REPRESENTATION OF INFINITELY DIFFERENTIABLE FUNCTIONS BY DIRICHLET SERIES

I.Kh. MUSIN

The problem of representation of elements of weighted space of infinitely differentiable functions on real line by exponential series is considered.

1. Introduction. Let $\alpha > 1$ and $\psi : \mathbb{R} \to [0, \infty)$ be a convex function satisfying the conditions:
   1. $\exists A_\psi > 0 \ \forall x_1, x_2 \in \mathbb{R}$
      \[|\psi(x_1) - \psi(x_2)| \leq A_\psi(1 + |x_1| + |x_2|)^{\alpha - 1}|x_1 - x_2|;\]
   2. $\lim_{x \to \infty} \frac{\psi(x)}{|x|} = +\infty.$

   Let $\mathcal{M}$ be a set of increasing sequences of numbers $L = (L_n)_{n=0}^{\infty}$ with $L_0 = 1$ satisfying the following conditions:
   $i_1). \ L_n^2 \leq L_{n-1}L_{n+1} \ \forall n \in \mathbb{N};$
   $i_2). \exists H_1 = H_1(L) > 0, \ H_2 = H_2(L) > 0 : \ \forall n \in \mathbb{Z}_+\ L_n \geq H_1H_2^n n!;$
   $i_3). \ \forall s > 1 \ \lim_{n \to +\infty} \left(\frac{L_{[sn]}}{L_n^s}\right)^{\frac{1}{s}} > 1;$
   $i_4). \ \forall \delta > 0 \ \exists p_\delta = p_\delta(L) > 0 \ \exists t_\delta = t_\delta(L) > 1 : \ \forall n \in \mathbb{Z}_+$
      \[\sup_{m \in \mathbb{N}} \frac{L_{m+n}}{L_m(1 + \delta)^m} \leq p_\delta t_\delta^n L_n.\]
It follows from \(i_4\) that for any sequence \((L_n)_{n=0}^{\infty}\) from \(\mathcal{M}\)

\[
\lim_{n \to \infty} \left( \frac{L_{n+1}}{L_n} \right)^{\frac{1}{n}} = 1.
\]  

(1)

We fix an arbitrary sequence \(M = (M_k)_{k=0}^{\infty} \in \mathcal{M}\).

Let \(w(r) = \sup_{k \in \mathbb{Z}_+} \frac{r^k}{M_k}, \ r > 0, \ w(0) = 0\). It is easy to see that \(w\) is continuous for \(r \geq 0\). Also \(w(r) = 0\) for \(r \in [0, M_1]\). From this and \(i_2\) it follows that there exists \(A_w > 0\) such that \(w(r) \leq A_w r, \ r \geq 0\). Clearly, \(w(|z|)\) is a subharmonic function in the complex plane.

Fix \(\sigma > 0\). Let \(\{\varepsilon_m\}_{m=1}^{\infty}\) be an arbitrary decreasing to zero sequence of positive numbers. Let \(\varphi(x) = \sup_{y \in \mathbb{R}} (xy - \psi(y)), \ \theta_m(x) = \exp(\varphi(x) - m \ln(1 + |x|)), \ x \in \mathbb{R}, m \in \mathbb{N}\). Let

\[
G_m = \{ f \in \mathcal{E}(\mathbb{R}) : p_m(f) = \sup_{x \in \mathbb{R}, k \in \mathbb{Z}_+} \frac{|f(k)(x)|}{(\sigma + \varepsilon_m)^k M_k \theta_m(x)} < \infty \}, \ m \in \mathbb{N}.
\]

We let \(G = \bigcap_{m=1}^{\infty} G_m\) and endow this vector space with its natural projective limit topology.

Let \(w_m(|z|) = w((\sigma + \varepsilon_m)^{-1}|z|), \ z \in \mathbb{C}, m \in \mathbb{N}\). Let

\[
P_m = \left\{ f \in H(\mathbb{C}) : \|f\|_m = \sup_{z \in \mathbb{C}} \frac{|f(z)|}{\exp(\psi(\Im z) + w_m(|z|))} < \infty \right\}, \ m \in \mathbb{N}.
\]

Let \(P\) be the union of these normed spaces. The vector space \(P\) endowed with a topology \(\tau\) of inductive limit of the spaces \(P_m\) is denoted by \(P_\tau\).

