$, we have $C_u \geq 0$ and

$$|A_u(v)| = |\varphi(u, v)| \leq C \|u\|_E \|v\|_E.$$  

Hence, from Definition 139 (bounded linear map $A_u$ is bounded), Definition 152 (topological dual), Definition 153 (dual norm), and Lemma 144 (finite operator norm is continuous $A_u \in E'$), we have

$$\|A_u\|_{E'} \leq C_u = C \|u\|_E.$$  

Let $A : E \to E'$ be the mapping defined by, for all $u \in E$, $A(u) = A_u$, i.e.

$$\forall u, v \in E, \quad \langle A(u)v, v \rangle_{E', E} = A(u)(v) = A_u(v) = \varphi(u, v).$$

Let $\lambda, \lambda' \in \mathbb{R}$ be scalars. Let $u, u', v \in E$ be vectors. Then, from Definition 65 (bilinear map $\varphi$ is left linear), Definition 88 (inherited vector operations on $E'$), and Lemma 154 (topological dual is complete normed space $E'$ is space), we have

$$A(\lambda u + \lambda' u')(v) = A_{\lambda u + \lambda' u'}(v) = \varphi(\lambda u + \lambda' u', v) = \lambda \varphi(u, v) + \lambda' \varphi(u', v) = \lambda A_u(v) + \lambda' A_{u'}(v) = \lambda A(u)(v) + \lambda' A_u'(v) = (\lambda A(u) + \lambda' A_{u'})(v).$$

Hence, from Lemma 92 (linear map preserves linear combinations), $A$ is a linear map from $E$ to $E'$.

Let $\xi \in S_1$ be a unit vector. Then, from Lemma 116 (equivalent definition of unit sphere $\|\xi\|_E = 1$), we have

$$\|A(\xi)\|_{E'} = \|A\xi\|_{E'} \leq C \|\xi\|_E = C.$$  

Hence, from Lemma 144 (finite operator norm is continuous $C$ is a finite upper bound for $\|A(S_1)\|_{E'}$), $A$ belongs to $L_c(E, E')$ and $\|A\|_{E', E} \leq C$.

**Uniqueness.** Let $A, A' \in L_c(E, E')$ be continuous linear maps such that

$$\forall u, v \in E, \quad \varphi(u, v) = \langle A(u)v, v \rangle_{E', E} = \langle A'(u)v, v \rangle_{E', E}.$$  

From Theorem 147 (normed space of continuous linear maps), Definition 104 (normed vector space $L_c(E, E')$ is a space), and Definition 70 (vector subtraction), let $B = A - A' \in L_c(E, E')$. Let $u, v \in E$ be vectors. Then, from Lemma 156 (bracket is bilinear map), Lemma 101 ($K$ is space), and Definition 58 (vector space $([K, +])$ is an abelian group), we have

$$\langle B(u)v, v \rangle_{E', E} = \langle (A(u) - A'(u))v, v \rangle_{E', E} = \varphi(u, v) - \varphi(u, v) = 0.$$  

Thus, from Definition 155 (bra-ket notation), $B(u) = 0_{E'}$, and then $B = 0_{L_c(E, E')}$. Hence, from Definition 58 (vector space $([L_c(E, E'), +])$ is an abelian group), $A = A'$.

**Definition 159 (coercive bilinear form).** Let $(E, \|\cdot\|_E)$ be a real normed space. A bilinear form $\varphi \in L_2(E)$ is coercive (or elliptic) iff

$$\exists \alpha > 0, \forall u \in E, \quad \varphi(u, u) \geq \alpha \|u\|^2_E.$$  

That $\alpha$ is called coercivity constant of $\varphi$.  

Lemma 160 (coercivity constant is less than continuity constant). Let $(E, \| \cdot \|_E)$ be a real normed space. Let $\varphi \in \mathcal{L}_2(E)$ be a bilinear form on $E$. Assume that $\varphi$ is continuous with constant $C \geq 0$, and coercive with constant $\alpha > 0$. Then, $\alpha \leq C$.

Proof. Let $u \in E$ be a vector. Assume that $u \neq 0_E$. Then, from Definition 102 (norm is definite, contrapositive), and Lemma 107 (norm is nonnegative for $\| u \|_E$), we have $\| u \|_E > 0$. From Definition 159 (coercive bilinear form), properties of the absolute value on $\mathbb{R}$, and Definition 157 (bounded bilinear form with $v = u$), we have

$$\alpha \| u \|_E^2 \leq \varphi(u, u) \leq |\varphi(u, u)| \leq C \| u \|_E^2.$$ 

Hence, from ordered field properties of $\mathbb{R}$, $\alpha \leq C$. □

4.5 Inner product space

Definition 161 (inner product). Let $G$ be a real space. A mapping $(\cdot, \cdot)_G : G \times G \to \mathbb{R}$ is an inner product on $G$ iff it is a bilinear form on $G$ that is symmetric, nonnegative, and definite:

$$\begin{align*}
\forall u, v \in G, & \quad (u, v)_G = (v, u)_G; \\
\forall u \in G, & \quad (u, u)_G \geq 0; \\
\forall u \in G, & \quad (u, u)_G = 0 \implies u = 0_G.
\end{align*}$$

Remark 162. Note that the symmetry property (95) implies the equivalence between left additivity (45) and right additivity (46) in the definition of a bilinear map.

Remark 163. Most results below are valid on a semi-inner space in which the definite property (97) is dropped. The associated norm is then a semi-norm (the separation property is dropped).

Remark 164. In the case of a complex space, the symmetry property becomes a conjugate symmetry property. In the sequel, we specify that the space is real only in the case where the very same statement does not hold in a complex space. When proofs differ, they are only given in the real case.

Definition 165 (inner product space). $(G, (\cdot, \cdot)_G)$ is an inner product space (or pre-Hilbert space) iff $G$ is a space and $(\cdot, \cdot)_G$ is an inner product on $G$.

Lemma 166 (inner product subspace). Let $(G, (\cdot, \cdot)_G)$ be an inner product space. Let $F$ be a subspace of $G$. Then, $F$ equipped with the restriction to $F$ of the inner product $(\cdot, \cdot)_G$ is an inner product space.

Proof. Direct consequence of Definition 74 (subspace $F$ is a subset of $G$ and $F$ is a space), Definition 161 (inner product), the restriction of $(\cdot, \cdot)_G$ to $F$ is trivially an inner product on $F$), and Definition 165 (inner product space).

Lemma 167 (inner product with zero is zero). Let $(G, (\cdot, \cdot)_G)$ be an inner product space. Then,

$$\forall u \in G, \quad (0, u)_G = (u, 0)_G = 0.$$ 

Proof. Let $u \in G$. From Definition 161 (inner product $(\cdot, \cdot)_G$ is symmetric and a bilinear map), Definition 58 (vector space $(G, +)$ is an abelian group), Definition 70 (vector subtraction), Definition 65 (bilinear map $(\cdot, \cdot)_G$ is right linear), and field properties of $\mathbb{R}$, we have

$$(0_G, u)_G = (u, 0_G)_G = (u, 0_G - 0_G)_G = (u, 0_G)_G - (u, 0_G)_G = 0.$$
Lemma 168 (square expansion plus). Let \((G, (\cdot, \cdot)_G)\) be a real inner product space. Then,
\[
\forall u, v \in G, \quad (u + v, u + v)_G = (u, u)_G + 2(u, v)_G + (v, v)_G. \tag{99}
\]

**Proof.** Let \(u, v \in G\) be vectors. From Definition 161 (inner product \((\cdot, \cdot)_G\) is a bilinear map and symmetric), Definition 65 (bilinear map), and field properties of \(\mathbb{R}\), we have
\[
(u + v, u + v)_G = (u, u)_G + (u, v)_G + (v, u)_G + (v, v)_G = (u, u)_G + 2(u, v)_G + (v, v)_G.
\]

Lemma 169 (square expansion minus). Let \((G, (\cdot, \cdot)_G)\) be a real inner product space. Then,
\[
\forall u, v \in G, \quad (u - v, u - v)_G = (u, u)_G - 2(u, v)_G + (v, v)_G. \tag{100}
\]

**Proof.** Let \(u, v \in G\) be vectors. From Definition 161 (inner product \((\cdot, \cdot)_G\) is a bilinear map), Definition 70 (vector subtraction), Lemma 168 (square expansion plus), and field properties of \(\mathbb{R}\), we have
\[
(u - v, u - v)_G = (u + (-v), u + (-v))_G = (u, u)_G + 2(u, -v)_G + (-v, -v)_G = (u, u)_G - 2(u, v)_G + (v, v)_G.
\]

Lemma 170 (parallelogram identity). Let \((G, (\cdot, \cdot)_G)\) be an inner product space. Then,
\[
\forall u, v \in G, \quad (u + v, u + v)_G + (u - v, u - v)_G = 2((u, u)_G + (v, v)_G). \tag{101}
\]

**Proof.** Let \(u, v \in G\) be vectors. From Lemma 168 (square expansion plus), Lemma 169 (square expansion minus), and field properties of \(\mathbb{R}\), we have
\[
(u + v, u + v)_G + (u - v, u - v)_G = (u, u)_G + 2(u, v)_G + (v, v)_G + (u, u)_G - 2(u, v)_G + (v, v)_G = 2((u, u)_G + (v, v)_G).
\]

Lemma 171 (Cauchy–Schwarz inequality). Let \((G, (\cdot, \cdot)_G)\) be a real inner product space. Then,
\[
\forall u, v \in G, \quad ((u, v)_G)^2 \leq (u, u)_G (v, v)_G. \tag{102}
\]

**Proof.** Let \(u, v \in G\) be vectors. Let \(\lambda \in \mathbb{R}\) be a scalar. From Lemma 168 (square expansion plus), Definition 161 (inner product \((\cdot, \cdot)_G\) is a bilinear map), Definition 65 (bilinear map), and field properties of \(\mathbb{R}\), we have
\[
(u + \lambda v, u + \lambda v)_G = \lambda^2 (v, v)_G + 2\lambda (u, v)_G + (u, u)_G.
\]
Let \(P(X) = (v, v)_G X^2 + 2(u, v)_G X + (u, u)_G\). It is a quadratic polynomial with real coefficients. From Definition 161 (inner product \((\cdot, \cdot)_G\) is nonnegative), the associated polynomial function \(P\) is nonnegative. Hence, since a quadratic polynomial function has a constant sign iff its discriminant is nonpositive, we have
\[
4((u, v)_G)^2 - 4(v, v)_G (u, u)_G \leq 0.
\]
Therefore, from ordered field properties of \(\mathbb{R}\), we have
\[
((u, v)_G)^2 \leq (u, u)_G (v, v)_G.
\]
Definition 172 (square root of inner square). Let $(G, ⟨·, ·⟩_G)$ be an inner product space. The associated square root of inner square is the mapping $N_G : G \rightarrow \mathbb{R}$ defined by

$$\forall u \in G, \quad N_G(u) = \sqrt{(u,u)_G}. \quad (103)$$

Remark 173. Mapping $N_G$ is well defined thanks to the nonnegativeness of the inner product. It will be proved below to be a norm.

Lemma 174 (squared norm). Let $(G, ⟨·, ·⟩_G)$ be an inner product space. Then,

$$\forall u \in G, \quad N_G(u)^2 = (u,u)_G. \quad (104)$$

Proof. Direct consequence of Definition 172 (square root of inner square), Definition 161 (inner product $⟨·, ·⟩_G$ is nonnegative), and properties of square and square root functions in $\mathbb{R}^+$. 

Lemma 175 (Cauchy–Schwarz inequality with norms). Let $(G, ⟨·, ·⟩_G)$ be an inner product space. Then,

$$\forall u, v \in G, \quad |(u,v)_G| \leq N_G(u) N_G(v). \quad (105)$$

Proof. Direct consequence of Lemma 171 (Cauchy–Schwarz inequality), Definition 172 (square root of inner square), Definition 161 (inner product $⟨·, ·⟩_G$ is nonnegative), and compatibility of the square root function with comparison in $\mathbb{R}^+$. 

Lemma 176 (triangle inequality). Let $(G, ⟨·, ·⟩_G)$ be an inner product space. Then,

$$\forall u, v \in G, \quad N_G(u + v) \leq N_G(u) + N_G(v). \quad (106)$$

Proof. Let $u, v \in G$ be vectors. From Lemma 174 (squared norm), Lemma 168 (square expansion plus), Lemma 175 (Cauchy–Schwarz inequality with norms), and field properties of $\mathbb{R}$, we have

$$N_G(u + v)^2 = (u + v, u + v)_G$$
$$= (u,u)_G + 2(u,v)_G + (v,v)_G$$
$$\leq (N_G(u))^2 + 2N_G(u)N_G(v) + (N_G(v))^2$$
$$= (N_G(u) + N_G(v))^2.$$

Therefore, from Definition 161 (inner product $⟨·, ·⟩_G$ is nonnegative), and compatibility of the square function with comparison in $\mathbb{R}^+$, we have

$$N_G(u + v) \leq N_G(u) + N_G(v).$$

Lemma 177 (inner product gives norm). Let $(G, ⟨·, ·⟩_G)$ be an inner product space. Let $∥∥_G$ be the associated square root of inner square. Then, $(G, ∥∥_G)$ is a normed space.

Proof. From Definition 172 (square root of inner square), and nonnegativeness of the square root function in $\mathbb{R}^+$, $∥∥_G$ is nonnegative.

From Definition 172 (square root of inner square), definiteness of the square root function in $\mathbb{R}^+$, and Definition 161 (inner product $⟨·, ·⟩_G$ is definite), $∥∥_G$ is definite.

Let $\lambda \in \mathbb{K}$ be a scalar. Let $u \in G$ be a vector. From Definition 172 (square root of inner square, Definition 161 (inner product $⟨·, ·⟩_G$ is a bilinear map), multiplicativity of the square root function in $\mathbb{R}^+$, and since for all $x \in \mathbb{R}$, $\sqrt{x^2} = |x|$, we have

$$∥\lambda u∥_G = \sqrt{(\lambda u, \lambda u)_G} = \sqrt{\lambda^2 (u,u)_G} = \sqrt{\lambda^2} \sqrt{(u,u)_G} = |\lambda| \cdot ∥u∥_G.$$
Thus, $\|\cdot\|_G$ is absolutely homogeneous of degree 1.

From Lemma 176 (triangle inequality), $\|\cdot\|_G$ satisfies triangle inequality. Therefore, from Definition 102 (norm), $\|\cdot\|_G$ is a norm over $G$, and from Definition 104 (normed vector space), $(G, \|\cdot\|_G)$ is a normed space.

**Remark 178.** Norm $\|\cdot\|_G$ is called norm associated with inner product $(\cdot, \cdot)_G$.

### 4.5.1 Orthogonal projection

**Definition 179 (convex subset).** Let $E$ be a real space. Let $K \subseteq E$. $K$ is convex iff

$$\forall u, v \in K, \forall \theta \in [0, 1], \theta u + (1 - \theta)v \in K.$$  \hfill (107)

**Theorem 180 (orthogonal projection onto nonempty complete convex).** Let $(G, (\cdot, \cdot)_G)$ be a real inner product space. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot, \cdot)_G$. Let $d_G$ be the distance associated with norm $\|\cdot\|_G$. Let $K \subset G$ be a nonempty convex subset which is complete for distance $d_G$. Then, for all $u \in G$, there exists a unique $v \in K$ such that

$$\|u - v\|_G = \min_{w \in K} \|u - w\|_G.$$  \hfill (108)

**Proof.** Let $u \in G$. From Lemma 107 (norm is nonnegative) for $\|\cdot\|_G$, function $w \mapsto \|u - w\|_G$ from $K$ to $\mathbb{R}$ admits 0 as finite lower bound. Thus, from Lemma 13 (finite infimum discrete), $\delta = \inf_{w \in K} \{\|u - w\|_G\}$ is finite and there exists a sequence $(w_n)_{n \in \mathbb{N}}$ in $K$ such that for all $n \in \mathbb{N}$, $\|u - w_n\|_G < \delta + \frac{1}{n}$. From Definition 155 (inner product space), $G$ is a space.

**Existence.** Let $p, q \in \mathbb{N}$. Let $a = u - w_q$ and $b = u - w_p$. From Definition 102 (norm), $\|\cdot\|_G$ is absolutely homogeneous of degree 1. Definition 71 (scalar division), Definition 58 (vector space) $(G, +)$ is an abelian group), Lemma 174 (squared norm), and Lemma 170 (parallelogram identity) for $\|\cdot\|_G$, we have

$$4 \left\| u - \frac{w_q + w_p}{2} \right\|^2_G + \left\| w_p - w_q \right\|^2_G = \|a + b\|^2_G + \|a - b\|^2_G = 2\|a\|^2_G + 2\|b\|^2_G = 2\|u - q\|^2_G + 2\|u - w_p\|^2_G.$$

From Definition 179 (convex subset), $\frac{w_q + w_p}{2}$ belongs to $K$. Thus, from Definition 9 (infimum) $\delta$ is a lower bound for $\{\|u - w\|_G \mid w \in K\}$, and field properties of $\mathbb{R}$, we have

$$\|w_p - w_q\|^2_G = -4 \left\| u - \frac{w_q + w_p}{2} \right\|^2_G + 2\|u - w_q\|^2_G + 2\|u - w_p\|^2_G < -4\delta^2 \left( \delta + \frac{1}{q + 1} \right)^2 + 2\left( \delta + \frac{1}{p + 1} \right)^2 = \frac{4\delta}{q + 1} + \frac{2}{(q + 1)^2} + \frac{4\delta}{p + 1} + \frac{2}{(p + 1)^2}.$$

Let $\varepsilon > 0$. Let $\eta = \max\left( \frac{16\delta^2}{\varepsilon^2}, \frac{2\sqrt{3}}{\varepsilon} \right)$. From the definition of the max function, and ordered field properties of $\mathbb{R}$, $\eta > 0$. Let $N = \lceil \eta \rceil - 1$. From the definition of the ceiling function, $N \geq 0$ and $N \geq \eta - 1$. Assume that $p, q \geq N$. Then, from ordered field properties of $\mathbb{R}$, we have $p, q \geq \eta - 1$ and $\frac{16\delta^2}{q + 1} - \frac{2\sqrt{3}}{\varepsilon} \leq \frac{\varepsilon^2}{4}$. Thus, from field properties of $\mathbb{R}$, Lemma 107 (norm is nonnegative) for $\|\cdot\|_G$, and compatibility of the square root function.
with comparison in $\mathbb{R}^+$, we have $\|w_p - w_q\|_G \leq \varepsilon$. Hence, from Definition \ref{Cauchy sequences} (Cauchy sequences), $(w_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $K$.

From hypothesis, and Definition \ref{complete subset} (complete subset), $K$ is complete. Let $v \in K$ be its limit. From Lemma \ref{norm is continuous for $\|\cdot\|_G$}, Lemma \ref{compatibility of limit with continuous functions}, and Definition \ref{minimum} (minimum), we have

$$
\|u - v\|_G = \lim_{n \to +\infty} \|u - w_n\|_G = \delta = \min_{w \in K} \|u - w\|_G.
$$

Uniqueness. Let $v, v' \in K$ such that $\|u - v\|_G = \|u - v'\|_G = \delta$. Then, from Definition \ref{norm} (norm), $\|\cdot\|_G$ is absolutely homogeneous of degree 1). From Definition \ref{parallelogram identity} (parallelogram identity), we have

$$
4\|u - v''\|^2_G + \|v - v''\|^2_G = 2\|a\|^2_G + 2\|b\|^2_G
$$

From Definition \ref{convex subset} (convex subset), $v''$ belongs to $K$. Thus, from Definition \ref{infimum} (infimum), $\delta$ is a lower bound for $\{\|u - w\|_G \mid w \in K\}$, and field properties of $\mathbb{R}$, we have

$$
0 \leq \|v - v''\|^2_G = -4\|u - v''\|^2_G + 4\delta^2 \leq -4\delta^2 + 4\delta^2 = 0.
$$

Hence, from Lemma \ref{norm is nonnegative} (norm is nonnegative) and compatibility of square root function with comparison in $\mathbb{R}^+$, $\|u - v\|^2_G = 0$. Therefore, from Definition \ref{norm} (norm), $\|\cdot\|_G$ is definite, Definition \ref{vector subtraction} (vector subtraction) and Definition \ref{vector space} (vector space), we have $v = v' = 0_G$ and $v = v'$.

