THE AUTOMORPHISM GROUP OF A VALUED FIELD
OF GENERALISED FORMAL POWER SERIES

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ABSTRACT. Let \( k \) be a field, \( G \) a totally ordered abelian group and \( \mathbb{K} = k((G)) \) the maximal field of generalised power series, endowed with the canonical valuation \( v \) \([Hah07]\). We study the group \( v\text{-Aut} \mathbb{K} \) of valuation preserving automorphisms of a subfield \( k(G) \subset \mathbb{K} \), where \( k(G) \) is the fraction field of the group ring \( k[G] \). Under the assumption that \( \mathbb{K} \) satisfies two lifting properties we are able to generalise and refine the decomposition of \( v\text{-Aut} \mathbb{K} \) \([Hof91]\) and prove a structure theorem decomposing \( v\text{-Aut} \mathbb{K} \) into a 4-factor semi-direct product of notable subgroups. We then identify a large class of Hahn fields satisfying the two aforementioned lifting properties. Next we focus on the group of strongly additive automorphisms of \( \mathbb{K} \). We give an explicit description of the group of strongly additive internal automorphisms in terms of the groups of homomorphisms \( \text{Hom}(G,k \times) \) of \( G \) into \( k \times \) and \( \text{Hom}(G,1 + I_\mathbb{K}) \) of \( G \) into the group of 1-units of the valuation ring of \( K \). Finally, we specialise our results to some relevant special cases. In particular, we extend the work of Schilling \([Sch44]\) on the field of Laurent series and that of \([Des05]\) on the field of Puiseux series.

1. Introduction

In his paper \([Sch44]\), Schilling studied the \( k \)-automorphisms of the field \( L = k((\mathbb{Z})) \) of formal Laurent series over a coefficient field \( k \). First, he observed \([Sch44, \text{Lemma } 1]\) that all \( k \)-automorphisms are necessarily valuation preserving. Moreover, since the ordered abelian group \( Z \) admits only the trivial automorphism, all \( k \)-automorphisms of \( L \) are internal (Definition 3.1.1).

Schilling then proves that the group of \( k \)-automorphisms is isomorphic to the group of units of the valuation ring of \( L \) endowed with a particular group operation (see Section 5.2 for more details).

In this paper, inspired by Schilling’s ideas, we study the group \( v\text{-Aut} \mathbb{K} \) of valuation preserving automorphisms of a Hahn field \( \mathbb{K} \), and its subgroups \( v\text{-Aut}_{(k)} \mathbb{K} \), \( v\text{-Aut}_{k} \mathbb{K} \) of \( k \)-stable resp. \( k \)-automorphisms (Notation 2.1.6(ii)). Hahn fields are distinguished subfields of fields of generalised power series \( \mathbb{K} = k((G)) \) for a field \( k \) and a totally ordered abelian group \( G \) (Definition 2.1.1). The group \( \text{Int Aut} \mathbb{K} \) of internal automorphisms is an important normal subgroup of \( v\text{-Aut} \mathbb{K} \). In order to further analyse \( v\text{-Aut} \mathbb{K} \), we impose an additional assumption on \( K \). A Hahn field...

Date: April 12, 2022
2020 Mathematics Subject Classification: 13J05 (12J20 16W60 06F20).
such that every pair \((\rho, \tau) \in \text{Aut } k \times o\text{-Aut } G\) lifts to an automorphism of \(K\) is said to have the **first lifting property** (Notation 2.1.6 and Definition 3.1.3). This allows us to identify another important subgroup, namely the group \(\text{Ext Aut } K\) of external valuation preserving automorphisms (Definition 3.1.6).

A result of Hofberger [Hof91, Satz 2.2] allows to present the group \(v\text{-Aut } K\) of the maximal Hahn field \(K\) as the semi-direct product \(\text{Int Aut } K \rtimes \text{Ext Aut } K\). In Theorem 3.2.1 we generalise Hofberger’s result and show that all Hahn fields which satisfy the first lifting property admit Hofberger’s decomposition.

We further refine Hofberger’s result by first noticing that the group \(\text{Ext Aut } K\) is isomorphic to the direct product \(\text{Aut } k \times o\text{-Aut } G\). A detailed study of the factor \(o\text{-Aut } G\) is provided in [DG97] and generalised in [KS21].

Secondly, we decompose the group \(\text{Int Aut } K\) (Theorem 3.7.1) of a Hahn field \(K\) which satisfies the **second lifting property** (Definition 3.3.5) by identifying two further subgroups of fundamental importance: the group \(1\text{-Aut } K\) of \(1\text{-automorphisms}\) and the group \(G\text{-Exp } K\) of \(G\text{-exponentiations}\) (Definitions 3.3.3 and 3.3.7). In Corollary 3.7.3 we deduce that \(v\text{-Aut } K\) is completely determined, up to isomorphism, by \(1\text{-Aut } K\). Similar results are obtained for \(v\text{-Aut}_k K\) and \(v\text{-Aut}_{(k)} K\).

An interesting large class of Hahn fields is that of **Rayner fields** (Definition 3.6.3). We characterise Rayner fields which fulfil the canonical first lifting property (Definition 3.5.5) and prove that all Rayner fields satisfy the second lifting property. We follow up our study on Rayner fields in [KKS20] and [KKS21].

Pursuing our generalisation of Schilling’s results, we study the group \(v\text{-Aut}^+ K\) of strongly additive (Definition 4.0.1) valuation preserving automorphisms (and its subgroups \(v\text{-Aut}^+_{(k)} K\), \(v\text{-Aut}^+_{(k)} K\)) of a Hahn field \(K\). In particular, in Theorem 4.2.8 we describe \(v\text{-Aut}^+_{(k)} K\), for a Hahn field \(K\) which satisfies the first and second lifting property, in terms of the valuation invariants of \(K\). Finally, to illustrate our results, we provide a detailed study of two special cases. If the group \(G\) is finitely generated we obtain a generalisation of a result of [Sch44] on the field \(L\) of Laurent series. If \(G\) is divisible and finite dimensional, we obtain increasingly precise results depending on our assumptions on \(k\). For the field \(P\) of Puiseux series we find descriptions of \(v\text{-Aut}_{(k)} P\) and \(v\text{-Aut}_k P\) that generalise results of [Des05].

The paper is organised as follows. In Section 2 we introduce the fundamental definitions and establish some notation that will be used throughout the paper. In this section Example 2.1.10 is of particular interest: we show that if the coefficient field \(k\) is not archimedean, order preserving automorphisms need not be valuation preserving.

In Section 3 we introduce two notions of lifting property, which allow us to obtain several decomposition theorems. Subsection 3.1 is devoted to the study of the first lifting property (Definition 3.1.3). For a Hahn field \(K\) satisfying the first lifting property, we introduce two important subgroups of \(v\text{-Aut } K\): the normal subgroup \(\text{Int Aut } K\) and its complement \(\text{Ext Aut } K\) (Definitions 3.1.1 and 3.1.6).
The main result of Subsection 3.2 is Theorem 3.2.1, which generalises Hofberger’s decomposition to any Hahn field with the first lifting property. This will be the starting point for a much deeper investigation of the structure of the automorphism group.

In Subsection 3.3 we introduce the canonical second lifting property. For a Hahn field $K$ satisfying the canonical second lifting property, we further introduce two important subgroups of $\text{Int Aut} K$: the normal subgroup $1\text{-Aut} K$ of 1-automorphisms and its complement, the group $G\text{-Exp} K$ of $G$-exponentiations (Definition 3.3.1).

In Subsection 3.4 we provide a decomposition of $\text{Int Aut} K$ into a semi-direct product of the subgroups $1\text{-Aut} K$ and $G\text{-Exp} K$, for a Hahn field $K$ satisfying the canonical second lifting property.

Subsection 3.5 is dedicated to a special version of the first lifting property: the canonical first lifting property. This allows to simplify the decomposition of the group $v\text{-Aut}_{(k)} K$ of $k$-stable automorphism, as will be shown in Subsection 3.7.

In Subsection 3.6 we present the class of Rayner fields, first introduced in [Ray68]. All these fields satisfy the canonical second lifting property and include, for example, the $\kappa$-bounded Hahn fields (Example 3.6.4) introduced in [All62] and, in particular, the maximal Hahn fields. The main result of this section is a characterisation of Rayner fields with the canonical first lifting property (Proposition 3.6.6).

In Subsection 3.7, combining results from the previous sections, we obtain two decompositions into a 4-factor semi-direct product. One for the groups $v\text{-Aut} K$ and $v\text{-Aut}_{(k)} K$ for a Hahn field $K$ with the first and canonical second lifting property (Theorem 3.7.1) and one for $v\text{-Aut}_{(k)} K$ under the extra assumption, that $K$ has the canonical first lifting property (Proposition 3.7.2).

Section 4 focuses on strong additivity.

In Subsection 4.1 we show that Hofberger’s decomposition, in its generalised and refined form, also holds if we restrict to the group $v\text{-Aut}^+ K$ (Proposition 4.1.1 and Proposition 4.1.3). This way we obtain a detailed description of $v\text{-Aut}^+ K$ and its subgroups $v\text{-Aut}_{(k)}^+ K$, $v\text{-Aut}_{(k)} K$, $v\text{-Aut}^+ K$ (Theorem 4.1.3), which is the main result of this subsection.

Subsection 4.2 is devoted to a deeper investigation of the component $1\text{-Aut}_{(k)}^+ K$ appearing in Theorem 4.1.3. We do this in terms of the $K$-summable homomorphisms $\text{Hom}^+(G, 1 + I_K)$ (Definition 4.2.3) from the value group into the group of 1-units of the valuation ring. This is the main result of the section (Theorem 4.2.8), which provides a decomposition of $v\text{-Aut}_{(k)}^+ K$ and $v\text{-Aut}_{(k)}^+ K$ purely in terms of the valuation invariants of $K$.

Section 5 is devoted to the explicit description of the automorphism groups in some special cases.

In Subsection 5.1 we study the case of a Hahn field $K \subseteq k[[G]]$ with a finitely generated exponent group, necessarily of the form $G = \mathbb{Z}^n$. In this case, if $K$ has the canonical second lifting property, we can explicitly describe $G\text{-Exp} K$ in terms
of $k$ and the number $n$ of generators of $G$ (Theorem 5.1.1). If $G = \mathbb{Z}^n$ is ordered lexicographically, we can moreover represent $o\text{-}\text{Aut} G$ as a group of matrices. Applying this, we give a description of the groups $v\text{-}\text{Aut}^{(k)}_+ K$ and $v\text{-}\text{Aut}^{(k)}_+ K$, which depends solely on $k$, $1 + I_K$ and $n$ (Theorem 5.1.4).

In Subsection 5.2 we apply the results from Subsection 5.1 to the field $L$ of Laurent series. We notice that $v\text{-}\text{Aut} \mathbb{L} = v\text{-}\text{Aut}^+ \mathbb{L}$ and obtain a precise description of $v\text{-}\text{Aut}^{(k)}_+ \mathbb{L}$ in terms of $1 + I_L$, $K^\times$ and $\text{Aut} k$ (Theorem 5.2.4). Schilling’s result on the $k$-automorphisms of $\mathbb{L}$ can be derived as a special case (Corollary 5.2.6). We also provide a sharpening of Theorem 5.2.1 for the case of an ordered coefficient field $k$ to characterize the group $o\text{-}\text{Aut}\mathbb{L}$ of order preserving automorphisms.

Subsection 5.3 is also dedicated to the case $G = \mathbb{Z}$, but focuses on the field $k(\mathbb{Z})$. The group $C_{\mathbb{Z}}(k) := \text{Aut}_k k(\mathbb{Z})$ is the Cremona\textsuperscript{1} group in dimension 1 over $k$. We identify the groups $v\text{-}\text{Aut}_k k(\mathbb{Z})$ and $1\text{-}\text{Aut}_k k(\mathbb{Z})$ as subgroups of $C_{\mathbb{Z}}(k)$.

In Subsection 5.4 we study the case of a Hahn field $K \subseteq k((G))$ where $G$ is a totally ordered, divisible, abelian group, of finite dimension $d$ over $\mathbb{Q}$ and $k$ a real closed field. In analogy to Subsection 5.1, we provide a description of $o\text{-}\text{Aut} G$ as a group of rational matrices and, under the extra assumption that $K$ is henselian, we give an explicit description of the groups $v\text{-}\text{Aut}^{(k)}_+ K$ and $v\text{-}\text{Aut}^{(k)}_+ K$, depending only on $k$, $1 + I_K$ and $d$ (Theorem 5.4.11).

In Subsection 5.5 we consider the field $\mathbb{P}$ of Puiseux series. After observing that $v\text{-}\text{Aut} \mathbb{P} = v\text{-}\text{Aut}^+ \mathbb{P}$ (Proposition 5.5.1), we apply our general results and retrieve decompositions analogous to those appearing in [Des05].

2. Definitions and notations

Let $(G, +, 0, <)$ be a totally ordered abelian group\textsuperscript{2} and $k$ a field. Consider the set $k^G$ of all functions from $G$ to $k$ and for an element $a \in k^G$ define the support of $a$ to be the set $\text{supp } a := \{ g \in G : a(g) \neq 0 \}$. We denote by $\mathbb{K} = k((G))$ the set of all elements of $k^G$ with well ordered support. Denote a function $a \in k((G))$ by the formal power series expression $a = \sum_{g \in G} a_g t^g$ where $a_g := a(g)$. In particular, the coefficient $a_0 := a(0)$ will play an important role and we call it the constant term of $a$. We define the following two operations: for $a = \sum_{g \in G} a_g t^g, b = \sum_{g \in G} b_g t^g \in \mathbb{K}$ set

\[a + b := \sum_{g \in G} (a_g + b_g) t^g \quad \text{and} \quad ab := \sum_{g \in G} c_g t^g \quad \text{where} \quad c_g = \sum_{r,s \in G, r + s = g} a_r b_s.\]

It was shown by Hahn [Hah07] that these two operations are well defined and make $\mathbb{K}$ into a field. We call $\mathbb{K}$ the maximal Hahn field over $k$ with exponents in $G$.

\textsuperscript{1}Cremona groups are of fundamental importance in algebraic geometry (see [Dés21]).

\textsuperscript{2}Unless explicitly specified otherwise, all orderings will be total.
The group ring $k[G]$ is the unitary subring of $\mathbb{K}$ consisting of series with finite support (generalised polynomials). It is an integral domain and we denote its field of fractions by $k(G) := \text{Frac}(k[G]) \subseteq \mathbb{K}$.

**Definition 2.1.1.** A Hahn field is a field $K$ such that $k(G) \subseteq K \subseteq \mathbb{K}$. We define a valuation $v^K_{\text{min}}$ on a Hahn field $K$ by setting $v^K_{\text{min}}(a) = \min \text{supp}(a)$ for $a \neq 0$ and $v^K_{\text{min}}(0) = \infty$. We will call this the canonical valuation on $K$. Whenever the context is clear we will simply write $v$ instead of $v^K_{\text{min}}$. For $a \in K^\times$ we define the first coefficient of $a$ to be $a_{v(a)} := a(v(a))$.

**Notation 2.1.2.** For a Hahn field $K$ we will denote by $v$ its canonical valuation and by $G = v(K^\times)$ its value group. We denote by $t^G$ the multiplicative group of monic monomials $t^G := \{t^g : g \in G\}$. The valuation ring $R^K$ is $k((G^{>0})) \cap K$ (elements with non-negative value) and the valuation ideal $I^K$ is $k((G^{>0})) \cap K$ (elements with positive value). The residue field is $\bar{K} = R^K/I^K$.

