Anticipatory Psychological Models for Quickest Change Detection: Human Sensor Interaction

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Abstract—We consider anticipatory psychological models for human decision makers and their effect on sequential decision making. From a decision theoretic point of view, such models are time inconsistent meaning that Bellman’s principle of optimality does not hold. The aim of this paper is to study how such an anxiety-based anticipatory utility can affect sequential decision making, such as quickest change detection, in multi-agent systems. We show that the interaction between anticipation-driven agents and sequential decision maker results in unusual (nonconvex) structure of the optimal decision policy. The methodology yields a useful mathematical framework for sensor interaction involving a human decision maker (with behavioral economics constraints) and a sensor equipped with automated sequential detector.

Keywords Time inconsistency, anticipatory decision making, subgame Nash equilibrium, quickest change detection, change blindness, Blackwell dominance

I. INTRODUCTION

This paper considers multi-agent sequential decision problems when each agent is an anticipatory decision maker (in a behavioral economics sense). The main question addressed is: If multiple anticipatory decision makers (each equipped with a sensing/computing device) interact sequentially, how can the resulting system solve a partially observed sequential detection problem? Extensive studies in behavioral economics, experimental psychology and neuroscience [1], [2], [3], [4] show that humans are anticipation-driven agents, and that even simple decisions involve sophisticated multi-stage planning. Human decision makers often postpone the realization of rewards, have a preference for uncertainty to be resolved earlier and tend to be over-optimistic about future results. In neural imaging experiments [5], certain reward circuit is over-activated in the presence of anticipatory feelings resulting in different types of planning fallacies, and provides evidence of anticipatory utility theory. Also [3] shows that mesolimbic reward circuitry in the human brain selectively treats the opportunity to gain knowledge about future favorable outcomes, but not unfavorable outcomes, as if it has positive utility; this explains why knowledge is not always preferred.

A. Anticipation and Time Inconsistent Decision Problems

In behavioral economics, Caplin & Leahy [6] propose a remarkable model for anticipatory human decision making via a horizon-2 dynamic decision process: the first stage involves choosing an action to minimize the sum of anticipatory reward and expected reward, while at the second stage the agent realizes its actual reward. In decision models with anticipatory reward, the dependence of current reward on future plan results in a deviation between planning and execution, and implies that Bellman’s principle of optimality no longer holds – this phenomenon is called time-inconsistency in economics1 [7].

Time inconsistency results in the so called planning fallacy dating back to Kahneman & Tversky [8]: people tend to underestimate the time required to complete a future task. Compared to rational agents, optimistic agents take higher risk of making the wrong decision but have higher anticipatory

Glossary of Symbols

**Anticipatory agent.** Sec.II and III
- $s_1, s_2$: physical state
- $z_1, z_2$: psychological state (5), (12)
- $a_1, a_2$: actions (4)
- $\mu_1^*, \mu_2^*$: Nash equilibrium policy (10), (8)
- $V_1(\cdot), V_2(\cdot)$: value function

**Quickest detection.** Sec.IV
- $n$: discrete time $n$ (also agent $n$)
- $x_n$: jump state (for quickest detection)
- $P$: transition matrix of $\{x_n, n \geq 0\}$ (24)
- $f, d$: false alarm and delay penalty parameters

**Anticipatory agents acting sequentially.** Sec.IV
- $s_n$: physical state
- $z_n$: psychological state
- $a_{n1}, a_{n2}$: local decision maker’s actions
- $\eta_n$: private belief of local decision maker $n$ (26)
- $\mu_{n1}^*, \mu_{n2}^*$: Nash equilibrium policy (10), (8)
- $y_n$: private observation of $x_n$ at time $n$
- $B_{x_n, y_n}$: observation likelihood $p(y_n|x_n)$ (27)
- $T(\pi, y)$: private belief update (28)
- $\sigma(\pi, y)$: normalization measure for private belief

**Global Decision maker.** Sec.IV
- $u_n$: action at time $n \in \{1(\text{stop}), 2(\text{cont})\}$
- $\phi^*(\pi, s)$: optimal policy for quickest detection
- $\pi_n$: public belief at $n$ (26)
- $R_{x,a}(s)$: action likelihood $p(a|x, \pi, s)$ (30), (31)
- $\bar{T}(\pi, a, s)$: public belief update (29)
- $\bar{\sigma}(\pi, a, s)$: normalization measure for public belief
- $\bar{V}(\pi, s)$: value function
- $C(\pi, u)$: costs incurred in quickest detection

1In game-theoretic terms, time-inconsistency arises when the optimal policy to the current multi-stage decision problem is sub-game imperfect.
reward at the same time. From this point of view, Brunnermeier and Parker [9] in 2005 show that it is optimal for agents with anticipatory reward to take irrational beliefs (referred to as subjective beliefs) deliberately. This provides a general explanation to the optimistic planning fallacy, in which people tend to overestimate future reward and make myopic decisions. In their recent paper [10], time inconsistency of beliefs is introduced to model the evolution of subjective beliefs over time in multi-stage tasks.

B. Main Results

The central theme of this paper involves partially observed sequential detection problems with multiple anticipatory agents that interact with each other. In such problems, agents have incomplete (noisy) information of the underlying state of nature. Specifically, we consider a multi-agent quickest change change detection problem where individual agents are anxiety-driven anticipatory reward decision makers with behavioral models specified by [6]. These agents act sequentially and are affected by the decisions of previous agents. A global decision maker monitors the decisions of these anticipatory agents. How can the global decision maker use the local decisions from these anticipatory agents to decide when a change has occurred in the underlying state of nature? The goal of the global decision maker is to achieve quickest change detection, namely, minimize the Kolmogorov-Shiryaev criterion involving the false alarm and decision delay penalty. Figure 1 shows the schematic setup.

![Fig. 1: Schematic of Quickest Change Detection Problem involving multiple anticipatory local decision makers and a global decision maker.](image)

(i) Anticipatory local decision maker \(n = 1, 2 \ldots\) observes state of nature \(x_n\) in noise as \(y_n\) and receives public belief \(\pi_{n-1}\) from the global decision maker. Given \(\pi_{n-1}\) and physical state \(s_n\), it makes anticipatory decision \(a_n\).

(ii) The global decision maker uses \(a_n\) to update the public belief \(\pi_n\) and makes decision \(u_n = \{1\text{(stop and declare change)}, 2\text{(continue)}\}\). If \(u_n = 1\), then the procedure continues with agent \(n + 1\), otherwise the process stops and a change is declared.

The anticipatory model for individual decision makers is discussed in Sec.II and Sec.III. The interaction of anticipatory agents with a global decision maker to achieve quickest detection of \(x_n\) is discussed in Sec.IV.

Examples: An example of our framework (involving local decision makers interacting with a global decision maker) arises in the social media based accommodation such as Airbnb. Individual anticipatory agents make local decisions sequentially whether to rent a property; these decisions are affected by the reviews (decisions) of previous agents. The global decision maker (e.g. Airbnb) monitors these local decisions and wishes to detect if there is a sudden change in the quality of the accommodation.

Another example arises in the measurement of the adoption of a new product using a micro-blogging platform like Twitter. The adoption of the technology diffuses through the market but its effects can only be observed through the tweets of select individuals of the population. These selected individuals act as sensors for estimating the diffusion. They interact and learn from the decisions (tweeted sentiments) of other members. Suppose the state of nature suddenly changes due to a sudden market shock or presence of a new competitor. The goal for a market analyst or product manufacturer is to detect this change as quickly as possible by minimizing a cost function that involves the sum of the false alarm and decision delay.

Organization and Main Results: The main result and organization of the paper can be summarized as follows: Sec.II reviews time inconsistent sequential decision problems and the framework for anticipatory decision making in [6] as a 2-stage stochastic optimization problem. Then Sec.III introduces sufficient conditions on the anticipatory decision model so that the subgame Nash equilibrium at the first stage is a bang-bang policy, and at the second stage is a threshold policy. This is illustrated by an example involving social media accommodations (such as Airbnb). Sec.IV formulates the quickest change detection protocol involving multiple anticipatory agents where a global decision maker uses the decisions of the anticipatory decision makers to decide if a state of nature has changed. The optimal policy that minimizes the Kolmogorov-Shiryaev criterion is formulated as the solution of a stochastic dynamic programming problem in the belief space. Then Sec.V describes various structural properties of the Bayesian updates of the local and global decision makers, constructs a lower bound for the optimal cost incurred using Blackwell dominance and presents numerical examples of the unusual structure of the optimal quickest change policy (the value function is non-concave and the stopping region is non-convex).

It is well known that in classical quickest detection, the optimal policy has a monotone threshold structure: when the posterior probability of change exceeds a threshold, it is optimal to declare a change. This is illustrated in Fig.2a. So the optimal stopping set (set of posteriors probabilities where it is optimal to declare change) is convex. In quickest detection with anticipatory agents, the remarkable feature is that the stopping set is disconnected, see Fig.2b. One sees the counter-intuitive property: the optimal detection policy switches from announce change to announce no change as the posterior probability of a change increases! Thus making a global decision as to whether a change has occurred based on local decisions of interacting agents is non-trivial.
Fig. 2: Optimal Quickest Change Detection Policy $\phi^*$ as a function of Bayesian belief (posterior) $\pi$. In classical quickest detection, the stopping set is convex (connected). In comparison, for quickest detection with anticipatory agents, the stopping set is nonconvex (disconnected) as indicated in red.

be interpreted as a form of change blindness, namely people fail to detect surprisingly large changes to scenes [11]. Even though the posterior probability of a change is higher than a fail to detect surprisingly large changes to scenes [11].

