A Complementary Resource Relation of Concurrence and Roughness for a two Qubits State

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Abstract

Quantum resources lie at the core of quantum computation as they are responsible for the computational advantage in many tasks. The complementary relations are a pathway to understanding how the physical quantities are related. Here it was shown that there is a complementary relationship that evolves quantumness of a two qubits state and the degree of entanglement, respectively measured by Roughness and Concurrence. It was shown that there is a \( \mathbb{R}^2 \) surface that characterizes these quantities and can be understood as a projection of the Bloch Sphere. The distribution of states on the surface shows a concentration in the region of lower roughness and Concurrence, i.e., in the subset of states of greater entropy.
Since its advent as a physical theory, Quantum Mechanics brought several new insights into how nature works in the microscopic domain. In addition to the impact on understanding the physics of microscopic objects such as particles, and fields, Quantum Mechanics is responsible for the next leap in computing, quantum computers. They will not only be faster or with greater storage capacity but will allow us to address problems that are inaccessible by any other previously known computational method. Some of these exploits are already a reality, such as quantum cryptography \[1\], generation of random numbers \[2\], and quantum teleportation \[4\]. Recently, Google has claimed that they have obtained the quantum supremacy \[3\], a group of researchers and Yin et al carried out a cryptography experiment based on entanglement over a distance of 1,120 km \[1\]. All of these advances in quantum technology have as a primary ingredient the use of quantum properties as a resource \[5\] and are directly connected with the Quantum Contextuality \[6\]. Although there is an established connection between quantum resources and non-locality, it is also possible to obtain them from a prepare-and-measure experiment approach, which explores the quantum nature of the state \[7\] as well practical application on metrology \[8\]. The classicality depends on many factors, on the chosen observables, on how the measurement is made, and on the criterion adopted, see \[9\] and references therein. To measure the degree of entanglement between two qubits, the Concurrence (\(C\)) is a good measure, it is zero when systems are not entangled and one for systems that are maximally entangled \[10\]. Along with Concurrence, other functions were proposed in order to measure and identify if features in a given physical system are best described by Quantum Mechanics. Some of those Quantumness quantifiers make use of the statistical properties in the Wigner representation of a state to spot differences with the classical description via probability distribution. An example of a largely used function for this purpose is the Negativity function \[11\]. Using a generalization of its ideas though, a new quantity called Roughness were proposed \[12\]. As one of desirable properties in a quantumness measure, Roughness has its values limited, being zero for classical states and becoming one in the limit of extreme quantum states. In a previous work \[13\] prior to this one, a numerical investigation corroborated a mathematical relationship between Concurrence and Roughness. Here it is shown that for a general two qubits states \(\rho\) there is a complementary relation

\[ R_{\rho}^2 + \tilde{C}^2 = \tilde{N}_{\rho} + \tilde{f}_C - \kappa, \]  

where \(R_{\rho}\) is the sum of squared Roughness of each subsystem, \(\tilde{C}\) is the Concurrence re-scaled by a constant factor, \(\tilde{N}_{\rho}\) is related to the total excitation number, \(\tilde{f}_C\) is an auto-correlating function...
and $\kappa$ is a constant, more details in the following. The right side of \ref{eq:1} defines a $\mathbb{R}^2$ surface, that can be understood as a projection of the two qubits Bloch Sphere \cite{14} in the $(R_+, \tilde{C})$ set, $RC$. The main advantage is that the axis variables can be considered such as quantum resources. This is similar to the complementarity relation for a bipartite system exposed to an interferometer \cite{15}, and its generalization for open systems \cite{16,17}. Fan and collaborators \cite{18} have shown a complementary relation for quantum coherence and quantum correlations. The Roughness is sensible to quantum coherence and also to any other quantum aspect of the state. Also, some proprieties of the $RC$ set were investigated thru a graphic approach.

