The Dilatation Operator of Conformal $\mathcal{N} = 4$ Super Yang-Mills Theory

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Abstract

We argue that existing methods for the perturbative computation of anomalous dimensions and the disentanglement of mixing in $\mathcal{N} = 4$ gauge theory can be considerably simplified, systematized and extended by focusing on the theory’s dilatation operator. The efficiency of the method is first illustrated at the one-loop level for general non-derivative scalar states. We then go on to derive, for pure scalar states, the two-loop structure of the dilatation operator. This allows us to obtain a host of new results. Among these are an infinite number of previously unknown two-loop anomalous dimensions, new subtleties concerning ’t Hooft’s large $N$ expansion due to mixing effects of degenerate single and multiple trace states, two-loop tests of various protected operators, as well as two-loop non-planar results for two-impurity operators in BMN gauge theory. We also put to use the recently discovered integrable spin chain description of the planar one-loop dilatation operator and show that the associated Yang-Baxter equation explains the existence of a hitherto unknown planar “axial” symmetry between infinitely many gauge theory states. We present evidence that this integrability can be extended to all loops, with intriguing consequences for gauge theory, and that it leads to a novel integrable deformation of the XXX Heisenberg spin chain. Assuming that the integrability structure extends to more than two loops, we determine the planar three-loop contribution to the dilatation operator.
1 Introduction and overview

Conformal quantum field theories have been fascinating theoretical physicists for a long time. In two dimensions they are arguably the most important class of field theories: Mathematically, they are quite tractable once one puts to full use the fact that the conformal algebra has an infinite number of generators. In consequence many exact results on the spectrum of operators and the structure of correlation functions may be derived. Apart from their pivotal importance to string theory, they are physically of great value due to their relationship with critical phenomena and integrable models of statistical mechanics. For example, in many cases the representation theory of 2D conformal field theories fixes the scaling dimensions of local operators, which in turn are often related to critical exponents of experimentally relevant systems of solid states physics.

In four dimensions conformal symmetry was long believed to play only a minor role. The group has only finitely many generators, the QFT's relevant to particle physics are certainly not conformal, and the only, trivial, example seemed to be free, massless field theory. However, after the discovery of supersymmetry, it has become clear that supersymmetric gauge theories can be exactly conformally invariant on the quantum level, and in many cases the phase diagram of such gauge theories contained conformal points or regions, quite analogously to phase diagrams in two dimensions. There is therefore obvious theoretical interest in increasing our understanding of these phases. In particular, the 4D gauge theory with the maximum possible number $\mathcal{N} = 4$ of rigid supersymmetries, discovered in 1976 \cite{1,2}, has a superconformal phase \cite{3,5}.

Supersymmetric gauge theories are intimately connected to superstring theory. In fact, the $\mathcal{N} = 4$ action was originally discovered \cite{1} by considering the low-energy limit of superstrings. Surprisingly, with the advent of the AdS/CFT correspondence (see \cite{6,7} for comprehensive reviews) it was argued that the $\mathcal{N} = 4$ gauge theory is, via duality, a superstring theory in a particular background. This conjecture woke the Sleeping Beauty and resulted in a large number of investigations into the structure of $\mathcal{N} = 4$. In particular the representation theory of the superconformal symmetry group $\text{SU}(2,2|4)$ \cite{8} was investigated more closely \cite{9,10}, and numerous non-renormalization theorems were derived, see e.g. \cite{11}. In addition, some unexpected non-renormalization theorems that do not follow from $\text{SU}(2,2|4)$ representation theory were found \cite{12}. Once thought to be somewhat boring, it gradually became clear that conformal $\mathcal{N} = 4$ is an extremely rich and non-trivial theory with many hidden secrets. We will see more examples for its intricate structure in the present paper.

While many of the recently obtained new results on $\mathcal{N} = 4$ in principle could have been derived a long time ago, the AdS/CFT correspondence clearly helped immensely in formulating the right questions. The latest chapter in exploiting this useful duality involved taking a certain limit on both sides of the correspondence: On the string side, after a process termed Penrose contraction of the AdS space, one obtains a new maximally supersymmetric background for the IIB superstring \cite{13,14} which has the great advantage that the spectrum of free, massive string modes can be found exactly \cite{15,16}. On the gauge theory side, one considers the so-called BMN limit \cite{17}. It involves studying infinite sequences of operators containing an increasing number of fields. The relation-
ship to plane wave string excitations is then made by comparing the conformal scaling
dimension of these high-dimension operators to the masses of string excitations [17].
One therefore had to develop techniques to efficiently compute in perturbation theory
the scaling dimensions of operators containing an arbitrary number of fields. This was
done in various papers, on the planar [17–21] and non-planar level [22–28], extending
earlier work on protected half BPS [29–32] and quarter BPS operators [33]. In [34] it was
realized, following important insights by [35,36] (see also [37]) that these well-established
techniques can be considerably simplified and extended, as we will now explain.

The standard way to find the scaling dimensions $\Delta_\alpha$ of a set of conformal fields $\hat{O}_\alpha$, employed by all of the just mentioned papers, is to consider the two-point functions

$$\langle \hat{O}_\alpha(x) \hat{O}_\beta(0) \rangle = \frac{\delta_{\alpha\beta}}{|x|^{2\Delta_\alpha}}. \tag{1.1}$$

Here the form of the two-point function is determined by conformal symmetry alone.
Finding the scaling dimension $\Delta_\alpha$ of a conformal field $O_\alpha$ is more subtle and, in general,
requires an understanding of the dynamics of the theory. In the free field limit (i.e. zero
coupling constant $g_{YM} = 0$) $\Delta$ equals the naive classical (tree level) scaling dimension ob-
tained by standard power counting. For weak coupling $g_{YM}$, we can compute corrections
to the naive dimension by perturbation theory. In practice, extracting these corrections
from eq.(1.1) is somewhat tedious and fraught with various technical complications. For
one, while the corrections are perfectly finite numbers, the perturbative computation of
two-point functions such as eq.(1.1) leads to spurious infinities, requiring renormaliza-
tion¹. Secondly, the definition of the scaling dimensions tacitly assumes that we have
already found the correct conformal operators $\hat{O}_\alpha$. However, if one starts with a set
of naive operators $O_\alpha$ with the same engineering (tree-level) dimension one generically
encounters the phenomenon of mixing: The two-point function is not diagonal in $\alpha, \beta$,
and one rather has a matrix $\langle O_\alpha(x) O_\beta(0) \rangle$ since a generic field does not have a definite
scaling dimension. It therefore seems that we have to diagonalize the two-point functions,
after renormalization, order by order in perturbation theory.

Here we would like to shift the attention away from the two-point functions eq.(1.1)
and rather focus on the dilatation operator acting on states at the origin of space-time
(in a radial quantization scheme) as was already done, for one-loop, in [34] (on the planar
level) and in [33] (in the BMN limit). Its eigensystem consists of the eigenvalues $\Delta_\alpha$ and
the eigenstates $\hat{O}_\alpha$. One thus has

$$D \hat{O}_\alpha = \Delta_\alpha \hat{O}_\alpha. \tag{1.2}$$

By way of example we will restrict the discussion to fields solely composed of the six
scalar fields $\phi_n = \phi_n^{(a)} T^a$ ($n = 1, \ldots, 6$, $a = 1, \ldots, N^2 - 1$) of the $\mathcal{N} = 4$ SU($N$) gauge
theory, but the extension to a larger class of fields is clearly possible. (Our notation and
conventions are explained in App. A) The classical dilatation operator $D_0$ then simply

¹An alternative method consists in extracting the anomalous dimensions from four-point functions.
This is how e.g. the two-loop anomalous dimension of the Konishi field was first found [32,33]. However,
this is not really different, as one can look at these four-point functions, via a double operator product
expansion, as a generator of two-point functions regulated by a different method, namely point-splitting.
counts the total number of scalar fields, as the engineering dimension of scalars is one. This can be formally written as

\[ D_0 = \text{Tr} \, \Phi \, \hat{\Phi}, \quad (1.3) \]

where we introduced the notation \( \hat{\Phi} \) for the variation with respect to the field \( \Phi \). \(^2\)

\[ \hat{\Phi}_m = \frac{\delta}{\delta \Phi_m} = T^a \frac{\delta}{\delta \Phi_m^{(a)}}. \quad (1.4) \]

In an interacting theory the scaling dimensions and therefore also the dilatation generator depend on the coupling constant. In perturbation theory the dilatation generator can be expanded in powers of the coupling constant

\[ D = \sum_{k=0}^{\infty} \left( \frac{g_{YM}^2}{16\pi^2} \right)^k D_{2k}, \quad (1.5) \]

and we shall denote \( D_{2k} \) as the \( k \)-loop dilatation generator. The full one-loop contribution \( D_2 \) was first worked out, in the disguise of an “effective vertex”, in appendix C of \( [25] \) (but see also \( [22, 23, 26, 20, 37, 34] \)): \(^3\)

\[ D_2 = - \text{Tr} [\Phi_m, \bar{\Phi}_n] [\hat{\Phi}_m, \hat{\Phi}_n] - \frac{1}{2} \text{Tr} [\Phi_m, \hat{\Phi}_n] [\hat{\Phi}_m, \bar{\Phi}_n]. \quad (1.6) \]

However, it was shown in \( [34] \) that one can bypass the consideration of two-point functions and directly diagonalize \( D_2 \) in any convenient, linearly independent and complete basis. In eq.\((1.6)\) the normal ordering notation indicates that the derivatives do not act on the fields enclosed by \( : \cdot : \). To see how \( D_2 \) acts, let us consider the simplest scalar fields of the theory, which have classical dimension two. One has the protected chiral primaries \( Q_{nm} \) as well as the Konishi field \( \mathcal{K} \)

\[ Q_{nm} = \text{Tr} \, \Phi_n \Phi_m - \frac{1}{6} \delta_{nm} \text{Tr} \, \Phi_k \Phi_k \quad \text{and} \quad \mathcal{K} = \text{Tr} \, \Phi_k \Phi_k. \quad (1.7) \]

One immediately verifies\(^4\), using eqs.\((1.3), (1.4), (1.5), (1.6)\) that, to one loop, as \( D_2 Q_{nm} = 0 \) and \( D_2 \mathcal{K} = 12N \, \mathcal{K} \),

\[ \Delta_Q = 2 \quad \text{and} \quad \Delta_{\mathcal{K}} = 2 + \frac{3g_{YM}^2 N}{4\pi^2}, \quad (1.8) \]

that is the one-loop anomalous dimension of \( Q_{nm} \) vanishes as expected, while the anomalous dimension of the Konishi scalar agrees with the well-known result \( [10] \).

\(^2\)In the language of canonical quantization the field \( \Phi \) and the variation \( \hat{\Phi} \) correspond to creation and annihilation operators, respectively. The dilatation generator corresponds to the Hamiltonian and a vacuum expectation value is obtained by setting the fields to zero, \( \ldots |_{\phi=0} \).

\(^3\)Note that this expression is perfectly valid also for \( \text{SO}(N), \text{Sp}(N) \) and exceptional gauge groups.

\(^4\)The fission rule \( \text{Tr} A \hat{\Phi}_m B \Phi_n = \delta_{mn} \text{Tr} A \text{Tr} B \) and the fusion rule \( \text{Tr} A \hat{\Phi}_m \text{Tr} \Phi_n B = \delta_{mn} \text{Tr} AB \) are useful when calculating the action of \( D_2 \). These are the \( \text{U}(N) \) rules but due to the commutators in \( D_2 \) there is no difference when using the \( \text{SU}(N) \) rules.
Let us next look at a less trivial example and consider the following set of unprotected \( SO(6) \) invariant operators of classical dimension four, spanning a four-dimensional space of states:

\[
\mathcal{O} = \begin{pmatrix}
\text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_m \Phi_n \\
\text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_m \Phi_n \\
\text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_n \Phi_n \\
\text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_n \Phi_n
\end{pmatrix}.
\] (1.9)

For the convenience of the reader we are optically separating double and single trace sectors by solid lines. This case was studied in detail in [41], using the standard procedure (see also [32]) of directly analyzing, employing \( N = 1 \) superspace Feynman rules, the set of two-point functions of the four composite fields, cf. our above discussion surrounding eq.(1.1). Using eq.(1.6), we find the following action of the one-loop dilatation generator

\[
D_2 = N \begin{pmatrix}
0 & 4 & -\frac{20}{N} & \frac{20}{N} \\
0 & 24 & -\frac{24}{N} & \frac{24}{N} \\
-\frac{2}{N} & \frac{4}{N} & 8 & -4 \\
-\frac{1}{N} & \frac{1}{N} & -4 & 18
\end{pmatrix}.
\] (1.10)

The matrix notation indicates the fields that are generated by the action of \( D_2 \) from the ones in eq.(1.9), e.g.

\[
D_2 \text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_m \Phi_n = 0 \text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_m \Phi_n + 4N \text{Tr} \Phi_m \Phi_n \text{Tr} \Phi_n \Phi_n - 20 \text{Tr} \Phi_m \Phi_n \Phi_m \Phi_n + 20 \text{Tr} \Phi_m \Phi_n \Phi_n \Phi_n.
\] (1.11)

We see that the states in eq.(1.9) are not eigenstates of the dilatation operator: The latter linearly mixes all operators in the four-dimensional vector space [41]. However, diagonalizing the dilatation matrix \( D_2 \) (we will throughout this paper take the freedom of using the same notation \( D_{2k} \) for both the dilatation operator and its matrix representations) in eq.(1.10) will in one go resolve the mixing problem and yield the anomalous dimensions: The conformal operators are given, up to normalization, by the eigenvectors of eq.(1.10) (in the basis eq.(1.9)), and their respective one-loop anomalous dimensions correspond to the eigenvalues of eq.(1.10). The latter are obtained by finding the roots of the characteristic polynomial \( \det(\omega - \frac{1}{2}D_2) \):

\[
\omega^4 - 25\omega^3 + \left(188 - \frac{160}{N^2}\right)\omega^2 - \left(384 - \frac{1760}{N^2}\right)\omega - \frac{7680}{N^2} = 0.
\] (1.12)

The dimensions of the four conformal operators are

\[
\Delta = 4 + \frac{g_{\text{YM}}^2 N}{8\pi^2} \omega,
\] (1.13)

with \( \omega \) being the four different roots of the above quartic equation. They are easily seen to agree with the ones obtained in [41]. Comparison with the methodology of [41] clearly demonstrates the great simplifications resulting from the direct use of the dilatation
In this paper we will find the two-loop contribution $D_4$ to the scalar dilatation operator $D$ eq.(1.5), with the restriction to operators in the SO(6) representations $[p,q,p]$ with tree-level scaling dimension $2p+q$. These fields may always be constructed out of two complex scalars, say $Z = \frac{1}{\sqrt{2}}(\Phi_5 + i\Phi_6)$ and $\phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$. We will argue that operators composed by these two fields form a closed set under dilatation. The idea is then to inspect all Feynman diagrams contributing to the dilatation operator and write down the possible linear combinations of terms contributing to $D_4$. We fix some unknown constants by using gauge invariance, and proven two-loop non-renormalization theorems [30,31]. To save us extra work, we finally use a result conjectured in the context of BMN gauge theory in [42] and derived in [18,43] to determine the remaining coefficients. All this is done without explicitly working out a single Feynman diagram! One finds

$$D_0 = \text{Tr}Z\tilde{Z} + \text{Tr}\phi\tilde{\phi},$$
$$D_2 = -2:\text{Tr}[\phi,Z][\tilde{\phi},\tilde{Z}];$$
$$D_4 = -2:\text{Tr}[[\phi,Z],[\tilde{Z}]][[\tilde{\phi},\tilde{Z}],Z];$$
$$- 2:\text{Tr}[[\phi,Z],[\tilde{\phi}]][[\tilde{\phi},\tilde{Z}],\phi];$$
$$- 2:\text{Tr}[[\phi,Z],T^a][[\tilde{\phi},\tilde{Z}],T^a];$$

(1.14)

where we have also explicitly rewritten the action of $D_0$ and $D_2$ on two complex scalars. Let us check eq.(1.14) and apply our result to work out the two-loop dimension of the Konishi field. We cannot directly apply $D_4$ to $\mathcal{K}$ in eq.(1.7) as we have not yet worked out the terms relevant to scalar fields containing SO(6) traces. However, it is well-known that the following field is a Konishi descendant, with identical anomalous correction to its classical dimension four:

$$\mathcal{K}' = \text{Tr}[\phi,Z][\phi,Z].$$

(1.15)

Applying eq.(1.14) to $\mathcal{K}'$ we find

$$\Delta_{\mathcal{K}'} = 4 + \frac{3g^2_{\text{YM}}N}{4\pi^2} - \frac{3g^4_{\text{YM}}N^2}{16\pi^4}.$$  

(1.16)

This agrees with the results of [38,39].

Recently it was discovered by Minahan and Zarembo that properties of pure scalar states in the planar, one-loop sector of $\mathcal{N} = 4$ Super Yang-Mills theory can be described by an integrable SO(6) spin chain [37]: The planar version of the one-loop dilatation operator maps onto the spin chain Hamiltonian. Interestingly, we are able to derive from this spin chain picture new results for the gauge theory. The integrability ensures, via the existence of an $R$-matrix satisfying the Yang-Baxter equation, the existence of

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5 Incidentally, one easily verifies that our dilatation matrix $D_2$ in eq.(1.14) is, up to an overall constant, precisely given by the product of the one-loop mixing matrix eq.(5.4) of [41] times the inverse of the tree-level mixing matrix eq.(5.3) of [11], in agreement with the results of [33], where the relation between the traditional method and the dilatation operator method was explained in detail. This furnishes a nice, explicit example illustrating that much of the combinatorial complications of the usual method cancel out from the physics.
further conserved charges in addition to the Hamiltonian. Here we shall find that the first non-trivial conserved charge of the spin chain translates into an “axial” symmetry of $\mathcal{N} = 4$ Super Yang-Mills theory at $N = \infty$. The latter transforms between states of opposite “parity” with respect to reflection of color traces.

What is more, this symmetry turns out to be conserved by the two loop interactions as well. This might indicate that $\mathcal{N} = 4$ gauge theory can be used as a source for new integrable spin chain Hamiltonians with interactions more subtle than just the standard nearest neighbor ones: At one-loop, we have the standard XXX$_{1/2}$ Heisenberg spin chain, which is exactly integrable. Higher loop quantum corrections constitute deformations of the spin chain with a smooth deformation parameter $g_{\text{YM}}^2$. At $\mathcal{O}(g_{\text{YM}}^{2k})$ the spin chain interactions involve $k$ nearest neighbors. When all interactions up to $\mathcal{O}(g_{\text{YM}}^{2k})$ are included, the model’s charges commute only up to terms of higher order. Therefore, in order for the deformed model to still be exactly integrable, all quantum corrections (any $k$) have to be taken into account. Our integrable deformation is thus, for finite $g_{\text{YM}}$, non-local. In a certain sense locality is recovered as $g_{\text{YM}}$ is tuned to zero.

