Star products and central extensions

Jouko Mickelsson
Department of Mathematics
University of Helsinki
and Mathematical Physics
Royal Institute of Technology, Stockholm

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Abstract
The purpose of the present note is two-fold. First, to show that deformations of algebras of smooth functions can be used to construct topologically nontrivial standard central extensions of loop groups. Second, to use noncommutative geometry as a regularization of current algebras in higher dimensions with the aim of constructing representations of current algebras.

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Dedicated to Krzysztof P. Wojciechowski on his 50th birthday

1 Introduction

The standard central extension of the algebra \( Lg \) of smooth loops in a Lie algebra \( g \) of a compact Lie group \( G \) defines a central extension by the circle of the smooth loop group \( LG \). An explicit geometric construction for the central extension \( \hat{LG} \) was given by Mickelsson \[8\]; for an alternative construction see Murray \[11\]. The method in \[8\] was to first define a topologically trivial central extension of the group \( DG \) of smooth \( G \) valued functions in the unit disk \( D \) and then take a quotient by a normal subgroup isomorphic with the group \( G \) of functions which take the value \( 1 \in G \) on the boundary circle. The
central extension of $DG$ is defined by a $S^1$ valued 2-cocycle. In Section 2 we shall see that we can dispense the 2-cocycle if we use a Moyal product for the functions in the disk. The structure of the loop group on the boundary circle remains undeformed but we need a determinant in $G$ associated to a trace functional on the Moyal algebra.

The second application of the use of Moyal product for function algebras is related to the problem of constructing nontrivial representations of current algebras arising from hamiltonian anomalies, [9]. The main difficulty comes from the missing Hilbert-Schmidt property of off-diagonal elements of the currents with respect to the energy polarization. This problem does not arise in the case of current algebras on the circle (the lowest energy representations are the highest weight representations of affine Lie algebras). However, in any dimension bigger than one the Hilbert-Schmidt condition fails; this is related to ultraviolet divergencies in perturbative Yang-Mills theory. In one space dimension the divergencies can be removed by normal ordering but in higher dimensions one needs additional subtractions. The (background field dependent) subtractions form an obstruction for constructing true Hilbert space representations; the best what one can achieve is a geometric action on sections of a Hilbert bundle over the space of background fields.

A deformation of the commutative algebra of smooth functions on a manifold can improve the short distance behaviour in quantum field theory. One of the examples is the fuzzy sphere which has been studied in great detail by Grosse and Madore, [3], [7]. In this case the algebra becomes finite-dimensional, avoiding any kind of ultraviolet divergencies. Consequences for the current algebra representations are illustrated in terms of three examples in Section 3.

The algebra of functions on the disk can be deformed in a variety of ways. A different construction can be found in an article by Lizzi, Vitale, and Zampini [5] which is more close in spirit to the fuzzy sphere algebra in [3], [7].

2 The disk algebra and central extensions of loop groups

Let $\omega$ be the standard symplectic form $\omega = dx \wedge dy$ in $\mathbb{R}^2$. Its restriction to the unit disk $D$ in $\mathbb{R}^2$ can be used to define a star product deformation of
the algebra $\mathcal{B}$ of complex $n \times n$ matrix valued smooth functions in $D$, with vanishing normal derivatives to all orders at the boundary $S^1$,

$$ (f * g)(x, y) = e^{\frac{\nu}{2}(-\partial_x \partial_{y'} + \partial_y \partial_{x'})}f(x, y)g(x', y')|_{x=x', y=y'}, \quad (1) $$

defined as a formal power series in $\nu$. Note that at the boundary the star product is just the pointwise product of functions. Thus the restriction to the boundary gives the trivial formal deformation of the loop algebra. For general background on Moyal product and deformation quantization see Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [1].

Integration over the disk defines a linear functional in $\mathcal{B}$,

$$ \text{TR}_\nu(f) = \frac{1}{2\pi \nu} \int_D \text{tr} f(x, y) dxdy, \quad (2) $$

where 'tr' is the matrix trace.

