A FINITE ANALOGUE OF FINE’S FUNCTION $F(a,b;t)$

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ABSTRACT. We initiate a systematic development of $F_N(a,b;t)$, a finite analogue of Fine’s function $F(a,b;t)$. Our results are transformations between $F_N(a,b;t)$ and $F_N(aq^\ell,bq^m;tq^n)$, where $\ell$, $m$ and $n$ take the values 0 or 1.

1. INTRODUCTION

In [5], Nathan J. Fine extensively studied the function $F(a,b;t)$ which he defined by

$$F(a,b;t) := \sum_{n=0}^{\infty} \frac{(aq)_n}{(bq)_n} t^n, \quad \text{(1.1)}$$

where, here as well as in the rest of the paper, the following standard $q$-series notation is used:

$$(A)_0 := (A;q)_0 = 1,$$
$$(A)_n := (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \quad n \geq 1,$$
$$(A)_\infty := (A;q)_\infty = \lim_{n \to \infty} (A;q)_n, \quad |q| < 1,$$

$$(A_1,A_2,\cdots,A_m;q)_n := (A_1;q)_n(A_2;q)_n\cdots(A_m;q)_n.$$

As shown by Fine, $F(a,b;t)$ satisfies nice properties galore and which have applications in basic hypergeometric series, theory of partitions, modular forms and mock theta functions. The first chapter of Fine’s book [5] is devoted to establishing functional equations satisfied by $F(a,b;t)$, for example, [5, p. 2, Equation (4.1)]

$$F(a,b;t) = \frac{1-atq}{1-t} + \frac{(1-aq)(b-atq)}{(1-bq)(1-t)} tq F(aq,bq; tq). \quad \text{(1.2)}$$

There have been attempts to generalize the theory of $F(a,b;t)$ by considering the more general function $\sum_{n=0}^{\infty} \frac{(aq)_n(bq)_n}{(cq)_n(dq)_n} t^n$ although they have not been quite successful, see, for example, [8].

Another direction is to study finite analogues of $F(a,b;t)$. Andrews and Bell [2] studied the finite analogue

$$F(a,b,t,N) := \sum_{n=0}^{N} \frac{(aq)_n}{(bq)_n} t^n. \quad \text{(1.3)}$$

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They constructed another function using $F(a,b,t,N)$, namely, $S_{M,N}(a,b,t)$, which specializes to the series $S_{M,N}$ used by Euler in his unpublished proof of his famous pentagonal number theorem. One of the properties Andrews and Bell proved in their paper [2] Lemma 2.1 is that for $N \geq 1$,

$$F(a,b,t,N) = \frac{1 - atq}{1-t} + \frac{(1-aq)(b-atq)}{(1-bq)(1-t)} tqF(a,b,t,N) + R_{1,N}(a,b,t),$$

(1.4)

where

$$R_{1,N}(a,b,t) = \sum_{j=0}^{N} \frac{(aq)_{j} (btq)_{j} (1-(b-atq)q^{j})}{(bq)_{j+1}} \left( \frac{b-1}{1-t} \right).$$

(1.5)

Clearly, we recover (1.2) from (1.4) upon letting $N \to \infty$ in the latter since $\lim_{N \to \infty} R_{1,N}(a,b,t) = 0$.

Fine iterated (1.2) to give a proof of the famous Rogers-Fine identity [5, p. 15] given by

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(aq)_{n}(atq/b)_{n}(1-atq^{2n+1})(bt)_{n}q^{2n}}{(bq)_{n+1}(tq)_{n}}.$$  

(1.7)

In the same vein, Andrews and Bell iterated (1.2) and used the resulting identity along with Euler’s method to give a stronger version of the Rogers-Fine identity from which they then deduce the Rogers-Fine identity.

A new finite analogue of $F(a,b;t)$ was studied by the authors of [4] Section 9. It is given for $N \in \mathbb{N}$ by

$$F_{N}(a,b;t) := \sum_{n=0}^{N} \left[ \frac{N}{n} \right] \frac{(aq)_{n}(atq/bq)_{n}(1-atq^{2n+1})(bt)_{n}q^{2n}}{(bq)_{n+1}(tq)_{n}}.$$  

(1.6)

Clearly, $\lim_{N \to \infty} F_{N}(a,b;t) = F(a,b;t)$.

