COMPLEXIFICATION AND HYPERCOMPLEXIFICATION OF
MANIFOLDS WITH A LINEAR CONNECTION

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Abstract. We give a simple interpretation of the adapted complex structure
of Lempert-Szöke and Guillemin-Stenzel: it is given by a polar decomposition
of the complexified manifold. We then give a twistorial construction of an
SO(3)-invariant hypercomplex structure on a neighbourhood of $X$ in $TTX$,
where $X$ is a real-analytic manifold equipped with a linear connection. We
show that the Nahm equations arise naturally in this context: for a connection
with zero curvature and arbitrary torsion, the real sections of the twistor space
can be obtained by solving Nahm’s equations in the Lie algebra of certain
vector fields. Finally, we show that, if we start with a metric connection, then
our construction yields an $SO(3)$-invariant hyperkähler metric.

Let $X$ be a manifold equipped with a linear connection $\nabla$. Then the tangent
bundle $TX$ of $X$ has a canonical foliation (nonsingular on $TX \setminus X$) by tangent
bundles to geodesics, i.e. by surfaces $T_{\gamma}\gamma$, where $\gamma$ is a geodesic.

A complex structure on $TX$ (or some neighbourhood of $X$) is called adapted (to
$\nabla$) if the leaves of the canonical foliation are complex (immersed) submanifolds of
$TX$.

For Riemannian connections the adapted complex structures were constructed
and studied by R. Szöke and L. Lempert [7, 11, 8]. An equivalent definition was
given by V. Guillemin and M. Stenzel [3]. The results of Lempert and Szöke can
be formulated as follows:

Theorem 0.1. [7, 8, 11, 3] Let $(X, g)$ be a Riemannian manifold. There exists a
(unique) adapted complex structure for the Levi-Civita connection on some neigh-
bourhood of $X$ in $TX$ if and only if $(X, g)$ is a real-analytic Riemannian manifold
(i.e. both $X$ and $g$ are real-analytic).

In this paper we shall give a simple construction of an adapted complex structure
which works for any real-analytic manifold $X$ with real-analytic linear connection
$\nabla$. It is sort of a polar decomposition of the complexified manifold and generalises
the basic example of the adapted complex structure on $TG$, where $G$ is a compact
Lie group equipped with the bi-invariant metric. Actually, this construction is
implicitly used by R. Szöke [11] in the proof that his (different) construction yields
the adapted complex structure.

In the Riemannian case, $(X, g)$, once we have the adapted complex structure $J$
on $TX$, we obtain a Kähler metric on $TX$ whose Kähler form is the canonical 2-form
on the tangent bundle of a Riemannian manifold. This metric has in particular the
property of being flat on leaves of the canonical foliation.

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We would like to have an analogous statement in the non-Riemannian case, i.e. obtain a Kählerian (an almost complex connection whose curvature is of type $(1, 1)$) connection $\nabla$ on $TX$, extending $\nabla$ and such that the leaves of the canonical foliation are flat and totally geodesic. This turns out to be a byproduct of our next construction: i.e. hypercomplexification of $X$. Again, the motivating example is a compact group with its bi-invariant metric: Kronheimer [6] showed that there exists a hyperkähler metric on $T^{1,0}G^C$. We would like to construct a hypercomplex structure on a neighbourhood of $X$ in $T^{1,0}X$ where $X$ is any real-analytic manifold with a real-analytic linear connection $\nabla$, and $TX$ is equipped with the adapted complex structure. We do this in sections 2 and 3. The Obata connection of this hypercomplex structure restricted to $TX$ is the desired connection $\nabla$ (more precisely, we have to change the torsion if $\nabla$ was not torsion-free).

If our original connection was (pseudo)-Riemannian, then we obtain a (pseudo)-hyperkähler metric a neighbourhood of $X$ in $T^{1,0}X$. The restriction of this metric to $TX$ is the Kähler metric of Lempert and Szöke.

We should remark here that D. Kaledin [4] and B. Feix [1] showed that, if $M$ is a complex manifold with a real-analytic Kählerian connection $\nabla'$, then there exists a canonical hypercomplex structure on a neighbourhood of $M$ in $T^{0,1}M$. By the uniqueness result of Kaledin [4], our construction gives the same hypercomplex structure as those of Feix or Kaledin when applied to $M = TX$ with the adapted complex structure and $\nabla' = \tilde{\nabla}$.

