ON CERTAIN SPECTRAL FEATURES
INHERENT TO SCALAR TYPE SPECTRAL OPERATORS

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Abstract. Important spectral features, such as the emptiness of the residual spectrum, countability of the point spectrum, provided the space is separable, and a characterization of spectral gap at 0, known to hold for bounded scalar type spectral operators, are shown to naturally transfer to the unbounded case.

Curiosity is the lust of the mind.
Thomas Hobbes

1. Introduction

As is known [4, Theorem 8] (see also [5, 8]), a bounded linear operator \(T\) on a complex Banach space \((X, \| \cdot \|)\) is spectral iff it allows the unique canonical decomposition

\[ T = S + N, \]

where \(S\) is a scalar type spectral operator and \(N\) is a quasinilpotent operator commuting with \(S\). The operators \(T\) and \(S\) have the same spectrum and spectral measure \(E(\cdot)\), with

\[ S = \int_{\sigma(T)} \lambda dE(\lambda), \tag{1.1} \]

where \(\sigma(\cdot)\) is the spectrum of an operator, and are called the scalar and radical parts of \(T\), respectively.

The operator \(N\) being nilpotent, \(T\) is called of finite type (cf. [4, 10]), in which case, in particular, for a bounded scalar type spectral operator \((T = S, N = 0)\), the residual spectrum is empty [10, Theorem 4.1] and, provided the space \(X\) is separable, the point spectrum is countable [10, Theorem 4.4].

Furthermore, [10, Theorem 3.4] describing the closedness of the range of a bounded spectral operator \(T\) on a complex Banach space \(X\), when applied to a bounded scalar type spectral operator \((T = S, N = 0)\), turns into a characterization of spectral gap at 0 acquiring the following form:

Theorem 1.1 ([10, Theorem 3.4], the scalar type case).
For a bounded scalar type spectral operator \(A\) on a complex Banach space \((X, \| \cdot \|)\)

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with $0 \in \sigma(A)$, $0$ is an isolated point of the spectrum $\sigma(A)$ iff the range of $A$ is closed.

The case of an unbounded spectral operator $T$ in a complex Banach space $(X, \| \cdot \|)$ appears to be essentially more formidable. Thus, $E(\cdot)$ being the spectral measure of $T$, the scalar part $S$ of $T$ defined by (1.1) is an unbounded scalar type spectral operator and the radical part $N := T - S$ need not be bounded, let alone quasinilpotent [1, 8].

A natural question is whether the discussed spectral features pass to unbounded spectral operators at least when they are of scalar type, which would include the important class normal operators. In this note, we are to show that the emptiness of the residual spectrum, the countability of the point spectrum, provided the space is separable, as well as the characterization of spectral gap at 0 are inherent to scalar type spectral operators, bounded or not.

2. Preliminaries

Recall that, the spectrum $\sigma(A)$ of a closed linear operator $A$ in a complex Banach space $(X, \| \cdot \|)$ is partitioned into disjoint components, $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$, called the point, continuous, and residual spectrum of $A$, respectively, as follows:

$\sigma_p(A) = \{ \lambda \in \mathbb{C} \mid A - \lambda I$ is not one-to-one, i.e., $\lambda$ is an eigenvalue of $A\}$,

$\sigma_c(A) = \{ \lambda \in \mathbb{C} \mid A - \lambda I$ is one-to-one and $R(A - \lambda I) \neq X$, but $\overline{R(A - \lambda I)} = X \}$,

$\sigma_r(A) = \{ \lambda \in \mathbb{C} \mid A - \lambda I$ is one-to-one and $\overline{R(A - \lambda I)} \neq X \}$,

where $I$ stands for the identity operator on $X$, $R(\cdot)$ is the range of an operator, and $\overline{\cdot}$ is the closure of a set in $\mathbb{C}$ (see, e.g., [10]).

The properties of spectral operators, spectral measures, and the Borel operational calculus underlying the subsequent discourse are exhaustively delineated in [5, 8]. Here, for the reader’s convenience, we give an outline of some particularly important facts.

