Short-time critical dynamics and universality on a two-dimensional Triangular Lattice

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Abstract

Critical scaling and universality in short-time dynamics for spin models on a two-dimensional triangular lattice are investigated by using Monte Carlo simulation. Emphasis is placed on the dynamic evolution from fully ordered initial states to show that universal scaling exists already in the short-time regime in form of power-law behavior of the magnetization and Binder cumulant. The results measured for the dynamic and static critical exponents, $\theta$, $z$, $\beta$ and $\nu$, confirm explicitly that the Potts models on the triangular lattice and square lattice belong to the same universality class. Our critical scaling analysis strongly suggests that the simulation for the dynamic relaxation can be used to determine numerically the universality.

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1 Introduction

The study of phase transitions and critical phenomena has been an attractive and important topic in statistical physics for a long time [1]. As is well known there appears critical universality due to the infinite spatial and time correlation lengths at the continuous phase transition points, and the universal behavior of a critical system is characterized by a number of critical exponents such that the models in the same universality class have the same values of the critical exponents. Therefore, an estimation of critical exponents to determine the universality class for different statistical models is an interesting challenge. However such research is difficult to be carried out analytically since only few statistical models can be exactly solved [2]. Therefore, numerical simulations supply a powerful method to estimate critical exponents which are measured by generating equilibrium states and averaging over independent configurations in Monte Carlo (MC) studies. For example, the dynamical exponent $z$ can be measured from the exponential decay of time correlation for finite systems in the long-time regime. Unfortunately, MC simulations near the critical point in the equilibrium suffer from critical slowing down (CSD) [3], because the auto-correlation length $\tau$ diverges as $\tau \sim L^z$, where $L$ denotes the linear size of a lattice and $z \geq 2$.

In the last decade, the exploration of critical phenomena has been greatly broadened and much progress has been made in critical dynamics. An important discovery is that there exist critical scaling and universality for systems far from equilibrium [4]. Traditionally, it was believed that universal behavior exists only in the equilibrium or in the long-time regime of dynamical evolutions. However, recent researches in critical dynamics for many statistical models have shown that universal scaling behavior also emerges within the macroscopic short-time regime of dynamic processes after a microscopic time scale $t_{\text{mic}}$ [5, 6]. The investigation of the short-time critical dynamics (STCD) not only exhibits the existence of universal dynamic scaling behavior within the short-time regime, but also supplies a very efficient method to determine the critical exponents.

In Ref.[4], the recent progress in short-time dynamical studies has been reviewed, where several simulation methods and the corresponding numerical results have been presented. Up to now, the values of critical exponents for a variety of statistical models, such as the two-dimensional (2d) Potts and $\phi^4$ models on a square lattice [7, 8, 9], the 2d quantum XY and the random-bond Potts models on the square lattices [10, 11], the 2d FFXY model and (2+1)d SU(2) lattice-gauge models [12, 13], have been calculated precisely. But only very
few attention has been concentrated on the 2d triangular lattices [14, 15].

By the STCD method we can estimate not only the dynamic critical exponent \( z \), but also the static critical exponents \( \theta \), \( \beta \) and \( \nu \). More important, the results for both the dynamic critical exponent \( z \) and static critical exponents, \( \beta \) and \( \nu \), are consistent with those obtained by the traditional MC simulations performed in equilibrium. Furthermore, similar to the measurements of critical exponents, the determination of critical temperatures is also difficult in equilibrium because of the CSD. Now we have an alternative approach, the STCD, by which the critical temperature can be also extracted from the power-law scaling behavior in the critical regime [8]. Therefore, we can perform comprehensive investigations on statistical models by the STCD approach which are independent of the study in equilibrium.

In this paper we present our numerical study for the \( q \)-state Potts model on a 2d triangular lattice by using the STCD method. Generally, in short-time MC simulations, there are two different schemes to calculate the critical exponent: one is based on the power-law behavior and the other on the finite size scaling collapse for different lattice sizes within the non-equilibrium relaxation processes. Our attention is specially paid to the universal short-time evolution from the fully ordered initial state. The calculations show evidence that there exists universal scaling already in the short-time regime where the power-law behavior of magnetization and auto-correlation is observed, as well as the finite size scaling collapse behavior. These results are then applied to estimate the critical exponents and check the corresponding results for different schemes. Our numerical results confirm explicitly the universality proposal in the short-time regime [16].