If $K$ is totally ordered by $<_K$, we order $\mathbb{K}$ lexicographically by setting $a >_{\text{lex}} 0 \iff a_{v(a)} > 0$. Then $(\mathbb{K}, >_{\text{lex}})$ is a totally ordered field and so is any subfield $K \subseteq \mathbb{K}$. When considering the lexicographic order on a Hahn field $K$ we will omit the subscript and simply denote the ordering by $\prec$.

**Remark 2.1.3.**

(i) The residue field is isomorphic to $k$ via the canonical isomorphism $f_c : \bar{K} \to k$, $a + I^K \mapsto a_0$, for all $a \in R^K$. Note that, if $f : \bar{K} \to k$ is any other isomorphism, then there exists a uniquely determined $\rho_f \in \text{Aut} k$, defined by $\rho_f(a_0) = f(a_0 + I^K)$, such that $f = \rho_f f_c$.

(ii) The units of the valuation ring are the elements with null value, so the group of units $R^K_\times$ is the direct sum $U_K := I_K + k^\times$.

(iii) A subgroup of $U_K$ that will be relevant in the sequel is that of 1-units (units with constant term 1), i.e., $1 + I_K$. In fact, $U_K$ is (isomorphic to) the direct product $U_K \simeq (1 + I_K) \times k^\times$.

(iv) In the sequel we will consider the group $\text{Hom}(G, U_K)$ of homomorphisms of $(G, +)$ into $(U_K, \cdot)$, endowed with pointwise multiplication $\cdot$ (and similarly for $\text{Hom}(G, 1 + I_K)$ and $\text{Hom}(G, k^\times)$). Since $U_K = (1 + I_K) \times k^\times$ we have

$$\text{Hom}(G, U_K), \cdot) \simeq \text{Hom}(G, 1 + I_K), \cdot) \times \text{Hom}(G, k^\times), \cdot).$$

**Definition 2.1.4.** An automorphism $\tau$ of the group $(G, +)$ is said to be order preserving if, for all $g \in G$ we have $g > 0 \Rightarrow \tau(g) > 0$. If $K$ is an ordered Hahn field, an order preserving automorphism of $K$ is defined in a similar way. An automorphism $\sigma \in \text{Aut} K$ is valuation preserving if there exists a (necessarily unique) order preserving automorphism $\tau$ of $G$ such that, for all $a \in K$, we have $\tau(v(a)) = v(\sigma(a))$. An automorphism $\sigma \in \text{Aut} K$ is a $k$-stable automorphism if $\sigma(k) = k$ and a $k$-automorphism if $\sigma|_k = \text{id}_k$. Notice that $\sigma \in \text{Aut} K$ is a $k$-automorphism if and only if it is $k$-linear (when we view $K$ as a $k$-vector space).
Let $f$ be a Hahn field. For the corresponding definition for a general valued field see, for example, [EP05, Section 4.1]

**Definition 2.1.5.** Let $K \subseteq k((G))$ be a Hahn field. For a polynomial $p = p_0 + p_1 X + \ldots + p_n X^n \in R_K[X]$ let $\bar{p} := (p_0) + (p_1)X + \ldots + (p_n)X^n \in k[X]$. Then $K$ is henselian if for every polynomial $p \in R_K[X]$ and every $a \in R_K$ such that $\bar{p}(a) = 0$ and $\bar{p}'(a) \neq 0$ there exists $b \in R_K$ such that $b_0 = a_0$ and $p(b) = 0$.

**Notation 2.1.6.**

(i) The $k$-stable automorphisms form a group under composition that we will denote by $\text{Aut}_{(k)} K$. The $k$-automorphisms form a group under composition that we will denote by $\text{Aut}_k K$.

(ii) The valuation preserving automorphisms of $K$ form a group under composition that we will denote by $v$-$\text{Aut}_k K$. We will also write $v$-$\text{Aut}_k K = v$-$\text{Aut} K \cap \text{Aut}_{(k)} K$ and $v$-$\text{Aut}_k K = v$-$\text{Aut} K \cap \text{Aut}_k K$.

(iii) The order preserving automorphisms of $G$ form a group under composition that we will denote by $o$-$\text{Aut} G$. If $K$ is an ordered Hahn field, $o$-$\text{Aut} K$ is defined in a similar way.

**Remark 2.1.7** (Characterisations of valuation preserving automorphisms). Various characterisations of valuation preserving automorphisms in terms of the valuation ring, valuation ideal and group of units are given in [KMP17, Theorem 4.2]. In particular, for all $\sigma \in \text{Aut}_K$:

$$\sigma \in v$-$\text{Aut} K \iff \sigma(R_K) = R_K \iff \sigma(I_K) = I_K \iff \sigma(U_K) = U_K \iff \sigma(1+I_K) = 1+I_K$$

(i) So, for $\sigma \in v$-$\text{Aut} K$ we have

$$\sigma(k^\times \times (1 + I_K)) = k^\times \times (1 + I_K) \text{ and } \sigma(1 + I_K) = 1 + I_K.$$ 

This however does not imply, in general, that $\sigma(k^\times) = k^\times$, i.e., does not imply, in general, that $\sigma \in v$-$\text{Aut}_{(k)} K$.

(ii) Furthermore $\sigma \in v$-$\text{Aut} K$ if and only if, for all $a, b \in K$ we have

$$v(a) < v(b) \iff v(\sigma(a)) < v(\sigma(b)).$$

Equivalently, $\sigma \in v$-$\text{Aut} K$ if and only if the map

$$\sigma_G : G \to G, \quad \sigma_G(v(a)) := v(\sigma(a))$$

is a well defined element of $\text{Aut}_G$, if and only if the map

$$\bar{\sigma} : \bar{K} \to \bar{K}, \quad a + I_K \mapsto \sigma(a) + I_K$$

is a well defined element of $\text{Aut}_{\bar{K}}$.

(iii) If we fix an isomorphism $f : K \to k$, the automorphism $\bar{\sigma}$ uniquely defines an automorphism $\sigma_k \in \text{Aut}_k$, given by $\sigma_k = f \sigma f^{-1}$. Note that, if we write $f = \rho_f f_c$ then $\sigma_k(x) = \rho_f(\sigma(\rho_f^{-1}(x))_0)$ for all $x \in k$. We call $\sigma_G$ and $\sigma_k$ the automorphisms induced by $\sigma$ on $G$ and $k$, respectively. $\square$
**Remark 2.1.8** (The automorphism group of $R_K$). Let $\text{Aut } R_K$ denote the automorphism group of the valuation ring. The map $v:\text{Aut } K \to \text{Aut } R_K$ given by $\sigma \mapsto \sigma|_{R_K}$ is well defined by Remark 2.1.7. Moreover, since $R_K$ is a valuation ring in $K$, then for all $a \in K$ there exist $c, d \in R_K$ with $d \neq 0$ and $a = c/d$. Then every automorphism $\nu \in \text{Aut } R_K$ extends uniquely to an automorphism $\sigma \in v:\text{Aut } K$ by $\sigma(a) = \sigma \left( \frac{c}{d} \right) = \frac{\nu(c)}{\nu(d)}$. We see that $v:\text{Aut } K \simeq \text{Aut } R_K$, therefore, our work also provides a study of $\text{Aut } R_K$.

If $k$ is an archimedean field, an order preserving automorphism of $K$ also preserves the valuation $v$, so we have $\sigma:\text{Aut } K \leq v:\text{Aut } K$ ([Kuh00b, Lemma 1.3]). If $k$ is not archimedean this is no longer true, as shown in Example 2.1.10 below.

**Notation 2.1.9.** Let $A, B$ be ordered abelian groups. We denote by $G = A \oplus B$ the group $A \oplus B$ endowed with the lexicographic order. More generally, if $\Gamma$ is a chain and $Q$ is an ordered abelian group, then $H = \prod_{\Gamma} Q$ is the ordered abelian group $\bigoplus_{\gamma \in \Gamma} Q$ endowed with the lexicographic order. We denote an element $h \in H$ by $h = \sum_{\gamma \in \Gamma} q_{\gamma}1_{\gamma}$ where $q_{\gamma} \in Q$, $1_{\gamma}$ is the characteristic function on the singleton $\{\gamma\}$ and $\supp(h) = \{\gamma \in \Gamma : q_{\gamma} \neq 0\}$ is finite.

**Example 2.1.10.** Let $A = \bigoplus_{\mathbb{Z} < 0} \mathbb{Q}$, $B = \bigoplus_{\mathbb{Z} \geq 0} \mathbb{Q}$ and $G = \bigoplus_{\mathbb{Z}} \mathbb{Q}$. The group $G$ is what is called a Hahn group. It is a valued group with valuation $v_G: G \neq 0 \to \mathbb{Z}$ given by $v_G(\sum_{n \in \mathbb{Z}} q_n1_n) = \min\{n : q_n \neq 0\}$ (for more on Hahn groups see [KS21; Ser21]).

Consider the ordered Hahn fields $k = \mathbb{R}(\langle B \rangle)$, $K = k(\langle A \rangle)$ and $F = \mathbb{R}(\langle G \rangle)$. The canonical valuation $v^K_{\text{min}}$ on $K$ has value group $v^K_{\text{min}}(K^\times) = A$ and non-archimedean residue field $k = \mathbb{R}(\langle B \rangle)$. The canonical valuation $v^K_{\text{min}}$ has value group $v^K_{\text{min}}(F^\times) = G$ and residue field $\bar{F} = \mathbb{R}$. This is the finest convex valuation on $F$ ([Kuh00b, p 17]). The field $F$ also admits a coarser valuation $w$ whose value group is $w(F^\times) = A = v^K_{\text{min}}(K^\times)$ and residue field is $\bar{F}w = \mathbb{R}(\langle B \rangle) = k$. The valuation $w$ is defined (for non-zero elements) by

$$w \left( \sum_{(q,r) \in G} \alpha_{(q,r)}t^{(q,r)} \right) = \min\{q \in A : \exists r \in B, \alpha_{(q,r)} \neq 0\}. \quad (2.1)$$

It is straightforward to verify that the map

$$\xi: (F, w) \to (K, v^K_{\text{min}}), \quad \sum_{(q,r) \in G} \alpha_{(q,r)}t^{(q,r)} \mapsto \sum_q \left( \sum_r \alpha_{(q,r)}x^r \right)y^q,$$

where $x$ is the variable in $\mathbb{R}(\langle B \rangle)$ and $y$ the variable in $k(\langle A \rangle)$, is an isomorphism of valued fields (i.e., for all $\alpha \in F$ we have $w(\alpha) = v^K_{\text{min}}(\xi(\alpha))$).

We will construct an order preserving automorphism $\sigma$ of $F$ that does not preserve $w$ and therefore $\xi^{-1} \sigma \xi$ will be an order preserving automorphism of $K$ that does not preserve $v^K_{\text{min}}$. 
Consider the automorphism $\rho$ of the chain $(\mathbb{Z}, <)$ given by $n \mapsto n + 1$. Since $G$ has the canonical first lifting property as a Hahn group (see [KS21]), the map $\rho$ lifts to an order preserving automorphism $\rho_G \in o\text{-}Aut G$ given by $\rho_G \left( \sum_{n \in \mathbb{Z}} q_n 1_n \right) = \sum_{n \in \mathbb{Z}} q_n 1_{\rho(n)}$. Now, since $F$ is a maximal Hahn field, by Example 3.6.7 it has the canonical first lifting property, so $\rho_G$ lifts to an automorphism $\sigma \in v_{\text{min}}^F\text{-}Aut F$. Since $G$ is divisible, $F$ is real closed and all its automorphisms are necessarily order preserving, so $\sigma \in o\text{-}Aut F$ [Pri83, Theorem 8.6]. Now we show that $\sigma$ does not preserve $w$.

Set $U_w = \{ a \in F^\times : w(a) = 0 \}$, the group of units of the valuation ring of $w$ and $G_w = v_{\text{min}}(U_w)$. From (2.1), it is straightforward to verify that $G_w = B$. Write $\Gamma_w = v_G(G_w^{>0}) = \mathbb{Z}^{>0}$. Then [KMP17, Theorem 4.7] implies that $\sigma$ preserves $w$ if and only if the induced chain automorphism $\rho$ of $\mathbb{Z}$ preserves $\Gamma_w$. But we have $\Gamma_w = \mathbb{Z}^{>0} \neq \mathbb{Z}^{\geq 0} = \rho(\Gamma_w)$ thus $\sigma$ does not preserve $w$. □

3. Decomposition theorems

3.1. The first lifting property. Let $K$ be a Hahn field. As announced in the introduction, our aim is to study $v\text{-}Aut K$. Let $\sigma \in v\text{-}Aut K$, $\sigma_G \in o\text{-}Aut G$ the induced automorphism of the value group and $\bar{\sigma} \in \text{Aut } \bar{K}$ the one induced on the residue field. This gives rise to a map

$$\Phi_K: v\text{-}Aut K \longrightarrow \text{Aut } \bar{K} \times o\text{-}Aut G, \quad \sigma \longmapsto (\bar{\sigma}, \sigma_G).$$

(3.1)

Let $f: \bar{K} \sim k$ be an isomorphism. We recall that the automorphism $\bar{\sigma}$ uniquely defines an automorphism $\sigma_k \in \text{Aut } k$, given by $\sigma_k = f \bar{\sigma} f^{-1}$ (see Remark 2.1.7). This defines a map

$$\Phi_{K,f}: v\text{-}Aut K \longrightarrow \text{Aut } k \times o\text{-}Aut G, \quad \sigma \longmapsto (\sigma_k, \sigma_G).$$

(3.2)

It is straightforward to verify that $\Phi_{K,f}$ is a group homomorphism and that, if $e: \bar{K} \rightarrow k$ is another isomorphism, then $\Phi_{K,e}$ and $\Phi_{K,f}$ are related by the formula

$$\Phi_{K,e} = \Phi_{K,\delta f} \quad \text{where } \delta := ef^{-1} \in \text{Aut } k.$$

(3.3)

Whenever the context is clear we will omit $K,f$ from the notation and write $\Phi$ instead of $\Phi_{K,f}$.

Definition 3.1.1. The kernel $\ker \Phi$ of the map (3.2) is a normal subgroup of $v\text{-}Aut K$ that we call the subgroup of internal automorphisms of $K$. We will denote it by $\text{Int Aut } K$. We write $\text{Int Aut}_{(k)} K := \text{Int Aut } K \cap v\text{-}Aut_{(k)} K$ and $\text{Int Aut}_k K := \text{Int Aut } K \cap v\text{-}Aut_k K$. Notice that, since $\sigma_k = f \bar{\sigma} f^{-1}$, then $\sigma_k = \text{id}_k \iff \bar{\sigma} = \text{id}_{\bar{K}}$. The definition of $\text{Int Aut } K$ is therefore independent of our choice of $f$.

We explicitly point out the following key properties of internal automorphisms:

Proposition 3.1.2. Let $\sigma \in v\text{-}Aut K$. Then $\sigma \in \text{Int Aut } K$ if and only if both of the following hold:
(i) \( v(a) = v(\sigma(a)) \), for all \( a \in K \);
(ii) if \( a \in R_K \) then \( \sigma(a)_0 = a_0 \).

Moreover, we have \( \text{Int Aut}_{(k)} K = \text{Int Aut}_K K \).

**Proof.** Let \( \sigma \in \text{Int Aut} K \). Then (i) holds by definition, because \( \sigma_G = \text{id}_G \). To prove (ii) let \( a \in R_K \) and let \( \bar{\sigma} \in \text{Aut}\bar{K} \) be defined as in Remark 2.1.7. Then, since \( \bar{\sigma} = \text{id}_{\bar{K}} \) we have
\[
a_0 + I_K = a + I_K = \bar{\sigma}(a + I_K) = \sigma(a) + I_K = \sigma(a)_0 + I_K
\]
which, since \( a_0, \sigma(a)_0 \in k \), implies \( a_0 = \sigma(a)_0 \).

