The Geometry of Deformed Boson Algebras

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Abstract

Phase-space realisations of an infinite parameter family of quantum deformations of the boson algebra in which the $q$– and the $qp$–deformed algebras arise as special cases are studied. Quantum and classical models for the corresponding deformed oscillators are provided. The deformation parameters are identified with coefficients of non-linear terms in the normal forms expansion of a family of classical Hamiltonian systems. These quantum deformations are trivial in the sense that they correspond to non-unitary transformations of the Weyl algebra. They are non-trivial in the sense that the deformed commutators consistently quantise a class of non-canonical classical Poisson structures.

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\footnote{This work is dedicated to John Kennedy my former teacher who died of cancer 20th of January 1995}
I Introduction

It has been pointed out that the quantum group is not an intrinsically quantum concept. Flato and Lu have provided a purely classical realisation of $U_q(sU(2))$ in the canonical Poisson algebra of functions on $\mathbb{R}^2$ [1], while Chang, Chen and Guo have done the same on $\mathbb{R}^4$ [2]. In particular they have constructed classical Poisson brackets which depend on the $q$-deformation parameter, such that the fundamental Poisson bracket relations are those of $U_q(sU(2))$. They conclude that although $q$-deformation is related to $\ast$-deformation, it is not the same as quantisation. Their results demonstrate a need to clarify the difference between quantum-deformation and quantisation, and to understand the essential role of the deformation parameter. In a recent study of the $q$-boson, Man’ko, Marmo, Solimeno and Zaccaria propose that the role of the deformation parameter is to introduce non-linearity in the deformed oscillator [3].

From a different point of view the Weyl algebra is known to be rigid [4]. It does not possess a non-trivial associative deformation in the sense that any associative deformation of the Weyl algebra is formally isomorphic to the Weyl algebra itself. This applies in particular to quantum deformations of the Weyl algebra related to the $q$- and the $qp$-bosons. The isomorphisms which relate the deformed algebras to the Weyl algebra cannot be unitary since they do not preserve the fundamental commutation relations. The real problem therefore is to understand these non-unitary equivalence transformations. We would like to know if quantum deformation has a natural geometrical meaning.

In §II we introduce families of deformations of the Weyl algebra and of the Heisenberg Lie algebra which depend on an infinite number of independent parameters. In §III we introduce the tools we will need from phase-space quantum mechanics. We extend these to the deformed phase-space in §III.1. The corresponding Moyal algebras are studied using bimodular bases of deformed oscillator eigenstates and bases of anti-standard ordered monomials in §IV.1 and §IV.2 respectively. In the eigenstate bases it is easy to see the isomorphism between the deformed algebras. In §V.2 we demonstrate an isomorphism between the deformed algebras and the algebra of infinite matrices. Quantum models of the deformed algebras and their associated oscillators are dealt with in §IV.4. In §VI we show how phase-space representations of the deformed algebras are related by non-linear non-unitary transformations. We discuss the provision of co-product and Hopf-algebra structures in §VI.

In §VII we consider the classical limit of the deformed quantum dynamics. The deformed commutators contract to non-canonical Poisson brackets. We identify the deformed Weyl algebras with non-linear deformations of the harmonic oscillator, and in §VIII we reveal a relationship with the classical method of normal forms.

The overall picture which emerges is that quantum deformations of the Weyl algebra are trivial in the sense that they correspond to non-linear non-unitary equivalence transformations of $U(W(1))$. The parameters which arise in quantum deformation of the Weyl algebra have a geometrical meaning in that they parameterise a family of non-canonical transformations of classical phase-space. The quantum deformations are non-trivial in the sense that they quantise a family of non-canonical classical Poisson brackets. In §IX we reinterpret the algebras introduced in §II as quantum algebras which consistently quantise the infinite number of Hamiltonian structures of the simple harmonic oscillator. This indicates a number of directions in which certain applications of quantum groups can naturally be extended.
II  Deformed boson algebras

The defining relations for the $q$-deformed boson algebra have been given in many different forms \[5\], \[6\]. Taking $A$, and $A^+$ to be the $q$-deformed annihilation and creation operators, the number operator $N$ always satisfies

\[[N, A] = -A \quad [N, A^+] = A^+ \]

while $A$ and $A^+$ are related as follows

\[AA^+-qA^+A = q^{-N}.\]

A generalisation known as the $qp$-boson \[7\], \[8\] satisfies

\[AA^+-qA^+A = q^{N+1} - p^{N+1} - \frac{q^N - p^N}{q - p}.\]

A further generalisation satisfies

\[AA^+-qA^+A = P_q(N)\]

where $P_q(N)$ is a polynomial of finite order in the number operator $N$ whose coefficients functionally depend on a single deformation parameter $q \[9\].

The relations (1) along with (2), (3) or (4) constitute re-ordering rules or equivalence relations for the infinite polynomial ring generated by $A$, $A^+$, $N$ and 1. They are deformations of the standard boson algebra in the sense that in the limit $q,p \to 1$, (2), (3) and (4) reduce to

\[AA^+-qA^+A = 1.\]

In any given representation these re-ordering rules can be replaced by equivalent commutation relations. In the case of the $q$-deformed boson algebra for example, (2) can be replaced by one of the following commutation relations, one for each of the three families of representations of $AA^+-qA^+A = q^{-N}$ provided by Rideau \[10\].

\[\begin{align*}
[A, A^+] & = q^{-\nu_0}([N + 1]_q - [N]_q) \\
[A, A^+] & = q^{-\nu_0}([N + 1]_q - [N]_q) + \lambda_0(q - 1)q^N \\
[A, A^+] & = -q^{-\nu_0}(1 + q)^{-1}q^{-N}.
\end{align*}\]

We have modified our notion slightly so that $N$ satisfies $N|n\rangle = n|n\rangle$ on the relevant Fock-space. The advantage to be gained by rewriting (2) in this way is that the equivalence relations now
have a natural dynamical interpretation. On the basis of (6) we propose that any representation of a deformed Weyl-algebra has an equivalent formulation in terms of commutators. From now on we will consider algebras which consist of the infinite ring of polynomials generated by $A$, $A^+$, $N$ and 1 modulo the following relations

\[
\begin{align*}
[A,N] & = A \\
[N,A^+] & = A^+ \\
[A,A^+] & = f_q(N).
\end{align*}
\]

We could have replaced the commutator of $A$ and $A^+$ with something of the form $AA^+ - qA^+A = f_q(N)$. Doing so makes no difference to our results and the form that we have chosen has the advantage of being easier to interpret dynamically. For simplicity we assume that $f_q$ is an entire function of the complex plane such that $f_q(\mathbb{R}^+) \geq 0$. Everything we do will be valid for functions which are real analytic and positive on $[0, \infty)$. In fact $f_q(x)$ is not so much a function as an equivalence class of functions defined by the sequence of values taken on the positive integers. In a neighbourhood of the origin we identify $f_q$ with a power series expansion $f_q(N) = q_0 + q_1 N + q_2 N^2 + \ldots$ whose coefficients satisfy $q_i \in \mathbb{R}$ for all $0 \leq i \in \mathbb{Z}$ and constitute independent deformation parameters. We denote this category of deformed algebras by $\mathbb{W}$.

