Convergence of diffusions and their discretizations: from continuous to discrete processes and back

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Abstract

In this paper, we establish new quantitative convergence bounds for a class of functional autoregressive models in weighted total variation metrics. To derive our results, we show that under mild assumptions, explicit minorization and Foster-Lyapunov drift conditions hold. The main applications and consequences of the bounds we obtain concern the geometric convergence of Euler-Maruyama discretizations of diffusions with identity covariance matrix. Second, as a corollary, we provide a new approach to establish quantitative convergence of these diffusion processes by applying our conclusions in the discrete-time setting to a well-suited sequence of discretizations whose associated stepsizes decrease towards zero.

1 Introduction

The study of the convergence of Markov processes in general state space is a very attractive and active field of research motivated by applications in mathematics, physics and statistics [48]. Among the many works on the subject, we can mention the pioneering results from [68, 66, 67] using the renewal approach. Then, the work of [73, 60] paved the way for the use of Foster-Lyapunov drift conditions [39, 6] which, in combination of an appropriate minorization condition, implies $(f,r)$-ergodicity on general state space, drawing links with control theory, see [83, 21, 46]. This approach was successively applied to the study of Markov chains in numerous papers [11, 15, 77] and was later extended and used in the case of continuous-time Markov processes in [52, 61, 62, 24, 40, 38, 20, 84, 19]. However, most of these results establish convergence in total variation or in $V$-norm and are non-quantitative. Explicit convergence bounds in the same metrics for Markov chains have been established in [80, 36, 23, 81, 50, 56, 63, 37], driven by the need for stopping rules for Markov Chain Monte Carlo (MCMC) simulations. To the authors’ knowledge, the techniques developed in these papers have not been adapted to continuous-time Markov processes, except in [78]. One of the main reason is that deriving quantitative minorization conditions for continuous-time processes

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seems to be even more difficult than for their discrete counterparts [33]. Indeed, in most cases, the constants which appear in minorization conditions are either really pessimistic or hard to quantify accurately [49, 74].

Since the last decade, in order to avoid the use of minorization conditions, other metrics than the total variation distance, or $V$-norm, have been considered. In particular, Wasserstein metrics have shown to be very interesting in the study of Markov processes and to derive quantitative bounds of convergence as well as in the study of perturbation bounds for Markov chains [82, 72]. For example, [69, 51, 71] introduced the notion of Ricci curvature of Markov chains and its use to derive precise bounds on variance and concentration inequalities for additive functionals. Following [42], [43] generalizes the Harris’ theorem for $V$-norms to handle more general Wasserstein type metrics. In the same spirit, [10] establishes conditions which imply subgeometric convergence in Wasserstein distance of Markov processes. In addition, the use of Wasserstein distance has been successively applied to the study of diffusion processes and MCMC algorithms. In particular, [32, 33] establish explicit convergence rates for diffusions and McKean Vlasov processes. Regarding analysis of MCMC methods, [44] establishes geometric convergence of the pre-conditioned Crank-Nicolson algorithm. Besides, [28, 18, 12, 1] study the computational complexity in Wasserstein distance to sample from a log-concave density on $\mathbb{R}^d$ using appropriate discretizations of the overdamped Langevin diffusion. One key idea introduced in [43] and [32] is the construction of an appropriate metric designed specifically for the Markov process under consideration. The approach of [32] has then been generalized in [16, 58]. While this approach leads to quantitative results in the case of diffusions or their discretization, we can still wonder if appropriate minorization conditions can be found to derive similar bounds using classical results cited above.

In this paper, we show that for a class of functional auto-regressive models, sharp minorization conditions hold using an iterated Markov coupling kernel. As a result new quantitative convergence bounds can be obtained combining this conclusion and drift inequalities for well-suited Lyapunov functionals. We apply them to the study of the Euler-Maruyama discretization of diffusions with identity covariance matrix under various curvature assumptions on the drift. The rates of convergence we derive in weighted total variation metric in this case improve the one recently established in [34]. Note that this study is significant to be able to bound the computational complexity of this scheme when it is applied to the overdamped Langevin diffusion to sample from a target density $\pi$ on $\mathbb{R}^d$. Indeed, while recent papers have established precise bounds between the $n$-th iterate of the Euler-Maruyama scheme and $\pi$ in different metrics (e.g. total variation or Wasserstein distances), the convergence of the associated Markov kernel is in general needed to obtain quantitative bounds on the mean square error or concentration inequalities for additive functionals, see [28, 51].

In the second part of the present paper, we show how the results we derive for functional auto-regressive models can be used to establish explicit convergence rates for diffusion processes. First, we show that, under proper conditions on a sequence of discretizations, the distance in some metric between the distributions of the diffusion at time $t$ with different starting points can be upper bounded by the limit of the distance between the corresponding discretizations, when the discretization stepsize decreases towards zero. Similarly, in [53] general Markov processes are approximated by hidden Markov models under a continuous Foster-Lyapunov assumption. Second, we design appropriate discretizations satisfying the necessary conditions we obtain and which belong to the class of functional autoregressive models we study. Therefore, under the same curvature conditions as in the discrete case, we get quantitative convergence rates for diffusions by taking the limit in the bounds we derived for the Euler-Maruyama discretizations. Finally, the rates we obtain scale similarly with respect to the parameters of the problem under consideration to the
ones given in [32, 33] for the Kantorovitch-Rubinstein distance, and improve them in the case of
the total variation norm. Note that in the diffusion case, earlier results were derived in [14, 13, 86].

The paper is organized as follows. For reader’s convenience and to motivate our results, we begin
in Section 2, with one of their applications to the specific case of a diffusion over $\mathbb{R}^d$ with identity
covariance matrix and its Euler-Maruyama discretization, in the case where the drift function is
strongly convex at infinity. In Section 3, we present our main convergence results regarding a class
of functional autoregressive models. We then specialize them to the Euler-Maruyama discretization
diffusions under various assumptions on the drift function in Section 4. Section 5 deals with the
convergence of diffusion processes with identity covariance matrix. More precisely, in Section 5.1,
we derive sufficient conditions for the convergence of such processes based on a sequence of well-
suited discretizations. In Section 5.2, we apply our results to the continuous counterparts of the
situations considered in Section 4. For ease of presentation, the proofs and generalizations of our
results are gathered in appendix.

Notation

Let $A, B$ and $C$ three sets with $C \subseteq B$ and $f : A \to B$, we set $f^-(C) = \{x \in A : f(x) \in C\}$. For
any $A \subseteq B$ and $f : B \to C$ we denote $f|_A$ the restriction of $f$ to $A$. Let $d \in \mathbb{N}^*$ and $(\cdot, \cdot)$
be a scalar product over $\mathbb{R}^d$, and $\| \cdot \|$ be the corresponding norm. Let $A \subseteq \mathbb{R}^d$ and $R \geq 0$,
we denote $\text{diam}(A) = \sup_{(x,y)\in A} \|x-y\|$ and $\Delta_{A,R} = \{(x,y) \in A : \|x-y\| \leq R\} \subseteq \mathbb{R}^{2d}$ and
$\Delta_A = \Delta_{A,0} = \{(x,x) : x \in A\}$. In this paper, we consider that $\mathbb{R}^d$ is endowed with the topology
of the norm $\| \cdot \|$. $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel $\sigma$-field of $\mathbb{R}^d$. Let $U$ be an open set of $\mathbb{R}^d$, $n \in \mathbb{N}^*$ and
set $\mathcal{C}^n(U)$ be the set of the $n$-differentiable functions defined over $U$. Let $f \in \mathcal{C}^1(U)$, we denote by
$\nabla f$ its gradient. Furthermore, if $f \in \mathcal{C}^2(U)$ we denote $\nabla^2 f$ its Hessian and $\Delta$ its Laplacian.
We also denote $\mathcal{C}(U)$ the set of continuous functions defined over $U$ and for any set $A \subseteq \mathbb{R}^d$ and $k \in \mathbb{N}$
we set $\mathcal{C}^k(A) = \{f|_A : f \in \mathcal{C}^k(U), \text{ with } A \subseteq U \text{ and } U \text{ open}\}$. Let $f : A \to \mathbb{R}^p$ with $p \in \mathbb{N}^*$. The
function $f$ is said to be $L$-Lipschitz with $L \geq 0$ if for any $x,y \in A$, $\|f(x) - f(y)\| \leq L \|x-y\|$.

Let $X \in \mathcal{B}(\mathbb{R}^d)$. $X$ is equipped with the trace of $\mathcal{B}(\mathbb{R}^d)$ over $X$ defined by $\mathcal{X} = \{A \cap X : A \in \mathcal{B}(\mathbb{R}^d)\}$. Let
$(Y, \mathcal{Y})$ be some measurable space, we denote by $\mathcal{F}(X,Y)$ the set of the $\mathcal{X}$-measurable
functions over $X$. For any $f \in \mathcal{F}(X,\mathbb{R})$ we define its essential supremum by $\text{esssup}(f) = \inf \{a \geq 0 : \lambda(|f^-(a, +\infty)) = 0\}$, where $\lambda$ is the Lebesgue measure. Let $\mathcal{M}(\mathcal{X})$ be the set of finite signed
measures over $\mathcal{X}$ and $\mu \in \mathcal{M}(\mathcal{X})$. For $f \in \mathcal{F}(X,\mathbb{R})$ a $\mu$-integrable function we denote by $\mu(f)$
the integral of $f$ w.r.t. to $\mu$. Let $V \in \mathcal{F}(\mathbb{R}^d,[1, +\infty))$. We define the $V$-norm for any $f \in \mathcal{F}(X,\mathbb{R})$ and the
$V$-total variation norm for any $\mu \in \mathcal{M}(\mathcal{X})$ as follows

$$\|f\|_V = \text{esssup}(|f|/V), \quad \|\mu\|_V = (1/2) \sup_{f \in \mathcal{F}(X,\mathbb{R}), \|f\|_V \leq 1} \left| \int_{\mathbb{R}^d} f(x) \text{d}\mu(x) \right|.$$ 

In the case where $V = 1$ this norm is called the total variation norm of $\mu$. Let $\mu, \nu$ be two probability
measures over $\mathcal{X}$, i.e. two elements of $\mathcal{M}(\mathcal{X})$ such that $\mu(\mathcal{X}) = \nu(\mathcal{X}) = 1$. A probability measure
$\zeta$ over $\mathcal{X}^{\otimes 2}$ is said to be a transference plan between $\mu$ and $\nu$ if for any $A \subseteq \mathcal{X}$, $\zeta(A \times \mathcal{X}) = \mu(A)$
and $\zeta(\mathcal{X} \times A) = \nu(A)$. We denote by $\mathcal{T}(\mu, \nu)$ the set of all transference plans between $\mu$ and $\nu$. Let
c $c \in \mathcal{F}(\mathcal{X} \times \mathcal{X},[0, +\infty))$. We define the Wasserstein metric/distance $W_c(\mu, \nu)$ between $\mu$ and $\nu$ by

$$W_c(\mu, \nu) = \inf_{\zeta \in \mathcal{T}(\mu, \nu)} \int_{\mathcal{X}^2} c(x,y) d\zeta(x,y).$$
Note that the term Wasserstein metric/distance is an abuse of terminology since $W_c$ is only a real metric on a subspace of probability measures on $X$ under additional conditions on $c$, e.g. if $c$ is a metric on $\mathbb{R}^d$, see [85, Definition 6.1]. If $c(x,y) = \|x - y\|^p$ for $p \geq 1$, the Wasserstein distance of order $p$ is defined by $W_p = W_c^{1/p}$. Assume that $c(x,y) = 1_{\Delta^c(x,y)} \nu(x,y)$ with $\nu' \in F(X \times X, [0, +\infty))$ such that $\nu'$ is symmetric, satisfies the triangle inequality, i.e. for any $x, y, z \in X$, $\nu'(x, z) \leq \nu'(x, y) + \nu'(y, z)$, and for any $x, y \in X$, $\nu'(x, y) = 0$ implies $x = y$. Then $c$ is a metric over $X^2$ and the associated Wasserstein cost, denoted by $W_c$, is an extended metric. Note that if $\nu'(x, y) = \{V(x) + V(y)\}/2$ then $W_c(\mu, \nu) = \|\mu - \nu\|_c$, see [22, Theorem 19.1.7].

Assume that $\mu \ll \nu$ and denote by $\frac{d\mu}{d\nu}$ its Radon-Nikodym derivative. We define the Kullback-Leibler divergence, $KL(\mu|\nu)$, between $\mu$ and $\nu$, by

$$KL(\mu|\nu) = \int_X \log \left(\frac{d\mu}{d\nu}(x)\right)d\mu(x).$$

Let $Z$ be a $\sigma$-field. We say that $P : X \times Z \to [0, +\infty)$ is a Markov kernel if for any $x \in X$, $P(x, \cdot)$ is a probability measure over $Z$ and for any $A \in Z$, $P(\cdot, A) \in F(X, [0, +\infty))$. Let $Y \in B(\mathbb{R}^d)$ be equipped with $Y$ the trace of $\mathcal{B}(\mathbb{R}^d)$ over $Z$, $P : X \times Z$ and $Q : Y \times Z$ be two Markov kernels. We say that $K : X \times Y \to Z^{\otimes 2}$ is a Markov coupling kernel if for any $(x, y) \in X \times Y$, $K((x, y), \cdot)$ is a transference plan between $P(x, \cdot)$ and $Q(y, \cdot)$.

## 2 Motivation and illustrative example

### 2.1 Non-contraction setting

In this section, we motivate our work with applications of our main results to one specific example. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a drift function, $(B_t)_{t \geq 0}$ be a $d$-dimensional Brownian motion and assume that the stochastic differential equation

$$dX_t = b(X_t)dt + dB_t,$$

admits a unique strong solution $(X_t)_{t \geq 0}$ on $\mathbb{R}^+$ for any starting point $X_0 = x \in \mathbb{R}^d$. We denote by $(P_t)_{t \geq 0}$ its associated Markov semigroup. We consider the Euler-Maruyama discretization of this stochastic differential equation, i.e. the homogeneous Markov chain $(X_k)_{k \in \mathbb{N}}$, starting from $X_0 = x \in \mathbb{R}^d$ and defined by the following recursion: for any $k \in \mathbb{N}$

$$X_{k+1} = X_k + \gamma b(X_k) + \sqrt{\gamma} Z_{k+1},$$

where $\gamma > 0$ is a stepsize and $(Z_k)_{k \in \mathbb{N}^*}$ is a sequence of i.i.d. $d$-dimensional Gaussian random variables with zero mean and identity covariance matrix. We denote by $R_\gamma$ its associated Markov kernel.

The first consequence of the results established in the present paper is the explicit convergence of the Markov chain defined by (2) in a distance which is a mix of the total variation distance and the Wasserstein distance of order 1, under the assumption that $b$ is Lipschitz and strongly convex at infinity.

**Theorem 1.** Assume that there exist $m \in \mathbb{R}$, $m^+ > 0$ and $L, R \geq 0$ such that for any $x, y \in \mathbb{R}^d$

$$\|b(x) - b(y)\| \leq L \|x - y\|, \quad \langle b(x) - b(y), x - y \rangle \leq -m \|x - y\|^2,$$

for any $x, y \in \mathbb{R}^d$. Then $\sup_{t \geq 0} \mathbb{E} \|X_t - x\|^p \leq C \mathbb{E} \|X_0 - x\|^p$, where $C$ is a constant. The proof of this theorem relies on a careful analysis of the Euler-Maruyama discretization of the stochastic differential equation.
and if \( \|x - y\| \geq R \),
\[
\langle b(x) - b(y), x - y \rangle \leq -m^+ \|x - y\|^2.
\] (4)

Then there exist \( \bar{\gamma} > 0 \), \( D_{\gamma,1}, D_{\gamma,2}, E_{\bar{\gamma}} \geq 0 \) and \( \lambda_{\gamma}, \rho_{\bar{\gamma}} \in [0,1] \) with \( \lambda_{\gamma} \leq \rho_{\bar{\gamma}} \), which can be explicitly computed, such that for any \( \gamma \in (0, \bar{\gamma}] \), \( x, y \in \mathbb{R}^d \) and \( k \in \mathbb{N} \)
\[
W_c(\delta_x R^k, \delta_y R^k) \leq \lambda_{\gamma}^{k^2/4}[D_{\gamma,1} c(x, y) + D_{\gamma,2} 1_{\Delta^c}(x, y)] + E_{\bar{\gamma}} \rho_{\bar{\gamma}}^{k^{\gamma}/4} 1_{\Delta^c}(x, y),
\] (5)

where \( c(x, y) = 1_{\Delta^c}(x, y)(1 + \|x - y\|/R) \), \( \Delta = \{(x, x) : x \in \mathbb{R}^d\} \) and \( R_{\gamma} \) is the Markov kernel associated with (2).

\textit{Proof.} The result is a direct consequence of Theorem 13 and the corresponding discussion in Section 4.2.1. \qed

This result is derived as a specific case of a more general theorem for a class of functional autoregressive models, see Theorem 8 and Section 3. Its proof relies on the use of an extended Foster-Lyapunov drift assumption as well as a minorization condition on the Markov chain (2). As an important consequence, curvature assumptions on the drift (such as strong convexity at infinity) can be omitted if we instead assume some Foster-Lyapunov condition, similarly to [32, Theorem 6.1] and [33, Theorem 2.1].

The result derived in Theorem 1 has several important applications which we gather in the following corollary.

\textbf{Corollary 2.} Assume that there exist \( m \in \mathbb{R}, m^+ > 0 \) and \( L, R \geq 0 \) such that (3) and (4) are satisfied. Then, there exist \( \bar{\gamma} > 0 \), \( E_{\bar{\gamma},1}, E_{\bar{\gamma},2} \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}] \), \( x, y \in \mathbb{R}^d \) and \( k \in \mathbb{N} \) we have
\[
\|\delta_x R^k - \delta_y R^k\|_{TV} \leq W_c(\delta_x R^k, \delta_y R^k) \leq E_{\bar{\gamma},1} \rho_{\bar{\gamma}}^{k^{\gamma}/4} c(x, y),
\] (6)
\[
W_1(\delta_x R^k, \delta_y R^k) \leq E_{\bar{\gamma},2} \rho_{\bar{\gamma}}^{k^{\gamma}/4} \|x - y\|,
\] (7)

where \( c(x, y) = 1_{\Delta^c}(x, y)(1 + \|x - y\|/R) \), \( \Delta = \{(x, x) : x \in \mathbb{R}^d\} \) and \( \rho_{\bar{\gamma}} \) is given in (5). In addition, for any \( p \in \mathbb{N} \) and \( \alpha \in (p, +\infty) \) there exists \( E_{\bar{\gamma},\alpha} \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}] \), \( x, y \in \mathbb{R}^d \) and \( k \in \mathbb{N} \) we have
\[
W_p(\delta_x R^k, \delta_y R^k) \leq E_{\bar{\gamma},\alpha} \rho_{\bar{\gamma}}^{k^{\gamma}/(4\alpha)} (\|x - y\| + \|x - y\|^{1/\alpha}).
\] (8)

The constants \( \bar{\gamma}, \{E_{\bar{\gamma},i} : i = 1, 2, 3\} \) and \( E_{\bar{\gamma},\alpha} \) can be explicitly computed.

\textit{Proof.} The estimate (6) is a direct consequence of Theorem 1. The two inequalities (7) and (8) follow from Corollary 14. \qed

Note that the same rate \( \rho_{\bar{\gamma}} \) appears in the inequalities (5), (6), (7) and (8). Section 5 is devoted to the extension of our discrete-time results to their continuous-time counterparts. Note also that Theorem 3 and its consequences still hold if we only assume a local Lipschitz assumption, see the condition B5.

\textbf{Theorem 3.} Assume that there exist \( m \in \mathbb{R}, m^+ > 0 \) and \( L, R \geq 0 \) such that (3) and (4) are satisfied. Then there exist \( D_1, D_2, E \geq 0 \) and \( \lambda, \rho \in [0,1] \) with \( \lambda \leq \rho \) such that for any \( x, y \in \mathbb{R}^d \) and \( t \geq 0 \)
\[
\|\delta_x P_t - \delta_y P_t\|_{TV} \leq W_c(\delta_x P_t, \delta_y P_t) \leq \lambda^{t/4}[D_1 c(x, y) + D_2 1_{\Delta^c}(x, y)] + E \rho^{t/4} 1_{\Delta^c}(x, y),
\] (9)
where \(c(x, y) = 1_{\Delta_{\gamma}}(x, y)(1 + \|x - y\|/R)\), \(\Delta = \{(x, x) : x \in \mathbb{R}^d\}\), \((P_t)_{t \geq 0}\) is the Markov semigroup associated with (1) and

\[
D_1 = \lim_{\gamma \to 0} D_{\gamma, 1}, \quad D_2 = \lim_{\gamma \to 0} D_{\gamma, 2}, \quad E = \lim_{\gamma \to 0} E_{\gamma}, \quad \lambda = \lim_{\gamma \to 0} \lambda_{\gamma}, \quad \rho = \lim_{\gamma \to 0} \rho_{\gamma},
\]

and \(D_{\gamma, 1}, D_{\gamma, 2}, E_{\gamma}, \lambda_{\gamma}, \rho_{\gamma}\) are given in Theorem 1.

**Proof.** This result follows from Theorem 21. □

Note that the constants \(D_1, D_2, E, \lambda\) and \(\rho\) have explicit expressions, see the corresponding discussion in Section 5.2 after Theorem 21. In addition, the rate \(\rho\) and \(\lambda\) in (9) are independent of the dimension \(d\). This is a significant improvement compared to the total variation results in total in [33, Theorem 2.1] which imply a convergence rate which scales exponentially in the dimension, under the setting we consider. Similarly, we derive a continuous counterpart of Corollary 2 in the continuous time setting, see Corollary 22.

As stated before, the convergence rates \(\rho_{\gamma}, \rho, \lambda_{\gamma}, \lambda\) given in Theorem 1 and Theorem 3 can be explicitly computed. More precisely, we obtain the following expressions (up to logarithmic terms) with respect to the parameters \(m, L\) and \(R\) in the case \(-mR^2 \gg 1\), see Section 4.2, Theorem 13, Equations (46) and (47):

\[
\log(\log^{-1}(\rho_{\gamma}^{-1})) \simeq -(mR^2/4) \sup_{\gamma \in [0, \bar{\gamma}]} \left\{ \left(1 - \frac{\gamma L^2}{2m}\right) \left(1 - \exp \left[ \frac{R^2(2m - \gamma L^2)}{1 - 2m \gamma + \gamma^2 L^2} \right] \right)^{-1} \right\}
\]

(10)

\[
\log(\log^{-1}(\rho^{-1})) \simeq -(mR^2/4) \times (1 - \exp(2mR^2))^{-1},
\]

(11)

\[
\log(\lambda_{\gamma}) = -\frac{m^+}{2} + \frac{\gamma L^2}{4}, \quad \log(\lambda) = -\frac{m^+}{2}.
\]

where \(\simeq\) denotes equality up to logarithmic factors.

It is sensible to obtain two different convergence rates \(\lambda_{\gamma}, \rho_{\gamma}\) (resp. \(\lambda, \rho\)) in Theorem 1 (resp. in Theorem 3), one characterizing the forgetting of the initial distance between the two starting points \(x, y \in \mathbb{R}^d\), corresponding to a burn-in period, and the other one characterizing the effective convergence. In addition, note that \(\lambda_{\gamma} \ll \rho_{\gamma}\) and \(\lambda \ll \rho\) if \(-mR^2 \gg 1\).

We now compare these results and the rates obtained in (10)-(11) with recent works studying the convergence of the Markov chain defined by (2) and/or the corresponding diffusion process (1) in the same framework, *i.e.* under the conditions (3) and (4). Note that the same conclusions hold under more general Foster-Lyapunov drift conditions but it would make the comparison more involved. Note also we are still able to derive convergence results under weaker curvature assumptions on the drift. The discussion is postponed to Section 4.2.2 for the discrete setting and Section 5.2.2 for the continuous setting.

First, a major difference between our work and the ones mentioned below is that we use a completely different technique to establish our results. Indeed, all of them follow the approach initiated in [31], designing a suitable coupling and distance function of the form \(c(x, y) = f(\|x - y\|)\), for any \(x, y \in \mathbb{R}^d\), with \(f : \mathbb{R}_+ \to \mathbb{R}_+\), to obtain a geometric contraction in \(\mathcal{W}_c\) for either the Markov chain (2) or the diffusion (1) under the conditions (3)-(4). In this paper, we follow a different path and derive convergence estimates using minorization and Foster-Lyapunov drift conditions, adapting the technique used in [22] and the references therein. It has been thought for a long time that such an approach only gives very pessimistic convergence bounds [33]. We now compare more specifically our results with the ones obtained following the work of [31] and show that in fact our technique
inspired by classical methods to establish geometric convergence of Markov chains gives very sharp estimates, improving and simplifying the results obtained in the existing literature. This discussion and its conclusion are summarized in Table 1. In the rest of this section, \( C \geq 0 \) stands for a positive constant which may be different at each occurrence.

First we compare our work with the results of [34] which extend to the discrete setting the estimates of [32]. The authors use the following cost function defined for any \( x, y \in \mathbb{R}^d \) by

\[
c_a(x, y) = a \mathbb{I}_{\Delta^c}(x, y) + f_a(\|x - y\|),
\]

where \( a \geq 0 \) and \( f_a \) is given in [34, Equation (2.53)]. Note that the cost \( c_a \) is close to the one introduced in Theorem 1. Then, [34, Theorem 2.12] states that if \( a \in [2\gamma^{1/2}, \Phi_E(R)] \) where \( \Phi_E \) is given in [34, Theorem 2.10], then there exist \( \bar{\gamma}_a > 0 \) and \( \rho_a \in [0, 1) \) such that for any \( \gamma \in (0, \bar{\gamma}_a] \), \( x, y \in \mathbb{R}^d \) and \( k \in \mathbb{N} \),

\[
W_{c_a}(\delta_x R^{k\gamma}_y, \delta_y R^{k\gamma}_x) \leq \rho_a^{k\gamma} c_a(x, y). \tag{12}
\]

Compared to our results Theorem 1, (12) only gives one convergence rate \( \rho_a \) and does not dissociate the forgetting of the initial distance between the starting points \( x, y \in \mathbb{R}^d \) from the long-term behavior. In addition, \( a \) may depend on \( \gamma \), since it is required that \( a \in [2\gamma^{1/2}, \Phi(R)] \) and \( \bar{\gamma}_a < \bar{\gamma} \) where \( \bar{\gamma} \) is given by Theorem 1. Omitting the dependency of \( a \) and \( \rho_a \) with respect to \( \gamma \) for the sake of simplicity, and applying [34, Theorem 2.10] yield

\[
\log(\log^{-1}(\rho_a^{-1})) \simeq -mR^2/c_1, \text{ with } c_1 = 16^{-1} \int_{1/4}^{3/8} (1 - e^{u-1/2})\varphi(u)du \leq 0.00051, \tag{13}
\]

where for any \( t \in \mathbb{R} \), \( \varphi(t) = (2\pi)^{-1/2} \exp(-t^2/2) \). It is worth noticing that in the case we are interested in, \( -mR^2 \gg 1 \), we obtain that our rate given by (10) satisfies \( \rho_\gamma \ll \rho_a \) (also omitting dependency of \( \rho_\gamma \) with respect to \( \gamma \)).

Let \( c_b \) be defined for any \( x, y \in \mathbb{R}^d \) by \( c_b(x, y) = f_b(\|x - y\|) \) with \( f_b \) given in [34, Equation (2.68)]. Then, [34, Theorem 2.12] implies that there exist \( \bar{\gamma}_b > 0 \) and \( \rho_b \in [0, 1) \) such that for any \( \gamma \in (0, \bar{\gamma}_b] \), \( x, y \in \mathbb{R}^d \) and \( k \in \mathbb{N} \),

\[
W_{c_b}(\delta_x R^{k\gamma}_y, \delta_y R^{k\gamma}_x) \leq \rho_b^{k\gamma} c_b(x, y). \tag{14}
\]

Note that (12) implies convergence bounds both with respect to \( W_1 \) and the total variation distance whereas (14) implies convergence bounds with respect to \( W_1 \) only. Once again, omitting the dependency with respect to \( \gamma \), we obtain that the rate satisfies

\[
\log(\log^{-1}(\rho_b^{-1})) \simeq -49mR^2/(6c_2),
\]

with

\[
c_2 = 4 \min \left( \int_0^{1/2} u^2(1 - e^{u-1/2})\varphi(u)du, (1 - e^{-1}) \int_0^{1/2} u^3\varphi(u)du \right) \leq 0.0072, \tag{15}
\]

and we obtain that our rate given by (10) satisfies \( \rho_\gamma \ll \rho_a \) when \( -mR^2 \gg 1 \).

We now compare our results with the ones derived in [58]. For fair comparison, since [58] does not assume a one-sided Lipschitz condition but only a global Lipschitz condition we set \( m = -L \) in the next paragraph. This paper extends the techniques of [34, 32] to deal with \( W_2 \). It is shown in
[58, Theorem 2.1] that there exist $\bar{\gamma}_c > 0$ and $\rho_c \in [0, 1)$ such that for any $\gamma \in (0, \bar{\gamma}_c]$, $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}$,

$$W_{c_e}(\delta_x R^k, \delta_y R^k) \leq \rho_c^{k\gamma} c_e(x, y),$$

with $c_e$ given for any $x, y \in \mathbb{R}^d$ by $c_e(x, y) = f_e(\|x - y\|)$ and $f_e$ given in [34, Equation (2.11)]. Note that this result implies convergence bounds with respect to $W_1$ and $W_2$. In particular, we have for any $\gamma \in (0, \bar{\gamma}]$, $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}$,

$$W_2(\delta_x R^k, \delta_y R^k) \leq C \rho_c^{k\gamma/2} c_e^{1/2}(x, y) \leq C \rho_c^{k\gamma/2}(\|x - y\| + \|x - y\|^{1/2}).$$

In addition, it holds that

$$\log(\log^{-1}(\rho_c^{-1})) \simeq LR^2/(6c_2)$$

where $c_2$ is defined by (15),

and therefore our rate also satisfies $\rho_\gamma \ll \rho_c$ when $LR^2 \gg 1$.

The results of [34, 58] both extend, and generalize, in the discrete-time setting the techniques used in [32]. In the latter, contraction results for the semigroup $(P_t)_{t \geq 0}$ are obtained with respect to $W_{c_e}$, where for any $x, y \in \mathbb{R}^d$, $c_e(x, y) = f_e(\|x - y\|)$ and $f_e$ is defined by [32, Equation (2.6)]. In particular, in [32, Corollary 2.3], it is shown that there exists $\rho_e \in [0, 1)$ such that for any $x, y \in \mathbb{R}^d$ and $t \geq 0$

$$W_{c_e}(\delta_x P_t, \delta_y P_t) \leq \rho_e^{t} c_e(x, y),$$

Note that this result implies convergence bounds in $W_1$, see [32, Corollary 2.3]. The rate is given [32, Lemma 2.9] and, in the case $-mR^2 \gg 1$, we have

$$\log(\log^{-1}(\rho_e^{-1})) \simeq -mR^2/4,$$

which is better than our rate in the continuous-time case$^1$. However, note that we derive our results in $W_1$ from our estimates with respect to $W_{c_e}$ with $c$ given in Theorem 1, which controls both $W_1$ and the total variation norm. Also, the discrepancy between our rate and the one of (17) is controlled by $(e^{-2mR^2} - 1)^{-1}$ which is small when $-mR^2$ is large.

Finally we compare our continuous-time results with the ones of [57]. It is shown in [57, Theorem 1.3] that for any $p > 1$ there exist $\rho_f \in [0, 1)$ and $C \geq 0$ such that for any $x, y \in \mathbb{R}^d$ and $t \geq 0$

$$W_p(\delta_x P_t, \delta_y P_t) \leq C \rho_f^{t} \left\{ \|x - y\| + \|x - y\|^{1/p} \right\},$$

and the rate is given in [57, Theorem 1.3] by

$$\log(\log^{-1}(\rho_f^{-1})) = (-m + m^+)R^2/4.$$

The additional term $m^+ R^2/4$ does not appear in our rates$^2$. As a consequence our rate is better as soon as

$$m^+ \geq -m/(e^{-2mR^2} - 1).$$

Table 1 gives a summary of the comparisons we made above.

---

$^1$Note that in [32, Lemma 2.9, Equation (2.18)] the stated result implies that $\log(\log^{-1}(\rho_c^{-1})) \simeq LR^2/8$ if $\kappa(r) \geq -Lr$ for any $r \geq 0$, where $\kappa$ is defined in [32, p.5]. However, note that if $b$ is $L$-Lipschitz then $\kappa(r) \geq -2L$ and (17) follows.

$^2$Similarly to [32], in [57, Theorem 1.3] the stated result implies that $\log(\log^{-1}(\rho_f^{-1})) \simeq LR^2/2$ if $\kappa(r) \leq -Lr$ for any $r \geq 0$ and $\kappa(r) \leq -m^+r$ for $r \geq R$, where $\kappa$ is defined in [57, Equation (1.4)]. However, note that if $b$ is $L$-Lipschitz and $m^+$ strongly convex outside of $B(0, R)$ we have $\kappa(r) \leq Lr/2$ for any $r \geq 0$ and $\kappa(r) \leq m^+r/2$ for any $r \geq R$ and (18) follows.
Table 1: Every line of the table reads as follows. Suppose “Wasserstein distance” reads $W_{e_1}$ and “distance bound” reads $c_2(x, y)$ then: if (D) is checked, there exist $C \geq 0$ and $\rho \in [0, 1)$ such that for any $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}$, $W_{e_1}(\delta_x R_k^k, \delta_y R_k^k) \leq C \rho^k c_2(x, y)$ for $\gamma$ small enough. If (C) is checked, there exist $C \geq 0$ and $\rho \in [0, 1)$ such that for any $x, y \in \mathbb{R}^d$ and $t \geq 0$, $W_{e_1}(\delta_x P_t, \delta_y P_t) \leq C \rho^t c_2(x, y)$. In addition, if the normalized rate “(NR)” reads $\beta$ we have $-4 \log((\log^{-1}(\rho^{-1}))/((mR^2)) \approx \beta$ (with $m$ replaced by $-L$ in the case of [58]). Note that for the sake of simplicity we omit the dependency with respect to $\gamma$ in the present analysis. The exact distances used in papers with which we compare our results, are given in [34, Equation (2.53)], [58, Equation (2.11)], [32, Equation (2.6)] and [57, Equation (2.4)]. Note that $p \in \mathbb{N}$ and $\alpha \in (p, +\infty)$.

