COMPLEX ASYSTATIC ACTIONS OF COMPACT LIE GROUPS

ANNA GORI AND FABIO PODESTÀ

Abstract. In the present paper we introduce the notion of complex asystatic Hamiltonian action on a Kähler manifold. In the algebraic setting we prove that if a complex linear group $G$ acts complex asystatically on a Kähler manifold then the $G$-orbits are spherical. Finally we give the complete classification of complex asystatic irreducible representations.

1. Introduction

A proper action of a compact Lie group $K$ on a manifold $M$ is called asystatic if the isotropy representation of a principal isotropy subgroup $L$ has no non-trivial fixed vector in the tangent space of a principal orbit $K/L$, or, equivalently, if the quotient group $N_K(L)/L$ is finite. Asystatic actions are particular cases of polar actions, namely, if we endow $M$ with any $K$-invariant Riemannian metric, there exists a properly embedded submanifold $\Sigma$, called a section, which intersects every $K$-orbit orthogonally; in the case of asystatic actions, a section is provided by the fixed point set of a principal isotropy. Asystatic actions, whose introduction goes back to Lie (see [LE]), have been treated in [AA] and [PT].

Now, given a $K$-action on a manifold $M$ with principal isotropy type $(L)$, the core $\overline{M} \subset M$ is defined as the closure of $M^L \cap M_{\text{reg}}$, where $M^L$ denotes the fixed point set of the subgroup $L$. The quotient group $N_K(L)/L$ acts effectively on $\overline{M}$ and the inclusion $\overline{M} \hookrightarrow M$ induces a
bijection at the orbit space level; this gives the primary motivation for referring to the induced transformation group $((N_K(L)/L), \overline{M})$ as the reduction of $(K, M)$ (see e.g. [GS] and [SS]).

In this paper we are interested in the fixed point set of a principal isotropy for a Hamiltonian action of a compact group of holomorphic isometries $K$ of a Kähler manifold $M$; in this case one of our results states the following

**Proposition 1.2.** Let $M$ be a Kähler manifold endowed with an effective Hamiltonian $K$-action. Let $(L)$ be a principal isotropy type, $\overline{M}$ a core and $c$ the cohomogeneity of the $K$-action. Then

1. $\dim \overline{M} \geq c + \text{rank}(K) - \text{rank}(L)$;
2. the equality in (1) holds if and only if the group $(N_K(L)/L)^o$ is abelian;
3. if $M = V$ is a linear space and $K$ acts linearly on $V$ or $M$ is locally irreducible, then $\text{rank}(K) - \text{rank}(L) \geq 1$.

It is therefore natural to investigate actions for which this equality holds. With this notation we say that a proper $K$-Hamiltonian action on a Kähler manifold $M$ by holomorphic isometries is called complex asystatic (or $C$-asystatic) if

$$\text{dim } \overline{M} = c + \text{rank } K - \text{rank } L$$

or, equivalently, if $(N_K(L)/L)^o$ is abelian.

In the algebraic setting we prove the following theorem, which connects the $C$-asystatic condition with the class of actions of reductive groups with spherical orbits; we recall here that a complex homogeneous space $N$ of a reductive complex group $G$ is called spherical if a Borel subgroup of $G$ has an open orbit in $N$ (see e.g. [BLV]).

**Theorem 1.3.** Let $G$ be a connected complex linear algebraic reductive group acting algebraically on a Kähler algebraic manifold $M$. If the action of a maximal compact Lie subgroup of $G$ on $M$ is Hamiltonian and $C$-asystatic, then every $G$-orbit is spherical.

