Kibble–Zurek mechanism in a trapped ferromagnetic Bose–Einstein condensate

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Abstract
Spontaneous spin vortex formation in a magnetic phase transition of a trapped spin-1 Bose–Einstein condensate is investigated based on mean-field theory. In a harmonic trapping potential, an inhomogeneous atomic density leads to spatial variations of the critical point, magnetization time, and spin correlation length. The Kibble–Zurek phenomena are shown to emerge even in such inhomogeneous spinor condensates, when the quench of the quadratic Zeeman energy is fast enough. For slow quench, the magnetized region gradually expands from the center of the trap, pushing out spin vortices, which hinders the Kibble–Zurek mechanism from occurring. The case of a toroidal trapping potential is also discussed.

1. Introduction
Symmetry breaking phase transitions are considered to play crucial roles in the early universe. As the hot universe cooled down, the phase transitions broke the symmetries of the vacuum fields. Since causally disconnected regions acquire independent values of the order parameter in the course of the phase transition, topological defects, such as monopoles, strings, and domain walls, may be left behind, if the independently developed order parameters cannot be connected smoothly [1]. It was proposed that this cosmological scenario of topological defect formation may be tested by the normal fluid–superfluid phase transition of liquid helium [2]. Such a mechanism of topological defect formation is referred to as the Kibble–Zurek (KZ) mechanism, which has been studied in a wide variety of systems [3–12].

Bose–Einstein condensates (BECs) of atomic gases are highly controllable quantum systems and suitable for studying the KZ mechanism in a controlled manner. The Bose–Einstein transition breaks the $U(1)$ symmetry for a single-component system, and quantized vortices can be formed by the KZ mechanism. This has been demonstrated in the experiments reported in [13, 14] and the numerical simulation in [15]. Spinor BECs (i.e., BECs of atoms with spin degrees of freedom) [16–18] have a rich variety of magnetic phases with different symmetries, and thus have various kinds of topological defects [19, 20]. Spin dynamics and texture formation in spinor BECs have been investigated experimentally [21–26] and theoretically [27–32]. In the experiments reported in [22, 23], the transition from the polar state to the ferromagnetic state in a spin-1 $^{87}$Rb was observed, where the transition was controlled by an external magnetic field. Formation of spin vortices by the KZ mechanism in this magnetic transition has been investigated theoretically in [33–36]. It is predicted that the KZ mechanism can also be tested in the Mott transition of cold atoms in an optical lattice [37], soliton formation in the BEC transition in a one-dimensional gas [38, 39], a miscible–immiscible transition in a binary BEC [40], and a magnetic transition in an antiferromagnetic spinor BEC [41].

In the present paper, we investigate how the inhomogeneity of the system due to trapping potentials affects the KZ mechanism in a spinor BEC. The KZ mechanism in inhomogeneous systems has been studied for a single-component BEC [38, 42], a binary BEC [40], and an ion chain [43]. In [36], we studied the KZ mechanism in the magnetic transition of a spin-1 $^{87}$Rb BEC and numerically demonstrated the KZ scaling properties. However, the numerical simulations in [36] were restricted to systems with uniform atomic density. Here we perform numerical simulations of the magnetization...
dynamics of a spin-1 BEC confined in a harmonic trapping potential to show that the KZ mechanism can be observed in realistic experiments. We will show that the inhomogeneity of the trapped system has two effects on the KZ properties. The first one is caused by the spatial dependence of the spin correlation length. The number of spin vortices created by the KZ mechanism depends on the spin correlation length, and therefore depends on the position. The second one originates from the competition between two velocities. Since the density is high around the center of the atomic cloud, the magnetization starts from the center and the magnetized region expands outward. If this expansion velocity is slower than the velocity of the spin wave, the magnetized region can be causally connected with the region that is going to become magnetized, and the KZ mechanism will not work efficiently. A plug potential applied to the center of the trap creates an annular potential landscape, which is shown to prevent the KZ mechanism from becoming ineffectual.