### 2.2 An illustrative example

We now consider a toy example to justify the setting under study in the previous section. Consider the following Gaussian mixture distribution $\pi$ whose Radon-Nikodym density with respect to the Lebesgue measure $\lambda$ is given for any $x \in \mathbb{R}$ by

$$(d\pi/d\lambda)(x) = (2\sqrt{2\pi\sigma^2})^{-1} \exp[-x^2/(2\sigma^2)] + (2\sqrt{2\pi\sigma^2})^{-1} \exp[-(x - m)^2/(2\sigma^2)],$$

where $\sigma > 0$ and $m \geq 0$. For any $x \in \mathbb{R}$, we have $(d\pi/d\lambda)(x) \propto e^{-U(x-m)/2}$ and for any $\bar{x} \in \mathbb{R}$

$$U(\bar{x}) = \bar{x}^2/(2\sigma^2) - \log \left(\cosh(m\bar{x}/(2\sigma^2))\right),$$

Note that $U'$ is L-Lipschitz with $L = \sigma^{-2} \max\{1, (m/(2\sigma))^2 - 1\}$ and that $U$ is convex if and only if $m \leq 2\sigma$. Also, we obtain that $b = -U'$ satisfies (3) with $L = \sigma^{-2} \max\{1, (m/(2\sigma))^2 - 1\}$, $R = 2m$, $m^+ = 1/(2\sigma^2)$.

We now consider the Markov chain (2) with $b = -U'$ and its associated Markov kernel $R_x$ for $\gamma > 0$. Let $x_0 \in \mathbb{R}$ and we define $\log(\log(\rho_{\text{exp}}))$ to be the slope of the function $n \mapsto \log(\|\delta_{x_0} R_n^\gamma - \pi\|_{\text{TV}})$. Note that this slope is computed only until $\log(\|\delta_{x_0} R_n^\gamma - \pi\|_{\text{TV}})$ reaches a given precision, since for $\gamma > 0$ small enough there exists a probability measure $\pi_\gamma$ such that $\|\delta_{x_0} R_n^\gamma - \pi_\gamma\|_{\text{TV}} \to 0$ and $\pi_\gamma \neq \pi$. In what follows we compare $\log(\log(\rho_{\text{exp}}))$ with our estimates.

Let $\theta = m/(2\sigma)$ and assume that $\theta \geq \sqrt{2}$. Note that in this case $LR^2 = 16\theta^2(\theta^2 - 1)$. Let $\rho$ be the rate we identify in (11). Up to logarithmic terms we have $\log(\log^{-1}(\rho^{-1})) \approx 4\theta^2(\theta^2 - 1)/(1 - e^{-32\theta^2(a^2 - 1)})$. In Figure 1 and Figure 2, we fix $\sigma = 2$ and study the behavior of $\log(\rho_{\text{exp}})$ and $\log(\rho)$ w.r.t. $m$. In particular, Figure 2-(b) illustrates that the rates we obtain are much closer to the ones estimated by our numerical simulations.
Figure 1: In (a) and (b), the blue curve is the theoretical log-partition and in orange the estimated log-partition of $\delta_{x_0}R^n$ at iteration $n = 10000$ with $\gamma = 0.1$. The estimation of the log-partition is performed using Gaussian kernels and 1000 points sampled from 1000000 points using a bootstrap procedure. In (a), $m = 6$ and $\sigma = 2$ and in (b) $m = 10$ and $\sigma = 2$. In (c) we illustrate the behavior of $-\log_{10}(\|\delta_{x_0}R^n - \pi\|_{TV})$ for $\sigma = 2$ and $m$ between 6 and 14 (color blue to red). Note that the precision saturates since $\pi \neq \pi_\gamma$.

Figure 2: In (a) we present $\log(\log^{-1}(\rho^{-1}_{\text{exp}}))$ as $m$ varies. In (b) we present $\log_{10}(\log(\log^{-1}(\rho^{-1})))$ with $\rho \leftarrow \rho_{\text{exp}}$ (red), $\rho$ given by (11) (blue), $\rho$ given by (16) (green) and $\rho$ given by (13) (orange).
3 Quantitative convergence bounds for a class of functional autoregressive models

Let $X \in \mathcal{B}(\mathbb{R}^d)$ endowed with the trace of $\mathcal{B}(\mathbb{R}^d)$ on $X$ denoted by $\mathcal{X} = \{A \cap X : A \in \mathcal{B}(\mathbb{R}^d)\}$. In this section we consider the Markov chain $(X_k)_{k \in \mathbb{N}}$ defined by $X_0 \in X$ and the following recursion: for any $k \in \mathbb{N}$

$$X_{k+1} = \Pi (T_\gamma(X_k) + \sqrt{\gamma} Z_{k+1}),$$

where $\{T_\gamma : \gamma \in (0, \bar{\gamma})\}$ is a family of measurable functions from $X$ to $\mathbb{R}^d$ with $\bar{\gamma} > 0$, $\gamma \in (0, \bar{\gamma}]$ is a stepsize, $(Z_k)_{k \in \mathbb{N}}$ is a sequence of i.i.d $d$-dimensional zero mean Gaussian random variables with covariance identity and $\Pi : \mathbb{R}^d \to X$ is a measurable function. The Markov chain $(X_k)_{k \in \mathbb{N}}$ defined by (19) is associated with the Markov kernel $R_\gamma$ defined on $X \times \mathcal{B}(\mathbb{R}^d)$ for any $\gamma \in (0, \bar{\gamma}]$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$R_\gamma(x, A) = (2 \pi \gamma)^{-d/2} \int_{\Pi^{-1}(A)} \exp \left[-(2\gamma)^{-1} \| y - T_\gamma(x) \|^2 \right] \, dy.$$

Note that for any $x \in X$, $R_\gamma(x, X) = 1$ and therefore, $R_\gamma$ given in (20) is also a Markov kernel over $X \times \mathcal{X}$.

In this section we state explicit convergence results for $R_\gamma$ for some Wasserstein distances and discuss the rates we obtain. These results rely on appropriate minorization and Foster-Lyapunov drift conditions. We first derive the minorization condition for the $n$-th iterate of $R_\gamma$. To do so, we consider a Markov coupling kernel $K_\gamma$ for $R_\gamma$ for any $\gamma \in (0, \bar{\gamma}]$, i.e. for any $x, y \in \mathbb{R}^d$, $K_\gamma((x, y), \cdot)$ is a transference plan between $R_\gamma(x, \cdot)$ and $R_\gamma(y, \cdot)$. Indeed, in that case, by [22, Theorem 19.1.6], we have for any $x, y \in X$, $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}^*$,

$$\| \delta_x R^n_\gamma - \delta_y R^n_\gamma \|_{TV} \leq K^n_\gamma((x, y), \Delta_x),$$

where $\Delta_x = \{(x, x) : x \in X\}$. In this paper, we consider a projected version of the discrete reflection coupling [9, 28] which is the discrete counterpart of the coupling introduced in [54]. For any $x, y, z \in \mathbb{R}^d$, $\gamma \in (0, \bar{\gamma}]$, let

$$e(x, y) = \begin{cases} \frac{E(x, y)/\|E(x, y)\|}{\|E(x, y)\|} & \text{if } T_\gamma(x) \neq T_\gamma(y) \\ 0 & \text{otherwise} \end{cases}, \quad E(x, y) = T_\gamma(y) - T_\gamma(x),$$

and

$$S_\gamma(x, y, z) = T_\gamma(y) + (\text{Id} - 2e(x, y)e(x, y)^\top)z, \quad p_\gamma(x, y, z) = 1 \wedge \varphi_\gamma(\|E(x, y)\| - (e(x, y), z)),\]$$

where $\varphi_\gamma$ is the one dimensional zero mean Gaussian distribution function with variance $\gamma$. Let $(U_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. uniform random variables on $[0, 1]$ independent of $(Z_k)_{k \in \mathbb{N}}$. Define the Markov chain $(X_k, Y_k)_{k \in \mathbb{N}}$ starting from $(X_0, Y_0) \in X^2$ by the recursion: for any $k \in \mathbb{N}$,

$$X_{k+1} = T_\gamma(X_k) + \sqrt{\gamma} Z_{k+1},$$

$$Y_{k+1} = \begin{cases} X_{k+1} & \text{if } T_\gamma(X_k) = T_\gamma(Y_k) \\ W_{k+1} X_{k+1} + (1 - W_{k+1}) S_\gamma(X_k, Y_k, \sqrt{\gamma} Z_{k+1}) & \text{otherwise} \end{cases},$$

if $T_\gamma(X_k) = T_\gamma(Y_k)$,
where \( W_{k+1} = \mathbb{1}_{(-\infty,0]}(U_{k+1} - p(X_k, Y_k, \sqrt{\gamma}Z_{k+1})) \) and finally set
\[
(X_{k+1}, Y_{k+1}) = (\Pi(\tilde{X}_{k+1}), \Pi(\tilde{Y}_{k+1})).
\]
(22)

The Markov chain \((X_k, Y_k)_{k \in \mathbb{N}}\) is associated with the Markov kernel \(K_\gamma\) on \(X^2 \times \mathcal{X}^\otimes 2\) given for all \(\gamma \in (0, \bar{\gamma}]\), \(x, y \in X\) and \(A \in \mathcal{X}^\otimes 2\) by
\[
K_\gamma((x, y), A) = \frac{\mathbb{1}_{\Delta_{\mathbb{R}^d}}(T_\gamma(x), T_\gamma(y))}{(2\pi \gamma)^{d/2}} \int_{\mathbb{R}^d} \mathbb{1}_\Pi_A(\tilde{x}, \tilde{x}) e^{-\frac{\|\tilde{x} - T_\gamma(x)\|^2}{2\gamma}} d\tilde{x} + \frac{\mathbb{1}_{\Delta_{\mathbb{R}^d}}(T_\gamma(x), T_\gamma(y))}{(2\pi \gamma)^{d/2}} \int_{\mathbb{R}^d} \mathbb{1}_\Pi_A(\tilde{x}, S_\gamma(x, y, \tilde{x}, -T_\gamma(x))) \{1 - p_\gamma(x, y, \tilde{x} - T_\gamma(x))\} e^{-\frac{\|\tilde{x} - T_\gamma(x)\|^2}{2\gamma^2}} d\tilde{x},
\]
(23)

where \(\Pi_A = (\Pi, \Pi)^\text{adj}(A)\) and \(\Delta_{\mathbb{R}^d} = \{(x, x) : x \in \mathbb{R}^d\}\). Note that marginally, by definition, the distribution of \(X_{k+1}\) given \(X_k\) is \(R_\gamma(X_k, \cdot)\). It is well-known (see e.g. \[9, Section 3.3\]) that \(\tilde{Y}_{k+1}\) and \(T_\gamma(Y_k) + \sqrt{\gamma}Z_{k+1}\) have the same distribution given \(Y_k\), and therefore the distribution of \(Y_{k+1}\) given \(Y_k\) is \(R_\gamma(Y_k, \cdot)\). As a result, for any \(\gamma \in (0, \bar{\gamma}], x, y \in X\), \(K_\gamma((x, y), \cdot)\) is a transfer function between \(R_\gamma(\cdot, \cdot)\) and \(R_\gamma(\cdot, \cdot)\).

As emphasized previously, based on (21), to study convergence of \(R_\gamma\) for \(\gamma \in (0, \bar{\gamma}]\), we first give upper bounds for \(K_\gamma((x, y), \Delta_{\mathbb{R}^d})\) for any \(x, y \in X\) and \(n \in \mathbb{N}\) under appropriate conditions on \(T_\gamma\) and \(\Pi\).

**A1.** The function \(\Pi : \mathbb{R}^d \to X\) is non expansive: i.e. for any \(x, y \in \mathbb{R}^d\), \(\|\Pi(x) - \Pi(y)\| \leq \|x - y\|\).

Note that A1 is satisfied if \(\Pi\) is the proximal operator \([3, Proposition 12.27]\) associated with a convex lower semi-continuous function \(f : \mathbb{R}^d \to (-\infty, +\infty]\). For example, if \(f(x) = \sum_{i=1}^d |x_i|\), the associated proximal operator is the soft thresholding operator \([70, Section 6.5.2]\). If \(f\) is the convex indicator of a closed convex set \(C \subset \mathbb{R}^d\), defined by \(f(x) = 0\) for \(x \in C\), \(f(x) = +\infty\) otherwise, the proximal operator is simply the orthogonal projection onto \(C\) by \([3, Example 12.21]\) and we define for any \(x \in \mathbb{R}^d\)
\[
\Pi_C(x) = \arg \min_{y \in C} \|y - x\|.
\]
(24)

First, the class of Markov chains defined by (19) contains Euler-Maruyama discretizations of diffusion processes with identity diffusion matrix and for which \(\Pi = \text{Id}\). Our results will be specified for this particular case in Section 4. Second, for the applications that we have in mind, the use of Markov chains defined by (19) with \(\Pi \neq \text{Id}\) satisfying A1, has been proposed based on optimization literature to sample non-smooth log-concave densities \([30, 8, 26, 4]\). Finally, we will also make use of (19) with \(\Pi = \Pi_{K_n}\), where \(\Pi_{K_n}\) is defined by (24) with \(C \leftarrow K_n\), and \((K_n)_{n \in \mathbb{N}}\) is a sequence of nested compact sets of \(\mathbb{R}^d\), to derive our results on diffusion processes in Section 5.2.

We now consider the following assumption on \(\{T_\gamma : \gamma \in (0, \bar{\gamma}]\}\). Let \(A \in \mathcal{B}(\mathbb{R}^{2d})\).

**A2 (A).** There exists \(\kappa : (0, \bar{\gamma}] \to \mathbb{R}\) such that for any \(\gamma \in (0, \bar{\gamma}]\) and \((x, y) \in A \cap X^2\)
\[
\|T_\gamma(x) - T_\gamma(y)\|^2 \leq (1 + \gamma \kappa(\gamma)) \|x - y\|^2.
\]
(25)

Further, one of the following conditions holds for any \(\gamma \in (0, \bar{\gamma}]\): (i) \(\kappa(\gamma) < 0\); (ii) \(\kappa(\gamma) \leq 0\); (iii) \(\kappa(\gamma) > 0\).
If $T_\gamma(x) = x + \gamma b(x)$ and $b$ is $L$-Lipschitz we have that $A^2(\mathbb{R}^d)$ holds for any with $\kappa(\gamma) = L(2 + \gamma L)$. Note that $A^2(X^2)$-(i) or $A^2(X^2)$-(ii) imply that for any $\gamma \in (0, \tilde{\gamma}]$, $T_\gamma$ is non-expansive itself (see $A^1$). For $\kappa : (0, \tilde{\gamma}] \to \mathbb{R}$ and $\ell \in \mathbb{N}^*$, $\gamma \in (0, \gamma]$ such that $\gamma \kappa(\gamma) \in (-1, +\infty)$, define

$$\Xi_n(\kappa) = \gamma \sum_{k=1}^{n} (1 + \gamma \kappa(\gamma))^{-k}.$$  \hfill (26)

The following theorem gives a generalization of a minorization condition on autoregressive models [28, Section 6].

**Theorem 4.** Let $A \in \mathcal{B}(\mathbb{R}^{2d})$ and assume $A^1$ and $A^2(A)$. Let $(X_k, Y_k)_{k \in \mathbb{N}}$ be defined by $A^2$ with $(X_0, Y_0) = (x, y) \in A \cap X^2$ and $\gamma \in (0, \gamma]$. Then for any $n \in \mathbb{N}^*$

$$\mathbb{P}(X_n \neq Y_n \text{ and for any } k \in \{1, \ldots, n-1\}, (X_k, Y_k) \in A) \leq \mathbb{I}_{\Delta_\gamma^\ast}(x, y) \left\{ 1 - 2 \Phi \left( -\frac{\|x - y\|}{2\Xi_n^{1/2}(\kappa)} \right) \right\},$$

where $\Phi$ is the cumulative distribution function of the Gaussian distribution with zero mean and unit variance on $\mathbb{R}$.

**Proof.** The proof is a simple application of Theorem 43.

Based on Theorem 4, since $\mathbb{P}(X_n \neq Y_n) = K^n((x, y), \Delta_\gamma^\ast)$ where $(X_k, Y_k)_{k \in \mathbb{N}}$ is defined by $A^2$ with $(X_0, Y_0) = (x, y) \in X^2$, we can derive minorization conditions for the Markov kernel $R_\gamma^n$ with $n \in \mathbb{N}^*$ for any $\gamma \in (0, \gamma]$ depending on the assumption we make on $\kappa$ in $A^2(X^2)$. More precisely, these minorization conditions are derived using $K^{[1/\gamma]}_\ell$ with $\ell \in \mathbb{N}^*$. This is a requirement to obtain sharp bounds in the limit $\gamma \to 0$. Indeed, for any $x, y \in X$, based only on the results of Theorem 4, we get that for any $\ell \in \mathbb{N}^*$, $\lim_{\gamma \to 0} \|\delta_x R_\gamma^\ell - \delta_y R_\gamma^\ell\|_{TV} < 1$, whereas the following proposition implies that for any $\ell \in \mathbb{N}^*$, $\lim_{\gamma \to 0} \|\delta_x R_\gamma^{[1/\gamma]} - \delta_y R_\gamma^{[1/\gamma]}\|_{TV} < 1$.

**Proposition 5.** Let $A \in \mathcal{B}(\mathbb{R}^{2d})$ and assume $A^1$ and $A^2(A)$ hold. Let $(X_k, Y_k)_{k \in \mathbb{N}}$ be defined by $A^2$ with $(X_0, Y_0) = (x, y) \in A \cap X^2$ and $\gamma \in (0, \gamma]$. Then for any $\ell \in \mathbb{N}^*$ and $\gamma \in (0, \gamma]$,

$$\mathbb{P}(X_{\ell[\gamma]} \neq Y_{\ell[\gamma]} \text{ and for any } k \in \{1, \ldots, n-1\}, (X_k, Y_k) \in A) \leq 1 - 2 \Phi \left( -\alpha^{-1}(\kappa, \gamma, \ell)\|x - y\|/2 \right),$$

where

(a) $\alpha(\kappa, \gamma, \ell) = -\kappa^{-1}(\gamma) \left[ \exp(-\ell \kappa(\gamma)) - 1 \right]$ if $A^2(A)$-(i) holds;

(b) $\alpha(\kappa, \gamma, \ell) = \ell$ if $A^2(A)$-(ii) holds;

(c) $\alpha(\kappa, \gamma, \ell) = \kappa^{-1}(\gamma) \left[ 1 - \exp(-\ell \kappa(\gamma)/(1 + \gamma \kappa(\gamma))) \right]$ if $A^2(A)$-(iii) holds.

**Proof.** The proof is postponed to Appendix A.1.

Depending on the conditions imposed on $\kappa$ defined in $A^2(X^2)$, we obtain the following consequences of Proposition 5 which establish, either an explicit convergence bound in total variation for $R_\gamma$, or a quantitative minorization condition satisfied by this kernel.
Corollary 6. Assume A1 and A2(X²).

(a) If A2(X²)-(i) holds and κ₋ = supγ∈(0,γ] κ(γ) < 0. Then, for any γ ∈ (0,γ], R₇ admits a unique invariant probability measure πγ and we have for any γ ∈ (0,γ], x ∈ ℝᵈ and ℓ ∈ N⁺,

\[ \|d_x \gamma_{ℓ}^{[1/γ]} - πγ\|_{TV} ≤ 1 - 2 \int_{ℝᵈ} \Phi \left\{ -(-κ₋)^{1/2} ||x - y|| / \left( 2(\exp(-ℓκ₋) - 1)^{1/2} \right) \right\} dπγ(y). \]

(b) If A2(X²)-(ii) holds and, in addition, assume that for any γ ∈ (0,γ], R₇ admits an invariant probability measure πγ, then we have for any γ ∈ (0,γ], x ∈ ℝᵈ and ℓ ∈ N⁺,

\[ \|d_x \gamma_{ℓ}^{[1/γ]} - πγ\|_{TV} ≤ 1 - 2 \int_{ℝᵈ} \Phi \left\{ -||x - y|| / (2ℓ^{1/2}) \right\} dπγ(y). \]

Proof. The proof is postponed to Appendix A.2.

In other words, if T₇ is a contractive mapping, see A2(X²)-(i), then for x ∈ ℝᵈ the convergence of (d_x R₇^{[1/γ]})_{ℓ ∈ N⁺} to πγ in total variation is exponential in ℓ. If T₇ is non expansive, see A2(X²)-(ii), and R₇ admits an invariant probability measure πγ, for any x ∈ ℝᵈ, the convergence of (d_x R₇^{[1/γ]})_{ℓ ∈ N⁺} to πγ in total variation is linear in ℓ^{1/2}. In the case where T₇ is non expansive, see A2(X²)-(ii), or simply Lipschitz, see A2(X²)-(iii) and no additional assumption is made, we do not directly obtain contraction in total variation but only minorization conditions.

Corollary 7. Assume A1 and A2(X²). Then, for any γ ∈ (0,γ],

(a) if A2(X²)-(i) holds, for any x,y ∈ X with ||x - y|| ≤ M with M ≥ 0 and ℓ ∈ N⁺ with ℓ ≥ \left[ M^2 \right],

\[ K^{[1/γ]}(x,y,\Delta) ≤ 1 - 2\Phi \left( -1/2 \right); \]

(b) if A2(X²)-(iii) holds, for any x,y ∈ X and ℓ ∈ N⁺,

\[ K^{[1/γ]}(x,y,\Delta) ≤ 1 - 2\Phi \left\{ -(1 + \gamma)^{1/2}(1 + κ₋)^{1/2} ||x - y|| / 2 \right\}, \]

where κ₊ = supγ∈(0,γ] κ(γ).

Proof. The proof is postponed to Appendix A.3.

In our application below, we are mainly interested in the case where R₇ satisfies a geometric drift condition. Let (Y, Y) be a measurable space, λ ∈ (0,1), A ≥ 0, V : Y → [1, +∞) be a measurable function and C ∈ Y.

D 1 (D₄(V,λ,A,C)). A Markov kernel R on Y × Y satisfies the discrete Foster-Lyapunov drift condition if for all y ∈ Y

\[ RV(y) ≤ λV(y) + AI_C(y). \]

The index d in D₄ stands for “discrete” as we will introduce the continuous-time counterpart of this drift condition, denoted by D₄c, in Section 5.1. Note that this drift condition implies the existence of an invariant probability measure if R is a Feller kernel and the level sets of V are
compact, see [22, Theorem 12.3.3]. In the sequel, we are interested in establishing convergence results in the Wasserstein metric $W_c$ associated with the cost $c : (x, y) \mapsto 1_{\Delta_c^k}(x, y)\psi(x, y)$ (30)

where $\psi : X \times X \to [0, +\infty)$ satisfies for any $x, y, z \in X$, $\psi(x, y) = \psi(y, x)$, $\psi(x, z) \leq \psi(x, y) + \psi(y, z)$ and $\psi(x, y) = 0$ implies that $x = y$. Note that under these conditions on $\psi$, $c$ defines a metric on $\mathbb{R}^d$. Let $\mu, \nu$ be two probability measures over $X$, we highlight three cases.

- total variation: if $\psi = 1$ then $W_c(\mu, \nu) = ||\mu - \nu||_{TV}$;
- $V$-norm: if $\psi(x, y) = \{\psi(x) + \psi(y)\}/2$ where $V : \mathbb{R}^d \to [1, +\infty)$ is measurable then $W_c(\mu, \nu) = ||\mu - \nu||_V$;
- total variation + Kantorovitch-Rubinstein metric: if $\psi(x, y) = 1 + \theta \|x - y\|$ with $\theta > 0$, then by definition of Wasserstein metrics, $W_c(\mu, \nu) \geq ||\mu - \nu||_{TV} + \partial W_1(\mu, \nu)$.

We now state convergence bounds for Markov kernels which satisfy one of the conclusions of Corollary 7. Indeed, in order to deal with the two assumptions $A2(X^2)$-(ii) and $A2(X^2)$-(iii) together, we provide a general result regarding the contraction of $R_\gamma$ in the metric $W_c$ for some cost function $c$ on $X$. This result is based on an abstract condition on $K^{[1/\gamma]}_\gamma 1_{\Delta_c^k}$, which is satisfied under $A2(X^2)$-(ii) or $A2(X^2)$-(iii) by Corollary 7 with $K_{\gamma} \leftarrow K_{\gamma}$, and a drift condition for $K_\gamma$, where $K_\gamma$ is a Markov coupling kernel for $R_\gamma$. We recall that for any $M \geq 0$,

$$\Delta_{X,M} = \{(x, y) \in X : \|x - y\| \leq M\}. \quad (31)$$

**Theorem 8.** Assume that there exist $\lambda \in (0, 1)$, $A \geq 0$, $M_\delta > 0$, a measurable function $\psi : X \times X \to [1, +\infty)$, $C \in \mathcal{X}^{<2}$ with $C \subset \Delta_{X,M_\delta}$ and for any $\gamma \in (0, \bar{\gamma}]$, $K_\gamma$ a Markov coupling kernel for $R_\gamma$ satisfying $D_d(\psi, \lambda^2, A\gamma, C)$. Further, assume that for any $\gamma \in (0, \bar{\gamma}]$, $\Delta_X$ is absorbing for $K_\gamma$, i.e. for any $x \in X$, $K_\gamma 1_{\Delta_X}(x, x) = 1$, and that there exists $\Psi : (0, \bar{\gamma}] \times \mathbb{N}^* \times \mathbb{R}_+ \to [0, 1]$ such that for any $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^*$ and $x, y \in X$

$$\bar{K}^{[1/\gamma]}_\gamma((x, y), \Delta_X^k) \leq 1 - \Psi(\gamma, \ell, \|x - y\|), \text{ and for any } M \geq 0, \inf_{(x,y) \in \Delta_{X,M}} \Psi(\gamma, \ell, \|x - y\|) > 0. \quad (32)$$

Then, for any $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^*$ and $x, y \in X$

$$W_c(\delta_x R_\gamma^\ell, \delta_y R_\gamma^\ell) \leq \bar{K}^{[\gamma]}_\gamma c(x, y) \leq \lambda^{k/4}[\bar{D}_1 c(x, y) + \bar{D}_2 1_{\Delta_Y^k}(x, y)] + \bar{C}_1 \bar{\rho}_1^{k/4} 1_{\Delta_X^k}(x, y), \quad (33)$$

where

$$\bar{D}_1 = 1 + 4A\log^{-1}(1/\lambda)/\bar{\gamma}^1, \quad \bar{D}_2 = \bar{D}_1 A\gamma^{-1}(1 + \gamma)^{\ell}(1 + \gamma)^{\ell}, \quad \bar{C}_1 = 8A \log^{-1}(1/\bar{\rho}_1)/\bar{\rho}_1^{\gamma},$$

$$\log(\bar{\rho}_1) = \{(\log(\lambda) \log(1 - \varepsilon_{d,1})) / \{-(\log(\bar{c}_1) + \log(1 - \varepsilon_{d,1}))\}, \quad \bar{c}_1 = \bar{B}_d + A\gamma^{-1}(1 + \gamma)^{\ell}(1 + \gamma), \quad \bar{\varepsilon}_{d,1} = \inf_{\gamma \in (0, \bar{\gamma}), \langle x, y \rangle \in \Delta_{X,M_\delta}} \Psi(\gamma, \ell, \|x - y\|)$$

$$\bar{B}_d = \sup_{\langle x, y \rangle \in C} \psi(x, y).$$

In addition, if $\bar{\gamma} \leq 1$ and $\varepsilon_{d,1} \leq 1 - c^{-1}$, then

$$\log^{-1}(\bar{\rho}_1^{-1}) \leq [1 + \log(\bar{B}_d) + \log(1 + 2A\ell) + 2\ell \log(\lambda^{-1})] / [\log(\lambda^{-1}) \varepsilon_{d,1}].$$
Proof. The proof is postponed to Appendix A.4. 

We emphasize that (32) is satisfied under A2(\mathcal{X})-(ii) or A2(\mathcal{X})-(iii) by Corollary 7 with \tilde{K}_\gamma \leftarrow K_\gamma.

Further, note that in (66), the leading term, \tilde{C}_1 \tilde{\rho}_1^{\gamma/4}, does not depend on \(x, y \in \mathcal{X}\). Indeed, the rate in front of the initial conditions \(\mathcal{V}(x, y)\) is given by \(\lambda^{\gamma/4}\) which is always smaller than \(\tilde{\rho}_1^{\gamma/4}\). Therefore, Theorem 8 implies in particular that for any \(\gamma \in (0, \overline{\gamma}]\), \(x, y \in \mathcal{K}\) and \(k \in \mathbb{N}\)

\[ W_c(\delta_x R^k, \delta_y R^k) \leq \tilde{\rho}_1^{\gamma/4}[D_1 + D_2 + \tilde{C}_1]c(x, y). \]  

(34)

We conclude this section with two propositions which highlight the usefulness of the conclusions of Theorem 8 to establish convergence estimates with respect to different metrics. First, in Proposition 9, under additional conditions on \(\Psi\) and on \(\mathcal{V}\) (which will be satisfied in our applications, see Corollary 14) we get a similar result to (34) replacing \(c\) by \((x, y) \mapsto \|x - y\|\), i.e. replacing \(W_c\) by \(W_1\).

**Proposition 9.** Assume that the conditions of Theorem 8 are satisfied with for any \(x, y \in \mathcal{X}\), \(\mathcal{V}(x, y) = 1 + \vartheta \|x - y\|\), where \(\vartheta > 0\). In addition, assume that the following conditions hold.

(i) For any \(\gamma \in (0, \overline{\gamma}]\), \(t \mapsto \Psi(\gamma, 1, t)\) is convex on \(\mathbb{R}_+\), admits a right-derivative at 0, denoted by \(\Psi'(\gamma, 1, 0)\), and \(a = \inf_{t \in (0, \overline{\gamma}]} \Psi'(\gamma, 1, 0) > -\infty\).

(ii) There exists \(\kappa \geq 0\) such that for any \(x, y \in \mathcal{X}\), \(\tilde{K}_\gamma \|x - y\| \leq (1 + \gamma \kappa) \|x - y\|\).

Then there exist \(\overline{D}_3 \geq 0\) and \(\tilde{\rho}_1 \in [0, 1)\) such that for any \(\gamma \in (0, \overline{\gamma}]\), \(x, y \in \mathcal{X}\) and \(k \in \mathbb{N}\)

\[ W_1(\delta_x R^k, \delta_y R^k) \leq \tilde{K}_\gamma \|x - y\| \leq \overline{D}_3 \tilde{\rho}_1^{\gamma/4} \|x - y\|, \]  

(35)

with \(\tilde{\rho}_1\) given in Theorem 8 and \(\overline{D}_3 \) explicit in the proof. 
Proof. The proof is postponed to Appendix A.5. 

As a consequence, if \(\mathcal{X}\) is closed, the Markov kernel \(R_\gamma\) admits a unique invariant probability measure \(\pi_\gamma\), i.e. \(\pi_\gamma = \pi_\gamma R_\gamma\), using [41, Chapter 1, 6, A.1], and since we have that \(\mathcal{P}_1(\mathcal{X}) = \{\mu\text{ probability measure on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) : \int_{\mathbb{R}^d} \|x\| \, d\mu(x) < +\infty\}\), endowed with \(W_1\) is complete, see [85, Theorem 6.18]. Further, for any \(\gamma \in (0, \overline{\gamma}]\), using [59, Theorem 1], there exists a distance \(d_\gamma\) on \(\mathcal{P}_1(\mathcal{X})\), topologically equivalent to \(W_1\), such that \(\mathcal{P}_1\) is complete and for any \(x, y \in \mathbb{R}^d\) and \(k \in \mathbb{N}\)

\[ d_\gamma(\delta_x R^k, \delta_y R^k) \leq \rho^{\gamma/4} d_\gamma(\delta_x, \delta_y). \]

A similar result to (35) in Proposition 9 can be derived when replacing \(W_1\) by \(W_p\) with \(p \in \mathbb{N}\), if we assume some Foster-Lyapunov condition with respect to \((x, y) \mapsto \|x - y\|^p\).

**Proposition 10.** Assume that there exist \(\bar{\rho} \in (0, 1)\), \(\overline{D} \geq 0\) and for any \(\gamma \in (0, \overline{\gamma}]\), \(\tilde{K}_\gamma\) a Markov coupling kernel for \(R_\gamma\) satisfying for any \(x, y \in \mathcal{X}\) and \(k \in \mathbb{N}\)

\[ \tilde{K}_\gamma \|x - y\| \leq \overline{D} \bar{\rho}^{k\gamma} \|x - y\|. \]

In addition, assume that for any \(\gamma \in (0, \overline{\gamma}]\) and \(q \in \mathbb{N}\), \(\tilde{K}_\gamma\) satisfies \(D_4((x, y) \mapsto \|x - y\|^q, \tilde{K}_\gamma, \tilde{A}_\gamma)\) with \(\tilde{\lambda}_q \in (0, 1]\) and \(\tilde{A}_\gamma \geq 0\). Then, for any \(p \geq 1\) and \(\alpha \in (p, +\infty)\) there exists \(\overline{D}_{4, \alpha} \geq 0\) such that

\[ W_p(\delta_x R^k, \delta_y R^k) \leq (\tilde{K}_\gamma \|x - y\|^p)^{1/p} \leq \overline{D}_{4, \alpha} \bar{\rho}^{k\gamma/\alpha} \left\{ \|x - y\| + \|x - y\|^{1/\alpha} \right\}, \]

with \(\overline{D}_{4, \alpha} \) explicit in the proof.
4 Application to the projected Euler-Maruyama discretization

Here we consider the case in which the operator $\mathcal{T}_\gamma$ in (19) is given by the discretization of a diffusion (1). More precisely, for $b : \mathbb{R}^d \to \mathbb{R}^d$, we study the projected Euler-Maruyama discretization associated to the diffusion with drift function $b$ and diffusion coefficient $\text{Id}$, i.e. we consider the following assumption for $X \subset \mathbb{R}^d$.

