SCALING LIMIT OF THE CONDUCTIVITY OF RANDOM RESISTOR NETWORKS ON SIMPLE POINT PROCESSES

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Abstract. We consider random resistor networks with nodes given by a simple point process on \( \mathbb{R}^d \) and with random conductances. The length range of the electrical filaments can be unbounded. We assume that the randomness is stationary and ergodic w.r.t. the action of the group \( G \), given by \( \mathbb{R}^d \) or \( \mathbb{Z}^d \). This action is covariant w.r.t. translations on the Euclidean space. Under minimal assumptions we prove that a.s. the suitably rescaled directional conductivity of the resistor network along the principal directions of the effective homogenized matrix \( D \) converges to the corresponding eigenvalue of \( D \) times the intensity of the simple point process. We also prove the a.s. scaling limit of the conductivity along any direction perpendicular to the kernel of \( D \) by suitably modifying the shape of the region of physical observation. Our results cover plenty of models including e.g. the standard conductance model on \( \mathbb{Z}^d \), the Miller-Abrahams resistor network for conduction in amorphous solids (to which we can now extend the bounds in agreement with Mott’s law previously obtained in [9, 20, 21] for Mott’s random walk), resistor networks on the supercritical cluster in lattice and continuum percolations, resistor networks on crystal lattices and on Delaunay triangulations.

Keywords: simple point process, resistor network, Miller-Abrahams random resistor network, random conductance model, supercritical percolation, homogenization, 2-scale convergence.

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1. Introduction

Random resistor networks in \( \mathbb{R}^d \) are an effective tool to investigate transport in disordered media and have been much investigated both in Physics and Probability (cf. e.g. [5, 27, 28, 37] and references therein). Randomness can affect both the conductances of the electric filaments and the location of the nodes. It describes micro-inhomogeneities which can be of different physical nature. For example, one can consider mixtures of conducting and non-conducting materials (cf. [28, Section II]), thus motivating the study of resistor networks on percolation clusters. One can also consider amorphous solids as doped semiconductors in the regime of strong Anderson localization. In this case the doping impurities have random positions \( x_i \) (described mathematically by a simple point process) and the Mott’s variable range hopping (v.r.h.) of the conducting electrons can be modeled by the Miller-Abrahams (MA)

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random resistor network. In this network nodes are given by the impurities and, between any pair of impurities located at \( x_i \) and \( x_j \), there is an electrical filament of conductance

\[
c_{x_i, x_j}(\omega) := \exp \left\{ -\frac{2}{\gamma} |x_i - x_j| - \frac{\beta}{2} (|E_i| + |E_j| + |E_i - E_j|) \right\},
\]

where \( \gamma \) is the localization length, \( \beta = 1/kT \) is the inverse temperature and the \( E_i \)'s are random energy marks associated to the impurities (cf. [4, 33, 36, 38] and Section 3 below). Typically, the energy marks are taken as i.i.d. random variables with distribution \( p(E) \propto |E|^{\alpha} dE \) on an interval containing the origin, for some \( \alpha \geq 0 \). Mott’s v.r.h. was introduced by Mott to explain the anomalous low-temperature conductivity decay, which (for isotropic materials) behaves as

\[
\sigma(\beta) \approx A \exp \left\{ -c_0^{\alpha+1} \right\},
\]

where \( A \) has a negligible \( \beta \)-dependence, while \( c > 0 \) is \( \beta \)-independent (Mott considered the case \( \alpha = 0 \), while Efros and Shklovskii introduced \( \alpha \) to model a possible Coulomb pseudogap in the density of states). Eq. (2) is usually named Mott’s law. We refer to Mott’s Nobel Lecture [32] and the monographs [36, 38] for more details.

As in Mott’s law, a fundamental physical quantity is given by the conductivity of the resistor network along a given direction. One considers a box centered at the origin of \( \mathbb{R}^d \) with two opposite faces orthogonal to the fixed direction, where the electric potential takes value 0 and 1, respectively. Then the conductivity is the total electric current flowing across any section orthogonal to the given direction and equals the total dissipated energy (cf. [5, Section 1.3], [27, Section 11] and Section 2 below).

In this work we consider generic random resistor networks on \( \mathbb{R}^d \) with random conductances and nodes at random positions, hence described by a simple point process. The electric filaments can be arbitrarily long. We assume that the randomness is stationary and ergodic w.r.t. the action of the group \( G \), given by \( \mathbb{R}^d \) or \( \mathbb{Z}^d \). This action is covariant w.r.t. the action of \( G \) by translations on the Euclidean space. Under minimal assumptions we prove that, as the box size diverges, a.s. the suitably rescaled directional conductivity of the resistor network along the principal directions of the effective homogenized matrix \( D \) converges to the corresponding eigenvalue of \( D \) times the intensity of the simple point process (cf. Theorems 1 and 2). \( D \) admits a variational characterization which can be used to get upper and lower bounds (cf. e.g. [12, 20, 21]). The conductivity scaling limit along non principal directions is treated in our Theorem 3. Information on the limiting behavior of the electrical potential is provided in Proposition 2.8. We point out that our finite-moment conditions (A7) in Section 2 are the optimal ones, as they are necessary to define the effective homogenized matrix (integrability of \( \lambda_2 \) in (A7)) and the space of square integrable forms (integrability of \( \lambda_0 \) in (A7)).

Our target has been to achieve a universal qualitative result, hence our theorems apply to a large variety of geometric structures. In particular we
obtain the scaling limit of the conductivity for the resistor network on the
supercritical percolation cluster on $\mathbb{Z}^d$, providing a solution to the open
Problem 1.18 in [5] which goes far beyond Bernoulli bond percolation (cf. also
[10, 24, 27] and references therein). As a byproduct, we get also a proof
of the strictly positivity of $D$ for this model alternative to the original one
in [12]. Our results cover also the MA resistor network allowing to extend to
its asymptotic directional conductivities the bounds in agreement with Mott’s
law [4] previously obtained in [9, 20, 21] for Mott’s random walk as detailed in
Corollary 3.1 in Section 3 (Theorem 1 will be used also to fully prove Mott’s
law for several environments in [17]). For the standard conductance model on
$\mathbb{Z}^d$ our results improve the existing ones (see [30] and the discussion below).
The above examples are discussed in Section 3. We also mention some other
examples, as resistor networks on crystal lattices, on Delaunay triangulations,
on supercritical clusters in continuum percolation. We point out that there
isn’t a prototype random resistor network to deal with as a leading example.
For example, the underlying graph of the MA resistor network is the complete
graph on an infinite simple point process, which is completely different from
the supercritical percolation cluster. This geometric heterogeneity requires a
geometric abstract setting to get the desired universality.

Our proof is based on stochastic homogenization via 2-scale convergence (cf.
[16, 40, 41] and references therein) and the theory of simple point process. In
[41] Piatnitski and Zhikov have proved homogenization for the massive Pois-
son equation $\lambda u + Lu = f$ by 2-scale convergence on bounded domains also
with mixed Dirichlet-Neumann b.c., $L$ being the generator of a diffusion in
random environments. In [41, Section 7] the above result has been applied
to get that the magnitude of the effective homogenized matrix $D$ equals the
limiting rescaled “directional conductivity” for a diffusion on the skeleton of
the supercritical percolation cluster in Bernoulli bond percolation. The proof
relies on the a priori check that $D > 0$, based on previous results on left–right
crossings valid in the Bernoulli case. We have developed here a direct proof
of the scaling limit of the direction conductivity, which avoids the constraint
$D > 0$ (whose check usually requires further assumptions) and previous in-
vestigations of the massive Poisson equation (which would require the cut-off
procedures developed in [16, Sections 14,16] in order to deal with arbitrarily
long conductances). In general, we avoid any assumption on the left-right
crossings of the resistor network (usually of difficult investigation if the FKG
inequality is violated). We stress that the existing proofs for random diffusions
also with different b.c. (cf. [26, 41]) do not adapt well to our general discrete
setting as the presence of arbitrarily long conductances in an amorphous set-
ting forces to deal with amorphous local gradients, which keep trace of the
function variation along any filament exiting from any given point and which
are very irregular objects.

Concerning previous results, we point out that the case of i.i.d. random
conductances between nearest-neighbor sites of $\mathbb{Z}^d$ with value in a fixed interval
$(\delta_0, 1 - \delta_0)$, $\delta_0 > 0$, has been considered by Kozlov in [30], but - as discussed
in Section 3 below - in \textsuperscript{30} some Neumann b.c. for the electric potential are imposed at the microscopic level, while in general they are not satisfied in the presence of micro-inhomogeneities. As stated for example in \textsuperscript{7}, in the case of stationary ergodic random conductances between nearest-neighbor sites of \( \mathbb{Z}^d \) having value in \((\delta_0, 1 - \delta_0)\) \((\delta_0 > 0)\) and with potential at the boundary of the box given by a fixed linear function, one can prove the scaling limit of the dissipated energy by adapting the methods developed for the continuous case (cf. \textsuperscript{26, 29, 35} and the technical results collected in \textsuperscript{31}). We also point out that previously, in \textsuperscript{39, p. 26}, Zhikov obtained the scaling limit of the “directional conductivity” of the standard diffusion with partial Dirichlet b.c. in a perforated domain built by fattening the supercritical percolation cluster. We point out that the b.c. in \textsuperscript{39, Eq. (1.22)] does not correspond to the effective one for the resistor network on the supercritical percolation cluster as the Neumann part is missing.

As a further step of investigation we plan to derive quantitative results on the scaling of the directional conductivity at cost of additional technical assumptions (cf. e.g. \textsuperscript{2} for some quantitative stochastic homogenization results on the supercritical percolation cluster on \( \mathbb{Z}^d \)). Finally, we point out that the present work is an extension and improvement of our unpublished notes \textsuperscript{15}.

Outline of the paper. In Section 2 we present our models and main results (i.e. Theorems \textsuperscript{1, 2, 3} and Proposition \textsuperscript{2.8}). In Section 3 we discuss relevant examples. The rest of the paper is devoted to proofs.

2. Models and main results

We start with a probability space \((\Omega, \mathcal{F}, \mathcal{P})\) encoding all the randomness of the system. Elements \(\omega\) of \(\Omega\) are called environments. We denote by \(\mathbb{E}[\cdot]\) the expectation associated to \(\mathcal{P}\).

We denote by \(\mathcal{N}\) the space of locally finite subsets \(\{x_i\} \subset \mathbb{R}^d, d \geq 1\). As common, we will identify the set \(\{x_i\}\) with the counting measure \(\sum_i \delta_{x_i}\). In particular, if \(\xi = \{x_i\} \in \mathcal{N}\), then \(\int d\xi(x)f(x) = \sum_i f(x_i)\) and \(\xi(A) = \#(\{x_i\} \cap A)\) for \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) and \(A \subset \mathbb{R}^d\). On \(\mathcal{N}\) one defines a special metric \(d\) (cf. \textsuperscript{11, App. A2.6}) such that a sequence \((\xi_n)\) converges to \(\xi\) in \(\mathcal{N}\) if \(\lim_{n \to \infty} \int d\xi_n(x)f(x) = \int d\xi(x)f(x)\) for any bounded continuous function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) vanishing outside a bounded set. Then the \(\sigma\)-algebra of Borel sets of \((\mathcal{N}, d)\) is generated by the sets \(\{\xi \in \mathcal{N} : \xi(A) = k\}\) with \(A\) and \(k\) varying respectively among the Borel sets of \(\mathbb{R}^d\) and in \(\mathbb{N}\). In what follows, we think of \(\mathcal{N}\) as measure space endowed with the \(\sigma\)-algebra of Borel sets.

We consider a simple point process on \(\mathbb{R}^d\) defined on \((\Omega, \mathcal{F}, \mathcal{P})\), i.e. a measurable map \(\Omega \ni \omega \mapsto \hat{\omega} \in \mathcal{N}\). We also consider the group \(\mathbb{G}\) given by \(\mathbb{R}^d\) or \(\mathbb{Z}^d\) acting on the Euclidean space \(\mathbb{R}^d\) by the translations \(\tau_g : \mathbb{R}^d \rightarrow \mathbb{R}^d\), where \(\tau_g x = x + g\).

Warning 2.1. To simplify here the presentation, when \(\mathbb{G} = \mathbb{Z}^d\) we assume that \(\hat{\omega} \subset \mathbb{Z}^d\) for all \(\omega \in \Omega\) (in Section 2.1 we will remove this assumption).
We assume that $G$ acts also on $\Omega$ and, with a slight abuse of notation made non ambiguous by the context, we denote by $(\tau_g)_{g \in G}$ also the action of $G$ on $\Omega$. In particular, this action is given by a family of $G$–parametrized maps $\tau_g : \Omega \to \Omega$ such that $\tau_0 = I$, $\tau_g \circ \tau_g' = \tau_{g+g'}$ for all $g, g' \in G$, $G \times \Omega \ni (g, \omega) \mapsto \tau_g \omega$ is measurable ($\mathbb{R}^d, \mathbb{Z}^d$ are endowed with the Euclidean metric and the discrete topology, respectively). As common, a subset $A \subset \Omega$ is called translation invariant if $\tau_g A = A$ for all $g \in G$. The name comes from the fact that the action of $G$ on $\Omega$ describes how the environment changes when applying translations on the Euclidean space (cf. Assumption (A4) below). We will assume that $P$ is stationary and ergodic w.r.t. the action of $G$ on $\Omega$. We recall that stationarity means that $P(\tau_g A) = A$ for any $A \in F$ and $g \in G$, while ergodicity means that $P(A) \in \{0,1\}$ for any translation invariant set $A \in F$.

Due to our assumptions stated below, the simple point process has finite positive intensity $m$, where

$$m := \begin{cases} \mathbb{E}[\hat{\omega}([0,1]^d)] & \text{if } G = \mathbb{R}^d, \\ P(0 \in \hat{\omega}) & \text{if } G = \mathbb{Z}^d. \end{cases}$$

As a consequence, the Palm distribution $P_0$ associated to the simple point process is well defined (cf. [16, Section 2] and references therein). We recall that $P_0$ is the probability measure on $(\Omega, F)$ with support in

$$\Omega_0 := \{\omega \in \Omega : 0 \in \hat{\omega}\},$$

such that, for any $A \in F$,

$$P_0(A) := \begin{cases} \frac{1}{m} \int_{\Omega} dP(\omega) \int_{[0,1]^d} d\hat{\omega}(x) \mathbb{I}_A(\tau_x \omega) & \text{if } G = \mathbb{R}^d, \\ P(A|\Omega_0) & \text{if } G = \mathbb{Z}^d. \end{cases}$$

In the rest, we will denote by $\mathbb{E}_0[\cdot]$ the expectation w.r.t. $P_0$.

We fix a measurable function (describing the random conductance field)

$$\mathbb{R}^d \times \mathbb{R}^d \times \Omega \ni (x, y, \omega) \mapsto c_{x,y}(\omega) \in [0, +\infty)$$

such that $c_{x,x}(\omega) = 0$ for all $x \in \mathbb{R}^d$. The value of $c_{x,y}(\omega)$ will be relevant only for $x \neq y$ in $\hat{\omega}$. For later use we define the function $\lambda_k : \Omega_0 \to [0, +\infty]$ as

$$\lambda_k(\omega) := \int_{\mathbb{R}^d} d\hat{\omega}(x)c_{0,x}(\omega)|x|^k,$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$.

Recall that temporary assumption in Warning 2.1

**Assumptions.** We make the following assumptions:

(A1) $P$ is stationary and ergodic w.r.t. the action of $G$ on $\Omega$;
(A2) the intensity $m$ given in (3) is finite and positive;
(A3) $P(\omega \in \Omega : \tau_g \omega \neq \tau_{g'} \omega \forall g \neq g' \in G) = 1$;

...
(A4) for all \( \omega \in \Omega, g \in G \) and \( x, y \in \tilde{\tau}_g \omega \), it holds
\[
\tilde{\tau}_g \omega = \tau_{-g}(\tilde{\omega}),
\]
\[
c_{x,y}(\tilde{\tau}_g \omega) = c_{g,x}(\omega);
\]
(7)
(A5) for all \( \omega \in \Omega \) the weights \( c_{x,y}(\omega) \) are symmetric, i.e. \( c_{x,y}(\omega) = c_{y,x}(\omega) \)
\( \forall x, y \in \tilde{\omega} \);
(A6) for \( \mathcal{P} - \text{a.a. } \omega \) the graph with vertex set \( \tilde{\omega} \) and edges given by \( \{ x, y \} \) with \( x \neq y \) in \( \tilde{\omega} \) and \( c_{x,y}(\omega) > 0 \) is connected;
(A7) \( \lambda_0, \lambda_2 \in L^1(\mathcal{P}_0) \);
(A8) \( L^2(\mathcal{P}_0) \) is separable.

We point out that the above assumptions are the same presented in [16, Section 2.4] when the rates \( r_{x,y}(\omega) \) there are symmetric (hence coinciding with our \( c_{x,y}(\omega) \)). We now comment the above assumptions (recalling also some remarks from [16]). Due to [11, Proposition 10.1.IV], (A1) and (A2), for \( \mathcal{P} - \text{a.a. } \omega \) the set \( \tilde{\omega} \) is infinite. (A3) implicitly includes that the event under consideration is measurable as common in the applications (measurability is automatic if \( G = \mathbb{Z}^d \)). (A3) is usually a rather fictitious assumption. Indeed, usually by free one can add some randomness enlarging \( \Omega \) to assure (A3) (similarly to [12, Remark 4.2-(i)]). For example, if \( (\Omega, \mathcal{F}, \mathcal{P}) \) is simply the space \( \mathcal{N} \) endowed with a probability measure and if it describes a simple point process on \( \mathbb{R}^d \) obtained by spatially periodizing a simple point process on \( [0,1)^d \), then to gain (A3) it would be enough to mark points by i.i.d. random variables with non-degenerate distribution. (A4) describes how the Euclidean translations influence the randomness. (A5) is natural due to the interpretation of conductance of \( c_{x,y}(\omega) \) discussed below. (A6) is a technical assumption assuring that a measurable function \( u \) on \( \Omega_0 \) such that, \( \mathcal{P}_0 - \text{a.s.}, u(\tau_x \omega) = u(\omega) \) for all \( x \in \tilde{\omega} \) with \( c_{0,x}(\omega) > 0 \) is constant \( \mathcal{P}_0 - \text{a.s.} \) (cf. [16][Lemma 8.5]). In Section 3.2 we will discuss a relevant example where (A6) does not hold but anyway the application of our Theorem 1 allows to derive the scaling limit of the directional conductivity. By [8, Theorem 4.13] (A8) is fulfilled if \( (\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \) is a separable measure space where \( \mathcal{F}_0 := \{ A \cap \Omega_0 : A \in \mathcal{F} \} \) (i.e. there is a countable family \( \mathcal{G} \subset \mathcal{F}_0 \) such that the \( \sigma \)-algebra \( \mathcal{F}_0 \) is generated by \( \mathcal{G} \)). For example, if \( \Omega_0 \) is a separable metric space and \( \mathcal{F}_0 = \mathcal{B}(\Omega_0) \) (which is valid if \( \Omega \) is a separable metric space and \( \mathcal{F} = \mathcal{B}(\Omega) \)) then (cf. [8 p. 98]) \( (\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \) is a separable measure space and (A8) is valid. Note that we are not assuming that \( \Omega \) is a compact metric space as in e.g. [41], hence (A8) becomes relevant to have countable families of test functions for the 2-scale convergence (we refer to [16] for further comments on this issue).

**Definition 2.1.** We define the effective homogenized matrix \( D \) as the \( d \times d \) nonnegative symmetric matrix such that
\[
a \cdot Da = \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int d\mathcal{P}_0(\omega) \int d\tilde{\omega}(x)c_{0,x}(\omega) (a \cdot x - \nabla f(\omega, x))^2,
\]
where \( \nabla f(\omega, x) := f(\tau_x \omega) - f(\omega) \).
Figure 1. (Left) The resistor network (RN)_{\ell}^{\omega}, conductances are omitted. The box and the stripe correspond to \(\Lambda_{\ell}\) and \(S_{\ell}\) respectively. (Right) Functions in \(H_{0,\omega}^{1,x}\) take any value on the nodes surrounded by an annulus, and have value zero on all other nodes (the box corresponds here to \(\Lambda\) and has side length 1, the underlying graph is \(G_{\varepsilon}^{x}\)).

By (A7) the above definition is well posed. In general, \(a \cdot b\) denotes the Euclidean scalar product of the vectors \(a\) and \(b\).

Given \(\ell > 0\) we consider the box, stripe and half-stripes:

\[
\begin{align*}
\Lambda_{\ell} &:= (-\ell/2,\ell/2)^d, \\
S_{\ell} &:= \mathbb{R} \times (-\ell/2,\ell/2)^{d-1}, \\
S_{\ell}^- &:= \{x \in S_{\ell} : x_1 \leq -\ell/2\}, \\
S_{\ell}^+ &:= \{x \in S_{\ell} : x_1 \geq \ell/2\}.
\end{align*}
\]

(10)

**Warning 2.2.** We denote by \(e_1, \ldots, e_d\) the canonical basis of \(\mathbb{R}^d\). In what follows we focus on the direction determined by \(e_1\). In the general case, when considering the direction determined by a unit vector \(e\), one has just to refer our results to the regions \(O(\Lambda_{\ell})\), \(O(S_{\ell})\), \(O(S_{\ell}^-)\) and \(O(S_{\ell}^+)\), where \(O\) is a fixed orthogonal linear map such that \(O(e_1) = e\).

We define \(\Omega_1\) as the set of \(\omega \in \Omega\) satisfying the connectivity property in (A6) and the bounds (cf. (6))

\[
\lambda_0(\tau_{x,\omega}) = \sum_{y \in \omega} c_{x,y}(\omega) < +\infty \quad \forall x \in \hat{\omega}.
\]

(11)

Note that \(\Omega_1\) is a translation invariant measurable set with \(\mathcal{P}(\Omega_1) = 1\) (use (A7) and Lemma 5.1 below).

**Definition 2.2** (Resistor network (RN)_{\ell}^{\omega}). Given \(\omega \in \Omega_1\) we consider the \(\ell\)–parametrized resistor network (RN)_{\ell}^{\omega} on \(S_{\ell}\) with node set \(\hat{\omega} \cap S_{\ell}\). To each unordered pair of nodes \(\{x,y\}\), such that \(\{x,y\} \cap \Lambda_{\ell} \neq \emptyset\) and \(c_{x,y}(\omega) > 0\), we associate an electrical filament of conductance \(c_{x,y}(\omega)\) (see Figure 1 (left)).

