Planar Graphs Without 4-Cycles Adjacent to Triangles are DP-4-Colorable

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Abstract

DP-coloring (also known as correspondence coloring) of a simple graph is a generalization of list coloring. It is known that planar graphs without 4-cycles adjacent to triangles are 4-choosable, and planar graphs without 4-cycles are DP-4-colorable. In this paper, we show that planar graphs without 4-cycles adjacent to triangles are DP-4-colorable, which implies the two results above.

Keywords Coloring · List-coloring · DP-coloring · Signed graph

1 Introduction

We use standard notation. For a set $S$, $\text{Pow}(S)$ denotes the power set of $S$, i.e., the set of all subsets of $S$. We denote by $[k]$ the set of integers from 1 to $k$. All graphs considered here are finite, undirected, and simple. For a graph $G$, $V(G)$, $E(G)$ and $F(G)$ denote the vertex set, edge set and face set of $G$, respectively. For a set $U \subseteq V(G)$, $G[U]$ is the subgraph of $G$ induced by $U$.

Recall that a proper $k$-coloring of a graph $G$ is a mapping $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ for any $uv \in E(G)$. The minimum integer $k$ such that $G$ admits a proper coloring is called the chromatic number of $G$, and denoted by $\chi(G)$.

List coloring was introduced independently by Vizing [15] and Erdős et al. [8] as a generalization of graph coloring. Let $C$ be a set of colors. A list assignment $L : V(G) \to \text{Pow}(C)$ of $G$ is a mapping that assigns a set of colors to each vertex. If $|L(v)| \geq k$ for all $v \in V(G)$, then $L$ is called a $k$-list assignment. A proper coloring
\[ f : V(G) \to C \] is called an \textit{L-coloring} of \( G \) if \( f(u) \in L(u) \) for any \( u \in V(G) \). The \textit{list-chromatic number} or the \textit{choice number} of \( G \), denoted by \( \chi_L(G) \), is the smallest \( k \) such that \( G \) admits an \( L \)-coloring for every \( k \)-list assignment \( L \) for \( G \).

Since a proper \( k \)-coloring corresponds to an \( L \)-coloring with \( L(u) = [k] \) for any \( u \in V(G) \), we have \( \chi(G) \leq \chi_L(G) \). It is well-known that there are infinitely many graphs \( G \) satisfying \( \chi(G) < \chi_L(G) \), and the gap can be arbitrarily large.

Dvořák and Postle [7] introduced DP-coloring as a generalization of list coloring to solve an interesting conjecture related to list coloring. They call it a \textit{correspondence coloring}, but we call it a \textit{DP-coloring} for short, following Bernshteyn et al. [4].

Let \( G \) be a graph and \( L \) be a list assignment of \( G \). For each edge \( uv \) in \( G \), let \( M_{L,uv} \) be an arbitrary matching (maybe empty) between \( \{u\} \times L(u) \) and \( \{v\} \times L(v) \). Without abuse of notation, we sometimes regard \( M_{L,uv} \) as a bipartite graph in which the edges are between \( \{u\} \times L(u) \) and \( \{v\} \times L(v) \), and the maximum degree is at most 1.

**Definition 1.1** Let \( M_L = \{M_{L,uv} : uv \in E(G)\} \), which is called a \textit{matching assignment over} \( L \). Then a graph \( H \) is said to be an \( M_L \)-cover of \( G \) if it satisfies all the following conditions:

(i) The vertex set of \( H \) is \( \bigcup_{u \in V(G)} (\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\} \).

(ii) For every \( u \in V(G) \), the graph \( H[\{u\} \times L(u)] \) is a clique.

(iii) For any edge \( uv \) in \( G \), \( \{u\} \times L(u) \) and \( \{v\} \times L(v) \) induce the graph \( M_{L,uv} \) obtained from \( H \).

**Definition 1.2** An \( M_L \)-\textit{coloring} of \( G \) is an independent set \( I \) in the \( M_L \)-cover with \( |I| = |V(G)| \). We say that a graph \( G \) is \textit{DP-k-colorable} if \( G \) admits an \( M_L \)-coloring for each \( k \)-list assignment \( L \) and each matching assignment \( M_L \) over \( L \). The \textit{DP-chromatic number}, denoted by \( \chi_{DP}(G) \), is the minimum integer \( k \) such that \( G \) is \( DP-k \)-colorable.