\textbf{B1 $(X)$}. $X$ is assumed to be a closed convex (non-empty) subset of $\mathbb{R}^d$, $\Pi = \Pi_X$ is the orthogonal projection onto $X$ defined in (24) and

$$\mathcal{T}_\gamma(x) = x + \gamma b(x) \text{ for any } \gamma > 0 \text{ and } x \in X,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

Note that if $X = \mathbb{R}^d$ and $\Pi = \text{Id}$, then this scheme is the classical Euler-Maruyama discretization of a diffusion with drift $b$ and diffusion coefficient $\text{Id}$. The application to the tamed Euler-Maruyama discretization of the results of Section 3 is given in Appendix G. In what follows, we show the convergence in weighted total variation for the projected Euler-Maruyama discretization and discuss the dependency of the constants appearing in the bounds we obtain with respect to the properties we assume on the drift $b$. We first derive minorization conditions or convergence in total variation depending on the regularity/curvature assumption on the drift $b$ in Section 4.1. Drift conditions and the ensuing convergence when combined with the minorization assumption are studied in Section 4.2.

4.1 Minorization condition

First, we show that some regularity/curvature conditions on the drift $b$ imply condition $\mathbf{A2}(X^2)$ for $\mathcal{T}_\gamma$ given by (36). Let $m \in \mathbb{R}$.

\textbf{B2}. There exists $L \geq 0$ such that $b$ is $L$-Lipschitz, i.e. for any $x,y \in X$, $\|b(x) - b(y)\| \leq L\|x - y\|$ and $b(0) = 0$.

\textbf{B3 $(m)$}. For any $x,y \in X$,

$$\langle b(x) - b(y), x - y \rangle \leq -m \|x - y\|^2.$$

Note that $\textbf{B2}$ implies $\textbf{B3}(-L)$. However, we are interested in the case where $|m|$ is possibly strictly smaller than $L$. If there exists $U \in C^1(X)$ such that for any $x \in X$, $b(x) = -\nabla U(x)$ and $\textbf{B3}(m)$ holds with $m = 0$, respectively $m > 0$ then $U$ is convex, respectively strongly convex. Note that $\textbf{B3}(0)$ does not imply that $\mathcal{T}_\gamma$ given by (36) is non-expansive, therefore we consider the following assumption.

\textbf{B4}. There exists $m_b > 0$ such that for any $x,y \in X$,

$$\langle b(x) - b(y), x - y \rangle \leq -m_b \|b(x) - b(y)\|^2.$$
Note that B4 implies that B2 with \( L = m^{-1} \) and B3(0) hold. Conversely, in the case where \( X = \mathbb{R}^d \) and there exists \( U \in C^d(\mathbb{R}^d) \) such that for any \( x \in \mathbb{R}^d, b(x) = -\nabla U(x) \), [64, Theorem 2.1.5] implies that under B2 and B3(0), B4 holds with \( m = L^{-1} \). Based on Proposition 11 and assuming B1, we obtain the following results on the Markov kernel \( R_\gamma \) defined by (20) with \( \gamma > 0 \).

**Proposition 11.** Assume B1(\( X \)) holds for \( X \subset \mathbb{R}^d \).

(a) If B2 and B3(m) hold with \( m \in \mathbb{R} \). Then (25) in A2(\( X^2 \)) holds for any \( \gamma > 0 \) with \( \kappa(\gamma) = -2m + L^2 \gamma \). In particular, if \( m > 0 \) then A2(\( X^2 \))-(i) holds for any \( \bar{\gamma} < 2m/L^2 \) and if \( m \leq 0 \) then A2(\( X^2 \))-(ii) holds for any \( \bar{\gamma} > 0 \);

(b) If B4 holds, then A2(\( X^2 \))-(ii) holds with \( \kappa(\gamma) = 0 \) for any \( \bar{\gamma} \leq 2m_b \).

**Proof.** The proof is postponed to Appendix B.1.

Combining Proposition 11 and Proposition 5 and/or Corollary 6, we can draw the following conclusions.

If B2 and B3(m) hold with \( m > 0 \), then we obtain, by Proposition 11-(a) and Proposition 5-(a), that for any \( \gamma \in (0, 2m/L^2) \) and \( \ell \in \mathbb{N}^* \), (27) holds with \( \alpha = \alpha_- \) given by

\[
\alpha_-(\kappa, \gamma, \ell) = -\frac{\exp(-\ell(-2m + L^2 \gamma)) - 1}{-2m + L^2 \gamma}.
\]

In addition, Corollary 6-(a) implies that for any \( \gamma \in (0, 2m/L^2) \) and \( x \in X \), \( \xi_x R^{1/\gamma \ell}_\gamma \) converges exponentially fast to its invariant probability measure \( \pi_\gamma \) in total variation, with a rate which does not depend on \( \gamma \), but only on \( m \).

Under B4, combining Proposition 11-(b) and Corollary 7-(a) we obtain that on any compact set \( K \subset X, R^{1/\gamma \ell}_\gamma \) satisfies the minorization condition (28) with \( \ell \geq \text{diam}(K)^2 \). In addition, if \( R_\gamma \) admits an invariant probability measure \( \pi_\gamma \), then Corollary 6-(b) implies that for any \( \gamma \in (0, 2m_b] \) and \( x \in X \), \( \xi_x R^{1/\gamma \ell}_\gamma \) converges linearly in \( \ell^{1/2} \) to \( \pi_\gamma \) in total variation.

In the case where B2 and B3(m) are satisfied with \( m \in \mathbb{R}^- \), we obtain that for any \( \gamma > 0 \) and \( \ell \in \mathbb{N}^* \), (27) holds with \( \alpha = \alpha_+ \) given by

\[
\alpha_+(\kappa, \gamma, \ell) = (-2m + L^2 \gamma)^{-1} \left(1 - \exp \left[-\ell(-2m + L^2 \gamma)/(1 + \gamma(-2m + L^2 \gamma))\right]\right) \leq (-2m + L^2 \gamma)^{-1},
\]

which implies that the bound given by Proposition 5-(c) does not go to 0 when \( \ell \) goes to infinity. Therefore we cannot directly conclude that the Markov chain converges in total variation. However, by Proposition 11-(a), Corollary 7-(b) shows that for any \( \gamma \in (0, \bar{\gamma}] \) with \( \bar{\gamma} > 0 \) and \( \ell \in \mathbb{N}^* \), \( R^{1/\gamma \ell}_\gamma \) satisfies the minorization condition (29), with constants which only depend on \( m \) and \( L \). Note however that in (37) the influence of \( m \) is different than the one of \( L \) and this result justifies the two assumptions B2 and B3(m).

### 4.2 Drift conditions and convergence

In the sequel of this section, we consider several assumptions on the drift function \( b \) which imply Foster-Lyapunov drift conditions on the Markov coupling kernel \( K_\gamma \) defined in (23). These results in combination with Proposition 11 will allow us to use Theorem 8, see also Theorem 46.
4.2.1 Strongly convex at infinity

First, we consider conditions on $b$ which imply that $R_\gamma$ for $\gamma \in (0, \bar{\gamma}]$, is geometrically convergent in a metric which dominates the total variation distance and the Wasserstein distance of order 1. This result will be an application of Theorem 8 and the constants we end up with are independent of the dimension $d$. To do so, we establish that there exists a Lyapunov function $\psi$ for which $K_\gamma$ satisfies for $\gamma \in (0, \bar{\gamma}]$, $D_\psi(\psi', \lambda \gamma, A \gamma, \Delta_X, M_\lambda)$ where $\Delta_X, M_\lambda$ is given by (31) and $M_\lambda$ is independent of the dimension $d$. Assume in addition either $B_3(\delta)$ for $\delta \in \mathbb{R}$ or $B_4$. Then the conditions and the conclusions of Theorem 8 hold with $\gamma$, $\lambda$ and $A$ given by Proposition 12-(b), $\bar{\lambda}_\lambda = R_1$, $K_\gamma$ given by (23) for any $\gamma \in (0, \bar{\gamma}]$, $\psi' = \psi_1$ defined in (38), and for any $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^*$ and $t > 0$.

**Proof.** The proof is postponed to Appendix B.2. \qed

**Theorem 13.** Assume $B_1(\delta)$ for $\delta \in \mathbb{R}$, $B_2$ and $C_1$. Assume in addition either $B_3(\delta)$ for $\delta \in \mathbb{R}$ or $B_4$. Then the conditions and the conclusions of Theorem 8 hold with $\gamma$, $\lambda$ and $A$ given by Proposition 12-(b), $\bar{\lambda}_\lambda = R_1$, $K_\gamma$ given by (23) for any $\gamma \in (0, \bar{\gamma}]$, $\psi' = \psi_1$ defined in (38), and for any $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^*$ and $t > 0$.

under $B_3(\delta)$, $\psi(\gamma, \ell, t) = 2\Phi\{-t/(2\Xi^{1/2}e^{1/\gamma}(\kappa))\}$, \hspace{1cm} (40)

under $B_4$, $\psi(\gamma, \ell, t) = \begin{cases} 2\Phi\{-1/2\} & \text{if } \ell \geq \lfloor R_1 \rfloor^2 \text{ and } t \leq R_1, \\ 2\Phi\{-t/(2\Xi^{1/2}e^{1/\gamma}(\kappa))\} & \text{otherwise}, \end{cases}$ \hspace{1cm} (41)

where $\kappa$ is given in Proposition 11-(a) and $\Xi^{1/2}e^{1/\gamma}$ in (26).
Proof. First, note that for any $\gamma > 0$, $\Delta_X$ is absorbing for $K_\gamma$ by definition of the reflection coupling, see (23). We assume that $B_3(m)$ holds. Let $\tilde{\gamma} \in (0, 2m^+_1/L^2)$. Using Proposition 12-(b) we obtain that $\mathcal{V}_1$ given by (38) satisfies $D_2(\mathcal{V}_1, \lambda, A_\gamma, \Delta_X, R_1)$ for any $\gamma \in (0, \tilde{\gamma}]$ with $\lambda$ and $A$ given in (39). Using Theorem 4, Proposition 11-(a), we have for any $\gamma \in (0, \tilde{\gamma}]$, $\ell \in \mathbb{N}^*$ and $x, y \in X$

$$K_\gamma^{1/\gamma}(\{x, y\}, \Delta_X) \leq 1 - 2\Phi\left(-\Xi_{1/\gamma}(k)\|x - y\|/2\right),$$

where $\Xi(k) = -2m + \gamma L^2$, which concludes the proof.

The proof under $B_4$ follows the same lines upon noting that $B_4$ implies that $B_3(0)$ holds and using Proposition 11-(b) instead of Proposition 11-(a).

Let $\tilde{\gamma} \in (0, \max(2m^+_1/L^2, 1))$, $\ell \in \mathbb{N}^*$ specified below, $\lambda_{\tilde{\gamma}, a}, \rho_{\tilde{\gamma}, a} \in (0, 1)$ and $D_{\tilde{\gamma}, 1.a}, D_{\tilde{\gamma}, 2.a}, C_{\tilde{\gamma}, a} \geq 0$ the constants given by Theorem 13, such that for any $k \in \mathbb{N}$, $\gamma \in (0, \tilde{\gamma}]$ and $x, y \in X$

$$W_{c_1}(\delta_x R^k_{\gamma}, \delta_y R^k_{\gamma}) \leq K_\gamma^{k/4}c_1(x, y) \leq \lambda_{\tilde{\gamma}, a}^{k/4}[D_{\tilde{\gamma}, 1.a}c_1(x, y) + D_{\tilde{\gamma}, 2.a}1_{\Delta_X} + C_{\tilde{\gamma}, a}1_{\Delta_X}], \quad (42)$$

with $c_1(x, y) = 1_{\Delta_X}(x, y)(1 + \|x - y\|/R_1)$ for any $x, y \in X$. Note that by (34), this result implies that for any $k \in \mathbb{N}, \gamma \in (0, \tilde{\gamma}]$ and $x, y \in X$

$$W_{c_1}(\delta_x R^k_{\gamma}, \delta_y R^k_{\gamma}) \leq \{D_{\tilde{\gamma}, 1.a} + D_{\tilde{\gamma}, 2.a} + C_{\tilde{\gamma}, a}\} \rho_{\tilde{\gamma}, a}c_1(x, y).$$

We now give upper-bounds on $\rho_{\tilde{\gamma}, a}$. Note that using Theorem 8, we obtain that the following limits exist and do not depend on $L$

$$D_{1.a} = \lim_{\tilde{\gamma} \to 0} D_{\tilde{\gamma}, 1.a}, \quad D_{2.a} = \lim_{\tilde{\gamma} \to 0} D_{\tilde{\gamma}, 2.a}, \quad C_a = \lim_{\tilde{\gamma} \to 0} C_{\tilde{\gamma}, a}, \quad \lambda_a = \lim_{\tilde{\gamma} \to 0} \lambda_{\tilde{\gamma}, a}, \quad \rho_a = \lim_{\tilde{\gamma} \to 0} \rho_{\tilde{\gamma}, a}. \quad (43)$$

Once again, we point out that $\lambda_{\tilde{\gamma}, a} \leq \rho_{\tilde{\gamma}, a}$ in Theorem 8. In the following discussion we assume that $B_1(X)$ for $X \subset \mathbb{R}^d$, $B_2$ and $C_1$ hold. We now give upper bounds on the rate $\rho_{\tilde{\gamma}, a}$ and $\rho_a$ using Theorem 8 depending on the assumptions in Theorem 13.

(a) If $B_4$ holds, set $\ell = \lceil R_1^a \rceil$. Using that $2\Phi(-1/2) \leq 1 - e^{-1}$ and choosing $m^+_1$ sufficiently small such that the conditions of Theorem 8 hold, we have

$$\log^{-1}(\rho_{\tilde{\gamma}, a}^{-1}) \leq \left[1 + \log(2) + \log(1 + 2(1 + R_1^a) m^+_1) + 2(1 + R_1^a)(m^+_1 - \tilde{\gamma} L^2/2)\right] / [(m^+_1 - \tilde{\gamma} L^2/2)\Phi\{-1/2\}] . \quad (44)$$

Taking the limit $\tilde{\gamma} \to 0$ in (44) and using that for any $t \geq 0$, $\log(1 + t) \leq t$, we get that

$$\log^{-1}(\rho_{\tilde{\gamma}, a}^{-1}) \leq \left[1 + \log(2)\right] / [(m^+_1 - \tilde{\gamma} L^2/2)] / [\Phi\{-1/2\}] + 4(1 + R_1^a)/\Phi\{-1/2\} . \quad (45)$$

The leading term in (45) is of order $\max(R_1^a, 1/m^+_1)$, which corresponds to the one identified in [34, Theorem 2.8] and is optimal, see [32, Remark 2.10].

(b) If $B_3(m)$ holds with $m \in \mathbb{R}_-$, set $\ell = \lceil R_1^a \rceil$. Choosing $m^+_1 > 0$ sufficiently small and $R_1, |m|$ sufficiently large such that the conditions of Theorem 8 hold, we have

$$\log^{-1}(\rho_{\tilde{\gamma}, a}^{-1}) \leq \left[1 + \log(2) + \log\left(1 + 2(1 + R_1^a)(m^+_1 - m)\right) + 2(1 + R_1^a)(m^+_1 - \tilde{\gamma} L^2/2)\right] / \left[(m^+_1 - \tilde{\gamma} L^2/2)\Phi\{-1/2\} (k)\right] . \quad (46)$$
Taking the limit $\tilde{\gamma} \to 0$ in this result and using \begin{equation} (48) \end{equation}
we get that
\begin{align}
\log^{-1}(\rho^{-1}) &\leq \left[ 1 + \log(2) + \log(1 + 2\{m^+ - m\}) + 2m^+ \right] / \left[ m^+ \Phi\{(-m)^{1/2}R_1/(2 - 2e^{2\alpha R_1^2})^{1/2}\} \right] . 
\end{align} \tag{47}

The comparison between this rate and the ones derived in recent works is conducted in Section 2.1. We extend our result to other Wasserstein metrics in the following proposition.

**Corollary 14.** Assume B\texttt{1}(X) for $X \subset \mathbb{R}^d$, B\texttt{2} and C\texttt{1}. Assume in addition either B\texttt{3}(m) for $m \in \mathbb{R}_-$ or B\texttt{4}. Then for any $p \in \mathbb{N}$, $\alpha \in (p, +\infty)$, $\gamma \in (0, \tilde{\gamma}]$, $x, y \in X$ and $k \in \mathbb{N}$ we have
\begin{align*}
W_1(\delta_x R_{\gamma}^k, \delta_y R_{\gamma}^k) &\leq D_{3,\tilde{\gamma},a}^{k\gamma/4} \|x - y\| , \\
W_p(\delta_x R_{\gamma}^k, \delta_y R_{\gamma}^k) &\leq D_{3,\tilde{\gamma},a}^{k\gamma/(4\alpha)} \left\{ \|x - y\| + \|x - y\|^{1/\alpha} \right\} ,
\end{align*}
where $\rho_{\tilde{\gamma},a}$, $D_{3,\tilde{\gamma},a}$ and $D_{\alpha,\tilde{\gamma},a}$ are given in \begin{equation} (49) \end{equation}, Proposition 9 and Proposition 10 respectively.

**Proof.** The proof is postponed to Appendix B.3.

**4.2.2 Other curvature conditions**

We now derive uniform ergodic convergence in $V$-norm under weaker conditions than C\texttt{1}. The following assumption ensures that the radial part of $b$ decrease faster than a linear function with slope $-m^+_2 < 0$.

**C2.** There exist $R_2 \geq 0$ and $m^+_2 > 0$ such that for any $x \in \overline{B}(0, R_2) \cap X$,
\begin{equation}
(b(x), x) \leq -m^+_2 \|x\|^2 . 
\end{equation}

In the next proposition we derive a Foster-Lyapunov drift condition for $V_2 : X^2 \to [1, +\infty)$ defined for any $x, y \in X$ by
\begin{equation}
V_2(x, y) = 1 + \|x\|^2 / 2 + \|y\|^2 / 2 , \quad c_2(x, y) = \mathbb{I}_{\Delta_b}(x, y) V_2(x, y) . 
\end{equation}

Note that for any $x, y \in X$, $V_2(x, y) = \{V(x) + V(y)/2 \}$ with $V(x) = 1 + \|x\|^2$.

**Proposition 15.** Assume B\texttt{1}(X) for $X \subset \mathbb{R}^d$, B\texttt{2}, B\texttt{3}(m) for $m \in \mathbb{R}_-$ and C\texttt{2}. Then $K_\alpha$ defined by \begin{equation} (23) \end{equation} satisfies $D_d(V_2, \lambda^\gamma, A\gamma, B(0, R) \times B(0, R))$ for any $\gamma \in (0, \tilde{\gamma})$ where $\tilde{\gamma} \in (0, 2m^+_2/L^2)$ and
\begin{equation}
\lambda = \exp\{-(m^+_2 - \tilde{\gamma}L^2/2)\} , \quad A = d + 2R^2(m^+_2 - m) + 2m^+_2 , \quad R = \sqrt{2}\lambda^{-\tilde{\gamma}} A^{1/2} \log^{-1/2}(1/\lambda) . 
\end{equation}

**Proof.** The proof is postponed to Appendix B.4.

**Theorem 16.** Assume B\texttt{1}(X) for $X \subset \mathbb{R}^d$, B\texttt{2} and C\texttt{2}. Assume in addition either B\texttt{3}(m) for $m \in \mathbb{R}_-$ or B\texttt{4}. Then the conditions and conclusions of Theorem 8 hold with $V = V_2$ defined in \begin{equation} (48) \end{equation}, $\tilde{\gamma}$, $\lambda$, $A$ and $M_d = 2R$ given by Proposition 15, and $\Psi$ given by \begin{equation} (40) \end{equation} or \begin{equation} (41) \end{equation}.

**Proof.** The proof is similar to the one of Theorem 13.
Let \( \tilde{\gamma} \in (0, \max(2m_2^+ / L^2, 1)) \), \( \ell \in \mathbb{N}_+ \) specified below, \( \lambda_{\tilde{\gamma}, b}, \rho_{\tilde{\gamma}, b} \in (0, 1) \) and \( D_{\tilde{\gamma}, 1, b}, D_{\tilde{\gamma}, 2, b}, C_{\tilde{\gamma}, b} \geq 0 \) the constants given by Theorem 16, such that for any \( k \in \mathbb{N}, \gamma \in (0, \tilde{\gamma}) \) and \( x, y \in X \)

\[
W_{c_2}(\delta_x R_{\tilde{\gamma}}^b, \delta_y R_{\tilde{\gamma}}^b) \leq K^b_{\tilde{\gamma}} c_2(x, y) \leq \lambda_{\tilde{\gamma}, b}^{k/4} [D_{\tilde{\gamma}, 1, b} c_2(x, y) + D_{\tilde{\gamma}, 2, b} \Delta_x] + C_{\tilde{\gamma}, b} \rho_{\tilde{\gamma}, b}^{k/4},
\]

with \( c_2(x, y) = 1_{\Delta_x}(x, y) \{V(x) + V(y)\}/2 \) for any \( x, y \in X \). Note that by (48), this result implies that for any \( k \in \mathbb{N}, \gamma \in (0, \tilde{\gamma}) \) and \( x, y \in X \)

\[
\|\delta_x R_{\tilde{\gamma}}^b - \delta_y R_{\tilde{\gamma}}^b\|_V \leq \{D_{\tilde{\gamma}, 1, b} + D_{\tilde{\gamma}, 2, b} + C_{\tilde{\gamma}, b}\} \rho_{\tilde{\gamma}, b}^{k/4} c_2(x, y).
\]

Note that using Theorem 8, we obtain that the following limits exist and do not depend on \( L \)

\[
D_{1, b} = \lim_{\tilde{\gamma} \to 0} D_{\tilde{\gamma}, 1, b}, \quad D_{2, b} = \lim_{\tilde{\gamma} \to 0} D_{\tilde{\gamma}, 2, b}, \quad C_{b} = \lim_{\tilde{\gamma} \to 0} C_{\tilde{\gamma}, b}, \quad \lambda_b = \lim_{\tilde{\gamma} \to 0} \lambda_{\tilde{\gamma}, b}, \quad \rho_b = \lim_{\tilde{\gamma} \to 0} \rho_{\tilde{\gamma}, b}.
\]

We now discuss the dependency of \( \rho_b \) with respect to the introduced parameters, depending on the sign of \( m \) and based on Theorem 8.

(a) If \( B4 \) holds, set \( \ell = \lceil m_2^+ \rceil \). Then, if we consider \( m_2^+ \) sufficiently small and \( |m| \) and \( R_2 \) sufficiently large such that the conditions of Theorem 8 hold, we have

\[
\log^{-1}(\rho_b^{-1}) \leq \left[ 1 + 2 \log(1 + R^2) + \log(1 + 2A) + 2(1 + 4R^2)m_2^+ \right] / \left[ m_2^+ \Phi(-1/2) \right].
\]

Note that the leading term on the right hand side of this equation is of order \( R^2 \), i.e. of order \( \max(R_2^2, d/m_2^+) \).

(b) If \( B3(m) \) with \( m \in \mathbb{R}_- \), set \( \ell = \lceil m_2^+ \rceil \). Then, if we consider \( m_2^+ \) sufficiently small and \( |m| \) and \( R_2 \) sufficiently large such that the conditions of Theorem 8 hold, we have

\[
\log^{-1}(\rho_b^{-1}) \leq \left[ 1 + 2 \log(1 + R^2) + \log(1 + 2A) + 2(1 + 4R^2)m_2^+ \right] / \left[ m_2^+ \Phi\{-2(-m)^{1/2}R/(2 - 2e^{2R^2})^{1/2}\} \right].
\]

Note that the right hand side of (52) is exponential in \(-mR^2\), i.e. exponential in \(-md/m_2^+ \) and \(-R_2^2(m_2^+ - m)m/m_2^+ \).

We now consider a condition which enforces weak curvature outside of a compact set.

**C3.** There exist \( R_3, a \geq 0, k_1, k_2 > 0, \) such that for any \( x \in \mathbb{R}^d \)

\[
\langle b(x), x \rangle \leq -k_1 \|x\| \int_{B(0, R_3)}(x) - k_2 \|b(x)\|^2 + a/2.
\]

In the case where \( X = \mathbb{R}^d \), \( \Pi_X = \text{Id} \) and there exists \( U \in C^1(\mathbb{R}^d, \mathbb{R}) \) such that \( B2 \) and \( B3(0) \) hold with \( b = -\nabla U \) and \( \int_{\mathbb{R}^d} e^{-U(x)}dx < +\infty \), then there exist \( R_3 \geq 0 \) and \( k_1 > 0 \) such that \( C3 \) holds with \( k_2 = a = 0 \), see [2, Lemma 2.2]. Define \( V : X \to [1, +\infty) \) for any \( x \in X \) by

\[
V(x) = \exp(m_2^+ \phi(x)), \quad \phi(x) = \sqrt{1 + \|x\|^2}, \quad m_2^+ \in (0, k_1/2] \.
\]

We also define for any \( x, y \in X \),

\[
\nu_3(x, y) = \{V(x) + V(y)\}/2, \quad c_3(x, y) = 1_{\Delta_x}(x, y)\nu_3(x, y).
\]
Proposition 17. Assume B1(X) for $X \subset \mathbb{R}^d$ and C3. Then for any $\gamma \in (0, \bar{\gamma}]$, $K_\gamma$ defined by (23) satisfies $D_d(\nu'_0, \lambda', A_\gamma, B(0, R) \times B(0, R))$ where $\bar{\gamma} \in (0, 2\kappa_2)$, $R_4 = \max(1, R_3, (d + a)/\kappa_1)$ and

$$\lambda = e^{-/(x'_+)^2/2},$$

$$A = \exp \left[ \bar{\gamma}(m'_+ (d + a) + (m'_+)^2)/2 + m'_+(1 + R'_4)^{1/2} \right] (m'_+(d + a)/2 + (m'_+)^2), \quad R = \log(2\lambda^{-2\gamma}A\log^{-1}(1/\lambda)).$$

Proof. The proof is postponed to Appendix B.5. □

Theorem 18. Assume B1(X) for $X \subset \mathbb{R}^d$, B2 and C3. Assume in addition either B3(m) for $m \in \mathbb{R}_-$ or B4. Then the conditions and conclusions of Theorem 8 hold with $\nu' = \nu_2$ defined in (48), $\bar{\gamma}$, $\lambda$, $A$ and $\bar{M}_4 = 2\bar{R}$ given by Proposition 17, and $\Psi$ given by (40) or (41).

Proof. The proof is similar to the one of Theorem 13. □

5.1 Main results

5 Quantitative convergence bounds for diffusions

In this section, we aim at deriving quantitative convergence bounds with respect to some Wasserstein metrics for diffusion processes under regularity and curvature assumptions on the drift $b$. Consider the following stochastic differential equation

$$dX_t = b(X_t)dt + dB_t,$$ (55)

where $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous drift.

When there exists a unique strong solution $(X_t)_{t \geq 0}$ of (55) for any starting point $X_0 = x$, with $x \in \mathbb{R}^d$, we define the semi-group $(P_t)_{t \geq 0}$ for any $A \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t \geq 0$ by $P_t(x, A) = \mathbb{P}(X_t \in A)$. We now turn to establishing that $(P_t)_{t \geq 0}$ converges for some Wasserstein metrics. In order to prove this result we will rely on discretizations of the stochastic differential equation (55). If the conditions of Theorem 8 are satisfied, these discretized processes are uniformly geometrically ergodic and taking the limit when the discretization stepsize goes to zero, we obtain the convergence of the associated diffusion processes.

First, assume that $b$ is Lipschitz regular. We establish in Theorem 19 that for any $T \geq 0$ and $x, y \in \mathbb{R}^d$, the Wasserstein distance $W_{\psi}(\delta_x P_T, \delta_y P_T)$ is upper-bounded by the upper limit when $m \to +\infty$ of $W_{\psi}(\delta_x R_{T/m}^m, \delta_y R_{T/m}^m)$, where $R_\gamma$ is given for any $\gamma > 0$ in (20) and $c$ is given in (30).

Second, this result is extended in Theorem 20 to cover the case where $b$ is no longer Lipschitz regular but only locally Lipschitz regular, see B5. Theorem 19 and Theorem 20 are applications of a more general theory developed in Section 5.3. Let $M \geq 0$, we consider for any $x \in \mathbb{R}^d$

$$V_M(x, y) = \exp[M\phi(x)], \quad \phi(x) = (1 + \|x\|)^{1/2}. \quad (56)$$

Theorem 19. Assume B2 and $\sup_{x \in \mathbb{R}^d}(x, b(x)) < +\infty$. Then, for any starting point $X_0 = x$, with $x \in \mathbb{R}^d$, there exists a unique strong solution to (55). In addition, for any $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to [1, +\infty)$
satisfying \( \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \{ \mathcal{V}(x,y)(V_M(x) + V_M(y))^{-1} \} < +\infty \) with \( M \geq 0 \) and \( V_M \) given in (56), we get that for any \( x, y \in \mathbb{R}^d \) and \( T \geq 0 \)
\[
W_c(\delta_x P_T, \delta_y P_T) \leq \limsup_{m \to +\infty} W_c(\delta_x R_{T/m}^m, \delta_y R_{T/m}^m),
\]
where \( c \) is given by (30), \((P_t)_{t \geq 0}\) is the semigroup associated with (55) and for any \( \gamma \in (0, \bar{\gamma}] \), \( R_{\gamma} \) is the Markov kernel associated with (19) where \( \mathcal{T}_\gamma(x) = x + \gamma b(x), X = \mathbb{R}^d \) and \( \Pi = \text{Id} \).

Proof. The proof is postponed to Appendix C.1.

We now weaken the Lipschitz regularity assumption and consider the following condition on the drift \( b \).

**B5.** \( b \) is locally Lipschitz, i.e. for any \( M \geq 0 \), there exists \( L_M \geq 0 \) such that for any \( x, y \in \overline{B}(0, M) \), \(|b(x) - b(y)| \leq L_M \|x - y\| \) and \( b(0) = 0 \).

As a consequence, under a mild integrability assumption, which will be satisfied in all of our applications, we obtain the following generalization of Theorem 19.

**Theorem 20.** Assume **B3(m), B5** and that \( \sup_{x \in \mathbb{R}^d} \langle x, b(x) \rangle < +\infty \). Then, for any starting point \( X_0 = x, \) with \( x \in \mathbb{R}^d \), there exists a unique strong solution of (55). In addition assume that for any \( x \in \mathbb{R}^d \) and \( T \geq 0 \) there exists \( \varepsilon_b > 0 \) such that
\[
\sup_{s \in [0, T]} \left\{ \delta_x P_s \|b(x)\|^{2(1 + \varepsilon_b)} \right\} < +\infty .
\]

Then, for any \( \mathcal{V} : \mathbb{R}^d \times \mathbb{R}^d \to [1, +\infty) \) satisfying \( \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \{ \mathcal{V}(x,y)(V_M(x) + V_M(y))^{-1} \} < +\infty \) with \( M \geq 0 \) and \( V_M \) given in (56), we get that for any \( x, y \in \mathbb{R}^d \) and \( T \geq 0 \)
\[
W_c(\delta_x P_T, \delta_y P_T) \leq \limsup_{m \to +\infty} \limsup_{n \to +\infty} W_c(\delta_x R_{T/m, n}^m, \delta_y R_{T/m, n}^m),
\]
where \( c \) is given by (30), \((P_t)_{t \geq 0}\) is the semigroup associated with (55) and for any \( \gamma \in (0, \bar{\gamma}] \), \( n \in \mathbb{N}, R_{\gamma, n} \) is the Markov kernel associated with (19) where \( \mathcal{T}_\gamma(x) = x + \gamma b(x), X = \overline{B}(0, n) \) and \( \Pi = \Pi_{\overline{B}(0, n)} \).

Proof. The proof is postponed to Appendix C.2.

Note that (57) holds under mild conditions on the drift function, see Proposition 30. In the next section we apply these results to diffusion processes and derive sharp convergence bounds in the case where \( b \) satisfies some curvature assumption, similarly to Section 4.

### 5.2 Applications

In this section, we combine the results of Theorem 20 with the convergence bounds for discrete processes derived in Section 4, in order to obtain convergence bounds for continuous processes that are solutions of (55).
5.2.1 Strongly convex at infinity

**Theorem 21.** Assume either B 3(m) for m \( \in \mathbb{R}_- \) or B 4. Assume C 1, B 5. In addition, assume 
\( \sup_{x \in \mathbb{R}^d} \{ \| b(x) \|^{2(1+\epsilon_b)} e^{-\pi_b^+ \| x \|^2} \} < +\infty \) for some \( \epsilon_b > 0 \). Then, for any \( T \geq 0 \), and \( x, y \in \mathbb{R}^d \)
\[
W_{c_1}(\delta_x P_T, \delta_y P_T) \leq \lambda_{T/4}^{T/4} (D_{1,a} c_1(x,y) + D_{2,a} \mathbb{I}_{\Delta^c}(x,y)) + C_a \rho_a^{T/4} \mathbb{I}_{\Delta^c}(x,y),
\]
with \( D_{1,a}, D_{2,a}, C_a \geq 0, \lambda_a, \rho_a \in (0, 1) \) given by (43) and for any \( x, y \in \mathbb{R}^d \), \( c_1(x,y) = \mathbb{I}_{\Delta^c}(x,y) \mathcal{V}_1(x,y) \) with \( \mathcal{V}_1(x,y) = 1 + \| x - y \| / R_1 \).

