Universality of Jamming Criticality in Overdamped Shear-Driven Frictionless Disks

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(Dated: September 19, 2014)

We investigate the criticality of the jamming transition for overdamped shear-driven frictionless disks in two dimensions for two different models of energy dissipation: (i) Durian’s bubble model with dissipation proportional to the velocity difference of particles in contact, and (ii) Durian’s “mean-field” approximation to (i), with dissipation due to the velocity difference between the particle and the average uniform shear flow velocity. By considering the finite-size behavior of pressure, the pressure analog of viscosity, and the macroscopic friction σ/p, we argue that these two models share the same critical behavior.

PACS numbers: 45.70.-n 64.60.-i 64.70.Q-

Many different physical systems, such as granular materials, suspensions, foams and emulsions, may be modeled in terms of particles with short ranged repulsive contact interactions. As the packing fraction φ of such particles is increased, the system undergoes a jamming transition from a liquid state to a rigid but disordered solid. It has been proposed that this jamming transition is a manifestation of an underlying critical point, “point J”, with associated scaling properties such as is found in equilibrium phase transitions [1, 2]. Scaling properties are indeed found when such systems are isotropically compressed, with pressure, elastic moduli, and contact number increasing as power laws as φ increases above the jamming φJ [3]. When such systems are sheared with a uniform strain rate γ, a unified critical scaling theory has successfully described both the vanishing of the yield stress as φ → φJ from above, the divergence of the shear viscosity as φ → φJ from below, and the non-linear rheology exactly at φ = φJ [4].

One of the hallmarks of equilibrium critical points is the notion of universality: the critical behavior, specifically the exponents describing the divergence or vanishing of observables, depend only on the symmetry and dimensionality of the system, and not on details of the specific interactions. For statically jammed states created by compression, where only the elastic contact interaction comes into play, it is understood that the relevant critical exponents are simply related to the form of the elastic interaction, and are all simple rational fractions [3]. In contrast, sheared-driven steady states are formed by a balance of elastic and dissipative forces, and it is thus an important question whether or not the specific form taken for the dissipation is crucial for determining the critical behavior.

In a recent work by Tighe et al. [5], it was claimed that changing the form of the dissipation can indeed alter the nature of the criticality for sheared overdamped frictionless disks. In contrast to earlier work [4], where particle dissipation was taken with respect to a uniformly sheared background reservoir, Tighe et al. used a collisional model for dissipation. They argued that this change in dissipation resulted in dramatically different behavior from that found previously, specifically (i) there is no length scale ξ that diverges upon approaching φJ, and so behavior can be described analytically with a mean-field type model; (ii) critical exponents are simple rational fractions; (iii) there is no single critical scaling, but rather several different flow regimes, each with a different scaling. In this work we numerically re-investigate the model of Tighe et al. and present results arguing against these conclusions. In particular we conclude that the two models have rheology that is characterized by the same critical exponents, and so are in the same critical universality class.

We simulate bidisperse frictionless disks in two dimensions (2D), with equal numbers of big and small disks with diameter ratio 1.4, at zero temperature. The interaction of disks i and j in contact is V_{ij} = k_e δ_{ij}^2/2, where the overlap is δ_{ij} = r_{ij} − d_{ij} − 1, with d_{ij} the sum of the disks’ radii. The elastic force on disk i is f_{di} = −∇_{i} \sum_j V_{ij}, where the sum is over all particles j in contact with i. We use Lees-Edwards boundary conditions [6] to introduce a time-dependent uniform shear strain γ(t) = γx in the x direction.

