Existence and Asymptotic Behavior of Large Axisymmetric Solutions for Steady Navier–Stokes System in a Pipe

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Abstract

In this paper, the existence and uniqueness of strong axisymmetric solutions with large flux for the steady Navier–Stokes system in a pipe are established even when the external force is also suitably large in $L^2$. Furthermore, the exponential convergence rate at far fields for the arbitrary steady solutions with finite $H^2$ distance to the Hagen–Poiseuille flows is established as long as the external forces converge exponentially at far fields. The results can be regarded as a key step toward a Liouville-type theorem for steady solutions of Navier–Stokes system in a pipe and global existence for Leray’s problem on steady solutions of Navier–Stokes system in general infinitely long nozzles. The key point to get the existence of these large solutions is the refined estimate for the derivatives in the axial direction of the stream function and the swirl velocity, which exploits the good effect of the convection term. An important observation for the asymptotic behavior of general solutions is that the solutions are actually small at far fields when they have finite $H^2$ distance to the Hagen–Poiseuille flows. This makes the estimate for the linearized problem play a crucial role in studying the convergence of general solutions at far fields.

1. Introduction and Main Results

An important physical problem in fluid mechanics is to study the flows in nozzles. Given an infinitely long nozzle $\tilde{\Omega}$, a natural problem is to investigate the well-posedness theory for the steady Navier–Stokes system

$$\begin{cases} (u \cdot \nabla)u - \Delta u + \nabla p = F \text{ in } \tilde{\Omega}, \\ \text{div } u = 0 \text{ in } \tilde{\Omega}, \end{cases}$$

supplemented with the no slip conditions, i.e.,

$$u = 0 \text{ on } \partial\tilde{\Omega},$$

(1)
where \( \mathbf{u} = (u^x, u^y, u^z) \) denotes the velocity field of the flows and \( \mathbf{F} \) is the external force. If \( \mathbf{F} = 0 \) and \( \Omega \) is a straight cylinder of the form \( \Sigma \times \mathbb{R} \) where \( \Sigma \) is a smooth two-dimensional domain, then there exists a solution \( \mathbf{u} = (0, 0, u^z(x, y)) \) satisfying (1) and (2). The solution is called the Poiseuille flow and is uniquely determined by the flow flux \( \Phi \) defined by

\[
\Phi = \int_{\Sigma} u^z(x, y) \, dS. \tag{3}
\]

In particular, if the straight cylinder is a circular pipe, i.e., \( \Sigma \) is the unit disk \( B_1(0) \), then the associated Poiseuille flows \( \bar{\mathbf{U}} = \bar{\mathbf{U}}(r)\mathbf{e}_z \) have the explicit form as follows:

\[
\bar{\mathbf{U}}(r) = \frac{2\Phi}{\pi} (1 - r^2) \quad \text{with} \quad r = \sqrt{x^2 + y^2}. \tag{4}
\]

These are also called Hagen–Poiseuille flows.

Given an infinitely long nozzle \( \tilde{\Omega} \) tending to a straight cylinder \( \Omega \) at far fields, the problem on the well-posedness theory for (1)–(2) together with the condition that the velocity field converges to the Posieuille flows in \( \Omega \) is called Leray’s problem nowadays, and it was first addressed by Leray ([17]) in 1933. The first significant contribution to the solvability of Leray’s problem is due to Amick [4,5], who reduced the proof of existence to the resolution of a well-known variational problem related to the stability of the Poiseuille flow in a straight cylinder. However, Amick left out the investigation of uniqueness and existence of the solutions with large flux. A rich and detailed analysis of the problem is due to Ladyzhenskaya and Solonnikov [16]. However, the asymptotic far field behavior of the solutions obtained in [16] is not very clear. Therefore, in order to get a complete resolution for Leray’s problem, the key issue is to study the asymptotic behavior for the solutions of the steady Navier–Stokes equations in infinitely long nozzles. The asymptotic behavior for steady Navier–Stokes system in a nozzle was studied extensively in the literature; see [3,13,14,16,24] and references therein. A classical and straightforward way to prove the asymptotic behavior for steady solutions of Navier–Stokes system is to derive a differential inequality for the localized energy [13]. This approach was later refined in [3,14,16,24] and the book by Galdi [8], etc. The asymptotic behavior obtained via this method is also only for solutions with small fluxes. A significant open problem posed in [8, p. 19] is the global well-posedness for Leray’s problem in a general nozzle when the flux \( \Phi \) is large.

Another approach to prove the convergence to Poiseuille flows of steady solutions for Navier–Stokes system is the blowup technique. In fact, with the aid of the compactness obtained in [16], global well-posedness for the Leray’s problem in a general nozzle tending to a straight cylinder could be established even when the flux \( \Phi \) is large, provided that we can prove global uniqueness or some Liouville-type theorem for Poiseuille flows in the straight cylinder. In order to study the global uniqueness of Poiseuille flows in a straight cylinder, an important step is to prove the uniqueness in a bounded set in a suitable metric space. The uniqueness of Hagen–Poiseuille flows in a uniformly small neighborhood (independent of the size of the flux) was obtained in [27]. More precisely, suppose that \( \Omega = B_1(0) \times \mathbb{R} \)
is a pipe and \( \mathbf{F} = (F^x, F^y, F^z) \) is external force, does the problem

\[
\begin{aligned}
(u \cdot \nabla)u - \Delta u + \nabla p &= \mathbf{F}, \quad \text{in } \Omega, \\
\text{div } u &= 0, \quad \text{in } \Omega,
\end{aligned}
\]

supplemented with no-slip boundary condition

\[
u = 0 \quad \text{on } \partial \Omega
\]

and the flux constraint

\[
\int_{B_1(0)} u^z(x, y, z) \, dx \, dy = \Phi
\]

have a unique solution in the neighborhood of the Hagen-Poiseuille flows when the external force is small? The uniform nonlinear structural stability of Hagen–Poiseuille flows was established in [27] in the axisymmetric setting. It was proved in [27] that the problem (5)–(7) has a unique axisymmetric solution \( u \) satisfying

\[
\| u - \bar{U} \|_{H^5(\Omega)} \leq C \| F \|_{L^2(\Omega)}
\]

and

\[
\| u - \bar{U} \|_{H^2(\Omega)} \leq C (1 + \Phi^{1/4}) \| F \|_{L^2(\Omega)}
\]

when the \( L^2 \)–norm of \( F \) is smaller than a uniform constant independent of the flux \( \Phi \).

The main goal of this paper contains two parts. The first one is to show the existence and uniqueness of strong solutions for the problem (5)–(7), where \( \| F \|_{L^2(\Omega)} \) can be larger than any fixed constant as long as the flux of the flow is sufficiently large. The second goal in this paper is to investigate the convergence rate of steady solutions of Navier–Stokes system in a pipe which have finite \( H^2 \) distance to the Hagen–Poiseuille flows, even when the flows have large fluxes.

Our first main result is stated as follows:

**Theorem 1.1.** Assume that \( \mathbf{F} \in L^2(\Omega) \) and \( \mathbf{F} = \mathbf{F}(r, z) \) is axisymmetric. There exists a constant \( \Phi_0 \geq 1 \), such that if the flux \( \Phi \) and the force \( \mathbf{F} \) satisfy

\[
\Phi \geq \Phi_0 \quad \text{and} \quad \| F \|_{L^2(\Omega)} \leq \Phi^{1/8},
\]

then the problem (5)–(7) admits a unique axisymmetric solution \( u \) satisfying

\[
\| u - \bar{U} \|_{H^2(\Omega)} \leq C_0 \Phi^{4/32} \quad \text{and} \quad \| u' \|_{L^2(\Omega)} \leq \Phi^{-15/32},
\]

where \( C_0 \) is a constant independent of \( \Phi \) and \( \mathbf{F} \). Moreover, the solution \( u \) satisfies that

\[
\| u - \bar{U} \|_{H^2(\Omega)} \leq C \Phi^{7/32}.
\]

We have the following remarks on Theorem 1.1:
Remark 1.1. Theorem 1.1 is equivalent to that for any given \( F \in L^2(\Omega) \) whose \( L^2 \)-norm could be arbitrarily large, if \( \Phi \geq \max\{ \Phi_0, \| F \|^{\frac{96}{L^2}(\Omega)} \} \), then there exists a unique axisymmetric solution of the problem (5)–(7). Furthermore, the solution \( u \) satisfies the estimate (11)–(12).

Remark 1.2. If \( \Phi \) is sufficiently large, \( F = 0 \) and \( \| u - \bar{U} \|_{H^\frac{19}{12}(\Omega)} \leq C_0 \Phi^\frac{1}{96} \), then \( u \equiv \bar{U} \). It means that \( \bar{U} \) is the unique solution in a bounded set with large radius \( C_0 \Phi^\frac{1}{96} \). This can be regarded as a step to get Liouville-type theorem for steady Navier–Stokes system in a pipe.

Remark 1.3. For the viscous incompressible flows, the usual physical parameter is Reynolds number \( Re \) defined by \( Re = \frac{VL}{\nu} \), where \( V \) is the speed of the mean flow, \( L \) is the characteristic length, and \( \nu \) is the viscosity of the flow. When the nozzle and the specific material of the fluid are fixed, the characteristic length and the viscosity are fixed correspondingly. Hence, the Reynolds number is proportional to speed of mean flow and thus the flux of the flows in nozzles. In this paper, we use the flux \( \Phi \) as a major parameter for the flow because this is the classical and natural formulation for the Leray’s problem for steady Navier–Stokes flows in an infinitely long nozzle (cf. [8]).

Remark 1.4. The bound \( \Phi^\frac{1}{96} \) appeared in (10) and the space \( H^\frac{19}{12}(\Omega) \) appeared in (11) are not in their optimal forms. We tried to avoid heavy technical estimates and instead catch the phenomenon that \( \| F \|_{L^2(\Omega)} \) could be large as \( \Phi \) is large. The space \( H^\frac{19}{12}(\Omega) \) is mainly used to give \( L^\infty(\Omega) \)-bound of the functions in three dimensional domains via the Sobolev embedding theorem.

In case that \( F \) has additional structure at far fields, we have the following asymptotic behavior for solutions of Navier–Stokes system in a pipe:

**Theorem 1.2.** Assume that \( F \in H^1(\Omega) \) and \( F = F(r, z) \) is axisymmetric. There exists a constant \( \alpha_0 \) depending only on \( \Phi \), such that if \( F = F(r, z) \) satisfies

\[
\| e^{\alpha |z|} F \|_{L^2(\Omega)} < +\infty
\]

with some \( \alpha \in (0, \alpha_0) \), and \( u \) is an axisymmetric solution to the problem (5)–(7), satisfying

\[
\| u - \bar{U} \|_{H^2(\Omega)} < +\infty,
\]

then one has

\[
\| e^{\alpha z} (u - \bar{U}) \|_{H^2(\Omega \cap \{z \geq 0\})} + \| e^{-\alpha z} (u - \bar{U}) \|_{H^2(\Omega \cap \{z \leq 0\})} < +\infty.
\]

There are a few remarks in order.

**Remark 1.5.** The key point of Theorem 1.2 is that there is neither smallness assumption on the flux \( \Phi \) nor the smallness on the deviation of \( u \) with \( \bar{U} \).
Remark 1.6. It follows from (9) and (12) that the solutions obtained in [27] and Theorem 1.1 satisfy the condition (14). Hence, if $F$ in [27] and Theorem 1.1 also satisfies (13), then the associated solutions must converge to Hagen–Poiseuille flows exponentially fast.

Remark 1.7. If $F$ decays to zero with an algebraic rate, i.e., $F$ satisfies

$$\|z^k F\|_{L^2(\Omega)} < +\infty,$$

with some $k \in \mathbb{N}$, then under the same idea of the proof for Theorem 1.2 yields that the axisymmetric solution $u$ of the problem (5)–(7) converges to the Hagen–Poiseuille flows with the same algebraic rates, i.e., $u$ satisfies

$$\|z^k (u - \bar{U})\|_{H^2(\Omega)} < +\infty.$$ (17)

Remark 1.8. The same asymptotic behavior also holds for the axisymmetric flows in semi-infinite pipes.

The structure of this paper is as follows: in Sect. 2, we introduce the stream function formulation for the axisymmetric Navier–Stokes system and recall the existence results obtained in [27] for the associated linearized problem. Some good estimates for the derivatives in the axial direction of the stream function and the swirl velocity are established in Sect. 3. These are the key ingredients to get the existence and uniqueness of solutions when $F$ is large. The existence of solutions for the nonlinear problem is obtained via standard iteration in Sect. 4. Section 5 devotes to the study on the convergence rates of the flows at far fields, where the key observation is that $u - \bar{U}$ must be small at far fields when the condition (14) holds so that the estimate for the linearized problem can be used. Some important inequalities are collected in Appendix A. A sketch of the uniform estimate with respect to the flux for the solutions of the linearized problem obtained in [27] is provided in Appendix 6.

