INVARIANT MEASURE OF STOCHASTIC FRACTIONAL BURGERS EQUATION WITH DEGENERATE NOISE ON A BOUNDED INTERVAL

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ABSTRACT. This work is concerned with the invariant measure of a stochastic fractional Burgers equation with degenerate noise on a one dimensional bounded domain. Due to the disturbance and influence of the fractional Laplacian operator on a bounded interval interacting with the degenerate noise, the study of the system becomes more complicated. In order to get over the difficulties caused by the fractional Laplacian operator, the usual Hilbert space does not fit the system, we introduce an appropriate weighted space to study it. Meanwhile, we apply the asymptotically strong Feller property instead of the usually strong Feller property to overcome the trouble caused by the degenerate noise, the corresponding Malliavin operator is not invertible. We finally derive the uniqueness of the invariant measure which further implies the ergodicity of the stochastic system.

1. Introduction. Burgers equation plays an important role in describing the interaction of dissipative and non-linear inertial terms in the motion of the turbulent fluid [1]. Through the Hopf-Cole transformation [4, 17], the interaction can be solved in an explicit expression, and the solution develops shock waves in the limit of vanishing viscosity. However, it is no longer available in most case of random forces.

In this paper, we consider the stochastic fractional Burgers equation with degenerate noise on one dimensional bounded domain (also called SFBE for short)

\[
\begin{aligned}
    u_t &= -(-\Delta)^s u - uu_x + \dot{W}(t), \quad t > 0, \quad x \in D, \\
    u|_{D^c} &= 0, \\
    u(x, 0) &= u_0(x),
\end{aligned}
\]

(1)

where \( D = (-1, 1) \subseteq \mathbb{R}^1 \), \( D^c = \mathbb{R}^1 \setminus D \), and \( \dot{W}(t) \) is a degenerate noise specified in the next section in detail. The fractional Laplacian operator \((-\Delta)^s\) is defined as

\[
(-\Delta)^s u(x) = C_s \int_{\mathbb{R}^1 \setminus \{0\}} \frac{u(x + y) - u(x)}{|y|^{1+2s}} dy, \quad 0 < s < 1, \quad x \in D,
\]

(2)

where \( C_s \) is a constant depending on \( s \).

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Notice that the fractional Laplacian operator \((-\Delta)^s\) is defined on a bounded interval \(D\), which is a nonlocal operator and is heavily different from the usual Laplacian operator \(-\Delta\). Kwaśnicki [18] obtained the eigenvalues of the fractional Laplacian operator on a bounded interval. Du et al. [8] analyzed the characteristics of the fractional Laplacian operator. When the definition of the fractional Laplacian operator in (1.2) is extended to the whole space \(\mathbb{R}^1\), it coincides with the definition by Fourier transform.

For the Burgers equation with the usual Laplacian operator driven by the normal noise, Bertini et al. [3] proved the existence of solutions by an adoption of the Hopf-Cole transformation. Da Prato et al. [6] also used semigroup method to establish the well-posedness of this system. E et al. [11] proved that there exists a unique stationary distribution. Gourcy [13] proved the large deviations principle. Goldys et al. [12] studied that strong Feller property and irreducibility to obtain the ergodicity of the system.

For the Burgers equation with the fractional Laplacian operator driven by the normal noise, Lv and Duan [19] established the existence of the martingale solution and the weak solution. Brzeźniak, Debbi and Goldys [2] further studied the ergodicity properties of the system.

However, there are few works of the Burgers equation with the fractional Laplacian operator on bounded domains driven by the degenerate noise. In this paper, we are especially interested in the ergodicity of SFBE (1). Due to the disturbance and influence of the fractional Laplacian operator on a bounded interval interacting with the degenerate noise, the study of SFBE (1) becomes more complicated. There are three key points to achieve our goal: the first is to get over the difficulties, the usual Hilbert space does not fit the system, caused by the fractional Laplacian operator; the second is to overcome the trouble, the corresponding Malliavin operator is not invertible, caused by the degenerate noise; and the third is to deal with the nonlinearity in the case of interacting of the fractional Laplacian operator with the degenerate noise.

For the first key point, since that the fractional Laplacian operator defined on a bounded domain is nonlocal, the usual fractional Sobolev space does not fit the system. Combining with the fractional operator theory of Du et al. [8], we introduce an appropriate weighted function to construct weighted spaces to study it. For the second key point, SFBE (1) is driven by degenerate noise, which causes the corresponding Malliavin operator is not invertible, such that the usually strong Feller property is invalidating. We adopt the argument of Hairer and Mattingly [14], asymptotically strong Feller property, to solve it. For the third key point, perturbing by the interaction of degenerate noise and fractional Laplacian operator on the bounded interval, the estimate of the nonlinear term \(uu_x\) of SFBE (1) becomes more complicated. We manage to use some subtle inequalities and techniques to conquer the troubles.

The rest of this paper is organized as follows. In the next section, we state the definition of classical fractional Sobolev spaces and define nonlocal weighted Sobolev spaces, then specify the degenerate white noise. In the last section, we prove the asymptotically strong Feller property and irreducibility which further implies the ergodicity of SFBE (1).

2. Preliminaries. In this paper, let \(L^2(D)\) be the usual Sobolev space, whose inner product and norm are \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\), respectively. Further, let \(\{e_i(x)\}_{i \geq 1}\)
be an orthonormal basis of \( L^2(D) \). Define \( L^2_1 := \text{span}\{e_1, \cdots, e_N\} \), the projector \( \pi_1 : L^2(D) \to L^2_1 \), and \( \pi_h := I - \pi_1 \), where \( I \) is the identity operator. Then \( L^2(D) = L^2_1 \oplus L^2_h \).

Let \([0, T] \subseteq \mathbb{R}^1\) and \( X \) be a Banach space. For \( r \) with \( 1 \leq r < +\infty \), we use \( L^r([0, T]; X) \) to denote the space of Lebesgue measurable function \( f : [0, T] \to X \) such that

\[
\|f\|_{L^r([0, T]; X)} = \left( \int_0^T \|f(t)\|_X^r dt \right)^{1/r} < \infty.
\]

Similarly, for a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a Banach space \( X \), the function space \( L^r(\Omega; X) \) can also be defined for \( r \) with \( 1 \leq r < +\infty \).

