Strongly Stable Automorphisms of the Categories of the Finitely Generated Free Algebras of the some Varieties of Linear Algebras.

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Abstract

In this paper we consider some classical varieties of linear algebras over the field \( k \) such that \( \text{char}(k) = 0 \). If we denote by \( \Theta \) one of these varieties, then \( \Theta^0 \) is a category of the finite generated free algebras of the variety \( \Theta \). In this paper we calculate for the considered varieties the quotient group \( A/Y \), where \( A \) is a group of the all automorphisms of the category \( \Theta^0 \) and \( Y \) is a subgroup of the all inner automorphisms of this category. The quotient group \( A/Y \) measures difference between the geometric equivalence and automorphic equivalence of algebras from the variety \( \Theta \). The results of this paper and of the [9] are summarized in the table in the Section [6].

1 Introduction.

This is a paper from universal algebraic geometry. All definitions of the basic notions of the universal algebraic geometry can be found, for example, in [2], [3] and [4].

This research is a continuation of the [9].

If we will compare the geometric equivalence and the automorphic equivalence of the one-sorted universal algebras from the some variety \( \Theta \), we must take a countable set of symbols \( X_0 = \{x_1, x_2, \ldots, x_n, \ldots\} \) and consider all free algebras \( F(X) \) of the variety \( \Theta \), generated by finitely subsets \( X \subset X_0 \). These algebras: \( \{F(X) \mid X \subset X_0, |X| < \infty \} \) - will be objects of the category \( \Theta^0 \). Morphisms of the category \( \Theta^0 \) will be homomorphisms of these algebras.
If our variety $\Theta$ possesses the IBN property: for free algebras $F(X), F(Y) \in \Theta$ we have $F(X) \cong F(Y)$ if and only if $|X| = |Y|$ - then we have [5, Theorem 2] the decomposition
\[ A = Y\mathcal{S}, \] (1.1)
of the group $A$ of all automorphisms of the category $\Theta^0$. Here $Y$ is a group of all inner automorphisms of the category $\Theta^0$ and $\mathcal{S}$ is a group of all strongly stable automorphisms of the category $\Theta^0$. The definitions of the notions of inner automorphisms and strongly stable automorphisms can be found, for example, in [5] and [9]. But we will give these definitions here.

**Definition 1.1** An automorphism $\Upsilon$ of a category $\mathcal{R}$ is inner, if it is isomorphic as a functor to the identity automorphism of the category $\mathcal{R}$.

It means that for every $A \in \text{Ob}\mathcal{R}$ there exists an isomorphism $s^\Upsilon_A : A \rightarrow \Upsilon(A)$ such that for every $\psi \in \text{Mor}_\mathcal{R}(A, B)$ the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{s^\Upsilon_A} & \Upsilon(A) \\
\downarrow \psi & & \downarrow \Upsilon(\psi) \\
B & \xrightarrow{s^\Upsilon_B} & \Upsilon(B)
\end{array}
\]
commutes.

**Definition 1.2.** An automorphism $\Phi$ of the category $\Theta^0$ is called strongly stable if it satisfies the conditions:

$StSt1)$ $\Phi$ preserves all objects of $\Theta^0$,

$StSt2)$ there exists a system of bijections $\{s^\Phi_F : F \rightarrow F \mid F \in \text{Ob}\Theta^0\}$ such that $\Phi$ acts on the morphisms $\psi : D \rightarrow F$ of $\Theta^0$ by this way:
\[ \Phi(\psi) = s^\Phi_D \psi (s^\Phi_D)^{-1}, \] (1.2)

$StSt3)$ $s^\Phi_F|_X = \text{id}_X$, for every free algebra $F = F(X)$.

The subgroup $Y$ is a normal in $A$.

By [4] only strongly stable automorphism $\Phi$ can provide us automorphic equivalence of algebras which not coincides with geometric equivalence of algebras. Therefore, in some sense, difference from the automorphic equivalence to the geometric equivalence is measured by the quotient group $A/Y \cong \mathcal{S}/\mathcal{S} \cap Y$.

2 Verbal operations and strongly stable automorphisms.

In this paper, as in [9] we use the method of verbal operations for the finding of the strongly stable automorphisms of the category $\Theta^0$. The explanation of this method there is in [5] and [7].
We denote the signature of our variety \( \Theta \) by \( \Omega \), by \( m_\omega \) we denote the arity of \( \omega \) for every \( \omega \in \Omega \). If \( \omega = \psi (x_1, \ldots, x_{m_\omega}) \in F(x_1, \ldots, x_{m_\omega}) \in \text{Ob} \Theta^0 \), then we can define in every algebra \( H \in \Theta \) by using of the this word \( \omega \) the new operation \( \omega^* \):

\[
\omega^* (h_1, \ldots, h_{m_\omega}) = w(h_1, \ldots, h_{m_\omega})
\]

for every \( h_1, \ldots, h_{m_\omega} \in H \). This operation we call the \textit{verbal operation} defined on the algebra \( H \) by the word \( \omega \). If we have a system of words \( W = \{ w_\omega \mid \omega \in \Omega \} \) such that \( w_\omega \in F(x_1, \ldots, x_{m_\omega}) \) then we denote by \( H^*_W \) the algebra which coincide with \( H \) as a set but instead the original operations \( \{ \omega \mid \omega \in \Omega \} \) it has the system of the verbal operations \( \{ \omega^* \mid \omega \in \Omega \} \) defined by words from the system \( W \).

We suppose that we have the system of words \( W = \{ w_\omega \mid \omega \in \Omega \} \) satisfies the conditions:

\begin{enumerate}
    \item[Op1)] \( w_\omega (x_1, \ldots, x_{m_\omega}) \in F(x_1, \ldots, x_{m_\omega}) \in \text{Ob} \Theta^0 \),
    \item[Op2)] for every \( F = F(X) \in \text{Ob} \Theta^0 \) there exists an isomorphism \( \sigma_F : F \to F^*_W \) such that \( \sigma_F \mid_X = \text{id}_X \).
\end{enumerate}

It is clear isomorphisms \( \sigma_F \) are defined uniquely by the system of words \( W \).

The set \( S = \{ \sigma_F : F \to F \mid F \in \text{Ob} \Theta^0 \} \) is a system of bijections which satisfies the conditions:

\begin{enumerate}
    \item[B1)] for every homomorphism \( \psi : A \to B \in \text{Mor} \Theta^0 \) the mappings \( \sigma_B \psi \sigma_A^{-1} \) and \( \sigma_B^{-1} \psi \sigma_A \) are homomorphisms;
    \item[B2)] \( \sigma_F \mid_X = \text{id}_X \) for every free algebra \( F \in \text{Ob} \Theta^0 \).
\end{enumerate}

So we can define the strongly stable automorphism by this system of bijections. This automorphism preserves all objects of \( \Theta^0 \) and acts on morphism of \( \Theta^0 \) by formula (1.2), where \( s_F^\psi = \sigma_F \).

Vice versa if we have a strongly stable automorphism \( \Phi \) of the category \( \Theta^0 \) then its system of bijections \( S = \{ s_F^\psi : F \to F \mid F \in \text{Ob} \Theta^0 \} \) defined uniquely. Really, if \( F \in \text{Ob} \Theta^0 \) and \( f \in F \) then

\[
s_F^\psi (f) = s_F^\psi (x) = \left(s_F^\psi \left(s_D^\psi \right)^{-1}\right)(x) = (\Phi (\psi))(x), \tag{2.1}
\]

where \( D = F(x) \) - 1-generated free linear algebra - and \( \psi : D \to F \) homomorphism such that \( \psi (x) = f \). Obviously that this system of bijections \( S = \{ s_F^\psi : F \to F \mid F \in \text{Ob} \Theta^0 \} \) fulfills conditions B1) and B2) with \( \sigma_F = s_F^\psi \).

If we have a system of bijections \( S = \{ \sigma_F : F \to F \mid F \in \text{Ob} \Theta^0 \} \) which fulfills conditions B1) and B2) than we can define the system of words \( W = \{ w_\omega \mid \omega \in \Omega \} \) satisfies the conditions Op1) and Op2) by formula

\[
w_\omega (x_1, \ldots, x_{m_\omega}) = \sigma_{F_\omega} (\omega ((x_1, \ldots, x_{m_\omega}))) \in F_\omega, \tag{2.2}
\]

where \( F_\omega = F(x_1, \ldots, x_{m_\omega}) \).

By formulas (2.1) and (2.2) we can check that there are
1. one to one and onto correspondence between strongly stable automorphisms of the category $\Theta^0$ and systems of bijections satisfied the conditions B1) and B2)

2. one to one and onto correspondence between systems of bijections satisfied the conditions B1) and B2) and systems of words satisfied the conditions Op1) and Op2).

Therefore we can calculate the group $\mathcal{S}$ if we can find the all system of words which fulfill conditions Op1) and Op2). For calculation of the group $\mathcal{S} \cap \mathcal{Y}$ we also have a

**Criterion 2.1** The strongly stable automorphism $\Phi$ of the category $\Theta^0$ which corresponds to the system of words $W$ is inner if and only if for every $F \in \text{Ob}\Theta^0$ there exists an isomorphism $c_F : F \to F^*_W$ such that $c_F \psi = \psi c_D$ fulfills for every $(\psi : D \to F) \in \text{Mor}\Theta^0$.

### 3 Verbal operations in linear algebras.

From now on, the variety $\Theta$ will be some specific variety of the linear algebras over infinite field $k$, which has the characteristic 0. We neither consider the vanished varieties, i., e., variety defined by identity $x = 0$ or variety defined by identity $x_1x_2 = 0$.

We consider linear algebras as one-sorted universal algebras, i. e., multiplication by scalar we consider as 1-ary operation for every $\lambda \in k$: $H \ni h \to \lambda h \in H$ where $H \in \Theta$. Hence the signature $\Omega$ of algebras of our variety contains these operations: 0-ary operation 0; $|k|$ 1-ary operations of multiplications by scalars; 2-ary operation $\cdot$ and 2-ary operation $\pm$. We will finding the system of words $W = \{w_\omega \mid \omega \in \Omega\}$ satisfies the conditions Op1) and Op2). We denote the words corresponding to these operations by $w_0, w_\lambda$ for all $\lambda \in k, w, w_+$. So

$$W = \{w_\omega \mid \omega \in \Omega\} = \{w_0, w_\lambda (\lambda \in k), w_+, w\} \quad (3.1)$$

in our case. From this on we consider only these systems of words.

Some time we denote by $\lambda*$ the operation defined by the word $w_\lambda (\lambda \in k)$, by $\bot$ the operation defined by the word $w_+$ and by $\times$ the operation defined by the word $w$.

We denote the group of all automorphisms of the field $k$ by Aut$k$.

We use in our research the familiar fact that every variety of the linear algebras over infinite field $k$ is multi-homogenous. So, for example, every $F(X) \in \text{Ob}\Theta^0$ can be decompose to the direct sum of the linear spaces of elements which are homogeneous according the sum of degrees of generators from the set $X$: $F(X) = \bigoplus_{i=1}^{\infty} F_i$. We also denote the two sided ideals $\bigoplus_{i=j}^{\infty} F_i = F^j$. $F_iF_j \subset F_{i+j}$ and $F^iF^j \subset F_{i+j}$ fulfills for every $1 \leq i, j < \infty$. From now on, the word ”ideal” means two sided ideal of linear algebra.
All our varieties $\Theta$ possess the IBN property, because $|X| = \dim F/F^2$ fulfills for all free algebras $F = F(X) \in \text{Ob}\Theta^0$. So we have the decomposition for group of all automorphisms of the category $\Theta^0$.

4 Classical varieties of linear algebras.

In this Section we consider as the variety $\Theta$ the varieties of the all commutative algebras, of the all power associative algebras, i., e., the variety of linear algebras defined by identities

$$x (x^2) = (x^2) x,$$

$$x (x (x^2)) = x ((x^2) x) = (x (x^2)) x = (x^2) (x^2)$$

and so on, of the all alternative algebras, of the all Jordan algebras and arbitrary subvariety defined by identities with coefficients from $\mathbb{Z}$ of the variety of the all anticommutative algebras.

For the calculating of the group $S$ we consider an arbitrary strongly stable automorphism $\Phi$ of the category $\Theta^0$ and we will find for the all possible forms of the system of words $W$ which corresponds to the automorphism $\Phi$.

For the all considered varieties $F(\emptyset) = \{0\}$, so $w_0 = 0$.

The crucial point is the finding of the words $w_\lambda (x) \in F(x)$, where $\lambda \in k$. The system of words $W$ must fulfills conditions Op1) and Op2). By condition Op2) all axioms of the variety $\Theta$ must hold in the $F^* W$ for every $F \in \text{Ob}\Theta^0$. For every $\lambda \in k^*$ must holds

$$w_{\lambda - 1} (w_\lambda (x)) = w_\lambda (w_{\lambda - 1} (x)) = x.$$  

So the mapping $F(x) \ni x \to w_\lambda (x) \in F(x)$ can by extended to the isomorphism.

By [6] the variety of the all commutative algebras is a Shraier variety and by [4] all automorphisms of the free algebras of these varieties are tame. So if $\Theta$ is the variety of the all commutative algebras, then for $\lambda \in k^*$ we have that

$$w_\lambda (x) = \varphi (\lambda) x,$$

where $\varphi (\lambda) \in k$. If $\lambda = 0$, then must fulfills $w_\lambda (x) = 0$, so in this case we also can write (4.3), where $\varphi (\lambda) = 0$.

