FORMALITY FOR ALGEBROIDS II: FORMALITY THEOREM FOR GERBES

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Abstract. We extend the formality theorem of M. Kontsevich from deformations of the structure sheaf on a manifold to deformations of gerbes.

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1. Introduction

This is the second of two articles devoted to extension of M. Kontsevich formality theorem [16] to deformations of gerbes. In the previous paper [3] we discuss extension of the notion of Deligne 2-groupoid to $L_\infty$ algebras. In particular, due to the work of J. Duskin [8], E. Getzler [9], and V. Hinich [11] one has a natural notion of Deligne 2-groupoid for an $L_\infty$ algebra $g$ which we denote $MC_2(g)$. In the present paper we use these results to construct $L_\infty$ algebra controlling the deformation theory of a gerbe.

In the fundamental paper [16] M. Kontsevich showed that the set of equivalence classes of formal deformations of the algebra of functions on a manifold is in one-to-one correspondence with the set of equivalence classes of formal Poisson structures on the manifold. This result was obtained as a corollary of the formality of the Hochschild complex of the algebra of functions on the manifold conjectured by M. Kontsevich (cf. [15]) and proven in [16]. Later proofs by a different method were given in [22] and in [6].

In this paper we extend the formality theorem of M. Kontsevich to deformations of gerbes on smooth manifolds or complex analytic manifolds, using the method of [6].

Consider first the case of a $C^\infty$ manifold $X$. We denote by $O_X$ the sheaf of complex valued $C^\infty$ functions on $X$. For a twisted form $S$ of $O_X$ regarded as an algebroid stack (see Section 6.2) we denote by $[S]_{dR} \in H_3^{dR}(X)$ the de Rham class of $S$. The main result of this paper establishes an equivalence of 2-groupoid valued functors on Artin $\mathbb{C}$-algebras between $\text{Def}(S)$ (the formal deformation theory of $S$, see [2]) and the Deligne 2-groupoid of the $L_\infty$-algebra of multivector fields on $X$ twisted by a closed three-form representing $[S]_{dR}$:

$$\text{Theorem 1.1. Let } X \text{ be a smooth manifold. Suppose that } S \text{ is a twisted form of } O_X. \text{ Let } H \text{ be a closed } 3\text{-form on } X \text{ which represents } [S]_{dR} \in H_3^{dR}(X). \text{ For any Artin algebra } R \text{ with maximal ideal } m_R \text{ there is an equivalence of 2-groupoids}$$

$$MC^2(s(O_X)_H \otimes m_R) \cong \text{Def}(S)(R)$$

natural in $R$.

Here, $s(O_X)_H$ denotes the $L_\infty$-algebra of multivector fields with the trivial differential, the binary operation given by Schouten bracket, the ternary operation given by $H$ (see 5.2) and all other operations equal to zero. As a corollary of this result we obtain that the isomorphism classes of formal deformations of $S$ are in a bijective correspondence with equivalence classes of the formal twisted Poisson structures defined by P. Severa and A. Weinstein in [21]. More precisely, equivalence classes of $R$-deformations of a gerbe are in a bijection with equivalence classes of $\pi \in \Gamma(X; \wedge^2 T_X) \otimes m_R$, satisfying the equation

$$[\pi, \pi] = \Phi(H)(\pi, \pi, \pi).$$

A construction of an algebroid stack associated to a twisted formal Poisson structure was proposed by P. Ševera in [20].

To give a treatment which also covers a complex analytic case and to incorporate other examples we work in the framework of the manifold with two transversal integrable polarizations. Let as before $X$ be a smooth manifold. We shall assume
that the manifold $X$ is equipped with two complementary integrable complex distributions $P$ and $Q$. We do suppose that $P \cap Q = 0$ and $P \oplus Q = T^c_X$. We denote by $\mathcal{O}_{X/P}$ the sheaf of complex valued $C^\infty$ functions on $X$ constant along $P$; see the section 3 for the details. In the case when $P = 0$ we recover the case of the smooth manifold $X$ with the sheaf $\mathcal{O}_{X/P} = \mathcal{O}_X$ of complex valued $C^\infty$ functions.

In the case when $P$ is a complex structure, i.e. $P = Q$ we recover the case of complex analytic manifold $X$ with the sheaf $\mathcal{O}_{X/P}$ of holomorphic functions.

The decomposition $P \oplus Q = T^c_X$ induces bigrading on forms $\Omega^n_X = \bigoplus_{p+q=n} \Omega^{p,q}_X$ with $\Omega^{p,q}_X = \Lambda^p P^\perp \otimes \Lambda^q Q^\perp$ and filtration: $F^{-i}_X \Omega^n = \bigoplus_{p \geq i} \Omega^{p,n-p}_X$. If $S$ is a twisted form of $\mathcal{O}_{X/P}$ its class $[S] \in H^3_{dR}(X)$ can be represented by a form $H \in \Gamma(X; F^{-1}_X \Omega^3_X)$. Such a form $H$ defines a structure of $L_\infty$-algebra on the $P$-Dolbeault resolution of $P$-holomorphic multivector fields, see Section 5.4. We denote this $L_\infty$-algebra by $s(\mathcal{O}_{X/P})$. Then more general version of the Theorem 1.1 is the following:

**Theorem 1.2.** Suppose that $X$ is a $C^\infty$-manifold equipped with a pair of complementary complex integrable distributions $P$ and $Q$, and $S$ is a twisted form of $\mathcal{O}_{X/P}$ (6.2). Let $H \in \Gamma(X; F^{-1}_X \Omega^3_X)$ be a representative of $[S]$ (6.2). Then, for any Artin algebra $R$ with maximal ideal $m_R$ there is an equivalence of bigroupoids

$$MC^2(s(\mathcal{O}_{X/P})_H \otimes m_R) \cong \mbox{Def}(S)(R),$$

where the left-hand side is defined as in (6.1), natural in $R$.

The proof of the Theorem proceeds along the following lines. As a starting point we use the construction of the Differential Graded Lie Algebra (DGLA) controlling the deformations of $S$, obtained in [1, 2]. Next we construct a chain of $L_\infty$-quasi-isomorphisms between this DGLA and $s(\mathcal{O}_{X/P})_H$, using the techniques of [6]. Then we use the results of [3] to deduce the equivalence of bigroupoids.

The paper is organized as follows.

Section 2 contains a short exposition of the proof of Kontsevich formality theorem given in [6].

Section 3 contains a short review of differential calculus and differential geometry of jets in the presence of an integrable distribution.

In section 4 the proof from [6] is modified to the twisted case.

The section 5 is devoted to constructions of $L_\infty$-structures on the algebra of multivectors and related algebras.

Finally, in the Section 6 we prove the main results on the deformations of algebroid stacks.

## 2. Formality

We give a synopsis of the results of [6] in the notation of loc. cit. Let $k$ be a field of characteristic zero. For a $k$-cooperad $\mathcal{C}$ and a complex of $k$-vector spaces $V$ we denote by $F_{\mathcal{C}}(V)$ the cofree $\mathcal{C}$-coalgebra on $V$.

We denote by $e_2$ the operad governing Gerstenhaber algebras. The latter is Koszul, and we denote by $e_2^\vee$ the dual cooperad.
2.1. **Hochschild cochains.** For a $k$-algebra $A$, $p = 0, 1, 2, \ldots$ the space $C^p(A) = C^p(A; A)$ of Hochschild cochains of degree $p$ with values in $(A \otimes_k A^{op})$-module $A$ is defined by $C^0(A) = A$ and $C^p(A) = \text{Hom}_k(A \otimes_k A^p, A)$ for $p \geq 1$. Let $g^p(A) = C^{p+1}(A)$. There is a canonical isomorphism of graded $k$-modules $g(A) = \text{Der}_k(\text{coAss}(A[1]))$, where coAss$(V)$ denotes the co-free co-associative co-algebra on the graded vector space $V$. In particular, $g(A) = C(A)[1]$ has a canonical structure of a graded Lie algebra under the induced Gerstenhaber bracket. Let $m \in C^2(A)$ and let $\delta$ denote the adjoint action of $m$ with respect to the Gerstenhaber bracket. Thus, $\delta$ is an endomorphism of degree one and a derivation of the Gerstenhaber bracket. The fact that $\delta \circ \delta = 0$ is equivalent to the associativity of $m$. The graded space $C^*(A)$ equipped with the Hochschild differential $\delta$ is called the Hochschild complex of $A$. The graded space $g(A) = C^*(A)[1]$ equipped with the Gerstenhaber bracket and the Hochschild differential is a DGLA which controls the deformation theory of $A$.

2.2. **Outline of the proof of formality for cochains.** For an associative $k$-algebra $A$ the Hochschild complex $C^*(A)$ has a canonical structure of a brace algebra, hence a structure of homotopy $\mathfrak{e}_2$-algebra. The latter structure is encoded in a differential (i.e. a coderivation of degree one and square zero) $M : F_{\mathfrak{e}_2}^\vee(C^*(A)) \to F_{\mathfrak{e}_2}^\vee(C^*(A))[1]$.

Suppose from now on that $A$ is regular commutative algebra over a field of characteristic zero (the regularity assumption is not needed for the constructions). Let $V^\bullet(A) = \text{Sym}_k(\text{Der}(A)[-1])$ viewed as a complex with trivial differential. In this capacity $V^\bullet(A)$ has a canonical structure of an $\mathfrak{e}_2$-algebra which gives rise to the differential $d_{V^\bullet(A)}$ on $F_{\mathfrak{e}_2}^\vee(V^\bullet(A))$: we have: $B_{\mathfrak{e}_2}^\vee(V^\bullet(A)) = (F_{\mathfrak{e}_2}^\vee(V^\bullet(A)), d_{V^\bullet(A)})$ (see [6], Theorem 1 for notation).

