Quantum view on contextual logic of composite intelligent devices

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Based on the ideas of quantum theory of open systems (QTOS) we propose the consistent approach to study probabilistic many-valued propositional logic of intelligent devices that are composed from separate but interconnected logical units. In this preliminary communication we consider only the simplest example of such systems, namely, four-valued probabilistic logical device composed of two logical subsystems. We demonstrate that similar devices can generate two classes of probabilistic propositions: 1) decomposable propositions, which in fact are equivalent to certain ordered pairs of propositions in device subsystems and 2) indecomposable propositions which are connected with inherent logical interaction between device units. The indecomposable propositions are undoubtedly of greatest interest since they, as shown in the paper, provide powerful additional logical resource compared to standard parallel processing in composite intelligent systems. The contextual logic of composite devices proposed in this paper, as we believe, can be also used for analysis of highly organized systems in cognitive sciences specifically in neuropsychology and linguistics.

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I. INTRODUCTION

In the last years close attention of quantum physicists community was focused on the possibility of creation of effectively working quantum computer (QC). Although certain progress has been achieved recently in this field (see for example [1]), nevertheless many essential difficulties yet to be overcome. The most serious obstacle to solve this problem is undoubtedly decoherence that is interaction of working part of QC with its environment that leads to the destruction of quantum states superpositions and thus to lower efficiency of quantum computations. Thereby as we believe it makes sense to pay attention also to other options to increase computing resource of intelligent devices which would not be so sensitive to destruction by inner and exterior noise. One such option connected with the study of composite probabilistic logical devices just considered in present paper. The approach proposed uses essentially the results of preceding authors paper [2] in which on the basis of quantum theory of open systems (QTOS) were considered both probabilistic logical propositions and all possible logical operations (that is logical connectives) with them. Thus it becomes possible by simple and unified method to formulate various logical calculuses: probabilistic logic (including usual Boolean logic), many-valued logic (including a modal logic) and so on. In the present paper our main goal with the help of ideas and methods of QTOS to state the logic of composite probabilistic devices that composed of two or more logical units mentally interacting with each other. Our study can not be considered as too much speculative. Indeed here is sufficient to point out the well-known fact that brains, both animals and humans, consist exactly from two hemispheres each of which is able to perform special mental functions and continuously exchange with partner by cognitive information. The rest of the paper is organized as follows. In Section I for the convinience of the reader we briefly describe the information from [2] concerning probabilistic logic which is necessary for the understanding of the given paper. In addition we introduce here the valuable concept of distance between two probabilistic propositions with the help of which one can define the degree of closeness of different propositions. In the Section II, that is the main in the paper, we consider the structure of logical propositions in composite logical devices (CLD). In particular the simplest example of four-valued logical device is studied more detail. We demonstrate that all logical propositions in similars system can be divided into two different classes: 1) decomposable propositions (DP) each of which is in fact equivalence to certain ordered pair of propositions in device logical subsystems and 2) indecomposable propositions (IP) which are connected with inherent mental interaction between device subsystems. We believe that the existence of IP and the possibility to operate with them represents additional powerful logical resource in composite systems as compared with usual parallel information processing. Let us now go to the presentation of concrete results of the paper.

II. PRELIMINARY INFORMATION

In this part we briefly remind main results of the paper [2] which are necessary for the understanding of the following text. As well as in [2] the main subject of our consideration will be a set of plausible propositions (PP), that is the propositions, the truth or falsity of which is known to recipient not exactly but only with certain probability. It is convinent to represent any
such proposition by normalized to unity diagonal matrix with positive elements. For example in the simplest case of two-valued logic any plausible proposition \( A \) can be represented by the matrix:

\[
\rho (A) = \begin{pmatrix} p_A & 0 \\ 0 & 1 - p_A \end{pmatrix},
\]

where \( p_A \) is the probability for \( A \) to be true. In what we follow everywhere where it does not lead to confusion we will identify propositions with their representative matrices. It turns out that various operations with \( PP \), that is logical connectives, under such approach can be realized as special transformations of their representative matrices. Referring the reader for details to (2) we represent here only final decisive result that we will need in the form of the next theorem.

