Chiral Ground States in a Frustrated Holographic Superconductor

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ABSTRACT: Frustration is an important phenomenon in condensed matter physics because it can introduce a new order parameter such as chirality. Towards understanding a mechanism of the frustration in strongly correlated systems, we study a holographic superconductor model with three scalar fields and an interband Josephson coupling, which is important for the frustration. We analyze free energy of solutions of the model to determine ground states. We find chiral ground states, which have nonzero chirality.
1 Introduction

Frustration is widely studied in condensed matter physics (see, for example, [1, 2]). This frustration means a situation where several constraints compete and it can cause degeneracy of vacuum. One famous example is antiferromagnets on a triangular lattice. Some frustrated systems have a characteristic order parameter such as chirality, which represents a property under a mirror operation. There is a strange phenomenon by nonzero chirality, for example, anomalous Hall effect [3]. Study of the frustration is aimed to create new materials by using a property of frustrated systems.

In condensed matter physics, there is a three-band superconductor model in which the frustration occurs [4]. This model has three interband Josephson coupling terms between three scalar fields and they lead the frustration. Furthermore, chiral ground states exist in this model. These chiral ground states are vacua, which have nonzero chirality and their origin is the frustration.

In this paper, we find a holographic superconductor model [5] in which the frustration occurs. From a viewpoint of the gauge/gravity correspondence [6–8], holographic superconductor models with several order parameters are well-studied [9–25]. In particular, a two-scalar model with the Josephson coupling was studied in Ref. [26]. However, we need more than two scalar fields for the frustration since we need some constraints of complex phases of the scalar fields. Therefore, we consider a three-scalar model with the Josephson coupling for the frustration. To determine ground states, we analyze free energy of solutions of this model. We add a perturbative potential term in order to obtain the chiral ground states. When we choose specific values of parameters, there are solutions corresponding to the chiral ground states. This model may be a help to understand a mechanism of the frustration in strongly correlated systems from the viewpoint of the gauge/gravity correspondence.

The organization of this paper is as follows. In section 2, we analyze a holographic two-scalar model and see that the frustration does not occur in this model. In section 3,
we analyze a holographic three-scalar model and see that there are the chiral ground states by comparing the free energy when we choose specific values of parameters. Section 4 is a summary and a discussion.

2 Two-scalar model

In this section, we study a two-scalar model as a minimal model with the Josephson coupling. This model was studied first in Ref. [26]\(^1\). We see that the two-scalar model can be solved and cannot describe the frustration.

In this paper, we use a four-dimensional AdS planar black hole space-time

\[
\begin{align*}
    ds^2 &= \frac{L^2}{z^2}(-f(z)dt^2 + dx^2 + dy^2 + \frac{dz^2}{f(z)}), \\
    f(z) &= 1 - \left(\frac{z}{z_h}\right)^3,
\end{align*}
\]

where \(z = 0\) is the AdS boundary and \(z = z_h\) is the horizon of the black hole. This metric is often used in the field of the holographic superconductor since a phase transition of the order parameter occurs by changing the Hawking temperature of the black hole. For simplicity, we fix the metric and use the probe limit.

For a two-scalar model with the Josephson coupling, we consider an action with a Maxwell field \(A_\mu\), two complex scalar fields \(\varphi_i\) and the nonzero Josephson coupling \(\epsilon\) as

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D_\mu \varphi_1|^2 - |D_\mu \varphi_2|^2 \\
- m_1^2 |\varphi_1|^2 - m_2^2 |\varphi_2|^2 - \epsilon (\varphi_1^* \varphi_2 + \varphi_1 \varphi_2^*) \right],
\]

where

\[
D_\mu = \nabla_\mu - iA_\mu.
\]

Because of the existence of the Josephson coupling, phase angles of the scalar fields are constrained as we will see in (2.14).

The equations of motion of \(\varphi_1\) are

\[
\begin{align*}
    D^\mu D_\mu \varphi_1 - m_1^2 \varphi_1 - \epsilon \varphi_2 &= 0, \\
    D^\mu D_\mu \varphi_2 - m_2^2 \varphi_2 - \epsilon \varphi_1 &= 0.
\end{align*}
\]

In these equations, two types of the solution are possible:

- \(\varphi_1 = \varphi_2 = 0\). This solution corresponds to the normal conducting phase.
- \(\varphi_1 \neq 0, \varphi_2 \neq 0\). This solution corresponds to the superconducting phase.

If \(\varphi_1 = 0\), we obtain \(\varphi_2 = 0\) from (2.5). Therefore, there are no solutions as \(\varphi_1 \neq 0, \varphi_2 = 0\) and \(\varphi_1 = 0, \varphi_2 \neq 0\).

