Coupled-wire construction of static and Floquet second-order topological insulators

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Second-order topological insulators (SOTI) exhibit protected gapless boundary states at their hinges or corners. In this paper, we propose a generic means to construct SOTIs in static and Floquet systems by coupling one-dimensional topological insulator wires along a second dimension through dimerized hopping amplitudes. The Hamiltonian of such SOTIs admits a Kronecker sum structure, making it possible for obtaining its features by analyzing two constituent one-dimensional lattice Hamiltonians defined separately in two orthogonal dimensions. The resulting topological corner states do not rely on any delicate spatial symmetries, but are solely protected by the chiral symmetry of the system. We further utilize our idea to construct Floquet SOTIs, whose number of topological corner states is arbitrarily tunable via changing the hopping amplitudes of the system. Finally, we propose to detect the topological invariants of static and Floquet SOTIs constructed following our approach in experiments by measuring the mean chiral displacements of wavepackets.

I. INTRODUCTION

Topological phases of matter have emerged as an active research topic studied by both theorists and experimentalists since the last decade. As the name suggests, such phases of matter are characterized by the topology of their bulk states, the latter of which manifests itself as physical observables at their boundaries. For example, topological insulators are distinguished from normal insulators by the value of the $Z_2$ topological invariant that their bulk states possess, which determines the presence or absence of the topologically protected helical edge states at the boundaries of the systems [1,3]. The edge properties of topological phases are thus robust to local perturbations that preserve their topology as well as the symmetries protecting them. Consequently, topological phases are considered as a promising platform for designing robust electronic/spintronic devices, offering (almost) dissipationless and faster charge transfers [4], as well as providing protections at the hardware level in the realization of fault-tolerant quantum computations [5].

In recent years, a new type of topological phases whose topology manifests itself at the boundaries of their boundaries has been discovered and termed higher-order topological phases [6–28]. In particular, a $d$-dimensional $n$th-order topological insulator (where $d \geq n$) is characterized by the existence of topologically protected $(d - n)$-dimensional boundary states and gapped higher-dimensional boundary and bulk states.

Unlike first-order topological phases, which can exist even in the absence of any symmetries, most of the existing proposals on the construction of higher-order topological phases relies on the presence of additional spatial (reflection, inversion, or rotational) symmetries. It thus raises a fundamental question regarding the existence of higher-order topological phases in the absence of any spatial symmetries, which may provide further insight into the similarity between first- and higher-order topological phases. This question has been explored recently in Ref. [26], which proposes the construction of higher-order topological insulators in square and cubic lattices by coupling together four and eight Su-Schrieffer-Heeger (SSH) systems [29], respectively. Each of them describes a one-dimensional (1D) topological insulating model characterized by a topological winding number that determines the presence or absence of zero energy states at each end of the system. By construction, such models are protected solely by chiral symmetry without the need for additional spatial symmetries. However, it remained an open question if a more general construction based on 1D topological insulating models other than the SSH model is possible.

In this paper, we propose a general framework for constructing second-order topological insulators (SOTI) in a square lattice by means of coupling an array of 1D topological insulators with dimerized inter-array hopping amplitude, as illustrated in Fig. 1. As will be shown below, the total Hamiltonian of such a system can be written as a Kronecker sum of two 1D topological insulating Hamiltonians, enabling one to characterize the topology of the full system from that of its 1D Hamiltonian constituents separately. In particular, we show that topological corner modes exist only if both 1D Hamiltonian constituents are topologically nontrivial, which persist even in the presence of perturbations breaking all but the chiral symmetry, as well as small perturbations breaking the Kronecker sum structure of the system.

By the same mechanism outlined above, Floquet (periodically driven) SOTIs can be obtained by coupling an array of 1D Floquet topological insulators with the same (static) dimerized inter-array hopping amplitude. It is noted that the studies of Floquet topological phases

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we show that the full Hamiltonian of the system can be written as a Kronecker sum of two 1D SSH Hamiltonians. As a result, the latter symmetry and topological properties can be obtained from those of its 1D Hamiltonian constituents separately. In Sec. [II], we discuss the difference between our proposal and that of Ref. [26], and the robustness of our proposal in the presence of small perturbations which destroy the Kronecker sum structure of the full Hamiltonian. In Sec. [IIA], we extend our proposal to construct Floquet SOTI which may host topological corner modes at quasienergy zero and $\pi$ (zero and $\pi$ modes). We present an explicit model of such Floquet SOTI in Sec. [IIIB] and show how arbitrarily many zero and $\pi$ modes can be systematically obtained by tuning some system parameters. In Sec. [IV] we suggest to detect the topological winding numbers of static and Floquet SOTIs by measuring the mean chiral displacement of wavepackets localized initially at the center of the lattice. We summarize our results and discuss some future directions in Sec. [V].

II. COUPLED-WIRE CONSTRUCTION OF STATIC SOTI

In this section, we introduce our scheme of constructing static SOTIs via coupling topological insulator wires, and present explicit model calculations to demonstrate our findings.

We start by considering a prototypical tight-binding Hamiltonian $H$, which describes particles hopping on a 2D lattice:

$$H = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left\{ \left[ J_{y} + (-1)^{i} \delta J_{y} \right] |i, j + 1\rangle \langle i, j| + \right.$$

$$\left. + \left[ J_{x} + (-1)^{i} \delta J_{x} \right] |i + 1, j\rangle \langle i, j| + h.c. \right\} .$$

(1)

Here $J_{x(y)} \pm \delta J_{x(y)}$ denote dimerized hopping amplitudes in the $x$-($y$)-direction. $|i,j\rangle$ denotes the basis state at lattice site $(x, y) = (i, j)$. $N_x$ and $N_y$ are the number of lattice sites in $x$- and $y$-directions, respectively. Eq. (1) can thus be understood as an array of SSH chains along the $x$-direction, coupled with each other by another SSH-type dimerized hopping along the $y$-direction.

