ON THE LOCUS OF 2-DIMENSIONAL CRYSSTALINE REPRESENTATIONS WITH A GIVEN REDUCTION MODULO $p$

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Abstract. We consider the family of irreducible crystalline representations of dimension 2 of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ given by the $V_{k,a_p}$ for a fixed weight integer $k \geq 2$. We study the locus of the parameter $a_p$ where these representations have a given reduction modulo $p$. We give qualitative results on this locus and show that for a fixed $p$ and $k$ it can be computed by determining the reduction modulo $p$ of $V_{k,a_p}$ for a finite number of values of the parameter $a_p$. We also generalize these results to other Galois types.

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Introduction

Let $p$ be a prime number. Fix a continuous representation $\overline{\rho}$ of $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with values in $\text{GL}_2(\mathbb{F}_p)$. In [Kis08], Kisin has defined local rings $R^\psi(k,\overline{\rho})$ that parametrize the deformations of $\overline{\rho}$ to characteristic 0 representations that are crystalline with Hodge-Tate weights $(0,k-1)$ and determinant $\psi$. These rings are very hard to compute, even for relatively small values of $k$. We are interested in this paper in the rings $R^\psi(k,\overline{\rho})[1/p]$. These rings lose some information from $R^\psi(k,\overline{\rho})$, but still retain all the information about the parametrization of deformations of $\overline{\rho}$ in characteristic 0.

We can relate the study of the rings $R^\psi(k,\overline{\rho})[1/p]$ to another problem: When we fix an integer $k \geq 2$ and set the character $\psi$ to be $\chi_{cycl}^{k-1}$, the set of isomorphism classes of irreducible crystalline representations of dimension 2, determinant $\psi$ and Hodge-Tate weights $(0,k-1)$ is in bijection with the set $D = \{x \in \mathbb{Q}_p, v_p(x) > 0\}$ via a parameter $a_p$, and we call $V_{k,a_p}$ the representation corresponding to $a_p$. So given a residual representation $\overline{\rho}$ we can consider the set $X(k,\overline{\rho})$ of $a_p \in D$ such that the semi-simplified reduction modulo $p$ of $V_{k,a_p}$ is equal to $\overline{\rho}^{ss}$.

It turns out that $X(k,\overline{\rho})$ has a special form. We say that a subset of $\mathbb{Q}_p$ is a standard subset if it is a finite union of rational open disks from which we have removed a finite union of rational closed disks. Then we show that under some hypotheses on $\overline{\rho}$ (including in particular the fact that is has trivial endomorphisms, so that the rings $R^\psi(k,\overline{\rho})$ are well-defined):

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**Theorem A.** The set $X(k, \overline{p})$ is a standard subset of $\overline{\mathbb{Q}_p}$, and $R^e(k, \overline{p})[1/p]$ is the ring of bounded analytic functions on $X(k, \overline{p})$.

This tells us that we can recover $R^e(k, \overline{p})[1/p]$ from $X(k, \overline{p})$. But we need to be able to understand $X(k, \overline{p})$ better.

We can define a notion of complexity for a standard subset $X$ which invariant under the absolute Galois group of $E$ for some finite extension $E$ of $\mathbb{Q}_p$. This complexity is a positive integer $c_E(X)$, which mostly counts the number of disks involved in the definition of $X$, but with some arithmetic multiplicity that measures how hard it is to define the disk on the field $E$. A consequence of this definition is that if an upper bound for $c_E(X)$ is given, then $X$ can be recovered from the sets $X \cap F$ for some finite extensions $F$ of $E$, and even from the intersection of $X$ with some finite set of points under an additional hypothesis (Theorems 4.5.1 and 4.5.2).

A key point is that this complexity, which is defined in a combinatorial way, is actually related to the Hilbert-Samuel multiplicity of the special fiber of the rings of analytic functions bounded by 1 on the set $X$ (Theorem 4.4.1). This is especially interesting in the case where the set $X$ is $X(k, \overline{p})$ as in this case this Hilbert-Samuel multiplicity can be bounded explicitly using the Breuil-Mézard conjecture. So, under some hypotheses on $\overline{p}$, we have:

**Theorem B** (Proposition 5.4.9). There is an explicit upper bound for the complexity of $X(k, \overline{p})$.

As a consequence we get:

**Theorem C** (Theorem 5.4.10). The set $X(k, \overline{p})$ can be determined by computing the reduction modulo $p$ of $V_{k,a_p}$ for $a_p$ in some finite set.

In particular, it is possible to compute the set $X(k, \overline{p})$, and also the ring $R^e(k, \overline{p})[1/p]$, by a finite number of numerical computations. We give some examples of this in Section 6. One interesting outcome of these computations is that when $\overline{p}$ is irreducible, in every example that we computed we observed that the upper bound for the complexity given by Theorem 5.4.9 is actually an equality. It would be interesting to have an interpretation for this fact and to know if it is true in general.

Finally, we could ask the same questions about more general rings parametrizing potentially semi-stable deformations of a given Galois type, instead of only rings parametrizing crystalline deformations. Our method relies on the fact that we work with rings that have relative dimension 1 over $\mathbb{Z}_p$, so we cannot use it beyond the case of 2-dimensional representations of $G_{\mathbb{Q}_p}$. But in this case we can actually generalize our results to all Galois types. In order to do this, we need to introduce a parameter classifying the representations that plays a role similar to the role the function $a_p$ plays for crystalline representations, and to show that it defines an analytic function on the rigid space attached to the deformation ring. This is the result of Theorem 5.3.1. Once we have this parameter, we show that an analogue of Theorem A holds, and an analogue of Theorem B (Theorem 5.3.3). However we get only a weaker analogue of Theorem C (Theorem 5.3.6). The main ingredient of this theorem that is known in the crystalline case, but missing the case of more general Galois types, is the fact that the reduction of the representation is locally constant with respect to the parameter $a_p$, with an explicit radius for local constancy.

**Plan of the article.** The first three sections contain some preliminaries. In Section 1 we prove some results on the smallest degree of an extension generated by a point of a disk in $\mathbb{C}_p$. These results may be of independant interest. In Section 2 we prove some results on Hilbert-Samuel multiplicities and how to compute them for some special rings.
of dimension 1. In Section 3 we introduce the notion of standard subset of $\mathbb{P}^1(\mathbb{Q}_p)$ and prove some results about some special rigid subspaces of the affine line.

Section 4 contains the main technical results. This is where we introduce the complexity of so-called standard subsets of $\mathbb{P}^1(\mathbb{Q}_p)$, and show that it can be defined in either a combinatorial or an algebraic way.

We apply these results in Section 5 to the locus of points parametrizing potentially semi-stable representations of a fixed Galois type with a given reduction. We also explain some particularities of the case of parameter rings for crystalline representations.

In Section 6 we report on some numerical computations that were made using the results of Section 5 in the case of crystalline representations, and mention some questions inspired by these computations.

Finally in Section 7 we explain the construction of a parameter classifying the representations on the potentially semi-stable deformation rings.

**Notation.** If $E$ is a finite extension of $\mathbb{Q}_p$, we denote its ring of integers by $\mathcal{O}_E$, with maximal ideal $\mathfrak{m}_E$, and its residue field by $k_E$. We write $\pi_E$ for a uniformizer of $E$, and $v_E$ for the normalized valuation on $E$ and its extension to $\mathbb{C}_p$. We write also $v_p$ for $v_{\mathbb{Q}_p}$. Finally, $G_E$ denotes the absolute Galois group of $E$.

If $R$ is a ring and $n$ a positive integer, we denote by $R[X]_{<n}$ the subspace of $R[X]$ of polynomials of degree at most $n − 1$.

If $a \in \mathbb{C}_p$ and $r \in \mathbb{R}$, we write $D(a, r)^+$ for the set $\{x \in \mathbb{C}_p, |x − a| ≤ r\}$ (closed disk) and $D(a, r)^−$ for the set $\{x \in \mathbb{C}_p, |x − a| < r\}$ (open disk).

We denote by $\chi_{cycl}$ the $p$-adic cyclotomic character, and $\omega$ its reduction modulo $p$. We denote by $\text{unr}(x)$ the unramified character that sends a geometric Frobenius to $x$.

1. Points in disks in extensions of the base field

   Let $D \subset \mathbb{C}_p$ be a disk (open or closed). It can happen that $D$ is defined over a finite extension $E$ of $\mathbb{Q}_p$ (that is, invariant by $G_E$), but $E \cap D$ is empty. For example, let $\pi$ be a $p$-th root of $p$ and let $D$ be the disk $\{x, v_p(x − \pi) > 1/p\}$. Then $D$ is defined over $\mathbb{Q}_p$, as it contains all the conjugates of $\pi$, that is the $\zeta_p^i\pi$ for a primitive $p$-th root $\zeta_p$ of $1$.

   On the other hand, $D$ does not contain any element of $\mathbb{Q}_p$. The goal of this section is to understand the relationship between the smallest ramification degree over $E$ of a field $F$ such that $F \cap D \neq \emptyset$, and the smallest degree over $E$ of such a field.

   In this Section a disk will mean either a closed or an open disk.

   The results of this Section are used in the proofs of Propositions 4.5.8 and 4.5.10.

1.1. Statements.

**Theorem 1.1.1.** Let $D$ be a disk defined over $E$. Let $e$ be the smallest integer such that there exists a finite extension $F$ of $E$ with $e_{F/E} = e$ and $F \cap D \neq \emptyset$. Then $e = p^s$ for some $s$, and there exists an extension $F$ of $E$ with $[F : E] \leq \max(1, p^{2s−1})$ such that $F \cap D \neq \emptyset$. For $s \leq 1$ any such $F/E$ is totally ramified.

   We can in fact do better in the case where $p = 2$. Note that this result proves Conjecture 2 of [Ben15] in this case.

**Theorem 1.1.2.** Let $p = 2$. Let $D$ be a disk defined over $E$. Let $e$ be the smallest integer such that there exists a finite extension $F$ of $E$ with $e_{F/E} = e$ and $F \cap D \neq \emptyset$. Then $e = p^s$ for some $s$, and there exists a totally ramified extension $F$ of $E$ with $[F : E] = p^s$ such that $F \cap D \neq \emptyset$. 

1.2. Preliminaries. We recall the following result, which is [Ben15, Lemma 2.6] (it is stated only for closed disks, but applies also to open disks).

**Lemma 1.2.1.** Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $D$ be a disk defined over $K$. Suppose that $D$ contains an $a \in \overline{\mathbb{Q}}_p$ of degree $n$ over $K$. Then $D$ contains an element $b \in \overline{\mathbb{Q}}_p$ of degree $\leq p^n$ over $K$ where $s = v_p(n)$.

**Proof.** It follows from Lemma 1.2.1 that the minimal degree over $K$ of an element of $D$ is a power of $p$. On the other hand, applying Lemma 1.2.1 to $a$, we get an element of degree at most $p^n$ for $s = v_p(n)$. Hence the minimal degree is of the form $p^t$ for some $t \leq s$. □

**Corollary 1.2.2.** Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $D$ be a disk defined over $K$. Suppose that $D$ contains an element $a$ such that $[K(a) : K] = n$. Then the minimal degree over $K$ of an element of $D$ is of the form $p^t$ for some $t \leq v_p(n)$.

**Proof.** We first apply Corollary 1.2.2 with $K = E^{nr}$ to see that the minimal ramification degree is a power of $p$. Let $b \in D$ be such that $e_{E(b)/E} = p^t$ is the minimal ramification degree.

Let $E(b)_0 = E^{nr} \cap E(b)$, and let $F$ be the maximal subextension of $E(b)_0$ of degree a power of $p$. Note that $v_p([E(b) : F]) = t$, as $[E(b)_0 : F]$ is prime to $p$. We apply Corollary 1.2.2 to $K = F$, and we get an element $a \in D$ of degree at most $p^t$ over $F$. By minimality of $t$, we get that in fact $[E(a) : F] = p^t$, and $E(a)/F$ is totally ramified. Finally, $[E(a) : E]$ is a power of $p$ and $e_{E(a)/E} = p^t$. □

Let $\pi_E$ be a uniformizer of $E$, and let $F$ be a finite unramified extension of $E$. For $x \in F$, we define the $E$-part of $x$, which we denote by $x^0$, as follows: write $x$ as $x = \sum_{n \geq 0} a_n \pi_E^n$ where the $a_n$ are Teichmüller lifts of elements of the residue field of $F$. Let $x^0 = \sum_{n = 0}^m a_n \pi_E^n$ with $a_n \in E$ for all $n \leq m$ and $a_{m+1} \not\in E$ (or $m = \infty$ if $a \in E$) so that $x^0 \in E$. We have that $v_F(x - x^0) = m + 1$. This definition depends on the choice of $\pi_E$.

**Proposition 1.2.4.** Let $D$ be a disk defined over $E$, and suppose that $F \cap D \neq \emptyset$ for some unramified extension $F$ of $E$. Then $E \cap D \neq \emptyset$.

**Proof.** Let $a \in F \cap D$. We fix $\pi_E$ a uniformizer of $E$, and let $a^0$ be the $E$-part of $a$. Let $\sigma$ be the Frobenius of $\text{Gal}(F/E)$. Then $v_E(a - \sigma(a)) = v_E(a - a^0)$. So any disk containing $a$ and $\sigma(a)$ also contains $a^0$. □

We also recall the well-known result:

**Lemma 1.2.5.** Let $f$ be a rational function. Then for any disk $D$, if $f$ does not have a pole in $D$ then $f(D)$ is also a disk. Moreover, if $D$ is defined over $E$ and $f \in E(X)$ then $f(D)$ is defined over $E$.

1.3. Proofs. The part that states that $e$ is a power of $p$ in Theorems 1.1.1 and 1.1.2 is a consequence of Corollary 1.2.3.

We start with the rest of the proof of Theorem 1.1.2 which is actually easier.

**Proof of Theorem 1.1.2.** By applying Corollary 1.2.3, we get an element $a \in D$ that generates a totally ramified extension $F$ of $K$ of degree $e = p^n$, where $K$ is an unramified extension of $E$ of degree a power of $p$, and we take $[K : E]$ minimal. If $K \neq E$, let $K' \subset K$ with $[K : K'] = p$. We will show that we can find $b \in D$ of degree $e$ over $K'$, which gives a contradiction by minimality of $K$ so in fact $K = E$. 

Let $\mu$ be the minimal polynomial of $a$ over $K$, so $\mu \in K[X]$ is monic of degree $e$. Now we use that $p = 2$: let $(1, u)$ be a basis of $K$ over $K'$, and write $\mu = \mu_0 + u\mu_1$ with $\mu_0$, $\mu_1$ in $K'[X]$. If $\mu_0$ has a root in $D$ we are finished, so we can assume that $\mu_0$ has no zero in $D$, and let $f = \mu_1/\mu_0 \in K'(X)$. Let $D' = f(D)$. It is a disk defined over $K'$, containing $-u \in K$, so by Lemma 1.2.4 $D'$ contains an element $c \in K'$. This means that $\mu_0 - c\mu_1$ has a root $b$ in $D$.

Then $b$ is of degree at most $e$ over $K'$. By minimality of $e$, it means that $b$ is of degree exactly $e$ over $K'$, and $K'(b)/K'$ is totally ramified. So this gives the contradiction we were looking for.

Now we turn to the proof of Theorem 1.1.1. We start with a special case.

**Proposition 1.3.1.** Let $D$ be a disk defined over $E$ and $a \in D$. Suppose that $v_E(a) = n/e$ where $e = e_{E(a)/E}$ and $n$ is prime to $e$. Then there exists an extension $F$ of $E$ of degree at most $e$ such that $F \cap D \neq \emptyset$.

**Proof.** Let $K = E(a) \cap E^{nr}$. Let $\mu$ be the minimal polynomial of $a$ over $K$, so that $\mu$ has degree $e$. We write $\mu = \sum b_iX^i$, $b_i \in K$. Define $\mu^0 = \sum b_i^0X^i$ where $b_i^0 \in E$ is the $E$-part of $b_i$. Let $x_1, \ldots, x_e$ be the roots of $\mu^0$. Then $v_E(\mu^0(a)) = \sum v_E(a - x_i)$. On the other hand, $\mu^0(a) = \mu^0(a) - \mu(a) = \sum_{i=0}^{e-1} (b_i^0 - b_i)a^i$. By the condition on $v_E(a)$, we get that $v_E(\mu^0(a)) = \min_{0 \leq i < e} v_E(b_i^0 - b_i) + \log(e)$. Let $\sigma$ be an element of $G_E$ that induces the Frobenius on $K$. Let $y_1, \ldots, y_e$ be the roots of $\sigma(\mu) = \sum \sigma(b_i)X^i$. Then as before, $v_E(\sigma(\mu)(a)) = \sum v_E(a - y_i)$, and $v_E(\sigma(\mu)(a)) = \min_{0 \leq i < e} v_E(\sigma(b_i) - b_i) + \log(e)$. As $v_E(b_i^0 - b_i) = v_E(b_i^0 - b_i)$ for all $i$, we get that $v_E(\mu^0(a)) = v_E(\sigma(\mu)(a))$.

Suppose first that $D$ is closed. Write $D$ as the set $\{z, v_E(z - a) \geq \lambda\}$ for some $\lambda$, then we get that $v_E(\sigma(\mu)(a)) \geq e\lambda$ as the $y_i$ are among the conjugates of $a$ over $E$ and hence are in $D$, so $v_E(\mu^0(a)) \geq e\lambda$ and so there exists an $i$ with $x_i \in D$. Let $F = E(x_i)$ then $F$ is an extension of $E$ of degree at most $e$. The case of an open disk is similar.

Note that if we take $e$ to be minimal, then necessarily $F/E$ is totally ramified and of degree $e$.

**Proof of Theorem 1.1.1** The case $e = 1$ is a consequence of Proposition 1.2.4.

Assume now that $e > 1$. Let $a \in D, F = E(a)$ with $e_{F/E} = e, K = E(a) \cap E^{nr}$. If $a$ is a uniformizer of $F$, the result follows from Proposition 1.3.1. Otherwise, let $f \in E[X] \setminus e$ be a monic polynomial such that $f(a)$ is a uniformizer of $F$.

Assume first that such a $f$ exists. Let $D' = f(D)$. Then $D'$ is a disk defined over $E$ by Lemma 1.2.5 containing an element $w = f(a)$ with $v_{E(a)/E} = e$ and $v_E(\sigma w) = 1/e$, so it satisfies the hypotheses of Proposition 1.3.1. Hence there exists a $c \in D'$ with $[E(c) : E] \leq e$. Let $b \in D$ such that $f(b) = c$, then $[E(b) : E] \leq e(e - 1)$ as $b$ is a root of $f(X) - c$, which is a polynomial of degree at most $e - 1$ with coefficients in an extension of degree $e$ of $E$. Now we apply again Lemma 1.2.4 but with $K$ the maximal subextension of $E(b) \cap E^{nr}$ with degree a power of $p$. Then $[K : E] \leq e + 1 - 1 = p^{e-1}$ where $e = p^s$, and $D$ contains a point $a'$ with $[K(a') : K] \leq p^{sp([E(b)/K])}$, that is $[K(a') : K] \leq p^s$. So finally $a' \in D$ and $[E(a') : E] \leq p^{2s-1}$.

We prove now the existence of such a polynomial $f$. Fix a uniformizer $\pi_F$ of $F$, and let $E$ be the set of pairs of $e$-uples $(a, P)$ where $a = \alpha_1, \ldots, \alpha_e$ are elements of $K$, $P = P_1, \ldots, P_e$ are elements of $E[X]_e$, and $\sum i_1P_1(a) = \pi_F$. Then $E$ is not empty: we can write $\pi_F = Q(a)$ for some $Q \in K[X]_e$; now let $\alpha_1, \ldots, \alpha_e$ be a basis of $F$ over $K$, and write $Q = \sum \alpha_i P_i$ with $P_i \in E[X]_e$. For each $(a, P) \in E$ let $m(a, P) = \inf v_E(a \sigma_i P_i(a))$, so $m(a, P) \leq 1/e$. It is enough to show that there is an $(\alpha, P)$ with $m(a, P) = 1/e$. Indeed, if $v_E(\alpha_i P_i(a)) = 1/e$, let $\beta_i \in E$ with $v_E(\alpha_i) = v_E(\beta_i)$ then $\beta_iP_i$ is the $f$ we are looking for.
So choose a \((\alpha, P) \in \mathcal{E}\) with \(m = m_{(\alpha, P)}\) minimal, and with minimal number of indices \(i\) such that \(v_E(\alpha, P_i(a)) = m\). Suppose that \(n < 1/e\). Then there are at least two indices \(i\) with \(v_E(\alpha_i P_i(a)) = m\). Say for simplicity that \(v_E(\alpha_1 P_1(a)) = v_E(\alpha_2 P_2(a)) = m\). By minimality of \(e\), \(P_1\) and \(P_2\) have no root in \(D\). Let \(f = P_1/P_2\), and \(D' = f(D)\). Then \(D'\) is defined over \(E\), and contains an element \(f(a)\) of valuation \(r = v_E(P_1(a)/P_2(a)) \in \mathbb{Z}\), as \(r = v_E(\alpha_2/\alpha_1)\). Consider \(\pi^{-1}_E D'\). It does not contain 0, so it is contained in a disk \(\{z, v_E(z - c) > 0\}\) for some element \(c\) that is the Teichmueller lift of an element of \(F_p\). So \(v_E(\pi^{-1}_E P_1(a)/P_2(a) - c) > 0\). As \(\pi^{-1}_E D'\) is defined over \(E\), we have that \(c \in E\). Let \(x = c \pi_E\), then \(v_E(P_1(a) - xP_2(a)) > r + v_E(P_2(a)) = v_E(P_1(a))\). We define an element \((\alpha', P')\) of \(\mathcal{E}\) by setting \(P_1' = P_1 - xP_2\) and \(\alpha'_2 = \alpha_2 + x \alpha_1\), and \(\alpha'_i = \alpha_i\) and \(P'_i = P_i\) for all other indices. We observe that \(v_E(\alpha'_i P'_i(a)) > m, v_E(\alpha'_2 P'_2(a)) \geq m\), and all other valuations are unchanged. This contradicts the choice we made for \((\alpha, P)\) at the beginning. So in fact \(m = 1/e\).

\[\square\]

2. Some results on Hilbert-Samuel multiplicities

2.1. Hilbert-Samuel multiplicity. Let \(A\) be a noetherian local ring with maximal ideal \(m\), and \(d\) be the dimension of \(A\). Let \(M\) be a finite-type module over \(A\). We recall the definition of the Hilbert-Samuel multiplicity \(e(A, M)\) (see [Mat86, Chapter 13]). For \(n\) large enough, \(\text{len}_A(M/m^n M)\) is a polynomial in \(n\) of degree at most \(d\). We can write its term of degree \(d\) as \(e(A, M)n^d/d!\) for an integer \(e(A, M)\), which is the Hilbert-Samuel multiplicity of \(M\) (relative to \((A, m))\). We also write \(e(A)\) for \(e(A, A)\).

If \(\dim A = 1\), it follows from the definition that \(e(A, M) = \text{len}_A(M/m^n+1 M) - \text{len}_A(M/m^n M) = \text{len}_A(M/m/M/m^{n+1} M)\) for \(n\) large enough.

We give some results that will enable us to compute \(e(A)\) for some special cases of rings \(A\) of dimension 1.

Lemma 2.1.1. Let \(k\) be a field, and \((A, m)\) be a local noetherian \(k\)-algebra of dimension 1, with \(A/m = k\). Suppose that there exists an element \(z \in m\) such that \(A\) has no \(z\)-torsion and for all \(n\) large enough, \(zm^n = m^{n+1}\). Then \(e(A) = \dim_k A/(z)\).

Proof. For \(n\) large enough, we have \(m^{n+1} \subset (z)\). So the surjective map \(A \to A/(z)\) factors through \(A/m^{n+1}\) (and in particular \(\text{len}_A(A/(z))\) is finite). We have an exact sequence:

\[
A^* \to A/m^{n+1} \to A/(z) \to 0
\]

For \(n\) large enough, the kernel of the first map is \(m^n\) by the assumptions on \(z\): it contains \(m^d\), and as multiplication by \(z\) is injective, it is exactly equal to \(m^n\). So we have an exact sequence:

\[
0 \to A/m^n \to A/m^{n+1} \to A/(z) \to 0
\]

This gives \(\text{len}_A(m^n/m^{n+1}) = \text{len}_A(A/(z)) = \dim_k A/(z)\) as stated.

\[\square\]

Corollary 2.1.2. Let \(k\) be a field, and \((A, m)\) be a local noetherian \(k\)-algebra of dimension 1, with \(A/m = k\). Suppose that there exist an element \(z \in m\) such that \(A\) has no \(z\)-torsion and a nilpotent ideal \(I\) such that \(m = (z, I)\). Then \(e(A) = \dim_k A/(z)\).

Proof. We need only show that \(zm^n = m^{n+1}\) for all \(n\) large enough, as we can then apply Lemma 2.1.1. Let \(m\) be an integer such that \(I^m = 0\). Then for \(n > m\) we have \(m^n = \sum_{i=0}^m I^i z^{n-i}\), which gives the result.

\[\square\]

Let \(k\) be a field. Let \(A_1, \ldots, A_s\) be a family of local noetherian complete \(k\)-algebras of dimension 1 with maximal ideals \(V_i\) and \(A_i/V_i = k\). Let \(A\) be a local noetherian complete \(k\)-algebra with \(A/m = k\). We say that \(A\) is nearly the sum of the family \((A_i)\) if \(A = k \oplus (\bigoplus_{i=1}^s V_i)\) as a \(k\)-vector space and \(m = \bigoplus_{i=1}^s V_i\), and for all \(i, A_i \subset A\) with image
Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}_{\geq 0}^s$, we denote by $V^\alpha$ the closure of the vector space generated by elements of the form $x_1 \ldots x_s$, where $x_i$ is an element of the image $V_i^{\alpha_i}$ of the ideal $V_i^{\alpha_i}$ of $A_i$. Note that this is not in general an ideal of $A$. We also denote by $V^\alpha V_i^n$ the set $V^\beta$ where $\beta_j = \alpha_j$, except for $\beta_i = \alpha_i + n$.

**Lemma 2.1.3.** Let $k$ be a field. Let $A_1, \ldots, A_s$ be a family of local noetherian complete $k$-algebras of dimension 1 with maximal ideals $V_i$ and $A_i/V_i = k$. Suppose that for all $i$, there is an element $z_i \in A_i$ such that $A_i$ has no $z_i$-torsion and that for all $n$ large enough, $z_i V_i^n = V_i^{n+1}$.

Let $A$ be a $k$-algebra with maximal ideal $\mathfrak{m}$ that is nearly the sum of the family $(A_i)$. Moreover, suppose that there exist integers $N_0 \geq t_0$ such that for all $i$ and $j$, $V_i V_j^n \subset V_j^{n-t_0} \cap V_i^{n-t_0}$ for all $n \geq N_0$.

Then $e(A) = \sum_{i=1}^s e(A_i)$.

Note that if we had the stronger property that $V_i V_j = 0$ for all $i \neq j$ the result would be trivial.

**Proof.** Observe first that there exist integers $N \geq t$ such that for all $\alpha$, for all $i$, $V^\alpha V_i^n \subset V_i^{n-t}$ for all $n \geq N$. Indeed, $V^\alpha \subset V_{j_1} \ldots V_{j_r}$, where $\{j_1, \ldots, j_r\} \subset \{1, \ldots, s\}$ is the set of indices with $\alpha_j > 0$. Then if $n \geq rN_0$, then $V_{j_1} \ldots V_{j_r} V_i^n \subset V_i^{n-t_0}$. So we can take $N = sN_0$ and $t = sN_0$.

If $\alpha_j > N$ then $V^\alpha \subset V_j^{n-t} \subset V_j$. So if there are two different indices $i, j$ with $\alpha_i > N$ and $\alpha_j > N$ then $V^\alpha = 0$ as it is contained in $V_i \cap V_j$. If $|\alpha| > sN$ then there exists at least one $i$ with $\alpha_i > N$ so $V^\alpha = \sum (V^\alpha \cap V_j)$.

