One-loop $F(R, P, Q)$ gravity in de Sitter universe

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Abstract

Motivated by the dark energy issue, the one-loop quantization approach for a class of relativistic higher order theories is discussed in some detail. A specific $F(R, P, Q)$ gravity model at the one-loop level in a de Sitter universe is investigated, extending the similar program developed for the case of $F(R)$ gravity. The stability conditions under arbitrary perturbations are derived.

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1. Introduction

It is well known that recent astrophysical data indicate that our universe is currently in a phase of accelerated expansion. This is one of the most important achievements in cosmology. The origin of this observation is substantially not completely understood and the related issue is called the dark energy problem.

Several possible explanations have been proposed in the literature; among them one of the most popular is based on the use of gravitational modified models, the simplest being Einstein gravity plus the inclusion of a small and positive cosmological constant, and this model works quite well but, however, has some drawbacks (see, for example, [1–3] and references therein).

Roughly, the idea is that Einstein gravity is only an approximate low energy contribution, and additional terms depending on quadratic curvature invariants should be included. The idea is quite old; one of the first proposals was contained in [4], where quantum $R^2$ gravity modifications were investigated (for a review, see [5]). The inclusion of general higher order contributions is also important from another aspect, since sometimes they give extra terms which may also realize the early time inflation [6].

In previous papers [7–10], $f(R)$ gravity models and a non-local Gauss–Bonnet gravity model at the one-loop level in a de Sitter background have been investigated. A similar program for the case of pure Einstein gravity was initiated in [11–13] (see also [14, 15]). Furthermore, such an approach also suggests a possible way of investigating the cosmological constant...
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issue [13]. Hence, the study of one-loop generalized modified gravity is a natural step to be undertaken for the completion of such a program, keeping always in mind, however, that a consistent quantum gravity theory is not yet available.

Making use of generalized zeta function regularization (see, for instance [16–20]), one may evaluate the one-loop effective action and then study the possibility of stabilization of the de Sitter background by quantum effects. Recall that in the one-loop approximation, the theory can be conveniently described by the (Euclidean) one-loop partition function (see [5]). For example, in the simplest case of a scalar field, one obtains

\[ Z = \int D\phi \, e^{-I[\phi] + \int d^d \phi \, L} = e^{-\Gamma(\phi)}. \]  

(1.1)

Here \( I[\phi] \) is the classical action, evaluated on the background field \( \phi_c \), while \( \Gamma \) is the one-loop effective action, which can be related to the determinant of the fluctuation operator \( L \) by

\[ \frac{1}{\Gamma_1} = -\ln Z = I + \frac{1}{2} \ln \det \frac{L}{\mu^2}, \]  

(1.2)

\( \mu^2 \) being a renormalization parameter, which appears for dimensional reasons. Of course, in dealing with gauge theories, one needs a gauge breaking term and the related F–P ghost contribution.

The functional determinant may be formally expressed by

\[ \ln \det \frac{L}{\mu^2} = -\int_0^\infty dt \, t^{-1} \text{Tr} \, e^{-t L/\mu^2}. \]  

(1.3)

Here the heat trace \( \text{Tr} \, e^{-t L} \) plays a preeminent role. In fact, for a second-order, elliptic non-negative differential operator \( L \) in a boundaryless compact \( d \)-dimensional manifold, one has the small-\( t \) asymptotic heat trace expansion

\[ \text{Tr} \, e^{-t L} \sim \sum_{j=0}^\infty A_j(L) t^{-j/2}, \]  

(1.4)

where \( A_j(L) \) are the Seeley–deWitt coefficients [21, 22]. As a result, expression (1.3) is divergent and regularization and renormalization are required. Zeta function regularization may be implemented by [16]

\[ \Gamma(\epsilon) = I - \frac{1}{2} \int_0^\infty dt \, t^{\epsilon-1} \text{Tr} \, e^{-t L/\mu^2} = I - \frac{1}{2\epsilon} \zeta(\epsilon|L/\mu^2), \]  

(1.5)

where the zeta function associated with \( L \) is defined by

\[ \zeta(\epsilon|L) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{Tr} \, e^{-t L}, \quad \zeta(s|L/\mu^2) = \mu^{2s} \zeta(s|L). \]  

(1.6)

For a second-order differential operator in four dimensions, the integral is convergent as soon as \( \text{Re } s > 2 \).

