PRESSURE STRAIN CORRELATION MODELING
FOR TURBULENT FLOWS

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Dedicated to my parents

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Abstract

Accurate and robust models for the pressure strain correlation are an essential component for the success of Reynolds Stress Models in turbulent flow simulations. However, replicating the non-local action of pressure using only local tensors places a large limitation on potential model performance. In this thesis, we outline an approach that extends the tensor basis used for pressure strain correlation modeling to formulate models with improved precision and robustness. This set of additional tensors is analyzed and justified based on physics-based arguments and analysis of simulation data. In the first part of the work, a higher degree nonlinear return-to-isotropy model has been developed for the slow pressure strain correlation, considering anisotropies in Reynolds stress, dissipation and length scales. In the second part of the work, the modeling of the rapid pressure strain correlation of turbulence is considered. The anisotropy of turbulence in the presence of mean strain is studied and a new model is formulated by calibrating the model constants at the rapid distortion limit. The resulting complete pressure strain correlation model is tested for a wide variety of turbulent flows, while being contrasted against the predictions of established models. It is shown that the new model provides significant improvement in prediction accuracy. In the last stage of the work, a series of experiments on decaying grid-generated turbulence and grid turbulence with mean strain were conducted. Experimental data of turbulence statistics including Reynolds stress anisotropies is collected, analyzed and then compared to the predictions of Reynolds Stress Models to assess their accuracy and is used to evaluate the variability in the coefficients of the rate of dissipation model and the pressure strain correlation models used in Reynolds Stress Modeling.
# TABLE OF CONTENTS

LIST OF FIGURES  ................................................................. IV
LIST OF TABLES  ................................................................. V
LIST OF TERMS AND ABBREVIATIONS  ..................................... XI

1  Introduction
   1.1 Background of turbulence  ........................................... 1
   1.2 Numerical treatment of turbulence  ................................ 1
   1.3 Turbulence closure models  ......................................... 2
   1.4 A brief overview of the present thesis  .......................... 6

2  Review of Literature
   2.1 General introduction  ................................................. 8
   2.2 Mathematical and modeling details  .............................. 10
   2.3 Slow pressure strain correlation models  ....................... 12
      2.3.1 Rotta Model: .................................................. 12
      2.3.2 Lumley Model: ............................................... 12
      2.3.3 Shih, Mansour and Moin Model: ............................ 13
      2.3.4 Sarkar and Speziale Model: ................................ 13
      2.3.5 Sjogren and Johansson Model: .............................. 14
   2.4 Rapid pressure strain correlation models  ....................... 15
      2.4.1 LRR Model: .................................................. 16
      2.4.2 SSG Model: .................................................. 17
      2.4.3 Johansson-Hallback Model: ................................ 18
      2.4.4 Rapid pressure strain correlation models with extended bases 20
   2.5 Summary  ............................................................... 28

3  Modeling Turbulence/Turbulence Interactions  .................... 29
LIST OF FIGURES

1.1 Turbulence generated in a wind tunnel behind a grid. .................. 2
1.2 Large and small scale structures in a plume, representing interaction of length scales. ................................................................. 3
1.3 The energy spectrum of turbulence with the dominant physics therein. . 5
2.1 Lumley triangle trajectories of the model predictions against the plane contraction experiment of Le Penven et al. (1985) ($III_b < 0$) ........... 15
2.2 Lumley triangle trajectories of model predictions against the plane distortion experiment of Choi and Lumley (1984) ............................. 16
2.3 Lumley triangle trajectories of model predictions against the experiment of Warhaft and Lumley (1978) ............................... 17
2.4 Comparisons of model predictions against the experiment of Uberoi (1963) .......................................................... 18
2.5 Rapid pressures train correlation model predictions for a plane strain mean flow at the rapid distortion limit, (a) $b_{11}$, (b) $log(k)$ .............. 23
2.6 Rapid pressures train correlation model predictions for a planar strained mean flow at the rapid distortion limit, (a) $b_{12}$, (b) $log(k)$ .............. 24
2.7 Rapid pressures train correlation model predictions for an elliptic streamline mean flow at the rapid distortion limit, (a) $b_{22}$, (b) $log(k)$ .............. 25
2.8 Rapid pressures train correlation model predictions for an elliptic streamline mean flow at the rapid distortion limit .......................... 26
2.9 Rapid pressures train correlation model predictions for an elliptic streamline mean flow at the rapid distortion limit .......................... 27
3.1 Phase space comparison with the axisymmetric expansion experiment of Choi and Lumley. New model prediction is shown by the solid line, SS model by dashed line and the experimental data are shown in plus symbols .......................... 37
3.2 Time evolution of the second invariant with the axisymmetric expansion experiment of Choi and Lumley. ........................................ 38
3.3 Time evolution of the third invariant with the axisymmetric expansion experiment of Choi and Lumley. ................................. 38
3.4 Time evolution of second invariant with the plane contraction experiment of Le Penven et al. ................................................. 39
3.5 Time evolution of third invariant with the plane contraction experiment of Le Penven et al. .................................................... 39
3.6 Time evolution of third invariant with the plane distortion experiment of Choi and Lumley. .................................................... 40
3.7 Time evolution of third invariant with the plane distortion experiment of Le Penven et al. .................................................... 40
4.1 Anisotropy evolution to a stationary state for (a) plane strain mean flow, (b) elliptic mean flow and (c) axisymmetric expansion mean flow at the rapid distortion limit. ........................................ 52
4.2 Identification of stationary states for elliptic flows: (a) schematic outlining the methodology, (b) Comparison of methodology with DNS data from Blaisdell and Shariff (1996) ........................................ 53
4.3 Calculated values of the model coefficients (a) $L_2$, (b) $L_3$ and (c) $L_4$ as functions of $\beta$. .............................................. 54
4.4 Evolution of the $b_{ij}$ in homogeneous shear mean flow. RDT evolution is shown by the solid line, new model predictions by the dashed line. LRR and SSG are shown by the dotted and dash-dot lines. (a) $b_{11}$, (b) $b_{22}$ and (c) $b_{12}$. .............................................. 55
4.5 Evolution of $b_{22}$ in elliptic mean flows at $\beta = 0.6, 0.7$ and 0.8. RDT evolution is shown by the solid line, new model predictions by the dashed line. LRR and SSG are shown by the dotted and dash-dot lines. (a) $\beta = 0.6$ (b) $\beta = 0.7$ (c) $\beta = 0.8$ ............................................. 56
4.6 Evolution of turbulent kinetic energy $k$ in elliptic mean flows at $\beta = 0.6, 0.7$ and 0.8. RDT evolution is shown by the solid line, new model predictions by the dashed line. LRR and SSG are shown by the dotted and dash-dot lines. ............................................. 57
4.7 Evolution of a) the Reynolds stress anisotropy $b_{11}$ b) turbulent kinetic energy for plane strain mean flow. The predictions of the present model are shown by the solid line. LRR and SSG are shown by the dotted and dash-dot lines. The data from the direct numerical simulation of Lee and Reynolds (1985b) is included for comparison.

4.8 Predictions for homogeneous shear flow a) $b_{12}$ evolution for the LES of Isaza and Collins (2009) with $\frac{S_k}{\epsilon} = 27$, b) the evolution for the LES of Isaza and Collins (2009) with $\frac{S_k}{\epsilon} = 27$, c) the evolution for the DNS of Bardina et al. (1983).

4.9 Turbulent kinetic energy evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff (1996) is included for comparison.

4.10 Reynolds stress anisotropy $b_{13}$ evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff (1996) is included for comparison.

5.1 Evolution of a) the Reynolds stress anisotropy $b_{11}$ b) turbulent kinetic energy $k$ for plane strain mean flow. The predictions of the present model are shown by the solid line. SSG and LRR model are shown by the dashed and dash-dot lines. The data from the direct numerical simulation of Lee and Reynolds (1985b) is included for comparison.

5.2 Turbulent kinetic energy evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff (1996) is included for comparison.
5.3 Reynolds stress anisotropy $b_{13}$ evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff (1996) is included for comparison.

5.4 Turbulence kinetic evolution for purely sheared flows a) $S^*=3$ b) $S^*=15$ c) $S^*=27$. The predictions of the present model are shown by the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Isaza and Collins (2009) is included for comparison.

5.5 Reynolds stress anisotropy $b_{12}$ for purely sheared flows a) $S^*=3$ b) $S^*=15$ c) $S^*=27$. The predictions of the present model are shown by the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Isaza and Collins (2009) is included for comparison.

5.6 Evolution of turbulent kinetic energy for the purely sheared flow. The predictions of the present model are shown by the solid line. SSG and LRR model are shown by the dashed and dash-dot lines. The data from the direct numerical simulation of Bardina et al. (1983) is included for comparison.

6.1 Schematic of the recirculating water tank and the wedge, all dimensions are in meter, the width of the wedge is same as width of the water tank.

6.2 Schematic diagram showing ADV probe and signal conditioning module.

6.3 Evolution of the streamwise mean velocity $U$ and the fluctuating velocity $u_{rms}$ downstream of the mesh.

6.4 Distribution of pressure coefficient along the streamwise direction for $Re_M = 25000$, the reference pressure was taken at the beginning of the contraction, the downstream distance is non-dimensionalized by the wedge step height.
6.5 Downstream evolution of turbulence kinetic energy, a) without mean strain b) with mean strain, solid lines corresponds to $Re_M$ of 39000, dashed lines 32000 and dashed dot lines 25000.

6.6 Downstream evolution of Reynolds stress, a) without mean strain b) with mean strain, solid lines corresponds to $Re_M$ of 39000, dashed lines 32000 and dashed dot lines 25000.

6.7 Downstream evolution of Reynolds stress, a) without mean strain b) with mean strain, solid lines corresponds to $Re_M$ of 39000, dashed lines 32000 and dashed dot lines 25000.