The Heisenberg algebra $\mathfrak{h}(1)$ is closely related to the Weyl-algebra. It is the Lie algebra with three generators $A$, $A^+$ and $E$ whose only non-vanishing bracket is given by $[A,A^+] = \hbar E$. From a physical point of view the Weyl-algebra corresponds to a phase-space realisation of $\mathfrak{h}(1)$, it is the quotient of $\mathfrak{h}(1)$ with the two-sided ideal $E = 1$, it is not a Lie algebra, and according to one authority is not known to possess a Hopf-algebra structure [11]. The Heisenberg Lie algebra does have a Hopf-algebra structure, and some authors have shown that it is instructive to consider In"on"u-Wigner contractions from $q$-deformed semi-simple quantum groups such as $U(q(2))$ to the $q$-deformed Heisenberg group $U(\mathfrak{h}(1))$ [12]. There are advantages to be gained from considering the deformation of boson algebras from the point of view of the Heisenberg Lie algebra, but there is no unique way in which to extend Weyl-type algebras satisfying relations such as (2) or (6) to Heisenberg-type algebras. For example one could replace (2) with $AA^+ - qE^+A = Eq^{-N}$, or any one of an infinite number of other choices. In any case we will keep in mind that it is possible to consider a family of deformed algebras of Heisenberg type generated by $A$, $A^+$, $E$ and $N$, whose non-vanishing commutation relations can be given in the following form

\[
\begin{align*}
[A,N] & = EA \\
[N,A^+] & = EA^+ \\
[A,A^+] & = f_q(N,E).
\end{align*}
\]

Various restrictions should apply to $f_q(x,y)$ so that it is analytic and positive on $\mathbb{R}^+ \times \mathbb{R}^+$. As before the coefficients of its power series expansion about the origin can be thought of as independent deformation parameters. The infinite polynomial ring generated by $A$, $A^+$, $E$ and $N$ modulo (8) is a deformation of $U(\mathfrak{h}(1))$ in the sense that for suitable values of these coefficients
we recover the commutation relations of the Heisenberg algebra with \( N = A^+ A \). We denote these deformations of the Heisenberg Lie algebra by \( \hat{\mathfrak{h}}_q \).

In the next section we summarise relevant results from the phase-space theory of the boson oscillator, which we adapt to study \( \mathcal{W}_q \in \mathcal{W} \), the related \( \hat{\mathfrak{h}}_q \) and the deformed phase-space dynamics of their associated classical and quantum oscillators.

### III Deformed phase-space dynamics

In the phase-space quantum mechanical formalism the space of states has a bimodular structure like that of the algebra of observables. By considering the eigenvalue problem of the harmonic oscillator it is possible to construct a basis for the space of states which consists of left-right eigenstates of the harmonic oscillator Hamiltonian. Eventually the algebra of observables can be thought of as lying in the closure of finite linear combinations of states with respect to the appropriate topology. We will use an extension of results such as these in developing the theory of deformed bosonic algebras. We start by describing in more detail the case of the harmonic oscillator.

In this formalism the vacuum state is required to satisfy

\[
a \ast \Omega_{00} = 0 = \Omega_{00} \ast a^+,
\]

where \( f \ast g \) stands for the Moyal product of \( f \) and \( g \). The solution to this equation is unique up to a constant and when properly normalised satisfies \( \Omega_{00} \ast \Omega_{00} = \Omega_{00} \). The left-right eigenstates of the harmonic oscillator are given by

\[
\Omega_{nm} = \frac{1}{(n! m! \hbar^{n+m})^{1/2}} a^+^n \ast \Omega_{00} \ast a^m
\]

and have the following remarkable properties

\[
\Omega_{nm}(q,p) = \hat{\Omega}_{nm}(q,p) \\
\Omega_{nm}(q,p) \ast \Omega_{nm'}(q,p) = \delta_{mm'} \Omega_{nm'}(q,p)
\]

as well as

\[
(2\pi\hbar)^{-1} \int_{\mathbb{R}^2} \Omega_{nm}(q,p) \Omega_{nm'}(q,p) \ dq \wedge dp = \delta_{nm} \delta_{mm'}
\]

\[
(2\pi\hbar)^{-1} \int_{\mathbb{R}^2} \Omega_{nm}(q,p) \ dq \wedge dp = \delta_{nm}.
\]

These functions were introduced by Hansen who showed that they provide a basis for the Schwartz space \( \mathcal{S}^{'}(\mathbb{R}^2) \), the Hilbert space of quantum states on phase-space \( L^2(\mathbb{R}^2, dq \wedge dp) \) and for \( \mathcal{S}^{'}(\mathbb{R}^2) \) the space of tempered distributions of \( \mathcal{S}(\mathbb{R}^2) \) \[13\]. It is customary to study \( \mathcal{W} \) as a quotient of the

5
infinite polynomial ring generated by $a$ and $a^+$. Instead of using a basis of Weyl-ordered monomials, we found it more convenient to work with Anti-Standard Ordered Monomials ASOM’s. In this basis the intertwiner of $W$ is given by

$$f(a, a^+) \cdot g(a, a^+) = \exp\left(h \cdot \partial^{(1)}_a \cdot \partial^{(2)}_a\right) f^{(1)}(a, a^+) \cdot g^{(2)}(a, a^+)$$

(13)

where $f$ and $g$ are the ASO representations of elements of $W$. An identity distribution in $S'(\mathbb{R}^2)$ which satisfies

$$I \ast f(q, p) = f(q, p)$$

$$\int_{\mathbb{R}^2} I \ast g(q, p) \ dq \wedge dp = \int_{\mathbb{R}^2} g(q, p) \ dq \wedge dp$$

(14)

for all $f(q, p) \in S'(\mathbb{R}^2)$, and $g(q, p) \in S(\mathbb{R}^2)$, and which is given by

$$I = \sum_{n=0}^{\infty} \Omega_{nn}(q, p)$$

(15)

was used to construct the linear transformation from the ASO basis for $W$ to the $\Omega_{nm}$ basis. The following expression for the vacuum transformation from the ASO basis for $W$ to the $\Omega_{nm}$ basis.

The following expression for the vacuum state

$$\Omega_{00} = 2 \exp\left(-\frac{2}{h}aa^+\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{h^n} a^+ \ast a^n$$

was used to construct the inverse transformation from the left-right eigenstate basis back to the basis of ASOMs.

By providing an algorithm for moving between them, these results establish the equivalence of the ASO monomial basis and the left-right eigenvector basis for the Moyal algebra of observables. We now turn to the infinite parameter family of deformations of the Weyl algebra [7], which we study in the monomial and oscillator eigenstate bases.

