Self-Weight Loading of Horizontal Hydraulic Cylinders with Axial Load

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Abstract. An iterative matrix method is described to determine the deformed shape of a horizontal stepped tube with large axial load, representing a heavy hydraulic cylinder. The method is applicable to both simply supported (pinned-pinned) and clamped-clamped end constraints. The clamped-clamped case is extended to include rotational compliance in the clamps. Using this analysis, radial loads on the piston seals of the cylinder are determined.

Comparison is made between the seal loading of a typical large cylinder in pinned-pinned and clamped-clamped configurations, respectively. It is shown that the seal loading can be considerably reduced by the use of the clamped-clamped configuration provided the ends can be well aligned, but that the load is sensitive to angular misalignment.

1. Introduction
Long telescopic hydraulic rams are restricted in their use by the large bending moment that occurs near the mid span as they start to buckle under the combined influence of self weight and axial load. This causes large radial loads on the seals, which in turn leads to rapid wear.

Figure 1. Deflection and bending moment of horizontal beams with self-weight loading
Mid-span deflection and bending moment are both greatly reduced by the use of clamped-clamped (built-in) end constraints.
Putting aside the effects of axial load and changing cross-section initially, the large bending moment in a horizontal beam due to self weight is of the form shown in Figure 1 for simply supported and clamped-clamped or built-in ends, respectively. This is reflected in the small normal deflection in the clamped-clamped case, also shown in Figure 1.

A method for determining bending moment and normal displacement of a stepped shaft under combined self weight and axial force with different end mounts is presented. This is a good idealisation of a hydraulic ram, and the seal loads are obtained from the bending moment at the step. Results are presented for a single stage ram, but the method is readily extended to multi-stage telescopic rams. The method used, based on iteration and numerical integration, accommodates any number of changes of section and/or material, and any combination of misalignment of the ends, and so goes beyond the closed form solution for a uniform bar without misalignment used for validation. Furthermore, it lends itself to extension to the dynamic problem of a long horizontal ram subject to imposed vertical motion, for instance in a heavy earthmoving vehicle.

2. Analysis of a stepped beam with self-weight loading

A single stage ram loaded by self-weight only may be idealised as a stepped beam as shown in Figure 2 for the clamped-clamped case. It may be analysed using the conventional double integration method [1] by making imaginary cuts and applying a shear force $F$ and bending moment $M$ to maintain equilibrium as shown in Figure 3. $M_A$ and $R_A$ are the reactions at the clamped left hand end. $w_1$ and $w_2$ are the self weights per unit length of the two sections.

![Figure 2. Stepped beam with self-weight](image)

![Figure 3. Loading on a two section beam with self weight](image)

Taking moments about the imaginary cut face, an expression may be obtained for bending moment as a function of the applied loads including the end constraints. The bending moment is related to the curvature by:

$$M = EI \frac{d^2 z}{dx^2}$$

where $M$ is the local bending moment
$E$ is young's modulus
$I$ is second moment of area
$x$ is distance from the end of the beam
$z$ is normal deflection of the beam (a function of $x$)
For the left hand region, Figure 3(a):

\[ M + M_A - R_A x + \frac{1}{2} w_1 x^2 = 0 \]  
\[ \therefore \frac{d^2 z}{dx^2} = \frac{1}{E_1 l_1} M = \frac{1}{E_1 l_1} \left( -M_A + R_A x - \frac{1}{2} w_2 x^2 \right) \]  
\[ \text{Integrating: } \frac{dz}{dx} = \frac{1}{E_1 l_1} \left( -M_A x + \frac{1}{2} R_A x^2 - \frac{1}{6} w_1 x^3 + C_1 \right) \]  
\[ \text{and } z = \frac{1}{E_1 l_1} \left( -\frac{1}{2} M_A x^2 + \frac{1}{2} R_A x^3 - \frac{1}{24} w_1 x^4 + C_1 x + C_2 \right) \]  

