Matter Perturbations in Scaling Cosmology

A. Romero Fuño∗, W.S. Hipólito-Ricaldi†, and W. Zimdahl‡

1Universidade Federal do Espírito Santo, Departamento de Física
Av. Fernando Ferrari, 514, Campus de Goiabeiras,
CEP 29075-910, Vitória, Espírito Santo, Brazil
2Universidade Federal do Espírito Santo,
Departamento de Ciências Naturais
Rodovia BR 101 Norte, km. 60, CEP 29932-540,
São Mateus, Espírito Santo, Brazil
(Dated: September 30, 2014)

Abstract

A suitable interaction between dark matter with an energy density $\rho_M$ and dark energy with an energy density $\rho_X$ is known to give rise to a non-canonical scaling $\rho_M \propto \rho_X a^{-\xi}$ where $\xi$ is a parameter which generally deviates from $\xi = 3$. Here we present a covariant generalization of this class of models and investigate the corresponding perturbation dynamics. The resulting matter power spectrum for the special case of a decaying Lambda model is compared with data from the SDSS DR7 catalogue. We find a large degeneracy in $\xi$, equivalent to a poor restriction of the interaction but our results are compatible with the LCDM model which corresponds to the noninteracting limit with $\xi = 3$ and an equation-of-state parameter $w = -1$. 

∗E-mail: alonsoromero.ufes@gmail.com
†E-mail: william.ricaldi@ufes.br
‡E-mail: winfried.zimdahl@pq.cnpq.br
I. INTRODUCTION

The currently preferred cosmological model, the \( \Lambda \)CDM model, is characterized by a pressureless dark-matter (DM) component with an energy density \( \rho_M \), which decays with the third power of the cosmic scale factor \( a \) and a constant energy density \( \rho_\Lambda \), attributed to a cosmological constant \( \Lambda \). Alternative models, as far as they remain in the context of Friedmann-Lemaître-Robertson-Walker (FLRW) models, replace \( \rho_\Lambda \) by a (not necessarily constant) \( \rho_X \), the energy density of dark energy (DE), equipped with an equation-of-state (EoS) parameter \( w \) which may be time dependent. Together, DM and DE form a “dark sector” which makes up about 95% of the present cosmic energy budget and which therefore dominates the cosmological dynamics. In the simplest case, following the \( \Lambda \)CDM paradigm, DM and DE are considered to be independent entities, governed by separate conservation laws. The more general case is, however, not to exclude the possibility of a non-gravitational coupling between these both components which results in a richer structure of the dark sector. Moreover, it has been demonstrated, that ignoring a potentially existing interaction may lead to an incorrect interpretation of cosmological observations \[1\]. Since neither the physical nature of DE nor that of DM are known, these models are necessarily phenomenological. Since they differ from the standard \( \Lambda \)CDM model they are useful to test potential deviations from the latter. While for the homogeneous and isotropic background dynamics a lot of models do fit the observations, their different perturbation dynamics may serve to limit the number of seriously competing approaches. There exists a large body of literature in which non-gravitational interactions between DE and DM are considered. A subclass of these activities is devoted to models of DE which keep an EoS parameter \( w = -1 \) as in the standard model, but generalize the latter insofar as \( \rho_X \) is admitted to be time dependent. These models are also called decaying \( \Lambda \) models \[2\]. Our aim here is to study in this context the perturbation dynamics of a model in which DM and DE interact in such a way that the ratio of their energy densities obeys a power-law in the scale factor, i.e. \( \rho_M/\rho_X \propto a^{-\xi} \), where \( \xi \) is a constant parameter. Such model was proposed by Dalal et al. \[3\] to address the coincidence problem. Independently of whether or not one considers this problem to be really a problem, the ansatz by Dalal et al. gives rise to a testable alternative cosmological dynamics which contains the standard model as a limiting case. Various aspects of the background dynamics of this model have been studied so far and were confronted with
observational data [5–8]. But in its original form its validity is restricted to the homogeneous and isotropic background. The dynamics depends directly on the scale factor which is not a covariant quantity. Here we present a covariant generalization of this model and complement previous investigations by a gauge-invariant perturbation analysis. To this purpose the scale factor is replaced by a general, covariantly defined length scale which under the conditions of homogeneity and isotropy reduces to the scale factor. This allows us to establish a covariant and gauge-invariant perturbation theory on the basis of which we calculate the matter power spectrum and discuss its dependence on the parameter $\xi$.

The paper is organized as follows. In Sec. II we present the basic framework for an interacting system of two perfect fluids. Our scaling model is established in Sec. III where we also study its dynamics in the spatially homogeneous and isotropic background. The perturbation analysis is the subject of Sec. IV. It provides us with an expression for the matter-density perturbation which is analyzed and observationally tested in Sec. V. A summary of the paper is given in Sec. VI.

II. INTERACTING TWO-COMPONENT SYSTEM

We assume the cosmic substratum to be dynamically dominated by a dark sector, made of DM and DE. The substratum as a whole is characterized by a conserved total perfect-fluid type energy-momentum tensor $T_{ik} = \rho u_i u_k + p h_{ik}$ with $T_{;k} = 0$. Here, $h_{ik} = g_{ik} + u_i u_k$ and $g_{ik} u^i u^k = -1$. The quantity $u^i$ is the total four-velocity of the cosmic substratum and Latin indices run from 0 to 3. Splitting the conservation laws into their timelike and spacelike parts, we have

$$\rho, a u^a + \Theta (\rho + p) = 0 \quad \text{and} \quad (\rho + p) u^a + p, i h^{ai} = 0,$$

respectively. Now we assume a split of $T_{ik}$ into a matter component (subindex M) and a dark energy component (subindex X), according to $T_{ik} = T_{ik}^M + T_{ik}^X$ with $(A = M, X)$

$$T_{ik}^A = \rho_A u_A^i u_A^k + p_A h_{ik}^A, \quad h_{ik}^A = g_{ik}^A + u_A^i u_A^k.$$

An interaction between the components is covariantly characterized by

$$T_{ik}^M; k = Q^i, \quad T_{ik}^X; k = -Q^i.$$
Then, the separate energy-balance equations are

\[- u_{Mi} T_{ik}^{M} = \rho_M a u_M^a + \Theta_M (\rho_M + p_M) = - u_M a Q^a \]  \(4\)

and

\[- u_{Xi} T_{ik}^{X} = \rho_X a u_X^a + \Theta_X (\rho_X + p_X) = u_X a Q^a . \]  \(5\)

The four-velocities, in general different for both components, obey \(g_{ik} u_i u_k = -1\). The quantities \(\Theta_A\) are \(\Theta_A \equiv u_A^a u_A^b\). In the homogeneous and isotropic background we assume all four-velocities to coincide: \(u^a_M = u^a_X = u^a\). The coupled momentum balances are

\[ h_{Mi}^a T_{ik}^{M} = (\rho_M + p_M) \dot{u}_M^a + p_M i h_{Mi}^{ai} = h_{Mi}^a Q^i \]  \(6\)

and

\[ h_{Xi}^a T_{ik}^{X} = (\rho_X + p_X) \dot{u}_X^a + p_X i h_{Xi}^{ai} = - h_{Xi}^a Q^i , \]  \(7\)

where \(\dot{u}_A^a \equiv u_A^a u_A^b\).

