SO(2,1) Covariant IIB Superalgebra

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Abstract

We propose a type IIB super-Poincaré algebra with SO(2,1) covariant central extension. Together with SO(2,1) and SO(9,1) generators, a SO(2,1) triplet (momenta), a Majorana-spinor doublet (supercharges) and a Rarita-Schwinger central charge generate a group, $G$. We consider a coset $G/H$ where $H=(SO(2)\times\text{Lorentz})$, and the SL(2,R) 2-form doublet is obtained by the coset construction. It is shown that U(1) connections, whose strengths are associated with 2-forms, are recognized as coordinates of the enlarged space. We suggest that this is the fundamental algebra governing the superstring theories which explains the IIB SL(2,R) duality and geometrical origin of U(1) fields.

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1 Introduction

Origin of the SL(2,Z) duality of type IIB supergravity theory [1, 2] is still an unsolved problem. Possible geometrical origins of the IIB SL(2,Z) duality in D-brane theories are examined in higher dimensional theories, such as 12-dimensional theory (F theory [3]) and 13 or 14-dimensional theories [4]. On the other hand, it is known that a hidden dimension (11-th direction) appears as worldvolume U(1) fields on a D2-brane [5, 6]. It is also expected that two hidden dimensions (11- and 12-th directions) are described as two transverse components of worldvolume U(1) gauge fields on a D3-brane [7]. Towards the higher dimensional theories the worldvolume U(1) fields will play important roles in exploration of the geometrical origin of the IIB SL(2,Z) duality.

In order to find better understanding on this point, the superalgebra and the Hamiltonian analyses are proper tools. The superalgebras (SUSY algebras) for Dp-branes contain central charges representing their NS/NS and R/R gauge couplings [8, 9]. SUSY central charges, in general, arise from the Wess-Zumino terms or topological terms [10]. The Wess-Zumino terms or topological terms in the Dp-brane theories come from NS/NS and R/R gauge couplings [11] where the NS/NS and R/R gauge fields appear as representations of the type IIB SL(2,R) symmetry (SL(2,Z) at the quantum level). Concrete expressions of SUSY central charges for Dp-branes are given as the p-form brane charges [10] and others depending on the worldvolume U(1) field [12, 13, 14, 15]. For a D1-brane the worldvolume electric field corresponds to the NS/NS charge and the D1-brane charge corresponds to the R/R charge [12]. For a D3 brane, the electric field and the magnetic field appear in the NS/NS and the R/R central charges respectively in its superalgebra, and in the Hamiltonian its BPS mass is obtained as the electro-magnetic energy [14, 16]. The analyses of the superalgebra in the Hamiltonian formalism explain the BPS mass property of Dp-branes as a consequence of worldvolume U(1) excitations [12, 14, 16]. It is appropriate to examine the IIB D-brane superalgebra to explore the geometrical origin of the worldvolume U(1) gauge fields and the IIB SL(2,Z) duality.

Siegel have constructed the Wess-Zumino term of N=1 fundamental string [17] using a central extension of super-Poincaré algebra proposed by Green [18]. The Wess-Zumino term is given in a bi-linear combination of invariant forms and the Lagrangian density is invariant under super transformations by itself. Recently Sakaguchi [19] discussed central extension of superalgebras inspired by works in [20] for p-branes. These extended superalgebras contain new bosonic generators corresponding to brane charges and new fermionic generators in addition to original super-Poincaré generators. Using with the algebras and the coset construction, he discussed the IIA, IIB and (p,q) string actions.

It is important to notice that the type IIB D-brane superalgebra naturally fits with the SL(2,R) covariantization [4]. The total momentum together with two vector central charges associated with two second rank anti-symmetric fields $B_{\mu\nu}$ form a SL(2,R) triplet. In this paper we propose an extension of the super-Poincaré algebra manifestly covariant
under $SO(2,1) \cong SL(2,R)$:

$$\{Q_{A\alpha}, Q_{B\beta}\} = 2i P_{jm} (C^m)_{\alpha\beta} (c\gamma^j)_{AB},$$  \hspace{1cm} (1.1)

$$[P_{im}, Q_{A\alpha}] = i Z_{iA\beta} (\Gamma_m)_{\beta\alpha},$$  \hspace{1cm} (1.2)

$$[P_{im}, P_{jn}] = 0.$$  \hspace{1cm} (1.3)

Here $Q_A$ is the $SO(2,1)$ doublet "supercharges", $P_j$ is the $SO(2,1)$ triplet "momenta" operators and $Z_{iA}$ is the $SO(2,1)$ spin 3/2 fermionic central charges. It will be discussed the identification of the $SO(2,1)$ with the $SL(2,R)$ symmetry of the type IIB supergravity theory. We suggest that this is the basic symmetry algebra of the superstring and the super D-brane theories.