If $\Theta$ is the variety of the all power associative algebras, then $F(x)$ is the algebra of the polynomials of degrees no less then 1. Hence from (4.2) we can conclude that $\deg w_\lambda (x) = 1$ and (4.3) holds. Similar result we have for the variety of the all alternative algebras and for the variety of the all Jordan algebras, because these varieties are subvarieties of the variety of the all power associative algebras (see [10] Chapter 2, Theorem 2 and [10] Chapter 3, Corollary from Theorem 8), so in these varieties $F(x)$ is also the algebra of the polynomials of degrees no less then 1.

We conclude (4.3) for the arbitrary subvariety defined by identities with coefficients from $\mathbb{Z}$ of the variety of the all anticommutative algebras from the
fact that in this variety \( \dim F(x) = 1 \). Therefore in all our varieties we have \( 1 \leq \lambda \leq k \).

\[ \lambda * (\mu * x) = (\lambda \mu) * x \]

must fulfills in \( F(x) \) for every \( \lambda, \mu \leq k \). We can conclude from this axiom as in [9] that \( \varphi (\lambda \mu) = \varphi (\lambda) \varphi (\mu) \). Also by using [7] Proposition 4.2] we can prove that \( \varphi : k \to k \) is a surjection.

After this we can conclude from axioms \( x_1 \bot 0 = x_1, 0 \bot x_2 = x_2, x_1 \bot x_2 = x_2 \bot x_1 \) and \( \lambda * (x_1 \bot x_2) = (\lambda * x_1) \bot (\lambda * x_2) \) as in [9] that in all our varieties the

\[ w_+ (x_1, x_2) = x_1 + x_2 \]

holds. Hear we must use the decomposition of \( F(x_1, x_2) \) to the direct sum of the linear spaces of elements which are homogeneous according the sum of degrees of generators, which was used in [9].

From axiom \( (\lambda + \mu) * x = \lambda * x + \mu * x \) for every \( \lambda, \mu \leq k \) we conclude that \( \varphi (\lambda + \mu) = \varphi (\lambda) + \varphi (\mu) \). So \( \varphi \in \text{Aut} k \).

Now we must to find the all possible forms of the word \( w \in F(x_1, x_2) \). Hear as in [9] we use the decomposition of \( F(x_1, x_2) \) to the direct sum of the linear spaces of elements which are homogeneous according the degree of \( x_1 \), and after this according the degree of \( x_2 \). From axioms \( 0 \times x_2 = x_1 \times 0 = 0 \) and \( \lambda * (x_1 \times x_2) = (\lambda * x_1) \times x_2 = x_1 \times (\lambda * x_2) \) for every \( \lambda \leq k \) we can conclude that \( w \in F_2(x_1, x_2), w. (x_1, 0) = w. (0, x_2) = 0 \). It means that for the variety of the all power associative algebras

\[ w. (x_1, x_2) = \alpha_{1,2} x_1 x_2 + \alpha_{2,1} x_2 x_1, \]  

where \( \alpha_{1,2}, \alpha_{2,1} \leq k \). By condition Op2) in this variety the multiplication in \( F^*_W \) can not by commutative or anticommutative, so \( \alpha_{1,2} \neq \pm \alpha_{2,1} \).

For the varieties of the all commutative algebras, of the all Jordan algebras and for the arbitrary subvariety defined by identities with coefficients from \( \mathbb{Z} \) of the variety of the all anticommutative algebras we have that

\[ w. (x_1, x_2) = \alpha_{1,2} x_1 x_2, \]

where \( \alpha_{1,2} \neq 0 \).

For the variety of the all alternative algebras we conclude from axiom \( (x_1 \times x_1) \times x_2 = x_1 \times (x_1 \times x_2) \) that

\[ w. (x_1, x_2) = \alpha_{1,2} x_1 x_2 \]

or

\[ w. (x_1, x_2) = \alpha_{2,1} x_2 x_1, \]

where \( \alpha_{1,2}, \alpha_{2,1} \neq 0 \).

Now we will prove for all our varieties that the systems of words \( W \) defined above fulfill condition Op2). First of all we will prove that if \( H \in \Theta \) then \( H^*_W \in \Theta \). It means that we will check that in the \( H^*_W \) the all axioms of the variety \( \Theta \) hold. All these checking can be made by direct calculations. For all our varieties we must check only these axioms of linear algebra: \( (x_1 + x_2) \times x_3 = \)
(x_1 \times x_3) + (x_2 \times x_3) and x_1 \times (x_2 + x_3) = (x_1 \times x_2) + (x_1 \times x_3), because other axioms are immediately concluded from the forms of the words of the system $W$.

For the varieties of the all commutative algebras and of the all Jordan algebras we must check the axiom $x_1 \times x_2 = x_2 \times x_1$.

Also for the variety of the all Jordan algebras we also must check the axiom $((x_1 \times x_1) \times x_2) \times x_1 = (x_1 \times x_1) \times (x_2 \times x_1)$. For the variety of the all alternative algebras we must check the axioms $x \times (y \times z) = (x \times y) \times z$.

For the variety of the all power associative algebras we must check the axioms $\sigma[9]$ where the original multiplication changed by operation $\times$.

For the varieties of the all commutative algebras and of the all Jordan algebras we must check the axiom $u \times (y \times z) = (u \times y) \times z$. Also for the variety of the all Jordan algebras we also must check the axiom $((x_1 \times x_1) \times x_2) \times x_1 = (x_1 \times x_1) \times (x_2 \times x_1)$.

For the arbitrary subvariety defined by identities with coefficients from $\mathbb{Q}$ all operations of the process of the homogenization of the identities can be made with coefficients from $\mathbb{Q}$. We consider a monomial $u \in F$ such that coefficient of $u$ is 1, and deg $u = n$. If in the monomial $u$ we change the original multiplication by operation $\times$ then we achieve $u^\times \in F_W^\ast$. It is easy to prove by induction that $u^\times (h) = (\alpha_{1,2} + \alpha_{2,1})^{n-1} u(h)$ holds for every $h \in H$. It finishes the checking of the necessary axioms.

For the arbitrary subvariety defined by identities with coefficients from $\mathbb{Z}$ of the variety of the all anticommutative algebras we must check the axiom $x_1 \times x_2 = -1 \times (x_2 \times x_1)$ and the specific axioms of this subvariety. Our field $k$ is infinite so all axioms of this subvariety can be presented in the homogeneous form: $\sum_{i=1}^m \lambda_i u_i = 0$, where $\lambda_i \in \mathbb{Z}$, $u_i \in F(x_1, \ldots, x_r)$, $F(x_1, \ldots, x_r) \in \text{Ob}_\Theta^0$, $u_i$ are monomials with coefficients 1, deg $u_i = n$ for every $i$. $\lambda_i \in \mathbb{Z}$ because all operations of the process of the homogenization of the identities can be made with coefficients from $\mathbb{Q}$. We must check that for every $H \in \Theta$ and every $h_1, \ldots, h_r \in H$ the $\sum_{i=1}^m \lambda_i \ast u_i^\times(h_1, \ldots, h_r) = 0$ holds. As above by induction we can prove that $u_i^\times(h_1, \ldots, h_r) = \alpha_{1,2}^{-1} u_i(h_1, \ldots, h_r)$, so $\sum_{i=1}^m \lambda_i \ast u_i^\times(h_1, \ldots, h_r) = \alpha_{1,2}^{-1} \sum_{i=1}^m \lambda_i u_i(h_1, \ldots, h_r) = 0$.

After all these checking we can conclude that for every $F = F(X) \in \text{Ob}_\Theta^0$ there exists a homomorphism $\sigma_F : F \to F_W^\ast$ such that $\sigma_F |_{X = id_X}$. As in the $[9]$ we can prove that $\sigma_F$ is an isomorphism, so the systems of words defined above fulfill condition Op2). It completes the calculation of the group $\mathcal{G}$.

For calculation of the group $\mathcal{G}$ we can prove as in the $[9]$ that for the all considered varieties the strongly stable automorphism $\Phi$ which corresponds to the defined above system of words $W$ is inner if and only if $\alpha_{2,1} = 0$ and $\varphi = id_k$.

So, as in the $[9]$ we can prove that for the variety of the all power associative algebras $\mathfrak{A}/\mathfrak{G} \cong (U(kS_2)/U(k\{e\})) \times \text{Aut} k$, where $S_2$ is the symmetric group of the set which has 2 elements, $U(kS_2)$ is the group of all invertible elements of the group algebra $kS_2$, $U(k\{e\})$ is a group of all invertible elements of the subalgebra $k\{e\}$, every $\varphi \in \text{Aut} k$ acts on the algebra $kS_2$ by natural way:
\[ \varphi (ae + b(12)) = \varphi (a)e + \varphi (b)(12). \] But there is an isomorphism of groups

\[ U(kS_2) \ni ae + b(12) \to (a + b, a - b) \in k^* \times k^* \]

so there is isomorphism

\[ U(kS_2) / U(k\{e\}) = U(kS_2) / k^*e \ni (ae + b(12))k^*e \to \frac{a + b}{a - b} \in k^*. \]

Hence we prove the

**Theorem 4.1** For variety of the all power associative algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong k^* \times \text{Aut} k \]

holds.

By similar way we prove

**Theorem 4.2** For the variety of the all alternative algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong S_2 \times \text{Aut} k \]

holds.

And for other considered varieties we achieve

**Theorem 4.3** For the variety of the all commutative algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong \text{Aut} k \]

holds.

**Theorem 4.4** For the variety of the all Jordan algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong \text{Aut} k \]

holds.

**Theorem 4.5** For the arbitrary subvariety defined by identities with coefficients from \( \mathbb{Z} \) of the variety of the all anticommutative algebras

\[ \mathfrak{A}/\mathfrak{Y} \cong \text{Aut} k \]

holds.
5 Varieties of nilpotent algebras.

We denote by $\mathcal{P}_n$ the set of all arrangements of the brackets on monomial which has $n$ factors. For every $p \in \mathcal{P}_n$ we denote by $p(x_1, \ldots, x_n)$ the monomial from absolutely free linear algebra $\tilde{F}(X)$, where $\{x_1, \ldots, x_n\} \subset X$. This monomial we obtain by putting brackets according the arrangement $p$ on associative word $x_1 \cdots x_n$. In this notation the variety of all nilpotent linear algebras of degree not more than $n$ is a variety of linear algebras defined by identities

$$p(x_1, \ldots, x_n) = 0,$$

where $p \in \mathcal{P}_n$. These varieties we denote in this section by $\Theta_n$, where $n \geq 3$.

The categories of the finitely generated free algebras of these varieties we denote by $\Theta^0_n$. By $\mathfrak{A}_n$ we denote the group of the all automorphisms of the category $\Theta^0_n$, by $\mathfrak{P}_n$ the group of the all inner automorphisms of this category and by $\mathfrak{S}_n$ the group of the all strongly stable automorphisms of the category $\Theta^0_n$.

5.1 Connection between groups $\mathfrak{A}_n/\mathfrak{P}_n$ of the categories $\Theta^0_n$.

In this subsection we develop the method of [8, Section 2].

Proposition 5.1 If $F \in \text{Ob}\Theta^0_n$ and the system of words $W$ fulfills conditions Op1) and Op2) in $\Theta^0_n$ then $F^i = (F_W^*)^i$ holds for $1 \leq i \leq n-1$.

Proof. By condition Op1) we have that $w_0 = 0$ and

$$w_\lambda(x) = \varphi(\lambda) x + t_2(x) \in F(x),$$

where $t_2(x) \in (F(x))^2$, $\varphi(\lambda) \in k$. Also

$$w_+(x_1, x_2) = \alpha x_1 + \beta x_2 + p_2(x_1, x_2) \in F(x_1, x_2),$$

where $p_2(x_1, x_2) \in (F(x_1, x_2))^2$, $\alpha, \beta \in k$. By Op2)

$$w_+(x_1, 0) = \alpha x_1 + p_2(x_1, 0) = x_1$$

and

$$w_+(0, x_2) = \beta x_2 + p_2(0, x_2) = x_2$$

must fulfill. So $\alpha = \beta = 1$. And

$$w_+(x_1, x_2) = x_1 + x_2 + p_2(x_1, x_2).$$

Analogously

$$w_+(x_1, x_2) = \gamma x_1 + \delta x_2 + q_2(x_1, x_2) \in F(x_1, x_2),$$

where $q_2(x_1, x_2) \in (F(x_1, x_2))^2$, $\gamma, \delta \in k$.

$$w_+(x_1, 0) = \gamma x_1 + q_2(x_1, 0) = 0$$
must fulfill. So $\gamma = \delta = 0$ and $q_2(x_1, 0) = q_2(0, x_2) = 0$. $q_2(x_1, 0)$ is a sum of monomials which contain only $x_1$ and $q_2(x_2, 0)$ is a sum of monomials which contain only $x_2$. Therefore

$$w. (x_1, x_2) = \alpha_{1,2}x_1x_2 + \alpha_{2,1}x_2x_1 + q_3(x_1, x_2)$$

where $q_3(x_1, x_2) \in (F(x_1, x_2))^3$, $\alpha_{1,2}, \alpha_{2,1} \in k$.

Now we can prove that

$$(F_W^i)^j \subseteq F^i$$

holds for every $i \geq 1$. We will use induction by $i$. For $i = 1$ (5.2) fulfills by definition of $F_W^n$. We assume that (5.2) proved for $i < j$. If $u \in (F_W^i)^j$, $v \in (F_W^i)^s$, such that $s + r = j$, $1 \leq s + r$, then by our assumption $u \in F^r$, $v \in F^s$. We have

$$u \times v = \alpha_{1,2}uv + \alpha_{2,1}vu + q_3(u, v) \in F^j.$$

If $u, v \in F^j$ then

$$u \perp v = u + v + p_2(u, v) \in F^j.$$

Also for every $\lambda \in k$

$$\lambda \ast u = \varphi (\lambda) u + l_2(u) \in F^j.$$

Therefore (5.2) is proved.