As a consequence, \( \zeta(s|L) \) is regular at the origin and one obtains the well-known result \( \zeta(0|L) = A_2(L) \). This quantity is computable (see, for example, [23]). Furthermore, one may perform a Taylor expansion of the zeta function

\[ \zeta(\epsilon|L) = \zeta(0|L) + \zeta'(0) \epsilon + O(\epsilon^2); \]  

(1.7)

thus,

\[ \Gamma(\epsilon) = I - \frac{1}{2\epsilon} \zeta(0|L) + \frac{\zeta(0|L)}{2} \log \mu^2 + \frac{\zeta'(0|L)}{2} + O(\epsilon). \]  

(1.8)

As a result, one obtains the one-loop divergences as well as finite contributions to the one-loop effective action in terms of the zeta function. With regard to this, a theory is one-loop renormalizable as soon as the divergences can be cancelled in a consistent way by the renormalization of the bare coupling constants present in the classical action \( I \).
In this paper, we shall investigate modified generalized models, described by a Lagrangian density \( F(R, P, Q) \), where \( R \) is the Ricci scalar, and \( P = R_{ij}R^{ij} \) and \( Q = R_{ijkl}R^{ijkl} \) are quadratic curvature invariants. We do not include the Gauss–Bonnet topological invariant because in four dimensions, it can be expressed as \( G = R^2 - 4P + Q \).

After some considerations at the classical level, the main part of this paper will deal with the one-loop evaluation of a particular but interesting \( F(R, P, Q) \) model on the de Sitter space, more exactly on its Euclidean version \( \mathbb{S}(4) \).

This paper ends with an application to the stability of the de Sitter space within the class of the modified gravitational models investigated.

2. Linear perturbation of \( F(R, P, Q) \) model at the classical level

As a warm up exercise, we shall begin with some considerations at the classical level. The equation of motion for general \( F(R, P, Q) \) model can be found in [24] and will not be reported here. In fact, for our purposes, in this section, it will be sufficient to consider only the trace of the equations of motion, which is trivial in Einstein gravity \( R = -\kappa^2 T \), but, for a general \( F(R, P, Q) \) model, reads

\[
\Delta (3f''_R + RF_R^0) + 2\nabla_i \nabla_j \left[ (f''_R + 2RF_R^0)R^{ij} - 2F + RF_R^0 + 2(PF_R^0 + QF_R^0) \right] = \kappa^2 T. \tag{2.1}
\]

Requiring \( R = R_0 \), constant and non-negative, \( P = P_0 \), and \( Q = Q_0 \) constant, one has the de Sitter existence condition in vacuum

\[
[2F - RF_R^0 - 2PF_R^0 - 2QF_R^0]_{R=R_0, P=P_0, Q=Q_0} = 0. \tag{2.2}
\]

As a particular but interesting model, let us make the choice

\[
F(R, P, Q) = f(R) + aP + bQ, \tag{2.3}
\]

namely a generic dependence on \( R \), but only linear in the two quadratic invariants \( P \) and \( Q \). With regard to this choice, as mentioned in the introduction, the full quadratic case

\[
F(R, P, Q) = R - 2\Lambda + cR^2 + aP + bQ, \tag{2.4}
\]

namely an Einstein gravity with cosmological constant with the inclusion of curvature square terms, is of particular interest and this model has been investigated in many papers, and was studied in the seminal paper [25] on flat space. Note that in this particular case, we may take \( b = 0 \), because the quadratic Gauss–Bonnet invariant \( G = R^2 - 4P + Q \) does not contribute to the equations of motion in four dimensions. Another interesting quadratic model is the Einstein plus conformal invariant quadratic term, i.e.

\[
F(R, P, Q) = R - 2\Lambda + \omega \left( \frac{R^2}{3} - 2P + Q \right) = R - 2\Lambda + \omega C_{ijrs}C^{ijrs}, \tag{2.5}
\]

where \( C_{ijrs} \) is the conformal invariant Weyl tensor. In the pure conformal case, one has the Weyl conformal gravity, and this is a quadratic model admitting exact black hole solutions (see, for example, [26–28]).

