A principal component analysis for LISA – the TDI connection

J. D. Romano* and G. Woan*

1School of Physics and Astronomy, Cardiff University, Cardiff, CF24 3YB, UK
2Department of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK

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Data from the Laser Interferometer Space Antenna (LISA) is expected to be dominated by frequency noise from its lasers. However the noise from any one laser appears more than once in the data and there are combinations of the data that are insensitive to this noise. These combinations, called time delay interferometry (TDI) variables, have received careful study, and point the way to how LISA data analysis may be performed. Here we approach the problem from the direction of statistical inference, and show that these variables are a direct consequence of a principal component analysis of the problem. We present a formal analysis for a simple LISA model and show that there are eigenvectors of the noise covariance matrix that do not depend on laser frequency noise. Importantly, these orthogonal basis vectors correspond to linear combinations of TDI variables. As a result we show that the likelihood function for source parameters using LISA data can be based on TDI combinations of the data without loss of information.

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I. INTRODUCTION

The Laser Interferometer Space Antenna (LISA) is a space-borne gravitational telescope currently under development by ESA and NASA. It is designed to detect gravitational radiation at frequencies between $\sim 10^{-5}$ Hz and $\sim 1$ Hz, a band that is not readily accessible from Earth due to local noise. The current design comprises a constellation of three spacecraft in circular orbits of radius 1 AU around the Sun, in a near-equilateral configuration of side $5 \times 10^6$ km. Gravitational waves passing through the telescope modulate the separation between the spacecraft on scales of picometres. This modulation is sensed by laser beams exchanged between spacecraft, and recorded as the difference in frequency between the locally-generated and received laser signals. In simple terms there are therefore three independent lasers in the system, and six raw signal streams (referred to as Doppler measurements in the literature, e.g.,[7]) corresponding to the six bidirectional baseline combinations of these lasers.

The main source of noise in these raw streams comes from the relative frequency stability of the reference lasers, giving a spectral density of $\sim 10^{-13}$ Hz$^{-1/2}$ in the millihertz band. This laser frequency noise results in a strain noise floor $\sim 10^4$ times higher than the target sensitivity of LISA and, at first sight, severely limits LISA’s performance. The scenario is very similar to that encountered in radio astronomical very long baseline interferometry (VLBI), where free-running local oscillators are used at each end of the interferometric baseline, giving the resultant fringes an unknown and varying fringe rate. The breakthrough in radio astronomy came with the realisation that with three or more telescopes, and therefore three or more baselines, one can form closure quantities that are insensitive to the relative phases of the local oscillators. The simplest of these relations is closure phase, comprising the sum of interferometric phases around a loop of baselines [13]. The phase of each local oscillator appears twice in such a sum and with opposite signs, so that closure phase contains only baseline-dependent (and therefore astronomical), rather than antenna-dependent contributions. This direct method was used in the early days of radio interferometry [5], in VLBI [11] and more recently in optical interferometry [1]. In modern radio astronomy these principles are encapsulated in the idea of self calibration, in which each antenna in the interferometer is allocated an unknown complex gain factor [11,12]. The interferometer data are then used to simultaneously determine both these gain factors and the sky map, so greatly increasing the performance of the instrument.

The situation for LISA is more complicated. Laser noise can be canceled only if the closure relations take account of the light travel time between spacecraft, and the presence of bidirectional beams along the baselines increases the number of possible relations. This has led to the development of Time-Delay Interferometry (TDI) variables for LISA [16]. These are linear combinations of the six LISA data streams, suitably offset in time to cancel the noise contributions from the three lasers. TDI variables have received detailed attention in the literature and there is now a sophisticated understanding of their generation and properties [2,3,4,5,14,21], extending to ‘second-generation’ TDI variables that take account of the slight relative motion of the spacecraft [13,17].