After this we can prove the inverse inclusion by [7, Proposition 4.2].

**Proposition 5.2** There is a homomorphism $S_n : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$.

**Proof.** We will define the mapping $S_n : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$ and after this we will prove that this mapping is a homomorphism of groups. By Section 2 the element $\Phi \in \mathfrak{S}_n$ which is a strongly stable automorphism of the category $\Theta_n^0$ can be defined by the system of words (3.1) which fulfills conditions Op1) and Op20 as well as by the system of bijections $\{ \sigma_{F(n)} : F(n) \to F(n) \mid F(n) \in \text{Ob} \Theta_n^0 \}$ which fulfills conditions B1) and B2). In this proof we actively use the second definition.

$F(n) = F(n) / (F(n))^{n-1}$ fulfills for every $F(n) \in \text{Ob} \Theta_n^0$, where $F(n-1) \in \text{Ob} \Theta_n^{0-1}$. We consider $\Phi \in \mathfrak{S}_n$. By Section 2 $\sigma_{F(n)}$ is an isomorphism $F(n) \to (F(n))_W^*$, where $W = \{ \sigma_{F(n)} \omega \mid \omega \in \Omega \}$. $\Omega$ is a signature of linear algebras, $F(n) = F(n)(x_1, \ldots, x_m) \in \text{Ob} \Theta_n^0$, $m$ is an arity of the operation $\omega$. We denote $\kappa_{F(n)}$ the homomorphism $F(n) \to F(n-1)$. $F(n-1) \in \Theta_n$, so in the set $F(n-1)$ we also can consider the verbal operations defined by the system of words $W$. This algebra we denote $(F(n-1))_W^*$. By [7] Remark 3.1 $\kappa_{F(n)}$ is also homomorphism from $(F(n))_W^*$ to $(F(n-1))_W^*$. 

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Now we will consider this diagram:

\[
0 \rightarrow (F^{(n)})^{n-1} \xleftarrow{\sigma_{F(n)}} F^{(n)} \xrightarrow{\kappa_{F(n)}} F^{(n-1)} \rightarrow 0
\]

(5.3)

\[
0 \rightarrow (F^{(n)})^{n-1} \xleftarrow{(F^{(n)})^*_W} (F^{(n-1)})^*_W \rightarrow 0
\]

Both rows of this diagram are exact because \( \ker \kappa_{F(n)} = (F^{(n)})^{n-1} \). \( \sigma_{F(n)} \) is an isomorphism according to the condition Op2. So

\[
\sigma_{F(n)} \left( (F^{(n)})^{n-1} \right) = \left( (F^{(n)})^{n-1}_W \right) = (F^{(n)})^{n-1},
\]

(5.4)

by Proposition \[\text{[5.1]}\]. Therefore the left square of (5.3) is commutative. So we can close the right square of (5.3) by uniquely defined homomorphism \( \sigma_{F(n-1)} \) which fulfills

\[
\sigma_{F(n-1)} \kappa_{F(n)} = \kappa_{F(n)} \sigma_{F(n)}.
\]

(5.5)

\( \sigma_{F(n)} \) is an isomorphism and (5.4) fulfills, so \( \sigma_{F(n-1)} \) is an isomorphism.

Therefore for we have a system of bijections

\[
\left\{ \sigma_{F(n-1)} : F^{(n)} \rightarrow F^{(n-1)} \mid F^{(n-1)} \in \text{Ob}\Theta_{n-1}^0 \right\}.
\]

(5.6)

We will check that \[\text{[5.6]}\] fulfills conditions B1) and B2). If \( X \) is a set of free generators of \( F^{(n)}(X) \equiv F^{(n)} \in \text{Ob}\Theta_{n}^0 \), then \( \{ \kappa_{F(n)}(x) \mid x \in X \} \) is a set of free generators of \( F^{(n-1)} \), \( \sigma_{F(n-1)} \kappa_{F(n)}(x) = \kappa_{F(n)}(\sigma_{F(n)}(x)) = \kappa_{F(n)}(x) \), so \( [5.6] \) fulfills condition B2).

We assume that \( \psi : A^{(n-1)} \rightarrow B^{(n-1)} \in \text{Mor}\Theta_{n-1}^0 \). \( A^{(n-1)}, B^{(n-1)} \in \Theta_n \), \( \kappa_{B^{(n)}} : B^{(n)} \rightarrow B^{(n-1)} \) is an epimorphism, \( A^{(n)} \) is a free algebra in the variety \( \Theta_n \), so there exists a homomorphism \( \tilde{\psi} : A^{(n)} \rightarrow B^{(n)} \), such that \( \kappa_{B^{(n)}} \tilde{\psi} = \psi \kappa_{A^{(n)}} \). Therefore we have that

\[
\sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \kappa_{A^{(n)}} = \sigma_{B^{(n-1)}} \tilde{\psi} \kappa_{A^{(n)}} \sigma_{A^{(n)}}^{-1} =
\]

\[
\sigma_{B^{(n-1)}} \kappa_{B^{(n)}} \tilde{\psi} \sigma_{A^{(n)}}^{-1} = \kappa_{B^{(n)}} \sigma_{B^{(n)}} \tilde{\psi} \sigma_{A^{(n)}}^{-1}.
\]

The system of bijections \( \{ \sigma_{F(n)} \mid F^{(n)} \in \text{Ob}\Theta_{n}^0 \} \) fulfills condition B1), so \( \sigma_{B^{(n)}} \tilde{\psi} \sigma_{A^{(n)}}^{-1} \) and \( \kappa_{B^{(n)}} \sigma_{B^{(n)}} \tilde{\psi} \sigma_{A^{(n)}}^{-1} \) are homomorphisms. Therefore \( \sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \kappa_{A^{(n)}} \) is a homomorphism. If \( \omega \in \Omega \) is a \( \text{m-ary} \) operation and \( a_1, \ldots, a_m \in A^{(n-1)} \) then there are \( f_1, \ldots, f_m \in A^{(n)} \) such that \( a_i = \kappa_{A^{(n)}}(f_i), 1 \leq i \leq m \). So

\[
\sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \omega(a_1, \ldots, a_m) = \sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \omega(\kappa_{A^{(n)}}(f_1), \ldots, \kappa_{A^{(n)}}(f_m)) =
\]

\[
\sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \kappa_{A^{(n)}}(f_1, \ldots, f_m) =
\]

\[
\omega(\sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \kappa_{A^{(n)}}(f_1), \ldots, \sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1} \kappa_{A^{(n)}}(f_m)) =
\]

\[
\omega(\sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1}(a_1), \ldots, \sigma_{B^{(n-1)}} \psi \sigma_{A^{(n-1)}}^{-1}(a_m)).
\]
Hence $\sigma_{B(n-1)}\psi\sigma_{A(n-1)}^{-1}$ is a homomorphism. Analogously we can prove that $\sigma_{B(n-1)}^{-1}\psi\sigma_{A(n-1)}$ is also a homomorphism. Therefore (5.5) fulfills condition B1).

We can consider $\mathcal{S}_n$ as set of bijections $\{\sigma_{F(n)} : F(n) \rightarrow F(n) \mid F(n) \in \text{Ob}\Theta_0^n\}$ which fulfills conditions B1) and B2). And we can consider $\mathcal{S}_{n-1}$ as set of bijections (5.6) which fulfills conditions B1) and B2). We prove that there is a mapping

$$\mathcal{S}_n : \mathcal{S}_n \ni \left\{ \sigma_{F(n)} : F(n) \rightarrow F(n) \mid F(n) \in \text{Ob}\Theta_0^n \right\} \rightarrow$$

$$\left\{ \sigma_{F(n-1)} : F(n-1) \rightarrow F(n-1) \mid F(n-1) \in \text{Ob}\Theta_0^{n-1} \right\} \in \mathcal{S}_{n-1}.$$ 

Now we will prove that this mapping is a homomorphism. We also will use notation $\mathcal{S}_n (\sigma_{F(n)}) = \sigma_{F(n-1)}$, where $\sigma_{F(n-1)}$ defined by condition (5.5). We assume that $\Phi_1, \Phi_2 \in \mathcal{S}_n$. $\Phi_1$ defined by system of bijection $\{\sigma_{F(n)}^{(i)} \mid F(n) \in \text{Ob}\Theta_0^n\}$, $i = 1, 2$. $\mathcal{S}_n (\Phi_i)$ is defined by system of bijection $\{\mathcal{S}_n (\sigma_{F(n)}^{(i)}) \mid F(n) \in \text{Ob}\Theta_0^n\}$, $i = 1, 2$. $\Phi_1 \Phi_2$ is defined by system of bijection $\{\mathcal{S}_n (\sigma_{F(n)}^{(1)}) \sigma_{F(n)}^{(2)} \mid F(n) \in \text{Ob}\Theta_0^n\}$ and $\mathcal{S}_n (\Phi_1 \Phi_2)$ is defined by system of bijection $\{\mathcal{S}_n (\sigma_{F(n)}^{(1)}) \sigma_{F(n)}^{(2)} \mid F(n) \in \text{Ob}\Theta_0^n\}$. By (5.5) we have that

$$\mathcal{S}_n (\sigma_{F(n)}^{(1)}) \mathcal{S}_n (\sigma_{F(n)}^{(2)}) \kappa_{F(n)} = \mathcal{S}_n (\sigma_{F(n)}^{(1)}) \mathcal{S}_n (\sigma_{F(n)}^{(2)}) \kappa_{F(n)} \sigma_{F(n)}^{(1)} \sigma_{F(n)}^{(2)} = \kappa_{F(n)} (\sigma_{F(n)}^{(1)} \sigma_{F(n)}^{(2)})$$

fulfills for every $F(n) \in \text{Ob}\Theta_0^n$. Therefore

$$\mathcal{S}_n (\sigma_{F(n)}^{(1)}) \mathcal{S}_n (\sigma_{F(n)}^{(2)}) = \mathcal{S}_n (\sigma_{F(n)}^{(1)}) \sigma_{F(n)}^{(2)}$$

and $\mathcal{S}_n (\Phi_1) \mathcal{S}_n (\Phi_2) = \mathcal{S}_n (\Phi_1 \Phi_2)$. ■

**Corollary 1** For every $\Phi \in \mathcal{S}_n$ which defined by system of words $W = \{w_\omega, n \mid \omega \in \Omega\}$ there exists $\Psi \in \mathcal{S}_{n-1}$ which defined by system of words $\bar{W} = \{\kappa_{F(n)} w_\omega, n-1 \mid \omega \in \Omega\}$ such that $w_\omega, n = w_\omega, n-1 + r_\omega, n$, $w_\omega, n \in F_{m_\omega}^{(n)}$, $w_\omega, n-1 \in \bigoplus_{i=1}^{n-2} F_{m_\omega}^{(n)}$, $r_\omega, n \in F_{m_\omega}^{(n)}$, $F_{m_\omega}^{(n)}(x_1, \ldots, x_{m_\omega}) \in \text{Ob}\Theta_0^m$, $m_\omega$ is arity of $\omega$.

**Proof.** We take $\Psi = \mathcal{S}_n (\Phi)$. The system of bijection $\{\sigma_{F(n)} \mid F(n) \in \text{Ob}\Theta_0^n\}$ corresponds to the automorphism $\Phi$, the system of bijection $\{\sigma_{F(n-1)} \mid F(n-1) \in \text{Ob}\Theta_0^{n-1}\}$ corresponds to the automorphism $\Psi$. The system of words $\{\sigma_{F(n-1)} \kappa_{F(n)} \omega \mid \omega \in \Omega\}$ defines the automorphism $\Psi$. For every $\omega \in \Omega$ we can decompose $\sigma_{F(n-1)} \omega = w_\omega, n = w_\omega, n-1 + r_\omega, n$. We have that

$$\kappa_{F(n)} w_\omega, n-1 = \kappa_{F(n)} w_\omega, n = \kappa_{F(n)} \sigma_{F(n-1)} \omega = \sigma_{F(n-1)} \kappa_{F(n)} \omega.$$

■
Proposition 5.3 If \( \Phi \in \mathcal{S}_n \cap \mathcal{Y}_n \), then \( \mathcal{S}_n (\Phi) \in \mathcal{S}_{n-1} \cap \mathcal{Y}_{n-1} \).

Proof. If \( \Phi \in \mathcal{S}_n \cap \mathcal{Y}_n \), then by definitions 1.11 and 1.12 there exists a system of isomorphisms \( \{ s_{F(n)} : F(n) \to F'(n) \mid F(n) \in \text{Ob}\Theta_n \} \) such that \( s_{B(n)} \psi = \Phi(\psi) s_{A(n)} \) fulfills for every \( \psi \in \text{Mor}_{\Theta_n}(A(n), B(n)) \). For every \( F(n) \in \text{Ob}\Theta_n \), we consider the diagram

\[
\begin{array}{ccc}
0 & \to & (F(n))^{n-1} \\
\downarrow s_{F(n)} & & \downarrow s_{F(n)} \\
0 & \to & (F(n))^{n-1}
\end{array}
\]

\[
\begin{array}{ccc}
\kappa_{F(n)} & \to & F(n) \\
\downarrow s_{F(n)} & & \downarrow s_{F(n)} \\
\kappa_{F(n)} & \to & F(n)
\end{array}
\]

and simpler than after the consideration of the diagram 5.3 conclude that we can close the right square of this diagram by isomorphism \( s_{F(n-1)} \kappa_{F(n)} = \kappa_{F(n)} s_{F(n)} \).