They naturally encountered this function in their work concerning the theory of the restricted partition function $p(n,N)$. The latter is defined as the number of partitions on $n$ whose parts are less than or equal to $N$. In this study, they derived some properties of $F_{N}(a,b;t)$ they needed in order to prove their finite analogue of a recent identity of Garvan [6 Equation (1.3)]. These properties are the partial fraction decomposition of $F_{N}(\alpha,\beta;t)$, namely,

$$F_{N}(\alpha,\beta;t) = \frac{(1-tq^{N})(aq)_{N}}{(bq)_{N}} \sum_{n=0}^{N} \left[ \frac{N}{n} \right] \frac{(b/a)_{n}(aq)_{N-n}(aq)_{na}}{(aq)_{N}(1-tq^{n})}.$$  

This generalizes Fine’s partial fraction decomposition of $F(a,b;t)$ [5, p. 18, Equation (16.3)]:

$$F(a,b;t) = \left( \frac{aq}{bq} \right)_{\infty} \sum_{n=0}^{\infty} \frac{(b/a)_{n}(aq)_{na}}{(aq)_{n}(1-tq^{n})}.$$  

(1.7)

They also derived a finite analogue of the Rogers-Fine identity [4, Lemma 9.2], that is, for $\beta \neq 0$,

$$F_{N}(\alpha,\beta;t) = (1 - tq^{N}) \sum_{n=0}^{N} \left[ \frac{N}{n} \right] \frac{(aq)_{n}(atq^{2})_{N-1}(btq^{2})^{n}q^{2n}}{(bq)_{n}(tq)_{n}(atq^{2})_{N+n}}.$$

These two properties were used to obtain the required finite analogue of Garvan’s identity [4, Theorem 1.3].
At the end of [1], it was remarked that it would be worthwhile to develop the theory of $F_N(a,b;t)$ considering the enormous impact and applications that the theory of $F(a,b;t)$ has. We set forth such a task with a goal of systematically developing the theory of $F_N(a,b;t)$.

It is to be noted that Andrews [1] has already embarked upon this task. Even though his results are expressed as transformations between 3φ2, they can be equivalently written as transformations or functional equations for $F_N(a,b;t)$. This rephrasing is done in Section 2. Then in Section 3 we present new results on $F_N(a,b;t)$ we have found so far.

It should be mentioned here that Bowman and Wesley [3] have recently obtained new transformations for Fine’s function $F(a,b;t)$ iterating what they call as Fine’s “seed” identities. A seed identity is an identity where $F(a,b;t)$ gets transformed to $F(aq,b;t)$ or $F(a,bq;t)$ or $F(a,b;q)$. One of the goals in this paper is obtain the seed identities for $F_N(a,b;t)$.

2. SOME RESULTS OF ANDREWS AND DEDUCTIONS FROM THEM

In [1], Andrews obtained a finite version of Heine’s transformation and then derived some corollaries from it. Andrews’ transformations are for the 3φ2 hypergeometric series

$$3\phi_2 \left[ \begin{array}{c} q^{-N}, \\ bq \end{array} ; q, q^{-1-N}/t \right] = F_N(a,b;t), \quad (2.1)$$

as can be easily seen from (1.6).

Thus, we can rephrase these transformations in terms of $F_N(a,b;t)$.

2.1. An identity of Andrews generalizing Equation (6.3) of [5]. From [1 Corollary 3],

$$3\phi_2 \left[ \begin{array}{c} q^{-N}, \\ abt/c, b \end{array} ; q, q^{-1-N}/c \right] = \frac{(c,t;q)_N}{(c/bt;q)_N} 3\phi_2 \left[ \begin{array}{c} q^{-N}, \\ a, b \end{array} ; q, q^{-1-N}/t \right] \quad (2.2)$$

If we let $b \to q, t \to b, c \to t q, a \to atq/b$ in the above identity, this gives

$$F_N(a,b,t) = \frac{(1-tq^N)(1-b)}{(1-bq^N)(1-t)} \sum_{n=0}^{N} \frac{(q^{-N})_n(atq/b)_n(q)_nq^n}{(tq)_n(q^{-1-N}/b)_n(q)_n}. \quad (2.3)$$