Assuming the symmetry to hold at the three-loop level allows us to fix the planar version of the three-loop dilatation operator for scalar operators of the type discussed above. E.g. the resulting dilatation operator implies the following result for the three-loop planar contribution to $\Delta_{K'}$:

$$\Delta_{K'} = 4 + \frac{3g_{\text{YM}}^2N}{4\pi^2} - \frac{3g_{\text{YM}}^4N^2}{16\pi^4} + \frac{21g_{\text{YM}}^6N^3}{256\pi^6}.$$ \hfill (1.17)

Our paper is organized as follows. We start by recalling the derivation of the one-loop dilatation generator in Sec. 2 and move on to determine its two-loop version in Sec. 3. Section 4 is devoted to the application of the resulting dilatation generator to lower dimensional operators and contains in particular a discussion of certain peculiarities of the large-$N$ expansion related to degeneracy and mixing of single and multiple trace states. Next follows in Sec. 5 a treatment of two-impurity operators of arbitrary finite dimension in the spirit of reference [20], leading in particular to an exact expression for the planar two-loop anomalous dimension for any such operator. As a corollary we show in Sec. 6 that the two-loop contribution to the quantum mechanical Hamiltonian describing BMN gauge theory [34] can be simply expressed in terms of its one-loop counterpart which suggests an all genera version of the celebrated all-loop BMN square root formula [17, 18, 43]. In Sec. 7 we discuss the spin chain description of $\mathcal{N} = 4$ Super Yang-Mills theory [37] and point out certain consequences thereof, cf. the discussion above. Finally, in Sec. 8 we discuss possible implications for arbitrarily high loop orders. In particular, we speculate about how fully exploiting the integrability might lead to exact planar anomalous dimensions. In addition we conjecture the existence of novel integrable deformations of the standard Heisenberg spin chain. We furthermore call the attention of the reader to App. D which contains a collection of previously unknown anomalous dimensions for lower dimensional operators.
2 One-loop revisited

The generator of dilatations on scalar fields at one-loop is given by eq. (1.6). At one-loop mixing between operators with different kinds of fields is irrelevant. Therefore (1.6) alone determines the one-loop anomalous dimensions of operators whose leading piece is purely made out of scalar fields.

In the following we will rederive this result to make the reader familiar with the methods and transformations in a rather simple context. All these steps will reappear at two-loops in a more involved fashion. We start by introducing the notation at tree-level.

We sketch the renormalization procedure at one-loop and apply it to obtain the one-loop dilatation generator.

2.1 Tree-level

We obtain the anomalous dimensions of a set of operators \( O_\alpha \) by formally evaluating their two-point functions. For that we distinguish between fields at points \( x \) and 0 by the superscript \( \pm \)

\[
\Phi^+ = \Phi_m(x), \quad \Phi^- = \Phi_m(0). \tag{2.1}
\]

The operators \( O_\alpha \) are constructed as traces of scalar fields

\[
O_\alpha(\Phi) = \text{Tr} \Phi_m \Phi_n \Phi_{m} \cdots \text{Tr} \Phi_n \cdots, \quad O_\alpha^\pm = O_\alpha(\Phi^\pm), \tag{2.2}
\]

where \( \alpha \) enumerates the operators. The tree-level two-point function can formally be written as

\[
\langle O_\alpha^+ O_\beta^- \rangle_{\text{tree}} = \exp \left( W_0(x, \Phi^+, \Phi^-) \right) O_\alpha^+ O_\beta^- |_{\Phi=0}, \tag{2.3}
\]

where \( W_0 \) is the tree-level Green function

\[
W_0(x, \Phi^+, \Phi^-) = I_{0x} \text{Tr} \Phi_m^+ \Phi^-_m. \tag{2.4}
\]

The scalar propagator \( I_{xy} \) is defined in (B.1). In order for the result to be non-vanishing all the scalar fields \( \Phi^- \) in \( O_\beta^- \) need to be contracted to fields \( \Phi^+ \) in \( O_\alpha^+ \) with the propagator \( I_{0x} \). In particular the number of fields of the two operators must be equal.

2.2 Renormalization

For the one-loop correlator we insert the (connected) one-loop Green function \( W_2 \) into the tree-level correlator (2.3)

\[
\langle O_\alpha^+ O_\beta^- \rangle_{\text{one-loop}} = \exp \left( W_0(x, \Phi^+, \Phi^-) \right) (1 + g^2 W_2(x, \Phi^+, \Phi^-)) O_\alpha^+ O_\beta^- |_{\Phi=0}, \tag{2.5}
\]

where \( g^2 = g_{YM}^2 / 16\pi^2 \). We now change the argument \( \Phi^+ \) of \( W_2 \) to \( I_{0x}^{-1} \Phi^- \). This can be done because the result vanishes unless every \( \Phi^- \) is contracted with some \( \Phi^+ \) before the fields \( \Phi \) are set to zero. Here, the only possibility is to contract with a term in \( W_0 \) which effectively changes the argument back to \( \Phi^+ \). In doing that we need to make sure that
no new contractions appear between the arguments $\Phi^-$ and $\Phi^-$ of $W_2$. Formally, this is achieved by ‘normal ordering’ $\mathbf{\ldots}$. The correlator becomes

$$
\langle O^+_\alpha O^-_{\beta} \rangle_{\text{one-loop}} = \exp(W_0(x, \Phi^+, \Phi^-))(1 + g^2V_2^-(x))O^+_\alpha O^-_{\beta}|_{\phi=0},
$$

(2.6)

with the one-loop effective vertex

$$V_2(x) = :W_2(x, I_{x^2}\Phi, \Phi):.$$

(2.7)

Instead of replacing $\Phi^+$ we could have replaced $\Phi^-$. This shows that in (2.6) $V_2^-$ is equivalent to $V_2^+$. In other words $V_2$ is self-adjoint with respect to the tree-level scalar product.

We renormalize the operators according to

$$\tilde{\mathcal{O}} = (1 - \frac{1}{2}g^2V_2(x_0))\mathcal{O},
$$

(2.8)

and find

$$\langle \tilde{\mathcal{O}}^+_\alpha \tilde{\mathcal{O}}^-_{\beta} \rangle_{\text{one-loop}} = \exp(W_0(x, \Phi^+, \Phi^-))(1 + g^2V_2^-(x) - g^2V_2^-(x_0))O^+_\alpha O^-_{\beta}|_{\phi=0},
$$

(2.9)

In the present renormalizable field theory in dimensional regularization the dependence of $V_2$ on $x$ is fixed, we write

$$V_2(x) = \frac{\Gamma(1-\epsilon)}{\frac{1}{2}\mu^2 x^2}V_2.
$$

(2.10)

We send the regulator to zero and find

$$\lim_{\epsilon \to 0}(V_2(x) - V_2(x_0)) = \log\left(\frac{x_0^2}{x^2}\right)D_2,
$$

(2.11)

with

$$D_2 = -\lim_{\epsilon \to 0}\epsilon V_2.
$$

(2.12)

The final answer for the renormalized correlator at $\epsilon = 0$ is

$$\langle \tilde{\mathcal{O}}^+_\alpha \tilde{\mathcal{O}}^-_{\beta} \rangle_{\text{one-loop}} = \exp(W_0)\exp(\log\left(\frac{x_0^2}{x^2}\right)g^2D_2^-)O^+_\alpha O^-_{\beta}|_{\phi=0},
$$

(2.13)

in agreement with conformal field theory. $D_2$ is the one-loop correction to the dilatation generator. Furthermore, the coefficient of the correlator is given by its tree-level value as $D_2$ multiplies the logarithm only. Although we are interested in correlators of renormalized operators as on the left-hand side of (2.13), we can work with bare operators as on the right hand side of (2.13). We will not distinguish between the two types of operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$. It will be understood that an operator in a correlator is renormalized.

### 2.3 Evaluation of diagrams

The one-loop connected Green functions are depicted in Fig. 1. To evaluate them we make use of the $\mathcal{N} = 4$ SYM action in (A.1) with the coupling constant $g$ which is...
related to the usual coupling constant $g_{YM}$ by $g^2 = g_{YM}^2/16\pi^2$. The one-loop Green functions evaluate to

\[
W_{2,a} = \frac{1}{4}X_{00xx}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^+_n][\hat{\Phi}^-_m, \hat{\Phi}^-_n],
\]

\[
W_{2,b} = \frac{1}{4}X_{00xx}(\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_m] + \text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^-_n, \hat{\Phi}^+_m]),
\]

\[
W_{2,c} = (-\frac{1}{2}\hat{H}_{0x,0x} - Y_{00x}I_{0x} + \frac{1}{4}X_{00xx})\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_m],
\]

\[
W_{2,d} = -Y_{00x}\text{Tr}[\hat{\Phi}^+_m, T^a][T^a, \hat{\Phi}^-_n].
\]

(2.14)

The functions $X, Y, \hat{H}$ are defined in (B.2). Diagram e vanishes by antisymmetry, its structure is $\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_m][\hat{\Phi}^+_n, \hat{\Phi}^-_n] = 0$. We use a Jacobi-Identity to transform the second structure in $W_{2,b}$

\[
\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^-_m, \hat{\Phi}^+_n] = \text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^-_m, \hat{\Phi}^-_n] - \text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_n].
\]

(2.15)

We shuffle some terms around and get

\[
W_{2,a} = X_{00xx}(\frac{1}{2}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^+_n][\hat{\Phi}^-_m, \hat{\Phi}^-_n] + \frac{1}{4}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_n]),
\]

\[
W_{2,b} = -\frac{1}{2}\hat{H}_{0x,0x}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_m],
\]

\[
W_{2,c} = -Y_{00x}(I_{0x}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_m] + \text{Tr}[\hat{\Phi}^+_m, T^a][T^a, \hat{\Phi}^-_n]).
\]

(2.16)

According to (2.17) we write

\[
V_{2,a} = X_{00xx}I^{-2}_{0x}(\frac{1}{2}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_n] + \frac{1}{4}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\hat{\Phi}^+_n, \hat{\Phi}^-_n]),
\]

\[
V_{2,b} = -\frac{1}{2}\hat{H}_{0x,0x}I^{-2}_{0x}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\Phi^+_n, \hat{\Phi}^-_m],
\]

\[
V_{2,c} = -Y_{00x}I^{-1}_{0x}(\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\Phi^+_n, \hat{\Phi}^-_m] + \text{Tr}[\hat{\Phi}^+_m, T^a][T^a, \hat{\Phi}^-_n]).
\]

(2.17)

The last term can now be written as

\[
V_{2,c} = -Y_{00x}I^{-1}_{0x}\text{Tr}[\hat{\Phi}^+_m, \hat{\Phi}^-_n][\Phi^+_n, \hat{\Phi}^-_m]
= -Y_{00x}I^{-1}_{0x}\text{Tr}([T^a, \hat{\Phi}^-_n][\Phi^+_n])\text{Tr}([T^a, \Phi^+_n]\hat{\Phi}^-_n)
= -Y_{00x}I^{-1}_{0x}G^aG^a.
\]

(2.18)

The generator $G^a = \text{Tr}[T^a, \Phi^+_n]\hat{\Phi}^-_m$ is the generator of gauge transformations for scalar fields. Therefore $V_{2,c}$ does not act on operators which are color singlets, it is irrelevant.
The remaining functions $X$ and $\tilde{H}$ have the following expansion in $\epsilon$, see (B.3),

\[
\frac{X_{00xx}}{I_{0x}^2} = \left( \frac{2}{\epsilon} + 2 + \mathcal{O}(\epsilon^2) \right) \frac{\Gamma(1 - \epsilon)}{\left[ \frac{1}{2} \mu^2 x^2 \right]^{-\epsilon}},
\]

\[
\frac{\tilde{H}_{0x,0x}}{I_{0x}^2} = (-48 \zeta(3) \epsilon + \mathcal{O}(\epsilon^2)) \frac{\Gamma(1 - \epsilon)}{\left[ \frac{1}{2} \mu^2 x^2 \right]^{-\epsilon}}.
\] (2.19)

When inserted in (2.12) this proves our initial statement about the one-loop dilatation generator

\[
D_2 = D_{2,A} = -\text{Tr}[\Phi_m, \Phi_n][\Phi_m, \Phi_n] - \frac{1}{2} \text{Tr}[\Phi_m, \Phi_n][\Phi_m, \Phi_n].
\] (2.20)

3 Two-loop

In this chapter we will derive the central technical result of the paper: We will find the two-loop correction to the dilatation generator, as announced in eq.(1.14). Written in SO(6) notation, it is given by

\[
D_4 = -2 \text{Tr} \Phi_m[\Phi_n, [\Phi_p, [\Phi_p, \Phi_p]]] + \text{Tr} \Phi_m[\Phi_n, [T^a, [T^a, [\Phi_m, \Phi_n]]]].
\] (3.1)

This result is valid for pure scalar operators. Here pure means those composite operators, made from the six fundamental scalar fields of the model, that do not mix with other, non-scalar types of fields (or fields containing covariant derivatives). These pure scalars are given by the SO(6) representations $[p, q, p]$ with $\Delta_0 = 2p + q$. We start with a group-theoretical argument why the dilatation generator closes on this class of operators. We proceed by extending the renormalization scheme to two-loops. Then we investigate the allowed structures in $D_4$ by considering the relevant Feynman diagrams. Finally we fix the coefficients of the structures to obtain (3.1).

3.1 Group-theoretical constraints on mixing

Restricting oneself to operators constructed from scalar fields $\Phi_m$ only is in general not possible due to operator mixing: Scalar operators can also contain scalar combinations of fermions $\Psi^A$, field strengths $F_{\mu\nu}$ and derivatives $D_\mu$. All these operators are space-time singlets and they must be treated on equal footing. Nevertheless, one can isolate certain classes of operators by employing certain representations of the flavor symmetry group SO(6) $\simeq$ SU(4). For example the representations $(a, b, c) \simeq [b + c, a - b, b - c]$ (Young tableau/Dynkin labels) with classical scaling dimension $\Delta_0 = a + b + |c|$ do not admit derivatives or field strengths. If furthermore $c = 0$, fermions are excluded and the operator must consist of only scalar fields. We will therefore consider operators in the SO(6) representation

\[
(a, b, 0) \simeq [b, a - b, b] \quad \text{with} \quad \Delta_0 = a + b.
\] (3.2)

These operators need not be superconformal primaries. If they are, they belong to the protected C series of unitary irreducible representations (UIR) of SU(2,2|4). In that
case they are quarter BPS operators (for \( b \neq 0 \)) or half BPS (for \( b = 0 \)). Otherwise they belong the A series unprotected scalar UIR \((a - 2, b - 2, 0) \simeq [b - 2, a - b, b - 2] \) with \( \Delta_0 = a + b - 2 \). This UIR is at both unitarity bounds and splits into four UIR’s at \( g_{YM} = 0 \). The operators under consideration are primary operators of the highest-lying submultiplet.

We will now prove the above two statements concerning absence of mixing. The SO(6) weights of the six scalars are \((\pm 1, 0, 0)\), \((0, \pm 1, 0)\), \((0, 0, \pm 1)\), the weights of the eight space-time pairs of fermions are \((\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})\) and derivatives as well as field strengths have trivial weight \((0, 0, 0)\). We see that the sum of labels \( a + b + c \) for all of these is bounded from above by the corresponding scaling dimension of the field. For a composite operator the labels of the constituents simply are summed and therefore \( a + b + c \) is bounded from above by the scaling dimension \( \Delta_0 \). If this bound is to be achieved, all constituent fields must be on the bound as well. This is true only for the three scalars with the weights \((1, 0, 0)\), \((0, 1, 0)\), \((0, 1, 1)\) and a pair of fermions with weight \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\). This proves the first statement. Furthermore if \( c = 0 \) for the composite operator the third scalar and the fermions are excluded as constituents. This proves the second statement.

As advertised, an operator with \( a + b = \Delta_0 \), \( c = 0 \) is constructed from two scalars only. These scalars are charged with respect to different \( U(1) \) subgroups of SO(6). We choose

\[
Z = \frac{1}{\sqrt{2}}(\Phi_5 + i\Phi_6), \quad \phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2).
\]  

(3.3)

We identify \( a, b \) with the number of \( Z, \phi \) fields and assume \( a \geq b \). It may be shown that any combination of the fields \( Z, \phi \) is traceless with respect to the original SO(6) symmetry. Written in terms of \( Z, \phi \) the dilatation generator (1.3), (1.6), (3.1) is

\[
D_0 = \text{Tr} \, Z \bar{Z} + \text{Tr} \, \phi \bar{\phi}, \\
D_2 = -2 : \text{Tr} \, [\phi, Z] [\bar{\phi}, \bar{Z}] : , \\
D_4 = -2 : \text{Tr} \, [[\phi, Z], \bar{Z}] [[\bar{\phi}, \bar{Z}], Z] : , \\
- 2 : \text{Tr} \, [[\phi, Z], \bar{\phi}] [[\bar{\phi}, \bar{Z}], \phi] : , \\
- 2 : \text{Tr} \, [[\phi, Z], T^a] [[\bar{\phi}, \bar{Z}], T^a] :.
\]

(3.4)

For the derivation of the two-loop result, however, we keep the SO(6) symmetry manifest and consider the correlator of two pure scalar SO(6)-traceless operators.
3.2 Three pairs of legs

At two loops the interaction can be disconnected (Fig. 2a) or connected. The disconnected diagrams can be shown to effectively yield \((\log)^2\) terms, necessary for the conformal structure of the correlator, plus further connected diagrams, see App. C. We classify the connected interactions by their number of legs. We can have three pairs of legs (Fig. 2b), two pairs of legs with one bulk loop (Fig. 2c,d,e,f) or one pair of legs with two bulk loops (Fig. 2g). We start by analyzing the diagrams with three pairs of legs.

The interactions with three pairs of legs are depicted in Fig. 3. Most of the diagrams correspond to two one-loop interactions (see Fig. 1) connected at one leg. We start by subtracting the connected part of two one-loop vertices, see (C.6). This gives rise to the same structures but with different coefficient functions. As in the one-loop case the type b sub-diagrams consist of two parts, see (2.14). The first part contains local SO(6) traces, \(\tilde{\Phi}_m^+\tilde{\Phi}_m^+\). We have restricted ourselves to traceless operators and drop the term. The second term can be transformed by a Jacobi identity to sub-diagrams of type a and c, see (2.15). This effectively doubles the sub-diagrams of type a. Then we notice that all type c sub-diagrams have a scalar line that is connected to the rest of the diagram only by one gluon-line. By removing some normal ordering we can write those diagrams as \(G^aX^a\) plus diagrams with two pairs of legs in analogy to (2.18). As \(G^a\) is the generator of gauge transformations and the operators are color singlets, we can drop \(G^aX^a\). The same holds for the d1 diagram. The d2 diagram has local SO(6) traces and will be dropped. The d3 diagram (as well as a number of diagrams with vertical gluon lines as in Fig. 1e, which have been omitted in Fig. 3) vanishes by symmetry: The diagram is proportional to \(:G^aG^bG^c:\text{Tr}[T^a,T^b]T^c\).

The only diagram with three pairs of legs that effectively remains is Fig. 3aa. It is multiplied by four due to contributions from ab, ba and bb. Its combinatorial structure
\[ :\text{Tr} \phi_m [\phi_n, [\tilde{\phi}_m, [\phi_p, [\tilde{\phi}_n, \tilde{\phi}_p]]]] :. \] (3.5)

### 3.3 One and two legs

There are many diagrams with one or two pairs of legs, see Fig. 2c,d,e,f,g. Additionally, the connected piece of two one-loop diagrams and some reductions of diagrams with three pairs of legs have to be taken into account. Instead of considering all these we investigate the possible algebraic structures. A generic diagram consists of one trace and four commutators. This can always be brought into the form

\[ :\text{Tr} \ldots [\ldots, [\ldots, [\ldots, [\ldots, \text{Tr} 1]]] :. \] (3.6)

For diagrams with two pairs of legs we need to distribute \( \phi_m, \phi_n, \tilde{\phi}_m, \tilde{\phi}_n \) and two \( T^a \) on the six slots. We start by distributing the \( T^a \). If both \( T^a \) are in the first two or the last two slots, the structure vanishes by antisymmetry. If we place a \( T^a \) in the first or the last slot, we can interchange it with the second or second-but-last. There are identities when the two \( T^a \) are next to each other (corresponding to Fig. 2c)

\[ [T^a, [T^a, [X, Y]]] = \text{Tr} [T^a T^a] [X, Y], \] (3.7)

or one step apart (corresponding to Fig. 2d)

\[ [T^a, [X, [T^a, Y]]] = (\text{Tr} T^a T^a / \text{Tr} 1) \text{Tr} [X, Y]. \] (3.8)

The combination \( \text{Tr} T^a T^a / \text{Tr} 1 \) depends only on the gauge group, for \( U(N) \) it equals \( N \). With these we can remove two commutators along with the \( T^a \) and get two independent structures

\[ \text{Tr} \phi_m [\phi_n, [T^a, [T^a, [\tilde{\phi}_m, [\tilde{\phi}_n, \tilde{\phi}_n]]]]] \sim N \text{Tr} \phi_m [\phi_n, [\tilde{\phi}_m, \tilde{\phi}_n]], \] (3.9)

and

\[ :\text{Tr} \phi_m [\tilde{\phi}_m, [T^a, [T^a, [\phi_n, [\tilde{\phi}_n, \tilde{\phi}_n]]]] :. \] (3.10)

By use of gauge invariance, see (2.18), we can write (3.10) as a one pair of legs diagram. The remaining structures have the two \( T^a \) two steps apart (corresponding to Fig. 2e,f). We write them as

\[ :\text{Tr} T^a [X_1, [X_2, [X_3, [X_4, T^a]]]] :. \] (3.11)

Using Jacobi identities on the nested commutators it is easy to see that we can interchange any of the \( X_i \) at the cost of terms like (3.8). The only remaining independent structure can thus be chosen as

\[ :\text{Tr} T^a [T^a, [T^a, [T^a, \tilde{\phi}_m]]]] :. \] (3.12)

By use of the identities (3.7) and (3.8) a diagram with one pair of legs (Fig. 2g) can always be written as

\[ \text{Tr} \phi_m [T^a, [T^a, [T^b, [T^b, \tilde{\phi}_m]]]] \sim N^2 \text{Tr} \phi_m \tilde{\phi}_m - N \text{Tr} \phi_m \text{Tr} \tilde{\phi}_m. \] (3.13)
3.4 Determining the coefficients

All in all there are four independent structures (3.5), (3.9), (3.12) and (3.13). The dilatation generator $D_4$ at two-loops must be a linear combination of these. By using known results we determine this linear combination.

