The electrical conductivity is calculated for regular inhomogeneous two component isotropic medium in which droplets of one phase with conductivity $\sigma_2$ are embedded in another, with conductivity $\sigma_1$. An expression is formulated which can be used in many different situations, and is of particular relevance in the case where the relative proportion of the components is temperature dependent and varies over a wide range. The behavior of effective conductivity depends on the spatial arrangements and the shape of the inclusions.

I. INTRODUCTION

The determination of the effective conductivity $\sigma_{\text{eff}}$ of spatially inhomogeneous heterophase systems is an old, yet increasingly important problem of theoretical physics. With the advent of new nanoscale probes of condensed matter systems, it has been apparent that many very diverse systems which were previously thought to be homogeneous are in fact either statically or dynamically inhomogeneous. The effective conductivity in such cases cannot be dealt with in terms of homogeneous medium theory, is not trivial, and solutions are known so far only in some rather special cases. Different aspects of the theory and different limiting cases are extensively discussed in the Ref. [1].

In this paper we focus on the problem of calculating the effective conductivity of an inhomogeneous two-dimensional (2D) plane. The classical problem could be formulated as follows. Let us assume a 2D system which contains a mixture of $N$ ($N \geq 2$) different phases or materials with different conductivities $\sigma_i$, $i = 1, 2, \ldots, N$. The arrangements of different phases could be random or regular. The question which we wish to address is how the effective conductivity of the plane depends on the conductivities of the phases, their concentration and the spatial arrangements of the phases.

In the past, a number of different approaches have been used to tackle this problem. The exact result for the effective conductivity of a two component system with a symmetric and isotropic distribution of components was obtained by Dykhne [2]. He found that effective conductivity of the system is determined by the simple relation $\sigma_{\text{eff}} = \sqrt{\sigma_1 \sigma_2}$. A "symmetric" distribution, applied to this problem is one in which the two components can be interchanged without changing the end result. Obviously, one requirement for a symmetric distribution is that the two components have equal proportions, but it also means that more general cases cannot be considered with this model.

Further investigations have shown that more general duality relation is valid for 2D heterogeneous conductors than that initially considered by Keller and Dykhne [1]. More recently it was shown that more general relation for the tensor of the effective conductivity exists which is valid for multicomponent and anisotropic systems [3,4]. The effective conductivity of several examples of ordered two-component systems was also calculated exactly [5–7]. It was shown [5,6] that for a chess-board plane and for a plane constructed of triangles, the relation derived by Dykhne is also valid.

A similar relation to the Dykhne formula for the effective conductivity of a system consisting of randomly distributed metallic and dielectric regions near a metal-to-insulator transition was derived by Efros and Shklovskii [8]. They generalized the expression of Dykhne on the basis of scaling arguments to the case of arbitrary concentrations of the two phases near a percolation threshold, so that the effective conductivity becomes:

$$\sigma_{\text{eff}} = \sigma_1 (\frac{\sigma_2}{\sigma_1})^s,$$

where $s$ is a universal scaling exponent. Critical exponents are relatively well known as well for this type of system [9]. This relation is not applicable when the system is driven away from the percolation threshold and the general solution of the effective conductivity of an inhomogeneous medium thus remains an open problem.

Sen and Torquato [10] derived an expression which allows for explicit calculation of the effective conductivity tensor from $n$-point probability functions $S_n(r_1, \ldots, r_n)$. These functions give the probability of points at $r_1, \ldots, r_n$ belonging to the same phase, and are thus uniquely determined by spatial distribution of the phases. Unfortunately, the application of this method is limited as the computations involving $n > 5$ are fairly time consuming.
Different expansions of the effective conductivity in terms of small parameter have been used in the past [1,11,12]. In most of the cases the low order terms weakly depend on microgeometry. A diagrammatic expansion for the effective conductivity was developed by Khalatnikov and Kamenshchik [13] which promises to give more generally applicable results. The perturbative approach seems to be quite effective since it allows us to analyze random and nonsymmetric distributions with different conductivity.

