Chiral theory of $\rho$-meson gravitational form factors

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The low-energy chiral effective field theory of vector mesons and Goldstone bosons in external gravitational field is considered. The energy-momentum tensor is obtained and the gravitational form factors of the $\rho$-meson are calculated up to next-to-leading order in the chiral expansion. This amounts to considering tree-level and one-loop order diagrams. The chiral expansion of the form factors at zero momentum transfer as well as of the slope parameters is worked out. Also, the long-range behaviour of the energy and internal force distributions is obtained and analysed.

During the preparation of this paper Maxim Polyakov passed away.
We dedicate this paper to the memory of Maxim.

I. INTRODUCTION

The linear response of a hadron to a change of the background space-time metric is described by the gravitational form factors (GFFs). For the first time, the GFFs for spin-0 and spin-1/2 particles were introduced and discussed in detail in Refs. [1, 2], for spin-1 particles in Ref. [3], and for hadrons with arbitrary spin in the recent work of Ref. [4]. The GFFs contain rich information about the internal structure of hadrons, such as the distribution of the spin [5], the energy distribution [6], as well as the elastic pressure and shear force distributions [7]. For recent reviews see Refs. [8, 9].

Our aim here is to study the GFFs of the spin-1 $\rho$-meson in chiral effective field theory (EFT). Hadrons with spin $S > 1/2$ are not spherically symmetric. The spin, energy, and force distributions acquire higher multipole components (quadrupole, etc.) [4, 10–12]. The higher multipole energy and force distributions carry valuable information about the mechanisms of the hadron’s binding. For example, the large-$N_c$ picture of baryons as chiral solitons implies certain relations between the quadrupole energy and the force distributions [11, 13]. Experimental checks of these relations would allow one to reveal the nature of higher spin baryons.

The GFFs of the $\rho$-meson were computed in the light-cone constituent quark model [14]. More recently, the gluon part of the GFFs was obtained in lattice QCD calculations [15]. Here, we investigate the dependence of the $\rho$-meson GFFs on the soft scales (pion mass, small momentum transfer) using chiral EFT. To this end, following the logic of Ref. [16], we first write down the chiral effective action for the $\rho$- and $\omega$-mesons and pions in an external gravitational field. Next we obtain the corresponding energy-momentum tensor (EMT) and compute the chiral corrections to the GFFs of the $\rho$-meson. The corresponding calculation, in particular, allows us to obtain the large distance behaviour of the energy and force distributions. The results of our study can be also used in chiral extrapolations of the lattice-QCD simulations down to the physical values of the pion masses.

Chiral EFTs with heavy degrees of freedom encounter a non-trivial power-counting problem [17]. In the one-nucleon sector of baryon chiral perturbation theory this problem can be solved by applying the heavy-baryon approach [18, 19] or a suitably chosen renormalization condition [20–23]. Because of the small nucleon-delta mass difference, the $\Delta$ resonance can also be consistently included in the framework of EFT [24–27].

The treatment of the $\rho$ meson in chiral EFT is complicated as it decays in two pions with masses that vanish in the chiral limit. Because of this, for energies of the order of the $\rho$-meson mass, loop diagrams develop large imaginary parts. In distinction to the baryonic sector, these large power-counting-violating contributions cannot be absorbed
in the redefinition of the parameters of the Lagrangian as long as the usual renormalization procedure is used. Still, the problem can be handled [28] by using the complex-mass renormalization scheme [29, 30], which is an extension of the on-mass-shell renormalization scheme to unstable particles. For more details on and different approaches to these problems, see e.g. Refs. [31–39].

Our work is organized as follows. In Section II we write down the action corresponding to the effective Lagrangian up to next-to-leading order and obtain the pertinent EMT. In Section III we briefly discuss the renormalization and the power counting. The definition and the calculations of the GFFs of the $\rho$ meson are presented in Section IV. It also contains various chiral expansions of the obtained results. Section V is devoted to the discussion of the energy and the force distributions, and we summarize the obtained results in Section VI.

II. EFFECTIVE LAGRANGIAN AND THE ENERGY-MOMENTUM TENSOR

Using the results of Refs. [28, 40] we consider the following action of $\rho$ and $\omega$ mesons and pions using the parametrization of the model III of Ref. [31] (where the $\rho$-meson vector fields transform inhomogeneously under chiral transformations), interacting with an external gravitational field $g^{\mu\nu}$:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{F^2}{4} g^{\mu\nu} \mathrm{Tr}(D_\mu U(D_\nu U)^\dagger) + \frac{F^2}{4} \mathrm{Tr}(\chi U^\dagger + U \chi^\dagger) - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \mathrm{Tr}(\rho_{\alpha\mu} \rho_{\nu\beta}) + g^{\mu\nu} \left[ M^2_R + \frac{c_x}{4} \mathrm{Tr}(\chi U^\dagger + U \chi^\dagger) \right] \mathrm{Tr} \left( \rho_{\mu\nu} - i \frac{\Gamma_{\mu\nu}}{g} \right) \right\} \mathrm{Tr} \left( \rho_{\nu\mu} - i \frac{\Gamma_{\nu\mu}}{g} \right),$$