All irreducible representations of a reductive algebraic group $G$ with spherical orbits have been classified by Arzhantsev ([A2]). In the last section we therefore use this result and our Theorem 1.3 in order to determine all the irreducible $C$-asystatic representations. Our result is the following

**Theorem 1.4.** Let $K$ be a connected compact Lie group and $\rho: K \to U(V)$ be a complex irreducible representation. Then $\rho$ is $C$-asystatic if and only if $\rho(K)$ is one of the following
(1) SU(n), A^2(SU(n)), S^2(SU(n)) (n \geq 3), SO(n), SU(n) \otimes SU(m), Spin(7), Spin(10), E_6;
(2) \rho(H) \cdot T^1, where \rho(H) appears in (1), or S^2(U(2)), T^1 \times Sp(n), T^1 \times G_2.

We note here that the actions appearing in the classification are either orbit equivalent to isotropy representations of Hermitian symmetric spaces or are obtained from these by deleting the one-dimensional torus.

The organization of the paper is as follows: definitions and basic useful facts on complex asystatic actions are given in Section 2, while in Section 3 we prove our main result in the algebraic setting. Applying our main result we give, in Section 4, the complete classification of irreducible complex asystatic representations.

2. Actions on Kähler manifolds and the \(\mathbb{C}\)-asystatic condition

Throughout the following \(K\) will denote a connected compact Lie group. A proper \(K\)-action on a manifold \(M\) is said to be asystatic if, given a principal point \(p \in M\) and its isotropy subgroup \(K_p := L\), the isotropy representation of \(L\) on \(T_p (K \cdot p)\) has no nontrivial fixed vector.

If \(M\) is a Kähler manifold and \(K\) acts on \(M\) by holomorphic isometries, the asystatic condition is rarely satisfied, at least if \(K\) acts in Hamiltonian fashion. We now give the proof of Prop. 1.1 (for some related see also [GP]).

**Proof of Proposition 1.1.** Let \(L\) be a principal isotropy subgroup and let \(\Sigma\) be the connected component of the fixed point set \(M^L\) passing through the principal point \(p\). Then \(T_p \Sigma \cap T_p (K \cdot p) = \{0\}\) and since \(\Sigma\) is complex, so is \(K \cdot p\). We now claim that every \(K\)-orbit is complex. Indeed, \(\Sigma\) intersects every \(K\)-orbit and for every \(q \in \Sigma\) we have that \(T_q \Sigma\) is a section for the linear \(K_q\)-action on the normal space \(\nu_q\) of \(K \cdot q\) (see e.g. [PT]); moreover if \(v \in \nu_q\), there exists \(g \in K_q\) so that \(gv \in T_q \Sigma\), hence \(Jv \in g^{-1} T_q \Sigma \subset \nu_q\) since \(\Sigma\) is complex. This means that \(\nu_q\), hence \(K \cdot q\) is complex.

Since the \(K\)-action is Hamiltonian, the \(Z\)-action on \(K \cdot p\) is also Hamiltonian, where \(Z\) denotes the connected component of the center of \(K\); this implies that \(Z\) acts trivially on \(K \cdot p\), hence on \(M\) by the effectiveness of the action, and therefore \(K\) is semisimple.

If \(\mu : M \to \mathfrak{k}^*\) is a moment map for the \(K\)-action, where \(\mathfrak{k}^*\) denotes the dual of the Lie algebra of \(K\), we first note that \(\mu\) is constant on \(\Sigma\), because \(\Sigma\) is complex and therefore \(\langle d\mu_q(X), Y \rangle = \omega(X, \hat{Y})|p = 0\).
for every \(X \in T_q \Sigma, Y \in \mathfrak{k}\) and \(q \in \Sigma\). We put \(\mu(\Sigma) = \{\beta\}\) for some \(\beta \in \mathfrak{k}^*\). If \(x \in \Sigma\), then \(K \cdot x\) is complex, hence \(\mu : K \cdot x \rightarrow K \cdot \beta\) is a covering and since \(K \cdot \beta\) is simply connected, we have \(K_x = K_\beta\) and every \(K\)-orbit in \(M\) is principal. This also implies that \(\Sigma\) intersects every \(K\)-orbit in a single point: indeed if \(x, gx \in \Sigma\) for some \(g \in K\), then \(g \in K_\beta = K_x\), hence \(x = gx\). The map \(\phi : K \cdot p \times \Sigma \rightarrow M\) given by \(\phi(kp, x) = kx\) where \(p\) is any fixed point in \(\Sigma\) is a well defined \(K\)-equivariant diffeomorphism, which can be proved to be an isometry using the same arguments as in the proof of the main result in [GP]. □

From now on we will always suppose that the \(K\)-action is also Hamiltonian.