If the hopping amplitude and dimerization parameter along $y$-direction satisfy $J_{y} = \delta J_{y} = 0$, the system described by Hamiltonian $H$ reduces to $N_y$ identical copies of 1D SSH chain. Each of them can be in either a topologically trivial ($\delta J_{x} < 0$) or a nontrivial ($\delta J_{x} > 0$) phase. In the topologically nontrivial regime, a pair of degenerate zero-energy edge states (also called zero modes) appears at the two ends of each chain, resulting in totally $2N_y$ such degenerate edge states in the whole system. When $J_{y}, \delta J_{y} \neq 0$, all these zero modes will in general be coupled together, lifting their degeneracy. However, if $J_{y} = \delta J_{y}$, the two pairs of zero modes appearing at the ends of the first ($j = 1$) and last ($j = N_y$) arrays will be decoupled, and therefore remaining degenerate.
In this case, four zero modes emerge as corner states in the whole system.

Away from the fully dimerized limit \( J_y = \delta J_x \) along \( y \)-direction, it can be analytically shown that there are four corner modes in the system if \( \delta J_y > 0 \) (in addition to \( \delta J_x > 0 \) as required for each SSH chain to host zero edge modes). To this end, we first write the zero energy eigenstate of Eq. (1) explicitly as

\[
|0(x,y)\rangle = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left( J_y - \delta J_y \right)^{y_j} \left( J_x - \delta J_x \right)^{x_i} |X_i, Y_j\rangle,
\]

where \( X = 1, N_x \), \( Y = 1, N_y \), \( X_i = |X - i - 1| \), and \( Y_j = |Y - j - 1| \). That Eq. (2) is a zero energy eigenstate of \( \mathcal{H} \) can be verified by applying Eq. (1) directly to \(|0(x,y)\rangle\).\[58\]. Furthermore, for these corner states to be normalizable, one should require \( \delta J_y, \delta J_x > 0 \), i.e., the intercell coupling should be stronger than intracell coupling along both \( x \) and \( y \) directions. In the following subsection, we discuss the symmetry protecting these corner modes and introduce topological invariants to characterize them.

### A. Symmetry analysis and topological invariant

Under periodic boundary conditions (PBC), the Hamiltonian \( \mathcal{H} \) in Eq. (1) can be rewritten in momentum space as

\[
\mathcal{H} = \sum_{k_x, k_y} h_{x,k} \oplus h_{y,k},
\]

\[
h_{x,k} = h_{a,S} \sigma_x^{(S)} + h_{b,S} \sigma_y^{(S)},
\]

where \( S = x, y \), \( h_{a,S} = [J_S - \delta J_S + (J_S + \delta J_S) \cos k_S] \), \( h_{b,S} = (J_S + \delta J_S) \sin k_S \), \( k_S \) and \( \sigma^{(S)} \)'s are respectively quasimomentum and Pauli matrices acting in the sublattice/pseudospin subspace in the \( S \)-direction. It is noted that each \( h_{x,k} \) is simply the momentum space Hamiltonian describing an SSH model, which possesses inversion, time-reversal, particle-hole, and chiral symmetries respectively dictated by the operators \( \mathcal{T}_S = \sigma_x^{(S)} \), \( \mathcal{P}_S = \sigma_z^{(S)} \), and \( \mathcal{C}_S = \sigma_z^{(S)} \), where \( \mathcal{C} \) is the complex conjugation operator. So it belongs to class BDI in the Altland-Zirnbauer (AZ) classification scheme \[59\], and is characterized by a winding number topological invariant.

Due to the Kronecker sum structure of Eq. (3), the full Hamiltonian \( \mathcal{H} \) also possesses time-reversal, particle-hole, and chiral symmetries described by the operators \( \mathcal{T} = \mathcal{K}, \mathcal{P} = \sigma_z^{(x)} \sigma_z^{(y)} \mathcal{K}, \) and \( \Gamma = \sigma_z^{(x)} \sigma_z^{(y)} \). Moreover, the former also suggests that while the winding number of \( h_{x,k} \) \((h_{y,k})\) still dictates the existence of edge states of \( \mathcal{H} \) under open boundary conditions (OBC) at the edges of the lattice in the \( x(y)\)-direction, they are no longer pinned at zero energy since such edge states can be expressed as the tensor product between the edge states of \( h_{x,k} \) \((h_{y,k})\) and the bulk states of \( h_{x,k} \) \((h_{x,k})\), which therefore have nonzero energies. However, if the winding number of \( h_{x,k} \) and \( h_{y,k} \) are both nonzero, zero energy eigenstates of \( \mathcal{H} \) under OBC can be constructed as a tensor product between the edge states of \( h_{x,k} \) and \( h_{y,k} \), both having zero energies. By construction, such states are localized at both edges of the lattice and are thus corner modes. Therefore, the existence of corner modes of \( \mathcal{H} \) or any Hamiltonian with similar Kronecker sum structures is determined by the product of the topological invariant (e.g., winding number) of each Kronecker sum component.