Within the class of modified models (2.3), the dS existence condition becomes

\[
2f_0 - R_0 f'_0 = 0, \tag{2.6}
\]

and the trace equation in vacuum reads

\[
\Delta (3f''_R + RF_R^0) + 2(a + 2b)\nabla_i \nabla_j R^{ij} - 2f + RF'_R = 0. \tag{2.7}
\]

Making use of the contracted Bianchi identity,

\[
\nabla_i \nabla_j R^{ij} = \frac{1}{2} \Delta R, \tag{2.8}
\]
one has
\[ \Delta (3f' + 2(a + b)R - 2f + Rf') = 0. \] (2.9)
Perturbing around dS space, namely \( R = R_0 + \delta R \), one arrives at the perturbation equation
\[ -\Delta \delta R + M_0^2 \delta R = 0, \] (2.10)
in which the scalar degree of freedom effective mass reads
\[ M_0^2 = \frac{f_0' - R_0 f_0''}{3 f_0' + 2(a + b)}. \] (2.11)
Thus, \( M_0^2 > 0 \) is a necessary condition for the stability of the dS solution. In these \( F(R, P, Q) \) models, besides the massless graviton, there exists also a massive spin-two field, as we shall see in the next section.

In the particular case of \( F(R, P, Q) = f(R) \) models, there is only the scalaron, and one recovers the well-known condition for the dS stability (see, for example, [29–31] and references therein)
\[ \frac{f_0'}{R_0 f_0''} > 1. \] (2.12)

3. Quantum field fluctuations around the maximally symmetric instantons

In this section we will discuss the one-loop quantization of the model on the maximally symmetric space. Of course, this should be considered only an effective approach (see, for instance, [5]). To start with, we consider the Euclidean gravitational model described by the action
\[ I_E[g] = -\int d^4x \sqrt{|g|} F(R, P, Q) = -\int d^4x \sqrt{|g|} [f(R) + aP + bQ], \] (3.1)
with \( a, b \) being dimensionless (bare) parameters, the Newton constant \( G \) being included in the \( f(R) \) contribution. We assume the function \( F(R) \) to satisfy condition (2.2) which ensures the existence of constant curvature solutions. This means that \( f(R) \) is not completely arbitrary, but it has to satisfy the equation
\[ f_0 - R_0 f_0'' = 0, \] (3.2)
where here and in the following for the sake of simplicity, we use the notation \( f_0 = f(R_0) \), \( f_0' = f'(R_0) \) and so on.

We are interested in the dS instanton \( S^4 \) with positive constant scalar curvature \( R_0 \). This is a maximally symmetric space having covariant conserved curvature tensors. Its metric may be written in the form
\[ ds_E^2 = dr^2 (1 - H_0^2 r^2) + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 dS_2^2, \] (3.3)
\( dS^2 \) being the metric of the two-dimensional sphere \( S^2 \). The finite volume reads
\[ V(S^4) = \frac{384\pi^2}{R_0^2}, \quad R_0 = 12H_0^2, \] (3.4)
while Riemann and Ricci tensors are given by
\[ R^{(0)}_{ijrs} = \frac{R_0}{12} (g^{(0)}_{ir} g^{(0)}_{js} - g^{(0)}_{is} g^{(0)}_{jr}), \quad R^{(0)}_{ij} = \frac{R_0}{4} g^{(0)}_{ij}. \] (3.5)
Now let us consider small fluctuations around the maximally symmetric instanton. In the action (3.1) then we set
\[ g_{ij} \rightarrow g_{ij} + h_{ij}, \quad g^{ij} \rightarrow g^{ij} - h^{ij} + h^{ij} \frac{1}{2} + \mathcal{O}(h^3), \quad h = g^{ij} h_{ij}, \] (3.6)
where from now on \( g_{ij} = g_{ij}^{(0)} \) is the metric of the maximally symmetric space and as usual, indices are lowered and raised by the means of such a metric. Up to second order in \( h_{ij} \) one obtains
\[ \sqrt{g} \rightarrow \sqrt{g} \left[ 1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{2} h_{ij} h^{ij} + \mathcal{O}(h^3) \right] \] (3.7)
and
\[ R \sim R_0 - \frac{R_0}{4} h + \nabla_i \nabla_j h^{ij} - \Delta h + \frac{R_0}{4} h^{jk} h_{jk} - \frac{1}{4} \nabla_i h^{ij} h - \frac{1}{4} \nabla_i h_{ij} \nabla^k h^{ij} + \nabla_i h^{ij} \nabla_j h^{ik} - \frac{1}{2} \nabla_j h_{ik} \nabla^l h^{ik}, \] (3.8)
where \( \nabla_i \) represents the covariant derivative in the unperturbed metric \( g_{ij} \). More complicated expressions are obtained for the other invariants \( P, Q \), but for our aim it is not necessary to write them explicitly.