C. Perspective. Multi-stage social learning

Humans very likely do not solve time inconsistent decision processes to make decisions - even graduate students and professors struggle with such formalisms! However time inconsistent behavioral economics models such as [6], [9], [10] are widely used because they provide generative models for the peculiarities of human decision making. This paper is an early step in understanding the interaction of signal processing with behavioral economics. Signal processing and behavioral economics are mature areas; yet their intersection, namely the interaction of human decision makers with sensing based estimation is relatively unexplored.

The anticipatory models used in this paper are from [6], [9], [10]; these behavioral economics models generalize classical social learning models that have been studied extensively in sociology, economics and more recently signal processing [12], [13], [14]. Classical social learning assumes that agents make one-shot (myopic) decisions to maximize their expected utility. The behavioral economics models considered here are useful generalizations of social learning since they involve multi-stage planning; as mentioned earlier, even simple human decisions involve multi-stage planning and can lead to time-inconsistency.

II. DECISION MAKING WITH ANTICIPATORY FEELINGS

This section defines time inconsistent decision problems and reviews the widely cited behavioral economics model for human decision making with anticipatory feelings from [6]. This model will be used in Sec.IV to formulate our human sensor interactive quickest change detection problem.

A. Time Inconsistent Sequential Decision Problems

We start with a brief discussion of time inconsistent decision problems; see [7] for an exposition. Let $\{s_k, k = 1, \ldots, N\}$ denote a controlled Markov chain evolving on a finite time horizon size $N$. The initial distribution for $s_1$ is denoted as $\pi_1$. Let $\mu_k : s_k \rightarrow a_k$ denote a (possibly randomized) decision policy that maps the state $s_k$ to an action $a_k$ at time $k$. For $n = 1, 2, \ldots, N$, define the expected utility-to-go

$$J_n(s_n, \mu_{n:N}) = \mathbb{E}_\mu \left\{ \sum_{k=n}^N r_{n,k}(s_n, s_k, \mu_k(s_k)) \right\}$$

The aim is to compute the policy sequence $\arg\max_{\mu} J_n(s_n, \mu_{n:N})$. As the reward $r_{n,k}$ depends on $n$ and $k$, and also $s_n, s_k$, this optimization problem is time inconsistent since the principle of optimality (and therefore Bellman’s dynamic programming equation) does not hold.

1) Subgame Perfect Nash Equilibria: As discussed in [7], an appropriate method of “solving” a time inconsistent problem is in game-theoretic terms. 2

1) Given state $s_N = s$, player $N$ chooses policy

$$\mu_N^*(s) = \arg\max_{\mu_N} J_N(s, \mu_{1:N-1}, a_N)$$

This yields the value function

$$V_N = J_N(s, \mu_{1:N-1}, \mu_N^*)$$

2) Given $s_{N-1} = s$, and that player $N$ is using policy $\mu_N^*$, player $N-1$ chooses policy

$$\mu_{N-1}^*(s) = \arg\max_{\mu_{N-1}} J_{N-1}(s, \mu_{1:N-2}, a_{N-1}, \mu_N^*)$$

$$V_{N-1}(s) = J_{N-1}(s, \mu_{1:N-2}, \mu_{N-1}^*)$$

3) Proceed by backward induction to compute policy $\mu_1^*$. The above procedure is called the extended Bellman equation in [7]. The sequence of policies $\mu^* = (\mu_1^*, \ldots, \mu_N^*)$ constitutes a subgame perfect Nash equilibrium; see [7] for details.

2) Remarks: (i) As might be expected, for the time consistent case where $r_{n,k}(s_n, s_k, a_k) = r_k(s_k, a_k)$ in (1), the extended Bellman’s equation becomes the standard Bellman’s dynamic programming equation.

(ii) For the time inconsistent case, neither the Nash equilibrium $\mu^*$ nor its value $J_n(\mu^*)$ are unique. This is in contrast to time consistent dynamic programming where the optimal policy may not be unique but the optimal value is always unique.

B. Anticipatory Model of Caplin & Leahy [6]

We now review the time inconsistent model for anticipatory human decision making in Caplin & Leahy’s seminal paper [6]. Their model uses the terminology of temporal lotteries in dynamic choice theory [15]. We translate their model to a more familiar Markov decision process framework. The final outcome is a time inconsistent problem of the form (1).
1) Anticipatory Model and Time Inconsistency: The anticipatory decision model in \cite{6} comprises two time steps indexed by \( k = 1, 2 \). The physical state \( s_k \in S, k = 1, 2 \), where \( S \) denotes the state space, evolves with Markov transition kernel \( p(s_2|s_1) \). Let \( a_1 \in \mathcal{A}_1 \) and \( a_2 \in \mathcal{A}_2 \) denote the actions taken by the agent (human) at time 1 and 2. These actions are determined by the non-randomized policies \( \mu_1 \) and \( \mu_2 \) where
\[
a_1 = \mu_1(s_1), \quad a_2 = \mu_2(s_2, a_1).
\] (4)

The first key idea in Caplin & Leahy \cite{6} is to define the psychological state \( z_k, k = 1, 2 \), which models the human decision maker’s state of mind (anxiety):
\[
z_1 = \phi(s_1, a_1, \{p(a_2 = a|s_1, a_1, \mu_2), a \in \mathcal{A}_2\}),
\]
\[
z_2 = (s_2, a_2, a_1),
\] (5)
for some pre-defined function \( \phi \). Note that the psychological state \( z_1 \) depends on the set of conditional probabilities \( \{p(a_2 = a|s_1, a_1, \mu_2), a \in \mathcal{A}_2\} \). These conditional probabilities model anxiety (anticipation)\(^3\) of the decision maker at time 1. The anxiety disappears at time 2 when physical state \( s_2 \) is observed and all uncertainty is resolved; hence the psychological state \( z_2 \) only contains physical state \( s_2 \) and realized action \( a_2 \).

The next key idea in \cite{6} is that the anticipatory agent makes decisions by maximizing the 2-stage psychological utility
\[
\sup_{\mu_1, \mu_2} J(\mu_1, \mu_2) = E_{\mu_1, \mu_2}\{r_1(z_1) + r_2(z_2)\}
\] (6)
Here \( r_k : z_k \rightarrow \mathbb{R} \) for \( k = 1, 2 \) denote the reward functions. The 2-stage psychological utility (6) looks just like a standard time separable utility except for the presence of the anxiety term \( \{p(a_2 = a|s_1, a_1, \mu_2), a \in \mathcal{A}_2\} \) in \( r_1(z_1) \). This \( \mu_2 \) dependency gives rise to time inconsistency in decision making. Indeed (6) is a special case of the general time inconsistent formulation (1) with
\[
r_{2,2} = r_2(s_2, a_2, a_1),
\]
\[
r_{1,1} + r_{1,2} = r_1(\phi(s_1, a_1, \{p(a_2 = a|s_1, a_1, \mu_2), a \in \mathcal{A}_1\}))
\] (7)

2) Subgame Perfect Nash Equilibrium: Caplin & Leahy \cite{6} ‘solve’ the time inconsistent decision problem (6) using the extended Bellman equation described in Sec.II-A. Indeed, the optimal policy at time 2 simply follows from (2) with \( N = 2 \):
\[
\mu_2^*)(s_2, a_1) = \arg\max_{a_2} r_2(s_2, a_2)
\] (8)
Note that by definition (4), \( \mu_2^* \) depends on \( a_1 \) and \( s_2 \).

To specify the optimal policy at time 1, we first introduce the following compact notation. Define the measure
\[
\lambda_a = \int_S I(s_2 : \mu_2^*(s_2, a_1) = a) p(s_2|s_1) ds_2,
\]
\[
\lambda = \{\lambda_a, a \in \mathcal{A}_2\}
\] (9)

At time 1, due to time inconsistency, the agent chooses a time consistent policy \( \mu_1^* \) based on extended Bellman equation (3):
\[
\mu_1^*(s_1) = \arg\max_{a_1} J_1(s_1, a_1, \mu_2^*),
\]
\[
V_1(s, a_1) = \max_{a_1} J_1(s, a_1, \mu_2^*),
\]
\[
J_1(s, a_1, \mu_2^*) = \int_S r_2(s_2, \mu_2^*(s_2, a_1)) p(s_2|s_1) ds_2
\] (10)
Recall \( p(s_2|s_1) \) is the transition kernel of the physical state.

Remarks:
(i) We can express (10) in more compact notation as
\[
\mu_1^*(s_1) = \arg\max_{a_1} r_1(\phi(s_1, a_1, \lambda)) + \mathbb{E}_\lambda\{r_2(s_2, a_2, a_1)\}
\] (11)
which is the same as the master equation \cite[Eq.2]{6} since
\[
\mathbb{E}_\lambda\{r_2(s_2, a_2, a_1)\} = \int_{A_2} \int_S r_2(s_2, a, a_1) \lambda_a ds_2 da
\]
\[
= \int_{A_2} \int_S r_2(s_2, a, a_1) I((a = \mu_2^*(s_2, a_1))) p(s_2|s_1) ds_2 da
\]
(ii) The psychological state \( z_1 \) in (5) consisted of the set of conditional probabilities \( \{p(a_2 = a|s_1, a_1, \mu_2), a \in \mathcal{A}_2\} \). Instead, one can formulate a more general psychological state with the conditional probabilities replaced by
\[
\{\mathbb{E}(\Psi(a_2 = a, s_2)|s_1, a_1, \mu_2), a \in \mathcal{A}_2\}
\] (12)
for some pre-defined function \( \Psi \) in psychological state \( z_1 \). As an example (which we will elaborate on below)
\[
z_1 = \max\{p(a_2 = 1|s_1, a_1, \mu_2), E\{s_2 I(a_2 = 2)|s_1, a_1, \mu_2\}\}
\]
(iii) We mentioned previously that the subgame Nash equilibrium approach to time inconsistency disregards the fact that \( \mu_2^* \) is no longer optimal at time 1. Another insightful way of viewing this is that the estimated anticipatory reward \( r_1(\phi(s_1, a_1, \lambda)) \) requires the agent to extrapolate what might happen at the second stage, plans are not optimal once an action is taken. As an example, people tend to assign higher future workload than what they will actually take on.

