Quantum renormalization of entanglement in an antisymmetric anisotropic and bond-alternating spin system

Xiang Hao

Department of Physics, School of Mathematics and Physics,
Suzhou University of Science and Technology,
Suzhou, Jiangsu 215011, People’s Republic of China

Abstract

The quantum renormalization group method is applied to study the quantum criticality and entanglement entropy of the ground state of the Ising chain in the presence of antisymmetric anisotropic couplings and alternating exchange interactions. The quantum phase transitions can be characterized by the discontinuity in the second derivative of the energy of renormalized ground state. The phase diagram is obtained by the critical boundary line. The first derivative of entanglement entropy also diverges at the same critical points after enough iterations of the renormalization of coupling constants. The antisymmetric anisotropy and alternating interaction can enhance the renormalized entanglement via the creation of quantum fluctuations. The scaling behavior of the derivative of the entropy around the critical points manifest the logarithm dependence on the size of the spin system.

PACS: 03.67.Mn, 03.65.Ud, 75.10.Pq, 73.43.Nq

*Corresponding author; Electronic address: 110523007@suda.edu.cn
I. INTRODUCTION

The resource of entanglement can play an important role in the quantum information processing such as quantum teleportation and algorithms for quantum computation [1, 2]. The entanglement is quantum correlation which can describe the property of the ground state for various spin systems [3]. It has been justified that there exist the close relations between the entanglement and the quantum phase transitions in condensed matter physics [4–10]. The quantum critical phenomena happen at zero temperature when the spin couplings and external fields are varied in the vicinity of critical points [11]. As is known, the quantum phase transitions can be represented by the discontinuities in the derivatives of some order parameters. However, for no a priori knowledge of suitable order parameters, some special physical quantities can be used to universally identify the quantum criticality at the ground state [12, 13]. The ground-state energy is considered as one characterizing the quantum phase transitions for some spin systems [14, 15]. So far some measures of quantum entanglement have been found out to be relevant to the quantum criticality for many spin systems [16–18]. During the theoretical investigation of low-dimensional spin systems, the numerical methods of density-matrix renormalization group and exact diagonalization have received much attention [19–21]. In many materials, the magnetic properties were studied by the spin models with the antisymmetric anisotropic interaction such as Dzyaloshinskii-Moriya interaction [22, 23]. Recently, the quantum renormalization group approach has provided a convenient way to analyze the quantum phase transitions for such systems [24–27]. By this method, the renormalized ground states can be described in terms of matrix product states [28]. The quantum phase diagram can be approximately obtained by the enough iterations of the renormalization of couplings and external fields. Meanwhile, rich quantum criticality can also be embodied by the bond-alternating spin models [29]. These respects motivate us to investigate the effects of antisymmetric anisotropy and alternating spin interactions on the entanglement and quantum phase transitions.

The paper is organized as follows. In section II, the quantum renormalization group method is introduced and used for the Ising chain with antisymmetric anisotropy and alternating exchange interactions. The renormalized ground state can be obtained analytically. In section III, the quantum phase transitions are approximately characterized by the energy of the renormalized ground state. The boundary line of critical points determines two
different phases. The effects of antisymmetric anisotropic interactions and alternating ones on the entanglement entropy are investigated. The divergence of the derivative of entropy occurs at the same critical points. The finite size scaling behavior of the entanglement are also studied. A simple discussion concludes the paper.

II. RENORMALIZATION OF THE SPIN MODEL

The implementation of quantum renormalization group is the perturbation method to reduce the degrees of freedom of the spin model. In the Kadanoff’s block approach, the original Hamiltonian of the model can be divided into the block Hamiltonian and the interacting Hamiltonian. The block Hamiltonian is the sum of some commuting items which act on different blocks. The effective Hamiltonian can be obtained by mapping the original Hamiltonian to the low energy subspace via the projector consisting of some lowest energy eigenstates of the block Hamiltonian.

