AdS/CFT with Tri-Sasakian Manifolds

Ho-Ung Yee

Korea Institute for Advanced Study,
207-43, Cheongryangri 2-dong, Dongdaemun-gu, Seoul 130-722, Korea

2006

Abstract

We consider generic toric Tri-Sasakian 7-manifolds $X_7$ in the context of M-theory on $AdS_4 \times X_7$ and study their AdS/CFT correspondence to $\mathcal{N} = 3$ SCFT in 3D spacetime. We obtain volumes of Tri-Sasakian manifolds and their supersymmetric 5-cycles via cohomological integration technique, and use this to calculate conformal dimensions of baryonic operators in the SCFT side. We also propose quiver-type gauge theories for UV description of the corresponding $\mathcal{N} = 3$ SCFT.

\textsuperscript{1}ho-ung.yee@kias.re.kr
1 Introduction

AdS/CFT correspondence is a conjectured duality between a field theory living on solitonic extended objects and a theory of gravity in the corresponding back-reacted space-time near the extended objects [1, 2, 3]. Its usefulness rests on the fact that the strong coupling regime of the field theory side is mapped to a weakly interacting regime of the gravity theory, allowing us to calculate non-trivial information about the strongly coupled field theory. In addition to the original example of D3 branes in flat (9+1)-dimensional space-time, corresponding to the duality between $\mathcal{N} = 4$ SYM in (3+1)-dimension and Type IIB string on $AdS_5 \times S^5$, there are many other examples with less number of supersymmetries as well as with different space-time dimensions. For example, when we consider D3 branes at the tip of a 6-dimensional cone, we would have a supersymmetric 4D CFT at low energy with the number of supersymmetries determined by the special holonomy of the cone. As we consider supergravity back-reaction of the D3 branes, the resulting near horizon geometry is $AdS_5 \times X_5$, where $X_5$ is the constant radius section of the cone. The AdS/CFT correspondence claims that the Type IIB string theory on this geometry describes the corresponding 4D SCFT on D3 branes [4, 5].

A well-studied class of examples is the case when our 6-dimensional cone is locally Calabi-Yau, or Ricci-flat Kähler. The constant radius section is then a Sasaki-Einstein manifold, and the corresponding 4D SCFT is $\mathcal{N} = 1$. The simplest example of these is provided by conifold, or its constant radius section $T^{1,1}$, whose dual $\mathcal{N} = 1$ SCFT was first studied by Klebanov and Witten [6]. In fact, this is the simplest example of toric CY cones that can be obtained through Kähler quotient from flat higher dimensional space. However, the Kähler quotient only dictates and guarantees the Kähler class where Ricci-flat metric belongs to, and the actual Ricci flat metric is not given by the induced metric from the original flat space. Finding the Ricci-flat metric is not an easy problem, and the recently found $Y^{p,q}$ and $L^{p,q,r}$ are among the examples [7, 8]. It is also closely related to the recent $Z$-minimization approach [9, 10]. There also appeared several interesting proposals for the corresponding $\mathcal{N} = 1$ SCFT’s in terms of quiver-type gauge theories [11, 12, 13, 14, 15, 16, 17]. Predictions on $U(1)_R$ symmetry via a-maximization [18] are found to match nicely to the corresponding Reeb vector in the local CY cone via $Z$-minimization, supporting these proposals in a convincing way [19, 20, 21].

In this paper, we consider another class of AdS/CFT examples in M-theory. Specifically, we study 3D $\mathcal{N} = 3$ SCFT’s that arise from M2 branes sitting at the tip of
8-dimensional toric hyper-Kahler cones. The corresponding M-theory dual is $AdS_4 \times X_7$ with $X_7$ being Tri-Sasakian manifolds. Our discussion will include generic toric hyper-Kahler cones obtained by arbitrary $U(1)^n$ hyper-Kahler quotients from flat spaces. Although these spaces are well-known mathematically [22, 23, 24, 25], the analysis in the context of AdS/CFT correspondence has not been done in full generality except simple special cases [26, 27, 28, 29, 30]. Contrary to local Calabi-Yau cases in both Type IIB and M-theory, the structure of hyper-Kahler manifold is sufficiently rigid such that the induced metric from the original flat space is automatically hyper-Kahler that we would need. This is essentially due to the rigidity of non-Abelian nature of $SU(2)_R$ structure of hyper-Kahler manifolds. In the $\mathcal{N} = 3$ SCFT side, this is reflected to the fact that $SU(2)_R$ R-symmetry does not change under RG-flows. This is much help to us since R-charges of chiral primary operators encode information on their conformal dimensions [31]. RG-rigidity may enable us to extract non-trivial information about far IR physics from a simple UV description that is supposed to flow to our SCFT at IR. This information on $\mathcal{N} = 3$ SCFT can then be compared to the results from its gravity dual description, that is, M-theory on $AdS_4 \times X_7$.

We will study M5 brane wrapping a supersymmetric cycle inside $X_7$ in $AdS_4 \times X_7$. (For a recent analysis for Type IIB case, see Ref.[32]). In view of $AdS_4$, this looks like a very heavy point-like excitation. In full second quantized quantum theory, this will be described by an effective field in $AdS_4$ with a very large mass term. Considering supersymmetry, there should actually be a super-multiplet arising from quantization of M5 world-volume theory. Via a standard AdS/CFT dictionary, this super-multiplet with a heavy mass corresponds to a chiral primary operator in $\mathcal{N} = 3$ SCFT whose conformal dimension is determined from the $AdS_4$ mass. For a heavy mass compared to the curvature scale of $AdS_4$, the conformal dimension is proportional to the mass of the super-multiplet in $AdS_4$, which is again proportional to the volume of the cycle $\Sigma_5$ that M5 brane is wrapping. The result of these calculations gives us

$$\Delta = \frac{\pi N \text{Vol}(\Sigma_5)}{6 \text{Vol}(X_7)},$$

where $\Delta$ is the conformal dimension of the corresponding chiral primary operator, and $N$ is the number of background M2 branes [33]. Therefore, from calculations on volumes of $\Sigma_5$ and $X_7$ we can extract $\Delta$. This is then compared to predictions from a UV description of our $\mathcal{N} = 3$ SCFT in terms of quiver-type gauge theory [27, 29, 30]. We stress that this is meaningful due to RG-rigidity of $SU(2)_R$ R-symmetry that encodes information on conformal dimensions at IR SCFT fixed point.
Though we will find a nice agreement later in this context of wrapping M5 brane, this UV description is not completely satisfactory. Some of chiral primary spectrum in the UV description is in fact shown to be absent in M-theory on $\text{AdS}_4 \times X_7$ for the simplest case of homogeneous Tri-Sasakian manifold $N(1,1)$ \cite{27, 29}. This suggests that chiral primary spectrum may jump as we flow to the strong coupling regime at IR. We don’t see any tool to extract information about these phenomena. This elimination of spectrum makes sense, since we know that the number of degrees of freedom at the IR fixed point scales mysteriously as $N^3$ while our UV description in terms of gauge theory has $N^2$ degrees of freedom. (For a recent account of $N^3$ scaling of M5 branes, see Ref.\cite{34}.)

Recently, there appeared an alternative description of $\mathcal{N} = 2$ SCFT’s arising from M2 branes at the apex of Calabi-Yau cones, in terms of crystal lattice of M5 branes \cite{35}. Since $\mathcal{N} = 3$ belongs to $\mathcal{N} = 2$, it would be interesting to study further in that context.

## 2 Quotient and Localization

In this section, we discuss necessary technical gadgets to calculate symplectic volume of toric tri-Sasakian manifolds. These manifolds will arise as hyper-kahler quotients from higher dimensional flat quaternion spaces. In general, the quotient manifold that we are interested in is a curved manifold, on which it is difficult to do explicit calculations. A basic motivation of equivariant cohomology is to develop a method that describes the usual cohomology of the symplectic quotient space in terms of a language of the flat ambient space where calculations may become much easier.

Perpendicular to this, there is a technique of localization. It is often the case that the integration of our interest, like symplectic volumes, has a fermionic nilpotent symmetry. We introduce concepts of cohomology with it. If there is a bosonic global symmetry, we can use it to deform the fermionic nilpotent symmetry and its cohomology in a specific way, parameterized by $\epsilon$’s. To keep invariance under the defomed fermionic symmetry, the integrand should also be modified. The defomed fermionic symmetry then allows us to add a cohomologically trivial term in the integrand without affecting the result, which contains, among other things, a positive definite purely bosonic term. By taking the overall coefficient infinitely large, the integration localizes to saddles points of the bosonic term, which turn out to be nothing but the fixed points of the global symmetry we started with.

For compact manifolds, we may take a continuous limit of turning off the deformation
parameter $\epsilon$’s to get the results for the original problem. For non-compact manifolds, the deformation often provides a regularization and we may get other information from the results of the deformed integral. The volume of tri-Sasakian manifolds that we will discuss belongs to the latter case.

In many cases, the symmetry we use for quotient is compatible with the symmetry of the quotient space for $\epsilon$-deformation. In other words, the symmetry for $\epsilon$-deformation in the quotient space is actually a symmetry in the ambient space, too. We are then able to describe $\epsilon$-deformation and the resulting localization for the quotient space in a language of equivariant cohomology in flat ambient space. This gives us a powerful handle over easy calculations in flat spaces of seemingly difficult integrals in curved quotient spaces.

The techniques presented in this section have been established in Ref.[30, 36, 37, 38], and for completeness we will expound them in more explicit detail. Readers familiar to it may skip this section.

A supermanifold $T[1]X$

Integrals of our interest are typically those of differential forms, and it is possible to rewrite them in a way that looks like a supersymmetric path integral. Though this is not a strictly necessary formulation, it may give us a comfortable understanding of some mathematical results in physics terms.

Given a bosonic manifold $X$ with a coordinate $\{x^\mu\}$, a tangent vector $V$ is canonically written as $V = V^\mu \frac{\partial}{\partial x^\mu}$. We can think of $\{x^\mu, V^\mu\}$ as a canonical coordinate system of the tangent bundle $TX$ associated to a coordinate $\{x^\mu\}$. The supermanifold $T[1]X$ is obtained by replacing the bosonic coordinates $\{V^\mu\}$ with fermionic ones $\{\psi^\mu\}$ to which we assign a degree number 1, hence explaining the notation. Functions on $T[1]X$ will be expanded as

$$f(x, \psi) = f^{(0)}(x) + f^{(1)}(x)\psi^\mu + \frac{1}{2!} f^{(2)}(x)\psi^\mu\psi^\nu + \cdots,$$

(2.2)

up to the dimension $n$ of the manifold $X$, and the space of functions on $T[1]X$ is easily identified as $\Omega^*(X)$, the space of differential forms on $X$. An integration over $T[1]X$ of a function $f(x, \psi)$ is

$$\int_{T[1]X} [dx^\mu][d\psi^\mu] f(x, \psi) = \int_{T[1]X} [dx^\mu][d\psi^\mu] \frac{1}{n!} f^{(n)}_{\mu_1\mu_2...\mu_n}(x)\psi^{\mu_1}\psi^{\mu_2}...\psi^{\mu_n}$$

$$= \int_X [dx^\mu] \frac{1}{n!} f^{(n)}_{\mu_1\mu_2...\mu_n}(x)\epsilon^{\mu_1\mu_2...\mu_n} = \int_X f^{(n)},$$

(2.3)

which is the usual integration of top differential form on $X$. Note that the top form in $f(x, \psi)$ is picked up automatically by the fermionic integration over $\psi^\mu$. In fact, the
measure $[dx^\mu][d\psi^\mu]$ is invariant under coordinate transformations. Since $\psi^\mu$ transforms as a vector under a coordinate change $x^\mu \to \tilde{x}^\mu$, we have $\tilde{\psi}^\nu = \psi^\mu \left( \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \right)$ and $[d\tilde{\psi}] = \det \left( \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \right)^{-1} [d\psi]$, which cancels the bosonic Jacobian.

