THE ROLE OF THE BESOV SPACE $B^{-1,\infty}_{\infty}$ IN THE CONTROL OF THE EVENTUAL EXPLOSION IN FINITE TIME OF THE REGULAR SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Abstract. This paper is essentially a translation from French of my article [9] published in 2003. Let $u \in C([0,T^*]; L^3(\mathbb{R}^3))$ be a maximal solution of the Navier-Stokes equations. We prove that $u$ is $C^\infty$ on $[0,T^*[\times \mathbb{R}^3$ and there exists a constant $\varepsilon_*>0$ independent of $u$ such that if $T^*$ is finite then, for all $\omega \in S(\mathbb{R}^3)$, we have $\lim_{t \to T^*} \|u(t)-\omega\|_{B^{-1,\infty}_{\infty}} \geq \varepsilon_*$. 

1. Introduction and results

In this note, we consider the integral Navier-Stokes equations:

$$u(t) = e^{t\Delta}u_0 + L(\mathbb{P}\nabla.(u \otimes u))(t),$$

where $u_0(x) = (u_{01}, u_{02}, u_{03})$ is a given initial data satisfying the divergence free condition $\nabla.u_0 = 0$ and $u(t, x) = (u_1, u_2, u_3)$, the velocity, is the unknown. The operator $\mathbb{P} = (\mathbb{P}_{ij})_{1\leq i,j \leq d}$ is the Leray projector and $L$ is the linear operator defined by:

$$L(f)(t) = -\int_0^t e^{(t-s)\Delta}f(s)ds.$$ 

Here $(e^{t\Delta})_{t>0}$ is the heat semi-group defined through the Fourier Transform $\mathcal{F}$

$$\mathcal{F}(e^{t\Delta}f)(\xi) = e^{-t|\xi|^2} \mathcal{F}(f)(\xi).$$

In the sequel, we denote by $L^3_{\mathbb{R}^3}$ the space of $f = (f_1, f_2, f_3) \in L^3(\mathbb{R}^3)$ that $\nabla.f = 0$. It is well known (see [5] and [4]) that for any initial data $u_0 \in L^3_{\mathbb{R}^3}$, the equation has a unique maximal solution $u \in C([0,T^*]; L^3(\mathbb{R}^3))$. Recently, by using the Caffarelli, Kohn et Nirenberg criterion, P. G. Lemarié-Rieusset [7] have proved that such solution $u$ is smooth on $Q_{T^*} \equiv [0,T^*] \times \mathbb{R}^3$. In this paper, we will give a direct and simple proof of this result.

Hereafter, we suppose that the maximal existence time $T^*$ of the solution $u$ is finite. The main purpose of this short paper, is to study the behavior of the solution near blowup time $T^*$. Let us first recall some known results in this direction: J. Leray [8] and Y. Giga [3] proved that for any $p$ in $[3, +\infty]$ there exists a constant $c_p > 0$ such that

$$\|u(t)\|_p \geq c_p(T^* - t)^{\frac{1}{2}}.$$

For the limit case $p = 3$, H. Shor and W. Von Wahl [11] proved that the solution $u$ can not be extended to a continuous function from $[0,T^*]$ into $L^3(\mathbb{R}^3)$. Later, H. Kozono and H. Shor
improved this result: they established that there exists a constant \(\varepsilon_{KS} > 0\) such that if \(\lim_{t \to T^*} u(t) = u^*\) in \(L^3(\mathbb{R}^3)\) with respect to the weak topology, then
\[
\lim_{t \to T^*} \|u(t)\|_3^3 - \|u^*\|_3^3 \geq \varepsilon_{KS}.
\]
As a consequence they deduced that \(u \notin BV([0, T^*[, L^3(\mathbb{R}^3))\). Recently, L. Escauriaza, G. Seregin et V. Šverák [2] have proved that if in addition the solution \(u\) belongs to the Leray-Hopf energy space \(L_{T^*} = L^\infty([0, T^*[, L^2(\mathbb{R}^3)) \cap L^2([0, T^*[, H^1(\mathbb{R}^3))\) then
\[
\lim_{t \to T^*} \|u(t)\|_3 = \infty.
\]
In the present paper, we aim to study the behavior of the solution \(u\) in the limit space space \(B^{-1,\infty}_\infty(\mathbb{R}^3)\) (we recall that for any \(p \geq 3\) we have \(L^p(\mathbb{R}^3) \subset B^{-1,\infty}_\infty(\mathbb{R}^3)\)).