Lemma \ref{characterization of orthogonal projection onto convex}. Let $(G, (\cdot, \cdot)_G)$ be a real inner product space. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot, \cdot)_G$. Let $K \subseteq G$ be a nonempty convex subset. Then, for all $u \in G$, for all $v \in K$,

$$
\|u - v\|_G = \inf_{w \in K} \|u - w\|_G \iff \forall w \in K, \quad (u - v, w - v)_G \leq 0.
$$

Proof. Let $u \in G$ and $v \in K$ be vectors.

“Left” implies “right”. Assume that $\|u - v\|_G = \inf_{w \in K} \|u - w\|_G$. Let $w \in K$. Let $\theta \in (0, 1]$. From Definition \ref{convex subset} (convex subset), $\theta w + (1 - \theta)v$ belongs to $K$. Thus, from Definition \ref{infimum} (infimum), $\|u - v\|_G$ is a lower bound for $\{\|u - w\|_G \mid w \in K\}$, compatibility of the square function with comparison in $\mathbb{R}^+$, Definition \ref{inner product space} (G is a space), Definition \ref{vector space} (vector space), (G, +) is an abelian group and scalar multiplication is compatible with scalar addition, \ref{vector subtraction} (vector subtraction), Lemma \ref{square norm} (square norm for $(\cdot, \cdot)_G$), Lemma \ref{square expansion plus for $(\cdot, \cdot)_G$}, Definition \ref{inner product} (inner product $(\cdot, \cdot)_G$ is a bilinear map), Definition \ref{bilinear map} (bilinear map), and Lemma \ref{square expansion plus} (square expansion plus), we have

$$
\|u - v\|^2_G \leq \|u - (\theta w + (1 - \theta)v)\|^2_G
$$

Let $a = (u - v, w - v)_G$ and $b = \|w - v\|^2_G$. Then, from ordered field properties of $\mathbb{R}$ (with $\theta > 0$), we have

$$
\forall \theta \in (0, 1], \quad 2a \leq \theta b.
$$
Assume that \( b = 0 \). Then, from ordered field properties of \( \mathbb{R} \), we have \((u - v, w - v)_G = a \leq 0\). Conversely, assume now that \( b \neq 0 \). Then, from nonnegativeness of the square function, \( b > 0 \). Assume that \( a > 0 \). Let \( \theta = \min\left(1, \frac{a}{b}\right) \). From the definition of the min function, and ordered field properties of \( \mathbb{R} \), we have \( \theta \leq \frac{a}{b} \) and \( 0 < \theta \leq 1 \). Thus, \( 2a \leq \theta b \leq a \). Hence, from ordered field properties of \( \mathbb{R} \), \( a \leq 0 \). Which is impossible. Therefore, we have \((u - v, w - v)_G = a \leq 0\).

“Right” implies “left”. Conversely, assume now that, for all \( w \in K \), \((u - v, w - v)_G \leq 0\). Let \( w \in K \). Then, from Definition 165 (inner product space) \( G \) is a space), Definition 58 (vector space) \((G, +)\) is an abelian group), Definition 70 (vector subtraction), Lemma 174 (squared norm for \((\cdot, \cdot)_G\), Lemma 168 (square expansion plus) for \((\cdot, \cdot)_G\), Definition 161 (inner product) \((\cdot, \cdot)_G\) is a bilinear map), Definition 65 (bilinear map) \((\cdot, \cdot)_G\) is right linear), and nonnegativeness of the square function in \( \mathbb{R}^+ \), we have

\[
\|u - w\|_G^2 = \|(u - v) + (v - w)\|_G^2 \\
\leq \|(u - v)\|_G^2 + 2\|(u - v)\|_G\|(v - w)\|_G + \|(v - w)\|_G^2 \\
\geq \|v - w\|_G^2 - 2\|(u - v)\|_G^2 - 2\|(u - v)\|_G\|(v - w)\|_G \\
\geq \|v - w\|_G^2.
\]

Hence, from Lemma 107 (norm is nonnegative for \(\|\cdot\|_G\), and compatibility of the square root function with comparison in \(\mathbb{R}^+\), we have \(\|u - v\|_G \leq \|u - w\|_G\). Therefore, from Lemma 15 (finite minimum), and Definition 14 (minimum), we have

\[
\|u - v\|_G = \min_{w \in K} \|u - w\|_G = \inf_{w \in K} \|u - w\|_G.
\]

\[\square\]

**Lemma 182 (subspace is convex).** Let \( E \) be a real space. Let \( F \) be a subspace of \( E \). Then, \( F \) is a convex subset of \( E \).

**Proof.** Let \( u, v \in F \) be vectors in the subspace. Let \( \theta \in [0, 1] \). Then, from Lemma 79 (closed under linear combination is subspace), the linear combination \( w = \theta u + (1 - \theta)v \) belongs to \( F \). Therefore, from Definition 179 (convex subset), \( F \) is a convex subset of \( E \).

\[\square\]

**Theorem 183 (orthogonal projection onto complete subspace).** Let \((G, (\cdot, \cdot)_G)\) be a real inner product space. Let \(\|\cdot\|_G\) be the norm associated with inner product \((\cdot, \cdot)_G\). Let \(d_G\) be the distance associated with norm \(\|\cdot\|_G\). Let \( F \) be a subspace of \( G \) which is complete for distance \(d_G\). Then, for all \( u \in G \), there exists a unique \( v \in F \) such that

\[
\|u - v\|_G = \min_{w \in F} \|u - w\|_G. \tag{110}
\]

**Proof.** Direct consequence of Definition 74 (subspace \( F \) is vector space), Definition 58 (vector space \( F \) is nonempty), Lemma 182 (subspace is convex), and Theorem 180 (orthogonal projection onto nonempty complete convex \( F \) is a nonempty convex subset of \( G \) which is complete for distance \(d_G\)).

\[\square\]

**Definition 184 (orthogonal projection onto complete subspace).** Assume hypotheses of Theorem 183 (orthogonal projection onto complete subspace). The mapping \( P_F : G \to F \) associating to any vector of \( G \) the unique vector of \( F \) satisfying (110) is called orthogonal projection onto \( F \).

**Lemma 185 (characterization of orthogonal projection onto subspace).** Let \((G, (\cdot, \cdot)_G)\) be a real inner product space. Let \(\|\cdot\|_G\) be the norm associated with inner product \((\cdot, \cdot)_G\). Let \( F \) be a subspace of \( G \). Then, for all \( u \in G \), for all \( v \in F \),

\[
\|u - v\|_G = \inf_{w \in F} \|u - w\|_G \iff \forall w \in F, (v, w)_G = (u, w)_G. \tag{111}
\]

RR n° 8934
Proof. Let $u \in G$ and $v \in F$ be vectors.

“Left” implies “right”. Assume that $\|u - v\|_G = \inf_{w \in F} \|u - w\|_G$. Then, from Definition 74 (subspace $F$ is a vector space), Definition 58 (vector space $F \ni 0_G$ is nonempty), Lemma 181 (characterization of orthogonal projection onto convex $F$ is a nonempty convex subset), we have for all $w \in F$, $(u - v, w - v)_G \leq 0$. Let $w \in F$. Let $w' = w + v$. Then, from Definition 74 (subspace $F$ is a vector space), Definition 58 (vector space $(F, +)$ is an abelian group), and Definition 70 (vector subtraction, $w'$ belongs to $F$ and $w = w' - v$. Thus, we have $(u - v, w)_G = (u - v, w' - v)_G \leq 0$. Similarly, $w'' = -w + v$ belongs to $F$ and $(u - v, w)_G = (u - v, w'' - v)_G \leq 0$. Hence, from Definition 161 (inner product $(\cdot, \cdot)_G$ is a bilinear map) Definition 65 (bilinear map $(\cdot, \cdot)_G$ is right linear), and ordered field properties of $\mathbb{R}$, we have $(u - v, w)_G = 0$. Therefore, from Definition 70 (vector subtraction, and Definition 65 (bilinear map $(\cdot, \cdot)_G$ is left linear), we have $(v, u)_G = (v, w)_G$.

“Right” implies “left”. Conversely, assume now that for all $w \in F$, $(v, w)_G = (u, w)_G$. Let $w \in F$. Let $w' = w - v$. Then, from Definition 74 (subspace $F$ is a vector space), and Definition 58 (vector space $(F, +)$ is an abelian group), $w'$ belongs to $F$. Hence, from Definition 161 (inner product $(\cdot, \cdot)_G$ is a bilinear map), Definition 65 (bilinear map $(\cdot, \cdot)_G$ is left linear), hypothesis, and ordered field properties of $\mathbb{R}$, we have

$$(u - v, w - v)_G = (u - v, w')_G = (u, w')_G - (v, w')_G = 0 \leq 0.$$ 

Therefore, from Definition 74 (subspace $F$ is a vector space), Definition 58 (vector space $F \ni 0_G$ is nonempty), Lemma 182 (subspace is convex, and Lemma 181 (characterization of orthogonal projection onto convex $F$ is a nonempty convex subset), we have $\|u - v\|_G = \inf_{w \in F} \|u - w\|_G$. \hfill \square

Lemma 186 (orthogonal projection is continuous linear map). Assume hypotheses of Theorem 183 (orthogonal projection onto complete subspace). Then, the orthogonal projection $P_F$ is a 1-Lipschitz continuous linear map from $G$ to $F$.

Proof. From Definition 184 (orthogonal projection onto complete subspace), and Theorem 183 (orthogonal projection onto complete subspace), $P_F$ effectively defines a mapping from $G$ to $F$.

Linearity. Let $u', u'' \in G$. Let $\lambda', \lambda'' \in \mathbb{R}$. From Definition 74 (subspace $F$ is a vector space), and Definition 58 (vector space $G$ and $F$ are closed under vector operations), $\lambda' u' + \lambda'' u''$ belongs to $G$ and $\lambda' P_F(u') + \lambda'' P_F(u'')$ belongs to $F$. Let $w \in F$. From Definition 161 (inner product $(\cdot, \cdot)_G$ is a bilinear map), Definition 65 (bilinear map $(\cdot, \cdot)_G$ is left linear), and Lemma 185 (characterization of orthogonal projection onto subspace), we have

$$\lambda' P_F(u') + \lambda'' P_F(u'') w)_G = \lambda' (P_F(u'), w)_G + \lambda'' (P_F(u''), w)_G = \lambda' (u', w)_G + \lambda'' (u'', w)_G = (\lambda' u' + \lambda'' u''), w)_G.$$ 

Hence, from Lemma 185 (characterization of orthogonal projection onto subspace), and Theorem 183 (orthogonal projection onto complete subspace, orthogonal projection is unique), we have $P_F(\lambda' u' + \lambda'' u'') = \lambda' P_F(u') + \lambda'' P_F(u'')$.

Therefore, from Lemma 92 (linear map preserves linear combinations), $P_F$ is a linear map.

Continuity. Let $u \in G$.

Case $P_F(u) = 0_G$. Then, from Lemma 107 (norm is nonnegative) for $\|\cdot\|_G$, we have $\|P_F(u)\|_G = 0 \leq \|u\|_G$.

Case $P_F(u) \neq 0_G$. Then, from Lemma 174 (squared norm). Lemma 185 (characterization of orthogonal projection onto subspace with $w = \bar{P}_F(u) \in F$), and Lemma 175 (Cauchy-Schwarz inequality with norms), we have

$$\|P_F(u)\|_G^2 = (P_F(u), P_F(u))_G = (u, P_F(u))_G \leq \|u\|_G \|P_F(u)\|_G.$$ 

Inria
Hence, from Definition \[\text{norm}\] (\|\cdot\|_G\text{ is definite}), and \textbf{ordered field properties of }\mathbb{R},\textbf{ we have }\|P_F(u)_G\| \leq \|u\|_G.\]

Therefore, from Definition \[\text{Lipschitz continuity}\] (\(P_F\) is \(1\)-Lipschitz continuous). \(\square\)

**Definition 187 (orthogonal complement).** Let \((G,\langle \cdot, \cdot \rangle_G)\) be an inner product space. Let \(F\) be a subspace of \(G\). The \textit{orthogonal complement} of \(F\) in \(G\), denoted \(F^\perp\), is defined by

\[F^\perp = \{u \in G | \forall v \in F, \langle u, v \rangle_G = 0\} \quad (112)\]

**Lemma 188 (trivial orthogonal complements).** Let \((G,\langle \cdot, \cdot \rangle_G)\) be an inner product space. Then, \(G^\perp = \{0_G\}\) and \(\{0_G\}^\perp = G\).

\[\begin{proof}
\text{Proof. From Lemma 167 (trivial subspaces), } G \text{ and } \{0_G\}\text{ are subspaces of } G. \text{ From Definition 187 (orthogonal complement), } G^\perp \text{ and } \{0_G\}^\perp \text{ are subsets of } G. \\

Let } u \in G \text{ be a vector. Then, from Lemma 167 (inner product with zero is zero) for } \langle \cdot, \cdot \rangle_G, \text{ we have } (0_G, u)_G = \langle u, 0_G \rangle_G = 0. \text{ Hence, from Definition 187 (orthogonal complement), } \{0_G\}\text{ is a subset of } G^\perp \text{ and } G \text{ is a subset of } \{0_G\}^\perp. \\

Let } u \in G^\perp \text{ be a vector in the orthogonal. Let } v = u \in G. \text{ Then, from Definition 187 (orthogonal complement), we have } \langle u, v \rangle_G = (u, u)_G = 0. \text{ Thus, from Definition 161 (inner product) } \langle \cdot, \cdot \rangle_G \text{ is definite}. \text{ Therefore, } G^\perp = \{0_G\}. \\

\text{ Therefore, } G^\perp = \{0_G\}\text{ and } \{0_G\}^\perp = G. \quad \square
\end{proof}\]

**Lemma 189 (orthogonal complement is subspace).** Let \((G,\langle \cdot, \cdot \rangle_G)\) be an inner product space. Let \(F\) be a subspace of \(G\). Then, \(F^\perp\) is a subspace of \(G\).

\[\begin{proof}
\text{Proof. Let } v \in F. \text{ Then, from Lemma 167 (inner product with zero is zero) for } \langle \cdot, \cdot \rangle_G, \text{ we have } (0_G, v)_G = 0. \text{ Hence, from Definition 187 (orthogonal complement), } 0_G \text{ belongs to } F^\perp. \\

Let } \lambda, \lambda' \in \mathbb{R}\text{. Let } u, u' \in F^\perp. \text{ Let } v \in F. \text{ From Definition 161 (inner product) } \langle \cdot, \cdot \rangle_G \text{ is a bilinear map}, \text{ Definition 65 (bilinear map) } \langle \cdot, \cdot \rangle_G \text{ is left linear}, \text{ and } \textbf{ordered field properties of } \mathbb{R},\textbf{ we have }

\[\langle \lambda u + \lambda' u', v \rangle_G = \lambda \langle u, v \rangle_G + \lambda' \langle u', v \rangle_G = \lambda 0 + \lambda' 0 = 0.\]

Thus, from Definition 187 (orthogonal complement), \(\lambda u + \lambda' u'\) belongs to \(F^\perp\). Hence, \(F^\perp\) is closed under linear combination. Therefore, from Lemma 79 (closed under linear combination is subspace), \(F^\perp\) is a subspace of \(G\). \quad \square
\end{proof}\]

**Lemma 190 (zero intersection with orthogonal complement).** Let \((G,\langle \cdot, \cdot \rangle_G)\) be an inner product space. Let \(F\) be a subspace of \(G\). Then,

\[F \cap F^\perp = \{0_G\}. \quad (113)\]

\[\begin{proof}
\text{Proof. Let } \|\cdot\|_G \text{ be the norm associated with inner product } \langle \cdot, \cdot \rangle_G. \text{ Let } u \in F \cap F^\perp. \text{ Then, } u \in F \text{ and } v = u \in F^\perp. \text{ Thus, from Definition 187 (orthogonal complement), we have }

\[\langle u, u \rangle_G = (u, v)_G = 0.\]

Thus, from Definition 161 (inner product) \(\langle \cdot, \cdot \rangle_G \text{ is definite}), \(u = 0_G\). \quad \square
\end{proof}\]
Theorem 191 (direct sum with orthogonal complement when complete). Assume hypotheses of Theorem 183 (orthogonal projection onto complete subspace). Then,

\[ G = F \oplus F^\perp. \] (114)

Moreover, for all \( u \in G \), the (unique) decomposition onto \( F \oplus F^\perp \) is

\[ u = P_F(u) + (u - P_F(u)) \] (115)

and we have the following characterizations of the orthogonal complements:

\[ u \in F \iff P_F(u) = u; \] (116)

\[ u \in F^\perp \iff P_F(u) = 0_G. \] (117)

Proof. Let \( u \in G \).

Then, from Definition 184 (orthogonal projection onto complete subspace), and Lemma 185 (characterization of orthogonal projection onto subspace), there exists a unique \( P_F(u) \in F \) characterized by, for all \( w \in F \), \( (P_F(u), w) \in (u, w) \). Thus, from Definition 161 (inner product \( \langle \cdot, \cdot \rangle \) is a bilinear map), Definition 65 (bilinear map \( \langle \cdot, \cdot \rangle \) is left linear), and Definition 187 (orthogonal complement), \( u - P_F(u) \) belongs to \( F^\perp \). From Definition 58 (vector space \( (G, +) \) is an abelian group), we have

\[ u = P_F(u) + (u - P_F(u)). \]

Hence, from Definition 81 (sum of subspaces), \( G = F + F^\perp \). Therefore, from Lemma 190 (zero intersection with orthogonal complement for \( F \)), and Lemma 84 (equivalent definitions of direct sum), we have \( G = F \oplus F^\perp \). From Definition 83 (direct sum of subspaces), the decomposition (115) with \( P_F(u) \in F \) and \( u - P_F(u) \in F^\perp \) is unique.

From Lemma 190 (zero intersection with orthogonal complement), \( 0_G \) belongs to both \( F \) and \( F^\perp \).

(116): “left” implies “right”. Assume that \( u \in F \). Then, from Definition 58 (vector space \( (G, +) \) is an abelian group), \( u = u + 0_G \) is a decomposition over \( F \oplus F^\perp \). From uniqueness of the decomposition, we have \( P_F(u) = u \).

(116): “right” implies “left”. Conversely, assume now that \( P_F(u) = u \). Then, from Definition 184 (orthogonal projection onto complete subspace), \( P_F \) is a mapping to \( F \), \( u = P_F(u) \) belongs to \( F \).

(117): “left” implies “right”. Assume that \( u \in F^\perp \). Then, from Definition 58 (vector space \( (G, +) \) is an abelian group), \( u = 0_G + u \) is a decomposition over \( F \oplus F^\perp \). From uniqueness of the decomposition, we have \( P_F(u) = 0_G \).

(117): “right” implies “left”. Conversely, assume now that \( P_F(u) = 0_G \). Let \( v \in F \). Then, from Lemma 185 (characterization of orthogonal projection onto subspace), and Lemma 167 (inner product with zero is zero), we have

\[ (u, v)_G = (P_F(u), v)_G = (0_G, v)_G = 0. \]

Hence, from Definition 187 (orthogonal complement), \( u \) belongs to \( F^\perp \).

Lemma 192 (sum is orthogonal sum). Assume hypotheses of Theorem 183 (orthogonal projection onto complete subspace). Let \( u \in G \) be a vector. Then, there exists \( u' \in F^\perp \) such that

\[ F + \text{span}(\{u\}) = F + \text{span}(\{u'\}). \]

Proof. Let \( u' = u - P_F(u) \). Then, from Theorem 191 (direct sum with orthogonal complement when complete), \( P_F(u) \) belongs to \( F \) and \( u' \) belongs to \( F^\perp \).

Let \( w \in F + \text{span}(\{u\}) \). Then, from Definition 58 (sum of subspaces), and Definition 80 (linear span), there exists \( v \in F \) and \( \lambda \in \mathbb{R} \) such that \( w = v + \lambda u \). From Lemma 79 (closed under linear
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

49

combination is subspace, with 1 and λ, we have v' = v + λP_F(u) ∈ F, and thus, from Definition 58
(vector space) (G, +) is an abelian group), we have

w = v + λu = v + λP_F(u) + λu' = v' + λu'

with v' ∈ F. Hence, w belongs to F + span({w'}). And thus F + span({u}) ⊂ F + span({w'}). Let w ∈ F + span({w'}). Similarly, from Definition 81 (sum of subspaces), and Definition 80 (linear span), there exists v ∈ F and λ ∈ R such that w = v + λu'; from Lemma 79 (closed under linear combination is subspace), with 1 and −λ, we have v' = v − λP_F(u) ∈ F, and thus, from Definition 58 (vector space) (G, +) is an abelian group), we have

w = v + λu = v − λP_F(u) + λu = v' + λu

with v' ∈ F. Hence, w belongs to F + span({u}), and thus F + span({u'}) ⊂ F + span({u}).

