Non-perturbative Renormalization Constants using Ward Identities*

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We extend the application of vector and axial Ward identities to calculate $b_A$, $b_P$, and $b_T$, coefficients that give the mass dependence of the renormalization constants of the corresponding bilinear operators in the quenched theory. The extension relies on using operators with non-degenerate quark masses. It allows a complete determination of the $O(a)$ improvement coefficients for bilinears in the quenched approximation using Ward Identities alone. Only the scale dependent normalization constants $Z^0_T$ (or $Z^0_A$) and $Z_T$ are undetermined. We present results of a pilot numerical study using hadronic correlators.

To remove errors of $O(a)$ from physical matrix elements, one must improve both the action and the operators [2]. The former requires the addition of the Sheikholeslami-Wohlert term [2]. The latter requires the introduction of a mass dependence, [2]. The bare WI mass

$$ S_{SW} = -a^3 c_{SW} \sum_x \bar{\psi}(x) \frac{i}{4} \sigma_{\mu \nu} F_{\mu \nu}(x) \psi(x). \tag{1} $$

Improvement of flavor off-diagonal bilinears requires [2] both the addition of extra operators,

$$ (A_I)_\mu = A_\mu + ac_{A} \partial_{\mu} P $$

$$ (V_I)_\mu = V_\mu + ac_{V} \partial_{\mu} T_{\mu} $$

$$ (T_I)_\mu = T_{\mu} + ac_{T} (\partial_{\mu} V_\nu - \partial_{\nu} V_\mu), \tag{4} $$

and the introduction of a mass dependence,

$$ (X_R)^{(ij)} = Z_X^0 (1 + b_{X am}^{ij}) (X^{(ij)}). \tag{5} $$

Here $X = A, V, P, S, T$, $Z_X^0$ are the renormalization constants in the chiral limit, and $m_{ij} \equiv (m_i + m_j)/2$ are the bare quark masses defined using the axial Ward Identity (WI), Eq. [2]. The bare unimproved bilinears are $A^{(ij)}_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi^j$, etc. The task is to determine the coefficients $c_{SW}$, $Z_X^0$'s, $c_X$'s, and $b_X$'s non-perturbatively.

Previous calculations have shown how $Z^0_V$, $Z^0_A$ and $Z^0_P/Z^0_S$ [3], $c_{SW}$, $c_A$ and $b_V$ [3], $c_V$ [3], $c_T$ [3], and $b_P - b_A$ and $b_S$ [1] can be determined non-perturbatively using axial and vector

WI. We discuss here an extension that yields $b_A$, $b_P$, and $b_T$. The two remaining constants $Z^0_T$ (or $Z^0_A$) and $Z^0_P$ are scale and scheme dependent, and so cannot be determined using WI. Note that the relations we derive do not extend directly to the unquenched theory, which requires additional improvement constants and a more complicated set of conditions [1].

We begin by recalling the ALPHA method for determining $c_{SW}$ and $c_A$ [1]. The bare WI mass

$$ 2m_{ij} = \frac{\sum_x (\bar{\psi}(0) [A_\mu + ac_{A} \partial_{\mu} P]^{(ij)}(0), (x), J^{(ij)}(0))}{\sum_x (P^{(ij)}(0), (x), J^{(ij)}(0))} \tag{6} $$

should be independent of $t$, up to corrections of $O(a^2)$, since it is proportional to the average renormalized quark mass

$$ m_{ij}^R = \frac{m_{ij}^R}{2} = m_{ij} \frac{Z^0_A (1 + b_A a m_{ij})}{Z^0_A (1 + b_P a m_{ij})}. \tag{7} $$

This is achieved by simultaneously tuning $c_{SW}$ and $c_A$. Our approach differs from the Schrödinger functional method of Ref. [1] in that we use standard 2-point correlation functions. Consistency of these estimates are checked by varying the initial state using $J = P$ or $A_4$ and with different types of sources for the quark propagators (Wuppertal smearing, Wall, point). In the following we use the abbreviation $m_i = m_{ii}$, where $m_{ii}$ refers to two degenerate flavors.
With $c_{SW}$ fixed, $Z_V^0$ and $b_V$ are obtained using charge conservation. We use the forward matrix elements of $(V_f)_{J4}$ between pseudoscalars,

$$\frac{1}{Z_V^0(1 + b_V am_2)} = \sum_y \frac{\langle P^{(12)}(\vec{x}, \tau)(V_f)_{J4}^{(22)}(\vec{y}, t) J^{(21)}(0) \rangle}{\langle \sum x P^{(12)}(\vec{x}, \tau) J^{(21)}(0) \rangle}.$$  

(8)

with $\tau > t > 0$ and $J = P$ or $A_4$. Note that the $c_V$ term in $V_1$ does not contribute.