\(^1\)In Ref. [26], effect of the Josephson coupling to boundary conditions of the scalar fields at the AdS boundary is not considered. We will derive this effect by diagonalization.
From now, we consider the solution with \( \varphi_1 \neq 0, \varphi_2 \neq 0 \) to describe the superconducting phase. This solution exists when the Hawking temperature of the black hole is low enough. In this case, we can rewrite \( \varphi_i \) by absolute values \( \psi_i \) and phase angles \( \theta_i \) as
\[
\varphi_i = \psi_i e^{i\theta_i},
\]
where \( \psi_i > 0 \). For homogeneous of the fields and simplicity of \( \theta_i \), our ansatz for the fields is
\[
A_t = A_t(z), \quad \psi_i = \psi_i(z), \quad \theta_i = \text{const.},
\]
and other components are zero. Under this ansatz, the equations of motion are
\[
\nabla_\mu F^{\mu\nu} - 2\psi_1^2 A' - 2\psi_2^2 A'' = 0, \tag{2.20}
\]
\[
\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - m_1^2 \psi_1 - \epsilon \psi_2 \cos \theta_1 = 0, \tag{2.21}
\]
\[
\nabla_\mu \nabla^\mu \psi_2 - A_\mu A^\mu \psi_2 - m_2^2 \psi_2 - \epsilon \psi_1 \cos \theta_2 = 0, \tag{2.22}
\]
\[
\epsilon \psi_1 \psi_2 \sin (\theta_1 - \theta_2) = 0, \tag{2.23}
\]
\[
\epsilon \psi_1 \psi_2 \sin (\theta_2 - \theta_1) = 0. \tag{2.24}
\]
From the equations of motion of \( \theta_i \), we obtain
\[
\theta_1 - \theta_2 = 0 \text{ or } \pi. \tag{2.25}
\]
This solution is not frustrated because it minimizes the coefficient of the Josephson coupling potential term \( 2\epsilon \cos (\theta_1 - \theta_2) \) in (2.3). Therefore, the two-scalar model cannot describe the frustration. To describe the frustration, we need more than two scalar fields as section 3.

Incidentally, the two-scalar model can be solved by diagonalization. To rewrite the equations of motion of \( \psi_i \) by a diagonal basis of the scalar fields, we define \( \epsilon', \lambda_i \) and \( \psi'_i \) as
\[
\epsilon' = \epsilon \cos (\theta_1 - \theta_2), \tag{2.26}
\]
\[
\lambda_1 = \frac{(m_1^2 + m_2^2) - \sqrt{(m_1^2 - m_2^2)^2 + 4\epsilon'^2}}{2}, \tag{2.27}
\]
\[
\lambda_2 = \frac{(m_1^2 + m_2^2) + \sqrt{(m_1^2 - m_2^2)^2 + 4\epsilon'^2}}{2}, \tag{2.28}
\]
\[
\psi'_1 = \frac{-\epsilon' \psi_1 + (m_2^2 - \lambda_1) \psi_2}{\sqrt{\epsilon'^2 + (m_2^2 - \lambda_1)^2}}, \tag{2.29}
\]
\[
\psi'_2 = \frac{\epsilon' \psi_1 + (m_1^2 - \lambda_1) \psi_2}{\sqrt{\epsilon'^2 + (m_1^2 - \lambda_1)^2}}. \tag{2.30}
\]
Using them, the equations of motion of \( A_\mu \) and \( \psi'_i \) are written as
\[
\nabla_\mu F^{\mu\nu} - 2\psi_1'^2 A' - 2\psi_2'^2 A'' = 0, \tag{2.31}
\]
\[
\nabla_\mu \nabla^\mu \psi'_1 - A_\mu A^\mu \psi'_1 - \lambda_1 \psi'_1 = 0, \tag{2.32}
\]
\[
\nabla_\mu \nabla^\mu \psi'_2 - A_\mu A^\mu \psi'_2 - \lambda_2 \psi'_2 = 0. \tag{2.33}
\]
These equations of motion are same as those of a two-scalar model without the Josephson coupling. We assume that there is no solution with \( \psi_1' \neq 0, \psi_2' \neq 0 \) since it was checked that there is not such solution if charges of the scalar fields are same and \( \lambda_1 \neq \lambda_2 \) by numerical calculations in Ref. [9].