The supermanifold $T[1]X$ has a nilpotent fermionic symmetry,

\begin{align}
Q x^\mu &= \psi^\mu, \\
Q \psi^\mu &= 0,
\end{align}

with $Q^2 = 0$. An inspection of its action to functions on $T[1]X$ shows that it is nothing but the usual de Rham differential acting on $\Omega^*(X)$, $Q \simeq d$. We define observables of our supersymmetric theory as $Q$-cohomology classes, and correlation functions of them are integrals on $T[1]X$, which are identical to intersection integrals of $X$.

The measure on $T[1]X$ is invariant under $Q$, and from this we can derive that

$$\int_{T[1]X} Q \Lambda(x, \psi) = 0,$$

for any $\Lambda$. This is a compact form of the Stokes theorem in the following sense. For a cycle $Y$ in $X$, there corresponds a Poincare dual form $\delta(Y)$ in $Q$-cohomology, such that

$$\int_{T[1]Y} f = \int_{T[1]X} f \cdot \delta(Y),$$

for any $f$. Let’s define the boundary of $Y$ such that its Poincare dual is $Q\delta(Y)$, that is, $\delta(\partial Y) = Q\delta(Y)$. It follows that

$$0 = \int_{T[1]X} Q (f \cdot \delta(Y)) = \int_{T[1]X} Q f \cdot \delta(Y) \pm \int_{T[1]X} f \cdot Q \delta(Y) = \int_{T[1]Y} Q f \pm \int_{T[1]\partial Y} f,$$

where $\pm$ depends on the degree numbers.

We are interested in calculating volumes of symplectic(Kahler) manifolds $(X, \omega)$ of dimension $2n$ with a non-degenerate symplectic 2-form $\omega = \frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu$. Since the volume element is simply $\frac{1}{n!} \omega^n$, it can be written as

$$\text{Vol}(X) = \int_{T[1]X} e^\omega = \int_{T[1]X} e^S,$$

with an action $S = \omega$ which is $Q$-symmetric, $QS = Q\omega = 0$, from $d\omega = 0$. The above looks like a $(0+0)$-dimensional supersymmetric partition function. From (2.5), we are free to add $Q$-exact terms to the action, $S \to S + Q\mathcal{O}$, without affecting the result. However, since $Q\mathcal{O}$ contains at least one $\psi^\mu$, there is no purely bosonic term we can add to facilitate
localization argument in this case. This is one motivation to introduce $\epsilon$-deformation using global symmetries of $\omega$.

**Hamiltonian Flow**

Symmetries of a symplectic manifold $(X,\omega)$ is locally isomorphic to the space of real functions on $X$ up to a constant function. More explicitly, for any function $H$ on $X$ we can define a vector field $V$ from $QH = i_V \omega$, where $i_V$ is the contraction by the vector $V$,

$$i_V = V^\mu \frac{\partial}{\partial \psi^\mu} .$$

(2.9)

In components, we have $\frac{\partial H}{\partial \psi^\mu} = V^\mu \omega_{\mu\nu}$, and $H$ determines $V$ because $\omega$ is non-degenerate and we can invert it. From $\mathcal{L}_V = \{i_V, Q\} = i_V Q + Qi_V$, where $\mathcal{L}_V$ is the Lie derivative by $V$, and $Q \omega = 0$, we have

$$0 = Q^2 H = Qi_V \omega = \{i_V, Q\} \omega = \mathcal{L}_V \omega ,$$

(2.10)

saying that $V$ generates a symmetry flow of $\omega$. Conversely, given a symmetry vector $V$,

$$0 = \mathcal{L}_V \omega = \{i_V, Q\} \omega = Qi_V \omega ,$$

(2.11)

and we can always find $H$ with $QH = i_V \omega$ at least locally up to an additive constant.

The space of symmetries of $\omega$ is closed under Lie bracket, and the corresponding operation in the space of functions turns out to be Poisson bracket with respect to $\omega$. In other words, letting $V_f$ and $V_g$ be symmetries obtained from functions $f$ and $g$, their Lie bracket $[V_f, V_g]$ is equal to $V_{\{f, g\}}$, a symmetry from the Poisson bracket $\{f, g\}$ with respect to $\omega$. Therefore, the correspondence is an algebra isomorphism. The Jacobi identity for Poisson bracket is a simple consequence of that for Lie bracket of vector fields.

**Deformation via Global Symmetries and Localization**

In (2.8), the symplectic volume is written in a $Q$-symmetric way, but a freedom of adding $Q$-exact terms to the action doesn’t help much for its calculation. However, by deforming $Q$-symmetry using global symmetries and also the action accordingly, there is a way to use this freedom to reduce our integration into a localized sum over discrete points. After an easy calculation of the deformed integral via localization, one may turn off the deformation parameter $\epsilon$ to get the answer for the original integral.

Suppose we have a well-defined symmetry flow $V = V^\mu \frac{\partial}{\partial x^\mu}$, and the corresponding function $H$ with $QH = i_V \omega$. This allows us to deform $Q$ into

$$Q_\epsilon x^\mu = \psi^\mu ,$$

$$Q_\epsilon \psi^\mu = \epsilon V^\mu (x) ,$$

(2.12)
with $\epsilon$ a real number. This can also be written as $Q_\epsilon = Q + \epsilon i_V$, with $i_V$ in (2.9). From $Q^2 = (i_V)^2 = 0$, it is readily seen that $Q_\epsilon^2 = \epsilon L_V$, which implies $Q_\epsilon$ is nilpotent only in the subspace of $V$-invariant functions on $T[1]X$. We henceforth must restrict to this $V$-invariant subspace when we discuss adding $Q_\epsilon$-exact terms later on.

The original action $S = \omega$ is not $Q_\epsilon$-invariant,  

$$Q_\epsilon S = Q\omega + \epsilon i_V \omega = \epsilon QH = Q_\epsilon(\epsilon H) \quad . \tag{2.13}$$

Instead, the deformed action $S_\epsilon = S - \epsilon H$ is $Q_\epsilon$-invariant. Note that $S_\epsilon$ is also $V$-invariant from $(i_V)^2 = 0$. Therefore, one considers the deformed volume which is $Q_\epsilon$-invariant,

$$\text{Vol}_\epsilon(X) = \int_{T[1]X} e^S = \int_{T[1]X} e^{S - \epsilon H} \quad , \tag{2.14}$$

and an arbitrary $Q_\epsilon$-exact term $Q_\epsilon O$ may be added to the action without changing the result as long as $O$ is $V$-invariant, $L_V O = 0$. One such $Q_\epsilon O$ of our interest is

$$- tQ_\epsilon (g_{\mu\nu}\psi^\mu V^\nu) = - t(\epsilon g_{\mu\nu} V^\mu V^\nu + \partial_\eta (g_{\mu\nu} V^\nu) \psi^\eta \psi^\mu) \quad \tag{2.15}$$

with a $V$-invariant positive definite metric $g_{\mu\nu}$ on $X$. We can always find it by averaging any metric along $V$-flows. Note that we have an expected bosonic term, $- t\epsilon g_{\mu\nu} V^\mu V^\nu$ which is negative definite, and taking $t \to +\infty$ limit reduces the whole integral to a saddle-point approximation around fixed points of $V$, which becomes strictly exact. Note that vanishing points of $V = 0$ is nothing but the extreme points of $H$. Performing quadratic expansion around a fixed point $x^\mu = 0$,

$$V^\mu = V^\mu_\alpha x^\alpha + \mathcal{O}(x^2) \quad , \quad g_{\mu\nu} = g_{\mu\nu}^{(0)} + \mathcal{O}(x) \quad , \tag{2.16}$$

and calculating Gaussian integration, we have a formula by Duistermaat-Heckmann,

$$\text{Vol}_\epsilon(X) = \sum_{\text{fixed points } p} (-1)^n \# e^{-\epsilon H(p)} \left( \frac{2\pi}{\epsilon} \right)^n \left( \frac{\det(g_{\mu\nu}^{(0)} V^\mu_\eta)}{\det(g_{\mu\nu} V^\mu_\alpha V^\nu_\beta)} \right)^{\frac{1}{2}} \ . \tag{2.17}$$

**Symplectic Quotient and Equivariant Cohomology**

In general, even localization calculation on our space $(X, \omega)$ is not easy if it is a curved manifold. If it can arise as a symplectic quotient from a larger flat space $(M, \omega)$, the language of equivariant cohomology provides a method for calculating things for $X$ in the ambient flat space $M$ more easily. Moreover, in the case where a global symmetry of $X$ is actually a symmetry of $M$, the $\epsilon$-deformation and the localization on $X$ can also
be described on $M$ equivariantly. Since localization probes only the tangent space of the geometry, the relation between $X$ and $M$ becomes a linear one around the fixed points, and the calculation on $M$ is a tractable one.

For simplicity we will discuss $U(1)$ quotient only, although generalizations to $U(1)^n$ as well as non-abelian groups are straightforward. Starting from an ambient space $(M, \omega)$, whether flat or not, with a $U(1)$-symmetry flow $V$, its symplectic quotient $M//U(1)$ is defined as follows. From $L_V \omega = 0$, there exists a function $\mu(x)$ with $Q \mu = i_V \omega$, which is called a moment map. Note that we have a freedom of adding constant function to $\mu(x)$, and it actually parameterizes a family of $M//U(1)$ that we define with $\mu(x)$. Since $V \mu \partial \mu \partial x^i = (i_V)^2 \omega = 0$, the level surface $\mu^{-1}(0)$ is invariant under $V$-flow, in other words, $V$ is a well-defined $U(1)$-flow on $\mu^{-1}(0)$. $M//U(1)$ is then defined as the usual quotient of $\mu^{-1}(0)$ by $U(1)$.

To analyze the situation more clearly, we introduce a coordinate system $\{x^i, x^v, x^n\}$, $i = 1, \ldots, (\dim M - 2)$, of $M$ around $\mu^{-1}(0)$ such that $\{x^i\}$ parameterize $\mu^{-1}(0)/U(1)$, and $V = \frac{\partial}{\partial x^v}$, that is, $x^v$ is the Gauss coordinate of the $V$-flow. $x^n$ parameterizes the normal direction to $\mu^{-1}(0)$. The equation $Q \mu = i_V \omega$ in components reads as

$$\omega_{vi} = \frac{\partial \mu}{\partial x^i}, \quad \omega_{vn} = \frac{\partial \mu}{\partial x^n}, \quad \omega_{ij}.$$(2.18)

and because $\mu = 0$ on $\mu^{-1}(0)$, its derivative with respect to $x^i$ is also zero, hence $\omega_{vi} = 0$ on $\mu^{-1}(0)$. Then the $ij$-component of $Q \omega = 0$ gives us

$$\partial_i \omega_{ij} = \partial_i \omega_{vj} - \partial_j \omega_{vi} = 0,$$(2.19)

and we see that $\omega_{ij}$ is $V$-invariant on $\mu^{-1}(0)$. Therefore, $\frac{1}{2} \omega_{ij} \psi^i \psi^j$ is a well-defined symplectic form on $M//U(1) = \mu^{-1}(0)/U(1)$ with its coordinates $\{x^i\}$, and this defines the symplectic quotient $(M//U(1), \omega)$. Its symplectic volume is then

$$\text{Vol}(M//U(1)) = \int_{T[1]M//U(1)} [dx^v][dx^i][d\psi^i] e^{\frac{1}{2} \omega_{ij} \psi^i \psi^j}.$$(2.20)

Our basic objective is to re-express (2.20) in a language of the ambient space $M$. Since everything is independent of $x^v$, we can simply extend the bosonic integration to include $\int dx^v$ and divide by $\text{Vol}(U(1)) = \int dx^v$. The integration is now over $\mu^{-1}(0)$ and we have

$$\text{Vol}(M//U(1)) = \frac{1}{\text{Vol}(U(1))} \int_{\mu^{-1}(0)} [dx^v][dx^i][d\psi^i] e^{\frac{1}{2} \omega_{ij} \psi^i \psi^j} \delta (\mu(x)) \left( \frac{\partial \mu(x)}{\partial x^n} \right).$$