Using the elementary identities

$$\left( \frac{q^{-N}}{c} \right)_n = (-1)^n \frac{(cq^{N-n+1})_nq^{-n}}{c^n q^{Nn}}, \quad (2.4)$$

$$\left( \frac{q^{-N}}{b} \right)_n = \frac{(q)_n(b)_n q^n}{(q^{-1-N}/b)_n q^{Nn}},$$

we are led to

$$F_N(a,b,t) = \frac{(1-b)^N}{(1-t)(1-bq^N)} \sum_{n=0}^{N} \frac{N}{n} \frac{(atq/b)_n(q)_nq^n}{(tq)_n(b)_n}. \quad (2.5)$$

Note that this identity is a finite analogue of Fine’s identity [5 Equation (6.3)]:

$$F(a,b;t) = \frac{1-b}{1-t} F\left( \frac{at}{b}; t; b \right). \quad (2.6)$$
2.2. Another $\phi_2$ transformation of Andrews. Lemma 1 of [1] gives

$$3\phi_2 \left[ q^{-N}, a, b ; q, q \right] = \frac{(at)_n}{(t)_n} 3\phi_2 \left[ q^{-N}, c/b, a \ ; q, btq^N \right]$$

(2.7)

Letting $a \rightarrow q, b \rightarrow aq, c \rightarrow bq$ in the above identity and using (2.1), we get

$$F_N(a, b; t) = \frac{(tq)_N}{(t)_N} 3\phi_2 \left[ q^{-N}, b/a, q \ ; q, atq^{N+1} \right]$$

$$= (1 - tq^N) \sum_{n=0}^{N} (-1)^n \left[ N \atop n \right] \frac{(b/a)_n (qt)^n q^{n(n+1)/2}}{(btq^N)_n (tq)_n}.$$

(2.8)

Letting $b = 0$ in the above result gives a finite analogue of [5] Equation (6.1):

$$F_N(a, 0; t) = (1 - tq^N) \sum_{n=0}^{N} (-1)^n \left[ N \atop n \right] \frac{(qt)^n q^{n(n+1)/2}}{(tq)_n}.$$

(2.9)

Also, if we let $a \rightarrow 0$ in (2.8), we obtain a finite analogue of [5] Equation (12.3):

$$(1 - t)F_N(0, b; t) = (1 - tq^N) \sum_{n=0}^{N} \left[ N \atop n \right] \frac{(bt)_n q^2}{(btq^N)_n (tq)_n}.$$

(2.10)

If we let $b = t^{-1} = e^{-i\theta}$ in (2.10), we get

$$(1 - e^{i\theta})F_N(0, e^{-i\theta}; e^{i\theta}) = (1 - e^{i\theta} q^N) + \sum_{n=1}^{N} \left[ N \atop n \right] \frac{(qt)^n q^{2n}}{(1 - 2q \cos \theta + q^2) \cdots (1 - 2q^n \cos \theta + q^{2n})}.$$

(2.11)

which is a finite analogue of [5] Equation (12.32).

Also, letting $b \rightarrow t$ in (2.10) gives a finite analogue of [5] Equation (12.33):

$$(1 - t)F_N(0, t; t) = (1 - tq^N) \sum_{n=0}^{N} \left[ N \atop n \right] \frac{(qt)_n t^{2n} q^{2n}}{(tq)_n (tq)_n}.$$

Let $r$ and $m$ be positive integers, and replace $q$ by $q^m$, $t$ by $q^r$ in finite analogue of [5] eqn(12.33). Then using the definition of finite analogue of Fine’s function for the left side, We obtain the finite analogue of [5] eqn(12.331):

$$\sum_{n=0}^{N} \left[ N \atop n \right] \frac{(q^m, q^m)_{N-n} q^{mn}}{(q^r)_n (1 - q^r) \cdots (1 - q^{mn+r})}$$

$$= (1 - q^{N+r}) \sum_{n=0}^{N} \left[ N \atop n \right] \frac{(q^m, q^m)_{n} q^{mn+2n}}{(1 - q^r) (1 - q^{m+r}) \cdots (1 - q^{nm+r})}.$$
2.3. Andrews’ finite Heine transformation in terms of $F_N(a,b;t)$. Andrews’ finite Heine transformation [1, Theorem 2] in terms of $F_N(a,b;t)$ is given by

$$3\phi_2\left[\begin{array}{c}
q^{-N}, \quad c/b, \quad t ; q,q \\
q^{-1-N}/b
\end{array}\right] = \frac{(c,t;q)_N}{(b,tq;q)_N} 3\phi_2\left[\begin{array}{c}
q^{-N}, \quad a, \quad b ; q,q \\
c, \quad q^{-1-N}/t
\end{array}\right]. \tag{2.12}
$$