Theorem 1. Let \( \rho \) be \( N \times N \) diagonal matrix with positive elements whose trace is equal to 1 and we assume that \( G \) is some \( M \times N \) matrix which possesses two defining properties: 1) all elements of \( G \) are equal 0 or 1 and 2) in each column of \( G \) only single element is equal to 1 and all the rest are equal to zero. In this case the transformation of the form \( \tilde{\rho} = GG_T \) maps \( \rho \) onto \( M \times M \) diagonal matrix \( \tilde{\rho} \) with positive elements \( \tilde{\rho}_{ii} \) that satisfy to the relation \( \sum_{i=1}^{M} \tilde{\rho}_{ii} = 1 \). Similar approach can be used in the general case of \( N \)-valued probabilistic (or modal) logic where every proposition has \( N \) logical alternatives with corresponding probabilities \( p_i (i = 1, \ldots, N) \) which satisfy to normalization condition \( \sum_{i=1}^{N} p_i = 1 \). We will call the above transformations as admissible (logical) transformations. Having in hands the above result one can by unifying way determine all \( n \)-place logical operations in \( N \)-valued probabilistic logic. For example the result of any one-place operation applied to \( PP \) \( A \) can be represented in the form:

\[
\tilde{A} = G_1 \cdot A \cdot G_1^T,
\]

where \( G_1 \) is some admissible \( N \times N \) matrix.

Similarly the result of any two-place logical operation applied to propositions \( B_1 \) and \( B_2 \) can be written as

\[
\tilde{B} = G_2(B_1 \otimes B_2)G_2^T,
\]

where \( G_2 \) is some admissible \( N^2 \times N \) matrix.

It is clear that with the help of admissible transformations one can determine logical operations for any number of \( PP \) as well. Note that in the case of \( N \)-place logical operation it is necessary to take \( N \) propositions \( A_1, A_2, \ldots, A_N \), and choose as initial state the tensor product of them, that is \( A_1 \otimes A_2 \otimes \ldots \otimes A_N \). It is clear also that total number of \( n \)-place logical operations in \( N \)-valued logic is equal to \( N^N \). The number of two-place operations is equal to \( N^{N^2} \) and so on. Note that the approach proposed also allows one to introduce the concept of distance between two \( N \) valued propositions. Indeed, according to natural reason, the distance \( D(A, B) \) between two propositions \( A \) and \( B \) can be defined as:

\[
D(A, B) = \frac{1}{2} \sum_{i=1}^{N} | p_i - q_i | (we \ have \ in \ mind \ here \ that \ propositions \ A \ and \ B \ have \ representative \ matrices: diag\{p_1, p_2, p_N \} \ and \ diag\{q_1, q_2, q_N \} \) correspondently. It is clear that above definition is satisfied to all relevant conditions of the concept. Using this definition of closeness and also the definitions of basic connectives between propositions introduced in [2], one can prove for example the relation \( D(A \implies B, A \land B) \geq D(B, A \land B) \) and many other similar relations. On the other hand from the physical point of view it is very important that one can think of representative matrix of proposition as density matrix of relevant quantum system, and hence should to realize all plausible propositions and logical operations with them as result of certain physical manipulations in correspondent quantum system. In addition this analogy allows one to propose concrete physical realizations of logical devices in order to improve their effectiveness of information processing.

III. THE LOGIC OF COMPOSITE LOGICAL DEVICE. THE SIMPLEST MODEL.

Till now we consider probabilistic logic and possible logical devices in which they can be generated as certain integral systems. However, using the analogy with quantum theory an essential element of which is investigation of composite systems it is fully justified to include in our focus also composite logic and correspondently composite logical devices (CLD) consisting of several subsystems that are logically interconnected with each other. Evidently such approach is not pure academical. To confirm it enough to point out that brains of humans and animals exactly consist of two hemispheres each of which is able to perform special cognitive functions. In this connection it is very essential to emphasize two facts: 1) a flow of afferent information from sensor organs to brain reaches both hemispheres almost simultaneously and 2) left and right hemispheres can continuously exchange by cognitive information with partner through special fiber system in the brain, so called the corpus callosum.

Of course the simplest and in many respects primitive model which we will study in this paper by no means can not represent to the right degree any realistic model of the brain. Nevertheless we believe the very idea about peculiar logic of CDL deserves close attention and further study. Now let us turn to the description of composite logic. Note that in this paper in view of maximal simplicity and clearness we restrict ourselves only the simplest case of the device in which composite logic should be occur. Namely we will discuss the four-valued composite logical device consisting of two-valued subsystems (units). It should be noted however that almost all results of our consideration can be generalized (with appropriate refinements) to the more complex situations. Returning to four-valued CDL remind, that according to Sect. 1 any proposition in four-valued logic \( A \)
can be represented in the form of diagonal matrix $4 \times 4$, namely: $A = \text{diag}\{p_1, p_2, p_3, p_4\}$ where probabilities of different logical outcomes $p_i$ ($i=1..4$) satisfy to the conditions: 1) $0 \leq p_i \leq 1$ and 2) $\sum_{i=1}^{4} p_i = 1$. Our next step is the statement of basic logical operations (connectives) with such propositions. According to theorem 1 from Sect. 1 this task can be realized with the help of relevant admissible matrices. Omitting intermediate and to some extent tedious expressions for admissible matrices we adduce here only the required results for the basic connectives. So, the negation of PP $A$ that is $\overline{A}$ can be written as:

$$
\overline{A} = \begin{pmatrix}
p_1 \\
p_3 \\
p_2 \\
p_1 
\end{pmatrix},
$$

(3)

conjunction of two propositions $A$ and $B = \text{diag}\{q_1, q_2, q_3, q_4\}$ is equal to

