Sparse Equidistribution of Unipotent Orbits in Finite-Volume Quotients of PSL(2, \mathbb{R})

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Abstract

In this note, we consider the orbits \( \{ pu(n^{1+\gamma}) \mid n \in \mathbb{N} \} \) in \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \), where \( \Gamma \) is a non-uniform lattice in \( \text{PSL}(2, \mathbb{R}) \) and \( u(t) \) is the standard unipotent group in \( \text{PSL}(2, \mathbb{R}) \). Under a Diophantine condition on the initial point \( p \), we can prove that \( \{ pu(n^{1+\gamma}) \mid n \in \mathbb{N} \} \) is equidistributed in \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \) for small \( \gamma > 0 \), which generalizes the work of Venkatesh in [V10].

1 Introduction

The theory of equidistribution of unipotent flows on homogeneous spaces has been studied extensively over the past few decades. Furstenberg [F73] first proved that the unipotent flow on \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \), where \( \Gamma \) is a uniform lattice, is uniquely ergodic. In [D78] Dani classified ergodic invariant measures for unipotent flows on finite volume homogeneous spaces of \( \text{PSL}(2, \mathbb{R}) \), and using this result Dani and Smillie [DS84] proved that any non-periodic unipotent orbit is equidistributed on \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \) for any lattice \( \Gamma \). The proof of Oppenheim Conjecture due to Margulis [M89] by proving a special case of Raghunathan’s conjecture drew a lot of attention to this subject. Soon afterwards, Ratner published her seminal work [R90a, R90b, R91a] proving measure classification theorem for unipotent actions on homogeneous spaces as conjectured by Raghunathan and Dani [D81]. Using these results, Ratner [R91b] proved that any unipotent orbit in a finite volume homogeneous space is equidistributed in its orbit closure; see also Shah [Sh91] for the case of Rank-1 semisimple groups.

Ratner’s work has led to many new extensions and number theoretic applications of ergodic theory of unipotent flows. One of these results, which is related to this paper, was the work by Shah [Sh94]. In that paper, Shah asked whether \( \{ pu(n^2) \mid n \in \mathbb{N} \} \) is equidistributed in a sub-homogeneous space of \( \text{PSL}(2, \mathbb{Z}) \backslash \text{PSL}(2, \mathbb{R}) \), where \( u : \mathbb{R} \to \text{PSL}(2, \mathbb{R}) \) is the standard unipotent 1-parameter subgroup

\[
 u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]

In this direction, recently Venkatesh published a result about sparse equidistribution ([V10], Theorem 3.1). There he introduced a soft technique of calculations by using discrepancy trick, and proved that if \( \Gamma \) is a cocompact lattice in \( \text{PSL}(2, \mathbb{R}) \) and \( \gamma > 0 \) is small, then for any point \( p \in \Gamma \backslash \text{PSL}(2, \mathbb{R}) \) we have

\[
 \frac{1}{N} \sum_{n=0}^{N-1} f(pu(n^{1+\gamma})) \to \int_{\Gamma \backslash \text{PSL}(2, \mathbb{R})} f \, d\mu.
\]

In other words, in the case of \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \) being compact, the equidistribution holds for the sparse subset \( \{ n^{1+\gamma} \mid n \in \mathbb{N} \} \).
In this paper, we will consider the sparse subset \( \{ n^{1+\gamma} | n \in \mathbb{N} \} \) and orbits of \( \{ u(n^{1+\gamma}) | n \in \mathbb{N} \} \) in \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) \), where \( \Gamma \) is a non-uniform lattice. We want to prove a sparse equidistribution theorem similar to Shah’s conjecture \[\text{(Sh94)}\] and the work of Venkatesh \[\text{(V10)}\]. To deal with the complexity caused by initial points of unipotent orbits, we will introduce a Diophantine condition (see section 3). Now we state the main theorem in this paper.

**Theorem 1.1** (Main theorem). Let \( \Gamma \) be a non-uniform lattice in \( \text{PSL}(2, \mathbb{R}) \) and \( k \) the number of inequivalent cusps of \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) \). Suppose that \( p \in \Gamma \setminus \text{PSL}(2, \mathbb{R}) \) is Diophantine of type \( (\kappa_1, \kappa_2, \ldots, \kappa_k) \). Then there exists a constant \( \gamma_0 > 0 \) such that for any \( 0 < \gamma < \gamma_0 \), we have

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(pu(n^{1+\gamma})) \to \int_{\Gamma \setminus \text{PSL}(2, \mathbb{R})} f \, d\mu.
\]

Here the constant \( \gamma_0 \) depends on \( \kappa_1, \kappa_2, \ldots, \kappa_k \).

**Remark 1.1.** From the proof of the main theorem, we will see that the constant

\[
\gamma_0 = \min\{ \beta/(2\kappa_j + 8) | j = 1, 2, \ldots, k \}
\]

for some constant \( \beta > 0 \) which comes from the spectrum information of the Laplacian on \( \Gamma \setminus \mathbb{H} \).

Now let \( \Gamma \) be a subgroup of finite index of \( \text{PSL}(2, \mathbb{Z}) \). Then we have the following corollary of the main theorem, which will be explained in section 3.

**Corollary 1.1.** Let \( \Gamma \) be a subgroup of finite index of \( \text{PSL}(2, \mathbb{Z}) \). Let \( p = \Gamma g \in \Gamma \setminus \text{PSL}(2, \mathbb{R}) \) with

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

If \( a/c \in \mathbb{R} \) is a Diophantine number of type \( \zeta \); that is, there exists \( C > 0 \) such that for all \( m/n \in \mathbb{Q} \), we have

\[
|n|^\zeta \left| n \cdot \left( \frac{a}{c} \right) - m \right| \geq C,
\]

then the orbit \( \{ pu(n^{1+\gamma}) | n \in \mathbb{N} \} \) is equidistributed in \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) \) for \( 0 < \gamma < \gamma_0 := \beta/(2\zeta + 8) \).

To prove the main theorem, we shall use the technique of Venkatesh in \[\text{(V10)}\] and Strömbergsson’s result in \[\text{(S13)}\] about effective version of Ratner’s theorem on \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) \). However, since \( \Gamma \setminus G \) is not compact, all the effective results we obtain in this note will depend on initial points, and hence the estimates get out of control when we combine these results. To overcome this difficulty, we will introduce a Diophantine condition. With the help of this Diophantine condition along with the notion of \( (C, \alpha; \rho) \)-good functions, we will be able to control the rates of these effective results. In section 2, we list the concepts and theorems that we need in this paper. In section 3, we define the Diophantine condition we need in our proofs and deduce Corollary 1.1 from the main theorem. In section 4, we will study dynamics of a special class of orbits in \( \Gamma \setminus G \). The dynamical properties of these orbits will help us control the rates of the effective results in this paper. Since we are dealing with the noncompact case of \( \Gamma \setminus G \), and also for the sake of completeness, we include the technique of \[\text{(V10)}\] in section 5. We will finish the proof of the main theorem in section 6.

It may be interesting to explore the relation between the techniques used in this work and those developed in the work of Sarnak and Ubis \[\text{(SU14)}\], where they have described the limiting distribution of horocycles at primes.

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2 Prerequisites

Throughout this note, if there exists an absolute constant $C > 0$ such that $f \leq Cg$, then we write $f \ll g$. If $f \ll g$ and $g \ll f$, then we use the notation $f \sim g$. We denote $G = \text{PSL}(2, \mathbb{R})$ and $\Gamma$ a non-uniform lattice in $G$. Let

$$N = \{u(t) | t \in \mathbb{R}\}, \ A = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} | s \in \mathbb{R}_+ \right\}.$$

For any element $a \in A$, we denote $\alpha(a) = s$.

