On the nonexistence of global weak solutions to the Navier-Stokes-Poisson equations in $\mathbb{R}^N$

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Abstract

In this paper we prove nonexistence of stationary weak solutions to the Euler-Poisson equations and the Navier-Stokes-Poisson equations in $\mathbb{R}^N$, $N \geq 2$, under suitable assumptions of integrability for the density, velocity and the potential of the force field. For the time dependent Euler-Poisson equations we prove nonexistence result assuming additionally temporal asymptotic behavior near infinity of the second moment of density. For a class of time dependent Navier-Stokes-Poisson equations this asymptotic behavior of the density can be proved if we assume the standard energy inequality, and therefore the nonexistence of global weak solution follows from more plausible assumption in this case.

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1 Introduction

We are concerned on the Navier-Stokes-Poisson equations in \( \mathbb{R}^N, \ N \geq 1 \).

\[
(\text{NSP, EP}) \begin{cases} 
\partial_t \rho + \text{div}(\rho v) = 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) = -\nabla p + k \rho \nabla \Phi + \mu \Delta v + (\mu + \lambda) \nabla \text{div} v, \\
\Delta \Phi = \rho, \\
\rho \geq 0, \ p = p(\rho) \geq 0 (p = 0 \text{ only if } \rho = 0). 
\end{cases}
\]

The system \((\text{NSP, EP})\) describes compressible gas flows, and \(\rho, v, \Phi\) and \(p\) denote the density, velocity, the potential of the underlying force and the pressure respectively. Here \(k\) is a physical constant, which signifies the property of the forcing, repulsive if \(k > 0\) and attractive if \(k < 0\). In this paper we consider only the case of repulsive forcing, \(k \geq 0\). We treat the viscous case \(\mu > 0\), which corresponds to the Navier-Stokes-Poisson equations(\text{NSP}), and the inviscid case \(\mu = \lambda = 0\), which corresponds to the Euler-Poisson equations(\text{EP}), simultaneously. Our aim here is to prove the nonexistence of global weak solutions to the system \((\text{NSP, EP})\) with \(N \geq 2\) under suitable integrability conditions for the solutions together with additional condition for the initial data. For the stationary case the standard finite energy condition already implies the integrability for the solutions. For the time dependent Euler-Poisson equations, however, we need an extra condition for the temporal asymptotic behavior near infinity of the second moment of density \(\rho(x,t)(\text{see (1.18) for more specification})\) to get desired nonexistence result. For the Navier-Stokes-Poisson equations describing the isothermal viscous fluids with \(p(\rho) = a \rho^\gamma, 1 < \gamma \leq N/4 + 1/2, N \geq 3\), the condition of the asymptotic behavior of density can be proved, thanks to a lemma due to Guo and Jiang[3], if we assume the energy inequality. Hence, in this case the finite energy condition together with \(v \in L^{\frac{N}{N-1}}(\mathbb{R}^N \times [0, T))\) for all \(T > 0\) imply the nonexistence of the global weak solutions satisfying the energy inequality for an initial data satisfying suitable sign condition. This implies that even if the finite blow-up happens for certain smooth initial data, it could not be continued as a physically meaningful global weak solution afterwards. The results derived in this paper for the nonexistence of global weak solutions could be regarded as Luouville type of theorems. The convection term and the forcing term have the “positivity” structure, in appropriate sense, which resembles the ellipticity in the elliptic partial differential equations. Those “positivity” structures, combined with the actual positivity of
the pressure term provides us the desired nonexistence results for the nontrivial global weak solutions. Earlier observations of similar feature for the convection term were made, and applied to the compressible Euler and the compressible Navier-Stokes equations in \cite{1, 2}, and the theorems obtained here are generalizations of those in \cite{1, 2}, which are not straightforward due to the nonlinear forcing term $k\rho \nabla \Phi$ in (NSP, EP). The sign condition for $k \geq 0$ and the restriction of spatial dimension $N \geq 2$ are crucially important to deduce the favorable positivity of the forcing term. At this moment we do not know if similar nonexistence results hold also for $k < 0$ or $N = 1$.

1.1 Nonexistence of stationary weak solutions

In this section we state precisely the nonexistence theorem for the stationary weak solutions to the system (NSP, EP). A stationary weak solutions of (NSP, EP) is defined as follows.

**Definition 1.1** We say that a triple $(\rho, v, \Phi) \in L^\infty_{\text{loc}}(\mathbb{R}^N) \times \left[ L^2_{\text{loc}}(\mathbb{R}^N) \right]^N \times W^{2,2}_{\text{loc}}(\mathbb{R}^N)$ is a stationary weak solution of (NSP, EP) if

\[ \int_{\mathbb{R}^N} \rho v \cdot \nabla \psi \, dx = 0 \quad \forall \psi \in C^\infty_0(\mathbb{R}^N), \quad (1.1) \]

\[ \int_{\mathbb{R}^N} \rho v \otimes v : \nabla \phi \, dx = - \int_{\mathbb{R}^N} p \, \text{div} \phi \, dx - k \int_{\mathbb{R}^N} \rho \nabla \Phi \cdot \phi \, dx - \mu \int_{\mathbb{R}^N} v \cdot \Delta \phi \, dx \]

\[ - (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \text{div} \phi \, dx \quad \forall \phi \in [C^\infty_0(\mathbb{R}^N)]^N, \quad (1.2) \]

\[ \Delta \Phi = \rho \quad \text{almost everywhere in } \mathbb{R}^N, \quad (1.3) \]

\[ p = p(\rho) \geq 0, \quad p = 0 \quad \text{only if } \rho = 0 \quad \text{almost everywhere on } \mathbb{R}^N. \quad (1.4) \]

In our proof of the following theorem we do not use the equation of continuity (1.1).

**Theorem 1.1** Let $N \geq 2$. Suppose $(\rho, v, \Phi)$ is a stationary weak solution to (NSP, EP) satisfying one of the following conditions depending on $\mu$ and $\lambda$.

(i) For (EP)$(\mu = \lambda = 0)$;
(i-a) (The case $N \geq 3$) There exists $w \in L_{loc}^1([0, \infty))$, which is positive, non-increasing on $[0, \infty)$ such that
\[
\int_{\mathbb{R}^N} \left( \rho |v|^2 + p + k |\nabla \Phi|^2 \right) \times \\
\times \left[ w(|x|) + \frac{1}{|x|} \int_0^{[x]} w(s) ds + \frac{1}{|x|^2} \int_0^{[x]} \int_0^{r} w(s) ds dr \right] dx < \infty.
\]
(1.5)

(i-b) (The case $N = 2$)
\[
\int_{\mathbb{R}^N} (\rho |v|^2 + p + k |\nabla \Phi|^2) dx < \infty.
\]
(1.6)

(ii) For (NSP) $(\mu > 0)$ with $N \geq 2$ ;

(a) if $2\mu + \lambda = 0$,
\[
\int_{\mathbb{R}^N} (\rho |v|^2 + p + k |\nabla \Phi|^2) \, dx < \infty.
\]
(1.7)

(b) if $2\mu + \lambda \neq 0$,
\[
\int_{\mathbb{R}^N} (\rho |v|^2 + |v|^{\frac{N}{N-1}} + p + k |\nabla \Phi|^2) \, dx < \infty.
\]
(1.8)

Then, $\rho(x) = 0, \nabla \Phi(x) = 0$ for almost every $x \in \mathbb{R}^N$.

