Fast Algorithms for Exact String Matching  
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Abstract

Given a pattern string $P$ of length $n$ and a query string $T$ of length $m$, where the characters of $P$ and $T$ are drawn from an alphabet of size $\Delta$, the exact string matching problem consists of finding all occurrences of $P$ in $T$. For this problem, we present algorithms that in $O(n\Delta^2)$ time preprocess $P$ to essentially identify $\text{sparse}(P)$, a rarely occurring substring of $P$, and then use it to find occurrences of $P$ in $T$ efficiently. Our algorithms require a worst case search time of $O(m)$, and expected search time of $O(m/\min(|\text{sparse}(P)|, \Delta))$, where $|\text{sparse}(P)|$ is at least $\delta$ (i.e. the number of distinct characters in $P$), and for most pattern strings it is observed to be $\Omega(n^{1/2})$.

Key words: Exact String Matching; Combinatorial Pattern Matching; Computational Biology; Bio-informatics; Analysis of Algorithms; Fast Heuristics.

1 Introduction

Given a pattern string $P$ of length $n$ and a query string $T$ of length $m$, where the characters of $P$ and $T$ are drawn from an alphabet of size $\Delta$, the exact string matching problem consists of finding all occurrences of $P$ in $T$. This is a fundamental problem with wide range of applications in Computer Science (used in parsers, word processors, operating systems, web search engines, image processing and natural language processing), Bioinformatics and Computational Biology (Sequence Alignment and Database Searches). The algorithms for exact string matching can be broadly categorized into the following categories: (1) character based comparison algorithms, (2) automata based algorithms, (3) algorithms based on bit-parallelism and (4) constant-space algorithms. In this paper, our focus is on designing efficient character based comparison algorithms for exact string matching. For a comprehensive survey of all categories of exact string matching algorithms, we refer the readers to Baeza-Yates[17], Gusfield[25], Charras et al [28], Chochemore et al [29] and Faro et al [30].

A Typical character based comparison algorithm can be described within the following general framework as follows:

(1) First, initialize the search window to be the first $n$ characters of the query string $T$ (i.e. align the $n$ characters of the pattern string $P$ with the first $n$ characters of $T$).

(2) Repeat the following until the search window is no longer contained within the query string $T$:

inspect the aligned pairs in some order until there is either a mis-match in an aligned pair or there is a complete match among all the $n$ aligned pairs. Then shift the search window to the right. The order in which the aligned pairs are inspected
and the length by which the search window is shifted differs from one algorithm to another.

The mechanism that the above framework provides is usually referred to as the *sliding window mechanism* [30, 31]. The algorithms that employ the sliding window mechanism can be further classified based on the order in which they inspect the aligned pairs into the following broad categories: (1) left to right scan; (2) right to left scan; (3) scan in specific order, and (4) scan in random order or scan order is not relevant. The algorithms that inspect the aligned pairs from left to right are the most natural algorithms; the algorithms that inspect the aligned pairs from right to left generally perform well in practice; the algorithms that inspect the aligned pairs in a specific order yield the best theoretical bounds. For a comprehensive description of the exact string matching algorithms and access to an excellent framework for development, testing and analysis of exact string matching algorithms, we refer the readers to the *smart tool* (string matching research tool) of Faro and T. Lecroq[31].