(A and $B$) = 

$$
\begin{pmatrix}
p_1q_1 \\
p_1q_2 + p_2q_1 + p_2q_2 \\
p_1q_3 + p_3q_1 + p_3q_3 \\
p_1q_4 + p_4 + q_4 - q_4q_4
\end{pmatrix},
$$

(4)

disjunction of the same two propositions $A$ and $B$ is equal to

(A or $B$) = 

$$
\begin{pmatrix}
p_3q_2 + p_2q_3 + p_1 + q_1 - p_1q_1 \\
p_4q_2 + p_2q_4 + p_2q_2 \\
p_4q_3 + p_3q_4 + p_3q_3 \\
p_4q_4
\end{pmatrix},
$$

(5)

and so on. Clearly that in the case of four-valued logic there are considerably more one-place and two-place connectives than in two-valued case. For example it is easy to see that we have $4^4 = 256$ one-place connectives in contrast of $2^2$ similar connectives in two-valued logic. Now using the analogy with quantum theory of composite systems we can make the next important step. Let us consider two logical projections of some four-valued proposition on corresponding two-valued propositions in each of the subsystem of CLD). For example arbitrary proposition $A = \text{diag}\{p_1, p_2, p_3, p_4\}$ has two logical projections: $A_1 = \begin{pmatrix} p_1 + p_2 \\ p_3 + p_4 \end{pmatrix}$ in the first subsystem and $A_2 = \begin{pmatrix} p_1 + p_3 \\ p_2 + p_4 \end{pmatrix}$ in the second one. On the language of the quantum theory one should say that the density matrix of composite quantum system uniquely determines corresponding density matrices of its subsystems. By means of this mapping we can set the correspondence between all logical operations in composite system and correspondent operations in its subsystems. It turns out that above definitions of basic logical connectives in four-valued logic imply the next simple rules between operations in composite system and corresponding operations in its subsystems. Indeed one can easily verify that the simple relations hold: 1) $(\overline{A})_1 = \overline{A}_1$ , $(\overline{A})_2 = \overline{A}_2$ for negation, and 2) for two-place operations: $(A$ and $B)_1 = (A_1$ and $B_1)$ , $(A$ or $B)_1 = (A_1$ or $B_1)$.

Now we intend to demonstrate that there are two distinct classes of propositions generated in composite device, namely 1) decomposable propositions (DP) each of which in fact is equivalent to some ordered pair of propositions in device subsystems and 2) indecomposable propositions (IP) that do not allow such a reduction. To prove this fact it is convinient to use the representation which is valid in fact for any four-valued proposition $A$. Namely let $A = \text{diag}\{p_1, p_2, p_3, p_4\}$ then it is easy to see that $A$ can be represented also as
\[ A = \begin{pmatrix} pq + C \\ p(1-q) - C \\ q(1-p) - C \\ (1-p)(1-q) + C \end{pmatrix}, \] (6)

where in Eq. (6) we use the notation: \( p = p_1 + p_2, \) \( q = p_1 + p_3 \) and \( C = p_1p_4 - p_2p_3. \)

The collection of propositions which satisfy to condition \( C = 0 \) will be called decomposable since for them the decomposition in the form of tensor product \( A = A_1 \otimes A_2 \) obviously holds. Thus such "composite" propositions are in fact coincided with tensor products of their logical projections. Note that in the quantum theory of composite systems analogues of these propositions are factorable pure states and correspondent quantity \( C \) (that is equal to zero in the case of factorable states) is called concurrence. However, with respect to the case of probabilistic many-valued logic, will call the quantity \( C \) from Eq. (6) the context variable or simply as context. Reasons for such designation will become clear a little later. Now let us point out two nearly obvious facts concerning DP: 1) if \( A \) is DP then \( \overline{T} \) is DP as well and 2) if \( A \) and \( B \) are two DP then propositions \( (A \otimes B) \) and \( (A \lor B) \) are decomposable also. Thereby the collection of all DP is closed with respect to basic logical transformations, namely one can prove that the following theorem has place. Theorem 2: let \( A \) is DP in composite system and \( G \) some one-place admissible transformation. Then \( A = GAG^T \) is decomposable proposition as well and can uniquely be represented in the form: \( \overline{A} = \overline{A_1} \otimes \overline{A_2} \) where \( \overline{A_1} = G_1(A_1 \otimes A_2)G_1^T \) and \( \overline{A_2} = G_2(A_1 \otimes A_2)G_2^T \) where \( (G_1, G_2) \) is some ordered pair of two-place admissible transformations and \( A_1, A_2 \) are logical projections of DP \( \overline{A} \) in device subsystems. Without stopping to prove theorem 2 note only that total number of admissible one-place transformations in CLD is equal \( 4^4 \) which is coincides with the number of ordered two-place pairs \( (G_1, G_2) \) which is equal to \( 2^4 \times 2^4 \).