Remark 1.1 Choosing, in particular,
\[
w(r) = 1/(1 + r^2),
\]
then for all $x \in \mathbb{R}^N$ we have
\[
\int_0^{[x]} w(r) dr \leq \frac{\pi}{2}, \quad \int_0^{[x]} \int_0^{r} w(s) ds dr \leq \frac{\pi |x|}{2}, \text{ and}
\]
\[
w(|x|) + \frac{1}{|x|} \int_0^{[x]} w(s) ds + \frac{1}{|x|^2} \int_0^{[x]} \int_0^{r} w(s) ds dr \leq \frac{C}{1 + |x|}.
\]

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for some constant $C$ independent of $x$. Thus the condition for the initial data (1.17) and (1.21) are implied by
\[ \int_{\mathbb{R}^N} \rho_0(x)|v_0(x)|dx < \infty, \] (1.10)
and
\[ \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \frac{x}{|x|} \arctan(|x|)dx \geq CK_1 \] (1.11)
respectively, while the condition (1.5) is implied by
\[ \int_{\mathbb{R}^N} \rho(x)|v(x)|^2 + p(x) + k|\nabla \Phi(x)|^2 \frac{1}{1 + |x|}dx < \infty \] (1.12)
respectively. Note that the condition (1.12) is even weaker than the finite energy condition, in the sense that it is implied by the finite energy condition that is also obtained from (1.5) by choosing $w = 1$.

### 1.2 Nonexistence of time dependent weak solutions

The definition of time dependent weak solutions for (NSP, EP) is follows.

**Definition 1.2** We say a triple

\[ (\rho, v, \Phi) \in L^1_{\text{loc}}((0, \infty); L^\infty_{\text{loc}}(\mathbb{R}^N)) \times [L^1_{\text{loc}}((0, \infty); L^2_{\text{loc}}(\mathbb{R}^N))]^N \times L^2_{\text{loc}}((0, \infty); W^{2,2}_{\text{loc}}(\mathbb{R}^N)) \]

is a global weak solution of (NSP) with initial data $(\rho_0, v_0)$ if

\[
\xi(0) \int_{\mathbb{R}^N} \rho_0(x)\psi(x)dx + \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)\psi(x)\xi'(t)dxdt \\
+ \int_0^\infty \int_{\mathbb{R}^N} \rho v(x,t) \cdot \nabla \psi(x)\xi(t) dx = 0 \\
\forall \psi \in C^\infty_0(\mathbb{R}^N), \xi \in C^1_0([0, \infty)), \] (1.13)

\[
\xi(0) \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \phi(x)dx + \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)v(x,t) \cdot \phi(x)\xi'(t)dxdt \\
+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)v(x,t) \otimes v(x,t) : \nabla \phi(x)\xi(t) dxdtdx = 0 \\
\forall \phi \in C^\infty_0(\mathbb{R}^N), \xi \in C^1_0([0, \infty)), \]
\[ -\int_0^\infty \int_{\mathbb{R}^N} p(x,t) \text{div} \phi(x) \xi(t) \, dxdt - k \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \nabla \Phi(x,t) \cdot \phi(x) \xi(t) \, dxdt \\
- \mu \int_0^\infty \int_{\mathbb{R}^N} v(x,t) \cdot \Delta \phi(x) \xi(t) \, dxdt - (\mu + \lambda) \int_0^\infty \int_{\mathbb{R}^N} v(x,t) \cdot \nabla \text{div} \phi(x) \xi(t) \, dxdt \]

\[ \forall \phi \in [C_0^\infty(\mathbb{R}^N)]^N, \xi \in C_0^1([0, \infty)), \]

(1.14)

\[ \Delta \Phi = \rho \text{ almost everywhere on } \mathbb{R}^N \times [0, \infty), \]

(1.15)

\[ \rho \geq 0, p = p(\rho) \geq 0(p = 0 \text{ only if } \rho = 0) \text{ almost everywhere on } \mathbb{R}^N \times [0, \infty). \]

(1.16)

In the above the derivatives of \( \xi \in C_0^1([0, \infty)) \) at \( t = 0 \) should be understood as \( \xi'(0) := \xi'(0^+). \)

**Theorem 1.2 (Conditional nonexistence for (EP))**

(i) The case \( N \geq 3 \):

Let the function \( w \in L^1_{\text{loc}}([0, \infty)) \) be given, which is positive, non-increasing on \([0, \infty)\), and let \((\rho_0, v_0)\) satisfy

\[ \int_{\mathbb{R}^N} \rho_0(x) v_0(x) \left[ \int_0^{\|x\|} w(r) \, dr \right] \, dx < \infty. \]

(1.17)

Suppose \((\rho, v, \Phi)\) is a global weak solution to (EP) with the initial data \((\rho_0, v_0)\) such that

\[ \limsup_{\tau \to \infty} \int_{\tau \leq t \leq 2\tau} \int_{\mathbb{R}^N} \rho(x,t) \left[ \int_0^{\|x\|} w(s) \, ds \right] \, dxdt \leq K_1 \]

(1.18)

for a constant \( K_1 \geq 0 \), satisfying

\[ \int_0^T \int_{\mathbb{R}^N} (\rho|v|^2 + p + k|\nabla \Phi|^2) \times \]

\[ \times \left[ w(|x|) + \frac{1}{|x|} \int_0^{\|x\|} w(s) \, ds + \frac{1}{|x|^2} \int_0^{\|x\|} \int_0^{\|x\|} w(s) \, ds \, dr \right] \, dxdt < \infty \]

(1.19)

for all \( T > 0 \). Then, necessarily the following inequality holds true.

\[ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \left[ w(|x|) \frac{(v \cdot x)^2}{|x|^2} + \frac{1}{|x|} \int_0^{\|x\|} w(r) \, dr \left( |v|^2 - \frac{(v \cdot x)^2}{|x|^2} \right) \right] \, dxdt \]

\[ + \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \left[ w(|x|) + \frac{N-1}{|x|} \int_0^{\|x\|} w(r) \, dr \right] \, dxdt \]
\[ + \frac{(N - 3)k}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} \left[ \int_0^{|x|} w(s) ds \right] dx dt \]
\[ + k \int_0^\infty \int_{\mathbb{R}^N} \left[ \frac{1}{|x|} \int_0^{|x|} w(s) ds - w(|x|) \right] \frac{(x \cdot \nabla \Phi)^2}{|x|^2} dx dt \]
\[ + \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 w(|x|) dx dt \]
\[ + \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \frac{x}{|x|} \left[ \int_0^{|x|} w(r) dr \right] dx \leq C K_1 \]

for a constant \( C \). Therefore, if
\[ \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \frac{x}{|x|} \left[ \int_0^{|x|} w(r) dr \right] dx > C K_1, \]

then there exists no global weak solution satisfying (1.18)-(1.19).

(ii) The case \( N = 2 \) : Let \((\rho_0, v_0)\) satisfy
\[ \int_{\mathbb{R}^N} \rho_0(x) |v_0(x)| |x| dx < \infty. \]
Suppose \((\rho, v, \Phi)\) is a global weak solution to (EP) with the initial data \((\rho_0, v_0)\) such that
\[ \limsup_{\tau \to \infty} \int_{\tau \leq t \leq 2\tau} \int_{\mathbb{R}^N} \rho(x, t) \frac{|x|^2}{1 + t^2} dx dt \leq K_1 \]

for a constant \( K_1 \geq 0 \), satisfying
\[ \int_0^T \int_{\mathbb{R}^2} (\rho |v|^2 + p + k|\nabla \Phi|^2) dx dt < \infty \]

for all \( T > 0 \). Then, necessarily the following inequality holds true.
\[ \int_0^\infty \int_{\mathbb{R}^2} (\rho |v|^2 + 2p) dx dt + \int_{\mathbb{R}^2} \rho_0(x)v_0(x) \cdot x dx \leq C K_1 \]

for a constant \( C \). Therefore, if
\[ \int_{\mathbb{R}^2} \rho_0(x)v_0(x) \cdot x dx > C K_1, \]

then there exists no global weak solution satisfying (1.23)-(1.25).
Remark 1.2 Similarly to Remark 1.1, choosing \( w(r) = 1/(1 + r^2) \), the conditions for the solution (1.18) and (1.19) are implied by

\[
\limsup_{\tau \to \infty} \int_{\tau \leq t \leq 2\tau} \int_{\mathbb{R}^N} \frac{\rho(x,t)|x|}{1 + t^2} dxdt \leq K_1, \tag{1.27}
\]

and

\[
\int_0^T \int_{\mathbb{R}^N} \frac{\rho(x,t)|v(x,t)|^2 + p(x,t) + k|\nabla \Phi(x,t)|^2}{1 + |x|} dxdt < \infty \quad \forall T > 0 \tag{1.28}
\]

respectively.