We can think of (RN)_{\ell}^{\omega} as a weighted non-oriented graph with vertex set \(\hat{\omega} \cap S_{\ell}\), edge set

\[
\mathbb{B}_{\omega} := \{\{x,y\} \subset (\hat{\omega} \cap S_{\ell}) : \{x,y\} \cap \Lambda_{\ell} \neq \emptyset, c_{x,y}(\omega) > 0\}
\]

(12)

and weight of the edge \(\{x,y\}\) given by the conductance \(c_{x,y}(\omega)\).

1The term stripe is appropriate for \(d = 2\). We keep the same terminology for all dimensions \(d\).
Definition 2.3 (Electric potential). Given \( \omega \in \Omega_1 \) we denote by \( V^\omega_\ell \) the electric potential of the resistor network \((\text{RN})^\omega_\ell\) with values 0 and 1 on \( S^-_\ell \) and \( S^+_\ell \), respectively, taken by convention equal to zero on the connected components of \( \sum \) and satisfying

\[
\sum_{y \in \hat{\omega} \cap S_\ell^c} c_{x,y}(\omega) (V^\omega_\ell(y) - V^\omega_\ell(x)) = 0 \quad \forall x \in \hat{\omega} \cap \Lambda_\ell, \tag{13}
\]

and

\[
\begin{cases}
V^\omega_\ell(x) = 0 & \text{if } x \in \hat{\omega} \cap \Lambda_\ell, \text{ } x \text{ is not connected to } \hat{\omega} \setminus \Lambda_\ell \text{ in } (\text{RN})^\omega_\ell, \\
V^\omega_\ell(x) = 0 & \text{if } x \in \hat{\omega} \cap S^-_\ell, \\
V^\omega_\ell(x) = 1 & \text{if } x \in \hat{\omega} \cap S^+_\ell.
\end{cases} \tag{14}
\]

Note that (13) corresponds to Kirchhoff’s law. As discussed in Section 6, the above electric potential exists and is unique and has values in \([0, 1]\). We recall that, given \((x, y)\) with \(\{x, y\} \in B^\ell \) (cf. (12)),

\[
i_{x,y}(\omega) := c_{x,y}(\omega) (V^\omega_\ell(y) - V^\omega_\ell(x)) \tag{15}
\]

is the electric current flowing from \(x\) to \(y\) under the electric potential \(V^\omega_\ell\), due to Ohm’s law. For simplicity, \(\ell\) is understood in the notation \(i_{x,y}(\omega)\).

Definition 2.4 (Directional effective conductivity). Given \(\omega \in \Omega_1\) we call \(\sigma(\omega)\) the effective conductivity of the resistor network \((\text{RN})^\omega_\ell\) along the first direction under the electric potential \(V^\omega_\ell\). More precisely, \(\sigma(\omega)\) is given by

\[
\sigma_\ell(\omega) := \sum_{x \in \hat{\omega} \cap S^-_\ell} \sum_{y \in \hat{\omega} \cap \Lambda_\ell} i_{x,y}(\omega) = \sum_{x \in \hat{\omega} \cap S^-_\ell} \sum_{y \in \hat{\omega} \cap \Lambda_\ell} c_{x,y}(\omega) (V^\omega_\ell(y) - V^\omega_\ell(x)).
\]

It is simple to check that, for any \(\gamma \in [-\ell/2, \ell/2]\), \(\sigma_\ell(\omega)\) equals the current flowing through the hyperplane \(\{x \in \mathbb{R}^d : x_1 = \gamma\}\):

\[
\sigma_\ell(\omega) = \sum_{x \in \hat{\omega} \cap S^-_\ell \cap \{x_1 \leq \gamma\}} \sum_{y \in \hat{\omega} \cap S^+_\ell \cap \{x_1 > \gamma\}} i_{x,y}(\omega). \tag{16}
\]

\(\sigma_\ell(\omega)\) also equals the total dissipated energy:

\[
\sigma_\ell(\omega) = \sum_{\{x, y\} \in B^\ell} c_{x,y}(\omega) (V^\omega_\ell(y) - V^\omega_\ell(x))^2. \tag{17}
\]

Indeed, by collapsing all nodes in \(\hat{\omega} \cap S^+_\ell\) into a single node and similarly for \(\hat{\omega} \cap S^-_\ell\), one reduces to the same setting of [13, Section 1.3] where (17) is proved.

We can now state our first main result concerning the infinite volume asymptotics of \(\sigma(\omega)\) (the proof is given in Sections 7 and 14):

**Theorem 1.** Suppose that \(e_1\) is an eigenvector of \(D\). Then there exists a translation invariant measurable set \(\Omega_{\text{typ}} \subset \Omega_1\) with \(\mathcal{P}(\Omega_{\text{typ}}) = 1\) such that for all \(\omega \in \Omega_{\text{typ}}\) it holds

\[
\lim_{\ell \to +\infty} \ell^{2-d} \sigma(\omega) = mD_{1,1}.
\]

The notation \(\Omega_{\text{typ}}\) refers to the fact that elements of \(\Omega_{\text{typ}}\) are typical environments, as \(\mathcal{P}(\Omega_{\text{typ}}) = 1\). From
Remark 2.5. From the description of \( \Omega_{\text{typ}} \) in Sections \( 7 \) and \( 10 \) and from the proof, it is simple to check that the same set \( \Omega_{\text{typ}} \) leads to the conclusion of Theorem \( 7 \) (and similarity of Proposition \( 2.8 \) and Theorem \( 2 \) below) for any eigenvector of \( D \). Similarly, the set \( \Omega_{\text{typ}} \) leads to the conclusion of Theorem \( 3 \) below for any vector in \( \text{Ker}(D)^\perp \).

To clarify the link with homogenization and state our further results, it is convenient to rescale space in order to deal with fixed stripe and box. More precisely, we set \( \varepsilon := 1/\ell \). Then \( \varepsilon > 0 \) is our scaling parameter. We set

\[
\begin{align*}
\Lambda := (-1/2, 1/2)^d, \\
S := \mathbb{R} \times (-1/2, 1/2)^{d-1}, \\
S^- := \{x \in S : x_1 \leq -1/2\}, \\
S^+ := \{x \in S : x_1 \geq 1/2\}.
\end{align*}
\]

(18)

Note that \( \Lambda_\ell = \ell \Lambda, \ S_\ell = \ell S, \ S_\ell^= = \ell S^\pm \). Here and below, \( \omega \in \Omega_1 \). We write \( V_\varepsilon : \varepsilon \omega \cap S \to [0, 1] \) for the function given by \( V_\varepsilon(x) := V_\varepsilon(x) \) (note that the dependence on \( \omega \) in \( V_\varepsilon \) is understood, as for other objects below).

We introduce the atomic measures

\[
\mu_{\omega, \Lambda}^\varepsilon := \varepsilon^d \sum_{x \in \varepsilon \omega \cap \Lambda} \delta_x, \quad \nu_{\omega, \Lambda}^\varepsilon := \varepsilon^d \sum_{(x, y) \in E_\varepsilon(\omega)} c_{x/y/\varepsilon}(\omega) \delta_{(x, (y-x)/\varepsilon)},
\]

where

\[
E_\varepsilon(\omega) := \{(x, y) : x, y \in \varepsilon \omega \cap S, c_{x, y}(\omega) > 0 \text{ and } \{x, y\} \cap \Lambda \neq \emptyset\}.
\]

(20)

Note that \( \mu_{\omega, \Lambda}^\varepsilon \) and \( \nu_{\omega, \Lambda}^\varepsilon \) have finite total mass (for the latter use that \( \Lambda \in \Omega \)).

Given a function \( f : \varepsilon \omega \cap S \to \mathbb{R} \), we define the amorphous gradient \( \nabla_\varepsilon f \) on pairs \( (x, z) \) with \( x \in \varepsilon \omega \cap S \) and \( x + \varepsilon z \in \varepsilon \omega \cap S \) as

\[
\nabla_\varepsilon f(x, z) = \frac{f(x + \varepsilon z) - f(x)}{\varepsilon}.
\]

(21)

Moreover, we define the operator

\[
\mathbb{L}_\varepsilon f(x) := \varepsilon^{-2} \sum_{y \in \varepsilon \omega \cap S} c_{x/y/\varepsilon}(f(y) - f(x)), \quad x \in \varepsilon \omega \cap \Lambda,
\]

(22)

whenever the series in the r.h.s. is absolutely convergent. Let \( f : \varepsilon \omega \cap S \to \mathbb{R} \) be a bounded function. Then \( \mathbb{L}_\varepsilon f(x) \) is well defined for all \( x \in \varepsilon \omega \cap \Lambda \) as \( \omega \in \Omega_1 \). As the amorphous gradient \( \nabla_\varepsilon f \) is bounded too, we have that \( \nabla_\varepsilon f \in L^2(\nu_{\omega, \Lambda}) \).

Moreover, if in addition \( f \) is zero outside \( \Lambda \), it holds (cf. Lemma \( 6.1 \))

\[
\langle f, -\mathbb{L}_\varepsilon f \rangle_{L^2(\nu_{\omega, \Lambda})} = \frac{1}{2} \langle \nabla_\varepsilon f, \nabla_\varepsilon f \rangle_{L^2(\nu_{\omega, \Lambda})} < +\infty.
\]

(23)

Definition 2.6 (Graph \( G_\omega^\varepsilon \) and sets \( C_{\omega, \Lambda}^\varepsilon, C_\omega^\varepsilon \)). Given \( \varepsilon > 0 \) and \( \omega \in \Omega_1 \), we consider the non-oriented graph \( G_\omega^\varepsilon \) with vertex set \( \varepsilon \omega \cap S \) and edges given by the unordered pairs \( \{x, y\} \) such that \( c_{x, y}(\omega) > 0 \) and \( \{x, y\} \) intersects \( \Lambda \). We write \( C_{\omega, \Lambda}^\varepsilon \) and \( C_\omega^\varepsilon \) for the union of the connected components in \( G_\omega^\varepsilon \) included in \( \Lambda \) and, respectively, intersecting \( S^- \cup S^+ \).
Suppose that Proposition 2.8.

Section 14. In the case for the scale converges weakly to $\psi_1$ A. FAGGIONATO in the weighted graph $(\mathbb{R}^N)$ defined by Proposition 2.8, noting that $C^\varepsilon_0$ has no edge between $\varepsilon \omega \cap S^-$ and $\varepsilon \omega \cap S^+$ and that $\varepsilon \omega \cap S = C^\varepsilon_{\omega \Lambda} \cup C^\varepsilon_{\omega \Lambda}$. The edges of $C^\varepsilon_{\omega \Lambda}$ coincide with the edges obtained from $E_{\varepsilon}(\omega)$ when disregarding the orientation.

**Definition 2.7 (Functional spaces $H^{1,\varepsilon}_{0,\omega}, K^{\varepsilon}_{\omega}$).** Given $\omega \in \Omega_1$, we define the Hilbert space

$$H^{1,\varepsilon}_{0,\omega} := \{ f : \varepsilon \omega \cap S \to \mathbb{R} \text{ such that } f(x) = 0 \ \forall x \in C^\varepsilon_{\omega \Lambda} \cup (C^\varepsilon_{\omega \Lambda} \setminus \Lambda) \}$$

endowed with the squared norm $\|f\|^2_{H^{1,\varepsilon}_{0,\omega}} := \|f\|^2_{L_2(\varepsilon \omega, \Lambda)} + \|\nabla f\|^2_{L_2(\varepsilon \omega, \Lambda)}$. In addition, we define $K^{\varepsilon}_{\omega}$ as the set of functions $f : \varepsilon \omega \cap S \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in (\varepsilon \omega \cap S^-) \cup C^\varepsilon_{\omega \Lambda}$ and $f(x) = 1$ for all $x \in \varepsilon \omega \cap S^+$.

We refer to Figure 1(right) for a graphical comment on the graph $G^\varepsilon_\omega$ and functions in $H^{1,\varepsilon}_{0,\omega}$. Given $\omega \in \Omega_1$, in Section 6 we will derive that, due to Definition 2.7 and (13) and (14), $V_\varepsilon$ is the unique function in $K^{\varepsilon}_{\omega}$ such that $L^\varepsilon_\omega V_\varepsilon(x) = 0$ for all $x \in \varepsilon \omega \cap \Lambda$ (cf. Lemma 6.2). We point out that, by (17), the rescaled conductivity $\ell^2\sigma_\varepsilon(\omega)$ equals the flow energy associated to $V_\varepsilon$:

$$\ell^2\sigma_\varepsilon(\omega) = \frac{1}{2} (\nabla V_\varepsilon, \nabla V_\varepsilon)_{L_2(\varepsilon \omega, \Lambda)}. \quad (24)$$

The limit in Theorem 1 can therefore be restated as

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2} (\nabla V_\varepsilon, \nabla V_\varepsilon)_{L_2(\varepsilon \omega, \Lambda)} = mD_{1,1}, \ \forall \omega \in \Omega_{\text{typ}}. \quad (25)$$

To prove Theorem 1 we will distinguish the cases $D_{1,1} = 0$ and $D_{1,1} > 0$. The proof for $D_{1,1} = 0$ (which is simpler) is given in Section 7, while the proof for $D_{1,1} > 0$ will take the rest of our investigation and will be concluded in Section 14. In the case $D_{1,1} > 0$ we can say more on the behavior of $V_\varepsilon$. To this aim we introduce the function $\psi : S \to [0, 1]$ as

$$\psi(x) := x_1 + \frac{1}{2} \text{ if } x \in \Lambda, \ \psi(x) := 0 \text{ if } x \in S^-, \ \psi(x) := 1 \text{ if } x \in S^+. \quad (26)$$

**Proposition 2.8.** Suppose that $e_1$ is an eigenvector of $D$ with eigenvalue $D_{1,1} > 0$. Then, for all $\omega \in \Omega_{\text{typ}}$, $V_\varepsilon \in L^2(\mu^\varepsilon_{\omega \Lambda})$ converges weakly and 2-scale converges weakly to $\psi \in L^2(\Lambda, dx)$.

The definition of the above convergences is recalled in Section 11.

As discussed in Section 8, $\psi$ is the unique weak solution on $\Lambda$ of the so-called effective equation given by $\nabla \cdot (D\nabla \psi) = 0$ with suitable mixed Dirichlet-Neumann boundary conditions, where $\nabla$ denotes the projection of the standard gradient $\nabla$ on the subspace of $\mathbb{R}^d$ orthogonal to the kernel of $D$ (cf. Definition 8.7). Due to Proposition 2.8, $\nabla \cdot (D\nabla \psi) = 0$ represents the effective macroscopic equation of the electric potential $V_\varepsilon$ in the limit $\varepsilon \downarrow 0$, when $D_{1,1} > 0$. 
2.1. Extensions. We detail here the extended setting where our Theorem 1 and Proposition 2.8 still hold, with suitable modifications.

First we point out that the action of the group $\mathbb{G}$ on $\mathbb{R}^d$ can be a generic proper action, i.e. given by translations $\tau_g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ of the form $\tau_g x := x + g_1v_1 + \cdots + g_dv_d$ for a fixed basis $v_1, v_2, \ldots, v_d$. Indeed, one can always reduce to the previous case $\tau_g x = x + g$ at cost of a linear isomorphism of the Euclidean space (we refer to [16] Section 2) for the form of $D$ when applying a generic proper action.

Let us keep in what follows $\tau$ proper action, i.e. given by translations $\tau : \mathbb{R}^d \rightarrow \mathbb{R}^d$ of the form $\tau x := x + g_1v_1 + \cdots + g_dv_d$ for a fixed basis $v_1, v_2, \ldots, v_d$. For the general case we have the following result:

**Theorem 2.** When $\mathbb{G} = \mathbb{Z}^d$ and not necessarily $\hat{\omega} \subset \mathbb{Z}^d$, the content of Theorem 1 and Proposition 2.8 remain valid under Assumptions (A1),...,(A8) with $m$, $P_0$, $\lambda_k$ and $D$ defined respectively as in [16, Eq. (10)], [16, Eq. (11)], [16, Eq. (15)] and [16, Def. 3.6].

For the reader’s convenience, we point out that in the above formulas from [16] one has to replace $\theta_g$ by $\tau_g$, $r_{x,y}(\omega)$ by $c_{x,y}(\omega)$, $\mu_\omega$ by $\hat{\omega}$, $\Delta$ by $[0,1)^d$.

2.2. Beyond principal directions. We explain here how to modify the definition of directional conductivity to get a quenched scaling limit along directions determined by unit vectors in $\ker(D)^\bot$, the subspace perpendicular to the kernel of $D$ ($D$ is thought of as linear map on $\mathbb{R}^d$). Obstructions when the unit vector does not belong to $\ker(D)^\bot$ are discussed in Section 8.1. Trivially, $\ker(D)^\bot$ is left invariant by the linear map $x \mapsto Dx$. On $\ker(D)^\bot$ this map is an isomorphism and the quadratic form $x \mapsto x \cdot Dx$ is strictly positive definite.

Suppose that $e_1 \in \ker(D)^\bot$. We fix an orthonormal basis $e_1, e_2, \ldots, e_d$ of $\ker(D)^\bot$ and an orthonormal basis $e_{d+1}, \ldots, e_d$ of $\ker(D)$, where $d_* := \dim(\ker(D)^\bot)$ (this notation will be in agreement with Definition 8.1 later).

We fix a unit vector $w \in \ker(D)^\bot$ orthogonal to $D e_2, \ldots, D e_d$. Then $w \cdot e_1 \neq 0$ (by contradiction, if $w = \alpha_2 e_2 + \cdots + \alpha_d e_d$, we would have $w \cdot D w = \alpha_1 \sum_{i=1}^{d_*} w \cdot D e_i = 0$, thus implying that $w \in \ker(D)$). Hence, without loss, we take $w \cdot e_1 > 0$. Note that, by this constraint, $w$ is unique as $D e_2, \ldots, D e_d$ are $d_*$-independent vectors of the $d_*$-dimensional space $\ker(D)^\bot$.

We define $\Lambda$ as the open parallelepiped

$$\Lambda := \{ x \in \mathbb{R}^d : |w \cdot x| < cw \cdot e_1 \text{ and } |x \cdot e_k| < 1/2 \text{ for } 2 \leq k \leq d \}$$

where $c$ is such that the volume of $\Lambda$ equals 1. Note that for each $x \in \Lambda$ the set $x + \mathbb{R} e_1$ is a segment between $F_-$ and $F_+$, where $F_+ := \{ x \in \Lambda : w \cdot x = \pm c w \cdot e_1 \}$. We define $S := \{ x \in \mathbb{R}^d : |x \cdot e_k| < 1/2 \text{ for } 2 \leq k \leq d \}$. We set $S^- := \{ x \in S : w \cdot x \leq -cw \cdot e_1 \}$, $S^+ := \{ x \in S : w \cdot x \geq cw \cdot e_1 \}$, $\Lambda_t := \ell \Lambda$, $S_t := \ell S$, $S^+_t := \ell S^+$. Keep Definitions 2.2, 2.3 and 2.4. We define $\psi : S \rightarrow \mathbb{R}$
by setting $\psi(x) := 0$ for $x \in S^-$, $\psi(x) := 1$ for $x \in S^+$ and $\psi(x) := a \cdot (x + ce_1)$ otherwise, where $a := (2cw \cdot e_1)^{-1}w$.

Comments motivating the above new setting are given in Section 8.1.

**Theorem 3.** Suppose that $e_1 \in \text{Ker}(D)^\perp$. Then the conclusion of Theorem 1 and Proposition 2.8 holds, with the exception that in Theorem 4 the limiting rescaled conductivity is now $m \cdot D$ instead of $mD_{1,1} = m \cdot D e_1$. Moreover, the analogous of Theorem 3 remains valid.

**Warning 2.3.** Since the geometrical objects involved in Theorem 3 are slightly different, in order to avoid some heavy notation, up to Section 14 we will deal with the setting of Theorems 1 and 2 and Proposition 2.8. In Appendix A we will comment how to modify our arguments to prove Theorem 3.

### 3. Examples

Our results can be applied to plenty of random resistor networks. Our assumptions (A1),...,(A8) (with the modifications indicated in Theorem 2 for the general case with $G = \mathbb{Z}^d$) equal the ones in [16] when $n_x(\omega) \equiv 1$ there (corresponding to the fact that $r_{x,y}(\omega)$ in [16] is symmetric and equals our $c_{x,y}(\omega)$). Hence, to all the examples discussed in [16, Section 5] with symmetric rates one can apply the present Theorems 1, 2, 3 and Proposition 2.8. Below we focus on three of these examples particularly relevant for transport in disordered media, adding some comments. As some other examples discussed in [16] we just mention resistor networks with random long conductances on $\mathbb{Z}^d$ and on crystal lattices. By the same techniques one can consider also e.g. resistor networks on supercritical percolation clusters in continuum percolation and on Delaunay triangulations [22].

#### 3.1. Nearest-neighbor random conductance model on $\mathbb{Z}^d$

We take $\Omega := (0, +\infty)^E_d$, where $E_d$ is the set of non-oriented edges of the standard lattice $\mathbb{Z}^d$. We take $\omega := \mathbb{Z}^d$ for all $\omega \in \Omega$ and set $c_{x,y}(\omega) := \omega_{\{x,y\}}$ if $\{x, y\} \in E_d$ and $c_{x,y}(\omega) = 0$ otherwise. The group $G = \mathbb{Z}^d$ acts in a natural way on $\mathbb{R}^d$ by translations and on $\Omega$ by the maps $\tau_{y} \omega = (\omega_{x-g,y-g})_{\{x,y\} \in E_d}$ if $\omega = (\omega_{\{x,y\}})_{\{x,y\} \in E_d}$. We recall that Assumption (A3) is fictitious. Then all the other Assumptions (A1),(A2) and (A4),..,(A8) are satisfied whenever $P$ is stationary and ergodic w.r.t. $\mathbb{Z}^d$-translations and $E[c_{0,e}] < +\infty$ for all $e$ in the canonical basis of $\mathbb{Z}^d$. A previous almost result for the directional conductivity in the random conductance model on $\mathbb{Z}^d$ was provided by Kozlov in [30, Section 3] for i.i.d. conductances with value in a strictly positive interval $(\delta_0, 1 - \delta_0)$, $\delta_0 > 0$. Indeed, as mentioned in the Introduction, in [30] the author introduced some Neumann boundary conditions at the microscopic level (cf. [30, Eq. (3.3)]) which in general are not fulfilled by the electric potential (compare for example with [13] and [27, Section 11.3]). To see why they can be violated, consider the resistor network in Figure 3.1. When $\varepsilon \ll 1$, the resistor network is well approximated by a linear series of 5 electrical filaments with conductance 1. Hence, for $\varepsilon \ll 1$, the electrical potential is around $1/5$ at node A and around
2/5 at node B, hence it cannot coincide at A and B as in [30] Eq. (3.3)] (note that in [30] vertical and horizontal faces are exchanged w.r.t. our setting). Finally we point out that, as D in [30] Section 3 is a multiple of the identity, the special role played by the principal directions of D does not emerge there.