**MEASURING THE QUANTUMNESS OF A PHYSICAL SYSTEM**

**Entanglement.** Quantum correlations between parts of a larger system are often referred to as entanglement between its parts. As a measure of entanglement of formation between parts of a 2-level quantum system represented by a density matrix $\rho$, Wooters and Hill \cite{10} proposed the function known as Concurrence, which can be obtained as the non-negative eigenvalues of a “flipped” form of $\rho$ given by: $\hat{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$, where $\sigma_y$ is the matrix representation of Pauli spin operator and $\rho^*$ is the complex conjugate of $\rho$. To obtain the Concurrence, one consider the operator $\rho_F = \sqrt{\rho \hat{\rho} \sqrt{\rho}}$. Writing the eigenvalues of $\rho_F$ in decreasing order as $\{\lambda_k\}_{k=1}^4$, the Concurrence of $\rho$ can then be formally written as: $C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$. Another useful function to be used for measuring entanglement of a bipartite state $\rho$ is the linear entropy, which can be obtained as: $\delta = 1 - \text{Tr}\{\rho_1^2\}$, where $\rho_1$ stands for the local state obtained by tracing over the other part from the global state, $\rho_1 = \text{Tr}\{\rho\}$. For pure global states, the two quantities are related: $C(\rho) = \sqrt{2\delta}\rho_1 = \sqrt{2\delta}\rho_2$.

The entanglement cost under operations preserving the positivity (PPT) of partial transpose can be measured by the negativity, although PPT entanglement cost has some correlation with the entanglement of formation. Miranowicz and Grudka have shown that concurrence and negativity can give different orderings of two-qubit states \cite{19}.

**Roughness.** The Roughness of a quantum system were defined by Lemos et al to be the overall distance between the Wigner and Husimi representation of the system. For the $h_4$ algebra of bosonic systems, it can be expressed by the formula:

$$ R(\Psi) = \sqrt{2\pi \int_{\mathbb{R}^2} dq dp \left\{ W_\Psi(q,p) - Q_\Psi(q,p) \right\}^2}, $$

(2)
where \( W_\Psi(q, p) \) and \( Q_\Psi(q, p) \) are the Wigner and Husimi functions for the quantum state \( \Psi \) according to the position \( q \) and momentum \( p \) phase space quadratures.

Following this definition, the Roughness of a density state is a real-valued function restricted to the \([0, 1]\) interval. Analytic expressions for the Roughness of traditionally physical systems with well known quantum features can be found in [12]. Departing from the Negativity’s behaviour, Roughness is able to capture quantum features in states represented by for strictly positive Wigner functions. For example, physical systems represented by squeezed gaussian states, which present well known quantum effects, tend to have large \( R \), since \(|W - Q|\) increases with the squeezing.

A COMPLEMENTARY RELATION BETWEEN ROUGHNESS AND ENTANGLEMENT FOR A 2-QUBIT SYSTEM

The proposal herein is to treat the quantum features of the general 2 qubit states represented by a density operator \( \rho \) in two parts: one which could be measured performing local operations in each part and other related to the quantum correlations between the parts measured by the linear entropy \( \delta(\rho) \). The first relations will be get in view of a single qubit state \( \rho_1 \), written in the \( \{|0\rangle, |1\rangle\} \) basis as:

\[
\rho_1 = \sum_{n, n' \in \{0, 1\}} A_{n,n'} |n\rangle \langle n'| \tag{3}
\]

with \( \sum |A_{n,n'}|^2 = 1 \) and \( A_{n,n'} = A_{n',n}^* \). Using this 2 conditions it is possible to represent this state operator as a dimension 3 vector given by:

\[
\begin{bmatrix}
A_{00} \\
\frac{1}{\sqrt{2}}A_{01} \\
A_{11}
\end{bmatrix}
\tag{4}
\]

Within this notation, the linear entropy of \( \rho \) can be obtained by a product:

\[
\delta(\rho) = 1 - \text{Tr} \{ \rho^2 \} = 1 - v^\dagger v, \tag{5}
\]

where \( v^\dagger \) the transposed and complex conjugated version of \( v \). Also, under this notation, the square of Roughness can be written as a quadratic form:

\[
R^2 = v^\dagger \Lambda v, \tag{6}
\]
where $\Lambda$ can be obtained by computing the corresponding entries from Appendix:

$$
\Lambda = \frac{1}{108}\begin{pmatrix}
18 & 0 & -21 \\
0 & 39 & 0 \\
-21 & 0 & 55
\end{pmatrix}.
$$

(7)

Putting $\Lambda$ in diagonal form and performing the corresponding change of basis in $v$ one can recast the relations previously stated to find how $R^2$ of a single qubit is related to its linear entropy:

$$
R^2(\rho) = \frac{55}{108} - \frac{37}{108}A_{00}(2 - A_{00}) - \frac{39}{108}\delta(\rho).
$$