It is convenient to decompose the source term \(Q^i\) into parts proportional and perpendicular to the total four-velocity,

\[ Q^i = u^i Q + \bar{Q}^i , \]  \(8\)

where \(Q \equiv - u_i Q^i\) and \(\bar{Q}^i \equiv h_i^a Q^a\) with \(u_i \bar{Q}^i = 0\).

III. THE MODEL AND ITS BACKGROUND DYNAMICS

A. The model

Following [4], we introduce a length scale \(l\) by

\[ \dot{l} \equiv \frac{1}{3} \Theta , \quad \dot{i} \equiv l_a u^a . \]  \(9\)

Our aim is to consider the dynamics of a class of models for which the ratio of the energy densities of both components, \(r \equiv \rho_M / \rho_X\), behaves as a power of the length scale \(l\),

\[ r = \frac{\rho_M}{\rho_X} \Rightarrow r = r_0 l^{-\xi } , \]  \(10\)

where \(\xi\) is a constant parameter and \(r_0\) is the present value of the ratio \(r\). The evolution of the ratio \(r\) is then given

\[ \frac{\dot{r}}{r} \equiv \frac{r_a u^a}{r} = - \frac{\xi}{3} \Theta . \]  \(11\)
Relations (9), (10) and (11) generalize a previous model introduced by Dalal et al. [3] which subsequently was studied in detail in Refs. [5–8]. In its original form, this model was restricted to the homogeneous and isotropic background dynamics. Our covariant generalization relying on the use of the length scale (9) opens the possibility to consider an inhomogeneous perturbation dynamics as well. In the background one has \[ \Theta = 3H = \frac{3}{a}, \] were \( a \) is the scale factor of the Robertson-Walker metric and (10) reduces to \[ r = r_0a^{-\xi} \] which defines the class of models considered in [3].

B. Background dynamics

Assuming a pressureless matter component, in the homogeneous and isotropic background the balance equations (4) and (5) reduce to

\[ \dot{\rho}_M + 3H \rho_M = Q \quad \text{and} \quad \dot{\rho}_X + 3H(1+w) \rho_X = -Q, \tag{12} \]

respectively, where \( w \equiv \frac{p_X}{\rho_X} \) is the EoS parameter of the DE and \( Q \) is the background value of the general source term \( Q \).

Combining the background ansatz \( r = r_0a^{-\xi} \Rightarrow \dot{r} = -\xi H r \) with the balances (12) yields

\[ Q = -3H \rho_M \frac{\xi + w}{r_0a^{-\xi} + 1}. \tag{13} \]

This relation determines the interaction that is necessary to generate a dynamics with \( r = r_0a^{-\xi} \). The interaction vanishes for the special cases \( \xi = -3w \). The \( \Lambda \)CDM model is recovered for \( \xi = 3 \) and \( w = -1 \). Every combination \( \frac{\xi}{3} + w \neq 0 \) gives rise to an alternative, testable model.

For a constant EoS parameter \( w \) the matter-energy balance in (12) can be integrated,

\[ \rho = \rho_{M0} \left[ 1 + \frac{1}{a} \right]^{3(1+w)+\xi} \left[ 1 + \frac{r_0(1+z)^{\xi}}{r_0 + 1 + \xi z} \right]^{-1 - \frac{w}{\xi}}, \tag{14} \]

where \( z = \frac{1}{a} - 1 \) is the redshift parameter. The total energy density becomes

\[ \rho = \rho_0 a^{-3} \left( \frac{r_0 + a^{\xi}}{r_0 + 1} \right)^{-\frac{w}{\xi}}, \quad \rho_0 = \frac{r_0 + 1}{r_0} \rho_{M0}, \tag{15} \]

where \( \rho_0 \) is the present value of the energy density \( \rho \). Restricting ourselves to a universe with spatially flat sections, we obtain the Hubble rate

\[ H = H_0 a^{-\frac{3}{2}} \left( \frac{r_0 + a^{\xi}}{r_0 + 1} \right)^{-\frac{w}{\xi}}, \tag{16} \]
and the deceleration parameter \( q = -1 - \frac{\dot{H}}{H^2} \),

\[
q = \frac{1}{2} + \frac{3}{2} \frac{w}{r_0 a^{-3} + 1}.
\]

The present matter fraction \( \Omega_{M0} \) of the Universe is related to the ratio \( r_0 \) by \( \Omega_{M0} = r_0/(1 + r_0) \). For the special case \( p_X = -\rho_X \) we have (in the background)

\[
T_X^{ik} = -\rho_X g^{ik} \quad \text{and} \quad \dot{\rho}_X = -Q,
\]

equivalent to a decaying cosmological term.

IV. PERTURBATION DYNAMICS

A. General setup

Let us denote first-order perturbations by a hat symbol. Assuming the equality of all four velocities in the background, \( u^a_M = u^a_X = u^a \), it follows from \( g_{ik} u^i_A u^k_A = -1 \) that the perturbed time components of the four-velocities are equal as well, i.e., \( \hat{u}_0 = \hat{u}^0 = \hat{u}_M^0 = \hat{u}_X^0 = \frac{1}{2} \hat{g}_{00} \). Because of this property, at first order, the expressions \( \rho_{M,a} u^a_M \) and \( \rho_{X,a} u^a_X \) in the energy balances (4) and (5), respectively, are

\[
(\rho_{M,a} u^a_M) = \hat{\rho}_{M,a} u^a_M + \rho_{M,a} \hat{u}_M^a = \hat{\rho}_M + \dot{\rho}_M \hat{u}_0 \quad (19)
\]

and

\[
(\rho_{X,a} u^a_X) = \hat{\rho}_{X,a} u^a_X + \rho_{X,a} \hat{u}_X^a = \hat{\rho}_X + \dot{\rho}_X \hat{u}_0, \quad (20)
\]

respectively. This implies that at first order

\[
\rho_{M,a} u^a_M = \rho_{M,a} u^a \quad \text{and} \quad \rho_{X,a} u^a_X = \rho_{X,a} u^a \quad (21)
\]

are valid. The timelike projections of the derivatives along the four velocities of the components coincide with the corresponding projections along the total four velocity. In other words, there is only one timelike derivative. Obviously, this is no longer valid at higher orders.