2. Stream Function Formulation and Linearized Problem

Suppose that $u$ is an axisymmetric solution of (5) and $\bar{U} = \bar{U} e_z$ is the Hagen–Poiseuille flow with $\bar{U}$ defined in (4). Let

$$v = u - \bar{U} = u^r(r, z) e_r + u^\theta(r, z) e_\theta + u^z(r, z) e_z.$$ (18)

Then, $v$ satisfies the nonlinear system

$$\begin{aligned}
\bar{U}(r) \frac{\partial v^r}{\partial z} + \frac{\partial P}{\partial r} - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^r}{\partial r} \right) + \frac{\partial^2 v^r}{\partial z^2} - \frac{v^r}{r^2} \right] + v^r \partial_r v^r + v^z \partial_z v^r - \frac{(v^\theta)^2}{r} = F^r,

v^r \frac{\partial \bar{U}}{\partial r} + \bar{U}(r) \frac{\partial v^z}{\partial z} + \frac{\partial P}{\partial z} - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^z}{\partial r} \right) + \frac{\partial^2 v^z}{\partial z^2} \right] + v^r \partial_r v^z + v^z \partial_z v^z = F^z,

\partial_r v^r + \partial_z v^z + \frac{v^r}{r} = 0,
\end{aligned}$$ (19)
and
\[
\bar{U}(r)\partial_z v^\theta - \left[ \frac{1}{r} \partial_r \left( r \frac{\partial v^\theta}{\partial r} \right) + \frac{\partial^2 v^\theta}{\partial z^2} - \frac{v^\theta}{r^2} \right] + v^r \partial_r v^\theta + v^z \partial_z v^\theta + \frac{v^r v^\theta}{r} = F^\theta
\] (20)
in \( D = \{ (r, z) : r \in (0, 1), z \in \mathbb{R} \} \). Here, \( F^r, F^z, \) and \( F^\theta \) are the radial, axial, and azimuthal component of \( F \), respectively. The no-slip boundary conditions and the flux constraint (6)–(7) become
\[
v^r(1, z) = v^z(1, z) = 0, \quad \int_0^1 r v^z(r, z) \, dr = 0,
\] (21)
and
\[
v^\theta(1, z) = 0.
\] (22)

2.1. Stream function formulation

It follows from the third equation in (19) that there exists a stream function \( \psi(r, z) \) satisfying
\[
v^r = \partial_z \psi \quad \text{and} \quad v^z = -\frac{\partial_r (r \psi)}{r}.
\] (23)
Let the azimuthal vorticities of \( v \) and \( F \) be defined as
\[
\omega^\theta = \partial_z v^r - \partial_r v^z = \frac{\partial}{\partial r} \left( \frac{1}{r} \partial_r (r \psi) \right) + \partial_z^2 \psi \quad \text{and} \quad f = \partial_z F^r - \partial_r F^z,
\]
respectively. Taking the curl of the first two equations of (19) gives
\[
\bar{U}(r)\partial_z \omega^\theta - (\mathcal{L} + \partial_z^2) \omega^\theta = \partial_z F^r - \partial_r F^z - \partial_r (v^r \omega^\theta) - \partial_z (v^z \omega^\theta) + \partial_z \left[ \frac{(v^\theta)^2}{r} \right],
\] (24)
where
\[
\mathcal{L} = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r \cdot) \right) = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}.
\] (25)
Hence, the stream function \( \psi \) satisfies the following equation:
\[
\bar{U}(r)\partial_z (\mathcal{L} + \partial_z^2) \psi - (\mathcal{L} + \partial_z^2)^2 \psi
= \partial_z F^r - \partial_r F^z - \partial_r (v^r \omega^\theta) - \partial_z (v^z \omega^\theta) + \partial_z \left[ \frac{(v^\theta)^2}{r} \right].
\] (26)
As discussed in [18], in order to get classical solutions, some compatibility conditions at the axis should be imposed. Assume that \( v \) and the vorticity \( \omega \) are continuous so that \( v^r(0, z) \) and \( \omega^\theta(0, z) \) should vanish. This implies
\[
\partial_z \psi(0, z) = (\mathcal{L} + \partial_z^2) \psi(0, z) = 0.
\]
Without loss of generality, one can assume that $\psi(0, z) = 0$. Hence, the following compatibility conditions hold at the axis,

$$\psi(0, z) = \mathcal{L}\psi(0, z) = 0.$$  

Therefore, the boundary conditions for $\psi$ are

$$\psi(0, z) = \psi(1, z) = \partial_r \psi(1, z) = \mathcal{L}\psi(0, z) = 0. \quad (27)$$

The swirl velocity $v^\theta = v^\theta e^\theta$ satisfies the equation

$$\bar{U}(r)\partial_z v^\theta - \Delta v^\theta = F^\theta - (v^r \partial_r + v^z \partial_z)v^\theta - \frac{v^r}{r}v^\theta, \quad (28)$$

supplemented with the homogeneous boundary condition

$$v^\theta = 0 \quad \text{on } \partial \Omega. \quad (29)$$

### 2.2. Linearized problem

To get the existence of nonlinear problem (26)–(27) and (28)–(29), we first investigate the following linearized system around Hagen–Poiseuille flows:

$$\begin{cases}
\bar{U} \cdot \nabla v + v \cdot \nabla \bar{U} - \Delta v + \nabla P = F & \text{in } \Omega, \\
\text{div } v = 0 & \text{in } \Omega,
\end{cases} \quad (30)$$

supplemented with no-slip boundary conditions and the flux constraint,

$$v = 0 \quad \text{on } \partial \Omega, \quad \int_{B_1(0)} v^z(\cdot, \cdot, z) \, dS = 0 \quad \text{for any } z \in \mathbb{R}. \quad (31)$$

Note that the system (30) for axisymmetric solutions can be written in the following form:

$$\begin{cases}
\bar{U}(r)\partial_z v^r + \partial_r P - \left[ \frac{1}{r} \partial_r \left( r \frac{\partial v^r}{\partial r} \right) + \frac{\partial^2 v^r}{\partial z^2} - \frac{v^r}{r^2} \right] = F^r & \text{in } D, \\
v^r \partial_r \bar{U} + \bar{U}(r)\partial_z v^z + \partial_z P - \left[ \frac{1}{r} \partial_r \left( r \frac{\partial v^z}{\partial r} \right) + \frac{\partial^2 v^z}{\partial z^2} \right] = F^z & \text{in } D, \\
\partial_r v^r + \partial_z v^z + \frac{v^r}{r} = 0 & \text{in } D 
\end{cases} \quad (32)$$

and

$$\bar{U}(r)\partial_z v^\theta - \left[ \frac{1}{r} \partial_r \left( r \frac{\partial v^\theta}{\partial r} \right) + \frac{\partial^2 v^\theta}{\partial z^2} - \frac{v^\theta}{r^2} \right] = F^\theta \quad \text{in } D. \quad (33)$$

Similarly, one can introduce the stream function $\psi$ for the solution $(v^r, v^z)$ of (32). Then, $\psi$ satisfies the fourth-order equation

$$\bar{U}(r)\partial_z (\mathcal{L} + \partial_z^2)\psi - (\mathcal{L} + \partial_z^2)^2 \psi = f, \quad (34)$$
where $\mathcal{L}$ is defined in (25) and $f = \partial_z F^r - \partial_r F^z$. Furthermore, the boundary conditions for $\psi$ are of the form

$$
\mathcal{L}\psi(0, z) = \psi(0, z) = \psi(1, z) = \frac{\partial \psi}{\partial r}(1, z) = 0. \quad (35)
$$

Meanwhile, if $v$ is continuous, then the compatibility conditions for $v$ obtained in [18] implies $v^\theta(0, z) = 0$. Hence, $v^\theta$ satisfies the following problem:

$$
\begin{align*}
\tilde{U}(r) \partial_z v^\theta - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^\theta}{\partial r} \right) \right] + \frac{\partial^2 v^\theta}{\partial z^2} - \frac{v^\theta}{r^2} & = F^\theta \quad \text{in } D, \\
v^\theta(1, z) & = v^\theta(0, z) = 0.
\end{align*}
\quad (36)
$$

Now, let us introduce some notations and recall the existence results for $\psi$ and $v^\theta$ obtained in [27]. For a given function $g(r, z)$, define its Fourier transform with respect to $z$ variable by

$$
\hat{g}(r, \xi) = \int_\mathbb{R} g(r, z) e^{-i\xi z} dz.
$$

Let $\Re g$ and $\Im g$ denote the real and imaginary part of a function or a number $g$, respectively.

**Definition 2.1.** Define a function space $C_s^\infty(D)$ as follows:

$$
C_s^\infty(D) = \left\{ \varphi(r, z) : \varphi \in C^\infty_c([0, 1] \times \mathbb{R}), \varphi(1, z) = \frac{\partial \varphi}{\partial r}(1, z) = 0, \quad \text{and } \lim_{r \to 0+} \mathcal{L}\varphi(r, z) = \lim_{r \to 0+} \frac{\partial}{\partial r}(r \mathcal{L}^k \varphi)(r, z) = 0, \quad k \in \mathbb{N} \right\}.
$$

The $H^3_r(D)$-norm is defined as follows:

$$
\|\varphi\|_{H^3_r(D)}^2 := \int_{-\infty}^{+\infty} \int_0^1 \left[ \frac{\partial}{\partial r} (r \mathcal{L} \hat{\varphi}) \frac{1}{r} + \xi^2 |\mathcal{L} \hat{\varphi}|^2 r + \xi^4 \frac{\partial}{\partial r} (r \hat{\varphi}) \frac{1}{r} + \xi^6 |\hat{\varphi}|^2 r \right] dr d\xi
$$

$$
+ \int_{-\infty}^{+\infty} \int_0^1 \left[ |\mathcal{L} \hat{\varphi}|^2 r + \xi^2 \frac{\partial}{\partial r} (r \hat{\varphi}) \frac{1}{r} + \xi^4 |\hat{\varphi}|^2 r \right] dr d\xi
$$

$$
+ \int_{-\infty}^{+\infty} \int_0^1 \left[ \frac{\partial}{\partial \xi} (r \hat{\varphi}) \frac{1}{r} + \xi^2 |\hat{\varphi}|^2 r + |\hat{\varphi}|^2 r \right] dr d\xi. \quad (37)
$$

$H^3_s(D)$ denotes the closure of $C_s^\infty(D)$ under the $H^3_r(D)$-norm. Furthermore, $L^2_r(D)$ is the completion of $C^\infty(D)$ under the $L^2_r(D)$-norm defined as follows:

$$
\|g\|_{L^2_r(D)}^2 = \int_{-\infty}^{+\infty} \int_0^1 |g|^2 r dr dz.
$$

The existence of solutions for the problems (34)–(35) and (36) has been established in [27].
Proposition 2.1. [27, Theorem 1.1] Assume that $F^* = F^r e_r + F^z e_z \in L^2(\Omega)$ and $F^*$ is axisymmetric. There exists a unique solution $\psi \in H^3_*(D)$ to the linear system (34)–(35), and a positive constant $C_1$ independent of $F^*$ and $\Phi$, such that

$$\|v^*\|_{H^{5/2}(\Omega)} \leq C_1 \|F^*\|_{L^2(\Omega)},$$

and

$$\|v^*\|_{H^2(\Omega)} \leq C_1 (1 + \Phi^{1/4}) \|F^*\|_{L^2(\Omega)},$$

where $v^* = v^r e_r + v^z e_z = \partial_z \psi e_r - \frac{\partial_r (r \psi)}{r} e_z$.

Proposition 2.2. [27, Proposition 4.6] Assume that $F^\theta = F^\theta e_\theta \in L^2(\Omega)$ and $F^\theta = F^\theta(r, z)$ is axisymmetric. There exist a unique solution $v^\theta$ to the linear problem (36) and a positive constant $C_2$ independent of $F^\theta$ and $\Phi$, such that

$$\|v^\theta\|_{H^2(\Omega)} \leq C_2 \|F^\theta\|_{L^2(\Omega)},$$

where $v^\theta = v^\theta e_\theta$.

3. Some Refined Estimates for Solutions of Linearized Problem

Propositions 2.1–2.2 provide some uniform estimates for $\psi$ and $v^\theta$ with respect to the flux $\Phi$. They are the key ingredients to get the existence and uniqueness of solutions of the steady Navier–Stokes system, when $F$ is uniformly small [27]. In this section, we give some refined estimates, especially for the $z$-derivatives of $\psi$ and $v^\theta$, which yield the existence of solutions of steady Navier–Stokes system even when the external force is large. These estimates also show the stabilizing effect of the linearized convection term when $\Phi$ is large.

We take the Fourier transform with respect to $z$ for the equation (34). For every fixed $\xi$, $\hat{\psi}$ satisfies

$$i \xi \hat{\psi} r (\mathcal{L} - \xi^2) \hat{\psi} - (\mathcal{L} - \xi^2)^2 \hat{\psi} = \hat{f} = i \xi \hat{F}^r - \frac{d}{dr} \hat{F}^z. \tag{41}$$

Furthermore, the boundary conditions (35) can be written as

$$\hat{\psi}(0) = \hat{\psi}(1) = \hat{\psi}'(1) = \mathcal{L} \hat{\psi}(0) = 0. \tag{42}$$

First, let us recall the a priori estimates obtained in [27], which hold for every $\xi \in \mathbb{R}$.

Proposition 3.1. [27, Section 6] Let $\hat{\psi}(r, \xi)$ be a smooth solution of the problem (41)–(42). Then, it holds

$$\int_0^1 |\mathcal{L} \hat{\psi}|^2 r dr + \xi^2 \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 r dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r dr \leq C \int_0^1 (|\hat{F}^r|^2 + |\hat{F}^z|^2) r dr. \tag{43}$$
The next two propositions give some further estimates for \( \psi \), especially the \( z \)-derivatives of \( \psi \).