We consider two different models for energy dissipation. The first, which we call “contact dissipation” (CD), is the model introduced by Durian for bubble dynamics in foams [7], and is the model used by Tighe et al. [5]. Here dissipation occurs due to velocity differences of disks in contact,

\[ f_{\text{CD},i}^{\text{dis}} = −k_d \sum_j (v_i − v_j), \quad \mathbf{v}_i = \mathbf{r}_i. \]  

In the second, which we call “reservoir dissipation” (RD), dissipation is with respect to the average shear flow of a background reservoir,

\[ f_{\text{RD},i}^{\text{dis}} = −k_d (\mathbf{v}_i − \mathbf{v}_R(\mathbf{r}_i)), \quad \mathbf{v}_R(\mathbf{r}_i) \equiv γy_i \hat{x}. \]  

RD was also introduced by Durian [7] as a “mean-field” [8] approximation to CD, and is the model used in many
The equation of motion for both models is
\[ m_i \dot{v}_i = f_i^{\text{el}} + f_i^{\text{dis}}. \tag{3} \]

Here we are interested in the overdamped limit, \( m_i \to 0 \) \([7]\). In RD it is straightforward to set \( m_i = 0 \), in which case the equation of motion becomes simply \( \dot{v}_i = v_R(r_i) + f_i^{\text{el}}/kd_i \); we call this limit RD\(_0\). In CD, because the dissipation couples velocities one to another, setting \( m_i = 0 \) effectively requires inverting the matrix of contacts to rewrite the equation of motion in a form suitable for numerical integration. Instead of that numerically difficult procedure, our approach here is to simulate particles with a finite mass, and verify that the mass is small enough for the system to be in the overdamped \( m_i \to 0 \) limit; we call this limit CD\(_0\). For our simulations we use units in which \( k_e = k_d = 1 \), length is in units such that the small disk diameter \( d_s = 1 \), time in units of \( \tau_0 \equiv k_d d_s^2/k_e = 1 \), and particles of equal mass density, such that \( m_i \) for a particle of diameter \( d_i \) is \( m_i = 2\pi rd_s^2/4 \), with \( m = 1 \). In our Supplemental Material \([11]\) we confirm that this choice is sufficient to be in the \( m_i \to 0 \) limit. For RD\(_0\) our simulations use \( N = 65536 \) particles, while for CD\(_0\) we use \( N = 262144 \), unless otherwise noted. For RD\(_0\) our slowest strain rate is \( \dot{\gamma} = 10^{-9} \), while for CD\(_0\) we can reach only \( \dot{\gamma} = 10^{-7} \).

Before presenting our evidence that the two models RD\(_0\) and CD\(_0\) have the same critical rheology, we first comment on one quantity that is clearly very different in the two models, the transverse velocity correlation, \( g_0(x) \equiv \langle v_y(0)v_y(x)\rangle \). In RD\(_0\) \( g_0(x) \) shows a clear minimum at a distance \( x = \xi \), and this length diverges as one approaches the critical point \((\phi_J, \dot{\gamma} \to 0)\) \([4]\). In CD\(_0\) however, it was found \([5]\) that \( g_0(x) \) decreases monotonically without any obvious strong dependence on either \( \phi \) or \( \dot{\gamma} \). This led Tighe et al. to conclude that there is no diverging length \( \xi \) in model CD\(_0\), that the only macroscopic length scale is the system length \( L \), and thus there are no critical fluctuations. In our own work we have confirmed this dramatic difference in the behavior of \( g_0(x) \), but see our Supplemental Material for further comments \([11]\).

However the apparent absence of a diverging \( \xi \) in \( g_0(x) \) for CD\(_0\) does not necessarily imply that such a diverging length does not exist. In the following we present evidence for such a diverging \( \xi \) in CD\(_0\) by considering the finite-size dependence of the pressure \( p \) as a function of strain rate \( \dot{\gamma} \) at \( \phi_J \). By a critical scaling analysis of the pressure analogue of viscosity, \( \eta_p \equiv p/\dot{\gamma} \), we further show that the rheology in CD\(_0\) is characterized by the same critical exponents as is RD\(_0\). Finally we consider the macroscopic friction \( \mu \equiv \sigma/p \) in the two models, with \( \sigma \) the shear stress, and show that they behave similarly. There is no sign of the roughly square root vanishing of \( \mu \) at \( \phi_J \) that would be expected from the model of Tighe et al., but rather \( \mu \) appears to be finite passing through \( \phi_J \), as found in recent experiments on foams \([12]\).

