Aspects of model dependence of $\eta'-\eta$ complex
treated by going beyond the isospin limit

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Abstract. Exploring the extent of model dependence, we study effects of certain Ansätze for the $T$-dependence of the correction term in the QCD topological susceptibility. The one producing unwanted effects on results at $T > 0$ in the $\eta'-\eta$ complex in the usual limit of isospin symmetry, is largely cured from its peculiar behavior and brought into agreement with the other, when breaking of isospin symmetry is allowed and the realistic quark mass ratio $m_u/m_d = 0.50$ is adopted.

1 Introduction

While lattice QCD calculations now mostly agree that for the physical quark masses, high temperatures lead to smooth, crossover restoration of chiral symmetry around the pseudocritical transition temperature, $T \sim T_{Ch}$ (for example, see Refs. [1,2,3] and references therein), the fate of the related $U_A(1)$ symmetry restoration is still not clear [4,5,6,1]. Since the anomalous $U_A(1)$ breaking strongly affects the mass of the $SU(3)$ flavor-singlet pseudoscalar meson $\eta'$, and to a lesser extent (through mixing) also the mass of its fellow isoscalar $\eta$ from the $SU(3)$ flavor octet of light pseudoscalar (almost-)Goldstone bosons, the temperature dependence of the $\eta'$ and $\eta$ masses are very indicative for the temperature dependence of the anomalous breaking and of the restoration of the $U_A(1)$ symmetry of QCD.

In Ref. [7], we obtained a crossover $U_A(1)$ transition, characterized by smooth, gradual melting of anomalous mass contributions. This resulted in a significant decrease of the $\eta'$-meson mass around the chiral transition temperature $T_{Ch}$, but no decrease of the $\eta$-meson mass. Both is consistent with the present experimental data, which indicate only the possibility of a strong $\eta'$ mass drop ($\gtrsim 200$ MeV) [8], but there is no sign of any $\eta$ mass drop whatsoever.

Ref. [7] used a phenomenologically successful effective model of non-perturbative, low-energy QCD which, since its original inception [9], has been, in several variants, successfully tested in many different phenomenological applications. This includes extending to $T > 0$ [10,11,12,13,14,15,16,17,18,19,20] our studies of $U_A(1)$ breaking through $\eta'$- and $\eta$-mesons at $T = 0$. 

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However, regardless of how well-tested and reputable a model can be, there is always the issue of model and parameterization dependence of its results: is this dependence very large, or results are largely model independent, or they are reasonably robust at least in the qualitative sense? Exploring the extent of model dependence was thus announced already in Ref. [7]. Since then, we have indeed found some examples of parameterizations which led to $T$-dependence of $\eta'$ and $\eta$, and consequently of $U_A(1)$ (non-)restoration, which are even qualitatively different than in Ref. [7], as we exemplify below. We will then also show how this is cured by adopting realistic isospin breaking [21].

2 Flavorless mesons: $\eta'$, $\eta$ and $\pi^0$ beyond isospin symmetry at $T > 0$

Ref. [7] relied on our previous work [20] which had demonstrated (at $T = 0$, even model-independently) the soundness of the approximate way of introducing and combining $[10,11,12,13,14,15,16,17,18,19]$ $U_A(1)$-anomaly contributions to the masses of light quark-antiquark ($qq$) flavorless (i.e., hidden-flavor) pseudoscalar mesons, with non-anomalous, chiral-limit-vanishing contributions $M_{qq}$ ($q = u, d, s$) to these masses. The latter can be calculated by solving a consistently truncated system of pertinent Dyson-Schwinger (DS) equations; first the “gap” equations for dressed-quark propagators and then consistently, to ensure the correct chiral behavior, bound-state $q\bar{q}$ equations, typically using some effective interaction to model low-energy, nonperturbative QCD. Other non-anomalous quantities like decay constants $f_{qq}$ and condensates $\langle \bar{q}q \rangle$, can also be calculated in such DS models. In the present paper, our model choice is the same as in Ref. [7] (and as in Refs. [10,11,12,13] preceding it and [14,15,16,17,18,19] following it). The model details and parameter values used in the isosymmetric limit are most conveniently looked up in the Appendix of our Ref. [14].

Nevertheless, the main point in the present paper will be considering the effects of the quark mass model parameters breaking the isospin symmetry. Concretely, we take from Table 3 of Ref. [21], which performed the isospin asymmetric refitting of our chosen model, the variants (a) $m_u = 0.67 m_d = 4.37$ MeV, $m_s = 115.34$ MeV, and (b) $m_u = 0.5 m_d = 3.40$ MeV, $m_s = 115.61$ MeV. (The variant (b) is perfectly consistent with the ratio of the QCD Lagrangian $u$- and $d$-quark masses: $0.493 \pm 0.019$ [22].)