Let $t \to q, b \to t, c \to atq$ and $a \to b$ in the above result. Then

$$F_N(a,b;t) = \frac{(atq,q;q)_N}{(t,bq,q;q)_N} \sum_{n=0}^{\infty} \frac{(b)_n(t)_nq^n}{(atq)_n(q)_n}. \tag{2.13}
$$

Letting $N \to \infty$ gives

$$F(a,b;t) = \frac{(atq)_{\infty}(q)_{\infty}}{(t)_{\infty}(bq)_{\infty}} \sum_{n=0}^{\infty} \frac{(b)_n(t)_nq^n}{(atq)_n(q)_n}. \tag{2.14}
$$

Letting $b = 1$ in (2.13) gives

$$F_N(a,1;t) = \frac{(atq)_N}{(t)_N}. \tag{2.15}
$$

An interesting result is obtained if we multiply both sides of (2.5) by $(1-t)$ and then let $t \to 1$:

$$\lim_{t \to 1}(1-t)F_N(a,b;t) = \frac{(1-b)(1-tq^N)}{(1-bq^N)}F_N(a/b,1;b)
= \frac{(1-b)(1-tq^N)}{(1-bq^N)} \cdot \frac{(aq)_N}{(b)_N}
= \frac{(1-q^N)(aq)_N}{(bq)_N}. \tag{2.16}
$$

where in the last step, we used (2.15). Letting $N \to \infty$ in the above result we get Equation (6.31) in Fine’s book [5]:

$$\lim_{t \to 1}(1-t)F(a,b;t) = \frac{(aq)_{\infty}}{(bq)_{\infty}}.
$$

If we let $t \to 1$ in (2.10) and compare the right-hand side of the resulting identity with $a = 0$ case of (2.16), we arrive at a finite analogue of [5] Equation (12.31):

$$\frac{1}{(bq)_{\infty}} = (1-q^N) \sum_{n=0}^{\infty} \frac{[N]}{n} \frac{(q)_n(b)_nq^n}{(bq)_n(q)_n}. \tag{2.17}
$$

The $b = 1$ case of the above identity generalizes [5] Equation (12.311)].

Next, replace $a$ by $b/t$ in (2.13) and then let $b \to 0$ to get

$$F_N(b/t,0;t) = \frac{(bq)_{\infty}(q)_{\infty}}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_{n}q^n}{(bq)_n(q)_n}, \tag{2.18}
$$

which is a finite analogue of [5] Equation (7.3)].

Further, if we replace $a$ by $b/t$ in (2.9), we have

$$F_N(b/t,0;t) = \frac{(1-tq^N)}{(1-t)} \sum_{n=0}^{\infty} \frac{[N]}{n} \frac{(q)_n(-b)_nq^{n(n+1)/2}}{(tq)_n}. \tag{2.19}
$$
Letting $t \to 0$ in (2.18) and (2.19) and comparing the right-hand sides, we get
\[
\sum_{n=0}^{N} \frac{q^n}{(bq)_n(q)_n} = \frac{1}{(bq)_N(q)_N} \sum_{n=0}^{N} \left[ \frac{N}{n} \right] (q)_n (-b)^n q^{n(1+n)/2}.
\] (2.20)

This is a finite analogue of [5, Equation (7.32)]. If we now put $b = 1$ in the above identity, we get a finite analogue of [5, Equation (7.322)].

Letting $b = q^{-1/2}$ in (2.20) and then replacing $q$ by $q^2$, we obtain a finite analogue of [5, Equation (7.323)]:
\[
\sum_{n=0}^{N} \frac{q^{2n}}{(q)_{2n}} = \frac{1}{(q)_{2n}} \sum_{n=0}^{N} \left[ \frac{N}{n} \right] q^2 (-1)^n (q^2)_n q^{n^2}.
\]

When $a = 0$, (2.13) reduces to
\[
\frac{(t)_{N}(bq)_{N}}{(q)_{N}} F_N(0,b;t) = \sum_{n=0}^{N} \frac{(b)_{n}(t)_n q^n}{(q)_n},
\] (2.21)
which is a finite analogue of [5, Equation (11.3)].