Finite-size dependence of pressure: We consider here only the elastic part of \( p \) which is computed from the elastic contact forces in the usual way \([3]\). If jamming behaves like a critical point, we expect \( p \) to obey finite-size-scaling at large system lengths \( L \) \([13]\),

\[
p(\phi, \dot{\gamma}, L) = L^{-\gamma/L^*}p((\phi - \phi_J)L^{1/\nu}, \dot{\gamma}L^{\xi/L^*}). \tag{4}\]

Exactly at \( \phi = \phi_J \) the above becomes \([14]\)

\[
p(\phi_J, \dot{\gamma}, L) = L^{-\gamma/L^*}p(0, \dot{\gamma}L^{\xi/L^*}). \tag{5}\]

For sufficiently small \( L \), where \( \dot{\gamma}L^z \ll 1 \), we get \( p \sim L^{-\gamma/L^*} \). For sufficiently large \( L \), where \( \dot{\gamma}L^z \gg 1 \), \( p \) becomes independent of \( L \) and so \( p \sim \dot{\gamma}^{z/L^*} \). The crossover occurs when \( L = \xi \) at \( \dot{\gamma}L^z \approx 1 \Rightarrow \xi \sim \dot{\gamma}^{-1/z} \), giving a diverging correlation length as \( \dot{\gamma} \to 0 \).

In Fig. 1(a) we plot \( p \) vs. \( L \) for RD\(_0\) and CD\(_0\) at \( \phi = 0.8433 \approx \phi_J \) for several different \( \dot{\gamma} \). Both models clearly behave similarly. To determine the crossover \( \xi \) we fit our data to the simple empirical form \( p = C(1+[\xi/L]^\nu) \) that interpolates between the two asymptotic limits. This fit gives the solid lines in Fig. 1(a). The resulting \( \xi \) is plotted in Fig. 1(b). We see that \( \xi \) is essentially identical in the two models, growing monotonically as \( \dot{\gamma} \) decreases, reaching values as large as \( \xi \approx 20 \) for our smallest \( \dot{\gamma} \). Such a large length, many times the microscopic length set by the particle size, is clear evidence for cooperative behavior \([15]\). Thus our results indicate a growing macroscopic length scale in CD\(_0\), just as was found for RD\(_0\). The exponent \( z \approx 5.6 \) found in Fig. 1(b) must, however, be viewed with caution since corrections-to-scaling are large at the sizes \( L \) considered here \([16]\), and the neglect of such corrections can skew the resulting effective exponents away from their true values at criticality. See our
Pressure analog of viscosity: We now seek to compute the critical exponents of the two models RD₀ and CD₀, to see if they are indeed in the same universality class. To do this we consider data at various packing fractions \( \phi \) and strain rates \( \dot{\gamma} \) close to the jamming transition. We use system sizes large enough \( (N = 65536 \text{ for } \text{RD}_0 \text{ and } N = 262144 \text{ for } \text{CD}_0) \), such that finite size effects are negligible for the data presented here. As in our recent work on \( \text{RD}_0 \) [13], we consider here the pressure analog of viscosity \( \eta_p \equiv p/\dot{\gamma} \), since corrections to scaling are smaller for \( p \) than for shear stress \( \sigma \) [13].

To extract the jamming fraction \( \phi_J \) and the critical exponents we use a mapping from our system of soft-core disks to an effective system of hard-core disks. We have previously shown [17] this approach to give excellent agreement with results from a more detailed two variable critical scaling analysis [13] for \( \text{RD}_0 \). We use it here because it requires no parameterization of an unknown crossover scaling function and so is better suited particularly to \( \text{CD}_0 \) where the range of our data is more limited \( (10^{-7} \leq \dot{\gamma} \leq 1) \) as compared to \( \text{RD}_0 \) \( (10^{-9} \leq \dot{\gamma} \leq 1) \).