Given a \(K\)-action on a manifold \(M\) with principal isotropy type \((L)\), recall that the core \(\overline{M} \subset M\) is defined as the closure of \(M^L \cap M_{reg}\) and it is known (see [GS] and [SS]) that all the components of \(\overline{M}\) are equi-dimensional. We now give the proof of Prop. 1.2.

**Proof of Proposition 1.2.**

(1) We denote by \(\mu : M \rightarrow \mathfrak{k}^* \cong \mathfrak{k}\) a moment map, where \(\mathfrak{k}\) is the Lie algebra of \(K\), identified with its dual \(\mathfrak{k}^*\) by means of a \(Ad(K)\)-invariant scalar product \(<\cdot, \cdot>\) on \(\mathfrak{k}\). We fix a point \(p \in M^L\) and, since principal isotropy subgroups are conjugate, we may suppose that \(p\) belongs also to the open dense subset \(M_\mu := \{q \in M \mid \dim K \cdot \mu(q) \geq \dim K \cdot \mu(w), \forall w \in M\}\). Then it is known (see [HW], p.267) that, given a \(K\)-regular point \(p \in M\) with \(K_p = L\), \(L\) is normal in \(K_{\mu(p)}\) with \(K_{\mu(p)}/L\) abelian; moreover the kernel \(\ker(d\mu_p) \cap T_p(K \cdot p)\) is a \(L\)-invariant subspace of \(T_p(K \cdot p)\), which can be identified with the tangent space \(T_p(K_{\mu(p)} \cdot p)\). This implies that

\[
\dim(\ker(d\mu_p) \cap T_p(K \cdot p)) = \text{rank}(K) - \text{rank}(L)
\]

and also that \(\ker(d\mu_p) \cap T_p(K \cdot p) \subset (T_pM)^L\). Since the normal space \(\nu_p = (T_p(K \cdot p))^\perp\) with respect to any \(K\)-invariant scalar product on \(M\) is obviously contained in \((T_pM)^L\) because \(p\) is regular, we obtain our first claim, recalling that \(\dim T_pM^L = \dim \overline{M}\).

(2) The fixed point set \((K \cdot p)^L\) clearly identifies with the orbit \(N_K(L) \cdot p\) and the equality holds if and only if \(\dim(T_p(N_K(L) \cdot p)) = \dim K_{\mu(p)} \cdot p\), i.e. if and only if \((N_K(L)/L)^\circ = K_{\mu(p)}/L\), which is abelian.

(3) Suppose that \(\text{rank}(K) = \text{rank}(L)\); this implies that \(L = K_{\mu(p)}\) because \(K_{\mu(p)}/L\) is abelian. Since \(L\) has maximal rank, \(V^L \cap T_p(K \cdot p) = \{0\}\); moreover since \(V^L\) contains the normal space of the orbit \(K \cdot p\) at \(p\), we see that \(V^L\) coincides with \((T_pK \cdot p)^\perp\) and therefore the orbit \(K \cdot p\) is complex. In particular, since the \(K\)-action is Hamiltonian, the group \(K\) is semisimple by the same arguments used in the proof of Prop. 1.1.