We start by applying the structures to the half BPS operator $\text{Tr} Z^J$. This operator is protected, it does not receive corrections to its scaling dimension. Consequently $D_4$ must annihilate it. Indeed the structures (3.5) and (3.9) do so. The structure (3.13) gives $JN^2 \text{Tr} Z^J$ while (3.12) gives

$$2J \sum_{p=1}^{J-1} \left( N \text{Tr} Z^p \text{Tr} Z^{J-p} + \text{Tr} Z^2 \text{Tr} Z^{p-1} \text{Tr} Z^{J-1-p} \right),$$

(3.14)

(up to terms involving $\text{Tr} Z$). Thus in order for $D_4$ to annihilate $\text{Tr} Z^J$ the coefficients of (3.13) and (3.12) must vanish. This vanishing was explicitly confirmed in references [30, 31]. Our two independent structures correspond directly to those appearing there.

We are now left with

$$D_4 = \alpha \text{Tr} \phi_m[\phi_n, [\phi_p, [\phi_{\bar{n}}, \phi_{\bar{p}}]]] + \beta \text{Tr} \phi_m[T^a, [T^a, [\phi_{\bar{m}}, \phi_{\bar{n}}]]]],$$

(3.15)

For the BMN-operators, see Sec. 6 (3.15) produces the two-loop planar anomalous dimension

$$-\frac{(2\beta + \alpha) g_{YM}^4 N^2 n^2}{16\pi^2 J^2} + \frac{\alpha g_{YM}^4 N^2 n^4}{8J^4}.$$ 

(3.16)

By the pp-wave/BMN correspondence [17] we would expect the first term to vanish in order for the BMN limit (6.1) to be well-defined. The second term should equal

$$-\frac{1}{4} \lambda^2 n^4 = -\frac{g_{YM}^4 N^2 n^4}{4J^4}.$$ 

(3.17)

This is also the result of an explicit calculation [18] and an investigation using superconformal symmetry [43]. We find

$$\alpha = -2, \quad \beta = 1.$$ 

(3.18)

This concludes our derivation of the two-loop dilatation generator (3.1).

4 Applications to lower dimensional operators

In this section we will apply the two-loop dilatation generator to a set of lower dimensional operators. We consider only those representations where mixing with fermionic and derivative insertions is prohibited, see Sec. 3.1. More explicitly, these are space-time scalars in the $\text{SO}(6)$ representations $[p, \Delta_0 - 2p, p]$. Furthermore, we will consider $\text{SU}(N)$ as the gauge group. This is equivalent to dropping all traces of single fields. For low dimensions the calculations are easily performed by hand; as the dimension increases it is useful to use an appropriate code in order to quickly and reliably work out the action of the dilatation operator$^6$.

$^6$Many results of this section and Sec. 8 have been obtained using Mathematica. Computing e.g. the exact dilatation matrix at two-loops for the 48 operators of dimension 8 took 90 sec. on a 366 MHz
Table 1: BPS and unprotected operators up to dimension 10. The unprotected operators of representation $\Delta_0$, $[p, q, p]$ belong to the SU(2,2)$\mid$ UIR $[p-2, q, p-2]$ of bare dimension $\Delta_0 - 2$. They are separated into two groups, + and − according to their parity, cf. Sec. 7.

### 4.1 Quarter BPS operators

Quarter BPS operators belong to the representations $[p, q, p]$ with $p \geq 1$ and have classical scaling dimension $\Delta_0 = 2p + q$. (For $p = 0$ the operators are half BPS.) In reference the explicit construction of lower dimensional 1/4 BPS operators was initiated. The strategy consisted in first writing down all linearly independent local polynomial scalar composite operators in a given representation and next diagonalizing the corresponding matrix of one-loop two-point functions in the inner product given by the matrix of tree-level two-point functions. The former part of the strategy was very recently systematized and completed in [44]. Our idea of focusing on the dilatation operator of the theory allows us to significantly simplify and extend the analysis of [33,44]. As explained in Sec. 1 one does not need to work with the tree-level inner product but can by use of the effective vertices (2.20) and (3.1) directly write down the action of the dilatation generator in any convenient basis of operators. The (possible) 1/4 BPS operators are then simply determined as the kernel of the dilatation operator. By our method we have confirmed the BPS nature of the operators found in [33,44] up to two-loop order (without any modification of mixing coefficients at $\mathcal{O}(g_{YM}^2)$). Furthermore, we have determined all 1/4 BPS operators of dimension 8 (to two-loop order) and of dimension 9 and 10 (one-loop order). Needless to say that our method also gives access to a host of previously unknown two-loop anomalous dimensions of unprotected operators. Our results on BPS operators are summarized in Tab. 1 and a collection of anomalous dimensions can be found in [PC]. Obviously it is also useful and straightforward to use Mathematica or similar to diagonalize the dilatation matrices and perform the $1/N$ expansion of the eigensystem.
the following sections and in App. D. It is known that operators in the representations \([1, p, 1]\) are always 1/4 BPS. For the operators considered in Tab. I it appears that exactly half of those belonging to representations of the type \([2, p, 2]\) are 1/4 BPS, see also Sec. 5.

In the following two sub-sections we shall discuss some further insights gained from our analysis.

### 4.2 Exact results for lower dimensional operators

From Tab. I we see that there is exactly one unprotected operator in each of the representations \((\Delta_0 = 4, [2, 0, 2]), (\Delta_0 = 5, [2, 1, 2])\) and \((\Delta_0 = 6, [3, 0, 3])\). These operators can be written as

\[
\mathcal{K}' = \mathcal{O}_a = \text{Tr}[\phi, Z][\phi, Z],
\]

\[
\mathcal{O}_b = \text{Tr}[\phi, Z][\phi, Z]Z,
\]

\[
\mathcal{O}_c = \text{Tr}[\phi, Z][\phi, Z][\phi, Z],
\]

and descend from the operators

\[
\mathcal{K} = \mathcal{O}_{a,0} = \text{Tr} \Phi_m \Phi_m,
\]

\[
\mathcal{O}_{b,0} = \text{Tr} \Phi_m \Phi_m \Phi_n,
\]

\[
\mathcal{O}_{c,0} = \text{Tr} \Phi_m \Phi_m[\Phi_n, \Phi_p],
\]

with \((\Delta_0 = 2, [0, 0, 0]), (\Delta_0 = 3, [0, 1, 0])\) and \((\Delta_0 = 4, [1, 0, 1])\). For the operators (4.1) the dilatation generator yields

\[
\Delta_{\mathcal{K}'} = \Delta_a = 4 + \frac{3g_{\text{YM}}^2 N}{4\pi^2} - \frac{3g_{\text{YM}}^4 N^2}{16\pi^4},
\]

\[
\Delta_b = 5 + \frac{g_{\text{YM}}^2 N}{2\pi^2} - \frac{3g_{\text{YM}}^4 N^2}{32\pi^4},
\]

\[
\Delta_c = 6 + \frac{3g_{\text{YM}}^2 N}{4\pi^2} - \frac{9g_{\text{YM}}^4 N^2}{64\pi^4},
\]

and these values do not receive any non-planar corrections at the present loop order. Note that the anomalous dimension of the Konishi descendant \(\mathcal{K}'\) agrees with previous results \([15, 38, 16]\) and that \(\Delta_{\mathcal{K}'}\) and \(\Delta_b\) are in agreement with the general formula (5.17), corresponding respectively to the cases \((J, n) = (2, 1)\) and \((J, n) = (3, 1)\). The operator \(\mathcal{O}_c\) is the first in an infinite sequence of operators with identical quantum anomalous dimensions, to be discussed in Sec. 8.4.

### 4.3 Peculiarities of the large \(N\) expansion

Most of the other single-trace operators in Tab. I cease to be eigenvectors of the one-loop dilatation generator when we include non-planar corrections, due to mixing with multiple trace operators. This mixing has important consequences for the very nature of the anomalous dimension of these operators as we shall illustrate by the examples of \((\Delta_0 = 6, [2, 2, 2])\) and \((\Delta_0 = 7, [2, 3, 2])\) from Tab. I.
The operators in the first example are descendants of the operators with \((\Delta_0 = 4, [0, 2, 0])\) studied in [47]. The three unprotected operators can be chosen as

\[
\mathcal{O} = \left( \frac{\text{Tr}[\phi, Z][\phi, Z]Z^2}{\text{Tr}[\phi, Z][\phi, Z]Z}, \frac{\text{Tr}[\phi, Z]Z[\phi, Z]}{\text{Tr} Z^2 \text{Tr}[\phi, Z][\phi, Z]} \right),
\]

(4.4)

the commutators conveniently projecting out the three protected ones. In the basis given by (4.4) the one- and two-loop dilatation generator take the form

\[
D_2 = N \left( \begin{array}{ccc}
+12 & -2 & +6 \\
-8 & +8 & -4 \\
+16 & -16 & +12
\end{array} \right),
\]

(4.5)

and

\[
D_4 = N^2 \left( \begin{array}{ccc}
-52 - \frac{24}{N^2} & +4 + \frac{24}{N^2} & -52 \\
+48 + \frac{24}{N^2} & -16 - \frac{16}{N^2} & +52 \\
-144 & +80 & -48 - \frac{48}{N^2}
\end{array} \right).
\]

(4.6)

Similarly, for \(\Delta_0 = 7\) a convenient basis of unprotected operators consists of the set of operators given by

\[
\mathcal{O} = \left( \frac{\text{Tr}[\phi, Z][\phi, Z]Z^3}{\text{Tr}[\phi, Z]Z[\phi, Z]Z^2}, \frac{\text{Tr}[\phi, Z][\phi, Z]}{\text{Tr} Z^3 \text{Tr}[\phi, Z][\phi, Z]} \right),
\]

(4.7)

and in this basis the one- and two-loop dilatation matrices read

\[
D_2 = N \left( \begin{array}{ccc}
+12 & 0 & 0 + \frac{6}{N} \\
-4 & +4 & 0 + \frac{4}{N} \\
+24 & -24 & +12
\end{array} \right),
\]

(4.8)

and

\[
D_4 = N^2 \left( \begin{array}{ccc}
-48 - \frac{36}{N^2} & -12 + \frac{36}{N^2} & 0 \\
+20 + \frac{12}{N^2} & -4 + \frac{4}{N^2} & \frac{16}{N} \\
-64 & +16 & \frac{48}{N}
\end{array} \right).
\]

(4.9)

Notice that the dilatation matrices (4.5), (4.6), (4.8) and (4.9) are exact to the given loop order. Not surprisingly, these matrices have a form compatible with the existence of a ‘t Hooft expansion, trace splitting and trace joining processes being suppressed by a power of \(\frac{1}{N}\) compared to trace conserving ones.

At the classical level all the operators in respectively (4.4) and (4.7) have conformal dimension respectively 6 and 7. Including quantum effects at one-loop and two-loop order the proper scaling operators consist of some operators which are mainly single trace and some which are mainly multiple trace. The corresponding quantum corrections to the conformal dimensions are the roots of the characteristic polynomial of the matrix \(g^2 D_2 + g^4 D_4\). It is easy to show that this characteristic polynomial has a well-defined ‘t
Hooft expansion with the dependence of $g_{\text{YM}}^2$ and $N$ organizing into the ’t Hooft coupling $\lambda = g_{\text{YM}}^2 N$ and the genus counting parameter $\frac{1}{N^2}$. Furthermore, its topological expansion terminates at a finite genus. However, the roots of the characteristic polynomial need not have a terminating genus expansion or even a well defined ’t Hooft expansion. Let us illustrate this by considering in more detail the operators (4.4), (4.7).

For $\Delta_0 = 6$ the relevant characteristic polynomial is of cubic order

$$
\omega^3 - 8\omega^2 + \left(20 - \frac{10}{N^2}\right) \omega - \left(15 - \frac{10}{N^2}\right),
$$

and the scaling dimensions up to two-loops are exactly

$$
\Delta = 6 + \frac{g_{\text{YM}}^2 N}{4\pi^2} \omega,
$$

with $\omega$ a root of the corresponding cubic equation. The discriminant of the cubic equation does not take the form of a perfect square but is easily seen to be expressible as

$$
\mathcal{D} = \sum_{i,j} c_{i,j} \frac{(g_{\text{YM}}^2 N)^i}{N^j},
$$

with the $c_{i,j}$ some constants and $c_{0,0} \neq 0$, implying that the roots will have a well-defined ’t Hooft expansion which however involves a non-terminating sum over genera. To three leading orders in the genus expansion the conformal dimension of the two mainly single trace operators read

$$
\Delta_{\text{single}} = 6 + \frac{g_{\text{YM}}^2 N}{\pi^2} \left(\frac{5 \pm \sqrt{5}}{8} + \frac{5 \pm 2\sqrt{5}}{2N^2} - \frac{75 \pm 34\sqrt{5}}{N^4}\right),
$$

$$
\Delta_{\text{double}} = 6 + \frac{g_{\text{YM}}^2 N^2}{\pi^4} \left(-\frac{17 \pm 5\sqrt{5}}{128} - \frac{131 \pm 57\sqrt{5}}{64N^2} + \frac{1675 \pm 751\sqrt{5}}{16N^4}\right),
$$

whereas that of the mainly double trace one equals

$$
\Delta_{\text{double}} = 6 + \frac{g_{\text{YM}}^2 N^2}{\pi^4} \left(-\frac{3}{16} + \frac{5}{16N^2} - \frac{1675\sqrt{5}}{8N^4}\right).
$$

The one-loop contributions to these conformal dimensions agree with those obtained in reference [47] and the planar versions of $\Delta_{\text{single}}$ are in accordance with the general formula (5.17) for $J = 4$ and $n = 1$ and $n = 2$ respectively. Furthermore the planar part of the quantum correction to $\Delta_{\text{double}}$ is identical to that of the operator $\mathcal{K}'$ in (4.1) as it should be.

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7This was also noticed in [33, 47, 41].
In the case $\Delta_0 = 7$ the situation is more involved. Here the relevant characteristic polynomial is of degree four. The discriminant of the associated cubic polynomial does also not here take the form of a perfect square but can be expressed as

$$D = \sum_{i,j} c_{i,j} \frac{(g^2 N)^i}{N^{2j}}, \quad (4.14)$$

where the $c_{i,j}$ are again some constants. However, this time $c_{0,0} = c_{1,0} = 0$. This implies that not all the wished for conformal dimensions have well defined 't Hooft expansions. (Of course they still exist as the exact roots of the characteristic polynomial.) Two of the conformal dimensions, associated with respectively a mostly single trace and a mostly double trace operator, do have well defined 't Hooft expansions and for these we find for the first few orders

$$\Delta_{\text{single}} = 7 + \frac{g^2_{\text{YM}}}{\pi^2} N \left( \frac{1}{4} + \frac{11}{16 N^2} + \frac{253}{64 N^4} \right) + \frac{g^4_{\text{YM}} N^2}{\pi^4} \left( -\frac{3}{128} - \frac{103}{1024 N^2} - \frac{5541}{8192 N^4} \right),$$

$$\Delta_{\text{double}} = 7 + \frac{g^2_{\text{YM}}}{\pi^2} N \left( \frac{1}{2} - \frac{1}{2 N^2} - \frac{17}{2 N^4} \right) + \frac{g^4_{\text{YM}} N^2}{\pi^4} \left( -\frac{3}{32} + \frac{7}{32 N^2} + \frac{129}{32 N^4} \right). \quad (4.15)$$

Here we notice that the planar contribution to $\Delta_{\text{single}}$ is in accordance with the general formula (5.17) for $(J, n) = (5, 1)$ and that the planar quantum correction to $\Delta_{\text{double}}$ as expected is the same as that of the operator $O_b$ in equation (4.1). We then remove the roots associated to these scaling dimensions and obtain from the quartic polynomial a quadratic one. This quadratic polynomial has the roots

$$\Delta_{\text{remaining}} = 7 + \frac{g^2_{\text{YM}} N}{\pi^2} \left( \mathcal{M} \pm \sqrt{\mathcal{D}} \right), \quad (4.16)$$

where

$$\mathcal{M} = \left( \frac{3}{4} - \frac{3}{32 N^2} + \frac{291}{128 N^4} \right) + \frac{g^2_{\text{YM}} N}{\pi^2} \left( -\frac{45}{256} - \frac{537}{2048 N^2} - \frac{27483}{16384 N^4} \right),$$

$$\mathcal{D} = \left( \frac{27}{32 N^2} - \frac{531}{1024 N^4} \right) + \frac{g^2_{\text{YM}} N}{\pi^2} \left( -\frac{3123}{4096 N^2} + \frac{29295}{32768 N^4} \right) + \frac{9 g^4_{\text{YM}} N^2}{65536 \pi^4}. \quad (4.17)$$

As we have determined the two well-defined scaling dimensions in a perturbative fashion, the coefficients $\mathcal{M}, \mathcal{D}$ of the new polynomial are series in $\frac{1}{N}$ and $g^2_{\text{YM}} N$. (Here we notice that the last coefficient in $\mathcal{D}$ is indeed determined by $D_2$ and $D_4$ alone and as we shall see it plays a crucial role for the analysis.) From the form of $\mathcal{D}$ it follows that we have a square root singularity in the $(\lambda, \frac{1}{N})$ plane with a branch point at the origin. Obviously, we can not expand $\sqrt{\mathcal{D}}$ in powers of $\lambda$ and $\frac{1}{N}$ at the same time. Thus for the remaining two operators it only makes sense to speak of either a singular anomalous dimension (4.16), a planar anomalous dimension ($N \to \infty$) or a one-loop anomalous dimension ($g \to 0$). For the planar anomalous dimension we find

$$\Delta_{\text{single, planar}} = 7 + \frac{3 g^2_{\text{YM}} N}{4 \pi^2} \frac{21 g^4_{\text{YM}} N^2}{128 \pi^4},$$

$$\Delta_{\text{double, planar}} = 7 + \frac{3 g^2_{\text{YM}} N}{4 \pi^2} \frac{3 g^4_{\text{YM}} N^2}{16 \pi^4}. \quad (4.17)$$
The first of these anomalous dimensions correctly reproduces the result encoded in the general formula (5.17) and the quantum correction of the second agrees with that of the Konishi operator $K$. For the one-loop anomalous dimension one gets to four leading orders in the large-$N$ expansion

$$\Delta_{\text{one-loop}} = 7 + \frac{g_{YM}^2 N}{\pi^2} \left( \frac{3}{4} \pm \frac{3\sqrt{6}}{8N} - \frac{3}{32N^2} + \frac{59\sqrt{6}}{512N^3} \right).$$

The absence of a well-defined ’t Hooft expansion for the conformal dimension of certain operators is a degeneracy effect and the appearance of odd powers of $\frac{1}{N}$ is a particular manifestation of this. From the form of the one-loop dilatation matrix (4.8) it follows that at the planar level a single trace state and a double trace state are degenerate. Treating the $\frac{1}{N}$ effects as a perturbation thus calls for the use of degenerate perturbation theory which exactly gives rise to the $\frac{1}{N}$ corrections in equation (4.18), lifting the degeneracy present at the one-loop level. The degeneracy is also lifted if instead the two-loop effects are treated perturbatively and it is this simultaneous lift of degeneracy by two non-commuting perturbations which is the origin of the problems with the ’t Hooft expansion. Note that these problems also renders problematic the perturbative evaluation of higher point functions in conjunction with the $\frac{1}{N}$ expansion. This question is however beyond the scope of this paper.