**Proposition 3.2.** Assume that \( F^r = F^r e_r \in L^2(\Omega) \), the solution \( \psi \) of the problem

\[
\begin{align*}
\bar{U}(r) \partial_z(L + \partial_z^2)\psi - (L + \partial_z^2)^2 \psi &= \partial_z F^r, \\
\psi(0) &= \psi(1) = \partial_z \psi(1) = L\psi(0) = 0
\end{align*}
\]

satisfies

\[
\|v^*\|_{L^2(\Omega)} \leq C \Phi^{-\frac{2}{3}} \|F^r\|_{L^2(\Omega)} \quad \text{and} \quad \|v^*\|_{H^\frac{19}{12}(\Omega)} \leq C \Phi^{-\frac{1}{30}} \|F^r\|_{L^2(\Omega)}.
\]

**Proof.** Taking the Fourier transform with respect to \( z \) for the system (44) yields that for every fixed \( \xi \), \( \hat{\psi}(r, \xi) \) satisfies

\[
i\xi \bar{U}(r)(L - \xi^2)\hat{\psi} - (L - \xi^2)^2\hat{\psi} = i\xi \hat{F^r}.
\]

(45)

Multiplying (45) by \( \hat{\psi} \) and integrating the resulting equation over \([0, 1]\) give

\[
\int_0^1 i\xi \bar{U}(r)(L - \xi^2)\hat{\psi}\hat{\psi}r \, dr - \int_0^1 (L - \xi^2)^2\hat{\psi}\hat{\psi}r \, dr = i\xi \int_0^1 \hat{F^r}\hat{\psi}r \, dr.
\]

It follows from the direct computations and integration by parts that one has

\[
\xi \int_0^1 \bar{U}(r) \left| \frac{d}{dr}(\hat{\psi}) \right|^2 dr + \xi^3 \int_0^1 \bar{U}(r)|\hat{\psi}|^2 r \, dr = -\Im \int_0^1 \xi \hat{F^r}\hat{\psi}r \, dr.
\]

(46)

This, together with Lemma A.2, gives

\[
\Phi \int_0^1 \left| \frac{d}{dr}(\hat{\psi}) \right|^2 \frac{1 - r^2}{r} dr \leq C \int_0^1 |\hat{F^r}| |\hat{\psi}| r \, dr
\]

\[
\leq C \left( \int_0^1 |\hat{F^r}|^2 r \, dr \right)^\frac{1}{2} \left( \int_0^1 \left| \frac{d}{dr}(\hat{\psi}) \right|^2 \frac{1 - r^2}{r} dr \right)^\frac{1}{2}.
\]

Hence, we have

\[
\int_0^1 \left| \frac{d}{dr}(\hat{\psi}) \right|^2 \frac{1 - r^2}{r} dr \leq C \Phi^{-2} \int_0^1 |\hat{F^r}|^2 r \, dr.
\]

(47)

It follows from Lemmas A.1 and A.3, and Proposition 3.1 that

\[
\int_0^1 \left| \frac{d}{dr}(\hat{\psi}) \right|^2 \frac{1}{r} \, dr \leq C \left( \int_0^1 \left| \frac{d}{dr}(\hat{\psi}) \right|^2 \frac{1 - r^2}{r} dr \right)^\frac{2}{3} \left( \int_0^1 |\hat{\psi}|^2 r \, dr \right)^\frac{1}{3}
\]

\[
\leq C \Phi^{-\frac{4}{3}} \int_0^1 |\hat{F^r}|^2 r \, dr.
\]

(48)
This implies
\[ \|v^z\|_{L^2(\Omega)}^2 = \int_{-\infty}^{+\infty} \int_{0}^{1} \left| \frac{d}{dr} (r \hat{\psi}) \right|^2 \frac{1}{r} dr d\xi \leq C \Phi^{-\frac{4}{3}} \|F^r\|_{L^2(\Omega)}^2. \]  
(49)

Similarly, the equality (46), together with Lemma A.2, gives
\[ \Phi \xi^2 \int_{0}^{1} (1 - r^2) |\hat{\psi}|^2 r dr \leq C \left( \int_{0}^{1} |F^r|^2 r dr \right)^{\frac{1}{2}} \left( \int_{0}^{1} \left| \frac{d}{dr} (r \hat{\psi}) \right|^2 \frac{1}{r} dr \right)^{\frac{1}{2}} \]
\[ \leq C \Phi^{-1} \int_{0}^{1} |F^r|^2 r dr. \]  
(50)

It follows from Lemma A.3, Lemma A.1, and Proposition 3.1 again that one has
\[ \xi^2 \int_{0}^{1} |\hat{\psi}|^2 r dr \leq C \left( \xi^2 \int_{0}^{1} (1 - r^2) |\hat{\psi}|^2 r dr \right)^{\frac{1}{2}} \left( \xi^2 \int_{0}^{1} \left| \frac{d}{dr} (r \hat{\psi}) \right|^2 \frac{1}{r} dr \right)^{\frac{1}{2}} \]
\[ \leq C \Phi^{-\frac{4}{3}} \int_{0}^{1} |F^r|^2 r dr. \]  
(51)

This implies
\[ \|v^r\|_{L^2(\Omega)}^2 = \int_{-\infty}^{+\infty} \int_{0}^{1} \xi^2 |\hat{\psi}|^2 r dr d\xi \leq C \Phi^{-\frac{4}{3}} \|F^r\|_{L^2(\Omega)}^2. \]  
(52)

By the interpolation between \(L^2(\Omega)\) and \(H^{\frac{5}{3}}(\Omega)\) (cf. Lemma A.4), one has
\[ \|v^*\|_{H^{\frac{19}{12}}(\Omega)} \leq C \|v^r\|_{L^2(\Omega)}^{\frac{19}{20}} \|v^x\|_{H^{\frac{5}{3}}(\Omega)}^{\frac{19}{20}} \leq C \Phi^{-\frac{1}{30}} \|F^r\|_{L^2(\Omega)}. \]  
(53)

This finishes the proof of Proposition 3.2. \(\Box\)

Remark 3.1. The problem (44) was addressed, though not in the same manner, in [7, Section 7]. The key issue in [7] is to analyze the spectrum of the linear problem. Our major aim is to give the resolvent bound for the resolvent 0 for any given frequency \(\xi \in \mathbb{R}\).

Next, we study the case when \(F\) has only axial component.

**Proposition 3.3.** Assume that \(F^z = F^z e_z \in L^2(\Omega)\), the solution \(\psi\) of the following problem
\[
\begin{aligned}
\bar{U}(r) \partial_z (\mathcal{L} + \partial_z^2) \psi - (\mathcal{L} + \partial_z^2)^2 \psi &= -\partial_r F^z, \\
\psi(0) &= \psi(1) = \partial_r \psi(1) = \mathcal{L} \psi(0) = 0,
\end{aligned}
\]  
(54)

satisfies
\[ \|v^r\|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{2}} \|F^z\|_{L^2(\Omega)} \quad \text{and} \quad \|v^r\|_{H^{\frac{19}{12}}(\Omega)} \leq C \Phi^{-\frac{1}{30}} \|F^z\|_{L^2(\Omega)}, \]
where \(v^r = v^r e_r = \partial_z \psi e_r\).
Proof. Taking the Fourier transform with respect to $z$ for the system (54) yields that for fixed $\xi$, $\hat{\psi}(r, \xi)$ satisfies

$$i \xi \hat{U}(r)(\mathcal{L} - \xi^2)\hat{\psi} - (\mathcal{L} - \xi^2)^2 \hat{\psi} = -\frac{d}{dr} \hat{F}_z.$$  \hfill (55)

Multiplying (55) by $r \hat{\psi}$ and integrating the resulting equation over $[0, 1]$ give

$$\xi \int_0^1 \hat{U}(r) \left| \frac{d}{dr} (r \hat{\psi}) \right|^2 dr + \xi^3 \int_0^1 \hat{U}(r) |\hat{\psi}|^2 r dr = \Im \int_0^1 \hat{F}_z \frac{d}{dr} (r \hat{\psi}) dr.$$ \hfill (56)

This, together with Proposition 3.1, implies

$$\Phi^2 \int_0^1 \left| \frac{d}{dr} (r \hat{\psi}) \right|^2 \frac{1 - r^2}{r} dr \leq C |\xi| \int_0^1 \left| \hat{F}_z \right| \left| \frac{d}{dr} (r \hat{\psi}) \right| dr \leq C \int_0^1 \left| \hat{F}_z \right|^2 r dr.$$ \hfill (57)

Applying Lemma A.2 yields

$$\Phi \int_0^1 \left| \hat{\psi} \right|^2 r dr \leq C \Phi^{-1} \left| \hat{F}_z \right| \int_0^1 \left| \frac{d}{dr} (r \hat{\psi}) \right| dr.$$ \hfill (58)

Therefore, one has

$$\| v^r \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{2}} \| F_z \|_{L^2(\Omega)}.$$ \hfill (59)

It follows from Lemma A.4 and Proposition 2.1 that one has

$$\| v^r \|_{H_{1/2}^1(\Omega)} \leq C \Phi^{-\frac{1}{4}} \| F_z \|_{L^2(\Omega)}.$$ \hfill (60)

This finishes the proof of Proposition 3.3. \hfill \square

Now, we are in position to analyze $v^\theta$. Taking the Fourier transform with respect to $z$ for the problem (36) gives

$$\begin{cases}
i \xi \hat{U}(r) \hat{v}^\theta - (\mathcal{L} - \xi^2) \hat{v}^\theta = \hat{F}^\theta, \\
\hat{v}^\theta(1) = \hat{v}^\theta(0) = 0.
\end{cases}$$ \hfill (61)

Let us first recall the uniform estimate for $v^\theta$ obtained in [27].

Proposition 3.4. [27, Proposition 3.1] Assume that $\hat{v}^\theta$ is a smooth solution to the problem (61). For every fixed $\xi$, it holds that

$$\int_0^1 |\mathcal{L} \hat{v}^\theta|^2 r dr + \xi^2 \int_0^1 \left| \frac{d}{dr} (r \hat{v}^\theta) \right|^2 \frac{1}{r} dr + \xi^4 \int_0^1 |\hat{v}^\theta|^2 r dr \leq C \int_0^1 |\hat{F}^\theta|^2 r dr.$$ \hfill (62)

The next two propositions give some further estimates for $v^\theta$. 
Proposition 3.5. Assume that $F^\theta = F^\theta \varepsilon_\theta \in L^2(\Omega)$. The solution $v^\theta$ to the linear problem (36) satisfies that

$$\|\partial_z v^\theta\|_{L^2(\Omega)} \leq C\Phi^{-\frac{1}{2}}\|F^\theta\|_{L^2(\Omega)}.$$

Proof. Multiplying the equation in (61) by $r\widehat{v^\theta}$ and integrating the resulting equation over $[0, 1]$ yield

$$\int_0^1 i\xi \bar{U}(r)|\widehat{v^\theta}|^2 r \, dr + \int_0^1 \left| \frac{d}{dr}(r\widehat{v^\theta}) \right|^2 \frac{1}{r} \, dr + \xi^2 \int_0^1 |\widehat{v^\theta}|^2 r \, dr = \int_0^1 \bar{F}^\theta r \, dr.$$  \hspace{1cm} (63)

It follows from Hölder inequality, Lemma A.1, and (63) that one has

$$\int_0^1 \left| \frac{d}{dr}(r\widehat{v^\theta}) \right|^2 \frac{1}{r} \, dr + \xi^2 \int_0^1 |\widehat{v^\theta}|^2 r \, dr \leq \int_0^1 |\bar{F}^\theta|^2 r \, dr$$  \hspace{1cm} (64)

and

$$\Phi|\xi| \int_0^1 (1 - r^2)|\widehat{v^\theta}|^2 r \, dr \leq C \int_0^1 |\bar{F}^\theta||\widehat{v^\theta}| r \, dr.$$  \hspace{1cm} (65)

The inequality (65), together with (62), gives

$$|\xi|^3 \int_0^1 (1 - r^2)|\widehat{v^\theta}|^2 r \, dr \leq C\Phi^{-1}\left( \int_0^1 |\bar{F}^\theta|^2 r \, dr \right)^{\frac{1}{2}} \left( \xi^4 \int_0^1 |\widehat{v^\theta}|^2 r \, dr \right)^{\frac{1}{2}}$$

$$\leq C\Phi^{-\frac{1}{2}} \int_0^1 |\bar{F}^\theta|^2 r \, dr.$$  \hspace{1cm} (66)

By Lemma A.3 and (64), one has

$$|\xi|^2 \int_0^1 |\widehat{v^\theta}|^2 r \, dr \leq C\left( |\xi|^3 \int_0^1 (1 - r^2)|\widehat{v^\theta}|^2 r \, dr \right)^{\frac{1}{2}} \left( \int_0^1 \left| \frac{d}{dr}(r\widehat{v^\theta}) \right|^2 \frac{1}{r} \, dr \right)^{\frac{1}{2}}$$

$$\leq C\Phi^{-\frac{1}{2}} \int_0^1 |\bar{F}^\theta|^2 r \, dr.$$  \hspace{1cm} (67)

This implies

$$\|\partial_z v^\theta\|_{L^2(\Omega)} \leq C\Phi^{-\frac{1}{2}}\|F^\theta\|_{L^2(\Omega)}.$$  \hspace{1cm} (68)

Hence, the proof of Proposition 3.5 is completed. $\Box$

Remark 3.2. If we divide the both sides of (61) by $\Phi$, then one has

$$- \frac{1}{\Phi} \mathcal{L}\widehat{v^\theta} + i \left[ \frac{\xi}{\pi}(1 - r^2) - i \frac{\xi^2}{\Phi} \right] \widehat{v^\theta} = \frac{1}{\Phi} \bar{F}^\theta.$$  \hspace{1cm} (69)

The problem (69) shares some similar features with Schrödinger operator as the Planck constant is small (cf. [12]). However, the associated potential is not a purely imaginary function and it also depends on $\Phi$. This makes the direct spectrum analysis for (69) more complicated.
Proposition 3.6. Assume that $G^\theta = G^\theta e_\theta \in H^1(\Omega)$. The solution $v^\theta$ to the following linear problem

$$\begin{cases}
\bar{U}(r)\partial_z v^\theta - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^\theta}{\partial r} \right) + \frac{\partial^2 v^\theta}{\partial z^2} - \frac{v^\theta}{r^2} \right] = \partial_z G^\theta & \text{in } D, \\
v^\theta(1, z) = v^\theta(0, z) = 0,
\end{cases}$$

(70)

satisfies

$$\|v^\theta\|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{3}} \|G^\theta\|_{L^2(\Omega)}$$

and

$$\|v^\theta\|_{H^{10/17}(\Omega)} \leq C \Phi^{-\frac{5}{21}} \|\partial_z G^\theta\|_{L^2(\Omega)}^{\frac{19}{24}} \|G^\theta\|_{L^2(\Omega)}^{\frac{5}{24}}.$$

Proof. Similar computations as that in the proof of Proposition 3.5 yield

$$\int_0^1 i \xi \bar{U}(r)|\hat{v}^\theta|^2 r dr + \int_0^1 \left| \frac{d}{dr} (r \hat{v}^\theta) \right|^2 \frac{1}{r} dr + \xi^2 \int_0^1 |\hat{v}^\theta|^2 r dr = i \xi \int_0^1 \hat{G}^\theta \hat{v}^\theta r dr.$$

(71)

It follows from Hölder inequality and Lemma A.1 that one has

$$\int_0^1 \left| \frac{d}{dr} (r \hat{v}^\theta) \right|^2 \frac{1}{r} dr + \xi^2 \int_0^1 |\hat{v}^\theta|^2 r dr \leq C \int_0^1 |\hat{G}^\theta|^2 r dr$$

(72)

and

$$\Phi \int_0^1 (1 - r^2) |\hat{v}^\theta|^2 r dr \leq C \int_0^1 |\hat{G}^\theta|^2 r dr.$$  

(73)

Hence, it holds that

$$\int_0^1 |\hat{v}^\theta|^2 r dr \leq C \left( \int_0^1 (1 - r^2) |\hat{v}^\theta|^2 r dr \right)^{\frac{3}{2}} \left( \int_0^1 \left| \frac{d}{dr} (r \hat{v}^\theta) \right|^2 \frac{1}{r} dr \right)^{\frac{1}{2}}$$

(74)

$$\leq C \Phi^{-\frac{3}{2}} \int_0^1 |\hat{G}^\theta|^2 r dr.$$

This implies

$$\|v^\theta\|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{3}} \|G^\theta\|_{L^2(\Omega)}.$$

(75)

By Lemma A.4 and Proposition 2.2, one has

$$\|v^\theta\|_{H^{10/17}(\Omega)} \leq C \|v^\theta\|_{L^2(\Omega)}^{\frac{5}{24}} \|\partial_z G^\theta\|_{L^2(\Omega)}^{\frac{19}{24}} \leq C \Phi^{-\frac{5}{21}} \|G^\theta\|_{L^2(\Omega)}^{\frac{5}{24}} \|\partial_z G^\theta\|_{L^2(\Omega)}^{\frac{19}{24}}.$$  

(76)

Hence, the proof of Proposition 3.6 is finished.
4. Existence and Uniqueness of Solutions for Nonlinear Problem

In this section, we prove the existence and uniqueness of strong axisymmetric solution of the nonlinear problem (5)–(7), in particular, when \( \| F \|_{L^2(\Omega)} \) is of the same order as \( \Phi^{\frac{1}{16}} \) for large \( \Phi \).