As will be seen below, for a class of isothermal viscous fluids the key condition (1.18) is really satisfied with \( K_1 = 0 \), if we assume the energy inequality, and we have the following stronger nonexistence results of the global weak solutions.

**Theorem 1.3 (Nonexistence for (NSP))** We fix \( N \geq 3, 1 < \gamma \leq N/4 + 1/2, \mu > 0, \mu + \lambda > 0 \), and the following form of pressure law,

\[
p = p(\rho) = a\rho^\gamma \tag{1.29}
\]

in (NSP). Let the initial data \((\rho_0, v_0)\) satisfy

\[
\int_{\mathbb{R}^N} \rho_0(x)|v_0(x)||x|dx < \infty. \tag{1.30}
\]

Suppose \((\rho, v, \Phi)\) is a global weak solution to (NSP) such that

\[
\int_0^T \int_{\mathbb{R}^N} \left[ \rho|v|^2 + p + |v|^{N-1} + k|\nabla \Phi|^2 \right] dxdt < \infty \tag{1.31}
\]

for all \( T > 0 \). We further assume that the following energy inequality holds.

\[
E(t) + \int_0^t \int_{\mathbb{R}^2} (\mu|\nabla v|^2 + (\mu + \lambda)|\text{div } v|^2) dxds \leq E(0) < \infty \quad \forall t \geq 0,
\]

where

\[
E(t) := \int_{\mathbb{R}^N} \left[ \frac{\rho}{2}|v|^2 + \frac{a\rho^\gamma}{\gamma - 1} + \frac{k}{2}|\nabla \Phi|^2 \right] dx. \tag{1.32}
\]
Then, necessarily we have the equality
\[
\int_0^\infty \int_{\mathbb{R}^N} \left[ \rho(x,t)v(x,t)^2 + Np(x,t) + \frac{N-2}{2} |\nabla \Phi(x,t)|^2 \right] \, dxdt
\]
\[
= - \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x \, dx
\]  
(1.33)

Therefore, if
\[
\int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x \, dx \geq 0,
\]  
(1.34)
then the only global weak solution corresponds to \( \rho = 0 \) and \( \nabla \Phi = 0 \) almost everywhere on \( \mathbb{R}^N \times [0, \infty) \). In particular, if the strict inequality holds in (1.34), then there exists no global weak solution satisfying (1.30)-(1.32).

2 Proof of the main theorems

Proof of Theorem 1.1 We suppose there exists a stationary weak solution \((\rho, v, \Phi)\). We begin the proof with the inviscid case.

(i) The case \( \mu = \lambda = 0 \) (EP): Let us consider a radial cut-off function \( \sigma \in C_0^\infty(\mathbb{R}^N) \) such that
\[
\sigma(|x|) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 2,
\end{cases}
\]  
(2.1)
and \( 0 \leq \sigma(x) \leq 1 \) for \( 1 < |x| < 2 \). We set
\[
W(u) := \int_0^u \int_0^s w(r) drds.
\]  
(2.2)
Then, for each \( R > 0 \), we define
\[
\varphi_R(x) = W(|x|)\sigma \left( \frac{|x|}{R} \right) = W(|x|)\sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).
\]  
(2.3)
We choose the vector test function \( \phi \) in (1.2) as
\[
\phi = \nabla \varphi_R(x).
\]  
(2.4)
Then, after routine computations, the equation (1.2) becomes
\[
0 = \int_{\mathbb{R}^N} \rho(x) \left[ W''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + \frac{\sigma''(|x|)}{|x|^2} \right) \right. \\
\left. + \frac{1}{R} \int_{\mathbb{R}^N} \rho(x) W''(|x|) \sigma' \left( \frac{|x|}{R} \right) \frac{(v \cdot x)^2}{R |x|^2} \right] \sigma_R(|x|) \, dx \\
+ \frac{1}{R} \int_{\mathbb{R}^N} \rho(x) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \sigma' \left( \frac{|x|}{R} \right) W(|x|) \, dx \\
+ \int_{\mathbb{R}^N} \rho(x) \frac{(v \cdot x)^2}{R^2 |x|^2} \sigma'' \left( \frac{|x|}{R} \right) W(|x|) \, dx \\
+ \int_{\mathbb{R}^N} p(x) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \sigma_R(|x|) \, dx \\
\left. + \frac{2}{R} \int_{\mathbb{R}^N} p(x) \frac{1}{R} \frac{|x|}{|x|^2} \sigma' \left( \frac{|x|}{R} \right) \right] \sigma_R(|x|) \, dx \\
+ \frac{N - 1}{R} \int_{\mathbb{R}^N} p(x) \frac{1}{|x|} \sigma' \left( \frac{|x|}{R} \right) W(|x|) \, dx \\
+ \int_{\mathbb{R}^N} p(x) \frac{1}{R^2} \sigma'' \left( \frac{|x|}{R} \right) W(|x|) \, dx \\
+ k \int_{\mathbb{R}^N} \rho \nabla \Phi \cdot \nabla \left[ W(|x|) \sigma_R(|x|) \right] \, dx \\
:= I_1 + \cdots + I_9. \tag{2.5}
\]
In terms of the function $W$ defined in (2.2) our condition (1.5) can be written as
\[
\int_{\mathbb{R}^N} (\rho(x)|v(x)|^2 + |p(x)| + k|\nabla \Phi|^2) \left[ W''(|x|) + \frac{1}{|x|} W'(|x|) + \frac{1}{|x|^2} W(|x|) \right] \, dx < \infty. \tag{2.6}
\]
Since
\[
\int_{\mathbb{R}^N} \rho(x) \left[ W''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + \frac{W'(|x|)}{|x|} \right) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \\
\leq 2 \int_{\mathbb{R}^N} \rho(x)|v(x)|^2 \left[ W''(|x|) + \frac{W'(|x|)}{|x|} \right] \, dx < \infty,
\]
we can use the dominated convergence theorem to show that
\[
I_1 \to \int_{\mathbb{R}^N} \rho(x) \left[ W''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \right) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \tag{2.7}
\]
as $R \to \infty$. Similarly,

$$I_5 \to \int_{\mathbb{R}^N} p(x) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \, dx$$

(2.8)
as $R \to \infty$. For $I_2$ we estimate

$$|I_2| \leq \int_{R < |x| < 2R} \rho(x)|v(x)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{W'(|x|)|x|}{|x|} \, dx$$

$$\leq 2 \sup_{1 < s < 2} \sigma'(s) \int_{R < |x| < 2R} \rho(x)|v(x)|^2 \frac{W'(|x|)}{|x|} \, dx$$

$$\to 0$$

(2.9)
as $R \to \infty$ by the dominated convergence theorem. Similarly

$$|I_3| \leq 2 \int_{R < |x| < 2R} \frac{|x|}{R} \rho(x)|v(x)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \, dx$$

$$\leq 4 \sup_{1 < s < 2} \sigma'(s) \int_{R < |x| < 2R} \rho(x)|v(x)|^2 \frac{W'(|x|)}{|x|} \, dx \to 0,$$

(2.10)
and

$$|I_4| \leq \int_{R < |x| < 2R} \frac{|x|^2}{R^2} \rho(x)|v(x)|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \, dx$$

$$\leq 4 \sup_{1 < s < 2} \sigma''(s) \int_{R < |x| < 2R} \rho(x)|v(x)|^2 \frac{W(|x|)}{|x|^2} \, dx \to 0$$