We will consider a homomorphism \( \psi \in \text{Mor}_{\Theta_n}(A(n-1), B(n-1)) \). As in the proof of the Proposition 5.2 we can conclude that there exists a homomorphism \( \tilde{\psi} \in \text{Mor}_{\Theta_n}(A(n), B(n)) \) such that \( \kappa_{B(n)} \tilde{\psi} = \psi \kappa_{A(n)} \) fulfills. The system of bijections which corresponds to the automorphism \( \Phi \) we denote by \( \{ \sigma_{F(n)} : F(n) \in \text{Ob}\Theta_n \} \) and the system of bijections which corresponds to the automorphism \( \mathcal{S}_n (\Phi) \) we denote by \( \{ \tilde{\sigma}_{F(n)} : F(n) \in \text{Ob}\Theta_n \} \).

By definition 1.12 the \( \Phi(\tilde{\psi}) = \sigma_{B(n)} \tilde{\psi} \sigma_{A(n)}^{-1} \) and \( (\mathcal{S}_n (\Phi))(\psi) = \mathcal{S}_n (\sigma_{B(n)} \psi \sigma_{A(n)}^{-1}) \).

So we have that

\[
((\mathcal{S}_n (\Phi))(\psi)) s_{A(n-1)} \kappa_{A(n)} = \mathcal{S}_n (\sigma_{B(n)} \psi \sigma_{A(n)}^{-1}) s_{A(n-1)} \kappa_{A(n)} =
\]

\[
\mathcal{S}_n (\sigma_{B(n)} \psi \sigma_{A(n)}^{-1}) s_{A(n)} = \mathcal{S}_n (\sigma_{B(n)} \psi \sigma_{A(n)}^{-1}) \mathcal{S}_n (\sigma_{A(n)} \psi \sigma_{A(n)}^{-1}) s_{A(n)}
\]

\[
\kappa_{B(n)} \sigma_{B(n)} \psi \sigma_{A(n)}^{-1} s_{A(n)} = \kappa_{B(n)} \sigma_{B(n)} \psi \sigma_{A(n)}^{-1} s_{A(n)}
\]

\[
\kappa_{B(n)} \sigma_{B(n)} \psi \sigma_{A(n)}^{-1} s_{A(n)} = \kappa_{B(n)} \psi \sigma_{A(n)}^{-1} s_{A(n)}
\]

\[
\kappa_{B(n)} \sigma_{B(n)} \psi \sigma_{A(n)}^{-1} s_{A(n)} = \kappa_{B(n)} \psi \sigma_{A(n)}^{-1} s_{A(n)}
\]

holds. Therefore \( ((\mathcal{S}_n (\Phi))(\psi)) s_{A(n-1)} = s_{B(n-1)} \psi \) and \( \mathcal{S}_n (\Phi) \) is an inner automorphism with the system of isomorphisms \( \{ s_{F(n-1)} \mid F(n-1) \in \text{Ob}\Theta_n \} \).

From Propositions 5.2 and 5.3 we can conclude the

Theorem 5.1 There is a homomorphism \( \mathcal{S}_n : \mathfrak{A}_n / \mathcal{Y}_n \to \mathfrak{A}_{n-1} / \mathcal{Y}_{n-1} \).

5.2 Group \( \mathfrak{A}_3 / \mathcal{Y}_3 \).

Theorem 5.2 \( \mathfrak{A}_3 / \mathcal{Y}_3 \cong k^* \cdot \text{Aut} k \).

Proof. We assume that the system of words \( W \) fulfills conditions Op1) and Op2) in the variety \( \Theta_3 \). We will find the specific form of these words. As in the proof of the Proposition 5.1 we have that \( w_{0,3} = 0, w_{0,3} (x_1, x_2) = x_1 + x_2 + p_2 (x_1, x_2) \), where \( p_2 (x_1, x_2) \in (F(3) (x_1, x_2))^2 \) and \( p_2 (x_1, 0) = p_2 (0, x_2) = 0 \).
fulfills. Therefore $p_2 (x_1, x_2) = \gamma_{1,2} x_1 x_2 + \gamma_{2,1} x_2 x_1$, where $\gamma_{1,2}, \gamma_{2,1} \in k$. By Op2) $w_{+,3} (x_1, x_2) = w_{+,3} (x_2, x_1)$ must fulfill, so $\gamma_{1,2} = \gamma_{2,1}$.

By Op1) $w_{3,3} (x) = \varphi (\lambda) x + \psi (\lambda) x^2 \in F (x)$, where $\varphi (\lambda), \psi (\lambda) \in k$, fulfills
for every $\lambda \in k$. 

$$
\lambda \ast (x_1 \perp x_2) = (\lambda \ast x_1) \perp (\lambda \ast x_2) \text{ must fulfill in } F^{(3)} (x_1, x_2).
$$

$$
\varphi (\lambda) \left( x_1 + x_2 + \gamma_{1,2} x_1 x_2 + \gamma_{2,1} x_2 x_1 \right) = \varphi (\lambda) x_1 + \varphi (\lambda) x_2 + \varphi (2 \lambda) \gamma_{1,2} x_1 x_2 + \varphi (\lambda) \gamma_{1,2} x_2 x_1 + \\
\psi (\lambda) x^2_1 + \psi (\lambda) x^2_2 + \psi (\lambda) x_1 x_2 + \psi (\lambda) x_2 x_1.
$$

$$(\lambda * x_1) \perp (\lambda * x_2) = (\varphi (\lambda) x_1 + \psi (\lambda) x^2_1) \perp (\varphi (\lambda) x_2 + \psi (\lambda) x^2_2) = \\
\varphi (\lambda) x_1 + \psi (\lambda) x^2_1 + \varphi (\lambda) x_2 + \psi (\lambda) x^2_2 + \\
\gamma_{1,2} \left( \varphi (\lambda) x_1 + \psi (\lambda) x^2_1 \right) \left( \varphi (\lambda) x_2 + \psi (\lambda) x^2_2 \right) + \\
\gamma_{1,2} \left( \varphi (\lambda) x_2 + \psi (\lambda) x^2_2 \right) \left( \varphi (\lambda) x_1 + \psi (\lambda) x^2_1 \right) = \\
\varphi (\lambda) x_1 + \psi (\lambda) x^2_1 + \varphi (\lambda) x_2 + \psi (\lambda) x^2_2 + \gamma_{2,1} \left( \varphi (\lambda) \right)^2 x_1 x_2 + \gamma_{1,2} \left( \varphi (\lambda) \right)^2 x_2 x_1.
$$

So $\varphi (\lambda) \gamma_{1,2} + \psi (\lambda) = \gamma_{1,2} \left( \varphi (\lambda) \right)^2$ and $\psi (\lambda) = \gamma_{1,2} \left( \left( \varphi (\lambda) \right)^2 - \varphi (\lambda) \right)$.

$$(\mu * x) \equiv (\mu * x) \perp (\lambda * x) \text{ must fulfill in } F^{(3)} (x) \text{ for every } \mu, \lambda \in k.
$$

$$(\mu + \lambda) \ast x \equiv \varphi (\mu + \lambda) x \left( \text{mod} \left( F^{(3)} (x) \right)^2 \right)$$

$$
(\mu * x) \perp (\lambda * x) \equiv (\varphi (\mu) + \varphi (\lambda)) x \left( \text{mod} \left( F^{(3)} (x) \right)^2 \right),
$$

so $\varphi (\mu + \lambda) = \varphi (\mu) + \varphi (\lambda)$. From $(\mu \lambda) * x = \mu * (\lambda * x)$ we conclude that $\varphi (\mu \lambda) = \varphi (\mu) \varphi (\lambda)$.

We consider $\varphi \in \text{Aut}_k$. By condition Op1) $F^{(3)} (x) \rightarrow \left( F^{(3)} (x) \right)^{*,w}$ is an isomorphism. So $\mu x = \lambda * x \perp \nu \ast (x \times x)$, where $\lambda, \nu \in k$.

As in the proof of the Proposition 5.1 we have that $w_1 (x_1, x_2) = \alpha_{1,2} x_1 x_2 + \alpha_{2,1} x_2 x_1$.

By Proposition 5.1 we have that $\mu x \equiv \lambda * x \equiv \varphi (\lambda) x \left( \text{mod} \left( F^{(3)} (x) \right)^2 \right)$, so $\mu = \varphi (\lambda)$. Therefore $\varphi \in \text{Aut}_k$.

We prove that if the system of words (5.1) fulfills conditions Op1) and Op2) then necessary

$$
w_{0,3} = 0, w_{+,3} (x_1, x_2) = x_1 + x_2 + \gamma_{1,2} x_1 x_2 + \gamma_{2,1} x_2 x_1,
$$

$$
w_{3,3} (x_1) = \varphi (\lambda) x_1 + \gamma_{1,2} \left( \left( \varphi (\lambda) \right)^2 - \varphi (\lambda) \right) x^2_1 (\lambda \in k), \quad (5.7)
$$

$$
w_{3,3} (x_1, x_2) = \alpha_{1,2} x_1 x_2 + \alpha_{2,1} x_2 x_1,
$$

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where \( \alpha_{1,2}, \alpha_{2,1}, \gamma_{1,2} \in k, \varphi \in \text{Aut}k \). If we will construct the isomorphism \( \sigma_F : F \to F_W^{\varphi} \) for every \( F \in \text{Ob}\Theta^0_3 \), then, as in the \( [9] \) we must demand \( \alpha_{1,2} \neq \pm \alpha_{2,1} \).

Now we will prove that if the system of words \( W \) has form \( (5.7) \) then this system of words fulfills conditions Op1) and Op2). First of all we must check \( \alpha \) where \( \alpha \in \Theta \).\( \varphi \) and \( \gamma \). If \( h \in H \) \( \in \Theta_3 \), \( \mu, \lambda \in k \) then

\[
(\mu \lambda) \ast h = \varphi (\mu \lambda) h + \gamma_{1,2} \left( (\varphi (\mu \lambda))^2 - \varphi (\mu \lambda) \right) h^2,
\]

\[
\mu \ast (\lambda \ast h) = \mu \ast \left( \varphi (\lambda) h + \gamma_{1,2} \left( (\varphi (\lambda))^2 - \varphi (\lambda) \right) h^2 \right) = \varphi (\mu) \left( \varphi (\lambda) h + \gamma_{1,2} \left( (\varphi (\lambda))^2 - \varphi (\lambda) \right) h^2 \right) + \gamma_{1,2} \left( (\varphi (\mu))^2 - \varphi (\mu) \right) (\varphi (\lambda))^2 h^2 = \varphi (\mu) \varphi (\lambda) h + \gamma_{1,2} \left( (\varphi (\mu))^2 - \varphi (\mu) \right) (\varphi (\lambda))^2 h^2.
\]

If \( h_1, h_2 \in H \) \( \in \Theta_3 \) then \( \lambda \in k \)

\[
(\lambda \ast h_1) \times h_2 = \left( \varphi (\lambda) h_1 + \gamma_{1,2} \left( (\varphi (\lambda))^2 - \varphi (\lambda) \right) h_2^2 \right) \times h_2 = \alpha_{1,2} \varphi (\lambda) h_1 h_2 + \alpha_{2,1} \varphi (\lambda) h_2 h_1,
\]

\[
\lambda \ast (h_1 \times h_2) = \lambda \ast (h_1 h_2 + \alpha_{2,1} h_1 h_2),
\]

\[
\varphi (\lambda) (\alpha_{1,2} h_1 h_2 + \alpha_{2,1} h_1 h_2).
\]

By similar way we can prove that \( h_1 \times (\lambda \ast h_2) = \lambda \ast (h_1 \times h_2) \). Other axioms of the variety \( \Theta_3 \) fulfill in \( H_W^{\varphi} \) by the constructions of the words \( (5.7) \).

It means that if the system of words \( W \) has form \( (5.7) \) then for every \( F = F (X) \in \text{Ob}\Theta^0_3 \) exists a homomorphism \( \sigma_F : F \to F_W^{\varphi} \) such that \( \sigma_F |_X = id_X \). Our goal is to prove that these homomorphisms are isomorphisms. For this purpose we will research the superpositions of these homomorphisms. If the system of words \( W \) has form \( (5.7) \) then it depends on the parameters \( \varphi, \gamma_{1,2}, \alpha_{1,2} \) \( \text{and} \alpha_{2,1} \), where \( \varphi \in \text{Aut}k, \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1} \in k, \alpha_{1,2} \neq \pm \alpha_{2,1} \).