One of the ingredients in our calculations is the effective version of the mixing property of unipotent flows in $\Gamma \backslash G$. The following effective version is proved by Kleinbock and Margulis [KM99]. For $f \in C^k(\Gamma \backslash G)$ we let $\|f\|_{p,k}$ be the Sobolev $L^p$-norm involving all the Lie derivatives of order $\leq k$ of $f$. Note that $\|f\|_{\infty,0}$ is the supreme norm of $f$.

**Theorem 2.1** (Kleinbock and Margulis [KM99]). There exists $\kappa > 0$ such that for any $f, g \in C^\infty(\Gamma \backslash G)$, we have

$$\left| (u(t) \cdot f, g) - \int_{\Gamma \backslash G} f \int_{\Gamma \backslash G} g \right| \ll (1 + |t|)^{-\kappa} \|f\|_{\infty,1} \|g\|_{\infty,1}.$$

Here $(u(t) \cdot f)(x) = f(xu(t))$ is the right translation of $f$ by $u(t)$.

Another ingredient in the calculations is the effective version of Ratner’s theorem proved by Strömbergsson [S13]. To state the result, we will introduce some notations in [S13]. We know that $G$ acts on the upper half plane $\mathfrak{H}$ by the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

and we have the standard projection of $\Gamma \backslash G$ to the fundamental domain of $\Gamma$ in $\mathfrak{H}$

$$\pi : \Gamma \backslash G \rightarrow \Gamma \backslash \mathfrak{H}$$

by sending $\Gamma g$ to $\Gamma g(i)$. We define the geodesic flows on $\Gamma \backslash G$ by

$$g_t(\Gamma g) = \Gamma g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$ 

Fix, once for all, a point $p_0 \in \Gamma \backslash \mathfrak{H}$. For $p \in \Gamma \backslash G$ let

$$\text{dist}(p) = d_{\mathfrak{H}}(p_0, \pi(p))$$

where $d_{\mathfrak{H}}(\cdot, \cdot)$ is the hyperbolic distance on $\Gamma \backslash \mathfrak{H}$.

**Theorem 2.2** (Strömbergsson [S13]). For all $p \in \Gamma \backslash G$, $T \geq 10$, and all $f \in C^4(\Gamma \backslash G)$ such that $\|f\|_{2,4} < \infty$ and $\|f\|_{\infty,0} < \infty$

$$\left| \frac{1}{T} \int_0^T f(h_t(p)) dt - \int_{\Gamma \backslash G} f dp \right| \leq O(\|f\|_{2,4}) r^{-s} + O(\|f\|_{\infty,0}) r^{-\frac{1}{2}}$$

where $s > 0$ is a number depending on the spectrum of the Laplacian on $\Gamma \backslash \mathfrak{H}$ and $r = r(p, T) = T \cdot e^{-\text{dist}(g_0, T(p))}$. The implied constants depend only on $\Gamma$ and $p_0$. 


3 The Diophantine Condition

In this section, we will introduce a Diophantine condition for points in $\Gamma \backslash G$.

For $G = \text{PSL}(2, \mathbb{R})$, we consider the sets $N_\Omega A_\alpha K$ where

$$N_\Omega = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \text{ is in a bounded subset } \Omega \subset \mathbb{R} \right\}, \quad A_\alpha = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \mid s \geq \alpha \right\}$$

and $K = \text{SO}(2)$. As is known, for the non-uniform lattice $\Gamma$, there exist $\sigma_j \in G$ and a connected bounded subset $\Omega_j \subset \mathbb{R}$ ($1 \leq j \leq k$) with the following property:

1. For some $\alpha > 0$, $G = \bigcup \Gamma \sigma_j N_\Omega A_\alpha K$.
2. $\sigma_j^{-1} \Gamma \sigma_j \cap N$ is a cocompact lattice in $N$.
3. $N_\Omega$ is a fundamental domain of $\sigma_j^{-1} \Gamma \sigma_j \cap N \backslash N$.

We will fix $\sigma_j$ ($1 \leq j \leq k$). Note that in the upper half plane $\mathfrak{H}$, each $\sigma_j$ corresponds to a cusp $\eta_j$, i.e. $\sigma_j(i \infty) = \eta_j$, and $\eta_1, \eta_2, \ldots, \eta_k$ are the inequivalent cusps of $\Gamma \backslash \mathfrak{H}$. Let $\Gamma_j = \Gamma \cap \sigma_j N \sigma_j^{-1}$. Let $\pi_j$ be the covering map

$$\pi_j : \Gamma_j \backslash G \to \Gamma \backslash G.$$ 

Now consider the usual action of $G$ on $\mathbb{R}^2$ and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For any $q = \Gamma_j g \in \Gamma_j \backslash G$, we can define a map

$$m_j : \Gamma_j \backslash G \to \mathbb{R}^2 \setminus \{0\}$$

by

$$m_j(q) = g^{-1} \sigma_j e_1.$$ 

In this way, we obtain $k$ maps $m_j$ ($j = 1, 2, \ldots, k$) whose images are all in $\mathbb{R}^2 \setminus \{0\}$. For our purposes, we will identify a vector in $v \in \mathbb{R}^2$ with its opposite vector $-v$. Using these notations, we can give the following definition of Diophantine condition of a point $p \in \Gamma \backslash G$.

**Definition 3.1.** Let $p \in \Gamma \backslash G$. We say that $p$ is Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$ for some $\kappa_j > 0$ ($j = 1, 2, \ldots, k$) if for each $j$, there exist $\mu_j, \nu_j > 0$ such that for every point $(a, b) \in m_j(\pi_j^{-1}(p))$, we have $a = 0$ or $|b| \geq \mu_j$ or $|a|^{\kappa_j} |b| \geq \nu_j$.

It is straightforward to verify that if $g \in AN$ then the Diophantine types of $p$ and $pg$ are the same; although the choices of $\mu_j, \nu_j > 0$ in the above definition may differ. The Hausdorff dimension of the complement of the set of points of the Diophantine type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$ is given in Theorem [11].

Now we can deduce Corollary [11] from the main theorem.

**Proof of Corollary [11].** If $\Gamma$ is a subgroup of finite index of $\text{PSL}(2, \mathbb{Z})$, then we can pick $\sigma_j \in \text{PSL}(2, \mathbb{Z})$ ($1 \leq j \leq k$). Now let $p = \Gamma g \in \Gamma \backslash G$ with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Note that for each $m_j$, we have

$$m_j(\pi_j^{-1}(p)) \subseteq g^{-1} \mathbb{Z}^2 \setminus \{0\} = \left\{ \begin{pmatrix} dm - bn \\ -cm + an \end{pmatrix} \mid (m, n) \in \mathbb{Z}^2 \setminus \{0\} \right\}.$$
If there exist constants $\zeta > 0, \mu, \nu > 0$ such that for any $(m, n) \in \mathbb{Z}^2 \setminus \{0\}$
\[
dm - bn = 0 \text{ or } |an - cm| \geq \mu \text{ or } |dm - bn| \zeta |an - cm| \geq \nu,
\]
then $p$ is Diophantine of type $(\zeta, \ldots, \zeta)$ by the definition above. In particular, if $a/c \in \mathbb{R}$ is a Diophantine number of type $\zeta$, i.e. there exists $C > 0$ such that for $m/n \in \mathbb{Q}$,
\[
|n| \zeta \left| n \cdot \frac{a}{c} - m \right| \geq C,
\]
then condition (I) holds because when $|an - cm|$ is sufficiently small,
\[
|dm - bn| = \left| \frac{cdm - bcn}{c} \right| = \left| \frac{cdm - (ad - 1)n}{c} \right| = \left| \frac{d(cm - an) + n}{c} \right| \sim |n|.
\]

Hence, Corollary (I) follows from the main theorem. \hfill \Box

In order to prove the main theorem, we have to analyze the map $m_\jmath : \Gamma_j \backslash G \to \mathbb{R}^2$ for each $\jmath$. The following lemma is well known. The reader may refer to [DS 84]. We will denote $B_d$ the ball of radius $d$ around the origin in $\mathbb{R}^2$.

