Hydrodynamics of a 5D Einstein-dilaton black hole solution and the corresponding BPS state

Song He, Ya-Peng Hu, Jian-Hui Zhang

Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, PRC
Center for High-Energy Physics, Peking University, Beijing 100871, China

E-mail: hesong@itp.ac.cn, huyp@pku.edu.cn, zhangjianhui@gmail.com

ABSTRACT: We apply the potential reconstruction approach to generate a series of asymptotically AdS (aAdS) black hole solutions, with a self-interacting bulk scalar field. Based on the method, we reproduce the pure AdS solution as a consistency check and we also generate a simple analytic 5D black hole solution. We then study various aspects of this solution, such as temperature, entropy density and conserved charges. Furthermore, we study the hydrodynamics of this black hole solution in the framework of fluid/gravity duality, e.g. the ratio of the shear viscosity to the entropy density. In a degenerate case of the 5D black hole solution, we find that the c function decreases monotonically from UV to IR as expected. Finally, we investigate the stability of the degenerate solution by studying the bosonic functional energy of the gravity and the Witten-Nester energy $E_{WN}$. We confirm that the degenerate solution is a BPS domain wall solution. The corresponding superpotential and the solution of the killing spinor equation are found explicitly.
1 Introduction

Five-dimensional black hole solutions of Einstein gravity coupled to dilaton field have been an avenue of intense research for many years. The gauge/gravity duality [1][2][3][4] promotes finding new kinds of solutions that are asymptotically $AdS_5$ ($aAdS_5$) in the UV region, which are very important in building models that can be used to describe the dual field theory. As we know, the vacuum solution with a negative cosmological constant is pure $AdS$, which is dual to $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. It is valuable to find new solutions with asymptotical AdS behavior that may be dual to non-conformal field theory. Such new solutions will help extend the application of gauge/gravity duality. Asymptotically AdS solutions can be found for example in Refs.[5][6][7][8][9]. To describe phenomena in realistic field theories, one needs to break the conformal symmetry. A simple way to do this is to introduce a nontrivial dilaton. Motivated by this, we introduce a general framework to generate a series of new $aAdS_5$ solutions from a bottom-up point of view.

From the phenomenological perspective, introducing the scalar field and its self interaction to build up effective gravity background is fairly important for investigating phenomena in field theories via holography. For example, in holographic QCD (hQCD) communities, nontrivial dilaton potentials have been used to build dual hQCD models [10–12]. The authors of [11] [12] make contribution to building up 5D gravity background in graviton with minimally coupled dilaton system to describe the QCD phenomena. All the solutions they found are numerical. Their strategy is to fix the dilaton potential and then find the gravity solutions. Staring from the 10D string theory or supergravity, one can use various ways to reduce the theory to the 5D low energy effective gravity theory. After
a consistent truncation or reduction, there could be various versions of black holes with the scalar fields, such as one charge $\mathcal{N} = 4$ black hole [13][14][15], $\mathcal{N} = 2^*$ black hole[16], 5D effective quenched background[17] of the Sakai-Sugimoto model [18]. It is a non-trivial task to find gravity solutions of these theories [19]. The procedure of solving the gravity background with fixing dilaton potential as in Refs [11] [12] is difficult, both analytically and numerically. Is it possible to reconstruct the whole background in an easy manner?

Different from the logic of Refs [11] [12], a bottom-up approach known as the potential reconstruction approach [20][21] is indeed a much easier way to obtain gravity solutions if the information of the metric is known well. Using this bottom-up approach, a new Schwarzschild-AdS black hole in five-dimensions coupled to a scalar field was discussed in [20], while dilatonic black hole solutions with a Gauss-Bonnet term in various dimensions were discussed in [23]. The new four dimensional gravity solution has been found in [24].

We will use the potential reconstruction approach to consistently work out the general 5D gravity solution, which is valuable for studying the phase transition in thermal QCD phase diagram. In [21], the authors use this method to construct a semianalytical gravity solution to study the thermodynamical quantities, their results agree with the numerical results from recent studies in lattice QCD. The authors also study loop operators in field theory from gauge/gravity duality and the results are consistent with the lattice QCD calculation. Ref. [21] provides an excellent example of reconstructing a holographic model using the potential reconstruction approach. It is worthwhile pushing and extending this approach to reconstruct gravity solutions.

In this paper, we extend the potential reconstruction approach and study various aspects of a 5D black hole solution generated by this approach. Motivated from understanding the aAdS solution from gauge/gravity duality perspective, we have investigated the related properties of the solution. We first study the thermodynamic quantities of the solution. In studying the conserved charges associated with this solution we find several different subtractions rendering the action finite, which correspond to introducing different counterterms [25, 26] to the action. We show how the appropriate subtraction can be singled out by looking at a special limit of the solution. In order to understand the dual hydrodynamical properties containing dilaton field in the bulk, we also study the ratio of the shear viscosity to the entropy density from fluid/gravity duality perspective [27]. Our result is consistent with the lowest bound given by using the Green-Kubo formula from AdS/CFT [28–39]. To investigate the field theory dual to the background, we then focus on a degenerate case of the 5D black hole solution with $V_1 = 0$ and calculate the $c$ function[41] characterizing the holographic RG flow [43][44] of the conformal anomaly. Finally, we discuss the stability of the degenerate solution from the bosonic functional energy [45] and the Witten-Nester energy $E_{WN}$[46]. The superpotential for the degenerate solution has been figured out, which means that the solution is a BPS domain wall solution. From the Witten-Nester energy $E_{WN}$ point of view, it is also a stable one. Furthermore, the solution of the killing spinor equation/Witten equation has been given explicitly. From the results given in this paper, the new method of reconstructing gravity promotes us to do further studies in this direction.

This paper is organized as follows. In Section 2 we briefly describe the effective ap-
approach to reconstruct the gravity background pointed out in Ref. [21]. All these gravity backgrounds satisfy the asymptotical AdS boundary condition near the UV region. We will point out two strategies to realize the reconstruction and comment on the approach. In Sections 3 and 4 we calculate, for a simple 5D black hole solution, the relevant thermodynamic quantities such as the temperature, the entropy and the conserved charges, where we also introduce the counterterm rendering the black hole energy finite in Section 4. In Section 5, we use the fluid/gravity duality to investigate the ratio of the shear viscosity to the black hole entropy density. In section 6, motivated from studies on the holographic RG flow of the conformal anomaly in an aAdS solution, we calculate the c function for the degenerate case of the 5D black hole solution and find that it decreases monotonically from UV to IR. In Section 7, in order to test the stability of the degenerate aAdS solution, we use the standard way to investigate the superpotential and work out the solution of the killing spinor equation. Section 8 is devoted for conclusion and discussions.

2 Generating black hole solution for Einstein-dilaton system

We start from the minimal non-critical 5D effective gravity action in the string frame [21]:

\[ S_{5D} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g^S} e^{-2\phi} \left( R^S + 4\partial_\mu \phi \partial^\mu \phi - V_S(\phi) \right), \quad (2.1) \]

where \( G_5 \) is the 5D Newton constant, \( g^S \) and \( V_S(\phi) \) are the determinant of the 5D metric and the dilaton potential in the string frame, respectively. We will list relations between geometrical quantities in the string frame and in the Einstein frame. One can easily link the two frames by using the following well-known relations.

If the metric in the Einstein frame \( g^E_{\mu\nu} \) and in the string frame \( g^S_{\mu\nu} \) are connected by the scaling transformation

\[ g^S_{\mu\nu} = e^{-2\Omega} g^E_{\mu\nu}, \quad (2.2) \]

then the scalar curvature in the Einstein frame and in the string frame has the following exact relation

\[ e^{-2\Omega} R^S = R^E - (D - 1)(D - 2)\partial_\mu \Omega \partial^\mu \Omega + 2(D - 1)\nabla^2 \Omega \quad (2.3) \]

with \( D \) dimension, and \( \nabla^2 \) is defined by \( \nabla^2 = \frac{1}{\sqrt{-g^E}} \nabla_\mu \sqrt{-g^E} \nabla^\mu \), which is useful to derive the exact relation between the actions in the string frame and in the Einstein frame.