This paper is organized as follows. Section 2 formulates the problem and provides mean-field and Bogoliubov analyses. Sections 3.1 and 3.2 show the numerical results for sudden quench and gradual quench of the magnetic field, respectively. Section 3.3 adds a plug potential to a harmonic trap and demonstrates how it helps the KZ mechanism work. Section 4 concludes this paper.

2. Mean-field analysis of spin correlations

We consider bosonic atoms with mass $M$ confined in an external potential $V_{\text{trap}}(r)$, whose hyperfine spin is $F = 1$. The magnetic field $B$ is applied in the $z$ direction, and the linear and quadratic Zeeman effects shift the energies of magnetic sublevels $m = \pm 1$ by [16, 44]

$$ p = \mp gf_B B, \quad q = \frac{\mu_B B^2}{4E_{\text{hf}}}, \tag{1} $$

where $g_F$ is the hyperfine $g$ factor, $\mu_B$ is the Bohr magneton, and $E_{\text{hf}}$ is the hyperfine splitting energy. For $^{87}$Rb atoms, $g_F = 1/2$ for $F = 1$ and $E_{\text{hf}} / h \simeq 6.8$ GHz. The interaction between atoms is spin-dependent and given by spin-independent and spin-dependent interaction coefficients given by [17, 44]

$$ c_0 = \frac{4\pi \hbar^2 a_0 + 2a_2}{3M}, \quad c_1 = \frac{4\pi \hbar^2 a_2 - a_0}{3M}, \tag{2} $$

respectively, where $a_2$ is the s-wave scattering length for two colliding atoms with total spin $S$. We use the values of $a_0 = 101.8a_0$ and $a_2 = 100.4a_0$ [45] for $F = 1^{87}$Rb atoms, where $a_0$ is the Bohr radius.

We employ the mean-field theory at zero temperature. The state of the system is described by the macroscopic wavefunctions $\psi_m(r, t)$ with $m = 1, 0, -1$. The mean-field energy is given by [16–18]

$$ E = \int \mathrm{d}r \left[ \sum_m \psi_m^* \left( -\frac{\hbar^2}{2M} \nabla^2 + V_{\text{trap}} + mp + m^2 q \right) \psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} \mathbf{F} \cdot \mathbf{F} \right], \tag{3} $$

where

$$ \rho(r, t) = |\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2, \tag{4} $$

$$ F(r, t) = \sum_{m, m'} \psi_m^{*} \partial_{m'} \psi_{m'}, \tag{5} $$

with $f = (f_x, f_y, f_z)$ being the spin-1 polar field. The transverse magnetization of the spin space ($\psi_{\pm 1} \rightarrow e^{\pm i\sigma_y/\sqrt{2}} \psi_{\pm 1}$), the linear Zeeman terms in equation (4) can be eliminated, and we neglect them in the following calculations.

To understand the behavior of the system analytically, we first consider a uniform system with density $\rho = n_0$. When $c_1 < 0$ and $q > 0$, which is the case of spin-1 $^{87}$Rb BECs, the ground state of equation (3) satisfying $F_z = 0$ is given by [16]

$$ \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \sqrt{n_0} e^{i\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{6} $$

for $q > q_c$ and

$$ \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \sqrt{n_0} e^{i\alpha} \begin{pmatrix} e^{i\beta} \frac{1}{2} \sqrt{1 - \frac{q}{q_c}} \\ 1 \\ e^{-i\beta} \frac{1}{2} \sqrt{1 - \frac{q}{q_c}} \end{pmatrix} \tag{7} $$

for $q \leq q_c$, where

$$ q_c = 2|c_1|n_0, \tag{8} $$

and $\alpha$ and $\beta$ are arbitrary phases. The states in equations (8) and (9) are called the polar state and broken axisymmetry state [46], respectively. The transverse magnetization of the polar state (8) is $\left(F_x^2 + F_y^2\right)^{1/2} = 0$ and that of the broken axisymmetry state (9) is $\left(F_x^2 + F_y^2\right)^{1/2} = \left(1 - q^2 / q_c^2\right)^{1/2}$.

