Phase transitions in systems of self-propelled agents and related network models

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An important characteristic of flocks of birds, school of fish, and many similar assemblies of self-propelled particles is the emergence of states of collective order in which the particles move in the same direction. When noise is added into the system, the onset of such collective order occurs through a dynamical phase transition controlled by the noise intensity. While originally thought to be continuous, the phase transition has been claimed to be discontinuous on the basis of recently reported numerical evidence. We address this issue by analyzing two representative network models closely related to systems of self-propelled particles. We present analytical as well as numerical results showing that the nature of the phase transition depends crucially on the way in which noise is introduced into the system.

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The collective motion of a group of autonomous particles is a subject of intense research that has potential applications in biology, physics and engineering. One of the most remarkable characteristics of systems such as a flock of birds, a school of fish or a swarm of locusts, is the emergence of ordered states in which the particles move in the same direction. When noise is added into the system, the onset of such collective order occurs through a dynamical phase transition controlled by the noise intensity. While originally thought to be continuous, the phase transition has been claimed to be discontinuous on the basis of recently reported numerical evidence. We address this issue by analyzing two representative network models closely related to systems of self-propelled particles. We present analytical as well as numerical results showing that the nature of the phase transition depends crucially on the way in which noise is introduced into the system.

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from the one presented in [4] to the one described in [5].

The first network model, which we will refer to as the vectorial network model, consists of a network of $N$ 2D-vectors (represented as complex numbers), $\{\sigma_1 = e^{i\theta_1}, \sigma_2 = e^{i\theta_2}, \ldots, \sigma_N = e^{i\theta_N}\}$, all of the same length $|\sigma_n| = v$ and whose angles $\{\theta_1(t), \theta_2(t), \ldots, \theta_N(t)\}$ can change in time. Each vector $\sigma_n$ interacts with a fixed set of $K$ other vectors, $\{\sigma_{n_1}, \ldots, \sigma_{n_K}\}$, randomly chosen from anywhere in the system. We will call this set of $K$ vectors the inputs of $\sigma_n$. Once each vector $\sigma_n$ has been provided with a fixed set of $K$ input connections, the dynamics of the network are then given by one of the two following interaction rules:

$$\theta_n(t+1) = \text{Angle} \left\{ \frac{1}{vK} \sum_{j=1}^{K} \sigma_{n_j}(t) \right\} + \eta \xi(t),$$  \hspace{1cm} (1)

$$\theta_n(t+1) = \text{Angle} \left\{ \frac{1}{vK} \sum_{j=1}^{K} \sigma_{n_j}(t) + \eta e^{i\xi(t)} \right\},$$  \hspace{1cm} (2)

where for any vector $\vec{v} = |v|e^{i\phi}$ we define the function $\text{Angle}(\vec{v}) = \phi$, and $\xi(t)$ is a random variable uniformly distributed in the interval $[-\pi, \pi]$. The dynamics of the network is fully deterministic for $\eta = 0$ and becomes more random as the parameter $\eta$ increases. In what follows, we will refer to the quantity $\left(1/vK\right) \sum_{j=1}^{K} \sigma_{n_j}$ as the average contribution of the inputs of $\sigma_n$.

To quantify the amount of order in the system we define the instantaneous order parameter $\psi(t)$ as

$$\psi(t) = \lim_{N \to \infty} \frac{1}{vN} \sum_{n=1}^{N} |\sigma_n(t)|.$$  \hspace{1cm} (3)

In the limit $t \to \infty$, the instantaneous order parameter $\psi(t)$ reaches a stationary value $\psi$ [4, 5, 6, 7]. Thus, in
the stationary state all the vectors are aligned if \( \psi \sim 1 \), whereas if \( \psi \sim 0 \) the vectors are in random directions.

The interaction rules given in Eqs. (1) and (2) were proposed by Vicsek et al. in Ref. [4], and by Grégoire and Chaté in Ref. [5], respectively. The difference between these two interaction rules consists in the way in which the noise is introduced: in Eq. (1) the noise is added outside the Angle function, i.e. after the Angle function has been applied to the average contribution of the inputs. On the other hand, in Eq. (2) the noise is added inside the Angle function, i.e. it is added directly to the average contribution of the inputs. In Ref. [5], Grégoire and Chaté posed the question as to whether these two rules lead to the same type of phase transition.

In this letter we show that the interaction rules in Eq. (1) and Eq. (2) produce different types of phase transitions in the network systems under consideration, which suggests that a similar effect is being observed in [5] for the self-propelled systems.

Obviously, in the self-propelled particle models the elements do not interact through a network. Instead, they move in a 2D space, each particle interacting locally with the particles that fall within a certain radius. This motion allows particles that are initially far apart to meet, interact, and separate again, giving rise to effective long range interactions. On the other hand, in our vectorial network model the particles are fixed to the nodes of a network. The long-range correlations produced by the motion of the particles in the self-propelled models are proxied in our network model through randomly choosing the inputs of each element from anywhere in the network. An underlying assumption of our work is that the existence and nature of the phase transition depends mostly on the occurrence of such long-range interactions, and less crucially on whether they are produced by the motion of the particles or by the network topology [6, 7]. While the exact relation between these two ways of establishing long-range interactions is not yet known, it has been shown that a strong parallel can be established between them [4, 9]. Further, below we show that there are at least two limits in which they are fully equivalent: for large particle speeds and for high densities (see Fig. 1 and Fig. 2).