**Lemma 3.1** ([DS 84] Lemma 2.2). For each $\jmath$ with the maps $\pi_j : \Gamma_j \backslash G \to \Gamma \backslash G$ and $m_\jmath : \Gamma_j \backslash G \to \mathbb{R}^2 \setminus \{0\}$, there exists a constant $d_\jmath > 0$ such that for any $p \in \Gamma \backslash G$ there exists at most one point of $m_\jmath(\pi_j^{-1}(p))$ which lies in $B_{d_\jmath}$.

**Remark 3.1.** We will fix these $d_\jmath$ for $j = 1, 2, \ldots, k$ throughout this note.

**Definition 3.2.** For $p \in \Gamma \backslash G$, we define
\[
||p||_j := \min\{||(a, b)|| |(a, b) \in m_\jmath(\pi_j^{-1}(p))\}\]
where $|| \cdot ||$ denotes the standard Euclidean norm in $\mathbb{R}^2$. Moreover, we define
\[
d(p) = \min\{||p||_j | j = 1, 2, \ldots, k\}.
\]

**Lemma 3.2.** For any $p \in \Gamma \backslash SL(2, \mathbb{R})$, we have
\[
e^{\text{dist}(p)} \sim \frac{1}{d(p)^2}.
\]
\*Proof.* Recall that $\eta_j$ $(1 \leq j \leq k)$ are the inequivalent cusps of $\Gamma \backslash \mathcal{H}$. For each $1 \leq j \leq k$, we fix a small neighborhood $C_j$ of $\eta_j$ in $\Gamma \backslash G$ such that $C_1, C_2, \ldots, C_k$ are pairwise disjoint. We observe that it suffices to prove the lemma for $p \in C_j$ $(j = 1, 2, \ldots, k)$ since the complement of $\bigcup C_j$ is compact. Let $p \in C_j$ for some $j \in \{1, 2, \ldots, k\}$. Let $a_\jmath > 0$ be such that $\pi_j$ maps $\sigma_j N_{\Omega_j} A_{a_\jmath} K$ isomorphically to $C_j$. Then we can pick a representative for $p$ in $\sigma_j N_{\Omega_j} A_{a_\jmath} K$, say $\sigma_j n_p a_p k_p$. By definition we know that
\[
d(p) = ||p||_j = ||k_p^{-1} a_p^{-1} e_1|| = \alpha(a_p)^{-1}.
\]
On the other hand, in the upper half plane, the point corresponding to $\sigma_j n_p a_p k_p$ is equal to
\[
\sigma_j n_p a_p k_p \cdot i = \sigma_j (n_p a_p \cdot i) = \sigma_j (n_p \cdot (\alpha(a_p)^2 i)).
\]
Since $\sigma_j$ is fixed and $n_p$ is in the compact set $N_{\Omega_j}$ of $N$, we obtain
\[
\text{dist}(p) \sim \ln \alpha(a_p)^2
\]
and hence
\[
e^{\text{dist}(p)} \sim e^{\ln \alpha(a_p)^2} = \alpha(a_p)^2 = \frac{1}{d(p)^2}.
\]
\hfill \Box
4  \((C, \alpha; \rho)\)-good functions in presence of the Diophantine condition

This section will be important in the proof of the main theorem. First, we need a modified version of the concept of \((C, \alpha)\)-good functions (see [KM98] for the definition of \((C, \alpha)\)-good functions).

**Definition 4.1.** A function \(f(x)\) is said to be \((C, \alpha; \rho)\)-good if for sufficiently small \(\epsilon > 0\) and any \(I = (x_1, x_2) \subset [1, \infty)\) with \(|f(x_1)| = \rho\), we have

\[
m(\{x \in I : |f(x)| \leq \epsilon\}) \leq C \left(\frac{\epsilon}{\rho}\right)^\alpha m(I).
\]

Now we shall begin to study a special class of functions and prove that they are \((C, \alpha; \rho)\)-good for some \(C, \alpha\) and \(\rho > 0\). Note that we restrict these functions to the domain \([1, \infty)\).

**Lemma 4.1.** Let \(k, \mu, \nu > 0\) and \(0 < \gamma < \frac{1}{\pi + 4}\). Let \((a, b) \in \mathbb{R}^2 \setminus \{0\}\) be such that

\[a = 0 \text{ or } |b| \geq \mu \text{ or } |a|^\kappa |b| \geq \nu.
\]

Then there exists \(C > 0\) such that

\[f(x) = (bx^{\frac{1}{2} + \gamma} - ax^{-\frac{1}{4}})(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2 + (bx^{\frac{1}{4}})^2(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2\]

is \((C, \frac{1}{2}; \rho)\)-good on \([1, \infty)\), where \(\rho\) is any fixed constant \(\leq f(1)\). Here the constant \(C\) depends only on \(\rho, k, \mu, \nu\) and \(\gamma\).

**Proof.** We observe that if \(|b| \geq \mu\), then \(f(x) \geq (bx^{\frac{1}{4}})^2(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2 \geq b^2 \geq \mu^2\) and \(f(x)\) is automatically \((C, \alpha; \rho)\)-good for any \(C, \alpha, \rho\). Therefore, in the following we assume that \(|b| < \mu\). We have three cases: \(ab = 0\), \(ab < 0\) and \(ab > 0\).

Case 1: \(ab = 0\). In this case, by the definition of Diophantine condition, we know that \(a = 0\) and hence

\[f(x) = (bx^{\frac{1}{2} + \gamma})^2(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2 + (bx^{\frac{1}{4}})^2(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2.
\]

Note that \(f(1) \geq \rho\), and this function is positive and increasing. Then it is automatically \((C_1, \alpha_1)\)-good for any \(C_1, \alpha_1 > 0\).

Case 2: \(ab < 0\). This implies that \(|a|^\kappa |b| \geq \nu\). Then our function \(f(x)\) becomes

\[(|b|x^{\frac{1}{4} + \gamma} + |a|x^{-\frac{1}{4}})(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2 + (bx^{\frac{1}{4}})^2(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2.
\]

If \(|a| > 1\), we have

\[
\begin{align*}
&(|b|x^{\frac{1}{4} + \gamma} + |a|x^{-\frac{1}{4}})(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2 + (bx^{\frac{1}{4}})^2(x^{\frac{1}{4} - \frac{1}{\pi + 4}})^2 \\
&= |b|x^{1 + \gamma - \frac{1}{\pi + 4}} + |a|x^{-\frac{1}{\pi + 4}} \\
&= |b|x^{1 + \gamma - \frac{1}{\pi + 4}} + \sum |\frac{a}{k}|x^{-\frac{1}{\pi + 4}} + \cdots + |\frac{a}{k}|^\kappa x^{-\frac{1}{\pi + 4}} \\
&\geq ([\kappa] + 1)^{[\kappa] + 1} \sqrt{[\kappa]|k|^\kappa}[b]^k|a|^{[\kappa]}x^{1 + \gamma - \frac{1}{\pi + 4}} \\
&= \frac{[\kappa] + 1}{[\kappa] + 1} \sqrt{[\kappa]|k|^\kappa}[b]^k|a|^{[\kappa]}x^{1 + \gamma - \frac{1}{\pi + 4}}.
\end{align*}
\]
\[ \geq \frac{[\kappa] + 1}{[\kappa + 1]^{\frac{1}{[\kappa + 1]}}} \sqrt{\nu}. \]

If \(|a| \leq 1\), by the assumption, we have \(|b| \geq \nu\) and hence \(f(x) \geq (bx\frac{3}{4} - ax\frac{1}{4})^2 \geq \nu^2\). Either way, we get \(f(x) \geq M\) for some absolute constant \(M > 0\) depending only on \(\kappa, \nu\). Again, this implies that \(f(x)\) is \((C_2, \alpha_2)\)-good for any \(C_2, \alpha_2 > 0\).