For algorithms that inspect aligned pairs from left to right, Morris and Pratt [1] proposed the first known linear time algorithm. This algorithm was improved by Knuth, Morris and Pratt [4] and requires $O(n)$ preprocessing time and a worst case search time of at most $2m - 1$ comparisons. For small pattern strings and reasonable probabilistic assumptions about the distribution of characters in the query string, hashing [2,9] provides an $O(n)$ preprocessing time and $O(m)$ worst case search time solution. For pattern strings that fit within a word of main memory, Shift-Or [16,21] requires $O(n + \Delta)$ preprocessing time and a search time of $O(m)$ and can also be easily adapted to solve approximate string matching problems. For algorithms that inspects aligned pairs from right to left, the Boyer-Moore [3] algorithm is one of the classic algorithms that requires $O(n + \Delta)$ preprocessing time and a worst case search time of $O(nm)$ but in practice is very fast. There are several variants that simplify the Boyer-Moore algorithm and mostly avoid its quadratic behaviour. Among the variants of Boyer-Moore, the algorithms of Apostolico and Giancarlo [7,24], Crochemore et al [13, 23] (Turbo BM), and Colussi (Reverse Colussi) [12, 22] have $O(m)$ worst case search time and are efficient in minimizing the number of character comparisons, whereas the Quick Search [10], Reverse Factor [19], Turbo Reverse Factor [24], Zhu and Takaoka [8] and Berry-Ravindran [27] algorithms are very efficient in practice. For Algorithms that inspects the aligned pairs in a specific order, Two Way algorithm [13], Colussi [12], Optimal Mismatch and Maximal Shift [10], Galil-Giancarlo [18], Skip Search , KMP Skip Search and Alpha Skip Search [26] are some of the well known algorithms. Two way algorithm was the first known linear time optimal space algorithm. The Colussi algorithm improves the Knuth-Morris-Pratt algorithm and requires at most $3/2n$ text character comparisons
in the worst case. The Galil-Giancarlo algorithm improves the Colussi algorithm in one special case which enables it to perform at most \(4/3n\) text character comparisons in the worst case. For Algorithms that inspects the aligned pairs in any order, the Horspool [5], Quick Search [10], Tuned Boyer-Moore [14], Smith [15] and Raita [20] algorithms are some of the well known algorithms. All these algorithms have worst case search time that is quadratic but are known to perform well in practice.

Our Results: In this paper, we present two similar algorithms \(A\) and \(B\) for exact string matching. Both these algorithms employ sliding window mechanism, preprocess \(P\) in \(O(n\Delta^2)\) time to essentially identify \(\text{sparse}(P)\), a rarely occurring substring of \(P\) characterized by two characters of \(P\), and then use it to find occurrences of \(P\) in \(T\) efficiently. Algorithms \(A\) and \(B\) have worst case search times of \(O(m)\) and \(O(mn)\) respectively. However, both of them have an expected search time of \(O(m/\min(|\text{sparse}(P)|, \Delta))\). The main difference between these two algorithms is that Algorithm \(A\) inspects the aligned pairs in the search window in the order specified by Apostolico-Giancarlo’s [7] Algorithm, whereas Algorithm \(B\) inspects the aligned pairs in random order. This makes algorithm \(B\) much simpler than \(A\) and equally effective in practice. In terms of preliminary empirical analysis, for most pattern strings \(P\), we observe that \(|\text{sparse}(P)|\) is \(\Omega(n^{1/2})\). We also believe that a tighter analysis of our algorithms can result in sub-linear worst case run-time from the perspective of randomized analysis.

The rest of this paper is organized as follows: In Section 2, we present our Algorithms \(A\) and \(B\). In Section 3, we present the analysis of these algorithms, and in Section 4 we present our conclusions and future work.

2 Algorithms for Exact String Matching

In this section, we present two similar Algorithms \(A\) and \(B\), that given a pattern string \(P\) and a query string \(T\), finds all occurrences of \(P\) within \(T\). First, we introduce some definitions that are essential for defining our Algorithms \(A\) and \(B\). Then, we present Algorithms \(A\) and \(B\).

Definitions 2.1 Given a pattern string \(P\) of length \(n\) and a query string \(T\) of length \(m\), we define \(N_i(P)\), \(i \in [1..n]\), denote the length of the longest suffix of \(P[1..i]\) that matches a suffix of \(P\), and \(M\) to be a \(m\) length vector whose \(j\)th entry \(M[j] = k\) indicates that a suffix of \(P\) of length at least \(k\) occurs in \(T\) and ends at position \(j\). (See Apostolico-Giancarlo Algorithm [7]).

Definitions 2.2 Given a pattern string \(P\), and an ordered pair of characters \(u, v \in \Sigma\) (not necessarily distinct), we define \(\text{sparse}^{(u,v)}(P)\), the 2-sparse pattern of \(P\) with respect to \(u\) and \(v\), to be the
rightmost occurrence of a substring of $P$ of longest length that starts with $u$, ends with $v$, but does not contain $u$ or $v$ within it. We define $\text{sparse}(P)$ to be the longest among the $2$-sparse patterns of $P$.