2 Model and STCD Scaling

The Hamiltonian for the \( q \)-state Potts model with ferromagnetic coupling \( (J > 0) \) defined on a 2d triangular lattice is given by

\[
H = -\frac{J}{2} \sum_{i=1}^{N} \sum_{l_{i}=1}^{6} \delta_{\sigma_{i},\sigma_{l_{i}}}, \quad \sigma_{i,j} = 1, \cdots, q,
\]

where \( \sum_{i} \) represents the sum over the lattice sites on a triangular lattice \( N = L \times L \), and \( l_{i} \) denotes the nearest-neighbors of site \( i \), whose coordinate number is six on the lattice. \( \sigma_{i} \) is a spin variable on every lattice site \( i \). It is known that in equilibrium the Potts model is exactly solvable, and the critical points are \( J_{c} = 0.5493061 \cdots \) and \( J_{c} = 0.6309447 \cdots \) for


$q = 2$ and $q = 3$ respectively \cite{2}. In this work we consider both the $q = 2$ and $q = 3$ cases and calculate their critical exponents ($\theta, z, \beta, \nu$) to investigate the universality explicitly.

As our starting point we consider an $O(n)$ vector model ($n = 1$ for the Ising model) with the dynamics of model A \cite{17} and let it suddenly quench from a very high temperature with small initial magnetization $m_0$ to the critical temperature $T_c$. Janssen, Schaub and Schmittmann \cite{5} argued that at the critical temperature dynamic scaling behavior already emerges within the short-time regime as,

$$M^{(k)}(t, \tau, L, m_0) = b^{-k\beta/\nu}M^{(k)}(b^{z-\beta/\nu}\tau, b^{-1}L, b^{x_0}m_0),$$  \hspace{1cm} (2)

where $M^{(k)}$ is $k$th moment of the magnetization, $\tau = (T - T_c)/T_c$ is the reduced temperature, $\beta$ and $\nu$ are the well known static critical exponents and $b$ is a scaling factor, while $x_0$, a new independent exponent, is the scaling dimension of the initial magnetization $m_0$. Here a MC sweep over all sites on the lattice is defined as the unit of MC time $t$.

For a sufficiently large lattice ($L \to \infty$) and setting $\tau = 0$, $b = t^{1/z}$ in the scaling form Eq.(2), the power-law behavior of time evolution of the magnetization at the critical temperature can be deduced,

$$M(t) \sim m_0 t^{\theta},$$  \hspace{1cm} (3)

where $\theta$ is a new dynamic exponent which characterizes the universality in the short-time regime and is related to $x_0$ by $\theta = (x_0 - \beta/\nu)/z$. The relation shows that, after the microscopic time $t_{mic}$, the magnetization undergoes an initial increase at the critical point and we can easily obtain the result of the exponent $\theta$ based on this power-law form.

On the other hand, one can also focus on dynamic processes starting from an ordered initial state to estimate exponents $\beta/\nu$ and $z$. Although no detailed analytic study has been made about this process, MC simulations in a variety of statistical spin models have shown that there also exists a similar scaling relation \cite{8, 18, 19},

$$M^{(k)}(t, \tau, L) = b^{-k\beta/\nu}M^{(k)}(b^{-z}\tau, b^{1/\nu}\tau, b^{-1}L).$$  \hspace{1cm} (4)

At the exact critical temperature($\tau = 0$) and setting $b = t^{1/z}$, Eq. (4) leads to a power-law decay for the magnetization,

$$M(t) \sim m_0 t^{-\beta/\nu z},$$  \hspace{1cm} (5)
where $M(t)$ is defined by

$$ M(t) = \frac{1}{N} \langle \sum_i \sigma_i(t) \rangle, \quad (q = 2), $$

and

$$ M(t) = \frac{3}{2N} \langle \sum_i (\delta_{\sigma_i(t),1} - \frac{1}{3}) \rangle, \quad (q = 3). $$

Here $N = L \times L$ is also the total number of spins defined on the lattices and $\langle \cdots \rangle$ denotes the average over independent initial configurations and/or random number sequences.