Case 3: \(ab > 0\). Without loss of generality, we assume that \(a > 0, b > 0\). Now let \(I = (x_1, x_2) \subseteq [1, \infty)\) be an interval \((x_1, x_2)\) where \(f(x_1) = \rho\). Since \(f(x_1) = \rho\), we know that

\[
\text{either } (bx_1^{\frac{3}{4}} - ax_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}} - x_2^{\frac{1}{4}})^2 \geq \frac{\rho}{2} \text{ or } (bx_1^{\frac{3}{4}} - ax_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}} - x_2^{\frac{1}{4}})^2 \geq \frac{\rho}{2}.
\]

If \((bx_1^{\frac{3}{4}} - ax_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}} - x_2^{\frac{1}{4}})^2 \geq \rho/2\), then there is nothing to prove because \(f(x) \geq \rho/2\) for all \(x \geq x_1\).

Otherwise, since \(x_1 \geq 1\) we have

\[
b \leq \sqrt[3]{\rho/2} \quad \text{and} \quad (bx_1^{\frac{3}{4}} - ax_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}} - x_2^{\frac{1}{4}})^2 \geq \frac{\rho}{2}.
\]

Note that

\[
\{x \in I \mid |f(x)| \leq \epsilon\} \subseteq \{x \in I \mid (bx_1^{\frac{3}{4}} - ax_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}} - x_2^{\frac{1}{4}})^2 \leq \epsilon\}.
\]

Therefore, to finish the proof of the lemma, it suffices to show that

\[
\frac{1}{x_2 - x_1} m(\{x \in (x_1, x_2) \mid |g(x)| \leq \sqrt[3]{\epsilon}\}) \leq C \left(\frac{\epsilon}{\rho}\right)^{\frac{1}{2}}
\]

where \(g(x) = (bx_1^{\frac{3}{4}} - ax_1^{\frac{1}{4}})x_1^{\frac{1}{4}} - x_2^{\frac{1}{4}} = bx_1^{1+\gamma} - ax_1^{\frac{1}{4}} - ax_1^{-\frac{1}{4}}\). Note that \(g(x)\) is increasing and \(|g(x_1)| \geq \sqrt[3]{\rho/2}\). Without loss of generality, we may assume that \(|g(x_1)| = \sqrt[3]{\rho/2}\). If \(g(x_1) = \sqrt[3]{\rho/2}\), since \(g(x)\) is increasing, the \((C, \alpha)\)-good property automatically holds in this case. Otherwise, we have \(g(x_1) = -\sqrt[3]{\rho/2}\). Since \(g(x)\) is increasing, the maximum of the ratio

\[
\frac{1}{x_2 - x_1} m(\{x \in (x_1, x_2) \mid |g(x)| \leq \sqrt[3]{\epsilon}\})
\]

occurs where \(g(x_2) = \sqrt[3]{\epsilon}\). So we will assume that \(g(x_2) = \sqrt[3]{\epsilon}\). To compute this maximal ratio, let \(z \in (x_1, x_2)\) such that \(g(z) = -\sqrt[3]{\epsilon}\), and by mean value theorem we obtain

\[
\frac{1}{x_2 - x_1} m(\{x \in (x_1, x_2) \mid |g(x)| \leq \sqrt[3]{\epsilon}\}) = \frac{x_2 - z}{x_2 - x_1} \frac{g'(\xi_2) \cdot g(x_2) - g(z)}{g'(\xi_1) \cdot g(x_2) - g(x_1)} = \frac{g'(\xi_2)}{g'(\xi_1)} \cdot \frac{2\sqrt[3]{\epsilon}}{\sqrt[3]{\epsilon + \sqrt[3]{\rho/2}}}
\]

where \(\xi_1\) is between \(x_2\) and \(z\), \(\xi_2\) is between \(x_1\) and \(x_2\).

Let \(x_3 \in [1, \infty)\) such that \(g(x_3) = \sqrt[3]{\rho/2}\). Then \((x_1, x_2) \subseteq (x_1, x_3)\). According to the discussion above, to prove formula (2), it suffices to prove that for any \(x, y \in (x_1, x_3)\) the ratio

\[
g'(x)/g'(y)
\]
is bounded above by constants depending only on \( \rho, \kappa, \mu, \nu \) and \( \gamma \). Observe that

\[
g'(x) = \left( 1 + \gamma - \frac{1}{\kappa + 4} \right) bx^{\gamma - \frac{1}{\kappa + 4}} + \frac{a}{\kappa + 4}x^{-\frac{\kappa+5}{\kappa + 4}}
\]
is decreasing. Therefore we only need to estimate the following ratio

\[
\frac{g'(x_1)}{g'(x_3)} = \frac{\left( 1 + \gamma - \frac{1}{\kappa + 4} \right) bx_1^{\gamma - \frac{1}{\kappa + 4}} + \frac{a}{\kappa + 4}x_1^{-\frac{\kappa+5}{\kappa + 4}}}{\left( 1 + \gamma - \frac{1}{\kappa + 4} \right) bx_3^{\gamma - \frac{1}{\kappa + 4}} + \frac{a}{\kappa + 4}x_3^{-\frac{\kappa+5}{\kappa + 4}}}
\]

\[
= \frac{\left( 1 + \gamma - \frac{1}{\kappa + 4} \right) bx_1^{\gamma - \frac{1}{\kappa + 4}}/x_1 + \frac{a}{\kappa + 4}x_1^{-\frac{\kappa+5}{\kappa + 4}}}{\left( 1 + \gamma - \frac{1}{\kappa + 4} \right) bx_3^{\gamma - \frac{1}{\kappa + 4}}/x_3 + \frac{a}{\kappa + 4}x_3^{-\frac{\kappa+5}{\kappa + 4}}}
\]

\[
\leq \frac{\left( 1 + \gamma - \frac{1}{\kappa + 4} \right) ax_1^{\gamma - \frac{1}{\kappa + 4}}/x_1 + \frac{a}{\kappa + 4}x_1^{-\frac{\kappa+5}{\kappa + 4}}}{\left( 1 + \gamma - \frac{1}{\kappa + 4} \right) ax_3^{\gamma - \frac{1}{\kappa + 4}}/x_3 + \frac{a}{\kappa + 4}x_3^{-\frac{\kappa+5}{\kappa + 4}}}
\]

Now let \( x_0 \in (x_1, x_2) \) such that \( g(x_0) = 0 \). \( (x_0 = (a/b)^{\frac{1}{\kappa + 4}} \) by solving the equation \( g(x) = 0 \). We set \( x_1 = x_0 + \delta_1 x_0 \) and \( x_3 = x_0 + \delta_2 x_0 \) for some \( \delta_1, \delta_2 \). Then \( \delta_1, \delta_2 \) satisfy the following equation

\[
|b(x_0 + x_0 \delta)^{1+\gamma - \frac{1}{\kappa + 4}} - a(x_0 + x_0 \delta)^{-\frac{1}{\kappa + 4}}| = \sqrt{\rho / 2}.
\]

For this equation, since \( bx_0^{1+\gamma - 1/(\kappa + 4)} = ax_0^{-1/(\kappa + 4)} \) and \( ba^\kappa \geq \nu \), we have

\[
|ax_0^{\gamma - \frac{1}{\kappa + 4}} (1 + \delta)^{1+\gamma - \frac{1}{\kappa + 4}} - ax_0^{-\frac{1}{\kappa + 4}} (1 + \delta)^{-\frac{1}{\kappa + 4}}| = \sqrt{\rho / 2}
\]

\[
|(1 + \delta)^{1+\gamma - \frac{1}{\kappa + 4}} - (1 + \delta)^{-\frac{1}{\kappa + 4}}| = \sqrt{\frac{\rho}{2} \frac{1}{a}} = \sqrt{\frac{\rho}{2} \frac{a+1 + \kappa a^\kappa}{(1 + \gamma)(\kappa + 4)}}
\]