The hypercomplex and hyperkähler structures obtained here have a bigger symmetry than the more general ones of Feix and Kaledin. Namely, they admit an $SO(3)$ action rotating the complex structures.

If the construction of the adapted complex structure should be viewed as a polar decomposition of $X^C$, then the construction of the adapted hypercomplex structure should be viewed as solving certain Riemann-Hilbert problem. We make this point of view precise in the case of connections with zero curvature and arbitrary torsion. We show that the hypercomplex structure on $TTX$ can be in this case obtained by solving a factorization problem in the loop group of the group of holomorphic diffeomorphisms of $TX$. This yields a construction of this hypercomplex structure through solving Nahm’s equations in the Lie algebra of vector fields on $TX$ invariant under the canonical involution $v \mapsto -v$ on $TX$.

1. Adapted complex structures

Let $X$ be a real-analytic manifold equipped with a real-analytic linear connection $\nabla$. We wish to construct an adapted complex structure on a neighbourhood of $X$ in $TX$. Here “adapted” means that the immersed surfaces $T_\gamma \gamma$, where $\gamma$ is the a geodesic are complex submanifolds.

Our construction is sort of a polar decomposition of the complexification of $X$.

Let $X^C$ be a complexification of $X$, i.e. a complex manifold equipped with an antiholomorphic involution $\tau$ such that $(X^C)^\tau \simeq X$. Since $X$ is real-analytic, we can construct such a complexification by a holomorphic extension of real-analytic transition functions. As the connection $\nabla$ is real-analytic and $X$ is totally real in $X^C$, we can extend $\nabla$ to a holomorphic linear connection $\nabla^C$ on $T^{1,0}X^C$ (i.e. a splitting of $T^{1,0}(T^{1,0}X^C)$ into vertical and horizontal holomorphic bundles). The exponential map for this connection, which we denote by $\exp^C$, is a holomorphic extension of the exponential map for $\nabla$. 
At points \( m \) of \( X \subset X^C \) we have the induced antiholomorphic linear map \( \tau_* : T_mX^C \to T_mX^C \). We denote by the same symbol the corresponding linear map on \( T_{m}^{1,0}X^C \), i.e.

\[
\tau_* (v - iIv) = \tau_* (v) - iI\tau_* (v),
\]

(1.1)

where \( I \) is the complex structure of \( X^C \) and \( v \in TX^C \) (real tangent vector). Let \( V^+_m \) and \( V^-_m \) denote the \( \pm 1 \) eigenspaces of \( \tau_* \). They are interchanged by the multiplication by \( i \). Thus we have two bundles, \( V^+_m \) and \( V^-_m \), over \( X \) which are subbundles of \( T_{m}^{1,0}X^C \). Moreover the restriction of \( \exp^C \) to \( V^- \) is the exponential map on \( TX \). On the other hand, as the differential of an exponential map is identity at the zero section of a manifold, \( \exp^C \) restricted to \( V^- \) is a diffeomorphism (near the zero section of \( V^- \)) between \( V^- \) and \( X^C \). We can define a complex structure on \( TX \) by pulling back the complex structure of \( X^C \) via this diffeomorphism. We have

**Proposition 1.1.** The complex structure on \( TX \simeq V^- \) obtained via the (local) diffeomorphism

\[
\exp^C_{|_{V^-}} : V^- \to X^C
\]

(1.2)

is a complex structure adapted to the connection \( \nabla \).

**Proof.** Since \( \exp^C \) is a holomorphic extension of \( \exp \), it satisfies

\[
\exp^C ((z_1 + z_2)v) = \exp^C \left( z_1 \left( \frac{\partial}{\partial z} \exp^C (zv) \right) \right) |_{z = z_2}
\]

(1.3)

for any complex numbers \( z_1 \) and \( z_2 \) and for any \( (1, 0) \)-tangent vector \( v \) (as long as the expressions are defined). Let us identify \( V^+ \) with \( TX \) and let \( \exp(tv) \) be the corresponding geodesic. The image of the tangent bundle to this geodesic under the map (1.2) is the set

\[
\exp^C \left( iq \frac{d\exp(tv)}{dt} \right) |_{t = p}
\]

where \( p, q \in \mathbb{R} \). As \( \exp \) and \( \exp^C \) coincide on \( V^+ \) and using (1.3), this is the same as \( \exp^C ((p + iq)v) \). Hence the immersion sending \( \mathbb{C} \) to the tangent bundle of a geodesic is holomorphic.

