A 3D Free Vibration Analysis of the Horn-Gear System through Chebyshev–Ritz Method in Ultrasonic Gear Honing

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1.Introduction

Gear is an important mechanical component to transmit power in most mechanical equipment [1, 2]. The trends of gear development are miniaturization and high speed [3]. The hardened surface gear machining is widely used for improving gears’ load capacity to satisfy the gear’s developing trends [4]. Hardened surface gear honing is applied for machining hardened tooth surface gear because it can improve surface quality and machining accuracy and reduce costs. However, the honing speed between honing wheel and gear is too slow to meet the required honing speed of hardened surface gear honing. The abrasive grit falls off from the honing wheel leading to block because of the cutting force increasing caused by the slow honing speed [5, 6].

The ultrasonic machining can improve honing speed, reduce cutting force, and avoid blocking to enhance the processing efficiency. Therefore, it is necessary to introduce the ultrasonic machining into hardened surface gear honing to avoid the disadvantages mentioned above [7–9]. Ultrasonic gear honing equipment consists of ultrasonic power supply, transducer, vibrating rod, ultrasonic horn, and gear. The horn-gear system is made of the ultrasonic horn and the gear. To ensure the ultrasonic gear honing equipment works
normally, the horn-gear system should vibrate with high frequency within the frequency modulation range of ultrasonic power supply. So, it is important to analyze the vibration characteristic of the horn-gear system accurately. There are already two theories analyzing the vibration characteristic of the horn-gear system. At first, the whole-resonant theory was used in the ultrasonic gear honing to analyze the vibration characteristics of the horn-gear system. Because of the large size of gear, the gear cannot be neglected in the analysis process. The whole-resonant theory requires that ultrasonic horn and gear should have the same resonant frequency within the frequency modulation range of ultrasonic power supply. The whole-resonant theory has limitation on the size of gear and ultrasonic horn. However, the size of gear is decided by the gear’s application requirements in manufacturing, not by the resonant frequency of the ultrasonic gear honing equipment.

The nonresonant theory can solve the deficiency of the whole-resonant theory. The gear of the horn-gear system can be simplified as a uniform annular plate whose outside diameter is the gear’s pitch diameter [10, 11]. The dynamic equation is derived based on force coupling of the plate and the ultrasonic horn in nonresonant theory. We can adjust the size of the ultrasonic horn to ensure the resonant frequency of the horn-gear system within the frequency modulation range of ultrasonic power supply. In past analyzing process of the nonresonant theory, one-dimensional longitudinal vibration beam theory was adopted for the ultrasonic horn, and thin-plate theory was adopted for the gear [12, 13]. The thin-plate theory based on three hypotheses is only applicable to the plate with the thickness-diameter ratio less than 0.2 [14–17]. The one-dimensional longitudinal vibration beam theory just gives the slender rods’ frequencies of longitudinal vibration mode. If we want to obtain more frequencies of other kinds of vibration mode, this theory will be no longer applicable [18]. The other theories for analyzing the vibration characteristics of beams still cannot conduct the three-dimensional researches [19–24]. The traditional nonresonant theory mentioned above cannot reflect the vibration of the horn-gear system comprehensively and cause the mismatching of boundary condition at the coupling location of ultrasonic horn and gear. The finite element method is also applied to analyze the horn-gear system, but this method cannot obtain a solution in theory.

Many scholars have conducted the three-dimensional research to analyze the vibration characteristics of rod and plate. Leissa and Kang conducted three-dimensional research to analyze the eigenfrequencies of thick, linearly, tapered, annular plates, and rods with arbitrary size using Algebraic–Ritz method. The admissible function of the Algebraic–Ritz method consists of algebraic polynomial multiplying boundary functions [25–27]. The poor numerical stability of the algebraic polynomial leads to the truncation order cannot be large enough to ensure enough eigenfrequencies converging to the accurate. Because the Chebyshev polynomial is as simple as the algebraic polynomial and has outstanding numerical stability, Zhou replaces the algebraic polynomial to the Chebyshev polynomial in admissible functions. This method, the so-called Chebyshev–Ritz method, can obtain more accurate eigenfrequencies of single plate [28–30]. The Ritz method above is still not adopted for analyzing the eigenfrequencies of the horn-gear system yet.

It is essential to apply Chebyshev–Ritz method into analyzing the eigenfrequencies of the horn-gear system to solve the problems owned by the traditional nonresonant theory. A free vibration analysis of the horn-gear system using Chebyshev–Ritz method based on three-dimensional elasticity theory was presented. The model of the horn-gear system was divided into four parts. The eigenvalue equations were derived, and the convergent study was conducted. The comparison study of the natural frequencies obtained by the traditional nonresonant theory, and the method mentioned in this paper was made. The experimental platform of hammering method for testing the horn-gear system’s frequencies in a completely free condition was established. The finite element method was also used to obtain the horn-gear system’s eigenfrequencies. The comparative analysis of the frequencies obtained by the three methods was made.

### 2. Method Application

The homogeneous, isotropic, and simplified model of horn-gear system is shown in Figure 1. A cylindrical coordinate $(r, \theta, z)$ is defined. The model of the horn-gear system is divided into four parts, namely, $\Omega_1$, $\Omega_2$, $\Omega_3$, and $\Omega_4$ shown in Figure 2. The first part $\Omega_1$ is the coupling part of the gear and the ultrasonic horn with radius $R_1$ and thickness $h$. The second part $\Omega_2$ is the gear except the coupling part, and it is simplified as an annular plate with pitch radius $R_3$ and thickness $h$. The third part $\Omega_3$ is a solid cylinder as a part of the ultrasonic horn; its left end is at $z=0$, and its right end is at $z=L$ with radius $R_1$, and $L$ is the length of the solid cylinder. The rest of the horn is the fourth part $\Omega_4$ which is regarded as a cone with hole. The cone’s left end is at $z=0$ with radius $R_1$, and right end at $z=L$ with radius $R_1$. The cone’s height between $r = R_1$ and $r = R_4$ can be expressed as $L_4(r)$. The zero point of radial ($r$) and axial ($z$) coordinates is measured from central axis and $\theta$ is the circumferential angle. The corresponding displacement components are $u$, $v$, and $w$ in the $r$, $\theta$, and $z$ direction, respectively. And, $L_4(r)$ can be expressed as follows:

$$L_4(r) = -\frac{L}{R_1 - R_3} r - \frac{LR_3}{R_1 - R_3}, \quad R_1 \leq r \leq R_3.$$