**Proof of Theorem 1.1.** Assume that \( \Phi \geq 1 \). We divide the proof into three steps.

**Step 1. Iteration scheme.** Let \( u = \bar{U} + v \). The existence of solutions is proved via an iteration method for the problem on \( v \). Let \( F^* = F^f e_r + F^z e_z \). For any given \( F(r, z) = F^* + F^\theta \in (L^2(\Omega))^3 \), there exists a unique axisymmetric solution \( (\psi, v^\theta) \) to the linear problem (34)–(35) and (36), and we denote this solution by \( Tf \). Let

\[
\psi_0 = (\psi_0, v_0^\theta) = Tf, \quad \psi_n = (\psi_n, v_n^\theta),
\]

and

\[
\psi_{n+1} = \psi_0 + T(F_n^*, F_n^\theta)
\]

with

\[
F_n^* = \left[ -v_n^r \omega_n^\theta + \frac{(v_n^\theta)^2}{r} \right] e_r + v_n^r \omega_n^\theta e_z, \quad F_n^\theta = -(v_n^r \partial_r + v_n^z \partial_z) v_n^\theta - \frac{v_n^r}{r} v_n^\theta
\]

where

\[
v_i^r = \partial_z \psi_i, \quad v_i^z = -\frac{\partial_r (r \psi_i)}{r}, \quad v_i^\theta = v_i^\theta \cdot e_\theta, \quad \omega_i^\theta = -\partial_r v_i^z + \partial_z v_i^r, \quad i \in \mathbb{N}.
\]

Set

\[
S = \left\{ (\psi, v^\theta) \in H^3(D) \times H^2(\Omega) \left| \begin{array}{c}
\| v^\theta \|_{H^6_{10}(\Omega)} \leq 2C_1 \Phi^{\frac{1}{16}}, \quad \| v^\theta \|_{L^2(\Omega)} \leq \Phi^{-\frac{15}{64}} \\
\| \omega^\theta \|_{H^6_{10}(\Omega)} \leq 2C_2 \Phi^{\frac{1}{16}}, \quad \| \partial_z v^\theta \|_{L^2(\Omega)} \leq \Phi^{-\frac{5}{16}}
\end{array} \right. \right\}.
\]

Here, \( C_1 \) and \( C_2 \) are the constants indicated in Propositions 2.1–2.2. Under the assumption that \( \| F \|_{L^2(\Omega)} \leq \Phi^{\frac{1}{16}} \), according to Propositions 2.1–2.2 and Propositions 3.2–3.5, one has \( \psi_0 \in S \) when \( \Phi \) is large enough. Assume that \( \psi_n \in S \), our aim is to prove \( \psi_{n+1} \in S \).

**Step 2. Estimate for the velocity field and existence.** Denote \( v^*_i = v_i^r e_r + v_i^z e_z \). It follows from Hölder inequality and Gagliardo–Nirenberg inequality (cf. Lemma A.5) that the estimates

\[
\| v_n^r \omega_n^\theta \|_{L^2(\Omega)} \leq C \| v_n^r \|_{L^{12}(\Omega)} \| \omega_n^\theta \|_{L^{12}(\Omega)} \leq C \| v_n^r \|_{L^2(\Omega)} \| v_n^r \|_{L^{12}(\Omega)} \| v_n^\theta \|_{W^{1, 12}(\Omega)} \leq C \| v_n^r \|_{H^{\frac{5}{6}}(\Omega)} \| v_n^r \|_{L^{\infty}(\Omega)} \| v_n^\theta \|_{H^{\frac{5}{6}}(\Omega)} \]

\[
\leq C \| v_n^r \|_{L^2(\Omega)} \| v_n^\theta \|_{L^{12}(\Omega)} \| v_n^\theta \|_{H^{\frac{5}{6}}(\Omega)} \| v_n^\theta \|_{H^{\frac{5}{6}}(\Omega)} \leq C \Phi^{-\frac{15}{52}} \Phi^{\frac{1}{16}} \Phi^{\frac{11}{6}} \leq C \Phi^{-\frac{17}{288}}
\]
and
\[ \| v_n^r \|_{L^2(\Omega)} \leq \| v_n^r \|_{L^\infty(\Omega)} \| \omega_n^\theta \|_{L^2(\Omega)} \leq C \| v_n^* \|_{H^{19}(\Omega)} \leq C \Phi^{-\frac{1}{28}}. \] (80)

hold. Moreover, one has
\[ \left\| \frac{v_n^\theta}{r} \right\|_{L^2(\Omega)} \leq C \| v_n^\theta \|_{L^\infty(\Omega)} \left\| \frac{v_n^\theta}{r} \right\|_{L^2(\Omega)} \leq C \| v_n^\theta \|_{H^{19}(\Omega)} \| v_n^\theta \|_{H^1(\Omega)} \leq C \Phi^{-\frac{1}{28}}. \] (81)

Similarly, using Gagliardo–Nirenberg inequality (cf. Lemma A.5) and Sobolev embedding inequalities gives
\[ \| v_n^r \partial_r v_n^\theta \|_{L^2(\Omega)} \leq C \| v_n^r \|_{L^\infty(\Omega)} \| \partial_r v_n^\theta \|_{L^2(\Omega)} \leq C \| v_n^r \|_{L^2(\Omega)} \| \partial_r v_n^\theta \|_{L^2(\Omega)} \leq C \Phi^{-\frac{15}{17}} \Phi^{-\frac{1}{19}} \| \omega \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{28}}. \] (82)

and
\[ \| v_n^r \partial_z v_n^\theta \|_{L^2(\Omega)} \leq C \| v_n^r \|_{L^\infty(\Omega)} \| \partial_z v_n^\theta \|_{L^2(\Omega)} \leq C \| v_n^r \|_{H^{19}(\Omega)} \| \partial_z v_n^\theta \|_{L^2(\Omega)} \leq C \Phi^{-\frac{15}{17}} \Phi^{-\frac{1}{19}} \| \omega \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{28}}. \] (83)

Furthermore, it holds that
\[ \left\| \frac{v_n^r v_n^\theta}{r} \right\|_{L^2(\Omega)} \leq C \| v_n^r \|_{L^\infty(\Omega)} \left\| \frac{v_n^r v_n^\theta}{r} \right\|_{L^2(\Omega)} \leq C \Phi^{-\frac{15}{32}} \Phi^{-\frac{1}{19}} \Phi^{-\frac{1}{16}} \| \omega \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{28}}. \] (84)

Combining the estimates (79)–(84) and applying Propositions 2.1–2.2, 3.2 yield
\[ \| v_{n+1}^\theta \|_{H^{19}(\Omega)} \leq C_1 \Phi^{-\frac{1}{96}} + C \left( \Phi^{-\frac{17}{288}} + \Phi^{-\frac{1}{30}} \Phi^{-\frac{1}{48}} \right) \leq C_1 \Phi^{-\frac{1}{96}} + C_3 \Phi^{-\frac{1}{80}} \] (85)

and
\[ \| v_{n+1}^\theta \|_{H^{19}(\Omega)} \leq C_2 \Phi^{-\frac{1}{96}} + C \left( \Phi^{-\frac{1}{32}} \Phi^{-\frac{1}{96}} + \Phi^{-\frac{1}{30}} \Phi^{-\frac{1}{48}} \right) \leq C_2 \Phi^{-\frac{1}{96}} + C_4 \Phi^{-\frac{1}{288}}. \] (86)

Moreover, it follows from Propositions 3.2–3.5 that one has
\[ \| v_{n+1}^r \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{2}} \left( \Phi^{-\frac{1}{96}} + \Phi^{-\frac{17}{288}} \right) + C \Phi^{-\frac{1}{3}} \left( \Phi^{-\frac{1}{96}} + \Phi^{-\frac{1}{30}} \right) \leq C_5 \Phi^{-\frac{47}{96}} \] (87)

and
\[ \| \partial_z v_{n+1}^\theta \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{3}} \left( \Phi^{-\frac{1}{96}} + \Phi^{-\frac{1}{32}} + \Phi^{-\frac{1}{96}} \right) \leq C_6 \Phi^{-\frac{31}{96}}. \] (88)
Choose a constant \( \Phi_0 (\geq 1) \) large enough such that
\[
C_3 \Phi_0^{-\frac{1}{80}} < C_1 \Phi_0^{\frac{1}{96}}, \quad C_4 \Phi_0^{-\frac{1}{228}} < C_2 \Phi_0^{\frac{1}{96}},
\]
and
\[
C_5 \Phi_0^{-\frac{47}{96}} \leq \Phi_0^{-\frac{15}{32}}, \quad C_6 \Phi_0^{-\frac{31}{96}} < \Phi_0^{-\frac{5}{32}}.
\]
When \( \Phi \geq \Phi_0 \), the estimates (85)–(88) imply that \( (\psi_{n+1}, \psi^\theta_{n+1}) \in S \). By mathematical induction, \( \Psi_n \in S \) for every \( n \in \mathbb{N} \). Note that \( v_n = v^*_n + \psi^\theta_n \). The above estimates show that
\[
\|v_n\|_{H^{\frac{19}{12}}(\Omega)} \leq C_0 \Phi_0^{\frac{1}{96}} \quad \text{with} \quad C_0 = 2C_1 + 2C_2.
\]
According to the regularity estimates in Propositions 2.1–2.2, it holds that
\[
\|v_{n+1}\|_{H^2(\Omega)} \leq C \Phi^{\frac{1}{3}} \|F\|_{L^2(\Omega)} + C \Phi^{\frac{1}{2}} \|v_n \cdot \nabla\|_{L^2(\Omega)} \leq C \Phi^{\frac{13}{32}}.
\]
Since \( \{v_n\} \) is uniformly bounded in \( H^2(\Omega) \), there exists a vector-valued function \( v \in H^2(\Omega) \) such that \( v_n \rightharpoonup v \) in \( H^2(\Omega) \) and
\[
\|v\|_{H^{\frac{19}{12}}(\Omega)} \leq C_0 \Phi_0^{\frac{1}{96}}, \quad \|v\|_{H^2(\Omega)} \leq C \Phi^{\frac{13}{32}}.
\]
Meanwhile, as proved in [27], the Eq. (77) implies that
\[
\text{curl } ((\bar{\nabla} \cdot \nabla)v_{n+1} + (v_{n+1} \cdot \nabla)\bar{U} - \Delta v_{n+1} + (v_n \cdot \nabla)v_n - F) = 0.
\]
Taking the limit of the Eq. (92) yields
\[
\text{curl } ((\bar{U} \cdot \nabla)v + (v \cdot \nabla)\bar{U} - \Delta v + (v \cdot \nabla)v + \nabla P = F).
\]
Hence, there exists a function \( P \) with \( \nabla P \in L^2(\Omega) \), satisfying
\[
(\bar{U} \cdot \nabla)v + (v \cdot \nabla)\bar{U} - \Delta v + (v \cdot \nabla)v + \nabla P = F.
\]
Clearly, \( u = v + \bar{U} \) is a solution of the problem (5)–(7).

**Step 3. Uniqueness.** Suppose there are two axisymmetric solutions \( u \) and \( \tilde{u} \) of the problem (5)–(7), satisfying
\[
\|u - \bar{U}\|_{H^{\frac{19}{12}}(\Omega)} \leq C_0 \Phi_0^{\frac{1}{96}}, \quad \|\tilde{u} - \bar{U}\|_{H^{\frac{19}{12}}(\Omega)} \leq C_0 \Phi_0^{\frac{1}{96}},
\]
and
\[
\|\tilde{u}'\|_{L^2(\Omega)} \leq \Phi^{-\frac{15}{32}}.
\]
Let
\[
v = u - \bar{U} = v^r e_r + v^\theta e_\theta + v^z e_z, \quad \tilde{v} = \tilde{u} - \bar{U} = \tilde{v}^r e_r + \tilde{v}^\theta e_\theta + \tilde{v}^z e_z.
\]
Suppose that $\psi$ and $\tilde{\psi}$ are the stream functions associated with the vector fields $v$ and $\tilde{v}$, respectively. Then, $\psi - \tilde{\psi}$ satisfies the following equation:

$$
\tilde{U}(r) \partial_z (L + \partial_z^2) (\psi - \tilde{\psi}) - (L + \partial_z^2)^2 (\psi - \tilde{\psi}) = - \partial_r (v^r \omega^0 - \tilde{v}^r \omega^0) - \partial_z (v^z \omega^0 - \tilde{v}^z \omega^0) + 2 \partial_z \left( \frac{v^\theta}{r} \psi^\theta - \frac{\tilde{v}^\theta}{r} \tilde{\psi}^\theta \right).
$$

It follows from Sobolev’s embedding inequalities and interpolation inequality (Lemma A.4) that one has

$$
\| v^r \omega^0 - \tilde{v}^r \omega^0 \|_{L^2(\Omega)} \leq \| v^r - \tilde{v}^r \|_{L^4(\Omega)} \| \omega^0 \|_{L^3(\Omega)} + \| \tilde{v}^r \|_{L^4(\Omega)} \| \omega^0 - \tilde{\omega}^0 \|_{L^3(\Omega)}
$$

$$
\leq C \| v^r - \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \| v^* \|_{H^{10}(\Omega)}^{19/12} + C \| \tilde{v}^r \|_{L^4(\Omega)}^{1/2} \| \tilde{v}^r \|_{L^4(\Omega)}^{3/2} \| v^* - \tilde{v}^* \|_{H^{10}(\Omega)}^{19/12} \leq C \| v^r - \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \left( \| v^r \|_{H^{10}(\Omega)}^{19/12} + \| \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \right) \leq C \| v^r - \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \Phi^{1/12}.
$$