The very existence of TDI variables clearly shows that LISA is capable of generating data that is sensitive to astronomical sources but not to laser noise. This is an important step, but it does not tell us how to use these TDI variables to do astronomy. For this we need a method of using these derived data to constrain the sky and make statements about the parameters of individual sources of

*graham@astro.gla.ac.uk
†joseph.romano@astro.cf.ac.uk

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gravitational radiation. The natural framework for such statements is that of statistical inference, in which we construct a Bayesian probability for a source parameter, or set of parameters, $a$, given the LISA data, $d$. Formally
\[ p(a|d) \propto p(a) \, p(d|a) \, , \quad (1) \]
where $p(d|a)$ is the probability of getting a certain set of data given a noise model and a particular value for $a$. This is familiar as Bayes’ Theorem, and requires the prior probability for $a$, $p(a)$, before it can be fully applied. Although priors form an important part of any analysis we will not concentrate on them here and they do not depend on the current data. The quantity $p(d|a)$ is usually called the likelihood of $a$ and, importantly, fully defines how the data enters into the calculation. Indeed the likelihood contains all that the data has to say on the matter, so that the heart of a parameter estimation problem is fully defined once a likelihood is written down.

We will show below that the likelihood function for LISA contains several insights into the meaning and role of TDI variables. We consider the noise covariance matrix for a series of LISA Doppler measurements over several time steps. For Gaussian noise, the inverse of this covariance matrix defines the log likelihood of the parameters, and its quadratic form defines contours of equal likelihood, in a space equal in dimension to the number of data points used. The eigenvectors of this covariance matrix are the principal axes of the equal-likelihood hyperellipsoidal surface, and correspond to linear combinations of the data that give maximal and minimal covariance. Principal component analysis (PCA) is simply the process of identifying these eigenvectors and using only a subset of them to characterise the data. Usually it is the components with the largest eigenvalues that are desired, as these are the data combinations that contain the majority of the covariance. These components are used in fields such as image recognition in order to generalise or compress data sets. For LISA however, these principal components are the ones that contain the highly correlated laser frequency noise. In contrast therefore, we are interested in the existence of eigenvalues that do not depend on the laser noise and are minimal. We will show that the eigenvalues of the LISA covariance matrix fall into two distinct groups, distinguished by their dependence on laser noise. The group that is independent of the common frequency noise corresponds directly to the TDI variables considered above, in fact the relationship is hinted at in \ref{eq:1}. Not all will necessarily be sensitive to gravitational wave signals, but as a group they orthogonally span the data sub-space that corresponds to LISA’s design noise floor. The other group still contain astrophysics, but are dominated by the effects of laser frequency fluctuations.

In Section II we will consider a simple example of PCA to highlight the essence of the method and demonstrate its applicability to LISA data analysis. In Section III we extend these ideas to tackle a real, but somewhat simplified, LISA model and show that minor principal components are TDI variables. In Section IV we discuss some of the immediate implications of this work for the design and data analysis of LISA.

\section{Simple Example}
Consider a simple sample of data from two generic detectors
\[ s_1 = p + n_1 + h_1 , \quad (2) \]
\[ s_2 = p + n_2 + h_2 , \quad (3) \]
where $n_1$, $n_2$ are uncorrelated noises in the individual detectors, $p$ is a common noise term, and $h_1$, $h_2$ are the astrophysical signals of interest. For simplicity, we assume that the noises are Gaussian-distributed with zero mean and variances
\[ \langle n_1^2 \rangle = \langle n_2^2 \rangle \equiv \sigma_n^2 \quad \text{and} \quad \langle p^2 \rangle \equiv \sigma_p^2 , \quad (4) \]
and that they are mutually uncorrelated, so that
\[ \langle n_1 n_2 \rangle = \langle n_1 p \rangle = \langle n_2 p \rangle = 0 . \quad (5) \]
In addition, we assume a simple model in which the astrophysical signals in the two detectors are
\[ h_1 = 2a \quad \text{and} \quad h_2 = a , \quad (6) \]
where $a$ is a fixed but unknown constant whose posterior distribution, $p(a|s_1, s_2)$, we want to compute.

