Bound entanglement in the Jaynes–Cummings model

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Abstract

We study in detail entanglement properties of the Jaynes–Cummings model assuming a two-level atom (qubit) interacting with the first $N$ levels of an electromagnetic field mode (qudit) in a cavity. In the Jaynes–Cummings model, the number operator is the conserved quantity that allows for the exact diagonalization of the Hamiltonian and thus we study states that commute with this conserved quantity and whose structure is preserved under the Jaynes–Cummings dynamics. Contrary to the common belief, we show that there are bound entangled states that satisfy the symmetries imposed by the conservation of the number of excitations when $N > 3$. Furthermore we show that the Jaynes–Cummings interaction can be used to generate bound entanglement between the atom and the mode.

1. Introduction

The Jaynes–Cummings (JC) model is one of the workhorses of quantum optics research [1, 2]. In this model the interaction of a two-level atom with a single electromagnetic field (EMF) mode is studied under the dipole and rotating wave approximations. Its rich dynamics has been used to model and understand phenomenology both in the realm of cavity [3–5] and circuit [6] QED and in trapped ions [7, 8], where the EMF is replaced by the quantized vibrations of the ions. There is also a significant amount of literature devoted to entanglement in the JC model. While some of the previous results concerned pure states [9–11], mixed state entanglement has been addressed using either entropic relations [12–16] or detection techniques drawn directly from the field of quantum information. In particular, projecting the Fock space of the EMF mode onto a two-dimensional subspace results in a combined Hilbert space that essentially reduces to a two-qubit system whose entanglement properties are well understood [17–22]. Also, the positive partial transpose (PPT) criterion introduced by Peres [23] has been used to this aim [24–28].

We note, however, that neither the use of entropic inequalities (which are a corollary of the majorization criterion [29]) nor the projection of the system onto $2 \times 2$ subspaces leads to entanglement criteria that are stronger than the PPT one [30, 31]. Thus, so far only ‘distillable’ (or free) entanglement has been detected in these systems [32].

In this paper, we first formalize the problem that most previous studies have analysed in terms of the conservation of the number of excitations that makes the JC Hamiltonian an exactly solvable problem. We then show that this symmetry gives rise to a super-selection rule [33] that severely constrains the structure of the density matrices describing the states. With these tools at hand we review entanglement detection techniques previously used and show that they are, in general, weaker than the Peres criterion. Further, we show numerically that other entanglement criteria that are not based on the use of positive maps but not completely positive ones are not able to supplement the Peres criterion. This is the case of the computable cross-norm or realignment (CCNR) criterion [34, 35] and some covariance matrix (CM) corollaries [36, 37]. We construct explicitly states that are PPT but entangled demonstrating that bound entangled states exist in the JC model that are not detected by any of the previous methods and that some of these states can be generated by the JC interaction. We emphasize that to the best of our knowledge, this is the first study that addresses the possibility of generating bound entanglement using the JC dynamics which is a commonly occurring interaction in different physical systems. We note, however, that several
theoretical [38–40] and experimental [41–44] investigations have been performed seeking bound entanglement in other physical systems.

This paper is organized as follows: in section 2, we introduce and define the class of states to be studied here and argue that those are the natural mixed states resulting from the dynamics of the JC model. Further, we introduce the notation that will be used in the rest of the paper in terms of qubit–qudit (2 × N) density operators. In section 3, we derive under which conditions these states are PPT, i.e. have a positive partial transpose (PT). In section 4, we focus at N = 4 and show that despite the high symmetry of the states considered, bound entangled states exist. Following some ideas presented in [45, 46], we demonstrated the existence of bound entanglement by using the range criterion [47]. Therefore, under the JC dynamics, the PPT criterion is necessary but not sufficient to ensure entanglement. In section 5, we show how using the JC interaction it is possible to generate bound entangled states starting from uncorrelated ones. Final remarks are given in section 6. For completeness, in appendix A we present two well known entanglement criteria that have been used to detect bound entanglement and numerically show that they are ineffective for the states considered here and in appendix B we construct the convex hull for the PPT separable states when N = 2, 3.

2. Motivation and definitions

Assume a cavity with a finite quality factor where a N − 1 photon state, |N − 1⟩, has been prepared. After such initial preparation an atom in its ground state |0⟩ enters the cavity and starts to interact with the photons. Typically, three processes will occur in the dynamics of the system:

• The atom and the cavity will reversibly exchange one excitation which under the assumptions of the JC model will cause the transition |0⟩ ⊗ |N − 1⟩ ↔ |1⟩ ⊗ |N − 2⟩ with |1⟩ being the excited state of the atom.

• The cavity might irreversibly lose one of its photons since it has a finite quality factor.

• Once the atom is excited it can spontaneously and irreversibly emit a photon that escapes the cavity.