Vice versa, let \( \sigma \in v\text{-Aut} K \) satisfy (i) and (ii). From (i) it follows that \( \sigma_G = \text{id}_G \).

Let \( a \in R_K \) and compute
\[
\bar{\sigma}(a + I_K) = \sigma(a) + I_K = \sigma(a)_0 + I_K = a_0 + I_K.
\]
Thus \( \bar{\sigma} = \text{id}_{\bar{K}} \) and so \( \sigma \in \text{Int Aut} K \). The last statement now follows immediately. \( \square \)

In Section 3.3 we will study the group \( \text{Int Aut} K \) of internal automorphisms of a Hahn field \( K \). Now we want to determine a complement of \( \text{Int Aut} K \) in \( v\text{-Aut} K \).

**Definition 3.1.3.** We say that a pair \( (\rho, \tau) \in \text{Aut} k \times o\text{-Aut} G \) lifts to \( K \) if there exists an automorphism \( \sigma \in v\text{-Aut} K \) such that \( \Phi(\sigma) = (\rho, \tau) \). In this case we call \( \sigma \) a lift of \( (\rho, \tau) \). If the map \( \Phi \) defined in (3.2) admits a section (in particular, \( \Phi \) is surjective), i.e., an injective group homomorphism \( \Psi: \text{Aut} k \times o\text{-Aut} G \rightarrow v\text{-Aut} K \) such that \( \Phi\Psi = \text{id} \), then every pair \( (\rho, \tau) \in \text{Aut} k \times o\text{-Aut} G \) lifts to an automorphism \( \Psi(\rho, \tau) \) of \( K \) and we say that \( K \) has the first lifting property with respect to \( \Psi \).

**Example 3.1.4.** (i) The maximal Hahn field \( K \) has the first lifting property. Indeed, for every pair \( (\rho, \tau) \in \text{Aut} k \times o\text{-Aut} G \) the map \( \sigma: \sum a_g t^g \mapsto \sum \rho(a_g) t^{\tau(g)} \) is an automorphism of \( K \) such that \( \Phi(\sigma) = (\rho, \tau) \) (see Corollary 3.6.7).

(ii) The field \( k(G) \) has the first lifting property. The map \( \sigma \) of part (i) restricts to an automorphism of \( k(G) \).

(iii) A large class of Hahn fields with the first lifting property will be described in Section 3.6. \( \square \)

**Remark 3.1.5.** The morphism \( \Phi_K \) admits a section if and only if \( \Phi_{K,f} \) admits a section, for every isomorphism \( f: \bar{K} \rightarrow k \). Indeed, if \( e: \bar{K} \rightarrow k \) is another isomorphism and \( \Psi_{K,f} \) is a section of \( \Phi_{K,f} \), then it follows from Equation (3.3) that a section \( \Psi_{K,e} \) of \( \Phi_{K,e} \) is given by the formula
\[
\Psi_{K,e}(\rho, \tau) = \Psi_{K,f}(\delta^{-1}\rho\delta, \tau) \quad \text{where} \quad \delta = ef^{-1}.
\]
(3.4) \( \square \)
Definition 3.1.6. Assume that $K$ has the first lifting property with respect to a fixed section $\Psi$ of $\Phi$. The subgroup $\Psi(\text{Aut}_k \times \text{o-Aut} G)$ of $\text{v-Aut} K$ will be called the subgroup of $\Psi$-external automorphisms of $K$ and denoted by $\Psi\text{-Ext Aut } K$. Therefore, for every $\sigma \in \Psi\text{-Ext Aut } K$ and every $(\rho, \tau) \in \text{Aut}_k \times \text{o-Aut} G$ we have

$$\Phi(\sigma) = (\rho, \tau) \iff \Psi(\rho, \tau) = \sigma.$$ 

Hence, for any section $\Psi$ we have $\Psi\text{-Ext Aut } K \simeq \text{Aut}_k \times \text{o-Aut} G$.

We will also use the notations $\Psi\text{-Ext Aut}_{(k)} K := \Psi\text{-Ext Aut } K \cap \text{Aut}_{(k)} K$ and $\Psi\text{-Ext Aut}_k K := \Psi\text{-Ext Aut } K \cap \text{Aut}_k K$.

Remark 3.1.7. (i) Let $\sigma \in \Psi\text{-Ext Aut } K$, say $\sigma = \Psi(\rho, \tau)$ for $(\rho, \tau) \in \text{Aut}_k \times \text{o-Aut} G$. Then $\sigma \in \text{v-Aut}_k K \iff \rho = \text{id}_k$. Thus $\Psi\text{-Ext Aut}_k K \simeq \text{o-Aut} G$. (ii) If $K$ has the first lifting property with respect to $\Psi$, the homomorphism $\Phi$ defined in (3.2) is surjective and every pair $(\rho, \tau) \in \text{Aut}_k \times \text{o-Aut} G$ lifts to an automorphism $\sigma = \Psi(\tau) \in \text{v-Aut} K$. The first isomorphism theorem then yields

$$\text{Aut}_k \times \text{o-Aut} G \simeq \frac{\text{v-Aut} K}{\text{Int Aut } K}.$$ 

In particular, the set of lifts of some pair $(\rho, \tau) \in \text{Aut}_k \times \text{o-Aut} G$ is the coset $\{\sigma \sigma' : \sigma' \in \text{Int Aut } K\}$, where $\sigma$ is any given lift of $(\rho, \tau)$.

(iii) Notice also that a $k$-automorphism is not necessarily internal. Let $\text{id}_G \neq \tau \in \text{o-Aut} G$, then the the pair $(\text{id}_k, \tau)$ lifts to an automorphism $\sigma \in \text{v-Aut}_k K \setminus \text{Int Aut } K$. □

Corollary 3.1.8. We have:

$$\Psi\text{-Ext Aut } K \simeq \text{Aut}_k \times \text{o-Aut} G \quad (3.5)$$

$$\Psi\text{-Ext Aut}_k K \simeq \text{o-Aut} G \quad (3.6)$$ 

□

Analogous results to Corollary 3.1.8 for $k$-stable automorphisms will appear in Section 3.5.

3.2. The group of valuation preserving automorphisms. In this section we assume that all Hahn fields under consideration have the first lifting property with respect to $\Psi$. Whenever the context is clear we will omit $\Psi$ from the notation and terminology. Recall our notation $\mathbb{K} = k((G))$. In [Hof91, Satz 2.2] Hofberger shows that $\text{v-Aut } \mathbb{K}$ can be decomposed into a semi-direct product of the groups of internal and external automorphisms (with respect to a specific section – see (3.18)). We generalise Hofberger’s result to a Hahn field $K \subseteq \mathbb{K}$ which has the first lifting property with respect to an arbitrary section.
Theorem 3.2.1. Let $K \subseteq \mathbb{K}$ be a Hahn field with the first lifting property. Then we have the following inner semi-direct product decompositions:

\begin{align*}
v \cdot \text{Aut} K &= \text{Int Aut } K \rtimes \text{Ext Aut } K \\
v \cdot \text{Aut}(k) K &= \text{Int Aut}_k K \rtimes \text{Ext Aut}_{(k)} K \\
v \cdot \text{Aut}_k K &= \text{Int Aut}_k K \rtimes \text{Ext Aut}_k K
\end{align*}

(3.7) (3.8) (3.9)

Proof. Let $\Phi$ be defined as in (3.2) and let $\Psi$ be a section. Consider the sequence

$$
\begin{array}{c}
\text{Int Aut } K \\
\downarrow \iota \\
v \cdot \text{Aut } K \\
\downarrow \Phi_f \\
\text{Aut } k \times o \cdot \text{Aut } G \\
\downarrow \Psi
\end{array}
$$

where $\iota$ is the canonical embedding. By definition of Int Aut $K$ we have $\text{im } \iota = \ker \Phi_f$ so the sequence is exact. Therefore (see [Mac95, p. 109]) we have:

$$
v \cdot \text{Aut } K = \text{im } \iota \rtimes \text{im } \Psi = \text{Int Aut } K \rtimes \text{Ext Aut } K.
$$

Equations (3.8) and (3.9) are obtained from (3.7) by taking intersections with $v \cdot \text{Aut}(k) K$ and $v \cdot \text{Aut}_k K$ respectively and using Proposition 3.1.2. \qed

Let $K$ have the first lifting property. By Theorem 3.2.1, describing $v \cdot \text{Aut } K$ consists of two tasks: describing the normal subgroup Int Aut $K$ and the subgroup Ext Aut $K$. By Corollary 3.1.8, we have Ext Aut $K \simeq \text{Aut } k \times o \cdot \text{Aut } G$. A detailed study of o-Aut $G$ in terms of its value set $v_G(G \neq 0)$ is carried out in [KS21]. In the next section we investigate the structure of Int Aut $K$.

3.3. The canonical second lifting property. In this section we study the group Int Aut $K$ in more detail and provide a decomposition into a semi-direct product of two notable subgroups. As announced in part (iv) of Remark 2.1.3 we work with the following:

Definition 3.3.1. Let $\text{Hom}(G, k^\times)$ be the set of all homomorphisms of the additive group $(G, +)$ into the multiplicative group $(k^\times, \cdot)$. Let $x \in \text{Hom}(G, k^\times)$. We will denote the image of a $g \in G$ under $x$ by $x^g := x(g)$, so, for all $g, h \in G$ we have $x^{g+h} = x^gx^h$. Let $1: (G, +) \to (k^\times, \cdot), g \mapsto 1$ be the trivial morphism. Then the set Hom$(G, k^\times)$ forms a group under the pointwise multiplication defined by $(xy)^g := x^gy^g$. The inverse of $x$ is the morphism $g \mapsto x^{-g}$ and $1$ is the neutral element.

Lemma 3.3.2. Let $K$ be a Hahn field. The map

$$X: \text{Int Aut } K \to \text{Hom}(G, k^\times), \quad \sigma \mapsto x_\sigma$$

(3.10)

where $x_\sigma$ is defined by

$$x_\sigma(g) := x^g := \sigma(t^g)_g$$

(3.11)

is a group homomorphism.

Footnote: For $\alpha_i \in \text{Int Aut } K$ and $\beta_i \in \text{Ext Aut } K$, $i = 1, 2$, we have $(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\beta_1^{-1} \alpha_1 \beta_1, \beta_1 \beta_2)$.
Let \( \sigma \in \text{Int Aut } K \) and \( g, h \in G \). The formula (3.11) defines an element of \( \text{Hom}(G, k^\times) \). Indeed, by Proposition 3.1.2, \( g = v(t^g) = v(\sigma(t^g)) \), so \( x_\sigma(g) = \sigma(t^g)_g \in k^\times \). Moreover, we have
\[
x_\sigma(g + h) = \sigma(t^{g+h})_{g+h} = (\sigma(t^g)\sigma(t^h))_{g+h} = \sigma(t^g)_g \sigma(t^h)_h = x_\sigma(g)x_\sigma(h).
\]
So \( X \) is a well defined map. To show that it is also a group homomorphism, let \( \sigma, \tau \in \text{Int Aut } K \). Let \( \alpha := \tau(t^g)_g \). Then
\[
x_{\sigma \tau}(g) = (\sigma \tau(t^g))_g = \sigma(\alpha t^g)_g = \alpha \sigma(t^g)_g = x_\sigma(t^g)x_\tau(t^g).
\]
\[\square\]

**Definition 3.3.3.** The kernel \( \text{ker } X \) will be called the group of \( 1\text{-automorphisms of } K \) and denoted by \( \text{1-Aut } K \). Hence \( \text{1-Aut } K \leq \text{Int Aut } K \). We use the notations \( 1\text{-Aut}_{(k)} K := 1\text{-Aut } K \cap \text{Aut}_{(k)} K \) and \( 1\text{-Aut}_k K := 1\text{-Aut } K \cap \text{Aut}_k K \).

**Lemma 3.3.4.** Let \( K \) be a Hahn field. The following hold:
(i) For all \( \tau \in 1\text{-Aut } K \) and all \( a \in K^\times \) we have \( \tau(a) v(a) = a v(a) \);
(ii) \( 1\text{-Aut}_{(k)} K = 1\text{-Aut}_k K \).

**Proof.** (i) By definition of \( 1\text{-Aut } K \), if \( \tau \in 1\text{-Aut } K \) and \( a \in K^\times \) with \( v(a) = h \) we have \( \tau(a) = \tau(a_h t^h + b) \) with \( v(b) > h \). Therefore \( \tau(a) = \tau(a_h t^h) + \tau(b) = a_h t^h + c + \tau(b) \) for some \( b, c \in K \) with \( v(\tau(b)) > h \) and \( v(c) > h \).

(ii) An automorphism that fixes the first coefficient of every series and keeps \( k \) invariant is necessarily trivial on \( k \).
\[\square\]

**Definition 3.3.5.** We say that a Hahn field \( K \) satisfies the **canonical second lifting property** if the map
\[
P: \text{Hom}(G, k^\times) \to \text{Int Aut } K, \quad x \mapsto \rho_x
\]
where \( \rho_x \) is given by
\[
\rho_x(\sum a_g t^g) = \sum a_g x^g t^g
\]
is a well defined group homomorphism.

**Proposition 3.3.6.** Let \( K \) satisfy the canonical\(^4\) second lifting property. Then the map \( P \) of Definition 3.3.5 is injective and a section of \( X \), that is \( XP = \text{id}_{\text{Hom}(G, k^\times)} \).

In particular, \( X \) is surjective.

**Proof.** Let \( x, y \in \text{Hom}(G, k^\times) \) and let \( a = \sum a_g t^g \in K \). If \( \rho_x = \rho_y \) then, for all \( g \in G \) we have \( x^g t^g = \rho_x(t^g) = \rho_y(t^g) = y^g t^g \), which implies \( x^g = y^g \) for all \( g \) and so \( x = y \). So \( P \) is injective.

Moreover, for all \( g \in G \) we have
\[
XP(x)(g) = X(\rho_x)(g) = (\rho_x(t^g))_g = (x^g t^g)_g = x^g,
\]
\(^4\)In analogy to Definition 3.1.3 one could define a **general** second lifting property. However, in this paper we only work with the canonical second lifting property.
thus \( XP(x) = x \) which proves \( XP = \text{id}_{\text{Hom}(G,k^\times)} \) and, in particular, \( X \) is surjective.

\[ \]

**Definition 3.3.7.** Let \( K \) satisfy the canonical second lifting property. The subgroup \( \{ \rho_x : x \in \text{Hom}(G,k^\times) \} = \text{im } P \leq \text{Int Aut } K \) is called the group of \( G \)-exponentiations on \( K \) and denoted by \( G\text{-Exp } K \).

Clearly we have \( G\text{-Exp } K \simeq \text{Hom}(G,k^\times) \), hence it only depends on \( G \) and \( k^\times \).

\[ \]

**Remark 3.3.8.** Let \( K \) be a Hahn field satisfying the canonical second lifting property. The following assertions hold.

(i) We could define a notion of general second lifting property along with the canonical one, similarly to what we did for the first lifting property. We refrain from doing so in this paper and refer the interested reader to [Ser21].

(ii) Composing the two maps from Proposition 3.3.6 we get a homomorphism

\[ PX : \text{Int Aut } K \to G\text{-Exp } K, \sigma \mapsto \rho_x, \]

that associates to an internal automorphism \( \sigma \) its \( G\text{-Exp } K \) component.