(1)

According to the theory of three-dimensional elasticity theory [31], the strain energy $V$ of the horn-gear system can be expressed as follows:
\[ V = \sum_{q=1}^{4} V^q = \sum_{q=1}^{4} \left( \frac{G}{2} \int_{R_0}^{R_1} \int_{L_H}^{L} \left( \frac{2\nu}{1-2\nu} (\varepsilon_{rr}^q + \varepsilon_{\theta\theta}^q + \varepsilon_{zz}^q) + 2 \left( (\varepsilon_{rr}^q)^2 + (\varepsilon_{\theta\theta}^q)^2 + (\varepsilon_{zz}^q)^2 \right) + (\varepsilon_{r\theta}^q)^2 + (\varepsilon_{r\theta}^q)^2 \right) r dz d\theta dr \right), \]
where \( G \) is the shear modulus and \( \nu \) is Poisson’s ratio and the other parameters are expressed as follows:

\[
\begin{align*}
R_{ul}^1 &= R_{ul}^2 = R_1, \\
R_{ul}^2 &= R_2, \\
R_{ul}^3 &= R_3, \\
R_{ul}^4 &= 0, \\
R_{ul}^5 &= R_1, \\
R_{ul}^6 &= R_1, \\
\text{Len}_{ul}^1 &= \text{Len}_{ul}^2 = h + L, \\
\text{Len}_{ul}^3 &= L, \\
\text{Len}_{ul}^4 &= L_4(r), \\
\text{Len}_{ul}^5 &= \text{Len}_{ul}^2 = L, \\
\text{Len}_{ul}^6 &= \text{Len}_{ul}^0 = 0.
\end{align*}
\]

The strain components \( \varepsilon_{ij}^q \) (\( i, j = r, \theta, z; q = 1, 2, 3, 4 \)) are written as follows [31]:

\[
T = \sum_{q=1}^{4} T^q = \sum_{q=1}^{4} \left( \frac{\rho}{2} \int_{\text{Len}_{ul}^q}^0 \int_{\text{Len}_{ul}^q} \left( \frac{\partial u_q}{\partial r} \right)^2 + \left( \frac{\partial v_q}{\partial \theta} \right)^2 + \left( \frac{\partial w_q}{\partial z} \right)^2 \right) rdz \sin \theta d\theta dr, \tag{5}
\]

where \( \rho \) is the density of each part and \( t \) is the time.

For the convenience of calculation, the dimensionless variables can be defined as follows:

\[
\begin{align*}
\bar{r}_1 &= \bar{r}_4 = \frac{2r}{R_4} - 1, \\
\bar{r}_2 &= \frac{2r}{R_2 - R_1} - \frac{R_2 + R_1}{R_2 - R_1}, \\
\bar{r}_3 &= \frac{2r}{R_3 - R_1} - \frac{R_3 + R_1}{R_3 - R_1}, \\
\bar{\theta}_1 &= \bar{\theta}_2 = \bar{\theta}_3 = \bar{\theta}_4 = \theta, \\
\bar{z}_1 &= \bar{z}_2 = \frac{2(z - L - (h/2))}{h}, \\
\bar{z}_3 &= \frac{2z}{L} - 1, \\
\bar{z}_4 &= \frac{2z}{L_4(r)} - 1.
\end{align*}
\]

Thus, equation (1) can be simplified as

\[
L_4(\bar{r}_4) = \frac{L_4}{2} \bar{r}_4 + \frac{L_4}{2}
\]

According to the free vibration analysis method, the vibration displacement at any parts of the horn-gear system can be assumed as follows:

\[
\begin{align*}
u_q(r, \theta, z, t) &= U_q(\bar{r}_q, \bar{\theta}, \bar{z}_q)e^{j\omega t}, \\
\nu_q(r, \theta, z, t) &= V_q(\bar{r}_q, \bar{\theta}, \bar{z}_q)e^{j\omega t}, \\
\omega_q(r, \theta, z, t) &= W_q(\bar{r}_q, \bar{\theta}, \bar{z}_q)e^{j\omega t},
\end{align*}
\]

where \( \omega \) is the angular frequency of the horn-gear system and \( i = \sqrt{-1} \).

Since each part of the horn-gear system is a rotator and has symmetry characteristics, the amplitude displacement of each part of the system can be expressed as

\[
\begin{align*}
U_q(\bar{r}_q, \bar{\theta}, \bar{z}_q) &= \overline{U}_q(\bar{r}_q, \bar{z}_q) \cos(s\bar{\theta}), \\
V_q(\bar{r}_q, \bar{\theta}, \bar{z}_q) &= \overline{V}_q(\bar{r}_q, \bar{z}_q) \sin(s\bar{\theta}), \\
W_q(\bar{r}_q, \bar{\theta}, \bar{z}_q) &= \overline{W}_q(\bar{r}_q, \bar{z}_q) \cos(s\bar{\theta}),
\end{align*}
\]

where \( s \) is the circumferential wave number, which is an integer (namely, \( s = 0, 1, 2, \ldots, \infty \)) to ensure the period of vibration in the \( \bar{\theta} \) direction.

