Recognizing the topology of the space of closed convex subsets of a Banach space

by

TARAS BANAKH (Kielce and Lviv), IVAN HETMAN (Lviv) and KATSURO SAKAI (Tsukuba)

Abstract. Let $X$ be a Banach space and $\text{Conv}_H(X)$ be the space of non-empty closed convex subsets of $X$, endowed with the Hausdorff metric $d_H$. We prove that each connected component $H$ of the space $\text{Conv}_H(X)$ is homeomorphic to one of the spaces: \{0\}, $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}_+$, $Q \times \mathbb{R}_+$, $l_2$, or the Hilbert space $l_2(\kappa)$ of cardinality $\kappa \geq c$. More precisely, a component $H$ of $\text{Conv}_H(X)$ is homeomorphic to:

1. Introduction.

In this paper we recognize the topological structure of the space $\text{Conv}_H(X)$ of non-empty closed convex subsets of a Banach space $X$. The space $\text{Conv}_H(X)$ is endowed with the Hausdorff metric

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\} \in [0, \infty],$$

where $\text{dist}(a, B) = \inf_{b \in B} \|a - b\|$ is the distance from the point $a$ to the subset $B$ in $X$. In fact, the topology of $\text{Conv}_H(X)$ can be defined directly without appealing to the Hausdorff metric: a subset $U \subset \text{Conv}_H(X)$ is open if and only if for every $A \in U$ there is an open neighborhood $U$ of the origin in $X$ such that $B(A, U) \subset U$, where $B(A, U) = \{A' \in \text{Conv}_H(X) : A' \subset A + U$ and $A \subset A' + U\}$. Here, as expected, $A + B = \{a + b : a \in A, b \in B\}$ stands for the pointwise sum of the sets $A, B \subset X$. In this way, for every linear
topological space $X$ we can define the topology on the space $\text{Conv}_H(X)$ of non-empty closed convex subsets of $X$. This topology will be called the\textit{ uniform topology} on $\text{Conv}_H(X)$ because it is generated by the uniformity whose base consists of the sets

$$2^U = \{(A, A') \in \text{Conv}_H(X)^2 : A \subset A' + U, A' \subset A + U\}$$

where $U$ runs over open symmetric neighborhoods of the origin in $X$.

We shall observe in Remark 4.8 that for a Banach space $X$ the space $\text{Conv}_H(X)$ is locally connected: two sets $A, B \in \text{Conv}_H(X)$ lie in the same connected component of $\text{Conv}_H(X)$ if and only if $d_H(A, B) < \infty$. So, in order to understand the topological structure of the hyperspace $\text{Conv}_H(X)$ it suffices to recognize the topology of its connected components. This problem is quite easy if $X$ is a 1-dimensional real space. In this case $X$ is isometric to $\mathbb{R}$ and a connected component $H$ of $\text{Conv}_H(X)$ is homeomorphic to:

1. $\{0\}$ iff $X \in H$;
2. $\mathbb{R}$ iff $H$ contains a closed ray;
3. $\mathbb{R} \times \mathbb{R}_+$ iff $H$ contains a bounded set.

Here $\mathbb{R}_+ = [0, \infty)$ stands for the closed half-line.

For arbitrary Banach spaces we shall add to this list two more spaces:

4. $Q \times \mathbb{R}_+$, where $Q = [0, 1]^\omega$ is the Hilbert cube;
5. $l_2(\kappa)$, the Hilbert space with an orthonormal basis of cardinality $\kappa$.

For $\kappa = \omega$ the separable Hilbert space $l_2(\omega)$ is usually denoted by $l_2$. By the famous Toruńczyk Theorem [15], [16], each infinite-dimensional Banach space $X$ of density $\kappa$ is homeomorphic to the Hilbert space $l_2(\kappa)$. In particular, the Banach space $l_\infty^\omega$ of bounded real sequences is homeomorphic to $l_2(\kappa)$. In what follows, we shall identify cardinals with the sets of ordinals of smaller cardinality and endow such sets with discrete topology. The cardinality of a set $A$ is denoted by $|A|$.

Let $X$ be a Banach space. As we shall see in Theorem 1, each non-locally compact connected component $H$ of $\text{Conv}_H(X)$ is homeomorphic to the Hilbert space $l_2(\kappa)$ of density $\kappa = \text{dens}(H)$. This reduces the problem of recognizing the topology of $\text{Conv}_H(X)$ to calculating the densities of its components. In fact, separable components $H$ of $\text{Conv}_H(X)$ have been characterized in [3] as components containing a polyhedral convex set.

We recall that a convex subset $C$ of a Banach space $X$ is\textit{ polyhedral} if $C$ can be written as the intersection $C = \bigcap F$ of a finite family $F$ of closed half-spaces. A \textit{half-space} in $X$ is a convex set of the form $f^{-1}((-\infty, a])$ for some real number $a$ and some non-zero linear continuous functional $f : X \to \mathbb{R}$. The whole space $X$ is a polyhedral set, being the intersection $X = \bigcap F$ of the empty family $F = \emptyset$ of closed half-spaces.

The principal result of this paper is the following classification theorem.
Theorem 1. Let $X$ be a Banach space. Each connected component $\mathcal{H}$ of the space $\text{Conv}_H(X)$ is homeomorphic to one of the spaces: $\{0\}$, $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}_+$, $Q \times \mathbb{R}_+$, $l_2$, or the Hilbert space $l_2(\kappa)$ of density $\kappa \geq c$. More precisely, $\mathcal{H}$ is homeomorphic to:

(1) $\{0\}$ iff $\mathcal{H}$ contains the whole space $X$;
(2) $\mathbb{R}$ iff $\mathcal{H}$ contains a half-space;
(3) $\mathbb{R} \times \mathbb{R}_+$ iff $\mathcal{H}$ contains a linear subspace of $X$ of codimension 1;
(4) $Q \times \mathbb{R}_+$ iff $\mathcal{H}$ contains a linear subspace of $X$ of finite codimension $\geq 2$;
(5) $l_2$ iff $\mathcal{H}$ contains a polyhedral convex subset of $X$ but contains no linear subspace and no half-space of $X$;
(6) $l_2(\kappa)$ for some cardinal $\kappa \geq c$ iff $\mathcal{H}$ contains no polyhedral convex subset of $X$.

Theorem 1 will be proved in Section 6 after some preliminary work in Sections 2–5.

In Corollary 2 below we shall derive from Theorem 1 a complete topological classification of the spaces $\text{Conv}_H(X)$ for Banach spaces $X$ with the Kunen–Shelah property and $|X^*| \leq c$.