We remark that this construction provides a natural interpretation of the result in [8] that global existence of the complex structure adapted to a metric connection \( \nabla \) implies that the curvature of \( \nabla \) is nonnegative. Indeed, global existence means that \( \exp^C \) doesn’t have conjugate points along geodesics starting at \( X \) and going in the imaginary directions. Thus, in particular, we expect that the Jacobi operator \( R(\cdot, iv)iv \) is nonpositive at points of \( X \).

2. ADAPTED HYPERCOMPLEX STRUCTURE

We wish to construct a canonical hypercomplex structure on a neighbourhood of \( X \) in \( TTX \). We shall do this by constructing a twistor space, i.e. a complex manifold \( Z \) fibering over \( \mathbb{C}P^1 \) equipped with a real structure, i.e. an antiholomorphic involution \( \sigma \) covering the antipodal map on \( \mathbb{C}P^1 \), and a family of \( \sigma \)-invariant sections whose normal bundle is a direct sum of \( \mathcal{O}(1) \)'s.
Let $M$ be the complex manifold defined in the previous section, i.e. $M$ is a neighbourhood of $X$ in $TX$ equipped with an adapted complex structure. $M$ has an antiholomorphic involution $\sigma = \tau^*$ such that $M^\sigma = X$ and a holomorphic connection $\nabla^C$ which is an extension of the original linear connection $\nabla$ on $X$. With these data we can construct a twistor space $Z$. In this section we shall assume only that we are given a complex connection on $M$; the real structure will be introduced in the next section.

$Z$ will be defined by gluing two open subsets of $\mathbb{C} \times T^{1,0}M$. For any number $r > 0$, let $D_r = \{ z \in \mathbb{C}; |z| \leq r \}$ and $U_r$ be an open subset of $T^{1,0}M$ defined as the set of these $v$ for which $\exp^C(zv)$ is defined for all $z \in D_r$. Here $\exp^C$ denotes the (holomorphic) exponential map for the connection $\nabla^C$.

$Z$ is obtained by taking two copies of $U_2 \times D_2$, parameterised by $\{ (\beta, y, \zeta); \beta \in T^1_y M, \zeta \in D_2 \}$ and $\{ (\tilde{\beta}, \tilde{y}, \tilde{\zeta}); \tilde{\beta} \in T^1_{\tilde{y}} M, \tilde{\zeta} \in D_2 \}$, and identifying over $1/2 < |\zeta| < 2$ via

$$\tilde{\zeta} = \zeta^{-1}, \quad \tilde{y} = \exp^C(-\beta/\zeta), \quad \tilde{\beta} = \zeta^{-2} \frac{\partial}{\partial z} \exp^C(z\beta) \big|_{z=\zeta^{-1}}.$$ (2.1)

We observe that the points of $M$ give rise to sections of $Z \to \mathbb{C}P^1$: $m \mapsto (0, m, \zeta), m \in M$. We have

**Proposition 2.1.** The normal bundle of sections corresponding to points $m$ of $M$ splits as the direct sum of line bundles of degree 1.

**Proof.** Choose local holomorphic coordinates $z_1, \ldots, z_n$ on $M$ and let $z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}$ be the induced coordinates on $T^{1,0}M$, i.e. $z_{n+i}(Y) = dz_i(Y)$. If $t \mapsto Y(t)$ is an integral (complex) curve of the geodesic flow, then in the above local coordinates

$$\dot{z}_k(t) = z_{n+k}(t),$$

$$\dot{z}_{n+k}(t) = -\sum_{i,j} \Gamma_{ij}^k (\pi \circ Y(t)) z_{n+i}(t) z_{n+j}(t).$$

The transition functions (2.1) identify $\dot{z}_k$ with $z_k(-1/\zeta)$ and $\dot{z}_{n+k}$ with $\zeta^{-2} z_{n+k}(-1/\zeta)$. It follows that at a section corresponding to a point of $m$, i.e. one where $z_{n+k}(t) \equiv 0$, $k = 1, \ldots, n$, the normal bundle has the transition function

$$\begin{pmatrix} 1 & -\zeta^{-1} \\ 0 & \zeta^{-2} \end{pmatrix},$$ (2.2)

where the blocks have size $n \times n$ (and correspond to the choice of coordinates $z_1, \ldots, z_n$ and $z_{n+1}, \ldots, z_{2n}$). This means that the normal bundle is isomorphic to $n$ copies of a rank 2 bundle on $\mathbb{C}P^1$ whose transition function is (2.2) with $(1 \times 1)$-blocks. Therefore this rank 2 bundle $E$ is a nontrivial extension