Combining the balances (4) and (5) with (21) it follows that, up to first order,

\[
\frac{\xi}{3} \Theta = \Theta_M + \frac{u_M a Q^a}{\rho_M} + \left[ -\Theta_X (1 + w) + \frac{u_X a Q^a}{\rho_X} \right].
\]

(22)
In the background ($\Theta_M = \Theta_X = \Theta$ and $u_{Ma}Q^a = u_{Xa}Q^a = u_aQ^a = -Q$) we recover relation (13). Equation (22) will be crucial to determine the perturbed source in the following subsection.

In a next step we define the scalar velocity potentials $v$, $v_M$ and $v_X$ for the spatial velocity perturbations by (Greek indices run from 1 to 3)

$$
\hat{u}_\alpha = v_{\alpha}, \quad \hat{u}_{Ma} = v_{M\alpha}, \quad \hat{u}_{Xa} = v_{X\alpha}.
$$

(23)

Directly from the definition of $\Theta$ it follows that

$$
\hat{\Theta} = \frac{1}{a^2} (\Delta v + \Delta \chi) - 3\dot{\psi} - 3H\phi,
$$

(24)

where $\Delta$ is the three-dimensional Laplacian and where we have introduced the line element for scalar perturbations,

$$
ds^2 = -(1 + 2\phi) dt^2 + 2a^2 F_\alpha dt dx^\alpha + a^2 [(1 - 2\psi) \delta_{\alpha\beta} + 2E_{\alpha\beta}] dx^\alpha dx^\beta,
$$

(25)

together with the abbreviation

$$
\chi \equiv a^2 \left( \dot{E} - F \right).
$$

(26)

Via the variable $F$ the spatial velocity components $\hat{u}^\mu$ are related to the potential $v$,

$$
a^2 \hat{u}^\mu + a^2 F_\mu = \hat{u}_\mu \equiv v_{\mu}.
$$

(27)

Analogous relations are valid for the velocity variables of the components. Similarly to (24) one has ($A = X, M$)

$$
\hat{\Theta}_A = \frac{1}{a^2} (\Delta v_A + \Delta \chi) - 3\dot{\psi} - 3H\phi.
$$

(28)

For the differences $\hat{\Theta}_A - \hat{\Theta}$ it follows that

$$
\hat{\Theta}_A - \hat{\Theta} = \frac{1}{a^2} (\Delta v_A - \Delta v).
$$

(29)

According to the perfect-fluid structure of both the total energy-momentum tensor and the energy-momentum tensors of the components in (2), and with $u^a_M = u^a_X = u^a$ in the background, we have first-order energy-density perturbations $\hat{\rho} = \hat{\rho}_M + \hat{\rho}_X$, pressure perturbations $\hat{p} = \hat{p}_M + \hat{p}_X$, $\hat{p}_X = \hat{p}_X$ and

$$
\hat{T}^0_\alpha = \hat{T}^0_{Ma} + \hat{T}^0_{Xa} \quad \Rightarrow \quad (\rho + p) \hat{u}_\alpha = \rho_M \hat{u}_{Ma} + (\rho_X + p_X) \hat{u}_{Xa}.
$$

(30)

With the definitions in (23) relation (30) implies

$$
v_M - v = (1 + w) \frac{\rho_X}{\rho + p} (v_M - v_X), \quad v_X - v = -\frac{\rho_M}{\rho + p} (v_M - v_X).
$$

(31)
B. The perturbed source term

From now on we shall restrict ourselves to the case \( p_X = -\rho_X \), i.e. to an EoS parameter \( w = -1 \). The departure of the background dynamics from the \( \Lambda \)CDM model is then quantified by the difference of the parameter \( \xi \) from \( \xi = 3 \). Under this condition it follows from (30) that

\[
p_X = -\rho_X \Rightarrow \rho + p = \rho_M \Rightarrow \hat{u}_{Ma} = \hat{u}_\alpha \Rightarrow v_M = v . \tag{32}
\]

Since the component \( M \) is supposed to describe matter, it is clear from (30) that the perturbed matter velocity \( \hat{u}_{Ma} \) coincides with the total velocity perturbation \( \hat{u}_\alpha \). With \( u_M^a = u^a \) up to first order, the energy balance in (31) (correct up to first order) can be written as

\[
\rho_{Ma} u^a = -\Theta \rho_M - u_a Q^a , \quad (w = -1) . \tag{33}
\]

On the other hand, the total energy balance (cf. (11)) is \( \rho_a u^a = -\Theta (\rho + p) \). For the difference we find

\[
\dot{\rho} - \dot{\rho}_M \equiv (\rho - \rho_M)_a u^a = u_a Q^a . \tag{34}
\]

Since, at least up to linear order, \( \rho - \rho_M = \rho_X \), Eq. (34) is equivalent (up to the first order) to

\[
\dot{\rho}_X \equiv \rho_{X,a} u^a = u_a Q^a . \tag{35}
\]

In zeroth order we consistently recover (18). The first order of Eq. (35) is

\[
\dot{\rho}_X + \dot{\rho}_X \hat{u}^0 = (u_a Q^a)^\hat{\cdot} . \tag{36}
\]

Notice that Eq. (36) results from a combination of the total energy conservation and the matter energy balance. It has to be consistent with the DE balance (5). At first order, the latter becomes

\[
\dot{\rho}_X + \dot{\rho}_X \hat{u}^0 = (u_{X,a} Q^a)^\hat{\cdot} . \tag{37}
\]