Next consider the generic axial WI

$$\left\langle \delta S^{(12)}^{(23)}(y) J^{(31)}(z) \right\rangle = \left\langle \delta O^{(13)}_R(y) J^{(31)}(z) \right\rangle$$

(9)

where $O^{23} = \bar{\psi}^2(\gamma_5 \gamma_5 \psi)^3$, and

$$\delta S^{(12)} = \int_V \left[ (m_R^1 + m_R^2) (P_R)^{(12)} - \partial_\mu (A_R)^{(12)}_\mu \right].$$

(10)

This results from a chiral rotation on flavors 1, 2 in the 4-volume $V$, with $y \in V$ and $z \not\in V$. By enforcing these identities in the chiral limit one can determine $c_V$ and $c_T$, as shown below.

Away from the chiral limit, operators $P$ and $O$ in the product $\int_V (m_R^1 + m_R^2) (P_R)^{(12)} x O^{(23)}(y)$ need off-shell improvement. This requires the addition of a contact term, of unknown normalization, having the same form as the RHS of (9). Our new observation is that the contact term is proportional to $m_1 + m_2$ and so can be removed by extrapolating $m_1$ and $m_2$ to zero. This leaves the freedom to examine the dependence on $m_3$, and from this one can determine certain combinations of the $b_V$. In the following, the extrapolation to $m_1 = m_2 = 0$ is implicit.

As a first application of this method we show how to obtain $b_A$, as well as $c_V$, using the AWI

$$r_1 = \frac{Z_A^0(1 + b_A am_3/2)}{Z_A^0 \cdot Z_V^0(1 + b_V am_3/2)} = \frac{\sum_y \delta S^{(12)}(V_f)^{(23)} (\vec{y}, y_4) J^{(31)}(0)}{\sum_y \langle A^{(13)}_4 (\vec{y}, y_4) J^{(31)}(0) \rangle}.$$  

(11)

$$= \frac{\sum_y e^{i \vec{p} \cdot \vec{y}} \delta S^{(12)}(V_f)^{(23)} (\vec{y}, y_4) A^{(31)}(0)}{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle A^{(13)}_4 (\vec{y}, y_4) A^{(31)}(0) \rangle}.$$  

(12)

where $J = P$ or $A_4$. Eq. (11) is independent of $c_V$ at $\vec{p} = 0$, and its $m_3$ dependence gives $b_A - b_V$. The intercept provides a second determination of $Z_V^0$. Eq. (12) is used to determine $c_V$.

Given $c_V$, an alternate determination of $b_A - b_V$ is obtained from

$$r_2 = \frac{Z_A^0(1 + b_A am_3/2)}{Z_A^0 \cdot Z_V^0} \cdot \frac{Z_{b-V}^0(1 + b_V am_3/2)}{Z_{b-V}^0} = \frac{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle \delta S^{(12)}(A^{(13)}_4 (\vec{y}, y_4) V^{(31)}_i(0)) \rangle}{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle \delta S^{(12)}_R (V^{(31)}_i(0)) \rangle}.$$  

This also yields $Z_A^0$. The same information can be obtained from the combinations

$$\frac{1}{\sqrt{r_1 \cdot r_2}} = Z_A^0,$$

(14)

$$\frac{\sqrt{r_2}}{r_2} = \frac{Z_A^0(1 + (b_A - b_V) am_3/2)}{Z_V^0}.$$  

(15)

Similarly, we determine $b_p - b_S$ and $Z_p^0/Z_S^0$ using

$$\frac{Z_p^0(1 + b_p am_3/2)}{Z_A^0 \cdot Z_V^0} = \frac{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle \delta S^{(12)}_R (A^{(13)}_4 (\vec{y}, y_4) T^{(31)}_{ik} (0)) \rangle}{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle \delta S^{(12)}_R (A^{(13)}_4 (\vec{y}, y_4) T^{(31)}_{ik} (0)) \rangle}.$$  

(16)

To get $c_T$ we use the WI with $O = T_{ij}$

$$1 + ac_T \frac{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle - \partial_\mu V^{(13)}_\mu (\vec{y}, y_4) T^{(31)}_{ik} (0) \rangle}{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle T^{(13)}_{ik} (\vec{y}, y_4) T^{(31)}_{ik} (0) \rangle} = Z_A^0 \frac{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle \delta S^{(12)}_R (T^{(13)}_{ij} (\vec{y}, y_4) T^{(31)}_{ik} (0)) \rangle}{\sum_y e^{i \vec{p} \cdot \vec{y}} \langle T^{(13)}_{ik} (\vec{y}, y_4) T^{(31)}_{ik} (0) \rangle}.$$  

(17)

At $\vec{p} = 0$, $(T_{ij})_{ik}$ has no contribution from the $c_T$ term, so the only $c_T$ dependence is on the LHS.