**Example 1.3** Figure 1 is an example of \( M_L \)-\textit{cover} of a graph \( G \). It is an well-known fact that \( \chi_L(C_4) = 2 \). In Fig. 1, \( H_1 \) and \( H_2 \) are two examples of \( M_L \)-covers of \( C_4 \) when \( |\{v\} \times L(v)| = 2 \) for every vertex \( v \) in \( C_4 \). Note that \( \alpha(H_1) = 4 = |V(C_4)| \), but \( \alpha(H_2) = 3 \). Thus \( C_4 \) is not DP-2-colorable. Actually, we can show that \( \chi_{DP}(C_4) = 3 \).

Note that when \( G \) is a simple graph and

\[ M_{L,uv} = \{(u, c)(v, c) : c \in L(u) \cap L(v)\} \]
for any edge $uv$ in $G$, then $G$ admits an $L$-coloring if and only if $G$ admits an $\mathcal{M}_L$-coloring. This implies $\chi_\ell(G) \leq \chi_{DP}(G)$. Thus DP-coloring is a generalization of the list coloring.

However, there are infinitely many simple graphs $G$ satisfying $\chi_\ell(G) < \chi_{DP}(G)$. As in Example 1.3, it is known that $\chi(C_n) = \chi_\ell(C_n) = 2 < 3 = \chi_{DP}(C_n)$ for each even integer $n \geq 4$ (see [4]). Furthermore, the gap $\chi_{DP}(G) - \chi_\ell(G)$ can be arbitrary large. For example, Bernshteyn [2] showed that for a simple graph $G$ with average degree $d$, we have $\chi_{DP}(G) = \Omega(d/\log d)$, while Alon [1] proved that $\chi_\ell(G) = \Omega(\log d)$ and the bound is sharp for $K_{d,d}$.

Since $\chi_\ell(G) \leq \chi_{DP}(G)$ for every graph $G$, given the fact $\chi_\ell(G) \leq k$, it is interesting to check whether $\chi_{DP}(G) \leq k$ or not. Dvořák and Postle [7] showed that $\chi_{DP}(G) \leq 5$ if $G$ is a planar graph, and $\chi_{DP}(G) \leq 3$ if $G$ is a planar graph with girth at least 5. Also, Dvořák and Postle [7] observed that $\chi_{DP}(G) \leq k + 1$ if $G$ is $k$-degenerate. Thomassen [14] showed that every planar graph is 5-choosable, and Voigt [16] showed that there are planar graphs which are not 4-choosable. Thus finding sufficient conditions for planar graphs to be 4-choosable is an interesting problem.

Recently, there are some other works on DP-colorings, see [3,5,10].

Two faces (cycles) are adjacent if they have at least one common edge, and two faces (cycles) are normally adjacent if they are adjacent and have exactly one common edge. Let $C_k$ be the cycle of length $k$. Lam et al. [12] verified that every planar graph without $C_4$ is 4-choosable. And Cheng et al. [6], and Kim and Ozeki [11] extended the result independently by certifying the following theorem.

Theorem 1.4 The following results hold.

(A) [6] If $G$ is a planar graph without 4-cycles adjacent to 3-cycles, then $\chi_\ell(G) \leq 4$.

(B) [11] If $G$ is a planar graph without 4-cycles, then $\chi_{DP}(G) \leq 4$.

In this paper, we extend Theorem 1.4 by proving the following theorem.

Theorem 1.5 If $G$ is a planar graph without 4-cycles adjacent to 3-cycles, then $\chi_{DP}(G) \leq 4$.

Even though $\chi_{DP}(G) = \chi_\ell(G)$ for some special graphs, Theorem 1.5 is not trivial from Theorem 1.4(A). We will give an explanation in Sect. 3.1.

2 Proof of Theorem 1.5

Suppose that Theorem 1.5 does not hold. In the rest of paper, let $G$ be a minimal counterexample to Theorem 1.5 with fewest vertices. From our hypothesis, graph $G$ has the following properties:

(a) $G$ is connected; and
(b) $G$ has no subgraph isomorphic to 4-cycles adjacent to 3-cycles; and
(c) $G$ is not DP-4-colorable; and
(d) every proper subgraph $G'$ of $G$ is DP-4-colorable.
Embedding $G$ into a plane, we obtain a plane graph $G = (V, E, F)$ where $V$, $E$, $F$ are the sets of vertices, edges and faces of $G$, respectively.