For \(T \in G^*\) we define the Fourier-Laplace transform \(\hat{T}\) of \(T\) by \(\hat{T}(z) = T(e^{-ixz}), \ z \in \mathbb{C}\).

Since the sequence \(M\) satisfies conditions \(i_1), i_2\) and (1) then the following theorem holds.

**Theorem A.** The Fourier-Laplace transform establishes topological isomorphism of the spaces \(G^*\) and \(P_\tau\).

In case \(\sum_{k=0}^{\infty} \frac{M_k}{M_k+1} < \infty\) theorem A is proved in [2]. In general case theorem A is obtained in [3].
In this paper we announce our result on representation of functions from $G$ by exponential series. We pay special attention to examples and properties of the sequences satisfying conditions $i_1) - i_4)$ and to properties of functions connected with these sequences.

For weighted spaces of infinitely differentiable functions on real line similar to $G$ the problem was not considered earlier.

**2. Examples of sequences belonging to class $M$.**

First introduce the class $V$ of nonnegative convex increasing functions $v$ defined on $[0, \infty)$ with $v(0) = 0$ and such that:

$V_1$. $\exists A_v \in \mathbb{R} \exists B_v \in \mathbb{R} : \forall x \geq 1 \ v(x) \geq x \ln x + A_v x + B_v$;

$V_2$. $\forall s > 1 \exists \eta_s = \eta_s(v) > 0 \exists m_s = m_s(v) \in \mathbb{R} : \forall x \geq 0 \ v(sx) \geq sv(x) + \eta_s x + m_s$;

$V_3$. $\forall \varepsilon > 0 \exists a_\varepsilon = a_\varepsilon(v) > 0 \exists b_\varepsilon = b_\varepsilon(v) \in \mathbb{R} : \forall y \geq 1$

$$\sup_{x \geq 1} (v(x+y) - v(x) - \varepsilon x) \leq v(y) + a_\varepsilon y + b_\varepsilon.$$

Clearly, for any increasing function $v$ on $[0, \infty]$ satisfying condition $V_3$ we have

$$\lim_{x \to +\infty} \frac{v(x+1) - v(x)}{x} = 0. \quad (2)$$

**Proposition 1.** Let $u : [0, \infty) \to [0, \infty)$ be a convex increasing twice continuously differentiable function such that:

1. $\lim_{x \to +\infty} \frac{u(x+1) - u(x)}{x} = 0$;

2. there exists a constant $C > 0$ such that $u''(x) \leq C x^{-1}$ for $x \geq 1$.

Then for any $\varepsilon \in (0, C)$ exists a constant $Q_\varepsilon$ such that for all $y \geq 1$

$$\sup_{x \geq 1} (u(x+y) - u(x) - \varepsilon x) < u(y) + \left( C \ln \frac{2C}{\varepsilon} + \frac{5C^4}{4} \right) y + Q_\varepsilon.$$

**Proof.** Let $y \geq 1$. Then $y \in [N, N+1)$ for some $N \in \mathbb{N}$. Let $\varepsilon \in (0, C)$. We shall find the upper estimate of $\sup_{x \geq 1} (u(x+y) - u(x) - \varepsilon x - u(y))$.

From the first condition on $u$ one can find a constant $q_\varepsilon > 0$ such that $u(x+1) < u(x) + \frac{\varepsilon x}{2} + q_\varepsilon, x \geq 0$. Further, we have

$$\sup_{x \geq 1} (u(x+y) - u(x) - \varepsilon x - u(y)) \leq \sup_{x \geq 1} (u(x+N+1) - u(x) - \varepsilon x - u(N)) \leq$$
\[
\sup_{x \geq 1} (u(x + N) - u(x) - \frac{\varepsilon x}{2} - u(N + 1)) + \varepsilon N + 2q_\varepsilon.
\]

Note that \( u(x + N) - u(x) = \sum_{k=1}^{N} (u(x + k) - u(x + k - 1)) \leq \sum_{k=1}^{N} u'(x + k); \)

\( u(N + 1) = \sum_{k=1}^{N} (u(k + 1) - u(k)) + u(1) > \sum_{k=1}^{N} u'(k). \) Consequently,

\[
u(x + N) - u(x) - u(N + 1) < \sum_{k=1}^{N} (u'(x + k) - u'(k)) = \end{equation}

\[
= \sum_{k=1}^{N} \int_{k}^{x+k} u''(t) \, dt \leq C \sum_{k=1}^{N} \int_{k}^{x+k} \frac{dt}{t} = C \sum_{k=1}^{N} \ln \left(1 + \frac{x}{k}\right).
\]