where

$$U = u^2 = \exp \left( \frac{i \vec{r} \cdot \vec{r}}{F} \right),$$

$$\rho^\mu = \frac{\bar{\rho}^\mu}{2},$$

$$\rho^{\mu\nu} = \partial^\mu \rho^\nu - \partial^\nu \rho^\mu - ig [\rho^\mu, \rho^\nu],$$

$$\chi = 2B_0(s + ip),$$

$$D_\mu U = \partial_\mu U - ir_\mu U + iU l_\mu,$$

$$F_{\mu\nu} = \frac{1}{2} \left[ u^\dagger \partial_\mu u + u \partial_\mu u^\dagger - i \left( u^\dagger r_\mu u + u l_\mu u^\dagger \right) \right],$$

$$u_{\mu} = i \left[ u^\dagger \partial_\mu u - u \partial_\mu u^\dagger - i(u^\dagger r_\mu u - u l_\mu u^\dagger) \right].$$

Terms involving the Riemann-tensor $R^{\mu\nu\rho\sigma}$, the Ricci-tensor $R^{\mu\nu}$ and the Ricci scalar $R$ (for definitions of these quantities see, e.g., Ref. [41]) are those with non-minimal coupling of the $\rho$-meson fields to gravity, which are relevant for the considered order of accuracy. The parameter $B_0$ is proportional to the scalar vacuum condensate and $s, p, l_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$ are external sources, while $F$ denotes the pion-decay constant in the chiral limit. Further, $M^2_R$ and $M^2_{\omega\rho\pi}$ are the (complex) pole positions of $\rho$ and $\omega$ propagators in chiral limit, and the $v_i \ (i = 1, \ldots, 6)$, $g, c_x$, and $g_{\omega^2\rho\pi}$ are coupling constants. For the $\rho\pi\pi$ coupling we use [42]

$$M^2_R = a g^2 F^2,$$

which in the case of $a = 2$ amounts to the KSFR relation [43–45]. Although phenomenologically $a \simeq 2$, in what follows we will keep this parame parameter explicitly. All parameters of the effective Lagrangian are to be interpreted as renormalized ones. We apply the complex-mass scheme and do not show counterterms explicitly, however, their contributions are taken into account in calculations of the quantum corrections to the physical quantities.
Applying the standard formula for the EMT of matter fields interacting with the metric fields [41] we obtain in flat spacetime:

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \delta S_{\mu\nu}. \] (4)

to the action of Eq. (1) we obtain:

\[
T_{\mu\nu} = \frac{F^2}{4} \text{Tr}(D_\mu U(D_\nu U)^\dagger + D_\nu U(D_\mu U)^\dagger) \\
- \eta_{\mu\nu} \left\{ \frac{F^2}{4} \text{Tr}(D^\alpha U(D_\alpha U)^\dagger) + \frac{F^2}{4} \text{Tr}(U^\dagger + U)^\dagger \right\} \\
- 2 \eta^{\alpha\beta} \text{Tr}(\rho_{\alpha\beta} \rho_{\nu\beta}) + 2 \left[ M_\rho^2 + \frac{c_\rho}{g} \text{Tr}(U^\dagger + U)^\dagger \right] \text{Tr}\left( \rho_\mu - \frac{i \Gamma_\mu}{g} \right) \left( \rho_\nu - \frac{i \Gamma_\nu}{g} \right) \\
- \eta^{\alpha\beta} (\partial_\mu \omega_\alpha - \partial_\nu \omega_\alpha) (\partial_\nu \omega_\beta - \partial_\mu \omega_\beta) + M_\rho^2 \omega_\mu \omega_\nu, \\
- \eta_{\mu\nu} \left\{ -\frac{1}{4} (\partial_\mu \omega_\alpha - \partial_\nu \omega_\alpha) (\partial^\alpha \omega^\beta - \partial^\beta \omega^\alpha) + \frac{M_\rho^2 \omega_\alpha \omega_\alpha}{2} \right\} \\
+ 2(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \left\{ [v_1 + v_2 \text{Tr}(U^\dagger + U)^\dagger] \text{Tr}(\rho_\alpha \rho_\alpha) + v_4 \text{Tr}(\rho_{\alpha\beta} \rho_{\beta\alpha}) \right\} \\
+ (\eta_{\mu\nu} \eta_{\rho\sigma} \partial^2 + \eta_{\mu\nu} \partial_\alpha \partial_\beta - \eta_{\rho\sigma} \partial_\alpha \partial_\beta - \eta_{\nu\sigma} \partial_\alpha \partial_\beta) [v_3 \text{Tr}(\rho^\alpha \rho^\beta) + v_5 \eta_{\alpha\sigma} \text{Tr}(\rho^\alpha \rho^\beta) \right\] \\
+ 4v_6 \eta^{\alpha\lambda} \eta^{\beta\sigma} \partial_\lambda \partial_\sigma \text{Tr}(\rho_{\alpha\beta} \rho_{\beta\alpha}), \] (5)

where \( \eta_{\mu\nu} \) is the metric tensor in Minkowski space.