Consistency then requires that

\[
(u_{X,a} Q^a)\hat{\cdot} = (u_a Q^a)\hat{\cdot} , \tag{38}
\]

i.e., the projections of \( Q^a \) along \( u_{X,a} \) and along \( u_a \) coincide. Explicitly,

\[
(u_a Q^a)\hat{\cdot} = (u_a u^a Q)\hat{\cdot} = -\hat{Q} . \tag{39}
\]
Under the conditions (38) and (39) relation (22) becomes (for $w = -1$)

$$\frac{\xi}{3} \dot{\Theta} = \dot{\Theta}_M - \dot{Q} \frac{\rho}{\rho_M \rho_X} - Q \left( \frac{\rho}{\rho_M \rho_X} \right).$$

(40)

Solving for $\dot{Q}$ we find, after some transformation,

$$\dot{Q} = Q \left[ \frac{\dot{\Theta}}{\Theta} + \delta_M (1 - r) + r \delta \right],$$

$$\delta \equiv \frac{\dot{\rho}}{\rho}, \quad \delta_M \equiv \frac{\dot{\rho}_M}{\rho_M},$$

(41)

where $Q$ is the background expression (13). Since the combination of interest will be $\dot{Q} - Q \delta_M$, it is useful to rewrite (11) as

$$\left( \frac{Q}{\rho_M} \right) = Q \left[ \frac{\dot{\Theta}}{\Theta} - r (\delta_M - \delta) \right].$$

(42)

With the perturbed source term explicitly known, we may now, in the following subsection, establish the basic set of perturbation equations.

C. Basic set of perturbation equations

To establish the basic set of perturbation equations it is convenient to introduce the gauge-invariant quantities

$$\delta^c = \delta + \frac{\dot{\rho}}{\rho} v, \quad \delta_M^c = \delta_M + \frac{\dot{\rho}_M}{\rho_M} v, \quad \delta_X^c = \delta_X + \frac{\dot{\rho}_X}{\rho_M} v, \quad \tilde{p}_X^c = \tilde{p}_X + \dot{\rho}_X v$$

(43)

as well as

$$\dot{\Theta}^c = \dot{\Theta} + \dot{\Theta} v, \quad \text{and} \quad \dot{Q}^c = \dot{Q} + \dot{Q} v.$$  

(44)

The superscript $c$ stands for comoving. All the symbols have their physical meaning on comoving hypersurfaces $v = 0$. In terms of these gauge-invariant variables the total energy and momentum conservations can be combined to yield (cf. [9])

$$\dot{\delta}^c - 2 \frac{\rho}{\rho} \delta^c + \dot{\Theta}^c \left( 1 + \frac{p}{\rho} \right) = 0.$$

(45)

The energy-density perturbations are coupled to the perturbations of the expansion scalar which are determined by the Raychaudhuri equation. At first-order this equation becomes (cf. [9])

$$\dot{\Theta}^c + \frac{2}{3} \Theta \dot{\Theta}^c + 4 \pi G \rho \delta^c + \frac{1}{a^2 \rho + p} \Delta \tilde{p}_X^c = 0.$$

(46)
Combining Eqs. (45) and (46), changing to $a$ as independent variable ($\delta' \equiv \frac{d\delta_c}{da}$) and transforming into the $k$-space, the resulting equation for the total density perturbations is

$$
\delta'' + \left[ \frac{3}{2} - \frac{15p}{2\rho} + 3' \right] \frac{\delta'}{a} - \left[ \frac{3}{2} + 12\frac{p}{\rho} - \frac{9p'^2}{2\rho^2} - 9\frac{p'}{\rho} \right] \frac{\delta}{a^2} + \frac{k^2}{a^2H^2}\hat{\delta}_X^c = 0 .
$$

(47)

Because of the scale-dependent pressure perturbation term this is not a closed equation for $\delta_c$. To clarify the role of this term in (47) we introduce the sound speed $c_s$ in the rest-frame $v = 0$ by $\hat{p}_X^c = c_s^2\hat{\rho}_X^c$. The sound speed is considered here as a free parameter. In an interacting two-component system the sound does not, in general, propagate with the adiabatic sound speed $c_{ad}$, given by $c_{ad}^2 = \dot{p}/\dot{\rho}$. Then, the first-order energy balance for the matter component takes the form

$$
\dot{\delta}_M^c + \Theta^c + c_s^2\frac{\hat{\rho}_X}{\hat{\rho}_M}\hat{\rho}_M \delta_M^c = \left( \frac{Q}{\rho_M} \right)^c ,
$$

(48)

where

$$
\left( \frac{Q}{\rho_M} \right)^c = \frac{\dot{Q}^c}{\hat{\rho}_M} - \frac{Q}{\hat{\rho}_M} \delta_M^c = \frac{\Theta^c}{\Theta} - r(\delta_M^c - \delta^c) .
$$

(49)

Since $\hat{\rho}_X^c = \hat{\rho}^c - \hat{\rho}_M^c$, we realize that the pressure perturbations can be written as

$$
\hat{\rho}_X^c = c_s^2S_M , \quad S_M \equiv D^c - \delta_M^c ,
$$

(50)

where we have introduced the fractional quantity

$$
D^c \equiv \frac{\rho^c}{\rho + p} = \frac{1 + r}{r} \delta^c \quad \leftrightarrow \quad \delta^c = \frac{r}{1 + r}D^c .
$$

(51)

It becomes obvious that via (50) the dynamics of $\delta^c$ in (47) is coupled to $S_M = D^c - \delta_M^c$. To describe the dynamics of $S_M$ we couple equations (45) and (48):

$$
\dot{S}_M + \frac{Q}{\rho_M}D^c - \frac{\hat{\rho}_X^c}{\hat{\rho}_M} = - \left( \frac{Q}{\rho_M} \right)^c ,
$$

(52)

with the right-hand side of this equation given by (49). Eliminating $\Theta^c$ in the expression (49) with the help of (45) provides us with

$$
\dot{S}_M + \frac{Q}{\rho_M}D^c + \left( \Theta - \frac{Q}{\rho_M} \right) \frac{\hat{\rho}_X^c}{\hat{\rho}_M} = \frac{Q}{\Theta\rho_M} \left[ \dot{\delta}_M^c + \left( \frac{Q}{\rho_M} + \Theta \frac{r}{1 + r} \right) D^c - \Theta r S_M \right] .
$$

(53)