Therefore, F + span({u}) = F + span({u'})

Lemma 193 (sum of complete subspace and linear span is closed). Assume hypotheses of Theorem 183 (orthogonal projection onto complete subspace). Let u be a nonzero vector in the orthogonal of F. Then, F ⊕ span({u}) is closed for distance d_G.

Proof. From Lemma 190 (zero with orthogonal complement), u does not belong to F, thus from Lemma 85 (direct sum with linear span), the sum F + span({u}) is direct.

From Lemma 186 (orthogonal projection is continuous linear map), F is complete for distance d_F, P_F is a continuous linear map. Then, from Lemma 146 (identity map is continuous), Theorem 147 (normed space of continuous linear maps), and Lemma 79 (closed under linear combination is subspace), Id − P_F is also a continuous linear map.

Let (w_n)_{n ∈ N} be a sequence in F ⊕ span({u}). Assume that this sequence is convergent with limit w ∈ G. From Definition 81 (sum of subspaces), and Definition 80 (linear span), for all n ∈ N, there exists v_n ∈ F and λ_n ∈ R such that w_n = v_n + λ_n u. Then, from Lemma 44 (compatibility of limit with continuous functions), P_F and Id − P_F are continuous), the sequences (w_n')_{n ∈ N} = P_F((w_n)_{n ∈ N}) and (w''_n)_{n ∈ N} = (Id − P_F)((w_n)_{n ∈ N}) are also convergent, respectively with limits w' = P_F(w) and w'' = (Id − P_F)(w) = w − w'. From Theorem 191 (direct sum with orthogonal complement when complete), we have, w' ∈ F and w'' ∈ F⊥, and for all n ∈ N,

w''_n = (Id − P_F)(w_n) = (Id − P_F)(v_n + λ_n u) = v_n + λ_n u − v_n = λ_n u.

Thus, (w''_n)_{n ∈ N} is also a sequence of span({u}). Then, from Lemma 122 (linear span is closed), span({u}) is closed), and Lemma 31 (closed is limit of sequences), the limit w'' actually belongs to span({u}). Hence, from Definition 80 (linear span), there exists λ ∈ R such that w'' = λu. And we have

w = w' + w'' = w' + λu ∈ F ⊕ span({u}).

Therefore, from Lemma 31 (closed is limit of sequences), F ⊕ span({u}) is closed for distance d_G.

4.6 Hilbert space

Definition 194 (Hilbert space). Let (H, ⟨·, ·⟩_H) be an inner product space. Let ‖·‖_H be the norm associated with inner product ⟨·, ·⟩_H through Definition 172 (square root of inner square), and Lemma 171 (inner product gives norm). Let d_H be the distance associated with norm ‖·‖_H through Lemma 111 (norm gives distance). (H, ⟨·, ·⟩_H) is an Hilbert space iff (H, d_H) is a complete metric space.

Lemma 195 (closed Hilbert subspace). Let (H, ⟨·, ·⟩_H) be a Hilbert space. Let H_b be a closed subspace of H. Then, H_b equipped with the restriction to H_b of the inner product ⟨·, ·⟩_H is a Hilbert space.
Proof. Direct consequence of Lemma 166 (inner product subspace, $H_k$ is a subspace of $H$), Definition 74 (subspace, $H_k$ is a subset of $H$), Lemma 194 (Hilbert space $H$ is complete), Lemma 39 (closed subset of complete is complete $F$ is closed), and Definition 194 (Hilbert space).

Theorem 196 (Riesz–Fréchet). Let $(H,⟨·,·⟩_H)$ be a Hilbert space. Let $∥·∥_H$ be the norm associated with inner product $⟨·,·⟩_H$. Let $ϕ ∈ H'$ be a continuous linear form on $H$. Then, there exists a unique vector $u ∈ H$ such that

$$∀v ∈ H, \quad ⟨ϕ|v⟩_H = (u,v)_H.$$  \hspace{1cm} (118)

Moreover, the mapping $τ : H' → H$ defined by

$$∀ϕ ∈ H', \quad τ(ϕ) = u,$$

where $u$ is characterized by (118), is a continuous isometric isomorphism from $H'$ onto $H$.

Proof. From Definition 194 (Hilbert space $(H,⟨·,·⟩_H)$ is an inner product space), and Definition 165 (inner product space), $H$ is a space.

Uniqueness. Let $u, u' ∈ H$ be two vectors such that

$$∀v ∈ H, \quad ⟨ϕ|v⟩_H = (u,v)_H = (u',v)_H.$$

Let $v ∈ H$ be a vector. Then, from Definition 161 (inner product $(H,⟨·,·⟩_H)$ is a bilinear map), Definition 65 (bilinear map $(H,⟨·,·⟩_H)$ is left linear), and Definition 70 (vector subtraction), we have $(u - u',v)_H = 0$. Thus, from Definition 187 (orthogonal complements), and Lemma 188 (trivial orthogonal complements), $u - u'$ belongs to $H^⊥ = \{0_H\}$. Hence, from Definition 58 (vector space $(H,+,0_H)$ is an abelian group), $u = u'$.

Existence.

Case $ϕ = 0_H'$. Then, from Definition 58 (vector space $0_H$ belongs to $H$), let $u = 0_H$ be the zero vector. Let $v ∈ H$ be a vector. Then, from Lemma 188 (trivial orthogonal complements $H^⊥ = \{0_H\}$), we have

$$⟨ϕ|v⟩_H = (0_H,v)_H = (u,v)_H = (0_H,v)_H.$$

Case $ϕ \neq 0_H'$. Then, let $u_0 ∈ H$ such that $ϕ(u_0) \neq 0$. Let $F$ be the kernel of $ϕ$. Then, from Definition 98 (kernel, $u_0 \notin F$). Moreover, from Lemma 149 (continuous linear maps have closed kernel for $ϕ$), and Lemma 39 (kernel is subspace), $F$ is a closed subspace of $H$. Thus, from Lemma 194 (closed Hilbert subspace), $F$ is a complete subspace of $H$. Hence, from Theorem 183 (orthogonal projection onto complete subspace), and Definition 184 (orthogonal projection onto complete subspace), let $P_F$ be the orthogonal projection onto $F$. Then, from Theorem 191 (direct sum with orthogonal complement when complete), decomposition and contrapositive of (116), we have

$$P_F(u_0) ∈ F, \quad u_0 - P_F(u_0) ∈ F^⊥ \quad \text{and} \quad P_F(u_0) \neq u_0.$$  \hspace{1cm} (119)

Thus, from Definition 98 (kernel $F = \ker(ϕ)$), Definition 64 (linear form $ϕ$ is a linear map), Definition 62 (linear map $ϕ$ is additive), and Definition 70 (vector subtraction), we have

$$ϕ(P_F(u_0)) = 0 \quad \text{and} \quad ϕ(u_0 - P_F(u_0)) = ϕ(u_0).$$

Let $v_0 = u_0 - P_F(u_0)$. Then,

$$v_0 ∈ F^⊥ \quad \text{and} \quad ϕ(v_0) = ϕ(u_0) \neq 0.$$  \hspace{1cm} (120)

Moreover, from Definition 58 (vector space $(H,+,0_H)$ is an abelian group), and Definition 70 (vector subtraction), we have $v_0 \neq 0_H$. Thus, from Definition 102 (norm $∥·∥_H$ is definite, contrapositive), we have $∥v_0∥_H \neq 0$. 

Inria
Let $\xi_0 = \frac{v_0}{\|v_0\|_H}$. Then, from Lemma 189 (orthogonal complement is subspace $F^\perp$ is subspace), Lemma 78 (closed under vector operations is subspace $F^\perp$ is closed under scalar multiplication), Definition 71 (scalar division), Definition 64 (linear form $\varphi$ is a linear map), Definition 62 (linear map $\varphi$ is homogeneous of degree 1), Definition 71 (scalar division), Lemma 78 (zero-product property contrapositive), and field properties of $\mathbb{R}$, we have

$$\xi_0 \in F^\perp, \quad \varphi(\xi_0) = \frac{\varphi(v_0)}{\|v_0\|_H} \neq 0 \quad \text{and} \quad \xi_0 \neq 0_H.$$  

Moreover, from Lemma 108 (normalization by nonzero with $\lambda = 1$), and field properties of $\mathbb{R}$, we have $\|\xi_0\|^2_H = 1$.

Let $u = \varphi(\xi_0)\xi_0$. Then, from Lemma 189 (orthogonal complement is subspace $F^\perp$ is subspace), and Lemma 78 (closed under vector operations is subspace $F^\perp$ is closed under scalar multiplication), $u \in F^\perp$.

Let $v \in H$ be a vector. Since $\varphi(\xi_0) \neq 0$, let $\lambda = \frac{\varphi(v)}{\varphi(\xi_0)}$ and $w = v - \lambda \xi_0$. Then, from Definition 64 (linear form $\varphi$ is a linear map), Lemma 92 (linear map preserves linear combinations), Definition 70 (vector subtraction), and Definition 71 (scalar division), we have

$$\varphi(w) = \varphi(v) - \lambda \varphi(\xi_0) = \varphi(v) - \frac{\varphi(v)}{\varphi(\xi_0)} \varphi(\xi_0) = 0.$$  

Thus, from Definition 98 (kernel $F = \ker(\varphi)$), $w$ belongs to $F$. Hence, from field properties of $\mathbb{R}$ (with $\varphi(\xi_0) \neq 0$), Lemma 174 (squared norm $\|\xi_0\|^2_H = 1$), Definition 161 (inner product $(\cdot,\cdot)_H$ is a bilinear map), Definition 65 (bilinear map $(\cdot,\cdot)_H$ is left linear), and Definition 187 (orthogonal complement $u \in F^\perp$ and $w \in F$), we have

$$(u, v)_H - \varphi(v) = (u, v)_H - \varphi(v) \frac{\varphi(\xi_0)}{\varphi(\xi_0)}(\xi_0, \xi_0)_H = (u, v)_H - \lambda(u, \xi_0)_H = (u, v)_H - \lambda \xi_0 = (u, w)_H = 0.$$

Hence, from Definition 155 (bra-ket notation), and field properties of $\mathbb{R}$, we have

$$(\varphi|v\rangle_{H',H} = \varphi(v) = (u, v)_H.$$  

**Linearity.** From Lemma 154 (topological dual is complete normed space for $H$), and Definition 104 (normed vector space $H'$ is a space).

Let $\tau : H' \to H'$ be the mapping defined by, for all $\varphi \in H'$, $\tau(\varphi) = u$ where $u$ is uniquely characterized by

$$\forall v \in H, \quad (\varphi|v\rangle_{H',H} = \varphi(v) = (u, v)_H.$$  

(120)

Let $\lambda', \lambda'' \in \mathbb{K}$ be scalars. Let $\varphi', \varphi'' \in H'$ be continuous linear forms on $H$. Then, $\tau(\varphi')$ and $\tau(\varphi'')$ belong to $H$. Thus, from Definition 68 (vector space $H'$ and $H$ are closed under vector operations), $\varphi = \lambda\varphi' + \lambda''\varphi''$ belongs to $H'$ and $u = \lambda\tau(\varphi') + \lambda''\tau(\varphi'')$ belongs to $H$. Let $v \in H$ be a vector. Then, from Lemma 156 (bra-ket is bilinear map), Definition 161 (inner product $(\cdot,\cdot)_H$ is a bilinear map), Definition 65 (bilinear map) bra-ket and $(\cdot,\cdot)_H$ are left linear), and characterization (120), we have

$$(\varphi|v\rangle_{H',H} = (\lambda\varphi' + \lambda''\varphi'')(v|v\rangle_{H',H}) = \lambda'(\varphi'|v\rangle_{H',H} + \lambda''(\varphi''|v\rangle_{H',H}) = \lambda'(\tau(\varphi'), v)_H + \lambda''(\tau(\varphi''), v)_H = (\lambda'\tau(\varphi') + \lambda''\tau(\varphi''), v)_H = (u, v)_H.$$
Hence, from unique characterization \( [120] \), we have
\[
\tau(\lambda'\varphi' + \lambda''\varphi'') = \tau(\varphi) = u = \lambda'\tau(\varphi') + \lambda''\tau(\varphi'').
\]

Therefore, from Definition \( [62] \) \( \text{(linear map)} \), \( \tau \) is a linear map from \( H' \) to \( H \).

**Isomorphism.** Let \( \varphi \in H' \) be a continuous linear form on \( H \). Assume that \( \tau(\varphi) = 0_H \). Let \( v \in H \) be a vector. Then, from characterization \( [120] \), and Lemma \( [167] \) \( \text{(inner product with zero is zero)} \), we have
\[
\varphi(v) = (\tau(\varphi), v)_H = (0_H, v)_H = 0.
\]
Thus, \( \varphi = 0_H' \) is the zero linear form. Hence, from Definition \( [98] \) \( \text{(kernel)} \) \( \ker(\tau) = \{0_H'\} \), and Lemma \( [100] \) \( \text{(injective linear map has zero kernel)} \), \( \tau \) is injective.

Let \( u \in H \) be a vector. Let \( \varphi : H \to \mathbb{R} \) be the mapping defined by, for all \( v \in H \), \( \varphi(v) = (u, v)_H \). Then, from Definition \( [161] \) \( \text{(inner product)} \), \( (\cdot, \cdot)_H \) is a bilinear map, Definition \( [65] \) \( \text{(bilinear map)} \), \( (\cdot, \cdot)_H \) is right linear, and Definition \( [64] \) \( \text{(linear form)} \), \( \varphi \) is a linear form on \( H \). Let \( v \in H \) be a vector. Then, from Lemma \( [173] \) \( \text{(Cauchy–Schwarz inequality with norms)} \), we have
\[
|\varphi(v)| = |(u, v)_H| \leq \|u\|_H \|v\|_H.
\]
Thus, from Definition \( [139] \) \( \text{(bounded linear map)} \), with \( C = \|u\|_H \geq 0 \), and Theorem \( [142] \) \( \text{(continuous linear map)} \), \( \varphi \) is continuous. Hence, from Definition \( [152] \) \( \text{(topological dual)} \), \( \varphi \) belongs to \( H' \). Moreover, from characterization \( [120] \), we have \( \tau(\varphi) = u \). Hence, from the definition of a surjective function, \( \tau \) is surjective.

Therefore, from the definition of a bijective function, \( \tau \) is bijective, and from Definition \( [97] \) \( \text{(isomorphism)} \), \( \tau \) is an isomorphism from \( H' \) onto \( H \).

**Isometry.** Let \( \varphi \in H' \) be a continuous linear form on \( H \). Let \( u = \tau(\varphi) \in H \).

Case \( \varphi = 0_H' \). Then, from Lemma \( [91] \) \( \text{(linear map preserves zero)} \), \( \tau \) is a linear map, we have \( u = 0_H \). Hence, from Lemma \( [106] \) \( \text{(norm preserves zero)} \), we have
\[
\|\tau(\varphi)\|_H = \|u\|_H = 0 = \|\varphi\|_{H'}.
\]

Case \( \varphi \neq 0_H' \). Then, from Definition \( [98] \) \( \text{(kernel)} \), \( \ker(\varphi) = \{0_H'\} \), we have \( u \neq 0_H \). Thus, from Definition \( [102] \) \( \text{(norm)} \), \( \|\cdot\|_H \) is definite, contrapositive, \( \|u\|_H \neq 0 \). Hence, from characterization \( [120] \), \( \|u\|_H \neq 0 \). Lemma \( [174] \) \( \text{(squared norm)} \), \( \|\cdot\|_H \), nonnegativity of the square function in \( \mathbb{R} \), and field properties of \( \mathbb{R} \) (with \( \|u\|_H \neq 0 \)), we have
\[
\frac{|\varphi(u)|}{\|u\|_H} = \frac{|(u, u)_H|}{\|u\|_H^2} = \frac{\|u\|_H^2}{\|u\|_H} = \|u\|_H.
\]
Hence, from Definition \( [153] \) \( \text{(dual norm)} \), Definition \( [135] \) \( \text{(operator norm)} \), and Definition \( [2] \) \( \text{(supremum)} \), \( \|\varphi\|_{H'} \) is an upper bound for \( \{\frac{|\varphi(v)|}{\|v\|_H^2} \mid v \in H, v \neq 0_H\} \), we have
\[
\|v\|_H \leq \|\varphi\|_{H'}.
\]
Finally, let \( v \in H \) be a vector. Assume that \( v \neq 0_H \). Then, from Definition \( [102] \) \( \text{(norm)} \), \( \|\cdot\|_H \) is definite, contrapositive, Lemma \( [107] \) \( \text{(norm is nonnegative for } \|\cdot\|_H) \), Lemma \( [175] \) \( \text{(Cauchy–Schwarz inequality with norms)} \), and \( \text{ordered field properties of } \mathbb{R} \) (with \( \|v\|_H > 0 \)), we have
\[
\frac{|\varphi(v)|}{\|v\|_H} = \frac{|(u, v)_H|}{\|v\|_H} \leq \|u\|_H.
\]
Thus, from Lemma \( [144] \) \( \text{(finite operator norm is continuous)} \), \( \|u\|_H \) is an upper bound for the subset \( \{\frac{|\varphi(v)|}{\|v\|_H} \mid v \in H, v \neq 0_H\} \), and Definition \( [153] \) \( \text{(dual norm)} \), we have
\[
\|\varphi\|_{H'} \leq \|u\|_H.
\]
Hence, \( \|\tau(\varphi)\|_H = \|u\|_H = \|\varphi\|_{H'} \).

Therefore, from Definition \[ \text{linear isometry} \], \( \tau \) is a linear isometry from \( H' \) to \( H \).

**Continuity.** From Lemma \[ \text{linear isometry is continuous} \] \( \tau \) is a linear isometry from \( H' \) to \( H \), \( \tau \) belongs to \( \mathcal{L}_c(H', H) \).

**Lemma 197 (compatible \( \rho \) for Lax–Milgram).** Let \( \alpha, C \in \mathbb{R} \). Assume that \( 0 < \alpha \leq C \).
Then,
\[
\forall \rho \in \mathbb{R}, \quad 0 < \rho < \frac{2\alpha}{C^2} \implies 0 \leq \sqrt{1 - 2\rho \alpha + \rho^2 C^2} < 1.
\] (121)

**Proof.** From hypothesis \( (0 < \alpha \leq C) \), ordered field properties of \( \mathbb{R} \), and increase of the square function over \( \mathbb{R}^+ \), we have \( 0 < \frac{\alpha}{C^2} \leq 1 \). Let \( \rho \in \mathbb{R} \). Then, from field properties of \( \mathbb{R} \), we have
\[
1 - 2\rho \alpha + \rho^2 C^2 = \left( \rho C - \frac{\alpha}{C} \right)^2 + 1 - \frac{\alpha^2}{C^2} \geq 0.
\]

Assume that \( 0 < \rho < \frac{\alpha}{C^2} \), Then, from ordered field properties of \( \mathbb{R} \) (with \( C > 0 \) and \( \rho > 0 \)), we successively have \( \rho C^2 < 2\alpha \), \( \rho^2 C^2 < 2\rho \alpha \), and \( 1 - 2\rho \alpha + \rho^2 C^2 < 1 \). Hence, from compatibility of the square root with comparison in \( \mathbb{R}^+ \), we have
\[
0 = \sqrt{0} \leq \sqrt{1 - 2\rho \alpha + \rho^2 C^2} < \sqrt{1} = 1.
\]

**Theorem 198 (Lax–Milgram).** Let \( (H, (\cdot, \cdot)_H) \) be a real Hilbert space. Let \( \|\cdot\|_H \) be the norm associated with inner product \( (\cdot, \cdot)_H \). Let \( H' \) be the topological dual of \( H \). Let \( \|\cdot\|_{H'} \) be the dual norm associated with \( \|\cdot\|_H \). Let \( a \) be a bounded bilinear form on \( H \). Let \( f \in H' \) be a continuous linear form on \( H \). Assume that \( a \) is coercive with constant \( \alpha > 0 \). Then, there exists a unique\( u \in H \) solution to Problem \[ \text{Problem (1)} \]. Moreover,
\[
\|u\|_H \leq \frac{1}{\alpha} \|f\|_{H'}.
\] (122)

**Proof.** Let \( d_H \) be the distance associated with norm \( \|\cdot\|_H \). Then, from Definition \[ \text{Hilbert space} \], \( (H, (\cdot, \cdot)_H) \) is an inner product space and \( (H, d_H) \) is a complete metric space. Thus, from Lemma \[ \text{inner product gives norm} \], \( (H, \|\cdot\|_H) \) is a normed space. Moreover, from Lemma \[ \text{topological dual is complete normed space} \], \( (H, \|\cdot\|_H) \) is also a normed space. Hence, from Definition \[ \text{normed vector space} \], \( H \) and \( H' \) are both spaces.