**Definitions 2.3** Given $\text{sparse}(P)$, the longest $2$-sparse pattern of $P$, we define $\text{startc}(P)$ and $\text{endc}(P)$ to be the respective first and last characters of $\text{sparse}(P)$, and $\text{startpos}(P)$ and $\text{endpos}(P)$ be the respective indices of the first and last characters of $\text{sparse}(P)$ in $P$. For $c \in \Sigma$, if $c \in \text{sparse}(P)$, $\text{shift}^c(P)$ is the distance between the rightmost occurrence of $c$ in $\text{sparse}(P)$ and the last character of $\text{sparse}(P)$. If $c$ is not present in $P$ then $\text{shift}^c(P)$ is set to $n$, the length of $P$. If $c$ is present in $P$ but not in $\text{sparse}(P)$ then $\text{shift}^c(P)$ is set to $|\text{sparse}(P)| + 1$.

**BASIC IDEA**: First, we preprocess $P$ to identify $\text{sparse}(P)$, a rarely occurring substring of $P$ characterized by two characters in $P$, and compute statistics of its occurrence relative to other characters in $P$. Then, during the search phase, we set the search window to be the first $n$ characters of $T$ (i.e. align the $n$ characters of $P$ to the first $n$ characters of $T$). Then, we do the following repeatedly until the search window reaches the end of $T$:

- Check whether there is a match between the first and last characters of $\text{sparse}(P)$ and their respective aligned characters in the search window. In the case of a match, Algorithm $A$ (Algorithm $B$) invokes Apostolico – Giancarlo Algorithm (Random – Match) to look for exact match between $P$ and the characters in the search window. If there is an exact match then it reports the match, shifts the search window by at least the length of $\text{sparse}(P)$ and then continues. However, if there is a mismatch in either the first or last character of $\text{sparse}(P)$ or during the invocation of Apostolico-Giancarlo (Random-Match), then it shifts the search window based on the statistics of $\text{sparse}(P)$’s occurrence relative to the mismatched character and then continues.

**DESCRIPTION OF ALGORITHM $A$**: We first describe the preprocessing and search phases of Algorithm $A$. Then, we present Algorithm $A$ formally.

**Preprocessing Phase**: In the pre-processing phase, we first compute $N_i(P)$, for $i \in [1..n]$, where $N_i(P)$ is the longest suffix of $P[1..i]$ that matches a suffix of $P$. Second, for each character $c \in \Delta$, we determine the list of indices in $P$ where it occurs, and from this we determine $\text{sparse}(P)$, the longest $2$-sparse pattern of $P$, as follows:

- For each ordered pair of characters $a, b \in P$ (not necessarily distinct), determine $\text{sparse}^{(a,b)}(P)$, the $2$-sparse pattern with respect to $a$ and $b$;
(b) Choose the longest among the 2-sparse patterns determined in (a).

Third, from \(\text{sparse}(P)\), we determine \(\text{startc}(P)\) and \(\text{endc}(P)\), the respective first and last characters of \(\text{sparse}(P)\), and \(\text{startpos}(P)\) and \(\text{endpos}(P)\), the respective indices within \(P\) of the first and last characters of \(\text{sparse}(P)\). Finally, for \(c \in \Sigma\), determine \(\text{shift}^c(P)\), the distance between the rightmost occurrence of \(c\) in \(\text{sparse}(P)\) and the last character of \(\text{sparse}(P)\).

**Search Phase**: In the search phase, we first set the search window to be the first \(n\) characters of \(T\) (i.e. align the \(n\) characters of \(P\) to the first \(n\) characters of \(T\)). Then, while the search window is contained within the query string \(T\), compare \(\text{endc}(P)\) and \(\text{startc}(P)\) (i.e the last and first characters of \(\text{sparse}(P)\)) with the respective characters at offset \(\text{endpos}(P)\) and \(\text{startpos}(P)\) within the search window. The following three disjoint scenarios (events) are possible:

(i) [Type-1 event] If the character \(c\) at offset \(\text{endpos}(P)\) within the search window is not \(\text{endc}(P)\) then we shift the search window to the right by \(\text{shift}^c(P)\);

(ii) [Type-2 event] If the character \(c\) at offset \(\text{endpos}(P)\) within the search window is \(\text{endc}(P)\) but the character \(d\) at offset \(\text{startpos}(P)\) is not \(\text{startc}(P)\) then we shift the search window to the right by \(|\text{sparse}(P)| + 1(|\text{sparse}(P)|)\) when \(c\) and \(d\) are different characters (\(c\) and \(d\) are the same characters).

