Generalized power domination in regular claw-free graphs

Hangdi Chen  Changhong Lu
School of Mathematical Sciences,
Shanghai Key Laboratory of PMMP,
East China Normal University,
Shanghai 200241, P. R. China

Email: 471798693@qq.com
Email: chlu@math.ecnu.edu.cn

Abstract

In this paper, we first show that if $k \geq 2$, the $k$-power domination number of a connected claw-free $(k + 3)$-regular graph on $n$ vertices is at most $\frac{n}{k+4}$, and this bound is tight. The statement partly prove the conjecture presented by Dorbec et al. in SIAM J. Discrete Math., 27:1559-1574, 2013.

1 Introduction

Electric power systems need to be continually monitored. One method of monitoring these systems is to place phase measurement units (PMUs) at selected locations. Because of the high cost of a PMU, the number of PMUs used to monitor the entire system must be minimized. Power domination was introduced in [3, 17] to model the problem of monitoring electrical systems. The problem was first described as a domination problem.

*Supported in part by National Natural Science Foundation of China (No. 11371008) and Science and Technology Commission of Shanghai Municipality (No. 18dz2271000)
in graph theory by Haynes et al. in [13]. The problem has a domination flavor to it, but in addition to domination properties there is the possibility of some propagation according to Kirschoff laws.

Let $G = (V(G), E(G))$ (abbreviated as $G = (V, E)$) be a simple graph. The open neighborhood $N(v)$ of a vertex $v$ consists of the vertices adjacent to $v$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$, denoted $d(v)$, is the size of its open neighborhood $|N(v)|$. A graph $G$ is $k$-regular if $d(v) = k$ for every vertex $v \in V(G)$. The open neighborhood of a subset $S \subseteq V(G)$ is the set $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is $N[S] = N(S) \cup S$. The complete bipartite graph with partite sets of cardinality $i$ and $j$ we denote by $K_{i,j}$. A claw-free graph is a graph that does not contain a claw, i.e. $K_{1,3}$, as an induced subgraph. Let $d(u, v)$ be the distance of $u$ and $v$ in graph $G$. We say a subset $S \subseteq V(G)$ is a packing if the vertices in $S$ are pairwise at distance at least three apart in $G$. For two graphs $G = (V, E)$ and $G' = (V', E')$, let $G \cup G' = (V \cup V', E \cup E')$ and $G \cap G' = (V \cap V', E \cap E')$. If $V \cap V' = \emptyset$, then $G$ and $G'$ are called disjoint. For a set $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by $S$.

The definition of power domination, originally asking to monitor both edges and vertices, was simplified to the following definition independently in [8, 9, 12, 15].

**Definition 1.1.** Let $G$ be a graph. A subset $S$ of $V(G)$ is a power dominating set (abbreviated as PDS) of $G$ if and only if all vertices of $V(G)$ are observed either by Observation Rule 1 (abbreviated as OR 1) initially or by Observation Rule 2 (abbreviated as OR 2) recursively.

**OR 1.** all vertices in $N_G[S]$ are observed initially.

**OR 2.** If an observed vertex $v$ has all neighbors observed except one neighbor $u$, then $u$ is observed (by $v$).

The power domination number $\gamma_p(G)$ is the minimum cardinality of a PDS. Power domination is now well-studied in graph theory. From the algorithmic and complexity point of view, the power domination problem is known to be NP-complete [1, 2, 11, 12, 13], and approximation algorithms were given in [2]. On the other hand, linear-time algorithms for the power domination problem were given for trees [13], for interval graphs [15] and for block graphs [22]. Parameterized results were given in [14]. The exact values for the power domination numbers were determined for various products of graphs in [8, 9] and some important graphs in [19, 24]. Bounds for the power domination numbers of connected graphs and of claw-free cubic graphs were given in [18], for planar or outerplanar graphs with bounded diameter in [23], for Knödel graphs in [19], and for generalized
Petersen graphs in [21, 24]. The Nordhaus-Gaddum problems for power domination were investigated in [4].

Chang et al. [5] generalized power domination to $k$-power domination. When $k = 1$, the $k$-power domination is usual power domination. When $k = 0$, their definition also generalized usual domination. $k$-power domination is now well-studied in graph theory. From the algorithmic point of view, the $k$-power domination problem was known to be NP-complete for chordal graphs and bipartite graphs [5]. On the other hand, linear-time algorithms for the $k$-power domination problem were given for trees [5] and block graphs [20]. The exact values for the $k$-power domination numbers were determined for Sierpiński graphs [7]. Bounds for the $k$-power domination numbers of some important graphs were given in [5, 6]. The relationship between the $k$-forcing and the $k$-power domination numbers of a graph were given in [10].

Let $G = (V, E)$ be a graph and let $S \subseteq V(G)$. For $k \geq 0$, we define the sets $(P^i_G(S))_{i \geq 0}$ of vertices observed by $S$ at step $i$ by the following rules:

1. $P^0_G(S) = N_G[S]$;
2. $P^{i+1}_G(S) = \cup\{N_G[v] : v \in P^i_G(S) \text{ such that } |N_G[v] \setminus P^i_G(S)| \leq k\}$.