Fix some index $i$. Let $n > 0$. Then $m^n = \sum_{|\alpha| = n} V^\alpha$. So if $n > Ns$ then $(m^n \cap V_i) = \sum_{|\alpha| = n} V^\alpha \cap V_i$ and the only contributing terms are those with $\alpha_j \leq N$ for all $j \neq i$, and $\alpha_i > N$. For such an $\alpha$, we have $V^\alpha \subset V_i^{n-sN}$ as $\alpha_i \geq n - (s - 1)N$. Let $r = sN$, so that $V_i^{n-r} \subset V_i$ for all $n > r$. So for all $n > r$ and all such $\alpha$ we have $V^\alpha \subset V_i$, so finally for $n > r$ we have:

$$1) \quad (m^n \cap V_i) = \sum_{|\alpha| = n, \alpha_j \leq N \text{ if } j \neq i} V^\alpha$$

We see that $V_i^n \subset (m^n \cap V_i) \subset V_i^{n-r}$ for all $n > r$.

Note that $(m^n \cap V_i)$ is an ideal of $A_i$, which we denote by $W_{i,n}$. We know that $z_i V_i^n = V_i^{n+1}$ for all $n$ large enough, so by the formula (1) for $W_{i,n}$ we see that $z_i W_{i,n} = W_{i,n+1}$ for all $n$ large enough. In $A_i$, multiplication by $z_i$ induces an isomorphism from $V_i^n$ to $V_i^{n+1}$ and from $W_{i,n}$ to $W_{i,n+1}$, so it also induces an isomorphism from $V_i^{n-r}/W_{i,n}$ to $V_i^{n+1-r}/W_{i,n+1}$ for all $n$ large enough. Note that these vector spaces are finite-dimensional, so they have the same dimension, as dim$_k V_i^{n-r}/V_i^n$ is finite for all $n$.

We consider the inclusions

$$V_i^n \subset W_{i,n} \subset V_i^{n-r} \subset W_{i,n-r} \subset V_i^{n-2r}$$

We know that, for all $n \gg 0$, dim$_k V_i^{n-r}/V_i^n = \dim_k V_i^{n-2r}/V_i^{n-r} = e(A_i)$ and dim$_k V_i^{n-r}/W_{i,n} = \dim_k V_i^{n-2r}/W_{i,n-r}$, which gives that dim$_k W_{i,n-r}/W_{i,n} = e(A_i)$.

We now go back to $A$. For all $n \gg 0$ we have that dim$_k (m^{n-r}/m^n) = e(A)$. On the other hand, we have seen that for all $n \gg 0$, $m^n = \oplus_i (m^n \cap V_i)$, so $m^{n-r}/m^n$ is isomorphic to $\oplus_i (m^{n-r} \cap V_i)/(m^n \cap V_i) = \oplus_i (W_{i,n-r}/W_{i,n})$. So $e(A) = \sum_{i=1}^s e(A_i)$, and so $e(A) = \sum_{i=1}^s e(A_i)$.

2.2. Hilbert-Samuel multiplicity of the special fiber. Let $R$ be a discrete valuation ring with uniformizer $\pi$ and residue field $k$. 

Let $A$ be a local $R$-algebra with maximal ideal $\mathfrak{m}$, and let $M$ be an $A$-module of finite type. We denote by $\tau_R(A, M)$ the Hilbert-Samuel multiplicity of $M \otimes_R k$ as an $A \otimes_R k$-module, with respect to the ideal $\mathfrak{m} \otimes_R k$. When $M = A$ we just write $\tau_R(A)$ instead of $\tau_R(A, A)$, and we omit the subscript $R$ when the choice of the ring is clear from the context.

**Lemma 2.2.1.** Let $(T, m_T) \to (S, m_S)$ be a local morphism of local noetherian rings of the same dimension, with residue fields $k_T$ and $k_S$ respectively, then $e(T, S) \geq \len_T(S/(m_T S)^n)$.

**Proof.** Let $n \geq 0$ be an integer. Then $S/m^n_S$ is a quotient of $S/(m_T S)^n$, so $\len_T(S/m^n_S) \leq \len_T(S/(m_T S)^n)$. Moreover,

$$\len_T(S/m^n_S) = \sum_{i=0}^{n-1} \dim_k m^i_S/m^{i+1}_S = [k_S : k_T] \sum_{i=0}^{n-1} \dim_k m^i_S/m^{i+1}_S = [k_S : k_T] \len_S(S/m^n_S)$$

so finally $\len_S(S/m^n_S) \leq [k_S : k_T] \len_T(S/(m_T S)^n)$ which gives the result. □

**Proposition 2.2.2.** Let $A$ be a local complete noetherian local $R$-algebra which is a domain. Let $B \subset A[1/\pi]$ be a finite $A$-algebra. Let $k_A$ and $k_B$ be the residue fields of $A$ and $B$ respectively. Then $\tau(A) \geq [k_B : k_A]\tau(B)$.

**Proof.** Note that $B$ is also a local complete noetherian local $R$-algebra which is a domain. Indeed, $A$ is henselian and $B$ is a finite $A$-algebra, so $B$ is a finite product of local rings, and so it is a local ring as it is a domain.

It is enough to prove the result when $\pi B \subset A$, as $B$ is generated over $A$ by a finite number of elements of the form $x/\pi^n$ for $x \in A$.

We have an exact sequence of $R$-modules:

$$0 \to A \to B \to B/A \to 0$$

After tensoring by $k$ over $R$ we get the exact sequence:

$$0 \to B/A \to A \otimes_R k \to B \otimes_R k \to B/A \to 0$$

Indeed, $(B/A) \otimes_R k = B/A$, and $(B/A)[\pi] = B/A$ and $B$ is $\pi$-torsion free so $B[\pi] = 0$.

Hence we get that $\tau(A, B) = \tau(A, A)$. So we only need to show that $\tau(A, B) \geq [k_B : k_A]\tau(B)$, which follows from Lemma 2.2.1 applied to $T = A \otimes_R k$ and $S = B \otimes_R k$. □

**Remark 2.2.3.** We give some examples: Let $R = \mathbb{Z}_p, C = R[[X]]$, $A_n = R[[pX, X^n]] \subset C$ for $n \geq 1, B_n = R[[pX, pX^2, \ldots, pX^{n-1}, X^n]] \subset C$ for $n \geq 1$. We check easily that $A_n \subset B_n \subset C$ and that $C$ is finite over $A_n$, and $A_n$ is not equal to $B_n$ if $n > 2$. We compute that $\tau(A_n) = \tau(B_n) = n$, and $\tau(C) = 1$. So we see that in Proposition 2.2.2 both possibilities $\tau(B) < \tau(A)$ and $\tau(B) = \tau(A)$ can happen for $A \neq B$. See also Paragraph 4.1.3 for more examples.

2.3. **Change of ring.** We suppose now that $R$ is the ring of integers of a finite extension $K$ of $\mathbb{Q}_p$. If $K'$ is a finite extension of $K$, we denote by $R'$ its ring of integers.

**Proposition 2.3.1.** Let $K'$ be a finite extension of $K$, with ramification degree $e_{K'/K}$. Let $A$ be a local noetherian $R'$-algebra. Then $\tau_R(A) = e_{K'/K}\tau_{R'}(A)$.

**Proof.** Suppose first that $K'$ is an unramified extension of $K$, and let $k$ and $k'$ be the residue fields of $K$ and $K'$ respectively, and let $\pi$ be a uniformizer of $R$ and $R'$. Then $A \otimes_{R'} k' = A \otimes_R k = A/\pi A$. So $\tau_R(A) = e(A/\pi A) = \tau_{R'}(A)$.

Suppose now that $K'$ is a totally ramified extension of $K$. Let $u$ be an Eisenstein polynomial defining the extension, so that $R' = R[X]/u(X)$, and $\tau(X) = X^s$ where $s = [K' : K]$. Then $A \otimes_R k = A \otimes_{R'} (R') \otimes_R k = A \otimes_{R'} (k[X]/X^s) = (A \otimes_{R'} k) \otimes_k k[X]/X^s$. So $\tau_R(A) = s\tau_{R'}(A) = [K' : K]\tau_{R'}(A)$. 


For the general case, let $R_0$ be the ring of integers of the maximal unramified extension $K_0$ of $K$ in $K'$, then $e_R(A) = e_{R_0}(A)$ and $e_{R_0}(A) = [K' : K_0]e_R(A)$ which gives the result. 

We recall the following result, which is [BM02, Lemme 2.2.2.6]:

**Lemma 2.3.2.** Let $A$ be a local noetherian $R$-algebra, with the same residue field as $R$ and $A$ is complete and topologically of finite type over $R$. Let $K'$ be a finite extension of $K$, and $A' = R' \otimes_R A$. Suppose that $A'$ is still a local ring. Then $\overline{e}_R(A) = \overline{e}_{R'}(A')$.

### 3. Rigid geometry and standard subsets of the affine line

#### 3.1. Quasi-affinoids.

We recall some definitions and results from [LR00].

Let $F$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $R$. We denote by $R_{n,m} = R[[x_1, \ldots, x_n]](y_1, \ldots, y_m)$ the $F$-algebra $R[[x_1, \ldots, x_n]](y_1, \ldots, y_m) \otimes_R F$, and we say that such an algebra is quasi-affinoid. We say that it is of closed type if $n = 0$, and of open type if $m = 0$.

In general, we call a $F$-algebra quasi-affinoid if it is a quotient of an $R_{n,m}$ for some $n$ and $m$. We can attach canonically to such an algebra $A$ a rigid space $X_A$, which has the property that $A$ is the ring of bounded functions on $X$. Such a rigid space is called quasi-affinoid. If $X$ is a quasi-affinoid rigid space, we denote by $A(X)$ the set of bounded functions on $X$, and by $A^0(X)$ the set of functions on $X$ bounded by 1.

**3.1.1. $R$-subdomains.** As in the case of affinoid algebras and rigid spaces, we define some special subsets.

Let $X$ be a quasi-affinoid rigid space. Let $h, f_1, \ldots, f_n, g_1, \ldots, g_m$ be elements of $A(X)$ that generate the unit ideal of $A(X)$. A quasi-rational subdomain of $X$ is a subset $U$ of the form $\{x, |f_i(x)| \leq |h(x)| \forall i \text{ and } |g_i(x)| < |h(x)| \forall i\}$. In contrast to the case of affinoid rigid spaces, it is not necessarily true that a quasi-rational subdomain of a quasi-rational subdomain of $X$ is itself a rational subdomain of $X$.

In particular, the set of rational subdomains of $X$ is defined as the smallest set of subsets of $X$ that contains $X$ and is closed by the operation of taking a quasi-rational subdomain of an element of this set.

An important result is the following:

**Theorem 3.1.1** (Theorem 6.2.2 of [LR00]). Let $X$ be a quasi-affinoid rigid space, and $U \subset X$ be the image of a map $Z \to X$ of quasi-affinoid spaces that is an open immersion. Then $U$ is a finite union of $R$-subdomains of $X$.

#### 3.2. Standard subsets and quasi-affinoid subdomains of the affine line.

**3.2.1. Definition of standard subsets.**

**Definition 3.2.1.** We say that a subset of $\overline{\mathbb{Q}}_p$ is a rational disk if it is a set of the form $\{x, |x - a| < |b|\}$ with $a, b \in \overline{\mathbb{Q}}_p$, $b \neq 0$ (open disk), or of the form $\{x, |x - a| \leq |b|\}$ with $a, b \in \overline{\mathbb{Q}}_p$, $b \neq 0$ (closed disk).

From now on, when we write "disk" we always mean "rational disk".

**Definition 3.2.2.** We say that a subset $X$ of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ is a connected standard subset if it is of one of the following forms:

1. $D_0 \setminus \bigcup_{i=1}^n D_i$ where the $D_i$ are rational disks, $\infty \notin D_0$, $D_0 \neq D_i$ for all $i > 0$, $D_i \subset D_0$, and $D_i$ and $D_j$ are disjoint if $i \neq j$ and $i, j > 0$ (bounded connected standard subset).
(2) $\mathbb{P}^1(\mathbb{Q}_p) \setminus \bigcup_{i=1}^n D_i$ where the $D_i$ are rational disks, and $D_i$ and $D_j$ are disjoint if $i \neq j$ (unbounded connected standard subset).

One checks easily that a bounded connected standard subset is an $R$-subdomain of the rigid affine line, and even a quasi-rational subset. Note that if $X$ is a connected standard subset, then the disks $D_i$, their radii, and the integer $n$ are entirely determined.

A connected standard subset is called of closed type if $D_0$ is a closed disk (in the bounded case), and the $D_i$ are open disks for $i > 0$. A connected standard subset is called of open type if $D_0$ is an open disk (in the bounded case), and the $D_i$ are closed disks for $i > 0$.

A subset of $\mathbb{P}^1(\mathbb{Q}_p)$ is called standard if it is a finite union of disjoint connected standard subsets. It is called a standard subset of open type if it is a finite union of disjoint connected standard subsets of open type, and we define similarly a standard subset of closed type.

If $X$ is a standard subset of open type of $\mathbb{P}^1(\mathbb{Q}_p)$ it can be written uniquely as a disjoint finite union of connected standard subsets of open type, which we call the connected components of $X$.

We check easily the following result:

**Lemma 3.2.3.** Let $X$ and $Y$ be two connected standard subsets of closed (resp. open) type. If $X \cap Y \neq \emptyset$ then $X \cap Y$ and $X \cup Y$ are connected standard subsets of closed (resp. open) type. As a consequence, any finite union of connected standard subsets of closed (resp. open) type is a standard subset of closed (resp. open) type.

Following [LR96, Definition 4.1], we define:

**Definition 3.2.4.** A special subset of $\mathbb{Q}_p$ is a subset of one of the following form:

1. $\{x, |b| < |x-a| < |c|\}$ for some $a, b, c \in \mathbb{Q}_p$ with $b \neq 0$
2. $\{x, |x-a| \leq |b|$ and for all $i \in \{1, \ldots, N\}, |x-\alpha_i| \geq |\beta_i|\}$ for some $a, b, \alpha_i, \beta_i \in \mathbb{Q}_p$ with $b \neq 0$ (that is, a connected standard subset of closed type).

Then we have the following result:

**Theorem 3.2.5** (Theorem 4.5 of [LR96].) An $R$-subdomain of $\mathbb{Q}_p$ is a finite union of special sets.

**Corollary 3.2.6.** Let $X$ be a quasi-affinoid space of open type. Let $\mathcal{D}$ be a quasi-affinoid space corresponding to some rational disk of $\mathbb{Q}_p$. Let $\phi : X \to \mathcal{D}$ be a quasi-affinoid map that is an open immersion. Let $U$ be the image of $X$ in $\mathbb{Q}_p$. Then $U$ is a bounded standard subset of open type.

If moreover $X$ is geometricaly Zariski-connected then $U$ is a connected standard subset of open type.

**Proof.** We recall the following property of quasi-affinoid spaces of open type ([LR00, Proposition 5.3.9]), which we will use repeatedly: let $f \in \mathcal{A}(X)$, then the set $\{x, \|f(x)\| = \|f\|_\infty\}$ is a union of Zariski components of $X$.

The set $U$ is an $R$-subdomain of a disk by Theorem 3.1.1, hence a finite union of special sets by Theorem 3.2.5. We write $U$ as $(\bigcup_{i=1}^m Y_i) \cup (\bigcup_{i=1}^n C_i)$ where the $Y_i$ are as in (2) of Definition 3.2.4 and the $C_i$ as in (1) of this definition. Note that the $Y_i$ are connected standard subsets of closed type, hence we can assume that they are pairwise disjoint by Lemma 3.2.3 and that each of them is non-empty. So there exists an $\varepsilon > 0$ such that for all $x \in Y_i, y \in Y_j$ with $i \neq j$, we have $|x-y| \geq \varepsilon$.

Fix an $i$, and write $Y_i = D(a, r_0) \cup \bigcup_{j=1}^n D(\alpha_j, r_j)$. Suppose that there is an $\eta > 0, \eta < \varepsilon$ such that $U \cap \{x, r_0 < |x-a| < r_0 + \eta\}$ is empty. Then $X' = \phi^{-1}(D(a, r_0 + \eta)^-)$
is union of Zariski components of $X$, hence a quasi-affinoid space of open type. Consider $f(x) = x - a'$ for any $a' \in D(a, r_0)^+$, then $\|\phi^* f\|_X = r_0$ (where $\phi^* f = f \circ \phi$). Then $Y_i \subset \{x, |x - a'| = r_i\}$ for all such $a'$, as $Y_i$ is Zariski connected. So $Y_i = \emptyset$ as we can take $a' \in Y_i$, a contradiction.

So for all $\eta > 0$ with $\eta < \varepsilon$, the set $U \cap \{x, r_0 < |x - a| < r_0 + \eta\}$ is not empty. It does not meet any of the $Y_j$ for $j \neq i$. This means that for some $\ell_0$, the set $C_{\ell_0}$ is of the form $\{x, p_1 < |x - a'| < p_2\}$ for some $p_1 \leq r_0 < p_2$ and $a' \in D(a, r_0)^+$. Similarly, we also have for all $j$ that for some $\ell_j$, the set $C_{\ell_j}$ is of the form $\{x, p_{1,j} < |x - a'| < p_{2,j}\}$ for some $a' \in D(a_j, r_j)^-$ and $p_{1,j} < r_j \leq p_{2,j}$. We set $Y'_i = D(a', p_2^{-} \cup \cup_{j=1}^{N} D(a'_j, p_{1,j})^+)$. Then $Y_i \subset Y_i' \subset U$, and $Y_i'$ is a standard subset of open type.

As we can do this for all $i$, we have written $U$ as $(\cup_i Y'_i) \cup (\cup_i C_i)$ which is finite union of standard subsets of open type, as stated.

Assume now that $X$ is Zariski connected. We can write $U$ uniquely as a disjoint union of connected standard subsets of open type. Let $Y$ be one of these subsets, which we write as $D(a_0, r_0)^- \setminus \cup_{i=1}^{n} D(a_i, r_i)^+$. It is enough to show that $U \subset D(a_0, r_0)^-$. The set $\{x, |x - a| = r_0\}$ is a disjoint union of an infinite number of open disks of radius $r_0$. Hence one of these disks, say $D(b, r_0)^-$, does not meet $U$. Let $f(x) = 1/(x - b)$. Then $\|\phi^* f\|_X = 1/r_0$, and $|\phi^* f(x)| = 1/r_0$ for all $x \in \phi^{-1}(D(a, r_0)^+)$, and $|\phi^* f(x)| < r_0$ if $|\phi(x) - a| > r_0$. So $X = \phi^{-1}(D(a, r_0)^+)$ as $X$ is Zariski connected. Now let $f(x) = x - a$. Then $\|\phi^* f\|_X = r_0$. For all $x \in \phi^{-1}(D(a_0, r_0)^-)$, we have $\phi^* f(x) < \|\phi^* f\|_X$. As $X$ is Zariski connected, this means that $X = \phi^{-1}(D(a_0, r_0)^-)$, what we wanted. \[\square\]

### 3.3. Rings of functions on standard subsets of open type.

From now on, we will be only interested in standard subsets that are of open type. So we will simply write standard subset and connected standard subsets for standard subsets of open type and connected standard subsets of open type.

#### 3.3.1. Rings of functions of standard subsets.

Let $X \subset \overline{\mathbb{Q}}$ be a bounded connected standard subset. Then $X$ is the set of points of a well-defined quasi-affinoid space $\mathcal{X}$ which is a rational subdomain of an open disk (defined by strict inequalities). We define by $\mathcal{A}(X)$ the set of bounded analytic functions on $\mathcal{X}$ and $\mathcal{A}^0(X)$ the set of analytic functions on $\mathcal{X}$ bounded by 1.

Let $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be an unbounded connected standard subset, which is not equal to all of $\mathbb{P}^1(\overline{\mathbb{Q}})$. Let $f$ be a homography with $\overline{\mathbb{Q}}$-coefficients with its pole outside of $X$, then $Y = f(X)$ is a bounded connected standard subset of $\overline{\mathbb{Q}}$, so $\mathcal{A}(Y)$ and $\mathcal{A}^0(Y)$ are well-defined. We define $\mathcal{A}(X)$ and $\mathcal{A}^0(X)$ to be the functions of $X$ of the form $u \circ f$ for $u \in \mathcal{A}(Y)$ and $\mathcal{A}^0(Y)$ respectively. It is clear that this does not depend on the choice of $f$, as different choices of $f$ give rise to bounded connected standard subsets coming from isomorphic quasi-affinoids.

Let now $X$ be a standard subset. It can be written uniquely as $X = \cup_{i=1}^{n} X_i$ where the $X_i$ are disjoint connected standard subsets. Then we set $\mathcal{A}(X) = \oplus_{i=1}^{n} \mathcal{A}(X_i)$ and $\mathcal{A}^0(X) = \oplus_{i=1}^{n} \mathcal{A}^0(X_i)$.

#### 3.3.2. Subsets defined over a field.

If $E$ is a finite extension of $\mathbb{Q}$, denote by $G_E$ its absolute Galois group. We say that $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ is defined over $F$ if $\sigma(X) = X$ for all $\sigma \in G_F$.

Let $E$ be a finite extension of $\mathbb{Q}$. The field of definition of $X$ over $E$ is the fixed field of $\{\sigma \in G_E, \sigma(X) = X\}$. The field of definition of $X$ is the field of definition of $X$ over $\mathbb{Q}$. Then $X$ is defined over $F$ if and only if $F$ contains the field of definition of $X$. 
Proposition 3.3.2. Let $X$ be a connected standard subset of $\mathbb{Q}_p$. Suppose that $X$ is defined over $\mathbb{Q}_p$. If $F$ is a finite extension of $E$, we write $A_F(X)$ and $A_F^0(X)$ for $A(X)^{Gr}$ and $A(X)^{Gr}$, respectively. So for example, if $X = D(0, 1)^+$, then $X$ is defined over $\mathbb{Q}_p$, and $A^0(X)$ is $O_{\mathbb{Q}_p}[x]$ with $G_{\mathbb{Q}_p}$ acting on the coefficients. So $A_F(X) = O_F[\mathbb{Q}_p]$ for any finite extension $F$ of $\mathbb{Q}_p$. 

Proposition 3.3.1. Let $X = D(a_0, r_0)^- \setminus \bigcup_{i=1}^n D(a_i, r_i)^+$ be a connected standard subset of $\mathbb{Q}_p$, with $a_i \in \mathbb{Q}_p$ for all $i$, and the sets $D(a_i, r_i)^+$ are pairwise disjoint for $i > 0$. For each $i$, let $t_i \in \mathbb{Q}_p$ be such that $|t_i| = r_i$. Let $E$ be the finite extension of $\mathbb{Q}_p$ generated by the elements $a_i$ and $t_i$. Then $X$ is defined over $E$, and for any finite extension $F/E$, we have:

$$A_F(X) = \{ f, f(x) = \sum_{i \geq 0} c_i,0 \left( \frac{x - a_0}{t_0} \right)^i + \sum_{j=1}^n \sum_{i \geq 0} c_{i,j} \left( \frac{t_j}{x - a_j} \right)^i \}$$

if $X$ is bounded and

$$A_F(X) = \{ f, f(x) = c_0 + \sum_{j=1}^n \sum_{i \geq 0} c_{i,j} \left( \frac{t_j}{x - a_j} \right)^i \}$$

if $X$ is unbounded.

Moreover, $\|f\|_X = \sup_{t_i,j} |c_{i,j}|$ if $f$ is written as above. If we write $f_0 = \sum_{i \geq 0} c_{i,0} \left( \frac{x - a_0}{t_0} \right)^i$ (or $f_0 = c_0$ in the unbounded case), and $f_j = \sum_{i \geq 0} c_{i,j} \left( \frac{t_j}{x - a_j} \right)^i$ for $j > 0$ so that $f = \sum_{i=0}^n f_i$, then $\|f\|_X = \max_{0 \leq i \leq n} \|f_i\|_X$.

In particular, $f \in A_F^0(X)$ if and only if $c_{i,j} \in O_F$ for all $i, j$.

Proof. The fact that any element of $A_F(X)$ can be written this way is a consequence of the description of the ring of functions of a quasi-rational subsets, as described in [LR00 Proposition 5.3.2].

Let $f$ be as in the statement of the Proposition, and let $M = \sup_{t_i,j} |c_{i,j}|$. Then it is clear from the formula that for all $x \in X$, the series defining $f(x)$ converges and that $|f(x)| \leq M$.

As $F$ has discrete valuation, the sup defining $M$ is in fact a maximum. Let us show that $\|f\|_X = M$. Fix first a $j$ such that there exists an $i$ with $|c_{i,j}| = M$. For simplicity of notation we will assume that $j = 0$, the other cases being similar. Let $i_0$ be the smallest index such that $|c_{i_0,0}| = M$. Let $\rho < r_0$ be such that $|a_j| < \rho$ for all $j > 0$. Note that for all $j > 0$, we have $r_j < \rho$ and $D(a_j, r_j)^+ \subset D(0, \rho')$. Fix $\rho'$ with $\rho' < \rho < r_0$. Let $x \in \mathbb{Q}_p$ with $\rho' < |x| < r_0$, so that $x \in X$.

For all $j > 0$ and all $i$ we have $|c_{i,j}(t_j/(x - a_j))| \leq M(\rho'/\rho)^i \leq M(\rho'/\rho) < M$. For $i < i_0$, we have $|c_{i,0}(x/t_0)| < M'$ for some $M' < M$ as $|c_{i,0}| < M$. For $i < i_0$ we have $|c_{i,j}(x/t_0)| < M(|x|/r_0)^{a_0}$. Finally $|c_{i_0,0}(x/t_0)| = M(|x|/r_0)^{a_0}$. By taking $|x|$ close enough to $r_0$, we get that $|c_{i_0,0}(x/t_0)| > |c_{i_0,0}(x/t_0)|$ for all $i \neq i_0$, and $|c_{i_0,0}(x/t_0)| > |c_{i_0,0}(x/(x - a_j))|$ for all $j > 0$ and all $i$. So $|f(x)| = |c_{i_0,0}(x/t_0)| = M(|x|/r_0)^{a_0}$ can get arbitrarily close to $M$. So finally $\|f\|_X = M$, and the result follows.

Remark 3.3.2. The description of $A_F(X)$ is similar to the result given by the Mittag-Leffler theorem (see [Kra83]) in the situations studied by Krasner. Our situation is slightly different as we are considering subspaces of $\mathbb{P}^1(\mathbb{Q}_p)$ that are 'open', and simpler as we have only a finite number of 'holes'.

\[\square\]
Proposition 3.3.3. Let $X$ be a standard subset defined over $E$. Let $F$ be a finite extension of $E$. Then $A_F(X) = F \otimes_E A_E(X)$, and $O_F \otimes_{O_E} A_F^0(X) \subset A_E^0(X)$, with $A_F^0(X)$ finite over $O_F \otimes_{O_E} A_E^0(X)$. If $F/E$ is unramified, then this inclusion is an isomorphism.

Note that we do not assume that the conditions of Proposition 3.3.1 are satisfied.

Proof. We define a map $\phi : F \otimes_E A_E(X) \to A_F(X)$ by $\phi(a \otimes f) = af$. Let us describe the inverse $\psi$ of $\phi$. Let $Q = G_E/G_F$. If $a \in F$ and $f \in A_F(X)$, $\sigma(a)$ and $\sigma(f)$ are well-defined for $\sigma \in Q$ as $a$ and $f$ are invariant by $G_F$. Moreover, for $a \in F$, we have that $\text{tr}_{F/E}(a) = \sum_{\sigma \in Q} \sigma(a)$.

