Entropic Principles

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Abstract

We discuss the evolution of radiation and Bekenstein-Hawking entropies in expanding isotropic universes. We establish a general relation which shows why it is inevitable that there is currently a huge difference in the numerical values of these two entropies. Some anthropic constraints on their values are given and other aspects of the cosmological 'entropy gap' problem are discussed. The coincidence of the classical and quantum entropies for black holes with Hawking lifetime equal to the age of the universe, and hence of radius equal to the proton size, is shown to be identical to the condition that we observe the universe at the main sequence lifetime.

1. Introduction

In this paper we discuss the inevitability of a number of simple relations between the matter and radiation entropies of Friedmann universes and the value of the Bekenstein-Hawking entropy of the visible universe. The huge difference in values of these two possible entropies of the observable universe has led some to the conclusion that the universe is in an extraordinarily improbable state compared to one in which its radiation and matter contents were reorganised. We will show that the entropy gap is a consequence of the gravitational field equations alone and is just another way of stating the cosmological 'flatness problem'. We also show how the weak anthropic principle places very strong lower bounds on the entropy gap.
Since the dimensionless numbers involved in the resolution of these questions are very large we shall provide an order of magnitude analysis which ignores particle spin weight factors and numerical contributions $O(2\pi)$. We begin by analysing the simplest situation of the flat, radiation-dominated universe to establish some conclusions and then extend the analysis to the open and matter plus radiation cases before making some points about black hole entropy coincidences.

2. Entropies in flat radiation universes

Consider the Bekenstein-Hawking entropy, $S$, of the cosmological particle horizon at comoving proper time, $t$. It is given by the area of the horizon size $\sim t$ in Planck units ($c = h = k = 1$); that is,

$$S \sim \frac{A}{A_p} \sim \left(\frac{t}{t_p}\right)^2. \quad (2.1)$$

The current value of this quantity, at $t = t_0 \sim 10^{17}s \sim 10^{60}t_p$ gives the cube of a Dirac large number:

$$S_0 \sim \left(\frac{t_0}{t_p}\right)^2 \sim 10^{120}. \quad (2.2)$$

We can regard $S$ as the maximum possible gravitational entropy for the visible universe inside a Hubble radius of size $\sim t$. It is related to the Bekenstein entropy bound $[1]$, that for any physical system of size $R$ and energy $E$ the entropy is bounded above by

$$S \leq S_{\text{max}} = ER. \quad (2.3)$$

This maximum value, $S_{\text{max}}$, is equal to the value (2.1) for a gravitating system with $R \sim E/m_p^2 \sim t$. Now consider the radiation entropy inside the horizon, $S_\gamma$. This is roughly equal to the number of photons inside the horizon of size $\sim t$; thus, if the number density of photons at temperature $T$ is $n_\gamma \sim T^3$, we have

$$S_\gamma \sim n_\gamma t^3 \sim T^3 t^3 \sim \left(\frac{T}{T_p}\right)^3 \left(\frac{t}{t_p}\right)^3. \quad (2.4)$$

If the universe is flat, isotropic, and always radiation dominated then the energy density of radiation, $\rho_\gamma$, is related to the comoving proper time by a relation of the approximate form
\[ \rho_\gamma \sim T^4 \sim \frac{1}{Gt^2} \]  
and so we always have

\[ S_\gamma \sim \left( \frac{t}{t_p} \right)^{3/2}. \]  

Hence today, at \( t_0 \), we have roughly

\[ S_\gamma \sim \left( \frac{t_0}{t_p} \right)^{3/2} \sim 10^{90} \]  

and the explicit dependence of the values of \( S \) and \( S_\gamma \) on the time when they are evaluated is clear.

### 3. The Entropy Gap

If we combine eqns. (2.1) and (2.6) we see that the following relation between \( S \) and \( S_\gamma \) must hold at all times:

\[ S \sim S_\gamma^{4/3}. \]  

