Analysis of Lackadaisical Quantum Walks*

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Abstract

The lackadaisical quantum walk is a quantum analogue of the lazy random walk obtained by adding a self-loop to each vertex in the graph. We analytically prove that lackadaisical quantum walks can find a unique marked vertex on any regular locally arc-transitive graph with constant success probability quadratically faster than the hitting time. This result proves several speculations and numerical findings in previous work, including the conjectures that the lackadaisical quantum walk finds a unique marked vertex with constant success probability on the torus, cycle, Johnson graphs, and other classes of vertex-transitive graphs. Our proof establishes and uses a relationship between lackadaisical quantum walks and quantum interpolated walks for any locally arc-transitive graph.

1 Introduction

Searching is one of the most important tasks in computer science, and searching algorithms have been well studied from both classical and quantum aspects. One of the most famous quantum algorithms, Grover’s search algorithm [Gro96], can search an \( N \)-item unstructured database in \( O(\sqrt{N}) \) steps, which is quadratically faster than classical searching. The method used in Grover’s algorithm is generalized as amplitude amplification in [BHMT02].

Searching structured databases can be modeled as spatial search problems on graphs, where the vertices of the graph represent the search space. A subset of the vertices are marked, and the goal is to find one of the marked vertices. One classical strategy is to use a random walk to traverse the graph along its edges until a marked vertex is reached. The expected number of steps \( \text{HT} \) required to reach a marked vertex by a random walk is called the hitting time. Quantum walks, which are quantum counterparts of random walks, are used to develop quantum algorithms for spatial search problems.

Szegedy introduced a generic method of constructing a quantum walk from a reversible random walk [Sze04]. The resulting quantum walk uses \( O(\sqrt{\text{HT}}) \) steps, which yields a quadratic speedup over the random walk. Szegedy’s algorithm does not necessarily find a marked vertex, but it can detect the presence of a marked vertex. Krovi et al. [KMOR16] later proposed a quantum algorithm based on the novel idea of interpolated walks. They applied Szegedy’s correspondence on the interpolated walks and called the resulting algorithms quantum interpolated walks. Quantum interpolated walks can find a marked vertex in \( O(\sqrt{\text{HT}}) \) steps for any reversible random

*Published in: Quantum Information and Computation, Vol. 20, No. 13–14 (2020) 1137–1152, by Rinton Press. http://www.rintonpress.com/xxqic20/qic-20-1314/1137-1152.pdf
walk, where $HT^+$ is the extended hitting time of the random walk. When there is a unique marked vertex, then $HT^+ = HT$ and this quantum walk thus achieves a quadratic speedup over the random walk. When there are multiple marked vertices, $HT^+$ may be asymptotically larger than $HT$ [AK15]. Dohotaru and Høyer [DH17] achieved the same result by introducing a different framework called controlled quantum walks.

Quantum walks are commonly applied on graphs without self-loops. The lackadaisical quantum walk proposed by Wong [Wong15], is a quantum analogue of the lazy random walk, which adds a self-loop of weight $\ell$ to each vertex. The lackadaisical quantum walk generalizes the three-state lazy quantum walk on the line [Won15, IKS05]. The idea of adding self-loops was first applied by Ambainis et al. [AKR05], who showed that a quantum walk on a complete graph with a self-loop on every vertex corresponds to Grover’s algorithm. The lackadaisical quantum walk can also be viewed as a coined quantum walk, and Wong uses this to demonstrate that the asymptotic behavior of some coined quantum walks can be improved by modifying the coin [Won18b].

The lackadaisical quantum walk, with a unique marked vertex, has been studied on several classes of graphs. First, the complete graph was studied by Wong [Won15, Won18a, Won17], who proved analytically that the lackadaisical quantum walk with $\ell = 1$ finds a unique marked vertex in $O(\sqrt{N})$ steps with probability close to 1.

Second, the $\sqrt{N} \times \sqrt{N}$ torus was independently studied by Wong [Won18b] and Wang et al. [WZWY17], who both showed numerically that the lackadaisical quantum walk finds a unique marked vertex with probability close to 1. The value of $\ell$ is $\frac{1}{N}$ in Wong [Won18b] and $\frac{\sqrt{\frac{1}{N}} - 1}{N}$ in Wang et al. [WZWY17]. The result is strongly supported by the experiments, but with no analytical proofs of the complexity and success probability, the result is stated as a conjecture.

Third, the cycle was studied by Giri and Korepin [GK19], who showed numerically that by setting $\ell = \frac{2}{N}$, the lackadaisical quantum walk finds a unique marked vertex with probability at least a constant.

Fourth, regular complete bipartite graphs were studied by Rhodes and Wong [RW19], who proved analytically that the lackadaisical quantum walk with $\ell = \frac{1}{2}$ finds a unique marked vertex in $O(\sqrt{N})$ steps with probability close to 1.

Fifth, in a recent paper, Rhodes and Wong [RW20] study a collection of graphs. Their collection is a rich sample of vertex-transitive graphs, comprised of the following instances and classes of graphs: arbitrary-dimensional cubic lattices, Paley graphs, the two Latin square graphs with strongly regular parameters $(9, 6, 3, 6)$ and $(1024, 93, 32, 6)$, triangular graphs, Johnson graphs, and the hypercube. They show numerically that by setting $\ell = \frac{d}{N}$, the lackadaisical quantum walk finds a unique marked vertex with probability at least a constant. Here $d$ is the degree of the vertices. They propose that this holds for all vertex-transitive graphs with a unique marked vertex, and they propose that the weight of self-loop $\ell = \frac{d}{N}$ optimally boosts the success probability.

In this work, we prove analytically that the lackadaisical quantum walk finds a unique marked vertex with probability at least a constant on all of the above-mentioned graphs when choosing the weight $\ell$ of the self-loops as listed above. More generally, we prove that for any $d$-regular locally arc-transitive graph, by adding a self-loop of weight $\ell = \frac{d}{N}$ on each vertex, the lackadaisical quantum walk finds a unique marked vertex with probability at least a constant.

Our main results are stated as Theorem 5 and Theorem 6 in Section 3. Theorem 5 states that the quantum hitting time of lackadaisical quantum walks and quantum interpolated walks are of the same order. Theorem 6 states that the $\ell_2$-distance between the two quantum states of the lackadaisical quantum walk and the quantum interpolated walk, respectively, remains negligible.
for any number of steps that is in the order of the quantum hitting time. The two theorems hold for any regular locally arc-transitive graph.

In Section 5, we prove Theorem 5 by introducing a variant of lackadaisical quantum walks as an intermediate walk operator and then giving an exact relationship between the quantum hitting times of all three quantum walk operators. In Section 6, we construct isometries and use them to upper bound the $\ell_2$-distance between the resulting states of the intermediate quantum walk operator and the quantum interpolated walk after any number of steps.