Summary. Maximizing the 2-stage psychological utility (6) is a time inconsistent problem. The anticipatory human makes decisions \( a_1, a_2 \) according to policies \( \mu_1^* \) in (10) and \( \mu_2^* \) in (8); these policies constitute a subgame perfect Nash equilibrium. Indeed (10) or equivalently (11) corresponds to the key master equation (2) in \cite{6}. The paper \cite{6} has received significant attention in behavioral economics (mindful economics \cite{1}), neuroscience and psychology \cite{3, 4}.

III. Characterizing the Nash Equilibrium Policy and Examples

The previous section gave a general setup of the anticipatory decision making model. In this section, to give a more concrete characterization, we make specific assumptions on this model. These assumptions result in a bang-bang structure for the subgame Nash equilibrium policy (Theorem 1 below). This structural result is then illustrated by an example motivated by social media based decision making.
Bayesian parametrization of reward

Recall $r_2$ is the reward at time 2; see (5), (6). In the rest of the paper, we will parametrize $r_2$ by a Bayesian parameter. This parameterized reward is constructed as follows: Define the reward $r_2(s_2, a_2, a_1, x)$ which now also depends on a state of nature $x$. The process $x \in \mathcal{X} = \{1, 2, \ldots, m\}$ will be formally defined and used in Sec.IV to model change in quickest detection. Then define the parametrized reward $r_{\eta, 2}$ as

$$r_{\eta, 2}(s_2, a_2, a_1) = \sum_{x \in \mathcal{X}} r_2(s_2, a_2, a_1, x) \eta(x) \tag{13}$$

Here $\eta$ is an $m$-dimensional Bayesian belief (posterior) vector that lies in the unit $m$-dimensional simplex $\Pi$ of probability mass functions:

$$\eta = [\eta(1), \ldots, \eta(m)]' \in \Pi$$
$$\Pi = \{ \eta : \eta(i) \in [0, 1], \sum_{i=1}^m \eta(i) = 1 \} \tag{14}$$

The posterior $\eta$ (which will be formally defined in (26)) will be used in the quickest change detection formulation in Sec.IV. For the purposes of this section, $\eta$ is simply a fixed vector and $r_{\eta, 2}(s_2, a_2, a_1)$ is the corresponding parametrized reward at stage 2 in the two-stage anticipatory decision model.

A. Structural Result of Nash equilibrium

With $r_{\eta, 2}$ defined in (13), for notational convenience, define

$$\Delta_\eta(s_2, a_1) = r_{\eta, 2}(s_2, 2, a_1) - r_{\eta, 2}(s_2, 1, a_1) \tag{15}$$

We make the following assumptions on the anticipatory decision model of Sec.II-B:

(A1) The action spaces are $A_1 = [0, 1]$, $A_2 = \{1, 2\}$. Recall $a_1 \in A_1$ and $a_2 \in A_2$.

(A2) $r_{\eta, 2}(s_2, a_2, a_1)$ is convex in $a_1$.

(A3) $\Delta_\eta(s_2, a_1)$ defined in (15) is increasing in $s_2$. That is, $r_{\eta, 2}(s_2, a_2, a_1)$ is supermodular in $(s_2, a_2)$.

(A4) The solution $s_2^*(a_1)$ of $\Delta_\eta(s_2, a_1) = 0$ is a continuously differentiable function on $(0, 1)$.

(A5) \[
\frac{\partial \Delta_{\eta}}{\partial a_1} \cdot \frac{\partial^2 \Delta_{\eta}}{\partial a_2 \partial a_1} - \frac{\partial \Delta_{\eta}}{\partial s_2} \frac{\partial^2 \Delta_{\eta}}{\partial a_2^2} \geq 0
\]

(A6) The anticipatory cost is $r_1(z_1) = \beta z_1$ where $\beta > 0$ and psychological state (see (12))

$$z_1 = \max\{ \mathbb{E}[\Psi(a_2 = a, s_2)|s_1, a_1, \mu_2], a \in A_2 \}$$

(A7) With $p(s_2|s_1)$ denoting the state transition density, $\Psi(a_2 = 1, s_2)p(s_2|s_1)$ is increasing in $s_2$

$\Psi(a_2 = 2, s_2)p(s_2|s_1)$ is decreasing in $s_2$.

The above assumptions will be clarified by an example below. The following structural result that characterizes the structure of the subgame Nash equilibrium. For subsequent reference, we will denote the explicit dependence of $\mu_1^*$ and $\mu_2^*$ on Bayesian parameter $\eta$ (see (14)) as $\mu_{1, \eta}^*$ and $\mu_{2, \eta}^*$.

Theorem 1. Consider the anticipatory decision model of Sec.II-B with action and state spaces specified by (A1). Then

1) Under (A3), (A4), the subgame perfect Nash equilibrium policy $\mu_2^*$ specified by (8) has the following threshold structure:

$$\mu_{2, \eta}^*(s_2, a_1) = \begin{cases} 1 & \text{if } s_2 \leq s_2^+(a_1) \\ 2 & s_2 > s_2^+(a_1) \end{cases} \tag{16}$$

for some threshold state $s_2^+(a_1) \in [0, 1]$.

2) Under (A4), (A5), the threshold state $s_2^+(a_1)$ is convex in $a_1$.

3) Under (A2)-(A7), the utility-to-go $J_1(s, a_1, \mu_2^*)$ defined in (10) is convex in $a_1$. Therefore, the subgame Nash equilibrium policy $\mu_1^*$ has the following bang-bang structure:

$$\mu_{1, s}^*(s_1) = \begin{cases} 1 & \text{if } \beta > \beta^* \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

for some positive constant $\beta^*$.

Remarks. (i) Relaxing Assumption (A4). Instead of (A4), the following weaker condition based on the classic implicit function theorem [17] can be used. Assume (i) $\Delta_\eta(s_2, a_1)$ has continuous first partial derivatives; (ii) $\Delta_\eta(s_2^+, a_1) = 0 \implies \partial \Delta_\eta(s_2^+, a_1)/\partial s_2 \neq 0$. Then by the implicit function theorem, the solution $s_2^+(a_1)$ is continuously differentiable on an open subset of $(0, 1)$. Furthermore, assume that this subset is convex.

(ii) Deliberate Avoidance of Information. Suppose $a_1$ denotes a non-refundable financial deposit made by the agent at time 1 in anticipation of choosing action $a_2 = 1$ at time 2. Due to the bang-bang structure of (17) the agent makes a full deposit $a_1 = 1$ if $\beta > \beta^*$. Yet this full non-refundable deposit does not guarantee that the agent will choose $a_2 = 1$ since if $s_2 > s_2^+(a_1)$, then the agent will choose $a_2 = 2$. Thus the agent would like to avoid observing $s_2$. There is an elegant interpretation of this in [6], namely, the agent might deliberately choose not to observe the state $s_2$ in order not to lose the deposit. “In this manner, anticipatory emotions may rationalize the deliberate avoidance of information” [6].

Proof. For convenience we omit parameter $\eta$ in the notation.

Statement 1. By (A3), $r_{\eta, 2}$ is supermodular in $(s_2, a_2)$. Thus by Topkis theorem [18], $\mu_{2, \eta}^*(s_2, a_1)$ is non-decreasing in $s_2$ for fixed $a_1$. So either $\mu_{2, \eta}^*(s_2, a_1)$ is constant wrt $s_2$ (in which case the theorem holds trivially); or for each $a_1$ there exists an indiffERENCE state $s_2^* \in [0, 1]$ such that $\mu_{2, \eta}^*(s_2, a_1)$ switches from 1 to 2 as $s_2$ increases. Clearly the indifference set $\{ s_2, a_1 : \Delta_\eta(s_2, a_1) = 0 \}$ determines where $\mu_{2, \eta}^*(s_2, a_1)$ switches from 1 to 2. By (A4), a solution $s_2^+(a_1)$ exists to $\Delta_\eta(s_2, a_1) = 0$ for $a_1 \in (0, 1)$. Hence, $\mu_{2, \eta}^*$ has the threshold structure (16).

Statement 2. Proving statement 2, is equivalent to showing convexity of the solution $s_2^+(a_1)$ of the algebraic equation

4The phrase “bang-bang controller” comes from classical optimal control theory. It characterizes a control policy with continuous-valued actions that switches between two extremes.
Δ_η(s_2, a_1) = 0. By (A4), s_2^*(a_1) is continuously differentiable in a_1. It is verified by elementary calculus that
\[
\frac{∂^2 s_2^*}{∂a_1^2} = \frac{1}{(∂Δ_η/∂s_2)^2} \left[ \frac{∂Δ_η}{∂a_1} \frac{∂^2 Δ_η}{∂a_2} - \frac{∂Δ_η}{∂s_2} \frac{∂^2 Δ_η}{∂a_1} \right]
\]
So s_2^*(a_1) is convex in a_1 iff (A5) holds; see [19] for a more general result.