A Banach space $X$ is defined to have the Kunen–Shelah property if each closed convex subset $C \subset X$ can be written as the intersection $C = \bigcap \mathcal{F}$ of an at most countable family $\mathcal{F}$ of closed half-spaces (in fact, this is one of seven equivalent Kunen–Shelah properties considered in [6] and [7, 8.19]). For a Banach space $X$ with the Kunen–Shelah property we get

$$|X^*| \leq |\text{Conv}_H(X)| \leq |X^*|^\omega.$$ 

The upper bound $\text{Conv}_H(X) \leq |X^*|^\omega$ follows from the definition of the Kunen–Shelah property, while the lower bound $|X^*| \leq |\text{Conv}_H(X)|$ follows from the observation that a functional $f \in X^*$ is uniquely determined by its polar half-space $H_f = f^{-1}((-\infty, 1])$.

It is clear that each separable Banach space has the Kunen–Shelah property. However there are also non-separable Banach spaces with that property. The first example of such a Banach space was constructed by S. Shelah [13] under $\diamondsuit_{\aleph_1}$. The second example is due to K. Kunen who used the Continuum Hypothesis to construct a non-metrizable scattered compact space $K$ such that the Banach space $X = C(K)$ of continuous functions on $K$ is hereditarily Lindelöf in the weak topology and thus has the Kunen–Shelah property; see [10] p. 1123. Kunen’s space $X = C(K)$ has the additional property that its dual space $X^* = C(X)^*$ has cardinality $|X^*| = c$ (this follows from the fact that each Borel measure on the scattered compact space $K$ has countable support). Let us remark that for every separable Banach space $X$ the dual space $X^*$ also has the cardinality of the continuum, $|X^*| = c$. It should
be mentioned that non-separable Banach spaces with the Kunen–Shelah property can be constructed only under certain additional set-theoretic assumptions: there are models of ZFC (see [14]) in which each Banach space with the Kunen–Shelah property is separable.

**Corollary 1.** For a separable Banach space (more generally, a Banach space with the Kunen–Shelah property and \(|X^*| \leq c\)), each connected component \(\mathcal{H}\) of the space \(\text{Conv}_H(X)\) is homeomorphic to \(\{0\}, \mathbb{R}, \mathbb{R} \times \overline{\mathbb{R}}_+, Q \times \overline{\mathbb{R}}_+, l_2\) or \(l_\infty\). More precisely, \(\mathcal{H}\) is homeomorphic to:

1. \(\{0\}\) iff \(\mathcal{H}\) contains the whole space \(X\);
2. \(\mathbb{R}\) iff \(\mathcal{H}\) contains a half-space;
3. \(Q \times \overline{\mathbb{R}}_+\) iff \(\mathcal{H}\) contains a linear subspace of \(X\) of codimension 1;
4. \(l_2\) iff \(\mathcal{H}\) contains a polyhedral convex set but contains no linear subspace and no half-space;
5. \(l_\infty\) iff \(\mathcal{H}\) contains no polyhedral convex set.

Since \(\text{Conv}_H(X)\) is homeomorphic to the topological sum of its connected components, we can use Corollary 1 to classify topologically the spaces \(\text{Conv}_H(X)\) for separable Banach spaces \(X\) (and more generally Banach spaces with the Kunen–Shelah property and \(|X^*| \leq c\)). In the following corollary the cardinal \(c\) is considered as a discrete topological space.

**Corollary 2.** For a separable Banach space \(X\) (more generally, a Banach space \(X\) with the Kunen–Shelah property and \(|X^*| \leq c\)) the space \(\text{Conv}_H(X)\) is homeomorphic to the topological sum:

1. \(\{0\} \oplus \mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R} \times \overline{\mathbb{R}}_+)\) iff \(\dim(X) = 1\);
2. \(\{0\} \oplus Q \times \overline{\mathbb{R}}_+ \oplus c \times (\mathbb{R} \oplus \mathbb{R} \times \overline{\mathbb{R}}_+ \oplus l_2 \oplus l_\infty)\) iff \(\dim(X) = 2\);
3. \(\{0\} \oplus c \times (\mathbb{R} \oplus \mathbb{R} \times \overline{\mathbb{R}}_+ \oplus Q \times \overline{\mathbb{R}}_+ \oplus l_2 \oplus l_\infty)\) iff \(\dim(X) \geq 3\).

Moreover, under \(2^{\omega_1} > c\), for a Banach space \(X\), the space \(\text{Conv}_H(X)\) has cardinality \(|\text{Conv}_H(X)| \leq c\) if and only if \(|X^*| \leq c\) and the Banach space \(X\) has the Kunen–Shelah property.

**Proof.** The statements (1)–(3) easily follow from the classification of the components of \(\text{Conv}_H(X)\) given in Corollary 1 and a routine calculation of the number of components of a given topological type.

Now assume that \(2^{\omega_1} > c\). If \(X\) is a Banach space with the Kunen–Shelah property and \(|X^*| \leq c\), then the definition of the Kunen–Shelah property yields the upper bound

\[|\text{Conv}_H(X)| \leq |X^*|^\omega \leq c^\omega = c.\]

If \(|\text{Conv}_H(X)| \leq c\), then \(|X^*| \leq c\) as \(|X^*| \leq |\text{Conv}_H(X)|\) (because each functional \(f \in X^*\) can be uniquely identified with its polar half-space \(f^{-1}((-\infty, 1]) \in \text{Conv}_H(X)\)). Assuming that \(X\) fails to have the Kunen–
Shelah property and applying Theorem 8.19 of [7] (see also [6]), we can find a sequence \( \{x_\alpha\}_{\alpha < \omega_1} \subset X \) such that for every \( \alpha < \omega_1 \) the point \( x_\alpha \) does not lie in the closed convex hull \( C_{\omega_1 \setminus \{\alpha\}} \) of the set \( \{x_\beta\}_{\beta \in \omega_1 \setminus \{\alpha\}} \). Now for every subset \( A \subset \omega_1 \) consider the closed convex hull \( C_A = \text{conv} \{x_\alpha\}_{\alpha \in A} \). We claim that \( C_A \neq C_B \) for any distinct subsets \( A, B \subset \omega_1 \). Indeed, if \( A \neq B \) then the symmetric difference \( (A \setminus B) \cup (B \setminus A) \) contains some ordinal \( \alpha \). Without loss of generality, we can assume that \( \alpha \in A \setminus B \). Then \( x_\alpha \in C_A \setminus C_B \) as \( C_B \subset C_{\omega_1 \setminus \{\alpha\}} \neq x_\alpha \). This implies that \( \{C_A : A \subset \omega_1\} \) is a subset of cardinality \( 2^{\omega_1} > \mathfrak{c} \) in \( \text{Conv}_H(X) \) and hence \( |\text{Conv}_H(X)| \geq 2^{\omega_1} > \mathfrak{c} \), which is the desired contradiction.