(iii) [Type-3 event] If the character at offset \(\text{startpos}(P)\) within the search window is \(\text{startc}(P)\) and at offset \(\text{endpos}(P)\) is \(\text{endc}(P)\) then we use Apostolico-GianCarlos Algorithm to look for an occurrence of \(P\) within the search window in \(T\) and then shift the search window to the right by \(|\text{sparse}(P)| + 1(|\text{sparse}(P)|)\) when \(c\) and \(d\) are different characters (\(c\) and \(d\) are the same characters).

**ALGORITHM A**

**Input(s)**: (1) Pattern string \(P\) of length \(n\);
(2) Query string \(T\) of length \(m\);

**Output(s)**: The starting positions of the occurrences of \(P\) in \(T\);

**Preprocessing**: (1) For \(i \in [1..n]\), compute \(N_i(P)\) using the Z Algorithm [25].
(2) For \(c \in \Delta\), determine \(\text{List}^c(P)\), the list of positions in \(P\) where \(c\) occurs.
   Then from these lists, compute
   [a] for each \(u, v \in \Delta\), \(\text{sparse}^{(u,v)}(P)\);
   [b] \(\text{sparse}(P) = |\text{sparse}^{(u,v)}(P)| = \max_{u, v \in \Delta} |\text{sparse}^{(u,v)}(P)|\).
(3) From \(\text{sparse}(P)\), compute \(\text{startc}(P)\), \(\text{endc}(P)\), \(\text{startpos}(P)\), and \(\text{endpos}(P)\).
(4) For \(c \in \Sigma\), compute \(\text{shift}^c(P)\).
Search:

[1] [a] Set $i = 0$ and $j = n$ [Search Window set to $[1..n]$]

[b] Set $\hat{j} = \text{endpos}(P)$ and $\hat{i} = \text{startpos}(P)$; [Indices of last and first characters of $\text{sparse}(P)$ in $P$]

[2] while $(j < m)$ [While the search window is contained in $T$]

[a] Let $c = T[i + \hat{j}]$; $d = T[i + \hat{i}]$; [Characters in search window aligned with the last and first characters of $\text{sparse}(P)$]

[i] if ($c \neq \text{endc}(P)$) [Type-1 event]

\[ i = i + \text{shift}^c(P) ; \ j = j + \text{shift}^c(P) \] [Shift search window by $\text{shift}^c(P)$]

[ii] elseif ($d \neq \text{startc}(P)$) [Type-2 event]

\[ \text{if} (\text{startc}(P) == \text{endc}(P)) \]
\[ i = i + |\text{sparse}(P)|; \ j = j + |\text{sparse}(P)| \] [Shift search window by $|\text{sparse}(P)|$]

[else]
\[ i = i + |\text{sparse}(P)| + 1; \ j = j + |\text{sparse}(P)| + 1 \] [Shift search window by $|\text{sparse}(P)| + 1$]

[iii]else [Type-3 event]

Call $\text{Apostolico – GianCarlo}(T, P, i, j)$; [Look for $P$ in $T[i + 1..j]$]

\[ \text{if} (\text{startc}(P) == \text{endc}(P)) \]
\[ i = i + |\text{sparse}(P)|; \ j = j + |\text{sparse}(P)| \] [Shift search window by $|\text{sparse}(P)|$]

[else]
\[ i = i + |\text{sparse}(P)| + 1; \ j = j + |\text{sparse}(P)| + 1 \] [Shift search window by $|\text{sparse}(P)| + 1$]

ALGORITHM B

We now define Algorithm B by making the following simple modification to Step [2][a][iii] of the search phase of Algorithm A:

Replace the statement ”Call $\text{Apostolico – GianCarlo}(T, P, i, j)$” by the statement ”Call $\text{Random – Match}(T, P, i, j)$”.

The $\text{Apostolico – Giancarlo}$ Algorithm determines an exact match between $P$ and $n$ characters in the search window by inspecting the aligned pairs in a specific order until it encounters a mismatch or finds a match in all $n$ characters. However, $\text{Random – Match}$ inspects the aligned pairs in a random order until it encounters a mismatch or finds a match in all $n$ characters.