Let $(e_1, \ldots, e_n)$ be a basis of $F$ over $E$, and $(u_1, \ldots, u_n)$ be the dual basis with respect to $\text{tr}_{F/E}$, that is, $\text{tr}_{F/E}(e_i u_j) = \delta_{i,j}$. One checks easily that for $\sigma \in G_E$, we have $\sum_{i=1}^n e_i \sigma(u_i) = 1$ if $\sigma \in G_F$, and 0 otherwise.

For $f \in A_F(X)$, we set $t_i(f) = \sum_{\sigma \in Q} \sigma(u_i)$. Let $\psi(f) = \sum_{i=1}^n e_i \otimes t_i(f)$. Let us check that $\psi$ is the inverse of $\phi$. Let $f \in A_F(X)$, and $f' = \phi(\psi(f))$. Then $f' = \sum_{i=1}^n e_i \sum_{\sigma \in Q} \sigma(u_i) f = \sum_{\sigma \in Q} \sigma(f) (\sum_{i=1}^n e_i \sigma(u_i))$, so $f' = f$. Let $f \in A_E(X)$, and $a \in F$. Let $g = \phi(a \otimes f)$. Then $t_i(g) = \text{tr}_{F/E}(au_i)f$, as $\sigma(f) = f$ for all $\sigma \in Q$. So $\psi(g) = \sum_{i=1}^n e_i \otimes \text{tr}_{F/E}(au_i)f = (\sum_{i=1}^n e_i \text{tr}_{F/E}(au_i)) \otimes f = \text{tr}_{F/E}(au_i) \in E$. Then we check that $\sum_{i=1}^n e_i \text{tr}_{F/E}(au_i) = a$, so $\psi(\phi(a \otimes f)) = a \otimes f$. So we see that $\psi$ is the inverse map of $\phi$, so $\phi$ is an isomorphism.

We see that $\phi$ induces a map $\phi^0$ from $O_F \otimes_{O_E} A_F^0(X)$ to $A_E^0(X)$. When $F/E$ is unramified, we can choose $(e_i)$ and $(u_i)$ to be in $O_F$, and in this case the restriction $\psi^0$ of $\psi$ to $A_F^0(X)$ maps into $O_F \otimes_{O_E} A_E^0(X)$, and so $\psi^0$ is the inverse map of $\phi^0$, and so $\phi^0$ is an isomorphism. $\square$

3.3.4. Some algebraic results. Let $X$ be a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}_p})$ that is defined over $E$ for some finite extension $E$ of $\mathbb{Q}_p$. Let $F$ be a finite extension of $E$. We say that $X$ is irreducible over $F$ if it can not be written as a finite disjoint union of standard subsets of $\mathbb{P}^1(\overline{\mathbb{Q}_p})$ that are defined over $F$. There exists a unique decomposition of $X$ as a finite disjoint union of standard subsets of $\mathbb{P}^1(\overline{\mathbb{Q}_p})$ that are irreducible over $F$. A standard subset is connected if and only if it is irreducible over any field of definition.

Lemma 3.3.4. Let $X$ be a connected standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}_p})$. Then $A(X)$ is a domain, and $A^0(X)$ is a local ring. Suppose that moreover $X$ is defined over $E$. Then $A_E^0(X)$ is a local ring which has the same residue field as $E$.

Proof. Let $\mathfrak{m}$ be the ideal in $A^0(X)$ of functions $f$ such that $|f(x)| < 1$ for all $x \in X$. Then $\mathfrak{m}$ is a closed ideal, and it is maximal. Indeed, consider the description of $A(X)$ given in Proposition 3.3.1. We see that $\mathfrak{m}$ contains the constant functions with values in $\mathfrak{m}_{C_p}$, $(x - a_0)/t_0$ and the $t_i/(x - a_i)$. So $A^0(X) \setminus \mathfrak{m}$ is the set of non-zero constant functions with values in $O_{C_p}^\times$, so all functions that are not in $\mathfrak{m}$ are units.

For $A_E^0(X)$, note that the set of constant functions on $X$ that are in $A_E^0(X)$ is $O_E$. $\square$

Lemma 3.3.5. Let $X$ be defined and irreducible over $E$, and let $X = \bigcup_{i=1}^n X_i$ its decomposition in a finite union of connected standard subsets. Let $F$ be the field of definition of $X_1$ over $E$. Then the restriction map $A^0(X) \to A^0(X_1)$ induces an $O_E$-linear isomorphism $A_E^0(X) \to A_E^0(X_1)$.

Note in particular that: $[F : E]$ is the number of connected components of $X$, and the isomorphism class of $A_E^0(X_1)$ as an $O_E$-algebra does not depend on the choice of $X_1$.

Proof. The group $G_E$ acts transitively on the set of the $(X_i)$ as $X$ is irreducible, and $G_F$ is the stabilizer of $X_1$. We fix a system $(\sigma_i)$ of representatives of $G_E/G_F$, numbered so that $\sigma_i(X_1) = X_i$ for all $i$. $\square$
Let $f$ be an element of $\mathcal{A}_E^0(X)$. First note that $f$ is invariant under the action of $G_F$, so $f|_{X_i}$ is in $\mathcal{A}_E^0(X_1)$.

Moreover, we have that for all $x \in X_i$,

$$f(x) = \sigma_i((\sigma_i^{-1}f)(\sigma_i^{-1}(x))) = \sigma_i(f|_{X_i}(\sigma_i^{-1}x))$$

So $f|_{X_i}$ is entirely determined by $f|_{X_1}$, so the restriction map is injective, and moreover for any $f \in \mathcal{A}_E^0(X_1)$ the formula above defines an element of $\mathcal{A}_E^0(F)$, so the restriction map is bijective.

**Corollary 3.3.6.** If $X$ is defined and irreducible over $E$ then $\mathcal{A}_E(X)$ is a domain, and $\mathcal{A}_E^0(X)$ is a local ring.

**Proof.** We apply Lemma 3.3.5 $\mathcal{A}_E^0(X)$ is isomorphic as a ring to $\mathcal{A}_E^0(X_1)$, which is local.

**Definition 3.3.7.** If $X$ is defined and irreducible over $E$, we denote by $k_{X,E}$ the residue field of $\mathcal{A}_E^0(X)$.

By construction, $k_{X,E}$ is a finite extension of $k_E$. In the notation of Lemma 3.3.5 we have $k_{X,E} = k_F$ (which does not depend on the choice of $X_1$).

### 4. Complexity of standard subsets

#### 4.1. Algebraic complexity of a standard subset over a field of definition.

**4.1.1. Definition.** Recall that we defined $\mathfrak{r}$ in Section 2.2

**Definition 4.1.1.** Let $X$ be a standard subset of $\mathbb{P}^1(\overline{k}_p)$ that is defined over $E$. If $X$ is irreducible over $E$, we define the complexity of $X$ over $E$ to be:

$$c_E(X) = [k_{X,E} : k_E]\mathfrak{r}_{O_E}(\mathcal{A}_E^0(X))$$

In general, let $X = \bigcup_{i=1}^r X_i$ be the decomposition of $X$ as a disjoint union of standard subsets that are defined and irreducible over $E$. We define the complexity of $X$ over $E$ to be $c_E(X) = \sum_{i=1}^r c_E(X_i)$.

The above definition makes sense as $\mathcal{A}_E^0(X)$ is a complete noetherian local $\mathcal{O}_E$-algebra if $X$ is irreducible over $E$ by Corollary 3.3.6.

Note that in particular if $X$ is connected then $c_E(X) = \mathfrak{r}_{O_E}(\mathcal{A}_E^0(X))$ as $k_{X,E} = k_E$ in this case.

**4.1.2. Some general results on algebraic complexity.** We now give explicit formulas for the complexity. It is enough to give such formulas for subsets $X$ that are irreducible over $E$.

**Proposition 4.1.2.** In the situation of Proposition 3.3.3, we have $c_E(X) = [F : E]c_F(X_1)$.

Note that $c_F(X_1)$ does not depend on the choice of $X_1$ among the connected components.

**Proof.** Let $e_{F/E}$ be the ramification degree of $F/E$. We have that $\mathcal{A}_E^0(X_1) = \mathcal{A}_E^0(X)$ as $\mathcal{O}_E$-algebras, and $k_{X,E} = k_{X_1,F} = k_F$. So $c_E(X) = [k_F : k_E]\mathfrak{r}_{O_E}(\mathcal{A}_E^0(X)) = [k_F : k_E]\mathfrak{r}_{O_E}(\mathcal{A}_E^0(X_1))$ which is equal to $[k_F : k_E]e_{F/E}\mathfrak{r}_{O_F}(\mathcal{A}_F^0(X_1)) = [F : E]c_F(X_1)$ by Proposition 2.3.1.

**Proposition 4.1.3.** Let $X$ be a connected standard subset defined over $E$, and $F$ a finite extension of $E$. Then $c_E(X) \geq c_F(X)$ with equality when $F/E$ is unramified.

**Proof.** From Proposition 2.3.2 we see that $\mathfrak{r}(\mathcal{O}_F \otimes_{\mathcal{O}_E} \mathcal{A}_E^0(X)) = \mathfrak{r}(\mathcal{A}_E^0(X)) = c_E(X)$, and from Propositions 3.3.3 and 2.2.2 we see that $\mathfrak{r}(\mathcal{O}_F \otimes_{\mathcal{O}_E} \mathcal{A}_E^0(X)) \geq \mathfrak{r}(\mathcal{A}_F^0(X))$ with equality when $F/E$ is unramified.
Proposition 4.1.4. Let $X$ be a standard subset defined over $E$, and $F$ a finite extension of $E$. Then $c_E(X) \geq c_F(X)$ with equality when $F/E$ is unramified.

Proof. By additivity of the complexity we can assume that $X$ is irreducible over $E$. Write $X = \bigcup_{i=1}^r X_i$ where each $X_i$ is connected. Let $E_i$ be the field of definition of $X_i$ over $E$, so that $c_{E_i}(X_i) = uc_{E_i}(X_i)$. Then $FE_i$ is the field of definition of $X_i$ over $F$. Suppose that the action of $G_F$ on the set of the irreducible components of $X$ has $r$ orbits, with representatives say $X_1, \ldots, X_r$. Then $c_{F}(X) = \sum_{i=1}^r [FE_i : F]c_{FE_i}(X_i)$. We have that $c_{FE_i}(X_j) \leq c_{E_i}(X_j)$ by Proposition 4.1.3 and $c_{E_i}(X_j)$ is independent of $j$, and equal to $(1/n)c_E(X_i)$. Moreover, $[FE_i : F]$ is the cardinality of the orbit of $X_j$, so $\sum_{j=1}^r [FE_i : F] = n$. Finally we get that $c_{F}(X) \leq c_{E}(X)$, with equality if and only if $c_{FE_i}(X_j) = c_{E_i}(X_j)$ for all $j$, which happens in particular if $F/E$ is unramified. □

4.1.3. Does $c_{E}(X)$ characterize $A^0_E(X)$? We ask the following question: Let $X$ be defined and irreducible over $E$. Let $R \subset A^0_E(X)$ be a local, noetherian, complete, $O_E$-flat $O_E$-subalgebra of $A^0_E(X)$, such that $R[1/p] = A_E(X)$. Suppose moreover that $R$ and $A^0_E(X)$ both have residue field $k_E$, and $\overline{\sigma}(R) = \overline{\sigma}(A^0_E(X))$, that is $\overline{\sigma}(R) = c_E(X)$. Do we have $R = A^0_E(X)$?

It follows from [BM02, Lemme 5.1.8] that the equality holds if $c_E(X) = 1$, and in this case both rings are isomorphic to $O_E[[x]]$, and $X$ is a disk of the form $\{x, |x - a| < |b|\}$ for some $a, b \in E$.

But as soon as $c_E(X) > 1$ there are counterexamples. We give a few, with $E = \mathbb{Q}_p$.

1. Let $X = \{x, 0 < v_p(x) < 1\}$. Then $A^0_{\mathbb{Q}_p}(X)$ is isomorphic to $\mathbb{Z}_p[[x,y]]/(xy - p)$. Let $R$ be the closure of the subring generated by $px, py$ and $x - y$. Here $\overline{\sigma}(R) = c_{\mathbb{Q}_p}(X) = 2$.

2. Let $X = \{x, v_p(x) > 1/2\}$. Then $A^0_{\mathbb{Q}_p}(X)$ is isomorphic to $\mathbb{Z}_p[[x,y]]/(x^2 - py)$. Let $R$ be the closure of the subring generated by $y$ and $px$. Here $\overline{\sigma}(R) = c_{\mathbb{Q}_p}(X) = 2$.

3. Let $X = \{x, |x - \pi| < |\pi|\}$ where $\pi^p = p$. Then $A^0_{\mathbb{Q}_p}(X)$ is isomorphic to $\mathbb{Z}_p[[x,y]]/(x^p - p(y + 1))$. Let $R$ be the closure of the subring generated by $y$ and $px$. Here $\overline{\sigma}(R) = c_{\mathbb{Q}_p}(X) = p$.

4.2. Computations of the algebraic complexity in some special cases.

4.2.1. Preliminaries. If $P \in E[X]$, and $a \in \mathbb{C}_p$, let $P_a(X) = P(X + a) \in \mathbb{C}_p[X]$.

Lemma 4.2.1. Let $D$ be an open disk defined over $E$, let $s$ be the smallest degree over $E$ of an element in $D$. Let $a$ be an element of $D$ of degree $s$ over $E$. Let $\lambda \in \mathbb{R}$ be such that $D = \{x, v_E(x - a) > \lambda\}$.

Let $P \in E[X]_{<s}$, and write $P_a(X) = \sum_{i=0}^{s-1} b_i X^i$. Then: $v_E(b_i) \geq v_E(b_0) - i\lambda$ for all $i$. In particular, if $v_E(b_0) \geq 0$, then $v_E(b_i) \geq -i\lambda$ for all $i > 0$, and if $v_E(b_0) > 0$, then $v_E(b_i) > -i\lambda$ for all $i > 0$.

Proof. Consider the Newton polygon of $P_a$: if the conclusion of the Lemma is not satisfied, then it has at least one slope $\mu$ which is $< -\lambda$. So $P_a$ has a root $y$ of valuation $-\mu > \lambda$. Let $b = a + y$, then $b$ is a root of $P$, so of degree $< s$ over $E$. On the other hand, $v_E(b - a) = v_E(y) > \lambda$ so $b$ is in $D$, which contradicts the definition of $s$. □

A similar proof shows:

Lemma 4.2.2. Let $D$ be a closed disk defined over $E$, let $s$ be the smallest degree over $E$ of an element in $D$. Let $a$ be in $D$ of degree $s$ over $E$. Let $\lambda \in \mathbb{R}$ be such that $D = \{x, v_E(x - a) \geq \lambda\}$.

Let $P \in E[X]_{<s}$, and write $P_a(X) = \sum_{i=0}^{s-1} b_i X^i$. Then: $v_E(b_i) > v_E(b_0) - i\lambda$ for all $i > 0$. In particular, if $v_E(b_0) \geq 0$, then $v_E(b_i) > -i\lambda$ for all $i > 0$. 

Let $L/Q_p$ be a finite extension. Let $f \in \mathcal{O}_L[[T]]$, $f = \sum_{i \geq 0} f_i T^i$. We say that $f$ is regular of degree $n$ if $f_n \in \mathcal{O}_L^*$ and $f_m \in \mathfrak{m}_L$ for all $m < n$. We recall the following result (see for example [Was97], Proposition 7.2):

**Lemma 4.2.3** (Weierstrass Division Theorem). Let $f \in \mathcal{O}_L[[T]]$ that is regular of degree $n$, and $g \in \mathcal{O}_L[[T]]$. Then there exists a unique pair $(q, r)$ with $q \in \mathcal{O}_L[[T]], r \in \mathcal{O}_L[T]_{<n}$ and $g = qf + r$.

4.2.2. Open disks.

**Proposition 4.2.4.** Let $D$ be an open disc of radius $r \in p^2$ defined over $E$. Let $s$ be the smallest ramification degree of $E(a)/E$ for $a \in D$. Let $t$ be the smallest positive integer such that $r^t \in [E(a)^\times]$. Then $c_E(D) = st$.

**Proof.** Let $a \in D$ be as in the statement. As the complexity does not change by unramified extensions by Proposition 4.1.1 we can enlarge $E$ so that $E(a)/E$ is totally ramified. Let $\mu$ be the minimal polynomial of $a$ over $E$, so that $\mu$ has degree $s$. Write $F = E(a)$. For $\nu \in \mathbb{Q}$, let $F_{\nu}$ be the set $\{x \in F, v_E(x) \geq \nu\}$ (so that $F_0 = \mathcal{O}_F$).

Let $\lambda$ be such that $D = \{x, v_E(x - a) > \lambda\}$. Let also $\rho \in F$ such that $v_E(\rho) = st\lambda$, which is possible by the condition on $r$.

Let $L$ be a Galois extension of $E$ containing $F$ and an element $u$ such that $v_E(u) = \lambda$. Then $A^n_0(L) = \mathcal{O}_L[[T]]$, with $T$ corresponding to $(x - a)/u$.

Let $\mathcal{E}_n$ be the subset of $E[X]_{<s}$ of polynomials that can be written as $\sum_{i=0}^{s-1} b_i (X - a)^i$ with $v_E(b_i) \geq -(i + ns)\lambda$. Note that by Lemma 4.2.1, $\mathcal{E}_n$ is the set of polynomials in $E[X]_{<s}$ with $v_E(b_i) \geq -ns\lambda$. In fact $\mathcal{E}_n$ is in bijection with the set $F_{-ns\lambda}$ by $P \mapsto P(a)$, as any element of $F$ can be written uniquely as $P(a)$ for some $P \in E[X]_{<s}$.

Note that $\rho^{-1} \in F_{-st\lambda}$. We fix $R \in \mathcal{E}_t$ the unique polynomial such that $R(a) = \rho^{-1}$.

We set $\alpha = R\mu^t$. We check that $\alpha$ is regular of degree $st$ when seen as an element of $A^n_0(L) = \mathcal{O}_L[[T]]$.

Let also $\mathcal{E}'$ be the subset of $E[X]_{<st}$ of polynomials that can be written as $\sum_{i=0}^{st-1} b_i (X - a)^i$ with $v_E(b_i) \geq -i\lambda$.

Then

$$A^n_E(D) = \{ \sum_{n \geq 0} P_n \alpha^n, P_n \in \mathcal{E}' \}$$

and any element of $A^n_0(D)$ can be written uniquely in such a way. Indeed: Let $f \in A^n_0(D)$, which we see as an element of $A^n_0(L) = \mathcal{O}_L[[T]]$. Applying repeatedly the Weierstrass Division Theorem, $f$ can be written uniquely as $\sum_{n \geq 0} P_n \alpha^n$ with $P_n \in \mathcal{O}_L[T]_{<st}$. The fact that $f$ is in $A^n_0(D)$ means that $f$ is invariant under $\text{Gal}(L/E)$. As $\alpha$ itself is invariant under this group, this means that each $P_n$ is invariant, and so $P_n \in \mathcal{E}'$ (where we see $\mathcal{E}' \subset \mathcal{O}_L[T]_{<st}$ by $T = (X - a)/u$).

We observe that $\mathcal{E}' = \bigoplus_{j=0}^{t-1} \mu^j E_j$. For $0 \leq i < t$, let $(U_{i,j})_{1 \leq j \leq s}$ be a basis of $\mathcal{E}_i$ as an $\mathcal{O}_E$-module, where we take $U_{0,1} = 1$, and $v_E(U_{0,j}(a)) > 0$ for $j > 1$. We can satisfy this condition as taking a basis of $\mathcal{E}_i$ is the same as taking a basis of $\mathcal{O}_F$ over $\mathcal{O}_E$, and $F$ is totally ramified over $E$.

Write $Y_{i,j} = U_{i,j} \mu^j$ and $Z = \alpha$ (note that $Y_{1,0} = 1$). Then $A^n_E(D)$ is a quotient of $\mathcal{O}_E[[Y_{i,j}, Z]]$, hence the ring $A = A^n_E(D)/\pi_E$ is a quotient of $k_E[[Y_{i,j}, Z]]$. Let $y_{i,j}$, $z$ be the images of $Y_{i,j}, Z$ in $A$, so that the maximal ideal $\mathfrak{m}$ of $A$ is generated by $z$ and the $y_{i,j}$ for $(i, j) \neq (1, 0)$.

Let $f \in \mathcal{E}'$, and suppose that when we write $f(X) = \sum_{i=0}^{st-1} b_i (X - a)^i$, we have for all $i$, that $v_E(b_i) > -i\lambda$. The condition implies that $f = \pi_L g$ for some $g \in A^n_0(L)$, where $\pi_L$ is a uniformizer of $L$. Let $n = e_{L/E}$, so that $v_{\pi_E}(\pi_L) \geq 1$, then $f^n/\pi_E \in A^n_0(D) \cap A^n_E(D) = A^n_E(D)$. So the image of $f^n$ in $A$ is zero, hence the image of $f$ in $A$ is nilpotent. We see
that the $Y_{i,j}$ for $(i,j) \neq (1,0)$ satisfy this condition, as $t$ is the smallest integer such that there exists an element of $F$ of valuation $st\lambda$, hence $y_{i,j}$ is nilpotent for all $(i,j) \neq (1,0)$. Let $I$ be the ideal generated by the $y_{i,j}$ for $(i,j) \neq (1,0)$. Then $I$ is nilpotent.

We deduce that the conditions of Lemma 2.2.1 are satisfied. So $e(A) = \dim_k A/(z)$, and we see easily that the $y_{i,j}$, $1 \leq i \leq s$ and $0 \leq i < t$ form a $k$-basis of $A/(z)$.

4.2.3. Holes.

Proposition 4.2.5. Let $X = \mathbb{P}^1(\mathbb{Q}_p) \setminus T$ where $T = \cup_{i=1}^N D_i$ is a $G_E$-orbit of closed disks of positive radius $r \in p^\mathbb{Q}$, with each disk defined over a totally ramified extension of $E$. Let $K$ be the field of definition of $D_1$. Let $s$ be the smallest ramification degree of $K(a)/K$ for $a \in D_1$. Let $t$ be the smallest positive integer such that $r^{st} \in |E(a)^x|$. Then $e(X) = Nst$.

Proof. Write $X' = \mathbb{P}^1(\mathbb{Q}_p) \setminus D_1$, so that $X'$ is defined over $K$, and let $a \in D_1$ as in the statement of the Proposition. Note that $[K : E] = N$.

Let $F = E(a)$. Note that $K \subset F$ so $E(a) = K(a)$. As the complexity does not change by unramified extensions, we can assume that $F/E$ is totally ramified. We write $[F : K] = s$.

Write $D_1$ as the set $\{x, v_E(x-a) \geq \lambda\}$ for some $\lambda \in \mathbb{Q}$. Let $\mu$ be the minimal polynomial of $a$ over $K$, so that $\mu$ has degree $s$. Let also $\rho \in F$ be such that $v_E(\rho) = st\lambda$, which is possible by the condition on $r$. Let $L$ be an extension of $E$ containing $a$ and an element $u$ such that $v_E(u) = \lambda$, and which is Galois over $E$.

Let $\mathbb{Q} = \{\sigma_1, \ldots, \sigma_N\}$ be a system of representatives in $G_E$ of $G_E/G_K$, numbered so that $\sigma_i D_1 = D_i$ (so we take $\sigma_1 = id$). For $f \in K(x)$, we denote by $tr f \in E(x)$ the element $\sum_{i=1}^N \sigma_i f$. Note that $A^0_{E}(X) = \{a + tr f, a \in O_E, f \in A^0_{E}(X')\}$. So we begin first by describing $A_k^0(X')$.

Let $R$ be the unique element of $F[x]_{<st}$ such that $R(a) = \rho$. Note that when we write $R(X) = \sum b_i(x-a)^i$, we have $v_E(b_i) > (st-i)\lambda$ for all $i > 0$ by Lemma 2.2.2. For $n > 1$, set $\alpha_n = \frac{\mu}{\mu^n} (\frac{\mu}{\mu^{n+1}})^{n-1}$.

Note that $A_k^0(X')$ is isomorphic to $O_L[[Y]]$, with $Y$ corresponding to the function $u/(x-a)$. In this isomorphism, observe that $\alpha_n$ is regular of degree $nst$ and is divisible by $Y^{st}$. Let $f = f g \in Y A^0_{L}(X')$. Then I can write $Y^{st-1}f = Y^{st}g$ as $\sum_{n \geq 1} P_n(Y)\alpha_n$ for $P_n \in \mathbb{O}_L[[Y]]_{<st}$ (there is no remainder as $Y^{st}$ and $\alpha_1$ differ by a unit). So $f = \sum_{n \geq 1} Y^{1-st}P_n(Y)\alpha_n$. Write $Y^{1-st}P_n(Y) = Q_n(1/Y)$, $Q_n(1/Y) \in O_{L}[1/Y]_{<st}$. Finally, any element of $A_k^0(X')$ can be written uniquely $f = a_0 + \sum_{n \geq 1} Q_n(1/Y)\alpha_n$. Note that $\beta_n = \rho^{-1}\alpha_n$ is in fact in $A_k^0(X')$. So the elements of $A_k^0(X')$ that are in $A_k^0(X')$ are those for which $a_0 \in O_K$ and $\rho Q_n(1/Y)$ (which is a polynomial in $x$ of degree $< st$) is in $K[x]$.

Let $\mathcal{E}'$ the set of elements $P \in K[x]_{<st}$ such that when we write $P(x) = \sum b_i(x-a)^i$, we have $v_E(b_i) \geq (st-i)\lambda$. Then we have shown that:

$$A_k^0(X') = \left\{ a_0 + \sum_{n \geq 1} P_n(Y) \frac{R(x)^{n-1}}{\mu(x)^n}, a_0 \in O_K, P_n \in \mathcal{E}' \right\}$$

For $0 \leq j < t$, let $\mathcal{E}_j$ be the subset of $K[x]_{<s}$ of polynomials that can be written as $\sum_{i=1}^j b_i(x-a)^i$ with $b_i \in F$, $v_E(b_i) \geq (s(t-j) - i)\lambda$. Note that by Lemma 2.2.2, $\mathcal{E}_j$ is the subset of elements of $K[x]_{<s}$ with $v_E(b_i) \geq (s(t-j) - i)\lambda$, and if $P \in \mathcal{E}_j$ then for all $i > 0$, $v_E(b_i) \geq (s(t-j) - i)\lambda$. Moreover, $\mathcal{E}_j$ is in bijection with the set $F_{s(t-j)\lambda} = \{ b \in F, v_E(b) \geq s(t-j)\lambda \}$ by $P \mapsto P(a)$. Indeed, if $b \in F$, it can be written uniquely as $b = P(a)$ for some $P \in K[x]_{<s}$ as $F = K(a)$. Note that by definition, for $0 < j < t$, $F_{s(t-j)\lambda}$ does not contain an element of valuation $s(t-j)\lambda$. We note that $\mathcal{E}' = \bigoplus_{j=0}^{t-1} \mu^j \mathcal{E}_j$. We define
bases for the $E_j$ as $\mathcal{O}_K$-modules as follows: fix $\delta_j$ in $F_{s(t-j)\lambda}$ of minimal valuation (take $\delta_0 = 1$, and note that $v_E(\delta_j) > s(t-j)\lambda$ if $j \neq 0$). Let $\varpi$ be a uniformizer of $F$, so that $(1, \varpi, \ldots, \varpi^{s-1})$ is a basis of $\mathcal{O}_F$ as an $\mathcal{O}_K$-module. Then let $Q_{i,j} \in E_j$ be the polynomial such that $Q_{i,j}(a) = \delta_j \varpi^{i-1}$ for $1 \leq i \leq s$. So we deduce a basis $(P_{i,j})_{0 \leq j < t, 1 \leq i \leq s}$ of $E_j'$ as an $\mathcal{O}_K$-module by taking $P_{i,j} = Q_{i,j}\mu^j$.