**Statement 3(a).** From (A6), the psychological state is
\[
z_1 = \max \left\{ \int S I(s_2 : μ_2^*(s_2, a_1) = a) Ψ(a, s_2) p(s_2|s_1) ds_2 \right\}
\]
\[
= \max \left\{ \left\{ F_1(s_2^*(a_1)) - F_1(0), F_2(1) - F_2(s_2^*(a_1)) \right\} \right\}
\]
where \( F_a(y) = \int_0^y Ψ(a, s_2) p(s_2|s_1) ds_2 \)

By (A7), \( F_1(y) \) and \(-F_2(y)\) are increasing convex functions of y. Since \( s_2^*(a_1) \) is convex in a_1 (by Statement 2), the composition functions \( F_1(s_2^*(a_1)) \) and \(-F_2(s_2^*(a_1))\) are convex. Thus \( r_1 = βz_2 \) being the max of two convex functions is convex in a_1. This together with (A2) implies that the reward-to-go \( J_z(s_1, a_1, μ_2^*) \) is convex in a_1.

**Statement 3(b).** Finally a convex function on a convex set (recall \( a_1 ∈ A_1 = [0, 1] \)) achieves its global maximum at an extreme point [20, Theorem 3, pp.181]. Hence the bang-bang structure (17) holds for \( μ_2^* \).

### B. Example. Social Media based Accommodation Choice

We now discuss an anticipatory decision making example involving choosing accommodation using a social-media based online agency such as Airbnb. The example is based on [6] and will be developed further in Section IV in the context of quickest time change detection.

Suppose an agent chooses between a vacation either at a previously known accommodation \( (K) \), or a new accommodation \( (N) \). The agent has initial wealth of \( w_0 \).

1) **Model:** The reviews of accommodation \( N \) posted at an online reputation website at times \( k = 1, 2 \) are review_k ∈ \{G (good), B (bad)\}. The physical states \( s_1 \) and \( s_2 \) denotes the probability of good review of \( N \) at times 1 and 2. For simplicity, assume \( s_2 \) is uniformly distributed in \([0, 1]\).

Regarding the actions, at the first stage the decision maker chooses action \( a_1 ∈ [0, 1] \) which denotes making a non-refundable deposit 2000 a_1 for booking \( K \). The second stage action is \( a_2 ∈ A_2 = \{K, N\} \). Given \( a_1 \), similar to [6], choose the psychological state \( z_1 \) at time 1 as (see (5))
\[
z_1 = \max\{6000 p(a_2 = N, \text{review}_2 = G|a_1, μ_2), 4000 p(a_2 = K|a_1, μ_2)\}
\]
(18)

Each accommodation costs 2000 units. After making a deposit of 2000a_1 for \( K \), if \( N \) is chosen, then the deposit of 2000a_1 is lost. The benefit accrued by staying in \( N \) when rating is \( s_2 \) is 6000s_2η; the reward for choosing \( K \) is 4000. Here η ∈ \([0, 1]\) is the posterior probability that accommodation \( N \) is suitable given the most recent review of \( N \). Then with \( β > 0 \) denoting the importance of anticipatory reward relative to the reward of the vacation, the rewards are
\[
r_1 = βz_2, \quad β > 0
\]
\[
r_2(s_2, a_2 = N, a_1) = 6000s_2η + w_0 - 2000(1 + a_1),
\]
\[
r_2(s_2, a_2 = K, a_1) = 4000 + w_0 - 2000
\]

2) **Structural Result for Nash equilibrium:** We can verify Assumptions (A1)-(A7) hold and therefore Theorem 1 holds. Specifically, (A1) holds by formulation; (A2) holds trivially since \( r_1 \) is linear in \( a_1 \); (A3) hold since \( r_2(s, a_2 = 1, a_1) \) is independent of \( s_2 \); (A5) holds trivially since \( Δ_η \) is linear in \( s_2 \) and \( a_1 \); (A6) holds by construction since it is easily shown that for optimal policy \( μ_2^* \), \( z_1 = 4000 p(a_2 = K|a_1, μ_2^*) \). Finally, (A7) holds since \( p(s_2) \) is the uniform density, so \( F(s_2) = s_2 \) which is trivially convex.

Therefore from Theorem 1 it follows that \( μ_2^* \) has a threshold structure (16), and \( μ_2^* \) has a bang-bang structure (17).

3) **Nash equilibrium:** Given the simple structure, can solve explicitly for the Nash equilibrium. From the extended Bellman equation, (8), \( μ_2^*(s_2, a_1) \) has threshold structure
\[
m_2^*(s_2, a_1) = \arg\max_{a_2} r_2(s_2, a_2, a_1) = \begin{cases} N & \text{if } s_2 ≥ \frac{2 + a_1}{3η}
K & s_2 < \frac{2 + a_1}{3η}
\end{cases}
\]
(19)

with associated value function
\[
V_2(s_2, a_1, μ_2^*) = \max_{a_2} \{r_2(s_2, a_2, a_1)\}
\]
\[
= \begin{cases} 6000s_2η + w_0 - 2000(1 + a_1) & s_2 ≥ \frac{2 + a_1}{3η}
4000 + w_0 - 2000 & s_2 < \frac{2 + a_1}{3η}
\end{cases}
\]
(20)

In order to determine the policy \( μ_2^* \) and value function \( V_1 \), let us first compute the psychological state \( z_1 \) in (18) under \( μ_2^* \). Since \( s_2 \) is uniformly distributed in \([0, 1]\), clearly
\[
p(a_2 = K|a_1, μ_2^*) = P\{s_2 : μ_2^*(s_2) = K\|a_1\}
\]
\[
= \int_S p(s_2|s_1) I(s_2 : μ_2^*(s_2) = K) ds_2
\]
(20)
\[
= \int_S p(s_2|s_1) I(s_2 ∈ [\frac{2 + a_1}{3η}, 1]) ds_2 = \frac{2 + a_1}{3η}
\]
(21)

Therefore,
\[
p(a_2 = N, \text{review}_2 = G|s_2, a_1, μ_2^*)
\]
\[
= p(\text{review}_2 = G)p(a_2 = N|s_2, a_1, μ_2^*) = s_2 I(s_2 ∈ [\frac{2 + a_1}{3η}, 1])
\]

Then in terms of notation (9) and (18), it is easily checked that the psychological state is
\[
z_1 = \max\{4000 p(a_2 = K|a_1, μ_2^*), 6000 p(a_2 = N, \text{review}_2 = G|a_1, μ_2^*)\}
\]
\[
= 4000 p(a_2 = K|a_1, μ_2^*)
\]
(22)
Then substituting $V_2$ computed in (19) into (10) yields

$$V_1(s_1) = \max_{a_1 \in A_1} \{ \beta z_1 + \int_0^1 V_2(s_2)ds_2 \}$$

It is easily verified that the expression within $\{ \cdot \}$ is convex in $s_1$. Since $A_1 = [0, 1]$ is convex, the maximum is achieved at an extreme point $a_1 = 0$ or $a_1 = 1$. Thus the optimal policy

$$\mu_1^* = \begin{cases} 1 \text{ (full deposit)} & \text{if } \beta > 1 - 3\eta + \frac{g\eta^2}{4} \\ 0 \text{ (no deposit)} & \text{if } \beta \leq 1 - 3\eta + \frac{g\eta^2}{4} \end{cases} \quad (23)$$

Recall $\eta$ is a Bayesian parameter (footnote 5) and $\beta > 0$ is a scaling constant (A6).

IV. ANTICIPATORY QUICKEST CHANGE DETECTION

Thus far we have described how a single anticipatory agent makes decisions over a two-period time horizon. In this section we consider multiple such anticipatory agents. These agents interact with each other sequentially and also with a global decision maker to achieve quickest change detection. Each agent observes the state of nature (Markov chain) in noise and makes local anticipatory decisions as described in Sec.II-B. How can a global decision maker use these local decisions to detect a change in the state of nature? Specifically the aim of the global decision maker is to achieve quickest change detection by minimizing the Kolmogorov-Shiryaev criterion (defined in (33) below) which involves the false alarm and decision delay.

Human-Sensor System. Quickest change detection involves decision making in a partially observed Bayesian setting. In the context of this paper, there are two interpretations.

(i) In human-sensor systems, each anticipatory agent is equipped with a sensing/computing device. The sensing device observes the state of nature (Markov chain) in noise. The computing device evaluates the posterior and provides the agent with these probabilities. The agent then makes anticipatory decisions $a_1, a_2$ as detailed in Sec.III.

(ii) In online reputation social networks, the reputation system provides the user with prior probabilities (e.g. histogram of reviews), and the Bayesian update is a useful idealization of the agent’s behavior; see Theorem 2 below. The agent then makes an anticipatory decision as detailed in Sec.III.

In both of the above cases, the anticipatory model seamlessly follows the framework of Sec.III where the rewards were parametrized by the posterior probability $\eta$.