3 Analysis of Algorithms A and B

In this section, we present the analysis of Algorithms A and B. We now present the main results in this paper. The proofs follow.
Theorem 1  Given any pattern string $P$ of length $n$ and a query string $T$ of length $m$, Algorithm $A$ finds all occurrences of $P$ in $T$ is $O(m)$ time.

Theorem 2  Given any pattern string $P$ of length $n$ and a query string $T$ of length $m$ where each character is drawn uniformly at random, Algorithms $A$ and $B$ find all occurrences of $P$ in $T$ in $O(m/\min(|\text{Sparse}(P)|, \Delta))$ expected time, where $|\text{sparse}(P)|$ is atleast $\delta$ (i.e the number of distinct characters in $P$).

Proof of Theorem 1: The Algorithm $A$ during its search phase searches for $P$ by essentially looking for a match for the first and last characters of $\text{Sparse}(P)$, a 2-sparse pattern of $P$, with the characters in the search window at offsets $\text{startpos}(P)$ and $\text{endpos}(P)$ respectively. Let $c = \text{endc}(P)$ and $d = \text{startc}(P)$ be the last and first characters of $\text{Sparse}(P)$. The following three scenarios/events are possible. (i) Type-1 event happens if there is a mis-match between $c$ and the character at offset $\text{endpos}(P)$ within the search window; (ii) Type-2 event happens if there is a match between $c$ and the character at offset $\text{endpos}(P)$ within the search window and a mis-match between $d$ and the character at offset $\text{startpos}(P)$ within the search window, and (iii) Type-3 event happens when there is a match between $c$ and the character at offset $\text{endpos}(P)$ within the search window and a match between $d$ and the character at offset $\text{startpos}(P)$ within the search window.

In the case of Type-1 event, there is exactly one character in $T$ that is looked at and the search window is shifted by $\text{shift}^c(P) \geq 1$. In the case of Type-2 event, there are two characters in $T$ that are looked at and the search window is shifted by $|\text{sparse}(P)| (|\text{sparse}(P)| + 1)$ when $c = d$ ($c \neq d$). In the case of type 3 event, we call the Apostolico-Giancarlo Algorithm where the query string is the $n$ characters in the current search window. However, we maintain the $N$ and $M$ vectors as global variables so that when we repeatedly invoke the Apostolico-Giancarlo Algorithm the M values computed for any particular position of $T$ during any invocation is available without recomputation for future invocations. This ensures that the total number of character comparisons done during type-3 events is $O(m)$. This bound follows from the analysis of Apostolico-Giancarlo Algorithm. Now to bound the total number of comparisons done by algorithm $A$, we only need to compute the number of comparisons performed by $A$ that are associated with $Type−1$ and $Type−2$ events.

We bound the number of comparisons made by $A$ due to Type-1 and Type-2 events by partitioning the search phase into sub-phases, where each sub-phase consists of maximal sequence of events that begins with any type of event and is terminated by a Type-3 event, and then account for the number of comparisons made during Type-1 and Type-2 events of the sub-phases. Notice that except the first sub-phase, every other sub-phase begins with either a Type-1 or a Type-2 event, and ends with
a contiguous run of one or more Type-3 events. Notice that across sub-phases there can be an overlap in the character comparisons only between the last event of a sub-phase and the first event of the next sub-phase. From an earlier observation, we notice that each Type-1 (Type-2) event requires 1 (2) character comparisons and the search window is shifted by at least 1. This implies that the total number of character comparisons associated with type-1 and type-2 events is at most $2m$. Therefore the total number of character comparisons due to Type-1, Type-2 and Type-3 events is $O(m)$.