5 Two impurity operators

Berenstein, Maldacena and Nastase suggested to investigate operators of a large dimension $\Delta_0$ and a nearly equally large charge $J$ under a subgroup SO(2) of SO(6). We identify the SO(2) subgroup with the group U(1) corresponding to the phase of the field $Z$ defined in (3.3). Then the relevant operators constitute long strings of $Z$-fields with $\Delta_0 - J$ impurities scattered in. As particular examples BMN investigated operators with zero, one and two impurities of type $\phi$. As shown in [20] it makes perfect sense to consider these operators also for arbitrary finite values of $J$. The primary single-trace operators with up to two impurities are in one-to-one correspondence with the SO(6) representations $[0, \Delta_0, 0]$ and $[0, \Delta_0 - 2, 0]$. These primary operators have supersymmetry descendants that can be written in terms of $J$ fields $Z$ and two impurities of type $\phi$. They belong to the representation $[2, \Delta_0 - 4, 2]$ for which we may apply the dilatation generator up to two-loops as given by (3.4). A generic multi-trace operator with two impurities is written as

$$O_p^{J_0; J_1, \ldots, J_k} = \text{Tr}(\phi Z^p \phi Z^{J_0 - p}) \prod_{i=1}^k \text{Tr} Z^{J_i},$$

$$Q^{J_0; J_1; J_2, \ldots, J_k} = \text{Tr}(\phi Z^{J_0}) \text{Tr}(\phi Z^{J_1}) \prod_{i=2}^k \text{Tr} Z^{J_i},$$

We observe that there seem to be as many unprotected operators of the type (5.1) as there are BPS operators of the type (5.2) in the SO(6) representation $[2, \Delta_0 - 4, 2]$, see also Tab. 1.
with $\sum_{i=0}^{k} J_i = J$. Both series of operators are symmetric under the interchange of sizes $J_k$ of traces $\text{Tr} Z^J_k$, $\mathcal{O}$ is symmetric under $p \rightarrow J_0 - p$ and $Q$ is symmetric under $J_0 \leftrightarrow J_1$.

The quantum dilatation generator (3.1) can be seen to act as

$$ (g^2 D_2 + g^4 D_4) \left( \begin{array}{c} \mathcal{O} \\ Q \end{array} \right) = \left( \begin{array}{cc} * & 0 \\ * & 0 \end{array} \right) \left( \begin{array}{c} \mathcal{O} \\ Q \end{array} \right), $$

i.e. operators of type $Q$ are never produced. This follows from the fact that all produced objects will contain a commutator $[\phi, Z]$ in some trace and this trace will vanish unless it contains another $\phi$. It immediately shows that for every $Q$ there is one protected (quarter BPS) operator. Its leading part is given by $Q$ itself, plus a $1/N$ correction from the operators $\mathcal{O}$ [3,14,25,26]. On the other hand the operators $\mathcal{O}$ are in general not protected and we will investigate their spectrum of anomalous dimensions in the following. From the form of the dilatation matrix we infer that operators of type $O$ do not receive corrections from operators of type $Q$; the latter therefore completely decouple as far as the consideration of the $O$'s is concerned.

5.1 The action of the dilatation generator

When acting on states of the type given in (5.1) the one- and two-loop dilatation operator take the form given in equation (3.4), i.e.

$$ D_2 = -2: \text{Tr}[Z, \phi][Z, \phi]; $$
$$ D_4 = 2: \text{Tr}[Z, \phi][Z, [Z, \hat{Z}, \hat{\phi}]]: $$
$$ + 4N: \text{Tr}[Z, \phi][Z, \hat{\phi}]: $$
$$ + 2: \text{Tr}[Z, \phi][\hat{\phi}, [\phi, [\hat{Z}, \hat{\phi}]]:]. $$

(5.4)

Here we shall prove a further simplification occurring when $D_4$ acts on (5.1), namely

$$ D_4 = -\frac{1}{4} (D_2)^2 + 2: \text{Tr}[Z, \phi][\hat{\phi}, [\phi, [\hat{Z}, \hat{\phi}]]]: $$
$$ \equiv -\frac{1}{4} (D_2)^2 + \delta D_4. $$

(5.5)

(5.6)

(5.7)

This relation explicitly shows that $D_2$ and $D_4$ do not commute due to the extra piece $\delta D_4$, involving an interaction of both impurities that was not present at the one-loop level. Eq. (5.6) is shown as follows: When applying $D_2$ to an operator $O_p$ of the type (5.1) there are two possible ways of contracting the $\hat{\phi}$ in $D_2$ with a $\phi$ in $O_p$. Each of these possibilities gives the same contribution to $D_2 O_p$. The operator $D_2 O_p$ is again a sum of operators of the type (5.1). Therefore, when evaluating $D_2(D_2 O_p)$ we get the same contribution from the two possible contractions of the $\hat{\phi}$ in the leftmost $D_2$ with a $\phi$ in $(D_2 O_p)$. We can thus determine $D_2(D_2 O_p)$ by contracting $\hat{\phi}$ in the leftmost $D_2$ with

This fact can be confirmed by direct computation. It can also be proven by use of the parity operation defined in Sec. 7. Let us tag one of the impurities, $\phi'$, $D_2$ will only contract to this impurity. Whatever the outcome of $D_2 \text{Tr} \phi' Z^p \phi Z^{J-p}$ will be, it is again of the type (5.1) and thus has positive parity. Therefore $D_2 \text{Tr} \phi' Z^p \phi Z^{J-p} = P D_2 \text{Tr} \phi' Z^p \phi Z^{J-p} = D_2 P \text{Tr} \phi' Z^p \phi Z^{J-p} = D_2 \text{Tr} Z^{J-p} \phi Z^p \phi' = D_2 \text{Tr} \phi Z^p \phi' \text{Tr} Z^{J-p}$. 21
the $\phi$ in the already applied $D_2$ and multiplying the result by two. Next, the $\tilde{Z}$ of the leftmost $D_2$ must be contracted with one of the $Z$’s of $(D_2 \mathcal{O}_p)$. This can either be a $Z$ of the original $\mathcal{O}_p$ or the $Z$ appearing in the already applied $D_2$. These two types of contractions are, up to an overall factor, described respectively by the vertices (5.4) and (5.5).

It is easy to write down the exact expression for $D_2 \mathcal{O}_p$. Let us define

$$D_2 \equiv N D_{2;0} + D_{2;+} + D_{2;-},$$

where $D_{2;0}$ is trace conserving and $D_{2;+}$ and $D_{2;-}$ respectively increases and decreases the trace number by one. Then we find

$$D_{2;0} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = -4 \left( \delta_{p \neq J_0} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} - (\delta_{p \neq J_0} + \delta_{p = J_0}) \mathcal{O}_p^{J_0;J_1,\ldots,J_k} \right),$$

$$D_{2;+} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4 \sum_{J_{k+1} = 1}^{p-1} \left( \mathcal{O}_p^{J_0;J_1,\ldots,J_k} - \mathcal{O}_p^{J_0-J_{k+1};J_1,\ldots,J_k+1} \right),$$

$$D_{2;-} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4 \sum_{J_{k+1} = 1}^{J_0-p} \left( \mathcal{O}_p^{J_0-J_{k+1};J_1,\ldots,J_k} - \mathcal{O}_p^{J_0-J_{k+1};J_1,\ldots,J_k+1} \right).$$

In analogy with (5.8) one can write $\delta D_4$ as

$$\delta D_4 = N^2 \delta D_{4;0} + N \delta D_{4;+} + N \delta D_{4;-} + \delta D_{4;++} + \delta D_{4;+} + \delta D_{4;++} + \delta D_{4;++},$$

where $\delta D_{4;++}$ stands for the genus one trace number conserving part. For $\delta D_{4;0}$ we get (for $J_0 > 0$)

$$\delta D_{4;0} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4 (\delta_{p,0} + \delta_{p,J_0} - \delta_{p,1} - \delta_{p,J_0-1}) \left( \mathcal{O}_1^{J_0;J_1,\ldots,J_k} - \mathcal{O}_0^{J_0;J_1,\ldots,J_k} \right).$$

The remaining contributions to $\delta D_4$ likewise always produce operators in the combination $\mathcal{O}_1 - \mathcal{O}_0$. We defer the exact expressions for $\delta D_{4;+}, \delta D_{4;++}$ and $\delta D_{4;+}$ to App. E. The contribution $\delta D_{4;-}$ is easily seen to vanish.
5.2 The planar limit

In the basis of symmetric two-impurity single trace states of the type (5.1) with \( k = 0 \) the planar one-loop dilatation matrix takes the form (for \( J \geq 1 \))

\[
D_{2;0} = 4 \cdot \begin{pmatrix}
+1 & -1 \\
-1 & +2 & -1 \\
& & \ddots & \ddots \\
& & & -1 & +2 & -1 \\
& & & & -1 & +1 \\
& & & & & 0
\end{pmatrix}.
\] (5.12)

Notice the special form of the boundary contributions which follows from the presence of the \( \delta \)-functions in equation (5.9). As exploited in reference [20] the matrix (5.12) has the following exact eigenvectors

\[
O^J_n = \frac{1}{J+1} \sum_{p=0}^{J} \cos \left( \frac{\pi n (2p+1)}{J+1} \right) O^J_p.
\] (5.13)

We emphasize that equation (5.13) thus gives for any value of \( J \) our one-loop planar eigenoperators. Note that \( O^J_n = O^J_{-n} = -O^J_{J+1-n} \). Thus the set of independent modes can be chosen as given by the mode numbers \( 0 \leq n \leq [J/2] \). The inverse transform is

\[
O^J_p = O^J_n = 0 + 2 \sum_{n=1}^{[J/2]} \cos \left( \frac{\pi n (2p+1)}{J+1} \right) O^J_n.
\] (5.14)

The corresponding exact planar one-loop anomalous dimension was found in [20]. Moving on to the two-loop analysis we infer from (5.7) and (5.10) that the matrix \( D_{4;0} \) can be expressed as the square of the one-loop matrix (5.12) plus contact-term contributions from (5.11)

\[
D_{4;0} = -\frac{1}{4}(D_{2;0})^2 + 4 \cdot \begin{pmatrix}
-1 & +1 \\
+1 & -1 \\
& & \ddots & \ddots \\
& & & 0 \\
& & & & -1 & +1 \\
& & & & & +1 & -1
\end{pmatrix}.
\] (5.15)

We now face the problem that the states (5.13) are no longer eigenstates of \( D_{4;0} \), since \( D_{2;0} \) and \( \delta D_{4;0} \) do not commute. We find

\[
\delta D_{4;0} O^J_n = -\frac{256}{J+1} \sin^2 \frac{\pi n}{J+1} \cos \frac{\pi n}{J+1} \sum_{m=1}^{[J/2]} \sin^2 \frac{\pi m}{J+1} \cos \frac{\pi m}{J+1} O^J_m.
\] (5.16)
However, we can treat \( D_{4;0} \) as a perturbation and thus find that the two-loop part of the planar anomalous dimension is the diagonal piece of \( D_{4;0} \) proportional to \( O_n \). Exploiting the relation (5.7), for the operators whose one-loop form is given by (5.13) we obtain the following planar anomalous dimension exact to two loops

\[
\Delta^J_n = J + 2 + \frac{g_{YM}^2 N}{\pi^2} \sin^2 \frac{\pi n}{J + 1} \left( 1 + \frac{1}{2} \frac{\cos^2 \frac{\pi n}{J + 1}}{\frac{\sin^2 \frac{\pi n}{J + 1}}{J + 1} - \frac{\sin^2 \frac{\pi m}{J + 1}}{J + 1}} \right). \tag{5.17}
\]

This formula comprises the results \( \Delta_{K'} \), \( \Delta_b \) in (4.3) as well as the planar piece in \( \Delta_{\text{single}} \) in (4.13), (4.14) and (4.17). In (8.10) we present a conjecture for the planar three-loop contribution to \( \Delta^J_n \). Furthermore, using standard perturbation theory we can also find the (planar) perturbative correction to the eigenstates (5.13): They involve the coupling constant dependent redefinition

\[
O_n^J \rightarrow O_n^J - \frac{g_{YM}^2 N}{\pi^2} \frac{1}{J + 1} \sum_{m=1}^{[J/2]} \delta_{m \neq n} \frac{\sin^2 \frac{\pi n}{J + 1} \cos \frac{\pi n}{J + 1} \sin^2 \frac{\pi m}{J + 1} \cos \frac{\pi m}{J + 1}}{\sin^2 \frac{\pi n}{J + 1} - \sin^2 \frac{\pi m}{J + 1}} O_m^J. \tag{5.18}
\]

This \( g_{YM} \)-dependent remixing is a complicating feature that we can expect at each further quantum loop order; remarkably, it is absent in the small \( J \) limit (i.e. for \( J = 2 \) (Konishi) and \( J = 3 \)) and in the large \( J \) (BMN) limit as discussed in the next section.

### 6 The BMN limit of two impurity operators

In the BMN limit one is supposed to send \( J \) to infinity and consider only operators for which \( D_0 - J \) is finite [17]. More precisely, the BMN limit is defined as the double scaling limit [22,23]

\[
J \rightarrow \infty, \quad N \rightarrow \infty, \quad \lambda' = \frac{g_{YM}^2 N}{J^2}, \quad g_2 = \frac{J}{N} \quad \text{fixed}. \tag{6.1}
\]

In reference [34] it was shown how the one-loop dilatation operator could be identified as a certain quantum mechanical Hamiltonian in the BMN limit. Since the same quantum mechanical Hamiltonian turns out to contain all information about the two-loop dilatation operator in the BMN limit we shall briefly recall its derivation.

#### 6.1 The one-loop Hamiltonian

With \( J \) being very large we can view \( x \equiv \frac{J}{J} \) and \( r_i \equiv \frac{J}{J} \) as continuum variables and we replace the discrete set of states in equation (5.1) by a set of continuum states

\[
O_{p;J_1,\ldots,J_k} \rightarrow |x; r_1, \ldots, r_k \rangle = |r_0 - x; r_1, \ldots, r_k \rangle, \tag{6.2}
\]

where

\[
x \in [0, r_0], \quad r_0, r_i \in [0, 1] \quad \text{and} \quad r_0 = 1 - (r_1 + \ldots + r_k), \tag{6.3}
\]

and where it is understood that \( |x; r_1, \ldots, r_k \rangle = |x; r_{\pi(1)}, \ldots, r_{\pi(k)} \rangle \) with \( \pi \in S_k \) an arbitrary permutation of \( k \) elements.
Defining
\[ g^2 D_2 = \frac{\lambda'}{4\pi^2} d_2, \] (6.4)
we can write \( d_2 = d_{2,0} + g_2 d_{2,+} + g_2 d_{2,-} \) and imposing the BMN limit (6.1) we get a continuum version of the equations (5.4)
\[
d_{2,0} |x; r_1, \ldots, r_k\rangle = -\partial_x^2 |x; r_1, \ldots, r_k\rangle, \quad (6.5)
\]
\[
d_{2,+} |x; r_1, \ldots, r_k\rangle = \int_0^x dr_{k+1} \partial_x |x - r_{k+1}; r_1, \ldots, r_{k+1}\rangle - \int_0^{x_{r_0}} dr_{k+1} \partial_x |x; r_1, \ldots, r_{k+1}\rangle,
\]
\[
d_{2,-} |x; r_1, \ldots, r_k\rangle = \sum_{i=1}^k r_i \partial_x |x + r_i; r_1, \ldots, x_i, \ldots, r_k\rangle - \sum_{i=1}^k r_i \partial_x |x; r_1, \ldots, x_i, \ldots, r_k\rangle.
\]

The \((k + 1)\)-trace eigenstates at \( g_2 = 0 \) are (with \( n \) integer)\(^{10}\)
\[
|n; r_1, \ldots, r_k\rangle = \frac{1}{r_0} \int_0^{x_{r_0}} dx \cos (2\pi nx/r_0) |x; r_1, \ldots, r_k\rangle. \quad (6.6)
\]
This is of course in accordance with the nature of the exact eigenstates at finite \( J \), cf. eqn. (5.13). The inverse transform of (6.6) reads
\[
|x; r_1, \ldots, r_k\rangle = |0; r_1, \ldots, r_k\rangle + 2 \sum_{m=1}^{\infty} \cos (2\pi mx/r_0) |m; r_1, \ldots, r_k\rangle. \quad (6.7)
\]
In the basis (6.6) the action of the operator \( d_2 \) reads
\[
d_{2,0} |n; r_1, \ldots, r_k\rangle = \left(\frac{2\pi n}{r_0}\right)^2 |n; r_1, \ldots, r_k\rangle, \quad (6.8)
\]
\[
d_{2,+} |n; r_1, \ldots, r_k\rangle = \frac{8}{r_0} \int_0^{r_0} dr_{k+1} \sum_{m=1}^{\infty} \frac{2\pi m}{r_0 - r_{k+1}} \sin^2 \left(\frac{\pi m}{r_0}\right) |m; r_1, \ldots, r_{k+1}\rangle,
\]
\[
d_{2,-} |n; r_1, \ldots, r_k\rangle = 8 \sum_{i=1}^k \sum_{m=1}^{\infty} \frac{2\pi m}{r_0 + r_i} \sin^2 \left(\frac{\pi m}{r_0 + r_i}\right) |m; r_1, \ldots, x_i, \ldots, r_k\rangle.
\]

Now, the scene is set for determining the spectrum of the full one-loop Hamiltonian order by order in \( g_2 \) by standard quantum mechanical perturbation theory and in reference 34 we carried out this program for single trace operators to three leading orders in \( g_2 \). Note the great simplicity of the expressions eq. (6.8) (no contact terms etc.), emerging directly from the study of the dilatation operator. Rather than comparing specific consequences of the above equations, and turning around the suggestion in 35, 48, 49, we feel that it would be important to derive the above Hamiltonian in the string field formulation of plane wave strings 50, 51 or the (interpolating) string bit formulation of 57, 59.