In addition, the authors introduce a sub-$\mathfrak{e}_2$-$\vee$-coalgebra $\Xi(A)$ of both $F_{\mathfrak{e}_2}^\vee(C^*(A))$ and $F_{\mathfrak{e}_2}^\vee(V^\bullet(A))$. We denote by $\sigma : \Xi(A) \to F_{\mathfrak{e}_2}^\vee(C^*(A))$ and $\iota : \Xi(A) \to F_{\mathfrak{e}_2}^\vee(V^\bullet(A))$ respective inclusions and identify $\Xi(A)$ with its image under the latter one. By [6], Proposition 7 the differential $d_{V^\bullet(A)}$ preserves $\Xi(A)$; we denote by $d_{V^\bullet(A)}$ its restriction to $\Xi(A)$. By Theorem 3, loc. cit. the inclusion $\sigma$ is a morphism of complexes. Hence, we have the following diagram in the category of differential graded $\mathfrak{e}_2$-$\vee$-coalgebras:

\[
\begin{array}{c}
(F_{\mathfrak{e}_2}^\vee(C^*(A)), M) \xleftarrow{\sigma} (\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\iota} (B_{\mathfrak{e}_2}^\vee(V^\bullet(A)))
\end{array}
\]

Applying the functor $\Omega_{\mathfrak{e}_2}$ (adjoint to the functor $B_{\mathfrak{e}_2}^\vee$, see [6], Theorem 1) to (2.1) we obtain the diagram

\[
\begin{array}{c}
\Omega_{\mathfrak{e}_2}(F_{\mathfrak{e}_2}^\vee(C^*(A)), M) \xleftarrow{\Omega_{\mathfrak{e}_2}(\sigma)} \Omega_{\mathfrak{e}_2}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\Omega_{\mathfrak{e}_2}(\iota)} \Omega_{\mathfrak{e}_2}(B_{\mathfrak{e}_2}^\vee(V^\bullet(A)))
\end{array}
\]

of differential graded $\mathfrak{e}_2$-algebras. Let $\nu = \eta_{\mathfrak{e}_2} \circ \Omega_{\mathfrak{e}_2}(\iota)$, where $\eta_{\mathfrak{e}_2} : \Omega_{\mathfrak{e}_2}(B_{\mathfrak{e}_2}^\vee(V^\bullet(A))) \to V^\bullet(A)$ is the counit of adjunction. Thus, we have the diagram

\[
\begin{array}{c}
\Omega_{\mathfrak{e}_2}(F_{\mathfrak{e}_2}^\vee(C^*(A)), M) \xleftarrow{\Omega_{\mathfrak{e}_2}(\sigma)} \Omega_{\mathfrak{e}_2}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\nu} V^\bullet(A)
\end{array}
\]

of differential graded $\mathfrak{e}_2$-algebras.

**Theorem 2.1** ([6], Theorem 4). The maps $\Omega_{\mathfrak{e}_2}(\sigma)$ and $\nu$ are quasi-isomorphisms.
Additionally, concerning the DGLA structures relevant to applications to deformation theory, deduced from respective \( e_2 \)-algebra structures we have the following result.

**Theorem 2.2** (\cite{6}, Theorem 2). The DGLA \( \Omega_{e_2}(\mathbb{F}_e^\vee(C^\bullet(A)), M)[1] \) and \( C^\bullet(A)[1] \) are canonically \( L_\infty \)-quasi-isomorphic.

**Corollary 2.3** (Formality). The DGLA \( C^\bullet(A)[1] \) and \( V^\bullet(A)[1] \) are \( L_\infty \)-quasi-isomorphic.

### 2.3. Some (super-)symmetries.

For applications to deformation theory of algebroid stacks we will need certain equivariance properties of the maps described in \( \text{(2.1)} \).

For \( a \in A \) let \( i_a : C^\bullet(A) \to C^\bullet(A)[-1] \) denote the adjoint action (in the sense of the Gerstenhaber bracket and the identification \( A = C^0(A) \)). It is given by the formula

\[
i_a D(a_1, \ldots, a_n) = \sum_{i=0}^n (-1)^i D(a_1, \ldots, a_i, a, a_{k+1}, \ldots, a_n).
\]

The operation \( i_a \) extends uniquely to a coderivation of \( \mathbb{F}_e^\vee(C^\bullet(A)) \); we denote this extension by \( i_a \) as well. Furthermore, the subcoalgebra \( \Xi(A) \) is preserved by \( i_a \).

Since the operation \( i_a \) is a derivation of the cup product as well as of all of the brace operations on \( C^\bullet(A) \) and the homotopy-\( e_2 \)-algebra structure on \( C^\bullet(A) \) is given in terms of the cup product and the brace operations it follows that \( i_a \) anticommutes with the differential \( M \). Hence, the coderivation \( i_a \) induces a derivation of the differential graded \( e_2 \)-algebra \( \Omega_{e_2}(\mathbb{F}_e^\vee(C^\bullet(A)), M) \) which will be denoted by \( i_a \) as well. For the same reason the DGLA \( \Omega_{e_2}(\mathbb{F}_e^\vee(C^\bullet(A)), M)[1] \) and \( C^\bullet(A)[1] \) are quasi-isomorphic in a way which commutes with the respective operations \( i_a \).

On the other hand, let \( i_a : V^\bullet(A) \to V^\bullet(A)[-1] \) denote the adjoint action in the sense of the Schouten bracket and the identification \( A = V^0(A) \). The operation \( i_a \) extends uniquely to a coderivation of \( \mathbb{F}_e^\vee(V^\bullet(A)) \) which anticommutes with the differential \( d_{V^\bullet(A)} \) because \( i_a \) is a derivation of the \( e_2 \)-algebra structure on \( V^\bullet(A) \). We denote this coderivation as well as its unique extension to a derivation of the differential graded \( e_2 \)-algebra \( \Omega_{e_2}(\mathbb{F}_e^\vee(V^\bullet(A))) \) by \( i_a \). The counit map \( \eta_{e_2} : \Omega_{e_2}(\mathbb{F}_e^\vee(V^\bullet(A))) \to V^\bullet(A) \) commutes with respective operations \( i_a \).

The subcoalgebra \( \Xi(A) \) of \( \mathbb{F}_e^\vee(C^\bullet(A)) \) and \( \mathbb{F}_e^\vee(V^\bullet(A)) \) is preserved by the respective operations \( i_a \). Moreover, the restrictions of the two operations to \( \Xi(A) \) coincide, i.e. the maps in (2.1) commute with \( i_a \) and, therefore, so do the maps in (2.2) and (2.3).

### 2.4. Extensions and generalizations.

The constructions and results of \[6\] apply in a variety of situations. First of all, observe that the constructions of all objects and morphisms involved can be carried out in any closed symmetric monoidal category such as for example the category of sheaves of \( k \)-modules, \( k \) a sheaf of commutative algebras (over a field of characteristic zero). As is pointed out in \[6\], Section 4, the proof of Theorem \( \text{(2.1)} \) is based on the flatness of the module \( \text{Der}(A) \) and the Hochschild-Kostant-Rosenberg theorem.

The considerations above apply to a sheaf of \( k \)-algebras \( K \) yielding the sheaf of Hochschild cochains \( C^p(K) \), the Hochschild complex (of sheaves) \( C^\bullet(K) \) and the (sheaf of) DGLA \( g(K) \).
Suppose that \( X \) is a \( C^\infty \)-manifold. The sheaf \( \mathcal{C}^p(\mathcal{O}_X) \) coincides with the sheaf of multilinear differential operators \( \mathcal{O}_X^{\otimes_p} \to \mathcal{O}_X \) by the celebrated theorem of J. Peetre [13, 19].

### 3.1. Complex distributions.

A \textit{(complex) distribution} on \( X \) is a sub-bundle\(^1\) of \( \mathcal{T}_X^\mathbb{C} \).

A distribution \( \mathcal{P} \) is called \textit{involutive} if it is closed under the Lie bracket, i.e. \([\mathcal{P}, \mathcal{P}] \subseteq \mathcal{P}\).

For a distribution \( \mathcal{P} \) on \( X \) we denote by \( \mathcal{P}^\perp \subseteq \Omega^1_X \) the annihilator of \( \mathcal{P} \) (with respect to the canonical duality pairing).

A distribution \( \mathcal{P} \) of rank \( r \) on \( X \) is called \textit{integrable} if, locally on \( X \), there exist functions \( f_1, \ldots, f_r \in \mathcal{O}_X \) such that \( df_1, \ldots, df_r \) form a local frame for \( \mathcal{P}^\perp \).

It is easy to see that an integrable distribution is involutive. The converse is true when \( \mathcal{P} \) is real, i.e. \( \overline{\mathcal{P}} = \mathcal{P} \) (Frobenius) and when \( \mathcal{P} \) is a \textit{complex structure}, i.e. \( \overline{\mathcal{P}} \cap \mathcal{P} = 0 \) and \( \overline{\mathcal{P}} \oplus \mathcal{P} = \mathcal{T}_X^\mathbb{C} \) (Newlander-Nirenberg). See [R] for generalizations.

### 3.2. The Hodge filtration.

We assume from now on that \( \mathcal{P} \) is an integrable distribution on \( X \). (Involutivity suffices for basic constructions.)

Let \( F_\mathcal{O}_X^i \) denote the filtration by the powers of the differential ideal generated by \( \mathcal{P}^\perp \), i.e. \( F_\mathcal{O}_X^i=\bigwedge^i \mathcal{P}^\perp \subseteq \Omega^i_X \). Let \( \mathcal{D} \) denote the differential in \( Gr^F_\mathcal{O}_X^i \).

The wedge product of differential forms induces a structure of a commutative DGA on \( (Gr^F_\mathcal{O}_X^i, \mathcal{D}) \).

In particular, \( Gr^F_0\mathcal{O}_X = \mathcal{O}_X \), \( Gr^F_0\Omega^1_X = \Omega^1_X/\mathcal{P}^\perp \) and \( \mathcal{D}: \mathcal{O}_X \to Gr^F_0\Omega^1_X \) is equal to the composition \( \mathcal{O}_X \xrightarrow{\partial} \Omega^1_X \to \Omega^1_X/\mathcal{P}^\perp \). Let \( \mathcal{O}_{X/\mathcal{P}} := \ker(\mathcal{O}_X \xrightarrow{\mathcal{D}} Gr^F_0\Omega^1_X) \);

\(^1\)A sub-bundle is an \( \mathcal{O}_X \)-submodule which is a direct summand locally on \( X \)
equivalently, $\mathcal{O}_{X/P} = (\mathcal{O}_X)^P \subset \mathcal{O}_X$, the subsheaf of functions constant along $P$. Note that $\overline{\partial}$ is $\mathcal{O}_{X/P}$-linear. Higher $\overline{\partial}$-cohomology of $\mathcal{O}_X$ vanishes, i.e.

$$H^i(Gr^F_0 \Omega^*_X, \overline{\partial}) = \begin{cases} \mathcal{O}_{X/P} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

### 3.3. $\overline{\partial}$-operators.

Suppose that $\mathcal{E}$ is a vector bundle on $X$, i.e. a locally free $\mathcal{O}_X$-module of finite rank. A connection along $\mathcal{P}$ on $\mathcal{E}$ is, by definition, a map $\nabla^\mathcal{P} : \mathcal{E} \to \Omega^1_X/\mathcal{P} \otimes \mathcal{O}_X \mathcal{E}$ which satisfies the Leibniz rule $\nabla^\mathcal{P}(fe) = f\nabla^\mathcal{P}(e) + \overline{\partial} f \cdot e$. Equivalently, a connection along $\mathcal{P}$ is an $\mathcal{O}_X$-linear map $\nabla^\mathcal{P} : \mathcal{P} \to \text{End}_C(\mathcal{E})$ which satisfies the Leibniz rule $\nabla^\mathcal{P}_\xi(fe) = f\nabla^\mathcal{P}_\xi(e) + \overline{\partial} f \cdot e$. In particular, $\nabla^\mathcal{P}_\xi$ is $\mathcal{O}_{X/P}$-linear.

The two avatars of a connection along $\mathcal{P}$ are related by $\nabla^\mathcal{P}_\xi(e) = \iota_\xi \nabla^\mathcal{P}(e)$.