This relation is instructive. It shows that if we have a universe containing radiation we are not at liberty to imagine that the radiation entropy could be as large as \( S \) except if the universe is of Planck size, in which case we must have \( S \sim S_\gamma \sim O(1) \). Even if explosive non-equilibrium behaviour were suddenly to erupt in the Universe, the eventual equilibrium state would have to satisfy (3.1) if the expansion were still isotropic. This observation is relevant to the argument of Penrose [2] that the present state of the universe is highly improbable because the maximum entropy that it could have, \( S(t_0) \), is so much larger than the observed radiation entropy, \( S_\gamma(t_0) \). This argument implies that the matter could be rearranged to make the radiation or matter entropy as large as \( S \sim 10^{120} \) today without changing \( S \). However this is clearly not the case. If anything were done to raise the radiation entropy to a value of order \( S(t_0) \), then Einstein’s equations would couple the new radiation density to the age and size of the horizon via eq. (2.3) and we would have to have \( S \sim 10^{160} \). In this respect, the gap between \( S \) and \( S_\gamma \) tells us nothing about the probability or improbability of the present state.
of the observable universe or of its initial data. The $S - S_\gamma$ gap is a direct consequence of the law of gravitation. Moreover, we see that the vast difference in the values of $S$ and $S_\gamma$ actually follows from an initial state at the Planck scale where $S_\gamma \sim S$ which would have to be judged highly probable if the same reasoning were applied to it by 'observers' existing at that time.

A similar relation to that given by (3.1) also applies to oscillating closed universes, with the entropy values evaluated at the expansion maxima of successive cycles. It shows very simply why an increase in thermal radiation entropy ($S_\gamma$) from cycle to cycle would require an increase in the size of each cycle’s expansion maximum (determined by $S$) and hence a closer approach of the closed universe to 'flatness'. This was first pointed out by Tolman [3] and recently generalised by Barrow and Dabrowski [4]. It is interesting to note that the presence of a positive cosmological constant always requires these oscillations to cease and be replaced by expansion towards a de Sitter state.

We note that the 'entropy gap' between $S$ and $S_\gamma$ is also the reason why the traditional 'heat death' of the universe does not occur in ever-expanding isotropic universes [3]. There is no approach to gravitational equilibrium. Although the radiation entropy increases (and may be augmented by other non-equilibrium processes) in accord the expectations of the second law, the maximum entropy defined by the gravitational horizon entropy $S$ increases faster and so the entropy gap $S - S_\gamma$ grows with time: the universe gets farther from the equilibrium of Helmholtz’s 'heat death' [3], [8], since $S/S_\gamma \sim (t/t_p)^{1/2}$ as $t \to \infty$. In fact, in more general anisotropic and inhomogeneous universes the 'heat death' is a more complicated question that involves the analysis of the asymptotic evolution of the anisotropic gravitational-wave distortions to the expansion dynamics. It appears that there need be no gravitational 'heat death' either, unless there is a positive cosmological constant [6].

We should remark that the total entropy of the Universe may not be a finite quantity. Without delving into the possible classical, dynamical, quantum gravitational, or stringy contributions to the total entropy we can see that if the Universe is infinite in volume and contains a uniform distribution of black holes then the total Bekenstein-Hawking entropy of the black holes will be infinite. Any evaluations of the total entropy of the Universe are therefore problematic in the infinite volume case, as are all other global aspects of such universes [10].

There is a clear anthropic aspect. If observers who evolve by natural selection can only be on the cosmic scene after a time of order the main-sequence stellar lifetime for hydrogen burning, $t_{ms} \sim m_p^2 m_N^{-3}$, then we must be observing the
universe at a value of $t_0$ that is bounded by \[11], \[12],

$$t_0 > t_{ms} \sim \left( \frac{m_p}{m_N} \right)^2 m_N^{-1} \sim 10^{38} \times 10^{-23} \text{s} \sim 10^9 \text{yrs}, \quad (3.2)$$

where $m_N = 1.67 \times 10^{-24} \text{gm}$ is the proton mass. If we fail to find ways of existing after the stars have died then we may be more strongly constrained to observe only when $t_0 \sim t_{ms}$. But assuming only (3.2), we would have to find values of $S$ and $S_\gamma$ which satisfy

$$S \geq \left( \frac{t_{ms}}{t_p} \right)^2 \sim \left( \frac{m_p}{m_N} \right)^6 \sim 10^{114}, \quad (3.3)$$

$$S_\gamma \geq \left( \frac{t_{ms}}{t_p} \right)^{3/2} \sim \left( \frac{m_p}{m_N} \right)^{9/2} \sim 10^{86}. \quad (3.4)$$