Finally let $U_{i,j} = P_{i,j}/\mu^j \in \mathcal{A}_K^0(X')$, and $V = R/\mu^t = U_{1,0}$, so that the elements of $\mathcal{A}_K^0(X')$ can be written uniquely as $a_0 + \sum_{n \geq 0} \left( \sum_{i,j,n} a_{i,j,n} U_{i,j} \right) V^n$, with $a_0$ and the $a_{i,j,n}$ in $\mathcal{O}_K$. Consider such a function $f$ with $a_0 = 0$ as an element of $\mathcal{A}_K^0(X') = \mathcal{O}_L[[Y]]$, its image $\bar{f} \in k_L[[Y]]$. If $(i,j) \neq (1,0)$, then $U_{i,j}$ goes to zero in $k_L[[Y]]$. So $\bar{f}$ is equal to $\sum_n \bar{a}_{i,j,n} \bar{V}^{n+1}$, and $\bar{V}$ has valuation $s$ as a series in $Y$. So the image is non-zero if and only if there exists an $n$ such that $a_{1,0,n}$ is in $\mathcal{O}_K^X$, and then $\bar{f}$ has valuation $st(n+1)$ for the smallest such $n$.

Let $\alpha$ be a uniformizer of $K$, so that $\mathcal{O}_K = \mathcal{O}_E[\alpha]$. Let $f_{i,j,\ell,n} = \alpha^\ell U_{i,j} V^n$, for $0 \leq \ell < N$, so that elements of $\mathcal{A}_K^0(X')$ can be written uniquely as $a_0 + \sum_{n \geq 0} \sum_{i,j,\ell,n} f_{i,j,\ell,n}$, with $a_0$ and the $a_{i,j,\ell,n}$ in $\mathcal{O}_E$. Let $f$ be such a function with $a_0 = 0$ and consider $f$ as an element of $\mathcal{O}_L[[Y]]$. We define the valuation of $f$ as the smallest valuation of the coefficients of $f$, and the leading term of $f$ as the smallest power of $Y$ where this valuation occurs. We compute easily that the valuation of $f_{i,j,\ell,n}$ is $\ell \cdot v_E(\alpha) + (i-1)v_E(\varpi) + v_E(\delta_j)$ and the leading term is $Y^{s(t(n+1)-\ell)}$. So we can determine $j$ and $n$ from the leading term. Note also that $v_E(\alpha) = 1/N$, $v_E(\varpi) = 1/sN$. As $0 \leq \ell < N$ and $0 \leq i - 1 < s$, we see that for a given $j$, the valuations of $f_{i,j,\ell,n}$ and $f_{i',j',\ell',n}$ are not equal modulo $\mathbb{Z}$ except if $i = i'$ and $\ell = \ell'$.

Using the description of $\mathcal{A}_E^0(X)$ from $\mathcal{A}_K^0(X')$ we see that:

$$\mathcal{A}_E^0(X) = \left\{ a_0 + \sum_{n \geq 0} \sum_{i,j} b_{i,j,\ell,n} \alpha^\ell U_{i,j} V^n \right\}, a_0 \in \mathcal{O}_E, b_{i,j,\ell,n} \in \mathcal{O}_E$$

and elements can be written uniquely in such a way. We deduce this from the previous description by setting $a_{i,j,n} = \sum_{\ell} b_{i,j,\ell,n} \alpha^\ell$.

We write $S_{i,j,\ell,n} = \alpha^\ell U_{i,j} V^n$. We also set $Y_{i,j,\ell} = S_{i,j,\ell,0}$, and $Z = S_{1,0,0,0} = Y_{1,0,0}$. We denote by lowercase letters their images in $A = \mathcal{A}_E^0(X)/\pi_E$. Recall that $L$ is an extension of $E$ containing $a$, an element $u$ such that $v_E(r) = \lambda$, and Galois over $E$. Also, note that if $i \neq j$ then $\sigma_i a$ and $\sigma_j a$ are not in the same disk, so $v_E(\sigma_i a - \sigma_j a) < \lambda$. We also assume that the uniformizer $\pi_L$ of $L$ satisfies $v_E(\pi_L) \leq \lambda - \sup v_E(\sigma_i a - \sigma_j a)$.

Let $I$ be the ideal of $A$ generated by the $s_{i,j,\ell,m}$ for $(i,j,\ell) \neq (1,0,0)$. Then $I$ is a nilpotent ideal. Indeed, consider $f$ one of the elements $s_{i,j,\ell,m}$, that is, $f = \pi \alpha^\ell U_{i,j} V^m$. We see $f$ as an element of $\mathcal{A}_L^0(X)$. When we write $\alpha^\ell U_{i,j} V^m$ as an element of $\mathcal{O}_L[[Y]]$, with $Y = u/(x-a)$ as before, we see that in fact it is in $\pi_L \mathcal{O}_L[[Y]]$, as either $\ell > 0$ or $(i,j) \neq (1,0)$. So $f$ is in $\pi_L \mathcal{A}_L^0(X)$. As in the proof of Proposition 1.2.3, this means that the image of $f$ in $A$ is nilpotent. As $I$ is generated by nilpotent elements, and $A$ is noetherian, we see that $I$ is nilpotent.

Let us show that $s_{1,0,0,m} - z^m \in I$ for all $m > 0$. We write $Z^m - S_{1,0,0,m}$ as $\pi f$ for some $f \in \mathcal{A}_L^0(X')$ (up to a constant, which goes to zero in $A$ anyway). To study $f$ we work in $\mathcal{A}_L^0(X)$, then $f$ is the part with poles in $D = D_i$. Consider a product $u_1 x_{i-a_1} u_2 x_{i-a_2}$ (with $u_i = \sigma_i u$, $a_i = \sigma_i a$). We see that if $i \neq j$ it can be written as $v_{i,j} \varepsilon_1 x_{i-a_1} + v_{i,j} \varepsilon_2 x_{i-a_2}$, with $v_{i,j}(\varepsilon_1) = v_{i,j}(\varepsilon_2) = v_E(u/(a_i - a_j)) \geq v_E(\pi_L)$. So when we compute $Z^m = (\pi V)^m$, all the parts coming from the product of terms with poles in different disks $D_i$s are in
\(\pi_L A^0_L(X)\). So \(Z^n - S_{i,0,0,m} = \text{tr} f\) with \(f \in \pi_L A^0_L(X)\). We see \(f\) as an element of \(O_L[[Y]]\) as before, then \(f \in \pi_L O_L[[Y]]\), which means that when we write \(f = \sum_{i,j,n} a_{i,j,n} U_{i,j} V^n\), we have \(a_{i,0,n} \in \pi_K O_K\) for all \(n\), and so the image of \(\text{tr} f\) in \(A\) is indeed in \(I\). From this we deduce that the maximal ideal \(m\) of \(A\) is generated by \(z\) and \(I\).

Let us show that \(A\) has no \(z\)-torsion. Let \(f \in A^0_K(X)\) which we write as \(\sum b_{i,j,\ell,n} S_{i,j,\ell,n}\), where we can assume that each coefficient is either \(0\) or in \(O^*_E\), and at least one coefficient is not zero. Let \(g = \sum b_{i,j,\ell,n} f_{i,j,\ell,n}\). Let \(Y^{s((n+1)t-j)}\) be the leading term. Then the leading coefficient comes from \(f_{i,j,\ell,n,0}\) for a well-determined \((i,j,\ell,n)\). Consider now \(Zf = \text{tr}h\) for some \(h \in A^0_K(X)\) (up to a constant in \(m_E\)). We write \(h = h_1 + h_2\) where \(h_1\) is the part coming from \((\sigma_1 h)(\sigma_1 V)\), and \(h_2\) the part coming from the \((\sigma_1 h)(\sigma_1 V)\) where either \(i\) or \(j\) is not \(1\). From the previous computations, we see that the valuation of \(h_2\) is strictly smaller than the valuation of \(g\). On the other hand, the valuation of \(h_1\) is the same as the valuation of \(g\) and its leading term is \(Y^{s((n+2)t-j)}\), as the valuation is the same it means that it comes from \(f_{i,j,\ell,n,0}\) which appears with a coefficient of the same valuation as the coefficient of \(f_{i,j,\ell,n,0}\) in \(g\), that is \(0\). So we see that when we write \(Zf = \sum b'_{i,j,\ell,n} S_{i,j,\ell,n}\), one of the coefficients at least is in \(O_E^*\), and so the image of \(Zf\) in \(A\) is not zero.

So we are in the conditions of Corollary 2.1.2 and so \(e(A) = \dim_k A/(z) = Nst.\) \(\square\)

4.2.4. Additivity formula.

Proposition 4.2.6. Let \(X\) be a connected standard subset defined over \(E\). Write \(X = D \setminus T\), where \(D\) is an open disk, \(T = \bigcup_{i=1}^m T_i\) where each \(T_i\) is a disjoint union of closed disks \(D_{i,j}\) such that the \(T_i\) are pairwise disjoint, with each defined and irreducible over \(E\), and the field of definition of each \(D_{i,j}\) is totally ramified over \(E\). Then \(c_E(X) = c_E(D) + \sum_{i=1}^m c_E(\mathbb{P}^1(\mathbb{Q}_p) \setminus T_i).\)

Proof. For simplicity we treat only the case where \(X = D_0 \setminus (D_1 \cup D_2)\), with \(D_1\) and \(D_2\) being disjoint disks defined over \(E\). The general case needs no new ideas but requires more complicated notation.

Write \(X_i = \mathbb{P}^1 \setminus D_i\) for \(i = 1, 2\). In this case, each \(X_i\) satisfies the conditions of Proposition 1.2.3. Also, denote \(D_0\) by \(X_0\), it satisfies the conditions of Proposition 1.2.3.

We fix a finite Galois extension \(L\) of \(E\) such that each of the disks that appear in the definition of \(X\) is defined over \(L\) and contains a point of \(L\), and each radius that appears is in \(|L^*|\). So we write \(D_i = D(a_i, |u_i|)^+\), with \(a_i\) and \(u_i\) in \(L\). Note that \(|u_i/(a_i - a_j)| < 1\) if \(\{i, j\} = \{1, 2\}\), so \(v_L(u_i/(a_i - a_j)) \geq 1\). Let \(Y_i = u_i/(x - a_i)\) for \(i = 1, 2\), and \(Y_0 = (x - a_0)/u_0\). Then \(A^0_E(X_i) \subset O_L[[Y_i]]\). Let \(t_0 = e_{L/E}\). If \(h \in A^0_E(X_i) \cap \pi_L^0 O_L[[Y_i]]\), then \(h\) is in \(\pi_E A^0_E(X_i)\).

For \(0 \leq i \leq 2\), denote by \(A_i\) the ring \(A^0_E(X_i)\), and by \(m_i\) its maximal ideal. From the descriptions of the rings \(A^0_E(X_i)\) given in Propositions 1.2.3 and 1.2.5, we see that we can write \(A^0_E(X_i) = O_E \oplus W_i\) for some \(O_E\)-module \(W_i\), such that \(m_i\) is the ideal generated by \(\pi_E\) and \(W_i\). We have then that \(A^0_E(X) = O_E \oplus (\bigoplus_{i=0}^2 W_i)\), and the maximal ideal of \(A^0_E(X)\) is the ideal generated by \(\pi_E\) and the submodules \(W_i\), \(0 \leq i \leq 2\). Denote by \(\alpha_i : A^0_E(X) \to W_i\) the projection with respect to \(O_E \oplus W_j \oplus W_k\) where \(\{i, j, k\} = \{0, 1, 2\}\).

Let \(A_i = A_i/\pi_E\) and \(V_i \subset A_i\) its maximal ideal. Note that \(A_i = k \oplus V_i\) and \(V_i\) is the image in \(A_i\) of \(m_i\), hence also of \(W_i\). Let \(A = A^0_E(X)/\pi_E\). Then we get that \(A = k \oplus (\bigoplus_{i=0}^2 V_i)\), and \(m = \bigoplus_{i=0}^2 V_i\) is the maximal ideal of \(A\). Indeed, \(m\) is the image of the maximal ideal of \(A^0_E(X)\), hence also the image of \(\sum_i W_i\). Moreover we have \(k\)-algebra inclusions \(A_i \subset A\). So \(A\) is nearly the sum of the family \((A_i)\) (see definition before Proposition 2.1.3).

We want to apply Proposition 2.1.3 which will give the result we want. Note first that the existence of the elements \(z_i \in A_i\) was established in the course of the proofs of
Propositions 1.2.4 and 1.2.5. So we only need to find integers $N, t$ such that $V_j V_i^n \subset V_i^{n-t}$ for all $n > N$ and all $i, j$.

Fix some $f \in W_i$ such that its image in $V_i$ is in $V_i^n$, and $g \in W_j$ for $j \neq i$. What we want to do is look at $\alpha_k(fg)$, and show that it goes to zero in $V_k$ if $k \neq i$, and to an element of $V_i^{n-t}$ in $V_i$ for $k = i$. For simplicity we do the proof only for $i = 1$ and $j = 2$, but there is no added difficulty when one of the indices is 0.

Denote by $Z_1$ the element that was called $Z$ in the proof of Proposition 1.2.5 applied to $X_1$ (which is also the element that was called $V$, as we are in the case where $N = 1$), and denote by $\tau$ the integer that was denoted by $st$. Then in $\mathcal{O}_L[[Y_1]]$, $Z_1$ is equal to $\pi L P + Y_1 U$ for some $P \in \mathcal{O}_L[Y_1]_{<7}$ and $U \in \mathcal{O}_L[[Y_1]]^\times$. For $m \geq 0$, write $Z_1^n = \sum_{ji \geq 0} u_{m,j} Y_i^j$ with $u_{m,j} \in \mathcal{O}_L$. Then we have that $v_L(u_{m,j}) \geq m - j/\tau$. On the other hand, we can write $Y_1^{m_1} = \sum_{ji \geq 0} Q_i Z_1^n$ with $Q_i \in \pi L_0\max(0,m-i)\mathcal{O}_L[[Y_1]]$.

Let $z_1$ be the image of $Z_1$ in $A_1$. Then as in the proof of Proposition 1.2.5, $V_1$ is generated by $z_1$ and a nilpotent ideal $I$ of $A_1$. Let $t_1$ be an integer such that $I^{t_1} = 0$. Then any element of $V_1^n$ for $n$ large enough is a multiple of $z_1^{n-t_1}$. Let $f \in W_1$ such that its image in $V_1$ is in $V_1^n$, then we can assume that $f$ is divisible by $Z_1^{n-t_1}$. So when we write $f$ as $\sum_j f_j Y^j$, we have $v_L(f_j) \geq n - t_1 - j/\tau$.

We see easily that for all integers $a, b$, we can write $Y_1^n Y_2^n = \sum_{i=1}^{n} \lambda_{a,b_i} Y_1^i + \sum_{i=1}^{n} \mu_{a,b_i} Y_2^i$ with $\lambda_{a,b_i}$ and $\mu_{a,b_i}$ in $\mathcal{O}_L$, and $v_L(\lambda_{a,b_i}) \geq a + b - i$ and $v_L(\mu_{a,b_i}) \geq a + b - i$.

Let $g \in W_2$, which we see as an element of $\mathcal{O}_L[[Y_2]]$.

We study first $\alpha_1(fg)$. We have $\alpha_1(fg) = \sum_{ji \geq 0} f_j \alpha_1(Y_i^j g)$. As $v_L(f_j) \geq n - t_1 - j/\tau$, all terms $f_j \alpha_1(Y_i^j g)$ for $j \leq (n - t_1 - t_1-i)$ contribute elements that are in $\pi L_0\mathcal{O}_L[[Y_1]]$. Consider now $\alpha_1(Y_i^j g)$ for $j > (n - t_1 - t_1)$. It contributes to $Y_1^n$ with a coefficient of valuation $\geq j - i$. So all terms in $Y_1^n$ with $i \leq (n - t_1 - t_1-i)$ are in $\pi L_0\mathcal{O}_L[[Y_1]]$. We see that $\alpha_1(fg)$ is in $([\pi L_0\mathcal{O}_L[[Y_1]]] + Y_1^{(n-t_1)} \mathcal{O}_L[[Y_1]]) \cap A_E(X_1)$ for $t_2 = t_1 + 2t_0$. We have that $Y_1^{(n-t_1)} = \sum_i Q_i Z_1^n$ with $Q_i \in \pi L_0\max(0,n-t_1-i)\mathcal{O}_L[[Y_1]]$.

We set $\alpha_1(fg)$ in $([\pi L_0\mathcal{O}_L[[Y_1]]] + Z_1^{n-t_1} \mathcal{O}_L[[Y_1]]) \cap A_E(X_1)$ for $t_3 = t_2 + t_0$. From this we deduce that the image is equal to $\alpha_3(fg)$, then $\alpha_3(fg)$ goes to 0 in $V_2$ (and also clearly $\alpha_0(fg) = 0$).

So we get the result we wanted by taking $t = t_3$ and $N = t$.

4.3. Combinatorial complexity of a standard subset with respect to a field. We give another definition of complexity of a standard subset. It is defined in more cases than the algebraic complexity, as we do not require $X$ to be defined over $E$ to define the complexity of $X$ with respect to $E$.

4.3.1. Definition. Let $X$ be a standard subset of $\mathcal{Q}_p$, and $E$ be a finite extension of $\mathcal{Q}_p$. We define an integer $\gamma_E(X)$ which we call combinatorial complexity of $X$.

Let $D$ be a disk (open or closed). Let $F$ be the field of definition of $D$ over $E$. Let $s$ be the smallest integer such that there exists an extension $K$ of $F$, with $e_{K/F} = s$, and $K \cap D \neq \emptyset$. Let $t$ be the smallest positive integer such that $D$ can be written as $\{x, s \in V(x-a) \geq v_E(b)\}$ or as $\{x, s \in V_E(x-a) > v_E(b)\}$ for elements $a, b$ in $K$. Then we set $\gamma_E(D) = st$. We also set $\gamma_E(\mathbb{P}^1(\mathcal{Q}_p)) = 0$.

If $X$ is a connected standard subset, it can be written uniquely as $D_0 \setminus \bigcup_{i=1}^n D_i$ with $D_0$ an open disk or $D_0 = P^1(I\mathcal{Q}_p)$, $D_j$ a closed disk for $j > 0$, and the $D_j$ are disjoint for $j > 0$. We set $\gamma_E(X) = \sum_{i=0}^n \gamma_E(D_j)$.

Now let $X$ be a standard subset. We can write uniquely $X = \bigcup_{i=1}^n X_i$ where $X_i$ is a connected standard subset and the $X_i$ are disjoint. Then we set $\gamma_E(X) = \sum_{i=1}^n \gamma_E(X_i)$.

We also define $\gamma_E(X)$ when $X = \bigcup_{i=1}^n D_i$ is a disjoint union of closed disks: in this case we set $\gamma_E(X) = \sum_i \gamma_E(D_i)$. 


4.3.2. Some properties of the combinatorial complexity.

Lemma 4.3.1. Let $X$ be a standard subset. Let $F/E$ be a finite extension. Then $\gamma_E(X) \geq \gamma_F(X)$, with equality when $F/E$ is unramified, or when $F$ is contained in the field of definition of $X$.

Proof. It suffices to show that $\gamma_E(D) \geq \gamma_F(D)$, with equality when $F/E$ is unramified, for any disk $D$ (open or closed), and then it is clear from the definition. □

Proposition 4.3.2. Let $X$ be a standard subset defined and irreducible over $E$, and write $X = \bigcup_{i=1}^s X_i$ its decomposition in connected standard subsets. Let $E_1$ be the field of definition of $X_1$ over $E$. Then $\gamma_E(X) = [E_1 : E] \gamma_{E_1}(X_1)$.

Proof. We have $\gamma_E(X) = \sum_{i=1}^s \gamma_{E_i}(X_i) = \sum_{i=1}^s \gamma_{E_1}(X_i)$. Observe first that $\gamma_{E_i}(X_i)$ does not depend on $i$. Indeed, for all $i$ there exists $\sigma \in G_E$ such that $\sigma(X_1) = X_i$ and $\sigma(E_1) = E_i$. Such a $\sigma$ transforms an equation $\{x, v_E(x - a) \geq v_E(b)\}$ (or $\{x, v_E(x - a) > v_E(b)\}$) of a disk appearing in the definition of $X_1$ to an equation defining the corresponding disk in $X_i$. Moreover, $s = [E_1 : E]$, as $G_E$ acts transitively on the set of $X_i$ because we have assumed $X$ to be irreducible over $E$. □

4.4. Comparison of complexities. The important result is that the two definitions of complexity actually coincide when both are defined.

Theorem 4.4.1. Let $X$ be a standard subset defined over $E$. Then $c_E(X) = \gamma_E(X)$.

Proof. We can assume that $X$ is irreducible over $E$, as both multiplicities are additive with respect to irreducible standard subsets.

Write now $X = \bigcup X_i$ where the $X_i$ are connected standard subsets, and let $E_i$ be the field of definition of $X_i$. Then $c_E(X) = [E : E_1] c_{E_i}(X_1)$ by Proposition 4.1.2 and $\gamma_E(X) = [E : E_1] \gamma_{E_1}(X_1)$ by Proposition 4.3.2.

So we can assume that $X$ is a connected standard subset defined over $E$. Note that $c_E(X) = c_F(X)$ and $\gamma_E(X) = \gamma_F(X)$ for any finite unramified extension $E'/E$ by Propositions 4.1.2 and 4.3.1. So we can enlarge $E$ if needed to an unramified extension, and we can assume that we have written $X = D \setminus \bigcup Y_i$ satisfying the hypotheses of Proposition 4.2.6. So we have $c_E(X) = c_E(D) + \sum_i c_{E}(\mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus Y_i)$ by Proposition 4.2.6 and the analogous result for $\gamma_E$ follows from the definition. So we need only prove the equality for these standard subsets.

Let $D$ be a disk defined over $E$, of the form $\{x, v_E(x - a) > \lambda\}$. Let $s$ be the minimal ramification degree of an extension $F$ of $E$ such that $F \cap D \neq \emptyset$, and $t > 0$ be the smallest integer such that $st \lambda \in (1/s) \mathbb{Z}$. Then $c_E(D) = \gamma_E(D) = st$. For $c_E(D)$ it follows from Proposition 4.2.4, and for $\gamma_E(D)$ it is the definition. So we get that $c_E(D) = \gamma_E(D)$.

Let now $X = \mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus T$, where $T$ is defined and irreducible over $E$, and $T = \bigcup_{i=1}^N D_i$ where the $D_i$ are disjoint closed disks defined over a totally ramified extension of $E$. We have $\gamma_E(X) = \sum \gamma_E(D_i) = N \gamma_E(D_1)$ as the $D_i$ are $G_E$-conjugates. Let $F$ be the field of definition of $D_1$. Then $\gamma_E(X) = N \gamma_F(D_1) = N \gamma_F(\mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus D_1)$. On the other hand, it follows from Proposition 4.2.6 that $c_E(X) = N \gamma_F(\mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus D_1)$. Now the proof that $\gamma_F(\mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus D_1) = c_E(\mathbb{P}^1(\bar{\mathbb{Q}}_p) \setminus D_1)$ is the same as in the case of a disk. So finally $c_E(X) = \gamma_E(X)$. □

From now on we only write $c_E$ to denote either $c_E$ or $\gamma_E$ (so we can consider $c_E(X)$ even for $X$ that is not defined over $E$, or for $X$ a disjoint union of closed disks).

Corollary 4.4.2. The complexity of $X$ is at least equal to the number of connected components of $X$. 
Corollary 4.4.3. Let $X = \mathbb{P}^1(\overline{\mathbb{Q}_p}) \setminus Z$, where $Z$ is defined over $E$ is a disjoint union of $d$ disks. Then $c_E(X) \geq d$.

4.5. Finding a standard subset from a finite set of points.

4.5.1. Approximations of a standard subset. Let $X = \bigcup_{n=1}^{N}(D_{n,0} \setminus \bigcup_{i=1}^{m_n}D_{n,i})$ be a standard subset, where the $D_{n,0} \setminus \bigcup_{i=1}^{m_n}D_{n,i}$ form the decomposition of $X$ as a disjoint union of connected standard subset. For $J \subset \{1, \ldots, N\}$ and $I_n \subset \{1, \ldots, m_n\}$ for $n \in J$, we set $Y_{J,I} = \bigcup_{n \in J}(D_{n,0} \setminus \bigcup_{i \in I_n}D_{n,i})$. This is a standard subset with $c_E(Y_{J,I}) \leq c_E(X)$ and equality if and only if $X = Y_{J,I}$. Such standard subsets are called approximations of $X$.

For a connected standard subset $Y$ of $D(0, 1)^-$, written as $D(a, r)^- \setminus \Delta$ for some finite union of closed disks $\Delta$, we define its outer part as $D(a, r)^-$. If $X$ is any standard subset, we define its outer part as the union of the outer parts of its connected components. Note that if $Y$ is defined over a field $E$, then so is its outer part $Y'$, and $Y'$ is an approximation of $Y$.

Let $Y$ be a connected standard subset. If the outer part of $Y$ contains 0, we define its circular part as follows: write $Y$ as $D(0, r)^- \setminus \bigcup_{i=1}^{n}D_i$ where the $D_i$ are disjoint closed disks. If none of the $D_i$ contains 0, we define the circular part of $Y$ as $D(0, r)^-$. If 0 is contained in one of the $D_i$, say $D_1$, then we define the circular part of $Y$ as $D(0, r)^- \setminus D_1$. Note that the circular part of $Y$ is defined over $\mathbb{Q}_p$, hence over the definition field of $Y$. The circular part of $Y$ is an approximation of $Y$.

4.5.2. Main results.

Theorem 4.5.1. Let $X$ be a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}_p})$ defined over $E$. Let $m$ be an integer such that $c_E(X) \leq m$. Then there exists a finite set $E$ of finite extensions of $E$, depending only on $E$ and $m$, such that $X$ is entirely determined by the sets $X \cap F$ for all extensions $F \in \mathcal{E}$.

We can actually take the set $\mathcal{E}$ to be the set of all extensions of $E$ of degree at most $N$ for $N$ depending only on $E$ and $m$.

Corollary 4.5.2. Let $X$ be a standard subset of $D(0, 1)^-$ defined over $E$. Let $m$ be an integer such that $c_E(X) \leq m$. Moreover suppose that there exists an $\varepsilon > 0$ such that for all $x \in X$, $D(x, \varepsilon)^- \subset X$, and for all $x \not\in X$, $D(x, \varepsilon)^- \cap X = \emptyset$. Then there exists a finite subset $\mathcal{P}$ of $D(0, 1)^-$, depending only on $E$, $m$, and $\varepsilon$, such that $X$ is entirely determined by $X \cap \mathcal{P}$.

Proof of Corollary 4.5.2. Let $N$ be the integer as in Theorem 4.5.1. For each extension $F$ of $E$ of degree at most $N$, $F \cap D(0, 1)^-$ can be covered by a finite number of open disks of radius $\varepsilon$, and we define a finite set $\mathcal{P}_F$ by taking an element in each of these disks. Then we set $\mathcal{P}$ to be the union of the sets $\mathcal{P}_F$, which is finite as there is only a finite number of extensions of $E$ of degree at most $N$.

Remark 4.5.3. As is clear from the proof, the set $\mathcal{P}$ can be huge. However in practice for a given $X$ we need only test points in a very small proportion of this subset.

We give the proof of Theorem 4.5.1 in Section 4.5.3. We work by constructing a sequence $(X_i)$ of approximations of $X$, such that each $X_i$ is defined over $E$ and is an approximation of $X_{i+1}$ and $c_E(X_{i+1}) > c_E(X_i)$, so that at some point we get $X_i = X$.

4.5.3. Notation. If $a < b$ are rational numbers, denote by $A(a, b)$ the annulus $\{x, a < v_E(x) < b\}$. If $c$ is a rational number, denote by $C(c)$ the circle $\{x, v_E(x) = c\}$. Sometimes we also write $C(r)$ to denote the circle $\{x, |x| = r\}$ when no confusion can arise.
If $t \in \mathbb{Q}$, we introduce $\mathrm{denom}(t)$ the denominator of $t$, which is the smallest integer $d$ such that $t \in (1/d)\mathbb{Z}$. Let $v$ be the valuation on $\mathcal{O}_p$ that extends the normalized valuation on $E$. If $x \in \mathcal{O}_p$, we write $\mathrm{denom}(x)$ for $\mathrm{denom}(v_E(x))$. Note that $[E(x) : E] \geq \mathrm{denom}(x)$.

