Near-feasible stable matchings with budget constraints

Yasushi Kawase†,♮ and Atsushi Iwasaki‡,♮

†Tokyo Institute of Technology, Tokyo, Japan. kawase.y.ab@m.titech.ac.jp
‡University of Electro-Communications, Tokyo, Japan. iwasaki@is.uec.ac.jp
♮RIKEN AIP Center, Tokyo, Japan.

Abstract

This paper deals with two-sided matching with budget constraints where one side (firm or hospital) can make monetary transfers (offer wages) to the other (worker or doctor). In a standard model, while multiple doctors can be matched to a single hospital, a hospital has a maximum quota: the number of doctors assigned to a hospital cannot exceed a certain limit. In our model, a hospital instead has a fixed budget: the total amount of wages allocated by each hospital to doctors is constrained. With budget constraints, stable matchings may fail to exist and checking for the existence is hard. To deal with the nonexistence of stable matchings, we extend the “matching with contracts” model of Hatfield and Milgrom, so that it handles near-feasible matchings that exceed each budget of the hospitals by a certain amount. We then propose two novel mechanisms that efficiently return such a near-feasible matching that is stable with respect to the actual amount of wages allocated by each hospital. In particular, by sacrificing strategy-proofness, our second mechanism achieves the best possible bound.

1 Introduction

This paper studies a two-sided, one-to-many matching model when there are budget constraints on one side (firm or hospital), i.e., the total amount of wages that it can pay to the other side (worker or doctor) is limited. The theory of two-sided matching has been extensively developed. See the book by Roth and Sotomayor [25] or Manlove [20] for a comprehensive survey. In this literature, rather than fixed budgets, maximum quotas are typically used, i.e., the total number of doctors that each hospital can hire is limited.

Some real-world examples are subject to matching with budget constraints: a college can offer stipends to students to recruit better students while the budget for admission is limited, a firm can offer wages to workers under the condition that employment costs depend on earnings in the previous accounting period, a public hospital can offer salaries to doctors in the case where the total amount relies on funds from the government, and so on. To establish our model and concepts, we use doctor-hospital matching as a running example.

However, most papers on matching with monetary transfers assume that budgets are unrestricted (e.g., [17]). When they are restricted, stable matchings may fail to exist [22]. In particular, Abizada [2] considers a subtly different model from ours and shows that (coalitional) stable matchings, where groups of doctors and hospitals have no profitable deviations, may not exist[1].

We construct and analyze our mechanisms on a “matching with contracts” model [13], which characterizes a class of mechanisms called the generalized Deferred Acceptance (DA) mechanism. If a mechanism—specifically, the choice function of every hospital—satisfies three properties, i.e., substitutability, irrelevance of rejected contracts, and law of aggregate demand, then it always finds a “stable” allocation and is strategy-proof for doctors. However, in the presence of budget constraints, the hospital’s choice function cannot satisfy these properties because stable matchings may not exist.

To deal with the nonexistence of stable matchings, we focus on a near-feasible matching that exceeds the budget of each hospital by a certain amount. This idea can be interpreted as one in which, for each instance of a matching problem, our mechanisms find a “nearby” instance with a stable matching. For a choice function that produces a near-feasible matching, the existing properties are not sufficient to ensure the optimality of the hospitals’ utilities. To resolve

[1]He also proposes a modified deferred acceptance mechanism that produces a pairwise stable matching, and that is strategy-proof for doctors.
this, we devise a new property, which we call compatibility, on the matching with contract model. Moreover, from a practical point of view, we need to compute choice functions efficiently. However, computing each hospital’s choice function is NP-hard because it is equivalent to solving a knapsack problem.

Building upon these ideas, we propose two novel mechanisms that efficiently return a near-feasible stable matching with respect to the actual amount of wages allocated by each hospital: one is strategy-proof for doctors and the other is not. Furthermore, we discuss some special cases.

The idea of near-feasible matchings closely relates to Nguyen and Vohra [23]. They examine matchings with couples where joint preference lists over pairs of hospitals are submitted (e.g., [18, 24]), and then develop an algorithm that outputs a stable matching, in which the number of doctors assigned to each hospital differs (up or down) from the actual maximum quota by at most three. Alternatively, Dean, Goemans, and Immorlica [8] examine a problem similar to ours, restricting each hospital to having only a lexicographic utility. We instead allow each hospital to have an additive utility. In any case, it must be emphasized that those studies discuss no strategic issue, i.e., misreporting a doctor’s preference may be profitable. The literature on matching has found strategy-proofness for doctors, i.e., each doctor has no incentive to misreport his or her preference, to be a key property in a wide variety of settings [1].

2 Preliminaries

This section describes a model for two-sided matchings with budget constraints. A market is a tuple \((D, H, X, \succ_D, f_H, B_H)\), where each component is defined as follows. There is a finite set of doctors \(D = \{d_1, \ldots, d_n\}\) and a finite set of hospitals \(H = \{h_1, \ldots, h_m\}\). Let \(X \subseteq D \times H \times \mathbb{R}_+\) denote a finite set of contracts where each contract \(x \in X\) is of the form \(x = (d, h, w)\). Here, \(\mathbb{R}_+\) is the set of positive real numbers. A contract means that hospital \(h \in H\) offers wage \(w \in \mathbb{R}_+\) to doctor \(d\). A hospital can choose a wage freely within \(\mathbb{R}_+\) and can offer a doctor multiple contracts with different wages. Each contract is acceptable for each hospital \(h\).

Furthermore, for any subset of contracts \(X' \subseteq X\), let \(X_d'\) denote \(\{(d', h', w') \in X' \mid d' = d\}\) and \(X_h'\) denote \(\{(d', h', w') \in X' \mid h' = h\}\). We use the notation \(x_D, x_H,\) and \(x_W\) to describe the doctor, the hospital, and the wage associated with a contract \(x \in X\), respectively.

Let \(\succ_D = (\succ_d)_{d \in D}\) denote the doctors’ preference profile, where \(\succ_d\) is the strict relation of \(d \in D\) over \(X_d \cup \{\emptyset\}\), i.e., \(x \succ_d x'\) means that \(d\) strictly prefers \(x\) to \(x'\). \(\emptyset\) indicates a null contract. Let \(f_H = (f_h)_{h \in H}\) denote hospitals’ utility profile, where \(f_h : X_h \rightarrow \mathbb{R}_+\) is a function such that for any two sets of contracts \(X', X'' \subseteq X_h\), hospital \(h\) prefers \(X'\) to \(X''\) if and only if \(f_h(X') > f_h(X'')\) holds. We further assume that \(f_h\) is additive for all \(h \in H\), i.e., \(f_h(X') = \sum_{x \in X_h} f_h(x)\) holds for any \(X' \subseteq X_h\). We assume, instead of priority orderings, cardinal utilities as used in some previous work [3,4,5]. However, this does not matter for our theoretical results. Indeed, a cardinal utility can be transformed into a priority ordering \(\succ_h\) over contracts \(X_h\) for each hospital \(h \in H\), where, for any two contracts \(x, x' \in X_h\), \(x \succ_h x'\) if and only if \(f_h(x) > f_h(x')\).

Each hospital \(h\) has a fixed budget \(B_h \in \mathbb{R}_+\) that it can distribute as wages to the doctors it admits. Let \(B_H = (B_h)_{h \in H}\) be the budget profile. We assume that, for any contract \((d, h, w), 0 < w \leq B_h\) holds. Given \(X\), let

\[
\underline{w}_h = \min_{x \in X_h} x_W \quad \text{and} \quad \overline{w}_h = \max_{x \in X_h} x_W.
\]

Moreover, we use the notation \(w_h(X')\) for any \(X' \subseteq X\) to denote the total wage that \(h\) offers in \(X'\), i.e., \(\sum_{x \in X_h} x_W\).

We call a subset of contracts \(X' \subseteq X\) a matching if \(|X'_h| \leq 1\) for all \(d \in D\). A matching \(X' \subseteq X\) is \(B'_H\)-feasible if \(w_h(X') \leq B'_h\) for all \(h \in H\). Let us next explain the notion of a blocking set (or coalition). Given a matching \(X'\), another matching \(X'' \subseteq X_h\) for hospital \(h\) is a blocking coalition if \(X'' \succ_D X'\) for all doctors \(x \in X'' \setminus X'\), \(f_h(X'') > f_h(X'_h)\), and \(w_h(X'') \leq B'_h\). If such \(X''\) exists, we say \(X''\) blocks \(X'\). Then we obtain a stability concept.

**Definition 1** \((B'_H\)-stability\). We say a matching \(X'\) is \(B'_H\)-stable if it is \(B'_H\)-feasible and there exists no blocking coalition.

As we will see, when no hospital is allowed to violate the given constraints \(B_H\), conventional stable matchings \((B'_H = B_H)\) may not exist. **Definition 1** for example, allows a central planner to add or redistribute the budgets and this planner finds a problem instance with \(B'_H(\geq B_H)\) whose \(B'_H\)-stable matching is guaranteed to exist. If each
contract has the same amount of wage \( w \), we can regard a budget constraint \( B_h \) for each hospital \( h \) as its maximum quota of \( \lfloor B_h/w \rfloor \).