6.8 Effect of mean strain on free stream turbulence, a) represents evolution of free stream turbulence without mean strain b) with mean strain, solid line represents $R_{11}$, dashed lines $R_{22}$ and dashed dot lines $R_{33}$ corresponding to $Re_M$ of 32000.

6.9 $b_{ij}$ evolution under mean strain at $Re_M =$a)32000 b)39000. Solid lines show the longitudinal component, dashed lines the transverse normal component.

6.10 Contrasting the relationship between $C_{\epsilon 2}$ and $n$ based on experimental studies (solid black line) and the values used in Reynolds Stress Modeling investigations.

6.11 Turbulent kinetic energy downstream of the mesh and the decay predicted by the model [Speziale et al. (1991)] with different values of the coefficients for the rate of dissipation equation.

6.12 Turbulent kinetic energy downstream of the mesh and the decay predicted by the model of [Lauder et al. (1975)] with different values of the coefficients for the rate of dissipation equation.

6.13 Turbulence kinetic energy evolution for decaying grid generated turbulence, $S^* = 1.43$. Predictions of [Rotta (1951)] are shown by solid line, results at $Re_M = 25000$ by unfilled circles.

6.14 Turbulence kinetic evolution for grid generated turbulence under axisymmetric contraction. Results at $Re_M = 25000$ are shown as unfilled circles.
LIST OF TABLES

2.1 Rapid pressure strain correlation models compared with respect to their non-zero coefficients and order of expression . . . . . . . . . . . . . . . . . [11]
CHAPTER 1

Introduction

1.1 Background of turbulence

Turbulence is the complex fluid motion, which appears in problems of interest to many fields of engineering sciences such as aeronautics, mechanical, chemical engineering and in oceanographic, meteorological and astrophysical sciences, besides others. Starting from flow past ships, automobiles, airplanes to rockets and exists in all scales form flow of plasma in a cell to motion of galaxies in space. Improved understanding of turbulence evolution would lead to important advances in these fields.

The specific characteristics of turbulent flows are: irregularity, unpredictability, diffusivity and dissipation. In turbulent flows eddies of different sizes are present which possess complex spatial and temporal behavior. The main source of production of these structures are the hydrodynamic instabilities from shear in the mean flow or buoyancy in the flow field. The largest size of these eddies bounded by the scale of the flow. The Reynolds number of these eddies is large. The largest eddies obtain energy from the mean flow and transport most of the momentum in a flow field. The large eddies are unstable and break up. Those transfer energy to comparatively smaller eddies. These smaller eddies again transfer energy to smaller and smaller eddies. The smallest eddies dissipate energy as heat through the process of eddy breakdown by the viscous action of the fluid medium.

1.2 Numerical treatment of turbulence

For incompressible turbulent flow of a viscous fluid, the continuity equation and Navier-Stokes equation has the form:

\[
\frac{\partial v_i}{\partial x_i} = 0
\]

\[
\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = - \frac{\partial P}{\partial x_i} + \nu \nabla^2 v_i
\]

(1.1)

where \( v_i \) and \( x_i \) are the velocity and position, \( P \) is pressure, \( t \) is time and \( \nu \) is the
After time averaging of equation 1 and 2, the Reynolds averaged Navier-Stokes equation (RANS) can be derived, which takes the form:

\[
\begin{align*}
\frac{\partial U_i}{\partial x_i} &= 0 \\
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} &= -\frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i - \frac{\partial R_{ij}}{\partial x_j},
\end{align*}
\]

(1.2)

Much more research is focused on computing the \( R_{ij} \) tensor which is the time averaged momentum transfer rate due to turbulence and is termed as Reynolds stress. The RANS equations are not closed system of equations, since number of equations are less than the number of unknowns, from which the turbulence closure problem arises. The motive behind turbulence closure modeling is to close the system of equations either by hypothesizing a relationship between the Reynolds stress tensor and the mean flow properties or by generating additional equations by taking moment of the RANS equations.

![Fig. 1.1 Turbulence generated in a wind tunnel behind a grid.](image)

**1.3 Turbulence closure models**

Turbulence modeling in computational fluid dynamics can be classified into four major approaches as eddy viscosity models, Reynolds stress models, large eddy simulations(LES) and direct numerical simulations(DNS)(Pope (2000)). Direct numerical simulations solve complete set of Navier-Stokes equations without any approximations, but in large eddy simulations the energy containing motions are resolved and
the small scale motions are modeled. The cost associated with such simulations is very high when used in larger flow domains with complex geometries. Reynolds averaged Navier-Stokes (RANS) based models are based on modeling of all scales of turbulence, the cost of simulations using RANS is very less but these are less accurate than DNS and LES. The accurate structure of turbulence can’t be predicted by use of such models. Reynolds stress models or stress transport models are more accurate than RANS based two equation models and computationally cheaper than DNS and LES.

Fig. 1.2 Large and small scale structures in a plume, representing interaction of length scales.

Reynolds Stress Models are based on the Reynolds Stress Transport Equation, that describes the evolution of individual components of the Reynolds stress tensor. This is in contrast to two-equation modeling approach where evolution equations for scalars like the turbulent kinetic energy $k$ and dissipation $\epsilon$ are solved and the eddy viscosity hypothesis is used to approximate the Reynolds stresses. The Reynolds Stress Transport Equations describe the production, dissipation and redistribution each of the components of the Reynolds stress tensor. Different physical mechanisms in this evolution are represented by the separate terms in this equation. The general form of the Reynolds
Stress Transport Equation is given by Pope (2000)

\[
\partial_t u_i u_j + U_k \frac{\partial u_i u_j}{\partial x_k} = P_{ij} - \frac{\partial T_{ijk}}{\partial x_k} - \epsilon_{ij} + \phi_{ij},
\]

where,

\[
P_{ij} = -\bar{u}_k \frac{\partial U_i}{\partial x_k} - \bar{u}_i \frac{\partial U_j}{\partial x_k},
\]

\[
T_{kij} = \bar{u}_i \bar{u}_j \bar{u}_k - \nu \frac{\partial u_i}{\partial x_k} + \delta_{jk} u_i \frac{p}{\rho} + \delta_{ik} u_j \frac{p}{\rho},
\]

\[
\epsilon_{ij} = -2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}
\]

\[
\phi_{ij} = \frac{p}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

The turbulence production process is represented by \( P_{ij} \) and is an inner product between the Reynolds stress tensor and the mean velocity gradient. In physical terms this mechanism represents the action of the mean velocity gradients working against the Reynolds stresses and represents a transfer of kinetic energy from the mean flow to the fluctuating velocity field. The production mechanism acts as a source of energy for the turbulent flow. \( \epsilon_{ij} \) represents the dissipation process and is the product of the fluctuating velocity gradients and the fluctuating rate of strain. Physically it represents the fluctuating velocity gradients working against the deviatoric fluctuating stresses transforming turbulence kinetic energy into internal energy. The dissipation mechanism acts as a sink of energy for the turbulent flow. The turbulent transport process is represented by \( T_{ijk} \) and represents the transfer of turbulent kinetic energy between different locations in the flow domain. This has contributions from viscous diffusion, pressure transport and turbulent convection. Finally \( \phi_{ij} \) represents the pressure strain correlation and re-distributes turbulent kinetic energy among the components of the Reynolds stresses. Of these terms, production is the only process that is closed at the single point level. The other terms require models for their closure. The accuracy of the Reynolds stress modeling approach depends on the quality of the models developed for these turbulence processes.
Fig. 1.3 The energy spectrum of turbulence with the dominant physics therein.
1.4 A brief overview of the present thesis

Besides this chapter on introduction to turbulence and turbulence modeling, there are six more chapters in the thesis. A brief summary of the remaining chapters is depicted below:

Chapter 2
Almost all investigations of turbulent flows in academia and in the industry utilize some degree of turbulence modeling. Of the available approaches to turbulence modeling Reynolds Stress Models have the highest potential to replicate complex flow phenomena. Due to its complexity and its importance in flow evolution modeling of the pressure strain correlation mechanism is generally regarded as the key challenge for Reynolds Stress Models. In this chapter, the modeling of the pressure strain correlation for complex turbulent flows is reviewed. Starting from the governing equations the theory behind models for both the slow and rapid pressure strain correlation are outlined. Established models for both these are introduced and their successes and shortcomings are illustrated using simulations and comparisons to experimental and numerical studies. Recent advances and developments in this context are presented. Finally, challenges and hurdles for pressure strain correlation modeling are outlined.

Chapter 3
In this chapter, the evolution of decaying homogeneous anisotropic turbulence without mean velocity gradients were considered, where only the slow pressure rate of strain is non zero. A higher degree nonlinear return- to - isotropy model has been developed for the slow pressure-strain correlation, considering anisotropies in Reynolds stress, dissipation rate and length scale tensor. Assumption of single length scale across the flow is not sufficient, from which stems the introduction of length scale anisotropy tensor, which has been assumed to be a linear function of Reynolds stress and dissipation tensor. The new model with anisotropy in length scale show better agreement with well accepted experimental results when contrasted against other established models.

Chapter 4
In the presence of mean strain or rotation, the anisotropy of turbulence increases due to the rapid pressure strain term. In this chapter, we consider the modeling of the rapid pressure strain correlation of turbulence. The anisotropy of turbulence in the presence of mean strain is studied and a new model is formulated by calibrating the model constants at the rapid distortion limit. This model is tested for a range of plane strain and elliptic flows and compared to DNS results. The present model shows agreement with DNS and improvements over the earlier models, that have been reported to give satisfactory performance for hyperbolic flows but not satisfactory for elliptic flows.
Chapter 5
Accurate and robust models for the pressure strain correlation are an essential component for the success of Reynolds Stress Models in turbulent flow simulations. However, replicating the non-local action of pressure using only local tensors places a severe limitation on potential model performance. In this chapter we outline an approach that extends the tensor basis used for pressure strain correlation modeling to formulate models with improved precision and robustness. This set of additional tensors is analyzed and justified based on physics based arguments and analysis of simulation data. Using these tensors models for the rapid and slow pressure strain correlation are developed. The resultant pressure strain correlation model is tested for a wide variety of turbulent flows, while being contrasted against the predictions of other popular models. It is shown that the new model provides significant improvement in predictive accuracy.