A finite analogue of [5, Equation (11.4)] is obtained if we let $b = t^{-1}$ in (2.21):
\[
\frac{(t)_{N}(t^{-1})_n q^n}{(q)_{N}} F_N(0,t^{-1};t) = \sum_{n=0}^{N} \frac{(t^{-1})_{n}(t)_n q^n}{(q)_n}.
\] (2.22)

3. NEW RESULTS ON $F_N(a,b;t)$

Equation (2.2) of [5] is
\[
F(a,b;t) = 1 + \frac{t(1-aq)}{1-bq} F(aq,bq;t).
\] (3.1)

We begin with a finite analogue of this result which takes $(a,b) \to (aq,bq)$.

**Theorem 3.1.**
\[
F_N(a,b;t) = 1 + \frac{t(1-aq)(1-q^{N+1})}{(1-bq)(1-tqN)} F_N(aq,bq;t).
\] (3.2)

**Proof.**
\[
F_N(a,b;t) = \sum_{n=0}^{N} \left[ \frac{N}{n} \right] \frac{(aq)_n(t)_{N-n}(q)_n t^n}{(bq)_n(t)_n}
= 1 + \frac{(1-aq)t}{1-bq} \sum_{n=1}^{N} \left[ \frac{N}{n} \right] \frac{(q^2)_n(a^{-1}(t)_{N-n}(q)_n t^{n-1})}{(bq^2)_{n-1}(t)_n}
= 1 + \frac{(1-aq)t}{1-bq} \sum_{n=0}^{N-1} \frac{(q)_n(aq^2)_n(t)_{N-n-1} t^n}{(q)_{N-n-1}(bq^2)_n(t)_n}
= 1 + \frac{(1-aq)t}{1-bq} \frac{(1-q^N)}{(1-tq^{N-1})} \sum_{n=0}^{N-1} \frac{(q)_{N-1}(t)_{N-n-1}(aq^2)_n t^n}{(q)_{N-n-1}(t)_{N-1}(bq^2)_n}.
\] (3.3)
Using the identity \[\text{[7, p. 351, (I.11)]}\]

\[
\frac{(b, q)_N (a, q)_{N-n} q^n}{(a, q)_N (b, q)_{N-n} b^n} = \frac{(q^{1-N}/b, q)_n}{(q^{1-N}/a, q)_n}
\]  
(3.4)

with \(b \to q, a \to t\) and replacing \(N\) by \(N - 1\), we have

\[
F_N(a, b; t) = 1 + \frac{(1 - aq)(1 - q^N)t}{(1 - bq)(1 - tq^{N-1})} \sum_{n=0}^{N-1} \frac{(q^{1-N})_n (aq^2)_n q^n}{(q^{2-N}/t)_n (bq^2)_n}
\]

\[
= 1 + \frac{(1 - aq)(1 - q^N)t}{(1 - bq)(1 - tq^{N-1})} \Phi_2 \left( \begin{array}{c} q^{1-N}, \ aq^2, \ bq^2 \\ q^{2-N}/t, \ b \end{array} \right)
\]

Replace \(N\) by \(N + 1\) to get

\[
F_N(a, b; t) = 1 + \frac{(1 - aq)(1 - q^{N+1})t}{(1 - bq)(1 - tq^N)} \Phi_2 \left( \begin{array}{c} q^{-N}, \ aq^2, \ bq^2 \\ q^{1-N}/t, \ b \end{array} \right)
\]

\[
= 1 + \frac{(1 - aq)(1 - q^{N+1})t}{(1 - bq)(1 - tq^N)} F_N(aq, bq, t),
\]
(3.5)

where in the last step we used \(2.1\). \(\square\)

Our next result allows us to advance the parameter \(t\) in \(F_N(a, b; t)\) to \(tq\).