For the special case \( D = 5 \), we have

\[ \int \sqrt{-g^S} e^{m\Omega} R^S = \int \sqrt{-g^E} e^{(m-3)\Omega} \left[ R^E - 12\partial_\mu \Omega \partial^\mu \Omega + 8\nabla^2 \Omega \right], \quad (2.4) \]

\[ \int \sqrt{-g^S} e^{m\Omega} \left( g^S_{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \right) = \int \sqrt{-g^E} e^{(m-5)\Omega} \left( e^{2\Omega} g^E_{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \right). \quad (2.5) \]

The dilaton potentials in two different frames are related by

\[ V_S(\Omega) = V_E(\Omega)e^{2\Omega}. \quad (2.6) \]
By setting $m = 3$ and $\Omega = -\frac{2}{3} \phi$, we have

$$V_S = V_E e^{-\frac{4\phi}{3}}$$  \hspace{1cm} (2.7)

and the following relation

$$\int \sqrt{-g_S} e^{-2\phi} \left( R^S + 4 \partial_{\mu} \phi \partial^{\mu} \phi - V_S(\phi) \right)
= \int \sqrt{-g_E} \left[ R^E - \frac{4}{3} \partial_{\mu} \phi \partial^{\mu} \phi - V_E(\phi) \right].$$  \hspace{1cm} (2.8)

Therefore, the action Eq. (2.1) becomes in the Einstein frame

$$S_{5D} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g_E} \left( R^E - \frac{4}{3} \partial_{\mu} \phi \partial^{\mu} \phi - V_E(\phi) \right).$$  \hspace{1cm} (2.9)

We can write down the general ansatz in the string frame which is similar to holographic QCD models given by [12]

$$ds^2_S = \frac{L^2 e^{2A_s}}{z^2} \left( -f(z) dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i \right)$$  \hspace{1cm} (2.10)

with $L$ the radius of AdS$_5$. The metric in the string frame is useful to calculate the loop operator in holographic QCD communities. To derive the Einstein equations and to study the thermodynamical quantities, we transform it to the Einstein frame

$$ds^2_E = \frac{L^2 e^{2A_s - \frac{4\phi}{3}}}{z^2} \left( -f(z) dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i \right).$$  \hspace{1cm} (2.11)

The general Einstein equations and equation of motion of scalar from the action (2.9) take the form of

$$E_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu} \left( \frac{4}{3} \partial_\mu \phi \partial^{\mu} \phi + V_E(\phi) \right) - \frac{4}{3} \partial_\mu \phi \partial_\nu \phi = 0.$$  \hspace{1cm} (2.12)

Where $\Box$ is the d’Alembertian operator in curved space. By using Eqs (2.9) and (2.11), we can derive the following nontrivial Einstein equations in $(t, t), (z, z)$ and $(x_1, x_1)$ spaces, respectively

$$b''(z) + \frac{b'(z) f'(z)}{2 f(z)} = \frac{2}{9} b(z) \phi'(z)^2 + \frac{b(z)^3 V_E(\phi(z))}{6 f(z)} = 0,$$  \hspace{1cm} (2.13)

$$\phi'(z)^2 - \frac{9b'(z)^2}{b(z)^2} - \frac{9b'(z) f'(z)}{4b(z) f(z)} - \frac{3b(z)^2 V_E(\phi(z))}{4 f(z)} = 0,$$  \hspace{1cm} (2.14)

$$f''(z) + \frac{6b'(z)f'(z)}{b(z)} = \frac{6 f(z) b'(z)}{b(z)} + \frac{4}{3} f(z) \phi'(z)^2 + b(z)^2 V_E(\phi(z)) = 0,$$  \hspace{1cm} (2.15)

where $b(z) = \frac{L e^{A_E}}{z}$ and $A_E(z) = A_s(z) - \frac{2}{3} \phi(z)$. 
One should notice that the above three equations are not independent. There are only two independent functions, one can solve them from two of the three equations, and the third equation can be used to check the consistency of the solutions. To simplify the procedure of finding solution, one can use the following two equations without dilaton potential $V_E$

\[
A''_s(z) + A'_s(z) \left( \frac{4 \phi'(z)}{3} + \frac{2}{z} \right) - A'_s(z)^2 - \frac{2 \phi''(z)}{3} - \frac{4 \phi'(z)}{3z} = 0, \tag{2.16}
\]

\[
f''(z) + f'(z) \left( 3A'_s(z) - 2 \phi'(z) - \frac{3}{z} \right) = 0. \tag{2.17}
\]

These two equations can be obtained from (2.13)(2.14)(2.15) by canceling terms involving $V_E$. Eq.(2.16) is our starting point to find gravity solution. The EOM of the dilaton field is given explicitly as following

\[
\frac{8}{3} \partial_z \left( \frac{L^3 e^{3A_s(z)} - 2 \phi f(z)}{z^3} \partial_z \phi \right) - \frac{L^5 e^{5A_s(z)} - \frac{10}{3} \phi}{z^5} \partial_y V_E = 0. \tag{2.18}
\]

Our first strategy of reconstructing the gravity background is using as an input the conformal factor $A_s(z)$ and determine the function $\phi(z)$. Having $A_s$ and $\phi$, we can solve for the unknown functions $f(z)$ and $V_E(z)$ from Eqs (2.16) and (2.18). The second strategy is to choose a special form for $\phi$ and then determine the conformal factor $A_s$ from Eq. (2.16). The remaining steps is the same as in the first strategy. Therefore, one can use the two kinds of generating function $A_s, \phi$ to reconstruct the gravity solution as we expect. Here we list the general formalism of generating gravity solution by using $A_s$ as an input. For the second strategy, we do not find general a formalism to generate solutions here. We should comment that it is also possible to find analytical or numerical solution by using the second strategy.

Starting with a given geometric structure $A_s(z)$, we can derive the general solutions to the Einstein equations, which take the following form

\[
\phi(z) = \phi_0 + \phi_1 \int_0^z \frac{e^{2A_s(x)}}{x^2} dx + \frac{3A_s(z)}{2} + 3 \int_0^z \frac{e^{2A_s(x)}}{x^2} \int_0^x y e^{-2A_s(y)} A'_s(y)^2 dy dx, \tag{2.19}
\]

\[
f(z) = f_0 + f_1 \left( \int_0^z x^3 e^{2\phi(x)} - 3A_s(x) dx \right), \tag{2.20}
\]

\[
V_E(\phi) = \frac{e^{4\phi(z) - 2A_s(z)}}{L^2} \left( z^2 f''(z) - 4 f(z) \left( 3z^2 A''_s(z) - 2z^2 \phi''(z) + z^2 \phi'(z)^2 + 3 \right) \right), \tag{2.21}
\]

where $\phi_0, \phi_1, f_0, f_1$ are constants of integration. As a consistency check, one can set $A_s = 0$, $\phi$ is then found to be $\phi = \phi_0$ with $\phi_0$ a constant, $\phi_1 = 0$ is indicated by requiring a regular solution for $\phi$. In terms of the (2.20), one can see the general system can reproduce the
pure AdS$_5$ vacuum solution if $f_0 = 1$ and $f_1 = 0$. Plugging these values to Eq. (2.21), one can easily get the scalar potential $V_s(z) = \frac{12}{L^2}$, as one expects for the pure AdS$_5$ case.

From the results above, one can easily obtain various gravity solutions by putting an $A_s(z)$ into Eqs (2.19)-(2.20)-(2.21) and imposing physical boundary conditions near UV and near IR region. Here we should point out that, in contrast to the ordinary procedure of finding the gravity solution after fixing the bulk potential $V_s$ or $V_E$, the framework motivated from the bottom-up point of view is good at producing lots of, and also helpful to reconstruct the gravity background. That is to say that one is able to extract suitable backgrounds to describe the phenomena on the field theory side. It is an efficient way of linking gravity and field theory. This approach will shed light on the application of gauge/gravity duality from a bottom-up viewpoint.

Before ending this section, we will comment on this approach. One should note from the solution in Eqs (2.19) (2.20) (2.21) that the geometric parameters such as $\phi_0, \phi_1, f_0, f_1$ contribute to the dilaton potential $V_E(\phi_0, \phi_1, f_0, f_1)$. Changing these parameters in the potential $V_E$ means that the theory is changed. In other words, different values of the parameters $\phi_0, \phi_1, f_0, f_1$ in $V_E$ correspond to different gravity theories. This is a main difference between this method and the ordinary procedure of finding the gravity solution with a fixed potential $V_E$. In some gravity solutions [21][22] reconstructed by this method, it seems possible that these different theories can be linked by the same form of action with different values of parameters in $V_E$. The different values of parameters correspond to different configurations of bulk field and different potentials, therefore these theories are not equivalent to each other. For one simple solution given in Appendix A of Ref. [21], $V_1 \neq 0$ in $f(z)$ can affect the form of potential through (2.21). For the dilaton potential with only a cosmological constant, there are two gravity solutions, the pure AdS$_5$ solution and the AdS Schwarzschild black hole. This means different parameter $f_1$ is dual to different states of one gravity theory in this case. Once one constructs the gravity background, one should confirm that the potential $V_E$ and $A_s(z), f(z)$ satisfy the constraints from other physical considerations, e.g. the Breitenlohner-Freedman (BF) bound of scalar field near AdS boundary, the finiteness of the action, a well-defined boundary of the system and so forth. In general, using this effective approach necessitates some physical checks to ensure that the solution is self-consistent.