As time goes by, the dynamics of the system is in general quite complicated and to treat it completely it is necessary to include the modes outside the cavity that gives rise to an irreversible behaviour. Nevertheless, under very general assumptions, such as the interaction with the outside modes preserves the number of photons (i.e. that each photon that disappears from the cavity becomes a photon in one of the outside modes) the state of the system commutes with the number of excitations of the atom plus cavity. Including the first N Fock states in the dynamics one can write the number of excitations operator as

\[ \Pi = \Pi_2 \otimes I + I \otimes \Pi_N, \]

\[ \Pi_2 = \sum_{i=0}^{1} |i⟩⟨i| = |1⟩⟨1|, \]

\[ \Pi_N = \sum_{n=0}^{N-1} n |n⟩⟨n|. \]  

At any time the state of the atom and photons ρ satisfies

\[ [ρ, \Pi] = 0. \]  

The above symmetry gives rise directly to the following super-selection rule [33]:

\[ \langle in|ρ|jm⟩ \propto δ_{i+n,j+m}, \]  

where as usual |in⟩ ≡ |i⟩ ⊗ |n⟩ and |jm⟩ are two atomic states and |m⟩, |n⟩ are Fock states with m and n photons respectively. It can be easily shown that if at t = 0 the state ρ(t = 0) satisfies (2) and if it evolves under the von Neumann equation with the JC Hamiltonian,

\[ \frac{d}{dt}ρ = i[ρ, H_{JC}], \]  

then at later times it also satisfies (2). In (4) the JC Hamiltonian is given by

\[ H_{JC} = ω_0 I \otimes a^d a + (ω_0 − Δ)σ^z \otimes I \]

\[ −ig(σ^y \otimes a^d − σ^y \otimes a), \]  

where a, (a†) is an annihilation (creation) bosonic operator satisfying [a, a†] = I, σ = |0⟩⟨1|, g is the light–matter coupling constant (the Rabi frequency), ω0 is the frequency of the mode and Δ is the detuning between the mode and the transition frequency between the two atomic states. We note that the JC Hamiltonian also commutes with the number operator when all the states in the Fock ladder are included (i.e. N → ∞ in (1)) and thus it is reasonable to think of the states ρ in (2) as the natural states of the JC dynamics. Even more interesting is the fact that when some phenomenological dissipation terms are added to the rhs of (4) the dynamics still preserves the structure of the density matrix describing the state. This is the case if Lindblad terms of the form:

\[ \mathcal{L}_{σO}ρ = γ_σ \left( OρO^+ − \frac{ρO^+O + O^+Oρ}{2} \right), \]  

with O = I ⊗ a for photon depopulation, O = I ⊗ a† for photon re-population, O = σ ⊗ I for atom depopulation, O = σ† ⊗ I for atom re-population, O = σ^z ⊗ I for atomic dephasing are added. Even if a microscopic treatment is used and a rigorous derivation of the system operator dynamics is performed by first coupling it to a bosonic reservoir and then tracing it out, if the dynamics of the system plus reservoir as a whole preserves the overall number of excitations the shape of the states satisfying (2) will be preserved [48].

For future convenience we name the non-zero elements of the density matrix that satisfy the super-selection rule (3) as follows. The two sets of N non-zero elements along the diagonal (populations) we label

\[ a_n ≡ ⟨0n|ρ|0n⟩ \quad b_n ≡ ⟨1n|ρ|1n⟩, \]  

and the two sets of N − 1 non-zero off-diagonal elements or coherences we parametrize as

\[ c_n ≡ ⟨0n|ρ|1n−1⟩ \quad c_n^* ≡ ⟨1n−1|ρ|0n⟩. \]  

In the above equation it is assumed that ρ is a state and thus it is automatically Hermitian. The state is explicitly given by

\[ ρ = \sum_{n=0}^{N−1} (a_n|0n⟩⟨0n| + b_n|1n⟩⟨1n|) + \sum_{n=1}^{N−1} c_n^*|1n−1⟩⟨0n| + c_n|0n⟩⟨1n−1|. \]
It can be represented as a $2 \times 2$ matrix in the atom–field basis whose entries are $N \times N$ sparse matrices:

$$\rho = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \quad (10)$$

more explicitly in the basis given by the tensor product of $|i\rangle_{1=0}^1$ and $|n\rangle_{n=0}^N$, the matrix has the following structure:

$$\rho = \begin{pmatrix} a_0 & c_1 \\ c_1^* & b_0 \end{pmatrix}. \quad (11)$$

In spite of its simplicity, the separability properties of such a state are intricate. Note that although the above matrix was constructed to be explicitly hermitian, to represent a physical state the density operator must also be positive semi-definite, which is equivalent to the requirement that

$$a_n, b_n \geq 0, \quad |c_n|^2 \leq a_nb_{n-1}. \quad (12)$$

The positivity conditions (12) guarantee that all the eigenvalues of $\rho$ will be non-negative. Moreover, unless at least one of the inequalities (12) is saturated $\rho$ is of full rank. Note that by applying a local unitary transformation to the density matrix it is possible to make all the coefficients in it non-negative. Such local unitary is given by

$$V = \sum_{n=0}^{N-1} e^{i\theta_n} |n\rangle \langle n|, \quad \theta_n - \theta_{n-1} = \arg(c_n), \quad (13)$$

and since the entanglement properties of the state are invariant under local unitaries it suffices to consider only the absolute values of the $c_n$. In other words, from now on we deal with density operators whose matrix elements are non-negative.