A connection along $\mathcal{P}$ on $\mathcal{E}$ is called flat if the corresponding map $\nabla^\mathcal{P}_\xi : \mathcal{P} \to \text{End}_C(\mathcal{E})$ is a morphism of Lie algebras. We will refer to a flat connection along $\mathcal{P}$ on $\mathcal{E}$ as a $\overline{\partial}$-operator on $\mathcal{E}$.

A connection on $\mathcal{E}$ along $\mathcal{P}$ extends uniquely to a derivation of the graded $Gr^F_0 \Omega^*_X$-module $Gr^F_0 \Omega^*_X \otimes_\mathcal{O}_X \mathcal{E}$. A $\overline{\partial}$-operator $\overline{\partial}_\xi$ satisfies $\overline{\partial}_\xi^2 = 0$. The complex $(Gr^F_0 \Omega^*_X \otimes_\mathcal{O}_X \mathcal{E}, \overline{\partial}_\xi)$ is referred to as the (corresponding) $\overline{\partial}$-complex. Since $\overline{\partial}_\xi$ is $\mathcal{O}_{X/P}$-linear, the sheaves $H^i(Gr^F_0 \Omega^*_X \otimes_\mathcal{O}_X \mathcal{E}, \overline{\partial}_\xi)$ are $\mathcal{O}_{X/P}$-modules. The vanishing of higher $\overline{\partial}$-cohomology of $\mathcal{O}_X$ generalizes easily to vector bundles.

**Lemma 3.1.** Suppose that $\mathcal{E}$ is a vector bundle and $\overline{\partial}_\xi$ is a $\overline{\partial}$-operator on $\mathcal{E}$. Then, $H^i(Gr^F_0 \Omega^*_X \otimes_\mathcal{O}_X \mathcal{E}, \overline{\partial}_\xi) = 0$ for $i \neq 0$, i.e. the $\overline{\partial}$-complex is a resolution of $\ker(\overline{\partial}_\xi)$. Moreover, $\ker(\overline{\partial}_\xi)$ is locally free over $\mathcal{O}_{X/P}$ of rank $\text{rk}_\mathcal{O}_X \mathcal{E}$ and the map $\mathcal{O}_X \otimes_\mathcal{O}_{X/P} \ker(\overline{\partial}_\xi) \to \mathcal{E}$ (the $\mathcal{O}_X$-linear extension of the inclusion $\ker(\overline{\partial}_\xi) \hookrightarrow \mathcal{E}$) is an isomorphism.

**Remark 3.2.** Suppose that $\mathcal{F}$ is a locally free $\mathcal{O}_{X/P}$-module of finite rank. Then, $\mathcal{O}_X \otimes_\mathcal{O}_{X/P} \mathcal{F}$ is a locally free $\mathcal{O}_X$-module of rank $\text{rk}_\mathcal{O}_X \mathcal{F}$ and is endowed in a canonical way with a $\overline{\partial}$-operator, namely $\overline{\partial} \otimes \text{Id}$. The assignments $\mathcal{F} \mapsto (\mathcal{O}_X \otimes_\mathcal{O}_{X/P} \mathcal{F}, \overline{\partial} \otimes \text{Id})$ and $(\mathcal{E}, \overline{\partial} \otimes \text{Id}) \mapsto \ker(\overline{\partial}_\xi)$ are mutually inverse equivalences of suitably defined categories.

### 3.4. Calculus.

The adjoint action of $\mathcal{P}$ on $\Omega^i_X$ preserves $\mathcal{P}$, hence descends to an action on $\Omega^i_X/\mathcal{P}$. The latter action defines a connection along $\mathcal{P}$, i.e. a canonical $\overline{\partial}$-operator on $\Omega^i_X/\mathcal{P}$ which is easily seen to coincide with the one induced via the duality pairing between the latter and $\mathcal{P}^\perp$. Let $\mathcal{X}_{\mathcal{P}} := (\Omega^i_X/\mathcal{P})^\perp$ (the subsheaf of $\mathcal{P}$ invariant section, equivalently, the kernel of the $\overline{\partial}$-operator on $\Omega^i_X/\mathcal{P}$). The Lie bracket on $\mathcal{X}_{\mathcal{P}}$ (respectively, the action of $\mathcal{X}_{\mathcal{P}}$ on $\mathcal{O}_X$) induces a Lie bracket on $\mathcal{X}_{\mathcal{P}}$ (respectively, an action of $\mathcal{X}_{\mathcal{P}}$ on $\mathcal{O}_{X/P}$). Together the two endow $\mathcal{X}_{\mathcal{P}}$ with a structure of an $\mathcal{O}_{X/P}$-Lie algebroid.

The action of $\mathcal{P}$ on $\Omega^i_X$ by Lie derivative restricts to a flat connection along $\mathcal{P}$, i.e. a canonical $\overline{\partial}$-operator on $\mathcal{P}^\perp$ and, therefore, on $\wedge^i \mathcal{P}^\perp$ for all $i$. It is easy to see that the multiplication map $Gr^F_0 \Omega^i \otimes \wedge^i \mathcal{P}^\perp \to Gr^F_0 \Omega^i[\iota]$ is an isomorphism which identifies the $\overline{\partial}$-complex of $\wedge^i \mathcal{P}^\perp$ with $Gr^F_0 \Omega^i[\iota]$. Let $\Omega^i_{X/P} :=$

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2In the case of a real polarization this connection is known as the Bott connection.
\[ H^i(Gr^F_i, \Omega^\bullet_X, \mathcal{D}) \) (so that \( \mathcal{O}_{X/P} := \Omega^0_{X/P} \)). Then, \( \Omega^i_{X/P} \subset \bigwedge^i \mathcal{P}^\perp \subset \Omega^i_X \). The wedge product of differential forms induces a structure of a graded-commutative algebra on \( \Omega^\bullet_{X/P} := \bigoplus \Omega^i_{X/P}[-i] = H^\bullet(Gr^F \Omega^\bullet_X, \mathcal{D}) \). The multiplication induces an isomorphism \( \bigwedge^i \mathcal{O}_{X/P} \cdot \Omega^1_{X/P} \to \Omega^i_{X/P} \). The de Rham differential \( d \) restricts to the map \( d : \Omega^i_{X/P} \to \Omega^{i+1}_{X/P} \) and the complex \( \Omega^\bullet_{X/P} := (\Omega^\bullet_{X/P}, d) \) is a commutative DGA. Moreover, the inclusion \( \bigwedge^\bullet \mathcal{O}_{X/P} \to \Omega^\bullet_X \) is a quasi-isomorphism.

The duality pairing \( \mathcal{T}^\perp_X \mathcal{P} \otimes \mathcal{P}^\perp \to \mathcal{O}_X \) restricts to a non-degenerate pairing \( \mathcal{T}_{X/P} \otimes \mathcal{O}_{X/P} \Omega^1_{X/P} \to \mathcal{O}_{X/P} \). The action of \( \mathcal{T}^\perp_X \mathcal{P} / \mathcal{O}_{X/P} \mathcal{O}_{X/P} \) on \( \mathcal{T}_{X/P} \) the pairing and the de Rham differential are related by the usual formula \( \xi(f) = \partial f \), for \( \xi \in \mathcal{T}_{X/P} \) and \( f \in \mathcal{O}_{X/P} \).

### 3.5. Jets

Let \( \text{pr}_i : X \times X \to X \), \( i = 1, 2 \), denote the projection on the \( i \)th factor. The restriction of the canonical map

\[ \text{pr}_i^\perp: \text{pr}_i^{-1} \mathcal{O}_X \to \mathcal{O}_{X \times X} \]

to the subsheaf \( \mathcal{O}_{X/P} \) takes values in the subsheaf \( \mathcal{O}_{X \times X/P \times P} \) hence induces the map

\[ \text{pr}_i^\perp: \mathcal{O}_{X/P} \to (\text{pr}_i)_* \mathcal{O}_{X \times X/P \times P} \]

Let \( \Delta_X : X \to X \times X \) denote the diagonal embedding. It follows from the Leibniz rule that the restriction of the canonical map

\[ \Delta_X^* : \mathcal{O}_{X \times X} \to (\Delta_X)_* \mathcal{O}_X \]

to the subsheaf \( \mathcal{O}_{X \times X/P \times P} \) takes values in the subsheaf \( (\Delta_X)_* \mathcal{O}_{X/P} \). Let

\[ \mathcal{I}_{X/P} := \ker(\Delta_X^*) \cap \mathcal{O}_{X \times X/P \times P} \]

The sheaf \( \mathcal{I}_{X/P} \) plays the role of the defining ideal of the “diagonal embedding \( X/P \to X/P \times X/P \)”: there is a short exact sequence of sheaves on \( X \times X \)

\[ 0 \to \mathcal{I}_{X/P} \to \mathcal{O}_{X \times X/P \times P} \to (\Delta_X)_* \mathcal{O}_{X/P} \to 0 \]

For a locally-free \( \mathcal{O}_{X/P} \)-module of finite rank \( \mathcal{E} \) let

\[ \mathcal{J}^k_{X/P}(\mathcal{E}) := (\text{pr}_1)_* \left( \mathcal{O}_{X \times X/P \times P} / \mathcal{T}_{X/P}^{k+1} \otimes \text{pr}_2^{-1} \mathcal{O}_{X/P} \text{pr}_2^{-1} \mathcal{E} \right) \]

\[ \mathcal{J}^k_{X/P} := \mathcal{J}^k_{X/P}(\mathcal{O}_{X/P}) \]

It is clear from the above definition that \( \mathcal{J}^k_{X/P} \) is, in a natural way, a commutative algebra and \( \mathcal{J}^k_{X/P}(\mathcal{E}) \) is a \( \mathcal{J}^k_{X/P} \)-module.

Let

\[ 1^{(k)}: \mathcal{O}_{X/P} \to \mathcal{J}^k_{X/P} \]

denote the composition

\[ \mathcal{O}_{X/P} \xrightarrow{\text{pr}_1^\perp} (\text{pr}_1)_* \mathcal{O}_{X \times X/P \times P} \to \mathcal{J}^k_{X/P} \]

In what follows, unless stated explicitly otherwise, we regard \( \mathcal{J}^k_{X/P}(\mathcal{E}) \) as a \( \mathcal{O}_{X/P} \)-module via the map \( 1^{(k)} \).

Let

\[ j^k: \mathcal{E} \to \mathcal{J}^k_{X/P}(\mathcal{E}) \]
denote the composition
\[ \mathcal{E} \overset{\varepsilon^1 \circ \varepsilon}{\longrightarrow} (\text{pr}_1)_* \mathcal{O}_{X \times X / P \times P} \otimes \mathcal{E} \rightarrow \mathcal{J}_{X/P}^k (\mathcal{E}) \]

The map \( j^k \) is not \( \mathcal{O}_{X/P} \)-linear unless \( k = 0 \).