The winding number associated with the Hamiltonian in the form of Eq. (3) is defined as

\[
\nu_{X} = \frac{1}{2\pi i} \int dk_x H_{S,k}^{-1} \frac{d}{dk_x} H_{S,k},
\]

where \( H_{S,k} \equiv h_{a,S} + i h_{b,S} \). It is well-known that in SSH model, the winding number \( \nu_{X} = 1 \) (\( \nu_{X} = 0 \)) when the dimerization parameter \( \delta J_S > 0 \) (\( \delta J_S < 0 \)). This again implies that corner states of \( \mathcal{H} \) exist only if both dimerization parameters \( \delta J_y, \delta J_x > 0 \). In order to check the generality of the above argument, we will now introduce a perturbation which amounts to modifying \( h_{b,S} \rightarrow h_{b,S} + \delta S \cos k_S \sigma_y^{(S)} \). This term breaks the time-reversal, particle-hole, and inversion symmetry of the system. However, since the chiral symmetry remains intact, Eq. (4) is still well-defined, which is plotted as a function of the perturbation strength \( \delta S \) in Fig. (2a). In particular, we find that even for moderate perturbation strength, \( h_{S,k} \) could still be topologically nontrivial. Consequently, by choosing different parameter values for \( h_{x,k} \) \((h_{y,k})\), assuming both the presence of the perturbations \( \delta x \) \((\delta y)\), we find that \( \mathcal{H} \) host topological corner modes only if \( h_{x,k} \) \((h_{y,k})\) are both topologically nontrivial, i.e., \( \nu_x = \nu_y = 1 \), as shown in Fig. (2b). Furthermore, the number of topological corner modes at zero-energy is given by \( \nu_0 = 4 \nu_x \cdot \nu_y \), which accounts the bulk-corner correspondence of our system. If either or both \( h_{x,k} \) \((h_{y,k})\) are topologically trivial, there is no such corner modes (see Fig. (2c)).

### B. Discussion

In contrast to many existing proposals on SOTIs so far, our construction above introduces an SOTI model that is protected solely by the chiral symmetry and does not rely on any spatial symmetries. Therefore, our proposed model is fundamentally different from other SOTI models, such as those studied in Ref. [6,25,27], which belong to a family of second order topological crystalline insulators. In fact, our model closely resembles that of Ref. [26], which also relies on the existence of chiral symmetry alone.

While the model proposed in Ref. [26] also describes a stack of 1D SSH models, it cannot be expressed as a
Kronecker sum in the spirit of Eq. \([3]\). However, since it can be broken down into four distinct 1D SSH models, the existence of corner modes is dictated by the four winding numbers of these SSH models. By contrast, the Kronecker sum structure of our model implies that only two winding numbers are needed to predict the existence of corner modes. Moreover, our model can be generalized beyond the Kronecker sum of two SSH models as described in Eq. \([3]\). For example, we will elucidate further in the next section, we may take \(h_{x,k}\) to be a 1D Floquet topological insulator, which enables the construction of Floquet SOTIs.

We end this section by discussing the fate of our proposal in the presence of perturbations breaking the Kronecker sum structure of Eq. \([3]\). To this end, we further add a perturbation of the form \(h_{xy,k} = -\delta_{xy,1}\sigma_x^{(x)}\sigma_y^{(y)} - \delta_{xy,2}(\cos(k_x)\sigma_x^{(x)} + \sin(k_x)\sigma_y^{(x)})\sigma_y^{(y)}\) to \(\mathcal{H}\), which preserves its chiral symmetry but breaking the Kronecker sum structure of Eq. \([3]\). In fact, we have also implemented such perturbations with \(\delta_{xy,1} = \delta_{xy,2} = 0.2\) in our results earlier presented in Figs. 2(b)-(d). In general, we observe that the presence of small perturbations does not affect the existence of the topological corner modes in the system. At moderate perturbation strengths, however, it is possible for the bulk or edge band gaps to close, resulting in the change of the number of topological corner modes in the system, which can no longer be captured by the individual topological invariants of \(h_x\) and \(h_y\). Nevertheless, our results demonstrate that if a general 2D Hamiltonian can be adiabatically deformed into a Hamiltonian that admits Kronecker sum structure without closing the bulk or edge band gaps in the process, its higher-order topology can still be studied from the bulk perspective by calculating the topological invariants of two 1D Hamiltonians.

### III. Constructing Floquet SOTI by Stacking 1D Floquet Topological Phases

#### A. General formulation

The idea we developed in Sec. II can also be applied to construct Floquet SOTIs. To this end, we may start with an array of chains of any 1D Floquet topological insulator in \(x\)-direction. Each of them is then coupled to adjacent chains by static dimerized couplings in \(y\)-direction (see Fig. 1 for an illustration). The full Hamiltonian of such a Floquet SOTI can then be written as

\[
\mathcal{H}(t) = -\sum_{j=1}^{N_y} \sum_{i=1}^{N_x} \left[ J_y + (\pm 1)\delta J_y \right] |i,j+1\rangle\langle i,j| + H_{1D}(t) \otimes |j\rangle\langle j| + \text{h.c.} \right] ,
\]

where \(J_y \pm \delta J_y\) again denote the dimerized hopping amplitudes in the \(y\)-direction, and \(H_{1D}(t)\) is a time-periodic Hamiltonian describing a 1D Floquet topological insulator. \(\mathcal{H}(t)\) in Eq. \([5]\) is thus time-periodic, and Floquet theory can be applied \([60, 61]\). To this end, we define a Floquet operator as the one-period propagator

\[
U_{\mathcal{H}} = U(t+T, t) = \mathcal{T} \exp \left( -i \int_t^{t+T} \mathcal{H}/\hbar dt \right) ,
\]

where \(T\) is the period of the system in time, and \(\mathcal{T}\) is the time-ordering operator. The topology of the system is then encoded in the quasienergy eigenstates \(|\varepsilon\rangle\) of \(U_{\mathcal{H}}\), which satisfies \(U_{\mathcal{H}}|\varepsilon\rangle = e^{-i\varepsilon T/\hbar}|\varepsilon\rangle\), where \(\varepsilon\) is the associated quasienergy.