The organization of this paper is the following: In section 2, we propose the $SO(2,1)$ covariant IIB SUSY algebra and examine a condition from the Jacobi identity. In section 3, the coset construction is applied to find invariant forms. In section 4, the type IIB $SL(2,R)$ representations are given in terms of $SO(2,1)/SO(2)$ coset representations. It is also shown the identification of the $SO(2,1)$ with the type IIB $SL(2,R)$. In section 5, the $SO(2,1)$ invariant D-string action is considered in terms of the invariant forms. Especially we show how the U(1) fields are described in terms of coordinates in the enlarged space. Some discussions are given in the last section.

2 SO(2,1) covariant central extension of Superalgebra

$SO(2,1)$, which is isomorphic to $SL(2,R)$ and $SU(1,1)$, is the Lorentz group in 2+1 dimensions. The generators $N_{ij}$, $(i, j = \hat{0}, \hat{1}, \hat{2})$ are satisfying the Lorentz algebra

$$[N_{ij}, N_{kl}] = i (\eta_{li} N_{jk} - \eta_{lj} N_{ik} - \eta_{ki} N_{jl} + \eta_{kj} N_{il}),$$  \hspace{1cm} (2.1)

where $\eta_{ij} = \text{diag}(-1, 1, 1)$. The IIB supersymmetry generators $Q_A$ ($A = 1, 2$) are 10D Majorana-Weyl spinors with the same chirality. We assign them as a $SO(2,1)$ Majorana spin 1/2 doublet $^2$

$$[Q_A, N_{jk}] = -\frac{i}{2} Q_B (\bar{\vartheta}_{jk})^{BA}, \hspace{1cm} \vartheta_{jk} \equiv \frac{1}{2} \vartheta_{[j} \vartheta_{k]}.$$  \hspace{1cm} (2.3)

where $(i/2)\vartheta_{jk}$ is the spinor representation of $N_{jk}$. As a consequence three 10D vector charges $P_i$ appearing in the superalgebra (1.1), form a $SO(2,1)$ vector triplet,

$$[P_i, N_{jk}] = i (\eta_{ij} P_k - \eta_{ik} P_j).$$  \hspace{1cm} (2.4)

$^2$The gamma matrices in 2+1 dimensions are satisfying $\{ \vartheta^i, \vartheta^j \} = 2\eta^{ij}$. In the Majorana representation

$$\vartheta^0 = -i \tau_2, \vartheta^1 = \tau_1, \vartheta^2 = -\tau_3 \hspace{1cm} \text{and} \hspace{1cm} c\vartheta^\dagger = 1, c\vartheta^i = \tau_3, c\vartheta^2 = \tau_1,$$  \hspace{1cm} (2.2)

where the charge conjugation matrix is $c = i\tau_2, (\vartheta^i)^\dagger = -c \vartheta^i c^{-1}$. 


The fermionic central charges $Z_{iA}$ appearing in the $P,Q$ commutator (1.2) transform as spinor-vector under SO(2,1),

$$[Z_{iA}, N_{jk}] = i \eta_{i[j} Z_{k]A} - \frac{i}{2} Z_{iB} (\theta_{jk})^B_A .$$

(2.5)

10D Lorentz transformation generators are $M_{mn}$, and $P_i$'s transform as vectors and $Q_A$'s and $Z_{iA}$'s as Majorana-Weyl spinors.

Jacobi identities are satisfied trivially except one of three $Q$'s,

$$\sum_{cyclic} \{[Q_{A\alpha}, Q_{B\beta}], Q_{C,\gamma}\} = 0.$$  It requires, using the 10 dimensional cyclic identity,

$$Z_{jB} (\theta^j)^B_A = 0$$

(2.6)

which tells $Z_{jB}$ is SO(2,1) irreducible spin 3/2 generator. With this irreducibility condition $(Q_{A\alpha}, P_{im}, M_{mn}, Z_{iA}, N_{ij})$ form a closed algebra $\mathcal{G}$.

The algebra $\mathcal{G}$ is a SO(2,1) covariant generalization of ones recently proposed by Sakaguchi [19]. Actually if one of fermionic generators $Z_i$ is eliminated explicitly by using (2.6) it reproduces algebras in ref.[19].

## 3 Coset construction

In this section we construct invariant forms using the nonlinear realization of the group $G$ of the algebra $\mathcal{G}$ introduced in the last section.