### III.1 Extension to the quantum deformation $W_q$

We assume the existence of a deformed $\ast$-product which satisfies (7), and postulate the existence of a vacuum state $\Omega_{00}$ with the following reasonable properties

$$A \ast \Omega_{00} = 0 \quad \Omega_{00} \ast A^+ = 0$$

$$N \ast \Omega_{00} = 0 \quad \Omega_{00} \ast N = 0$$

$$\Omega_{00} \ast \Omega_{00} = \Omega_{00}.$$
When considering $\tilde{q}$ it will also be necessary to add $\Omega_{\tilde{q}}^{00} \ast E = E \ast \Omega_{\tilde{q}}^{00} = \Omega_{\tilde{q}}^{00}$. From (7) we define the following real functions on the strictly positive integers

\[ F_{\tilde{q}}(n) = \sum_{i=1}^{n} f_{\tilde{q}}(i - 1) \]

\[ F_{\tilde{q}}!(n) = \prod_{i=1}^{n} F_{\tilde{q}}(i), \]

and extend them to $n = 0$ by putting $F_{\tilde{q}}(0) = 0$ and $F_{\tilde{q}}!(0) = 1$. Due to the restrictions imposed on $f_{\tilde{q}}$ in §II, $F_{\tilde{q}}(n)$ is positive and monotonic increasing function on $0 \leq n \in \mathbb{Z}$. From the re-ordering relations (7), and the vacuum postulate (16), one can derive the following identities

\[ A \ast A^{+n} \ast \Omega_{\tilde{q}}^{00} = F_{\tilde{q}}(n) A^{+n-1} \ast \Omega_{\tilde{q}}^{00}, \]

\[ \Omega_{\tilde{q}}^{00} \ast A^{n} \ast A^{+m} \ast \Omega_{\tilde{q}}^{00} = \delta_{nm} F_{\tilde{q}}!(n) \Omega_{\tilde{q}}^{00}. \]

(17)

The $\tilde{q}$-states are defined as follows

\[ \Omega_{nm}^{\tilde{q}} = (F_{\tilde{q}}!(n) F_{\tilde{q}}!(m))^{-1/2} A^{+n} \ast \Omega_{\tilde{q}}^{00} \ast A^{m}. \]

(18)

From (17) it is easily seen that these are left-right eigenvectors of the $\tilde{q}$-deformed oscillator Hamiltonian

\[ H = \frac{1}{2} (A^{+} \ast A + A \ast A^{+}) \]

(19)

since

\[ H \ast \Omega_{nm}^{\tilde{q}} = E_{n}^{\tilde{q}} \Omega_{nm}^{\tilde{q}}, \]

\[ \Omega_{nm}^{\tilde{q}} \ast H = E_{n}^{\tilde{q}} \Omega_{nm}^{\tilde{q}}, \]

\[ E_{n}^{\tilde{q}} = \frac{1}{2} (F_{\tilde{q}}(n + 1) + F_{\tilde{q}}(n)). \]

(20)

The standard boson corresponds to $f_{\tilde{q}}(n) = \hbar$, in which case $F_{\tilde{q}}(n) = n\hbar$, $F_{\tilde{q}}!(n) = n!\hbar^n$, and the eigenvalues $E_{n}^{\tilde{q}}$ are the familiar $\hbar(n + 1/2)$. Later we will use the following relations which can be derived from the identities (17) and the definitions (18)

\[ E \ast \Omega_{nm}^{\tilde{q}} = \Omega_{nm}^{\tilde{q}}, \]

\[ A \ast \Omega_{nm}^{\tilde{q}} = F_{\tilde{q}}(n)^{1/2} \Omega_{nm}^{\tilde{q}}, \]

\[ A^{+} \ast \Omega_{nm}^{\tilde{q}} = F_{\tilde{q}}(n + 1)^{1/2} \Omega_{nm+1}^{\tilde{q}}, \]

\[ N \ast \Omega_{nm}^{\tilde{q}} = n \Omega_{nm}^{\tilde{q}}. \]
\[ \begin{align*}
\Omega^n_{nm} \ast E &= \Omega^n_{nm} \\
\Omega^n_{nm} \ast A^+ &= F^q_{m+1/2} \Omega^n_{nm+1} \\
\Omega^n_{nm} \ast A &= F^q_{m+1/2} \Omega^n_{nm-1} \\
\Omega^n_{nm} \ast N &= m \Omega^n_{nm}.
\end{align*} \] (21)

Using the projection property of the vacuum (16) and the identities (17), one can now show that in this basis the deformed Moyal products are given by

\[ \Omega^n_{nm} = \Omega^n_{mn}, \quad \Omega^n_{nm} \ast \Omega^{n'}_{m'} = \delta_{mn'} \Omega^n_{m}. \] (22)

Just as the Weyl algebra (5) was replaced by (11), the defining relations of the deformed Weyl algebras (7) have been replaced by (22). In these oscillator eigenstate bases the deformed Moyal products do not depend on values of the deformation parameters. The expected isomorphism between the Weyl algebra and its deformations becomes immediately apparent. In the case of phase-space realisations of (6) which satisfy the vacuum postulate (16), this turns out to be simply an isomorphism with the algebra of infinite matrices.

By analogy with (12) the natural inner product is given by

\[ \langle \Omega^n_{nm}, \Omega^{n'}_{m'} \rangle = \delta_{nn'} \delta_{mm'}. \] (23)

From Hansen’s work [13], we know that the closure of the space of finite linear combinations of \( \Omega^n_{nm} \) with respect to appropriate topologies will provide the deformed versions of \( S(\mathbb{R}^2) \), \( L^2(\mathbb{R}^2, dq \wedge dp) \) and \( S'(\mathbb{R}^2) \). These can now be identified with the undeformed spaces.

In view of the rigidity of the Weyl algebra the equivalence of phase-space realisations of the infinite parameter family of \( \bar{q} \)-boson algebras to that of the standard boson algebra does not come as a surprise. However this equivalence does not preserve the defining relations (5), and so is different from the unitary equivalence of quantum mechanics. In §V we apply elements from a theory of non-linear representations of Lie algebras to understand this equivalence better. In the next section we establish the existence of concrete representations of \( \bar{q} \), demonstrate the equivalence of oscillator eigenstate bases to bases of ASO monomials and we provide classical and quantum models for the deformed oscillators.

IV Realisations and Representations of \( \mathcal{H}_q \) and \( W_q \)

IV.1 Realisation of \( \mathcal{H}_q \) and \( W_q \) as \( \left\{ \Omega_q, * \right\} \).

We denote by \( \Omega_q \) the set of formal linear combinations of \( \Omega^q_{nm} \), and by \( \Omega \) the set of formal linear combinations of left-right eigenstates of the standard boson. Equipped with a \( * \)–product obeying (22) we know from Hansen’s work that suitably restricted subsets of \( \left\{ \Omega_q, * \right\} \) are Moyal algebras [13]. We will look at a family of projections from \( \mathcal{H}_q \) (\( W_q \)) to \( \left\{ \Omega_q, * \right\} \) which are \( \mathcal{H}_q \) (\( W_q \)) algebra homomorphisms.
In particular we consider $\Sigma_\bar{q}$ from $\mathfrak{H}_\bar{q}$ to $\{\Omega_\bar{q}, \ast\}$ based on $\mathcal{I}$ a resolution of the identity on $\{\Omega_\bar{q}, \ast\}$. $\mathcal{I}$ satisfies $\mathcal{I} \ast \omega_\bar{q} = \omega_\bar{q} \ast \mathcal{I} = \omega_\bar{q}$ for all $\omega_\bar{q} \in \Omega_\bar{q}$, and is given by