(2.21)
where we have extended the bosonic integration over the whole $M$ by introducing $\delta$-function of $\mu(x)$ with an appropriate Jacobian factor for $x^n$ integration. With an auxiliary variable $\phi$ to write $\delta(\mu(x)) = \frac{1}{(2\pi)^n} \int [d\phi] e^{i\phi\mu(x)}$, and also introducing $\psi^v$ and $\psi^n$ to write

$$
\frac{\partial}{\partial x^n} = \int [d\psi^n][d\psi^n] e^{\frac{\partial}{\partial x^n}} = \int [d\psi^n][d\psi^n] e^{\omega_{vn}\psi^v\psi^n}, \quad (2.22)
$$

where we have used (2.18), we arrive at

$$
\text{Vol}(M//U(1)) = \frac{1}{(2\pi)^n} \frac{1}{\text{Vol}(U(1))} \int_M [d\phi][dx^n][dx^n][dx^n][d\psi^n] [d\psi^n] e^{\frac{1}{2} \omega_{ij}\psi^i\psi^j + \omega_{vn}\psi^v\psi^n + i\phi\mu(x)}
$$

$$
= \frac{1}{(2\pi)^n} \frac{1}{\text{Vol}(U(1))} \int_{T[1]M \otimes \phi} [d\phi][dx^n] e^{\frac{1}{2} \omega_{ij}\psi^i\psi^j + i\phi\mu(x)}
$$

$$
= \frac{1}{(2\pi)^n} \frac{1}{\text{Vol}(U(1))} \int_{T[1]M \otimes \phi} e^S, \quad (2.23)
$$

where we have a complete $\omega = \frac{1}{2} \omega_{ij}\psi^i\psi^j$ and the measure on $T[1]M$ in the ambient space $M$. Note that there is no contribution from $\omega_{in}\psi^i\psi^n$ in the fermionic integration. Though we have used a specific coordinate system to arrive at the above, we see that the end result is coordinate invariant.

Our integration (2.23) is now over a supermanifold $T[1]M \otimes \phi$ and our action $S = \omega + i\phi\mu(x)$ is a function on $T[1]M \otimes \phi$. We find that $S$ is invariant under the following fermionic symmetry acting on $T[1]M \otimes \phi$,

$$
\tilde{Q} x^\mu = \psi^\mu, \quad \tilde{Q} \psi^\mu = -i\phi V^\mu(x), \quad \tilde{Q} \phi = 0, \quad (2.24)
$$

with $\tilde{Q} = -i\phi\mathcal{L}_V \simeq 0$ on the subspace of $V$-invariant functions. We can also write it as $\tilde{Q} = Q - i\phi \mathcal{L}_V$, with $Q$ the usual de Rham differential.

What we have done is to replace the symplectic form in the quotient space $T[1]M//U(1)$ with a $V$-invariant $\tilde{Q}$-closed object $S = \omega + i\phi\mu(x)$ in the ambient space $T[1]M \otimes \phi$. Up to a numerical factor in front, we see that the original symplectic volume integral over $T[1]M//U(1)$ is equal to the integral of the corresponding $\tilde{Q}$-closed form $e^S$ over the ambient space $T[1]M \otimes \phi$. The claim of equivariant cohomology is a more general statement of the above relation between $T[1]M//U(1)$ and $T[1]M \otimes \phi$. The usual $Q$-cohomology, or de Rham cohomology, of functions on $T[1]M//U(1)$ is identical to the equivariant $\tilde{Q}$-cohomology of $V$-invariant functions on $T[1]M \otimes \phi$, and their correlation functions also
match with each other. This provides an easy way of calculating things about the quotient space in the flat ambient space.

*Equivariant $\epsilon$-deformation*

Suppose that there is an additional $U(1)$ global symmetry on $M$ generated by $R$ with $QH = i_R \omega$, which is compatible with $V$, the symmetry by which we perform the quotient. This means that $[V, R] = 0$ and $i_R Q \mu = i_R i_V \omega = -i_V i_R \omega = -i_V QH = 0$, that is, $\mu(x)$ is invariant under $R$-flow and $R$ is well-defined on $\mu^{-1}(0)$. Under this assumption, since $R$ is a well-defined vector field on $\mu^{-1}(0)$,

$$R = R^i \frac{\partial}{\partial x^i} + R^v \frac{\partial}{\partial x^v},$$

and from $V = \frac{\partial}{\partial x^v}$, the component equation of $[V, R] = 0$ reads as

$$\left( \frac{\partial R^i}{\partial x^v} \right) \frac{\partial}{\partial x^i} + \left( \frac{\partial R^v}{\partial x^v} \right) \frac{\partial}{\partial x^v} = 0,$$

and we see that $R^i$ is $V$-invariant on $\mu^{-1}(0)$ and defines a vector field $R^i \frac{\partial}{\partial x^i}$ on $M//U(1)$. Moreover, $i$-component of the equation $i_R \omega = QH$ on $\mu^{-1}(0)$ is

$$R^j \omega_{ji} + R^v \omega_{vi} = \frac{\partial H}{\partial x^i},$$

and from $\omega_{vi} = 0$ on $\mu^{-1}(0)$ as before, we have $R^j \omega_{ji} = \frac{\partial H}{\partial x^i}$, which is a simple statement that $i_R \omega = QH$ still holds on the quotient space with the same function $H$, and $R$ is also a symmetry on $M//U(1)$.

We are then able to perform $\epsilon$-deformation with a symmetry $R$ in the quotient space $M//U(1)$ by simply replacing $\omega$ with $\omega - \epsilon H$ as before, and the usual $Q$ in $T[1]M//U(1)$ is now replaced by $Q_\epsilon = Q + \epsilon i_R$. Using equivariant cohomology, we next try to reformulate these in terms of the ambient space $T[1]M \otimes \phi$. The fermionic symmetry $\tilde{Q}$ in the ambient space will be modified to $\tilde{Q}_\epsilon = \tilde{Q} + \epsilon i_R = Q - i \phi i_V + \epsilon i_R$, and the $V$-invariant $\tilde{Q}_\epsilon$-closed object that corresponds to the deformed action will naturally be $\omega + i \phi \mu(x) - \epsilon H$ on $T[1]M \otimes \phi$. We also restrict to both $V$- and $R$-invariant subspace to ensure nilpotency of $\tilde{Q}_\epsilon^2 = 0$. Therefore, we arrive at

$$\text{Vol}_\epsilon(M//U(1)) = \frac{1}{(2\pi)\text{Vol}(U(1))} \int_{T[1]M \otimes \phi} e^{\omega + i \phi \mu(x) - \epsilon H(x)}.$$

We may add an arbitrary $V, R$-invariant $\tilde{Q}_\epsilon$-exact term to the action to facilitate localization in $T[1]M \otimes \phi$. The result usually boils down to contour integrations over $\phi$. 

10
Extension to Hyperkahler Quotient

We can generalize the previous discussions to hyperkahler quotients, and the idea is similar in spite of a few technical complications. A hyperkahler manifold $M$ has three kahler forms $\omega$, and its symplectic volume is simply defined in terms of one of them, say, $\omega^3 = \omega$. Once we pick up one kahler form $\omega$, its $\epsilon$-deformation by global symmetries of $\omega$ and localization are same as in the kahler manifolds. It is not important whether the other $\omega^1$ and $\omega^2$ may or may not be invariant under the global symmetry.

The difference arises in the quotient because hyperkahler quotient is something more than kahler quotient. Suppose there is a $V$-flow which is a symmetry of the three kahler forms $\omega$, such that $i_V \omega = Q \tilde{\mu}(x)$ with three moment maps $\tilde{\mu}(x)$. As before $V$-flow leaves invariant the codimension 3 submanifold $\tilde{\mu}^{-1}(0)$ and the hyperkahler quotient is defined as $M/\!/U(1) = \tilde{\mu}^{-1}(0)/U(1)$. Introducing a local coordinate system $\{x^i, x^n, x^m\}$ of $M$ around $\tilde{\mu}^{-1}(0)$, where $x^i$ parametrize $\tilde{\mu}^{-1}(0)/U(1)$, $V = \partial_{x^n}$, and $x^n, n = 1, 2, 3$, are three coordinates normal to $\tilde{\mu}^{-1}(0)$, it is easily verified in the exactly same manner as before that $\tilde{\omega}_{ij} = 0$, $\tilde{\omega}_{vm} = \partial_n \tilde{\mu}$, and $\tilde{\omega}_{ij}$ is $V$-invariant on $\tilde{\mu}^{-1}(0)$. The quotient space $M/\!/U(1)$ then naturally inherits $\tilde{\omega}_{ij}$ as its tri-holomorphic kahler forms of hyperkahler structure.

After picking up $\omega^3 = \omega$ to define the symplectic volume, it is written as

$$\text{Vol}(M/\!/U(1)) = \int_{T[1]M/\!/U(1)} e^{\frac{i}{2} \omega_{ij} \psi^i \psi^j}, \quad \text{(2.29)}$$

and to rewrite the above in a language of the ambient space $M$ as before, we have

$$\int_{T[1]M/\!/U(1)} = \frac{1}{\text{Vol}(U(1))} \int_{\tilde{\mu}^{-1}(0)} [dx^n][dx^i][d\psi^j] \frac{1}{\text{Vol}(U(1))} \int_M [dx^v][dx^n][dx^i][d\psi^j] \prod_{a=1}^3 \delta(\mu^a(x)) \det \left( \frac{\partial \mu^a(x)}{\partial x^n} \right) = \frac{1}{(2\pi)^3 \text{Vol}(U(1))} \int_M [d\tilde{\phi}][d\tilde{x}^i][dx^n][d\psi^j][d\psi^n] e^{i\tilde{\phi} \cdot \tilde{\mu} + \tilde{x}^i \cdot (\partial_n \tilde{\mu}) \psi^n} \quad \text{(2.30)}$$

where we introduce bosonic auxiliary variables $\tilde{\phi}$ and fermionic $\tilde{\chi}$ as well as $\psi^n$. Using $\partial_n \mu^3 = \omega^3_{vn} = \omega_{vn}$, and $\omega_{vi} = 0$, as well as calling $\chi_3 = \psi^n$, we have $\chi_3(\partial_n \mu^3) \psi^n = \psi^n \omega_{vn} \psi^n = \psi^n \omega_{vm} \psi^m$, where $\mu$ runs now over all coordinates of $M$. Similarly, $\chi_a(\partial_n \mu^a) \psi^n = \chi_a(\partial_n \mu^a) \psi^\mu = \chi_a Q \mu^a$ for $a = 1, 2$. Inserting these to (2.29), we obtain

$$\text{Vol}(M/\!/U(1)) = \frac{1}{(2\pi)^3 \text{Vol}(U(1))} \int_{T[1]M/\!/U(1)} e^{\omega_{ij} \psi^i \psi^j + \frac{i}{2} \omega_{vm} \psi^m \psi^n} \quad \text{(2.31)}$$

where $a = 1, 2$ and in completing $\omega = \frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu$ in the action, we have used the fact that the missing piece $\omega_{in} \psi^i \psi^n + \frac{1}{2} \omega_{mn} \psi^m \psi^n$ can be absorbed by shifting $\psi^i$ and $\chi_a$ variables.
Note that the end result is a coordinate-invariant expression. Therefore, we see that the symplectic volume integral in the quotient space can be written as an integral in the ambient supermanifold $T[1]M \otimes \tilde{\phi} \otimes \chi_a$.

The action $S = \omega + i\tilde{\phi} \cdot \tilde{\mu}(x) + \chi_a Q \mu^a(x)$ is easily seen to be invariant under a fermionic symmetry on $T[1]M \otimes \tilde{\phi} \otimes \chi_a$,

$$
\bar{Q} x^\mu = \psi^\mu,
\bar{Q} \psi^\mu = -i \phi_3 V^\mu(x),
\bar{Q} \phi_3 = 0,
\bar{Q} \chi_a = -i \phi_a, a = 1, 2,
$$

which is also written as $\bar{Q} = Q - i \phi_3 i V - i \phi_a \frac{\partial}{\partial \chi_a}$, with $\bar{Q}^2 = -i \phi_3 \mathcal{L}_V \simeq 0$ in the subspace of $V$-invariant functions. The natural expectation of equivariant cohomology for hyperkahler quotient would be that the usual $Q$-cohomology on $T[1]M//U(1)$ is identical to the $\bar{Q}$-cohomology on $V$-invariant functions of $T[1]M \otimes \tilde{\phi} \otimes \chi_a$.