The Hamiltonian of the Ising chain with the antisymmetric anisotropic and alternating interactions can be written by

$$H = J \sum_{i=1}^{N} [1 - (-1)^i \lambda] S_i^z S_{i+1}^z + \vec{D} \cdot (\vec{S}_i \times \vec{S}_{i+1}). \quad (1)$$

The periodic boundary condition of $N + 1 = 1$ is considered. Here $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ denotes the Pauli operator acting on the $i$-th spin and $S_i^z | \uparrow (\downarrow) \rangle = \pm \frac{1}{2} | \uparrow (\downarrow) \rangle$. The case of $J > 0$ represents the nearest-neighboring antiferromagnetic coupling and $0 \leq \lambda \leq 1$ describes the relative strength of alternating exchange coupling. The antisymmetric anisotropic interaction is described by the vector of the Dzyaloshinskii-Moriya interaction, $\vec{D} = (D_x, D_y, D_z)$, arising from the spin-orbit coupling. In the following analysis, the case of $D_x = D_y = 0, D_z = D$ is taken into account.

According to the renormalization group method, the original Hamiltonian is divided into two parts

$$H = H^0 + H^I = \sum_{k=1}^{L} h_k^0 + h_k^I. \quad (2)$$

The part of $H^0 = \sum_{k=1}^{L} h_k^0$ is the block Hamiltonian where $L = N/3$ represents the number of blocks and each item has three sites, $h_k^0 = J \sum_{l=1}^{2} [1 + (-1)^{k+l} \lambda] S_{l,k}^z S_{l+1,k}^z$ +
The renormalization of the couplings satisfy the equation $D(S^x_{l,k}S^y_{l+1,k} - S^y_{l,k}S^x_{l+1,k})$. Here $S^\alpha_{l,k} (\alpha = x, y, z)$ denotes the $\alpha$-component pauli operator on the $l$-th spin of the $k$-th block. The other part of $H'$ represents the interacting Hamiltonian between two neighboring blocks, $H' = \sum_{k=1}^{L} h^I_{k,k+1}$ where $h^I_{k,k+1} = J\{[1 - (-1)^k\lambda]S^x_{3,k}S^x_{1,k+1} + D(S^y_{3,k}S^y_{1,k+1} - S^y_{3,k}S^y_{1,k+1})\}$. We need the degenerate lowest energy eigenstates $|\psi^\pm_k\rangle$ of $h^0_k$ to construct the projector $P_k = |\uparrow_k\rangle \langle \psi^+_k | + |\downarrow_k\rangle \langle \psi^-_k |$ where $|\uparrow_k\rangle$ (or $|\downarrow_k\rangle$) is the renamed basis in the effective Hilbert space for each block. When $k$ is odd, the analytic expression of $|\psi^\pm_k\rangle$ can be given in the product Hilbert space of a block

$$|\psi^+_k\rangle = a|\uparrow\uparrow\downarrow\rangle + ib|\uparrow\downarrow\uparrow\rangle + c|\downarrow\downarrow\uparrow\rangle \quad (3)$$

$$|\psi^-_k\rangle = a|\downarrow\downarrow\uparrow\rangle - ib|\downarrow\uparrow\downarrow\rangle + c|\uparrow\uparrow\downarrow\rangle.$$

Here the real parameters of $a, b, c$ satisfy the condition of the normalization where $a = bD/(\lambda - 2\epsilon_0), b = 1/\sqrt{1 + 2D^2(4\epsilon_0^2 + \lambda^2)/(4\epsilon_0^2 - \lambda^2)^2}$. Here $\epsilon_0$ is the scale of the lowest energy by the parameter $J$ and calculated by the minimal root of the equation $8\epsilon_0^3 + 4\epsilon_0^2 - 2(\lambda^2 + 2D^2)\epsilon_0 - \lambda^2 = 0$. For the simplest example of $\lambda = 0$, $\epsilon_0 = -(1 + \sqrt{1 + 8D^2})/4$.