If \(M = V\) is a linear space, then \(K \cdot p\) is a compact connected complex
submanifold, hence a point and the $K$-action is trivial. 
Suppose now that $M$ is a locally irreducible Kähler manifold. We write $\mathfrak{k} = \mathfrak{l} + \mathfrak{m}$, an $\text{Ad}(L)$-invariant decomposition, where $\mathfrak{m}$ identifies with $T_p(K \cdot p)$; we decompose $\mathfrak{m} = \bigoplus_i \mathfrak{m}_i$ into irreducible $\text{Ad}(L)$-invariant subspaces, which are mutually inequivalent, because $K/L$ is a generalized flag manifold of a semisimple compact Lie group (see e.g. [Wo]). If $\xi \in V^L$, then the shape operator $A_\xi$, viewed as an element of $\text{End}(\mathfrak{m})$, commutes with $\text{Ad}(L)$ and therefore maps each $\mathfrak{m}_i$ into itself; being self-adjoint, we have that $A_\xi|_{\mathfrak{m}_i} = \lambda_i I$ for some $\lambda_i \in \mathbb{R}$ by Schur’s Lemma. Moreover, the complex structure $J$ of $V$ leaves each $\mathfrak{m}_i$ invariant and therefore commutes with $A_\xi$; on the other hand it is well-known (see [KN]) that $A_\xi$ anti-commutes with $J$, hence $A_\xi = 0$ for all $\xi \in V^L$, meaning that every principal orbit is totally geodesic. The open subset of principal points admits two complementary, totally geodesic foliations, given by the $K$-orbits and by the fixed point sets of principal isotropies; it is well-known that two complementary totally geodesic foliations are parallel, contradicting the local irreducibility of $M$. □

**Definition 2.1.** A proper $K$-Hamiltonian action on a Kähler manifold $M$ by holomorphic isometries is called complex asystatic (or $\mathbb{C}$-asystatic) if

$$\dim \overline{M} = c + \text{rank } K - \text{rank } L$$

or, equivalently, if $(N_K(L)/L)^o$ is abelian.

**Example.** On the quadric $Q_n = \text{SO}(n+2)/\text{SO}(2) \times \text{SO}(n)$, we consider the action of $K = \text{SO}(2) \times \text{SO}(n)$. A principal isotropy subgroup $L$ is given by $\mathbb{Z}_2 \times \text{SO}(n-2)$ and $(N_K(L)/L)^o = (\text{SO}(2))^2$, hence the action is $\mathbb{C}$-asystatic.

**Remark.** The example $K = \text{SO}(n)$ acting on $\mathbb{C}^n$ shows that $N_K(L)/L$ is in general not abelian, even if the action is $\mathbb{C}$-asystatic; indeed in this case $N_K(L)/L = T^1 \cdot \mathbb{Z}_2$.

The following proposition shows that the condition of $\mathbb{C}$-asystatic is preserved in the slice representation at complex orbits.

**Proposition 2.2.** If the $K$-action on $M$ is $\mathbb{C}$-asystatic then the slice representation at a complex $K$-orbit is $\mathbb{C}$-asystatic.

**Proof.** Given a complex orbit $K \cdot x$, we have $\text{rank}(K) = \text{rank}(K_x)$; let $L \subset K_x$ be a principal isotropy subgroup and denote by $\nu_x$ the normal space at $x$ of the orbit $K \cdot x$.

$$\dim(\nu_x(T_x(K \cdot x)))^{K_p} \leq \dim \overline{M} = c + \text{rank } K - \text{rank } K_p$$

this is equal to $c + \text{rank}(K_x) - \text{rank } (K_x)_p$ and we get the claim.
We now close this section with the following lemma that might be useful in order to classify \(\mathbb{C}\)-asystatic actions.

**Lemma 2.3.** Let \(K\) be a compact connected Lie group acting on a Kähler manifold \(M\) and \(K' \subset K\) a compact connected subgroup whose action on \(M\) is \(\mathbb{C}\)-asystatic, then the action of \(K\) is \(\mathbb{C}\)-asystatic, provided one of the following conditions holds:

1. \(K\) and \(K'\) have the same orbits;
2. \(K = K' \cdot T^1\), where \(T^1\) is a one dimensional torus.