2.1. Classical fractional Sobolev space. For \( s \in (0, 1) \) and a bounded domain \( D \) in \( \mathbb{R}^1 \), we define

\[
W^{s, 2}(D) := \left\{ u \in L^2(D) \middle| \frac{u(x) - u(y)}{|x - y|^{\frac{1}{2} + s}} \in L^2(D \times D) \right\},
\]

endowed with the norm

\[
\|u\|_{W^{s, 2}(D)} := \left( \int_D |u(x)|^2dx + \int_D \int_D \frac{|u(x) - u(y)|^2}{|x - y|^{1 + 2s}}dx dy \right)^{\frac{1}{2}}. \tag{3}
\]

When \( 1 < s < 2 \), we write \( s = 1 + \sigma \) with \( \sigma \in (0, 1) \). Define

\[
W^{s, 2}(D) := \left\{ u \in W^{1, 2}(D) \middle| \nabla u \in W^{s, 2}(D) \right\}
\]

with the norm

\[
\|u\|^2_{W^{s, 2}(D)} = \|u\|^2_{W^{1, 2}(D)} + \|\nabla u\|^2_{W^{s, 2}(D)}.
\]

We then have the following embedding inequalities.

**Lemma 2.1** ([21], Propositions 2.1 and 2.2). (i). For \( 0 < s \leq s' < 1 \), let \( D \) be an open set in \( \mathbb{R}^1 \) and \( u \) be a measurable function from \( D \) to \( \mathbb{R}^1 \). Then

\[
\|u\|_{W^{s, 2}(D)} \leq C\|u\|_{W^{s', 2}(D)}
\]

for some suitable positive constant \( C \). In particular, \( W^{s', 2}(D) \subseteq W^{s, 2}(D) \).

(ii). For \( 0 < s < 1 \), let \( D \) be an open set in \( \mathbb{R}^1 \) of class \( C^{0, 1} \) with bounded boundary and \( u \) be a measurable function from \( D \) to \( \mathbb{R}^1 \). Then, the following inequality holds

\[
\|u\|_{W^{s, 2}(D)} \leq C\|u\|_{W^{1, 2}(D)}
\]

for some suitable positive constant \( C \). In particular, \( W^{1, 2}(D) \subseteq W^{s, 2}(D) \).

**Lemma 2.2** ([21], Theorem 6.7 and Theorem 8.2). (i). For \( 0 < s < 1 \), let \( D \subset \mathbb{R}^1 \) be an extension domain for \( W^{s, 2}(D) \). Then there exists a positive constant \( C = C(s, D) \) such that, for any \( u \in W^{s, 2}(D) \), we have

\[
\|u\|_{L^k(D)} \leq C\|u\|_{W^{s, 2}(D)}
\]

for any \( k \in \left[ 2, \frac{2}{1 - 2s} \right] \); i.e. the space \( W^{s, 2}(D) \) is continuously embedded in \( L^k(D) \) for any \( k \in \left[ 2, \frac{2}{1 - 2s} \right] \). If, in addition, \( D \) is bounded, then the space \( W^{s, 2}(D) \) is continuously embedded in \( L^k(D) \) for any \( k \in \left[ 2, \frac{2}{1 - 2s} \right] \).

(ii). For \( 0 < s < 1 \), let \( D \subset \mathbb{R}^1 \) be an extension domain for \( W^{s, 2}(D) \) with no external cusps. Then there exists a positive constant \( C = C(s, D) \), such that

\[
\|u\|_{C^{0, 2}(D)} \leq C\|u\|_{W^{s, 2}(D)}
\]
for any $u \in L^2(D)$, with $\beta = \frac{2s-1}{2}$.

**Lemma 2.3** ([16], Theorem 1.1). Let $\dot{H}^s_p(\mathbb{R}^d)$ be the homogeneous Sobolev space and assume that $p, p_0, p_1 \in (1, +\infty)$, $s, s_1 \in \mathbb{R}$, $\theta \in [0, 1]$ satisfy

$$
\frac{d}{p} - s = (1 - \theta) \frac{d}{p_0} + \theta \left( \frac{d}{p_1} - s_1 \right) \quad \text{and} \quad s \leq \theta s_1.
$$

Then

$$
\|u\|_{\dot{H}^s_p(\mathbb{R}^d)} \leq C \|u\|_{L^p(\mathbb{R}^d)}^{1-\theta} \|u\|_{\dot{H}^s_{p_0}(\mathbb{R}^d)}^\theta.
$$

### 2.2. Weighted fractional Sobolev space

For $s \in (0, 1)$, from Du et al. [8, 9], we define the nonlocal divergence operator

$$
\mathcal{D}(\mathcal{V})(x) := -\int_{\mathbb{R}^1} (\mathcal{V}(x, y) + \mathcal{V}(y, x)) \cdot \beta(x,y)dy,
$$

for $x \in \mathbb{R}^1$, and the adjoint operator

$$
\mathcal{D}^*(u)(x, y) = -(u(y) - u(x))\beta(x,y), \quad \text{for } x, y \in \mathbb{R}^1.
$$

Furthermore,

$$
\mathcal{D}(\Theta \cdot \mathcal{D}^* u)(x) = -2 \int_{\mathbb{R}^1} (u(y) - u(x))\beta(x,y) \cdot (\Theta(x,y) \cdot \beta(x,y))dy, \quad \text{for } x, y \in \mathbb{R}^1.
$$

Here the vector mapping $\mathcal{V}(x, y)$, $\beta(x,y) : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^k$ with $\beta$ antisymmetric and $u(x) : \mathbb{R}^1 \to \mathbb{R}^1$, the second-order tensor $\Theta(x,y)$ satisfies $\Theta = \Theta^T$.

Let $2|\beta(x,y)|^2 = \frac{C'_s}{|x-y|^{1+2s}}$ and $\Theta$ be an identity matrix, then

$$
\mathcal{D}(\Theta \cdot \mathcal{D}^* u)(x) = C_s \int_{\mathbb{R}^1} \frac{u(x) - u(y)}{|x-y|^{1+2s}} dy,
$$

which implies

$$
(-\Delta)^s u(x) = \mathcal{D}(\Theta \cdot \mathcal{D}^* u)(x) = \mathcal{D}(\mathcal{D}^* u)(x),
$$

where $C_s = -C'_s = -2$.