For \( 0 \leq k \leq l \) the inclusion \( \mathcal{I}_{X/P}^{l+1} \rightarrow \mathcal{I}_{X/P}^{k+1} \) induces the surjective map \( \pi_{l,k} : \mathcal{J}_{X/P}^l (\mathcal{E}) \rightarrow \mathcal{J}_{X/P}^k (\mathcal{E}) \). The sheaves \( \mathcal{J}_{X/P}^k (\mathcal{E}) \), \( k = 0, 1, \ldots \) together with the maps \( \pi_{l,k} \), \( k \leq l \) form an inverse system. Let \( \mathcal{J}_{X/P}^\infty (\mathcal{E}) := \lim_{\leftarrow} \mathcal{J}_{X/P}^k (\mathcal{E}) \). Thus, \( \mathcal{J}_{X/P}^\infty (\mathcal{E}) \) carries a natural topology.

The maps \( 1^{(k)} \) (respectively, \( j^k \), \( k = 0, 1, 2, \ldots \) are compatible with the projections \( \pi_{l,k} \), i.e. \( \pi_{l,k} \circ 1^{(l)} = 1^{(k)} \) (respectively, \( \pi_{l,k} \circ j^l = j^k \)). Let \( 1 := \lim_{\leftarrow} 1^{(k)} \), \( j^\infty := \lim_{\leftarrow} j^k \).

Let
\[
d_1 : \mathcal{O}_{X \times X / P \times P} \otimes_{\text{pr}_2^{-1} \mathcal{O}_{X/P}} \text{pr}_2^{-1} \mathcal{E} \rightarrow \nabla_1 \mathcal{O}_{X/P} \otimes_{\text{pr}_1^{-1} \mathcal{O}_{X/P}} \mathcal{O}_{X \times X / P \times P} \otimes_{\text{pr}_2^{-1} \mathcal{O}_{X/P}} \text{pr}_2^{-1} \mathcal{E}
\]
denote the exterior derivative along the first factor. It satisfies
\[
d_1 (\mathcal{I}_{X/P}^{k+1} \otimes_{\text{pr}_2^{-1} \mathcal{O}_{X/P}} \text{pr}_2^{-1} \mathcal{E}) \subset \nabla_1 \mathcal{O}_{X/P} \otimes_{\text{pr}_1^{-1} \mathcal{O}_{X/P}} \mathcal{I}_{X/P}^k \otimes_{\text{pr}_2^{-1} \mathcal{O}_{X/P}} \text{pr}_2^{-1} \mathcal{E}
\]
for each \( k \) and, therefore, induces the map
\[
d_1^{(k)} : \mathcal{J}_{X/P}^k (\mathcal{E}) \rightarrow \Omega_{X/P}^1 \otimes \mathcal{O}_{X/P} \mathcal{J}_{X/P}^{k-1} (\mathcal{E})
\]
The maps \( d_1^{(k)} \) for different values of \( k \) are compatible with the maps \( \pi_{l,k} \) giving rise to the canonical flat connection
\[ \nabla^\text{can}_{\mathcal{E}} : \mathcal{J}_{X/P}^\infty (\mathcal{E}) \rightarrow \Omega_{X/P}^1 \otimes \mathcal{O}_{X/P} \mathcal{J}_{X/P}^\infty (\mathcal{E}) \]

Let
\[
\mathcal{J}_{X,P} (\mathcal{E}) := \mathcal{O}_X \otimes \mathcal{O}_{X/P} \mathcal{J}_{X/P} (\mathcal{E}) \nabla_\mathcal{E} = \mathcal{J}_{X,P} (\mathcal{O}_{X/P}) \nabla_\mathcal{E} = \mathcal{J}_{X,P} / \mathcal{I} (\mathcal{O}_X) \nabla_\mathcal{E} = \ker (\mathcal{J}_{X,P} (\mathcal{O}_{X/P}) \rightarrow \mathcal{O}_X)
\]
Here and below by abuse of notation we write \((\bullet) \otimes \mathcal{O}_{X/P} \mathcal{J}_{X/P} (\mathcal{E})\) for \( \lim (\bullet) \otimes \mathcal{O}_{X/P} \mathcal{J}_{X/P}^\infty (\mathcal{E}) \).

The canonical flat connection extends to the flat connection
\[ \nabla^\text{can}_{\mathcal{E}} : \mathcal{J}_{X,P} (\mathcal{E}) \rightarrow \Omega_{X}^1 \otimes \mathcal{O}_X \mathcal{J}_{X,P} (\mathcal{E}) \]

3.5.1. **De Rham complexes.** Suppose that \( \mathcal{F} \) is an \( \mathcal{O}_X \)-module and \( \nabla : \mathcal{F} \rightarrow \Omega_{X}^1 \otimes \mathcal{O}_X \mathcal{F} \) is a flat connection. The flat connection \( \nabla \) extends uniquely to a differential \( \nabla^\flat \) on \( \Omega_{X}^1 \otimes \mathcal{O}_X \mathcal{F} \) subject to the Leibniz rule with respect to the \( \Omega_{X}^1 \)-module structure. We will make use of the following notation:
\[ (\Omega_{X}^1 \otimes \mathcal{O}_X \mathcal{F})^\text{cl} := \ker (\Omega_{X}^1 \otimes \mathcal{O}_X \mathcal{F} \nabla \rightarrow \Omega_{X}^{1+1} \otimes \mathcal{O}_X \mathcal{F}) \]
Suppose that \((\mathcal{F}^\bullet, d)\) is a complex of \(\mathcal{O}_X\)-modules with a flat connection \(\nabla = (\nabla^i)_{i \in \mathbb{Z}}\), i.e. for each \(i \in \mathbb{Z}\), \(\nabla^i\) is a flat connection on \(\mathcal{F}^i\) and \([d, \nabla] = 0\). Then, 
\((\Omega_X^0 \otimes \mathcal{O}_X \mathcal{F}^\bullet, \nabla, \text{Id} \otimes d)\) is a double complex. We denote by \(\text{DR}(\mathcal{F})\) the associated simple complex.

\(\mathcal{O}_X\)-modules equipped with flat connections form a closed monoidal category. In particular, the sheaf of \(\mathcal{O}_X\)-multilinear maps \(\mathcal{J}_X^{p} \to \mathcal{J}_X\) is endowed with a canonical flat connection induced by \(\nabla^{\text{can}}\). It follows directly from the definitions that \(\nabla^{\text{can}}\) preserves \(C^p(\mathcal{J}_X)\).

4. Formality for the algebroid Hochschild complex

Until further notice we work with a fixed \(C^\infty\) manifold \(X\) equipped with an integrable complex distribution \(\mathcal{P}\) and denote \(J_{X, \mathcal{P}}\) (respectively, \(J_{X, \mathcal{P}}, J_{X, \mathcal{P}}(\mathcal{E})\)) by \(J\) (respectively, \(\mathcal{J}, J(\mathcal{E})\)).

4.1. Hochschild cochains in formal geometry. In what follows we will be interested in the Hochschild complex in the context of formal geometry. To this end we define \(C^p(\mathcal{J})\) to be the sheaf of continuous (with respect to the adic topology) \(\mathcal{O}_X\)-multi-linear Hochschild cochains on \(\mathcal{J}\). The Gerstenhaber bracket endows \(\mathfrak{g}(\mathcal{J}) = C^\bullet(\mathcal{J})[1]\) with a structure of a graded Lie algebra. The product on \(\mathcal{J}\) is a global section of \(C^2(\mathcal{J})\), hence the Hochschild differential preserves \(C^\bullet(\mathcal{J})\).

The complex \(\Gamma(X; \text{DR}(C^\bullet(\mathcal{J}))) = (\Gamma(X; \Omega_X^0 \otimes C^\bullet(\mathcal{J}_X)), \nabla^{\text{can}} + \delta)\) is a differential graded brace algebra in a canonical way. The abelian Lie algebra \(J = C^0(\mathcal{J})\) acts on the brace algebra \(C^\bullet(\mathcal{J}_X)\) by derivations of degree \(-1\) via the restriction of the adjoint action with respect to the Gerstenhaber bracket. The above action factors through an action of \(\mathcal{J}\). Therefore, the abelian Lie algebra \(\Gamma(X; \Omega_X^0 \otimes \mathcal{J})\) acts on the brace algebra \(\Omega_X^0 \otimes C^\bullet(\mathcal{J})\) by derivations of degree \(+1\); the action of an element \(\alpha\) is denoted by \(\iota_\alpha\).

Due to commutativity of \(\mathcal{J}\), for any \(\omega \in \Gamma(X; \Omega_X^2 \otimes \mathcal{J})\) the operation \(\iota_\omega\) commutes with the Hochschild differential \(\delta\). If, moreover, \(\omega\) satisfies \(\nabla^{\text{can}}\omega = 0\), then \(\nabla^{\text{can}} + \delta + \iota_\omega\) is a square-zero derivation of degree one of the brace structure. We refer to the complex
\[
\Gamma(X; \text{DR}(C^\bullet(\mathcal{J})))\omega := (\Gamma(X; \Omega_X^0 \otimes C^\bullet(\mathcal{J})), \nabla^{\text{can}} + \delta + \iota_\omega)
\]
as the \(\omega\)-twist of \(\Gamma(X; \text{DR}(C^\bullet(\mathcal{J})))\).

Let
\[
\mathfrak{g}_{\text{DR}}(\mathcal{J})_\omega := \Gamma(X; \text{DR}(C^\bullet(\mathcal{J}_X))[1])_\omega
\]
regarded as a DGLA.

4.2. Formality for jets. Let \(V^\bullet(\mathcal{J}) = \text{Sym}_{\mathcal{J}}(\text{Der}_{\Omega_X^0(\mathcal{J})}^\text{cont}(\mathcal{J})[-1])\).

Working now in the category of graded \(\mathcal{O}_X\)-modules we have the diagram
\[
(\Omega_{e_2}(C^\bullet(\mathcal{J})), M) \xleftarrow{\Omega_{e_2}(\sigma)} \Omega_{e_2}(\Xi(\mathcal{J}), d_{V^\bullet(\mathcal{J})}) \xrightarrow{\nu} V^\bullet(\mathcal{J})
\]
of sheaves of differential graded \(\mathcal{O}_X\cdot e_2\)-algebras. According to the Theorem 2.1 the morphisms \(\Omega_{e_2}(\sigma)\) and \(\nu\) in (4.1) are quasi-isomorphisms. The sheaves of DGLA \(\Omega_{e_2}(C^\bullet(\mathcal{J})), M)[1]\) and \(C^\bullet(\mathcal{J})[1]\) are canonically \(L_\infty\)-quasi-isomorphic.

The canonical flat connection \(\nabla^{\text{can}}\) on \(\mathcal{J}\) induces a flat connection which we denote \(\nabla^{\text{can}}\) as well on each of the objects in the diagram (4.1). Moreover, the maps
The maps \((4.5)\)
\[
\Omega_{\mathcal{E}_2}(\sigma) \quad \text{and} \quad \nu \Rightarrow \nabla^\text{can} \quad \text{hence induce the maps of respective de Rham complexes}
\]
\[
(4.2) \quad \text{DR}(\Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M)) \xleftarrow{\text{DR}(\Omega_{\mathcal{E}_2}(\sigma))} \text{DR}(\Omega_{\mathcal{E}_2}(\Xi(\mathcal{J}), d_{V^*(\mathcal{J})})) \xrightarrow{\text{DR}(\nu)} \text{DR}(V^*(\mathcal{J})).
\]
All objects in the diagram \((4.2)\) have canonical structures of differential graded \(\mathfrak{e}_2\)-algebras and the maps are morphisms of such.