### 3 Black hole solution and thermodynamic quantities

In this section, we just introduce a simple analytical gravity solution and then focus on the properties of the solution. If one is interested in other solutions, please refer to the appendix A of Ref. [21]. Before giving the 5D black hole solution generated by the above method from Eqs (2.19)(2.20)(2.21), we present the relation of the geometric factors $A_s$ and $A_E$ in the string frame and in the Einstein frame

\[
\frac{ds_E^2}{z^2} = \frac{L^2 e^{2A_s}}{z^2} \left( - f(z) dt^2 + \frac{d^2 z^2}{f(z)} + dx^i dx^i \right) \\
= \frac{L^2 e^{2A_E}}{z^2} \left( - f(z) dt^2 + \frac{d^2 z^2}{f(z)} + dx^i dx^i \right)
\]  

(3.1)
with \( A_E(z) = A_s(z) - \frac{2\phi(z)}{3} \).

We now give a simple analytical 5D black hole solution which is first obtained in Ref. [21]

\[
A_E(z) = \log \left( \frac{z}{z_0 \sinh \left( \frac{z}{z_0} \right)} \right),
\]

\[
\phi(z) = \frac{3z}{2z_0},
\]

\[
f(z) = 1 - \frac{V_1}{3} \left( 2 - 3 \cosh \left( \frac{z}{z_0} \right) + \cosh^3 \left( \frac{z}{z_0} \right) \right).
\]

The non-trivial dilaton potential in Eq. (2.8) is given by

\[
V_E(\phi) = -\frac{16V_1 \sinh^6 \left( \frac{\phi}{3} \right) + 9 \sinh^2 \left( \frac{2\phi}{3} \right) + 12}{L^2},
\]

where \( z_0 \) is an integration constant and \( V_1 \) is the constant parametrizing the dilaton potential. The special solution with \( V_1 = 0 \) thus \( f(z) = 1 \) is a degenerate aAdS solution without black hole, we will discuss it later and show that it corresponds to a BPS domain wall solution in the context of supergravity.

We now discuss the relevant thermodynamic quantities for the black hole solution (3.2). The first thing is how to parameterize the Hawking temperature, which is defined as \( \left. \frac{f'(z)}{4\pi} \right|_{z_h} \). A black-hole solution with a regular horizon is characterized by the existence of a surface \( z = z_h \) satisfying \( f(z_h) = 0 \). The Euclidean version of the solution is defined only for \( 0 < z < z_h \), in order to avoid the conical singularity, the periodicity of the Euclidean time can be fixed by

\[
\tau \rightarrow \tau + \frac{4\pi}{|f'(z_h)|}.
\]

This determines the temperature of the solution as

\[
T = \frac{|f'(z_h)|}{4\pi}
\]

\[
= \frac{1}{4\pi} \left| V_1 \sinh \left( \frac{z_h}{z_0} \right) - V_1 \cosh^2 \left( \frac{z_h}{z_0} \right) \sinh \left( \frac{z_h}{z_0} \right) \right|,
\]

where \( V_1 \) can be expressed by the horizon of the black hole \( z_h \) as

\[
V_1 = \frac{3}{2 - 3 \cosh \left( \frac{z_h}{z_0} \right) + \cosh^3 \left( \frac{z_h}{z_0} \right)}.
\]

Following the standard Bekenstein-Hawking formula [47], one can easily read the black-hole entropy density \( s \) from the geometry given in Eq. (2.11), which is defined by the area \( A_{\text{area}} \) of the horizon as

\[
s = \frac{A_{\text{area}}}{4G_5 V_3} = \frac{L^3}{4G_5} \left| \left. e^{\frac{A_s}{2} - \frac{2\phi}{3}} \right| \right|_{z_h}.
\]
where $G_5$ is the Newton constant in 5D curved space and $V_3$ is the volume of the spatial directions. It is noticed that the entropy density is closely related to the metric in the Einstein frame.

For later convenience, we set $L = 1$ and make the coordinate transformations $z = 1/r, z_0 = 1/r_0$. The metric (3.1) then becomes

$$ds^2 = r_0^2 \text{csch}^2 \left( \frac{r_0}{r} \right) \left( -f(r) dt^2 + \frac{1}{r^4 f(r)} + dx_i^2 \right),$$  \hspace{1cm} (3.8)

Setting $R = r_0 \text{csch} \left( \frac{r_0}{r} \right)$, one has

$$ds^2 = R^2 \left( -f(R) dt^2 + \frac{1}{R^2 (R^2 + r_0^2)} \right) + dx_i^2.$$

$$\hspace{1cm} (3.9)$$

\section{The conserved charges and counterterms}

In this section, we will calculate the conserved charges of the solution (3.9) by using the counterterm method [26]. This method is based on the Brown-York quasilocal stress tensor $T_{ab}$ which is first proposed in [25]. According to this method, for a $d+1$ dimensional spacetime $M$ with the boundary geometry $\partial M$, the metric $\gamma_{ab}$ on the boundary $\partial M$ can be written with the help of an ADM decomposition as

$$\gamma_{ab} dx^a dx^b = -N_\Sigma^2 dt^2 + \sigma_{ij}(dx^i + N_i^\Sigma dt)(dx^j + N_j^\Sigma dt),$$

$$\hspace{1cm} (4.1)$$

where $\sigma_{ij}$ is the induced metric on the spacelike hypersurface $\Sigma$ of the boundary $\partial M$, and $N_\Sigma$ is the lapse function related to the timelike normal vector $u^a$ of $\Sigma$ as $u^a = (\frac{\partial}{\partial t})^a / N_\Sigma$, $N_i^\Sigma$ is the shift vector field. Therefore, the conserved charge $Q_\xi$ associated with the killing vector $\xi^a$ is defined by the quasilocal stress tensor $T_{ab}$ as [25, 26, 48]

$$Q_\xi = \int_\Sigma d^{d-1} x \sqrt{\sigma} (u^a T_{ab} \xi^b),$$

$$\hspace{1cm} (4.2)$$

where $\sigma$ is the determinant of $\sigma_{ij}$, the energy associated with the timelike killing vector $\xi^a$ and the momentum could be defined respectively as

$$E = \int_\Sigma d^{d-1} x \sqrt{\sigma} N_\Sigma (u^a T_{ab} u^b),$$

$$\hspace{1cm} (4.3)$$

$$P_i = \int_\Sigma d^{d-1} x \sqrt{\sigma} \sigma_{ij} u_a T^{ja}.$$ 

$$\hspace{1cm} (4.4)$$

We have used $a, b$ to denote indices on the boundary $\partial M$, to avoid confusion from $\mu, \nu$ used in previous sections denoting indices in the $d+1$ bulk spacetime $M$. $i, j$ are indices on the spacelike hypersurface $\Sigma$.

For the asymptotically $AdS_5$ solution, the quasilocal stress tensor usually is [26]

$$T_{ab} = \frac{1}{8\pi} (\theta_{ab} - \theta \gamma_{ab} - 3 \gamma_{ab} - G_{ab}),$$

$$\hspace{1cm} (4.5)$$

where $\theta_{ab} = -\frac{1}{2} (\nabla_a n_b + \nabla_b n_a)$ is the extrinsic curvature of the boundary $\partial M$, $n^a$ is the normal vector of $\partial M$, and $G_{ab}$ is the Einstein tensor of $\gamma_{ab}$, and the last two terms in
come from the counterterm action [26]. However, in our case in (3.9), we find that the quasilocal stress tensor (4.5) defined as above would be divergent, which is common in Einstein-dilaton gravity. There have been many references [49–51] attacking the issue of removing the divergence and achieving a well-defined quasilocal stress tensor. In our case (3.9), we found two renormalizations that remove the divergence. In the first scheme we have the following well-defined quasilocal stress tensor

\[
T_{ab} = \frac{1}{8\pi} \left( \theta_{ab} - \theta_{\gamma ab} - (1 - \frac{V_E(\phi)}{6}) \gamma_{ab} - G_{ab} \right), \tag{4.6}
\]

while in the second

\[
T_{ab} = \frac{1}{8\pi} \left( \theta_{ab} - \theta_{\gamma ab} - (-1 + \sqrt{-4V_E(\phi)} \gamma_{ab} - G_{ab}) \right). \tag{4.7}
\]