To do this, in accordance with Eq. \ref{eq:1}, we need to calculate the likelihood $p(s_1, s_2|a)$. For Gaussian noise,
\[ p(s_1, s_2|a) \propto \exp \left[ -\frac{1}{2} Q \right] , \quad (7) \]
where
\[ Q \equiv (s - h)^T C^{-1} : (s - h) \quad (8) \]
\[ \equiv \sum_{i,j=1}^{2} (s_i - h_i) C^{-1}_{ij} (s_j - h_j) \quad (9) \]
is a quadratic form, involving the inverse of the noise covariance matrix $C$ whose elements are
\[ C_{ij} \equiv \langle (s_i - h_i)(s_j - h_j) \rangle . \quad (10) \]
Using Eqs. \ref{eq:2}, \ref{eq:3}, we find
\[ C = \left( \begin{array}{ccc} \sigma_n^2 & \sigma_n^2 & \sigma_n^2 \\ \sigma_n^2 & \sigma_p^2 & \sigma_p^2 \\ \sigma_n^2 & \sigma_p^2 & \sigma_n^2 \end{array} \right) . \quad (11) \]

Principal component analysis simplifies the calculation of the likelihood $p(s_1, s_2|a)$, by identifying the eigenvectors of this covariance matrix $C$. Note that the eigenvectors of $C$ are also the eigenvectors of $C^{-1}$, but with
reciprocal eigenvalues. The eigenvectors of $C$ in Eq. (11) are
\[ e_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad e_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \]
with eigenvalues
\[ \lambda_+ = 2\sigma_p^2 + \sigma_n^2 \quad \text{and} \quad \lambda_- = \sigma_n^2 \]
respectively. If we form the matrix of eigenvectors
\[ E = (e_+ \ e_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]
and use the facts that
\[ E \cdot E^T = E^T \cdot E = I, \]
\[ E \cdot C^{-1} \cdot E^T = \begin{pmatrix} 1/\lambda_+ & 0 \\ 0 & 1/\lambda_- \end{pmatrix} = \Lambda^{-1}, \]
and
\[ E \cdot (s - h) = \frac{1}{\sqrt{2}} \begin{pmatrix} (s_1 + s_2) - (h_1 + h_2) \\ (s_1 - s_2) - (h_1 - h_2) \end{pmatrix}, \]
we can rewrite $Q$ as
\[ Q = (s - h)^T \cdot C^{-1} \cdot (s - h) \]
\[ = (s - h)^T \cdot (E^T \cdot E) \cdot C^{-1} \cdot (E^T \cdot E) \cdot (s - h) \]
\[ = (E \cdot (s - h))^T \cdot \Lambda^{-1} \cdot (E \cdot (s - h)) \]
\[ = \frac{1}{2} \frac{1}{2\sigma_p^2 + \sigma_n^2} (s_+ - 3a)^2 + \frac{1}{2} \frac{1}{\sigma_n^2} (s_- - a)^2, \]
where we have defined $s_+$, $s_-$ to be the eigencombinations
\[ s_+ \equiv s_1 + s_2, \]
\[ s_- \equiv s_1 - s_2. \]
Thus, the likelihood $p(s_1, s_2|a)$ factorises into a product of likelihoods
\[ p(s_1, s_2|a) \propto p(s_+|a)p(s_-|a), \]
where
\[ p(s_+|a) \propto \exp \left[ -\frac{1}{2} \frac{(s_+ - 3a)^2}{4\sigma_p^2 + 2\sigma_n^2} \right], \]
\[ p(s_-|a) \propto \exp \left[ -\frac{1}{2} \frac{(s_- - a)^2}{2\sigma_n^2} \right]. \]