**Proof.** If (1) holds, let \(N := K/L = K'/L'\) be a common principal orbit through a point \(p\) with \(L' \subset L\); if \(c\) denotes the common cohomogeneity then

\[
c + \text{rank}(K) - \text{rank}(L) \leq \dim(T_p M)^K = \dim(T_p M)^{K'} = c + \text{rank}(K') - \text{rank}(L').
\]

Our claim follows from the fact that \(\text{rank}(K) - \text{rank}(L) = \text{rank}(K') - \text{rank}(L')\) (see e.g. [O], p. 207).

If (2) holds, we consider the moment maps \(\mu, \mu'\) relative to \(K, K'\) respectively; the Lie algebra \(\mathfrak{k}\) of \(K\) splits as \(\mathfrak{k} = \mathfrak{k}' \oplus \mathbb{R} \cdot Z\), where \(Z\) is a generator of the Lie algebra of \(T^1\) and the moment map \(\mu' = \pi \circ \mu\), where \(\pi : \mathfrak{k} \rightarrow \mathfrak{k}'\) is the projection. Let \(p \in M\) be a principal point, then \(\mu(K \cdot p) = K/H\), where \(H \subset K\) is the centralizer of a torus, hence of the form \(H = H' \cdot T^1\) with \(H' \subset K'\) the centralizer of some torus in \(K'\); moreover \(\mu'(K' \cdot p) = \pi \circ \mu(K' \cdot p) = \pi(K/H) = K'/H'\), because \(K'\) acts transitively on \(K/H\).

Let \(L' \subset L\) be principal isotropy subgroups of \(K', K\) respectively; by (1), we can suppose that \(K' \cdot p \neq K \cdot p\), hence \(l = l'\), where \(l, l'\) are the Lie algebras of \(L, L'\) resp. We decompose \(\mathfrak{k} = l + a + n\), where \(l + a = h\), \(Z \in a\) and \(\{l, a\} = 0\) and \(n = (l + a)^\perp\) w.r.t. an \(\text{Ad}(K)\)-invariant scalar product \(\langle \cdot , \cdot \rangle\) with \(\langle \mathfrak{k}', Z \rangle = 0\). Now, \(n^{\text{Ad}(L)} \subseteq n^{\text{Ad}(L')} = \{0\}\) because the \(K'\)-action is \(\mathbb{C}\)-asystatic; it then follows that

\[
\dim(T_p K \cdot p)^L = \dim a = \dim(\mathfrak{h}'/l) + 1 = \text{rank}(K') - \text{rank}(L) + 1 = \text{rank}(K) - \text{rank}(L)
\]

hence our claim.

**Remark.** We remark that if \(K' \subset K\) have the same orbits and the \(K\)-action is \(\mathbb{C}\)-asystatic, then the \(K'\)-action is not necessarily \(\mathbb{C}\)-asystatic, as the example of the linear action of \(K' = \text{Sp}(n) \subset K = \text{U}(2n)\) shows.

### 3. \(\mathbb{C}\)-Asystatic Actions and Spherical Orbits

In this section we prove our main Theorem 1.3. We recall here that a complex homogeneous space \(X = G/H\), where \(G\) is a complex algebraic
group, is called \textit{spherical} if a Borel subgroup of $G$ has an open orbit in $X$. We recall also that if $(M,g)$ is a Kähler manifold with Kähler form $\omega$ and $K$ is a compact connected Lie subgroup of its full isometry group, then the $K$-action is called \textit{coisotropic} if the principal $K$-orbits are coisotropic with respect to $\omega$.