Since the first and second terms of Eq. \([5]\) commute, we may write the Floquet operator as

\[
U_{\mathcal{H}} = U_{H_{1D}} \otimes U_{H_y} ,
\]

where \(U_{H_{1D}}\) and \(U_{H_y}\) are 1D Floquet operators associated with \(H_{1D}\) and \(H_y = \sum_{j=1}^{N_y} \left[ J_y + (\pm 1)\delta J_y \right] |j\rangle\langle j| + \text{h.c.} \right]\), respectively. The tensor product structure of Eq. \([7]\) enables us to systematically study the emergence of Floquet SOTIs from the properties of the underlying 1D Floquet system described by \(H_{1D}\). Indeed, let’s assume that \(|0_{1(N_y)}\rangle\) and \(|\pi_{1(N_y)}\rangle\) are the quasienergy zero and \(\pi\) eigenstates of \(U_{H_{1D}}\) localized near the left (right) end of the 1D lattice along \(x\)-direction. Topological corner modes of...
Eq. (7) at quasienergies zero and \( \pi \) can then be obtained as

\[
|0(x,y)\rangle = \sum_{j=1}^{N_x} \left( \frac{J_y - \delta J_y}{J_y + \delta J_y} \right)^{Y_j} |0_x\rangle \otimes |Y_j\rangle ,
\]

\[
|\pi(x,y)\rangle = \sum_{j=1}^{N_x} \left( \frac{J_y - \delta J_y}{J_y + \delta J_y} \right)^{Y_j} |\pi_x\rangle \otimes |Y_j\rangle ,
\]

where \( X = 1, N_x \), \( Y = 1, N_y \), and \( Y_j = |Y - j - 1| \). Equation (8) thus shows that topological corner modes exist provided \(|0_x\rangle\) and/or \(|\pi_x\rangle\) exist and \(\delta J_y > 0\), i.e., both \(U_{H_{1D}}\) and \(U_{H_D}\) are topologically nontrivial.

### B. Floquet SOTI with arbitrarily many topological corner modes

To elucidate the application of our construction outlined in Sec. IIIA, we consider a specific \(H_{1D}(t)\) hereinafter as given by

\[
H_{1D}(t) = \begin{cases} 
    h_1 & \text{for } (m-1)T < t \leq (m-1/2)T, \\
    h_2 & \text{for } (m-1/2)T < t \leq mT ,
\end{cases}
\]

where \(J_1 \) and \(J_2\) are hopping amplitudes, \(\sigma = A,B\) is a sublattice or pseudospin index, and \(\tilde{\sigma}\) is its complement. The model in Eq. (9) is first proposed in Ref. [35] as a quantum-walk realization of spin-1/2 double kicked rotor, and later also extended to non-Hermitian [36] and 2D [37] systems. It is capable of hosting a controllable number of edge states. This can be shown by first writing down \(h_1\) and \(h_2\) in Eq. (9) in momentum space as

\[
h_{1,k} = J_1 \cos(k_x) \sigma_{z}^{(x)},
\]

\[
h_{2,k} = J_2 \sin(k_x) \sigma_{y}^{(x)}. 
\]

The momentum space Floquet operator of Hamiltonian Eq. (9) can then be found as [35] (we take \(\hbar = T/2 = 1\) from here onwards)

\[
U_{H_{1D,k}} = \exp(-ih_{2,k}) \exp(-ih_{1,k}) = \exp(-ih_{\text{eff},k}),
\]

where

\[
h_{\text{eff},k} \propto \varepsilon = \arccos \left[ \cos(J_1 \cos(k_x)) \cos(J_2 \sin(k_x)) \right].
\]

It follows that the quasienergy gap closes at \(\varepsilon = 0(\pi)\) when \(\cos(J_1 \cos(k_x)) \cos(J_2 \sin(k_x)) = 1\). Consequently, as one fixes \(J_1\) (\(J_2\)), the two quasienergy gaps of \(U_{H_{1D,k}}\) close and reopen alternately at \(\varepsilon = 0\) and \(\varepsilon = \pi\) when \(J_2\) (\(J_1\)) increases by \(\pi\). Every time the gap closes and reopens, a topological phase transition happens and new pairs of degenerate edge states at quasienergy zero or \(\pi\) (i.e., Floquet zero or \(\pi\) edge modes) emerge at both ends of the lattice [35].

By implementing \(H_{1D}(t)\) defined above to Eq. (9), the discussion of Sec. IIIA implies the generation of Floquet SOTIs with arbitrarily many zero and \(\pi\) corner modes satisfying Eq. (8), whose number is also controllable via tuning the system parameters \(\delta J_y\) and \(J_1\) or \(J_2\). This is also evidenced by our numerics as shown in Fig. 3 in which the existence of zero and \(\pi\) edge modes of \(U_{H_{1D}}\) in panels (a) and (b) directly translates into a pair of the same number of Floquet zero and \(\pi\) corner modes of \(U_{H_D}\) in panels (c) and (d), provided \(\delta J_y > 0\) (panel (f) vs. (g)). It is thus expected that most properties of \(U_{H_D}\) can be obtained by analyzing \(U_{H_{1D}}\) and \(U_{H_D}\) separately.