The problem was discussed for the case \( N = 2 \) and \( N = 3 \) on the basis of numerical calculations [3,14] as well. It was shown that the effective conductivity for \( N = 3 \) is not universal and depends on the spatial arrangements of the phases. We have employed boundary element method for efficient numerical treatment of two-dimensional multi-phase systems with arbitrary arrangement of phases. More details on the method and its results can be found in [15].

In this paper we consider the conductivity of a two-phase system in two dimensions for a wide range of concentration and conductivities. One phase is assumed to be composed of droplets (of different shapes) with conductivity \( \sigma_2 \) embedded within a medium of conductivity \( \sigma_1 \) (see Fig. 1). We begin by calculating the effective conductivity \( \sigma_{eff} \) using a perturbation theory approach with the two phases having volume fractions \( (1 - \nu) \) and \( \nu \) respectively. Since the problem is linear, we can introduce a dimensionless conductivity \( \sigma \), measured in units of \( \sigma_1 = 1 \), and the effective conductivity \( \sigma_{eff} \) is a function of \( \sigma = \sigma_2/\sigma_1 = \sigma_2 \) and \( \nu \). The volume-averaged conductivity \( \bar{\sigma} = \frac{1}{V} \int \sigma dV \) is given by:

\[
\bar{\sigma} = (1 - \nu) + \nu \sigma
\]  

(2)

Assuming that the conductivity of the two phases is not vastly different \( |\sigma - 1| \ll 1 \), the effective conductivity could be calculated by perturbation theory [13]. To apply perturbation theory we rewrite the spatial dependence of conductivity as:

\[
\sigma(\mathbf{r}) = \bar{\sigma}(1 - \alpha(\mathbf{r})),
\]  

(3)

where \( \alpha(\mathbf{r}) = \frac{\sigma(\mathbf{r}) - \bar{\sigma}}{\bar{\sigma}} \). Then, assuming that the spatial distribution of conductivity is uncorrelated, we obtain:

\[
\int d\mathbf{r} \alpha(\mathbf{r}) \alpha(\mathbf{r} + \mathbf{r}') = \frac{(\sigma - 1)^2 \nu (1 - \nu)}{\bar{\sigma}^2} \delta(\mathbf{r}')
\]  

(4)

After a straightforward calculation, the conductivity up to second order in \( \alpha \) is given by:

\[
\sigma_{eff} = \bar{\sigma} - \frac{(\sigma - 1)^2 \nu (1 - \nu)}{2 \bar{\sigma}}
\]  

(5)

This result has been known for many years and was derived for the dielectric function of dielectric mixtures [16]. In Refs. [11,12] it was also derived using a systematic perturbation expansion which showed it to be exact to second order in \( \alpha \). The second term in Eq.(5) represents the first non-vanishing contribution due to the inhomogeneity of the distribution of the phases. For the case \( \nu = 0.5 \), the result coincides with the expansion of the exact expression for the conductivity up to the second order in \( (\sigma_2 - \sigma_1) \) [2]:

\[
\sigma_{eff} = \sqrt{\sigma_1 \sigma_2}.
\]  

(6)

II. CONDUCTIVITY OF A REGULAR ISOTROPIC TWO-COMPONENT SYSTEM IN 2D.

Next, we calculate exactly the effective conductivity of the plane with different regular isotropic distributions. As before, let us consider a 2D plane which is constructed from two different phases with different conductivities \( \sigma_1 = 1 \) and \( \sigma_2 = \sigma \). The regions with conductivity \( \sigma_2 \) have a circular shape with radius \( R \) and form a regular square lattice with the period \( a \) as shown in Fig.1a. Changing the radius \( R \) from 0 to \( R = a/2 \) we can change the volume fraction of the second phase from \( \nu = 0 \) to a critical concentration \( \nu_c = 0.785 \) whereafter the regions with conductivity \( \sigma_2 \) start to overlap and a percolation threshold is reached. In the case of metallic droplets, the total charge density should be zero, while a finite charge density can accumulate on the surface between different phases. This allows us to formulate the integral equation for the surface charge density [5,6]. Let us define a surface charge density by the relation \( \rho(\theta) Rd\theta = d\rho(\theta) \), where \( d\rho(\theta) \) is the charge on the small part of the surface between the two components with the length \( dl = Rd\theta \). Taking into account that the scalar potential at the point \( \mathbf{r} \) is determined by the relation
\[ \phi = E_0 x - 2 \int d^2 r' \ln |r - r'| \rho(r') \]  