\(^{10}\)Notice that as opposed to in reference 34 we use symmetrized operators and therefore the cosine transform appears.
6.2 Two loops

Proceeding to two loops we define

\[ g^4 D_4 = \left( \frac{\lambda'}{4\pi^2} \right)^2 d_4, \]  

(6.9)

and we see from equation (5.7) that the only new element in the analysis is the determination of the BMN limit of the operator \( \delta D_4 \). Now, it follows from the equations in App. [E] that acting with \( \delta D_4 \) on a state of the type (5.1) produces at the discrete level only states in the combination \( O_j - O_0 \). Expressing this quantity in terms of the exact eigenstates at \( g_2 = 0 \), (5.13), it is easily seen that in the BMN limit we have\(^{11}\)

\[ O_j - O_0 \sim \frac{1}{J_0^2}. \]  

(6.10)

Using this result a straightforward scaling analysis of the relations in App. [E] shows that the operator \( \delta D_4 \) becomes irrelevant in the BMN limit. Thus, in this limit we have

\[ d_4 = -\frac{1}{4}(d_2)^2, \]  

(6.11)

and at any given order in the genus expansion, the eigenstates of the one-loop Hamiltonian are also eigenstates of the two-loop Hamiltonian\(^{12}\). For the BMN limit of \( D - J \) one gets up to and including two loops

\[ D - J \rightarrow 2 + \left( \frac{\lambda'}{4\pi^2} \right) d_2 - \frac{1}{4} \left( \frac{\lambda'}{4\pi^2} \right)^2 (d_2)^2. \]  

(6.12)

Writing the BMN double expansion of the conformal dimension of an operator as

\[ \Delta(\lambda', g_2) - \Delta_0 = \sum_{k=1}^{\infty} \left( \frac{\lambda'}{4\pi^2} \right)^k \Delta_k(g_2), \]  

(6.13)

with \( \Delta_0 \) the tree-level conformal dimension, what we learned above can be expressed as

\[ \Delta_2(g_2) = -\frac{1}{4}(\Delta_1(g_2))^2, \quad \forall \ g_2. \]  

(6.14)

6.3 Degeneracies

From equation (6.8) it follows that at the planar, one-loop level we have a degeneracy between states \( |n; r_1, \ldots, r_k \rangle \) and \( |n'; r'_1, \ldots, r'_k \rangle \) for which \( \frac{n}{r_0} = \frac{n'}{r'_0} \). Focusing on the simplest case a single trace state \( |n \rangle \) is degenerate with double trace states of the type \( |m; r \rangle \) for which \( n = \frac{m}{1-r} \), with triple trace states \( |k; r_1, r_2 \rangle \) for which \( n = \frac{k}{1-r_1-r_2} \) and so

\(^{11}\) Notice that \( O_1 - O_0 \) is down by a factor of \( \frac{1}{4} \) compared to \( O_p - O_{p-1} \) for general \( p \). In the continuum language this is reflected by the fact that \( \partial_x |x; r_1, \ldots, r_k \rangle \big|_{x=0} = 0 \) which follows from the relation (6.7).

\(^{12}\) This proves that the \( O(\lambda') \) contribution to the three-point function of symmetric-traceless operators \( O^{(J)}_{(ij),n} \) in \([25]\) is indeed correct. For the singlet or antisymmetric operators this remains an open question.
on [23]. Notice that degeneracy is excluded for the special case $|n\rangle = |1\rangle$. Determining the energy shift of the state $|n\rangle$ with $n > 1$ to genus one by treating the term $g_2(d_{2,+} + d_{2,-})$ as a perturbation thus in principle requires the use of degenerate perturbation theory. It can be shown that at order $g_2$ matrix elements between single trace states and double trace states vanish [25,26]. In contrast, matrix elements between states $|n\rangle$ and $|k; r_1, r_2\rangle$ for $n = \frac{k}{1-r_1-r_2}$ are finite [34] at order $g_2^2$. So far these degeneracy effects have not been rigorously taken into account due to the technical obstacles caused by the fact that a single trace state is degenerate with a continuum of triple trace states$^{13}$. Bearing in mind the implications of the finite-$J$ degeneracies, cf. Sec. 4.3, it is most desirable to stringently carry through analysis in the large-$J$ case as well.

### 6.4 All loop conjecture

Assuming that the effective vertex idea works at any loop order one would expect that $d_2$ encodes all information about the BMN limit of the dilatation operator in the basis (5.1). In the BMN limit we single out those terms of the dilatation operator which produce the maximum number of $J$ factors. These are the terms with the maximum number of $\hat{Z}$ constituents. Therefore, only terms of the dilatation operator which have only one impurity will survive the limit. Any such term will appear in some higher power of $d_2$ and it has to appear with a weight which ensures that the planar BMN limit $\Delta - \rightarrow J + 2\sqrt{1 + \lambda'n^2}$, is conserved. Thus, it is tempting to conjecture that the all loop version of the formula (6.12) reads

$$D - D_0 \longrightarrow 2\sqrt{1 + \left(\frac{\lambda'}{4\pi^2}\right) d_2}.$$  

(6.16)

We now construct a possible all-loop expression for the one impurity part of the full dilatation generator which manifestly gives rise to the above formula in the case of two impurity operators. Iterating the argument given in the beginning of Sec. 5.1 we have the following simplification when $(D_2)^l$ acts on two-impurity operators

$$(D_2)^l \sim 2^{l-1}\phi^{(ao)} D_2^{aoa_1} D_2^{a_1a_2} \ldots D_2^{a_{l-1}a_l}\phi^{(a_l)},$$  

(6.17)

where

$$D_2^{ab} = -2:\text{Tr}[Z, T^a][\hat{Z}, T^b]:.$$  

(6.18)

Thus, a possible form for the one-impurity part of the full $l$-loop dilatation generator is

$$D_{2l} = (-2)^{1-l} \frac{\phi^{(ao)} D_2^{aoa_1} D_2^{a_1a_2} \ldots D_2^{a_{l-1}a_l}\phi^{(a_l)} + \ldots}{(l-1)!!},$$  

(6.19)

up to terms irrelevant in the BMN limit. There might, however, be other combinations which also give rise to (6.16). At the planar level (6.19) is equivalent to the conjecture

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$^{13}$One way to treat the degeneracy might be to consider the finite $J$ operators discussed in Sec. 5 as regularized BMN operators and then ensure that all relevant degeneracies are resolved before taking $J$ to infinity. For single-trace operators this would require working with $n$ being a divisor of $J + 1$.  

27
of \[17,18\] and the result of \[13\]. However, our possible formula would apply to arbitrary genus. If it was correct, one would be able to perform all-loop computations not only for two-impurity BMN operators, but also for more than two impurities. There, we do not find a reason why \(D_{2l}\) should be proportional to \((D_2)^l\) and an individual genus expansion would be required each loop level.

The formula \(\text{(6.19)}\), if true, would also have implications for \(\mathcal{N} = 4\) SYM beyond the BMN limit. It successfully predicts the dilatation generator at two-loops completely: \(D_4\) consists of three parts which can be labelled as \((1,1)\), \((2,1)\) and \((1,2)\), the numbers representing the number of interacting background and impurity fields \(Z, \phi\). The parts with one impurity are predicted directly, the part \((1,2)\) is related to \((2,1)\) by symmetry. At the three-loop level all terms except the terms with two impurities and two background fields, \((2,2)\), can be predicted. As this represents a three-loop interaction between four fields, the corresponding Feynman-diagrams do not have bulk loops. The complexity of such diagrams is comparatively low and a direct computation of coefficients may still be feasible.

7 Symmetry enhancement at the planar level and integrability

Let us define a parity operation \(P\) acting on traces of group generators \(T^a\) by inverting the order of generators within a trace or, alternatively, by transposing all generators \(T^a\). For example

\[
P \text{Tr} Z^3 \phi Z^3 \phi = \text{Tr} Z^3 \phi \phi Z^3 = \text{Tr} Z^3 \phi \phi Z^3.
\]  

(7.1)

It is easily seen that the dilatation generators \(D_0\), \(D_2\), \(D_4\) commute with the parity operation \(PD_{2l}P = D_{2l}\). In fact, parity is related to complex conjugation on group generators, \(PT^a = (T^a)^\dagger = (T^a)^*\). As the dilatation generator \(D\) is real it conserves parity in general

\[
PD = DP.
\]  

(7.2)

This implies that there is no mixing between operators of different parity\(^{14}\), which was also noticed in \[44\]. This parity operation is specific to the \(\text{SU}(N)\) series. In the \(\text{SO}(N)\) and \(\text{Sp}(N)\) series the parity of a trace of \(n\) generators \(T^a\) is given by \((-1)^n\) due to the identity \(PT^a = (T^a)^\dagger = -T^a\)\(^{15}\).

As parity is conserved one should a priori expect that the spectra of positive and negative parity operators are not related. An example of a system with conserved parity is the infinite potential well. There, the even and odd modes have distinct spectra which are not related to each other in any obvious way. This is what happens in \(\mathcal{N} = 4\) SYM in general. If we however consider the strict planar limit, \(\mathcal{N} = \infty\), we find that the two sectors are related. We observe that, whenever parity even and odd states have equal quantum numbers (except parity), they form pairs of operators with degenerate scaling

\(^{14}\)The reason why the separation of parities has not been an issue so far is that at least three impurities are required to allow both parities within the same \(\text{SO}(6)\) representation.

\(^{15}\)A consequence of this is that many of the operators discussed so far are incompatible with the \(\text{SO}(N)\) and \(\text{Sp}(N)\) series.
dimension. As a systematic degeneracy almost inevitably indicates a symmetry there seems to be a symmetry enhancement in the planar limit of $\mathcal{N} = 4$ SYM.

We start by discussing the most simple example of such a planar parity pair and afterwards we explain the degeneracy at the one-loop level by means of an integrable spin chain [37].

### 7.1 Planar parity pairs

There are three unprotected operators in the representation $[3, 1, 3]$ with $\Delta_0 = 7$. One has negative parity

$$O_\pm = \text{Tr}[\phi, Z][\phi, Z][\phi, Z]Z,$$

and two have positive parity

$$O_+ = \left(2 \text{Tr} Z^4 \phi^3 + 2 \text{Tr} Z^2 \phi Z^2 \phi^2 + 2 \text{Tr} Z^2 \phi Z \phi Z \phi - 3 \text{Tr} Z^3 \phi \{\phi, Z\} \phi \right).$$

The scaling dimension for the first is readily obtained

$$\Delta_- = 7 + \frac{5 g_{YM}^2 N}{8 \pi^2} - \frac{15 g_{YM}^4 N^2}{128 \pi^4}.\quad (7.5)$$

The dilatation generator acts on the other two as

$$D_2 = N \left(\frac{10}{20} \frac{8}{N} \frac{8}{8} \right), \quad D_4 = N^2 \left(-\frac{30}{N} - \frac{40}{N^2} \frac{-140}{N^2} - \frac{-40}{N} - \frac{24}{N} - \frac{64}{N^2} \right),\quad (7.6)$$

corresponding to the scaling dimensions (the one loop part of which has been found in [33])

$$\Delta_+ = 7 + \frac{g_{YM}^2 N}{16 \pi^2} \left(9 \pm \sqrt{1 + \frac{169}{N^2}} \right) - \frac{g_{YM}^4 N^2}{256 \pi^4} \left(27 + \frac{52}{N^2} \pm \frac{3 + \frac{948}{N^2}}{\sqrt{1 + \frac{160}{N^2}}} \right).\quad (7.7)$$

Intriguingly we find that the scaling dimension of the two single-trace operators, $O_-$ and $O_{+,1}$ approach each other for $N \to \infty$ at one-loop as well as at two-loop order. We hence find a pair of operators with opposite parity which have degenerate scaling dimensions. This is actually not an exception, it is rather the rule. Among the operators in the representations $[3, \Delta_0 - 6, 3]$ with $6 \leq \Delta_0 \leq 10$ we find 8 pairs of operators and only 3 unpaired ones, cf. Tab. [2] Sec. [3] and App. [4]. And they are not the only examples, we find them in the representations $(\Delta_0 = 9, [4, 1, 4])$, $(\Delta_0 = 10, [4, 2, 4])$, $(\Delta_0 = 6, [0, 2, 0])$ and $(\Delta_0 = 6, [1, 0, 1])$. In fact, every representation that admits both parities seems to have such pairs of operators.

This phenomenon is a signal of an enhanced symmetry in the planar limit of the SU($N$) series[16]. It cannot be supersymmetry (this time) for a simple reason: Supersymmetry is independent of the choice of gauge group whereas at finite $N$ the operators $O_+$

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[16]: The groups SO($N$), Sp($N$), although admitting a planar limit, do not show this behavior: One of the parity partners is always absent.
Table 2: Single-trace pure scalar operators of dimension $\Delta_0 \leq 12$. The operators are distinguished by their representation and parity. The protected operators of representation $[0, \Delta_0, 0]$ were omitted. The operators correspond to states of a SU(2) spin chain of length $L = \Delta_0$.

mix and have modified anomalous dimensions. Furthermore, for gauge groups other than SU($N$) the pairs cannot even exist. Therefore the operators $O^{+,1}$ and $O^{-}$ cannot be part of the same supermultiplet. As the multiplets of this symmetry seem to be either singlets or doublets the symmetry is most likely abelian. In the next subsection we proceed by proving the degeneracy at the one-loop level.

## 7.2 A conserved charge

In order to prove the existence of an abelian symmetry in the planar sector we will find a generator $U$ and show that it commutes with the generator of dilatations $D$. Furthermore, we demand that $U$ anticommutes with the parity operation $P$ so that it may relate operators of opposite parities.

$$[D, U] = \{ P, U \} = 0. \quad (7.8)$$

We will restrict ourselves to the one-loop level, where we can allow all six scalar fields; mixing with other fields is a higher-order effect. The degeneracy at the two-loop level will be investigated in the next section.

We represent a single-trace operator by a cyclic SO(6)-vector spin chain of length $L = \Delta_0$ with zero (angular) momentum as suggested by Minahan and Zarembo [37]. The one-loop dilatation operator is then given by

$$D_2 = N \sum_{k=1}^{L} D_{2,k(k+1)}, \quad (7.9)$$

where $D_{2,k(k+1)}$ is a local interaction linking sites $k$ and $k + 1$ (the sites are periodically
identified). The local interaction $D_{2,k(k+1)}$ is given by

$$D_{2,k(k+1)} = 2I_{k(k+1)} - 2P_{k(k+1)} + K_{k(k+1)}. \quad (7.10)$$

Here, $I_{k(k+1)}$ is the identity, $P_{k(k+1)}$ exchanges sites $k, k+1$ and $K_{k(k+1)}$ is a SO(6) trace over sites $k, k+1$. Graphically this may be represented as in Fig. 4.

We introduce a generator

$$U_2 = \sum_{k=1}^L U_{2,k(k+1)(k+2)}, \quad U_{2,k(k+1)(k+2)} = [D_{2,(k+1)(k+2)}, D_{2,k(k+1)}]. \quad (7.11)$$

Graphically it may be represented as in Fig. 5. This generator is easily seen to anticommute with parity (it has negative mirror symmetry w.r.t. the vertical axis, see Fig. 5). A straightforward but tedious calculation shows that $U_2$ indeed commutes with $D_2$. We will explain this fact in terms of integrability in the next subsection, cf. (7.17).

We note that $U_2$ interchanges $O_-$ of (7.3) with $O_{+,1}$ of (7.4),

$$U_2 O_{+,1} = -60 O_-, \quad U_2 O_- = +4 O_{+,1}, \quad (7.12)$$

so the generator $U_2$ is indeed responsible for the degeneracy of their scaling dimensions. The same is true for all the other pairs of operators we observed. The unpaired operators, for example (4.1), are annihilated by $U_2$. As an aside we note that the eigenoperators are $\frac{1}{\sqrt{60}}O_{+,1} \pm i O_-$. The corresponding eigenvalues of $U_2$ are $\pm 4i \sqrt{15}$ in this case.

Thus we have proven the existence of an additional abelian symmetry in the planar sector at the one-loop level. The symmetry generated by $U_2$, however, cannot be compact: The eigenvalues of $U_2$ are not integer multiples of a common number. Nevertheless, one is led to believe that there exists a SO(2) symmetry (not generated by $U_2$) which has uncharged singlets and charged doublets. Together with the parity operation $P$ it would form the group $O(2)$. In this scenario, when $\frac{1}{N}$ corrections are included, the group $O(2)$ breaks to the parity $\mathbb{Z}_2$. It would be very desirable to understand this...
degeneracy/symmetry better, in terms of $N = 4$ SYM as well as in terms of the AdS/CFT correspondence. Another peculiarity of the planar sector is a recently found degeneracy in four-point functions \[60\]. There it was observed that for $N \to \infty$ a four-point function at one-loop could be described by a single function, although by superconformal symmetry, two functions would be allowed. Four-point functions are related to three-point functions and anomalous dimensions by means of the operator product expansion. Therefore these issues might be related in some way. It would also be of interest to study whether the degeneracy can be observed as a symmetry of planar $n$-point functions.

7.3 Higher charges of the spin chain

A much more enlightening way to prove $[D_2, U_2] = 0$ is to make use of integrability of the spin chain. In \[37\] it was shown that the interaction \[7.10\] has just the right relative coefficients to exhibit integrability. The corresponding $R$ matrix is

$$R_{0k}(u) = P_{0k}((1 - \frac{3}{2}u + \frac{1}{2}u^2)I_{0k} + u(1 - \frac{3}{2}u)\frac{1}{2}D_{2,0k} + \frac{1}{2}u^2(\frac{1}{2}D_{2,0k})^2),$$ \[7.13\]

where $D_{2,0k}$ is the local dilatation generator \[7.10\] acting on an auxiliary site 0, see \[37\] for details. This $R$ matrix satisfies the Yang-Baxter equation for the SO(6) case \[61, 62\]

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u),$$ \[7.14\]

and thus gives rise to an integrable spin chain. Integrability predicts the existence of a tower of commuting charges $t_n$. The zeroth charge is the cyclic shift, it equals the identity in the zero (angular) momentum sector. The first charge is identified with the dilatation generator

$$t_1 = \frac{1}{2N}D_2 - \frac{3}{2}L,$$ \[7.15\]

up to a constant. Using the expressions in \[37\] we find for the second charge

$$t_2 = -\frac{1}{8}U_2 + \frac{1}{2}(t_1)^2 - \frac{5}{8}L.$$ \[7.16\]

From the fact that $t_1$ and $t_2$ commute we easily derive

$$[D_2, U_2] = 0.$$ \[7.17\]

We have seen that the second charge of the integrable spin chain has important consequences. It is thus natural to investigate the third charge $t_3$. Up to polynomials in $t_1$, $t_2$ and $L$ we find some generator $Q_{3,2}$ which commutes with $D_2$, $U_2$ and parity $P$. In contrast to $U_2$ we find that $Q_{3,2}$ does not pair up operators, it simply assigns a number (charge) to each operator. This is in fact what is to be expected. The reason why $U_2$ was interesting is that it anticommutes with $P$ while $D_2$ commutes thus giving rise to pairs. The next charge, $t_4$, will again give rise to some new generator, $Q_{4,2}$, that anticommutes with parity. This generator will relate the same pairs, only with different coefficients (charges). Due to \[7.15\] we know that the spectrum of $t_1$ is related to the spectrum of one-loop anomalous dimensions. A natural question to ask is whether the spectra of the higher charges $Q_{2,2} = U_2$, $Q_{3,2}$, \ldots, have a physical meaning in the gauge theory. Except for special classes of operators, the two-impurity operators for example, we find no obvious relation between the spectra of $D_2$ and $Q_{3,2}$. Thus, there might be some non-trivial information contained in the higher charges. Clearly, the deep question is why integrability emerges from the planar gauge theory.
8 Integrability at higher loops

In the previous section we have found planar parity pairs and justified their existence using integrability at the one-loop level. At the two-loop level integrability has not been established yet. An obstacle in doing so is that two-loop interactions are interactions of next-to-nearest neighbors, whereas an integrable spin chain usually involves nearest neighbor interactions only. Although some next-to-nearest neighbor interactions are included in the tower of higher charges, these cannot be related to the two-loop dilatation generator, because $D_2$ and $D_4$ do not commute while the spin chain charges do. In order to construct an integrable spin chain with non-nearest neighbor interactions we cannot make direct use of the R-matrix formalism and the Yang-Baxter equation. However, we may use our discovery of degenerate pairs in the previous section as the starting point. Integrability at one-loop gives rise to a conserved charge $U_2$. The charge anticommutes with parity $\{P, U_2\} = 0$ and thus pairs up operators. By promoting $D_2$ and $U_2$ to their full counterparts $D$ and $U$ we may generalize the integrable spin chain to higher loops or non-nearest neighbor interactions. The observed degeneracy at the two-loop level is a clear indication that a conserved $U$ exists up to two-loops.