By performing a Taylor expansion of the Lagrangian around the de Sitter metric, up to second order in \( h_{ij} \), we obtain
\[ I_{\mathcal{L}}[g] \sim - \int d^4 x \sqrt{g} \left[ F(R_0, P_0, Q_0) + \frac{hX}{2} + \mathcal{L}_2 \right], \] (3.9)
where \( \mathcal{L}_2 \) represents the second-order contribution and \( X = f_0 - R_0 f_0'/2 \) vanishes when \( f(R) \) satisfies the de Sitter existence solution (3.2).

It is convenient to carry out the standard expansion of the tensor field \( h_{ij} \) in irreducible components [13], namely
\[ h_{ij} = \tilde{h}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{2} g_{ij}(h - \Delta_0 \sigma), \] (3.10)
where \( \sigma \) is the scalar component, while \( \xi_i \) and \( \tilde{h}_{ij} \) are the vector and tensor components with the properties
\[ \nabla_i \xi^i = 0, \quad \nabla_i \tilde{h}^{ij} = 0, \quad \tilde{h}^i_i = 0. \] (3.11)

In terms of the irreducible components of the \( h_{ij} \) field, the Lagrangian density, disregarding total derivatives, becomes
\[ \mathcal{L}_2 = \mathcal{L}_{hh} + 2 \mathcal{L}_{hr} + \mathcal{L}_{\sigma \sigma} + \mathcal{L}_V + \mathcal{L}_T, \] (3.12)
where \( \mathcal{L}_{hh} \) and \( \mathcal{L}_{hr} \) represent the scalar contribution (a \( 2 \times 2 \) matrix), while \( \mathcal{L}_V \) and \( \mathcal{L}_T \) represent the vector and tensor contributions, respectively. One obtains
\[ \mathcal{L}_{hh} = h \left[ \frac{9 f_0'' \Delta^2}{32} - \frac{3 f_0' \Delta}{32} + \frac{a R_0 \Delta}{16} + \frac{b R_0 \Delta}{16} + \frac{3 f_0'' R_0 \Delta}{16} + \frac{f_0'' R_0^2}{32} - \frac{f_0' R_0}{32} \right] \] (3.13)
\[ \mathcal{L}_{hr} = h \left[ - \frac{9 f_0'' \Delta^3}{32} + \frac{3 f_0' \Delta^2}{32} - \frac{3}{16} f_0' R_0 \Delta^2 - \frac{1}{32} f_0'' R_0^2 \Delta + \frac{f_0' R_0}{32} \right] - \frac{3 a \Delta^3}{16} + \frac{3 b \Delta^3}{16} - \frac{1}{16} a R_0 \Delta^2 - \frac{1}{16} b R_0 \Delta^2 \right] \sigma, \] (3.14)
\[ \mathcal{L}_{\sigma} = \sigma \left[ \frac{9 f_0^4 \Delta^4}{32} - \frac{3 f_0^2 \Delta^3}{32} + \frac{3}{16} f_0^2 R_0 \Delta^2 + \frac{1}{32} f_0^2 R_0^2 \Delta^2 - \frac{1}{32} f_0^2 R_0 \Delta^2 \right] + \frac{3 \chi^2}{16} \frac{\Delta^2}{R_0 \chi^2} + \frac{3 \alpha \Delta^4}{16} + \frac{1}{16} a R_0 \Delta^3 + \frac{1}{16} b R_0 \Delta^3 \right] \sigma, \]  