(7)

where \( \ln |r - r'|/2\pi \) is the 2D Green’s function. The boundary conditions on the surface between two phases are [16]:

\[ E_n^1 - E_n^2 = 4\pi \rho(\theta) \]  

(8)

\[ \sigma_1 E_n^1 = \sigma_2 E_n^2 \]  

(9)

Substituting \( r' = (ma + R \cos(\theta'))i + (na + R \sin(\theta'))j \) and \( r = R \cos(\theta)i + R \sin(\theta)j \) to Eqs.(7-9) we obtain integral equation for the surface charge density in the following form:

\[ \rho(\theta) = \frac{\kappa}{2\pi} \left[ E_0 \cos(\theta) + 2r \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} d\theta' \rho(\theta') \right. \]  

\[ \left. \Re \left( \frac{\exp(i\theta)}{m + r(\cos(\theta') - \cos(\theta)) + i(n + r(\sin(\theta') - \sin(\theta)))} \right) \right] \]  

(10)

where \( r = R/a \), and \( \kappa = \frac{(1-\sigma)}{1+\sigma} \). As it is shown in the Appendix the sum over \( n \) can be calculated exactly and the integral equation for surface charge density will be reduced to the following form:

\[ \rho(\theta) = \frac{\kappa}{2\pi} \left[ E_0 \cos(\theta) + 2r \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} d\theta' K(n, \theta, \theta') \rho(\theta') \right] \]  

(11)

where

\[ K(n, \theta, \theta') = \frac{\pi}{\cos(\theta) \sin(2\pi r(\cos(\theta') - \cos(\theta))) + \sin(\theta) \sinh(2\pi r(\sin(\theta') - \sin(\theta)))} \]  

(12)

\[ \cosh(2\pi r(\sin(\theta') - \sin(\theta))) - \cosh(2\pi r(\cos(\theta') - \cos(\theta))) \]

Expanding the surface density \( \rho(\theta) \) in terms of Legendre polynomials \( P_l(\cos(\theta)) \), and taking into account that \( \rho(-\theta) = \rho(\theta) \) and \( \rho(\pi - \theta) = -\rho(\theta) \):

\[ \rho(\theta) = \sum_{l=1}^{\infty} c_{2l-1} P_{2l-1}(\cos(\theta)) \]  

(13)

we obtain the following linear set of algebraic equations for the coefficients \( c_{2l-1} \)

\[ \frac{2c_{2l-1}}{4l-1} = \frac{\kappa}{2\pi} \left[ E_0 \delta_{l,1} + 2r \sum_{k=1}^{\infty} c_{2k-1} K_{l,k} \right] \]  

(14)

where

\[ K_{l,k} = \sum_{n=-\infty}^{\infty} \int_{0}^{\pi} d\theta' \int_{0}^{\pi} d\theta K(n, \theta, \theta') \sin \theta P_{2l-1}(\cos(\theta)) P_{2k-1}(\cos(\theta')) \]  

(15)

Solving Eq.(14) taking into account a finite number of Legendre polynomials we obtain surface charge density Eq.(13). As a result the effective conductivity is evaluated by calculating the total current \( j = \sigma_1 E_n = E_n \) through the semicircular surface with the radius \( R' = a/2 \) (see Fig.1a). Calculations, similar to that of Eq.(12), lead to the expression for effective conductivity:

\[ \sigma_{eff} = \frac{\kappa}{4\pi} \int_{-\pi/2}^{\pi/2} d\theta \cos(\theta) + \frac{2r}{E_0} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} d\theta' K'(n, \theta, \theta') \rho(\theta') \]  