Note that the $s_-$ data combination, Eq. (29), corresponds to the minimum eigenvalue $\lambda_- \equiv \sigma_n^2$, which is independent of the common noise variance $\sigma_p^2$. In this sense, $s_-$ is the preferred data combination for this example, since it is least affected by the various noise terms (see Fig. 1). Moreover, if $\sigma_p^2 \gg \sigma_n^2$, as is the case for the common laser frequency noise for LISA, then there is (effectively) no loss in information by doing statistical inference with only the $s_-$ data combination. This is most easily seen by writing down the posterior distribution for $a:
\[ p(a|s_1, s_2) \propto p(s_1, s_2|a)p(a) \]
\[ \propto p(s_+|a)p(s_-|a)p(a), \]
where the last proportionality follows from the fact that the Gaussian $p(s_+|a)$ is effectively constant over the range of $a$-values for which $p(s_-|a)$ is peaked (cf. Eqs. (25-26) with $\sigma_p^2 \gg \sigma_n^2$). Thus PCA has simplified the analysis of this particular problem by identifying a combination of the original data (in this case $s_- \equiv s_1 - s_2$) that captures nearly all the available information on our parameter $a$.

III. LISA EXAMPLE

As mentioned in Sec. I, the current design of LISA comprises three spacecraft in circular orbits of radius 1 AU around the Sun, in a near equilateral configuration of side $5 \times 10^6$. The tiny (picometre) modulation of the separation between the spacecraft produced by the passage of gravitational waves is sensed by Doppler measurements of laser beams exchanged between spacecraft, recorded as the difference in frequency between the locally-generated and received laser signals at each spacecraft. Since each spacecraft receives laser signals sent from the other two spacecraft, there are six raw data streams in total, denoted $s_1$, $s'_1$, $s_2$, $s'_2$, $s_3$, $s'_3$, where the subscript indicates the spacecraft receiving the laser beam, and is unprimed or primed depending on whether the beam is traveling...
counter-clockwise or clockwise around the LISA triangle, as viewed from above (see Fig. 2).

![Diagram of LISA configuration](image)

**FIG. 2:** Schematic of LISA configuration, following the conventions in [5]. The spacecraft are labeled 1, 2, 3. The separation between spacecraft are denoted by \( L_i, L'_i \), where the index \( i \) corresponds to the opposite spacecraft. The beam arriving at spacecraft \( i \) has subscript \( i \) and is unprimed or primed depending on whether the beam is traveling counter-clockwise or clockwise around the LISA triangle, as viewed from above.

Following the notation in [5], we write the six LISA data streams as

\[
s_1 = D_3p_2 - p_1 + n_1 + h_1, \quad (30)
\]

\[
s'_1 = D_2p_3 - p_1 + n'_1 + h'_1, \quad (31)
\]

together with their cyclic permutations \((1 \rightarrow 2 \rightarrow 3 \rightarrow 1)\) at spacecrafts 2 and 3. Here \( p_i \) is the frequency noise associated with the two lasers (assumed for now to be locked to one another) on spacecraft \( i \) \((i = 1, 2, 3)\); \( n_i \) and \( n'_i \) are all other noises associated with the transmission of the signal to spacecraft \( i \) in the counter-clockwise and clockwise directions, and \( h_i \) and \( h'_i \) are the frequency modulations produced by the astrophysical signals. \( D_i \) is a delay operator that takes a data stream \( x(t) \) and delays it by the light travel time down the arm \( L_i \):

\[
D_i x(t) = x(t-L_i), \quad (32)
\]

in units where the speed of light \( c = 1 \). Explicitly,

\[
s_1(t) = p_2(t-L_3) - p_1(t) + n_1(t) + h_1(t), \quad (33)
\]

\[
s'_1(t) = p_3(t-L_2) - p_1(t) + n'_1(t) + h'_1(t), \quad (34)
\]

and similarly for the data streams at spacecrafts 2 and 3. Note that we are restricting ourselves in this example to the case of a non-rotating LISA configuration with fixed arm-lengths. We assume that the light travel time down an arm is independent of the direction in which it is moving (counter-clockwise or clockwise) and is independent of the time of emission. This corresponds to \( L_i = L'_i \) in Fig. 2.