The coset we are going to use is $G/H$. The numerator is the SO(2,1) covariant IIB super-Poincaré group with the central extension introduced in previous sections. The subgroup $H$ is product of the homogeneous Lorentz group and SO(2) which is a subgroup of SO(2,1). We parameterize the coset in the following form

$$g = g_N g_Z(\xi) g_P(y) g_Q(\theta),$$

(3.1)

$$\begin{align*}
g_Z &= e^{-iZ_{iA} \xi^{iA}} = e^{-iZ\xi} \\
g_P &= e^{iP_{im} y^{im}} = e^{iPy} \\
g_Q &= e^{-iQ_{A\alpha} \theta^{A\alpha}} = e^{-iQ\theta}
\end{align*}$$

(3.2)

The SO(2,1)/SO(2) part $g_N$ is usually parameterized by two scalars, dilaton and axion. Due to the irreducibility condition of $Z_{iA}$ (2.6) the coordinates $\xi^{iA}$ are not independent and $g$ is invariant under

$$\delta\xi^{iA} = (\theta^j)^A_B \Lambda^{Ba} .$$

(3.3)

We call $Z_{iA\beta}$'s “central charges” in the sense that they (anti-)commute with $P,Q$ and $Z$, i.e. $\{Z,Z\} = \{Z,Q\} = [Z,P] = 0$. 

3
The transformation of the coset element $g$ under the group $G$ is

$$g \rightarrow \Lambda g h^{-1}, \quad \Lambda \in G, \ h \in H, \quad (3.4)$$

where $h$ is (induced local) subgroup transformation parameterized as

$$h = e^{M_{mn} \omega_{mn}} e^{iN_{12} \psi}. \quad (3.5)$$

It determines the transformation of the coset coordinates. Under SO(2,1) transformations $y^{\hat{0}} m$ is a scalar, $y^{im}, (\hat{i} = \hat{1}, \hat{2})$ is SO(2) vector doublet $\theta^{A\alpha}$ and $\xi^{iA\alpha} (A = 1, 2)$ are SO(2) spinor doublets, $\xi^{iA\alpha}$ is SO(2) vector-spinor. It is noted that they are not transformed as SO(2,1) multiplets but of SO(2) due to the parameterization (3.1).

Left invariant one form is constructed as

$$\Omega \equiv -ig^{-1}dg = \frac{1}{2} N_{ij} L^j_N + Q L_Q + P L_P + Z L_Z. \quad (3.6)$$

Under the transformation of $G$ the M-C form transforms as

$$\Omega \rightarrow h \Omega h^{-1} - ihdh^{-1}, \quad h \in H. \quad (3.7)$$

$L^{12}_N$ transforms as the SO(2) gauge connection while other one forms $L$’s transform homogeneously as SO(2) covariant quantities. That is $L^{\hat{0}} m$ is a SO(2) scalar, $L^{im}, (\hat{i} = \hat{1}, \hat{2})$ is SO(2) vector doublet, $L^A_Q$ and $L^A_Z, (A = 1, 2)$ are SO(2) spinor doublets and $L^{A\alpha}_Z$ is SO(2) vector-spinor. The one form coefficients are

$$L^A_Q^\alpha = D\theta^{A\alpha},$$
$$L^m_P = Dy^{im} + \bar{\theta} \Gamma^m (c \theta) D\theta,$$
$$L^{iA\alpha}_Z = D \xi^{iA\alpha} + (\Gamma^m \theta)^{A\alpha} \{ D y^i_m + \frac{1}{3} (\bar{\theta} \Gamma_m (c \theta) D\theta) \}. \quad (3.8)$$

where $D$ is the SO(2) covariant derivative in which $-i g_N d g^{-1}_N$ plays a role of the gauge connection. The M-C equations, $d \Omega + i \Omega^2 = 0$, holds as

$$DL_Q = 0, \quad (3.9)$$
$$DL_P^i - (L_Q \Gamma (c \theta^i) L_Q) = 0, \quad (3.10)$$
$$Z_i (DL^i_Z - \Gamma L_Q L_Z^i ) = 0. \quad (3.11)$$

In the last one $Z$ is kept multiplying since $Z$’s are not independent, (2.6), and there is an ambiguity in defining $L_Z$’s.