Chapter 6
This chapter presents experimental and numerical analysis of grid generated turbulence with and without the effects of applied mean strain. We conduct a series of experiments on decaying grid generated turbulence and grid turbulence with mean strain. Experimental data of turbulence statistics including Reynolds stress anisotropies is collected, analyzed and then compared to the predictions of Reynolds Stress Models to assess their accuracy. The experimental data is used to evaluate the variability in the coefficients of the rate of dissipation model and the pressure strain correlation models used in Reynolds Stress Modeling. For both models we recommend optimal values of coefficients that should be used for experimental studies of grid generated turbulence.

Chapter 7
This is the concluding chapter which summarizes the whole work in the thesis. In it, the advantages of the new formulation of the pressure strain correlation over other established models are described. This also focuses on the on the contribution of the present thesis to the turbulence modeling literature. Other possibilities for further extending the present research are kept as course of future work.

Chapter 8
This chapter deals with the summary of work done in the thesis. Various conclusions drawn in different Chapters of the thesis are summarized in this Chapter. The future scope of the present research work is also highlighted.

Following the closure are the References, a brief of author’s educational journey in the form of author’s biography and list of publications.
CHAPTER 2

Review of Literature

2.1 General introduction

In academic and industrial applications, most investigations into turbulent flow problems use turbulence models. Turbulence models are simplified relations that express quantities that are difficult to compute in terms of simpler flow parameters. They relate higher-order unknown correlations to lower-order quantities. These unknown correlations represent the actions of viscous dissipation, pressure-velocity interactions, etc. For example pressure strain correlation is a non-local phenomenon and is difficult to compute. Using models for pressure strain correlation, it is expressed as a function of Reynolds stresses, dissipation and mean velocity gradients which are local quantities. This enables us to estimate the pressure strain correlation and its effects on flow evolution in a simpler manner that is computationally inexpensive. Turbulence models are an essential component of all computational fluid dynamics software and are used in almost all simulations into real life fluid flows of engineering importance.

A majority of industrial applications use simple two-equation turbulence models like the $k – \epsilon$ and $k – \omega$ models. However recent emphasis in the scientific community has markedly shifted to Reynolds stress models (Hanjalić and Launder (2011), Durbin (2017), Klifi and Lili (2013), Mishra and Girimaji (2014), Jakirić and Maduta (2015), Manceau (2015), Eisfeld et al. (2016), Schwarzkopf et al. (2016), Moosaie and Manhart (2016), Lee et al. (2016), Mishra and Girimaji (2017), Sun et al. (2017)). Reynolds stress models have the potential to provide better predictions than turbulent viscosity based models at a computational expense significantly lower than DNS studies. They may be able to model the directional effects of the Reynolds stresses and additional complex interactions in turbulent flows (Johansson and Hallbäck (1994)). They have the ability to accurately model the return to isotropy of decaying turbulence and the behavior of turbulence in the rapid distortion limit (Pope (2000)).

In the Reynolds Stress Transport Equation, of the terms that require models for their closure is the pressure strain correlation term is generally considered to be the most important. There are three reasons behind this. Firstly the pressure strain correlation term is active in all turbulent flows. For instance in homogeneous turbulence,
transport is absent due to spatial homogeneity. Similarly in decaying turbulence, turbulence production is zero due to the absence of mean velocity gradients. In the rapidly distorted turbulent flows, the dissipation mechanism is negligible as its time scale is much larger than the applied distortion. However in all these flows, the pressure strain correlation is present and actively transforming the evolution of the turbulent flow.

The second reason is due to the action of pressure strain correlation being very important in the evolution of turbulent flows. In important flow regimes like elliptic streamline flow the flow instability is initiated by pressure action Kerswell (2002). In strained mean flows like plane strain, axisymmetric strained mean flows pressure action stabilizes the flow instability Mishra and Girimaji (2015). The pressure strain correlation term determines if turbulence grows or decays in many turbulent flows. Hence its accurate modeling is highly important.

The final reason is due to the complexity of the pressure strain correlation mechanism and the challenges in its modeling. The central ideas for pressure strain correlation modeling were introduced by Chou (1945). The first model for the pressure strain correlation was formulated by Rotta (1951). Since their foundational investigations, many researchers have developed more advanced and complex closure models for the pressure strain correlation term Launder et al. (1975), Speziale et al. (1991), Johansson and Hallbäck (1994), Reynolds and Kassinos (1995). However all the available pressure strain correlation models have notable shortcomings. These shortcomings exist in the accuracy of their predictions and the realizability of their predictions. For example in rotation dominated flows the predictions of available models is incorrect both quantitatively and qualitatively. While Direct Numerical Simulations show that turbulence should be growing, available models predict that turbulence decays for these flows. Such rotation dominated flows include many flows of aerospace engineering like the trailing vortex, flap edge vortices and leading edge vortex flows. Another type of flows where the available pressure strain correlation models are unsatisfactory is non-equilibrium turbulent flows. Non equilibrium turbulent flows include highly strained flows for example flows with shock turbulence interactions. The shortcomings of available models in accurately predicting such important classes of engineering flows is limits engineering investigations in such flows. In addition to accuracy in predictions the available pressure strain correlation models are unable to provide realizable predictions. The realizability condition Schumann (1977), du Vachat (1977) tries to guarantee that the predictions of the turbulence model are not unphysical and correspond to flows that can exist in real life. In mathematical terms, the realizability condition requires turbulence models to predict a positive semi-definite Reynolds stress tensor. Unrealizable turbulence models may lead to issues in numerical convergence and numerical instability. Many investigators have found that the available pressure strain correlation models
can guarantee realizable predictions only for low to moderate levels of Reynolds stress anisotropy [Mishra and Girimaji (2014)].

The rest of this chapter is organized as follows. In Section II we outline the mathematical details of pressure strain correlation modeling. In Section III we discuss the action of the slow pressure strain correlation term, introduce established models for this term and compare their performance for different flows. In Section IV we discuss the action of the rapid pressure strain correlation term, introduce established rapid pressure strain correlation models and compare their performance for different flows. This chapter concludes with a summary and concluding remarks in Section V.

2.2 Mathematical and modeling details

In incompressible flows fluctuating pressure is governed by a Poisson equation:

$$\frac{1}{\rho} \nabla^2 p = -2 \frac{\partial U_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j - \bar{u}_i \bar{u}_j)$$  \hspace{1cm} (2.1)

Poisson equation is an elliptic partial differential equation. The Laplacian $\Delta^2$ is an elliptic operator and because of this the Poisson equation has no real characteristic directions. This elliptic nature of the operator leads to the non-local nature of the solution for fluctuating pressure. This elliptic nature of the governing equation indicates that the pressure at a single point in the flow is affected by changes to the flow at all points in the flow domain. This non-local character is inherited by the pressure strain correlation as well.

In literature this fluctuating pressure is decomposed into two components, rapid and slow pressure.

$$p = p^S + p^R$$  \hspace{1cm} (2.2)

Rapid pressure corresponds to the linear part of the source term in the Poisson equation. This term is directly and instantaneously affected by any changes in the mean velocity gradient and is referred to as rapid pressure. It is governed by

$$\frac{1}{\rho} \nabla^2 p^R = -2 \frac{\partial U_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}$$  \hspace{1cm} (2.3)

Slow pressure corresponds to the nonlinear part of the source term in the Poisson equation. This term is not directly affected by changes in the mean velocity gradient and is referred to as slow pressure. It is governed by

$$\frac{1}{\rho} \nabla^2 p^S = - \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j - \bar{u}_i \bar{u}_j)$$  \hspace{1cm} (2.4)
The most general form of the slow pressure strain correlation which represents the turbulence/turbulence interactions, has the form (Sjögren and Johansson (2000)):

$$\phi^{(S)}_{ij} = \beta_1 b_{ij} + \beta_2 (b_{ik} b_{kj} - \frac{1}{3} I_{I_b} \delta_{ij})$$

(2.5)

\(\beta_1\) and \(\beta_2\) can be the functions of second and third invariants of Reynolds stress anisotropy or can be a function of turbulent Reynolds number. \(b_{ij} = \frac{\nu W_{ij}}{2k} - \frac{\delta_{ij}}{3}\) is the Reynolds stress anisotropy tensor, \(I_{I_b}\) and \(III_{I_b}\) are the second and third invariants of the Reynolds stress anisotropy respectively.

The most general form of the rapid pressure strain correlation which represents the mean-strain/turbulence interactions, has the form:

$$\phi^{(R)}_{ij} = S_{pq} [Q_1 \delta_{ip} \delta_{jq} + Q_2 (b_{ip} \delta_{jq} + b_{jp} \delta_{iq} - 2/3 b_{pq} \delta_{ij}) + Q_3 b_{pq} b_{ij} + Q_4 (b_{iq} b_{jp} - 1/3 b_{pk} b_{kj} \delta_{ij}) + Q_5 b_{pl} b_{lq} b_{ij} + (Q_5 b_{pq} + Q_6 b_{pl} b_{lq}) (b_{ik} b_{kj} - 1/3 I_{I_b} \delta_{ij}) + \Omega_{pq} [Q_7 (b_{ip} \delta_{jq} + b_{jp} \delta_{iq}) + Q_8 b_{pk} (b_{jk} \delta_{iq} + b_{ik} \delta_{jq} + Q_9 b_{pk} (b_{jk} b_{ik} + b_{ik} b_{jq})]$$

(2.6)

where, \(S_{ij}\) is the mean rate of strain, \(W_{ij}\) is the mean rate of rotation and \(K\) is the turbulent kinetic energy. \(I_{I_b} = b_{ij} b_{ji}\) is the second invariant of the Reynolds stress anisotropy tensor. Different models differ in the choice of the values of the model coefficients. Choosing specific sets of coefficients to be non-zero determines the order of the model with respect to the Reynolds stress tensor. This is outlined in Table 1.