**Proof.** Let \( T \geq 0 \) and \( x, y \in \mathbb{R}^d \). Using Theorem 19 or Proposition 30 and Theorem 20 we have
\[
W_{c_1}(\delta_x P_T, \delta_y P_T) \leq \limsup_{n \to +\infty} \limsup_{m \to +\infty} W_{c_1}(\delta_x R_{T/m,n}^m, \delta_y R_{T/m,n}^m).
\]
Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N}^* \) such that \( x, y \in \mathbb{B}(0,n) \) and \( T/m \leq 2m_1^+/L_n^2 \). Since B 1(\( \mathbb{B}(0,n) \)) holds and B 5 implies B 2 on \( \mathbb{B}(0,n) \), we can apply Theorem 13 and we get
\[
W_{c_1}(\delta_x R_{T/m,n}^m, \delta_y R_{T/m,n}^m) \leq \lambda_{T/4}^{T/4} (D_{T/m,1,a} c_1(x,y) + D_{T/m,2,a} \mathbb{I}_{\Delta^c}(x,y)) + C_{T/m,a} \rho_{T/m,a}^{T/4} \mathbb{I}_{\Delta^c}(x,y),
\]
where \( D_{T/m,1,a}, D_{T/m,2,a}, C_{T/m,a}, \lambda_{T/m,a} \) and \( \rho_{T/m,a} \) are given in (49). In addition, these quantities admit limits \( D_{1,a}, D_{2,a}, C_a \geq 0 \) and \( \lambda_a, \rho_a \in (0, 1) \) when \( m \to +\infty \) which do not depend on \( L_n \), hence on \( n \), see (43). \( \Box \)

Note that B 2 implies B 5 and \( \sup_{x \in \mathbb{R}^d} \{ \| b(x) \|^{2(1+\epsilon_b)} e^{-\pi_b^+ \| x \|^2} \} < +\infty \) for some \( \epsilon_b > 0 \).

**Corollary 22.** Assume either B 3(m) for m \( \in \mathbb{R}_- \) or B 4. Assume C 1, B 5 and also that 
\( \sup_{x \in \mathbb{R}^d} \{ \| b(x) \|^{2(1+\epsilon_b)} e^{-\pi_b^+ \| x \|^2} \} < +\infty \) for some \( \epsilon_b > 0 \). Then, for any \( p \in \mathbb{N}, \alpha \in (p, +\infty), T \geq 0, \) and \( x, y \in \mathbb{R}^d \) we have
\[
W_1(\delta_x P_T, \delta_y P_T) \leq D_{3,a} \rho_a^{T/4} \| x - y \|,
\]
\[
W_p(\delta_x P_T, \delta_y P_T) \leq D_{\alpha,a} \rho_a^{T/(4\alpha)} \{ \| x - y \| + \| x - y \|^{1/\alpha} \},
\]
where \( \rho_a \) is given in (43), \( D_{3,a} = \lim_{\gamma \to 0} D_{3,\gamma,a} \) and \( D_{\alpha,a} = \lim_{\gamma \to 0} D_{\alpha,\gamma,a} \) with \( D_{3,\gamma,a} \) and \( D_{\alpha,\gamma,a} \) given in Corollary 14.

**Proof.** The proof is similar to the one of Theorem 21. \( \Box \)

The discussion on the dependency of \( \rho_a \) with respect to the parameters of the problem conducted in Section 4 still holds. We distinguish the following cases, assuming that the conditions of Theorem 8 are satisfied.

(a) If B 4 holds, we have
\[
\log^{-1}(\rho_a^{-1}) \leq (1 + \log(2))/(\Phi\{1/2\}m_1^+) + 4R_1^2/\Phi\{-1/2\}.
\]

The leading term in (58) is of order \( \max(R_1^2, 1/m_1^+) \), which corresponds to the one identified in [32, Lemma 2.9] and is optimal, see [32, Remark 2.10].
(b) If $\text{B}3(m)$ holds with $m \in \mathbb{R}_-$, we have
\[
\log^{-1}(\rho^{-1}_n) \leq \left[ 1 + \log(2) + \log(1 + 2\{m_1^+ - m\} (1 + R_1^2)) + 2m_1^+ (1 + R_1^2) \right] \left( m_1^+ \Phi_{(-m)^{1/2}} R_1 / (2 - 2e^{2mR_1^2})^{1/2} \right).
\] (59)

We now give an upper-bound for (59) when both $R$ and $m$ are large. For any $t \geq C$ with $C \geq 0$ we have
\[
\Phi(-t)^{-1} \leq \sqrt{2\pi}(1 + C^{-2})te^{t^2/2}.
\] (60)

As a consequence if we also have $R_1 \geq 2$, $1 \leq -mR_1^2$ and using that for any $t \in (0,1)$, $-\log(1 - t) \leq t$ as well as (60) we get that $\log^{-1}(\rho^{-1}_n) \leq \log^{-1}(\rho^{-1}_\text{max})$
\[
\log^{-1}(\rho^{-1}_\text{max}) = C \left[ 1 + \log(1 + 2\{m_1^+ - m\} (1 + R_1^2)) + 2m_1^+ (1 + R_1^2) \right] R_1 (-m)^{1/2} \times \exp \left[ -mR_1^2 / (4 - 4e^{2mR_1^2}) \right] \left( m_1^+ (1 - e^{2mR_1^2})^{1/2} \right),
\] with $C = 2(1 + \log(2))\sqrt{\pi} \approx 6.00$.

For a comparison of our results with recent works, see Section 2.1.

5.2.2 Other curvature conditions

**Theorem 23.** Assume either $\text{B}3(m)$ for $m \in \mathbb{R}_-$ or $\text{B}4$. Assume $\text{C}2$, $\text{B}5$. In addition, assume $\sup_{x \in \mathbb{R}^d} \{b(x)\|x\|^{2(1 + \varepsilon_b)} e^{-\varepsilon_b x^2} \|x\|^2 \} < +\infty$ for some $\varepsilon_b > 0$. Then for any $T \geq 0$ and $x, y \in \mathbb{R}^d$
\[
\| \delta_x P_T - \delta_y P_T \|_\mathbb{V} \leq (D_{1,b} + D_{2,b} + C_b) \rho_b^T c_2(x, y),
\]
with $D_{1,b}, D_{2,b}, C_b \geq 0$ and $\rho_b \in (0,1)$ given by (50) and $c_2$ defined in (48).

**Proof.** The proof is identical to the one of Theorem 21 upon replacing Theorem 13 by Theorem 16. \qed

**Theorem 24.** Assume either $\text{B}3(m)$ for $m \in \mathbb{R}_-$ or $\text{B}4$. Assume $\text{C}3$, $\text{B}5$. In addition, assume $\sup_{x \in \mathbb{R}^d} \{b(x)\|x\|^{2(1 + \varepsilon_b)} e^{-\varepsilon_b x^2} (1 + \|x\|) \|x\|^2 \} < +\infty$ for some $\varepsilon_b > 0$. Then for any $T \geq 0$ and $x, y \in \mathbb{R}^d$
\[
\| \delta_x P_T - \delta_y P_T \|_\mathbb{V} \leq (D_{1,c} + D_{2,c} + C_c) \rho_c^T c_3(x, y),
\]
with $D_{1,c}, D_{2,c}, C_c \geq 0$ and $\rho_c \in (0,1)$ given by Appendix H and $c_3$ defined in (54).

**Proof.** The proof is postponed to Appendix C.3. \qed

The rates we obtain in Theorem 23, respectively Theorem 24, are identical to the ones derived taking the limit $\varepsilon \to 0$ in Theorem 16, respectively Theorem 18. An upper bound on $\rho_b$, respectively $\rho_c$, is provided in (51) and (52), respectively Appendix H.

5.3 From discrete to continuous processes

In this section we present the general theory which leads to Theorem 19 and Theorem 20. First, we derive bounds between the discrete and continuous process given a family of approximating drift functions in Section 5.3.1. Second, we show in Section 5.3.2 that under mild regularity assumptions on $b$ such families can be explicitly constructed.
5.3.1 Quantitative convergence bounds for diffusion processes

We recall that the stochastic differential equation under study is given by
\[ dX_t = b(X_t)dt + dB_t, \]
where \((B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion and \(b : \mathbb{R}^d \to \mathbb{R}^d\) is a continuous drift. In the sequel we will always consider the following assumption.

**L1.** There exists a unique strong solution of (55) for any starting point \(X_0 = x, x \in \mathbb{R}^d\).

Under **L1**, the Markov semigroup \(P_t\), whose definition is given in Section 5.1, exists for any time \(t \geq 0\). Consider the extended infinitesimal generator \(\mathcal{A}\) associated with \((P_t)_{t \geq 0}\) and defined for any \(f \in C^2(\mathbb{R}^d, \mathbb{R})\) by
\[ \mathcal{A}f = (1/2)\Delta f + \langle \nabla f, b \rangle. \]

Let \(V \in C^2(\mathbb{R}^d, [1, +\infty)), \zeta \in \mathbb{R}\) and \(B \geq 0\).

**D2** \((D_c(V, \zeta, B))\). The extended infinitesimal generator \(\mathcal{A}\) satisfies the continuous Foster-Lyapunov drift condition if for all \(x \in \mathbb{R}^d\)
\[ \mathcal{A}V(x) \leq -\zeta V(x) + B. \]

This assumption is the continuous counterpart of \(D_d(V, \lambda, A, \mathbb{R}^d)\). We start by drawing a link between the continuous drift condition \(D_d(V, \zeta, B)\) and the discrete drift condition \(D_d(V, \lambda, A, \mathbb{R}^d)\). The result and its proof are standard [62, Theorem 2.1] but are given here for completeness. Denote by \((\mathcal{F}_t)_{t \geq 0}\) the filtration associated with \((B_t)_{t \geq 0}\) satisfying the usual conditions [45, Chapter I, Section 5].

**Lemma 25.** Let \(\zeta \in \mathbb{R}, B \geq 0\) and \(V \in C^2(\mathbb{R}^d, [1, +\infty))\) such that \(\lim_{\|x\| \to +\infty} V(x) = +\infty\).
Assume **L1** and **D2**.

(a) If \(B = 0\), then for any \(x \in \mathbb{R}^d\), \((V(X_t)e^{\zeta t})_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-supermartingale where \((X_t)_{t \geq 0}\) is the solution of (55) starting from \(X_0 = x\).

(b) For any \(t_0 > 0\), \(P_{t_0}\) satisfies \(D_d(V, \exp(-\zeta t_0), B(1 - \exp(-\zeta t_0))/\zeta, \mathbb{R}^d)\).

**Proof.** The proof is postponed to Appendix C.4.

Consider a family of drifts \(\{b_{\gamma, n} : \mathbb{R}^d \to \mathbb{R}^d: \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}\) for some \(\bar{\gamma} > 0\). For all \(\gamma \in (0, \bar{\gamma}]\) and \(n \in \mathbb{N}\), we denote by \(\bar{R}_{\gamma, n}\) the Markov kernel associated with (19) where \(T_{\gamma}(x) = x + \gamma b_{\gamma, n}(x), X = \mathbb{R}^d\) and \(\Pi = \text{Id}\). We will show that under the following assumptions the family \(\{\bar{R}_{\gamma, n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}\) approximates \(P_T\) for \(T \geq 0\) as \(\gamma \to 0\) and \(n \to +\infty\).

**L2.** There exist \(\beta > 0\) and \(C_1 \geq 0\) such that for any \(\gamma \in (0, \bar{\gamma}], n \in \mathbb{N}, b_{\gamma, n} \in C(\mathbb{R}^d, \mathbb{R}^d)\) and for any \(x \in \mathbb{R}^d\),
\[ \|b(x) - b_{\gamma, n}(x)\|^2 \leq C_1 \gamma^2 \|b(x)\|^2. \]

The following assumption is mainly technical and is satisfied in our applications.

**L3.** There exists \(\varepsilon_b > 0\) such that \(\sup_{x \in [0, T]} \{\delta_x P_s \|b(x)\|^2(1 + \varepsilon_b)\} < +\infty\), for any \(x \in \mathbb{R}^d\) and \(T \geq 0\).
By Lemma 25-(a), if \( D_\epsilon (V, \zeta, 0) \) is satisfied with \( \zeta \in \mathbb{R} \), it holds that for any starting point \( x \in \mathbb{R}^d \),

\[ \sup_{t \in [0,T]} E[V(X_t)] \leq e^{-cT} V(x) \]

where \( (X_t)_{t \geq 0} \) is solution of (55) starting from \( x \). Therefore, if \( \|b(x)\|^{2(1+\epsilon)} \leq V(x) \) for any \( x \in \mathbb{R}^d \), L3 is satisfied.

The proof of the next result relies on the combination of the Girsanov theorem with estimates on the drift functions, adapting [29, Theorem 10]. Similar strategies have also been used in [17, 35, 76].

**Proposition 26.** Assume L1, L2 and L3. Let \( V : \mathbb{R}^d \to [1, +\infty) \). In addition, assume that for any \( n \in \mathbb{N} \), \( T \geq 0 \) and \( x \in \mathbb{R}^d \)

\[ P_T V^2(x) < +\infty \quad \lim_{m \to +\infty} \sup_{T/m, n} V^2(x) < +\infty. \]

Then for any \( n \in \mathbb{N} \), \( T \geq 0 \) and \( x \in \mathbb{R}^d \)

\[ \lim_{m \to +\infty} \|\delta_x P_T - \delta_x \tilde{R}_{T/m, n}\|_V = 0, \]

where \( (P_t)_{t \geq 0} \) is the semigroup associated with (55) and for any \( \gamma \in (0, \bar{\gamma}) \) and \( n \in \mathbb{N} \), \( \tilde{R}_{\gamma,n} \) is the Markov kernel associated with (19) where \( \bar{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x) \) and \( \Pi = \text{Id} \).

**Proof.** The proof is postponed to Appendix C.5. \( \square \)

If \( V = 1 \), Proposition 26 implies that \( \lim_{m \to +\infty} \|\delta_x P_T - \delta_x \tilde{R}_{T/m, n}\|_V = 0. \) Let \( V : \mathbb{R}^d \to [1, +\infty) \) and \( c : \mathbb{R}^d \times \mathbb{R}^d \to [1, +\infty) \) a distance such that for any \( x, y \in \mathbb{R}^d \), \( c(x, y) \leq \{V(x) + V(y)\}/2 \). Then, under the conditions of Proposition 26, we obtain that for any \( T \geq 0, n \in \mathbb{N} \) and \( x, y \in \mathbb{R}^d \)

\[ W_e(\delta_x P_T, \delta_y P_T) \leq \lim_{m \to +\infty} \sup_{T/m, n} W_e(\delta_x \tilde{R}_{T/m, n}, \delta_y \tilde{R}_{T/m, n}), \]

Therefore, if for any \( T \geq 0 \), \( W_e(\delta_x \tilde{R}_{T/m, n}, \delta_y \tilde{R}_{T/m, n}) \) can be bounded uniformly in \( m \) using Theorem 8, we obtain an explicit bound for \( W_e(\delta_x P_T, \delta_y P_T) \) for any \( T \geq 0 \). As a consequence, this result easily implies non-asymptotic convergence bounds of \( (P_t)_{t \geq 0} \) to its invariant measure if it exists. However, in our applications, global Lipschitz regularity on \( b_{T/m, n} : \mathbb{R}^d \to \mathbb{R}^d \) is needed in order to apply Theorem 8 to \( \tilde{R}_{T/m, n} \) for \( T \geq 0, m \in \mathbb{N} \) and \( n \in \mathbb{N} \). To be able to deal with the fact that \( b_{T/m, n} \) is non necessarily globally Lipschitz, we consider an appropriate sequence of projected Euler-Maruyama schemes associated to a sequence of subsets of \( \mathbb{R}^d \), \( (K_n)_{n \in \mathbb{N}} \) satisfying the following assumption.

**L4.** For any \( n \in \mathbb{N} \), \( K_n \) is convex and closed, and \( \bar{B}(0, n) \subset K_n \).

Consider for any \( \gamma \in (0, \bar{\gamma}) \) and \( n \in \mathbb{N} \) the Markov chain associated (19), where for any \( x \in \mathbb{R}^d \), \( \bar{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x) \), \( X = K_n \) and \( \Pi = \Pi_{K_n} \), the projection on \( K_n \). The Markov kernel associated with this chain is denoted \( R_{\gamma,n} \) for any \( \gamma \in (0, \bar{\gamma}) \) and \( n \in \mathbb{N} \). Assuming only local Lipschitz regularity we can apply Theorem 8 to the projected version of the Markov chain associated with \( R_{T/m, n} \). Therefore we want to replace \( \tilde{R}_{T/m, n} \) by \( R_{T/m, n} \) in (61). In order to do so we consider the following assumption on the family of drifts \( \{b_{\gamma,n} : \gamma \in (0, \bar{\gamma}) \}, n \in \mathbb{N} \}.

**L5.** There exist \( \bar{A} > 0 \) and \( \bar{V} : \mathbb{R}^d \to [1, +\infty) \) such that for any \( n \in \mathbb{N} \) there exist \( \bar{E}_n \geq 0, \bar{\varepsilon}_n > 0 \) and \( \bar{\gamma}_n \in (0, \bar{\gamma}) \) satisfying for any \( \gamma \in (0, \bar{\gamma}_n) \) and \( x \in \mathbb{R}^d \),

\[ \bar{R}_{\gamma,n} \bar{V}(x) \leq \exp \left[ \log(\bar{A}) \gamma (1 + \bar{E}_n \gamma \bar{\varepsilon}_n) \right] \bar{V}(x) , \quad \sup_{x \in \mathbb{R}^d} \left\{ \|x\| / \bar{V}(x) \right\} \leq 1, \]
where for any $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}$, $\bar{R}_{\gamma,n}$ is the Markov kernel associated with (19) where $T_{\gamma}(x) = x + \gamma b_{\gamma,n}(x)$ and $\Pi = \text{Id}$.

**Proposition 27.** Let $V : \mathbb{R}^d \to [1, +\infty)$. Assume $L_1, L_4, L_5$ and that for any $T \geq 0$, $x \in \mathbb{R}^d$

$$\lim_{n \to +\infty} \limsup_{m \to +\infty} \left( R_{T/m,n}^m + \bar{R}_{T/m,n}^m \right) V^2(x) < +\infty.$$  

Then for any $T \geq 0$ and $x \in \mathbb{R}^d$

$$\lim_{n \to +\infty} \limsup_{m \to +\infty} \| \delta_x R_{T/m,n}^m - \delta_x \bar{R}_{T/m,n}^m \| V = 0;$$

**Proof.** The proof is postponed to Appendix C.6. \qed

Based on Proposition 26 and Proposition 27, we have the following result which establishes a clear link between the convergence of the family of the projected Euler-Maruyama scheme $\{R_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}$ and the semigroup $(P_t)_{t \geq 0}$ associated with (55).

**Theorem 28.** Let $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to [1, +\infty)$ and $V : \mathbb{R}^d \to [1, +\infty)$ satisfying for any $x, y \in \mathbb{R}^d$, $\sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \varphi(x, y) \{V(x) + V(y)\}^{-1} < +\infty$. Assume $L_1, L_2, L_3, L_4$ and $L_5$. In addition, assume that for any $T \geq 0$ and $x \in \mathbb{R}^d$

$$P_T V^2(x) < +\infty, \quad \limsup_{n \to +\infty} \limsup_{m \to +\infty} \left( R_{T/m,n}^m + \bar{R}_{T/m,n}^m \right) V^2(x) < +\infty.$$  

Then,

$$W_c(\delta_x P_T, \delta_y P_T) \leq \limsup_{n \to +\infty} \limsup_{m \to +\infty} W_c(\delta_x R_{T/m,n}^m, \delta_y R_{T/m,n}^m),$$

where for any $x, y \in \mathbb{R}^d$, $c(x, y) = 1_{\Delta_\delta_x}(x, y) \varphi(x, y)$, $(P_t)_{t \geq 0}$ is the semigroup associated with (55) and for any $\gamma \in (0, \bar{\gamma}]$, $n \in \mathbb{N}$, $R_{\gamma,n}$ is the Markov kernel associated with (19) where $T_{\gamma}(x) = x + \gamma b_{\gamma,n}(x)$, $X = K_n$ and $\Pi = \Pi_{K_n}$, $\bar{R}_{\gamma,n}$ is the Markov kernel associated with (19) where $T_{\gamma}(x) = x + \gamma b_{\gamma,n}(x)$, $X = \mathbb{R}^d$ and $\Pi = \text{Id}$.

**Proof.** Let $T \geq 0$, $x, y \in \mathbb{R}^d$ and

$$C_V = 2 \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \varphi(x, y) \{V(x) + V(y)\}^{-1} < +\infty.$$  

We have for any $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$ such that $T/m \leq \bar{\gamma}$

$$W_c(\delta_x P_T, \delta_y P_T) \leq C_V \| \delta_x P_T - \delta_x \bar{R}_{T/m,n}^m \| V + C_V \| \delta_x R_{T/m,n}^m - \delta_x \bar{R}_{T/m,n}^m \| V$$

$$+ W_c(\delta_x R_{T/m,n}^m, \delta_y R_{T/m,n}^m) + C_V \| \delta_y P_T - \delta_y \bar{R}_{T/m,n}^m \| V + C_V \| \delta_y R_{T/m,n}^m - \delta_y \bar{R}_{T/m,n}^m \| V;$$

which concludes the proof upon combining Proposition 26 and Proposition 27. \qed
5.3.2 Explicit approximating family of drifts

In this section we show that under regularity and curvature assumptions on the drift function \( b \) we can construct explicit families of approximating drift functions satisfying the assumptions of Theorem 28. The section is divided into two parts. First, we show under regularity conditions \( L_1, L_2, L_3, L_4 \) and \( L_5 \) are satisfied. Second, we show, under similar, that the summability assumptions (62) in Theorem 28 hold for \( V \leftarrow V_M \) with \( V_M : \mathbb{R}^d \rightarrow [1, +\infty) \) given by (56) for \( M \geq 0 \). We start with the case where \( b \) satisfies \( B_2 \).

**Proposition 29.** Assume \( B_2 \). Let \( \{ b_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N} \} \) be given for any \( \gamma > 0 \), \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \) by \( b_{\gamma,n}(x) = b(x) \). Let \( K_n = \mathbb{R}^d \) for any \( n \in \mathbb{N} \). Then, \( L_1, L_2, L_3, L_4 \) and \( L_5 \) are satisfied.

**Proof.** The proof is postponed to Appendix C.7.

We now consider the more challenging case where \( B_2 \) does not hold and is replaced by the weaker condition \( B_5 \). In this setting, by \([45, \text{Chapter 4, Theorem 2.3}]\), (55) admits a unique solution \((X_t)_{t \in [0, +\infty]} \) with \( X_0 = x \in \mathbb{R}^d \) and let \( e = \inf \{ s \geq 0 : \|X_s\| = +\infty \} \). In particular, the condition \( e = +\infty \) is met a.s. if we assume that \( b \) is sub-linear \([45, \text{Chapter 4, Theorem 2.3}]\) or that the condition \( D_c(V, \zeta, 0) \) holds with \( \zeta \in \mathbb{R} \) and \( \lim_{\|x\| \to +\infty} V(x) = +\infty \) \([52, \text{Theorem 3.5}]\). This last condition is satisfied for all the applications we consider in Section 5.2.

**Proposition 30.** Assume \( B_3(\bar{m}) \) with \( \bar{m} \in \mathbb{R} \) and \( B_5 \), then \( L_1 \) holds. In addition:

(a) if there exists \( \varepsilon_b > 0 \) and \( p \in \mathbb{N}^* \) such that \( \sup_{x \in \mathbb{R}^d} \{ \|b(x)\|^{2(1 + \varepsilon_b)} (1 + \|x\|^{2p})^{-1} \} < +\infty \) then \( L_3 \) holds;

(b) assume that \( C_2 \) holds and \( \sup_{x \in \mathbb{R}^d} \{ \|b(x)\|^{2(1 + \varepsilon_b)} e^{-\varepsilon_b \|x\|^p} \} < +\infty \) for some \( \varepsilon_b > 0 \) satisfying then \( L_3 \) holds.

**Proof.** The proof is postponed to Appendix C.8.

Proposition 30 gives conditions under which \( L_1 \) and \( L_3 \) hold. In addition, \( L_4 \) is satisfied if we take for any \( n \in \mathbb{N} \), \( K_n = \bar{B}(0, n) \). Therefore, it only remains to find a family of drift functions which satisfies \( L_2 \) and \( L_5 \). To this end, consider the following family of drift functions \( \{ b_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N} \} \) defined for any \( \gamma > 0 \), \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \) by

\[
b_{\gamma,n}(x) = \varphi_n(x)b(x) + (1 - \varphi_n(x)) \frac{b(x)}{1 + \gamma^\alpha \|b(x)\|}, \tag{63}
\]

with \( \alpha < 1/2 \) and \( \varphi_n \in C(\mathbb{R}^d, \mathbb{R}) \) such that for any \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \),

\[
\varphi_n(x) = \begin{cases} 
1 & \text{if } x \in \bar{B}(0, n), \\
0 & \text{if } x \in \bar{B}(0, n+1)^c 
\end{cases} . \tag{64}
\]

An example of such a family is displayed in Figure 3.

**Proposition 31.** Assume \( B_3(\bar{m}) \) for \( \bar{m} \in \mathbb{R} \) and \( B_5 \), then \( L_2 \) and \( L_5 \) hold for the family \( \{ b_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N} \} \) defined by (63).

**Proof.** The proof is postponed to Appendix C.9.
Figure 3: In this figure we illustrate the approximation properties of the family of drift functions defined by (63). Let \( b(x) = |x|^{1.5} \sin(x) \) and for any \( n \in \mathbb{N} \), \( \varphi_n(x) = d(x, B(0, n+1)^c)^2/(d(x, B(0, n))^2 + d(x, B(0, n+1)^c)^2) \). In both figures, the original drift is displayed in cyan and we fix \( \alpha = 0.3 \). In (a), we fix \( n = 1 \), represented by the black dashed lines, and observe the behavior of the drift functions for different values of \( \gamma > 0 \). In (b), we plot the drift for different \( \gamma > 0 \) and \( n \in \mathbb{N} \).

The following proposition is a generalization of Proposition 29.

Proposition 32. Assume \( B_3(m) \) with \( m \in \mathbb{R} \) and \( B_5 \). Let \( \{b_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\} \) be given for any \( \gamma > 0 \), \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \) by (63). Let \( K_n = B(0, n) \) for any \( n \in \mathbb{N} \). Then, \( L_1, L_2, L_4 \) and \( L_5 \) are satisfied.

Proof. The proof is a straightforward combination of Proposition 30 and Proposition 31.

In Proposition 33 and Proposition 34 we show that the second part of (62) holds under regularity assumptions on the drift function \( b \).

Proposition 33. Assume \( B_3(m) \) for \( m \in \mathbb{R} \) and \( B_2 \), then for any \( T, M \geq 0 \) and \( x \in \mathbb{R}^d \)

\[
\limsup_{m \to +\infty} \limsup_{m \to +\infty} \left( R_T^{m,n} + R_T^{m,n} \right) V_M(x) < +\infty ,
\]

with \( V_M \) given in (56) and where for any \( \gamma \in (0, \bar{\gamma}] \), \( R_\gamma \) is the Markov kernel associated with (19) where \( T_\gamma(x) = x + \gamma b(x) \), \( \mathbb{X} = \mathbb{R}^d \) and \( \Pi = \text{Id} \).

Proof. The proof is postponed to Appendix C.10.

Proposition 34. Assume \( B_3(m) \) for \( m \in \mathbb{R} \) and \( B_5 \), then for any \( T, M \geq 0 \) and \( x \in \mathbb{R}^d \)

\[
\limsup_{n \to +\infty} \limsup_{m \to +\infty} \left( R_T^{m,n} + R_T^{m,n} \right) V_M(x) < +\infty ,
\]

with \( V_M \) given in (56) and where for any \( \gamma \in (0, \bar{\gamma}] \), \( n \in \mathbb{N} \), \( R_{\gamma,n} \) is the Markov kernel associated with (19) where \( T_{\gamma}(x) = x + \gamma b_{\gamma,n}(x) \), \( \mathbb{X} = \mathbb{R}^d \) and \( \Pi = \text{Id} \), \( \tilde{R}_{\gamma,n} \) is the Markov kernel associated with (19) where \( T_{\gamma}(x) = x + \gamma b_{\gamma,n}(x) \), \( \mathbb{X} = \mathbb{R}^d \) and \( \Pi = \text{Id} \).
Proof. The proof is postponed to Appendix C.11.

Finally, we show that under mild curvature assumption on the drift function $b$, the first part of (62) holds.

**Proposition 35.** Assume $L^1$ and that $\sup_{x \in \mathbb{R}^d} \langle b(x), x \rangle < +\infty$. Then for any $M \geq 0$, there exists $\zeta \in \mathbb{R}$ such that $D_c(V_M, \zeta, 0)$ holds with $V_M$ given in (56). In particular, for any $T, M \geq 0$, $P_T V_M(x) < +\infty$.

Proof. The proof is postponed to Appendix C.12.

References

[1] J. Baker, P. Fearnhead, E. B. Fox, and C. Nemeth. Control variates for stochastic gradient mcmc. Statistics and Computing, pages 1–17.

[2] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. Electron. Commun. Probab., 13:60–66, 2008.

[3] H. H. Bauschke and P. L. Combettes. Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer Publishing Company, Incorporated, 1st edition, 2011.

[4] E. Bernton. Langevin monte carlo and jko splitting. arXiv preprint arXiv:1802.08671, 2018.

[5] S. Boucheron, G. Lugosi, and P. Massart. Concentration inequalities. Oxford University Press, Oxford, 2013. A nonasymptotic theory of independence, With a foreword by Michel Ledoux.

[6] P. Brémaud. Markov chains, volume 31 of Texts in Applied Mathematics. Springer-Verlag, New York, 1999. Gibbs fields, Monte Carlo simulation, and queues.

[7] N. Brosse, A. Durmus, Éric Moulines, and S. Sabanis. The tamed unadjusted langevin algorithm. Stochastic Processes and their Applications, 2018.

[8] S. Bubeck, R. Eldan, and J. Lehec. Finite-time analysis of projected langevin monte carlo. In Proceedings of the 28th International Conference on Neural Information Processing Systems, NIPS’15, pages 1243–1251, Cambridge, MA, USA, 2015. MIT Press.

[9] R. Bubley, M. Dyer, and M. Jerrum. An elementary analysis of a procedure for sampling points in a convex body. Random Structures Algorithms, 12(3):213–235, 1998.

[10] O. Butkovsky. Subgeometric rates of convergence of Markov processes in the Wasserstein metric. Ann. Appl. Probab., 24(2):526–552, 2014.

[11] K. S. Chan. Asymptotic behavior of the Gibbs sampler. J. Amer. Statist. Assoc., 88(421):320–326, 1993.

[12] N. Chatterji, N. Flammarion, Y. Ma, P. Bartlett, and M. Jordan. On the theory of variance reduction for stochastic gradient monte carlo. In International Conference on Machine Learning, pages 763–772, 2018.
[13] M. F. Chen and F. Y. Wang. Estimation of the first eigenvalue of second order elliptic operators. *J. Funct. Anal.*, 131(2):345–363, 1995.

[14] M.-F. Chen and F.-Y. Wang. Estimation of spectral gap for elliptic operators. *Trans. Amer. Math. Soc.*, 349(3):1239–1267, 1997.

[15] R. Chen and R. S. Tsay. On the ergodicity of TAR(1) processes. *Ann. Appl. Probab.*, 1(4):613–634, 1991.

[16] X. Cheng, N. S. Chatterji, Y. Abbasi-Yadkori, P. L. Bartlett, and M. I. Jordan. Sharp convergence rates for langevin dynamics in the nonconvex setting. *arXiv preprint arXiv:1805.01648*, 2018.

[17] A. S. Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. *J. R. Stat. Soc. Ser. B. Stat. Methodol.*, 79(3):651–676, 2017.

[18] A. S. Dalalyan and A. Karagulyan. User-friendly guarantees for the langevin monte carlo with inaccurate gradient. *Stochastic Processes and their Applications*, 2019.

[19] A. Devraj, I. Kontoyiannis, and S. Meyn. Geometric ergodicity in a weighted sobolev space, 2017.

[20] R. Douc, G. Fort, and A. Guillin. Subgeometric rates of convergence of $f$-ergodic strong Markov processes. *Stochastic Process. Appl.*, 119(3):897–923, 2009.

[21] R. Douc, G. Fort, E. Moulines, and P. Soulier. Practical drift conditions for subgeometric rates of convergence. *Ann. Appl. Probab.*, 14(3):1353–1377, 2004.

[22] R. Douc, E. Moulines, P. Priouret, and P. Soulier. *Markov Chains*. Springer, 2019.

[23] R. Douc, E. Moulines, and J. S. Rosenthal. Quantitative bounds on convergence of time-inhomogeneous Markov processes. *Ann. Appl. Probab.*, 14(4):1643–1665, 2004.

[24] D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of Markov processes. *Ann. Probab.*, 23(4):1671–1691, 1995.

[25] A. Durmus, G. Fort, and E. Moulines. Subgeometric rates of convergence in Wasserstein distance for Markov chains. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(4):1799–1822, 2016.

[26] A. Durmus, S. Majewski, and B. Miasojedow. Analysis of langevin monte carlo via convex optimization. *arXiv preprint arXiv:1802.09188*, 2018.

[27] A. Durmus and É. Moulines. Quantitative bounds of convergence for geometrically ergodic markov chain in the wasserstein distance with application to the metropolis adjusted langevin algorithm. *Statistics and Computing*, 25(1):5–19, 2015.

[28] A. Durmus and E. Moulines. High-dimensional Bayesian inference via the Unadjusted Langevin Algorithm. *ArXiv e-prints*, May 2016.

[29] A. Durmus and E. Moulines. Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *Ann. Appl. Probab.*, 27(3):1551–1587, 2017.
[30] A. Durmus, E. Moulines, and M. Pereyra. Efficient bayesian computation by proximal markov chain monte carlo: when langevin meets moreau. *SIAM Journal on Imaging Sciences*, 11(1):473–506, 2018.

[31] A. Eberle. Reflection coupling and Wasserstein contractivity without convexity. *C. R. Math. Acad. Sci. Paris*, 349(19-20):1101–1104, 2011.

[32] A. Eberle. Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields*, 166(3-4):851–886, 2016.

[33] A. Eberle, A. Guillin, and R. Zimmer. Quantitative harris-type theorems for diffusions and mckean–vlasov processes. *Transactions of the American Mathematical Society*, 2018.

[34] A. Eberle and M. B. Majka. Quantitative contraction rates for markov chains on general state spaces. *arXiv preprint arXiv:1808.07033*, 2018.