\textbf{Proof of Theorem 1.3.} Let $K$ be a maximal compact Lie subgroup of $G$ whose action on $M$ is Hamiltonian. Let $\mu : M \to \mathfrak{k}^*$ be a moment map for the $K$-action and let $M_{\mu} := \{v \in M; \dim K(\mu(v)) \geq \dim K(\mu(w)) \forall w \in M\}$. Since $M_{\mu}$ is an open set, the intersection $M_{\mu} \cap M_{\text{reg}}$ is nonempty and open; we fix $p$ in $M_{\mu} \cap M_{\text{reg}}$ and we denote by $O$ the orbit $G \cdot p$ under the complexified group $G$. We now prove that the orbit $K \cdot p$ is coisotropic in $O$, hence every principal orbit of $K$ in $O$ will be coisotropic by Theorem 3 in [HW]. By a result due to Wurzbacher (Theorem 2.1, p. 542 [Wu]) this means that the $G$-orbit $O$ is spherical.

In order to prove our claim, we denote by $a_p$ the tangent space $T_p(K_{\mu(p)} \cdot p)$ and by $n_p$ the orthogonal complement $n_p = a_p^\perp \cap T_p(K \cdot p)$; by the assumption of $\mathbb{C}$-asystatic, the fixed point set $(T_pM)^L$ is given by $(T_pM)^L = \nu_p \oplus a_p$, where $\nu_p$ denotes the normal space $(T_p(K \cdot p))^\perp$; since $T_pM^L$ is a complex subspace, we have that $n_p$ is complex too. This means that $T_pO$ is given by $T_p(K \cdot p) \oplus J(a_p)$, where $J$ denotes the complex structure of $M$, so that the normal space of $T_p(K \cdot p)$ inside $T_pO$ is $Ja_p$; hence the orbit $K \cdot p$ is coisotropic inside $O$.

This argument shows that for any point $x$ in the open set $\in M_{\mu} \cap M_{\text{reg}}$ the orbit $G \cdot x$ is spherical; our claim now follows from [A1], where it is proved that every $G$-orbit is spherical if this happens in an open dense subset. \hfill $\square$

In the following we will denote by $G/H$ a $G$-spherical orbit and by $N_G(H)$ the normalizer of $H$ in $G$, using [A1] (Proposition 10, p. 19), under the same hypotheses of Theorem 1.3 we get

\textbf{Corollary 3.1.} If the action of a compact Lie subgroup of $G$ on $M$ is $\mathbb{C}$-asystatic then $N_G(H)/H$ is abelian.

\section{The Linear Case}

All representations of a reductive algebraic group $G$ with spherical orbits have been classified in [A2]. In order to have the classification of $\mathbb{C}$-asystatic representations, we have to go through the list in [A2] and select which are $\mathbb{C}$-asystatic. Since the computation of a principal isotropy is not easily accessible in the literature, we provide a table of all irreducible representations with spherical orbits, where we also indicate
a principal isotropy $L$. We now give the proof of the classification Theorem.

Proof of Theorem 1.4. By Theorem 1.3, we know that the $G$-orbits are spherical. By Arzhantsev’s result [A2], an irreducible representation of a reductive algebraic group has spherical orbits if and only if it appears in Table I (p. 291 in [A2]) or is obtained from any of these by a torus extension; here in Table 1, we indicate a compact real form $K$ with the corresponding representation, a principal isotropy subgroup $L$, the difference $f = \text{rank}(K) - \text{rank}(L)$ and the dimension $d$ of the fixed point set for the isotropy action of $L$ on the tangent space of the orbit $K/L$. Clearly, the representation is $\mathbb{C}$-asystatic if and only if $f = d$. According to Lemma 2.3, we then need only consider the cases which are not $\mathbb{C}$-asystatic and compute the same invariants $f$ and $d$ for $K \cdot T^1$; this is done in Table 2 and the full classification is then obtained by taking all representations which appear in Table 1 or 2 and are $\mathbb{C}$-asystatic. \hfill \Box