If one expands the contribution of two different counterterms in Eqs (4.6) and (4.7), one finds

\[
1 - \frac{V_E(\phi)}{6} \sim 3 + \frac{3r_0^2}{2R^2} + \mathcal{O}(\frac{1}{R^4}),
\]

\[
-1 + \sqrt{-4V_E(\phi)} \sim 3 + \frac{3r_0^2}{2R^2} - \frac{9r_0^4}{32R^4} + \mathcal{O}(\frac{1}{R^5}). \tag{4.8}
\]

Actually we can write a most general well-defined quasilocal stress tensor as

\[
T_{ab} = \frac{1}{8\pi} \left( \theta_{ab} - \theta_{\gamma ab} - X(\phi) \gamma_{ab} - G_{ab} \right), \tag{4.9}
\]

where \( X(\phi) \)  is associated to the counterterm, and behaves as \( X(\phi) \rightarrow 3 + \frac{3r_0^2}{2R^2} + C_4 + \mathcal{O}(\frac{1}{R^4}) \) as \( R \) goes to infinity (\( C_4 \) is a constant), in order to achieve a finite \( T_{ab} \). The quasilocal stress tensor in Eq. (4.6) clearly corresponds to the minimal subtraction with \( C_4 = 0 \). In terms of the general form of the stress tensor Eq. (4.9), the useful components of quasilocal stress tensor as \( R \) goes to infinity are

\[
T_{tt} = \frac{(3 + 3V_1)r_0^4 + 8C_4}{8R^2},
\]

\[
T_{xx} = T_{yy} = T_{zz} = \frac{(-3 + V_1)r_0^4 - 8C_4}{8R^2}. \tag{4.10}
\]

Where \( V_3 \) is the volume of space including \( x_1, x_2, x_3 \). Therefore, the energy of black hole solution (3.9) can be expressed as

\[
E = \frac{(3 + 3V_1)r_0^4 + 8C_4}{8} V_3. \tag{4.11}
\]

At first sight, the \( C_4 \) term seems to introduce an ambiguity. However, as we will show in the next section, this ambiguity can actually be removed by looking at some special limit of our solution.
5 Hydrodynamics of dual boundary fluid via AdS/CFT correspondence

In this section we study the hydrodynamics of the boundary fluid dual to the bulk gravitational solution \((3.9)\), following the method proposed by Ref. [27]. In Ref. [27] the authors established a systematic way to map the hydrodynamic expansion of the boundary theory to the gradient expansion of the bulk gravity. One starts from a static black brane solution boosted along the translational invariant spatial directions, which corresponds to the boundary dynamics in global thermal equilibrium. The parameters characterizing the bulk solution such as the black hole temperature, velocities correspond precisely to the hydrodynamic degrees of freedom. To study the first order hydrodynamics, one moves away from the equilibrium by promoting the parameters to slowly-varying functions of the boundary coordinates, the original black brane solution no longer fulfills the equation of motion of the bulk gravity. An exact solution of the bulk gravity can be achieved only if the parameters now satisfy a set of equations of motion, which turn out to be the equations of the boundary fluid dynamics. In this way one establishes a one-to-one mapping between the bulk gravitational solution and the boundary fluid dynamics.

To start with, we transform the metric in Eq. \((3.9)\) into the Eddington-Finkelstein coordinates \(v = t + r_*\) with \(dr_* = \frac{dR}{R\sqrt{R^2 + r_0^2}}\).

\[
\begin{align*}
    ds^2 &= -R^2 f(R)dv^2 + \frac{2R}{R^2 + r_0^2}dv dR + R^2 dx^i dx_i. \\
\end{align*}
\]

(5.1)

Following the spirit of Ref. [27], we start from the boosted form of the above metric

\[
    ds^2 = -R^2 f(R)(u_a dx^a)^2 - \frac{2R}{R^2 + r_0^2}u_a dx^a dR + R^2 P_{ab} dx^a dx^b
\]

(5.2)

with

\[
    u^v = \frac{1}{\sqrt{1 - \beta_i^2}}, \quad u^i = \frac{\beta_i}{\sqrt{1 - \beta_i^2}}, \quad P_{ab} = \eta_{ab} + u_a u_b,
\]

(5.3)

where \(x^a = (v, x_i)\) denote the coordinates on the boundary, boost velocities \(\beta^i\) are constants, \(P_{ab}\) is the projector onto spatial directions, and the indices in the boundary are raised and lowered with the Minkowski metric \(\eta_{ab}\).

According to the procedure outlined above, we now lift the parameters in the metric in Eq. \((5.2)\) to slowly-varying functions of \(x^a\). The metric no longer fulfills the equations of motion \((2.12)\). To find the solutions, we can define the following quantities

\[
    W_{\mu\nu} = R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} V_E(\phi) - \frac{4}{3} \partial_\mu \phi \partial_\nu \phi,
\]

(5.4)

\[
    W = \Box \phi - \frac{3}{8} \frac{\partial V_E}{\phi},
\]

(5.5)

where vanishing of the RHS of Eqs \((5.4)\) \((5.5)\) will give the equations of motion \((2.12)\). After lifting the parameters \(\beta^i, r_0\) to functions of the boundary coordinates \(x^a\), \(W_{\mu\nu}\) and \(W\) are proportional to the derivatives of \(\beta^i, r_0\). They give rise to the source terms on the LHS of \((5.4)\) \((5.5)\). We should introduce corrections to the metric and to the dilaton field,
such that these corrections compensate for the effect of the source terms \( W_{\mu \nu}, W \). Then a solution of the bulk equations of motion can still be achieved. We will consider the derivative expansion to the first order. As usual, there exists a gauge redundancy in the metric. To eliminate this redundancy, we choose the background field gauge for the metric following Ref. [27]. Note that the background metric has a spatial \( SO(3) \) symmetry, it allows us to write the corrections to the metric in terms of the \( SO(3) \) scalar, vector and traceless symmetric tensor components.

To the first order, we have

\[
\begin{align*}
\frac{ds}{R^2} &= k(R) dv^2 + \frac{2R h(R)}{\sqrt{R^2 + r_0^2}} dv dR + \frac{2 j_i(R)}{R^2} dv dx^i + R^2 \left( \alpha_{ij}(R) - \frac{2}{3} h(R) \delta_{ij} \right) dx^i dx^j, \\
\phi (1) &= a(R),
\end{align*}
\]

(5.6)

where \( h(R), k(R) \) are the scalar components, \( j_i \) and \( \alpha_{ij}(R) \) the vector and tensor components, respectively. \( a(R) \) corresponds to correction term of dilaton field.

The equations of motion for the correction terms can be derived by requiring that the bulk equations of motion are fulfilled after adding the correction terms to the metric. We are interested in extracting the shear viscosity of the dual boundary fluid. For this purpose, we only need the solution for the non-diagonal components of the traceless symmetric tensor \( \alpha_{ij} \). The equation for such components \( \alpha_{ij}(i \neq j) \) can be obtained by putting (5.6) into (5.4)(5.5), which reads

\[
\begin{align*}
c_2(R) \alpha_{ij}''(R) + c_1(R) \alpha_{ij}'(R) - 9(\partial_i \beta_j + \partial_j \beta_i) &= 0 \\
\end{align*}
\]

(5.7)

with

\[
\begin{align*}
c_2(R) &= R^2 \sqrt{R^2 + r_0^2}(-3 + 2 V_1) - \frac{(2R^4 + R^2 r_0^2 - r_0^4)V_1}{R}, \\
c_1(R) &= -10 R^2 V_1 - 3 r_0^2 V_1 + \frac{r_0^4 V_1}{R^2} + \frac{R(5R^2 + 4r_0^2)(-3 + 2 V_1)}{\sqrt{R^2 + r_0^2}}.
\end{align*}
\]

(5.8)

Note that \( \alpha_{ij}(R) \) has the following form

\[
\alpha_{ij}(R) = \alpha(R)(\partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \partial_k \beta_k),
\]

(5.9)

where the tensor structure on the RHS is the traceless symmetric tensor structure constructed from the derivative of \( \beta_i \).