(97)

and

$$
\| v^z \omega^0 - \tilde{v}^z \omega^0 \|_{L^2(\Omega)} \leq \| v^z - \tilde{v}^z \|_{L^4(\Omega)} \| \omega^0 \|_{L^3(\Omega)} + C \| \tilde{v}^z \|_{L^4(\Omega)} \| \omega^0 - \tilde{\omega}^0 \|_{L^3(\Omega)}
$$

$$
\leq C \| v^z - \tilde{v}^z \|_{H^{10}(\Omega)}^{19/12} \left( \| v^z \|_{H^{10}(\Omega)}^{19/12} + \| \tilde{v}^z \|_{H^{10}(\Omega)}^{19/12} \right) \leq C \| v^z - \tilde{v}^z \|_{H^{10}(\Omega)}^{19/12} \Phi^{1/12}.
$$

(98)

Similarly, it holds that

$$
\left\| \frac{v^\theta}{r} \psi^\theta - \frac{\tilde{v}^\theta}{r} \tilde{\psi}^\theta \right\|_{L^2(\Omega)} \leq C \| v^\theta - \tilde{v}^\theta \|_{H^{10}(\Omega)}^{19/12} \left( \| v^\theta \|_{H^{10}(\Omega)}^{19/12} + \| \tilde{v}^\theta \|_{H^{10}(\Omega)}^{19/12} \right) \leq C \| v^\theta - \tilde{v}^\theta \|_{H^{10}(\Omega)}^{19/12} \Phi^{1/12}.
$$

(99)

Combining the above estimates (97)–(99), it follows from Propositions 3.2–3.3 that one has

$$
\Phi^{1/12} \| v^r - \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \leq C \Phi^{1/12} \| v^r - \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \left( \| v^r \|_{H^{10}(\Omega)}^{19/12} + \| \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} \right)
$$

$$
+ C \Phi^{1/12} \| v^* - \tilde{v}^* \|_{H^{10}(\Omega)}^{19/12} \left( \| v^\theta \|_{H^{10}(\Omega)}^{19/12} + \| \tilde{v}^\theta \|_{H^{10}(\Omega)}^{19/12} \right)
$$

$$
\leq C \Phi^{-1/12} \| v^r - \tilde{v}^r \|_{H^{10}(\Omega)}^{19/12} + C \Phi^{-1/12} \| v^* - \tilde{v}^* \|_{H^{10}(\Omega)}^{19/12} + C \Phi^{-1/12} \| v^\theta - \tilde{v}^\theta \|_{H^{10}(\Omega)}^{19/12}.
$$

(100)
and

\[
\| v^z - \tilde{v}^z \|_{H^{19}(\Omega)} \leq C \Phi^{\frac{1}{10}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \Phi^{-\frac{1}{10}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \Phi^{-\frac{1}{10}} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)} + C \Phi^{-\frac{1}{10}} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)}.
\]

(101)

On the other hand, \( v^\theta - \tilde{v}^\theta \) satisfies

\[
\bar{u}(r) \partial_r (v^\theta - \tilde{v}^\theta) - \Delta (v^\theta - \tilde{v}^\theta) = - \left( v^r \partial_r v^\theta - \tilde{v}^r \partial_r \tilde{v}^\theta \right) - 2 \left( v^r \frac{\partial}{r} v^\theta - r \left( \partial_r v^\theta \right) - \partial_r (v^z v^\theta - \tilde{v}^z \tilde{v}^\theta) \right).
\]

It follows from Sobolev’s embedding inequalities and Gagliardo–Nirenberg inequality (Lemma A.5) that one has

\[
\| v^r \partial_r v^\theta - \tilde{v}^r \partial_r \tilde{v}^\theta \|_{L^2(\Omega)} \leq \| v^r - \tilde{v}^r \|_{L^\infty(\Omega)} \| \partial_r v^\theta \|_{L^2(\Omega)} + \| \tilde{v}^r \|_{L^\infty(\Omega)} \| \partial_r v^\theta - \partial_r \tilde{v}^\theta \|_{L^2(\Omega)} \\
\leq C \Phi^{\frac{1}{20}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \| \tilde{v}^r \|_{L^2(\Omega)} \| \tilde{v}^r \|_{H^{19}(\Omega)} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)} \\
\leq C \Phi^{\frac{1}{20}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \Phi^{-\frac{10}{19}} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)} \\
\leq C \Phi^{\frac{1}{20}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \Phi^{-\frac{10}{19}} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)}.
\]

(102)

Similar computations give

\[
\left\| \frac{v^r v^\theta}{r} - \frac{\tilde{v}^r \tilde{v}^\theta}{r} \right\|_{L^2(\Omega)} \leq C \Phi^{\frac{1}{20}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \Phi^{-\frac{10}{19}} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)}
\]

(103)

and

\[
\| \partial_r v^r v^\theta - \partial_r \tilde{v}^r \tilde{v}^\theta \|_{L^2(\Omega)} \leq C \| \partial_r v^r - \partial_r \tilde{v}^r \|_{L^2(\Omega)} \| v^\theta \|_{L^\infty(\Omega)} + C \| \partial_r \tilde{v}^r \|_{L^2(\Omega)} \| v^\theta - \tilde{v}^\theta \|_{L^\infty(\Omega)} \\
\leq C \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} \| v^\theta \|_{H^{19}(\Omega)} + C \| \tilde{v}^r \|_{L^2(\Omega)} \| \tilde{v}^r \|_{H^{19}(\Omega)} \| v^\theta - \tilde{v}^\theta \|_{L^\infty(\Omega)} \\
\leq C \Phi^{\frac{1}{20}} \| v^r - \tilde{v}^r \|_{H^{19}(\Omega)} + C \Phi^{-\frac{10}{19}} \| v^\theta - \tilde{v}^\theta \|_{H^{19}(\Omega)}.
\]

(104)
Moreover,
\[
\|v^z v^\theta - \tilde{v}^z \tilde{v}^\theta\|_{L^2(\Omega)} \leq C \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} \|v^\theta\|_{H^{10}(\Omega)} + C \|\tilde{v}^z\|_{H^{10}(\Omega)} \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)}
\]
\[
\leq C \Phi^{\frac{1}{10}} \left( \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} + \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)} \right),
\]
(105)
and
\[
\|\partial_z (v^z v^\theta) - \partial_z (\tilde{v}^z \tilde{v}^\theta)\|_{L^2(\Omega)}
\leq C \Phi^{\frac{1}{10}} \left( \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} + \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)} \right).
\]
(106)
Combining the estimates (102)–(106), it follows from Propositions 3.5–3.6 that
\[
\|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)}
\leq C \Phi^{\frac{1}{10}} \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} + C \Phi^{-\frac{1}{100}} \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)} + C \Phi^{-\frac{1}{100}} \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)}
\]
\[
\quad + C \Phi^{-\frac{1}{100}} \Phi^{\frac{1}{10}} \left( \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} + \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)} \right)
\]
\[
\leq C \Phi^{\frac{1}{10}} \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} + C \Phi^{-\frac{17}{100}} \|v^z - \tilde{v}^z\|_{H^{10}(\Omega)} + C \Phi^{-\frac{9}{100}} \|v^\theta - \tilde{v}^\theta\|_{H^{10}(\Omega)}.
\]
(107)
Combining the estimates (100)–(101) and (107) together gives the uniqueness of the solution when \(\Phi\) is large enough. Thus, the proof of Theorem 1.1 is finished.
\[\square\]

5. Asymptotic Behavior

In this section, we investigate the asymptotic behavior of solutions to (5)–(7) and prove Theorem 1.2. The proof consists of two steps. In the first step, the asymptotic behavior of the solution which is a small perturbation of Hagen–Poiseuille flow is established. In the second step, the smallness requirement is removed since the solution can be regarded as a small perturbation of Hagen–Poiseuille flow at far fields when the condition (14) is satisfied.

Before giving the detailed proof of Theorem 1.2, we first state the uniform estimate of \(\psi\) for the linear problem (34)–(35) when \(f \in L^2_r(D)\).

**Proposition 5.1.** Assume that \(f(r, z) \in L^2_r(D)\). The solution \(\psi\) obtained in Proposition 2.1 belongs to \(H^3(D)\) and satisfies
\[
\|\psi\|_{H^3(\Omega)} \leq C \|f\|_{L^2_r(D)}, \quad \text{and} \quad \|\psi\|_{H^2(\Omega)} \leq C (1 + \Phi^{\frac{1}{4}}) \|f\|_{L^2_r(D)},
\]
(108)
where the constant \(C\) is independent of \(\Phi\).
Proof. Let $F(r, z) = -\int_0^1 f(r, z) \, dz$. Hence, $f(r, z) = \partial_r F(r, z)$. Moreover, for every $(r, z) \in D$, by Hölder inequality,

$$|F(r, z)| \leq \left( \int_0^1 |f(s, z)|^2 \, ds \right)^{1/2} |\ln r|^{1/2},$$

which implies that

$$\|F\|_{L^2_r(D)} \leq C \int_{-\infty}^{+\infty} \left( \int_0^1 |f(s, z)|^2 \, ds \cdot \int_0^1 |\ln r| \, dr \right) \, dz \leq C \|f\|_{L^2_r(D)}^2.$$

This, together with the regularity estimates in Proposition 2.1, finishes the proof of Proposition 5.1.

The next lemma on the estimate between the stream function and the velocity field is needed in the proof of Theorem 1.2.

Lemma 5.2. Assume that $v^* = v^r e_r + v^z e_z \in H^2(\Omega)$ and $v^*$ is axisymmetric. Let $\psi \in H^3(D)$ be the corresponding stream function of the velocity field $v^*$. It holds that

$$\|\mathcal{L} \psi\|_{L^2_r(D)} + \|\partial^2_z \psi\|_{L^2_r(D)} + \left\| \frac{1}{r} \frac{\partial}{\partial r} (r \psi) \right\|_{L^2_r(D)} + \|\partial_z \psi\|_{L^2_r(D)} + \|\psi\|_{L^2_r(D)} \leq C \|v^*\|_{H^1(\Omega)} \tag{109}$$

and

$$\|\psi e_r\|_{H^2_r(\Omega)} \leq C \|v^*\|_{H^1(\Omega)} \tag{110}$$

Proof. Recall that the stream function $\psi$ of the vector field $v^*$ satisfies

$$(\mathcal{L} + \partial^2_z) \psi = \partial_z v^r - \partial_r v^z \quad \text{in } D. \tag{111}$$

Multiplying (111) by $(\mathcal{L} + \partial^2_z) \psi r$ and integrating over $D$, integration by parts gives

$$\int_{-\infty}^{+\infty} \int_0^1 \left[ |\mathcal{L} \psi|^2 r^2 + 2|\partial_z \partial_r (r \psi)|^2 + |\partial^2_z \psi|^2 r^2 \right] \, dr \, dz \leq C \int_{-\infty}^{+\infty} \int_0^1 |\partial_z v^r - \partial_r v^z|^2 \, r \, dr \, dz \leq C \|v^*\|_{H^1(\Omega)}^2. \tag{112}$$

Furthermore, by Lemma A.1 and (112), one has

$$\left\| \frac{1}{r} \frac{\partial}{\partial r} (r \psi) \right\|_{L^2_r(D)} + \|\partial_z \psi\|_{L^2_r(D)} + \|\psi\|_{L^2_r(D)} \leq C \|\mathcal{L} \psi\|_{L^2_r(D)} + C \left\| \frac{1}{r} \frac{\partial^2}{\partial r \partial z} (r \psi) \right\|_{L^2_r(D)} \leq C \|v^*\|_{H^1(\Omega)}. \tag{113}$$

The straightforward computations yield

$$\begin{cases}
\Delta(\psi e_r) = (\mathcal{L} + \partial^2_z) \psi e_r \quad \text{in } \Omega, \\
\psi = 0 \quad \text{on } \partial \Omega.
\end{cases}$$
It follows from the regularity theory for elliptic equations ([10]) and (112) that one has
\[
\| \psi e_r \|_{H^2(\Omega)} \leq C \| (\mathcal{L} + \partial_z^2) \psi e_r \|_{L^2(\Omega)} \leq C \| v^* \|_{H^1(\Omega)}. \tag{114}
\]
The proof of Lemma 5.2 is completed. □

**Proposition 5.3.** Assume that \( \mathbf{F} \in L^2(\Omega), \mathbf{F} = \mathbf{F}(r, z) \) is axisymmetric, and \( \mathbf{u} \in H^2(\Omega) \) is an axisymmetric solution to the problem (5)–(7). There exist a positive constant \( \varepsilon_0 \), independent of \( \mathbf{F} \) and \( \Phi \), and a positive constant \( \alpha_0 (\leq 1) \) depending on \( \Phi \), such that if
\[
\| e^{\alpha |z|} \mathbf{F} \|_{L^2(\Omega)} < + \infty
\]
with some \( \alpha \in (0, \alpha_0) \) and
\[
\| \mathbf{u} - \tilde{\mathbf{U}} \|_{H^\frac{5}{2}(\Omega)} \leq \varepsilon_0,
\]
then the solution \( \mathbf{u} \) satisfies
\[
\| e^{\alpha z} (\mathbf{u} - \tilde{\mathbf{U}}) \|_{H^\frac{5}{2}(\Omega \cap z \geq 0)} + \| e^{-\alpha z} (\mathbf{u} - \tilde{\mathbf{U}}) \|_{H^\frac{5}{2}(\Omega \cap z \leq 0)} \leq C \| e^{\alpha |z|} \mathbf{F} \|_{L^2(\Omega)}. \tag{117}
\]
Here, \( C \) is a uniform constant independent of \( \mathbf{F} \) and \( \Phi \).