This method assumes that the soft-core disks at \( \phi \) and \( \dot{\gamma} \) can be described as effective hard-core disks at \( \phi_{\text{eff}}(\dot{\gamma}) \), by modeling overlaps as an effective reduction in particle radius [17]. Measuring the overlap via the average energy per particle \( E \), we take

\[
\phi_{\text{eff}} = \phi - cE^{1/2y},
\]

where \( y \) is the exponent with which the pressure rises as \( \phi \) increases above \( \phi_J \) along the yield stress curve \( \dot{\gamma} \rightarrow 0 \), as in Eq. (4), and \( c \) is a constant. We can then express the viscosity of this effective hard-core system as,

\[
\eta_p(\phi, \dot{\gamma}) = \eta_p^{\text{hd}}(\phi_{\text{eff}}) = A(\phi_J - \phi_{\text{eff}})^{-\beta}.
\]

Our analysis then consists of adjusting \( \phi_J \), the exponents \( y \) and \( \beta \), and the constants \( c \) and \( A \) in Eqs. (6) and (7), to get the best possible fit to our data.

In Fig. 2 we show the results of such an analysis for \( \text{CD}_0 \). Panel (a) shows our raw data for \( \eta_p \) vs. \( \phi \), for several different \( \dot{\gamma} \), in the narrow density interval around \( \phi_J \) that is used for the analysis. Panel (b) shows the result from fitting \( \eta_p \) for \( \dot{\gamma} \leq 10^{-6} \) to Eq. (7). Our fitted values \( \phi_J = 0.8434, \beta = 2.5 \pm 0.2, y = 1.07 \pm 0.05 \) for \( \text{CD}_0 \) are all very close to our earlier results for \( \text{RD}_0 \) \( (\phi_J = 0.8433, \beta = 2.58 \pm 0.10, y = 1.09 \pm 0.01) \) [17] thus suggesting that the critical behavior in \( \text{CD}_0 \) is the same as in \( \text{RD}_0 \).

In quoting the fitted values of \( \phi_J \), \( \beta \) and \( y \) we note that our results for \( \text{RD}_0 \) include data to much lower strain rates \( 10^{-9} \leq \dot{\gamma} \leq 1 \) than for \( \text{CD}_0 \) where \( 10^{-7} \leq \dot{\gamma} \leq 1 \). For a more accurate comparison of the two models, we should fit our data over the same range of strain rates \( \dot{\gamma} \).

We therefore carry out a fitting to Eqs. (6) and (7) using data in the interval \( \dot{\gamma}_{\text{min}} \leq \dot{\gamma} \leq \dot{\gamma}_{\text{max}} \). In Fig. 3 we show our results for \( \phi_J \) and \( \beta \), where we plot the fitted values vs. \( \gamma \) for two different fixed values of \( \dot{\gamma} \). For \( \text{CD}_0 \) we can extend this procedure down to \( \dot{\gamma}_{\text{min}} = 1 \times 10^{-9} \), while for \( \text{CD}_0 \) we are limited to \( \dot{\gamma}_{\text{min}} = 1 \times 10^{-7} \). We see that for equivalent ranges of \( \dot{\gamma} \), the fitted values of \( \beta \) agree nicely for the two models, for the smaller value of \( \dot{\gamma}_{\text{max}} \). We see that \( \phi_J \) for \( \text{CD}_0 \) is just slightly higher than for \( \text{RD}_0 \). We cannot say whether this is a systematically significant difference, or whether \( \phi_J \) would decrease slightly to match the value found for \( \text{RD}_0 \) were we able to study \( \text{CD}_0 \) down to comparably small \( \dot{\gamma} \). Whether or not the \( \phi_J \) of the two models are equal, or just slightly different, the equality of the exponents \( \beta \) strongly argues that models \( \text{RD}_0 \) and \( \text{CD}_0 \) are in the same critical universality class.