Now we turn to the question of equivariant description of an $\epsilon$-deformation in the quotient space in terms of the ambient space $T[1]M \otimes \tilde{\phi} \otimes \chi_a$. Contrary to the previous kahler case, we can have two distinct situations. The first case is where we have an additional symmetry $R$ in $M$ compatible with $V$, which preserves all three kahler forms $\omega, \tilde{\omega}$. In this case, the modification of $\bar{Q}$ is simply $\bar{Q}_\epsilon = \bar{Q} + \epsilon \mathcal{L}_R$ with the modification of the action $S_\epsilon = S - \epsilon H$ with $i_R \omega = QH$. The structure is essentially identical to that for the kahler case. The more interesting case is where the symmetry $R$ preserves only $\omega^3 = \omega$ and it acts as an $U(1)_R$ rotation for the other $\omega^1$ and $\omega^2$, that is, $\mathcal{L}_R(\omega^1 - i \omega^2) = 2i (\omega^1 - i \omega^2)$, $\mathcal{L}_R(\mu^1 - i \mu^2) = R^\mu \partial_\mu (\mu^1 - i \mu^2) = 2i (\mu^1 - i \mu^2)$ and $i_R \omega = QH$. In this case, the deformed action $S_\epsilon = S - \epsilon H$ is invariant under the symmetry

$$
\bar{Q}_\epsilon x^\mu = \psi^\mu,
\bar{Q}_\epsilon \psi^\mu = -i \phi_3 V^\mu(x) + \epsilon R^\mu(x),
\bar{Q}_\epsilon \phi_3 = 0,
\bar{Q}_\epsilon \phi_1 = 2i \epsilon \chi_2,
\bar{Q}_\epsilon \phi_2 = -2i \epsilon \chi_1,
\bar{Q}_\epsilon \chi_a = -i \phi_a, a = 1, 2,
$$

with $\bar{Q}_\epsilon^2 = -i \phi_3 \mathcal{L}_V + \epsilon \mathcal{L}_R$, where $\mathcal{L}_R$ acts on $\phi_a$ and $\chi_a$ as

$$
\mathcal{L}_R(\phi_1 - i \phi_2) = 2i (\phi_1 - i \phi_2), \quad \mathcal{L}_R(\chi_1 - i \chi_2) = 2i (\chi_1 - i \chi_2).
$$

(2.33)
Then, we have to restrict to $R$-invariant subspace of $T[1]M \otimes \bar{\phi} \otimes \chi_a$, taking into account of the $R$-action on $\phi_a$ and $\chi_a$, when we discuss adding $\bar{Q}_\epsilon$-exact terms to the action.

An interesting point is that we can always have one such term

$$-it\bar{Q}_\epsilon(\chi_a \phi_a) = -t\phi_a \phi_a - (4\epsilon t)\chi_1 \chi_2 \quad ,$$

and by taking $t \to +\infty$ limit, $\phi_a$, $\chi_a$ integration is dominated by this term and we can simply integrate them out leaving

$$\int [d\phi_1][d\phi_2][d\chi_1][d\chi_2] e^{-t\phi_a \phi_a - (4\epsilon t)\chi_1 \chi_2} = \frac{\pi}{t} \cdot (4\epsilon t) = 4\pi \epsilon \quad .$$

Performing this, our $\epsilon$-deformed symplectic volume ends up to

$$\text{Vol}_\epsilon(M//U(1)) = \frac{4\pi \epsilon}{(2\pi)^3 \text{Vol}(U(1))} \int_{T[1]M \otimes \phi_3} e^{\omega + i\phi_3 \mu^2(x) - \epsilon H} \quad ,$$

which looks just like that of the kahler case (2.28). This formula will be our starting point in the next section.

### 3 Volumes of Toric Tri-Sasakian Manifolds and Their Supersymmetric Cycles

A simple definition of Tri-Sasakian manifold is that its metric cone is hyper-Kahler. In other words, a $(4n - 1)$-dimensional manifold $X_{4n-1}$ is Tri-Sasakian when its cone with the metric

$$ds_{4n}^2 = dr^2 + r^2 ds_{4n-1}^2 \quad ,$$

is a $(4n)$-dimensional hyper-Kahler cone, where $ds_{4n-1}^2$ is normalized to satisfy $R_{ij} = (4n - 2)g_{ij}$. Starting from flat quaternion spaces, we can construct non-trivial hyper-Kahler cones by the process of hyper-Kahler quotient, and we subsequently obtain Tri-Sasakian manifolds as the constant radius section of the cones [22,23,24,25]. As we will consider only abelian symmetries $U(1)^r$ of flat quaternion spaces in performing hyper-Kahler quotient, the resulting hyper-Kahler cones or Tri-Sasakian manifolds are toric.

We expect that we can obtain generic toric Tri-Sasakian manifolds in this way, as in the case of toric Kahler manifolds.

A single quaternion $q$, which is equivalent to a flat $R^4$, is given by $q = q^4 I_2 + i\bar{\sigma} \cdot \bar{q}$ with four real numbers $(q^4, \bar{q})$. It is also useful to introduce two complex variables $u$ and $v$ to write

$$q = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \quad .$$
The flat metric is \( ds^2 = \frac{1}{2} \text{tr} (dq d\bar{q}) = dud\bar{u} + dvd\bar{v} \), and the triplet of Kahler forms of hyper-Kahler structure is given by \( \bar{\omega} \cdot \bar{\sigma} = \frac{1}{2} dq \wedge d\bar{q} \), or more explicitly

\[
\begin{align*}
\omega^3 &= -\frac{i}{2} (du \wedge d\bar{u} + dv \wedge d\bar{v}) , \\
(\omega^1 - i \omega^2) &= i (du \wedge dv) .
\end{align*}
\] (3.40)

Under a \( U(1) \)-action of charge \( Q \) defined by \( q \rightarrow q \cdot e^{iQ\sigma_3 \xi} \), where \( \xi \) is an angle variable, we easily see that the triplet of Kahler forms \( \omega \) is invariant, and we will use these actions later to perform hyper-Kahler quotients. In components, this corresponds to \( u \rightarrow u \cdot e^{iQ} \) and \( v \rightarrow v \cdot e^{-iQ} \). Under the \( SU(2) \) \(_R\) symmetry given by

\[
q \rightarrow \exp \left( -\frac{i}{2} \bar{\epsilon} \cdot \bar{\sigma} \right) q ,
\] (3.41)

the Kahler forms transform as a triplet.

We will start from a \((n + r)\)-dimensional flat quaternion space \((q_a)\), \( a = 1, \ldots, (n + r) \), or flat \( R^{4(n+r)} \), and consider \( G = U(1)^r \) action under which \( q_a \) has integer charges \( Q^i_a, i = 1, \ldots, r \). After performing hyper-Kahler quotient by \( G = U(1)^r \), we obtain a \((4n)\)-dimensional hyper-Kahler cone and its \((4n - 1)\)-dimensional Tri-Sasakian cross-section, which is labeled by the charges \( Q^i_a \). We will calculate its volume as well as the volumes of its co-dimension 1 supersymmetric cycles via the equivariant localization technique in the previous section. The results are simple integration formulae in terms of \( Q^i_a \), and in many cases they can be explicitly calculated into compact expressions, as we will see in several examples.

It is worth commenting a difference between our case of Tri-Sasakian manifolds and the case of Sasaki-Einstein manifolds from Calabi-Yau cones. In the case of toric Calabi-Yau cones or Sasaki-Einstein manifolds obtained from Kahler quotient of flat spaces, the induced metric on the quotient space from the ambient flat space is not in general Calabi-Yau, although the Kahler class coincides. The actual Ricci-flat Kahler metric must be found in other ways, and this is all about recent development of Z-minimization [9, 10]. In the dual \( \mathcal{N} = 1 \) SCFT side, this corresponds to non-rigidity of \( U(1)_R \) symmetry under RG-flow, and the fixing of the \( U(1)_R \) symmetry by a-maximization [18] at IR fixed point is equivalent to finding Ricci-flat Kahler metric in the gravity side. On the contrary, hyper-Kahler structure that we are interested in for Tri-Sasakian manifolds is more rigid, essentially because of its non-abelian \( SU(2)_R \) structure, and the induced metric from the flat ambient space is already hyper-Kahler that we need. This is a basic reason behind our ability to calculate the volumes via equivariant localization. In 3D \( \mathcal{N} = 3 \) SCFT...
side, this is related to the fact that $SU(2)_R$ symmetry is rigid under RG-flow due to its non-abelian nature.

**Toric Tri-Sasakian Manifolds**

The $G = U(1)^r$-action on $(n + r)$-dimensional quaternion space $H^{(n+r)}$, $(q_a), a = 1, \ldots, (n + r)$ with charges $Q^i_a, i = 1, \ldots, r$, is given by

$$q_a \rightarrow q_a \cdot e^{iQ^i_a \sigma^i} , \quad (3.42)$$

or in terms of complex coordinates $u_a \rightarrow u_a \cdot e^{iQ^i_a \xi_i}$ and $v_a \rightarrow v_a \cdot e^{-iQ^i_a \xi_i}$. The generating vector fields of $G$ commuting with each other are

$$V^i = \frac{\partial}{\partial \xi_i} = i \sum_a Q^i_a \left( u_a \frac{\partial}{\partial u_a} - \bar{u}_a \frac{\partial}{\partial \bar{u}_a} - v_a \frac{\partial}{\partial v_a} + \bar{v}_a \frac{\partial}{\partial \bar{v}_a} \right) , \quad (3.43)$$

and with (3.40), the corresponding moment maps defined by $i_{V^i} \bar{\omega} = Q^i \bar{\mu}_i$ are easily calculated to be

$$\mu^3 = \frac{1}{2} \sum_a Q^i_a (|u_a|^2 - |v_a|^2) , \quad (\mu^1 - i \mu^2) = - \sum_a Q^i_a u_a v_a . \quad (3.44)$$

It is also easy to find a $U(1)_R \subset SU(2)_R$ symmetry with the properties $\mathcal{L}_R \omega^3 = 0$, $\mathcal{L}_R (\omega^1 \omega^2) = 2i(\omega^1 \omega^2)$, and $\mathcal{L}_R (\mu^1 - i \mu^2) = R[i \partial \mu^1 - i \mu^2] = 2i(\mu^1 - i \mu^2)$ for all $i$, to implement equivariant $\epsilon$-deformation that we discussed in the previous section,

$$R = i \sum_a \left( u_a \frac{\partial}{\partial u_a} - \bar{u}_a \frac{\partial}{\partial \bar{u}_a} + v_a \frac{\partial}{\partial v_a} - \bar{v}_a \frac{\partial}{\partial \bar{v}_a} \right) . \quad (3.45)$$