Solving Eq. (5.7), we find the following asymptotic form for \( \alpha(R) \)

\[
\alpha(R) \sim \frac{1}{R} - \frac{r_0^2}{6 R^3} - \frac{r_0^3}{4 R^4} + \mathcal{O}(\frac{1}{R^5}).
\]

(5.10)

The equations for the remaining correction terms in Eq. (5.6) are rather lengthy, we do not list those equations here, but we give the asymptotic form of those remaining correction...
terms for completeness

\[ a(R) \sim \left( \frac{3}{2} \frac{r_0^2}{R^2} - \frac{3r_0^2}{R^2} \right) (\partial_v r_0 + \frac{r_0 \partial_i \beta^i}{3}) + O\left( \frac{1}{R^6} \right), \]

\[ h(R) \sim \left( - \frac{4r_0}{R^3} + \frac{42r_0^3}{R^5} \right) (\partial_v r_0 + \frac{r_0 \partial_i \beta^i}{3}) + O\left( \frac{1}{R^6} \right), \]

\[ k(R) \sim \frac{2R^3 \partial_i \beta^i}{3} - R r_0 (r_0 \partial_i \beta^i + 2 \partial_v r_0) - \frac{r_0^3 (109r_0 \partial_i \beta^i + 372 \partial_v r_0)}{60R} + O\left( \frac{1}{R^6} \right), \]

\[ j_i(R) \sim R^3 \partial_v \beta^i + \frac{R r_0 (2 \partial_i r_0 - r_0 \partial_v \beta^i)}{6} + O\left( \frac{1}{R} \right), \quad (5.11) \]

where in obtaining these results and the asymptotic form of \( \alpha_{ij} \) we have used the same boundary conditions and gauge choice as in Refs. [27, 52, 53], e.g. non-deformation of the field theory metric and working in the Landau frame.

Having the results for the correction terms, we can derive the boundary stress tensor from the corrected metric. This can be done by varying the total action with respect to the boundary metric \( \gamma_{ab} \). From (4.9), the boundary stress tensor is given by

\[ T_{ab} = \frac{1}{8\pi} (\theta_{ab} - \theta \gamma_{ab} - X(\phi) \gamma_{ab} - G_{ab}). \quad (5.12) \]

Note that the background metric upon which the dual field theory resides is \( h_{ab} = \lim_{R \to \infty} \frac{1}{R^2} \gamma_{ab} \), which is the Minkowski metric, the expectation value of the stress tensor of the dual fluid \( \tau_{ab} \) can then be computed from [54]

\[ \sqrt{-h} h^{ab} < \tau_{bc} > = \lim_{R \to \infty} \sqrt{-\gamma} \gamma^{ab} T_{bc}, \quad (5.13) \]

where \( T_{ab} \) is the boundary stress tensor in Eq. (5.12). On the other hand, the stress tensor of the boundary fluid has the following form

\[ \tau_{ab} = P(\eta_{ab} + u_a u_b) + \rho u_a u_b - 2\eta \sigma_{ab}, \quad (5.14) \]

where

\[ \sigma^{ab} = \frac{1}{2} P^{ac} P^{bd} \left( \partial_c u_d + \partial_d u_c \right) - \frac{1}{3} P^{ab} \partial_c u^c, \]

\[ P = \frac{(-3 + V_1) r_0^4 - 8C_4}{8}, \]

\[ \rho = \frac{(3 + 3V_1) r_0^4 + 8C_4}{8}. \quad (5.15) \]

From the above equation it is clear that, to read off the shear viscosity \( \eta \), we need only the non-diagonal components of \( T_{ab} \), which give

\[ T_{ij} = -\frac{r_0^3 (\partial_i \beta_j + \partial_j \beta_i)}{16\pi R^2}. \quad (5.16) \]

Thus from (5.13), (5.14) and (5.16), the shear viscosity can be read off \( \eta = \frac{r_0^3}{16\pi} \), and the ratio of the shear viscosity and the entropy density is

\[ \frac{\eta}{s} = \frac{r_0^3 h/(16\pi)}{r_0^3 h/4} = \frac{1}{4\pi}, \quad (5.17) \]
which agrees with the result obtained previously from other considerations\[28, 29\].

A few comments on the $C_4$ term are in order. First of all, we do not include the bulk viscosity term in Eq. (5.14), since we have checked that it vanishes in our case for all choices of $C_4$. It should be pointed out that this result is nontrivial, because usually the bulk viscosity term can appear when there is a dilaton field in the bulk. Whether the zero bulk viscosity is related to our special black hole solution (3.2) or to the special dilaton potential (3.3) is unclear at this stage, and deserves further studies. Second, the potential ambiguity related to the choice of $C_4$ can be removed by investigating a special limit of our solution. Note that when $r_0 = 0$, our solution becomes pure $AdS_5$. If one agrees that the energy-momentum of the field theory dual to pure $AdS_5$ vanishes, e.g. as in Ref. [26], $C_4$ has to be zero. This corresponds to the minimal subtraction. The trace of the energy-momentum tensor is then given by $\tau = -\frac{3V_E}{2}$, which implies a nonzero conformal anomaly unless $r_0 = 0$, which is pure $AdS$.

6 Degenerate aAdS, domain wall and the c-function

Our purpose in this section is to show that the degenerate aAdS solution with $V_1 = 0$ is a domain wall solution, and to demonstrate a simple property of the bulk that translates via holography into a c-theorem [41] for the boundary theory. Based on the action (2.9), the general equations of motion [55] for graviton and scalar field are

$$R_{\mu\nu} = \frac{4}{3} \partial_\mu \phi \partial_\nu \phi + \frac{1}{3} g_{\mu\nu} V_E(\phi),$$  \hspace{1cm} (6.1)

$$\Box \phi = \frac{3}{8} \frac{\partial V_E}{\partial \phi},$$  \hspace{1cm} (6.2)

where we have used the fact $R = \frac{4}{3} (\partial \phi)^2 - \frac{5}{3} V_E(\phi)$ to simplify the formula (2.12). In order to consider the holographic RG flow of conformal anomaly, we will discuss the c function [42], under the domain wall ansatz for convenience. Let us begin by making the domain wall ansatz [55]

$$ds^2 = e^{2B(r)} ds^2(\mathbb{R}^{1,3}) + dr^2$$

$$= U^2 ds^2(\mathbb{R}^{1,3}) + \frac{1}{(\partial_r B)^2} dU^2$$  \hspace{1cm} (6.3)

with dilaton field $\phi(r)$. An important aspect of the AdS/CFT correspondence is the notion that the radial coordinate $U$ of AdS can be regarded as a measure of energy. In order to connect with our previous ansatz (2.11), we make the coordinate transformation as $dr = -\frac{L e^{A_E(z)}}{z} dz$ and impose the condition $B(r) = (A_E(z) - \log(\frac{z}{L}))$. The new form of the Einstein-dilaton equations for the metric (6.3) can be reduced to the following form

$$12(\partial_r B)^2 - \frac{4}{3} (\partial_r \phi)^2 + V_E(\phi) = 0,$$  \hspace{1cm} (6.4)

$$3\partial_r^2 B + 6(\partial_r B)^2 + \frac{2}{3} (\partial_r \phi)^2 + V_E(\phi) = 0,$$  \hspace{1cm} (6.5)

$$\partial_r^2 \phi + 4\partial_r B \partial_r \phi - \frac{3}{8} V'_E(\phi) = 0,$$  \hspace{1cm} (6.6)
where the prime indicates the differentiation with respect to $\phi$. One can use the coordinate transformation to reproduce the equations (2.13)(2.14)(2.15) as a consistency check. One should note that the above three equations are not independent, as in our previous case given in section 2. The above equations (6.4)(6.5)(6.6) imply the following simple exact relation

$$\partial_r^2 B = -\frac{4}{9} (\partial_r \phi)^2$$

$$= -\frac{4}{9} \left( \frac{\partial z}{\partial r} \partial_z \phi(z) \right)^2 = -\frac{4}{9 L^2} \left( \frac{z}{e^{A_E(z)}} \partial_z \phi(z) \right)^2. \tag{6.7}$$

The $c$ function proposed by the author of [42] is useful for studying the conformal anomaly in this system. In terms of the definition, the $c$ function is a positive function of the coupling constants that is non-increasing along the RG flow from the UV region ($U \to \infty, z \to 0$) to the IR region ($U \to 0, z \to \infty$). The definition of the $c$ function is [41][43][44]

$$C \equiv \int \sqrt{-T} = C_0 \left( \partial_r B(r) \right)^2, \tag{6.8}$$

where $T$ is the the trace of energy momentum tensor of dual field theory and $\gamma$ is the determinant of the metric on the boundary. The important thing is that once one has fixed $C_0$ within a particular supergravity theory, the formula (6.8) gives a form for computing the anomaly coefficients in any conformal theory dual to an AdS vacuum of that supergravity theory. The sign of $C_0$ ensures that $C$ is a positive function. Now we consider a special spacetime (3.2) with $V_1 = 0$ which is asymptotic to AdS spaces for $z \to 0$ or $r \to \infty$. Here we can get that

$$U \frac{\partial}{\partial U} C = -3C \frac{\partial^2 B}{(\partial_r B)^2}$$

$$= \frac{4C}{3L^2} \left( \frac{\partial_z \phi(z)}{\exp(A_E(z))} \right)^2$$

$$= \frac{4}{3L^2 \coth^3 \left( \frac{z}{2z_0} \right)} \left( \frac{\partial_z \phi(z)}{\exp(A_E(z))} \right)^2 \geq 0. \tag{6.9}$$

by using the function (6.7). It is easy to see that the $C(U)$ is a non-increasing function from UV to IR as expected for our special solution. Therefore, the boundary theory dual to the degenerate aAdS solution does not violate the c-theorem [44] in our case. Its monotonicity has been checked for the flows considered here.