(16)

where

\[ K'(n, \theta, \theta') = \frac{\cos(\theta) \sin(2\pi r(\cos(\theta') - \cos(\theta)/2) + \sin(\theta) \sinh(2\pi r(\sin(\theta') - \sin(\theta)/2))}{\cosh(2\pi r(\sin(\theta') - \sin(\theta)/2)) - \cosh(2\pi r(\cos(\theta') - \cos(\theta)/2))} \]  

(17)

The result above applies to the case of a uniform distribution of circular droplets within the plane. To see how the effective conductivity depends on the shape of the regions with conductivity \( \sigma_2 \), we have performed calculations for the case where of circular droplets we substituted with squares, triangles and rhombuses with the ratio of diagonals \( \tan \alpha = a/b \) where \( a \) and \( b \) translation vectors along \( x \) and \( y \) respectively (see Fig.1b,c,d). In all these cases Eqs.(10-17) are slightly modified since in polar coordinate system \( r(\theta) \) is a function of angle. Contrary to the case of circles, percolation threshold for cases b,c, and d is \( \nu_c = 0.5 \). Note that in the case of rhombuses the lattice is anisotropic and \( \sigma_{eff}^{11} \neq \sigma_{eff}^{22} \).
III. DISCUSSION.

The results of the calculations of the effective conductivity are presented in Fig.2 as a function of \( \sigma^{\nu} \) for different values of volume fraction \( \nu \). It is easy to check that the results satisfy the generalized duality relation \([3,4]\):

\[
(\sigma^{\nu}_{\text{eff}}(1,1/\sigma_2) = 1.
\]

(18)

For the cases of circles, squares and triangles \( \sigma^{22} = \sigma^{11} \). In the case of rhombuses \( \sigma^{22}(\alpha) = \sigma^{11}(\pi/2 - \alpha) \). Fig.2 (a,b,c,d) shows that for small \( \kappa \), perturbation theory \([11–13]\) (Eq.5) gives the correct result independent of geometry.

A. Approximate expression for the effective conductivity.

Although the predictions in Fig.2 represent the results of a precise numerical calculation, they are not very tractable when it comes to comparing with experimental data, being the result of numerical calculations. It is therefore helpful to try and obtain a functional form for describing the behavior predicted in Figure 2, which also includes all the relevant parameters, such as volume fraction \( \nu \), and the two conductivities \( \sigma_1 \) and \( \sigma_2 \). Such an expression can then be used for a wide range of problems, provided the range of validity is taken into account. We describe the properties of such a heuristically determined function and determine its range of validity in terms of parameters \( \nu, \sigma_1 \) and \( \sigma_2 \).

As can be seen from the Figure 2, the dependence of the effective conductivity on \( \sigma \) shows similar behavior, independent on the particular geometry of the phases. First, we observe that when \( \kappa \) is small all the curves are linear in \( \sigma^{\nu} \) with the same slope. In the relatively wide interval of sigma (0.1 < \( \sigma \) < 10) the effective conductivity is determined by the equation:

\[
\sigma^{\nu}_{\text{eff}}(\sigma) = \sigma_1^{1-\nu} \sigma_2^{\nu}.
\]

(19)

The range of applicability of this formula becomes wider as we approach the percolation threshold \( \nu_c \). When \( \sigma = \sigma_2/\sigma_1 \gg 1 \) effective conductivity saturates at \( \sigma_{\text{sat}} \). \( \sigma_{\text{sat}} \) is not universal and depends on the geometry. Recently it was pointed out that for the case of circles for \( \nu < 0.5 \) in the whole range of \( \sigma \) effective conductivity may be approximated by the formula \([18]\):

\[
\sigma^{\nu}_{\text{eff}}(\kappa) = (1 - \nu\kappa)/(1 + \nu\kappa).
\]