Next we use the M-C one forms (3.8) to obtain bosonic two forms which satisfy two requirements: One is that the new coordinates $y^i$ and $\xi^{iA}$ appear only in exact forms [17]. The other is the invariance under (1.3). From now on we assume constant dilaton and

$^4$In the adjoint representation $h$ represents a rotation of three vectors around the $\hat{0}$th axis.
axion and the SO(2) covariant derivatives are replaced by the ordinary derivatives. There exists only three such two forms. One of them is an exact form

$$F^{12} \equiv \frac{1}{2} \left[ L^1_P L^2_P - L^1_Q \{ c \hat{g}^1 (L^2_Z - \hat{g}^2 L^0_Z) - c \hat{g}^2 (L^1_Z - \hat{g}^1 L^0_Z) \} \right].$$

(3.12)

Other two are $F^i_{\hat{i}} (\hat{i} = \hat{1}, \hat{2})$ defined as

$$F^i_{\hat{i}} \equiv \frac{1}{2} \left[ L^0_P L^i_P - L^0_Q (c \hat{g}^0 L^i_Z + c \hat{g}^i L^0_Z) \right]$$

$$= \frac{1}{2} \left[ dx dy^{\hat{i}} - d\theta d(c \hat{g}^0 \xi^i + c \hat{g}^i \xi^0) \right] - (\partial \Gamma c \hat{g}^i d\theta)(dx + \frac{1}{2} \bar{\theta} \Gamma d\theta)$$

$$= dA^i_{\hat{i}} - B^i_{\hat{i}},$$

(3.13)

where

$$B^i_{\hat{i}} \equiv (\bar{\theta} \Gamma c \hat{g}^i d\theta)(dx + \frac{1}{2} \bar{\theta} \Gamma d\theta),$$

(3.14)

and

$$F^i_{\hat{i}} \equiv dA^i_{\hat{i}} = d\left[ - \frac{1}{2} \{ y^{\hat{i}} dx - (\xi^i c \hat{g}^0 + \xi^0 c \hat{g}^i) d\theta \} \right].$$

(3.15)

$B^i_{\hat{i}} (\hat{i} = \hat{1}, \hat{2})$ are pullbacks of the NS-NS and R-R two forms of the flat background [6]. $A^i_{\hat{i}} (\hat{i} = 1, 2)$ are corresponding to the DBI world volume U(1) potential and its counterpart in the R-R sector. In this way the world volume U(1) potentials acquire an interpretation in terms of coordinates in the extended superspace [9]. $F^i_{\hat{i}}, A^i_{\hat{i}}$ and $B^i_{\hat{i}}$ transform as SO(2) vector doublets under $G$.

4 IIB SL(2,R) multiplet

Next we use the M-C one forms (3.8) to describe the type IIB multiplet and examine their relation to the NS-NS and R-R two forms following SL(2,R) $\cong$ SO(2,1) transformation rules.

The SL(2,R) representations of the type IIB supergravity multiplet are the followings: the metric and the antisymmetric rank four tensor fields are singlet and the dilaton $\phi$ and axion $\chi$ are coordinates in the nonlinear realization transforming as

$$\mu = e^\phi \left( \chi^2 + e^{-2\phi} \chi \right), \quad \mu \to \Lambda \mu \Lambda^t, \quad \Lambda \in \text{SL}(2,\mathbb{R}).$$

(4.1)

Two kinds of second rank antisymmetric tensor fields $B = (B^{NS}, B^R)$ are SL(2,R) doublet

$$B \to (\Lambda^t)^{-1} B.$$  

(4.2)
Since $y^\hat{b}m$ is a 10D vector coordinate invariant under SO(2,1), we identify $y^\hat{b}m$ to the position coordinate of branes $x^m$ and the SO(2,1) and SUSY invariant one forms are

$$L^\hat{b}m_P = dy^\hat{b}m + \hat{\theta} \Gamma^m c q^\hat{b} d\theta = dx^m + \hat{\theta} \Gamma^m d\theta \equiv \Pi^m. \quad (4.3)$$

The induced metric of the branes is

$$G^\mu\nu = \eta^m_n (\Pi^\mu)^m_n, \quad \Pi^m = d\sigma^\mu (\Pi^\mu)^m_n \quad (4.4)$$

and is the SO(2,1) invariant Einstein metric. The $\Pi^m$ and

$$L^A_{Q\alpha} = d\theta^A_{\alpha} \quad (4.5)$$

are SUSY invariant one forms in the usual space. The latter transforms as a SO(2) spinor doublet under the SO(2,1).