(8)

Now, a general 2 qubit state $\rho$ can be written as:

$$
\rho = \sum_{mm'kk'\in\{0,1\}} \Psi_{mm'kk'} |mk\rangle \langle m'k'|,
$$

(9)

where the each $|mm'\rangle$ is a short form for $|m\rangle \otimes |m'\rangle$, the product state of two independent Fock states for each subsystem, and $\langle kk'|$ the corresponding linear functional. Consider now the physical properties which would be detected locally, i.e., performing local measurements on each subsystem alone. Such operations can be described using partial trace operation acting over $\rho$:

$$
\rho_1 = \sum_{mm'kk'\in\{0,1\}} \Psi_{mm'kk} |mk\rangle \langle m'k'|.
$$

(10)

Equation (10) defines a state of the form (3) where $A_{00} = \Psi_{0000} + \Psi_{0011}$, $A_{01} = \Psi_{0100} + \Psi_{0111}$, $A_{10} = \Psi_{1000} + \Psi_{1011}$ and $A_{11} = \Psi_{1100} + \Psi_{1111}$.

Changing the system of interest and applying the corresponding operations, analogous coefficients for $\rho_2$ are to be found. The states represented by $\rho_1$ and $\rho_2$ are two mixed states in the same form of the general 1-qubit states given by (3). It is a matter of straightforward calculation to obtain the Roughness of such states and express them in terms of the linear entropy:

$$
R^2(\rho_i) = \frac{1}{108}[37\mu_i + 55 - 39\delta(\rho_i)],
$$

(11)

where $\mu_1 = (\Psi_{0000} + \Psi_{0011})(\Psi_{0000} + \Psi_{0011} - 2)$ and $\mu_2 = (\Psi_{0000} + \Psi_{1100})(\Psi_{0000} + \Psi_{1100} - 2)$.

In view of the arguments given by [12], expression (11) and its counterpart for $\rho_2$ represents a scale of quantum features of a 2 qubit state which could be measured and detected locally. Combining expressions given by (11) and its corresponding expression for $\rho_2$, the following equation can be written:
\[
\frac{R^2(\rho_1) + R^2(\rho_2)}{2} + \frac{39}{216} [\delta(\rho_1) + \delta(\rho_2)] = \frac{37}{216} \left\{ z^2 + w^2 - 2(z + w) \right\} + \frac{55}{108} \tag{12}
\]

where \( z = \Psi_{0000} + \Psi_{0011} \) and \( w = \Psi_{0000} + \Psi_{1100} \).

The equation (12) expresses a non-negative quantity depending only on the diagonal coefficients in (9), a general 2 qubit system. Since only normalized states are being considere, \( 0 \leq z \leq 1 \) and \( 0 \leq w \leq 1 \), so the quantity enclosed by braces in equation (12) is bound to the interval \([-2, 0]\), with its maximum being achieved when \( z = w = 0 \). According to this, the combined sum from the left hand side of (12) is bound to values in the range: \([1/6, 55/108]\).

The quantities on the right hand side can be interpreted as measurable properties of the physical system. On the other hand, it is possible to estimate the entanglement from randomized measurements [20], thus is possible to infer the mean Roughness using equation (12). Considering two separate measures of the mean photon number, one should find:

\[
N_e = \text{Tr} \{ (\hat{n}_1 + \hat{n}_2) \rho \} = 2 - z - w. \tag{13}
\]

Now, \( z \) and \( w \) are the coefficients of \(|0\rangle \langle 0|\) in each reduced state operator \( \rho_{1,2} \) obtained via partial trace operation. Due to this, is convenient to define the auto-correlation function of the localized fundamental states as:

\[
f_C(z, w) = \frac{z^2 + w^2}{2}. \tag{14}
\]

Using the above definitions, expression (12) can be rewritten as:

\[
\frac{R^2(\rho_1) + R^2(\rho_2)}{2} + \frac{39}{216} [\delta(\rho_1) + \delta(\rho_2)] = \frac{37}{108} (N_e + f_C) - \frac{19}{108}. \tag{15}
\]

Both terms in the left hand side of (15) are affected by quantum features in \( \rho \), and they are being related to the number of localized excitation in \( \rho \) and the self correlation function \( f_C \) in a complementary way: lowering \( N_e \) by increasing \( z \) and \( w \) leaves a residual quantum effect. This can be interpreted as a collective effect in the ground state projector of the reduced density operators obtained \textit{via} partial trace operation over \( \rho \).