For a vertex $v \in V$, the degree of $v$ in $G$ is denoted by $d_G(v)$. A vertex of degree $d$ (at least $d$, at most $d$, respectively) is called a $d$-vertex ($d^+$-vertex, $d^-$-vertex, respectively). The notions of $d$-face, $d^+$-face, $d^-$-face are similarly defined. According to (b), if a 5-face is adjacent to a 3-face, then they are normally adjacent.

For a face $f \in F$, if the vertices on $f$ in a cyclic order are $v_1, v_2, \ldots, v_k$, then we write $f = [v_1 v_2 \ldots v_k v_1]$, and call $f$ a $(d_G(v_1), d_G(v_2), \ldots, d_G(v_k))$-face.

### 2.1 Structure of $G$

Using the properties of $G$ above, we can obtain several local structure of $G$.

**Lemma 2.1** Graph $G$ has no $3^-$-vertex.

**Proof** Suppose to the contrary that there exists a $3^-$-vertex $w$ in $G$. Let $L$ be a list assignment of $G$ with $|L(v)| \geq 4$ for any $v \in V$, and let $M_L$ be a matching assignment over $L$. Define that $G' := G - \{w\}$ and $L'(v) = L(v)$ for $v \in V(G')$. According to (d), $G'$ admits an $M_L$-coloring. Thus there is an independent set $I'$ in $M_L$-cover with $|I'| = |V(G')| - 1$. For $w$, we define

$$L^*(w) = L(w) \setminus \bigcup_{uw \in E(G)} \{c' \in L(w) : (u, c)(w, c') \in M_{L,uw} \text{ and } (u, c) \in I'\}.$$ 

Since $|L(w)| \geq 4$ and $w$ is a $3^-$-vertex, we have $|L^*(w)| \geq 1$. Assume that $c^*(w)$ is a color in $L^*(w)$. Obviously, $I = I' \cup \{(w \times c^*(w))\}$ is an independent set in the $M_L$-cover with $|I| = |V(G')| + 1 = |V(G)|$, which is a contradiction to (c). \qed

Next we define several different 5-faces. Let $f$ be a 5-face of $G$.

(i) If $f$ is a $(4, 4, 4, 4, 4)$-face, then we call $f$ a small 5-face. If $f$ is adjacent to a 3-face $f' = [v_1 v_2 v]$ with the common edge $v_1 v_2$, then we call $v$ a source of $f$. Equivalently, the face $f$ is called a sink of $v$ (see $F_1$ in Fig. 2).

(ii) If $f$ is a $(5^+, 4, 4, 4, 4)$-face, and $f$ is adjacent to four 3-faces and one $4^+$-face, and the $4^+$-face is incident to the $5^+$-vertex on $f$, then we call $f$ a bad 5-face (see $F_2$ in Fig. 2).
Lemma 2.3 Every source is a $5^+$-vertex.

Proof Let $f = \{v_1 v_2 v_3 v_4 v_5 v_1\}$ be a small 5-face, and let $z$ be a source of $f$. By Lemma 2.1, there exists no 3-face. Meanwhile, we call the $5^+$-vertex on $f$ a special vertex (see $F_3$ in Fig. 2).

Remark 2.2 (1) A special vertex is incident to at least one special 5-face, and a special 5-face is incident to exactly one special vertex.

(2) If there exist two $5^+$-faces on a 5-face $f$, then $f$ is neither special nor bad.

Lemma 2.5 Let $v$ be a $5^+$-vertex of $G$. Then the following hold:

(1) $v$ is incident to at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces; and

(2) if $v$ is incident to $t$ 3-faces and $2t < d_G(v)$, then $v$ is incident to at most $t - 1$ special 5-faces.

Proof Note that (1) holds obviously from (b).

And from the definition of a special 5-face, if $v$ is a $5^+$-vertex of $G$, then each of the special 5-faces incident to $v$ is adjacent to two 3-faces, and both of the two 3-faces are incident to $v$, which implies (2).
Lemma 2.6 Let $v$ be a $5^+$-vertex of $G$. Assume that $v$ is incident to a special or bad 5-face $f_1$ and a 3-face $f_2$ such that $f_1$ and $f_2$ are adjacent. Then $v$ has no sink adjacent to $f_2$.