By combining the two main Theorems with the analysis of quantum interpolated walks given in [KMOR16] and the analysis of controlled quantum walks given in [DH17], we complete the analytical proofs of the complexity and success probability of lackadaisical walks on regular locally arc-transitive graphs.

The main technical contribution in our work is the use of locally arc-transitivity to establish a connection between lackadaisical quantum walks and quantum interpolated walks. The definition of locally arc-transitivity is given in Section 3. We discuss the relationship between locally arc-transitivity, vertex transitivity and other graph properties in Section 7, and we conclude in Section 8.

Our results are for the case when there is a unique marked vertex. When there are multiple marked vertices, the lackadaisical quantum walk may fail in finding a marked vertex. Nahimovs [Nah19] proves that, on a $\sqrt{N} \times \sqrt{N}$ torus with two marked vertices placed adjacent to each other, the lackadaisical quantum walk has a stationary state that is close to the initial state, which implies that the walk finds a marked vertex with probability no bigger than $O(1/N)$.

2 Two Quantum Walks

The graphs that we apply the quantum walks on, are regular undirected connected graphs with a unique marked vertex. A graph is said to be $d$-regular if every vertex has degree $d$. We will interchangeably consider an undirected graph as a directed graph, where we consider each edge $\{x, y\}$ between two distinct vertices as two arcs $(x, y)$ and $(y, x)$. When introducing self-loops below, we also interchangeably consider each edge $\{x, x\}$ as a single arc $(x, x)$. For each vertex $x$ in turn, fix any ordering of the $d$ neighbors $y_1, y_2, \ldots, y_d$ of $x$. We refer to $y_i$ as the $i$th neighbor of $x$, and the arc $(x, y_i)$ as the $i$th outgoing arc of $x$.

Let $N$ denote the number of vertices, and let $\mathcal{H}_N$ be the Hilbert space spanned by the vertices of the graph. To each vertex we associate a coin register in the Hilbert space $\mathcal{H}_d$ spanned by the basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_d\rangle\}$. The coined quantum walk takes place in the Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_d$, in which the state $|x\rangle|e_i\rangle$ represents the arc from $x$ to its $i$th neighbor $y_i$.

**Definition 1 (Lackadaisical quantum walks [Won15])** Given a $d$-regular graph with a unique marked vertex $m$, by adding a self-loop of weight $\ell$ to every vertex, the coined Hilbert space $\mathcal{H}_{d+1}$ is spanned by $\{|e_1\rangle, |e_2\rangle, \ldots, |e_d\rangle, |\bigcirc\rangle\}$. The lackadaisical quantum walk is defined as

$$A_{\text{lazy}} = W \cdot G.$$  \hspace{1cm} (1)

Here $W$ is the quantum walk operator (without searching) defined as $W = S_{ff} \cdot (I_N \otimes C)$, where operator
\[ C = 2|c)(c| - 1_{d+1} \text{ with} \]
\[ |c\rangle = \frac{1}{\sqrt{d+1}} (|e_1\rangle + |e_2\rangle + \cdots + |e_d\rangle + \sqrt{\ell}|\Diamond\rangle) \tag{2} \]

is the diffusion coin for a weighted graph and \( \mathcal{S}_{\mathcal{H}} \) is the flip-flop shift operator [AKR05] defined as

\[
\mathcal{S}_{\mathcal{H}}: \begin{cases} 
|y, e_i\rangle &\mapsto |x, e_i\rangle \\
|x, \Diamond\rangle &\mapsto |x, \Diamond\rangle,
\end{cases}
\]

where \( y \) is the \( i\)th neighbor of \( x \), and \( x \) is the \( j\)th neighbor of \( y \). A query to the oracle is defined as

\[
G = (1_N - 2|m\rangle\langle m|) \otimes 1_{d+1}, \tag{3}
\]

where \( |m\rangle \) denotes the unique marked vertex.

The lackadaisical quantum walk \( \mathcal{A}_{\text{lazy}} \) begins in the state

\[
|m\text{init}_{\text{lazy}}\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq m} |x\rangle \otimes |c\rangle,
\]

which is a uniform superposition over all unmarked vertices.

Given a graph \( G \), define the random walk \( \mathcal{P} = \mathcal{P}(G) \), where \( \mathcal{P}_{xy} \) is the transition probability from vertex \( x \) to vertex \( y \). If vertex \( y \) is a neighbor of vertex \( x \), then \( \mathcal{P}_{xy} = \frac{1}{\deg(x)} \), where \( \deg(x) \) denotes the degree of \( x \), and otherwise \( \mathcal{P}_{xy} = 0 \). The stationary distribution of \( \mathcal{P} \) is denoted by \( \pi \), and \( \pi_v \) denotes the probability of being in vertex \( v \) in the stationary distribution. The absorbing walk \( \mathcal{P}' \) is obtained from \( \mathcal{P} \) by replacing all outgoing transitions from any marked vertex with self-loops, that is, \( \mathcal{P}'_{xy} = \mathcal{P}_{xy} \) for all unmarked vertices \( x \) and all \( y \), and, for any marked vertex \( m \), \( \mathcal{P}'_{mm} = 1 \) and \( \mathcal{P}'_{my} = 0 \) for \( y \neq m \).

Given \( 0 \leq s \leq 1 \), the interpolated walk \( \mathcal{P}(s) \) is defined as

\[
\mathcal{P}(s) = (1-s)\mathcal{P} + s\mathcal{P}'.
\]

Note that \( \mathcal{P}(s)_{xy} \) is the transition probability from vertex \( x \) to vertex \( y \) in \( \mathcal{P}(s) \).

**Definition 2 (Quantum interpolated walks [KMOR16])** Applying Szegedy’s correspondence [Sze04] on the interpolated walk \( \mathcal{P}(s) \), we construct the quantum interpolated walk

\[
\mathcal{W}(\mathcal{P}(s)) = \text{SWAP} \cdot \text{Ref}(\mathcal{A}).
\]

The operator \( \text{Ref}(\mathcal{A}) \) is a reflection about the subspace \( \mathcal{A} \) spanned by \( \{|x, \mathcal{P}(s)_x\rangle\} \) for all vertices \( x \), where

\[
|\mathcal{P}(s)_x\rangle = \sum_y \sqrt{\mathcal{P}(s)_{xy}} |y\rangle
\]

is a superposition over the neighbors of \( x \). The operator \( \text{SWAP} \) swaps the two registers.

The initial state for the quantum interpolated walk \( \mathcal{W}(\mathcal{P}(s)) \) is

\[
|m\text{init}_{ip}\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq m} |x\rangle \otimes |\mathcal{P}(s)_x\rangle.
\]
Definition 3 (The cotangent quantum hitting time [DH17]) The cotangent quantum hitting time of a quantum walk \( U \) on a state \(|w\rangle\) is

\[
\text{QHT}_{\cot}(U, |w\rangle) = \sqrt{\sum_{\phi_k \neq 1} |\langle \phi_k^\pm |w\rangle|^2 \cot^2 \theta_k / 2}
\]

where \(|\phi_k^\pm\rangle\) are the eigenvectors of \( U \) corresponding to the eigenvalues \( \phi_k^\pm = e^{\pm i\theta_k} \).