Among the connected components of \( \text{Conv}_H(X) \) there is a special one, namely, the component \( \mathcal{H}_0 \) containing the singleton \( \{0\} \). This component coincides with the space \( \text{BConv}_H(X) \) of all non-empty bounded closed convex subsets of a Banach space \( X \). The spaces \( \text{BConv}_H(X) \) have been intensively studied both by topologists [9], [12] and analysts [5]. In particular, S. Nadler, J. Quinn and N. M. Stavrakas [9] proved that for a finite \( n \geq 2 \) the space \( \text{BConv}_H(\mathbb{R}^n) \) is homeomorphic to \( Q \times \mathbb{R}_+ \), while K. Sakai proved in [12] that for an infinite-dimensional Banach space \( X \) the space \( \mathcal{H}_0 = \text{BConv}_H(X) \) is homeomorphic to a non-separable Hilbert space. Moreover, if \( X \) is separable or reflexive, then \( \text{dens}(\mathcal{H}_0) = 2^{\text{dens}(X)} \). If \( X \) is reflexive, then the density \( \text{dens}^*(X^*) \) of the dual space \( X^* \) in the weak* topology is equal to the density \( \text{dens}(X) \) of \( X \). Banach spaces \( X \) with \( \text{dens}^*(X^*) = \text{dens}(X) \) are called \textit{DENS Banach spaces} (see [7 5.39]). By Proposition 5.40 of [7], the class of DENS Banach spaces includes all weakly Lindelöf determined spaces, and hence all weakly countably generated and all reflexive Banach spaces.

Applying Theorem 1 to describing the topology of the component \( \mathcal{H}_0 = \text{BConv}_H(X) \), we obtain the following classification.

**Corollary 3.** The space \( \mathcal{H}_0 = \text{BConv}_H(X) \) of non-empty bounded closed convex subsets of a Banach space \( X \) is homeomorphic to one of the spaces: \( \{0\}, \mathbb{R} \times \mathbb{R}_+, Q \times \mathbb{R}_+ \) or the Hilbert space \( l_2(\kappa) \) of density \( \kappa \geq \mathfrak{c} \). More precisely, \( \text{BConv}(X) \) is homeomorphic to:

1. \( \{0\} \) iff \( \dim(X) = 0 \);
2. \( \mathbb{R} \times \mathbb{R}_+ \) iff \( \dim(X) = 1 \);
3. \( Q \times \mathbb{R}_+ \) iff \( 2 \leq \dim(X) < \infty \);
4. \( l_2(\kappa) \) for some cardinal \( \kappa \in [2^{\text{dens}^*(X^*)}, 2^{\text{dens}(X)}] \) iff \( \dim(X) = \infty \);
5. \( l_2(2^{\text{dens}(X)}) \) if \( X \) is an infinite-dimensional DENS Banach space.

**Proof.** This corollary will follow from Theorem 1 as soon as we check that \( 2^{\text{dens}^*(X^*)} \leq \text{dens}(\mathcal{H}_0) \leq |\mathcal{H}_0| \leq |\text{Conv}_H(X)| \leq 2^{\text{dens}(X)} \) for each infinite-dimensional Banach space \( X \).
In fact, the inequality $|\text{Conv}_H(X)| \leq 2^{\text{dens}(X)}$ has general-topological nature and follows from the known fact that the number of closed subsets (equal to the number of open subsets) of a topological space $Y$ does not exceed $2^w(Y)$, where $w(Y)$ is the weight of $Y$ (which is equal to $\text{dens}(Y)$ if the space $Y$ is metrizable; see [4, 4.1.15]).

To prove that $2^{\text{dens}^*(X^*)} \leq \text{dens}(\mathcal{H}_0)$ we shall use a result of Plichko [11] (see also Theorem 4.12 of [7]) saying that for each infinite-dimensional Banach space $X$ there is a bounded sequence $\{(x_\alpha, f_\alpha)\}_{\alpha<\kappa} \subset X \times X^*$ of length $\kappa = \text{dens}^*(X^*)$, which is biorthogonal in the sense that $f_\alpha(x_\beta) = 0$ for any distinct ordinals $\alpha, \beta < \kappa$. Let $L = \sup\{|x_\alpha|, |f_\alpha| : \alpha < \kappa\}$.

For every subset $A \subset \kappa$ consider $C_A = \overline{\text{conv}}(\{x_\alpha\}_{\alpha \in A})$, the closed convex hull of the set $\{x_\alpha\}_{\alpha \in A}$. We claim that for any distinct subsets $A, B \subset \kappa$ we get $d_H(C_A, C_B) \geq 1/L$. Indeed, since $A \neq B$ the symmetric difference $(A \setminus B) \cup (B \setminus A)$ contains some ordinal $\alpha$. Without loss of generality, we can assume that $\alpha \in A \setminus B$. Then $C_B \subset f_\alpha^{-1}(0)$ and hence for each $c \in C_B$ we get

$$\|x_\alpha - c\| \geq \frac{|f_\alpha(x_\alpha) - f_\alpha(c)|}{\|f_\alpha\|} \geq \frac{|1 - 0|}{L},$$

which implies $\text{dist}(x_\alpha, C_B) \geq 1/L$ and hence $d_H(C_A, C_B) \geq 1/L$ as $x_\alpha \in C_A$.

Now we see that $\mathcal{C} = \{C_A : A \subset \kappa\}$ is a closed discrete subspace in $\mathcal{H}_0$ and hence $\text{dens}(\mathcal{H}_0) \geq |\mathcal{C}| = 2^\kappa = 2^{\text{dens}^*(X^*)}$. ■

Corollaries 1 and 2 motivate the following problem.

**Problem 1.1.** Is $|X^*| \leq c$ for each Banach space $X$ with the Kunen–Shelah property?

Another problem concerns possible densities of the components of the space $\text{Conv}_H(X)$.

**Problem 1.2.** Let $X$ be an infinite-dimensional Banach space. Is it true that each component $\mathcal{H}$ (in particular, $\mathcal{H}_0$) of $\text{Conv}_H(X)$ has density $2^\kappa$ or $2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa\}$ for some cardinal $\kappa$?

Observe that under GCH (the Generalized Continuum Hypothesis) the answer to Problem 1.2 is trivially “yes” as under GCH all cardinals are of the form $2^{<\kappa}$ for some $\kappa$.

**2. $\infty$-Metric spaces.** Because the Hausdorff distance $d_H$ on $\text{Conv}_H(X)$ can take the infinite value we should work with generalized metrics called $\infty$-metrics.

By an $\infty$-metric on a set $X$ we understand a function $d : X \times X \to [0, \infty]$ satisfying the three axioms of a usual metric:
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- \( d(x, y) = 0 \) iff \( x = y \),
- \( d(x, y) = d(y, x) \),
- \( d(x, z) \leq d(x, y) + d(y, z) \).

Here we extend the addition operation from \((-\infty, \infty)\) to \([-\infty, \infty]\) letting

\[ \infty + \infty = \infty, \quad -\infty + (-\infty) = -\infty, \quad \infty + (-\infty) = -\infty + \infty = 0 \]

and

\[ x + \infty = \infty + x = \infty, \quad x + (-\infty) = -\infty + x = -\infty \]

for every \( x \in (-\infty, \infty) \).