Furthermore, for $D = (-1, 1)$ and $u = 0$ in $D^c$, we deduce that

$$
\langle (-\Delta)^s u, u \rangle_{L^2(D)} = \langle \mathcal{D}^* u(x), \mathcal{D}^* u(x) \rangle_{L^2(D)}
$$

$$
= 2 \int_D \int_{D^c} \frac{|u(x)|^2}{|x-y|^{1+2s}} dydx + \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} dydx.
$$

(4)

Put $\rho(x) := \int_{D^c} \frac{2}{|x-y|^{1+2s}} dy$, which is in $(0, 4)$ for $x \in D$. Hence, we introduce a weighted fractional Sobolev space

$$
W^{s,2}_p(D) := \left\{ u \in L^2(\mathbb{R}^1) \mid \|u\|_{W^{s,2}_p(D)} < \infty \quad \text{and} \quad u|_{D^c} \equiv 0 \right\},
$$

equipped with the norm

$$
\|u\|_{W^{s,2}_p(D)} := \left( \int_D \rho(x)|u(x)|^2 dx + \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} dydx \right)^{\frac{1}{2}}.
$$

(5)

It is obvious that

$$
W^{s,2}_p(D) \subseteq W^{s,2}_p(D) \subseteq L^2(D).
$$

Furthermore, it follows from (4) that $\langle (-\Delta)^s u, u \rangle_{L^2(D)} = \|u\|^2_{W^{s,2}_p(D)}$ and

$$
\langle (-\Delta)^s u, v \rangle_{L^2(D)} \leq \|\mathcal{D}^* u\|_{L^2(D)} \|\mathcal{D}^* v\|_{L^2(D)}
$$

$$
= \|u\|_{W^{s,2}_p(D)} \|v\|_{W^{s,2}_p(D)}.
$$

(6)
When $1 < s < 2$, let $s = 1 + \sigma$ with $\sigma \in (0, 1)$. We can also define
\[ W^{s, 2}_\rho(D) = \{ u \in W^{1, 2}_\rho(D) | \nabla u \in W^{s, 2}_\rho(D) \} \]
with the norm
\[ \| u \|^2_{W^{s, 2}_\rho(D)} = \int_D \rho(x) |\nabla u|^2 dx + \| \nabla u \|^2_{W^{s, 2}_\rho(D)}. \] (7)

2.3. **Degenerate white noise.** Let $L^{0,0}_2$ denote the space of Hilbert-Schmidt operators from $L^2(D)$ to $L^2(D)$, endowed with the norm $\| \phi \|_{L^{0,0}_2}^2 = \sum_{k=1}^{\infty} \| \phi_k \|_{L^2}^2$, where $\{ e_k \}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(D)$ as above. Similarly, we can define the space $L^{0,1}_2$, which is the space of Hilbert-Schmidt operators from $L^2(D)$ to $H^1(D)$ with the norm $\| \phi \|^2_{L^{0,1}_2} = \sum_{k=1}^{\infty} \| \phi_k \|_H^2$.

Let $W(t)$ be a Wiener process, which is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{ \mathcal{F}_t \}_{t \geq 0}$ and takes value in the separable Hilbert space $L^2(D)$ with zero value on $D^c$. Its covariance operator $Q$ is a symmetric nonnegative operator satisfying $\text{Tr}Q < +\infty$. Furthermore, it satisfies the expansion as follows
\[ W(t) = \sum_{k=1}^{N} \beta_k(t) e_k, \]
where $N$ is a finite positive integer, the operator $\phi$ is in $L^{0,1}_2$, and $\{ \beta_k(t) \}$ is a sequence of mutually independent standard scalar Wiener process in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\text{Tr}Q = \text{Tr}\phi \phi^* = \sum_{k=1}^{N} \| \phi_k \|_H^2 = \| \phi \|_{L^{0,1}_2}^2$. The white noise $W(t)$ is degenerate in the sense that it drives the system only in the first $N$ Fourier modes (essentially, only a finite Fourier modes of the noise are nonzero).

2.4. **Well-posedness.** Denote $u(t, \omega; u_0)$ the solution of SFBE (1) with the initial value $u_0$. Obviously, $u(0, \omega; u_0) = u_0$. Furthermore, for SFBE (1), we define
\[ \mathcal{M}(L^2(D)) := \{ \mu | \text{all the probability measures on } L^2(D) \}. \]
In addition, we use $C_b(L^2(D))$ to denote $C_b(L^2(D); \mathbb{R})$, the set of bounded continuous function mapping $L^2(D)$ to $\mathbb{R}$. As in Da Prato and Zabczyk [7], for $\mu \in \mathcal{M}(L^2(D))$ and $\varphi \in C_b(L^2(D))$, let $P_t^\mu$ denote a measure semigroup mapping from $\mathcal{M}(L^2(D))$ to $\mathcal{M}(L^2(D))$, and $P_t$ be a transition probability semigroup, which are defined as
\[ \int_{L^2(D)} \varphi(u_0)(P_t^\mu)(du_0) = \int_{L^2(D)} P_t \varphi(u_0)(\mu)(du_0) = \int_{L^2(D)} E(\varphi(u(t, \omega; u_0))) \mu(du_0). \] (8)

A probability measure $\mu \in \mathcal{M}(L^2(D))$ is called an *invariant measure* if
\[ P_t^\mu \mu = \mu \text{ for all } t \geq 0. \]

Using the same arguments as Lv and Duan [19], and Brzeźniak, Debbi and Goldys [2], it is easily to derive the well-posedness and the existence of invariant measure of SFBE (1) as follows.
Proposition 2.4. For $\frac{1}{2} < s < 1$, Assume that the initial condition $u_0$ is $\mathcal{F}_0$-measurable in $L^2(\Omega, W^{s,2}(D))$. Then, for any $\tau > 0$, SFBE (1) has a unique solution $u(t)$ in $C([0, \tau], W^{s,2}(D))$. Furthermore, it almost surely satisfies
$$u(t) \in L^\infty(0, \tau; L^2(D)) \cap L^2(0, \tau; W^{s,2}_p(D)).$$

Proposition 2.5. There exists an invariant measure $\mu \in \mathcal{M}(L^2(D))$ of SFBE (1).