For example, we consider the solution in the case of \( \epsilon' < 0 \). In this case, the solution is \( \psi_1' \neq 0, \psi_2' = 0 \) since \(-\epsilon' \psi_1 + (m_1^2 - \lambda_1) \psi_2 \) is positive and we obtain

\[
\begin{align*}
\nabla_\mu F^{\mu\nu} - 2\psi_1'^2 A^\nu &= 0, \\
\nabla_\mu \nabla_\nu \psi_1' - A_\mu A_\nu \psi_1' - \lambda_1 \psi_1' &= 0, \\
\psi_1' &= -\frac{\epsilon'^2 + (m_1^2 - \lambda_1)^2}{\epsilon'} \psi_1 = \frac{\epsilon'^2 + (m_1^2 - \lambda_1)^2}{m_1^2 - \lambda_1} \psi_2,
\end{align*}
\]

(2.23) and (2.24) are same as the equations of motion of a one-scalar model [5]. It is well-known that the solution with \( \psi_1' \neq 0 \) exists if the Hawking temperature of the black hole is low enough. Therefore, we conclude that the two-scalar model can be solved. Similarly, we can obtain the solution in the case of \( \epsilon' > 0 \).

In the gauge/gravity correspondence, the free energy is related to on-shell Euclidean action. Generally, the free energy of the solution in the case of \( \epsilon' < 0 \) is smaller than that of \( \epsilon' > 0 \) since \( \lambda_1 < \lambda_2 \). It is reasonable because \( \epsilon' \) is a coefficient of the Josephson coupling potential term in (2.3). From (2.24), the power of \( z \) in \( \psi_1' \) at the AdS boundary is determined as

\[
\psi_1' = \psi_1'^{(1)} z^{\Delta_1} + \psi_1'^{(2)} z^{\Delta_2}, \\
\Delta_1 = \frac{3 - \sqrt{9 + 4\lambda_1}}{2}, \quad \Delta_2 = \frac{3 + \sqrt{9 + 4\lambda_1}}{2}.
\]

If \( |\epsilon| \) is large enough, \( \lambda_1 \) is below Breitenlohner-Freedman bound [27] and the theory is unstable.

### 3 Three-scalar model

In this section, we study a three-scalar model as a model in which the frustration occurs. When we choose specific values of parameters, the frustration is realized and the solutions which correspond to the chiral ground states exist.

For a three-scalar model with the Josephson coupling, we consider an action with a Maxwell field \( A_\mu \), three complex scalar fields \( \varphi_i \), three nonzero Josephson coupling \( \epsilon_{ij} \) and a perturbative coupling \( \eta \) as

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D_\mu \varphi_1|^2 - |D_\mu \varphi_2|^2 - |D_\mu \varphi_3|^2 \\
- m_1^2 |\varphi_1|^2 - m_2^2 |\varphi_2|^2 - m_3^2 |\varphi_3|^2 \\
- \epsilon_{12} (\varphi_1 \varphi_2 + \varphi_1 \varphi_2^*) - \epsilon_{23} (\varphi_2 \varphi_3 + \varphi_2 \varphi_3^*) - \epsilon_{31} (\varphi_3 \varphi_1 + \varphi_3 \varphi_1^*) \\
- \eta (|\varphi_1|^4 + |\varphi_2|^4 + |\varphi_3|^4) \right],
\]

(3.1)
where $\eta$ is nonnegative and small enough for perturbative analysis. If $\epsilon_{12}\epsilon_{12}\epsilon_{31} > 0$, there is no configuration of $\theta_i$ which minimizes each coefficient of the Josephson coupling potential terms. In this case, the chiral ground states can exist. The perturbative potential of this action is symmetric about $\varphi_i$ and it is one example of the potential that the chiral ground states can exist. We use the metric (2.1) and the probe limit for simple calculation.

### 3.1 Analysis of the solutions with $\eta = 0$

First, we ignore the perturbative potential and analyze the equations of motion with $\eta = 0$. The equations of motion of $\varphi_i$ are

$$D^\mu D_\mu \varphi_1 - m_1^2 \varphi_1 - \epsilon_{12} \varphi_2 - \epsilon_{31} \varphi_3 = 0, \quad (3.2)$$
$$D^\mu D_\mu \varphi_2 - m_2^2 \varphi_2 - \epsilon_{23} \varphi_3 - \epsilon_{12} \varphi_1 = 0, \quad (3.3)$$
$$D^\mu D_\mu \varphi_3 - m_3^2 \varphi_3 - \epsilon_{31} \varphi_1 - \epsilon_{23} \varphi_2 = 0. \quad (3.4)$$

In these equations, three types of the solution are possible:

Sol.1 $\varphi_1 = \varphi_2 = \varphi_3 = 0$.

Sol.2 One scalar field is zero and others are nonzero as $\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0$.