Thus,

\[
\sup_{x \geq 1} \left( u(x + N) - u(x) - u(N + 1) - \frac{\varepsilon x}{2} \right) \leq \sup_{x \geq 1} \left( C \sum_{k=1}^{N} \ln \left(1 + \frac{x}{k}\right) - \frac{\varepsilon x}{2} \right) \leq \sum_{k=1}^{N} \sup_{x \geq 1} \left( C \ln \left(1 + \frac{x}{k}\right) - \frac{\varepsilon x}{2N} \right) = NC \ln \frac{2C}{\varepsilon} + CN \ln N - C \sum_{k=1}^{N} \ln k + \frac{\varepsilon(N + 1)}{4}.
\]

Since \( \sum_{k=1}^{N} \ln k \geq \int_{1}^{N} \ln x \, dx = N \ln N - N + 1 \) then

\[
\sup_{x \geq 1} \left( u(x + N) - u(x) - u(N + 1) - \frac{\varepsilon x}{2} \right) \leq NC \ln \frac{2C}{\varepsilon} - C + \frac{\varepsilon(N + 1)}{4}.
\]

Hence, for \( y \geq 1 \)

\[
\sup_{x \geq 1} (u(x + y) - u(x) - \varepsilon x - u(y)) < \left( C \ln \frac{2C}{\varepsilon} + \frac{5C}{4} \right) y - C + \frac{\varepsilon}{4} + 2q_\varepsilon.
\]

This proves the lemma.

It is easy to see that for arbitrary sequence \( (L_k)_{k=0}^{\infty} \in \mathcal{M} \) an increasing function \( v \) on \([0, \infty]\) such that \( v(k) = \ln L_k, \ k \in \mathbb{Z}_+, \) satisfies conditions
$\forall 1-\forall 3$. Thus, for arbitrary sequence $L = (L_k)_{k=0}^{\infty} \in \mathcal{M}$ an increasing convex function $v$ on $[0, \infty]$ such that $v(k) = \ln L_k$, $k \in \mathbb{Z}_+$, is in $\mathcal{V}$. In particular, function $v_L$ such that $v_L(tk + (1-t)(k+1)) = t \ln L_k + (1-t) \ln L_{k+1}$, where $k \in \mathbb{Z}_+, t \in [0,1]$, is in $\mathcal{V}$.

Obviously, if increasing function $v$ on $[0, \infty]$ satisfies the condition $\forall 3$ then the sequence $(\exp(v(k)))_{k=0}^{\infty} \in \mathcal{V}$ satisfies the condition $i_4$. Also it is clear that if increasing function $v$ on $[0, \infty]$ satisfies the conditions $\forall 2$ and (2) then the sequence $(\exp(v(k)))_{k=0}^{\infty}$ satisfies the condition $i_3$. Thus, for each function $v \in \mathcal{V}$ we have $(\exp(v(k)))_{k=0}^{\infty} \in \mathcal{M}$.

**Proposition 2.** Let $v$ satisfies conditions $\forall 1, \forall 2$ and conditions of Proposition 1. Then the sequence $(\exp(v(k)))_{k=0}^{\infty} \in \mathcal{M}$.

Now we give some examples of sequences belonging to $\mathcal{M}$.

1. Consider the function $v_1(x) = \rho x \ln(x+1), \rho \geq 1, x \geq 0$. $v_1$ is increasing and nonnegative on $[0, \infty)$, $v_1(0) = 0$. It is easy to verify that $v_1$ satisfies to conditions $\forall 1, \forall 2$ and to the first condition of Proposition 1. Since $v_1''(x) = \frac{\rho}{x+1} + \frac{\rho}{(x+1)^2} > 0$ for $x \geq 0$ then $v_1$ is a convex function on $[0, \infty)$. Obviously the second condition of Proposition 1 holds. By Proposition 2 the sequence $M^* = ((n+1)^{\rho n})_{n=0}^{\infty}$ is in $\mathcal{M}$.