**Theorem 3.2.**

\[
F_N(a, b; t) = \frac{(1 - b)(1 - tq^{N+1})}{(1 - t)(1 - bq^{N+1})} + \frac{(1 - q^{N+1})(b - atq)}{(1 - bq^{N+1})(1 - t)} F_N(a, b; tq).
\]

**Proof.** Using \(2.3\), we have

\[
F_N(a, b; t) = \frac{(1 - b)(1 - tq^{N})}{(1 - t)(1 - bq^{N})} \sum_{n=0}^{N} \left( \frac{N}{n} \right) \frac{(atq/b)_n (q)_n (b)_{N-n} b^n}{(tq)_n (b)_N}
\]

\[
= \frac{(1 - b)(1 - tq^N)}{(1 - t)(1 - bq^N)} + \frac{(1 - b)(1 - tq^N)(1 - atq/b)_b}{(1 - t)(1 - bq^N)(1 - t)} \sum_{n=1}^{N} \left( \frac{N}{n} \right) \frac{(atq^2/b)_{n-1} (q)_n (b)_{N-n} b^{n-1}}{(tq^2)_{n-1} (b)_N}
\]

\[
= \frac{(1 - b)(1 - tq^N)}{(1 - t)(1 - bq^N)} + \frac{(1 - b)(1 - tq^N)(1 - atq/b)_b}{(1 - t)(1 - bq^N)(1 - t) (1 - tq^N)} \sum_{n=1}^{N-1} \left( \frac{N}{n} \right) \frac{(atq^2/b)_{n+1} (q)_{N-1} (atq^2/b)_{n} (b)_{N-n-1} b^n}{(tq^2)_{n} (b)_{N-1}}
\]

Let \(b \to q, a \to b\) in \(3.4\), replace \(N\) by \(N - 1\) and substitute the resulting expression in the sum on the right-hand side of the above equation so that

\[
F_N(a, b; t) = \frac{(1 - b)(1 - tq^N)}{(1 - t)(1 - bq^N)} + \frac{(1 - b)(1 - q^N)(b - atq)(1 - q^N)}{(1 - t)(1 - bq^N)(1 - bq^{N-1})(1 - t) (1 - tq^N)} \sum_{n=0}^{N-1} \left( \frac{N}{n} \right) \frac{(atq^2/b)_{n} (q^{1-N})_n q^n}{(q^{2-N}/b)_{n} (tq^N)_n}
\]

\[
= \frac{(1 - b)(1 - tq^N)}{(1 - t)(1 - bq^N)} + \frac{(1 - b)(1 - q^N)(b - atq)(1 - q^N)}{(1 - t)(1 - bq^N)(1 - bq^{N-1})(1 - t) (1 - tq^N)} \Phi_2 \left( \begin{array}{c} q^{1-N}, \ atq^2/b, \ bq \ 2-N/b \\ q^{1-N}/tq^2, \ q^{2-N}/b \end{array} \right)
\]
Theorem 3.3. \[ \text{Finally replace } a \text{ by } atq^2, b \text{ by } btq^2, t \text{ by } b \text{ in (2.2), then replace } N \text{ by } N-1 \text{ and use the resulting identity to transform the } 3\phi_2 \text{ on the right-hand side of the above equation so that} \]

\[
F_N(a,b;t) = \frac{(1-b)(1-tq^N)}{(1-t)(1-bq^N)} + \frac{(1-b)(1-tq^N)(b-atq)(1-q^N)(1-tq)(1-bq^{N-1})}{(1-t)(1-bq^N)(1-bq^{N-1})(1-tq)(1-b)(1-tq^N)} \cdot 3\phi_2 \left[ q^{1-N}, aq, q ;q,q \right]_N \sum_{n=0}^{N-1} \frac{(q^1-N)_n(aq)_n(q)_n}{(bq)_n(q)_n(q^{2-N}/tq)_n}.
\]

Finally replace \( N \) by \( N+1 \) and use (2.4) to obtain

\[
F_N(a,b;t) = \frac{(1-b)(1-tq^{N+1})}{(1-t)(1-bq^{N+1})} + \frac{(1-q^{N+1})(b-atq)(1-bq^N)}{(1-bq^{N+1})(1-bq^N)(1-t)(1-bq)(1-tq)} \sum_{n=0}^{N} \frac{(q)_N(aq)_n(q)_n(tq)_n}{(q)_N-bq_n(q)_n(q)_n(tq)_n} N_n.
\]

\[
= \frac{(1-b)(1-tq^{N+1})}{(1-t)(1-bq^{N+1})} + \frac{(1-q^{N+1})(b-atq)}{(1-bq^{N+1})(1-t)} F_N(a,b;btq).
\]