If the functions $f, g$ are constant on the boundary then by integration by parts one observes that

$$ \text{TR}_\nu(f * g - g * f) = 0. \quad (3) $$

Otherwise, one has

$$ \text{TR}_\nu(f * g - g * f) = \frac{1}{4\pi i} \int_D \text{tr} (df dg - dg df) + \cdots = \frac{1}{2\pi i} \int_{S^1} \text{tr} fdg, \quad (4) $$

where the dots denote terms containing higher derivatives in the radial direction which integrate to zero through integration by parts due to the boundary conditions. Thus $\text{TR}_\nu$ is a true trace only in the subalgebra $\mathcal{B}_0$ of functions constant on the boundary. We shall also use the complex trace 'TR' defined as the zeroth order term in the formal Laurent series $\text{TR}_\nu$. This is likewise a true trace on the algebra of functions vanishing at the boundary.

Any multiple of (2) by a Laurent series in $\nu$ is also a trace on the star subalgebra of constant functions on the boundary. However, the choice of the normalization will become apparent later. Actually, any trace is proportional, up to a factor in $\mathbb{C}[\nu^{-1}, \nu]]$, to the trace above, Fedosov [2]. (For a short proof in the manifold case see Gutt and Rawnsley [3].)

Let $G$ be a compact matrix group and $DG \subset \mathcal{B}$ be the group of $n \times n$ matrix valued functions on $D$, formal power series in $\nu$, which are invertible.
with respect to the star product and matrix multiplication and such that the boundary values belong to the matrix group $G$. Note that an inverse exists if and only if the zeroth order term in $\nu$ is invertible as an ordinary matrix valued function.

The group $DG$ factorizes to a product of two spaces. The first factor is the set $D_0G$ of zero order functions in $DG$ and the second factor is the group $K$ of functions of the form

$$f = 1 + \nu f_1 + \nu^2 f_2 + \ldots.$$ 

Note that any $f$ of this type has an inverse as a formal power series in $\nu$. The group $K$ is contractible and it has a uniquely defined logarithmic function taking values in the formal power series without constant term.

We denote by $G$ the subgroup of $DG$ consisting of functions which are constants equal to the neutral element of $G$ on the boundary circle.

Writing a general element $f \in G$ as $f = gk$ with $g \in D_0G$ and $k \in K$ we can define the determinant as

$$\det(f) = \det(g) \cdot e^{\text{TR} \log(k)} = \det(g)e^{\frac{1}{2\pi} \int_0^1 \nu f_1^0 dt}, \quad (5)$$

The determinant $\det(g)$ is defined as

$$\log \det(g) = \int_0^1 \text{TR}(g(t)^{-1} \ast \partial_t g(t)) \, dt, \quad (6)$$

where $g(t)$ (with $0 \leq t \leq 1$) is a homotopy in $D_0G$ joining the neutral element $g(0)$ to $g = g(1)$. One should remember that the inverse $g(t)^{-1}$ is defined with respect to the star and matrix product, so it contains terms of higher order in $\nu$. This determinant for the star product algebra was introduced by Melrose and Rochon in [10] in connection with a construction of determinant line bundles over pseudodifferential operators.

The expression $\text{TR}(g^{-1} \ast dg)$ is a closed form on $G$ by the tracial property of TR and for this reason $\log \det(g)$ depends only on the homotopy class of the path $g(t)$. In order that the determinant is well-defined independent of the path one only needs to check that the integral for generators of $\pi_1(G)$ is equal to a multiple of $2\pi i$:

**Theorem 1** Let $G$ be connected and simply connected compact simple matrix Lie group and $f : S^1 \to G$ be a closed smooth loop. Then the winding number
of the determinant $\det(f(t, \cdot))$ around the loop is equal to the integer
\[ -\frac{1}{24\pi^2} \int_{S^1 \times D} \text{tr} (f^{-1}df)^3. \]

Here we have identified the parameter space $D$ as a unit sphere $S^2$ since on the boundary of $D$ all the functions $f \in G$ take the constant value 1.

**Proof:** The proof is by a direct computation. We need to select a generator for $\pi_1(G) = \mathbb{Z}$. Since the topology of the group is determined by the constant part of formal power series in $\nu$, we can assume that $f(t, \cdot)$ is zero order in $\nu$.