In practice, the data streams will be discretely-sampled on-board the spacecraft, with sampling period \( \Delta t \). For simplicity, we assume that the light travel times down the arms are related by simple integers—in particular,

\[
L_1 = \Delta t, \quad L_2 = 2\Delta t, \quad L_3 = 3\Delta t. \quad (35)
\]

This restriction is made only to minimise the number of data points needed to illustrate the PCA method. It should be relatively straightforward to extend our analysis to the case of more complicated light travel times, as well as to situations where there is relative motion between the spacecraft (i.e., \( L_i \neq L'_i \)).

If we denote the discrete time stamps by \( t_\alpha \equiv \alpha \Delta t \) and the value of data stream \( x(t) \) at \( t = t_\alpha \) by

\[
x[\alpha] \equiv x(t_\alpha), \quad (36)
\]

then the values of the six LISA data streams at time stamp \( \alpha = 1 \) become

\[
s_1[1] = p_2[-2] - p_1[1] + n_1[1] + h_1[1], \quad (37)
\]

\[
s'_1[1] = p_3[-1] - p_1[1] + n'_1[1] + h'_1[1], \quad (38)
\]

\[
s_2[1] = p_3[0] - p_2[1] + n_2[1] + h_2[1], \quad (39)
\]

\[
s'_2[1] = p_1[-2] - p_2[1] + n'_2[1] + h'_2[1], \quad (40)
\]

\[
s_3[1] = p_1[-1] - p_3[1] + n_3[1] + h_3[1], \quad (41)
\]

\[
s'_3[1] = p_2[0] - p_3[1] + n'_3[1] + h'_3[1], \quad (42)
\]

where we used Eq. (30) to explicitly evaluate the arguments of the time-delayed laser frequency noise. To obtain expressions for the data streams evaluated at other time stamps, we simply increment or decrement the arguments of the data streams by the appropriate number of sampling periods.

To simplify the calculation further, we will only consider data streams having time stamps \( \alpha = 1, 2, 3, 4, 5 \). Since there are six data streams in total, this corresponds to a 30-dimensional vector space of data points. The noise covariance \( C \) is thus a 30 \( \times \) 30 matrix, whose \( \alpha \beta \)th element is itself a 6 \( \times \) 6 matrix:

\[
C_{\alpha\beta} = \begin{pmatrix}
(\langle s_1[\alpha] - h_1[\alpha] \rangle (s_1[\beta] - h_1[\beta])) & (\langle s_1[\alpha] - h_1[\alpha] \rangle (s'_1[\beta] - h'_1[\beta])) & \text{etc.} \\
(\langle s'_1[\alpha] - h'_1[\alpha] \rangle (s_1[\beta] - h_1[\beta])) & (\langle s'_1[\alpha] - h'_1[\alpha] \rangle (s'_1[\beta] - h'_1[\beta])) & \text{etc.} \\
(\langle s_2[\alpha] - h_2[\alpha] \rangle (s_1[\beta] - h_1[\beta])) & (\langle s_2[\alpha] - h_2[\alpha] \rangle (s'_1[\beta] - h'_1[\beta])) & \text{etc.} \\
(\langle s'_2[\alpha] - h'_2[\alpha] \rangle (s_1[\beta] - h_1[\beta])) & (\langle s'_2[\alpha] - h'_2[\alpha] \rangle (s'_1[\beta] - h'_1[\beta])) & \text{etc.} \\
(\langle s_3[\alpha] - h_3[\alpha] \rangle (s_1[\beta] - h_1[\beta])) & (\langle s_3[\alpha] - h_3[\alpha] \rangle (s'_1[\beta] - h'_1[\beta])) & \text{etc.} \\
(\langle s'_3[\alpha] - h'_3[\alpha] \rangle (s_1[\beta] - h_1[\beta])) & (\langle s'_3[\alpha] - h'_3[\alpha] \rangle (s'_1[\beta] - h'_1[\beta])) & \text{etc.}
\end{pmatrix}, \quad (43)
\]
As the matrix $C_{\alpha\beta}$ depends only on the difference between $\alpha$ and $\beta$, (i.e., $C_{11} = C_{22}, C_{12} = C_{23}$, etc.), the full covariance matrix $C$ is block Toeplitz (i.e., it is constant along the $\alpha\beta$ diagonals). Since $C$ is also symmetric, we need only calculate $C_{11}, C_{12}, C_{13}, C_{14},$ and $C_{15}$ to fully determine $C$.