$$\mathcal{I} = \sum_{n=0}^{\infty} \Omega_{\bar{q}nn}.$$ 

The $\mathfrak{H}_\bar{q}$–homomorphism $\Sigma_\bar{q}$ is defined by

$$\Sigma_\bar{q}(u) = \mathcal{I} \ast u \ast \mathcal{I}$$

for all $u \in \mathfrak{H}_\bar{q}$. Its action on $u \in \mathfrak{H}_\bar{q}$ can be computed explicitly using the ladder relations (24). These relations can be used to show that for $u_1$ and $u_2$ in $\mathfrak{H}_\bar{q}$ we have

$$\Sigma_\bar{q}(u_1) \ast \Sigma_\bar{q}(u_2) = \Sigma_\bar{q}(u_1 \ast u_2),$$

so $\Sigma_\bar{q}$ is a deformed algebra homomorphism from $\mathfrak{H}_\bar{q}$ into $\{\Omega_\bar{q}, \ast\}$. Given a realisation of the $\ast$-product (24) which admits a vacuum state satisfying the requirements (22), $\Sigma_\bar{q}$ provides a representation of $\mathfrak{H}_\bar{q}$, and therefore of $W_\bar{q}$ in $\Omega_\bar{q}$ acting on $\Omega_\bar{q}$. We have

$$E \to \sum_{n=0}^{\infty} \Omega_{\bar{q}nn}(q, p)$$

$$A^+ \to \sum_{n=0}^{\infty} F_\bar{q}(n+1)^{1/2} \Omega_{\bar{q}n+1n}(q, p)$$

$$A \to \sum_{n=0}^{\infty} F_\bar{q}(n+1)^{1/2} \Omega_{\bar{q}n+1n}(q, p)$$

$$N \to \sum_{n=0}^{\infty} n \Omega_{\bar{q}nn}(q, p)$$

$$H \to \frac{1}{2} \sum_{n=0}^{\infty} (F_\bar{q}(n+1) + F_\bar{q}(n)) \Omega_{\bar{q}nn}(q, p).$$

These infinite sums are meaningful in the sense of distributions on $S'(\mathbb{R}^2)$. The action of these operators on the Hilbert space spanned by $\Omega_\bar{q}$ can be computed directly from (22).

IV.2 Isomorphism of $\{\Omega_\bar{q}, \ast\}$ and $\{W_\bar{q}^{\text{AS}}, \bullet\}$.

We denote by $W_\bar{q}^{\text{AS}} \subseteq \mathfrak{H}_\bar{q}$ the linear space spanned by $A^+^n \ast A^m$ for $n, m \geq 0$, the ASO monomial functions of co-ordinates on a $\bar{q}$-deformed phase-space. The restriction of $\Sigma_\bar{q}$ to $W_\bar{q}^{\text{AS}}$ will be denoted $\sigma_\bar{q}$ and $\sigma$ refers to its action on $W^{\text{AS}}$ the space of formal linear combinations of ASO monomials in the standard boson algebra. In particular we will show that $\Sigma_\bar{q} : \mathfrak{H}_\bar{q} \to \{W_\bar{q}^{\text{AS}}, \bullet\}$ is an algebra homomorphism which will lead to a phase-space representation of the deformed Weyl
algebra (7) analogous to that of the Weyl algebra (5) provided by (13). By explicit computation using (21) and (24) we get

\[
\sigma_{\bar{q}}(A^{+n} A^{m}) = \sum_{i=0}^{\infty} C(n, m, i) \Omega_{n+i}^{\bar{q}} \Omega_{m+i}^{\bar{q}}
\]

This transformation can be inverted using

\[
\sigma^{-1}_{\bar{q}}(\Omega_{n,m}^{\bar{q}}) = \sum_{i=0}^{\infty} D(n, m, i) A^{+n+i} A^{m+i}
\]

where the coefficients \(D(n, m, i)\) are computed recursively according to

\[
D(n, m, 0) = C(n, m, 0)^{-1}
\]

\[
D(n, m, k) = -C(n + k, m + k, 0)^{-1} \sum_{i=0}^{k-1} C(n + i, n + i, k - i) D(n, m, i).
\]

\(F_{\bar{q}}(n) \neq 0\) for \(0 \leq n \in \mathbb{Z}\) so \(C(n, m, k)\) never vanishes and the coefficients \(D(n, m, k)\) are well-defined. The product \(\bullet: W_{\bar{q}}^{\text{AS}} \times W_{\bar{q}}^{\text{AS}} \rightarrow W_{\bar{q}}^{\text{AS}}\) intertwines the deformed Moyal product on \(\Omega_{\bar{q}}\) and can easily be computed using (22) as follows

\[
w_1 \bullet w_2 = \sigma^{-1}_{\bar{q}}(\sigma_{\bar{q}}(w_1) * \sigma_{\bar{q}}(w_2))
\]

for \(w_1, w_2 \in W_{\bar{q}}^{\text{AS}}\). It is a simple matter to show that \(\sigma^{-1}_{\bar{q}} \circ \sigma_{\bar{q}}\) is the identity on \(W_{\bar{q}}\), and that \(\sigma_{\bar{q}} \circ \sigma^{-1}_{\bar{q}}\) is the identity on \(\Omega_{\bar{q}}\). This establishes the equivalence of \(\{W_{\bar{q}}^{\text{AS}}, \bullet\}\) and \(\{\Omega_{\bar{q}}, *, \}\) as phase-space representations of \(W_{\bar{q}}\) or \(\Omega_{\bar{q}}\).

Suzuki has considered \(*\)-products associated with the q-boson which deform the Moyal \(*\)-product [14]. By contrast we have provided \(\bullet\)-products (24) which deform the \(\bullet\)-product of \(W^{\text{AS}}\) given by (13). Our construction does not require the introduction on phase-space of internal degrees of freedom. By constructing the left-right eigenstate basis for the deformed ring we saw explicitly in § III.1 that the Moyal product is not really deformed at all.

IV.3 Infinite matrix representations of \(\Omega_{\bar{q}}\) and \(W_{\bar{q}}\)

Now we define a family of maps \(\pi_{\bar{q}}: \Omega_{\bar{q}} \rightarrow M_{\infty}\), where \(M_{\infty}\) is the ring of infinite matrices over the complex numbers. These maps are \(*\)-algebra isomorphisms of \(\{\Omega_{\bar{q}}, *\}\) based on (22). We will use the unscripted \(\pi: \Omega \rightarrow M_{\infty}\) to denote the action on the standard boson algebra. Each \(\pi_{\bar{q}}\) is linear and is defined by its action on \(\Omega_{n,m}^{\bar{q}}\) according to
\[ \pi_\Omega(\Omega^\dagger_{nm}) = e(n, m) \]  
\[ (27) \]

where \( e(n, m) \) is the infinite matrix with zero everywhere except for the number one in row \( n \) and column \( m \). The action of \( \pi_\Omega \) or \( \pi_\Omega \circ \Sigma_\Omega \) provides infinite matrix representations of elements of \( W_\Omega \) or \( \mathcal{H}_\Omega \) respectively in \( M_\infty \).