**Existence and uniqueness.** From Lemma \[ \text{representation for bounded bilinear form} \] for \( a \), let \( A \in \mathcal{L}_c(H, H') \) be the (unique) continuous linear map from \( H \) to \( H' \) such that
\[
\forall u, v \in H, \quad a(u, v) = \langle A(u)|v \rangle_{H', H}.
\]

Then, from Definition \[ \text{coercive bilinear form} \] for \( a \), we have
\[
\forall u \in H, \quad \langle A(u)|u \rangle_{H', H} = a(u, u) \geq \alpha \|u\|_H^2.
\] (123)

From Definition \[ \text{bounded bilinear form} \], let \( C \geq 0 \) be a continuity constant of \( a \). Then, from Lemma \[ \text{operator norm estimation} \] in \( \mathcal{L}_c(H, H') \), and Lemma \[ \text{representation for bounded bilinear form} \] for \( a \), we have
\[
\forall u \in H, \quad \|A(u)\|_{H'} \leq \|A\|_{H', H} \|u\|_H \leq C \|u\|_H.
\] (124)

Let \( u \in H \) be a vector. Then, from Theorem \[ \text{Riesz–Fréchet} \] for \( \varphi = A(u) \) and \( \varphi = f \), \( \tau(A(u)), \tau(f) \in H \) are the (unique) vectors such that
\[
\forall v \in H, \quad a(u, v) = \langle A(u)|v \rangle_{H', H} = \langle \tau(A(u)), v \rangle_H;
\]
\[
\forall v \in H, \quad f(v) = \langle f|v \rangle_{H', H} = \langle \tau(f), v \rangle_H.
\]
Moreover, from (123), ordered field properties of \( \mathbb{R} \), (124), and Definition 122 (linear isometry), we have
\[
\forall u \in H, \quad - (\tau(A(u)), u)_H = -(A(u))u_{H', H} \leq -\alpha ||u||^2_H; \quad (125)
\]
\[
\forall u \in H, \quad ||\tau(A(u))||_H = ||A(u)||_{H'} \leq C ||u||_H. \quad (126)
\]

Let \( u, v \in H \) be vectors. Then, from Definition 161 (inner product \((\cdot, \cdot)_H \) is a bilinear map), and Definition 65 (bilinear map \((\cdot, \cdot)_H \) is left linear), we have the equivalences
\[
a(u, v) = f(v) \iff (\tau(A(u)), v)_H = (\tau(f), v)_H \iff (\tau(A(u)) - \tau(f), v)_H = 0. \]

Hence, from Definition 187 (orthogonal complement \( \tau(A(u)) - \tau(f) \) belongs to \( H^\perp \)), Lemma 188 (trivial orthogonal complements \( H^\perp = \{0\}_H \)), and Definition 58 (vector space \((H, +)\) is an abelian group), we have the equivalence
\[
\text{Problem 1} \iff \text{find } u \in H \text{ such that: } \tau(A(u)) = \tau(f). \quad (127)
\]

From Lemma 150 (compatibility of composition with continuity, \( \tau \) belongs to \( \mathcal{L}_c(H', H) \), \( \tau \circ A \) belongs to \( \mathcal{L}_c(H, H) \). From Lemma 160 (coercivity constant is less than continuity constant), we have \( 0 < \alpha \leq C \), hence \( \frac{\alpha}{C} > 0 \). Let \( \rho \in \mathbb{R} \) be a number. Assume that \( 0 < \rho < \frac{\alpha}{C} \). Then, from Theorem 147 (normed space of continuous linear maps \( \mathcal{L}_c(H, H), ||\cdot||_H \) is a normed space), Definition 104 (normed vector space \( \mathcal{L}_c(H, H) \) is a space), Definition 58 (vector space \( \mathcal{L}_c(H, H) \) is closed under vector operations), Definition 70 (vector subtraction), and Lemma 146 (identity map is continuous), \( g_0 = \text{Id}_H - \rho \circ A \) belongs to \( \mathcal{L}_c(H, H) \).

From Definition 58 (vector space \( H \) is closed under vector operations and \( \tau(f) \in H \)), let \( g : H \to H \) be the mapping defined by
\[
\forall v \in H, \quad g(v) = g_0(v) + \rho \tau(f). \]

Let \( u \in H \) be a vector. Then, from the definition of mappings \( g \) and \( g_0 \), Definition 58 (vector space \((H, +)\) is an abelian group and scalar multiplication is distributive wrt vector addition), and Lemma 73 (zero-product property, with \( \lambda = \rho \neq 0 \)), we have
\[
g(u) = u \iff g_0(u) + \rho \tau(f) = u \iff u - \rho(A(u)) + \rho \tau(f) = u \iff \rho(\tau(A(u)) - \tau(f)) = 0_H \iff \tau(A(u)) = \tau(f). \quad (128)
\]

Hence, from (127), we have the equivalence
\[
\text{Problem 1} \iff \text{find } u \in H \text{ such that: } g(u) = u. \]

Let \( v, v' \in H \) be vectors. Then, from Definition 58 (vector space \((H, +)\) is an abelian group), and Definition 70 (vector subtraction), let \( z = v - v' \in H \). Then, from Definition 70 (vector subtraction), Lemma 69 (minus times yields opposite vector with \( \lambda = 1 \)), and Definition 58 (vector space \((H, +)\) is an abelian group and scalar multiplication is distributive wrt vector addition), we have
\[
g(v) - g(v') = g_0(v) + \rho \tau(f) - (g_0(v') + \rho \tau(f)) = g_0(v - v') = g_0(z).
\]

Thus, from Lemma 168 (square expansion plus for \( ||\cdot||_H \)), Definition 161 (inner product \((\cdot, \cdot)_H \) is a symmetric bilinear map), Definition 65 (bilinear map \((\cdot, \cdot)_H \) is right linear), Definition 102 (norm \( ||\cdot||_H \) is absolutely homogeneous of degree 1), ordered field properties of \( \mathbb{R} \), (125), and
we have
\[ \|g(v) - g(v')\|_H^2 = \|g_0(z)\|_H^2 \]
\[ = \|z - \rho \tau(A(z))\|_H^2 \]
\[ = \|z\|_H^2 + 2\rho \tau(A(z))\|_H^2 + \rho^2 \|\tau(A(z))\|_H^2 \]
\[ \leq \|z\|_H^2 + 2\rho \|z\|_H^2 + \rho^2 C^2 \|z\|_H^2 \]
\[ = (1 - 2\rho + \rho^2 C^2)\|v - v'\|_H^2. \]

Hence, from compatibility of the square root function with comparison in $\mathbb{R}^+$, Definition 46 (Lipschitz continuity) with $k = \sqrt{1 - 2\rho + \rho^2 C^2}$, Lemma 107 (compatible $\rho$ for Lax-Milgram), since $0 < \alpha < C$ and $0 < \rho < \frac{1}{2}$, and Definition 48 (nonnegative for $g$ contraction), there exists a unique fixed point $u \in H$ such that $g(u) = u$. Hence, from 128, there exists a unique solution to Problem 1.

**Estimation.** Let $u \in H$ be the solution to Problem 1.

**Case $u = 0_H$.** Then, from Lemma 106 (norm preserves zero for $\|\cdot\|_H$), Lemma 107 (norm is coercive for $\|\cdot\|_H$), and ordered field properties of $\mathbb{R}$ (with $\alpha > 0$), we have
\[ \|u\|_H = 0 \leq \frac{1}{\alpha} \|f\|_H'. \]

**Case $u \neq 0_H$.** Then, from Definition 102 (norm $\|\cdot\|_H$ is definite, contrapositive), and Lemma 107 (norm is coercive for $\|\cdot\|_H$), we have $\|u\|_H > 0$. Moreover, from Definition 159 (coercive bilinear form for $a$), properties of the absolute value on $\mathbb{R}$, 1 with $v = u$, and Lemma 148 (operator norm estimation for $f \in H'$), we have
\[ \alpha \|u\|_H^2 \leq a(u, u) \leq |a(u, u)| = |f(u)| \leq \|f\|_H' \|u\|_H. \]

Hence, from ordered field properties of $\mathbb{R}$ (with $\|u\|_H, \alpha > 0$), we have the estimation
\[ \|u\|_H \leq \frac{1}{\alpha} \|f\|_H'. \]

\[ \square \]

**Lemma 199 (Galerkin orthogonality).** Let $(H, (\cdot, \cdot)_H)$ be a real Hilbert space. Let $a$ be a bounded bilinear form on $H$. Let $f \in H'$ be a continuous linear form on $H$. Let $H_h$ be a subspace of $H$. Let $u \in H$ be a solution to Problem 1. Let $u_h \in H_h$ be a solution to Problem 2. Then,
\[ \forall v_h \in H_h, \quad a(u - u_h, v_h) = 0. \]

**Proof.** Let $v_h \in H_h$ be a vector. Then, from Definition 74 (subspace $H_h$ is a subset of $H$), $v_h$ also belongs to $H$. Hence, from 1 with $v = v_h$, 2. Definition 65 (bilinear map $a$ is left linear), and field properties of $\mathbb{R}$, we have
\[ a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = f(v_h) - f(v_h) = 0. \]

\[ \square \]

**Theorem 200 (Lax–Milgram, closed subspace).** Assume hypotheses of Theorem 198 (Lax–Milgram). Let $H_h$ be a closed subspace of $H$. Then, there exists a unique $u_h \in H_h$ solution to Problem 2. Moreover,
\[ \|u_h\|_H \leq \frac{1}{\alpha} \|f\|_H'. \]

\[ (130) \]
Proof. Direct consequence of Lemma 195 (closed Hilbert subspace) \( H_k \) is a closed subspace of \( H \), and Theorem 198 (Lax–Milgram) \( (H, (\cdot,\cdot)_H) \) is a Hilbert space where the restriction to \( H_k \) of the norm associated to \( (\cdot,\cdot)_H \) is still denoted \( \|\cdot\|_H \).

**Lemma 201 (Céa).** Assume hypotheses of Theorem 200 (Lax–Milgram, closed subspace). Let \( C \geq 0 \) be a continuity constant of the bounded bilinear form \( a \). Let \( u \in H \) be the unique solution to Problem 1. Let \( u_h \in H_h \) be the unique solution to Problem 2. Then,

\[
\forall v_h \in H_h, \quad \|u - u_h\|_H \leq C \alpha \|u - v_h\|_H.
\]

Proof. Let \( v_h \in H_h \) be a vector in the subspace.

**Case** \( u = u_h \). Then, from Definition 58 (vector space) \((H,+)\) is an abelian group, Lemma 106 (norm preserves zero) \( u - u_h = 0_H \), Lemma 107 (norm is nonnegative) \( \|\cdot\|_H \), and ordered field properties of \( \mathbb{R} \) with \( \alpha > 0 \) and \( C \geq 0 \), we have

\[
\|u - u_h\|_H = 0 \leq C \alpha \|u - v_h\|_H.
\]

**Case** \( u \neq u_h \). Then, from Definition 58 (vector space) \((H,+)\) is an abelian group, and Definition 102 (norm \( \|\cdot\|_H \) is definite, contrapositive) we have \( \|u - u_h\|_H \neq 0 \). Moreover, from Definition 70 (vector subtraction), Definition 161 (inner product) \((\cdot,\cdot)_H \) is a bilinear map, Definition 65 (bilinear map) \((\cdot,\cdot)_H \) is right linear, and Lemma 199 (Galerkin orthogonality) \( a(u - u_h, u - v_h) = a(u, u - v_h) - a(u - u_h, u) \).

Thus, from Definition 159 (coercive bilinear form) \( a \) with \( u = u - u_h \), properties of the absolute value on \( \mathbb{R} \), (132) with \( u_h \) and \( v_h \) in \( H_h \), compatibility of the absolute value with comparison in \( \mathbb{R} \), and Definition 157 (bounded bilinear form), we have

\[
\alpha \|u - u_h\|^2_H \leq a(u - u_h, u - u_h) \\
\leq |a(u - u_h, u - u_h)| \\
= |a(u - u_h, u)| \\
= |a(u - u_h, u - v_h)| \\
\leq C \|u - u_h\|_H \|u - v_h\|_H.
\]

Hence, from ordered field properties of \( \mathbb{R} \) with \( \alpha, \|u - u_h\|_H > 0 \), we have

\[
\|u - u_h\|_H \leq C \alpha \|u - v_h\|_H.
\]

**Lemma 202 (finite dimensional subspace in Hilbert space is closed).** Let \((H, (\cdot,\cdot)_2)\) be a real Hilbert space. Let \( \|\cdot\|_H \) be the norm associated with inner product \((\cdot,\cdot)_H\). Let \( d_H \) be the distance associated with norm \( \|\cdot\|_H \). Let \( F \) be a subspace of \( H \). Assume that \( F \) is a finite dimensional subspace. Then, \( F \) is closed for distance \( d_H \).

Proof. From Definition 52 (finite dimensional subspace), let \( n \in \mathbb{N} \), and let \( u_1, \ldots, u_n \in H \) such that \( F = \text{span}(\{u_1, \ldots, u_n\}) = \text{span}(\{u_1\}) + \ldots + \text{span}(\{u_n\}) \). For \( i \in \mathbb{N} \) with \( 1 \leq i \leq n \), let \( F_i = \text{span}(\{u_1, \ldots, u_i\}) \). Then, for \( 2 \leq i \leq n \), we have \( F_i = F_{i-1} + \text{span}(\{u_i\}) \). Let \( P(i) \) be the property “\( F_i \) is closed for distance \( d_H \)”.

**Induction:** \( P(1) \). From Lemma 112 (linear span is closed), \( F_1 = \text{span}(\{u_1\}) \) is closed for distance \( d_H \).


**Induction:** $P(i - 1)$ implies $P(i)$. Assume that $2 \leq i \leq n$. Assume that $P(i - 1)$ holds. Then, from Lemma 192 (sum is orthogonal sum), there exists $u'_i \in F^i_{i-1}$ such that

$$F_i = F_{i-1} + \text{span}(\{u_i\}) = F_{i-1} + \text{span}(\{u'_i\}).$$

**Case $u'_i = 0$**. Then, $F_i = F_{i-1}$ is closed for distance $d_G$. **Case $u'_i \neq 0$**. Then, from Definition 194 (Hilbert space $H$ is complete for distance $d_H$), and Lemma 39 (sum of complete subspace and linear span is closed), $F_i = F_{i-1} + \text{span}(\{u'_i\})$ is closed for distance $d_G$.

Hence, by (finite) induction on $i \in \mathbb{N}$ with $1 \leq i \leq n$, we have $P(n)$. Therefore, $F = F_n$ is closed for distance $d_G$.

**Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace).** Assume hypotheses of Theorem 198 (Lax–Milgram). Let $C \geq 0$ be a continuity constant of the bounded bilinear form $a$. Let $u \in H$ be the unique solution to Problem (1). Let $H_h$ be a finite dimensional subspace of $H$. Then, there exists a unique $u_h \in H_h$ solution to Problem (2). Moreover,

\[
\|u_h\|_H \leq \frac{1}{\alpha} \|f\|_{H'}; \quad \text{(133)}
\]

\[
\forall v_h \in H_h, \quad \|u - u_h\|_H \leq \frac{C}{\alpha} \|u - v\|_H. \quad \text{(134)}
\]

**Proof.** Direct consequence of Lemma 202 (finite dimensional subspace in Hilbert space is closed), Theorem 200 (Lax–Milgram, closed subspace), and Lemma 201 (Céa).

## 5 Conclusions, perspectives

We have presented a very detailed proof of the Lax–Milgram theorem for the resolution on a Hilbert space of linear (partial differential) equations set under their weak form. Among the various proofs available in the literature, we have chosen a path using basic notions. In particular, we have avoided to obtain the result from a more general one, e.g. set on a Banach space. The proof uses the following main arguments: the Riesz–Fréchet representation theorem, the characterization of the projection onto a closed subspace, and the fixed point theorem on a complete metric space.

The short-term purpose of this work is to help the formalization of such a result in the Coq formal proof assistant. One of the key issues will then be to deal with the embedded algebraic structures: group, vector space (an external operation is added), normed vector space (a norm is added), inner vector space (an inner product is added), Hilbert space (completeness is added). New structures should be extensions of the previous ones: the addition operation in the Hilbert space should be the very same addition operation from the initial group structure. The other tricky part will be to deal with functional spaces.

The long-term purpose of this activity is the formal proof of programs implementing the finite element method. As a consequence, we will also have to write very detailed pen-and-paper proofs for the following notions and results: large parts of the integration and distribution theories, define Sobolev spaces (at least $L^2(\Omega)$, $H^1(\Omega)$ and $H^1_0(\Omega)$ for some bounded domain $\Omega$ of $\mathbb{R}^d$ with $d = 1$, 2, or 3), and prove that they are Hilbert spaces, and finally some aspects of the interpolation theory.
References

[1] Robert A. Adams. *Sobolev spaces*. Academic Press (A subsidiary of Harcourt Brace Jovanovich, Publishers), New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] Sylvie Boldo, François Clément, Jean-Christophe Filliâtre, Micaela Mayero, Guillaume Melquiond, and Pierre Weis. Trusting computations: a mechanized proof from partial differential equations to actual program. *Comput. Math. Appl.*, 68(3):325–352, 2014.

[3] Haïm Brezis. *Analyse fonctionnelle* [Functional analysis]. Collection Mathématiques Appliquées pour la Maîtrise [Collection of Applied Mathematics for the Master’s Degree]. Masson, Paris, 1983. Théorie et applications [Theory and applications].

[4] Philippe G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[5] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[6] Bernard Gostiaux. *Cours de mathématiques spéciales - Tome 1* [Lecture notes in Special Mathematics - Tome 1]. Mathématiques [Mathematics]. Presses Universitaires de France, Paris, 1993. Algèbre [Algebra], With a preface by Paul Deheuvels.

[7] Bernard Gostiaux. *Cours de mathématiques spéciales - Tome 2* [Lecture notes in Special Mathematics - Tome 2]. Mathématiques [Mathematics]. Presses Universitaires de France, Paris, 1993. Topologie, analyse réelle [Topology, real analysis].

[8] Bernard Gostiaux. *Cours de mathématiques spéciales - Tome 3* [Lecture notes in Special Mathematics - Tome 3]. Mathématiques [Mathematics]. Presses Universitaires de France, Paris, 1993. Analyse fonctionnelle et calcul différentiel [Functional analysis and differential calculus].

[9] P. D. Lax and A. N. Milgram. Parabolic equations. In *Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, no. 33, pages 167–190. Princeton University Press, Princeton, N. J., 1954.

[10] Alfio Quarteroni and Alberto Valli. *Numerical approximation of partial differential equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1994.