The DGLA \(\Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M)[1]\) and \(C^*(\mathcal{J})[1]\) are canonically \(L_\infty\)-quasi-isomorphic in a way compatible with \(\nabla^\text{can}\). Hence, the DGLA \(\text{DR}(\Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M)[1])\) and \(\text{DR}(C^*(\mathcal{J})[1])\) are canonically \(L_\infty\)-quasi-isomorphic.

4.3. Formality for jets with a twist. Suppose that \(\omega \in \Gamma(X; \Omega^2_X \otimes \mathcal{J})\) satisfies \(\nabla^\text{can} \cdot \omega = 0\).

For each of the objects in \((4.2)\) we denote by \(i_\omega\) the operation which is induced by the one described in \(2.3\) and the wedge product on \(\Omega^\bullet\). Thus, for each differential graded \(\mathfrak{e}_2\)-algebra \((N^\bullet, d)\) in \((4.2)\) we have a derivation of degree one and square zero \(i_\omega\) which anticommutes with \(d\) and we denote by \((N^\bullet, d_\omega)\) the \(\omega\)-twist of \((N^\bullet, d)\), i.e., the differential graded \(\mathfrak{e}_2\)-algebra \((N^\bullet, d + i_\omega)\). Since the morphisms in \((4.2)\) commute with the respective operations \(i_\omega\), they give rise to morphisms of respective \(\omega\)-twists
\[
(4.3) \quad \text{DR}(\Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M))_{\omega} \xleftarrow{\text{DR}(\Omega_{\mathcal{E}_2}(\sigma))} \text{DR}(\Omega_{\mathcal{E}_2}(\Xi(\mathcal{J}), d_{V^*(\mathcal{J})}))_{\omega} \xrightarrow{\text{DR}(\nu)} \text{DR}(V^*(\mathcal{J}))_{\omega}.
\]

Let \(G_*\Omega_X^\bullet\) denote the filtration given by \(G_i\Omega_X^\bullet = \Omega_X^\bullet = 1\). The filtration \(G_*\Omega_X^\bullet\) induces a filtration denoted \(G_i\text{DR}(K^\bullet, d)\) for each object \((K^\bullet, d)\) of \((4.3)\) defined by \(G_i\text{DR}(K^\bullet, d)_{\omega} = G_i\Omega_X^\bullet \otimes K^\bullet\). As is easy to see, the associated graded complex is given by
\[
(4.4) \quad G_i^{\text{GR}} \text{DR}(K^\bullet, d)_{\omega} = (\Omega_X^\bullet [-p] \otimes K^\bullet, \text{Id} \otimes d).
\]
It is clear that the morphisms \(\text{DR}(\Omega_{\mathcal{E}_2}(\sigma))\) and \(\text{DR}(\nu)\) are filtered with respect to \(G_*\).

**Theorem 4.1.** The morphisms in \((4.3)\) are filtered quasi-isomorphisms, i.e., the maps \(G_i^{\text{GR}} \text{DR}(\Omega_{\mathcal{E}_2}(\sigma))\) and \(G_i^{\text{GR}} \text{DR}(\nu)\) are quasi-isomorphisms for all \(i \in \mathbb{Z}\).

**Proof.** We consider the case of \(\text{DR}(\Omega_{\mathcal{E}_2}(\sigma))\) leaving \(G_i^{\text{GR}} \text{DR}(\nu)\) to the reader.

The map \(G_{-p} \text{DR}(\Omega_{\mathcal{E}_2}(\sigma))\) induced by \(\text{DR}(\Omega_{\mathcal{E}_2}(\sigma))\) on the respective associated graded objects in degree \(-p\) is equal to the map of complexes
\[
(4.5) \quad \text{Id} \otimes \Omega_{\mathcal{E}_2}(\sigma): \Omega_X^p \otimes \Omega_{\mathcal{E}_2}(\Xi(\mathcal{J}), d_{V^*(\mathcal{J})}) \to \Omega_X^p \otimes \Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M).
\]
The map \(\sigma\) is a quasi-isomorphism by Theorem 2.3, therefore so is \(\Omega_{\mathcal{E}_2}(\sigma)\). Since \(\Omega_X^\bullet\) is flat over \(\mathcal{O}_X\), the map \((4.5)\) is a quasi-isomorphism.

**Corollary 4.2.** The maps \(\text{DR}(\Omega_{\mathcal{E}_2}(\sigma))\) and \(\text{DR}(\nu)\) in \((4.3)\) are quasi-isomorphisms of sheaves of differential graded \(\mathfrak{e}_2\)-algebras.

Additionally, the DGLA \(\text{DR}(\Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M)[1])\) and \(\text{DR}(C^*(\mathcal{J})[1])\) are canonically \(L_\infty\)-quasi-isomorphic in a way which commutes with the respective operations \(i_\omega\), which implies that the respective \(\omega\)-twists \(\text{DR}(\Omega_{\mathcal{E}_2}(\mathbb{F}\mathcal{E}_2^\vee(C^*(\mathcal{J})), M)[1])_{\omega}\) and \(\text{DR}(C^*(\mathcal{J})[1])_{\omega}\) are canonically \(L_\infty\)-quasi-isomorphic.
5. $L_\infty$-STRUCTURES ON MULTIVECTORS

5.1. $L_\infty$-deformation complex. Recall that a graded vector space $W$ gives rise to the graded Lie algebra $\text{Der}(\text{coComm}(W[1]))$. An element $\mu \in \text{Der}(\text{coComm}(W[1]))$ of degree one is of the form $\mu = \sum_{i=0}^{\infty} \mu_i$ with $\mu_i : \wedge^i W \to W[2 - i]$. If $\mu_0 = 0$ and $[\mu, \mu] = 0$, then $\mu$ defines a structure of an $L_\infty$-algebra on $W$. (If $\mu_0$ is non-trivial, one obtains a "curved" $L_\infty$-algebra.)

An element $\mu$ as above determines a differential $\partial_\mu := [\mu, \cdot]$ on $\text{Der}(\text{coComm}(W[1]))$, such that $(\text{Der}(\text{coComm}(W[1])), \partial_\mu)$ is a differential graded Lie algebra.

If $g$ is a graded Lie algebra and $\mu$ is the element of $\text{Der}(\text{coComm}(g[1]))$ corresponding to the bracket on $g$, then $(\text{Der}(\text{coComm}(g[1])), \partial_\mu)$ is equal to the shifted Chevalley cochain complex $C^\bullet(g; g)[1]$.

5.2. $L_\infty$-structures on multivectors. The canonical pairing $(\cdot, \cdot) : \Omega^1_{X/\mathcal{P}} \otimes T_{X/\mathcal{P}} \to \mathcal{O}_X$ extends to the pairing

$$(\cdot, \cdot) : \Omega^1_{X/\mathcal{P}} \otimes V^\bullet(\mathcal{O}_{X/\mathcal{P}}) \to V^\bullet(\mathcal{O}_{X/\mathcal{P}})[-1]$$

For $k \geq 1$, $\omega = \alpha_1 \wedge \ldots \wedge \alpha_k$, $\alpha_i \in \Omega^1_{X/\mathcal{P}}$, $i = 1, \ldots, k$, let

$$\Phi(\omega) : \text{Sym}^k V^\bullet(\mathcal{O}_{X/\mathcal{P}})[2] \to V^\bullet(\mathcal{O}_{X/\mathcal{P}})[k]$$

denote the map given by the formula

$$\Phi(\omega)(\pi_1, \ldots, \pi_k) = \sum_{\sigma} \text{sgn}(\sigma) \langle \alpha_{\sigma(1)}, \pi_1 \rangle \wedge \ldots \wedge \langle \alpha_{\sigma(k)}, \pi_k \rangle,$$

where $|\pi| = l$ for $\pi \in V^l(\mathcal{O}_X)$. For $\alpha \in \mathcal{O}_X$ let $\Phi(\alpha) = \alpha \in V^0(\mathcal{O}_X)$.

In what follows we consider the (shifted) de Rham complex $\Omega^\bullet_{X/\mathcal{P}}[2]$ as a differential graded Lie algebra with the trivial bracket.

Lemma 5.1. The map $\omega \mapsto \Phi(\omega)$ defines a morphism of sheaves of differential graded Lie algebras

$$(5.1)\quad \Phi : \Omega^\bullet_{X/\mathcal{P}}[2] \to C^\bullet(V^\bullet(\mathcal{O}_{X/\mathcal{P}})[1]; V^\bullet(\mathcal{O}_{X/\mathcal{P}})[1])[1].$$

Proof. Recall the explicit formulas for the Schouten bracket. If $f$ and $g$ are functions and $X_i, Y_j$ are vector fields, then

$$[fX_1 \ldots X_k, gY_1 \ldots Y_l] = \sum_i (-1)^{i+1} fX_i(g)X_1 \ldots \hat{X}_i \ldots X_k Y_1 \ldots Y_l +$$

$$\sum_j (-1)^{j} gY_j(f)X_1 \ldots X_k Y_1 \ldots \hat{Y}_j \ldots Y_l +$$

$$\sum_{i,j} (-1)^{i+j} fX_i, Y_j X_1 \ldots \hat{X}_i \ldots X_k Y_1 \ldots \hat{Y}_j \ldots Y_l$$

Note that for a one-form $\omega$ and for vector fields $X$ and $Y$

$$(5.2)\quad \langle \omega, [X, Y] \rangle = \langle [\omega, X], Y \rangle - \langle X, [\omega, Y] \rangle = \Phi(d\omega)(X, Y)$$

From the two formulas above we deduce by an explicit computation that

$$\langle \omega, [\pi, \rho] \rangle = \langle [\omega, \pi], \rho \rangle - \langle \pi, [\omega, \rho] \rangle = \Phi(d\omega)(\pi, \rho)$$

Note that Lie algebra cochains are invariant under the symmetric group acting by permutations multiplied by signs that are computed by the following rule: a
permutation of \( \pi_i \) and \( \pi_j \) contributes a factor \((-1)^{|\pi_i|+|\pi_j|+|\pi_i|} \). We use the explicit formula for the bracket on the Lie algebra complex.

\[
[\Phi, \Psi] = \Phi \circ \Psi - (\Phi \circ \Psi) \circ \Phi
\]

\[
(\Phi \circ \Psi)(\pi_1, \ldots, \pi_{k+l-1}) = \sum_{I,J} \epsilon(I,J) \Phi(\Psi(\pi_i, \ldots, \pi_{i_k}), \pi_{j_1}, \ldots, \pi_{j_{l-1}})
\]