An example of this setup (involving local decision makers interacting with a global decision maker) arises in the social media based accommodation example of Sec.III-B. Individual anticipatory agents make local decisions sequentially whether to rent the property $K$ or $N$. The Bayesian parameter $\eta$ determines the reward and policy of each local decision maker and is updated by each agent based on the decisions of previous agents (as described below). The global decision maker (e.g. Airbnb) monitors these local decisions and wishes to detect if there is a sudden change in the quality of the accommodation. (Other examples were discussed in Sec.1).

A. Multiagent Bayesian Quickest Detection Formulation

Notation. Since we will consider sequential interaction of anticipatory decision makers, the notation of Sec.III is modified to the sequential setup as follows:

- Each anticipatory agent acts in a predetermined sequential order $7$ indexed by $n = 1, 2, \ldots$
- The physical states $s_1, s_2$ encountered by agent $n$ are now denoted by $s_{n,1}, s_{n,2}$.
- The anticipatory decisions $a_1, a_2$ taken by agent $n$ are denoted as $a_{n,1}, a_{n,2}$.
- The Bayesian belief parameter $\eta$ is replaced by $\eta_n$.
- Recall due to the bang-bang structure (17) of the optimal policy, $a_{1,n}$ is independent of $s_{n,1}$. Also from (16), $a_{n,2}$ depends on $s_{n,2}$ and not $s_{n,1}$. So for notational convenience we denote $s_{n,2}$ as $s_n$. The physical state process $\{s_n, n \geq 1\}$ on state space $S$ is assumed to be Markov over time $n$ with transition density $p(s_{n+1}|s_n)$.

Model. In Bayesian quickest detection, the change event is modeled by a random process $8$ $\{x_n, n \geq 0\}$ that starts at state 2 at time 0 and jumps to state 1 at some random time $\tau^0$. We assume that $\tau^0$ is geometrically distributed with mean $1/(1-\theta)$, for some prescribed $\theta \in [0,1]$.

Equivalently, the state of nature $\{x_n\}$ is a 2-state Markov chain with initial distribution $\pi_0 = [0 1]'$, where $\pi_0(x) = \mathbb{P}(x_0 = x), x \in \{1, 2\}$ and absorbing transition matrix

$$P = \begin{bmatrix} 1 & 0 \\ 1 - \theta & \theta \end{bmatrix} \quad (24)$$

with change time

$$\tau^0 = \inf\{n : x_n = 1\}. \quad (25)$$

Clearly the transition matrix $P$ implies that $\mathbb{E}\{\tau^0\} = 1/(1-\theta)$. Quickest detection involves detecting the change time $\tau^0$ with minimal cost. The multiagent formulation considered here comprises of interacting local decision makers (anticipatory agents) and a global decision maker:

1) Each anticipatory agent acts in a predetermined sequential order $9$ indexed by $n = 1, 2, \ldots$. Agent $n$ observes $x_n$ in noise and makes a local decisions $a_n = \{a_{n,1}, a_{n,2}\}$ corresponding to actions $a_1, a_2$ in Sec.III.

2) Based on the history of local actions $a_1, \ldots, a_n$, the global decision maker chooses action $u_n \in \{1 \text{ (stop and announce change)}, 2 \text{ (continue)}\}$

$7$Equivalently, a single agent acts repeatedly and chooses its local decision using the current public belief.

$8$We assume the state-of-nature $\{x_n, n \geq 0\}$ is statistically independent of physical state process $\{s_n, n \geq 1\}$.

$9$Equivalently, a single agent acts repeatedly and chooses its local decision using the current public belief.
Define the public belief $\pi_n$ and private belief $\eta_n$ at time $n$ as the posteriors
\[
\begin{align*}
\pi_n(x) &= \mathbb{P}(x_n = x | a_1, \ldots, a_n), \quad x = 1, 2, \\
\eta_n(x) &= \mathbb{P}(x_n = x | a_1, \ldots, a_{n-1}, y_n), \\
\eta_0 &= \pi_0 = [0 \ 1]'
\end{align*}
\] (26)
Note $\eta = [1 - \eta(2), \eta(2)]'$ and $\pi = [1 - \pi(2), \pi(2)]'$ are two-dimensional beliefs; they lie in the one dimensional simplex $\Pi = [0, 1]$.

With the above framework and notation, we are now ready to describe the multiagent quickest detection protocol, see also Figure 1 for a visual schematic setup:

Protocol 1. Multiagent Bayesian Quickest Detection

1) **Local anticipatory decision maker** $n$ equipped with sensing/computing device
a) Obtain public belief $\pi_{n-1}$ from global decision maker.
b) The sensing functionality records private noisy observation $y_n \in \mathcal{Y}$ of state $x_n$ with conditional density
\[
B_{x,y} = p(y_n = y | x_n = x)
\] (27)
c) **Private Belief.** The computing functionality of the agent evaluates the private belief
\[
\eta_n = T(\pi_{n-1}, y_n)
\] (28)
where, $T(\pi, y) = B_y P'\pi \sigma(\pi, y)$, $\sigma(\pi, y) = 1' B_y P'\pi$

\[
B_y = \text{diag}(B_{1,y}, B_{2,y})
\]
d) **Local decision.** The agent uses $\eta_n$ to make anticipatory decisions $a_n = (a_{n,1}, a_{n,2})$ according to (17), (16).

2) **Global decision maker.** Based on the decisions $a_n$ of local decision maker $n$, the global decision maker:
a) Updates the public belief from $\pi_{n-1}$ to $\pi_n$ as
\[
\begin{align*}
\pi_n &= T(\pi_{n-1}, a_n, s_n) \\
T(\pi, a, s) &= \frac{R^\pi_n(s)}{\sigma(\pi, a, s)}, \quad \sigma(\pi, a, s) = 1' R^\pi_n(s) P'\pi
\end{align*}
\] (29)
where
\[
\begin{align*}
R^\pi_n(s) &= \text{diag}(R^\pi_{1,n}(s), R^\pi_{2,n}(s)), \\
R^\pi_{x,a}(s) &= \mathbb{P}(a_n = a | x_n = x, \pi_{n-1}, s_n = s)
\end{align*}
\] (30)
The action probabilities $R^\pi_{x,a}$ are computed as
\[
R^\pi_{x,a}(s) = \int_I f_\phi^\pi(\mu_{x,T(y)} \omega(s), a) B_{x,y} d\eta dy
\] (31)
Recall $\mu_{x,T}$ is the local decision maker’s subgame Nash equilibrium policy (16).
b) Chooses global action $u_n$ using optimal policy $\phi^*$:
\[
\begin{align*}
u_n &= \phi^*(\pi_n, s_n) \\
&\in \{1 \text{ (announce change (stop))}, 2 \text{ (continue)}\}
\end{align*}
\] (32)
c) If $u_n = 2$, then set $n = n + 1$ and go to Step 1. If $u_n = 1$, then stop and announce change.

In (32), the optimal stationary policy $\phi^*$ of the global decision maker is defined as the minimizer of the Kolmogorov–Shiryaev criterion for detection of disorder [21]:
\[
\begin{align*}
J_{\phi^*}(\pi, s) &= \inf_{\phi} J_{\phi}(\pi, s), \\
J_{\phi}(\pi, s) &= d\mathbb{E}_\phi \{(\tau - \tau^0)^+ \} + f \mathbb{P}_\phi(\tau < \tau^0)
\end{align*}
\] (33)
Here $\tau = \inf\{n : u_n = 1\}$ is the time at which the global decision maker announces the change. The non-negative parameters $d$ and $f$ specify the delay penalty and false alarm penalty, respectively. So waiting too long to announce a change incurs a delay penalty $d$ at each time instant after the system has changed, while declaring a change before it happens, incurs a false alarm penalty $f$. $\mathbb{P}_\phi$ and $\mathbb{E}_\phi$ are the probability measure and expectation of the evolution of the local decisions, observations and Markov state which are strategy dependent.

Remarks: We now discuss several aspects of Protocol 1.
1) **Sensor-human Interface.** Step 1 details the sensor-human decision interface. Each anticipatory human decision maker is equipped with a sensing/computing device that performs Steps 1a to 1c. Specifically, the noisy observation $y_n$ in Step 1b is obtained by a sensor/computing device which then uses Bayes rule to evaluate the private belief $\eta_n$ in Step 1c according to (28). The sensing functionality then provides $\eta_n$ to the anticipatory human decision maker. Recall that $\eta_n$ enters the parametrized rewards of the anticipatory decision maker as discussed in (14). Finally, the anticipatory decision maker chooses action $a_n$ in Step 1d according to the framework in Sec.III. Thus Step 1 preserves the structural simplicity of the anticipatory human decision modeling in [6].

2) **Global decision maker.** Step 2 details the decision making framework of the global decision maker. The global decision maker has access to the physical state $s_2$ and the actions $a_{n,1}, a_{n,2}$ of the local decision maker. These are used by the global decision maker in Step 2a to update the public belief in (29). The action likelihoods in (31) follow from (28) and the fact that
\[
R^\pi_{x,a}(s) = \int_I f_\phi^\pi(\mu_{x,T(y)} \omega(s), a) p(\eta | \pi_{n-1}, y) B_{x,y} d\eta dy
\]
Then in Step 2b, the global decision maker applies the optimal policy $\phi^*$ to the updated public belief $\pi_n$, to chooses whether to continue or stop (announce change).

3) **Comparison with Bayesian social learning.** Protocol 1 generalizes classical Bayesian social learning [12] in two ways. First, the public belief update (29) is a generalization of the Bayesian social learning filter [22], where the local decision maker is a myopic optimizer (in comparison, we now have a two-stage anticipatory local decision maker). Second, the local decision makers operate in closed loop; they are controlled by the global decision maker as discussed below.