We remark that $P^i_G(S) \subseteq P^{i+1}_G(S) \subseteq V(G)$ for any $i$. Moreover, every time a vertex of the set $P^i_G(S)$ has at most $k$ neighbors outside the set, then $P^{i+1}_G(S)$ contains $N_G[v]$. If $P^0_G(S) = P^{i_0+1}_G(S)$ for some $i_0$, then $P^j_G(S) = P^{i_0}_G(S)$ for every $j \geq i_0$ and we accordingly define $P^\infty_G(S) = P^{i_0}_G(S)$. If the graph $G$ is clear from the context, then we will omit the subscripts $G$ for convenience. Now we state the definition of a $k$-power dominating set in a graph first defined by Chang et al. [5].

**Definition 1.2.** Let $G$ be a graph, let $S \subseteq V(G)$, and let $k \geq 0$ be an integer. If $P^\infty_G(S) = V(G)$, then the set $S$ is called a $k$-power dominating set of $G$, abbreviated $k$-PDS. The $k$-power domination number of $G$, denoted by $\gamma_{p,k}(G)$, is the minimum cardinality of a $k$-PDS in $G$.

If $G$ is a connected $(k + 1)$-regular graph, we know $\gamma_{p,k}(G) = 1 \leq \frac{n}{k+2}$ is trivial. Therefore, some scholars began to study the $k$-power domination number of connected $(k + 2)$-regular graphs. Zhao et al. [18] showed that if $G$ is a connected claw-free 3-regular graph of order $n$, then $\gamma_{p,1}(G) \leq \frac{n}{4}$. Chang et al. [5] generalized these result to connected claw-free $(k + 2)$-regular graphs and proved that if $G$ is a connected claw-free $(k + 2)$-regular graph on $n$ vertices, then $\gamma_{p,k}(G) \leq \frac{n}{k+3}$. And after that Dorbec et al. [6] showed the claw-free condition can be removed and presented the Conjecture 1.3. Recently, Lu
et al. [16] studied the $k$-power domination number of connected claw-free $(k+3)$-regular graphs when $k=1$. In this paper, we further studied the case of $k \geq 2$.

**Conjecture 1.3.** For $k \geq 1$ and $r \geq 3$, if $G \not\cong K_{r,r}$ is a connected $r$-regular graph of order $n$, then $\gamma_{p,k}(G) \leq \frac{n}{r+1}$.

We know if the conjecture holds for $k = 1$, then it also holds for all $k \geq 2$. Hence, many scholars would like to check the case of $k = 1$. Dorbec et al. [6] proved that Conjecture 1.3 holds for $k = 1$ and $r = 3$. For $k = 1$ and each even $r \geq 4$, Lu et al. [16] showed that Conjecture 1.3 does not always hold. Nowadays, we find that the Conjecture 1.3 does not always hold for $k = 1$ and each odd $r \geq 5$ (see Section 2).

Remark that Conjecture 1.3 may hold for all $k \geq 2$. Hence, we pay attention to the case of $k \geq 2$. The main result of this paper is as follows.

**Theorem 1.4.** For $k \geq 2$, if $G$ is a connected claw-free $(k+3)$-regular graph of order $n$, then $\gamma_{p,k}(G) \leq \frac{n}{k+4}$ and the bound is tight.

## 2 Counterexample of Conjecture 1.3

In this section, for $k = 1$ and each odd $r \geq 5$, we show that Conjecture 1.3 does not always hold. Suppose that $r = 2s + 1$ and $s \geq 2$.

When $s = 2$, we construct the connected 5-regular graphs in this way. For any integer $t \geq 1$, let $I_{3t}$ be the graph obtained from $3t$ disjoint copies of $K_4$ in linear order, say $D_1, D_2, ..., D_{3t}$. First, we link any two adjacent copies $(D_i, D_{i+1})$ with one common edge and two common vertices, where $i = 1, ..., 3t-1$. Second, for each vertex $u \in V(D_1) \setminus V(D_2)$ and each vertex $v \in V(D_{3t}) \setminus V(D_{3t-1})$, we link $u$ and $v$ with one edge (see Figure 1).

![Figure 1. $I_{3t}$ for $t = 2$.](image)

**Observation 2.1.** For any integer $t \geq 1$, $I_{3t}$ is a connected 5-regular graph of order $n = 6t + 2$ and $\gamma_p(G) = t + 1 > \frac{n}{6}$. 

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When $s = 3$, we construct the connected 7-regular graph in this way. Let $L$ be the graph obtained from 6 disjoint copies of $K_5$ in linear order, say $J_1, J_2, ..., J_6$. First, we link any two adjacent copies ($J_i, J_{i+1}$) with one common edge and two common vertices, where $i = 1, ..., 5$. Second, for each vertex $u \in V(J_1) \setminus V(J_2)$ and each vertex $v \in V(J_5) \setminus V(J_6)$, we link $u$ and $v$ with one edge. After the second operation, the current graph is denoted by $L_0$. It is clear that there are four vertices in $L_0$ which degree is not 7. Third, we add edges to make these four vertices become a complete graph $K_4$ (see Figure 2).