Here \( I = \{i_1, \ldots, i_k\} \); \( J = \{j_1, \ldots, j_{l-1}\} \); \( i_1 < \ldots < i_k \); \( j_1 < \ldots < j_{l-1} \); \( I \prod J = \{1, \ldots, k + l - 1\} \); the sign \( \epsilon(I,J) \) is computed by the same sign rule as above. The differential is given by the formula

\[
\partial \Phi = [m, \Phi]
\]

where \( m(\pi, \rho) = [\pi, \rho] \). Let \( \alpha = \alpha_1 \ldots \alpha_k \) and \( \beta = \beta_1 \ldots \beta_l \). We see from the above that both cochains \( \Phi(\alpha) \circ \Phi(\beta) \) and \( \Phi(\beta) \circ \Phi(\alpha) \) are antisymmetrizations with respect to \( \alpha_i \) and \( \beta_j \) of the sums

\[
\sum_{I,J,p} \pm \langle \alpha_1 \beta_1, \pi_p \rangle \langle \alpha_2, \pi_{i_1} \rangle \ldots \langle \alpha_k, \pi_{i_k-1} \rangle \langle \beta_2, \pi_{j_1} \rangle \ldots \langle \beta_1, \pi_{j_{l-1}} \rangle
\]

over all partitions \( \{1, \ldots, k + l - 1\} = I \prod J \prod \{p\} \) where \( i_1 < \ldots < i_{k-1} \) and \( j_1 < \ldots < j_{l-1} \); here \( \langle \alpha, \pi \rangle = \langle \alpha, \langle \beta, \pi \rangle \rangle \). After checking the signs, we conclude that \( [\Phi(\alpha), \Phi(\beta)] = 0 \). Also, from the definition of the differential, we see that \( \partial \Phi(\alpha, \pi_k, \ldots, \pi_{k+1}) \) is the antisymmetrizations with respect to \( \alpha_i \) and \( \beta_j \) of the sum

\[
\sum_{i < j} \pm \langle \alpha_1, [\pi_i, \pi_j] \rangle - \langle \alpha_1, \pi_i \rangle \pi_j - [\pi_i, \langle \alpha_1, \pi_j \rangle] \rangle.
\]

We conclude from this and (5.2) that \( \partial \Phi(\alpha) = \Phi(d\alpha) \).

5.3. \( L_\infty \)-structures on multivectors via formal geometry. Let \( C^\bullet(V^\bullet(\mathcal{J})[1]; V^\bullet(\mathcal{J})[1]) \) denote the complex of continuous \( \mathcal{O}_X \)-linear multilinear cochains.

Let \( \hat{\Omega}_J^k \) denote the (\( \mathcal{O}_X \)-linear) differential in \( \hat{\Omega}_J^\bullet \) induced by the de Rham differential in \( \Omega^\bullet_X \). The differential \( \hat{\partial}_R \) is horizontal with respect to the canonical flat connection \( \nabla^{can} \) on \( \hat{\Omega}_J^\bullet \), hence we have the double complex \( \Omega_X^\bullet \otimes \hat{\Omega}_J^\bullet, \nabla^{can}, \text{Id} \otimes \hat{\partial}_R \) whose total complex is denoted \( \text{DR}(\hat{\Omega}_J^\bullet) \).

The map of differential graded Lie algebras

\[
\hat{\Phi}: \hat{\Omega}_J^\bullet[2] \rightarrow C^\bullet(V^\bullet(\mathcal{J})[1]; V^\bullet(\mathcal{J})[1])[1]
\]

defined in the same way as (5.1) is horizontal with respect to the canonical flat connection \( \nabla^{can} \) and induces the map

\[
\text{DR}(\hat{\Phi}): \text{DR}(\hat{\Omega}_J^\bullet)[2] \rightarrow \text{DR}(C^\bullet(V^\bullet(\mathcal{J})[1]; V^\bullet(\mathcal{J})[1])[1])
\]

There is a canonical morphism of sheaves of differential graded Lie algebras

\[
\text{DR}(C^\bullet(V^\bullet(\mathcal{J})[1]; V^\bullet(\mathcal{J})[1])[1]) \rightarrow C^\bullet(\text{DR}(V^\bullet(\mathcal{J})[1]); \text{DR}(V^\bullet(\mathcal{J})[1])[1])
\]

Therefore, a degree three cocycle in \( \Gamma(X; \text{DR}(F_{-1} \hat{\Omega}_J^\bullet)) \) determines an \( L_\infty \)-structure on \( \text{DR}(V^\bullet(\mathcal{J})[1]) \) and cohomologous cocycles determine \( L_\infty \)-isomorphic structures.

Notation. For a section \( B \in \Gamma(X; \Omega^2_X \otimes \mathcal{J}) \) we denote by \( \hat{B} \) its image in \( \Gamma(X; \Omega^2_X \otimes \hat{\mathcal{J}}) \).
Lemma 5.2. If $B \in \Gamma(X;\Omega^2_X \otimes \mathcal{J})$ satisfies $\nabla^{can}\overline{B} = 0$, then

1. $\hat{d}_R B$ is a (degree three) cocycle in $\Gamma(X;DR(F_{-1}\hat{\Omega}^\mathcal{J}_{/O}))$.
2. The $L_\infty$-structure induced by $\hat{d}_R B$ is that of a differential graded Lie algebra equal to $DR(V^\bullet(\mathcal{J})[1])_{\overline{B}}$.

Proof. For the first claim it suffices to show that $\nabla^{can}\hat{d}_R B = 0$. This follows from the assumption that $\nabla^{can}\overline{B} = 0$ and the fact that $\hat{d}_R : \Omega^\bullet_X \otimes \mathcal{J} \to \Omega^\bullet_X \otimes \hat{\Omega}^1_{\mathcal{J}/O}$ factors through $\Omega^\bullet_X \otimes \mathcal{J}$.

The proof of the second claim is left to the reader. □

Notation. For a 3-cocycle

$$\omega \in \Gamma(X;DR(F_{-1}\hat{\Omega}^\mathcal{J}_{/O}))$$

we will denote by $\text{DR}(V^\bullet(\mathcal{J})[1])_{\omega}$ the $L_\infty$-algebra obtained from $\omega$ via (5.4) and (5.5). Let

$$\text{sDR}(\mathcal{J})_\omega := \Gamma(X;\text{DR}(V^\bullet(\mathcal{J})[1]))_{\omega}.$$

Remark 5.3. Lemma 5.2 shows that this notation is unambiguous with reference to the previously introduced notation for the twist. In the notation introduced above, $\hat{d}_R B$ is the image of $\overline{B}$ under the injective map $\Gamma(X;\Omega^2_X \otimes \mathcal{J}) \to \Gamma(X;\Omega^2_X \otimes \hat{\Omega}^1_{\mathcal{J}/O})$ which factors $\hat{d}_R$ and “allows” us to “identify” $\overline{B}$ with $\hat{d}_R B$.

5.4. Dolbeault complexes. We shall assume that the manifold $X$ admits two complementary integrable complex distributions $\mathcal{P}$ and $\mathcal{Q}$. We do suppose that $\mathcal{P} \cap \mathcal{Q} = 0$ and $\mathcal{P} \oplus \mathcal{Q} = \mathcal{T}_X^C$. The latter decomposition induces a bi-grading on differential forms: $\Omega^p_X = \bigoplus_{p+q=n} \Omega^p_X$ with $\Omega^p_X = \bigwedge^p \mathcal{P}^\perp \otimes \bigwedge^q \mathcal{Q}^\perp$. The bi-grading splits the Hodge filtration: $F_{-i}\Omega^n = \bigoplus_{p\geq i} \Omega^{p,n-p}_X$.

Two cases of particular interest in applications are

- $\mathcal{P} = 0$
- $\mathcal{P}$ is a complex structure, i.e. $\overline{\mathcal{P}} = \mathcal{Q}$.

The map (5.4) extends to the morphism of sheaves of DGLA

$$\Phi : \Omega^\bullet_X[2] \to C^\bullet(\Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p})[1]; \Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p})[1])[1].$$

Let $F_\bullet(\Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p}))$ denote the filtration defined by $F_{-i}(\Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p})) = \bigoplus_{p\geq i} \Omega^0_X \otimes \mathcal{O}_{X/p} V^p(\mathcal{O}_{X/p})$. The complex

$$C^\bullet(\Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p})[1]; \Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p})[1])[1]$$

carries the induced filtration.

We leave the verification of the following claim to the reader.

Lemma 5.4. The map (5.6) is filtered.

Thus, the image under (5.6) of a closed 3-form $H \in \Gamma(X;F_{-1}\Omega^3_X)$, $dH = 0$, gives rise to a structure of an $L_\infty$-algebra on $\Omega^0_X \otimes \mathcal{O}_{X/p} V^\bullet(\mathcal{O}_{X/p})[1]$ (whereas general closed 3-forms give rise to curved $L_\infty$-structures). Moreover, cohomologous closed 3-forms give rise to gauge equivalent Maurer-Cartan elements, hence to $L_\infty$-isomorphic $L_\infty$-structures.
Notation. For $H$ as above we denote by $\mathfrak{s}(\mathcal{O}_{X/P})_H$ the $\mathcal{P}$-Dolbeault complex of the sheaf of multi-vector fields equipped with the corresponding $L_\infty$-algebra structure:

\[
\mathfrak{s}(\mathcal{O}_{X/P})_H = \Gamma(X; \Omega^0_X \otimes_{\mathcal{O}_{X/P}} V^\bullet (\mathcal{O}_{X/P}))[1]
\]

Remark 5.5. In the case when $\mathcal{P} = 0$, in other words, $X$ is a plain $C^\infty$ manifold, the map \((5.6)\) simplifies to

\[
\Phi : \Omega_X^*[2] \to C^\bullet (V^\bullet (\mathcal{O}_X)[1]; V^\bullet (\mathcal{O}_X)[1])[1]
\]

and $\mathfrak{s}(\mathcal{O}_{X/P}) = \mathfrak{s}(\mathcal{O}_X) = \Gamma(X; V^\bullet (\mathcal{O}_X))[1]$, a DGLA with the Schouten bracket and the trivial differential. These are the unary and the binary operations in the $L_\infty$-structure on $\mathfrak{s}(\mathcal{O}_X)_H$, $H$ a closed 3-form on $X$; the ternary operation is induced by $H$ and all operations of higher valency are equal to zero. The $L_\infty$-structure on multi-vector fields induced by a closed three-form appeared earlier in \([20]\) and \([21]\). \hfill \Box

5.5. Formal geometry vs. Dolbeault. Compatibility of the two constructions, one using formal geometry, the other using Dolbeault resolutions, is the subject of the next theorem.

Theorem 5.6. Suppose given $B \in \Gamma(X; \Omega^3_X \otimes \mathcal{J})$ and $H \in \Gamma(X; F^{-1} \Omega^3_X)$ such that $dH = 0$ and $j^\infty(H)$ is cohomologous to $\mathfrak{d}_{\hat{\mathcal{R}}} B$ in $\Gamma(X; \mathcal{D}_{\hat{\mathcal{R}}}(F^{-1} \hat{\Omega}^*_X))$. Then, the $L_\infty$-algebras $\mathfrak{g}_{\hat{\mathcal{R}}}(\mathcal{J}/B)$ and $\mathfrak{s}(\mathcal{O}_{X/P})_H$ are $L_\infty$-quasi-isomorphic.