2. Let $v_2(x) = \rho \ln \Gamma(x+2), \rho \geq 1, x \geq 0$, where $\Gamma(x)$ is Euler’s Gamma Function. From the definition and properties of Gamma Function [5], [6, pp. 755, 763] it follows that $v_2(0) = 0$ and $v_2$ is increasing and convex on $[0, \infty)$. Using the Stirling’s formula it is easy to verify that the conditions $\forall 1, \forall 2$ are fulfilled. The first condition of Proposition 1 is fulfilled obviously. Since $(\ln \Gamma(x+2))'' = \sum_{k=2}^{\infty} \frac{1}{(x+k)^2}$ (see, for example, [5], [6, p. 774, formula (28)]) then $v_2''(x) < \rho \int_{1}^{\infty} \frac{dt}{(x+t)^2} = \frac{\rho}{x+1}$. Hence, the second condition of Proposition 1 is fulfilled too. By Proposition 1 the sequence $(\Gamma^\rho(n+2))_{n=0}^{\infty}$ belongs to $\mathcal{M}$.

3. For function $v_3(x) = (x+1) \ln(x+1) \arctg(x+1)$ considered on $[0, \infty)$ we have $v_3(0) = 0$ and $v_3'(x) = (\ln(x+1) + 1) \arctg(x+1) + \frac{1}{x+1}$.
\[
\frac{(x+1)\ln(x+1)}{1+(x+1)^2} > 0 \text{ for } x \geq 0. \text{ Hence, } v_3 \text{ is a nonnegative increasing function on } [0, \infty). \text{ Obviously, condition } \mathcal{V}_1 \text{ for } v_3 \text{ holds. Since } v_3''(x) = \frac{\arctg(x+1) + 2 \ln(x+1)}{x+1} + \frac{2}{1+(x+1)^2} > 0 \text{ for } x \geq 0 \text{ then } v_3 \text{ is convex on } [0, \infty). \text{ Conditions of Proposition 1 are fulfilled since } \lim_{x \to +\infty} \frac{v'(x)}{x} = 0 \text{ and } v_3''(x) < \frac{6}{x} \text{ for } x \geq 1. \text{ Next, for all } s, x > 1
\]

\[
v_3(sx) - sv_3(x) = sx \ln(sx+1)\arctg(sx+1) - sx \ln(x+1)\arctg(x+1) + \ln(sx+1)\arctg(sx+1) - s \ln(x+1)\arctg(x+1) \geq \frac{\pi sx}{4} \ln \frac{s+1}{2} + \frac{\pi}{4} \ln(sx+1) - \frac{\pi s}{2} \ln(x+1).
\]

So condition \(\mathcal{V}_2\) for \(v_3\) is fulfilled. By Proposition 2 the sequence \((n+1)\arctg(n+1)\)\(\in\mathcal{M}\).

3. Auxiliary results. In this section \(v\) is an arbitrary function in \(\mathcal{V}\) such that \(M_k = \exp(v(k))\), \(k \in \mathbb{Z}_+\). As we know \(v\) satisfies conditions \(\mathcal{V}_1 - \mathcal{V}_3\). Note that conditions \(\mathcal{V}_2, \mathcal{V}_3\) impose some conditions on growth of \(v\). For example, from \(\mathcal{V}_2\) it follows that for some \(a > 0, b, c \in \mathbb{R}\) depending on \(v\) we have \(v(x) > ax \ln x + bx + c, \ x \geq 1\). \(\mathcal{V}_3\) implies that for any \(\varepsilon > 0, x, y \geq 1\)

\[
v(x+y) \leq v(x) + \varepsilon x + v(y) + a_\varepsilon y + b_\varepsilon, \quad (3)
\]

where the numbers \(a_\varepsilon > 0, b_\varepsilon\) depend on \(v\) and \(\varepsilon\). In particular, for any \(x \geq 1, \varepsilon > 0\) \(v(2x) \leq 2v(x) + (a_\varepsilon + \varepsilon)x + b_\varepsilon\). From this inequality it easily follows that

\[
v(x) \leq (2v(1) + a_\varepsilon + 2b_\varepsilon + \varepsilon)x + \frac{(a_\varepsilon + \varepsilon)x \ln x}{\ln 2} - b_\varepsilon. \quad (4)
\]

Set

\[
h_v(s) = \lim_{x \to +\infty} \left(\frac{v(x)}{x} - \frac{v(sx)}{sx}\right), \ s > 0.
\]

Lemma 1. Function \(h_v\) has the following properties:

1. for all \(s \in (0, +\infty)\) \(-\infty < h_v(s) < +\infty\);
2. \(h_v(s) > 0\) for \(s \in (0, 1)\) and \(h_v(s) < 0\) for \(s > 1\);
3. \( h_v \) is nonincreasing in \((0, \infty)\);
4. \( \lim_{s \to 0, s > 0} h_v(s) = +\infty \);
5. \( h_v \) is continuous at the point \( s = 1 \);
6. for any \( s > 0 \) \( h_v(s) + h_v(s^{-1}) \leq 0 \).