![Figure 2. L.](image)

Observation 2.2. $L$ is a connected 7-regular graph of order $n = 20$ and $\gamma_p(G) = 3 > \frac{n}{8}$.

When $s \geq 4$, we construct the connected $(2s + 1)$-regular graphs in this way. For any integer $t \geq 1$, let $Q_{(s+1)t+2}$ be the graph obtained from $(s + 1)t + 2$ disjoint copies of $K_{s+2}$ in linear order, say $X_1, X_2, ..., X_{(s+1)t+2}$. First, we link any two adjacent copies ($X_i, X_{i+1}$) with one common edge and two common vertices, where $i = 1, ..., (s + 1)t + 1$. Second, for each vertex $u \in V(X_1) \setminus V(X_2)$ and each vertex $v \in V(X_{(s+1)t+2}) \setminus V(X_{(s+1)t+1})$, we link $u$ and $v$ with one edge. After the second operation, the current graph is denoted by $Q_0$. Third, every time from each of $X_j{(s+1)t+2}, X_j{(s+1)t+3}, ..., X_j{(s+1)t+s+2}(j = 0, 1, ..., t-1)$, take a vertex which degree is not $2s + 1$ in $Q_0$, and let these vertices form a complete graph $K_{s+1}$.

Observation 2.3. For any integer $t \geq 1$, $Q_{(s+1)t+2}$ is a connected $(2s + 1)$-regular graph of order $n = (ts + 2)(s + 1)$ and $\gamma_p(G) = \left\lceil \frac{(s+1)t+2}{2} \right\rceil > \frac{n}{2s+2}$.

3 Structure of a minimal counterexample $G$

If the statement of Theorem 1.4 fails, then we suppose that $G$ is a counterexample with minimal $|V(G)|$. In other words, for each $k \geq 2$, $G$ is a connected claw-free $(k+3)$-regular
graph of order \( n \) and \( \gamma_{p,k}(G) > \frac{n}{k+4} \). Hence, we have the following result.

**Observation 3.1.** \( G \) is not isomorphic to \( K_{k+4} \).

Let \( H \) be a subgraph of \( G \). We call \( u \in V(H) \) is a saturated vertex of \( H \) if \( d_H(u) = k+3 \). Otherwise, \( u \) is called an unsaturated vertex of \( H \). Then we introduce two important structure, that is, \( A \) and \( B \).

**Definition 3.2.** Let \( A \) be the graph obtained from \( K_{k+4} \) by removing two edges which share a common vertex in \( K_{k+4} \). Then \( A \) contains a subgraph which is isomorphic to \( K_{k+3} \). We call this subgraph is the \( K_{k+3} \)-structure of \( A \) (see Figure 3).

**Definition 3.3.** Let \( B \) be the graph obtained from \( K_{k+4} \) by removing one edge (see Figure 4).

![Figure 3. A for \( k = 2 \).](image)

![Figure 4. B for \( k = 2 \).](image)

Now we present some useful Lemmas.

**Lemma 3.1.** Let \( H_1 \cong A \) and \( H_2 \cong A \) be two different subgraphs of \( G \). If \( V(H_1) \cap V(H_2) \neq \phi \), then \( |V(H_1) \cap V(H_2)| = k + 4 \).

**Proof.** Let \( |V(H_1) \cap V(H_2)| = t \). If \( t \neq k + 4 \), then we have \( 1 \leq t \leq k + 3 \). Now we consider the following cases.

1. **Case 1.** There exists a vertex \( x \in V(H_1) \cap V(H_2) \) satisfying \( d_{H_1}(x) = k + 1 \) (resp. \( d_{H_2}(x) = k+1 \)). Since \( d_{H_1}(x) \geq k+1 \) and \( d_G(x) = k+3 \), we obtain \( |N_{H_1}(x) \cap N_{H_2}(x)| \geq k-1 \). Let \( y \in N_{H_1}(x) \cap N_{H_2}(x) \), then \( d_{H_1}(y) = k+3 \). We claim that \( d_{H_2}(y) \leq t-1 \). Otherwise, if \( d_{H_2}(y) \geq t \), then \( |V(H_1) \cup V(H_2)| \geq t + 1 \), a contradiction. Since \( d_{H_2}(y) \geq k + 1 \), \( t \geq k+2 \geq 4 \). By the configuration of \( A \), there exists a vertex \( z \in V(H_1) \cap V(H_2) \) satisfying \( d_{H_2}(z) = k + 3 \). If \( z = x \), then \( d_{H_2}(z) = k+1 \). By Observation 3.1, \( |N_G(z) \setminus V(H_1)| \geq 1 \). Since \( d_{H_2}(z) = k+3 \), there is a vertex \( y' \in (N_G(z) \setminus V(H_1)) \cap V(H_2) \). By the configuration of \( A \), we deduce all vertices in \( N_{H_1}(z) \) are not adjacent to \( y' \) and \( N_{H_1}(z) \subseteq V(H_2) \). It
means that \(d_{H_2}(y') \leq (k + 3) - (k + 1) \leq 2\), a contradiction. Hence, we suppose \(z \neq x\). Then \(d_{H_1}(z) \geq k + 2\) and \(N_{H_1}(z) \subseteq V(H_2)\). Thus, \(t = k + 4\), a contradiction.