$$0 \to \mathcal{O} \to E \to \mathcal{O}(2) \to 0$$

and hence $E \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$. \qed

In the above situation, i.e. when given a complex manifold $Z^{2n+1}$ fibering over $\mathbb{C}P^1$, it is well known that the parameter space $W$ of sections of $Z$ with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$ is a complex $4n$-dimensional manifold equipped with a canonical torsion-free holomorphic connection known as the Obata connection (see, e.g., [2]). Since, by the previous lemma, $M$ is a complex submanifold of $W$, we can restrict the Obata connection to $M$. The following comes as no surprise:
Proposition 2.2. The restriction of the Obata connection to $M$ is the unique torsion-free connection with the same geodesics as the original connection $\nabla^C$.

Proof. It is enough to prove that every $\nabla^C$-geodesic in $M$ is also a geodesic in $W$ (equipped with the Obata connection). Let $\gamma \subset M$ be a smooth geodesic disc (i.e. the image of a disc in $\mathbb{C}$ under a $\nabla^C$-geodesic) containing a point $m \in M$. We can glue together two copies of $T^{1,0}\gamma \times D_\gamma$ using transition functions (2.1) and so obtain a twistor subspace $Z_\gamma$ of the twistor space $Z$. Any section of $Z_\gamma$ is a section of $Z$. Moreover such a section has the correct normal bundle in $Z$, at least in a neighbourhood of a section $m(\zeta)$ corresponding to a point in $M$, since $W$ is a complete family of sections at $m(\zeta)$. This means that the space of sections of $Z_\gamma$ is a (complex) hypercomplex submanifold of $W$ and hence totally geodesic for the Obata connection. However as $\gamma$ is a flat manifold (being a geodesic), the hypercomplex structure of $T^{1,0}\gamma$ obtained from (2.1) is simply that of $\gamma \times \mathbb{C}$ and hence $\gamma$ is a geodesic in $W$. \hfill \square

Remark 2.3. The proof shows that $M$ is totally geodesic in $W$. This follows also from the existence of a $\mathbb{C}^*$ action on $Z$: $\lambda \cdot (\beta, y, \zeta) = (\lambda \beta, y, \lambda \zeta)$ (more precisely the action may exist only for $\log |\lambda| \leq c$, for some $c$), which induces an action on the space of sections, i.e. $W$, whose fixed point set is $M$.

Remark 2.4. The above proof also shows that the twistor space $Z$ has a property analogous to the one of adapted complex structure. Namely it is foliated by 3-dimensional complex manifolds $Z_\gamma$, fibered over $\mathbb{C}P^1$ and corresponding to geodesics of $M$. These $Z_\gamma$ are "trivial" 3-dimensional twistor spaces, i.e. isomorphic to the twistor space of an open subset of $\mathbb{C}^2$ with its canonical hypercomplex structure. They give rise to flat 4-dimensional submanifolds of the space of sections $W$, but $W$ is not foliated by these.

It is the last property that gives rise to a $\text{PSL}(2, \mathbb{C})$-action on $W$.

Proposition 2.5. The twistor space $Z$ admits a (local) action of $\text{PSL}(2, \mathbb{C})$ covering the standard action on $\mathbb{C}P^1$. For each point $\zeta \in \mathbb{C}P^1$, the subgroup $\mathbb{C}^*$ fixing $\zeta$ acts on the fiber $Z_\zeta \simeq T^{1,0}M$ by multiplications on the fibers of $T^{1,0}M$.

Proof. On the patch $\zeta \neq \infty$, the group of linear fractional transformations acts, where defined, by

$$ (m, \beta, \zeta) \mapsto \left( \exp^C_m \left( -\frac{c}{c\zeta + d} \beta \right), \frac{\beta}{(c\zeta + d)^2}, \frac{a\zeta + b}{c\zeta + d} \right). \quad (2.3) $$

Here $1 = ad - bc$ and this is helpful when checking that we do have a $\text{PSL}(2, \mathbb{C})$ action on the first term. This action arises from expressing the canonical (diagonal) action of $\text{PSL}(2, \mathbb{C})$ on $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (the twistor space of a $Z_\gamma$ in Remark 2.4) as the lift of the canonical action on $\mathcal{O}(2)$ in the extension

$$ 0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(2) \to 0 $$

given by (2.2). \hfill \square

Corollary 2.6. There exists a neighbourhood of $X$ in $W$ which admits an action of $\text{SO}(3, \mathbb{R})$ induced from the local action of $\text{PSL}(2, \mathbb{C})$ on $Z$.