A mechanism is a function that takes a profile of doctors’ preferences as input and returns matching \( X' \). We say a mechanism is stable if it always produces a \( B'_H \)-stable matching for certain \( B'_H \). We also say a mechanism is strategy-proof for doctors if no doctor ever has any incentive to misreport her preference, regardless of what the other doctors report.

Next, we briefly describe a class of mechanisms called the generalized DA mechanism \([13]\) and its properties. This mechanism uses choice functions \( Ch_H : 2^X \rightarrow 2^X \) and \( Ch_D : 2^X \rightarrow 2^X \). For each doctor \( d \), its choice function \( Ch_d(X') \) chooses \( \{x\} \), where \( x = (d, h, w) \in X'_d \) such that \( x \) is the most preferred contract within \( X'_d \) (we assume \( Ch_d(X') = \emptyset \) if \( 0 > d \) \( x \) for all \( x \in X'_d \)). Then, the choice function of all doctors is given as: \( Ch_D(X') := \bigcup_{d \in D} Ch_d(X'_d) \). Similarly, the choice function of all hospitals \( Ch_H(X') \) is \( \bigcup_{h \in H} Ch_h(X'_h) \) where \( Ch_h \) is a choice function of \( h \). There are alternative ways to define the choice function of each hospital \( Ch_h \). As we discuss later, the mechanisms considered in this paper can be expressed by the generalized DA with different formulations of \( Ch_H \).

Formally, the generalized DA is given as Algorithm 1.

```
Algorithm 1: Generalized DA

input: \( X, Ch_D, Ch_H \) output: matching \( X' \subseteq X \)
1 \( R^{(0)} \leftarrow \emptyset \);
2 for \( i = 1, 2, \ldots \) do
3 \( Y^{(i)} \leftarrow Ch_D(X \setminus R^{(i-1)}), Z^{(i)} \leftarrow Ch_H(Y^{(i)}) \);
4 \( R^{(i)} \leftarrow R^{(i-1)} \cup (Y^{(i)} \setminus Z^{(i)}) \);
5 if \( Y^{(i)} = Z^{(i)} \) then return \( Y^{(i)} \);
```

Here, \( R^{(i)} \) is a set of rejected contracts at the \( i \)th iteration. Doctors cannot choose contracts in \( R^{(i)} \). Initially, \( R^{(0)} \) is empty. Thus, each doctor can choose her most preferred contract. The chosen set by doctors is \( Y^{(i)} \). Then, hospitals choose \( Z^{(i)} \), which is a subset of \( Y^{(i)} \). If \( Y^{(i)} = Z^{(i)} \), i.e., no contract is rejected by the hospitals, the mechanism terminates. Otherwise, it updates \( R^{(i)} \) and repeats the same procedure.

Hatfield and Milgrom \([13]\) define a notion of stability, which we refer to as HM-stability.

**Definition 2. (HM-stability)** A matching \( X' \subseteq X \) is said to be HM-stable if \( X' \) satisfies (i) \( X' = Ch_D(X') = Ch_H(X') \) and (ii) there is no set of matchings \( X'' \subseteq X \) such that \( X'' \subseteq Ch_D(X' \cup X'') \) and \( X'' = Ch_H(X' \cup X'') \).

HM-stability unifies stability concepts that are designed for each context of (standard) matching problems without constraints. Indeed, it implies \( B_H \)-stability if we require the choice functions of hospitals to strictly satisfy budget constraints \( B_H \). Let us next see the properties for \( Ch_H \).

**Definition 3. (Substitutability, SUB)** For any \( X', X'' \subseteq X \) with \( X'' \subseteq X', X'' \setminus Ch_H(X'') \subseteq X' \setminus Ch_H(X') \).

**Definition 4. (Irrelevance of Rejected Contracts, IRC)** For any \( X' \subseteq X \) and \( X'' \subseteq X \setminus X' \), \( Ch_H(X') = Ch_H(X' \cup X'') \) holds if \( Ch_H(X' \cup X'') \) is a set of rejected contracts.

**Definition 5. (Law of Aggregate Demand, LAD)** For any \( X', X'' \subseteq X \) with \( X'' \subseteq X' \subseteq X \), \( |Ch_H(X'')| \leq |Ch_H(X')| \).

Hatfield and Milgrom \([13]\) proved that if \( Ch_H \) satisfies SUB and IRC, the generalized DA always produces a matching that is HM-stable, i.e., \( B_H \)-stable. If \( Ch_H \) further satisfies LAD, it is strategy-proof for doctors.

### 2.1 Impossibility and Intractability

When no hospital is allowed to violate the given constraints, it is known that stable matchings may not exist \([22][21][2]\).

For readers’ convenience, let us describe an example that no stable matching exists in our model.

**Example 1.** Consider a market with three doctors \( D = \{d_1, d_2, d_3\} \) and two hospitals \( H = \{h_1, h_2\} \). Hospital \( h_1 \) offers wage 9 (e.g., nine hundred thousand dollars) to doctor \( d_1 \), 6 and 4 to \( d_2 \) and \( d_3 \), while \( h_2 \) offers 6 and 4 to \( d_2 \) and \( d_3 \), respectively.
and \( d_3 \), respectively. Then, the set of offered contracts \( X \) is
\[
\{(d_1, h_1, 9), (d_2, h_1, 6), (d_3, h_1, 4), (d_2, h_2, 6), (d_3, h_2, 4)\}.
\]

The doctors’ preferences are given as follows:
\[
\succ_{d_1} : (d_1, h_1, 9),
\succ_{d_2} : (d_2, h_1, 6) \succ_{d_2} (d_2, h_2, 6),
\succ_{d_3} : (d_3, h_2, 4) \succ_{d_3} (d_3, h_1, 4).
\]

Next, each utility of the hospitals is given from the amount of wages. For example, doctor \( d_2 \) prefers contract \((d_2, h_1, 6)\) to \((d_2, h_2, 6)\). Hospital \( h_1 \) has the utility of 9, 6, and 4 for \((d_1, h_1, 9)\), \((d_2, h_1, 6)\), and \((d_3, h_1, 4)\), respectively, i.e.,
\[
f_{h_1} (\{(d_1, h_1, 9)\}) = 9, f_{h_1} (\{(d_2, h_1, 6)\}) = 6, \text{ and } f_{h_1} (\{(d_3, h_1, 4)\}) = 4.
\]

For each single contract, it prefers \((d_1, h_1, 9)\) to \((d_2, h_1, 6)\) to \((d_3, h_1, 4)\). Finally, hospital \( h_1 \) has a fixed budget of \( B_{h_1} = 10 \) and \( h_2 \) has \( B_{h_2} = 6 \).

We claim that there exists no \( B_H \)-stable matching in this situation. First, assume \( d_1 \) is assigned to \( h_1 \) with wage 9. Then no more doctor is assigned to this hospital due to the budget constraint. If \( d_2 \) is assigned to \( h_2 \), \( \{(d_2, h_1, 6), (d_3, h_1, 4)\} \) is a blocking coalition because \( d_2 \) prefers \( h_1 \) to \( h_2 \), \( d_3 \) prefers \( h_1 \) to being unmatched, and \( h_1 \) prefers \( \{(d_2, h_1, 6), (d_3, h_1, 4)\} \) to \( \{(d_1, h_1, 9)\} \). If \( d_3 \) is assigned to \( h_2 \), \((d_2, h_2, 6)\) blocks this matching.

Second, assume \( d_1 \) is not assigned to \( h_1 \). If \( d_2 \) and \( d_3 \) are simultaneously assigned to \( h_1 \), \( d_3 \) prefers \( h_2 \) to \( h_1 \) and \( h_2 \) prefers \( d_3 \) to being unmatched. If they are assigned to different hospitals, respectively, \((d_1, h_1, 9)\) blocks this matching, regardless of which doctor is assigned to \( h_1 \). For the remaining cases, since either hospital is being unmatched, some contract or coalition always block the matching.

Note that \( f_{h} (\{(d, h, w)\}) = w \) holds for any \((d, h, w) \in X\) in the above example. Thus, stable matching may not exist even when the utility of each hospital over a set of contracts is the total amount of their wages.

This raises the issue of the complexity of deciding the existence of a \( B_H \)-stable matching. McDermid and Manlove [21] considered a special case of our model and proved NP-hardness. Hamada et al. [12] examined a similar case to ours and proved that the existence problem is \( \Sigma_2^P \)-complete.

To deal with the nonexistence of stable matchings, we focus on a near-feasible matching that exceeds each budget by a certain amount. For each instance of a matching problem, our mechanisms find a nearby instance with a stable matching. The following theorem implies that, to obtain a stable matching, at least one hospital \( h \) needs to increase its budget by nearly \( \bar{w}_h \).

**Theorem 1.** For any positive reals \( \alpha < \beta < 1 \), there exists a market \((D, H, X, \succ_D, f_H, B_H)\) such that \( \bar{w}_h \leq \beta : B_h \) and no stable matching exists in any inflated market \((D, H, X, \succ_D, f_H, B'_H)\) if \( B_h \leq B'_h \leq (1 + \alpha)B_h \) for all \( h \in H \).