Proof Let $f_1 = [v_1v_2v_3v_4v]$ be a special or a bad 5-face. According to the definition of special 5-face and bad 5-face, there are at least four 3-faces adjacent to $f_1$. Assume that $f_2 = [v_4v_5v_6v_7v_8v]$ and $f_2$ is adjacent to $f_1$ (see Fig. 3). Suppose to the contrary that $v$ has a sink $f_3 = [v_4v_5v_6v_7v_8v]$ adjacent to $f_2$ with a common edge $v_4v_5$. Then $v_3$ is a source of $f_3$. However, $v_3$ is a 4-vertex, contrary to Lemma 2.3.

Lemma 2.7 Let $f_1$ and $f_2$ be two bad 5-faces. Then they cannot normally adjacent with one common edge $vv_1$, where $v$ is the $5^+$-vertex on $f_1$ and $f_2$.

Proof Assume that two bad 5-faces, say $f_1$ and $f_2$, are normally adjacent, and they have one common $5^+$-vertex $v$ and one common edge $vv_1$ (see Fig. 4). Since both $f_1$ and $f_2$ are bad, then $v_1$ is a 4-vertex, and $f_3 = [v_1v_2v_3v_1]$ and $f_4 = [v_1v_2v_3v_1]$ are two 3-faces. Hence $f_3$ and $f_4$ are adjacent, which is contrary to (b). Thus the assumption is false.

2.2 Discharging

First, we define a configuration as shown in Fig. 5. In Fig. 5, the degrees of white vertices are at least the number of edges incident to them, but the degrees of the black vertices are equal to the number of edges incident to them. The red vertex $v$ is a special 5-vertex, which is incident to a special 5-face $f_1$, a bad 5-face $f_2$, and a 5-face $f_3$ satisfying the following:

- $f_3$ is neither special nor bad; and
- $f_3$ is incident to a special 5-vertex $v$, and $v$ is incident to a bad 5-face $f_2$; and
- $f_3$ is normally adjacent to $f_2$.

We call $F_5$ the family of $f_3$, that is, $F_5$ is a family of 5-faces that have the same properties of $f_3$. We call such a special 5-vertex $v$ a poor special 5-vertex.
Remark 2.8 Let \( w \) be a vertex on \( f_3 \) as shown in Fig. 5. If \( d_G(w) = 4 \), then \( w \) is incident to at most one 3-face according to (b).

It follows that

\[
\sum_{v \in V} (2d_G(v) - 6) + \sum_{f \in F} (d_G(f) - 6) = -12
\]

from Euler’s formula \(|V| - |E| + |F| = 2\) and the equality \( \sum_{v \in V} d_G(v) = 2|E| = \sum_{f \in F} d_G(f) \). Now we define an initial charge function \( ch(x) \) for each \( x \in V \cup F \) by letting \( ch(v) = 2d_G(v) - 6 \) for each \( v \in V \) and \( ch(f) = d_G(f) - 6 \) for each \( f \in F \).

We are going to design several discharging rules. Since the sum of total charge is fixed during the discharging procedure, if we can change the initial charge function \( ch(x) \) to the final charge function \( ch'(x) \) such that \( ch'(x) \geq 0 \) for each \( x \in V \cup F \), then

\[
0 \leq \sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} ch'(x) = -12,
\]

which is a contradiction. It means that no counterexample to Theorem 1.5 exists. Thus Theorem 1.5 holds. We denote by \( c(v \rightarrow f) \) the discharged value from \( v \) to \( f \), where \( v \in V(G) \) and \( f \in F(G) \). The discharging rules are designed as follows:

R1 If \( v \) is a 4\(^+\)-vertex and \( f \) is its incident 3-face, then \( c(v \rightarrow f) = 1 \).

R2 If \( v \) is a 4\(^+\)-vertex and \( f \) is its incident 4-face, then \( c(v \rightarrow f) = \frac{1}{2} \).