3.2. Resistor network on infinite percolation clusters. We take $d \geq 2$ and $\Omega = \{0, 1\}^{\mathbb{Z}^d}$. $E_d$ and the action of $G := \mathbb{Z}^d$ are as in the previous example. Let $\mathcal{P}$ be a probability measure on $\Omega$ stationary, ergodic and fulfilling (A3). We assume that for $\mathcal{P}$–a.a. $\omega$ there exists a unique infinite connected component $C(\omega) \subset \mathbb{Z}^d$ in the graph given by the edges $\{x, y\}$ in $E_d$ with $\omega_{x,y} = 1$. Given $\omega \in \Omega$ we set $\hat{\omega} := C(\omega)$ when $C(\omega)$ exists and $\hat{\omega} = \emptyset$ otherwise. We set $c_{x,y}(\omega) := 1$ for all $x, y \in \hat{\omega}$ such that $\{x, y\} \in E_d$ and $\omega_{x,y} = 1$, and $c_{x,y}(\omega) := 0$ otherwise. Then all assumptions (A1),..., (A8) are satisfied.

To simplify the notation, we suppose that $\mathcal{P}$ is invariant by coordinate permutations. Then by symmetry $D = c I$ and Theorem 1 implies that, along any direction of $\mathbb{R}^d$, $\mathcal{P}$–a.s. $\lim_{\ell \to +\infty} \ell^{-d} \sigma_\ell(\omega) = cm$, where $m$ is the probability that the origin belongs to the infinite cluster, i.e. $m = \mathcal{P}(0 \in \hat{\omega})$. Since $\mathcal{P}_0 := \mathcal{P}(\cdot | 0 \in \hat{\omega})$, we can rewrite the variational characterization of limiting value $cm$ as

$$cm = \inf f \frac{1}{2} \int d\mathcal{P}(\omega) \sum_{e : |e| = 1, \omega_{(0,e)} = 1} \mathbf{1}_{0 \in \hat{\omega}} (e \cdot e_1 - \nabla f(\omega, e))^2,$$

where $f$ varies among all bounded Borel functions on $\Omega_0 = \{0 \in \hat{\omega}\}$ and $\nabla f(\omega, e) := f(\tau_e \omega) - f(\omega)$. This result solves the open problem stated in [5, Problem 1.18], even without restricting to Bernoulli bond percolation.

From now on, here and in the next paragraph, we focus on the case that $\mathcal{P}$ is a Bernoulli bond percolation on $\mathbb{Z}^d$ with supercritical parameter $p$ (which fulfills all the above properties). It is known that $c > 0$ (cf. [6, 12, 34]). The proof originally provided in [12, Section 4.2] can now be simplified due to our result and due to [25]. Indeed, calling $N_\ell(\omega)$ the maximal number of vertex-disjoint left-right crossings of $\Lambda_\ell$ and calling $s_j$ the number of bonds in the
In fact, the first bound follows by computing exactly the conductivity of $N_\ell(\omega)$ linear resistor chains, while the second bound follows from Jensen’s inequality. Since, for some positive constant $C$, $\mathcal{P}$–a.s. $N_\ell(\omega) \geq C \ell^{d-1}$ for $\ell$ large (combine \cite{25} Remark (d) in Section 5 with Borel-Cantelli lemma) and since $\sum_{j=1}^{N_\ell(\omega)} s_j$ is upper bounded by the total number of bonds in $\Lambda_\ell$, we conclude that $c = m^{-1} \lim_{\ell \to +\infty} \ell^{2-d} \sigma_\ell(\omega) > 0$.

Differently from \cite{5} Problem 1.18, where the resistor network in $\Lambda_\ell$ is made only by the bonds in the supercritical percolation cluster, in e.g. \cite{10} the resistor network in $\Lambda_\ell$ is made by all bonds $\{x,y\} \in \mathbb{E}_d$ with $\omega_{\{x,y\}} = 1$. On the other hand, we claim that $\mathcal{P}$–a.s. the two resistor networks have the same conductivity along a given direction for $\ell$ large enough (hence the scaling limit is the same). Indeed, by combining Borel-Cantelli lemma with Theorems (8.18) and (8.21) in \cite{23}, we get that the following holds for some $C > 0$ $\mathcal{P}$–a.s. eventually in $\ell$: for all $x \in \Lambda_\ell \cap \mathbb{Z}^d$, if the cluster in the bond percolation $\omega$ containing $x$ is finite, then it has radius at most $C \ln \ell$. As a consequence of this result, fixed a given direction, $\mathcal{P}$–a.s. the bonds not belonging to the infinite percolation cluster do not contribute to the directional conductivity if the box side length $\ell$ is large enough. As known, by similar arguments (apply \cite{23} Theorem (5.4))) one gets that $\mathcal{P}$–a.s. the directional conductivity $\sigma_\ell(\omega)$ is zero for $\ell$ large when considering the subcritical bond percolation and the resistor network on $\Lambda_\ell$ given by all bonds $\{x,y\} \in \mathbb{E}_d$ with $\omega_{\{x,y\}} = 1$.

Results similar to the above ones can be derived for site percolation or for positive random conductances on the supercritical cluster as in \cite{16} Section 5.

### 3.3. Miller-Abrahams (MA) random resistor network

The MA random resistor network plays a fundamental role in the study of electron transport in amorphous media as doped semiconductors in the regime of strong Anderson localization (cf. \cite{4} \cite{33} \cite{36} \cite{38}). With our notation, it is obtained by taking as $\Omega$ the space of configurations of marked simple point processes \cite{11}. In particular, an element of $\Omega$ is a countable set $\omega = \{(x_i, E_i)\}$ with $\{x_i\} \in \mathcal{N}$, such that if $x_i = x_j$ then $E_i = E_j$. $E_i \in \mathbb{R}$ is a real number, called energy mark of $x_i$. The simple point process $\Omega \ni \omega \mapsto \hat{\omega} \in \mathcal{N}$ is given by $\hat{\omega} := \{x_i\}$ if $\omega = \{(x_i, E_i)\}$. The group $\mathbb{G} := \mathbb{R}^d$ acts on $\Omega$ as $\tau_x \omega := \{(x_i - x, E_i)\}$ if $\omega = \{(x_i, E_i)\}$.

Moreover, between any pair of nodes $x_i$ and $x_j$, there is a filament with conductance \cite{11}, in the physical applications (cf. \cite{21} \cite{38} and references therein), $\mathcal{P}$ is obtained by starting with a simple point process $\{x_i\}$ with law $\mathcal{P}$ and marking points with i.i.d. random variables $\{E_i\}$ (independently from the rest) with common distribution of the form $\nu(dE) = c |E|^\alpha dE$ with finite support $[-A,A]$, for some exponent $\alpha \geq 0$ and some $A > 0$. Then (A1) and (A2) are satisfied if $\bar{\mathcal{P}}$ is stationary and ergodic (see \cite{11}) with positive finite intensity. (A3), (A4), (A5), (A6) are automatically satisfied. As
discussed in [16] Section 5.4] (A7) is satisfied when $E[\hat{\omega}([0, 1]^d)] < +\infty$, while (A8) is always satisfied.

As by Theorem 1 the asymptotic rescaled directional conductivities can be expressed in terms of $D = D(\beta)$, which coincides with the diffusion matrix of Mott’s random walk, we get the following:

**Corollary 3.1.** The asymptotic rescaled directional conductivities of the MA resistor network, along the principal directions of the effective homogenized $D(\beta)$, satisfy for $\beta \uparrow +\infty$ the bounds in agreement with Mott’s law obtained for the diffusion matrix of Mott’s random walk stated in [20, Thm. 1] and [21, Thm. 1] for $d \geq 2$ and in [9, Thm. 1.2] for $d = 1$.

For other rigorous results on the MA resistor network we refer to [18] [19].

4. **Reduction to the Case $G = \mathbb{R}^d$**

In this section we explain how to prove Theorem 2 (and therefore also Theorem 1 for $G = \mathbb{Z}^d$) once proved Theorem 1 and Proposition 2.8 for $G = \mathbb{R}^d$. To this aim we point out that there exists a canonical procedure to pass from the model with group $\mathbb{Z}^d$ to a related model with group $\mathbb{R}^d$ that we now briefly recall, referring to [16] Section 6] for a detailed discussion.

Let us consider the context of Theorem 2 with $G = \mathbb{Z}^d$. Again, without loss of generality, we take $\tau_g x = x + g$ for all $x \in \mathbb{R}^d$ and $g \in \mathbb{Z}^d$. Similarly, we have the translation action of the group $\mathbb{R}^d$ given by $\tau_g x := x + g$ for all $x, g \in \mathbb{R}^d$.

We consider the probability space $\tilde{\Omega} := \Omega \times [0, 1)^d$ with probability $P := \mathcal{P} \times dx$ and $\sigma$-field $\mathcal{F} := \mathcal{F} \otimes \mathcal{B}([0, 1)^d)$, $\mathcal{B}([0, 1)^d)$ being the family of Borel sets of $[0, 1)^d$. Given $y \in \mathbb{R}^d$, we define the integer part $[y]$ of $y$ as the unique element $z \in \mathbb{Z}^d$ such that $y \in z + [0, 1)^d$. We then set $\beta(y) := y - [y] \in [0, 1)^d$. The group $\mathbb{R}^d$ acts on $\tilde{\Omega}$ by means of the maps $(\tau_g)_g \in \mathbb{R}^d$ with $\tau_g(\omega, a) := (\tau_{y[a]}(\omega), \beta(g + a))$.

Finally, to $(\omega, a) \in \tilde{\Omega}$ we associate the locally finite set $(\omega, a) := \omega - a$ and consider the conductances $\bar{c}_{x-y,a}(\omega, a) := c_{x,y}(\omega)$ for $x, y \in \omega$.

Then, under the assumptions of Theorem 2, by the observations collected in [16] Section 6] the setting given by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, the group $\tilde{G} := \mathbb{R}^d$ with the above two actions, the simple point process $\tilde{\omega}$ and the conductance field $\bar{c}_{x,y}(\omega)$ satisfies assumptions (A1),...,(A8). Hence, once proven Theorem 1 and Proposition 2.8 for $G = \mathbb{R}^d$, one gets the following (taking $e_1$ as principal direction of the effective homogenized matrix $\tilde{D}$ for the new setting). There exists a translation invariant measurable set $\tilde{\Omega}_{typ}$ (for the $\mathbb{R}^d$–action) with $\tilde{P}(\tilde{\Omega}_{typ}) = 1$ such that for all $(\omega, a) \in \tilde{\Omega}_{typ}$, it holds $\lim_{m \to +\infty} \ell^{\mathbb{R}^d, D, \ell} = m D_{1,1}$. As discussed in [16] Section 6], $\tilde{m} = m$ and $\tilde{D} = D$. On the other hand, since $\tilde{\Omega}_{typ}$ is translation invariant, for $(\omega, a) \in \tilde{\Omega}_{typ}$ we have $(\omega, 0) = \tau_{-a}(\omega, a) \in \tilde{\Omega}_{typ}$. Note also that $\sigma_{1}(\omega, 0) = \sigma_{1}(\omega) = \hat{\omega}$ and $\bar{c}_{x,y}(\omega, 0) = c_{x,y}(\omega)$ for $x, y \in \hat{\omega}$. Hence, if $(\omega, a) \in \tilde{\Omega}_{typ}$, then $\lim_{m \to +\infty} \ell^{\mathbb{R}^d, D, \ell} = m D_{1,1}$. One can then check that Theorem 1 is fulfilled with $\Omega_{typ} := \{ \omega \in \Omega : (\omega, a) \in \tilde{\Omega}_{typ} \text{ for some } a \in [0, 1]^d \}$ ($\Omega_{typ}$ is translation invariant as $(\tau_g \omega, a) = \tau_g(\omega, a)$ for all $g \in \mathbb{Z}^d$, while the
issue concerning $\Omega_1$ can be settled using that $\tilde{\lambda}_k(\omega, a) = \lambda_k(\omega)$ as observed in [16, Section 6]). By similar arguments one obtains Proposition 2.8.

**Warning 4.1.** Due to the above result, in the rest we just take $\mathbb{G} = \mathbb{R}^d$ and focus on Theorem 3 and Proposition 2.8. We treat Theorem 3 in Appendix A.

5. **Preliminary facts on the Palm distribution $\mathcal{P}_0$**

In this short section we recall some basic facts on the Palm distribution $\mathcal{P}_0$ used in what follows. Recall (4).

**Lemma 5.1.** [16, Lemma 7.1] Given a measurable subset $A \subset \Omega_0$, the following facts are equivalent: (i) $\mathcal{P}_0(A) = 1$; (ii) $\mathcal{P}(\omega \in \Omega : \tau_x \omega \in A \forall x \in \hat{\omega}) = 1$; (iii) $\mathcal{P}_0(\omega \in \Omega_0 : \tau_x \omega \in A \forall x \in \hat{\omega}) = 1$.

By ergodicity (cf. (A1)), given a measurable function $g : \Omega \to [0, \infty)$ it holds

$$\lim_{s \to \infty} \frac{1}{(2s)^d} \int_{(-s,s)^d} d\hat{x} g(x) = \mathbb{E}_0[g] \quad \text{P-a.s.}$$

(29)

One can indeed refine the above result. To this aim we define $\mu^\varepsilon_\omega$ as the atomic measure on $\mathbb{R}^d$ given by $\mu^\varepsilon_\omega := \varepsilon^d \sum_{x \in \hat{\omega}} \delta_\varepsilon x$. Then it holds:

**Proposition 5.2.** [16, Prop. 3.1] Let $g : \Omega_0 \to \mathbb{R}$ be a measurable function with $\|g\|_{L^1(\mathcal{P}_0)} < +\infty$. Then there exists a translation invariant measurable subset $\mathcal{A}[g] \subset \Omega$ such that $\mathcal{P}(\mathcal{A}[g]) = 1$ and such that, for any $\omega \in \mathcal{A}[g]$ and any $\varphi \in C_c(\mathbb{R}^d)$, it holds

$$\lim_{\varepsilon \downarrow 0} \int dm^\varepsilon_\omega(\varphi(x)g(\tau_x \omega)) = \int dx m(\varphi(x)\mathbb{E}_0[g]).$$

(30)

The above proposition (which is the analogous e.g. of [11, Theorem 1.1]) is at the core of 2-scale convergence. It follows again from ergodicity. The variable $x$ appears in the l.h.s. of (30) at the macroscopic scale in $\varphi(x)$ and at the microscopic scale in $g(\tau_x \omega)$.

**Definition 5.3.** Given a function $g : \Omega_0 \to [0, +\infty)$ such that $\mathbb{E}_0[g] < +\infty$, we define $\mathcal{A}[g]$ as $\mathcal{A}[g_0]$ (cf. Proposition 5.2), where $g_0 : \Omega_0 \to \mathbb{R}$ is defined as $g$ on $\{g < +\infty\}$ and as $0$ on $\{g = +\infty\}$.

**Remark 5.4.** By playing with suitable test functions $\varphi$, one derives from Proposition 5.3 that the limit (29) holds indeed for all $\omega$ varying in a suitable translation invariant measurable subset of $\Omega$ with $\mathcal{P}$–probability 1.

6. **The Hilbert space $H^{1,\varepsilon}_{0,\omega}$ and the amorphous gradient $\nabla^\varepsilon f$**

Let $\omega \in \Omega_1$. Recall Definition 2.6 of $G^\varepsilon_\omega$, $C^\omega_{\omega,\Lambda}$ and $C^\varepsilon_\omega$ and Definition 2.7 of $H^{1,\varepsilon}_{0,\omega}$ and $K^\varepsilon_\omega$. For later use, we point out that $K^\varepsilon_\omega = H^{1,\varepsilon}_{0,\omega} + \psi^\varepsilon$, where $\psi^\varepsilon : \varepsilon \hat{\omega} \cap S \to [0, 1]$ is the function (depending also on $\omega$)

$$\psi^\varepsilon(x) := \begin{cases} x_1 + \frac{1}{2} & \text{if } x \in C^\varepsilon_\omega \cap \Lambda, \\ 0 & \text{if } x \in (\varepsilon \hat{\omega} \cap S^-) \cup C^\varepsilon_{\omega,\Lambda}, \\ 1 & \text{if } x \in \varepsilon \hat{\omega} \cap S^+. \end{cases}$$

(31)
As discussed in Section \cite{2}, if \( f : \varepsilon \omega \cap S \to \mathbb{R} \) is bounded, then \( f \in L^2(\mu_{\omega,\Lambda}^\varepsilon) \), \( \nabla \varepsilon f \in L^2(\nu_{\omega,\Lambda}^\varepsilon) \) and \( \mathbb{L}_\omega^\varepsilon f \in L^2(\mu_{\omega,\Lambda}^\varepsilon) \). By definition of \( \nu_{\omega,\Lambda}^\varepsilon \), given bounded functions \( f, g : \varepsilon \omega \cap S \to \mathbb{R} \), we have

\[
\langle \nabla \varepsilon f, \nabla \varepsilon g \rangle_{L^2(\nu_{\omega,\Lambda}^\varepsilon)} = \varepsilon^{d-2} \sum_{(x,y) \in \mathcal{E}_\varepsilon(\omega)} c_{x/\varepsilon,y/\varepsilon}(\omega) \left( f(y) - f(x) \right) \left( g(y) - g(x) \right).
\]  

(32)

**Lemma 6.1.** Let \( \omega \in \Omega_1 \). Given \( f, g : \varepsilon \omega \cap S \to \mathbb{R} \) with \( f \equiv 0 \) on \( \varepsilon \omega \setminus \Lambda \) and \( g \) bounded, it holds

\[
\langle f, -\mathbb{L}_\omega^\varepsilon g \rangle_{L^2(\mu_{\omega,\Lambda}^\varepsilon)} = \frac{1}{2} \langle \nabla \varepsilon f, \nabla \varepsilon g \rangle_{L^2(\nu_{\omega,\Lambda}^\varepsilon)}.
\]  

(33)

**Proof.** Since \( f \equiv 0 \) outside \( \Lambda \) we have

\[
\langle f, -\mathbb{L}_\omega^\varepsilon g \rangle_{L^2(\mu_{\omega,\Lambda}^\varepsilon)} = - \sum_{x \in \varepsilon \omega \cap S} \varepsilon^{d-2} f(x) \sum_{y \in \varepsilon \omega \cap S} c_{x/\varepsilon,y/\varepsilon}(\omega) \left( g(y) - g(x) \right).
\]  

(34)

The r.h.s. is an absolutely convergent series as \( \omega \in \Omega_1 \), \( f \) and \( g \) are bounded and \( f \equiv 0 \) on \( S \setminus \Lambda \), hence we can freely permute the addenda. Due to the symmetry of the conductances, the r.h.s. of (34) equals

\[
- \sum_{y \in \varepsilon \omega \cap S} \varepsilon^{d-2} f(y) \sum_{x \in \varepsilon \omega \cap S} c_{x/\varepsilon,y/\varepsilon}(\omega) \left( g(x) - g(y) \right).
\]

By summing the above expression with the r.h.s. of (34), we get

\[
\langle f, -\mathbb{L}_\omega^\varepsilon g \rangle_{L^2(\mu_{\omega,\Lambda}^\varepsilon)} = \frac{\varepsilon^{d-2}}{2} \sum_{x \in \varepsilon \omega \cap S} \sum_{y \in \varepsilon \omega \cap S} c_{x/\varepsilon,y/\varepsilon}(\omega) \left( f(y) - f(x) \right) \left( g(y) - g(x) \right).
\]

As the generic addendum in the r.h.s. is zero if \( (x,y) \notin \mathcal{E}_\varepsilon(\omega) \) since \( f \equiv 0 \) on \( S \setminus \Lambda \), by (32) we get (33). \( \square \)

**Lemma 6.2.** Given \( \omega \in \Omega_1 \), the following holds:

(i) \( V_{\varepsilon} \) is the unique function \( v \in K_\omega^\varepsilon \) such that \( \mathbb{L}_\omega^\varepsilon v(x) = 0 \) for all \( x \in \varepsilon \omega \cap \Lambda \);

(ii) \( V_{\varepsilon} \) is the unique function \( v \in K_\omega^\varepsilon \) such that \( \langle \nabla \varepsilon u, \nabla \varepsilon v \rangle_{L^2(\nu_{\omega,\Lambda}^\varepsilon)} = 0 \) for all \( u \in H_{0,\omega}^{1,\varepsilon} \);

(iii) \( V_{\varepsilon} \) is the unique minimizer of the following variational problem:

\[
\inf \{ \langle \nabla \varepsilon v, \nabla \varepsilon v \rangle_{L^2(\nu_{\omega,\Lambda}^\varepsilon)} : v \in K_\omega^\varepsilon \}.
\]

(35)

**Proof.** On the finite dimensional Hilbert space \( H_{0,\omega}^{1,\varepsilon} \) we consider the bilinear form \( a(f,g) := \frac{1}{2} \langle \nabla \varepsilon f, \nabla \varepsilon g \rangle_{L^2(\nu_{\omega,\Lambda}^\varepsilon)} \). Trivially, \( a(\cdot,\cdot) \) is a continuous symmetric bilinear form. If \( a(f,f) = 0 \), then \( f \) is constant on the connected components of the graph \( \mathcal{G}_\varepsilon \). This fact and the definition of \( H_{0,\omega}^{1,\varepsilon} \) imply that \( f \equiv 0 \). As \( H_{0,\omega}^{1,\varepsilon} \) is finite dimensional, we conclude that \( a(\cdot,\cdot) \) is also coercive. By writing \( v = f_\varepsilon + \psi_\varepsilon \), the function \( v \in K_\omega^\varepsilon \) in Item (i) is the one such that \( f_\varepsilon \in H_{0,\omega}^{1,\varepsilon} \) and

\[
\mathbb{L}_\omega^\varepsilon f_\varepsilon(x) = -\mathbb{L}_\omega^\varepsilon \psi_\varepsilon(x) \quad \forall x \in \varepsilon \omega \cap \Lambda.
\]

(36)
As \( f_\varepsilon \equiv 0 \equiv \psi_\varepsilon \) on \( C_{\varepsilon, \Lambda}^e \), which is a union of connected components of \( G_{\varepsilon, \Lambda}^e \), (36) is automatically satisfied for \( x \in C_{\varepsilon, \Lambda}^e \). Hence in (36) we can restrict to \( x \in C_{\varepsilon, \Lambda}^e \cap \Lambda \). On the other hand, functions in \( H_{0, \varepsilon, \Lambda}^1 \) are free on \( C_{\varepsilon, \Lambda}^e \cap \Lambda \), and zero elsewhere. Due to the above observations, (36) is equivalent to requiring that \( \langle u, -L_\varepsilon^e f_\varepsilon \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)} = \langle u, L_\varepsilon^e \psi_\varepsilon \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)} \) for any \( u \in H_{0, \varepsilon, \Lambda}^1 \). Hence, by Lemma 6.1, \( f_\varepsilon \in H_{0, \varepsilon, \Lambda}^1 \) satisfying (36) can be characterized also as the solution in \( H_{0, \varepsilon, \Lambda}^1 \) of the problem

\[
a(f_\varepsilon, u) = -\frac{1}{2} \langle \nabla \psi_\varepsilon, \nabla u \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)} \quad \forall u \in H_{0, \varepsilon, \Lambda}^1.
\]

By the Lax–Milgram theorem we conclude that there exists a unique function \( f_\varepsilon \) satisfying (37), thus implying Item (i). Since \( a(f_\varepsilon, u) = \frac{1}{2} \langle \nabla \psi_\varepsilon, \nabla u \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)} \), the uniqueness of the solution \( f_\varepsilon \) of (37) leads to Item (ii). Moreover, always by the Lax–Milgram theorem, \( f_\varepsilon \) is the unique minimizer of the functional \( H_{0, \varepsilon, \Lambda}^1 \ni v \mapsto \frac{1}{2} a(v, v) + \frac{1}{2} \langle \nabla \psi_\varepsilon, \nabla v \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)\times L^2(\mu_{\varepsilon, \Lambda}^e)} \), and therefore of the functional \( H_{0, \varepsilon, \Lambda}^1 \ni v \mapsto \frac{1}{2} \langle \nabla \psi_\varepsilon(v + \psi_\varepsilon), \nabla (v + \psi_\varepsilon) \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)} \). This proves Item (iii). \( \square \)

**Remark 6.3.** As \( V_\varepsilon \) is harmonic on \( \varepsilon \hat{\omega} \cap \Lambda \) (cf. Lemma 6.2–(i)), \( V_\varepsilon \) has values in \([0, 1]\).