When $k$ is even, the degenerate lowest energy eigenstates have the similar expression in Eq. (3) and the real parameters $b_{even} = b, a_{even} = -c, c_{even} = -a$. The effective Hamiltonian mapped onto the low energy subspace can be expressed by

$$H_{eff} = \prod_{k=1}^{L} P_k H(\prod_{k=1}^{L} P_k^\dagger). \quad (4)$$

By the effective spin operators $\vec{\tau}_k$ for the $k$-th block, the definite form in the effective Hilbert space can be written by $H_{eff} = J' \sum_{k=1}^{L} [1 - (-1)^k\lambda] \tau^x_k \tau^x_{k+1} + D'(\tau^y_k \tau^y_{k+1} - \tau^y_k \tau^y_{k+1})$ where the operator $\tau^z = \frac{1}{2}(|\uparrow\rangle \langle \uparrow | - |\downarrow\rangle \langle \downarrow |)$. For the case of $k$ = odd, the relations between the spin operator $\vec{S}_{l,k} = (S^x_{l,k}, S^y_{l,k}, S^z_{l,k})$ and the effective one $\vec{\tau}_k = (\tau^x_k, \tau^y_k, \tau^z_k)$ are given by

$$P_k \vec{S}_{1,k} P_k^\dagger = (2ab\tau^y_k, -2ab\tau^x_k, (1 - 2c^2)\tau^z_k)$$

$$P_k \vec{S}_{2,k} P_k^\dagger = (2ac\tau^y_k, 2ac\tau^x_k, (1 - 2b^2)\tau^z_k)$$

$$P_k \vec{S}_{3,k} P_k^\dagger = (2bc\tau^y_k, -2bc\tau^x_k, (1 - 2a^2)\tau^z_k). \quad (5)$$

The renormalization of the couplings satisfy the equation

$$J' = J(1 - 2a^2)(1 - 2c^2), D' = -\frac{4Da^2c}{(1 - 2a^2)(1 - 2c^2)}. \quad (6)$$

In general, after the $n$-th renormalization group iteration step, the chain of $3^{n+1}$ sites can be represented by the effective three-site Hamiltonian with the renormalized couplings. Accord-
ing to the renormalization of ground state given by the expression of Eq. (3), the quantum phase transitions and the entanglement properties can be studied.

III. PHASE DIAGRAM AND ENTANGLEMENT ENTROPY

The quantum phase transitions at the ground state are driven by the quantum fluctuations. It is the key to select the suitable order parameters. To obtain the phase diagram, we choose the ground-state energy as one universal measure. From the renormalization equation of couplings in Eq. (7), it is seen that the quantum phase transitions will happen by the change of the antisymmetric anisotropic interaction $D$ and relative strength of alternating spin exchange $\lambda$. For a certain value of $\lambda$, the obvious discontinuity in the second derivative of ground-state energy $\frac{\partial^2 \varepsilon_0}{\partial D^2}$ can be shown at the quantum critical point $D_c$ in Fig. 1(a). It means the feature of the second-order quantum phase transition. With the change of the alternating interaction $\lambda$, the phase diagram is obtained in Fig. 1(b). It is shown that two different phases are divided by the critical points $D_c \approx \sqrt{1 - \lambda^2}$.

It is interesting to investigate the entanglement properties of the spin system in the two different phases. Using the renormalization group method, we calculate the entanglement between some degrees of freedom and the rest of the system. The renormalization of entanglement in the large enough iteration step can be used to estimate the large size behavior of the spin system. As one successful measure of entanglement, the block entropy can be implemented to describe the global property of the entanglement. The density matrix of the renormalized ground state is $\rho_0 = |\psi_0\rangle\langle\psi_0|$ where $|\psi_0\rangle = |\psi_0^+\rangle$ or $|\psi_0^-\rangle$ is expressed by Eq. (3). Therefore, the reduced density matrix for the middle site can be obtained by tracing over the rest sites

$$\rho = (1 - b^2)|\uparrow\rangle\langle\uparrow| + b^2|\downarrow\rangle\langle\downarrow|.$$