(iii) By (3.13), the inverse of \( \rho_x \) is given by

\[ \rho_x^{-1} \left( \sum a_g t^g \right) = \sum a_g x^{-g} t^g. \]  

(iv) All \( G \)-exponentiations are trivial on \( k \), so we have \( G\text{-Exp } K \leq \text{Aut}_k K \).

(v) For all \( \rho \in G\text{-Exp } K \) and for all \( a \in K \) we have \( \text{supp } \rho(a) = \text{supp } a \). \( \square \)

**Examples 3.3.9.** (i) The maximal Hahn field \( \mathbb{K} \) satisfies the canonical second lifting property. Indeed, for all \( a = \sum a_g t^g \in \mathbb{K} \) and all \( x \in \text{Hom}(G,k^\times) \) the element \( \rho_x(a) = \sum a_g x^{-g} t^g \) has the same support as \( a \) (see Remark 3.3.8(iv)), hence \( \rho_x(a) \in \mathbb{K} \) and \( P \) is well defined, as required.

(ii) The field \( k(G) \) satisfies the canonical second lifting property. Indeed, let \( a = c/d \in k(G) \) for \( c,d \in \mathbb{K} \) with finite support. Let \( x \in \text{Hom}(G,k^\times) \) and consider \( \rho_x \) as an automorphism of \( \mathbb{K} \). Then

\[ \rho_x \left( \frac{c}{d} \right) = \frac{\rho_x(c)}{\rho_x(d)} \in k(G) \]

because \( \text{supp}(c) = \text{supp}(\rho_x(c)) \) and \( \text{supp}(d) = \text{supp}(\rho_x(d)) \) are finite.

(iii) A large class of Hahn fields satisfying the canonical second lifting property will be described in Section 3.6. \( \square \)

The next result characterises internal automorphisms as products of \( G \)-exponentiations and 1-automorphisms.

**Lemma 3.3.10.** Let \( K \) satisfy the canonical second lifting property and let \( \sigma \in v\text{-Aut } K \). Then \( \sigma \in \text{Int Aut } K \) if and only if there exist \( \rho \in G\text{-Exp } K \) and \( \tau \in 1\text{-Aut } K \) such that \( \sigma = \rho \tau \).
Proof. Since Int Aut $K$ is a group, a composition of internal automorphisms is internal. So if there exist $\rho, \tau$ as in the statement, then in particular $\rho, \tau \in \text{Int Aut } K$, therefore $\sigma = \rho \tau \in \text{Int Aut } K$.

Conversely, let $\sigma \in \text{Int Aut } K$. For all $g \in G$ let $x^g := \sigma(t^g)_g$ be the first coefficient of $\sigma(t^g)$. Notice that the elements $\{x^g : g \in G\}$ have the property that $x^gx^h = x^{g+h}$.

Indeed $x^{g+h}$ is the first coefficient of $\sigma(t^{g+h}) = \sigma(t^g)\sigma(t^h)$ and the first coefficient of the last series is the product of the first coefficients of the factors. Hence the map $x : G \to k^\times, g \mapsto x^g$ is an element of $\text{Hom}(G, k^\times)$, and the corresponding $\rho_x$ defined as in (3.12) is a $G$-exponentiation on $K$. Set $\rho = \rho_x$ and let $\tau := \rho^{-1}\sigma$. Obviously we have $\sigma = \rho \tau$, so we just need to show that $\tau \in 1-\text{Aut } K$.

Let $a \in K$ and let $h = v(a)$. Then we have

$$
\tau(a)_h = (\rho^{-1}\sigma(a))_h = (\rho^{-1}\sigma(a_hk^h))_h = (\rho^{-1}\sigma(a_h) \cdot \rho^{-1}\sigma(t^h))_h
$$

$$
= (\rho^{-1}\sigma(a_h)_0 \cdot (\rho^{-1}\sigma(t^h))_h = a_h(x^h)^{-1} \sigma(t^h)_h = a_h.
$$

So $\tau \in 1-\text{Aut } K$ and the proof is complete. \qed

3.4. The group of internal automorphisms. The next proposition gives a decomposition of $\text{Int Aut } K$ that will be used to further refine Theorem 3.2.1.

**Theorem 3.4.1.** Let $K$ satisfy the canonical second lifting property. Then the group $\text{Int Aut } K$ (resp. $\text{Int Aut}_k K$) admits the following semi-direct product decomposition:

\[ \text{Int Aut } K = 1-\text{Aut } K \rtimes \text{G-Exp } K \] (3.16)

\[ \text{Int Aut}_k K = 1-\text{Aut}_k K \rtimes \text{G-Exp } K \] (3.17)

**Proof.** Consider the sequence

\[ 1-\text{Aut } K \xrightarrow{\iota} \text{Int Aut } K \xrightarrow{X} \text{Hom}(G, k^\times) \xrightarrow{P} - 
\]

where $\iota$ is the canonical embedding. By Lemma 3.3.4 we have $\ker X = 1-\text{Aut } K = \text{im } \iota$ so the sequence is exact and, by Proposition 3.3.6, $P$ is a section of $X$. Hence (3.16) follows.

By Remark 3.3.8 we have $\text{G-Exp } K \leq v-\text{Aut}_k K$ and by Lemma 3.3.4 we have $1-\text{Aut}_k K = 1-\text{Aut}_k K$, so (3.17) follows. \qed

**Remark 3.4.2.** We noted (Remark 2.1.3) that $\text{Hom}(G, k^\times)$ is a direct factor of the group $\text{Hom}(G, U) = \text{Hom}(G, 1 + I_K) \times \text{Hom}(G, k^\times)$. Under some further assumptions, in Section 4.2 we will be able to relate $1-\text{Aut } K$ to $\text{Hom}(G, 1 + I_K)$ (with a twisted group operation), thereby relating $\text{Int Aut } K$ to $\text{Hom}(G, U)$. \qed
3.5. **The canonical first lifting property.** Recall that (Remark 2.1.3) for any Hahn field $K$ there is an isomorphism $f : \hat{K} \to k$. We can choose this isomorphism canonically to be $f_c : a + I_K \mapsto a_0$, for all $a \in R_K$. We call $f_c$ the *canonical or coefficient isomorphism* between $K$ and $k$. The homomorphism $\Phi_{K,f}$ defined in (3.2) implicitly depends on the choice of $f$: from now on we fix this to be the coefficient isomorphism $f_c$. Then $\Phi_{K,f}$ assumes the special form

$$
\Phi_c : v-\text{Aut} \ K \to \text{Aut} \ k \times o-\text{Aut} \ G,
\sigma \mapsto (\sigma_k, \sigma_G) \quad (3.18)
$$

where $\sigma_k = f_c \sigma f_c^{-1}$. Computing gives $f_c \sigma f_c^{-1}(a_0) = f_c \sigma(a_0 + I_K) = f_c(\sigma(a_0) + I_K) = \sigma(a_0)_0$, for all $a_0 \in k$. Thus

$$
\sigma_k(a_0) = \sigma(a_0)_0 \quad \text{for all } a_0 \in k. \quad (3.19)
$$

**Remark 3.5.1.** Let $\sigma \in v-\text{Aut}(k) K$. Then $\sigma|_k = \sigma_k$. Indeed, for $a_0 \in k$, from $\sigma \in v-\text{Aut}(k) K$ it follows that $\sigma(a_0) \in k$ so Equation (3.19) gives $\sigma_k(a_0) = \sigma(a_0)_0 = \sigma(a_0) = \sigma|_k(a_0)$. Moreover, let $\pi_1 : \text{Aut} k \times o-\text{Aut} G \to \text{Aut} k$, $(\rho, \tau) \mapsto \rho$ be the projection on the first component. Then the restriction $\pi_1 \Phi_c : v-\text{Aut}(k) K \to \text{Aut} k$ is a homomorphism with kernel $v-\text{Aut}(k) K$. Thus $v-\text{Aut}(k) K \triangleright v-\text{Aut}(k) K$. □

**Definition 3.5.2.** Let $(\rho, \tau) \in \text{Aut} k \times o-\text{Aut} G$. The automorphism $\tilde{\rho} \tau \in v-\text{Aut} K$ given by

$$
\tilde{\rho} \tau \left( \sum_{g \in G} a_g t^g \right) = \sum_{g \in G} \rho(a_g) t^{\tau(g)} \quad (3.20)
$$

is a lift of the pair $(\rho, \tau)$ to $K$ that we call the *canonical lift*. Indeed, we have $\Phi_c(\tilde{\rho} \tau) = (\rho, \tau)$.

We will denote by $\tilde{\rho}$ the lift of $(\rho, \text{id}_G)$ and simply refer to it as the lift of $\rho \in \text{Aut} k$. Similarly for $\tilde{\tau}$.

**Remark 3.5.3.** If $k$ is an ordered field, $\rho \in o-\text{Aut} k$ and we take the induced lexicographic ordering on $K$, then the canonical lift of a pair $(\rho, \tau) \in o-\text{Aut} k \times o-\text{Aut} G$ preserves the lexicographic ordering on $K$. □

**Lemma 3.5.4.** Let $K$ be a Hahn field such that

$$
\forall (\rho, \tau) \in \text{Aut} k \times o-\text{Aut} G : \quad \tilde{\rho} \tau(K) = K. \quad (3.21)
$$

Then the map $\Psi_c : \text{Aut} k \times o-\text{Aut} G \to v-\text{Aut} K$, $(\rho, \tau) \mapsto \tilde{\rho} \tau|_K$ is a section of $\Phi_c$.

**Proof.** Let $(\rho, \tau)$ and $(\rho', \tau')$ be elements of $\text{Aut} k \times o-\text{Aut} G$ and let $\tilde{\rho} \tau, \tilde{\rho'} \tau'$ be the respective canonical lifts. Then, for all $a = \sum a_g t^g \in K$, we have

$$
\Psi_c(\rho \rho', \tau \tau')(a) = \sum \rho(\rho'(a_g)) t^{\tau'(g)}
= \Psi_c(\rho, \tau) \left( \sum \rho'(a_g) t^{\tau'(g)} \right)
= \Psi_c(\rho, \tau) \left( \Psi_c(\rho', \tau') (a) \right).
$$
For injectivity, let \((\rho, \tau) \neq (\rho', \tau')\). Then if \(\rho(\alpha) \neq \rho'(\alpha)\) for some \(\alpha \in k\) then \(\tilde{\rho}(\alpha) = \rho(\alpha) \neq \rho'(\alpha) = \tilde{\rho}'(\alpha)\); similarly, if \(\tau(g) \neq \tau'(g)\) for some \(g \in G\) then \(\tilde{\rho}(\tau^g) = \tilde{\rho}'(\tau^g)\).

Finally, we prove that \(\Phi_c \Psi_c = \text{id}_{\text{Aut}_k \times o-\text{Aut}_G}\). Let \((\rho, \tau) \in \text{Aut}_k \times o-\text{Aut}_G\) and let \(\sigma = \Psi_c(\rho, \tau)\). Then, for all \(a = \sum a_g \tau^g \in K\) we have \(\sigma(a) = \sum \rho(a_g) \tau^g\). Then \(\Phi_c(\sigma) = (\sigma_k, \sigma_G)\) where \(\sigma_k\) is defined by \(\sigma_k(a_0) = \sigma(a_0) = \rho(a_0)\) and \(\sigma_G\) is defined by \(\sigma_G(v(a)) = v(\sigma(a)) = v\left(\sum_{g \geq v(a)} \rho(a_g) \tau^g\right) = \tau(v(a))\). So \((\sigma_k, \sigma_G) = (\rho, \tau)\) and thus \(\Phi_c \Psi_c = \text{id}\). □

**Definition 3.5.5.** Let \(K\) be a Hahn field satisfying (3.21). We say that \(K\) has the canonical first lifting property and we call \(\Psi_c\) the canonical section (on \(K\)) of \(\Phi_c\).

Whenever \(K\) has the canonical first lifting property, we will assume our chosen section to be the canonical one.

**Remark 3.5.6.**

1. For every pair \((\rho, \tau) \in \text{Aut}_k \times o-\text{Aut}_G\) we have \(\Psi_c(\rho, \tau) \in v-\text{Aut}_k K\).

2. Assume that \(K\) has the canonical first lifting property and let \(f = \rho_f \in \text{Aut}_k\) (see Remark 2.1.3). Then an explicit section \(\Psi_{K,f}\) of \(\Phi_{K,f}\) is given by the formula:

\[
\Psi_{K,f}(\rho, \tau) = \Psi_{K,f}(\rho_f^{-1} \rho \rho_f, \tau)
\]

\[
\Psi_{K,f}(\rho, \tau) \left(\sum_{g \in G} a_g \tau^g\right) = \sum_{g \in G} \rho_f^{-1} \rho_f(a_g) \tau^g
\]

(3.22)

**Lemma 3.5.7.** Let \(K\) be a Hahn field with the canonical first lifting property. Then \(v-\text{Aut}_k K \simeq v-\text{Aut}_k K \times \text{Aut}_k\).

**Proof.** By Remark 3.5.1 we have \(v-\text{Aut}_k K = \ker \pi_1 \Phi_c\), so the sequence

\[
v-\text{Aut}_k K \hookrightarrow v-\text{Aut}_k K \xrightarrow{\pi_1 \Phi_c} \text{Aut}_k
\]

is exact. Because \(\Psi_c\) is a section of \(\Phi_c\) it follows that the map \(\text{Aut}_k \rightarrow v-\text{Aut}_k K\), \(\rho \mapsto \tilde{\rho} = \Psi_c(\rho, \text{id}_G)\) is a section of \(\pi_1 \Psi_c\). The statement follows. □

3.6. **Rayner fields.** Now we are going to study a class of Hahn fields, which satisfy the canonical first and second lifting property.

**Definition 3.6.1.** Let \(G\) be a non-trivial ordered abelian group. A family \(\mathcal{F} \neq \emptyset\) of subsets of \(G\) is said to be a field family (with respect to \(G\)) (see [Ray68, Section 2]) if the following six properties are satisfied:

(R1) The elements of \(\mathcal{F}\) are well ordered subsets of \(G\).
(R2) The union of the elements of $\mathcal{F}$ generates $G$ as a group.

(R3) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

(R4) $A \in \mathcal{F}, B \subset A \Rightarrow B \in \mathcal{F}$.

(R5) $A \in \mathcal{F}, g \in G \Rightarrow A + g \in \mathcal{F}$.

(R6) if $A \in \mathcal{F}$ and $A \subseteq G^{\geq 0}$ then the set of all finite sums of elements of $A$ belongs to $\mathcal{F}$.

**Theorem 3.6.2** ([Ray68, Theorem 1]). If $\mathcal{F}$ is a field family then the set $k((\mathcal{F}))$ of elements of $K$ whose support belongs to $\mathcal{F}$ is a subfield of $K$.

**Definition 3.6.3.** The fields $k((\mathcal{F}))$ obtained in Theorem 3.6.2 are Hahn fields\(^5\) that will be called Rayner fields.

**Examples 3.6.4.** (i) A general class of Rayner fields is described in [Ray68, Section 3]. In particular, the field of Puiseux series: let $K = k((\mathbb{Q}))$ and consider the family $\mathcal{F}$ of sets of the form $\frac{1}{d}A$ where $d$ is a positive integer and $A$ is a well ordered subset of $\mathbb{Z}$. It is clear that $\mathcal{F}$ is a field family and that the field $k((\mathcal{F}))$ thus obtained is the field $\mathbb{P}$ of Puiseux series (see Section 5.5). We describe further examples of interest to us.