Our next result is a generalization of [5 Equation (4.3)]:

\[
F(a,b;t) = \frac{1}{1-t} + \frac{(b-a)btq}{(1-t)(1-bq)} F(a,bq;tbq).
\]

Theorem 3.3. We have

\[
F_N(a,b;t) = \frac{(1-tq^{N+1})}{(1-t)} + \frac{(1-tq^{N+1})(b-atq)}{(1-bq)(1-t)} F_N(a,bq;tbq).
\]

Proof. From (2.8),

\[
F_N(a,b;t) = \frac{1-tq^N}{1-t} + \frac{(1-tq^N)}{(1-t)} \sum_{n=0}^{N} (-1)^n \left[ \frac{N}{n} \left( \frac{b/a}{n}q_n(at)_nq^{n(n+1)/2} \right) \right] \frac{(bq)_n(tq)_n}{(q)_n-bq_n(tq)_n} \]

\[
= \frac{1-tq^N}{1-t} + \frac{(1-tq^N)(b-atq)}{(1-t)(1-bq)(1-tq)} \sum_{n=0}^{N} (-1)^n \frac{(q)_Nq_n(at)_nq^{n+2(n+1)/2}}{(q)_N-1-bq^2_n(tq^2)_n}.
\]

Now (2.4) with \( c = 1/q \) gives

\[
(q^{1-N})_n = \frac{(-1)^nq^{n(n-1)/2}(q)_{N-1-n}}{(q)_N-1q^{(N-1)n}}.
\]

Hence

\[
F_N(a,b;t) = \frac{1-tq^N}{1-t} + \frac{(1-tq^N)(b-atq)}{(1-t)(1-bq)(1-tq)} \sum_{n=0}^{N} (-1)^n \frac{(q)_Nq^{n+n+1}(bq/a)_n(1q)^n}{(q)_N-1-bq^2_n(tq^2)_n}.
\]

\[
= \frac{1-tq^{N+1}}{1-t} + \frac{(1-tq^{N+1})(b-atq)}{(1-t)(1-bq)(1-tq)} \sum_{n=0}^{N} (-1)^n \frac{(q)_Nq^{(N+1)n+n}(bq/a)_n(1q)^n}{(q)_N-1-bq^2_n(tq^2)_n}.
\]
Now let \( b \to aq, c \to bq^2, a \to q \) and replace \( t \) by \( tq \) in (2.7) to obtain
\[
\sum_{n=0}^{N} \frac{(q^{-N})_n(q^{N+1})_n(q/a)_n(at)_n}{(bq^2)_n(tq^2)_n} = \frac{(1-tq)}{(1-tq^{N+1})} F_N(a, bq; tq).
\]
Finally substitute the above equation in (3.11) to complete the proof.

In the next result, we transform \( F_N(a, b; t) \) to \( F_N(a, bq; t) \).

**Corollary 3.4.** We have
\[
F_N(a, b; t) = \frac{1-tq^{N+1}}{1-t} - \frac{(b-a)(1-tq^{N+1})t}{(1-t)(b-at)} + \frac{(b-a)(1-bq^{N+2})t}{(1-bq)(b-at)} F_N(a, bq; t).
\]  
(3.12)

**Proof.** Replace \( b \) by \( bq \) in Theorem 3.2 and then substitute the resulting expression for \( F_N(a, bq; tq) \) in (3.8).

Letting \( N \to \infty \) in Corollary 3.4 gives [5] Equation (4.4)
\[
F(a, b; t) = \frac{b}{b-at} + \frac{(b-a)t}{(1-bq)(b-at)} F(a, bq; t).
\]

Our next results relates \( F_N(a, b; t) \) with \( F_N(aq, b; t) \).

**Corollary 3.5.** We have
\[
F_N(a, b; t) = 1 - \frac{b(1-aq)(1-q^{N+1})(1-tq^{N+1})}{(1-tq)(1-bq^{N+2})(b-aq)} + \frac{(1-aq)(1-q^{N+1})(b-aqt)}{(1-tq^2)(b-aq)(1-bq^{N+2})} F_N(aq, b; t).
\]

**Proof.** Replace \( a \) by \(aq \) in Corollary 3.4 and then substitute the resulting expression for \( F_N(aq, bq; t) \) in Theorem 3.1.