We will denote the homomorphism \( \sigma_F \) which corresponds to the system of words \( W \) with parameters \( \varphi, \gamma_{1,2}, \alpha_{1,2} \) \( \text{by} \) \( \sigma_F = \sigma_F (\varphi, \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1}) \), or for shortness, \( \sigma_F (\varphi, \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1}) \). It is clear that \( \sigma_F (id_k, 0, 1, 0) = id_F \) for every \( F \in \text{Ob}\Theta^0_3 \). We consider two system of words \( W^{(i)}, i = 1, 2 \). Both these systems have form \( (5.7) \) and defined the system of homomorphisms \( \{ \sigma_F (\varphi, \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1}) \mid F \in \text{Ob}\Theta^0_3 \}, i = 1, 2 \). We have in \( F (x) \in \text{Ob}\Theta^0_3 \) for every \( \lambda \in k \)

\[
\sigma \left( \varphi (2), \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1} \right) \sigma \left( \varphi (1), \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1} \right) (\lambda x) = \]
\[
\sigma\left(\varphi^{(2)}, \gamma^{(2)}_{1,2}, \alpha^{(2)}_{1,2}, \alpha^{(2)}_{2,1}\right) \left(\varphi^{(1)}(\lambda) x + \gamma^{(1)}_{1,2}\left(\left(\varphi^{(1)}(\lambda)\right)^2 - \varphi^{(1)}(\lambda)\right)x^2\right) = \\
\varphi^{(1)}(\lambda) \star x \perp_{(2)} \gamma^{(1)}_{1,2}\left(\left(\varphi^{(1)}(\lambda)\right)^2 - \varphi^{(1)}(\lambda)\right) \star_{(2)} x \times x = \\
\left(\varphi^{(2)}\left(\varphi^{(1)}(\lambda)\right)x + \gamma^{(2)}_{1,2}\left(\left(\varphi^{(2)}\left(\varphi^{(1)}(\lambda)\right)\right)^2 - \varphi^{(2)}\left(\varphi^{(1)}(\lambda)\right)\right)x^2\right) \perp_{(2)} = \\
\varphi^{(2)}\left(\gamma^{(1)}_{1,2}\left(\left(\varphi^{(1)}(\lambda)\right)^2 - \varphi^{(1)}(\lambda)\right)\right)\left(\alpha^{(2)}_{1,2} + \alpha^{(2)}_{2,1}\right)x^2 = \\
\left(\varphi^{(2)}\varphi^{(1)}(\lambda)\right)x + \\
\left(\gamma^{(2)}_{1,2} + \varphi^{(2)}\left(\gamma^{(1)}_{1,2}\left(\alpha^{(2)}_{1,2} + \alpha^{(2)}_{2,1}\right)\right)\left(\left(\varphi^{(2)}\varphi^{(1)}(\lambda)\right)^2 - \left(\varphi^{(2)}\varphi^{(1)}(\lambda)\right)x^2.\right)\right)\]

Also we have in \( F (x_1, x_2) \in \text{Ob} \Theta^3_{(2)}, \)

\[
\sigma\left(\varphi^{(2)}, \gamma^{(2)}_{1,2}, \alpha^{(2)}_{1,2}, \alpha^{(2)}_{2,1}\right) \sigma\left(\varphi^{(1)}, \gamma^{(1)}_{1,2}, \alpha^{(1)}_{1,2}, \alpha^{(1)}_{2,1}\right) (x_1 + x_2) = \\
\sigma\left(\varphi^{(2)}, \gamma^{(2)}_{1,2}, \alpha^{(2)}_{1,2}, \alpha^{(2)}_{2,1}\right) x_1 + x_2 + \gamma^{(1)}_{1,2}x_1x_2 + \gamma^{(1)}_{1,2}x_2x_1 = \\
x_1 + x_2 + \gamma^{(1)}_{1,2}\varphi^{(2)}\left(\gamma^{(1)}_{1,2}\left(\alpha^{(2)}_{1,2} + \alpha^{(2)}_{2,1}\right)\right) x_1x_2 + \\
\gamma^{(2)}_{1,2} + \varphi^{(2)}\left(\gamma^{(1)}_{1,2}\left(\alpha^{(2)}_{1,2} + \alpha^{(2)}_{2,1}\right)\right)x_2x_1. \\
\sigma\left(\varphi^{(2)}, \gamma^{(2)}_{1,2}, \alpha^{(2)}_{1,2}, \alpha^{(2)}_{2,1}\right) \sigma\left(\varphi^{(1)}, \gamma^{(1)}_{1,2}, \alpha^{(1)}_{1,2}, \alpha^{(1)}_{2,1}\right) (x_1x_2) = \\
\sigma\left(\varphi^{(2)}, \gamma^{(2)}_{1,2}, \alpha^{(2)}_{1,2}, \alpha^{(2)}_{2,1}\right) \left(\alpha^{(1)}_{1,2}x_1x_2 + \alpha^{(1)}_{2,1}x_2x_1\right) = \\
\alpha^{(1)}_{1,2}\varphi^{(2)}\left(\alpha^{(1)}_{1,2}\left(\alpha^{(2)}_{1,2} x_1 + \alpha^{(2)}_{2,1} x_2\right)\right) = \\
\left(\varphi^{(2)}\left(\alpha^{(1)}_{1,2}\right)\left(\alpha^{(2)}_{1,2}x_1x_2 + \alpha^{(2)}_{2,1}x_2x_1\right) + \varphi^{(2)}\left(\alpha^{(2)}_{1,2}\right)x_1x_2 + \varphi^{(2)}\left(\alpha^{(2)}_{2,1}\right)x_2x_1\right) = \\
\varphi^{(2)}\left(\alpha^{(1)}_{1,2}\right)\left(\alpha^{(2)}_{1,2}x_1x_2 + \alpha^{(2)}_{2,1}x_2x_1\right) + \varphi^{(2)}\left(\alpha^{(2)}_{1,2}\right)x_1x_2 + \varphi^{(2)}\left(\alpha^{(2)}_{2,1}\right)x_2x_1.
\]

Therefore
\[
\sigma\left(\varphi^{(2)}, \gamma^{(2)}_{1,2}, \alpha^{(2)}_{1,2}, \alpha^{(2)}_{2,1}\right) \sigma\left(\varphi^{(1)}, \gamma^{(1)}_{1,2}, \alpha^{(1)}_{1,2}, \alpha^{(1)}_{2,1}\right) = \sigma\left(\varphi^{(3)}, \gamma^{(3)}_{1,2}, \alpha^{(3)}_{1,2}, \alpha^{(3)}_{2,1}\right) .
\]
where
\[ \varphi^{(3)} = \varphi^{(2)} \varphi^{(1)}, \gamma_{1,2}^{(3)} = \gamma_{1,2}^{(2)} + \varphi^{(2)} \left( \gamma_{1,2}^{(1)} \right) (\alpha_{1,2}^{(2)} + \alpha_{2,1}^{(2)}) \]
\[ \alpha_{1,2}^{(3)} = \varphi^{(2)} \left( \alpha_{1,2}^{(1)} \right) (\alpha_{1,2}^{(2)} + \varphi^{(2)} \left( \gamma_{2,1}^{(1)} \right) \alpha_{2,1}^{(2)}), \quad (5.8) \]
\[ \alpha_{2,1}^{(3)} = \varphi^{(2)} \left( \alpha_{2,1}^{(1)} \right) (\alpha_{2,1}^{(2)} + \varphi^{(2)} \left( \gamma_{1,2}^{(1)} \right) \alpha_{1,2}^{(2)}). \]

Hence we have a decomposition
\[ \sigma \left( \varphi, \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1} \right) = \sigma \left( \text{id}_k, \gamma_{1,2}, 1, 0 \right) \sigma \left( \varphi, 0, \alpha_{1,2}, \alpha_{2,1} \right). \quad (5.9) \]

Also we have for every \( F \in \text{Ob} \Theta_3^0 \) by (5.8)
\[ \sigma_F \left( \text{id}_k, -\gamma_{1,2}, 1, 0 \right) = \sigma_F \left( \text{id}_k, \gamma_{1,2}, 1, 0 \right)^{-1}, \]
\[ \sigma_F \left( \varphi^{-1}, 0, \beta_{1,2}, \beta_{2,1} \right) = \sigma_F \left( \varphi, 0, \alpha_{1,2}, \alpha_{2,1} \right)^{-1}, \]
where
\[ \left( \begin{array}{cc} \beta_{1,2} & \beta_{2,1} \\ \beta_{1,2} & \beta_{2,1} \end{array} \right) = \left( \begin{array}{cc} \varphi^{-1} (\alpha_{1,2}) & \varphi^{-1} (\alpha_{2,1}) \\ \varphi^{-1} (\alpha_{2,1}) & \varphi^{-1} (\alpha_{1,2}) \end{array} \right)^{-1}. \]

Therefore all homomorphisms \( \sigma_F \left( \varphi, \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1} \right) \) for every \( F \in \text{Ob} \Theta_3^0 \), every \( \varphi \in \text{Aut} k \) and \( \gamma_{1,2}, \alpha_{1,2}, \alpha_{2,1} \in k \) such that \( \alpha_{1,2} \neq \pm \alpha_{2,1} \) are isomorphisms. So all systems of words \( W \) which have form (5.7) fulfill conditions Op1 and Op2) and all automorphisms of \( \Theta_3 \) which corresponds to these systems of words are strongly stable.

Now we must calculate the group \( \mathfrak{G}_3 \cap \mathfrak{S}_3 \). If \( \Phi \in \mathfrak{S}_3 \) then the system of words \( W \) which corresponds to \( \Phi \) fulfills conditions Op1 and Op2. From Proposition 5.1 we can conclude that \( \sigma_F \left( F^i \right) = F^i \) for every \( F \in \text{Ob} \Theta_3^0 \) and \( i = 1, 2 \). Therefore as in the Lemma 4.1 and Proposition 4.1 we can prove that if \( \Phi \) is inner then \( \alpha_{2,1} = 0, \varphi = \text{id}_k \). Now we will prove that this conditions are sufficient. We consider \( \Phi \in \mathfrak{S}_3 \) which corresponds to the system of words \( W \) which has form (5.7) such that \( \alpha_{2,1} = 0 \) and \( \varphi = \text{id}_k, \alpha_{1,2} \neq \pm \alpha_{2,1} \), so \( \alpha_{1,2} \neq 0 \). For every \( F \in \text{Ob} \Theta_3^0 \) we define the mapping \( c_F : F \to F \) by this formula:
\[ c_F (f) = \alpha_{1,2}^{-1} f + \alpha_{1,2}^{-2} \gamma_{1,2} f^2, \]
where \( f \in F \). We will prove by direct calculation that \( c_F : F \to F_W \) is an isomorphism. For every \( f \in F \) and \( \lambda \in k \) we have
\[ c_F (\lambda f) = \alpha_{1,2}^{-1} \lambda f + \alpha_{1,2}^{-2} \gamma_{1,2} \lambda^2 f^2, \]
\[ \lambda * c_F (f) = \lambda \left( \alpha_{1,2}^{-1} f + \alpha_{1,2}^{-2} \gamma_{1,2} f^2 \right) + \gamma_{1,2} \alpha_{1,2}^{-2} (\lambda^2 - \lambda) f^2 = c_F (\lambda f). \]
For every \( f_1, f_2 \in F \) we have
\[ c_F (f_1 + f_2) = \alpha_{1,2}^{-1} (f_1 + f_2) + \alpha_{1,2}^{-2} \gamma_{1,2} (f_1 + f_2)^2 = \]
\[ \alpha_{1,2}^{-1} f_1 + \alpha_{1,2}^{-1} f_2 + \alpha_{1,2}^{-2} \gamma_{1,2} f_1 f_2 + \alpha_{1,2}^{-2} \gamma_{1,2} f_1 f_2 + \alpha_{1,2}^{-2} \gamma_{1,2} f_1 f_2 + \alpha_{1,2}^{-2} \gamma_{1,2} f_1 f_2. \]
Therefore we have

\[
\begin{align*}
&\text{Corollary 1 from Proposition 5.2 we have} \\
&\quad \left(\text{conclude that } W^* \left(\sum_{i=1}^{\lambda} \left(\sum_{j=1}^{\kappa} F_{i,j,i,j} x_i x_j + \sum_{j=1}^{\lambda} F_{i,j,i,j} x_i x_j\right)\right)\right) F_{i,j,i,j} F_{i,j,i,j} = 0 \\
&\quad \text{we have for every } F \in \text{Ob}\Theta_3^0 \text{ that } \operatorname{sp}\{c_F(x) + (F_W^*)^2 \mid x \in X\} = F_W^*/(F_W^*)^2. \quad \text{By Proposition 4.1} \\
&\quad F_W^* \in \Theta_3 \text{. So } \left\{c_F(x) + (F_W^*)^2 \mid x \in X\right\}_{\text{alg}} = F_W^* \text{ and } c_F \text{ is an epimorphism.} \\
&\quad \text{By the dimensional reason } c_F \text{ is an isomorphism. It is clear that } c_F \psi = \psi c_D \text{ fulfills for every } F, D \in \text{Ob}\Theta_3^0 \text{ and every } (\psi : D \to F) \in \text{Mor}\Theta_3^0. \quad \text{Therefore } \Phi \text{ is inner.} \\
&\quad \text{After this by using of the } (5.9) \text{ we prove as in the } [9] \text{ that} \\
&\quad \mathfrak{A}_3/\mathfrak{Y}_3 \approx (U(kS_2)/U(k\{e\})) \times \text{Aut} k \approx k^* \times \text{Aut} k. \\
\end{align*}
\]

\section*{5.3 Group $\mathfrak{A}_4/\mathfrak{Y}_4$.}

\textbf{Theorem 5.3 $\mathfrak{A}_4/\mathfrak{Y}_4 \approx k^* \times \text{Aut} k$.}

\textbf{Proof.} We assume that the system of words $W$ fulfills conditions Op1) and Op2) in the variety $\Theta_4$. We will find the specific form of these words. By Corollary [3] from Proposition 5.2 we have $w_{+,4} = w_{+,3} + r_{+,4}$, where $w_{+,4} \in F(4)(x_1, x_2)$, $w_{+,3} = x_1 + x_2 + \gamma_{1,2} x_1 x_2 + \gamma_{1,2} x_2 x_1 \in \sum_{i=1}^{2} F(4)(x_1, x_2)_{i}$, $r_{+,4} \in (F(4)(x_1, x_2))_3$. We denote $r_{+,4} = \sum_{i_1,i_2,i_3=1}^2 \gamma_{i_1,i_2,i_3} (x_{i_1,i_2}) x_{i_3} + \sum_{i_1,i_2,i_3=1}^2 \gamma_{i_1,i_2,i_3} (x_{i_1,i_2}) x_{i_3}$, where $\gamma_{i_1,i_2,i_3}, \gamma_{i_1,i_2,i_3} \in k$. From $w_{+,4}(x_1, 0) = x_1$ and $w_{+,4}(0, x_2) = x_2$ we conclude that

\[r_{+,4}(x_1, 0) = r_{+,4}(0, x_2) = 0.\]  

\[\text{(5.10)}\]