Sol.3 $\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 \neq 0$.

To analyze the frustration between three scalar fields, we consider the solution 3. We rewrite $\varphi_i$ as (2.7) and use the ansatz as (2.8) in the same way of section 2. Under this ansatz, the equations of motion are

$$\nabla_\mu F^{\mu\nu} - 2\psi_1^2 A^\nu - 2\psi_2^2 A^\nu - 2\psi_3^2 A^\nu = 0, \quad (3.5)$$
$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - m_1^2 \psi_1 - \epsilon_{12}' \psi_2 - \epsilon_{31}' \psi_3 = 0, \quad (3.6)$$
$$\nabla_\mu \nabla^\mu \psi_2 - A_\mu A^\mu \psi_2 - m_2^2 \psi_2 - \epsilon_{23}' \psi_3 - \epsilon_{12}' \psi_1 = 0, \quad (3.7)$$
$$\nabla_\mu \nabla^\mu \psi_3 - A_\mu A^\mu \psi_3 - m_3^2 \psi_3 - \epsilon_{31}' \psi_1 - \epsilon_{23}' \psi_2 = 0, \quad (3.8)$$
$$\epsilon_{12} \psi_1 \psi_2 \sin (\theta_1 - \theta_2) + \epsilon_{31} \psi_1 \psi_3 \sin (\theta_1 - \theta_3) = 0, \quad (3.9)$$
$$\epsilon_{23} \psi_2 \psi_3 \sin (\theta_2 - \theta_3) + \epsilon_{12} \psi_2 \psi_1 \sin (\theta_2 - \theta_1) = 0, \quad (3.10)$$
$$\epsilon_{31} \psi_3 \psi_1 \sin (\theta_3 - \theta_1) + \epsilon_{23} \psi_3 \psi_2 \sin (\theta_3 - \theta_2) = 0, \quad (3.11)$$
$$\epsilon_{12}' = \epsilon_{12} \cos (\theta_1 - \theta_2), \quad \epsilon_{23}' = \epsilon_{23} \cos (\theta_2 - \theta_3), \quad \epsilon_{31}' = \epsilon_{31} \cos (\theta_3 - \theta_1). \quad (3.12)$$

In the equations of $\theta_i$, two types of the solution are possible:

Sol.3a $\sin (\theta_1 - \theta_2) \neq 0, \sin (\theta_2 - \theta_3) \neq 0, \sin (\theta_3 - \theta_1) \neq 0$.

Sol.3b $\sin (\theta_1 - \theta_2) = \sin (\theta_2 - \theta_3) = \sin (\theta_3 - \theta_1) = 0$.

In the three-scalar model, there is the solution 3a with $\sin (\theta_i - \theta_j) \neq 0$ unlike the two-scalar model. This solution has nonzero chirality.
We derive a condition that the solution 3a is possible. From (3.10) and (3.11), we obtain
\[
\psi_2 = - \frac{\epsilon_{31} \sin (\theta_3 - \theta_1)}{\epsilon_{23} \sin (\theta_3 - \theta_2)} \psi_1, \quad \psi_3 = - \frac{\epsilon_{12} \sin (\theta_2 - \theta_1)}{\epsilon_{23} \sin (\theta_2 - \theta_3)} \psi_1. \tag{3.13}
\]
Since \(\psi_i > 0\), we need
\[
\frac{\epsilon_{31} \sin (\theta_3 - \theta_1)}{\epsilon_{23} \sin (\theta_3 - \theta_2)} < 0, \quad \frac{\epsilon_{12} \sin (\theta_2 - \theta_1)}{\epsilon_{23} \sin (\theta_2 - \theta_3)} < 0. \tag{3.14}
\]
Substituting (3.13) for (3.6), (3.7) and (3.8), we obtain
\[
\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_1^2 - \frac{\epsilon_{12} \epsilon_{31}}{\epsilon_{23}} \right) \psi_1 = 0, \tag{3.15}
\]
\[
\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_2^2 - \frac{\epsilon_{23} \epsilon_{12}}{\epsilon_{31}} \right) \psi_1 = 0, \tag{3.16}
\]
\[
\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_3^2 - \frac{\epsilon_{31} \epsilon_{23}}{\epsilon_{12}} \right) \psi_1 = 0. \tag{3.17}
\]
Therefore, the solution 3a is possible only if
\[
m_1^2 - \frac{\epsilon_{12} \epsilon_{31}}{\epsilon_{23}} = m_2^2 - \frac{\epsilon_{23} \epsilon_{12}}{\epsilon_{31}} = m_3^2 - \frac{\epsilon_{31} \epsilon_{23}}{\epsilon_{12}}. \tag{3.18}
\]
To compare the free energy of the solutions, we analyze other solutions with (3.18).