By the uncertainty principle in quantum field theory, the more the position of a particle is definite, the less the momentum is known. From a relativistic point of view, when the momentum is large, particles can be created making their numbers uncertain. These particles interact, so the more you know about the location the less you know about the interactions. The result presented
here is similar, although the number of particles is constant, the interaction between them can be determined by the state and measured by concurrence. Those which have a well-defined location of the excitation are states with no interaction, whereas in states of maximum entanglement, the excitation is not local.

**The Globally-pure states case**

If the global state $\rho$ is pure, the relationship between Concurrence and the Linear Entropy of the reduced states $C(\rho) = \sqrt{2\delta(\rho_1)} = \sqrt{2\delta(\rho_2)}$, can be used. Also, just for aesthetic and simplicity reasons it is worthwhile to define:

$$R^2_+(\rho) = \frac{R^2(\rho_1) + R^2(\rho_2)}{2}$$  \hspace{1cm} (16)

and

$$\tilde{C}(\rho) = \frac{39}{216} C(\rho).$$ \hspace{1cm} (17)

Then equation (15) becomes:

$$R^2_+(\rho) + \tilde{C}^2(\rho) = \tilde{N}_e(\rho) + \tilde{f}_C(\rho) - \kappa,$$ \hspace{1cm} (18)

where $\tilde{N}_e(\rho) = \frac{37}{108} N_e$, $\tilde{f}_C(\rho) = \frac{37}{108} f_C(\rho)$ and $\kappa = \frac{19}{108}$.

Equation (18) suggests the possibility of using $R^2_+$ and $C^2$ to address complementary quantum features in the 2-qubit subspace. Figure 1 shows a sampling of states considering $R^2$ and $C^2$ as parameters and illustrates the trade-off of local and global quantum features. The sampling were done using the algorithm available in [21], implemented by the Qutip Python package [22, 23].

The map of possible pure 2-qubit states are restricted to a region in blade-like shape. At least 3 important points can be related to well known quantum states. At the tip of the blade, for $C^2 = 1$, lies the family of Bell states, $|\Phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. They all have the same $R^2_+$ value of $(31/432) \approx 0.07176$. For states where $C^2 = 0$, the values of $R^2_+$ varies from a minimum of $1/6$ for $\rho = |00\rangle \langle 00|$ and the maximum of $55/108$ for $\rho = |11\rangle \langle 11|$. In this case, though no entanglement between parts exists, it is still possible to have quantum features detected locally. For a better understanding of what those quantum features might mean, one could think about the nature of a pure vacuum state. Statistically, such a state is related to cooperative effects like Bose-Einstein condensates which exists at zero temperature. As for the product of two Fock states with a single excitation, the locally detectable quantum feature can be related to
FIG. 1. Numerical sampling of $1.57 \times 10^6$ 2-qubit pure states. Gray (color online) scale indicates multiplicity of states in the vicinity of each point.

FIG. 2. Numerical sampling of $1.26 \times 10^7$ 2-qubit rank-2 states. Gray (color online) scale indicates multiplicity of states in the vicinity of each point.

having a pure state with a definite number of excitation. This can be compared with the sampling of non-pure states which follows next.

The Globally non-pure states case

Figure 2 shows the sampling in the 2-qubits subspace without requiring the global states to be pure. It was also done with the algorithm given in [21], through the implementation of [22]. The density matrix representing the global quantum state were expected to have rank 2. It means the states could have the structures like $\rho = p \ket{\psi_0}\bra{\psi_0} + (1 - p) \ket{\psi_1}\bra{\psi_1}$ i.e., they could be written as a sum of up to 2 projectors. Though pure states were still possible to be obtained, they would
have a negligible proportion in this subspace. Still, sampling in this way show how the loss of purity affects the quantum features mapped in the \((\mathbb{C}^2, R_2^+\)) parameter space. As can be seen in the figure, there is a concentration of states in the lower left corner, indicating the great majority of the states in this sampling have much lower quantum features, be it in the form of entanglement or local quantum features. But it also indicates the existance of a large region where global states with similar quantum features analogous of those in the globally pure state cases. Further numerical investigation indicates the states in the intercepting region of the two cases have higher purity.