### 3. Entanglement detection

To determine if the state (11) is entangled or not i.e. whether it can be written as

$$\rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| \otimes |\xi_i\rangle \langle \xi_i|, \quad (14)$$

is not a trivial question since the state acts in a Hilbert space of dimension $2N$ where only sufficient but not necessary separability criteria are known. Furthermore, one would like to quantify the extent to which the state cannot be written as a separable mixture. One often used metric is the concurrence $C(\rho)$ [49]. For a pure state $\psi = |\psi\rangle \langle \psi|$ in a $2 \times N$ system the concurrence is simply

$$C(\psi) = \sqrt{2(1 - \text{tr}(\mu^2))} \quad (15)$$

$$\mu = \text{tr}_{A/B}(\psi), \quad (16)$$

where $\text{tr}_{A/B}$ denotes the partial trace over subsystem $A$ or $B$. For mixed states the concurrence is extended via the convex roof construction:

$$C(\rho) = \min_{\sum p_i \psi_i = \rho} \sum p_i C(\psi_i), \quad (17)$$

with $p_i$ being probabilities and the $\psi_i$ being pure states. This quantity ranges from zero (for separable states) to one for maximally entangled ones.

A possible approach to answer whether a state is separable or not is to obtain the PT with respect to one of its subsystems (does not matter which one) and see if it has negative eigenvalues [23]: if there is at least one negative eigenvalue then $\rho$ must be entangled. The PT of $\rho$ with respect to subsystem $B$ reads

$$\rho^B = \begin{pmatrix} A & C^T \\ C & B \end{pmatrix}. \quad (18)$$

We point out that once the local unitary (13) has been applied to $\rho$ then $\rho^{\Gamma^1} = \rho^{\Gamma^2} = \rho^{\Gamma}$, where $\rho^{\Gamma}$ is the PT with respect to the $p$th party. For $\rho^{\Gamma}$ to be positive semi-definite one new constraint is needed:

$$|c_n|^2 \leq a_{n-1}b_n. \quad (19)$$

The violation of the constraints above gives a sufficient condition for $\rho$ to be entangled. One reaches the same condition by finding an upper bound to the concurrence of the state projecting the qudit into the subspace spanned by two consecutive Fock states $|n\rangle, |n+1\rangle$ [17]. Another way to arrive to the constraints given by (19) is by using the bound on the concurrence for a $2 \times N$ dimensional system derived in [50] in which a set of $N(N-1)/2$ quantities are proposed to bound the concurrence of the system. Of these only $N - 1$ turn out to be not trivial and they express the same bounds contained in (19). Using the results from [50] the following bound for the concurrence is found:

$$C(\rho) \geq G_{\rho} = \sqrt{\sum_{i=1}^{N-1} G_i(\rho)^2} \quad (20)$$

$$G_i(\rho) = 2 \max(0, |c_i| - \sqrt{a_{i-1}b_i}). \quad (21)$$

It was pointed out in [31] that the above bound cannot have more entanglement detection capabilities than the positive PT criterion, it is interesting to notice that for these states they have exactly the same entanglement detection capabilities. The negativity [51] can also be used to quantify the failure of $\rho^B$ to be positive semi-definite:

$$N(\rho) = \sum_{n=1}^{N} \min \left\{ 0, \lambda_{n-1}^\Gamma \right\}, \quad (22)$$

$$\lambda_{n-1}^\Gamma = a_{n-1} + b_n - \sqrt{(a_{n-1} - b_n)^2 + 4|c_n|^2}. \quad (23)$$

The normalization is such that it takes the value 1 when $\rho$ is a maximally entangled pure state $\rho = |\psi\rangle \langle \psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|0_n\rangle + |1_{n-1}\rangle)$. It was pointed out in [52] that the negativity with the normalization used above is another lower bound for the concurrence:

$$C(\rho) \geq N(\rho). \quad (24)$$

When the negativity (or for these states the bound (20)) is zero and $N > 3$ the criterion is inconclusive [53], i.e. it is known that in the general case of $2 \times N$ there are entangled states
that are not detected by the negativity criterion [47]. These PPT states were detected using the so-called range criterion. It states that if \( \rho \) is separable then a separable vector exists \( |\psi f\rangle \) such that \( |\psi f\rangle \in R(\rho) \) and \( |\psi f\rangle \in R(\rho^{T_1}) \), with \( |\psi f\rangle \) indicating complex conjugation in the first subsystem and \( R \) is used to indicate the range of an operator. Another way of expressing this criterion is to say that if there is no separable vector \( |\psi f\rangle \) such that

\[
\langle k_i|e f\rangle = 0 \forall |k_i\rangle \in K(\rho) \quad \text{and} \quad \langle k_i|e^* f\rangle = 0 \forall |k_i\rangle \in K(\rho^{T_1}),
\]

(25)

with \( K \) representing the kernel of an operator, then \( \rho \) must be entangled. Bound entangled states satisfying (25) are called edge states [54].