Our existence would not be possible if the values of $S$ and $S_\gamma$ were significantly smaller. A slightly weaker (but more fundamental) anthropic constraint can be derived by arguing that observers can only exist when the temperature falls below the ionisation energy of atoms; that is, when the radiation temperature satisfies

$$T < T_{\text{ion}} \sim \alpha^2 m_e, \quad (3.5)$$

where $\alpha = 1/137.04$ is the fine structure constant and $m_e = 9.1 \times 10^{-28} \text{gm}$ is the electron mass. Using (2.5), this means that we must be observing the universe when its age satisfies

$$t > t_{\text{ion}} \sim \frac{m_p}{T_{\text{ion}}^2} \sim \alpha^{-4} \left( \frac{m_p}{m_e} \right)^2 t_p \sim 10^{12} \text{ s},$$

which would require atom-based observers to witness

$$S \geq \left( \frac{t_{\text{ion}}}{t_p} \right)^2 \sim \alpha^{-8} \left( \frac{m_p}{m_e} \right)^4 \sim 10^{110}, \quad (3.6)$$

$$S_\gamma \geq \left( \frac{t_{\text{ion}}}{t_p} \right)^{3/2} \sim \alpha^{-6} \left( \frac{m_p}{m_e} \right)^3 \sim 10^{83}. \quad (3.7)$$

If we were to require the observers to have molecular structure then the ionisation temperature would be replaced by the dissociation temperature and the lower bounds on the entropies would only be slightly increased by a geometrical factor.
4. Open universes

So far, for simplicity, we have assumed that the universe is spatially flat and contains only radiation. This is a good approximation to the actual state of affairs because so many decades of evolution occurred in the radiation era. But we can easily generalise our results to the cases where these assumptions do not hold. Consider first the situation of an open radiation universe. To an excellent approximation the dynamics are described by an expansion scale factor $a(t) \propto t^{1/2} \propto T^{-1}$ for the flat universe up until some characteristic time $t_c$, after which the expansion is curvature dominated with $a(t) \propto t \propto T^{-1}$. Hence, the maximum entropy, $S$, inside a scale of size $\sim t$ is still given by (2.1), but the radiation entropy is given by (2.4) or (2.6) only at times $t \leq t_c$, before the universe becomes curvature dominated. At later times we have

$$S_\gamma \sim \left(\frac{t_c}{t_p}\right)^{3/2} ; t \geq t_c,$$

independent of time $t$. Thus $S_\gamma$ remains of order the value it had at the time when the expansion first became curvature dominated. The earlier results for the flat universe are obviously recovered by putting $t_c = t_0$. Thus we have

$$\frac{S}{S_\gamma} \sim \left(\frac{t}{t_p}\right)^2 \left(\frac{t_p}{t_c}\right)^{3/2} ; t \geq t_c.$$

Again, we see there is no heat death: the entropy gap grows even faster than in the flat universe as $t \to \infty$. As in the flat case, we see that the fact that we observe $S \gg S_\gamma$ is a direct consequence of the cosmological evolution equations and nothing to do with the likelihood of particular initial conditions or the present state of the visible universe. In order to have a universe with $S \sim S_\gamma$, we would need it to have begun expanding right from the quantum era in a curvature-dominated state, that is $t_c \sim t_p$, and to observe it at a time $t_0 \sim t_p$. This is a situation that cannot be observed by beings like ourselves. The same general conclusions hold regarding anthropic constraints on the observed values of $S$ and $S_\gamma$ and we can supplement them with the requirement that we need $t_c > t_{ms}$ if stars are to form by the process of gravitational instability from small primordial density perturbations [4, 8].

The relations (2.1), (2.6), and (3.1) also reveal the source of the large present-day values today of $S$ and $S_\gamma$. The large numbers reflect the large value of the present age of the universe, $t_0$, in Planck units. Thus the largeness of $S$ and $S_\gamma$ is
a way of restating the flatness problem of cosmology for which inflation provides a possible solution by supplying a mechanism for enlarging the size of a universe that begins expanding with \( t_c \) very close to \( t_p \).

5. Matter and Radiation Entropies

Now consider the addition of matter to the flat radiation universe. There will appear a characteristic time, \( t_{eq} \), determined by the relative number densities of protons and photons such that at times earlier than \( t_{eq} \), the dynamics are radiation dominated and eqns. (2.1)-(2.6) apply; but in the matter-dominated era after \( t_{eq} \) the expansion scale factor evolves as \( a(t) \propto t^{2/3} \). Thus \( S \) will be given by (2.1) and \( S_\gamma \) by

\[
S_\gamma \sim \left( \frac{t_{eq}}{t_p} \right)^{1/2} \left( \frac{t}{t_p} \right) ; t \geq t_{eq}.
\] (5.1)

In general, therefore, we have a simple relation between \( S_\gamma \) and \( S \):

\[
S_\gamma \sim \left( \frac{t_{eq}}{t_p} \right)^{1/2} S^{1/2} ; t \geq t_{eq}.
\] (5.2)

The previous results for the flat radiation universe are recovered by putting \( t_{eq} \sim t_0 \). In our universe we have a matter-radiation balance which implies that \( t_{eq} \sim 10^{10} s \sim 10^{53} t_p \).

