Quantum many-body attractor with strictly local dynamical symmetries

Berislav Buča, Archak Purkayastha, Giacomo Guarnieri, Mark T. Mitchison, Dieter Jaksch, and John Goold

1 Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom
2 School of Physics, Trinity College Dublin, College Green, Dublin 2, Ireland

(Dated: August 26, 2020)

Dynamical symmetries are algebraic constraints on quantum dynamical systems, which are often responsible for persistent temporal periodicity of observables. In this work, we discuss how an extensive set of strictly local dynamical symmetries can exist in an interacting many-body quantum system. These strictly local dynamical symmetries lead to spontaneous breaking of continuous time-translation symmetry, i.e. the formation of extremely robust and persistent oscillations when an infinitesimal time-dependent perturbation is applied to an arbitrary initial (stationary) state. Observables which do not overlap with the local (dynamical) symmetry operators can relax, losing memory of their initial conditions. The remaining observables enter highly robust non-equilibrium limit cycles, signaling the emergence of a non-trivial quantum many-body attractor. We provide an explicit recipe for constructing Hamiltonians featuring local dynamical symmetries. As an example, we introduce the XYZ spin-lace model, which is a model of a quasi-1D quantum magnet.

Introduction— In a stark rebuttal of the reductionist hypothesis of science, the late P. W. Anderson, in his influential essay ‘More is different’ [1], discussed the importance of broken symmetry for the emergence of fundamental physical laws. Time periodicity is one such symmetry-breaking phenomenon, which Anderson remarked “is either universal or surprisingly common” in complex systems and conjectured that temporal regularity could be Nature’s “means of handling information, similar to information bearing spatial regularity”. Almost half a century since Anderson’s manifesto, we are seeing an explosion of interest in the emergence of periodicity in the long-time dynamics of complex quantum systems. In particular, advances in quantum simulation, e.g. in ultra-cold atomic systems, now allow for the study of coherent quantum dynamics over timescales that are unprecedented in traditional condensed matter physics [2–4].

The generic behaviour of most quantum many-body systems is that local observables eventually reach thermal equilibrium, as formalised by the eigenstate thermalisation hypothesis [5–9]. However, it is known that if a system possesses some extensive set of local or quasi-local conserved quantities then the system will non thermalise, but will instead relax to a thermal-like stationary state, called a generalised Gibbs ensemble [10–13]. The latest experiments have shown that integrability breaking perturbations can be used to controllably alternate between these two behaviours [14].

In this work we explore another type of asymptotic dynamics in complex quantum many-body systems where certain observables, rather than relax, display temporal periodicity that is stable to local perturbations and changes in initial conditions. In addition, these systems may also have other observables which relax to stationarity. Generic observables which have finite overlap with the above two initially partially relax then enter a robust limit cycle, which we term a quantum many-body attractor. This behaviour, which is one of the hallmarks of complex attractive long-time dynamics found in Nature, is in clear contrast with relaxation dynamics expected for generic macroscopic quantum many-body systems and simple persistent oscillations set by the initial value seen in isolated few-body (or non-interacting) quantum systems. We provide an algebraic microscopic prescription for such a robust and spontaneous breaking of time-translation symmetry to occur and then show that it can arise in an explicit model of a quantum many-body system. Periodic long-time dynamics in quantum many-body systems is now under scrutiny in a variety of different scenarios such as time crystals [15–43], models with quantum many-body scars [44–59] and confinement [60–65]. The role of dynamical symmetries [18] in the appearance of periodic dynamics has recently been discussed [19, 66–70], as well as their weaker versions [55–57, 59, 71]. Here we show that the presence of an extensive set of strictly local dynamical symmetries can lead to spontaneous breaking of time translation symmetry and extraordinarily robust time-periodicity of observables.

Dynamical symmetries— Consider a quantum many-body system on an arbitrary lattice of \(L\) sites, in which there is no site which is disconnected from the rest of the system. A dynamical symmetry is the property

\[
[\hat{H}, \hat{E}] = \omega \hat{E},
\]

which defines an eigenoperator \(\hat{E}\) of the Hamiltonian \(\hat{H}\). Suppose that we take an out of equilibrium initial state \(\hat{\rho}\), i.e., \([\hat{\rho}, \hat{H}] \neq 0\). The expectation value of the operator \(\hat{E}\) will then oscillate periodically in time:

\[
\langle \hat{E}(t) \rangle = e^{i\omega t} \langle \hat{E}(0) \rangle.
\]