Here \( \frac{1}{(1 + \gamma)(\kappa + 4)} < 1 \). Since \( b \leq \sqrt{\rho/2} \) and hence \( a \geq \sqrt{\sqrt{2}/\nu} \), the above inequality implies that \( |(1 + \delta)^{1+\gamma - \frac{1}{\kappa + 4}} - (1 + \delta)^{-\frac{1}{\kappa + 4}}| \) is bounded above by a constant, which shows that \( \delta_1, \delta_2 \) are also bounded by constants. Therefore \( g'(x_1)/g'(x_3) \leq (x_3/x_1)^{(\kappa+5)/(\kappa + 4)} = (1 + \delta_2)/(1 + \delta_1)^{(\kappa+5)/(\kappa + 4)} \) is bounded above by a constant. Note that these constants depend only on \( \rho, \kappa, \mu, \nu \) and \( \gamma \). This completes the proof of the lemma.

\[
\square
\]

For the rest of this section, we turn to the dynamics on \( \Gamma \setminus G \). For later use, we give the following definition.
Definition 4.2. For any $\delta > 0$ and any $j \in \{1, 2, \ldots, k\}$, we define the subset of $\Gamma \setminus G$

$$S_{j,\delta} := \{q \in \Gamma \setminus G | \|q\|_j \leq \delta\}.$$  

Moreover, we define

$$S_{\delta} := \bigcup_j S_{j,\delta} = \{q \in \Gamma \setminus G | d(q) \leq \delta\}.$$ 

Lemma 4.2. Let $p \in \Gamma \setminus G$ be Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$. We fix $j \in \{1, 2, \ldots, k\}$ and let $0 < \gamma < 1/(\kappa_j + 4)$. Consider the following orbit

$$\left\{ p \left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right) \bigg| x \geq 1 \right\}. $$

Let $\rho = \min \left\{ d \left( p \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \right), d_1, d_2, \ldots, d_k \right\}$. Then for sufficiently small $\epsilon > 0$, we have

$$\frac{1}{T} m \left( \left\{ x \in [1, T] | p \left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right) \in S_{j,\epsilon x^{-\frac{1}{4} + \frac{1}{\kappa_j + 4}}} \right\} \right) \leq C \left( \frac{\epsilon}{\rho} \right) T,$$

where $C$ is as in the lemma 4.1.

Proof. We consider the model in section 2 with the maps $m_j$ and $\pi_j$. Then the image of $p \left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right)$ under $m_j$ is equal to

$$\left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right)^{-1} m_j(\pi_j^{-1}(p)) = \left\{ \left( x^\frac{1}{4} - x^\frac{3}{4} + \gamma, x^\frac{1}{4} \right) \left( \begin{array}{cc} a \\ b \end{array} \right) \right\} = \left\{ \left( \frac{ax^\frac{1}{4} - bx^\frac{3}{4} + \gamma}{bx^\frac{1}{4}} \right) \right\}$$

where $(a, b)$ runs over all points in $m_j(\pi_j^{-1}(p))$. By definition, what we need to prove is equivalent to the following

$$m \left( \left\{ x \in [1, T] \bigg| p \left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right) \bigg|_{j} \leq \epsilon x^{-\frac{1}{4} + \frac{1}{\kappa_j + 4}} \right\} \right) \leq C \left( \frac{\epsilon}{\rho} \right) T,$$

which is equivalent to the following

$$m \left( \left\{ x \in [1, T] \bigg| \exists \text{ a point in } m_j \left( \pi_j^{-1} \left( p \left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right) \right) \right) \text{ with length } \leq \epsilon x^{-\frac{1}{4} + \frac{1}{\kappa_j + 4}} \right\} \right) \leq \epsilon \left( \frac{\epsilon}{\rho} \right) T.$$

We denote by $P$ the subset $m_j(\pi_j^{-1}(p))$. For $(a, b) \in P$, let $I^j_{(a, b)}(j = 1, 2, \ldots)$ be all the maximal connected subintervals in $[1, T]$ such that for any $x \in I^j_{(a, b)}$ the point of $p \left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right)$ corresponding to $(a, b)$

$$\left( \begin{array}{cc} x^\frac{1}{4} & x^\frac{3}{4} + \gamma \\ 0 & x^\frac{1}{4} \end{array} \right)^{-1} \left( \begin{array}{cc} a \\ b \end{array} \right) = \left( \begin{array}{cc} ax^{-\frac{1}{4}} - bx^\frac{3}{4} + \gamma \\ bx^\frac{1}{4} \end{array} \right).$$
has norm $\leq px^{-\frac{1}{4}+\frac{1}{\kappa_j+1}}$. Since $x \geq 1$, Lemma 3.1 implies that all the intervals

$$\{I_j^{(a,b)}|(a, b) \in P, j = 1, 2, \ldots\}$$

are pairwise disjoint. Therefore, we have

$$m \left( \left\{ x \in [1, T] \mid \exists \text{ a point in } m_j \left( \frac{n^{\frac{1}{4}+\gamma}}{n^{\frac{1}{4}}} \right) \right\} \right) \leq \epsilon x^{-\frac{1}{4}+\frac{1}{\kappa_j+1}}$$

Because of this, to prove the lemma, it suffices to prove the following

$$m \left( \left\{ x \in I_j^{(a,b)} \left| \left( a x^{-\frac{1}{4}} - b x^{\frac{1}{\kappa_j+1}} \right) \right| \leq \epsilon x^{-\frac{1}{4}+\frac{1}{\kappa_j+1}} \right\} \right) \leq C \left( \frac{\epsilon}{\rho} \right) m(I_j^{(a,b)})$$

or to prove that the function

$$f(x) = (b x^{\frac{1}{\kappa_j+1}} - a x^{-\frac{1}{4}})^2 (x^{\frac{1}{\kappa_j+1}})^2 + (b x^{\frac{1}{\kappa_j+1}})^2 (x^{\frac{1}{\kappa_j+1}})^2$$

has $(C, 1/2; \rho)$-good property. This follows immediately from Lemma 4.1.

To conclude this section, we give the following proposition, which is crucial in our proof of the main theorem. It is the discrete version of Lemma 4.2.

**Proposition 4.1.** Let $p \in \Gamma \setminus G$ be Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$. Let $0 < \gamma < \min\{1/(\kappa_j+4) : j = 1, 2, \ldots, k\}$. Consider the following orbit

$$\left\{ p \left( \begin{array}{cc} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{array} \right) \mid n \in \mathbb{N} \right\}.$$

Then there exists a constant $C_0 > 0$ depending only on $p$ and $\gamma$ such that for any $\epsilon > 0$,

$$\frac{1}{N} \left| \left\{ n \in [1, N] \cap \mathbb{N} \mid p \left( \begin{array}{cc} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{array} \right) \in S_{\theta(n)} \right\} \right| \leq C_0 \left( \frac{\epsilon}{\rho} \right)$$

where

$$\theta(x) = \epsilon \min \{ x^{-\frac{1}{4}+\frac{1}{\kappa_j+1}} \mid j = 1, 2, \ldots, k \}. \quad (3)$$

**Proof.** By the definition of $S_\theta$, it suffices to prove that there exists a constant $C_0 > 0$ depending only on $p$ and $\gamma$ such that for each $j$ and any $\epsilon > 0$,

$$\frac{1}{N} \left| \left\{ n \in [1, N] \cap \mathbb{N} \mid p \left( \begin{array}{cc} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{array} \right) \in S_{\theta(n) - \frac{1}{4}+\frac{1}{\kappa_j+1}} \right\} \right| \leq C_0 \left( \frac{\epsilon}{\rho} \right).$$