**Proof.** First note that since \( v \) is convex and \( v(0) = 0 \) then the function \( v(x) \) is nondecreasing on \((0, \infty)\). So \( h_v(s) \leq 0 \) for \( s > 1 \) and \( h_v(s) \geq 0 \) for \( s < 1 \).

Let \( s > 1 \). Then \( s \in (N, N + 1] \) for some \( N \in \mathbb{N} \). From (3) we have for all \( x \geq (s - N)^{-1}, \varepsilon > 0 \)

\[
v(sx) \leq Nv(x) + \varepsilon N \frac{a_s N (2s - N - 1)}{2} + N b_s + v((s - N)x).
\]

From this taking into account that \( v(tx) \leq tv(x) \) for all \( t \in [0, 1], x \geq 0 \), we get

\[
v(sx) \leq sv(x) + \left( \varepsilon + \frac{a_s (2s - N - 1)}{2} \right) sx + b_s s, \quad (5)
\]

for all \( N \in \mathbb{N}, s \in (N, N + 1], x \geq (s - N)^{-1}, \varepsilon > 0 \). In particular, one can find a constant \( \tilde{c}_s > 0 \) such that for all \( x \geq 0 \)

\[
v(sx) \leq sv(x) + \left( \varepsilon + \frac{a_s s}{2} \right) sx + b_s s + \tilde{c}_s. \quad (6)
\]

Using the inequality (6) we obtain \( h_v(s) \geq -\varepsilon - 0.5 a_s s \) for all \( s > 1, \varepsilon > 0 \).

From the representation

\[
h_v(s) = \lim_{x \to +\infty} \left( \frac{v(s^{-1}x)}{s^{-1}x} - \frac{v(x)}{x} \right), \quad s > 0, \quad (7)
\]

and the inequality (6) we have \( h_v(s) \geq \varepsilon + 0.5 a_s s^{-1} \) for all \( s \in (0, 1), \varepsilon > 0 \).

Since by the definition \( h_v(1) = 0 \) then the first property is completely proved.

From the condition \( V2 \) it follows that \( h_v(s) < 0 \) for \( s > 1 \). Using the representation (7) and the condition \( V2 \) we get \( h_v(s) > 0 \) for \( s \in (0, 1) \).

The third property of \( h_v \) follows from nonincreasing of \( \frac{v(x)}{x} \) on \((0, \infty)\).
Since \( v \) satisfies the condition \( \mathcal{V}2 \) then one can find numbers \( \eta_s(v) > 0, m_s(v) \) such that \( \forall x \geq 0, s > 1 \) \( v(sx) \geq sv(x) + \eta_s(v)x + m_s(v) \).

\( \text{From this we have for all } s > 1, x \geq 0, n \in \mathbb{N} \) \( v(s^n x) \geq s^n v(x) + \eta_s(v) ns^{n-1} x + m_s(v)(s^n - 1)(s - 1)^{-1} \). Consequently, \( h_v(s^{-n}) \geq s^{-1} \eta_s n \) for all \( s > 1, n \in \mathbb{N} \).

\( \text{From this and nonincreasing of function } h_v \) we obtain the fourth property of \( h_v \).

We shall prove that function \( h_v(s) \) is continuous at \( s = 1 \). Let \( \varepsilon > 0 \) be arbitrary. Using (5) we have \( h_v(s) \geq -\varepsilon - a_s(s - 1) \) for \( s \in (1, 2) \). Thus, if \( 0 < s - 1 < \min(1, \varepsilon a_s^{-1}) \) then \( -2\varepsilon < h_v(s) - h_v(1) \leq 0 \). Therefore,

\[
\lim_{s \to 1, s > 1} h_v(s) = h_v(1). \]

For \( 0 < s < 1 \) according to (5) \( v(s^{-1} x) \leq s^{-1} v(x) + (\varepsilon + a_s(s^{-1} - 1)) s^{-1} x + b_s s^{-1} \), so \( h_v(s) \leq \varepsilon + a_s(s^{-1} - 1) \). Therefore, if \( 0 < 1 - s < \min(2^{-1}, \varepsilon(2a_s)^{-1}) \) then \( 0 \leq h_v(s) - h_v(1) < 2\varepsilon \). Therefore,

\[
\lim_{s \to 1, s < 1} h_v(s) = h_v(1). \]

Thus, function \( h_v \) is continuous at point \( s = 1 \).