It is easy to see what will happen in the most general case of an open universe containing matter and radiation. The gravitational entropy will always obey (2.1) but the radiation entropy will have a value today (where \( t_0 > t_c \)) given by (5.2) evaluated at \( t = t_c \), that is by

\[
S_\gamma \sim \left( \frac{t_{eq}}{t_p} \right)^{1/2} \left( \frac{t_c}{t_p} \right) ; t \geq t_c \geq t_{eq}.
\] (5.3)

We recover the radiation-universe results if we put \( t_0 = t_{eq} \) in (5.3). Again, we see that there is a gravitationally determined link between the values of \( S \) and \( S_\gamma \). In order for them to be similar in magnitude we would require the universe to have \( t_0^2 \sim t_p^{1/2} t_c t_{eq}^{1/2} \).

We can also consider the fate of the classical matter entropy, \( S_m \), which is determined by the total number of particles. If we count particles of mass \( m \) inside the scale \( t \), then
Hence, in a flat matter-dominated evolution we have $S_m \sim$ constant, but in an open curvature-dominated universe we approach a zero-entropy asymptote with $S_m \propto t^{-1}$. We may also write

$$S_m \sim n_m t^3 \propto a^{-3} t^2.$$  

so the observed entropy per proton is given in the flat case (or at $t_0 < t_c$ in an open universe) by

$$\frac{S_\gamma}{S_m} \sim \left(\frac{t_0}{t_p}\right)\left(\frac{m}{m_p}\right)^{1/2} \sim 10^9.$$  

If we determine the classical entropy by counting particles other than protons then we can alter the absolute numerical value. Conventionally, we count protons, but if the universe is dominated by non-baryonic dark matter particles — for example by the lightest supersymmetric particle or by axions — then they may be more appropriate entropy counters and the particle-entropy numerology will be changed.

6. A Black Hole Coincidence

We may also determine classical and quantum (Bekenstein-Hawking) entropies for a Schwarzschild black hole of mass $M$ and radius $R_{BH} = 2M m_p^{-2}$. From (2.1), the quantum entropy of the black hole is

$$S_{BH} \sim \left(\frac{M}{m_p}\right)^2$$

and its Hawking lifetime \[13\] is given by

$$t_{BH} \sim \frac{M^3}{m_p^4}.$$  

The classical entropy of the black hole, $S_{cl}$, is determined by the number of particles of mass $m$ it ‘contains’. This is given by
We see that there is an interesting coincidence \cite{4}: black holes which have a Hawking lifetime equal to the present age of the universe, $t_{BH} \sim t_0$, have classical and quantum entropies that are similar in value (if $m \sim m_N \sim 10^{-24}gm$) and a Schwarzschild radius equal to the proton radius $\sim m^{-1}$ because

$$S_{BH} \sim S_{cl} \sim \frac{M}{m}.$$ 

However, we note that the coincidence that $S_{BH} \sim S_{cl}$ for $t_{BH} \sim t_0$, reduces to the condition that the present epoch is

$$t_0 \sim \frac{m_p^2}{m^3}.$$ 

By reference to eq. (3.2), we see that this is precisely the condition that we are living at about the main-sequence age: $t_0 \sim t_{ms}$. If we chose to count the classical entropy of the black hole using a reference particle mass, $m$, not equal to the proton mass then this coincidence would no longer hold but the fact that for black holes with $S_{BH} \sim S_{cl}$ we must have $R_{BH} \sim m^{-1}$ would still hold regardless of the identity of the particle with mass $m$.

\textbf{7. Conclusions}

We have shown that various measures of the entropy of the observable universe can be estimated by simple order-of-magnitude analysis. This reveals the necessary connection between the radiation and matter entropies and the Bekenstein-Hawking (BH) entropy of the mass density contained within the Hubble radius at any time. This BH entropy is generally taken to define the maximum entropy that the observable universe could possess. The smallness of the radiation entropy with respect to the BH entropy is shown to be a necessary consequence of the Friedmann equations governing the expansion of the universe. Its relative smallness should not therefore be interpreted as telling us that the material in the observable universe is (or was) in an extraordinarily low entropy state. The entropy gap between the radiation entropy and the BH entropy necessarily grows.
with time and the magnitude of the gap is a way of measuring the size and age of the observable universe. The gap could only be negligible in a universe that was too young and dense for living observers to exist: it is another way of stating the flatness problem.

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