A NOTE ON SURJUNCTIVE GROUPS

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Abstract. In this article, we prove that a semidirect product of a locally finite group with a surjunctive group is also surjunctive. We also prove that a surjunctive-by-locally finite group is again surjunctive.

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Consider a topological space $X$ which is compact, Hausdorff, and totally disconnected. If $T : X \to X$ is a continuous map, then the pair $(X, T)$ is called a symbolic dynamical system. We are interested to know when injectivity of $T$ implies its surjectivity. The most important case for us is the case when $X$ is a full shift space over a group. Let $G$ be a group and $A$ be a finite set. Elements of $A$ are called letters and elements of $G$ are cells. The set $A$ is also called alphabet. Consider the space $A^G$ which is the set of all maps $G \to A$. If we consider the set $A$ as a discrete topological space, then clearly $A^G$ will be a compact, Hausdorff, and totally disconnected space. The group $G$ acts on $A^G$ by shift, i.e. for any $x \in A^G$ and $g \in G$, we have the new map $g \cdot x$, which sends any element $h \in G$ to the new symbol $x(g^{-1}h)$. A map $T : A^G \to A^G$ is called $G$-invariant, if $T(g \cdot x) = g \cdot T(x)$, for all $g \in G$ and $x \in A^G$. A cellular automaton is just a continuous $G$-invariant map $T : A^G \to A^G$. Hence, corresponding to every cellular automaton $T$, we have a symbolic dynamical system $(A^G, T)$. The group $G$ is called surjunctive, if for any finite set $A$, any injective cellular automaton $T : A^G \to A^G$ is also surjective. Since the year 1973, when the surjunctive groups introduced by Gottschalk (see [3]), many groups have been known to be surjunctive: locally finite groups, residually finite groups, solvable groups, sofic groups, and so on. Also, it is known that subgroups of surjunctive groups are surjunctive too, and a group is surjunctive, if and only if, its finitely generated subgroups are surjunctive. But, the amazing fact about this class of groups is that no example of a non-surjunctive group is discovered until now. It is also
not known that if the direct product of two surjunctive group is surjunctive or not. In this article, we consider two special cases: the first one is the case where the group is a semidirect product $H \rtimes K$, with $H$ locally finite and $K$ surjunctive. We will show that the resulting group is again surjunctive. The second one is the case where the group is surjunctive-by-locally finite group. We will show that such a group is also surjunctive.

Before ending this introduction, we must say that surjunctive groups are important from the perspective of Kaplansky’s direct finiteness conjecture: consider a group $G$ and a field $F$. It is conjectured that for any two elements $x$ and $y$ in the group ring $F[G]$, the equality $xy = 1$ implies $yx = 1$. It is known that all surjunctive groups satisfy this conjecture (see [1]). Therefore, our results can be viewed as constructing new groups which satisfy Kaplansky’s direct finiteness conjecture.

All notations in this article are standard and the same as in [2], where in the same time, for detailed discussion of notions, the reader can consult it. I would like to dedicate this paper to professor Mahmut Kuzucuoğlu (from Middle East Technical University of Ankara) for the occasion of his 60th birthday.

1. Main results

Consider a group $G = KH$ which is decomposed into the product of two subgroups; $H$ is finite, and $H \cap K = 1$. Let $A$ be a finite alphabet. Then $A^H$ is also finite and it can be considered as a new alphabet.

**Lemma 1.** The spaces $A^G$ and $(A^H)^K$ are homeomorphic.

**Proof.** Let $y \in (A^H)^K$. Then for all $k \in K$ we have $y(k) \in A^H$, and hence for all $h \in H$, the letter $y(k)(h)$ is an element of $A$, which we denote it by $u_y(k, h)$. Therefore, for any $g \in G$, with its unique decomposition $g = kh$, we define $\overline{y}(g) = u_y(k, h)$. Hence $\overline{y} \in A^G$. Now, define a map $\lambda : (A^H)^K \to A^G$ by $\lambda(y) = \overline{y}$. We prove that this map is a homeomorphism. Let $\lambda(y_1) = \lambda(y_2)$, so for all $h$ and $k$, we have $y_1(k)(h) = y_2(k)(h)$, so $y_1(k) = y_2(k)$, and hence $y_1 = y_2$, showing that $\lambda$ is injective. For $x \in A^G$ and for any element $g = kh$, the letter $x(g)$ depends on $x$, $h$ and $k$. Hence $x(g) = w_x(k, h)$, where $w_x : K \times H \to A$ is map, depending on $x$. Now, define an element $y \in (A^H)^K$ by