Finally, we assume that the laser frequency noise $p_t[\alpha]$ and individual noise terms $n_i[\alpha], n'_i[\alpha]$ are Gaussian-distributed with zero-mean and variances

$$\langle n_i[\alpha]n_j[\beta]\rangle = \langle n'_i[\alpha]n'_j[\beta]\rangle = \delta_{\alpha\beta}\delta_{ij}\sigma_n^2,$$  \hspace{0.5cm} (44)

$$\langle p_t[\alpha]p_j[\beta]\rangle = \delta_{\alpha\beta}\delta_{ij}\sigma_p^2,$$  \hspace{0.5cm} (45)

(i.e., the random processes are white), and that they are mutually uncorrelated, so that

$$\langle n_i[\alpha]n'_j[\beta]\rangle = \langle n_i'[\alpha]p_j[\beta]\rangle = \langle n'_i[\alpha]p_j[\beta]\rangle = 0.$$  \hspace{0.5cm} (46)

Given these assumptions, it follows that

$$C_{11} = \begin{pmatrix}
2\sigma_p^2 + \sigma_n^2 & \sigma_p^2 & 0 & \ldots & 0 \\
\sigma_p^2 & 2\sigma_p^2 + \sigma_n^2 & 0 & \ldots & 0 \\
0 & 0 & 2\sigma_p^2 + \sigma_n^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2\sigma_p^2 + \sigma_n^2
\end{pmatrix},$$  \hspace{0.5cm} (47)

is

$$C = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\
C_{12}^T & C_{12} & C_{13} & C_{14} & C_{15} \\
C_{13}^T & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{14}^T & C_{14} & C_{15} & C_{16} & C_{17} \\
C_{15}^T & C_{15} & C_{16} & C_{17} & C_{18}
\end{pmatrix},$$  \hspace{0.5cm} (52)

and is shown schematically in Fig. 3.

FIG. 3: Schematic representation of the covariance matrix, $C$, for the simple LISA model. Blocks of increasing grey represent values of $2\sigma_p^2 + \sigma_n^2, \sigma_p^2, 0$ and $-\sigma_p^2$.

It is now an exercise in linear algebra to compute the eigenvectors and eigenvalues of $C$. We use MAPLE to do this calculation for us. There are 22 distinct eigenvalues, the smallest of which is independent of the laser frequency noise. This minimal eigenvalue

$$\lambda_{\min} = \sigma_n^2,$$  \hspace{0.5cm} (53)

In terms of these sub-matrices, the full covariance matrix
is 9-fold degenerate. The nine eigenvectors corresponding to $\lambda_{\text{min}}$ orthogonally span a 9-dimensional vector subspace of the 30-dimensional vector space of data points, elements of which do not depend on the laser frequency noise (see Table I). Thus, these eigencombinations correspond to the TDI-like variables for this particular example. In principle one could compute the covariance matrix for the entire set of LISA data ($\sim 10^8$ data points), though in practice this should not be necessary. In our example we need only four time stamps to fully characterise the covariance matrix, and a long time sequence of data from the example can be generated using the same set of eigencombinations of Doppler measurements. We will refer to the time series generated by these combinations as an eigenstreams. One can show e.g., that the Sagnac combination for spacecraft 1 (denoted $\alpha(t)$ in F) is a linear combination of eigenvectors: $-e_4 - e_7 + e_8$. Explicitly,

$$\alpha[5] \equiv s'_1[5] + s'_3[3] + s'_2[2] - s_1[5] - s_2[2] - s_3[1],$$

(54)

which is just the discretised-version of

$$\alpha(t) \equiv [s'_1(t) + D_2 s'_3(t) + D_1 D_2 s'_2(t)] - [s_1(t) + D_3 s_2(t) + D_1 D_3 s_3(t)]$$

$$= [s'_1(t) + s'_3(t - L_2) + s'_2(t - L_1 - L_2)]$$

$$- [s_1(t) + s_2(t - L_3) + s_3(t - L_1 - L_3)],$$

(55)

(56)

(cf. Eq. (42) in Ref. F) appropriate for the arm lengths given in Eq. G.