R3 Let \( v \) be a 4-vertex incident to at most one 3-face and let \( f \) be a 5-face incident to \( v \). Then \( c(v \rightarrow f) = \)

\[
\begin{cases} 
\frac{1}{2}, & \text{if } v \text{ is incident to exactly one 3-face and one 4-face;} \\
\frac{5}{2}, & \text{otherwise.}
\end{cases}
\]
R4 By Remark 2.2(1), a special 5-face is incident to exactly one special vertex. Let $v$ be a special $5^+$-vertex and $f$ be a 5-face incident to $v$. Then $c(v \rightarrow f) =$

\[
\begin{cases}
1, & \text{if } f \text{ is special;} \\
2, & \text{if } d_G(v) = 5 \text{ and } f \text{ is bad (see } f_2 \text{ in Fig. 5)}; \\
\frac{1}{3}, & \text{if } d_G(v) = 5, v \text{ is poor, and } f \in F_5 \text{ (see } f_3 \text{ in Fig. 5).}
\end{cases}
\]

R5 If $v$ is a non-special 5-vertex or $6^+$-vertex, and $f$ is a bad 5-face incident to $v$, then $c(v \rightarrow f) = \frac{3}{4}$.

R6 If $v$ is a source of $f$, then $c(v \rightarrow f) = \frac{1}{2}$.

R7 If $v$ is a $6^+$-vertex or a non-special 5-vertex, or a special 5-vertex which is not incident to a bad 5-face, and if $f$ is a 5-face which is neither special nor bad and $f$ is incident to $v$, then $c(v \rightarrow f) = \frac{1}{2}$.

To complete the proof of Theorem 1.5, it remains to check that the final charge of every element in $V \cup F$ is nonnegative. This will be shown by the following two claims.

**Claim 2.9** For all $v \in V(G)$, $ch'(v) \geq 0$.

**Proof** According to Lemma 2.1, there exists no $3^-$-vertex in $G$, thus we need to consider the 4$^+$-vertices in the following.

**Case 1:** $d_G(v) = 4$.

In this case, we have $ch(v) = 4 \times 2 - 6 = 2$. According to Lemma 2.3, $v$ has no any sink. If $v$ is incident to two 3-faces, then $ch'(v) = ch(v) - 2 \times 1 = 0$ by R1.

And, if $v$ is incident to exactly one 3-face, then $v$ is incident to at most one 4-face according to (b). Assume that $v$ is incident to one 4-face, then it is incident to at most two 5-faces. Thus $ch'(v) \geq ch(v) - 1 - \frac{1}{2} - \frac{1}{3} \times 2 = 0$ by R1, R2 and R3. Otherwise, $v$ is incident to at most three 5-faces. Therefore, $ch'(v) \geq ch(v) - 1 - \frac{1}{2} \times 3 > 0$ by R1 and R3.

If $v$ is not incident to any 3-face, then $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$ by R2 and R3.

**Case 2:** $d_G(v) = 5$.

In this case, we have $ch(v) = 5 \times 2 - 6 = 4$. Assume that $v$ is special, that is, $v$ is incident to a special 5-face by Remark 2.2(1). From Lemma 2.5 and the definition of special 5-face, it holds that $v$ is incident to exactly two 3-faces and one special 5-face. According to Lemma 2.6, $v$ has no sink. If $v$ is incident to a bad 5-face, then $v$ is incident to at most one bad 5-face by Lemma 2.7. Therefore, we have $ch'(v) \geq ch(v) - 1 \times 2 - 1 - \frac{2}{3} - \frac{1}{3} = 0$ from R1 and R4. Otherwise $v$ is not incident to any bad 5-face. It holds that $ch'(v) \geq ch(v) - 1 \times 2 - 1 - \frac{1}{2} \times 2 = 0$ by R1, R4 and R7.

We next assume that $v$ is not special, that is, $v$ is not incident to a special 5-face by Remark 2.2(1). The vertex $v$ is incident to at most two 3-faces by Lemma 2.5(1). According to Lemma 2.7, $v$ is incident to at most two bad 5-faces.
Case 1: If \( v \) is incident to two 3-faces, then \( v \) is incident to at most one bad 5-face from Lemma 2.7, the definition of bad 5-face and (b), and \( v \) is not incident to any 4-face by (b). Thus we have \( ch'(v) > ch(v) - 1 \times 2 - \frac{3}{4} - \frac{1}{2} \times 2 > 0 \) from R1, R4 (2b), R4 (2c), R6 and R7.

Case 2: If \( v \) is incident to exactly one 3-face, then \( v \) is incident to at most two bad 5-faces by the definition of bad 5-face. Thus the final charge \( ch'(v) > ch(v) - 1 - \frac{1}{5} - \frac{3}{4} \times 2 - \frac{1}{2} \times 2 > 0 \) from R1, R4 (2b), R4 (2c), R6 and R7.