To further demonstrate the flexibility of generating many Floquet zero and \(\pi\) corner modes following our construction, we show in Fig. 4 the quasienergy spectrum of \(U_{H_D}\) vs. \(J_1\), with the number of corner modes \(n_0\) and \(n_{\pi}\) denoted explicitly in the figure. Other system parameters are chosen as \(J_y = \delta J_y = \pi/40\) and \(J_2 = \pi/2\), which with the increase of \(J_1\), the number of Floquet corner states \(n_0\) (or \(n_{\pi}\)) at quasienergy zero (\(\pi\)) increases by 8 every time when \(J_1\) passes through an even (odd) multiple of \(\pi\). The same pattern is also observed in the Floquet spectrum of \(U_{H_D}\) vs. \(J_2\). The combination of these observations with the results of Ref. [35] suggests that there is no upper bound for such a trend of increasing the number of Floquet corner states at both zero and \(\pi\) quasienergies. Furthermore, we found from numerical calculations that with the above choice of system parameters, the number of Floquet corner modes at \(0(\pi)\) quasienergy is given by \(n_0 = 4|\nu_0| (n_{\pi} = 4|\nu_{\pi}|)\) for \((\nu_0 - 1)\pi < J_1 < (\nu_0 + 1)\pi\) \((|\nu_{\pi} - 1|\pi < J_1 < |\nu_{\pi} + 1|\pi)\), where \(\nu_0, \nu_{\pi}\) are two topological invariants of \(U_{H_D}\) that will be introduced in the following subsection.

Note in passing that edge states are also found at quasienergies \(\pm 2(J_y + \delta J_y)\) and \(\pm (\pi - 2(J_y + \delta J_y))\) in Fig. 4. However, these edge states are gapped and their numbers depend on the size of the lattice. Therefore, they do not belong to the type of Floquet topological corner states that is the focus of our study.

### C. Symmetry analysis and topological invariant

In this subsection, we introduce the topological invariants characterizing the Floquet SOTIs, and discuss their relations to the number of Floquet corner states. By transforming Eq. (11) to symmetric time frames [30],

\[
\tilde{U}_{H_{1D,k}}^{(1)} = \hat{F}_k \hat{G}_k , \quad \tilde{U}_{H_{1D,k}}^{(2)} = \hat{G}_k \hat{F}_k ,
\]

\[
\hat{F}_k = \exp(-ih_{2,k}/2) \times \exp(-ih_{1,k}/2) , \\
\hat{G}_k = \exp(-ih_{1,k}/2) \times \exp(-ih_{2,k}/2). 
\]

The full 2D momentum space Floquet operator can be
written as,
\[
\tilde{U}^{(1,2)}_{H,k} = \tilde{U}^{(1,2)}_{H_{1D},k} \exp(-i2n_{y,k}) ,
\]
where \( h_{y,k} \) is defined in Eq. (3). In particular, it is easy to verify that both \( \tilde{U}^{(1)}_{H_{1D},k} \) and \( \tilde{U}^{(2)}_{H_{1D},k} \) possess inversion, time-reversal, particle-hole, and chiral symmetries, respectively, given by the same operators defined in Sec. II A which satisfy \( T_\sigma h_{1D}(k_x,t)T_\sigma^{-1} = h_{1D}(-k_x,t) \), \( T_x h_{1D}(k_x,t)T_x^{-1} = h_{1D}(-k_x,2-t) \), \( \mathcal{P}_x U_{H_{1D},k} \mathcal{P}_x^{-1} = U_{H_{1D},-k_x} \), and \( \hat{\Gamma}_x \hat{\Gamma}_x^{-1} = \hat{G}_k \), \( \hat{G}_k \) \( 30 \), \( 31 \), \( 32 \), \( 33 \), \( 34 \), \( 35 \) \( 36 \), where \( \hat{h}_{1D}(k_x,t) \) is the momentum space time-dependent Hamiltonian in the symmetric time frame associated with Eq. (9). This implies that \( \tilde{U}^{(1,2)}_{H_{1D},k} \) also belong to class BDI in the AZ classification scheme, which is now characterized by two winding numbers \( \nu_0 \) and \( \nu_\pi \) associated with the number of Floquet zero and \( \pi \) edge modes respectively \( 62 \). By writing \( \tilde{U}^{(1)}_{H_{1D},k} \) explicitly as a matrix in the \( \sigma_z^{(x)} \) basis,
\[
\tilde{U}^{(1)}_{H_{1D},k} = \begin{pmatrix}
\hat{a}(k_x) & \hat{b}(k_x) \\
\hat{c}(k_x) & \hat{d}(k_x)
\end{pmatrix} ,
\]
the winding numbers \( \nu_0 \) and \( \nu_\pi \) can be obtained as \( 30 \) \( 62 \)
\[
\nu_0 = \frac{1}{2\pi i} \int dk_x b^{-1} \frac{d}{dk_x} b ,
\nu_\pi = \frac{1}{2\pi i} \int dk_x d^{-1} \frac{d}{dk_x} d .
\]
We plot \( \nu_0 \) and \( \nu_\pi \) of \( \tilde{U}^{(1)}_{H_{1D},k} \) as a function of \( J_1 \) (\( J_2 \)) in Fig. 3(a) (Fig. 3(b)), where \( J_2 = 1 \) (\( J_1 = 1 \)) is fixed. Consistent with the argument presented in Sec. III B, either \( \nu_0 \) or \( \nu_\pi \) increases as \( J_1 \) or \( J_2 \) increases by an integer multiple of \( \pi \). In the presence of perturbation \( h_{2,k} \rightarrow h_{2,k} - \delta_x \cos(k_x)\sigma_y^{(x)} \), which breaks all but chiral symmetry of the system, the winding numbers \( \nu_0 \) and \( \nu_\pi \)
and $\pi$ leads to the generation of arbitrarily many Floquet zero
and $J$ and/or $\delta J$ different Floquet SOTI phases in the parameter space.
Other system parameters are chosen as $J_y = \delta J_y = \pi/40$ and $J_z = \pi/2$. Dashed lines represent boundaries separating different Floquet topological corner states at quasienergy zero ($\pi$).