(3.15)

\[ \mathcal{L}_V = \xi^k \left[ \frac{1}{8} X R_0 + \frac{1}{2} X \Delta \right] \xi_k. \]  

(3.16)

\[ \mathcal{L}_F = \hat{h}^{ij} \left[ + \frac{f_0 \Delta}{4} - \frac{f_0 R_0}{24} - \frac{X \alpha \Delta^2}{4} + \frac{b \Delta^2}{4} + \frac{a R_0 \Delta}{24} - \frac{b R_0 \Delta}{3} - \frac{a R_0^2}{72} + \frac{b R_0^2}{36} \right] \hat{h}_{ij}, \]  

(3.17)

where \( \Delta = g^{ij} \nabla_i \nabla_j \) is the Laplace–Beltrami operator in the unperturbed metric \( g_{ij} \), which is a solution of field equations, but only if \( X = 0 \). We have written the above expansions around a maximally symmetric space which in principle could not be a solution. This means that the function \( f(R) \) can be arbitrary.

As is well known, invariance under diffeomorphisms renders the operator in the \((h, \sigma)\) sector not invertible. One needs a gauge fixing term and a corresponding ghost compensating term. Here we choose the harmonic gauge, that is,

\[ \chi_j = - \nabla_j h' - \frac{1}{2} \nabla_j h = 0, \]  

(3.18)

and the gauge fixing term

\[ \mathcal{L}_{\text{gf}} = \frac{1}{2} \chi^k G_{ij} \chi^j, \quad G_{ij} = \gamma g_{ij}. \]  

(3.19)

The corresponding ghost Lagrangian reads [5]

\[ \mathcal{L}_{\text{gh}} = B^k G_{ik} \frac{\delta \chi^k}{\delta \epsilon^i} C_i, \]  

(3.20)

where \( C_i \) and \( B_k \) are the ghost and anti-ghost vector fields, respectively, while \( \delta \chi^k \) is the variation of the gauge condition due to an infinitesimal gauge transformation of the field. In this case it reads

\[ \delta h_{ij} = \nabla_i \epsilon_j + \nabla_j \epsilon_i \implies \frac{\delta \chi^i}{\delta \epsilon^i} = g_{ij} \Delta + R_{ij}. \]  

(3.21)

Neglecting total derivatives, one has

\[ \mathcal{L}_{\text{gh}} = B^k \gamma \left( \Delta + \frac{R_0}{4} \right) C_i. \]  

(3.22)

In irreducible components one finally obtains

\[ \mathcal{L}_{\text{gf}} = \frac{\gamma}{2} \left[ \left( \Delta + \frac{R_0}{4} \right)^2 \hat{C}_k + \frac{3 \rho}{8} h \left( \Delta_0 + \frac{R_0}{3} \right) \Delta_0 \sigma \right] \]  

\[ - \frac{\rho}{16} h \Delta_0 h - \frac{R_0}{16} \sigma \left( \Delta_0 + \frac{R_0}{3} \right)^2 \Delta_0 \sigma \right] \]  

(3.23)

\[ \mathcal{L}_{\text{gh}} = \gamma \left[ \frac{1}{2} B_0 \left( \Delta + \frac{R_0}{4} \right) \hat{C}_k + \frac{\rho - 3}{2} b \left( \Delta_0 - \frac{R_0}{\rho - 3} \right) \Delta_0 \right] \]  

(3.24)

where ghost irreducible components are defined by

\[ C_k = \hat{C}_k + \nabla_k \hat{\epsilon}, \quad \nabla_k \hat{\epsilon} = 0, \]  

(3.25)

\[ B_k = \hat{B}_k + \nabla_k \hat{b}, \quad \nabla_k \hat{b} = 0. \]
4. One-loop effective action