**Remark 6.4.** Given \((x, y) \in E_\varepsilon(\omega)\), \( \{x, y\} \) is included either in \( C_{\varepsilon, \Lambda}^e \) or in \( C_{\varepsilon}^e \). In the first case we have \( \nabla V_\varepsilon(x, (y - x)/\varepsilon) = 0 \). This observation and Lemma 6.2–(ii) imply that \( \langle \nabla \psi_\varepsilon f, \nabla V_\varepsilon \rangle_{L^2(\mu_{\varepsilon, \Lambda}^e)} = 0 \) for all \( f : \varepsilon \hat{\omega} \cap S \to \mathbb{R} \) such that \( f(x) = 0 \ \forall x \in \varepsilon \hat{\omega} \cap (S \setminus \Lambda) = C_{\varepsilon}^e \setminus \Lambda \).

For the next result, recall (26) and (31).

**Lemma 6.5.** There exists a translation invariant measurable subset \( \Omega_2 \subset \Omega_1 \) such that \( P(\Omega_2) = 1 \) and, for all \( \omega \in \Omega_2 \),

\[
\lim_{\varepsilon \downarrow 0} \| \psi_\varepsilon \|_{L^2(\mu_{\varepsilon, \Lambda}^e)} < +\infty, \quad \lim_{\varepsilon \downarrow 0} \| \nabla \psi_\varepsilon \|_{L^2(\mu_{\varepsilon, \Lambda}^e)} < +\infty, \quad (38)
\]
\[
\lim_{\varepsilon \downarrow 0} \| \psi_\varepsilon \|_{L^2(\mu_{\varepsilon, \Lambda}^e)} < +\infty, \quad \lim_{\varepsilon \downarrow 0} \| \nabla \psi_\varepsilon \|_{L^2(\mu_{\varepsilon, \Lambda}^e)} < +\infty, \quad (39)
\]
\[
\lim_{\varepsilon \downarrow 0} \| V_\varepsilon \|_{L^2(\mu_{\varepsilon, \Lambda}^e)} < +\infty, \quad \lim_{\varepsilon \downarrow 0} \| \nabla V_\varepsilon \|_{L^2(\mu_{\varepsilon, \Lambda}^e)} < +\infty. \quad (40)
\]

**Proof.** By Proposition 5.2 and Remark 5.4 there exists a translation invariant measurable set \( \Omega_2 \subset \Omega_1 \) such that, for any \( \omega \in \Omega_2 \), \( \lim_{\varepsilon \downarrow 0} \mu_{\varepsilon, \Lambda}^e(\Lambda) = m \) and \( \lim_{\varepsilon \downarrow 0} \int_\Lambda d\mu_{\varepsilon, \Lambda}^e(x) \lambda_2(\mathbb{R}^2/\varepsilon\Lambda) = m\mathbb{E}[\lambda_2] \).

Let us take \( \omega \in \Omega_2 \). Since \( \psi_\varepsilon, \psi_\varepsilon, V_\varepsilon \) have value in \([0, 1]\) and \( \mu_{\varepsilon, \Lambda}^e \) has mass \( \mu_{\varepsilon, \Lambda}^e(\Lambda) \to m \), we get the first bounds in (38), (39) and (40).
Let us prove that \( \lim_{\varepsilon \to 0} \| \nabla_{\varepsilon} \psi \|_{L^2(\Omega_{\varepsilon})} < +\infty \). We have (recall (32))
\[
\| \nabla_{\varepsilon} \psi \|_{L^2(\Omega_{\varepsilon})}^2 = \varepsilon^{d-2} \sum_{(x,y) \in E_{\varepsilon}^\varepsilon(\omega)} c_{x,y} \varepsilon(\omega) \left( \psi(y) - \psi(x) \right)^2
\]
\[
\leq \varepsilon^{d-2} \sum_{(x,y) \in E_{\varepsilon}^\varepsilon(\omega)} c_{x,y} \varepsilon(\omega) (y_1 - x_1)^2
\]
\[
\leq 2\varepsilon^{d-2} \sum_{x \in \hat{\omega} \cap \Lambda} \sum_{y \in \hat{\omega} \cap S} c_{x,y} \varepsilon(\omega) (y_1 - x_1)^2.
\]
We can rewrite the last expression as \( 2\varepsilon^d \sum_{x \in \hat{\omega} \cap (\varepsilon^{-1}\Lambda)} \sum_{y \in \hat{\omega} \cap (\varepsilon^{-1}\Lambda)} c_{x,y} \varepsilon(\omega)(y_1 - x_1)^2 \), which is bounded by \( 2\varepsilon^d \sum_{x \in \hat{\omega} \cap (\varepsilon^{-1}\Lambda)} \lambda_2(\tau_x \omega) = 2 \int_{\Lambda} d\mu_0^\varepsilon(x) \lambda_2(\tau_x \omega) \).

The last integral converges to \( 2m\mathbb{E}_0[\lambda_2] < +\infty \) as \( \varepsilon \downarrow 0 \) since \( \omega \in \Omega_2 \). This concludes the proof that \( \lim_{\varepsilon \to 0} \| \nabla_{\varepsilon} \psi \|_{L^2(\Omega_{\varepsilon})} < +\infty \).

To prove that \( \lim_{\varepsilon \to 0} \| \nabla_{\varepsilon} \psi \|_{L^2(\Omega_{\varepsilon})} < +\infty \) it is enough to use the above result and note that \( c_{x,y}(\omega) |\psi(y) - \psi(x)| \leq c_{x,y}(\omega) |\psi(y) - \psi(x)| \forall x, y \in \hat{\omega} \).

Since \( V_\varepsilon \) minimizes (35) and \( \psi_\varepsilon \in K_{\varepsilon}^\varepsilon \), we have \( \| \nabla_{\varepsilon} V_\varepsilon \|_{L^2(\Omega_{\varepsilon})} \leq \| \nabla_{\varepsilon} \psi_\varepsilon \|_{L^2(\Omega_{\varepsilon})} \).

Hence \( \lim_{\varepsilon \to 0} \| \nabla_{\varepsilon} V_\varepsilon \|_{L^2(\Omega_{\varepsilon})} < +\infty \) by the second bound in (39).

6.1. Some properties of the amorphous gradient \( \nabla_{\varepsilon} \). In Section 2 we have defined \( \nabla_{\varepsilon} f \) for functions \( f : \hat{\omega} \cap S \to \mathbb{R} \). Definition (21) can be extended by replacing \( S \) with any set \( A \subseteq \mathbb{R}^d \) by requiring that \( x, x + \varepsilon z \in \hat{\omega} \cap A \). Given \( f, g : \hat{\omega} \to \mathbb{R} \), it is simple to check the following Leibniz rule:
\[
\nabla_{\varepsilon}(fg)(x,z) = \nabla_{\varepsilon}f(x,z)g(x) + f(x + \varepsilon z) \nabla_{\varepsilon}g(x,z).
\]

Given \( \varphi \in C_c^1(\mathbb{R}^d) \), take \( \ell \) such that \( \varphi(x) = 0 \) if \( |x| \geq \ell \) and fix \( \phi \in C_c^1(\mathbb{R}^d) \) with values in \([0,1]\) such that \( \phi(x) = 1 \) for \( |x| \leq \ell \) and \( \phi(x) = 0 \) for \( |x| \geq \ell + 1 \). Then, by applying respectively the mean value theorem and Taylor expansion, we get for some positive constant \( C(\varphi) \)
\[
\| \nabla_{\varepsilon} \varphi(x,z) \| \leq \| \nabla \varphi \|_{\infty} |z| \left( \phi(x) + \phi(x + \varepsilon z) \right) \text{ if } \varphi \in C_c^1(\mathbb{R}^d),
\]
\[
\| \nabla_{\varepsilon} \varphi(x,z) - \nabla \varphi(x) \cdot z \| \leq \varepsilon C(\varphi) |z|^2 \left( \phi(x) + \phi(x + \varepsilon z) \right) \text{ if } \varphi \in C_c^2(\mathbb{R}^d).
\]

7. Proof of Theorem 1 when \( D_{1,1} = 0 \)

Due to (25) we need to prove that \( \lim_{\varepsilon \to 0} \langle \nabla_{\varepsilon} V_\varepsilon, \nabla_{\varepsilon} V_\varepsilon \rangle_{L^2(\Omega_{\varepsilon})} = 0 \) for all \( \omega \) varying in a good set, where \( \text{good set} \) means a translation invariant measurable subset of \( \Omega_1 \) with \( \mathcal{P} \)-probability equal to 1. Trivially it is enough to prove the following claim: given \( \delta > 0 \), \( \lim_{\varepsilon \to 0} \frac{1}{2} \langle \nabla_{\varepsilon} V_\varepsilon, \nabla_{\varepsilon} V_\varepsilon \rangle_{L^2(\Omega_{\varepsilon})} \leq \delta \) for all \( \omega \) in a good set. Let us prove our claim.

As \( D_{1,1} = 0 \) and by (3), given \( \delta > 0 \) we can fix \( f \in L^\infty(P_0) \) such that
\[
m\mathbb{E}_0 \left[ \int d\omega(x)c_{0,x}(\omega) (x_1 - \nabla f(\omega, x))^2 \right] \leq \delta.
\]

Given \( \varepsilon > 0 \) we define the functions \( v_\varepsilon, \tilde{v}_\varepsilon : \hat{\omega} \cap S \to \mathbb{R} \) as \( v_\varepsilon(x) := \psi(x) - \varepsilon f(\tau_x \omega) \) if \( x \in \Lambda \); \( v_\varepsilon(x) := 0 \) if \( x \in S^- \); \( v_\varepsilon(x) := 1 \) if \( x \in S^+ \) and \( \tilde{v}_\varepsilon(x) := v_\varepsilon(x)1_{S \setminus \Omega_{\varepsilon}}(x) \) for all \( x \). As \( \tilde{v}_\varepsilon(x) \in K_{\varepsilon}^\varepsilon \), by Lemma 6.2-(iii) it is enough to
prove that \( \lim_{\epsilon \downarrow 0} \langle \nabla \epsilon \tilde{v}, \nabla \epsilon \tilde{v} \rangle_{L^2(\nu_{\epsilon, A})} \leq \delta \) for all \( \omega \) in a good set. To this aim we bound

\[
\frac{1}{2} \langle \nabla \epsilon \tilde{v}, \nabla \epsilon \tilde{v} \rangle_{L^2(\nu_{\epsilon, A})} \leq \epsilon^{d-2} \sum_{x \in \Omega_{\epsilon \Lambda}^{-1}} \sum_{y \in \Omega_{\epsilon S}^{-1}} c_{x,y}(\omega) (v_{\epsilon}(y) - v_{\epsilon}(x))^2.
\]

We split the sum in the r.h.s. into three contributions \( C(\epsilon), C_-(\epsilon) \) and \( C_+(\epsilon) \), corresponding respectively to the cases \( y \in \tilde{\Omega} \cap \epsilon^{-1} \Lambda, y \in \tilde{\Omega} \cap \epsilon^{-1} S^- \) and \( y \in \tilde{\Omega} \cap \epsilon^{-1} S^+ \), while in all the above contributions \( x \) varies among \( \tilde{\Omega} \cap \epsilon^{-1} \Lambda \).

If \( x, y \in \tilde{\Omega} \cap \epsilon^{-1} \Lambda \), then \( v_{\epsilon}(y) - v_{\epsilon}(x) = \epsilon(y_1 - x_1 - \nabla f(\tau_\omega, y - x)) \). Hence, we can bound

\[
C(\epsilon) \leq \epsilon^d \sum_{x \in \Omega_{\epsilon \Lambda}^{-1}} \sum_{y \in \Omega_{\epsilon S}^{-1}} c_{x,y}(\omega) (y_1 - x_1 - \nabla f(\tau_\omega, y - x))^2. \tag{46}
\]

By ergodicity (cf. (29) and Remark 5.4) for all \( \omega \) varying in a suitable good set the r.h.s. of (46) converges to the l.h.s. of (45), and therefore it is bounded by \( \delta \). Hence, \( \lim_{\epsilon \downarrow 0} C(\epsilon) \leq \delta \).

We now consider \( C_-(\epsilon) \) and prove that \( \lim_{\epsilon \downarrow 0} C_-(\epsilon) = 0 \). If \( x \in \tilde{\Omega} \cap \epsilon^{-1} \Lambda \) and \( y \in \tilde{\Omega} \cap \epsilon^{-1} S^- \), then \( (v_{\epsilon}(y) - v_{\epsilon}(x))^2 = \epsilon^2(x_1 - f(\tau_\omega,y))^2 \leq 2\epsilon^2 x_1^2 + 2\epsilon^2 \| f \|_\infty^2 \leq 2\epsilon^2 (x_1 - y_1)^2 + 2\epsilon^2 \| f \|_\infty^2. \) Hence it remains to show that

\[
\epsilon^d \sum_{x \in \Omega_{\epsilon \Lambda}^{-1}} \sum_{y \in \Omega_{\epsilon S}^{-1}} c_{x,y}(\omega) [(x_1 - y_1)^2 + 1] \tag{47}
\]

goes to zero as \( \epsilon \downarrow 0 \). Given \( \rho \in (0,1) \) we set \( \Lambda_\rho := \rho \Lambda \). We denote by \( A_1(\rho, \epsilon) \) the sum in (47) restricted to \( x \in \tilde{\Omega} \cap \epsilon^{-1} \Lambda_\rho \) and \( y \in \tilde{\Omega} \cap \epsilon^{-1} S^- \). We denote by \( A_2(\rho, \epsilon) \) the sum coming from the remaining addenda. We observe that \( A_1(\rho, \epsilon) \leq \epsilon^d \sum_{x \in \Omega_{\epsilon \Lambda_\rho}^{-1}} g_{1-\rho}(\tau_\omega) \), where \( g_{1-\rho}(\omega) := \sum_{z \in \Omega} c_{0,\omega}(\omega)[z^2 + 1][|z|_\infty \geq \ell]. \) By (29) and Remark 5.4 for all \( \omega \) varying in a good set \( \Omega' \) it holds \( \lim_{\epsilon \downarrow 0} \epsilon^d \sum_{x \in \Omega_{\epsilon \Lambda_\rho}^{-1}} g_{1-\rho}(\tau_\omega) = m \rho^d \mathbb{E}_0[g_1] \) for all \( \ell \in \mathbb{N} \). As \( g_{1-\rho}(\omega) \) is bounded and by (A7) and dominated convergence, we obtain for all \( \omega \in \Omega' \) that \( \lim_{\epsilon \downarrow 0} A_1(\rho, \epsilon) \leq \lim_{\ell \uparrow \infty} m \rho^d \mathbb{E}_0[g_1] = 0 \). We move to \( A_2(\rho, \epsilon) \). We can bound \( A_2(\rho, \epsilon) \) by

\[
\epsilon^d \sum_{x \in \Omega_{\epsilon \Lambda_\rho}^{-1}} \sum_{y \in \tilde{\Omega}} c_{x,y}(\omega) [(x_1 - y_1)^2 + 1]. \tag{48}
\]

By Proposition 5.2 with suitable test functions, for all \( \omega \) varying in a suitable good set we get that (48) converges as \( \epsilon \downarrow 0 \) to \( m \mathbb{E}_0[\lambda_2+\lambda_0] \ell(\Lambda \setminus \Lambda_\rho) \), where here \( \ell(\cdot) \) denotes the Lebesgue measure. To conclude the proof that \( \lim_{\epsilon \downarrow 0} C_-(\epsilon) = 0 \), it is therefore enough to take the limit \( \rho \uparrow 1 \).

By the same arguments used for \( C_-(\epsilon) \), one proves that \( \lim_{\epsilon \downarrow 0} C_+(\epsilon) = 0 \).

8. Effective equation with mixed boundary conditions

In this section we assume that \( d_* := \dim(\text{Ker}(D)\downarrow) \geq 1 \) and, given a domain \( A \subset \mathbb{R}^d \), \( L^2(A) \) and \( H^1(A) \) refer to the Lebesgue measure \( dx \), which will be omitted from the notation. Here, and in the other sections, given a unit vector
we write $\partial_v f$ for the weak derivative of $f$ along the direction $v$ (if $v = e_i$, then $\partial_v f$ is simply the standard weak derivative $\partial_i f$).

We are interested in elliptic operators with mixed (Dirichlet and Neumann) boundary conditions. By denoting $\bar{\Lambda}$ the closure of $\Lambda$, we set

$$F_+ := \{ x \in \bar{\Lambda} : x_1 = 1/2 \}, \quad F_- := \{ x \in \bar{\Lambda} : x_1 = -1/2 \}, \quad F := F_- \cup F_+ .$$

**Definition 8.1.** We fix an orthonormal basis $e_1, e_2, \ldots, e_{d_\star}$ of $\text{Ker}(D)^\perp$ and an orthonormal basis $e_{d_\star+1}, \ldots, e_d$ of $\text{Ker}(D)$, where $d_\star := \dim(\text{Ker}(D)^\perp)$ (when $D$ is non degenerate, one can simply take $e_k := e_k$).

**Definition 8.2.** We introduce the following three functional spaces:

- We define $H^1(\Lambda, d_\star)$ as the Hilbert space given by functions $f \in L^2(\Lambda)$ with weak derivative $\partial_v f$ in $L^2(\Lambda)$ for any $i = 1, \ldots, d_\star$, endowed with the squared norm $\|f\|_{H^1(\Lambda)}^2 := \|f\|_{L^2(\Lambda)}^2 + \sum_{i=1}^{d_\star} \|\partial_i f\|_{L^2(\Lambda)}^2$. Moreover, given $f \in H^1(\Lambda, d_\star)$, we define

$$\nabla_s f := \sum_{i=1}^{d_\star} (\partial_i f) e_i .$$

- We define $H_0^1(\Lambda, F, d_\star)$ as the closure in $H^1(\Lambda, d_\star)$ of

$$\{ \varphi|_\Lambda : \varphi \in C_\infty^0(\mathbb{R}^d \setminus F) \} .$$

- We define the functional set $K$ as (cf. (26))

$$K := \{ \psi|_\Lambda + f : f \in H_0^1(\Lambda, F, d_\star) \} .$$

We stress that, if $e_i = e_i$ for all $i = 1, \ldots, d_\star$ (as in many applications), then

$$\nabla_s f := (\partial_1 f, \ldots, \partial_{d_\star} f, 0, \ldots, 0) \in L^2(\Lambda)^d .$$

The definition of $H^1(\Lambda, d_\star)$ is indeed intrinsic, i.e. it does not depend on the particular choice of $e_1, e_2, \ldots, e_{d_\star}$ as orthonormal basis of $\text{Ker}(D)^\perp$: by linearity, $f \in H^1(\Lambda, d_\star)$ if and only if $f \in L^2(\Lambda)$ and $\partial_v f \in L^2(\Lambda)$ for any unit vector $v \in \text{Ker}(D)^\perp$ ($\|f\|_{H^1(\Lambda)}$ is also basis-independent). Moreover, when $f$ is smooth, $\nabla_s f$ is simply the orthogonal projection of $\nabla f$ on $\text{Ker}(D)^\perp$.

**Remark 8.3.** Suppose that $e_i = e_i$ for all $1 \leq i \leq d_\star$. Let $f \in H^1(\Lambda, d_\star)$. Given $1 \leq i \leq d_\star$, by integrating $\partial_i f$ times $\varphi(x_1, \ldots, x_{d_\star})\phi(x_{d_\star+1}, \ldots, x_d)$ with $\varphi \in C_\infty^0(\mathbb{R}^{d_\star})$ (varying in a suitable countable subset) and $\phi \in C_\infty^0(\mathbb{R}^{d-d_\star})$, one gets that the function $f(y_1, \ldots, y_{n-d_\star})$ belongs to $H^1((-1/2, 1/2)^{d_\star})$ for a.e. $(y_1, \ldots, y_{n-d_\star}) \in (-1/2, 1/2)^{n-d_\star}$.