![Figure 4](image)

**FIG. 4.** The Floquet spectrum $\epsilon$ of $\mathcal{H}_\Gamma$ versus the hopping amplitude $J_1$. The size of the 2D lattice is $N_x = N_y = 50$. Other system parameters are chosen as $J_y = \delta J_y = \pi/40$ and $J_z = \pi/2$. Dashed lines represent boundaries separating different Floquet SOTI phases in the parameter space. $n_0$ ($n_\pi$) denotes the number of Floquet topological corner states at quasienergy zero ($\pi$).

remain well-defined, as depicted in Fig. 3(c) at $J_1 = 5$ and $J_2 = 1$. A large number of zero and $\pi$ edge modes can therefore be generated in a controlled manner by simply setting $J_1$ or $J_2$ to be large \cite{33}. Consequently, this leads to the generation of arbitrarily many Floquet zero and $\pi$ corner modes, tunable via the parameters $J_1$, $J_2$, and/or $\delta J_y$. Furthermore, their numbers $n_0$ and $n_\pi$ are related to the values of $\nu_0$ and $\nu_\pi$ as $n_0 = 4|\nu_0 \cdot w|$ and $n_\pi = 4|\nu_\pi \cdot w|$, respectively, where $w$ is the winding number of $h_{y,k}$. These relations establish the “bulk-corner correspondence” of Floquet SOTIs belonging to the same symmetry class of our system.

In Figs. 3(c)-(g), we have also included the presence of small perturbations of the form $h_{1,k} \rightarrow h_{1,k} - \delta_{x,y,1} \cos(k_x) \sigma_x^{(x)} \sigma_y^{(y)}$, $h_{2,k} \rightarrow h_{2,k} - \delta_{x,y,2} \sin(k_x) \sigma_y^{(x)} \sigma_y^{(y)} - \delta_x \cos(k_y) \sigma_y^{(y)}$, and $h_{y,k} \rightarrow h_{y,k} - \delta_y \cos(k_y) \sigma_y^{(y)}$, where $\delta_x$ and $\delta_y$ terms break all but chiral symmetry of the system, while $\delta_{x,y,1}$ and $\delta_{x,y,2}$ terms break the tensor product structure of Eq. \cite{7}. As expected, such perturbations do not qualitatively affect the existence of zero and $\pi$ corner modes in the system, provided the former do not induce edge or bulk gap closing of the quasienergy bands. This shows that our Floquet SOTI proposal does not rely on any spatial symmetry protection and its topological characterization presented above also provide insights into more general Floquet SOTI models, whose Floquet operator can be adiabatically deformed to the form of Eq. \cite{7}.

**IV. MEAN CHIRAL DISPLACEMENT**

In this section, we propose to detect the topological winding numbers of Floquet SOTIs in the symmetry class BDI by measuring the mean chiral displacement (MCD) of a wavepacket.

The MCD is proposed and applied in \cite{33, 34, 35} as a dynamical probe of winding numbers for 1D topological insulators. The tensor product structure of Floquet operator $\hat{U}_\Gamma$, together with its chiral symmetry allow us to extend the definition of MCD straightforwardly to two-dimensional dynamics.

We first introduce the chiral displacement operator $\hat{C}_\alpha$, which in Heisenberg representation is given by

$$\hat{C}_\alpha(t) = \left[U^{(\alpha)}_{\mathcal{H}} \right]^t (\hat{x} \otimes \Gamma_x) \otimes (\hat{y} \otimes \Gamma_y) \left[U^{(\alpha)}_{\mathcal{H}} \right]^t.$$ \hspace{1cm} (17)

Here $U^{(\alpha)}_{\mathcal{H}}$ is the full Floquet operator given by Eq. \cite{7} in the symmetric time frame $\alpha$. $t$ denotes the number of driving periods. $\hat{x}$ and $\hat{y}$ are quantized unit-cell position operators. For the model we investigated in Sec. III, the chiral symmetry operators $\Gamma_x$ and $\Gamma_y$ are explicitly given by $\Gamma_x = \sigma_x^{(x)}$ and $\Gamma_y = \sigma_x^{(y)}$. For a wavepacket $|\psi_0\rangle$ prepared at time $t = 0$, the expectation value $\langle\psi_0 | \hat{C}_\alpha(t) |\psi_0\rangle$ thus describes the chirality-resolved shift of $|\psi_0\rangle$ over $t$ driving periods.

Let’s now choose the initial state $|\psi_0\rangle$ to be a fully polarized state located at the center ($x = 0, y = 0$) of the lattice \cite{67}. Explicitly it has the form

$$|\psi_0\rangle = |\nu_0 \rangle \otimes |\pm_x \rangle \otimes |\nu_y \rangle \otimes |\pm_y \rangle,$$ \hspace{1cm} (18)

where $|\nu_x \rangle$ ($|\nu_y \rangle$) is the eigenstate of $\hat{x}$ ($\hat{y}$) with eigenvalue 0, and $|\pm_x \rangle$ ($|\pm_y \rangle$) is the eigenstate of $\Gamma_x$ ($\Gamma_y$) with eigenvalue $+1$ or $-1$. The MCD of such a wavepacket
over $t$ driving periods is then given by:

$$
C_\alpha(t) = \langle 0_x \rangle \otimes (\pm_x \otimes \langle 0_y \rangle \otimes \langle \pm_y \rangle
\times \left[U^\dagger_H(\alpha)\right]^{-t} \left(\hat{\varepsilon} \otimes \Gamma_x\right) \otimes \left(\hat{y} \otimes \Gamma_y\right) \left[U^\dagger_H(\alpha)\right]^t \langle 0_x \rangle \otimes |\pm_x\rangle \otimes |0_y\rangle \otimes |\pm_y\rangle.
$$