Substituting equations (4), (6)–(9) into equations (2) and (5), the maximum potential energy \( V_{\text{max}}^q \) and kinetic energy \( T_{\text{max}}^q \) of the horn-gear system can be calculated, respectively \((q = 1, 2, 3, \text{and } 4)\)
\[
V^q_{\text{max}} = \frac{GA^2}{4} \int_{-1}^{1} \left( \frac{2v}{1 - 2v} \Gamma_1 \left( \frac{\varphi^q_{\text{max}}}{\varphi_{\text{max}}} + \varphi_{\text{max}} \right)^2 + 2V_1 \left( \frac{\varphi^q_{\text{max}}}{\varphi_{\text{max}}} \right)^2 + \Gamma_1 \left( \frac{\varphi^q_{\text{max}}}{\varphi_{\text{max}}} \right)^2 + \Gamma_2 \left( \frac{\varphi^q_{\text{max}}}{\varphi_{\text{max}}} \right)^2 \right)
\cdot (\varphi_q + \delta^q) \, d\varphi_q \, d\varphi_q,
\]
when \( q = 1, 2, 3 \),

\[
T^q_{\text{max}} = \frac{\rho A^3 (A^q_{\text{max}})^2 \omega^2}{16} \int_{-1}^{1} \left( \Gamma_1 (U_q)^2 + \Gamma_2 (V_q)^2 + \Gamma_3 (W_q)^2 \right) (\varphi_q + \delta^q) \, d\varphi_q \, d\varphi_q,
\]
when \( q = 1, 2, 3 \),

\[
T^4_{\text{max}} = \frac{\rho (R_3 - R_1) \omega^2}{16} \int_{-1}^{1} \left( \Gamma_1 (U_4)^2 + \Gamma_2 (V_4)^2 + \Gamma_3 (W_4)^2 \right) (\varphi_4 (R_3 - R_1) + (R_3 + R_1) f (\varphi_4) \, d\varphi_4 \, d\varphi_4,
\]

\[
V^4_{\text{max}} = \frac{G (R_3 - R_1)}{16} \int_{-1}^{1} \left( \frac{2v}{1 - 2v} \Gamma_1 (U_4)^2 + \frac{2s}{R_3 - R_1} + (R_3 + R_1) (U_4)^2 + \frac{2s}{L_4 (\varphi_4) \, d\varphi_4 \, d\varphi_4} \right)
\cdot (\varphi_4 (R_3 - R_1) + (R_3 + R_1) V_4)^2 + \left( \frac{2s}{L_4 (\varphi_4) \, d\varphi_4 \, d\varphi_4} \right)^2
\]

where

\[
\Gamma_1 = \int_{0}^{2\pi} \cos^2 (s \bar{\theta}_q) \, d\bar{\theta}_q = \begin{cases} 
2\pi, & s = 0, \\
\pi, & s > 0,
\end{cases}
\]

\[
\Gamma_2 = \int_{0}^{2\pi} \sin^2 (s \bar{\theta}_q) \, d\bar{\theta}_q = \begin{cases} 
0, & s = 0, \\
\pi, & s > 0,
\end{cases}
\]

\[
\Lambda^\dagger_{\Phi^q} = \Lambda^\dagger_{\Phi^q} = h,
\]

\[
\Lambda^\dagger_{\Phi^q} = L,
\]
Each part’s vibration displacement function of the horn-gear system can be expressed as the two Chebyshev polynomials multiplied by corresponding boundary conditions:

\[ U_q(r_q, z_q) = \eta_u^q(r_q) \eta_u^q(z_q) \sum_{i=1}^{L_u} \sum_{j=1}^{L_u} A_{ij}^q P_i(r_q) P_j(z_q), \]

\[ V_q(r_q, z_q) = \eta_v^q(r_q) \eta_v^q(z_q) \sum_{k=1}^{L_v} \sum_{l=1}^{L_v} B_{kl}^q P_k(r_q) P_l(z_q), \]

\[ W_q(r_q, z_q) = \eta_w^q(r_q) \eta_w^q(z_q) \sum_{m=1}^{L_w} \sum_{n=1}^{L_w} C_{mn}^q P_m(r_q) P_n(z_q), \]

(11)

where \( L_u, L_v, L_w, M_u, \) and \( N_q \) are the truncation orders of the Chebyshev polynomial, \( A_{ij}^q, B_{kl}^q, C_{mn}^q \) are to be determined parameter, \( P_p(\chi) (p = 1, 2, 3, \ldots, \chi = r_q, z_q) \) is the \( p \)th order Chebyshev polynomial in the one-dimensional which can be described as follows:

\[ P_p(\chi) = \cos((p-1)\arccos(\chi)). \]

(12)

\( \eta_u^q(\tau_q), \eta_v^q(\tau_q), \eta_u^q(z_q), \eta_v^q(z_q), \eta_w^q(\tau_q), \) and \( \eta_w^q(z_q) \) are the boundary condition functions for the four parts of the horn-gear system. In the improved method, each part of the horn-gear system is recognized as completely free, so the displacement parameters \( u_q, v_q, \) and \( w_q \) should satisfy the geometric boundary conditions of each part, then the boundary characteristic functions \( \eta_u^q(\tau_q), \eta_u^q(z_q), \eta_v^q(\tau_q), \eta_w^q(\tau_q), \eta_w^q(z_q) = 1. \) Each part’s vibration displacement function can be obtained by substituting the boundary conditions into equation (11).