**Proof.** Let \( m \) be a positive integer larger than \( 1/(\beta - \alpha) + 1/(1 - \beta) \). We consider a market with \( m^2 \) doctors
\[
D = \{d^1, d^0_1, \ldots, d^0_m, d^1_2, \ldots, d^m_2, \ldots, d^1_m, \ldots, d^m_m\}
\]
and \( m \) hospitals \( H = \{h_1, h_2, \ldots, h_m\} \). The set of offered contracts is a union of
\[
X_{d^*} = \{(d^*, h, \beta)\},
X_{d^1} = \{(d^1_i, h_i, 1/m)\} \text{ for all } i \in [m-1],
X_{d^0_1} = \{(d^0_i, h_i, 1/m)\} \text{ for all } i \in [m-1],
X_{d^m} = \{(d^m_i, h_i, 1)\} \text{ for all } i \in [m-1],
\]
where \([m-1]\) indicates \(1, 2, \ldots, m-1\). We assume that the preferences of the doctors are
\[
\succ_{d^*} : (d^*, h, \beta),
\succ_{d^1_i} : (d^1_i, h_i, 1/m) \succ_{d^1_j} (d^0_i, h_i, \beta) \text{ for all } i \in [m-1],
\succ_{d^0_i} : (d^0_i, h_i, 1/m) \text{ for all } i \in [m-1],
\succ_{d^m_i} : (d^m_i, h_i, \beta) \succ_{d^m_j} (d^m_i, h_i, 1/m) \text{ for all } i \in [m-1].
\]
The utilities of each hospital are

\[ f_{h_m}((d^*, h_m, \beta)) = 1, \]
\[ f_{h_m}((d^0_i, h_m, 1/m)) = 2^{-i} \text{ for all } i \in [m-1], \]
\[ f_{h_i}((d^0_i, h_i, \beta)) = 2^m \text{ for all } i \in [m-1], \]
\[ f_{h_i}((d^0_i, h_i, 1/m)) = 2^{m-j} \text{ for all } i, j \in [m-1], \]
\[ f_{h_i}((d^m_i, h_i, \beta)) = 1 \text{ for all } i \in [m-1], \]
\[ f_{h_m}((d^m_i, h_m, 1/m)) = 2^{m-i} \text{ for all } i \in [m-1]. \]

Every hospital has the fixed budget 1 \((B_{h_1} = \cdots = B_{h_m} = 1)\). The condition \(\bar{v}_h \leq \beta \cdot B_h\) holds since \(1/m \leq \beta = \beta \cdot B_h\) and \((1 - \beta)/(m - 1) \leq \beta = \beta \cdot B_h\).

We are going to show that there exists no \(B_H\)-stable matching such that \(B_h \leq B_h' \leq (1 + \alpha) \cdot B_h\) for all \(h \in H\) by contradiction. Let us assume that \(X'\) is a \(B_{H'}\)-stable matching, namely, a stable matching for a market \((D, H, X, r_D, f_H, B'_{H'})\) such that \(B_h \leq B_h' \leq (1 + \alpha)B_h\) for all \(h \in H\).

First, let us consider a case where, for all \(i \in [m-1]\), doctor \(d^0_i\) is assigned to \(h_m\) with \(1/m\), i.e.,

\[ Y_{m}^0 = \{(d^0_1, h_m, 1/m), \ldots, (d^0_{m-1}, h_m, 1/m)\} \subseteq X'. \]

In this case, \(d^*\) must be assigned to \(h_m\) because we assume \(X'\) is \(B'_{H'}\)-stable. Precisely, doctor \(d^*\) prefers \(h_m\) to being unmatched. Hospital \(h_m\) prefers the contract with \(d^*\), whose utility is one, to \(Y_{m}^0\), whose utility is \(\sum_{i=1}^{m-1} 2^{-i}\). In addition, the wage to \(d^*\), that is, \(\beta\), is smaller than the total wages to \(Y_{m}^0\), that is, \((m-1)/m\). Thus, unless \(d^*\) is assigned to \(h_m\) with \(\beta\), she can form a blocking coalition. Since \(X'\) contains at least \((d^*, h_m, \beta)\) and \(Y_{m}^0\), we derive

\[ w_{h_m}(X') \geq \beta + (m - 1)/m \]
\[ > 1 + \beta - \frac{(\beta - \alpha)(1 - \beta)}{1 - \alpha} \]
\[ > (1 + \beta) - (\beta - \alpha) = (1 + \alpha) \]

from the assumptions of \(m, \alpha, \beta\). Thus, \(w_{h_m}(X')\) is strictly greater than \((1 + \alpha)B_{h_m}\). This contradicts that \(X'\) is \(B'_{H'}\)-feasible, which is implied by \(B_{H'}\)-stability.

Second, let us consider the other case where, for some \(i \in [m-1]\), \(d^0_i\) is not assigned to \(h_m\), i.e., \(Y_{m}^0 \not\subseteq X'\). In this case, such \(d^0_i\) must be assigned to \(h_i\). To show this, it is sufficient to consider a situation where \((d^m_i, h_i, \beta)\) and

\[ Y_i = \{(d^1_i, h_i, 1/m), \ldots, (d^{m-1}_i, h_i, 1/m)\} \]

are chosen by \(h_i\), i.e., \(\{(d^m_i, h_i, \beta)\} \cup Y_i \subseteq X'\). Hospital \(h_i\) obtains the utility of \(1 + \sum_{i=1}^{m-1} 2^{-i} = 2^m - 1\) on the assignment. On the other hand, if \(d^0_i\) is assigned to \(h_i\) with \(\beta\), \(h_i\) obtains the utility of \(2^m\). Evidently, \(h_i\) prefers \((d^0_i, h_i, \beta)\) to \(\{(d^m_i, h_i, \beta)\} \cup Y_i\) and \(d^0_i\) can form a blocking coalition unless she is assigned to \(h_i\). To consider a set of doctors which are not assigned to \(h_m\), we introduce a set of indexes

\[ I = \{i \in [m-1] \mid (d^0_i, h_m, 1/m) \not\in X'\}. \]

By the assumption, \(I\) is not the empty set. In what follows, we concentrate on matchings where doctor \(d^0_i\), for all \(i \in I\), is assigned to \(h_i\), instead of \(h_m\). Note that \(d^m_i\) for \(i \in [m-1] \setminus I\) must be assigned to \(h_i\) because, if \(d^m_i\) is not assigned to \(h_i\), \(\{(d^m_i, h_i, \beta)\} \cup Y_i\) is a blocking coalition. In addition, \(d^m_i\) for \(i \in I\) must be assigned to \(h_i\) or \(h_m\) because, if \(d^m_i\) is unmatched, \(X_h \setminus \{(d^*, h_m, \beta)\} \cup \{(d^m_i, h_i, 1/m)\}\) is a blocking coalition.

Let us first examine a case where, for all \(i \in I\), \(d^m_i\) are assigned to \(h_m\). In this case, \(d^*\) must be assigned to \(h_m\). Unless \(d^*\) is assigned to \(h_m\), \(X'' = X''_{h_m} \cup \{(d^0_i, h_m, 1/m)\}\) is a blocking coalition for \(i \in I\). This is because \(d^0_i\) prefers \(h_m\) the most and \(w_{h_m}(X'') = 1\). Then, since \(X''_{h_m}\) contains

\[ \{(d^0_i, h_m, 1/m) \mid i \in [m-1] \setminus I\}, \{(d^m_i, h_m, 1/m) \mid i \in I\}, \text{ and } (d^*, h_m, \beta) \].
we derive \( w_{h_m}(X') = (m - 1)/m + \beta \), which is strictly greater than \( 1 + \alpha \) from the assumptions of \( m, \alpha, \) and \( \beta \). Thus, \( w_{h_m}(X') \) is strictly greater than \( (1 + \alpha)B_{h_m} \). This contradicts that \( X' \) is \( B_{H'} \)-stable.

Next, let us examine the other case where, for some \( i \in I \), doctor \( \delta^m_i \) is assigned to \( h_i \). In this case, \( Y_i \) must be chosen by \( h_i \) because, if \( \delta^m_i \) is not assigned to \( h_i \) for some \( j \in [m - 1] \), \( X'_{h_i} \setminus \{(\delta^m_i, h_i, \beta)\} \cup \{(\delta^m_i, h_i, 1 - \beta/m - 1)\} \) is a blocking coalition. In fact, doctor \( \delta^m_i \) prefers \( h_i \) to being unmatched. Hospital \( h_i \) prefers \( (\delta^m_i, h_i, 1 - \beta/m - 1) \), whose utility is \( 2^{m-j} \geq 2 \), to \( (\delta^m_i, h_i, \beta) \), whose utility is 1. In addition, the wage to \( \delta^m_i \), that is, \( 1 - \beta/m - 1 \), is smaller than the wage to \( \delta^m_i \), that is, \( \beta \). Then, since \( X'_{h_i} \) contains \( (\delta^m_i, h_i, \beta) \), \( (\delta^m_i, h_i, \beta) \), and \( Y_i \), we derive \( w_{h_i}(X') = 1 + \beta > 1 + \alpha \) from the assumptions of \( m, \alpha, \) and \( \beta \). Thus, \( w_{h_i}(X') \) is strictly greater than \( (1 + \alpha)B_{h_i} \). This contradicts that \( X' \) is \( B_{H'} \)-stable. \( \square \)

### 3 New property: Compatibility

This section introduces a new property, which we call compatibility, to extend Hatfield and Milgrom’s framework for a situation where budget constraints may be violated. Let us first consider the following choice function for a hospital \( h \):

\[
Ch^*_h(X') = \arg \max_{X'' \subseteq X', w_h(X'') \subseteq B_h} f_h(X'')
\]

for each \( X' \subseteq X_h \).\(^2\) In this case, evaluating \( Ch^*_h \) is computationally hard because the problem is equivalent to the well-known knapsack problem, which is an NP-hard problem (see e.g., [16]). Hence, the choice function is not practical. Even worse, the generalized DA does not always produce a \( B_{H'} \)-stable matching because such a matching need not exist. Furthermore, even if there exists a \( B_{H'} \)-stable matching, the generalized DA with the choice function may produce an unstable matching.