### IV.4 Bosonisation of \( \mathcal{H}_\Omega \) and \( W_\Omega \)

It is a simple matter to construct bosonic representations of \( \mathcal{H}_\Omega \) or \( W_\Omega \) either in the space of left-right eigenstates \( \{ \Omega_\Omega, * \} \) or in \( \{ W_{AS}, * \} \). In the first case this can be achieved by the action of \( \pi^{-1} \circ \pi_\Omega \circ \Sigma_\Omega \) on \( u \in \mathcal{H}_\Omega \). Applying this to the generators of \( \mathcal{H}_\Omega \) yields

\[ E \rightarrow \sum_{n=0}^{\infty} \Omega_{nn} \]
\[ A^+ \rightarrow \sum_{n=0}^{\infty} F_\Omega(n + 1)^{1/2} \Omega_{n+1, n} \]
\[ A \rightarrow \sum_{n=0}^{\infty} F_\Omega(n + 1)^{1/2} \Omega_{n, n+1} \]
\[ N \rightarrow \sum_{n=0}^{\infty} n \Omega_{nn} \]
\[ H \rightarrow \frac{1}{2} \sum_{n=0}^{\infty} (F_\Omega(n + 1) + F_\Omega(n)) \Omega_{nn}. \]

A bosonisation of the \( \tilde{q} \)-oscillator can immediately be provided by

\[ A = K_{\tilde{q}}(N + 1) * a \]
\[ A^+ = a^+ * K_{\tilde{q}}(N + 1) \]
\[ H_{\tilde{q}} = \frac{1}{2} (F_{\tilde{q}}(N + 1) + F_{\tilde{q}}(N)) \]  
\[ (28) \]

where \( K_{\tilde{q}}(N) = F_{\tilde{q}}(N + 1)^{1/2}(\hbar(N + 1))^{-1/2} \) with \( \hbar N = a^+ * a \). There are no ordering problems in these expressions, but we have to check that objects such as \( K_{\tilde{q}}(N) \) are well defined. From \( \S \text{II.1} \) we know that \( F_{\tilde{q}} \) satisfies \( 0 < F_{\tilde{q}}(n + 1) < \infty \) for all \( 0 \leq n \in \mathbb{Z} \). The action of \( K_{\tilde{q}}(N) \) on \( \Omega_{nn} \) is given by \( K_{\tilde{q}}(n) = F_{\tilde{q}}(n + 1)^{1/2}(\hbar(n + 1))^{-1/2} \). The transformation (28) is a well defined invertible map from \( \{ \Omega_{\tilde{q}}, * \} \) to \( \{ \Omega, * \} \).

Now we return to the main issue of understanding the non-unitary equivalence of the \( \tilde{q} \)-bosons.
Non-linear representations of boson algebras.

A non-linear representation theory of Lie groups and Lie algebras developed by Flato, Pinczon and Simon [15] introduces a concept of non-linear equivalence which is appropriate in this context. Their work refers specifically to representations of Lie groups or Lie algebras and so they focus on transformations which are Lie structure homomorphisms. We are not dealing with Lie algebras, but with categories of deformed Weyl or Heisenberg Lie algebras. We therefore consider equivalence transformations which preserve these categories. In what follows we stick closely to the intuitive framework provided by Flato et al.

Consider Banach algebras \( U \) and \( V \). Let \( L_n(U, V) \) denote the \( n \)-linear maps from \( U \times \cdots \times U \to V \). The space of non-linear transformations \( S: U \to V \) consists of maps \( s \in S \) such that

\[ s(u) = \sum_{n=1}^{\infty} f_n(u) \]

where \( u \in U \) and each \( f_n \in L_n(U, V) \). Algebras or representations of algebras which are related by invertible transformations of this type will be considered equivalent up to non-linear transformation. In particular we would like to consider \( \{ \Omega, \ast \} \) or \( \{ W_q, \cdot \} \) for \( U \) and \( V \), in which case the Banach space structure will be derived from the weak topology associated with the Hilbert space structure [23].

Let us take two deformed Weyl algebras \( W_A, W_B \in \mathcal{W} \) with generators \( A, A^+, N \) and \( B, B^+, N \) as in § II. From (28) we can equate

\[ A * A^+ = F_A(N + 1) \quad B * B^+ = F_B(N + 1). \]

Now we consider the following non-linear transformation

\[ A = K_B^A(N) * B \]
\[ A^+ = B^+ * K_B^A(N) \]

where

\[ K_B^A(N) = \left[ \frac{F_{\overline{q}}(N + 1)}{F_{\overline{q}}(N + 1)} \right]^{\frac{1}{2}}. \]

\( K_B^A(N) \) satisfies \( 0 < K_B^A(n) < \infty \) for all \( 0 \leq n \in \mathbb{Z} \) so this transformation leaves \( N \) invariant, and (29) provides an invertible map from \( W_A \) to \( W_B \). Phase-space realisations of \( W_A, W_B \in \mathcal{W} \) are therefore equivalent up to transformations of this form. These transformations are not unitary since they do not preserve the commutation relations (7). We expect such transformations to exist because of the rigidity of the Weyl algebra [4]. This cohomological result guarantees the existence of a formal transformation from one structure to another, but it must be verified in each case that this formal transformation is well defined and invertible. An example where this is so, is the standard Fock-space representation of the \( q \)-deformed boson for \( q \in \mathbb{R}^+ \). However if \( q \) lies at a root of unity the transformation (29) cannot be inverted since \( K_B^A(n) \) vanishes for some positive integer \( n \). In this case the equivalence is only a formal one and two algebras will differ in a fundamental way.

Although we do not elaborate in this paper, it is straightforward to extend this notion of equivalence to deformations of the Heisenberg Lie algebra \( \mathfrak{H}_q \) [8].
VI Coproduct structures for $\mathfrak{h}\bar{q}$ and $W_{\bar{q}}$

Any Lie algebra can be provided with a Hopf algebra structure \([16]\). This amounts to providing a pair of linear algebra homomorphisms called the co-product $\Delta : A \rightarrow A \otimes A$, and the evaluation map $\epsilon : A \rightarrow C$, as well as a linear algebra anti-homomorphism called the antipode $S : A \rightarrow A$. $\Delta$ and $\epsilon$ are required to satisfy a number of consistency conditions \([16]\). It is a simple matter to provide $U(\mathfrak{h}(1))$ with a Hopf-algebra structure \([17]\). This is achieved by defining the action of the co-product $\Delta$, the evaluation map $\epsilon$ and the antipode $S$ on the generators of the Lie algebra and on the unit as follows

$$\Delta(1) = 1 \otimes 1, \quad \epsilon(1) = 1, \quad S(1) = -1$$

$$\Delta(E) = E \otimes 1 + 1 \otimes E, \quad \epsilon(E) = 0, \quad S(E) = -E$$

$$\Delta(A) = A \otimes 1 + 1 \otimes A, \quad \epsilon(A) = 0, \quad S(A) = -A$$

$$\Delta(A^+) = A^+ \otimes 1 + 1 \otimes A^+, \quad \epsilon(A^+) = 0, \quad S(A^+) = -A^+.$$  

One could try to include the number operator $N$ in the canonical way by assigning $\Delta(N) = N \otimes 1 + 1 \otimes N$. However this is not compatible with the definition $\bar{h}N = A^+ \cdot A$ since $\bar{h}\Delta(N) \neq \Delta(a^+) \cdot \Delta(a)$. To ensure compatibility we must make the following non-canonical assignment

$$\Delta(N) = N \otimes 1 + 1 \otimes N + A \otimes A^+ + A^+ \otimes A.$$ 

In either case the evaluation map and the antipode will act according to $\epsilon(N) = 0$ and $S(N) = -N$. The definitions can now be extended from the generators $A$, $A^+$, $N$ and $E$ to the entire ring using the homomorphism properties $\Delta(x \cdot y) = \Delta(x) \cdot \Delta(y)$, $\epsilon(x \cdot y) = \epsilon(x) \cdot \epsilon(y)$ and the anti-homomorphism property $S(x \cdot y) = S(y) \cdot S(x)$.