3 Main Theorems

Rhodes and Wong [RW20] and the other earlier work consider lackadaisical quantum walks on certain instances and classes of regular vertex-transitive graphs. One common property of these instances and classes of graphs is that they are locally arc-transitive. A graph \( G \) is said to be locally arc-transitive if for any vertex \( u \) with neighbors \( v_1 \) and \( v_2 \), there exists an automorphism \( \sigma \) of \( G \) that maps the arc \((u, v_1)\) to the arc \((u, v_2)\). That is, there is an automorphism that fixes \( u \) while mapping any one of \( u \)'s neighbors to any other of \( u \)'s neighbors.

Any connected locally arc-transitive graph must be biregular, since for any two vertices \( u \) and \( v \) connected by a path of even length, there exists an automorphism that maps \( u \) to \( v \). By the same argument, if the graph contains an odd cycle, then it must be regular. For simplicity, in this paper, we consider only locally arc-transitive graphs that are regular.

We prove that the lackadaisical quantum walk \( A_{\text{lazy}} \) searches regular locally arc-transitive graphs for a unique marked vertex in \( O(\sqrt{\text{HT}}) \) steps with constant success probability.

Theorem 4 Let \( G \) be a \( d \)-regular locally arc-transitive graph with \( N \) vertices and a unique marked vertex \( m \). The lackadaisical quantum walk with selfloop of weight \( \ell = \frac{d}{N} \) can find \( m \) with constant success probability in \( O(\sqrt{\text{HT}(P(G), \{m\})}) \) steps.

Here \( \text{HT}(P(G), \{m\}) \) denotes the hitting time on \( G \) when the unique marked vertex is \( m \). Theorem 4 follows from Theorems 5 and 6. Theorem 5 shows that the quantum hitting time of lackadaisical quantum walks is of the same order as the quantum hitting time of quantum interpolated walks. Theorem 6 then shows that the \( \ell^2 \)-distance between the resulting states of \( A_{\text{lazy}} \) and \( W(P(s)) \) is small for any number of steps in \( O(\sqrt{\text{HT}}) \).

Theorem 5 Consider a \( d \)-regular locally arc-transitive graph with \( N \) vertices and a unique marked vertex \( m \). For the lackadaisical quantum walk \( A_{\text{lazy}} \), add self-loops of weight \( \ell = \frac{d}{N} \) on every vertex. For the quantum interpolated walk \( W(P(s)) \), choose \( s = 1 - \frac{\ell}{d} \). Then

\[
\text{QHT}_{\cot}^2(A_{\text{lazy}}, |\text{init}_{\text{lazy}}\rangle) = \frac{N+1}{N} \text{QHT}_{\cot}^2(W(P(s)), |\text{init}_{\text{ip}}\rangle) + \frac{1}{2N-1},
\]

and

\[
\text{QHT}_{\cot}(W(P(s)), |\text{init}_{\text{ip}}\rangle) \in O\left(\sqrt{\text{HT}(P, \{m\})}\right).
\]

Theorem 6 Set \( T_0 = \left\lceil c \cdot \sqrt{\text{HT}(P, \{m\})} \right\rceil \) for any fixed constant \( c \geq 1 \). For all \( t \leq T_0 \),

\[
\|A_{\text{lazy}}^t|\text{init}_{\text{lazy}}\rangle - W(P(s))^t|\text{init}_{\text{ip}}\rangle\|_2 \in O\left(\frac{1}{N^{1/4}}\right).
\]
Our main results show that the lackadaisical quantum walk $A_{\text{lazy}}$ is closely related to the quantum interpolated walk $W(P(s))$. This relationship permits us to analyze the quantum hitting time and behavior of the lackadaisical quantum walk $A_{\text{lazy}}$ using known results about quantum interpolated walks. It is shown in [KMOR16] that $W(P(s))$ finds a unique marked element in $O(QHT_{\text{cot}}(W(P(s)), [\text{init}]_N))$ steps with constant success probability, where we use the tight bounds on the cotangent quantum hitting time given in Appendix A in [DH17]. This proves the conjectures and numerical findings in [RW20] and the other earlier work on the complexity and success probability of lackadaisical quantum walks on those graphs. We prove Theorem 5 in Sec. 5 and Theorem 6 in Sec. 6.

Throughout the remaining sections, we fix $\ell = \frac{d}{N}$ for the lackadaisical quantum walk and $s = 1 - \frac{\ell}{d}$ for the quantum interpolated walk.

### 4 Technical Preliminaries

Define the lazy random walk on $G$ as
\[
\hat{P} = \frac{d}{d + \ell} \cdot P + \frac{\ell}{d + \ell} \cdot I_N,
\]
(4)

obtained by adding a self-loop of weight $\ell$ to every vertex. The interpolation of a lazy random walk is then denoted
\[
\hat{P}(s) = (1 - s) \cdot \hat{P} + s \cdot \hat{P}',
\]
(5)

where $\hat{P}' = (\hat{P})'$ is the absorbing walk derived from the lazy random walk $\hat{P}$.

We apply Szegedy’s correspondence on $P(s)$ and $\hat{P}(s)$. For convenience, we only show the details on constructing $W(P(s))$. Applying Szegedy’s correspondence on $\hat{P}(s)$ is similar, except we use ‘$\sim$’ when referring to $\hat{P}(s)$. The discriminant [Sze04] of the interpolated walk $P(s)$ is
\[
D(P(s)) = \sqrt{P(s) \circ P(s)^T},
\]
where the Hadamard product “$\circ$” and the square root are taken entry-wise, and the $T$ denotes matrix transposition. We denote the corresponding eigenvalues of $D(P(s))$ by $\lambda_k$, where $k = 1, \ldots, N$. Let $-\pi/2 \leq \theta_k \leq \pi/2$ be angles so that $\lambda_k = \cos \theta_k$.

The interpolated hitting time [KMOR16] of an interpolated walk $P(s)$ is
\[
\text{HT}_{ip}(P(s)) = \sum_{\lambda_k \neq 1} \frac{|\langle \lambda_k | \sqrt{\pi} \rangle|^2}{1 - \lambda_k},
\]
(6)

where $|\lambda_k\rangle$ are the corresponding eigenvectors and $\sqrt{\pi}$ is the uniform distribution over all unmarked vertices.