Being a closed subspace of the Hilbert space $H^1(\Lambda, d_\star)$, $H_0^1(\Lambda, F, d_\star)$ is a Hilbert space. We also point out that in the definition of $K$ one could replace $\psi|_\Lambda$ by any other function $\phi \in H^1(\Lambda, d_\star) \cap C(\bar{\Lambda})$ such that $\phi \equiv 0$ on $F_-$ and $\phi \equiv 1$ on $F_+$ as follows from the next lemma:

**Lemma 8.4.** Let $u \in H^1(\Lambda, d_\star) \cap C(\bar{\Lambda})$ satisfy $u \equiv 0$ on $F$. Then $u \in H_0^1(\Lambda, F, d_\star)$. 
Proof. For simplicity of notation we suppose that \( c_i = c_i \) for all \( 1 \leq i \leq d_s \) (in the general case, replace \( \partial_i \) by \( \partial_i \) below). We use some idea from the proof of [8, Theorem 9.17]. We set \( u_n(x) := G(nu(x))/n \), where \( G \in C^1(\mathbb{R}) \) satisfies: \( |G(t)| \leq |t| \) for all \( t \geq 0 \) and \( G(t) = 0 \) for \( |t| \leq 1 \) and \( G(t) = t \) for \( |t| \geq 2 \). Note that \( \partial_i u_n(x) = G'(nu(x)) \partial_i u(x) \) for \( 1 \leq i \leq d_s \) (cf. [8, Prop. 9.5]). Hence, \( u_n \to u \) a.e. and \( \partial_i u_n \to \partial_i u \) a.e. In the last identity, we have used that \( \partial_i u = 0 \) a.e. on \( \{u = 0\} \) which follows as a byproduct of Remark 8.3 and Stampacchia’s theorem (see Thereom 3 and Remark (ii) to Theorem 4 in [14, Section 6.1.3]). By dominated convergence one obtains that, by adapting [8, Cor. 9.8] or [14, Thm. 1, Sec. 4.4], there exists a sequence of functions \( \phi \). Hence, for simplicity of notation we suppose that \( \phi \) is defined by extending \( \partial \) to \( \Lambda \). Given a function \( u \in L^2(\Lambda) \), the following properties are equivalent:

(i) \( u \in H_0^1(\Lambda, F, d_s) \);

(ii) there exists \( C > 0 \) such that

\[
\left| \int_{\Lambda} u \partial_i \varphi dx \right| \leq C \| \varphi \|_{L^2(\Lambda)} \quad \forall \varphi \in C^\infty_c(S), \forall i : 1 \leq i \leq d_s ;
\]

(iii) the function

\[
\bar{u}(x) := \begin{cases} 
  u(x) & \text{if } x \in \Lambda, \\
  0 & \text{if } x \in S \setminus \Lambda,
\end{cases}
\]

belongs to \( H^1(S, d_s) \) (which is defined similarly to \( H^1(\Lambda, d_s) \) by replacing \( \Lambda \) with \( S \)). Moreover, in this case it holds \( \partial_i \bar{u} = \partial_i u \) for \( 1 \leq i \leq d_s \), where \( \partial_i u \) is defined by extending \( \partial_i \) as zero on \( S \setminus \Lambda \).

Lemma 8.6 (Poincaré inequality). It holds \( \| f \|_{L^2(\Lambda)} \leq \| \partial_1 f \|_{L^2(\Lambda)} \) for any \( f \in H_0^1(\Lambda, F, d_s) \).

Proof. Given \( f \in C^\infty_c(\mathbb{R}^d \setminus F) \), by Schwarz inequality, for any \( (x_1, x') \in \Lambda \) with \( x_1 \in (-\frac{1}{2}, \frac{1}{2}) \) we have

\[
|f(x_1, x')|^2 = \left( \int_{-1/2}^{1/2} \partial_1 f(s, x') ds \right)^2 \leq \int_{-1/2}^{1/2} \partial_1 f(s, x')^2 ds.
\]

By integrating over \( \Lambda \) we get the desired estimate for \( f \in C^\infty_c(\mathbb{R}^d \setminus F) \). Since \( C^\infty_c(\mathbb{R}^d \setminus F) \) is dense in \( H_0^1(\Lambda, F, d_s) \), we get the thesis.

Definition 8.7. We say that \( v \) is a weak solution of the equation

\[
\nabla_s \cdot (D \nabla_s v) = 0
\]

(54)
on $\Lambda$ with boundary conditions
\begin{align*}
  v(x) &= 0 & \text{if } x \in F^-, \\
  v(x) &= 1 & \text{if } x \in F^+, \\
  D\nabla_v(x) \cdot n(x) &= 0 & \text{if } x \in \partial\Lambda \setminus F,
\end{align*}
(55)
for all $v \in K$ (cf. (50)) and if $\int_\Lambda \nabla_v u \cdot D\nabla_v dx = 0$ for all $u \in H_0^1(\Lambda, F, d_s)$.

Above $\mathbf{n}$ denotes the outward unit normal vector to the boundary in $\partial\Lambda$.

Remark 8.8. In the above definition it would be enough to require that $\int_\Lambda \nabla_v u \cdot D\nabla_v dx = 0$ for all $u \in C^\infty_c(\mathbb{R}^d \setminus F)$ since the functional $H_0^1(\Lambda, F, d_s) \ni u \mapsto \int_\Lambda \nabla_v u \cdot D\nabla_v dx \in \mathbb{R}$ is continuous.

We shortly motivate the above Definition 8.7. Just to simplify the notation we suppose that $e_i = e_i$ for $1 \leq i \leq d_s$. By Green’s formula we have
\[
  \int_\Lambda (\partial_i f) g dx = - \int_\Lambda f (\partial_i g) dx + \int_{\partial\Lambda} f g (n \cdot e_i) dS, \quad \forall f, g \in C^1(\bar{\Lambda}) , \tag{56}
\]
where $dS$ is the surface measure on $\partial\Lambda$. By taking $1 \leq i, j \leq d_s$, $f = \partial_j v$ and $g = u$ in (56), we get
\[
  \int_\Lambda u \nabla_v \cdot (D\nabla_v) dx = - \int_\Lambda \nabla_v u \cdot (D\nabla_v) dx + \int_{\partial\Lambda} u (\nabla_v \cdot (D\mathbf{n})) dS, \tag{57}
\]
for all $v \in C^2(\bar{\Lambda})$ and $u \in C^1(\bar{\Lambda})$. Hence, $v \in C^2(\bar{\Lambda})$ satisfies $\nabla_v \cdot (D\nabla_v) = 0$ on $\Lambda$ and $\nabla_v v \cdot (D\mathbf{n}) \equiv 0$ on $\partial\Lambda \setminus F$ if and only if $\int_\Lambda \nabla_v u \cdot (D\nabla_v) dx = 0$ for any $u \in C^1(\bar{\Lambda})$ with $u \equiv 0$ on $F$. Such a set $C$ of functions $u$ is dense in $H_0^1(\Lambda, F, d)$. Indeed $C \subseteq H_0^1(\Lambda, F, d)$ by Lemma 8.1 while $C^\infty_c(\mathbb{R}^d \setminus F) \subseteq C$. Hence, we conclude that $v \in C^2(\bar{\Lambda})$ satisfies $\nabla_v \cdot (D\nabla_v) = 0$ on $\Lambda$ and $\nabla_v v \cdot (D\mathbf{n}) \equiv 0$ on $\partial\Lambda \setminus F$ if and only if $\int_\Lambda \nabla_v u \cdot (D\nabla_v) dx = 0$ for any $u \in H_0^1(\Lambda, F, d)$. We have therefore proved that $v \in C^2(\bar{\Lambda})$ is a classical solution of (54) and (55) if and only if it is a weak solution in the sense of Definition 8.7.

Lemma 8.9. Suppose that $e_1 \in \text{Ker}(D)^\perp$. Then there exists a unique weak solution $v \in K$ of the equation $\nabla_v \cdot (D\nabla_v) = 0$ with boundary conditions (55). Furthermore, $v$ is the unique minimizer of $\inf_{h \in K} \int \nabla_v h \cdot D\nabla_v h dx$.

Proof. To simplify the notation, in what follows we write $\psi$ instead of $\psi_K$. We define the bilinear form $a(f, g) := \int_\Lambda \nabla_v f \cdot D\nabla_v g dx$ on the Hilbert space $H_0^1(\Lambda, F, d_s)$. The bilinear form $a(\cdot, \cdot)$ is symmetric and continuous. Due to the Poincaré inequality (cf. Lemma 8.1) and since $e_1 \in \text{Ker}(D)^\perp$, $a(\cdot, \cdot)$ is also coercive (use that $\int |\partial_i f|^2 dx = \int |\nabla_v f \cdot e_i|^2 dx \leq \int |\nabla_v f|^2 dx \leq a(f, f)/C$, where $C$ is the minimal positive eigenvalue of $D$).

By definition we have that $v \in K$ is a weak solution of equation $\nabla_v \cdot (D\nabla_v) = 0$ with b.c. (55) if and only if, setting $f := v - \psi$, $e \in H_0^1(\Lambda, F, d_s)$ and $f$ satisfies
\[
  \int \nabla_v f \cdot D\nabla_v u dx = - \int \nabla_v \psi \cdot D\nabla_v u dx \quad \forall u \in H_0^1(\Lambda, F, d_s). \tag{58}
\]
Note that the r.h.s. is a continuous functional in \( u \in H^1_0(\Lambda, F, d_*). \) Due to the above observations and by Lax–Milgram theorem, we conclude that there exists a unique such function \( f \), hence there is a unique weak solution \( v \) of equation \( \nabla_* \cdot (D\nabla_* v) = 0 \) with b.c. (55). Moreover \( f \) satisfies
\[
\frac{1}{2} a(f, f) + \int \nabla_\psi \cdot D\nabla_* f \, dx = \inf_{g \in H^1_0(\Lambda, F, d_*)} \left\{ \frac{1}{2} a(g, g) + \int \nabla_\psi \cdot D\nabla_* g \, dx \right\}.
\]
By adding to both sides \( \frac{1}{2} \int \nabla_\psi \cdot D\nabla_\psi \, dx \), we get that \( \frac{1}{2} \int \nabla_\psi \cdot D\nabla_\psi \, dx = \inf_{h \in K} \frac{1}{2} \int \nabla_\psi h \cdot D\nabla_\psi h \, dx. \)

As \( \nabla_\psi \psi_\Lambda(x) = e_1 \) and \( De_1 \) is proportional to \( e_1 \) if \( e_1 \) is a \( D \)-eigenvector, from the above lemma we immediately get:

**Corollary 8.10.** If \( e_1 \) is an eigenvector of \( D \) with eigenvalue \( D_{1,1} > 0 \), then the function \( \psi_\Lambda \) is the unique weak solution \( v \in K \) of the equation \( \nabla_\psi \cdot (D\nabla_\psi v) = 0 \) with boundary conditions (55).

### 8.1. Comments on the effective equation (specific for the case \( \Lambda = (-1/2, 1/2)^d \))

When \( e_1 \cdot De_1 = 0 \), automatically \( e_1 \in \text{Ker}(D) \) and Theorems 1 and 2 hold. On the other hand, if \( e_1 \cdot De_1 > 0 \) but \( e_1 \notin \text{Ker}(D)^\perp \), then the conclusion of Lemma 8.9 can fail. Take for example \( d = 2 \), \( d_* = 1 \). Then, if \( |e_1 \cdot e_1| \ll 1 \), functions of the form \( f(x) := \varphi(x \cdot e_2) \), with \( \varphi \in C^1(\mathbb{R}) \), \( \varphi(u) \equiv 0 \) for \( u < -\varepsilon \), \( \varphi(u) \equiv 1 \) for \( u > \varepsilon \) and \( \varepsilon > 0 \) small, satisfy \( f \in K \) and \( \nabla_\psi f \equiv 0 \). Hence Lemma 8.9 fails.

From now on let us restrict to \( e_1 \in \text{Ker}(D)^\perp \). Then Lemma 8.9 holds. The problem is now that the unique weak solution \( v \) of (54) with b.c. (55) is not necessarily affine. To be affine one should have \( v(x) = e_1 \cdot x + b \) for some \( b \in \mathbb{R} \) (because of the values on \( F_- \) and \( F_+ \)), but then we would need \( De_1 \) to be orthogonal to \( e_2, \ldots, e_d \) due to (55), and therefore we would need again that \( e_1 \) is an eigenvector. To overcome this obstruction one has to change the shape of \( \Lambda \) as discussed in Section 2.2 for Theorem 3.

### 9. Square integrable forms and effective homogenized matrix

As common in homogenization theory, the variational formula (9) defining the effective homogenized matrix \( D \) admits a geometrical interpretation in the Hilbert space of square integrable forms. For later use we recall here this interpretation. We also collect some facts taken from [16].

We define \( \nu \) as the Radon measure on \( \Omega \times \mathbb{R}^d \) such that
\[
\int d\nu(\omega, z)g(\omega, z) = \int dP_0(\omega) \int d\hat{\omega}(z)c_{0,z}(\omega)g(\omega, z)
\]
for any nonnegative measurable function \( g(\omega, z) \). We point out that \( \nu \) has finite total mass since \( \nu(\Omega \times \mathbb{R}^d) = \mathbb{E}_0[\lambda_0] < +\infty \). Elements of \( L^2(\nu) \) are called square integrable forms.

Given a function \( u : \Omega_0 \to \mathbb{R} \), its gradient \( \nabla u : \Omega \times \mathbb{R}^d \to \mathbb{R} \) is defined as
\[
\nabla u(\omega, z) := u(\tau_z \omega) - u(\omega).
\]

\( \nonumber \)
If $u$ is defined $\mathcal{P}_0$–a.s., then $\nabla u$ is well defined $\nu$–a.s. by Lemma 9.1. If $u$ is bounded and measurable, then $\nabla u \in L^2(\nu)$. The subspace of potential forms $L^2_{\text{pot}}(\nu)$ is defined as the following closure in $L^2(\nu)$:

$$L^2_{\text{pot}}(\nu) := \{\nabla u : u \text{ is bounded and measurable}\}.$$ 

The subspace of solenoidal forms $L^2_{\text{solenoid}}(\nu)$ is defined as the orthogonal complement of $L^2_{\text{pot}}(\nu)$ in $L^2(\nu)$.

**Definition 9.1.** Given a square integrable form $v \in L^2(\nu)$ its divergence $\text{div} \ v \in L^1(\mathcal{P}_0)$ is defined as $\text{div} \ v(\omega) = \int d\omega(z)c_{a,z}(\omega)(v(\omega, z) - v(\tau_z\omega, -z))$.

The r.h.s. in the above identity is well defined since it corresponds $\mathcal{P}_0$–a.s. to an absolutely convergent series (cf. [16, Section 8] where (A3) is used). For any $v \in L^2(\nu)$ and any bounded and measurable function $u : \Omega_0 \to \mathbb{R}$, it holds (cf. [16, Section 8]) $\int d\mathcal{P}_0(\omega) \text{div} \ v(\omega) u(\omega) = - \int d\nu(\omega, z) v(\omega, z) \nabla u(\omega, z)$. As a consequence we have that, given $v \in L^2(\nu)$, $v \in L^2_{\text{solenoid}}(\nu)$ if and only if $\text{div} \ v = 0 \mathcal{P}_0$–a.s. By using (A6) one also gets (cf. [16, Section 8]):

**Lemma 9.2.** The functions $g \in L^2(\mathcal{P}_0)$ of the form $g = \text{div} \ v$ with $v \in L^2(\nu)$ are dense in $\{w \in L^2(\mathcal{P}_0) : E_0[w] = 0\}$.

As $\lambda_2 \in L^1(\mathcal{P}_0)$, given $a \in \mathbb{R}^d$ the form $u_a(\omega, z) := a \cdot z$ belongs to $L^2(\nu)$. We note that the effective homogenized matrix $D$ defined in [9] satisfies, for any $a \in \mathbb{R}^d$,

$$q(a) := a \cdot D a = \inf_{v \in L^2_{\text{pot}}(\nu)} \frac{1}{2} \|u_a + v\|^2_{L^2(\nu)} = \frac{1}{2} \|u_a + v^a\|^2_{L^2(\nu)},$$

(61)

where $v^a = -\Pi u_a$ and $\Pi : L^2(\nu) \to L^2_{\text{pot}}(\nu)$ denotes the orthogonal projection of $L^2(\nu)$ on $L^2_{\text{pot}}(\nu)$. Hence $v^a$ is characterized by the properties

$$v^a \in L^2_{\text{pot}}(\nu), \quad v^a + u_a \in L^2_{\text{solenoid}}(\nu).$$

(62)

Moreover it holds (cf. [16, Section 9]):

$$Da = \frac{1}{2} \int d\nu(\omega, z) z(a \cdot z + v^a(\omega, z)) \quad \forall a \in \mathbb{R}^d.$$  

(63)

By (61) the kernel $K(q)$ of the quadratic form $q$ is given by $\text{Ker}(q) := \{a \in \mathbb{R}^d : q(a) = 0\} = \{a \in \mathbb{R}^d : u_a \in L^2_{\text{pot}}(\nu)\}$. Note that $\text{Ker}(q) = \text{Ker}(D)$. The following result is the analogous of [41, Lemma 5.1] (cf. [16, Section 9]):

**Lemma 9.3.** $\text{Ker}(q)^\perp = \{\int d\nu(\omega, z)b(\omega, z)z : b \in L^2_{\text{solenoid}}(\nu)\}$.

It is simple to check that Definition 8.1 and Lemma 9.3 imply the following:

**Corollary 9.4.** Span$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d\} = \{\int d\nu(\omega, z)b(\omega, z)z : b \in L^2_{\text{solenoid}}(\nu)\}$.

**Definition 9.5.** Let $b(\omega, z) : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function with $\|b\|_{L^1(\nu)} < +\infty$. We define the measurable function $c_b : \Omega_0 \to [0, +\infty]$ as

$$c_b(\omega) := \int d\mathcal{L}(z)c_{0,z}(\omega)|b(\omega, z)|,$$  

(64)
the measurable function \( \hat{b} : \Omega_0 \to \mathbb{R} \) as

\[
\hat{b}(\omega) := \begin{cases} 
\int \hat{\omega}(z)c_{0,z}(\omega)b(\omega, z) & \text{if } c_b(\omega) < +\infty, \\
0 & \text{if } c_b(\omega) = +\infty,
\end{cases}
\]

and the measurable set \( A_1[b] := \{ \omega \in \Omega : c_b(\tau_z\omega) < +\infty \forall z \in \hat{\omega} \} \).

\( A_1[b] \) is a measurable translation invariant set and \( \mathcal{P}(A_1[b]) = \mathcal{P}_0(A_1[b]) = 1 \) if \( \| b \|_{L^1(\nu)} < +\infty \) (cf. [16, Section 10]). Trivially \( \| \hat{b} \|_{L^1(\nu)} \leq \| b \|_{L^1(\nu)} \).

**Definition 9.6.** Given a measurable function \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) we define \( \tilde{b} : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) as

\[
\tilde{b}(\omega, z) := \begin{cases} 
b(\tau_z\omega, -z) & \text{if } z \in \hat{\omega}, \\
0 & \text{otherwise}.
\end{cases}
\]

**Definition 9.7.** Let \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a measurable function with \( \| b \|_{L^1(\nu)} < +\infty \). If \( \omega \in A_1[b] \cap A_1[\tilde{b}] \cap \Omega_0 \), we set \( \text{div}_b(\omega) := \hat{b}(\omega) - \tilde{b}(\omega) \in \mathbb{R} \).

One can check (cf. [16, Section 11]) that given a measurable function \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) with \( \| b \|_{L^2(\nu)} < +\infty \), then \( \| b \|_{L^2(\nu)} = \| \hat{b} \|_{L^2(\nu)} \), \( \mathcal{P}_0(A_1[b] \cap A_1[\tilde{b}]) = 1 \), \( \text{div}_b = \text{div} b = -\text{div} \tilde{b} \) in \( L^1(\mathcal{P}_0) \). In particular, for \( \| b \|_{L^2(\nu)} < +\infty \), \( b \in L^2_{\text{sol}}(\nu) \) if and only if \( \hat{b} \in L^2_{\text{sol}}(\nu) \).

**Definition 9.8.** Let \( b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a measurable function with \( \| b \|_{L^2(\nu)} < +\infty \) and such that its class of equivalence in \( L^2(\nu) \) belongs to \( L^2_{\text{sol}}(\nu) \). Then we set

\[
A_d[b] := \{ \omega \in A_1[b] \cap A_1[\tilde{b}] : \text{div}_b(\tau_z\omega) = 0 \ \forall z \in \hat{\omega} \}.
\]

\( A_d[b] \) is a translation invariant measurable set and \( \mathcal{P}(A_d[b]) = 1 \) (cf. [16, Section 11]).

Recall the set \( A[g] \) introduced in Proposition 5.2 and Definition 5.3. The following lemma corresponds to [16, Lemma 18.2]. Here we have isolated the properties on \( \omega \) used in the proof there.

**Lemma 9.9.** Suppose that \( \omega \) belongs to the sets \( A_1[1], \ A_1[\lambda_0], \ A_1[|z|^2] \) and \( A[\int d\hat{\omega}(z)c_{0,z}(\omega)|z|^21(|z| \geq \ell)] \) for all \( \ell \in \mathbb{N} \). Then \( \forall \varphi \in C_c^\infty(\mathbb{R}^d) \) we have

\[
\lim_{\varepsilon \downarrow 0} \int \nu^\varepsilon(x, z) \left[ \nabla \varphi(x, z) - \nabla \varphi(x) \cdot z \right]^2 = 0.
\]

10. The set \( \Omega_{\text{typ}} \) of typical environments

In this section we describe the set \( \Omega_{\text{typ}} \) of typical environments for which Theorem 1 and Proposition 2.8 hold when \( D_{1,1} > 0 \), and for which Theorem 3 holds when \( a \cdot D a > 0 \). To define \( \Omega_{\text{typ}} \) and also \( 2 \)-scale convergence in the next section, we need to isolate suitable countable dense sets of \( L^2(\mathcal{P}_0) \) and \( L^2(\nu) \).