Lemma 2.6 (5), Proposition 6.1. Let $D \subseteq \mathbb{R}^n$ be a bounded, open set of class $C^1$, $s_1, s_2, s_3 \in \mathbb{R}$, such that $0 \leq s_1 \leq l$, $0 \leq s_2 \leq l - 1$, $0 \leq s_3 \leq l$ and assume that
$$(i) \quad s_1 + s_2 + s_3 \geq \frac{n}{2}, \text{ if } s_i \neq \frac{n}{2}, i = 1, 2, 3;$$
or
$$(ii) \quad s_1 + s_2 + s_3 > \frac{n}{2}, \text{ if } s_i \neq \frac{n}{2}, \text{ for at least some } i \in \{1, 2, 3\}; \text{ i.e., } s_1 + s_2 + s_3 \geq \frac{n}{2} \text{ and } (s_1, s_2, s_3) \neq (0, 0, \frac{n}{2}), (0, \frac{n}{2}, 0), (\frac{n}{2}, 0, 0);$$
then
$$\|\langle u, \nabla v, w \rangle\| \leq C|D|^\frac{s_1+s_2+s_3}{n} \frac{1}{2} \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \|w\|_{H^{s_3}}.$$

Lemma 2.7 (10), Lemma A.4. Let $\pi_i$ be the projector from $L^2(D)$ to $L^2_i$ (defined as the above). If $u, v \in L^2(D)$, then there exists a positive constant $C$ such that
$$\|\pi_i (u \cdot \nabla v)\| \leq C\|u\|\|v\|.$$

For simplicity in this paper, we denote $H := L^2(D)$, $V := W^{s,2}(D)$ and $V_i := W^{s_i,2}(D)$ with $s \in (1/2, 1)$. In addition, we will also use $C$ to denote various positive constants, which may be different from line to line.

3. Ergodicity. Since SFBE (1) is driven by the degenerate noise, we study asymptotically strong Feller property originated from Hairer and Mattingly [14], instead of the strong Feller property of the transition probability semigroup $P_t$ defined as (8). In this section, we also denote $u(t, \omega; u_0) = \Phi_t(\omega, u_0)$ of SFBE (1).

For $0 \leq r < t$, let $\mathcal{J}_{r,t}\xi$ be the solution of the linearized equation
$$\begin{cases}
\partial_t \mathcal{J}_{r,t}\xi = -(-\Delta)^s \mathcal{J}_{r,t}\xi - (u \mathcal{J}_{r,t}\xi)_x, \\
\mathcal{J}_{r,r}\xi = \xi.
\end{cases}
$$

When $r = 0$, for simplicity, we write $\mathcal{J}_t\xi = \mathcal{J}_{0,t}\xi$. It is not difficult to show that for every $\omega$,
$$(\mathcal{J}_t\xi)(\omega) = \lim_{\varepsilon \to 0} \frac{\Phi_t(\omega, u_0 + \varepsilon \xi) - \Phi_t(\omega, u_0)}{\varepsilon}.$$

Given $v \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^1)$, The Malliavin derivative of $\Phi_t(\omega, u_0)$ with respect to $\omega$ in the direction $v$ is given by
$$\mathcal{D}^v u(t, \omega; u_0) = \lim_{\varepsilon \to 0} \frac{\Phi_t(\omega + \varepsilon \tilde{V}, u_0) - \Phi_t(\omega, u_0)}{\varepsilon},$$
where $\tilde{V}(t) = \int_0^t v(r)dr$. If we set $\mathcal{A}_tv = \mathcal{D}^v u(t, \omega; u_0)$, then
$$\begin{cases}
\partial_t \mathcal{A}_t v = -(-\Delta)^s \mathcal{A}_t v - (u \mathcal{A}_t v)_x + Qv(t), \\
\mathcal{A}_0v = 0.
\end{cases}
$$

By Duhamel principle, $\mathcal{A}_tv = \int_0^t \mathcal{J}_{t,r}Qv(r)dr$. Set $\varrho(t) = \mathcal{J}_t\xi - \mathcal{A}_t v(t)$, where $v(t)$ is the restriction of $v$ to the internal $[0, t]$. Then
$$\begin{cases}
\partial_t \varrho = -(-\Delta)^s \varrho - (u \varrho)_x - Qv(t), \\
\varrho(0) = \xi.
\end{cases}
$$
Lemma 3.1. Let $s$ be in $(\frac{1}{2},1)$ and $u_0$ be in $V$. There then exist two positive constants $C$ and $\eta_s > 0$ such that for every $t > 0$ and every $\eta \in (0, \eta_s]$, 
\begin{equation}
\mathbb{E}e^{\eta}\|u(t)\|^2 \leq Ce^{\nu \eta t}\|u_0\|^2.
\end{equation}

Proof. Using Itô’s formula and noticing that $\langle (-\Delta)^s u, u \rangle = \|u\|_V^2$ and $\langle uu_x, u \rangle = 0$, we have 
\begin{align*}
\|u(t)\|^2 &= \|u_0\|^2 - 2 \int_0^t \int_D u(r)(-\Delta)^s u(r)dxdr - 2 \int_0^t \int_D uu_xu(r)dxdr \\
&\quad + 2 \int_0^t \int_D u(r)dW(r)dx + \|\phi\|_{L^2}^2 t.
\end{align*}

Further using the inequality \cite[Lemma A.1]{20} and taking the parameters $\beta \to \infty$ and $\gamma = 1$ in the inequality, we drive 
\begin{align*}
\|u(t)\|^2 &\leq \|u_0\|^2 - 2 \int_0^t e^{-(t-r)}\|u(r)\|^2 dr \\
&\quad + 2 \int_0^t \int_D e^{-(t-r)}u(r)dW(r)dx + \|\phi\|_{L^2}^2 t.
\end{align*}

Since there exists a positive constant $\alpha > 0$ such that $\|u(r)\|^2 \geq \frac{\alpha}{2} \|Q^* u(r)\|^2$, it yields that 
\begin{align*}
\|u(t)\|^2 - e^{-t}\|u_0\|^2 - \|\phi\|_{L^2}^2 t &\leq 2 \int_0^t \int_D e^{-(t-r)}u(r)dW(r)dx \\
&\quad - \frac{\alpha}{2} \int_0^t e^{-(t-r)}\|Q^* u(r)\|^2 dr.
\end{align*}

Further using the inequality \cite[Lemma A.1]{20} and taking the parameters $\beta \to \infty$ and $\gamma = 1$ in the inequality, we drive 
\begin{align*}
\mathbb{P}(2 \int_0^t \int_D e^{-(t-r)}u(r)dW(r)dx - \frac{\alpha}{2} \int_0^t e^{-(t-r)}\|Q^* u(r)\|^2 dr > \frac{K}{\alpha}) \leq e^{-K},
\end{align*}

which, similarly as the proof of Lemma A.1 in Hairer and Mattingly’s \cite{14}, immediately implies 
\begin{align*}
\mathbb{P}(\|u(t)\|^2 - e^{-t}\|u_0\|^2 - C > \frac{K}{\alpha}) \leq e^{-K}.
\end{align*}

Further note that if $\mathbb{P}(X \geq C) \leq \frac{1}{C^2}$ for any $C > 0$, then $\mathbb{E}X \leq 2$. Taking $\eta_s = \frac{\alpha}{2}$ and $C = 2e^{\frac{K}{\alpha}}$, the proof is completed. \hfill \Box