An \( \infty \)-metric space is a pair \((X, d)\) consisting of a set \( X \) and an \( \infty \)-metric \( d \) on \( X \). It is clear that each metric is an \( \infty \)-metric and hence each metric space is an \( \infty \)-metric space.

In some respects, the notion of an \( \infty \)-metric is more convenient than the usual notion of a metric. In particular, for any family \((X_i, d_i), i \in \mathcal{I}\), of \( \infty \)-metric spaces it is trivial to define a nice \( \infty \)-metric \( d \) on the topological sum \( X = \bigoplus_{i \in \mathcal{I}} X_i \). Just let

\[ d(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in X_i, \\ \infty & \text{otherwise}. \end{cases} \]

The resulting \( \infty \)-metric space \((X, d)\) will be called the direct sum of the family of \( \infty \)-metric spaces \((X, d_i), i \in \mathcal{I}\).

In fact, each \( \infty \)-metric space \((X, d)\) decomposes into the direct sum of metric subspaces of \( X \) called metric components of \( X \). More precisely, a metric component of \( X \) is an equivalence class of \( X \) by the equivalence relation \( \sim \) defined by \( x \sim y \) iff \( d(x, y) < \infty \). So, the metric component of a point \( x \in X \) coincides with the set \( \mathbb{B}_{<\infty}(x) = \{ x' \in X : d(x, x') < \infty \} \). The restriction of the \( \infty \)-metric \( d \) to each metric component is a metric. Therefore \( X \) is the direct sum of its metric components, and hence understanding the (topological) structure of a \( \infty \)-metric space reduces to studying the metric (or topological) structure of its metric components.

A typical example of an \( \infty \)-metric is the Hausdorff \( \infty \)-metric \( d_H \) on the space \( \operatorname{Cld}(X) \) of non-empty closed subsets of a (linear) metric space \( X \) (and the restriction of \( d_H \) to the subspace \( \operatorname{Conv}(X) \subset \operatorname{Cld}(X) \) of non-empty closed convex subsets of \( X \)). So both \( \operatorname{Cld}_H(X) = (\operatorname{Cld}(X), d_H) \) and \( \operatorname{Conv}_H(X) = (\operatorname{Conv}(X), d_H) \) are \( \infty \)-metric spaces.

A much simpler (but still important) example of an \( \infty \)-metric space is the extended real line \( \mathbb{R} = [-\infty, \infty] \) with the \( \infty \)-metric

\[ d_\infty(x, y) = \begin{cases} |x - y| & \text{if } x, y \in (-\infty, \infty), \\ 0 & \text{if } x = y \in (-\infty, \infty), \\ \infty & \text{otherwise}, \end{cases} \]
which will be denoted by $|x - y|$ again. The $\infty$-metric space $\mathbb{R}$ has three metric components: $\{-\infty\}, \mathbb{R}, \{\infty\}$.

This example allows us to construct another important example of an $\infty$-metric space. Namely, for a set $\Gamma$ consider the space $\bar{\mathbb{R}}^\Gamma$ of functions from $\Gamma$ to $\mathbb{R}$ endowed with the $\infty$-metric

$$d_\infty(f, g) = \|f - g\|_\infty = \sup_{\gamma \in \Gamma} |f(\gamma) - g(\gamma)|.$$  

The resulting $\infty$-metric space $(\bar{\mathbb{R}}^\Gamma, d_\infty)$ will be denoted by $\bar{l}_\infty(\Gamma)$. Observe that the topology of $\bar{l}_\infty(\Gamma)$ is different from the Tikhonov product topology of $\bar{\mathbb{R}}^\Gamma$. Another reason for using the notation $\bar{l}_\infty(\Gamma)$ is that the metric component of $\bar{l}_\infty(\Gamma)$ containing the zero function coincides with the classical Banach space $l_\infty(\Gamma)$ of bounded functions on $\Gamma$. More generally, for each $f_0 \in \bar{l}_\infty(\Gamma)$ its metric component

$$\mathbb{B}_{\infty}(f_0) = \{f \in \bar{l}_\infty(\Gamma) : \|f - f_0\|_\infty < \infty\}$$

is isometric to the Banach space $l_\infty(\Gamma_0)$ where $\Gamma_0 = \{\gamma \in \Gamma : \|f_0(\gamma)\| < \infty\}$. This fact will be used later in Corollary 4.5.

It turns out that for every normed space $X$ the space $\text{Conv}_H(X)$ nicely embeds into the $\infty$-metric space $\bar{l}_\infty(\mathbb{S}^*)$ where

$$\mathbb{S}^* = \{x^* \in X^* : \|x^*\| = 1\}$$

stands for the unit sphere of the dual Banach space $X^*$.

Namely, consider the function

$$\delta : \text{Conv}_H(X) \to \bar{l}_\infty(\mathbb{S}^*), \quad C \mapsto \delta_C,$$

where $\delta_C(x^*) = \sup x^*(C)$ for $x^* \in \mathbb{S}^*$. The function $\delta$ will be called the canonical representation of $\text{Conv}_H(X)$.

**Proposition 2.1.** For every normed space $X$ the canonical representation $\delta : \text{Conv}_H(X) \to \bar{l}_\infty(\mathbb{S}^*)$ is an isometric embedding.

**Proof.** Let $A, B \in \text{Conv}_H(X)$ be two convex sets. We should prove that

$$d_H(A, B) = \|\delta_A - \delta_B\|,$$

where

$$\|\delta_A - \delta_B\| = \sup_{x^* \in \mathbb{S}^*} |\delta_A(x^*) - \delta_B(x^*)| = \sup_{x^* \in \mathbb{S}^*} |\sup x^*(A) - \sup x^*(B)|.$$  

The inequality $\|\delta_A - \delta_B\| \leq d_H(A, B)$ will follow as soon as we check that $|\sup x^*(A) - \sup x^*(B)| \leq d_H(A, B)$ for each functional $x^* \in \mathbb{S}^*$. This is trivial if $d_H(A, B) = \infty$. So we assume that $d_H(A, B) < \infty$. To obtain a contradiction, assume that $|\sup x^*(A) - \sup x^*(B)| > d_H(A, B)$. Then either $\sup x^*(A) - \sup x^*(B) > d_H(A, B)$ or $\sup x^*(B) - \sup x^*(A) > d_H(A, B)$. In the first case $\sup x^*(B) \neq \infty$, so we can find a point $a \in A$ with $x^*(a) - \sup x^*(B) > d_H(A, B)$. It follows from the definition of the Hausdorff metric $d_H(A, B) \geq \text{dist}(a, B)$ that $\|a - b\| < x^*(a) - \sup x^*(B)$ for some point
b ∈ B. Then $x^*(a) - x^*(b) \leq \|x^*\| \cdot \|a - b\| < x^*(a) - \sup x^*(B)$ and hence $x^*(b) > \sup x^*(B)$, which is a contradiction.