### III. Renormalization and Power Counting

To perform the renormalization we express the bare quantities in terms of renormalized ones and counterterms and apply the complex-mass renormalization scheme [29, 30]. We parameterize the pole of the \( \rho \)-meson dressed propagator in the chiral limit as \( M_R^2 = (M_\rho - i \Gamma_\rho/2)^2 \), where \( M_\rho \) and \( \Gamma_\rho \) are the pole mass and width of the \( \rho \) meson in the chiral limit, respectively. Both are input parameters within our formalism.

Following Ref. [28], we fix the mass counterterm and the wave function renormalization constant by requiring that in the chiral limit, \( M_R^2 \) coincides with the pole position of the dressed propagator and the residue is equal to unity. The renormalized complex mass \( M_R \) appears in the propagator, and the counterterms are included perturbatively. Notice that in the complex-mass renormalization scheme, the counterterms are also complex quantities. This does, however, not lead to a violation of unitarity as one might naively expect [46, 47]. Let us demonstrate this using the example of the renormalization of the \( \rho \)-meson mass. The Lagrangian is given in terms of bare parameters, and physical quantities also can be calculated in terms of these parameters within some ultraviolet regularization scheme. The physical mass of a stable particle as well as the mass and width of an unstable particle can be obtained from the corresponding two-point function by finding its pole position. Defining the self-energy of the \( \rho \)-meson as the sum of all one-particle irreducible diagrams contributing to the two-point function of the \( \rho \)-meson field operators we parameterize this quantity as

\[ i \Pi^{\mu\nu}(p) = i \left( g^{\mu\nu} \Pi_1(p^2) + p^\mu p^\nu \Pi_2(p^2) \right), \] (6)

The equation determining the pole position \( z \) of the two-point function written in terms of the bare parameters has the form:

\[ z - M_0^2 - \Pi_1(z, M_0, M_\pi, \cdots) = 0, \] (7)

where \( M_0 \) is the bare mass of the \( \rho \)-meson, \( M_\pi \) is the pion mass, and the ellipses denote other parameters of the Lagrangian and also the ultraviolet regulator. The solution to Eq. (7) has the form:

\[ z = f(M_0, M_\pi, \cdots) \equiv M_0^2 + \text{corrections}, \] (8)
We denote the quantity \( z \) in chiral limit by \( M_R^2 \) and invert Eq. (8) for \( M_\pi = 0 \) to obtain
\[
M_0^2 = f^{-1}(M_R^2, 0, \cdots) \equiv M_R^2 + \delta M^2(h, M_R^2, \cdots),
\]
where \( M_R^2 \) and \( \delta M^2 \) are both complex, while \( M_0^2 \) is real. We have indicated the explicit factor of \( h \) to emphasize that within the formalism employed in the current work, after substituting Eq. (9) in the Lagrangian, we treat the second term on the right-hand side (after further expanding it in powers of \( h \)) together with the loop diagrams, i.e. perturbatively.

For the effective Lagrangian, we apply the standard rules counting the pion mass and the derivatives acting on pion fields as small quantities, while the derivatives acting on the heavy vector mesons count as \( \mathcal{O}(q^{-2}) \) if it does not carry large external momenta and as \( \mathcal{O}(q^0) \) otherwise. A vector-meson propagator counts as \( \mathcal{O}(q^0) \) if it does not carry large external momenta and as \( \mathcal{O}(q) \) if it does. The vector-meson mass counts as \( \mathcal{O}(q^1) \), the width of the vector mesons as well as the pion mass count as \( \mathcal{O}(q^2) \).

Interaction vertices generated by the effective Lagrangian of the order \( n \) do not automatically count as \( \mathcal{O}(q^n) \) but rather need to be assigned orders according to a given flow of large and small external momenta. As the contributions of loops involving only vector meson propagators can be absorbed systematically in the redefinition of the parameters of the effective Lagrangian, such loop diagrams need not be included at low energies.

IV. GRAVITATIONAL FORM FACTORS OF THE \( \rho \) MESON: DEFINITIONS AND CALCULATION

The gravitational form factors (GFFs) of a spin-1 particle were defined for the first time in Ref. [3]. Here, we follow the conventions and notations of Ref. [12], in which the GFFs of a spin-1 particle were defined as:
\[
\langle p', \sigma' | T_{\mu \nu} | p, \sigma \rangle = \epsilon^{a\alpha'}(p', \sigma')\epsilon^a(p, \sigma) \left[ 2P_\mu P_\nu \left( -\eta_{a\alpha'} A_0(t) + \frac{P_\alpha P_{a'}}{m^2} A_1(t) \right) + 2 [P_\mu (\eta_{b\alpha'} P_\alpha + \eta_{c\alpha'} P_{a'}) + P_\nu (\eta_{b\alpha'} P_\alpha + \eta_{c\alpha'} P_{a'})] J(t) \right. \\
\left. + \frac{1}{2} (\Delta_\mu \Delta_\nu - \eta_{\mu\nu} \Delta^2) \left( \eta_{a\alpha'} D_0(t) + \frac{P_\alpha P_{a'}}{m^2} D_1(t) \right) + \left[ \frac{1}{2} (\eta_{b\alpha} \eta_{b\alpha'} + \eta_{c\alpha} \eta_{c\alpha'}) \Delta^2 - (\eta_{b\alpha} \Delta_\mu + \eta_{c\alpha} \Delta_\nu) P_\alpha \right. \\
\left. + (\eta_{b\alpha} \Delta_\mu + \eta_{c\alpha} \Delta_\nu) P_{a'} - 4 \eta_{\mu\nu} P_\alpha P_{a'} \right] E(t) \right],
\]
where \( \Delta = p_f - p_i, P = (p_f + p_i)/2, m \) is the mass (note that we reserve the symbol \( M \) for the \( \rho \) and \( \omega \) masses) of the spin-1 particle and the polarization vector \( \epsilon_\alpha(p, \sigma) \) satisfies the condition
\[
\sum_\sigma \epsilon_\alpha(p, \sigma) \epsilon_\beta(p, \sigma) = -\eta_{\alpha\beta} + \frac{P_\alpha P_\beta}{m^2}.
\]
For the reader’s convenience we collect in Table I other notations for the GFFs of spin-1 particles used in the literature.