The total mass matrix (squared) $M^2$ of the flavorless, $q\bar{q}$ light pseudoscalars has the anomalous contribution $M^2_A$. Due to the $U_A(1)$-anomaly suppression as $1/N_c$ in the limit of the large number of colors $N_c$, $M^2_A$ is formally treated $[10,11,12,13,14,15,16,17,18,19,20,7]$ as a perturbation to the non-anomalous contribution $M^2_{NA}$, and, in the first order, simply added to it: $M^2 = M^2_{NA} + M^2_A$. In the hidden-flavor basis of the unphysical pseudoscalar states $|qq\rangle$ ($q = u, d, s$),

$$M^2_{NA} = \begin{bmatrix} M^2_{uu} & 0 & 0 \\ 0 & M^2_{dd} & 0 \\ 0 & 0 & M^2_{ss} \end{bmatrix}, \quad M^2_A = \beta \begin{bmatrix} 1 & Y & X \\ Y & Y^2 & XY \\ X & XY & X^2 \end{bmatrix}, \quad \text{where} \quad \beta = \frac{2A}{f^2_\pi}, \quad (1)$$

since $\beta$ is related to the amplitudes of the $U_A(1)$ anomaly-induced hidden-flavor sector transitions, i.e., $|qq\rangle \rightarrow |q'q'\rangle$; see, e.g., Figure 1 in Ref. [21]. It is given by the matrix elements of the anomalous mass matrix (squared): $\langle qq| M^2_A |q'q'\rangle = b_q b_{q'}$, where $b_q \equiv \sqrt{3}$ for $q = u$. However, the amplitudes for the transitions from, and into, heavier $s\bar{s}$ pairs are smaller than those of lighter flavors. As in earlier papers $[7,10,11,12,13,14,15,16,17,18,19,20]$, the effects of this breaking of the $SU(3)$ flavor symmetry for $q, q' = s$ are given by $b_s = X \sqrt{3}$, where the flavor-breaking factor $X < 1$ is given
by the ratio of the pertinent $q\bar{q}$ pseudoscalar decay constants, $X = f_{uu}/f_{ss} < 1$. Of course, $f_{uu} = f_{dd} = f_\pi$ in the limit of the $SU(2)$ isospin symmetry, which is an excellent approximation for almost all purposes in hadronic physics. Nevertheless, we shall now consider also its breaking, $m_d - m_u \neq 0$, however small. We allow unequal lightest quark mass parameters and define $Y = f_{uu}/f_{dd}$. Then, $b_d = Y \sqrt{\beta}$ is (at least at $T = 0$) just slightly smaller than $b_u = \sqrt{\beta}$, just like $M_{uu} < M_{ud} \equiv M_{\pi \pm} < M_{dd}$, even though all these differences are very small even on the scale of the pion masses, which are the smallest of hadronic masses. But due to $Y \neq 1$, it is no longer possible to decouple $\pi^0$ as a pure isovector without any $U_A(1)$ anomaly influence and $s\bar{s}$ admixture, and to isolate $\eta' - \eta$ complex in their simple $2 \times 2$ mass matrix; now one has to diagonalize the $3 \times 3$ matrix $M^2 = M^2_{NA} + M^2_{A(1)}$ at every $T$.

The reason why we anyway consider the breaking of the isospin symmetry will become clear further below. It is related to the way $U_A(1)$ symmetry breaking is tied to the chiral symmetry breaking in our approach [20,7,14,21] – through the light-quark expression for the QCD topological susceptibility $\chi$ and the corresponding full QCD topological charge parameter $A$,

$$A = \frac{1}{1 + \chi} \sum_{q=u,d,s} \frac{1}{m_q} \frac{1}{\langle qq \rangle}, \quad \text{with} \quad \chi = \frac{-1}{\sum_{q=u,d,s} \frac{1}{m_q} \langle qq \rangle} + C_m,$$

where $A$ is the quantity which, in Shore’s generalization [23,24] of the Witten-Veneziano relation [25,26], takes place of the Yang-Mills topological susceptibility

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**Fig. 1.** Temperature dependence of the condensates $\langle \bar{q}q \rangle$ ($q = u, d, s$) for the model quark mass parameters $m_q$ in the variants [21] (a) and (b) – see text. They exhibit smooth, crossover $T$-dependence, in contrast to the massless, chiral-limit condensate $\langle \bar{q}q \rangle_0$. 