In the next sections we discuss the slow and rapid pressure strain correlation models respectively. We find a similar trend in their development where the first few models
are simpler and attempt to replicate some basic details of the pressure strain correlation. The succeeding models attempt to address shortcomings by incorporating more complex expressions, such as higher degrees of nonlinearity with respect to the Reynolds stress anisotropy terms. The final models attempt to add additional tensors to the modeling basis that can admit missing non-local information to the model.

2.3 Slow pressure strain correlation models

Numerous experimental and numerical investigations into decaying turbulent flows (Lumley and Newman 1977, Warhaft and Lumley 1978, Le Penven et al. 1985) have observed that along with a decay in the turbulent kinetic energy the anisotropy of the Reynolds stresses reduces towards an isotropic state. This is also observed in experimental investigations (Choi and Lumley 1984, 2001, Hallbäck et al. 1995) that initially anisotropic Reynolds stresses relax towards an isotropic state in the absence of external mean velocity gradients. This is known as the return to isotropy phenomenon of turbulence. In turbulence modeling the slow pressure strain correlation is chiefly responsible for the return to isotropy of turbulence.

2.3.1 Rotta Model:

The first model for the slow pressure strain correlation was proposed by Rotta (1951). The form of this model is given by

$$\phi_{ij} = -2C_R \epsilon_{ij}$$

(2.7)

The evolution equation for the Reynolds stress anisotropy for the Rotta model can be written as Pope (2000)

$$\frac{db_{ij}}{dt} = (-C_R - 1) \frac{\epsilon}{k} b_{ij}$$

(2.8)

The slow pressure strain correlation model of Rotta (1951) is linear in the Reynolds stresses. While it captures the return to anisotropy it is unable to capture the nonlinear nature of this return to isotropy process. For example on the Lumley triangle the paths predicted by the Rotta model are straight lines. Experimental data clearly shows that the return to isotropy is via curved trajectories (Chung and Kim 1995). The dependence of the rate of return to isotropy on the invariants of the Reynolds stress anisotropies is also not accounted for in the Rotta model.

2.3.2 Lumley Model:

The nonlinear effects were incorporated in the model of Lumley (1979), in which the nonlinearities were introduced in the model through the functions of Reynolds stress
anisotropy or the invariants of the Reynolds stress anisotropy. The coefficients of the model were taken as the function of turbulence Reynolds number

\[ \beta_1 = 2.0 + \frac{F}{9} \exp \left( \frac{-7.7}{\sqrt{Re_t}} \left( \frac{72}{\sqrt{Re_t}} + 80.1 \ln[1 + 62.4(-II_b + 2.3III_b)] \right) \right), \text{ and} \]
\[ \beta_2 = 0 \]  

(2.9)

where, \( q^2 = 2K \), \( II_b \) and \( III_b \) are the second and third invariants of the Reynolds stress anisotropy tensor. \( F \) is the determinant of the normalized Reynolds stress tensor.

2.3.3 Shih, Mansour and Moin Model:

The nonlinear model has similar form as that of Lumley model, the only difference is the non zero second coefficient. Sarkar and Speziale (1990) have reported that, the prediction of the return to isotropy of both the models is almost identical.

\[ \beta_1 = 2.0 + \frac{F^{0.85}}{9} \exp \left( \frac{-7.7}{\sqrt{Re_t}} \left( \frac{72}{\sqrt{Re_t}} + 80.1 \ln[1 + 62.4(-II + 2.3III)] \right) \right), \text{ and} \]
\[ \beta_2 = -2(1 - F^{0.5}) \]  

(2.10)

2.3.4 Sarkar and Speziale Model:

The model of Sarkar and Speziale [Sarkar and Speziale (1990)] is a quadratic model, the coefficients of the model are constants.

\[ \beta_1 = 3.4 \quad \text{and} \quad \beta_2 = 3(\beta_1 - 2) \]  

(2.11)

The transport equations for the Reynolds stress anisotropy were considered as follows:

\[ \frac{dII_b}{d\tau} = 1.4II_b + 8.4III_b \]
\[ \frac{dIII_b}{d\tau} = -4.2II_b + 2.1II_b^2 \]  

(2.12)

This simple quadratic model with only one independent constant is able to account for most of the nonlinear character of the return to isotropy phenomenon. It has been widely adopted in engineering simulations of turbulent flows.
2.3.5 Sjogren and Johansson Model:

Sjogren and Johansson (2000) used a similar model for the dissipation anisotropy developed by Hallback et al. (1990):

\[
e_{ij} = [2 + 0.75(0.5II_b - 2/3)]b_{ij} - 0.75(b_{ik}b_{kj} - 1/3II_b\delta_{ij})
\]

(2.13)

The model coefficients of the Sjogren and Johansson (2000) model has the form: \(\beta_1 = c_1 + c_3II_b + c_5III_b + c_7II_b^2\) and \(\beta_1 = c_2 + c_4II_b + c_5III_b + c_6III_b\).

The anisotropy states in the turbulent flows can be characterized by two variables \(\xi\) and \(\eta\) given by

\[
\xi = (III_b/2)^{1/3}, \quad \eta = (-III_b/3)^{1/2},
\]

(2.14)

In a turbulent flow, \(\xi\) and \(\eta\) can be determined at any point and time from the Reynolds stresses. The \(\xi-\eta\) phase space is bounded by two straight line segments denoting axisymmetric turbulence and from above by a curved line representing two-dimensional turbulence. This representation is referred to as the Lumley triangle [Pope (2000)] and is a simple manner to visualize the anisotropy of the Reynolds stress tensor. All realizable states of the Reynolds stress tensor lie inside the Lumley triangle.

We compare the predictions of the slow pressure strain correlation models introduced in this section and contrast them to experimental data. Here we focus on three models specifically: the slow pressure strain correlation models of Rotta (1951), Sarkar and Speziale (1990) and Panda et al. (2017). The model of Rotta (1951) is chosen as due to its simplicity it is widely used in turbulence simulations. The model of Sarkar and Speziale (1990) attempts to address the deficiencies of the linear Rotta (1951) model by adding nonlinear terms in the Reynolds stress anisotropy. This is one methodology to address the limitations in the models and is thus included. The model of Panda et al. (2017) attempts to address the deficiencies of slow pressure strain correlation models by adding additional tensors to the modeling basis. This represents another approach to address the limitations in the models and is thus included in the comparisons.

Figure 1 represents the evolution of trajectories for the plane contraction experiment of Le Penven et al. (1985), for this case the initial value of third invariant of Reynolds stress anisotropy is negative. The experimental data shows mild curvature in the plane. The nonlinear quadratic model of Sarkar and Speziale (1990) has predictions that better fit the experimental results, in comparison to Rotta (1951) and Panda et al. (2017) models. The trajectories on the Lumley triangle in figure 2 are strongly curved indicating the nonlinear effects in the return to isotropy behavior. It is noticed in figure 2 that model of Sarkar and Speziale (1990) has predictions that are better in comparison...
other two models. The phase space comparison for the experiments of Warhaft and Lumley (1978) is presented in figure 3. The curvature in the experimental results is very small. Both models of Sarkar and Speziale (1990) and Rotta (1951) have predictions that are very similar same. This indicates that the nonlinear effects were not dominant in the flows.

Figure 4 represents the temporal evolution of second ($II_b$) and third ($III_b$) invariants for the experimental results of Uberoi (1963). The Panda et al. (2017) model has predictions that are better in comparison to the model of Sarkar and Speziale (1990).

From the comparison against experimental data the predictions of the model of Sarkar and Speziale (1990) show best agreement with data across different experimental studies.

2.4 Rapid pressure strain correlation models

The rapid component of the pressure strain correlation accounts for the linear interactions between the fluctuating velocity field and the mean velocity gradient. This rapid pressure strain correlation has behavior that is very dependent on the mean velocity field. As an illustrative example we can demarcate the behavior of the rapid pressure strain correlation in two regimes of planar flows: hyperbolic streamline flow and elliptic streamline flow. In elliptic streamline flows the elliptical flow instability is initiated by the rapid pressure strain correlation Kerswell (2002). In hyperbolic streamline flows the rapid pressure strain correlation stabilizes the flow instability Mishra and Girimaji (2015). The effect of the rapid pressure strain correlation is highly dependent on the mean gradient and substantially varies between different flows.
Besides giving accurate predictions of the flow evolution and ensuring realizable Reynolds stresses there are additional properties required of the ideal RPSC model. These include

1. The RPSC model \((\phi_{ij}^{(R)})\) should have a model expression linear in the Reynolds stresses [Reynolds (1976), Pope (2000)].

2. The RPSC model should have a model expression linear in the mean velocity gradient [Johansson and Hallbäck (1994), Pope (2000)].

3. The RPSC model should obey the Crow constraint (from isotropic initial conditions) [Crow (1968)].

An ideal model is expected to conform to these properties. However no available model is able to meet all these properties and still produce accurate predictions. While there are many available models for the rapid pressure strain correlation, we discuss three established popular models. These include the models by [Lauder et al. (1975)](term the LRR model), [Speziale et al. (1991)](termed the SSG model) and [Johansson and Hallbäck (1994)](termed the Johansson-Hallback model).