[35] W. Fang and M. B. Giles. Multilevel Monte Carlo method for ergodic SDEs without contractivity. *J. Math. Anal. Appl.*, 476(1):149–176, 2019.

[36] G. Fort. *Contrôle explicite d’ergodicité de chaînes de Markov : Applications à l’analyse de convergence de l’algorithme Monte-Carlo EM*. PhD thesis, Université Pierre et Marie Curie, Paris, Paris, 2001.

[37] G. Fort. Computable bounds for V-geometric ergodicity of Markov transition kernels. *Rapport de Recherche, Univ. J. Fourier, RR 1047-M.*, https://www.math.univ-toulouse.fr/%7Egfort/Preprints/fort:2002.pdf, 2002.

[38] G. Fort and G. O. Roberts. Subgeometric ergodicity of strong Markov processes. *Ann. Appl. Probab.*, 15(2):1565–1589, 2005.

[39] F. G. Foster. On the stochastic matrices associated with certain queuing processes. *Ann. Math. Statistics*, 24:355–360, 1953.

[40] B. Goldys and B. Maslowski. Lower estimates of transition densities and bounds on exponential ergodicity for stochastic PDE’s. *Ann. Probab.*, 34(4):1451–1496, 2006.

[41] A. Granas and J. Dugundji. *Fixed point theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[42] M. Hairer and J. C. Mattingly. Yet another look at harris’ ergodic theorem for markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, pages 109–117. Springer, 2011.

[43] M. Hairer, J. C. Mattingly, and M. Scheutzow. Asymptotic coupling and a general form of Harris’ theorem with applications to stochastic delay equations. *Probab. Theory Related Fields*, 149(1-2):223–259, 2011.

[44] M. Hairer, A. M. Stuart, and S. J. Vollmer. Spectral gaps for a Metropolis-Hastings algorithm in infinite dimensions. *Ann. Appl. Probab.*, 24(6):2455–2490, 2014.
[45] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.

[46] S. F. Jarner and G. O. Roberts. Polynomial convergence rates of Markov chains. *Ann. Appl. Probab.*, 12(1):224–247, 2002.

[47] S. F. Jarner and R. L. Tweedie. Locally contracting iterated functions and stability of Markov chains. *J. Appl. Probab.*, 38(2):494–507, 2001.

[48] J. E. Johndrow and J. C. Mattingly. Error bounds for approximations of markov chains used in bayesian sampling. *arXiv preprint arXiv:1711.05382*, 2017.

[49] G. L. Jones and J. P. Hobert. Honest exploration of intractable probability distributions via Markov chain Monte Carlo. *Statist. Sci.*, 16(4):312–334, 2001.

[50] G. L. Jones and J. P. Hobert. Sufficient burn-in for Gibbs samplers for a hierarchical random effects model. *Ann. Statist.*, 32(2):784–817, 2004.

[51] A. Joulin and Y. Ollivier. Curvature, concentration and error estimates for markov chain monte carlo. *The Annals of Probability*, 38(6):2418–2442, 2010.

[52] R. Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.

[53] I. Kontoyiannis and S. P. Meyn. Approximating a diffusion by a finite-state hidden Markov model. *Stochastic Process. Appl.*, 127(8):2482–2507, 2017.

[54] T. Lindvall and L. C. G. Rogers. Coupling of multidimensional diffusions by reflection. *Ann. Probab.*, 14(3):860–872, 1986.

[55] R. Liptser and A. N. Shiryaev. *Statistics of random Processes: I. general Theory*, volume 5. Springer Science & Business Media, 2013.

[56] R. B. Lund and R. L. Tweedie. Geometric convergence rates for stochastically ordered Markov chains. *Math. Oper. Res.*, 21(1):182–194, 1996.

[57] D. Luo and J. Wang. Exponential convergence in $L^p$-Wasserstein distance for diffusion processes without uniformly dissipative drift. *Math. Nachr.*, 289(14-15):1909–1926, 2016.

[58] M. B. Majka, A. Mijatović, and L. Szpruch. Non-asymptotic bounds for sampling algorithms without log-concavity. *arXiv preprint arXiv:1808.07105*, 2018.

[59] P. R. Meyers. A converse to Banach’s contraction theorem. *J. Res. Nat. Bur. Standards Sect. B, 71B*:73–76, 1967.

[60] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes. I. Criteria for discrete-time chains. *Adv. in Appl. Probab.*, 24(3):542–574, 1992.

[61] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes. II. Continuous-time processes and sampled chains. *Adv. in Appl. Probab.*, 25(3):487–517, 1993.
[62] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*, 25(3):518–548, 1993.

[63] S. P. Meyn and R. L. Tweedie. Computable bounds for geometric convergence rates of Markov chains. *Ann. Appl. Probab.*, 4(4):981–1011, 1994.

[64] Y. Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Applied Optimization. Springer, 2004.

[65] J. Neveu. *Discrete-parameter martingales*. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, revised edition, 1975. Translated from the French by T. P. Speed, North-Holland Mathematical Library, Vol. 10.

[66] E. Nummelin and P. Tuominen. Geometric ergodicity of Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process. Appl.*, 12(2):187–202, 1982.

[67] E. Nummelin and P. Tuominen. The rate of convergence in Orey’s theorem for Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process. Appl.*, 15(3):295–311, 1983.

[68] E. Nummelin and R. L. Tweedie. Geometric ergodicity and $R$-positivity for general Markov chains. *Ann. Probability*, 6(3):404–420, 1978.

[69] Y. Ollivier. Ricci curvature of markov chains on metric spaces. *Journal of Functional Analysis*, 256(3):810–864, 2009.

[70] N. Parikh and S. Boyd. *Proximal Algorithms*. Foundations and Trends(r) in Optimization. Now Publishers, 2013.

[71] D. Paulin. Mixing and concentration by Ricci curvature. *J. Funct. Anal.*, 270(5):1623–1662, 2016.

[72] N. S. Pillai and A. Smith. Ergodicity of approximate mcmc chains with applications to large data sets. *arXiv preprint arXiv:1405.0182*, 2014.

[73] N. N. Popov. Geometric ergodicity conditions for countable Markov chains. *Dokl. Akad. Nauk SSSR*, 234(2):316–319, 1977.

[74] Q. Qin and J. P. Hobert. Wasserstein-based methods for convergence complexity analysis of mcmc with application to albert and chib’s algorithm. *arXiv preprint arXiv:1810.08826*, 2018.

[75] Q. Qin and J. P. Hobert. Geometric convergence bounds for markov chains in wasserstein distance based on generalized drift and contraction conditions. *arXiv preprint arXiv:1902.02964*, 2019.

[76] M. Raginsky, A. Rakhlin, and M. Telgarsky. Non-convex learning via stochastic gradient langevin dynamics: a nonasymptotic analysis, 2017.

[77] G. O. Roberts and N. G. Polson. On the geometric convergence of the gibbs sampler. *Journal of the Royal Statistical Society, Series B*, 56:377–384, 1994.
A Proofs of Section 3

A.1 Proof of Proposition 5

First, we prove the following technical lemma.

\textbf{Lemma 36.} Let \( \bar{\gamma} > 0 \) and \( \kappa : (0, \bar{\gamma}) \to \mathbb{R} \), with \( \kappa(\gamma) \gamma \in (-1, +\infty) \) for any \( \gamma \in (0, \bar{\gamma}] \). We have for any \( \gamma \in (0, \bar{\gamma}] \) such that \( \kappa(\gamma) \neq 0 \) and \( \ell \in \mathbb{N}^* \)

\[
\Xi_{(1/\gamma)}(\kappa) = -\kappa^{-1}(\gamma) \left\{ \exp \left[ -\ell \left[ 1 + \gamma \kappa(\gamma) \right] \right] - 1 \right\},
\]

where \( \Xi_{(1/\gamma)}(\kappa) \) is defined by (26). In addition, for any \( \ell \in \mathbb{N}^* \) and \( \gamma \in (0, \bar{\gamma}] \)

(a) \( \Xi_{(1/\gamma)}(\kappa) \geq \alpha_-(\kappa, \gamma, \ell) = -\kappa^{-1}(\gamma) \left\{ \exp(-\ell \kappa(\gamma)) - 1 \right\} \) if for any \( \gamma \in (0, \bar{\gamma}] \), \( \kappa(\gamma) < 0 \); 

(b) \( \Xi_{(1/\gamma)}(\kappa) \geq \alpha_0(\kappa, \gamma, \ell) = \ell \) if for any \( \gamma \in (0, \bar{\gamma}] \), \( \kappa(\gamma) \leq 0 \); 

(c) \( \Xi_{(1/\gamma)}(\kappa) \geq \alpha_+(\kappa, \gamma, \ell) = \kappa^{-1}(\gamma) \left[ 1 - \exp \left\{ -\frac{\ell \kappa(\gamma)}{1 + \gamma \kappa(\gamma)} \right\} \right] \) if for any \( \gamma \in (0, \bar{\gamma}] \), \( \kappa(\gamma) > 0 \).
Proof. Let \( \ell \in \mathbb{N}^* \) and \( \gamma \in (0, \bar{\gamma}] \). First note that the following equalities hold if \( \kappa(\gamma) \neq 0 \)

\[
\Xi_{\ell/\gamma}(\kappa) = \gamma \sum_{i=1}^{\ell/\gamma} (1 + \gamma \kappa(\gamma))^{-i} 
= \gamma (1 + \gamma \kappa(\gamma))^{-1} \frac{1 - (1 + \gamma \kappa(\gamma))^{-\ell/\gamma}}{1 - (1 + \gamma \kappa(\gamma))^{-1}} 
= -\kappa^{-1}(\gamma) \left\{ [1 + \gamma \kappa(\gamma)]^{-\ell/\gamma} - 1 \right\} 
= -\kappa^{-1}(\gamma) \left\{ \exp \left[ -\ell \left( \frac{1}{\gamma} \right) \log \left( 1 + \gamma \kappa(\gamma) \right) \right] - 1 \right\} .
\]

(65)

We now give a lower-bound on \( \Xi_{\ell/\gamma}(\kappa) \) depending on the condition satisfied by \( \gamma \mapsto \kappa(\gamma) \).

(a) Assume that for any \( \tilde{\gamma} \in (0, \bar{\gamma}] \), \( \kappa(\tilde{\gamma}) < 0 \). Using that \( \log(1 - t) \leq -t \) for \( t \in (0, 1) \), we obtain that

\[
\exp \left[ -\ell \left( \frac{1}{\gamma} \right) \log \left( 1 + \gamma \kappa(\gamma) \right) \right] \geq \exp \left[ -\ell \left( \frac{1}{\gamma} \right) \gamma \kappa(\gamma) \right] \geq \exp \left[ -\ell \kappa(\gamma) \right] ,
\]

which together with (65) concludes the proof for Proposition 5-(a).

(b) Assume that for any \( \tilde{\gamma} \in (0, \bar{\gamma}] \), \( \kappa(\tilde{\gamma}) \leq 0 \). Then,

\[
\Xi_{\ell/\gamma}(\kappa) = \gamma \sum_{i=1}^{\ell/\gamma} (1 + \gamma \kappa(\gamma))^{-i} \geq \gamma \left\{ \frac{1}{\gamma} \right\} \ell \geq \ell .
\]

(c) Assume that for any \( \tilde{\gamma} \in (0, \bar{\gamma}] \), \( \kappa(\tilde{\gamma}) > 0 \). Using that \( \log(1 + t) \geq t/(1 + t) \) for \( t > 0 \), we obtain that

\[
\exp \left[ -\ell \left( \frac{1}{\gamma} \right) \log \left( 1 + \gamma \kappa(\gamma) \right) \right] \leq \exp \left[ -\ell \left( \frac{1}{\gamma} \right) \gamma \kappa(\gamma) \right] \leq \exp \left[ -\ell \kappa(\gamma)/(1 + \gamma \kappa(\gamma)) \right] ,
\]

which concludes the proof for Proposition 5-(a).

\( \square \)

Proof of Proposition 5. The proof is a direct application of Theorem 4 and Lemma 36 with \( \kappa(\gamma) = \kappa(\gamma) \).

A.2 Proof of Corollary 6

(a) Consider \( V: X \to [1, +\infty] \) given for any \( x \in X \) by \( V(x) = 1 + \|x\| \). Then since \( A^2(X^2) \) with \( \sup_{\gamma \in (0, \bar{\gamma}]} \kappa(\gamma) \leq \kappa_- < 0 \) holds, using the triangle inequality and the Cauchy-Schwarz inequality, we have for any \( \gamma \in (0, \bar{\gamma}] \) and \( x \in X \)

\[
R_\gamma V(x) \leq \|T_\gamma(x)\| + \sqrt{d} \leq (1 + \kappa_-) \|x\| + \|T_\gamma(0)\| + \sqrt{d} + 1 \leq \lambda V(x) + A ,
\]

with \( \lambda \in (0, 1) \) and \( A \geq 0 \). As a result, since for any \( \gamma \in (0, \bar{\gamma}] \), \( R_\gamma \) is a Feller kernel and the level sets of \( V \) are compact, \( R_\gamma \) admits a unique invariant probability measure \( \pi_\gamma \) for any \( \gamma \in (0, \bar{\gamma}] \) by [22, Theorem 12.3.3]. Then the last result is a straightforward consequence of Proposition 5-(a), (21) and the fact that for any \( \ell \in \mathbb{N}^* \) and \( \gamma \in (0, \bar{\gamma}] \), \( \alpha_- (\kappa, \gamma, \ell) \geq -\ell (\exp(-\ell \kappa_-) - 1)/\kappa_- \) since \( t \mapsto (\exp(\ell t) - 1)/t \) is increasing on \( \mathbb{R} \).

(b) This result is a direct consequence of Proposition 5-(b), (21) and the fact that \( R_\gamma \) admits an invariant probability measure \( \pi_\gamma \).
A.3 Proof of Corollary 7

(a) The proof is a direct application of Proposition 5-(b), the fact that \((X_k, Y_k) \in \mathcal{X}^2\) for any \(k \in \mathbb{N}\) and that \(K_\gamma\) is the Markov kernel associated with \((X_k, Y_k)_{k \in \mathbb{N}}\).

(b) Consider the case where \(A(2)\mathcal{X}^2\)-(iii) holds. Using that for any \(t \geq 0, 1 - e^{-t} \geq t/(t + 1)\) we obtain that for any \(\gamma \in (0, \bar{\gamma}]\) and \(\ell \in \mathbb{N}^*\)

\[
\alpha_+ (\kappa, \gamma, \ell) \geq \ell / (1 + (\ell + \bar{\gamma})\kappa (\gamma)) \geq (1 + (1 + \bar{\gamma})\kappa)^{-1} \geq (1 + \bar{\gamma})^{-1}(1 + \kappa)^{-1},
\]

where \(\alpha_+\) is given in Lemma 36-(c). Then, combining this result and Proposition 5-(c) complete the proof.

A.4 Proof of Theorem 8

We start with the following theorem.

**Theorem 37.** Under the assumptions of Theorem 8, we have for any \(\gamma \in (0, \bar{\gamma}], x, y \in \mathcal{X}\) and \(k \in \mathbb{N}\)

\[
W_c(\delta x, R^k, \delta y, R^k) \leq \tilde{K}_{\gamma}^k c(x, y) \leq \lambda^{k/4} [D_{\gamma, 1} c(x, y) + D_{\gamma, 2} \mathbb{I}_{\Delta^V_k(x, y)}] + \tilde{C}_{\gamma} \tilde{\rho}_\gamma^{k/4} \mathbb{I}_{\Delta^V_k(x, y)}, \tag{66}
\]

where \(W_c\) is the Wasserstein metric associated with \(c\) defined by (30),

\[
D_{\gamma, 1} = 1 + 4A [\log(1/\lambda) \lambda^n]^{-1}, \quad D_{\gamma, 2} = D_{\gamma, 1} \left[ A \lambda^{-n/\gamma} [1/\gamma] [1/\ell] \right],
\]

\[
\tilde{C}_{\gamma} = 8A \log^{-1}(1/\tilde{\rho}_\gamma),
\]

\[
\log(\tilde{\rho}_\gamma) = \{ \log(\lambda) \log(1 - \tilde{\varepsilon}_{d, \gamma}) / \{ - \log(\tilde{\varepsilon}_{d, \gamma}) + \log(1 - \tilde{\varepsilon}_{d, \gamma}) \} , \quad \tilde{\varepsilon}_{d, \gamma} = \sup_{(x, y) \in \mathcal{C}} \mathcal{V}(x, y), \quad \tilde{\varepsilon}_{d, \gamma} = \tilde{\varepsilon}_{d, \gamma}, \quad \tilde{\varepsilon}_{d, \gamma} = \in \Delta_{\mathcal{X}, d}\],

\[
\gamma \in (0, \bar{\gamma}], \quad \tilde{\varepsilon}_{d, \gamma} = \inf_{(x, y) \in \Delta_{\mathcal{X}, d}} \mathcal{V}(x, y, \ell, \| x - y \|), \quad \text{then H2(K_\gamma)-(i) is satisfied since for any x, y} \in \mathbb{C} \subset \Delta_{\mathcal{X}, d},
\]

\[
\tilde{K}_{\gamma}^{[1/\gamma] \ell} \mathbb{I}_{\Delta^V_k(x, y)} \leq \left\{ 1 - \inf_{(x, y) \in \Delta_{\mathcal{X}, d}} \mathcal{V}(x, y, \ell, \| x - y \|) \right\} \mathbb{I}_{\Delta^V_k(x, y)} \leq (1 - \tilde{\varepsilon}_{d, \gamma}) \mathbb{I}_{\Delta^V_k(x, y)}.
\]

**Proof.** The proof of this proposition is an application of Theorem 40 in Appendix D with \(d \leftarrow \tilde{\mathbb{I}}_{\Delta^V_k}\) which satisfies H1. Let \(\gamma \in (0, \bar{\gamma}]\). Then, since \(\tilde{K}_{\gamma}\) and \(\Psi\) satisfy \(D_d(\mathcal{V}, \lambda^n, A_\gamma, C)\) and (32) respectively, and \(\Delta_{\mathcal{X}}\) is absorbing for \(K_{\gamma}\), H2(K_\gamma) and H3(K_\gamma) are satisfied. More precisely, for any \(\gamma \in (0, \bar{\gamma}]\) setting \(\tilde{\varepsilon}_{d, \gamma} = \inf_{(x, y) \in \Delta_{\mathcal{X}, d}} \mathcal{V}(x, y, \ell, \| x - y \|), \quad \text{then H2(K_\gamma)-(i) is satisfied since for any x, y} \in \mathbb{C} \subset \Delta_{\mathcal{X}, d},
\]

\[
\tilde{K}_{\gamma}^{[1/\gamma] \ell} \mathbb{I}_{\Delta^V_k(x, y)} \leq \left\{ 1 - \inf_{(x, y) \in \Delta_{\mathcal{X}, d}} \mathcal{V}(x, y, \ell, \| x - y \|) \right\} \mathbb{I}_{\Delta^V_k(x, y)} \leq (1 - \tilde{\varepsilon}_{d, \gamma}) \mathbb{I}_{\Delta^V_k(x, y)}.
\]

H2(K_\gamma)-(ii) is satisfied for any \(\gamma \in (0, \bar{\gamma}]\) and \(x, y \in \mathcal{X}, K_{\gamma} \mathbb{I}_{\Delta^V_k(x, y)} \leq \mathbb{I}_{\Delta^V_k(x, y)}. \quad \text{Finally, the conditions H2(K_\gamma)-(iii) and H3(K_\gamma) hold using D_d(\mathcal{V}, \lambda^n, A_\gamma, \tilde{C}) with Q_1 \leftarrow \mathcal{V}, Q_2 \leftarrow \mathcal{V}d, \lambda_1 = \lambda_2 \leftarrow \lambda^n, A_1 = A_2 \leftarrow A_\gamma, n_0 \leftarrow [1/\gamma]. \quad \text{Applying Theorem 40, we obtain that for any } k \in \mathbb{N},
\]

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\[ \gamma \in (0, \bar{\gamma}] \] and \( x, y \in X \)

\[
\begin{align*}
W_c(\delta_x R^k_\gamma, \delta_y R^k_\gamma) &\leq \lambda^{k\gamma/4} \mathcal{Q}(x, y) + A_1 \left[ (1 + 1) \Delta^\gamma(\mathcal{E}(x, y)) + \lambda\lambda^{k\gamma/4} \right] \\
&\leq \lambda\lambda^{k\gamma/4} \mathcal{Q}(x, y) + 2r_1 A_1 \gamma_\lambda^{k\gamma/4} + A_1 \gamma_\lambda^{k\gamma/4} \mathcal{E}(x, y, \ell \left[ 1/\gamma \right] ) \\
&\leq \lambda\lambda^{k\gamma/4} (1 + 2r_2) \left[ \mathcal{Q}(x, y) + A_1 \gamma_\lambda^{k\gamma/4} \mathcal{E}(x, y, \ell \left[ 1/\gamma \right] ) \right] + 2r_1 A_1 \gamma_\lambda^{k\gamma/4},
\end{align*}
\]

where

\[
r_1 = 4 \log^{-1}(1/\rho_1) / (\gamma \rho_1^2), \quad r_2 = 4 \log^{-1}(1/\lambda) / (\gamma \lambda) .
\]

This concludes the proof of (66) upon using that \( \Delta X \) is absorbing for \( \bar{K}_\gamma \). \( \square \)

**Proof of Theorem 8.** The first part of the proof is straightforward using Theorem 37 and that \( \lambda \lambda \geq \lambda \lambda \).

By assumption on \( \bar{\gamma} \) and \( \lambda \), we have \( \lambda^{\gamma[1/\gamma]} \lambda \lambda \left[ 1/\gamma \right] \ell \leq \lambda^{-(1+\bar{\gamma})}(1 + \gamma) \ell \). As a result and using the fact that \( \log(1 - t) \leq -t \) for any \( t \in (0, 1) \), \( \log((1 - \bar{\varepsilon}_{d,1})^{-1}) \leq 1 \) and \( \mathcal{Q}(x, y) \geq 1 \) for any \( x, y \in X \), we obtain that

\[
\log^{-1}(\bar{\rho}_1) \leq \log((\lambda^{\gamma}) \log((1 - \bar{\varepsilon}_{d,1})^{-1})^{-1} [1 + \log(\bar{\varepsilon}_{d,1})] \\
\leq \log((1 + \bar{\varepsilon}_{d,1})^{-1} [1 + \log(\bar{\varepsilon}_{d,1}) + \log(1 + 2A\lambda^{-2\bar{\gamma}})] , \\
\leq \log((1 + \bar{\varepsilon}_{d,1})^{-1} [1 + \log(\bar{\varepsilon}_{d,1}) + \log(1 + 2A\lambda) + 2\ell \log(\lambda^{-1})] ,
\]

which completes the proof. \( \square \)

### A.5 Proof of Proposition 9

Let \( \gamma \in (0, \bar{\gamma}] \), \( x, y \in X \) and \( k \in \mathbb{N} \). We divide the proof into two parts.

(a) If \( k \leq [1/\gamma] \). Then using we get that

\[
\begin{align*}
\bar{K}_\gamma \|x - y\| &\leq (1 + \gamma \gamma) \|x - y\| \leq (1 + \gamma \gamma) \|x - y\| \leq \exp[\varphi(1 + \gamma)] \|x - y\|.
\end{align*}
\]

(b) If \( k > [1/\gamma] \) then using Theorem 8, (32) and \( \rho_1 \geq \lambda \) we get that

\[
\begin{align*}
W_c(\delta_x R^k_\gamma, \delta_y R^k_\gamma) &\leq \bar{K}_\gamma^{[1/\gamma]} \lambda^{k-1/\gamma} c(x, y) \\
&\leq \bar{K}_\gamma^{[1/\gamma]} \lambda^{k-1/\gamma} \left[ (\bar{D}_1 + \bar{D}_2 + \bar{C}_1)(1 - \varphi(1 + \gamma) \|x - y\|) + \bar{D}_1 \exp[\varphi(1 + \gamma)] \|x - y\| \right] / \rho_1^{(1+\bar{\gamma})/4} \\
&\leq -a(\bar{D}_1 + \bar{D}_2 + \bar{C}_1) \lambda^{k-1/\gamma} \|x - y\| / \rho_1^{1/4} + \bar{D}_1 \exp[\varphi(1 + \gamma)] \|x - y\| / \rho_1^{(1+\bar{\gamma})/4},
\end{align*}
\]

which concludes the proof upon noting that \( W_c(\delta_x R^k_\gamma, \delta_y R^k_\gamma) \geq \partial W_1(\delta_x R^k_\gamma, \delta_y R^k_\gamma) \).
A.6 Proof of Proposition 10

Let $q \in \mathbb{N}$ and $\gamma \in (0, \gamma]$. Using that $\tilde{K}_\gamma$ satisfies $D_4((x, y) \mapsto \|x - y\|^q, \tilde{\lambda}_q^\gamma, \tilde{A}_q^\gamma)$, we get that for any $x, y \in X$ and $k \in \mathbb{N}$ we have

$$\tilde{K}_\gamma^k \|x - y\|^q \leq \|x - y\|^q + \tilde{A}_q^\gamma \sum_{t=0}^{k-1} \tilde{\lambda}_q^\gamma \leq \|x - y\|^q + \tilde{A}_q \log^{-1}(1/\tilde{\lambda}_q) \tilde{\lambda}_q^{-\gamma}. \quad (67)$$

Let $p \geq 1, \alpha \in (p, +\infty), x, y \in X$ and $k \in \mathbb{N}$ and consider $q = p(\alpha - 1)/(\alpha - p)$. Note that we have

$$(1 - 1/\alpha)p/q \leq (1 - 1/\alpha)p/q \leq 1 - 1/\alpha \leq 1.$$  

Using this result, (67), Hölder’s inequality, Jensen’s inequality and that for any $a, b \geq 0$ and $r \geq 1, (a + b)^{1/r} \leq a^{1/r} + b^{1/r}$, we have

$$\tilde{K}_\gamma^k \|x - y\|^p \leq \tilde{K}_\gamma^k \left\{ \|x - y\|^{p(1-1/\alpha)} \|x - y\|^{p/\alpha}\right\}$$

$$\leq \left( \tilde{K}_\gamma^k \|x - y\|^{p(1-1/\alpha)}(1-1/\alpha) \right)^{1-p/\alpha} \left( \tilde{K}_\gamma^k \|x - y\|^{p/\alpha}\right)^{p/\alpha}$$

$$\leq \left( \tilde{K}_\gamma^k \|x - y\|^q \right)^{1-p/\alpha} \tilde{D}^{p}\tilde{A}_q^{k\gamma} \|x - y\|^{p/\alpha}$$

$$\leq \left( \|x - y\|^{p(1-1/\alpha)} + \tilde{A}_q \log^{-1}(1/\tilde{\lambda}_q) \tilde{\lambda}_q^{-\gamma} \right)^{1-p/\alpha} \tilde{D}^{p} \tilde{A}_q^{k\gamma} \|x - y\|^{p/\alpha}$$

$$\leq \left( \|x - y\|^{p(1-1/\alpha)} + \tilde{A}_q \log^{-1}(1/\tilde{\lambda}_q) \tilde{\lambda}_q^{-\gamma} \right)^{1-p/\alpha} \tilde{D}^{p} \tilde{A}_q^{k\gamma} \|x - y\|^{p/\alpha}$$

$$\leq \left( \|x - y\|^{(1-1/\alpha)p} + \tilde{A}_q \log^{-1}(1/\tilde{\lambda}_q) \tilde{\lambda}_q^{-\gamma} \right)^{1-p/\alpha} \tilde{D}^{p} \tilde{A}_q^{k\gamma} \|x - y\|^{p/\alpha}$$

$$\leq \left( \|x - y\|^p + \tilde{A}_q \log^{-1}(1/\tilde{\lambda}_q) \tilde{\lambda}_q^{-\gamma} \right)^{1-p/\alpha} \tilde{D}^{p} \tilde{A}_q^{k\gamma} \|x - y\|^{p/\alpha}$$

which completes the proof upon using that for any $a, b \geq 0$ and $p \geq 1, (a + b)^{1/p} \leq a^{1/p} + b^{1/p}.$

B Proofs of Section 4

B.1 Proof of Proposition 11

(a) By B.2 and B.3(m) we have for any $\gamma > 0$ and $x, y \in X, \|T_\gamma(x) - T_\gamma(y)\|^2 \leq (1 - 2\gamma \mathbb{L} + \gamma^2 \mathbb{L}^2) \|x - y\|^2 \leq (1 + \gamma \kappa(\gamma)) \|x - y\|^2$, which concludes the proof.

(b) We have for any $\gamma > 0$ and $x, y \in X, \|T_\gamma(x) - T_\gamma(y)\|^2 \leq \|x - y\|^2 + \gamma(-2\mathbb{m} + \gamma) \|b(x) - b(y)\|^2$. Then if $\gamma \leq 2\mathbb{m}, \|T_\gamma(x) - T_\gamma(y)\|^2 \leq \|x - y\|^2$, which concludes the proof.
B.2 Proof of Proposition 12

Let $\gamma \in (0, \bar{\gamma}]$, $x, y \in X$ and set $E = \mathcal{T}_\gamma(y) - \mathcal{T}_\gamma(x)$. We divide the proof into three parts.

(a) First, we show that Proposition 12-(a) holds. If $E = 0$ then the proposition is trivial, therefore we suppose that $E \neq 0$ and let $e = E / \|E\|$. Consider $Z_1$, a $d$-dimensional Gaussian random variable with zero mean and covariance identity. By (23) and the fact that $\Pi_X$ is non-expansive, we have for any $\gamma \in (0, \bar{\gamma}]$

\[
K_\gamma \|x - y\| \leq E \left[(1 - p_\gamma(x, y, \sqrt{\gamma}Z_1)) \left\|\left(\mathcal{T}_\gamma(x) + \sqrt{\gamma}Z_1\right) - \left(\mathcal{T}_\gamma(y) + \sqrt{\gamma}(\text{Id} - 2ee^\top)Z_1\right)\right\|\right]
\]

\[
= E \left[\left\|E - 2\sqrt{\gamma}ee^\top Z_1\right\| (1 - p_\gamma(x, y, \sqrt{\gamma}Z_1))\right]
\]

\[
= \int_{\mathbb{R}} \|E - 2ze\| \{\varphi_\gamma(z) - (\varphi_\gamma(z) \wedge \varphi_\gamma(\|E\| - z))\} \, dz
\]

\[
= \int_{\|E\|/2}^{\infty} (\|E\| - 2z) \{\varphi_\gamma(z) - \varphi_\gamma(\|E\| - z)\} \, dz \leq \|E\| ,
\]

where we have used the change of variable $z \mapsto \|E\| - z$ for the last line. We conclude this part of the proof upon using B2 and B3(m).

(b) Second, we show that Proposition 12-(b) holds. Consider the case $(x, y) \in \Delta^c_{X,R_1}$. By B2, C1, and since for any $t \in [-1, +\infty)$, $\sqrt{1 + t} \leq 1 + t/2$, we have that

\[
\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\| \leq (1 - 2\gamma m_1^+ + \gamma^2 L^2)^{1/2} \|x - y\| \leq (1 - \gamma m_1^+ + \gamma^2 L^2/2) \|x - y\| .
\]

Combining (68) and (69) and since $\gamma < 2m_1^+/L^2$, we obtain that for any $(x, y) \in \Delta^c_{X,R_1}$,

\[
K_\gamma \psi_1^0(x, y) \leq (1 - \gamma m_1^+ + \gamma^2 L^2/2) \|x - y\| / R_1 + 1
\]

\[
\leq (1 - \gamma m_1^+/2 + \gamma^2 L^2/4)(1 + \|x - y\| / R_1) \leq \lambda^\gamma \psi_1(x, y) .
\]

Similarly, we obtain using Proposition 11-(a) that for any $(x, y) \in \Delta_{X,R_1}$

\[
K_\gamma \psi_1 \leq (1 - \gamma m + \gamma^2 L^2/2) \|x - y\| / R_1 + 1
\]

\[
\leq (1 - \gamma m_1^+/2 + \gamma^2 L^2/4) \|x - y\| / R_1 + 1 + \gamma \left\{m_1^+/2 - m + \gamma L^2/4\right\}
\]

\[
\leq (1 - \gamma m_1^+/2 + \gamma^2 L^2/4) \psi_1(x, y) + \gamma \left[m_1^+ - m\right] \leq \lambda^\gamma \psi_1(x, y) + A\gamma .
\]

We conclude the proof upon combining (70) and (71).