(ii) Let $\kappa$ be an uncountable regular cardinal. The family $\mathcal{F}_\kappa$ of well ordered subsets of $G$ with cardinality smaller than $\kappa$ is clearly a field family. The resulting field, denoted by $K_\kappa$, is called the $\kappa$-bounded subfield of $K$. It consists of all elements of $K$ whose support has cardinality less then $\kappa$ (see [All62] or [KS05]).

(iii) Consider the set $S$ of finitely generated subgroups of $G$ and let $\mathcal{F}$ consist of all well ordered subsets of elements of $S$. Then $\mathcal{F}$ is a field family and thus $k((\mathcal{F}))$ is a Hahn field containing $k(G)$. □

**Lemma 3.6.5.** Let $K$ be a Rayner field. Then $K$ satisfies the canonical second lifting property.

**Proof.** Let $a \in K$ and $x \in \text{Hom}(G, k^\times)$. By part (iv) of Remark 3.3.8 we have $\text{supp}(a) = \text{supp}(\rho_x(a))$. Since $K$ is a Rayner field this implies $\rho_x(a) \in K$. □

The following proposition characterises Rayner fields with the canonical first lifting property.

**Proposition 3.6.6.** Let $\mathcal{F}$ be a field family and let $F = k((\mathcal{F}))$ be the corresponding Rayner field. Then $F$ has the canonical first lifting property if and only if $\mathcal{F}$ is stable under $o$-Aut $G$, by which we mean that if $A \in \mathcal{F}$ and $\tau \in o$-Aut $G$ then $\tau(A) \in \mathcal{F}$.

**Proof.** Assume that $\mathcal{F}$ be stable under $o$-Aut $G$, let $a = \sum a_g t^g \in F$ and $(\rho, \tau) \in \text{Aut } k \times o$-Aut $G$. Then the support of $\widetilde{\rho \tau}(a) = \sum \rho(a_g) t^{\tau(g)}$ is $\{\tau(g) : g \in$ \[\text{See [KKS20, Theorem 3.15].}\]
supp(a) = τ(supp(a)) ∈ F, by assumption (since supp(a) ∈ F). Hence \( \tilde{\rho} \tau(a) \in F \). So F has the canonical first lifting property.

Vice versa, assume that F has the canonical first lifting property and let A ∈ F. Take any element a = ∑ aₙtⁿ ∈ F such that supp(a) = A. By assumption, for all \((\rho, \tau) \in \text{Aut } k \times o\text{-Aut } G\) we have b = \( \tilde{\rho} \tau(a) \in F \) so, in particular, supp(b) = \( \tau(A) \in F \) and so F is stable under o-Aut G. □

**Corollary 3.6.7.** (i) The field \( \mathbb{P} \) of Puiseux series has the canonical first lifting property.

(ii) The \( \kappa \)-bounded subfields of \( K \) (Example 3.6.4) have the canonical first lifting property.

**Proof.**

(i) Let \( \frac{1}{\alpha} A \) be as in Example 3.6.4 above and let \( \tau \) be an order preserving automorphism of \( \mathbb{Q} \). Now \( o\text{-Aut } (\mathbb{Q}, +) \simeq (\mathbb{Q}^{>0}, \cdot) \) so \( \tau \) is multiplication by some positive rational \( \frac{m}{n} \). Then it is clear that \( \tau(\frac{1}{\alpha} A) = \frac{1}{\alpha}(mA) \) also belongs to the same field family.

(ii) An order preserving automorphism of \( G \) maps any well ordered subset onto another one of the same cardinality, hence \( F_\kappa \) is stable under \( o\text{-Aut } G \). □

**Example 3.6.8.** Not all Hahn fields are Rayner. Let \( k = \mathbb{Q}, G = \mathbb{Z} \), so \( \mathbb{K} = \mathbb{Q}(t) \) and consider \( K = \mathbb{Q}(t) \). Then \( a := (1-t)^{-1} = \sum_{n \in \mathbb{N}} t^n \in K \) and supp a = \( \mathbb{N} \). But for \( \exp(t) := \sum_{n \in \mathbb{N}} \frac{1}{n!} t^n \) we have supp \( \exp(t) = \mathbb{N} \) and yet \( \exp(t) \notin K \) (see [Eis75, pp 765–767]). So \( K \) contains some but not all elements of \( \mathbb{K} \) with support equal to \( \mathbb{N} \) and hence it is not a Rayner field.

However \( k(G) \) has the canonical first lifting property for every ordered abelian group \( G \). Indeed \( k(G) \) is the set of elements of \( \mathbb{K} \) of the form \( a = \frac{p}{q} \) where \( p \in \mathbb{K} \) and \( q \in \mathbb{K}^\times \) have finite support. Now let \( (\rho, \tau) \in \text{Aut } k \times o\text{-Aut } G \). Then the canonical lift \( \tilde{\rho} \tau \) is an automorphism of \( \mathbb{K} \), so \( \tilde{\rho} \tau(a) = \frac{\tilde{\rho}(p)}{\tau(q)} \). The supports supp \( \tilde{\rho} \tau(p) = \tau(\text{supp } p) \) and supp \( \tilde{\rho} \tau(q) = \tau(\text{supp } q) \) are finite, so \( \tilde{\rho} \tau(a) \in k(G) \). □

### 3.7. General decomposition theorem

Combining Theorem 3.2.1 with Theorem 3.4.1 and Corollary 3.1.8 we get the following decomposition theorem.

**Theorem 3.7.1.** Let \( K \) be a Hahn field with the first and canonical second lifting property. Then

\[
\begin{align*}
\nu\text{-Aut } K &\simeq (1\text{-Aut } K \times \text{Hom}(G,k^\times)) \rtimes (\text{Aut } k \times o\text{-Aut } G) \\
\nu\text{-Aut } k \simeq (1\text{-Aut } k \times \text{Hom}(G,k^\times)) \rtimes o\text{-Aut } G
\end{align*}
\]

The canonical first lifting property allows us to refine some of the results that we proved above under the weaker assumption of a general first lifting property.

**Proposition 3.7.2.** Let \( K \) be a Hahn field with the canonical first and second lifting property. Then
(i) $\text{Ext Aut } K = \text{Ext Aut}_{(k)} K$
(ii) $v\text{-Aut}_{(k)} K = \text{Int Aut}_k K \rtimes \text{Ext Aut } K$
(iii) $v\text{-Aut}_{(k)} K \simeq (1\text{-Aut}_k K \rtimes \text{Hom}(G, k^\times)) \rtimes (\text{Aut } k \times o\text{-Aut } G)$.
(iv) 
\[ v\text{-Aut}_{(k)} K \simeq \text{Int Aut}_k K \rtimes (o\text{-Aut } G \rtimes \text{Aut } k) \]
\[ \simeq (\text{Int Aut}_k K \rtimes o\text{-Aut } G) \rtimes \text{Aut } k \]

Proof. Part (i) follows immediately from Remark 3.5.6. Parts (ii) and (iii) are analogous to Theorems 3.2.1 and 3.7.1 respectively, for the case where $K$ has the canonical first lifting property. Finally, for part (iii): The first line follows from part (ii) and 3.1.8, noticing that we have $o\text{-Aut } G \rtimes \text{Aut } k = o\text{-Aut } G \times \text{Aut } k$. The second line follows by combining Lemma 3.5.7, Theorem 3.2.1 and Proposition 3.1.8. □

Corollary 3.7.3. Let $k(G) \subseteq K, F \subseteq K$ be two Hahn fields with the first and canonical second lifting property. Then
\[ v\text{-Aut } K \simeq v\text{-Aut } F \iff 1\text{-Aut } K \simeq 1\text{-Aut } F \] (3.25)
\[ v\text{-Aut}_{(k)} K \simeq v\text{-Aut}_{(k)} F \iff 1\text{-Aut}_k K \simeq 1\text{-Aut}_k F \] (3.26)
\[ v\text{-Aut}_k K \simeq v\text{-Aut}_k F \iff 1\text{-Aut}_k K \simeq 1\text{-Aut}_k F \] (3.27)

The following diagram summarises the information on the group structure of $v\text{-Aut } K$, for a Hahn field $K$ with the first and canonical second lifting property. The double line means that the smaller group is normal in the larger and, in general, all the inclusions are strict.

Analogous diagrams hold for $v\text{-Aut}_{(k)} K$ and $v\text{-Aut}_k K$. Comparing the groups appearing in different diagrams, many open questions remain, which we intend to address in future publications.
4. STRONGLY ADDITIVE AUTOMORPHISMS

Inspired by the work of Schilling [Sch44], in this section we study automorphisms of a Hahn field \( K \) which have the very powerful property of commuting with infinite sums.

**Definition 4.0.1.** Let \( A = \{a(i) : i \in I\} \subseteq K \) be a family of elements of \( K \) indexed by a set \( I \). Let \( \text{Supp} A := \bigcup_{i \in I} \text{supp} a(i) \) and for \( g \in G \) define \( S_g = \{i \in I : g \in \text{supp} a(i)\} \). We say that \( A \) is summable if

(a) \( \text{Supp} A \) is well ordered;
(b) for all \( g \in \text{Supp} A \) the set \( S_g \) is finite.

**Lemma 4.0.2.** Let \( A = \{a(i) : i \in I\} \subseteq K \) and for all \( g \in \text{Supp} A \) let \( a_g := \sum_{i \in S_g} (a(i))_g \). Then

(i) \( A \) is summable if and only if

\[
\sum_{i \in I} a(i) := \sum_{g \in \text{Supp} A} a_g g
\]

is a well defined element of \( K \).

(ii) Assume \( A \) is summable and set \( \nu = \min\{v(a(i)) : i \in I\} \). Then \( v(a) \geq \nu \).

(ii) If \( |S_{\nu}| = 1 \), i.e., \( \exists! j \in I : v(a(j)) = \nu \), then \( v(a) = \nu \) and \( a_{\nu} = (a(j))_{\nu} \).

**Definition 4.0.3.** Let \( A = \{a(i) : i \in I\} \subseteq K \) be a family of elements of \( K \) indexed by a set \( I \).

(i) If \( A \) is a summable family we call \( a = \sum_{i \in I} a(i) \) the sum of \( A \).

(ii) Let \( K \) be a Hahn field and \( A = \{a(i) : i \in I\} \subseteq K \) a summable family. We say that \( A \) is \( K \)-summable if \( a \in K \).

(iii) Let \( C = \{c_i : i \in I\} \subseteq \mathbb{K} \) be a family of coefficients. We define the family \( CA = \{c_i a(i) : i \in I\} \subseteq K \). We call \( CA \) the scalar multiple of \( A \) by \( C \).

(iv) Let \( B = \{b(j) : j \in J\} \subseteq \mathbb{K} \) be another family. Assume, without loss of generality, that \( I = J \). We define the sum \( A + B = \{a(i) + b(j) : i \in I\} \) and the product \( AB = \{a(i) b(j) : i, j \in I\} \).

**Remark 4.0.4.** (i) It follows from Lemma 4.0.2 that a scalar multiple of a summable family, the sum of two summable families and the product of two summable families are all summable.

(ii) The maximal Hahn field \( \mathbb{K} \) is the only Hahn field that is closed under taking sums of arbitrary summable families. Indeed, if \( K \) is a Hahn field such that every summable family \( A \subseteq K \) is \( K \)-summable, then for all \( a = \sum a_g g \in \mathbb{K} \) the family \( \{a_g g : g \in \text{supp} a\} \) is \( K \)-summable. Thus \( a \in K \) and so \( K = \mathbb{K} \).

**Definition 4.0.5.** Let \( K \) be a Hahn field
(i) A map $\sigma : K \to K$ is $K$-summable if, for all $a = \sum a_g t^g \in K$,
(a) the family $\{\sigma(a_g t^g) : g \in \text{supp}(a)\}$ is $K$-summable;
(b) $\sigma(a) = \sum \sigma(a_g t^g)$.

(ii) An automorphism $\sigma \in \nu\text{-Aut} K$ is strongly additive if both $\sigma$ and $\sigma^{-1}$ are $K$-summable maps.

**Notation 4.0.6.** The set of strongly additive, valuation preserving automorphisms will be denoted by $\nu\text{-Aut}^+ K$. We will also use the superscript “$+$” on the other groups of automorphisms, to denote the corresponding subset of strongly additive automorphisms: for example $\text{Int Aut}^+ K = \text{Int Aut} K \cap \nu\text{-Aut}^+ K$.

**Remark 4.0.7.**
(i) A strongly additive automorphism needs not be valuation preserving: the non-valuation preserving automorphism constructed in Example 2.1.10 is strongly additive.

(ii) A strongly additive automorphism needs not be a $k$-automorphism: let $\alpha \in \text{Aut} k$ be a non-trivial automorphism of $k$ (for example, we can choose $k = \mathbb{Q}(\sqrt{2})$ and $\alpha : \sqrt{2} \mapsto -\sqrt{2}$) and let $\sigma \in \text{Aut} K$ be defined by $\sigma(\sum a_g t^g) = \sum \alpha(a_g)t^g$. This is a strongly additive automorphism that does not fix $k$. □

The following is an example of an automorphism that is not strongly additive. We wish to thank L. S. Krapp for suggesting the idea.

**Example 4.0.8.** Let $\omega$ be the first infinite ordinal, let $G = \coprod_{n \in \omega} \mathbb{Q}$ (see Notation 2.1.9) and let $K = \mathbb{C}(G)$. Elements of $G$ have the form $\sum_{n \in \omega} q_n \mathbb{1}_n + q_\omega \mathbb{1}_\omega$ and the set $\{\mathbb{1}_n : n \in \mathbb{N}\} \cup \{\mathbb{1}_\omega\}$ is a $\mathbb{Q}$-valuation basis for $G$ (see [Kuh00b, Page 4]).

The set $\{t^{-1^n} : n \in \mathbb{N}\}$ is algebraically independent over $\mathbb{C}$ (see [EP05, Theorem 3.4.2]), therefore, it extends to a transcendence basis $\mathcal{B}$ of $K$ over $\mathbb{C}$. The set $\{t^{-1^n} + t_\omega^n : n \in \mathbb{N}\}$ is also algebraically independent so it also extends to a transcendence basis $\mathcal{B}'$. There exists a bijection $f : \mathcal{B} \to \mathcal{B}'$ such that $f(t^{-1^n}) = t^{-1^n} + t_\omega^n$ for all $n \in \mathbb{N}$. Since $\mathbb{C}$ is algebraically closed, a bijection of transcendence bases extends, in turn, to an isomorphism of fields: that is, there exists a field automorphism $f \in \text{Aut} K$ such that $f(t^{-1^n}) = t^{-1^n} + t_\omega^n$ for all $n \in \mathbb{N}$.

Now consider the element $a = \sum_{n \in \mathbb{N}} t^{-1^n}$. Notice that the sequence $(\mathbb{1}_n)_{n \in \mathbb{N}}$ is anti-well ordered in $G$, so the support $(-\mathbb{1}_n)_{n \in \mathbb{N}}$ of $a$ is well ordered, so $a \in K$. However, the family $A = \{f(t^{-1^n}) : n \in \mathbb{N}\}$ is not summable. Indeed $\text{Supp} A = \{-\mathbb{1}_n : n \in \mathbb{N}\} \cup \{\mathbb{1}_\omega\}$ and $\mathbb{1}_\omega \in \text{supp} f(t^{-1^n})$ for all $n \in \mathbb{N}$ hence $S_{\mathbb{1}_\omega} = \mathbb{N}$ violates condition (b) of Definition 4.0.1. □

**Proposition 4.0.9.** Let $K$ be a Hahn field. Then
(i) $\nu\text{-Aut}^+ K$ is a subgroup of $\nu\text{-Aut} K$.
(ii) Assume that $K$ has the first lifting property. Then $\text{Ext Aut} K \leq \nu\text{-Aut}^+ K$.
(iii) Assume, moreover, that $K$ satisfies the canonical second lifting property. Then we have $G\text{-Exp} K \leq \text{Int Aut}^+_K K$.