Proof. From the previous proposition there exists a Lie subalgebra of vector fields on $W$ isomorphic to $\text{so}(3, \mathbb{R})$. These vector fields vanish at points of $X$. By the
properties of ODE’s we can find a neighbourhood of $X$ such that the integral curves of vector fields corresponding to vectors in the unit sphere of $\mathfrak{so}(3, \mathbb{R})$ exist for time $\pi$. We obtain action of $SU(2)$ with the center acting trivially because of (2.3). □

3. REAL STRUCTURES

We shall now assume that the complex manifold $M$ admits an anti-holomorphic involution $\tau$ whose fixed points set is a manifold $X$ with $\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} M$. We can extend $\tau_*$ to $T^{1,0}M$ by the formula (1.1). Then $\tau_*$ is an antiholomorphic involution on the complex manifold $T^{1,0}M$. We assume that the complex connection $\nabla^C$ is compatible with $\tau_*$, which in particular means that the exponential mapping $\exp^C$ commutes with $\tau_*$. These conditions imply that $M$ is a neighbourhood of $X$ in $TX$ equipped with the adapted complex structure for the restriction of $\nabla^C$ to $X$. In particular, the geodesics of $\nabla^C$ passing through points of $X$ are leaves of the canonical foliation of $TX$.

Using $\tau_*$ we define a real structure on $Z$, i.e. an antiholomorphic involution $\sigma$ covering the antipodal map on $\mathbb{C}P^1$. With the notation of (2.1), we put

$$\sigma(\beta, y, \zeta) = \left(-\tau_*(\bar{\beta}), \tau_*(\bar{y}), -1/\bar{\zeta}\right). \quad (3.1)$$

Proposition 2.1 implies that points of $X$ give rise to $\sigma$-invariant sections of $Z$ with the normal bundle splitting into line bundles with first Chern class 1. Thus we obtain a hypercomplex structure on a neighbourhood $W$ of $X$ in $T^{1,0}M$. We can summarize the properties of this hypercomplex structure as follows:

**Proposition 3.1.**  
(1) There exists an $SO(3, \mathbb{R})$-action on $W$ rotating the complex structures. The fixed point set of this action is $X$.

(2) With respect to any complex structure, $W$ is biholomorphic to a neighbourhood of $X$ in $T^{1,0}M$, where $M$ is a neighbourhood of $X$ in $TX$ equipped with the adapted complex structure.

(3) For any leaf $L$ of the canonical foliation of $M$, its holomorphic tangent bundle $T^{1,0}L$ is a hypercomplex submanifold of $W$, and the hypercomplex structure of $T^{1,0}L$ is the one of an open subset of $\mathbb{C}^2$.

(4) $X$ is totally geodesic in $W$ and the Obata connection restricted to $X$ is the unique torsion-free connection with the same geodesics as the original connection $\nabla$ on $X$.

We are not able to describe the hypercomplex structure on $T^{1,0}M$ directly in terms of the geometry of $X$ (apart from the case considered in the next section). Such a description involves finding the full $4n$-dimensional family of real sections. We can, however, describe the $2n$-dimensional family corresponding to points of $M$ (where $M$ is a complex submanifold of the fiber of $Z$ over $\zeta = 0$). Indeed, as $M$ is $TX$ with the adapted complex structure, the description of section 1 shows that any point of $m$ can be written uniquely as $\exp^C_x(iv)$, where $x \in X$ and $v \in T_xX$. Define tangent vectors $V$ at $m$ and $\hat{V}$ at $\tau(m)$ by $V = \frac{d}{dt}\exp^C_x(itv)|_{t=1}$, $\hat{V} = \frac{d}{dt}\exp^C_x(itv)|_{t=-1}$. Then the following is a real section of $Z$:

$$\zeta \mapsto (2\zeta V, m, \zeta), \quad \bar{\zeta} \mapsto (-2\bar{\zeta} \hat{V}, \tau(m), \bar{\zeta)). \quad (3.2)$$

We now restrict the Obata connection to $M$, i.e. to the sections of the above form. We have
Proposition 3.2. The restriction of the Obata connection to \( M \) is a Kählerian connection (i.e. an almost complex connection whose curvature is a \((1,1)\)-form) such that the leaves of the canonical foliation are flat and totally geodesic.