However, the independent determination of $1/\nu$ seems slightly more complicated than $z$ and $\beta/\nu$. We should differentiate $\ln M(t, \tau)$ with respect to $\tau$ at the critical point

$$ \partial_\tau \ln M(t, \tau) |_{\tau=0} = t^{1/\nu} \partial_\tau' \ln M(t', \tau') |_{\tau'=0}. $$

The exponent $1/\nu$ can be determined from this power-law behavior if the value of $z$ is known. In the practice of our numerical calculations, the differential to $\tau$ is substituted with a reasonable small difference $\Delta \tau$.

Furthermore the dynamic exponent $z$ can be determined independently. In order to do so, we introduce a Binder cumulant

$$ U(t, L) = \frac{M^{(2)}(t)}{(M(t))^2} - 1. $$

Here the second moment of the magnetization $M^{(2)}(t, L)$ is defined as

$$ M^{(2)}(t) = \frac{1}{N^2} \langle \left( \sum_i S_i(t) \right)^2 \rangle, \quad (q = 2), $$

and

$$ M^{(2)}(t) = \frac{9}{4N^2} \langle \left( \sum_i (\delta_{\sigma_i(t),1} - \frac{1}{3}) \right)^2 \rangle, \quad (q = 3), $$

respectively. For sufficiently large lattices, finite size scaling analysis shows a power-law increase of the Binder cumulant,

$$ U(t, L) \sim t^{d/z}. $$

Based on this feature the dynamic exponent $z$ can be determined and then be used to calculate the exponent $1/\nu$ through Eq.(8).
As done by many authors [7, 11, 19], the critical exponents can be measured both from disordered initial states and ordered initial states. We perform our investigations involving these two different initial states for MC measurements to obtain more reliable results. Particularly we paid much attention to the scaling forms which describe the dynamic processes from the ordered initial state instead of the disordered initial states to avoid the finite $m_0$ effect on the measured results. There is also an advantage to reduce the statistical fluctuations for determination of critical exponents and critical temperatures from ordered initial dynamic processes.

Then, there is another scheme to estimate these critical exponents, $\beta/\nu$ and $z$, based on the finite size scaling collapse [7]. The results estimated should be consistent with those obtained by the power-law behavior described above. This scheme gives an independent check whether our dynamic MC simulations are reliable or not. Set $\tau = 0$ and $m_0 = 0$, according to the dynamic scaling behavior Eq. (2), the simple scaling relation for the second moment magnetization $M^{(2)}(t, L)$ is easily obtained

$$M^{(2)}(t, L) = b^{-2\beta/\nu}M^{(2)}(b^{-z}t, b^{-1}L).$$ (13)

Using this scaling relation Eq.(13) for a pair of curves of $M^{(2)}(t, L)$ and $M^{(2)}(t', L')$ with $t' = t/2^z$ and $L' = L/2$, we can estimate the critical exponents $\beta/\nu$ and $z$ by searching for the best fit of collapse of the curve $M^{(2)}(t', L')$ through a global scaling factor $2^{2\beta/\nu}$. Also if we input the value of $z$ got from the power-law the Eq. (12), the value of $\beta/\nu$ can also be estimated by the scaling relation Eq.(13).

3 Simulation and Results

It has been demonstrated in previous works [8, 20, 21] that the results obtained by the Metropolis and Heat-bath algorithms are consistent with each other and the latter is somewhat more efficient than the former. Therefore, in the present paper, the MC simulation is only performed by the Heat-bath algorithm. Samples for average are taken over 150,000 independent initial configurations on the $L^2$ triangular lattices with the periodic boundary conditions using $L = 32, 64$ and 128. Statistical errors are simply estimated by performing three groups of averages selecting different random seeds for the initial configurations. Our simulation is performed at or near the critical temperature.

We begin the study by considering the dynamic process starting from random initial states
with small $m_0$ to determine the critical exponent $\theta$. These disordered initial configurations with given values of $m_0$ are prepared by the sharp preparation method [8], and the evolutions of the magnetization $M(t)$ are measured as a function of MC time $t$ for $m_0 = 0.04, 0.02$ and $0.01$ on the $N = 128^2$ lattice. The curves of $M(t)$ in a log-log scale are shown in Figs. 1 and 2 for the $q = 2$ and $q = 3$ model, respectively. Obviously, there exist very nice power-law increases of $M(t)$ after $t_{\text{mic}} \simeq 10$ and all the curves are almost parallel to each other. Thus the $\theta$ can be estimated from the slopes of the curves in the regime of $t=[10, 200]$. In Table 1, the values of $\theta$ as a function of initial magnetizations $m_0$ are presented for $q = 2$ and $q = 3$. Then the final results of $\theta$, after an extrapolation to $m_0 \to 0$, are summarized in Table 2 both for the models on the triangular and square lattice. It is found that the values of $\theta$ on the triangular lattice are the same as those on the square lattice. This calculation shows first evidence of the dynamic universality in the short-time regime.