What choice function \( Ch_h \) can we construct when we allow it to violate budget constraints? Strategy-proofness is still characterized by SUB, IRC, and LAD because changing the budgets of hospitals does not affect doctors’ preferences. However, SUB and IRC are not sufficient to admit a stable matching in our sense.

Intuitively, to admit such a stable matching, the set of contracts chosen by the choice function does maximize the hospital’s utility. Otherwise, a hospital with non-optimal utility can form a blocking coalition. To prevent this, we need to introduce a new property.

**Definition 6. (Compatibility, COM)** Consider a hospital \( h \) with a utility function \( f_h \), a budget \( B_h \), and contracts \( X_h \). For any \( X'' \subseteq X' \subseteq X_h \) such that \( w_h(X'') \leq \max\{B_h, w_h(Ch_h(X'))\} \), it holds that

\[
f_h(Ch_h(X')) \geq f_h(X'').
\]

With this property, the output of the choice function \( Ch_h \) is guaranteed to be the optimal solution for a knapsack problem with a certain capacity that is greater than or equal to the predefined capacity.

We next prove that COM together with SUB and IRC characterizes stable matchings when budget constraints may be violated.

**Theorem 2.** Suppose that for every hospital the choice function satisfies SUB, IRC, and COM. The generalized DA produces a matching \( X' \) that is \( B_{H'} \)-stable where \( B_{H'} = \{\max\{B_h, w_h(X')\}\}_{h \in H} \).

**Proof.** Let the mechanism terminate at the \( l \)th iteration, i.e., \( X' = Y^{(l)} = Z^{(l)} \). From its definition, it is immediately derived that the union of \( Y^{(i)} \) and \( R^{(i)} \) is nondecreasing in \( i (\leq l) \), i.e., for any \( i \in \{2, 3, \ldots, l\} \),

\[
Y^{(i)} \cup R^{(i)} \supseteq Y^{(i-1)} \cup R^{(i-1)}.
\] (1)

For notational simplicity, we refer to \( Y^{(i)} \cup R^{(i)} \) as \( T^{(i)} \).\(^2\)

---

\(^2\)When ties occur in the argmax above, we break ties arbitrarily, for example, by choosing the lexicographically smallest one with respect to a fixed order of doctors.
Next, to obtain \( \text{Ch}_H(X' \cup R^{(i)}) = X' \) for any \( X' \), we claim that \( \text{Ch}_H(T^{(i)}) = Z^{(i)} \) for any \( i \in \{1, 2, \ldots, l\} \). For the base case \( i = 1 \), we have \( \text{Ch}_H(T^{(1)}) = \text{Ch}_H(Y^{(1)}) = Z^{(1)} \) since \( R^{(1)} = Y^{(1)} \setminus Z^{(1)} \subseteq Y^{(1)} \). For the general case \( i > 1 \), we suppose \( \text{Ch}_H(T^{(i-1)}) = Z^{(i-1)} \). From [1], we rewrite the SUB condition as \( T^{(i-1)} \setminus \text{Ch}_H(T^{(i-1)}) \subseteq T^{(i)} \setminus \text{Ch}_H(T^{(i)}) \). By the inductive hypothesis, we transform the left side of the equation \( (Y^{(i-1)} \cup R^{(i-1)}) \setminus Z^{(i-1)} \) to

\[
\begin{align*}
R^{(i-1)} \setminus Z^{(i-1)} & \cup (Y^{(i-1)} \setminus Z^{(i-1)}) \\
& = R^{(i-2)} \cup (Y^{(i-1)} \setminus Z^{(i-1)}) = R^{(i-1)}.
\end{align*}
\]

Hence, it holds that \( R^{(i-1)} \subseteq T^{(i)} \setminus \text{Ch}_H(T^{(i)}) \) and thus \( \text{Ch}_H(T^{(i)}) \) includes no contract in \( R^{(i-1)} \). Together with the IRC condition, \( \text{Ch}_H(T^{(i)}) \) is equal to

\[
\text{Ch}_H(T^{(i)} \setminus R^{(i-1)}) = \text{Ch}_H((Y^{(i)} \cup R^{(i)}) \setminus R^{(i-1)}) = \text{Ch}_H(Y^{(i)}) = Z^{(i)}.
\]

The third equality holds because \( Y^{(i)} \cap R^{(i-1)} = \emptyset \) and \( Y^{(i)} \setminus R^{(i)} = Y^{(i)} \cup (R^{(i-1)} \cup (Y^{(i)} \setminus Z^{(i)})) = Y^{(i)} \cup R^{(i-1)} \). Consequently, we obtain the claim and since \( X' = Y^{(l)} = Z^{(l)} \), we get \( \text{Ch}_H(X' \cup R^{(l)}) = X' \).

Next, since \( \text{Ch}_H \) is COM, \( f_h(\text{Ch}_H(X'_h \cup R^{(l)}_h)) \geq f_h(X''_h) \) holds for any \( X''_h \subseteq X'_h \cup R^{(l)}_h \) such that \( w_h(X''_h) \leq \max\{B_h, w_h(\text{Ch}_H(X'_h \cup R^{(l)}_h))\} \). Here, as \( \text{Ch}_H(X'_h \cup R^{(l)}_h) = X'_h \), we have \( f_h(\text{Ch}_H(X'_h)) \geq f_h(X''_h) \) holds for any \( X''_h \subseteq X'_h \cup R^{(l)}_h \) such that \( w_h(X''_h) \leq B'_h \).

Suppose, contrary to our claim, that \( X''_h \subseteq X'_h \) is a blocking coalition for a hospital \( h \). By the definition of blocking coalition, (i) \( X''_h \supseteq x_D \) \( X' \) for all \( x \in X''_h \), (ii) \( f_h(X''_h) > f_h(X'_h) \), and (iii) \( w_h(X''_h) \leq B'_h \). The condition (i) implies \( x \in X' \cup R^{(l)}_h \) for all \( x \in X''_h \), and hence, \( X''_h \subseteq X'_h \cup R^{(l)}_h \) holds. This contradicts the fact that \( \text{Ch}_H(X' \cup R^{(l)}) = X' \) for any \( X' \). Therefore, \( X' \) is \( B'_h \)-stable where \( B'_h = \max\{B_h, w_h(X')\} \) for each hospital.

Note that this theorem does not specify how much budget a hospital may exceed. Here, one can define a choice function such that the hospital affords to hire all of the doctors who have accepted its contracts much beyond the predefined budgets. The theorem simply ensures that if a choice function satisfies COM, in addition to SUB and IRC, the generalized DA admits a \( B'_h \)-stable matching \( X' \) with \( B'_h = \max\{B_h, w_h(X')\} \) for each hospital.

We also remark that if each hospital \( h \) knows the selectable contracts, i.e., \( Y^{(l)}_h \cup R^{(l)}_h \), in advance, it only needs to select \( \arg \max\{f_h(X''_h) \mid X''_h \subseteq X'_h \cup R^{(l)}_h, w_h(X''_h) \leq B'_h\} \) for a certain budget \( B'_h \geq B_h \). However, the selectable contracts are difficult to predict because the resulting set depends on the choice function itself. It is not so straightforward to design or find a choice function such that it satisfies the required properties and only violates budget constraints to an acceptable extent.

4 Near-feasible stable mechanisms

In matching with constraints [15] [11] [19], designing a desirable mechanism essentially tailors choice functions for hospitals to satisfy necessary properties and constraints simultaneously. We tackle this challenging task as an analogue to approximation or online algorithms for knapsack problems.