One can provide the Weyl-ring $W$ with a co-product which is an homomorphism of the associative product as follows

$$\Delta(1) = 1 \otimes 1$$

$$\Delta(A) = (A \otimes 1 + 1 \otimes A) / \sqrt{2}$$

$$\Delta(A^+) = (A^+ \otimes 1 + 1 \otimes A^+) / \sqrt{2}.$$ 

(30)

This can then be extended to the number operator using $\bar{h} \Delta(N) = \Delta(A^+) \cdot \Delta(A)$, and then to the whole of $W$ in the natural way.

The only co-unit which is homomorphic to the associative product on $W$ is the zero map. This follows since $[\epsilon(a), \epsilon(a^+)] = 0 = \bar{h}\epsilon(1)$, so that $\epsilon(A) = \epsilon(1 \cdot A) = \epsilon(1)\epsilon(A) = 0$ and similarly $\epsilon(A^+) = 0$. As a consequence the Weyl algebra does not have a bi-algebra or a Hopf-algebra structure although it does have a co-algebra structure which is an associative algebra homomorphism. It is possible to use the non-linear non-canonical equivalence transformations in \([24]\) to provide each of the $\bar{q}$-deformed Weyl algebras with a co-algebra structure as follows

$$\Delta(1) = 1 \otimes 1.$$ 

(31)
\[
\Delta(A) = \Delta \dot{K}(N) \left[ (K(N)^{-1} A) \otimes 1 + 1 \otimes (K(N)^{-1} A) \right] / \sqrt{2}
\]
\[
\Delta(A^+) = \left[ (A \otimes K(N)) \otimes 1 + 1 \otimes (A^+ K^{-1}(N)) \right] \Delta \cdot K(N) / \sqrt{2}
\]
\[
\Delta(N) = N \otimes 1 + 1 \otimes N + \left( (K(N)^{-1} A) \otimes (A^+ K(N)^{-1}) + (A^+ K(N)^{-1}) \otimes (K(N)^{-1} A) \right).
\]

This smoothly deforms the natural co-product structure of the Weyl algebra \((\mathfrak{w})\) in such a way that it respects the identity \(\hbar N = A^+ A\) in the undeformed limit. In exactly the same way one can deform the Hopf-algebra structure of the Heisenberg Lie algebra whenever the transformation function \(K(N,E)\) corresponding to \((\mathfrak{w})\) is known.

VII The classical limit of deformed quantum dynamics

The quantum phase-space dynamics of the \(\bar{q}\)-deformed oscillator is easy to study. It suffices to take \(\omega \in L^2(\mathbb{R}^2, dq \wedge dp)\) and construct from this the corresponding density \(\rho = \omega * \bar{\omega} = \sum_{nm} c_{nm} \Omega_{nm}\). Time evolution of this density will be given by

\[
\rho(t,q,p) = \exp\left(-\imath \frac{t}{\hbar} H^*\right) * \rho_0(q,p) * \exp\left(+\imath \frac{t}{\hbar} H^*\right) = \sum_{n,m=0} c_{nm} \exp(t \omega_{nm}) \Omega_{nm}(q,p),
\]

where \(\omega_{nm} = (F_q(m + 1) + F_q(m) - F_q(n + 1) - F_q(n))/2\). The expectation of observables will then be given by

\[
\langle O \rangle_t = (2\pi\hbar)^{-1} \int_{\mathbb{R}^2} O(q,p) \rho(t,q,p) \, dq \wedge dp.
\]

The non-linear transformation \((\mathfrak{w})\) which relates the algebra of the standard oscillator to that of the \(\bar{q}\)-deformed oscillator induces the following transformation in the defining relations of the deformed Weyl algebra

\[
\begin{align*}
[N,A] &= -A \\
[N,A^+] &= A^+ \\
[A,A^+] &= f_{\bar{q}}(N)
\end{align*}
\]

\rightarrow

\[
\begin{align*}
[N,a] &= -a \\
[N,a^+] &= a^+ \\
[a,a^+] &= \hbar
\end{align*}
\]

where

\[
f_{\bar{q}}(N) = F_{\bar{q}}(N+1) - F_{\bar{q}}(N).
\]

At the same time the Hamiltonian is transformed according to

\[
H(A,A^+) = \frac{1}{2}(A^+ A + AA^+) \rightarrow H(a,a^+) = \frac{1}{2} (F_{\bar{q}}(N+1) + F_{\bar{q}}(N))
\]
where $\hbar N = a^+ a$. To understand the geometrical meaning of these transformations we need to consider the classical limit of all of these objects. To start with we put $N = H_0/\hbar - 1/2$ where $H_0 = (aa^+ + a^+ a)/2$ is the Hamiltonian of the simple harmonic oscillator. Equation (28) becomes

$$H_\bar{q} = \frac{1}{2} \left[ F_\bar{q} \left( \frac{H_0^* + \hbar}{2} \right) + F_\bar{q} \left( \frac{H_0^* - \hbar}{2} \right) \right].$$

The limit $\hbar \to 0$ does not make any sense unless the coefficients of $F_\bar{q}$ scale appropriately with $\hbar$. This is achieved if we replace

$$q_n \to q_n \hbar^n$$

in $F_\bar{q}$. The quantum Hamiltonian now becomes

$$H_\bar{q} = \frac{1}{2} \left[ F_\bar{q} \left( H_0^* + \frac{\hbar}{2} \right) + F_\bar{q} \left( H_0^* - \frac{\hbar}{2} \right) \right]$$

and the classical limit of the $\bar{q}$-deformed oscillator is given by

$$H^C_{\bar{q}} = F_\bar{q}(H_0) = q_0 + q_1 H_0 + q_2 H_0^2 + \ldots$$

From (33) and (34) the commutator of $A$ and $A^+$ in (3) becomes

$$A * A^+ - A^+ * A = F_\bar{q} \left( H_0^* + \frac{\hbar}{2} \right) - F_\bar{q} \left( H_0^* - \frac{\hbar}{2} \right) = F'_\bar{q}(H_0) \hbar + O(\hbar^3).$$

In the classical limit this corresponds to the following non-canonical Poisson bracket

$$\{f, g\}_\bar{q} = F'_\bar{q}(H_0) \{f, g\}.$$  \hspace{1cm} (37)