We study the stability of the polar state (8) using the Bogoliubov analysis. Substituting

$$ \psi_0(r, t) = e^{-i\sigma_y/\sqrt{2}} \sqrt{n_0}, \tag{9} $$

$$ \psi_{\pm 1}(r, t) = e^{-i\sigma_y/\sqrt{2}} \sum_k \frac{1}{\sqrt{V}} e^{ikr} a_{\pm 1, k}(t), \tag{10} $$

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into the GP equation (7), where \( V \) is the volume of the system, and keeping the first-order terms in \( a_{\pm 1,k} \), we obtain
\[
\hbar \frac{d a_{\pm 1,k}(t)}{dt} = (\varepsilon_k + q + c_1 n_0) a_{\pm 1,k}(t) + c_1 n_0 a_{\pm 1,-k}(t),
\]
where \( \varepsilon_k = h^2 k^2 / (2M) \). The solution is given by
\[
a_{\pm 1,k}(t) = \left( \cos \frac{E_k t}{\hbar} - i \varepsilon_k + q + c_1 n_0 \sin \frac{E_k t}{\hbar} \right) a_{\pm 1,k}(0) - \left( c_1 n_0 \sin \frac{E_k t}{\hbar} \right) a_{\pm 1,-k}(0),
\]
where
\[
E_k = \sqrt{(\varepsilon_k + q)(\varepsilon_k + q - q_c)}.
\]
When \( q \geq q_c \), \( E_k \) is real for all \( k \), and equation (14) is an oscillating function. In this case, the polar state (8) is stable against small deviations. When \( q < q_c \), \( E_k \) is imaginary for \( 0 < \varepsilon_k < q_c - q \). The modes with imaginary \( E_k \) exponentially grow, rendering the polar state (8) dynamically unstable.

If the initial state is prepared in the stable polar state (8) with \( q \geq q_c \), and \( q \) is decreased to \( q < q_c \), the system becomes dynamically unstable and the transverse magnetization \( F \) grows, rendering the polar state (8) dynamically unstable.

The correlation function of transverse magnetization is calculated to be
\[
\langle F_+(r,t)F_-(r',t) \rangle \propto \int dk J_0(k|r-r'|) \exp \left( \frac{t}{\tau} - \frac{\tau |r-r'|^2}{\xi^2_{corr}} \right),
\]
where \( \xi_{corr} \) have the meanings of the growth time and the correlation length of the transverse magnetization, respectively.

3. Numerical results

We restrict ourselves to two-dimensional (2D) systems confined in a harmonic potential \( V_{\text{trap}} = M a^2 (x^2 + y^2) / 2 \) with \( \omega / (2\pi) = 2 \). We take a sufficiently large size of the system, which is comparable to the spin healing length (typically a few micrometers), and the spin dynamics are effectively 2D. We assume that the thickness in the \( z \) direction is \( \gtrsim 1 \mu m \), and introduce the 2D interaction coefficients as \( c_{ij}^{2D} = c_j / \mu m \).

We numerically solve the 2D GP equation using the pseudospectral method [47]. We prepare the ground state of equation (3) with \( \psi_{\pm 1} = 0 \), which is obtained by the imaginary time propagation method. We add small random noises to the initial state of \( \psi_{\pm 1} \) to trigger the magnetization.

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We also define the transverse autocorrelation function along a circle of radius $r$ as

$$G_T(r) = \frac{\int_0^{2\pi} |F_+|^2(r, \theta) \, d\theta}{\int_0^{2\pi} \rho^2(r, \theta) \, d\theta}. \quad (28)$$

The transverse spin winding number along the circle with radius $r$ is defined as

$$w(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \theta} \arg F_+(r, \theta) \, d\theta. \quad (29)$$

### 3.1. Sudden quench

We first investigate the magnetization dynamics for a sudden quench of the quadratic Zeeman energy to $q = 0$. This corresponds to the situation in which the stable polar state is prepared at sufficiently large $q$, and the magnetic field is suddenly switched off at $t = 0$. Spin texture formation in trapped systems for nonzero $q$ has been studied in [28, 31, 32].