Let us start from Dantzig’s greedy algorithm for fractional knapsack problems [7]. It greedily selects contracts with respect to utility per wage and then outputs an optimal but fractional solution. We need to develop an algorithm that always provides an integral solution. Roughly speaking, we have to provide an algorithm (choice function) that satisfies the necessary properties, e.g., SUB and COM, for any set of contracts \( X' \) given at each round of the generalized DA. At the same time, we need to let the algorithm determine how much budget should be exceeded beyond the predefined one (how many contracts should be chosen). Indeed, at each round, it is difficult to predict the amount of excess over the budgets without violating the necessary properties. In what follows, we propose two choice functions that adaptively specify how much budgets should be spent within the generalized DA process.
4.1 Strategy-proof stable mechanism

This subsection proposes a strategy-proof mechanism that outputs a matching \( X' \) that is \( B'_h \)-stable where \( B'_h \) is at most \( \bar{w}_h \cdot \lceil B_h/\bar{w}_h \rceil \) for any \( h \in H \). Let \( k_h = \lceil B_h/\bar{w}_h \rceil \). The choice function greedily takes the top \( \min\{k_h, |X'|\} \) contracts according to utility per wage. Formally, it is given as Algorithm 2.

| Algorithm 2: |
|---|
| **input:** \( X' \subseteq X_h \)  
**output:** \( Ch_h(X') \)  
1 Initialize \( Y \leftarrow \emptyset \)  
2 Sort \( X' \) in descending order of utility per wage;  
3 for \( i = 1, 2, \ldots, \min\{k_h, |X'|\} \) do  
4 add the \( i \)th contract in \( X' \) to \( Y \);  
5 return \( Y \);  

We remark that we can implement the mechanism to run in \( O(|X| \log |X|) \) time by using heaps. Let us prepare a min-heap with respect to utility per wage for each hospital. We derive the time complexity from an “amortized” analysis and can add a newly proposed contract \( x \in X_h \) to the heap for \( h \in O(\log(|X_h|)) \) time. When a hospital \( h \) rejects a contract, we can delete it in \( O(\log(|X_h|)) \) time. Hence, we obtain that the total time complexity is \( O(\sum_{h \in H} |X_h| \log(|X_h|)) = O(|X| \log(|X|)) \).

Next, let us illustrate this mechanism via an example.

**Example 2.** Consider a market with five doctors \( D = \{d_1, d_2, d_3, d_4, d_5\} \) and two hospitals \( H = \{h_1, h_2\} \). The set of offered contracts is

\[
X = \{(d_1, h_1, 57), (d_2, h_1, 50), (d_3, h_1, 42), (d_4, h_1, 55), (d_5, h_1, 50), (d_1, h_2, 100), (d_2, h_2, 100), (d_3, h_2, 100), (d_4, h_2, 100), (d_5, h_2, 100)\}.
\]

Here, \( h_1 \) offers the doctors wages from 42 to 57, while \( h_2 \) offers each of them wage 100. We assume that the preferences of the doctors are

\[
\succ_d: \quad (d_1, h_1, 57) \succ_d (d_1, h_2, 100),  
\succ_{d_2}: \quad (d_2, h_1, 50) \succ_{d_2} (d_2, h_2, 100),  
\succ_{d_3}: \quad (d_3, h_1, 42) \succ_{d_3} (d_3, h_2, 100),  
\succ_{d_4}: \quad (d_4, h_1, 55) \succ_{d_4} (d_4, h_2, 100),  
\succ_{d_5}: \quad (d_5, h_2, 100) \succ_{d_5} (d_5, h_1, 50).
\]

The utilities of the hospitals are given in Table 1. Each hospital has a common fixed budget \( 100 (B_{h_1} = B_{h_2} = 100) \).

First, each doctor chooses her most preferred contract:

\[
X' = \{(d_1, h_1, 57), (d_2, h_1, 50), (d_3, h_1, 42), (d_4, h_1, 55), (d_5, h_2, 100)\}.
\]

Since \( \lceil B_{h_1}/\bar{w}_{h_1} \rceil = 3 \), \( Ch_{h_1}(X') \) chooses the top three contracts according to the ranking of utilities per wage shown in Table 1, i.e., \( \{(d_4, h_1, 55), (d_3, h_1, 42), (d_2, h_1, 50)\} \). As well, \( Ch_{h_2}(X') \) chooses the top \( \lceil B_{h_2}/\bar{w}_{h_2} \rceil = 1 \) contract, i.e., \( \{(d_5, h_2, 100)\} \).

| \( x \in X_{h_1} \) | \( f_{h_1} \) | \( f_{h_1}/w \) | \( x \in X_{h_2} \) | \( f_{h_2} \) | \( f_{h_2}/w \) |
|---|---|---|---|---|---|
| \((d_1, h_1, 57)\) | 111 | 1.95 | \((d_1, h_2, 100)\) | 50 | 0.50 |
| \((d_2, h_1, 50)\) | 98 | 1.96 | \((d_2, h_2, 100)\) | 30 | 0.30 |
| \((d_3, h_1, 42)\) | 83 | 1.98 | \((d_3, h_2, 100)\) | 20 | 0.20 |
| \((d_4, h_1, 55)\) | 110 | 2.00 | \((d_4, h_2, 100)\) | 10 | 0.10 |
| \((d_5, h_1, 50)\) | 101 | 2.02 | \((d_5, h_2, 100)\) | 40 | 0.40 |
Then, $d_1$ chooses her second preferred contract,

$$X' = \{(d_1, h_2, 100), (d_2, h_1, 100), (d_3, h_1, 42), (d_4, h_1, 55), (d_5, h_2, 100)\}.$$  

$\text{Ch}_{h_2}(X')$ is $\{(d_1, h_2, 100)\}$, whose utility per wage is larger than $(d_5, h_2, 100)$.

Next, $d_5$ chooses her second preferred contract, i.e., $(d_2, h_2, 100)$, whose utility per wage is 2.02. Since this is higher than the other contract in $X'_{h_1}$, $(d_2, h_1, 50)$ is rejected. Thus, $d_2$ chooses her second preferred contract, i.e., $(d_2, h_2, 100)$:

$$X' = \{(d_1, h_2, 100), (d_2, h_2, 100), (d_3, h_1, 42), (d_4, h_1, 55), (d_5, h_1, 50)\}.$$  

$\text{Ch}_{h_2}(X') = \{(d_1, h_2, 100)\}$, since it has a higher utility per wage than $(d_2, h_2, 100)$. Finally, since $d_2$ no longer has a preferred contract:

$$X' = \{(d_1, h_2, 100), (d_3, h_1, 42), (d_4, h_1, 55), (d_5, h_1, 50)\}.$$  

No contract is rejected and the mechanism terminates.

We claim that the choice function satisfies the following properties.

**Lemma 1.** For each hospital $h$, the choice function defined in Algorithm 2 is SUB, IRC, LAD, and COM.

**Proof.** It is straightforward that the choice function is SUB, IRC, and LAD because it simply picks at most the top $\min\{k_h, |X'|\}$ contracts. Next, let us turn to COM. Let $X'' \subseteq X' \subseteq X_h$. If $|X'| \leq k_h$, since the choice function picks all contracts in $X'$, $f_h(\text{Ch}_h(X')) = f_h(X') \geq f_h(X'')$ clearly holds. On the other hand, if $|X'| > k_h$, since it picks $k_h$ contracts, we have $w_h(\text{Ch}_h(X')) \geq w_h \cdot k_h \geq B_h$. Thus, it is sufficient to claim that

$$f_h(\text{Ch}_h(X')) \geq f_h(X'') \text{ if } w_h(X'') \leq w_h(\text{Ch}_h(X'))$$

for any $X'' \subseteq X' \subseteq X$.

Since the choice function greedily picks $k_h$ contracts with respect to the utility per wage, the chosen contracts yield the optimal utility of a fractional knapsack problem. Also, to maximize the utility of hospital $h$ with $X''$, we need to solve an integral knapsack problem. Therefore, $f_h(\text{Ch}_h(X'))$ is at least

$$\max_{\substack{x \in [0, 1]^X'}} \left\{ \sum_{x \in X'} f_h(x) \cdot z_x \mid \sum_{x \in X'} x_W \cdot z_x \leq w_h(\text{Ch}_h(X')) \right\}$$

$$\geq \max_{x \in \{0, 1\}^X'} \left\{ \sum_{x \in X'} f_h(x) \cdot z_x \mid \sum_{x \in X'} x_W \cdot z_x \leq w_h(\text{Ch}_h(X')) \right\}$$

$$= \max_{Y \subseteq X} \left\{ \sum_{x \in Y} f_h(x) \mid \sum_{x \in Y} x_W \leq w_h(\text{Ch}_h(X')) \right\} \geq f_h(X'').$$

Note that the first inequality is derived from the fact that the optimal value of the fractional knapsack problem is never worse than that of the integral one. Thus, the choice function $\text{Ch}_h$ satisfies COM. The proof is complete.

Next, we show the upper bound of the increment of the budgets. The following lemma clearly holds from $|\text{Ch}_h(X')| \leq k_h$ and $x_W \leq \overline{w}_h$ for all $x \in X_h$.

**Lemma 2.** For each choice function defined in Algorithm 2 and a set of contracts $X' \subseteq X_h$, it holds that

$$w_h(\text{Ch}_h(X')) \leq \overline{w}_h \cdot k_h = \overline{w}_h \cdot \lceil B_h / \overline{w}_h \rceil.$$  

Now, we summarize the arguments on the above in the following theorem:
Theorem 3. The generalized DA mechanism with the choice functions defined in Algorithm 2 is strategy-proof for doctors and it produces a \(B'_H\)-stable matching such that \(B_h \leq B'_h \leq \overline{w}_h \cdot k_h\) for any \(h \in H\). In addition, the mechanism can be implemented to run in \(O(|X| \log |X|)\) time.