By going to the two-loop level we again restrict ourselves to the operators described in Sec. 3.1 Effectively this means we consider a XXX$\frac{1}{2}$ (SU(2) spin $\frac{1}{2}$) spin chain which does not have the trace term. We can now represent all interactions by means of permutations of adjacent sites. A generic term will be written as

$$\{n_1, n_2, \ldots\} = \sum_{k=1}^{L} P_{k+n_1,k+n_1+1} P_{k+n_2,k+n_2+1} \ldots \quad (8.1)$$

In this notation the one-loop dilatation generator (we will drop factors of $N$ in this section) and the charge $U_2$ are given by

$$D_2 = 2(\{\} - \{0\}), \quad U_2 = 4(\{1, 0\} - \{0, 1\}). \quad (8.2)$$

We start by showing that the degeneracy of parity pairs holds at two-loop. From the assumption that planar $\mathcal{N} = 4$ SYM is integrable at three-loops we derive the corresponding dilatation generator. Then we investigate the constraints on the four-loop dilatation operator and conclude with an analysis of a tower of special three-impurity operators.

8.1 Two-loops

At the two-loop level we have found that the dilatation generator $D$ receives the correction

$$D_4 = 2(-4\{0\} + 6(0) - (\{0, 1\} + \{1, 0\})), \quad (8.3)$$

and we expect that also $U$ receives some correction $U_4$. For the degeneracy to hold at the two-loop level we have to find a $U_4$ such that

$$[D_4, U_2] + [D_2, U_4] = 0, \quad \text{and} \quad \{P, U_4\} = 0. \quad (8.4)$$
We find that this is satisfied by
\[ U_4 = 8 (\{2,1,0\} - \{0,1,2\}). \]  
(8.5)

This proves the degeneracy of anomalous dimensions at two-loops. In performing the commutator we were required to make use of the impossibility of three antisymmetric spins in \( SU(2) \). This indicates that for spin chains with next-to-nearest neighbor interactions, integrability requires that for each site there can only be two states, i.e. a \( SU(2) \) spin chain. This, however, does not exclude that integrability extends to all six scalars of SYM. When the other scalars are included, we have to consider fermions and derivative insertions as well. This might be, roughly speaking, a \( SU(2,2|4) \) spin chain. The interactions involving all relevant fields might then be integrable. At least for the superpartners of the operators under consideration this must indeed be the case due to supersymmetry.

In addition we have found the extension of the third charge \( Q_3 \) to two-loops, see App. F. This suggests that also the higher charges will generalize to at least two-loops. If this is indeed so it justifies our claim that planar \( \mathcal{N} = 4 \) SYM at two-loops is integrable. It is thus natural to conjecture that integrability holds at all-loops. This would mean there exist infinitely many commuting charges \( Q_k \) with loop expansion
\[ Q_k = \sum_{l=0}^{\infty} \left( \frac{g_{YM}^2}{16\pi^2} \right)^l Q_{k,2l}, \]  
(8.6)
where \( Q_1 = D \) and \( Q_2 = U \). Considering the structure of the first few known terms we assume that \( Q_{k,2l} \) is composed of up to \( k + l - 1 \) permutations involving up to \( k + l \) adjacent sites. For \( k \) odd the interactions are symmetric and parity-conserving, and for \( k \) even they are antisymmetric and parity-inverting. Note that in the latter case this implies a vanishing constant piece.

### 8.2 Three-loops

In this subsection we investigate the constraints on the planar dilatation generator at three-loops due to integrability. This requirement together with the correct behavior in the BMN limit, (6.15), fixes \( D_6 \) completely. This is not supposed to imply that the \( D_6 \) we derive is the correct planar dilatation generator of \( \mathcal{N} = 4 \) SYM. It may well be that the degeneracy of pairs is broken at three-loops.

As building blocks for the three-loop interaction we allow symmetric (with respect to the order of permutations \( \{n_1, \ldots, n_k\} \mapsto \{n_k, \ldots, n_1\} \) to ensure a real spectrum) and parity-conserving (positive symmetry under \( \{n_1, n_2, \ldots\} \mapsto \{-n_1, -n_2, \ldots\} \)) structures. These should act on only four adjacent sites and have no more than three permutations. This follows from the general structure of a connected three-loop vertex. We find exactly six structures that satisfy these constraints. Using the planar BMN limit (6.15) we can fix the coefficients of four of these structures. The requirement of pairing for three-impurity operators at dimension 7 and 8, see Sec. 7.1 and (D.13) fixes the remaining two. We find
\[ D_6 = 4 (15\{\} - 26\{0\} + 6 (\{0,1\} + \{1,0\}) + \{0,2\} - (\{0,1,2\} + \{2,1,0\})). \]  
(8.7)
The associated correction to $U$ can be found in much the same way analyzing all possible antisymmetric, parity-inverting structures and requiring that $D$ and $U$ commute. This fixes $U_6$ up to a contribution proportional to the fourth charge at one-loop $Q_{4,2}$. Acting on the Konishi descendant $\text{Tr}[\phi, Z][\phi, Z]$ we find the three-loop planar contribution to its scaling dimension

$$\Delta_{K'} = 4 + \frac{3g_{YM}^2 N}{4\pi^2} - \frac{3g_{YM}^4 N^2}{16\pi^4} + \frac{21g_{YM}^6 N^3}{256\pi^6},$$

subject to the assumption made at the beginning of this subsection. It is also worth noting the contribution to a general two-impurity operator. Up to a piece proportional to $(D_2)^3$ we find some contribution in the upper-left $3 \times 3$ block (and equivalently in the lower-right $3 \times 3$ block)

$$D_6 = \frac{1}{8}(D_2)^3 + 4 \cdot \begin{pmatrix} +9 & -10 & +1 \\ -10 & +10 & +1 \\ +1 & -1 & 0 \\ \end{pmatrix},$$

in analogy to (5.15). Considering now both $D_4$ and $D_6$ as perturbations of $D_2$ (cf. Sec. 5.2) we can obtain the planar three-loop contribution to the anomalous dimension of the operators (5.13). We get one contribution from the diagonal part of $D_6$ and one contribution from the off-diagonal part of $D_4$ as encoded in the usual formula for second order non-degenerate perturbation theory. The final result reads

$$\delta \Delta_n^J = \frac{g_{YM}^6 N^3}{\pi^6} \sin^6 \frac{\pi n}{J+1} \left( \frac{1}{8} + \frac{\cos^2 \frac{\pi n}{J+1}}{4(J+1)^2} \left( 3J + 2(J+6) \cos^2 \frac{\pi n}{J+1} \right) \right),$$

and it produces the correct result for all values of $J, n$, even for the Konishi (8.8) at $J = 2, n = 1$.

Using the planar three-loop dilatation generator it might be possible derive the non-planar version by analyzing all possible diagrams and fitting their coefficients to non-renormalization theorems and the planar version in analogy to Sec. 3.

### 8.3 Four-loops

Having convinced ourselves of the usefulness of the constraints from the pairing and the BMN limit we cheerfully proceed to four-loops, just to find that all coefficients but one are fixed. Unfortunately, the left-over coefficient does influence most four-loop anomalous dimensions, in particular the one of the Konishi descendant $K'$ \(^{18}\). The consideration of higher charges might fix this remaining parameter. Nevertheless, we will see below that we are able to find special operators for which the four-loop scaling dimensions turn out to be independent of the so far undetermined parameter.

\(^{18}\)It is not even clear how to treat $K'$, because the interaction is longer than the spin chain in this case.
At four-loops we find in total twelve independent structures. Five coefficients can be fixed using the planar BMN limit \(6.15\). Further five coefficients are fixed by demanding degeneracy of pairs. An expression for \(D_8\) involving the remaining two coefficients \(\alpha, \beta\) can be found in App. F. Evaluating some eigenvalues of \(D\) up to four-loops one observes that they do not depend on the parameter \(\beta\). This behavior is however expected. At four loops we have the freedom to rotate the space of operators with the orthogonal transformation generated by the antisymmetric generator \([D_2, D_4]\). This gives rise to the following similarity transformation

\[
D' = (1 + \beta' g^6[D_2, D_4])^{-1} D (1 + \beta' g^6[D_2, D_4]).
\]  

(8.11)

The first term in \(D'\) due to the transformation is

\[
\beta' g^8 [D_2, [D_2, D_4]],
\]  

(8.12)

and this is proportional to what multiplies \(\beta\) in \(D_8\). Now, as \(D\) and \(D'\) are related by a similarity transformation, their eigenvalues are equal and \(\beta\) only affects the eigenvectors, but not the eigenvalues.

### 8.4 A tower of three impurity operators

For \(\Delta_0\) odd we observe that there are as many even operators in the representation \([3, \Delta_0 - 6, 3]\) as there are odd ones. For a complete table of spin chain states, up to dimension 12, see Tab. 2. If \(\Delta_0\) is even the same is true except for one additional operator with negative parity. This additional operator is

\[
\mathcal{O} = \sum_{k=1}^{\Delta_0-4} (-1)^k \text{Tr} \phi Z^k \phi Z^{\Delta_0-3-k} \phi,
\]  

(8.13)

which is an exact planar eigenoperator of \(D_2\) with eigenvalue \(12N\). The lowest dimensional example at \(\Delta_0 = 6\) was discussed in Sec. 4.2. It is annihilated by the extra symmetry generator \(U_2\), i.e. it is unpaired. Note, that for all of these operators two of the impurities are always next to each other.

The operators \(\mathcal{O}\) are not exact planar eigenoperators of \(D_4\). Nevertheless, we can project \(D_4 \mathcal{O}\) to the piece proportional to \(\mathcal{O}\) and find \(-36N^2\) as the coefficient. Consequently we have found a sequence of operator with planar dimensions

\[
\Delta = \Delta_0 + \frac{3g_{YM}^2N}{4\pi^2} - \frac{9g_{YM}^4N^2}{64\pi^4}.
\]  

(8.14)

Interestingly, the anomalous dimension does not depend on the bare dimension \(\Delta_0\). In that sense, this operator behaves much like the highest mode in the series of two impurity operators discussed in Sec. 5. Assuming the highest mode number were \(n = (J + 1)/2\) \(^{19}\), the anomalous dimension (5.17) would equal \(g_{YM}^2N/\pi^2 - g_{YM}^4N^2/4\pi^4\), which is also

\(^{19}\)The corresponding operator does not exist. The highest mode number is \([J/2]\), for which, however, the anomalous dimension does not differ considerably.
independent of $\Delta_0$. Furthermore, the frequency of the phase factor in (5.13) would also be extremal as in (8.13).

For a few of the lower dimensional operators we can also evaluate the anomalous dimensions up to four-loops (subject to the assumptions in deriving the corresponding vertices). Here, the yet undetermined coefficient $\alpha$ does not contribute. Intriguingly we find

$$\Delta = \Delta_0 + 3 \frac{g_{YM}^2 N}{4\pi^2} - \frac{9 g_{YM}^4 N^2}{64\pi^2} + \frac{66 g_{YM}^6 N^3}{1024\pi^6} - \frac{645 g_{YM}^8 N^4}{16384\pi^8},$$

for $\Delta_0 = 10, 12, 14$. For $\Delta_0 = 8$ the four-loop contribution differs

$$\Delta = 8 + 3 \frac{g_{YM}^2 N}{4\pi^2} - \frac{9 g_{YM}^4 N^2}{64\pi^2} + \frac{66 g_{YM}^6 N^3}{1024\pi^6} - \frac{648 g_{YM}^8 N^4}{16384\pi^8},$$

while for $\Delta_0 = 6$ also the three-loop contribution is modified

$$\Delta = 6 + 3 \frac{g_{YM}^2 N}{4\pi^2} - \frac{9 g_{YM}^4 N^2}{64\pi^2} + \frac{63 g_{YM}^6 N^3}{1024\pi^6} - \frac{621 g_{YM}^8 N^4}{16384\pi^8}.$$

So the following picture emerges: The $k$-loop anomalous dimensions for the operators of dimensions $\Delta_0 = 2 + 2k + n, n \geq 0$ seem to be equal.

### 9 Conclusions and outlook

The main message of this paper is the proposal that perturbative scaling dimensions in 4D conformal gauge theories should not be computed on a case-by-case basis, using the standard, very laborious procedure. The latter consists in first working out classical and quantum two-point functions of a set of fields, subsequently recursively diagonalizing and renormalizing the fields at each order in the gauge coupling $g_{YM}$ in order to find “good” conformal fields satisfying the expected diagonal form in eq. (1.1) to the desired order, and finally extracting the scaling dimensions from eq. (1.1). Instead one should focus on the dilatation operator relevant to the general (i.e. possessing an arbitrary engineering dimension) class of operators under study. Once it is found, to the desired order in $g_{YM}$, the computation of the dilatation matrix becomes a straightforward, purely algebraic exercise. The subsequent calculation of the eigenvalues of the matrix then yields the scaling dimensions, while the eigenvectors resolve the mixing problem.

In the present work we illustrated this proposal in a specific example: We obtained the dilatation operator up to $O(g_{YM}^4)$ of $\mathcal{N} = 4$ SU($N$) Yang-Mills theory for arbitrary traceless pure scalar fields, cf. eqs. (1.14). However, we certainly believe that our methodology is rather general and should therefore be equally applicable to other four-dimensional conformal gauge theories with less supersymmetry.

In order to save us a significant amount of additional work we did use non-renormalization theorems as well as recent results on the scaling dimensions of the so-called BMN two-impurity operators $[17,18,43]$ to fix constants in our two-loop dilatation operator. Incidentally, this may serve as an interesting illustration how the AdS/CFT correspondence (here in its latest reincarnation, i.e. the plane wave/BMN correspondence) can lead to new insights into gauge theory. We then went on to apply our example to a large
number of situations of interest, obtaining with ease a host of novel specific results for anomalous dimensions and for the resolution of mixing of scalar operators. This yields important new information on the generic structure of $\mathcal{N} = 4$: E.g. the degeneracy of certain single and double-trace operators leads to unexpected $1/\mathcal{N}$ terms in the associated anomalous dimensions, and to the breakdown of a well-defined double expansion in the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ and the genus counting parameter $1/\mathcal{N}^2$. One surely very interesting issue we did not yet address is the interpretation of these effects in the light of the AdS/CFT correspondence.

Our new results on $\mathcal{N} = 4$ SYM already lead to new insights into the plane wave strings/BMN correspondence. We were able to extend our quantume mechanical description of BMN gauge theory \cite{34}, deriving from the two-loop dilatation operator the two-loop contribution to the quantum Hamiltonian. This contribution strongly hinted at an all genera version of the celebrated all loop BMN square root formula, cf. eqn. \eqref{6.16}. Whether this generalization withstands closer scrutiny remains to be seen, of course.

It would also be very helpful to develop techniques similar to the ones presented here for the efficient evaluation of correlation functions of more than two local fields. It is e.g. well known that conformal invariance completely fixes the space-time form of three-point functions ($x_{ij} = x_i - x_j$)

$$\langle O_\alpha(x_1) O_\beta(x_2) O_\gamma(x_3) \rangle = \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha + \Delta_\beta - \Delta_\gamma} |x_{23}|^{\Delta_\beta + \Delta_\gamma - \Delta_\alpha} |x_{31}|^{\Delta_\gamma + \Delta_\alpha - \Delta_\beta}. \quad (9.1)$$

Therefore the quantities of interest are the finite structure constants $C_{\alpha\beta\gamma}$ alone. These constants appear in the operator product expansion and apart from the scaling dimensions they are the other central quantity in a conformal field theory of local operators\cite{20}. However, in order to really obtain this form one has to use the correct eigenstates of the dilatation operator. Furthermore, standard perturbative computations are “contaminated” by useless finite and divergent contributions from the perturbative expansion of the weights $\Delta_\alpha$. One therefore wonders whether one may generalize our methodology and develop purely algebraic techniques for directly finding the structure constants. Of equal interest are four-point functions, which also seem to possess many unexpected, simplifying features waiting to be explored (see e.g. \cite{25, 60}).

By exploiting the recently discovered description of the planar limit of the one-loop dilatation operator as the Hamiltonian of an integrable spin chain \cite{37} we pointed out the existence of a new axial symmetry of $\mathcal{N} = 4$ Super Yang-Mills theory at $N = \infty$, linking fields of opposite parity (w.r.t. reversing the order of fields inside SU($N$) traces). We were able to prove the symmetry as a direct consequence of the presence of non-trivial charges commuting with the Hamiltonian. We furthermore derived this to follow immediately from the existence of an $R$-matrix satisfying the Yang-Baxter equation, i.e. from the integrability discovered in \cite{37}. An exciting open question is the interpretation of this symmetry (and of the integrability in general) from the point of view of the planar gauge theory, or possibly from the point of view of its dual string description. This is even more

\footnote{A number of structure constants for two and three impurity BMN operators \cite{25, 64, 26, 27, 64} have been obtained so far. In terms of the BMN correspondence it would be important to understand their dual on the string theory side \cite{65}}.
pressing as we found much evidence that this integrable structure extends to two loops (we did prove that at least two further charges commute with the two loop dilatation operator) and it is obviously tempting to conjecture that it holds to all loops. One could therefore hope that a proper understanding of the integrability might lead to the exact construction of the all-order planar dilatation generator, see also the discussion below.

It would be important to extend the methods of this work to fix the remaining terms in the dilatation operator eq.(1.5) pertaining to further classes of operators such as scalars with SO(6) traces, fermions, field strengths or covariant derivatives. The latter would be interesting in order to study the high spin limit of \[66\], see also \[67\]. Here we expect the superconformal symmetry of the model to be helpful. It would also be fascinating to investigate whether the reformulation of the planar theory as an integrable spin chain can be extended to include the other fields.

It is striking that, once the dilatation operator is found, the calculations of anomalous dimension matrices become purely algebraic. However, in order to fully justify its derivation we did need to take a look at two-point functions (cf. Sec. 2.3). It is natural to wonder whether there are further shortcuts that leads to the determination of the terms in the perturbative expansion of the dilatation operator eq.(1.5). For the \( \mathcal{N} = 4 \) model this is not an unreasonable expectation: First of all, the theory is scale-invariant on the quantum level and therefore possesses a finite dilatation operator. Thus one could expect that the renormalization procedure of Sec. 2.3 is only a scaffold that one might be able to avoid. Secondly, the theory’s action is unique, and entirely determined by the maximal superconformal symmetry SU(2, 2|4). Is there a way to use the symmetry algebra, possibly paired with some further insights into the \( \mathcal{N} = 4 \) model, to completely fix the structure of the dilatation operator (1.5)?

We would certainly like to push our procedure to higher loops. This is not a pointless exercise. E.g. little seems to be known about the analytic structure of the exact anomalous dimension of low dimensional fields such as Konishi. All we currently have is the two-loop result eq.(1.6), the planar three-loop conjecture eq.(1.7) and the conjectured (from AdS/CFT) strong coupling behavior \( \Delta \sim (g_\text{YM}^2 N)^{1/4} \). Knowing further terms in the perturbative expansion might give essential clues about the convergence structure of the series (is the radius of convergence zero or finite?) and might allow us to estimate the strong coupling result by Padé approximants. Incidentally, the Konishi field eq.(1.7) is particularly interesting as it cannot mix with any other fields and is therefore known to be an exact eigenstate of the dilatation operator to all orders in perturbation theory.