The previous method fails for $b_T$ since both sides of (16) have the same dependence on $b_T$ if $m_1 = m_2 = 0$. The cure is to consider three non-degenerate masses. The contact term required to improve the RHS of (12) is proportional to $m_1 + m_2$, while the relative dependence of the two sides on $b_T$ is proportional to $m_1 - m_2$. Thus the two terms can be separated and $b_T$ determined. More details will be given in [11]. We have not yet implemented this proposal.

Thus far we do not have a separate determination of $b_P$ or $b_S$. This can be accomplished with
the method of Ref. [10], which uses 2-point correlation functions for non-degenerate masses. In addition to using that method we present a variant which avoids the need to study quantities as a function of the underlying hopping parameter κ. We first note that if we use Eq. (18) and enforce 

\( (2m_1)^R + (2m_2)^R = 2(m_1 + m_2)^R \), we find

\[
b_p - b_A = -\frac{4m_{12} - 2[m_{11} + m_{22}]}{a[m_{11} - m_{22}]^2}, \tag{18}\]

We next make use of the vector two-point WI

\[
\Delta m_{12} = \sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \langle \partial_\mu V_{ij}^{(12)}(\vec{x}, t) J^{(21)}(0) \rangle \\
\sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \langle S^{(12)}(\vec{x}, t) J^{(21)}(0) \rangle
\]

\[
m_1^R - m_2^R = \frac{Z_0^0 [1 + b_V a (m_1 + m_2)/2]}{Z_0^0 [1 + b_S a (m_1 + m_2)/2]} \Delta m_{12}
\]

where the source \( J \) is either \( J^{(21)} \) or \( \sum_{\vec{x}} P^{(2)}(\vec{x}, z_t) P^{(3)}(0) \) for \( 0 < t < z_t \). Enforcing \( 2(m_1^R - m_2^R) = (2m_1)^R - (2m_2)^R \), we find

\[
b_S - b_V = \frac{\Delta m_{12} - R_Z [m_{11} - m_{22}]}{2 a R_Z [m_{11} - m_{22}]}, \tag{19}\]

where \( R_Z \equiv Z_0^0 Z_A^0 / (Z_0^0 Z_P^0) \). Since \( b_A \) and \( b_V \) are already known, \( b_P \) and \( b_S \) are given by Eqs. (18) and (19).

The above discussion shows that in principle one can determine all the constants, except \( Z_P^0 \) (or \( Z_A^0 \) and \( Z_P^0 \)), in the quenched theory using Ward identities. The results of an exploratory study are summarized in Table 1. These were obtained on 83 lattices of size \( 16^2 \times 48 \) at \( \beta = 6.0 \). Since the action is only tree-level tadpole improved (\( c_{SW} = 1.4755 \)), the results do not represent full \( O(a) \) improvement, but they indicate the efficacy of the method. More details will appear in [11].

We draw two preliminary conclusions. First, even though we have found channels in which the statistical and systematic errors on the determination of \( b_A - b_V \) and \( b_P - b_S \) are fairly small, the magnitude of these differences are still comparable to their error. This rough equality between all the \( b_X \) is consistent with perturbative results [12]. Second, the determination of \( c_V \) has a large uncertainty, which accounts for a substantial fraction of the errors in \( Z_A^0, Z_P^0 / Z_S^0 \), and \( c_T \). For \( c_A \) and \( c_V \), the Schrödinger functional method [8] gives results with much smaller errors, and may prove to be the method of choice.

| Eq. # | observable | intercept |
|-------|------------|-----------|
| (4)   | \( c_A \)  | -0.016(11)|
| (5)   | \( Z_V^0 \) | +0.746(1) |
| (8)   | \( b_V \)  | +1.55(2)  |
| (11)  | \( Z_V^0 \) | +0.752(7) |
| (11)  | \( b_A - b_V \) | +0.34(21) |
| (12)  | \( c_V \)  | +0.46(29) |
| (13)  | \( Z_V^0 / (Z_A^0)^2 \) | +1.32(12) |
| (14)  | \( b_A - b_V \) | +1.8(1.1) |
| (15)  | \( Z_A^0 / Z_P^0 \) | +0.78(2) |
| (15)  | \( Z_A^0 / Z_P^0 \) | +1.00(5) |
| (15)  | \( b_A - b_V \) | +1.2(8) |
| (16)  | \( b_P - b_S \) | -0.08(9) |
| (16)  | \( c_T \)  | -0.14(7) |
| (16)  | \( Z_P^0 / Z_S^0 \) | +0.96(1) |
| (16)  | \( b_A - b_P + b_S / 2 \) | +0.49(1) |
| (16)  | \( b_P - b_A \) | +0.4(1) |
| (16)  | \( b_S - b_V - 2(b_P - b_A) \) | -0.5(5) |

Table 1

Constants extracted from the different WI.

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