After some transformations the final equation for $S_M$ is

$$
S_M'' + \frac{3}{1 + r} \left[ \left( \frac{\xi}{3} + r \right) c_s^2 - r \left( \frac{\xi}{3} - 1 \right) \right] \frac{S_M^c}{a} = \left( 1 - \frac{\xi}{3} \right) \frac{\delta''}{r} .
$$

(54)
Equation $\delta''$ is coupled to the equation $\delta'$ for $\delta c$, which together with (50) becomes
\[
\delta'' + \left[ \frac{3}{2} - \frac{15}{2} \frac{p}{\rho} + 3 \frac{p'}{\rho} \right] \frac{\delta'}{a} - \left[ \frac{3}{2} + \frac{12}{\rho} - \frac{9}{2} \frac{p^2}{\rho^2} - \frac{9}{\rho} \right] \frac{\delta c}{a^2} + \frac{k^2}{a^2 H^2} c^2 \frac{r}{1 + r} \frac{S c_M}{a^2} = 0. \tag{55}
\]
The coupled set of equations (54) and (55) is the main result of this paper. The explicit structure of the coefficients is
\[
\frac{p}{\rho} = -\frac{1}{1 + r_0 a^{-\xi}}, \quad \frac{p'}{\rho} = \frac{\xi - 1}{1 + r_0 a^{-\xi}}, \quad H = H_0 a^{-\frac{3}{2} \left( \frac{r_0 + a^\xi}{r_0 + 1} \right)^{\frac{1}{2}}}. \tag{56}
\]
The total EoS parameter approaches zero at high redshifts. The adiabatic sound speed square is positive for $\xi > 3$ and it is negative for $\xi < 3$. For $\xi = 3$ one consistently recovers the $\Lambda$CDM model with $\frac{p}{\rho} = Q = 0$. Only if the pressure perturbations of the dark energy are negligible, the equation for $\delta c$ decouples from that for $S c_M$.

D. Matter perturbations

The set (54) and (55) for $S c_M$ and $\delta c$, respectively, describes the entire perturbation dynamics of the system. With this system solved, the matter perturbations $\delta c_M$ are then obtained as the combination
\[
\delta_M = \frac{1 + r}{r} \delta c - S c_M. \tag{57}
\]
To evaluate the set (54) and (55) we have to consider its behavior in the high-redshift limit. Since $r \gg 1$ for $a \ll 1$ one has
\[
\frac{p}{\rho} \ll 1, \quad \frac{p'}{\rho} \ll 1, \quad a^2 H^2 \gg H_0^2 \quad (a \ll 1). \tag{58}
\]
Under this condition Eq. (55) reduces to
\[
\delta'' + \frac{3 \delta'}{2 a} - \frac{3 \delta c}{2 a^2} = 0 \quad (a \ll 1), \tag{59}
\]
which coincides with the corresponding equation for the Einstein-de Sitter universe. It has the growing solution $\delta c = c_1 a$ where $c_1$ is a constant. Equation (54) also decouples and reduces to
\[
S' c_M + \left( 3 - \xi + 3 c_s^2 \right) \frac{S c_M}{a} = 0 \quad (a \ll 1). \tag{60}
\]
It has the solution
\[
S c_M \propto a^{-(3-\xi+3 c_s^2)} \quad (a \ll 1), \tag{61}
\]
which is constant exactly only in the $\Lambda$CDM limit $\xi = 3$ and $c_s^2 = 0$. It decays for $\xi < 3(1 + c_s^2)$. For the matter perturbations at $a \ll 1$ this implies
\[ \delta_M \approx \delta \quad (a \ll 1) \] (62)
as expected for $r \gg 1$.

For $\xi > 3(1 + c_s^2)$ the quantity $S_M^\xi$ may also grow. But as long as it remains close to $\xi = 3$ the growth will be weaker than the growth of $\delta$ and (62) is still valid.

V. COMPARISON WITH OBSERVATIONAL RESULTS AND DISCUSSIONS

Now we look for observational consequences of the model based on the ansatz (10) both for the homogeneous background dynamics and for structure formation. Since any deviation from the standard model is accompanied by a non-gravitational coupling between DE and DM this amounts to check the viability of the existence of such type of interaction in the

FIG. 1: Two-dimensional contour lines (1$\sigma$, 2$\sigma$ and 3$\sigma$ CL) in the $\Omega_{M0}$-$\xi$ plane, based on the Union 2.1 data set. The point with the best-fit values $\Omega_{M0} = 0.275^{+0.020}_{-0.019}$ and $\xi = 3.06^{+0.43}_{-0.42}$ for the scaling model is almost indistinguishable from the point that characterizes the $\Lambda$CDM model.
dark sector and to put limits on its strength. To this purpose we shall perform \( \chi^2 \)-statistics both for SNIa and for LSS data. The \( \chi^2 \)-statistics is based on the expression

\[
\chi^2(\theta) = \sum_{i=1}^{N} \frac{[y_i - y(x_i|\theta)]^2}{\sigma_i^2},
\]

where \( y_i \) represents the observational data (SNIa or LSS) which are compared with the theoretical predictions \( y(x_i|\theta) \) with a set of parameters \( \theta \). From (63) one obtains the probability distribution function (PDF) \( P \propto \exp\left(-\frac{\chi^2(\theta)}{2}\right) \). The quantity \( \sigma_i^2 \) represents the error bars of the data.

Let us start with the 580 data points of the Union 2.1 sample [11]. This updates a previous analysis in [7] based on the Constitution data set [12]. In this case \( y \) represents the luminosity-distance modulus, which is theoretically calculated as

\[
\mu = 5 \log d_L(z) + \mu_0,
\]

with \( \mu_0 = 42,384 - 5 \log h \), where

\[
d_L = (z + 1) H_0 \int_0^z \frac{dz'}{H(z')}
\]

and \( h \) is defined by \( H_0 = 100h \text{km}s^{-1}\text{Mpc}^{-1} \). The Hubble rate \( H(z) \) is given by eq. (16) with \( a \) replaced by \( a = (1 + z)^{-1} \).