### VIII  Relationship to method of normal forms

In classical Hamiltonian dynamics on the plane an Hamiltonian $H(q, p)$ in the neighbourhood of a point of stable equilibrium can be transformed by a series of canonical transformations to a co-ordinate system in which it takes the form $H(p^2 + q^2)$. These transformations preserve the Poisson bracket and their quantum analogues are unitary transformations in the quantum version of the method of normal forms [18]. Effectively they smooth the invariant curves of the flow into Euclidean circles. Expansion in normal forms is an important technique to study the non-linear dynamics of a system in the neighbourhood of a critical point [19, Chapter 4, eqn 4.96]. The truncated expansion can provide a useful integrable approximation to a non-integrable system. This provides the starting point for an important semi-classical approach to the quantisation of a range of non-integrable Hamiltonian systems such as small molecules.
\( H_{\bar{q}} \) is a power series expansion in terms of the classical Hamiltonian of the simple harmonic oscillator. It is the expansion in normal forms of a one-dimensional classical Hamiltonian system in the neighbourhood of a point of stable equilibrium. Up to a simple linear transformation the coefficients of powers of \( H_0 \) in \( H_{\bar{q}} \) are just the \( \bar{q} \)-deformation parameters which appeared originally in (3). Hamiltonian systems of this form are very like linear oscillators in the sense that the orbits are confined to circles about the origin. Non-linearity is exhibited in the fact that the fundamental frequency of each orbit depends on the circle to which it is confined.

It is possible to make a further change of co-ordinates so that a Hamiltonian of the form \( H(p^2 + q^2) \) is transformed into the even simpler form \( (p^2 + q^2)/2 \). This last transformation is not canonical since the Poisson bracket changes form. It is in fact the classical counterpart of (29) which relates the deformed Weyl algebras (22) to the Weyl algebra itself (5). The inverse of (29) which in the classical limit would transform \( H(q,p) \) into the form \((p^2 + q^2)/2\), seems closely related to the factorisation method for linear differential equations on the line due to Infeld and Hull [20].

IX Quantisation of non-canonical Poisson structures

In previous work on the existence of Hamiltonian structures for arbitrary vector fields [21], [22], one of us raised the question as to whether the equivalence between the infinite number of Hamiltonian structures of a given classical Hamiltonian system extends to its quantisation. The simple harmonic oscillator is not just Hamiltonian with Hamiltonian function \( H_0 = aa^+ \) with respect to the Poisson bracket \{a, a+\} = 1, but also with Hamiltonian function \( H = F(H_0) \) with respect to the Poisson bracket \{a, a+\} = \( (F'(H_0))^{-1} \). It has an infinite number of non-canonical Hamiltonian structures. A natural question arises as to whether there exists a theory of quantisation which consistently applies to these classically equivalent formulations. On the basis of (34), (35) and (37) we can recast our understanding of the classical limit of the deformed commutation relations

\[
\{a, a^+\} \rightarrow \Theta_{\bar{q}}(H_0) \rightarrow [a, a^+] = \int_{H_0 - \hbar/2}^{H_0 + \hbar/2} \Theta_{\bar{q}}(x) \, dx
\]

(38)

Looking at the deformed Weyl algebras [1] from this point of view we see that the fundamental commutation relation is not only a deformation of the classical Poisson bracket, but also a coarse graining of the classical Poisson bracket whose coarseness is controlled by Planck's constant. The quantum commutator is also the spectral–gap operator for the energy levels of a non-linear oscillator.
Discussion

From cohomological arguments we know that formally there are no non-trivial associative-deformations of the Weyl-algebra. This raises the problem of how to understand quantum deformations of the Weyl algebra such as the $q$- and $qp$-deformed boson algebras.

To tackle this we introduce a family of deformed Weyl and Heisenberg Lie algebras which depend on an infinite number of independent deformation parameters. Using phase-space quantum mechanical methods we studied phase-space realisations of these algebras using ASO monomials and bimodular bases of deformed-oscillator eigenstates. In the monomial bases the intertwiner of the deformed Moyal product was given by a $\bullet$-product which deforms that associated with the standard boson. We provided an algorithm for transforming to the eigenstate bases. In the eigenstate bases the deformed Moyal–products are immediately seen to be isomorphic to that of the standard oscillator, and to the algebra of infinite matrices. These results are best summarised in the following diagram and by equations (22) and (23). The diagram relates the quantum deformed Heisenberg Lie algebra $\tilde{H}_q$, to the phase-space representation of the deformed Weyl algebra $W_q$ given by $\{W_{\tilde{A}S}, \bullet\}$ and $\{\tilde{\Omega}_q, \ast\}$, to $M_\infty$ the algebra of infinite matrices, and to representations of the Weyl algebra given by $\{W_{AS}, \bullet\}$ and $\{\Omega, \ast\}$. The vertical arrows are algebra homomorphisms, whereas the horizontal arrows are isomorphisms of phase-space representations of the deformed Weyl algebras which satisfy the vacuum postulate.

\[
\begin{array}{c}
\tilde{H}_q \\
\downarrow \quad \downarrow \quad \downarrow \\
W_q \\
\downarrow \\
\{W_{\tilde{A}S}, \bullet\} \quad \Leftrightarrow \quad \{\tilde{\Omega}_q, \ast\} \quad \Leftrightarrow \quad M_\infty \quad \Leftrightarrow \quad \{\Omega, \ast\} \quad \Leftrightarrow \quad \{W_{AS}, \bullet\}
\end{array}
\]

This enables us to construct phase-space quantum mechanical models of the deformed Weyl algebras and their associated oscillators. The equivalence of all realisations of satisfying the vacuum postulate, is not a unitary equivalence since it does not preserve the form of the commutation relations. These algebras are equivalent in the sense that they are related by nonlinear non-unitary transformations. This allows us to provide the deformed Weyl-algebra with a co-product structure which is a deformed Weyl algebra homomorphism and which smoothly deforms the co-product structure of the Weyl algebra.

Finally we considered the classical limit of phase-space dynamics of the deformed oscillators. We found that the deformation parameters have a natural interpretation as coefficients of nonlinear terms in a normal forms expansion of the deformed oscillator. The situation is best summed up by considering a fully integrable classical system whose Hamiltonian is an entire complex function of the harmonic oscillator $F_q(H_0) = q_0 + q_1 H_0 + q_2 H_0^2 + \ldots$ with respect to the Poisson bracket $\{A, A^+\} = 1$. The parameters $q_n$ for $2 \leq n \in \mathbb{Z}$ parameterise non-linear deformations of the harmonic oscillator. The quantum Hamiltonian is given by $(F_q(H_0 + \hbar/2) + \ldots$
It can be transformed into \( H_0 = (AA^+ + A^+A)/2 \) by a non-linear non-canonical transformation of co-ordinates. Under this transformation the fundamental quantum commutator \([A, A^+] = \hbar\) becomes \([A, A^+] = (F_q(H_0 + \hbar/2) - F_q(H_0 - \hbar/2)\) which we recognise as a deformation of the Weyl algebra of the form (7). In the classical limit the transformed Hamiltonian is \( AA^+ \) and the Poisson bracket is given by \( \{A, A^+\} = F_q'\bar{q}(H_0) \). Eventually we are lead to consider the deformed Weyl algebras (8) as providing fundamental commutation relations (38) which quantise the non-canonical classical Poisson brackets given in (37). Other families of deformations of the Weyl-algebra can easily be constructed. This would lead to a quantisation of the infinite family of non-canonical Hamiltonian structures associated with more general one dimensional systems.

