QUANTUM DEFORMATIONS OF MULTI-INSTANTON SOLUTIONS IN THE TWISTOR SPACE

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We consider the quantum-group self-duality equation in the framework of the gauge theory on a deformed twistor space. Quantum deformations of the Atiyah-Drinfel’d-Hitchin-Manin and t’Hooft multi-instanton solutions are constructed.

The quantum-group gauge theory was considered in the framework of the algebra of local differential complexes \[\Pi\] or as a noncommutative generalization of the fibre bundles over the classical or quantum basic spaces \[\mathfrak{g} \mathfrak{l}(4)\].

We prefer to use local constructions of the noncommutative connection forms or gauge fields as a deformed analogue of the local gauge fields. In particular, the quantum-group self-duality equation (QGSDE) has been considered in the deformed 4-dimensional Euclidean space, and an explicit formula for the corresponding one-instanton solution has been constructed \[\mathfrak{g} \mathfrak{l}(4)\]. This solution can be treated as \(q\)-deformation of the BPST-instanton \[\mathfrak{g} \mathfrak{l}(4)\]. We shall discuss here quantum deformations of the general multi-instanton solutions \[\mathfrak{g} \mathfrak{l}(4)\].

The conformal covariant description of the classical ADHM solution was considered in Ref\[8\]. We shall study the quantum deformation of this version of the twistor formalism. It is convenient to discuss firstly the deformations of the complex conformal group \(\mathfrak{g} \mathfrak{l}(4,\mathbb{C})\), complex twistors and the complex linear gauge groups.

Let \(R_{\alpha\beta}^{ab}, (a, b, c, d \ldots = 1 \ldots 4)\) be the solution of the 4D Yang-Baxter equation satisfying also the Hecke relation

\[
R R' R = R' R R' \quad (1)
\]
\[
R^2 = I + (q - q^{-1}) R \quad (2)
\]

where \(q\) is a complex parameter. Note that the standard notation for these \(R\)-matrices is \(R = \hat{R}_{12}, R' = \hat{R}_{23}\) \[\mathfrak{g} \mathfrak{l}(4)\].

Consider also the \(SL_q(2,\mathbb{C})\) \(R\)-matrix

\[
R_{\mu\nu}^{\alpha\beta} = q\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} + \varepsilon^{\alpha\beta}(q)\varepsilon_{\mu\nu}(q) \quad (3)
\]

where \(\varepsilon(q)\) is the deformed antisymmetric symbol.

Noncommutative twistors were considered in Ref\[10\]. We shall use the \(R\)-matrix approach to define the differential calculus on the deformed twistor space.

Let \(z_\alpha^a\) and \(d z_\alpha^a\) be the components of the \(q\)-twistor and their differentials

\[
R_{\mu\nu}^{\alpha\beta} z_\alpha^a z_\nu^b = z_\alpha^c z_\nu^d R_{\mu\nu}^{cd} \quad (4)
\]
\[
z_\alpha^a d z_\beta^b = R_{\mu\nu}^{\alpha\beta} d z_\mu^c z_\nu^d R_{\mu\nu}^{cd} \quad (5)
\]
\[
d z_\alpha^a d z_\beta^b = -R_{\mu\nu}^{\alpha\beta} d z_\mu^c d z_\nu^d R_{\mu\nu}^{cd} \quad (6)
\]
One can define also the algebra of partial derivatives \( \partial_\alpha \)

\[
R_{\alpha \beta}^c \partial_\alpha \partial_\beta^d = \partial_\mu \partial_\nu^{\mu} R_{\beta \alpha}^{\nu} \\
\partial_\alpha^\beta z_b^\mu = \delta_b^\alpha \delta_\alpha^\beta + R_{\alpha \beta}^{\mu} R_{\beta \alpha}^{\nu} z_d^\nu \partial_\mu
\]  

(7)

Consider the 4D deformed \( \varepsilon_q \)-symbol

\[
R_{fe}^{ba} \varepsilon_{fcd} = -\frac{1}{q} \varepsilon_{abcd}
\]

(9)

The \( q \)-twistors satisfy the following identity:

\[
\varepsilon_{abcd} z_b^\beta z_c^\mu z_d^\nu = 0
\]

(10)

The \( SL_q(2) \)-invariant bilinear function of twistors has the zero length in the projective 6D vector space

\[
y_{ab} = \varepsilon_{\alpha \beta}(q) z_\alpha^a z_\beta^b = [P(-)]_{ba} y_{cd}
\]

(11)

\[
(y, y) = \varepsilon_{abcd} y_{ab} y_{cd} = 0
\]

(12)

Consider a duality transformation \( * \) of the basic differential 2-forms

\[
* dz \ dz' = dz \ dz' P^{(+)} - dz \ dz' P^{(-)}
\]

(13)

where \( P^{(\pm)} \) are the projection operators of \( GL_q(4) \). Note that the self-dual part \( dz \ dz' P^{(+)} \) is proportional to

\[
\varepsilon_{\alpha \beta}(q) d z_\alpha^a d z_\beta^b
\]

(14)

Let \( T^i_k \) be matrix elements of the \( GL_q(N) \) quantum group

\[
R_G T^i_k = T^i_k R_G
\]

(15)

where \( R_G \) is the \( R \)-matrix of \( GL_q(N) \).