Recall that \( L^2(\mathcal{P}_0) \) is separable due to (A8). Then one can prove that also \( L^2(\nu) \) is separable (cf. [16, Section 12]). The separability of \( L^2(\mathcal{P}_0) \) and \( L^2(\nu) \) will be used below to construct suitable countable dense subsets. The
functional sets $G_0, H_1, G_2, H_2, W$ presented below are as in [16, Section 12] (we recall their definition for the reader’s convenience).

- **The functional sets $G_1, H_1$.** We fix a countable set $H_1$ of measurable functions $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ such that $\|b\|_{L^2(\omega)} < +\infty$ for any $b \in H_1$ and such that $\{\text{div } b : b \in H_1\}$ is a dense subset of $\{w \in L^2(\mathcal{P}_0) : \mathbb{E}_0[w] = 0\}$ when thought of as set of $L^2$–functions (recall Lemma [9.2]). For each $b \in H_1$ we define the measurable function $g_b : \Omega_0 \to \mathbb{R}$ as (cf. Definition [9.7])

$$g_b(\omega) := \begin{cases} \text{div}_z b(\omega) & \text{if } \omega \in A_1[b] \cap A_1[\hat{b}] ; \\ 0 & \text{otherwise}. \end{cases}$$

(69)

Note that $g_b = \text{div } b, \mathcal{P}_0$–a.s. (see Section [9]). We set $G_1 := \{g_b : b \in H_1\}$.

- **The functional sets $G_2, H_2$.** We fix a countable set $G_2$ of bounded measurable functions $g : \Omega_0 \to \mathbb{R}$ such that the set $\{\nabla g : g \in G_2\}$, thought in $L^2(\nu)$, is dense in $L^2_{\text{sol}}(\nu)$. We define $H_2$ as the set of measurable functions $h : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ such that $h = \nabla g$ for some $g \in G_2$.

- **The functional set $W$.** We fix a countable set $W$ of measurable functions $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ such that, thought of as subset of $L^2(\nu)$, $W$ is dense in $L^2_{\text{sol}}(\nu)$. Since $\hat{b} \in L^2_{\text{sol}}(\nu)$ for any $b \in L^2_{\text{sol}}(\nu)$, at cost to enlarge $W$ we assume that $\hat{b} \in W$ for any $b \in W$.

We introduce the following functions on $\Omega_0$ (note that they are $\mathcal{P}_0$–integrable), where $n, i, b$ vary respectively in $\mathbb{N}, \{1, \ldots, d\}, W$:

$$f_n(\omega) := \int d\omega(z)c_0(z)|z|^21(|z| \geq n),$$

(70)

$$u_{b,i}(\omega) := \int d\omega(z)c_0(z)z_i b(\omega, z),$$

(71)

$$u_{b,i,n}(\omega) := |u_{b,i}(\omega)|1(|u_{b,i}(\omega)| \geq n).$$

(72)

**Definition 10.1 (Functional set $G$).** We define $G$ as the union of the following countable sets of measurable functions on $\Omega_0$, which are $\mathcal{P}_0$–square integrable: $\{1\}, G_1, G_2$ and $\{u_{b,i}(1(|u_{b,i}| \leq n)) \}$ with $b \in W$, $i \in \{1, \ldots, d\}$, $n \in \mathbb{N}$.

**Definition 10.2 (Functional set $H$).** We define $H$ as the union of the following countable sets of measurable functions on $\Omega_0 \times \mathbb{R}^d$, which are $\nu$–square integrable: $H_1, H_2, W$, $\{(\omega, z) \mapsto z_i : 1 \leq i \leq d\}$, $\{b1(|b| \leq n) : b \in W, n \in \mathbb{N}\}$.

**Definition 10.3 (Set $\Omega_{\text{typ}}$ of typical environments).** The set $\Omega_{\text{typ}} \subset \Omega$ of typical environments is the intersection of the following sets:

- $A[gg']$ for all $g, g' \in G$ (recall that $1 \in G$);
- $A_1[bb'] \cap A[\hat{bb'}]$ as $b, b' \in H$;
- $\Omega_2$ (cf. Lemma [6.5]);
- $A_1[|z|^k] \cap A[\lambda_k]$ for $k = 0, 2$;
- $A[f_n]$ for all $n \in \mathbb{N}$;
- $A_1[b] \cap A_1[\hat{b}] \cap A_1[b^2] \cap A_1[\hat{b}^2]$ for all $b \in H$;
- $A[\hat{b}^2] \cap A[\hat{b}^2] \cap A[|b|] \cap A[|b|]$ for all $b \in H$;
\( \bullet \) \( A_1[\hat{b}(\omega, z)] \) for \( 1 \leq i \leq d \) for all \( b \in \mathcal{W} \);
\( \bullet \) \( A_i[\hat{b}, t, n] \) for all \( b \in \mathcal{W} \), \( 1 \leq i \leq d \) and \( n \in \mathbb{N} \);
\( \bullet \) \( A_i[b] \) for all \( b \in \mathcal{W} \) (recall (67)).

As \( \lambda_0, \lambda_2 \in L^1(\mathcal{P}_0) \), due to Proposition 5.2 and the discussion in Section 9, and since we are dealing with a countable family of constraints, \( \Omega_{\text{typ}} \) is a measurable subset of \( \Omega = \Omega_2 \subset \Omega_1 \) with \( \mathcal{P}(\Omega_{\text{typ}}) = 1 \). Since moreover \( \Omega_2, A[], A_1[] \) and \( A_d[] \) are translation invariant sets as already pointed out, we conclude that \( \Omega_{\text{typ}} \) is also translation invariant.

Above we have listed the properties characterizing \( \Omega_{\text{typ}} \) as they will be used along the proof (anyway, we will point out the specific property used in each step under consideration).

## 11. Weak Convergence and 2-scale convergence

Recall \( \mu^\varepsilon_{\omega, \Lambda} \) and \( \nu^\varepsilon_{\omega, \Lambda} \) given in (19). We also set (recalling the definition of \( \mu^\varepsilon_\omega \) given before Proposition 5.2)

\[
\mu^\varepsilon_\omega := \varepsilon^d \sum_{x \in \omega} \delta_x, \quad \nu^\varepsilon_\omega := \varepsilon^d \sum_{x \in \omega} \sum_{y \in \omega} \sum_{\tau \in \mathcal{F}} c_{x, y}^\varepsilon(\omega) \delta_{(x, \tau_x y)}, \quad (73)
\]

\[
\mu^\varepsilon_{\omega, S} := \varepsilon^d \sum_{x \in \omega \cap S} \delta_x, \quad \nu^\varepsilon_{\omega, S} := \varepsilon^d \sum_{x \in \omega \cap S} \sum_{y \in \omega \cap S} \sum_{\tau \in \mathcal{T}} c_{x, y}^\varepsilon(\omega) \delta_{(x, \tau_x y)}, \quad (74)
\]

In this section \( \Delta \) equals \( S \) or \( \Lambda \).

**Definition 11.1.** Fix \( \omega \in \Omega \) and a family of \( \varepsilon \)-parametrized functions \( v_\varepsilon \in L^2(\mu^\varepsilon_{\omega, \Delta}) \). We say that the family \( \{v_\varepsilon\} \) converges weakly to the function \( v \in L^2(\Delta, m dx) \), and write \( v_\varepsilon \rightharpoonup v \), if the family \( \{v_\varepsilon\} \) is bounded (in the sense: \( \lim_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\omega, \Delta})} < +\infty \)) and \( \lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_{\omega, \Delta}(x) v_\varepsilon(x) \varphi(x) = \int_\Delta dx m v(x) \varphi(x) \) for all \( \varphi \in C_c(\Delta) \).

**Definition 11.2.** Fix \( \omega \in \Omega_{\text{typ}} \), an \( \varepsilon \)-parametrized family of functions \( v_\varepsilon \in L^2(\mu^\varepsilon_{\omega, \Delta}) \) and a function \( v \in L^2(\Delta \times \Omega, m dx \times \mathcal{P}_0) \). We say that \( v_\varepsilon \) is weakly 2-scale convergent to \( v \), and write \( v_\varepsilon \overset{2}{\rightharpoonup} v \), if the family \( \{v_\varepsilon\} \) is bounded, i.e. \( \lim_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\omega, \Delta})} < +\infty \), and

\[
\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_{\omega, \Delta}(x) v_\varepsilon(x) \varphi(x) g(\tau_x \varepsilon_\omega) = \int d\mathcal{P}_0(\omega) \int_\Delta dx m v(x, \omega) \varphi(x) g(\omega),
\]

(75)

for any \( \varphi \in C_c(\Delta) \) and any \( g \in \mathcal{G} \).

As \( \omega \in \Omega_{\text{typ}} \subseteq A[g] \) for all \( g \in \mathcal{G} \), by Proposition 5.2 one gets that \( v_\varepsilon \overset{2}{\rightharpoonup} v \) where \( v_\varepsilon := \varphi \in L^2(\mu^\varepsilon_{\omega, \Delta}) \) and \( v := \varphi \in L^2(\Delta, m dx) \) for any \( \varphi \in C_c(\Delta) \).

One can prove the following fact by using the first item in Definition 10.3 and by adapting the proof of Lemma 13.5:

**Lemma 11.3.** Let \( \omega \in \Omega_{\text{typ}} \). Then, given a bounded family of functions \( v_\varepsilon \in L^2(\mu^\varepsilon_{\omega, \Delta}) \), there exists a subsequence \( \{v_{\varepsilon_k}\} \) such that \( v_{\varepsilon_k} \overset{2}{\rightharpoonup} v \) for some \( v \in L^2(\Delta \times \Omega, m dx \times \mathcal{P}_0) \) with \( \|v\|_{L^2(\Delta \times \Omega, m dx \times \mathcal{P}_0)} \leq \lim_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\omega, \Delta})} \).
Recall the definition of the measure \( \nu \) given in (59).

**Definition 11.4.** Given \( \hat{\omega} \in \Omega_{\text{typ}} \), an \( \varepsilon \)-parametrized family of functions \( w_\varepsilon \in L^2(\nu_{\hat{\omega}, \Delta}) \) and a function \( w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times d\nu) \), we say that \( w_\varepsilon \) is weakly 2-scale convergent to \( w \), and write \( w_\varepsilon \xrightarrow{2} \Delta \) \( w \), if \( \{w_\varepsilon\} \) is bounded in \( L^2(\nu_{\hat{\omega}, \Delta}) \), i.e. \( \lim_{\varepsilon \downarrow 0} \|w_\varepsilon\|_{L^2(\nu_{\hat{\omega}, \Delta})} < +\infty \), and

\[
\lim_{\varepsilon \downarrow 0} \int \nabla_{\varepsilon}^\Delta (x,z) w_\varepsilon(x,z) \varphi(x) b(\tau_{x/\varepsilon} \hat{\omega}, z) = \int_{\Delta} dx \int d\nu(\omega,z) w(x,\omega,z) \varphi(x) b(\omega, z), \tag{76}
\]

for any \( \varphi \in C_c(\Delta) \) and any \( b \in \mathcal{H} \).

One can prove the following fact by using the second item in Definition [10.3] and by adapting the proof of [10, Lemma 13.7]:

**Lemma 11.5.** Let \( \hat{\omega} \in \Omega_{\text{typ}} \). Then, given a bounded family of functions \( w_\varepsilon \in L^2(\nu_{\hat{\omega}, \Delta}) \), there exists a subsequence \( \{w_{\varepsilon_k}\} \) such that \( w_{\varepsilon_k} \xrightarrow{2} \Delta \) \( w \) for some \( w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times d\nu) \) with \( \|w\|_{L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times d\nu)} \leq \lim_{\varepsilon \downarrow 0} \|w_\varepsilon\|_{L^2(\nu_{\hat{\omega}, \Delta})} \).

12. 2-scale limits of uniformly bounded functions

We fix \( \hat{\omega} \in \Omega_{\text{typ}} \) and we assume that \( d_* = \dim(\text{Ker}(D^\perp)) \geq 1 \). The domain \( \Delta \) below can be \( \Lambda, S \). We consider a family of functions \( \{f_\varepsilon\} \) with \( f_\varepsilon : \varepsilon \hat{\omega} \cap S \rightarrow \mathbb{R} \) such that

\[
\lim_{\varepsilon \downarrow 0} \|f_\varepsilon\|_\infty < +\infty, \tag{77}
\]

\[
\lim_{\varepsilon \downarrow 0} \|f_\varepsilon\|_{L^2(\nu_{\hat{\omega}, \Delta})} < +\infty, \tag{78}
\]

\[
\lim_{\varepsilon \downarrow 0} \|\nabla_{\varepsilon} f_\varepsilon\|_{L^2(\nu_{\hat{\omega}, \Delta})} < +\infty. \tag{79}
\]

Due to Lemmas [11.3] and [11.5] along a subsequence \( \{\varepsilon_k\} \) we have

\[
L^2(\mu_{\hat{\omega}, \Delta}) \ni f_{\varepsilon} \xrightarrow{2} v \in L^2(\Delta \times \Omega, mdx \times \mathcal{P}_0), \tag{80}
\]

\[
L^2(\nu_{\hat{\omega}, \Delta}) \ni \nabla_{\varepsilon} f_{\varepsilon} \xrightarrow{2} w \in L^2(\Delta \times \Omega \times \mathbb{R}^d, mdx \times d\nu), \tag{81}
\]

for suitable functions \( v, w \).

**Warning 12.1.** In this section (with exception of Lemma [12.1] and Claim [12.4]), when taking the limit \( \varepsilon \downarrow 0 \), we understand that \( \varepsilon \) varies along the subsequence \( \{\varepsilon_k\} \) satisfying (80) and (81). Moreover, we set

\[
f_{\varepsilon}(x) := 0 \text{ for } x \in \varepsilon \hat{\omega} \setminus S. \tag{82}
\]

The structural results presented below (cf. Propositions 12.2 and 12.3) correspond to a general strategy in homogenization by 2-scale convergence (see Propositions 15.1 and 17.1 in [10], Lemmas 5.3 and 5.4 in [41], Theorems 4.1 and 4.2 in [29]). Condition (77) would not be strictly necessary, but it allows
important technical simplifications, and in particular it allows to avoid the cut-off procedures developed in [16 Sections 14,16] in order to deal with the long jumps in the Markov generator (22). Differently from [16] here we have to control also several boundary contributions. We will apply Propositions 12.2 and 12.3 only to the following cases: $\Delta = \Lambda$ and $f = V_\varepsilon$. \[ \text{In what follows we will use the following control on long filaments:} \]

**Lemma 12.1.** Given $\tilde{\omega} \in \Omega_{\text{typ}}$, $\ell > 0$ and $\varphi \in C_c(\mathbb{R}^d)$, it holds

\[ \lim_{\varepsilon \downarrow 0} \varepsilon^{-2} \int d\nu^c_\varepsilon(x,z) |\varphi(x)| 1(|z| \geq \ell/\varepsilon) = 0. \]  

**Proof.** Recall the definition (70) of $f_n$. Given $n \in \mathbb{N}$ we take $\varepsilon$ small so that $\ell/\varepsilon > n$. Then we can bound

\[ \varepsilon^{-2} \int d\nu^c_\varepsilon(x,z) |\varphi(x)| 1(|z| \geq \ell/\varepsilon) \leq \varepsilon^{-2} \int d\nu^c_\varepsilon(x,z)|\varphi(x)||1(|z| \geq \ell/\varepsilon)|z|^2 \leq \varepsilon^{-2} \int d\nu^c_\varepsilon(x,z)|\varphi(x)| 1(|z| \geq n)|z|^2 \leq \varepsilon^{-2} \int d\mu^c_\varepsilon(x)|\varphi(x)|f_n(\tau_{x/\varepsilon}\tilde{\omega}). \]

As $\omega \in \Omega_{\text{typ}} \subset A[f_n] \cap A_1(|z|^2)$, we have (cf. (A7))

\[ \lim_{\varepsilon \downarrow 0} \int d\mu^c_\varepsilon(x)|\varphi(x)|f_n(\tau_{x/\varepsilon}\omega) = \int dxm|\varphi(x)|E_0[f_n] < +\infty. \]

The last expression goes to zero as $n \to +\infty$ by dominated convergence. \[ \square \]

**Proposition 12.2.** For $dx$–a.e. $x \in \Delta$, the map $v(x, \cdot)$ given in (80) is constant $P_0$–a.s.

**Proof.** Recall the definition of the functional sets $\mathcal{G}_1$, $\mathcal{H}_1$ given in Section 10. We claim that $\forall \varphi \in C_c^1(\Delta)$ and $\forall \psi \in \mathcal{G}_1$ it holds

\[ \int_{\Delta} dx m \int dP_0(\omega)v(x,\omega)\varphi(x)\psi(\omega) = 0. \]  

Having (84) it is standard to conclude. Indeed, (84) implies that, $dx$–a.e. on $\Delta$, $\int dP_0(\omega)v(x,\omega)\psi(\omega) = 0$ for any $\psi \in \mathcal{G}_1$. We conclude that, $dx$–a.e. on $\Delta$, $v(x,\cdot)$ is orthogonal in $L^2(P_0)$ to $\{w \in L^2(P_0) : E_0[w] = 0\}$. Hence $v(x,\omega) = E_0[v(x,\cdot)]$ for $P_0$–a.a. $\omega$.

It now remains to prove (84). We first note that, by (75), (80) and since $\tilde{\omega} \in \Omega_{\text{typ}}$ and $\psi \in \mathcal{G}_1$, \[ \text{l.h.s. of (84)} = \lim_{\varepsilon \downarrow 0} \int d\mu^c_{\tilde{\omega},\Delta}(x)f_\varepsilon(x)\varphi(x)\psi(\tau_{x/\varepsilon}\tilde{\omega}). \]

Let us take $\psi = g_b$ with $b \in \mathcal{H}_1$ as in (69). By [16 Lemma 11.7] and since $\tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b] \cap A_1[\tilde{b}]$, we have

\[ \int d\mu^c_{\tilde{\omega},\Delta}(x)f_\varepsilon(x)\varphi(x)\psi(\tau_{x/\varepsilon}\tilde{\omega}) = \int d\mu^c_{\tilde{\omega}}(x)f_\varepsilon(x)\varphi(x)\psi(\tau_{x/\varepsilon}\tilde{\omega}) = -\varepsilon \int d\nu^c_\varepsilon(x,z)\nabla_\varepsilon(f_\varepsilon\varphi)(x,z)b(\tau_{x/\varepsilon}\tilde{\omega}, z). \]
By (42) we have

\[- \varepsilon \int \nu^\varepsilon(x, z) \nabla \varphi(f^\varepsilon)(x, z)b(\tau_{x/\varepsilon}\tilde{\omega}, z) = -\varepsilon C_1(\varepsilon) + \varepsilon C_2(\varepsilon),\]

\[C_1(\varepsilon) := \int \nu^\varepsilon(x, z) \nabla \varphi(f^\varepsilon(x, z)b(\tau_{x/\varepsilon}\tilde{\omega}, z),\]

\[C_2(\varepsilon) := \int \nu^\varepsilon(x, z) f^\varepsilon(x + \varepsilon z) \nabla \varphi(x, z)b(\tau_{x/\varepsilon}\tilde{\omega}, z) .\]

To get (84) we only need to show that \(\lim_{\varepsilon \downarrow 0} \varepsilon C_1(\varepsilon) = 0, \lim_{\varepsilon \downarrow 0} \varepsilon C_2(\varepsilon) = 0.\)

We start with \(C_1(\varepsilon).\) By Schwarz inequality and since \(\tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2]\)

\[|C_1(\varepsilon)| \leq \left[ \int \nu^\varepsilon(x, z)|\varphi(x)|\nabla \varphi f^\varepsilon(x, z)^2 \right]^{1/2} \left[ \int \mu^\varepsilon(x)|\varphi(x)| \left( \tilde{\omega}^2(\tau_{x/\varepsilon}\tilde{\omega}) \right) \right]^{1/2} .\]

Since \(\tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2] \cap A[\tilde{b}^2],\) the last integral in the r.h.s. converges to a finite constant as \(\varepsilon \downarrow 0.\) It remains to prove that \(\int \nu^\varepsilon(x, z)|\varphi(x)|\nabla \varphi f^\varepsilon(x, z)^2\)

remains bounded from above as \(\varepsilon \downarrow 0.\) We call \(\ell\) the distance between the support of \(\varphi\) (which is contained in \(\Delta\) as \(\varphi \in C^1_\ell(\Delta)\)) and \(\partial \Delta.\) Then, between the pairs \((x, z)\) with \(x + \varepsilon z \notin S\) contributing to the above integral, only the pairs \((x, z)\) such that \(x \in \Delta\) and \(|z| \geq \ell/\varepsilon\) can give a nonzero contribution. In both cases \(\Delta = \Lambda\) and \(\Delta = S\) we can estimate

\[\int \nu^\varepsilon(x, z)|\varphi(x)|\nabla \varphi f^\varepsilon(x, z)^2 \leq \int \nu^\varepsilon,\Delta(x, z)|\varphi(x)|\nabla \varphi f^\varepsilon(x, z)^2\]

\[+ \int \nu^\varepsilon(x, z)|\varphi(x)|\nabla \varphi f^\varepsilon(x, z)^2 1(|z| \geq \ell/\varepsilon) .\]

The first addendum in the r.h.s. of (85) is bounded due to (79). The second addendum goes to zero due to (77) (implying that \(\nabla f \leq C/\varepsilon\) for small \(\varepsilon\)) and Lemma 12.1. Hence the l.h.s. of (85) remains bounded as \(\varepsilon \downarrow 0.\) This completes the proof that \(\lim_{\varepsilon \downarrow 0} \varepsilon C_1(\varepsilon) = 0.\)

We move to \(C_2(\varepsilon).\) Let \(\phi\) be as in (43). Using (77), (43) and afterwards [16] Lemma 11.3–(i), for some \(\varepsilon\)-independent constants \(C\)'s (which can change from line to line), for \(\varepsilon\) small we can bound

\[|C_2(\varepsilon)| \leq C \int \nu^\varepsilon(x, z)|\nabla \varphi(x, z)b(\tau_{x/\varepsilon}\tilde{\omega}, z)|\]

\[\leq C \int \nu^\varepsilon(x, z)|z||b(\tau_{x/\varepsilon}\tilde{\omega}, z)||\phi(x) + \phi(x + \varepsilon z)||\]

\[\leq C \int \nu^\varepsilon(x, z)|\phi(x)|z||b| + |\tilde{b}||(\tau_{x/\varepsilon}\tilde{\omega}, z)\]

\[\leq C \left[ \int \nu^\varepsilon(x, z)|\phi(x)|z|^2 \right]^{1/2} \left[ 2 \int \nu^\varepsilon(x, z)|\phi(x)|b^2 + \tilde{b}^2(\tau_{x/\varepsilon}\tilde{\omega}, z) \right]^{1/2} .\]

The first integral in the last line of (86) equals \(\int d\mu^\varepsilon(x, z)\phi(\lambda_2(\tau_{x/\varepsilon}\tilde{\omega})\). Since \(\tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[|z|^2] \cap A[\lambda_2],\) this integral converges to a finite constant as \(\varepsilon \downarrow 0.\)

The second integral in the last line of (86) equals \(\int d\mu^\varepsilon(x, z)\phi(\tilde{b}^2 + \tilde{b}^2)(\tau_{x/\varepsilon}\tilde{\omega})\)
as \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2] \cap A_1[\tilde{b}^2] \). Since also \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A[\tilde{b}^2] \cap A[\tilde{b}^2] \), the last integral converges to a finite constant. This implies that \( \lim_{\varepsilon \downarrow 0} \varepsilon C_2(\varepsilon) = 0 \). \( \square \)

Due to Proposition 12.2 we can write \( v(x) \) instead of \( v(x,\omega) \), where \( v \) is given by (80). Recall Definition 8.1. We extend the notation (49) for \( \nabla_* \) also to functions with domain different from \( \Lambda \).