In order to compute the one-loop contributions to the effective action, one has to consider the path integral for the bilinear part $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{\phi f} + \mathcal{L}_{\phi h}$ of the total Lagrangian and take into account the Jacobian due to the change of variables with respect to the original ones. In this way one obtains [13, 5]

$$Z^{(1)} = \left( \det G_{ij} \right)^{-1/2} \int D[h_{ij}] D[G] D[B^i] \exp \left( -\int d^4x \sqrt{g} \mathcal{L} \right)$$

$$= \left( \det G_{ij} \right)^{-1/2} \det J^{-1} \det J^{1/2} \int D[h] D[\xi] D[\phi] D[\phi^i] D[B^i] D[c] D[h] \exp \left( -\int d^4x \sqrt{g} \mathcal{L} \right),$$

(4.1)

where $J_1$ and $J_2$ are the Jacobians due to the change of variables in the ghost and tensor sectors, respectively [13]. They read

$$J_1 = \Delta_0, \quad J_2 = \left( \Delta_0 + \frac{R_0}{3} \right) \Delta_0,$$

(4.2)

and the determinant of the operator $G_{ij}$ in this case is trivial. Due to the presence of curvature, the Euclidean gravitational action is not bounded from below, because arbitrary negative contributions can be induced on $\mathcal{R}$ by conformal rescaling of the metric. For this reason, we have also used the Hawking prescription of integrating over imaginary scalar fields. Furthermore, the problem of the presence of additional zero modes introduced by the decomposition (3.10) can be treated making use of the method presented in [13].

Now, a straightforward computation leads to the following off-shell one-loop contribution to the ‘partition function’

$$e^{-\Gamma^{(1)}} \equiv Z^{(1)} = \det \left[ \left( \Delta_1 + \frac{R_0}{3} \right) \left( \Delta_0 + \frac{R_0}{2} \right) \left( \Delta_1 + \frac{R_0}{4} \right) \right]^{-1/2} \det \left[ \left( \Delta_0 + \frac{R_0}{2} \right) \left( \Delta_1 + \frac{R_0}{4} \right) \right],$$

(4.3)

where

$$L_0^+ = \Delta_0 - \frac{2f_0 - 5R_0 f_0'' - 2R_0(a + b)}{4(3f_0'' + 2(a + b))} \pm \sqrt{\left[2f_0 - 5R_0 f_0'' - 2R_0(a + b)\right]^2 - 8[3f_0'' + 2(a + b)][2X - R_0 f_0'(f_0 - R_0 f_0')]}$$

$$\times \frac{4(3f_0'' + 2(a + b))}{4(a + b)},$$

$$L_2^+ = \Delta_2 + \frac{6f_0' + R_0(a - 8b)}{12(a + 4b)} \pm \sqrt{\left(2f_0' + aR_0\right)^2 + 16X(a + 4b)}$$

$$\times \frac{4(a + 4b)}{4(a + 4b)},$$

$\Delta_0, \Delta_1, \Delta_2$ being, respectively, the Laplacian operators acting on scalars, transverse vectors and transverse, traceless tensors and of course $a + 4b \neq 0$.

The partition function in (4.3) explicitly depends on the gauge parameter $\gamma$, but it is known that when one goes ‘on-shell’, that is, when one imposes the background metric $g_{ij}$ to be a solution of the field equation, the one-loop partition function becomes gauge independent. In our case we have simply to perform the limit $X \rightarrow 0$ obtaining

$$Z_{\text{on-shell}}^{(1)} = \left\{ \det \left[ -\Delta_0 + \frac{f_0 - R_0 f_0'}{3f_0'' + 2(a + b)} \right] \right\}^{-1/2} \left\{ \det \left( -\Delta_1 - \frac{R_0}{4} \right) \right\}^{1/2} \times \left\{ \det \left[ -\Delta_2 + \frac{R_0}{6} \right] \left( -\Delta_2 + \frac{(2b - a)R_0 - 3f_0'}{3(a + 4b)} \right) \right\}^{-1/2}.$$  