The energy equation of the horn-gear system can be defined as follows:

\[ \Pi_q = V_{\text{max}}^q - T_{\text{max}}^q. \]

(13)

The minimum of the coefficient can be calculated from the deriving the energy equation:

\[ \frac{\partial \Pi_q}{\partial A_{ij}^q} = 0, \]

\[ \frac{\partial \Pi_q}{\partial B_{kl}^q} = 0, \]

\[ \frac{\partial \Pi_q}{\partial C_{mn}^q} = 0. \]

(14)

Then, the following eigenvalue equation can be obtained:

\[ ([K^q] - \Omega^2[M^q])\{X^q\} = 0, \]

where
\[ \Omega = \omega \sqrt{\frac{\rho}{G}} \]

\[ \begin{bmatrix} [\mathbf{K}_{\text{uq}}] & [\mathbf{K}_{\text{uvq}}] & [\mathbf{K}_{\text{uwq}}] \\ [\mathbf{K}_{\text{vq}}] & [\mathbf{K}_{\text{vq}}] & [\mathbf{K}_{\text{wq}}] \\ \text{sym} & [\mathbf{K}_{\text{wq}}] \end{bmatrix} \]

\[ [\mathbf{M}_{\text{uq}}] = \begin{bmatrix} [\mathbf{M}_{\text{uq}}] & [\mathbf{0}] & [\mathbf{0}] \\ \text{sym} & [\mathbf{M}_{\text{wq}}] \end{bmatrix} \]

\[ [\mathbf{X}_q] = \{[\mathbf{A}_q][\mathbf{B}_q][\mathbf{C}_q]\}^T, \quad \text{for } s > 1, \]

\[ [\mathbf{K}_q] = \begin{bmatrix} [\mathbf{K}_{\text{uq}}] & [\mathbf{K}_{\text{vq}}] \\ \text{sym} & [\mathbf{K}_{\text{wq}}] \end{bmatrix} \]

\[ [\mathbf{M}_q] = \begin{bmatrix} [\mathbf{0}] \\ \text{sym} & [\mathbf{M}_{\text{wq}}] \end{bmatrix} \]

\[ [\mathbf{X}_q] = \{[\mathbf{A}_q]\}^T, \quad s = 0 \text{ for the axisymmetric vibration}, \]

\[ [\mathbf{K}_q] = [\mathbf{K}_{\text{vq}}]; \quad [\mathbf{M}_q] = [\mathbf{M}_{\text{vq}}]; \quad [\mathbf{X}_q] = \{[\mathbf{B}_q]\}^T, \quad s = 0 \text{ for the torsional vibration}. \]

\[ \text{(16)} \]

In equations (15) and (16), \([\mathbf{K}_{\text{uq}}] \) and \([\mathbf{M}_{\text{uq}}] \) (\(\alpha, \beta = u, v, w\)) are the stiffness submatrices and the mass submatrices of each part of the horn-gear system.

The vectors \([\mathbf{A}_q]\), \([\mathbf{B}_q]\), and \([\mathbf{C}_q]\) can be described as follows:

\[ [\mathbf{A}_q] = [A_{11}^q \ldots A_{1j}^q \ldots A_{11}^q \ldots A_{1j}^q]^T, \]

\[ [\mathbf{B}_q] = [B_{11}^q \ldots B_{1l}^q \ldots B_{12}^q \ldots B_{1K}^q \ldots B_{KL}^q]^T, \]

\[ [\mathbf{C}_q] = [C_{11}^q \ldots C_{1N}^q \ldots C_{21}^q \ldots C_{2N}^q \ldots C_{3M}^q \ldots C_{3NN}^q]^T, \]

\[ \text{(17)} \]

when \( q = 1, 2, 3 \), are respectively, the submatrices \([\mathbf{K}_{\text{uq}}] \) and \([\mathbf{M}_{\text{uq}}] \) (\(\alpha, \beta = u, v, w\)) when can be described as follows:
\[
\begin{align*}
K^{uq} &= \Lambda_V^q \left( \frac{1 - \nu}{1 - 2\nu} (D^q_{111} + D^q_{001}) + \frac{\nu}{1 - 2\nu} (D^q_{010} + D^q_{100}) + \frac{\nu^2}{2} (D^q_{000}) \right) H^q_{\nu \mu j} + \frac{1}{2} \left( \frac{\Lambda_V^q}{\Lambda_V^T} \right)^2 D^q_{001} H^q_{111} \right), \\
K^{va} &= \Lambda_V^q \left( \frac{\nu}{1 - 2\nu} D^q_{000} + \frac{1 - \nu}{1 - 2\nu} D^q_{010} \right) H^q_{\nu \mu j} + \frac{1}{2} \left( \frac{\Lambda_V^q}{\Lambda_V^T} \right)^2 D^q_{001} H^q_{111} \right), \\
K^{wq} &= \Lambda_V^q \left( \frac{1 - \nu}{1 - 2\nu} D^q_{000} + \frac{\nu}{1 - 2\nu} D^q_{010} \right) H^q_{\nu \mu j} + \frac{1}{2} \left( \frac{\Lambda_V^q}{\Lambda_V^T} \right)^2 D^q_{001} H^q_{111} \right), \\
M^{uq} &= \Lambda_T^q (\Lambda_V^q)^2 D^q_{001} H^q_{\nu \mu j}, \\
M^{va} &= \Lambda_T^q (\Lambda_V^q)^2 D^q_{001} H^q_{\nu \mu j}, \\
M^{wq} &= \Lambda_T^q (\Lambda_V^q)^2 D^q_{001} H^q_{\nu \mu j},
\end{align*}
\]

where

\[
\begin{align*}
D^q_{\alpha \beta \gamma \delta} &= \int_{-1}^{1} \frac{d^4 \left( P_\alpha \left( \tau_q \right) \right)}{d \tau_q^4} \frac{d^4 \left( P_\beta \left( \tau_q \right) \right)}{d \tau_q^4} \left( \tau_q + 1 \right)^\alpha d \tau_q, \\
H^q_{\alpha \beta \gamma \delta} &= \int_{-1}^{1} \frac{d^4 \left( P_\alpha \left( \tau_q \right) \right)}{d \tau_q^4} \frac{d^4 \left( P_\beta \left( \tau_q \right) \right)}{d \tau_q^4} d \tau_q, \quad \alpha, \beta = u, v, w, \quad \sigma = i, j, k, l, m, n, \quad \tau = i, j, k, l, m, n,
\end{align*}
\]
When $q = 4$, the submatrices $[\mathbf{K}^{\alpha\beta}]$ and $[\mathbf{M}^{\alpha\beta}] (\alpha, \beta = u, v, w)$ can be described as follows:

\[
[K^{uvq}] = \left( \frac{1}{R_3 - R_1} \right) \left[ \begin{array}{ccc}
1 - \nu & \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvvi} \\
\nu & 1 - \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\end{array} \right] H^{\alpha\beta\gamma\delta}_{uv}\]

\[
\cdot [K^{uvq}](\nu) = \left( \frac{1}{1 - 2\nu} \right) \left( \frac{1}{R_3 - R_1} \right) \left[ \begin{array}{ccc}
1 - \nu & \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvvi} \\
\nu & 1 - \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\end{array} \right] H^{\alpha\beta\gamma\delta}_{uv}\]

\[
[M^{uvq}] = \left( \frac{1}{R_3 - R_1} \right) \left( \frac{1}{R_3 - R_1} \right) \left[ \begin{array}{ccc}
1 - \nu & \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvvi} \\
\nu & 1 - \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\end{array} \right] H^{\alpha\beta\gamma\delta}_{uv}\]

\[
\cdot [M^{uvq}](\nu) = \left( \frac{1}{1 - 2\nu} \right) \left( \frac{1}{R_3 - R_1} \right) \left[ \begin{array}{ccc}
1 - \nu & \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvvi} \\
\nu & 1 - \nu & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} & \frac{1 - \nu}{1 - 2\nu} (R_3 - R_1) D^{\alpha\beta\gamma\delta}_{uvji} \\
\end{array} \right] H^{\alpha\beta\gamma\delta}_{uv}\]

(20)
where

\[ D_{\text{ AFC}} = \int_{-1}^{1} \left( \frac{\partial L_{4}(\bar{r})}{\partial \bar{r}} \right)^{a} \left( L_{4}(\bar{r}) \right)^{b} \left( \bar{r}_{4}(R_{3} - R_{1}) + (R_{3} + R_{1}) \right) \frac{d^{r}}{d\bar{r}_{4}} \frac{d^{r}}{d\bar{r}_{4}} d\bar{r}_{4}, \]

\[ H_{\text{ AFC}} = \int_{-1}^{1} (\bar{z}_{4} + 1) \frac{d^{r}}{d\bar{z}_{4}} \frac{d^{r}}{d\bar{z}_{4}} d\bar{z}_{4} \]

According to the above formulas, equation (15) can be further simplified to

\[ ([K] - \Omega^{2}[M])\{X\} = \{0\}, \]  

(22)

where

\[ [K] = \begin{pmatrix}
[K'] & [0] & [0] & [0] \\
[0] & [K^2] & [0] & [0] \\
[0] & [0] & [K^3] & [0] \\
[0] & [0] & [0] & [K^4]
\end{pmatrix}, \]

\[ [M] = \begin{pmatrix}
[M'] & [0] & [0] & [0] \\
[0] & [M^2] & [0] & [0] \\
[0] & [0] & [M^3] & [0] \\
[0] & [0] & [0] & [M^4]
\end{pmatrix}, \]

\[ \{X\} = \begin{pmatrix}
\{X'\} \\
\{X^2\} \\
\{X^3\} \\
\{X^4\}
\end{pmatrix}. \]

It is obvious that the four parts of the horn-gear system are coupled with each other, and it results that the matrix \{X\} are also not independent of each other, and the eigenfrequencies of the horn-gear system could not be obtained directly from equation (22). However, the four parts of the horn-gear system need to meet the conditions of equal displacement conditions at each coupling, and it can be defined as follows:

\[ U_{1}(r_{1}, z_{1}) = U_{2}(r_{2}, z_{2}), \quad V_{1}(r_{1}, z_{1}) = V_{2}(r_{2}, z_{2}), \quad W_{1}(r_{1}, z_{1}) = W_{2}(r_{2}, z_{2}), \quad \text{at } r_{1} = 1 \text{ and } r_{2} = -1, \]

\[ U_{3}(r_{3}, z_{3}) = U_{4}(r_{4}, z_{4}), \quad V_{3}(r_{3}, z_{3}) = V_{4}(r_{4}, z_{4}), \quad W_{3}(r_{3}, z_{3}) = W_{4}(r_{4}, z_{4}), \quad \text{at } r_{3} = 1 \text{ and } r_{4} = -1. \]

(24)
Substituting equation (11) into (24),

\[
\sum_{i=1}^{L_1} \sum_{j=1}^{L_1} A_{ij} P_i(1) P_j(\bar{z}_1) = \sum_{i=1}^{L_1} \sum_{j=1}^{L_1} A_{ij} P_i(-1) P_j(\bar{z}_2),
\]

\[
\sum_{k=1}^{K_1} \sum_{l=1}^{L_1} B_{kl} P_k(1) P_l(\bar{z}_1) = \sum_{k=1}^{K_1} \sum_{l=1}^{L_1} B_{kl} P_k(-1) P_l(\bar{z}_2),
\]

\[
\sum_{m=1}^{M_1} \sum_{n=1}^{N_1} C_{mn} P_m(1) P_n(\bar{z}_1) = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} C_{mn} P_m(-1) P_n(\bar{z}_2),
\]

\[
\sum_{i=1}^{L_1} \sum_{j=1}^{L_1} A_{ij} P_i(\bar{p}_1) P_j(-1) = \sum_{i=1}^{L_1} \sum_{j=1}^{L_1} A_{ij} P_i(\bar{p}_3) P_j(1),
\]

\[
\sum_{k=1}^{K_1} \sum_{l=1}^{L_1} B_{kl} P_k(\bar{p}_1) P_l(-1) = \sum_{k=1}^{K_1} \sum_{l=1}^{L_1} B_{kl} P_k(\bar{p}_3) P_l(1),
\]