Recall that a spectral operator is a densely defined closed linear operator $A$ in a complex Banach space $(X, \| \cdot \|)$ with an associated spectral measure (resolution of the identity) $E_A(\cdot)$, i.e., a strongly $\sigma$-additive operator function, which assigns to each set $\delta$ from the $\sigma$-algebra $\mathcal{B}$ of Borel sets in $\mathbb{C}$ a projection operator $E_A(\delta) = E^A_A(\delta)$ on $X$ and has the following properties:

$E_A(\emptyset) = 0$, $E_A(\mathbb{C}) = I$, $E_A(\delta \cap \sigma) = E_A(\delta)E_A(\sigma) = E_A(\sigma)E_A(\delta)$, $\delta, \sigma \in \mathcal{B}$,

where 0 stands for the zero operator on $X$, and

$E_A(\delta)X \subseteq D(A)$, for each bounded $\delta \in \mathcal{B}$,

$E_A(\delta)D(A) \subseteq D(A)$, $AE_A(\delta)f = E_A(\delta)Af$, $\delta \in \mathcal{B}$, $f \in D(A)$,

$\sigma(A|E_A(\delta)X) \subseteq \delta$, $\delta \in \mathcal{B}$,

where $D(\cdot)$ is the domain of an operator and $\cdot|\cdot$ is the restriction of an operator (left) to a subspace (right).
Due to its strong countable additivity, the spectral measure $E_A(\cdot)$ is bounded $[6, 8]$, i.e.,

$$\exists M > 0 \forall \delta \in \mathcal{B} : \|E_A(\delta)\| \leq M.$$  

The notation $\| \cdot \|$ has been recycled here to designate the norm in the space $\mathcal{L}(X)$ of all bounded linear operators on $X$, such an economy of symbols being rather conventional.

A spectral operator $A$ in a complex Banach space $(X, \| \cdot \|)$ with spectral measure $E_A(\cdot)$ is said to be of scalar type if

$$A = \int \lambda dE_A(\lambda),$$

which is imbedded into the structure of the Borel operational calculus associated with such operators $[5, 8]$ and assigning to any Borel measurable function $F : \mathbb{C} \rightarrow \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int \mathbb{C} F(\lambda) dE_A(\lambda)$$

defined as follows:

$$F(A)f := \lim_{n \rightarrow \infty} F_n(A)f, \; f \in D(F(A)),$$

$$D(F(A)) := \left\{ f \in X \mid \lim_{n \rightarrow \infty} F_n(A)f \text{ exists} \right\},$$

where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda \in \mathbb{C} \mid |F(\lambda)| \leq n\}}(\cdot), \; n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the characteristic function of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \ldots \}$ is the set of natural numbers) and

$$F_n(A) := \int \mathbb{C} F_n(\lambda) dE_A(\lambda), \; n \in \mathbb{N},$$

are bounded scalar type spectral operators on $X$ defined in the same manner as for a normal operator (see, e.g., $[7, 18]$).

The spectrum $\sigma(A)$ of a scalar type spectral operator $A$ being the support of its spectral measure $E_A(\cdot)$, $\mathbb{C}$ can be replaced with $\sigma(A)$ in the above definitions whenever appropriate $[5, 8]$.

In a complex Hilbert space, the scalar type spectral operators are precisely those similar to the normal ones $[19]$.

3. Spectral Features Inherent to Scalar Type Spectral Operators

In $[14]$, the following generalization of the well-known orthogonal decomposition for a normal operator in a complex Hilbert space (see, e.g., $[7, 18]$) is found:

Theorem 3.1 ($[14$, Theorem$]$).

For a scalar type spectral operator $A$ in a complex Banach space $(X, \| \cdot \|)$ with spectral measure $E_A(\cdot)$, the direct sum decomposition

$$X = \ker A \oplus \overline{R(A)}$$  

(3.2)
(ker $\cdot$ is the kernel of an operator) holds with
$$\ker A = E_A(\{0\})X \quad \text{and} \quad \overline{R(A)} = E_A(\sigma(A) \setminus \{0\})X.$$ Decomposition (3.2) has the following immediate implication generalizing the well-known fact for normal operators (see, e.g., [7, 18]).