### 4.5.4. Preliminaries.

**Lemma 4.5.4.** Let $Z$ be an irreducible standard subset defined over $E$, which is contained in the set $C(\lambda)$ for $\lambda \in (1/d)\mathbb{Z}$ for $d$ minimal. Then $c_E(Z) \geq d$.

**Proof.** We can replace $Z$ by the union of the outer parts of its connected components, as in can only lower the multiplicity. Write $Z = \bigcup_{i=1}^n Z_i$ with each $Z_i$ connected, so $Z_i$ is a disk. Let $F$ be the field of definition of $Z_1$, then $c_E(Z) = [F : E]c_F(Z_1)$ by Corollary 4.1.2.

As $Z \subset C(\lambda)$, we see that for all $y \in Z_1$, $e_{F(y)/E} \geq e_{E(y)/E} \geq d$, so $e_{F(y)/E} \geq d/e_{F/E}$. We have that $c_F(Z_1) \geq e_{F(y)/F} \geq d/e_{F/E}$, and so $[F : E]c_F(Z_1) \geq d$, that is, $c_E(Z) \geq d$. □

**Lemma 4.5.5.** Let $X = \mathbb{P}^1(\mathcal{O}_p) \setminus T$ where $T$ is a disjoint union of closed disks defined over $E$ and contained in $C(\lambda)$ for $\lambda \in (1/d)\mathbb{Z}$ for $d$ minimal. Then $c_E(X) \geq d$.

**Proof.** Write $T = \bigcup_{i=1}^n T_i$ with each $T_i$ a closed disk. Let $F$ be the field of definition of $T_1$, then $c_E(X) = [F : E][c_F(T_1)$ by Proposition 4.3.2.

As $T \subset C(\lambda)$, we see that for all $y \in T_1$, $e_{F(y)/E} \geq e_{E(y)/E} \geq d$, so $e_{F(y)/F} \geq d/e_{F/E}$. By definition of $\gamma_E$, we have that $c_F(T_1) \geq e_{F(y)/F} \geq d/e_{F/E}$, and so $[F : E]c_F(T_1) \geq d$, that is, $c_E(T) \geq d$. □

**Lemma 4.5.6.** Let $X$ be a standard open subset defined over $E$ and contained in $C(r)$ for some $r > 0$, and suppose that $c_E(X) \leq m$. Then $X$ is contained in a union of at most $m$ open disks of radius $r$ contained in $C(r)$.

**Proof.** Let $Y$ be the union of the outer parts of the connected components of $X$, so that $X \subset Y, Y$ is defined over $E$ and is a disjoint union of open disks, and $c_E(Y) \leq c_E(X) \leq m$. So it is enough to prove the result for $Y$, but it is clear in this case. □

**Lemma 4.5.7.** Let $X$ be a standard open subset defined over $E$ and of the form $\mathbb{P}^1(\mathcal{O}_p) \setminus Z$ with $Z \subset C(r)$ for some $r > 0$, and suppose that $c_E(X) \leq m$. Then $Z$ is contained in a union of at most $m$ open disks of radius $r$ contained in $C(r)$.

**Proof.** Let $Y$ be the connected component of $X$ containing $\mathbb{P}^1(\mathcal{O}_p) \setminus C(r)$, so that $X \subset Y$, $Y$ is defined over $E$ and is of the form $\mathbb{P}^1(\mathcal{O}_p) \setminus T$ where $T$ is a disjoint union of closed disks, and $c_E(Y) \leq c_E(X) \leq m$. So it is enough to prove the result for $Y$, but it is clear in this case. □

**Proposition 4.5.8.** Let $E$ be a finite extension of $\mathbb{Q}_p$. There exists a function $\psi_E$ such that for any standard subset $X$ of $\mathcal{O}_p$ defined over $E$, if $c_E(X) \leq m$ then there exists an extension $F$ of $E$ with $[F : E] \leq \psi_E(m)$ and $X \cap F \neq \emptyset$.

**Lemma 4.5.9.** Let $E$ be a finite extension of $\mathbb{Q}_p$. There exists a function $\psi_E^0$ such that for any open disk $D$ of $\mathcal{O}_p$ defined over $E$, if $c_E(D) \leq m$ then there exists an extension $F$ of $E$ with $[F : E] \leq \psi_E^0(m)$ and $D \cap F \neq \emptyset$ and the radius of $D$ is in $|F^*|$. For $m < p^2$ or $p = 2$ we can take $\psi_E^0(m) = m$ and consider only extensions $F/E$ that are totally ramified.

**Proof.** Let $s$ be the minimal ramification degree of an extension $K$ of $E$ with $K \cap D \neq \emptyset$, and let $t$ be the smallest positive integer such that $D$ can be written as $\{x, stv_E(x - a) > v_E(b)\}$ for a $b \in K$. So by definition $c_E(D) = st$. By Theorem 4.1.1 there exists an extension $K$ of $E$ with $e_{K/E} = s$ and $|K : E| \leq s^2$ and $K \cap D \neq \emptyset$. Then if $F$ is a totally ramified extension of degree $t$ of $K$, then $F$ satisfies the conditions, and we have $[F : E] \leq s^2t$. As $st \leq m$, this means that we can take $\psi_E^0(m) = m^2$. 


Note that $s$ is a power of $p$ by Theorem 1.1.1 and $s \leq m$. So if $m < p^2$ then $s = 1$ or $s = p$ so we can take $[K : E] \leq s$ and $K/E$ totally ramified instead of $[K : E] \leq s^2$, and so we can take $[F : E] \leq m$.

When $p = 2$ the result comes from applying Theorem 1.1.2 instead of Theorem 1.1.1.

Proof of Proposition 4.5.8. We show first that there exists a function $\psi_E^1$ such that for all $X$ a standard connected subset defined over $E$ with $c_E(X) \leq m$, there exists an extension $F$ of $E$ with $[F : E] \leq \psi_E^1(m)$ and $X \cap F \neq \emptyset$.

We can write $X$ as $D \setminus Y$ for some open disk $D$. By Lemma 4.5.9, there exists an extension $K$ of $E$ of degree at most $\psi_E^0(m)$ such that $D$ contains a point in $K$ and has a radius in $|K^s|$. Moreover, $c_K(X) \leq c_E(X) \leq m$. By doing an affine transformation in $K$, we can assume that $X$ is of the form $D(0, 1)^c \setminus Y$. Then: either $0 \notin Y$, in which case $K \cap X \neq \emptyset$, or $0 \in Y$. In the latter case, $m > 1$ and $X$ is contained in a standard subset $X'$ of the form $D(0, 1)^c \setminus (D(0, r^c) \cup Z)$ for some $r \in p\mathbb{Z}$, $r < 1$, with $c_K(X') \leq c_K(X)$ (we take $D(0, r^c)$ to be the outer part of the irreducible component of $Y$ containing 0). So we have $c_K(\mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, r^c)) + c_K(\mathbb{P}^1(\mathbb{Q}_p) \setminus Z) \leq m - 1$. Let $s$ be the smallest integer such that $r^c \in [K^s]$, then $c_K(\mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, r^c)) = m - s$ and $s \leq m - 1$. There exists an extension $L$ of $K$ of degree $2s$ such that $L$ contains an element of norm $\rho = \sqrt{r}$. The circle $C = C(\psi_E(\rho))$ is contained in $L \cap D(0, 1)^c \setminus D(0, r^c)$. Then $Z \cap C$ meets at most $c_K(\mathbb{P}^1 \setminus Z) \leq 2m - 2$ open disks of radius $\rho$ in $C$ by Lemma 4.5.6. Let $q$ be the cardinality of the residue field of $E$, so that $qL \geq qE$. By replacing if necessary $L$ by an unramified extension of $L$ of degree $1 + \max(0, \lfloor \log_q(m-1) \rfloor)$, we can assume that $qL - 2 > m - 2$, so that $L \cap C$ is not contained in the union of these open disks, and so $(C \cap L) \setminus (C \cap Z)$ is not empty, hence $L \cap X$ is not empty. Finally, we notice that $[L : E] \leq 2(m-1)\psi_E^0(m)(1 + \max(0, \lfloor \log_q(m-1) \rfloor))$. So we can take $\psi_E^1(m) = 2(m-1)\psi_E^0(m)(1 + \max(0, \lfloor \log_q(m-1) \rfloor))$ if $m > 1$, and $\psi_E^0(1) = 1$.

Now we go back to the general case. Write $X$ as a disjoint union of irreducible components over $E$. Each of them has complexity at most $m$, and it is enough to find a point in one of them. So we can assume that $X$ is irreducible over $E$.

Suppose now that $X$ is irreducible over $E$: write $X = \cup_{i=1}^s X_i$ where the $X_i$ form a $G_E$-orbit. Let $F$ be the field of definition of $X_1$, and $s = [F : E]$. Then $c_E(X) = s c_F(X_1)$, so $c_F(X) \leq m' = \lfloor m/s \rfloor$. There exists an extension $K$ of $F$ of degree at most $\psi_E^1(m')$ such that $K \cap X_1 \neq \emptyset$. As $K$ is an extension of $E$ of degree at most $s\psi_E^1(m')$, we see that we can take $\psi_E(m) = \sup_{1 \leq s \leq m} s \psi_E^1([m/s])$, which is finite as $E$ has only a finite number of extensions of a given degree.

Proposition 4.5.10. Let $E$ be a finite extension of $\mathbb{Q}_p$. There exists a function $\phi_E$ such that for any standard subset $X$ of $\mathbb{Q}_p$ defined over $E$ and different from $\mathbb{P}^1(\mathbb{Q}_p)$, if $c_E(X) \leq m$ then there exists an extension $F$ of $E$ with $[F : E] \leq \phi_E(m)$ and $F \not\subseteq X$.

Lemma 4.5.11. Let $E$ be a finite extension of $\mathbb{Q}_p$. There exists a function $\phi_E^0$ such that for any closed disk $D$ of $\mathbb{Q}_p$ defined over $E$, if $c_E(D) \leq m$ then there exists an extension $F$ of $E$ with $[F : E] \leq \phi_E^0(m)$ and $D \cap F \neq \emptyset$ and the radius of $D$ is in $|F^s|$. For $m < p^2$ or $p = 2$ we can take $\phi_E^0(m) = m$ and consider only extensions $F/E$ that are totally ramified.

The proof is the same as the proof of Lemma 4.5.9.

Proof of Proposition 4.5.10. The proof is very similar to the proof of 4.5.8. We first define $\phi_E^1$ for $X$ of the form $\mathbb{P}^1 \setminus Y$ for $Y$ connected, which we can take to be $\phi_E^1(m) = \phi_E^0(m)(1 + \max(0, \lfloor \log_q(m) \rfloor))$ if $m > 1$, and $\phi_E^0(1) = 1$. Indeed, after introducing $K$ of degree at most $\phi_E^1(m)$ as before, and transforming $Y$ to $D(0, 1)^c \setminus Z$ for some standard open subset $Z$, we can look for points of $Y$ that are in the circle of radius 1 so we do not need to introduce the ramified extension $L$.

We then take as before $\phi_E(m) = \sup_{1 \leq s \leq m} s \phi_E^1([m/s])$. □
4.5.5. Proof of Theorem 4.5.1. We work by constructing a sequence \((X_i)\) of approximations of \(X\), such that each \(X_i\) is defined over \(E\) and is an approximation of \(X_{i+1}\) and \(c_E(X_{i+1}) > c_E(X_i)\), so that at some point \(X_i = X\) and we stop.

We divide \(X\) in two parts \(Y\) and \(Z\), each being defined over \(E\). The first part \(Y\) is the union of connected components such that their outer part contains \(0\). The other part \(Z\) is the union of the other connected components. We have that \(c_E(X) = c_E(Y) + c_E(Z)\), and the outer part of \(Z\) does not contain \(0\).

Let \(Y_0\) be the circular part of \(Y\), so that \(Y \subset X_0\). It is clear from the definition that \(Y_0\) is an approximation of \(X\) (and of \(Y\)). We write \(Y = Y_0 \setminus T\), so that \(T\) is a union of closed disks that do not contain \(0\).

Our first approximation of \(X\) will be \(X_0 = Y_0\). We now explain how to compute \(Y_0\).

Observe first:

**Lemma 4.5.12.** The set \(Z\) is contained in \(\bigcup_{\lambda \in \mathbb{Q}, \text{denom}(\lambda) \leq m} C(\lambda)\).

**Proof.** By definition of \(Z\), it is equal to the union of the \(Z_\lambda = Z \cap C(\lambda)\) for \(\lambda \in \mathbb{Q}\), each \(Z_\lambda\) being a standard open subset. Suppose that there exists a \(\lambda \in \mathbb{Q}\) with \(\text{denom}(\lambda) > m\) and \(Z_\lambda\) is not empty. By Lemma 4.5.4 we see that \(c_E(Z) \geq c_E(Z_\lambda) \geq \text{denom}(\lambda) > m\), which is not possible. \(\square\)

Similarly to Lemma 4.5.12 but using Lemma 4.5.5 instead of Lemma 4.5.4 we see that:

**Lemma 4.5.13.** The set \(T\) is contained in \(\bigcup_{\lambda \in \mathbb{Q}, \text{denom}(\lambda) \leq m} C(\lambda)\).

As a consequence of Lemmas 4.5.12 and 4.5.13 we have:

**Lemma 4.5.14.** Let \(x \in D(0,1)^-\) such that \(\text{denom}(x) > m\). Then \(x \in Y_0\) if and only if \(x \in X\).

**Lemma 4.5.15.** Write \(Y_0 = \bigcup_{i=1}^n A(a_i, b_i)\) or \(Y_0 = D(0, b_0)^- \cup \bigcup_{i=1}^n A(a_i, b_i)\), with \(b_{i-1} \leq a_i < b_i\) for all \(i\). Then \(\sum_i \text{denom}(a_i) + \sum_i \text{denom}(b_i) = c_E(Y_0)\).

**Corollary 4.5.16.** Let \(a < b\) be two rational numbers such that for all rational numbers \(c\) strictly between \(a\) and \(b\), we have that \(\text{denom}(c) > m\). Then either \(A(a, b) \subset X\) or \(A(a, b) \cap X = \emptyset\).

**Proof.** By Lemma 4.5.14 \(A(a, b) \cap X = A(a, b) \cap Y_0\). So we can work with \(Y_0\). The result then follows from Lemma 4.5.15. \(\square\)

Fix a sequence of rationals \(0 = t_0 < t_1 < \cdots < t_n = 1\) such that for any rational number \(c\) strictly between \(t_i\) and \(t_{i+1}\), we have \(\text{denom}(c) > m\). Extend this sequence to \((t_i)_{i \in \mathbb{Z}}\) by setting \(t_{i+n} = t_i + 1\). Choose for each \(i \in \mathbb{Z}\) an element \(x_i\) with \(t_i < v_E(x_i) < t_{i+1}\). We can do this by taking the elements \(x_i\) in some totally ramified extension \(L_m\) of \(E\), of degree bounded in terms of \(m\). Then for each annulus \(A(t_i, t_{i+1})\), we know whether it is contained in \(X\) (if \(x_i \in X\)), or if it does not meet \(X\) (if \(x_i \notin X\)) by considering only \(X \cap L_m\).

Note that \(X\) being a standard subset, then if \(0 \in X\) then there is an open disk around \(0\) contained in \(X\), and otherwise there is an open disk around \(0\) that does not meet \(X\); and likewise with \(\infty\) instead of \(0\).

Moreover, we only need to understand additionally whether \(C(t_i) \subset Y_0\) for \(i \in \mathbb{Z}\) in order to understand \(Y_0\). Let \(I\) be the set of indices such that both \(A(t_{i-1}, t_i)\) and \(A(t_i, t_{i+1})\) are contained in \(X\). If \(C(t_i) \subset Y_0\), then \(t_i \in I\), but the converse is not necessarily true.

Let \(Y_1 = Y_0 \cup \bigcup_{i \in I} C(t_i)\) (so \(Y_1\) is entirely known at this step). Then \(Y_1\) is an approximation of \(Y_0\) and \(c_E(Y_1) \leq c_E(Y_0)\). Let \(m_1 = m - c_E(Y_1)\), then \(c_E(T) + c_E(Z) \leq m_1\), as \(c_E(X) = c_E(Y_0) + c_E(T) + c_E(Z)\).
Lemma 4.5.17. Let $i \in I$, and let $x_1, \ldots, x_{m+1}$ be such that $v_E(x_j) = t_i$ for all $j$, and $v_E(x_j - x_j) = t_i$ if $j \neq j'$. Then $C(t_i) \subset Y_0$ if $x_j \in X$ for all $j$, and $C(t_i)$ does not meet $Y_0$ if none of the $x_j$ are in $X$.

Proof. Suppose that $x_j \in X$ for all $j$, but $C(t_i)$ is not contained in $Y_0$. Then it means that $x_j \in Z$ for all $j$. But this is a contradiction by Lemma 4.5.10. Suppose that none of the $x_j$ are in $X$, but that $C(t_i) \subset Y_0$. This means that $x_j$ is in $T$ for all $j$. But this is a contradiction by Lemma 4.5.7.

So we see how to determine whether $C(t_i) \subset Y_0$ for $i \in I$: choose an element $x$ of valuation $t_i$, compute if $x$ is in $X$ or not. After a finite number of such computations, one of the hypotheses Lemma 4.5.17 is satisfied, so we can conclude. Moreover, we can speed this up by noting that if $\text{denom}(t_i) \geq m/2$ and $i \in I$, then $C(t_{i-1}, t_{i+1}) \subset Y_0$, by Lemma 4.5.13. So for such $t_i$ we do not have to do the computations.

So finally we have computed $Y_0 = X_0$ our first approximation of $X$. From the method we used to compute $X_0$, we see that for each $E$ there is a non-decreasing function $f_E$ such that if $c_E(X) \leq m$, then we can compute $X_0$ by testing only if $x \in X$ for elements $x$ with $[E(x) : E] \leq f_E(m)$.

We now assume that we have computed an approximation $X_i$ of $X$ defined over $E$, and we explain how to compute another approximation $X_{i+1}$ of $X$ such that $X_i$ is an approximation of $X_{i+1}$. Note that if $c_E(X_i) = m$ then $X_i = X$ so we are finished.

We can write uniquely $X = (X_i \setminus T_i) \cup Z_i$ where $T_i$ is a disjoint union of closed disks and $Z_i$ is a disjoint union of connected standard subsets that do not meet $X_i \setminus T_i$, and $T_i$ and $Z_i$ are defined over $E$. Let $m_i = m - c_E(X_i)$. Note that $c_E(X) = c_E(X_i) + c_E(Z_i) + c_E(T_i)$, so that $c_E(Z_i) + c_E(T_i) \leq m_i$.

If there exists a point that is in $X$ but not in $X_i$, then $Z_i$ is not empty. By Proposition 4.5.8, it means that there exists an extension $F/E$ with $[F : E] \leq \psi_E(m_i)$ such that $Z_i \cap F \neq \emptyset$.

Let $Y_i = (\mathbb{P}^1(\overline{\mathbb{Q}}) \setminus T_i) \cup Z_i$. If there exists a point $x$ that is in $X_i$, but not in $X$, then $x$ is in $T_i$ but not in $Z_i$, so $x$ is not in $Y_i$ and so $Y_i$ is not $\mathbb{P}^1(\overline{\mathbb{Q}})$. We see that $c_E(Y_i) \leq c_E(Z_i) + c_E(T_i) \leq m_i$. So if $Y_i$ is not $\mathbb{P}^1(\overline{\mathbb{Q}})$, then by Proposition 4.5.10 there exists an extension $F/E$ with $[F : E] \leq \phi_E(m_i)$ and $F \not\subset Y_i$.

So we see that we can determine whether $X = X_i$ by doing computations only in extensions of $E$ of degree at most $\max(\psi_E(m_i), \phi_E(m_i))$. If $X \neq X_i$, we explain how to compute an $X_{i+1}$.

Suppose first that we have found some $a \not\in Y_i$, and let $F = E(a)$. We have that $(X \setminus Y_i) \cap D(a, |a|)^- \subset D(a, r)^+$ for some $r < |a|$, as $T_i \cap D(a, |a|)^-$ is a closed disk. Consider $X' = Y_i \cap D(a, |a|)^-$. Then it is a standard subset defined over $F$, with $c_E(X') \leq c_E(Y_i) + 1 \leq m_i + 1$. We can compute an approximation $X'_0$ of $X'$ defined over $F$ in the same way that we computed the approximation $X_0$ of $X$. Then we define a standard subset $X_{i+1}$ as follows: $X_{i+1}$ coincides with $X_i$ outside of the $G_E$-orbit of $D(a, |a|)^-$; $D(a, |a|)^- \cap X_{i+1} = D(a, |a|)^- \cap X_0$; and $X_{i+1}$ is defined over $E$. We check that $X_{i+1}$ is an approximation of $X$, $X_i$ is an approximation of $X_{i+1}$ and $c_E(X_{i+1}) > c_E(X_i)$.

Suppose now that we have found some $a \in Z_i$, and let $F = E(a)$. Let $X' = Z_i \cap D(a, |a|)^-$. It is an approximation of $Z_i$ and defined over $F$ so $c_E(X') \leq m_i$. We can compute an approximation $X'_0$ of $X'$ defined over $F$ in the same way that we computed the approximation $X_0$ of $X$. Then we define a standard subset $X_{i+1}$ as follows: $X_{i+1}$ coincides with $X_i$ outside of the $G_E$-orbit of $D(a, |a|)^-$; $D(a, |a|)^- \cap X_{i+1} = D(a, |a|)^- \cap X'_0$; and $X_{i+1}$ is defined over $E$. We check that $X_{i+1}$ is an approximation of $X$, $X_i$ is an approximation of $X_{i+1}$ and $c_E(X_{i+1}) > c_E(X_i)$. 


In both cases, we see that in order to compute \( X_{i+1} \) we needed only to test if \( x \in X \) for elements \( x \) with \( |E(x) : E| \leq |F : E|f_F(m_i) \leq |F : E|f_E(m) \), where \( F = E(a) \) satisfies \( |F : E| \leq \max(\psi_E(m), \phi_E(m)) \).

So we see how to compute the sequence of approximations of \( X \). From the construction, we see that we need only to test if \( x \in X \) for elements \( x \) such that \( |E(x) : E| \leq \max_F |F : E|f_F(m), \) where the max is taken over extensions \( F \) such that \( |F : E| \leq \max(\psi_E(m), \phi_E(m)) \).

5. Application to potentially semi-stable deformation rings

5.1. Definition of the potentially semi-stable deformation rings. We recall the definition and some properties of the rings defined by Kisin in [Kis08] (see also [Kis10]).

Let \( \rho : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_p) \) be a potentially semi-stable representation. Then we know from [Fon94] that we can attach to \( \rho \) a Weil-Deligne representation WD(\( \rho \)), that is, a smooth representation \( \sigma : W_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_p) \), and an endomorphism \( N \) of \( \overline{\mathbb{Q}}_p^2 \), such that \( N \sigma(x) = \rho^{\text{deg} x} \sigma(x)N \) for all \( x \in W_{\mathbb{Q}_p} \). We say that \( \sigma \) is the extended type of \( \rho \), and \( \sigma|_{I_{\mathbb{Q}_p}} \) the inertial type of \( \rho \), where \( I_{\mathbb{Q}_p} \) is the inertia subgroup of \( W_{\mathbb{Q}_p} \). We make the following definition:

**Definition 5.1.1.** A Galois type of dimension 2 is one of the following representations with values in \( GL_2(\overline{\mathbb{Q}}_p) \):

1. a scalar smooth representation \( \tau = \chi \oplus \chi \) of \( I_{\mathbb{Q}_p} \), such that \( \chi \) extends to a character of \( W_{\mathbb{Q}_p} \).
2. a smooth representation \( \tau = \chi_1 \oplus \chi_2 \) of \( I_{\mathbb{Q}_p} \), where both \( \chi_1 \) and \( \chi_2 \) extend to characters of \( W_{\mathbb{Q}_p} \).
3. if \( p > 2 \), a smooth representation \( \tau = \chi_1 \oplus \chi_2 \) of \( W_{\mathbb{Q}_p} \), such that \( \chi_1 \) and \( \chi_2 \) have the same restriction to inertia, and \( \chi_1(F) = p \chi_2(F) \) for any Frobenius element \( F \) in \( W_{\mathbb{Q}_p} \).
4. if \( p > 2 \), a smooth irreducible representation \( \tau \) of \( W_{\mathbb{Q}_p} \).

We call Galois types of the form (1) and (2) inertial types, and those of the forms (3) and (4) discrete series extended types. If \( \rho \) is a potentially semi-stable representation of \( G_{\mathbb{Q}_p} \) of dimension 2 and \( p > 2 \), then we know from the classification of 2-dimension smooth representations of \( W_{\mathbb{Q}_p} \) that either its inertial type is isomorphic to a Galois type of the form (1) or (2), or its extended type is isomorphic to a Galois type of the form (3) or (4) (if \( p = 2 \) there are other possibilities). Note that if the Galois type of \( \rho \) is of the form (2) and (4) then it is potentially crystalline (that is, the endomorphism \( N \) of the Weil-Deligne representation is zero), and that if \( \rho \) is potentially semi-stable but not potentially crystalline (that is, \( N \neq 0 \)) then its Galois type is of the form (3).

**Definition 5.1.2.** A deformation data \((k, \tau, \overline{\rho}, \psi)\) is the data of:

1. an integer \( k \geq 2 \).
2. a Galois type \( \tau \).
3. an continuous representation \( \overline{\rho} \) of \( G_{\mathbb{Q}_p} \) of dimension 2, with trivial endomorphisms, over some finite extension \( F \) of \( \mathbb{F}_p \).
4. a continuous character \( \psi : G_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_p^\times \) lifting \( \overline{\rho} \) such that \( \psi \) and \( \chi_{\text{cycl}}^{k-1} \det \tau \) coincide.

If the type \( \tau \) is a discrete series extended type, we will assume that \( p > 2 \).

Let \((k, \tau, \overline{\rho}, \psi)\) be a deformation data, and let \( E \) be a finite extension of \( \mathbb{Q}_p \) over which \( \tau \) and \( \psi \) are defined, and such that its residue field contains \( F \). Let \( R(\overline{\rho}) \) be the universal
deformation ring of \( \overline{\rho} \) over \( \mathcal{O}_E \), it is a local noetherian complete \( \mathcal{O}_E \)-algebra. Let \( R^\psi(\overline{\rho}) \) the quotient of \( R(\overline{\rho}) \) that parametrizes deformations of determinant \( \psi \).

Then Kisin in [Kis08] defines deformation rings \( R^\psi(k, \tau, \overline{\rho}) \) that are quotients of \( R^\psi(\overline{\rho}) \). We will use a refinement of these rings introduced in [Roz15], which are better for our purposes in view of Theorem 5.3.1. If the Galois type \( \tau \) is an inertial type, we denote by \( R^\psi(k, \tau, \overline{\rho}) \) the ring classifying potentially crystalline representations with Hodge-Tate weights \((0, k-1)\), inertial type \( \tau \), determinant \( \psi \) with reduction isomorphic to \( \overline{\rho} \), as defined by Kisin in [Kis08]. If the Galois type \( \tau \) is a discrete series extended type, we denote by \( R^\psi(k, \tau, \overline{\rho}) \) the complete local noetherian \( \mathcal{O}_E \)-algebra which is a quotient of \( R^\psi(\overline{\rho}) \), classifying potentially semi-stable representations with Hodge-Tate weights \((0, k-1)\), extended type \( \tau \), determinant \( \psi \) with reduction isomorphic to \( \overline{\rho} \) defined in [Roz15] 2.3.3.