Case 3: If \( v \) is not incident to any 3-face, then \( v \) is not incident to a bad 5-face. It holds that \( ch'(v) \geq ch(v) - \frac{1}{5} \times 5 > 0 \) by R2 and R7.

Case 3: \( d_G(v) = 6 \).

Note that we have \( ch(v) = 2 \times 6 - 6 = 6 \). According to Lemma 2.5, \( v \) is incident to at most three 3-faces and at most three special 5-faces.

Assume that \( v \) is incident to three 3-faces. If \( v \) is incident to three special 5-faces, then \( v \) has no any sink by Lemma 2.6. Thus we have \( ch'(v) \geq ch(v) - 3 - 3 = 0 \) from R1 and R4 (2a). Otherwise \( v \) is incident to at most two special 5-faces. Then \( v \) has no any sink by Lemma 2.6. Hence we have \( ch'(v) \geq ch(v) - 3 - 2 - \frac{3}{4} > 0 \) from R1, R4 (2a) and R5.

Assume that \( v \) is incident to at most two 3-faces, then \( v \) is incident to at most one special 5-face by Lemma 2.5 (2). If \( v \) is incident to one special 5-face, then \( ch'(v) > ch(v) - 2(1 + \frac{1}{5}) - 1 - \frac{3}{4} \times 3 > 0 \) from R1, R4 (2a), R5 and R6. Otherwise, by Lemma 2.7, \( v \) is incident to at most two bad 5-faces. Therefore, it holds that \( ch'(v) > ch(v) - 2(1 + \frac{1}{5}) - \frac{3}{4} \times 2 - \frac{1}{2} \times 2 > 0 \) from R1, R4 (2a), R5, R6 and R7.

Case 4: \( d_G(v) = 7 \).

In this case, we have \( ch(v) = 2 \times 7 - 6 = 8 \). By Lemma 2.5, \( v \) is incident to at most three 3-faces and at most two special 5-faces. The smallest final charge \( ch'(v) > ch(v) - 3(1 + \frac{1}{5}) - 2 - \frac{3}{4} \times 2 > 0 \) from R1, R4 (2a), R5 and R6.

Case 5: \( d_G(v) = k \geq 8 \).

Observe that \( ch(v) = 2k - 6 \). By Lemma 2.5, \( v \) is incident to at most \( \left\lfloor \frac{d_G(v)}{2} \right\rfloor \) 3-faces and at most \( \left\lfloor \frac{d_G(v)}{2} \right\rfloor \) special 5-faces. It holds that \( ch'(v) \geq ch(v) - (1 + \frac{1}{5}) \times \left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1 \times \left\lfloor \frac{d_G(v)}{2} \right\rfloor > 0 \) by R1, R4 (2a) and R6. \( \square \)

Claim 2.10 For all \( f \in F(G) \), \( ch'(f) \geq 0 \).

Proof Let \( f \) be a face of \( G \). Because \( G \) is simple, \( G \) has no loops and multi-edges. Thus \( d_G(f) \geq 3 \). If \( d_G(f) \geq 6 \), no charge is discharged from or to \( f \), thus \( ch'(f) = ch(f) = d_G(f) - 6 \geq 0 \). If \( d_G(f) = 3 \), then every vertex incident to \( f \) gives 1 to \( f \) according to R1. Therefore, we have \( ch'(f) = ch(f) + 3 \times 1 = d_G(f) - 6 + 3 = 0 \).

If \( d_G(f) = 4 \), then every vertex incident to \( f \) gives \( \frac{1}{2} \) to \( f \) according to R2. Hence the final charge \( ch'(f) = ch(f) + 4 \times \frac{1}{2} = d_G(f) - 6 + 2 = 0 \). Next we assume that \( d_G(f) = 5 \). Note that \( ch(f) = 5 - 6 = -1 \).

Case 1: \( f \) is small, that is, all the vertices incident to \( f \) are 4-vertices.
For $0 \leq t \leq 5$, let $t$ be the number of 4-vertices which are incident to two 3-faces on $f$. Then $f$ has $(5 - t)$ 4-vertices which are incident to at most one 3-face, and $f$ has at least $t + 1$ sources. Thus we have $ch'(f) \geq -1 + \frac{1}{4} \times (5 - t) + \frac{1}{3} \times (t + 1) = \frac{9 - t}{20} > 0$ for every $t \in \{0, 1, 2, 3, 4, 5\}$ by R3 and R6.