Quantum deformation of the \( GL_q(N) \) gauge connection can be treated in terms of the noncommutative algebra for the components \( A^i_k \) of the connection 1-form

\[
(A R_G A + R_G A R_G A R_G)^{ijkl} = 0
\]

(16)

where \( i, k, l, m, n, p = 1 \ldots N \). These relations generalize the anticommutativity conditions for components of the classical connection form.

The restriction on the quantum trace of the connection \( \alpha = \text{Tr}_q A = 0 \) is inconsistent with (16), but we can use the gauge-covariant relations \( \alpha^2 = 0 \), \( \text{Tr}_q A^2 = 0 \) and \( d \alpha = 0 \). The curvature 2-form \( F = d A - A^2 \) is \( q \)-traceless for this model.

Consider the explicit realization of this gauge algebra in terms of \( z, dz \) and the set \( B \) of additional noncommutative parameters

\[
A^i_k(z, dz, B) = d z_\alpha^a A_{\alpha k}(z, B)
\]

(17)

The analogous realizations were considered on the \( GL_q(2) \) and \( E_q(4) \) quantum spaces. We shall treat the representation (17) as a local gauge field on the \( q \)-twistor space.
Let us consider the quantum deformation of the GL(2) t’Hooft solution \[8\]

\[A^\alpha_\beta = q^{-3}dz^\alpha_a(\partial^\mu_\mu\Phi)\Phi^{-1}\varepsilon^\sigma\mu(q)\varepsilon_{\sigma\beta}(q)\]

\[\Phi = \sum_a(X^a)^{-1}, \quad X^i = (y, b^i) = \varepsilon_{abcd}y_{ab}b^i_{cd}\]

where \(b^i_{cd}\) are the noncommutative isotropic 6D vectors

\[db^i_{cd} = 0, \quad (b^i, b^j) = 0\]

\[[y_{ab}, X^i] = [b^i_{cd}, X^i] = 0\]

The central elements \(X^i\) of the \((B, z)\)-algebra do not commute with \(dz\)

\[X^i dz^\alpha_a = q^{2}dz^\alpha_a X^i\]

Stress that \(A^\alpha_\beta\) satisfies Eq(16) and its quantum trace is a \(U(1)\)-gauge field with the zero field-strength

\[\text{Tr}_q A = -q^{-3}d\Phi\Phi^{-1}, \quad \text{Tr}_q dA = 0\]

The QGSDE for \(A^\alpha_\beta\) is equivalent to the finite-difference Laplace equation for the function \(\Phi\) on the \(q\)-twistor space

\[\Delta^{ba}(X^i) = \sum_i \Delta^{ba}_{i}X^i = 0\]

\[\Delta^{ba}(X^i)^{-1} = \frac{-q^{2}}{2}y_{ab}(X^i)^{-2}(b^i)^{ab}\]

The ADHM-twistor functions of Ref[7] can be connected with some \(GL(N + 2k)\) matrix function. Let us introduce the notation for indices of different types: \(I, K, L, M = 1 \ldots N + 2k\) and \(A, B = 1 \ldots k\). The Ansatz for the general self-dual \(GL_q(N, C)\) field contains the deformed twistors \(u(z)\) and \(\tilde{u}(z)\)

\[A^i_k = du^i_k\tilde{u}^l_k, \quad u^i_k\tilde{u}^l_k = \delta^i_k\]

The commutation relations for the \(u\) and \(\tilde{u}\) twistors are

\[(R_G)^{ik}_{lm}u^i_lu^m_k = u^i_lu^k_mR^{LM}_{IK}\]

\[R^{KL}_{ML}\tilde{u}^l_k\tilde{u}^M_k = \tilde{u}^l_i\tilde{u}^K_m(R_G)^{mi}_{Kl}\]

\[\tilde{u}^l_i(R_G)^{li}_{mk}u^m_K = u^i_lR^{IL}_{KM}\tilde{u}^M_k\]

where the \(R\)-matrices for \(GL_q(N, C)\) and \(GL_q(N + 2k, C)\) are used.