(19)

To proceed, we express $U^\dagger_H(\alpha)$ as

$$
U^\dagger_H(\alpha) = U^\dagger_x(\alpha) \otimes U^\dagger_y(\alpha),
$$

(20)

where $U^\dagger_x(\alpha)$ and $U^\dagger_y(\alpha)$ are 1D Floquet operators associated with Hamiltonians $H_{1D}$ and $H_y$ in Eq. (7), respectively. Note that for the time-independent Hamiltonian $H_y$, we have $U^\dagger_y(1) = U_y(2)$. With the help of Eq. (20), we can rewrite $C_\alpha(t)$ as a product of two MCDs along two orthogonal dimensions, i.e.,

$$
C_\alpha(t) = C_{\alpha x}(t) \cdot C_{\alpha y}(t),
$$

(21)

where

$$
C_{\alpha x}(t) = \langle 0_x \rangle \otimes (\pm_x |[U^\dagger_x(\alpha)]^{-t} (\hat{x} \otimes \Gamma_x) [U^\dagger_x(\alpha)]^t \langle 0_x \rangle \otimes |\pm_x\rangle,
$$

(22)

$$
C_{\alpha y}(t) = \langle 0_y \rangle \otimes (\pm_y |[U^\dagger_y(\alpha)]^{-t} (\hat{y} \otimes \Gamma_y) [U^\dagger_y(\alpha)]^t \langle 0_y \rangle \otimes |\pm_y\rangle.
$$

(23)

Now performing a Fourier transform from position to momentum representations, we find

$$
C_{\alpha x}(t) = \int_0^\pi \frac{dk_x}{2\pi} (\pm_x |[U^\dagger_x(k_x)]^{-t} \Gamma_x i\partial_{k_x} [U^\dagger_x(k_x)]^t |\pm_x\rangle),
$$

(24)

$$
C_{\alpha y}(t) = \int_0^\pi \frac{dk_y}{2\pi} (\pm_y |[U^\dagger_y(k_y)]^{-t} \Gamma_y i\partial_{k_y} [U^\dagger_y(k_y)]^t |\pm_y\rangle),
$$

(25)

where $U^\dagger_x(k_x)$ and $U^\dagger_y(k_y)$ are $2 \times 2$ matrices satisfying $U^\dagger_x(\alpha) = \sum_{k_x} U^\dagger_x(k_x) |k_x\rangle \langle k_x|$ and $U^\dagger_y(\alpha) = \sum_{k_y} U^\dagger_y(k_y) |k_y\rangle \langle k_y|$ in momentum representations. Then following the derivations detailed in Ref. 85, we obtain

$$
C_{\alpha x}(t) = \frac{v_\alpha}{2} - \int_{-\pi}^\pi \frac{dk_x}{2\pi} \frac{\cos(2\varepsilon t)}{2} (n^x_{x\alpha} \partial_{k_x} n^y_{y\alpha} - n^x_{y\alpha} \partial_{k_x} n^y_{x\alpha}),
$$

(26)

$$
C_{\alpha y}(t) = \frac{w_\alpha}{2} - \int_{-\pi}^\pi \frac{dk_y}{2\pi} \frac{\cos(2E t)}{2} (d^x_{x\alpha} \partial_{k_y} d^y_{y\alpha} - d^x_{y\alpha} \partial_{k_y} d^y_{x\alpha}).
$$

(27)

Here, for the Floquet model we studied in the last section, $v_\alpha$ is the winding number of the 1D Floquet operator $\hat{U}^\dagger_H(\alpha)$ in symmetric time frame $\alpha$, and $w_\alpha = w$ is the winding number of SSH model associated with the propagator $e^{-i2\varepsilon_{\alpha}K_{\alpha}}$, $\varepsilon$ is the eigenphase of $\hat{U}^\dagger_H(\alpha)$ as defined in Eq. (12), and $E = \pm4J_y$ is the eigenphase of $e^{-i2\varepsilon_{\alpha}K_{\alpha}}$. The components of unit vectors $(d^x_{x\alpha}, d^y_{y\alpha})$ and $(n^x_{x\alpha}, n^y_{y\alpha})$ are given by

$$
d^x_{x\alpha} = \cos k_y, \quad d^y_{y\alpha} = \sin k_y,
$$

(28)

and

$$
n^1_x = \frac{\sin(J_1)}{\sin(\varepsilon)}, \quad n^2_x = \frac{\sin(J_2)}{\sin(\varepsilon)},
$$

$$
n^1_y = \frac{\sin(J_2) \cos(J_1)}{\sin(\varepsilon)}, \quad n^2_y = \frac{\sin(J_2)}{\sin(\varepsilon)},
$$

(29)

where $J_1 = J_1 \cos k_x$ and $J_2 = J_2 \sin k_x$. It is clear that both $C_{\alpha x}(t)$ and $C_{\alpha y}(t)$ are composed of a time-independent topological part and a time-dependent oscillating term. For general dispersion relations, the oscillating terms tend to decay at large $t$ under the integral over corresponding quasimomenta.