Figure 1(a) shows the time evolution of the autocorrelation functions $G_T(t)$ and $G_L(t)$. The transverse magnetization starts to grow at $t \simeq 100$ ms and the longitudinal magnetization follows. The time scale of magnetization coincides with that in the experiments [22, 23]. Figures 1(b)–(e) show the profiles of transverse magnetization. The transverse magnetization emerges around the center of the system and grows outward. This is because the growth time in equation (23) is inversely proportional to the atomic density and the magnetization grows faster where the density is larger. The total density distribution $\rho(r)$ is almost unchanged during the time evolution, since $c_0$ is much larger than $c_1$. The mean-field approximation is applicable even in such a sudden parameter change, since the state does not undergo the sudden change and the time scale of the spin dynamics is $\sim 100$ ms. In fact, the experimental and mean-field results agree well over this time scale of spin dynamics [24, 32].

Many spin vortices can be seen in figures 1(c)–(e) (the holes in the $|F_+|$ profiles, around which $\arg F_+$ rotates by $\pm 2\pi$). In terms of the spin components in equation (9), $\beta$ changes by $\pm 2\pi$ around the vortex core, which is occupied by the $m = 0$ component. Such a spin vortex is called a polar-core vortex. The spin winding number $w(r)$ defined in equation (29) represents the difference between the numbers of polar-core vortices with opposite circulations within the radius $r$.

We note that the spin vortices are produced by two distinct mechanisms in figure 1 with $q = 0$: the KZ mechanism and the spin conservation dynamics [36]. Since the spin correlation function in equation (26) has a finite correlation length $\xi_{corr}$, the directions of magnetization at $r$ and $r'$ are independent for $|r - r'| \gg \xi_{corr}$, giving rise to the KZ mechanism. On the other hand, when $q = 0$, the total magnetization $\int Fdr$ must be conserved to be zero, since equation (3) is invariant with respect to spin rotation in the rotating frame $\psi_{\pm 1} \rightarrow e^{i\varphi t}h \psi_{\pm 1}$. In the present case, however, the conservation law holds not just globally but also locally due to the finite spin correlation length. Since the spin directions at $r$ and $r'$ are independent for $|r - r'| \gg \xi_{corr}$, not only the total magnetization but also the local magnetization $\int_{local} Fdr$ integrated over the region of radius $\sim \xi_{corr}$ must...
be conserved in each spatial region. The magnetization thus occurs in such a way that the local magnetization is conserved to be zero; consequently, spin textures must be formed [28–30]. Among various spin textures, the polar-core vortices are most likely, since the excess energy at the defect can be minimized [48]. This is the second mechanism of the spin vortex formation in figure 1. Thus, to see the effect of the KZ mechanism, we must take the linear dimension of the spatial region to be much larger than $\xi_{\text{corr}}$. The correlation length is $\xi_{\text{corr}} \simeq 10 \, \mu\text{m}$ around the center of the trap and $\xi_{\text{corr}} \simeq 20 \, \mu\text{m}$ at $r = 200 \, \mu\text{m}$ for figure 1.

We consider the $r$-dependence of the spin winding number $w(r)$. According to the KZ theory [2], the number of domains along the circle of radius $r$ is $\sim r/\xi_{\text{corr}}$ and hence the phase ‘random walk’ along the circumference leads to the variance of the winding number on the order of

$$w^2(r) \sim r/\xi_{\text{corr}}. \quad (30)$$

In the Thomas–Fermi approximation, the density distribution is given by [44]

$$n_{\text{TF}}(r) \propto R_{\text{TF}}^2 - r^2, \quad (31)$$

where $R_{\text{TF}}$ is the Thomas–Fermi radius. Substituting $q = 0$ and $q_c = 2|c_1|n_{\text{TF}}(r)$ into $\xi_{\text{corr}}$ in equation (24), we obtain

$$w^2(r) \propto \sqrt{R_{\text{TF}}^2 - r^2}. \quad (32)$$

To compare equation (32) with the numerical simulation, we perform many runs of time evolution with different initial random noises, and take the average of $w^2(r)$ over those runs, with the result shown in figure 2. Since the time at which the magnetization emerges depends on $r$, each $w(r)$ is calculated when $G_T(t)$ in equation (28) exceeds a certain value (0.1 in figure 2 and plots in the following figures). The numerical result and equation (32) (circles and dashed curve in figure 2, respectively) are in good agreement, where the fitting parameter is only the proportionality coefficient in equation (32).