Case 2. For each \(u \in V(H_1) \cap V(H_2)\), it satisfies \(d_{H_1}(u) \geq k + 2\) and \(d_{H_2}(u) \geq k + 2\). Suppose there exists a vertex \(x \in V(H_1) \cap V(H_2)\) satisfying \(d_{H_1}(x) = k + 2\) (resp. \(d_{H_2}(x) = k + 2\)). If \(d_{H_2}(x) = k + 3\), then \(N_{H_1}[x] \subseteq V(H_2)\) and \(t = k + 3\). By the configuration of \(A\), there are at most four different vertices which degree are \(k + 2\) in \(H_1\) or \(H_2\). Since \(t = k + 3 \geq 5\), there exists a vertex \(v \in V(H_1) \cap V(H_2)\) satisfying \(d_{H_1}(v) = d_{H_2}(v) = k + 3\). So, \(t = k + 4\), a contradiction. If \(d_{H_2}(x) = k + 2\), then \(|N_{H_1}(x) \cap N_{H_2}(x)| \geq k + 1 \geq 3\). By the configuration of \(A\), there are at most two different vertices in \(|N_{H_1}(x) \cap N_{H_2}(x)|\) which degree are \(k + 2\) in \(H_1\) or \(H_2\). It means that there also exists a vertex \(v' \in V(H_1) \cap V(H_2)\) satisfying \(d_{H_1}(v') = d_{H_2}(v') = k + 3\). Hence, \(t = k + 4\), a contradiction. Now we suppose that every vertex \(x \in V(H_1) \cap V(H_2)\) satisfies \(d_{H_1}(x) = d_{H_2}(x) = k + 3\). We deduce that \(t = k + 4\), a contradiction.

**Lemma 3.2.** Let \(H_1 \cong A\) and \(H_2 \cong A\) be two different subgraphs of \(G\). If \(V(H_1) \cap V(H_2) \neq \emptyset\), then \(H_1 \cup H_2 \cong B\).

**Proof.** Let \(V(H_1) = \{x, v_1, v_2, ..., v_{k+3}\}\), \(d_{H_1}(x) = k + 1\) and \(N_{H_1}(x) = \{v_1, v_2, ..., v_{k+1}\}\). By Lemma 3.1, we have \(|V(H_1) \cap V(H_2)| = k + 4\). We claim that \(d_{H_2}(x) \neq k + 3\). Otherwise, if \(d_{H_2}(x) = k + 3\), then \(xv_{k+2} \in E(G)\) and \(xv_{k+3} \in E(G)\). It means that \(G \cong K_{k+4}\), contradicting Observation 3.1. So \(d_{H_2}(x) \neq k + 3\). If \(d_{H_2}(x) = k + 1\), then \(|N_{H_1}(x) \cap N_{H_2}(x)| \geq k - 1\). Next, we show that \(|N_{H_1}(x) \cap N_{H_2}(x)| = k\). Otherwise, suppose \(|N_{H_1}(x) \cap N_{H_2}(x)| = k - 1\), then \(xv_{k+2} \in E(G)\) and \(xv_{k+3} \in E(G)\). It implies \(G \cong K_{k+4}\), contradicting Observation 3.1. Hence, \(|N_{H_1}(x) \cap N_{H_2}(x)| = k\). By the configuration of \(A\), we obtain \(H_1 \cup H_2 \cong B\). If \(d_{H_2}(x) = k + 2\), then \(|N_{H_1}(x) \cap N_{H_2}(x)| \geq k\). Similar to the above proof, we deduce that \(|N_{H_1}(x) \cap N_{H_2}(x)| = k + 1\). Therefore, \(H_1 \cup H_2 \cong B\).

**Lemma 3.3.** Let \(H_1 \cong A\) and \(H_2 \cong A\) be two different subgraphs of \(G\). If \(V(H_1) \cap V(H_2) \neq \emptyset\), then the saturated vertex of \(H_1\) (resp. \(H_2\)) must be in the \(K_{k+3}\)-structure of \(H_2\) (resp. \(H_1\)).

**Proof.** Suppose that \(x \in V(H_1) \cap V(H_2)\). By Lemma 3.2, \(H_1 \cap H_2 \cong B\). According to the proof of Lemma 3.2, we obtain if \(d_{H_1}(x) = k + 1\), then \(d_{H_2}(x) \neq k + 3\). It means that the saturated vertex of \(H_2\) must be in the \(K_{k+3}\)-structure of \(H_1\). By the symmetry, Lemma 3.3 holds.
4 The proof of Theorem 1.4

In this section, we present a proof of our main result, namely, Theorem 1.4.

We give the following algorithm to choose a packing \( P_0 \) for \( G \).

**Initialize.** \( P_0 = \emptyset \).

**Step 1.** If \( G \) contains a subgraph which is isomorphic to \( A \) and none saturated vertex of \( A \) is observed, then we add one saturated vertex of \( A \) to \( P_0 \). Process the step till \( G \) contains no such a subgraph.