\[
\sum_{m=1}^{M_1} \sum_{n=1}^{N_1} C_{mn} P_m(\bar{p}_1) P_n(-1) = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} C_{mn} P_m(\bar{p}_3) P_n(1),
\]

\[
\sum_{i=1}^{L_1} \sum_{j=1}^{L_1} A_{ij} P_i(1) P_j(\bar{z}_3) = \sum_{i=1}^{L_1} \sum_{j=1}^{L_1} A_{ij} P_i(-1) P_j(\bar{z}_4),
\]

\[
\sum_{k=1}^{K_1} \sum_{l=1}^{L_1} B_{kl} P_k(1) P_l(\bar{z}_3) = \sum_{k=1}^{K_1} \sum_{l=1}^{L_1} B_{kl} P_k(-1) P_l(\bar{z}_4),
\]

\[
\sum_{m=1}^{M_1} \sum_{n=1}^{N_1} C_{mn} P_m(1) P_n(\bar{z}_3) = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} C_{mn} P_m(-1) P_n(\bar{z}_4).
\]

According to the properties of Chebyshev polynomial, it can be obtained that

\[
P_x(1) = 1,
\]

\[
P_x(-1) = (-1)^{x-1},
\]

\[
G_{xy} = \int_{-1}^{1} \frac{P_x(\bar{z}_q) P_y(\bar{z}_q)}{\sqrt{1-z_q^2}} d\bar{z}_q = \begin{cases} 0, & x \neq y, \\ 1, & x = y, \end{cases}
\]

\[
H_{xy} = \int_{-1}^{1} \frac{P_x(\bar{r}_q) P_y(\bar{r}_q)}{\sqrt{1-r_q^2}} d\bar{r}_q = \begin{cases} 0, & x \neq y, \\ 1, & x = y, \end{cases}
\]

and then the two sides of equation (25) are multiplied by the Chebyshev polynomial, it can be changed to
\[
\sum_{i=1}^{I_2} (-1)^{i-1} A_{ij}^2 = \sum_{j=1}^{J_2} \sum_{i=1}^{I_2} A_{ij}^2 G_{jj} (j = 1, 2, \ldots, J_2),
\]
\[
K_2 \sum_{k=1}^{K_2} (-1)^{k-1} B_{ik}^2 = \sum_{i=1}^{I_2} B_{ik}^2 G_{ii} (l = 1, 2, \ldots, L_2),
\]
\[
M_2 \sum_{m=1}^{M_2} (-1)^{m-1} C_{mn}^2 = \sum_{n=1}^{N_2} C_{mn}^2 G_{nn} (n = 1, 2, \ldots, N_2),
\]
\[
\sum_{i=1}^{I_2} (-1)^{(i-1)} A_{ij}^2 = \sum_{j=1}^{J_2} \sum_{i=1}^{I_2} (-1)^{(i-1)} A_{ij}^2 H_{ij} (i = 1, 2, \ldots, I_2),
\]
\[
L_2 \sum_{k=1}^{L_2} (-1)^{(k-1)} B_{ik}^2 = \sum_{k=1}^{K_2} B_{ik}^2 H_{kk} (k = 1, 2, \ldots, K_2),
\]
\[
N_2 \sum_{n=1}^{N_2} (-1)^{(n-1)} C_{mn}^2 = \sum_{m=1}^{M_2} C_{mn}^2 H_{mn} (m = 1, 2, \ldots, M_2),
\]
\[
\sum_{i=1}^{I_2} (-1)^{(i-1)} A_{ij}^2 = G_{ij} \sum_{i=1}^{I_2} \sum_{j=1}^{J_2} (-1)^{(i-1)} G_{ij} A_{ij} + \sum_{i=1}^{I_2} \sum_{j=1}^{J_2} (-1)^{(i-1)} A_{ij}^2 (G_{ij} - G_{ij}) (j = 1, 2, \ldots, J_2),
\]
\[
K_2 \sum_{k=1}^{K_2} (-1)^{(k-1)} B_{ik}^2 = G_{ij} \sum_{k=1}^{K_2} \sum_{k=1}^{K_2} (-1)^{(k-1)} B_{ik}^2 (G_{ij} - G_{ij}) (k = 1, 2, \ldots, K_2),
\]
\[
M_2 \sum_{m=1}^{M_2} (-1)^{(m-1)} C_{mn}^2 = G_{ij} \sum_{m=1}^{M_2} \sum_{n=1}^{N_2} (-1)^{(m-1)} C_{mn}^2 (G_{mn} - G_{mn}) (m = 1, 2, \ldots, M_2).
\]

From equation (27), the number of independent unknown parameters is
\[ F = \sum_{i=1}^{I} (I_q \times J_q) + K_q \times L_q + M_q \times N_q - (J_2 + L_2 + N_2 + I_2 + K_2 + M_2 + J_4 + L_4 + N_4). \]
The independent unknown parameters extracted from \([X]\) can form a new matrix \([\bar{X}]\). It can be described as
\[ [X] = [S][\bar{X}], \]
where the coefficients \(s\) of the matrix \([S]\) can be calculated by equation (27). Finally, equation (22) can be simplified to
\[ ([\bar{K}] - \Omega^2 [\bar{M}]) [\bar{X}] = [0], \]
where
\[ [\bar{K}] = [S]^T [K][S], \]
\[ [\bar{M}] = [S]^T [M][S]. \]

3. Convergence and Comparison Analysis

3.1. Convergence Analysis. It is necessary to check the convergence of the eigenfrequencies using Chebyshev polynomial as admissible function. Firstly, a convergence study is performed for the completely free horn-gear system with \( h = 0.012, R_1 = 0.014, R_2 = 0.040, R_3 = 0.028, L = 0.174, \) and \( s = 0.3 \). In the following study, Poisson’s ration is taken as \( \nu = 0.3 \). For convenience, the displacement amplitude functions \( \bar{U}, \bar{V}, \) and \( \bar{W} \) in each coordinate direction have equivalent Chebyshev polynomial terms which means \( I_q = K_q = M_q \) and \( J_q = L_q = N_q \). In the following article, all the eigenvalues have four significant figures. In order to express and compare easily, the eigenvalue parameter \( \Omega = \omega \sqrt{\rho/G} \) is adopted.