Proof. The first part is obvious since the Obata connection is Kählerian and the second part follows from Proposition 3.1(3).

Remark 3.3. It follows now, from the uniqueness result of Kaledin, that the hypercomplex structure on \( T^{1,0}M \) arises also via the construction of Kaledin [4] or that of Feix [2], if we start with the above connection on \( M \).

Remark 3.4. Since we know the sections corresponding to points of \( M \), we can, in principle, compute the restriction of the Obata connection to \( M \) by the method of Merkulov [9]. This involves computing, for a section of the form (3.2), its second formal neighbourhood \( N^{(2)} \) in \( Z \), and of the (unique) splitting of the sequence \( 0 \to N \to N^{(2)} \to S^2(N) \to 0 \), where \( N \) is the normal bundle of the section.

4. Flat connections and Nahm’s equations

We shall describe explicitly sections of the twistor space and hence the hypercomplex structure in the case of a linear connection with zero curvature (and non-trivial torsion). Again, let \( M \) denote a neighbourhood of \( X \) in \( TX \) equipped with the adapted complex structure and the complexified connection \( \nabla^C \). Let \( \exp^C \) be the exponential map for this connection. For a tangent vector \( v \in T^{1,0}_M \) we denote by \( \hat{v} \) the holomorphic vector field in a neighbourhood of \( m \) obtained by translating \( v \) parallelwise along geodesics of \( \nabla^C \) (as the curvature is zero, this is the same as parallel translation along arbitrary curves). For a holomorphic vector field \( V \) on \( M \), we denote by \( e^V \) the (local) diffeomorphism of \( M \) defined by \( e^V(m) = m(1) \), where \( m(t) \) is the curve with \( m(0) = m \) and \( m(t) = V_{|m(t)} \). Vanishing of the curvature of \( \nabla^C \) implies the following key fact:

\[
\exp^C_p(v_p) = e^\hat{v}(p),
\]

where \( v \) is a tangent vector in \( T^{1,0}_p \) and \( p \) is in a simply-connected neighbourhood of \( x \).

Because of this identity, the problem of finding real sections of \( Z \) can be solved via factorization in the (local) group of holomorphic diffeomorphisms of \( M \). The sections of the twistor space \( Z \) will be parameterised by points \( x \) of \( X \) and triples of vectors \( V_1, V_2, V_3 \) in \( T_xX \) (usually only triples close to zero). Let us denote by the same symbol \( V_k \) the corresponding vector in \( T^{1,0}_k \), i.e. \( V_k = iV_k \). The idea is that the section of \( Z \) will be of the form

\[
\zeta \mapsto (\beta(\zeta), g_+(1, \zeta)x, \zeta), \quad \tilde{\zeta} \mapsto (\tilde{\beta}(\tilde{\zeta}), g_-(1, \tilde{\zeta})x, \tilde{\zeta})
\]

where \( g_\pm \) are certain holomorphic diffeomorphisms of \( M \) and \( \beta(\zeta), \tilde{\beta}(\tilde{\zeta}) \) are vector fields on \( M \) defined by parallel translation (with respect to the connection \( \nabla^C \)) of vectors

\[
(V_2 + iV_3) + 2iV_1\zeta + (V_2 - iV_3)\zeta^2, \quad (V_2 + iV_3)\tilde{\zeta}^2 + 2iV_1\tilde{\zeta} + (V_2 - iV_3)
\]

which are in \( T^{1,0}_k \). The diffeomorphisms \( g_\pm \) are defined as solutions of the following Riemann-Hilbert problem:

\[
e^{-t\beta(\zeta)/\zeta}g_+(t, \zeta) = g_-(\tilde{\zeta}, t)
\]
for $t \in [0, 1]$ with $g_+(\zeta, 0) = g_-(\zeta, 0) = 1$. This equation, together with (4.1), shows that (4.2) really defines a section of $Z$. The $R$-matrix method, cf. [10], implies that $g_\pm$ are obtained from vector field-valued solutions $B_0, B_1, B_2$ to the Nahm equations:

$$\dot{B}_0 = [B_0, B_1], \quad \dot{B}_1 = [B_0, B_2], \quad \dot{B}_2 = [B_1, B_2]$$