**Proof.** (i) Let $\sigma$ and $\tau$ be two strongly additive automorphisms. Then, for all $a = \sum a_g t^g \in K$, applying subsequently the strong additivity of $\sigma$ and $\tau$ we...
get
\[(\sigma \tau)(a) = \sigma \left( \sum \tau(a_g t^g) \right) = \sum \sigma(\tau(a_g t^g))\]
so \(\sigma \tau\) is strongly additive.

(ii) Let \(\sigma \in \text{Ext Aut} K\). Then there are \(\rho \in \text{Aut} k\) and \(\tau \in o\text{-Aut} G\) such that, for all \(a = \sum a_g t^g \in K\) we have \(\sigma(a) = \sum \rho(a_g) t^{\tau(g)}\). In particular, for every term \(a_g t^g\) we also have \(\sigma(a_g t^g) = \rho(a_g) t^{\tau(g)}\). Thus
\[
\sigma \left( \sum_{g \in G} a_g t^g \right) = \sum_{g \in G} \rho(a_g) t^{\tau(g)} = \sum_{g \in G} \sigma(a_g t^g).
\]
So \(\sigma\) is strongly additive.

(iii) By (3.13) and (3.14), a \(G\)-exponentiation \(\rho_x\) on \(K\) is a strongly additive automorphism. \(\square\)

4.1. The structure of \(v\text{-Aut}^+ K\). Since normality is preserved by taking intersections, it follows that \(\text{Int Aut}^+ K = \text{Int Aut} K \cap v\text{-Aut}^+ K \leq v\text{-Aut}^+ K\). Decomposition results analogous to those obtained in Section 3.1 hold for the group of strongly additive automorphisms and its subgroups:

**Proposition 4.1.1.** Let \(K\) be a Hahn field with the first lifting property. Then
\[
v\text{-Aut}^+ K = \text{Int Aut}^+ K \rtimes \text{Ext Aut} K.
\] (4.1)

**Proof.** Let \(\sigma \in v\text{-Aut}^+ K\). By Theorem 3.2.1 there exist \(\tau \in \text{Ext Aut} K\) and \(\rho \in \text{Int Aut} K\) such that \(\sigma = \rho \tau\). Now, by Remark 4.0.9, \(\tau\) is strongly additive, and since \(\rho = \sigma \tau^{-1}\) then \(\rho\) is also strongly additive. So \(\text{Ext Aut} K\) and \(\text{Int Aut}^+ K\) generate \(v\text{-Aut}^+ K\). Moreover, \(\text{Ext Aut} K \cap \text{Int Aut}^+ K \subseteq \text{Ext Aut} K \cap \text{Int Aut} K = \{\text{id}_K\}\). Finally, as remarked above, \(\text{Int Aut}^+ K\) is a normal subgroup of \(v\text{-Aut}^+ K\). The statement follows. \(\square\)

**Lemma 4.1.2.** Let \(K\) be a Hahn field satisfying the canonical second lifting property. The following hold.
\[
\begin{align*}
\text{Int Aut}^+ K &= 1\text{-Aut}^+ K \rtimes G\text{-Exp} K; \quad (4.2) \\
\text{Int Aut}_{(k)}^+ K &= \text{Int Aut}_k^+ K = 1\text{-Aut}_k^+ K \rtimes G\text{-Exp} K. \quad (4.3)
\end{align*}
\]

**Proof.** Follows from Theorem 3.4.1 taking intersections with \(v\text{-Aut}^+ K\) and applying part (iii) of Proposition 4.0.9. \(\square\)

Finally we have
Theorem 4.1.3. Let $K$ be a Hahn field with the first and canonical second lifting property. Then

\[ v-Aut^+ K = (1-Aut^+ K \times G-Exp K) \times \text{Ext Aut } K \]
\[ \cong (1-Aut^+ K \times \text{Hom}(G, k^\times)) \times (\text{Aut } k \times o-Aut G); \]
\[ v-Aut_{(k)}^+ K = (1-Aut_{(k)}^+ K \times G-Exp K) \times \text{Ext Aut } K \]
\[ \cong (1-Aut_{(k)}^+ K \times \text{Hom}(G, k^\times)) \times (\text{Aut } k \times o-Aut G); \]
\[ v-Aut_k^+ K = (1-Aut_k^+ K \times G-Exp K) \times \text{Ext Aut}_k K \]
\[ \cong (1-Aut_k^+ K \times \text{Hom}(G, k^\times)) \times o-Aut G. \]

\[ \square \]

Proof. Proposition 4.1.1 and Lemma 4.1.2 yield the equalities. The isomorphisms are now a consequence of Lemma 3.3.4 Definition 3.3.7 and Remark 3.1.7. \[ \square \]

From now we will focus on the subgroups $v-Aut_{(k)}^+ K$ and $v-Aut_k^+ K$. In Theorem 4.1.3 we see that all components, except possibly $1-Aut_k^+ K$, only depend on $k$ and $G$. The next section is devoted to the study of the remaining component: $1-Aut_k^+ K$. Recall that, by Lemma 3.3.4, we have $1-Aut_{(k)}^+ K = 1-Aut_k^+ K$.

4.2. Description of Int Aut$^+$ $K$ and 1-Aut$^+$ $K$. Schilling [Sch44] describes the group $v-Aut_k K$, for $K = k((\mathbb{Z}))$, in terms of $U_K$, the group of units of the valuation ring of $K$. Drawing inspiration from his work, we aim at an explicit description of the groups $v-Aut_{(k)}^+ K$ and $v-Aut_k^+ K$, for an arbitrary Hahn field $K$, in terms of the fundamental objects connected to $K$. Let $U = U_K$. We will further describe the group Int Aut$_k^+$ $K$ in terms of (a subgroup of) the group Hom($G, U$). Then we will deduce a description of 1-Aut$_{(k)}^+$ $K$ in terms of a subgroup of Hom($G, 1 + I_K$). In Section 5.2 we will retrieve Schilling’s result as a special case of ours.

Let $K$ be a Hahn field and $\sigma \in \text{Int Aut}_k K$. Recall that $\sigma$ satisfies the following conditions: for all $a \in K$ we have $v(\sigma(a)) = v(a)$ and $\sigma|_k = \text{id}_k$. These properties of $\sigma$ imply that $\sigma(t^g) = u_\sigma(g)t^g$ for some $u_\sigma(g) \in U$ depending on $g$. For all $\sigma \in \text{Int Aut}_K K$ define $u_\sigma: G \rightarrow U$ by $g \mapsto t^{-g}\sigma(t^g)$.

Lemma 4.2.1. Let $K$ be a Hahn field. For all $\sigma \in \text{Int Aut}_K K$ the map $u_\sigma$ is a group homomorphism.

Proof. Let $\sigma \in \text{Int Aut}_K K$ and $g, h \in G$. Then we have

\[ u_\sigma(g + h) = t^{-g-h}\sigma(t^{g+h}) \]
\[ = t^{-g}t^{-h}\sigma(t^g)\sigma(t^h) \]
\[ = t^{-g}\sigma(t^g)t^{-h}\sigma(t^h) \]
\[ = u_\sigma(g)u_\sigma(h). \]
Lemma 4.2.4. We have
\[K\text{summable elements of }\text{Hom}(\{G\})\] so \(\text{im}S\) is (since \(\sigma\) and \(\tau\) are strongly additive \(k\)-automorphisms) implies that \(\sigma(a) = \tau(a)\) for all \(a \in K\), hence \(\sigma = \tau\). \(\square\)

Now we determine the image of \(S\).

Definition 4.2.3. An element \(u \in \text{Hom}(G, U)\) is \(K\)-summable if, for every \(a \in K\), the family \(\{a_gu(g) : g \in \text{supp} a\}\) is \(K\)-summable. Let us denote the set of summable elements of \(\text{Hom}(G, U)\) by \(\text{Hom}^+(G, U)\).

Lemma 4.2.4. We have \(\text{im}S = \text{Hom}^+(G, U)\). Therefore \(S\) corestricts to a bijection
\[S : \text{Int Aut}_k^+ K \to \text{Hom}^+(G, U).\]

Proof. Let \(u \in \text{im}S\). Then \(u = u_\sigma\) for some \(\sigma \in \text{Int Aut}_k^+ K\). Now let \(a = \sum a_gt^g \in K\). Since \(\sigma \in \text{Int Aut}_k^+ K\) we have \(\sigma(a) = \sum a_g\sigma(t^g) = \sum a_gu_\sigma(g)t^g\), hence the family \(\{a_gu_\sigma(g)t^g : g \in \text{supp} a\}\) is \(K\)-summable. Therefore \(u \in \text{Hom}^+(G, U)\) and so \(\text{im}S \subseteq \text{Hom}^+(G, U)\).

Conversely, let \(u \in \text{Hom}^+(G, U)\) and define \(\sigma\) by \(\sigma(\sum a_gt^g) = \sum a_gu(g)t^g\) for all \(\sum a_gt^g \in K\). Since \(u\) is \(K\)-summable, \(\sigma \in v\text{-Aut}_k^+ K\) is well defined. Now we show \(\sigma \in \text{Int Aut}_k K\). For \(g \in G\) let \(u(g) = u_0 + \varepsilon(g)\) with \(u_0 \in k^\times\), \(\varepsilon(g) \in I_K\), let \(a = \sum a_g\varepsilon(g) \in K\) and set \(v(a) = h\). Note that \(v(a_gu(g)t^g) = g\) for all \(g \in \text{supp} a\). Moreover, \(\bar{\sigma}(a_0 + I_K) = \sigma(a_0) + I_K = a_0u(0) + I_K = a_0 + I_K\) so \(\bar{\sigma} = \text{id} (\bar{\sigma}\) was defined in Remark 2.1.7). So \(\sigma \in \text{Int Aut}_k^+ K\) and, by definition of \(\sigma\) we have \(\sigma(t^g) = u(g)t^g\) thus \(u = u_\sigma \in \text{im}S\). Hence \(\text{Hom}^+(G, 1 + I_K) \subseteq \text{im}S\), which completes the proof. \(\square\)

Definition 4.2.5. We define an operation\(^6\)
\[\times : \text{Hom}^+(G, U) \times \text{Hom}^+(G, U) \rightarrow \text{Hom}^+(G, U)\]
\[(u_\tau, u_\sigma) \mapsto [u_\tau \times u_\sigma : g \mapsto \tau(u_\sigma(g))u_\tau(g)].\]

Proposition 4.2.6. The map \(S : \text{Int Aut}_k^+ K \to \text{Hom}^+(G, U)\) defined in (4.5) is a group isomorphism, if we equip \(\text{Hom}^+(G, U)\) with the new operation \(\times\):
\[S : (\text{Int Aut}_k^+ K, \circ) \cong (\text{Hom}^+(G, U), \times).\]

\(^6\)This operation corresponds to the crossed representation defined by Schilling for \(\text{Aut}_k \mathbb{L}\) where \(\mathbb{L} = k(\langle \mathbb{Z} \rangle)\). See Section 5.2 for more details.
Proof. By Lemmas 4.2.2 and 4.2.4 the map (4.6) is bijective. It remains to show
that it is a group homomorphism. Let $\sigma, \tau \in \text{Int Aut}_K^+ K$ and let $g \in G$. Then we have

$$u_{\tau \sigma}(g) = t^{-g}(\tau \sigma)(t^g) = t^{-g} \tau(u_\sigma(g)t^g) = t^{-g} \tau(u_\sigma(g)) \tau(t^g)$$

$$= t^{-g}(u_\sigma(g)t^g)u_\tau(g) = \tau(u_\sigma(g))u_\tau(g) = (u_\tau \times u_\sigma)(g).$$

□

Corollary 4.2.7. Restricting $S$ we get

$$(G-\text{Exp } K, \circ) \simeq (\text{Hom}(G, k^\times), \times)$$

(4.7)

and thus

$$(\text{Hom}^+(G, U), \times) \simeq (\text{Hom}^+(G, 1 + I_K), \times) \rtimes (\text{Hom}(G, k^\times), \cdot).$$

(4.8)

Proof. Let $\rho = \rho_x \in G-\text{Exp } K$. Then $S(\rho) = u_\rho = x \in \text{Hom}(G, k^\times)$, so $S|_{G-\text{Exp } K} = P^{-1}$, where $P$ is the map given in Proposition 3.3.6. We therefore have an isomorphism

$$S : G-\text{Exp } K \xrightarrow{\sim} (\text{Hom}(G, k^\times), \times).$$

(4.10)

To prove (4.7) we notice that, for $x, y \in \text{Hom}(G, k^\times)$, corresponding to $\sigma_x, \sigma_y \in \text{Int Aut}_K^+ K$ we have

$$(x \times y)(g) = x(g) \cdot \sigma_x(y(g)) = x(g)y(g)$$

because $y(g) \in k^\times$ and $\sigma_x \in G-\text{Exp } K \leq \text{Aut}_K K$. So $(\text{Hom}(G, k^\times), \times) = (\text{Hom}(G, k^\times), \cdot)$, and (4.7) follows.

Similarly, if $\tau \in 1-\text{Aut}_K^+ K$ then $u_\tau \in \text{Hom}(G, 1 + I_K)$. So $\text{Hom}^+(G, 1 + I_K) := (\text{Hom}^+(G, U)) \cap \text{Hom}(G, 1 + I_K)$. Then we have

$$S : 1-\text{Aut}_K^+ K \xrightarrow{\sim} \text{Hom}^+(G, 1 + I_K)$$

(4.11)

which proves (4.8). Equation (4.9) now follows from Proposition 4.2.6 applying (4.3), (4.7) and (4.8). □

Combining Theorem 4.1.3 and Proposition 4.2.6 we find

Theorem 4.2.8. Let $K$ be a Hahn field with the first and canonical second lifting property. Then

$$v-\text{Aut}_{(k)}^+ K \simeq (\text{Hom}^+(G, 1 + I_K) \times \text{Hom}(G, k^\times)) \rtimes (\text{Aut } k \times o-\text{Aut } G);$$

$$v-\text{Aut}_K^+ K \simeq (\text{Hom}^+(G, 1 + I_K) \times \text{Hom}(G, k^\times)) \rtimes o-\text{Aut } G.$$
Theorem 4.2.8 thus provides a decomposition of \( v\text{-Aut}^+_k K \) and \( v\text{-Aut}_k^+ K \) purely in terms of the valuation invariants of \( K \). In the next section we are going to apply the results obtained so far under some further assumptions on the group \( G \) and the field \( k \). This will allow to retrieve results of Schilling [Sch44] on the field of Laurent series and of Deschamps [Des05] on the field of Puiseux series.

5. Explicit examples in special cases

5.1. Finitely generated exponent group.

Let \( k \) be an arbitrary field and let \( G \) be a totally ordered, finitely generated abelian group. Without loss of generality, we can assume \( G = \mathbb{Z}^n = \prod_{i=1}^n \mathbb{Z} \), for some \( n \in \mathbb{N} \). Then \( \text{Hom}(G, k^\times) \simeq (k^\times)^n \). Let \( K \subseteq k((G)) \) be a Hahn field satisfying the first and canonical second lifting property \(^7\).