**Output.** \( P_0 \).

**Observation 4.1.** For each vertex \( x \in P_0 \), it can be contained in a subgraph \( H \) of \( G \) which is isomorphic to \( A \) and \( x \) is a saturated vertex of \( H \).

**Lemma 4.1.** Let \( P_0 \) be the vertex subset of \( G \) obtained by the above algorithm. Then \( P_0 \) is a packing of \( G \).

**Proof.** Suppose there are two different vertices \( x \) and \( y \) of \( P_0 \). Let \( H_1 \cong A \) and \( H_2 \cong A \) be two different subgraphs of \( G \). Without loss of generality, we suppose \( x \) and \( y \) are vertices added to \( P_0 \) when dealing with subgraphs \( H_1 \) and \( H_2 \), respectively. By Observation 4.1, \( x \) (resp. \( y \)) is a saturated vertex of \( H_1 \) (resp. \( H_2 \)). We claim that \( V(H_1) \cap V(H_2) = \emptyset \). Otherwise, suppose to the contrary that \( V(H_1) \cap V(H_2) \neq \emptyset \). By Lemma 3.2, \( H_1 \cup H_2 \cong B \). Without loss of generality, we suppose \( x \) is added to \( P_0 \) before \( y \). When \( x \) is added to \( P_0 \), by Lemma 3.3, all saturated vertex of \( H_2 \) are observed. It means that \( y \notin P_0 \), a contradiction. So \( V(H_1) \cap V(H_2) = \emptyset \). It is clear that \( d_G(x, y) \geq 3 \). Hence, \( P_0 \) is a packing of \( G \).

We extend the packing \( P_0 \) of \( G \) to a maximal packing and denote the resulting packing by \( S_0 \).

**Lemma 4.2.** \( G \) has a sequence \( S_0 \subset S_1 \subset \cdots \subset S_l \) such that the following holds:

- (a) For all \( 0 \leq i \leq l - 1 \), \( |S_{i+1}| = |S_i| + 1 \) and \( |P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + k + 4 \).
- (b) \( P^\infty(S_l) = V(G) \).

**Proof.** If \( P^\infty(S_0) = V(G) \), then there is nothing to prove. Hence, we may assume that \( P^\infty(S_0) \neq V(G) \). Let \( i \geq 0 \) and suppose that \( S_i \) exists and \( P^\infty(S_i) \neq V(G) \). Denote
\[M = P^\infty(S_i) \text{ and } \overline{M} = V(G) \setminus M.\] Let \(U = \{u \mid u \in M, N_G(u) \setminus M \neq \emptyset\}.\) For each vertex \(u \in U,\) since \(N_G[u] \not\subseteq M,\) we note that \(d_M(u) \geq 1\) and \(k + 1 \leq d_M(u) \leq k + 2.\) We have the following results.

**Claim 1.** Let \(H \cong A\) be a subgraph of \(G,\) then all saturated vertices of \(H\) are contained in \(M.\)

*Proof.* By the choice of \(P_0,\) \(P^1(P_0)\) contains at least one saturated vertex of \(H.\) Without loss of generality, we assume \(x \in P^1(P_0)\) is a saturated vertex of \(H.\) If \(y\) is a saturated vertex of \(H\) and \(y \not\in P^1(P_0),\) then we deduce that \(x \not\in P_0\) and \(x\) is observed by a vertex \(z\) of \(H.\) So \(y \in P^1(P_0),\) a contradiction. Hence, \(P^1(P_0)\) contains all saturated vertices of \(H.\) It means Claim 1 holds. \(\square\)

**Claim 2.** For each \(u \in U,\) \(N_G(u) \setminus M\) induces a clique in \(G.\)

*Proof.* Suppose \(x_1\) and \(x_2\) are two neighbors of \(u\) in \(N_G(u) \setminus M\) and \(u\) is observed by \(v\) in \(M.\) Then \(x_1v, x_2v \notin E(G).\) If \(x_1x_2 \notin E(G),\) then \(\{u, x_1, x_2, v\}\) induces a claw, a contradiction. Therefore, \(N_G(u) \setminus M\) induces a clique in \(G.\) \(\square\)

**Claim 3.** For each vertex \(x \in \overline{M},\) \(d_{\overline{M}}(x) \geq k + 1.\)

*Proof.* If \(N_G[x] \subseteq \overline{M},\) then we have proved it. Now we assume \(x\) has a neighbor \(x' \in M,\) then \(x' \in U.\) Since \(k + 1 \leq d_{\overline{M}}(x') \leq k + 2\) and Claim 2, \(d_{\overline{M}}(x) \geq k.\) If \(d_{\overline{M}}(x) = k,\) then let \(N_{\overline{M}}(x) = \{x_1, x_2, \ldots, x_k\}\) and \(N_M(x) = \{y_1, y_2, x'\}.\) Since none of vertices in \(\{y_1, y_2, x'\}\) can observe \(x\) and \(N_G(y_1) \setminus M\) induces a clique for each \(i \in \{1, 2\},\) we obtain \(N_M(x) = \{y_1, y_2, x'\}\) for each \(j \in \{1, 2, \ldots, k\}.\) Since \(G\) is claw-free, \(G'[\{y_1, y_2, x'\}]\) contains at least one edge. Hence, \(G'[[x, y_1, y_2, x', x_1, x_2, \ldots, x_k]]\) is isomorphic to \(A.\) However, all saturated vertices \(\{x, x_1, x_2, \ldots, x_k\}\) of \(G'[[x, y_1, y_2, x', x_1, x_2, \ldots, x_k]]\) are not contained in \(M,\) contradicting Claim 1. Hence, \(d_{\overline{M}}(x) \geq k + 1.\) \(\square\)