From

\[w_{+,4}(x_1, x_2) = w_{+,4}(x_2, x_1)\]

\[\text{(5.11)}\]

we conclude that $\gamma_{(1,1,2)} = \gamma_{(2,2,1)}$ and so on.

\[w_{+,4}(x_1, w_{+,4}(x_2, x_3)) = w_{+,4}(w_{+,4}(x_1, x_2), x_3)\]

\[\text{(5.12)}\]

must fulfills in $F(4)(x_1, x_2, x_3)$. We can write the left side of this equation as

\[w_{+,3}(x_1, w_{+,4}(x_2, x_3)) + r_{+,4}(x_1, w_{+,4}(x_2, x_3)).\]

By nilpotence $r_{+,4}(x_1, w_{+,4}(x_2, x_3)) = r_{+,4}(x_1, x_2 + x_3)$. The terms of degree 3 of the $w_{+,3}(x_1, w_{+,4}(x_2, x_3))$ are equal

\[r_{+,4}(x_2, x_3) + \gamma_{1,2}^2 x_1 (x_2 x_3 + x_3 x_2) + \gamma_{1,2}^2 (x_2 x_3 + x_3 x_2) x_1.\]
Reciprocally the terms of degree 3 of the right side of equation (5.12) are equal
\[ r_{+,4} (x_1, x_2) + \gamma_{1,2}^2 (x_1 x_2 + x_2 x_1) x_3 + \gamma_{2,2}^2 (x_1 x_2 + x_2 x_1) + r_{+,4} (x_1 + x_2, x_3). \]
Therefore from (5.12) we can conclude
\[ \gamma_{1,2}^2 x_1 (x_2 x_3 + x_3 x_2) + \gamma_{1,2}^2 (x_2 x_3 + x_3 x_2) x_1 + \]
\[ (r_{+,4} (x_1, x_2 + x_3) - r_{+,4} (x_1, x_2) - r_{+,4} (x_1, x_3)) = \]
\[ \gamma_{1,2}^2 (x_1 x_2 + x_2 x_1) x_3 + \gamma_{2,2}^2 x_3 (x_1 x_2 + x_2 x_1) + \]
\[ (r_{+,4} (x_1 + x_2, x_3) - r_{+,4} (x_1, x_3) - r_{+,4} (x_2, x_3)). \]
By (5.10) all terms of (5.13) have entries of \( x_1, x_2 \) and \( x_3 \). If we compare the coefficients of \( (x_1 x_2) x_3 \) in (5.13), then we achieve
\[ \gamma_{(1,2)2} = \gamma_{1,2}^2 + \gamma_{(1,1)2}. \]
From comparison of the coefficients of \( (x_2 x_1) x_3 \) we achieve by using of (5.11)
\[ \gamma_{(1,2)1} = \gamma_{1,2}^2 + \gamma_{(1,1)2}. \]
By consideration of the coefficients of \( (x_1 x_3) x_2 \) and of the coefficients of \( (x_3 x_1) x_2 \) we conclude \( \gamma_{(1,2)2} = \gamma_{(1,2)1} \), but this is a corollary of (5.14) and (5.15). If we compare the coefficients of \( (x_2 x_3) x_1 \) we conclude the equation (5.14), and if we compare the coefficients of \( (x_3 x_2) x_1 \) we conclude the equation (5.14). From comparison of the coefficients of \( x_1 (x_2 x_3) \) we achieve
\[ \gamma_{1,2}^2 + \gamma_{1,(2,2)} = \gamma_{1,(2,1)}. \]
By consideration of the coefficients of \( x_1 (x_3 x_2) \) we achieve
\[ \gamma_{1,2}^2 + \gamma_{1,(2,2)} = \gamma_{1,(2,1)}. \]
If we consider the coefficients of \( x_2 (x_1 x_3) \) or the coefficients of \( x_2 (x_3 x_1) \), then we have \( \gamma_{1,(2,1)} = \gamma_{1,(2,1)} \), but this is a corollary of (5.16) and (5.17). If we compare the coefficients of \( x_3 (x_1 x_2) \), then we achieve the equation (5.16), and if we compare the coefficients of \( x_3 (x_2 x_1) \), then we achieve the equation (5.16). By (5.14) - (5.17) we have that
\[ r_{+,4} (x_1, x_2) = \gamma_{(1,1)2} x_1^2 x_2 + \left( \gamma_{1,2}^2 + \gamma_{(1,1)2} \right) (x_1 x_2) x_2 + \left( \gamma_{1,2}^2 + \gamma_{(1,1)2} \right) (x_1 x_2) x_1 + \]
\[ \gamma_{1,(1,2)2} x_1^2 x_2 + \left( \gamma_{1,2}^2 + \gamma_{(1,1)2} \right) (x_2 x_1) x_1 + \left( \gamma_{1,2}^2 + \gamma_{(1,1)2} \right) (x_2 x_1) x_2 + \]
\[ \gamma_{1,(2,2)} x_1 x_2^2 + \left( \gamma_{1,2}^2 + \gamma_{(1,2,2)} \right) x_1 (x_1 x_2) + \left( \gamma_{1,2}^2 + \gamma_{(1,2,2)} \right) x_1 (x_2 x_1) + \]
\[ \gamma_{1,(2,2)} x_2 x_1^2 + \left( \gamma_{1,2}^2 + \gamma_{(1,2,2)} \right) x_2 (x_2 x_1) + \left( \gamma_{1,2}^2 + \gamma_{(1,2,2)} \right) x_2 (x_1 x_2). \]
By Corollary \(|1|\) from Proposition \(|5.2|\) we have \(w_{.,4} = w_{.,3} + r_{.,4}\), where
\[w_{.,4} \in F^4(\alpha, x_2), \quad w_{.,3} = \alpha_{1,2} x_2 x_1 + \alpha_{2,1} x_2 x_1 \in \bigoplus_i (F^4(\alpha, x_2)), \quad r_{.,4} \in (F^4(\alpha, x_2))_4.\]
We denote \(r_{.,4} = \sum_{i_1, i_2, i_3=1}^2 \alpha_{(i_1, i_2), i_3} (x_{i_1} x_{i_2}) x_{i_3} + \sum_{i_1, i_2, i_3=1}^2 \alpha_{(i_1, i_2), i_3} (x_{i_2} x_{i_3}),\)
where \(\alpha_{(i_1, i_2), i_3}, \alpha_{i_1, (i_2, i_3)} \in k\). From \(w_{.,4}(x_1, 0) = w_{.,4}(0, x_2) = 0\) we conclude that
\[r_{.,4}(x_1, 0) = r_{.,4}(0, x_2) = 0.\]
(5.19)
\[w_{.,4}(w_{+, 4}(x_1, x_2), x_3) = w_{+, 4}(w_{., 4}(x_1, x_3), w_{., 4}(x_2, x_3))\]
(5.20)
must fulfills in \(F^4(x_1, x_2, x_3)\) by Op2. By nilpotence
\[w_{+, 4}(w_{., 4}(x_1, x_3), w_{., 4}(x_2, x_3)) = w_{., 4}(x_1, x_3) + w_{., 4}(x_2, x_3).\]

If we compare the terms of degree 3 of the equation (5.20), then we have
\[w_{., 3}(\gamma_{1,2} x_1 x_2 + \gamma_{1,2} x_2 x_1, x_3) + r_{., 4}(x_1 + x_2, x_3) = r_{., 4}(x_1, x_3) + r_{., 4}(x_2, x_3)\]
or
\[\alpha_{1,2} \gamma_{1,2} (x_1 x_2) x_3 + \alpha_{1,2} \gamma_{1,2} (x_2 x_1) x_3 + \alpha_{2,1} \gamma_{1,2} x_3 (x_1 x_2) + \alpha_{2,1} \gamma_{1,2} x_1 (x_2 x_3) + \]
\[(r_{., 4}(x_1 + x_2, x_3) - r_{., 4}(x_1, x_3) - r_{., 4}(x_2, x_3)) = 0.\]
(5.21)

By \(|5.10|\) all terms of \(|5.13|\) have entries of \(x_1, x_2\) and \(x_3\). If we compare the coefficients of \((x_1 x_2) x_3\) we achieve
\[\alpha_{1,2} \gamma_{1,2} + \alpha_{(1,1)2} = 0.\]
(5.22)

Other comparisons in the order, which was described above, give us
\[\alpha_{(1,2)1} = 0,\]
(5.23)
\[\alpha_{(2,1)1} = 0,\]
(5.24)
\[\alpha_{1(1,2)} = 0,\]
(5.25)
\[\alpha_{1(2,1)} = 0,\]
(5.26)
\[\alpha_{2(1,1)} = 0.\]
(5.27)

Here we omit the repeated equations.

Also
\[w_{., 4}(w_{+, 4}(x_1, x_2), w_{., 4}(x_1, x_3)) = w_{+, 4}(w_{., 4}(x_1, x_2), w_{., 4}(x_1, x_3))\]
(5.28)
must fulfills in \(F^3_3\). As above we conclude from this equation
\[\alpha_{1,2} \gamma_{1,2} x_1 (x_2 x_3) + \alpha_{1,2} \gamma_{1,2} x_1 (x_3 x_2) + \alpha_{2,1} \gamma_{1,2} x_1 (x_2 x_3) x_1 + \alpha_{2,1} \gamma_{1,2} x_1 (x_3 x_2) x_1 = \]
\[(r_{., 4}(x_1, x_2 + x_3) - r_{., 4}(x_1, x_2) - r_{., 4}(x_1, x_3)) = 0.\]
(5.29)
After this, we have by comparisons of the coefficients of suitable monomials
\[\alpha_{(1,2)2} = 0,\]  
\[\alpha_{(2,1)2} = 0,\]  
\[\alpha_{2,1}\gamma_{1,2} + \alpha_{(2,2)1} = 0,\]  
\[\alpha_{1,2}\gamma_{1,2} + \alpha_{1(2,2)} = 0,\]  
\[\alpha_{2(1,2)} = 0,\]  
\[\alpha_{2(2,1)} = 0.\]  

Therefore
\[r_{a}(x_1, x_2) = -\alpha_{1,2}\gamma_{1,2}x_1^2x_2 - \alpha_{2,1}\gamma_{1,2}x_2^2x_1 - \alpha_{2,1}\gamma_{1,2}x_1x_2^2 - \alpha_{1,2}\gamma_{1,2}x_1x_2^2.\]  

Also by Corollary 1 from Proposition 5.2 we have
\[w_{\lambda,4}(x) = w_{\lambda,3}(x) + r_{\lambda,4}(x),\]  
where \(w_{\lambda,3}(x) = \varphi(\lambda)x + \gamma_{1,2}\left((\varphi(\lambda))^2 - \varphi(\lambda)\right)x^2, r_{\lambda,4}(x) = \psi_1(\lambda)x(x^2) + \psi_2(\lambda)(x^2)x.\]  

By Op2
\[w_{\lambda,4}(w_{+,4}(x_1, x_2)) = w_{+,4}(w_{\lambda,4}(x_1), w_{\lambda,4}(x_2))\]

must fulfills in \(F^{(4)}(x_1, x_2).\) If we compare the terms of degree 3 of this equation, we achieve
\[\gamma_{1,2}^2\left((\varphi(\lambda))^2 - \varphi(\lambda)\right)(x_1(x_1x_2) + x_1(x_2x_1) + x_2(x_1x_2) + x_2(x_2x_1) + (x_1x_2)x_1 + (x_1x_2)x_2 + (x_2x_1)x_1 + (x_2x_1)x_2)\]
\[\left((\varphi(\lambda))^3 - \varphi(\lambda)\right)r_{+,4}(x_1, x_2) + \gamma_{1,2}^2\varphi(\lambda)\left((\varphi(\lambda))^2 - \varphi(\lambda)\right)(x_1x_2^2 + x_1^2x_2 + x_2x_1^2 + x_2^2x_1).\]

It is clear that in \(r_{\lambda,4}(x_1 + x_2) - r_{\lambda,4}(x_1) - r_{\lambda,4}(x_2)\) there are no terms in which there are only entries of \(x_1\) or only entries of \(x_2.\) If we compare coefficients of \(x_1^2x_2\) in the equation \(5.38\) we achieve
\[\psi_2(\lambda) = \gamma_{1,2}^2\varphi(\lambda)\left((\varphi(\lambda))^2 - \varphi(\lambda)\right) + \gamma_{(1,1)2}\left((\varphi(\lambda))^3 - \varphi(\lambda)\right).\]  

if we compare coefficients of \(x_1x_2^2\) in the equation \(5.38\) we achieve
\[\psi_1(\lambda) = \gamma_{1,2}^2\varphi(\lambda)\left((\varphi(\lambda))^2 - \varphi(\lambda)\right) + \gamma_{1(2,2)}\left((\varphi(\lambda))^3 - \varphi(\lambda)\right).\]  

Other comparisons give us only repetitions of the equations \(5.39\) and \(5.40.\)

So
\[r_{\lambda,4}(x) = \left(\gamma_{1,2}^2\left((\varphi(\lambda))^3 - \varphi(\lambda)^2\right) + \gamma_{1(2,2)}\left((\varphi(\lambda))^3 - \varphi(\lambda)\right)\right)x(x^2) + \gamma_{1,2}\varphi(\lambda)x^2.\]