To relate $C_\alpha(t)$ to the topological invariants of Floquet SOTIs, we consider its average over $t$ driving periods, given by

$$
\overline{C_\alpha(t)} = \frac{1}{t} \sum_{t'=1}^t C_\alpha(t).
$$

(30)

With the help of Eqs. (21), (26) and (27), we see that the oscillating parts of $\overline{C_\alpha(t)}$ decay in time at least of order $1/t$. Therefore, in long time limit ($t \to \infty$), we obtain

$$
\overline{C_\alpha} = \lim_{t \to \infty} \overline{C_\alpha(t)} = \frac{v_\alpha w_\alpha}{4}.
$$

(31)

For the model we considered in the last section, the winding numbers $w_\alpha = w = 1$. Furthermore, the winding numbers $v_0$ and $v_\pi$ are related to $v_0$ and $v_\pi$ through

$$
v_0 = \frac{v_2 + v_1}{2}, \quad v_\pi = \frac{v_2 - v_1}{2}.
$$

(32)

Combining Eqs. (31) and (32) then yields the relations between time-averaged MCDs and topological winding numbers $v_0, v_\pi$, i.e.,

$$
v_0 = 2(\overline{C_2} + \overline{C_1}),
$$

(33)

$$
v_\pi = 2(\overline{C_2} - \overline{C_1}).
$$

Therefore, by measuring the long-time averaged MCDs in two complementary symmetric time frames, we would be able to obtain the topological invariants characterizing the Floquet SOTIs introduced in the last section. The number of Floquet corner states can also be indirectly deduced from bulk dynamics through the relations

$$
n_0 = 8|\overline{C_1} + \overline{C_2}|,
$$

(34)

$$
n_\pi = 8|\overline{C_1} - \overline{C_2}|.
$$

(35)

It is also not hard to extend these results to other Floquet SOTIs protected by chiral symmetry, for which evolutions in four symmetric time frames may need to be executed. The formalism presented here could also be
applied to static SOTIs protected by chiral symmetry, where the number of driving periods \( t \) should be interpreted as the duration of evolution time, and the sum over \( t \) replaced by an integral over the continuous time duration \( t \).

In Fig. 6, we present the \( C_\alpha(t) \) and the winding numbers \( \nu_0, \nu_\pi \) vs. \( J_1 \). Up to \( t = 20 \), we already find good convergence of \( 2C_2(t) + 2C_1(t) \) and \( 2C_2(t) - 2C_1(t) \) to their corresponding winding numbers \( \nu_0 \) and \( \nu_\pi \), respectively. When \( J_1 \) is close to an integer multiple of \( \pi \), the MCD combinations \( 2C_2(t) + 2C_1(t) \) and \( 2C_2(t) - 2C_1(t) \) deviate from quantization due to the topological phase transitions happening there. Other small deviations from quantization are originated from the oscillating terms in Eqs. \((26)\) and \((27)\).

In previous studies, the MCD has been measured in 1D photonic and cold atom systems. An experimental proposal for detecting the MCDs of \( U^{(\alpha)}_x \) by implementing quantum walks in BECs has also been discussed. According to the tensor product structure of \( U^{(\alpha)}_x \), one may implement the dynamics governed by \( U^{(\alpha)}_x \) and \( U^{(\alpha)}_y \) in two decoupled 1D systems separately following the proposals of Ref. [35], and therefore detecting the topological invariants \( \nu_0 \) and \( \nu_\pi \) of Floquet SOTIs proposed in this study. The topological winding numbers of static SOTIs as introduced in Sec. II could also be extracted from MCDs in a similar manner.

V. CONCLUDING REMARKS

In this paper, we report a theoretical proposal for constructing static and Floquet SOTI by stacking 1D topological phases and coupling them with dimerized hopping amplitude. The total Hamiltonian can then be written as a Kronecker sum of two 1D Hamiltonian describing a static 1D SSH model in the \( y \)-direction and another 1D static or Floquet topological insulating model in the \( x \)-direction, allowing one to characterize the existence of the topological corner modes of the whole system by separately analyzing the topology of the 1D model in the \( x \)- and \( y \)-direction.

Although the explicit models presented in this paper possess all inversion, time-reversal, particle-hole, and chiral symmetries, their topological corner modes are protected solely by the chiral symmetry alone, as demonstrated by our numerical results in the presence of perturbations breaking all but the chiral symmetry. The SOTI proposed in this paper is thus fundamentally different from most other existing SOTI proposals, which rely on the presence of spatial symmetries. It is also expected that our proposal can be generalized to a class of 2D systems whose Hamiltonian can be broken down into a Kronecker sum of two 1D Hamiltonian describing any static and/or Floquet topological phases. Moreover, we have also numerically verified that the presence of small perturbations breaking the Kronecker sum structure of the Hamiltonian does not qualitatively affect the existence of the corner modes, provided such perturbations do not close the bulk or edge gap of the system.

It is expected that our proposal above can be extended to higher dimensional systems for constructing static and Floquet higher-order topological phases, which is left for future studies. To this end, one may start with a Kronecker sum of several 1D and/or 2D static or time-periodic Hamiltonian, then tune each of these Hamiltonian in its topologically nontrivial regime. By a similar mechanism elucidated in Sec. IIIA, the resulting system is expected to host topological corner and/or hinge states at its boundaries.

Finally, we have demonstrated the capability of Floquet SOTI to host arbitrarily many topological corner modes at quasienergy zero and \( \pi \), which may find its potential applications in quantum information processing (see e.g. Ref. [63]). It is expected that there are other interesting and unique features of Floquet SOTI with no static analogue that have yet to be explored in this paper. Further exploration on the physics of Floquet SOTI and higher-order topological phases is thus imagined to be an interesting aspect to pursue in the future.

Note added. During the finalization of our manuscript, we became aware of a recent preprint which also discusses a proposal for constructing Floquet SOTI protected solely by chiral symmetry. The model introduced in Ref. [69] however does not admit a Kronecker sum structure in the spirit of Eq. (3) and is thus different from ours.
As we are working under PBC, we can, without loss of generality, shift the center of the lattice to $x = y = 0$ for simplicity.

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