**Claim 4.** Suppose \(i\) is an integer such that \(1 \leq i \leq k - 1.\) Let \(H \cong K_{k+2}\) be a subgraph of \(G\) and \(x_j\) be a vertex of \(H\) for each \(j \in \{1, 2, \ldots, i\}.\) If \(\{x_1, x_2, \ldots, x_i\} \subseteq \overline{M}\) and there is a vertex \(v\) such that \(N_H(v) = \{x_1, x_2, \ldots, x_i\},\) then \(v \in \overline{M}.\)

*Proof.* If \(v \in M,\) then \(v \in U.\) So \(d_{\overline{M}}(v) \geq k + 1.\) By Claim 2, \(|N_G(x_1) \cap (N_G(v) \setminus \{x_1, x_2, \ldots, x_i\})| \geq k + 1 - i \geq 2.\) Since \(H \cong K_{k+2}\) and \(N_H(v) = \{x_1, x_2, \ldots, x_i\},\) we have \(|N_G(x_1) \cap (N_G(v) \setminus \{x_1, x_2, \ldots, x_i\})| \leq 1,\) a contradiction. \(\square\)
Claim 5. Suppose \( i \) is an integer such that \( 1 \leq i \leq k \). Let \( H \cong K_{k+3} \) be a subgraph of \( G \) and \( x_j \) be a vertex of \( H \) for each \( j \in \{1, 2, ..., i\} \). If \( \{x_1, x_2, ..., x_i\} \subseteq \overline{M} \) and there is a vertex \( v \) such that \( N_H(v) = \{x_1, x_2, ..., x_i\} \), then \( v \in \overline{M} \).

Proof. Since \( G \) is claw-free and \( N_H(v) = \{x_1, x_2, ..., x_i\} \), \( N_G[v] \setminus \{x_1, x_2, ..., x_i\} \) induces a subgraph \( H' \) which is isomorphic to \( K_{k+4-i} \). If \( v \in M \), then \( v \) is observed from one of its neighbors in \( H' \). Since \( H' \cong K_{k+4-i} \), all vertices of \( H' \) are observed. Since \( i \leq k \), \( v \) can observe \( x_i \), contradicting that \( x_i \in \overline{M} \). \( \square \)

Note that \( k + 1 \leq d_{\overline{M}}(u) \leq k + 2 \) for each vertex \( u \in \mathcal{U} \). If there is a vertex \( u \in \mathcal{U} \) such that \( d_{\overline{M}}(u) = k + 2 \), then \( u \) together with its \( k + 2 \) neighbors in \( \overline{M} \), say \( u_1, u_2, ..., u_{k+2} \), induces a \( K_{k+3} \). If \( k + 1 \) vertices of the \( k + 2 \) vertices, say \( u_1, u_2, ..., u_{k+1} \), have a common neighbor \( u_{k+3} \) other than \( u \) or \( u_{k+2} \), then \( \{u, u_1, u_2, ..., u_{k+3}\} \) induces an \( A \) with its saturated vertices \( u_1, u_2, ..., u_{k+1} \). But \( u_1, u_2, ..., u_{k+1} \notin M \), contradicting Claim 1. Now suppose there are at most \( k \) vertices in \( \{u_1, u_2, ..., u_{k+2}\} \) having a common neighbor which is not in \( \{u, u_1, u_2, ..., u_{k+2}\} \). By Claim 5, there exists at least two vertices in \( \overline{M} \cap (N(\{u_1, u_2, ..., u_{k+2}\}) \setminus \{u_1, u_2, ..., u_{k+2}\}) \). Then let \( S_{i+1} = S_i \cup \{u_i\} \). We have \( |P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + k + 4 \).

Then we assume \( d_{\overline{M}}(u) = k+1 \) for each \( u \in \mathcal{U} \). Let \( u \in \mathcal{U} \), \( N_G(u) = \{v, v', u_1, u_2, ..., u_{k+1}\} \), \( v \) observe \( u \) but \( u_1, u_2, ..., u_{k+1} \in \overline{M} \). We consider the following cases.

Case 1. \( vv' \notin E(G) \).