**Sol.1** \(\varphi_1 = \varphi_2 = \varphi_3 = 0\).

Generally, the free energy of the solution with \(\varphi_1 = \varphi_2 = \varphi_3 = 0\) is larger than that with the nonzero scalar fields. Therefore, this solution is not a ground state if the Hawking temperature of the black hole is low enough.

**Sol.2** \(\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0\).

From (3.4), we obtain
\[
\psi_2 = - \frac{\epsilon_{31}}{\epsilon_{23}} \psi_1 e^{i(\theta_1 - \theta_2)}, \tag{3.19}
\]
and
\[
\theta_1 - \theta_2 = 0 \text{ or } \pi, \tag{3.20}
\]
are the solutions. Substituting (3.19) for (3.6) and (3.7), we obtain
\[
\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_1^2 - \frac{\epsilon_{12} \epsilon_{31}}{\epsilon_{23}} \right) \psi_1 = 0, \tag{3.21}
\]
\[
\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_2^2 - \frac{\epsilon_{23} \epsilon_{12}}{\epsilon_{31}} \right) \psi_1 = 0, \tag{3.22}
\]
and the solution with \(\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0\) is possible if (3.18) holds. Similarly, other solutions with \(\varphi_1 = 0, \varphi_2 \neq 0, \varphi_3 \neq 0\) and \(\varphi_1 \neq 0, \varphi_2 = 0, \varphi_3 \neq 0\) are possible. Free energy of these solutions is same as that of solution 3a because the mass squared of \(\psi_i\) in (3.15) and (3.21) is same.
\[ \sin(\theta_1 - \theta_2) = \sin(\theta_2 - \theta_3) = \sin(\theta_3 - \theta_1) = 0. \]

In this case, the diagonalization as section 2 is useful. We diagonalize a matrix as
\[
\begin{pmatrix}
  m_1^2 & \epsilon'_{12} & \epsilon'_{31} \\
  \epsilon'_{12} & m_2^2 & \epsilon'_{23} \\
  \epsilon'_{31} & \epsilon'_{23} & m_3^2
\end{pmatrix}.
\] (3.23)

If (3.18) holds, this matrix can be transformed to a diagonal matrix as
\[
\begin{pmatrix}
  m_1^2 & 0 & 0 \\
  0 & m_2^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} & 0 \\
  0 & 0 & m_3^2 + \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}
\end{pmatrix}.
\] (3.24)

If \(\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0\), a minimum value of the free energy of this solution is same as that of the solution 3a since \(m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} < m_1^2 + \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\). However, if \(\epsilon_{12}\epsilon_{23}\epsilon_{31} < 0\), the free energy of the solution corresponds to \(m_1^2 + \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\) is smaller than that of the solution 3a and the frustration does not occur.

Summarizing the above, if \(\eta = 0\), \(\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0\) and (3.18) hold, there are several solutions whose free energy is same by the frustration.

### 3.2 Analysis of the free energy with \(\eta > 0\)

Next, we consider the three-scalar model with \(\eta > 0\), \(\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0\) and (3.18). Substituting the solutions with \(\eta = 0\) for the action, we analyze the free energy with \(\eta > 0\). Especially, we compare the perturbative potential terms. For simplicity, we set
\[
m_1^2 = m_2^2 = m_3^2, \quad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} > 0.
\] (3.25)

In this case, difference of \(\theta_i\) of the chiral ground states becomes symmetric, but we lose generality of the model.

Sol.2 \(\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0\).

Substituting (3.19) for (3.1), we obtain
\[
S_{\text{on-shell}} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu
u} F^{\mu\nu} - 2|D_\mu \psi_1|^2 - 2 \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi_1^2 - 2\eta \psi_4^2 \right] = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu
u} F^{\mu\nu} - |D_\mu \psi'||^2 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi'^2 - \frac{\eta}{2} \psi'^4 \right],
\] (3.26)

where we redefine a new scalar field \(\psi'\) to compare the free energy as
\[
\psi'^2 = 2\psi_1^2.
\] (3.27)

In this solution, the coefficient of the perturbative potential of \(\psi'\) is \(\eta/2\). The solutions with \(\varphi_1 = 0, \varphi_2 \neq 0, \varphi_3 \neq 0\) and \(\varphi_1 \neq 0, \varphi_2 = 0, \varphi_3 \neq 0\) are also the same.