Similarly, for the right hand section in Figure 3(b):

\[ M + M_A - R_A x + w_1 l_1 \left( x - \frac{1}{2} l_1 \right) + \frac{1}{2} w_2 l_1 \left( x - l_1 \right)^2 = 0 \]  
\[ \therefore \frac{d^2 z}{dx^2} = \frac{1}{E_2 l_2} M = \frac{1}{E_2 l_2} \left( -M_A + R_A x - w_1 l_1 \left( x - \frac{1}{2} l_1 \right) - \frac{1}{2} w_2 \left( x - l_1 \right)^2 \right) \]  
\[ \therefore \frac{dz}{dx} = \frac{1}{E_2 l_2} \left( -M_A x + \frac{1}{2} R_A x^2 - \frac{1}{6} w_1 l_1 \left( x - \frac{1}{2} l_1 \right)^2 - \frac{1}{6} w_2 \left( x - l_1 \right)^3 + D_1 \right) \]  
\[ \therefore z = \frac{1}{E_2 l_2} \left( -\frac{1}{2} M_A x^2 + \frac{1}{2} R_A x^3 - \frac{1}{6} w_1 l_1 \left( x - \frac{1}{2} l_1 \right)^3 - \frac{1}{24} w_2 \left( x - l_1 \right)^4 + D_1 x + D_2 \right) \]  

In equations 1-8 \( E, I \) and \( w \) take the suffix 1 or 2 to indicate values for the 1\(^{st} \) (left hand) and 2\(^{nd} \) (right hand) sections of the ram.  

There are four constants of integration and two unknown reactions \( M_A \) and \( R_A \). These may be evaluated from six simultaneous equations: four boundary conditions \( z \) and \( \frac{dz}{dx} \) at \( x = 0 \) and \( x = L \), and equating displacement \( z \) and slope \( \frac{dz}{dx} \) at \( x = l_1 \) in the two sections.  

The values of the end fixing constraints \( M_d \) and \( R_d \) and the constants of integration can thus be obtained for any mounting configuration. If required, the constraints at the right hand end can then be obtained from equilibrium.

It is convenient to set up these simultaneous equations in matrix form:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & \frac{1}{6} l_1^3 & l_1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
M_A \\
R_A \\
C_1 \\
C_2 \\
D_1 \\
D_2 \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
-\frac{1}{2} w_1 l_1^3 \\
-\frac{1}{24} w_1 l_1^4 \\
\end{pmatrix}
\begin{pmatrix}
E_1 l_1 \\
E_2 l_2 \\
\end{pmatrix}
\]

The left hand side is the left hand region (equations 3-4) and the right hand side is the right hand region (equations 7-8).

Rows 1 and 2 are the slope and displacement at the left hand end \( x = 0 \) (equations 3 and 4).

Rows 3 and 4 are the slope and displacement at the right hand end \( x = L \) (equations 7 and 8).

Rows 5 and 6 are the slope and displacement at the discontinuity \( x = l_1 \) (equations 3 and 7).
This matrix equation may be written as \( [K_1] [V] + [K_2] = [K_5] [V] + [K_6] \) \( (9) \)
Rearranging:
\( ([K_1] - [K_5]) [V] = [K_6] - [K_2] \)
\([K_1] - [K_4]\) is a square matrix and is readily inverted, and \([K_6] - [K_2]\) is a column vector, so the equation \( (9) \) becomes:
\[ [V] = ([K_1] - [K_5])^{-1} ([K_6] - [K_2]) \] \( (10) \)
and solved for \( M_A, R_A \) and the constants of integration \( C_1, C_2, D_1, D_2 \) using MatLab [2].

For any more complex stepped beam, the same analysis may be used. Two new constants of integration are introduced for each step, but there are also two new equilibrium equations, and so the system may be solved in the same way.

It should be noted that this problem of the stepped beam cannot be analysed by Macaulay’s method. This makes use of equating slope and displacement at loading discontinuities and uses a mathematical trick to make the constants of integration the same in all regions. This breaks down for a stepped bar because the modulus of rigidity \( EI \) is different in each region.