Since \( G \) is claw-free, \( \{u, v', u_1, u_2, ..., u_{k+1}\} \) induces a \( K_{k+3} \). Let \( H' = G[\{u_1, u_2, ..., u_{k+1}\}] \). If \( u_1, u_2, ..., u_{k+1} \) share a common neighbor other than \( v' \) or \( u \), say \( w \), then \( \{u, v', w, u_1, u_2, ..., u_{k+1}\} \) induces an \( A \) such that its saturated vertices \( u_1, u_2, ..., u_{k+1} \) are not observed, contradicting Claim 1. Now we suppose \( u_1, u_2, ..., u_{k+1} \) do not share a common neighbor other than \( v' \) or \( u \). Without loss of generality, there exists a vertex \( w_1 \) such that \( N_H(w_1) = \{u_1, u_2, ..., u_i\} \) and \( 2i \leq k + 1 \). By Claim 3, \( d_{\overline{M}}(w_1) \geq k + 1 \) and \( w_1 \in \overline{M} \).

Then \( d_{\overline{M}}(w_1) \geq k + 1 \). Hence, there are at least \( (k + 2) + (k + 1 - i) \) vertices in \( \overline{M} \cap (N_G[w_1] \cup \{u_{i+1}, u_{i+2}, ..., u_{k+1}\}) \). Let \( S_{i+1} = S_i \cup \{w_1\} \), we have \( |P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + (k + 2) + (k + 1 - i) \). When \( i = 1 \), \( (k + 2) + (k + 1 - i) = k + 2 + k \geq k + 4 \). When \( i \geq 2 \), \( (k + 2) + (k + 1 - i) \geq k + 2 + i \geq k + 4 \). So \( |P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + k + 4 \).

Case 2. \( vv' \in E(G) \).

If all vertices in \( \{u_1, u_2, u_{k+1}\} \) are adjacent to \( v' \), then \( \{v', u, u_1, u_2, ..., u_{k+1}\} \) induces a \( K_{k+3} \). Similar to the proof of Case 1, we can prove that (a) holds.
Next, we assume that none vertices in \(\{u_1, u_2, ..., u_{k+1}\}\) are adjacent to \(v'\). Let \(T = G[\{u_1, u_2, ..., u_{k+1}\}]\) and \(T' = N_G(T) \setminus (T \cup \{w\})\). If all vertices in \(T\) share two common neighbors other than \(u\), say \(w_1, w_2\). Since \(G\) is claw-free, \(w_1w_2 \in E(G)\). So \(G[N[u_1]] \equiv A\) and all saturated vertices of \(G[N[u_1]]\) are not contained in \(M\), a contradiction. If all vertices in \(T\) share exactly one common neighbor other than \(u\), say \(w_1\), then \(|T' \setminus \{w_1\}| \geq 2\). We claim that \(|T' \setminus \{w_1\}| = 2\). Otherwise, suppose \(|T' \setminus \{w_1\}| \geq 3\) and let \(w_2, w_3, w_4 \in T' \setminus \{w_1\}\). Since \(G\) is claw-free, we have \(w_1w_2 \in E(G)\) for each \(i \in \{2, 3, 4\}\). It means \(d_G(w_i) \geq |T| + 3 \geq k + 4\), a contradiction. So \(|T' \setminus \{w_1\}| = 2\). Let \(w_2, w_3 = T' \setminus \{w_1\}\), \(N_T(w_2) = \{u_1, u_2, ..., u_i\}\) and \(N_T(w_3) = \{u_{i+1}, u_{i+2}, ..., u_{k+1}\}\), where \(2i \leq k + 1\). Since \(G\) is claw-free, \(w_1w_2 \in E(G)\) and \(w_1w_3 \in E(G)\). So \(N_G(w_1) = \{w_2, w_3, u_1, u_2, ..., u_{k+1}\}\). We claim that \(w_1 \in \overline{M}\). Otherwise, \(w_1\) is observed by \(w_2\) or \(w_3\), then \(u_j \in M\) for some \(j \in \{1, 2, ..., k + 1\}\), a contradiction. Moreover, we claim that \(w_2 \in \overline{M}\). Otherwise, suppose to the contrary that \(w_2 \in M\), then \(w_2\) is observed by some vertices of \(N_G(w_2) \setminus \{w_1, u_2, ..., u_i\}\). By the claw-freeness of \(G\), \(N_G[w_2]) \setminus \{w_1, u_2, ..., u_i\}\) induces a \(K_{k+3-i}\). Hence, all vertices of \(N_G(w_2) \setminus \{w_1, u_2, ..., u_i\}\) are observed. When \(i = 1, i + 1 = 2 \leq k\). When \(i \geq 2, i + 1 \leq 2i - 1 \leq (k + 1) - 1 \leq k\). It means all vertices of \(\{w_1, u_1, u_2, ..., u_i\}\) are observed by \(w_2\), a contradiction. So \(w_2 \in \overline{M}\). Let \(S_{i+1} = S_i \cup \{w_1\}\), similar to the proof of Case 1, we have \(|P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + (k + 2) + (k + 1) - i \geq |P^\infty(S_i)| + k + 4\). If all vertices in \(T\) do not have other common neighbors than \(u\), then we claim that there exists a vertex \(w \in T'\) such that \(|N_T(w)| \leq k - 1\). Otherwise, suppose all vertices of \(T'\) have \(k\) neighbors in \(T\). But there are \(2k + 2\) edges between \(T\) and \(T'\), a contradiction. Without loss of generality, we assume \(w \in T'\) and \(|N_T(w)| \leq k - 1\). By Claim 4, \(w \in \overline{M}\). Since Claim 3, \(d_{\overline{M}}(w) \geq k + 1\). Hence, let \(S_{i+1} = S_i \cup \{w\}\), we have \(|P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + k + 4\).