**Corollary 3.1** (Emptiness of Residual Spectrum).
*For a scalar type spectral operator $A$ in a complex Banach space $(X, \|\cdot\|), \sigma_r(A) = \emptyset.$*

**Proof.** Whenever, for $\lambda \in \mathbb{C}$, the scalar type spectral operator $A - \lambda I$ is one-to-one, $\ker(A - \lambda I) = \{0\}$, and hence, by (3.2), $R(A - \lambda I) = X$, which implies that $\sigma_r(A) = \emptyset$. \hfill $\square$

**Example 3.1.** In $l_2$, the unbounded linear operator
$$A(x_1, x_2, \ldots) = (0, x_1, 2x_2, \ldots, nx_n+1, \ldots)$$ with the domain $D(A) = \{ (x_1, x_2, \ldots) \in l_2 \mid (0, x_1, 2x_2, \ldots, nx_n+1, \ldots) \in l_2 \}$ is densely defined and closed, but, by Corollary 3.1, is *not spectral of scalar type* since $0 \in \sigma_r(A)$.

In respect that $\sigma_r(A) = \emptyset$, the proof of [10, Theorem 4.4] can be used verbatim to prove the following

**Proposition 3.1** (Countability of Point Spectrum).
*For a scalar type spectral operator $A$ in a complex separable Banach space $(X, \|\cdot\|), \sigma_p(A)$ is a countable set.*

**Example 3.2.** In the separable Banach space $C([a, b], \mathbb{C}) (-\infty < a < b < \infty)$ with the maximum norm, the differentiation operator
$$C^1[a,b] \ni x \mapsto [Ax](t) = x'(t), \quad a \leq t \leq b,$$ is densely defined, linear, and closed, but, by Proposition 3.1, *not spectral of scalar type* since $\sigma_p(A) = \mathbb{C}$.

Now, let us stretch Theorem 1.1 to the unbounded case.

**Theorem 3.2** (Characterization of Spectral Gap at 0).
*For a scalar type spectral operator $A$ in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $0 \in \sigma(A)$, $0$ is an isolated point of the spectrum $\sigma(A)$ iff the range $R(A)$ of $A$ is closed, i.e., $\overline{R(A)} = R(A).$*

**Proof.**

*“Only if” part.* Suppose that $0$ is an *isolated point* of $\sigma(A)$.

Considering that
$$\sigma(A) \setminus \{0\} = \sigma(A) \setminus \{\lambda \in \mathbb{C} \mid |\lambda| < \gamma\},$$ with some $\gamma > 0$, to the *bounded* Borel measurable function
$$F(\lambda) := \begin{cases} 0 & \text{for } \lambda \in \mathbb{C} \text{ with } |\lambda| < \gamma \\ \frac{1}{\lambda} & \text{for } \lambda \in \mathbb{C} \text{ with } |\lambda| \geq \gamma, \end{cases}$$
by the properties of the operational calculus ([8, Theorem XVIII.2.11]), there corresponds a bounded scalar type spectral operator

\[ F(A) = \int_{\mathbb{C}} F(\lambda) \, dE_A(\lambda) \]

and, for each \( f \in X \),

\[ E_A(\sigma(A) \setminus \{0\})f = E_A(\sigma(A) \setminus \{\lambda \in \mathbb{C} \mid |\lambda| < \gamma\})f \]

\[ = \int_{\{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\}} 1 \, dE_A(\lambda)f = \int_{\mathbb{C}} |\lambda| \, dE_A(\lambda)f = AF(A)f \in R(A). \]

Since, by Theorem 3.1, \( E_A(\sigma(A) \setminus \{0\}) \) is the projection onto \( \overline{R(A)} \) along \( \ker A \) [14], we infer that \( R(A) = \overline{R(A)} \).

"If" part. Suppose that \( \overline{R(A)} = R(A) \), which, considering \( \sigma_r(A) = \emptyset \), implies that \( 0 \in \sigma_p(A) \), i.e., \( \ker A \neq \{0\} \).