2.4.1 LRR Model:

The model proposed by Lauder, Reece and Rod [Lauder et al. (1975)] has the form

\[
\phi_{ij}^{(R)} = C_1 K S_{ij} + C_2 K (b_{ik} S_{jk} + b_{jk} S_{ik} - 2/3 b_{mn} S_{mn} \delta_{ij}) + C_3 K (b_{jk} W_{jk} + b_{jk} W_{ik})
\]  

(2.15)
The closure coefficients are given as $C_1 = 0.8$, $C_2 = 1.75$ and $C_3 = 1.31$.

The model proposed by Launder et al. (1975) conforms to all the properties for a RPSC model. It is linear in the Reynolds stresses and the mean velocity gradient. It also conforms to the Crow constraint. However it is not able to show accurate predictions for complex flows for example flows dominated by rotational effects. It is also not able to maintain realizability of the Reynolds stress or their evolution for moderate to high levels of anisotropy in the flow (Mishra and Girimaji 2014).

The LRR model has been widely adopted in turbulence simulations and is available in most computational fluid dynamics software. Many variants of this model have also been developed. For example the model of Jones and Musonge (1988) retains the model form of the LRR model but changes the coefficient values to improve performance in turbulent flows with high rates of shear.

2.4.2 SSG Model:

The model proposed by Speziale et al. (1991) has the form

$$\phi_{ij}^{(R)} = (C_1 - C_1^* I^0.5) K S_{ij} + C_2 K (b_{ik} S_{jk} + b_{jk} S_{ik} - 2/3 b_{mn} S_{mn} \delta_{ij}) + C_3 K (b_{ik} W_{jk} + b_{jk} W_{ik})$$

(2.16)

The closure coefficients are given as $C_1 = 0.8$, $C_1^* = 1.3$, $C_2 = 1.25$ and $C_3 = 0.4$.

The model expression does not conform to all the properties for RPSC models stated above and is quadratic in the Reynolds stresses. However it is able to show much improved accuracy in predictions and better realizability behavior than other linear models. For turbulent flows in non-inertial frames of reference this model is much better.
than other RPSC models.

2.4.3 Johansson-Hallback Model:

Johansson and Hallback [Johansson and Hallbäck (1994)] derived the most general expression for the RPSC model. This is given by

\[
\phi_{ij}^{(R)} = S_{ij} + Q_1 \delta_{ip} \delta_{jq} + Q_2 (b_{ip} \delta_{jq} + b_{jp} \delta_{iq} - 2/3 b_{pq} \delta_{ij}) + Q_3 b_{pq} b_{ij} + Q_4 (b_{iq} b_{jp} - 1/3 b_{pk} b_{lq} \delta_{ij}) \]

\[
+ \Omega_{pq} [Q_7 (b_{ip} \delta_{jq} + b_{jp} \delta_{iq}) + Q_8 b_{pk} (b_{jk} \delta_{iq} + b_{ik} \delta_{jq} + Q_9 b_{pk} (b_{jk} b_{ik} + b_{lk} b_{jq})]
\]

(2.17)

Here \( Q_i \) are scalar functions of the invariants of the Reynolds stress anisotropy and
the mean velocity gradient. These can in turn be expressed in terms of scalars $B_\alpha$ as

\[ Q_1 = \frac{4}{5} - \frac{2}{5}(4B_2 + 15B_3)II_\alpha - \frac{2}{5}B_3III_\alpha - \frac{1}{220}(19B_6 - 120B_7)II^2_\alpha, \]
\[ Q_2 = -12B_3 - \frac{1}{2}B_3II_\alpha - \frac{1}{2}(B_6 - 8B_7)III_\alpha, \]
\[ Q_3 = -8B_2 + 36B_3 + \frac{1}{2}B_6 - 72B_7)II_\alpha, \]
\[ Q_4 = 96B_2 - 36B_3 - \frac{1}{2}B_6 - 72B_7)III_\alpha, \]
\[ Q_5 = B_5, \]
\[ Q_6 = B_6, \]
\[ Q_7 = -\frac{4}{3} - \frac{28}{3}B_1 + \frac{1}{6}(2B_4 - B_3)II_\alpha - \frac{1}{18}(3B_6 - 56B_7)III_\alpha, \]
\[ Q_8 = -16B_2 + 28B_3 + \frac{1}{2}B_6 - 56B_7)II_\alpha, \]
\[ Q_9 = -\frac{4}{3} - \frac{28}{3}B_1 + \frac{1}{6}(2B_4 - B_3)II_\alpha - \frac{1}{18}(3B_6 - 56B_7)III_\alpha, \]

(2.18)

Based on the choices for the $B_\alpha$, [Johansson and Hallbäck (1994)] outlined models that were second-, third- and fourth-order with respect to the Reynolds stresses. All these models conform to the strong realizability condition. The fourth-order model shows high agreement with data from experiments for strain-dominated mean flows. The performance of this model for rotation-dominated mean flows is still lacking.

In figures 5, 6 and 7, we show the performance of these models in the rapid distortion limit by comparing them to the results of RDT simulations. For purely strained flows such as the plane strain mean flow shown in figure 5 the models have good agreement with the trends observed in the RDT simulation. The fourth-order model of [Johansson and Hallbäck (1994)] shows better performance as compared to the models of [Launder et al. (1975)] and [Speziale et al. (1991)].

This trend is observed again in figure 6 for a planar strained mean flow at the rapid distortion limit. All the models show acceptable agreement with the RDT simulation for the Reynolds stress anisotropy. The agreement of the fourth-order model of [Johansson and Hallbäck (1994)] for the evolution of the turbulent kinetic energy is very accurate.

In figure 7 we compare the predictions of these models for an elliptic streamline flow at the rapid distortion limit. The figure shows that none of the models give satisfactory predictions for elliptic streamline flows. For example in figure 7 (b) the RDT simulations suggest that the turbulent kinetic energy of the flow is growing exponentially. All the RPSC models predict otherwise. For elliptic streamline flows RPSC models are inexact.
2.4.4 Rapid pressure strain correlation models with extended bases

One of the primary challenges in RPSC modeling is to replicate the non-local dynamics of pressure while using local tensors such as the Reynolds stresses and mean velocity gradients. The models of Launder et al. (1975), Speziale et al. (1991), Johansson and Hallbäck (1994) attempt to do this but have unsatisfactory performance in rotation dominated flows, etc. Some investigators have tried to formulate RPSC models by appending additional tensors to the modeling basis. We discuss a few such notable models here and analyze one of these in detail.

Reynolds and Kassinos (1995) attempted to formulate a RPSC model using additional tensors in the modeling basis including as stropholysis, circulicity, etc. This was justified by differentiating between the componentiality and the dimensionality of turbulent flow field. Using single point (or local) tensors such as the Reynolds stresses informs the model about the componentiality of the turbulent flow field but not about the dimensionality of the turbulent flow field. Reynolds and Kassinos (1995) define the structure dimensionality tensor $D_{ij} = M_{kkiij}$, where the Reynolds stress is given by $R_{ij} = M_{ijkk}$. Addition of this tensor to the modeling basis would bring in important information and improve predictions. The final model in Reynolds and Kassinos (1995) did not show much improved predictions for rotation dominated flows. The model also had realizability issues Mishra and Girimaji (2014). The structure dimensionality tensor $D_{ij}$ is non-local and is not available in most engineering simulations. This made the usage of this model more problematic.

Cambon et al. (1992) posited that using just the deviatoric component of the Reynolds stresses was unable to describe the turbulent flow field completely especially in the presence of mean rotation. They decomposed the Reynolds stress anisotropy tensor into two components: directional and polarization anisotropy ($b_{ij} = b_{ij}^e + b_{ij}^z$). Transport equations for these two components separately were developed. This model was able to show some improvements in rotation dominated flows. The transport equations for the decomposed anisotropy components were not unique and the closure coefficients were tuned to give agreement with the experiments used in the investigation. Above all in a real life engineering problem there is no clear manner on how to decompose the Reynolds stress anisotropy as information about the decomposition is non-local.

Mishra and Girimaji (2017) developed an illustrative model where the model closure coefficients were functions of the mean velocity gradient invariants. Mishra and Girimaji (2010, 2013) have illustrated the details of the intercomponent energy transfer caused by the rapid pressure strain correlation. Using spectral analysis, they establish a most likely evolution based on the statistics of the turbulent velocity that models should aim to reproduce. They have shown that including the mean velocity gradient in the modeling basis would lead to the addition of missing physics and improved model pre-
dictions. This is in agreement with [Lee (1989)] where it is shown that adding the mean strain rate information would improve the predictions of the pressure strain correlation model. In [Mishra and Girimaji (2014)] a new approach to realizability is developed. Using this realizability approach it was shown that addition of the mean velocity gradient information would lead to better realizability behavior. Finally [Mishra and Girimaji (2017), Mishra (2014)] have shown that including the mean gradient information by making the model coefficients functions of the mean velocity gradient would lead to a simple model structure, better realizability behavior and improved accuracy of predictions.

Instead of adding non-local tensors to the modeling basis the model of [Mishra and Girimaji (2017)] uses the mean velocity gradient invariants to add missing information to the model expression. This model may be considered as compliant to use in real life engineering problems as it does not require the estimation of non-local tensors. The model expression is given by

$$\phi^{(R)}_{ij} = \frac{4}{5}K S_{ij} + 6A_5\beta S_{pq}K(b_{ip}\delta jq + b_{jp}\delta iq + 2/3b_{pq}\delta ij) + 2/3(4 + 7A_5(\beta))W_{pq}(b_{ip}\delta jq + b_{jp}\delta iq)$$

(2.19)

where

$$A_5(\beta) = 0.22\beta - 0.44, \beta \in [0, 0.5] \quad \text{and} \quad A_5(\beta) = -0.83\beta^2 - 0.44, \beta \in [0.5, 1]$$

(2.20)

where $\beta$ is the ellipticity parameter and is defined as

$$\beta = \frac{W_{mn}W_{mn}}{W_{mn}W_{mn} + s_{mn}s_{mn}}$$

(2.21)

In figures 8 and 9 we compare the predictions of this model to the DNS investigation of [Blaisdell and Shariff (1996)] for different elliptic flows. The predictions of the LRR model are included for contrast. [Blaisdell and Shariff (1996)] have simulated homogeneous turbulence subjected to elliptic mean flows:

$$\frac{\partial U_i}{\partial x_j} = \begin{bmatrix} 0 & 0 & -\gamma - e \\ 0 & 0 & 0 \\ \gamma - e & 0 & 0 \end{bmatrix}$$

(2.22)

where $e = \sqrt{1-\beta^2}$ and $\gamma = \sqrt{\frac{\beta}{2}}$. For $e > \gamma$ the mean flow has elliptic streamlines of aspect ratio $E = \sqrt{(\gamma + e)(\gamma - e)}$. We use this data from two simulations with mean flows having aspect ratios $E = 2$ and $1.5$. The turbulent velocity field is initially isotropic and the initial $\frac{\eta}{s_k} = 0.167$.