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\[ \left( \gamma_{1,2} \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) + \gamma_{1,1;2} \left( \varphi(\lambda)^3 - \varphi(\lambda) \right) \left( x^2 \right) x. \quad (5.41) \]

Now we will prove that if the system of words \( W \) defined by formulas \((5.7), (5.13), (5.14) \) and \((5.26)\) then this system of words fulfills conditions Op1 and Op2). First of all we must check that if \( H \in \Theta_3 \) then \( H_W \in \Theta_4 \). We can check it by direct calculation. For example, if \( h \in H \in \Theta_3, \mu, \lambda \in k \) then

\[
(\mu \lambda) h = \varphi(\mu \lambda) h + \gamma_{1,2} \left( \varphi(\mu \lambda)^2 - \varphi(\mu \lambda) \right) h^2 +
\]

\[
\left( \gamma_{1,2} \left( \varphi(\mu \lambda)^3 - \varphi(\mu \lambda)^2 \right) + \gamma_{1,1,2} \left( \varphi(\mu \lambda)^3 - \varphi(\mu \lambda) \right) \right) h (h^2) +
\]

\[
\left( \gamma_{1,2} \left( \varphi(\mu \lambda)^3 - \varphi(\mu \lambda)^2 \right) + \gamma_{1,1,2} \left( \varphi(\mu \lambda)^3 - \varphi(\mu \lambda) \right) \right) (h^2) h =
\]

\[
\varphi(\mu) \varphi(\lambda) h + \gamma_{1,2} \varphi(\mu) \left( \varphi(\lambda)^2 - \varphi(\lambda) \right) h^2 +
\]

\[
\left( \gamma_{1,2} \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) + \gamma_{1,1,2} \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda) \right) \right) h (h^2) +
\]

\[
\left( \gamma_{1,2} \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) + \gamma_{1,1,2} \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda) \right) \right) (h^2) h +
\]

\[
\gamma_{1,2} \varphi(\mu) \left( \varphi(\lambda)^2 - \varphi(\mu) \right) h^2 +
\]

\[
\gamma_{1,2} \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) h (h^2) + \gamma_{1,2} \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) (h^2) h +
\]

\[
\left( \gamma_{1,2} \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) + \gamma_{1,1,2} \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda) \right) \right) \varphi(\lambda)^3 h (h^2) +
\]

\[
\left( \gamma_{1,2} \left( \varphi(\mu)^3 - \varphi(\mu)^2 \right) + \gamma_{1,1,2} \left( \varphi(\mu)^3 - \varphi(\mu) \right) \right) \varphi(\lambda)^3 (h^2) h =
\]

\[
\varphi(\mu) h + \gamma_{1,2} \left( \varphi(\mu) \left( \varphi(\lambda)^2 - \varphi(\lambda) \right) + \left( \varphi(\mu)^2 - \varphi(\mu) \right) \varphi(\lambda)^2 \right) h^2 +
\]

\[
\gamma_{1,2} \left( \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) + \left( \varphi(\mu)^2 - \varphi(\mu) \right) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) +
\]

\[
\left( \varphi(\mu)^3 - \varphi(\mu)^2 \right) \varphi(\lambda)^3 +
\]

\[
\gamma_{1,1,2} \left( \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda) \right) + \left( \varphi(\mu)^3 - \varphi(\mu) \right) \varphi(\lambda)^3 \right) h (h^2) +
\]

\[
\gamma_{1,2} \left( \varphi(\mu) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) + \left( \varphi(\mu)^2 - \varphi(\mu) \right) \left( \varphi(\lambda)^3 - \varphi(\lambda)^2 \right) +
\]

\[
\left( \varphi(\mu)^3 - \varphi(\mu)^2 \right) \varphi(\lambda)^3 +\]
Also by direct calculation we can check that for every homomorphisms are isomorphisms. For this purpose we will research the superficial cases of superpositions. If the system of words the superposition, similar to the formula (5.8). Hear we will consider only superpositions of these homomorphisms. We will not lead out the general formula of words of the system every λ depend on parameters γ 4, 2, 2, 1. So we can conclude that for every h1, h2 ∈ H ∈ Θ3, and every λ ∈ k the
\[
\lambda \ast (h_1 \times h_2) = (\lambda \ast h_1) \times h_2 = h_1 \times (\lambda \ast h_2)
\]
holds. Other axioms of the variety Θ4 fulfill in H_W by the constructions of the words of the system W.

So we can conclude that for every F = F(X) ∈ ObΘk there exists a homomorphism σ_F : F → F_W such that σ_F |X = id_X. Our goal is to prove that these homomorphisms are isomorphisms. For this purpose we will research the superpositions of these homomorphisms. We will not lead out the general formula of the superposition, similar to the formula (5.8). Hear we will consider only special cases of superpositions. If the system of words W defined by formulas (5.7), (5.18), (5.41) and (5.36) then its words and homomorphisms F, σ depend on parameters ϕ ∈ Autk and γ1,2, γ1(2,2), γ(1,1)2, α1,2, α2,1 ∈ k such that α1,2 ≠ ±α2,1. So we denote
\[
\sigma_F = \sigma_F (\varphi, \gamma_{1,2}, \gamma_{1(2,2)}, \gamma_{(1,1)2}, \alpha_{1,2}, \alpha_{2,1})
\]
or
\[
\sigma_F = \sigma (\varphi, \gamma_{1,2}, \gamma_{1(2,2)}, \gamma_{(1,1)2}, \alpha_{1,2}, \alpha_{2,1})
\]
Sometimes we will give the indexes to these parameters. We have that
\[
\sigma (id_k, \gamma_{1,2}, \gamma_{1(2,2)}, \gamma_{(1,1)2}, 1, 0) \sigma (\varphi, 0, 0, 0, \alpha_{1,2}, \alpha_{2,1}) (x_1 + x_2) =
\]
\[
\sigma (id_k, \gamma_{1,2}, \gamma_{1(2,2)}, \gamma_{(1,1)2}, 1, 0) (x_1 + x_2) =
\]
x_1 + x_2 + γ_{1,2}x_1x_2 + γ_{1,2}x_2x_1 +
\[
(\gamma_{1,2} + \gamma_{1(2,2)}) x_1 (x_1x_2) + (\gamma_{1,2} + \gamma_{1(2,2)}) x_1 (x_2x_1) +
\]
\[
\gamma_{1(2,2)}x_1 (x_2^2) + \gamma_{1(2,2)}x_2 (x_2^2) +
\]
\[
(\gamma_{1,2} + \gamma_{1(2,2)}) x_2 (x_1x_2) + (\gamma_{1,2} + \gamma_{1(2,2)}) x_2 (x_2x_1) +
\]
\[
(\gamma_{1,2} + \gamma_{(1,1)2}) (x_1x_2) x_1 + (\gamma_{1,2} + \gamma_{(1,1)2}) (x_2x_1) x_1 +
\]
\[
(\gamma_{1,2} + \gamma_{(1,1)2}) (x_1x_2) x_2 + (\gamma_{1,2} + \gamma_{(1,1)2}) (x_2x_1) x_2 +
\]
\[
(\gamma_{1,2} + \gamma_{(1,1)2}) (x_1x_2) x_1 + (\gamma_{1,2} + \gamma_{(1,1)2}) (x_2x_1) x_2 +
\]
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Therefore we have the decomposition

$$\gamma_{(1,1)2}(x_2^2)x_1 + \gamma_{(1,1)2}(x_1^2)x_2 + \left(\gamma_{1,2}^2 + \gamma_{(1,1)2}\right)(x_1x_2)x_2 + \left(\gamma_{1,2}^2 + \gamma_{(1,1)2}\right)(x_2x_1)x_2.$$

$$\sigma \left(id_k, \gamma_{1,2}, \gamma_{(1,2)}, \gamma_{(1,1)2}, 1, 0\right) \sigma \left(\varphi, 0, 0, 0, \alpha_{1,2}, \alpha_{2,1}\right)(x_1x_2) =$$

$$\sigma \left(id_k, \gamma_{1,2}, \gamma_{12}, \gamma_{(1,1)2}, 1, 0\right)(\alpha_{1,2}x_1x_2 + \alpha_{2,1}x_2x_1) =$$

$$\alpha_{1,2} \ast (x_1 \times x_2) \perp \alpha_{2,1} \ast (x_2 \times x_1),$$

where operations $\perp, \times$ and $\ast$ defined by the system of words which depend on parameters $id_k, \gamma_{1,2}, \gamma_{12}, \gamma_{(1,1)2}, 1, 0$. So

$$\alpha_{1,2} \ast (x_1 \times x_2) = \alpha_{1,2}x_1x_2 - \alpha_{1,2}\gamma_{1,2}x_1 \left(x_2^2\right) - \alpha_{1,2}\gamma_{1,2} \left(x_1^2\right)x_2$$

and

$$\sigma \left(id_k, \gamma_{1,2}, \gamma_{12}, \gamma_{(1,1)2}, 1, 0\right) \sigma \left(\varphi, 0, 0, 0, \alpha_{1,2}, \alpha_{2,1}\right)(x_1x_2) =$$

$$\alpha_{1,2}x_1x_2 - \alpha_{1,2}\gamma_{1,2}x_1 \left(x_2^2\right) - \alpha_{1,2}\gamma_{1,2} \left(x_1^2\right)x_2 +$$

$$\alpha_{2,1}x_2x_1 - \alpha_{2,1}\gamma_{1,2}x_2 \left(x_2^2\right) - \alpha_{2,1}\gamma_{1,2} \left(x_1^2\right)x_1.$$
\[
\begin{align*}
&\left(id_k, \gamma_{1,2}^{(1)}, \gamma_{1(2),2}^{(1)}, \gamma_{(1,2),2}^{(1)}, 1, 0 \right) (x_1 + x_2 + \gamma_{1,2}^{(1)}x_1x_2 + \gamma_{1,2}^{(1)}x_1x_2 + \\
&\left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) x_1 (x_1x_2) + \left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) x_1 (x_2x_1) + \\
&\gamma_{1(2),2}^{(1)} x_1 (x_2^2) + \gamma_{1(2),2}^{(1)} x_2 (x_1^2) + \\
&\left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) x_2 (x_1x_2) + \left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) x_2 (x_2x_1) + \\
&\left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) (x_1x_2) x_1 + \left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) (x_2x_1) x_1 + \\
&\gamma_{1(2),2}^{(1)} (x_2^2) x_1 + \gamma_{1(2),2}^{(1)} (x_1^2) x_2 + \\
&\left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) (x_1x_2) x_2 + \left(\gamma_{1,2}^{(1)} + \gamma_{1(2),2}^{(1)}\right) (x_2x_1) x_2.
\end{align*}
\]