We compute that for any $\delta \in (-1, 1)$ and $n \geq 1$

$$\left( \begin{array}{cc} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{array} \right)^{-1} \left( \begin{array}{cc} (n + \delta)^{\frac{1}{4}} & (n + \delta)^{\frac{3}{4}+\gamma} \\ 0 & (n + \delta)^{-\frac{1}{4}} \end{array} \right)$$

10
\[
\begin{pmatrix}
  n^{-\frac{1}{4}} & -n^{\frac{3}{4}+\gamma} \\
  0 & n^{\frac{1}{4}}
\end{pmatrix}
\begin{pmatrix}
  (n + \delta)^{\frac{1}{4}} & (n + \delta)^{\frac{3}{4}+\gamma} \\
  0 & (n + \delta)^{-\frac{1}{4}}
\end{pmatrix}
= \\
\begin{pmatrix}
  (1 + \delta/n)^{\frac{1}{4}} & n^{-\frac{1}{4}}(n + \delta)^{\frac{3}{4}+\gamma} - n^{\frac{3}{4}+\gamma}(n + \delta)^{-\frac{1}{4}} \\
  0 & (1 + \delta/n)^{-\frac{1}{4}}
\end{pmatrix}
= \\
\begin{pmatrix}
  (1 + \delta/n)^{\frac{1}{4}} & (n + \delta)^{1+\gamma} - n^{1+\gamma}(n(n + \delta))^{-\frac{1}{4}} \\
  0 & (1 + \delta/n)^{-\frac{1}{4}}
\end{pmatrix}
\]
which lies in a compact neighborhood \(U\) of identity in \(\text{PSL}(2, \mathbb{R})\). Let
\[
L = \max\{\|g\| \mid g \in U\}
\]
where \(\|g\|\) denotes the operator norm of \(g\) on \(\mathbb{R}^2\). Then by the computations above, we know that
\[
\frac{1}{N} \left| \left\{ n \in [1, N] \cap \mathbb{N} \mid \frac{1}{n^{\frac{1}{4}}} \begin{pmatrix}
  n^{\frac{3}{4}+\gamma} \\
  0
\end{pmatrix} \in S_{j,\text{lex}}^{n^{\frac{1}{4}+\frac{1}{4}+\gamma}} \right\} \right| \leq \frac{1}{N} \left| \left\{ x \in [1, N] \mid \frac{1}{x^{\frac{1}{4}}} \begin{pmatrix}
  x^{\frac{3}{4}+\gamma} \\
  0
\end{pmatrix} \in S_{j,\text{lex}}^{x^{\frac{1}{4}+\frac{1}{4}+\gamma}} \right\} \right|.
\]
Now the proposition follows immediately from Lemma 4.2.

5 Calculations

In this section, we shall apply the technique of Venkatesh to obtain some effective results about averaging over arithmetic progressions. It is very similar to [V10], where Venkatesh proved the sparse equidistribution theorem for \(\Gamma\) being cocompact. Since in our setting \(\Gamma\) is non-uniform, and for the sake of self-containedness, we include the details of the calculations in this section. We will follow the notations in [V10]. Throughout this section, we fix an arbitrary point \(q \in \Gamma \setminus G\). For a character \(\psi : \mathbb{R} \to S^1\), we define
\[
\mu_{T, \psi}(f) = \frac{1}{T} \int_0^T \psi(t) f(qu(t)) dt
\]
for \(f\) on \(\Gamma \setminus G\).

Lemma 5.1. Suppose that \(\int_{\Gamma \setminus G} f \, d\mu = 0\). Then there exists a constant \(\beta > 0\) such that
\[
|\mu_{T, \psi}(f)| \ll r^{-\beta} \|f\|_{\infty, 4}.
\]
Here
\[
r = r(q, T) = T \cdot e^{-\text{dist}(q, \mathbb{R})}
\]
and the implicit constant is independent of \(\psi\).

Proof. (C.f. [V10] Lemma 3.1) We define
\[
\sigma_H(f)(x) = \frac{1}{H} \int_0^H \psi(s) f(xu(s)) ds.
\]
We estimate $\mu_{T,\psi}(f)$ as follows. First, we compute $\mu_{T,\psi}(\sigma_H(f))$.

\[
\mu_{T,\psi}(\sigma_H(f)) = \frac{1}{T} \int_0^T \psi(t)\sigma_H(f)(q(t))dt = \frac{1}{T} \int_0^T \psi(t) \frac{1}{H} \int_0^H \psi(s)f(q(t)u(s))dsdt = \frac{1}{TH} \int_0^H \int_s^{T+s} \psi(t)f(q(t))dt ds.
\]

Therefore the difference between $\mu_{T,\psi}(f)$ and $\mu_{T,\psi}(\sigma_H(f))$ is equal to

\[
|\mu_{T,\psi}(f) - \mu_{T,\psi}(\sigma_H(f))| = \left| \frac{1}{T} \int_0^T \psi(t)f(xu(t))dt - \frac{1}{TH} \int_0^H \int_s^{T+s} \psi(t)f(q(t))dt ds \right|
\]

\[
= \left| \frac{1}{TH} \int_0^H \int_0^T \psi(t)f(xu(t))dt ds - \frac{1}{TH} \int_0^H \int_s^{T+s} \psi(t)f(q(t))dt ds \right|
\]

\[
\leq \frac{1}{TH} \int_0^H \int_0^s \psi(t)f(xu(t))dt ds + \frac{1}{TH} \int_0^H \int_s^{T+s} \psi(t)f(xu(t))dt ds \leq \frac{H}{T} \|f\|_{\infty,0}.
\]

Then we estimate $\mu_{T,\psi}(\sigma_H(f))$.

\[
|\mu_{T,\psi}(\sigma_H(f))| = \left| \frac{1}{T} \int_0^T \psi(t)\sigma_H(f)(q(t))dt \right|
\]

\[
\leq \frac{1}{T} \left( \int_0^T |\psi(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |\sigma_H(f)(q(t))|^2 dt \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{1}{T} \int_0^T |\sigma_H(f)(q(t))|^2 dt \right)^{\frac{1}{2}}.
\]

Now we estimate $\frac{1}{T} \int_0^T |\sigma_H(f)(q(t))|^2 dt$ as follows.

\[
\left| \frac{1}{T} \int_0^T |\sigma_H(f)(q(t))|^2 dt \right|
\]

\[
= \left| \frac{1}{T} \int_0^T \left( \frac{1}{H} \int_0^H \psi(s)f(q(t)u(s))ds \right)^2 dt \right|
\]

\[
= \left| \frac{1}{T} \int_0^T \left( \frac{1}{H} \int_0^H \psi(y)f(q(t)u(y))dy \right) \left( \frac{1}{H} \int_0^H \psi(z)f(q(t)u(z))dz \right) dt \right|
\]

\[
= \left| \frac{1}{H^2} \int_0^H \int_0^H \psi(z-y) \left( \frac{1}{T} \int_0^T f(q(t)u(y))f(q(t)u(z))dt \right) dy dz \right|
\]

\[
\leq \frac{1}{H^2} \int_0^H \int_0^H \left| \frac{1}{T} \int_0^T f^y(z)q(t)dt \right| dy dz.
\]