Before embarking upon a proof of Theorem 5.6, we introduce some notations. Let $
abla^0, \nabla^1 = \mathcal{J}_X(\Omega^0_X)$, $\nabla^n = \bigoplus_{p+q=n} \Omega^p_X \otimes \mathcal{J}_X(\Omega^q_X)$. The differentials $\hat{\partial}$ and $\hat{\delta}$ induce, respectively, the differentials $\partial$ and $\delta$ in $\hat{\Omega}^\bullet_X$ which are horizontal with respect to the canonical flat connection. The complex $\hat{\Omega}^\bullet_X$ with the differential $\hat{\delta}$ is a resolution of $\nabla^{0,\partial}_\mathcal{O}$ and $\hat{\Omega}^\bullet_{\mathcal{O} \otimes \mathcal{J}} \hat{\Omega}^\bullet_X = \hat{\Omega}^\bullet_X$.

The filtration on $\hat{\Omega}^\bullet_X$ is defined by $F_i \hat{\Omega}^\bullet_X = \mathcal{J}_X(F_i \hat{\Omega}^\bullet_X)$. The map $j^\infty : \Omega^\bullet_X \to DR(\hat{\Omega}^\bullet_X)$ is a filtered quasi-isomorphism.

The map \((5.3)\) extends to the map of DGLA

\[
\hat{\Phi} : \hat{\Omega}^\bullet_X[2] \to C^\bullet (\hat{\Omega}^0_X \otimes \mathcal{J} V^\bullet (\mathcal{J}))[1]; \hat{\Omega}^0_X \otimes \mathcal{J} V^\bullet (\mathcal{J})[1])[1]
\]

which gives rise to the map of DGLA

\[
\mathfrak{D}(\hat{\Phi}) : \mathfrak{D}(\hat{\Omega}^\bullet_X[2]) \to C^\bullet (\mathfrak{D}(\hat{\Omega}^0_X \otimes \mathcal{J} V^\bullet (\mathcal{J}))[1]); \mathfrak{D}(\hat{\Omega}^0_X \otimes \mathcal{J} V^\bullet (\mathcal{J})[1])[1]
\]

Therefore, a degree three cocycle in $\Gamma(X; \mathfrak{D}(F^{-1} \hat{\Omega}^\bullet_X[2])$ determines an $L_\infty$-structure on $\mathfrak{D}(\hat{\Omega}^\bullet_X \otimes \mathcal{J} V^\bullet (\mathcal{J})[1]$ and cohomologous cocycles determine $L_\infty$-quasi-isomorphic structures.

Notation. For a degree three cocycle $\omega$ in $\Gamma(X; \mathfrak{D}(F^{-1} \hat{\Omega}^\bullet_X[2])$ we denote by $\mathfrak{D}(\hat{\Omega}^\bullet_X \otimes \mathcal{J} V^\bullet (\mathcal{J})[1])_\omega$ the $L_\infty$-algebra obtained via \((5.7)\).

Proof of Theorem 5.6. The map

\[
j^\infty : \Omega^0_X \otimes_{\mathcal{O}_{X/P}} V^\bullet (\mathcal{O}_{X/P})[1) \to \mathfrak{D}(\hat{\Omega}^\bullet_X \otimes \mathcal{J} V^\bullet (\mathcal{J})[1])
\]
induces a quasi-isomorphism of sheaves of $L_\infty$-algebras

\[ j^\infty: (\Omega^{0,\bullet}_X \otimes_{\mathcal{O}_X/\mathcal{P}} V^\bullet(\mathcal{O}_X/\mathcal{P})[1])_H \rightarrow \mathcal{DR}(\hat{\Omega}^{0,\bullet}_X \otimes_{\mathcal{J}} V^\bullet(\mathcal{J})[1])_{j^\infty(H)}. \]

Since, by assumption, $j^\infty(H)$ is cohomologous to $\hat{d}_R B$ in $\Gamma(X; \mathcal{DR}(\hat{F}_1 \hat{\Omega}^{\bullet}_X))$ the $L_\infty$-algebras $\mathcal{DR}(\hat{\Omega}^{0,\bullet}_X \otimes_{\mathcal{J}} V^\bullet(\mathcal{J})[1])_{j^\infty(H)}$ and $\mathcal{DR}(\hat{\Omega}^{0,\bullet}_X \otimes_{\mathcal{J}} V^\bullet(\mathcal{J})[1])_{\hat{d}_R B}$ are $L_\infty$-quasi-isomorphic.

The quasi-isomorphism $V^\bullet(\mathcal{J})[1] \rightarrow \hat{\Omega}^{0,\bullet}_X \otimes_{\mathcal{J}} V^\bullet(\mathcal{J})[1]$ induces the quasi-isomorphism of sheaves of $L_\infty$-algebras

\[ \mathcal{DR}(V^\bullet(\mathcal{J})[1])_{\hat{d}_R B} \rightarrow \mathcal{DR}(\hat{\Omega}^{0,\bullet}_X \otimes_{\mathcal{J}} V^\bullet(\mathcal{J})[1])_{\hat{d}_R B}. \]

The former is equal to the DGLA $\mathcal{DR}(V^\bullet(\mathcal{J})[1])_{\hat{B}}$ by Lemma 5.2

According to Corollary 4.2 the sheaf of DGLA $\mathcal{DR}(V^\bullet(\mathcal{J})[1])_{\hat{B}}$ is $L_\infty$-quasi-isomorphic to the DGLA deduced form the differential graded $e_2$-algebra $\mathcal{DR}(\Omega_{e_2}(\mathcal{F}_{e_2}(\mathcal{C}^\bullet(\mathcal{J})), M))_{\hat{B}}$.

The latter DGLA is $L_\infty$-quasi-isomorphic to $\mathcal{DR}(\mathcal{C}^\bullet(\mathcal{J})[1])_{\hat{B}}$.

Passing to global sections we conclude that $\mathcal{G}_{\mathcal{DR}(\mathcal{J})}^{j^\infty(H)}$ and $\mathcal{G}_{\mathcal{DR}(\mathcal{J})_{\hat{B}}}$ are $L_\infty$-quasi-isomorphic. Together with (5.9) this implies the claim. \qed

6. Deformations of algebroid stacks

6.1. Algebroid stacks. Here we give a very brief overview, referring the reader to [11 12] for the details. Let $k$ be a field of characteristic zero, and let $R$ be a commutative $k$-algebra.

**Definition 6.1.** A stack in $R$-linear categories $\mathcal{C}$ on $X$ is an $R$-algebroid stack if it is locally nonempty and locally connected, i.e. satisfies

1. any point $x \in X$ has a neighborhood $U$ such that $\mathcal{C}(U)$ is nonempty;
2. for any $U \subseteq X$, $x \in U$, $A, B \in \mathcal{C}(U)$ there exits a neighborhood $V \subseteq U$ of $x$ and an isomorphism $A|_V \cong B|_V$.

For a prestack $\mathcal{C}$ we denote by $\tilde{\mathcal{C}}$ the associated stack.

For a category $C$ denote by $iC$ the subcategory of isomorphisms in $C$; equivalently, $iC$ is the maximal subgroupoid in $C$. If $\mathcal{C}$ is an algebroid stack then the stack associated to the substack of isomorphisms $i\mathcal{C}$ is a gerbe.

For an algebra $K$ we denote by $K^+$ the linear category with a single object whose endomorphism algebra is $K$. For a sheaf of algebras $\mathcal{K}$ on $X$ we denote by $K^+$ the prestack in linear categories given by $U \mapsto \mathcal{K}(U)^+$. Let $\tilde{K}^+$ denote the associated stack. Then, $\tilde{K}^+$ is an algebroid stack equivalent to the stack of locally free $\mathcal{K}^{op}$-modules of rank one.

By a twisted form of $K$ we mean an algebroid stack locally equivalent to $\tilde{K}^+$. The equivalence classes of twisted forms of $K$ are in bijective correspondence with $H^2(X; \mathbb{Z}(K)^+)$, where $\mathbb{Z}(K)$ denotes the center of $K$. To see this note that there is a canonical monoidal equivalence of stacks in monoidal categories $\alpha: i\mathbb{Z}(K)^+ \rightarrow \mathbb{A}ut(\tilde{K}^+)$. Here, $\mathbb{Z}(K)^+$ is the stack of locally free modules of rank one over the commutative algebra $\mathbb{Z}(K)$ and isomorphisms thereof with the monoidal structure given by the tensor product; $\mathbb{A}ut(\tilde{K}^+)$ is the stack of auto-equivalences of $\tilde{K}^+$. The functor $\alpha$ is given by $\alpha(a)(L) = a \otimes \mathbb{Z}(K) L$ for $a \in \mathbb{Z}(K)^+$ and $L \in \tilde{K}^+$. The inverse associates to an auto-equivalence $F$ the $\mathbb{Z}(K)$-module $\text{Hom}(\text{Id}, F)$. 

6.2. Twisted forms of $\mathcal{O}$. Twisted forms of $\mathcal{O}_{X/P}$ are in bijective correspondence with $\mathcal{O}_{X/P}^\times$-gerbes: if $\mathcal{S}$ is a twisted form of $\mathcal{O}_{X/P}$, the corresponding gerbe is the substack $\mathcal{iS}$ of isomorphisms in $\mathcal{S}$. We shall not make a distinction between the two notions. The equivalence classes of twisted forms of $\mathcal{O}_{X/P}$ are in bijective correspondence with $H^2(X; \mathcal{O}_{X/P}^\times)$.

The composition

$$
\mathcal{O}_{X/P}^\times \to \mathcal{O}_{X/P}^\times / \mathbb{C}^\times \xrightarrow{\log} \mathcal{O}_{X/P} / \mathbb{C} \xrightarrow{j^\infty} \text{DR}(\mathcal{J})
$$

induces the map $H^2(X; \mathcal{O}_{X/P}^\times) \to H^2(X; \text{DR}(\mathcal{J})) \cong H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}), \nabla^{\text{can}})$. We denote by $[\mathcal{S}]$ the image in the latter space of the class of $\mathcal{S}$. Let $\mathcal{B} \in \Gamma(X; \Omega_X^2 \otimes \mathcal{J})$ denote a representative of $[\mathcal{S}]$. Since the map $\Gamma(X; \Omega_X^2 \otimes \mathcal{J}) \to \Gamma(X; \Omega_X^\bullet \otimes \mathcal{J})$ is surjective, there exists a $B \in \Gamma(X; \Omega_X^2 \otimes \mathcal{J})$ lifting $\mathcal{B}$.