Table 1 shows the convergence of the first four eigenfrequencies. The first four eigenfrequencies can basically meet the requirements of ultrasonic gear honing. The values of \( I_q \) and \( J_q \) start at six and ten, respectively, increasing \( I_q \) from 6 to 16 and \( J_q \) from 10 to 16. The results showed that when \( I_q \times J_q = 6 \times 10 \), the first, second, and fourth eigenfrequencies were close to the final values when \( I_q \times J_q = 16 \times 16 \). This phenomenon indicated the outstanding numerical stability of this method. The eigenfrequencies which were the easiest to converge to the final value are the first and second eigenfrequencies, \( I_q = 15 \) and \( J_q = 14 \) were the smallest terms to obtain accurate values in the \( r \) and \( z \) directions. If we want to obtain higher order eigenfrequencies, the numbers of Chebyshev polynomial terms \( I_q \) and \( J_q \) should be larger. The first four eigenfrequencies all converging to the final values were at \( I_q \times J_q = 16 \times 14 \). This result proved that the Chebyshev–Ritz method can ensure the first four eigenfrequencies converge to the final values at least. It can be adopted for analyzing the vibration characteristic of the horn-gear system in Figure 1.

![Figure 1](image-url)
3.2. Experiment Analysis. In order to verify the accuracy of the eigenfrequencies achieved by Chebyshev–Ritz method, we designed one ultrasonic horn and three gears for the modal experiment. The ultrasonic horn can be assembled with the three gears separately to form three horn-gear systems. The ultrasonic horn and the three gears are made of 45# steel. Poisson’s ratio of the horn-gear system was $\nu = 0.31$, Young’s modulus was $E = 210 \text{ GPa}$, and density $\rho = 7850 \text{ kg/m}^3$. The one ultrasonic horn and three gears are shown in Figure 3. The dimension parameters of the horn-gear systems are listed in Table 2.

In Table 2, the parameter $h/2R_3$ changed from 0.15 to 0.35 representing the plates became thicker and thicker. The plates in HGS 2 and HGS 3 were not the thin plates anymore. The parameter $L/(R_1 + R_3) = 4.14$ meant the ultrasonic horn was not thin rod. If we needed the natural frequencies of other kinds of vibration mode of the ultrasonic horn, the one-dimensional longitudinal vibration beam theory was no longer applicable.

We established an experimental platform to acquire the natural frequencies of the horn-gear system through hammering method. To simulate the completely free condition, we hung the horn-gear system with two ropes located at both ends of the horn-gear system. A microphone connected to computer was hung near the horn-gear system to receive the

### Table 1: Convergence of the first four eigenvalues in $\Omega = \omega \sqrt{\rho/G}$ of a completely free horn-gear system with $h = 0.012, R_1 = 0.014, R_2 = 0.040, R_3 = 0.028, L = 0.174, s = 0.3$, and $s = 0'$.

| $I_q \times I_q$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ |
|-----------------|------------|------------|------------|------------|
| 6 x 10          | 6.701      | 25.85      | 33.05      | 44.48      |
| 6 x 12          | 6.674      | 25.84      | 32.84      | 44.11      |
| 6 x 14          | 6.664      | 25.83      | 32.74      | 44.09      |
| 6 x 16          | 6.657      | 25.82      | 32.68      | 44.07      |
| 10 x 10         | 6.623      | 25.73      | 30.68      | 44.06      |
| 10 x 12         | 6.606      | 25.72      | 30.49      | 44.04      |
| 10 x 14         | 6.592      | 25.70      | 30.34      | 44.02      |
| 10 x 16         | 6.582      | 25.68      | 30.22      | 44.00      |
| 14 x 10         | 6.594      | 25.67      | 29.91      | 44.04      |
| 14 x 12         | 6.567      | 25.65      | 29.53      | 44.01      |
| 14 x 14         | 6.550      | 25.64      | 29.32      | 43.99      |
| 14 x 16         | 6.538      | 25.63      | 29.17      | 43.98      |
| 15 x 10         | 6.591      | 25.66      | 29.83      | 44.04      |
| 15 x 12         | 6.561      | 25.63      | 29.41      | 44.01      |
| 15 x 14         | 6.522      | 25.60      | 28.85      | 43.98      |
| 15 x 16         | 6.522      | 25.59      | 28.85      | 43.97      |
| 16 x 10         | 6.554      | 25.66      | 29.77      | 44.04      |
| 16 x 12         | 6.522      | 25.60      | 29.31      | 44.00      |
| 16 x 14         | 6.522      | 25.59      | 28.85      | 43.97      |
| 16 x 16         | 6.522      | 25.59      | 28.85      | 43.97      |

### Table 2: Dimension parameters of the three horn-gear systems.

| Parameters (mm) | $h$ | $R_1$ | $R_2$ | $R_3$ | $L$ |
|-----------------|-----|-------|-------|-------|-----|
| Horn-gear system 1 (HGS1) | 12  | 14    | 40    | 28    | 174 |
| Horn-gear system 2 (HGS2) | 20  | 14    | 40    | 28    | 174 |
| Horn-gear system 3 (HGS3) | 28  | 14    | 40    | 28    | 174 |

![Figure 3: (a) The three gears and (b) the one horn.](image-url)
The frequencies of the three horn-gear system obtained by Chebyshev–Ritz method, the experiment, and the finite element method are listed in Table 3–5. The frequencies obtained by the Chebyshev–Ritz method are expressed by FC. The frequencies obtained by the experiment are expressed by FE and frequencies obtained by the finite element method are expressed FF. The finite element analysis type was a modal analysis with Block Lanczos. The selected unit type is 20-node solid 95, and the meshing method is intelligent meshing. The index FCR represents the relative error between the natural frequency FC obtained from Chebyshev–Ritz method and the natural frequency FF obtained by the finite element method. The index FER represents the natural frequency FE obtained by the hammering experiment and the natural frequency FF obtained by the finite element method. Poisson’s ratio of the ultrasonic horn and gear is \( \nu = 0.31 \), Young’s modulus is \( E = 210 \text{ GPa} \), and density \( \rho = 7850 \text{ kg/m}^3 \).