4. Bound entanglement in the JC model

In this section, we show that there are bound entangled states that satisfy (2) and thus from now on, we only consider states that are PPT. We will assume that the local unitary operation in equation (13) has been applied and thus \( \epsilon_n = \zeta_n^\star \geq 0 \). To reduce the complexity of the problem and facilitate the proof it is convenient to apply the following local filtering operation to the state:

\[
F = I \otimes F_{\theta}
\]

\[
F_{\theta} = \sum_{n=0}^{N-1} \frac{1}{\sqrt{b_n}} |n\rangle \langle n|.
\]

(26)

Notice that if there is some \( b_n = 0 \) then from (12) and (19) \( \epsilon_{m+1} = \epsilon_m = 0 \). In such a case the density matrix can be split as follows:

\[
\rho = \rho_{[0,m-1]} + \rho_{[m]} + \rho_{[m+1:N-1]}
\]

(27)

\[
\rho_{[0,m-1]} = \sum_{n=0}^{m-1} (a_n |0n\rangle \langle 0n| + b_n |1n\rangle \langle 1n|)
\]

\[
+ \sum_{n=1}^{m-1} c_n (|1n-1\rangle \langle 0n| + |0n\rangle \langle 1n-1|)
\]

(28)

\[
\rho_{[m]} = a_m |0m\rangle \langle 0m|
\]

(29)

\[
\rho_{[m+1:N-1]} = \sum_{n=m+1}^{N-1} (a_n |0n\rangle \langle 0n| + b_n |1n\rangle \langle 1n|)
\]

\[
+ \sum_{n=m+2}^{N-1} c_n (|1n-1\rangle \langle 0n| + |0n\rangle \langle 1n-1|),
\]

(30)

where \( \rho_{[0,m-1]} \) is a state supported in \( 2 \times m - 1 \), \( \rho_{[m+1,N-1]} \) is a state supported in \( 2 \times (N - m) \) and \( \rho_{[m]} \) is a separable state. Because of this the problem is reduced to two lower rank density matrices with the same structure as the original one plus a trivially separable state. Now, without loss of generality we assume that all the populations of the density matrix are non-zero and after applying the above filtering operation to \( \rho \) one obtains

\[
\rho \rightarrow \sigma = F \rho F^\dagger = \sum_{n=0}^{N-1} (\frac{1}{\sqrt{b_n}} |0n\rangle \langle 0n| + |1n\rangle \langle 1n|)
\]

\[
+ \sum_{n=1}^{N-1} y_n (|1n-1\rangle \langle 0n| + |0n\rangle \langle 1n-1|)
\]

\[
\quad + \sum_{n=1}^{N-1} y_n (|0n-1\rangle \langle 1n| + |1n\rangle \langle 0n-1|)
\]

(31)

\[
x_n^2 = a_n/b_n
\]

(32)

\[
y_n = c_n/\sqrt{b_n-1}b_n.
\]

(33)

Local filtering operations map entangled states to entangled states and separable states to separable states and thus \( \sigma \) is still positive and is PPT. The positivity and PPT conditions equivalent to equations (12) and (19) are now

\[
y_i \leq \min(x_i, x_{i+1})
\]

(34)

or equivalently

\[
x_i \geq \max(y_i, y_{i+1}).
\]

(35)

Further, notice that the above matrix can be split as a trivially separable part \( \sigma \) plus a PPT state \( \tau \) that satisfies \( r(\tau)+r(\tau^{T_1}) \leq 3N \) with \( r(\omega) \) being the dimensionality of the range of \( \omega \):

\[
\sigma = \sigma_\tau + \tau
\]

(36)

\[
\sigma_\tau = \sum_{n=0}^{N-1} (x_n^2 - \max(y_n^2, y_{n+1}^2)) |0n\rangle \langle 0n|
\]

(37)

\[
\tau = \sum_{n=0}^{N-1} (\max(y_n^2, y_{n+1}^2)) |0n\rangle \langle 0n| + |1n\rangle \langle 1n|
\]

\[
+ \sum_{n=1}^{N-1} y_n (|1n-1\rangle \langle 0n| + |0n\rangle \langle 1n-1|).
\]

(38)

From the above it is clearly seen that the separability properties of \( \sigma \) are the same as those of \( \tau \).