In order to find the relation between the SO(2,1) considered above and the type IIB SL(2,R), we examine the transformation rules by using concrete expression of the coset SO(2,1)/SO(2). It is convenient to use the SL(2,R) algebra $L_n$

$$[L_n, L_m] = (n-m) L_{n+m}, \quad (n, m = -1, 0, 1). \quad (4.6)$$

rather than the SO(2,1) algebra $N_{ij}$. They are relating by

$$L_{+1} = -i(N_{\hat{0}\hat{2}} + N_{\hat{1}\hat{2}}), \quad L_{-1} = i(N_{\hat{0}\hat{2}} - N_{\hat{1}\hat{2}}), \quad L_0 = iN_{\hat{0}\hat{1}}. \quad (4.7)$$

We parameterize the coset element $g_N \in SO(2,1)/SO(2)$ also by introducing redundant SO(2) gauge degrees of freedom $\varphi$ \footnote{In the matrix representation}

$$g_N = e^{L_{+1}x} e^{L_0 \varphi} e^{i N_{\hat{1}\hat{2}} \varphi} \quad (4.8)$$

and it transforms under SL(2,R) $\cong$ SO(2,1) as

$$g_N \rightarrow \Lambda g_N h^{-1}, \quad \Lambda \in SL(2, R), \quad h \in SO(2). \quad (4.9)$$

The SO(2) transformation $h$ is determined up to that of $\varphi$. The $2 \times 2$ matrix representation of $g_N$ is written as \footnote{In the matrix representation}

$$g_N|_{2 \times 2} = K = VR_\varphi, \quad \begin{bmatrix} V & \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\varphi} & 0 \\ 0 & e^{\varphi} \end{pmatrix} = \begin{pmatrix} e^{-\varphi} & \frac{e^{\varphi}}{2} \chi \\ 0 & e^{\varphi} \end{pmatrix} \\ R_\varphi = \begin{pmatrix} \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix} \end{bmatrix}.$$
These coset coordinates $\phi$ and $\chi$ are identified with the ones of the IIB supergravity; dilaton and axion, since the bi-linear SO(2) invariant combination of $K (4.11)$ gives $\mu$ in (4.1)

$$KK^t = VV^t = \mu$$

and it transforms in the same manner as (4.1).

This 2x2 matrix representation $K$ of the SO(2,1)/SO(2) makes SO(2) spinor doublets to be SO(2,1) doublets, e.g. $\theta^A = K^A_B\theta^B$ transforms as $\tilde{\theta} \rightarrow \Lambda \tilde{\theta}$. In order to define SL(2,R) doublet two forms $B^A$ from SO(2) vector doublet $B^\perp$ in (5.13) it is necessary to convert the vector index of $B^\perp$ to the spinor index. It is possible if there exists an SO(2) spinor $\Psi = (\sin \tilde{\varphi}/2, \cos \tilde{\varphi}/2)$. We define

$$K^{(v)}_A \perp_j \equiv K^A_B(\theta^j_\perp) C \Psi^C = (V(\phi, \chi) R_{\phi} R_{\tilde{\varphi}} A_j \perp).$$

Then the SO(2) vector doublet of two forms (3.15) can be lifted to SO(2,1) doublet

$$B^A = (K^{(v)})_A \perp_j B^\perp_j.$$ (4.14)

We can take the SO(2) gauge degrees of freedom $\varphi$ satisfying $\varphi + \tilde{\varphi} = 0$, so that the SO(2,1)/SO(2) coset element $K^{(v)}$ takes a conventional form $V(\phi, \chi)$ expressed in terms of a dilaton and an axion,

$$B^A = V(\phi, \chi) A \perp_j B^\perp_j, \quad V = \begin{pmatrix} e^{-\frac{\varphi}{2}} & e^{\frac{\varphi}{2}} \chi \\ 0 & e^{\frac{\varphi}{2}} \end{pmatrix}. \quad (4.15)$$

5 IIB Dp-branes

In this section we consider SL(2,R) invariant Dp-branes. The action have been discussed in SL(2,R) covariant forms in [21]. H invariant Lagrangians which are constructed from $L_Q$ in (4.3), $L_p^0$ in (4.3) and $F^j$ in (3.13) have the SO(2,1) symmetry. As the H invariant Lagrangian we take the Lagrangian similar form as that in [21]. For a D1 case,

$$\mathcal{L} = \frac{1}{2e} \left( \det G_{\mu\nu} + F^i F_i \right)$$

(5.1)

with $F^i$ given (3.13). The difference is that the U(1) potentials $A^i$ are not fundamental worldvolume fields but given in terms of new superspace coordinates in (5.1). In [21] the same form of the Lagrangian is examined and is shown to have the kappa invariance. Here we clarify the canonical structure and show how the U(1) degrees of freedoms are incorporated in terms of new superspace coordinates.