For comparison, a similar method was used to analyse a simply supported stepped beam. For this case the slope and deflection at \( x = l_1 \) are equated for the two regions as before and so rows 5 and 6 of the matrix equation are unchanged. The boundary condition \( z = 0 \) at the right hand and left hand ends \( (x = 0 \text{ and } x = L) \) and so rows 2 and 4 are also unchanged. The slopes at the ends are unknown, so rows 1 and 3 are replaced by the bending moment equations 2 and 6 with \( M \) set to zero at \( x = 0 \) and \( x = L \).

This method was validated by making the stepped beam appear as a uniform beam and comparing the results with standard solutions. This was done in three ways:
1. the values of \( E, I \) and \( w \) were made the same in regions 1 and 2;
2. the length of the left hand section \( l_1 \) was set to zero;
3. the length of the left hand section \( l_1 \) was set to \( L \).
In each case the bending moments at the ends and mid-span, and the deflection at mid-span, agreed with standard solutions for the equivalent uniform beam.

3. Analysis of a heavy hydraulic ram with end loading
Adding axial load \( F_A \) as shown in Figure 4 changes the loading on an imaginary cut face as shown in Figure 5.

This has the effect of adding an extra term \(+ F_A z(x)\) to equations (1) and (5). This means that the integrations cannot be done directly because the displacement \( z(x) \) is a function of those integrations. The best we can say is that the slope equations (3) and (7) take an extra term \(+ F_A \int_0^x zdx\) and the displacement equations (4) and (8) an extra term \(+ F_A \int_0^x zdx\).

An iterative method has been found to solve these equations effectively. Starting from a “reasonable” deformed shape \( z(x) \), the integral terms \( F_A \int_0^x zdx \) and \( F_A \int_0^x zdx \) were evaluated by...
numerical integration (trapezium rule). These integral values were added to the matrix equation (9) by adding to appropriate rows in vectors \{K_3\} and \{K_7\}, forming the equation:

\[ [K_1]\{V\} + \{K_2\} + \{K_3\} = [K_5]\{V\} + \{K_6\} + \{K_7\} \] (11)

It was found that the equations converged provided the end load was less than the buckling load. The method was validated by degenerating the stepped beam into a uniform beam by the three methods described above and comparing the results for bending moment and deflection at the centre span with the results given in Roark’s “Formulas for Stress and Strain” table 10.2e [3].

Results are presented for a 5m long single stage hydraulic ram. The theory above was embodied in a MatLab program which calculates the deformation and bending moment, and hence radial load on the seals at the change of section. The ram has dimensions:
cylinder O/D 200mm,
piston O/D 188mm,
wall thickness of cylinder and piston 6mm.

The cylinder and piston material is steel of density 7860kg/m\(^3\) and Young’s modulus 210GPa, filled with oil of density 910kg/m\(^3\). The axial force is 10\(^6\)N (1.0MN), which is approaching the buckling load for the pin jointed cylinder.

The overlap between piston and cylinder at full extension is taken as 200mm with a seal at each end. From this, the radial load on the seals is easily calculated from the bending moment at the change of section.

Figure 6 shows the effect of this large axial load of 1MN, close to the buckling load for the pin jointed beam. It may be seen that it has a large effect on the pin jointed beam, increasing the maximum deflection from 1.3mm to 5.4mm. The mid-span bending moment is also greatly increased, raising the radial load on the seal from 3.9kN to 35kN. However, the axial load has very little effect on the clamped-clamped beam, increasing the maximum deflection by only 0.1mm from 0.2 to 0.3mm. The seal load is increased from 2.6 to 3.4kN.

It is evident that the use of the clamped-clamped arrangement has a big effect, reducing both mid-span deflection and seal loading by an order of magnitude.

4. The effects of misalignment
Pin jointed cylinders are unaffected by misalignment of the ends because they are self-aligning. However, clamped ends force the cylinder to take up a shape determined by the slope and position of the ends, and this is reflected by the bending moment distribution, and hence the radial loads on the seals.