Theorem 3.7.1 thus yields

**Theorem 5.1.1.** Let \( G = \mathbb{Z}^n \). Let \( k \) be a field and \( K \subseteq k((G)) \) a Hahn field with the first and canonical second lifting property. Then we have

\[
\text{v-Aut}^+ K \simeq (1\text{-Aut} K \times (k^\times)^n) \times (\text{Aut} k \times o\text{-Aut} G)
\]

\[
\text{v-Aut}_k^+ K \simeq (1\text{-Aut} K \times (k^\times)^n) \times o\text{-Aut} G.
\]

If, moreover, \( K \) satisfies the canonical first lifting property, Proposition 3.7.2 yields

\[
\text{v-Aut}_{(k)} K \simeq (1\text{-Aut} K \times (k^\times)^n) \times (\text{Aut} k \times o\text{-Aut} G).
\]

□

Now we will provide a description of \( v\text{-Aut}^+_k K \) and \( v\text{-Aut}_k^+ K \). For \( G = \mathbb{Z}^n \) we have \( \text{Hom}(G, 1 + I_K) \simeq (1 + I_K)^n \). More precisely, this isomorphism is given as follows. Let \( g_1, \ldots, g_n \) be generators of \( G \), let \( \bar{u} \in \text{Hom}(G, 1 + I_K) \) and let \( u_i := \bar{u}(g_i) \in 1 + I_K \), for \( i = 1, \ldots, n \). Then

\[
\xi : \text{Hom}(G, 1 + I_K) \to (1 + I_K)^n, \quad \bar{u} \mapsto (u_1, \ldots, u_n)
\]

is a group isomorphism. Under \( \xi \), a summable automorphism \( \bar{u} \in \text{Hom}^+(G, 1 + I_K) \) (Definition 4.2.3) corresponds to a tuple \( \xi(\bar{u}) = (u_1, \ldots, u_n) \) such that, for all \( a \in K \) the family

\[
\left\{ a_g \left( \sum r_i u_i \right) t^g : r_i \in \mathbb{Z}, \sum r_i u_i = g, \ g \in \text{supp} \ a \right\}
\]

is \( K \)-summable. Let us denote by \( (1 + I_K)^n^+ := \xi \left( \text{Hom}^+(G, 1 + I_K) \right) \). On \( \text{Hom}^+(G, 1 + I_K) \) we defined the operation \( \times \) (Definition 4.2.5). We can define an operation on \( (1 + I_K)^n^+ \), also denoted by \( \times \), by setting \( u_1 \times u_2 := \xi^{-1}(u_1) \times \xi^{-1}(u_2) \), for all \( u_1, u_2 \in (1 + I_K)^n^+ \). We thus obtain

**Lemma 5.1.2.** \( \text{Hom}^+(G, 1 + I_K) \simeq ((1 + I_K)^n^+, \times) \). □

\(^7\)This applies, in particular, to \( K = k(\mathbb{Z}^n) \). Then \( v\text{-Aut}_k K \) is an interesting subgroup of \( \text{Aut}_k K \), which is the Cremona group \( \text{Cr}_n(k) \) (for more on the Cremona group, see [Dés21]).
Now assume that $G = \mathbb{Z}^n$ is equipped with the lexicographic order $<_{\text{lex}}$. We can explicitly describe $o$-$\text{Aut} G$. Let $\text{UUT}_n(\mathbb{Z})$ be the multiplicative group of upper uni-triangular $n \times n$-matrices with integer coefficients.

**Lemma 5.1.3** ([Con58, Lemma 1]9). Let $G = (\mathbb{Z}^n, <_{\text{lex}})$. Then $o$-$\text{Aut} G \simeq \text{UUT}_n(\mathbb{Z})$. □

Now Lemmas 5.1.2 and 5.1.3 applied to Theorem 5.1.1 provide the following refinement of Theorem 4.2.8.

**Theorem 5.1.4.** Let $G = (\mathbb{Z}^n, <_{\text{lex}})$. Let $k$ be a field and $K \subseteq k((G))$ a Hahn field with the first and canonical second lifting property. Then we have

$$v$-$\text{Aut}^+_k K \simeq (((1 + I_K)^{n^+}, \times) \rtimes (k^\times)^n) \rtimes (\text{Aut} k \times \text{UUT}_n(\mathbb{Z}))$$

$$v$-$\text{Aut}_k K \simeq (((1 + I_K)^{n^+}, \times) \rtimes (k^\times)^n) \rtimes \text{UUT}_n(\mathbb{Z}).$$

□

In the next two sections we investigate in more detail the case $G = \mathbb{Z}$ and provide a more explicit description of the automorphism groups of the field $L = k((Z))$ of Laurent series and of the function field $k(Z)$.

### 5.2. Laurent series

Let $k$ be a field and let $L := k((Z))$ be the field of formal Laurent series with coefficients in $k$. This is a maximal Hahn field, thus it has the canonical first and second lifting properties. On this field the valuation $v$ has residue field $k$ and value group $\mathbb{Z}$. In [Sch44] Schilling studies the group $v$-$\text{Aut}_k L$ of $k$-automorphisms of $L$. In this section we prove Theorem 5.2.4, which is both a generalisation and a refinement of Schilling’s result. We also provide a refinement in order to describe the group $o$-$\text{Aut} L$, in the case of $k$ an ordered field (Corollary 5.2.7).

We recall that, by Remark 2.1.3, the group of units is $U := U_L \simeq (1 + I_L) \times k^\times$.

**Lemma 5.2.1.** We have $\text{Hom}(\mathbb{Z}, U) = \text{Hom}^+(\mathbb{Z}, U)$.

*Proof.* Let $\bar{u} \in \text{Hom}(\mathbb{Z}, U)$, $u = \bar{u}(1)$ and $a \in L$. By Neumann’s Lemma [Pri83, p. 57] the family $\{a_n u^n t^n : n \in \text{supp} a\}$ is $L$-summable. So $u \in \text{Hom}^+(\mathbb{Z}, U)$. □

By Lemma 5.2.1 we can use the group structure $(\text{Hom}(\mathbb{Z}, U), \times)$ described in Definition 4.2.5 to induce an alternative group structure on $U$. Call $\vartheta : \text{Hom}(\mathbb{Z}, U) \to U$ the isomorphism given by $\vartheta(\bar{u}) = u := \bar{u}(1)$. Set, for all $u_1, u_2 \in U$

$$u_1 \times u_2 = \vartheta(\vartheta^{-1}(\bar{u}_1) \times \vartheta^{-1}(\bar{u}_2)).$$

(5.1)

---

8For $g = (g_1, \ldots, g_n) \in G$ we set $g >_{\text{lex}} 0$ if and only if $g \neq 0$ and for the smallest index $i$ such that $g_i \neq 0$ we have $g_i > 0$.

9[Con58] uses upper triangular matrices because he takes the anti-lexicographic ordering on $G$. 

---
Lemma 5.2.2. We have
\[ (\text{Hom}(\mathbb{Z}, U), \times) \simeq (U, \times_s) \quad (5.2) \]
\[ (\text{Hom}(\mathbb{Z}, k^\times), \times) \simeq (\text{Hom}(\mathbb{Z}, k^\times), \cdot) \simeq (k^\times, \cdot) \quad (5.3) \]
\[ (\text{Hom}(\mathbb{Z}, 1 + I_L), \times) \simeq (1 + I_L, \times_s) \quad (5.4) \]
and thus
\[ (\text{Hom}(\mathbb{Z}, U), \times) \simeq (1 + I_L, \times_s) \times (k^\times, \cdot). \quad (5.5) \]

Proof. Equation (5.2) follows immediately from (5.1). Equations (5.3), (5.4) and (5.5) are now special cases of Corollary 4.2.7. □

Next we show that all automorphisms of \( L \) are strongly additive.

Lemma 5.2.3. We have \( v\text{-Aut} L = v\text{-Aut}^+ L \).

Proof. Let \( \sigma \in v\text{-Aut} L \). By Lemma 5.1.3 with \( n = 1 \) it follows that \( o\text{-Aut} \mathbb{Z} \) is trivial so for all \( a \in L \) we have \( v(a) = v(\sigma(a)) \). Now let \( a = \sum_{i=m}^\infty a_i t^i \in L \) with \( m = v(a) \). Then the family \( \{ \sigma(a_i t^i) : i \in \text{supp} a \} \) is summable and, for all \( n \in \mathbb{Z} \) we have
\[
v(\sigma(a) - \sum a_i t^i) = v\left(\sigma\left(\sum_{i=m}^\infty a_i t^i\right) - \sum_{i=m}^\infty \sigma(a_i t^i)\right)
\]
\[
= v\left(\sigma\left(\sum_{i=m}^n a_i t^i\right) + \sum_{i>n} \sigma(a_i t^i) - \sum_{i=n}^\infty \sigma(a_i t^i) + \sum_{i>n} \sigma(a_i t^i)\right)
\]
\[
= v\left(\sum_{i>n} \sigma(a_i t^i) + \sum_{i>n} \sigma(a_i t^i)\right) > n
\]
hence \( v(\sigma(a) - \sum a_i t^i) = \infty \) which implies \( \sigma(a) = \sum a_i t^i \). □

The following theorem is now a consequence of Theorem 5.1.4 and Lemmas 5.2.2 and 5.2.3.

Theorem 5.2.4. We have
\[ v\text{-Aut}(k) L \simeq ((1 + I_L, \times_s) \times (k^\times, \cdot)) \rtimes \text{Aut} k. \]

□

Remark 5.2.5. We can now show explicitly how an automorphism \( \sigma \in v\text{-Aut}(k) K \) acts. Let \( a = \sum_{i=m}^\infty a_i t^i \in L \). We know that \( \sigma \) is strongly additive, so \( \sigma(a) = \sum \sigma(a_i) t^i \). For all \( i \in \text{supp} a \) we have \( \sigma(a_i) \in k \). Moreover, because \( v(t) = v(\sigma(t)) = 1 \) we have \( u_\sigma := t^{-1} \sigma(t) \in U \). Then \( \sigma \) is uniquely determined by \( u_\sigma \) and \( \sigma|_k \):
\[ \sigma\left(\sum_{i=m}^\infty a_i t^i\right) = \sum_{i=m}^\infty \sigma|_k(a_i)(u_\sigma t)^i. \]
Conversely, to every unit \( u \in U \) and every \( \tau \in \text{Aut} \, k \) we have the corresponding \( \sigma_{u,\tau} \in v-\text{Aut}_{(k)} \mathbb{L} \) defined by

\[
\sigma_{u,\tau} \left( \sum_{i \geq m} a_i t^i \right) = \sum_{i \geq m} \tau(a_i)(ut)^i.
\]

\[ \square \]

**Corollary 5.2.6.** We have \( v-\text{Aut}_k \, \mathbb{L} \simeq (U, \times_s) \simeq (1 + I_L, \times_s) \rtimes (k^\times, \cdot). \]

With Corollary 5.2.6 we retrieve Schilling’s result [Sch44, Theorem 1]. To conclude this subsection we sharpen Theorem 5.2.4 in the case where \( k \) is an ordered field, to characterise the group \( o-\text{Aut}_k \, \mathbb{L} \) of order preserving \( k \)-automorphisms of \( \mathbb{L} \) (Definition 2.1.4).

**Corollary 5.2.7.** The \( k \)-automorphisms preserving the lexicographic order on \( \mathbb{L} \) are exactly those corresponding to positive units: \( o-\text{Aut}_k \, \mathbb{L} \simeq (U^{>0}, \times_s) \). More precisely, we have

\[
o-\text{Aut}_{(k)} \, \mathbb{L} \simeq \left( (1 + I_L, \times_s) \rtimes k^{>0} \right) \rtimes o-\text{Aut} \, k. \tag{5.6}\]

**Proof.** Let \( a = \sum_{i=m}^{\infty} a_i t^i \) with \( v(a) = m \in \mathbb{Z} \) and \( u = \sum_{i=0}^{\infty} u_i t^i \) a unit \( (u_0 \neq 0) \). We can write \( a = t^m \sum_{i=0}^{\infty} b_i t^i \) with \( b_i = a_{i-m} \). Let us assume \( a > 0 \), that is \( b_0 > 0 \). Let us write \( \sigma_a := \sigma_{u,\text{id}} \) (see Remark 5.2.5). Then

\[
\sigma_a(a) = \sigma_a \left( t^m \sum_{i=0}^{\infty} b_i t^i \right) = \sigma_u(t^m) \sigma_a \left( \sum_{i=0}^{\infty} b_i t^i \right) = (tu)^m \left( b_0 + \sigma_u \left( \sum_{i=1}^{\infty} b_i t^i \right) \right)
\]

\[ = t^m u_0^m b_0 + \text{[higher order terms]} \]

hence \( \sigma(a)_m = u_0^m a_m > 0 \) if and only if \( u_0 > 0 \) or \( m \) is even. Thus we have \( o-\text{Aut}_k \, \mathbb{L} \simeq (U^{>0}, \times_s) = (1 + I_L, \times_s) \rtimes k^{>0} \) (the 1-units are all positive). Now (5.6) follows immediately. \[ \square \]

5.3. **The Cremona group in dimension one.** Let \( k \) be an arbitrary field. Consider the Hahn field \( k(\mathbb{Z}) \subseteq \mathbb{L} \). Theorem 3.7.1 applies to this field, so we have

\[
v-\text{Aut}_k \, k(\mathbb{Z}) = \text{Int} \, \text{Aut}_k \, k(\mathbb{Z}) \simeq 1-\text{Aut} \, k(\mathbb{Z}) \rtimes k^\times. \tag{5.7}\]

Note that \( v-\text{Aut}_k \, k(\mathbb{Z}) \) is a subgroup of \( \text{Aut}_k \, k(\mathbb{Z}) \), which is the Cremona group \( \text{Cr}_1(k) \). It is well known that \( \text{Cr}_1(k) \simeq \text{PGL}_2(k) \) (see, for example, [Can18, § 1.2]). Indeed, \( \sigma \in \text{Cr}_1(k) \) is completely determined by the invertible matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in k^{2 \times 2} \) such that \( \sigma(t) = (at + b)/(ct + d) \). We characterise \( v-\text{Aut}_k \, k(\mathbb{Z}) \) as a subgroup
of \( \text{Cr}_1(k) \) as follows:

\[
\text{v-Aut}_k k(\mathbb{Z}) = \left\{ \sigma \in \text{Cr}_1(k) : \sigma(t) = \frac{at}{ct + d}, \ a, c, d \in k \text{ with } ad \neq 0 \right\}
\]

\[
= \left\{ \sigma \in \text{Cr}_1(k) : \sigma(t) = \frac{at}{ct + d}, \text{ with } v\left(\frac{a}{ct + d}\right) = 0 \right\}. \tag{5.8}
\]

Indeed if \( \sigma \in \text{Int Aut}_k k(\mathbb{Z}) \) then \( 1 = v(t) = v(\sigma(t)) = v\left(\frac{at + b}{ct + d}\right) = v(at + b) - v(ct + d) \). This implies \( v(at + b) = 1 \) and therefore \( a \neq 0 \) and \( b = 0 \). Conversely, let \( a, c, d \in k \) with \( ad \neq 0 \). Then \( u := \frac{a}{ct + d} \) is a unit in the valuation ring of \( \mathbb{L} \), because \( v(u) = 0 \). Therefore, by Lemma 5.2.3, \( t \mapsto ut = \frac{at}{ct + d} \) determines a \( \sigma_u \in \text{v-Aut}_k \mathbb{L} \). Thus the restriction \( \sigma_u|_{k(\mathbb{Z})} \in \text{v-Aut}_k k(\mathbb{Z}) \), as required.

Notice that what we just showed implies, in particular, that every \( \sigma \in \text{v-Aut}_k k(\mathbb{Z}) \) extends to an automorphism in \( \text{v-Aut}_k \mathbb{L} \). Moreover, since the group of lower triangular matrices is not normal inside \( \text{PGL}_2(k) \), it follows that \( \text{v-Aut}_k k(\mathbb{Z}) \) is not a normal subgroup of \( \text{Cr}_1(k) \).