So

\[
\sigma \left(id_k, \gamma_{1,2}^{(1)}, \gamma_{1,2}^{(1)}, \gamma_{1(2),2}^{(1)}, 1, 0 \right) \sigma \left(id_k, \gamma_{1,2}^{(2)}, \gamma_{1(2),2}^{(2)}, \gamma_{(1,2),2}^{(2)}, 1, 0 \right) (x_1 + x_2) = \\
(x_1 + x_2 + \gamma_{1,2}^{(2)}x_1x_2 + \gamma_{1,2}^{(2)}x_1x_2 + \\
\left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) x_1 (x_1x_2) + \left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) x_1 (x_2x_1) + \\
\gamma_{1(2),2}^{(2)} x_1 (x_2^2) + \gamma_{1(2),2}^{(2)} x_2 (x_1^2) + \\
\left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) x_2 (x_1x_2) + \left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) x_2 (x_2x_1) + \\
\left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) (x_1x_2) x_1 + \left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) (x_2x_1) x_1 + \\
\gamma_{1(2),2}^{(2)} (x_2^2) x_1 + \gamma_{1(2),2}^{(2)} (x_1^2) x_2 + \\
\left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) (x_1x_2) x_2 + \left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) (x_2x_1) x_2 + \\
\left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) (x_1x_2) x_2 + \left(\gamma_{1,2}^{(2)} + \gamma_{1(2),2}^{(2)}\right) (x_2x_1) x_2 = 25.
\]
\[
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_1 (x_1 x_2) + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_1 (x_2 x_1) + \\
\gamma_{1(2,2)}^{(1)} x_1 x_2^2 + \gamma_{1(2,2)}^{(1)} x_2 x_1^2 + \\
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_2 (x_1 x_2) + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_2 (x_2 x_1) + \\
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{(1,1)2}^{(1)} \right) (x_1 x_2) x_1 + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{(1,1)2}^{(1)} \right) (x_2 x_1) x_1 + \\
\gamma_{(1,1)2}^{(1)} x_2 x_1 + \gamma_{(1,1)2}^{(1)} x_1 x_2 + \\
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{(1,1)2}^{(1)} \right) (x_1 x_2) x_2 + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{(1,1)2}^{(1)} \right) (x_2 x_1) x_2 = \\
x_1 + x_2 + \gamma_{1(2,2)}^{(2)} x_1 x_2 + \gamma_{1(2,2)}^{(2)} x_2 x_1 + \\
\left( (\gamma_{1,2}^{(2)})^2 + \gamma_{1(2,2)}^{(2)} \right) x_1 (x_1 x_2) + \left( (\gamma_{1,2}^{(2)})^2 + \gamma_{1(2,2)}^{(2)} \right) x_1 (x_2 x_1) + \\
\gamma_{1(2,2)}^{(2)} x_1 (x_2^2) + \gamma_{1(2,2)}^{(2)} x_2 (x_1^2) + \\
\left( (\gamma_{1,2}^{(2)})^2 + \gamma_{1(2,2)}^{(2)} \right) x_2 (x_1 x_2) + \left( (\gamma_{1,2}^{(2)})^2 + \gamma_{1(2,2)}^{(2)} \right) x_2 (x_2 x_1) + \\
\left( (\gamma_{1,2}^{(2)})^2 + \gamma_{(1,1)2}^{(2)} \right) (x_1 x_2) x_1 + \left( (\gamma_{1,2}^{(2)})^2 + \gamma_{(1,1)2}^{(2)} \right) (x_2 x_1) x_1 + \\
\gamma_{(1,1)2}^{(2)} (x_2 x_1) x_1 + \gamma_{(1,1)2}^{(2)} (x_1 x_2) x_2 + \\
\left( (\gamma_{1,2}^{(2)})^2 + \gamma_{(1,1)2}^{(2)} \right) (x_1 x_2) x_2 + \left( (\gamma_{1,2}^{(2)})^2 + \gamma_{(1,1)2}^{(2)} \right) (x_2 x_1) x_2 + \\
\gamma_{1(2,2)}^{(2)} x_1 x_2 - \gamma_{1,2}^{(2)} \gamma_{1(2,2)}^{(2)} x_1 (x_2) - \gamma_{1,2}^{(2)} \gamma_{1(2,2)}^{(2)} (x_2) x_1 + \\
\gamma_{1(2,2)}^{(2)} x_2 x_1 - \gamma_{1,2}^{(2)} \gamma_{1(2,2)}^{(2)} (x_1) - \gamma_{1,2}^{(2)} \gamma_{1(2,2)}^{(2)} (x_1) x_1 + \\
\gamma_{1(2,2)}^{(2)} (x_1 + x_2) (x_1 x_2 + x_2 x_1) + \gamma_{1(2,2)}^{(2)} (x_1 x_2 + x_2 x_1) (x_1 + x_2) + \\
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_1 (x_1 x_2) + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_1 (x_2 x_1) + \\
\gamma_{1(2,2)}^{(1)} x_1 x_2^2 + \gamma_{1(2,2)}^{(1)} x_2 x_1^2 + \\
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_2 (x_1 x_2) + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{1(2,2)}^{(1)} \right) x_2 (x_2 x_1) + \\
\left( (\gamma_{1,2}^{(1)})^2 + \gamma_{(1,1)2}^{(1)} \right) x_1 x_2 + \left( (\gamma_{1,2}^{(1)})^2 + \gamma_{(1,1)2}^{(1)} \right) x_2 x_1 + \\
\gamma_{(1,1)2}^{(1)} x_2 x_1 + \gamma_{(1,1)2}^{(1)} x_1 x_2 + \\
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\]
\[
\left( \left( \gamma_{(1)}^{(1)} \right)^2 + \gamma_{(1,1)}^{(1)} \right) (x_1 x_2) x_2 + \left( \left( \gamma_{(1)}^{(1)} \right)^2 + \gamma_{(1,1)}^{(1)} \right) (x_2 x_1) x_2 = \\
\left( \gamma_{(1,2)}^{(1)} + \gamma_{(1,2)}^{(2)} \right) x_1 + x_2 + \left( \gamma_{(1,2)}^{(2)} + \gamma_{(1,1)}^{(2)} \right) x_1 x_2 + \left( \gamma_{(1,2)}^{(2)} + \gamma_{(1,1)}^{(1)} \right) x_2 x_1 = \\
\left( \gamma_{(1,2)}^{(1)} + \gamma_{(1,1)}^{(2)} \right) x_1 (x_1 x_2) + \left( \gamma_{(1,2)}^{(2)} + \gamma_{(1,1)}^{(1)} \right) x_1 (x_2 x_1) + \\
\left( \gamma_{(1,2)}^{(1)} + \gamma_{(1,1)}^{(2)} \right) x_2 (x_1 x_2) + \left( \gamma_{(1,2)}^{(2)} + \gamma_{(1,1)}^{(1)} \right) x_2 (x_2 x_1) + \\
\left( \gamma_{(1,2)}^{(1)} + \gamma_{(1,1)}^{(2)} \right) x_2 x_1 x_1 + \left( \gamma_{(1,2)}^{(2)} + \gamma_{(1,1)}^{(1)} \right) x_2 x_1 x_1 = \\
\left( \gamma_{(1,1,2)}^{(1)} + \gamma_{(1,1,2)}^{(2)} \right) (x_1 x_2) x_1 + \left( \gamma_{(1,1,2)}^{(1)} + \gamma_{(1,1,2)}^{(2)} \right) (x_1 x_2) x_2 + \left( \gamma_{(1,1,2)}^{(1)} + \gamma_{(1,1,2)}^{(2)} \right) (x_2 x_1) x_2 x_1 \\
\sigma \left( id_k, \gamma_{(1,1,2)}^{(1)}, \gamma_{(1,1,2)}^{(2)} \right) \sigma \left( id_k, \gamma_{(1,1,2)}^{(1)}, \gamma_{(1,1,2)}^{(2)} \right) (x_1 x_2) \equiv \\
\sigma \left( id_k, \gamma_{(1,1,2)}^{(1)}, \gamma_{(1,1,2)}^{(2)} \right) (x_1 x_2) \equiv x_1 x_2 \pmod{F^3},
\]
where \( F = F^4 (x_1, x_2) \). Therefore
\[
\sigma \left( id_k, \gamma_{(1,1,2)}^{(1)}, \gamma_{(1,1,2)}^{(2)} \right) \sigma \left( id_k, \gamma_{(1,1,2)}^{(1)}, \gamma_{(1,1,2)}^{(2)} \right) = \\
\sigma \left( id_k, \gamma_{(1,1,2)}^{(1)} + \gamma_{(1,2,1)}^{(2)} + \gamma_{(1,1,2)}^{(1)} \right) \gamma_{(1,1,2)}^{(1)} - \gamma_{(1,2,1)}^{(2)} + \gamma_{(1,1,2)}^{(1)} - \gamma_{(1,2,1)}^{(2)} + \gamma_{(1,1,2)}^{(1)} + \gamma_{(1,1,2)}^{(2)}.
\]
From this formula we can conclude that
\[
\sigma \left( id_k, \gamma_{(1,1,2)}, \gamma_{(1,1,2)} \right) = \\
\sigma \left( id_k, \gamma_{(1,1,2)}, 0, 0, 1 \right) \sigma \left( id_k, 0, \gamma_{(1,1,2)} \right) \sigma \left( id_k, 0, 0, \gamma_{(1,1,2)} \right).
\]
\[ \begin{align*}
\sigma \left( \text{id}_k, 0, 0, \gamma_{(1,1)2}, 1, 0 \right)^{-1} &= \sigma \left( \text{id}_k, 0, 0, -\gamma_{(1,1)2}, 1, 0 \right), \\
\sigma \left( \text{id}_k, 0, \gamma_{(1,2)2}, 0, 1, 0 \right)^{-1} &= \sigma \left( \text{id}_k, 0, -\gamma_{(1,2)2}, 0, 1, 0 \right), \\
\sigma \left( \text{id}_k, \gamma_{1,2}, 0, 0, 1, 0 \right)^{-1} &= \sigma \left( \text{id}_k, -\gamma_{1,2}, -\gamma_{1,2}^2, -\gamma_{1,2}^2, 1, 0 \right).
\end{align*} \]

These formulas and (5.42) allow to conclude that the all systems of words \( W \) defined by formulas (5.7), (5.18), (5.41) and (5.36) fulfill conditions Op1) and Op2). So we calculated the group \( \mathcal{S}_1 \).

Now we will calculate the group \( \mathfrak{F}_1 \cap \mathfrak{S}_4 \). We will consider the strongly stable automorphism \( \Phi \) defined by the system of words \( W \) which fulfills conditions Op1) and Op2). We will find when this automorphism is inner. We can prove as in the Subsection 5.2 that if \( \Phi \) is inner then \( \alpha_{2,1} = 0 \), \( \phi = \text{id}_k \). Now we will prove that this conditions are sufficient. For this purpose for every \( F \in \text{Ob}\Theta_4^0 \) we define the mapping \( c_F : F \to F \) by this formula:

\[ c_F (f) = \alpha_{1,2}^{-1} f + \alpha_{1,2}^{-2} \gamma_{1,2} f^2 + \alpha_{1,2}^{-3} \left( \gamma_{1,2}^2 + \gamma_{(1,1)2} \right) f (f^2 ) + \alpha_{1,2}^{-4} \left( \gamma_{1,2}^2 + \gamma_{(1,1)2} \right) (f^2) f, \]

where \( f \in F \). We can prove by direct calculation that \( c_F : F \to F_W \) is an isomorphism. For example, we can check that \( c_F (f_1 f_2) = c_F (f_1) \times c_F (f_2) \), where \( f_1, f_2 \in F \). We have by nilpotence that

\[ c_F (f_1 f_2) = \alpha_{1,2}^{-1} f_1 f_2. \]

\[ c_F (f_1) \times c_F (f_2) = \alpha_{1,2}^{-1} f_1 + \alpha_{1,2}^{-2} \gamma_{1,2} f_2^2 + \alpha_{1,2}^{-3} \left( \gamma_{1,2}^2 + \gamma_{(1,2)2} \right) f_1 (f_2^2) + \alpha_{1,2}^{-3} \left( \gamma_{1,2}^2 + \gamma_{(1,2)2} \right) (f_2^2) f_1 \]

\[ = \alpha_{1,2}^{-1} f_1 + \alpha_{1,2}^{-2} \gamma_{1,2} f_2^2 + \alpha_{1,2}^{-3} \left( \gamma_{1,2}^2 + \gamma_{(1,2)2} \right) (f_1 f_2^2) = \alpha_{1,2}^{-1} f_1 f_2 + \alpha_{1,2}^{-2} \gamma_{1,2} f_1 f_2^2. \]

Also by direct calculations we can prove that \( c_F (f_1 + f_2) = c_F (f_1) \perp c_F (f_2) \), where \( f_1, f_2 \in F \), and \( c_F (\lambda f) = \lambda \ast f \), where \( f \in F \), \( \lambda \in k \). We can prove as in the Subsection 5.2 that \( c_F \) is an isomorphism. It is clear that \( c_F \psi = \psi c_D \) fulfills for every \( F, D \in \text{Ob}\Theta_4^0 \) and every \( (\psi : D \to F) \in \text{Mor}\Theta_4^0 \). Therefore, as in the [9] we can prove that

\[ \mathfrak{A}/\mathfrak{Y} \cong \left( U \left( k\mathfrak{S}_2 \right) / U \left( k \left\{ e \right\} \right) \right) \times \text{Aut} k \ast \times \text{Aut} k. \]
6 Summary.

The results of this paper and of the [9] can be summarized in this table:

| Variety of the all | $\mathfrak{A}/\mathfrak{J}$ |
|-------------------|-------------------------|
| 1 linear algebras  | $k^\ast \cdot \text{Aut}_k$ |
| 2 commutative algebras | $\text{Aut}_k$ |
| 3 power associative algebras | $k^\ast \cdot \text{Aut}_k$ |
| 4 alternative algebras | $S_2 \times \text{Aut}_k$ |
| 5 Jordan algebras | $\text{Aut}_k$ |
| 6 arbitrary subvariety of anticommutative algebras defined by identities with coefficients from $\mathbb{Z}$ | $\text{Aut}_k$ |
| 7 nilpotent algebras with degree of nilpotence no more than 3 | $k^\ast \cdot \text{Aut}_k$ |
| 8 nilpotent algebras with degree of nilpotence no more than 4 | $k^\ast \cdot \text{Aut}_k$ |

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References

[1] J. Lewin, On Schreier varieties of linear algebras. *Trans. Amer. Math. Soc.* 132 (1968), pp. 553–562.

[2] B. Plotkin, Varieties of algebras and algebraic varieties. Categories of algebraic varieties. *Siberian Advanced Mathematics, Allerton Press, 7:2*, (1997), pp. 64 – 97.

[3] B. Plotkin, Some notions of algebraic geometry in universal algebra, *Algebra and Analysis*, 9:4 (1997), pp. 224 – 248, *St. Petersburg Math. J.*, 9:4, (1998), pp. 859 – 879.

[4] B. Plotkin, Algebras with the same (algebraic) geometry, *Proceedings of the International Conference on Mathematical Logic, Algebra and Set Theory, dedicated to 100 anniversary of P.S. Novikov, Proceedings of the Steklov Institute of Mathematics, MIAN, 242*, (2003), pp. 17 – 207.

[5] B. Plotkin, G. Zhitomirski, On automorphisms of categories of free algebras of some varieties, *Journal of Algebra*, 306:2, (2006), pp. 344 – 367.

[6] A. I. Shirshov, Subalgebras of the free commutative and free anticommutative algebras. *Matematicheskiy Sbornik*, 34(76):1, (1954), pp. 81-88. (In Russian.)
[7] A. Tsurkov, Automorphic equivalence of algebras. *International Journal of Algebra and Computation.* 17:5/6, (2007), pp. 1263–1271.

[8] A. Tsurkov, Automorphisms of the category of the free nilpotent groups of the fixed class of nilpotency. *International Journal of Algebra and Computation.* 17:5/6, (2007), pp. 1273–1281.

[9] A. Tsurkov, Automorphic equivalence of linear algebras, [http://arxiv.org/abs/1106.4853](http://arxiv.org/abs/1106.4853) Submitted to the *Journal of Algebra and Its Applications.*

[10] K. A. Zhevlakov, A. M. Slin’ko, I. P. Shestakov, A. I. Shirshov, Almost associative rings. *Moscow, ”Nauka”, 1978.* (In Russian.)