**Proof.** Let \( \mathbf{v} = \mathbf{u} - \tilde{\mathbf{U}} \) and \( \psi \) be the stream function associated with the vector field \( \mathbf{v} \). Then, \( (\psi, \mathbf{v}^\theta) \) satisfies the problem (26)–(29). Multiplying (26) and (28) by \( e^{\alpha z} \) gives
\[
\tilde{U}(r) \partial_z (\mathcal{L} + \partial_z^2)(e^{\alpha z} \psi) - (\mathcal{L} + \partial_z^2)(e^{\alpha z} \psi)
\]
\[
= \partial_z \left[ \frac{v^\theta}{r} (e^{\alpha z} v^\theta) \right] + \tilde{U}(r) \left[ \alpha \mathcal{L}(e^{\alpha z} \psi) + \alpha^2 e^{\alpha z} \psi - 3\alpha^2 \partial_z(e^{\alpha z} \psi) + 3\alpha \partial_z^2(e^{\alpha z} \psi) \right]
\]
\[
- \partial_z \left[ 4\alpha \partial_z^2(e^{\alpha z} \psi) - 2\alpha^2 \partial_z(e^{\alpha z} \psi) + 4\alpha^3(e^{\alpha z} \psi) \right] + \alpha^4 e^{\alpha z} \psi
\]
\[
- \left[ 4\alpha \mathcal{L}(e^{\alpha z} \psi) - 2\alpha^2 \mathcal{L}(e^{\alpha z} \psi) \right] - \alpha e^{\alpha z} F^r
\]
\[
+ \partial_r \left[ v^r \left[ 2\alpha \partial_z(e^{\alpha z} \psi) - \alpha^2 e^{\alpha z} \psi \right] \right] + \partial_z \left[ v^z \left[ 2\alpha \partial_z(e^{\alpha z} \psi) - \alpha^2 e^{\alpha z} \psi \right] \right]
\]
\[
+ \alpha v^z (\mathcal{L} + \partial_z^2)(e^{\alpha z} \psi) - \alpha v^r \left[ 2\alpha \partial_z(e^{\alpha z} \psi) - \alpha^2 e^{\alpha z} \psi \right] - \alpha \frac{v^\theta}{r} (e^{\alpha z} v^\theta),
\]
(118)
and
\[
\tilde{U}(r) \partial_z (e^{\alpha z} v^\theta) - \Delta (e^{\alpha z} v^\theta) = e^{\alpha z} F^\theta - (v^* \cdot \nabla)(e^{\alpha z} v^\theta) - \frac{v^r}{r} (e^{\alpha z} v^\theta)
\]
\[
+ \tilde{U}(r) \alpha e^{\alpha z} v^\theta - 2\alpha \partial_z(e^{\alpha z} v^\theta) + \alpha^2 e^{\alpha z} v^\theta + v^z \alpha e^{\alpha z} v^\theta.
\]
(119)

Denote
\[
v^r_\alpha = \partial_z (e^{\alpha z} \psi), \quad v^z_\alpha = -\frac{\partial_r (r e^{\alpha z} \psi)}{r}, \quad v^r_\alpha e_r + v^z_\alpha e_z, \quad \text{and} \quad v^\theta_\alpha = e^{\alpha z} v^\theta e_\theta.
\]
Regarding the terms on the right hand of (118), by Sobolev embedding inequalities and Lemma 5.2, one has

\[ \| v^{\prime} (\mathcal{L} + \partial_{z}^{2}) (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} \leq C \| v^{\prime} \|_{L^{\infty}(\Omega)} \| (\mathcal{L} + \partial_{z}^{2}) (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} \]

\[ \leq C \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{1}(\Omega)} \]

(120)

and

\[ \| v^{\ast} (\mathcal{L} + \partial_{z}^{2}) (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} \leq C \| v^{\ast} \|_{L^{\infty}(\Omega)} \| (\mathcal{L} + \partial_{z}^{2}) (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} \]

\[ \leq C \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{1}(\Omega)} \leq C \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} . \]

(121)

Similarly, one has

\[ \left\| \frac{v^{\theta}}{r} e^{\alpha z} v^{\theta} \right\|_{L^{2}_{r}(D)} \leq \left\| \frac{v^{\theta}}{r} \right\|_{L^{2}(\Omega)} \| e^{\alpha z} v^{\theta} \|_{L^{\infty}(\Omega)} \leq C \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} . \]

(122)

Note that

\[ \| \tilde{U} (r) \left[ \alpha \mathcal{L} (e^{\alpha z} \psi) + \alpha^{3} e^{\alpha z} \psi - 3 \alpha^{2} \partial_{z} (e^{\alpha z} \psi) + 3 \alpha \partial_{z}^{2} (e^{\alpha z} \psi) \right] \|_{L^{2}_{r}(D)} \]

\[ \leq C \Phi |\alpha| \| v^{\alpha} \|_{H^{1}(\Omega)} \leq C \Phi |\alpha| \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} . \]

(123)

and

\[ \| 4 \alpha \partial_{z}^{2} (e^{\alpha z} \psi) - 6 \alpha^{2} \partial_{z} (e^{\alpha z} \psi) + 4 \alpha^{3} (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} + \| \alpha^{4} e^{\alpha z} \psi \|_{L^{2}_{r}(D)} \]

\[ + \| 4 \alpha \mathcal{L} (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} + \| 2 \alpha^{3} \mathcal{L} (e^{\alpha z} \psi) \|_{L^{2}_{r}(D)} \]

\[ \leq C |\alpha| \| v^{\alpha} \|_{H^{1}(\Omega)} + C |\alpha| \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} \leq C |\alpha| \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} . \]

(124)

Furthermore, it holds that

\[ \| v^{\prime} \left[ 2 \alpha \partial_{z} (e^{\alpha z} \psi) - \alpha^{2} e^{\alpha z} \psi \right] \|_{L^{2}_{r}(D)} + \| v^{\ast} \left[ 2 \alpha \partial_{z} (e^{\alpha z} \psi) - \alpha^{2} e^{\alpha z} \psi \right] \|_{L^{2}_{r}(D)} \]

\[ + \| \alpha v^{\ast} (\mathcal{L} + \partial_{z}^{2}) (e^{\alpha z} \psi) - \alpha v^{\ast} \left[ 2 \alpha \partial_{z} (e^{\alpha z} \psi) - \alpha^{2} e^{\alpha z} \psi \right] \|_{L^{2}_{r}(D)} \]

\[ \leq C |\alpha| \| v^{\prime} \|_{L^{\infty}(\Omega)} \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} + C |\alpha| \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} \]

\[ \leq C |\alpha| \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} . \]

(125)

Similarly to (122), one has

\[ \left\| \frac{\alpha v^{\theta}}{r} (e^{\alpha z} v^{\theta}) \right\|_{L^{2}_{r}(D)} \leq C |\alpha| \| v \|_{H^{\frac{5}{3}}(\Omega)} \| v^{\alpha} \|_{H^{\frac{5}{3}}(\Omega)} . \]

(126)

It follows from the Sobolev embedding inequalities that one has
Choose a small constant \( \epsilon \) then the inequality (131) implies that
\[
\text{It follows from Lemma 5.2 that one has}
\]

Furthermore, it holds that
\[
\text{Hence, collecting the estimates (120)–(130) and applying Propositions 2.1–2.2, Proposition 5.1 yield}
\]

Choose a small constant \( \epsilon_0 \) such that \( C_7 \epsilon_0 \leq \frac{1}{4} \). If \( u \) and \( \alpha \) satisfy
\[
\|u\|_{H^\frac{3}{5}(\Omega)} = \|u - \tilde{U}\|_{H^\frac{3}{5}(\Omega)} \leq \epsilon_0 \quad \text{and} \quad |\alpha| \leq \alpha_0 \leq \frac{1}{2C_8(1 + \Phi + \epsilon_0)},
\]
then the inequality (131) implies that
\[
\|v^\alpha\|_{H^\frac{3}{5}(\Omega)} + \|v^\beta\|_{H^\frac{3}{5}(\Omega)} \leq C_8 \|e^{\alpha z} F\|_{L^2(\Omega)}.
\]

Choose a small constant \( \epsilon_0 \) such that \( C_7 \epsilon_0 \leq \frac{1}{4} \). If \( u \) and \( \alpha \) satisfy
\[
\|u\|_{H^\frac{3}{5}(\Omega)} = \|u - \tilde{U}\|_{H^\frac{3}{5}(\Omega)} \leq \epsilon_0 \quad \text{and} \quad |\alpha| \leq \alpha_0 \leq \frac{1}{2C_8(1 + \Phi + \epsilon_0)},
\]
then the inequality (131) implies that
\[
\|v^\alpha\|_{H^\frac{3}{5}(\Omega)} + \|v^\beta\|_{H^\frac{3}{5}(\Omega)} \leq C_8 \|e^{\alpha z} F\|_{L^2(\Omega)}.
\]

Note that
\[
e^{\alpha z} v^r = \partial_z (e^{\alpha z} \psi) - \alpha e^{\alpha z} \psi = v^a - \alpha e^{\alpha z} \psi \quad \text{and} \quad e^{\alpha z} v^z = -\frac{\partial_r (r e^{\alpha z} \psi)}{\partial r} = v^a.
\]

Decompose \( e^{\alpha z} v^* \) into two parts as follows:
\[
e^{\alpha z} v^* = v^a - \alpha e^{\alpha z} \psi e_r.
\]

It follows from Lemma 5.2 that one has
\[
\|e^{\alpha z} \psi e_r\|_{H^\frac{3}{5}(\Omega)} \leq \|e^{\alpha z} \psi e_r\|_{H^\frac{3}{5}(\Omega)} \leq C \|v^a\|_{H^1(\Omega)} \leq C \|v^a\|_{H^\frac{3}{5}(\Omega)}.
\]
This, together with (132), implies
\[ \|e^{\alpha z}(u - \tilde{U})\|_{H^\delta(\Omega \cap \{z \geq 0\})} \leq C\|e^{\alpha z}F\|_{L^2(\Omega)}. \] (134)
Similarly, one has
\[ \|e^{-\alpha z}(u - \tilde{U})\|_{H^\delta(\Omega \cap \{z \leq 0\})} \leq C\|e^{\alpha z}F\|_{L^2(\Omega)}. \] (135)
Hence, the proof of Proposition 5.3 is completed. □

Next, we remove the smallness requirement for \( u - \tilde{U} \) in Proposition 5.3. The key observation is that \( \|u - \tilde{U}\|_{H^2(\Omega_L)} \) with \( \Omega_L = B_1(0) \times (L, \infty) \) is sufficiently small for sufficiently large \( L \), provided that \( \|u - \tilde{U}\|_{H^2(\Omega)} \) is bounded. This implies that \( u \) satisfies the assumptions of Proposition 5.3 in the domain \( \Omega_L \), and hence Theorem 1.2 can be proved in the similar way.

Proof of Theorem 1.2. Let \( v = u - \tilde{U} \) and \( \psi \) be the corresponding stream function for \( v \). Let \( L \) be a positive constant to be determined. Denote \( \Omega_L = B_1(0) \times (L, +\infty) \). Choose a smooth cut-off function \( \eta(z) \) satisfying
\[ \eta(z) = \begin{cases} 0, & z \leq L, \\ 1, & z \geq L + 1. \end{cases} \]
Note that \((\psi, v^\theta)\) is a solution to the problem (26)–(29). Multiplying (26) by \( \eta(z) \) yields
\begin{align*}
\tilde{U}(r)\partial_z(\mathcal{L} + \partial_z^2)(\eta\psi) - (\mathcal{L} + \partial_z^2)^2(\eta\psi) \\
= \partial_z(\eta F^r + \tilde{F}^r) - \partial_r(\eta F^z + \tilde{F}^z) + \tilde{f} - \partial_r \left[ v^\theta(\mathcal{L} + \partial_z^2)(\eta\psi) \right] \\
- \partial_z \left[ v^z(\mathcal{L} + \partial_z^2)(\eta\psi) \right] + \partial_z \left[ \frac{v^\theta}{r}\eta v^\theta \right],
\end{align*}
(136)
where
\begin{align*}
\tilde{F}^r &= \tilde{U}(r)\left[ 2\eta'(z)\partial_z\psi + \eta''(z)\psi \right] - 4\eta'(z)\mathcal{L}\psi - 4\eta'(z)\partial_z^2\psi + 2v^\theta\eta'(z)\partial_z\psi + v^z\eta''(z)\psi, \\
\tilde{F}^z &= -\left[ 2v^\theta\eta'(z)\partial_z\psi + v^z\eta''(z)\psi \right],
\end{align*}
and
\begin{align*}
\tilde{f} &= -\eta'(z)F^r + \tilde{U}(r)\eta'(z)(\mathcal{L} + \partial_z^2)\psi + 2\eta''(z)\mathcal{L}\psi \\
&\quad - \left[ \eta^{(4)}(z)\psi + 4\eta^{(3)}(z)\partial_z\psi + 2\eta''(z)\partial_z^2\psi \right] \\
&\quad + v^z\eta'(z)(\mathcal{L} + \partial_z^2)\psi - \frac{v^\theta}{r}\eta'(z)v^\theta.
\end{align*}
Similarly, one has
\begin{align*}
\tilde{U}(r)\partial_z(\eta v^\theta) - \Delta(\eta v^\theta) &= \eta F^\theta + \tilde{F}^\theta - (v^r\partial_r + v^z\partial_z)(\eta v^\theta) - \frac{v^r}{r}\eta v^\theta, \\
\end{align*}
(137)
where
\[ \tilde{F}^\theta = U(r)\eta'(z)v^\theta - \left[2\eta'(z)\partial_z v^\theta + \eta''(z)v^\theta\right] + v^z\eta'(z)v^\theta. \]

Denote
\[ v_{\alpha, \eta}^* = \partial_z (e^{\alpha z} \eta \psi), \quad v_{\alpha, \eta}^\alpha = -\frac{\partial_r (r e^{\alpha z} \eta \psi)}{r}, \quad \text{and} \quad v_{\alpha, \eta}^* = v_{\alpha, \eta} e_r + v_{\alpha, \eta} e_z. \]