Next, we have

\[
h_v(s^{-1}) = \lim_{x \to \infty} \left( \frac{v(sx)}{sx} - \frac{v(x)}{x} \right) = -\lim_{x \to \infty} \left( \frac{v(x)}{x} - \frac{v(sx)}{sx} \right) \leq
\]

\[\leq -\lim_{x \to \infty} \left( \frac{v(x)}{x} - \frac{v(sx)}{sx} \right) = -h_v(s), \quad s > 0.
\]

\( \text{From this we obtain the sixth property of function } h_v. \)

\textbf{Lemma 2.} \textbf{The following equality holds}

\[
h_v(s) = \lim_{k \to +\infty} \left( \frac{v(k)}{k} - \frac{v([sk] + 1)}{sk} \right), \quad s > 0.
\]

\textbf{Proof.} Let \( s > 0 \). For \( x \in [k, k + 1) \), where \( k \in \mathbb{N} \), we have

\[
\frac{v(x)}{x} - \frac{v(sx)}{sx} \geq \frac{v(k)}{k} - \frac{v(sk + s)}{sk} = \frac{v(k)}{k} - \frac{v([sk] + 1)}{sk} + \frac{v([sk] + 1) - v(sk + s)}{sk};
\]

\[
\frac{v(x)}{x} - \frac{v(sx)}{sx} \leq \frac{v(k + 1)}{k + 1} - \frac{v(sk)}{sk} \leq \frac{v(k)}{k} - \frac{v([sk] + 1)}{sk} + \frac{v(sk + 1) - v(sk)}{sk} + \frac{v(k + 1) - v(k)}{k + 1} - \frac{v(k)}{k(k + 1)}.
\]
Since \( v \) satisfies the condition of the form (2) and the inequality (4) then from the last estimates the assertion of lemma follows.

Thus, according to lemma 2 for any \( v \in \mathcal{V} \) such that \( \exp(v(k)) = M_k, \ k \in \mathbb{Z}_+ \), the function \( h_v(s) \) coincides with

\[
h(s) = \lim_{k \to +\infty} (sk)^{-1} \ln \frac{M_k^s}{M_{|sk|+1}}, \ s > 0.
\]

We set \( l(s) = \exp(h(s)), s > 0 \). From the properties of function \( h \) it follows that function \( l \) has the following properties:

1. for all \( s \in (0, +\infty) \) \( 0 < l(s) < +\infty \);
2. \( l(s) \) is continuous at the point \( s = 1 \);
3. \( l(s) > 1 \) for \( s \in (0, 1) \), \( 0 < l(s) < 1 \) for \( s > 1 \);
4. \( \lim_{s \to 0, s > 0} l(s) = +\infty \);
5. \( l(s)l(s^{-1}) \leq 1 \).
6. \( l \) is a nonincreasing function on \((0, \infty)\).

**Lemma 3.** For each \( m \in \mathbb{N} \) and \( A > 0 \) there exists a positive constant \( Q \) such that

\[
w_m(|z|) + A \ln(1 + |z|) \leq w_{m+1}(|z|) + Q, \ z \in \mathbb{C}.
\]

The proof of this lemma is given in [2], [3].

4. Weakly sufficient sets for \( P \). Let \( \mathcal{K} \) denote a set of all positive continuous functions \( k \) on the complex plane such that for each \( m \in \mathbb{N} \)

\[
\sup_{z \in \mathbb{C}} \frac{\exp(\psi(Im z) + w_m(|z|))}{k(z)} < \infty.
\]

For each closed subset \( S \) of \( \mathbb{C} \) that is an uniqueness set for \( P \) we define topologies \( \tau_S \) and \( \mu_S \) in \( P \) in the following manner. The topology \( \tau_S \) is an inductive limit topology of the normed spaces

\[
P_{S,m} = \left\{ f \in P : \|f\|_{S,m} = \sup_{z \in S} \frac{|f(z)|}{\exp(\psi(Im z) + w_m(|z|))} < \infty \right\}, \ m \in \mathbb{N}.
\]

The topology \( \mu_S \) is defined in \( P \) with the help of the norms

\[
\|f\|_{S,k} = \sup_{z \in S} \frac{|f(z)|}{k(z)} < \infty, \ k \in \mathcal{K}.
\]
Call the subset $S$ sufficient (weakly sufficient) for $P$ if $\mu_C = \mu_S(\tau_C = \tau_S)$.