The crucial fact that makes it possible to extract the volume of the transverse Tri-Sasakian section out of the $\epsilon$-deformed volume of the hyper-Kahler cone is that the function $H$ defined by $i_{R^i \omega^3} = QH$ turns out to be simply $H = \frac{1}{2} \sum_a (|u_a|^2 + |v_a|^2) = \frac{1}{2} r^2$, where $r$ is the radial coordinate of the hyper-Kahler cone. Therefore the resulting expression (2.37) of the $\epsilon$-deformed volume of our hyper-Kahler cone labeled by charges $Q^i_a$,

$$\text{Vol}_\epsilon \left( H^{(n+r)}/U(1)^r \right) = \frac{(4\pi \epsilon)^r}{(2\pi)^{3r} \text{Vol}(U(1)^r)} \int_{\mathcal{T}|H^{(n+r)} \otimes \phi^a} e^{\omega + i\phi^a \mu^i_a(x) - \epsilon H} , \quad (3.46)$$

must be equal to the regularized volume of the cone $ds^2_{4n} = dr^2 + r^2 ds^2_{4n-1}$ with a damping factor $e^{-\frac{\epsilon}{2} r^2}$, which is nothing but

$$\text{Vol}_\epsilon \left( H^{(n+r)}/U(1)^r \right) = \frac{2^{2n-1} \Gamma(2n)}{\epsilon^{2n}} \cdot \text{Vol}(X_{4n-1}) , \quad (3.47)$$
with the normalized volume of our Tri-Sasakian space $X_{4n-1}$. The integration over $T[1]H^{(n+r)}$ in the right-hand side of (3.46) is simple Gaussian integration for both bosonic and fermionic variables, and we can calculate it easily. This is the pay-off that we receive for all the previous technicalities of reformulating things in terms of the flat ambient space where calculations become tractible. What remains is an integration over $\phi^i, i = 1, \ldots, r$, and the resulting expression for the right-hand side of (3.46) is

$$\text{Vol}_{k} \left( H^{(n+r)}//U(1)^r \right) = \frac{2^r(2\pi)^{2n}}{\epsilon^{2n}\text{Vol}(U(1)^r)} \int \prod_{i=1}^{r} d\phi^i \prod_{a=1}^{n+r} \frac{1}{1 + (\sum_{i=1}^{r} Q_a^i \phi^i)^2}, \quad (3.48)$$

and by comparing with (3.47), we finally obtain the result for the normalized volume of our Tri-Sasakian manifold labeled by $Q_a^i$

$$\text{Vol} (X_{4n-1}) = \frac{2^{r+1} \pi^{2n}}{\Gamma(2n)\text{Vol}(U(1)^r)} \int \prod_{i=1}^{r} d\phi^i \prod_{a=1}^{n+r} \frac{1}{1 + (\sum_{i=1}^{r} Q_a^i \phi^i)^2}. \quad (3.49)$$

Note that $\text{Vol} (U(1)^r)$ that is defined in the previous section is the coordinate volume of the $r$-dimensional torus defined by the angle variables $\xi_i$. That is, we identify $\xi_i \sim \xi_i + \eta_i$ whenever $\sum_{i=1}^{r} Q_a^i \eta_i \in 2\pi Z$ for all $a = 1, \ldots, (n + r)$, and $\text{Vol} (U(1)^r)$ is the volume of an unit cell in the $r$-dimensional $\xi_i$ space. As it should be, our result (3.49) is invariant under an overall rescaling of $Q_a^i$ because of this $\text{Vol} (U(1)^r)$ factor, and we can consider only relatively prime set of charges $Q_a^i$.

**Examples**

Let us consider the simplest example with $n = r = 1$, which is a 3-dimensional Tri-Sasakian manifold obtained by $U(1)$ hyper-Kahler quotient from $H^2$, labeled by a relatively prime pair of integer charges $(Q_1, Q_2)$. Explicitly, in terms of complex variables of two flat quaternions $q_1 \sim (u_1, v_1)$ and $q_2 \sim (u_2, v_2)$, the moment map equations are

$$Q_1 (|u_1|^2 - |v_1|^2) + Q_2 (|u_2|^2 - |v_2|^2) = 0, \quad Q_1 u_1 v_1 + Q_2 u_2 v_2 = 0, \quad (3.50)$$

with the identification $(u_1, v_1, u_2, v_2) \sim (u_1 e^{iQ_1 \xi}, v_1 e^{-iQ_1 \xi}, u_2 e^{iQ_2 \xi}, v_2 e^{-iQ_2 \xi})$. The first equation in the above with this $U(1)$-quotient is a usual Kahler quotient of $C^4$ with charges $(Q_1, -Q_1, Q_2, -Q_2)$ which is also equivalent to a holomorphic quotient of $C^4$ by a $C^*$ action

$$(u_1, v_1, u_2, v_2) \sim (\lambda^{Q_1} u_1, \lambda^{-Q_1} v_1, \lambda^{Q_2} u_2, \lambda^{-Q_2} v_2), \quad \lambda \in C^*. \quad (3.51)$$

Introducing $Z_1 = u_1 v_1$, $Z_2 = u_2 v_2$, $Z_3 = u_1^Q_2 v_2^Q_1$ and $Z_4 = u_2^Q_1 v_1^Q_2$ satisfying an algebraic equation $Z_1^Q_2 Z_2^Q_1 = Z_3 Z_4$ that defines a hypersurface in $C^4$, it is not difficult to show that
the above map from \((u_i, v_i)\) to \(Z_i\) is a bijection from \(C^4/C^*\) to the hypersurface \(Z_1^{Q_1}Z_2^{Q_2} = Z_3Z_4\) in \(C^4\). The fact that \((Q_1, Q_2)\) are relatively prime integers is used in showing the above equivalence. In terms of \(Z_i\), the remaining equation \(Q_1u_1v_1 + Q_2u_2v_2 = 0\) becomes \(Q_1Z_1 + Q_2Z_2 = 0\), and therefore our hyper-Kahler cone is identified as an algebraic variety in \(C^4\) defined by two equations

\[
Z_1^{Q_1}Z_2^{Q_2} = Z_3Z_4, \quad Q_1Z_1 + Q_2Z_2 = 0, \tag{3.52}
\]

and after solving for \(Z_2 = -\frac{Q_1}{Q_2}Z_1\) and inserting into the first equation, it becomes up to constant rescaling, a variety in \(C^3\) described by \(Z_1^{Q_1+Q_2} = Z_3Z_4\). This is nothing but an ALE space with \(A_{(Q_1+Q_2-1)}\)-singularity or \(C^2/Z(Q_1+Q_2)\) embedded into \(C^3\), which is indeed a hyper-Kahler cone. The corresponding Tri-Sasakian manifold is then simply \(S^3/Z(Q_1+Q_2)\) or the Lens space \(L(Q_1+Q_2)\), whose normalized volume must be

\[
\frac{\text{Vol}(S^3)}{(Q_1 + Q_2)} = \frac{2\pi^2}{(Q_1 + Q_2)} \tag{3.53}
\]

To check that our formula \(\text{(3.49)}\) indeed reproduces this, note that \(\text{Vol}(U(1))\) is the minimal number \(\eta\) with \(Q_1\eta, Q_2\eta \in 2\pi Z\), which is simply \(2\pi\) for relatively prime \((Q_1, Q_2)\). We then have

\[
\text{Vol}(X_3) = 2\pi \int d\phi \frac{1}{(1 + Q_1^2\phi^2)(1 + Q_2^2\phi^2)} = 2\pi \cdot \frac{\pi}{(Q_1 + Q_2)} = \frac{2\pi^2}{(Q_1 + Q_2)} \tag{3.54}
\]

which agrees with the previous result.

In the context of M-theory in \(AdS_4 \times X_7\) that is dual to a 3D \(\mathcal{N} = 3\) SCFT, the next example is \(n = 2\) with \(r = 1\) \[30\]. The resulting 7-dimensional space is labeled by three integer charges \((Q_1, Q_2, Q_3)\) being relatively prime. With

\[
\text{Vol}(U(1)) = (2\pi) l.c.m. \left(\frac{1}{Q_i}\right) = \frac{2\pi}{Q_1Q_2Q_3} l.c.m.(Q_1Q_2, Q_2Q_3, Q_3Q_1) = 2\pi \tag{3.55}
\]

where \(l.c.m.\) stands for least common multiple, the right-hand side of \(\text{(3.49)}\) is readily calculated to be

\[
\text{Vol}(X_7(Q_1, Q_2, Q_3)) = \frac{\pi^4}{3} \frac{(Q_1Q_2 + Q_2Q_3 + Q_3Q_1)}{(Q_1 + Q_2)(Q_2 + Q_3)(Q_3 + Q_1)} \tag{3.56}
\]

which includes the unique homogeneous 7-dimensional Tri-Sasakian manifold \(N(1, 1) = SU(3)/U(1)\) as a special case of \(Q_1 = Q_2 = Q_3 = 1\) with volume \(\frac{\pi^4}{8}\).

**Codimension 1 Cycles**
In M theory on $AdS_4 \times X_7$, an M5-brane wrapping a supersymmetric codimension 1 cycle $\Sigma_5$ in $X_7$ looks like a very heavy point-like excitation in $AdS_4$ spacetime. Its mass in $AdS_4$ is proportional to the volume of the cycle, and by the standard AdS/CFT relation between mass in AdS and the conformal dimension $\Delta$ of the corresponding operator in the dual CFT, we have

$$\Delta = \frac{\pi N \text{Vol}(\Sigma_5)}{6 \text{Vol}(X_7)},$$

(3.57)

where $N$ is the number of background M2-brane charge in $AdS_4 \times X_7$ \[33\]. We are therefore interested in volumes of codimension 1 cycles.

We will calculate volumes of supersymmetric codimension 1 cycles defined by a holomorphic constraint $u_a = 0$ or $v_a = 0$ for some $a$, for generic toric Tri-Sasakian manifolds labeled by $Q_i^a$. Since the result turns out to be independent of $a$, we will simply consider the cycle defined by $u_1 = 0$. In the flat ambient space $H^{(n+r)}$ before performing hyper-Kahler quotient, the hypersurface $u_1 = 0$ is Poincare dual to the 2-form

$$\Gamma_2 = \delta(u_1) \delta(\bar{u}_1) \psi^{u_1} \bar{\psi}^{\bar{u}_1},$$

(3.58)

with $Q \Gamma_2 = 0$. Though it is divergent, the symplectic volume of the hypersurface is represented formally by the expectation value of the dual 2-form $\Gamma_2$ in the previous superspace formalism of the symplectic volume of $H^{(n+r)}$;

$$\text{Vol}(\Sigma_{u_1=0}) = \langle \Gamma_2 \rangle = \int_{T[1]H^{(n+r)}} \Gamma_2 e^S,$$

(3.59)

with $S = \omega$. The delta functions in $\Gamma_2$ restrict the integration onto the hypersurface $u_1 = 0$ and the fermionic factor takes care of reduction of the degree of the volume form on $u_1 = 0$. The fact $Q \Gamma_2 = 0$ is interpreted as $\Gamma_2$ being a nice observable in the $Q$-cohomology.

Getting back to our hyper-Kahler cone obtained by a hyper-Kahler quotient of $G = U(1)^r$ with charges $Q_i^a$, we previously described the quotient space in terms of the ambient superspace $T[1]H^{(n+r)} \otimes \tilde{\phi}^i \otimes \chi_a^i$, $i = 1, \ldots, r, a = 1, 2$, with a fermionic symmetry $\tilde{Q}$ in \[232\], and the action $S = \omega + i\tilde{\phi}^i \cdot \tilde{\mu}_i(x) + \chi_a^i Q \mu_i^a(x)$. According to the spirit of equivariant cohomology, the usual $Q$-cohomology of the quotient space is equivalent to the $\tilde{Q}$-cohomology in the ambient space. Moreover, expectation values of cohomology elements are also expected to agree with each other. The hypersurface defined by $u_1 = 0$ in the quotient space should be described by some Poincare dual 2-form, and to describe it in terms of the ambient superspace, we need to find a suitable generalization of $\Gamma_2$.
satisfying \( \tilde{Q}\tilde{\Gamma}_2 = 0 \). Fortunately, due to delta function factor in \( \Gamma_2 \), we simply have \( \tilde{\Gamma}_2 = \Gamma_2 \).