Finally, note that, this mechanism is almost tight as long as we use the choice functions that satisfy LAD and COM.

Theorem 4. For any \(\overline{w}_h, \underline{w}_h\), and \(B_h\) \((0 < \underline{w}_h \leq \overline{w}_h \leq B_h)\), there exists a set of contracts \(X_h\) and an additive utility function \(f_h : X_h \to \mathbb{R}^+\) such that any choice function \(Ch_h : 2^{X_h} \to 2^{X_h}\) satisfies \(Ch_h(X') > \overline{w}_h \cdot (B_h - \overline{w}_h)/\underline{w}_h\) for some \(X' \subseteq X_h\) if \(Ch_h\) is LAD and COM.

Proof. Let \(k = \lfloor B_h/\underline{w}_h \rfloor\), \(X_h = \{x_1, \ldots, x_{2k}\}\), and

\[
x_i = \begin{cases} (d_i, h, \underline{w}_h) & (i = 1, \ldots, k), \\ (d_i, h, \overline{w}_h) & (i = k + 1, \ldots, 2k).
\end{cases}
\]

\(f_h(x_i) = \begin{cases} \underline{w}_h & (i = 1, \ldots, k), \\ 2 \cdot \overline{w}_h & (i = k + 1, \ldots, 2k).
\end{cases}\)

Since \(Ch_h\) is COM, \(Ch_h(\{x_1, \ldots, x_k\}) = \{x_1, \ldots, x_k\}\) holds. Thus, we have \(|Ch_h(X'_h)| \geq k\) if \(\{x_1, \ldots, x_k\} \subseteq X'_h \subseteq X_h\) because \(Ch_h\) is LAD. Let \(|Ch(X_h) \cap \{x_1, \ldots, x_k\}| = t\) and \(|Ch(X_h) \cap \{x_{k+1}, \ldots, x_{2k}\}| = s\). Here, \(s + t \geq k\). Without loss of generality, we may assume that \(Ch(X_h) = \{x_1, \ldots, x_s, x_{k+1}, \ldots, x_{k+t}\}\).

If \(s \geq \overline{w}_h/\underline{w}_h\) and \(t < k\), \(Ch_h\) is not COM as \(f_h(Ch(X_h)) < f_h(X')\) and \(w_h(Ch_h(X_h)) \geq w_h(X')\) for \(X' = \{x_s \overline{w}_h/\underline{w}_h + 1, \ldots, x_s, x_{k+1}, \ldots, x_{k+t}\}\). Thus, we have \(s < \overline{w}_h/\underline{w}_h\) or \(t = k\). If \(t = k\), we obtain that

\[
w_h(Ch_h(X_h)) = w_h \cdot s + \overline{w}_h \cdot t = w_h \cdot \frac{B_h}{\overline{w}_h} > w_h \cdot \frac{B_h}{\underline{w}_h} - 1 = w_h \cdot \frac{B_h - \overline{w}_h}{\underline{w}_h}.
\]

On the other hand, if \(s < \overline{w}_h/\underline{w}_h\), we get that

\[
w_h(Ch_h(X_h)) = w_h \cdot s + \overline{w}_h \cdot t
\geq w_h \cdot k - (\overline{w}_h - \underline{w}_h) \cdot s
\geq w_h \cdot \frac{B_h}{\overline{w}_h} - (\overline{w}_h - \underline{w}_h) \cdot \frac{\underline{w}_h}{\overline{w}_h}
\geq w_h \cdot \frac{B_h - \overline{w}_h}{\underline{w}_h}.
\]

\[\square\]

4.2 Non-strategy-proof stable mechanism

This subsection proposes a stable mechanism that is not strategy-proof, but improves the budget bound, i.e., this mechanism outputs a matching \(X'\) that is \(B'_H\)-stable where \(B'_h\) is at most \(B_h + \overline{w}_h\) for any \(h \in H\). This bound is best possible from Theorem 1.

As with the first one, the second choice function greedily picks the top \(\min\{k_h, |X'|\}\) contracts. However, \(k_h\) is defined as \(\min\{k \mid \sum_{i=1}^{k} x^{(i)} \geq B_h\}\), where \(x^{(i)}\) denotes the \(i\)th highest contract with respect to utility per wage. Formally, it is given as Algorithm 3. Note that the running time is \(O(|X| \log |X|)\), as with the first mechanism. Let us illustrate this mechanism via an example.

Algorithm 3:

```
input: X' ⊆ X_h  output: Ch_h(X')
1 Initialize Y ← 0;
2 Sort X' in descending order of utility per wage;
3 for i = 1, 2, ..., |X'| do
4     let x be the i-th contract in X';
5     if w_h(Y) < B_h then Y ← Y ∪ \{x\};
6 return Y;
```

10
Example 3. We consider a situation that is identical to Example 2 and the first two rounds are the same.

At the third round, when \(d_5\) chooses \((d_5, h_1, 50)\):

\[
X' = \{(d_1, h_2, 100), (d_2, h_1, 50), (d_3, h_1, 42), (d_4, h_1, 55), (d_5, h_1, 50)\}.
\]

The number of contracts \(Ch_h(X')\) chooses changes from three to two; the total wage of the first two contracts for \(h_1\), i.e., 105, exceeds the budget limit of 100. Thus, \((d_2, h_1, 50)\) and \((d_3, h_1, 42)\) are rejected.

Next, \(d_2\) and \(d_3\) choose their second preferred contracts, i.e., \((d_2, h_2, 100)\) and \((d_3, h_2, 100)\), but those contracts are also rejected. Finally, since they have no longer preferred contracts,

\[
X' = \{(d_1, h_2, 100), (d_4, h_1, 55), (d_5, h_1, 50)\}.
\]

No contract is rejected and the mechanism terminates.

Here, we show the properties that this mechanism satisfies.

Lemma 3. For each hospital, the choice function defined in Algorithm 3 is SUB, IRC, and COM.

Proof. IRC clearly follows from the definition of the choice functions. Next, we claim that the choice functions satisfy SUB. Let \(X'' \subseteq X' \subseteq X_h\). By definition, the utility per wage of any contract in \(Ch_h(X') \subseteq Ch_h(X'') \cap X''\) is higher than that of any contract in \(X' \setminus Ch_h(X') \subseteq X'' \setminus Ch_h(X')\). Hence, we can partition \(X''\) into two subsets: \(H = Ch_h(X') \cap X''\) and \(L = X'' \setminus Ch_h(X')\). Any contract in \(H\) has higher utility per wage than any contract in \(L\). When \(Ch_h\) takes \(X''\) as an input, it first picks all of the contracts in \(H\) and some contracts in \(L\). Therefore, we obtain \(Ch_h(X') \cap X'' \subseteq Ch_h(X'')\) and derive the SUB property:

\[
X'' \setminus Ch(X'') \subseteq X'' \setminus (Ch_h(X') \cap X'' \subseteq X' \setminus Ch_h(X')).
\]

Finally, we prove COM. Let \(X' = \{x^{(1)}, \ldots, x^{(\ell(X'))}\} \subseteq X_h\), where the contracts are arranged in decreasing order of the utility per wage. If \(w_h(X') \leq B_h\), then it is clear that \(Ch_h(X') = X'\) and \(f_h(Ch_h(X')) \geq f_h(X'')\) hold for any \(X'' \subseteq X'\). Otherwise, let \(Ch_h(X') = \{x^{(1)}, \ldots, x^{(k)}\}\). Here,

\[
w_h\{x^{(1)}, \ldots, x^{(k-1)}\} < B_h \leq w_h\{x^{(1)}, \ldots, x^{(k)}\}
\]

holds. As described in Lemma 1, since the greedy solution \(Ch_h(X')\) is optimal, we have \(f_h(Ch_h(X')) \geq f_h(X'')\) for any \(X'' \subseteq X'\) such that \(w_h(X'') \leq w_h(Ch_h(X'))\). Thus, the lemma holds.

It is straightforward to demonstrate that Algorithm 3 does not satisfy LAD. In Example 2 when a set of contracts \(\{(d_2, h_1, 50), (d_1, h_1, 42), (d_4, h_1, 55)\}\) is given, the choice function chooses all the three contracts. Here, if \((d_1, h_1, 57)\) is further added, it chooses only two contracts, i.e., \(\{(d_1, h_1, 57), (d_2, h_1, 50)\}\). Thus, the second mechanism fails to satisfy LAD.

Lemma 4. For each choice function defined in Algorithm 3 and a set of contracts \(X' \subseteq X_h\), it holds that

\[
w_h(Ch_h(X')) < B_h + \overline{w}_h.
\]

Now, we summarize the results for our second mechanism.

Theorem 5. The generalized DA mechanism with the choice functions defined in Algorithm 3 produces a set of contracts \(X'\) that is \(B'_h\) stable where \(B_h \leq B'_h < B_h + \overline{w}_h\) for any \(h \in H\). In addition, the mechanism can be implemented to run in \(O(|X| \log |X|)\) time.