4) **Information Structure.** Protocol 1 depicts three types of interactions. Local decision makers learn from previous local decision makers (as in social learning). Second, the local decisions $a_n$ determine the global decisions $u_n$.
Finally, the global decision maker’s public belief $\pi_{n-1}$ affects the local decision makers’ decision $a_n$, and the global decision $u_n$ determines whether the local decision makers continue or stop.

B. Stochastic Dynamic Programming Formulation of Quickest Change Detection

The aim of this section is to formulate the global decision maker’s optimal change detection policy $\phi^*(\pi, s)$ (defined in (33)) as the solution of a stochastic dynamic programming equation. The quickest detection problem (33) is an example of a stopping-time partially observed Markov decision process (POMDP) problem with a stationary optimal policy [22].

1) Costs: To present the dynamic programming equation, we first formulate the false alarm and delay costs incurred by the global decision maker in terms of the public belief (also called the information state [23]).

(i) False alarm penalty: If global decision $u_n = 1$ (stop) is chosen at time $n$, then the protocol terminates. If $u_n = 1$ is chosen before the change point $\tau^0$, then a false alarm penalty is incurred. The false alarm event $\{x_n = 2, u_n = 1\}$ represents the event that a change is announced before the change happens at time $\tau^0$. The expected false alarm penalty is $\mathbb{E}I(x_n = 2, u_n = 1)\mathbb{P}_n$ where $f \geq 0$ and

$$G_n = \sigma\text{-algebra generated by } (a_1, \ldots, a_n) \quad (34)$$

In terms of the public belief, the expected false alarm penalty is

$$C(\pi_n, u_n = 1) = f e_2^T \pi_n, \quad \text{where } e_2 = [0 \ 1]^T. \quad (35)$$

(ii) Delay cost of continuing: If global decision $u_n = 2$ is taken then the multiagent protocol of Sec.IV-A continues to the next time. A delay cost is incurred when the event $\{x_n = 1, u_n = 2\}$ occurs, i.e., no change is declared at time $n$, even though the state has changed at time $n$. The expected delay cost is $\mathbb{E}I(x_n = 1, u_n = 2)\mathbb{P}_n$ where $d > 0$ denotes the delay cost. In terms of the public belief, the delay cost is

$$C(\pi_n, u_n = 2) = d e_1^T \pi_n, \quad \text{where } e_1 = [1 \ 0]^T. \quad (36)$$

We can re-express Kolmogorov-Shiryaev criterion (33) as

$$J_\phi(\pi, s) = \mathbb{E}_\phi \left\{ \sum_{n=0}^{\tau-1} C(\pi_n, 2) + C(\pi_\tau, 1) \right\} \quad (37)$$

where $\tau = \inf \{n : u_n = 1\}$ is adapted to the $\sigma$-algebra $G_n$. Since $C(\pi, 1), C(\pi, 2)$ are non-negative and bounded for $\pi \in \Pi$, stopping is guaranteed in finite time, i.e., $\tau$ is finite with probability $1$.

2) Bellman’s equation: Consider the costs (35), (36) defined in terms of the public belief $\pi$. Then the optimal stationary policy $\phi^*(\pi, s)$ defined in (32), (33), and associated value function $V(\pi, s)$ are the solution of Bellman’s dynamic programming functional equation [22]

$$Q(\pi, s, 1) \equiv C(\pi, 1),$$

$$Q(\pi, s, 2) \equiv C(\pi, 2)$$

$$+ \int_S \sum_{a \in A_1 \times A_2} V(T(\pi, a, s), s) \sigma(\pi, a, s) p(s|s) d\bar{s} \quad (38)$$

$$\phi^*(\pi, s) = \arg\min_{u \in U} \{Q(\pi, s, 1), Q(\pi, s, 2)\},$$

$$V(\pi, s) = \min_{u \in U} \{Q(\pi, s, 1), Q(\pi, s, 2)\} = J_\phi^*(\pi, s)$$

The public belief update $\bar{T}$ and normalization measure $\bar{\sigma}$ were defined in (29). Recall (32) that $u_n = \phi^*(\pi_n, s_n)$ is the global decision maker’s action whether to continue or stop.

The goal of the global decision-maker is to solve for the optimal quickest change policy $\phi^*$ in (38) or equivalently, determine the optimal stopping set $S$

$$S = \{\pi, s : \phi^*(\pi, s) = 1\} = \{\pi, s : Q(\pi, s, 1) \leq Q(\pi, s, 2)\} \quad (39)$$

3) Value Iteration Algorithm: The optimal policy $\phi^*(\pi, s)$ and value function $V(\pi, s)$ can be constructed as the solution of a fixed point iteration of Bellman’s equation (38) – the resulting algorithm is called the value iteration algorithm. The value iteration algorithm proceeds as follows: Initialize $V_0(\pi, s) = 0$ and for iterations $k = 1, 2, \ldots$

$$V_{k+1}(\pi, s) = \min_{u \in U} Q_{k+1}(\pi, s, u),$$

$$\phi_{k+1}(\pi, s) = \arg\min_{u \in U} Q_{k+1}(\pi, s, u) \quad \pi \in \Pi,$$

$$Q_{k+1}(\pi, s, 2) = C(\pi, 2)$$

$$+ \int_S \sum_{a \in A_1 \times A_2} V_k(T(\pi, s, a), s) \sigma(\pi, s, a) p(s|s) d\bar{s},$$

$$Q_{k+1}(\pi, s, 1) = C(\pi, 1).$$

Let $B$ denote the set of bounded real-valued functions on $\Pi$. Then for any $V, \widehat{V} \in B$, define the sup-norm metric $\|V - \widehat{V}\|, \pi \in \Pi, s \in S$. Then $B$ is a Banach space. Since $C(\pi, 1), C(\pi, 2), \pi \in \Pi$, are bounded, the value iteration algorithm (40) will generate a sequence of lower semi-continuous value functions $\{V_k\} \subset B$ that will converge pointwise as $k \to \infty$ to $V(\pi, s) \in B$, the solution of Bellman’s equation, see [25, Prop.1.3, Chap 3, Vol.2].

Summary. Protocol 1 describes the setup for quickest detection protocol involving anticipative agents acting sequentially. Each local decision maker (agent) $n = 1, 2, \ldots$ makes anticipatory decisions $a_{n,1}, a_{n,2}$ according to the framework in Sec.III. The global decision maker uses these actions to make decision $u_n = \phi^*(\pi_n, s_n) \in \{1, 2\}$. Here the optimal detection policy $\phi^*$ of the global decision maker satisfies Bellman’s equation (38) and can be constructed by the value iteration algorithm (40).
V. STRUCTURAL RESULTS FOR QUICKEST DETECTION WITH ANTICIPATORY AGENTS

As the belief space \( \Pi \) in (14) is a unit simplex, the value iteration algorithm (40) does not directly yield a practical solution for computing stopping set \( S \) since \( \nu(x) \) needs to be evaluated on the continuum \( \pi \in \Pi \). In quickest detection, since \( x_k \in \{1,2\} \), the belief space \( \Pi \) is a 1-dimensional simplex comprising 2-dimensional beliefs of the form \( \pi = [1-\pi(2), \pi(2)]' \). The value iteration algorithm can be solved numerically by one-dimensional grid discretization of \( \Pi \).

The aim of this section is to characterize the structure of the belief updates and achievable optimal cost without brute force computations.

Specifically we discuss 5 important structural results below:

1) The private belief update of individual anticipatory agents follows simple rules justifying human decision-making.
2) Even though the public belief update depends on the action probabilities \( R^\pi \) (30) where \( \pi \in \Pi \) is continuum, there are only a finite number of such action probabilities.
3) In stark contrast to classical quickest detection, the value function (38) in Bellman’s equation for quickest detection with anticipative agents is not necessarily concave.
4) We give numerical examples of the optimal quickest detection policy to highlight the unusual structure of non-convex stopping regions. These illustrate changeblindness.
5) Finally, by using Blackwell dominance, we show that the cumulative cost incurred is always larger than classical quickest change detection.

A. Private Belief Update follows simple monotone rules

As discussed at the beginning of Sec.IV, the agent either uses a sensing/computing device to evaluate its private Bayesian belief or constructs an approximation to the private belief in order to make an anticipative decision. Below we show that the Bayesian update for the private belief is monotone in the observation and prior; thus it follows simple rules and is a useful idealization of an agent’s behavior.

Recall Theorem 1 asserted monotonicity of the anticipatory decision maker’s policy \( \mu^\pi_{2,\eta}(s_2, a_1) \) wrt physical state \( s_2 \). Here we show monotonicity wrt the Bayesian parameter \( \pi \) (recall \( \pi \) is the prior for \( \eta \) in the Bayesian update (28)) and observation \( y \). We make the following assumptions

(A8) The observation likelihoods \( B_{x,y} \) (27) are TP2 (totally positive of order 2); that is, \( B_{x,y}B_{x,y} \leq B_{x,y}B_{x,y} \), \( x > x', y > y' \).

(A9) \( r_2(s_2, a_2, a_1, x) \) (see (13)) is supermodular in \( (x, a_2) \), i.e., \( r_2(s_2, a_2, a_1, x) - r_2(s_2, a_2, a_1, x) \) is increasing in \( a_2 \).

(A8) is widely studied in monotone decision making; see the classical book by Karlin [26] and [27]; numerous examples of noise distributions are TP2. As described in [28], observation \( y \) is said to be more “favorable news” than observation \( y' \) if (A8) holds. (A9) is the supermodularity condition on the rewards.