[11] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[12] Kösaku Yosida. *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

[13] O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu. *The finite element method: its basis and fundamentals*. Elsevier/Butterworth Heinemann, Amsterdam, seventh edition, 2013.
A  Lists of statements

List of Definitions

2  Definition (supremum) ................................................. 4
7  Definition (maximum) .................................................. 9
9  Definition (infimum) .................................................. 9
14 Definition (minimum) ............................................... 11
16 Definition (distance) .................................................. 11
17 Definition (metric space) ......................................... 11
19 Definition (closed ball) ........................................... 11
20 Definition (sphere) ................................................... 11
21 Definition (open subset) .......................................... 11
22 Definition (closed subset) ....................................... 11
25 Definition (closure) .................................................. 12
26 Definition (convergent sequence) ................................ 12
32 Definition (stationary sequence) ................................ 13
34 Definition (Cauchy sequence) .................................... 14
37 Definition (complete subset) ..................................... 14
38 Definition (complete metric space) ............................ 14
42 Definition (continuity in a point) ............................... 15
43 Definition (pointwise continuity) ............................... 15
45 Definition (uniform continuity) ................................. 15
46 Definition (Lipschitz continuity) ................................. 15
48 Definition (contraction) ........................................... 16
52 Definition (iterated function sequence) ......................... 16
58 Definition (vector space) ........................................... 19
61 Definition (set of mappings to space) ......................... 19
62 Definition (linear map) ............................................. 19
63 Definition (set of linear maps) .................................. 19
64 Definition (linear form) ............................................ 19
65 Definition (bilinear map) ........................................... 20
66 Definition (bilinear form) ........................................ 20
67 Definition (set of bilinear forms) ............................... 20
70 Definition (vector subtraction) ................................... 20
71 Definition (scalar division) ...................................... 20
74 Definition (subspace) ............................................... 21
80 Definition (linear span) .......................................... 22
81 Definition (sum of subspaces) ................................... 22
82 Definition (finite dimensional subspace) ....................... 22
83 Definition (direct sum of subspaces) ........................... 22
86 Definition (product vector operations) ......................... 23
88 Definition (inherited vector operations) ....................... 23
94 Definition (identity map) ......................................... 24
97 Definition (isomorphism) ......................................... 25
98 Definition (kernel) .................................................. 25
102 Definition (norm) ................................................... 26
104 Definition (normed vector space) .............................. 26
109 Definition (distance associated with norm) .................. 27
113 Definition (closed unit ball) ................................... 28
115 Definition (unit sphere) .......................................... 28
122 Definition (linear isometry) .................................... 29
124 Definition (product norm) ...................................... 29
List of Lemmas

3  Lemma (finite supremum)                        8
4  Lemma (discrete lower accumulation)            8
5  Lemma (supremum is positive scalar multiplicative) 8
8  Lemma (finite maximum)                         9
10 Lemma (duality infimum-supremum)              9
11 Lemma (finite infimum)                        9
12 Lemma (discrete upper accumulation)            9
13 Lemma (finite infimum discrete)                9
15 Lemma (finite minimum)                        9
18 Lemma (iterated triangle inequality)           11
23 Lemma (equivalent definition of closed subset) 12
24 Lemma (singleton is closed)                   12
27 Lemma (variant of point separation)           12
28 Lemma (limit is unique)                       12
29 Lemma (closure is limit of sequences)         12
30 Lemma (closed equals closure)                 13
31 Lemma (closed is limit of sequences)          13
33 Lemma (stationary sequence is convergent)     13
35 Lemma (equivalent definition of Cauchy sequence) 14
36 Lemma (convergent sequence is Cauchy)         14
39 Lemma (closed subset of complete is complete) 14
44 Lemma (compatibility of limit with continuous functions) 15
49 Lemma (uniform continuous is continuous)      16
50 Lemma (zero-Lipschitz continuous is constant) 16
51 Lemma (Lipschitz continuous is uniform continuous) 16
53 Lemma (stationary iterated function sequence) 17
54 Lemma (iterate Lipschitz continuous mapping)  17
55 Lemma (convergent iterated function sequence) 17
68 Lemma (zero times yields zero)                20
69 Lemma (minus times yields opposite vector)    20
72 Lemma (times zero yields zero)                20
73 Lemma (zero-product property)                 20
77 Lemma (trivial subspaces)                     21
| Lemma                                                                 | Page |
|----------------------------------------------------------------------|------|
| (closed under vector operations is subspace)                        | 21   |
| (closed under linear combination is subspace)                       | 21   |
| (equivalent definitions of direct sum)                              | 22   |
| (direct sum with linear span)                                        | 22   |
| (product is space)                                                   | 23   |
| (space of mappings to a space)                                       | 23   |
| (linear map preserves zero)                                          | 24   |
| (linear map preserves linear combinations)                           | 24   |
| (space of linear maps)                                               | 24   |
| (identity map is linear map)                                         | 24   |
| (composition of linear maps is bilinear)                             | 25   |
| (kernel is subspace)                                                 | 25   |
| (injective linear map has zero kernel)                               | 25   |
| (K is space)                                                         | 26   |
| (K is normed space)                                                  | 26   |
| (norm preserves zero)                                                | 26   |
| (norm is nonnegative)                                                | 26   |
| (normalization by nonzero)                                           | 26   |
| (norm gives distance)                                                | 27   |
| (linear span is closed)                                              | 27   |
| (equivalent definition of closed unit ball)                          | 28   |
| (equivalent definition of unit sphere)                               | 28   |
| (zero on unit sphere is zero)                                        | 28   |
| (reverse triangle inequality)                                         | 28   |
| (norm is one-Lipschitz continuous)                                   | 28   |
| (norm is uniformly continuous)                                       | 29   |
| (norm is continuous)                                                 | 29   |
| (identity map is linear isometry)                                    | 29   |
| (product is normed space)                                            | 29   |
| (vector addition is continuous)                                      | 30   |
| (scalar multiplication is continuous)                                | 30   |
| (norm of image of unit vector)                                       | 31   |
| (norm of image of unit sphere)                                       | 31   |
| (operator norm is nonnegative)                                       | 32   |
| (finite operator norm is continuous)                                 | 32   |
| (linear isometry is continuous)                                      | 33   |
| (identity map is continuous)                                         | 33   |
| (operator norm estimation)                                           | 35   |
| (continuous linear maps have closed kernel)                          | 35   |
| (compatibility of composition with continuity)                       | 36   |
| (complete normed space of continuous linear maps)                   | 36   |
| (topological dual is complete normed space)                          | 37   |
| (bra-ket is bilinear map)                                            | 37   |
| (representation for bounded bilinear form)                           | 37   |
| (coercivity constant is less than continuity constant)               | 37   |
| (inner product subspace)                                             | 38   |
| (inner product with zero is zero)                                    | 38   |
| (square expansion plus)                                              | 38   |
| (square expansion minus)                                             | 38   |
| (parallelogram identity)                                             | 38   |
| (Cauchy–Schwarz inequality)                                          | 38   |
| (Cauchy–Schwarz inequality with norms)                               | 38   |
List of Theorems

47 Theorem (equivalent definition of Lipschitz continuity) ........................................ 15
56 Theorem (fixed point) ................................................................................................. 18
142 Theorem (continuous linear map) ............................................................................. 33
147 Theorem (normed space of continuous linear maps) .................................................. 35
180 Theorem (orthogonal projection onto nonempty complete convex) ......................... 43
183 Theorem (orthogonal projection onto complete subspace) ...................................... 45
191 Theorem (direct sum with orthogonal complement when complete) ...................... 48
196 Theorem (Riesz–Fréchet) ......................................................................................... 50
198 Theorem (Lax–Milgram) ......................................................................................... 53
200 Theorem (Lax–Milgram, closed subspace) ............................................................... 55
203 Theorem (Lax–Milgram–Céa, finite dimensional subspace) .................................... 57
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

Figure 1: Dependency graph (both ways). All dependencies are detailed in Appendices B and C.
B  Depends directly from...

Definition 2 (supremum) has no direct dependency.

Lemma 3 (finite supremum) has no direct dependency.

Lemma 4 (discrete lower accumulation) has no direct dependency.

Lemma 5 (supremum is positive scalar multiplicative) depends directly from:
  Definition 2 (supremum).

Definition 7 (maximum) has no direct dependency.

Lemma 8 (finite maximum) depends directly from:
  Definition 2 (supremum),
  Lemma 3 (finite supremum),
  Definition 7 (maximum).

Definition 9 (infimum) has no direct dependency.

Lemma 10 (duality infimum-supremum) depends directly from:
  Definition 2 (supremum),
  Definition 9 (infimum).

Lemma 11 (finite infimum) depends directly from:
  Lemma 3 (finite supremum),
  Lemma 10 (duality infimum-supremum).

Lemma 12 (discrete upper accumulation) depends directly from:
  Lemma 4 (discrete lower accumulation).

Lemma 13 (finite infimum discrete) depends directly from:
  Lemma 11 (finite infimum),
  Lemma 12 (discrete upper accumulation).

Definition 14 (minimum) has no direct dependency.

Lemma 15 (finite minimum) depends directly from:
  Lemma 8 (finite maximum),
  Lemma 10 (duality infimum-supremum),
  Definition 14 (minimum).

Definition 16 (distance) has no direct dependency.

Definition 17 (metric space) has no direct dependency.

Lemma 18 (iterated triangle inequality) depends directly from:
  Definition 16 (distance).

Definition 19 (closed ball) has no direct dependency.

Definition 20 (sphere) has no direct dependency.

Definition 21 (open subset) has no direct dependency.

Definition 22 (closed subset) has no direct dependency.

Lemma 23 (equivalent definition of closed subset) depends directly from:
  Definition 21 (open subset),
  Definition 22 (closed subset).
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

Lemma 24 (singleton is closed) depends directly from:
  Definition 16 (distance).
  Lemma 23 (equivalent definition of closed subset).

Definition 25 (closure) has no direct dependency.

Definition 26 (convergent sequence) has no direct dependency.

Lemma 27 (variant of point separation) depends directly from:
  Definition 16 (distance).

Lemma 28 (limit is unique) depends directly from:
  Definition 16 (distance).
  Definition 26 (convergent sequence).
  Lemma 27 (variant of point separation).

Lemma 29 (closure is limit of sequences) depends directly from:
  Definition 16 (distance).
  Definition 19 (closed ball).
  Definition 25 (closure).
  Definition 26 (convergent sequence).

Lemma 30 (closed equals closure) depends directly from:
  Definition 22 (closed subset).
  Lemma 23 (equivalent definition of closed subset).
  Definition 25 (closure).

Lemma 31 (closed is limit of sequences) depends directly from:
  Definition 25 (closure).
  Lemma 29 (closure is limit of sequences).
  Lemma 30 (closed equals closure).

Definition 32 (stationary sequence) has no direct dependency.

Definition 34 (Cauchy sequence) has no direct dependency.

Lemma 35 (equivalent definition of Cauchy sequence) depends directly from:
  Definition 16 (distance).
  Definition 34 (Cauchy sequence).

Lemma 36 (convergent sequence is Cauchy) depends directly from:
  Definition 16 (distance).
  Definition 26 (convergent sequence).
  Lemma 28 (limit is unique).
  Definition 34 (Cauchy sequence).

Definition 37 (complete subset) has no direct dependency.

Definition 38 (complete metric space) has no direct dependency.

Lemma 39 (closed subset of complete is complete) depends directly from:
  Lemma 29 (closure is limit of sequences).
  Lemma 30 (closed equals closure).
  Definition 37 (complete subset).
  Definition 38 (complete metric space).
Definition 42 (continuity in a point) has no direct dependency.

Definition 43 (pointwise continuity) has no direct dependency.

Lemma 44 (compatibility of limit with continuous functions) depends directly from:
Definition 26 (convergent sequence),
Definition 42 (continuity in a point).

Definition 45 (uniform continuity) has no direct dependency.

Definition 46 (Lipschitz continuity) has no direct dependency.

Theorem 47 (equivalent definition of Lipschitz continuity) depends directly from:
Definition 16 (distance),
Definition 46 (Lipschitz continuity).

Definition 48 (contraction) has no direct dependency.

Lemma 49 (uniform continuous is continuous) depends directly from:
Definition 42 (continuity in a point),
Definition 43 (pointwise continuity),
Definition 45 (uniform continuity).

Lemma 50 (zero-Lipschitz continuous is constant) depends directly from:
Definition 16 (distance),
Definition 46 (Lipschitz continuity).

Lemma 51 (Lipschitz continuous is uniform continuous) depends directly from:
Definition 45 (uniform continuity),
Definition 46 (Lipschitz continuity),
Lemma 50 (zero-Lipschitz continuous is constant).

Definition 52 (iterated function sequence) has no direct dependency.

Lemma 53 (stationary iterated function sequence) depends directly from:
Definition 32 (stationary sequence),
Definition 52 (iterated function sequence).

Lemma 54 (iterate Lipschitz continuous mapping) depends directly from:
Definition 46 (Lipschitz continuity),
Definition 52 (iterated function sequence).

Lemma 55 (convergent iterated function sequence) depends directly from:
Definition 26 (convergent sequence),
Lemma 28 (limit is unique),
Definition 32 (stationary sequence),
Lemma 33 (stationary sequence is convergent),
Definition 46 (Lipschitz continuity),
Lemma 50 (zero-Lipschitz continuous is constant),
Definition 52 (iterated function sequence).

Theorem 56 (fixed point) depends directly from:
Definition 16 (distance),
Lemma 18 (iterated triangle inequality),
Definition 32 (stationary sequence),
Lemma 33 (stationary sequence is convergent),
Lemma 35 (equivalent definition of Cauchy sequence),
Definition 37 (complete subset),
Definition 38 (complete metric space).
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

Definition 46 (Lipschitz continuity),
Definition 48 (contraction),
Lemma 50 (zero-Lipschitz continuous is constant),
Lemma 53 (stationary iterated function sequence),
Lemma 54 (iterate Lipschitz continuous mapping),
Lemma 55 (convergent iterated function sequence).

Definition 58 (vector space) has no direct dependency.
Definition 61 (set of mappings to space) has no direct dependency.
Definition 62 (linear map) has no direct dependency.
Definition 63 (set of linear maps) has no direct dependency.
Definition 64 (linear form) has no direct dependency.
Definition 65 (bilinear map) has no direct dependency.
Definition 66 (bilinear form) has no direct dependency.
Definition 67 (set of bilinear forms) has no direct dependency.

Lemma 68 (zero times yields zero) depends directly from:
Definition 58 (vector space).

Lemma 69 (minus times yields opposite vector) depends directly from:
Definition 58 (vector space),
Lemma 68 (zero times yields zero).

Definition 70 (vector subtraction) has no direct dependency.
Definition 71 (scalar division) has no direct dependency.

Lemma 72 (times zero yields zero) depends directly from:
Definition 58 (vector space),
Definition 70 (vector subtraction).

Lemma 73 (zero-product property) depends directly from:
Definition 58 (vector space),
Lemma 68 (zero times yields zero),
Lemma 72 (times zero yields zero).

Definition 74 (subspace) has no direct dependency.

Lemma 77 (trivial subspaces) depends directly from:
Definition 74 (subspace).

Lemma 78 (closed under vector operations is subspace) depends directly from:
Definition 58 (vector space),
Lemma 69 (minus times yields opposite vector),
Definition 74 (subspace).

Lemma 79 (closed under linear combination is subspace) depends directly from:
Definition 58 (vector space),
Lemma 73 (zero-product property),
Lemma 78 (closed under vector operations is subspace).

Definition 80 (linear span) has no direct dependency.
Definition 81 (sum of subspaces) has no direct dependency.
Definition 82 (finite dimensional subspace) has no direct dependency.

Definition 83 (direct sum of subspaces) has no direct dependency.

Lemma 84 (equivalent definitions of direct sum) depends directly from:
Definition 58 (vector space),
Lemma 69 (minus times yields opposite vector),
Definition 70 (vector subtraction),
Lemma 78 (closed under vector operations is subspace),
Definition 83 (direct sum of subspaces).

Lemma 85 (direct sum with linear span) depends directly from:
Lemma 73 (zero-product property),
Lemma 78 (closed under vector operations is subspace),
Definition 80 (linear span),
Lemma 84 (equivalent definitions of direct sum).

Definition 86 (product vector operations) has no direct dependency.

Lemma 87 (product is space) depends directly from:
Definition 58 (vector space),
Definition 86 (product vector operations).

Definition 88 (inherited vector operations) has no direct dependency.

Lemma 90 (space of mappings to a space) depends directly from:
Definition 58 (vector space),
Definition 61 (set of mappings to space),
Definition 88 (inherited vector operations).

Lemma 91 (linear map preserves zero) depends directly from:
Definition 62 (linear map),
Lemma 68 (zero times yields zero).

Lemma 92 (linear map preserves linear combinations) depends directly from:
Definition 58 (vector space),
Definition 62 (linear map),
Lemma 68 (zero times yields zero).

Lemma 93 (space of linear maps) depends directly from:
Definition 58 (vector space),
Definition 62 (linear map),
Definition 63 (set of linear maps),
Lemma 72 (times zero yields zero),
Lemma 79 (closed under linear combination is subspace),
Definition 88 (inherited vector operations),
Lemma 90 (space of mappings to a space).

Definition 94 (identity map) has no direct dependency.

Lemma 95 (identity map is linear map) depends directly from:
Definition 62 (linear map),
Definition 94 (identity map).

Lemma 96 (composition of linear maps is bilinear) depends directly from:
Definition 62 (linear map),
Definition 65 (bilinear map),
Lemma 87 (product is space),
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

Definition 88 (inherited vector operations).
Lemma 92 (linear map preserves linear combinations).
Lemma 93 (space of linear maps).

Definition 97 (isomorphism) has no direct dependency.
Definition 98 (kernel) has no direct dependency.

Lemma 99 (kernel is subspace) depends directly from:
Definition 58 (vector space).
Lemma 72 (times zero yields zero).
Lemma 79 (closed under linear combination is subspace).
Lemma 91 (linear map preserves zero).
Lemma 92 (linear map preserves linear combinations).
Definition 98 (kernel).

Lemma 100 (injective linear map has zero kernel) depends directly from:
Definition 58 (vector space).
Definition 62 (linear map).
Definition 70 (vector subtraction).
Lemma 91 (linear map preserves zero).
Definition 98 (kernel).

Lemma 101 (K is space) has no direct dependency.

Definition 102 (norm) has no direct dependency.

Definition 104 (normed vector space) has no direct dependency.

Lemma 105 (K is normed space) depends directly from:
Definition 102 (norm).
Definition 104 (normed vector space).

Lemma 106 (norm preserves zero) depends directly from:
Definition 58 (vector space).
Lemma 68 (zero times yields zero).
Definition 102 (norm).
Definition 104 (normed vector space).

Lemma 107 (norm is nonnegative) depends directly from:
Definition 58 (vector space).
Definition 102 (norm).
Definition 104 (normed vector space).

Lemma 108 (normalization by nonzero) depends directly from:
Definition 71 (scalar division).
Definition 102 (norm).
Lemma 107 (norm is nonnegative).

Definition 109 (distance associated with norm) has no direct dependency.

Lemma 111 (norm gives distance) depends directly from:
Definition 16 (distance).
Definition 17 (metric space).
Definition 70 (vector subtraction).
Definition 102 (norm).
Lemma 107 (norm is nonnegative).
Definition 109 (distance associated with norm).
Lemma 112 \textit{(linear span is closed)} depends directly from:
- Lemma 24 \textit{(singleton is closed)}
- Definition 26 \textit{(convergent sequence)}
- Lemma 28 \textit{(limit is unique)}
- Lemma 31 \textit{(closed is limit of sequences)}
- Definition 34 \textit{(Cauchy sequence)}
- Lemma 36 \textit{(convergent sequence is Cauchy)}
- Definition 37 \textit{(complete subset)}
- Definition 58 \textit{(vector space)}
- Definition 70 \textit{(vector subtraction)}
- Definition 80 \textit{(linear span)}
- Definition 102 \textit{(norm)}
- Lemma 107 \textit{(norm is nonnegative)}
- Definition 109 \textit{(distance associated with norm)}

Definition 113 \textit{(closed unit ball)} has no direct dependency.

Lemma 114 \textit{(equivalent definition of closed unit ball)} depends directly from:
- Definition 19 \textit{(closed ball)}
- Definition 109 \textit{(distance associated with norm)}
- Lemma 111 \textit{(norm gives distance)}
- Definition 113 \textit{(closed unit ball)}

Definition 115 \textit{(unit sphere)} has no direct dependency.

Lemma 116 \textit{(equivalent definition of unit sphere)} depends directly from:
- Definition 20 \textit{(sphere)}
- Definition 109 \textit{(distance associated with norm)}
- Lemma 111 \textit{(norm gives distance)}
- Definition 115 \textit{(unit sphere)}

Lemma 117 \textit{(zero on unit sphere is zero)} depends directly from:
- Definition 58 \textit{(vector space)}
- Definition 62 \textit{(linear map)}
- Lemma 72 \textit{(times zero yields zero)}
- Lemma 91 \textit{(linear map preserves zero)}
- Lemma 108 \textit{(normalization by nonzero)}
- Lemma 116 \textit{(equivalent definition of unit sphere)}

Lemma 118 \textit{(reverse triangle inequality)} depends directly from:
- Definition 102 \textit{(norm)}
- Definition 104 \textit{(normed vector space)}

Lemma 119 \textit{(norm is one-Lipschitz continuous)} depends directly from:
- Definition 46 \textit{(Lipschitz continuity)}
- Definition 109 \textit{(distance associated with norm)}
- Lemma 118 \textit{(reverse triangle inequality)}

Lemma 120 \textit{(norm is uniformly continuous)} depends directly from:
- Lemma 51 \textit{(Lipschitz continuous is uniform continuous)}
- Lemma 119 \textit{(norm is one-Lipschitz continuous)}

Lemma 121 \textit{(norm is continuous)} depends directly from:
- Lemma 49 \textit{(uniform continuous is continuous)}
- Lemma 120 \textit{(norm is uniformly continuous)}

Definition 122 \textit{(linear isometry)} has no direct dependency.
Lemma 123 (identity map is linear isometry) depends directly from:
- Definition 94 (identity map),
- Lemma 95 (identity map is linear map),
- Definition 122 (linear isometry).