(20)

To derive approximate expression for effective conductivity we assume that Eq.(20) remains correct if we substitute instead of \( \nu \) the effective volume fraction \( \nu^{\text{eff}}(\kappa,\nu) \). We require that \( \nu^{\text{eff}}(\kappa,\nu) \approx \nu \) for \( \kappa \rightarrow 0 \) or \( \nu \rightarrow 0 \), and \( \nu^{\text{eff}}(\kappa,\nu) \approx \nu_c^{\text{eff}}(\kappa,\nu) \) for \( \nu \rightarrow \nu_c \) to satisfy Eq.(19) which is valid at \( \nu = \nu_c \). It is easy to see that the function:

\[
\nu^{\text{eff}}(\kappa,\nu) = \nu + \frac{1 - \nu}{\kappa} \left( 1 - \frac{\nu}{\nu_c} \right)^{\nu_c/(1 - \nu)} - (1 - \nu)\nu_c,
\]

(21)

where \( p(\nu) \rightarrow 0 \) as \( \nu \rightarrow \nu_c \) and \( p(\nu) \rightarrow 1 \) as \( \nu \rightarrow 0 \), satisfies all of the above requirements. Function \( p(\nu) \) is not universal and depends on the geometric shape of the region with conductivity \( \sigma \) and on the particular arrangement of these inclusions in the 2D plane. In Fig.3 we plot \( p(\nu) \) as function of \( 1 - \nu/\nu_c \) for the cases a,b, and c respectively. The case d is different, since the effective conductivity is anisotropic. As it is clearly seen from Figure 3 the behavior of the function \( p(\nu) \) for circles (case a) is different from the cases of squares and triangles (b,c). On the other hand in the cases b and c \( p(\nu) \) shows similar behavior.

B. Shape dependence of the effective conductivity.

The function \( p(\nu) \) is connected with the value of \( \sigma_{\text{sat}} = (1 + \nu^{\text{eff}}(\kappa = -1,\nu))/(1 - \nu^{\text{eff}}(\kappa = -1,\nu)) \). Therefore the behavior of the function \( p(\nu) \) close to percolation threshold should be different for different geometries. In Fig.4 we plot the value of \( \sigma_{\text{sat}} \) as a function of \( (1 - \nu/\nu_c) \) for the case of circles, squares and triangles. There is an important difference between these two cases. In the case of circles \( \sigma_{\text{sat}} \) diverges as a power of \( (1 - \nu/\nu_c)^{-k} \) (\( k \approx 0.5 \)) for the case of squares and triangles this behavior is logarithmic. In both cases close to percolation threshold \( \sigma_{\text{sat}} \) is proportional to average inverse distance between boundaries of the neighboring circles or squares \( \sigma_{\text{sat}} \propto \int dy/(1 - 2f(y)) \) and
\[ f(y) = \sqrt{r^2 - y^2} \] for circles and \[ f(y) = r - |y| \]. Here we assume that the period of the system is 1, and the dimensionless size of the circle and the square is \( r \). Direct integration leads to the following results:

\[
\sigma_{\text{sat}} \propto \left( \pi/2 - \arccos((1 - \nu/\nu_c)^{1/2}) \right) / (1 - \nu/\nu_c)^{1/2} - \pi/2
\]

(22)

for the case of circles and

\[
\sigma_{\text{sat}} \propto -\ln \left( 1 - (\nu/\nu_c)^{1/2} \right)
\]

(23)

for the case of squares (Fig.4). For the case of triangles (c) the asymptotic is similar to Eq.(23) with different numeric coefficients. Interestingly this observation suggests that behavior of the function \( p(\nu) \) is different depending on the curvature of the embedded regions.