7 BPS domain wall

In this section, we follow the well-known procedure [45][55] to discuss the 5D gravity solution with $V_1 = 0$ and show that it is a BPS domain wall solution. We will introduce the bosonic differentiable functional by using Euler-Lagrange equation method to study the stability. In Lagrangian mechanics, evolution of a physical system is described by the
solutions to the Euler-Lagrange equation (equations of motion (6.1)(6.2)) for the action of the system. The Euler-Lagrange functional [45] is as below

\[
E[A_E, \phi] = \int_0^\infty dz \frac{L e^{A_E(z)}}{z} \left[ \frac{4}{3} \left( \frac{z}{L e^{A_E(z)}} \partial_z \phi \right)^2 - 12 \left( \frac{z}{L e^{A_E(z)}} \partial_z \left( A_E - \log \left( \frac{z}{L} \right) \right) \right)^2 + V_E \right].
\]

(7.1)

We just use the conformal frame ansatz (3.1) to write down the functional \(E\). Note that we have restored the AdS radius \(L\). One can differentiate with respect to \(A_E\) or \(\phi\) to obtain the equation of motion (6.5) (6.6) as a consistency check. The function (7.1) can be rewritten, à la Bogomol’nyi, as

\[
E = \int_0^\infty dz \frac{L e^{A_E(z)}}{z} \left\{ \frac{z}{L e^{A_E(z)}} \sqrt{\frac{4}{3}} \partial_z \phi \mp 3 \sqrt{3} W'(\phi) \right\}^2 - 12 \left( \frac{z}{L e^{A_E(z)}} \partial_z \left( A_E - \log \left( \frac{z}{L} \right) \right) \right)^2 \pm 6 e^{A_E - \log \left( \frac{z}{L} \right)} W'[\infty]_0
\]

(7.2)

After introducing an auxiliary field \(W(\phi)\) which is called superpotential later, the Einstein-dilaton equations can be replaced by the following first-order equations. Here we should require that the following equations hold

\[
\frac{z}{L e^{A_E(z)}} \partial_z \left( A_E - \log \left( \frac{z}{L} \right) \right) = \mp 2 W(\phi),
\]

\[
\frac{z}{L e^{A_E(z)}} \partial_z \phi = \pm \frac{9}{2} W'(\phi).
\]

(7.3)

We just consider the degenerate solution given by taking \(V_1 = 0\) in (3.2). Luckily we obtain a superpotential analytically from the above formula. The superpotential \(W(\phi)\) is given by

\[
W(\phi) = \frac{1}{2L} \cosh \left( \frac{2\phi}{3} \right).
\]

(7.4)

The result of [56] tells us that stability constrains the form of the scalar potential to be

\[
V_E = 12 \left[ \frac{9}{4} \left( W'(\phi) \right)^2 - 4W^2 \right],
\]

(7.5)

where \(W(\phi)\) is the superpotential, which is an arbitrary function and the prime indicates the differentiation with respect to \(\phi\). As a consistency check, one can plug (7.4) into (7.5) to obtain \(V_E(\phi)\). If there exists a real solution of linear perturbation of \(\phi\), it will define a bound of stability named as the BF bound [56]. Therefore the fact that the dilaton potential can be expressed as the form of (7.5) is equivalent to that mass of the scalar field does not violate the BF bound.

Another possibility to obtain the first order equations (7.3) is to calculate the Witten-Nester energy [46]. From the Witten-Nester energy, one can obtain the same superpotential (7.4), as we will see below.

The Witten-Nester positive energy theorem [46] is motivated by the fact that the Hamiltonian in supersymmetric theories is the square of supercharges. This implies that
there is an expression for the energy in terms of spinors and that the energy is positive definite. The construction below imitates the supersymmetric argument but does not require supersymmetry. The spinor formalism of Witten and Nester provides a generalized “energy” $E_{WN}$ \cite{44}\cite{57}

$$E_{WN} = \int_{\partial \Sigma} * \hat{E},$$

(7.6)

where $\partial \Sigma$ is the 3D spatial hypersurface in the whole manifold and $*$ corresponds to the hodge star in 3D space. The Witten-Nester spinorial energy $E_{WN}$ \cite{44}\cite{57} is derived using the following antisymmetric tensor constructed from a spin or fields $\hat{E}^{\mu\nu} = \overline{\epsilon} \Gamma^{\mu\rho} \hat{\nabla}_\mu \epsilon - \overline{\nabla}_\mu \Gamma^{\mu\rho} \epsilon$

(7.7)

with

$$\hat{\nabla}_\mu = \nabla_\mu + W(\phi) \Gamma_\mu,$$

(7.8)

where $\nabla$ is the covariant derivative associated with the metric $g$ and $W(\phi)$ the superpotential. $\Gamma^{\mu\rho}$ is the 5D antisymmetric gamma matrix. Stokes theorem states that

$$E_{WN} = \int_{\Sigma} \nabla_\nu \hat{E}^{\mu\nu}.$$

(7.9)

Note that this step requires that the background and deformed solutions are non-singular. The singular case is beyond the scope of this paper. Readers interested in this issue may refer to \cite{57}. Just imposing the covariant derivative on $E_{WN}$ and making use of the equations of motion (6.1) (6.2), one can obtain that

$$E_{WN} = \int_{\Sigma} \delta \epsilon \left[ \Gamma^{\mu\rho} \hat{\nabla}_\mu \delta \psi_\rho - \frac{1}{2} \delta \chi \Gamma^{\mu} \delta \chi \right]$$

(7.10)

with

$$\delta \psi_\mu = \hat{\nabla}_\mu \epsilon,$$

(7.11)

$$\delta \chi = \left( \frac{2}{\sqrt{3}} \Gamma^{\mu} \nabla_\mu \phi - 3 \sqrt{3} W'(\phi) \right) \epsilon,$$

(7.12)

where $W'(\phi) = \partial_\phi W$. One can check that (7.10) is consistent with the result in \cite{58}. Here we do not calculate the final result step by step, but just list the most important formulas leading us to (7.10)(7.11)(7.12).

In obtaining the above results we have used the following formula

$$\overline{\nabla}_\mu \epsilon \Gamma^{\mu\rho} \nabla_\rho (\omega) \epsilon = - \overline{\tau} \Gamma^{\mu\rho} \nabla_\mu \nabla_\rho (\omega) \epsilon,$$

(7.13)

where $w$ is the spin connection in curved space. The relation $\nabla_\mu \nabla_\nu \epsilon = \frac{1}{8} R^{ab}_{\mu\nu} \Gamma_{ab} \epsilon$ is helpful to realize (7.10), where $\mu, \nu, \rho$ stand for the indices of curved space and $a, b$ denote
the coordinates of local flat space. The right hand side of (7.13) can be expanded using the clifford algebra and gives

\[ R_{ab}^{\mu\nu}\bar{\epsilon} \{ \Gamma_{\mu\nu\rho}^{ab} + \frac{1}{2} \Gamma_{\nu}^{ab} \} \epsilon = R_{ab}^{\mu\nu}\bar{\epsilon} \left\{ \frac{1}{2} \Gamma_{\nu}^{ab} \right\} \]

where we just impose the covariant derivative on (7.7), and we used (7.14) to get the second term in (7.10). One can make use of Einstein equation to replace \( R^{\mu\rho} - \frac{1}{2} R g^{\mu\rho} \) appearing in (7.14) to obtain (7.8). The first term in (7.14) will be canceled by \( \bar{\epsilon} \{ \Gamma_{\nu}^{ab} \} \epsilon \) automatically. Through straightforward calculation, one can get the the final form (7.10).