It follows from the same lines as in the proof of Proposition 5.3 that one has
\[
\|v_{\alpha, \eta}^\alpha\|_{H^2(\Omega)} + \|e^{\alpha z}(\eta v^\theta)\|_{H^2(\Omega)} \\
\leq C(1 + \Phi \frac{1}{2})(\|e^{\alpha z} \eta F\|_{L^2(\Omega)} + \|e^{\alpha z} \tilde{F}^r\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{F}^z\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{f}\|_{L^2_1(\Omega)}) \\
+ C_9(1 + \Phi \frac{1}{2})\|v\|_{H^\frac{5}{2}(\Omega)} \left(\|v_{\alpha, \eta}^\alpha\|_{H^2(\Omega)} + \|e^{\alpha z}(\eta v^\theta)\|_{H^2(\Omega)}\right) \\
+ C_{10}(1 + \Phi \frac{1}{2})|\alpha| \left(\Phi + 1 + \|v\|_{H^\frac{5}{2}(\Omega)}\right) \left(\|v_{\alpha, \eta}^\alpha\|_{H^2(\Omega)} + \|e^{\alpha z}(\eta v^\theta)\|_{H^2(\Omega)}\right).
\]

Since \( \|v\|_{H^\frac{5}{2}(\Omega)} < +\infty \), there exists an \( L \) large enough such that
\[
\|v\|_{H^\frac{5}{2}(\Omega)} \leq \min \left\{ \frac{1}{4C_9(1 + \Phi \frac{1}{2})}, 1 \right\}.
\]

Choose an \( \alpha_0 > 0 \) small enough such that
\[
C_{10}(1 + \Phi \frac{1}{2})(\Phi + 2)\alpha_0 \leq \frac{1}{4}.
\]

For every \( \alpha \in (0, \alpha_0) \), it holds that
\[
\|v_{\alpha, \eta}^\alpha\|_{H^2(\Omega)} + \|e^{\alpha z}(\eta v^\theta)\|_{H^2(\Omega)} \\
\leq C(1 + \Phi \frac{1}{2}) \left(\|e^{\alpha z} \tilde{F}^r\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{F}^z\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{f}\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{F}^\theta\|_{L^2_1(\Omega)}\right) \\
+ C(1 + \Phi \frac{1}{2})\|e^{\alpha z} \eta F\|_{L^2(\Omega)}.
\]

Note that \( \text{supp} \tilde{F}^r, \text{supp} \tilde{F}^z, \text{supp} \tilde{f}, \text{supp} \tilde{F}^\theta \subseteq B^2_1(0) \times [L, L + 1] \), then
\[
\|e^{\alpha z} \tilde{F}^r\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{F}^z\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{f}\|_{L^2_1(\Omega)} + \|e^{\alpha z} \tilde{F}^\theta\|_{L^2_1(\Omega)} \leq C(1 + \Phi)\|v\|_{H^1(\Omega)}.
\]

Hence, one has
\[
\|v_{\alpha, \eta}^\alpha\|_{H^2(\Omega)} + \|e^{\alpha z}(\eta v^\theta)\|_{H^2(\Omega)} < +\infty. \tag{140}
\]

Similar to the proof of Proposition 5.3, one can rewrite \( e^{\alpha z} \eta v^* \) as follows,
\[
e^{\alpha z} \eta v^* = v_{\alpha, \eta}^\alpha - (\alpha e^{\alpha z} \eta \psi + e^{\alpha z} \eta'(z)\psi) e_r.
\]

Thus it follows from Lemma 5.2 that one has
\[
\|(\alpha e^{\alpha z} \eta \psi)e_r\|_{H^2(\Omega)} + \|e^{\alpha z} \eta'(z)\psi e_r\|_{H^2(\Omega)} \leq C\|v_{\alpha, \eta}^\alpha\|_{H^1(\Omega)} + C\|v\|_{H^1(\Omega)} < +\infty. \tag{141}
\]

This, together with (140), completes the proof of Theorem 1.2. \( \square \)
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Appendix A: Some Elementary Lemmas

In this appendix, we collect some basic lemmas which play important roles in the paper. The proof of Lemmas A.1–A.3 can be found in [27, Appendix A], so we omit the details here.

The following lemma is about Poincaré-type inequalities:

**Lemma A.1.** For a function $g \in C^2([0, 1])$ it holds that

$$\int_0^1 |g|^2 r \, dr \leq \int_0^1 \left| \frac{d}{dr} (rg) \right|^2 \frac{1}{r} \, dr.$$  \hspace{1cm} (142)

If, in addition, $g(0) = g(1) = 0$, then one has

$$\int_0^1 \left| \frac{d}{dr} (rg) \right|^2 \frac{1}{r} \, dr \leq \left( \int_0^1 |\mathcal{L}g|^2 r \, dr \right)^{\frac{1}{2}} \left( \int_0^1 |g|^2 r \, dr \right)^{\frac{1}{2}} \leq \int_0^1 |\mathcal{L}g|^2 r \, dr.$$  \hspace{1cm} (143)

The following lemma is a variant of Hardy–Littlewood–Pólya-type inequality [11], which plays an important role in many estimates in the paper:

**Lemma A.2.** Let $g \in C^1([0, 1])$ satisfy $g(0) = 0$, one has

$$\int_0^1 |g(r)|^2 \, dr \leq \frac{1}{2} \int_0^1 |g'(r)|^2 (1 - r^2) \, dr,$$  \hspace{1cm} (144)

and

$$\int_0^1 |g|^2 r \, dr \leq C \int_0^1 \left| \frac{d}{dr} (rg) \right|^2 \frac{1 - r^2}{r} \, dr.$$  \hspace{1cm} (145)

The following lemma is about a weighted interpolation inequality, which is quite similar to [9, (3.28)].

**Lemma A.3.** Let $g \in C^2[0, 1]$, then one has

$$\int_0^1 |g|^2 r \, dr \leq C \left\{ \int_0^1 (1 - r^2)|g|^2 r \, dr \right\}^\frac{2}{3} \left\{ \int_0^1 \left| \frac{d}{dr} (rg) \right|^2 \frac{1}{r} \, dr \right\}^\frac{1}{3} + C \int_0^1 (1 - r^2)|g|^2 r \, dr.$$  \hspace{1cm} (146)
and

\[ \int_0^1 \left| \frac{d}{dr} (rg) \right|^2 r dr \leq C \left[ \int_0^1 \frac{1 - r^2}{r} \left| \frac{d}{dr} (rg) \right|^2 dr \right]^{2/3} \left( \int_0^1 |\mathcal{L}g|^2 r dr \right)^{1/3} \]

(147)

The following lemma is about the interpolation inequality which was used frequently in the paper.

**Lemma A.4.** [1, Theorem 5.2] Let \( \Omega \) be a domain in \( \mathbb{R}^n \) satisfying the cone condition. There exists a constant \( C \), depending on \( n, m, p, \) and \( \Omega \), such that if \( 0 \leq j \leq m \) and \( u \in W^{m, p}(\Omega) \), then

\[ \| u \|_{W^{j, p}(\Omega)} \leq C \| u \|_{L^p(\Omega)}^{1 - \frac{j}{m}} \| u \|_{W^{m, p}(\Omega)}^{\frac{j}{m}}. \]

(148)

The following Gagliardo–Nirenberg inequality can be regarded as a more general interpolation inequality:

**Lemma A.5.** [8, Lemma II.3.3, Theorem II.3.3] Let \( \Omega \) be a locally Lipschitz domain in \( \mathbb{R}^n \). There exists a constant \( C \), depending on \( n, m, r, q, j, \alpha, \) and \( a \), such that if \( u \in W^{m, r}(\Omega) \cap L^q(\Omega) \), then

\[ \| u \|_{W^{j, s}(\Omega)} \leq C \| u \|_{W^{m, r}(\Omega)}^{a} \| u \|_{L^q(\Omega)}^{1-a}, \]

(149)

where

\[ \frac{1}{s} = \frac{j}{n} + a \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - a) \frac{1}{q}, \]

for all \( a \) in the interval

\[ \frac{j}{m} \leq a \leq 1, \]

with the following exceptional cases: (1) If \( j = 0, rm < n, q = \infty \); (2) if \( 1 < r < \infty \), and \( m - j - n/r \) is a nonnegative integer, then (149) holds only for \( a \) satisfying \( j/m \leq a < 1 \).

**Appendix B: Uniform Estimate for the Solutions**

In this appendix, we give a sketch of the proof for Propositions 2.1 and 2.2.
Multiplying the both sides of the Eq. (34) with $\frac{1}{r}$ and integrating over $(0, 1)$ yield

$$
\int_0^1 |\mathcal{L} \hat{\psi}|^2 r \, dr + 2\xi^2 \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} \, dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r \, dr
= -\Re \int_0^1 \left( \mathcal{F}^r (i \xi \hat{\psi}) r + \mathcal{F}^\varphi \partial_r (r \hat{\psi}) \right) dr - \frac{4\Phi}{\pi} \xi \int_0^1 \left[ \frac{d}{dr}(r \hat{\psi}) r \hat{\psi} \right] \, dr
$$

(150)

and

$$
\xi \int_0^1 \frac{\tilde{U}(r)}{r} \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \, dr + \xi^3 \int_0^1 \tilde{U}(r) |\hat{\psi}|^2 r \, dr = -\Im \int_0^1 \left( \mathcal{F}^r (i \xi \hat{\psi}) r + \mathcal{F}^\varphi \partial_r (r \hat{\psi}) \right) dr.
$$

(151)

Applying Lemma A.2 for (151) gives

$$
\Phi |\xi| \int_0^1 |\hat{\psi}|^2 r \, dr \leq C \left( \int_0^1 |\mathcal{F}^r| |\xi| |r \hat{\psi}| \, dr + \int_0^1 |\mathcal{F}^\varphi| \left| \frac{d}{dr}(r \hat{\psi}) \right| \, dr \right).
$$

(152)

This, together with Cauchy–Schwarz inequality and Lemma A.1, gives

$$
\frac{4\Phi}{\pi} \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right| |r \hat{\psi}| \, dr
$$

$$
\leq C \Phi \frac{2}{\xi} \left[ |\xi| \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} \, dr \right]^{\frac{1}{2}} \left( \Phi |\xi| \int_0^1 |\hat{\psi}|^2 r \, dr \right)^{\frac{1}{2}}
$$

$$
\leq \frac{1}{4} (1 + \xi^2) \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} \, dr + C \Phi \left( \int_0^1 |\mathcal{F}^r| |\xi| |r \hat{\psi}| \, dr + \int_0^1 |\mathcal{F}^\varphi| \left| \frac{d}{dr}(r \hat{\psi}) \right| \, dr \right)
$$

(153)

By virtue of Lemma A.1 and Cauchy–Schwarz inequality, one has

$$
\int_0^1 \left( |\mathcal{L} \hat{\psi}|^2 r + \xi^2 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} + \xi^4 |\hat{\psi}|^2 r \right) \, dr \leq C (1 + \Phi^2) \int_0^1 (|\mathcal{F}^r|^2 + |\mathcal{F}^\varphi|^2) r \, dr.
$$

(154)

Integrating (154) with respect to $\xi$ yields

$$
\int_{-\infty}^{+\infty} \int_0^1 \left( (|\mathcal{L} \hat{\psi}|^2 + \xi^2 |\hat{\psi}|^2) r + \xi^2 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} \right) \, dr \, d\xi \leq C (1 + \Phi^2) \| F^* \|^2_{L^2(\Omega)}
$$

(155)

where $F^* = F^r e_r + F^\varphi e_\varphi$. Note that $\omega^\theta = (\mathcal{L} + \partial^2_\varphi) \psi e_\theta$, one has

$$
\| v^* \|_{H^1(\Omega)} \leq C \| \nabla v^* \|_{L^2(\Omega)} = C \| \omega^\theta \|_{L^2(\Omega)} \leq C (1 + \Phi) \| F^* \|_{L^2(\Omega)},
$$
where $v^* = v^r e_r + v^z e_z$. Applying the regularity theory for Stokes equations ([8]) gives
\[
\|v^*\|_{H^2(\Omega)} \leq C[\|F^*\|_{L^2(\Omega)} + \Phi \|\partial_z v^*\|_{L^2(\Omega)} + \Phi \|v^r\|_{L^2(\Omega)} + \|v^*\|_{H^1(\Omega)}]
\leq C(1 + \Phi^2)\|F^*\|_{L^2(\Omega)}.
\] (156)

**B.2. Case with large flux and low frequency ($|\xi| \leq \frac{1}{\epsilon_1 \Phi}$)**

It follows from the energy estimate as (154) gives
\[
\int_0^1 |\hat{L}\hat{\psi}|^2 r dr + 2\xi^2 \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r dr \leq C(\epsilon_1) \int_0^1 |\hat{F}^*|^2 r dr.
\] (157)

Let
\[
\chi_1(\xi) = \begin{cases} 
1, & |\xi| \leq \frac{1}{\epsilon_1 \Phi}, \\
0, & \text{otherwise},
\end{cases}
\]
and $\psi_{low}$ be the function such that $\hat{\psi}_{low} = \chi_1(\xi) \hat{\psi}$. Define
\[
v^r_{low} = \partial_z \psi_{low}, \quad v^z_{low} = -\frac{\partial_r (r \psi_{low})}{r}, \quad \text{and} \quad v^*_{low} = v^r_{low} e_r + v^z_{low} e_z.
\]
Similarly, one can define $F^r_{low}, F^z_{low}, F^*_{low}, \omega^\theta_{low}$. It follows from the regularity estimate for the Stokes equations ([8]) that one has
\[
\|v^*_{low}\|_{H^2(\Omega)} \leq C\|F^*_{low}\|_{L^2(\Omega)},
\] (158)
where $C$ is a uniform constant independent of $\Phi$ and $F$.