To return to the results of Tighe et al. [5], we note that their value \( \phi_J = 0.8423 \) for \( \text{CD}_0 \) is clearly different from our above value of 0.8434. We believe that this difference is due to two main effects: (i) their data is restricted to \( 10^{-5} \leq \dot{\gamma} \) and so does not probe as close to the critical point as we do here, and (ii) their analysis was based on the scaling of shear viscosity \( \eta \equiv \sigma/\dot{\gamma} \) rather than the pressure viscosity \( \eta_p \). As we have noted previously [13] corrections to scaling for \( \sigma \) are significantly larger than they are for \( p \), and without taking these corrections into account, one generally finds a lower value for \( \phi_J \), such as was also found in the original scaling analysis of \( \text{RD}_0 \) [4]. Their lower value of \( \phi_J \), and their higher window of strain rates \( \dot{\gamma} \), we believe are also responsible for the different value they find for the exponent describing non-linear rheology exactly at \( \phi_J \), \( \sigma \sim p \sim \dot{\gamma}^q \); they claim \( q = 1/2 \) whereas our present result finds a clearly different value \( q = y/(\beta + y) = 0.30 \) [17].

Macroscopic friction: Finally we consider the macroscopic friction, \( \mu \equiv \sigma/p \). In Fig. 4 we plot \( \mu \) vs \( \phi \) for several different values of strain rate \( \dot{\gamma} \). We also show results from quasistatic simulations [16, 18], representing the \( \dot{\gamma} \rightarrow 0 \) limit [19]. We use a system with \( N = 1024 \) particles to more explicitly compare with the results of Tighe.
et al., who used a similar size system. While $\mu$ for the models RD$_0$ and CD$_0$ differ slightly at the lower $\phi$, we see that near $\phi_J$ they become essentially equal at the smaller $\dot{\gamma}$, and both RD$_0$ and CD$_0$ approach the quasistatic limit as $\dot{\gamma} \to 0$. We thus conclude that $\mu$ is finite as $\phi$ passes through $\phi_J$, consistent with recent experiments on foams by Lespiat et al. [12].

From our fit to $\eta_p$ in Fig. 2 we conclude that for both models CD$_0$ and RD$_0$ the pressure along the yield stress line, i.e. $\dot{\gamma} \to 0$, $\phi > \phi_J$, vanishes upon approaching $\phi_J$ as $p_0 \sim (\phi - \phi_J)^\gamma$ with $\gamma \approx 1.08$. Our results in Fig. 4 then argue that the shear stress along the yield stress line, $\sigma_0$, vanishes similarly, so that $\mu$ stays finite. However the prediction of Tighe et al. is that the yield shear stress vanishes as $\sigma_0 \sim (\phi - \phi_J)^{3/2}$. Were this conclusion correct, we would expect $\mu \sim (\phi - \phi_J)^{0.42}$, 

vanishing as $\phi \to \phi_J$ from above. Nothing in Fig. 4, where we see that $\mu = \sigma/p$ is a monotonically increasing function as $\phi$ decreases, suggests any such vanishing of $\mu(\phi_J)$. We thus conclude from Fig. 4 that the predicted scaling of $\sigma_0$ by Tighe et al. is not correct, and moreover the two models CD$_0$ and RD$_0$ behave qualitatively the same for both pressure $p$ and shear stress $\sigma$.

To conclude, we have examined the issue of the universality of the jamming transition for overdamped shear-driven frictionless soft-core disks in 2D. We have considered two different dissipative models that have been widely used in the literature: the collisional Durian bubble model CD$_0$ and its mean field approximation RD$_0$. Contrary to previous claims [5] we find clear evidence that CD$_0$ does exhibit a growing macroscopically large length $\xi$ that appears to diverge as the jamming critical point is approached. We further provide strong evidence that CD$_0$ and RD$_0$ are in fact in the same universality class with the same critical exponents at jamming, and have qualitatively the same rheological behavior more generally.