We also characterise \( 1\text{-Aut}_k k(\mathbb{Z}) \) as a subgroup of \( \text{Cr}_1(k) \) as follows:

\[
1\text{-Aut}_k k(\mathbb{Z}) = \left\{ \sigma \in \text{Cr}_1(k) : \sigma(t) = \frac{at}{ct + a}, \ a, c \in k \text{ with } a \neq 0 \right\}
\]

\[
= \left\{ \sigma \in \text{Cr}_1(k) : \sigma(t) = \frac{at}{ct + a}, \text{ with } v\left(\frac{a}{ct + a} - 1\right) > 0 \right\}. \tag{5.9}
\]

Indeed, \( \sigma \in \text{v-Aut}_k k(\mathbb{Z}) \) (and its extension to \( \mathbb{L} \)) is defined by \( t \mapsto ut \) with \( u = \frac{a}{ct + d} \) and \( ad \neq 0 \). By Corollary 5.2.6 we know that \( \sigma \in 1\text{-Aut}_k \mathbb{L} \) if and only if \( u \in 1 + I_{\mathbb{L}} \) which is indeed equivalent to the condition \( a = d \).

5.4. Divisible and finite dimensional exponent group. In this subsection we consider the special case of a Hahn field \( K \subseteq k((G)) \) where \( G \) is uniquely divisible and finite dimensional (as a \( \mathbb{Q} \)-vector space).

- If \( G \) is ordered lexicographically, we know precisely what \( o\text{-Aut} G \) is.
- If \( k \) is real closed, we get an explicit description of the group \( \text{Hom}(G, k^\times) \).
- If \( K \) is henselian of characteristic 0, we can explicitly describe \( \text{Hom}(G, 1 + I_K) \).

**Definition 5.4.1.** A divisible group is an abelian group \( H \) such that, for every \( h \in H \) and every \( n \in \mathbb{Z} \) there exists \( h' \in H \) such that \( h = nh' \). If the choice of \( h' \) is unique then \( H \) is called uniquely divisible (i.e., \( H \) is uniquely divisible if and only if it is divisible and torsion free).

A uniquely divisible group is canonically a vector space over \( \mathbb{Q} \). Throughout this subsection, let \( G \) be a divisible, totally ordered, abelian group which is finite dimensional as a vector space over \( \mathbb{Q} \). In particular \( G \) is uniquely divisible. Set \( d = \dim_G \mathbb{Q} \). Without loss of generality we can assume \( G = \mathbb{Q}^d \).

**Remark 5.4.2.** If \( H \) is a uniquely divisible group then every group homomorphism \( \vartheta \in \text{Hom}(G, H) \) is \( \mathbb{Q} \)-linear. It follows that \( \text{Hom}(G, H) \simeq H^d \). \( \square \)
5.4.1. Lexicographically ordered exponent group. Assume that $G = \mathbb{Q}^d$ is equipped with the lexicographic ordering. Let $\text{UPT}_d(\mathbb{Q})$ be the multiplicative group of upper triangular $d \times d$-matrices over $\mathbb{Q}$ with positive diagonal entries:

$$ \text{UPT}_d(\mathbb{Q}) = \left\{ (q_{i,j})_{i,j=1}^d : q_{i,j} \in \mathbb{Q} \text{ and } \begin{cases} q_{i,j} = 0 \text{ for } i > j \\ q_{i,j} > 0 \text{ for } i = j \end{cases} \right\}.$$ 

Since $o\text{-Aut} \mathbb{Q} \simeq (\mathbb{Q}^{>0}, \cdot)$ we have

**Lemma 5.4.3** ([Con58, Lemma 1]). We have $o\text{-Aut} G \simeq \text{UPT}_d(\mathbb{Q})$. □

5.4.2. Real closed coefficient field. Let $k$ be a real closed field and let $k^{>0}$ be the multiplicative subgroup of positive elements of $k$.

**Lemma 5.4.4.** We have $\text{Hom}((\mathbb{Q}, +), (k^\times, \cdot)) = \text{Hom}((\mathbb{Q}, +), (k^{>0}, \cdot))$.

**Proof.** Let $\vartheta \in \text{Hom}((\mathbb{Q}, +), (k^\times, \cdot))$. We need to show that $\vartheta(\mathbb{Q}) \subseteq k^{>0}$. Let $q \in \mathbb{Q}$. Then $q = 2^\frac{a}{2}$. Therefore

$$ \vartheta(q) = \vartheta \left( 2^\frac{q}{2} \right) = \vartheta \left( 2^\frac{q}{2} \right)^2 > 0. $$

□

**Lemma 5.4.5.** The group $(k^{>0}, \cdot)$ is uniquely divisible.

**Proof.** Let $x \in k^{>0}$ and $n \in \mathbb{N}$ with $n > 0$. Since $k$ is real closed, there exists $y \in k^{>0}$ such that $y^n = x$, so $k^{>0}$ is divisible. Moreover, assume that $z \in k^{>0}$ is such that $z^n = x = y^n$ and that we have $y \neq z$. We may assume $y > z$ (the case $y < z$ is identical). Because $y$ and $z$ are both positive, it follows that $y^n > z^n$. A contradiction. So $y$ is unique and the proof is complete. □

**Corollary 5.4.6.** We have $\text{Hom}(G, (k^\times, \cdot)) \simeq (k^{>0}, \cdot)^d$. In particular, we also have $\text{Hom}((\mathbb{Q}, +), (k^\times, \cdot)) \simeq (k^{>0}, \cdot)$. □

5.4.3. Henselian Hahn field. Let $G$ be an arbitrary ordered abelian group. Let $k$ be an arbitrary field with char $k = 0$ and let $K \subseteq k((G))$ be a henselian Hahn field (Definition 2.1.5). Denote by $\mu(K) = \{ x \in K^\times : x^n = 1 \text{ for some } n \in \mathbb{N} \setminus \{0\} \}$ the multiplicative group of roots of unity in $K$. Note that $\mu(K) \subseteq U_K$. Indeed, if $a \in \mu(K)$ there exists $n \in \mathbb{N} \setminus \{0\}$ such that $a^n = 1$. Therefore $0 = v(1) = v(a^n) = nv(a)$ implies $v(a) = 0$. So $a \in U_K$. Moreover $a^n = 1$ implies $(a_0)^n = (a^n)_0 = 1$. Thus the map $\mathcal{R} : (\mu(K), \cdot) \rightarrow (\mu(k), \cdot)$, $a \mapsto a_0$ is a well defined group homomorphism.

**Lemma 5.4.7.** The map $\mathcal{R}$ is an isomorphism.

**Proof.** Let us prove the surjectivity: let $\alpha \in k$ be such that $\alpha^n = 1$ for some $n \in \mathbb{N} \setminus \{0\}$. Then $\alpha$ is a simple root of $X^n - 1 \in k[X]$. Because $K$ is henselian there exists $a \in R_K$ such that $a_0 = \alpha$ and $a^n - 1 = 0$. Thus $\mathcal{R}(a) = \alpha$, which proves surjectivity.
Now let $u \in \ker \mathcal{R}$. Then $u_0 = 1$ and there exists $n \in \mathbb{N} \setminus \{0\}$ such that $u^n = 1$. So $u \in 1 + I_K$, that is, $u$ is of the form $u = 1 + \varepsilon$ for some $\varepsilon \in I_K$. Then the binomial expansion gives $1 = u^n = (1 + \varepsilon)^n = \sum_{j=0}^{n} \binom{n}{j} \varepsilon^j = 1 + \sum_{j=1}^{n} \binom{n}{j} \varepsilon^j$. Thus
\[ \varepsilon \sum_{j=1}^{n} \binom{n}{j} \varepsilon^{j-1} = 0. \] (5.10)
Because $v(\varepsilon^j) \neq v(\varepsilon^j)$ for $i \neq j$ the strict ultrametric inequality implies that $v \left( \sum_{j=1}^{n} \binom{n}{j} \varepsilon^{j-1} \right) = v(n) = 0$ and so $\sum_{j=1}^{n} \binom{n}{j} \varepsilon^{j-1} \neq 0$. Since char $K = 0$ we deduce from (5.10) that $\varepsilon = 0$ and thus $u = 1$ as required.

**Proposition 5.4.8.** Let $k = 0$ and let $K \subseteq k((G))$ be a henselian Hahn field. The multiplicative group $(1 + I_K, \cdot)$ is uniquely divisible.

**Proof.** Let $a \in 1 + I_K$ and let $n \in \mathbb{N} \setminus \{0\}$. We want to show the existence and the uniqueness of a $b \in 1 + I_K$ such that $b^n = a$. Consider the polynomial $P = X^n - a \in R_K[X]$. Because $a \in 1 + I_K$ we have $a_0 = 1$. The polynomial $\bar{P} = X^n - 1 \in k[X]$ has the root $x = 1$ in $k$, which is simple because $\bar{P}'(1) = n$ and char $k = 0$. Since $(K, v)$ is henselian there exists $b \in R_K$ such that $P(b) = 0$ and $b_0 = a_0 = 1$. So $b^n = a$ and $b \in 1 + I_K$. This proves the existence. To prove uniqueness, let $c \in 1 + I_K$ be such that $c^n = a$. Then $b^n = c^n$ and so $(b/c)^n = 1$. Therefore, $b/c \in 1 + I_K$ is an $n$-th root of unity and $(b/c)_0 = 1$. But 1 is also an $n$-th root of unity with $(1)_0 = 1$. By Lemma 5.4.7 we must have $b/c = 1$ and therefore $b = c$. \hfill \Box

Assume now that $G = \mathbb{Q}^d$.

**Corollary 5.4.9.** We have $\operatorname{Hom}((G, +), (1 + I_K, \cdot)) \simeq (1 + I_K)^d$. In particular, $\operatorname{Hom}((\mathbb{Q}, +), (1 + I_K, \cdot)) \simeq 1 + I_K$. \hfill \Box

A summable automorphism $u \in \operatorname{Hom}^+(G, 1 + I_K)$ (Definition 4.2.3) corresponds to a tuple $(u_1, \ldots, u_d) \in (1 + I_K)^d$ such that, for all $a \in K$ the family
\[ \left\{ a_g \left( \sum q_i u_i \right) t^g : q_i \in \mathbb{Q}, \sum r_i u_i = g, g \in \supp a \right\} \]
is $K$-summable. Let us denote by $(1 + I_K)^d^+$ the subgroup of $(1 + I_K)^d$ corresponding to $\operatorname{Hom}^+(G, 1 + I_K)$, equipped with the operation $\times$ induced by that on $\operatorname{Hom}^+(G, 1 + I_K)$ (same exact procedure as in Subsection 5.1). We therefore have

**Corollary 5.4.10.** We have $\operatorname{Hom}(G, 1 + I_K) \simeq ((1 + I_K)^d^+, \times)$. \hfill \Box

Combining Lemmas 5.4.3 and 5.4.6 and Corollary 5.4.9 we obtain the following refinement of Theorem 4.2.8.

**Theorem 5.4.11.** Let $k$ be a real closed field, $G = (\mathbb{Q}^d, <_{\text{lex}})$ and $K \subseteq k((G))$ a henselian Hahn field satisfying the first and canonical second lifting property.
Then
\[ v\text{-Aut}^+_K K \simeq (((1 + I_K)^d, \times) \rtimes (k^\times)^d) \rtimes \text{UPT}_d(k) \]
\[ v\text{-Aut}_K^+ K \simeq (((1 + I_K)^d, \times) \rtimes (k^\times)^d) \rtimes \text{UPT}_d(Q) \]

In the next section we analyse in further detail a special case for \( G = Q \), namely the field \( P \) of Puiseux series.

5.5. Puiseux series. Let \( k \) be a real closed field and let \( P \) be the field of Puiseux series in the indeterminate \( t \) over \( k \). These are power series with coefficients in the field \( k \) and exponents in \( Q \) with the restriction that all the exponents of a given power series have a common denominator. A general Puiseux series has the form:

\[ a = \sum_{n=m}^{\infty} a_n t^{n/m} \]  (5.11)

where \( m \in \mathbb{Z} \) and \( n_a \in \mathbb{Z}^{>0} \) is a positive integer depending on \( a \). The field \( P \) is a subfield of the Hahn field \( k((Q)) \) (Example 3.6.4). It therefore has value group \((Q, +, <)\). By Lemma 3.6.5 and Corollary 3.6.7 the field \( P \) has the canonical first and second lifting property. Moreover, like in the case of Laurent series, all the valuation preserving automorphisms of \( P \) are strongly additive:

Proposition 5.5.1. We have \( v\text{-Aut} P = v\text{-Aut}^+ P \).

Proof. Let \( \sigma \in v\text{-Aut}^+_k P \). We showed earlier (Remark 4.0.9) that external automorphisms are always strongly additive. So let us assume \( \sigma \) to be internal. By Theorem 3.7.2 we have \( \sigma \in v\text{-Aut}^+_k \). Since \( \sigma \) is internal, then for all \( a \in P \) we have \( v(\sigma(a)) = v(a) \). In particular, \( v(\sigma(t^{1/n})) > 0 \) for all \( n \in \mathbb{N} \) and by Neumann’s lemma [Pri83, Lemma 15, p. 57], the family \( \left\{ \sigma \left( a_n t^{1/n} \right)^n : n \in \mathbb{N} \right\} \) is summable, hence \( \sum \sigma(a_n t^{1/n}) \) is well defined. To prove our statement we show that the value \( v(\sum_{n=m}^{\infty} (\sigma(a_n t^{1/n}))^n - \sigma(a)) \) is greater than \( \frac{s}{n_a} \) for all \( s \in \mathbb{Z} \). Indeed:

\[
v \left( \sum_{n=m}^{\infty} \sigma \left( a_n t^{1/n} \right)^n - \sigma \left( \sum_{n=m}^{\infty} a_n t^{1/n} \right) \right) \geq v \left( \sum_{n>s} \sigma \left( a_n t^{1/n} \right) - \sigma \left( \sum_{n>s} a_n t^{1/n} \right) \right) > \frac{s}{n_a}.
\]

Thus \( v(\sum_{n=m}^{\infty} (\sigma(a_n t^{1/n}))^n - \sigma(a)) = \infty \) and therefore \( \sigma(a) = \sum_{n=m}^{\infty} \sigma(a_n t^{1/n}) \). □

The field \( P \) is henselian [Kuh00a, Lemma 10.1], so Theorem 5.4.11 applies. Combining this with Proposition 5.5.1 we get
Theorem 5.5.2. Let $k$ be a real closed field. Then
\[
v\text{-Aut}_{(k)} \mathbb{P} \cong \left( (1 + I \mathbb{P}, x_s) \rtimes k^\times \right) \rtimes (\text{Aut} k \times (\mathbb{Q}^{>0}, \cdot)) \tag{5.12}
\]
\[
v\text{-Aut}_k \mathbb{P} \cong \left( (1 + I \mathbb{P}, x_s) \rtimes k^\times \right) \rtimes (\mathbb{Q}^{>0}, \cdot) \tag{5.13}
\]
\hfill \Box

The case where $k$ is an algebraically closed field of characteristic 0 was treated by Deschamps [Des05, Théorème 10]. Under this assumption, he proves that $1\text{-Aut}_k \mathbb{P}$ and $\mathbb{Q}\text{-Exp} \mathbb{P}$ can be described, respectively, as $1\text{-Aut}_k \mathbb{P} \cong \lim\leftarrow (1 + I \mathbb{P})$ and $\mathbb{Q}\text{-Exp} k \cong \lim \kappa^\times$, where the limits are taken over the directed system given by the natural numbers with divisibility.

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