An exponentially large set of eigenoperators can always trivially be found, yet generally such an \(\hat{E}\) is completely non-local in the site basis of the lattice. If the system is macroscopic, \(L \to \infty\), such operators are however not experimentally accessible. The measurable quantities are
extensive sums of strictly local operators. A strictly local operator $\hat{O}$ is one that has support only on a nonextensive number of contiguous sites of the lattice. The situation becomes non-trivial when a strictly local eigenoperator of the Hamiltonian can be found, such that

$$[\hat{H}, \hat{A}] = \omega_A \hat{A}, \quad \hat{A} \to \text{strictly local.}$$

(3)

In this case, the system is said to possess a strictly local dynamical symmetry. Since $\hat{A}$ is supported on only few sites of the system, we must have $\hat{H} = \hat{H}_A + \hat{H}'_A$ where

$$[\hat{H}_A, \hat{A}] = \omega_A \hat{A}, \quad [\hat{H}'_A, \hat{A}] = 0.$$  \hspace{1cm} (4)

To ensure the subparts are not disconnected, we have

$$[\hat{H}_A, \hat{H}'_A] \neq 0.$$  \hspace{1cm} (5)

Here, $\hat{H}_A$ is a local Hamiltonian defined on a few contiguous sites of the lattice, such that its support is strictly larger than that of $\hat{A}$ (see Fig. 1(a)). That is, the support of $\hat{A}$ does not include the boundary of the support of $\hat{H}_A$. On the contrary, $\hat{H}'_A$ shares the boundary of its support with $\hat{H}_A$, and also covers the rest of the lattice. By virtue of strict locality, the persistent oscillations associated with $\hat{A}$ are protected from any change in $\hat{H}'_A$. Furthermore, it can be easily checked that $A^\dagger \hat{A}$ is a strictly local conserved quantity of the system, i.e.,

$$[\hat{H}, A^\dagger \hat{A}] = [\hat{H}_A, A^\dagger \hat{A}] = 0.$$  \hspace{1cm} (6)

This oscillation with frequency $\omega_A$, as well as the conserved quantity $A^\dagger \hat{A}$, are properties of the strictly local constituent Hamiltonian $\hat{H}_A$ which are preserved even in the macroscopic system. Eq 3 gives a quadratic set of equations that may be solved to generate candidate $\hat{H}_A$ and $\hat{A}$ provided that we demand it holds for an arbitrary $\hat{H}'_A$.

Since $\hat{A}$ is not supported on the boundary of $\hat{H}_A$, we can take several such local Hamiltonians and connect them via the boundary sites, thereby building a macroscopic system (see Fig. 1(b)). Such a system will have an extensive number of strictly local dynamical symmetries. The full Hamiltonian of such a system can be written as a sum of strictly local Hamiltonians on a lattice $\hat{H} = \sum_p \hat{H}_p$, each $\hat{H}_p$ being associated with its own local dynamical symmetry $\hat{A}_p$.

$$\hat{H} = \lim_{L \to \infty} \sum_{p=-fL}^{fL} \hat{H}_p, \quad [\hat{H}_p, \hat{H}_{p+1}] \neq 0, \quad 0 \leq f \leq 1,$$

$$[\hat{H}_p, \hat{A}_q] = \omega_A \hat{A}_q \delta_{pq},$$  \hspace{1cm} (7)

where $\delta_{pq}$ is the Kronecker delta symbol. We stress that we are restricting to lattices in which there are no sites that are disconnected from the rest of the system. As we discuss below, this will result in relaxation of some local observables, which provides a clear distinction between the models we study and a bunch of disconnected small systems. Satisfying Eq. (6) in such a lattice is highly non-trivial. Remarkably, as we show later, by explicit example, Eq. (6) can be satisfied in a realistic physical system.

This macroscopic system possesses an extensive number of strictly local operators that will persistently oscillate at frequency $\omega_A$ and never relax to a steady value. Observables that have overlap with these operators will undergo partial relaxation and be lead to a limit cycle at long times. Note that each $\hat{A}_q$ is associated with the frequency $\omega_A$. Further, an arbitrary change in any local Hamiltonian $\hat{H}_p \to \hat{H}'_p$, can, at worst, destroy the local dynamical symmetry associated with it, i.e, so that $[\hat{H}'_p, \hat{A}_q] \neq \omega_A \hat{A}_q$, leaving all the other strictly local dynamical symmetries unaffected. This is also true if one or more of the sites in the support of $\hat{H}_p$ is coupled to an arbitrary environment. Therefore, the persistent oscillations are extremely robust.