IV. CONCLUSION

From calculations of the effective conductivity of inhomogeneous two-phase systems in two dimensions we find that the results of precise numerical calculations can be approximated by a universal function for \( \sigma_{\text{eff}} \) Eqs.(20,21), where the function \( p(\nu) \) depends on the spatial arrangements of the 2D plane and on the shape of the inclusions with conductivity \( \sigma \). It is shown that in a large interval of conductivity \( \sigma \), the effective conductivity \( \sigma_{\text{eff}} \) is determined by the spatial average of the logarithm of individual conductivities. The closer the system is to the percolation threshold, the larger the range of validity of this result. For large values of conductivity \( \sigma \), \( \sigma_{\text{eff}} \) saturates at the value \( \sigma_{\text{sat}} \).

The value of \( \sigma_{\text{sat}} \) near percolation threshold is determined by the average inverse distance between boundaries of neighboring regions with conductivity \( \sigma \) in the direction of the field (Eqs.22,23).

The model which we have developed is quite generally applicable and can be applied in some interesting situations, such as cuprates and other two-dimensional complex transition metal oxides which exist near a phase-separation threshold. Importantly, there appears to be a significant amount of experimental evidence that many anomalous properties of oxides are associated with the coexistence of two or more phases. The application of the presented model may help in understanding the transport properties of such systems.

V. APPENDIX

Here we show how the sum over \( m \) in the Eq.(10) could be calculated exactly. Let us represent the sum in the form:

\[
S = \sum_{m=-\infty}^{\infty} \Re \left( \frac{\exp(i\theta)}{m + \beta + i\alpha} \right)
\]

where \( \beta = r(\cos(\theta') - \cos(\theta)) \), \( \alpha = n + r(\sin(\theta') - \sin(\theta)) \). Sum over \( m \) is calculated using the definition of di-gamma function. As a result we obtain expression for the sum:

\[
S = \Re \left( \exp(i\theta)[\psi(-\beta - i\alpha) - \psi(1 + \beta + i\alpha)] \right) = \pi \Re \left( \exp(i\theta) \cot(\pi(\beta + i\alpha)) \right)
\]

Calculating imaginary part of previous equation we arrive to the results Eq.(12):

\[
S = \pi \frac{\cos(\theta) \sin(2\pi\beta) + \sin(\theta) \sinh(2\pi\alpha)}{\cosh(2\pi\alpha) - \cos(2\pi\beta)}
\]

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[3] L.G. Fel, V.S. Machavariani, D.J. Bergman, J.Phys. A, 33, 6669, (2000).
Figure 1. Spatial arrangements of phases with conductivities $\sigma_1 = 1$ and $\sigma_2 = \sigma$ for four considered cases.

Figure 2. Effective conductivity of the plane as a function of $\sigma''$ for different volume fractions and four considered geometries.

Figure 3. Dependence of the function $p(\nu)$ on $1 - \nu/\nu_c$ for the cases a, b and c.

Figure 4. Saturated effective conductivity as $\sigma \to \infty$ for the case of a, b and c. Full, dotted and dashed lines show different analytical asymptotical behavior for these cases.
|   |   |   |
|---|---|---|
| σ | σ | σ |
| σ | σ | σ |
| σ | σ | σ |

Fig. 1a
Fig. 1d
Fig. 2b
\[ \sigma_{eff} \]

\[ \nu = 0.5 \]
\[ \nu = 0.495 \]
\[ \nu = 0.485 \]
\[ \nu = 0.45 \]
\[ \nu = 0.4 \]
\[ \nu = 0.3 \]
\[ \sigma_{\text{eff}} \]

\[ \frac{\pi}{2} + \arcsin \left( \frac{v}{v_c} \right)^{1/2} \left( \frac{1 - \frac{1}{v_c} \cdot v}{\left( \frac{1}{v_c} \cdot v - 1 \right)^{1/2}} \right) - \frac{\pi}{2} \] for circles

\[ 1.3 \ln \left( \frac{1}{1 - \left( \frac{1}{v_c} \cdot v \right)^{1/2}} \right) \] for squares

\[ 2^{1/2} \ln \left( \frac{2/3^{1/2}}{1 - \left( \frac{1}{v_c} \cdot v \right)^{1/2}} \right) \] for triangles

Fig. 3