As the \( \rho \)-meson is an unstable particle we extract its gravitational form factors from the residue at the complex double-pole of the three-point correlation function of the EMT and the vector meson fields [52]. In this case, \( m^2 \) in the above formulas is the complex pole position of the corresponding dressed propagator.

In the current work we consider contributions of tree-level and one-loop diagrams to the gravitational form factors of the \( \rho \)-meson, see Fig. 1. At tree level, there are contributions of higher-order terms in the effective Lagrangian. As the higher-order Lagrangian is not available yet, we include these contributions to the form factors parametrized in
we apply dimensional regularization and use the program FeynCalc \[\text{the general form as polynomials of the pion mass and the momentum transfer squared. To calculate the loop diagrams}
\]

we take $M^2 = M_R^2$, which is a good approximation given the accuracy of this work. Below, we specify the chiral

expansions of the form factors at $t = 0$ and of the slope parameters and also provide expressions for the form factors

in the small-$t$ region in the chiral limit. Within the accuracy of our calculations, the pion mass at leading order in the

chiral expansion can be replaced by its full expression $M_\pi$.

The chiral expansion of the form factors at zero momentum-transfer has the form:

$$A_0(0) = 1,$$

$$A_1(0) = 8\epsilon_0 M_R^2 + X_A M_\pi^2 + \frac{g_{\omega\rho\pi}^2 M_R}{4\pi F^2} M_\pi - \frac{g_{\omega\rho\pi}^2 (1 + 4\epsilon_0 M_R^2)}{8\pi^2 F^2} M_\pi \ln \frac{M_\pi}{M_R} + O(M_\pi^3),$$

$$J(0) = 1,$$

$$D_0(0) = 1 + 4\epsilon_1 + 8\epsilon_3 M_R^2 + X_{D_0} M_\pi^2 - \frac{(a + 3g_{\omega\rho\pi}^2(v_1 + 2v_3 M_R^2))}{12\pi^2 F^2} M_\pi^2 \ln \frac{M_\pi}{M_R} + O(M_\pi^3);$$

$$D_1(0) = -8(4\epsilon_4 + \epsilon_5 + \epsilon_6) M_R^2 + \frac{g_{\omega\rho\pi}^2 M_R^3}{60\pi^2 F^2} \ln \frac{M_\pi}{M_R} + O(M_\pi^3);$$

$$J(0) = 1 - v_3 - v_5 M_R^2 + X_E M_\pi^2 + \frac{g_{\omega\rho\pi}^2 M_R}{96\pi^2 F^2} M_\pi^2 + \frac{(6 a_3 + 6 v_5 M_R^2 - 5 g_{\omega\rho\pi}^2 - 4 a)}{96\pi^2 F^2} M_\pi^2 \ln \frac{M_\pi}{M_R} + O(M_\pi^3).$$

Here, $X_{F_i}$ (as well as $Y_{F_i}$, $Z_{F_i}$, and $W_{F_i}$ below) are some linear combinations of renormalized complex-valued low-energy constants from the higher-order effective Lagrangian.

The above equations provide the dependence of the GFFs at zero momentum transfer on the pion mass. They can be used for extrapolations of the lattice-QCD results for the GFFs to the physical values of the pion masses. In
recent lattice calculations of the gluon part of GFFs for the $\rho$-meson [15], it was found, in particular, that the value of $D_1(0)$ is compatible with zero, albeit with large error bars ($D_1(0) = 0.0 \pm 0.7$). From our calculations we see that $D_1(0) \sim 1/M_\pi$ for small pion masses. This singular contribution alone leads to a large value of $D_1(0) \approx 4$ for the physical pion mass. This value of the singular part of the GFF can, unfortunately, not be directly compared to the results of the lattice simulations of Ref. [15] as only the gluon part of the GFF $D^g_1$ was computed in that paper. However, we expect that the singular $\sim 1/M_\pi$ part is also present in $D^g_1(0)$ with a slightly modified coefficient. This suggests that the chiral extrapolation of the lattice results of Ref. [15] for the pion mass to its physical point should be studied with great care.