Spontaneously broken continuous time-translation symmetry— Even if the initial state is stationary, i.e, $[\hat{\rho}, \hat{H}] = 0$, an arbitrarily small perturbation will initiate persistent oscillations. This is one way of understanding spontaneous breaking of continuous time translational symmetry. According to the theory of linear response, the change in an observable $\hat{O}$, after a small instantaneous perturbation over the a stationary state via a perturbing Hamiltonian $\hat{H}_{\text{pert}}$,

$$\hat{H}_{\text{tot}} = \hat{H} + \epsilon \delta(t) \hat{H}_{\text{pert}},$$  \hspace{1cm} (8)

is given by

$$\frac{\delta O(t)}{\epsilon} = \frac{\langle \hat{O}_{\text{tot}}(t) \rangle - \langle \hat{O}(0) \rangle}{\epsilon} = -i \langle [\hat{\hat{\hat{O}}}(t), \hat{H}_{\text{pert}}] \rangle.$$  \hspace{1cm} (9)

Here $\epsilon \ll 1$ is the strength of the perturbation, $\hat{O}_{\text{tot}}(t) = e^{i\hat{H}_{\text{tot}}} \hat{O} e^{-i\hat{H}_{\text{tot}}}$, $\hat{O}(t) = e^{i\hat{H}_0 t} \hat{O} e^{-i\hat{H}_0 t}$, and $\delta(t)$ is the Dirac delta function. The response of the operator $\hat{A}_p$ is then, by construction, given by

$$\frac{\delta \hat{A}_p(t)}{\epsilon} = -ie^{-i\omega_A t} \langle [\hat{\hat{\hat{A}}}_p, \hat{H}_{\text{pert}}] \rangle.$$  \hspace{1cm} (10)
Hence, an arbitrarily small perturbation which satisfies \(\langle [\hat{A}_p, \hat{H}_{\text{pert}}] \rangle \neq 0\) will lead to persistent oscillations in \(\hat{A}_p\) with frequency \(\omega_A\). It is not just the operators \(\hat{A}_p\), and their extensive sums that will show these oscillations. It can be shown that any operator \(\hat{O}\) which has overlap with the \(\hat{A}_p\), i.e., which satisfies, \(\langle \hat{O}\hat{A} \rangle \neq 0\), will also inherit these oscillations with frequency \(\omega_A\) [19, 60]. These observables will partially lose the memory of initial conditions, while retaining complex dynamics in the long-time limit and show a fully robust limit cycle-like behavior that may be best described as an emerging quantum many-body attractor.

It is important to understand why our system is not a continuous time crystal according to the definition put forward in Ref. [72]. Our system does not have long-range spatial order in the sense of \(\lim_{|p-q| \to \infty} \lim_{L \to \infty} \langle \hat{O}_p(t)\hat{O}_q \rangle = f(t)\), where \(\hat{O}_p\) is a strictly local operator around site \(p\), \(f(t)\) is a periodic function of time and \(\langle \ldots \rangle\), refers to the connected part of the correlation function. However, this property is typically required in order to distinguish the system from a collection of disconnected small systems, e.g., a collection of isolated spins. In our system, there exist local observables that do not overlap with any conserved quantities or dynamical symmetry operators, i.e., \(\langle \hat{O}\hat{A} \rangle = \langle \hat{O}\hat{A}^\dagger\hat{A} \rangle = 0\). In the absence of any further conservation laws, their response to an instantaneous perturbation over a stationary state relaxes to zero,

\[
\lim_{t \to \infty} \frac{\delta \hat{O}(t)}{\epsilon} = -i \lim_{t \to \infty} \langle \{\hat{O}(t), \hat{H}_{\text{pert}}\} \rangle = 0. \tag{10}
\]

Thus, following the perturbation, the expectation value of these operators returns to its original value. This behaviour, which is expected in generic non-integrable quantum systems, explicitly distinguishes the system we have in mind from a trivial bunch of disconnected spins.

The system we are considering here also has strictly local charges, given by \(\hat{A}^\dagger_p\hat{A}_p\). They exist as a consequence of the strictly local dynamical symmetries and, like the dynamical symmetries, they are robust against any local change in the Hamiltonian, no matter how strong. The response of \(\hat{A}^\dagger_p\hat{A}_p\) to an arbitrary instantaneous perturbation over a stationary state is given by

\[
\delta \langle \hat{A}^\dagger_p\hat{A}_p \rangle(0) = -i \langle \{\hat{A}^\dagger_p\hat{A}_p, \hat{H}_{\text{pert}}\} \rangle. \tag{11}
\]