In the most general case there are four free parameters: \( \theta = (h, \Omega_{M0}, \xi, w) \). Then, marginalizing over \( h \) and minimizing the \( \chi^2 \)-function we find the best-fit values for the three
remaining free parameters. Now, the results based on the Union 2.1 data indicate an EoS parameter very close to \( w = -1 \) \[11\]. Moreover, our perturbation analysis was performed for this case as well and thus hereinafter we will use \( w = -1 \). Then we are left with the parameters \( \xi \) and \( \Omega_{M0} \). On this basis we find the two-dimensional curves in the \( \xi - \Omega_{M0} \) plane shown in FIG. 11. The continuous curves represent the confidence levels (CL) at 1\(\sigma\), 2\(\sigma\) and 3\(\sigma\). The point characterizes the best-fit values for our scaling model. It is almost indistinguishable from the point of best-fit values for the \( \Lambda \)CDM model. Note that the values for \( \xi \) are highly degenerate.

| Data                  | \( \Omega_{M0} \)       | \( \xi \)       | \( \chi^2_\nu \) | \( Q \)          |
|-----------------------|--------------------------|-----------------|-----------------|-----------------|
| SDSS(MLCS2k2)         | 0.393\(^{+0.084}_{-0.074}\) | 3.123\(^{+1.645}_{-1.322}\) | 0.851           | -0.088\(^{+0.87}_{-1.63}\) |
| SDSS (SALT II)        | 0.243\(^{+0.063}_{-0.055}\) | 3.649\(^{+1.357}_{-1.219}\) | 0.874           | -0.358\(^{+0.62}_{-1.13}\) |
| Constitution(MLCS2k2) | 0.363\(^{+0.052}_{-0.049}\) | 2.199\(^{+1.011}_{-0.932}\) | 1.098           | +0.556\(^{+0.56}_{-0.18}\) |
| Constitution (SALT II)| 0.236\(^{+0.051}_{-0.045}\) | 4.109\(^{+1.261}_{-1.131}\) | 0.990           | -0.599\(^{+0.61}_{-1.05}\) |
| Union 2.1             | 0.275\(^{+0.040}_{-0.038}\) | 3.060\(^{+0.875}_{-0.829}\) | 0.979           | -0.036\(^{+0.45}_{-0.64}\) |
| DR7                  | 0.285\(^{+0.055}_{-0.043}\) | 3.043\(^{+1.637}_{-1.627}\) | 0.718           | -0.026\(^{+0.90}_{-1.27}\) |
| DR7+Union 2.1        | 0.286\(^{+0.008}_{-0.007}\) | 3.016\(^{+0.231}_{-0.264}\) | 0.956           | -0.010\(^{+0.14}_{-0.20}\) |

**TABLE I:** Best-fit values for the different tests of the scaling model with errors at 2\(\sigma\) CL.

The source term \( Q \) is given in units of \( H_0^3 M_p^2 \), where \( M_p \) is the Planck mass.

For a more complete analysis of the SNIa data and to test the robustness of the results, we have also considered data from the samples SDSS \[13\] and Constitution \[12\] with both the fitters Multicolor Light Curve Shapes (MLCS2k2) \[14\] and Spectral Adaptive Lightcurve Template (SALT II) \[15, 16\]. All results are summarized in TABLE I. The curve in FIG. 2 shows the redshift dependence of the best-fit luminosity-distance modulus of our model compared with the data from the Union 2.1 sample.

In order to illustrate the consequences of the best-fit values of the model for the background dynamics, we plot the fractional densities of DE and DM and of the deceleration parameter as functions of the redshift in Figures 3 and 4 respectively. In FIG. 3 we also included the results of a previous analysis in \[7\] (dot dashed curves). The dashed curves represent the best-fit values for our model, the solid curves those for the \( \Lambda \)CDM model. The gray color indicates the region at 1\(\sigma\) CL . Note that the behavior of the scaling model is
FIG. 3: Fractional densities of DE and DM as functions of the redshift. The dot-dashed curves represent the best-fit values found in [7], dashed curves correspond to the best-fit values of our model based on the Union 2.1 sample and continuous curves denote the $\Lambda$CDM model. The gray color indicates the region at 1σ CL. Practically the same as that of the $\Lambda$CDM model, i.e., the current SNIa samples can not discriminate between these models. The same indistinguishability happens for the deceleration parameter as seen in FIG. 4 where we plotted $q(z)$ for our model, for the model in [7] and for the $\Lambda$CDM model. Again, the gray color indicates the region at 1σ CL.

Now let us perform an analysis of the perturbation dynamics. Here we have the sound speed square $c_S^2$ as an additional parameter. To get an idea of the role of this parameter we solved the system (54) and (55) for different values of $c_S^2$ with initial conditions provided by the limits of the model for $a \ll 1$. In this limit eq. (55) reduces to eq. (59), while the initial condition for eq. (54) is found from eq. (61) which governs the evolution of $S_M$ at early times. This procedure to choose initial conditions is similar to that described in more detail in [17, 19]. The results for the integration of the system (54)-(55) for a typical case, we have chosen here $\Omega_{M0} = 0.275$ and $\xi = 2.99$, are shown in FIG. 5 which visualizes the evolution of total density contrast, the relative density contrast and the matter density contrast as...
FIG. 4: Deceleration parameter as function of the redshift. The dot-dashed curve represents the best-fit values in [7], the dashed curve corresponds to the best-fit values of our model for the Union 2.1 sample and the continuous curve characterizes the $\Lambda$CDM model. The gray color indicates the region at $1\sigma$ CL.

functions of the scale factor for three different scales. Figures 5a, 5b and 5c present the curves for $k = 45\ hMpc^{-1}$, figures 5d, 5e and 5f those for $k = 10\ hMpc^{-1}$ and figures 5g, 5h and 5i those for $k = 0.3\ hMpc^{-1}$. Similar plots can be obtained for different values of $\Omega_{M0}$ and $\xi$.