Defining the slopes $s_F$ as the coefficients of linear terms in the Taylor expansion of the form factors, $F(t) = F(0) + s_F t + \ldots$, we obtain for their chiral expansions the following results:

\begin{align}
\text{for } s_{A_0} & = \frac{v_5}{2} + Z_{A_0} M_\pi^2 - \frac{g_{\omega \rho \pi}^2}{64 \pi^2 F^2 M_R} M_\pi - \frac{g_{\omega \rho \pi}^2 (6v_5 M_R^2 + 7)}{192 \pi^2 F^2 M_R^2} M_\pi^2 \ln \frac{M_\pi}{M_R} + O(M_\pi^2), \\
\text{for } s_{A_1} & = Y_{A_1} - \frac{g_{\omega \rho \pi}^2 M_R}{480 \pi^2 F^2} \frac{1}{M_\pi} + O(M_\pi), \\
\text{for } s_{J} & = \frac{1}{2} (v_5 + 2v_6) + Z_J M_\pi^2 - \frac{17g_{\omega \rho \pi}^2 + a}{1152 \pi^2 F^2} \ln \frac{M_\pi}{M_R} + \frac{5g_{\omega \rho \pi}^2}{768 \pi^2 F^2 M_R} M_\pi, \\
\text{for } s_{D_0} & = -4v_4 - \frac{v_5}{2} + Z_{D_0} M_\pi^2 + \frac{(35 + 24i\pi) a - 36g_{\omega \rho \pi}^2}{1440 \pi^2 F^2} \ln \frac{M_\pi}{M_R} + \frac{5g_{\omega \rho \pi}^2}{160 \pi^2 F^2} M_\pi + \frac{5g_{\omega \rho \pi}^2 + 4a}{240 \pi^2 F^2} \ln \frac{M_\pi}{M_R}, \\
\text{for } s_{D_1} & = \frac{43g_{\omega \rho \pi}^2}{3840 \pi^2 F^2 M_R^2} M_\pi + \frac{g_{\omega \rho \pi}^2 (11g_{\omega \rho \pi}^2 - 12a) M_\pi^2}{840 \pi^2 F^2 M_R} - \frac{131g_{\omega \rho \pi}^2 M_\pi}{13440 \pi^2 F^2 M_R} - \frac{35g_{\omega \rho \pi}^2 + 64a}{840 \pi^2 F^2} \ln \frac{M_\pi}{M_R} + O(M_\pi^3), \\
\text{for } s_{E} & = \frac{1}{2} (v_5 + 2v_6) + Z_E M_\pi^2 + \frac{(35 + 12i\pi) a - 92g_{\omega \rho \pi}^2}{5760 \pi^2 F^2} \ln \frac{M_\pi}{M_R} + \frac{g_{\omega \rho \pi}^2 M_\pi}{1920 \pi^2 F^2 M_R} + \frac{4a - 5g_{\omega \rho \pi}^2}{960 \pi^2 F^2} \ln \frac{M_\pi}{M_R} + \frac{19g_{\omega \rho \pi}^2}{15360 \pi^2 F^2 M_R^2} M_\pi - \frac{5v_5 g_{\omega \rho \pi}^2 M_R^2 + 10v_6 g_{\omega \rho \pi}^2 M_R^2 + 2a}{160 \pi^2 F^2 M_R^2} \ln \frac{M_\pi}{M_R} + O(M_\pi^3). \tag{13}
\end{align}

We see from the above expressions that some of the slopes have strong singularities for small pion masses. This again underlines the need of a careful analysis of the chiral extrapolation of lattice data.

It is also instructive to study the $t$-dependence of GFFs in the chiral limit $M_\pi = 0$. The corresponding results will allow us in the next section to derive the large distance asymptotics of the energy and force distributions. The expressions of the form factors in the small-$t$ region in the chiral limit has the form:

\begin{align}
A_0(t) & = 1 + \frac{v_5}{2} t + W_{A_0} t^3 + \frac{7g_{\omega \rho \pi}^2}{4096 \pi^2 F^2 M_R} (-t)^{3/2} + \frac{5g_{\omega \rho \pi}^2}{768 \pi^2 F^2 M_R^2} t^2 \ln \left( -\frac{t}{M_R^2} \right), \\
A_1(t) & = 8v_6 M_R^2 + Y_{A_1} t + \frac{3g_{\omega \rho \pi}^2 M_R}{1024 F^2} \sqrt{-t}, \\
J(t) & = 1 + \frac{1}{2} (v_5 + 2v_6) t - \frac{9g_{\omega \rho \pi}^2 + a}{1152 \pi^2 F^2} t \ln \left( -\frac{t}{M_R^2} \right), \\
D_0(t) & = 1 + 4v_4 + 8v_4 M_R^2 \left( 4v_4 + \frac{v_5}{2} \right) t - \frac{5g_{\omega \rho \pi}^2 M_R}{1024 F^2} \sqrt{-t} + \frac{5g_{\omega \rho \pi}^2 + 4a}{480 \pi^2 F^2} t \ln \left( -\frac{t}{M_R^2} \right) + \frac{3(7 + 40i\pi) a - 200g_{\omega \rho \pi}^2}{7200 \pi^2 F^2} t, \\
D_1(t) & = -8 (4v_4 + v_5 + v_6) M_R^2 + \frac{M_R^2}{450 \pi^2 F^2} (47a - 20g_{\omega \rho \pi}^2) + \frac{3g_{\omega \rho \pi}^2 M_R^3}{256 F^2} \sqrt{-t} + \frac{(5g_{\omega \rho \pi}^2 - 8a) M_R^2}{120 \pi^2 F^2} \ln \left( -\frac{t}{M_R^2} \right), \tag{14}
\end{align}