Therefore, so long as \(\langle \{\hat{A}^\dagger_p\hat{A}_p, \hat{H}_{\text{pert}}\} \rangle \neq 0\), there will be a constant non-zero response. This behavior will also be captured by any operator \(\hat{O}\) which has overlap with \(\hat{A}^\dagger_p\hat{A}_p\), \(\langle \hat{O}\hat{A}^\dagger_p\hat{A}_p \rangle \neq 0\). The response of such operators will show fluctuations about a constant value in the long time limit. Thus, following a small instantaneous perturbation, the expectation values of these operators will never relax to their values before the perturbation. This is similar to the behaviour seen in integrable systems. Indeed, since the dynamical symmetries and their corresponding charges are strictly local, models generated using the above conditions are essentially quasi-(super)integrable [73]. Despite this, they are also robustly stable to local perturbations, unlike standard quantum integrability [74–77]. Furthermore, due to the purely local nature of the charges, they are also partially localized [78]. These non-trivial and interesting properties deserve a fully detailed investigation which we delegate to future works.

**Example: The XYZ spin lace model** — We are now in a position to construct an explicit spin model that exhibits an extensive number of strictly local dynamical symmetries. This class of models is defined on a quasi-one-dimensional lattice, depicted in Fig. 2. We label the sites in the lattice as shown in the figure. The red sites, which we call nodes, are labelled by odd integers, \(2r-1\). The next adjacent green sites, are labelled by \(2r, 1\) and \(2r, 2\), where the second index refers to top and bottom sites respectively. We call a top and bottom pair a double-site. We define the total spin operators on the top and the bottom green sites,

\[
\hat{S}^x_{2r,1} = (\hat{\sigma}^x_{2r,1} + \hat{\sigma}^x_{2r,2}), \tag{12}
\]

where \(\sigma^x, y, z\) are the standard Pauli spin operators. The general spin lace model is given by the following Hamiltonian,

\[
\hat{H} = \sum_{r=-\infty}^{\infty} \left( \hat{h}_{2r-1} + \hat{h}_{2r} + \hat{h}_{2r-1,2r} + \hat{h}_{2r-1,2r-2} \right),
\]

\[
\hat{h}_{2r-1} = B\hat{\sigma}^z_{2r-1}, \quad \hat{h}_{2r} = B_\sigma \hat{S}^z_{2r}, \quad \hat{h}_{2r-1,2r} = J_r \hat{\sigma}^x_{2r-1} \hat{S}^x_{2r} + J^y_r \hat{\sigma}^y_{2r-1} \hat{S}^y_{2r} + J^z_r \hat{\sigma}^z_{2r-1} \hat{S}^z_{2r}, \tag{13}
\]

In order to write down the strictly local dynamical symmetries, we define the projection operator into the singlet sector of the top and bottom green sites,

\[
\hat{P}_{2r} = \frac{1}{2} \left( |\uparrow\downarrow\rangle_{2r} - |\downarrow\uparrow\rangle_{2r} \right) \left( (|\uparrow\downarrow\rangle_{2r} - (|\downarrow\uparrow\rangle_{2r}) \right),
\]

\[
= \frac{1}{2} \left( |I_4 - \hat{\sigma}^z_{2r,1} \hat{\sigma}^z_{2r,2} - \hat{\sigma}^y_{2r,1} \hat{\sigma}^y_{2r,2} - \hat{\sigma}^x_{2r,1} \hat{\sigma}^x_{2r,2} \right), \tag{14}
\]

where \(I_4\) is the identity operator on the two sites. The strictly local dynamical symmetries can now be written
and the associated local Hamiltonian is given by the Hamiltonian of two adjacent plaquettes,
\[ \hat{H}_p = \hat{h}_{2r-1} + \hat{h}_{2r} + h_{2r-1,2r} + \hat{h}_{2r-1,2r-2} + \hat{h}_{2r-3,2r-2} + \hat{h}_{2r+1} + \hat{h}_{2r+1,2r}. \]

The supports of \( \hat{A}_p \) and \( \hat{H}_p \) are shown in Fig. 2. The index \( p \) refers to the two plaquettes around the red site \( 2r - 1 \). The dynamical symmetry is given by
\[ [\hat{H}_p, \hat{A}_p] = 2B \hat{A}_p. \]

and the properties enforced by Eq.(6) are satisfied. In fact the system satisfies Eq.(6) even when all couplings \( J^{x,y,z}_r \) and all fields at the double-sites, \( B_{2r} \) are different. As long as each node has the same field \( B \), a class of macroscopic experimental observables show persistent oscillations with a single frequency. If each node has a different field, then the collective dynamics of a sum of local operators instead shows complex behavior due to the interference of multiple frequencies. However, even then, the local dynamical symmetries are preserved. The only two ways to destroy a local dynamical symmetry in this model are (i) to destroy the parity between the top and the bottom of a double site, for example, by changing the magnetic field on the top site, or (ii) by connecting the middle node in the doubled plaquette with the leftmost/rightmost one. Nevertheless, as mentioned before, only the local dynamical symmetry in the associated two plaquettes would be destroyed, leaving all others unchanged. If an extensive number of nodes have the same field while others have different fields, then a class of macroscopic observables show predominant contributions from a single frequency and the system still behaves like a non-trivial quantum many-body attractor.