By analogy, we can derive a contradiction from the assumption $\sup x^*(B) - \sup x^*(A) > d_H(A, B)$ and thus prove the inequality $\|\delta_A - \delta_B\| \leq d_H(A, B)$.

To prove the reverse inequality $\|\delta_A - \delta_B\| \geq d_H(A, B)$ let us consider two cases:

(i) $d_H(A, B) = \infty$. To prove that $\infty = \|\delta_A - \delta_B\|$, it suffices given any number $R < \infty$ to find a linear functional $x^* \in S^*$ such that $|\sup x^*(A) - \sup x^*(B)| \geq R$.

The equality $d_H(A, B) = \infty$ implies that either $\inf_{a \in A} \delta_a \geq R$ or $\inf_{b \in B} \delta_b = \infty$. In the first case we can find a point $a \in A$ with $\inf_{a \in A} \delta_a \geq R$ and using the Hahn–Banach Theorem construct a linear functional $x^* \in S^*$ that separates the convex set $B$ from the closed $R$-ball $\bar{B}(a, R) = \{x \in X : \|x - a\| \leq R\}$ in the sense that $\sup x^*(B) \leq \inf x^*(\bar{B}(a, R))$. For this functional $x^*$ we get $\sup x^*(A) \geq x^*(a) \geq R + \inf x^*(\bar{B}(a, R)) \geq R + \sup x^*(B)$ and thus $\sup x^*(A) - \sup x^*(B) \geq R$.

In the second case, we can repeat the preceding argument to find a linear functional $x^* \in S^*$ with

$$\sup x^*(A) - \sup x^*(B) \geq \sup x^*(B) - \sup x^*(A) \geq R.$$ 

(ii) $d_H(A, B) < \infty$. To prove that $\|\delta_A - \delta_B\| \geq d_H(A, B)$ it suffices given any number $\varepsilon > 0$ to find a linear functional $x^* \in S^*$ such that $|\sup x^*(A) - \sup x^*(B)| \geq d_H(A, B) - \varepsilon$. It follows from the definition of $d_H(A, B)$ that either there is a point $a \in A$ with $\inf_{a \in A} \delta_a > d_H(A, B) - \varepsilon$ or else there is a point $b \in B$ with $\inf_{b \in B} \delta_b > d_H(A, B) - \varepsilon$. In the first case we can use the Hahn–Banach Theorem to find a linear functional $x^* \in S^*$ which separates the convex set $B$ from the closed $R$-ball $\bar{B}(a, R)$, where $R = d_H(A, B) - \varepsilon$, in the sense that $\sup x^*(B) \leq \inf x^*(\bar{B}(a, R))$. Then $\sup x^*(B) \leq \inf x^*(\bar{B}(a, R)) = x^*(a) - R \leq \sup x^*(A) - R$ and hence $|\sup x^*(A) - \sup x^*(B)| \geq \sup x^*(A) - \sup x^*(B) \geq R = d_H(A, B) - \varepsilon$.

The second case can be considered by analogy.

3. Assigning cones to components of $\text{Conv}_H(X)$. In this section to each convex set $C$ of a normed space $X$ we assign two cones: the recession cone $V_C \subset X$ and the dual recession cone $V_C^* \subset X^*$.

We recall that a subset $V$ of a linear space $L$ is called a convex cone if $ax + by \in V$ for any points $x, y \in W$ and any non-negative real numbers $a, b \in [0, \infty)$. 

For a convex subset \( C \) of a normed space \( X \) its recession cone is the convex cone
\[
V_C = \{ v \in X : \forall c \in C, c + \mathbb{R}_+ v \subset C \},
\]
lying in the normed space \( X \), and its dual recession cone \( V_C^* \) is the closed convex cone
\[
V_C^* = \{ x^* \in X^* : \sup x^*(C) < \infty \},
\]
which is contained in the dual Banach space \( X^* \).

It turns out that the recession cone \( V_C \) of a convex set \( C \) is uniquely determined by its dual recession cone \( V_C^* \).

**Lemma 3.1.** For any non-empty closed convex set \( C \) in a normed space \( X \) we get
\[
V_C = \bigcap_{f \in V_C^*} f^{-1}(\langle -\infty, 0 \rangle).
\]

**Proof.** Fix any vector \( v \in V_C \) and a functional \( f \in V_C^* \). Observe that for each point \( c \in C \) and each number \( t \in \mathbb{R}_+ \), we get \( c + tv \in C \) and hence \( f(c) + tf(v) \leq \sup f(C) < \infty \), which implies that \( f(v) \leq 0 \). This proves the inclusion \( V_C \subset \bigcap_{f \in V_C^*} f^{-1}(\langle -\infty, 0 \rangle) \).

To prove the reverse inclusion, fix any vector \( v \in X \setminus V_C \). Then for some point \( c \in C \) and some positive real number \( t \) we get \( c + tv \notin C \). Using the Hahn–Banach Theorem, find a functional \( f \in X^* \) that separates the convex set \( C \) and the point \( x = c + tv \) in the sense that \( \sup f(C) < f(c + tv) \). Then \( f \in V_C^* \). Moreover, \( f(c) \leq \sup f(C) < f(c) + tf(v) \) implies that \( f(v) > 0 \) and \( v \notin f^{-1}(\langle -\infty, 0 \rangle) \).

Let \( X \) be a normed space. It is easy to see that for each metric component \( \mathcal{H} \) of the \( \infty \)-metric space \( \text{Conv}_\mathcal{H}(X) \) and any two convex sets \( A, B \in \mathcal{H} \) we get \( V_A^* = V_B^* \). In this case Lemma 3.1 implies that \( V_A = V_B \) as well. This allows us to define the recession cone \( V_\mathcal{H} \) and the dual recession cone \( V_\mathcal{H}^* \) of the metric component \( \mathcal{H} \) letting \( V_\mathcal{H} = V_C \) and \( V_\mathcal{H}^* = V_C^* \) for any convex set \( C \in \mathcal{H} \). Lemma 3.1 guarantees that
\[
V_\mathcal{H} = \bigcap_{f \in V_\mathcal{H}^*} f^{-1}(\langle -\infty, 0 \rangle),
\]
so the recession cone \( V_\mathcal{H} \) of \( \mathcal{H} \) is uniquely determined by its dual recession cone \( V_\mathcal{H}^* \).

4. The algebraic structure of \( \text{Conv}_\mathcal{H}(X) \). In this section given a normed space \( X \) we study the algebraic properties of the canonical representation \( \delta : \text{Conv}_\mathcal{H}(X) \to \tilde{l}_\infty(S^*) \).