Note that the hypersurface \( u_1 = 0 \) in our \( 4n \)-dimensional hyper-Kahler cone is another cone with dimension \( 4n - 2 \). As our Tri-Sasakian manifold \( X_{4n-1} \) is the unit radius section of the hyper-Kahler cone, its codimension 1 cycle \( \Sigma_{4n-3} \) that we are interested in is the unit radius section of the hypersurface \( u_1 = 0 \). We have previously introduced an \( \epsilon \)-deformation using \( U(1)_R \subset SU(2)_R \) symmetry, which is nothing but a damping factor \( -\frac{\epsilon}{2}r^2 \), where \( r \) is the radial coordinate of the cone. From the resulting \( \epsilon \)-regularized volume of the cone, we were able to extract the volume of the unit radius section of the cone. After we deform the fermionic symmetry from \( \tilde{Q} \) to \( \tilde{Q}_\epsilon \) in (2.33) in addition to \( S_\epsilon = S - \epsilon H = S - \frac{\epsilon}{2}r^2 \), the \( \epsilon \)-regularized volume of the hypersurface \( u_1 = 0 \) is still expected to be an expectation value of an observable \( \tilde{\Gamma}_2 \) with \( \tilde{Q}_\epsilon \tilde{\Gamma}_2 = 0 \). Again due to the delta function factor, we easily find that \( \Gamma_2 \) satisfies \( \tilde{Q}_\epsilon \Gamma_2 = 0 \).

In summary, the regularized volume of the cone \( u_1 = 0 \) with dimension \( 4n - 2 \) inside our quotient space is simply obtained by inserting \( \Gamma_2 \) in the partition function (3.46),

\[
\langle \Gamma_2 \rangle_\epsilon = \frac{(4\pi \epsilon)^r}{(2\pi)^{3n}\text{Vol}(U(1)^r)} \int_{\pi \mathbb{H}^{(n+r)}} \Gamma_2 e^{i \omega + i \phi_i \mu_i^3(x) - \epsilon H} \cdot \text{Vol}(\Sigma_{4n-3}),
\]

(3.60)

Since the regularization is a simple factor \( -\frac{\epsilon}{2}r^2 \), the above must be equal to

\[
2^{2n-2}(2n-1) \cdot \epsilon^{2n-1} \cdot \text{Vol}(\Sigma_{4n-3}),
\]

(3.61)

where \( \Sigma_{4n-3} \) is the unit radius section, which is our codimension 1 cycle inside \( X_{4n-1} \).

The Gaussian integration is readily calculated as before to arrive at

\[
\text{Vol}(\Sigma_{4n-3}) = \frac{2^{r+1}\pi^{2n-1}}{\Gamma(2n-1)\text{Vol}(U(1)^r)} \int \prod_{i=1}^r d\phi^i \frac{1}{1 + i \sum_{i=1}^r Q_i^i \phi_i} \cdot \prod_{a=2}^{n+r} \frac{1}{1 + (\sum_{i=1}^r Q_a^i \phi_i)^2}.
\]

(3.62)

where in the second line the imaginary part vanishes under the integration over \( \phi^i \).

An interesting fact is that the above looks very similar to the volume expression of
\(X_{4n-1}\) in (3.49), in fact, the complicated integration over \(\phi^i\) is identical. Their ratio is
\[
\frac{\text{Vol}(\Sigma_{4n-3})}{\text{Vol}(X_{4n-1})} = \frac{1}{\pi} \frac{\Gamma(2n)}{\Gamma(2n-1)} = \frac{(2n-1)}{\pi},
\]
which depends only on the dimension \((4n-1)\) of the Tri-Sasakian space without regard to the quotient group \(G = U(1)^r\). It would be interesting to find an underlying mathematical reason behind this universality. Its consequence in AdS/CFT correspondence of M-theory in \(AdS_4 \times X_7\) is that the conformal dimension of chiral primary baryonic operators in \(\mathcal{N} = 3\) SCFT is always \(\Delta = \frac{N}{2}\).

4 \((2+1)D\) \(\mathcal{N} = 3\) Field Theories

Some facts about \((2+1)D\) field theories

Let us first recall some unusual subtleties in \((2+1)D\) field theories, as we will need those in discussing \(\mathcal{N} = 3\) supersymmetric theory. For a massive excitation, we can go to its rest frame and its spin \(s\) is defined as the charge of \(U(1) = SO(2)\) spatial rotation of \(R^2\). Note that the signature of \(s\) is meaningful as it is impossible to flip its sign using Lorentz transformation, contrary to \(3(1)D\) case. In fact, \(s\) is invariant under CPT and particles and anti-particles have the same spin \(s\),
\[
s \longrightarrow P -s \longrightarrow T s \longrightarrow C s.
\]
Since \(s\) flips its sign under parity \(P\), a theory with particles of spin \(s\) without particles of \(-s\) breaks parity. Because we can exchange two identical excitations on the spatial \(R^2\) plane, there still exists the concepts of statistics, and the usual spin-statistics theorem holds true in \((2+1)D\). A statistical phase under exchange of two identical particles must form a representation of \(\pi_1(RP^1) = \pi_1(S^1) = Z\) and can take an arbitrary \(U(1)\) phase, though we will only concern about bosons and fermions in the usual sense.

An example of massive, parity breaking theories is the Maxwell-Chern-Simons theory \cite{39},
\[
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon_{\mu\nu\lambda}A_\mu F_{\nu\lambda} - \frac{\xi}{2}(\partial_{\mu}A_\mu)^2,
\]
which describes spin \(s = \frac{\kappa}{|\xi|}\) excitations with \(m^2 = \kappa^2\) upon quantization. To see this briefly, the operator equation of motion in \(\xi = 1\) gauge is
\[
\left(\partial^2 \eta^{\mu\nu} - \kappa \epsilon^{\mu\nu\lambda}\partial_\lambda\right)A_\nu = 0,
\]
and expanding in terms of creation/annihilation operators $a_\dagger(\vec{p})$ and $a(\vec{p})$,

$$A_\mu(x) = \int \frac{d^2\vec{p}}{(2\pi)^2 \sqrt{2p_0}} \left[ \varepsilon_\mu(\vec{p}) e^{-ip\cdot x} a(\vec{p}) + \text{h.c.} \right] , \quad (4.67)$$

its polarization $\varepsilon_\mu(\vec{p})$ for particle excitations satisfies

$$\left( p^2 \eta^{\mu\nu} - i\kappa \varepsilon^{\mu\nu\lambda} p_\lambda \right) \varepsilon_\nu = 0 , \quad (4.68)$$

whose non-trivial solution exists uniquely when $p^2 = -\kappa^2$. To determine the spin, we go to the rest frame of particle excitation $p_\mu = (|\kappa|, \vec{0})$ where the polarization becomes

$$\varepsilon_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \frac{\kappa}{|\kappa|} \end{pmatrix} , \quad (4.69)$$

which has the charge $s = \frac{\kappa}{|\kappa|}$ under spatial $SO(2)$ rotation. Note that there is no additional anti-particle excitations in the theory.

Another example of parity breaking theories is a massive Majorana fermion in (2+1)D. In (2+1)-dimension, two-components Majorana representation is possible with

$$\gamma^0 = i\sigma^2 , \quad \gamma^1 = \sigma^1 , \quad \gamma^2 = \sigma^3 . \quad (4.70)$$

A Majorana spinor $\psi = (\psi_1, \psi_2)^T$ transforms as $(\psi_1, \psi_2)^T \rightarrow (e^{\frac{i\pi}{2}} \psi_1, e^{-\frac{i\pi}{2}} \psi_2)^T$ under a boost, and $(\psi_1, \psi_2)^T \rightarrow (\cos(\frac{\phi}{2})\psi_1 - \sin(\frac{\phi}{2})\psi_2, \sin(\frac{\phi}{2})\psi_1 + \cos(\frac{\phi}{2})\psi_2)^T$ under a rotation of angle $\phi$, which are consistent with the reality of $\psi$. The massive Lagrangian is

$$\mathcal{L} = i\bar{\psi} \gamma^\mu \partial_\mu \psi + im\bar{\psi} \psi , \quad (4.71)$$

whose equation of motion $i\gamma^\mu \partial_\mu \psi + im\psi = 0$ in components becomes

$$(\partial_2 + m)\psi_1 + (\partial_0 + \partial_1)\psi_2 = 0 \quad , \quad (-\partial_0 + \partial_1)\psi_1 + (-\partial_2 + m)\psi_2 = 0 . \quad (4.72)$$

The non-trivial solution exists only when $p^2 = -m^2$ as expected, with the form

$$\psi = \left( \begin{array}{c} 1 \\ -\frac{p_2 + im}{p_0 + p_1} \end{array} \right) e^{-ip\cdot x} a(\vec{p}) + \text{h.c.} , \quad (4.73)$$

for the mode with momentum $\vec{p}$. In the rest frame $p_\mu = (|m|, \vec{0})$, the wave function looks like $(1, -i\frac{m}{|m|})^T$, which has the charge $s = -\frac{1}{2} \frac{m}{|m|}$ under spatial rotation of angle $\phi$. Therefore, the spin depends on the sign of the mass term, and indeed the mass term in the Lagrangian is odd under parity. The usual parity conserving Dirac mass term for
a Dirac spinor can be considered as two Majorana fermions with opposite sign of mass terms.

If we take massless limit, the relevant concept would be helicity rather than spin as there doesn’t exist a rest frame at all, but since there is no little group in (2+1)-dimension, helicity does not exist either. This can be understood in an example of the duality between $U(1)$ gauge theory and a periodic real scalar field theory in (2+1)D. The statistics whether excitations are bosons or fermions remains meaningful in the massless theory from (anti)commutator algebra of operators. This means that the Hilbert space allows an operator $(-1)^F$ which anti-commutes with all fermionic operators.

A warm-up with $\mathcal{N} = 1$

The basic unit of supercharges in (2+1)D is a Majorana spinor with two real components. From the Lorentz transformation of the $\mathcal{N} = 1$ supercharge $Q_\alpha = (Q_1, Q_2)^T$, we see that $Q_+ = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$ has spin $\frac{1}{2}$, and $Q_- = (Q_+)^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$ has spin $-\frac{1}{2}$. Since $(p_1 \pm ip_2)$ have spin $\pm 1$ and $p_0 = E$ has spin 0, we expect

$$\{Q_+, Q_-\} = E \quad \{Q_+, Q_+\} = (p_1 + ip_2) \quad \{Q_-, Q_-\} = (p_1 - ip_2) \quad (4.74)$$

which can be written in a covariant way as

$$\{Q_\alpha, Q_\beta\} = -\left(\gamma^0 \gamma^\mu p_\mu\right)_{\alpha\beta} \quad (4.75)$$

For a massless excitation of a $\mathcal{N} = 1$ theory, we go to the frame with $E = p_1$ and $p_2 = 0$, where we have

$$\{Q_1, Q_1\} = E \quad \{Q_2, Q_2\} = \{Q_1, Q_2\} = 0 \quad (4.76)$$

The minimal representation is a pair of bosonic/fermionic excitations $|b\rangle, |f\rangle$ with $|f\rangle = Q_1 |b\rangle$ and $|b\rangle = Q_1 |f\rangle$. This is easily realized by the minimal super Yang-Mills theory of gauge field and Majorana gaugino,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\lambda} \gamma^\mu \partial_\mu \lambda \quad (4.77)$$

which is invariant under $\delta A_\mu = i \bar{\epsilon} \gamma_\mu \lambda = -i \bar{\lambda} \gamma_\mu \epsilon$ and $\delta \lambda = -\frac{1}{4} F_{\mu\nu} \gamma^{\mu\nu} \epsilon$. For $U(1)$ theory, it is equivalent to a theory of real scalar field and a Majorana fermion using abelian duality,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + i \bar{\lambda} \gamma^\mu \partial_\mu \lambda \quad (4.78)$$

with a supersymmetry $\delta \phi = i \bar{\epsilon} \lambda$ and $\delta \lambda = -\frac{1}{2} \partial_\mu \phi \gamma^\mu \epsilon$. The two supersymmetry transformations agree with each other under the duality transformation.
This duality breaks down under mass perturbations. For massive excitations, we go to the rest frame $p_{\mu} = (E, 0, 0)$ where

$$\{Q_+, Q_-\} = E \quad , \quad \{Q_+, Q_+\} = \{Q_-, Q_-\} = 0 \quad . \quad (4.79)$$

Starting from a state $|s\rangle$ with spin $s$, the state $Q_+|s\rangle = |s + \frac{1}{2}\rangle$ has spin $s + \frac{1}{2}$. Up to parity transformation, there are only two cases for field theory; for $s = 0$ we have a massive theory of real scalar and a Majorana fermion, and for $s = \frac{1}{2}$ we have a massive version of minimal super Yang-Mills theory. For Yang-Mills theory, there are two ways of introducing masses. The one is through a Higgs mechanism which does not break parity. This means that elementary bosonic excitations consist of both signs of spin $s = \pm 1$. Since we then have two bosonic degrees of freedom, we must have two Majorana fermions with a parity conserving Dirac mass term, in other words, two Majorana mass terms with opposite signs. Note that we have to double the degrees of freedom to preserve parity. To avoid doubling of excitations, we have to break parity. The Chern-Simons term for gauge field and a Majorana mass term for gaugino that we have discussed before do the job. The supersymmetry can be easily checked with these terms. The massive theory of a real scalar field and a Majorana fermion is also simply obtained by adding usual mass terms to the massless Lagrangian, and it is physically different from the massive super Yang-Mills theory, although it also breaks parity.