To analyze the quantum analogue $W(P(s))$ of the interpolated walk $P(s)$, define the isometry
\[
T(s) = \sum_x |x, P(s)_x\rangle \langle x|.
\]

The quantum walk $W(P(s))$ has a unique eigenvector $|\phi_N\rangle = T(s)|\lambda_N\rangle$ with eigenvalue $\phi_N = 1$. The remaining $2(N - 1)$ eigenvalues and eigenvectors are
\[
\phi_k^\pm = e^{\pm i \theta_k}, \quad |\phi_k^\pm\rangle = \frac{T(s)|\lambda_k\rangle \pm i(T(s)|\lambda_k\rangle)^\perp}{\sqrt{2}}
\]
for $k = 1, \ldots, N - 1$. The phases of the eigenvectors can be chosen so that they satisfy that

$$T(s)|\lambda_k\rangle = \frac{1}{\sqrt{2}}(|\phi_k^+\rangle + |\phi_k^-\rangle).$$

We decompose $\sqrt{\bar{\pi}}$ into the basis of $D(P(s))$ for scalars $a_k$,

$$\sqrt{\bar{\pi}} = \sum_{k=1}^{N} a_k |\lambda_k\rangle,$$

and write the initial state as

$$|\text{init}_{\text{ip}}\rangle = T(s)\sqrt{\bar{\pi}} = a_N |\phi_N\rangle + \frac{1}{\sqrt{2}} \sum_{k=1}^{N-1} a_k (|\phi_k^+\rangle + |\phi_k^-\rangle).$$

Applying the quantum walk $W(P(s))$ for $t$ times on $|\text{init}_{\text{ip}}\rangle$, yields the state

$$W(P(s))^t|\text{init}_{\text{ip}}\rangle = a_N |\phi_N\rangle + \frac{1}{\sqrt{2}} \sum_{k=1}^{N-1} a_k ((e^{i\theta_k})^t |\phi_k^+\rangle + (e^{-i\theta_k})^t |\phi_k^-\rangle).$$

### 5 Proof of Theorem 5

To prove Theorem 5, we use a variant of lackadaisical quantum walk as an intermediate quantum walk operator. The lackadaisical quantum walk $A_{\text{lazy}}$ in Definition 1 uses a query to the oracle $G$. We define a query to a different oracle as

$$\hat{G} = I_{d+1} - 2(|m, \odot\rangle\langle m, \odot| + |m, \oplus\rangle\langle m, \oplus|),$$

where

$$|\pm\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle$$

denotes an equally weighted superposition over all the $d$ outgoing arcs of any vertex. Using the query $\hat{G}$, we define the following variant of lackadaisical quantum walks,

$$\hat{A}_{\text{lazy}} = W \cdot \hat{G}.$$

We first show that for any locally arc-transitive graph, we can replace the query $G$ by the modified query $\hat{G}$ without altering the evolution of the walk.

Let $G$ be a $d$-regular locally arc-transitive graph with a unique marked vertex $m$. Write the state of the system after $t$ steps of the walk $A_{\text{lazy}}$ on the initial state $|\text{init}_{\text{lazy}}\rangle$,

$$A_{\text{lazy}}^t|\text{init}_{\text{lazy}}\rangle = \left( \sum_{u \neq m, i} a_{u,i} |u,e_i\rangle \right) + \left( \sum_{i} a_{m,i} |m,e_i\rangle \right) + \left( \sum_{v \in V(G)} a_{v,\odot} |v,\odot\rangle \right),$$

for some amplitudes $a$. We first show that locally arc-transitivity implies that the amplitudes of the outgoing arcs of the marked state $m$ remain equal after any number of iterations.
Lemma 7  For all \( t \geq 0 \), \( \alpha_{m,i} = \alpha_{m,j} \) for all outgoing arcs \((m, y_i)\) and \((m, y_j)\) of the marked vertex \( m \).

Proof  We define an application of an automorphism \( \sigma \) of \( G \) on quantum states and operators acting on \( \mathcal{H}_N \otimes \mathcal{H}_{d+1} \) as follows. For any vertex \( u \) and its \( i \)-th outgoing arc, let \( \sigma(\langle u, e_i \rangle) = \langle u', e_i \rangle \), where \( \sigma(u) = u', v \) is the endpoint of the \( i \)-th arc from \( u \), \( \sigma(v) = v' \), and \( v' \) is the endpoint of the \( j \)-th arc from \( u \). For any vertex \( u \) and its self-loop, let \( \sigma(\langle u, \circ \rangle) = \langle u', \circ \rangle \). Generalize and define the action on the adjoint as \( \sigma(\langle u, e_i \rangle) = \langle u', e_i \rangle \) and \( \sigma(\langle u, \circ \rangle) = \langle u', \circ \rangle \), and extend to operators and the entire Hilbert space by composition and linearity.

For any automorphism \( \sigma \), then \( \sigma(S_U) = S_U \), since \( S_U \) changes the direction of every arc. Similarly, \( \sigma(I_N \otimes C) = (I_N \otimes C) \), since an automorphism preserves the neighborhood of any vertex.

Let \( v_1 \) and \( v_2 \) be the endpoints of any two distinct outgoing arcs of \( m \) (excluding the self-loop). Since \( G \) is locally arc-transitive, there exists an automorphism \( \sigma_m \) that fixes \( m \) and that maps the arc \((m, v_1)\) to \((m, v_2)\). Let \( \Delta_m \) be a set of \( d(d - 1) \) such automorphisms, one for each pair \( v_1, v_2 \) of distinct neighbors of \( m \). For any automorphism \( \sigma_m \in \Delta_m \) we have that \( \sigma_m(G) = G \) and \( \sigma_m(\overline{\text{init}_{\text{lazy}}}^t) = \overline{\text{init}_{\text{lazy}}}^t \).

The proof of the lemma now follows readily by mathematical induction on \( t \). Assume that \( \sigma_m(\overline{A_l}_{\text{lazy}}^t \overline{\text{init}_{\text{lazy}}}^t) = \overline{A_l}_{\text{lazy}}^t \overline{\text{init}_{\text{lazy}}}^t \) for any \( \sigma_m \in \Delta_m \) immediately prior to the \((t + 1)\)-th application of \( \overline{A_l}_{\text{lazy}} \). Then by the above arguments, it holds immediately after the \((t + 1)\)-th application of \( \overline{A_l}_{\text{lazy}} \).

Lemma 8  For all \( t \geq 0 \),

\[
\overline{A_l}_{\text{lazy}}^t \overline{\text{init}_{\text{lazy}}}^t = \overline{A_{\text{lazy}}}_{\text{lazy}}^t \overline{\text{init}_{\text{lazy}}}^t.
\]

Proof  The difference between \( A_{\text{lazy}} \) and \( \overline{A_{\text{lazy}}} \) is that \( A_{\text{lazy}} \) uses a query to the oracle \( G \) in Eq. (3) and \( \overline{A_{\text{lazy}}} \) uses a different query operator \( \overline{G} \) given by Eq. (9). By Lemma 7, whenever we apply \( G \) or \( \overline{G} \), we have a state where the second summand in Eq. (10) can be written in the form \( \alpha_{m,\circ} |m, \circ\rangle + \alpha_{m,\circ} |m, \circ\rangle \). On such states, \( G \) and \( \overline{G} \) act identically. \( \square \)

Consider the two quantum walks \( \overline{A_{\text{lazy}}} \) and \( W(\overline{\mathcal{P}}(s)) \). The search space of \( \overline{A_{\text{lazy}}} \) is the Hilbert space \( \mathcal{H}_N \otimes \mathcal{H}_{d+1} \). The search space of \( W(\overline{\mathcal{P}}(s)) \) is the Hilbert space \( \mathcal{H}_N \otimes \mathcal{H}_N \). By Szegedy’s correspondence, the quantum interpolated walk \( W(\overline{\mathcal{P}}(s)) \) takes place within a smaller subspace \( C^{(d+1)N} \) of the full Hilbert space \( \mathcal{H}_N \otimes \mathcal{H}_N \). We identify the subspace \( C^{(d+1)N} \) with \( \mathcal{H}_N \otimes \mathcal{H}_c \) by defining an isometry \( E: C^{(d+1)N} \to \mathcal{H}_N \otimes \mathcal{H}_{d+1} \) as follows.