In figure 8 the case with $E = 1.5$ is shown. In figure 8 (a) The model of [Mishra and Girimaji (2017)] shows better agreement with DNS data than the predictions of the LRR.
model. In figure 8 (b) the LRR model does predict turbulent kinetic energy growth but at a rate much smaller than DNS. The rate of turbulent kinetic energy growth predicted by the model of [Mishra and Girimaji (2017)] is in agreement with the DNS data.

In figure 9 the case with $E = 2$ is shown. (In this case the effect of rotation on flow evolution has increased over $E = 1.5$). In figure 9 (b) while the DNS simulations predict the turbulent kinetic energy to be growing the model of LRR predicts decay. The rate of turbulent kinetic energy growth predicted by the model of [Mishra and Girimaji (2017)] is in agreement with the DNS data.
Fig. 2.5 Rapid pressures train correlation model predictions for a plane strain mean flow at the rapid distortion limit, (a) $b_{11}$, (b) $\log(k)$
Fig. 2.6 Rapid pressures train correlation model predictions for a planar strained mean flow at the rapid distortion limit, (a) $b_{12}$, (b) $\log(k)$
Fig. 2.7 Rapid pressures train correlation model predictions for an elliptic streamline mean flow at the rapid distortion limit, (a) $b_{22}$, (b) $\log(k)$
Fig. 2.8 Rapid pressures train correlation model predictions for an elliptic streamline mean flow at the rapid distortion limit
Fig. 2.9 Rapid pressures train correlation model predictions for an elliptic streamline mean flow at the rapid distortion limit
2.5 Summary

In this chapter we provide a thorough review of pressure strain correlation modeling for turbulent flows. Starting from the Reynolds stress transport equations the numerical and mathematical foundations of pressure strain correlation modeling are established. The key challenges in this modeling effort arising due to the non-local nature of the behavior of the pressure strain correlation are established.

Established slow pressure strain correlation models are introduced. Their predictions are compared and contrasted against experimental data from a range of experiments.

Popular rapid pressure strain correlation were introduced. Their predictions were contrasted against rapid distortion theory based simulations. It was shown that most rapid pressure strain correlation models have satisfactory behavior in strain dominated turbulent flows but unsatisfactory predictions in rotation dominated flows. Alternative models that add to the modeling basis were introduced and their predictions for elliptic streamline flows were shown.
CHAPTER 3

Modeling Turbulence/Turbulence Interactions

3.1 General introduction

Prandtl can be considered as the pioneer of turbulence modeling, because the concept of turbulence modeling came after he proposed his successful mixing length theory (Prandtl (1925), Prandtl and Tietjens (1934a,b)). This was followed by a host of other models in the following years with more elaborate and accurate models replacing the simple mixing length models. The Fluid Dynamics community started devoting more time to turbulence modeling after the development of algebraic model of (Cebeci and Smith (1974)) followed by the subsequently the zero, one-equation and finally the two-equation models of [Launder et al.] (1975), Sarkar and Speziale (1990), Harlow and Nakayama (1968, 1967) in the late 60’s derived the transport equations of eddy viscosity and decay rate of turbulence kinetic energy. Later B. Daly (1970) derived turbulence transport equations describing flow of incompressible fluid in arbitrary geometry. Various mathematical models of turbulence were described by Launder and Spalding (1972). Chou (1945) derived velocity corrections and presented solutions of the equations of the turbulent fluctuation.

A lot of effort and research has gone into second-order turbulence modeling in the last few decades and there have been many successful attempts at closing the turbulence equations [Hanjalic et al., (1997)] at the second or third moment [Launder (2005), Lumley (1979), Launder et al. (1975)]. Closing the equations at the second moment would imply that the third order moments are represented by simple algebraic equations. Though solving a partial differential equation is more complicated, the advantages greatly overshadow the difficulties, since it is much more accurate and once programmed, it is easy to simulate. Recently many Reynolds stress models have been developed, with these more complicated equations replacing the two-equation models of which the most famous one is the $k-\epsilon$ model (Launder and Spalding (1974)).

If an initial disturbance (arising from shear or buoyancy) excites a flow in ordinary grid turbulence into high Reynolds numbers, the flow develops an anisotropy (Sagaut and Cambon (2008)). The return of such a flow to an isotropic state after the production of turbulence kinetic energy is switched off, has been identified. This return to isotropy...
has been noticed by scientists and has been validated experimentally (Choi and Lumley (2001), Warhaft (2000)). In the first of its kind paper, Rotta (1951) proposed that the return to isotropy is a linear function of the anisotropy of the system. The model has enjoyed wide success and has been applied in many fluid flow problems. But later it was proposed by Champagne and Corrsin (1970) that this assumption is erroneous and that the return to isotropy is a function of Reynolds number. They showed that Rotta’s model does not distribute turbulent kinetic energy equally in all directions.

Many features in the experiments and simulations seemed to point to a non-linear return to isotropy. The most significant of them is the fact that different components of the anisotropy tensor return to isotropy at different rates. A lot of research was carried out in this topic (especially in the aerodynamics field) by studying grid turbulence after plane distortion, axisymmetric contraction, axisymmetric expansion etc. They used these experiments to develop turbulence models to study the return to isotropy problem.

The most relevant of these works are those of Le Penven et al. (1985), Choi and Lumley (1984), Warhaft (2000), Sarkar and Speziale (1990), developed the first quadratic closure for the return to isotropy, and calibrated it with the well-established experiments of Choi and Lumley (1984). In their seminal paper, they assumed that the slow pressure strain depends on $b_{ij}, q^2$ and $\epsilon$. Their model performed consistently well in many and varied types of flows - especially the high vorticity flows (Basara and Younis (1995)). This was followed up by Warrior et al. (2014), in which they included the dissipation tensor as the second parameter on which slow pressure strain rate depends.

For proper characterization of the non local phenomenon of turbulence various authors Sarkar and Speziale (1990), Chung and Kim (1995) have tried to model the anisotropy of the field in terms of the anisotropy of Reynolds stress. However only Reynolds stress anisotropy is not sufficient, so additional one point statistical measures of anisotropy are required to describe the anisotropic turbulence field. (Kassinos and Rogers (2001), Hallback (1990)) formulated algebraic models for anisotropic turbulent dissipation rate and successfully formulated a relationship between anisotropic dissipation rate and the Reynolds stress anisotropy. A fifth order model for the anisotropy of the dissipation rate was formulated by Sjögren and Johansson (2000). Mishra and Girimaji (2010, 2013) have discussed the need for additional tensors to capture the non local effects of pressure. However in this paper authors have tried to implement the anisotropy of length scales in the single point closures and have modeled the anisotropic length scales in terms Reynolds stress anisotropy and anisotropic dissipation rate which can appropriately predict the turbulence structures in strain dominated flow fields. The anisotropic length scales assumed as a linear function of Reynolds stress anisotropy and anisotropic dissipation rate as described by Breuer and Peters (2005), where authors
have studied the anisotropic length scales in a motored piston engine. Sarkar and Speziale (1990) proposed one of the first model of the slow pressure strain correlation in which he discussed algorithms for the derivation of Reynolds stress models, the details of the algorithms can be found in Speziale et al. (1991) in which he described the concepts of realizability, which physically means that the modeled equations should not give any physically unrealizable solutions, such as negative kinetic energy or dissipation (Pope (2000)). The route of such algorithms are strong realizability concept described in Lumley (1979), weak realizability of Pope (2000) and the process realizability framework of Mishra and Girimaji (2014).

In this paper, we use the Sarkar and Speziale (1990) model as the yard stick and compare our results to theirs. The fact that different components of turbulent kinetic energy decay at different rates in different directions indicates the return to isotropy must be non-linear. Since we are applying compression and expansion for introducing strain rates, it is not sufficient to assume a single length scale across the flow. At least we need to define two length scales one in the plane normal to the compression and another perpendicular to it. This brings in the concept of length scale anisotropy tensor. The anisotropy in length scale is introduced in this paper for the first time, which is assumed to be a linear function of Reynolds stress anisotropy and dissipation anisotropy tensor. The results will be compared with the experimental data of Le Penven et al. (1985), Choi and Lumley (1984) and also with SS nonlinear quadratic model (Sarkar and Speziale (1990)).

3.2 Background for the problem

In this chapter we adopt the Reynolds stress model in which an equation for Reynolds stress is solved;

Traditionally it was assumed that the rate of dissipation tensor is nearly isotropic because of which it can be represented as;

\[ \epsilon_{ij} = \frac{2}{3} \epsilon \delta_{ij} \quad (3.1) \]

where the isotropic dissipation rate, \( \epsilon \), is defined as

\[ \epsilon = \nu u_{i,j} u_{i,j} \quad (3.2) \]

A lot of models were developed using the idea that dissipation tensor is isotropic or constant in all directions. However, recent DNS studies have shown otherwise and is thus adopted in this paper. Based on that Warrior et al. (2014) showed that using a dissipation rate tensor is a better estimate of turbulence than isotropic dissipation. The
results were better compared to the Sarkar and Speziale (1990) model.