Our main result reads as follows:

**Theorem 1.** There exists constant \(\varepsilon_* > 0\) independent on \(u\) such that, for any vectorial distribution \(\omega = (\omega_1, \omega_2, \omega_3)\) in \(S(\mathbb{R}^3)B^{-1,\infty}_\infty\), we have
\[
\lim_{t \to T^*} \|u(t) - \omega\|_{B^{-1,\infty}_\infty} \geq \varepsilon_*.
\]

**Remark 1.** This result remains true if we replace the space \(L^3(\mathbb{R}^3)\) by any Lebesgue space \(L^p(\mathbb{R}^3)\) with \(p \geq 3\) or any Sobolev space \(H^s(\mathbb{R}^3)\) with \(s \geq \frac{3}{2}\) [10].

**Remark 2.** Theorem [7] jointed to Weak-Strong uniqueness result of W. Von Wahl allows to prove that any Leray-Hopf weak solution (see [7] for the definition) to the Navier-Stokes equations belonging to the space \(C([0, T], B^{-1,\infty}_\infty)\) is regular on \([0, T] \times \mathbb{R}^3\) [10].

The following result is a direct consequence of Theorem 1

**Corollary 1.** The solution \(u\) does not belong to the space \(BV([0, T^*[, B^{-1,\infty}_\infty(\mathbb{R}^3)\).

**Proof.** By using the embedding \(L^3(\mathbb{R}^3) \subset S(\mathbb{R}^3)B^{-1,\infty}_\infty\) and Theorem 1 one can easily construct by induction an increasing sequence \((t_j)\) in \([0, T^*[\) such that
\[
\forall j, \|u(t_{j+1}) - u(t_j)\|_{B^{-1,\infty}_\infty} \geq \varepsilon_*.
\]
Therefore,
\[
\sum_{j} \|u(t_{j+1}) - u(t_j)\|_{B^{-1,\infty}_\infty} = \infty,
\]
which implies the desired result.

2. Preliminaries

In this section, we recall some definitions and results that will be useful in the proof of Theorem 1. First, we define the nonhomogeneous Besov spaces \(B^{s}_p\). To do so, we need to introduce the Littlewood-Paley decomposition: Let \(\varphi\) be in the Schwartz class \(S(\mathbb{R}^3)\) such that its Fourier Transform \(\mathcal{F}(\varphi)\) is identically equal to 1 on the ball \(B(0, 1)\) and vanishes outside the ball \(B(0, 2)\). For \(j \in \mathbb{N}, k \in \mathbb{N}^*\) and \(f \in S'(\mathbb{R}^3)\), we set
\[
S_j f \equiv \varphi_j * f, \quad \Delta_k f \equiv S_k f - S_{k-1} f
\]
where \( \varphi_j \equiv 2^{3j} \varphi(2^j \cdot) \). Hence, for any \( f \in L^p(\mathbb{R}^3) \), we have the identity
\[
f = S_0 f + \sum_{k \geq 0} \Delta_k f,
\]
which is called the Littlewood-Paley decomposition of \( f \). In the sequel we often denote the operator \( S_0 \) by \( \Delta_0 \).

**Definition 1.** Let \( s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). The Besov space \( B^{s, \infty}_p \) is defined by:
\[
B^{s, \infty}_p = \{ f \in L^p(\mathbb{R}^3); \| f \|_{B^{s, \infty}_p} \equiv \sup_{k \in \mathbb{N}} 2^{sk} \| \Delta_k f \|_p < \infty \}.
\]
The space \( \tilde{B}^{s, \infty}_p \) is the closure of \( S(\mathbb{R}^3) \) in \( B^{s, \infty}_p \).

In order to study the pointwise product in the Besov space, we will use the following weak version of the Bony decomposition: For \( f \) and \( g \) in \( B^s(\mathbb{R}^3) \), we define
\[
\pi_0(f, g) = \sum_{k=0}^{\infty} S_k f \Delta_k g,
\]
\[
\pi_1(f, g) = \sum_{k=0}^{\infty} S_{k+1} f \Delta_k g.
\]
Formally,
\[
fg = \pi_0(f, g) + \pi_1(g, f).
\]
The following elementary lemma will play a crucial role for the proof of Theorem 1.

**Lemma 1.** Let \( s > 0 \). The bilinear operators \( \pi_0 \) and \( \pi_1 \) are bounded from \( B_\infty^{1, \infty} \times B_\infty^{s+1, \infty} \) (respectively, \( L^\infty(\mathbb{R}^3) \times B_\infty^{s+1, \infty} \)) into \( B_\infty^{s, \infty} \) (respectively, \( B_\infty^{s+1, \infty} \)).