(4.4)

with initial conditions $B_0(0) = \hat{V}_2 + i\hat{V}_3$, $B_1(0) = 2i\hat{V}_1$, $B_2(0) = \hat{V}_2 - i\hat{V}_3$. We have

$$g_+^{-1}\dot{g}_+ = \frac{1}{2}B_1(t) + B_2(t)\zeta, \quad g_-^{-1}\dot{g}_- = -\frac{1}{2}B_1(t) - B_0(t)\zeta, \quad g_+(\zeta, 0) = g_-(\zeta, 0) = 1.$$ (4.5)

Since the antiholomorphic involution $\tau_\ast$ on $T^{1,0}M$ is compatible with the vector field bracket, i.e. $\tau_\ast[X, Y] = [\tau_\ast(X), \tau_\ast(Y)]$, the solutions $B_t$ to (4.4) (with given initial conditions) satisfy $\tau_\ast(B_0) = B_2$, $\tau_\ast(B_1) = -B_1$. Consequently the sections of $Z$ obtained this way are real, i.e. $\sigma$-invariant.

Thus it remains to show that this construction works in our given infinite-dimensional setting, i.e. that for $V_1, V_2, V_3$ in a neighbourhood of 0 in $T_xX$, there exists a solution to the equations (4.4) and (4.5) on all of $t \in [0, 1]$ (for the equation (4.5) we assume that $\frac{1}{2} \leq |\zeta| \leq 2$). Alternatively we can prove that solutions exist for small $t$ for any $V_1, V_2, V_3$. The existence of solutions to (4.4) follows from the Cauchy-Kovalevskaya theorem, since the Nahm equations for holomorphic vector fields become in local coordinates a Cauchy’s problem for first-order PDE’s.

It remains to show that there exists a solution to (4.5) for small $t$.

**Lemma 4.1.** Let $M$ be a real (resp. complex) manifold and $V(t)$, $t \in [0, \epsilon]$ be a one-parameter family of real (resp. holomorphic) vector fields in a neighbourhood of $x \in M$. Then there exists a one-parameter family $g(t), t \in [0, \epsilon] \subset [0, \epsilon]$ of real (resp. holomorphic) diffeomorphisms of a (smaller) neighbourhood of $x$ satisfying

$$g^{-1}\dot{g} = V(t), \quad g(0) = 1.$$  

**Proof.** We observe that solving the equation $\dot{h}h^{-1} = -V(t)$, $h(0) = 1$ is simply equivalent to finding, for each point $p$ near $x$, a curve $p(t)$ with $p(0) = p$ and $\dot{p} = -V(t)$. Such a solution exists locally by the usual theorems (e.g. Peano’s) on existence and uniqueness of solutions to ODE’s. Now $g = h^{-1}$ is the desired family of diffeomorphisms.

Thus we can indeed find the full $4n$-dimensional family of sections of $Z$ by solving the Nahm equations (4.4) in the Lie algebra of holomorphic vector fields on $M$.

## 5. Hyperkähler metrics

In this section we consider the case of a Riemannian connection $\nabla$ on $X$. We shall show that the Obata connection of section 3 is given by a hyperkähler metric and that the restriction of this metric to $M \subset X^\mathbb{C}$ is the Kähler metric of Lempert and Szőke mentioned in the introduction.

Let $X$ be a real-analytic manifold with a real-analytic metric $g$ and let $\nabla$ be the Levi-Civita connection of $g$. We recall that the tangent bundle $\pi : TX \to X$ of a Riemannian manifold has a canonical 1-form $\Theta$ defined by

$$\Theta(a) = g(z, d\pi(a)), \quad a \in T_z(TX).$$ (5.1)
As Lempert and Szőke show in [8], if $TX$ is equipped with the adapted complex structure, then there is a Kähler metric on $TX$ whose Kähler form is $d\Theta$.

We now wish to construct a hyperkähler metric on a neighbourhood of $X$ in $TTX$. Let $M$ and $Z$ be as in sections 2 and 3. We need to give a holomorphic $O(2)$-valued $\sigma$-invariant 2-form on fibers of the twistor space $Z$. We extend metric $g$ to a complex metric $\hat{g}$ on a neighbourhood $M$ of $X$ in $X^C$. In the trivialisation $(2.1)$ we define a 1-form on each fibre $T^{1,0}M$ of $Z$ by the formula

$$\hat{\Theta}(a) = \hat{g}(z, d\pi(a)), \quad a \in T_z^{1,0}(T^{1,0}M).$$

This gives rise to a holomorphic $O(2)$-valued and $\sigma$-invariant 1-form on the fibers of $Z$ and hence we obtain a 2-form $\Omega = d\hat{\Theta}$. From this we obtain a triple $\omega_1, \omega_2, \omega_3$ of 2-forms on $W$ such that

$$\Omega = (\omega_2 + i\omega_3) + 2i\omega_1\zeta + (\omega_2 - i\omega_3)\zeta^2.$$  \hfill (5.3)