5 Two special cases

This section examines two special cases of hospitals’ utilities.
5.1 Proportional case

This subsection examines a special case of hospitals’ utilities where each hospital has utility over a set of contracts that is proportional to the total amount of wages. Formally, for every \( h \in H, X' \in X_h \), and a constant \( \gamma_h > 0 \),

\[
f_h(X') = \gamma_h \cdot w_h(X')
\]

holds. In this case, we can make the second mechanism strategy-proof without sacrificing the budget bound although stable matching may not exist as Example 1. Specifically, we modify Algorithm 3 so that i) sort \( X' \) in increasing order of wage, instead of decreasing order of utility per wage; ii) pick the contracts in the order while keeping the total wage less than \( B_h \); iii) add the contract with the highest wage unless it is already chosen. Formally, we define as follows.

Algorithm 4:

\begin{algorithm}
\begin{algorithmic}
\State \textbf{input:} \( X' \subseteq X_h \)
\State \textbf{output:} \( Ch_h(X') \)
\State Initialize \( Y \leftarrow \emptyset \);
\State Sort \( X' \) according to increasing order of wages;
\For {\( i = 1, 2, \ldots, |X'| - 1 \)}
\Statex \hspace{1cm} let \( x \) be the \( i \)th contract in \( X' \);
\If {\( w_h(Y) < B_h \)}
\Statex \hspace{1cm} \( Y \leftarrow Y \cup \{x\} \);
\Else
\Statex \hspace{1cm} add the \( |X'| \)th contract (highest wage contract) to \( Y \);
\EndIf
\EndFor
\State \textbf{return} \( Y \);
\end{algorithmic}
\end{algorithm}

Lemma 5. For each hospital \( h \), the choice function defined in Algorithm 4 is SUB, IRC, and COM.

Proof. Let \( X' = \{x^{(1)}, \ldots, x^{(|X'|)}\} \subseteq X_h \) such that

\[
w_h(\{x^{(1)}\}) < \cdots < w_h(\{x^{(|X'|)}\})
\]

and let \( k \) (\(< |X'| \)) be the largest integer that satisfy

\[
\sum_{i=1}^{k} x^{(i)}_W \leq B_h.
\]

Also, let \( X'' = \{x^{(\sigma(1))}, \ldots, x^{(\sigma(s))}\} \subseteq X' \)

such that \( \sigma(1) < \cdots < \sigma(s) \) and let \( k' \) (\(< |X''| \)) be the largest index that satisfy

\[
\sum_{i=1}^{k'} x^{(\sigma(i))}_W \leq B_h.
\]

Then, we have \( k \leq \sigma(k') \) by \( x^{(|X'|)}_{W} \geq x^{(\sigma(s))}_{W} \).

\( Ch_h \) is IRC because \( Ch_h(X') = Ch(X'') \) holds for any \( Ch_h(X') \subseteq X'' \subseteq X' \) by the definition of the choice function.

\( Ch_h \) is SUB because

\[
Ch_h(X') \cap X'' = \{x^{(1)}, \ldots, x^{(k)}, x^{(|X'|)}\} \cap \{x^{(\sigma(1))}, \ldots, x^{(\sigma(s))}\}
\subseteq \{x^{(\sigma(1))}, \ldots, x^{(\sigma(k'))}\} \cup \{x^{(|X'|)}\} \cap \{x^{(\sigma(s))}\}
\subseteq \{x^{(\sigma(1))}, \ldots, x^{(\sigma(k'))}, x^{(\sigma(s))}\} = Ch_h(X'').
\]

Thus, \( Ch_h \) is SUB and IRC.
Next we claim that $\text{Ch}_h$ is COM. Let $t$ satisfy that

$$w_h(\{x^{(1)}\}) < \cdots < w_h(\{x^{(t)}\}) < 1/2 < w_h(x^{(h+1)}) < \cdots < w_h(\{x^{(|X'|)}\}).$$

We consider the three following cases.

**Case 1:** $|\text{Ch}_h(X') \cap \{x^{(h+1)}, \ldots, x^{(|X'|)}\}| \geq 2$. In this case, since $w_h(\text{Ch}_h(X')) > 2 \cdot B_h/2 = B_h$, we have

$$f_h(\text{Ch}_h(X')) = \max \{f_h(X''') \mid X''' \subseteq X', w_h(X''') \leq w_h(\text{Ch}_h(X'))\}.$$

**Case 2:** $k < t$. In this case, since $w_h(\text{Ch}_h(X')) + w_h(\{x^{(l)}\}) > 1.5 \cdot B_h$, we have

$$w_h(\text{Ch}_h(X')) > 1.5 \cdot B_h - w_h(\{x^{(l)}\}) \geq B_h.$$

Thus, it holds that

$$f_h(\text{Ch}_h(X')) = \max \{f_h(X''') \mid X''' \subseteq X', w_h(X''') \leq w_h(\text{Ch}_h(X'))\}.$$

**Case 3:** $k \geq t$ and $|\text{Ch}_h(X') \cap \{x^{(t+1)}, \ldots, x^{(|X'|)}\}| \leq 1$. If $w_h(\text{Ch}_h(X')) \geq B_h$, we have

$$f_h(\text{Ch}_h(X')) = \max \{f_h(X''') \mid X''' \subseteq X', w_h(X''') \leq w_h(\text{Ch}_h(X'))\}.$$

If $h = |X'|$, we have $\text{Ch}_h(X') = X'$, and hence

$$f_h(\text{Ch}_h(X')) = \max \{f_h(X''') \mid X''' \subseteq X', w_h(X''') \leq \max \{B_h, w_h(\text{Ch}_h(X'))\}\}.$$

If $w_h(\text{Ch}_h(X')) < B_h$ and $h < |X'|$, we obtain

$$f_h(\text{Ch}_h(X')) = \max \{f_h(X''') \mid X''' \subseteq X', w_h(X''') \leq B_h\}$$

because $|X'' \cap \{x^{(t+1)}, \ldots, x^{(|X'|)}\}| \leq 1$ holds for any $X'' \subseteq X'$ such that $w_h(X''') \leq B_h$.

The proof is complete. \hfill \qed

**Lemma 6.** For each choice function defined in Algorithm 4 and a set of contracts $X' \subseteq X_h$, it holds that

$$w_h(\text{Ch}_h(X')) \leq 1.5 B_h.$$

Here, $w_h(\text{Ch}_h(X')) \leq 1.5 \cdot B_h$ is obvious by the definition of the choice functions. Thus, $B_h \leq B'_h \leq 1.5 \cdot B_h$ holds for any $h \in H$.

Now, we summarize the arguments on the above in the following theorem.

**Theorem 6.** The generalized DA mechanism with the choice functions defined in Algorithm 4 is strategy-proof for doctors and it produces a $B'_h$-stable matching such that $B_h \leq B'_h \leq 1.5 \cdot B_h$ for any $h \in H$, when each hospital has utility over a set of contracts that is proportional to the total amount of wages.

### 5.2 Equal profit case

We next consider the case that each hospital has the same utility across contracts. Formally, for every $h, X' \subseteq X_h$, and a constant $\gamma_h (> 0)$,

$$f_h(X') = \gamma_h \cdot |X'|$$

holds. In this case, we obtain a strategy-proof mechanism that always produces a conventional stable matching, which never violates the given budget constraints. The choice function greedily chooses contracts in an increasing order of wage until just before the total wage of chosen contracts exceeds the constraint.

**Theorem 7.** The generalized DA with the choice functions defined in Algorithm 5 is strategy-proof and it produces a stable matching.
Algorithm 5:

\[
\begin{align*}
\textbf{input: } & X' \subseteq X_h & \textbf{output: } & Ch_h(X') \\
1 & \text{Initialize } Y \leftarrow \emptyset; \\
2 & \text{Sort } X' \text{ according to increasing order of wages}; \\
3 & \text{for } i = 1, 2, \ldots, |X'| \text{ do} \\
4 & \quad \text{let } x \text{ be the } i\text{th contract in } X'; \\
5 & \quad \text{if } w_h(Y \cup \{x\}) \leq B_h \text{ then } Y \leftarrow Y \cup \{x\}; \\
6 & \text{return } Y;
\end{align*}
\]

\[\textbf{Proof.} \text{ Since the choice functions greedily choose contracts in increasing order of wages, we have}

\[|Ch_h(X')| = \max\{|X''| \mid X'' \subseteq X', \ w_h(X'') \leq B_h\}\]

\[\text{for all } h \text{ and } X' \subseteq X_h. \text{ In other words,}

\[Ch_h(X') \in \arg \max\{f_h(X'') \mid X'' \subseteq X', \ w_h(X'') \leq B_h\}\]

\[\text{holds. Thus, for each } h \text{ and } X' \subseteq X_h, Ch_h(X') \text{ picks the smallest } \max\{|X''| \mid X'' \subseteq X', \ w_h(X') \leq B_h\}\text{ contracts}
\]

\[\text{according to the wage. Therefore, each } Ch_h \text{ satisfies \text{SUB, IRC, and LAD, and hence the mechanism is strategy-proof for doctors. Also, it is clear that the mechanism produces a } B_H\text{-feasible set of contracts.} \]

\]

6 Discussion

Our model assumes that the amount of predefined budgets is flexible up to a certain amount. There is certainly some realistic situation where this assumption is justified. Indeed, in firm-worker matchings, if a firm finds an application from some worker who is appropriate for the business, the CEO would agree to increasing the employment cost. In doctor-hospital matchings, hospitals can make an association that pools some funds in advance and subsidizes the expense of salaries according to matching results. Alternatively, even when budgets must not be exceeded, we can let our mechanisms work by setting the budget \(B_h\) to \(B_h - \overline{w}_h\) in advance. Our second mechanism produces a \(B'_H\)-stable matching such that \(B_h - \overline{w}_h \leq B'_h \leq B_h\) for any \(h \in H\).