The general arguments [1, chapter 1] show that if $S$ is a sufficient set for $P$ and $\tau = \mu_C$ then every function $f \in G$ can be represented as an absolutely convergent integral

$$f(x) = \int_S e^{-izx} \frac{d\lambda(z)}{k(z)}, \quad x \in \mathbb{R},$$

where the complex measure $\lambda$ on $\mathbb{C}$ is supported by the set $S$ and satisfies the condition $\int_C |d\lambda(z)| = C_\lambda < \infty$, $k$ is some function from $\mathcal{K}$. If the sufficient set $S$ is a set of points $\nu_j \in \mathbb{C}$, $j = 1, 2, \ldots$, then from the integral representation we get the representation of $f$ in the form of the series

$$f(x) = \sum_{j=1}^{\infty} c_j e^{-i\nu_j x}$$

moreover, from the estimates (see [2], [3]): $|c_j| \leq \frac{C_\lambda}{k(\nu_j)}$ for each $j \in \mathbb{N}$,

$$p_m(\exp(-izx)) \leq K_m \exp(\psi(Im z) + w_{m+1}(|z|)) \quad \text{for each} \ m \in \mathbb{N}, z \in \mathbb{C},$$

where $K_m > 0$ is some constant independent of $z$, and lemma 3 it follows that this series absolutely converges in the space $G$.

By the main result of [7] and lemma 3 we have $\tau = \mu_C$.

According to [8], [9] there exists an entire function $N(z)$ such that:

1. all the zeros $\{\lambda_j\}_{j=1}^{\infty}$ of $N(z)$ are simple and the discs $D_j = \{z \in \mathbb{C} : |z - \lambda_j| < d\}$ are disjoint for some $d > 0$;

2. outside the set $\bigcup_{j=1}^{\infty} D_j$

$$|H_D(z) + w^{\sigma^{-1}}|\ln|N(z)|| \leq A \ln(1 + |z|) + C_0,$$

where $A, C_0$ are some positive numbers.

**Theorem 1.** The set $\tilde{S} = \{\lambda_j\}_{j=1}^{\infty}$ of zeros of $N$ is a weakly sufficient set for $P_{\tau}$.

**Theorem 2.** Every function $f \in G$ can be represented in the form of a series

$$f(x) = \sum_{j=1}^{\infty} c_j e^{-i\lambda_j x},$$

absolutely convergent in $G$. 
The proof of Theorem 1 is based on the representation of entire functions in the space $P$ by Lagrange series, on the key

**Lemma 4.** For all $s > 0, \delta \in (0, 1)$ there exists a constant $Q(s, \delta) \geq 0$ such that $sw(r) \leq w\left(\frac{r}{l(s)(1-\delta)}\right) + Q(s, \delta)$ for all $r \geq 0$

and will be given in [4].
Bibliography

1. L. Ehrenpreis. *Fourier analysis in several complex variables.* Wiley – Interscience, New York, 1970.
2. I. Kh. Musin. Paley-Wiener type theorem for a weighted space of infinitely differentiable functions. Izv. Akad. Nauk SSSR. Ser. Mat., 64:6 (2000), 181-204.
3. I. Kh. Musin. On the Fourier-Laplace transform of functionals on a weighted space of infinitely differentiable functions. Pbb: funct-an@xxx.lanl.gov N 9911067.
4. I. Kh. Musin. On representation of infinitely differentiable functions by Dirichlet seies. (submitted to ”Matematicheskie zametki”)
5. F. W. J. Olver. Introduction to asymptotics and special functions. Academic Press, NY and London, 1974.
6. G. M. Phihtengoltc. Course of differential and integral calculus. V. II. M.- L.: GITTL, 1948.
7. V. V. Napalkov. On comparison of the topologies in some spaces of entire functions. Dokl. Akad. Nauk SSSR, 264:4 (1982), 827-830.
8. R. S. Youlmukhametov. Approximation of subharmonic functions. Math. Sb., 1984. 124:3(1984), 393-415.
9. R. S. Youlmukhametov. Approximation of subharmonic functions. Analysis Mathematica, 11:3 (1985), 257-282.