### 3.2. Gradual quench

We next consider the case of a gradual quench of the magnetic field. The quadratic Zeeman energy is linearly decreased as

$$q(t) = q_0(1 - t/\tau_Q), \quad (33)$$

for $0 < t < \tau_Q$ and $q(t) = 0$ for $t \geq \tau_Q$. In equation (33), the quench time $\tau_Q$ is a parameter that determines the time scale of the quench of $q(t)$. As seen in section 3.1, the critical value $q_c(r)$ for magnetization depends on the position $r$, and $q_0$ in equation (33) is chosen to be the maximum of $q_c(r)$. We define the dimensionless quadratic Zeeman energy as

$$\epsilon(r, t) = \frac{q_c(r) - q(t)}{q_c(r)}, \quad (34)$$

which also depends on $r$. We assume that $\epsilon(r, t)$ is much smaller than unity, when we consider the freeze-out time. The local freeze-out time $\tilde{t}(r)$ is defined by [38]

$$\frac{\epsilon(r, \tilde{t})}{\epsilon(r, t)} \sim \tau(r, \tilde{t}), \quad (35)$$

and hence $\tilde{t}$ is defined as the time at which the characteristic time scale of the change of $\epsilon$ is comparable to the growth time of the magnetization. The local domain size of the magnetization and hence the winding number around the circumference of radius $r$ is determined at the instant $\tilde{t}(r)$. Using equations (19) and (35) with $\epsilon \ll 1$, we obtain

$$\epsilon(r, \tilde{t}) \sim \left[ \frac{\hbar q_0}{q_c^2(r) \tau_Q} \right]^{2/3}. \quad (36)$$

Substituting equation (36) into equation (20), we obtain the $\tau_Q$-dependence of the correlation length as

$$\bar{\xi}_{\text{corr}}(r) \sim \left[ \frac{\hbar^2}{M} \left( \frac{q_c(r)}{\bar{q}_0^2} \right)^{1/2} \right]^{1/3} \tau_Q^{1/3}. \quad (37)$$

The winding number in equation (30) thus obeys

$$w^2(r) \sim r/\bar{\xi}_{\text{corr}}(r) \propto \tau_Q^{-1/3}. \quad (38)$$

This $\tau_Q$-dependence of the winding number is the same as that in homogeneous systems [33, 35, 36]. The 1/3-power law in the magnetization of spin-1 BECs in homogeneous systems has been confirmed by numerical simulations for 1D [35] and 2D [36]. The same 1/3-power law also emerges in the Mott transition in a lattice [37], phase separation in a binary BEC [40], and another type of magnetic transition in a spinor BEC [41].
to magnetize is causally disconnected from the magnetized region, and therefore the KZ mechanism works. When $v_f$ is slower than $v_s$, on the other hand, the above two regions are causally connected and the KZ mechanism breaks down.

The radius $r_f(t)$ of the magnetization front at $t$ is determined by

$$q(t) = q_c(r_f(t)).$$

(39)

Using the Thomas–Fermi density distribution (31), the right-hand side is $q_c(r_f(t)) = 2\left|c_1\right|^2 |r_f(t)| \approx q_0[1 - r_f^2(t)/R_{TF}^2]$, giving

$$r_f(t) = R_{TF} \sqrt{\frac{t}{\tau_Q}}.$$  

(40)

The velocity $v_f$ is thus given by

$$v_f = \frac{d r_f(t)}{d t} = \frac{R_{TF}}{2 \sqrt{\tau_Q}} = \frac{R_{TF}^2}{2 q_0 r_f(t)}.$$  

(41)

The transverse spin wave for the broken axisymmetry state (9) is a phonon-like mode in the limit of $k \to 0$, whose velocity is given by [19, 46]

$$v_s = \sqrt{\frac{q(t)}{2M}} \simeq q_0 \frac{R_{TF}^2}{2M} \left[1 - \frac{r_f^2(t)}{R_{TF}^2}\right].$$