\begin{align}
E(t) & = 1 - a_3 - v_5 M_R^2 + \frac{1}{2} (v_5 + 2v_6) t + \frac{a(60i\pi - 9) - 245g_{\omega \rho \pi}^2}{28800 \pi^2 F^2} t + \frac{g_{\omega \rho \pi}^2 M_R}{1024 F^2} \sqrt{-t} \ln \left( -\frac{t}{M_R^2} \right) + \frac{4a - 5g_{\omega \rho \pi}^2}{1920 \pi^2 F^2} \log \left( -\frac{t}{M_R^2} \right) \log \left( -\frac{t}{M_R^2} \right) \log \left( -\frac{t}{M_R^2} \right).}
\end{align}
V. LARGE DISTANCE BEHAVIOUR OF THE ENERGY AND FORCE DISTRIBUTIONS

It is particularly interesting to look at the energy distribution and mechanical properties such as the elastic pressure and shear force distributions inside the \( p \)-meson. These fundamental distributions are encoded in the static EMT defined in the Breit frame as \([7]\):

\[
T^{\mu\nu}(\vec{r}', \sigma', \sigma) = \int \frac{d^3\Delta}{(2\pi)^3} \frac{2E}{2E} e^{-i\vec{\Delta}\cdot\vec{r}'} \langle p', \sigma' | \hat{T}^{\mu\nu}_{\text{QCD}}(0) | p, \sigma \rangle.
\]  

(15)

Here, \( \hat{T}^{\mu\nu}_{\text{QCD}}(0) \) is the QCD EMT operator of the matrix element that is computed between hadron states with spin projections \( \sigma, \sigma' \) and momenta \( p^0 = p'^0 = E = \sqrt{m^2 + \Delta^2/4} \), and \( p'^i = -p^i = \Delta^i/2 \). The 00-component of the static EMT contains the information about the energy distribution, the 0\( i \)-components encode the spin distribution while the \( i k \)-components provide us with the distributions of elastic pressure and shear forces inside the hadron \([7]\).

In relativistic quantum field theory it is impossible to localize an one-particle state with an accuracy better than its Compton wave length \( \lambda = \hbar/(mc) \). Therefore, at distances smaller or of the order of \( \lambda \), one has to interpret the Breit frame static EMT of Eq. (15) from a quasi-probabilistic phase-space perspective \([9, 56]\). The phase-space picture connecting the Breit frame to the light front has been obtained, for the first time, in the study of the angular distributions of forces inside the nucleon in its rest-frame. See the detailed discussion in Ref. \([9]\) for the EMT and in Ref. \([56]\) for the charge densities. Due to Heisenberg’s uncertainty principle, the Wigner distributions only have a quasi-probabilistic interpretation. For large distances \( r \gg 1/(2m) \) in Eq. (15), the Wigner distributions acquire a strict probabilistic interpretation, see the detailed discussion in Refs. \([9, 56, 57]\).

If one insists on a strict probabilistic interpretation of distributions, the static EMT of Eq. (15) for \( r \sim 1/(2m) \) acquires the so-called relativistic corrections discussed since the 1950ties \([60]\). See Refs. \([61-63]\) for more recent discussion. The relativistic corrections can be kinematically suppressed if one considers the distributions in the infinite momentum frame (IMF) or if one uses quantization on the light front, see, e.g., Refs. \([61, 62]\). In the recent study \([56]\), the natural interpolation between the Breit frame and IMF charge distributions was obtained using the phase-space Wigner distributions. Such an analysis can also be repeated for the case of the force distributions.

The densities for the internal force distributions in the nucleon in the IMF and on the light front were derived first in Ref. \([9]\) with the help of the Wigner phase-space distribution. More recently, light-front force distributions were also obtained in Ref. \([64]\) using light-cone quantization methods. These densities possess a strict probabilistic interpretation (no relativistic corrections) and they come out identical in both approaches.

In this paper we are interested in the large-distance behaviour of the energy and the force distributions \( (r \gg 1/(2m)) \). The Breit-frame distributions do possess a probabilistic interpretation in this limit. As the relativistic corrections are parametrically suppressed in the range of applicability of the chiral expansion, other types of energy and force distributions (light-cone, IMF, etc.) can easily be obtained from the Breit frame ones using the method of Abel transformations (and its generalisation for spin-1 particles \([65]\)) \([66, 67]\).