For turbulent flows, length scale tensor is also found to be varying in various directions. It is important to consider this length scale anisotropy to get better results.

We introduce a dissipation anisotropy tensor $e_{ij}$ defined as

$$e_{ij} = \frac{\epsilon_{ij}}{\epsilon} - \frac{2}{3} \delta_{ij} \quad (3.3)$$

Similarly, we define the parameter $l_{ij}$ as the anisotropy of length scales and derive it as (Breuer and Peters (2005)) follows;

$$L_{ij} = 0.75 \frac{k^{\frac{4}{3}}}{\epsilon} (c_{1}^* b_{ij} + c_{2}^* e_{ij})$$

$$l = \frac{k^{\frac{4}{3}}}{\epsilon}$$

$$l_{ij} = \frac{L_{ij}}{l} = 0.75 (c_{1}^* b_{ij} + c_{2}^* e_{ij}) \quad (3.4)$$

The total pressure-strain rate $\phi_{ij}$ can be assumed to be a sum of a slow pressure term $\phi_{ij}^{S}$ which is a slow pressure strain rate and a rapid pressure strain term $\phi_{ij}^{R}$ which is linear in $\frac{\partial U_{i}}{\partial x_{j}}$.

The rapid pressure strain correlation is explicitly and linearly dependent on the mean velocity gradients, as discussed in Cambon et al. (1992). In this investigation we consider turbulent flows where the mean gradients are absent, In such a scenario, it is only the slow pressure strain correlation that is non-zero.

Following the same line of thinking, for the return to isotropy case, the production of turbulent kinetic energy term in 1 can be put to zero (thus there is a decay).

$$\frac{\partial \bar{u}_{i} \bar{u}_{j}}{\partial t} + U_k \frac{\partial \bar{u}_{i} \bar{u}_{j}}{\partial x_k} = -\epsilon_{ij} + \phi_{ij} \quad (3.5)$$

In order to model the pressure strain correlation in equation (12), the Poisson equation for fluctuating pressure should be solved for determining the pressure fluctuations (Wilcox (1998))

$$\frac{1}{\rho} \nabla^2 (P) = -2 \frac{\partial U_{j}}{\partial x_{i}} \frac{\partial u_{i}}{\partial x_{j}} - \frac{\partial^2 u_{i} u_{j}}{\partial x_{i} \partial x_{j}} \quad (3.6)$$

Above equation can be solved through the decomposition, as in Sjögren and Johansson (2000), Mons and Sagaut (2016) of the pressure fluctuation as

$$P = P^{S} + P^{R} \quad (3.7)$$
Slow and rapid pressure fluctuations satisfy the following equations:

$$\frac{1}{\rho} \nabla^2 (p^S) = -\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}$$

(3.8)

$$\frac{1}{\rho} \nabla^2 (p^R) = -2 \frac{\partial U_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}$$

(3.9)

In this chapter only the slow pressure strain is modeled and the pressure strain correlation is assumed as independent of mean strain rate. The slow pressure strain correlation tensor is trace-less and that is why it may be removed via the contraction of the indices operation, upon contraction, the equation gives,

$$\partial_t u_i u_j = \partial_t q^2 = -\epsilon_{ii} = -2\epsilon$$

(3.10)

where, $q^2 = 2k$. The slow pressure strain term has effect on the turbulent kinetic energy. Its effect on the distribution of energy among the Reynolds stresses, which can be analyzed through normalized anisotropy tensor

$$b_{ij} = \frac{u_i u_j}{2k} - \frac{\delta_{ij}}{3}$$

(3.11)

The second and third invariants of the Reynolds stress anisotropy tensor are defined as

$$II_b = b_{ij} b_{ji} \quad \text{and} \quad III_b = b_{ij} b_{jk} b_{ki}$$

(3.12)

Using (1) and (2), the evolution equation for $b_{ij}$ is obtained:

$$\partial_t b_{ij} = \frac{\phi_{ij}}{q^2} - (\epsilon_{ij} - \frac{2}{3}\epsilon \delta_{ij}) + 2 \frac{\epsilon}{q^2} b_{ij}$$

(3.13)

A dimensionless tensor $\phi_{ij}$ can be defined as:

$$\Phi_{ij} = \frac{\phi_{ij}}{\epsilon}$$

(3.14)

Through an algebraic model for turbulent dissipation rate [Hallback (1990)] established a relationship between the dissipation rate and the Reynolds stress anisotropy by the comparison of the model with numerical simulations of strained homogeneous turbulence [Hallback (1990)]. The dissipation anisotropy was defined as:

$$\epsilon_{ij} = k_1 b_{ij} + k_2 (b_{ik} b_{kj} - \frac{1}{3} III \delta_{ij})$$

(3.15)

where $k_1$ and $k_2$ are the functions of strain, vorticity and the invariants of Reynolds Stress anisotropy. By imposing the symmetry conditions, zero trace and Cayley-Hamilton
Theorem for Reynolds stress anisotropy tensor, the expression reduces to:

\[ e_{ij} = [1 + 0.75(0.5II_b - \frac{2}{3})]b_{ij} - 0.75(b_{ik}b_{kj} - \frac{1}{3}II_b\delta_{ij}) \quad (3.16) \]

However for the sake of simplicity and for the incorporation of near wall effects into the slow pressure strain term in this paper the dissipation tensor is defined as

\[ e_{ij} = 2f_s b_{ij} \quad (3.17) \]

where, \( f_s = 1 - \sqrt{A} \) is a blending or coordinating function ([Warrior et al., 2014]).

\[ A = 1 - \frac{9}{8}(II_b - III_b) \quad (3.18) \]

The blending function can be expanded in Taylor series to get:

\[ f_s = \frac{9}{4}(II_b - 2III_b) \quad (3.19) \]

Where \( A \) is the Lumley’s Flatness Parameter ([Lumley, 1979]). This assumption even though somewhat erroneous, has proved to be a reasonable approximation provided \( f_s \) is specified adequately.

### 3.3 Model formulation

The new model will be formulated on the basis of the SS model ([Sarkar and Speziale, 1990]). In our assumptions in this paper, pressure strain correlation \( \Phi_{ij} \) is dependent on 1. Reynolds stress anisotropy \( (b_{ij}) \), 2. Dissipation anisotropy \( (e_{ij}) \), 3. Length scale anisotropy \( (l_{ij}) \). Since these three principal values of \( b_{ij}, e_{ij} \) and \( l_{ij} \) are smaller than unity, it is possible to expand \( \Phi_{ij} \) in a Taylor series expansion about the point \( b_{ij} = 0, e_{ij} = 0 \) and \( l_{ij} = 0 \) (at the point of perfect isotropy). Taylor series expansion reduces to:

\[
F(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = F(x_0, y_0, z_0) + (F_x(x_0, y_0, z_0)\Delta x + F_y(x_0, y_0, z_0)\Delta y + F_z(x_0, y_0, z_0)\Delta z) + 0.5(F_{xx}(x_0, y_0, z_0)(\Delta x)^2 + F_{yy}(x_0, y_0, z_0)(\Delta y)^2 + F_{zz}(x_0, y_0, z_0)(\Delta z)^2) + \quad (3.20)
\]

Here \((x_0, y_0, z_0) = (0, 0, 0)\), \( \Delta x = b_{ij}, \Delta y = e_{ij} \) and \( \Delta z = l_{ij} \) and \( F = \Phi_{ij} \), since \( \Phi_{ij} = 0 \) when anisotropy is zero, \( F(0, 0, 0) = 0 \).

In this paper, The model for the slow pressure-strain \( \Phi_{ij} \) is proposed as:
\[\phi_{ij} = c_1 b_{ij} + c_2 e_{ij} + c_3 l_{ij} + c_4 (b_{ik} b_{kj} - 1/3 b_{mn} b_{mn} \delta_{ij}) + c_5 (b_{ik} e_{kj} - 1/3 b_{mn} e_{mn} \delta_{ij}) + c_6 (e_{ik} e_{kj} - 1/3 e_{mn} e_{mn} \delta_{ij}) + c_7 (e_{ik} l_{kj} - 1/3 e_{mn} l_{mn} \delta_{ij}) + c_8 (l_{ik} l_{kj} - 1/3 l_{mn} l_{mn} \delta_{ij}) \]  

(3.21)

where \( c_1, c_2, c_3, c_4, c_5, c_7, c_8 \) and \( c_9 \) are model constants.