We now focus on an ordered XYZ spin-lace model with
\[ B_{2r} = B, \quad J^{x,y,z}_r = J^{x,y,z}, \]
and consider the response of the system to a small instantaneous kick in the \( x \)-direction at one of the nodes (red sites in Fig. 2) over the thermal state, \( \rho = e^{-\beta H}/Z \), of the system,
\[ \hat{H}_{pert} = \epsilon \hat{\sigma}_{2r-1}^x. \]

Thus, the behavior of the correlation function \( \langle \hat{O}(t)\hat{\sigma}_{2r-1}^x \rangle \) governs the linear response of the observable \( \hat{O} \). We will look at such correlation functions for some strictly local operators. From the expression for \( \hat{A}_p \) (Eq.(14)), it can be seen that \( \langle \hat{A}_p \hat{\sigma}_{2r-1}^x \rangle \neq 0 \), i.e., \( \hat{A}_p \) has overlap with \( \hat{\sigma}_{2r-1}^x \). So, we analytically know that \( \langle \hat{A}_p(t)\hat{\sigma}_{2r-1}^x \rangle \) will show persistent oscillations with frequency \( 2B \). Due to the overlap with \( \hat{A}_p \), these oscillations will also be picked up in \( \langle \hat{\sigma}_{2r-1}^x(t)\hat{\sigma}_{2r-1}^x \rangle \). However, in these observables the oscillations will not be perfect due to the presence of other frequencies. Conversely, the operator \( \hat{S}_{2r}^z \) has no overlap with either \( \hat{A}_p \) or \( \hat{A}_p^\dagger \). As a result, the correlation function \( \langle \hat{S}_{2r}^z(t)\hat{\sigma}_{2r-1}^x \rangle \) relaxes to zero. Plots of \( \langle \hat{\sigma}_{2r}^z(t)\hat{\sigma}_{2r-1}^z \rangle \), \( \langle \hat{A}_p(t)\hat{\sigma}_{2r}^z \rangle \) and \( \langle \hat{S}_{2r}^z(t)\hat{\sigma}_{2r-1}^z \rangle \) are shown in Fig. 3(a).

In order to analyze the results in more detail, we look at the frequency response from the finite-time results. To this end, we define
\[ F_{AB}(\omega) = \frac{1}{T} \int_0^T dt e^{i\omega t} \langle \hat{A}(t)\hat{B} \rangle, \]
where \( T \) is the maximum time for the numerical simulation. In Fig. 3(b), we show plots of \( F_{\sigma_{2r-1}^x\sigma_{2r-1}^z(\omega)} \) and \( F_{\hat{S}_{2r}^z\sigma_{2r-1}^z(\omega)} \). In \( F_{\hat{S}_{2r}^z\sigma_{2r-1}^z(\omega)} \), the major peak at \( 2B \) is completely clear. The presence of several other subleading frequencies is also clear. In \( F_{\sigma_{2r-1}^x\sigma_{2r-1}^z(\omega)} \), there is no peak at \( 2B \). There is a small contribution from
a range of frequencies about that value which may be a numerical artifact. However, due to the highly integrable nature of the model, we cannot completely rule out existence of some other non-trivial dynamical symmetries, and conserved charges. Nevertheless, the spontaneous time-translation symmetry breaking in one class of local observables, while relaxation in another class of local observables, is completely clear and is a robust signature leading to a quantum many-body attractor.

**Conclusion** — The existence of an extensive set of strictly local dynamical symmetries guarantee that a quantum subsystem may display robust quantum coherent dynamics. We gave a simple example of such a system, the spin-lace model. The lattice of the spin-lace model is similar to existing quasi-one-dimensional magnetic materials with tetramer unit cells [79–81] and the diamond lattice compounds much studied for their interesting low-temperature properties, e.g. [82–89].

Remarkably, due to their strictly local nature, the dynamical symmetries are completely stable to all perturbations of arbitrary strength (including dissipative ones) with the only caveat that the perturbation does not act everywhere in the system (i.e. not on every plaquette). Those observables that have overlap with the dynamical symmetries enter fully robust limit cycles that may be best described as quantum many-body attractors due to loss of memory of their initial conditions while retaining complex dynamics in the long-time limit. Due to their simple definition and remarkable stability, we postulate that such quantum many-body attractors may arise naturally in strongly interacting quantum many-body systems and that their existence could be the fundamental microscopic explanation for the emergence of temporal regularity in complex structures, hinted at by Anderson [1], and which is so universal in all living things.