The quasi-isomorphism $j^\infty : F_{-1}\Omega^\bullet \to \text{DR}(F_{-1}\Omega^\bullet)$ induces the isomorphism

$$
H^2(X; \text{DR}(F_{-1}\Omega^\bullet)[1]) \cong H^2(X; F_{-1}\Omega_X^\bullet[1]) = H^3(\Gamma(X; F_{-1}\Omega_X^\bullet)).
$$

Let $H \in \Gamma(X; F_{-1}\Omega_X^2)$ denote the closed form which represents the class of $\tilde{a}_R B$.

6.3. Deformations of linear stacks. Here we describe the notion of 2-groupoid of deformations of an algebroid stack. We follow \cite{2} and refer the reader to that paper for all the proofs and additional details.

For an $R$-linear category $\mathcal{C}$ and homomorphism of algebras $R \to S$ we denote by $\mathcal{C} \otimes_R S$ the category with the same objects as $\mathcal{C}$ and morphisms defined by $\text{Hom}_{\mathcal{C} \otimes_R S}(A, B) = \text{Hom}_\mathcal{C}(A, B) \otimes_R S$.

For a prestack $\mathcal{C}$ in $R$-linear categories we denote by $\mathcal{C} \otimes_R S$ the prestack associated to the fibered category $U \mapsto \mathcal{C}(U) \otimes_R S$.

**Lemma 6.2** (\cite{2}, Lemma 4.13). Suppose that $\mathcal{A}$ is a sheaf of $R$-algebras and $\mathcal{C}$ is an $R$-algebroid stack. Then $\mathcal{C} \otimes_R S$ is an algebroid stack.

Suppose now that $\mathcal{C}$ is a stack in $k$-linear categories on $X$ and $R$ is a commutative Artin $k$-algebra. We denote by $\text{Def}(\mathcal{C})(R)$ the 2-category with

- objects: pairs $(\mathcal{B}, \varpi)$, where $\mathcal{B}$ is a stack in $R$-linear categories flat over $R$ and $\varpi : \mathcal{B} \otimes_R k \to \mathcal{C}$ is an equivalence of stacks in $k$-linear categories
- 1-morphisms: a 1-morphism $(\mathcal{B}^{(1)}, \varpi^{(1)}) \to (\mathcal{B}^{(2)}, \varpi^{(2)})$ is a pair $(F, \theta)$ where $F : \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}$ is a $R$-linear functor and $\theta : \varpi^{(2)} \circ (F \otimes_R k) \to \varpi^{(1)}$ is an isomorphism of functors
- 2-morphisms: a 2-morphism $(F', \theta') \to (F'', \theta'')$ is a morphism of $R$-linear functors $\kappa : F' \to F''$ such that $\theta'' \circ (\text{Id}_{\varpi^{(2)}} \otimes (\kappa \otimes_R k)) = \theta'$

The 2-category $\text{Def}(\mathcal{C})(R)$ is a 2-groupoid.

Let $\mathcal{B}$ be a prestack on $X$ in $R$-linear categories. We say that $\mathcal{B}$ is flat if for any $U \subseteq X$, $A, B \in \mathcal{B}(U)$ the sheaf $\text{Hom}_\mathcal{B}(A, B)$ is flat (as a sheaf of $R$-modules).

**Lemma 6.3** (\cite{2}, Lemma 6.2). Suppose that $\mathcal{B}$ is a flat $R$-linear stack on $X$ such that $\mathcal{B} \otimes_R k$ is an algebroid stack. Then $\mathcal{B}$ is an algebroid stack.
6.4. **Deformations of twisted forms of \( \mathcal{O} \).** Suppose that \( \mathcal{S} \) is a twisted form of \( \mathcal{O}_X \). We will now describe the DGLA controlling the deformations of \( \mathcal{S} \).

Recall the DGLA

\[
\mathfrak{g}_{\text{DR}}(\mathcal{J}) := \Gamma(X; \mathcal{DR}(C^\bullet(\mathcal{J}))[1]) \omega
\]

introduced in (4.1). It satisfies the vanishing condition \( \mathfrak{g}_{\text{DR}}(\mathcal{J})^i \omega = 0 \) for \( i \leq -2 \).

For a nilpotent DGLA \( \mathfrak{g} \) which satisfies \( \mathfrak{g}^i = 0 \) for \( i \leq -2 \), following Deligne and Getzler one associates the (strict) Deligne-Getzler 2-groupoid \( \text{MC}^2(\mathfrak{g}) \) (see [3] 3.3.2).

The following theorem is proved in [2] (Theorem 1 of loc. cit.):

**Theorem 6.4.** For any Artin algebra \( R \) with maximal ideal \( m_R \) there is an equivalence of 2-groupoids

\[
\text{MC}^2(\mathfrak{g}_{\text{DR}}(\mathcal{J}) \otimes m_R) \cong \text{Def}(\mathcal{S})(R)
\]

natural in \( R \).

The main result of the present paper (Theorem 6.5 below) is a description of \( \text{Def}(\mathcal{S})(R) \) in terms of the \( L_\infty \)-algebra \( \mathfrak{s}(\mathcal{O}_X/P) \) defined in [5,3] in the situation when \( X \) is a \( C^\infty \)-manifold which admits a pair of complementary integrable complex distributions \( \mathcal{P} \) and \( \mathcal{Q} \). The statement of the result, which is analogous to that of Theorem 6.4, necessitates an extension of the domain of the Deligne-Getzler 2-groupoid functor to nilpotent \( L_\infty \)-algebras satisfying the same vanishing condition.

To this end we set, for a nilpotent \( L_\infty \)-algebra \( \mathfrak{g} \) which satisfies \( \mathfrak{g}^i = 0 \) for \( i \leq -2 \),

\[
(6.1) \quad \text{MC}^2(\mathfrak{g}) := \text{Bic} \Pi_2(\Sigma(\mathfrak{g}))
\]

where \( \Sigma(\mathfrak{g}) \) is the Kan simplicial set defined for any nilpotent \( L_\infty \)-algebra (see [3] 3.2 for the definition and properties) and \( \Pi_2 \) is the projector on Kan simplicial sets of Duskin ([8]) which is supplied with a natural transformation \( \text{Id} \to \Pi_2 \). The latter transformation induces isomorphisms on sets of connected components as well as homotopy groups in degrees one and two (component by component), while higher homotopy groups of a simplicial set in the image of \( \Pi_2 \) vanish (component by component).

In [8] the image of \( \Pi_2 \) is characterized as the simplicial sets arising as simplicial nerves of bi-groupoids (see [3] 2.1.3 and 2.2) and \( \text{Bic} \) denotes the functor which “reads the bi-groupoid off” the combinatorics of its simplicial nerve, i.e., for example, the simplicial set \( \Pi_2(\Sigma(\mathfrak{g})) \) is the simplicial nerve of \( \text{Bic} \Pi_2(\Sigma(\mathfrak{g})) \).

For a nilpotent DGLA \( \mathfrak{g} \) which satisfies \( \mathfrak{g}^i = 0 \) for \( i \leq -2 \), \( \text{MC}^2(\mathfrak{g}) \) was defined earlier as the Deligne-Getzler 2-groupoid of \( \mathfrak{g} \) ([3], 3.3.2). The definition (6.1) is justified by the principal result of [3] (Theorem 3.7, alternatively, Theorem 6.6) which states that, for a nilpotent DGLA \( \mathfrak{g} \) which satisfies \( \mathfrak{g}^i = 0 \) for \( i \leq -2 \), \( \Sigma(\mathfrak{g}) \) and the simplicial nerve of the Deligne-Getzler 2-groupoid of \( \mathfrak{g} \) are canonically homotopy equivalent. This implies that the Deligne-Getzler 2-groupoid of \( \mathfrak{g} \) is canonically equivalent to \( \text{Bic} \Pi_2(\Sigma(\mathfrak{g})) \).

**Theorem 6.5.** Suppose that \( X \) is a \( C^\infty \)-manifold equipped with a pair of complementary complex integrable distributions \( \mathcal{P} \) and \( \mathcal{Q} \), and \( \mathcal{S} \) is a twisted form of \( \mathcal{O}_{X/P}(\mathcal{J}) \). Let \( H \in \Gamma(X; F_{-1}\Omega^3_X) \) be a representative of \( [\mathcal{S}] \) (6.2). Then, for any Artin algebra \( R \) with maximal ideal \( m_R \) there is an equivalence of bi-groupoids

\[
\text{MC}^2(\mathfrak{s}(\mathcal{O}_{X/P})_H \otimes m_R) \cong \text{Def}(\mathcal{S})(R),
\]

where the left-hand side is defined as in (6.1), natural in \( R \).
Proof. We refer the reader to [3] for notations.

By Theorem 5.6, \(\mathfrak{s}(O_{X/P})_H\) is \(L_\infty\)-quasi-isomorphic to \(\mathfrak{g}_{\text{DR}}(J)\). Proposition 3.4 of [3] implies that \(\Sigma(\mathfrak{s}(O_{X/P})_H \otimes m_R)\) is weakly equivalent to \(\Sigma(\mathfrak{g}_{\text{DR}}(J) \otimes m_R)\). In particular, \(\Sigma(\mathfrak{s}(O_{X/P})_H \otimes m_R)\) is a Kan simplicial set with homotopy groups vanishing in dimensions larger than two. By Duskin (cf. [8]), the natural transformation \(\text{Id} \to \Pi_2\) induces a homotopy equivalence between \(\Sigma(\mathfrak{s}(O_{X/P})_H \otimes m_R)\) and \(\mathfrak{M}\mathcal{C}^2(\mathfrak{s}(O_{X/P})_H \otimes m_R)\), the nerve of the two-groupoid \(\mathfrak{M}\mathcal{C}^2(\mathfrak{s}(O_{X/P})_H) \otimes m_R\).

On the other hand, by Theorem 3.7 (alternatively, Theorem 6.6) of [3], \(\Sigma(\mathfrak{g}_{\text{DR}}(J) \otimes m_R)\) is equivalent to \(\mathfrak{M}\mathcal{G}^2(\mathfrak{g}_{\text{DR}}(J)) \otimes m_R\). Combining all of the above equivalences we obtain an equivalence of 2-groupoids

\[
\mathfrak{M}\mathcal{C}^2(\mathfrak{s}(O_{X/P})_H \otimes m_R) \cong \mathfrak{M}\mathcal{G}^2(\mathfrak{g}_{\text{DR}}(J) \otimes m_R)
\]

The result now follows from Theorem 6.4. \(\square\)

Remark 6.6. In the case when \(P = 0\), i.e. \(X\) is a plain \(C^\infty\)-manifold isomorphism classes of formal deformations of \(S\) are in bijective correspondence with equivalence classes of Maurer-Cartan elements of the \(L_\infty\)-algebra \(\mathfrak{g}_{\text{DR}}(O_X) \otimes m_R\). These are the formal twisted Poisson structures in the terminology of [21], i.e. elements \(\pi \in \Gamma(X; \wedge^2 \mathcal{T}_X) \otimes m_R\), satisfying the equation

\[
[\pi, \pi] = \Phi(H)(\pi, \pi, \pi).
\]

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