| n. | $K$ | $\rho$ | $L$ | $f$ | $d$ |
|----|-----|-------|-----|-----|-----|
| 1  | $\text{SU}(n)$ | $\rho_1$ | $\text{SU}(n-1)$ | 1   | 1   |
| 2  | $\text{SU}(2n)$ | $\Lambda^2 \rho_1$ | $\text{SU}(2)^n$ | $n-1$ | $n-1$ |
| 3  | $\text{SU}(2n+1)$ | $\Lambda^2 \rho_1$ | $\text{SU}(2)^n$ | $n$  | $n$  |
| 4  | $\text{SU}(n), n \geq 3$ | $S^2 \rho_1$ | $(\mathbb{Z}_2)^{n-1}$ | $n-1$ | $n-1$ |
| 5  | $\text{SU}(2)$ | $S^2 \rho_1$ | $\mathbb{Z}_2$ | 1   | 3   |
| 6  | $\text{SO}(n)$ | $\rho_1$ | $\text{SO}(n-2)$ | 1   | 1   |
| 7  | $\text{Sp}(n)$ | $\rho_1$ | $\text{Sp}(n-1)$ | 1   | 3   |
| 8  | $\text{SU}(n) \times \text{SU}(m), n < m$ | $\rho_1 \otimes \rho_1$ | $\text{U}(1)^{n-1} \times \text{SU}(m-n)$ | $n$ | $n$ |
| 9  | $\text{SU}(n) \times \text{SU}(n)$ | $\rho_1 \otimes \rho_1$ | $\text{U}(1)^{n-1}$ | $n-1$ | $n-1$ |
| 10 | $\text{SU}(2) \times \text{Sp}(n), n \geq 2$ | $\rho_1 \otimes \rho_1$ | $\text{U}(1) \times \text{Sp}(n-2)$ | 2   | 4   |
| 11 | $\text{SU}(3) \times \text{Sp}(n), n \geq 3$ | $\rho_1 \otimes \rho_1$ | $\text{Sp}(n-3)$ | 4   | 29  |
| 12 | $\text{SU}(3) \times \text{Sp}(2)$ | $\rho_1 \otimes \rho_1$ | $\{e\}$ | 4   | 18  |
| 13 | $\text{SU}(4) \times \text{Sp}(2)$ | $\rho_1 \otimes \rho_1$ | $\mathbb{Z}_2$ | 5   | 25  |
| 14 | $\text{SU}(n) \times \text{Sp}(2), n \geq 5$ | $\rho_1 \otimes \rho_1$ | $\text{SU}(n-4) \times \mathbb{Z}_2$ | 6   | 26  |
| 15 | $\text{Spin}(7)$ | $\rho_3$ | $\text{SU}(3)$ | 1   | 1   |
| 16 | $\text{Spin}(9)$ | $\rho_4$ | $\text{SU}(3)$ | 2   | 4   |
| 17 | $\text{Spin}(10)$ | $\rho_4$ | $\text{SU}(4)$ | 2   | 2   |
| 18 | $\text{G}_2$ | $\rho_1$ | $\text{SU}(2)$ | 1   | 3   |
| 19 | $\text{E}_6$ | $\rho_1$ | $\text{Spin}(8)$ | 2   | 2   |