Several features of FIG. 5 are worth discussing. For example, according eq. (61) the initial conditions for $S_M$ depend on the sound velocity square $c_S^2$. However, the behavior of $S_M$ is similar for all the cases in FIG. 5. It has constant values (or almost constant values for $k = 0.3hMpc^{-1}$) in early times and starts oscillating between $a \approx 0.08$ and $a \approx 0.12$. The amplitude of these oscillations is small and contributes only marginally to the matter density contrast $\delta_M$ (see eq. (57)). Thus, $\delta$ and $\delta_M$ (we have omitted the superscripts c in this figure) show a very similar behavior, i.e., relation (62) remains approximately valid. The evolution of both the total density contrast $\delta$ and of the matter density contrast $\delta_M$ depends crucially on $c_S^2$. Our calculations show that for higher values of the sound velocity square
FIG. 5: Total density contrast $\delta$, relative density contrast $S_M$ and matter density contrast $\delta_M$ as functions of the scale factor for three different scales. Figures 5a, 5b and 5c represent the curves for $k = 45 \, h \, \text{Mpc}^{-1}$, figures 5d, 5e and 5f those for $k = 10 \, h \, \text{Mpc}^{-1}$ and figures 5g, 5h and 5i those for $k = 0.3 \, h \, \text{Mpc}^{-1}$. We have assumed here $\Omega_{M0} = 0.275$ and $\xi = 2.99$. In all cases high values of the sound speed lead to oscillations while smaller values do not. (Note that we have omitted here the superscripts $c$.)

c_s^2$ there appear oscillations both in $\delta$ and in $\delta_M$ at about $a \approx 0.1 - 0.2$. At values of $c_s^2$ lower than a threshold value $c_{s0}^2$, i.e., for $c_s^2 < c_{s0}^2$ the oscillations disappear. We confirmed numerically that $c_{s0}^2$ is inversely proportional to $k$. For example, for $k = 45 \, h \, \text{Mpc}^{-1}$ we have $c_{s0}^2 = 1.7 \times 10^{-4}$ (see figures 5a, 5b and 5c), for $k = 10 \, h \, \text{Mpc}^{-1}$ the threshold changes to
$c_{S0}^2 = 1.8 \times 10^{-2}$ (see figures 5d, 5e and 5f) and for $k = 0.3h\, Mpc^{-1}$ the threshold value is $c_{S0}^2 = 3.2 \times 10^{-1}$ (see figures 5g, 5h and 5i).

Since with the solution of our basic system the matter density contrast $\delta_M(a)$ is known according to relation (57), we can construct the matter power spectrum ($P_k \propto |\delta_M|^2$) and study the influence of $c_s^2$ on this spectrum. In FIG. 6 we show the matter power spectrum for different values of the sound velocity. As to be expected, also the spectrum exhibits oscillations for larger values of $c_s^2$. Such type of behavior is similar to what is known from unified models of the dark sector, notably from Chaplygin-gas models. Since oscillations of the matter distribution are not observed, those models have temporarily fallen out of favor [18]. However, non-adiabatic pressure perturbations may reduce the effective sound speed to very small values which may cure the problem of oscillations [19, 20]. As a consequence, unified models continue to be discussed as potential alternatives to the $\Lambda$CDM model. In our case $c_s^2$ is a free parameter and we shall discard models for which $c_s^2$ is larger than the mentioned threshold value $c_{S0}^2$.

FIG. 6: Matter power spectrum for different values of the sound velocity. There are oscillations for large values of $c_s^2$ but not for smaller values.
Our observational analysis relies on the LSS DR7 data which represent the $y_i$ in the expression (63) for $\chi^2$, while $y$ now is the theoretically calculated matter power spectrum $P_k$. The set of free parameters is $\theta = (c_S^2, \Omega_{M0}, \xi)$. To avoid the mentioned unwanted and non-observed oscillations very small values of $c_S^2$ are required. For our calculations we take $c_S^2 \sim 10^{-4}$. Such value will ensure the absence of oscillations until $k \sim 45 \, h \, Mpc^{-1}$, equivalent to a scale of $\sim 0.02 \, h^{-1} \, Mpc$. Hence, we continue our analysis with only two free parameters, namely $\Omega_{M0}$ and $\xi$. Minimizing the $\chi^2$-function under this condition we find the best-fit values visualized in the $\Omega_{M0}$-$\xi$ plane of FIG. 7. The dashed contour lines represent the $1\sigma$ and $2\sigma$ and confidence levels. We performed also a joint analysis by using the combined SNIa + LSS data where $\chi^2 = \chi^2_{SN1a} + \chi^2_{LSS}$. The results are presented in FIG. 7 as well, where the grey region represents the $1\sigma$ and $2\sigma$ CL. The best-fit values for this analysis are also included in TABLE I. In FIG. 8 we confront the matter power spectrum with $\Omega_{M0} = 0.285$ for several values of $\xi$, including the best-fit value, with the DR7 data. Finally, with the known best-fit values we can infer the upper and lower bounds for the interaction strength. These values are presented in the outer right column of TABLE I. There is a slight tendency
to values $\xi > 3$ which corresponds to $Q < 0$, equivalent to a transfer of energy from DM to DE. But the error bars are large. Thermodynamic considerations prefer $Q$ to be positive [22]. Note that more recent data in TABLE I (Union 2.1 and DR7) prefer very low values for the interaction strength.

![FIG. 8: Matter power spectrum with $\Omega_{M0} = 0.285$ for several values of $\xi$, including the best-fit value, for the DR7 data. Note the high degeneracy concerning $\xi$.](image)

**VI. CONCLUSIONS**

With the intention to quantify and, possibly, to soften the coincidence problem, Dalal et al. [3] introduced a phenomenological parameter $\xi$ that governs the dynamics of the ratio of the energy densities of DM and DE. Independently of whether one takes the coincidence problem seriously, the resulting cosmological dynamics represents a simple, testable modification of the standard model. For $\xi = 3$ and $w = -1$ the $\Lambda$CDM model is recovered. Any combination $w + \xi/3 \neq 0$ corresponds to a non-gravitational interaction between DM and DE. Any value $\xi < 3$ is considered to make the coincidence problem less severe. This model was tested against observational data in [5–8]. However, in its original form and in the studies so far its validity is restricted to the homogeneous and isotropic background.
dynamics. Here we generalized this model to enable a study of the perturbation dynamics as well. To this purpose we replaced the scale factor in the model defining relation by a more general covariantly defined length scale which reduces to the scale factor in the appropriate limit. We performed a gauge-invariant first-order perturbation analysis which enabled us to obtain the fractional matter density perturbations as a combination from a coupled system of equations for the total and relative perturbations. Like in other DE models, the effective non-adiabatic sound velocity has to be smaller than a certain threshold value to avoid (un-observed) oscillations in the matter perturbations. The resulting matter power spectrum was confronted with data from the SDSS DR7 survey. We studied the dependence of the power spectrum on the values of the parameter $\xi$. Both the separate and the combined Bayesian analysis, including also background Union2.1 data are compatible with the $\Lambda$CDM model. The data show a slight preference for $\xi \gtrsim 3$ but with a high degeneration. For the Constitution background data the result depends on the choice of the fitter. Consequently, there is no clear indication for a deviation from the $\Lambda$CDM model or, if one wishes, for an alleviation of the coincidence problem on the basis of the presented scaling cosmology model. Finally we remark that for a more realistic study of matter clustering on smaller scales in the presence of DE and for a better understanding of the role of $c_S^2$ in structure formation a nonlinear treatment is necessary. Processes as virialization and spherical collapse which were outside the scope of the present paper have necessarily to be considered. We hope to come back to this in future work.