In the theorem below recall that \( \mu^\pi_{2,\eta}(s_2, a_1) \) is the subgame Nash equilibrium of the local anticipatory decision maker.

Theorem 2. The following properties hold for the anticipatory action \( a_{n,2} = \mu^\pi_{2,T}(s, a_{n,1}) \) in (16) made by agent \( n \):

1) Under (A8) and (A9), \( a_{n,2} \) is increasing and ordinal in observation \( y \). That is for any monotone function \( \phi \), it follows that \( \phi(a_{n,2}) \) is also increasing in \( y \).

2) Under (A8), \( \mu^\pi_{2,T}(s, a_{n,1}) \) is increasing in belief \( \pi \) with respect to the monotone likelihood ratio (MLR) stochastic order\(^\text{11} \) for any observation \( y_n \).

We can interpret the above theorem as follows. If anticipative agent \( n \) makes recommendations that are monotone and ordinal in the observations and monotone in the prior, then they mimic the Bayesian social learning model. Even if the agent does not exactly follow a Bayesian social learning model, its monotone ordinal behavior implies that such a Bayesian model is a useful idealization. Humans typically make monotone decisions - the more favorable the private observation, the higher the recommendation. Humans make ordinal decisions\(^\text{12} \) since humans tend to think in symbolic ordinal terms.

We now briefly discuss (A9). To simplify notation denote the reward vector \( r_a \) where

\[
\begin{array}{l}
\begin{bmatrix}
\begin{array}{c}
\frac{|B_y}{B_{x,y}P_{y}^\pi} \leq r \frac{B_{y}P_{y}^\pi}{1B_{y}P_{y}^\pi}
\end{array}
\end{bmatrix}
\end{array}
\end{equation}
\]

Then (A9) is a stronger version of the following more general single-crossing condition [18], [29]: For \( \bar{y} > y \),

\[
(r_{a+1} - r_a)^{y} B_{y}^\pi \leq 0 \implies (r_{a+1} - r_a)^{y} B_{y}^\pi \leq 0.
\]

This single crossing condition is ordinal, since for any monotone function \( \phi \), it is equivalent to

\[
\phi((r_{a+1} - r_a)^{y} B_{y}^\pi) \leq 0 \implies \phi((r_{a+1} - r_a)^{y} B_{y}^\pi) \leq 0.
\]

Proof. The proof uses MLR stochastic dominance (defined in footnote 11) and the following single crossing condition:

Definition 3 (Single Crossing [29]), \( g : \mathbb{Y} \times \mathbb{A} \rightarrow \mathbb{R} \) satisfies a single crossing condition in \((y, a)\) if \( g(y, a) - g(y, \bar{a}) \geq 0 \) implies \( g(y, a) - g(y, \bar{a}) \geq 0 \) for \( a < a' \) and \( y > y' > y \). Then \( a^*(y) = \arg\min_a g(y, a) \) is increasing in \( y \).

By (A8) it follows that [22] the Bayesian update satisfies

\[
\frac{B_{y}P_{y}^\pi}{1B_{y}P_{y}^\pi} \leq r \frac{B_{y}P_{y}^\pi}{1B_{y}P_{y}^\pi}, \quad \bar{y} > y
\]

where \( \leq_r \) is the MLR stochastic order. (Indeed, the MLR order is closed under conditional expectation.) By supermodularity (A9) \( r_{a+1} - r_a \) is a vector with increasing elements. Therefore

\[
(r_{a+1} - r_a)^{y} B_{y}^\pi \leq 0 \implies (r_{a+1} - r_a)^{y} B_{y}^\pi \leq 0.
\]

Since the denominator is non-negative, it follows that \( (r_{a+1} - r_a)^{y} B_{y}^\pi \leq 0 \). This implies that \( r_{a}^{y} B_{y}^\pi \) satisfies a single crossing condition in \((y, a)\). Therefore \( a_n^\pi(y, a) = \arg\max_a r_a^y B_{y}^\pi \) is increasing in \( y \) for any belief \( \pi \).

\(^{11}\) Given probability mass functions \( \{p_i\} \) and \( \{q_i\} \), \( i = 1, \ldots, X \) then \( p \) MLR dominates \( q \) if \( \log p_i - \log p_{i+1} \leq \log q_i - \log q_{i+1} \).
B. Structure of Public Belief Update

We assume in this section that
\[ \mathcal{Y} = \{1, \ldots, Y\}, \quad \mathcal{A}_2 = \{1, 2\} \]  
(42)

Even though the public belief \( \pi \in \Pi \) is continuum, it turns out that there are only \( Y + 1 \) possible distinct action likelihood probability matrices \( R^\pi \).

Specially, define the following \( Y \) points in the one-dimensional simplex \( \Pi \):
\[ \pi_y^* = \{ \pi : (r_1 - r_2)'B_yP'\pi = 0 \}, \quad y = 1, \ldots, Y \]

Note that each 2-dimensional probability vector \( \pi_y^* = [1 - \pi_y^*(2), \pi_y^*(2)]' \) depends on \( a_1, s \).

**Theorem 4.** Under (A8), (A9), it follows that
\[ \pi^*_1(2) \leq \pi^*_2(2) \cdots \leq \pi^*_Y(2) \]  
(43)

Thus the belief space \( \Pi \) can be partitioned into at most \( Y + 1 \) non empty intervals denoted \( \mathcal{P}_1, \ldots, \mathcal{P}_{Y+1} \) where
\[ \mathcal{P}_1 = [0, \pi^*_1(2)], \mathcal{P}_2 = (\pi^*_1(2), \pi^*_2(2)], \ldots, \mathcal{P}_{Y+1} = (\pi^*_Y(2), 1] \]  
(44)

On each such interval, the action likelihood \( R^\pi \) is a constant with respect to belief \( \pi \). Specifically, for fixed \( a_1, s \)
\[ R^\pi(s) = \begin{bmatrix} \sum_{i=0}^{Y-l} B_{1i} & \sum_{i=1}^{Y-l} B_{1i} \\ \sum_{i=0}^{Y} B_{1i} & \sum_{i=1}^{Y} B_{1i} \end{bmatrix}, \quad \pi \in \mathcal{P}_l \]  
(45)

**Proof.** The single crossing property (41) implies
\[ \{ \pi : (r_1 - r_2)'B_yP'\pi \leq 0 \} \subseteq \{ \pi : (r_1 - r_2)'B_yP'\pi \leq 0 \} \]
for \( y < \bar{y} \). This implies (43). From (31) we can write
\[ R^\pi(s) = \sum_y B_{x,y}M^\pi_{y,a,s}, \]  
(46)

where \( M^\pi_{y,a,s} = \text{def} \left( I(\mu_{2,T}(\pi,s)) (s,a_{n,1}) = a \right) \)

where \( B \) and \( M^\pi \) are stochastic matrices. This yields (45). \( \square \)

**Example.** For \( Y = 3 \), the 4 possible action likelihood matrices \( R^\pi \) are
\[ R^1(s) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad R^2(s) = \begin{bmatrix} B_{11} & B_{12} + B_{13} \\ B_{21} & B_{22} + B_{23} \end{bmatrix}, \]  
\[ R^3(s) = \begin{bmatrix} B_{11} + B_{12} & B_{13} \\ B_{21} + B_{22} & B_{23} \end{bmatrix}, \quad R^4(s) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \]  
(47)

Although somewhat tangential to this paper, it is easily shown that the agents deploying Protocol 1 exhibit herding behavior, i.e., there exists regimes where agents choose actions independent of their private observations; please see [12], [13] for the subtle distinction between herds and information cascades.

C. Why Quickest Detection with Anticipatory Agents is non-trivial

In classical quickest change detection, the value function is always concave and the optimal stopping region is convex, see [22] for a partially observed Markov decision formulation and proof of this. The aim of this section is to show that due to the interaction of local and global decision makers, quickest detection with anticipatory agents exhibits non-trivial behavior: the value function is not necessarily concave and the stopping region is not necessarily a convex set.

Consider the value iteration algorithm (40) which is used as a basis for mathematical induction to prove properties associated with Bellman’s equation (38). Note that from (40), \( V_k(\pi, s) \) is positively homogeneous, that is, for any \( \alpha > 0, V_k(\alpha \pi, s) = \alpha V_k(\pi, s) \). So choosing \( \alpha = \sigma(\pi, a) \) yields
\[ V_{k+1}(\pi, s) = \min \{ C(\pi, 1) \]  
(48)

Recall \( C(\pi, 1) \) and \( C(\pi, 2) \) are linear in \( \pi \). However, it is clear from (48) that if \( V_k(\pi, s) \) is assumed to be concave on \( \Pi \), \( V_{k+1}(\pi, s) \) is not necessarily concave on \( \Pi \); since patching together convex functions on different intervals does not necessarily yield a convex function. The key point is that the action likelihoods \( R^\pi \) are explicit and discontinuous functions of \( \pi \). This results in a possibly non-concave value function \( V(\pi) \) making determining \( S \) non-convex.

D. Multi-threshold Quickest Detection Policy: Change-Blindness

The non-concave value function in quickest detection with anticipatory agents leads to unusual multi-threshold behavior in the optimal policy, as we now discuss via a numerical example.