**IV. DISCUSSION**

Although the LISA model considered above is greatly simplified, only considering fixed arm-lengths related by small integers and a basic noise model, we believe that the principal component approach is suitable for more sophisticated LISA models. In particular, we have indicated that the likelihood is the natural generating function for orthogonal LISA data streams that are not dominated by laser frequency noise. This reduced set of eigenstreams is a sufficient basis (and possibly more than sufficient) to carry out all possible astronomy with LISA. The remaining eigenstreams contain information on the laser stability and are therefore important for instrumental diagnostics rather than astrophysics.

As we have emphasised, the frequency-noise-free eigenstreams are directly equivalent to TDI variables, but emerge naturally as a direct consequence of the likelihood analysis and so have a meaning that is directly relevant to subsequent data analysis. For example, if we take a model $M$ of how an astrophysical source, or a number of sources, would appear in the LISA data and assume this model depends on a set of parameters $a$ (such as sky position, polarisation angle, etc.), then the joint posterior probability of these parameters is simply

$$p(a|\{e_i\}, M) \propto p(a|M) \prod_i p(e_i|a, M)$$

(57)

where $e_i$ are the orthogonal frequency-noise-free eigenstreams generated from the data. In this way the LISA data analysis problem is cast in the powerful framework of classic inference, suitable for attack by standard search and exploration algorithms including Markov Chain Monte Carlo methods (e.g. 2, 10, 21).

More complex models of the LISA spacecraft will of course increase the complexity and size of the covariance matrix, but the basic principle will remain the same. For example, differing laser shot noise in the arms will break the eigenvalue degeneracy in the above example. In addition, the lengths $L_i$ and $L'_i$ cannot be assumed equal when there is relative spacecraft motion and as a result we expect the larger covariance matrix to yield 2nd generation TDI variables from its eigenstreams.

The eigenstreams are defined in terms of their minimal variance rather than the amount of astronomy they contain, and there is no reason why they should all contain astronomical information. The eigenstreams, or combinations of eigenstreams, that are devoid of astronomical data are ‘zero-signal solutions’ 13, and can help in instrument diagnostics in circumstances when the astronomical signals would dominate (such as in the low frequency confusion limit of LISA).

We are reminded here of the importance of careful data acquisition and sampling for LISA. It is well-understood that only identically represented samples of the laser frequency noise will cancel effectively, and there could be times when the covariance between Doppler channels from this noise is reduced. This would be encoded as an increase in the baseline-dependent noises terms, $n_i$ and $n'_i$, and in the limit as $\sigma_\pi^2$ approached $\sigma_n^2$ the eigenvalues of the covariance matrix would cease to break into two groups.

Finally, although PCA is often used for data compression and, as we have shown, the TDI-like eigenstreams contain all the ‘good quality’ astronomical data from LISA, we are not proposing that these be generated on the spacecraft and relayed back to Earth. The saving in telemetry bandwidth would be clear, but having the raw Doppler data on Earth would greatly enhance the flexibility of the analysis and the robustness of the mission.

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TABLE I: Eigenvectors corresponding to the 9-fold degenerate eigenvalue \( \lambda_{\text{min}} \equiv \sigma_n^2 \).

| Data | \( e_1 \) | \( e_2 \) | \( e_3 \) | \( e_4 \) | \( e_5 \) | \( e_6 \) | \( e_7 \) | \( e_8 \) | \( e_9 \) |
|------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( s_1 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_1' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_2 \) | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| \( s_2' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_3 \) | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| \( s_3' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_4 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| \( s_4' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_5 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( s_5' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_6 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| \( s_6' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_7 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( s_7' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_8 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( s_8' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

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