Having transformed the original states (9) to the form (36) in what follows we will show that there are bound entangled states that satisfy (2). Since we are only interested in showing the existence of bound entangled states of the type (9) we will consider the smallest dimension in which this is possible, i.e., \( N = 4 \). The study of the entanglement properties of these states for arbitrary \( N \) will be presented elsewhere [55], nevertheless we mention that based on the same ideas that will be presented in the following paragraphs it is possible to show that there are bound entangled states for arbitrary \( N \). For the sake of concreteness we will only examine the cases in which

\[
y_i \geq y_{i+1};
\]

(39)

4 We note that there is a slight abuse of notation in writing (35) since there are only \( N-1 \) variables \( y_i \) with the index \( 1 \leq i \leq N-1 \) (see (33) in which the \( y_i \) are defined in terms of the \( c_j \)). Because of this when \( i = 0 \) or \( i = N-1 \) in (35) the equation becomes meaningless since \( y_0 \) or \( y_N \) are not defined (they are not part of the problem) and indeed going back to equation (34), \( y_0 \) and \( y_{N-1} \) only need to satisfy \( y_0 \geq y_1 \) and \( y_{N-1} \geq y_N \). To solve this inconvenience one could simply define \( y_0 = y_1 \) and \( y_{N-1} = y_N \) or introduce two fictitious or auxiliary (in the sense that they are not part of the problem) quantities \( y_0 \) and \( y_N \) with the values \( y_0 = y_1 \) and \( y_N = y_{N-1} \).
Thus,

\[
\tau = \begin{pmatrix}
y_1^2 & y_1^2 & y_1 \\
y_2^2 & y_2^2 & y_2 \\
y_3^2 & y_3^2 & y_3 \\
y_1 & 1 & y_1 \\
y_2 & 1 & y_2 \\
y_3 & 1 & y_3 \\
\end{pmatrix}.
\]  \hspace{1cm} (40)

The dimensions of the ranges of \(\tau\) and \(\tau^\top\), are given by \(r(\tau) = 5\) and \(r(\tau^\top) = 7\) if none of the inequalities (39) are saturated. It is known [45] that the matrix is separable if \(r(\tau) = 4\) or \(r(\tau^\top) = 4\); therefore, one should prove if there are enough product vectors in the range so that the rank of the matrices diminishes appropriately. The three vectors in the kernel of \(\tau\), are given by

\[
|\chi_1\rangle = - |01\rangle + y_1 |10\rangle,
|\chi_2\rangle = - |02\rangle + y_2 |11\rangle,
|\chi_3\rangle = - |03\rangle + y_3 |12\rangle,
\]  \hspace{1cm} (41)

whereas the vector in the kernel of \(\tau^\top\) is simply

\[
|\phi_1\rangle = - |00\rangle + y_1 |11\rangle. \hspace{1cm} (42)
\]

Now we need to find a separable vector \(|ef\rangle\) such that

\[
\langle\phi_1|ef\rangle = 0 \quad \text{and} \quad \langle\chi_n|ef\rangle = 0 \quad \forall n.
\]  \hspace{1cm} (43)

It is found that the unique (up to a phase and normalization) separable vector that is orthogonal to (41) and that upon complex conjugation in the qubit system is orthogonal to (42) is

\[
|ef\rangle_{\theta} = \left( 0 + \frac{1}{y_1} e^{i\theta} |1\rangle \right) \otimes \left( y_1 |0\rangle + y_1 e^{i\theta} |1\rangle + y_2 e^{2i\theta} |2\rangle + \frac{y_3 y_2}{y_1} e^{3i\theta} |3\rangle \right). \hspace{1cm} (44)
\]

Now, to show that there are bound entangled states of the type defined by (9) we will look at a subset of the states defined by (40). Up to now we assumed that none of the inequalities (39) was saturated. To construct our bound entangled state we will look at the case when one of them is saturated, to be precise, we fix \(y_2 = y_3\) and thus our state is simply

\[
\tau(y_1, y_2) = \begin{pmatrix}
y_1^2 & y_1^2 & y_1 \\
y_2^2 & y_2^2 & y_2 \\
y_3^2 & y_3^2 & y_3 \\
y_1 & 1 & y_1 \\
y_2 & 1 & y_2 \\
y_3 & 1 & y_3 \\
\end{pmatrix}. \hspace{1cm} (45)
\]

With this new constraint, the dimension of the range of \(\tau^\top\) is decreased by 1 and a new vector appears in the kernel of \(\tau^\top\):

\[
|\phi_3\rangle = - |02\rangle + y_3 |13\rangle. \hspace{1cm} (46)
\]

As we mentioned before, the vector (44) is the only separable vector that is orthogonal to (41) and (42) and such constraints are not modified by assuming \(y_2 = y_3\). Nevertheless, this last assumption also implies that vector (44) must also be orthogonal to (46) for the state \(\tau\) to be separable. The inner product between the separable vector (44) and (46) is

\[
\langle\phi_3|e^*f\rangle_{\theta} = e^{2i\theta} y_2 \left( \frac{y_2^2}{y_1^2} - 1 \right) \neq 0. \hspace{1cm} (47)
\]

Since we are free to take \(y_1 > y_2\) we conclude that in such a case there is no separable vector \(|ef\rangle\) that satisfies the hypothesis of the range criterion (25) and hence we conclude that there exist bound entangled states of the type defined by (2) for \(N > 3\).