For all vertices \( x \) in the \( d \)-regular arc-transitive graph, and all neighbors \( y_i \) of \( x \), let

\[
E|x, y_i\rangle = |x, e_i\rangle
\]

and let

\[
E|x, x\rangle = \begin{cases} 
|x, \circ\rangle & \text{if } x \text{ is unmarked} \\
-|x, \circ\rangle & \text{if } x \text{ is marked}.
\end{cases}
\]

Lemma 9  \( \overline{A_{\text{lazy}}} = E \cdot W(\overline{\mathcal{P}}(s)) \cdot E^\dagger \).
Proof The quantum circuit of the quantum walk $\hat{A}_{\text{lazy}} = W \cdot \hat{G}$ is given in Figure 1.

The two states $|\bigcirc\rangle$ and $|\bigtriangledown\rangle$ are orthogonal, and they span a two-dimensional subspace of the coin space $\mathcal{H}_c$. The coin state $|c\rangle = \sqrt{\frac{d}{d + \ell}} |\bigcirc\rangle + \sqrt{\frac{\ell}{d + \ell}} |\bigtriangledown\rangle$ is in this two-dimensional subspace. Let $|c^\perp\rangle = \sqrt{\frac{\ell}{d + \ell}} |\bigtriangledown\rangle - \sqrt{\frac{d}{d + \ell}} |\bigcirc\rangle$ be the state that is orthogonal to the coin state $|c\rangle$ in this two-dimensional subspace. In Figure 1, we apply three reflections on the coin register if the vertex $x$ is marked, and we apply a single reflection if the vertex is unmarked. In either case, we apply an odd number of reflections on the coin register. We can therefore rewrite the circuit in Figure 1 as the equivalently acting circuit given in Figure 2.

For the lazy random walk $\hat{P}$ with self-loops of weight $\ell$, the neighborhood state of any vertex $x$ on a regular graph is $|\hat{P}x\rangle = \sqrt{\frac{d}{d + \ell}} |P_x\rangle + \sqrt{\frac{\ell}{d + \ell}} |x\rangle$. The interpolation of a random walk changes the neighborhood state of any marked vertex. With our choice of $s = 1 - \frac{\ell}{d}$, they become

$$
|\hat{P}(s)\rangle_u = \sqrt{\frac{d}{d + \ell}} |P_u\rangle + \sqrt{\frac{\ell}{d + \ell}} |u\rangle \\
|\hat{P}(s)\rangle_m = \sqrt{\frac{d}{d + \ell}} |P_m\rangle + \sqrt{\frac{\ell}{d + \ell}} |m\rangle.
$$

Here $u$ denotes any unmarked vertex, and $m$ denotes the unique marked vertex. Applying the isometry $E$ yields that

$$
E|u, \hat{P}(s)\rangle_u = |u, c\rangle \\
E|m, \hat{P}(s)\rangle_m = |m, c^\perp\rangle. \tag{11}
$$

By definition, the $S_{ff}$ operator is equivalent to the SWAP operator under the isometry,

$$
S_{ff} = E \cdot \text{SWAP} \cdot E^\dagger. \tag{12}
$$

Eqs. 11 and 12 permit us to write the coined quantum walk circuit in Figure 2 as a circuit of the quantum interpolated walk $E \cdot W(\hat{P}(s)) \cdot E^\dagger$, as in Figure 3. \qed
now follows from Lemmas 8 and 9 and the definition of the interpolated hitting time. In this work, we pick the value of $\ell$ to be equal to $\frac{d}{N}$, so that the correspondence with [RW20] in Lemma 9 is exact. The value of $s$ used in [KMOR16], and in the simulation in Section 7 in [DH17], is $1 - \frac{d}{2N}$, which corresponds to a value of $\ell$ equal to $\frac{d}{N}$. By Eqs. (192) and (21) in [KMOR16], our slightly different choice of $\ell$ implies that the $\text{HT}_{ip}(P(s))$ used in our paper is a factor of order $\frac{1}{N}$ larger than the $\text{HT}_{ip}(P(s))$ used in [KMOR16]. This negligible factor does not change the results stated in this paper.

By Lemma 8, $\text{QHT}_{\cot}(A_{\text{lazy}}, \text{init}_{\text{lazy}}) = \text{QHT}_{\cot}(\hat{A}_{\text{lazy}}, \text{init}_{\text{lazy}})$. By Lemma 9 and the definition of the isometry $E$, $\text{QHT}_{\cot}(A_{\text{lazy}}, \text{init}_{\text{lazy}}) = \text{QHT}_{\cot}(W(\hat{P}(s)), \text{init}_{ip})$. We next show the exact relationship between $\text{QHT}_{\cot}(W(\hat{P}(s)), \text{init}_{ip})$ and $\text{QHT}_{\cot}(W(P(s)), \text{init}_{ip})$.

**Lemma 10** For $\ell = \frac{d}{N}$ and $s = 1 - \frac{d}{2} = 1 - \frac{1}{N}$,

$$\text{QHT}_{\cot}^2(W(\hat{P}(s)), \text{init}_{ip}) = \frac{N+1}{N} \text{QHT}_{\cot}^2(W(P(s)), \text{init}_{ip}) + \frac{1}{2N-1}.$$ 

**Proof** By definitions of $P(s)$ and $\hat{P}(s)$, given by Eqs. 4 and 5, respectively,

$$\hat{P}(s) = \frac{N}{N+1} \cdot P(s) + \frac{1}{N+1} \cdot I_N.$$ 

Hence $\hat{P}(s)$ and $P(s)$ have the same eigenvectors $|\hat{\lambda}_k\rangle = |\lambda_k\rangle$ and corresponding eigenvalues

$$\hat{\lambda}_k = \frac{N}{N+1} \lambda_k + \frac{1}{N+1}. \quad (13)$$

Then by the definition of the interpolated hitting time in Eq. (6),

$$\text{HT}_{ip}(\hat{P}(s)) = \frac{N+1}{N} \cdot \text{HT}_{ip}(P(s)). \quad (14)$$

Given an interpolated walk $P(s)$ and its Szegedy’s correspondence $W(P(s))$, we have

$$\text{QHT}_{\cot}^2(W(P(s)), \text{init}_{ip}) = 2\text{HT}_{ip}(P(s)) - \frac{p_M}{1 - s(1 - p_M)}, \quad (15)$$

by direct calculation using Definition 3 and Eq. (6). Here $p_M$ is the probability of drawing a marked vertex from the stationary distribution $\pi$. Since our graph is regular and there is a unique marked vertex, $p_M = \pi_m = \frac{1}{N}$. Lemma 10 follows from plugging Eq. (15) into Eq. (14) on both sides.