Now let \(i\) be an integer and \(1 \leq i \leq k\), we suppose that \(i\) vertices in \(\{u_1, u_2, ..., u_{k+1}\}\) are adjacent to \(v'\). Without loss of generality, we assume \(v'u_j \in E(G)\) for each \(j \in \{1, 2, ..., i\}\). We claim that \(i = k\). Otherwise, suppose to the contrary that \(i < k\). Since \(vv', uv' \in E(G)\) and \(v, u \in M\), \(d_{\overline{M}}(v') \leq k + 1\). Since \(v' \in U\), \(d_{\overline{M}}(v') = k + 1\). By Claim 2, \(d_G(u_i) \geq |\{u, v', u_2, u_3, ..., u_{k+1}\}| + (k + 1 - i) \geq 2k + 3 - i \geq k + 4\), a contradiction. So \(i = k\). Suppose that \(N_{\overline{M}}(v') = \{u_1, u_2, ..., u_k, w_1\}\). By Claim 2, \(\{u_1, u_2, ..., u_k, w_1\}\) induces a clique. By Claim 3, we have \(d_{\overline{M}}(u_1) \geq k + 1\), implying that \(w_1 \in \overline{M}\). Let \(\{w_2, w_3\} = N_G(u_{k+1}) \setminus \{u_1, u_2, ..., u_k\}\), it is possible that \(w_3 = w_1\) or \(w_2 = w_1\). We assume that \(w_2 \neq w_1\). Since Claim 4, \(w_2 \in \overline{M}\). By Claim 3, \(d_{\overline{M}}(w_1) \geq k + 1\). Hence, there are at least \(k+4\) vertices in \(\overline{M} \cap (N_G[w_1] \cup \{w_2, u_{k+1}\})\). Let \(S_{i+1} = S_i \cup \{u_i\}\), we have \(|P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + k + 4\).

Since \(|V(G)|\) is finite, there exists an integer \(l\) such that \(P^\infty(S_l) = V(G)\). Then we
complete the proof.

We are now in a position to prove our main result, namely, Theorem 1.4. Recall its statement.

**Proof.** Let $G$ be a counterexample such that $|V(G)|$ is minimal. Let $S_0, S_1, ..., S_l$ be a sequence satisfying properties (a)-(b) in the statement of Lemma 4.2 with $l$ as small as possible. By Lemma 4.2(b), the set $S_l$ is a $k$-PDS in $G$, and so $\gamma_{p,k}(G) \leq |S_l|$. Since $S_0$ is a packing in $G$, we have that $|P^0(S_0)| = |N[S_0]| = (k + 4)|S_0|$. If $l = 0$, then $(k + 4)|S_0| \leq n$ and $\gamma_{p,k}(G) \leq |S_0| \leq \frac{n}{k+4}$, a contradiction. Now we suppose that $l \geq 1$. By Lemma 4.2(a), $|S_l| = |S_0| + l$. By our choice of $l$, we decuce that $|P^\infty(S_{i+1})| \geq |P^\infty(S_i)| + 1$ for $0 \leq i \leq l - 1$. Thus,

$$n = |P^\infty(S_l)| \geq |P^0(S_0)| + l(k + 4) = (|S_0| + l)(k + 4) = |S_l|(k + 4).$$

Hence, $\gamma_{p,k}(G) \leq |S_l| \leq \frac{n}{k+4}$, a contradiction. This proves the desired upper bound.

Next, we show this bound is tight. For positive integers $k \geq 2$ and $t$, we define the graph $C_{k,t}$ as follows. Take $t$ disjoint copies $C_i \cong K_{k+4} - x_iy_i \cong B$, a complete graph on $k + 4$ vertices minus one edge $x_iy_i$, where $1 \leq i \leq t$. Add the edges $y_ix_{i+1}(1 \leq i \leq t)$ where $x_{t+1} = x_1$ (see Figure 5). Then, $C_{k,t}$ is a connected claw-free $(k + 3)$-regular graph of order $n = t(k + 4)$. Suppose that $S$ be an arbitrary $k$-PDS in $C_{k,t}$. If $S \cap V(C_i) = \phi$, then no vertex in $V(C_i \setminus \{x_i, y_i\})$ belongs to the set $P^\infty(S)$, contradicting the assumption that $S$ is a $k$-PDS in $C_{k,t}$. Therefore, $|S \cap V(C_i)| \geq 1$ for all $i$, where $1 \leq i \leq t$. It means that $\gamma_{p,k}(C_{k,t}) \geq t = \frac{n}{k+4}$. Since the above proof, we obtain $\gamma_{p,k}(C_{k,t}) \leq \frac{n}{k+4}$. Hence, $\gamma_{p,k}(C_{k,t}) = \frac{n}{k+4}$. □

**Figure 5.** $C_{k,4}$. 12
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