We point out that the above argument requires at least three \(y_i\) and thus not surprisingly will only work for \(N > 3\). In appendix B we construct the convex hulls for the PPT separable states for \(N = 2, 3\). In appendix A we explicitly evaluate two often used criteria that have been shown to detect some bound entangled states and numerically show that they are unable to detect bound entanglement for the states considered here.

5. Generation of bound entanglement using the JC interaction

In the last section it was shown that there are bound entangled states compatible with the JC symmetries. In this section we shall show that the JC interaction can be used to generate bound entanglement from uncorrelated states. We will study the interaction of an atom prepared in the unpolarized state:

\[
\rho(0)_A = (1 - \lambda) |0\rangle \langle 0| + \lambda |1\rangle \langle 1|, \hspace{1cm} (48)
\]

where \(\lambda\) represents the probability of having the atom in the excited state and a field prepared in the thermal state:

\[
\rho(0)_B = \sum_{n=0}^{\infty} p_n |n\rangle \langle n| \hspace{1cm} (49)
\]

with

\[
p_n = \frac{m^n}{(1 + m)^{n+1}} \hspace{1cm} (50)
\]

and \(m = \langle a^\dagger a \rangle\) is the mean number of photons in the field. The atom–cavity system starts in the product state:

\[
\rho(0) = \rho(0)_A \otimes \rho(0)_F. \hspace{1cm} (51)
\]

The dynamics of this type of state under the resonant JC Hamiltonian \((\Delta = 0\) in equation (5)) has been studied by Scheel et al in [25]. They find that in the \((\lambda, m)\) parameter space there are three separate regions:

(i) In the first region the state becomes free entangled immediately after the interaction between the atom and the field starts, i.e., for \(t > 0\) the state does not have a positive PT.
(ii) In the second region the state becomes free entangled only after some finite \(t\), i.e. for some finite \(t\) the state has positive PT and then after this time the state becomes free entangled.
(iii) In the third region they find that the state is PPT for all times.
In this section we will be interested in the cases for which the state remains PPT for a finite time after the interaction starts, i.e. regions (ii) and (iii).

To understand how bound entanglement is generated we first write explicitly the time evolution generated by $H_{JC}$ in the resonant case:

$$\rho(t) = \exp(-iH_{JC}t)\rho(0)\exp(iH_{JC}t)$$

$$= \gamma_0^\dagger \langle 00|00\rangle + \sum_{n=1}^\infty f_n(\gamma_n^\dagger |0n\rangle\langle 0n|$$

$$+ \alpha_n^+ |1n-1\rangle\langle 1n-1|$$

$$+ \beta_n |0n\rangle\langle 1n-1| + |1n-1\rangle\langle 0n|)$$

(52)

with

$$f_n = \frac{1}{2}m^{n-1}(m + 1)^{-n-1}$$

(53)

$$\alpha_n^\pm = m + \lambda \pm (\lambda + m(2\lambda - 1)) \cos(2g\sqrt{n}t)$$

(54)

$$\beta_n = (m(2\lambda - 1) + \lambda) \sin(2g\sqrt{n}t).$$

(55)

Remembering the notation introduced in equations (7) and (8) it is found that the non-zero elements of $\rho$ are

$$a_n = f_n\alpha_n^+, \quad b_n = f_{n+1}\alpha_{n+1}^+, \quad c_n = |f_n\beta_n|,$$

(56)

and from them the $y_n$ defined in equation (33) can be found:

$$y_n = \frac{|f_n\beta_n|}{\sqrt{f_n\alpha_n^+ f_{n+1}\alpha_{n+1}^+}}.$$  

(57)

Finally, since we are only interested in the existence of bound entanglement we will truncate the photon ladder in three photons thus including only the lowest four Fock states. This corresponds to a local operation in the photonic system and thus cannot create entanglement. In particular, that means that if the truncated density operator is entangled, then it can be concluded that the full state is entangled.