(2.11)
as $R \to \infty$. The estimates for $I_6, I_7$ and $I_8$ are similar to the above, and we find

$$|I_6| \leq 2 \int_{R < |x| < 2R} |p(x)| \frac{|x|}{R} \left| \frac{W'(|x|)}{|x|} \right| \sigma' \left( \frac{|x|}{R} \right) \, dx$$

$$\leq 4 \sup_{1 < s < 2} \sigma'(s) \int_{R < |x| < 2R} |p(x)| \frac{W'(|x|)}{|x|} \, dx \to 0,$$

(2.12)
By similar computations to (2.7)-(2.14), using (2.6), we find that

\[ |I_7| \leq (N - 1) \int_{R < |x| < 2R} |p(x)| \left| \frac{|x|}{R} \right| \sigma' \left( \frac{|x|}{R} \right) \frac{W(|x|)}{|x|^2} dx \]

\[ \leq 2 \sup_{1 < s < 2} |\sigma'(s)| \int_{R < |x| < 2R} |p(x)| \frac{W(|x|)}{|x|^2} dx \to 0, \quad (2.13) \]

and

\[ |I_8| \leq \int_{\mathbb{R}^N} |p(x)| \frac{|x|^2}{R^2} \left| \frac{|x|}{R} \right| \frac{W(|x|)}{|x|^2} dx \]

\[ \leq 4 \sup_{1 < s < 2} |\sigma''(s)| \int_{R < |x| < 2R} |p(x)| \frac{W(|x|)}{|x|^2} dx \to 0 \quad (2.14) \]

as \( R \to \infty \) respectively. Using the relation (1.15), and integrating by parts we compute

\[ I_9 = k \int_{\mathbb{R}^N} \Delta \Phi \nabla \Phi \cdot \nabla [W(|x|)\sigma_R(|x|)] dx \]

\[ = -k \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i \Phi \partial_j \Phi \partial_i \partial_j [W(|x|)\sigma_R(|x|)] dx - k \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i \Phi \partial_j \Phi \partial_i \partial_j [W(|x|)\sigma_R(|x|)] dx \]

\[ = -k \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i \Phi \partial_j \Phi \partial_i \partial_j [W(|x|)\sigma_R(|x|)] dx + \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 \Delta [W(|x|)\sigma_R(|x|)] dx \]

\[ = -k \int_{\mathbb{R}^N} \left[ \frac{|\nabla \Phi|^2}{|x|} - \frac{(x \cdot \nabla \Phi)^2}{|x|^3} \right] \left[ W'(|x|)\sigma_R(|x|) + \frac{W(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \right] dx \]

\[-k \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \left[ W''(|x|)\sigma_R(|x|) + \frac{W(|x|)}{R^2} \sigma'' \left( \frac{|x|}{R} \right) + \frac{2W'(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \right] dx \]

\[ + \frac{(N - 1)k}{2} \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} \left[ W'(|x|)\sigma_R(|x|) + \frac{W(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \right] dx \]

\[ + \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 \left[ W''(|x|)\sigma_R(|x|) + \frac{W(|x|)}{R^2} \sigma'' \left( \frac{|x|}{R} \right) + \frac{2W'(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \right] dx. \]

\[ := J_1 + \ldots + J_4. \quad (2.15) \]

By similar computations to (2.7)-(2.14), using (2.6), we find that

\[ J_1 = -k \int_{\mathbb{R}^N} \left[ \frac{|\nabla \Phi|^2}{|x|} - \frac{(x \cdot \nabla \Phi)^2}{|x|^3} \right] W'(|x|) dx + o(1), \]

\[ J_2 = -k \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} W''(|x|) dx + o(1), \]

\[ := J_1 + \ldots + J_4. \quad (2.15) \]
\[ J_3 = \frac{(N-1)k}{2} \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} W'(|x|) \, dx + o(1), \]
\[ J_4 = \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 W''(|x|) \, dx + o(1) \]
as \( R \to \infty \). Therefore, taking the limit \( R \to \infty \), and rearranging the remaining terms, we have
\[ I_9 \to -k \int_{\mathbb{R}^N} \left[ \frac{|\nabla \Phi|^2}{|x|} - \frac{(x \cdot \nabla \Phi)^2}{|x|^3} \right] W'(|x|) \, dx - k \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} W''(|x|) \, dx \]
\[ + \frac{(N-1)k}{2} \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} W'(|x|) \, dx + \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 W''(|x|) \, dx \]
\[ = \frac{(N-3)k}{2} \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} W'(|x|) \, dx + k \int_{\mathbb{R}^N} \left[ \frac{W'(|x|)}{|x|} - W''(|x|) \right] \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \, dx \]
\[ + \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 W''(|x|) \, dx \]
(2.16)
as \( R \to \infty \). Thus passing \( R \to \infty \) in (2.5), and using (2.7)-(2.16), we finally obtain
\[ \int_{\mathbb{R}^N} \rho(x) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \]
\[ + \int_{\mathbb{R}^N} p(x) \left[ W''(|x|) + (N-1) \frac{W'(|x|)}{|x|} \right] \, dx \]
\[ + \frac{(N-3)k}{2} \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} W'(|x|) \, dx + k \int_{\mathbb{R}^N} \left[ \frac{W'(|x|)}{|x|} - W''(|x|) \right] \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \, dx \]
\[ + \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 W''(|x|) \, dx = 0, \]
(2.17)
which can be written, in terms of the function \( w(r) \), as
\[ \int_{\mathbb{R}^N} \rho(x) \left[ w(|x|) \frac{(v \cdot x)^2}{|x|^2} + \frac{1}{|x|} \int_0^{|x|} w(s) ds \left( \frac{|v|^2}{|x|^2} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \]
\[ + \int_{\mathbb{R}^N} p(x) \left[ w(|x|) + \frac{N-1}{|x|} \int_0^{|x|} w(s) ds \right] \, dx \]
\[ + \frac{(N-3)k}{2} \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} \left[ \int_0^{|x|} w(s) ds \right] \, dx \]
\[ + k \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \int_0^{|x|} w(s) |x| - w(|x|) \right) \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \, dx \]
\[ + \frac{k}{2} \int_{\mathbb{R}^N} |\nabla \Phi|^2 w(|x|) \, dx = 0. \tag{2.18} \]

We note that
\[ w(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + \frac{1}{|x|} \int_0^{|x|} w(s) \left( |v|^2 - \frac{(v \cdot x)^2}{|x|^2} \right) \right) \geq 0, \]
and
\[ w(|x|) + \frac{N - 1}{|x|} \int_0^{|x|} w(s) \, ds > 0. \]

Moreover, since \( w(r) \) is non-increasing a.e. on \([0, \infty)\) by hypothesis, we have
\[ \frac{1}{|x|} \int_0^{|x|} w(s) \, ds - w(|x|) \geq 0 \quad \text{for almost every } x \in \mathbb{R}^N. \]

Thus all of the terms in (2.18) are nonnegative for \( N \geq 3 \), and we need to have
\[ p(x) = p(\rho(x)) = 0, \nabla \Phi(x) = 0 \quad \text{almost every } x \in \mathbb{R}^N. \]

Therefore \( \rho(x) = 0, \nabla \Phi(x) = 0 \) for almost every \( x \in \mathbb{R}^N \).

If \( N = 2 \), then we fix \( w(r) = 1 \) on \([0, \infty)\) in all of the computations leading to (2.18). Then (2.18) reduces to
\[ \int_{\mathbb{R}^2} \left[ \rho(x) |v(x)|^2 + 2p(x) \right] \, dx = 0, \tag{2.19} \]
from which we have \( \rho = 0 \) on \( \mathbb{R}^2 \). From (1.3) \( \nabla \Phi \) is harmonic in \( \mathbb{R}^N \), and this combined with the condition (1.6) implies \( \nabla \Phi = 0 \) on \( \mathbb{R}^N \).