In Table 3, 6 FERs are not greater than 2%, and no FER is greater than 5% in the 10 data sets. In Figure 5(a), the percentage of FER no more than 2% which means \((6/10) \times 100\% = 60\%\) is 60%. At the same time, in Table 3, 4 FCRs are not greater than 2%, and no FCR is greater than 5%. In Figure 5(a), the percentage of FCR no more than 2% which means \((4/10) \times 100\% = 40\%\) is 40%. The percentage of FCR and FER in Figures 5(b) and 5(c) are the same as that of Figure 5(a). In Table 4, 4 FCRs and 5 FERs are no more than 2%, and 1 FCR and 0 FER are greater than 5%. In Table 5, there are 2 FCRs and 7 FERs no more than 2%, and 2 FCRs and 1 FER greater than 5%. As can be seen from the results in Table 3, there is no FCR greater than 5% which means the simplified model of the horn-gear system is reasonable, the coupling conditions of each part are accurate, and the Chebyshev–Ritz method is effective for multimodule coupling. This phenomenon indicates that the Chebyshev–Ritz method based on the three-dimensional elastic theory not only has higher precision for analyzing a horn-gear system consists of the thinner plate but
Table 5: Frequencies between 1 kHz and 20 kHz of HGS3 obtained by the Chebyshev–Ritz method and the experiment compared with those obtained by the finite element method.

|       | Frequency (Hz) |       |       |       |
|-------|----------------|-------|-------|-------|
|       | f<sub>1</sub>  | f<sub>2</sub> | f<sub>3</sub> | f<sub>4</sub> |
| s = 1 |
| FC   | 2777.8         | 12760 | 14489 |       |
| FE   | 2739.5         | 13158 | 14546 |       |
| FF   | 2818.6         | 13388 | 15153 |       |
| FCR  | 1.45           | 4.69  | 4.58  |       |
| FER  | 2.81           | 1.72  | 4.01  |       |
| s = 2 |
| FC   | 7567.4         | 14998 |       |       |
| FE   | 7688.5         | 15156 |       |       |
| FF   | 7726.6         | 15378 |       |       |
| FCR  | 2.06           | 2.47  |       |       |
| FER  | 0.49           | 1.44  |       |       |
| s = 3 |
| FC   | 2088.2         | 11692 | 16859 | 19330 |
| FE   | 2092.6         | 11609 | 17033 | 19745 |
| FF   | 2398.4         | 11443 | 17260 | 19968 |
| FCR  | 12.93          | 2.18  | 2.32  | 3.20  |
| FER  | 12.75          | 1.45  | 1.32  | 1.12  |

Figure 5: Continued.
also has sufficient accuracy to analyze the horn-gear system composed of thicker plates.

When \( L/(R_1 + R_3) \) is less than ten, the accuracy of the horn’s eigenfrequencies become unacceptable if we use the classical one-dimensional beam theory. Thus, it is necessary to use three-dimensional beam theory to analyze the eigenfrequencies of the ultrasonic horn. It can be seen from Figure 5, the percentage of FCRs greater than 5% of each horn-gear system was low. This phenomenon indicated that the accuracy rate of frequencies obtained by Chebyshev–Ritz method were acceptable. The Chebyshev–Ritz method can be adopted for analyzing the horn-gear system consist of thick rod and thick plate. Because the percentage of FERs greater than 5% was low, the experimental results can be considered reasonable and reliable. In Figure 5, the FERs were mainly concentrated in the range of 0% to 2% and rarely beyond the range of 5%. It indicated that the system error of the experimental platform has little effect on the measurement accuracy of frequencies. And, among the total of 30 sets of data, 33% of the FCRs is no more than 2% and less than 7% of the FCRs is more than 5%. This phenomenon proved that the frequencies obtained by Chebyshev–Ritz method, and the finite element method are not much different. The accuracy of frequencies obtained by the two methods was basically at the same level.

### 4. Conclusion

In this paper, the three-dimensional coupling model of the horn-gear system is established, and the vibration characteristics are analyzed in three dimensions. The natural frequencies of the horn-gear system are achieved accurately through applying the Chebyshev–Ritz method. The frequencies are compared with the results obtained by the experiment of hammering method and the finite element method, respectively. It is proved that the Chebyshev–Ritz method based on the three-dimensional elasticity theory is feasible to analyze the vibration characteristics of the horn-gear system. The application of the Chebyshev–Ritz method further extends the traditional nonresonant theory of mixed using 1D and 2D to 3D analysis, which fully reflects the vibration characteristics of the horn-gear system and solves the mismatching of the coupling boundary conditions and the size limitation. This is proved by the index FCR that the percentage of FCR, in total, greater than 5% is low. Since Chebyshev–Ritz method is easy to parameterize, we only need to input the different parameters in the program to get different frequencies in different vibration modes by adjusting the value of \( s \). The comparison of frequencies obtained by the experiment shows that the frequencies obtained by the Chebyshev–Ritz method are reasonable and reliable. It realizes the three-dimensional analysis of the horn-gear system used in ultrasonic gear honing.

### Data Availability

The raw data and processed data required to reproduce these findings are available within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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