Applying the truncation and decomposing the state according to equations (31) and (36), it is found that the bound entanglement properties of the state $\rho(t)$ are equivalent to those of the state (40) with the $y_n$ given in equation (57). To show that such a state is bound entangled we note that one can always find a time $t^*$ for which $y_2 = y_3$. Indeed, it is found that in a finite region $\mathcal{S}$ inside the rectangle $0 < m \leq 1$ and $0 < \lambda \leq 1/2$ and $gt = \pi/4$ the difference $y_2 - y_3 > 0$, whereas in the same region $\mathcal{S}$ but for $gt = \pi/2$ the difference $y_2 - y_3 < 0$ and thus by continuity of the $y_n$ at some time between $gt = \pi/4$ and $gt = \pi/2$ the equality $y_2 = y_3$ is satisfied. We have just shown that times $t^*$ exist for which the state of the system $\rho$ can be decomposed as a mixture of an edge-bound entangled state $\tau$ and a separable state $\sigma_\tau$ (see equation (36)). One might wonder if this mixture is still entangled. Indeed it is, and this is related to the fact that the separable state $\sigma_\tau$ only has support in the subspace spanned by $|0\rangle$ in the atomic subsystem [55]. This result implies that at least for a finite region of the $(m, \lambda)$ parameter space (the intersection of $\mathcal{S}$ and region (iii) which is not empty) the state always becomes bound entangled for some time $t$ after the JC interaction between the atom and field starts.

6. Conclusion

In this paper we have analysed the entanglement properties in the JC model. To this aim we have first used the conservation of the number operator to formalize the mixed states that naturally occur when a two-level atom interacts with an electromagnetic field in a cavity evolving under the JC dynamics. We have first examined the limitations of the entanglement criteria used so far to study these states. Then, to the best of our knowledge, we have for the first time demonstrated (analytically) that bound entanglement exists in such a model and the failure of some of the criteria used so far to detect such a type of entanglement. Finally, we showed that the JC interaction can be used to generate bound entangled states starting from mixed uncorrelated ones. Our results have implications for all systems whose dynamics can be approximated by a Jaynes–Cummings Hamiltonian, such as ion traps and cavity/circuit QED.

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Appendix A. Bound entanglement detection criteria

In the entanglement detection criteria to be evaluated in this appendix it is assumed that the states are normalized:

$$\text{tr}(\rho) = \sum_{n=0}^{N-1} (a_n + b_n) = 1.$$  

(A.1)

It will also be convenient to write the marginals (reduced density matrices) of the qubit and the qudit:

$$\rho_A = \text{tr}_B(\rho) = \sum_{i=0}^{1} \alpha_i |i\rangle\langle i|, \quad \rho_B = \text{tr}_A(\rho) = \sum_{n=0}^{N-1} \beta_n |n\rangle\langle n|, $$

(A.2)

where the coefficients are given by

$$\alpha_0 = \sum_{n=0}^{N-1} a_n, \quad \alpha_1 = \sum_{n=0}^{N-1} b_n, \quad \beta_n = a_n + b_n.$$  

(A.3)

They satisfy the normalization $\alpha_1 + \alpha_0 = \text{tr}(\rho) = \sum_{n=0}^{N-1} \beta_n$.

The first criterion that we evaluate is the CCNR criterion. The CCNR criterion states that if $\rho$ is separable, then the following inequality must hold [34, 35],

$$\|\mathcal{R}(\rho)\| \leq 1, $$

(A.4)

where $\| \cdot \|$ stands for the trace norm (i.e. the sum of the singular values). The realignment operation is defined as

$$\mathcal{R}(A \otimes B) = |A\rangle\langle B^*|,$$

with scalar product $\langle B|A\rangle = \text{tr}(B^*A)$.
in the Hilbert–Schmidt space of operators. For the states considered here the singular values are
\[ s(\rho) = \{ |c|, |c|, x_+, x_- \}, \tag{A.5} \]
with
\[ x_\pm = \sqrt{\frac{|\vec{a}|^2 + |\vec{b}|^2 \pm \sqrt{(|\vec{a}|^2 - |\vec{b}|^2)^2 + 4(\vec{a} \cdot \vec{b})^2}}{2}, \]
and where the following real vectors are defined in terms of the populations and coherences of the density matrix:
\[ \vec{a} = [a_0, \ldots, a_{N-1}] \in \mathbb{R}^N \]
\[ \vec{b} = [b_0, \ldots, b_{N-1}] \in \mathbb{R}^N \]
\[ \vec{c} = [c_1, \ldots, c_{N-1}] \in \mathbb{R}^{N-1}. \tag{A.6} \]
and \[ |\vec{a}| = \sqrt{x_+ x_-}. \] Note that the four singular values will be nonzero unless \[ |\vec{c}| = 0 \] i.e. \[ c_0 = 0 \] or \[ (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 \] i.e. \[ a_0 = w b_0 \] \[ N. \] If \[ \|\rho\| > 1 \] the state \( \rho \) will be entangled, explicitly one finds that
\[ \|\rho\| = 2|c| + x_+ + x_- \]
\[ = 2|c| + \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2\sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}}. \tag{A.7} \]
In \[ [52] \] it is shown that \( \|\rho\| = 1 \) will provide a lower bound of the concurrence that is in principle independent of whether the state has positive PT or not. We generated more than 10 million states for \( N = 4 \) of which a million and a half were PPT. In all cases \( N(\rho) \geq \max\{\|\rho\|, 1, 0\} \). Finally, one can improve the entanglement detection capabilities of the CCNR criterion by the following corollary of the CM method \[ [36, 37] \]. The corollary states that if \( \rho \) is separable then
\[ \|\rho\| = 2|c| + \sqrt{2\sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}}. \tag{A.7} \]
For the states considered here one can show that
\[ \|\rho - \rho_A \otimes \rho_B\| = 2|c| + \sqrt{2\sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}}. \tag{A.7} \]
The coefficients \( a_k \) are given in equations (A.3). To test if this criterion could detect any bound entanglement we generated a million and a half PPT states in \( 2 \times 4 \) and none of them violated inequality (A.8). To confirm that indeed the CM corollary (which is always stronger than the CCNR criterion \[ [37] \]) does not detect the states (45) we considered \( N = 4 \) and looked at the two parameter family of normalized states \( \rho(y_2, y_3) = \tau(y_2, y_3)/\text{tr}(\tau(y_2, y_3)) \) in the range \( 0 < y_2 < y_3 \leq 10 \) and in none of the parameter values the inequality (A.8) was violated.