**B.3. Case with large flux and high frequency ($|\xi| \geq \epsilon_1 \sqrt{\Phi}$)**

The energy estimate as (154) yields
\[
\int_0^1 |\hat{L}\hat{\psi}|^2 r dr + 2\xi^2 \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 \frac{1}{r} dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r dr \leq C(\epsilon_1)|\xi|^{-2} \int_0^1 |\hat{F}^*|^2 r dr
\] (159)
and
\[
\Phi|\xi| \int_0^1 \frac{1 - r^2}{r} \left| \frac{d}{dr}(r \hat{\psi}) \right|^2 dr + \Phi|\xi|^3 \int_0^1 (1 - r^2)|\hat{\psi}|^2 r dr \leq C(\epsilon_1)|\xi|^{-2} \int_0^1 |\hat{F}^*|^2 r dr.
\] (160)
Thus, one has
\[ \| v_{\text{high}}^* \|_{H^1(\Omega)} \leq C \| \omega_{\text{high}}^\theta \|_{L^2(\Omega)} \leq C \Phi^{-\frac{1}{2}} \| F_{\text{high}}^* \|_{L^2(\Omega)}, \quad (161) \]
where
\[ v_{\text{high}}^* = v_{\text{high}}^r e_r + v_{\text{high}}^z e_z \]
with
\[ v_{\text{high}}^r = \partial_z \psi_{\text{high}}, \quad v_{\text{high}}^z = -\frac{\partial_r(r\psi_{\text{high}})}{r}, \]
and \( \hat{\psi}_{\text{high}} = \chi_2(\xi) \hat{\psi} \) with
\[ \chi_2(\xi) = \begin{cases} 1, & |\xi| \geq \epsilon_1 \sqrt{\Phi}, \\ 0, & \text{otherwise}. \end{cases} \]

Note that one can define \( F_{\text{high}}^r, F_{\text{high}}^z, F_{\text{high}}^*, \) and \( \omega_{\text{high}}^\theta \) in the similar way. Hence, the regularity estimate for the Stokes equations (8) gives
\[ \| v_{\text{high}}^* \|_{H^2(\Omega)} \leq C (1 + \Phi^{\frac{1}{4}}) \| F_{\text{high}}^* \|_{L^2(\Omega)}. \quad (162) \]
Using the interpolation for (161) and (162) gives
\[ \| v_{\text{high}}^* \|_{H^{\frac{5}{3}}(\Omega)} \leq C \| F_{\text{high}}^* \|_{L^2(\Omega)}. \quad (163) \]

**B.4. Large flux and intermediate frequency \((\frac{1}{\epsilon_1 \Phi} \leq |\xi| \leq \epsilon_1 \sqrt{\Phi})\)**

When the frequency belongs to the intermediate regime, i.e., \( \frac{1}{\epsilon_1 \Phi} \leq |\xi| \leq \epsilon_1 \sqrt{\Phi} \), inspired by the analysis in [9], \( \psi \) can be decomposed into four parts,
\[ \hat{\psi}(r) = \hat{\psi}_s(r) + a I_1(|\xi| r) + b(\chi \hat{\psi}_{BL} + \hat{\psi}_e). \quad (164) \]
In what follows, we give the detailed explanations for \( \hat{\psi}_s, \hat{\psi}_{BL}, \hat{\psi}_e, \) and \( I_1 \).
\( \hat{\psi}_s \) is a solution to the problem
\[ \begin{cases} i\xi \tilde{U}(r)(\mathcal{L} - \xi^2) \hat{\psi}_s - (\mathcal{L} - \xi^2)^2 \hat{\psi}_s = \hat{f}, \\ \hat{\psi}_s(0) = \hat{\psi}_s(1) = \mathcal{L} \hat{\psi}_s(0) = \mathcal{L} \hat{\psi}_s(1) = 0. \end{cases} \quad (165) \]
Multiplying the both sides of the equation in (165) with \( r \hat{\psi}_s \) and \( r \mathcal{L} \hat{\psi}_s \), respectively, and using the integration by parts, yields
\[ \int_0^1 |\hat{\psi}_s|^2 r \, dr \leq C (\Phi |\xi|)^{-\frac{3}{2}} \int_0^1 |F^2|^2 r \, dr, \quad (166) \]
\[ \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}_s) \right|^2 r \, dr + \xi^2 \int_0^1 |\hat{\psi}_s|^2 r \, dr \leq C (\Phi |\xi|)^{-\frac{3}{2}} \int_0^1 |F^2|^2 r \, dr, \quad (167) \]
\[ \int_0^1 |\mathcal{L} \hat{\psi}_s|^2 r \, dr + \xi^2 \int_0^1 \left| \frac{d}{dr}(r \hat{\psi}_s) \right|^2 r \, dr + \xi^4 \int_0^1 |\hat{\psi}_s|^2 r \, dr \leq C (\Phi |\xi|)^{-\frac{3}{2}} \int_0^1 |F^2|^2 r \, dr, \quad (168) \]
\[ \int_0^1 \left| \frac{d}{dr} (r \mathcal{L} \tilde{\psi}_s) \right|^2 \frac{1}{r} + \xi^2 |\mathcal{L} \tilde{\psi}_s|^2 r + \xi^4 \left| \frac{d}{dr} (r \mathcal{L} \tilde{\psi}_s) \right|^2 \frac{1}{r} + \xi^6 |\mathcal{L} \tilde{\psi}_s|^2 r \, dr \leq C \int_0^1 |\tilde{F}|^2 r \, dr. \]  

\( \tilde{\psi}_{BL} \) is the boundary layer profile, which is the exact solution (exponentially decay away from \( r = 1 \)) of the equation

\[ \left( i \frac{\xi \Phi}{\pi} 4(1 - r) - \frac{d^2}{dr^2} + \xi^2 \right) \left( \frac{d^2}{dr^2} - \xi^2 \right) \tilde{\psi}_{BL} = 0. \]  

In fact, one can represent \( \tilde{\psi}_{BL} \) in terms of the Airy function. Let \( |\beta| = \left( \frac{4|\xi|\Phi}{\pi} \right)^{\frac{1}{3}} \) and

\[ \tilde{G}_{\xi, \Phi}(\rho) = \begin{cases} Ai \left( C_+ (\rho + \frac{\pi |\beta| \xi}{4i \Phi}) \right), & \text{when } \xi > 0, \\ Ai \left( C_- (\rho + \frac{\pi |\beta| \xi}{4i \Phi}) \right), & \text{when } \xi < 0, \end{cases} \]  

where \( C_+ = e^{i \pi \theta} \), \( C_- = e^{-i \pi \theta} \), and \( Ai(z) \) denotes the Airy function satisfying

\[ \frac{d^2}{dz^2} - z Ai = 0 \text{ in } \mathbb{C}. \]

Define

\[ G_{\xi, \Phi}(\rho) = \int_\rho^{+\infty} e^{-|\xi| \rho} \int_\tau^{+\infty} e^{-|\xi| \tau} \tilde{G}_{\xi, \Phi}(\sigma) d\sigma d\tau \]  

and

\[ C_{0, \xi, \Phi} = \begin{cases} \frac{1}{G_{\xi, \Phi}(0)}, & \text{if } |G_{\xi, \Phi}(0)| \geq 1, \\ 1, & \text{otherwise}. \end{cases} \]

Then,

\[ \tilde{\psi}_{BL}(r) := C_{0, \xi, \Phi} G_{\xi, \Phi}(|\beta| (1 - r)). \]

satisfies \( |\tilde{\psi}_{BL}(1)| \leq 1 \) and solves the equation (170). \( \tilde{\psi}_e \) is a remainder term, which satisfies the problem

\[ \begin{cases} i \xi \tilde{U}(r)(\mathcal{L} - \xi^2)(\mathcal{L} \tilde{\psi}_{BL} + \tilde{\psi}_e) - (\mathcal{L} - \xi^2)^2 (\mathcal{L} \tilde{\psi}_{BL} + \tilde{\psi}_e) = 0, \\ \tilde{\psi}_e(0) = \tilde{\psi}_e(1) = \mathcal{L} \tilde{\psi}_e(0) = \mathcal{L} \tilde{\psi}_e(1) = 0, \end{cases} \]

where \( \chi \) is a smooth cut-off function satisfying

\[ \chi(r) = 1 \text{ if } r \geq \frac{1}{2} \text{ and } \chi(r) = 0 \text{ if } r \leq \frac{1}{4}. \]
Then, one can have the estimates

\[
\int_0^1 |\hat{\psi}|^2 r \, dr \leq C|\beta|^5 (\Phi|\xi|)^{-2} \leq C(\Phi|\xi|)^{-\frac{5}{3}},
\]

(175)

\[
\int_0^1 |L\hat{\psi}|^2 r + \xi^2 \left( \frac{d}{dr}(r\hat{\psi}) \right)^2 \frac{1}{r} + \xi^4 |\hat{\psi}|^2 r \, dr \leq C|\beta|^5 (\Phi|\xi|)^{-\frac{7}{3}} \leq C(\Phi|\xi|)^{\frac{10}{3}},
\]

(176)

\[
\int_0^1 \left( \frac{d}{dr}(rL\hat{\psi}) \right)^2 \frac{1}{r} + \xi^2 |L\hat{\psi}|^2 r + \xi^4 \int_0^1 \left( \frac{d}{dr}(r\hat{\psi}) \right)^2 \frac{1}{r} + \xi^6 |\hat{\psi}|^2 r \, dr \leq C(\Phi|\xi|)^{\frac{5}{3}}.
\]

(177)

\[I_1(z)\] is the modified Bessel function of the first kind, which satisfies

\[
\begin{cases}
z^2 \frac{d^2}{dz^2} I_1(z) + z \frac{d}{dz} I_1(z) - (z^2 + 1)I_1(z) = 0, \\
I_1(0) = 0, \quad I_1(z) > 0 \text{ for } z > 0.
\end{cases}
\]

This implies

\[(L - \xi^2)I_1(|\xi|r) = 0.
\]

The no-slip boundary conditions (35) can be recovered if \(a\) and \(b\) satisfy

\[
\begin{cases}
aI_1(|\xi|) + b\hat{\psi}_{BL}(1) = 0, \\
|\xi|I'_1(|\xi|) + b \frac{d}{dr}\hat{\psi}_{BL}(1) + b \frac{d}{dr}\hat{\psi}_{e}(1) = -\frac{d}{dr}\hat{\psi}_s(1).
\end{cases}
\]

(178)

With the aid of the estimate for \(\hat{\psi}_s, \hat{\psi}_{BL},\) and \(\hat{\psi}_e,\) and the properties of \(I_1,\) one has

\[
|b| \leq C(\Phi|\xi|)^{-\frac{5}{6}} \left( \int_0^1 |\hat{F}^\ast|^2 r \, dr \right)^{\frac{1}{2}}
\]

(179)

and

\[
|a| \leq C(\Phi|\xi|)^{-\frac{5}{6}} |I_1(|\xi|)|^{-1} \left( \int_0^1 |\hat{F}^\ast|^2 r \, dr \right)^{\frac{1}{2}}.
\]

(180)

Therefore, \(\hat{\psi}\) defined in (164) satisfies

\[
\int_0^1 |L\hat{\psi}|^2 r + \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 \frac{1}{r} + |\hat{\psi}|^2 r \, dr \\
+ \int_0^1 \left| \frac{d}{dr}(rL\hat{\psi}) \right|^2 \frac{1}{r} + \xi^2 |L\hat{\psi}|^2 r + \xi^4 \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 \frac{1}{r} + \xi^6 |\hat{\psi}|^2 r \, dr \\
\leq C \int_0^1 |\hat{F}^\ast|^2 \, dr.
\]

(181)
Combining the estimates (154), (157), (159), and (181) together gives Proposition 3.1.
Let
\[
\chi_3(\xi) = \begin{cases} 
1, & \frac{1}{\epsilon_1 \Phi} \leq |\xi| \leq \epsilon_1 \sqrt{\Phi}, \\
0, & \text{otherwise},
\end{cases}
\]
and \(\psi_{med}\) be the function such that \(\hat{\psi}_{med} = \chi_3(\xi) \hat{\psi}\). Define
\[
v^r_{med} = \partial_z \psi_{med}, \quad v^z_{med} = -\frac{\partial_r (r \psi_{med})}{r}, \quad \text{and} \quad v^*_m = v^r_{med} e_r + v^z_{med} e_z.
\]
Similarly, we define \(F^r_{med}, F^z_{med}\), and \(F^*_m\). The solution \(v^*_m\) satisfies
\[
\|v^*_m\|_{H^2(\Omega)} \leq C \|F^*_m\|_{L^2(\Omega)},
\]
where \(C\) is a uniform constant independent of \(\Phi\) and \(F\). Combining the estimates (156), (158), (162), (163), and (182) together gives the estimates (38) and (39) in Proposition 2.1.

**B.5. Estimate for the swirl velocity**

Multiplying the equation in (36) by \(r \hat{v}^\theta\) yields
\[
\int_0^1 \left| \frac{d}{dr} (r \hat{v}^\theta) \right|^2 \frac{1}{r} dr + \xi^2 \int_0^1 |\hat{v}^\theta|^2 r dr = \Re \int_0^1 \hat{F}^\theta \hat{v}^\theta r dr + \frac{2 \Phi}{\pi} \xi^2 \int_0^1 \frac{2}{r} \hat{v}^\theta d \left( r \hat{v}^\theta \right) dr.
\]
and
\[
\frac{2 \Phi}{\pi} \xi^2 \int_0^1 (1 - r^2) |\hat{v}^\theta|^2 r dr = \Im \int_0^1 \hat{F}^\theta \hat{v}^\theta r dr.
\]
It follows from Lemma A.2 that one has
\[
\int_0^1 \left| \frac{d}{dr} (r \hat{v}^\theta) \right|^2 \frac{1}{r} dr + \xi^2 \int_0^1 |\hat{v}^\theta|^2 r dr \leq \int_0^1 |\hat{F}^\theta|^2 r dr.
\]
Using Lemma A.3 gives
\[
\int_0^1 |\hat{v}^\theta|^2 r dr \leq C (\Phi |\xi|)^{-\frac{1}{2}} \int_0^1 |\hat{F}^\theta|^2 r dr.
\]

Multiplying the equation in (36) by \(r (\mathcal{L} - \xi^2) \hat{v}^\theta\) yields
\[
- \int_0^1 ( \mathcal{L} - \xi^2 ) |\hat{v}^\theta|^2 r dr = \Re \int_0^1 \hat{F}^\theta (\mathcal{L} + \xi^2) \hat{v}^\theta r dr + \frac{2 \Phi}{\pi} \xi \Im \int_0^1 2 r \hat{v}^\theta \frac{d}{dr} (r \hat{v}^\theta) dr.
\]
Thus, one has
\[
\int_0^1 \left( |\mathcal{L} \hat{v}^\theta|^2 + 2 \xi^2 \left| \frac{1}{r} \frac{d}{dr} (r \hat{v}^\theta) \right|^2 + \xi^4 |\hat{v}^\theta|^2 \right) r \, dr \leq C \int_0^1 |F^\theta|^2 r \, dr.
\]
This proves the estimate (62) in Proposition 3.4. Then, the estimate (40) can be easily obtained via the estimate for elliptic equation.

With the aid of these estimates, the existence of the solutions for the problems (34)–(35) and (36) can be established via the Galerkin method. The detailed construction for the bases needed for the Galerkin method is given in [27].

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