**Case 2:** $f$ is a $(5^+, 4, 4, 4)$-face.

Denote the $5^+$-vertex by $v$. If $f$ is special, then $ch'(f) \geq -1 + 1 = 0$ according to R4 (2a). Next we assume that $f$ is a non-special 5-face. It means that there exists a 4-vertex incident to at most one 3-face of $f$ by the definition of special 5-face.

- Assume that $v$ is a special 5-vertex and $v$ is incident to a bad 5-face. If $f$ is bad, then there is one 4-vertex (the 4-vertex in Fig. 5 which is adjacent to $v$) which is incident to exactly one 3-face and is not incident to any 4-face on $f$. Hence $ch'(f) \geq -1 + \frac{2}{3} + \frac{1}{3} = 0$ according to R3 and R4 (2b). Otherwise $f$ is not bad, then $f \in F_5$ and $v$ is a poor special 5-vertex. Note that $f$ is $f_3$ in Fig. 5. Therefore, $f$ is incident to at least two 4-vertices which are incident to at most one 3-face, and are not incident to a 3-face and a 4-face at the same time by Remark 2.8, respectively. Therefore, we have $ch'(f) \geq -1 + \frac{1}{3} \times 2 + \frac{1}{3} = 0$ according to R3, R4 (2c) and R7.

- Otherwise, $v$ is a $6^+$-vertex, or $v$ is not a special 5-vertex, or $v$ is a special 5-vertex and $v$ is not incident to a bad 5-face. If $f$ is bad, then there are at least two 4-vertices incident to $f$, and each of them is incident to at most one 3-face. We can conclude that $ch'(f) \geq -1 + \frac{1}{4} \times 2 + \frac{1}{2} = 0$ according to R3 and R7.

**Case 3:** there exist at least two $5^+$-vertices on $f$.

From Remark 2.2(2), we have $f$ is neither special nor bad.

- If $f \in F_5$, then $ch'(f) \geq -1 + \frac{1}{3} \times 2 + \frac{1}{3} = 0$ by Remark 2.8, R3, R4 (2c) and R7.

- Otherwise $f \notin F_5$, that is, $f$ is not incident to a special 5-vertex which is on a special 5-face and is incident to a bad 5-face. We can conclude that $ch'(f) \geq -1 + \frac{1}{4} \times 2 = 0$ according to R7.

The proof of Theorem 1.5 is completed.

### 3 Remarks

#### 3.1 Difference Between DP-Coloring and List Coloring

A $\theta$-graph is a graph consisting of two 3-vertices and three pairwise internally disjoint paths between the two 3-vertices. A $\theta$-subgraph of $G$ is an induced subgraph that is isomorphic to a $\theta$-graph. We use $S\theta$ to denote such a special $\theta$-subgraph of $G$ in which one of the ends of the internal chord is a $5^-$-vertex and all of the other vertices are 4-vertices in $G$. 

\[ \Box \]
Let $G$ be a planar graph without 4-cycles adjacent to 3-cycles. In the proof of Theorem 1.4(A), the authors showed that if $G$ is not 4-choosable with fewest vertices, then $G$ contains no subgraph isomorphic to $S\theta$ (see Lemma 4 in [6]). But we cannot claim that if $G$ is not DP-4-colorable with fewest vertices, then $G$ contains no subgraph isomorphic to $S\theta$. Next we give an explanation.

In Fig. 6a, assume that $d_G(v_1) = 5, d_G(v_i) = 4$ for $i \in \{2, 3, 4, 5\}$ and $d_G(z) = 4$. Let $G[S]$ be the subgraph of $G$ induced by $S = \{z, v_1, v_2, v_3, v_4, v_5\}$. Let $L$ be a list assignment of $G$ with $|L(v)| \geq 4$ for all $v \in V$, and let $M_L$ be a matching assignment over $L$. Set $G' := G - S$ and $L'(v) = L(v)$ for $v \in V(G')$. By (d), $G'$ admits an $M_{L'}$-coloring. Thus there is an independent set $I'$ in the $M_{L'}$-cover with $|I'| = |V(G)| - |S| = |V(G)| - 6$. For $v \in \{v_1, v_2, v_3, v_4, v_5, z\}$, we define that

$$L^*(v) = L(v) \setminus \bigcup_{u,v \in E(G)} \{ c' \in L(v) : (u, c)(v, c') \in M_{L,uv} \text{ and } (u, c) \in I' \}.$$ 

Because $|L(v)| \geq 4$ for all $v \in V(G)$, we have $|L^*(v_5)| \geq 3$, and $|L^*(v)| \geq 2$ for $v \in \{z, v_1, v_2, v_3, v_4\}$. We denote by $M_{L^*}$ the restriction of $M_L$ into $G[S]$ and $L^*$.