**Acknowledgements** — BB warmly acknowledges V. Jukić Buča for the name 'spin lace'. BB thanks M. Medenjak for useful discussions and collaboration on related work. We acknowledge support from the European Research Council Starting Grant ODYSSEY (G. A. 758403). JG would like to thank A. Silva and M. Dalmonte for discussions. BB and DJ acknowledge funding from EPSRC programme grant EP/P009565/1, EPSRC National Quantum Technology Hub in Networked Quantum Information Technology (EP/M013243/1), and the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement no. 319286, Q-MAC. JG is supported by a SFI-Royal Society University Research Fellowship. AP acknowledges funding from European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 890884.

[1] P. W. Anderson, Science **177**, 393 (1972).
[2] T. Langen, R. Geiger, and J. Schmiedmayer, Annu. Rev. Condens. Matter Phys. **6**, 201 (2015).
[3] I. Bloch, J. Dalibard, and S. Nascimbene, Nat. Phys. **8**, 267 (2012).
[4] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen, and U. Sen, Adv. Phys. **56**, 243 (2007).
[5] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
[6] M. Srednicki, Phys. Rev. E **50**, 888 (1994).
[7] M. Srednicki, J. Phys. A **32**, 1163 (1999).
[8] M. Rigol, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).
[9] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. **65**, 239 (2016).
[10] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).
[11] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007).
[12] F. H. L. Essler and M. Fagotti, J. Stat. Mech. (2016), 064002.
[13] F. H. L. Essler, G. Mussardo, and M. Panfil, Phys. Rev. A **91**, 051602(R) (2015).
[14] Y. Tang, W. Kao, K.-Y. Li, S.seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Phys. Rev. X **8**, 021030 (2018).
[15] F. Wilczek, Phys. Rev. Lett. **109**, 160401 (2012).
[16] K. Sacha and J. Zakrzewski, Reports on Progress in Physics **81**, 016401 (2017).
[17] D. V. Else, C. Monroe, C. Nayak, and N. Y. Yao, Annual Review of Condensed Matter Physics **11**, 467 (2020).
[18] B. Buča, J. Tindall, and D. Jaksch, Nature Communications **10**, 1730 (2019).
[19] M. Medenjak, B. Buča, and D. Jaksch, Phys. Rev. B **102**, 041117 (2020).
[20] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. **116**, 250401 (2016).
[21] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. Lett. **117**, 090402 (2016).
[22] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. Lett. **112**, 150401 (2014).
[23] P. Bordia, H. Lütschen, U. Schneider, M. Knap, and I. Bloch, Nature Physics **13**, 460 EP (2017).
[24] K. Sacha and J. Zakrzewski, Reports on Progress in Physics **81**, 016401 (2017).
[25] W. C. Yu, J. Tangpanitanon, A. W. Glaetzle, D. Jaksch, and D. G. Angelakis, Phys. Rev. A **99**, 033618 (2019).
[26] B. Zhu, J. Marino, N. Y. Yao, M. D. Lukin, and E. A. Demler, arXiv preprint arXiv:1904.01026 (2019).
[27] F. M. Gambetta, F. Carolina, A. Lazarides, I. Lesanovskii, and J. P. Garrahan, Phys. Rev. A **99**, 043618 (2019).
[28] K. Giorgi, A. Dauphin, M. Lewenstein, J. Zakrzewski, and K. Sacha, New Journal of Physics **21**, 052003 (2019).
[29] F. M. Gambetta, F. Carolina, M. Maruzzi, J. P. Garrahan, and I. Lesanovskii, Phys. Rev. Lett. **122**, 015701 (2019).
[30] K. Sacha and J. Zakrzewski, Reports on Progress in Physics **81**, 016401 (2017).
[31] D. A. Ivanov, T. Y. Ivanova, S. F. Caballero-Benítez, and I. B. Mekhov, Phys. Rev. Lett. **124**, 016003 (2020).
[32] G. Homann, J. G. Cosme, and L. Mathey, “Higgs time crystal in a high-$t_c$ superconductor,” (2020), arXiv:2004.13383 [cond-mat.supr-con].

[33] R. R. W. Wang, B. Xing, G. G. Carlo, and D. Poletti, Phys. Rev. E 97, 020202 (2018).

[34] E. I. R. Chiacchio and A. Nunnenkamp, Phys. Rev. Lett. 122, 193605 (2019).