Consider also the linear twistor functions \(v\) and \(\tilde{v}\)

\[v^A_\alpha = z^\alpha_a b^A_a\]

\[\tilde{v}^{IA}_\alpha = \tilde{b}^I_{Aa}z^\alpha_a\]

Introduce the following condition for these functions:

\[v^A_\alpha v^{IB}_\beta = g^{AB}(z)\varepsilon^{\alpha\beta}(q)\]
where \( g(z) \) is the nondegenerate \((k \times k)\) matrix with the central elements

\[
g_{AB}(z) = \frac{q}{1 + q^2} b^A_I b^B_b y_{ab}
\]  

(34)

The condition \((33)\) is equivalent to the restriction on the elements of the \(B\)-algebra

\[
[P^{(+)}]_{ab}^{cd} b^A_I b^B_b = 0
\]  

(35)

Write the basic commutation relations of the \(B\)-algebra

\[
P^{ab}_{cd} b^C_I b^D_K = b^D_L b^C_M R^{ML}_{KI}
\]  

(36)

\[
R^{IK}_{LM} b^{K_Aa} b^{MBb} = R^{ab}_{cd} b^{I_Bc} b^{K_Ad}
\]  

(37)

\[
P^{ab}_{cd} b^C_I b^{K_Bd} = R^{KL}_{IM} b^{MBb} b^A_B
\]  

(38)

Remark that a formal permutation of the indices \(A\) and \(B\) is commutative. It is not difficult to define the relations between \(b, \bar{b}\) and \(z, dz\).

Consider the new functions

\[
\tilde{v}^I_{A\alpha} = g_{AB}(z) \varepsilon_{\alpha\beta}(q) \tilde{v}^{IB\beta}
\]  

(39)

where we use the inverse matrix with respect to the matrix \((34)\).

Now one can construct the full quantum \(GL_q(N + 2k, C)\) matrices

\[
U = \left( \begin{array}{c} u^I_i \\ v^I_{A\alpha} \end{array} \right), \quad U^{-1} = \left( \begin{array}{c} \tilde{u}^I_i \\ \tilde{v}^I_{A\alpha} \end{array} \right)
\]  

(40)

The standard \(GL_q(N + 2k, C)\) commutation relations for these matrices contain Eqs(28-30) and the relations for the \(v\) and \(\tilde{v}\) functions.

Write explicitly the orthogonality and completeness conditions for the deformed ADHM-twistors

\[
u^I_i \tilde{v}^{IA\alpha} = 0
\]  

(41)

\[
v^I_{A\alpha} \tilde{u}^I_i = 0
\]  

(42)

\[
\delta^I_K = \tilde{u}^I_i u^I_K + \tilde{v}^{IA\alpha} g_{AB}(z) \varepsilon_{\alpha\beta}(q) v^{B\beta}_K
\]  

(43)

The gauge-field algebra \((16)\) for the deformed ADHM-Ansatz \((27)\) can be generated by the differential algebra on the \(GL_q(N + 2k, C)\) matrices \(U, \ U^{-1}, \ dU\) which contains the following relations:

\[
\tilde{u}^I_i \ (R_G)^{ik}_{lm} du^l_K = du^l_K \ (R^{-1})^{IL}_{KM} \tilde{u}^M_m
\]  

\[
du^l_K \ du^k_M \ (R^{-1})^{LM}_{IK} = -(R^{-1})^{ik}_{lm} du^l_i du^m_K
\]  

(44)

(45)

These relations are consistent with the commutation relation \((28-30)\).

The self-duality of the connection \((27)\) follows from Eqs(31,32,41-43).

\[
dA^i_k = (R_G)^{ik}_{lm} \ (\tilde{u}^l_i u^M_M - \delta^l_M) d\tilde{u}^M_k =
\]

\[
= -u^I_i b^{IA\alpha} g_{AB}(z) \varepsilon_{\alpha\beta}(q) d\varepsilon_{\alpha}^\alpha d\varepsilon^\beta_{\beta} b^{Bb}_M \tilde{u}^M_k
\]  

(46)
This curvature contains the self-dual 2-form (14) only.

It should be stressed that all $R$-matrices of our deformation scheme satisfy the Hecke relation with the common parameter $q$. The other possible parameters of different $R$-matrices are independent. The case $q = 1$ corresponds to the unitary deformations ($R^2 = I$) of the twistor space and the gauge groups. It is evident that the trivial deformation of the $z$-twistors is consistent with the nontrivial unitary deformation of the gauge sector and vice versa.

The Euclidean conformal $q$-twistors are a representation of the $U^*(4) \times SU_q(2)$ group. The antiinvolution for these twistors has the following form:

$$(z_a^\alpha)^* = \varepsilon_{\alpha\beta}(q) \ z_b^\beta \ C_b^a$$

(47)

where $C$ is the charge conjugation matrix for $U^*(4)$. We can use the gauge group $U_q(N)$ in the framework of our approach.

An analogous construction can be considered for the real twistors and the gauge group $GL_q(N, R)$.

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