Note that the space \( \text{Conv}_\mathcal{H}(X) \) has a rich algebraic structure, namely three interrelated algebraic operations: multiplication by a real number, ad-
dition, and taking maximum. More precisely, for a real number \( t \in \mathbb{R} \) and convex sets \( A, B \in \operatorname{Conv}_H(X) \) let

\[
t \cdot A = \{ ta : a \in A \}; \\
A \oplus B = A + B; \\
\max\{A, B\} = \operatorname{conv}(A \cup B), \text{ where} \\
\operatorname{conv}(Y) \text{ stands for the closed convex hull of a subset } Y \subset X.
\]

The \( \infty \)-metric space \( \mathbb{R} \) also has the corresponding three operations (multiplication by a real number, addition and taking maximum), which induces the tree operations on \( \bar{l}_\infty(\Gamma) = \mathbb{R}^\Gamma \).

**Proposition 4.1.** The canonical representation \( \delta : \operatorname{Conv}_H(X) \to \bar{l}_\infty(S^*) \) has the following properties:

1. \( \delta(A \oplus B) = \delta(A) + \delta(B) \),
2. \( \delta(\max\{A, B\}) = \max\{\delta(A), \delta(B)\} \),
3. \( \delta(rA) = r\delta(A) \),

for every non-negative real number \( r \) and convex sets \( A, B \in \operatorname{Conv}_H(X) \).

**Proof.** The three items of the proposition follow from the three obvious equalities

\[
\sup x^*(A \oplus B) = \sup x^*(A + B) = \sup x^*(A) + \sup x^*(B), \\
\sup x^*(\operatorname{conv}(A \cup B)) = \sup x^*(A \cup B) = \max\{\sup x^*(A), \sup x^*(B)\}, \\
\sup x^*(rA) = r \sup x^*(A),
\]

holding for every functional \( x^* \in X^* \).

**Remark 4.2.** Easy examples show that the last item of Proposition 4.1 does not hold for negative real numbers \( r \). This means that the operator \( \delta : \operatorname{Conv}_H(X) \to \bar{l}_\infty(S^*) \) is positively homogeneous but not homogeneous.

The operations of addition and multiplication by a real number allow us to define another important operation on \( \operatorname{Conv}_H(X) \) preserved by the canonical representation \( \delta \), namely the **Minkowski operation**

\[
\mu : \operatorname{Conv}_H(X) \times \operatorname{Conv}_H(X) \times [0, 1] \to \operatorname{Conv}_H(X), \\
(A, B, t) \mapsto (1 - t)A \oplus tB,
\]

of producing a convex combination. Proposition 4.1 implies that the canonical representation \( \delta : \operatorname{Conv}_H(X) \to \bar{l}_\infty(S^*) \) is **affine** in the sense that

\[
\delta((1 - t)A \oplus tB) = (1 - t)\delta(A) + t\delta(B)
\]

for every \( A, B \in \operatorname{Conv}_H(X) \) and \( t \in [0, 1] \).

Propositions 2.1 and 4.1 will help us to establish the metric properties of the algebraic operations on \( \operatorname{Conv}_H(X) \).

**Proposition 4.3.** Let \( A, B, C, A', B' \in \operatorname{Conv}_H(X) \) be five convex sets and \( r \in \mathbb{R}, t, t' \in [0, 1] \) be three real numbers. Then
(1) $d_H(A \oplus B, A' \oplus B') \leq d_H(A, A') + d_H(B, B')$;
(2) $d_H(A \oplus B, A \oplus C) = d_H(B, C)$ provided $V_A^* \supset V_B^* \cup V_C^*$;
(3) $d_H(\max\{A, B\}, \max\{A', B'\}) \leq \max\{d_H(A, A'), d_H(B, B')\}$;
(4) $d_H(r \cdot A, r \cdot B) = |r| \cdot d_H(A, B)$;
(5) $d_H((1-t)A \oplus tB, (1-t')A \oplus t'B) = |t - t'|d_H(A, B)$.

**Proof.** All the items easily follow from Propositions 2.1, 4.1, and metric properties of algebraic operations on the $\infty$-metric space $\bar{l}_\infty(S^*)$.

Observe that the metric components of the $\infty$-metric space $\bar{l}_\infty(S^*)$ are closed with respect to taking the maximum and producing a convex combination. Moreover those operations are continuous on metric components of $\bar{l}_\infty(S^*)$. With the help of the canonical representation those properties of $\bar{l}_\infty(S^*)$ transform into the corresponding properties of $\text{Conv}_H(X)$. In this way we obtain

**Corollary 4.4.** Each metric component $\mathcal{H}$ of $\text{Conv}_H(X)$ is closed under the operations of taking maximum and producing a convex combination. Moreover those operations are continuous on $\mathcal{H}$.

**Corollary 4.5.** Each metric component $\mathcal{H}$ of $\text{Conv}_H(X)$ is isometric to a convex max-subsemilattice of the Banach lattice $l_\infty(S^*)$.

A subset of a Banach lattice is called a max-subsemilattice if it is closed under the operation of taking maximum.

By a recent result of Banakh and Cauty [1], each non-locally compact closed convex subset of a Banach space is homeomorphic to an infinite-dimensional Hilbert space. This result combined with Corollary 4.5 implies:

**Corollary 4.6.** Let $X$ be a Banach space. Then a metric component $\mathcal{H}$ of $\text{Conv}_H(X)$ is homeomorphic to an infinite-dimensional Hilbert space if and only if $\mathcal{H}$ is not locally compact.

This corollary reduces the problem of recognition of the topology of non-locally compact components of $\text{Conv}_H(X)$ to calculating their densities. This problem was considered in [3] where the following characterization was proved.

**Proposition 4.7.** For a Banach space $X$ and a metric component $\mathcal{H}$ of the space $\text{Conv}_H(X)$ the following conditions are equivalent:

1. $\mathcal{H}$ is separable;
2. $\text{dens}(\mathcal{H}) < c$;
3. $\mathcal{H}$ contains a polyhedral convex set;
4. the recession cone $V_\mathcal{H}$ is polyhedral and belongs to $\mathcal{H}$.

**Remark 4.8.** By Corollary 4.5 each metric component of $\text{Conv}_H(X)$, being homeomorphic to a convex set, is (locally) connected, and, being
closed-and-open in Conv$_H(X)$, coincides with a connected component of Conv$_H(X)$. Hence there is no difference between metric and connected components of Conv$_H(X)$, so using the term component of Conv$_H(X)$ (without an adjective “metric” or “connected”) will not lead to misunderstanding.

5. Operators between spaces of convex sets. Each linear continuous operator $T : X \to Y$ between normed spaces induces a map $\overline{T} : \text{Conv}_H(X) \to \text{Conv}_H(Y)$ assigning to each closed convex set $A \in \text{Conv}_H(X)$ the closure $\overline{T}(A)$ of its image $T(A)$ in $Y$. In this section we study properties of the induced operator $\overline{T}$. We start with algebraic properties that trivially follow from the linearity and continuity of $T$.