**Closing Marks.** The relationship of Roughness and linear entropy of the parts, in a 2-qubit system, were cast in a way all quantum correlation terms of the system were bound to be expressed through two other quantities with physical meaning which could be measured locally in each of the parts. It allows to take the two quantities as a complementary pair for describing quantum features of a physical system. Most of the world observed macroscopically appears to be absent of quantum features. Yet, even when considering a single part system, quantum effects can still be a remarkable feature. And it is a well known fact that the interaction of a physical system with a thermal bath leads to a decoherence process which makes quantum information to flow from system to environment in an irreversible way. Quantum mechanics of open systems describe such a process by the loss of state’s purity and this is highlighted by the linear entropy of the state. In a sense, physical systems represented by pure states are expected to have more quantum features than the non-pure states. Complementary to this, Roughness measure quantum features through the differences between Wigner and Husimi representations of the states. For globally pure states, those relationships leads to a bounded region in the \((\mathbb{C}^2, R_2^+)\) parameter space. For the non pure states of rank-2 density matrices the otherwise bound region were broken, with the great majority of states being found to have fewer quantum features. This is in line with general knowledge of what is observed for macroscopic physical systems. Low Roughness and low Concurrence states are the ones obtained most frequently, while the region of high values of \(R_+\) and \(C\) are inaccessible for states described by \([10]\). This corroborates the hypothesis that some states are more robust and form the pointer states, used in the construction of the Quantum Darwinism hypothesis \([24]\). Thus, Quantum Darwinism is not only the result of natural selection caused by decoherence, but it is also a consequence of the high density of states with low values of \(R_+\) and \(C\). This set of states is also the one with the highest entropy, since for non-pure states there is a strong correlation between Roughness and Shannon Entropy \([12]\).

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Appendix

In order to calculate the coefficients of (7), it has been used the Appendix C of [12]. The terms of $R^2$ for a general state $\rho$ the Roughness can be written as:

$$R^2(\rho) = \sum_{nm,n'm'} A^*_n A_{n'm'} \left( R^2_{\Pi_{nm} \Pi_{n'm'}} + R^2_{\Psi_{nm} \Psi_{n'm'}} - R^2_{\Pi_{nm} \Psi_{n'm'}} - R^2_{\Psi_{nm} \Pi_{n'm'}} \right)$$

(19)

where the various terms within the brackets are results of the evaluation of integrals. According to the referenced article, it is:

$$R^2_{\Pi_{nm} \Pi_{n'm'}} = \delta_{nn'} \delta_{mm'};$$

(20)

$$R^2_{\Psi_{nm} \Psi_{n'm'}} = \frac{\delta_{n-m,n'-m'}}{\sqrt{n!m!n'!m'!}} \left( \frac{1}{2} \right)^{1+u} v!;$$

(21)

$$R^2_{\Pi_{nm} \Psi_{n'm'}} = \frac{2\delta_{n-m,n'-m'}}{3} (-1)^y \sqrt{\frac{Y!}{X'Y'Y'!}} 2^{X-Y} \left( \frac{1}{3} \right)^u 2F1 \left[ -Y+1; \frac{X+X'-Y'+1}{4} ; \right]$$

(22)

$$R^2_{\Psi_{nm} \Pi_{n'm'}} = \frac{2\delta_{n-m,n'-m'}}{3} (-1)^{y'} \sqrt{\frac{Y'!}{X'Y'X'!}} 2^{X'-Y'} \left( \frac{1}{3} \right)^{u'} 2F1 \left[ -X+1; \frac{X'-X'+Y'+1}{4} ; \right]$$

(23)

where $X$ and $Y$ stands for max$(n,m)$ and min$(n,m)$, with the corresponding relations for the primed indexes, $u = (X - Y + X' + Y')/2$, $v = (n + m + n' + m')/2$ and $2F1$ is the Generalized Hypergeometric Function using the bracketed arguments. Evaluating the expressions above for the case of a density state $\rho$ as in [3] leads to the following expression for the squared Roughness:

$$R^2(\rho) = \frac{55}{108} - \frac{37}{108} A_{00}(2 - A_{00}) - \frac{39}{108} \delta(\rho).$$

(24)
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