\[\text{We assume that there are the solutions as those of a one-scalar model only as section 2. We will explain this assumption later.}\]
Substituting (3.13) for (3.1), we obtain
\[
S_{\text{on-shell}} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\sin^2(\theta_1 - \theta_2) + \sin^2(\theta_2 - \theta_3) + \sin^2(\theta_3 - \theta_1)}{\sin^2(\theta_2 - \theta_3)} |D_\mu \psi_1|^2 \\
- \left( m_1^2 - \frac{\epsilon_{12\epsilon_{31}}}{\epsilon_{23}} \right) \frac{\sin^2(\theta_1 - \theta_2) + \sin^2(\theta_2 - \theta_3) + \sin^2(\theta_3 - \theta_1)}{\sin^2(\theta_2 - \theta_3)} \psi_1^2 \\
- \eta \left( \frac{\sin^4(\theta_1 - \theta_2) + \sin^4(\theta_2 - \theta_3) + \sin^4(\theta_3 - \theta_1)}{\sin^4(\theta_2 - \theta_3)} \right) \frac{\psi^4}{\psi_1^4} \right],
\]
where we redefine a new scalar field \( \psi' \) as
\[
\psi'^2 \equiv \frac{\sin^2(\theta_1 - \theta_2) + \sin^2(\theta_2 - \theta_3) + \sin^2(\theta_3 - \theta_1)}{\sin^2(\theta_2 - \theta_3)} \psi_1^2.
\]
The equation of motion of \( \psi' \) with \( \eta = 0 \) does not depend on \( \theta_i \). Therefore, the solution of \( \theta_i \) which minimizes the free energy is a minimum point of the coefficient of the perturbative potential
\[
\frac{\sin^4(\theta_1 - \theta_2) + \sin^4(\theta_2 - \theta_3) + \sin^4(\theta_3 - \theta_1)}{\sin^2(\theta_1 - \theta_2) + \sin^2(\theta_2 - \theta_3) + \sin^2(\theta_3 - \theta_1))^2}.
\]
Figure 1 is a plot of (3.30) and the range of the plot is (3.14). There are two minimum points:
\[
\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{2\pi}{3},
\]
and
\[
\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{4\pi}{3}.
\]
At these points, the value of (3.30) is 1/3 and the coefficient of the perturbative potential of \( \psi' \) is \( \eta/3 \). These solutions are chiral as figure 2.

For example, we consider the solution with \( \cos(\theta_2 - \theta_3) = 1, \cos(\theta_1 - \theta_2) = \cos(\theta_3 - \theta_1) = -1 \). For the diagonalization, we define \( \psi'_i \) as
\[
\psi'_1 = \frac{2}{\sqrt{6}} \psi_1 + \frac{1}{\sqrt{6}} \psi_2 + \frac{1}{\sqrt{6}} \psi_3,
\]
\[
\psi'_2 = -\frac{1}{\sqrt{2}} \psi_2 + \frac{1}{\sqrt{2}} \psi_3,
\]
\[
\psi'_3 = -\frac{1}{\sqrt{3}} \psi_1 + \frac{1}{\sqrt{3}} \psi_2 + \frac{1}{\sqrt{3}} \psi_3.
\]
Using them, we can rewrite (3.1) as

\[
S_{\text{on-shell}} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu}^2 - |D_\mu \psi'_1|^2 - |D_\mu \psi'_2|^2 - |D_\mu \psi'_3|^2 
- \left(m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}\right) \psi'^2_1
- \left(m_2^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}\right) \psi'^2_2
- \left(m_3^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\right) \psi'^2_3
- \eta(\psi'^4_1 + \psi'^4_2 + \psi'^4_3) \right].
\]

(3.36)

It is difficult to compare the free energy of this solution with others. To compare the free energy, we assume that there are only four types of the solutions with \(\eta = 0\):

- \(\psi'_2 \neq 0, \psi'_1 = \psi'_3 = 0\) or \(\psi'_1 \neq 0, \psi'_2 = \psi'_3 = 0\).
- \(\psi'_1 = A\psi'_2 \neq 0, \psi'_3 = 0\) and \(A\) is constant.
- \(\psi'_1 = \psi'_2 = 0, \psi'_3 \neq 0\).
\[ \psi_1 = \psi_2 = \psi_3 = 0. \]

This assumption is natural because the mass squared of \( \psi_1' \) and \( \psi_2' \) is same, and the scalar fields with different mass squared do not have nonzero values together in two-scalar model [9] as we explained in section 2. Under this assumption, the solution with \( \eta = 0 \) is

\[ \psi_1' = \psi, \psi_2' = A\psi, \psi_3' = 0, \tag{3.37} \]

\[ \psi_1 = \frac{2}{\sqrt{6}}\psi, \psi_2 = \frac{1 - \sqrt{3}A}{\sqrt{6}}\psi, \psi_3 = \frac{1 + \sqrt{3}A}{\sqrt{6}}\psi, \tag{3.38} \]

\[ \psi > 0, -\frac{1}{\sqrt{3}} < A < \frac{1}{\sqrt{3}}. \tag{3.39} \]