The last term in (7.10) will be canceled, if we require that

\[ V_E(\phi) = 36 \left( \frac{2}{3} W'(\phi)^2 - \frac{4}{3} W^2(\phi) \right). \]  

This constraint is consistent with the previous statement (7.5). Starting from (7.10), one can read out the Witten equation:

\[ (\nabla_\mu + \Gamma_\mu W(\phi)) \epsilon = 0, \]

\[ (\Gamma^\mu \nabla_\mu \phi - \frac{9}{2} W'(\phi)) \epsilon = 0, \]

where \( W(\phi) \) is given by (7.4). The spinor solutions of (7.16) are the background Killing spinors similar to those given in [45] [58]:

\[ \epsilon = \epsilon_{0} = \Gamma^z \epsilon_{0}, \]

where \( \epsilon_{0} \) is a constant spinor which is chiral with respect to the radial component of \( r \) or \( z \). In the last equation we have used the coordinate transformation \( dr = \frac{z}{L} \frac{dz}{\epsilon_{0}} \). We confirm that our solution (3.2) with \( V_1 = 0 \) is a BPS one. For more complicated case, the authors of [59] have discussed the asymptotical solution of the killing spinor equation.

8 Conclusion and discussion

We apply the potential reconstruction approach to study the graviton coupled to dilaton system and generate 5D aAdS black hole solutions in a semi-analytical way. In our approach, there are two scenarios to obtain the solution. The first is, we choose the special conformal factor \( A_s \) or \( A_E \) and then determine the dilaton configuration \( \phi(z) \) from the tt
component of the Einstein equation. The black hole solution \( f(z) \) can subsequently be defined and the corresponding self-interacting potential \( V_s(\phi) \) or \( V_E(\phi) \) then determined. In the second scenario, one chooses a dilation configuration \( \phi(z) \) to define the conformal factor \( A_s \) or \( A_E \). The remaining steps of reconstructing the gravity background are the same as in the first scenario. We study the related properties of a simple 5D gravity solution (3.2) in the remaining part.

We also investigate the hydrodynamics of the boundary fluid dual to our bulk gravitational solution, following the spirit of the fluid/gravity correspondence. We worked out the asymptotic behavior of the correction terms that need to be added to achieve an exact solution of the bulk equation of motion after promoting the parameters of the original solution to slowly-varying functions of the boundary coordinates. In particular, we presented the asymptotic form of the \( SO(3) \) tensor components of the correction terms, which is useful for extracting the ratio of the shear viscosity to the entropy density of the boundary fluid. Another interesting observation is that, the dual boundary fluid is conformally anomalous, and the anomaly is related to the temperature of the black hole. This might be a potential useful feature of our solution in describing physical situations where conformal symmetry is broken.

Furthermore, we study the holographic conformal anomaly characterized by the \( c \) function and the stability of the degenerate aAdS solution as a special case of our 5D black hole solution. We check that the degenerate solution does not violate the \( c \) theorem. The dual RG flow of the holographic conformal anomaly owns the monotonicity and this situation is a special case of [45]. Following [45] [55], we list the formalism for the bosonic functional energy and the Witten-Nester energy in conformal coordinates. We check that the degenerate aAdS solution is a BPS domain wall solution.

For future studies, it is valuable and feasible to use the potential reconstruction approach to reconstruct the gravity solutions relevant for phenomenology. For example, to describe more realistic case, one can impose constraints on the metric from experimental data and use the method to reconstruct the gravity system corresponding to the metric. It is an efficient way to link the gravity and phenomena in field theory directly. It will apply to various fields, such as: AdS/QCD, AdS/CMT, fluid/gravity duality and so forth. It opens a new window to test the gauge/gravity duality.

**Acknowledgments:** The authors thank Rong-Gen Cai, Mei Huang, Bin Hu, Jen-Chi Lee, Hong Lu, Danning Li, Li Li, JunBao Wu, QiShu Yan, Yi Yang, YunLong Zhang, HaiQing Zhang for valuable discussions. S.H. appreciates the hospitality of the institute of high energy physics, CAS during the initial stage of this work. Y.P Hu is supported by China Postdoctoral Science Foundation under Grant No.20110490227 and National Natural Science Foundation of China (NSFC) under grant No.11105004. This work is also supported partially by grants from NSFC, No. 10975168 and No. 11035008.
References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[5] C. Martinez, R. Troncoso, J. Zanelli, “Exact black hole solution with a minimally coupled scalar field,” Phys. Rev. D 70, 084035 (2004). [hep-th/0406111].

[6] G. Dotti, R. J. Gleiser, C. Martinez, “Static black hole solutions with a self interacting conformally coupled scalar field,” Phys. Rev. D 77, 104035 (2008). [arXiv:0710.1735 [hep-th]].

[7] M. Hatsuda, M. Sakaguchi, “Wess-Zumino term for AdS superstring,” Phys. Rev. D 66, 045020 (2002). [hep-th/0205092].

[8] D. -f. Zeng, “An Exact Hairy Black Hole Solution for AdS/CFT Superconductors,” [arXiv:0903.2620 [hep-th]].

[9] T. Torii, K. Maeda, M. Narita, “Scalar hair on the black hole in asymptotically anti-de Sitter space-time,” Phys. Rev. D 64, 044007 (2001).

[10] E. Megias, H. J. Pirner and K. Veschgini, “QCD-Thermodynamics using 5-dim Gravity,” Phys. Rev. D 83, 056003 (2011) [arXiv:1009.2953 [hep-ph]]; K. Veschgini, E. Megias and H. J. Pirner, “Trouble Finding the Optimal AdS/QCD,” Phys. Lett. B 696, 495 (2011) [arXiv:1009.4639 [hep-th]]; E. Megias, H. J. Pirner and K. Veschgini, “Thermodynamics of AdS/QCD within the 5D dilaton-gravity model,” Nucl. Phys. Proc. Suppl. 207-208, 333 (2010) [arXiv:1008.4505 [hep-th]]; B. Galow, E. Megias, J. Nian and H. J. Pirner, “Phenomenology of AdS/QCD and Its Gravity Dual,” Nucl. Phys. B 834, 330 (2010) [arXiv:0911.0627 [hep-ph]].

[11] S. S. Gubser and A. Nellore, “Mimicking the QCD equation of state with a dual black hole,” Phys. Rev. D 78, 086007 (2008); S. S. Gubser, A. Nellore, S. S. Pufu and F. D. Rocha, “Thermodynamics and bulk viscosity of approximate black hole duals to finite temperature quantum chromodynamics,” Phys. Rev. Lett. 101, 131601 (2008); S. S. Gubser, S. S. Pufu and F. D. Rocha, “Bulk viscosity of strongly coupled plasmas with holographic duals,” JHEP 0808, 085 (2008).

[12] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Deconfinement and Gluon Plasma Dynamics in Improved Holographic QCD,” Phys. Rev. Lett. 101, 181601 (2008); U. Gursoy, E. Kiritsis, G. Michalogiorgakis and F. Nitti, “Thermal Transport and Drag Force in Improved Holographic QCD,” JHEP 0912, 056 (2009).

[13] K. Behrndt, M. Cvetic, W. A. Sabra, “Nonextreme black holes of five-dimensional N=2 AdS supergravity,” Nucl. Phys. B 553, 317-332 (1999). [hep-th/9810227].

[14] R. -G. Cai, K. -S. Soh, “Critical behavior in the rotating D-branes,” Mod. Phys. Lett. A 14, 1895-1908 (1999). [hep-th/9812121].
[15] S. S. Gubser, “Thermodynamics of spinning D3-branes,” Nucl. Phys. B551, 667-684 (1999). [hep-th/9810225].

[16] P. Benincasa, A. Buchel, A. O. Starinets, “Sound waves in strongly coupled non-conformal gauge theory plasma,” Nucl. Phys. B733, 160-187 (2006). [hep-th/0507026].

[17] P. Benincasa, A. Buchel, “Hydrodynamics of Sakai-Sugimoto model in the quenched approximation,” Phys. Lett. B640, 108-115 (2006). [hep-th/0605076].

[18] T. Sakai, S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113, 843-882 (2005). [arXiv:hep-th/0412141 [hep-th]].

[19] C. Chiou-Lahanas, G. A. Diamandis and B. C. Georgalas, “Five-Dimensional Black Hole String Backgrounds and Brane Universe Phys. Lett. B 678, 485 (2009) [arXiv:0904.1484 [hep-th]].

[20] K. Farakos, A. P. Kouretsis, P. Pasipoularides, “Anti de Sitter 5D black hole solutions with a self-interacting bulk scalar field: A Potential reconstruction approach,” Phys. Rev. D80, 064020 (2009). [arXiv:0905.1345 [hep-th]].

[21] D. Li, S. He, M. Huang and Q. S. Yan, “Thermodynamics of deformed AdS5 model with a positive/negative quadratic correction in graviton-dilaton system,” JHEP 1109, 041 (2011) [arXiv:1103.5389 [hep-th]].