Now, we establish that Algorithm $A$ finds all occurrences of $P$ in $T$. In the case of Type-1 event, we know that $c \neq \text{end}(P)$ and the search window is shifted to the right by $\text{shift}(P)$. Recall that $\text{shift}(P)$ indicates the number of positions to the left of $\text{end}(P)$ in $P$ where the earliest occurrence of $c$ happens. Now, by shifting the search window by $\text{shift}(P)$, we will show that no occurrence of $\text{sparse}(P)$ will be skipped and hence no occurrence of $P$ will be skipped. There are two situations possible depending on whether or not $\text{end}(P)$ occurs within the shifted interval. If $\text{end}(P)$ did not occur within the shifted interval of $T$ then we can see that shifting the search window to the right by $\text{shift}(P)$ will not result in skipping $\text{sparse}(P)$. Now, we consider the situation when $\text{end}(P)$ occurs within the shifted interval. From the definition of $\text{shift}(P)$, we can see that earliest occurrence of $c$ in $P$ will be $\text{shift}(P)$ positions to the left of $\text{endpos}(P)$, whereas in this situation $c$ occurs in the search window less than $\text{shift}(P)$ positions to the left of $\text{endpos}(P)$. Therefore, we can conclude that no occurrence of $\text{sparse}(P)$ can start within the shifted portion of the search window. In the case of Type-2 event, we can observe that $c = \text{end}(P)$ but $d \neq \text{start}(P)$ and the search window is shifted to the right by either $|\text{sparse}(P)| (|\text{sparse}(P)| + 1)$ if $c = d$ ($c \neq d$). Notice in this case, since characters $c$ and $d$ do not occur within $\text{sparse}(P)$ (i.e. occur only at the start and end of $\text{sparse}(P)$), the next occurrence of $\text{sparse}(P)$ in $T$ cannot start within the shifted portion of the search window. In the case of Type-3 event, we can observe that $c = \text{end}(P)$ and $d = \text{start}(P)$ and after invoking Apostolico- Giancarlo Algorithm we shift the window to the right by either $|\text{sparse}(P)| (|\text{sparse}(P)| + 1)$ if $c = d$ ($c \neq d$). Notice in this case also, since characters $c$ and $d$ do not occur within $\text{sparse}(P)$, the next occurrence of $\text{sparse}(P)$ cannot start within the shifted portion of the search window. Hence, Algorithm $A$ finds all occurrences of $P$ in $T$ correctly.

\[ \square \]

**Proof of Theorem 2**: We bound the expected number of comparisons performed by Algorithm $A$ during its search phase by looking at the expected number of comparisons performed in comparison to the expected length by which the search window is shifted during each of the three type of events it encounters. For Type-1 and Type-2 events the number of character comparisons with the query string $T$ is at most 2. For Type-3 events, we are invoking the Apostolico-Giancarlo Algorithm. From
Lemma 6, we know that when invoking Apostolico-Giancarlo Algorithm for the pattern string $P$ and the query string $T$ whose characters are drawn uniformly at random, the expected number of matches before a mismatch is $O(1)$. Therefore the number of character comparisons for Type-3 event is also $O(1)$. Now to bound the total number of comparisons, it is sufficient to bound the number of events. From Lemma 5, we know that the expected length by which the search window is shifted after encountering a Type-1, Type-2 or Type-3 event is at least $O(\min(|\text{sparse}(P)|, \Delta))$. Since the total amount by which the search window can be shifted is $m$, we can therefore see that the total number of events is $O(m/\min(|\text{sparse}(P)|, \Delta))$. The proof for Algorithm $B$ is almost the same.

**Lemma 3** For any pattern string $P$, the length of $\text{sparse}(P)$, the longest $2$-sparse pattern of $P$, is at least $\delta$, where $\delta$ is the number of distinct characters in $P$.

**Proof** From definition, we know that $P$ has $\delta$ distinct characters. Now, let $a$ be the character in $P$ whose last occurrence has the smallest index and let its index in $P$ be denoted by $\text{start}$. Let $b$ be the character in $P$ whose first occurrence in $P$ to the right of $\text{start}$ has the highest index and let its position in $P$ be denoted by $\text{end}$. Now, we can observe that every character in $P$ appears at least once within the interval $[\text{start}, \text{end}]$ and characters $a$ and $b$ do not appear within the interval $[\text{start}, \text{end}]$. Since there are $\delta$ distinct characters in $P$, the length of $\text{sparse}^{(a,b)}(P)$ is at least $\delta$. Hence the length of $\text{sparse}(P)$ is at least $\delta$.

**Lemma 4** For any pattern string $P$, Algorithm $A$ preprocesses $P$ in $O(n\Delta^2)$ time to determine (i) $N_i(P)$, for $i \in [1..n]$, (ii) $\text{sparse}(P)$, and (iii) $\text{shift}^c(P)$, for $c \in \Sigma$, where $\Delta$ is the number of characters in its alphabet $\Sigma$.