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6 The SL(2,R) covariant algebra considered in the ref. [19] is not (graded) Lie algebra since the structure constants depend on the dilaton and axion explicitly.
In order to discuss it the explicit form of the Lagrangian is not necessary we only assume the Lagrangian is described as a $H$ invariant function of form\footnote{Lagrangians may depend on the higher rank tensors also in case of $p > 1.$}

\[ \mathcal{L} = \mathcal{L}(G_{\mu\nu}, F^i). \] (5.2)

The new variables $y^i$ and $\xi^i$ enter in the Lagrangian only through $F^i$ in $\mathcal{F}^i$

\[ F^i_{\mu\nu} = \partial_{\mu} A^i_{\nu} = \partial_{\mu} \left[ -\frac{1}{2} (y^i \partial_{\nu} x + (\xi^i c\tilde{\theta} + \tilde{\xi}^i c\tilde{\theta}) \partial_{\nu} \theta) \right]. \] (5.3)

The variation of the action with respect to $y^i$, $\tilde{\xi}^0$, and $\xi^i$ gives the equations of motion,

\[ \partial_{\mu} \left( \partial \frac{L}{\partial F^i_{\mu\nu}} \right) \partial_{\nu} x^m = \partial_{\mu} \left( \partial \frac{L}{\partial F^i_{\mu\nu}} \right) c\tilde{\theta} \partial_{\nu} \theta = \partial_{\mu} \left( \partial \frac{L}{\partial F^i_{\mu\nu}} \right) c\tilde{\theta} \partial_{\nu} \theta = 0. \] (5.4)

If the induced metric is not singular the independent equations are

\[ \partial_{\mu} \left( \partial \frac{L}{\partial F^i_{\mu\nu}} \right) = 0. \] (5.5)

It is the Maxwell equation, which would be obtained when $A^i$ were independent world volume fields. It shows that $y^i$, $\xi^0$ and $\xi^i$ are not dynamically independent but have same dynamical modes of $U(1)$ potentials $A^i$. The $U(1)$ gauge transformation of $A^i$ is induced by that of the new variables, for example,

\[ \delta y^m = -2 \partial_{\mu} \lambda^i \Pi^m_{\mu a} \theta, \quad \delta \xi^i = 2 \partial_{\mu} \lambda^i \Pi^m_{\mu a} \theta, \quad \rightarrow \delta A^i_{\mu} = \partial_{\mu} \lambda^i \] (5.6)

where $\lambda^i$ are $U(1)$ gauge parameters.

In the canonical formalism the canonical variables are $x^m$, $\theta$, $y^i = (y^1, y^2)$, $\xi^i = (\xi^0, \xi^1, \xi^2)$ and their canonical conjugates $p_m$, $\zeta$, $p^i_\perp = (p^1_\perp, p^2_\perp)$, $\pi^i_\perp = (\pi^0_\perp, \pi^1_\perp, \pi^2_\perp)$ respectively. Defining equations of canonical conjugate momenta for new variables are

\[ p^i_\perp = \partial \frac{\mathcal{L}}{\partial \dot{y}^i} = -\frac{1}{2} (E^i_\perp)^a \partial_a x, \] (5.7)

\[ \pi^i_0 = \partial \frac{\mathcal{L}}{\partial \dot{\xi}^0} = -\frac{1}{2} (E^i_\perp)^a \partial_a \dot{\theta} c\tilde{\theta}, \] (5.8)

\[ \pi^i_\perp = \partial \frac{\mathcal{L}}{\partial \dot{\xi}^i} = -\frac{1}{2} (E^i_\perp)^a \partial_a \dot{\theta}, \] (5.9)

where

\[ (E^i_\perp)^a \equiv \partial \frac{\mathcal{L}}{\partial \mathcal{F}^i_{ba} (x, \dot{\theta}, \dot{y}, \dot{\xi})}, \quad (a = 1, ..., p). \] (5.10)

If we define the inverse of the spatial metric $G^{ab}$ and the following operator

\[ G_{ab} G^{bc} = \delta^c_a, \quad \Upsilon_{mn} \equiv (\Pi_a)_m G^{ab} (\Pi_b)_n, \] (5.11)
then $Y$ is the projection operator to spatial tangential direction of a p-brane; $Y^m \cdot Y^n = Y^m \cdot (\Pi_a)^m Y^n = (\Pi_a)^n$, \text{rank}(Y_m^n) = p.$

Combining (5.7) and (5.3) $E^a$ is solved in terms of the canonical variables as

$$\begin{align*}
(E^a_m)^2 &= -2 \left( \tilde{p}_m^y + \tilde{\pi}_m^\xi \Gamma_m \theta \right) (\Pi_b)^m G^{ba}.
\end{align*}$$