\textbf{(ii) The case of } \mu > 0, N \geq 2 \text{ with either } 2\mu + \lambda = 0 \text{ or } 2\mu + \lambda \neq 0 \text{ (NSP):} \quad \text{In this case we choose the function } w(r) \equiv 1 \text{ on } [0, \infty) \text{ in the proof of (i) above, which is equivalent to the choice of the vector test function, } \phi = \frac{1}{2} \nabla [\sigma_R(|x|)] \text{ instead of } (2.4). \quad \text{We just need to show the vanishing of the viscosity term}
\[ \mu \int_{\mathbb{R}^N} v \cdot \Delta \phi \, dx + (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \text{div} \, v \, dx = o(1) \tag{2.20} \]
as } R \to \infty . \\

If } 2\mu + \lambda = 0, \\

then

\begin{align*}
J &= \mu \int_{\mathbb{R}^N} v \cdot \nabla (|x|^2 \sigma_R) \, dx + (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \left[ \text{div} \, \nabla (|x|^2 \sigma_R) \right] \, dx \\
&= (2\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \Delta (|x|^2 \sigma_R) \, dx = 0,
\end{align*}

and (2.20) holds true.

If } 2\mu + \lambda \neq 0, \\

then we compute and estimate

\begin{align*}
|J| &= |2\mu + \lambda| \left| \int_{\mathbb{R}^N} v \cdot \nabla \Delta (|x|^2 \sigma_R) \, dx \right| \\
&\leq |2\mu + \lambda| \left| \int_{\mathbb{R}^N} (N + 5) \left[ \frac{(v \cdot x)}{R|x|} \sigma'(\frac{|x|}{R}) + \frac{(v \cdot x)}{R^2} \sigma'' \left( \frac{|x|}{R} \right) \right] \, dx \right| \\
&\quad + |2\mu + \lambda| \left| \int_{\mathbb{R}^N} \frac{|x|(v \cdot x)}{R^3} \sigma'' \left( \frac{|x|}{R} \right) \, dx \right| \\
&\leq \frac{C}{R} \int_{R \leq |x| \leq 2R} |v(x)| \, dx \\
&\leq C \left( \int_{R \leq |x| \leq 2R} |v(x)| \frac{N}{N-1} \, dx \right)^{\frac{N-1}{N}} \to 0
\end{align*}

as } R \to \infty , \\

since } v \in L^N_{N-1}(\mathbb{R}^N) \\

by the hypothesis in the viscous case. Thus (2.20) holds true. \qed

**Proof of Theorem 1.2** 

Suppose there exists a global weak solution } (\rho, v, \Phi) \\

satisfying (1.13)-(1.16) with } \mu = \lambda = 0. \\

We choose vector test function

\[ \phi = \nabla \varphi_R(x), \]

where } \varphi_R \\

is defined in (2.1)-(2.3). \\

We also introduce } \eta \in C_0^\infty([0, \infty)) \\

as follows.

\[ \eta(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
0 & \text{if } t > 2,
\end{cases} \quad (2.21) \]

and } 0 \leq \eta(t) \leq 1 \\

for all } t \geq 0. \\

Then, we set

\[ \eta_r(t) = \eta \left( \frac{t}{r} \right). \quad (2.22) \]
We substituting $\phi(x) = \nabla \varphi_R(x)$, $\xi(t) = \eta_r(t)$ into (1.14). Then, substituting $\rho = \Delta \Phi$, and following similar the computations in (2.15), we obtain

\[
0 = \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \frac{x}{|x|} W''(|x|) \sigma_R(|x|) dx \\
+ \frac{1}{R} \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \frac{x}{|x|} W(|x|) \sigma' \left( \frac{|x|}{R} \right) dx \\
+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)v(x,t) \cdot \nabla \varphi_R(x) \sigma_R(|x|) \eta_r(t) dx dt \\
+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + \right. \\
\left. + W'(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \sigma_R(|x|) \eta_r(t) dx dt \\
+ \frac{1}{R} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)W''(|x|) \sigma' \left( \frac{|x|}{R} \right) \frac{(v \cdot x)^2}{|x|^2} \eta_r(t) dx dt \\
+ \frac{1}{R} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \left( \frac{(v \cdot x)^2}{|x|^2} - \frac{(v \cdot x)^2}{|x|^3} \right) \sigma' \left( \frac{|x|}{R} \right) W(|x|) \eta_r(t) dx dt \\
+ \frac{1}{R^2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \frac{(v \cdot x)^2}{|x|^2} \sigma'' \left( \frac{|x|}{R} \right) W(|x|) \eta_r(t) dx dt \\
+ \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \sigma_R(|x|) \eta_r(t) dx dt \\
+ \frac{2}{R} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)W'(|x|) \sigma' \left( \frac{|x|}{R} \right) \eta_r(t) dx dt \\
+ \frac{N - 1}{R} \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \frac{1}{|x|} \sigma' \left( \frac{|x|}{R} \right) W(|x|) \eta_r(t) dx dt \\
+ \frac{1}{R^2} \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \sigma'' \left( \frac{|x|}{R} \right) W(|x|) \eta_r(t) dx dt \\
- k \int_0^\infty \int_{\mathbb{R}^N} \left[ \frac{(|\nabla \Phi|^2}{|x|} \right. \\
\left. - \frac{(x \cdot \nabla \Phi)^2}{|x|^3} \right] W'(|x|) \sigma_R(|x|) \eta_r(t) dx dt \\
- k \int_0^\infty \int_{\mathbb{R}^N} \left[ \frac{(|\nabla \Phi|^2}{|x|} \right. \\
\left. - \frac{(x \cdot \nabla \Phi)^2}{|x|^3} \right] \frac{W(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \eta_r(t) dx dt \\
- k \int_0^\infty \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} W''(|x|) \sigma_R(|x|) \eta_r(t) dx dt \\
- k \int_0^\infty \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \left[ \frac{W(|x|)}{R^2} \sigma'' \left( \frac{|x|}{R} \right) + \frac{2W'(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \right] \eta_r(t) dx dt
\]
\begin{align}
&+ \frac{(N-1)k}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{\nabla \Phi^2}{|x|} W'(|x|) \sigma_R(|x|) \eta_R(t) \, dx \, dt \\
&+ \frac{(N-1)k}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2 W(|x|)}{|x|} \sigma' \left( \frac{|x|}{R} \right) \eta_R(t) \, dx \, dt \\
&+ \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} \nabla \Phi^2 W''(|x|) \sigma_R(|x|) \eta_R(t) \, dx \, dt \\
&+ \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} \nabla \Phi^2 \left[ \frac{W(|x|)}{R^2} \sigma'' \left( \frac{|x|}{R} \right) + \frac{2W'(|x|)}{R} \sigma' \left( \frac{|x|}{R} \right) \right] \eta_R(t) \, dx \, dt \\
&:= I_1 + \cdots + I_{19}. \tag{2.23}
\end{align}