Definition 124 (product norm) has no direct dependency.

Lemma 127 (product is normed space) depends directly from:
- Definition 86 (product vector operations),
- Lemma 87 (product is space),
- Definition 102 (norm),
- Definition 104 (normed vector space),
- Lemma 107 (norm is nonnegative),
- Definition 124 (product norm).

Lemma 128 (vector addition is continuous) depends directly from:
- Definition 42 (continuity in a point),
- Definition 43 (pointwise continuity),
- Definition 58 (vector space),
- Definition 66 (product vector operations),
- Definition 102 (norm),
- Definition 109 (distance associated with norm),
- Definition 124 (product norm),
- Lemma 127 (product is normed space).

Lemma 129 (scalar multiplication is continuous) depends directly from:
- Definition 16 (distance),
- Definition 42 (continuity in a point),
- Definition 43 (pointwise continuity),
- Definition 58 (vector space),
- Lemma 68 (zero times yields zero),
- Definition 102 (norm),
- Definition 109 (distance associated with norm).

Lemma 133 (norm of image of unit vector) depends directly from:
- Definition 62 (linear map),
- Definition 102 (norm),
- Lemma 107 (norm is nonnegative),
- Lemma 108 (normalization by nonzero).

Lemma 134 (norm of image of unit sphere) depends directly from:
- Definition 102 (norm),
- Lemma 106 (norm preserves zero),
- Lemma 116 (equivalent definition of unit sphere),
- Lemma 133 (norm of image of unit vector).

Definition 135 (operator norm) has no direct dependency.

Lemma 137 (equivalent definition of operator norm) depends directly from:
- Lemma 134 (norm of image of unit sphere),
- Definition 135 (operator norm).

Lemma 138 (operator norm is nonnegative) depends directly from:
- Definition 2 (supremum),
- Lemma 107 (norm is nonnegative),
- Lemma 137 (equivalent definition of operator norm).
Definition 139 (bounded linear map) has no direct dependency.
Definition 140 (linear map bounded on unit ball) has no direct dependency.
Definition 141 (linear map bounded on unit sphere) has no direct dependency.

Theorem 142 (continuous linear map) depends directly from:
Definition 2 (supremum),
Lemma 3 (finite supremum),
Definition 42 (continuity in a point),
Definition 43 (pointwise continuity),
Definition 46 (Lipschitz continuity),
Lemma 49 (uniform continuous is continuous),
Lemma 51 (Lipschitz continuous is uniform continuous),
Definition 62 (linear map),
Definition 70 (vector subtraction),
Lemma 91 (linear map preserves zero),
Definition 102 (norm),
Lemma 106 (norm preserves zero),
Lemma 114 (equivalent definition of closed unit ball),
Lemma 116 (equivalent definition of unit sphere),
Definition 135 (operator norm),
Lemma 137 (equivalent definition of operator norm),
Lemma 138 (operator norm is nonnegative),
Definition 140 (linear map bounded on unit ball),
Definition 141 (linear map bounded on unit sphere).

Definition 143 (set of continuous linear maps) has no direct dependency.

Lemma 144 (finite operator norm is continuous) depends directly from:
Definition 2 (supremum),
Definition 139 (bounded linear map),
Definition 140 (linear map bounded on unit ball),
Definition 141 (linear map bounded on unit sphere),
Theorem 142 (continuous linear map),
Definition 143 (set of continuous linear maps).

Lemma 145 (linear isometry is continuous) depends directly from:
Definition 122 (linear isometry),
Lemma 144 (finite operator norm is continuous).

Lemma 146 (identity map is continuous) depends directly from:
Lemma 123 (identity map is linear isometry),
Lemma 145 (linear isometry is continuous).

Theorem 147 (normed space of continuous linear maps) depends directly from:
Definition 2 (supremum),
Lemma 5 (supremum is positive scalar multiplicative),
Lemma 78 (closed under vector operations is subspace),
Definition 88 (inherited vector operations),
Definition 102 (norm),
Definition 104 (normed vector space),
Lemma 107 (norm is nonnegative),
Lemma 117 (zero on unit sphere is zero),
Lemma 137 (equivalent definition of operator norm),
Definition 141 (linear map bounded on unit sphere).
Definition 143 (set of continuous linear maps).
Lemma 144 (finite operator norm is continuous).

Lemma 148 (operator norm estimation) depends directly from:
Definition 2 (supremum),
Lemma 91 (linear map preserves zero),
Definition 102 (norm),
Lemma 106 (norm preserves zero),
Lemma 107 (norm is nonnegative),
Definition 135 (operator norm),
Theorem 147 (normed space of continuous linear maps).

Lemma 149 (continuous linear maps have closed kernel) depends directly from:
Lemma 24 (singleton is closed),
Definition 98 (kernel).

Lemma 150 (compatibility of composition with continuity) depends directly from:
Lemma 96 (composition of linear maps is bilinear),
Lemma 116 (equivalent definition of unit sphere),
Lemma 144 (finite operator norm is continuous),
Lemma 148 (operator norm estimation).

Lemma 151 (complete normed space of continuous linear maps) depends directly from:
Definition 26 (convergent sequence),
Lemma 33 (stationary sequence is convergent),
Definition 34 (Cauchy sequence),
Definition 37 (complete subset),
Definition 38 (complete metric space),
Lemma 44 (compatibility of limit with continuous functions),
Definition 58 (vector space),
Definition 62 (linear map),
Definition 88 (inherited vector operations),
Lemma 91 (linear map preserves zero),
Lemma 92 (linear map preserves linear combinations),
Definition 102 (norm),
Definition 109 (distance associated with norm),
Lemma 111 (norm gives distance),
Lemma 121 (norm is continuous),
Lemma 128 (vector addition is continuous),
Lemma 129 (scalar multiplication is continuous),
Definition 135 (operator norm),
Definition 139 (bounded linear map),
Theorem 142 (continuous linear map),
Lemma 144 (finite operator norm is continuous),
Theorem 147 (normed space of continuous linear maps).

Definition 152 (topological dual) has no direct dependency.

Definition 153 (dual norm) has no direct dependency.

Lemma 154 (topological dual is complete normed space) depends directly from:
Lemma 105 (K is normed space),
Theorem 147 (normed space of continuous linear maps),
Lemma 151 (complete normed space of continuous linear maps),
Definition 153 (dual norm).

Definition 155 (bra-ket notation) has no direct dependency.
Lemma 156 (bra-ket is bilinear map) depends directly from:

Definition 62 (linear map),
Definition 65 (bilinear map),
Lemma 87 (product is space),
Definition 88 (inherited vector operations),
Lemma 101 (K is space),
Lemma 154 (topological dual is complete normed space),
Definition 155 (bra-ket notation).

Definition 157 (bounded bilinear form) has no direct dependency.

Lemma 158 (representation for bounded bilinear form) depends directly from:

Definition 68 (vector space),
Definition 64 (linear form),
Definition 65 (bilinear map),
Definition 66 (bilinear form),
Definition 67 (set of bilinear forms),
Definition 70 (vector subtraction),
Definition 88 (inherited vector operations),
Lemma 92 (linear map preserves linear combinations),
Lemma 101 (K is space),
Definition 104 (normed vector space),
Lemma 107 (norm is nonnegative),
Lemma 116 (equivalent definition of unit sphere),
Definition 130 (bounded linear map),
Lemma 144 (finite operator norm is continuous),
Theorem 147 (normed space of continuous linear maps),
Definition 152 (topological dual),
Definition 153 (dual norm),
Lemma 154 (topological dual is complete normed space),
Definition 155 (bra-ket notation),
Lemma 156 (bra-ket is bilinear map),
Definition 157 (bounded bilinear form).

Definition 159 (coercive bilinear form) has no direct dependency.

Lemma 160 (coercivity constant is less than continuity constant) depends directly from:

Definition 102 (norm),
Lemma 107 (norm is nonnegative),
Definition 157 (bounded bilinear form),
Definition 159 (coercive bilinear form).

Definition 161 (inner product) has no direct dependency.

Definition 165 (inner product space) has no direct dependency.

Lemma 166 (inner product subspace) depends directly from:

Definition 74 (subspace),
Definition 161 (inner product),
Definition 165 (inner product space).

Lemma 167 (inner product with zero is zero) depends directly from:

Definition 58 (vector space),
Definition 65 (bilinear map),
Definition 70 (vector subtraction),
Definition 161 (inner product).
Lemma 168 (square expansion plus) depends directly from:
- Definition 65 (bilinear map).
- Definition 161 (inner product).

Lemma 169 (square expansion minus) depends directly from:
- Definition 65 (bilinear map).
- Definition 70 (vector subtraction).
- Definition 161 (inner product).
- Lemma 168 (square expansion plus).

Lemma 170 (parallelogram identity) depends directly from:
- Lemma 168 (square expansion plus).
- Lemma 169 (square expansion minus).

Lemma 171 (Cauchy–Schwarz inequality) depends directly from:
- Definition 65 (bilinear map).
- Definition 161 (inner product).
- Lemma 168 (square expansion plus).

Definition 172 (square root of inner square) has no direct dependency.

Lemma 174 (squared norm) depends directly from:
- Definition 161 (inner product).
- Definition 172 (square root of inner square).

Lemma 175 (Cauchy–Schwarz inequality with norms) depends directly from:
- Definition 161 (inner product).
- Lemma 171 (Cauchy–Schwarz inequality).
- Definition 172 (square root of inner square).

Lemma 176 (triangle inequality) depends directly from:
- Definition 161 (inner product).
- Lemma 168 (square expansion plus).
- Lemma 174 (squared norm).
- Lemma 175 (Cauchy–Schwarz inequality with norms).

Lemma 177 (inner product gives norm) depends directly from:
- Definition 102 (norm).
- Definition 104 (normed vector space).
- Definition 161 (inner product).
- Definition 172 (square root of inner square).
- Lemma 176 (triangle inequality).

Definition 179 (convex subset) has no direct dependency.

Theorem 180 (orthogonal projection onto nonempty complete convex) depends directly from:
- Definition 9 (infimum).
- Lemma 13 (finite infimum discrete).
- Definition 14 (minimum).
- Definition 34 (Cauchy sequence).
- Definition 37 (complete subset).
- Lemma 44 (compatibility of limit with continuous functions).
- Definition 58 (vector space).
- Definition 70 (vector subtraction).
- Definition 71 (scalar division).
- Definition 102 (norm).
Lemma 107 (norm is nonnegative), Lemma 121 (norm is continuous), Definition 165 (inner product space), Lemma 170 (parallelogram identity), Lemma 174 (squared norm), Definition 179 (convex subset).

Lemma 181 (characterization of orthogonal projection onto convex) depends directly from:
- Definition 9 (infimum),
- Definition 14 (minimum),
- Lemma 15 (finite minimum),
- Definition 58 (vector space),
- Definition 65 (bilinear map),
- Definition 70 (vector subtraction),
- Lemma 107 (norm is nonnegative),
- Definition 163 (inner product),
- Definition 165 (inner product space),
- Lemma 168 (square expansion plus),
- Lemma 174 (squared norm),
- Definition 179 (convex subset).

Lemma 182 (subspace is convex) depends directly from:
- Lemma 79 (closed under linear combination is subspace),
- Definition 179 (convex subset).

Theorem 183 (orthogonal projection onto complete subspace) depends directly from:
- Definition 58 (vector space),
- Definition 74 (subspace),
- Theorem 180 (orthogonal projection onto nonempty complete convex),
- Lemma 182 (subspace is convex).

Definition 184 (orthogonal projection onto complete subspace) depends directly from:
- Theorem 183 (orthogonal projection onto complete subspace).

Lemma 185 (characterization of orthogonal projection onto subspace) depends directly from:
- Definition 58 (vector space),
- Definition 65 (bilinear map),
- Definition 70 (vector subtraction),
- Definition 74 (subspace),
- Definition 161 (inner product),
- Lemma 181 (characterization of orthogonal projection onto convex),
- Lemma 182 (subspace is convex).

Lemma 186 (orthogonal projection is continuous linear map) depends directly from:
- Definition 46 (Lipschitz continuity),
- Definition 58 (vector space),
- Definition 65 (bilinear map),
- Definition 74 (subspace),
- Lemma 92 (linear map preserves linear combinations),
- Definition 102 (norm),
- Lemma 107 (norm is nonnegative),
- Definition 161 (inner product),
- Lemma 174 (squared norm),
- Lemma 175 (Cauchy–Schwarz inequality with norms),
- Theorem 183 (orthogonal projection onto complete subspace).
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

Definition 184 (orthogonal projection onto complete subspace).
Lemma 185 (characterization of orthogonal projection onto subspace).

Definition 187 (orthogonal complement) has no direct dependency.

Lemma 188 (trivial orthogonal complements) depends directly from:
Lemma 77 (trivial subspaces),
Definition 161 (inner product),
Lemma 167 (inner product with zero is zero),
Definition 187 (orthogonal complement).

Lemma 189 (orthogonal complement is subspace) depends directly from:
Definition 65 (bilinear map),
Lemma 79 (closed under linear combination is subspace),
Definition 161 (inner product),
Lemma 167 (inner product with zero is zero),
Definition 187 (orthogonal complement).

Lemma 190 (zero intersection with orthogonal complement) depends directly from:
Definition 161 (inner product),
Definition 187 (orthogonal complement).

Theorem 191 (direct sum with orthogonal complement when complete) depends directly from:
Definition 58 (vector space),
Definition 65 (bilinear map),
Definition 81 (sum of subspaces),
Definition 83 (direct sum of subspaces),
Lemma 84 (equivalent definitions of direct sum),
Definition 161 (inner product),
Lemma 167 (inner product with zero is zero),
Theorem 183 (orthogonal projection onto complete subspace),
Definition 184 (orthogonal projection onto complete subspace),
Lemma 185 (characterization of orthogonal projection onto subspace),
Definition 187 (orthogonal complement),
Lemma 190 (zero intersection with orthogonal complement).

Lemma 192 (sum is orthogonal sum) depends directly from:
Definition 58 (vector space),
Lemma 79 (closed under linear combination is subspace),
Definition 80 (linear span),
Definition 81 (sum of subspaces),
Theorem 183 (orthogonal projection onto complete subspace),
Theorem 191 (direct sum with orthogonal complement when complete).

Lemma 193 (sum of complete subspace and linear span is closed) depends directly from:
Lemma 31 (closed is limit of sequences),
Lemma 44 (compatibility of limit with continuous functions),
Lemma 79 (closed under linear combination is subspace),
Definition 80 (linear span),
Definition 81 (sum of subspaces),
Lemma 85 (direct sum with linear span),
Lemma 112 (linear span is closed),
Lemma 146 (identity map is continuous),
Theorem 147 (normed space of continuous linear maps),
Theorem 183 (orthogonal projection onto complete subspace).
Lemma 186 (Orthogonal projection is continuous linear map).
Lemma 190 (Zero intersection with orthogonal complement).
Theorem 191 (Direct sum with orthogonal complement when complete).

Definition 194 (Hilbert space) depends directly from:
Lemma 111 (Norm gives distance).
Definition 172 (Square root of inner square).
Lemma 177 (Inner product gives norm).

Lemma 195 (Closed Hilbert subspace) depends directly from:
Lemma 39 (Closed subset of complete is complete).
Definition 74 (Subspace).
Lemma 166 (Inner product subspace).
Definition 194 (Hilbert space).

Theorem 196 (Riesz–Fréchet) depends directly from:
Definition 2 (Supremum).
Definition 58 (Vector space).
Definition 62 (Linear map).
Definition 64 (Linear form).
Definition 65 (Bilinear map).
Definition 70 (Vector subtraction).
Definition 71 (Scalar division).
Lemma 73 (Zero-product property).
Lemma 78 (Closed under vector operations is subspace).
Lemma 91 (Linear map preserves zero).
Lemma 92 (Linear map preserves linear combinations).
Definition 97 (Isomorphism).
Definition 98 (Kernel).
Lemma 99 (Kernel is subspace).
Lemma 100 (Injective linear map has zero kernel).
Definition 102 (Norm).
Definition 104 (Normed vector space).
Lemma 106 (Norm preserves zero).
Lemma 107 (Norm is nonnegative).
Lemma 108 (Normalization by nonzero).
Definition 122 (Linear isometry).
Definition 135 (Operator norm).
Definition 139 (Bounded linear map).
Theorem 142 (Continuous linear map).
Lemma 144 (Finite operator norm is continuous).
Lemma 145 (Linear isometry is continuous).
Lemma 146 (Continuous linear maps have closed kernel).
Definition 152 (Topological dual).
Definition 153 (Dual norm).
Lemma 154 (Topological dual is complete normed space).
Definition 155 (Bra-ket notation).
Lemma 156 (Bra-ket is bilinear map).
Definition 163 (Inner product).
Definition 165 (Inner product space).
Lemma 167 (Inner product with zero is zero).
Lemma 174 (Squared norm).
Lemma 175 (Cauchy–Schwarz inequality with norms).
Theorem 183 (Orthogonal projection onto complete subspace).
Definition 184 (Orthogonal projection onto complete subspace).
A detailed proof of the Lax-Milgram Theorem to be formalized in Coq

Definition 187 (orthogonal complement),
Lemma 188 (trivial orthogonal complements),
Lemma 189 (orthogonal complement is subspace),
Theorem 191 (direct sum with orthogonal complement when complete),
Definition 194 (Hilbert space),
Lemma 195 (closed Hilbert subspace).

Lemma 197 (compatible $\rho$ for Lax–Milgram) has no direct dependency.

Theorem 198 (Lax–Milgram) depends directly from:
Definition 46 (Lipschitz continuity),
Definition 48 (contraction),
Theorem 56 (fixed point),
Definition 58 (vector space),
Definition 65 (bilinear map),
Lemma 69 (minus times yields opposite vector),
Definition 70 (vector subtraction),
Lemma 73 (zero-product property),
Definition 102 (norm),
Definition 104 (normed vector space),
Lemma 106 (norm preserves zero),
Lemma 107 (norm is nonnegative),
Definition 122 (linear isometry),
Lemma 146 (identity map is continuous),
Theorem 147 (normed space of continuous linear maps),
Lemma 148 (operator norm estimation),
Lemma 150 (compatibility of composition with continuity),
Lemma 154 (topological dual is complete normed space),
Definition 157 (bounded bilinear form),
Lemma 158 (representation for bounded bilinear form),
Definition 159 (coercive bilinear form),
Lemma 160 (coercivity constant is less than continuity constant),
Definition 161 (inner product),
Lemma 168 (square expansion plus),
Lemma 177 (inner product gives norm),
Definition 187 (orthogonal complement),
Lemma 188 (trivial orthogonal complements),
Definition 194 (Hilbert space),
Theorem 196 (Riesz–Fréchet),
Lemma 197 (compatible $\rho$ for Lax–Milgram).

Lemma 199 (Galerkin orthogonality) depends directly from:
Definition 65 (bilinear map),
Definition 74 (subspace).

Theorem 200 (Lax–Milgram, closed subspace) depends directly from:
Lemma 195 (closed Hilbert subspace),
Theorem 198 (Lax–Milgram).

Lemma 201 (Céa) depends directly from:
Definition 58 (vector space),
Definition 65 (bilinear map),
Definition 70 (vector subtraction),
Definition 102 (norm),
Lemma 106 (norm preserves zero),
Lemma 107 (norm is nonnegative),
Definition 157 (bounded bilinear form),
Definition 159 (coercive bilinear form),
Definition 161 (inner product),
Lemma 199 (Galerkin orthogonality),
Theorem 200 (Lax–Milgram, closed subspace).

Lemma 202 (finite dimensional subspace in Hilbert space is closed) depends directly from:
Lemma 39 (closed subset of complete is complete),
Definition 82 (finite dimensional subspace),
Lemma 112 (linear span is closed),
Lemma 192 (sum is orthogonal sum),
Lemma 193 (sum of complete subspace and linear span is closed),
Definition 194 (Hilbert space).

Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace) depends directly from:
Theorem 198 (Lax–Milgram),
Theorem 200 (Lax–Milgram, closed subspace),
Lemma 201 (Céa),
Lemma 202 (finite dimensional subspace in Hilbert space is closed).
C Is a direct dependency of . . .

Definition 2 (supremum) is a direct dependency of:
- Lemma 5 (supremum is positive scalar multiplicative).
- Lemma 8 (finite maximum).
- Lemma 10 (duality infimum-supremum).
- Lemma 138 (operator norm is nonnegative).
- Theorem 142 (continuous linear map).
- Lemma 144 (finite operator norm is continuous).
- Theorem 147 (normed space of continuous linear maps).
- Lemma 148 (operator norm estimation).
- Theorem 196 (Riesz–Fréchet).

Lemma 3 (finite supremum) is a direct dependency of:
- Lemma 8 (finite maximum).
- Lemma 11 (finite infimum).
- Theorem 142 (continuous linear map).

Lemma 4 (discrete lower accumulation) is a direct dependency of:
- Lemma 12 (discrete upper accumulation).

Lemma 5 (supremum is positive scalar multiplicative) is a direct dependency of:
- Theorem 147 (normed space of continuous linear maps).

Definition 7 (maximum) is a direct dependency of:
- Lemma 8 (finite maximum).

Lemma 8 (finite maximum) is a direct dependency of:
- Lemma 15 (finite minimum).

Definition 9 (infimum) is a direct dependency of:
- Lemma 10 (duality infimum-supremum).
- Theorem 180 (orthogonal projection onto nonempty complete convex).
- Lemma 181 (characterization of orthogonal projection onto convex).

Lemma 10 (duality infimum-supremum) is a direct dependency of:
- Lemma 11 (finite infimum).
- Lemma 15 (finite minimum).

Lemma 11 (finite infimum) is a direct dependency of:
- Lemma 13 (finite infimum discrete).

Lemma 12 (discrete upper accumulation) is a direct dependency of:
- Lemma 13 (finite infimum discrete).

Lemma 13 (finite infimum discrete) is a direct dependency of:
- Theorem 180 (orthogonal projection onto nonempty complete convex).

Definition 14 (minimum) is a direct dependency of:
- Lemma 15 (finite minimum).
- Theorem 180 (orthogonal projection onto nonempty complete convex).
- Lemma 181 (characterization of orthogonal projection onto convex).

Lemma 15 (finite minimum) is a direct dependency of:
- Lemma 181 (characterization of orthogonal projection onto convex).
Definition 16 (distance) is a direct dependency of:
Lemma 18 (iterated triangle inequality),
Lemma 24 (singleton is closed),
Lemma 27 (variant of point separation),
Lemma 28 (limit is unique),
Lemma 29 (closure is limit of sequences),
Lemma 33 (stationary sequence is convergent),
Lemma 35 (equivalent definition of Cauchy sequence),
Lemma 36 (convergent sequence is Cauchy),
Theorem 47 (equivalent definition of Lipschitz continuity),
Lemma 50 (zero-Lipschitz continuous is constant),
Theorem 56 (fixed point),
Lemma 111 (norm gives distance),
Lemma 129 (scalar multiplication is continuous).

Definition 17 (metric space) is a direct dependency of:
Lemma 111 (norm gives distance).

Lemma 18 (iterated triangle inequality) is a direct dependency of:
Theorem 56 (fixed point).

Definition 19 (closed ball) is a direct dependency of:
Lemma 29 (closure is limit of sequences),
Lemma 114 (equivalent definition of closed unit ball).

Definition 20 (sphere) is a direct dependency of:
Lemma 116 (equivalent definition of unit sphere).

Definition 21 (open subset) is a direct dependency of:
Lemma 23 (equivalent definition of closed subset).

Definition 22 (closed subset) is a direct dependency of:
Lemma 23 (equivalent definition of closed subset),
Lemma 30 (closed equals closure).

Lemma 23 (equivalent definition of closed subset) is a direct dependency of:
Lemma 24 (singleton is closed),
Lemma 30 (closed equals closure).

Lemma 24 (singleton is closed) is a direct dependency of:
Lemma 112 (linear span is closed),
Lemma 149 (continuous linear maps have closed kernel).

Definition 25 (closure) is a direct dependency of:
Lemma 29 (closure is limit of sequences),
Lemma 30 (closed equals closure),
Lemma 31 (closed is limit of sequences).

Definition 26 (convergent sequence) is a direct dependency of:
Lemma 28 (limit is unique),
Lemma 29 (closure is limit of sequences),
Lemma 33 (stationary sequence is convergent),
Lemma 36 (convergent sequence is Cauchy),
Lemma 44 (compatibility of limit with continuous functions),
Lemma 55 (convergent iterated function sequence),
Lemma 112 (linear span is closed),
Lemma 151 (complete normed space of continuous linear maps).
Lemma 27 (variant of point separation) is a direct dependency of:

Lemma 28 (limit is unique).

Lemma 28 (limit is unique) is a direct dependency of:

Lemma 36 (convergent sequence is Cauchy),
Lemma 55 (convergent iterated function sequence),
Lemma 112 (linear span is closed).

Lemma 29 (closure is limit of sequences) is a direct dependency of:

Lemma 31 (closed is limit of sequences),
Lemma 39 (closed subset of complete is complete).

Lemma 30 (closed equals closure) is a direct dependency of:

Lemma 31 (closed is limit of sequences),
Lemma 39 (closed subset of complete is complete).

Lemma 31 (closed is limit of sequences) is a direct dependency of:

Lemma 112 (linear span is closed),
Lemma 195 (sum of complete subspace and linear span is closed).

Definition 32 (stationary sequence) is a direct dependency of:

Lemma 33 (stationary sequence is convergent),
Lemma 53 (stationary iterated function sequence),
Lemma 55 (convergent iterated function sequence),
Theorem 56 (fixed point).

Lemma 33 (stationary sequence is convergent) is a direct dependency of:

Lemma 55 (convergent iterated function sequence),
Theorem 56 (fixed point),
Lemma 151 (complete normed space of continuous linear maps).

Definition 34 (Cauchy sequence) is a direct dependency of:

Lemma 35 (equivalent definition of Cauchy sequence),
Lemma 36 (convergent sequence is Cauchy),
Lemma 112 (linear span is closed),
Lemma 151 (complete normed space of continuous linear maps),
Theorem 180 (orthogonal projection onto nonempty complete convex).

Lemma 35 (equivalent definition of Cauchy sequence) is a direct dependency of:

Theorem 56 (fixed point).

Lemma 36 (convergent sequence is Cauchy) is a direct dependency of:

Lemma 112 (linear span is closed).

Definition 37 (complete subset) is a direct dependency of:

Lemma 39 (closed subset of complete is complete),
Theorem 56 (fixed point),
Lemma 112 (linear span is closed),
Lemma 151 (complete normed space of continuous linear maps),
Theorem 180 (orthogonal projection onto nonempty complete convex).

Definition 38 (complete metric space) is a direct dependency of:

Lemma 39 (closed subset of complete is complete),
Theorem 56 (fixed point),
Lemma 151 (complete normed space of continuous linear maps).

Lemma 39 (closed subset of complete is complete) is a direct dependency of:

Lemma 195 (closed Hilbert subspace),
Lemma 202 (finite dimensional subspace in Hilbert space is closed).
Definition 42 (continuity in a point) is a direct dependency of:

- Lemma 44 (compatibility of limit with continuous functions)
- Lemma 49 (uniform continuous is continuous)
- Lemma 128 (vector addition is continuous)
- Lemma 129 (scalar multiplication is continuous)
- Theorem 142 (continuous linear map)

Definition 43 (pointwise continuity) is a direct dependency of:

- Lemma 49 (uniform continuous is continuous)
- Lemma 128 (vector addition is continuous)
- Lemma 129 (scalar multiplication is continuous)
- Theorem 142 (continuous linear map)

Lemma 44 (compatibility of limit with continuous functions) is a direct dependency of:

- Lemma 151 (complete normed space of continuous linear maps)
- Theorem 180 (orthogonal projection onto nonempty complete convex)
- Lemma 193 (sum of complete subspace and linear span is closed)

Definition 45 (uniform continuity) is a direct dependency of:

- Lemma 49 (uniform continuous is continuous)
- Lemma 51 (Lipschitz continuous is uniform continuous)

Definition 46 (Lipschitz continuity) is a direct dependency of:

- Theorem 47 (equivalent definition of Lipschitz continuity)
- Lemma 50 (zero-Lipschitz continuous is constant)
- Lemma 51 (Lipschitz continuous is uniform continuous)
- Lemma 54 (iterate Lipschitz continuous mapping)
- Lemma 55 (convergent iterated function sequence)
- Theorem 56 (fixed point)
- Lemma 119 (norm is one-Lipschitz continuous)
- Theorem 142 (continuous linear map)
- Lemma 186 (orthogonal projection is continuous linear map)
- Theorem 198 (Lax–Milgram)

Theorem 47 (equivalent definition of Lipschitz continuity) is a direct dependency of:

- Theorem 56 (fixed point)
- Theorem 198 (Lax–Milgram)

Lemma 49 (uniform continuous is continuous) is a direct dependency of:

- Lemma 121 (norm is continuous)
- Theorem 142 (continuous linear map)

Lemma 50 (zero-Lipschitz continuous is constant) is a direct dependency of:

- Lemma 51 (Lipschitz continuous is uniform continuous)
- Lemma 55 (convergent iterated function sequence)
- Theorem 56 (fixed point)

Lemma 51 (Lipschitz continuous is uniform continuous) is a direct dependency of:

- Lemma 120 (norm is uniformly continuous)
- Theorem 142 (continuous linear map)

Definition 52 (iterated function sequence) is a direct dependency of:

- Lemma 53 (stationary iterated function sequence)
- Lemma 54 (iterate Lipschitz continuous mapping)
- Lemma 55 (convergent iterated function sequence)
Lemma 53 (stationary iterated function sequence) is a direct dependency of:
Theorem 56 (fixed point).

Lemma 54 (iterate Lipschitz continuous mapping) is a direct dependency of:
Theorem 56 (fixed point).

Lemma 55 (convergent iterated function sequence) is a direct dependency of:
Theorem 56 (fixed point).

Theorem 56 (fixed point) is a direct dependency of:
Theorem 198 (Lax–Milgram).

Definition 58 (vector space) is a direct dependency of:
Lemma 68 (zero times yields zero),
Lemma 69 (minus times yields opposite vector),
Lemma 72 (times zero yields zero),
Lemma 73 (zero-product property),
Lemma 78 (closed under vector operations is subspace),
Lemma 79 (closed under linear combination is subspace),
Lemma 84 (equivalent definitions of direct sum),
Lemma 87 (product is space),
Lemma 90 (space of mappings to a space),
Lemma 92 (linear map preserves linear combinations),
Lemma 93 (space of linear maps),
Lemma 99 (kernel is subspace),
Lemma 100 (injective linear map has zero kernel),
Lemma 106 (norm preserves zero),
Lemma 107 (norm is nonnegative),
Lemma 112 (linear span is closed),
Lemma 117 (zero on unit sphere is zero),
Lemma 128 (vector addition is continuous),
Lemma 129 (scalar multiplication is continuous),
Lemma 151 (complete normed space of continuous linear maps),
Lemma 158 (representation for bounded bilinear form),
Lemma 167 (inner product with zero is zero),
Theorem 180 (orthogonal projection onto nonempty complete convex),
Lemma 181 (characterization of orthogonal projection onto convex),
Theorem 183 (orthogonal projection onto complete subspace),
Lemma 185 (characterization of orthogonal projection onto subspace),
Lemma 186 (orthogonal projection is continuous linear map),
Theorem 191 (direct sum with orthogonal complement when complete),
Theorem 192 (sum is orthogonal sum),
Theorem 196 (Riesz–Fréchet),
Theorem 198 (Lax–Milgram),
Lemma 201 (Céa).

Definition 61 (set of mappings to space) is a direct dependency of:
Lemma 90 (space of mappings to a space).

Definition 62 (linear map) is a direct dependency of:
Lemma 91 (linear map preserves zero),
Lemma 92 (linear map preserves linear combinations),
Lemma 93 (space of linear maps),
Lemma 95 (identity map is linear map),
Lemma 96 (composition of linear maps is bilinear),
Lemma 100 (injective linear map has zero kernel),
Lemma 117 (zero on unit sphere is zero),
Lemma 133 (norm of image of unit vector),
Theorem 142 (continuous linear map),
Lemma 151 (complete normed space of continuous linear maps),
Lemma 156 (bra-ket is bilinear map),
Theorem 196 (Riesz–Fréchet).

Definition 63 (set of linear maps) is a direct dependency of:
Lemma 93 (space of linear maps).

Definition 64 (linear form) is a direct dependency of:
Lemma 158 (representation for bounded bilinear form).
Theorem 196 (Riesz–Fréchet).

Definition 65 (bilinear map) is a direct dependency of:
Lemma 96 (composition of linear maps is bilinear),
Lemma 156 (bra-ket is bilinear map),
Lemma 158 (representation for bounded bilinear form),
Lemma 167 (inner product with zero is zero),
Lemma 168 (square expansion plus),
Lemma 169 (square expansion minus),
Lemma 171 (Cauchy–Schwarz inequality),
Lemma 181 (characterization of orthogonal projection onto convex),
Lemma 185 (characterization of orthogonal projection onto subspace),
Lemma 186 (orthogonal projection is continuous linear map),
Lemma 189 (orthogonal complement is subspace),
Theorem 191 (direct sum with orthogonal complement when complete),
Theorem 196 (Riesz–Fréchet),
Theorem 198 (Lax–Milgram),
Lemma 199 (Galerkin orthogonality),
Lemma 201 (Céa).

Definition 66 (bilinear form) is a direct dependency of:
Lemma 158 (representation for bounded bilinear form).

Definition 67 (set of bilinear forms) is a direct dependency of:
Lemma 158 (representation for bounded bilinear form).

Lemma 68 (zero times yields zero) is a direct dependency of:
Lemma 69 (minus times yields opposite vector),
Lemma 73 (zero-product property),
Lemma 91 (linear map preserves zero),
Lemma 92 (linear map preserves linear combinations),
Lemma 106 (norm preserves zero),
Lemma 129 (scalar multiplication is continuous).

Lemma 69 (minus times yields opposite vector) is a direct dependency of:
Lemma 78 (closed under vector operations is subspace),
Lemma 84 (equivalent definitions of direct sum),
Theorem 198 (Lax–Milgram).

Definition 70 (vector subtraction) is a direct dependency of:
Lemma 72 (times zero yields zero),
Lemma 84 (equivalent definitions of direct sum),
Lemma 100 (injective linear map has zero kernel),
Lemma 111 (norm gives distance),
Lemma 112 (linear span is closed).
Theorem 142  (continuous linear map).
Lemma 158  (representation for bounded bilinear form).
Lemma 167  (inner product with zero is zero).
Lemma 169  (square expansion minus).
Theorem 180  (orthogonal projection onto nonempty complete convex).
Lemma 181  (characterization of orthogonal projection onto convex).
Lemma 185  (characterization of orthogonal projection onto subspace).
Theorem 196  (Riesz–Fréchet).
Theorem 198  (Lax–Milgram).
Lemma 201  (Céa).

Definition 71  (scalar division) is a direct dependency of:
Lemma 108  (normalization by nonzero).
Theorem 180  (orthogonal projection onto nonempty complete convex).
Theorem 196  (Riesz–Fréchet).

Lemma 72  (times zero yields zero) is a direct dependency of:
Lemma 73  (zero-product property).
Lemma 93  (space of linear maps).
Lemma 99  (kernel is subspace).
Lemma 117  (zero on unit sphere is zero).

Lemma 73  (zero-product property) is a direct dependency of:
Lemma 79  (closed under linear combination is subspace).
Lemma 85  (direct sum with linear span).
Theorem 196  (Riesz–Fréchet).
Theorem 198  (Lax–Milgram).

Definition 74  (subspace) is a direct dependency of:
Lemma 77  (trivial subspaces).
Lemma 78  (closed under vector operations is subspace).
Lemma 166  (inner product subspace).
Theorem 183  (orthogonal projection onto complete subspace).
Lemma 185  (characterization of orthogonal projection onto subspace).
Lemma 186  (orthogonal projection is continuous linear map).
Lemma 195  (closed Hilbert subspace).
Lemma 199  (Galerkin orthogonality).

Lemma 77  (trivial subspaces) is a direct dependency of:
Lemma 188  (trivial orthogonal complements).

Lemma 78  (closed under vector operations is subspace) is a direct dependency of:
Lemma 79  (closed under linear combination is subspace).
Lemma 84  (equivalent definitions of direct sum).
Lemma 85  (direct sum with linear span).
Theorem 147  (normed space of continuous linear maps).
Theorem 196  (Riesz–Fréchet).

Lemma 79  (closed under linear combination is subspace) is a direct dependency of:
Lemma 93  (space of linear maps).
Lemma 99  (kernel is subspace).
Lemma 182  (subspace is convex).
Lemma 189  (orthogonal complement is subspace).
Lemma 192  (sum is orthogonal sum).
Lemma 193  (sum of complete subspace and linear span is closed).
Definition 80 \textit{(linear span)} is a direct dependency of:

Lemma 85 \textit{(direct sum with linear span)},
Lemma 112 \textit{(linear span is closed)},
Lemma 192 \textit{(sum is orthogonal sum)},
Lemma 193 \textit{(sum of complete subspace and linear span is closed)}.

Definition 81 \textit{(sum of subspaces)} is a direct dependency of:

Theorem 191 \textit{(direct sum with orthogonal complement when complete)},
Lemma 192 \textit{(sum is orthogonal sum)},
Lemma 193 \textit{(sum of complete subspace and linear span is closed)}.

Definition 82 \textit{(finite dimensional subspace)} is a direct dependency of:

Lemma 202 \textit{(finite dimensional subspace in Hilbert space is closed)}.

Definition 83 \textit{(direct sum of subspaces)} is a direct dependency of:

Lemma 84 \textit{(equivalent definitions of direct sum)},
Theorem 191 \textit{(direct sum with orthogonal complement when complete)}.

Lemma 84 \textit{(equivalent definitions of direct sum)} is a direct dependency of:

Lemma 85 \textit{(direct sum with linear span)},
Theorem 191 \textit{(direct sum with orthogonal complement when complete)}.

Lemma 85 \textit{(direct sum with linear span)} is a direct dependency of:

Lemma 193 \textit{(sum of complete subspace and linear span is closed)}.

Definition 86 \textit{(product vector operations)} is a direct dependency of:

Lemma 87 \textit{(product is space)},
Lemma 127 \textit{(product is normed space)},
Lemma 128 \textit{(vector addition is continuous)}.

Lemma 87 \textit{(product is space)} is a direct dependency of:

Lemma 96 \textit{(composition of linear maps is bilinear)},
Lemma 127 \textit{(product is normed space)},
Lemma 156 \textit{(bra-ket is bilinear map)}.

Definition 88 \textit{(inherited vector operations)} is a direct dependency of:

Lemma 90 \textit{(space of mappings to a space)},
Lemma 93 \textit{(space of linear maps)},
Lemma 96 \textit{(composition of linear maps is bilinear)},
Theorem 147 \textit{(normed space of continuous linear maps)},
Lemma 151 \textit{(complete normed space of continuous linear maps)},
Lemma 156 \textit{(bra-ket is bilinear map)},
Lemma 158 \textit{representation for bounded bilinear form}.

Lemma 90 \textit{(space of mappings to a space)} is a direct dependency of:

Lemma 93 \textit{(space of linear maps)}.

Lemma 91 \textit{(linear map preserves zero)} is a direct dependency of:

Lemma 99 \textit{(kernel is subspace)},
Lemma 100 \textit{(injective linear map has zero kernel)},
Lemma 117 \textit{(zero on unit sphere is zero)},
Theorem 142 \textit{(continuous linear map)},
Lemma 148 \textit{(operator norm estimation)},
Lemma 151 \textit{(complete normed space of continuous linear maps)},
Theorem 196 \textit{Riesz–Fréchet}
Lemma 92 (linear map preserves linear combinations) is a direct dependency of:

- Lemma 96 (composition of linear maps is bilinear),
- Lemma 99 (kernel is subspace),
- Lemma 131 (complete normed space of continuous linear maps),
- Lemma 158 (representation for bounded bilinear form),
- Lemma 186 (orthogonal projection is continuous linear map),
- Theorem 196 (Riesz–Fréchet).

Lemma 93 (space of linear maps) is a direct dependency of:

- Lemma 96 (composition of linear maps is bilinear).