Lemma 3.2. Let $s$ be in $(\frac{1}{2},1)$ and $u_0$ be in $V_1$. For any $T > 0$, there exists a positive constant $C$ and $\eta_s > 0$ such that every $\eta \in (0, \eta_s]$, 
\begin{align*}
\mathbb{E}\|\nabla u(t)\|^2 &\leq C, \quad \mathbb{E}\int_0^T \|u(r)\|^2_{V_1} dr \leq C, \\
\mathbb{E}e^{\eta}\|\nabla u(t)\|^2 &\leq Ce^{\nu \eta t}\|\nabla u_0\|^2, \quad t \in [0, T].
\end{align*}
Proof. It follows from Itô’s formula that
\begin{align*}
\| \nabla u(t) \|^2 = & \| \nabla u_0 \|^2 - 2 \int_0^t \int_D (-\Delta) \nabla u(r) \nabla u(r) dx dr + 2 \int_0^t \int_D u(r) uu_x dx dr \\
& - 2 \int_0^t \int_D \Delta u(r) dW(r) dx + \sum_{i=1}^N \int_0^t (\nabla \phi_{\epsilon,i})^2 dr \\
= & \| \nabla u_0 \|^2 - 2 \int_0^t \int_D (-\Delta) \nabla u(r) \nabla u(r) dx dr - \int_0^t \| \nabla u(r) \|_{L^2} dx dr \\
& - 2 \int_0^t \int_D \Delta u(r) dW(r) dx + \sum_{i=1}^N \int_0^t (\nabla \phi_{\epsilon,i})^2 dr.
\end{align*}
Noting that \( u(t, x) = 0 \), as \( x \in D^c \), we have
\begin{align*}
((-\Delta)^s \nabla u(x), \nabla u(x))_{L^2(D)} = & ((-\Delta)^s \nabla u(x), \nabla u(x))_{L^2(\mathbb{R})} \\
= & 2 \int_0^t \int_D \nabla u(x) dx dy + \int_0^t \int_D \nabla u(y) dx dy.
\end{align*}
Meanwhile, from the Hölder’s inequality and the Young’s inequality, we get
\begin{equation}
\| \nabla u \|^2_{L^2} \leq C \| \nabla u \|^2_{L^2} \leq c \| \nabla u \|^3_{L^3} + C \epsilon, \quad \forall \epsilon > 0.
\end{equation}
Hence, we can deduce that
\begin{align*}
\| \nabla u(t) \|^2 \leq & \| \nabla u_0 \|^2 - 2 \int_0^t \| u(r) \|^2_{V_1} dr - C \int_0^t \| \nabla u(r) \|^2 dr \\
& - 2 \int_0^t \int_D \Delta u(r) dW(r) dx + Ct \\
\leq & \| \nabla u_0 \|^2 - 2 \int_0^t \| u(r) \|^2_{V_1} dr - C \int_0^t \| \nabla u(r) \|^2 dr \\
& - 2 \int_0^t \int_D \Delta u(r) dW(r) dx + Ct,
\end{align*}
which, similarly as the proof of Lemma 3.1, immediately implies the third result holds. Furthermore, if taking the expectation, we obtain
\begin{equation*}
\frac{d\mathbb{E}\| \nabla u(t) \|^2}{dt} \leq -C \mathbb{E}\| \nabla u(t) \|^2 + C,
\end{equation*}
and
\begin{equation*}
\mathbb{E}\| \nabla u(t) \|^2 + C \mathbb{E} \int_0^t \| u(r) \|^2_{V_1} dr \leq \mathbb{E}\| \nabla u_0 \|^2 + Ct.
\end{equation*}
which implies from the Gronwall’s inequality that the first and the second results hold. The proof is completed.

Lemma 3.3. Let \( s \) be in \( (\frac{1}{2}, 1) \) and \( u_0 \) be in \( V_1 \). For any \( T > 0 \), there exists a positive constant \( C \) and \( \eta_* > 0 \) such that every \( \eta \in (0, \eta_*] \) and \( t \in (0, T] \)
\begin{equation}
\mathbb{E}\exp(\eta \| \nabla u(t) \|^2 + C \int_0^t \| \nabla u(r) \|^2 dr) \leq \mathbb{E}\exp(C\| \nabla u_0 \|^2 + Ct).
\end{equation}
Proof. From the proof of Lemma 3.2, we can deduce that
\[\eta \| \nabla u(t) \|^2 \leq \eta \| \nabla u_0 \|^2 - 2\eta \int_0^t \| u(r) \|_V^2 \, dr - C\eta \int_0^t \| \nabla u(r) \|^2 \, dr
\]
\[-2\eta \int_0^t \int_D \Delta u(r) \, dW(r) \, dx + C\eta t \]
\[\leq \eta \| \nabla u_0 \|^2 - 2\eta \int_0^t \| \nabla u(r) \|^2 \, dr - C\eta \int_0^t \| \nabla u(r) \|^2 \, dr
\]
\[+ 2\eta \int_0^t \langle u(r), dW(r) \rangle_{H^1} + C\eta t ,\]
which implies that
\[\exp[\eta \| \nabla u(t) \|^2 + 2\eta \int_0^t \| \nabla u(r) \|^2 \, dr]
\[\leq \exp[C \int_0^t \| \nabla u(r) \|^2 \, dr] \times \exp[\eta \| \nabla u_0 \|^2 + C\eta t]
\times \exp[2\eta \int_0^t \langle u(r), dW(r) \rangle_{H^1} - \eta \int_0^t \| u(r) \|^2_{H^1} \, dr].\]

Let \( M_t := 2\eta \int_0^t \langle u(r), dW(r) \rangle_{H^1} - \eta \int_0^t \| u(r) \|^2_{H^1} \, dr \). Then taking the expectation, we get
\[\mathbb{E} \exp[\eta \| \nabla u(t) \|^2 + 2\eta \int_0^t \| \nabla u(r) \|^2 \, dr]
\[\leq \mathbb{E}[\exp(C \int_0^t \| \nabla u(r) \|^2 \, dr) \cdot \exp(M_t) \cdot \exp(\eta \| \nabla u_0 \|^2 + C\eta t)].\] (15)

If put \( N_t := 2\eta \int_0^t \langle u(r), dW(r) \rangle_{H^1} \), then \( M_t = N_t - \frac{1}{2} \langle N \rangle_t \), where \( \langle N \rangle_t \) is the quadratic variation of \( N_t \). Noticing that Lemma 3.2, we know \( \mathbb{E}e^{\frac{1}{2} \langle N \rangle_t} = \mathbb{E}e^{\eta \int_0^t \| u(r) \|^2_{H^1} \, dr} \leq C \), satisfying the Novikov condition [22], which implies that \( e^{M_t} \) is an exponential martingale. Then
\[\mathbb{E}e^{M_t} = \mathbb{E}e^{M_0} = 1,\]
which, therefore, from (15) and Lemma 3.2 that Lemma 3.3 holds. The proof is completed. \( \square \)

Now, we begin to prove the asymptotical gradient estimate for the transition probability semigroup \( P_t \).