On the other hand, substituting \( \phi(x) = \nabla \varphi_R(x), \xi(t) = \eta'_R(t) \) into (1.13), we find that

\[ I_3 = \int_0^\infty \int_{\mathbb{R}^N} \rho v(x,t) \cdot \nabla \varphi_R(x) \eta'_R(t) \, dx \, dt \]
\[ = -\int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \varphi_R(x) \eta''_R(t) \, dx \, dt \]
\[ = -\int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \sigma_R(|x|) W(|x|) \eta''_R(t) \, dx \, dt \]
\[ \rightarrow -\int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) W(|x|) \eta''_R(t) \, dx \, dt \tag{2.24} \]

as \( R \to \infty \) by the dominated convergence theorem. In terms of the function \( W(\cdot) \) the condition (1.19) can be written as

\[ \int_0^\infty \int_{\mathbb{R}^N} (\rho(x,t) |v(x,t)|^2 + |p(x,t)| + k |\nabla \Phi(x,t)|^2) W''(|x|) + \]
\[ + \frac{1}{|x|} W'(|x|) + \frac{1}{|x|^2} W(|x|) \right] \, dx \, dt < \infty \tag{2.25} \]

for all \( T > 0 \). Since

\[ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \left[ W''(|x|) \left( \frac{|v(x,t)|^2}{|x|^2} \right) + \right. \]
\[ + W'(|x|) \left( \frac{|v(x,t)|^2}{|x|} - \frac{(v(x,t) \cdot x)^2}{|x|^3} \right) \right] \eta_R(t) \, dx \, dt \]
\[ \leq 2 \int_0^{2r} \int_{\mathbb{R}^N} \rho(x,t) |v(x,t)|^2 \left[ W''(|x|) + \frac{W'(|x|)}{|x|} \right] \, dx \, dt < \infty, \]
we can use the dominated convergence theorem to show that

\[
I_4 \to \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \left[ W''(|x|) \frac{(v(x,t) \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v(x,t)|^2}{|x|} - \frac{(v(x,t) \cdot x)^2}{|x|^3} \right) \right] \eta_r(t) \, dx \, dt
\]

as \( R \to \infty \). Similarly,

\[
I_8 \to \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \eta_r(t) \, dx \, dt,
\]  

\[
I_{12} \to -k \int_0^\infty \int_{\mathbb{R}^N} \frac{\sigma'(|x|)}{|x|^3} \left( \frac{x \cdot \nabla \Phi}{|x|^2} \right) W'(|x|) \eta_r(t) \, dx \, dt,
\]

\[
I_{14} \to -k \int_0^\infty \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} W''(|x|) \eta_r(t) \, dx \, dt,
\]

\[
I_{16} \to \frac{(N - 1)k}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{\sigma'(|x|)}{|x|^3} \nabla \Phi |x|^2 \eta_r(t) \, dx \, dt,
\]

and

\[
I_{18} \to \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{\sigma'(|x|)}{|x|^3} \nabla \Phi W''(|x|) \eta_r(t) \, dx \, dt
\]

as \( R \to \infty \). For \( I_5 \) we estimate

\[
|I_5| \leq \int_0^{2\tau} \int_{R<|x|<2R} \rho(x,t)|v(x,t)|^2 \left| \frac{\sigma'}{R} \right| \frac{W'(|x|)}{|x|} \, dx \, dt
\]

\[
\leq 2 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x)|v(x,t)|^2 \frac{W'(|x|)}{|x|} \, dx \, dt \to 0
\]

as \( R \to \infty \) by the dominated convergence theorem. Similarly

\[
|I_6| \leq 2 \int_0^{2\tau} \int_{R<|x|<2R} \frac{|x|}{R} \rho(x)|v(x,t)|^2 \left| \frac{\sigma'}{R} \right| \left( \frac{W(|x|)}{|x|^2} \right) \, dx
\]

\[
\leq 4 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x)|v(x,t)|^2 \frac{W'(|x|)}{|x|} \, dx \, dt \to 0,
\]

(2.33)
\[ |I_7| \leq \int_0^{2\tau} \int_{R<|x|<2R} \frac{|x|^2}{R^2} \rho(x,t)|v(x,t)|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \, dx \, dt \]
\[
\leq 4 \sup_{1<s<2} |\sigma''(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x,t)|v(x,t)|^2 \frac{W(|x|)}{|x|^2} \, dx \, dt \to 0, \quad (2.34)
\]

and
\[
|I_2| \leq 2 \sup_{1<s<2} |\sigma'(x)| \int_{R<|x|<2R} \rho_0(x)|v_0(x)| \frac{|W(|x|)|}{|x|} \, dx \to 0 \quad (2.35)
\]
as \( R \to \infty \). The estimates for \( I_9, I_{10} \) and \( I_{11} \) are similar to the above, and we find
\[
|I_9| \leq 2 \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \frac{|x|}{R} \frac{|W'(|x|)|}{|x|} \left| \sigma' \left( \frac{|x|}{R} \right) \right| \, dx \, dt
\]
\[
\leq 4 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \frac{W'(|x|)}{|x|} \, dx \, dt \to 0, \quad (2.36)
\]

\[
|I_{10}| \leq (N-1) \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \frac{|x|}{R} \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \, dx \, dt
\]
\[
\leq 2 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \frac{W(|x|)}{|x|^2} \, dx \, dt \to 0, \quad (2.37)
\]

and
\[
|I_{11}| \leq \int_0^{2\tau} \int_{R^N} |p(x,t)| \frac{|x|^2}{R^2} \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \, dx \, dt
\]
\[
\leq 4 \sup_{1<s<2} |\sigma''(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \frac{W(|x|)}{|x|^2} \, dx \, dt \to 0, \quad (2.38)
\]
as \( R \to \infty \) respectively. By similar estimates we can show easily that
\[
|I_{13}| + |I_{15}| + |I_{17}| + |I_{19}| \to 0. \quad (2.39)
\]
as $R \to \infty$.

Thus, passing $R \to \infty$ in (2.23), we obtain

\[
\int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \frac{x}{|x|} W'(|x|) \, dx
\]

\[
+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \eta_r(t) \, dx \, dt
\]

\[
+ \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \left[ W''(|x|) + (N-1) \frac{W'(|x|)}{|x|} \right] \eta_r(t) \, dx \, dt
\]

\[
+ \frac{(N-3)k}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{|\nabla \Phi|^2}{|x|} W'(|x|) \eta_r(t) \, dx \, dt
\]

\[
+ k \int_0^\infty \int_{\mathbb{R}^N} \left[ W''(|x|) - W''(0) \right] \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \eta_r(t) \, dx \, dt
\]

\[
+ \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 W''(|x|) \eta_r(t) \, dx \, dt
\]

\[
= \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) W(|x|) \eta''_r(t) \, dx \, dt
\]

(2.40)

The hypothesis (1.18) implies that

\[
\left| \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) W(|x|) \eta''_r(t) \, dx \, dt \right| \leq \frac{1}{\tau^2} \int_0^{2\tau} \int_{\mathbb{R}^N} \rho(x,t) W(|x|) \eta'' \left( \frac{t}{\tau} \right) \, dx \, dt
\]

\[
\leq \frac{1 + 4\tau^2}{\tau^2} \sup_{1 < t < 2} |\eta''(t)| \int_\tau^{2\tau} \int_{\mathbb{R}^N} \frac{\rho(x,t)}{1 + t^2} \left[ \int_0^{|x|} \int_0^r w(s) \, ds \, dr \right] \, dx \, dt
\]

\[
\leq CK_1
\]

(2.41)

as $\tau \to \infty$. Next, we observe that, by our definition on $W(|x|)$ and the hypothesis on $w(r)$, we have

\[
W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \geq 0,
\]

and

\[
W''(|x|) + (N-1) \frac{W'(|x|)}{|x|} > 0.
\]

Moreover, by the non-increasing assumption on the function $w(r)$ on $[0, \infty)$, we have

\[
\frac{W'(|x|)}{|x|} - W''(|x|) = \frac{1}{|x|} \int_0^{|x|} w(r) \, dr - w(|x|) \geq 0
\]

(2.42)
for almost every $x \in \mathbb{R}^N$. Thus, we can apply the monotone convergence theorem to obtain

$$
\int_0^\infty \int_{\mathbb{R}^N} \rho(x, t) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \eta_r(t) \, dx \, dt \\
\rightarrow \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \, dt,
$$

(2.43)

$$
\int_0^\infty \int_{\mathbb{R}^N} p(x, t) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \eta_r(t) \, dx \, dt \\
\rightarrow \int_0^\infty \int_{\mathbb{R}^N} p(x, t) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \, dt,
$$