(42)

where we used equation (39) and the Thomas–Fermi approximation. Thus, an equality $v_f = v_s$ is always satisfied, and therefore the KZ mechanism always works. This condition is given by

$$\tau_Q < \sqrt{\frac{2M}{q_0} R_{TF}}.$$  

(44)

For the parameters in figure 3, the right-hand side of this inequality is $\simeq 1.6$ s, which agrees well with the time at which the plots in figure 3(a) deviate from $\tau_Q^{-1/3}$. In contrast with the present case ($v_f \ls v_s$), the defect creation is shown to be suppressed for $v_f \ls 0.37 v_s$ in the miscible–immiscible transition of a binary BEC [40].

When the quench time $\tau_Q$ is larger than the right-hand side of equation (44), the KZ mechanism breaks down at the radius given by the smaller solution of equation (43), which we define as $r_{\text{break}}$. Since the right-hand side of equation (43) is $\ls 0.1$ for $\tau_Q \ls 2$ s, we can approximate it as

$$r_{\text{break}} \simeq \frac{R_{TF}^2}{\tau_Q} \sqrt{\frac{M}{2q_0}.}$$

(45)

The following scenario is expected. When the magnetization front $r_f$ reaches $r_{\text{break}}$, the magnetized region ($r < r_f$) and the region that is going to magnetize ($r > r_f$) are causally connected and no new spin vortices are created at $r > r_{\text{break}}$. Therefore, when the radius $r$ of the circle along which the

Figure 3. (a) Variance of the winding number $\langle w^2(r) \rangle$ along the circumference of a circle of radius $r = 200\,\mu m$ (circles), $r = 250\,\mu m$ (squares), and $r = 300\,\mu m$ (triangles) for gradual quench given by equation (33). The data for each $r$ is taken when $G_t(r) > 0.1$. The average $\langle \cdots \rangle$ is taken over 100 runs of simulations for the different initial states produced by random numbers. The dotted line and dot-dashed line are proportional to $\tau_Q^{-1/3}$ and $\tau_Q^{-2/7}$, respectively. (b), (c) Snapshots of the transverse magnetization $|F_+(r, t)|$ for $\tau_Q = 1$ s and 3 s. The arrows indicate how each vortex moves. The unit of $|F_+(r, t)|$ is $3.4 \times 10^{-34}\text{ m}^2$. The field of view of each panel is $300\,\mu m \times 300\,\mu m$. See the supplementary data files for the movies showing the dynamics (available at stacks.iop.org/JPhysCM/25/404212/mmedia).
winding number is calculated is larger than \( r_{\text{break}} \), the winding number is determined not by \( r \) but by \( r_{\text{break}} \). From this assumption, the winding number for \( r > r_{\text{break}} \) is expected to be

\[
w^2(r) \sim r_{\text{break}}/\tilde{\xi}_{\text{corr}} \propto \tau_0^{-4/3} \ ,
\]

where we used equations (38) and (45). However, in figure 3(a), the slope of the plots for \( \tau_0 \gtrsim 2 \text{ s} \) is about \(-2.7\) and significantly smaller than \(-4/3\). This discrepancy is attributed to the fact that the spin vortices created at \( r < r_{\text{break}} \) dynamically escape outward. In fact, for \( \tau_0 \gtrsim 2 \text{ s} \), \( w^2(r) \) is smaller for larger \( r \) in figure 3(a), which indicates that the spin vortices have escaped outward before the magnetization front arrives at \( r \). Thus, the steep drop of the winding number for \( \tau_0 \gtrsim 2 \text{ s} \) in figure 3(a) is caused by the two effects: the causal connection between both sides of the magnetization front and the dynamical escape of spin vortices.