**Proof.** One can consult the book [7] of P.G. Lemarié-Rieusset.

The next lemma recalls an important regularizing property of the heat kernel

**Lemma 2.** Let \( T \in [0, 1], \alpha \in \{1; 2\} \) and \( r \in \mathbb{R} \). The linear operator \( L \), defined by (.), is continuous from \( L^\infty([0, T], B_\infty^{r, \infty}) \) into \( L^\infty([0, T], B_\infty^{r+\alpha, \infty}) \) and its norm is bounded by \( CT^{2-\alpha} \) where \( C \) is a constant independent of \( T \).

**Proof.** See for example [1].

We conclude this section by setting a slightly modified version of the well-known existence theorem of T. Kato:

**Theorem 2.** Let \( v_0 \in L^3_\sigma \). Then, there exists a unique \( T_\sigma \equiv S_\sigma^*(v_0) \in [0, \infty] \) and a unique solution \( v \equiv S_\sigma^*(v_0) \) to the integral Navier-Stokes equations with initial data \( v_0 \) belonging to the space \( \cap_{\sigma<\sigma'<\infty} L^{\infty}_\sigma(Q_T) \), where \( L^{\infty}_\sigma(Q_T) \) is the space of function \( w \in C([0, T]; L^3_\sigma) \) satisfying \( \sqrt{t} w \in C([0, T]; C_0(\mathbb{R}^3)) \) and \( \lim_{t \to 0} \sqrt{t} \| w(t) \|_\infty = 0 \). Moreover, \( v \) is smooth on \( [0, T_\sigma] \times \mathbb{R}^3 \), more precisely, \( v \in \cap_{j, i \in \mathbb{N}} C^i([0, T_\sigma], B_\alpha^{1, \infty}). \) Finally, there exists a constant \( \varepsilon_3 > 0 \) independent of \( v_0 \) such that
\[
T_\sigma(v_0) \geq \sup\{ T \in [0, 1]; (1 + \| v_0 \|_3) \sup_{0 < t < T} \sqrt{t} \| e^{t \Delta} v_0 \|_\infty \leq \varepsilon_3 \}.
\]

An immediate consequence of this theorem is the following important result.
We argue by opposition, we suppose that $T^* = T^*_K(u_0)$ and $u = S^*_K(u_0)$ (this implies in particular, thanks to Theorem 2, that the solution $u$ is regular on $[0, T^*] \times \mathbb{R}^3$). The uniqueness theorem of solutions to the Navier-Stokes equations in the space $C([0, T]; L^3_B)$ [3] ensures that $T^* \geq T^*_K(u_0)$ and $u = S^*_K(u_0)$ on the interval $[0, T^*_K(u_0)]$. Thus, we conclude once we show that $T^* \leq T^*_K(u_0)$.

We argue by opposition, we suppose that $T^*_K(u_0) < T^*$. Hence, the set $S^*_K(u_0)([0, T^*_K(u_0)]) = u ([0, T^*_K(u_0)])$ is relatively compact in the space $L^3_B$. Therefore, by using the inequality

$$\forall f \in L^3(\mathbb{R}^3), \sup_{s > 0} \sqrt{s} \|e^{s \Delta} f\|_\infty \leq C \|f\|_3$$

and the fact

$$\forall f \in L^3(\mathbb{R}^3), \lim_{s \to 0} \sqrt{s} \|e^{s \Delta} f\|_\infty = 0,$$

one can easily deduce that there exists $\lambda \in [0, 1]$ such that, for all $t_0 \in [0, T^*_K(u_0)]$, we have

$$(1 + \|S^*_K(u_0)(t_0)\|_3) \sup_{0 < t < \lambda} \sqrt{t} \|e^{t \Delta} S^*_K(u_0)(t_0)\|_\infty \leq \varepsilon_3.$$ 

Choosing $t_0$ so that $0 < T^*_K(u_0) - t_0 < \lambda$, we get $I_*(u_0, t_0) \leq \varepsilon_3$, which contradicts (2.1).