(2.44)

$$
\int_0^\infty \int_{\mathbb{R}^N} \frac{|
abla \Phi|^2}{|x|} W'(|x|) \eta_r(t) \, dx \, dt \\
\rightarrow \int_0^\infty \int_{\mathbb{R}^N} \frac{|
abla \Phi|^2}{|x|} W'(|x|) \, dx \, dt
$$

(2.45)

$$
\int_0^\infty \int_{\mathbb{R}^N} \left[ \frac{W'(|x|)}{|x|} - W''(|x|) \right] \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \eta_r(t) \, dx \, dt \\
\rightarrow \int_0^\infty \int_{\mathbb{R}^N} \left[ \frac{W'(|x|)}{|x|} - W''(|x|) \right] \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \, dx \, dt,
$$

(2.46)

$$
\int_0^\infty \int_{\mathbb{R}^N} |
abla \Phi|^2 W''(|x|) \eta_r(t) \, dx \, dt \\
\rightarrow \int_0^\infty \int_{\mathbb{R}^N} |
abla \Phi|^2 W''(|x|) \, dx \, dt
$$

(2.47)

as $\tau \to \infty$. Thus, passing $\tau \to \infty$ in (2.40), we find that

$$
\int_{\mathbb{R}^N} \rho_0(x) v_0(x) \cdot \frac{x}{|x|} W'(|x|) \, dx \\
+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t) \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \, dt \\
+ \int_0^\infty \int_{\mathbb{R}^N} p(x, t) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \, dx \, dt
$$
\[\begin{align*}
+ \frac{(N-3)}{2} & \int_0^\infty \int_{\mathbb{R}^N} \frac{\vert \nabla \Phi \vert^2}{\vert x \vert} W'(|x|) \, dx \, dt \\
+ k & \int_0^\infty \int_{\mathbb{R}^N} \left[ \frac{W'(|x|)}{\vert x \vert} - W''(|x|) \right] \frac{(x \cdot \nabla \Phi)^2}{\vert x \vert^2} \, dx \, dt \\
+ \frac{k}{2} & \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 W''(|x|) \, dx \, dt \leq CK_1, \quad (2.48)
\end{align*}\]

which proves (1.24) for \(N \geq 1\). In the case \(N = 2\), we choose \(w(r) \equiv 1\) on \([0, \infty)\). Then, the inequality (2.48) reduces to

\[\begin{align*}
\int_{\mathbb{R}^2} \rho_0(x)v_0(x) \cdot x \, dx + \int_0^\infty \int_{\mathbb{R}^2} \left[ \rho(x, t) |v(x, t)|^2 + 2p(x, t) \right] \, dx \, dt \leq CK_1. \quad (2.49)
\end{align*}\]

\(\square\)

In order to establish Theorem 1.2 we use the following lemma, which is proved in [3].

**Lemma 2.1** Suppose \((\rho, v)\) is a global weak solution of \((NS)\) with the setting given by Theorem 1.2. We suppose that the energy inequality (1.32) holds. Then,

\[\begin{align*}
\int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)(1 + |x|^2)^{\frac{N+2}{4\gamma}} \, dx \, dt \leq CE(0). \quad (2.50)
\end{align*}\]

Since \(\frac{N+2}{4\gamma} \geq 1\) in our setting of Theorem 1.2, one immediate consequence of (2.50) is the following fact

\[\begin{align*}
\lim_{\tau \to \infty} \int_{\tau}^{2\tau} \int_{\mathbb{R}^N} \frac{\rho(x, t)}{1 + t^2} |x|^2 \, dx \, dt = 0. \quad (2.51)
\end{align*}\]

Indeed, using (2.50), we deduce

\[\lim_{\tau \to \infty} \int_{\tau}^{2\tau} \int_{\mathbb{R}^N} \frac{\rho(x, t)}{1 + t^2} |x|^2 \, dx \, dt \leq \lim_{\tau \to \infty} \int_{\tau}^{2\tau} \int_{\mathbb{R}^N} \frac{\rho(x, t)(1 + |x|^2)^{\frac{N+2}{4\gamma}}}{t^2} \, dx \, dt = 0,\]

where the last step follows from the dominated convergence theorem.

**Proof of Theorem 1.3** Suppose there exists a global weak solution \((\rho, v, \Phi)\) satisfying (1.13)-(1.16) (with \(\mu \neq 0\)). Here, we choose the vector test function as

\[\phi = \nabla \varphi_R, \quad \varphi_R(x) = \frac{|x|^2}{2} \sigma \left( \frac{|x|}{R} \right) = \frac{|x|^2}{2} \sigma_R(|x|), \quad (2.52)\]
where $\sigma$ is the cut-off function defined in (2.1). Similarly to the proof of Theorem 1.2 we also use the same temporal cut-off function $\eta_r(t)$ defined in (2.21)-(2.22). Substituting $\phi(x) = \nabla \varphi_R(x), \xi(t) = \eta_r(t)$ into (1.14), we have

$$0 = \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x\sigma_R(|x|)dx + \frac{1}{2R} \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x|x|\sigma'\left(\frac{|x|}{R}\right)dx$$

$$+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)v(x,t) \cdot \nabla \varphi_R(x)\eta'_r(t)dxdt$$

$$+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2\sigma_R(|x|)\eta_r(t)dxdt$$

$$+ \frac{1}{2R} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)\sigma'(\frac{|x|}{R}) (v(x,t) \cdot x)^2 \eta_r(t)dxdt$$

$$+ \frac{1}{2R} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2|x|\sigma'\left(\frac{|x|}{R}\right) \eta_r(t)dxdt$$

$$+ \frac{1}{2R^2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)(v(x,t) \cdot x)^2\sigma''\left(\frac{|x|}{R}\right) \eta_r(t)dxdt$$

$$+N \int_0^\infty \int_{\mathbb{R}^N} p(x,t)\sigma_R(|x|)\eta_r(t)dxdt$$

$$+ \frac{2}{R} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)|x|\sigma'\left(\frac{|x|}{R}\right) \eta_r(t)dxdt$$

$$+ \frac{N-1}{2R} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)|x|\sigma'\left(\frac{|x|}{R}\right) \eta_r(t)dxdt$$

$$+ \frac{1}{2R^2} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)|x|^2\sigma''\left(\frac{|x|}{R}\right) \eta_r(t)dxdt$$

$$-k \int_0^\infty \int_{\mathbb{R}^N} \left[\nabla \Phi_t^2 - \frac{(x \cdot \nabla \Phi_t)^2}{|x|^2}\right] \sigma_R(|x|)\eta_r(t)dxdt$$

$$-k \int_0^\infty \int_{\mathbb{R}^N} \left[\nabla \Phi_t^2 - \frac{(x \cdot \nabla \Phi_t)^2}{|x|^2}\right] \frac{|x|}{2R} \sigma'\left(\frac{|x|}{R}\right) \eta_r(t)dxdt$$

$$-k \int_0^\infty \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi_t)^2}{|x|^2} \sigma_R(|x|)\eta_r(t)dxdt$$

$$-k \int_0^\infty \int_{\mathbb{R}^N} \left(1 - \frac{2R^2}{|x|} \sigma''\left(\frac{|x|}{R}\right) + \frac{2}{|x|R} \sigma'\left(\frac{|x|}{R}\right)\right) \eta_r(t)dxdt$$

$$+\frac{(N-1)k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi_t|^2 \sigma_R(|x|)\eta_r(t)dxdt$$

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\[ + \frac{(N-1)k}{4} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \frac{|x|}{R} \sigma' \left( \frac{|x|}{R} \right) \eta_r(t) \, dx \, dt \]
\[ + \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \sigma_R(|x|) \eta_r(t) \, dx \, dt \]
\[ + \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \left[ \frac{|x|^2}{2R^2} \sigma'' \left( \frac{|x|}{R} \right) + \frac{2|x|}{R} \sigma' \left( \frac{|x|}{R} \right) \right] \eta_r(t) \, dx \, dt \]
\[ + (2\mu + \lambda) \int_0^\infty \int_{\mathbb{R}^N} v \cdot \nabla \Delta (|x|^2 \sigma) \left( \frac{|x|}{R} \right) \eta_r(t) \, dx \, dt, \]
\[ := I_1 + \cdots + I_{20}. \quad (2.53) \]