We thank A. J. Liu, B. Tighe, M. van Hecke and M. Wyart for helpful discussions. This work was supported by NSF Grant No. DMR-1205800 and the Swedish Research Council Grant No. 2010-3725. Simulations were performed on resources provided by the Swedish National Infrastructure for Computing (SNIC) at PDC and HPC2N.
SUPPLEMENTAL MATERIAL

Transverse Velocity Correlation Function

The one quantity for which models RD0 and CD0 are clearly different is the transverse velocity correlation function, \(g_y(x) \equiv \langle v_y(0)v_y(x) \rangle\). Defining the normalized correlation, \(G_y(x) \equiv g_y(x)/g_y(0)\), we plot in Fig. S1(a) \(G_y(x)\) vs \(x\), for several different values of strain rate \(\dot{\gamma}\), for model RD0 at \(\phi = 0.8433 \approx \phi_f\) in a system of \(N = 4096\) particles. We see that \(G_y(x)\) has a clear minimum at a distance \(x = \ell\), and that \(\ell\) increases as \(\dot{\gamma} \to 0\) and one approaches the critical point. In Ref. [4] \(\ell\) was interpreted as the diverging correlation length \(\xi\). In CD0, however, it was found [5] that \(G_y(x)\) decreases monotonically without any obvious strong dependence on either \(\phi\) or \(\dot{\gamma}\). In Fig. S1(b) we plot \(G_y(x)\) vs \(x\), for several different \(\dot{\gamma}\), at \(\phi = 0.8433 \approx \phi_f\) in a system of \(N = 4096\) particles, confirming this result.

As an alternative way to consider the difference in this correlation between the two models, we now consider the Fourier transformed correlation \(g_y(k_x) = \int dx g_y(x)e^{ik_xx}\), which we show in Figs. S2(a) and S2(b) for RD0 and CD0 respectively at packing fraction \(\phi = 0.8433 \approx \phi_f\). For RD0 we see a maximum in \(g_y(k_x)\) at a \(k_x^*\) that decreases for decreasing \(\dot{\gamma}\): \(\ell \sim 1/k_x^*\) gives the corresponding minimum of the real-space correlation. For CD0 we show results only for the single strain rate \(\dot{\gamma} = 10^{-6}\) since from Fig. S1(a) we expect no observable difference as \(\dot{\gamma}\) varies. We see an algebraic divergence \(g_y(k_x) \sim k_x^{-1.3}\) as \(k_x \to 0\). It is this algebraic divergence that causes the real space \(G_y(x)\) in CD0 to become solely a function of \(x/L\) as the system length \(L\) increases, as was reported in Ref. [5].

To try and give a qualitative understanding of this differing behavior of \(g_y(k_x)\), we can consider how energy is dissipated in each model. In RD0 the dissipation is \((1/N) \sum_i \langle |\delta v_i|^2 \rangle \approx \int dk \langle \delta v(k) \cdot \delta v(-k) \rangle\). For CD0, however, the dissipation is \((1/N) \sum_{i,j} \langle |v_i - v_j|^2 \rangle \approx \int dk \langle \delta v(k) \cdot \delta v(-k) \rangle |k|^2\), where the sum is over only neighboring particles \(i,j\) in contact. Here \(\delta v\) is the non-affine part of the particle velocity. If we make an equipartition-like ansatz, and assume that as \(k \to 0\) all modes \(k\), and both spatial directions \(x, y\), contribute equally to the dissipation, we would then conclude that for RD0 \(\langle v_y(k)v_y(-k) \rangle \propto \text{constant}\), while for CD0 \(\langle v_y(k)v_y(-k) \rangle \propto 1/k^2\). Noting that \(g_y(k_x) = \int dk_y \langle v_y(k)\rangle \delta(k_x - k_y)\), we then conclude that for RD0 we have \(g(k_x) \propto \text{constant}\) as \(k_x \to 0\), while for CD0 we have the divergence \(g(k_x) \propto 1/k_x\). This saturation of \(g_y(k_x)\) for RD0, as compared to the algebraic divergence of \(g_y(k_x)\) for CD0, is what is qualitatively seen in Fig. S2. The physical reason for this dramatic difference can be viewed as follows. For CD0, since the dissipation depends only on velocity differences, uniform translation of a large cluster of particles with respect to the ensemble average flow has little cost, thus enabling long wavelength fluctuations. For RD0 the dissipation is with respect to a fixed background, so uniform translation of a large cluster causes dissipation that scales with the cluster size; long wavelength fluctuations are suppressed.