Thus we come to the important conclusion: all logical operations with DP in CLD can be entirely reduced to the logical operations produced in device subsystems that is in this case CLD works exactly according to principles of parallel processing information. On the other hand we argue that for indecomposable propositions the situation is quite different. It turns out that existence of IP and the possibility to operate with them to a large degree increase logical resources of CLD. This important topic undoubtedly deserves a special and detail investigation. But in this paper which has preliminary nature we restrict ourselves only to single but very good example illustrating the above thesis. Let us consider in CLD one parameter collection of indecomposable propositions of the next form:

\[ A(C) = \begin{pmatrix} \frac{1}{4} + C \\ \frac{1}{4} - C \\ \frac{1}{4} - C \\ \frac{1}{4} + C \end{pmatrix}, \] (7)

where the variable \( C \) satisfies to inequalities: \(-\frac{1}{4} \leq C \leq \frac{1}{4}\). Evidently that if \( C \neq 0 \) then all propositions \( A(C) \) are indecomposable and the context variable for proposition \( A(C) \) coincides exactly with \( C. \) Note in passing that for any IP \( A \) its distance to the nearest DP that is \( A_1 \otimes A_2 \) is equal exactly to \( 4|C| \) where \( C \) is the value of context variable for \( A. \) It is easy to see also that logical projections for proposition \( A(C) \) from (7) are independent from context \( C \) and equal to each other, namely \( A_1 = A_2 = \left( \frac{1}{2}, \frac{1}{2} \right). \) Now let us perform the concrete one-place logical operation: \( \overline{A}(C) = GA(C)G^T \), where the admissible matrix \( G \) has the form:

\[ G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (8)

Note that above operation is isometric that is it does not change the distances between propositions from collection of interest. Really, it is easy to verify that: \( D[A(C_1), A(C_2)] = D[\overline{A}(C_1), \overline{A}(C_2)] = 4|C_1 - C_2|. \) In explicit form the matrix \( A(C) \) reads as

\[ \overline{A}(C) = \begin{pmatrix} \frac{1}{4} - C \\ \frac{1}{4} + C \\ \frac{1}{4} - C \\ \frac{1}{4} + C \end{pmatrix}, \] (9)

The expression Eq. (9) implies that logical projections of \( \overline{A}(C) \) in device subsystems has the form: \( \overline{A_1} = \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( \overline{A_2} = \left( \frac{1}{2} - 2C, \frac{1}{2} + 2C \right). \) We see that logical projection of transformed proposition in the first subsystem does not change, while its projection in the second subsystem essentially depends from context value \( C. \) Moreover the very plausibility of proposition \( \overline{A_2} \) may be determined by context and depending on \( C \) can change its value to the contrary. Indeed, the value of \( C = -\frac{1}{4} \)
implies that  $\widetilde{A}_2\left(\frac{1}{4}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and value of $C = \frac{1}{4}$ implies that  $\widetilde{A}_2\left(\frac{1}{4}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In this connection one can say that the applied logical operation is connected with recognition of context by second subsystem (agent). Of course the similar transformation can be applied with respect to first subsystem (agent) as well. We are convinced that application of IP and logical operations with them allows one significantly expend the class of deduced propositions in subsystems of CLD comparing with ordinary Boolean functions from $p$ and $q$ that are realized in the case of decomposable propositions.

Thus already this preliminary analysis of simple CLD clearly demonstrates the advantages of such devices in logical processing in comparance with ordinary parallel processing systems.

In conclusion we emphasize that many important and interesting issues have remained outside of our scope. Among the most important topics we may call such as: 1) the need to generalize the approach proposed on the case of composite system consisting of arbitrary number of subsystems with different dimensions 2) the need to classify logical operations in CDL from the point of view of their possibility to transform DP into IP and vica versa. Besides it is necessary to study in detail the problem of realization CDL in concrete quantum physical systems. In particular in the case of simplest four-valued CDL we should talk about two-qubit quantum system interacting with environment by special way. In part we consider this issue in [2], but this question needs much more detail investigation.

All these and some other questions we hope to discuss in later publications.

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