The entropy $E$ for the middle site can represent the entanglement between the degrees of freedom of the middle site and ones of the rest sites. The expression of $E$ is given by

$$E = -(1 - b^2) \log_2(1 - b^2) - b^2 \log_2(b^2).$$

In the first step of renormalization group, the chain of $3^2$ sites can be described by the effective Hamiltonian $H_{eff}$. The change of entanglement entropy $E$ is depicted as a function of the antisymmetric anisotropic coupling $D$ and alternating exchange interaction $\lambda$ in Fig.
2. It is shown that the values of $E$ are increased with $D$ and $\lambda$. The Dzyaloshinskii-Moriya interaction in the $XY$ plane can induce the planar quantum fluctuations which enhance the quantum correlation of the ground state. The existence of alternating couplings can spoil the form of the Néel ordered state which has no entanglement. The values of the entanglement tend to one maximum when $\lambda$ is close to one.

To analyze the connection of the entanglement to the quantum phase transition, we need study the scaling property. By means of increasing the iteration step, the entropy $E$ can be calculated for the large-size spin system. From Fig. 3, it is seen that the rate of the change of entanglement entropy will be increased as the size becomes large through the iteration step. In the phase $I$ for the case of $D < D_c$, the entropy is always less than one. The maximal entanglement entropy of $E = 1$ occurs in the phase $II$ for the case of $D > D_c$. After the large enough iteration step, the appearance of nonanalytic behavior of the entanglement entropy verifies the feature of the quantum phase transition. In this spin system, the measure of entanglement entropy can be applied to estimate the quantum criticality.

Furthermore, it is also necessary to evaluate the singularity of the derivative of the renormalized entanglement entropy. As depicted in Fig. 4(a), the drop of the first derivative of entropy will become more pronounced with the increase of the spin sites. The minimal values of $-\frac{\partial E}{\partial D}$ at some points $D = D_m$ are decreased rapidly with the size. The points of $D_m$ for a large iteration step are closely near to the critical ones $D_c$. The scaling of the points $D_m$ is also shown in Fig. 4(b). The logarithm dependence on the sites $N$ is numerically given by $D_m = D_c - N^{-\gamma}$ where the parameter $\gamma = \gamma(\lambda)$ is increased with the decrease of the alternating coupling $\lambda$. Through the numerical fit, $\gamma(1) \approx 0.46$ and $\gamma(0.5) \approx 0.52$. This respect manifests that the divergence of the derivative of the entropy appears more rapidly for the smaller values of alternating interaction $\lambda$ in the condition of the same size.

IV. DISCUSSION

The entanglement property and quantum phase transitions in the Ising chain with the Dzyaloshinskii-Moriya interaction and alternating coupling are studied by the method of quantum renormalization group. Based on the analytic renormalization equation, the renormalized ground state can be obtained. The ground-state energy is considered as one universal measure of quantum phase transitions. The nonanalytic behavior of the second derivative
of the renormalized ground-state energy can justify the feature of second-order phase transition. The critical boundary line divides the ground state into two different phases. It is demonstrated that the antisymmetric anisotropy and alternating couplings can help for the increase of the entanglement entropy. The discontinuities in the entanglement entropy happen at the critical points which are same to those provided via the order parameter of the ground-state energy. With the decrease of the alternating coupling, the divergence of the derivative of the entropy appears more rapidly in the vicinity of the quantum criticality. In this one-dimensional spin system, the renormalized entanglement entropy is the efficient measure of the quantum phase transition.

V. ACKNOWLEDGEMENT

X.H. was supported by the Research Program of Natural Science for Colleges and Universities in Jiangsu Province Grant No. 09KJB140009 and the National Natural Science Foundation Grant No. 10904104.