We know that \( R^\psi(k, \tau, \overline{\rho}) \) is a complete flat \( \mathcal{O}_E \)-algebra, such that \( \text{Spec } R^\psi(k, \tau, \overline{\rho})[1/p] \) is formally smooth of dimension 1.

A consequence of the properties of these potentially semi-stable deformation rings is the following: There is a bijection between the maximal ideals of \( R^\psi(k, \tau, \overline{\rho})[1/p] \) and the set of isomorphism classes of lifts \( \rho \) of \( \overline{\rho} \) of determinant \( \psi \), potentially crystalline of inertial type \( \tau \) (resp. potentially semi-stable of extended type \( \tau \)) and Hodge-Tate weights 0 and \( k-1 \). In this bijection, a maximal ideal \( x \), corresponding to a finite extension \( E_x \) of \( E \), corresponds to \( \rho_x : G_{Q_p} \to GL_2(E_x) \).

The Breuil-Mézard conjecture gives us some information about these rings ([BM02], proved in [Kis09], [Pas15], [Pas16], and [Roz15] for the cases of discrete series extended type):

**Theorem 5.1.3.** Let \( \overline{\rho} \) be a continuous representation of \( G_{Q_p} \) of dimension 2, with trivial endomorphisms. If \( p = 3 \), assume that \( \overline{\rho} \) is not a twist of an extension of 1 by \( \omega \), and let \((k, \tau, \overline{\rho}, \psi)\) be a deformation data. Then there is an explicit integer \( \mu_{out}(k, \tau, \overline{\rho}) \) such that \( \overline{\tau}(R^\psi(k, \tau, \overline{\rho})/\pi_E) = \mu_{out}(k, \tau, \overline{\rho}) \).

For our purposes, what is important to know about \( \mu_{out}(k, \tau, \overline{\rho}) \) is that it can be easily computed in a combinatorial way. For more details on the formula for this integer see the introduction of [BM02].

**Definition 5.1.4.** We will say that a representation \( \overline{\rho} \) with trivial endomorphisms is good if it satisfies the hypothesis of Theorem 5.1.3, that is, if \( p = 3 \) then \( \overline{\rho} \) is not a twist of an extension of 1 by \( \omega \).

Note that the condition of trivial endomorphisms implies that \( \overline{\rho} \) is not reducible with scalar semi-simplification.

### 5.2. Rigid spaces attached to deformation rings

We denote by \( \mathcal{X}^\psi(k, \tau, \overline{\rho}) \) the rigid space attached to \( R^\psi(k, \tau, \overline{\rho})[1/p] \) by the construction of Berthelot (see [dJ95, Section 7]).

Let \( p_1, \ldots, p_s \) the minimal prime ideals of \( R^\psi(k, \tau, \overline{\rho}) \), and let \( R_i = R^\psi(k, \tau, \overline{\rho})/p_i \). As \( R^\psi(k, \tau, \overline{\rho}) \) has no \( p \)-torsion, the set of ideals \( (p_i) \) is bijection with the set of minimal prime ideals \( (p'_i) \) of \( R^\psi(k, \tau, \overline{\rho})[1/p] \), with \( R_i[1/p] = R^\psi(k, \tau, \overline{\rho})[1/p]/p'_i \). Let \( \mathcal{X}_i \) be the rigid space attached to \( R_i[1/p] \), then \( \mathcal{X}^\psi(k, \tau, \overline{\rho}) = \bigcup_{i=1}^s \mathcal{X}_i \).

Let \( R_i^0 \) be the integral closure of \( R_i[1/p] \), so that \( R_i \subset R_i^0 \subset R_i[1/p] \) and \( R_i^0 \) is finite over \( R_i \). As \( R_i[1/p] \) is formally smooth, it is normal, hence so is \( R_i^0 \). From [dJ95, Theorem 7.4.1], we deduce that \( R_i^0 \) is equal to the ring \( \Gamma(\mathcal{X}_i, \mathcal{O}^0_{\mathcal{X}_i}) \) of analytic functions on \( \mathcal{X}_i \) that are bounded by 1, and that \( R_i[1/p] \) is equal to the ring of bounded analytic functions on \( \mathcal{X}_i \).

### 5.3. Results
5.3.1. Parameters on deformation spaces.

**Theorem 5.3.1.** Let \((k, \tau, \overline{p}, \psi)\) be a deformation data. There exists a finite extension \(E = E(k, \tau, \overline{p}, \psi)\) of \(\mathbb{Q}_p\), such that if \(\mathcal{X}\) is the rigid space attached to a deformation ring over \(E\), then there exists an analytic function \(\lambda : \mathcal{X} \to \mathbb{P}^1_{E^{\mathrm{rig}}}\) defined over \(E\) that is injective on \(\mathcal{X}(\overline{\mathbb{Q}}_p)\).

This will be proved as Propositions 7.4.1, 7.5.3, 7.6.1 and 7.7.4 with an explanation of the choice of the field \(E(k, \tau, \overline{p}, \psi)\).

**Corollary 5.3.2.** In the conditions of Theorem 5.3.1, the map \(\lambda\) defines an open immersion of analytic spaces. The image of \(\mathcal{X}(\overline{\mathbb{Q}}_p)\) by \(\lambda\) is a standard subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\) that is defined over \(E\).

**Proof.** Let \(\mathcal{X}\) be a rigid analytic space that is smooth of dimension 1, and \(f : \mathcal{X} \to \mathbb{P}^1_{\mathbb{Q}_p}^{\mathrm{rig}}\) a rigid map that induces an injective map \(\mathcal{X}(\overline{\mathbb{Q}}_p) \to \mathbb{P}^1(\overline{\mathbb{Q}}_p)\). Then \(f\) is an open immersion. Indeed, this follows from the well-known fact that an analytic function \(f\) from some open disk \(D\) to \(\mathbb{Q}_p\), that is injective satisfies \(f'(x) \neq 0\) for all \(x \in D\). Now we apply this to \(\mathcal{X} = \mathcal{X}^\psi(k, \tau, \overline{p})\) and \(f = \lambda\). Let \(X = X^\psi(k, \tau, \overline{p})\) be the image of \(\mathcal{X}(\overline{\mathbb{Q}}_p)\) by \(\lambda\). It is clear that \(X\) is defined over \(E\).

Assume first that \(X\) is contained in some bounded subset of \(\mathbb{Q}_p\) (this is automatic when \(\tau\) is an inertial type, see Paragraphs 7.4 and 7.5). Then \(\lambda\) is an analytic open immersion from the quasi-affinoid space \(\mathcal{X}\) to some quasi-affinoid space \(D\) attached to an open disk in \(\mathbb{A}^1_{\mathbb{Q}_p}^{\mathrm{rig}}\). By Corollary 3.2.6, \(X\) is a bounded standard subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\).

We do not assume anymore that \(X\) is contained in some bounded subset of \(\mathbb{Q}_p\). By the Breuil-Mézard conjecture, there is an infinite number of \(\overline{p}'\) with trivial endomorphisms such that \(X' = X^\psi(k, \tau, \overline{p}')\) is non-empty. For such a \(\overline{p}'\), \(X'\) contains a disk \(D(a, r)^-\) for some \(r > 0\) as it is open. For any \(\overline{p}'\) with trivial endomorphisms such that semi-simplification is not the same as the semi-simplification of \(\overline{p}\), we have that the intersection of \(X\) and \(X'\) is empty. So there exists some \(a \in \mathbb{P}^1(\overline{\mathbb{Q}}_p)\) and \(r > 0\) such that \(D(a, r)^- \cap X = \emptyset\). Let \(u\) be an homography sending \(a\) to \(\infty\), then \(u(X)\) is a bounded subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\). This means that \(u \circ \lambda\) is a bounded analytic function on \(\mathcal{X}\). So we can apply the same reasoning as before to show that \(u(X)\) is a bounded standard subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\), and so \(X\) is a standard subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\).

We denote by \(X^\psi(k, \tau, \overline{p})\) the subset \(\lambda(\mathcal{X}^\psi(k, \tau, \overline{p})(\overline{\mathbb{Q}}_p))\) of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\).

5.3.2. Complexity bounds. Now we give more information on the sets \(X^\psi(k, \tau, \overline{p})\).

**Theorem 5.3.3.** Let \((k, \tau, \overline{p}, \psi)\) be a deformation data. Then \(X^\psi(k, \tau, \overline{p})\) is a standard subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\), defined over \(E = E(k, \tau, \overline{p}, \psi)\), with \(c_E(X^\psi(k, \tau, \overline{p})) \leq e(R^\psi(k, \tau, \overline{p})/\pi_E)\). In particular, \(c_E(X^\psi(k, \tau, \overline{p})) \leq \mu_{\mathrm{aut}}(k, \tau, \overline{p})\) if \(\overline{p}\) is good.

**Remark 5.3.4.** Note that the right-hand side of the inequality does not depend on the choice of \(E\), whereas the left-hand side can get smaller when \(E\) has more ramification. In particular, to get a statement as strong as possible we want to take \(E\) with as little ramification as possible.

**Proof.** Let \(p_1, \ldots, p_n\) be the minimal prime ideals of \(R^\psi(k, \tau, \overline{p})\), \(R_i = R^\psi(k, \tau, \overline{p})/p_i\) and \(R^0_i\) be the integral closure of \(R_i\) in \(R_i[1/p]\) as in Section 5.2. Let \(\mathcal{X}_i\) be the rigid space attached to \(R_i[1/p]\), then \(X^\psi(k, \tau, \overline{p})\) is the disjoint union of the \(X_i = \lambda(\mathcal{X}_i(\overline{\mathbb{Q}}_p))\), and each of the \(X_i\) is a standard subset of \(\mathbb{P}^1(\overline{\mathbb{Q}}_p)\) which is defined over \(E\). Then \(A^0_E(X_i) = R^0_i\), so \(c_E(X_i) = [k_{X_i,E} : k_E]\mathcal{O}(R^0_i)\) by definition. Note that \(k_{X_i,E}\) is the residue field of \(R^0_i\), while \(k_E\) is the residue field of \(R_i\). So by Proposition 2.2.2 we have \(c_E(X_i) \leq \tau(R_i)\). So we
Note that in the proof above, the decomposition $X^\psi(k, \tau, \overline{\rho}) = \cup_i X_i$ is the decomposition of $X^\psi(k, \tau, \overline{\rho})$ in standard subsets that are defined and irreducible over $E$. So we also have the following result:

**Proposition 5.3.5.** Let $X^\psi(k, \tau, \overline{\rho}) = \cup_i X_i$, the decomposition of $X^\psi(k, \tau, \overline{\rho})$ in standard subsets that are defined and irreducible over $E$. Then $R^\psi(k, \tau, \overline{\rho})[1/p] = \oplus_i A_E(X_i)$.

Finally, we have the following result:

**Theorem 5.3.6.** Let $(k, \tau, \overline{\rho}, \psi)$ be a deformation data, and assume that $\overline{\rho}$ is good. There exists a finite set $\mathcal{E}$ of finite extensions of $E = E(k, \tau, \overline{\rho}, \psi)$, depending only on $\mu_{\text{aut}}(k, \tau, \overline{\rho})$, such that $X^\psi(k, \tau, \overline{\rho})$ is determined by the sets $X^\psi(k, \tau, \overline{\rho}) \cap F$ for $F \in \mathcal{E}$.

**Proof.** This is a consequence of Theorem 5.1.3 and Corollary 4.5.1, where we take $m = \mu_{\text{aut}}(k, \tau, \overline{\rho})$.

5.4. The case of crystalline deformation rings. We are interested here in the case of the deformation ring of crystalline representations, that we take $\tau$ to be the trivial representation. This case is of particular interest as we are able to deduce additional information.

In this case $R^\psi(k, \text{triv}, \overline{\rho})$ is zero unless $\psi$ is a twist of $\chi_{\text{cyc}}^{k-1}$ by an unramified character. Note that $R^\psi(k, \text{triv}, \overline{\rho})$ and $R^\psi(k, \text{triv}, \overline{\mu})$ are isomorphic as long as $\psi/\psi'$ is an unramified character with trivial reduction modulo $p$. So without loss of generality we will assume from now on that $\psi = \chi_{\text{cyc}}$ and $\det \overline{\rho} = \omega^{k-1}$.

We denote by $R(k, \overline{\rho})$ the ring $R^{k-1}_{\text{cyc}}(k, \text{triv}, \overline{\rho})$. It parametrizes the set of crystalline lifts of $\overline{\rho}$ with determinant $\chi_{\text{cyc}}^{k-1}$ and Hodge-Tate weights $0$ and $k - 1$. We also write $\mu_{\text{aut}}(k, \overline{\rho})$ for $\mu_{\text{aut}}(k, \text{triv}, \overline{\rho})$.

Let $F$ be the extension of $F$ over which $\overline{\rho}$ is defined (so $F = F_\rho$ when $\overline{\rho}$ is irreducible), and $E$ the unramified extension of $Q_p$ with residue field $\mathbb{F}$ (so $E = Q_p$ when $\overline{\rho}$ is irreducible). Then $R(k, \overline{\rho})$ is an $O_E$-algebra with residue field $\mathbb{F}$.

5.4.1. Classification of filtered $\phi$-modules. For $a_p \in \mathbb{Z}_p$ and $F$ a finite extension of $Q_p$ containing $a_p$, we define a filtered $\phi$-module $D_{k,a_p}$ as follows:

$$D_{k,a_p} = Fe_1 \oplus Fe_2$$

$$\phi(e_1) = p^{k-1}e_2, \quad \phi(e_2) = -e_1 + a_pe_2$$

$$\text{Fil}^i D_{k,a_p} = D_{k,a_p} \quad \text{if} \quad i \leq 0$$

$$\text{Fil}^i D_{k,a_p} = Fe_1 \quad \text{if} \quad 1 \leq i \leq k - 1$$

$$\text{Fil}^i D_{k,a_p} = 0 \quad \text{if} \quad i \geq k$$

Denote by $V_{k,a_p}$ the crystalline representation such that $D_{\text{cris}}(V_{k,a_p}) = D_{k,a_p}$. Then: $V_{k,a_p}$ has Hodge-Tate weights $(0, k-1)$ and determinant $\chi_{\text{cyc}}^{k-1}$. Moreover, $V_{k,a_p}$ is irreducible if $v_p(a_p) > 0$, and a reducible non-split extension of an unramified character by the product of an unramified character by $\chi_{\text{cyc}}^{k-1}$ if $v_p(a_p) = 0$. We have the following well-known result:

**Lemma 5.4.1.** Let $V$ be a crystalline representation with Hodge-Tate weights $(0, k-1)$ and determinant $\chi_{\text{cyc}}^{k-1}$. If $V$ is irreducible there exists a unique $a_p \in m_\rho$ such that $V$ is isomorphic to $V_{k,a_p}$. If $V$ is reducible non-split there exists a unique $a_p \in \mathbb{Z}_p$ such that $V$ is isomorphic to $V_{k,a_p}$.
5.4.2. The parameter \(a_p\). We show in Proposition 5.4.1 that the parameter \(a_p\) actually defines a rigid analytic function. This is the function that plays the role of \(\lambda\) of Theorem 5.3.1 for crystalline representations.

From Theorem 5.3.1, we can already deduce some results. It is a well-known conjecture (see [BG16, Conjecture 4.1.1]) that if \(p > 2\), \(k\) is even, and \(v(a_p) \not\in \mathbb{Z}\) then \(\mathcal{T}_{k,a_p}\) is irreducible. From this we get:

**Proposition 5.4.2.** Let \(p > 2\), \(k\) even, \(n \in \mathbb{Z}_{\geq 0}\). If the conjecture above is true, then there is an irreducible representation \(\overline{\rho}\) such that the set \(\{x, n < v_p(x) < n + 1\}\) is contained in \(X(k, \overline{\rho})\).

**Proof.** If the conjecture holds, then the set \(C = \{x, n < v_p(x) < n + 1\}\) is the union of the \(C \cap X(k, \overline{\rho})\) for \(\overline{\rho}\) irreducible. So we have written \(C\) as a finite disjoint union of standard subsets, which means that one of these subsets is equal to \(C\).

5.4.3. Reduction and semi-simplification. We want to show that the case of crystalline deformation rings is accessible to numerical computations. However, we must change slightly our setting: indeed, we can compute numerically only the semi-simplified reduction of \(V_{k,a_p}\). For this we need to express the result of Theorem 5.3.3 in terms of semi-simple representations instead of the conjecture of representations with trivial endomorphisms.

Let \(\rho\) be a semi-simple representation of \(G_{\mathbb{Q}_p}\) with values in \(\text{GL}_2(\overline{\mathbb{F}}_p)\). We define \(Y(k, \rho)\) to be the set \(\{a_p \in D(0, 1)^{\times}, \mathcal{V}_{k,a_p} = \rho\}\). Let \(\overline{\rho}\) be a representation of \(G_{\mathbb{Q}_p}\) with trivial endomorphisms with semi-simplification isomorphic to \(\rho\). Let \(X'(k, \overline{\rho}) = X(k, \overline{\rho}) \cap D(0, 1)^{\times}\). This means that we are only interested in elements in \(X(k, \overline{\rho})\) that correspond to irreducible representations \(V_{k,x}\). Then we have that \(X'(k, \overline{\rho}) \subset Y(k, \overline{\rho})\).

**Proposition 5.4.3.** Suppose that either \(\rho\) is irreducible, or \(\overline{\rho}\) is an extension of \(\alpha\) by \(\beta\) where \(\beta/\alpha \not\in \{1, \omega\}\). Then \(X'(k, \overline{\rho}) = Y(k, \overline{\rho})\).

**Proof.** The result is clear when \(\overline{\rho}\) is irreducible. Recall that \(\dim \text{Ext}^1(\alpha, \beta) > 1\) if and only if \(\beta/\alpha \not\in \{1, \omega\}\). Suppose that \(\overline{\rho}\) is an extension of \(\alpha\) by \(\beta\) where \(\beta/\alpha \not\in \{1, \omega\}\).

Let \(x \in Y(k, \overline{\rho})\). There exists a \(G_{\mathbb{Q}_p}\)-invariant lattice \(T \subset V_{k,x}\) such that \(T\) is a non-split extension of \(\alpha\) by \(\beta\), and so isomorphic to \(\overline{\rho}\). This means that \(x \in X'(k, \overline{\rho})\).

**Definition 5.4.4.** We say that \(\overline{\rho}\) is nice if it has trivial endomorphisms and either \(\overline{\rho}\) is irreducible, or \(\overline{\rho}\) is a non-split extension of \(\alpha\) by \(\beta\) where \(\beta/\alpha \not\in \{1, \omega\}\).

We say that a semi-simple representation \(\tau\) is nice if \(\tau\) is not scalar, and in addition when \(p = 3\) if \(\tau\) is not of the form \(\alpha \oplus \beta\) with \(\alpha/\beta \in \{\omega, \omega^{-1}\}\).

Note that any \(\overline{\rho}\) with trivial endomorphisms that is nice is also good, hence satisfies the hypotheses of Theorem 5.1.3. If \(\tau\) is semi-simple and nice, then there exists a nice \(\overline{\rho}\) with trivial endomorphisms such that \(\overline{\rho}^s = \tau\), so we have \(Y(k, \tau) = X'(k, \overline{\rho})\). Note that we can choose such a \(\overline{\rho}\) so that in addition, \(E(\overline{\rho}) = E(\tau)\).

We know some information about the difference between \(X(k, \overline{\rho})\) and \(X'(k, \overline{\rho})\):

**Proposition 5.4.5.** Let \(\overline{\rho}\) be a representation of \(G_{\mathbb{Q}_p}\) with trivial endomorphisms. If \(\overline{\rho}\) is not an extension \(\text{unr}(u)\) by \(\text{unr}(u^{-1})\omega^n\) for some \(n\) which is equal to \(k - 1\) modulo \(p - 1\), and \(u \in \mathbb{F}_p^{\times}\), then \(X(k, \overline{\rho}) \subset D(0, 1)^{\times}\). If \(\overline{\rho}\) is an extension of \(\text{unr}(u)\) by \(\text{unr}(u^{-1})\omega^n\) for some \(u \in \mathbb{F}_p^{\times}\) and \(0 \leq n < p - 1\), and \(n = k - 1\) modulo \(p - 1\), and \(u \not\in \{\pm 1\}\) if \(n = 0\) or \(n = 1\), then \(X(k, \overline{\rho}) \cap \{x, |x| = 1\}\) is the disk \(\{x, |x| = 1\}\).

**Proof.** For \(a_p \in \mathbb{Z}_p^{\times}\), the representation \(V_{k,a_p}\) is the unique crystalline non-split extension of \(\text{unr}(u)\) by \(\text{unr}(u^{-1})\chi_{cyl}^{k-1}\), where \(u \in \mathbb{Z}_p^{\times}\) and \(u\) and \(u^{-1}p^{k-1}\) are the roots of \(X^2 - a_p X + p^{k-1}\). In particular, for any invariant lattice \(T \subset V_{k,a_p}\) such that \(T\) is non-split, we get that \(T\)
is an extension of $\text{unr}(\overline{u})$ by $\text{unr}(\overline{u}^{-1})\omega^{k-1}$. So $X(k, \overline{p})$ does not meet $\{x, |x| = 1\}$ unless $\overline{p}$ has the specific form given. Moreover, $\overline{u} = \overline{u}p$. So $X(k, \overline{p}) \cap \{x, |x| = 1\} \subset \{x, \overline{p} = u\}$.

If $\overline{p}$ is an extension of $\text{unr}(u)$ by $\text{unr}(u^{-1})\omega^n$ for some $u \in \mathbb{F}_p$ and $0 \leq n < p - 1$, the conditions on $(n, u)$ imply there is a unique non-split extension of $\text{unr}(u)$ by $\text{unr}(u^{-1})\omega^n$, and so $X(k, \overline{p}) \cap \{x, |x| = 1\} = \{x, \overline{p} = u\}$.

**Remark 5.4.6.** We could actually also determine $X(k, \overline{p}) \cap \{x, |x| = 1\}$ when $n = 1$ and $u \in \{\pm 1\}$. However, we will have to exclude this case later (see Proposition 5.4.3), so we do not need it.

### 5.4.4. Local constancy results

We recall the following results:

**Proposition 5.4.7.** Let $a_p \in n_n$. If $a_p \neq 0$, then for all $a_p'$ such that $v_p(a_p - a_p') > 2v_p(a_p) + \left\lfloor p(k - 1)/(p - 1)^2 \right\rfloor$, we have $\overline{V}_{k,a_p} \simeq \overline{V}_{k,a_p'}$. Moreover, $\overline{V}_{k,a_p} \simeq \overline{V}_{k,0}$ for all $a_p$ with $v_p(a_p) > \left\lfloor (k - 2)/(p - 1) \right\rfloor$.

**Proof.** The result for $a_p \neq 0$ is Theorem A of [Ber12]. The result for $a_p = 0$ is the main result of [BLZ04].

**Corollary 5.4.8.** Let $X'(k, \overline{p}) = X(k, \overline{p}) \cap D(0, 1)^{-1}$. If $\overline{p}$ is not an extension $\text{unr}(u)$ by $\text{unr}(u^{-1})\omega^n$ for some $n$ which is equal to $k - 1$ modulo $p - 1$, then $X'(k, \overline{p}) = X(k, \overline{p})$ and $c_E(X'(k, \overline{p})) \leq c(R(k, \overline{p}))$. If $\overline{p}$ is good and is an extension $\text{unr}(u)$ by $\text{unr}(u^{-1})\omega^n$ for some $n$ which is equal to $k - 1$ modulo $p - 1$, and $u \notin \{\pm 1\}$ if $n = 0$ or $n = 1$, then $c_E(X'(k, \overline{p})) \leq \overline{c}(R(k, \overline{p})) - 1$.

**Proof.** The first part is clear by Proposition 5.4.3.

For the second part, we can write $X(k, \overline{p})$ as a disjoint union of $X'(k, \overline{p})$ and $X^+(k, \overline{p}) = X(k, \overline{p}) \cap \{x, |x| = 1\}$, and both are standard subsets defined over $E$, so $c_E(X(k, \overline{p})) = c_E(X'(k, \overline{p})) + c_E(X^+(k, \overline{p}))$. By Proposition 5.4.5, $c_E(X^+(k, \overline{p})) = 1$ under the hypotheses, hence the result.

### 5.4.5. Computation of $Y(k, \tau)$

We explain now how we can compute numerically the sets $Y(k, \tau)$ for $\tau$ nice (and hence the sets $X(k, \overline{p})$ for $\overline{p}$ with nice semi-simplification).

From Corollary 5.4.8 we deduce:

**Proposition 5.4.9.** Suppose that $\tau$ is nice, and let $\overline{p}$ be nice with $\overline{p}^{ss} = \tau$. Then $Y(k, \overline{p})$ is a standard subset of $D(0, 1)^{-1}$ defined over $E = E(\overline{p})$, with $c_E(Y(k, \overline{p})) \leq \mu_{\text{aut}}(k, \overline{p})$. Moreover if $\overline{p}$ is an extension of an unramified character by another character then $c_E(Y(k, \overline{p})) \leq \mu_{\text{aut}}(k, \overline{p}) - 1$.

**Theorem 5.4.10.** Suppose that the semi-simple representation $\tau$ is nice. Then there exists a finite set $\mathcal{E}$ of finite extensions of $E = E(\overline{p})$, depending only on $k$ and $\tau$, such that $Y(k, \tau)$ is determined by the sets $Y(k, \tau) \cap F$ for $F \in \mathcal{E}$.

**Proof.** This is Corollary 4.5.1 where we take for $E$ the field $E(\overline{p})$, and for $m$ the bound given by Proposition 5.4.9 that is $m = \mu_{\text{aut}}(k, \overline{p})$ or $\mu_{\text{aut}}(k, \overline{p}) - 1$ where $\overline{p}$ is some nice representation with $\overline{p}^{ss} = \tau$.

**Theorem 5.4.11.** Suppose that the semi-simple representation $\tau$ is nice. Then there exists a finite set of points $\mathcal{P} \subset D(0, 1)^{-1}$, depending only on $k$ and $\tau$, such that $Y(k, \tau)$ is determined by $Y(k, \tau) \cap \mathcal{P}$.

**Proof.** This is Corollary 4.5.2 where we take for $E$ the field $E(\overline{p})$, for $m$ the bound given by Proposition 5.4.9, and for $\varepsilon$ we can take the norm of an element of valuation $\left\lfloor 3p(k - 1)/(p - 1)^2 \right\rfloor$ by Proposition 5.4.7.
As a consequence, we see that if we are able to compute $V_{k,a_p}^{ss}$ for given $p$, $k$, $a_p$, then we can compute $Y(k,\tau)$ for $\tau$ nice in a finite number of such computations, bounded in terms of $E(\tau)$ and $k$. We give some examples of such computations in Section 6.

We give a last application of these results: It follows from the formula giving $\mu_{aut}(k,\overline{\rho})$ that there exists an integer $m(k)$, depending only on $k$, such that $\mu_{aut}(k,\overline{\rho}) \leq m(k)$ for all $\overline{\rho}$. The optimal value for $m(k)$ is of the order of $4k/p^2$ when $k$ is large.

In general, the value of $V_{k,a_p}^{ss}$ depends on more information than just the valuation of $a_p$. But there are some cases where it depends only on $v_p(a_p)$:

**Corollary 5.4.12.** Fix $k$, and let $m$ be an integer such that $m \geq \overline{c}(R(k,\overline{\rho}))$ for all nice $\overline{\rho}$ with trivial endomorphisms. Let $a$ and $b$ be rational numbers such that for all rational $c$ between $a$ and $b$, the denominator of $c$ is strictly larger than $m$. Then either for all $a_p$ with $a < v_p(a_p) < b$, $V_{k,a_p}^{ss}$ is not nice, or $V_{k,a_p}^{ss}$ is constant on the annulus $A(a,b)$.

In particular, let $c \in \mathbb{Q}$ with denominator strictly larger than $m$. Then either for all $a_p$ with $v_p(a_p) = c$, $V_{k,a_p}^{ss}$ is not nice, or $V_{k,a_p}^{ss}$ is constant on the circle $C(c)$.