** Massive $\mathcal{N} = 3$ theory**

We first discuss massive $\mathcal{N} = 3$ theory. In the rest frame of an excitation, the supersymmetry algebra becomes

$$\{Q^I_+, Q^J_-\} = E \delta^{I J} \quad , \quad I, J = 1, 2, 3 \quad ,$$

with others vanish. Starting from a state $|s\rangle$ of spin $s$ with $Q^I_+|s\rangle = 0$, the resulting multiplet is shown in the table below, where $SO(3)_R$ is the R-symmetry of three Majorana supercharges. Up to parity, the unique field theory multiplet is given by $s = -\frac{1}{2}$ for which

| spin | $|s\rangle$ | $Q^I_+|s\rangle$ | $Q^I_+ Q^J_-|s\rangle$ | $Q^I_+ Q^J_- Q^K_-|s\rangle$ |
|------|-------------|-----------------|-----------------|-----------------|
| $SO(3)_R$ | 1 | 3 | 3 | 1 |

we have one spin 1 state, three spin $\frac{1}{2}$ states, three spin 0 states and one spin $-\frac{1}{2}$ state. The field theory therefore has a gauge field, four Majorana fermions and three real scalars.

---

1 We are neglecting central charges.
Incidentally the field content is identical to the $\mathcal{N} = 4$ massless vector multiplet, which is obtained from a dimensional reduction of 6D $\mathcal{N} = (1, 0)$ vector multiplet. This indicates a possibility that the massive $\mathcal{N} = 3$ theory may be obtained by a mass perturbation to the massless $\mathcal{N} = 4$ super Yang-Mills theory. Since we should not double the elementary excitations while giving masses to them, the resulting theory will break parity. The only known way to achieve this is to introduce a Chern-Simons term for massive spin 1 particles and Majorana mass terms for three spin $\frac{1}{2}$ fermions and single spin $-\frac{1}{2}$ fermion with opposite sign of coefficient, and the usual mass terms for three scalars. In an interacting theory, we need to add necessary coupling terms consistent with $\mathcal{N} = 3$. This has been obtained in Ref. [40, 41] for non-abelian gauge theory,

$$\mathcal{L}_{\mathcal{N}=3} = \kappa \cdot \text{tr} \left\{ \frac{1}{2} \epsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho \right) - i \bar{\lambda}_a \lambda_a + i \bar{\chi} \chi - \kappa C_a^2 - \frac{i}{3} \epsilon^{abc} C_a [C_b, C_c] \right\},$$

where $a, b, c = 1, 2, 3$ are $SO(3)_R$ vector indices. Note that the signs of Majorana mass terms are consistent with our expectation. The Lagrangian of the massive $\mathcal{N} = 3$ theory is thus $\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \mathcal{L}_{\mathcal{N}=3}$.

In $\mathcal{N} = 4$, the R-symmetry is $SU(2)_1 \times SU(2)_2$, one of which is the original R-symmetry of 6D $\mathcal{N} = (1, 0)$ and the other comes from the spatial $R^3$ rotation of the reduced dimensions when we dimensionally reduce to 3D. $(\lambda_a, \chi)$ is doublet under both $SU(2)$'s, while $C_a$ is a triplet under the second $SU(2)_2$. The $\mathcal{N} = 3$ deformation preserves only the diagonal $SU(2)_D$ of $SU(2)_1 \times SU(2)_2$, which can be seen in the Majorana mass terms. The remaining $SU(2)_D$ is our R-symmetry $SO(3)_R$ of $\mathcal{N} = 3$.

**Massless $\mathcal{N} = 3$ theory**

For a massless excitation, the super-algebra in the frame $p_\mu = (E, E, 0)$ is

$$\{Q^I_1, Q^J_1\} = E \delta^{IJ} , \quad I, J = 1, 2, 3 \quad , \quad (4.81)$$

with $Q^I_2 = 0$. Since $Q^I_1$'s are real, we seem to have an $E^3$-Clifford algebra, and the massless multiplet must be a representation of it. However, this is not a whole story. Since statistics is relevant in (2+1)D, there must exist an operator $(-1)^F$ which anti-commutes with $Q^I_1$'s. Including $(-1)^F$ in the algebra, we actually have an $E^4$-Clifford algebra, whose minimal representation has dimension 4. Realizing the Clifford algebra by

$$Q^{I=1}_1 = \sigma^1 \otimes \sigma^1 , \quad Q^{I=2}_1 = \sigma^2 \otimes \sigma^1 , \quad Q^{I=3}_1 = \sigma^3 \otimes \sigma^1 \quad , \quad (4.81)$$

the representation automatically allows another operator $Q^{I=4}_1 = 1 \otimes \sigma^2$ which anti-commutes with $Q^I_1$, $I = 1, 2, 3$ and $(-1)^F$. This implies that a massless $\mathcal{N} = 3$ multiplet
is automatically completed to a massless $\mathcal{N} = 4$ multiplet. Note however that this does not imply that a massless $\mathcal{N} = 3$ theory is also $\mathcal{N} = 4$, because interactions do not necessarily preserve $\mathcal{N} = 4$ supersymmetry. Since it is known that $\mathcal{N} = 3$ sigma model with a hyper-Kahler manifold is automatically $\mathcal{N} = 4$ [42], a strict massless $\mathcal{N} = 3$ model should be something else. One possibility is to have non-dynamical vector multiplets with only $\mathcal{N} = 3$ Chern-Simons terms coupled with massless $\mathcal{N} = 4$ sigma model [42, 43].

The dimension of minimal representation of $E^4$-Clifford algebra is 4, and we expect two bosonic and two fermionic degrees of freedom. Note that this is half of the well-known $\mathcal{N} = 4$ vector/hyper multiplet from 6D $\mathcal{N} = (1,0)$. From the previous discussion of the massive $\mathcal{N} = 3$ theories, we also see that it would be impossible to introduce mass perturbation without doubling the multiplet. From the above explicit realization of the algebra (4.82), the $SO(3)_R$ R-symmetry generators are realized as

$$R^I = \frac{\sigma^I}{2} \otimes 1 \quad , \quad I = 1, 2, 3 \quad ,$$

with $[R^I, Q^K] = i\epsilon^{IKL} Q^L$ and $[R^I, (-1)^F] = 0$ as required. This implies that two bosons as well as two fermions are both doublets under $SO(3)_R$. If we would like to realize these in field theory, since a doublet of $SO(3)_R$ must be complex, we would end up with four bosonic degrees of freedom, contradicting to the above. This case is actually $\mathcal{N} = 4$ hyper-multiplet with two complex scalars and two Dirac fermions which are doublets under $SU(2)_1 \otimes SU(2)_2$ respectively. They both become doublets under the diagonal $SO(3)_R$ in $\mathcal{N} = 3$. The massless $\mathcal{N} = 4$ vector multiplet is identified by choosing somewhat complicated realization of $SO(3)_R$ generators, under which one real boson out of two complex bosons is a singlet and the remaining three form a triplet. These considerations indicate that in field theory level, there is probably no minimal massless $\mathcal{N} = 3$ Lagrangian possible, although it does not exclude the possibility of an intrinsic quantum theory realizing the above minimal representation.

5 Dual $\mathcal{N} = 3$ SCFT Proposal

In M-theory context, we consider N stack of M2 branes at the apex of a 8-dimensional hyper-Kahler cone, whose 3D world-volume theory is $\mathcal{N} = 3$ SCFT at low energy. Its supergravity solution has the near horizon geometry of $AdS_4 \times X_7$ with a background M2 charge flux N, where $X_7$ is the constant radius Tri-Sasakian section of the original hyper-Kahler cone. AdS/CFT correspondence conjectures a duality between these two
descriptions. The strongly coupled SCFT on M2 branes is mysterious. From the analyses
in the supergravity side, it has been found that the number of degrees of freedom in
3D SCFT scales as \( \sim N^{3/2} \), and its explanation is still missing. Because this seems true
without any regard to supersymmetry, its understanding may lie in a new but general
characteristic of strongly coupled 3D theories.

Although having supersymmetries does not help much to identify the nature of IR
conformal field theory, it can tell us some specific things that are protected by supersym-
metries, such as scale dimensions of chiral primaries, \( SU(2)_R \) symmetry, etc. The \( \mathcal{N} = 3 \)
superconformal algebra dictates that for chiral primaries, \( \Delta = j_R \) where \( j_R \) is the spin
number of the \( SU(2)_R \) representation [31]. Knowing charges under \( SU(2)_R \) thus helps
us to identify conformal dimensions. An advantage over the case of \( \mathcal{N} = 2 \) where the
R-symmetry \( U(1)_R \) is not rigid under RG flow is that for \( \mathcal{N} = 3 \), the non-abelian \( SU(2)_R \)
is invariant under RG flow, and can be fixed in UV before flowing to a SCFT in IR.

This may allow us to propose a UV description in terms of quiver-type gauge theory,
which is supposed to flow into our \( \mathcal{N} = 3 \) SCFT in far infrared [27]. This gauge theory
in UV is not intended to give any dynamical information about the IR SCFT, such as
fundamental degrees of freedom or interactions between them. Along the RG flow, the
theory becomes strongly coupled and we no longer expect gauge fields as our dynamical
degrees of freedom. The usefulness of the proposal for UV description, if it indeed flows
to our SCFT in IR, rests on the ability of identifying \( SU(2)_R \) symmetry and the spectrum
of chiral primary operators, which should coincide with those in the IR SCFT fixed point
due to rigidity under RG flow.

In attempts to propose a UV description of our \( \mathcal{N} = 3 \) SCFT that is dual to M-theory
on \( \text{AdS}_4 \times X_7 \), there is one necessary condition we need to consider. In the original M2
brane picture, there exists a Coulomb branch where M2 branes move apart from each other
around the tip of the hyper-Kahler cone. Our proposal then should have a moduli space
of vacua corresponding to the Coulomb branch, which is nothing but the N-symmetric
product of our hyper-Kahler cone over the Tri-Sasakian manifold. The simplest possibility
as in Ref.([6]) is to consider a non-abelian gauge theory whose vacuum conditions from
D/F-terms become a symmetric product of non-linear sigma model with target space given
by our hyper-Kahler cone. The symmetrization is a part of the original gauge symmetry.
The D/F-term equations from vector multiplets of \( \mathcal{N} = 3(\mathcal{N} = 4) \) gauge theory are in
fact moment-map equations of a hyper-Kahler quotient, and this matches with the fact
that our hyper-Kahler cone is also obtained from a hyper-Kahler quotient. Note that this
is a simple guideline, and the actual proposal is not completely determined by this.