Here $f^y$ and $f^z$ denote the right translation of $f$ by $u(y)$ and the right translation of $f$ by $u(z)$, respectively. According to Strömbergsson’s effective version of Ratner’s theorem for $\text{PSL}(2,\mathbb{R})$, we
know that
\[
\left| \frac{1}{T} \int_0^T \overline{f^y} f^z(qu(t)) dt - \int_{\Gamma \setminus G} \overline{f} f^z d\mu \right| \leq O(\|\overline{f^y} f^z\|_{\infty, A}) r^{-\alpha}
\]
for some \( \alpha > 0 \) and \( r \) is as in (1) (see Theorem 2.2). Hence we have
\[
\left| \frac{1}{T} \int_0^T |\sigma_H(f)(qu(t))|^2 dt \right| 
\leq \frac{1}{H^2} \int_0^H \int_0^H \left( \frac{1}{T} \int_0^T \overline{f^y} f^z(qu(t)) dt - \int_{\Gamma \setminus G} \overline{f} f^z d\mu \right) dydz 
\leq \frac{1}{H^2} \int_0^H \int_0^H O(\|\overline{f^y} f^z\|_{\infty, A}) r^{-\alpha} dydz + \frac{1}{H^2} \int_0^H \int_0^H |(f^{y-z}, f)| dydz.
\]
By mixing property of unipotent flows (Theorem 2.1), we know that
\[(f^{h}, f) \ll (1 + |h|)^{-\kappa} \|f\|_{\infty, 1}^2.\]
Also by product rule and chain rule in Calculus (see [V10] Lemma 2.2 for details), we know that
\[O(\|\overline{f^y} f^z\|_{\infty, A}) \ll O(\|\overline{f^y}\|_{\infty, A} \|f^z\|_{\infty, A}) \ll y^4 z^4 O(\|f\|_{\infty, A}^2).\]
Therefore
\[
\left| \frac{1}{T} \int_0^T |\sigma_H(f)(qu(t))|^2 dt \right| 
\ll \frac{1}{H^2} \int_0^H \int_0^H O(\|\overline{f^y} f^z\|_{\infty, A}) r^{-\alpha} dydz + \frac{1}{H^2} \int_0^H \int_0^H |(f^{y-z}, f)| dydz 
\ll (H^{8r^{-\alpha}} + H^{-\kappa}) \|f\|_{\infty, A}^2
\]
Combining the computations above, we obtain
\[
|\mu_{T, \psi}(f)| \ll |\mu_{T, \psi}(f) - \mu_{T, \psi}(\sigma_H(f))| + |\mu_{T, \psi}(\sigma_H(f))| 
\ll \frac{H}{T} \|f\|_{\infty} + (H^8 r^{-\alpha} + H^{-\kappa})^2 \|f\|_{\infty, A} 
\ll \frac{H}{r} \|f\|_{\infty} + (H^8 r^{-\alpha} + H^{-\kappa})^2 \|f\|_{\infty, A}.
\]
Let \( H \) be a small positive power of \( r \) and we complete the proof of the lemma.

We deduce the following proposition from Lemma 5.1. It will be used in the proof of the main theorem in section 6.

**Proposition 5.1.** Let \( T > K > 2 \) and \( f \in C^\infty(\Gamma \setminus G) \) be such that \( \int_{\Gamma \setminus G} f d\mu = 0 \). Then we have
\[
\left| \frac{1}{T/K} \sum_{0 \leq K_j < T} f(qu(K_j)) \right| \ll \left( \frac{1}{K} + \frac{K^4}{r^\beta} + \frac{1}{r^\beta} \right) \|f\|_{\infty, A}.
\]
Proof. (C.f. [V10] Theorem 3.1) Let $\delta > 0$ and $g_\delta(x) = \max\{\delta^{-2}(\delta - |x|), 0\}$. Let

$$g(x) = \sum_{j \in \mathbb{Z}} g_\delta(x + Kj).$$

For any $T > 0$, we consider the following

$$\int_0^T g(t)f(qu(t))dt.$$  

On the one hand, since $g(x)$ has most mass on the points $\{Kj|j \in \mathbb{Z}\}$, we know that

$$\left| \int_0^T g(t)f(qu(t))dt - \sum_{0 \leq Kj < T} f(qu(Kj)) \right| \leq \frac{T}{K} \delta \|f\|_{\infty,1}.$$  

On the other hand, since $g(x)$ is periodic, we have the Fourier expansion

$$g(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi ikx/K}.$$  

A simple calculation shows that

$$\sum_{k \in \mathbb{Z}} |a_k| \leq \frac{K}{\delta^2} + \frac{1}{K}.$$  

By Lemma 5.1 with characters $\psi_k = e^{2\pi ikx/K}$, we have

$$\left| \int_0^T g(t)f(qu(t))dt \right| \leq \sum_{k \in \mathbb{Z}} |a_k| \left| \int_0^T e^{2\pi ikx/K} f(qu(t))dt \right| \ll \left( \frac{K}{\delta^2} + \frac{1}{K} \right) \frac{T}{r^\beta} \|f\|_{\infty,4}.$$  

Combining the calculations above, we have

$$\left| \sum_{0 \leq Kj < T} f(qu(Kj)) \right| \ll \left( \frac{T}{K} \delta + \left( \frac{K}{\delta^2} + \frac{1}{K} \right) \frac{T}{r^\beta} \right) \|f\|_{\infty,4}$$  

and hence

$$\left| \frac{1}{T/K} \sum_{0 \leq Kj < T} f(qu(Kj)) \right| \ll \left( \delta + \frac{K^2}{\delta^2 r^\beta} + \frac{1}{r^\beta} \right) \|f\|_{\infty,4}.$$  

Let $\delta = 1/K$ and we complete the proof of the proposition. \[\square\]
6 Proof of the Main Theorem

Proof of the main theorem. Without loss of generality, we assume
\[ \int_{\Gamma \setminus G} f \, d\mu = 0. \]

We want to find \( \gamma_0 > 0 \) depending on \( \kappa_1, \ldots, \kappa_k \) such that for any \( 0 < \gamma < \gamma_0 \), the main theorem holds. Note that by Taylor expansion, for any \( M \in \mathbb{N} \) and \( k \in \mathbb{N} \),
\[ (M + k)^{1+\gamma} = M^{1+\gamma} + (1 + \gamma) M^\gamma k + O(M^{\gamma-1} k^2). \]

Therefore, if \( M \) is sufficiently large and \( \gamma < 1/2 \), then the sequence
\[ \left\{ (M + k)^{1+\gamma} \mid 0 \leq k \leq \frac{1}{1+\gamma} M^{\frac{1}{2}-\gamma} \right\} \]
is approximately equal to the arithmetic progression
\[ \left\{ M^{1+\gamma} + (1 + \gamma) M^\gamma k \mid 0 \leq k \leq \frac{1}{1+\gamma} M^{\frac{1}{2}-\gamma} \right\} \]
since
\[ O(M^{\gamma-1} k^2) \leq O(M^{\gamma-1}(M^{\frac{1}{2}-\gamma})^2) = O(M^{-\gamma}) \to 0 \]
as \( M \to \infty \).

By Proposition 4.1 we know that for any \( \epsilon > 0 \) and any \( N > 0 \),
\[ \frac{1}{N} \left| \left\{ n \in [1, N] \mid p \left( \begin{array}{cc} \frac{n}{M} & \frac{n^{\frac{4}{3}} + \gamma}{M} \\ 0 & \frac{n^{-\frac{4}{3}} + \gamma}{M} \end{array} \right) \in S_{\theta(n)} \right\} \right| \leq C_0 \left( \frac{\epsilon}{\rho} \right), \]
where \( \theta(n) = \epsilon \min \{ n^{-\frac{4}{3} + \gamma} \mid j = 1, 2, \ldots, k \} \). Set
\[ B = \left\{ n \in \mathbb{N} \mid p \left( \begin{array}{cc} \frac{n^{\frac{1}{2}}}{M} & \frac{n^{\frac{4}{3} + \gamma}}{M} \\ 0 & \frac{n^{-\frac{4}{3}}}{M} \end{array} \right) \in S^c_{\theta(n)} \right\}. \]

We proceed as follows. We pick the first element \( M_1 \in \mathbb{N} \) which lies in \( B \). Then we take
\[ P_1 = \left\{ M_1 + k \mid 0 \leq k \leq \frac{1}{1+\gamma} M_1^{\frac{1}{2}-\gamma} \right\}. \]