(c) Finally we show that Proposition 12-(c) holds. Let $p \in \mathbb{N}$ with $p \geq 2$. Similarly to Proposition 12-(a), we have

\[
K_\gamma \|x - y\|^p = \int_{\mathbb{R}} (\|E\| - 2z)^p \varphi_\gamma(z) \, dz .
\]

For any $k \in \mathbb{N}$, let $c_k = \int_{\mathbb{R}} z^k \varphi_1(z) \, dz$ and

\[
\kappa_{1, \gamma} = 1 - \gamma m_1^+ + \gamma^2 L^2/2 , \quad \kappa_{2, \gamma} = \max(1, 1 - \gamma m + \gamma^2 L^2/2) , \quad \bar{R} = \max(1, R_1) .
\]

\[
\]
Note that for any $k \in \mathbb{N}$, $c_{2k+1} = 0$. Consider the case $\|x - y\| \geq \bar{R}$. Using (72), (73), B2, C1 we have

$$K_\gamma \|x - y\|^p \leq \|E\|^p + \sum_{k=2}^{p} \left( \frac{p}{k} \right) \|E\|^{p-k} (2\gamma)^k c_k$$

$$\leq \kappa_{1,\gamma} \|x - y\|^p + \sum_{k=2}^{p} \left( \frac{p}{k} \right) \|x - y\|^{p-k} (2\gamma)^k c_k$$

$$\leq \kappa_{1,\gamma} \|x - y\|^p + \gamma c_2 2^p \max(1, \gamma)^p \|x - y\|^{p-2}$$

$$\leq \kappa_{2,\gamma/2} \|x - y\|^p + \gamma \left\{ c_2 2^p \max(1, \gamma)^p \|x - y\|^{p-2} - m_1^+ \|x - y\| / 2 \right\}$$

$$\leq \kappa_{2,\gamma/2} \|x - y\|^p + \gamma \sup_{t \in [0, \infty)} \left\{ c_2 2^p \max(1, \gamma)^p t^{p-2} - m_1^+ t^p / 2 \right\}.$$ 

Note that we have for any $a \geq b \geq 0$ and $t \geq 0$

$$(1 + ta)^p - (1 + tb) \leq t \left\{ -b + \max(1, t)^p \sum_{k=1}^{p} \left( \frac{p}{k} \right) a^k \right\} \leq t \left\{ \max(1, t)^p (1 + a)^p - b \right\}.$$ 

(74)

Now, consider the case $\|x - y\| \leq \bar{R}$. Using (72), (73), (74), B2, B3(m) we have

$$K_\gamma \|x - y\|^p - \kappa_{1,\gamma/2} \|x - y\|^p \leq (\kappa_{2,\gamma} - \kappa_{1,\gamma/2}) \|x - y\|^p + \gamma c_2 2^p \max(1, \gamma)^p \kappa_{2,\gamma} \bar{R}^{p-2}$$

$$\leq \gamma c_2 2^p \max(1, \gamma)^p \kappa_{2,\gamma} \bar{R}^{p-2} + (\kappa_{2,\gamma} - \kappa_{1,\gamma/2}) \bar{R}^p$$

$$\leq \gamma c_2 2^p \max(1, \gamma)^p \kappa_{2,\gamma} \bar{R}^{p-2} + \bar{R}^p \left\{ \max(1, \gamma)^p (1 - m/2 + L^2 \bar{\gamma}/4)^p + m_1^+ \right\},$$

which concludes the proof upon setting

$$\lambda_p = \exp[-m_1^+/2 + \gamma L^2/4],$$

$$A_p = \max \{ A_{p,1}, A_{p,2} \},$$

$$A_{p,1} = \sup_{t \in [0, \infty)} \left\{ c_2 2^p \max(1, \gamma)^p t^{p-2} - m_1^+ t^p / 2 \right\},$$

$$A_{p,2} = c_2 2^p \max(1, \gamma)^p \kappa_{2,\gamma} \bar{R}^{p-2} + \bar{R}^p \left\{ \max(1, \gamma)^p (1 - m/2 + L^2 \bar{\gamma}/4)^p + m_1^+ \right\}.$$ 

B.3 Proof of Corollary 14

Let $\bar{\gamma} > 0$. Then for any $\gamma \in (0, \bar{\gamma}]$, $\Psi : t \mapsto 2\Phi \{-t/(2\Xi_{\bar{\gamma}}(\nu))\}$ is convex on $[0, +\infty)$, differentiable on $\mathbb{R}$, and for any $\gamma \in (0, \bar{\gamma}]

$$\Psi'(0) \geq -\pi \inf_{\gamma \in (0, \bar{\gamma}]} \Xi_{\bar{\gamma}}(\nu))^{-1/2}.$$ 

(75)

We divide the rest of the proof into two parts.

(a) First combining (75), Proposition 12-(a), (49), Theorem 13 and Proposition 9 shows that

$$W_1(\delta_x R_{\gamma}^k, \delta_y R_{\gamma}^k) \leq D_{\delta_x, \alpha, \rho_{\bar{\gamma}, a}^{k))^4} \|x - y\|.$$

(b) Second, combining Proposition 12-(c) and Proposition 10 shows that

$$W_p(\delta_x R_{\gamma}^k, \delta_y R_{\gamma}^k) \leq D_{\delta_x, \alpha, \rho_{\bar{\gamma}, a}^{k)^4/4\alpha)} \left\{ \|x - y\| + \|x - y\|^{1/\alpha} \right\}.$$ 

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B.4 Proof of Proposition 15

We preface the proof by a technical result.

**Lemma 38.** Let $\tilde{\gamma} > 0$, such that for any $\gamma \in (0, \tilde{\gamma}]$, $P_\gamma$ is a Markov kernel and $Q_\gamma$ is a Markov coupling kernel for $P_\gamma$. Assume that there exist $V : X \to [1, +\infty)$ measurable, $\lambda \in (0, 1)$ and $A \geq 0$ such that for any $\gamma \in (0, \tilde{\gamma}]$, $P_\gamma$ satisfies $D_\lambda(V, \lambda^\gamma, A\gamma, X)$. Let $\mathcal{V} : X^2 \to [1, +\infty)$ given for any $x, y \in X$ by $\mathcal{V}(x, y) = \{V(x) + V(y)\}/2$. The following properties hold.

(a) $Q_\gamma$ satisfies $D_\lambda(\mathcal{V}, \lambda^\gamma, A\gamma, X^2)$

(b) if $\lim_{||x|| \to +\infty} V(x) = +\infty$, $Q_\gamma$ satisfies $D_\lambda(\mathcal{V}, \lambda^\gamma/2, A\gamma, \bar{B}(0, R) \times \bar{B}(0, R))$ where $R = \inf\{r \geq 0 : \text{for any } x \in \bar{B}(0, r)^c, V(x) \geq 2(\lambda^{1/2})^{-2}\gamma A \log^{-1}(1/\lambda^{1/2})\}$ and $\bar{B}(0, R) \times \bar{B}(0, R) \subset \Delta_{X, 2R}$.

**Proof.** Let $\gamma \in (0, \tilde{\gamma}]$ and $x, y \in X$.

(a) Since $\delta_{(x,y)} Q_\gamma$ is a transference plan between $\delta_x P_\gamma$ and $\delta_y P_\gamma$ we have

$$Q_\gamma(\mathcal{V}(x, y)) = Q_\gamma(V(x) + V(y))/2 = P_\gamma(V(x)/2 + P_\gamma V(y))/2 \leq \lambda^\gamma \mathcal{V}(x, y) + A\gamma.$$}

(b) Let $x, y \in X$. If $(x, y) \in \bar{B}(0, R) \times \bar{B}(0, R)$ then the result is immediate using Lemma 38-(a). Now, assume that $(x, y) \notin \bar{B}(0, R) \times \bar{B}(0, R)$. By definition of $R$, $\max(V(x), V(y)) \geq 4A^{-\gamma} A \log^{-1}(1/\lambda)$. Without loss of generality assume that $V(x) \geq V(y)$. Using this result, Lemma 38-(a) and that for any $b \geq a$, $(e^b - e^a) \geq e^a(b - a)$, we have

$$Q_\gamma(\mathcal{V}(x, y)) \leq \lambda^\gamma \mathcal{V}(x, y) + A\gamma$$

$$\leq \lambda^\gamma/2 \mathcal{V}(x, y) + \gamma [A + \lambda^\gamma \{\log(\lambda) - \log(\lambda)/2\} \mathcal{V}(x, y)]$$

$$\leq \lambda^\gamma/2 \mathcal{V}(x, y) + \gamma [A - \lambda^\gamma \log(\lambda^{-1}) V(x)/4] \leq \lambda^\gamma/2 \mathcal{V}(x, y).$$

**Proof of Proposition 15.** Let $\gamma \in (0, \tilde{\gamma}]$ and $x \in X$. We divide the proof into two parts. Using (19), B2, B3(m), C2, that the projection $\Pi_X$ is non expansive and $\gamma < 2m^+_{\lambda}/L^2$, we obtain for any $x \in X$

$$R_\gamma V(x) \leq 1 + ||x + \gamma b(x)||^2 + \gamma d$$

$$\leq 1 + ||x||^2 + 2\gamma(x, b(x)) + \gamma^2 ||b(x)||^2 + \gamma d$$

$$\leq (1 + ||x||^2) [1 - \gamma(2m^+_{\lambda} - \gamma L^2)] + \gamma (d + 2R^2_{\lambda}(m^+_{\lambda} - m) + 2m^+_{\lambda}).$$

In addition, for any $x \in X$, such that $||x|| \geq 2A^{1/2} \log^{-1/2}(1/\lambda)$, we have $V(x) \geq 4A \log^{-1}(1/\lambda)$. We conclude the proof using Lemma 38-(b).

**B.5 Proof of Proposition 17**

Let $\gamma \in (0, \tilde{\gamma}]$. Using the fact that $\Pi_X$ is non expansive, the Log-Sobolev inequality, the fact that $\pi$ is 1-Lipschitz, [5, Theorem 5.5] and the Jensen inequality we obtain for any $x \in \mathbb{R}^d$

$$R_\gamma V(x) \leq \exp \left[ m^+_{\lambda} R_\gamma \phi(x) + \gamma (m^+_{\lambda})^2 / 2 \right] \leq \exp \left[ m^+_{\lambda} \sqrt{1 + R_\gamma \|x\|^2 + \gamma (m^+_{\lambda})^2 / 2} \right]$$

$$\leq \exp \left[ m^+_{\lambda} \sqrt{1 + \|T_\gamma(x)\|^2 + \gamma d + \gamma (m^+_{\lambda})^2 / 2} \right].$$ (76)
Let $x \in \mathbb{R}^d$. The rest of the proof is divided in two parts.

(a) In the first case, $\|x\| \geq R_4$. Since $\|x\| \geq R_3$ and $\gamma \leq 2k_2$, we have using $C_3$

$$\|T_\gamma(x)\|^2 \leq \|x\|^2 - 2\gamma k_1 \|x\| + \gamma(\gamma - 2k_2) \|b(x)\|^2 + \gamma a \leq \|x\|^2 - 2\gamma k_1 \|x\| + \gamma a . \tag{77}$$

Since $\|x\| \geq 1$ we have $2\|x\| \geq \phi(x)$ and therefore, using that $\|x\| \geq (d + a)/k_1$, $2k_1 \|x\| \geq 2m_3^+ \phi(x) + d + a$. This inequality, combined with the fact that for any $t \in (-1, +\infty)$, $\sqrt{1 + t} \leq 1 + t/2$, yields

$$\sqrt{1 + \|x\|^2 + \gamma(-2k_1 \|x\| + d + a) - \phi(x)} \leq \gamma(-2k_1 \|x\| + d + a)/(2\phi(x)) \leq -\gamma m_3^+ . \tag{78}$$

Combining (76), (77) and (78) we get

$$R_\gamma V(x) \leq \lambda^\gamma V(x) .$$

(b) In the second case $\|x\| \leq R_4$. We have the following inequality using $C_3$ and that $\gamma \leq 2k_2$

$$\|T_\gamma(x)\|^2 \leq \|x\|^2 + \gamma(\gamma - 2k_2) \|b(x)\|^2 + \gamma c \leq \|x\|^2 + \gamma a . \tag{79}$$

Combining (76), (79) and the fact that for any $t \in (-1, +\infty)$, $\sqrt{1 + t} \leq 1 + t/2$ we get

$$R_\gamma V(x) \leq \exp \left[ \gamma m_3^+(d + a)/(2\phi(x)) + \gamma(m_3^+)^2/2 \right] V(x) \leq \exp \left[ \gamma(m_3^+(d + a) + (m_3^+)^2)/2 \right] V(x) . \tag{80}$$

Note that for any $c_1 \geq c_2$ and $t \in [0, \bar{t}]$ we have the following inequality

$$e^{c_1 t} \leq e^{c_2 t} + e^{c_1 t}(c_1 - c_2)t . \tag{81}$$

Combining (80) and (81) we get

$$R_\gamma V(x) \leq \lambda^\gamma V(x) + \exp \left[ \gamma(m_3^+(d + a) + (m_3^+)^2)/2 \right] C_2 \gamma ,$$

with $C_2 = (m_3^+(d + a)/2 + (m_3^+)^2) \exp(m_3^+(1 + R_2^2)^{1/2})$, which concludes the proof using Lemma 38.

C Proofs of Section 5

C.1 Proof of Theorem 19

Combining Proposition 29, Proposition 33 and Proposition 35 in Theorem 28 concludes the proof.

C.2 Proof of Theorem 20

Combining Proposition 29, Proposition 33 and Proposition 35 in Theorem 28 concludes the proof.
C.3 Proof of Theorem 24

Let $T \geq 0$ and $x \in \mathbb{R}^d$. First, using Proposition 32 we have that $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ and $\mathbf{L}_5$ are satisfied. In addition, using Proposition 35 we get

$$P_T V_k(x) < \infty,$$

where $V_k = V_M$ with $M \leftarrow k_1$ and $V_M$ given in (56). Since, $\sup_{x \in \mathbb{R}^d} \|b(x)\| 2^{(1+\varepsilon_k)} e^{-k_1(1+\|x\|)^{1/2}} < +\infty$, we get that $\mathbf{L}_3$ is satisfied.

In addition, using that $2m_3^+ \leq k_1$ we have

$$P_T V^2(x) < \infty.$$

Thus, the first part of (62) is satisfied. Second using Proposition 17 and replacing $m_3^+ \leftarrow 2m_3^+$ (which is valid since $m_3^+ \leq k_1/4$), we obtain that for any $n, m \in \mathbb{N}$, with $T/m < 2k_2$, $R_{T/m,n}$ and $R_{T/m}$ satisfy $D_{d}(V^2, \lambda^{T/m}, A T/m, X^2)$. Hence, for any $m \in \mathbb{N}$ with $T/m < 2k_2$ we have

$$ R_{T/m,n} + \tilde{R}_{T/m,n} V^2(x) \leq V^2(x) + AT m^{-1} \sum_{k \in \mathbb{N}} \lambda^{T/m} \leq V^2(x) + A \log^{-1}(1/\lambda) \lambda^{-\gamma}. $$

Therefore, the second part of (62) is satisfied and we can apply Theorem 28. Using Theorem 18 and we get that for any $m, n \in \mathbb{N}$ with $x, y \in \mathbb{R}^d$ and $T/m \in (0, 2k_2)$$

$$\|\delta_x R_{T/m,n} - \delta_y R_{T/m,n}\| \leq C_{1/m,c} \rho_{1/m,c} \{V(x) + V(y)\},$$

where $C_{1/m,c} \geq 0$ and $\rho_{1/m,c} \in (0, 1)$, see Appendix H. We conclude upon noting that $C_{1/m,c}$ and $\rho_{1/m,c}$ admit limits $C_c$ and $\rho_c$ when $m \to +\infty$ which do not depend on $n$.

C.4 Proof of Lemma 25

(a) Let $x \in \mathbb{R}^d$ and let $(X_t)_{t \geq 0}$ a solution of (55) starting from $x$. Define for any $k \in \mathbb{N}^*$, $\tau_k = \inf\{t \geq 0 : \|X_t\| \geq k\}$ and for any $t \geq 0$, $M_t = \int_0^t \langle VV(X_u), dB_u \rangle$. Using the Itô formula we obtain that for every $t \geq 0$ and $k \in \mathbb{N}^*$

$$V(X_{t \wedge \tau_k}) e^{\zeta (t \wedge \tau_k)} = \int_0^{t \wedge \tau_k} \left[ e^{\zeta (t \wedge \tau_k)} AV(X_u) + \zeta e^{\zeta u} V(X_u) \right] du + M_{t \wedge \tau_k} + V(x)$$

$$= V(X_{s \wedge \tau_k}) e^{\zeta (s \wedge \tau_k)} + M_{s \wedge \tau_k} - M_{s \wedge \tau_k} + \int_{s \wedge \tau_k}^{t \wedge \tau_k} \left[ e^{\zeta (t \wedge \tau_k)} AV(X_u) + \zeta e^{\zeta u} V(X_u) \right] du$$

$$\leq V(X_{s \wedge \tau_k}) e^{\zeta (s \wedge \tau_k)} + M_{t \wedge \tau_k} - M_{s \wedge \tau_k}.$$

Therefore since for any $k \in \mathbb{N}^*$, $(M_{t \wedge \tau_k})_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$-martingale, we get for every $t \geq s \geq 0$ and $k \in \mathbb{N}^*$

$$E \left[ V(X_{t \wedge \tau_k}) e^{\zeta (t \wedge \tau_k)} | \mathcal{F}_s \right] \leq V(X_{s \wedge \tau_k}) e^{\zeta (s \wedge \tau_k)},$$

which concludes the first part of the proof taking $k \to +\infty$ and using Fatou’s lemma.

(b) Similarly we have that $(V(X_t)e^{\zeta t} - B(1 - \exp(-\zeta t))/\zeta)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$-supermartingale which concludes the proof upon taking the expectation of $V(X_t)e^{\zeta t} - B(1 - \exp(-\zeta t))/\zeta$.
C.5 Proof of Proposition 26

Let $T \geq 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$ with $T/m \leq \bar{\gamma}$. Using \cite[Lemma 24]{29}, we obtain

$$
\|\delta x P_T - \delta x \tilde{R}^m_{T/m,n}\|_V
\leq (1/\sqrt{2}) \left(\delta x P_T V^2(x) + \delta x \tilde{R}^m_{T/m,n} V^2(x)\right)^{1/2} \text{KL} \left(\delta x P_T | \delta x \tilde{R}^m_{T/m,n}\right)^{1/2}.
$$

Let $M \geq 0$, $n \in \mathbb{N}^*$ with $n^{-1} < \bar{\gamma}$, $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$. Therefore, we only need to show that

$$
\lim_{m \to +\infty} \text{KL}(\delta x P_T | \delta x \tilde{R}^m_{T/m,n}) = 0.
$$

Consider the two processes $(X_t)_{t \in [0,T]}$ and $(\tilde{X}_t)_{t \in [0,T]}$ defined by \eqref{55} with $X_0 = \tilde{X}_0 = x$ and

$$
d \tilde{X}_t = \tilde{b}_{T/m,n}(t, X_t) dt + dB_t,
$$

where for any $(w_s)_{s \in [0,T]} \in C([0,T], \mathbb{R}^d)$, $t \in [0,T]$,

$$
\tilde{b}_{T/m,n}(t, (w_s)_{s \in [0,T]}) = \sum_{i=0}^{m-1} b_{T/m,n}(w_{iT/n}) I_{[iT/m, (i+1)T/m]}(t).
$$

Note for any $i \in \{0, \ldots, m\}$, the distribution of $X_{iT/m}$ is $\delta x \tilde{R}^i_{T/m,n}$. Using that $b$ and $b_{T/m,n}$ are continuous and that $(X_t)_{t \in [0,T]}$ and $(\tilde{X}_t)_{t \in [0,T]}$ take their values in $C([0,T], \mathbb{R}^d)$, we obtain that

$$
P \left( \int_0^T \|b(X_t)\|^2 dt < +\infty \right) = 1,
$$

and

$$
P \left( \int_0^T \|b(B_t)\|^2 dt < +\infty \right) = 1,
$$

where $(B_t)_{t \in [0,T]}$ is the $d$-dimensional Brownian motion associated with \eqref{55}. Therefore by \cite[Theorem 7.7]{55} the distributions of $(X_t)_{t \in [0,T]}$ and $(\tilde{X}_t)_{t \in [0,T]}$, denoted by $\mu^x$ and $\tilde{\mu}^x$ respectively, are equivalent to the distribution of the Brownian motion $\mu^x_B$ starting at $x$. In addition, $\mu^x$ admits a Radon-Nikodym density w.r.t. to $\mu^x_B$ and $\tilde{\mu}^x$ admits a Radon-Nikodym density w.r.t. to $\tilde{\mu}^x$; given $\mu^x_B$-almost surely for any $(w_t)_{t \in [0,T]} \in C([0,T], \mathbb{R}^d)$ by

$$
\frac{d\mu^x}{d\mu^x_B}((w_t)_{t \in [0,T]}) = \exp \left( (1/2) \int_0^T \langle b(w_s), dw_s \rangle - (1/4) \int_0^T \|b(w_s)\|^2 ds \right),
$$

and

$$
\frac{d\tilde{\mu}^x}{d\tilde{\mu}^x_B}((w_t)_{t \in [0,T]}) = \exp \left( -(1/2) \int_0^T \langle \tilde{b}_{T/m,n}(s, (w_u)_{u \in [0,T]}), dw_s \rangle + (1/4) \int_0^T \|\tilde{b}_{T/m,n}(s, (w_u)_{u \in [0,T]}\|^2 ds \right).
$$

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Finally we obtain that $\mu_B^x$-almost surely for any $(w_s)_{s \in [0,T]} \in C([0,T], \mathbb{R}^d)$

$$\frac{d\mu^x}{d\mu^x}(\omega_{[0,T]}) = \exp \left( \frac{1}{2} \int_0^T \langle b(w_s) - \bar{b}_{T/m,n}(s,(w_u)_{u \in [0,T]}), dw_s \rangle + \frac{1}{4} \int_0^T \| \bar{b}_{T/m,n}(s,(w_u)_{u \in [0,T]}) \|^2 - \| b(w_s) \|^2 ds \right).$$ \quad (83)

Now define for any $(w_s)_{s \in [0,T]} \in C([0,T], \mathbb{R}^d)$ and $t \in [0,T]$

$$b_{T/m}(t,(w_s)_{s \in [0,T]}) = \sum_{i=0}^{m-1} b(w_{iT/m}) \mathbbm{1}_{[iT/m,(i+1)T/m]}(t).$$ \quad (84)

Using (55), (82), (83), L2, and for any $a_1, a_1 \in \mathbb{R}^d$, $\|a_1 - a_2\|^2 \leq 2(\|a_1\|^2 + \|a_2\|^2)$, we obtain that

$$2KL \left( \delta_x P_T | \delta_x \tilde{R}_{T/m,n} \right) \leq 2^{-1} \mathbb{E} \left[ \int_0^T \| b(X_s) - \bar{b}_{T/m,n}(s,(X_u)_{u \in [0,T]}) \|^2 ds \right]$$

$$\leq \mathbb{E} \left[ \int_0^T \| b(X_s) - b_{T/m}(s,(X_u)_{u \in [0,T]}) \|^2 ds \right]$$

$$+ \sum_{i=0}^{m-1} \mathbb{E} \left[ \int_{iT/m}^{(i+1)T/m} \| b(X_{iT/m}) - b_{T/m,n}(X_{iT/m}) \|^2 ds \right]$$

$$\leq \mathbb{E} \left[ \int_0^T \| b(X_s) - b_{T/m}(s,(X_u)_{u \in [0,T]}) \|^2 ds \right]$$

$$+ C_1 T^{1+\beta} m^{-\beta} \sup_{s \in [0,T]} \mathbb{E} \left[ \| b(X_s) \|^2 \right].$$ \quad (85)

It only remains to show that the first term goes to 0 as $m \to +\infty$. Note that since $(X_s)_{s \in [0,T]}$ is almost surely continuous and $b$ is continuous on $\mathbb{R}^d$, $\lim_{m \to +\infty} \| b(X_s) - b_{T/m}(s,(X_u)_{u \in [0,T]}) \|^2 = 0$ for any $s \in [0,T]$ almost surely. Then, using the Lebesgue dominated convergence theorem and the continuity of $b$, we obtain that for any $M \geq 0$,

$$\lim_{m \to +\infty} \mathbb{E} \left[ \mathbbm{1}_{[0,M]} \left( \sup_{s \in [0,T]} \| X_s \| \right) \int_0^T \| b(X_s) - b_{T/m}(s,(X_u)_{u \in [0,T]}) \|^2 ds \right] = 0.$$ \quad (86)
On the other hand, using Hölder’s inequality and the definition of \( b_{T/m} \) (84), we obtain that for any \( M \geq 0 \),

\[
\begin{align*}
\mathbb{E} \left[ \mathbb{I}_{(M, +\infty)} \left( \sup_{s \in [0, T]} \| X_s \| \right) \int_0^T \| b(X_s) - b_{T/m}(s, (X_s)_{s \in [0, T]}) \|^2 ds \right] \\
\leq 2 \left( \mathbb{P} \left( \sup_{s \in [0, T]} \| X_s \| > M \right) \right) \epsilon_k/(1 + \epsilon_k) \\
\int_0^T \mathbb{E}^{1/(1 + \epsilon_k)} \left( \| b(X_s) \|^{2(1 + \epsilon_k)} \right) + \mathbb{E}^{1/(1 + \epsilon_k)} \left( \| b_{T/m}(s, (X_u)_{u \in [0, T]}) \|^{2(1 + \epsilon_k)} \right) ds \\
\leq 4T \left( \mathbb{P} \left( \sup_{s \in [0, T]} \| X_s \| > M \right) \right) \epsilon_k/(1 + \epsilon_k) \left( \sup_{s \in [0, T]} \mathbb{E} \left[ \| b(X_s) \|^{2(1 + \epsilon_k)} \right] \right)^{1/(1 + \epsilon_k)} .
\end{align*}
\]

Combining this result, L3, and (86) in (85), we obtain that for any \( M \geq 0 \),

\[
\limsup_{m \to +\infty} \text{KL} \left( \delta_x \mathbb{P}_T | \delta_x \tilde{R}_{T/m, n} \right) \\
\leq 2T \left( \mathbb{P} \left( \sup_{s \in [0, T]} \| X_s \| > M \right) \right) \epsilon_k/(1 + \epsilon_k) \left( \sup_{s \in [0, T]} \mathbb{E} \left[ \| b(X_s) \|^{2(1 + \epsilon_k)} \right] \right)^{1/(1 + \epsilon_k)} .
\]

Since \((X_s)_{s \in [0, T]}\) is a.s. continuous, we get by the monotone convergence theorem and L3, taking \( M \to +\infty \), that \( \lim_{m \to +\infty} \text{KL}(\delta_x \mathbb{P}_T | \delta_x \tilde{R}_{T/m, n}) = 0 \), which concludes the proof.

### C.6 Proof of Proposition 27

For any \( n \in \mathbb{N} \) and \( \gamma \in (0, \bar{\gamma}] \), we consider the synchronous Markov coupling \( Q_{\gamma, n} \) for \( R_{\gamma, n} \) and \( \tilde{R}_{\gamma, n} \) defined for any \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\) and \( A \in \mathcal{B}(\mathbb{R}^d)\) by

\[
Q_{\gamma, n}((x, y), A) = \frac{1}{(2\pi \gamma)^{d/2}} \int_{\mathbb{R}^d} \mathbb{I}_{(-\Pi_{n}, \Pi_{n})}(-A)(T_{\gamma}(x) + \sqrt{\gamma}z, T_{\gamma}(y) + \sqrt{\gamma}z)e^{-\|z\|^2/2} dz ,
\]

with \( T_{\gamma}(x) = x + \gamma b(x) \). Let \( T \geq 0 \), \( n \in \mathbb{N} \), \( m \in \mathbb{N}^* \) such that \( T/m \leq \bar{\gamma} \). Consider \((X_j, \tilde{X}_j)_{j \in \mathbb{N}}\) a Markov chain with Markov kernel \( Q_{T/m, n} \) and started from \( X_0 = \tilde{X}_0 = x \) for a fixed \( x \in \mathbb{R}^d \). Note that by definition and L4, we have that for \( k < \tau \), \( X_k = \tilde{X}_k \) where \( \tau = \inf\{j \in \mathbb{N} : \tilde{X}_j \notin B(0, n)\} \). Using L5, \((V(X_j) \exp \left[ -j \log(A)(T/m)(1 + E_{\gamma}(T/m)^{\tau}) \right])_{j \in \mathbb{N}}\) is a positive supermartingale. Combining (87), the Cauchy-Schwarz inequality, L5 and the Doob maximal
inequality for positive supermartingale [65, Proposition II-2-7], we get for any \( x \in \mathbb{R}^d \)
\[
\| \delta_x R_{T/m,n} - \delta_x \bar{R}_{T/m,n} \| \leq \mathbb{E} \left[ \mathbb{1}_{\Delta^c_{\epsilon d}} (X_m, \bar{X}_m) (V(X_m) + V(\bar{X}_m))/2 \right]
\leq (1/2) \mathbb{P} \left( \sup_{j \in \{0, \ldots, m \}} \| \bar{X}_j \| \geq n \right) \left( \mathbb{E} [V^2(X_m)]^{1/2} + \mathbb{E} [V^2(\bar{X}_m)]^{1/2} \right)
\leq (1/2) \mathbb{P} \left( \sup_{j \in \{0, \ldots, m \}} \bar{V}(\bar{X}_j) \geq n \right) \left( \mathbb{E} [V^2(X_m)]^{1/2} + \mathbb{E} [V^2(\bar{X}_m)]^{1/2} \right)
\leq (2n)^{-1} \exp \left[ \log(\tilde{A}) (T/m)(1 + E_n(T/m)^e n) \right] \bar{V}(x)
\times \left( (R_{T/m,n}^V V^2(x))^{1/2} + (\bar{R}_{T/m,n}^V V^2(x))^{1/2} \right),
\]
which concludes the proof upon taking \( m \to +\infty \) then \( n \to +\infty \).

C.7 Proof of Proposition 29
Let \( p \in \mathbb{N}^* \) and \( V \in C^2(\mathbb{R}^d, [1, +\infty)) \) be defined for any \( x \in \mathbb{R}^d \) by \( V(x) = 1 + \|x\|^{2p} \). For any \( x \in \mathbb{R}^d \), \( \nabla V(x) = 2p \|x\|^{2(p-1)} x \) and \( \Delta V(x) = (4p(p-1) + 2pd) \|x\|^{2(p-1)} \). Therefore, using \( B2 \) and the definition of \( \mathcal{A} \) we obtain that for any \( x \in \mathbb{R}^d \)
\[
\mathcal{A} V(x) \leq [2p(p-1) + p(d + 2L)] V(x) .
\]
Hence, using (88) and [52, Theorem 3.5], we obtain that \( L1 \) holds. Using that for any \( \sup_{x \in \mathbb{R}^d} \|b(x)\|(1 + \|x\|^{2p})^{-1} < +\infty \), (88) and Lemma 25-(b) we obtain that \( L3 \) holds.

\( L2 \) and \( L4 \) are trivially satisfied. Finally, using once again \( B2 \), we have that for any \( x \in \mathbb{R}^d \) and \( \gamma \in (0, \tilde{\gamma}] \) we have
\[
R_{\gamma}(1 + \|x\|^2) \leq 1 + \|x + \gamma b(x)\|^2 + \gamma d
\leq 1 + \|x\|^2 + 2\gamma \|b(x)\| \|x\| + \gamma^2 \|b(x)\|^2 + \gamma d
\leq 1 + \|x\|^2 + 2\gamma L \|x\|^2 + \gamma^2 L^2 \|x\|^2 + \gamma d
\leq (1 + 2\gamma L + \gamma^2 L^2 + \gamma d)(1 + \|x\|^2),
\]
which implies that \( L5 \) holds.

C.8 Proof of Proposition 30
Let \( p \in \mathbb{N}^* \) and \( V \in C^2(\mathbb{R}^d, [1, +\infty)) \) be defined for any \( x \in \mathbb{R}^d \) by \( V(x) = 1 + \|x\|^{2p} \). For any \( x \in \mathbb{R}^d \), \( \nabla V(x) = 2p \|x\|^{2(p-1)} x \) and \( \Delta V(x) = (4p(p-1) + 2pd) \|x\|^{2(p-1)} \). Therefore, using \( B3(m) \) and the definition of \( \mathcal{A} \) we obtain that for any \( x \in \mathbb{R}^d \)
\[
\mathcal{A} V(x) \leq [2p(p-1) + p(d - 2m)] V(x) .
\]
Hence, using (89) and [52, Theorem 3.5], we obtain that \( L1 \) holds.

(a) If there exists \( \varepsilon_0 > 0 \) such that \( \sup_{x \in \mathbb{R}^d} \|b(x)\|^{2(1+\varepsilon_0)} (1 + \|x\|^{2p})^{-1} < +\infty \), using (89) and Lemma 25-(b) we obtain that \( L3 \) holds.
(b) If there exists $\varepsilon_b > 0$ such that $\sup_{x \in \mathbb{R}^d} \|b(x)\|^2 e^{-\varepsilon_b \|x\|^2} < +\infty$, and $C2$ holds, then consider for any $x \in \mathbb{R}^d$, $V(x) = e^{\varepsilon_b \|x\|^2}$. We have for any $x \in \mathbb{R}^d$, $\nabla V(x) = 2m_2^+ e^{\varepsilon_b \|x\|^2} x$ and $\Delta V(x) = 4m_2^+ e^{\varepsilon_b \|x\|^2} \|x\|^2 + 2m_2^+ e^{\varepsilon_b \|x\|^2} \|x\|^2 d$. Therefore, using $C2$ we have for any $x \in \mathbb{B}(0, R_2)^c$

$$\mathcal{A} V(x) \leq m_2^+ \left[ d + (4m_2^+ / 2 - 2m_2^+) \|x\|^2 \right] V(x) \leq m_2^+ d V(x). \quad (90)$$

Setting $\zeta = (m_2^+ d) \vee \sup_{x \in \mathbb{B}(0, R_2)} \mathcal{A} V(x)$, we obtain that $V$ satisfies $D_{\varepsilon}(V, \zeta, 0)$. Therefore using (90) and Lemma 25-(b), we obtain that $L3$ holds.

C.9 Proof of Proposition 31

We preface the proof by a preliminary computation. Let $n \in \mathbb{N}$, $\gamma \in (0, \gamma]$, $x \in \mathbb{R}^d$ and $X = x + \gamma b_{\gamma, n}(x) + \sqrt{\gamma} Z$, where $Z$ is a $d$-dimensional Gaussian random variable with zero mean and covariance identity. We have using $B3(\gamma)$ and (63)

$$\mathbb{E} \left[ \|X\|^2 \right] \leq \|x\|^2 - 2\gamma \Phi_n(x) \|x\|^2 + \gamma^2 \Phi_n(x) \|b(x)\|^2 + \gamma d, \quad (91)$$

with $\Phi_n(x) = \varphi_n(x) + (1 - \varphi_n(x))(1 + \gamma ^{\|b(x)\|})^{-1}$. We recall that

$$\varphi_n(x) \in [0, 1] \quad \text{and} \quad \varphi_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{B}(0, n), \\ 0 & \text{if } x \in \mathbb{B}(0, n+1)^c. \end{cases} \quad (92)$$

Using $B5$ and (92), we have

$$\Phi_n(x) \|b(x)\| \leq L_{n+1} \|x\| + \gamma^{-\alpha}. \quad (93)$$

Combining (91) and (93) and since $\Phi_n(x) \leq 1$ by (92), we obtain

$$\mathbb{E} \left[ 1 + \|X\|^2 \right] \leq (1 + \|x\|^2) \left[ 1 + 2\gamma |m| + 2\gamma L_{n+1}^2 \right] + 2\gamma^2 - 2\alpha + \gamma d. \quad (94)$$

We are now able to complete the proof of Proposition 31. It is easy to check that $L2$ holds with $\beta = 2\alpha$. It only remains to show that $L5$ holds. Consider for any $x \in \mathbb{R}^d$, $V(x) = 1 + \|x\|$. By (94), for any $\gamma \in (0, \gamma]$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we have using for any $s \geq \mathbb{R}$, $1 + s \leq e^{s}$ we obtain

$$R_{\gamma, n} \tilde{V}(x) \leq \tilde{V}(x) \left[ 1 + 2\gamma |m| + 2\gamma L_{n+1}^2 + 2\gamma^2 - 2\alpha + \gamma d \right]$$

$$\leq \tilde{V}(x) \exp \left\{ \gamma \left\{ 2|m| + d + 2\gamma^{1-2\alpha}(\gamma^{2\alpha} L_{n+1} + 1) \right\} \right\}$$

$$\leq \tilde{V}(x) \exp \left\{ 2\gamma \left\{ 2|m| + d \right\} \left\{ 1 + \gamma^{1-2\alpha} \left(\gamma^{2\alpha} L_{n+1} + 1\right) \right\} \right\}.$$

As a result using that $d \geq 1$, $L5$ holds upon taking $\tilde{A} = \exp(4|m| + 2d)$, $\tilde{e}_n = 1 - 2\alpha$ and $\tilde{E}_n = 2(L_{n+1}^{2\alpha} + 1)$.