**Second step:** We will prove that for all $a \in [0, T^*]$, $u \notin L^\infty([a, T^*], L^\infty(\mathbb{R}^3))$. We argue by opposition. Let $a \in [0, T^*]$ such that $M \equiv \sup_{t \leq T^*} \|u(t)\|_\infty < \infty$. Let $b \in [a, T^*]$ to be chosen later. Set $v_0 = u(b)$ and $v = S^*_K(v_0)$. Using Lemma 3, the Young inequality and the fact that the $L^1(\mathbb{R}^3)$ norm of the kernel $K_t$ of the operator $e^{t \Delta \mathbb{P} \nabla}$ is equal to $\frac{C}{\sqrt{t}}$, we obtain, for all $t$ in $[0, T^*_K(v_0)]$, the following estimates

$$\|v(t)\|_3 \leq \|v_0\|_3 + C \int_0^t \|v(s)\|_\infty \|v(s)\|_3 \frac{ds}{\sqrt{t-s}}$$

$$\leq \|v_0\|_3 + 2CM \sqrt{t} \sup_{0 \leq s \leq t} \|v(s)\|_3$$

$$\leq \|v_0\|_3 + 2CM \sqrt{T^*_K(v_0)} \sup_{0 \leq s \leq t} \|v(s)\|_3$$

$$= \|v_0\|_3 + 2CM \sqrt{T^*_K(v_0) - b} \sup_{0 \leq s \leq t} \|v(s)\|_3.$$ 

Therefore, by taking $b$ closed enough to $T^*$, we get

$$N \equiv \sup_{0 \leq s < T^*_K(v_0)} \|v(s)\|_3 < \infty.$$ 

In conclusion, for all $t_0$ in $[0, T^*_K(v_0)]$, we have

$$I_*(v_0, t_0) \leq (1 + N)M \sqrt{T^*_K(v_0) - t_0},$$

which contradicts (2.1).
**Third step:** Let $\varepsilon > 0$. Suppose that there exists $\omega \in S(\mathbb{R}^3)$ such that

$$\lim_{t \to T^*} \|u(t) - \omega\|_{B^{-1,\infty}} < \varepsilon.$$ 

Then, there exists $\delta_0 \in ]0, T^*[ \,$ so that

$$\sup_{t \in [T^* - \delta_0, T^*]} \|u(t) - \omega\|_{B^{-1,\infty}} < \varepsilon.$$ 

Let $\delta \in ]0, \delta_0[ \,$ to be chosen later. Set $w_0 = u(T^* - \delta)$ and $w = S^*_K(w_0)$. According to Lemma 3, $T^*_K(w_0) = \delta$ and $w = u(\cdot + T^* - \delta)$. Thus,

$$\sup_{0 < t < \delta} \|w(t) - \omega\|_{B^{-1,\infty}} < \varepsilon.$$ 

Fix $s > 0$. Kato’s Theorem ensures that $w \in C([0, \delta[; \tilde{B}^{s+1,\infty}_{\infty})$. On the other hand, we have

$$w(t) = e^{tA}w_0 + \sum_{j=0}^{1} L (\mathbb{P} \nabla \pi_j [(w - \omega) \otimes w]) (t).$$

Therefore, applying Lemma 1 and 2 and using the fact that $\mathbb{P} \nabla$ maps boundedly $B^{r,\infty}_{\infty}$ into $B^{r-1,\infty}_{\infty}$ ($r \in \mathbb{R}$), yields for all $\delta_1 < \delta$

$$\sup_{0 < t < \delta_1} \|w(t)\|_{B^{s+1,\infty}_{\infty}} \leq \|w_0\|_{B^{s+1,\infty}_{\infty}} + C\{\varepsilon + \|\omega\|_{\infty} \sqrt{\delta}\} \sup_{0 < t < \delta_1} \|w(t)\|_{B^{s+1,\infty}_{\infty}},$$

where the constant $C$ depends only on $s$. Now, suppose that $\varepsilon \leq \varepsilon_* = \frac{1}{4C}$. Then, one can choose $\delta$ small enough so that $C\{\varepsilon + \|\omega\|_{\infty} \sqrt{\delta}\} \leq \frac{1}{2}$. Hence, the estimate (3.1) implies that

$$\sup_{0 < t < \delta_1} \|w(t)\|_{B^{s+1,\infty}_{\infty}} \leq 2 \|w_0\|_{B^{s+1,\infty}_{\infty}}.$$ 

Now using the embedding $B^{s+1,\infty}_{\infty} \hookrightarrow L^\infty(\mathbb{R}^3)$ and the fact that $\delta_1$ is arbitrary in $]0, \delta[$, we get

$$\sup_{t \in [T^* - \delta, T^*[} \|u(t)\|_{\infty} = \sup_{0 < t < \delta} \|w(t)\|_{\infty} < \infty,$$

which contradicts the conclusion of the second step. Then, we conclude that for all $\omega \in S(\mathbb{R}^3)$ we have

$$\lim_{t \to T^*} \|u(t) - \omega\|_{B^{-1,\infty}} \geq \varepsilon_*.$$ 

By density, this inequality remains true for all $\omega \in S(\mathbb{R}^3)^{B^{-1,\infty}_{\infty}}$.

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