That the observed divergence in CD0 is \(\sim k_x^{-1.3}\) rather than the simple \(k_x^{-1}\) predicted above, suggests that our equipartition ansatz is not quite correct, and that the different modes interact in a non-trivial way. That the exponent of this divergence is not an integer or simple rational fraction suggests the signature of underlying critical fluctuations, even though the correlation \(g_y(x)\) itself does not yield any obvious diverging length scale.

Finite-Size-Scaling of Pressure

In Fig. 1 of the main article we showed data for the dependence of pressure \(p\) on system size \(L\) at different strain rates \(\dot{\gamma}\), at the jamming fraction \(\phi_f \approx 0.8433\). We argued that these results provided evidence for a similar growing, macroscopically large, correlation length \(\xi\).
in both models RD₀ and CR₀. Here we attempt a finite-size-scaling analysis of this data. We must note at the outset, however, that our earlier work [13] demonstrated that it is important to consider corrections-to-scaling to get accurate values for the exponents at criticality, and that corrections-to-scaling are in fact large at the smaller sizes L considered in Fig. 1 of the main article [16]. Since our data for p(L) is not extensive enough to try a scaling analysis including corrections-to-scaling, our results based on a fit to Eq. (5) must be viewed as providing only effective exponents describing the data over the range of parameters considered, rather than the true exponents asymptotically close to criticality. We restate Eq. (5),

\[ p(\phi, \gamma, L) = L^{-y/\nu} \mathcal{P}(0, \gamma L^z). \]  

(S1)

We can equivalently write the above in the form

\[ p(\phi, \gamma, L) = z^{y/z\nu} f(\gamma^{1/z}), \]  

(S2)

using \( f(x) \equiv x^{-y/\nu} \mathcal{P}(0, x^z) \). We can now adjust the parameters \( q \equiv y/z\nu \) and \( z \) to try and collapse the data to a single common scaling curve. Plotting \( p/\gamma^q \) vs \( L^{1/z} \) we show the results for RD₀ and CD₀ in Figs. S3(a) and (b). For RD₀ we find the effective exponents \( z = 6.5 \) and \( q = 0.290 \), while for CD₀ we find \( z = 6.0 \) and \( q = 0.317 \). The values of \( z \) found in the present analysis are comparable to the value \( z = 5.6 \) found in the cruder analysis in Fig. 1(b) of the main article. Note that for both models the scaling function \( f(x) \) → constant as \( x \to \infty \), which gives \( p \sim \gamma^q, q \equiv y/z\nu \), in the limit of an infinite sized system.

![FIG. S3. Scaling collapse of pressure according to Eq. (S2) for models RD₀ and CD₀.](image)

The closeness of these fitted effective exponents for the two models is one more piece of evidence that RD₀ and CD₀ behave qualitatively the same, and do not have dramatically different rheology as was claimed by Tighe et al. in Ref. [5].

Finally we consider how the effective exponents found here compare to the true exponents asymptotically close to criticality. From our most accurate analysis [13] of the critical behavior in RD₀, using a large system size \( N = 65536 \) and including the leading corrections-to-scaling, we have found the critical exponents \( q = y/z\nu = 0.28 \pm 0.02 \) and \( y = 1.08 \pm 0.03 \), yielding \( z\nu = 3.9 \pm 0.4 \). This value of \( q \) is in reasonable agreement with that found above from the finite-size-scaling analysis of \( p(\phi, \gamma, L) \).

If we take the value of \( z \approx 6 \) found in the finite-size-scaling analysis, we would then conclude \( \nu \approx 0.65 \). We note that earlier scaling analyses [3, 4] that similarly ignored corrections-to-scaling found similar values for \( \nu \). However our recent [16] more detailed finite-size-scaling analysis of the correlation length exponent, which included corrections-to-scaling, found that \( \nu \approx 1 \), therefore implying \( z \approx 3.9 \) as the true critical value. We thus conclude that the larger than expected value of \( z \) found here from the finite-size-scaling of \( p \) is due to the strong corrections-to-scaling that effect the correlation length at small \( L \).