**Proposition 12.3.** Let \( v \) and \( w \) be as in (80) and (81). Then it holds:

(i) \( v \) has weak derivatives \( \partial_{\xi_j} v \in L^2(\Delta, dx) \) for \( 1 \leq j \leq d_* \);

(ii) \( w(x,\omega,z) = \nabla_* v(x) \cdot z + \nu_1(x,\omega,z) \), where \( \nu_1 \in L^2(\Delta, dx; L^2_{\text{pol}}(\nu)) \).

Above, \( L^2(\Delta, dx; L^2_{\text{pol}}(\nu)) \) denotes the space of square integrable maps \( f : \Delta \rightarrow L^2_{\text{pol}}(\nu) \), where \( \Delta \) is endowed with the Lebesgue measure.

**Proof.** Given a square integrable form \( b \), we define \( \eta_b := \int d\nu(\omega,z)z(b(\omega,z)). \) Note that \( \eta_b \) is well defined as \( b \in L^2(\nu) \) and by (A7). We observe that \( \eta_b = -\eta_b \) as derived in [16, Proof of Prop. 17.1] by using (A3). We claim that for each solenoidal form \( b \in L^2_{\text{sol}}(\nu) \) and each function \( \varphi \in C_c^2(\Delta) \), it holds

\[
\int_{\Delta} dx m \varphi(x) \int d\nu(\omega,z)w(x,\omega,z)b(\omega,z) = -\int_{\Delta} dx m \varphi(x) \nabla \varphi(x) \cdot \eta_b.
\]  

(87)

Having (87) one can conclude the proof of Proposition 12.3 by rather standard arguments. Indeed, having Corollary 9.4, it is enough to apply the same arguments presented in [16] to derive [16, Prop. 17.1] from [16, Eq. (131)]. The notation is similar, one has just to restrict to \( x \in \Delta \).

Let us prove (87). Since both sides of (87) are continuous as functions of \( b \in L^2_{\text{sol}}(\nu) \), it is enough to prove it for \( b \in W \). Since \( \tilde{\omega} \in \Omega_{\text{typ}} \), along \( \{ \varepsilon_k \} \) it holds \( \nabla_\varepsilon f_\varepsilon \xrightarrow{\varepsilon \downarrow 0} w \) as in (81) and since \( b \in W \subset H \) (cf. (78)) we can write

\[
\text{l.h.s. of (87)} = \lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(\Delta)(x,z)\nabla_\varepsilon(f_\varepsilon(x,z))\varphi(x)b(\tau_{x/\varepsilon}\tilde{\omega},z).
\]  

(88)

Since \( b \in W \subset L^2_{\text{sol}}(\nu) \) and \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_0[b] \), from [16, Lemma 11.7] we get

\[
\int d\nu_\varepsilon(\Delta)(x,z)\nabla_\varepsilon(f_\varepsilon(x,z))\varphi(x)b(\tau_{x/\varepsilon}\tilde{\omega},z) = 0.
\]  

Using the above identity and (42), we get

\[
\int d\nu_\varepsilon(\Delta)(x,z)\nabla_\varepsilon(f_\varepsilon(x,z))\varphi(x)b(\tau_{x/\varepsilon}\tilde{\omega},z)
\]  

\[
= -\int d\nu_\varepsilon(\Delta)(x,z)(f_\varepsilon(x+\varepsilon z)\nabla_\varepsilon\varphi(x,z)b(\tau_{x/\varepsilon}\tilde{\omega},z)).
\]  

As \( \omega \in \Omega_{\text{typ}} \subset A_1[b] \cap A_1[\tilde{b}] \), by applying now [16, Lemma 11.3-(ii)] to the above r.h.s., we get

\[
\int d\nu_\varepsilon(\Delta)(x,z)\nabla_\varepsilon(f_\varepsilon(x,z))\varphi(x)b(\tau_{x/\varepsilon}\tilde{\omega},z) = \int d\nu_\varepsilon(\Delta)(x,z)f_\varepsilon(x)\nabla_\varepsilon\varphi(x,z)b(\tau_{x/\varepsilon}\tilde{\omega},z).
\]
By combining (88) with the above identity, we obtain
\[
\text{l.h.s. of (87)} = \lim_{\varepsilon \downarrow 0} \left( -R_1(\varepsilon) + R_2(\varepsilon) \right),
\]
where
\[
R_1(\varepsilon) := \int d \left[ \nu^\varepsilon - \nu^\varepsilon,\Delta \right] (x, z) \nabla_x f^\varepsilon(x, z) \varphi(x) b(\tau_{x/\varepsilon}\tilde{\omega}, z),
\]
\[
R_2(\varepsilon) := \int d\nu^\varepsilon(x, z) \nabla_x \varphi(x, z) \tilde{b}(\tau_{x/\varepsilon}\tilde{\omega}, z).
\]
We claim that \( \lim_{\varepsilon \downarrow 0} R_1(\varepsilon) = 0 \). We call \( \ell \) the distance between the support \( \Delta_\varphi \subset \Delta \) of \( \varphi \) and \( \partial \Delta \). Then in \( R_1(\varepsilon) \) the contribution comes only from pairs \( (x, z) \) such that \( x \in \Delta_\varphi \) and \( x + \varepsilon z \notin S \) and therefore from pairs \( (x, z) \) such that \( x \in \Delta \) and \( |z| \geq \ell/\varepsilon \):
\[
|R_1(\varepsilon)| \leq \int d\nu^\varepsilon \varphi(x, z) |\nabla_x f^\varepsilon(x, z) \varphi(x) b(\tau_{x/\varepsilon}\tilde{\omega}, z)| \mathbf{1}(x \in \Delta, |z| \geq \ell/\varepsilon).
\]
By Schwarz inequality we have therefore that \( R_1(\varepsilon)^2 \leq I_1(\varepsilon) I_2(\varepsilon) \), where
\[
I_1(\varepsilon) := \int d\nu^\varepsilon \varphi(x, z) |\nabla_x f^\varepsilon(x, z)| \varphi(x) |\mathbf{1}(|z| \geq \ell/\varepsilon)|,
\]
\[
I_2(\varepsilon) := \int d\nu^\varepsilon \varphi(x, z) |\varphi(x)| b(|\tau_{x/\varepsilon}\tilde{\omega}|, z) = \int d\mu^\varepsilon \varphi(x) |\tilde{b}(\tau_{x/\varepsilon}\tilde{\omega})|.
\]
The last identity concerning \( I_2(\varepsilon) \) uses that \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_1[b^2] \). It holds \( \lim_{\varepsilon \downarrow 0} I_1(\varepsilon) = 0 \) due to Lemma 12.1 and (77), while \( \lim_{\varepsilon \downarrow 0} I_2(\varepsilon) < +\infty \) since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A[\tilde{b}^2] \). This proves that \( R_1(\varepsilon) \to 0 \).

We now move to \( R_2(\varepsilon) \). To treat this term we will use the following fact:

**Claim 12.4.** We have
\[
\lim_{\varepsilon \downarrow 0} \int d\nu^\varepsilon \varphi(x, z) \hat{f}^\varepsilon(x) \left[ \nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z \right] \tilde{b}(\tau_{x/\varepsilon}\tilde{\omega}, z) = 0.
\]

**Proof of Claim 12.4.** We fix \( k \in \mathbb{N} \) and \( \phi \in C_c(\mathbb{R}^d) \) with values in \([0, 1]\) such that \( \varphi(x) = 0 \) if \( |x| \geq k \), \( \phi(x) = 1 \) for \( |x| \leq k \) and \( \phi(x) = 0 \) for \( |x| \geq k + 1 \). Given \( \ell \in \mathbb{N} \) we write the integral in (90) as \( A_\ell(\varepsilon) + B_\ell(\varepsilon) \), where \( A_\ell(\varepsilon) \) is the contribution coming from \( z \) with \( |z| \leq \ell \) and \( B_\ell(\varepsilon) \) is the contribution coming from \( z \) with \( |z| > \ell \). Due to \([44], (77), \Gamma\) Lemma 11.3-(i)] we have
\[
A_\ell(\varepsilon) \leq C \ell^2 \varepsilon \int d\nu^\varepsilon \varphi(x)(\phi(x) + \phi(x + \varepsilon z)) |\tilde{b}(\tau_{x/\varepsilon}\tilde{\omega}, z)|
\]
\[
\leq C \ell^2 \varepsilon \int d\nu^\varepsilon \varphi(x)(|b| + |\tilde{b}|)(\tau_{x/\varepsilon}\tilde{\omega}, z).
\]
Since \( \omega \in \Omega_{\text{typ}} \subset A_1[b] \cap A_1[\tilde{b}] \), the last expression above can be written as
\( C \ell^2 \varepsilon \int d\mu^\varepsilon \phi(x)(|\tilde{b}| + |\tilde{b}|)(\tau_{x/\varepsilon}\tilde{\omega}) \). Since \( \omega \in \Omega_{\text{typ}} \subset A[\tilde{b}] \cap A[\tilde{b}] \), the above integral converges to a finite constant as \( \varepsilon \downarrow 0 \). This allows to conclude that \( \lim_{\varepsilon \downarrow 0} A_\ell(\varepsilon) = 0 \).
It remains to prove that \( \lim_{\varepsilon \downarrow 0} \lim_{n \uparrow \infty} B_{t}(\varepsilon) = 0 \). We argue as above but now we apply (43). Due to (43), (77) and \([16, \text{Lemma 11.3-(i)}]\), we can bound
\[
B_{t}(\varepsilon) \leq C \int d\nu_{\varepsilon}^{\omega}(x,z)\phi(x)(|b| + |\tilde{b}|)(\tau_{x/\varepsilon} \omega, z)|z| I(|z| \geq \ell) .
\]
Recall (70). Then, by Schwarz inequality \( B_{t}(\varepsilon) \leq C C_{t}(\varepsilon)^{1/2} D_{t}(\varepsilon)^{1/2} \), where (as \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A_{1}[\tilde{b}^{2}] \cap A_{1}[\tilde{b}^{2}] \))
\[
C_{t}(\varepsilon) := 2 \int d\nu_{\varepsilon}^{\omega}(x,z)\phi(x)(b^{2} + \tilde{b}^{2})(\tau_{x/\varepsilon} \omega, z) = 2 \int d\mu_{\varepsilon}^{\omega}(x)\phi(x)(\tilde{b}^{2} + \tilde{b}^{2})(\tau_{x/\varepsilon} \omega)
\]
\[
D_{t}(\varepsilon) := \int d\nu_{\varepsilon}^{\omega}(x,z)\phi(x)|z|^{2} I(|z| \geq \ell) = \int d\mu_{\varepsilon}^{\omega}(x)\phi(x)f_{\ell}(\tau_{x/\varepsilon} \omega) .
\]
As \( \tilde{\omega} \in \Omega_{\text{typ}} \subset \{A[\tilde{b}^{2}] \cap A[\tilde{b}^{2} \cap A[\tilde{b}^{2}] \cap A[f_{\ell}] \}, we conclude that \( \lim_{\varepsilon \downarrow 0} B_{t}(\varepsilon) \leq C' E_{0} |\tilde{b}^{2} + \tilde{b}^{2}|^{1/2} E_{0} |f_{\ell}|^{1/2} \), and the r.h.s. goes to zero as \( \ell \rightarrow \infty \) by dominated convergence and (A7).

We come back to (87). By combining (89), (90) and the limit \( R_{t}(\varepsilon) \rightarrow 0 \), we conclude that
\[
\text{l.h.s. of } (87) = \lim_{\varepsilon \downarrow 0} \int d\nu_{\varepsilon}^{\omega}(x,z)\bar{f}_{\varepsilon}(x)\nabla \varphi(x) \cdot z\tilde{b}(\tau_{x/\varepsilon} \omega, z) . \tag{91}
\]
Due to (91) and since \( \eta_{b} = -\eta_{b} \) as already observed, to prove (87) we only need to show that
\[
\lim_{\varepsilon \downarrow 0} \int d\nu_{\varepsilon}^{\omega}(x,z)\bar{f}_{\varepsilon}(x)\nabla \varphi(x) \cdot z\tilde{b}(\tau_{x/\varepsilon} \omega, z) = \int dx \, m v(x) \nabla \varphi(x) \cdot \eta_{b} . \tag{92}
\]
To this aim we observe that (recall (71))
\[
\int d\nu_{\varepsilon}^{\omega}(x,z)\bar{f}_{\varepsilon}(x)\partial_{i} \varphi(x)z_{i}\tilde{b}(\tau_{x/\varepsilon} \omega, z) = \int d\mu_{\varepsilon}^{\omega}(x)\bar{f}_{\varepsilon}(x)\partial_{i} \varphi(x)u_{b,i}(\tau_{x/\varepsilon} \omega) . \tag{93}
\]
We claim that
\[
\lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^{\omega}(x)\bar{f}_{\varepsilon}(x)\partial_{i} \varphi(x)u_{b,i}(\tau_{x/\varepsilon} \omega) = \int dx \, m v(x) \partial_{i} \varphi(x) E_{0}[u_{b,i}] . \tag{94}
\]
Since the r.h.s. equals \( \int_{\Delta} dx \, m v(x) \partial_{i} \varphi(x)(\eta_{b} \cdot e_{i}) \), our target (92) then would follow as a byproduct of (93) and (94). It remains therefore to prove (94). Recall (72). Due to Proposition 5.2 (recall that \( \tilde{b} \in \mathcal{W} \) for any \( b \in \mathcal{W} \) and that \( \tilde{\omega} \in \Omega_{\text{typ}} \subset A[u_{b,i,n}] \cap A[\tilde{b}(\omega, z)z] \) for all \( b \in \mathcal{W} \))
\[
\lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^{\omega}(x)\partial_{i} \varphi(x)|u_{b,i,n}(\tau_{x/\varepsilon} \omega)| = \int dx \, m v(x) |\partial_{i} \varphi(x)| E_{0}[u_{b,i,n}] .
\]
Since \( u_{b,i} \in L^{1}(P_{0}) \), then the above r.h.s. goes to zero as \( n \uparrow +\infty \). Hence, using also (77), to get (94) it is enough to show that
\[
\lim_{n \uparrow +\infty} \lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^{\omega}(x)\bar{f}_{\varepsilon}(x)\partial_{i} \varphi(x)u_{b,i}(\tau_{x/\varepsilon} \omega) 1(|u_{b,i}(\tau_{x/\varepsilon} \omega)| \leq n) = \int dx \, m v(x) \partial_{i} \varphi(x) E[u_{b,i}] . \tag{95}
\]
Note that in (95) we can replace $d\mu^c_{\omega}(x)f_\varepsilon(x)\partial_i\varphi(x)$ by $d\mu^c_{\omega,\Delta}(x)f_\varepsilon(x)\partial_i\varphi(x)$.

Due to (80) and since $u_{k,i}1_{|u_{k,i}| \leq n} \in \mathcal{G}$, by (75) we have

$$
\lim_{\varepsilon \downarrow 0} \int d\mu^c_{\omega,\Delta}(x)f_\varepsilon(x)\partial_i\varphi(x)u_{k,i}(\tau_{x/\varepsilon}\hat{\omega})1_{|u_{k,i}(\tau_{x/\varepsilon}\hat{\omega})| \leq n}
$$

$$
= \int_{\Delta} dx dv(x)\partial_i\varphi(x)\mathbb{E}[u_{k,i}1_{|u_{k,i}| \leq n}].
$$

By dominated convergence, we get (95) from (96). \hfill \Box

\section{3-scale limit points of $V_\varepsilon$ and $\nabla_i V_\varepsilon$}

We fix $\hat{\omega} \in \Omega_{\text{typ}}$ and we assume that $d_\varepsilon = \dim(\ker(D)^\perp) \geq 1$. As $\Omega_{\text{typ}} \subset \Omega_2$, due to Lemmas 6.5, 11.3 and 11.5 along a subsequence $\varepsilon_k$ we have that

$$L^2(\mu^c_{\omega,\lambda}) \ni V_\varepsilon \cdot \varepsilon \ni L^2(\Lambda \times \Omega, m dx \times \mathcal{P}_0),$$

$$L^2(\nu^c_{\omega,\lambda}) \ni \nabla_i V_\varepsilon \cdot \varepsilon \ni L^2(\Lambda \times \Omega \times \mathbb{R}^d, m dx \times \nu),$$

for suitable functions $\nu$ and $\nu$. In the rest of this section, when considering the limit $\varepsilon \downarrow 0$, we understand that $\varepsilon$ varies in the sequence $\{\varepsilon_k\}$.

\begin{proposition}
Let $v$ be as in (97). Then $v - \psi|_\Lambda \in H_0^1(\Omega, F, d_\varepsilon)$.
\end{proposition}

\begin{proof}
We apply the results of Section 12 to the case $\Delta = S$ and $f_\varepsilon := V_\varepsilon - \psi$. Since $f_\varepsilon$ is zero on $S \setminus \Lambda$ and takes values in $[-1,1]$ on $\Lambda$, conditions (77) and (78) are satisfied. In addition, we have $\nabla_i f_\varepsilon(x,z) = 0$ if $\{x,x+\varepsilon z\}$ does not intersect $\Lambda$ and therefore $\|f_\varepsilon\|_{L^2(\nu^c_{\omega,s})} = \|f_\varepsilon\|_{L^2(\nu^c_{\omega,\lambda})}$. By Lemma 6.5 we therefore conclude that also (79) is satisfied.

At cost to refine the subsequence $\{\varepsilon_k\}$, by Lemmas 11.3 and 11.5 without loss of generality we can assume that along $\{\varepsilon_k\}$ itself we have

$$L^2(\mu^c_{\omega,s}) \ni f_\varepsilon \ni L^2(S \times \Omega, m dx \times \mathcal{P}_0),$$

$$L^2(\nu^c_{\omega,s}) \ni \nabla_i f_\varepsilon \ni L^2(S \times \Omega \times \mathbb{R}^d, m dx \times \nu),$$

for suitable functions $\hat{v}, \hat{\omega}$. By Proposition 12.2 we have $\hat{v} = \hat{v}(x)$. Since $f_\varepsilon \equiv 0$ on $S \setminus \Lambda$, it is simple to derive from the definition of 2-scale convergence that $\hat{v}(x) \equiv 0$ a.e. on $S \setminus \Lambda$. Let us now prove that also $\hat{\omega}(x) \equiv 0$ a.e. on $S \setminus \Lambda$. To this aim we take $\varphi \in C_c(S \setminus \Lambda)$ and write $\ell$ for the distance between the support of $\varphi$ and $\Lambda$. As $\|f_\varepsilon\|_{\infty} \leq 1$ and since $f_\varepsilon \equiv 0$ on $S \setminus \Lambda$, we have

$$|\int dv^c_{\nu,\omega}(x,z)\varphi(x)\nabla_i f_\varepsilon(x,z)| \leq \varepsilon^{-1} \int dv^c_{\nu,\omega}(x,z)|\varphi(x)|1_{|z| \geq \ell}/\varepsilon).$$

By Lemma 12.1 the r.h.s. goes to zero as $\varepsilon \downarrow 0$. By the above observation, (76) and (100), we conclude that for all $\varphi \in C_c(S \setminus \Lambda)$ and $b \in \mathcal{H}_2$ or $b = h1_{|b| \leq n}$ for some $h \in \mathcal{W}$ and $n \in \mathbb{N}$ (in both cases $b \in L^\infty(\nu)$ and $b \in \mathcal{H}$), it holds

$$\int d\omega \int dv(\omega,z)\varphi(x)\hat{\omega}(x,\omega,z)b(\omega,z) = 0.$$

\end{proof}
When $b = h \mathbb{1}(|h| \leq n)$, by taking the limit $n \uparrow \infty$ and using dominated convergence, we conclude that (101) holds also for $b = h$. As $\mathcal{H}_2$ is dense in $L^2_{\text{pot}}(\nu)$ and $\mathcal{W}$ is dense in $L^2_{\text{sol}}(\nu)$, (101) implies that $\hat{w}(x, \cdot, \cdot) \equiv 0$ $dx$–a.e. on $S \setminus \Lambda$.