We consider the \( r \) dependence of \( (w^2(r)) \) also for \( \tau_0 \lesssim 2 \text{ s} \). From equation (37), the \( r \) dependence of the correlation length is estimated to be

\[
\tilde{\xi}_{\text{corr}}(r) \propto Q_{\text{corr}}^{3/8} \propto \left(1 - \frac{r^2}{R_{\text{TF}}} \right)^{1/6} .
\]

The \( r \) dependence of the winding number is therefore given by

\[
w^2(r) \sim r/\tilde{\xi}_{\text{corr}} \propto \frac{r}{R_{\text{TF}}} \left(1 - \frac{r^2}{R_{\text{TF}}} \right)^{-1/6} \equiv f(r) .
\]

This \( r \) dependence is slightly modified from that in the homogeneous case in which \( w^2 \propto r \) \cite{2,15}. Using the Thomas–Fermi radius \( R_{\text{TF}} \simeq 410 \mu m \) in figure 3, \( f(r) \) in equation (48) at \( r = 200 \mu m, r = 250 \mu m, \) and \( r = 300 \mu m \) gives 0.51, 0.66, and 0.83, respectively. These values are consistent with the plots in figure 3(a) for \( \tau_0 \lesssim 2 \text{ s} \), in that \( w^2(r) \) increases with \( r \).

3.3. Gradual quench with a plug potential

In the previous subsection, we showed that the KZ scenario breaks down when the magnetized region expands slowly. This is because the region that is going to magnetize is causally connected to the initially magnetized region at the trap center. To eliminate this effect, we deplete the atoms around the trap center by adding a plug potential as

\[
V_{\text{trap}}(r) = \frac{1}{2} M o^2 r^2 + A e^{-r/d} ,
\]

where the values of the parameters are chosen to be \( A = 1500\hbar o \) and \( d = 222 \mu m \). For these parameters, the potential \( V_{\text{trap}}(r) \) has a minimum at \( r \simeq 250 \mu m \) and the atomic density becomes maximal at this radius. As a result, the magnetization starts from the annulus around \( r \simeq 250 \mu m \). Therefore, magnetic domains on this radius are always causally disconnected, and the KZ mechanism is expected to work even for large \( \tau_0 \).

Figure 4 shows the results of the numerical simulations for this setting. The density at \( r \simeq 250 \mu m \) is almost the same as the density at the same radius in the system of figure 3, and \( (w^2(r)) \) is similar to the corresponding data in figure 3(a) (squares) for \( \tau_0 \lesssim 1 \text{ s} \). The difference from figure 3(a) is that \( (w^2(r)) \) in figure 4 obeys the KZ scaling law \( Q_{\text{corr}}^{-1/3} \) even for \( \tau_0 \gtrsim 1 \text{ s} \), as expected from the causal disconnection along the annulus geometry. This feature does not depend on the detail of the potential in equation (49); the KZ scaling always appears when the atomic density is maximal in an annular region whose circumference is larger than the magnetic domain size.

4. Conclusions

We have investigated the spin vortex formation due to the KZ mechanism in a quenched ferromagnetic BEC confined in a trapping potential. Since the atomic density is inhomogeneous in a harmonic trap, the spin correlation length depends on the radius \( r \). In fact, the numerical simulations have demonstrated that the spin winding number depends on \( r \), in good agreement with the theoretical prediction (figure 2). When the quadratic Zeeman energy \( q(t) \) is gradually quenched over the time scale \( \tau_0 \) as in equation (33), the magnetized region gradually expands from the center to the periphery of the atomic cloud. If the expansion velocity \( v_f \) of the magnetization front is much faster than the spin wave velocity \( v_s \), the magnetized region and the region that is going to magnetize are causally disconnected and the system exhibits the KZ scaling law, as shown in figure 3(a) for \( \tau_0 \lesssim 2 \text{ s} \). If \( v_f \) is slower than \( v_s \), on the other hand, both sides of the magnetization front are causally connected and the KZ scenario breaks down (figure 3), which results in a significant decrease in the winding number, as shown in figure 3(a) for \( \tau_0 \gtrsim 2 \text{ s} \). The spin vortices created around the center dynamically escape outward as the magnetization front expands, which also decreases the
winding number in figure 3(a) for $t_Q \gtrsim 2 \text{ s}$. When a plug potential is added to the harmonic trap, the geometry of the system is changed in such a way that the magnetization starts from an annular region, and the KZ power law can be observed over a wide range of $t_Q$ (figure 4).

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