As another way to see the effect of corrections-to-scaling on the correlation length, in Fig. S4 we plot our results for \( p \) vs \( L \) at \( \phi = 0.8433 \approx \phi_c \), as obtained from quasistatic simulations [16, 18] representing the \( \gamma \to 0 \) limit. From Eq. (S1) we expect as \( \gamma \to 0 \) the behavior, \( p \sim L^{-y/\nu} \). If we fit the data at small \( L \) in Fig. S4 to a power law, we then find the exponent, \( y/\nu \approx 1.79 \). Using \( y = 1.08 \) this then gives \( \nu \approx 0.60 \), in rough agreement with the value of \( \nu \) obtained from the measured \( z \) of our finite-size-scaling of \( p \) with \( \gamma \). If, however, we fit the data at only the largest \( L \) to a power law, we then find the exponent \( y/\nu \approx 1.11 \). Again using \( y = 1.08 \), we then get \( \nu \approx 0.97 \), in better agreement with the expected \( \nu \approx 1 \). Fig. S4 thus shows in a very direct way that corrections-to-scaling are significant for small system lengths \( L \).

![FIG. S4. Pressure \( p \) vs system length \( L \) at \( \phi = 0.8433 \) for quasistatic shearing. Dashed line is a power law fit to the data at the smallest \( L \), giving an exponent \( y/\nu \approx 1.79 \); solid line is a power law fit to the data at the largest \( L \), giving an exponent \( y/\nu \approx 1.11 \).](image)

To conclude this section, although our finite-size-scaling of the pressure data in Fig. 1(a) of the main article is effected by corrections-to-scaling, and so gives a larger value for the dynamic exponent \( z \) than we believe is actually the case at criticality, nevertheless the correlation length \( \xi \) extracted from this data and shown in Fig. 1(b) demonstrates that RD₀ and CD₀ are behaving qualitatively the same, and that both have a macroscopic length scale \( \xi \) that is growing (and we would argue diverging) as the jamming transition is approached.
We wish to verify that the mass parameter \( m = 1 \), which we use in model CD, is indeed sufficiently small so as to put our results in the overdamped \( m \to 0 \) limit corresponding to model CD\(_0\), for the range of parameters studied here. In Fig. S5(a) we show results for the elastic part of the pressure \( p \) vs \( \dot{\gamma} \) for model CD, with \( N = 262144 \) particles, at three different packing fractions: \( \phi = 0.80, \phi = 0.8433 \approx \phi_J \), and \( \phi = 0.85 \). We compare results for two different mass parameters, \( m = 1 \) and \( m = 10 \). We see that the results agree perfectly for small \( \dot{\gamma} \); significant differences are only found for \( \dot{\gamma} \geq 10^{-3} \) which is higher than the largest shear rate used in our scaling analysis. In Fig. S5(b) we similarly compare results for model CD\(_0\) with \( m = 1 \) with explicit results for model CD\(_0\), as obtained from simulations using the more costly matrix inversion dynamics for \( m = 0 \). In this case we are restricted to \( N = 1024 \) particles because our algorithm for CD\(_0\) scales as \( N^2 \). We see that in all cases there is no observed difference between the two models. Thus we conclude that our results from CD with \( m = 1 \) are indeed in the overdamped \( m \to 0 \) limit.

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[19] Quasistatic simulations only accurately compute \( \sigma/p \) for jammed states, where \( \sigma \) and \( p \) are both finite. The contribution from flowing, unjammed states, where \( \sigma/p = \eta/\eta_p \) (where \( \eta = \sigma/\dot{\gamma} \) is the shear viscosity), is not properly accounted for in a naive calculation since \( \sigma \) and \( p \) separately vanish as \( \dot{\gamma} \to 0 \) and so do not contribute to the average. Hence we only show quasistatic results for \( 0.842 \leq \phi \), where for \( N = 1024 \) we find roughly half or more of the configurations sampled during quasistatic shearing are jammed [16]. We note that the model of Tighe et al. [5] applies only above \( \phi_J \), and so is in the region where the quasistatic calculation is valid.