Moreover, if $J_1, J_2, J_3$ denotes the hypercomplex structure on $W$, then $\omega_1(\cdot, J_1 \cdot) = \omega_2(\cdot, J_2 \cdot) = \omega_3(\cdot, J_3 \cdot)$ and this is a non-degenerate (as $\omega_2$ is non-degenerate) symmetric tensor on $W$, i.e. a (possibly indefinite) hyperkähler metric on $W$. We wish to show that this hyperkähler metric restricted to $M$ is the metric of Lempert and Szőke. From (5.3), we need to compute (5.2) on sections of the form (3.2) and show that the term linear in $\zeta$ is equal to (5.1) (up to the factor of $2i$ and up to adding a closed form). Let $b$ be a $(1, 0)$ tangent vector to $M$ at $m$. Then the corresponding variation of the section (3.2) is given by $s(\zeta) = (b, 2\zeta A, \zeta)$ where $A$ is the resulting variation of $V$ and, for each $\zeta$, $d\pi(s(\zeta)) = b$. Hence

$$\hat{\Theta}(s(\zeta)) = \hat{g}(2\zeta V, b).$$ \hfill (5.4)

Now recall that $m$ is given as $\exp_C^C(i \nu)$ where $x \in X$ and $v \in T_x X$. Therefore $b = u(1)$, where $u(t)$ is the Jacobi vector field along the (complex) geodesic $\gamma(t) = \exp_C^C(itv)$. The vector $V$ is simply $\gamma'(1)$. Thus the $\zeta$-linear term in (5.4) is simply $2\hat{g}(\gamma'(1), u(1))$.

Now recall that a Jacobi vector field can be written (also in the complex case) as $u(t) = \lambda \gamma(t) + t\mu \gamma(t) + U(t)$, where $U(t)$ is orthogonal to $\gamma(t)$ for any $t$. Therefore

$$\hat{g}(\gamma'(1), u(1)) = (\lambda + \mu)\hat{g}(\gamma'(1), \gamma'(1)) = \hat{g}(\gamma'(0), \gamma'(0)) = \hat{g}(\gamma(0), u(0)) + \hat{g}(\gamma(0), \mu \gamma'(1)) = \hat{g}(\gamma(0), u(0)) + \hat{g}(\gamma(0), \dot{u}(0)).$$ \hfill (5.5)

Since $\gamma(0) = x \in X$, the last expression is simply

$$\hat{g}(v, u(0)) + \hat{g}(v, \dot{u}(0)).$$ \hfill (5.6)

The Jacobi vector field $u$ was obtained by varying geodesics of the form $\exp_C^C(itv)$ and hence we conclude that $u(0) = w$ is real and $\dot{u}(0)$ is purely imaginary. On the other hand, (5.3) implies that the exterior derivative of (5.6) is purely imaginary. Therefore the second term in (5.6) is a closed 1-form and $\omega_1 = d\Theta$.

We finally compute the signature of our hyperkähler metric. Observe first that the above construction works perfectly well for a metric $g$ of any signature $(p, q)$.

**Proposition 5.1.** If the metric $g$ on $X$ has signature $(p, q)$, then the $SO(3)$-invariant hyperkähler metric $G$ on $W$ has signature $(4p, 4q)$.

**Proof.** It is enough to compute the signature at points of $X \subset W$. As $X$ is $SO(3)$-invariant and totally real for any complex structure and the 2-forms $\omega_i$ are non-degenerate, $T_x W = T_x X \oplus J_1 T_x X \oplus J_2 T_x X \oplus J_3 T_x X$ at points $x \in X$. As the
hyperkähler metric $G$ is $J_i$-invariant and its restriction to $T_xX$ is $g$, the signature of $G$ restricted to each direct summand is $(p,q)$. Finally, the summands are mutually orthogonal for $G$, as $T_xX$ is isotropic for each 2-form $\omega_i$.

\[ \square \]

**Note.** After this paper has been completed I received from Robert Szőke a preprint “Canonical complex structures assoviated to connections and complexifications of lie groups”, in which he also introduces the construction given in section 1 of this paper.

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