(4.4)
As a consequence, the on-shell one-loop effective action reads
\[ \Gamma_{\text{on-shell}}^{(1)} = \frac{1}{2} \log \det \left( \frac{1}{\mu^2} \left[ -\Delta_0 + \frac{f_0' - R_0 f''}{3 f_0'' + 2(a + b)} \right] \right) \]
\[ - \frac{1}{2} \log \det \left( \frac{1}{\mu^2} \left[ -\Delta_1 - \frac{R_0}{4} \right] \right) \]
\[ + \frac{1}{2} \log \det \left( \frac{1}{\mu^2} \left[ -\Delta_2 + \frac{R_0}{6} \right] \right) \]
\[ + \frac{1}{2} \log \det \left( \frac{1}{\mu^2} \left[ -\Delta_2 + \frac{(2b - a)R_0 - 3f_0''}{3(a + 4b)} \right] \right). \] (4.5)

As usual an arbitrary renormalization parameter $1/\mu^2$ has been introduced for dimensional reasons.

The one-loop contribution $\Gamma^{(1)}$ to the effective action can be computed by using zeta function techniques. The eigenvalues of Laplacian operators on $S^3$ are explicitly known and so the determinant of all operators appearing in (4.5) in principle can be calculated. We refer the interested reader to [7], where all details of computation can be found.

5. Discussion and conclusions

We conclude this paper with several remarks. First, the one-loop effective action result is in agreement with a similar one given in [9] and in the limit $a \to 0, b \to 0$ becomes identical to those in [7], where only $f(R)$ modified gravity has been considered.

Equations (4.3) and (4.4) have been derived by assuming $a + 4b \neq 0$. As a useful check, we note that when $a = -4b$, the original classical Lagrangian density can be written in the form $f(R) + aP + bQ = \tilde{f}(R) + bG$, $\tilde{f}(R) = f(R) - bR^2$, $G = R^2 - 4P + Q$, where $G$ is the Gauss–Bonnet topological invariant which does not contribute to the field equations. In such a case, as expected, the one-loop contribution to the partition function becomes
\[ \tilde{Z}_{\text{on-shell}}^{(1)} = \left\{ \det \left[ -\Delta_0 + \frac{\tilde{f}_0' - R_0 \tilde{f}''}{3 \tilde{f}_0'' + 2(\omega + \tilde{b})} \right] \right\}^{-1/2} \]
\[ \times \left\{ \det \left[ -\Delta_1 - \frac{R_0}{4} \right] \right\}^{1/2} \left\{ \det \left[ -\Delta_2 + \frac{R_0}{6} \right] \right\}^{-1/2}, \]
which again is the result obtained for a pure $\tilde{f}(R)$ modified gravity [7].

Another interesting particular case is the one in which the Einstein–Hilbert Lagrangian density is modified by a term proportional to the square of the Weyl tensor. Such a classical model has a de Sitter solution and in the absence of Einstein–Hilbert term, the on-shell, one-loop contribution to the partition function trivially vanishes, but if $a = -4b, b = \omega$ and, as in (2.5)
\[ f(R) + aP + bQ = \tilde{f}(R) + \omega \left( \frac{R^2}{3} - 2P + Q \right), \quad \tilde{f}(R) = \frac{R}{16\pi G_N} = M_{\tilde{P}}^2 \tilde{R}, \]
$G_N, M_{\tilde{P}}$ being, respectively, the Newton constant and Planck mass, the on-shell one-loop contribution reads
\[ \tilde{Z}_{\text{on-shell}}^{(1)} = \left\{ \det \left[ -\Delta_1 - \frac{R_0}{4} \right] \right\}^{1/2} \left\{ \det \left[ \Delta_2 + \frac{R_0}{6} \right] \left( -\Delta_2 + \frac{R_0}{3} - \frac{\tilde{f}_0''}{2\omega} \right) \right\}^{-1/2}. \]
In contrast with previous cases, here the contribution due to scalar components is vanishing.
As an important application we discuss the stability of de Sitter space. To this aim we have to recall that the eigenvalues \( \lambda_n \) of Laplacian–Beltrami \(-\Delta\) operators on \( S^4 \) have the form

\[
\lambda_n = \frac{R_0}{12} (n + 1)^2 - \alpha, \\
g_n = c_1 (n + 1) + c_3 (n + 1)^3, \quad n = 0, 1, 2, \ldots
\]