By the definition of ‘TR’, we need to compute the term first order in $\nu$ in the integral (the zeroth order term vanishes identically since $f^{-1}df$ is traceless)
\[ \int_0^1 \text{TR}_\nu(f(t, \cdot)^{-1} * f(t, \cdot))dt. \]

The inverse $f^{-1}$, as defined with respect to the star product, can be written as
\[ g_0 + \nu g_1 + \nu^2 g_2 + \ldots, \]
where $g_0$ is the pointwise matrix inverse of the function $f(t, \cdot)$ and
\[ g_1 = \frac{i}{2} df^{-1}df f^{-1}. \]

Thus
\[ \int \text{TR}(f^{-1} * \partial_t f) dt = \frac{1}{2\pi} \int \int_{D} \frac{i}{2} \text{tr} \left( df^{-1}df f^{-1}\partial_t f + df^{-1}d(\partial_t f) \right) dt \]
\[ = -\frac{1}{12\pi i} \int_{[0,1] \times D} \text{tr} (f^{-1}df)^3 \]
which proves the Theorem. □

We define
\[ \hat{LG} = (DG \times S^1)/N, \quad (7) \]
where $N$ is the normal subgroup consisting of pairs $(g, \lambda)$ such that $g \in G$ and $\lambda = \det(g)$.

This is a central extension by the circle $S^1$ of the loop group $LG$. 5
Theorem 2: The Lie algebra of $\hat{LG}$ is isomorphic as a vector space to the direct sum $L g \oplus i \mathbb{R}$ with the commutator $[(f, \alpha), (g, \beta)] = ([f, g], c(f, g))$ where $[f, g]$ is the point-wise commutator of Lie algebra valued functions and $c$ is the 2-cocycle

$$c(f, g) = \frac{1}{2\pi i} \int_{S^1} tr f dg.$$  \hspace{1cm} (8)

Proof: Let $\psi$ be the local section of the circle bundle $\hat{LG} \to LG$, defined in a neighborhood of the unit element in the loop group, given by

$$\psi(e^X) = e^{\tilde{X}},$$

where $X : S^1 \to g$ and $\tilde{X} \in \mathcal{B}$ is equal to $X$ on the boundary. For example, we can fix a smooth function $f(r)$ of the radius $r$ such that $f(0) = 0, f(1) = 1$ and all the derivatives of $f$ vanish at $r = 1$ and put $\tilde{X} = f(r)X$. The exponential is defined by the star product,

$$e^Z = \sum_n \frac{1}{n!} Z \ast Z \ast \cdots \ast Z, \text{ } n \text{ factors.}$$

The section $\psi$ is well-defined in an open set of $G$ valued of loops where the logarithm is defined.

Locally, near the unit element, the central extension $\hat{LG}$ is a product of an open set of $LG$ with $S^1$. The local $S^1$ valued group cocycle is evaluated from

$$\det(\psi(e^{\tilde{X}}) \ast \psi(e^{\tilde{Y}}) \ast \psi(e^{-\tilde{X}}) \ast \psi(e^{-\tilde{Y}})).$$  \hspace{1cm} (9)

The Lie algebra cocycle $c(X, Y)$ is then the bilinear term in the expansion of (9) in powers of $X, Y$. Using the definition (5) of the determinant and the Baker-Campbell-Hausdorff formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+...}$$

we obtain

$$c(X, Y) = \text{TR}[\tilde{X}, \tilde{Y}] = \frac{1}{2\pi i} \int_{S^1} tr X dY.$$  \hspace{1cm} (10)

□
The canonical connection on the loop group $LG$ is given through the $S^1$ invariant 1-form $\theta$ on $\hat{LG}$,

$$\theta = pr_c(g^{-1} dg),$$

where $pr_c$ is the projection onto the center of the Lie algebra $\hat{Lg}$. The curvature form $\Omega$ of this connection is the left invariant 2-form on $LG$ which at the identity element is given by the cocycle $c : Lg \times Lg \rightarrow \mathbb{C}$. The winding number in Theorem 1 is then $1/2\pi$ times the integral of $\Omega$ over the set of loops $t \mapsto f(t, x)$ parametrized by $x \in D$.

### 3 Generalization to higher dimensions

The discussion above cannot directly be generalized to higher dimensions. The obstruction is the noninvariance of the boundary conditions. If we have a symplectic manifold with boundary of dimension $2d$ then the space of smooth functions with vanishing normal derivatives at the boundary is not closed in general. This happens already in the case of a disk in $\mathbb{R}^{2d}$ with the standard constant symplectic form in $\mathbb{R}^{2d}$. For this reason we focus only on a special case. Let $M = D \times S$ where $S$ is a closed manifold of dimension $2d-2$ and $D$ is the unit disk in $\mathbb{R}^2$. We assume that the algebra of functions $S$ on $S$ is equipped with a star product and $D$ comes with a star product as in Section 2. The star product on $S$ does not need to come from a bidifferential operator related to a symplectic form as in the case of the Moyal product. In fact, we can consider as well a product coming from quantum groups or quantum homogeneous spaces. However, what we need is an 'algebra of functions' possessing a trace functional $tr_S$. In this case the star product algebra of matrix valued functions on $M$ is replaced by the tensor product of the star algebra of matrix valued functions on the disk and a star algebra $S$. We can now impose vanishing normal derivatives at the boundary of $D$.