**Proposition 5.1.** If $T : X \to Y$ is a linear continuous operator between Banach spaces, and $\overline{T} : \text{Conv}_H(X) \to \text{Conv}_H(Y)$ is the induced operator, then

1. $\overline{T}(\max\{A, B\}) = \max\{\overline{T}(A), \overline{T}(B)\}$,
2. $\overline{T}(r \cdot A) = r \cdot \overline{T}(A)$,
3. $\overline{T}(A \oplus B) = \overline{T}(A) \oplus \overline{T}(B)$,
4. $\overline{T}((1 - t)A \oplus tB) = (1 - t)\overline{T}(A) \oplus t\overline{T}(B)$,

for any sets $A, B \in \text{Conv}_H(X)$ and real numbers $r \in \mathbb{R}$ and $t \in [0, 1]$.

We shall be mainly interested in the operators $\overline{T}$ induced by quotient operators $T$. We recall that for a closed linear subspace $Z$ of a normed space $X$ the quotient normed space $X/Z = \{x + Z : x \in X\}$ carries the quotient norm

$$\|x + Z\| = \inf_{y \in x + Z} \|y\|.$$ 

We shall denote by $q : X \to X/Z$, $x \mapsto x + Z$, the quotient operator and by $\overline{q} : \text{Conv}_H(X) \to \text{Conv}_H(X/Z)$ the induced operator between the spaces of closed convex sets.

For a closed convex set $C \subset X$ we let $C/Z$ denote the image $q(C) \subset X/Z$. So, $\overline{q}(C) = \overline{C}/Z$. If $Z \subset V_C$, then the set $C/Z$ is closed in $X/Z$ and hence $\overline{q}(C) = C/Z$. Indeed, $Z \subset V_C$ implies that $C + Z = C$ and hence $C/Z = (X/Z) \setminus q(X \setminus C)$ is closed in $X/Z$, being the complement of the set $q(X \setminus C)$, which is open as the image of the open set $X \setminus C$ under the open map $q : X \to X/Z$.

We shall need the following simple reduction lemma:

**Lemma 5.2.** Let $Z$ be a closed linear subspace of a normed space $X$ and let $A, B$ be non-empty closed convex subsets of $X$. If $Z \subset V_A \cap V_B$, then $d_H(A, B) = d_H(A/Z, B/Z)$.

**Proof.** The inequality $d_H(A/Z, B/Z) \leq d_H(A, B)$ follows from $\|q\| \leq 1$. Assuming that $d_H(A/Z, B/Z) < d_H(A, B)$, we can find a point $a \in A$ with
dist\((a, B) > d_H(A/Z, b/Z)\) or a point \(b \in B\) with \(\text{dist}(b, A) > d_H(A/Z, B/Z)\). Without loss of generality, we deal with the former case. Consider the image \(a' = q(a) \in A/Z\) under the quotient operator \(q : X \to X/Z\). Since \(d_H(A/Z, B/Z) < \text{dist}(a, B)\), there is a point \(b' \in B/Z\) such that \(\|b' - a'\| < \text{dist}(a, B)\). By the definition of the quotient norm, there is a vector \(x \in q^{-1}(b' - a')\) such that \(\|x\| < \text{dist}(a, B)\). Now consider the point \(b = a + x\) and observe that \(q(b) = q(a) + q(x) = a' + b' - a' = b' \in B/Z\) and hence \(b \in q^{-1}(B/Z) = B + Z \subseteq B + V_B \subseteq B\). So, \(\text{dist}(a, B) \leq \|a - b\| = \|x\| < \text{dist}(a, B)\), which is a desired contradiction that completes the proof of the equality \(d_H(A, B) = d_H(A/Z, B/Z)\). ■

**Corollary 5.3.** Let \(X\) be a normed space \(X\), \(H\) be a component of the space \(\text{Conv}_H(X)\), and \(Z\) be a closed linear subspace of \(X\). If \(Z \subseteq V_H\), then the quotient operator

\[
\tilde{q} : H \to H/Z, \quad C \mapsto C/Z,
\]

maps isometrically the component \(H\) of \(\text{Conv}_H(X)\) onto the component \(H/Z\) of \(\text{Conv}_H(X/Z)\) containing some (equivalently, each) convex set \(C/Z\) with \(C \in H\).

**6. Proof of Theorem** [1]. Let \(X\) be a Banach space and \(H\) be a component of the space \(\text{Conv}_H(X)\).

If \(H\) contains no polyhedral convex set, then by Proposition [4.7] it has density \(\text{dens}(H) \geq c\). Consequently, \(H\) is not locally compact and, by Corollary [4.6] \(H\) is homeomorphic to the non-separable Hilbert space \(l_2(\kappa)\) of density \(\kappa = \text{dens}(H) \geq c\).

It remains to analyze the topological structure of \(H\) if it contains a polyhedral convex set. In this case Proposition [4.7] guarantees that the recession cone \(V_H\) belongs to \(H\) and is polyhedral in \(X\). If \(V_H = X\), then \(H = \{X\}\) is a singleton. So, we assume that \(V_H \neq X\). Since the cone \(V_H\) is polyhedral, the closed linear subspace \(Z = -V_H \cap V_H\) has finite codimension in \(X\). Then the quotient Banach space \(\tilde{X} = X/Z\) is finite-dimensional. Let \(q : X \to \tilde{X}\) be the quotient operator.

By Corollary [5.3] the component \(H\) is isometric to the component \(\tilde{H} = H/Z\) of the space \(\text{Conv}_H(\tilde{X})\) of closed convex subsets of the finite-dimensional Banach space \(\tilde{X}\). The component \(\tilde{H}\) contains the polyhedral convex cone \(V_{\tilde{H}} = q(V_H)\), which has the property \(-V_{\tilde{H}} \cap V_{\tilde{H}} = \{0\}\).

The cone \(V_{\tilde{H}}\) can be of two types.

1. The cone \(V_{\tilde{H}} = \{0\}\) is trivial. In this case \(H\) contains the closed linear subspace \(Z = V_H\) of finite codimension in \(X\). Taking into account that \(V_H \neq X\), we conclude that \(\text{dim}(\tilde{X}) \geq 1\). Depending on the value of \(\text{dim}(\tilde{X})\), we have two subcases.
1a. The dimension \( \dim(\tilde{X}) = 1 \) and hence \( \mathcal{H} \) contains the linear subspace \( Z = V_{\tilde{H}} \) of codimension 1 in \( X \). In this case \( \tilde{H} \) coincides with the space \( \text{BConv}_H(\tilde{X}) \) of non-empty bounded closed convex subsets of the one-dimensional Banach space \( \tilde{X} \) and hence \( \tilde{H} \) is homeomorphic to the half-plane \( \mathbb{R} \times \mathbb{R}_+ \).

1b. The dimension \( \dim(\tilde{X}) \geq 2 \) and hence \( \mathcal{H} \) contains the linear subspace \( Z \) of codimension \( \geq 2 \) in \( X \). In this case \( \tilde{H} \) coincides with the space \( \text{BConv}_H(\tilde{X}) \) of non-empty bounded closed convex subsets of the Banach space \( \tilde{X} \) of finite dimension \( \dim(\tilde{X}) \geq 2 \). By the result of Nadler, Quinn and Stavrakas [9], the space \( \text{BConv}_H(\tilde{X}) \) is homeomorphic to the Hilbert cube manifold \( Q \times \mathbb{R}_+ \).