Inserting (5.12) back to (5.7), (5.8) and (5.9) we get first class constraints

$$\begin{align*}
\phi_y^m &= \left( (p_y^m)^m + \frac{1}{2} \pi_2^m \Gamma_m \theta \right) (\eta_{mn} - Y_{mn}) = 0,
\phi_0^m &= \pi_0^m - (p_y^m + \pi_2^m \Gamma_\theta) \cdot \Pi_b G^{ba} \partial_a \hat{\theta}(c q^i) = 0,
\phi_{\xi}^m &= \pi_{\xi}^m - (p_y^m + \pi_2^m \Gamma_\theta) \cdot \Pi_b G^{ba} \partial_a \hat{\theta} = 0.
\end{align*}$$

The first constraint restricts that the dynamical degrees of freedom of $p_y^m$ is lying on the $p$-brane. The latter two tells that $\xi^i$'s are gauge degrees of freedom. The second one is written using the third one as

$$\begin{align*}
\phi_{\xi}^m \theta^i &= \tilde{\pi}_{\xi}^m \theta^i = 0.
\end{align*}$$

This is a canonical realization of the irreducibility condition of $Z_{IA}$ in (2.6), i.e. (5.14) is the generator of the gauge transformation (5.3). It guarantees the (on-shell) closure of the SO(2,1) covariant super-Poincaré algebra $G$.

It is possible to find a canonical transformation in which $(E^a_\tilde{z}, A^\tilde{z}_a)$ are new (tilde) canonical pairs. The generating function is

$$\begin{align*}
\mathcal{W}(q, \tilde{p}) &= \int \left( \tilde{p} \cdot x \, + \, \zeta \hat{\theta} \, + \, \tilde{p}_{\tilde{z}} \left[ \eta_{mn} - Y_{mn} \right] y^{\tilde{\xi}} \, + \, \tilde{\pi}_{\tilde{z}} \xi^i \\
&\quad + \ E_{\tilde{z}}^a \left[ - \frac{1}{2} \left( y^{\tilde{\xi}} \partial_\alpha x + (\tilde{\xi} c q^i + \tilde{\xi} c q^i) \partial_\alpha \theta \right) \right] \right),
\end{align*}$$

where tilde coordinates are ($\tilde{x}, \tilde{\theta}, \tilde{y}^2, A^\tilde{z}_a, \tilde{\xi}^i$) and the conjugate momenta are ($\tilde{p}, \tilde{\zeta}, \tilde{p}_{\tilde{z}}^y, E_{\tilde{z}}^a, \tilde{\pi}_{\tilde{z}}^\xi$). $(\tilde{y}^{\tilde{\xi}}, \tilde{p}_{\tilde{z}}^y)$ are $(10-p) \times 2$ independent canonical pairs. Using tilde variables the constraints (5.13) take simple forms

$$\begin{align*}
\tilde{p}_{\tilde{z}}^y &= \tilde{\pi}_{\tilde{z}}^\xi = 0.
\end{align*}$$

The Gauss law constraints are obtained as the secondary constraints as

$$\begin{align*}
\partial_a E_{\tilde{z}}^a &= 0.
\end{align*}$$

In the case of $p = 1$ the D1-brane Lagrangian (5.11) gives following constraints in addition to (5.13) in terms of the original variables ($y^i, \theta, \xi^i$) and ($p_y^j, \zeta, \pi_{\xi}^j$);

$$\begin{align*}
H &= \frac{1}{2} \left[ (\tilde{p}_{\tilde{y}}^y)^2 + 4(\tilde{p}_{\tilde{z}}^y)^2 \right] + (\zeta \theta^0 \hat{\theta} \theta' + \pi_0^\xi \theta^0 \xi^0' + \pi_{\tilde{z}}^\xi \theta^0 \xi_{\tilde{z}}') = 0, \quad (5.18)
\end{align*}$$

$$\begin{align*}
T_1 &= \tilde{p}_{\tilde{z}}^y \cdot \tilde{x}' + (\zeta \theta' + \pi_0^\xi \xi_{\tilde{z}}' + \pi_{\tilde{z}}^\xi \xi_{\tilde{z}}') = 0, \quad (5.19)
\end{align*}$$

$$\begin{align*}
F &= \zeta - p_{\tilde{z}}^y \cdot \frac{1}{(x^0)} \left( (\tilde{\xi} c q^0 + \tilde{\xi} c q^0) + \tilde{\theta} \tilde{\Gamma} + (\tilde{p}_{\tilde{0}}^y + \frac{1}{2} (\tilde{\theta} \tilde{\Gamma} c q^0) c q^0 \\
&\quad - (x' - \frac{1}{2} \tilde{\theta} \tilde{\Gamma} c q^0 \theta') \cdot \tilde{\theta} \tilde{\Gamma} c q^0 \right) = 0, \quad (5.20)
\end{align*}$$

$9$
with
\[ \hat{p}_0^y = p_0^y + (p_j^y \cdot x' \frac{1}{(x')^2 y'^2})', \quad \hat{\varrho} = -2 p_j^y \cdot x' \frac{1}{(x')^2} \varrho \].