Theorem 5 now follows from Lemmas 8, 9 and 10. The fact that $\text{QHT}_{\cot}(W(P(s)), \text{init}_{ip})$ is in $O(\sqrt{\text{HT}(P, \{m\})})$ follows from [KMOR16] and [DH17].
6 Proof of Theorem 6

Lemmas 8 and 9 in the previous section give exact relationships between the lackadaisical quantum walk operator $A_{\text{lazy}}$ and the two intermediate walk operators $\hat{A}_{\text{lazy}}$ and $W(\hat{P}(s))$. In this section, we then show that the distance between the intermediate walk operator $W(\hat{P}(s))$ and the quantum interpolated walk operator $W(P(s))$ is bounded, when applied on their respective initial states. The bound is given in Lemma 15 below, and it follows from Lemmas 11 and 14. Theorem 6 follows from Lemmas 9 and 15.

By Szegedy’s correspondence, the eigenvectors $|\phi_k^\pm\rangle$ of $W(P(s))$ are in the subspace
\[
\text{span}\{T(s)|\lambda_k\rangle, \text{SWAP} \cdot T(s)|\lambda_k\rangle\} = \text{span}\{(T(s)|\lambda_k\rangle, (T(s)|\lambda_k\rangle)^\perp\},
\]
and the eigenvectors $|\hat{\phi}_k^\pm\rangle$ of $W(\hat{P}(s))$ are in the subspace
\[
\text{span}\{\hat{T}(s)|\lambda_k\rangle, \text{SWAP} \cdot \hat{T}(s)|\lambda_k\rangle\} = \text{span}\{\hat{T}(s)|\lambda_k\rangle, (\hat{T}(s)|\lambda_k\rangle)^\perp\}.
\]

We define an isometry
\[
R_1 = \sum_x (|x, \hat{P}(s)_x\rangle\langle x, P(s)_x| + |x, \hat{P}(s)_x\rangle\langle x, P(s)_x|),
\]
where $|x, P(s)_x\rangle$ is orthogonal to $|x, P(s)_x\rangle$ in the subspace spanned by $\{|x, P(s)_x\rangle, |P(s)_x, x\rangle\}$ and $|x, \hat{P}(s)_x\rangle$ is orthogonal to $|x, \hat{P}(s)_x\rangle$ in the subspace spanned by $\{|x, \hat{P}(s)_x\rangle, |\hat{P}(s)_x, x\rangle\}$. The isometry $R_1$ satisfies that
\[
R_1: \begin{aligned}
|\phi_k^+\rangle &\mapsto |\hat{\phi}_k^+\rangle \\
|\phi_k^-\rangle &\mapsto |\hat{\phi}_k^-\rangle \\
|\phi_N\rangle &\mapsto |\hat{\phi}_N\rangle.
\end{aligned}
\]

By Eq. (8), applying $R_1$ on the state $W(P(s))^t\langle \text{init}_{ip}\rangle$ changes from the eigenspace of $W(P(s))$ to the eigenspace of $W(\hat{P}(s))$,
\[
R_1 \cdot W(P(s))^t\langle \text{init}_{ip}\rangle = \alpha_N|\phi_N\rangle + \frac{1}{\sqrt{2}} \sum_{k=1}^{N-1} \alpha_k ((e^{i\theta_k})^t|\hat{\phi}_k^+\rangle + (e^{-i\theta_k})^t|\hat{\phi}_k^-\rangle).
\]

In the proof of Lemma 11 below, we require a second isometry, which is also a projection,
\[
R_2 = \sum_x (|x, P(s)_x\rangle\langle x, P(s)_x| + |x, P(s)_x\rangle\langle x, P(s)_x|).
\]

Applying $R_2$ on $W(P(s))^t\langle \text{init}_{ip}\rangle$ does not change the state itself,
\[
R_2 \cdot W(P(s))^t\langle \text{init}_{ip}\rangle = W(P(s))^t\langle \text{init}_{ip}\rangle.
\]

Note that since the states $|P(s)_x\rangle$ and $|\hat{P}(s)_x\rangle$ are close for all vertices $x$, the $\ell_2$-distance between $R_1$ and $R_2$ is small, which is $O\left(\frac{1}{\sqrt{N}}\right)$ by direct calculation.

Lemma 11 For all $t \geq 0$,
\[
\|R_1 \cdot W(P(s))^t \cdot \langle \text{init}_{ip}\rangle - W(P(s))^t \cdot \langle \text{init}_{ip}\rangle\|_2 \in O\left(\frac{1}{\sqrt{N}}\right).
\]
Proof

\[ \| R_1 \cdot W(P(s))^t \cdot [\text{init}_\text{ip}] - W(P(s))^t \cdot [\text{init}_\text{ip}] \|_2 = \| R_1 \cdot W(P(s))^t \cdot [\text{init}_\text{ip}] - R_2 \cdot W(P(s))^t \cdot [\text{init}_\text{ip}] \|_2 \leq \| R_1 - R_2 \|_2 \cdot \| W(P(s))^t \|_2 = \| R_1 - R_2 \|_2 \in O(\frac{1}{\sqrt{N}}). \]

Next consider the following $\ell_2$-distance

\[ \| R_1 \cdot W(P(s))^t \cdot [\text{init}_\text{ip}] - W(\check{P}(s))^t \cdot [\text{init}_\text{ip}] \|_2 = \left\| \frac{1}{\sqrt{2}} \sum_{k=1}^{N-1} \alpha_k \left( (e^{it\theta_k} - e^{it\hat{\theta}_k})|\tilde{\phi}_k^+\rangle + (e^{-it\theta_k} - e^{-it\hat{\theta}_k})|\tilde{\phi}_k^-\rangle \right) \right\|_2. \]

We pick a threshold angle $\theta_0$ satisfying that $0 < \theta_0 \leq \frac{\pi}{2}$ and separate the sum into two parts, where the first part is for angles $0 < \theta_k \leq \theta_0$ and the second part is for $\theta_0 < \theta_k \leq \pi$. We give an upper bound on the $\ell_2$-norm for each of these two parts in Facts 12 and 13, respectively.