**Proposition 3.4** (Asymptotical gradient estimate). Let \( P_t \) be the transition probability semigroup of SFBE (1). There exist some finite integer \( N_* \in \mathbb{N} \) and constants \( C, \delta > 0 \) such that for any Fréchet differentiable function \( \varphi \),
\[\| \nabla P_t \varphi(u_0) \| \leq Ce^{C\|u_0\|^2}(\| \varphi \|_\infty + e^{-\delta t}\| \nabla \varphi \|_\infty).\] (16)

**Proof.** For any \( \xi = \xi_t + \xi_h \in H \) with \( \| \xi \| = 1 \). We further define
\[\zeta_t(t) = \begin{cases} \xi_t \left( 1 - \frac{1}{2\|\xi_t\|} \right), & t \in [0,2\|\xi_t\|], \\ 0, & t > 2\|\xi_t\|. \end{cases}\]
Let \( \zeta_h(t) \) satisfy the equation
\[
\begin{align*}
\partial_t \zeta_h &= -(-\Delta)^s \zeta_h - \pi_h (u(\xi_t + \zeta_h))_x, \\
\zeta_h(0) &= \xi_h.
\end{align*}
\] (17)

Set \( \zeta(t) := \zeta(t) + \zeta_h(t) \). It is clear that \( \zeta(t) \) and \( g(t) \) satisfy the same equation (11) with the same initial data \( g(0) = \zeta(0) = \xi \), which implies that \( g = \zeta \).

For simplicity, we also put \( G(t) := \frac{\zeta(t)}{2\|G(t)\|} - (-\Delta)^s \zeta(t) - \pi (u(\xi(t) + \zeta_h(t)))_x \). Then \( v(t) = Q^{-1} G(t) \).

In the following, we firstly show that there exist constants \( \delta > 0 \) and \( C > 0 \) such that
\[
E\|\zeta(t)\|^2 \leq Ce^{C\|u\|^2-\delta t}. 
\] (18)

For the low frequency \( \zeta_l(t) \), it obviously satisfies \( \|\zeta_l(t)\| \leq 1 \) for \( 0 \leq t \leq 2 \) and \( \|\zeta_l(t)\| = 0 \) for \( t \geq 2 \).

For the high frequency \( \zeta_h(t) \), we have from (17) that
\[
\frac{d}{dt} \|\zeta_h(t)\|^2 = 2\langle (-\Delta)^s \zeta_h(t) - \pi_h (u(t)(\zeta(t) + \zeta_h(t)))_x, \zeta_h(t) \rangle
\]
\[
= -2\|\zeta_h(t)\|^2 + 2\pi_h \langle u(t)\zeta(t), \nabla \zeta_h(t) \rangle + 2\pi_h \langle u(t)\zeta_h(t), \nabla \zeta_h(t) \rangle
\]
\[
: = R_0 + R_1 + R_2.
\]

Next, we will estimate \( R_1 \) and \( R_2 \). Firstly, we estimate \( R_1 \).
\[
|R_1| = |2\pi_h \langle u\zeta_l, \nabla \zeta_h \rangle| = |2\pi_h \langle \nabla (u\zeta_l), \zeta_l \rangle|
\]
\[
= 2\|\nabla u\zeta_l, \zeta_l \rangle + 2\langle u\nabla \zeta_l, \zeta_l \rangle
\]
\[
= J_1 + J_2.
\]

For \( J_1 \), by the Hölder’s inequality and the Young’s inequality, we get that
\[
J_1 = 2\|\nabla u\zeta_l, \zeta_l \rangle \leq 2\|\nabla u\zeta_l\|_{L^2} \cdot \|\zeta_l\|_{L^2} \leq 2 \cdot \left( \frac{e_1}{2} \|\nabla u\zeta_l\|_{L^2}^2 + \frac{1}{2e_1} \|\zeta_l\|_{L^2}^2 \right)
\]
\[
\leq e_1 \|\nabla u\zeta_l\|_{L^2}^2 + 1 \|\zeta_l\|_{L^2}^2 \leq C e_1 \|\nabla u\zeta_l\|_{L^2}^2 + \frac{1}{e_1} \|\zeta_l\|_{L^2}^2.
\]

For \( J_2 \), applying Lemma 2.6 with \( s_1 = 1, s_2 = 0, s_3 = s \in (0,1) \) and \( d = 1 \), and the Young’s inequality, noticing that \( W^{s,2}_\rho (D) \subseteq W^{s,2}(D) \), it infers that
\[
J_2 = 2\langle u\nabla \zeta_l, \zeta_l \rangle \leq C\|\nabla u\zeta_l\|_{L^2} \cdot \|\zeta_l\|_{L^2} \leq C \|\nabla u\zeta_l\|_{L^2}^2 + \frac{1}{2e_2} \|\zeta_l\|_{L^2}^2
\]
\[
\leq \frac{Ce_2}{2} \|\nabla u\zeta_l\|_{L^2}^2 + \frac{C}{2e_2} \|\zeta_l\|_{L^2}^2.
\]

Therefore, it holds that
\[
R_1 \leq \frac{C}{2e_2} \|\zeta_l\|_{L^2}^2 + \left( C e_1 + \frac{Ce_2}{2} \right) \|\nabla u\|_{L^2}^2 + \frac{1}{e_1} \|\zeta_l\|_{L^2}^2.
\]