A massless $\mathcal{N} = 3$ multiplet is automatically $\mathcal{N} = 4$ multiplet as we see in the previous section. To have strict $\mathcal{N} = 3$ supersymmetry, the only known way is to add a $\mathcal{N} = 3$ Chern-Simons term, which makes vector multiplet massive with mass proportional to the coupling constant $e^2$. We keep $\mathcal{N} = 4$ matter hyper-multiplets massless. As we flow to IR, the gauge coupling constant blows up and the gauge fields become heavy and decouple. An equivalent way of saying is that we can neglect kinetic terms for fields in the vector multiplets and they become non-dynamical. However, the Chern-Simons term does not disappear. Their effect in abelian case is to induce a non-zero statistical phase for matter excitations to make them anyons [44]. Therefore, a naive picture of physics in IR is to have a non-linear sigma model from hyper-multiplets with target space obtained by D/F-term equations of vector multiplets, whose dynamics is affected by non-dynamical vector multiplets with $\mathcal{N} = 3$ Chern-Simons terms. However, the full IR dynamics will be more than the above, including loop excitations from M2 branes, which presumably account for $N^{3/2}$-scaling of degrees of freedom.

Having said the purposes and limitations of the UV proposals in terms of quiver gauge theories, we give a simple proposal for the $\mathcal{N} = 3$ theory that corresponds to a 7D Tri-Sasakian manifold obtained by an arbitrary $G = U(1)^r$ hyper-Kahler quotient labelled by $Q_a^i$, with $i = 1, \ldots, r$ and $a = 1, \ldots, r + 2$. The gauge group is a $r$-copy of $U(N) \times U(N)$,

$$G = \prod_{i=1}^r [U(N) \times U(N)]_i.$$  \hspace{1cm} (5.84)

Under the $i$’th group $[U(N) \times U(N)]_i$, the $(r + 2)$ hyper-multiplets $U_a = (u_a, \bar{v}_a)$ are charged as $\left(Sym^{Q_a^i}, S^{\bar{y}m} Q_a^i\right)$, where $Sym^Q$ stands for the $Q$-symmetric representation of fundamental representation of $U(N)$. This is a naive generalization of the previous proposal for the $r = 1$ case [27, 30].

The fundamental chiral primary fields $U_a = (u_a, \bar{v}_a)$ are doublet under $SU(2)_R$. We can find baryonic gauge invariant chiral primary operators composed of $N U_a$’s for a fixed $a$ by contracting their gauge indices by $\epsilon^{12\ldots N}$-tensors. We need $2 \sum_{a=1}^r Q_a^a$ number of $\epsilon$-tensors to contract all gauge indices. As this number is even, the resulting operator is totally symmetric under an exchange of any two $U_a$ components. Therefore, the baryonic operators for any fixed $a$ form a spin $j_R = N/2$ representation under $SU(2)_R$ and their conformal dimension must be $\Delta = N/2$ for all $a$. The baryonic operators with given $a$ are mapped to a M5 brane wrapping the cycle $u_a = 0$ or $v_a = 0$. Any other cycles that are obtained from this by $SU(2)_R$ action are all relevant. We have to quantize the M5
brane moving along flat directions of supersymmetric cycles connecting \( u_a = 0 \) and \( v_a = 0 \) through \( SU(2)_R \) orbit. As \( u_a = 0 \) or \( v_a = 0 \) is invariant under \( U(1)_R \subset SU(2)_R \), this orbit will be nothing but \( SU(2)/U(1) = S^2 \), and the M5 brane quantization is identical to a problem of point-like particle on \( S^2 \). Due to the background M2 charge flux of \( N \) units, this problem boils down to a charged particle on \( S^2 \) moving in a background monopole charge \( N \). The resulting spectrum agrees with the spin \( j_R = \frac{N}{2} \) multiplet of our baryon operators. As we calculated in the previous sections, the conformal dimension from the geometry side perfectly matches with this expectation.

As a final comment, there recently appeared a description of \( \mathcal{N} = 2 \) SCFT, arising from M2 branes at the tip of toric CY_4 cones, in terms of crystals of M5 branes after T-duality [35]. Since our \( \mathcal{N} = 3 \) SCFT’s belong to \( \mathcal{N} = 2 \) SCFT, there must be a similar description. It would also be interesting to think about the relation between our UV proposal of quiver-type gauge theories and M5 crystals.

Acknowledgement

The author is indebted to Kimyeong Lee, Sangmin Lee and Jae-Suk Park for valuable discussions. He also appreciates kind invitations to Kobe University and Chuo University in Japan by Chong-Sa Lim and Takeo Inami respectively, where part of the work has been done. This work was supported by the Korea Research Foundation Grant. (KRF-2005-070-c00030)

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113 [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].
[4] B. S. Acharya, J. M. Figueroa-O’Farrill, C. M. Hull and B. J. Spence, “Branes at conical singularities and holography,” Adv. Theor. Math. Phys. 2 (1999) 1249 [arXiv:hep-th/9808014].

[5] D. R. Morrison and M. R. Plesser, “Non-spherical horizons. I,” Adv. Theor. Math. Phys. 3 (1999) 1 [arXiv:hep-th/9810201].

[6] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) [arXiv:hep-th/9807080].

[7] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Sasaki-Einstein metrics on $\text{S}(2) \times \text{S}(3)$,” Adv. Theor. Math. Phys. 8, 711 (2004) [arXiv:hep-th/0403002].

[8] M. Cvetic, H. Lu, D. N. Page and C. N. Pope, “New Einstein-Sasaki spaces in five and higher dimensions,” Phys. Rev. Lett. 95, 071101 (2005) [arXiv:hep-th/0504225].

[9] D. Martelli, J. Sparks and S. T. Yau, “The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds,” Commun. Math. Phys. 268, 39 (2006) [arXiv:hep-th/0503183].

[10] D. Martelli, J. Sparks and S. T. Yau, “Sasaki-Einstein manifolds and volume minimisation,” [arXiv:hep-th/0603021].

[11] D. Martelli and J. Sparks, “Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals,” Commun. Math. Phys. 262, 51 (2006) [arXiv:hep-th/0411238].

[12] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, “An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals,” JHEP 0506, 064 (2005) [arXiv:hep-th/0411264].

[13] S. Benvenuti, A. Hanany and P. Kazakopoulos, “The toric phases of the $Y(p,q)$ quivers,” JHEP 0507, 021 (2005) [arXiv:hep-th/0412279].

[14] A. Hanany, P. Kazakopoulos and B. Wecht, “A new infinite class of quiver gauge theories,” JHEP 0508, 054 (2005) [arXiv:hep-th/0503177].

[15] S. Benvenuti and M. Kruczenski, “From Sasaki-Einstein spaces to quivers via BPS geodesics: $L(p,q-r)$,” JHEP 0604, 033 (2006) [arXiv:hep-th/0505206].
[16] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, “Gauge theories from toric geometry and brane tilings,” JHEP 0601, 128 (2006) [arXiv:hep-th/0505211].

[17] A. Butti, D. Forcella and A. Zaffaroni, “The dual superconformal theory for L(p,q,r) manifolds,” JHEP 0509, 018 (2005) [arXiv:hep-th/0505220].

[18] K. Intriligator and B. Wecht, “The exact superconformal R-symmetry maximizes a,” Nucl. Phys. B 667, 183 (2003) [arXiv:hep-th/0304128].

[19] M. Bertolini, F. Bigazzi and A. L. Cotrone, “New checks and subtleties for AdS/CFT and a-maximization,” JHEP 0412, 024 (2004) [arXiv:hep-th/0411249].

[20] A. Butti and A. Zaffaroni, “R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization,” JHEP 0511, 019 (2005) [arXiv:hep-th/0506232].

[21] S. Lee and S. J. Rey, “Comments on anomalies and charges of toric-quiver duals,” JHEP 0603, 068 (2006) [arXiv:hep-th/0601223].

[22] C. P. Boyer and K. Galicki, “3-Sasakian Manifolds,” Surveys Diff. Geom. 7 (1999) 123 [arXiv:hep-th/9810250].

[23] C. P. Boyer, K. Galicki, and B. M. Mann, ”Quaternionic reduction and Einstein manifolds,” Comm. Anal. Geom. 1 (1993), 1-51.

The geometry and topology of 3-Sasakian manifolds, T. reine angew. Math. 455 (1994), 183-220

[24] J.H. Eschenburg, “New examples of manifolds with strictly positive curvature”, Invent. Math. 66 (1982) 469-480.

Cohomology of biquotients, Manuscripta Math. 75 (1992), 151-166.

[25] R. Bielawski and A. Dancer, “The geometry and the topology of toric hyperkahler manifolds”, Comm. Anal. Geom. 8 (200) 727.

[26] P. Fre’, L. Gualtieri and P. Termonia, “The structure of N = 3 multiplets in AdS(4) and the complete Osp(3|4) x SU(3) spectrum of M-theory on AdS(4) x N(0,1,0),” Phys. Lett. B 471, 27 (1999) [arXiv:hep-th/9909188].

[27] D. Fabbri, P. Fre’, L. Gualtieri, C. Reina, A. Tomasiello, A. Zaffaroni and A. Zampa, “3D superconformal theories from Sasakian seven-manifolds: New nontrivial evidences for AdS(4)/CFT(3),” Nucl. Phys. B 577, 547 (2000) [arXiv:hep-th/9907219].
[28] M. Billo, D. Fabbri, P. Fre, P. Merlatti and A. Zaffaroni, “Rings of short $N = 3$ superfields in three dimensions and M-theory on AdS(4) $\times$ N(0,1,0),” Class. Quant. Grav. 18, 1269 (2001) [arXiv:hep-th/0005219].

[29] J. P. Gauntlett, S. Lee, T. Mateos and D. Waldram, “Marginal deformations of field theories with AdS(4) duals,” JHEP 0508, 030 (2005) [arXiv:hep-th/0505207].

[30] K. M. Lee and H. U. Yee, “New AdS(4) $\times$ X(7) geometries with $N = 6$ in M theory,” [arXiv:hep-th/0605214].

[31] S. Minwalla, “Restrictions imposed by superconformal invariance on quantum field theories,” Adv. Theor. Math. Phys. 2, 781 (1998) [arXiv:hep-th/9712074].

[32] A. Butti, D. Forcella and A. Zaffaroni, “Counting BPS Baryonic Operators in CFTs with Sasaki-Einstein duals,” [arXiv:hep-th/0611229].

[33] S. S. Gubser and I. R. Klebanov, “Baryons and domain walls in an $N = 1$ superconformal gauge theory,” Phys. Rev. D 58, 125025 (1998) [arXiv:hep-th/9808075].

[34] K. M. Lee and H. U. Yee, “BPS string webs in the 6-dim (2,0) theories,” [arXiv:hep-th/0606150].

[35] S. Lee, “Superconformal field theories from crystal lattices,” [arXiv:hep-th/0610204].

[36] E. Witten, “Supersymmetry and Morse theory,” J. Diff. Geom. 17, 661 (1982).

[37] G. W. Moore, N. Nekrasov and S. Shatashvili, “Integrating over Higgs branches,” Commun. Math. Phys. 209, 97 (2000) [arXiv:hep-th/9712241].

[38] Jae-Suk Park, private communications.

[39] S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” Annals Phys. 140, 372 (1982) [Erratum-ibid. 185, 406.1988 APNYA,281,409 (1988 APNYA,281,409-449.2000)].

[40] H. C. Kao and K. M. Lee, “Selfdual Chern-Simons systems with an N=3 extended supersymmetry,” Phys. Rev. D 46, 4691 (1992) [arXiv:hep-th/9205115].

[41] H. C. Kao, “Selfdual Yang-Mills Chern-Simons Higgs systems with an N=3 extended supersymmetry,” Phys. Rev. D 50, 2881 (1994).
[42] A. Kapustin and M. J. Strassler, “On mirror symmetry in three dimensional Abelian gauge theories,” JHEP 9904, 021 (1999) [arXiv:hep-th/9902033].

[43] J. H. Schwarz, “Superconformal Chern-Simons theories,” JHEP 0411, 078 (2004) [arXiv:hep-th/0411077].

[44] F. Wilczek and A. Zee, “Linking Numbers, Spin, And Statistics Of Solitons,” Phys. Rev. Lett. 51, 2250 (1983).