Next we give an $M_{L^*}$-cover as shown in (b) of Fig. 6. But we cannot find an independent set $I^*$ with $|I^*| = 6$ in this $M_{L^*}$-cover. Thus we cannot claim that if $G$ is not DP-4-colorable with fewest vertices, then $G$ contains no subgraph isomorphic to $S\theta$. Therefore, Theorem 1.5 is not trivial from Theorem 1.4(A).

### 3.2 Relationship with Signed Coloring

A concept of signed coloring was first defined by Zaslavsky [17] in slightly different form, and then modified by Mácajová et al. [13] so that it would be a natural extension of an ordinary vertex coloring. For detailed story about signed coloring, we refer readers to [9,11,13].
An observation in [11] is that the signed coloring of a signed graph \((G, \sigma)\) is a special case of a DP-coloring of \(G\). Thus Theorem 1.5 implies the following corollary, which is an extension of the result in [9].

**Corollary 3.1** Every planar graph with no 4-cycles adjacent to 3-cycles is signed 4-choosable.

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**References**

1. Alon, N.: Degrees and choice numbers. Random Struct. Algorithms **16**, 364–368 (2000)
2. Bernshteyn, A.: The asymptotic behavior of the correspondence chromatic number. Discret. Math. **339**, 2680–2692 (2016)
3. Bernshteyn, A., Kostochka, A.: On differences between DP-coloring and list coloring. arXiv:1705.04883v2
4. Bernshteyn, A., Kostochka, A., Pron, S.: On DP-coloring of graphs and multigraphs. Sib. Math. J. **58**, 28–36 (2017)
5. Bernshteyn, A., Kostochka, A., Zhu, X.: DP-colorings of graphs with high chromatic number. Eur. J. Comb. **65**, 122–129 (2017)
6. Cheng, P., Chen, M., Wang, Y.: Planar graphs without 4-cycles adjacent to triangles are 4-choosable. Discret. Math. **339**(12), 3052–3057 (2016)
7. Dvořák, Z., Postle, L.: Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. J. Comb. Theory Ser. B **129**, 38–54 (2018)
8. Erdős, P., Rubin, A.L., Taylor, H.: Choosability in graphs. In: Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State University, Arcata, California, 1979), pp. 125–157, Congress. Numer., XXVI, Utilitas Mathematica Publishing Inc. Winnipeg, Manitoba (1980)
9. Jin, L., Kang, Y., Steffen, E.: Choosability in signed planar graphs. Eur. J. Comb. **52**, 234–243 (2016)
10. Kim, S.-J., Ozeki, K.: A note on a Brooks’ type theorem for DP-coloring. J. Graph Theory (2018). https://doi.org/10.1002/jgt.22425
11. Kim, S.-J., Ozeki, K.: A Sufficient condition for DP-4-colorability. Discret. Math. **341**(7), 1983–1986 (2018)
12. Lam, P.C.-B., Xu, B., Liu, J.: The 4-choosability of plane graphs without 4-cycles. J. Comb. Theory Ser. B **76**(1), 117–126 (1999)
13. Mácajová, E., Raspaud, A., Škoviera, M.: The chromatic number of a signed graph. Electron. J. Comb. **23**, #P1.14 (2016)
14. Thomassen, C.: Every planar graph is 5-choosable. J. Comb. Theory Ser. B **62**(1), 180–181 (1994)
15. Vizing, V.G.: Vertex colorings with given colors (in Russian). Diskret. Analiz. **29**, 3–10 (1976)
16. Voigt, M.: A not 3-choosable planar graph without 3-cycles. Discret. Math. **146**, 325–328 (1995)
17. Zaslavsky, T.: Signed graph coloring. Discret. Math. **39**, 215–228 (1982)

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