On the other hand, substituting \( \phi(x) = \nabla \varphi_R(x) \), \( \xi(t) = \eta'_r(t) \) into (1.13), then similarly as before, we find that (note that \( \xi(0) = \eta'_r(0) = 0 \))
\[ I_3 = \int_0^\infty \int_{\mathbb{R}^N} \rho v(x,t) \cdot \nabla \varphi_R(x) \eta'_r(t) \, dx \, dt \]
\[ = - \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) \varphi_R(x) \eta''_r(t) \, dx \, dt \]
\[ = - \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) |x|^2 \sigma_R(|x|) \eta''_r(t) \, dx \, dt \]
\[ \to - \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) |x|^2 \eta''_r(t) \, dx \, dt \quad (2.54) \]
as \( R \to \infty \) by the dominated convergence theorem. We also have
\[ I_4 \to \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t) |v(x,t)|^2 \eta_r(t) \, dx \, dt \quad (2.55) \]
as \( R \to \infty \). Similarly,
\[ I_1 \to \int_{\mathbb{R}^N} \rho_0(x) v_0(x) \cdot x \, dx, \quad (2.56) \]
\[ I_8 \to N \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \eta_r(t) \, dx \, dt, \quad (2.57) \]
\[ I_{12} \to -k \int_0^\infty \int_{\mathbb{R}^N} \left[ |\nabla \Phi|^2 - \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \right] \eta_r(t) \, dx \, dt, \quad (2.58) \]
\[ I_{14} \to -k \int_0^\infty \int_{\mathbb{R}^N} \frac{(x \cdot \nabla \Phi)^2}{|x|^2} \eta_r(t) \, dx \, dt, \quad (2.59) \]
\( I_{16} \to \frac{(N - 1)k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \eta_r(t) \, dx \, dt, \) \hfill (2.60)

and

\( I_{18} \to \frac{k}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \eta_r(t) \, dx \, dt \) \hfill (2.61)

as \( R \to \infty \). For \( I_5, I_6 \), we estimate

\[
|I_5| + |I_6| \leq \int_0^{2\tau} \int_{R<|x|<2R} \rho(x,t)|v(x,t)|^2 \left| \frac{|x|}{R} \right| \left| \rho \left( \frac{|x|}{R} \right) \right| \, dx \, dt
\]

\[
\leq 2 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x)|v(x,t)|^2 \, dx \, dt \to 0
\]

as \( R \to \infty \) by the dominated convergence theorem. Similarly

\[
|I_2| \leq \int_{R<|x|<2R} \rho_0(x)|x| \, dx \to 0,
\] \hfill (2.63)

and

\[
|I_7| \leq \frac{1}{2} \int_0^{2\tau} \int_{R<|x|<2R} \left| \frac{|x|^2}{R^2} \rho(x,t)|v(x,t)|^2 \right| \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \, dx \, dt
\]

\[
\leq 2 \sup_{1<s<2} |\sigma''(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x)|v(x,t)|^2 \, dx \to 0
\] \hfill (2.64)

as \( R \to \infty \). The estimates for \( I_9, I_{10} \) and \( I_{11} \) are similar to the above, and we find

\[
|I_9| \leq 2 \int_0^{2\tau} \int_{R<|x|<2R} \left| p(x,t) \right| \left| \frac{|x|}{R} \right| \left| \sigma' \left( \frac{|x|}{R} \right) \right| \, dx \, dt
\]

\[
\leq 4 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \, dx \, dt \to 0,
\] \hfill (2.65)

\[
|I_{10}| \leq \frac{N - 1}{2R} \int_0^{2\tau} \int_{R<|x|<2R} \left| p(x,t) \right| \left| |x| \right| \left| \sigma' \left( \frac{|x|}{R} \right) \right| \, dx \, dt
\]

\[
\leq (N - 1) \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \, dx \, dt \to 0,
\] \hfill (2.66)
and
\[ |I_{11}| \leq \frac{1}{2R^2} \int_0^{2\tau} \int_{\mathbb{R}^N} |p(x, t)||x|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \, dx \, dt \]
\[ \leq 2 \sup_{1 < s < 2} |\sigma''(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x, t)| \, dx \, dt \to 0 \]
(2.67)
as \( R \to \infty \) respectively. Similarly,
\[ |I_{13}| + |I_{15}| + |I_{17}| + |I_{19}| \to 0 \]
(2.68)
as \( R \to \infty \). Now we show the vanishing of the viscosity term as \( R \to \infty \). Similarly to stationary case we estimate
\[ |I_{20}| = (2\mu + \lambda) \left| \int_0^\infty \int_{\mathbb{R}^N} v \cdot \nabla \Delta (|x|^2 \sigma \left( \frac{|x|}{R} \right)) \eta_r(t) \, dx \, dt \right| \]
\[ \leq (2\mu + \lambda) \left| \int_0^\infty \int_{\mathbb{R}^N} (N + 5) \left( \frac{v \cdot x}{|x|} \sigma' \left( \frac{|x|}{R} \right) + \frac{(v \cdot x)^2}{|x|^2} \sigma'' \left( \frac{|x|}{R} \right) \right) \eta_r(t) \, dx \, dt \right| \]
\[ + (2\mu + \lambda) \left| \int_0^\infty \int_{\mathbb{R}^N} \frac{|x|(v \cdot x)}{|x|^2} \sigma'' \left( \frac{|x|}{R} \right) \eta_r(t) \, dx \, dt \right| \]
\[ \leq \frac{C}{R} \int_0^{2\tau} \int_{R<|x|<2R} |v(x, t)| \, dx \, dt \]
\[ \leq C \left( \int_0^{2\tau} \int_{R<|x|<2R} |v(x)|^\frac{N}{N-1} \, dx \right)^{\frac{N-1}{N}} dt \to 0 \]
(2.69)
as \( R \to \infty \). Thus passing \( R \to \infty \) in (2.53), we obtain
\[ \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \eta_r''(t) \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|v(x, t)|^2 \eta_r(t) \, dx \, dt \]
\[ + N \int_0^\infty \int_{\mathbb{R}^N} p(x, t)\eta_r(t) \, dx \, dt + \frac{N-2}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \eta_r(t) \, dx \, dt \]
\[ + \int_{\mathbb{R}^N} \rho_0(x) v_0(x) \cdot x \, dx \]
(2.70)
for any \( \tau > 0 \). Note that
\[ \left| \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \eta_r''(t) \, dx \, dt \right| \leq \frac{1}{\tau^2} \int_\tau^{2\tau} \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \left| \eta'' \left( \frac{t}{\tau} \right) \right| \, dx \, dt \]
\[ \leq \frac{1}{\tau^2} \int_\tau^{2\tau} \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \left| \eta''(s) \right| \, dx \, dt \to 0, \]
(2.71)
as $\tau \to \infty$ by (2.51). On the other hand, by the monotone convergence theorem we deduce

\[
\int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2 \eta_\tau(t) \, dx\, dt \to \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2 \, dx\, dt, \\
\int_0^\infty \int_{\mathbb{R}^N} p(x,t) \eta_\tau(t) \, dx\, dt \to \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \, dx\, dt \\
\int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \eta_\tau(t) \, dx\, dt \to \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \, dx\, dt
\]

(2.72)
as $\tau \to \infty$. Thus, passing $\tau \to \infty$ in (2.70) we have

\[
0 = \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2 \, dx\, dt + N \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \, dx\, dt \\
+ \frac{N-2}{2} \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 \, dx\, dt + \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x \, dx.
\]

(2.73)

\[\square\]

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