Classically we can consider the family of quantum–deformations of the Weyl algebra considered in § II, as being equivalent to non-linear deformations of the classical oscillator (36), or non-canonical transformations of the related Poisson structure (37). Classically the deformed Poisson algebra is equivalent to the undeformed algebra as long as \( F_q\bar{q}(z) \) is an invertible map on \( \mathbb{R}^+ \). By contrast phase-space realisations of the related quantum algebras are equivalent as long as \( 0 < F_q(n+1) < \infty \). The \( q \)-deformed oscillator for \( q \) a root of unity is of considerable interest in physics [23], and provides a good example of how formal or local equivalence guaranteed by cohomology does not extend to global equivalence between the deformed and undeformed systems.

Realisations of the \( q \)-boson exist in terms of standard boson eigenstates on configuration space \[1\], on complexified configuration space \[2\], or in terms of \( q \)-difference operators acting on the functions of configuration space \[22\]. We think that phase-space quantum mechanical techniques can lead to a clarification of the similarities and differences between the standard boson and the \( q \)-boson which arise in the configuration-space formalism of standard quantum mechanics. We have considered deformations of the Weyl algebra which satisfy the vacuum postulate. This is the smallest category within which to deform the Weyl algebra. Rideau has shown that there exist infinite matrix representations of the \( q \)-boson for which the vacuum postulate is not satisfied \[10\]. In view of the existence of these other representations it is of interest to extend our results to the larger category of representations which do not necessarily satisfy the vacuum postulate.

It should be straightforward to provide quantum deformations of semi-simple Lie algebras such as \( \mathfrak{su}(2) \) using the Jordan-Schwinger construction. It will be of interest to extend the results presented in this paper to the higher dimensional Weyl algebras \( W(n) \). This should provide some new perspectives on the role of the \( R \)-matrix in quantum group constructions \[20\]. Quantum groups now find application in nuclear physics, molecular dynamics, and mesoscopic systems as well as in the quantum field theory of integrable models \[8, 11, 16\]. We expect that our results on deformed boson algebras provide a physically intuitive way to understand quantum deformation, which will be of use in generalising applications based on standard \( q \)- and \( qp \)-deformed boson algebras.

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References

[1] M. Flato & Z. Lu, Remarks on Quantum Groups, Lett. Math. Phys. 21, 85–88 (1991).

[2] Zhe Cheng, Wei Chen & Han-Ying Guo, $SU_{q,h\rightarrow 0}(2)$ and $SU_{q,h}$ the classical and quantum $q$-deformations of the $SU(2)$ algebra, J. Phys. A: Math. & Gen. 23, 4185–4190 (1990).

[3] V. I. Man’ko, G. Marmo, S. Solimeno & F. Zaccaria, Physical non-linear aspects of classical and quantum $q$-oscillators, Int. J. Mod. Phys. A 8, 3577–3597 (1993).

[4] G. Pinczon, On the rigidity of the Weyl-algebra, preprint (1995).

[5] P. P. Kulish & E. V. Damaskinsky, On the $q$-oscillator and the quantum algebra $su_q(1,1)$, J. Phys. A: Math. & Gen. 23, L415–419 (1990).

[6] A. J. Mcfarlane, On the $q$-analogues of the quantum harmonic oscillator and the quantum group $SU(2)$, J. Phys. A: Math. & Gen. 22, 4581–4588 (1989).

[7] R. Chakrabarti & R. Jagannathan, A $(p,q)$-oscillator realisation of two-parameter quantum algebras, J. Phys. A: Math. & Gen. 24, L711–718 (1990).

[8] M. R. Kibler: Introduction to Quantum Algebras, in Symmetry and Structural Properties of Condensed Matter 445–464. Eds. W. Florek, D. Lipinski & T. Lulek - World Scientific Singapore 1993. Also in hep-th/9409013.

[9] Won-Sang Chung, New deformed boson algebra, J. Math. Phys. 35 3631–3635 (1994).

[10] Guy Rideau, On representations of quantum oscillator algebra, Lett. Math. Physics 24 147–153 (1992).

[11] E. V. Damaskinsky & P. P. Kulish, Quantum Groups, Deformed Oscillators and their Interrelations, q-alg/9501006 (1995).

[12] E. Celeghini, R. Giachetti, E. Sorace & M. Tarlini, The quantum Heisenberg group $\mathfrak{g}(1)_q$, J. Math. Phys. 32, 1155–1158 (1991).

[13] F. Hansen, The Moyal product and spectral theory for a class of infinite dimensional matrices, Publ. RIMS, Kyoto Univ. 26, 885–933 (1990).

[14] Takashi Suzuki, Deformations of Poisson algebra in terms of star products and q-deformed mechanics, J. Math. Phys. 34, 3453–3462 (1993).
[15] M. Flato, G. Pinczon & J. Simon, Non-linear Representations of Lie Groups, Ann. Scient. Éc. Norm. Sup. 4e série, t.10, 405–418 (1977). See also D. Sternheimer, Recent Developments in Non-linear Representations and Evolution Equations, Lecture Notes in Physics 313 65–73, Eds. H-D. Doebner, J-D Hennig and T. D. Palev, 1988. Non-linear Group Representations and the Linearizability of Nonlinear Equations, Lecture Notes in Physics 116 300–302, Ed. K. Osterwalder, 1980.

[16] F.A. Smirnov & L. A. Takhtajan, Lectures on Quantum Group in Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory, Nankai Lectures on Mathematical Physics. Eds. Mo-Lin Ge and Bao-Heng Zhao - World Scientific 1990.

[17] Chang-Pu Sun & M. L. Ge, Quantum Double of Heisenberg Weyl Algebra its Universal R-Matrix and their Representations, [hep-th 9301096].

[18] P. Crehan, The proper quantum analogue of the Birkhoff–Gustavson method of normal forms, J. Phys. A: Math & Gen. 23, 5815–5828 (1990).

[19] Alfredo M. Ozorio De Almeida, Hamiltonian Systems - Chaos and Quantisation, Cambridge Monographs on Mathematical Physics (C) 1989.

[20] L. Infeld & T. E. Hull, The Factorization Method, Rev. Mod. Phys. 23, 21–68 (1951).

[21] P. Crehan, On the local Hamiltonian structure of vector fields, Modern Phys. Letters A 9, 1399–1405 (1994).

[22] P. Crehan, When are Vector Fields Hamiltonian?, to appear in Proceedings of the International Conference on Chaos and Dynamical Systems - Tokyo Metropolitain University 23rd-27th May 1994.

[23] Jens-U. H. Petersen, Representations at a Root of Unity of q-oscillators and Quantum Kac-Moody Algebras Ph.D. Thesis, University of London 1994. Also in [hep-th/9409079].

[24] Achim Kempf, Quantum group-symmetric Fock spaces with Bargmann–Fock representation, Lett. Math. Physics 26, 1–12 (1992).

[25] Natig M. Atakishiyev, Alejandro Frank and Kurt Bernardo Wolf, A simple difference realisation of the Heisenberg q-algebra, J. Math. Phys. 35, 3253–3260 (1994).

[26] J. Van der Jeugt, R-matrix formulation of deformed boson algebra, J. Phys. A: Math. & Gen 26 L405–411 (1993).