[35] D. Barberena, R. J. Lewis-Swan, J. K. Thompson, and A. M. Rey, Phys. Rev. A 99, 053605 (2019).

[36] G. Homann, J. G. Cosme, and L. Mathey, arXiv preprint arXiv:1909.00266 (2019).

[37] K. Seibold, R. Rota, and V. Savona, Phys. Rev. A 101, 033839 (2020).

[38] A. Riera-Campeny, M. Moreno-Cardoner, and A. Sanpera, “Time crystallinity in open quantum systems,” (2019), arXiv:1908.11339 [quant-ph].

[39] H. Keler, J. G. Cosme, C. Georges, L. Mathey, and A. Hemmerich, Phys. Rev. A 99, 053605 (2019).

[40] O. Scarlatella, R. Fazio, and M. Schirò, Phys. Rev. B 99, 064511 (2019).

[41] H. Keler, J. G. Cosme, C. Georges, L. Mathey, and A. Hemmerich, “From a continuous to a discrete time crystal,” (2020), arXiv:2004.14633 [cond-mat.quant-gas].

[42] F. Minganti, I. I. Arkhipov, A. Miranowicz, and F. Nori, “Correspondence between dissipative phase transitions of light and time crystals,” (2020), arXiv:2008.08075 [quant-ph].

[43] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, Nature Physics 14, 745 (2018).

[44] S. Moudgalya, S. Rachel, B. A. Bernevig, and N. Regnault, Physical Review B 98 (2018), 10.1103/physrevb.98.235155.

[45] S. Choi, C. J. Turner, H. Pichler, W. W. Ho, A. A. Michailidis, Z. Papić, M. Serbyn, M. D. Lukin, and D. A. Abanin, Phys. Rev. Lett. 122, 220603 (2019).

[46] C.-J. Lin and O. I. Motrunich, Phys. Rev. Lett. 122, 173401 (2019).

[47] T. Iadecola and M. Schecter, Physical Review B 101 (2020), 10.1103/physrevb.101.243306.

[48] S. Moudgalya, N. Regnault, and B. A. Bernevig, Physical Review B 98 (2018), 10.1103/physrevb.98.235156.

[49] C. J. Turner, J.-Y. Desaules, K. Bull, and Z. Papić, “Correspondence principle for many-body scars in ultracold rydberg atoms,” (2020), arXiv:2006.13207 [quant-ph].

[50] M. Schecter and T. Iadecola, Physical Review Letters 123 (2019), 10.1103/physrevlett.123.147201.

[51] A. A. Michailidis, C. J. Turner, Z. Papić, D. A. Abanin, and M. Serbyn, Physical Review Research 2 (2020), 10.1103/physrevresearch.2.022065.

[52] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, Nature 551, 579 EP (2017).

[53] K. Bull, J.-Y. Desaules, and Z. Papić, “Quantum scars as embeddings of weakly "broken" lie algebra representations,” (2020), arXiv:2001.08232 [cond-mat.str-el].

[54] S. Moudgalya, N. Regnault, and B. A. Bernevig, “Eta-pairing in hubbard models: From spectrum generating algebras to quantum many-body scars,” (2020), arXiv:2004.13727 [cond-mat.str-el].

[55] D. K. Mark and O. I. Motrunich, “Eta-pairing states as true scars in an extended hubbard model,” (2020), arXiv:2004.13800 [cond-mat.str-el].

[56] D. K. Mark, C.-J. Lin, and O. I. Motrunich, “Unified structure for exact towers of scar states in the aklt and other models,” (2020), arXiv:2001.03839 [cond-mat.str-el].

[57] K. Pakrouski, P. N. Pallegar, F. K. Popov, and I. R. Klebanov, “Many body scars as a group invariant sector of hilbert space,” (2020), arXiv:2007.00845 [cond-mat.str-el].

[58] N. O’Dea, F. Burnell, A. Chandran, and V. Khemani, “From tunnels to towers: quantum scars from lie algebras and q-deformed lie algebras,” (2020), arXiv:2007.16207 [cond-mat.stat-mech].

[59] A. Lerose, F. M. Surace, P. P. Maaza, G. Perfetto, M. Collura, and A. Gambassi, Phys. Rev. B 102, 041118 (2020).

[60] A. Lerose, F. M. Surace, P. P. Maaza, G. Perfetto, M. Collura, and A. Gambassi, “Quasilocalized dynamics from confinement of quantum excitations,” (2019), arXiv:1911.07877 [cond-mat.stat-mech].

[61] A. C. Cubero and N. J. Robinson, “Lack of thermalization in (1+1)-d qcd at large $\epsilon_c$,” arXiv:1908.00270 [hep-th].