2. The recession cone \( V_{\tilde{H}} \neq \{0\} \) is not trivial. Again there are two subcases.

2a. \( \dim(\tilde{X}) = \dim(V_{\tilde{H}}) = 1 \). In this case the component \( \tilde{H} \) (and its isometric copy \( \mathcal{H} \)) is isometric to the real line \( \mathbb{R} \).

2b. \( \dim(\tilde{X}) \geq 2 \). In this case we shall prove that the component \( \tilde{H} \) (and its isometric copy \( \mathcal{H} \)) is homeomorphic to the separable Hilbert space \( l_2 \). This will follow from the separability of \( \mathcal{H} \) and Corollary 4.6 as soon as we check that the space \( \mathcal{H} \) is not locally compact. To prove this fact, it suffices for every positive \( \varepsilon < 1 \) to construct a sequence of closed convex sets \( \{C_n\}_{n \in \mathbb{N}} \subset \tilde{H} \) such that \( d_\mathcal{H}(C_n, V_{\tilde{H}}) \leq \varepsilon \) and \( \inf_{n \neq m} d_\mathcal{H}(C_n, C_m) > 0 \).

The cone \( V_{\tilde{H}} \) is polyhedral and hence is generated by some finite set \( E \subset \tilde{X} \setminus \{0\} \); see [8] or Theorem 1.1 of [17]. For every \( e \in E \) the vector \( -e \) does not belong to \( V_{\tilde{H}} \). Then the Hahn–Banach Theorem yields a linear functional \( h_e \in X^* \) such that \( h_e(-e) < \inf h_e(V_{\tilde{H}}) = 0 \). It can be shown that the functional \( h = \sum_{e \in E} h_e \) has the property \( h(v) > 0 \) for all \( v \in V_{\tilde{H}} \setminus \{0\} \). Multiplying \( h \) by a suitable positive constant, we can additionally assume that \( ||h|| = 1 \).

Since \( \dim(\tilde{X}) \geq 2 \) and \( V_{\tilde{H}} \neq \tilde{X} \), we can find a linear continuous functional \( f : \tilde{X} \to \mathbb{R} \) such that \( ||f|| = 1 \), \( \sup f(V_{\tilde{H}}) = 0 \) and the intersection \( f^{-1}(0) \cap V_{\tilde{H}} \) contains a non-zero vector \( x \in \tilde{X} \). Multiplying \( x \) by a suitable positive constant, we can assume that \( h(x) = 1 \). Since \( h^{-1}(0) \cap V_{\tilde{H}} = \{0\} \neq f^{-1}(0) \cap V_{\tilde{H}} \), the functionals \( h \) and \( f \) are distinct and hence there is a vector \( y \in h^{-1}(0) \setminus f^{-1}(0) \) with norm \( ||y|| = \varepsilon \). Replacing \( y \) by \( -y \) if necessary, we can assume that \( f(y) > 0 \).

For every \( n \in \mathbb{N} \) consider the point \( c_n = 3^nx + y \) and the closed convex set

\[
C_n = \max\{V_{\tilde{H}}, \{c_n\}\} = \overline{\text{conv}}(V_{\tilde{H}} \cup \{c_n\}) \subset \tilde{X}.
\]

It follows from \( x \in V_{\tilde{H}} \) and \( \text{dist}(c_n, V_{\tilde{H}}) \leq \text{dist}(3^nx + y, 3^n x) = ||y|| = \varepsilon \) that \( d_\mathcal{H}(C_n, V_{\tilde{H}}) \leq \varepsilon \).
We claim that \( \inf_{n \neq m} d_H(C_n, C_m) \geq \delta \) where
\[
\delta = \frac{1}{2} f(y) \leq \frac{1}{2} \|y\| = \frac{1}{2} \varepsilon < \frac{1}{2}.
\]
This will follow as soon as we check that \( \text{dist}(c_n, C_m) \geq \delta \) for any numbers \( n < m \).

Assuming conversely that \( \text{dist}(c_n, C_m) < \delta \) and taking into account that the convex set \( \text{conv}(V_{\mathcal{H}} \cup \{c_m\}) \) is dense in \( C_m \), we can find a point \( c \in \text{conv}(V_{\mathcal{H}} \cup \{c_m\}) \) such that \( \text{dist}(c_n, c) < \delta \). The point \( c \) belongs to the convex hull of the set \( V_{\mathcal{H}} \cup \{c_m\} \) and hence can be written as a convex combination
\[
c = tc_m + (1 - t)v = t(3^m x + y) + (1 - t)v
\]
for some \( t \in [0, 1] \) and \( v \in V_{\mathcal{H}} \).

Observe that
\[
h(c_n) = h(3^n x + y) = 3^n h(x) + h(y) = 3^n \cdot 1 + 0 = 3^n
\]
while
\[
h(c) = th(c_m) + (1 - t)h(v) \geq th(c_m) = 3^m t.
\]
Then
\[
3^m t - 3^n \leq h(c) - h(c_n) \leq |h(c) - h(c_n)| \leq \|h\| \cdot \|c - c_n\| < 1 \cdot \delta
\]
and hence
\[
t < 3^{n-m} + 3^{-m} \delta \leq \frac{1}{3} + \frac{1}{3} \delta < \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.
\]

Next, we apply the functional \( f \) to the points \( c_n \) and \( c \). Since \( f(x) = 0 \), we get \( f(c_n) = f(3^n x + y) = f(y) = 2\delta \). On the other hand, \( f(V_{\mathcal{H}}) \subset (-\infty, 0] \) implies \( f(v) \leq 0 \) and hence
\[
f(c) = f(tc_m + (1 - t)v) = tf(3^m x + y) + (1 - t)f(v)
\]
\[
= tf(y) + (1 - t)f(v) \leq tf(y) = 2\delta t.
\]
Then
\[
\delta = 2\delta (1 - 1/2) < 2\delta (1 - t) \leq |f(c_n) - f(c)| \leq \|f\| \cdot \|c_n - c\| < \delta,
\]
which is the desired contradiction. \( \blacksquare \)

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Taras Banakh
Jan Kochanowski University
25-406 Kielce, Poland, and
Department of Mathematics
Ivan Franko National University of Lviv
Lviv, 79000 Ukraine
E-mail: t.o.banakh@gmail.com

Ivan Hetman
Department of Mathematics
Ivan Franko National University of Lviv
Lviv, 79000 Ukraine
E-mail: ihromant@gmail.com

Katsuro Sakai
Institute of Mathematics
University of Tsukuba
Tsukuba, 305-8571, Japan
E-mail: sakaiktr@sakura.cc.tsukuba.ac.jp

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