We estimate \( R_2 \) in the following. Because of that
\[
-\langle u\zeta_h, \nabla \zeta_h \rangle = \langle \nabla(u\zeta_h), \zeta_h \rangle = \langle u\nabla \zeta_h, \zeta_h \rangle + \langle \nabla u \zeta_h, \zeta_h \rangle.
\]
using the Hölder’s inequality, the Gagliardo-Nirenberg’s inequality and the Young’s inequality, taking to $W^{p,2}_0(D) \subseteq W^{k,2}(D)$, we have

\[
|R_2| = |2\pi_0(\nabla \zeta_h, \nabla \zeta_h)| = |(\nabla u \zeta_h, \zeta_h)| \leq \|\nabla u\|_{L^2} \|\zeta_h\|_{L^4} \|\zeta_h\|_{L^4}
\leq C\|\nabla u\|_{L^2} \|\zeta_h\|^2_{L^2} \|\nabla \zeta_h\|_{W^{k,2}(D)}
\leq C \epsilon_3 \|\nabla u\|_{L^2} \|\zeta_h\|^2_{L^2} + \frac{C}{4s} \|\zeta_h\|^2_{W^{k,2}(D)}
\leq C \epsilon_3 \|\nabla u\|_{L^2} \|\zeta_h\|^2_{L^2} + \frac{C}{4s} \|\zeta_h\|^2_{\dot{V}}.
\]

Combining $R_1$ with $R_2$, it immediately implies that

\[
\frac{d}{dt} \|\zeta_h(t)\|^2 \leq (-2 + \frac{C}{2\epsilon_2} + \frac{C}{4s} \epsilon_3^{-1}) \|\zeta_h(t)\|^2_V
+ \frac{1}{\epsilon_1} + \frac{C \epsilon_3}{1-\frac{1}{4s}} \|\nabla u(t)\| \|\zeta_h(t)\|^2 + (C \epsilon_1 + \frac{C \epsilon_2}{2}) \|\nabla u(t)\|^2.
\]

Especially, take $\frac{C}{2\epsilon_2} + \frac{C}{4s} \epsilon_3^{-1} \leq 2$. Then by the Gronwall’s inequality, we obtain

\[
\|\zeta_h(t)\|^2 \leq \|\zeta_h(0)\|^2 \exp\left[\int_0^t (-2 + \frac{C}{2\epsilon_2} + \frac{C}{4s} \epsilon_3^{-1}) \frac{1}{\epsilon_1} + \frac{C \epsilon_3}{1-\frac{1}{4s}} \|\nabla u(\tau)\| \|\zeta_h(\tau)\|^2 d\tau \right]
+ \int_0^t (C \epsilon_1 + \frac{C \epsilon_2}{2}) \|\nabla u(\tau)\|^2 \exp\left[\int_0^\tau (-2 + \frac{C}{2\epsilon_2} + \frac{C}{4s} \epsilon_3^{-1}) \frac{1}{\epsilon_1} + \frac{C \epsilon_3}{1-\frac{1}{4s}} \|\nabla u(\tau)\| \|\zeta_h(\tau)\|^2 d\tau \right].
\]

Furthermore, it holds that

\[
\|\zeta_h(t)\|^2 \leq \|\zeta_h(0)\|^2 \exp\left[\int_0^t (-2 + \frac{C}{2\epsilon_2} + \frac{C}{4s} \epsilon_3^{-1}) t + \int_0^t \frac{1}{\epsilon_1} + \frac{C \epsilon_3}{1-\frac{1}{4s}} \|\nabla u(\tau)\| \|\zeta_h(\tau)\|^2 d\tau \right]
+ \int_0^t (C \epsilon_1 + \frac{C \epsilon_2}{2}) \|\nabla u(\tau)\|^2 \exp\left[\int_0^\tau (-2 + \frac{C}{2\epsilon_2} + \frac{C}{4s} \epsilon_3^{-1}) (t - \tau) \right]
+ \int_0^t \frac{1}{\epsilon_1} + \frac{C \epsilon_3}{1-\frac{1}{4s}} \|\nabla u(\tau)\| \|\zeta_h(\tau)\|^2 d\tau \right].
\]

Taking expectation and using Lemma 3.1 and Lemma 3.3, it infers that

\[
E\|\zeta_h(t)\|^2 \leq C e^{C \|u_0\|^2 - \delta t}.
\]

Therefore, combining with $E\|\zeta_h(t)\|^2 \leq C$, then the above estimate (18) holds.

Secondly, we show that

\[
\int_0^t E\|v(\tau)\|^2 \leq C e^{C \|u_0\|^2}.
\]

Indeed, since $v(t) = Q^{-1} G(t)$, we only need to show $\int_0^t E\|G(\tau)\|^2 d\tau \leq C e^{C \|u_0\|^2}$.

By definition of $G(t)$, it deduces that

\[
E\|G(t)\|^2 \leq C \{1_{t \leq 2} + E\|(-\Delta)^s \zeta_t\|^2 + E\|\pi_1(u_\zeta)_x\|^2\}.
\]
Further, it follows from Lemma 2.7 that
\[ \|\pi_t(u\zeta_x)\| \leq \|\pi_t(u\zeta_x)\| + \|\pi_t(\zeta u_x)\| \leq C\|u\|\|\zeta\|, \]
which implies from Lemma 3.1, (18) and (21) that (20) holds.

Finally, let’s prove the main result (16) in Proposition 3.4. Using the chain rule and the integration by parts formula, we have
\[ \langle \nabla P_t \varphi(u_0), \xi \rangle = \mathbb{E}((\nabla \varphi)(u(t)) \cdot \mathcal{J}_t \xi) \]
\[ = \mathbb{E}((\nabla \varphi)(u(t))\cdot \mathcal{A}_t v) + \mathbb{E}((\nabla \varphi)(u(t))\varphi(t)) \]
\[ = \mathbb{E}((\mathcal{D}^{\nu} \varphi)(u(t))) + \mathbb{E}((\nabla \varphi)(u(t))\varphi(t)) \]
\[ = \mathbb{E}(\varphi(u(t)) \int_0^t v(r)dW(r)) + \mathbb{E}((\nabla \varphi)(u(t))\varphi(t)) \]
\[ \leq \|\varphi\|_\infty \mathbb{E}\int_0^t v(r)dW(r) + ||\nabla \varphi||_\infty \mathbb{E}||\varphi(t)||. \]

Now, since that \( v_{[0,t]} \) is adapted to the Wiener path, it follows from (20) that
\[ \mathbb{E}\int_0^t v(r)dW(r) \leq \left( \int_0^t \mathbb{E}||v(r)||^2d\tau \right)^{\frac{1}{2}} \leq C \|u_0\|^2. \quad (22) \]
Consequently, it yields from (18) and (22) that (16) holds, which completes the proof. \( \Box \)

In the following, we consider the irreducibility of the transition probability semigroup \( P_t \) of SFBE (1).