Secondly we study the evolution of magnetizations in the initial stage of the dynamic relaxation starting from fully ordered initial states. In Figs.3 and 4, the power-law behavior of the Binder cumulant $U(t)$ for $q=2$ and $q=3$ on a lattice with $N = 128^2$ is displayed on a log-log scale, and the nice power-law of $U(t)$ is clearly exhibited. The exponent $z$ can be estimated from the slope of the curves: $z = 2.145(3)$ for $q=2$ and $z = 2.148(4)$ for $q=3$. In Table 2, the values of $z$ measured for the two models are presented. For a comparison, we also list in Table 2 the $z$ values of the corresponding results on the square lattice. Then, we investigate the short-time behavior of $M(t)$ starting from the fully ordered initial state. The time evolution of $M(t)$ for the two models is displayed in Fig.5 and Fig.6 respectively. Here it is shown that the finite size effect is small, since there is almost no difference for the curves of the different lattice sizes. From the slopes of these power-law decay curves the values of the index $\beta/\nu z$ is determined. Then the values of $\beta/\nu$ are estimated by input of the $z$ values. The results are listed in Table 2 for both $q=2$ and $q=3$. Our measured results of $\beta/\nu$ are more accurate than those obtained in the previous works, compared with the exact values of $\beta/\nu = 1/8$ for the $q=2$ (Ising) and $\beta/\nu = 2/15$ for the $q=3$ Potts model.

Next the MC simulations for $\partial_t \ln M(t)$ are carried out by taking the difference $\Delta \tau = 0.02$ on $L=128, 64$ and $32$ lattices. In Figs.7 and 8 we plot the evolution of $\partial_t \ln M(t)$ for the two models respectively. The nearly complete overlap of these curves on the different lattice sizes indicates that the finite size effect can be ignored. We also notice that the power-law behavior is not yet seen before $t \approx 20$, as the effect of microscopic time scale $t_{\text{mic}}$. 
Table 1: The measured values of $\theta$ versus the initial $m_0$ for the Ising model ($q = 2$) and Potts model ($q = 3$) on a $N = 128^2$ lattice. The last column gives the results of $\theta$ after extrapolation to $m_0 = 0$.

| $m_0$ | 0.04 | 0.02 | 0.01 | 0.00 |
|-------|------|------|------|------|
| $q = 2$ | 0.183(1) | 0.185(1) | 0.189(2) | 0.191(2) |
| $q = 3$ | 0.101(1) | 0.089(1) | 0.082(1) | 0.076(2) |

Therefore, we measure the exponent $1/\nu z$ from the interval [50,500], and the measurements yield the results $1/\nu=1.027(6)$ for the $q=2$ model and $1/\nu=1.223(5)$ for $q=3$. These results are included in Table 2 and compared with those on the square lattice.

Finally, we study the time evolution of $M^{(2)}(t, L)$ with completely random initial states ($m_0=0$) on lattices up to $L = 128$ to estimate the critical exponent $\beta/\nu$ and $z$ by the finite size scaling collapse scheme, Eq.(13), as an alternative way. Figs.9 and 10 show the time evolution of the second moment of the magnetization $M^{(2)}(t, L)$ for $L = 128, 64, 32, 16$ by solid lines. The dots fitted to the lines show the data collapse for the rescaled variables $L' = L/2$, $t' = t/2^z$ and global scaling factor $2^{2\beta/\nu}$, with the standard least square fitting algorithm. In other words, we rescaled the $M^{(2)}(t, L)$ lines on a larger lattice to a smaller lattice $L' = L/2$ by the scaling factor $2^z$ and the $2^{2\beta/\nu}$ for the pair of different lattice sizes. The average values for $2\beta/\nu$ and $z$ of $q = 2$ and $q = 3$ are also given in the Table 2. This step gives a further independent check that the STCD method is efficient and the results of our dynamic MC simulations are reliable.