**Proof** First, we would like to recall that $N_i(P)$, $i \in [1..n]$, is the length of the longest suffix of $P[1..i]$ that is also a suffix of $P$. We compute $N_i(P)$, for $i \in [1..n]$ in $O(n)$ time using the Z Algorithm []. Second, for each ordered pair $a, b \in P$ of characters, we can scan $P$ in $O(n)$ time to find the maximum length substring of $P$ starting with $a$ and ending with $b$ such that there is no occurrence of $a$ or $b$ in between. Since there are $\Delta^2$ ordered pairs, we can trivially find the maximum length for all pairs of characters in $P$ and from them choose the longest in $O(n\Delta^2)$ time. Finally, we would like to recall that $\text{shift}^c(P)$, for $c \in \Sigma$, is the distance between the last character of $\text{sparse}(P)$ and the rightmost occurrence of $c$ in $\text{sparse}(P)$. If $c$ is not present in $P$ then $\text{shift}^c(P)$ is set to $n$, the length of the pattern string $P$. Therefore, in $O(n)$ time, we can scan $\text{sparse}(P)$ to find $\text{shift}^c(P)$, for $c \in \Delta$.

**Lemma 5** For any pattern string $P$, during the search phase of Algorithm $A$, the expected length of shift of $P$ after a Type-1, Type-2 or Type-3 event is at least $O(\min(|\text{sparse}(P)|, \Delta))$. 

Proof Notice that the query string $T$ is of length $m$ and each of its characters are drawn uniformly at random from the alphabet $\Sigma$ of size $\Delta$. In the case of a Type-1 event, we can observe that the search window is shifted by $shift^c(P)$. Notice that the mismatch character $c$ is equally likely to be any character in $\Sigma$ other than $endc(P)$. So, we can observe that if $c \in \text{sparse}(P)$, then $shift^c(P)$ is equally likely to be any value in the interval $[1..\delta]$ and if $c \notin \text{sparse}(P)$ then $shift^c(P)$ is at least $|\text{sparse}(P)|$. Therefore, the expected shift value will be at least $(1 + 2 + \ldots + \delta) + (\Delta - /\delta)|\text{sparse}(P)|$. Now, based on whether $\delta < \Delta/2$ or $\delta \geq \Delta/2$, we can evaluate the above sum. If $\delta < \Delta/2$, we can observe that this sum is $O(|\text{sparse}(P)|)$, otherwise the sum is $O(|\Delta|)$. Therefore, the above sum is at least $O(\min(|\text{sparse}(P)|, \Delta))$. In the case of Type-2 event, we can observe that the search window is shifted by the length of $\text{sparse}(P)$. Similarly after a type-3 event, Apostolico-Giancarlo Algorithm is called and after that the search window is shifted by the length of $\text{sparse}(P)$. Hence in all three types of events the expected search window shift is at least $O(\min(|\text{sparse}(P)|, \Delta)) \geq \delta$. Hence the result. 

Lemma 6 For any given pattern string $P$ of length $n$ from $\Sigma$ and a query string $T$ whose characters are drawn independently and uniformly from $\Sigma$, the expected number of matches before a mismatch when invoking Apostolico-Giancarlo or Random-Match Algorithm is $O(1)$.

Proof In $A$, each time we invoke Apostolico-Giancarlo/Random-Match Algorithm, we attempt to match $P$ with the search window consisting of $n$ length substring of $T$, where each character is drawn independently and uniformly from $\Sigma$. So, the expected length of a match $= (1/\Delta + 1/\Delta^2 + 1/\Delta^3 + \ldots)(\Delta - 1/\Delta) = O(1)$. 

4 Conclusions and Future Work

In this paper, we present algorithms for exact string matching that require a worst case search time of $O(m)$, and sub-linear expected search time of $O(m / \min(|\text{sparse}(P)|, \Delta))$, where $|\text{sparse}(P)|$ is at least $\delta$ (i.e. the number of distinct characters in $P$), and for most pattern strings is observed to be $\Omega(n^{1/2})$. We believe that a tighter analysis of our algorithms can establish sub-linear worst case run-time from the perspective of randomized analysis. We also believe that for a large class of pattern strings it seems plausible that one can theoretically establish that $|\text{sparse}(P)| = \Omega(n^{1/2})$.

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