**Example 1**

The product of the symplectic disk $D$ and a fuzzy sphere $S^2_N$. The fuzzy sphere is defined as the quotient by an ideal $I$ of the noncommutative associative polynomial algebra in three variables $x, y, z$ with relations $x \cdot y - y \cdot x = z, y \cdot z - z \cdot y = x, z \cdot x - x \cdot z = y$. The two-sided ideal $I$ is generated by the single element $x^2 + y^2 + z^2 + N(N+1)$ where $N$ is a nonnegative integer. Since $x, y, z$ define the Lie algebra of $SU(2)$ the trace is defined as the matrix
trace in an irreducible representation of dimension $N(N + 1)$. The algebra is simply the algebra of square matrices in dimension $N(N + 1)$.

**Example 2**

We can take as $\mathcal{S}$ the algebra of smooth $n \times n$ matrix valued functions in $\mathbb{R}^{2d-2}$ which decay faster than any inverse power of $|x|$ at infinity. The star product is defined as the Moyal product and the trace is the integral of a function over $\mathbb{R}^{2d-2}$. In this case the product can actually be defined analytically, not only as a formal power series in $\nu$. This is because the functions can be interpreted as symbols of infinitely smoothing pseudodifferential operators in $\mathbb{R}^{d-1}$. This is achieved by selecting a Lagrangian polarization $\mathbb{R}^{d-1} \oplus \mathbb{R}^{d-1}$ and interpreting the first $d-1$ variables as momenta and the last $d-1$ variables as coordinates. The algebra $\Psi^{-\infty}$ is a subalgebra of the algebra $\mathfrak{g}_1$ of trace-class operators in the Hilbert space $H = L^2(\mathbb{R}^{d-1}, \mathbb{C}^N)$.

The linear functional

$$
\text{TR}(f) = \frac{1}{2\pi} \int_D dx \, dy \, \text{tr}_S f
$$

is a trace in the subalgebra of functions which vanish on the boundary of $D$. Here $\text{tr}_S$ denotes the combined matrix trace and a trace in in the algebra $\mathcal{S}$.

The determinants are defined by straight-forward generalization of (12). The Lie algebra cocycle for $\text{Map}(S^1, \mathcal{S} \otimes \mathfrak{g})$ becomes

$$
c(f, g) = \frac{1}{2\pi} \int_{S^1} \text{tr}_S f \, dg. \tag{13}
$$

In the case of Example 1 we get the standard central extension of the loop algebra of smooth maps from $S^1$ to matrices of size $nN(N + 1) \times nN(N + 1)$ whereas in the example 2 we have a central extension of the loop algebra $L\Psi^{-\infty}$ in the algebra $\Psi^{-\infty}$ of infinitely smoothing $n \times n$ matrix pseudodifferential operators over $\mathbb{R}^{d-1}$.

The Lie algebra cocycle (13) extends to the loop algebra $L\mathfrak{g}_1$. A representation for $\widehat{L\mathfrak{g}_1}$ is obtained essentially in the same way as for central extensions of the loop algebra $L\mathfrak{g}$ based on a finite-dimensional Lie algebra $\mathfrak{g}$. A highest weight representation of the Lie algebra $\mathfrak{g}_1$ is given by an infinite increasing sequence of integers $\lambda_i$, $i \in \mathbb{Z}$, with $\lambda_i = \lambda_{\infty}$ for $i >> 0$ and $\lambda_i = \lambda_{-\infty} \leq \lambda_{\infty}$ for $i << 0$. The irreducible integrable highest weight representation corresponding to $\lambda$ is then characterized by the existence of a cyclic vector $v_\lambda$ such that

$$
e_{ii}v_\lambda = \lambda_i v_\lambda \quad \text{and} \quad e_{ij}v_\lambda = 0 \quad \text{for} \quad i > j,$$
where the $e_{ij}$’s are the Weyl basis vectors in $\mathfrak{g}$,