4 Summary and Conclusion

We have systematically investigated the short-time critical dynamics of the $q = 2$ (Ising) and $q = 3$ Potts models on the 2d triangular lattices by a large-scale dynamic MC simulation. Firstly, by observing the power-law increase of the magnetization $M(t)$ from random initial states, the new dynamic exponent $\theta$ has been calculated and its values are the same as those for the corresponding models on the square lattice. Then we paid our attention particularly to the time evolutions of the Binder cumulant $U(t)$, magnetization $M(t)$ and the derivative $\partial_t \ln M(t)$ from the fully ordered initial state to determine the dynamic exponent $z$ and the static critical exponents $\beta$ and $\nu$. Finally we checked our results independently by using the
Table 2: Results for the critical exponents $\theta$, $2\beta/\nu$, $1/\nu$ and $z$ for $q=2$ and $q=3$ on 2d triangular lattice. For comparison, the values on the square lattice are also listed.\(^4\)

| $q$=2 (Triangular) | $\theta$       | $2\beta/\nu$ | $1/\nu$ | $z$       |
|---------------------|----------------|--------------|---------|-----------|
| power law results    | 0.191(2)       | 0.250(5)     | 1.027(6)| 2.145(3)  |
| scaling collapse results | 0.252(2) | 2.153(2)   |         |           |
| (Square, Ref.[4])    | 0.191(1)       | 0.240(15)    | 1.03(2) | 2.155(3)  |
| exact               | $1/4$          | $1$          |         |           |

| $q$=3 (Triangular) | $\theta$       | $2\beta/\nu$ | $1/\nu$ | $z$       |
|---------------------|----------------|--------------|---------|-----------|
| power law results    | 0.076(2)       | 0.256(6)     | 1.223(8)| 2.148(4)  |
| scaling collapse results | 0.266(2) | 2.191(1)   |         |           |
| (Square, Ref.[4])    | 0.075(3)       | 0.269(7)     | 1.24(3) | 2.196(8)  |
| exact               | $4/15$         | $1.2$        |         |           |

finite-size scaling collapse scheme to the time evolutions of $M^{(2)}(t, L)$ from the completely random initial states. The advantage of the STCD method, by our experience, is that the CSD is eliminated because the measurements are carried out at the beginning of the evolution process, rather than in the equilibrium where all the correlation lengths are divergent.

In conclusion, the dynamic relaxation in the short-time regime for the $q = 2$ and $q = 3$ Potts models is studied on the 2d triangular lattice, and it is confirmed that, by comparing our estimated results $\theta$, $\beta$ and $\nu$ to those on the 2d square lattice, they belong to the same universality class for the corresponding models on different lattices respectively. Our critical scaling analysis strongly suggests that the simulation for the dynamic relaxations can be used to confirm numerically the universality. Further application of STCD is interesting to investigate the spin systems with quenched disorder\(^{[11]}\) and quantum phase transitions for their universal behavior.

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Figure 1: Plot of $M(t)$ versus $t$ on a log-log scale for the Ising model on a $N = 128^2$ lattice. The lines present the different initial magnetizations $m_0=0.04,0.02,0.01$ (from top to bottom).

Figure 2: The same as Fig.1, but for the $q=3$ Potts model.
Figure 3: Time evolution of Binder cumulant $U(t)$ on a log-log scale for the Ising model on a $N = 128^2$ lattice.

Figure 4: The same as Fig.3, but for the $q=3$ Potts model.
Figure 5: Power-law decay of $M(t)$ versus $t$ on a log-log scale for the Ising model with $m_0 = 1$. The lines for different lattices of $L=128,64,32$ almost overlap.

Figure 6: The same as Fig.5, but for the $q=3$ Potts model.
Figure 7: Curves of $\partial_t \ln M(t)$ on a log-log scale for the Ising model with $m_0 = 1$ for different lattices of $L=128,64,32$. It is obvious that the finite size effect seems small, and the power-law behavior exists after $t_{mic} \simeq 20$.

Figure 8: The same as Fig.7, but for the $q=3$ Potts model.
Figure 9: Curves of the second moment of the magnetization $M^{(2)}(t, L)$ for the Ising model with $m_0 = 0$. The solid lines present $M^{(2)}(t, L)$ for different lattices of $L=16, 32, 64, 128$ (from top to bottom) The dotted lines present the rescaled $M^{(2)}(t', L')$. (◇ denote rescaling 32 → 16, + denote 64 → 32 and □ denote 128 → 64)

Figure 10: The same as Fig.9, but for the $q=3$ Potts model.