[1] D. Loss and D. P. DiVincenzo, Phys. Rev. A57, 120(1998).
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2000).
[3] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys.80, 517(2008).
[4] T. J. Osborne and M. A. Nielsen, Phys. Rev. A66, 032110(2002).
[5] I. Bose and E. Chattopadhyay, Phys. Rev. A66, 062320(2002).
[6] S. J. Gu, S. S. Deng, Y. Q. Li, and H. Q. Lin, Phys. Rev. Lett.93, 086402(2004).
[7] J. Vidal, G. Palacios, and R. Mosseri, Phys. Rev. A69, 022107(2004).
[8] L. A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett.93, 250404(2004).
[9] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett.92, 027901(2004).
[10] A. Anfossi, P. Giorda, and A. Montorsi, Phys. Rev. B75, 165106(2007).
[11] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, UK, 2000).
[12] M. Cozzini, P. Giorda, and P. Zanardi, Phys. Rev. B75, 014439(2007).
[13] M. F. Yang, Phys. Rev. B76, 180403(R)(2007).
[14] P. Lou, W. C. Wu, and M. C. Chang, Phys. Rev. B70, 064405(2004).
[15] P. Lou, Phys. Rev. B72, 064435(2005).
[16] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature(London) 416, 608(2002).
[17] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett.90, 227902(2003).
[18] P. Lou and J. Y. Lee, Phys. Rev. B74, 134402(2006).
[19] U. Schollwöck, Rev. Mod. Phys.77, 259(2005).
[20] Ö. Legeza, J. Röder, and B. A. Hess, Phys. Rev. B67, 125114(2003).
[21] Y. C. Li and S. S. Li, Phys. Rev. A79, 032338(2009).
[22] D. C. Dender, P. R. Hammar, D. H. Reich, C. Broholm, and G. Aeppli, Phys. Rev. Lett.79, 1750(1997).
[23] P. Fulde, B. Schmidt, and P. Thalmeier, Europhys. Lett.31, 323(1995).
[24] M. Kargarian, R. Jafari, and A. Langari, Phys. Rev. A76, 060304(R)(2007).
[25] M. Kargarian, R. Jafari, and A. Langari, Phys. Rev. A77, 032346(2008).
[26] R. Jafari, M. Kargarian, A. Langari, and M. Siahatgar, Phys. Rev. B78, 214414(2008).
[27] M. Kargarian, R. Jafari, and A. Langari, Phys. Rev. A79, 042319(2009).
[28] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, Phys. Rev. Lett.94, 140601(2005).
[29] J. Almeida, M. A. Martin-Delgado, and G. Sierra, Phys. Rev. B76, 184428(2007).
[30] M. A. Marti-Delgado and G. Sierra, Int. J. Mod. Phys. A.11, 3145(1997).
Figure caption

Figure 1
(a). The second derivative of the energy of the renormalized ground state diverges at the critical point $D_c \simeq 0.85$ in the iteration step $n = 9$ when $\lambda = 0.5$. (b). The dash line denotes the critical boundary characterizing two different phases $I$ and $II$.

Figure 2
The renormalization of the entanglement entropy $E$ is plotted as a function of the antisymmetric anisotropy $D$ and alternating coupling $\lambda$ in the iteration step $n = 1$.

Figure 3
The evolution of the renormalized entanglement entropy as the size is shown for $\lambda = 0.5$. The square data denote the case of the step $n = 1$, the circle ones represent the case of $n = 5$ and the triangles are the data for the case of $n = 9$.

Figure 4
(a). The negative values of the first derivative of the entropy are calculated when $\lambda = 0.5$. The dot line describes the case of iteration step $n = 1$, the dash line denotes the result of $n = 5$ and the solid one represent the case of $n = 9$. (b). The data characterize the finite size scaling property of the singular points $D_m$ where the values of $-\frac{\partial E}{\partial D}$ are minimal. The squares represent the case of $\lambda = 0.5$ and the circles denote the case of $\lambda = 1$. 
Fig. 1
Fig. 2
Fig. 3
Fig. 4