Note that if $p > 3$ and $k$ is even, $V_{k,a_p}^{ss}$ is always nice.

*Proof.* Suppose that there exists at least an $a_p$ in $A(a,b)$ such that $\tau = V_{k,a_p}^{ss}$ is nice. Then $c_E(Y(k,\tau)) \leq m$ for $E = E(\overline{\rho})$ which is an unramified extension of $\mathbb{Q}_p$. So we can apply Corollary 4.5.16 the annulus $A(a,b)$ is a subset of $Y(k,\tau)$.

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6. Numerical examples

We give some numerical examples for the deformations rings of crystalline representations. We have computed some examples of $X(k,\overline{\rho})$ using Theorem 5.4.11 and a computer program written in SAGE ([SAGE]) that implements the algorithm described in [Roz].

We also used the fact that $V_{k,a_p}^{ss}$ is known for $v_p(a_p) < 2$ in almost all cases, by the results of [BG09, BG13, GG15, BG15, BGR15], which reduces the number of computations that are necessary to determine $X(k,\overline{\rho})$.

We make the following remark: let $\overline{\rho}$ be a representation such that $\overline{\rho} \otimes \text{unr}(-1)$ is isomorphic to $\overline{\rho}$. Then $X(k,\overline{\rho})$ is invariant by $x \mapsto -x$. Indeed, $V_{k,-a_p}$ is isomorphic to $V_{k,a_p} \otimes \text{unr}(-1)$. This applies in particular when $\overline{\rho}$ is irreducible.

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6.1. Observations for $p = 5$. We have computed $X(k,\overline{\rho})$ for $p = 5$, $k$ even, $k \leq 102$, or $k$ odd and $k \leq 47$, and $\overline{\rho}$ irreducible (so in this case we have $E(\overline{\rho}) = \mathbb{Q}_p$).

We summarize here some observations from these computations:

1. In each case, we have $V_{k,a_p}^{ss} = V_{k,0}^{ss}$ for all $a_p$ with $v_p(a_p) > [(k-2)/(p+1)]$, and not only $v_p(a_p) > [(k-2)/(p-1)]$ which is the value predicted by [BLZ04].
2. In each case, we have $c_{Q_p}(X(k,\overline{\rho})) = \overline{c}(R(k,\overline{\rho}))$, that is, the inequality of Proposition 5.4.9 is an equality.
3. Each disk $D$ appearing in the description of a $X(k,\overline{\rho})$ has $\gamma_{Q_p}(D) = 1$.
4. Each disk $D$ appearing in a $X(k,\overline{\rho})$ is defined over an extension of $\mathbb{Q}_p$ of degree at most 2, which is unramified if $k$ is even and totally ramified if $k$ is odd.
5. For each disk $D$ appearing in a $X(k,\overline{\rho})$, either $0 \in D$, or $D$ is included in the set \{ $x$, $v_p(x) = n$ \} for some $n \in \mathbb{Z}_{\geq 0}$ if $k$ is even, and in the set \{ $x$, $v_p(x) = n + 1/2$ \} for some $n \in \mathbb{Z}_{\geq 0}$ if $k$ is odd.

It would be interesting to know which of these properties hold in general. Property (1) is expected to be in fact true for all $p$ and $k$, but nothing is known about the other properties. We comment further on Property (2) in Section 6.4.
6.2. Some detailed examples. Let $p = 5$. Let $\tau_0 = \text{ind} \omega_2$ and $\tau_1 = \text{ind} \omega_3^p$, and for all $n$, $\tau(n) = \tau \otimes \omega^n$. We describe a few examples of sets $X(k, \tau)$. In each case, the sets given contain all the values of $a_p$ for which $V_{k,a_p}^s$ is irreducible. We also give the generic fibers of the deformation rings.

6.2.1. The case $k = 26$. We get that:
- $X(26, \tau_0) = \{ x, v_p(x) < 2 \} \cup \{ x, v_p(x) > 2 \}$, with $c_{Q_p}(X(26, \tau_0)) = 3$, and $R(26, \tau_0)[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p) \times (\mathbb{Z}_p[[X,Y]]/(XY - p) \otimes \mathbb{Q}_p)$.
- $X(26, \tau_1(2)) = \{ x, v_p(x - a) > 3 \} \cup \{ x, v_p(x + a) > 3 \}$, where $a = 4 \cdot 5^2$, with $c_{Q_p}(X(26, \tau_1(2))) = 2$, and $R(26, \tau_1(2))[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p)^2$.
- $X(26, \tau_1(1)) = \{ x, 2 < v_p(x-a) < 3 \} \cup \{ x, 2 < v_p(x+a) < 3 \}$, with $c_{Q_p}(X(26, \tau_1(1))) = 4$, and $R(26, \tau_1(1))[1/p] = (\mathbb{Z}_p[[X,Y]]/(XY - p) \otimes \mathbb{Q}_p)^2$.

Here we see an example where the geometry begins to be a little complicated, with annuli that do not have 0 as a center.

6.2.2. The case $k = 28$. We get that:
- $X(28, \tau_1) = \{ x, 0 < v_p(x) < 1 \} \cup \{ x, v_p(x) > 2 \} \cup \{ x, v_p(x - a) < 4 \} \cup \{ x, v_p(x + a) < 4 \}$, with $c_{Q_p}(X(28, \tau_1)) = 5$, and $R(28, \tau_1)[1/p] = (\mathbb{Z}_p[[X,Y]]/(XY - p) \otimes \mathbb{Q}_p) \times (\mathbb{Z}_p[[X,Y,Z]]/(XY - p - a/p^2) \otimes \mathbb{Q}_p)^2$.
- $X(28, \tau_0(1)) = \{ x, 1 < v_p(x) < 2 \}$, with $c_{Q_p}(X(28, \tau_0(1))) = 2$, and $R(28, \tau_0(1))[1/p] = (\mathbb{Z}_p[[X,Y]]/(XY - p) \otimes \mathbb{Q}_p)$.
- $X(28, \tau_0(3)) = \{ x, v_p(x-a) > 4 \} \cup \{ x, v_p(x+a) > 4 \}$, with $c_{Q_p}(X(28, \tau_0(3))) = 2$, and $R(28, \tau_0(3))[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p)^2$.

Here we see an example with an irreducible component that has complexity 3.

6.2.3. The case $k = 30$. We get that:
- $X(30, \tau_0) = \{ x, 0 < v_p(x) < 1 \} \cup \{ x, v_p(x) > 4 \}$, with $c_{Q_p}(X(30, \tau_0)) = 3$, and $R(30, \tau_0)[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p) \times (\mathbb{Z}_p[[X,Y]]/(XY - p) \otimes \mathbb{Q}_p)$.
- $X(30, \tau_0(2)) = \{ x, v_p(x - a) > 3 \} \cup \{ x, v_p(x + a) > 3 \}$, where $a = 5^3 \cdot 2$, with $c_{Q_p}(X(30, \tau_0(2))) = 2$, and $R(30, \tau_0(2))[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p)^2$.
- $X(30, \tau_1(1)) = \{ x, 1 < v_p(x) < 3 \} \cup \{ x, 3 < v_p(x) < 4 \}$, with $c_{Q_p}(X(30, \tau_1(1))) = 4$, and $R(30, \tau_1(1))[1/p] = (\mathbb{Z}_p[[X,Y]]/(XY - p) \otimes \mathbb{Q}_p) \times (\mathbb{Z}_p[[X,Y]]/(XY - p^2) \otimes \mathbb{Q}_p)$.

The interesting part here is $X(30, \tau_0(2))$: we see that $\mathcal{A}^0_{Q_p}(X(30, \tau_0(2)))$, which is a domain, has residue field $\mathbb{F}_{p^2}$, whereas $R(30, \tau_0(2))$ has residue field $\mathbb{F}_p$. So $R(30, \tau_0(2)) \neq \mathcal{A}^0_{Q_p}(X(30, \tau_0(2)))$.

6.3. Criteria for non-normality. Recall the notation of Section 5.2 Then we see, by Proposition 5.3.5 if we know $X(k, \overline{\tau})$ then we know $R(k, \overline{\tau})[1/p] = \oplus_i R_i[1/p] = \oplus_i \mathcal{A}_E(X_i)$. We can ask whether we can recover each $R_i$, that is, if $R_i = \mathcal{A}_E(X_i)$, or equivalently if $R_i = R_i^0$ for all $i$ (the description of $X(k, \overline{\tau})$ gives no indication about how the $R_i$ glue together so we can not hope for complete information on $R(k, \overline{\tau})$ anyway if it is not irreducible). We do not expect this to hold, as this would mean that each of the $R_i$ is a normal ring. So we can ask instead, how can we recognize when $R_i$ is not $R_i^0$?

A first criterion is when they have different residue fields, as in the example of $R(30, \tau_0(2))$ in Paragraph 6.2.3. Another criterion is when $R_i$ and $R_i^0$ have the same residue field (a situation that we can always obtain by replacing $E$ by an unramified extension, which does not change the complexities), but $\overline{\tau}(R_i^0) < \overline{\tau}(R_i)$. This is a situation that does not seem to arise often, see Section 6.4.

We give a last, more subtle criterion. Let $X_i$ be one of the components of $X(k, \overline{\tau})$, and assume that each of the disks that appears in the description of $X_i$ is defined over
\( \mathbb{Q}_p \), and has complexity 1. In this case, a closer look at the proof of Proposition 4.2.6 show that \( \text{Spec}(A_{q_p}(X_i)/p) \) has exactly \( c_{Q_p}(X_i) \) distinct irreducible components. On the other hand, the geometric version of the Breuil-Mézard conjecture, proved in [EG14], shows that if \( \overline{\tau} \) is irreducible then \( \text{Spec}(R(k, \overline{\tau})/p) \) has at most two irreducible components (which can have large multiplicity), and so \( \text{Spec}(R_i/p) \) also has at most two irreducible components. So if \( c_{Q_p}(X_i) > 2 \) then we certainly have that \( R_i \neq R_0^i \). This happens for example for the second irreducible component of \( X(28, \tau_i) \). It would be interesting in this case to understand how the irreducible components of \( \text{Spec}(R_i^i/p) \) map to the irreducible components of \( \text{Spec}(R_i/p) \).

6.4. Complexity and multiplicity. An interesting result coming from our computations is the following: for \( p = 5 \), for all irreducible representation \( \overline{\tau} \), for all \( k \leq 47 \) and all even \( k \leq 102 \), we have that \( c_{Q_k}(X(k, \overline{\tau})) = \overline{\tau}(R(k, \overline{\tau})) \), instead of simply the inequality \( c_{Q_k}(X(k, \overline{\tau})) \leq \overline{\tau}(R(k, \overline{\tau})) \). Given this, it is tempting to make the following conjecture: For all \( p > 2 \), for all \( k \geq 2 \) and for all irreducible \( \overline{\tau} \), we have that \( c_{Q_k}(X(k, \overline{\tau})) = \overline{\tau}(R(k, \overline{\tau})) \).

Note that this equality between complexity and multiplicity does not necessarily hold when \( \overline{\tau} \) is reducible. However, it may be true that for all \( p > 2 \), for all \( k \geq 2 \), there is only a finite number of reducible (nice) representations \( \overline{\tau} \) for which the equality does not hold.

We can also reformulate this equality in a different way: recall the notation of Section 5.2. So \( R(k, \overline{\tau}) \) has a family of quotients \( R_i \) that are integral domains, and \( \overline{\tau}(R(k, \overline{\tau})) = \sum_i \overline{\tau}(R_i) \). On the other hand, \( c_{Q_k}(X(k, \overline{\tau})) = \sum_i [k_{R_i^i} : \mathbb{F}_p][\overline{\tau}(R_i^i)] \) where \( k_{R_i^i} \) is the residue field of \( R_i^i \). The equality between complexity and multiplicity can be reformulated as saying that for all \( i \), \( \overline{\tau}(R_i) = [k_{R_i^i} : \mathbb{F}_p][\overline{\tau}(R_i^i)] \). Written in this way without any reference to the sets \( X(k, \overline{\tau}) \), the equality can be generalized to any potentially semi-stable deformation ring, including those that are of dimension larger than 1, such as the deformation rings classifying representations of dimension \( \geq 2 \) or representations of \( G_K \) for some finite extension \( \mathbb{K}/\mathbb{Q}_p \).

7. Parameters classifying potentially semi-stable representations

7.1. Results on Weil representations.

7.1.1. Field of definition. Let \( W_{Q_p} \) be the Weil group of \( \mathbb{Q}_p \). A Weil representation is a representation of \( W_{Q_p} \) with coefficients in \( \mathbb{Q}_p \), that is trivial on an open subgroup of \( I_{Q_p} \).

Let \( \tau \) be a Weil representation. The field of definition of \( \tau \), denoted by \( E(\tau) \), is the subfield of \( \mathbb{Q}_p \) generated by the tr \( \tau(x) \), \( x \in W_{Q_p} \). This is a finite extension of \( \mathbb{Q}_p \), as a Weil representation factors through a finitely generated group.

Let \( E \) be a finite extension of \( \mathbb{Q}_p \). We say that \( \tau \) is realizable over \( E \) if there is a representation \( \tau' : W_{Q_p} \to \text{GL}_n(E) \) that is isomorphic to \( \tau \). Then we have:

**Lemma 7.1.1.** Let \( \tau \) be an irreducible Weil representation. Then there exists a finite unramified extension \( E \) of \( E(\tau) \) such that \( \tau \) is realizable over \( E \).

**Proof.** From the results of [Kra83, 1.4], we see that the obstruction to realizing \( \tau \) over \( E(\tau) \) is in the Brauer group of \( E(\tau) \). An element of the Brauer group can be killed by taking a finite unramified extension, hence the result.

7.1.2. \((\phi, \text{Gal}(F/Q_p))-\text{modules.}\) We fix a finite Galois extension \( F \) of \( \mathbb{Q}_p \), and denote by \( F_0 \) the maximal subextension of \( F \) that is unramified over \( \mathbb{Q}_p \).

Let \( A \) be a \( \mathbb{Q}_p \)-algebra. Then a \((\phi, \text{Gal}(F/Q_p))-\text{module} \) \( M \) over \( F_0 \otimes_{\mathbb{Q}_p} A \) is a free \( F_0 \otimes_{\mathbb{Q}_p} A \)-module of finite rank, endowed with commuting actions of an automorphism \( \phi \) and the group \( \text{Gal}(F/Q_p) \). The action of \( \phi \) is \( A \)-linear and \( F_0 \)-semi-linear (with respect to
the Frobenius automorphism of $F_0$, and the action of $\text{Gal}(F/Q_p)$ is $F_0$-semi-linear (with respect to the action of $\text{Gal}(F/Q_p)$ on $F_0$) and $A$-linear.

Then:

**Proposition 7.1.2.** Let $A$ be an $F_0$-algebra. Then there is an equivalence of categories between $(\phi, \text{Gal}(F/Q_p))$-modules over $F_0 \otimes_{Q_p} A$ and Weil representations over a free $A$-module that are trivial on $I_F$, and this equivalence preserves rank. Moreover this construction is functorial in $A$ (in the category of $F_0$-algebras).

**Proof.** For a given $A$, the construction of the Weil representation from the $(\phi, \text{Gal}(F/Q_p))$-module is explained in [BM02], and the converse construction is immediate. $\square$

We will make use of this equivalence as some things are more naturally expressed in terms of $(\phi, \text{Gal}(F/Q_p))$-modules, whereas others are more easily proved in terms of representations of the Weil group (for example Proposition 7.3.2).

In the same situation, we also define a $(\phi, N, \text{Gal}(F/Q_p))$-module over $F_0 \otimes_{Q_p} A$ to be a $(\phi, \text{Gal}(F/Q_p))$-module over $F_0 \otimes_{Q_p} A$ that is additionally endowed with a $F_0 \otimes_{Q_p}$-linear endomorphism $N$ satisfying $N\phi = p\phi N$ that commutes with the action of $\text{Gal}(F/Q_p)$.

### 7.2. Universal (filtered) $(\phi, N)$-modules with descent data

We recall a few definitions concerning objects attached to $p$-adic representations of $G_{Q_p}$. If $F/Q_p$ is a finite extension, we denote by $F_0$ be maximal unramified extension of $Q_p$, contained in $F$.

Let $V$ be a continuous representation of $G_{Q_p}$ over an $E$-vector space for some finite $E/Q_p$. Let $F$ be a finite Galois extension of $Q_p$. We denote by $D^{\text{crys}}(V)$ the $F_0 \otimes_{Q_p} E$-module $(B_{\text{crys}} \otimes_{Q_p} (V)^{G_F}$. It is a $(\phi, \text{Gal}(F/Q_p))$-module over $F_0 \otimes_{Q_p} E$. If $V$ becomes crystalline over $F$ then $D^{\text{crys}}(V)$ is a free $F_0 \otimes_{Q_p} E$-module of rank $\dim_E(V)$. We denote by $D^{\text{st}}(V)$ the $F_0 \otimes_{Q_p} E$-module $(B_{\text{st}} \otimes_{Q_p} V)^{G_F}$. It is endowed with a structure of $(\phi, N, \text{Gal}(F/Q_p))$-module over $F_0 \otimes_{Q_p} E$. If $V$ becomes semi-stable over $F$ then it is a free $F_0 \otimes_{Q_p} E$-module of rank $\dim_E(V)$. If $V$ becomes crystalline over $F$ then $D^{\text{crys}}(V)$ and $D^{\text{st}}(V)$ coincide as $(\phi, \text{Gal}(F/Q_p))$-modules, and $N = 0$. We denote by $D^{\text{dR}}(V)$ the $F \otimes_{Q_p} E$-module $(B_{\text{dR}} \otimes_{Q_p} V)^{G_F}$. It is a $F \otimes_{Q_p} E$-module with a semi-linear action of $\text{Gal}(F/Q_p)$, and is endowed with a separated exhaustive decreasing filtration by sub-$F \otimes_{Q_p} E$-modules that is stable under the action of $\text{Gal}(F/Q_p)$, and satisfies an additional condition called admissibility. If $V$ is potentially semi-stable, then $D^{\text{dR}}(V)$ is an $E$-vector space of dimension $\dim_E V$. Moreover, we have that $D^{\text{dR}}(V) = F \otimes_{F_0} D^{\text{st}}(V)$ as $F \otimes_{Q_p} E$, so this endows $F \otimes_{F_0} D^{\text{st}}(V)$ with a filtration as above, that is, a structure of filtered $(\phi, N, \text{Gal}(F/Q_p))$-module.

**Theorem 7.2.1.** Let $F$ be a finite Galois extension of $Q_p$. Let $X$ be a reduced rigid analytic space, let $V$ be a locally free $O_X$-module of rank $n$ with a continuous action of $G_{Q_p}$. Assume that for all $x \in X$, $V_x$ is potentially semi-stable with weights independent of $x$, and becomes semi-stable on $F$. Then there exists a projective $F_0 \otimes_{Q_p} O_X$-module $D$ of rank $n$, endowed with a structure of $(\phi, N, \text{Gal}(F/Q_p))$-module over $F_0 \otimes_{Q_p} O_X$, such that for all $x$, $D_x$ is isomorphic, as a $(\phi, N, \text{Gal}(F/Q_p))$-module, to $D^{\text{st}}(V_x)$.

**Proof.** This follows immediately from [Bel15 Theorem 5.1.2]: we take the module $D$ to be the module called $D_{B_{\text{st}}}(V)$ there, considering $V$ as a representation of $G_F$ (see also [EC08 Théorème C]). $\square$

**Theorem 7.2.2.** Let $F$ be a finite Galois extension of $Q_p$. Let $X$ be a reduced rigid analytic space, let $V$ be a locally free $O_X$-module of rank $n$ with a continuous action of $G_{Q_p}$. Assume that for all $x \in X$, $V_x$ is potentially semi-stable with weights independent of $x$, and becomes semi-stable on $F$. Then $F \otimes_{F_0} D$ is endowed of a filtration by locally
free sub-$F\otimes_{Q_p}\mathcal{O}_X$-modules, such that the graded parts are also locally free, such that for all $x$, $(F\otimes_{F_0}D)_x$ is isomorphic, as a filtered $(\phi, N, \text{Gal}(F/Q_p))$-module, to $D_{\text{dR}}(V_x)$.

Proof. This follows from [Bel15 Theorem 5.1.7], as $F\otimes_{F_0}D$ is the $F\otimes_{Q_p}\mathcal{O}_X$-module that is called $D_{\text{dR}}(V)$ there, considering $V$ as a representation of $G_F$. Indeed the filtration, and the graded parts, are given by the modules called $D_{\text{dR}}^{[a,b]}(V)$). The point that we need to check is that for all $[a,b]$, the $F\otimes_{Q_p}E_x$-modules $D_{\text{dR}}^{[a,b]}(V_x)$ are actually free (then their rank is independent of $x$ by the condition on the weights). This comes from [Sav05 Lemma 2.1], and here we use the fact that we start from a representation of $G_{Q_p}$.

Let now $(k,\tau,\overline{\tau},\psi)$ be a deformation data, as defined in Definition 5.1.2. Let $E$ be a finite extension of $Q_p$ satisfying the following conditions:

1. the residual representation $\overline{\tau}$ can be realized on the residue field of $E$
2. the type $\tau$ can be realized on $E$
3. the character $\psi$ takes its values in $E^\times$

Let $R^\psi(k,\tau,\overline{\tau})[1/p]$ be the ring defined by Kisin attached to this data, as recalled in Section 5.1. It is an $\mathcal{O}_E$-algebra. We can apply Theorems 7.2.1 and 7.2.2 to the rigid analytic space $X = X^\psi(k,\tau,\overline{\tau})$ attached to the Kisin ring $R^\psi(k,\tau,\overline{\tau})[1/p]$. Indeed, we know that these rings are reduced, and the hypotheses come from the definition of the rings.

7.3. Working in families.

7.3.1. Reduction of an endomorphism.

Proposition 7.3.1. Let $K$ be a field and $A$ be a $K$-algebra. Let $\phi$ be an $A$-linear endomorphism of $A^2$, and assume that the characteristic polynomial of $\phi$ is in fact in $K[X]$, and that it is split over $K$ with distinct eigenvalues. Then, Zariski-locally on $A$, $\phi$ is diagonalizable.

Proof. Let $\lambda$ and $\mu$ be the roots of the characteristic polynomial of $\phi$, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of $\phi$ in the canonical basis of $A^2$ (so that $a + d = \lambda + \mu$ and $ad - bc = \lambda\mu$).

We are looking for a basis $(f_1, f_2)$ of $A^2$, with $f_1 = xe_1 + ye_2$, $f_2 = e_2$, such that the matrix of $\phi$ in this basis is upper triangular. The new basis is as wanted if $x, y$ satisfy one of the following systems of equations:

$$(a - \lambda)x + by = 0 \quad \text{and} \quad cx + (d - \lambda)y = 0$$

or

$$(a - \mu)x + by = 0 \quad \text{and} \quad cx + (d - \mu)y = 0$$

Assume that $u = d - \lambda$ is invertible. We solve the first system by setting $x = 1, y = -c/(d - \lambda)$. In the first case, in our new basis $\phi$ has a matrix of the form $\begin{pmatrix} \lambda & b \\ 0 & d \end{pmatrix}$, and actually $d = \mu$ by the trace condition. As $\lambda - \mu$ is invertible, we can change the basis again so that in the new basis, $\phi$ has matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

Assume now that $v = a - \lambda$ is invertible. Then so is $d - \mu = -v$. We solve the second system by setting $x = 1, y = -c/(d - \mu)$. In this case we do the same thing after exchanging $\lambda$ and $\mu$.

Note that $u + v = \mu - \lambda$ is invertible by assumption. We set $f = (d - \lambda)/(\mu - \lambda)$, $A_1 = A[f^{-1}]$, $A_2 = A[(1 - f)^{-1}]$. Then as we just saw in $A_1$ and $A_2$ there is a basis in which the matrix of $\phi$ is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, which gives the result.
7.3.2. Isomorphism of group representations.

Theorem 7.3.2. Let $K$ be a field of characteristic zero, and $A$ a $K$-algebra.

Let $G$ be a group. Let $\rho : G \to \text{GL}_n(K)$ be a representation that is absolutely irreducible. Let $\rho' : G \to \text{GL}_n(A)$ be a representation. Assume that for all $g \in G$, we have $\text{tr} \rho(g) = \text{tr} \rho'(g)$.

Then, Zariski-locally on $A$, there is an $M \in \text{GL}_n(A)$ such that $\rho'(g) = M \rho(g) M^{-1}$ for all $g \in G$.

Proof. By [Ron96] Théorème 5.1, there is an $A$-algebra automorphism $\tau$ of $M_n(A)$ such that for all $g \in G$, $\rho'(g) = \tau \rho(g)$. By [KO74] IV. Proposition 1.3, there is a family $(f_i)$ in $A$ generating the unit ideal such that for all $i$, the automorphism of $M_n(A[1/f_i])$ induced by $\tau$ is inner. Hence the result. \hfill \Box

7.3.3. Variations on Hilbert 90.

Proposition 7.3.3. Let $K$ be an infinite field, and $L/K$ be a finite Galois extension of fields.

1. Let $M$ be a finite $K$-algebra. Then $H^1(\text{Gal}(L/K), (L \otimes_K M)^\times) = 0$.

2. Let $A$ be a $K$-algebra. Assume that for every maximal ideal $m$ of $A$, $A/m$ is a finite extension of $K$. Let $c \in H^1(\text{Gal}(L/K), (L \otimes_K A)^\times)$. There exists a family of elements $(f_i)$ in $A$ that generate the unit ideal such that the image of $c$ in $H^1(\text{Gal}(L/K), (L \otimes_K A[f_i^{-1}]^\times))$ is zero for all $i$.

Proof. Let $M$ be a $K$-algebra, and $c \in H^1(\text{Gal}(L/K), (L \otimes_K M)^\times)$. Let $x \in L$. We set $\phi(c, x) = \sum_{\gamma \in \text{Gal}(L/K)} \gamma(x)c(\gamma) \in L \otimes_K M$. We have for all $g \in \text{Gal}(L/K)$, $c(g)g(\phi(c, x)) = \phi(c, x)$, so $c = 0$ as soon as we can find an $x$ such that $\phi(c, x)$ is invertible in $L \otimes_K M$.

Point (1) is well-known, and is proved by showing that if $M$ is finite over $K$ then such an $x$ exists, with a proof similar to the case where $M = M_n(K)$ (here we do not need $M$ to be commutative).

For any commutative $K$-algebra $M$, the $M$-algebra $L \otimes_K M$ is finite. We denote by $N_M$ the norm map $L \otimes_K M \to M$, so that for all $x \in L \otimes_K M$, we have $x \in (L \otimes_K M)^\times$ if and only if $N_M(x) \in M^\times$.

Moreover the norm map commutes with base change: let $u : M \to M'$ be a map of $K$-algebras, then $N_{M'}(1 \otimes u)(x) = u(N_M(x))$ for all $x \in L \otimes_K M$.

Let $A$ be as in point (2) and let $c \in H^1(\text{Gal}(L/K), (L \otimes_K A)^\times)$. For an extension $A'$ of $A$, denote by $c_{A'}$ the image of $c$ in $H^1(\text{Gal}(L/K), (L \otimes_K A')^\times)$.

Let $m$ be a maximal ideal of $A$, and $K_m = A/m$. Then $K_m$ is a finite extension of $K$. So there exists an $x \in L$ such that $\phi(c_{K_m}, x)$ is invertible in $L \otimes_K K_m$. Let $f = N_{A'}(\phi(c, x)) \in A$. Then $D_f$ is a neighborhood of $m$ in Spec $A$. Moreover the image of $\phi(c, x)$ in $L \otimes_K A[f^{-1}]$ is invertible, so $c_{A[f^{-1}]} = 0$.