Figure 6. Displacement of stepped beam with self-weight and axial loading
To investigate this effect, equations 3, 4, 7 and 8 (slope and deflection at \( x = 0 \) and \( x = L \)) must be modified. Equations 3 and 4 refer to slope and deflection at the left hand end, so the expressions are equated to imposed values rather than zero. According to the convention that the calculated values for the left hand section of the ram (section 1) go on the left hand side of the matrix equation, the imposed values are added to rows 1 and 2 of the right hand side.

Similarly, in equations 7 and 8 the imposed slope and deflection (\( \theta_B \) and \( z_B \)) at the far right of the ram (\( x = L \)) are added to rows 3 and 4 on the left hand side of the matrix equation. Equating slope and deflection at the change of section (\( x = l_1 \)) is unaffected by misalignment and so rows 5 and 6 of the matrix equation are unchanged.

For convenience, these misalignment values are placed in new column vectors \( \{ K_4 \} \) and \( \{ K_8 \} \) to form an enlarged matrix equation:

\[
[K_1]\{V\} + \{K_2\} + \{K_3\} + \{K_4\} = [K_5]\{V\} + \{K_6\} + \{K_7\} + \{K_8\}
\]

(12)

where \( \{K_4\} = \begin{bmatrix} 0 \\ 0 \\ \theta_B \\ z_B - z_A \\ 0 \\ 0 \end{bmatrix} \) and \( \{K_8\} = \begin{bmatrix} \theta_A \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \)

Note that for convenience, the origin of the co-ordinate system is taken as the position of the left hand end of the cylinder irrespective of its misalignment.

As before, equation 12 may be rearranged to:

\[
([K_1] - [K_5])\{V\} = \{K_6\} + \{K_7\} - \{K_3\} - \{K_4\}
\]

(13)

and solved for \( \{V\} \) to obtain the constants of integration and the constraint moments and reactions, and hence the distribution of bending moment from which the loads on the seals are calculated.

The effect of an axial misalignment of 9mm is shown in Figure 7(a) for the same 5m cylinder with 1MN axial load. It may be seen that deformed shape is hardly affected by the axial load, which is well below the buckling load for the clamped beam. The calculated radial seal load is 3.8kN. As shown in Figure 7(b), radial seal load is proportional to axial misalignment, but for reasonable offsets below about 25mm there is very little effect, and the seal load only approaches that of the pinned-pinned case at offset misalignments greater than 800mm.

![Figure 7](a) displacement (b) seal loading vs axial offset

Figure 7. Stepped beam with axial (parallel) misalignment. Loading by self-weight and axial force.
The effect of angular misalignment of 0.4° at one end, shown in Figure 8(a), has more dramatic effect. Again the radial seal load is proportional to misalignment, but as shown in Figure 4(b), the seal load rises very quickly with increased misalignment angle and reaches the value for the pin jointed case at this misalignment of only 0.4°.

5. Compliance

A further complication is the impossibility of providing truly “clamped” conditions, in which there is no displacement or rotation of the cylinder ends under load. From the analysis above, displacement parallel to the cylinder axis due to the end reactions will have negligible effect, but rotation due to the fixing moments is likely to have a significant effect on the seal loads. Estimates of this effect may be obtained from Roark’s “Formulas for Stress and Strain” table 24.20b [3]. For the 200mm diameter cylinder used in the examples above, welded to a circular 8mm thick steel plate of 400mm diameter simply supported at its edge, the compliance would be 1.6×10^-6 rad/Nm or 90×10^-6 degrees/Nm.

For rotationally compliant end fixings the slope of the cylinder at the ends becomes a function of both angular misalignment and fixing moment. At the left hand end (x = 0), equation 1 becomes

\[ M + M_A = 0 \]

and so the bending moment at the left hand end of the cylinder is \(-M_A\). Angular compliance \(c\) is angle rotated per unit applied moment, so the slope of the cylinder due to compliance

\[ \alpha_A = -cM_A \]

and this must be added to the slope due to angular misalignment. This is achieved by adding a term \(-c\) to location (1,1) in the matrix \([K_5]\) of equation 13.