**Proposition 3.5.** Zero belongs to the support of any invariant measure of \( \{ P_t \}_{t \geq 0} \).

**Proof.** Set \( v(t) = u(t) - W(t) \), then
\[ v_t = -((-\Delta)^s(v(t) + W(t)) - (v(t) + W(t))(v(t) + W(t))_.) \quad (23) \]
Give arbitrary \( T > 0 \) and \( \varepsilon > 0 \), which are choosed in the later. We assume that
\[ \sup_{t \in [0,T]} \|W(t)\|_{V_1} < \varepsilon. \quad (24) \]

It follows from (23) that
\[ \frac{d}{dt}\|v(t)\|^2 = -2((-\Delta)^s(v(t) + W(t)), v(t)) - 2((v(t) + W(t))(v(t) + W(t))_., v(t)) \]
\[ = -2((-\Delta)^s v(t), v(t)) - 2((-\Delta)^s W(t), v(t)) \]
\[ - 2((v(t) + W(t))(v(t) + W(t))_., v(t)) \]
\[ = -2\|v(t)\|^2 - 2((-\Delta)^s W(t), v(t)) \]
\[ - 2((v(t) + W(t))(v(t) + W(t))_., v(t)). \]
By the Hölder’s inequality and that \( W_{p+1,s/2}(D) \subseteq W_{p^*}^{s,2}(D) \), using the Young’s inequality, it implies that
\[ \|((-\Delta)^s W(t), v(t))\|_{L^2(D)} = \|D^{s} W(t), D^{s} v(t))\|_{L^2(D)} \leq \|D^{s} W(t)\|_{L^2} \cdot \|D^{s} v(t)\|_{L^2} \]
\[ = \|W(t)\|_{V} \cdot \|v(t)\|_{V} \leq C\varepsilon \|v(t)\|^2_V + C\varepsilon. \]
Applying integration by parts, we have
\[ ((v + W)\nabla(v + W), W) = -\langle v + W, \nabla(W(v + W)) \rangle \]
\[ = -\langle v + W, (v + W)\nabla W \rangle - \langle v + W, W\nabla(v + W) \rangle. \]
Applying the Hölder’s inequality, the Young’s inequality and integration by parts, one has
\[-2\langle (v + W)\nabla(v + W), v \rangle = -2\langle (v + W)\nabla(v + W), u \rangle + 2\langle (v + W)\nabla(v + W), W \rangle\]
\[= -2(\langle uu_x, u \rangle - \langle v + W, (v + W)\nabla W \rangle = -\langle (v + W)^2, \nabla W \rangle.\]

Furthermore, it implies that
\[| - \langle (v + W)^2, \nabla W \rangle | \leq \|
abla W \| \|(v + W)^2\| \leq \|W\|_{\text{V}_1}\|v + W\|^2_{L_2} \leq \|W\|_{\text{V}_1}(\|v\|^2_{L^4} + \|W\|^2_{L^4}) \leq \varepsilon\|v\|^2_{L^4} + \varepsilon^4.\]

Hence, we get the following estimate from the above estimates,
\[\frac{d}{dt}\|v(t)\|^2 \leq -\gamma\|v(t)\|^2 + C\varepsilon, \tag{25}\]
where \(\gamma = 2 - C\varepsilon - \varepsilon.\)

Therefore, we obtain that for any \(\varepsilon', h > 0,\) there exist \(T' > 0\) and \(\varepsilon\) small enough such that
\[\sup_{t \in [0, T']} \|v(t)\|^2 \leq 2r_1, \tag{26}\]
and
\[\sup_{t \in [T', T' + h]} \|v(t)\|^2 \leq \varepsilon'. \tag{27}\]

By the Gronwall’s inequality, one has
\[\|v(t)\|^2 \leq e^{-\gamma(t-r)}\|v(r)\|^2 + C\varepsilon.\]

Setting \(r = 0\) and \(t = T'\) we have
\[\|v(T')\|^2 \leq C.\]
Then setting \(r = T'\) and \(t = T' + h,\) we get that
\[\|v(T' + h)\|^2 \leq C e^{-\gamma h} + C\varepsilon,\]
which together with (27) yields that there exist a \(T\) large enough and a \(\varepsilon\) small enough such that
\[\|v(T)\| \leq \frac{r_2}{2}.\]
Therefore, it implies that
\[\|u(T, \omega, u_0)\| \leq r_2.\]

Let \(\Omega_\varepsilon = \{\omega : \sup_{t \in [0, T]} \|W(t)\|_{\text{V}_1} < \varepsilon\},\) then \(\Omega_\varepsilon \subset \bigcap_{\|u_0\| \leq r_1} \{\omega : \|u(T, \omega; u_0)\| \leq r_2\}.\)

Since \(\Omega_\varepsilon\) is an open set and \(P(\Omega_\varepsilon) > 0,\) we have
\[\inf_{\|u_0\| \leq r_1} P\{\omega : \|u(T, \omega; u_0)\| \leq r_2\} > 0. \tag{28}\]

We define \(B_\varepsilon = \{x \in H : \|x\| \leq r\}.\) For any invariant measure \(\mu \in \mathcal{M}(H)\) of SFBE (1), we choose some \(r_1 > 0\) such that \(\mu(B_{r_1}) \geq \frac{1}{2}.\) Therefore, for any \(r_2 > 0,\) there holds
\[\mu(B_{r_2}) = \mathcal{P}_t^\mu(B_{r_2}) = \int_H \mathcal{P}_t 1_{B_{r_2}}(x)\mu(dx) \geq \int_{B_{r_1}} \mathcal{P}_t 1_{B_{r_2}}(x)\mu(dx)
\geq \inf_{x \in B_{r_1}} \mathcal{P}_t 1_{B_{r_2}}(x) \cdot \mu(B_{r_1}) > 0,\]
which implies that 0 belongs to the support of $\mu$. This completes the proof. \hfill $\square$

**Theorem 3.6.** SFBE (1) has a unique invariant probability measure in $\mathcal{M}(H)$.

**Proof.** From Proposition 2.5, Proposition 3.4 and Proposition 3.5, it is easy to obtain this result. \hfill $\square$

**Remark 1.** The exponential ergodicity of the system SFBE (1) holds.

Similarly as Hairer and Mattingly’s work [15], combining with Proposition 3.4 (asymptotical gradient estimate) and Proposition 3.5 (irreducibility), we only need to look for a continuous function $V(u)$ to satisfy the Assumption 4 in the work [15]. In fact, if taking $V(u) = \exp(\eta \|\nabla u\|^2)$, using Lemma 5.1 of [15], and combining the proof of Lemma 3.2 with the estimating of Eq.(9), we easily examine it. Consequently, the exponential ergodicity of the system SFBE (1) still holds.

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