[62] S. Pai and M. Pretko, Physical Review Research 2 (2020), 10.1103/physrevresearch.2.013094.

[63] M. Kormos, M. Collura, G. Takács, and P. Calabrese, Nature Physics 13, 246 (2017).

[64] N. J. Robinson, A. J. James, and R. M. Konik, Physical Review B 99, 054303 (2019).

[65] M. Medenjak, T. Prosen, and L. Zadnik, “Rigorous bounds on dynamical response functions and time-translation symmetry breaking,” (2020), arXiv:2003.01035 [cond-mat.stat-mech].

[66] J. Tindall, C. S. Muñoz, B. Buča, and D. Jaksch, New Journal of Physics 22, 013026 (2020).

[67] K. Chinzei and T. N. Ikeda, “Time crystals protected by floquet dynamical symmetry in hubbard models,” (2020), arXiv:2003.13315 [cond-mat.stat-mech].

[68] B. Buča and D. Jaksch, Phys. Rev. Lett. 123, 260401 (2019).

[69] J. Tindall, C. S. Muñoz, B. Buča, and D. Jaksch, New Journal of Physics 22, 013026 (2020).

[70] K. Bull, J.-Y. Desaules, and Z. Papić, “Quantum scars as embeddings of weakly "broken" lie algebra representations,” (2020), arXiv:2001.08232 [cond-mat.str-el].

[71] V. Khemani, C. W. von Keyserlingk, and S. L. Sondhi, Phys. Rev. B 96, 115127 (2017).

[72] M. Fogatti, Journal of Statistical Mechanics: Theory and Experiment 2014, P03016 (2014).

[73] M. Brenes, T. LeBlond, J. Goold, and M. Rigol, Phys. Rev. Lett. 125, 070605 (2020).

[74] M. Znidaric, arXiv preprint arXiv:2006.09793 (2020).

[75] L. F. Santos, F. Pérez-Bernal, and E. J. Torres-Herrera, arXiv e-prints , arXiv:2006.10779 (2020), arXiv:2006.10779 [cond-mat.stat-mech].

[76] A. Smith, J. Knolle, D. L. Kovrizhin, and R. Moessner, Phys. Rev. Lett. 118, 266601 (2017).

[77] M. A. McGuire, V. O. Garlea, S. KC, V. R. Cooper, J. Yan, H. Cao, and B. C. Sales, Phys. Rev. B 95, 144421
[80] M. Hagiwara, Y. Narumi, A. Matsuo, H. Yashiro, S. Kimura, and K. Kindo, *New Journal of Physics* **8**, 176 (2006).

[81] H. Sakiyama, M. Kato, S. Sasaki, M. Tasaki, E. Asato, and M. Koikawa, *Polyhedron* **111**, 32 (2016).

[82] H. Kikuchi, Y. Fujii, M. Chiba, S. Mitsudo, T. Idehara, T. Tonegawa, K. Okamoto, T. Sakai, T. Kuwai, and H. Ohta, *Physical review letters* **94**, 227201 (2005).

[83] K. Rule, A. Wolter, S. Süllov, D. Tennant, A. Brühl, S. Köhler, B. Wolf, M. Lang, and J. Schreuer, *Physical review letters* **100**, 117202 (2008).

[84] F. Aimo, S. Krämer, M. Klanjšek, M. Horvatić, C. Berthier, and H. Kikuchi, *Physical review letters* **102**, 127205 (2009).

[85] K. Morita, M. Fujihala, H. Koorikawa, T. Sugimoto, S. Sota, S. Mitsuda, and T. Tohyama, *Phys. Rev. B* **95**, 184412 (2017).

[86] Y. Mizuno, T. Tohyama, S. Maekawa, T. Osafune, N. Motoyama, H. Eisaki, and S. Uchida, *Physical Review B* **57**, 5326 (1998).

[87] H. Jeschke, I. Opahe, H. Kandpal, R. Valentí, H. Das, T. Saha-Dasgupta, O. Janson, H. Rosner, A. Brühl, B. Wolf, *et al.*, *Physical review letters* **106**, 217201 (2011).

[88] M. Fujihala, H. Koorikawa, S. Mitsuda, K. Morita, T. Tohyama, K. Tomiyasu, A. Koda, H. Okabe, S. Itoh, T. Yokoo, *et al.*, *Scientific reports* **7**, 1 (2017).

[89] G. Radtke, A. Saúl, H. A. Dabkowska, M. B. Salamon, and M. Jaime, *Proceedings of the National Academy of Sciences* **112**, 1971 (2015).