Letting \( N \to \infty \) in Corollary 3.5 gives [5] Equation (4.5)
\[
F(a, b; t) = -\frac{(1-b)aq}{b-aq} + \frac{(1-aq)(b-atq)}{b-aq} F(aq, b; t).
\]

Our next theorem transforms \( F_N(a, b; t) \) to \( F_N(aq, b; tq) \).

**Theorem 3.6.**
\[
F_N(a, b; t) = 1 + \frac{(1-aq)(1-q^{N+1})(1-tq^{N+1})}{(1-tq)(1-bq^{N+2})(b-atq^2)(1-t)} \left( t - \frac{(b-aqt)}{(1-b)} \right) + \frac{(b-aqt)(1-b)}{(1-bq^{N+1})} \right) \right) + \frac{(1-aq)(1-q^{N+1})(b-aqt)(b-atq^2)}{(1-tq^2)(b-aq)(1-bq^{N+2})(1-bq^{N+1})(1-t)} F_N(aq, b; tq).
\]

**Proof.** Replace \( a \) by \(aq \) in Theorem 3.2 and then substitute the resulting expression for \( F_N(aq, b; t) \) in Corollary 3.5.

The identity in the above theorem leads to the following identity when we let \( N \to \infty \):
\[
F(a, b; t) = \frac{1-b}{1-t} \left( 1 - \frac{(b-atq)}{(b-aq)} \right) aq + \frac{(1-aq)(b-atq)(b-atq^2)}{(1-t)(b-aq)} F(aq, b; tq),
\]
which is Equation (4.6) from [5].

Our final result is the transformation between \( F_N(a, b; t) \) and \( F_N(aq, bq; tq) \).
Theorem 3.7. We have
\[
F_N(a, b; t) = 1 + \frac{(1 - at)(1 - q^{N+1})(1 - tq^{N+1})t}{(1 - bq^N)(1 - bq^{N+2})(1 - t)} + \frac{(1 - at)(1 - q^{N+1})^2(b - at)qt}{(1 - bq^N)(1 - bq^{N+2})(1 - t)}F_N(aq, bq; tq).
\]

Proof. Replace \(a\) and \(b\) in Theorem 3.2 by \(aq\) and \(bq\) respectively and then substitute the resulting expression for \(F_N(aq, bq; t)\) in Theorem 3.1.

Letting \(N \to \infty\) in Theorem 3.7 gives [5, Equation (4.1)]:
\[
F(a, b; t) = \frac{(1 - at)q}{(1 - t)} + \frac{(1 - at)(b - at)qt}{(1 - bq)(1 - t)}F(aq, bq; tq).
\]

REFERENCES

[1] G. E. Andrews, The finite Heine transformation, Bruce Landman (ed.) et al., Combinatorial number theory. Proceedings of the 3rd Integers Conference 2007, Carrollton, GA, USA, October 24–27, 2007. Berlin: Walter de Gruyter, Integers 9, Suppl., Article A1, 1–6.
[2] G.E. Andrews and J. Bell, Euler’s pentagonal number theorem and the Rogers-Fine identity, Ann. Comb. 16 (2012), 411–420.
[3] D. Bowman and S. Wesley, General Fine transformations I., Int. J. Number Theory 17 (2021), no. 2, 267–283.
[4] A. Dixit, P. Eyyunni, B. Maji and G. Sood, Unrodden pathways in the theory of the restricted partition function \(p(n, N)\), J. Combin. Theory Ser. A 180 (2021), 105423 (49 pages)
[5] N. J. Fine, Basic Hypergeometric Series and Applications, Mathematical Surveys and Monographs, Amer. Math. Soc., 1989.
[6] F. G. Garvan, Weighted partition identities and divisor sums, Ch. 12 in Frontiers in Orthogonal Polynomials and \(q\)-Series, Contemporary Mathematics and Its Applications: Monographs, Expositions and Lecture Notes: Vol. 1, M. Z. Nashed and X. Li, eds., World Scientific, 2018, pp. 239–249.
[7] G. Gasper, M. Rahman, Basic Hypergeometric series, Second Edition, (Encyclopedia of Mathematics and Its applications), 2004.
[8] A. Gupta, A study of certain transformations for a special type of \(3\phi_2\) series, Ganita, 44 no. 1-2 (1993), 87-100.

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