Substituting (3.37) and (3.38) for (3.36), we obtain

\[ S_{\text{on-shell}} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (1 + A^2)|D_\mu\psi'|^2 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) (1 + A^2)\psi^2 - \frac{\eta}{2}(1 + A^2)^2\psi'^4 \right] \]

\[ = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D_\mu\psi'|^2 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi'^2 - \frac{\eta}{2}\psi'^4 \right], \tag{3.40} \]

where we redefine a new scalar field \( \psi' \) as

\[ \psi'^2 \equiv (1 + A^2)\psi^2. \tag{3.41} \]

The coefficient of the perturbative potential of \( \psi' \) is \( \eta/2 \). This result is consistent with a limit of (3.28). The solutions with \( \cos(\theta_1 - \theta_2) = 1, \cos(\theta_2 - \theta_3) = \cos(\theta_3 - \theta_1) = -1 \) and \( \cos(\theta_3 - \theta_1) = 1, \cos(\theta_1 - \theta_2) = \cos(\theta_2 - \theta_3) = -1 \) can be analyzed in the same way. The solution with \( \cos(\theta_1 - \theta_2) = \cos(\theta_2 - \theta_3) = \cos(\theta_3 - \theta_1) = 1 \) corresponds to the solution with \( \psi_1' = \psi_2' = 0, \psi_3' \neq 0 \).

If \( \eta = 0 \), the values of the action (3.26), (3.28) and (3.40) are same\(^3\). Therefore, difference of the free energy comes from the coefficient of the perturbative potential of \( \psi' \). Comparing the coefficients of the perturbative potential of (3.26), (3.28) and (3.40), we can see that the coefficients of the perturbative potential of the solutions (3.31) and (3.32) are minimum. From this analysis, we conclude that if \( \eta > 0 \) and (3.25) hold, the free energy of the solutions (3.31) and (3.32) is minimum. These solutions correspond to the chiral ground states.

Summarizing the above, we found the chiral ground states (3.31) and (3.32) by comparing the free energy of the solutions of the three-scalar model (3.1) with \( \eta > 0 \) and (3.25). Therefore, the three-scalar model can describe the frustration in curved space-time and introduce chirality as an order parameter.

\(^3\)If the Hawking temperature of the black hole is low enough, the solutions exist [5].
4 Summary and discussion

In this paper, we have analyzed the holographic superconductor model with some scalar fields and the Josephson coupling from the viewpoint of gauge/gravity correspondence. We have seen that the frustration does not occur in the holographic two-scalar model because there is only one Josephson coupling term. On the other hand, we have found that there are several solutions whose free energy is the same if the frustration in the holographic three-scalar model with three Josephson coupling terms if $\eta = 0$, $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$ and (3.18) hold. Furthermore, we have analyzed the free energy of the solutions with $\eta > 0$. We have found that there are the solutions which correspond to the chiral ground states with (3.25).

To compare the free energy of the solutions, we reduced the solutions of the three-scalar model to those of the one-scalar model. It is important to check whether there are other solutions which do not reduce to those of the one-scalar model. Moreover, analysis with a back reaction or large $\eta$ is also important.

For anomalous Hall effect from chirality, a coupling between an electron and localized spins is important [3]. Therefore, the three-scalar model with fermions coupled with complex scalar fields may have strange property.

In Ref. [4], the existence of a chiral domain wall is discussed in the context of condensed matter physics. It is interesting to check whether there is a solution which corresponds to the chiral domain wall in a holographic model. In this paper, we considered specific values of the parameters as (3.25). Comparing the free energy with other values of the parameters which satisfy (3.18) is also interesting because there is some possibility of a phase transition by changing the parameters. They are future directions of this study.