In Ref. [5] it was proven that, as the noise amplitude \( \eta \) increases, the vectorial network model with the interaction rule given as in Eq. (1) undergoes a continuous phase transition from ordered states where \( \psi > 0 \), to disordered states where \( \psi = 0 \). Fig. 1 shows this phase transition obtained numerically for \( N = 20000 \) and \( K = 5 \). It also displays the phase transition in the Vicsek model for a system with the same \( N \), a density such that the average number of interactions per particle is also \( K = 5 \), and increasing particle speeds. As can be seen from Fig. 1 the Vicsek model curves approach continuously the network model curve as \( v \to \infty \). This supports the idea that in both cases a second order phase transition is observed when the noise is introduced as in Eq. (1), albeit the finite size effects observed near the critical point.

The probability distribution function (PDF) of the sum \( \frac{1}{N} \sum_{j=1}^{N} \sigma_{n_j}(t) + \eta \psi(t) \) that appears in Eq. (2) is computed as for a random walk assuming that all the terms are statistically independent. By projecting this PDF onto the unit circle we can establish a recursion relation for the order parameter, which for \( K \gg 1 \) becomes

\[
\psi(t + 1) = M_\eta(\psi(t)),
\]

and \( M_\eta(\psi) \) is shown in Fig. 2 for different values of \( \eta \) (solid curves). This figure also displays with symbols the numerical dynamical mapping computed for the self-propelled model with the interaction rule given in Eq. (2), \( N = 20000 \) particles, and an average number of interactions per particle \( K = 100 \). Clearly, the numerical mapping coincides with the theoretical result for \( M_\eta(\psi) \), showing that the network and self-propelled systems are also equivalent in the high density limit case considered here. The numerical mappings for the self-propelled system were obtained by placing the particles in various random initial conditions constrained to produce every order parameter value \( \psi(t) \) in the \( x \)-axis, and then computing one time step using Eq. (2) to obtain the corresponding value \( \psi(t + 1) \) in the \( y \)-axis.

The fixed points of the dynamical mapping \( \psi(t + 1) = M_\eta(\psi(t)) \) give the stationary values of the order parameter. From Eq. (4) it is clear that \( \psi = 0 \) is always a fixed point. However, the stability of this fixed point changes depending upon the value of \( \eta \). By numerically solving Eq. (4) to obtain the fixed point, we find that...
for $0.672 < \eta$ the only stable fixed point is $\psi = 0$. As $\eta \to 0.672$ from above, the graph of $\mathcal{M}_\eta(\psi)$ moves closer to the identity and eventually another non zero stable fixed point $\psi'$ appears discontinuously as $\eta$ decreases, whereas in (b) it appears continuously.

The validity of these results is corroborated by numerical simulations carried out for networks with $N = 20000$ and an average number of interactions per particle $K = 100$. (b) The interaction rule is as in Eq. (4). The solid and dotted-dashed curves are the results of the numerical simulation starting out the dynamics from initial conditions for which $\psi(0) \approx 1$ and $\psi(0) \approx 0$, respectively. The phase transition in this case is clearly discontinuous.

The second model that we consider is a majority voter model in which the network elements $\sigma_n$ can acquire only two values, +1 or -1. We can think of this system as a society in which every individual $\sigma_n$ has to make a decision about an issue with two possible alternatives, either +1 or -1. Again, each element $\sigma_n$ receives inputs from a set of $K$ other elements randomly chosen from anywhere in the system. Let us first consider the case in which the interaction between $\sigma_n$ and its $K$ inputs, $\{\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_K}\}$, is given by

$$
\sigma_n(t + 1) = \text{Sign} \left[ \frac{1}{K} \sum_{j=1}^{K} \sigma_{n_j}(t) + \xi(t) \right] + \xi \left( \frac{1}{1 - \eta} \right)
$$

where $\text{Sign}[x] = 1$ if $x > 0$, $\text{Sign}[x] = -1$ if $x < 0$, $\eta$ is a parameter that takes a constant value in the interval $[0, 1]$, and $\xi(t)$ is a random variable uniformly distributed between $[-1, 1]$. For the Sign function to be well defined we choose $K$ as an odd integer. Eq. (5) is similar to Eq. (4) in that the noise is added to the sign of the average contribution of the inputs. Since in this case $\sigma_n$ is a discrete variable that takes only the two values +1 or -1, the Sign function has to be applied again. This interaction rule reflects the fact that an individual in a society usually tends to be of the same opinion as the majority of his “friends” (inputs), though, with probability $\eta/2$ he can have the opposite opinion.

The instantaneous order parameter $\psi(t)$ is defined as in Eqs. (3), but now the vertical bars represent the absolute value instead of the norm of a vector.

In Ref. [3] it has been shown that the majority voter model with the interaction rule given by Eq. (5) under-
In summary, we have analyzed numerically and analytically the phase transition from ordered to disordered states in two network models that capture some of the main aspects of the interactions in systems of self-propelled particles. In particular, the self-propelled model becomes equivalent to the vectorial network model in the limit of large speeds or high densities. We have shown that for the two network models, the phase transition changes from second order to first order depending on the way in which the noise is introduced into the system. This change is consistent with the results reported by Vicsek et al. in [4], and by Grégoire and Chaté in [5]. This consistency suggests that a similar effect is being observed in the self-propelled model and motivates a deeper analysis in order to determine the nature of its phase transition. Clearly, the two ways of introducing noise correspond to different physical situations. On the one hand, with the Vicsek type of noise the uncertainty falls on the decision mechanism. On the other, introducing the noise à la Grégoire-Chaté, the decision function is perfectly determined and the uncertainty falls on the arguments of this function. There is no reason to expect, a priori, similar behaviors under these two different physical situations.

Note added in proof: While this manuscript was being reviewed, Ref. [10] appeared, in which it is shown that the first-order phase transition found in Ref. [8] by means of the Binder cumulant is a numerical artifact.

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