**Fact 12** For $0 < \theta_0 \leq \frac{\pi}{2}$ and all $t \geq 0$,

\[ \left\| \frac{1}{\sqrt{2}} \sum_{0 < \theta_k \leq \theta_0} \alpha_k \left( (e^{it\theta_k} - e^{it\hat{\theta}_k})|\tilde{\phi}_k^+\rangle + (e^{-it\theta_k} - e^{-it\hat{\theta}_k})|\tilde{\phi}_k^-\rangle \right) \right\|_2 \leq \frac{8t}{N-1} \sin\left(\frac{\theta_0}{2}\right). \]

Proof

\[ \left\| \frac{1}{\sqrt{2}} \sum_{0 < \theta_k \leq \theta_0} \alpha_k \left( (e^{it\theta_k} - e^{it\hat{\theta}_k})|\tilde{\phi}_k^+\rangle + (e^{-it\theta_k} - e^{-it\hat{\theta}_k})|\tilde{\phi}_k^-\rangle \right) \right\|_2 = \frac{1}{\sqrt{2}} \sqrt{\sum_{0 < \theta_k \leq \theta_0} |\alpha_k|^2 \left( |e^{it\theta_k} - e^{it\hat{\theta}_k}|^2 + |e^{-it\theta_k} - e^{-it\hat{\theta}_k}|^2 \right)} \]

\[ \leq \max_{\theta_k \leq \theta_0} |e^{it\theta_k} - e^{it\hat{\theta}_k}| \sqrt{\sum_{0 < \theta_k \leq \theta_0} |\alpha_k|^2} \]

\[ \leq \max_{\theta_k \leq \theta_0} |e^{it\theta_k} - e^{it\hat{\theta}_k}| \]

\[ = \max_{\theta_k \leq \theta_0} \left| 2 \sin\left( t \cdot \frac{\theta_k - \hat{\theta}_k}{2} \right) \right| \]

\[ = \max_{\theta_k \leq \theta_0} \left| 2 \sin\left( \frac{2t}{N+1} \cdot \frac{\theta_k - \hat{\theta}_k}{2} \right) \right| \]

\[ \leq \frac{8t}{N-1} \sin\left( \frac{\theta_0}{2} \right). \]

In the last inequality, by Eq. (13), since $0 < \theta_k \leq \theta_0 \leq \frac{\pi}{2}$, then $(1 - \frac{2}{N+1})\theta_k \leq \hat{\theta}_k \leq \theta_k$, which implies that $0 \leq \frac{N+1}{2} \cdot \frac{\theta_k - \hat{\theta}_k}{2} \leq \frac{\theta_0}{2}$. Finally $\sin(ax) \leq 2a \sin(x)$ for all $0 \leq x \leq \pi/4$ and all $a \geq 0$. □
Fact 13 For $0 < \theta_0 \leq \frac{\pi}{2}$ and all $t \geq 0$,

$$
\left\| \frac{1}{\sqrt{2}} \sum_{\theta_0 < \theta_k \leq \pi} \alpha_k \left( (e^{i\theta_k} - e^{\hat{i}\theta_k})|\hat{\phi}_k^+ \rangle + (e^{-i\theta_k} - e^{-i\hat{i}\theta_k})|\hat{\phi}_k^- \rangle \right) \right\|_2 \leq \frac{2}{\sqrt{(1 - \cos \theta_0)(N - 1)}}.
$$

Proof First simplify by bounding each factor $(e^{i\theta_k} - e^{-i\hat{i}\theta_k})$ by its trivial upper bound of 2.

$$
\left\| \frac{1}{\sqrt{2}} \sum_{\theta_0 < \theta_k \leq \pi} \alpha_k \left( (e^{i\theta_k} - e^{\hat{i}\theta_k})|\hat{\phi}_k^+ \rangle + (e^{-i\theta_k} - e^{-i\hat{i}\theta_k})|\hat{\phi}_k^- \rangle \right) \right\|_2 \leq \max_{\theta_0 < \theta_k} |e^{i\theta_k} - e^{\hat{i}\theta_k}| = 2 \sqrt{\sum_{\theta_0 < \theta_k \leq \pi} |\alpha_k|^2} \leq 2 \sqrt{\sum_{\theta_0 < \theta_k \leq \pi} |\alpha_k|^2}.
$$

We next upper bound the sum of the scalars $\alpha_k^2$ for the large angles $\theta_0 < \theta_k \leq \pi$ by $\frac{1}{(1 - \cos \theta_0)(N - 1)}$. For this, consider the quantum walk $W(P(s))$,

$$
\langle \text{init}_{ip} | W(P(s)) | \text{init}_{ip} \rangle = \sqrt{\pi}^t \cdot D(P(s)) \cdot \sqrt{\pi} = 1 - \frac{1}{N - 1}.
$$

By Eq. (7), we have $\sqrt{\pi}^t \cdot D(P(s)) \cdot \sqrt{\pi} = \sum_{k=1}^{N} \lambda_k \alpha_k^2$. Using that $\sum_{k=1}^{N} \alpha_k^2 = 1$, we infer that

$$
1 - \sqrt{\pi}^t \cdot D(P(s)) \cdot \sqrt{\pi} = \sum_{k=1}^{N} \lambda_k \alpha_k^2 = \sum_{k=1}^{N} (1 - \lambda_k) \alpha_k^2
$$

$$
= \sum_{0 < \theta_k < \theta_0} (1 - \lambda_k) \alpha_k^2 + \sum_{\theta_0 < \theta_k \leq \pi} (1 - \lambda_k) \alpha_k^2 = \frac{1}{N - 1}.
$$

Since $-1 \leq \lambda_k \leq 1$ for all $k$, we have $\sum_{0 < \theta_k < \theta_0} (1 - \lambda_k) \alpha_k^2 \geq 0$. For $\theta_0 < \theta_k \leq \pi$, i.e. $\lambda_k < \cos \theta_0$, we conclude that

$$
\sum_{\theta_0 < \theta_k \leq \pi} \alpha_k^2 \leq \frac{1}{1 - \cos \theta_0} \sum_{\theta_0 < \theta_k \leq \pi} (1 - \lambda_k) \alpha_k^2 \leq \frac{1}{(1 - \cos \theta_0)(N - 1)}.
$$

Lemma 14 Fix a constant $c \geq 1$, then for all $t \leq c \sqrt{HT(P, \{m\})}$,

$$
\| R_1 \cdot W(P(s))^t \cdot |\text{init}_{ip}\rangle - W(\hat{P}(s))^t \cdot |\text{init}_{ip}\rangle \|_2 \in O\left( \frac{1}{N^{1/4}} \right).
$$

Proof Choose the threshold angle $\theta_0$ such that $\cos \theta_0 = 1 - 2 \sqrt{\frac{N - 1}{2t \cdot HT(P, \{m\})}}$. Since the hitting time $HT(P, \{m\})$ for a connected regular graph is at least $N - 1$, the threshold angle is well-defined and satisfies that $0 < \theta_0 \leq \pi/2$, and thus $0 \leq \cos \theta_0 < 1$.  