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References

[1] R. Moessner and A. P. Ramirez, “Geometrical frustration,” Phys. Today 59(2), 24 (2006).
[2] H. T. Diep, “Frustrated spin systems,” World Scientific, (2004)
[3] K. Ohgushi, S. Murakami and N. Nagaosa, “Spin anisotropy and quantum Hall effect in the kagomé lattice: Chiral spin state based on a ferromagnet,” Phys. Rev. B62, R6065 (2000) [arXiv:cond-mat/9912206 [cond-mat.str-el]].
[4] Y. Tanaka and T. Yanagisawa, “Chiral ground state in three-band superconductors,” J. Phys. Soc. Jpn. 79, 114706 (2010).
[5] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, “Building a Holographic Superconductor,” Phys. Rev. Lett. 101, 031601 (2008) [arXiv:0803.3295 [hep-th]].
[6] J. M. Maldacena, “The Large $N$ limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200].
[7] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[8] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[9] P. Basu, J. He, A. Mukherjee, M. Rozali and H.-H. Shieh, “Competing Holographic Orders,” JHEP 1010, 092 (2010) [arXiv:1007.3480 [hep-th]].

[10] W.-Y. Wen, “Quantum Criticality in Einstein-Maxwell-Dilaton Gravity,” Phys. Lett. 707, 398 (2012) [arXiv:1009.3952 [hep-th]].

[11] C.-Y. Huang, F.-L. Lin and D. Maity, “Holographic Multi-Band Superconductor,” Phys. Lett. B 703, 633 (2011) [arXiv:1102.0977 [hep-th]].

[12] A. Krikun, V. P. Kirilin and A. V. Sadofyev, “Holographic model of the $S^\pm$ multiband superconductor,” JHEP 1307, 136 (2013) [arXiv:1210.6074 [hep-th]].

[13] D. Musso, “Competition/Enhancement of Two Probe Order Parameters in the Unbalanced Holographic Superconductor,” JHEP 1306, 083 (2013) [arXiv:1302.7295 [hep-th]].

[14] R.-G. Cai, L. Li, L.-F. Li and Y.-Q. Wang, “Competition and Coexistence of Order Parameters in Holographic Multi-Band Superconductors,” JHEP 1309, 074 (2013) [arXiv:1307.2768 [hep-th]].

[15] F. Nitti, G. Policastro and T. Vanel, “Dressing the Electron Star in a Holographic Superconductor,” JHEP 1310, 019 (2013) [arXiv:1307.4558 [hep-th]].

[16] Y. Liu, K. Schalm, Y.-W. Sun and J. Zaanen, “Bose-Fermi competition in holographic metals,” JHEP 1310, 064 (2013) [arXiv:1307.4572 [hep-th]].

[17] Z.-Y. Nie, R.-G. Cai, X. Gao and H. Zeng, “Competition between the s-wave and p-wave superconductivity phases in a holographic model,” JHEP 1311, 087 (2013) [arXiv:1309.2204 [hep-th]].

[18] I. Amado, D. Arean, A. Jimenez-Alba, L. Melgar and I. S. Landea, “Holographic s+p Superconductors,” Phys. Rev. D 89, 026009 (2014) [arXiv:1309.5086 [hep-th]].

[19] A. Amoretti, A. Braggio, N. Maggiore, N. Magnoli and D. Musso, “Coexistence of two vector order parameters: a holographic model for ferromagnetic superconductivity,” JHEP 1401, 054 (2014) [arXiv:1309.5093 [hep-th]].

[20] A. Donos, J. P. Gauntlett and C. Pantelidou, “Competing p-wave orders,” Class. Quant. Grav. 31, 055007 (2014) [arXiv:1310.5741 [hep-th]].

[21] M. Nishida, “Phase Diagram of a Holographic Superconductor Model with s-wave and d-wave,” JHEP 1409, 154 (2014) [arXiv:1403.6070 [hep-th]].

[22] L. F. Li, R. G. Cai, L. Li and Y. Q. Wang, “Competition between s-wave order and d-wave order in holographic superconductors,” JHEP 1408, 164 (2014) [arXiv:1405.0382 [hep-th]].

[23] F. Nitti, G. Policastro and T. Vanel, “Polarized solutions and Fermi surfaces in holographic Bose-Fermi systems,” arXiv:1407.0410 [hep-th].

[24] P. Chaturvedi and P. Basu, “Holographic quantum phase transitions and interacting bulk scalars,” Phys. Lett. B 739, 162 (2014) [arXiv:1409.4959 [hep-th]].

[25] Z. Y. Nie, R. G. Cai, X. Gao, L. Li and H. Zeng, “Phase transitions in a holographic s+p model with backreaction,” arXiv:1501.00004 [hep-th].
[26] W. -Y. Wen, M. -S. Wu and S. -Y. Wu, “A Holographic Model of Two-Band Superconductor,” Phys. Rev. D 89, 066005 (2014) [arXiv:1309.0488 [hep-th]].

[27] P. Breitenlohner and D. Z. Freedman, “Stability in Gauged Extended Supergravity,” Annals Phys. 144, 249 (1982).