We have taken first steps towards constraining the complete dilatation operator, building directly on the consequences of the spin chain picture: Assuming the above mentioned axial symmetry to hold also beyond two loops we were able to derive a planar version of the three-loop dilatation operator which in turn allowed us to obtain further results on anomalous dimensions. A preliminary study of the possible planar four-loop structures showed that the information which completely fixed the three-loop case leaves one coefficient of the planar four-loop dilatation operator undetermined. A speculation of, potentially, tremendous importance would be that imposing the full integrability structure completely fixes, at each loop order, the exact planar dilatation operator.
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A Conventions

We use the following $\mathcal{N} = 4$ supersymmetric action in components

$$S = \frac{1}{2} \int \frac{d^{4-2\epsilon} x}{(2\pi)^{2-\epsilon}} \text{Tr} \left( \frac{i}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu \Phi_m D_\mu \Phi_m - \frac{1}{4} g^2 \mu^{2 \epsilon} \Phi_m [\Phi_m, \Phi_n] \right. \left. + \frac{1}{2} \Psi^T \Sigma_\mu D_\mu \Psi - \frac{1}{2} g \mu \epsilon \Psi^T \Sigma_\mu [\Phi_m, \Psi] \right),$$

$$D_\mu X = \partial_\mu X - ig \mu \epsilon [A_\mu, X],$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig \mu \epsilon [A_\mu, A_\nu].$$

(A.1)

The coupling constant $g$ is related to the common coupling constant of $\mathcal{N} = 4$ SYM by

$$g^2 = \frac{g_{\text{YM}}^2}{4(2\pi)^{2-\epsilon}} \rightarrow \frac{g_{\text{YM}}^2}{16\pi^2}.$$  

(A.2)

This normalization turns out convenient when evaluating space-time integrals and when operators with fermions and derivative insertions are considered.

The fields carry a color structure, $\Phi_m = T^a \phi^{(a)}_m$. We will consider the gauge group $U(N)$ whose generators $T^a$ we have normalized such that

$$\text{Tr} T^a T^b = \delta^{ab}, \quad \sum_a (T^a)_\alpha^\beta (T^a)^\gamma_\delta = \delta^\alpha_\delta \delta^\gamma_\beta.$$  

(A.3)

This implies the $U(N)$ fusion and fission rules

$$\text{Tr} T^a A \text{Tr} T^a B = \text{Tr} AB, \quad \text{Tr} T^a AT^a B = \text{Tr} A \text{Tr} B.$$  

(A.4)

In this work we make extensive use of variations with respect to a field $\Phi_m$, which we will denote by

$$\delta \Phi_m = \frac{\delta}{\delta \Phi_m} = T^a \frac{\delta}{\delta \phi^{(a)}_m}.$$  

(A.5)

It is therefore understood when the variation hits a field, both the variation symbol and the field are replaced by a color generator $T^a$. Note that the variation symbols $\delta \Phi_m$ act only to the right. In a normal ordered word of fields and variations, $\delta \Phi \delta \Phi \delta \Phi \ldots$, it is understood that the variations do not contract to any of the fields within the normal ordering.
B Scalar space-time integrals

We have normalized the scalar propagator to

\[ I_{xy} = \frac{\Gamma(1 - \epsilon)}{|\frac{1}{2}(x - y)^2|^{1-\epsilon}}, \]  

(B.1)

it is a rather convenient normalization when derivatives are taken (e.g. for fermions). The following integrals are required at one-loop

\[
Y_{x_1x_2x_3} = \mu^{2\epsilon} \int \frac{d^{4-2\epsilon} z}{(2\pi)^{2-\epsilon}} I_{x_1z} I_{x_2z} I_{x_3z},
\]

(B.2)

\[
X_{x_1x_2x_3x_4} = \mu^{2\epsilon} \int \frac{d^{4-2\epsilon} z}{(2\pi)^{2-\epsilon}} I_{x_1z} I_{x_2z} I_{x_3z} I_{x_4z},
\]

(B.2)

\[
\tilde{H}_{x_1x_2x_3x_4} = \frac{1}{2} \mu^{2\epsilon} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right)^2 \int \frac{d^{4-2\epsilon} z_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} z_2}{(2\pi)^{4-2\epsilon}} I_{x_1z_1} I_{x_2z_1} I_{x_3z_2} I_{x_4z_2}. \]

When evaluated in two-point functions they yield [68]

\[
\frac{Y_{00x}}{I_{0x}} = \frac{1}{\epsilon(1 - 2\epsilon)} \xi, \quad \frac{X_{00xx}}{I_{0x}^2} = \frac{2(1 - 3\epsilon)\gamma}{\epsilon(1 - 2\epsilon)^2} \xi, \quad \frac{\tilde{H}_{00xx}}{I_{0x}^2} = -\frac{2(1 - 3\epsilon)(\gamma - 1)}{\epsilon^2(1 - 2\epsilon)} \xi, \]

(B.3)

where

\[
\xi = \frac{\Gamma(1 - \epsilon)}{|\frac{1}{2}\mu^{2}\cdot x^2 |^{\epsilon}}, \quad \gamma = \frac{\Gamma(1 - \epsilon)\Gamma(1 + \epsilon)^2 \Gamma(1 - 3\epsilon)}{\Gamma(1 - 2\epsilon)^2 \Gamma(1 + 2\epsilon)} = 1 + 6\zeta(3)\epsilon^3 + O(\epsilon^4). \]

(C.1)

C Renormalization at higher loops

To obtain the arbitrary loop correlator we insert all \(l\)-loop connected Green functions \(W_{2l}\) in the correlator

\[
\langle \mathcal{O}_\alpha^+ \mathcal{O}_\beta^- \rangle = \exp(W_0) \exp \left( \sum_{l=1}^{\infty} g^{2l} W_{2l}(x, \hat{\phi}^+, \hat{\phi}^-) \right) \mathcal{O}_\alpha^+ \mathcal{O}_\beta^- |_{\phi_0 = 0}. \]

(C.1)

In analogy to (2.6) we change the argument \(\hat{\phi}^+\) of \(W_{2l}(x, \hat{\phi}^+, \hat{\phi}^-)\) to \(I_{0x}^{-1} \hat{\phi}^-\)

\[
\langle \mathcal{O}_\alpha^+ \mathcal{O}_\beta^- \rangle = \exp(W_0) : \exp \left( \sum_{l=1}^{\infty} g^{2l} W_{2l}(x, I_{0x}^{-1} \hat{\phi}^-, \hat{\phi}^-) \right) : \mathcal{O}_\alpha^+ \mathcal{O}_\beta^- |_{\phi_0 = 0}. \]

(C.2)

Alternatively, we could change the argument \(\hat{\phi}^-\) to \(I_{0x}^{-1} \hat{\phi}^+\). This does not make a difference as \(W_{2l}(x, \hat{\phi}^+, \hat{\phi}^-)\) is symmetric in the arguments \(\hat{\phi}^+\) and \(\hat{\phi}^-\). We would then like to rewrite (C.2) in a convenient form for the conformal structure of the correlator:

\[
\langle \mathcal{O}_\alpha^+ \mathcal{O}_\beta^- \rangle = \exp(W_0) \exp(V^-(x)) \mathcal{O}_\alpha^+ \mathcal{O}_\beta^- |_{\phi_0 = 0}, \]

(C.3)

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where
\[
V(x) = \sum_{l=1}^{\infty} g^{2l} V_{2l}(x) - \frac{1}{48} g^8 [V_2(x), [V_2(x), V_4(x)]] + \ldots. \quad (C.4)
\]

The terms \( V_{2l} \) are defined by the equality of (C.2) and (C.3)
\[
\exp(V(x)) = \exp(\sum_{l=1}^{\infty} g^{2l} W_{2l}(x, I^{-1}_0 \Phi, \bar{\Phi})). \quad (C.5)
\]
which will have to be solved perturbatively. All the terms that arise due to normal ordering of the exponential and the commutator terms in (C.4) need to be absorbed into the definition of higher order vertices. For example, the two-loop effective vertex is
\[
V_4(x) = W_4(x, I^{-1}_0 \Phi, \bar{\Phi}) : -\frac{1}{2} (V_2(x) V_2(x) - V_2(x) V_2(x)) :. \quad (C.6)
\]
The commutator terms in (C.4) were included for convenience, we will explain this issue below. The symmetry of \( W_{2l} \) is translated into the effective equality
\[
\exp(W_0) V_{2l}^-(x) = \exp(W_0) V_{2l}^+(x). \quad (C.7)
\]
In fact we can introduce a transpose operation on a generator \( X \) by the definition
\[
\exp(W_0) X^- = \exp(W_0) X^{T+}, \quad (C.8)
\]
In other words, letting \( X \) act on \( \Phi^- \) is equivalent to letting \( X^T \) act on \( \Phi^+ \). The symmetry of the vertices translates to
\[
V_{2l}^T(x) = V_{2l}(x). \quad (C.9)
\]
We renormalize the operators according to
\[
\tilde{O} = \exp\left(-\frac{1}{2} Z(x_0)\right)O, \quad (C.10)
\]
with
\[
Z(x_0) = \sum_{l=1}^{\infty} g^{2l} V_{2l}(x_0) - \frac{1}{12} g^6 [V_2(x_0), V_4(x_0)] + \ldots \quad (C.11)
\]
This gives
\[
\langle O^+_\alpha O^-_\beta \rangle = \exp(W_0) \exp\left(-\frac{1}{2} Z^+(x_0)\right) O^+_\alpha \exp\left(-\frac{1}{2} Z^-(x_0)\right) O^-_\beta |_{\Phi=0}. \quad (C.12)
\]
We can commute objects with a + and a – index freely and use the transpose operation (C.8) to make \( Z^+ \) act on \( \Phi^- \) instead. We get
\[
\langle O^+_\alpha O^-_\beta \rangle = \exp(W_0) \exp\left(-\frac{1}{2} Z^+(x_0)\right) \exp(V^-(x)) \exp\left(-\frac{1}{2} Z^-(x_0)\right) O^+_\alpha O^-_\beta |_{\Phi=0}. \quad (C.13)
\]
The vertices \( V_{2l}(x_0) \) in \( Z(x_0) \) are symmetric, (C.9), only the commutator in (C.11) requires special care, because \( V_2 \) and \( V_4 \) need to be transformed consecutively. This effectively inverts their order and flips the sign of the commutator:
\[
Z^T(x_0) = \sum_{l=1}^{\infty} g^{2l} V_{2l}(x_0) + \frac{1}{12} g^6 [V_2(x_0), V_4(x_0)] + \ldots \quad (C.14)
\]
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In a renormalizable field theory the dependence of $V_{2l}$ on $x$ is determined, we write

$$V_{2l}(x) = \xi^l V_{2l}, \quad \xi = \frac{\Gamma(1-\epsilon)}{\frac{1}{2} \mu^2 x^2 |\epsilon|}. \quad (C.15)$$

We combine the last three exponentials in (C.13) into one with exponent

$$\sum_{l=1}^{\infty} (\xi^l - \xi_0^l) g^{2l} V_{2l}^r - \frac{1}{48} g^8 (\xi - \xi_0)^4 [V_{2r}^-, [V_{2r}^-, V_{4r}^-]] + \ldots \quad (C.16)$$

The $l$-loop Green function $W_{2l}$ is expected to have multiple poles at $\epsilon = 0$. In a conformal field theory, however, these poles must have cancelled in the combination $V_{2l}$ as given by (C.4), (C.5). If so, we can finally send the regulator to zero and find

$$\langle \tilde{\mathcal{O}}_\alpha^+ \tilde{\mathcal{O}}^-_\beta \rangle = \exp(W_0) \exp\left(\log(x_0^2/x^2) \sum_{l=1}^{\infty} g^{2l} D_{2l}^- \tilde{\mathcal{O}}_\alpha^+ \tilde{\mathcal{O}}^-_\beta \right|_{\Phi=0}; \quad (C.17)$$

with

$$D_{2l} = -l \lim_{\epsilon \to 0} \epsilon V_{2l}. \quad (C.18)$$

Note that the commutator term in (C.16) vanishes due to four powers of $\epsilon$ from $(\xi - \xi_0)^4$ opposed to only three powers of $1/\epsilon$ from the $V_{2l}$. For this cancellation to happen the commutator terms in (C.4) and (C.11) are necessary\(^{21}\).

Some comments about the renormalization program are in order. The effective vertices $V_{2l}$ are connected diagrams. They are generated from the Green functions $W_{2l}$ by removing the normal ordering of an exponential (C.5) and adding commutators (C.4). One can easily convince oneself that these operations produce connected diagrams. The same is then true also for the dilatation generator $D$. Secondly, the program ensures that the coefficient of the two-point function is given by free-contractions of the unrenormalized operators. Thirdly, the effective vertices $V_{2l}$ are symmetric with respect to the scalar product induced by free contractions, see (C.7). The same holds for the dilatation generator which consequently has real eigenvalues\(^{22}\).

**D** \quad **A collection of anomalous dimensions**

**D.1** \quad **Away from the unitarity bounds**

The following operators have a form similar to the ones discussed in Sec. 1. As they contain all six scalar fields and SO(6) traces they mix with operators containing fermions and derivatives, see Sec. 3.1. This mixing, however, only becomes relevant beyond one-loop and we can determine their one-loop anomalous dimensions using (1.6).

\(^{21}\)We have investigated all possible terms that arise in a four-loop computation. We found that exactly the commutator structure in (C.4) was required to obtain a finite, conformally covariant correlator.

\(^{22}\)Some eigenvalues may appear to be complex. This can only happen if the corresponding eigenvector has zero norm. In this case the operator in fact does not exist. This happens if the rank of the group is small compared to the size of the operator. Then group identities make some operators linearly dependent.
D.1.1 Dimension 5, [0, 1, 0]

We find one single trace operator with dimension
\[ \Delta = 5 + \frac{5g_{YM}^2 N}{4\pi^2}, \] (D.0)
three single-trace and three double-trace operators with dimensions
\[ \Delta = 5 + \frac{g_{YM}^2 N}{4\pi^2} \omega, \] (D.1)
where \( \omega \) is a root of the sixth-order equation
\[ \omega^6 - 17\omega^5 + \left(110 - \frac{50}{N^2}\right)\omega^4 - \left(335 - \frac{565}{N^2}\right)\omega^3 + \left(475 - \frac{2440}{N^2} + \frac{400}{N^4}\right)\omega^2 
- \left(250 - \frac{4850}{N^2} + \frac{1600}{N^4}\right)\omega - \left(\frac{3750}{N^2} - \frac{4000}{N^4}\right) = 0. \] (D.2)

Two of these operators, a single-trace and a double-trace one, are degenerate in the planar limit. The degeneracy is lifted by \( \frac{1}{N} \) corrections and the t’ Hooft expansion of their scaling dimension is subject to the issues discussed in Sec. 4.3.

D.1.2 Dimension 6, [0, 2, 0], planar

We find one pair of operators with dimension
\[ \Delta = 6 + \frac{7g_{YM}^2 N}{8\pi^2}, \] (D.3)
two operators with dimension
\[ \Delta = 6 + \frac{g_{YM}^2 N}{\pi^2}, \] (D.4)
and six operators with dimensions
\[ \Delta = 6 + \frac{g_{YM}^2 N}{8\pi^2} \omega, \] (D.5)
where \( \omega \) is a root of the sixth-order equation
\[ \omega^6 - 43\omega^5 + 731\omega^4 - 6238\omega^3 + 27936\omega^2 - 61776\omega + 52272 = 0. \] (D.6)

D.1.3 Dimension 6, [1, 0, 1], planar

We find two pairs of operators with dimension
\[ \Delta = 6 + \frac{g_{YM}^2 N}{\pi^2}, \] (D.7)
two operators with dimension
\[ \Delta = 6 + \frac{5g_{YM}^2 N}{4\pi^2}, \] (D.8)
and three operators with dimensions

\[ \Delta = 6 + \frac{g_{YM}^2 N}{8\pi^2} \omega, \quad (D.9) \]

where \( \omega \) is a root of the cubic equation

\[ \omega^3 - 23\omega^2 + 158\omega - 308 = 0. \quad (D.10) \]

**D.2 Further away from the unitarity bounds**

**D.2.1 Dimension 6, [0, 0, 0], planar**

We find five operators with dimensions

\[ \Delta = 6 + \frac{g_{YM}^2 N}{8\pi^2} \omega, \quad (D.11) \]

where \( \omega \) is a root of the quintic equation

\[ \omega^5 - 43\omega^4 + 701\omega^3 - 5338\omega^2 + 18480\omega - 21960 = 0. \quad (D.12) \]

**D.3 Three impurities**

The following operators are in the SO(6) representations for which mixing with fermions and derivative insertions is prohibited, see Sec. 3.1. We may apply the dilatation generator at two-loops (6.1).

**D.3.1 Dimension 8, [3, 2, 3], planar**

We find a pair of operators with dimension

\[ \Delta = 8 + \frac{g_{YM}^2 N}{2\pi^2} - \frac{5g_{YM}^4 N^2}{64\pi^4}, \quad (D.13) \]

and a single operator with dimension (see Sec. 8.4)

\[ \Delta = 8 + \frac{3g_{YM}^2 N}{4\pi^2} - \frac{9g_{YM}^4 N^2}{64\pi^4}. \quad (D.14) \]

**D.3.2 Dimension 9, [3, 3, 3], planar**

We find three pairs of operators with dimensions

\[ \Delta = 9 + \frac{g_{YM}^2 N}{16\pi^2} \omega, \quad (D.15) \]

where \( \omega \) is a root of the cubic equation

\[ \omega^3 - 34\omega^2 + 360\omega - 1176 + \frac{g_{YM}^2 N}{16\pi^2} (102\omega^2 - 2100\omega + 9912) = 0. \quad (D.16) \]
D.3.3 Dimension 10, [3, 4, 3], planar
We find one operator with dimension (see Sec. 8.4)

\[ \Delta = 10 + \frac{3g_{YM}^2 N}{4\pi^2} - \frac{9g_{YM}^4 N^2}{64\pi^4}, \]  

\hspace{1cm} (D.17)

and three pairs of operators with dimensions

\[ \Delta = 10 + \frac{g_{YM}^2 N}{16\pi^2} \omega, \]  

\hspace{1cm} (D.18)

where \( \omega \) is a root of the cubic equation

\[ \omega^3 - 30\omega^2 + 276\omega - 768 + g_{YM}^2 N (86\omega^2 - 1516\omega + 5984) = 0. \]  

\hspace{1cm} (D.19)

D.4 Four impurities
D.4.1 Dimension 8, [4, 0, 4], planar
We find three operators with dimensions

\[ \Delta = 8 + \frac{g_{YM}^2 N}{16\pi^2} \omega, \]  

\hspace{1cm} (D.20)

where \( \omega \) is a root of the cubic equation

\[ \omega^3 - 40\omega^2 + 464\omega - 1600 + \frac{g_{YM}^2 N}{16\pi^2} (128\omega^2 - 2720\omega + 12800) = 0. \]  

\hspace{1cm} (D.21)

D.4.2 Dimension 9, [4, 1, 4], planar
We find a pair of operators with dimension\(^{23}\)

\[ \Delta = 9 + \frac{5g_{YM}^2 N}{8\pi^2} - \frac{15g_{YM}^4 N^2}{128\pi^4}, \]  

\hspace{1cm} (D.22)

and two operators with dimensions

\[ \Delta = 9 + \frac{g_{YM}^2 N (3 \pm \sqrt{3})}{4\pi^2} - \frac{g_{YM}^4 N^2 (18 \pm 9\sqrt{3})}{128\pi^4} \]  

\hspace{1cm} (D.23)

D.4.3 Dimension 10, [4, 2, 4], planar
We find two pairs of operators with dimension

\[ \Delta = 10 + \frac{g_{YM}^2 N (11 \pm \sqrt{5})}{16\pi^2} - \frac{g_{YM}^4 N^2 (31 \pm 3\sqrt{5})}{256\pi^4}, \]  

\hspace{1cm} (D.24)

\(^{23}\)This dimension matches \( \text{red} \).
and six operators with dimensions
\[ \Delta = 10 + \frac{g_{\text{YM}}^2 N}{4\pi^2} \omega, \]
where \( \omega \) is a root of the sixth order equation
\[ \omega^6 - 21\omega^5 + 173\omega^4 - 711\omega^3 + 1525\omega^2 - 1603\omega + 637 = 0. \]

**D.5 Five Impurities**

**D.5.1 Dimension 10, \([5, 0, 5]\), planar**

We find four operators with dimensions
\[ \Delta = 10 + \frac{g_{\text{YM}}^2 N}{4\pi^2} \omega, \]
where \( \omega \) is a root of the quartic equation
\[ \omega^4 - 15\omega^3 + 78\omega^2 - 165\omega + 120 = 0. \]

\[ + \frac{g_{\text{YM}}^2 N}{16\pi^2} (47\omega^3 - 472\omega^2 + 1430\omega - 1305) = 0. \]