13
Apply the triangle inequality on Facts 12 and 13, and substitute \( \sin(\frac{\theta_0}{2}) = \left(\frac{N-1}{16\text{HT}(P, \{m\})}\right)^{1/4} \).

\[
\| R_1 \cdot W(P(s))^t \cdot |\text{init}_ip\rangle - W(\hat{P}(s))^t \cdot |\text{init}_ip\rangle \|_2 \\
= \left\| \frac{1}{\sqrt{2}} \sum_{0<\theta_k \leq \pi} \alpha_k \left( (e^{i\theta_k} - e^{i\hat{\theta}_k}) |\phi_k^+\rangle + (e^{-i\theta_k} - e^{-i\hat{\theta}_k}) |\phi_k^-\rangle \right) \right\|_2 \\
\leq \frac{8t}{N-1} \sin\left(\frac{\theta_0}{2}\right) + \frac{2}{\sqrt{(1-\cos\theta_0)(N-1)}} \\
\leq \frac{8t}{N-1} \left( \frac{N-1}{16\text{HT}(P, \{m\})} \right)^{1/4} + \frac{\sqrt{2}}{\sqrt{N-1}} \left( \frac{16\text{HT}(P, \{m\})}{N-1} \right)^{1/4} \\
\leq \frac{1}{(N-1)^{3/4}} \left( \frac{8t}{(16\text{HT}(P, \{m\}))^{1/4}} + \sqrt{2}(16\text{HT}(P, \{m\}))^{1/4} \right) \\
\leq \text{HT}(P, \{m\})^{1/4} \left( \frac{4c + 2\sqrt{2}}{4(N-1)^{3/4}} \right) \\
\leq \frac{2^{1/4}}{(N-1)^{3/4}} \sqrt{N} \left( 4c + 2\sqrt{2} \right) \\
\leq 9c \sqrt{\frac{N}{(N-1)^{3/4}}} \\
\leq O\left( \frac{1}{N^{1/4}} \right).
\]

In the second last inequality, we apply the upper bound on the hitting time of random walks on regular graphs given in [Fei96], which shows that \( \text{HT}(P, \{m\}) \leq 2N^2 \).

\[\square\]

Lemma 15 Fix a constant \( c \geq 1 \), then for all \( t \leq c\sqrt{\text{HT}(P, \{m\})} \),

\[
\| W(P(s))^t \cdot |\text{init}_ip\rangle - W(\hat{P}(s))^t \cdot |\text{init}_ip\rangle \|_2 \in O\left( \frac{1}{N^{1/4}} \right).
\]

Proof Applying the triangle inequality on Lemmas 11 and 14.

Theorem 6 follows from Lemmas 9 and 15.

7 On Locally Arc-Transitivity

The main property that we have used in our proofs is locally arc-transitivity. The graphs considered in [Won15, Won18b, WZWY17, GK19, RW19, RW20] are all locally arc-transitive, as well as vertex-transitive. This implies that these graphs are also symmetric, as we now show.

A graph is symmetric if for any two arcs \((u_1, v_1)\) and \((u_2, v_2)\), there exists an automorphism that maps \((u_1, v_1)\) to \((u_2, v_2)\). As discussed in Sec. 3, a graph \( G \) is locally arc-transitive if for any vertex \( u \) with neighbors \( v_1 \) and \( v_2 \), there exists an automorphism of \( G \) that maps the arc \((u, v_1)\) to the arc \((u, v_2)\). Let us say that a graph \( G \) is locally arc-transitive at vertex \( u \) if for any two neighbors \( v_1 \) and
of $u$, there exists an automorphism of $G$ that maps the arc $(u, v_1)$ to the arc $(u, v_2)$. A locally arc-transitive graph is then, by definition, a graph that is locally arc-transitive at every vertex.

We now show that a graph $G$ is symmetric if and only if it is locally arc-transitive at some vertex $u$ and vertex-transitive. Trivially, if a graph is symmetric, it satisfies the latter conditions. Now, consider the converse. Let $(x_1, y_1)$ and $(x_2, y_2)$ be two arcs in $G$. Using vertex-transitivity, let $\sigma_1$ be an automorphism that maps $x_1$ to $u$, and let $\sigma_2$ be an automorphism that maps $x_2$ to $u$. Using locally arc-transitivity at $u$, let $\sigma_3$ be an automorphism that maps $u$ to $u$, and that maps $\sigma_1(y_1)$ to $\sigma_2(y_2)$. Then $\sigma_2^{-1}\sigma_3\sigma_1$ maps $(x_1, y_1)$ to $(x_2, y_2)$, and thus $G$ is symmetric.

Locally arc-transitivity and vertex-transitivity are two distinct graph properties, even when restricting to regular graphs. There are regular graphs that are locally arc-transitive but not vertex-transitive, such as the Folkman graph [Fol67]. Conversely, there are regular graphs that are vertex-transitive but not locally arc-transitive, such as the Möbius ladder on 8 vertices, or the Cartesian product of a 3-cycle and a 2-path.

There are two obvious cases of graph properties that are not included in our main theorems. The first case is graphs that are locally arc-transitive but not regular, which includes the bipartite graphs considered in [RW19]. In this case, if we equip each vertex with a self-loop of weight proportional to its degree, the equivalence to quantum interpolated walks still holds, though the corresponding interpolated walk $P(s)$ may use a value of $s$ different from the value $s = 1 - \frac{\pi_m}{1+\pi_m}$ picked in a quantum interpolated walk, depending on the proportionality factor.

The second case is graphs that are regular and vertex-transitive, but not locally arc-transitive. In this case, our theorems do not apply, but a partial analytical answer may be obtained by bounding the variation between the amplitudes of the neighbors of the marked vertex after each iteration of the walk.

8 Conclusion

We analytically prove that lackadaisical quantum walks can find a unique marked vertex on any regular locally arc-transitive graph with constant success probability in $O(\sqrt{HT})$ steps. In our proof, we establish and use relationships between lackadaisical quantum walks and quantum interpolated walks for any regular locally arc-transitive graph. We also prove that self-loops of weight $\ell$ correspond to an interpolation parameter of $s = 1 - \frac{\ell}{d}$.

Our results prove several speculations and numerical findings in previous work, including the conjectures that lackadaisical quantum walks can find a unique marked vertex with constant success probability on the torus [Won18b, WZY17], the cycle [GK19], Paley graphs, some Latin square graphs, Johnson graphs, and the hypercube [RW20].

Acknowledgments

The authors are grateful to Dante Bencivenga, Xining Chen, Janet Leahy and Shang Li for discussions. We are grateful to Mason Rhodes and Thomas Wong for discussions on the results in their work [RW20], and to an anonymous referee for helpful comments. This work was supported in part by the Alberta Graduate Excellence Scholarship program (AGES), the Natural Sciences and Engineering Research Council of Canada (NSERC), and the University of Calgary’s Program for Undergraduate Research Experience (PURE).
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