\( g_n \) being the corresponding degeneracy and \( v, c_1, c_3 \) being dimensionless quantities, which depend on the operator one is dealing with. In particular one has

\[
\begin{array}{l}
- \Delta_0 \implies v = \frac{3}{2}, \quad \alpha = \frac{9}{4}, \quad c_1 = -\frac{1}{4}, \quad c_3 = \frac{1}{4}, \\
- \Delta_1 \implies v = \frac{5}{2}, \quad \alpha = \frac{13}{4}, \quad c_1 = -\frac{9}{4}, \quad c_3 = 1, \\
- \Delta_2 \implies v = \frac{7}{2}, \quad \alpha = \frac{17}{4}, \quad c_1 = -\frac{125}{17}, \quad c_3 = \frac{5}{7}.
\end{array}
\]

We see that only the scalar Laplacian \(-\Delta_0\) has a null eigenvalue, while the minimum eigenvalue of \(-\Delta_1\) is \( R_0/4 \) and the minimum eigenvalue of \(-\Delta_2\) is \( 2R_0/3 \).

In equation (4.5) we are dealing with operators of the kind \( L = -\Delta + M^2 \) and so, in order to have stability of the de Sitter solution, we have to assume all eigenvalues of \( L \) to be positive. In this way we obtain restrictions on the function \( F(R, P, Q) \).

Looking at (4.5) we see that, independently on the classical action, a zero mode is present coming from the Laplacian-like operator \(-\Delta_1 - R_0/4\). In principle, other zero modes may be present and all of these can be treated according to [13].

The other operators in (4.5) which could have vanishing or negative eigenvalues are

\[
\begin{align*}
L_0 &= -\Delta_0 + M^2_0, \\
M^2_0 &= \frac{f_0^2 - R_0 f_0''}{3 f_0' + 2 (a + b)}, \\
L_2 &= -\Delta_2 + M^2_2, \\
M^2_2 &= \frac{R_0 (2b - a) - 3 f_0'}{3 (a + 4b)},
\end{align*}
\]

but in the case in which

\[
\begin{align*}
M^2_0 > 0 \quad &\implies \frac{f_0'' - R_0 f_0''}{3 f_0' + 2 (a + b)} > 0, \\
M^2_2 + \frac{2}{3} R_0 > 0 \quad &\implies \frac{(a + 10b) R_0 - 3 f_0' }{3 (a + 4b)} > 0.
\end{align*}
\]

In the particular cases in which \( M^2_0 = 0 \) and/or \( M^2_2 = -2/3R_0 \) there are other zero modes which have to be treated as the previous ones.

In the interesting case \( f(R) = M^2_F (R - 2\Lambda) \), linear in the curvature, the dS stability conditions become

\[
\begin{align*}
M^2_0 &= \frac{M^2_F}{2 (a + b)} > 0, \\
M^2_2 &= \frac{4 \Lambda (a + 10b) - 3 M^2_F}{3 (a + 4b)} > 0, \quad \implies \begin{cases} 
& a + b > 0, \\
& a + 4b > 0, \\
& a + 10b > \frac{3 M^2_F}{4 \Lambda} \end{cases}, \quad \implies \begin{cases} 
& a + b > 0, \\
& a + 4b > 0, \\
& a + 10b > \frac{3 M^2_F}{4 \Lambda}.
\end{align*}
\]

We see that depending on the arbitrary parameters \( a, b \), the solution can be stable or unstable and in order to have a stable solution at least one of the two parameters has to be positive. In particular, in the special cases \( a = 0 \) or \( b = 0 \) one obtains the stability conditions

\[
\begin{align*}
a > \frac{3 M^2_F}{4 \Lambda} > 0, & \quad b = 0, \\
b > \frac{3 M^2_F}{40 \Lambda} > 0, & \quad a = 0.
\end{align*}
\]
In summary, here we have evaluated the one-loop effective action for a specific modified gravity model in de Sitter space. Generalized zeta regularization could be used to obtain a finite answer for the functional determinants in the effective action, what has proven to be a very convenient procedure.

The important lesson to be drawn from this calculation, generalizing the previous program for one-loop Einstein gravity and $f(R)$ modified gravity in the de Sitter background, is that quantum corrections may destabilize the classical de Sitter universe, as we have explicitly verified in the examples.

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