Next we pick the first element \( M_2 \in \mathbb{N} \) which appears after \( P_1 \) and lies in \( B \), and we take
\[ P_2 = \left\{ M_2 + k \mid 0 \leq k \leq \frac{1}{1+\gamma} M_2^{\frac{1}{2}-\gamma} \right\}. \]

Then we pick the first element \( M_3 \in \mathbb{N} \) which appears after \( P_2 \) and lies in \( B \), and so on. In this manner, we get pieces \( P_1, P_2, \ldots \) in \( \mathbb{N} \) and by our choices of \( M_1, M_2, \ldots \), we know that
\[ B \subset P_1 \cup P_2 \cup \ldots \]
and hence for any \( N > 0 \)
\[
\frac{1}{N} \left| [1, N] \setminus (P_1 \cup P_2 \cup \ldots) \right| \leq C_0 \left( \frac{\epsilon}{\rho} \right). \tag{5}
\]

Now we consider each of the pieces \( P_i \). From the discussion above, we know that \( \{n^{1+\gamma} | n \in P_i\} \) is approximated by the arithmetic progression
\[
\tilde{P}_i = \left\{ M_i^{1+\gamma} + (1 + \gamma)M_i^\gamma k \; \big| \; 0 \leq k \leq \frac{1}{1+\gamma}M_i^{\frac{1}{\gamma}+\gamma} \right\}.
\]
By Proposition \(5.1\) with \( T = M_i^{1/2} \) and \( K = (1 + \gamma)M_i^\gamma \), we have
\[
\left| \frac{1}{|P_i|} \sum_{n \in P_i} f(pu(n)) \right| = \left| \frac{1}{[M_i^{1/2}/(1+\gamma)M_i^\gamma]} \sum_{0 \leq (1+\gamma)M_i^\gamma k < M_i^{1/2}} f(q_iu((1+\gamma)M_i^\gamma k)) \right| \\
\leq \left( \frac{1}{(1+\gamma)M_i^\gamma} + \frac{((1+\gamma)M_i^\gamma)^4}{r_i^\beta} + \frac{1}{r_i^\beta} \right) \|f\|_{\infty,4} \tag{6}
\]
where \( q_i = pu(M_i^{1+\gamma}) \) and \( r_i = r(M_i^2, q_i) = M_i^{1/2}e^{-\text{dist}(g(\log M_i)/2(q_i))} \). We compute
\[
g(\log M_i)/2(q_i) = p \begin{pmatrix} 1 & M_i^{1+\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_i^{\frac{3}{2}} & 0 \\ 0 & M_i^{-\frac{1}{2}} \end{pmatrix} = p \begin{pmatrix} M_i^{\frac{1}{2}} & M_i^{\frac{3}{2}+\gamma} \\ 0 & M_i^{-\frac{1}{2}} \end{pmatrix} \in S_{0(M_i)},
\]
by our choice of \( M_i \in B \). By definition \(4.2\) and equation \(3\), we have
\[
d(g(\log M_i)/2(q_i)) \geq \theta(M_i) = \epsilon \min\{M_i^{-\frac{1}{\gamma}+\frac{1}{\gamma}+\gamma} \; | \; j = 1, 2, \ldots, k\}.
\]
By Lemma \(5.2\) \( e^{-\text{dist}(q)} \sim d(q)^2 \). Hence
\[
r_i = M_i^{1/2}e^{-\text{dist}(g(\log M_i)/2(q_i))} \gg \epsilon^2 \min\{M_i^{2/(\kappa_j+4)} \; | \; j = 1, 2, \ldots, k\}. \tag{7}
\]
According to inequalities \(6\) and \(7\), as long as \( 4\gamma < \min\{2\beta/(\kappa_j+4) \; | \; j = 1, 2, \ldots, k\} \), we have
\[
\left| \frac{1}{|P_i|} \sum_{n \in P_i} f(pu(n)) \right| \to 0
\]
and hence by the fact that \( P_i \) is approximated by \( \tilde{P}_i \), we obtain
\[
\left| \frac{1}{|P_i|} \sum_{n \in P_i} f(pu(n^{1+\gamma})) \right| \to 0 \tag{8}
\]
as \( i \to \infty \). By formula \(5\), the proportion in \([1, N]\) which is not covered by \( P_i \)'s is small relative to \( N \). Also observe that for the \( P_i \)'s which intersect \([1, N]\), their lengths are small relative to \( N \). Therefore, by \(8\) we have
\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(pu(n^{1+\gamma})) \right| = 0
\]
\[
\leq \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \in [1,N] \setminus (\bigcup P_i)} f(pu(n^{1+\gamma})) \right| + \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \in [1,N] \cap (\bigcup P_i)} f(pu(n^{1+\gamma})) \right|
\leq \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \in [1,N] \setminus (\bigcup P_i)} f(pu(n^{1+\gamma})) \right| + \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{[1,N] \cap P_i \neq \emptyset} \sum_{n \in P_i} f(pu(n^{1+\gamma})) \right|
\leq C_0 \left( \frac{\epsilon}{\rho} \right) \| f \|_{\infty,0} + 0 = C_0 \left( \frac{\epsilon}{\rho} \right) \| f \|_{\infty,0}.
\]

Let $\epsilon \to 0$ and we complete the proof of the main theorem with $\gamma_0 = \min\{\beta/(2\kappa_j + 8) | j = 1,2,\ldots,k\}$.

\[ \square \]

### 7 Discussions

There are two things we want to discuss. In section 3, we define the Diophantine condion on a point in $\Gamma \setminus \text{PSL}(2,\mathbb{R})$. Let

\[ S_{\kappa_1,\kappa_2,\ldots,\kappa_k} = \{ p \in \Gamma \setminus \text{PSL}(2,\mathbb{R}) | p \text{ is Diophantine of type } (\kappa_1,\kappa_2,\ldots,\kappa_k) \}. \]

Then we can compute the Hausdorff dimension of the complement of $S_{\kappa_1,\kappa_2,\ldots,\kappa_k}$. In fact, we have

**Theorem 7.1.**

\[ \dim_H S_{\kappa_1,\kappa_2,\ldots,\kappa_k}^c = 2 + \frac{2}{\min\{\kappa_j + 1 | 1 \leq j \leq k\}}. \]

Since this is not the main focus of the paper, we will not give a proof here.

Note that the Diophantine type remains constant on any weak unstable leaf of $\{g_t\}_{t>0}$. Therefore by Theorem 7.1, the set of non Diophantine points on any strong stable leaf has zero hausdorff dimension.

The other thing is that using the same argument as in section 4, we can actually prove that if $p$ is Diophantine of type $(\kappa_1,\kappa_2,\ldots,\kappa_k)$ with all $\kappa_j < 3$ and $0 \leq \gamma < 1/4$, for any $\epsilon > 0$, there exists a compact subset $K_\epsilon \subset \Gamma \setminus \text{PSL}(2,\mathbb{R})$ such that for all $T \geq 0$,

\[ \frac{1}{T} \left\{ x \in [1,T] \left| p \left( \begin{array}{c} x^\frac{1}{4} \\ 0 \\ x^\frac{3}{4} + \gamma \\ x^{-\frac{1}{4}} \end{array} \right) \in K_\epsilon \right\} \geq 1 - \epsilon. \]

Then using the arguments of [DS84] and [Sh94, Proposition 4.1], we get

**Theorem 7.2.** If $p$ is Diophantine of type $(\kappa_1,\kappa_2,\ldots,\kappa_k)$ with all $\kappa_j < 3$ and $0 \leq \gamma < 1/4$, then the trajectory

\[ \left\{ p \left( \begin{array}{c} x^\frac{1}{4} \\ 0 \\ x^\frac{3}{4} + \gamma \\ x^{-\frac{1}{4}} \end{array} \right) | x \geq 1 \right\} \]

is equidistributed in $\Gamma \setminus \text{PSL}(2,\mathbb{R})$.  

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