C.10 Proof of Proposition 33

The proof is similar to the one of Proposition 34 upon replacing (94) by

$$\mathbb{E} \left[ 1 + \|X\|^2 \right] \leq (1 + \|x\|^2)(1 + 2\gamma L + 2\gamma^2 L^2) + \gamma d.$$
C.11 Proof of Proposition 34

Let $M \geq 0$, $n \in \mathbb{N}$ and $p \geq 1$. Using the Log-Sobolev inequality [5, Theorem 5.5], the fact that $\phi$ is 1-Lipschitz and that $\Pi_{B(0,n)}$ is non expansive, as well as the Jensen inequality we obtain for any $\gamma \in (0,\bar{\gamma}]$ and $x \in \mathbb{R}^d$, 

$$R_{\gamma,n} V_M^p(x) \leq \exp \left[ pM \bar{R}_{\gamma,n} \phi(x) + (pM)^2 \gamma / 2 \right] \leq \exp \left[ pM \sqrt{\bar{R}_{\gamma,n} \phi^2(x) + (pM)^2 \gamma / 2} \right].$$

Using (94) and that $\sqrt{1 + t} \leq 1 + t/2$ for any $t \in (-1, +\infty)$ we get for any $\gamma \in (0,\bar{\gamma}]$ and $x \in B(0,n)$ 

$$R_{\gamma,n} V_M^p(x) \leq \exp \left[ pM \left\{ \phi(x)^2 (1 + 2\gamma |x| + 2\gamma^2 t^2 x_n^2) + 2\gamma^2 - 2\alpha \gamma d \right\}^{1/2} + (pM)^2 \gamma / 2 \right] \leq \exp \left[ \left( 1 + \gamma |x|^2 + \gamma^2 t^2 x_n + 1 \right) pM \phi(x) \right] \exp \left[ (1 + pM)^2 \left\{ \gamma (d + 1/2) + 2\gamma^2 - 2\alpha \right\} \right] \leq V_M^p(1 + C_1 \gamma + C_2 \gamma^2) \exp \left[ p^2 C_3 \gamma \right],$$

with $C_1 = |m|$, $C_2 = L^2_{n+1}$ and $C_3 = (1 + M)^2 (d + 3)/2$. By recursion, we obtain that for any $m,n \in \mathbb{N}$ with $m^{-1} \in (0,\bar{\gamma}]$, $T \geq 0$ and $x \in B(0,n)$ 

$$R_{T/m,n}^m V_M(x) \leq V_M(x)^a m \exp \left[ T C_3 \sum_{j=0}^{m-1} (1 + TC_1/m + C_2, n(T/m)^2)^{2j} / m \right] \leq V_M(x)^a m \exp \left[ T C_3 (1 + TC_1/m + C_2, n(T/m)^2)^{2m} \right],$$

with $a_m = (1 + TC_1/m + C_2, n(T/m)^2)^m$. Since $\lim_{m \to +\infty} (1 + TC_1/m + C_2, n(T/m)^2)^m = \exp(tTC_1)$ for any $t, T \geq 0$, we get that for any $n \in \mathbb{N}$, $T \geq 0$ and $x \in B(0,n)$

$$\limsup_{m \to +\infty} R_{T/m,n}^m V_M(x) \leq \exp(tTC_3 \exp(2TC_1)) V_M^{\exp(TC_1)}(x).$$

We conclude the proof upon remarking that the right-hand side quantity in (95) does not depend on $n$ and that the same inequality holds replacing $R_{T/m,n}$ by $R_{T/m,n}$ in (95).

C.12 Proof of Proposition 35

We have for any $x \in \mathbb{R}^d$, 

$$\nabla \phi(x) = x/\phi(x), \quad \nabla^2 \phi(x) = \text{Id} / \phi(x) - xx^T / \phi^2(x),$$

and therefore since $V_M(x) = \exp(M\phi(x))$, 

$$\nabla V_M(x) = M \nabla \phi(x) V_M(x),$$

$$\nabla^2 V_M(x) = \{ M^2 \nabla \phi(x)(\nabla \phi(x))^\top + M \nabla^2 \phi(x) \} V_M(x).$$
Therefore, for any $x \in \mathbb{R}^d$,

$$
(AV_M(x))/V_M(x) \\
\leq \left[ M^2 \|x\|^2 / \phi^2(x) + M \left\{ d/\phi(x) - \|x\|^2 / \phi^2(x) \right\} \right]/2 + M \sup_{x \in \mathbb{R}^d} \langle b(x), x \rangle_+ .
$$

Hence, for any $x \in \mathbb{R}^d$, $AV_M(x) \leq \zeta V_M(x)$ with $\zeta = M\{\sup_{x \in \mathbb{R}^d} \langle b(x), x \rangle_+ + d/2\} + M^2$. We conclude using Lemma 25-(a) and the Doob maximal inequality.

## D Quantitative bounds for geometric convergence of Markov chains in Wasserstein distance

In this section, we establish new quantitative bounds for Markov chains in Wasserstein distance. We consider a Markov kernel $P$ on the measurable space $(Y, \mathcal{Y})$ equipped with the bounded semi-metric $d : Y \times Y \rightarrow \mathbb{R}_+$, i.e. which satisfies the following condition.

**H1.** For any $x, y \in Y$, $d(x, y) \leq 1$, $d(x, y) = d(y, x)$ and $d(x, y) = 0$ if and only if $x = y$.

Let $K$ be a Markov coupling kernel for $P$. In this section, we assume the following condition on $K$.

**H2 (K).** There exists $C \in \mathcal{Y}^\otimes 2$ such that

(i) there exist $n_0 \in \mathbb{N}^*$ and $\varepsilon > 0$ such that for any $x, y \in C$, $K^n d(x, y) \leq (1 - \varepsilon)d(x, y)$;

(ii) for any $x, y \in Y$, $K d(x, y) \leq d(x, y)$;

(iii) there exist $\varphi_1 : Y^2 \rightarrow [1, +\infty)$ measurable, $\lambda_1 \in (0, 1)$ and $A_1 \geq 0$ such that $K$ satisfies $D_d(\varphi_1, \lambda_1, A_1, C)$.

We consider the Markov chain $(X_n, Y_n)_{n \in \mathbb{N}}$ associated with the Markov kernel $K$ defined on the canonical space $((Y \times Y)^\mathbb{N}, (Y^\otimes 2)_{y, n})$ and denote by $P_{(x, y)}$ and $E_{(x, y)}$ the corresponding probability distribution and expectation respectively when $(X_0, Y_0) = (x, y)$. Denote by $(\mathcal{G}_n)_{n \in \mathbb{N}}$ the canonical filtration associated with $(X_n, Y_n)_{n \in \mathbb{N}}$. Note that for any $n \in \mathbb{N}$ and $x, y \in Y$, under $\mathbb{P}_{(x, y)}$, $(X_n, Y_n)$ is by definition a coupling of $\delta_x P^n$ and $\delta_y P^n$. The main result of this section is the following which by the previous observation implies quantitative bounds on $W_d(\delta_x P^n, \delta_y P^n)$.

**Theorem 39.** Let $K$ be a Markov coupling kernel for $P$ and assume H1 and H2(K). Then for any $n \in \mathbb{N}$ and $x, y \in Y$,

$$
E_{(x, y)}[d(X_n, Y_n)] \leq \min \left[ \rho^n(M_{C, n_0} \Xi(x, y, n_0) + d(x, y)), \rho^{n/2}(1 + d(x, y)) + \lambda_1^{n/2} \Xi(x, y, n_0) \right],
$$

where

$$
\Xi(x, y, n_0) = \varphi_1(x, y) + A_1 \lambda_1^{-n_0} n_0 \\
\log(\rho) = \log(1 - \varepsilon) \log(1/\lambda_1)/[-\log(M_{C, n_0}) + \log(1 - \varepsilon)] ,
$$

$$
M_{C, n_0} = \sup_{(x, y) \in C} \Xi(x, y, n_0) = \sup_{(x, y) \in C} [\varphi_1(x, y)] + A_1 \lambda_1^{-n_0} n_0 .
$$

(96)
In Theorem 39, we obtain geometric contraction for $P$ in bounded Wasserstein metric $W_d$ since $d$ is assumed to be bounded. To obtain convergence associated with unbounded Wasserstein metric associated with $\nu_2 : Y^2 \to [0, +\infty)$, we consider the next assumption which is a generalized drift condition linking $\nu_2$ and the bounded semi-metric $d$.

**H3 (K).** There exist $\nu_2 : Y^2 \to [0, +\infty)$ measurable, $\lambda_2 \in (0, 1)$ and $A_2 \geq 0$ such that for any $x, y \in Y$,

$$K\nu_2(x, y) \leq \lambda_2 \nu_2(x, y) + A_2 d(x, y).$$

In the special case where $d(x, y) = 1_{\Delta x} (x, y)$, $\nu_2(x, y) = 1_{\Delta x} (\nu_2(x, y) = 1_{\Delta} (x, y)$ and for any $x \in Y$, $K((x, x), \Delta_Y) = 1$, we obtain that $D_d(\nu_1, \lambda_1, A, Y)$ implies $H3(K)$. The following result implies quantitative bounds on the Wasserstein distance $W_{\nu_2}(\delta_x P^n, \delta_Y^n)$ for any $x, y \in Y$ and $n \in \mathbb{N}^*$.

**Theorem 40.** Let $K$ be a Markov coupling kernel for $P$ and assume $H1$, $H2(K)$ and $H3(K)$. Then for any $n \in \mathbb{N}$ and $x, y \in Y$,

$$E_{(x, y)} [\nu_2(X_n, Y_n)] \leq \lambda_2^n \nu_2(x, y) + A_2 \min \left[ \rho^{n/4} r_\rho(d(x, y) + \Xi(x, y, n_0)), \rho^{n/2} r_\rho(1 + d(x, y)) + \lambda^{n/2} r_\rho \Xi(x, y, n_0) \right],$$

where

$$\hat{\rho} = \max(\lambda_2, \rho) \in (0, 1) \ , \ \lambda = \max(\lambda_1, \lambda_2) \in (0, 1) \ , \ r_\rho = 4 \log^{-1}(1/\hat{\rho})/\hat{\rho} \ , \ r_\lambda = 4 \log^{-1}(1/\lambda)/\lambda,$$

and $\Xi(x, y, n_0)$, $M_{\lambda, n_0}$ and $\rho$ are given in (96).

Theorem 39 and Theorem 40 share some connections with [80, Theorem 5], [43] and [27] but hold under milder assumptions than the ones considered in these works. Compared to [43] and [27], the main difference is that we allow here only a contraction for the $n_0$-th iterate of the Markov chain (condition $H2$-(i)) which is necessary if we want to use Theorem 4 to obtain sharp quantitative convergence bounds for (19). Finally, [80, Theorem 5] also considers minorization condition for the the $n_0$-th iterate, however our results compared favourably for large $n_0$. Indeed, Theorem 39 implies that the rate of convergence $\min(\rho, \lambda_1)$ is of the form $Cn_0^{-1}$ for $C \geq 0$ independent of $n_0$. Applying [80, Theorem 5], we found a rate of convergence of the form $Cn_0^{-2}$. Finally, a recent work [75] has established new results based on the technique used in [43]. However, we were not able to apply them since they assume as in [43], a contraction for $n_0 = 1$ which does not imply sharp bounds on the situations we consider.

The rest of this section is devoted to the proof of Theorem 39 and Theorem 40. Denote by $\theta : (Y \times Y)^N \to (Y \times Y)^N$ the shift operator defined for any $(x_n, y_n)_{n \in \mathbb{N}} \in (Y \times Y)^N$ by $\theta((x_n, y_n))_{n \in \mathbb{N}} = (x_{n+1}, y_{n+1})_{n \in \mathbb{N}}$. Define by induction, for any $m \in \mathbb{N}$, the sequence of $(G_m)_{m \in \mathbb{N}}$-stopping times $(T_{C, n_0})_{m \in \mathbb{N}}$, with $T_{C, n_0}^{(0)} = 0$ and for any $m \in \mathbb{N}^*$

$$T_{C, n_0}^{(m)} = \inf \left\{ k \geq T_{C, n_0}^{(m-1)} + n_0 : (x_k, y_k) \in C \right\}$$

$$= T_{C, n_0}^{(m-1)} + n_0 + \hat{T}_C \circ \theta^{T_{C, n_0}^{(m-1)} + n_0} = T_{C, n_0}^{(1)} + \sum_{i=1}^{(m-1)} T_{C, n_0}^{(i)} \circ \theta^{T_{C, n_0}^{(i)}} ,$$

where

$$\hat{T}_C = \inf \left\{ k \geq 0 : (x_k, y_k) \in C \right\}.$$
Lemma 41 ([25, Proposition 14]). Let K be a Markov coupling kernel for P and assume \( H_2(K) \)-(i)-(ii). Then for any \( n, m \in \mathbb{N} \), \( x, y \in \mathcal{Y} \),

\[
\mathbb{E}(x, y) \left[ d(X_n, Y_n) \right] \leq (1 - \varepsilon)^m d(x, y) + \mathbb{E}(x, y) \left[ d(X_n, Y_n) \mathbb{1}_{[n, +\infty)}(T_{\xi_n}^{(m)}) \right].
\]

Proof. Using \( H_2(K)-(ii) \), we have that \( (d(X_n, Y_n))_{n \in \mathbb{N}} \) is a \((\mathcal{G}_n)_{n \in \mathbb{N}}\)-supermartingale and therefore using the strong Markov property and \( H_2(K)-(i) \) we obtain for any \( m \in \mathbb{N} \) and \( x, y \in \mathcal{Y} \) that

\[
\mathbb{E}(x, y) \left[ d(X_{T_{\xi_n}^{(m+1)}}, Y_{T_{\xi_n}^{(m+1)}}) \right] \leq \mathbb{E}(x, y) \left[ \mathbb{E} \left[ d(X_{T_{\xi_n}^{(m+1)}}, Y_{T_{\xi_n}^{(m+1)}}) \big| \mathcal{G}_{T_{\xi_n}^{(m)}} \right] \right] \\
\leq (1 - \varepsilon) \mathbb{E}(x, y) \left[ d(X_{T_{\xi_n}^{(m)}}, Y_{T_{\xi_n}^{(m)}}) \right].
\]

Therefore by recursion and using (98) we obtain that for any \( m \in \mathbb{N} \) and \( x, y \in \mathcal{Y} \)

\[
\mathbb{E}(x, y) \left[ d(X_{T_{\xi_n}^{(m)}}, Y_{T_{\xi_n}^{(m)}}) \right] \leq (1 - \varepsilon)^m d(x, y).
\]

For any \( n, m \in \mathbb{N} \) we have using (99) and that \( (d(X_n, Y_n))_{n \in \mathbb{N}} \) is a supermartingale,

\[
\mathbb{E}(x, y) \left[ d(X_n, Y_n) \right] \leq \mathbb{E}(x, y) \left[ d(X_n \wedge T_{\xi_n}^{(m)}, Y_n \wedge T_{\xi_n}^{(m)}) \right] \\
\leq \mathbb{E}(x, y) \left[ d(X_{T_{\xi_n}^{(m)}}, Y_{T_{\xi_n}^{(m)}}) \mathbb{1}_{[0, n]}(T_{\xi_n}^{(m)}) \right] + \mathbb{E}(x, y) \left[ d(X_n, Y_n) \mathbb{1}_{[n, +\infty)}(T_{\xi_n}^{(m)}) \right] \\
\leq (1 - \varepsilon)^m d(x, y) + \mathbb{E}(x, y) \left[ d(X_n, Y_n) \mathbb{1}_{[n, +\infty)}(T_{\xi_n}^{(m)}) \right].
\]

By Lemma 41 and since \( d \) is bounded by 1, we need to obtain a bound on \( \mathbb{P}(x, y)(T_{\xi_n}^{(m)} \geq n) \) for \( x, y \in \mathcal{Y} \) and \( m, n \in \mathbb{N}^* \). To this end, we will use the following proposition which gives an upper bound on exponential moment of the hitting times \((T_{\xi_n}^{(m)})_{m \in \mathbb{N}^*}\).

Lemma 42. Let K be a Markov coupling kernel for P and assume \( H_2(K)-(iii) \). Then for any \( x, y \in \mathcal{Y} \) and \( m \in \mathbb{N}^* \),

\[
\mathbb{E}(x, y) \left[ \lambda_1^{-T_{\xi_n}^{(1)}} \right] \leq \Xi(x, y, n_0), \quad \mathbb{E}(x, y) \left[ \lambda_1^{-T_{\xi_n}^{(1)}(m)} \right] \leq M_{\xi_n}^{m-1}, \quad \mathbb{E}(x, y) \left[ \lambda_1^{-T_{\xi_n}^{(m)}} \right] \leq \Xi(x, y, n_0) M_{\xi_n}^{m-1},
\]

where \( \Xi(x, y, n_0) \) and \( M_{\xi_n} \) are defined in (96).

Proof. We first show that for any \( x, y \in \mathcal{Y} \) we have that \( \mathbb{P}(x, y)(T_{\xi} < +\infty) \). Let \( x, y \in \mathcal{Y} \). For any \( n \in \mathbb{N} \) we have using \( H_2(K)-(iii) \) and the Markov property

\[
\mathbb{E}(x, y) \left[ \mathbb{V}_1(X_{n+1}, Y_{n+1}) \mathbb{I}_{\mathcal{G}_n} \right] \leq \lambda_1 \mathbb{V}_1(X_n, Y_n) + A_1 \mathbb{I}_{\mathcal{G}}(X_n, Y_n).
\]

Therefore applying the comparison theorem [22, Theorem 4.3.1] we get that

\[
(1 - \lambda_1) \mathbb{E}(x, y) \left[ \sum_{k=0}^{T_{\xi}-1} \mathbb{V}_1(X_k, Y_k) \right] + \mathbb{E}(x, y) \left[ \mathbb{I}_{[0, +\infty)}(T_{\xi}) \mathbb{V}(X_{T_{\xi}}, Y_{T_{\xi}}) \right] \leq \mathbb{V}(x, y).
\]
Since for any \( \bar{x}, \bar{y} \in \mathcal{Y}, 1 \leq \mathcal{P}(\bar{x}, \bar{y}) < +\infty \) we obtain that \((1 - \lambda_1)E_{(x,y)}[\mathcal{T}_C] \leq \mathcal{Q}(x, y)\) which implies \(\mathcal{P}_{(x,y)}(\mathcal{T}_C) < +\infty\) since \(\lambda_1 \in (0, 1)\). We now show the stated result. Let \(x, y \in \mathcal{Y}\) and \((\mathcal{S}_n)_{n \in \mathbb{N}}\) be defined for any \(n \in \mathbb{N}\) by \(\mathcal{S}_n = \lambda_1^{-n}\mathcal{Q}_1(X_n, Y_n)\). For any \(n \in \mathbb{N}\) we have using H2(K)-(iii) and the Markov property

\[
E[\mathcal{S}_{n+1}|\mathcal{G}_n] \leq \lambda_1^{-n}\mathcal{Q}_1(X_n, Y_n) + A_1\lambda_1^{-(n+1)}I_C(X_n, Y_n)
\]

\[
\leq \mathcal{S}_n + A_1\lambda_1^{-(n+1)}I_C(X_n, Y_n) .
\]

(100)

Using the Markov property, the definition of \(\mathcal{T}_{C,n_0}^{(1)}\) given in (97), the comparison theorem [22, Theorem 4.3.1], (100) and H2(K)-(iii) we obtain that

\[
E_{(x,y)}\left[\mathcal{T}_{C,n_0}^{(1)}\right] = E_{(x,y)}\left[E_{(x,y)}\left[\mathcal{T}_{C,n_0}^{(1)}|\mathcal{G}_{n_0}\right]\right] = E_{(x,y)}\left[E_{(x,y)}\left[\mathcal{T}_{C,n_0}^{(1)}|\mathcal{G}_{n_0}\right]\right]
\]

\[
= E_{(x,y)}\left[\lambda_1^{-n_0}E_{(x,y)}\left[\mathcal{Q}_1(X_{n_0}+\mathcal{T}_{C,n_0}^{(1)}, Y_{n_0}+\mathcal{T}_{C,n_0}^{(1)})\lambda_1^{-\mathcal{T}_{C,n_0}^{(1)}}|\mathcal{G}_{n_0}\right]\right]
\]

\[
\leq E_{(x,y)}\left[\lambda_1^{-n_0}E_{(x,y)}\left[S_{\mathcal{T}_{C,n_0}^{(1)}}\right]\right]
\]

\[
\leq E_{(x,y)}\left[\lambda_1^{-n_0}E_{(x,y)}\left[S_{\mathcal{T}_{C,n_0}^{[0,\infty]}(\mathcal{T}_C)}\right]\right]
\]

\[
\leq E_{(x,y)}\left[\lambda_1^{-n_0}E_{(x,y)}\left[S_0 + A_1 \sum_{k=0}^{\mathcal{T}_{C,n_0}^{(1)}-1} \lambda_1^{-(k+1)}I_C(X_k, Y_k)\right]\right]
\]

\[
\leq E_{(x,y)}\left[\lambda_1^{-n_0}\mathcal{Q}_1(X_{n_0}, Y_{n_0})\right] \leq \mathcal{Q}_1(x, y) + A_1\lambda_1^{-n_0} .
\]

(101)

Combining (101) and the fact that for any \(x, y \in \mathcal{Y}\), \(\mathcal{Q}_1(x, y) \geq 1\), we obtain that

\[
E_{(x,y)}\left[\lambda_1^{-\mathcal{T}_{C,n_0}^{(1)}}\right] \leq \mathcal{Q}_1(x, y) + A_1\lambda_1^{-n_0} .
\]

(102)

We conclude by a straightforward recursion and using (102), the definition of \(\mathcal{T}_{C,n_0}^{(m)}\) (97) for \(m \geq 1\), the strong Markov property and the fact that for any \(m \in \mathbb{N}\), \((X_{\mathcal{T}_{C,n_0}^{(m)}}, Y_{\mathcal{T}_{C,n_0}^{(m)}}) \in \mathcal{C}\).

\(\square\)

**Proof of Theorem 39.** Let \(x, y \in \mathcal{Y}\) and \(n \in \mathbb{N}\). By Lemma 41, Lemma 42, H1, the fact that \(\mathcal{M}_{C,n_0} \geq 1\) and the Markov inequality, we have for any \(m \in \mathbb{N}\),

\[
E_{(x,y)}[d(X_n, Y_n)] \leq (1 - \varepsilon)^m d(x, y) + \mathcal{P}_{(x,y)}[\mathcal{T}_{C,n_0}^{(m)} \geq n]
\]

\[
\leq (1 - \varepsilon)^m d(x, y) + \lambda_1^{-n_0} E_{(x,y)}[\lambda_1^{-\mathcal{T}_{C,n_0}^{(m)}}]
\]

\[
\leq (1 - \varepsilon)^m d(x, y) + \lambda_1^{-n_0} \mathcal{M}_{C,n_0}^{\mathcal{E}(x,y,n_0)} ,
\]

where \(\mathcal{E}(x,y,n_0)\) is given in Theorem 39. Combining this result and Lemma 42, we can conclude that \(E_{(x,y)}[d(X_n, Y_n)] \leq \rho^m(\mathcal{M}_{C,n_0} \mathcal{E}(x,y,n_0) + d(x, y))\) setting

\[
m = \left\lfloor n \log(\lambda_1)/\{\log(1 - \varepsilon) - \log(\mathcal{M}_{C,n_0})\} \right\rfloor .
\]

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Applying Theorem 39 we obtain

\[ \mathbb{E}_{(x,y)} [d(X_n, Y_n)] \leq \rho^n/2 (1 + d(x, y)) + \lambda_n^{n/2} \Xi(x, y, n_0), \]

first note that Lemma 41 and \( \textbf{H1} \) imply that for any \( m \in \mathbb{N}, \)

\[
\mathbb{E}_{(x,y)} [d(X_n, Y_n)] \\
\leq (1 - \varepsilon) m d(x, y) + \mathbb{P}_{(x,y)} \left[ T_{\mathbb{C}, n_0}^{(m)} - T_{\mathbb{C}, n_0}^{(1)} \geq n/2 \right] + \mathbb{P}_{(x,y)} \left[ T_{\mathbb{C}, n_0}^{(1)} \geq n/2 \right] \\
\leq (1 - \varepsilon) m d(x, y) + \lambda_1^{n/2} \mathbb{E}_{(x,y)} \left[ \lambda_1^{-T_{\mathbb{C}, n_0}^{(m)}} + T_{\mathbb{C}, n_0}^{(1)} \right] + \lambda_1^{n/2} \mathbb{E}_{(x,y)} \left[ \lambda_1^{-T_{\mathbb{C}, n_0}^{(1)}} \right],
\]

where we have used the Markov inequality in the last line. Combining this result and Lemma 42, we can conclude that \( \mathbb{E}_{(x,y)} [d(X_n, Y_n)] \leq \rho^n/2 (1 + d(x, y)) + \lambda_n^{n/2} \Xi(x, y, n_0) \)

setting \( m = \left[ n \log(\lambda_1) / \{2 \log(1 - \varepsilon) - 2 \log(M_{\mathbb{C}, n_0})\} \right] \).

\[ \Box \]

\textbf{Proof of Theorem 40.} Let \( x, y \in Y \) and \( n \in \mathbb{N} \). Using \( \textbf{H3}(K) \), we obtain by recursion

\[
\mathbb{E}_{(x,y)} [\Psi_2(X_n, Y_n)] \leq \lambda_2^n \Psi_2(x, y) + A_2 \sum_{k=0}^{n-1} \lambda_2^{n-1-k} \mathbb{E}_{(x,y)} [d(X_k, Y_k)].
\]

(103)

Applying Theorem 39 we obtain

\[
\sum_{k=0}^{n-1} \lambda_2^{n-1-k} \mathbb{E}_{(x,y)} [d(X_k, Y_k)] \\
\leq \sum_{k=0}^{n-1} \lambda_2^{n-1-k} \min \left[ \rho^k(M_{\mathbb{C}, n_0} \Xi(x, y, n_0) + d(x, y)), \rho^{k/2} (1 + d(x, y)) + \lambda^{k/2} \Xi(x, y, n_0) \right] \\
\leq \min \left[ n \rho^{n-1} (d(x, y) + \Xi(x, y, n_0)), n \rho^{n/2-1} (1 + d(x, y)) + n \lambda^{n/2-1} \Xi(x, y, n_0) \right].
\]

We conclude plugging this result in (103) and using that for any \( n \in \mathbb{N} \) and \( t \in (0,1) \), \( n t^n/2 \leq 4 \log^{-1}(1/t) t^n/4 \).

\[ \Box \]

\section{Minorization conditions for functional autoregressive models}

In this section, we extend and complete the results of [28, Section 6] on functional autoregressive models. Let \( X \in \mathcal{B}(\mathbb{R}^d) \) equipped with its trace \( \sigma \)-field \( \mathcal{X} = \{ A \cap X : A \in \mathcal{B}(\mathbb{R}^d) \} \). In fact, we consider a slightly more general class of models than [28] which is associated with non-homogeneous Markov chains \( (X_k^{(a)})_{k \in \mathbb{N}} \) with state space \( (X, \mathcal{X}) \) defined for \( k \geq 0 \) by

\[
X_{k+1}^{(a)} = \Pi \left( T_{k+1}(X_k^{(a)}) + \sigma_{k+1} Z_{k+1} \right),
\]

where \( \Pi \) is a measurable function from \( \mathbb{R}^d \) to \( X \), \( (T_k)_{k \geq 1} \) is a sequence of measurable functions from \( X \) to \( \mathbb{R}^d \), \( (\sigma_k)_{k \geq 1} \) is a sequence of positive real numbers and \( (Z_k)_{k \geq 1} \) is a sequence of i.i.d. \( d \) dimensional standard Gaussian random variables. We assume that \( \Pi \) satisfies \( \textbf{A1} \). We also assume some Lipschitz regularity on the operator \( T_k \) for any \( k \in \mathbb{N}^* \).
AR 1 (A). For all \( k \geq 1 \) there exists \( \varpi \in \mathbb{R} \) such that for all \( (x, y) \in A \),
\[
||T_k(x) - T_k(y)||^2 \leq (1 + \varpi) ||x - y||^2 .
\]

The sequence \( \{X_k, k \in \mathbb{N}\} \) is an inhomogeneous Markov chain associated with the family of
Markov kernels \( (P_k^{(a)})_{k \geq 1} \) on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) given for all \( x \in \mathbb{R}^d \) and \( A \in \mathbb{R}^d \) by
\[
P_k^{(a)}(x, A) = \frac{1}{(2\pi \sigma_k^{2})^{d/2}} \int_{\Pi^{-1}(A)} \exp \left( -\frac{||y - T_k(x)||^2}{(2\sigma_k^{2})} \right) dy .
\]

We denote for all \( n \geq 1 \) by \( Q_n^{(a)} \) the marginal distribution of \( X_n^{(a)} \) given by \( Q_n^{(a)} = P_1^{(a)} \cdots P_n^{(a)} \). To
obtain an upper bound of \( ||\delta_x Q_n^{(a)} - \delta_y Q_n^{(a)}||_{TV} \) for any \( x, y \in \mathbb{R}^d \), \( n \in \mathbb{N}^* \), we introduce a Markov
coupling \( (X_k^{(a)}, Y_k^{(a)})_{k \in \mathbb{N}} \) such that for any \( n \in \mathbb{N}^* \), the distribution of \( X_n^{(a)} \) and \( Y_n^{(a)} \) are \( \delta_x Q_n^{(a)} \) and
\( \delta_y Q_n^{(a)} \) respectively, exactly as we have introduced in the homogeneous setting the Markov coupling
with kernel \( \mathcal{K}_\gamma \) defined by (23) for \( R_\gamma \) defined in (20). For completeness and readability, we recall
the construction of \( (X_k^{(a)}, Y_k^{(a)})_{k \in \mathbb{N}} \). For all \( k \in \mathbb{N}^* \) and \( x, y, z \in \mathbb{R}^d \), define
\[
c_k(x, y) = \begin{cases} E_k(x, y)/||E_k(x, y)|| & \text{if } E_k(x, y) \neq 0, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\varphi = \begin{cases} 0 & \text{if } T_k(y) - T_k(x), \\ \varphi_{\sigma_k^2}^{(a)}((||E_k(x, y)|| - (c_k(x, y), z))/\varphi_{\sigma_k^2}^{(a)}((c_k(x, y), z)), \end{cases}
\]
(104)

where \( \varphi_{\sigma_k^2}^{(a)} \) is the one-dimensional zero mean Gaussian distribution function with variance \( \sigma_k^2 \). Let
\( (U_k)_{k \in \mathbb{N}^*} \) be a sequence of i.i.d. uniform random variables on \([0, 1]\) and define the Markov chain
\( (X_k^{(a)}, Y_k^{(a)})_{k \in \mathbb{N}} \) starting from \( (X_0^{(a)}, Y_0^{(a)}) \in \mathbb{X}^2 \) by the recursion: for any \( k \geq 0 \)
\[
\hat{X}_{k+1}^{(a)} = T_{k+1}(X_k^{(a)}) + \sigma_{k+1} Z_{k+1} ,
\]
\[
\hat{Y}_{k+1}^{(a)} = \begin{cases} \hat{X}_{k+1}^{(a)} & \text{if } T_{k+1}(X_k^{(a)}) = T_{k+1}(Y_k^{(a)}), \\ W_{k+1}^{(a)} \hat{X}_{k+1}^{(a)} + (1 - W_{k+1}^{(a)}) S_{k+1}(X_k^{(a)}, Y_k^{(a)}, \sigma_{k+1} Z_{k+1}) & \text{otherwise}, \end{cases}
\]
(106)

where \( W_{k+1}^{(a)} = \mathbb{1}_{(-\infty, 0]}(U_{k+1} - p_{k+1}(X_k^{(a)}, Y_k^{(a)}) \sigma_{k+1} Z_{k+1})) \) and finally set
\[
(X_k^{(a)}, Y_k^{(a)}) = (\Pi(\hat{X}_{k+1}^{(a)}), \Pi(\hat{Y}_{k+1}^{(a)})) .
\]
(107)