[22] S. He, M. Huang, Q. -S. Yan, “Logarithmic correction in the deformed AdS5 model to produce the heavy quark potential and QCD beta function,” Phys. Rev. D83, 045034 (2011). [arXiv:1004.1880 [hep-ph]].

[23] N. Ohta, T. Torii, “Black Holes in the Dilatonic Einstein-Gauss-Bonnet Theory in Various Dimensions IV: Topological Black Holes with and without Cosmological Term,” Prog. Theor. Phys. 122, 1477-1500 (2009). [arXiv:0908.3918 [hep-th]].

[24] T. Kolyvaris, G. Koutsoumabras, E. Papantonopoulos, G. Siopsis, “A New Class of Exact Hairy Black Hole Solutions,” Gen. Rel. Grav. 43, 163-180 (2011). [arXiv:0911.1711 [hep-th]].

[25] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D 47, 1407 (1993) [arXiv:gr-qc/9209012]; J. D. Brown, J. Creighton and R. B. Mann, “Temperature, energy and heat capacity of asymptotically anti-de Sitter black holes,” Phys. Rev. D 50, 6394 (1994) [arXiv:gr-qc/9405007].

[26] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP 9807, 023 (1998) [arXiv:hep-th/9806087]; V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. 208, 413 (1999) [arXiv:hep-th/9902121]; K. Skenderis, “Lecture notes on holographic renormalization,” Class. Quant. Grav. 19, 5849 (2002) [arXiv:hep-th/0209067]; R. C. Myers, “Stress tensors and Casimir energies in the AdS/CFT correspondence,” Phys. Rev. D 60, 046002 (1999) [arXiv:hep-th/9903203]; S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. 217, 595 (2001) [arXiv:hep-th/0002230].

[27] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP 0802, 045 (2008) [arXiv:0712.2456 [hep-th]].

[28] N. Iqbal and H. Liu, “Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm,” arXiv:0809.3808 [hep-th].

[29] R. G. Cai, Z. Y. Nie and Y. W. Sun, “Shear Viscosity from Effective Couplings of Gravitons,” arXiv:0811.1665 [hep-th]; R. G. Cai, Z. Y. Nie, N. Ohta and Y. W. Sun, “Shear
Viscosity from Gauss-Bonnet Gravity with a Dilaton Coupling,” Phys. Rev. D 79, 066004 (2009) [arXiv:0901.1421 [hep-th]].

[30] G. Policastro, D. T. Son and A. O. Starinets, “The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma,” Phys. Rev. Lett. 87, 081601 (2001) [arXiv:hep-th/0104066].

[31] P. Kovtun, D. T. Son and A. O. Starinets, “Holography and hydrodynamics: Diffusion on stretched horizons,” JHEP 0310, 064 (2003) [arXiv:hep-th/0309213].

[32] A. Buchel and J. T. Liu, “Universality of the shear viscosity in supergravity,” Phys. Rev. Lett. 93, 090602 (2004) [arXiv:hep-th/0311175].

[33] P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” Phys. Rev. Lett. 94, 111601 (2005) [arXiv:hep-th/0405231].

[34] J. Mas, “Shear viscosity from R-charged AdS black holes,” JHEP 0603, 016 (2006) [arXiv:hep-th/0601144].

[35] D. T. Son and A. O. Starinets, “Hydrodynamics of R-charged black holes,” JHEP 0603, 052 (2006) [arXiv:hep-th/0601157].

[36] O. Saremi, “The viscosity bound conjecture and hydrodynamics of M2-brane theory at finite chemical potential,” JHEP 0610, 083 (2006) [arXiv:hep-th/0601159].

[37] K. Maeda, M. Natsume and T. Okamura, “Viscosity of gauge theory plasma with a chemical potential from AdS/CFT,” Phys. Rev. D 73, 066013 (2006) [arXiv:hep-th/0602010].

[38] R. G. Cai and Y. W. Sun, “Shear Viscosity from AdS Born-Infeld Black Holes,” JHEP 0809, 115 (2008) [arXiv:0807.2377 [hep-th]].

[39] A. Buchel, R. C. Myers, M. F. Paulos and A. Sinha, “Universal holographic hydrodynamics at finite coupling,” Phys. Lett. B 669, 364 (2008) [arXiv:0808.1837 [hep-th]]; 

[40] R. C. Myers, M. F. Paulos and A. Sinha, Phys. Rev. D 79, 041901 (2009) [arXiv:0806.2156 [hep-th]].

[41] E. Alvarez, C. Gomez, “Geometric holography, the renormalization group and the c theorem,” Nucl. Phys. B541, 441-460 (1999). [hep-th/9807226].

[42] M. Henningson and K. Skenderis, “The Holographic Weyl anomaly,” JHEP 9807, 023 (1998) [arXiv:hep-th/9806087]. M. Henningson, K. Skenderis, “Holography and the Weyl anomaly,” Fortsch. Phys. 48, 125-128 (2000). [hep-th/9812032].

[43] J. Distler, F. Zamora, “Nonsupersymmetric conformal field theories from stable anti-de Sitter spaces,” Adv. Theor. Math. Phys. 2, 1405-1439 (1999). [hep-th/9810206].

[44] D. Z. Freedman, S. S. Gubser, K. Pilch, N. P. Warner, “Renormalization group flows from holography supersymmetry and a c theorem,” Adv. Theor. Math. Phys. 3, 363-417 (1999). [hep-th/9904017].

[45] K. Skenderis, P. K. Townsend, “Gravitational stability and renormalization group flow,” Phys. Lett. B468, 46-51 (1999). [hep-th/9909070].

[46] E. Witten, “A Simple Proof of the Positive Energy Theorem,” Commun. Math. Phys. 80, 381 (1981). J. Nester, A new gravitational expression with a simple positivity proof, Phys. Lett. 83A, 241 (1981).

[47] J. D. Bekenstein, “Black holes and entropy,” Phys. Rev. D 7, 2333 (1973); S. W. Hawking,
“Particle Creation By Black Holes,” Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)].

[48] Y. P. Hu, “Tension term, interchange symmetry, and the analogy of energy and tension energy,” JHEP 0905, 096 (2009) [arXiv:0904.1250 [hep-th]].

[49] S. Nojiri and S. D. Odintsov, “Conformal anomaly for dilaton coupled theories from AdS/CFT correspondence,” Phys. Lett. B 444, 92 (1998) [arXiv:hep-th/9810008].

[50] R. G. Cai and N. Ohta, “Surface counterterms and boundary stress-energy tensors for asymptotically non-anti-de Sitter spaces,” Phys. Rev. D 62, 024006 (2000) [arXiv:hep-th/9912013].

[51] S. Nojiri, S. D. Odintsov and S. Ogushi, “Finite action in d5 gauged supergravity and dilatonic conformal anomaly for dual quantum field theory,” Phys. Rev. D 62, 124002 (2000) [arXiv:hep-th/0001122].

[52] J. Hur, K. K. Kim and S. J. Sin, “Hydrodynamics with conserved current from the gravity dual,” JHEP 0903, 036 (2009) [arXiv:0809.4541 [hep-th]].

[53] Y. -P. Hu, P. Sun, J. -H. Zhang, “Hydrodynamics with conserved current via AdS/CFT correspondence in the Maxwell-Gauss-Bonnet gravity,” Phys. Rev. D83 (2011) 126003. [arXiv:1103.3773 [hep-th]]; Y. -P. Hu, H. -F. Li, Z. -Y. Nie, “The first order hydrodynamics via AdS/CFT correspondence in the Gauss-Bonnet gravity,” JHEP 1101, 123 (2011). [arXiv:1012.0174 [hep-th]].

[54] R. C. Myers, “Stress tensors and Casimir energies in the AdS / CFT correspondence,” Phys. Rev. D60, 046002 (1999). [hep-th/9903203].

[55] M. Cvetic, H. Lu and C. N. Pope, “Domain walls and massive gauged supergravity potentials,” Class. Quant. Grav. 17, 4867 (2000) [arXiv:hep-th/0001002].

[56] P. K. Townsend, “Positive energy and the scalar potential in higher dimensional super gravity theorems,” Phys. Lett. B 148, 55 (1984).

[57] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, M. J. Perry, “Positive Mass Theorems For Black Holes,” Commun. Math. Phys. 88, 295 (1983).

[58] D. Z. Freedman, C. Nunez, M. Schnabl, K. Skenderis, “Fake supergravity and domain wall stability,” Phys. Rev. D69, 104027 (2004). [hep-th/0312055].

[59] M. C. N. Cheng, K. Skenderis, “Positivity of energy for asymptotically locally AdS spacetimes,” JHEP 0508, 107 (2005). [hep-th/0506123].