Definition 94 (identity map) is a direct dependency of:

- Lemma 95 (identity map is linear map),
- Lemma 123 (identity map is linear isometry).

Lemma 95 (identity map is linear map) is a direct dependency of:

- Lemma 123 (identity map is linear isometry).

Lemma 96 (composition of linear maps is bilinear) is a direct dependency of:

- Lemma 150 (compatibility of composition with continuity).

Definition 97 (isomorphism) is a direct dependency of:

- Theorem 196 (Riesz–Fréchet).

Definition 98 (kernel) is a direct dependency of:

- Lemma 99 (kernel is subspace),
- Lemma 100 (injective linear map has zero kernel),
- Lemma 149 (continuous linear maps have closed kernel),
- Theorem 196 (Riesz–Fréchet).

Lemma 99 (kernel is subspace) is a direct dependency of:

- Theorem 196 (Riesz–Fréchet).

Lemma 100 (injective linear map has zero kernel) is a direct dependency of:

- Theorem 196 (Riesz–Fréchet).

Lemma 101 (K is space) is a direct dependency of:

- Lemma 156 (bra-ket is bilinear map),
- Lemma 158 (representation for bounded bilinear form).

Definition 102 (norm) is a direct dependency of:

-Lemma 105 (K is normed space),
- Lemma 106 (norm preserves zero),
- Lemma 107 (norm is nonnegative),
- Lemma 108 (normalization by nonzero),
- Lemma 111 (norm gives distance),
- Lemma 112 (linear span is closed),
- Lemma 118 (reverse triangle inequality),
- Lemma 127 (product is normed space),
- Lemma 128 (vector addition is continuous),
- Lemma 129 (scalar multiplication is continuous),
- Lemma 133 (norm of image of unit vector),
- Lemma 134 (norm of image of unit sphere),
- Theorem 142 (continuous linear map),
- Theorem 147 (normed space of continuous linear maps),
- Lemma 148 (operator norm estimation),
- Lemma 151 (complete normed space of continuous linear maps).
Lemma 160 (coercivity constant is less than continuity constant),
Lemma 177 (inner product gives norm),
Theorem 180 (orthogonal projection onto nonempty complete convex set),
Lemma 186 (orthogonal projection is continuous linear map),
Theorem 196 (Riesz–Fréchet),
Theorem 198 (Lax–Milgram),
Lemma 201 (Céa).

Definition 104 (normed vector space) is a direct dependency of:
Lemma 105 (K is normed space),
Lemma 106 (norm preserves zero),
Lemma 107 (norm is nonnegative),
Lemma 118 (reverse triangle inequality),
Lemma 127 (product is normed space),
Theorem 147 (normed space of continuous linear maps),
Lemma 158 (representation for bounded bilinear form),
Lemma 177 (inner product gives norm),
Theorem 196 (Riesz–Fréchet),
Theorem 198 (Lax–Milgram).

Lemma 105 (K is normed space) is a direct dependency of:
Lemma 154 (topological dual is complete normed space).

Lemma 106 (norm preserves zero) is a direct dependency of:
Lemma 134 (norm of image of unit sphere),
Theorem 142 (continuous linear map),
Lemma 148 (operator norm estimation),
Theorem 196 (Riesz–Fréchet),
Theorem 198 (Lax–Milgram),
Lemma 201 (Céa).

Lemma 107 (norm is nonnegative) is a direct dependency of:
Lemma 108 (normalization by nonzero),
Lemma 111 (norm gives distance),
Lemma 112 (linear span is closed),
Lemma 127 (product is normed space),
Lemma 133 (norm of image of unit vector),
Lemma 138 (operator norm is nonnegative),
Theorem 147 (normed space of continuous linear maps),
Lemma 148 (operator norm estimation),
Lemma 158 (representation for bounded bilinear form),
Lemma 160 (coercivity constant is less than continuity constant),
Theorem 180 (orthogonal projection onto nonempty complete convex set),
Lemma 183 (characterization of orthogonal projection onto convex set),
Lemma 186 (orthogonal projection is continuous linear map),
Theorem 196 (Riesz–Fréchet),
Theorem 198 (Lax–Milgram),
Lemma 201 (Céa).

Lemma 108 (normalization by nonzero) is a direct dependency of:
Lemma 117 (zero on unit sphere is zero),
Lemma 133 (norm of image of unit vector),
Theorem 196 (Riesz–Fréchet).

Definition 109 (distance associated with norm) is a direct dependency of:
Lemma 111 (norm gives distance).
Lemma 112 (linear span is closed).
Lemma 114 (equivalent definition of closed unit ball).
Lemma 116 (equivalent definition of unit sphere).
Lemma 119 (norm is one-Lipschitz continuous).
Lemma 128 (vector addition is continuous).
Lemma 129 (scalar multiplication is continuous).
Lemma 151 (complete normed space of continuous linear maps).

Lemma 111 (norm gives distance) is a direct dependency of:
Lemma 114 (equivalent definition of closed unit ball).
Lemma 116 (equivalent definition of unit sphere).
Definition 194 (Hilbert space).

Lemma 112 (linear span is closed) is a direct dependency of:
Lemma 193 (sum of complete subspace and linear span is closed).
Lemma 202 (finite dimensional subspace in Hilbert space is closed).

Definition 113 (closed unit ball) is a direct dependency of:
Lemma 114 (equivalent definition of closed unit ball).

Lemma 114 (equivalent definition of closed unit ball) is a direct dependency of:
Theorem 142 (continuous linear map).

Definition 115 (unit sphere) is a direct dependency of:
Lemma 116 (equivalent definition of unit sphere).

Lemma 116 (equivalent definition of unit sphere) is a direct dependency of:
Lemma 117 (zero on unit sphere is zero).
Lemma 134 (norm of image of unit sphere).
Theorem 142 (continuous linear map).
Lemma 150 (compatibility of composition with continuity).
Lemma 158 (representation for bounded bilinear form).

Lemma 117 (zero on unit sphere is zero) is a direct dependency of:
Theorem 147 (normed space of continuous linear maps).

Lemma 118 (reverse triangle inequality) is a direct dependency of:
Lemma 119 (norm is one-Lipschitz continuous).

Lemma 119 (norm is one-Lipschitz continuous) is a direct dependency of:
Lemma 120 (norm is uniformly continuous).

Lemma 120 (norm is uniformly continuous) is a direct dependency of:
Lemma 121 (norm is continuous).

Lemma 121 (norm is continuous) is a direct dependency of:
Lemma 151 (complete normed space of continuous linear maps).
Theorem 180 (orthogonal projection onto nonempty complete convex).

Definition 122 (linear isometry) is a direct dependency of:
Lemma 123 (identity map is linear isometry).
Lemma 145 (linear isometry is continuous).
Theorem 196 (Riesz–Fréchet).
Theorem 198 (Lax–Milgram).

Lemma 123 (identity map is linear isometry) is a direct dependency of:
Lemma 146 (identity map is continuous).
Definition 124 (product norm) is a direct dependency of:

Lemma 127 (product is normed space).
Lemma 128 (vector addition is continuous).

Lemma 127 (product is normed space) is a direct dependency of:

Lemma 128 (vector addition is continuous).

Lemma 128 (vector addition is continuous) is a direct dependency of:

Lemma 151 (complete normed space of continuous linear maps).

Lemma 129 (scalar multiplication is continuous) is a direct dependency of:

Lemma 151 (complete normed space of continuous linear maps).

Lemma 133 (norm of image of unit vector) is a direct dependency of:

Lemma 134 (norm of image of unit sphere).

Lemma 134 (norm of image of unit sphere) is a direct dependency of:

Lemma 137 (equivalent definition of operator norm).

Definition 135 (operator norm) is a direct dependency of:

Lemma 137 (equivalent definition of operator norm).
Theorem 142 (continuous linear map).
Lemma 148 (operator norm estimation).
Lemma 151 (complete normed space of continuous linear maps).
Theorem 196 (Riesz–Fréchet).

Lemma 137 (equivalent definition of operator norm) is a direct dependency of:

Lemma 138 (operator norm is nonnegative).
Theorem 142 (continuous linear map).
Theorem 147 (normed space of continuous linear maps).

Lemma 138 (operator norm is nonnegative) is a direct dependency of:

Theorem 142 (continuous linear map).

Definition 139 (bounded linear map) is a direct dependency of:

Theorem 142 (continuous linear map).
Lemma 144 (finite operator norm is continuous).
Lemma 151 (complete normed space of continuous linear maps).
Lemma 158 (representation for bounded bilinear form).
Theorem 196 (Riesz–Fréchet).

Definition 140 (linear map bounded on unit ball) is a direct dependency of:

Theorem 142 (continuous linear map).
Lemma 144 (finite operator norm is continuous).

Definition 141 (linear map bounded on unit sphere) is a direct dependency of:

Theorem 142 (continuous linear map).
Lemma 144 (finite operator norm is continuous).
Theorem 147 (normed space of continuous linear maps).

Theorem 142 (continuous linear map) is a direct dependency of:

Lemma 144 (finite operator norm is continuous).
Lemma 151 (complete normed space of continuous linear maps).
Theorem 196 (Riesz–Fréchet).

Definition 143 (set of continuous linear maps) is a direct dependency of:

Lemma 144 (finite operator norm is continuous).
Theorem 147 (normed space of continuous linear maps).
Lemma 144 (finite operator norm is continuous) is a direct dependency of:
  Lemma 145 (linear isometry is continuous),
  Theorem 147 (normed space of continuous linear maps),
  Lemma 150 (compatibility of composition with continuity),
  Lemma 151 (complete normed space of continuous linear maps),
  Lemma 158 (representation for bounded bilinear form),
  Theorem 196 (Riesz–Fréchet).

Lemma 145 (linear isometry is continuous) is a direct dependency of:
  Lemma 146 (identity map is continuous),
  Theorem 196 (Riesz–Fréchet).

Lemma 146 (identity map is continuous) is a direct dependency of:
  Lemma 193 (sum of complete subspace and linear span is closed),
  Theorem 198 (Lax–Milgram).

Theorem 147 (normed space of continuous linear maps) is a direct dependency of:
  Lemma 148 (operator norm estimation),
  Lemma 151 (complete normed space of continuous linear maps),
  Lemma 154 (topological dual is complete normed space),
  Lemma 158 (representation for bounded bilinear form),
  Lemma 193 (sum of complete subspace and linear span is closed),
  Theorem 198 (Lax–Milgram).

Lemma 148 (operator norm estimation) is a direct dependency of:
  Lemma 150 (compatibility of composition with continuity),
  Theorem 198 (Lax–Milgram).

Lemma 149 (continuous linear maps have closed kernel) is a direct dependency of:
  Theorem 196 (Riesz–Fréchet).

Lemma 150 (compatibility of composition with continuity) is a direct dependency of:
  Theorem 198 (Lax–Milgram).

Lemma 151 (complete normed space of continuous linear maps) is a direct dependency of:
  Lemma 154 (topological dual is complete normed space).

Definition 152 (topological dual) is a direct dependency of:
  Lemma 158 (representation for bounded bilinear form),
  Theorem 196 (Riesz–Fréchet).

Definition 153 (dual norm) is a direct dependency of:
  Lemma 154 (topological dual is complete normed space),
  Lemma 158 (representation for bounded bilinear form),
  Theorem 196 (Riesz–Fréchet).

Lemma 154 (topological dual is complete normed space) is a direct dependency of:
  Lemma 156 (bra-ket is bilinear map),
  Lemma 158 (representation for bounded bilinear form),
  Theorem 196 (Riesz–Fréchet),
  Theorem 198 (Lax–Milgram).

Definition 155 (bra-ket notation) is a direct dependency of:
  Lemma 156 (bra-ket is bilinear map),
  Lemma 158 (representation for bounded bilinear form),
  Theorem 196 (Riesz–Fréchet).
Lemma 156 (bra-ket is bilinear map) is a direct dependency of:

Lemma 158 (representation for bounded bilinear form),

Theorem 196 (Riesz–Fréchet).

Definition 157 (bounded bilinear form) is a direct dependency of:

Lemma 158 (representation for bounded bilinear form),

Lemma 160 (coercivity constant is less than continuity constant),

Theorem 198 (Lax–Milgram),

Lemma 201 (Cea).

Lemma 158 (representation for bounded bilinear form) is a direct dependency of:

Theorem 198 (Lax–Milgram).

Definition 159 (coercive bilinear form) is a direct dependency of:

Lemma 160 (coercivity constant is less than continuity constant),

Theorem 198 (Lax–Milgram),

Lemma 201 (Cea).

Lemma 160 (coercivity constant is less than continuity constant) is a direct dependency of:

Theorem 198 (Lax–Milgram).

Definition 161 (inner product) is a direct dependency of:

Lemma 166 (inner product subspace),

Lemma 167 (inner product with zero is zero),

Lemma 168 (square expansion plus),

Lemma 169 (square expansion minus),

Lemma 171 (Cauchy–Schwarz inequality),

Lemma 174 (squared norm),

Lemma 175 (Cauchy–Schwarz inequality with norms),

Lemma 176 (triangle inequality),

Lemma 177 (inner product gives norm),

Lemma 181 (characterization of orthogonal projection onto convex),

Lemma 185 (characterization of orthogonal projection onto subspace),

Lemma 186 (orthogonal projection is continuous linear map),

Lemma 188 (trivial orthogonal complements),

Lemma 189 (orthogonal complement is subspace),

Lemma 190 (zero intersection with orthogonal complement),

Theorem 191 (direct sum with orthogonal complement when complete),

Theorem 196 (Riesz–Fréchet),

Theorem 198 (Lax–Milgram),

Lemma 201 (Cea).

Definition 165 (inner product space) is a direct dependency of:

Lemma 166 (inner product subspace),

Theorem 180 (orthogonal projection onto nonempty complete convex),

Lemma 181 (characterization of orthogonal projection onto convex),

Theorem 196 (Riesz–Fréchet).

Lemma 166 (inner product subspace) is a direct dependency of:

Lemma 195 (closed Hilbert subspace).

Lemma 167 (inner product with zero is zero) is a direct dependency of:

Lemma 188 (trivial orthogonal complements),

Lemma 189 (orthogonal complement is subspace),

Theorem 191 (direct sum with orthogonal complement when complete),

Theorem 196 (Riesz–Fréchet).
Lemma 168 (square expansion plus) is a direct dependency of:
Lemma 169 (square expansion minus),
Lemma 170 (parallelogram identity),
Lemma 171 (Cauchy–Schwarz inequality),
Lemma 176 (triangle inequality),
Lemma 181 (characterization of orthogonal projection onto convex),
Theorem 198 (Lax–Milgram).

Lemma 169 (square expansion minus) is a direct dependency of:
Lemma 170 (parallelogram identity),

Lemma 170 (parallelogram identity) is a direct dependency of:
Theorem 180 (orthogonal projection onto nonempty complete convex).

Lemma 171 (Cauchy–Schwarz inequality) is a direct dependency of:
Lemma 175 (Cauchy–Schwarz inequality with norms).

Definition 172 (square root of inner square) is a direct dependency of:
Lemma 174 (squared norm),
Lemma 175 (Cauchy–Schwarz inequality with norms),
Lemma 177 (inner product gives norm),
Definition 194 (Hilbert space).

Lemma 174 (squared norm) is a direct dependency of:
Lemma 176 (triangle inequality),
Theorem 180 (orthogonal projection onto nonempty complete convex),
Lemma 181 (characterization of orthogonal projection onto convex),
Lemma 186 (orthogonal projection is continuous linear map),
Theorem 196 (Riesz–Fréchet).

Lemma 175 (Cauchy–Schwarz inequality with norms) is a direct dependency of:
Lemma 176 (triangle inequality),
Lemma 186 (orthogonal projection is continuous linear map),
Theorem 196 (Riesz–Fréchet).

Lemma 176 (triangle inequality) is a direct dependency of:
Lemma 177 (inner product gives norm).

Lemma 177 (inner product gives norm) is a direct dependency of:
Definition 194 (Hilbert space),
Theorem 198 (Lax–Milgram).

Definition 179 (convex subset) is a direct dependency of:
Theorem 180 (orthogonal projection onto nonempty complete convex),
Lemma 181 (characterization of orthogonal projection onto convex),
Lemma 182 (subspace is convex).

Theorem 180 (orthogonal projection onto nonempty complete convex) is a direct dependency of:
Theorem 183 (orthogonal projection onto complete subspace).

Lemma 181 (characterization of orthogonal projection onto convex) is a direct dependency of:
Lemma 185 (characterization of orthogonal projection onto subspace).

Lemma 182 (subspace is convex) is a direct dependency of:
Theorem 183 (orthogonal projection onto complete subspace),
Lemma 185 (characterization of orthogonal projection onto subspace).
Theorem 183 (orthogonal projection onto complete subspace) is a direct dependency of:
- Definition 184 (orthogonal projection onto complete subspace),
- Lemma 186 (orthogonal projection is continuous linear map),
- Theorem 191 (direct sum with orthogonal complement when complete),
- Lemma 192 (sum is orthogonal sum),
- Lemma 193 (sum of complete subspace and linear span is closed),
- Theorem 196 (Riesz–Fréchet).

Definition 184 (orthogonal projection onto complete subspace) is a direct dependency of:
- Lemma 186 (orthogonal projection is continuous linear map),
- Theorem 191 (direct sum with orthogonal complement when complete),
- Theorem 196 (Riesz–Fréchet).

Lemma 185 (characterization of orthogonal projection onto subspace) is a direct dependency of:
- Lemma 186 (orthogonal projection is continuous linear map),
- Theorem 191 (direct sum with orthogonal complement when complete).

Lemma 186 (orthogonal projection is continuous linear map) is a direct dependency of:
- Lemma 193 (sum of complete subspace and linear span is closed).

Definition 187 (orthogonal complement) is a direct dependency of:
- Lemma 188 (trivial orthogonal complements),
- Lemma 189 (orthogonal complement is subspace),
- Lemma 190 (zero intersection with orthogonal complement),
- Theorem 191 (direct sum with orthogonal complement when complete),
- Theorem 196 (Riesz–Fréchet),
- Theorem 198 (Lax–Milgram).

Lemma 188 (trivial orthogonal complements) is a direct dependency of:
- Theorem 196 (Riesz–Fréchet),
- Theorem 198 (Lax–Milgram).

Lemma 189 (orthogonal complement is subspace) is a direct dependency of:
- Theorem 196 (Riesz–Fréchet).

Lemma 190 (zero intersection with orthogonal complement) is a direct dependency of:
- Theorem 191 (direct sum with orthogonal complement when complete),
- Lemma 193 (sum of complete subspace and linear span is closed).

Theorem 191 (direct sum with orthogonal complement when complete) is a direct dependency of:
- Lemma 192 (sum is orthogonal sum),
- Lemma 193 (sum of complete subspace and linear span is closed),
- Theorem 196 (Riesz–Fréchet).

Lemma 192 (sum is orthogonal sum) is a direct dependency of:
- Lemma 202 (finite dimensional subspace in Hilbert space is closed).

Lemma 193 (sum of complete subspace and linear span is closed) is a direct dependency of:
- Lemma 202 (finite dimensional subspace in Hilbert space is closed).

Definition 194 (Hilbert space) is a direct dependency of:
- Lemma 195 (closed Hilbert subspace),
- Theorem 196 (Riesz–Fréchet),
- Theorem 198 (Lax–Milgram),
- Lemma 202 (finite dimensional subspace in Hilbert space is closed).
Lemma 195 (closed Hilbert subspace) is a direct dependency of:
  - Theorem 196 (Riesz–Fréchet),
  - Theorem 200 (Lax–Milgram, closed subspace).

Theorem 196 (Riesz–Fréchet) is a direct dependency of:
  - Theorem 198 (Lax–Milgram).

Lemma 197 (compatible $\rho$ for Lax–Milgram) is a direct dependency of:
  - Theorem 198 (Lax–Milgram).

Theorem 198 (Lax–Milgram) is a direct dependency of:
  - Theorem 200 (Lax–Milgram, closed subspace),
  - Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace).

Lemma 199 (Galerkin orthogonality) is a direct dependency of:
  - Lemma 201 (Céa).

Theorem 200 (Lax–Milgram, closed subspace) is a direct dependency of:
  - Lemma 201 (Céa),
  - Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace).

Lemma 201 (Céa) is a direct dependency of:
  - Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace).

Lemma 202 (finite dimensional subspace in Hilbert space is closed) is a direct dependency of:
  - Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace).

Theorem 203 (Lax–Milgram–Céa, finite dimensional subspace)