Similarly, at the right hand end (x = L), equation 5 gives

\[ M + M_A - R_A L + w_1 l_1 \left( L - \frac{1}{2} l_1 \right) + \frac{1}{2} w_2 l_1 (L - l_1)^2 = 0 \]

where the bending moment \(M\) becomes the fixing moment \(M_0\). Thus

\[ M_B = -M_A + R_A L - w_1 l_1 \left( L - \frac{1}{2} l_1 \right) - \frac{1}{2} w_2 l_1 (L - l_1)^2 \]

The slope due to compliance is \(\alpha_B = -cM_B\). This is incorporated into row 3 of the matrix equation by inserting \(c\) and \(-cL\) to locations (3,1) and (3,2), respectively, of the matrix \([K_1]\) and inserting \(c \left( w_1 l_1 \left( L - \frac{1}{2} l_1 \right) + \frac{1}{2} w_2 l_1 (L - l_1)^2 \right)\) into row 3 of vector \(\{K_2\}\).

This formulation was tested within the MatLab program and validated by degenerating the stepped beam into a uniform beam as before. With the compliance \(c\) set to zero the beam behaved as a clamped-clamped beam and with it set to any large value greater than 0.5°/Nm it behaved as a simply supported beam.

Figure 8. Stepped beam with angular misalignment. Loading by self-weight and axial force.
The MatLab program was used to investigate the effect of moderate compliance on the behaviour of the misaligned 5m × 200mm diameter stepped cylinder with clamped-clamped ends. As might be expected, the compliance tends to influence the seal loading towards that in the simply supported case. For large misalignment, where the alignment angle is greater than the natural angle taken up in the simply supported case (0.4° for the case studied), this reduces the loading on the seal, but gives little improvement below 10⁻³ degree/Nm. For small misalignment less than the simply supported slope, the increase in seal load is modest (<7%) for compliance below 10⁻⁵ degree/Nm.

For this case of the 200mm diameter cylinder considered here, a compliance of 10⁻⁵ degree/Nm is very small (i.e. it is very stiff), being provided by a 400mm diameter 12mm thick steel plate clamped around its rim, or a 15mm thick plate simply supported. 10⁻³ degree/Nm is still quite small (i.e. quite stiff), being provided by a 650mm diameter 5mm thick steel plate with clamped support.

6. Discussion of results
It is clear from the results presented that a significant reduction in loading on the seals can be obtained by the use of clamped ends, providing the angular misalignment can be kept small and the clamping is stiff (low compliance).

If the compliance cannot be kept low, then clamping might still be a lower cost solution than pined joints and would still give a somewhat reduced seal loading.

If angular alignment cannot be maintained, then a pinned solution will always be required.

7. Conclusions
1. An iterative method has been established to analyse the internal loading and distortion of a stepped beam loaded with combined self weight and axial force.
2. It is demonstrated that a significant reduction in maximum bending moment is obtained by constraining the ends (clamped-clamped support).
3. This reduced bending moment is translated directly into reduced radial seal loading in hydraulic cylinders compared with the conventional pin jointed ends.
4. This seal load reduction is largely independent of offset misalignment.
5. The load reduction is highly sensitive to angular misalignment, increasing to that in the pin-pin cylinder when the misalignment angle equals the slope taken up in the pin-pin case, and exceeding it at higher misalignment angles.
6. Compliance in the clamped supports increases the seal loads at small misalignment angles, but small compliance coupled with small angular misalignment gives significantly reduced seal loading.

References
[1] Ryder G H 1969 Strength of Materials MacMillan
[2] Biran Adrian and Breiner Moshe 1995 MATLAB for Engineers Adison-Wesley
[3] Young Warren C 1989 Roark’s Formulas for Stress and Strain (6th edition) McGraw-Hill