$$y(k) = w_x(k, -) \in A^H.$$
We have \( \mathcal{Y}(kh) = y(k)(h) = w_x(k, h) = x(g) \). So, \( \lambda(y) = x \), proving the surjectivity of \( \lambda \). Note that, as \( A \) and \( A^H \) are finite the spaces \( A^G \) and \( (A^H)^K \) are compact, so it is enough to prove that the map \( \lambda \) is continuous. To do this, let \( a \in A \) and \( g \in G \) be fixed elements. Consider the set
\[
W = \{ x \in A^G : x(g) = a \}.
\]
We know that the collection of such sets is a subbasis of clopen sets for the topology of \( A^G \). So, we prove that \( \lambda^{-1}(W) \) is open in \( (A^H)^K \). Let \( g = kh \) be the decomposition of \( g \). Then
\[
W = \{ x \in A^G : \lambda^{-1}(x)(k)(h) = a \}.
\]
Consider the new set
\[
B = \{ u \in A^H : u(h) = a \} \subseteq A^H.
\]
Then, clearly we have
\[
\lambda^{-1}(W) = \{ y \in (A^H)^K : y(k) \in B \}.
\]
Since \( B \) is finite, so \( \lambda^{-1}(W) \) is a cylinder and hence it is open. This proves that \( \lambda \) is continuous. □

During the rest of this section, we keep the same notations as in the above proof.

**Lemma 2.** The above map \( \lambda \) is \( K \)-equivariant, i.e. for any \( y \in (A^H)^K \) and \( k \in K \), we have \( \lambda(k \cdot y) = k \cdot \lambda(y) \).

**Proof.** Consider an element \( g = lh \), with \( l \in K \) and \( h \in H \). Then we have
\[
\lambda(k \cdot y)(lh) = \overline{k \cdot y(lh)} = (k \cdot y)(l)(h) = y(k^{-1}l)(h).
\]
On the other hand, we have
\[
(k \cdot \lambda(y))(lh) = (k \cdot \overline{y})(lh) = \overline{y(k^{-1}l)}(h) = y(k^{-1}l)(h).
\]
This proves the assertion. □

**Lemma 3.** If \( K \) is surjunctive, the so is \( G \).

**Proof.** Consider an injective cellular automaton \( T : A^G \to A^G \). We define a new map \( T_0 : (A^H)^K \to (A^H)^K \), by \( T_0(y) = \lambda^{-1}(T(\lambda(y))) \). In other words \( T_0 = \lambda^{-1}T\lambda \), and hence it is continuous. For all \( k \in K \) we have \( T_0(k \cdot y) = k \cdot T_0(y) \) as \( T \) and \( \lambda \) are \( K \)-equivariants. Hence, \( T_0 \) is a
cellular automaton over $K$ with alphabet $A^H$. Since $K$ is assumed to be surjunctive, the map $T_0$ is surjective. So, clearly $T$ is also surjective, showing that $G$ is surjunctive.

Now, we are ready to state our first result:

**Theorem 1.** Let $G = KH$ be the semidirect product of a locally finite group $H$ with a surjunctive group $K$ (i.e. $K \cap H = 1$ and $K \lhd G$). Then $G$ is surjunctive.

**Proof.** We have to show that every finitely generated subgroup of $G$ is surjunctive. So, let $L = \langle g_1, g_2, \ldots, g_n \rangle$ be a subgroup of $G$. Let $g_i = k_i h_i$ be the unique decomposition of $g_i$. Then we have clearly

$$L \subseteq \langle k_1, k_2, \ldots, k_n \rangle \langle h_1, h_2, \ldots, h_n \rangle.$$ 

Let $K_0 = \langle k_1, k_2, \ldots, k_n \rangle$ and $H_0 = \langle h_1, h_2, \ldots, h_n \rangle$. Hence $L \subseteq K_0 H_0 \subseteq KH_0$, with $H_0$ finite. Since $K$ is a normal subgroup, $KH_0$ is a subgroup of $G$. By the lemma 3, $KH_0$ is surjunctive and hence its subgroup $L$ is surjunctive too. This proves the theorem.