Table 1. Semisimple $K$ such that $G$ has spherical orbits.
| n. | K | $\rho$ | $L$ | $f$ | $d$ |
|---|---|---|---|---|---|
| 1 | $U(2)$ | $S^2 \rho_1$ | $(\mathbb{Z}_2)^2$ | 2 | 2 |
| 2 | $\text{Sp}(n) \times T^1$ | $\rho_1 \otimes \epsilon$ | $\text{Sp}(n-1) \times T^1$ | 1 | 1 |
| 3 | $U(2) \times \text{Sp}(n)$ | $\rho_1 \otimes \rho_1$ | $T^1 \times \text{Sp}(n-2)$ | 3 | 5 |
| 4 | $U(3) \times \text{Sp}(n), n \geq 3$ | $\rho_1 \otimes \rho_1 \otimes \rho_1$ | $\mathbb{Z}_2 \times \text{Sp}(n-3)$ | 6 | 30 |
| 5 | $U(3) \times \text{Sp}(2)$ | $\rho_1 \otimes \rho_1$ | $\mathbb{Z}_2$ | 5 | 19 |
| 6 | $U(n) \times \text{Sp}(2), n \geq 4$ | $\rho_1 \otimes \rho_1$ | $U(n-4) \times \mathbb{Z}_2$ | 6 | 26 |
| 7 | $\text{Spin}(9) \times T^1$ | $\rho_4 \otimes \epsilon$ | $\text{SU}(3) \times \mathbb{Z}_2$ | 3 | 5 |
| 8 | $G_2 \times T^1$ | $\rho_1 \otimes \epsilon$ | $\text{SU}(2) \cdot \mathbb{Z}_2$ | 2 | 2 |

Table 2. Non-semisimple $K$ such that $G_{ss}$ acts with spherical orbits, but $K_{ss}$ acts non-$C$-asystatically

**Notation:** In the above Tables $\rho_i$ denote the standard representations.

**References**

[A1] Arzhantsev I.V.: *On SL$_2$-actions of complexity one*, Izvestiya Mathematics **61** (1997) 685–698

[A2] Arzhantsev I.V.: *A classification of reductive linear groups with spherical orbits*, J. of Lie Theory **12** (2002) 289–299

[AA] Alekseevsky D., Alekseevsky A.: *Asystatic G-manifolds* in Proc. Workshop Differential Geometry and Topology, Alghero, Italy, World Scientific (1993)

[Ak] Akhiezer D.: *Actions with a finite number of orbits*, Funct. Anal. Appl. **19** (1985) 1–4

[BLV] Brion M., Luna D., Vust Th.: *Espaces homogènes sphériques*, Invent. Math. **84** (1986) 617–632

[GP] Gori A., Podestà F.: *A note on the moment map on Compact Kähler manifolds*, to appear in Ann. Glob. Anal. Geom. (2004)

[GS] Grove K., Searle C.: *Global G-manifold Reduction and Resolutions*, Ann. Glob. Anal. Geom. **18** (2000) 437–446

[HW] Huckleberry A., Wurzbacher T.: *Multiplicity-free complex manifolds*, Math. Ann. **286** (1999) 261–280

[LE] Lie S., Engel F.: *Theorie der Transformationsgruppen* Vol. I., Teubner, Leipzig (1888)

[KN] Kobayashi S., Nomizu K.: *Foundations of differential geometry. Vol. II.* John Wiley & Sons, Inc., New York (1996)

[O] Onishchik A.L.: *Topology of Transitive Transformation Groups* Johann Ambrosius Barth (1994)

[PT] Palais R., Terng, C.L.: *General theory of canonical forms*, Trans. A.M.S. **300** (1987) 771–789

[SS] Skjelbred T., Straume E.: *A note on the reduction principle for compact transformation groups*, preprint (1995)
[Vi] Vinberg E.B.: *Commutative homogeneous spaces and co-isotropic symplectic actions*, Russ. math. Surv. 56 (2001) 1–60
[Wo] Wolf J.A.: *Spaces of constant curvature*, 5th ed, Publish or Perish, Houston (1984)
[Wu] Wurzbacher T.: *On a Conjecture of Guillemin and Sternberg in geometric quantization of multiplicity-free symplectic spaces*, Jour. Geom. Phis. 4 (1990) 537–552

Dipartimento di Matematica e Appl. per l’Architettura, Piazza Ghiberti 27, 50100 Firenze, Italy
E-mail address: gori@math.unifi.it

Dipartimento di Matematica e Appl. per l’Architettura, Piazza Ghiberti 27, 50100 Firenze, Italy
E-mail address: podesta@math.unifi.it