For any \( k \in \mathbb{N}^* \), marginally, the distribution of \( X_k^{(a)} \) given \( X_k^{(a)} \) is \( P_{k+1}(X_k^{(a)}, \cdot) \), and it is well-known
(see e.g. [9, Section 3.3]) that \( \hat{Y}_{k+1}^{(a)} \) and \( T_{k+1}(Y_k^{(a)}) + \sigma_{k+1} Z_{k+1} \) have the same distribution given \( Y_k \),
and therefore the distribution of \( Y_{k+1}^{(a)} \) given \( Y_k \) is \( P_{k+1}^{(a)}(Y_k, \cdot) \). As a result for any \( (x, y) \in \mathbb{X}^2 \)
and \( n \in \mathbb{N}^* \), \( (X_n^{(a)}, Y_n^{(a)}) \) with \( (X_0^{(a)}, Y_0^{(a)}) = (x, y) \) is a coupling between \( \delta_x Q_n^{(a)} \) and \( \delta_y Q_n^{(a)} \).
Therefore, we obtain that \( ||\delta_x Q_n^{(a)} - \delta_y Q_n^{(a)}||_{TV} \leq \mathbb{P}(X_n^{(a)} \neq Y_n^{(a)}) \). Therefore to get an upper
bound on \( ||\delta_x Q_n^{(a)} - \delta_y Q_n^{(a)}||_{TV} \), it is sufficient to obtain a bound on \( \mathbb{P}(X_n^{(a)} \neq Y_n^{(a)}) \) which is a simple consequence of the following more general result.
Theorem 43. Let $A \in \mathcal{B}(\mathbb{R}^{2d})$ and assume A1 and ARl(A). Let $(X^{(a)}_k, Y^{(a)}_k)_{k \in \mathbb{N}}$ be defined by (107), with $(X^{(a)}_0, Y^{(a)}_0) = (x, y) \in A$. Then for any $n \in \mathbb{N}^*$,

$$
P \left[ X^{(a)}_n \neq Y^{(a)}_n \text{ and for any } k \in \{1, \ldots, n-1\}, (X^{(a)}_k, Y^{(a)}_k) \in A^2 \right] \leq \mathbb{I}_{\Delta}(x, y) \left\{ 1 - 2 \Phi \left( -\frac{\|x - y\|}{2(\Xi^{(a)}_n)^{1/2}} \right) \right\},
$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution on $\mathbb{R}$ and $(\Xi^{(a)}_i)_{i \geq 1}$ is defined for all $k \geq 1$ by $\Xi^{(a)}_k = \sum_{i=1}^{k} \{ \sigma_i^2 / \prod_{j=1}^{i} (1 + \omega_j) \}$. Let $n \geq 1 \text{ and } (x, y) \in A^2$. We show by backward induction that for all $k \in \{0, \ldots, n-1\}$,

$$
P\{(X^{(a)}_n) \neq Y^{(a)}_n \} \cap \mathcal{A}_{n-1} \leq \mathbb{E} \left[ \mathbb{I}_{\Delta}(X^{(a)}_n, Y^{(a)}_n) \mathbb{I}_{\mathcal{A}_{n-1}} \left\{ 1 - 2 \Phi \left( -\frac{\|X^{(a)}_n - Y^{(a)}_n\|}{2(\Xi^{(a)}_{k+1,n})^{1/2}} \right) \right\} \right]. \quad (108)
$$

Note that the inequality for $k = 0$ will conclude the proof. Using (106) that $\bar{X}^{(a)}_n = \bar{Y}^{(a)}_n$ if $X^{(a)}_{n-1} = Y^{(a)}_{n-1}$ or $W_n = \mathbb{I}_{(-\infty, 0)} (U_n - p_n (X^{(a)}_{n-1}, Y^{(a)}_{n-1}, \sigma_n Z_n)) = 1$, where $p_n$ is defined by (105), and $(U_n, Z_n)$ are independent random variables independent of $\mathcal{F}^{(a)}_{n-1}$, we obtain on $\{X^{(a)}_{n-1} \neq Y^{(a)}_{n-1}\}$,

$$
\mathbb{E} \left[ \mathbb{I}_{\Delta}(\bar{X}^{(a)}_n, \bar{Y}^{(a)}_n) \mathcal{F}^{(a)}_{n-1} \right] = \mathbb{E} \left[ p_n (X^{(a)}_{n-1}, Y^{(a)}_{n-1}, \sigma_n Z_n) \mathcal{F}^{(a)}_{n-1} \right] = 2 \Phi \left\{ -\left( (2\sigma_n)^{-1} \mathbb{E}_{n} (X^{(a)}_{n-1}, Y^{(a)}_{n-1}) \right) \right\}.
$$

Since $\{X^{(a)}_{n-1} \neq Y^{(a)}_{n-1}\} \subset \{ \bar{X}^{(a)}_{n-1} \neq \bar{Y}^{(a)}_{n-1} \} \subset \{X^{(a)}_n \neq Y^{(a)}_n\}$ by (107) and (106), we get

$$
P \{X^{(a)}_n \neq Y^{(a)}_n\} \cap \mathcal{A}_{n-1} \leq \mathbb{E} \left[ \mathbb{I}_{\Delta}(X^{(a)}_n, Y^{(a)}_n) \mathbb{I}_{\mathcal{A}_{n-1}} \mathbb{E} \left[ \mathbb{I}_{\Delta}(\bar{X}^{(a)}_n, \bar{Y}^{(a)}_n) \mathcal{F}^{(a)}_{n-1} \right] \right]
= \mathbb{E} \left[ \mathbb{I}_{\Delta}(X^{(a)}_n, Y^{(a)}_n) \mathbb{I}_{\mathcal{A}_{n-1}} \left\{ 1 - 2 \Phi \left( -\frac{\|X^{(a)}_n - Y^{(a)}_n\|}{2(\Xi^{(a)}_{k+1,n})^{1/2}} \right) \right\} \right],
$$

Using that $(X^{(a)}_{n-1}, Y^{(a)}_{n-1}) \in A^2$ on $\mathcal{A}_{n-1}$, AR1(A) and (104), we obtain that

$$
\|E_n (X^{(a)}_{n-1}, Y^{(a)}_{n-1})\|^2 \leq (1 + \omega_n) \|X^{(a)}_{n-1} - Y^{(a)}_{n-1}\|^2,
$$

showing (108) holds for $k = n-1$ since $\mathcal{A}_{n-2} \subset \mathcal{A}_{n-1}$. Assume that (108) holds for $k \in \{1, \ldots, n-1\}$.

On $\{\bar{X}^{(a)}_k \neq \bar{Y}^{(a)}_k\}$, we have

$$
\|\bar{X}^{(a)}_k - \bar{Y}^{(a)}_k\| = -\left\| \mathbb{E}_k (X^{(a)}_{k-1}, Y^{(a)}_{k-1}) + 2\sigma_k \mathbb{E}_k (X^{(a)}_{k-1}, Y^{(a)}_{k-1})^T Z_k \right\|,
$$

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which implies by (107) and since \( \Pi \) is non expansive by \( A1 \)

\[
\mathbb{I}_{\Delta_{k}}(X_k^{(a)}, Y_k^{(a)}) \left( 1 - 2\Phi \left\{ \frac{\|X_k^{(a)} - Y_k^{(a)}\|}{2(\varepsilon_{k+1,n})^{1/2}} \right\} \right)
\]

\[
\leq \mathbb{I}_{\Delta_{k}}(X_k^{(a)}, Y_k^{(a)}) \left( 1 - 2\Phi \left\{ \frac{\|X_k^{(a)} - Y_k^{(a)}\|}{2(\varepsilon_{k+1,n})^{1/2}} \right\} \right)
\]

\[
\leq \mathbb{I}_{\Delta_{k}}(X_k^{(a)}, Y_k^{(a)}) \left( 1 - 2\Phi \left\{ \frac{\|X_k^{(a)} - Y_k^{(a)}\|}{2(\varepsilon_{k+1,n})^{1/2}} \right\} \right).
\]

Since \( Z_k \) is independent of \( \mathcal{F}_k^{(a)} \), \( \sigma_k e_k(X_k^{(a)}, X_k^{(a)})^T Z_k \) is a Gaussian random variable with zero mean and variance \( \sigma_k^2 \). Therefore by [28, Lemma 20] and since \( \mathcal{A}_{k-1} \) is \( \mathcal{F}_k^{(a)} \)-measurable, we get

\[
\mathbb{E} \left[ \mathbb{I}_{\Delta_{k}}(X_k^{(a)}, Y_k^{(a)}) \mathbb{I}_{\mathcal{A}_{k-1}} \left( 1 - 2\Phi \left\{ \frac{\|X_k^{(a)} - Y_k^{(a)}\|}{2(\varepsilon_{k+1,n})^{1/2}} \right\} \right) \right]
\]

\[
\leq \mathbb{I}_{\mathcal{A}_{k-1}} \mathbb{I}_{\Delta_{k}}(X_{k-1}^{(a)}, Y_{k-1}^{(a)}) \left( 1 - 2\Phi \left\{ \frac{\|E_k(X_{k-1}^{(a)}, Y_{k-1}^{(a)})\|}{2(\sigma_k + \varepsilon_{k+1,n})^{1/2}} \right\} \right).
\]

Since \( A2(A) \) implies that \( \|E_k(X_{k-1}^{(a)}, Y_{k-1}^{(a)})\|^2 \leq (1 + \varepsilon_{k-1}) \|X_{k-1}^{(a)} - Y_{k-1}^{(a)}\|^2 \) on \( \mathcal{A}_{k-1} \) and \( \mathcal{A}_{k-2} \subset \mathcal{A}_{k-1} \) concludes the induction of (108). \( \square \)

**F  Quantitative convergence results based on [22, 23]**

We start by recalling the following lemma from [22] which is inspired from the results of [23].

**Lemma 44 ([22, Lemma 19.4.2]).** Let \((\mathcal{Y}, \mathcal{Y})\) be a measurable space and \( R \) be a Markov kernel over \((\mathcal{Y}, \mathcal{Y})\). Let \( Q \) be a Markov coupling kernel for \( R \). Assume there exist \( C \in \mathcal{Y}^{\otimes 2}, M \geq 0, \) a measurable function \( \mathcal{V} : \mathcal{Y} \times \mathcal{Y} \to [1, +\infty], \lambda \in [0, 1) \) and \( c \geq 0 \) such that for any \( x, y \in \mathcal{Y} \),

\[
\mathcal{V}(x, y) \leq \lambda \mathcal{V}(x, y) + c \mathcal{I}_C(x, y).
\]

In addition, assume that there exists \( \varepsilon > 0 \) such that for any \( (x, y) \in \mathcal{C} \),

\[
\mathcal{Q}((x, y), \Delta_{\varepsilon}) \leq 1 - \varepsilon,
\]

where \( \Delta_{\varepsilon} = \{(y, y) : y \in \mathcal{Y}\} \). Then there exist \( \rho \in [0, 1) \) and \( C \geq 0 \) such that for any \( x, y \in \mathcal{Y} \) and \( n \in \mathbb{N}^* \)

\[
\int_{\mathcal{Y} \times \mathcal{Y}} \mathbb{I}_{\Delta_{\varepsilon}}(\bar{x}, \bar{y}) \mathcal{V}(\bar{x}, \bar{y}) \mathcal{Q}^n((x, y), d(\bar{x}, \bar{y})) \leq C \rho^n \mathcal{V}(x, y),
\]

where

\[
\rho = 2(1 + c/\{(1 - \varepsilon)(1 - \lambda)\})
\]

\[
\log(\rho) = \{\log(1 - \varepsilon) \log(\lambda)\}/\{\log(1 - \varepsilon) + \log(\lambda) - \log(c)\}.
\]

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Theorem 45. Let \((Y, \mathcal{Y})\) be a measurable space and \(R\) be a Markov kernel over \((Y, \mathcal{Y})\). Let \(Q\) be a Markov coupling kernel of \(R\). Assume that there exist \(\lambda \in [0, 1)\), \(A \geq 0\) and a measurable function \(\varphi : Y \times Y \to [1, +\infty)\), such that \(Q\) satisfies \(D_\lambda(\varphi, \lambda, A, Y)\). In addition, assume that there exist \(\ell \in \mathbb{N}^*\), \(\varepsilon > 0\) and \(M \geq 1\) such that for any \((x, y) \in C_M = \{(x, y) \in Y \times Y, \ \varphi(x, y) \leq M\}\),

\[
Q^\ell((x, y), \Delta_y) \leq 1 - \varepsilon,
\]

with \(\Delta_y = \{(x, y) \in Y^2 : x = y\}\) and \(M \geq 2A/(1 - \lambda)\). Then, there exist \(\rho \in [0, 1)\) and \(C \geq 0\) such that for any \(n \in \mathbb{N}\) and \(x, y \in Y\)

\[
W_c(\delta_y R^n, \delta_y R^n) \leq C \rho^{n/\ell} \varphi(x, y),
\]

with

\[
C = 2(1 + A\ell)(1 + c\ell/\{(1 - \varepsilon)(1 - \lambda\ell)\})^\ell, \\
\lambda \ell = (\lambda^\ell + 1)/2, \ c\ell = \lambda^\ell M + A\ell, \ A\ell = A(1 - \lambda^\ell)/(1 - \lambda), \\
\log(\rho\ell) = \{\log(1 - \varepsilon) \log(\lambda\ell)\}/\{\log(1 - \varepsilon) + \log(\lambda\ell) - \log(c\ell)\}.
\]

Proof. We first show that for any \((x, y) \in C_M\),

\[
Q^\ell(x, y) \leq \lambda^\ell \varphi(x, y) 1_{C_M^c}(x, y) + c\ell 1_{C_M}(x, y),
\]

in order to apply Lemma 44 to \(R^\ell\) with the Markov coupling kernel \(Q^\ell\). By a straightforward induction, for any \(x, y \in Y\) we have

\[
Q^\ell \varphi(x, y) \leq \lambda^\ell \varphi(x, y) + A(1 - \lambda^\ell)/(1 - \lambda).
\]

We distinguish two cases. If \((x, y) \notin C_M\), using that \(A/M \geq (1 - \lambda)/2\) we have that

\[
Q^\ell \varphi(x, y) \leq \lambda^\ell \varphi(x, y) + A(1 - \lambda^\ell)/(M(1 - \lambda)) \leq 2^{-1}(\lambda^\ell + 1) \varphi(x, y).
\]

If \((x, y) \in C_M\), we have

\[
Q^\ell \varphi(x, y) \leq \lambda^\ell M + A(1 - \lambda^\ell)/(1 - \lambda).
\]

Therefore (111) holds. As a result and since by assumption we have (109), we can apply Lemma 44 to \(R^\ell\). Then, we obtain that for any \(i \in \mathbb{N}\) and \(x, y \in Y\)

\[
\int_{Y \times Y} 1_{\Delta_y}(\tilde{x}, \tilde{y}) \varphi(\tilde{x}, \tilde{y}) Q^\ell_i((x, y), d(\tilde{x}, \tilde{y})) \leq C\ell \rho^{i/\ell} \varphi(x, y),
\]

with \(\rho\ell\) defined by (110) and \(C\ell = 2 \left\{1 + c\ell/[(1 - \lambda\ell)(1 - \varepsilon)]^{-1}\right\}\). In addition, using (112), for any \(k \in \{0, \ldots, \ell - 1\}\) and \(x, y \in Y\), \(Q^k \varphi(x, y) \leq (1 + A\ell) \varphi(x, y)\). Therefore, for any \(n \in \mathbb{N}\), since \(n = i_n\ell + k_n\) with \(i_n = [n/\ell]\) and \(k_n \in \{0, \ldots, \ell - 1\}\), we obtain for any \(x, y \in Y\) that

\[
W_c(\delta_x R^n, \delta_y R^n) \leq \tilde{C}\ell \rho^{i_n/\ell} \sum_{Y \times Y} 1_{\Delta_y}(\tilde{x}, \tilde{y}) \varphi(\tilde{x}, \tilde{y}) Q^k(\Delta_y, d(\tilde{x}, \tilde{y})) \leq (1 + A\ell)\tilde{C}\ell \rho^{i_n/\ell} \varphi(x, y),
\]

which concludes the proof. \(\Box\)
We now state an important consequence of Theorem 45. The comparison between Theorem 46 and Theorem 8 is conducted in the remarks which follow Theorem 8.

**Theorem 46.** Assume that there exists a measurable function \( \Psi : X \times X \to [1, +\infty) \) such that for any \( C \geq 0 \),
\[
\text{diam} \left\{ (x, y) \in X^2 : \Psi(x, y) \leq C \right\} < +\infty .
\]
Assume in addition that there exist \( \lambda \in [0, 1] \), \( A \geq 0 \) such that for any \( \gamma \in [0, \bar{\gamma}] \), there exists \( \tilde{K}_\gamma \), a Markov coupling kernel for \( R_\gamma \), satisfying \( \mathbf{D}_d(\Psi, \lambda^\gamma, A_\gamma, X^2) \). Further, assume that there exists \( \Psi : (0, \bar{\gamma}] \times \mathbb{N}^* \times \mathbb{R}_+ \to [0, 1] \) such that for any \( \gamma \in (0, \bar{\gamma}] \), \( \ell \in \mathbb{N}^* \) and \( x, y \in X \), (32) is satisfied. Then the following results hold.

(a) For any \( \gamma \in (0, \bar{\gamma}] \), \( M_d \geq\) \( \text{diam} \left\{ (x, y) \in X^2 : \Psi(x, y) \leq B_d \right\} \) with \( B_d = 2A(1 + \bar{\gamma}) \{1 + \log^{-1}(1/\lambda)\} \), \( \ell \in \mathbb{N}^* \), \( x, y \in X \) and \( k \in \mathbb{N} \)
\[
\mathbf{W}_c(\delta_x R^k_x, \delta_y R^k_y) \leq C_\gamma \bar{\rho}_\gamma \left\lfloor k(1/\gamma - 1) \right\rfloor \Psi(x, y) ,
\]
where \( \mathbf{W}_c \) is the Wasserstein metric associated with \( c \) defined by (30),
\[
C_\gamma = 2\{1 + A_\gamma \{(1 + c_1)/(1 - \bar{\lambda}1)\} \}, \\
\log(\bar{\rho}_\gamma) = \{(1 - \bar{\varepsilon}_{d,2}) \log(\lambda)\}/\{(1 - \bar{\varepsilon}_{d,2}) \log(\lambda)\} < 0 , \\
A_\gamma = A\gamma(1 - \lambda^\gamma(1/\gamma))/(1 - \lambda^\gamma) , \\
c_\gamma = \lambda^{\ell(1/\gamma - 1)} A_\gamma + B_d , \\
\bar{\varepsilon}_{d,2} = \inf_{\gamma \in (0, \bar{\gamma}], \ (x, y) \in \Delta_x, M_d} \Psi(\gamma, \ell, \|x - y\|) , \\
\lambda_\gamma = (\lambda^\gamma(1/\gamma) + 1)/2 .
\]

(b) For any \( \gamma \in (0, \bar{\gamma}] \), \( M_d \geq\) \( \text{diam} \left\{ (x, y) \in X^2 : \Psi(x, y) \leq B_d \right\} \) with \( B_d = 2A(1 + \bar{\gamma}) \{1 + \log^{-1}(1/\lambda)\} \) and \( \ell \in \mathbb{N}^* \), it holds that
\[
C_\gamma \leq \bar{C}_1 , \\
\log(\rho_\gamma) \leq \log(\bar{\rho}_2) \leq 0 , \\
\bar{C}_1 = 2\{1 + A_1 \{(1 + c_1)/(1 - \bar{\lambda}1)\} \}, \\
\log(\bar{\rho}_2) = \{(1 - \bar{\varepsilon}_{d,2}) \log(\lambda)\}/\{(1 - \bar{\varepsilon}_{d,2}) \log(\lambda)\} < 0 , \\
A_1 = A(1 + \bar{\gamma}) \min(\ell, 1 + \log^{-1}(1/\lambda)) , \\
c_1 = \bar{A}_1 + B_d , \\
\bar{\lambda}_1 = (\lambda + 1)/2 .
\]

(c) In addition, if \( \bar{\gamma} \leq 1 \), \(-\log(\lambda) \in [0, \log(2)]\), \( A \geq 1 \) and \( 0 < \bar{\varepsilon}_{d,4} \leq 1 - e^{-1} \), then
\[
\log^{-1}(1/\bar{\rho}_2) \leq 12 \log(2) \log(6A \{1 + \log^{-1}(1/\lambda)\})/(\log(1/\lambda)\bar{\varepsilon}_{d,\bar{\gamma}}) .
\]

**Proof.** First, note that \( 1 - \lambda^\ell = \int_0^\ell \log(\lambda) e^s \log(\lambda) ds \geq -\log(\lambda) t e^t \log(\lambda) \) for any \( t \in (0, \bar{t}] \), for \( \bar{t} > 0 \), and therefore
\[
t/(1 - \lambda^\ell) = \bar{t} + t \lambda^\ell/(1 - \lambda^\ell) \leq \bar{t} + \log^{-1}(\lambda^{-1}) .
\]

(a) To establish (113), we apply Theorem 45. For any \( x, y \in X \) such that \( \Psi(x, y) \leq B_d \) we have
\[
\tilde{K}_{\gamma}^{(1/\gamma)}((x, y), \Delta_x) \leq 1 - \bar{\varepsilon}_{d,2} .
\]

Using that \( \tilde{K}_\gamma \) satisfies \( \mathbf{D}_d(\Psi, \lambda^\gamma, A_\gamma, X^2) \), we can apply Theorem 45 with \( M \leftarrow B_d \) \( \geq 2A\gamma/(1 - \lambda^\gamma) \) by (115), which completes the proof of (a).
(b) We now provide upper bounds for $C_γ$ and $ρ_γ$. These constants are non-decreasing in $c_γ$ and $λ_γ$. Therefore it suffices to give upper bounds on $c_γ, ε_{d,γ}$ and $λ_γ$. The result is then straightforward using that $(1 - λ^{1/γ})/(1 - λ^γ) ≤ [1/γ], γ(1 - λ^{1/γ})/(1 - λ^γ) ≤ \tilde{γ} + \log^{-1}(1/λ)$ and $λ^{[γ]} ≤ \lambda$.

(c) By assumption on $\tilde{γ}$, $λ$ and $ε_{d,1}$ we have that $\log((1 - ε_{d,2})^{-1}) ≤ 1$ and

$$\log(λ^{-1}) ≤ \log(λ^{-1}) ≤ \log(2), \quad e ≤ 2(1 + 1/\log(2)) ≤ B_d ≤ \overline{c}_1.$$  

As a result, we obtain that $\log(\overline{λ}^{-1})/\log(ε_1) ≤ 1, \log((1 - ε_{d,2})^{-1})/\log(ε_1) ≤ 1$. Therefore we have

$$\log^{-1}(1/\overline{ρ}_2) = [\log(λ^{-1}) + \log((1 - ε_{d,2})^{-1}) + \log(ε_1)] \quad / \quad [\log(\overline{λ}^{-1}) \log((1 - ε_{d,2})^{-1})] \leq 3\log[6A(1 + \log^{-1}(1/λ))/\log(\overline{λ}^{-1}) \log((1 - ε_{d,2})^{-1})].$$

Using that $\log(1 - t) ≤ -t$ for any $t ∈ (0, 1]$ and the definition of $\overline{λ}_1$, we obtain that

$$\log^{-1}(\overline{ρ}_2^{-1}) ≤ 6ε_{d,2}^{-1}(1 - λ)^{-1}\log[6A(1 + \log^{-1}(1/λ))].$$

Finally, we get (114) using that for any $t ∈ [0, \log(2)], 1 - e^{-t} ≥ (2 \log(2))^{-1}t$.

□

Note that Theorem 46 gives an upper bound on the rate of convergence $\overline{ρ}_2$ in the worst case scenario for which the minorization constant $ε_{d,2}$ is small and the constant $λ$ in $D_d(V, λ^γ, Aγ, X^2)$ is close to one.

Some remarks are in order here concerning the bounds obtained in Theorem 46 and Theorem 8. Assume that $ℓ = 1$, we will see in Section 4 that the leading term in the upper bound in Theorem 8, respectively Theorem 46, is given by $\log(A)/\log(λ^{-1})ε_{d,1}$, respectively $\log(A)/\log(λ^{-1})ε_{d,2}$. In addition, in our applications, $ε_{d,1}$ is larger than $ε_{d,2}$. Therefore, in these cases the bounds provided in Theorem 8 yield better rates than the ones in Theorem 46-(c). The main difference between the two results is that in the proof of Theorem 46 a drift condition on the iterated coupling kernel $K_γ^{[γ]}$ is required. However, even if such drift conditions can be derived from a drift condition on $K_γ$, the constants obtained using this technique are not sharp in general. On the contrary, the proof of Theorem 8 uses the iterated minorization condition and a drift condition on the original coupling $K_γ$.

**G Tamed Euler-Maruyama discretization**

In this subsection we consider the following assumption.

**T1.** $X = \mathbb{R}^d$ and $Π = \text{Id}$ and

$$T_γ(x) = x + γb(x)/(1 + γ∥b(x)∥) \text{ for any } γ > 0 \text{ and } x ∈ \mathbb{R}^d.$$ 

Here, we focus on drift $b$ which is no longer assumed to be Lipschitz. Therefore, the ergodicity results obtained in Section 4 no longer hold since the minorization condition we derived relied heavily on one-sided Lipschitz condition or Lipschitz regularity for $b$. We now consider the following assumption on $b$. 

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T2. There exists $\tilde{L}, \tilde{\ell} \geq 0$ such that for any $x, y \in \mathbb{R}^d$

$$\|b(x) - b(y)\| \leq \tilde{L}(1 + \|x\|^{\tilde{\ell}} + \|y\|^{\tilde{\ell}}) \|x - y\|.$$ 

In addition, assume that $b(0) = 0$ and $M_\ell = \sup_{x \in \mathbb{R}^d} (1 + \|x\|^{\tilde{\ell}})(1 + \|b(x)\|)^{-1} < +\infty$.

Proposition 47. Assume T1 and T2 then $A2(\mathbb{R}^{2d})$-(iii) holds with $\bar{\gamma} > 0$ and for any $\gamma \in (0, \bar{\gamma}]$, $\kappa(\gamma) = 2\tilde{L}_\gamma + \gamma \tilde{L}_\gamma^2$ where

$$\tilde{L}_\gamma = 2\gamma^{-1}M_\ell(1 + M_\ell)\tilde{L}.$$ 

Proof. Let $x, y \in \mathbb{R}^d$ and assume that $\|x\| \geq \|y\|$. We have the following inequalities

$$\left\| \frac{b(x)}{1 + \gamma \|b(x)\|} - \frac{b(y)}{1 + \gamma \|b(y)\|} \right\| \leq \frac{\|b(x) - b(y)\|}{1 + \gamma \|b(x)\|} + \frac{\|b(y)\|}{1 + \gamma \|b(y)\|} \left( \frac{\|b(y)\|}{1 + \gamma \|b(y)\|} \right) \|x - y\|$$

$$\leq \gamma^{-1}2M_\ell\tilde{L}\|x - y\| + \gamma \frac{\|b(y)\| \|b(x) - b(y)\|}{(1 + \gamma \|b(x)\|)(1 + \gamma \|b(y)\|)}$$

$$\leq \gamma^{-1}M_\ell(1 + M_\ell)\tilde{L}\|x - y\|.$$ 

The same inequality holds with $\|y\| \geq \|x\|$. Therefore, we have

$$\|T_\gamma(x) - T_\gamma(y)\|^2 \leq (1 + 2\gamma\tilde{L}_\gamma + \gamma^2\tilde{L}_\gamma^2) \|x - y\|^2,$$

which concludes the proof.

Proposition 47 implies that the conclusions of Proposition 5-(c) hold. Note that contrary to the conclusion of Proposition 11, we do not get that $\sup_{\gamma \in (0, \bar{\gamma}]} \kappa(\gamma) < +\infty$. Hence we have for any $\tilde{\ell} \in \mathbb{N}^*$, $\inf_{\gamma \in (0, \bar{\gamma}]} \alpha_+(\kappa, \gamma, \tilde{\ell}) = 0$.

T3. There exist $\bar{R}$ and $\bar{m}^+$ such that for any $x \in \bar{B}(0, \bar{R})^c$,

$$\langle b(x), x \rangle \leq \bar{m}^+ \|x\| \|b(x)\|.$$ 

Under T2 and T3 it is shown in [7] that there exists $\bar{\gamma} > 0$, $\lambda \in (0, 1)$ and $A \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}]$, $R_\gamma$ satisfies $D_{\delta}(V, \lambda^2, A\gamma^2, X)$ with $V(x) = \exp(a(1 + \|x\|^2)^{1/2})$ for some fixed $a$.

Theorem 48. Assume T2 and T3 then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (0, \bar{\gamma}]$ there exist $C_\gamma \geq 0$ and $\rho_\gamma \in (0, 1)$ with for any $\gamma \in (0, \bar{\gamma}]$, $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}$

$$\|\delta_x R_\gamma^k - \delta_y R_\gamma^k\|_V \leq C_\gamma \rho_\gamma^k \{V(x) + V(y)\}.$$ 

Proof. The proof is a direct application of Theorem 46-(a).

It is shown in [7, Theorem 4] that the following result holds: there exists $V : \mathbb{R}^d \to [1, +\infty)$, $\bar{\gamma} > 0$, $C, D \geq 0$ and $\rho \in (0, 1)$ such that for any $k \in \mathbb{N}$, $\gamma \in (0, \bar{\gamma}]$ and $x \in \mathbb{R}^d$

$$\|\delta_x R_\gamma^k - \pi\|_V \leq C \rho^k V(x) + D \sqrt{\gamma},$$

where $\pi$ is the invariant measure for the diffusion with drift $b$ and diffusion coefficient $\text{Id}$.
H Explicit rates and asymptotics in Theorem 18

We recall that \( b \) satisfies

\[
\langle b(x), x \rangle \leq -k_1 \|x\| \mathbf{1}_{B(0,R_3)}(x) - k_2 \|b(x)\|^2 + a/2 ,
\]

with \( k_1, k_2 > 0 \) and \( R_3, a \geq 0 \) and that We recall that

\[
V(x) = \exp(m^+_3 \phi(x)) , \quad \phi(x) = \sqrt{1 + \|x\|^2} , \quad m^+_3 \in (0,k_1/2] .
\]

(116)

Let \( \mathcal{C}_2(x,y) = (V(x) + V(y))/2 \) with \( V(x) = \exp[m^+_3 \sqrt{1 + \|x\|^2}] \) and \( m^+_3 \in (0,k_1/2] \). Therefore, by Proposition 17, \( K_\gamma \) satisfies \( D_4(\mathcal{C}_2,\lambda^\gamma, \Delta \gamma, X^2) \) for any \( \gamma \in (0,\bar{\gamma}) \) where \( \bar{\gamma} \in (0,2k_2) \), \( R_4 = \max(1,R_3,(d+a)/k_1) \) and

\[
\lambda = e^{-2\bar{\gamma}^2/2} , \quad A = \exp \left[ \gamma(m^+_3 (d + a) + (m^+_3)^2)/2 + m^+_3(1 + R_4^2)^{1/2} \right] \left( m^+_3 (d + a)/2 + (m^+_3)^2 \right) ,
\]

\[
R = \log(2\lambda^{-2\bar{\gamma}}) \lg^{-1}(1/\lambda) .
\]

Let \( \bar{\gamma} \in (0,2k_2) \), \( \ell \in \mathbb{N}^* \) specified below, \( \lambda_{\bar{\gamma},c}, \rho_{\bar{\gamma},c} \in (0,1) \) and \( D_{\bar{\gamma},c} \), \( D_{\bar{\gamma},2,c}, C_{\bar{\gamma},c} \geq 0 \) the constants given by Theorem 16, such that for any \( k \in \mathbb{N}, \gamma \in (0,\bar{\gamma}) \) and \( x, y \in X \)

\[
\|\delta_x R^k_{\gamma} - \delta_y R^k_{\gamma}\|_V \leq \{D_{\bar{\gamma},1,c} + D_{\bar{\gamma},2,c} + C_{\bar{\gamma},c}\} \rho_{\bar{\gamma},c} \gamma^{k/4} ,
\]

with \( \mathcal{C}_2(x,y) = \mathbf{1}_{\Delta^2}(x,y) \{V(x) + V(y)/2 \} \) for any \( x, y \in X \). Note that by (116), this result implies that for any \( k \in \mathbb{N}, \gamma \in (0,\bar{\gamma}) \) and \( x, y \in X \)

\[
\|\delta_x R^k_{\gamma} - \delta_y R^k_{\gamma}\|_V \leq \{D_{\bar{\gamma},1,c} + D_{\bar{\gamma},2,c} + C_{\bar{\gamma},c}\} \rho_{\bar{\gamma},c} \gamma^{k/4} \}
\]

Note that using Theorem 8, we obtain that the following limits exist and do not depend on \( L \)

\[
\begin{aligned}
D_1,c &= \lim_{\bar{\gamma} \to 0} D_{\bar{\gamma},1,c} , \quad D_2,c = \lim_{\bar{\gamma} \to 0} D_{\bar{\gamma},2,c} , \quad C_c = \lim_{\bar{\gamma} \to 0} C_{\bar{\gamma},c} , \\
\lambda_c &= \lim_{\bar{\gamma} \to 0} \lambda_{\bar{\gamma},c} , \quad \rho_c = \lim_{\bar{\gamma} \to 0} \rho_{\bar{\gamma},c} .
\end{aligned}
\]

We now discuss the dependency of \( \rho_h \) with respect to the introduced parameters, depending on the sign of \( a \) and based on Theorem 8.

(a) If \( B \) holds, set \( \ell = [M^2_{\bar{\ell}}] \). Then, if we consider \( k_1,k_2 \) sufficiently small and \( a \) sufficiently large such that the conditions of Theorem 8 hold, we have

\[
\log^{-1}(\rho^{-1}) \leq 2 \left[ 1 + m^+_3 (1 + R^2)^4 + \log(1 + 2a) + (1 + 4R^2)m^+_3 \right] / [m^+_3 \Phi(-1/2)] .
\]

Note that the leading term on the right hand side of this equation is of order \( R^2 \).

(b) If \( B \) holds, with \( m \in \mathbb{R}_- \), set \( \ell = [M^2_{\bar{\ell}}] \). Then, if we consider \( k_1,k_2 \) sufficiently small and \( a \) sufficiently large such that the conditions of Theorem 8 hold, we have

\[
\log^{-1}(\rho^{-1}) \leq 2 \left[ 1 + m^+_3 (1 + R^2)^4 + \log(1 + 2a) + (1 + 4R^2)m^+_3 \right] \left[ m^+_3 \Phi(-2(-m^{1/2}R^2)/(2 - 2e^{2m}R^2)^{1/2}) \right] ,
\]

with \( \Phi \) as defined above.

(117)

A similar result was already obtained in [29, Theorem 10] but the scheme of the proof was different as the authors compared the discretization scheme to the associated diffusion process and used the contraction of the continuous process.