(a) $m_u/m_d = 2/3$, (b) $m_u/m_d = 1/2$
Fig. 2. Temperature dependence of $q\bar{q}$ pseudoscalar decay constants for $u$- and $d$-quark mass parameters breaking isospin symmetry in the variants [21] (a) and (b), see text.

$\chi_{YM}$. At $T = 0$, $A = \chi_{YM} + \mathcal{O}(1/N_c)$, so that inserting everywhere in $A$ [2] the chiral-limit condensate $(\bar{q}q)_0$ instead of the massive condensates $(\bar{q}q)$ $(q = u, d, s)$, returns the (inverted) Leutwyler-Smilga relation [27], used in our original [13] extension of our $\eta' - \eta$ approach to $T > 0$. But then the abrupt drop to zero of the chiral-limit condensate $(\bar{q}q)_0(T)$ at $T = T_{Ch}$ (see Fig. 1) causes an equally abrupt drop of not only $\eta'$-mass, but also of $\eta$-mass, for which no indication has been found. Contrary to that, Shore’s generalization [23,24] mandates employing the “massive” condensates in Eqs. (2), which gave us, in our earlier isosymmetric $\eta' - \eta$ study [7], the smooth crossover $T$-dependence of $\eta$ and $\eta'$, with the empirically supported [8] strong drop of the $\eta'$-mass, but no drop of the $\eta$-mass.

The isosymmetric analysis [7] was performed with the values of model parameters which had been fixed years ago in various $T = 0$ applications. Thus, the only new parameterization was trying Ansätze for the unknown $T$-dependence of the small (being of higher order in small quark masses) correction term $C_m$ in Eq. (2). But, its $T = 0$ value $C_m(0)$ is fixed, as it always was [13,20,21], by using Shore’s [23] $1/N_c-$ approximation $A = \chi_{YM}$. (As before, we adopt the lattice result $\chi_{YM} = (191 \text{ MeV})^4$ [28].) The results in Ref. [7] were quite insensitive to the tried parameterizations of $C_m(T)$; i.e., the resulting $T$-dependences of the $\eta'$ and $\eta$ obtained in Ref. [7] were rather similar, although the Ansätze used for $C_m(T)$ were quite different: the constant $C_m(T) \equiv C_m(0)$ vs. the fit smoothly joining $C_m(0)$ at $T \lesssim T_{Ch}$ with the power-law $T^{-5.17}$ when $\chi(T)$ enters this regime.
Fig. 3. Temperature dependence of the masses of $\eta'$, $\eta$ and $\pi^0$ for $C_m(T)$ Ansätze with the $\langle \bar{s}s \rangle$ condensate and $\delta = 1$. The isospin symmetry breaking variants (a) and (b) are described in the text, and (c) is the usual isosymmetric variant used also in Ref. [7].

Nevertheless, in our subsequent exploration of model dependences, surprising (at first) were some of the effects of employing $C_m(T)$ Ansätze of the form (3) from Ref. [13].

$$C_m(T) = C_m(0) \left( \frac{\langle qq \rangle(T)}{\langle qq \rangle(0)} \right)^{\delta},$$

with the cases considered presently:

$$\begin{cases} q = s, \delta = 1 & \text{in Fig. [5]} \\ q = u, \delta = 1/5 & \text{in Fig. [4]} \end{cases}$$

but with the massive condensates of the lightest flavors.
(a) \( m_u/m_d = 2/3 \), (b) \( m_u/m_d = 1/2 \), (c) \( m_u/m_d = 1 \)

Fig. 4. Temperature dependence of the masses of \( \eta' \), \( \eta \) and \( \pi^0 \) for \( C_m(T) \) \( \frac{\langle \bar{u}u \rangle}{\text{condensate}} \) and \( \delta = 1/5 \). The isospin symmetry breaking variants \( \frac{\langle \bar{u}u \rangle}{\text{condensate}} \) (a) and (b) are described in the text, and (c) is the usual isosymmetric variant used also in Ref. [7].

Namely, if we start with the most massive and most slowly melting one, \( \langle \bar{s}s \rangle(T) \) and \( \delta = 1 \) in Eq. (3), the thick and thin solid curves in Fig. 5 (the case (c)) give the respective \( T \)-dependences of \( \eta' \) and \( \eta \) masses in the isosymmetric limit. The change with respect to Ref. [13] is qualitative and even drastic, but it has been already explained: it just reflects going from sharp transitions of both \( \eta' \) and \( \eta \) (dictated by \( \langle \bar{q}q \rangle_0(T) \) dropping sharply to 0 at \( T = T_{Ch} \)), to the smooth crossover transition due to condensates of massive quarks. The difference with respect to Ref. [7] also exists,
but it is just quantitative (at least in the most interesting region, around $T \sim T_{\text{Ch}}$), e.g., just somewhat smaller $q'$-mass drop, so no significant surprises here, in Fig. 3.