Then, by Theorem 3.1, the direct sum decomposition

\[ X = \ker A \oplus R(A), \]

where \( \ker A = E_A(\{0\})X \) and \( R(A) = E_A(\sigma(A) \setminus \{0\})X \), holds, and hence, \( A \) can be treated as the matrix operator

\[ \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix}, \]

in \( \ker A \oplus R(A) \), where \( A_1 : D(A) \cap R(A) \to R(A) \) is the restriction of \( A \) to \( R(A) \). Such a consideration makes apparent the fact that

\[ \sigma(A) = \{0\} \cup \sigma(A_1). \]

Since \( \ker A \cap R(A) = \{0\} \), the closed linear operator \( A_1 : D(A) \cap R(A) \to R(A) \) is bijective and has an inverse defined on \( R(A) \), which, in respect that \( (R(A), \| \cdot \|) \) is a Banach space, by the Closed Graph Theorem (see, e.g., [6]), is bounded.

Hence, 0 is a regular point of \( A_1 \). Considering the fact that the resolvent set of a closed operator is open in \( \mathbb{C} \) (see, e.g., [6]), we infer that, there is a neighborhood of 0 not containing points of \( \sigma(A_1) \), i.e., other points of \( \sigma(A) \), which makes 0 to be an isolated point of \( \sigma(A) \).

**Remark 3.1.** Observe that, the fact that \( \lambda_0 \) is an isolated point of the spectrum \( \sigma(A) \) of a scalar type spectral operator \( A \), necessarily implies that \( \lambda_0 \in \sigma_p(A) \).

Indeed, the spectrum being the support of the operator’s spectral measure \( E_A(\cdot) \), we immediately infer that

\[ E_A(\{\lambda_0\}) \neq 0, \]

which makes \( \lambda_0 \) to be an eigenvalue of \( A \) with the eigenspace \( E_A(\{\lambda_0\})X \) [5, 8]. The converse, however, is not true.

**Example 3.3.** In \( l_2 \), for the self-adjoint operator

\[ l_2 \ni (x_1, x_2, \ldots) \mapsto A(x_1, x_2, \ldots) = (0, x_2, x_3/2, x_4/3, \ldots) \in l_2, \]

the eigenvalue 0 is not an isolated point of \( \sigma(A) = \sigma_p(A) = \{0, 1, 1/2, 1/3, \ldots\} \).
Corollary 3.2. If, for a scalar type spectral operator $A$ in a complex Banach space $(X, \| \cdot \|)$, 0 is a regular point or an isolated point of the spectrum $\sigma(A)$, direct sum decomposition (3.3) holds and the operator $A + E_A(\{0\})$ has a bounded inverse defined on $X$, i.e., $0 \in \rho(A + E_A(\{0\}))$ ($\rho(\cdot)$ is the resolvent set of an operator).

Proof. The validity of decomposition (3.3) follows immediately from Theorem 3.2.

Since, by Theorem 3.1, the projection $E_A(\{0\})$ is onto ker $A$ along $R(A)$, the rest follows from a more general statement concerning the existence of a bounded inverse defined on $X$ of $A + P$ with a closed linear operator $A$, for which decomposition (3.3) holds, and $P$ is the projection onto ker $A$ along $R(A)$ ([3, 12, 13], cf. also [15–17]). Such operators are naturally called reducibly invertible. □

Thus, a scalar type spectral operator $A$, for which 0 is a regular point or an isolated point of spectrum, is reducibly invertible.

4. Final Remarks

As Examples 3.1 and 3.2 demonstrate, Corollary 3.1 and Proposition 3.1 are ready tests for disqualifying an operator from being scalar type spectral.

Theorem 3.2 relates a peculiar topological property of the spectrum of a scalar type spectral operator to a rather natural topological property of its range.

Observe also that decompositions (3.2) and (3.3) are essential in the context of the asymptotic behavior of weak/mild solutions of the associated abstract evolution equation

$$y'(t) = Ay(t), \ t \geq 0,$$

[2, 9, 11, 15–17].

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