So we see that there is a covering of Spec $A$ by open subsets of the form $D_f$ with $c_{A[f^{-1}]} = 0$, which is what we wanted. \hfill \Box

7.4. The crystalline case. We want to prove Theorem 5.3.1 for the case where the Galois type is of the form (1), that is, $\tau = \chi \otimes \chi$ for some smooth character $\chi$ of $I_{Q_p}$ that extends to $W_{Q_p}$. By twisting by the character $\chi$, we can reduce to the case where $\tau$ is the trivial representation of $I_{Q_p}$, that is, the case of crystalline deformation rings. Recall from Section 5.4.1 the definition of the parameter $a_p$.

Proposition 7.4.1. There is an element $a_p \in R(k, \overline{p})[1/p]$ such that for any finite extension $E_x$ of $E$ and $x : R(k, \overline{p})[1/p] \to E_x$ corresponding to a representation $\rho_x$, $a_p(x)$ is the value of $a_p$ corresponding to $\rho_x$ by the classification of Lemma 5.4.1.
In particular, we can see $a_p$ as an analytic map from $\mathcal{X}(k, \overline{\rho})$ to $\mathbb{A}^1_{\text{rig}}$. Moreover, $a_p$ induces an injective map from $\mathcal{X}(k, \overline{\rho})(\overline{\mathbb{Q}}_p)$ to $D(0, 1)^+$.

**Proof.** Consider the $\phi$-module $D$ which is obtained from applying Theorem 7.2.1 to the rigid space $\mathcal{X}(k, \overline{\rho})$ attached to the ring $R(k, \overline{\rho})[1/p]$. It is a projective module of rank 2 over $R(k, \overline{\rho})[1/p]$ and is such that for all $x : R(k, \overline{\rho})[1/p] \to E_x$ corresponding to a representation $\rho_x$, $D \otimes_{R(k, \overline{\rho})[1/p]} E_x$ is the $\phi$-module $D_x$ attached to $\rho_x$ (forgetting the filtration). Now observe that $a_p$, as defined in Lemma 5.4.1, is the trace of $\phi$ on the dual of $D$, so it is an element of $R(k, \overline{\rho})[1/p]$, and $a_p(x)$ is the evaluation at $x$ of the trace of $\phi$ on the dual of $D$. 

7.5. The crystalline case. We suppose here that $\tau = \chi_1 \oplus \chi_2$, where $\chi_1$ and $\chi_2$ are distinct characters of $I_{Q_p}$ with finite image that extend to characters of $W_{Q_p}$, so that the representations classified by $R^\psi(k, \tau, \overline{\rho})$ become crystalline on an abelian extension of $Q_p$. In this case we show the existence of a function $\lambda$ as in Proposition 5.3.1 when $\chi_1 \neq \chi_2$. We make use of the results of [GM09], which classifies the filtered $\phi$-modules with descent data that give rise to a Galois representation of inertial type $\tau$ and Hodge-Tate weights $(0, k - 1)$. We summarize their results for such a $\tau$.

The characters $\chi_i$ factor through $F = \mathbb{Q}_p(\zeta_m)$ for some $m \geq 1$, so the Galois representations we are interested in become crystalline on $F$, and so are given by filtered $(\phi, \text{Gal}(F/Q_p))$-modules. Note that here $F_0 = \mathbb{Q}_p$.

Let $E$ be a finite extension of $Q_p$ containing the values of $\chi_1$ and $\chi_2$. Let $\alpha, \beta$ be in $O_E$ with $v_p(\alpha) + v_p(\beta) = k - 1$. We define a $(\phi, \text{Gal}(F/Q_p))$-module $\Delta_{\alpha, \beta}$ as follows: let $\Delta_{\alpha, \beta} = Ee_1 \oplus Ee_2$, with $g(e_1) = \chi_1(g)e_1$ and $g(e_2) = \chi_2(g)e_2$ for all $g \in \text{Gal}(F/Q_p)$. The action of $\phi$ is given by: $\phi(e_1) = \alpha^{-1}e_1$ and $\phi(e_2) = \beta^{-1}e_2$. We are looking at filtrations on $\Delta_{\alpha, \beta,F} = F \otimes_{Q_p} \Delta_{\alpha, \beta}$ satisfying $\text{Fil}^i \Delta_{\alpha, \beta,F} = 0$ if $i \leq 1 - k$, $\text{Fil}^i \Delta_{\alpha, \beta,F} = \Delta_{\alpha, \beta}$ if $i > 0$, and $\text{Fil}^i \Delta_{\alpha, \beta,F} = \text{Fil}^0 \Delta_{\alpha, \beta,F}$ for $1 - k < i \leq 0$ is a $F \otimes_{Q_p} E$-line.

We summarize now the results that are given in [GM09, Section 3].

**Proposition 7.5.1.** Fix $\alpha, \beta$ in $O_E$ with $v_p(\alpha) + v_p(\beta) = k - 1$. Then there exists a way to choose $\text{Fil}^0(\Delta_{\alpha, \beta,F}) \subset \Delta_{\alpha, \beta,F} = \Delta_{\alpha, \beta} \otimes F$ that makes it an admissible filtered $(\phi, \text{Gal}(F/Q_p))$-module.

If neither $\alpha$ nor $\beta$ is a unit, then all such choices give rise to isomorphic filtered $(\phi, \text{Gal}(F/Q_p))$-modules, which are irreducible.

If $\alpha$ or $\beta$ is a unit, the choices give rise to two isomorphism classes of filtered $(\phi, \text{Gal}(F/Q_p))$-modules, one being reducible split and the other reducible non-split.

We denote by $D_{\alpha, \beta}$ the isomorphism class of admissible filtered $(\phi, \text{Gal}(F/Q_p))$-module given by a choice of filtration that makes it into either an irreducible module (if neither $\alpha$ nor $\beta$ is a unit) or a reducible non-split module (if $\alpha$ or $\beta$ is a unit).

Then it follows from the computations of [GM09, Section 3] that:

**Proposition 7.5.2.** Let $V$ be a potentially crystalline representation with coefficients in $E$, of inertial type $\tau$ and Hodge-Tate weights $(0, k - 1)$ that is not reducible split. Then there exists a unique pair $(\alpha, \beta) \in O_E$ with $v_p(\alpha) + v_p(\beta) = k - 1$ such that $D^crys_i(V)$ is isomorphic to $D_{\alpha, \beta}$ as a filtered $(\phi, \text{Gal}(F/Q_p))$-module.

Let $E = E(k, \tau, \phi, \psi)$ be a finite extension of $Q_p$ such that $\phi$ can be defined over the residue field of $E$, $E$ contains the images of $\chi_1$ and $\chi_2$ and of the character $\psi$. Then the ring $R^\psi(k, \tau, \phi)$ can be defined over $E$. Moreover:
Proposition 7.5.3. Let \( \overline{p} \) be an element with trivial endomorphisms. There are elements \( \alpha, \beta \in R^0(k, \tau, \overline{p})[1/p] \) such that for each closed point \( x \) of \( \text{Spec } R^\psi(k, \tau, \overline{p})[1/p] \) corresponding to a representation \( \rho_x \), \( D^F_{\text{crys}}(\rho_x) \) is isomorphic to \( \Delta_\alpha(x), \beta(x) \) as a \( (\phi, \text{Gal}(F/Q_p)) \)-module.

Proof. By Theorem 7.2.1 applied to the rigid analytic space \( X^\psi(k, \tau, \overline{p}) \) attached to \( R^\psi(k, \tau, \overline{p})[1/p] \), there exists a \( \phi \)-module \( D \) with descent data by \( \text{Gal}(F/Q_p) \), where \( D \) is a projective module of rank 2 over \( R^\psi(k, \tau, \overline{p})[1/p] \), such that for each closed point \( x \) of \( \text{Spec } R^\psi(k, \tau, \overline{p})[1/p] \), \( D^F_{\text{crys}}(\rho_x) \) is isomorphic to \( D \otimes_R E_x \) (where \( E_x \) is the field of coefficients of \( \rho_x \)) as a \( (\phi, \text{Gal}(F/Q_p)) \)-module.

Applying Proposition 7.3.1, we see that the action of \( \text{Gal}(F/Q_p) \) on \( D \) is given as the action of \( \text{Gal}(F/Q_p) \) on each \( \Delta_{\alpha, \beta} \): that is, Zariski-locally on \( \text{Spec } R^\psi(k, \tau, \overline{p})[1/p] \), we can write \( D = Re_1 \oplus Re_2 \), with \( g(e_1) = \chi_1(g)e_1 \) and \( g(e_2) = \chi_2(g)e_2 \).

As the action of \( \phi \) on \( D \) commutes with the action of \( \text{Gal}(F/Q_p) \), this shows that the eigenvalues of \( \phi \) acting on \( D \) are in fact in \( R^\psi(k, \tau, \overline{p})[1/p] \), that is, \( \alpha \) and \( \beta \) are elements of \( R^\psi(k, \tau, \overline{p})[1/p] \).

Moreover, if we fix the determinant of the Galois representation corresponding to \( D_{\alpha, \beta} \) then we fix \( \alpha \beta \). So the function \( \alpha \) is injective on points, so it can play the role of the function \( \lambda \) of Theorem 5.3.1.

Let \( X^\psi(k, \tau, \overline{p}) \) be the image of \( X^\psi(k, \tau, \overline{p})(\overline{Q}_p) \) in \( \overline{Q}_p \), then we see that \( X^\psi(k, \tau, \overline{p}) \) is contained in the set \( \{x, 0 < v_p(x) < k - 1\} \), with the irreducible representations corresponding the subset of elements that are in \( \{x, 0 < v_p(x) < k - 1\} \).

7.6. Semi-stable representations. We now assume \( p > 2 \) and we study the case of the deformation rings attached to a discrete series extended type of the form \( \tau = \chi_1 \oplus \chi_2 \), where \( \chi_1 \) and \( \chi_2 \) are characters of \( W_{\overline{Q}_p} \) that have the same reduction to inertia, and such that \( \chi_1(F) = p \chi_2(F) \) for any Frobenius element \( F \). As in the case of crystalline representations, we can twist by a smooth character of \( W_{\overline{Q}_p} \) and reduce to the case where \( \chi_1 \) and \( \chi_2 \) are trivial on inertia. Then the deformation rings \( R^e(k, \tau, \overline{p}) \) classify representations that are semi-stable, and only a finite number of the representations that appear can be crystalline.

Let \( \rho \) be a semi-stable, non-crystalline representation of dimension 2 of \( G_{\overline{Q}_p} \), with Hodge-Tate weights \( (0, k - 1) \) for some \( k \geq 2 \). Then we know (see for example [GM09, Section 3.1], that the filtered \( (\phi, N) \)-module \( D_{\alpha, \beta}(\rho) \) is isomorphic to exactly one \( D_{\alpha, \beta} \) for some \( \alpha \) with \( v(\alpha) = k/2 \), some \( \alpha \in \overline{Q}_p \) and some finite extension \( E \) containing \( \alpha \) and \( L \), for \( (\phi, N) \)-modules \( D_{\alpha, \beta} \) defined as follows: \( D_{\alpha, \beta} = Ee_1 \oplus Ee_2 \), \( \phi(e_1) = p\alpha^{-1}e_1 \), \( \phi(e_2) = \alpha e_2 \), \( Ne_1 = e_2 \), \( \text{Fil}^1 D_{\alpha, \beta} = E(e_1 - Le_2) \). Then \( L \) is the \( L \)-invariant of Fontaine, as defined in [Maz94, §9]. Let \( \rho \) be a crystalline representation of dimension 2 of \( G_{\overline{Q}_p} \), we set its \( L \)-invariant to be \( \infty \).

Proposition 7.6.1. Let \( X \) be a rigid analytic space defined over some finite extension \( E \) of \( Q_p \). Assume that \( X \) is endowed with a 2-dimensional representation \( \rho \) of \( G_{\overline{Q}_p} \), such that for all \( x \in X \), \( \rho_x \) is semi-stable with Hodge-Tate weights \( (0, k - 1) \), the Weil representation attached to \( \rho_x \) is independent of \( x \), there exists at least one \( x \) such that \( \rho_x \) is not crystalline, and none of the \( \rho_x \) is reducible split. Then there exists a rigid analytic map \( L : X \to \mathbb{P}^1_E \), defined over \( E \), such that for all \( x \), \( L(x) \) is the \( L \)-invariant of \( \rho_x \).

Note that under these conditions, the \( \alpha \) of \( D_{\alpha, \beta} \) is independent of \( x \), and is in \( E \).

This proposition applies in the following situation: let \( p > 2 \), let \( X = X^\psi(k, \tau, \overline{p}) \) be the deformation space for the extended type \( \tau \), and \( \overline{p} \) is not reducible split. Then the function \( L \) can play the role of \( \lambda \) of Proposition 5.3.1.
Proof. In order to prove this result, it is enough to prove it for an admissible covering of \( \mathcal{X} \). Indeed, the condition that \( \mathcal{L}(x) \) is the \( \mathcal{L} \)-invariant of \( \rho_x \) ensures that the functions defined on each subset of the covering will glue. In particular, we can assume that \( \mathcal{X} \) is affinoid, coming from a Tate algebra \( A \) over \( E \).

By Theorems 7.2.1 and 7.2.2, there is a projective \( A \)-module \( D \) of rank 2 over \( A \), endowed with a structure of filtered \((\phi, N)\)-module, such that for all \( x \in \text{Max}(A) \), \( D_x \) is \( D_{\text{Gal}}(\rho_x) \). Consider the action of \( \phi \) on \( D \): it has eigenvalues \( px^{-1} \) and \( \alpha^{-1} \). By Proposition 7.3.1, we can assume, after replacing \( A \) by a Zariski covering, that \( D \) is free over \( A \), with a basis \( e_1, e_2 \) such that \( \phi(e_1) = px^{-1}e_1 \) and \( \phi(e_2) = \alpha^{-1}e_2 \). By the commutation relations between \( \phi \) and \( N \), there is a \( \lambda \in A \) such that \( Ne_1 = \lambda e_2 \). Moreover, we can assume that there is a free \( A \)-module \( L \) of rank 1 in \( D \), with quotient that is also free of rank 1, that gives the non-trivial step of the filtration. We fix a basis \( f \) of \( L \).

Let \( h = \det(f, \phi(f)) \). Let us show that \( N \) and \( h \) do not vanish simultaneously. If this is the case, let \( x \) be a point where they both vanish. Then \( \rho_x \) is crystalline, as \( N_x = 0 \), and the filtration of the associated filtered \( \phi \)-module is generated by an eigenvector of \( \phi \), as \( h_x = 0 \). Then the representation \( \rho_x \) is necessarily split reducible. But by hypothesis this can not happen. So by replacing \( \text{Max}(A) \) by a Zariski cover, we can assume that either \( N \) never vanishes, or \( h \) in a unit in \( A \).

Assume first that \( N \) never vanishes, that is, \( \rho_x \) is never crystalline. Then the \( \lambda \) as defined above is actually a unit in \( A \), so we can modify the basis \((e_1, e_2)\) so that \( \lambda = 1 \). Write \( f \) in this basis as \( ae_1 + be_2 \), with \( a, b \in A \). By specializing at each \( x \in \text{Max}(A) \), we see that \( a(x) \neq 0 \) for all \( x \), as this would contradict the admissibility condition of the filtered module. So \( a \in A^{\times} \). Then by definition of the \( \mathcal{L} \)-invariant, we have \( \mathcal{L}(x) = -(b/a)(x) \) for all \( x \in \text{Max}(A) \). So the function \( \mathcal{L} \) is indeed an analytic function on \( \text{Max}(A) \).

Assume now that \( h \) is a unit in \( A \). Let \((e_1, e_2)\) be the basis of \( D \) defined above such that each \( e_i \) is an eigenvector for \( \phi \). We can write \( f = ae_1 + be_2 \) for some \( a, b \in A \). Then the condition on \( h \) implies that \( a \) and \( b \) are in \( A^{\times} \), that is, \((ae_1, be_2)\) is also a basis of \( D \) over \( A \). So we can modify the basis so that we have moreover \( f = e_1 + e_2 \). After specializing at \( x \in \text{Max}(A) \) an easy computation shows that \( \lambda(x) = -1/\mathcal{L}(x) \) (and in particular the condition on \( h \) implies that \( \mathcal{L} \) does not take the value 0). So we have defined an analytic function \( \text{Max}(A) \to \mathbb{P}^1 \) by taking \( \mathcal{L} = 1/\lambda \).

\[
7.7. \textbf{Supersingular types.} \text{ In this Section, assume that } p > 2. \text{ We consider now the case where the type is supersingular, that is, the Weil representation is (absolutely) irreducible.}
\]

7.7.1. \textit{Defining the generalized \( \mathcal{L} \)-invariant.} We fix once and for all a supersingular extended type \( \tau \), that is, a smooth absolutely irreducible representation \( \tau: W_{\mathbb{Q}_p} \to \text{GL}_2(E_0) \) for some finite extension \( E_0 \) of \( \mathbb{Q}_p \). This corresponds to cases (2) and (3) of the classification of types of [GM09, Lemma 2.1]. Note that we can take \( E_0 \) to be an unramified extension of the definition field of \( \tau \) by Lemma 7.1.1.

Let \( F \) be a finite Galois extension of \( \mathbb{Q}_p \) such that \( \tau \) is trivial on \( I_F \), and let \( F_0 \) be the maximal unramified extension of \( \mathbb{Q}_p \) contained in \( F \). We assume, after taking an unramified extension of \( E_0 \) if necessary, that \( F_0 \subset E_0 \).

Let \( D_{\text{crys}, 0} \) be the \((\phi, \text{Gal}(F/\mathbb{Q}_p))\)-module corresponding to \( \tau \) via the correspondence of Proposition 7.1.2. Let \( D_{\text{dr}, 0} = F \otimes_{F_0} D_{\text{crys}, 0} \). It is endowed with an action of \( \text{Gal}(F/\mathbb{Q}_p) \) coming from the one on \( D_{\text{crys}, 0} \). Then:

\[
\textbf{Lemma 7.7.1.} \text{ Assume that there exists at least one potentially crystalline representation } \rho \text{ with coefficients in } E \text{ for some finite extension } E \text{ of } E_0, \text{ such that } D_{\text{dr}}(\rho) \text{ is isomorphic to } D_{\text{dr}, 0}(\text{Gal}(F/\mathbb{Q}_p)) \otimes_{E_0} E \text{ as a } F \otimes_{\mathbb{Q}_p} E \text{-module with an action of } \text{Gal}(F/\mathbb{Q}_p). \text{ Then } D_{\text{dr}, 0}(\text{Gal}(F/\mathbb{Q}_p)) \text{ is an } E_0 \text{-vector space of dimension 2.}
\]
Proof. Let \( D = D_{\text{dR},0} \otimes_{E_0} E \), with its action of \( \text{Gal}(F/\mathbb{Q}_p) \), which is isomorphic to the \( \phi \)-module \( D^F_{\text{dR}}(\rho) \) with its action of \( \text{Gal}(F/\mathbb{Q}_p) \) for some potentially crystalline representation \( \rho \). Then \( D^F_{\text{dR}}(\rho)^{\text{Gal}(F/\mathbb{Q}_p)} = D^Q_{\text{dR}}(\rho) \) is an \( E \)-vector space of dimension 2, as \( \rho \) is de Rham as a \( G_{\mathbb{Q}_p} \)-representation. The action of \( \text{Gal}(F/\mathbb{Q}_p) \) on \( D_{\text{dR},0} \) is \( E_0 \)-linear. So the dimension of its subspace of fixed elements is invariant by extension of scalars. Hence the result. \( \square \)

Remark 7.7.2. We could also make use of the results of \cite{GM09}, which give an explicit basis of the \( E \)-vector space \( (D_{\text{dR},0} \otimes_{E_0} E)^{\text{Gal}(F/\mathbb{Q}_p)} \) for some extension \( E \) of \( E_0 \).

We denote by \( V_\tau \) the \( E_0 \)-vector space of dimension 2 given by Lemma 7.7.1.

Any potentially semi-stable representation of extended type \( \tau \) becomes crystalline when restricted to \( G_F \). For any such representation \( \rho \), with coefficients in an extension \( E \) of \( E_0 \), \( D^F_{\text{crys}}(\rho) \) is a \( (\phi, \text{Gal}(F/\mathbb{Q}_p)) \)-module over \( F_0 \otimes_{\mathbb{Q}_p} E \). We have that \( D^F_{\text{dR}}(\rho) \) is canonically isomorphic to \( F \otimes_{F_0} D^F_{\text{crys}}(\rho) \), and is endowed with an admissible filtration. Moreover, \( D^F_{\text{dR}}(\rho)^{\text{Gal}(F/\mathbb{Q}_p)} = D^Q_{\text{dR}}(\rho) \) is an \( E \)-vector space of dimension 2.

We also fix an integer \( k \geq 2 \), a continuous character \( \psi : G_{\mathbb{Q}_p} \rightarrow E_0^* \). Note that there is no loss of generality in considering only characters with values in \( E_0 \), as the compatibility condition between type and determinant shows that if \( R^\circ(k, \tau, \overline{\psi}) \) is non-zero then \( \psi \) takes its values in \( E_0 \).

Let \( E_\tau \) be the set of Galois representations \( \rho : G_{\mathbb{Q}_p} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p) \) that are potentially crystalline of extended type \( \tau \), Hodge-Tate weights \((0, k-1)\), and determinant \( \psi \). Then:

**Theorem 7.7.3.** There exists a map \( E_\tau : E_\tau \rightarrow \mathbb{P}(V_\tau \otimes_{E_0} \overline{\mathbb{Q}}_p) \) such that two elements \( \rho, \rho' \) of \( E_\tau \) are isomorphic if and only if \( E_\tau(\rho) = E_\tau(\rho') \).

**Proof.** We can assume that \( E_\tau \) is not empty, otherwise the statement is trivially true. Let \( \rho : G_{\mathbb{Q}_p} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p) \) be an element of \( E_\tau \). Then \( WD(\rho) \), the Weil-Deligne representation attached to \( \rho \), is actually a Weil representation as \( \rho \) is potentially crystalline. By definition, \( WD(\rho) \) is isomorphic to \( \tau \otimes_{E_0} \overline{\mathbb{Q}}_p \) as a representation of \( W_\mathbb{Q}_p \). We fix such an isomorphism \( u \), it is unique up to a scalar by the irreducibility of \( \tau \). Then \( u \) gives us an isomorphism between \( D^F_{\text{crys}}(\rho) \) and \( D_{\text{crys},0} \otimes_{E_0} \overline{\mathbb{Q}}_p \) as \( \phi \)-modules with an action of \( \text{Gal}(F/\mathbb{Q}_p) \), by Proposition 7.1.2. This also gives us an isomorphism, that we still call \( u \), between \( D^F_{\text{dR}}(\rho) \) and \( D_{\text{dR},0} \otimes_{E_0} \overline{\mathbb{Q}}_p \).

The isomorphism class of \( \rho \) is entirely determined by the filtration on \( D^F_{\text{dR}}(\rho) \). As the Hodge-Tate weights of \( \rho \) are known, the only necessary information is the \( F \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \)-line corresponding to the non-trivial steps of the filtration. This line is invariant by the action of \( \text{Gal}(F/\mathbb{Q}_p) \). By the isomorphism \( u \), this gives rise to a \( \text{Gal}(F/\mathbb{Q}_p) \)-invariant \( F \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \)-line in \( D_{\text{dR},0} \otimes_{E_0} \overline{\mathbb{Q}}_p \). This line is generated by an element of \( D_{\text{dR},0} \otimes_{E_0} \overline{\mathbb{Q}}_p \) that is invariant by \( \text{Gal}(F/\mathbb{Q}_p) \) by (1) of Proposition 7.3.3, hence by an element of \( D_{\text{dR},0}^{\text{Gal}(F/\mathbb{Q}_p)} \otimes_{E_0} \overline{\mathbb{Q}}_p \).

We define \( E_\tau(\rho) \in \mathbb{P}(D_{\text{dR},0}^{\text{Gal}(F/\mathbb{Q}_p)} \otimes_{E_0} \overline{\mathbb{Q}}_p) \) to be the line generated by this element in \( D_{\text{dR},0}^{\text{Gal}(F/\mathbb{Q}_p)} \otimes_{E_0} \overline{\mathbb{Q}}_p \). This does not depend on the choices made, as \( u \) is unique up to multiplication by a scalar, and the invariant element generating the line is well-defined up to multiplication by a scalar. \( \square \)

7.7.2. Making it into an analytic function. Let \( X \) be the rigid analytic space corresponding to the deformation ring \( R^\circ(k, \tau, \overline{\psi}) \) for some representation \( \overline{\psi} \) with trivial endomorphisms and some supersingular extended type \( \tau \). Let \( E = E(k, \tau, \overline{\psi}, \psi) \) be the field \( E_0 \) defined above.

**Proposition 7.7.4.** There exists a rigid analytic map \( E_\tau : X \rightarrow \mathbb{P}(V_\tau) \), defined over \( E \), such that for all \( x, E_\tau(x) \) is the \( E_\tau \)-invariant of \( \rho_x \) as defined in 7.7.3.
By fixing a basis of the 2-dimensional $E$-vector space $V_\tau$, we then get a map $L_\tau : \mathcal{X} \to \mathbb{P}^1_E$, which plays the role of $\lambda$ in Theorem 5.3.1

**Proof.** It is enough to do this on an admissible covering of $\mathcal{X}$ by affinoid subspaces. So we can assume that $\mathcal{X} = \operatorname{Max}(A)$ for some affinoid algebra $A$, and replace $\mathcal{X}$ by an admissible covering by affinoid subspaces as needed.

Let $D^F_{\text{crys}}(A)$ be the $(\phi, \Gal(F/Q_p))$-module corresponding to the representation $\rho$. We can assume that $D^F_{\text{crys}}(A)$ is a free $A$-module of rank 2. Using the correspondence between $(\phi, \Gal(F/Q_p))$-modules and representations of the Weil group as in Section 7.1.2 and Theorem 7.3.2, we can assume that $D^F_{\text{crys}}(A) = D^F_{\text{crys,0}} \otimes_E A$ as a $(\phi, \Gal(F/Q_p))$-module over $F_0 \otimes_{Q_p} A$.

Consider now $D^F_{\text{dir}}(A)$. It is isomorphic to $F \otimes_{F_0} D^F_{\text{crys}}(A)$, so to $D^F_{\text{dir,0}} \otimes_E A$ as a $\phi$-module with action of $\Gal(F/Q_p)$. In particular, it is trivial as an $F \otimes_{Q_p} A$-module with an action of $\Gal(F/Q_p)$. Also, it has a basis as an $A$-module given by the chosen basis of $D^F_{\text{dir,0}}$. $D^F_{\text{dir}}(A)$ contains a locally free sub-$F \otimes_{Q_p} A$-module $\mathcal{F}$ of rank 1, such that $D^F_{\text{dir}}(A)/\mathcal{F}$ is also locally free of rank 1, that gives at each point $x$ the filtration on $D^F_{\text{dir}}(\rho_x)$. We can assume that $\mathcal{F}$ and $D^F_{\text{dir}}(A)$ are free of rank 1 over $F \otimes_{Q_p} A$. Moreover, this submodule is invariant by the action of $\Gal(F/Q_p)$. Consider a basis $f$ of $\mathcal{F}$. Then the action of $\Gal(F/Q_p)$ on $f$ gives rise to an element $c \in H^1(\Gal(F/Q_p), (F \otimes_{Q_p} A)^{\times})$. Using Theorem 7.3.3 and replacing $\operatorname{Max}(A)$ by an admissible covering if necessary, we can assume that $f$ itself is fixed by the action of $\Gal(F/Q_p)$.

So we get that $f$ is in $D^F_{\text{dir}}(A)^{\Gal(F/Q_p)}$, which is canonically isomorphic to $D^F_{\text{dir,0}} \otimes_E A$. So $f$ defines an analytic map over $\operatorname{Max}(A)$ with values in $\mathbb{P}(D^F_{\text{dir,0}}) = \mathbb{P}(V_\tau)$, which is what we wanted.

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