Nevertheless, when Eq. (1) employs the lightest and the fastest melting one, $(\bar{u}u)(T)$, which is “closest” to the chiral-limit condensate $(\bar{q}q)_{\text{u}}(T)$ used in Ref. [13], we encounter the blow-up $q'$-mass behavior unless the exponent parameter $\delta$ is an order of magnitude below 1, although Ref. [13], like the case with $(\bar{s}s)(T)$ in the previous paragraph, had no problem with values as high as $\delta = 1$. Thus we employ much smaller $\delta = 1/5$ in Fig. 4, but the thick solid curve, depicting $q'$ mass in the isospin limit (again denoted as the (c)-case), shows significantly smaller drop around $T_{\text{Ch}}$ than for the $C_m(T)$ parameterizations in Ref. [7].

The blow-up $q'$-mass behavior for Eq. (3) with $(\bar{a}u)(T)$ is reminiscent of that encountered in Ref. [11], due to the delayed melting of the $U_A(1)$-anomalous contribution. The cure should thus be sought in faster melting of the leading term of the QCD topological susceptibility $\chi(T)$ which determines the anomalous mass contributions in the $\eta'$-$\eta$ complex. As announced in the Introduction, this can be achieved by going realistically out of the isospin symmetry limit. Namely, the expressions (2) for $\chi(T)$, and consequently for $A(T)$, are dominated by the harmonic average of the products of the light quark masses and condensates, and a harmonic average is dominated by its smallest term. Here, it is the product $m_u(\bar{u}u)$. While the difference between the lightest condensates $(\bar{u}u)$ and $(\bar{d}d)$ is barely noticeable in Fig. 1 at small $T/T_{\text{Ch}}$, the lighter condensate $(\bar{u}u)(T)$ melts noticeably faster at $T \gtrsim T_{\text{Ch}}$. Even more important is multiplying $(\bar{u}u)$ by $m_u$ now around two times smaller than $m_d$. Thus, the isospin breaking can significantly affect the $T$-dependence of the $U_A(1)$ anomaly mass contribution through the harmonic averages in Eqs. (2).

Indeed, the dash-dotted curves and the dashed curves in both Figs. 3 and 4 correspond, respectively, to the intermediate case (a) $m_u = \frac{2}{3} m_d$ and to the case (b) of the quite realistic ratio $\frac{22}{22}$ of the lightest quark mass parameters, $m_u = \frac{1}{3} m_d$. Thus, this improvement of the model not only brought in a good agreement the results from two different present Ansätze (3), but also in reasonable agreement with the previous work [7].

3 Summary and discussion

Although it may seem puzzling at first, it is not difficult to understand why the “dominance” (actually, exclusive presence) of the fastest-melting condensate $(\bar{u}u)$ in the Ansatz for the correction term $C_m(T)$ delayed the melting of the anomalous contribution. The point is that $C_m$ is always negative [13,7], so that its particularly quick decrease left too much of the positive leading term of $\chi(T)$, and hence of $A(T)$. This gets divided by $f_2(T)^2$, which falls with $T$ faster, blowing up the $\eta'$ mass. However, this is avoided by introducing appropriate breaking of isospin symmetry.

Namely, we have also explained how refining [21] our model by letting the $u$-quark mass parameter be a half of $d$-quark one, in accordance with data [22], can compensate the effects of dominance of $(\bar{u}u)$ in the Ansatz for $C_m(T)$ – because $m_u(\bar{u}u)(T)$, being the smallest, will then dominate the harmonic averages in $\chi(T)$ and $A(T)$ (2), and they decrease almost like with the previous Ansatz, and similarly to Ref. [7].

The results from the two different Ansätze (3) were brought in a good agreement by taking $m_u = \frac{2}{3} m_d$ [22] since in our approach the mass of $\eta'$ has the lower limit [13,7] at every temperature $T$, and it is $M_{ss}(T)$, the non-anomalous mass of the $s\bar{s}$ pseudoscalar. When $m_u = \frac{1}{3} m_d$, the both Ansätze (3) are not very far from saturating this limit a little above $T_{\text{Ch}}$, hence mutual similarity in the case (b).
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