**Acknowledgements**

ARF acknowledges support from CAPES (Brazil). WSHR is thankful to FAPES by the grant (BPC No 476/2013) under which this work was carried out. WZ was supported be CNPq and FAPES.

[1] C.-G. Park, J. Hwang, J. Lee, and H. Noh, Phys. Rev. Lett. 103, 151303 (2009).
[2] M. Özer and M. O. Taha, Phys. Lett. B171, 363 (1986); Nucl. Phys. B287, 776 (1987); O. Bertolami, Nuovo Cimento Soc. Ital. Fis. B93, 36 (1986); K. Freese, F.C. Adams, J.A. Frieman and E. Mottola, Nucl. Phys. B287, 797 (1987); W. Chen and Y-S. Wu, Phys. Rev. D41, 695
M. S. Berman, Phys. Rev. **D46**, 2404 (1992); A. I. Arbab and A. M. M. Abdel-Rahman, Phys. Rev. **D50**, 7725 (1994); J. A. S. Lima and M. Trodden, Phys. Rev. **D53**, 4280 (1996); J. M. Overduin and F. I. Cooperstock, Phys. Rev. **D58**, 043506 (1998); J. M. Overduin, Astrophys. J. **517**, L1 (1999); M. V. John and K.B. Joseph, Phys. Rev. **D61**, 087304 (2000); O. Bertolami and P. J. Martins, Phys. Rev. **D61**, 064007 (2000); R. G. Vishwakarma, Gen. Rel. Grav. **33**, 1973 (2001); A. S. Al-Rawaf, Mod. Phys. Lett. **A14**, 633 (2001); M. K. Mak, J. A. Belinchón, and T. Harko, IJMP **D11**, 1265 (2002); W. Zimdahl and D. Pavón, Gen. Rel. Grav. **35**, 413 (2003); M. R. Mbonye, IJMP **A18**, 811 (2003); J. S. Alcaniz and J. M. F. Maia, Phys. Rev. **D67**, 043502 (2003); I. L. Shapiro, J. Solà, C. España-Bonet, and P. Ruiz-Lapuente, Phys. Lett. **B574**, 149 (2003); J. V. Cunha and R. C. Santos, IJMP **D13**, 1321 (2004); R. Opher and A. Pelinson, Phys. Rev. **D70**, 063529 (2004); R. Horvat, Phys. Rev. **D70**, 087301 (2004); P. Wang and X. Meng, Class. Quant. Grav. **22**, 283 (2005); I. L. Shapiro, J. Solà, and H. Štefančík, JCAP **0501**, 012 (2005); E. Elizalde, S. Nojiri, S. D. Odintsov, and P. Wang, Phys. Rev. **D71**, 103504 (2005); R. Aldrovandi, J. P. Beltrán Almeida, and J. G. Pereira, Grav. & Cosmol. **11**, 277 (2005); F. Bauer, Class. Quant. Grav. **22**, 3533 (2005); B. Wang, Y. Gong, and E. Abdalla, Phys. Lett. **B624**, 141 (2005); J. D. Barrow and T. Clifton, Phys. Rev. D73, 103520 (2006); B. Wang, C. Y. Lin, and E. Abdalla, Phys. Lett. **B637**, 357 (2006); A. E. Montenegro Jr. and S. Carneiro, Class. Quant. Grav. **24**, 313 (2007).

[3] N. Dalal, K. Abazajian, E. Jenkins, and A.V. Manohar, Phys. Rev. Lett. **86**, 1939 (2001).

[4] G.F.R. Ellis, Gen.Relativ.Grav. **41**, 581 (2009) (reprint of G.F.R. Ellis in: R.K. Sachs (ed.), Proceedings of the International School of Physics “Enrico Fermi", Course 47: General relativity and cosmology, pp. 104 - 182. Academic Press, New York and London (1971).)

[5] W. Zimdahl and D. Pavón, Gen.Rel.Grav. **35**, 413 (2003).

[6] D. Pavón, S. Sen and W. Zimdahl, JCAP **0405** (2004) 009.

[7] Yun Chen, Zong-Hong Zhu, J.S. Alcaniz and Yungui Gong, Astrophys.J. **711**, 439 (2010); arXiv:1001.1489.

[8] D.R. Castro, H.E.S. Velten and W. Zimdahl, *Scaling cosmology with variable dark-energy equation of state*, JCAP **1206**, 024 (2012).

[9] W.S. Hipólito-Ricaldi, H.E.S. Velten and W. Zimdahl, JCAP **0906** (2009) 016.

[10] W. Zimdahl, H.A. Borges, S. Carneiro, J.C. Fabris and W.S. Hipólito-Ricaldi, JCAP **1104**
(2011) 028.

[11] N. Suzuki et al., Astrophys. J. 746, 85 (2012).
[12] M. Hicken et al., Astrophys. J. 700, 1097 (2009).
[13] J. Sollerman et al., Astrophys. J., 703, 1374 (2009).
[14] A. G. Riess, W. H. Press and R. P. Kirshner, Astrophys. J. 473, 88 (1996).
[15] J. Guy et al., Astron. Astrophys. 443, 781 (2005).
[16] J. Guy et al., Astron. Astrophys. 466, 11 (2007).
[17] J.C. Fabris, I.L. Shapiro and J. Solá, JCAP 0702, (2007) 016.
[18] H.B. Sandvik, M. Tegmark, M. Zaldariaga and I. Waga, Phys. Rev. D 69, 123524 (2004).
[19] W.S. Hipólito-Ricaldi, H.E.S. Velten, W. Zimdahl, Phys. Rev. D 82, 063507(2010).
[20] H. A. Borges, S. Carneiro, J. C. Fabris and W. Zimdahl, Phys. Lett. B727, 37 (2013).
[21] K. N. Abazajian et al., Astrophys. J. Suppl. 182, 543 (2009), arXiv:0812.0649, 
http://www.sdss.org/dr7/
[22] D. Pavón and B. Wang, Gen.Rel.Grav. 41,1 (2009).