Our model can also handle a matching problem with inseparable couples, which was studied by McDermid and Manlove [21]. In this problem, there are single doctors and couples. Each single doctor and couple has a preference over hospitals and each hospital has a maximum quota. Then, we can regard the problem as ours with each wage 1 or 2 and budgets correspond to maximum quotas. Thus, our second mechanism outputs a stable matching that exceeds each maximum quota up to 1.

Let us discuss about the validity of increasing budgets. In the case that firms offer wages to workers, the firm may be able to increase its budget if it can employ superior workers. In the case that public hospitals offer salaries to doctors, some association may subsidize the expense of the salaries. When budgets are hard constraints, \(B_H\)-stable matchings may not exist but we can get a stable matching by decreasing budgets. In fact, by using our second mechanism with reducing the budgets of hospitals to \(B_h - \overline{w}_h\) in advance, we can obtain a \(B'_H\)-stable matching such that \(B_h - \overline{w}_h \leq B'_h \leq B_h\) holds for any \(h \in H\).

One might think that we could handle budget constraints together with maximum quota constraints. However, it is not so straightforward to design an appropriate choice function that handles both constraints simultaneously. Suppose, for example, that a hospital has the maximum quota of one and offers two contracts. One contract \(x\) has lower wage than the other contract \(x' (x_W < x'_W).\ The hospital has lower utility for the former than the latter \((f_h(x) < f_h(x'))\), and conversely it has higher utility per wage for the former than the latter \((f_h(x)/x_W > f_h(x')/x'_W).\ \{x'\} is the unique stable matching. Simply if the choice function additionally checks whether the current number of chosen contracts exceeds the maximum quota in line 5 in Algorithm 3 it chooses \{x\} and it fails to provide the stable matching. In general, this problem is known as cardinality constrained knapsack problem [6][16]. Building upon techniques for the problem, whether we can construct a proper choice function or not is still an open question.
Let us discuss about doctor-optimality of the mechanisms. Unfortunately, our both mechanisms do not always produce a doctor-optimal stable matching. To see this, consider a market with four doctors $D = \{d_1, d_2, d_3, d_4\}$ and two hospitals $H = \{h_1, h_2\}$. The set of offered contracts is $X = \{(d_1, h_1, 1), (d_2, h_2, 1), (d_3, h_1, 1), (d_4, h_1, 1), (d_3, h_2, 1), (d_4, h_2, 1)\}$. Suppose that $d_3$ prefers $(d_3, h_1, 1)$ to $(d_3, h_2, 1)$ and $d_4$ prefers $(d_4, h_2, 1)$ to $(d_4, h_1, 1)$. Also, let $B_{h_1} = 2$, $B_{h_2} = 1$, and $f_{h_1}((d_1, h_1, 1)) = 7$, $f_{h_2}((d_2, h_1, 2)) = 6$, $f_{h_3}((d_3, h_1, 1)) = 1$, $f_{h_3}((d_3, h_1, 1)) = 4$, $f_{h_3}((d_4, h_2, 1)) = 2$, $f_{h_3}((d_4, h_2, 1)) = 1$. Then, our mechanisms produce a stable matching $X' = \{(d_1, h_1, 1), (d_3, h_1, 1), (d_4, h_1, 1)\}$. However, $X'$ is not a doctor-optimal stable matching because another stable matching $X'' = \{(d_1, h_1, 1), (d_3, h_1, 1), (d_4, h_2, 1)\}$ Pareto-dominates $X'$.

Let us finally note that there is a certain amount of recent studies on two-sided matchings in the AI and multi-agent systems community, although this literature has been established mainly in the field across algorithms and economics. Drummond and Boutilier \cite{DBLP:conf/ijcai/DrummondB15, DBLP:conf/ijcai/DrummondB09} examine preference elicitation procedures for two-sided matching. In the context of mechanism design, Hosseini, Larson, and Cohen \cite{DBLP:conf/ijcai/HosseiniLC14} consider a mechanism for a situation where agents’ preferences dynamically change. Kurata et al. \cite{DBLP:conf/ijcai/KurataBM16} deal with strategy-proof mechanisms for affirmative action in school choice programs (diversity constraints), while Goto et al. \cite{DBLP:conf/ijcai/GotoMC16} handle regional constraints, e.g., regional minimum/maximum quotas are imposed on hospitals in urban areas so that more doctors are allocated to rural areas.

7 Conclusion

This paper deals with matching with budget constraints, introduced a concept of near-feasible matchings and proposed two novel mechanisms that return a stable matching in polynomial time: one is strategy-proof and the other is not. Furthermore, we derived the bound of increment of the budgets. In particular, the best possible bound is obtained by sacrificing strategy-proofness. In future work, we would like to derive the lower bound for strategy-proof mechanisms and extend our results to matching problems with other constraints.

References

\[1\] A. Abdulkadiroğlu and T. Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

\[2\] A. Abizada. Stability and incentives for college admissions with budget constraints. *Theoretical Economics*, 11(2):735–756, 2016.

\[3\] S. Barberà, W. Bossert, and P. Pattanaik. Ranking sets of objects. In *Handbook of Utility Theory*, volume 2. Kluwer Academic Publishers, 2004.

\[4\] S. Bouveret and J. Lang. Elicitation-free protocol for allocating indivisible goods. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 73–78, 2011.

\[5\] E. Budish and E. Cantillon. The multi-unit assignment problem: Theory and evidence from course allocation at Harvard. *American Economic Review*, 102(5):2237–2271, 2012.

\[6\] A. Caprara, H. Kellerer, U. Pferschy, and D. Pisinger. Approximation algorithms for knapsack problems with cardinality constraints. *European Journal of Operational Research*, 123(2):333–345, 2000.

\[7\] G. B. Dantzig. Discrete-variable extremum problems. *Operations Research*, 5(2):266–277, 1957.

\[8\] B. C. Dean, M. X. Goemans, and N. Immorlica. The unsplittable stable marriage problem. In *Proceedings of the 5th IFIP International Conference on Theoretical Computer Science*, pages 65–75, 2006.

\[9\] J. Drummond and C. Boutilier. Elicitation and approximately stable matching with partial preferences. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 97–105, 2013.

\[10\] J. Drummond and C. Boutilier. Preference elicitation and interview minimization in stable matchings. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence*, pages 645–653, 2014.
[11] M. Goto, A. Iwasaki, Y. Kawasaki, R. Kurata, Y. Yasuda, and M. Yokoo. Strategyproof matching with regional minimum and maximum quotas. *Artificial Intelligence*, 235:40 – 57, 2016.

[12] N. Hamada, A. Ismaili, T. Suzuki, and M. Yokoo. Weighted matching markets with budget constraints. In *Proceedings of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 317–325, 2017.

[13] J. W. Hatfield and P. R. Milgrom. Matching with contracts. *American Economic Review*, 95(4):913–935, 2005.

[14] H. Hosseini, K. Larson, and R. Cohen. Matching with dynamic ordinal preferences. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence*, pages 936–943, 2015.

[15] Y. Kamada and F. Kojima. Efficient matching under distributional constraints: Theory and applications. *American Economic Review*, 105(1):67–99, 2015.

[16] H. Kellerer, U. Pferschy, and D. Pisinger. *Knapsack Problems*. Springer, 2004.

[17] A. Kelso and V. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.

[18] F. Kojima, P. A. Pathak, and A. E. Roth. Matching with couples: Stability and incentives in large markets. *The Quarterly Journal of Economics*, 128(4):1585–1632, 2013.

[19] R. Kurata, N. Hamada, A. Iwasaki, and M. Yokoo. Controlled school choice with soft bounds and overlapping types. *Journal of Artificial Intelligence Research*, 58:153–184, 2017.

[20] D. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific Publishing Company, 2013.

[21] E. J. McDermid and D. F. Manlove. Keeping partners together: algorithmic results for the hospitals/residents problem with couples. *Journal of Combinatorial Optimization*, 19:279–303, 2010.

[22] S. J. Mongell and A. E. Roth. A note on job matching with budget constraints. *Economics Letters*, 21(2):135–138, 1986.

[23] T. Nguyen and R. Vohra. Near feasible stable matchings. In *Proceedings of the 16th ACM Conference on Economics and Computation*, pages 41–42, 2015.

[24] A. Perrault, J. Drummond, and F. Bacchus. Strategy-proofness in the stable matching problem with couples. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 132–140, 2016.

[25] A. E. Roth and M. A. O. Sotomayor. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis (Econometric Society Monographs)*. Cambridge University Press, 1990.