Various components of the static EMT for hadrons with arbitrary spin can be decomposed into multipoles of the hadron’s spin operator. The expansion to the quadrupole order has the following form \([10-12, 14]\):

\[
T^{00}(r) = \varepsilon_0(r) + \varepsilon_2(r) \hat{Q}^{pq} Y_2^{pq} + \ldots,
\]

(16)

\[
T^{ik}(r) = p_0(r) \delta^{ik} + s_0(r) Y_2^{ik} + \left( p_2(r) + \frac{1}{3} p_3(r) - \frac{1}{9} s_3(r) \right) \hat{Q}^{ik}
\]

\[
+ \left( \frac{s_2(r)}{6} - \frac{p_3(r)}{2} + \frac{1}{6} s_3(r) \right) \hat{Q}^{pk} Y_2^{pi} - \hat{Q}^{ik} \hat{Q}^{pq} Y_2^{pq}
\]

\[
+ \hat{Q}^{pq} Y_2^{pq} \left[ \left( \frac{1}{3} p_3(r) + \frac{1}{9} s_3(r) \right) \hat{Q}^{ik} + \left( \frac{1}{2} p_3(r) + \frac{5}{6} s_3(r) \right) Y_2^{ik} \right] + \ldots.
\]

(17)

\[1\] In what follows, we shall suppress the hadron’s spin indices \( \sigma, \sigma' \) when their position is obvious. Also, we employ here the parametrization of the static stress tensor that differs from that of Refs. \([12, 14]\) by a simple redefinition. The corresponding relations are given in the appendix of Ref. \([11]\).
Here, the ellipses denote the contributions of 2\(n\)th multipoles with \(n > 2\). They are absent for a spin-1 particle. The quadrupole operator is a \((2J + 1) \times (2J + 1)\) matrix:

\[
\hat{Q}^{jk} = \frac{1}{2} \left( \hat{j}^i \hat{j}^k + \hat{j}^k \hat{j}^i - \frac{2}{3} J(J + 1) \delta^{ik} \right),
\]

which is expressed in terms of the spin operator \(\hat{j}^i\). The spin operator can be expressed in terms of SU(2) Clebsch-Gordan coefficients (in the spherical basis):

\[
\hat{j}^\mu_{\sigma \sigma'} = \sqrt{J(J + 1)} C_{J\sigma \sigma'}^{\mu}.
\]

Furthermore, we introduce the irreducible (symmetric and traceless) tensor of rank \(n\) made out of \(r\):

\[
Y_{ni}^{i_2...i_n} = \frac{(-1)^n}{(2n - 1)!!} r^{n+1} \partial^{i_2} \partial^{i_n} \frac{1}{r},
\]

i.e.

\[
Y_0 = 1, \quad Y_1 = \frac{r^1}{r}, \quad Y_2 = \frac{r^2}{r^2} - \frac{3}{2} \delta^{ik}, \quad \text{etc.}
\]

Note that only the monopole quantities \(\varepsilon_0(r)\), \(p_0(r)\), and \(s_0(r)\) are left after spin averaging. The functions \(\varepsilon_0(r)\) and \(\varepsilon_2(r)\) correspond to the spin-averaged energy density and to the quadrupole deformation of the energy density inside the hadron, respectively. There is an obvious relation \(\int d^4 \varepsilon_0(r) = m\). Also, it is obvious that \(\varepsilon_2(r) = 0\) for hadrons with spin 0 and 1/2 (that is why such hadrons can be called spherically symmetric).

From the equilibrium condition for the stress tensor, \(\partial_k T^{rk}(r) = 0\), one can easily obtain the equations for the functions \(p_n(r)\) and \(s_n(r)\):

\[
\frac{d}{dr} \left( p_n(r) + \frac{2}{3} \varepsilon_n(r) \right) + \frac{2}{r} s_n(r) = 0, \quad \text{for}\ n = 0, 2, 3.
\]

To see the physical meaning of the quadrupole force distributions \(p_{2,3}(r)\) and \(s_{2,3}(r)\), it is instructive to look at the force acting on the infinitesimal radial area element \(dS_r\) \((dS = dS_r \varepsilon_r + dS_\theta \varepsilon_\theta + dS_\phi \varepsilon_\phi)\). With the help of the parameterization of Eq. (17) and the relation of the force to the stress tensor, \(dF_i = T_{ik} dS_k\), we obtain:

\[
\frac{dF_r}{dS_r} = p_0(r) + \frac{2}{3} s_0(r) + \hat{Q}^{rr} \left( p_2(r) + \frac{2}{3} s_2(r) + p_3(r) + \frac{2}{3} s_3(r) \right),
\]

\[
\frac{dF_\theta}{dS_r} = \hat{Q}^{r\theta} \left( p_2(r) + \frac{2}{3} s_2(r) \right),
\]

\[
\frac{dF_\phi}{dS_r} = \hat{Q}^{r\phi} \left( p_2(r) + \frac{2}{3} s_2(r) \right).
\]

We see that in contrast to spherically symmetric hadrons, the radial area element experiences not only normal forces but also tangential ones. The strengths of the tangential forces are governed by \(p_2(r)\) and \(s_2(r)\), the quadrupole force distributions \(p_3(r)\) and \(s_3(r)\) contribute to the spin-dependent part of the radial force.