Appendix B. Convex hulls for \( N = 2, 3 \)

As it is well known in these cases being PPT implies separability. In this appendix we explicitly construct the convex hull of the PPT states. For future convenience we will introduce the following projector CP-map that acts on states \( \omega \):
\[ \mathcal{P}(\omega) = \frac{1}{2N} \sum_{k=0}^{2N-1} \exp \left( \frac{\pi k \Pi}{N} \right) \omega \exp \left( -\frac{\pi k \Pi}{N} \right). \tag{B.1} \]
The superoperator \( \mathcal{P} \) has the property that for any \( \omega \):
\[ \{ \mathcal{P}(\omega), \Pi \} = 0. \tag{B.2} \]
Thus, \( \mathcal{P} \) projects onto the subspace defined by equation (2), it satisfies \( \mathcal{P}(\mathcal{P}(\omega)) = \mathcal{P}(\omega) \), cannot generate entanglement since it can be implemented with local operations and classical communication and it maps density operators to density operators.

In the \( N = 2 \) case one has
\[ \tau = \begin{pmatrix} y_1^2 & y_1 y_2 & y_1 & 1 \\ y_1 y_2 & y_2^2 & y_2 & 1 \\ y_1 & y_2 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{B.3} \]
It is easy to see that there are separable products in vectors \( |ef \rangle \) in the range of \( \tau, R(\tau) \) such that \( |ef \rangle \) is in the range of \( \tau^T, R(\tau^T) \). One of such product vectors is
\[ |gh \rangle = \left( |0\rangle + \frac{1}{y_1} |1\rangle \right) \otimes (y_1 |0\rangle + y_1 |1\rangle). \tag{B.4} \]
It is easily shown that \( \mathcal{P}(|gh\rangle \langle gh|) \) is precisely \( \tau \) thus showing the convex hull of the separable state. Not surprisingly the four vectors that are added to construct \( \mathcal{P}(|gh\rangle \langle gh|) \) and obtain (B.3) are the same four vectors that one would obtain by using the Wooters formula for a system of two qubits \[ [49] \].

For \( N = 3 \) one has two possibilities for \( \tau \):
\[ \tau = \left( \begin{array}{cccc} y_1^2 & y_1 y_2 & y_1 & 1 \\ y_1 y_2 & y_2^2 & y_2 & 1 \\ y_1 & y_2 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right) \tag{B.5} \]
\[ \text{or} \]
\[ \text{depending on whether } y_1 \geq y_2 \text{ or } y_2 \geq y_1. \]
In the first case a vector that is in \( R(\tau) \) and \( R(\tau^T) \) is simply
\[ |gh\rangle = \left( |0\rangle + \frac{1}{y_1} |1\rangle \right) \otimes (y_1 |0\rangle + y_1 |1\rangle + y_2 |2\rangle) \tag{B.6} \]
upon subtracting \( \mathcal{P}(|gh\rangle \langle gh|) \) one obtains \( (1 - \frac{y_1^2}{2}) |0\rangle \otimes |2\rangle |2\rangle \). In the case where \( y_2 \geq y_1 \) one finds a vector product in \( R(\tau) \) and \( R(\tau^T) \) by swapping \( y_1 \) and \( y_2 \) in (B.6) upon subtracting \( \mathcal{P}(|gh\rangle \langle gh|) \) one obtains \( (1 - \frac{y_1^2}{2}) |0\rangle \otimes |2\rangle |2\rangle \) thus completing the construction for \( N = 2, 3 \). We note that the technique presented here using the projector \( \mathcal{P} \) is equivalent to the uniform mixing/averaging over all phases used to find separability proofs for symmetric mixed states of \( N \) qubits \[ [56] \].

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