It is also possible to use almost the same argument for the case of surjunctive-by-locally finite groups. Consider a group $G$ with a surjunctive normal subgroup $K$ and suppose $G/K$ is finite. Let $A$ be a finite set. So the new set $A^{G/K}$ is also finite and we can consider it as a new alphabet.

**Lemma 4.** Two spaces $A^G$ and $(A^{G/K})^K$ are homeomorphic.

**Proof.** Consider a fixed right transversal for $K$, like $T = \{h_1, h_2, \ldots, h_n\}$, so $G = \cup_{i=1}^n K h_i$, and any $g \in G$ has a decomposition of the form $g = k h_i$ for a unique $k \in K$. Now, for any $y \in (A^{G/K})^K$ and any $k \in K$, we have $y(k) \in A^{G/K}$ and hence for any $g \in G$, we have $y(k)(Kg) \in A$. So, we are going to define a corresponding $\overline{y}$. Let $g = kh_i$ be the decomposition of $g$ and define $\overline{y}(g) = y(k)(K h_i)$. Note that this is the same as $\overline{y}(g) = y(gh_i^{-1})(Kg)$. Now, consider the map $\lambda : (A^{G/K})^K \to A^G$, defined by $\lambda(y) = \overline{y}$. We claim that this map $\lambda$ is a homeomorphism. The injectivity is obvious, because if $\lambda(y_1) = \lambda(y_2)$, then for any $k \in K$ and any $i$, we will have $y_1(k)(K h_i) = y_2(k)(K h_i)$, which means that $y_1(k) = y_2(k)$, and hence $y_1 = y_2$. To prove surjectivity, let $x \in A^G$ and consider an element $g = kh_i$. Then clearly $x(g)$ depends on $x$, $k$ and $h_i$, so we can denote it by a notation like $w_x(k; h_i)$. So, we have a map $w_x : K \times T \to A$. Now, define $y \in (A^{G/K})^K$ by
y(k) = w_x(k, −) ∈ A^{G/K}. Then we have
\[ \overline{\gamma}(g) = y(gh_i^{-1})(Kg) = y(k)(Kh_i) = w_x(k, h_i) = x(g). \]

This proves that \( \lambda \) is injective. As in the previous case, we have to show that \( \lambda \) is continuous. Let \( a \in A \) and \( g \in G \). Suppose
\[ W = \{ x \in A^G : x(g) = a \}. \]
Let \( g = kh_i \) be the decomposition of \( g \). Then
\[ W = \{ x \in A^G : \lambda^{-1}(x)(k)(Kh_i) = a \}. \]
Define a new set
\[ B = \{ u \in A^{G/K} : u(Kh_i) = a \}, \]
which is clearly a finite set. Then we have
\[ \lambda^{-1}(W) = \{ y \in (A^{G/K})^K : y(k) \in B \}, \]
and so it is open. □

**Lemma 5.** The map \( \lambda \) is \( K \)-equivariant.

**Proof.** Consider an element \( k \in K \) and an element \( g \in G \) with decomposition \( g = lh_i \), where \( l \in K \). We have
\[ \lambda(k \cdot y)(g) = (k \cdot y)(lh_i) = (k \cdot y)(l)(kh_i) = y(k^{-1}l)(kh_i). \]
On the other hand
\[ (k \cdot \lambda(y))(g) = (k \cdot \overline{\gamma})(lh_i) = \overline{\gamma}(k^{-1}lh_i) = y(k^{-1}l)(Kh_i). \]
This proves the lemma. □

Now, we can prove the next lemma, exactly as the lemma 3.

**Lemma 6.** Let \( G \) be a surjunctive-by-finite group. Then \( G \) is surjunctive.

Finally, we prove our second result.

**Theorem 2.** A surjunctive-by-locally finite group is surjunctive.
Proof. Let $K$ be a surjunctive normal subgroup of a group $G$, and let $G/K$ be locally finite. Let $L$ be a finitely generated subgroup. Then

$$\frac{LK}{K} \cong \frac{L}{K \cap L},$$

so it is finitely generated and consequently the quotient $LK/K$ is finite. Since $K$ is surjunctive by the lemma 6, $LK$ is surjunctive and hence so is $L$. □

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