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. $$

Given the irreducible highest weight representation $(\lambda)$ of $\mathfrak{g}$, one obtains an irreducible highest weight representation of the central extension of the loop algebra $L\mathfrak{g}$ by induction. The representation has a highest weight vector $v_{\lambda, k}$ characterized by

$$e_{ij}v_{\lambda, k} = 0 \text{ for } i > j \text{ and } x^{(n)}v_{\lambda, k} = 0 \text{ for } n < 0,$$

where $x^{(n)} = e^{in\phi}x \in L\mathfrak{g}$ and $k$ is the value of central element in the representation,

$$c(f, g) = \frac{k}{2\pi} \int_{S^1} \text{tr}(f dg).$$

The representation integrates to an unitary representation of the group $\hat{L}G$ if $k$ is an integer with $\lambda_\infty - \lambda_{-\infty} \leq k$, see the monograph by Kac [6].

The construction of the central group extension $\hat{L}G$ for the case of a compact matrix group $G$ can now be extended without any changes to the case when $G$ is the infinite-dimensional Lie group of unitary pseudodifferential operators $A$ such that $A - 1$ is trace class.

**Example 3**

We deform the gauge current algebra in 3 space dimensions. First, let $\mathfrak{n}$ be the ideal of pseudodifferential operators, on a compact spin manifold $M$ of dimension 3, of degree less or equal to $-2$. All pseudodifferential operators are taken with matrix coefficients. The matrices act in the tensor product of the spinor bundle and a trivial vector bundle $V$ over $M$. The finite-dimensional Lie algebra $\mathfrak{g}$ of a gauge group $G$ acts in the fibers of $V$ through a matrix representation. For each smooth map $X : M \to \mathfrak{g}$ we define a deformed operator

$$\tilde{X} = X + \frac{1}{4(D^2 + 1)}[D, [D, X]],$$

where $D$ is the Dirac operator on $M$ defined by a fixed metric and spin structure. The difference $\tilde{X} - X$ is a pseudodifferential operator of order $-1$. One easily checks that

$$[\tilde{X}, \tilde{Y}] = [\tilde{X}, \tilde{Y}] \mod \mathfrak{n}. $$

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Denote by $\mathfrak{p}$ the Lie algebra of pseudodifferential operators such that the leading symbols of order 0 and $-1$ are given by the leading symbols of symbols of the deformed operators $\{\tilde{X}|X \in \text{Map}(M, g)\}$. Let $\epsilon = D/|D|$. Then $[\epsilon, T]$ is Hilbert-Schmidt for all $T \in \mathfrak{p}$.

We have the exact sequence

$$0 \to \mathfrak{n} \to \mathfrak{p} \to \text{Map}(M, g),$$

where the second map is embedding of Lie algebras and the third map extracts the zero order part $X$ of an element $T = \tilde{X} + z \in \mathfrak{p}$, where $z \in \mathfrak{n}$.

The Lie algebra $\mathfrak{p}$ is a subalgebra of $\mathfrak{gl}_{res}$ where the latter consists of bounded operators $T$ in the Hilbert space $H$ such that $[\epsilon, T]$ is Hilbert-Schmidt. The algebra $\mathfrak{gl}_{res}$ has a canonical central extension $\hat{\mathfrak{gl}}_{res}$ defined by the cocycle

$$c(X, Y) = \frac{1}{4} \text{tr} \epsilon [\epsilon, X][\epsilon, Y].$$

The restriction to $\mathfrak{p}$ gives a central extension $\hat{\mathfrak{p}}$ of $\mathfrak{p}$. Likewise, we have a central extension $\hat{\mathfrak{n}}$ of $\mathfrak{n} \subset \mathfrak{gl}_{res}$. Putting these together we have the extension

$$0 \to \hat{\mathfrak{n}} \to \hat{\mathfrak{p}} \to \text{Map}(M, g).$$

The algebra $\hat{\mathfrak{p}}$ has unitary highest weight representations. For example, the Fermionic Fock space $\mathcal{F}$ based on the polarization $H = H_+ \oplus H_-$ carries through canonical quantization a represenation of $\hat{\mathfrak{gl}}_{res}$ and thus of $\hat{\mathfrak{p}}$. However, this representation does not preserve the domain of the quantization $\tilde{D}$ of the Dirac operator $D$.

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