We recall that in the proof of Proposition 12.3 we have in particular derived (87): for each solenoidal form $b \in L^2_{\text{sol}}(\nu)$ and each function $\varphi \in C^2_c(S)$, it holds

$$\int_S dx \varphi(x) \int d\nu(\omega, z) \hat{w}(x, \omega, z) b(x, \omega) = - \int_S dx \hat{v}(x) \nabla \varphi(x) \cdot \eta_b. \tag{102}$$

As $\hat{v}(x) \equiv 0$ $dx$–a.e. on $S \setminus \Lambda$ and $\hat{w}(x, \cdot, \cdot) \equiv 0$ $dx$–a.e. on $S \setminus \Lambda$, (102) implies that

$$|\int_\Lambda dx \hat{v}(x) \nabla \varphi(x) \cdot \eta_b| = \left| \int_\Lambda dx \varphi(x) \int d\nu(\omega, z) \hat{w}(x, \omega, z) b(x, \omega) \right|. \tag{103}$$

By Schwarz inequality we can bound

$$C^2 := \int_\Lambda dx \left[ \int d\nu(\omega, z) \hat{w}(x, \omega, z) b(x, \omega) \right]^2 \leq \int_\Lambda dx \left( \int d\nu(\omega, z) \hat{w}(x, \omega, z)^2 \right) \int d\nu(\omega, z) b(x, \omega)^2 = \|\hat{w}\|^2_{L^2(\Lambda \times \Omega \times \mathbb{R}^d, dx \times d\nu)} \|b\|^2_{L^2(\nu)} < \infty.$$ 

By applying now Schwarz inequality to the r.h.s. of (103) we conclude that

$$|\int_\Lambda dx \hat{v}(x) \nabla \varphi(x) \cdot \eta_b| \leq C \|\varphi\|_{L^2(\Lambda, dx)}. \tag{103}$$

Due to Corollary 9.4 for each $i : 1 \leq i \leq d_\star$ there exists $b \in L^2_{\text{sol}}(\nu)$ such that $\eta_b = \epsilon_i$. As a byproduct, we get $|\int_\Lambda dx \hat{v}(x) \partial_{\epsilon_i} \varphi(x)| \leq C \|\varphi\|_{L^2(\Lambda, dx)}$. The above bound and Proposition 8.5 imply that $\hat{v} \in H^1_2(\Lambda, F, d_\star)$. To conclude it remains to observe that $\hat{v} = v - \psi_{|\Lambda}$ $dx$–a.e. on $\Lambda$.

**Proposition 13.2.** Let $w$ be as in (98). For $dx$–a.e. $x \in \Lambda$, the map $(\omega, z) \mapsto w(x, \omega, z)$ belongs to $L^2_{\text{sol}}(\nu)$.

**Proof.** As common when proving similar statements, we take as test function $u(x) := \epsilon \varphi(x) g(\tau_{x/\epsilon} \hat{\omega})$, where $\varphi \in C_c(\Lambda)$ and $g \in \mathcal{G}_2$ (cf. Section 10). We use that $\langle \nabla_x u, \nabla_\tau V_\epsilon \rangle_{L^2(\nu_{\epsilon, \Lambda})} = 0$ (cf. Remark 6.4). Due to (42), this identity can be rewritten as

$$\epsilon \int d\nu_{\epsilon, \Lambda}(x, z) \nabla_\epsilon \varphi(x, z) g(\tau_{x/\epsilon} \hat{\omega}) \nabla_\epsilon V_\epsilon(x, z) + \int d\nu_{\epsilon, \Lambda}(x, z) \varphi(x) \nabla g(\tau_{x/\epsilon} \hat{\omega}, z) \nabla_\epsilon V_\epsilon(x, z) = 0. \tag{104}$$

We first show that the first integral in (104) remains uniformly bounded as $\epsilon \downarrow 0$. By applying Schwarz inequality, using that $g$ is bounded as $g \in \mathcal{G}_2$ and that $\lim_{\epsilon \downarrow 0} \|\nabla_\epsilon V_\epsilon\|_{L^2(\nu_{\epsilon, \Lambda})} = +\infty$ due to (40) and since $\hat{\omega} \in \Omega_{\text{typ}} \subset \Omega_2$, it is enough to show that $\lim_{\epsilon \downarrow 0} \|\nabla_\epsilon \varphi\|_{L^2(\nu_{\epsilon, \Lambda})} = +\infty$. As $\hat{\omega} \in \Omega_{\text{typ}}$, by Lemma 9.9
it remains to prove that \( \overline{\lim}_{\varepsilon \downarrow 0} \| \nabla \varphi(x) \cdot z \|_{L^2(\mu^\varepsilon)} < +\infty \), which follows from the fact that \( \tilde{\omega} \in \Omega_{\text{typ}} \subset \mathcal{A}_I[z^2] \cap \mathcal{A}[\lambda_2] \).

Coming back to (104), using that the first addendum goes to zero as \( \varepsilon \downarrow 0 \) and applying the 2-scale convergence \( \nabla_\varepsilon V_\varepsilon \overset{\text{w}}{\rightarrow} w \) in (98) to treat the second addendum, we conclude that \( \int_A f \, d\nu(\omega, z) \varphi(x) \nabla g(\omega, z) w(x, \omega, z) = 0 \) for all \( g \in G_2 \) (note that we have used (76) as \( \nabla g \in \mathcal{H}_2 \subset \mathcal{H} \)). Since \( \{ \nabla g : g \in G_2 \} \) is dense in \( L^2_{\text{pot}}(\nu) \), the above identity implies that, for \( dx \)-a.e. \( x \in \Lambda \), the map \( (\omega, z) \mapsto w(x, \omega, z) \) belongs to \( L^2_{\text{sol}}(\nu) \).

\[
\begin{align*}
14. \text{ Proof of Theorem 1 and Proposition 2.8 for } D_{1,1} > 0
\end{align*}
\]

We recall that at the end of Section 10 we have proved that \( \Omega_{\text{typ}} \) is a measurable translation invariant subset of \( \Omega_1 \) with \( \mathcal{P}(\Omega_{\text{typ}}) = 1 \). As \( e_1 \) is an eigenvector of \( D \) and \( D_{1,1} > 0 \), we have \( e_1 \in \text{Ker}(D)^\perp \).

14.1. Proof of Proposition 2.8. We fix \( \tilde{\omega} \in \Omega_{\text{typ}} \) and prove the convergence in Proposition 2.8 for \( \tilde{\omega} \) instead of \( \omega \) there. As \( \Omega_{\text{typ}} \subset \Omega_2 \) and due to Lemmas 11.3 and 11.5 along a subsequence \( \{\varepsilon_k\} \) we have that \( L^2(\mu^\varepsilon_\Lambda) \ni V_\varepsilon \overset{\text{w}}{\rightarrow} v \in L^2(\Lambda \times \Omega, dx \times \mathcal{P}_0) \) and \( L^2(\nu^\varepsilon_\Lambda) \ni \nabla_\varepsilon V_\varepsilon \overset{\text{w}}{\rightarrow} w \in L^2(\Lambda \times \Omega \times \mathbb{R}^d, dx \times \nu) \) (cf. (97) and (98)). From Proposition 13.2 it is standard to conclude that for \( dx \)-a.e. \( x \in \Lambda \) it holds

\[
\int d\nu(\omega, z) w(x, \omega, z) = 2D V_\varepsilon(x).
\] (105)

Indeed, by Proposition 13.2 for \( dx \)-a.e. \( x \in \Lambda \), the map \( (\omega, z) \mapsto w(x, \omega, z) \) belongs to \( L^2_{\text{sol}}(\nu) \). By Proposition 12.3 we know that \( w(x, \omega, z) = \nabla_\varepsilon V_\varepsilon(x) \cdot z + v_1(x, \omega, z) \), where \( v_1 \in L^2(\Lambda, L^2_{\text{pot}}(\nu)) \). Hence, by (62), for \( dx \)-a.e. \( x \in \Lambda \) we have that \( v_1(x, \cdot, \cdot) = v^a \), where \( a := \nabla_\varepsilon V_\varepsilon(x) \). As a consequence (using also (63)), for \( dx \)-a.e. \( x \in \Lambda \), we can rewrite the l.h.s. of (105) as \( \int d\nu(\omega, z)[\nabla_\varepsilon V_\varepsilon(x) \cdot z + v^a(\omega, z)] = 2D V_\varepsilon(x) \), thus proving (105).

We now take a function \( \varphi \in C^2(\mathbb{R}^d) \) which is zero on \( S \setminus \Lambda \) (note that we are not taking \( \varphi \in C^2(\mathbb{R}^d) \)). By Remark 6.4 we have the identity \( \langle \nabla_\varepsilon \varphi, \nabla_\varepsilon V_\varepsilon \rangle_{L^2(\nu^\varepsilon_\Lambda)} = 0 \). The above identity, (40) and Lemma 9.9 (use that \( \tilde{\omega} \in \Omega_{\text{typ}} \)) imply that

\[
0 = \langle \nabla_\varepsilon \varphi, \nabla_\varepsilon V_\varepsilon \rangle_{L^2(\nu^\varepsilon_\Lambda)} = \int d\nu^\varepsilon_\Lambda(x, z) \nabla \varphi(x) \cdot z \nabla_\varepsilon V_\varepsilon(x, z) + o(1) \quad (106)
\]

For each positive \( n \in \mathbb{N} \) let \( A_n := (1 - 1/n)\Lambda \) and let \( \phi_n \in C_c(\Lambda) \) be a function with values in \([0, 1]\) such that \( \phi_n \equiv 1 \) on \( A_n \). By Schwarz inequality

\[
\left| \int d\nu_{\mu^\varepsilon_\Lambda}(x, z)(\phi_n(x) - 1) \nabla \varphi(x) \cdot z \nabla_\varepsilon V_\varepsilon(x, z) \right| \leq \| \nabla \varphi \|_\infty \| \nabla_\varepsilon V_\varepsilon \|_{L^2(\nu^\varepsilon_\Lambda)} \left[ \int_{\Lambda \setminus A_n} d\mu^\varepsilon_\Lambda(x) \lambda_2(\tau_{x/\varepsilon}\tilde{\omega}) \right]^{1/2} \quad (107)
\]

Since \( \tilde{\omega} \in \Omega_{\text{typ}} \subset \mathcal{A}[\lambda_2] \cap \mathcal{A}_I[z^2] \), we get that \( \lim_{\varepsilon \downarrow 0} \int_{\Lambda \setminus A_n} d\mu^\varepsilon_\Lambda(x) \lambda_2(\tau_{x/\varepsilon}\tilde{\omega}) = \ell(\Lambda \setminus A_n) \mathcal{E}_0[\lambda_2] \). As a byproduct with (40) we get \( \lim_{n \uparrow \infty} \lim_{\varepsilon \downarrow 0} \text{h.s. of } (107) = \)
0. Due to (106) we then obtain
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \int d\nu_{\varepsilon,A}(x, z) \phi_n(x) \nabla \varphi(x) \cdot z \nabla \varphi(x) = 0. \tag{108}
\]

On the other hand, due to (98) and since \( \tilde{\omega} \in \Omega_{\text{typ}} \) (recall that the form \( \omega, z \mapsto z_i \) belongs to \( H \) and apply (76)), we can rewrite (108) as
\[
\lim_{n \to \infty} \int \nu_\varepsilon(x, z) \phi_n(x) \nabla \varphi(x) \cdot z \nabla \varphi(x) = 0. \tag{109}
\]

By Schwarz inequality, the above integral differs from the same expression with \( \phi_n(x) \) replaced by \( 1 \) by at most \( \|w\|\|\phi_n - 1\|_{L^2(\Lambda, dx)}\|\nabla \varphi\|_{\supL[\varepsilon]}^{1/2} \), where \( \|w\| \) is the norm of \( w \) in \( L^2(\Lambda \times \Omega \times \mathbb{R}^d, mdx \times \nu) \). Hence, due to (109), \( \int \nu_\varepsilon(x, z) \nabla \varphi(x) \cdot z \nabla \varphi(x) = 0 \). As a byproduct with (105), we conclude that \( 0 = \int \nabla \varphi(x) \cdot D \nabla \varphi(x) \cdot D \nabla \varphi(x) \) for any \( \varphi \in C^2(\mathbb{R}^d) \) with \( \varphi \equiv 0 \) on \( S \setminus \Lambda \) (we write \( \varphi \not\equiv 0 \)). If we take \( \varphi \in C^\infty_C(\mathbb{R}^d \setminus F) \), then \( \varphi_\Lambda \) can be approximated in the space \( H^1(\Lambda, dx) \) by functions \( \varphi_\Lambda \) with \( \varphi \not\equiv 0 \). Hence by density we conclude that \( 0 = \int \nabla \varphi_\Lambda \cdot D \nabla \varphi_\Lambda \) for any \( \varphi \in H^1(\Lambda, F, dx) \). Due to Proposition 13.1 we also have that \( v \in K \) (cf. [50]) in Definition 8.2. Hence, by Definition 8.7 and Lemma 8.9 \( v \) is the unique weak solution of the equation \( \nabla \varphi \cdot (D \nabla \varphi) = 0 \) with boundary conditions (50).

By Corollary 8.10 we conclude that \( v = \psi_\Lambda \).

Since the limit point is always \( \psi_\Lambda \) whatever the subsequence \( \{\varepsilon_k\} \), we get that \( V_\varepsilon \in L^2(\mu_\varepsilon^*) \) weakly 2-scale converges to \( \psi_\Lambda \in L^2(\Lambda \times \Omega, mdx \times \mathcal{P}_0) \) as \( \varepsilon \downarrow 0 \), and not only along some subsequence. As \( \psi_\Lambda \) does not depend from \( \omega \) and since \( \varepsilon \in \mathcal{G} \), we derive from (75) that \( L^2(\mu_{\varepsilon,A}) \ni V_\varepsilon \to \psi \in L^2(\Lambda, mdx) \) according to Definition 11.1.

14.2. Proof of Theorem 1
Let us show that, given \( \tilde{\omega} \in \Omega_{\text{typ}} \), it holds
\[
\lim_{\varepsilon \to 0} \frac{1}{2} \langle \nabla_x \psi_\varepsilon(x, z), \nabla_x \psi_\varepsilon(x, z) \rangle_{L^2(\nu_{\varepsilon,A})} = mD_{1,1} \tag{cf. (25)}
\]
To this aim we apply Remark 6.4 to get that \( \langle \nabla_x \psi_\varepsilon(x, z), \nabla_x \psi_\varepsilon(x, z) \rangle_{L^2(\nu_{\varepsilon,A})} = 0 \). This implies that
\[
\langle \nabla_x \psi_\varepsilon(x, z), \nabla_x \psi_\varepsilon(x, z) \rangle_{L^2(\nu_{\varepsilon,A})} = \langle \nabla_x \psi_\varepsilon(x, z), \nabla_x \psi_\varepsilon(x, z) \rangle_{L^2(\nu_{\varepsilon,A})}. \tag{110}
\]

Claim 14.1. It holds \( \lim_{\varepsilon \to 0} \int d\nu_{\varepsilon,A}(x, z)|\nabla_x \psi(x, z) - z_1|^2 = 0 \).

Proof. If \( x, x + \varepsilon z \in \Lambda \), then \( \nabla_x \psi(x, z) = z_1 \). We have only 4 relevant alternative cases: (a) \( x \in \Lambda, x + \varepsilon z \in S^+ \); (b) \( x \in S^+, x + \varepsilon z \in \Lambda \); (c) \( x \in \Lambda, x + \varepsilon z \in S^- \); (d) \( x \in S^-, x + \varepsilon z \in \Lambda \). Below we treat only case (a), since the other cases can be treated similarly. Hence we assume (a) to hold. Then \( x + \frac{1}{2} = \psi(x) \leq \psi(x + \varepsilon z) \leq x + \varepsilon z + \frac{1}{2} \) and therefore \( 0 \leq \nabla_x \psi(x, z) \leq z_1 \). This implies that \( |\nabla_x \psi(x, z) - z_1| \leq z_1^2 \). Fix \( \delta \in (0, 1/2) \) and set \( \Lambda_\delta := (-1/2 + \delta, 1/2 - \delta)^d \). Given \( n \in \mathbb{N} \), for \( \varepsilon \) small we can bound
\[
\int d\nu_{\varepsilon,A}(x, z)|\nabla_x \psi(x, z) - z_1|^2 \mathbb{1}(x \in \Lambda_\delta, x + \varepsilon z \in S^+) \leq \int d\nu_{\varepsilon,A}(x, z)z_1^2 \mathbb{1}(x \in \Lambda_\delta, z_1 \geq \delta/\varepsilon) \leq \int_{\Lambda_\delta} d\mu_{\varepsilon}(x)f_{\tau_\delta \tilde{\omega}}. \tag{111}
\]
where \( f_n(\omega) = \sum_{z \in \hat{\omega}} c_{0, z} |z|^2 \mathbf{1}(|z| \geq n) \) (recall (70)). By Proposition 5.2 and since \( \Omega_{\text{typ}} \subset \mathcal{A}[\mathcal{E}_n] \cap \mathcal{A}_1[|z|^2] \), as \( \varepsilon \downarrow 0 \) the last integral in (111) converges to \( m(1 - 2\delta)\mathcal{E}_0[\mathcal{E}_n] \), which goes to zero as \( n \uparrow +\infty \) due to (A7). This allows to conclude that the l.h.s. of (111) converges to zero as \( \varepsilon \downarrow 0 \).

We can bound
\[
\int \nu_{\delta, \Lambda}(x, z) |\nabla_x \psi(x, z) - z_1|^2 \mathbf{1}(x \in \Lambda \setminus \Lambda_\delta, x + \varepsilon z \in \mathcal{S}^+) \leq \int \nu_{\delta, \Lambda}(x, z) z_1^2 \mathbf{1}(x \in \Lambda \setminus \Lambda_\delta) \leq \int_{\Lambda \setminus \Lambda_\delta} d\mu_{\delta}(x) \lambda_2(\tau_{x/\varepsilon} \hat{\omega}).
\]  

(112)

By Proposition 5.2 and since \( \Omega_{\text{typ}} \subset \mathcal{A}[\mathcal{E}_2] \cap \mathcal{A}_1[|z|^2] \), as \( \varepsilon \downarrow 0 \) the last integral in (112) converges to \( \ell(\Lambda \setminus \Lambda_\delta)\mathcal{E}_0[\mathcal{E}_2] \), which goes to zero as \( \delta \downarrow 0 \) by (A7).

The above results allow to conclude the proof.

As a byproduct of Claim 14.1, Lemma 6.5 and (110), we get
\[
\lim_{\varepsilon \downarrow 0} \langle \nabla_x V_\varepsilon, \nabla_x V_\varepsilon \rangle_{L^2(\nu_{\delta, \Lambda}^\varepsilon)} = \lim_{\varepsilon \downarrow 0} \int \nu_{\delta, \Lambda}(x, z) z_1 \nabla_x V_\varepsilon(x, z).
\]

(113)

By applying Schwarz inequality as in (107), we get that
\[
\lim_{\varepsilon \downarrow 0} \int \nu_{\delta, \Lambda}(x, z) z_1 \nabla_x V_\varepsilon(x, z) = \lim_{n \uparrow \infty} \lim_{\varepsilon \downarrow 0} \int \nu_{\delta, \Lambda}(x, z) \phi_n(x) z_1 \nabla_x V_\varepsilon(x, z).
\]

(114)

We know that \( L^2(\mu_{\delta, \Lambda}^\varepsilon) \ni V_\varepsilon^{\frac{2}{\lambda}} \mathbf{1} v \in L^2(\Lambda \times \Omega, m dx \times P_0) \). By Lemma 11.5 from any vanishing sequence \( \{\varepsilon_k\} \) we can extract a sub-sequence \( \{\varepsilon_{k_n}\} \) such that \( \nabla_x V_{\varepsilon_k}^{\frac{2}{\lambda}} w \) for some \( w \) along the sub-sequence, similarly to (98). Since \( \phi_n \in C_0(\Lambda) \), as a byproduct of (113) and (114) we obtain that
\[
\lim_{\varepsilon \downarrow 0} \langle \nabla_x V_\varepsilon, \nabla_x V_\varepsilon \rangle_{L^2(\nu_{\delta, \Lambda}^\varepsilon)} = \lim_{n \uparrow \infty} \int_{\Lambda} dx m \phi_n(x) \int \nu(\omega, z) z_1 w(x, \omega, z)
\]
\[
= \int_{\Lambda} dx m \int \nu(\omega, z) z_1 w(x, \omega, z)
\]

(115)

along \( \{\varepsilon_{k_n}\} \). Due to (105) the last term equals \( m \int_{\Lambda} 2(D\nabla_x v(x)) \cdot e_1 dx \). Since \( v = \psi_{\Lambda} \) as derived in Section 14.1, we get that \( \nabla_x v(x) = e_1 \). As a consequence, the last term of (115) equals \( 2mD_{11} \), thus allowing to conclude the proof.

**APPENDIX A. PROOF OF THEOREM 3**

We just give some comments on how to adapt the arguments in the preceding sections. By the same arguments of Section 4, we just need to consider the case \( \mathbb{G} = \mathbb{R}^d \). From now on we use the notation of Section 2.2.

One has to adapt the definition of \( \psi_\varepsilon \) to the present new geometry: \( \psi_\varepsilon \) equals \( \psi \) on \( C_\omega^\varepsilon \) and is zero on \( C_\omega^{\varepsilon, \Lambda} \). Then Lemma 6.5 still holds as \( |\psi(x) - \psi(y)| \leq C|x - y| \) for all \( x, y \in S \) (apply this modification in (41)).
The proof in Section 7 is now for the case $a \cdot D_a = 0$ and remains valid essentially by replacing $x_1$ with $x \cdot a$ in (45), $y_1 - x_1$ with $(y - x) \cdot a$ in (46), (47), (48) and $1(|z|_\infty \geq \ell)$ with $1(|z|_\infty \geq C\ell)$ in the definition of $g_\ell(\omega)$.

From now on we focus on the case $a \cdot D_a > 0$. Apart from Remark 8.3 and the proof of Lemma 8.6 (which have to be slightly adapted), Section 8 up to Lemma 8.9 included remains unchanged in our new setting. We stress that, as for each $x \in \Lambda$ the set $x + \mathbb{R}e_1$ is a segment between $F_-$ and $F_+$, one can prove Lemma 8.6 by slight modifications. Instead of Corollary 8.10 we now have that $\psi|_\Lambda$ is the unique solution of (54) with boundary conditions (55) (indeed, $\pm n(x) \in \{e_2, \ldots, e_d\}$ and $D_w \cdot e_i = w \cdot D e_i = 0$ for $2 \leq i \leq d$ by construction).

Sections 9, 10, 11, 12 and 13 do not need any modification. The same holds for Section 14.1 by using the updated form of Corollary 8.10. Main differences emerge in Section 14.2. Now the limit in Claim 14.1 becomes $\lim_{\varepsilon \downarrow 0} \int d\nu(\omega, z) |\nabla_\perp \psi(x, z) - a \cdot z|^2 = 0$, and the proof is similar. Repeating the arguments leading to (115) we conclude that $\lim_{\varepsilon \downarrow 0} \int d\nu(\omega, z) a \cdot z v(x, \omega, z) = \int dx m \int d\nu(\omega, z) a \cdot z v(x, \omega, z)$, which equals $m a \cdot \int_\Lambda 2(Dv_\perp(x))dx$ due to (105). As $v = \psi|_\Lambda$, the above expression equals $ma \cdot D_a$.

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