Consider the quickest detection problem where the state of nature \( \{x_n, n \geq 0\} \) jumps according to transition matrix
\[ P = \begin{bmatrix} 0.605 & 0.395 \\ 0.95 & 0.05 \end{bmatrix}. \]

The global decision maker’s delay and false alarm penalties are \( d = 1.05, f = 3 \); these specify the costs (35), (36) in Bellman’s equation (38).

The local anticipative decision maker’s reward matrix is
\[ (r_2(x, a_2), x \in \{1, 2\}, a \in \{1, 2\}) = \begin{bmatrix} 5 & 4 \\ 6.5 & 9 \end{bmatrix}. \]

Also its observation likelihood matrix is \( B = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \).

Figure 3 displays the value function and optimal policy for classical quickest detection. Figure 4 displays the value function and optimal policy for quickest detection with anticipatory agents. The policy and value function were obtained by running the value iteration algorithm for 1000 iterations with \( \Pi = [0, 1] \) grid quantized uniformly to 1000 values.

For classical quickest detection, Figure 3 shows that, as expected, the value function is concave and the optimal policy
is a threshold. So the stopping region $\{ \pi : \phi^{*}(\pi) = 1 \}$ is the interval $\pi(2) \in [0, 0.466]$. In contrast for quickest detection involving anticipatory agents, Figure 4 shows that the value function is not concave. Also the optimal policy has an unusual multi-threshold structure: if it is optimal to declare a change for a particular posterior probability, it may not be optimal to declare a change when the posterior probability of change is larger! (Recall $1 - \pi(2)$ is the posterior probability of change). In this sense, Figure 4 depicts two forms of change-blindness. First, a human global decision maker might choose to ignore the optimal policy $\phi^{*}(\pi)$ and simply use the classical quickest detection policy $\phi^{*}(\pi)$. A second, and more interesting form of change-blindness occurs when the human global decision maker chooses the “simple” stopping set as $\pi(2) \in [0, a]$ and ignore the important regions between $[a, b]$ where it is optimal to stop.

**E. Blackwell Dominance Implications for Optimal Cost**

In this section we show that quickest detection with anticipatory agents (Protocol 1) results in a cumulative Kolmogorov Shiryaev cost $J_{\phi^{*}}(\pi, s)$ (defined in (33) or equivalently (37)) that is always larger than that of classical quickest detection. In Protocol 1, agents have access to the public belief (which depends on local decisions of previous agents) instead of the actual observations. Thus one would expect intuitively that this information loss results in less efficient quickest time change detection compared to classical quickest detection. Here we confirm this intuition. The main idea involves Blackwell dominance of observation measures.

First define the optimal policy and cost in classical quickest change detection. Similar to (38), the optimal policy $\phi^{*}(\pi)$ and cost $\mathcal{V}(\pi)$ incurred in classical quickest detection, satisfy the following stochastic dynamic programming equation:

$$
\phi^{*}(\pi) = \arg \min_{u \in \mathcal{U}} Q(\pi, u), \quad \mathcal{V}(\pi) = \min_{u \in \mathcal{U}} Q(\pi, u), \quad (49)
$$

where $Q(\pi, 2) = C(\pi, 2) + \sum_{y \in \mathcal{Y}} \mathcal{V}(T(\pi, y)) \sigma(\pi, y), \quad \mathcal{J}_{\phi^{*}}(\pi) = \mathcal{V}(\pi)$.

Here $T(\pi, y)$ is the Bayesian filter update defined in (28) and $\mathcal{J}_{\phi^{*}}(\pi)$ is the cumulative cost of the optimal policy starting with initial belief $\pi$. Note that unlike Protocol 1, in classical quickest detection, there is no public belief update (29) or interaction between the public and private beliefs.

The following theorem says that for any initial belief, the optimal detection policy with anticipative agents acting sequentially (Protocol 1) incurs a higher cumulative cost than that of classical quickest detection.

**Theorem 5.** Consider the quickest time detection problem involving anticipatory agents described in Protocol 1 and associated value function $\mathcal{V}(\pi, s)$ in (38). Consider also the classical quickest detection problem with value function $\mathcal{V}(\pi)$ in (49). Then for any initial belief $\pi \in \Pi$, the optimal cost incurred by classical quickest detection is smaller than that of quickest detection with anticipatory agents. That is, $\mathcal{V}(\pi) \leq \mathcal{V}(\pi, s)$ for all $\pi \in \Pi, s \in \mathcal{S}$. 

![Fig. 4: Quickest Detection with Anticipative Agents. The optimal policy $\phi^{*}(\pi)$ has a multi-threshold structure implying that the optimal stopping set $\mathcal{S} = \{ \pi : \phi^{*}(\pi) = 2 \}$ is not convex (comprises of disconnected regions). The global decision maker exhibits rational change blindness. As the posterior probability of change $\pi(1) = 1 - \pi(2)$ increases from $b$ to $a$, the global decision maker declares there is no change in several regions. Also the value function $\mathcal{V}(\pi)$ is not concave.](image)
Since the theorem holds when \( \mathcal{A} = \mathcal{Y} = \{1, 2\} \), i.e., equal number of local decision choices and observation symbols, a naive explanation that information is lost due to using fewer symbols in \( \mathcal{A} \) compared to \( \mathcal{Y} \) is not true.

The intuition behind the proof is as follows. From (31) we can write

\[
R^\pi_{x,a}(s) = \int_\mathcal{Y} B_{x,y} M^\pi_{y,a,s} dy,
\]

where \( M^\pi_{y,a,s} \) are stochastic kernels. Thus observation \( y \) with conditional distribution specified by \( B \) is said to be more informative than (Blackwell dominates) observation \( a \) with conditional distribution \( R^\pi \), see [22]. The main idea in the proof is that under the assumptions of Theorem 5, the value function \( \mathcal{V}(\pi) \) is concave for \( \pi \in \Pi \). Then the result is established using Jensen’s inequality together with Blackwell dominance on the Bellman’s equation.

The first instance of a similar proof using Blackwell dominance for partially observed stochastic control problems was given in [30], see also [31], where it was used to show optimality of certain myopic policies. Our use of Blackwell dominance in Theorem 5 is somewhat different since we are using it to compare the value functions of two different dynamic programming problems. A useful consequence of Theorem 5 is that performance analysis of standard quickest detection problems [32] readily applies to form a lower bound for the cost incurred in quickest detection with anticipatory agents in Protocol 1.

**Proof.** It is well known [22] (and straightforwardly demonstrated by induction) that the value function \( \mathcal{V}_k(\pi) \) for classical quickest detection is concave over \( \Pi \) for any \( k \). Then we use the Blackwell dominance condition (50). The public belief update (29) can be expressed in terms of the private belief update (28) as

\[
\bar{T}(\pi, a, s) = \sum_{y \in \mathcal{Y}} T(\pi, y) \frac{\sigma(y, a, s)}{\sigma(\pi, a, s)} M^\pi_{y,a,s}
\]

and

\[
\bar{\sigma}(\pi, a, s) = \sum_{y \in \mathcal{Y}} \sigma(y, a, s) M^\pi_{y,a,s}
\]

Note that \( \frac{\sigma(y, a, s)}{\sigma(\pi, a, s)} M^\pi_{y,a,s} \) is a probability measure wrt \( y \). Since \( \mathcal{V}_k(\cdot) \) is concave for \( \pi \in \Pi \), using Jensen’s inequality it follows that

\[
\mathcal{V}_k(\bar{T}(\pi, a, s)) = \mathcal{V}_k \left( \sum_{y \in \mathcal{Y}} T(\pi, y) \frac{\sigma(y, a, s)}{\sigma(\pi, a, s)} M^\pi_{y,a,s} \right)
\geq \sum_{y \in \mathcal{Y}} \mathcal{V}_k(T(\pi, y)) \frac{\sigma(y, a, s)}{\sigma(\pi, a, s)} M^\pi_{y,a,s}
\]

Therefore for each \( \bar{s} \in \mathcal{S} \),

\[
\sum_a \mathcal{V}_k(T(\pi, a, \bar{s})) \bar{\sigma}(\pi, a, \bar{s}) \geq \sum_y \mathcal{V}_k(T(\pi, y)) \sigma(\pi, y).
\]

Therefore multiplying by \( p(\bar{s}|s) \) and integrating we have

\[
\int_\mathcal{S} \sum_a \mathcal{V}_k(T(\pi, a, \bar{s})) \bar{\sigma}(\pi, a, \bar{s}) p(\bar{s}|s) d\bar{s} \geq \sum_y \mathcal{V}_k(T(\pi, y)) \sigma(\pi, y).
\]

The proof of Theorem 5 then follows by mathematical induction using the value iteration algorithm (40). Assume \( \mathcal{V}_k(\pi, s) \geq \mathcal{V}_k(\pi) \) for \( \pi \in \Pi \). Then

\[
C(\pi, 2) + \int_\mathcal{S} \sum_a \mathcal{V}_k(T(\pi, a, s), \bar{s}) \bar{\sigma}(\pi, a, \bar{s}) p(\bar{s}|s) d\bar{s} \geq C(\pi, 2) + \int_\mathcal{S} \sum_a \mathcal{V}_k(T(\pi, a, s)) \bar{\sigma}(\pi, a, s) p(\bar{s}|s) d\bar{s}
\]

where the second inequality follows from (52). Thus \( \mathcal{V}_{k+1}(\pi, s) \geq \mathcal{V}_{k+1}(\pi) \). This completes the induction step. Since value iteration converges pointwise, \( \mathcal{V}(\pi, s) \geq \mathcal{V}(\pi) \) thus proving the theorem.

\[\square\]

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