Using the result for the \(t\)-dependence of GFFs in the chiral limit of Eq. (15) obtained in the previous section, we can easily calculate the analytic expressions for the large distance behaviour (in the chiral limit) of the energy and force distributions defined in Eqs. (16), (17):

\[
\varepsilon_0(r) = \frac{g_{\omega \rho \pi}}{32\pi^2 F^2} \frac{1}{r^6} - \frac{3(2a + 5g_{\omega \rho \pi})}{32\pi^3 F^2 M_R} \frac{1}{r^7} + O \left( \frac{1}{r^8} \right),
\]

\[
\varepsilon_2(r) = \frac{3g_{\omega \rho \pi}}{128\pi^2 F^2} \frac{1}{r^6} + \frac{21(4a - 5g_{\omega \rho \pi})}{256\pi^3 F^2 M_R} \frac{1}{r^7} + O \left( \frac{1}{r^8} \right),
\]

\[
p_0(r) = -\frac{g_{\omega \rho \pi}}{96\pi^2 F^2} \frac{1}{r^6} + \frac{(16a + 15g_{\omega \rho \pi})}{144\pi^3 F^2 M_R} \frac{1}{r^7} + O \left( \frac{1}{r^8} \right),
\]

\[
s_0(r) = \frac{g_{\omega \rho \pi}}{32\pi^2 F^2} \frac{1}{r^6} - \frac{7(16a + 15g_{\omega \rho \pi})}{384\pi^3 F^2 M_R} \frac{1}{r^7} + O \left( \frac{1}{r^8} \right),
\]

\[
(25)
\]
\[ p_2(r) = \frac{g_{\rho\pi}^2}{32\pi^2 F^2} \frac{1}{r^6} + \frac{5(20a - 13g_{\rho\pi}^2)}{192\pi^3 F^2 M_R} \frac{1}{r^7} + O\left(\frac{1}{r^8}\right), \]
\[ s_2(r) = -\frac{3g_{\rho\pi}^2}{32\pi^2 F^2} \frac{1}{r^6} - \frac{35(20a - 13g_{\rho\pi}^2)}{512\pi^3 F^2 M_R} \frac{1}{r^7} + O\left(\frac{1}{r^8}\right), \]
\[ p_3(r) = -\frac{9g_{\rho\pi}^2}{128\pi^2 F^2} \frac{1}{r^6} - \frac{7(8a - 5g_{\rho\pi}^2)}{48\pi^3 F^2 M_R} \frac{1}{r^7} + O\left(\frac{1}{r^8}\right), \]
\[ s_3(r) = \frac{27g_{\rho\pi}^2}{128\pi^2 F^2} \frac{1}{r^6} + \frac{49(8a - 5g_{\rho\pi}^2)}{128\pi^3 F^2 M_R} \frac{1}{r^7} + O\left(\frac{1}{r^8}\right). \]

In Refs. [8, 9, 68] it was conjectured that for the stability of a mechanical system the spin averaged pressure and shear forces should satisfy the inequality

\[ \frac{2}{3} s_0(r) + p_0(r) \geq 0, \tag{26} \]

which corresponds to positivity of the radial pressure. From the derived large distance behaviour of the \( p_0(r) \) and \( s_0(r) \), we see that the inequality Eq. (26) is indeed satisfied. However, the \( \rho \)-meson decays in our theory. The terms in the large distance expansion (25) which “know” about the instability of the particle are proportional to the \( \rho \pi \pi \) coupling constant squared \( \sim a \). It is interesting to note that the corresponding terms violate the stability condition of Eq. (26), also the corresponding terms in the spin averaged energy density \( \varepsilon_0(r) \) violate its positivity. A detailed study of the relations between mechanical stability conditions and the decay of unstable particles will be given elsewhere.

VI. SUMMARY

To summarize, we have applied chiral EFT to vector mesons and Goldstone bosons in the presence of an external gravitational field. Using standard definitions, we obtained the expressions of the EMT in flat background metric. As first noticed in Ref. [40], terms in the effective Lagrangian involving the gravitational curvature, which vanishes in the flat background, do give non-trivial contributions to the EMT. This also happens for the case at hand. Therefore, in order to keep track of all relevant contributions to the EMT in flat spacetime, it is necessary to consider effective Lagrangian in curved spacetime. In the next step, we calculated the gravitational form factors of the \( \rho \)-meson at next-to-leading order. This involves the calculation of tree-level and one-loop diagrams. To get rid of ultraviolet divergences and power counting violating pieces we applied the complex-mass renormalization scheme, which allows one to subtract also the large imaginary parts from loop diagrams. The matrix element of the EMT for this spin-1 hadrons is parameterized using six independent structures. We do not give the rather lengthy expressions of the obtained expressions of the gravitational form factors of the \( \rho \)-meson in terms of standard loop functions\(^2\), but focus on the chiral expansion of the form factors at zero momentum transfer and of their slopes. These expressions should be useful for lattice extrapolations of the corresponding results by taking the pion mass to its physical value. We also presented the expansion of the form factors in the small-\( t \) region in the chiral limit. Using these expression, we further calculated the large-distance behaviour of the energy distribution and the internal forces. The obtained results are consistent with the stability condition of a mechanical system.

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