(5.21)

For a D-brane its static property \((x')^2 \neq 0\) is assumed. Performing the canonical transformation (5.15), above constraints (5.22), (5.23) and (5.24) are rewritten in terms tilde coordinates in addition to (5.16) and (5.17),

\[ H = \frac{1}{2} \left[ \hat{p}^2 + (E_1^j \xi)^2 \right] + \tilde{\zeta} \varrho \varrho_E \tilde{\varrho}' = 0, \]
\[ T_1 = \tilde{\varrho} x' + \tilde{\zeta} \tilde{\varrho}' = 0, \]
\[ F = \tilde{\zeta} + \tilde{\varrho} \Gamma \cdot (\varrho + \frac{1}{2} (\tilde{\varrho} \Gamma c \xi_E \varrho)) c \varrho_E - (x' - \frac{1}{2} \tilde{\varrho} \Gamma c \xi_E \varrho) \cdot \tilde{\varrho} \Gamma c \varrho_E = 0, \]

(5.22)

(5.23)

(5.24)

where
\[ \varrho_E \equiv E_1^j \varrho. \]

(5.25)

In the tilde canonical coordinates the extra variables \(\tilde{\varrho}\) and \(\tilde{\zeta}\) and their conjugate momenta are decoupled and the set of constraints (5.22)-(5.24) have the same forms as ones of (5.1) in which the U(1) potentials are regarded as independent fields. Comparing with (5.18) and (5.22), it can be seen that U(1) momenta contribute to \(H\) in a same form as the brane momentum. From the construction there exists the SO(2,1) symmetry which mixes brane coordinates and U(1) modes, i.e. they are coordinates of the enlarged space. Therefore the SO(2,1) covariant superalgebra with central extension explain geometrical origin not only of IIB SL(2,R) symmetry but also of world volume U(1) modes.

6 Discussions

In this paper we have proposed a SO(2,1) \(\cong\) SL(2,R) covariant IIB superalgebra with central extension \(G\). It contains SO(2,1) triplet momenta, doublet SUSY charges and spin 3/2 central charges. In order to describe Dp-brane systems the coset, \(G/(\text{SO(2)} \times \text{Lorentz})\), is considered. Our parametrization of the coset naturally leads to the brane coordinates; a singlet position \(x^m\) and SO(2) doublet fermionic coordinates \(\theta^{A \alpha}\). The dilaton and axion also appear as the coordinates of the coset. The SUSY invariant two form doublet is constructed and the U(1) potentials acquire an interpretation in terms of coordinates in the enlarged space.

The SO(2,1)/SO(2) coset element \(g_N\) naturally produce SO(2,1) covariant coordinates; SO(2,1) doublet from SO(2) doublet and SO(2,1) triplet from SO(2) triplet. However in order to obtain the IIB SL(2,R) supergravity doublet \((B^{NS}, B^R)\) from the SO(2) vector doublet the SO(2,1)/SO(2) coset element \(K\) is not sufficient. In order to solve this puzzle
we have introduced one more auxiliary field $\varphi$ in addition to the SO(2) gauge parameter $\varphi$. It gives an alternative representation of the coset SO(2,1)/SO(2). We showed that in a gauge $\varphi + \varphi = 0$ the SO(2,1)/SO(2) coset element coincides with that of SL(2,R)/SO(2) expressed in terms of the dilaton and the axion. In a similar context, Bars looked for the origin of auxiliary degrees of freedom in the extra space; 14 dimensions [4].

We performed a central extension of IIB superalgebras not only in a SO(2,1) covariant manner but it fits with the Siegel’s mechanism [17, 18]. An advantage of the central extension is that new bosonic coordinates $y^j$ and new fermionic coordinates $\xi^j$ appear in supersymmetric combinations to form the simple Wess-Zumino terms. Dynamical modes of the new coordinates are restricted only on a brane as (5.13) and they play a role of worldvolume U(1) gauge potentials. Therefore the worldvolume U(1) fields can be recognized as the coordinates of the enlarged superspace. It suggests that the 2 dynamical modes of the worldvolume U(1) field of D3-brane can be recognized as coordinates in the 10+2 dimensions [7]. The SO(2,1) covariant superalgebra may give us new insight of the geometrical origin of the IIB SL(2,R) duality from higher dimensional view point.

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