**E The operator \( \delta D_4 \)**

Here we list the exact contributions to the generator \( \delta D_4 \) acting on \( O_{J_0:J_1:\ldots:J_k} \), cf. equations (5.7) and (5.10). We see that in strong contrast to \( (D_2)^2 \), the operator \( \delta D_4 \) only creates states in the combination \( O_1 - O_0 \); we write in short
\[ O_{J_0:J_1:\ldots:J_k} = O_{1:J_1:\ldots:J_k} - O_{0:J_1:\ldots:J_k}. \]

The operators with \( J_0 = 0 \) are always annihilated, we assume \( J_0 > 1 \). The planar part is
\[ \delta D_{4:0} O_{J_0:J_1:\ldots:J_k} = 4(\delta_{p,0} + \delta_{p,J_0} - \delta_{p,1} - \delta_{p,J_0-1}) O_{J_0:J_1:\ldots:J_k}. \]

The operator can split off one trace
\[ \delta D_{4:+} O_{J_0:J_1:\ldots:J_k} = 4\delta_{p\neq 0,p\neq J_0} (O_{1:J_1:\ldots:J_k,J_0-p} + O_{1:J_1:\ldots:J_k,J_0-p}) \\
- 8\delta_{p>1} O_{1:J_1:\ldots:J_k,J_0-p-1} \\
- 8\delta_{p<J_0-1} O_{1:J_1:\ldots:J_k,J_0-p-1} \\
+ 4(\delta_{p,0} + \delta_{p,J_0}) \sum_{J_{k+1}=1}^{J_0-1} O_{1:J_1:\ldots:J_k,J_0,J_{k+1}}, \]

\[ \text{(E.3)} \]
or two traces

\[
\delta D_{4;+} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\delta_{p\neq 0} \sum_{J_{k+1}=1}^{J_0-p-1} \mathcal{O}_{1-0}^{J_0-p-J_{k+1};J_1,\ldots,J_{k+1}p} \\
+ 4\delta_{p\neq J_0} \sum_{J_{k+1}=1}^{p-1} \mathcal{O}_{1-0}^{p-J_{k+1};J_1,\ldots,J_{k+1}J_0-p} \\
- 4 \sum_{J_{k+1}=1}^{J_0-p-2} \mathcal{O}_{1-0}^{p+1;J_1,\ldots,J_{k+1}J_0-p-J_{k+1}-1} \\
- 4 \sum_{J_{k+1}=1}^{p-2} \mathcal{O}_{1-0}^{J_0-p+1;J_1,\ldots,J_{k+1}J_0-J_{k+1}-1},
\]

(E.4)

it can join two traces

\[
\delta D_{4;+} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4(\delta_{p,0} + \delta_{p,J_0}) \sum_{i=1}^{k} J_i \mathcal{O}_{1-0}^{J_0+J_i;J_1,\ldots,J_{i-1},J_{i+1},\ldots,J_k},
\]

(E.5)

or it can let two traces interact without changing the number of traces

\[
\delta D_{4;+} \mathcal{O}_p^{J_0;J_1,\ldots,J_k} = 4\delta_{p\neq 0} \sum_{i=1}^{k} J_i \mathcal{O}_{1-0}^{J_0+J_i-p;J_1,\ldots,J_{i-1},J_{i+1},\ldots,J_k} \\
+ 4\delta_{p\neq J_0} \sum_{i=1}^{k} J_i \mathcal{O}_{1-0}^{J_i+p;J_1,\ldots,J_{i-1},J_{i+1},J_0-p} \\
- 4 \sum_{i=1}^{k} J_i \mathcal{O}_{1-0}^{p+1;J_1,\ldots,J_{i-1},J_{i+1},J_0+J_i-p-1} \\
- 4 \sum_{i=1}^{k} J_i \mathcal{O}_{1-0}^{J_0-p+1;J_1,\ldots,J_{i-1},J_{i+1},J_0+J_i+p-1}.
\]

(E.6)

F Charges of the spin chain

In this appendix we present some of the commuting charges of the non-nearest neighbor SU(2) spin chain investigated in Sec. \(\square\). We use the notation

\[
\{n_1, n_2, \ldots\} = \sum_{k=1}^{L} P_{k+n_1,k+n_1+1}P_{k+n_2,k+n_2+1} \ldots
\]

(F.1)

We make extensive use of the identity

\[
\{\ldots, n, n \pm 1, n, \ldots\} = \{\ldots, n, \ldots\} - \{\ldots, n, \ldots\} - \{\ldots, n \pm 1, \ldots\} \\
+ \{\ldots, n, n \pm 1, \ldots\} + \{\ldots, n \pm 1, n, \ldots\},
\]

(F.2)
due the impossibility of antisymmetrizing three sites in SU(2). The following expression for the higher charges are unique up lower charges multiplied by powers of the coupling constant.

The first charge $D = Q_1$

\[
D_0 = \{\}, \\
D_2 = 2\{\} - 2\{0\}, \\
D_4 = -8\{\} + 12\{0\} - 2\{0, 1\} + \{1, 0\}, \\
D_6 = 60\{\} - 104\{0\} + 24\{0, 1\} + \{1, 0\} + 4\{0, 2\} - 4\{0, 1, 2\} + \{2, 1, 0\}, \\
D_8 = +(-572 + 4\alpha)\{\} + (1072 - 12\alpha + 4\beta)\{0\} \\
+ (-278 + 4\alpha - 4\beta)\{0, 1\} + \{1, 0\} + (-84 + 6\alpha - 2\beta)\{0, 2\} - 4\{0, 3\} \\
+ 4\{0, 1, 3\} + \{0, 2, 3\} + \{0, 3, 2\} + \{1, 0, 3\}) \\
+ (78 + 2\beta)(\{0, 1, 2\} + \{2, 1, 0\}) + (-6 - 4\alpha + 2\beta)(\{0, 2, 1\} + \{1, 0, 2\}) \\
+ (1 - \beta)(\{0, 1, 3\} + \{0, 2, 3\} + \{0, 3, 2\} + \{1, 0, 2\} + \{2, 1, 0\}) \\
+ (2\alpha - 2\beta)\{1, 0, 2, 1\} + 2\beta(\{0, 2, 1, 3\} + \{1, 0, 3, 2\}) \\
- 10\{0, 1, 2, 3\} + \{3, 2, 1, 0\}). \tag{F.3}
\]

The second charge $U = Q_2$

\[
U_2 = 4\{1, 0\} - \{0, 1\}, \\
U_4 = 8\{2, 1, 0\} - \{0, 1, 2\}, \\
U_6 = 8\{0, 1, 3\} + \{0, 2, 3\} - \{0, 3, 2\} - \{1, 0, 3\} \\
+ 40\{0, 1, 2\} - \{2, 1, 0\} + 16\{3, 2, 1, 0\} - \{0, 1, 2, 3\}. \tag{F.4}
\]

The third charge $Q_3$

\[
Q_{3,2} = -2\{0\} + (\{0, 1\} + \{1, 0\}) \\
+ (\{0, 2, 1\} + \{1, 0, 2\}) - (\{0, 1, 2\} + \{2, 1, 0\}), \\
Q_{3,4} = -2\{0\} + (\{0, 1\} + \{1, 0\}) - 4\{0, 2\} \\
- 3\{0, 1, 2\} + \{2, 1, 0\} + 5\{0, 2, 1\} + \{1, 0, 2\} + 2\{1, 0, 2, 1\} \\
- 3\{0, 1, 2, 3\} + \{3, 2, 1, 0\} + (\{0, 2, 1, 3\} + \{1, 0, 3, 2\}) \\
+ (\{0, 1, 3, 2\} + \{2, 1, 0, 3\} + \{0, 3, 2, 1\} + \{1, 0, 2, 3\}). \tag{F.5}
\]

The fourth charge $Q_4$

\[
Q_{4,2} = -2\{0, 1, 2\} - \{2, 1, 0\} \\
- (\{0, 2, 1, 3\} - \{1, 0, 3, 2\}) + (\{0, 1, 2, 3\} - \{3, 2, 1, 0\}) \\
+ (\{0, 3, 2, 1\} - \{0, 1, 3, 2\} - \{1, 0, 2, 3\} + \{2, 1, 0, 3\}). \tag{F.6}
\]

References

[1] F. Gliozzi, J. Scherk and D. I. Olive, “Supersymmetry, Supergravity theories and the dual spinor model”, Nucl. Phys. B122 (1977) 253.
[2] L. Brink, J. H. Schwarz and J. Scherk, “Supersymmetric Yang-Mills Theories”, Nucl. Phys. B121 (1977) 77.

[3] M. F. Sohnius and P. C. West, “Conformal invariance in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory”, Phys. Lett. B100 (1981) 245.

[4] P. S. Howe, K. S. Stelle and P. K. Townsend, “Miraculous ultraviolet cancellations in supersymmetry made manifest”, Nucl. Phys. B236 (1984) 125.

[5] L. Brink, O. Lindgren and B. E. W. Nilsson, “$\mathcal{N} = 4$ Yang-Mills theory on the light cone”, Nucl. Phys. B212 (1983) 401.

[6] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large $N$ field theories, string theory and gravity”, Phys. Rept. 323 (2000) 183, hep-th/9905111.

[7] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence”, hep-th/0201253.

[8] V. K. Dobrev and V. B. Petkova, “All positive energy unitary irreducible representations of extended conformal supersymmetry”, Phys. Lett. B162 (1985) 127.

[9] L. Andrianopoli and S. Ferrara, “Short and long $SU(2,2/4)$ multiplets in the AdS/CFT correspondence”, Lett. Math. Phys. 48 (1999) 145, hep-th/9812067.

[10] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, “Shortening of primary operators in $\mathcal{N}$-extended SCFT and harmonic-superspace analyticity”, Adv. Theor. Math. Phys. 3 (1999) 1149, hep-th/9912007.

[11] S.-M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in $D = 4, \mathcal{N} = 4$ SYM at large $N$”, Adv. Theor. Math. Phys. 2 (1998) 697, hep-th/9806074.

[12] G. Arutyunov, S. Frolov and A. C. Petkou, “Operator product expansion of the lowest weight CPOs in $\mathcal{N} = 4$ SYM$_4$ at strong coupling”, Nucl. Phys. B586 (2000) 547, hep-th/0005182.

[13] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory”, JHEP 0201 (2002) 047, hep-th/0110242.

[14] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “Penrose limits and maximal supersymmetry”, Class. Quant. Grav. 19 (2002) L87, hep-th/0201081.

[15] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background”, Nucl. Phys. B625 (2002) 70, hep-th/0112044.

[16] R. R. Metsaev and A. A. Tseytlin, “Exactly solvable model of superstring in plane wave Ramond-Ramond background”, Phys. Rev. D65 (2002) 126004, hep-th/0202109.

[17] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills”, JHEP 0204 (2002) 013, hep-th/0202021.

[18] D. J. Gross, A. Mikhailov and R. Roiban, “Operators with large $R$ charge in $\mathcal{N} = 4$ Yang-Mills theory”, Annals Phys. 301 (2002) 31, hep-th/0205066.

[19] A. Parnachev and A. V. Ryzhov, “Strings in the near plane wave background and AdS/CFT”, JHEP 0210 (2002) 066, hep-th/0208010.

[20] N. Beisert, “BMN Operators and Superconformal Symmetry”, Nucl. Phys. B659 (2003) 79, hep-th/0211032.
[21] T. Klose, “Conformal Dimensions of Two-Derivative BMN Operators”, JHEP 0303 (2003) 012, hep-th/0301150
[22] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “A new double-scaling limit of $\mathcal{N} = 4$ super Yang-Mills theory and PP-wave strings”, Nucl. Phys. B643 (2002) 3, hep-th/0205033
[23] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, “PP-wave string interactions from perturbative Yang-Mills theory”, JHEP 0207 (2002) 017, hep-th/0205089
[24] U. Gürsoy, “Vector operators in the BMN correspondence”, JHEP 0307 (2003) 048, hep-th/0208041
[25] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “BMN correlators and operator mixing in $\mathcal{N} = 4$ super Yang-Mills theory”, Nucl. Phys. B650 (2003) 125, hep-th/0208178.
[26] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, “Operator mixing and the BMN correspondence”, JHEP 0210 (2002) 068, hep-th/0209002
[27] B. Eynard and C. Kristjansen, “BMN correlators by loop equations”, JHEP 0210 (2002) 027, hep-th/0209244
[28] U. Gürsoy, “Predictions for pp-wave string amplitudes from perturbative SYM”, JHEP 0310 (2003) 027, hep-th/0212118
[29] E. D’Hoker, D. Z. Freedman and W. Skiba, “Field theory tests for correlators in the AdS/CFT correspondence”, Phys. Rev. D59 (1999) 045008, hep-th/9807098
[30] S. Penati, A. Santambrogio and D. Zanon, “Two-point functions of chiral operators in $\mathcal{N} = 4$ SYM at order $g^4$”, JHEP 9912 (1999) 006, hep-th/9910197
[31] S. Penati, A. Santambrogio and D. Zanon, “More on correlators and contact terms in $\mathcal{N} = 4$ SYM at order $g^4$”, Nucl. Phys. B593 (2001) 651, hep-th/0005223
[32] S. Penati and A. Santambrogio, “Superspace approach to anomalous dimensions in $\mathcal{N} = 4$ SYM”, Nucl. Phys. B614 (2001) 367, hep-th/0107071
[33] A. V. Ryzhov, “Quarter BPS operators in $\mathcal{N} = 4$ SYM”, JHEP 0111 (2001) 046, hep-th/0109064.
[34] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, “BMN gauge theory as a quantum mechanical system”, Phys. Lett. B558 (2003) 229, hep-th/0212269
[35] D. J. Gross, A. Mikhailov and R. Roiban, “A calculation of the plane wave string Hamiltonian from $\mathcal{N} = 4$ super-Yang-Mills theory”, JHEP 0305 (2003) 025, hep-th/0208231
[36] R. A. Janik, “BMN operators and string field theory”, Phys. Lett. B549 (2002) 237, hep-th/0209263
[37] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills”, JHEP 0303 (2003) 013, hep-th/0212208
[38] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, “Anomalous dimensions in $\mathcal{N} = 4$ SYM theory at order $g^4$”, Nucl. Phys. B584 (2000) 216, hep-th/0003203
[39] G. Arutyunov, B. Eden, A. C. Petkou and E. Sokatchev, “Exceptional non-renormalization properties and OPE analysis of chiral four-point functions in $\mathcal{N} = 4$ SYM$_4$”, Nucl. Phys. B620 (2002) 380, hep-th/0103230.

[40] D. Anselmi, M. T. Grisaru and A. Johansen, “A Critical Behaviour of Anomalous Currents, Electric-Magnetic Universality and CFT$_4$”, Nucl. Phys. B491 (1997) 221, hep-th/9601023.

[41] G. Arutyunov, S. Penati, A. C. Petkou and E. Sokatchev, “Non-protected operators in $\mathcal{N} = 4$ SYM and multiparticle states of AdS$_5$ SUGRA”, Nucl. Phys. B643 (2002) 49, hep-th/0206020.

[42] D. Berenstein and H. Nastase, “On lightcone string field theory from super Yang-Mills and holography”, hep-th/0205048.

[43] A. Santambrogio and D. Zanon, “Exact anomalous dimensions of $\mathcal{N} = 4$ Yang-Mills operators with large $R$ charge”, Phys. Lett. B545 (2002) 425, hep-th/0206079.

[44] E. D’Hoker, P. Heslop, P. Howe and A. V. Ryzhov, “Systematics of quarter BPS operators in $\mathcal{N} = 4$ SYM”, JHEP 0304 (2003) 038, hep-th/0301104.

[45] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, “On the logarithmic behavior in $\mathcal{N} = 4$ SYM theory”, JHEP 9908 (1999) 020, hep-th/9906188.

[46] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, “Properties of the Konishi multiplet in $\mathcal{N} = 4$ SYM theory”, JHEP 0105 (2001) 042, hep-th/0104016.

[47] M. Bianchi, B. Eden, G. Rossi and Y. S. Stanev, “On operator mixing in $\mathcal{N} = 4$ SYM”, Nucl. Phys. B646 (2002) 69, hep-th/0205321.

[48] J. Gomis, S. Moriyama and J. Park, “SYM description of SFT Hamiltonian in a pp-wave background”, Nucl. Phys. B659 (2003) 179, hep-th/0210153.

[49] J. Gomis, S. Moriyama and J. Park, “SYM description of pp-wave string interactions: Singlet sector and arbitrary impurities”, Nucl. Phys. B665 (2003) 49, hep-th/0301250.

[50] M. Spradlin and A. Volovich, “Superstring interactions in a pp-wave background”, Phys. Rev. D66 (2002) 086004, hep-th/0204146.

[51] I. R. Klebanov, M. Spradlin and A. Volovich, “New effects in gauge theory from pp-wave superstrings”, Phys. Lett. B548 (2002) 111, hep-th/0206221.

[52] M. Spradlin and A. Volovich, “Superstring interactions in a pp-wave background. II”, JHEP 0301 (2003) 036, hep-th/0206073.

[53] A. Pankiewicz, “More comments on superstring interactions in the pp-wave background”, JHEP 0209 (2002) 056, hep-th/0208209.

[54] A. Pankiewicz and B. Stefani´ski, Jr., “PP-Wave Light-Cone Superstring Field Theory”, Nucl. Phys. B657 (2003) 79, hep-th/0210246.

[55] Y.-H. He, J. H. Schwarz, M. Spradlin and A. Volovich, “Explicit formulas for Neumann coefficients in the plane-wave geometry”, Phys. Rev. D67 (2003) 086005, hep-th/0211198.

[56] R. Roiban, M. Spradlin and A. Volovich, “On light-cone SFT contact terms in a plane wave”, JHEP 0310 (2003) 055, hep-th/0211220.

[57] H. Verlinde, “Bits, matrices and 1/N”, JHEP 0312 (2003) 052, hep-th/0206059.
[58] D. Vaman and H. Verlinde, “Bit strings from $N = 4$ gauge theory”, JHEP 0311 (2003) 041, hep-th/0209215

[59] J. Pearson, M. Spradlin, D. Vaman, H. Verlinde and A. Volovich, “Tracing the String: BMN correspondence at Finite $J^2/N$”, JHEP 0305 (2003) 022, hep-th/0210102

[60] G. Arutyunov and E. Sokatchev, “On a large $N$ degeneracy in $N = 4$ SYM and the AdS/CFT correspondence”, Nucl. Phys. B663 (2003) 163, hep-th/0301058

[61] N. Y. Reshetikhin, “A method of functional equations in the theory of exactly solvable quantum system”, Lett. Math. Phys. 7 (1983) 205.

[62] N. Y. Reshetikhin, “Integrable models of quantum one-dimensional magnets with $O(N)$ and $Sp(2K)$ symmetry”, Theor. Math. Phys. 63 (1985) 555.

[63] C.-S. Chu, V. V. Khoze and G. Travaglini, “Three-point functions in $N = 4$ Yang-Mills theory and pp-waves”, JHEP 0206 (2002) 011, hep-th/0206005

[64] G. Georgiou and V. V. Khoze, “BMN operators with three scalar impurities and the vertex correlator duality in pp-wave”, JHEP 0304 (2003) 015, hep-th/0302064

[65] C.-S. Chu and V. V. Khoze, “Correspondence between the 3-point BMN correlators and the 3-string vertex on the pp-wave”, JHEP 0304 (2003) 014, hep-th/0301036

[66] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence”, Nucl. Phys. B636 (2002) 99, hep-th/0204051

[67] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, “Anomalous dimensions of Wilson operators in $N = 4$ SYM theory”, Phys. Lett. B557 (2003) 114, hep-ph/0301021

[68] D. I. Kazakov, “Calculation of Feynman integrals by the method of ‘uniqueness’”, Theor. Math. Phys. 58 (1984) 223.