After simplification we obtain the general model for the slow pressure-strain correlation as:

\[\phi_{ij} = \beta_1 b_{ij} + \beta_2 (b_{ik} b_{kj} - 1/3 II_b \delta_{ij})\]  

(3.22)

\[\beta_1 = \beta_1 (f_s), \text{ and } \beta_2 = \beta_2 (f_s)\]  

(3.23)

\( \beta_1 \) and \( \beta_2 \) in terms of blending function were derived as:

\[\beta_1 = c_1 + 2 f_s c_2 + \frac{3}{4} c_3 c_4^* + \frac{3}{2} f_s c_5 c_3\]

\[\beta_2 = c_4 + 2 f_s c_5 + 4 f_s^2 c_5 + \frac{3}{4} c_7 c_7^* + \frac{3}{2} f_s c_8 c_7^* + \frac{9}{16} c_8^* 2 c_9 + \frac{9}{8} f_s c_1 c_2^* c_9 + \frac{3}{2} f_s c_5 c_7^* + 3 f_s^2 c_2 c_7^* + \frac{9}{8} f_s c_1 c_2^* c_9 + \frac{9}{4} f_s^2 c_2 c_9^*\]  

(3.24)

Equation for non dimensional time (Sarkar and Speziale (1990)) is considered as:

\[d\tau = \epsilon q^2 dt\]  

(3.25)

The evolution of \( II_b \) and \( III_b \) can be obtained by multiplying equation (26) by \( b_{ij} \) and \( b_{ik} b_{kj} \) respectively as described in Sarkar and Speziale (1990):

\[\frac{dII_b}{d\tau} = -2((\beta_1 + 2 f_s - 2) II_b + \beta_2 III_b)\]

\[\frac{dIII_b}{d\tau} = -3((\beta_1 + 2 f_s - 2) III_b + \beta_2 II_b^2)\]  

(3.26)

The model constants used in our simulations are presented in table.1 and are calibrated by using experiments of Le Penven et al. (1985), Choi and Lumley (1984) as described in Sarkar and Speziale (1990).
3.4 Model performance

The coupled system of nonlinear equations 34 are solved by Runge- kutta method (Patankar (1980)). The current nonlinear model incorporating the length scale anisotropy is compared with Sarkar and Speziale (1990) using experimental data of Le Penven et al. (1985), Choi and Lumley (1984). Figure 1 shows phase space comparison graph of the axisymmetric expansion experiment of Choi and Lumley (1984). Due to the nature of nonlinearity in the evolution equations of second and third invariants, the present model with length scale anisotropy seems to have results with greater curvature and shows better agreements with experimental data. Figures 2-3 represent the axisymmetric expansion experiment of Choi and Lumley (1984), which indicates the variation with time during decay of the second and third invariants of anisotropy. In both the figures the present model is seen to do better than the SS quadratic model. The decay of both the invariants are slightly slower than the decay rate predicted by the SS quadratic model. Figure 4- 5 consider the plane contraction experiment of Le Penven et al. (1985). In the temporal evolution of second invariant in Fig. 4 the graph of the present model almost coincides with the experimental data of Le Penven et al. (1985), resulting in better result in comparison to the SS quadratic model but in fig. 5 there is slight improvement over the existing result. Figures 6 and 7 pertain to the plane distortion experiment of (Le Penven et al. (1985), Choi and Lumley (1984)) respectively. Both the figures concern to the temporal evolution of the third invariants. Figure. 6 shows that the present improved model performs better than the SS quadratic model. The decay rate of third invariant with present model is slower than the decay rate predicted by the SS quadratic model. Figure. 7 shows faster decay rate of the third invariant in comparison to the decay rate of the SS quadratic model.
Fig. 3.1 Phase space comparison with the axisymmetric expansion experiment of Choi and Lumley. New model prediction is shown by the solid line, SS model by dashed line and the experimental data are shown in plus symbols.
Fig. 3.2 Time evolution of the second invariant with the axisymmetric expansion experiment of Choi and Lumley.

Fig. 3.3 Time evolution of the third invariant with the axisymmetric expansion experiment of Choi and Lumley.
Fig. 3.4 Time evolution of second invariant with the plane contraction experiment of Le Penven et al.

Fig. 3.5 Time evolution of third invariant with the plane contraction experiment of Le Penven et al.
Fig. 3.6 Time evolution of third invariant with the plane distortion experiment of Choi and Lumley.

Fig. 3.7 Time evolution of third invariant with the plane distortion experiment of Le Penven et al.
3.5 Summary

A new return to isotropy model has been developed considering the anisotropy of length scales. In almost all the figures shown in this paper, the present model is an improvement over the SS quadratic model. In the slow pressure-strain correlation tensor higher degree nonlinearities were incorporated by using suitable combinations of Reynolds stress, dissipation and length scale anisotropy tensors. The length scale anisotropy tensor was taken as a linear function of Reynolds stress and dissipation anisotropy tensor as was found in literature. The present model developed can be easily incorporated into various standard CFD codes and also into geophysical turbulence models. In future course of work focus can be placed on the non-localness of the various properties by incorporating other non-linear features in vorticity fluctuations.
CHAPTER 4

Modeling Mean-strain/Turbulence Interactions

4.1 General introduction

Turbulence models are simplified constitutive equations that model quantities of interest that are difficult to compute in terms of known flow parameters. While the initial focus of turbulence modeling emphasized on simpler two equation models like the $k - \epsilon$ and $k - \omega$ models, recent emphasis has shifted to Reynolds stress models (Hanjalić and Launder (2011), Durbin (2017), Klič and Lili (2013), Mishra and Girimaji (2014), Jakirlić and Maduta (2015), Manceau (2015), Eisfeld et al. (2016), Schwarzkopf et al. (2016), Moosaie and Manhart (2016), Lee et al. (2016), Mishra and Girimaji (2017), Sun et al. (2017)). Reynolds stress models have the potential to provide better predictions than turbulent viscosity based models at a computational expense significantly lower than DNS studies. They may be able to model the directional effects of the Reynolds stresses and additional complex interactions in turbulent flows (Johansson and Hallbäck (1994)). They have the ability to accurately model the return to isotropy of decaying turbulence and the behavior of turbulence in the rapid distortion limit (Pope (2000)).

The accuracy of the Reynolds stress modeling approach depends on the quality of the models developed for the individual turbulence processes: turbulence transport, turbulence dissipation and pressure strain correlation. Out of these the modeling of the pressure strain correlation is often considered to be the most important.

The pressure strain correlation of turbulence consists of two components: the slow pressure strain correlation modeling the non-linear interactions in between the fluctuating velocity field and the rapid pressure strain correlation modeling the interactions between the mean velocity and the fluctuating velocity field. The slow pressure strain correlation is considered to have a return to isotropy effect where it causes the Reynolds stress to evolve toward an isotropic state. Rotta (1951) modeled the slow pressure strain correlation as a linear expression of the components of the Reynolds stress anisotropy tensor. Sarkar and Speziale (1990) developed a non-linear expression for the slow pressure strain correlation model. This non-linear expression was able to show improved agreement with experimental data. Warrior et al. (2014), Maity and Warrior (2011) and
Sasmal et al. (2014) expanded the modeling basis of the slow pressure strain correlation and included dissipation as an additional parameter. This was able to show additional improvements in performance over earlier slow pressure strain models. Unlike the slow pressure strain, the rapid pressure strain correlation does not have a universal behavior and its behavior depends heavily on the mean velocity gradients and the initial state of the turbulent velocity field. There have been many investigators that have developed advanced rapid pressure strain correlation models such as Johansson and Hallbäck (1994) and Reynolds and Kassinos (1995). These have shown improved agreement with experiments. In spite of these efforts, rapid pressure strain correlation models still have many shortcomings (Launder et al. (1987), Speziale et al. (1990)). For example, it is documented that the popular models like those developed by Launder et al. (1975) and Speziale et al. (1991) do not give satisfactory performance in many elliptic streamline flows (Speziale et al. (1990)).

Recently, Mishra and Girimaji (2010, 2013, 2014, 2017) have made fundamentally important insights about the behavior of the rapid pressure strain correlation and the structure of the pressure strain correlation model. Mishra and Girimaji (2010) have illustrated the details of the intercomponent energy transfer caused by the rapid pressure strain correlation. Using spectral analysis, they establish a most probable evolution based on the statistics of the turbulent velocity that models should aim to reproduce. They have shown that including the mean velocity gradient in the modeling basis would lead to the addition of missing physics and improved model predictions. This is in agreement with Lee (1989) where it is shown that adding the mean strain rate information would improve the predictions of the pressure strain correlation model. In Mishra and Girimaji (2014) a new approach to realizability is developed. Using this realizability approach it was shown that addition of the mean velocity gradient information would lead to better realizability behavior. Finally Mishra and Girimaji (2017) have shown that including the mean gradient information by making the model coefficients functions of the mean velocity gradient would lead to a simple model structure, better realizability behavior and improved accuracy of predictions.

Mishra and Girimaji (2017) developed an illustrative model to show the improvements in model predictions based on this approach. To compute the relationship between the model coefficients and the mean velocity gradient, Mishra and Girimaji (2017) used a least square error approach and used DNS results where they computed the values of the pressure strain correlation tensor components and fit approximate values of the model expression coefficients to give the minimum error. While the results of their illustrative model showed some improvement, they use a numerically intensive approach to evaluate their model coefficients. The methodology to compute their model coefficients is based more on numerics and not as much on purely physical arguments.
A more physics based approach was also explored in Mishra (n.d.). This approach based on physics and not numerics may show improvements on the Mishra Girimaji model.

In this chapter we introduce an approach using physics based postulates and representation theory to derive the coefficients for the rapid pressure strain correlation which represents the mean strain/turbulence interactions. Using numerical simulations we show that the Reynolds stress anisotropies reach stationary states after a short transient. This equilibrium state represents a stage in turbulence development where the effect of turbulence production is balanced by the rapid pressure strain correlation (Pope (2000), Mishra and Girimaji (2010)). Using representation theory we express these stationary values of the Reynolds stress anisotropy in terms of the mean strain and mean rotation rates. Using this relationship with the Reynolds stress anisotropy evolution equation evaluated at the stationary states, we derive a relationship expressing the model coefficients as functions of the invariants of mean strain and rotation rates. This develops a linear model for the rapid pressure strain correlation. This model is tested for a range of mean flows while compared to DNS results. In this investigation, we use the popular models of Launder et al. (1975) and et al. Speziale et al. (1991) for comparison. The present model shows improved agreement with DNS results and significant improvements over these earlier pressure strain correlation models.

4.2 Background for the problem

The poisson equation for fluctuating pressure \( p = p^R + p^S \) is (Pope (2000))

\[
\frac{1}{\rho} \nabla^2 (p^R + p^S) = -2 \frac{\partial U_j \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j}}{\partial x_i \partial x_j} - \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}
\] (4.1)

Here \( p^R \) and \( p^S \) are the rapid and slow components of pressure. On the right-hand side of equation (2), the first term represents linear interactions between the fluctuating velocity field with the mean velocity gradient and the second term represents the non-linear interactions in between the fluctuating velocity field.

In the rapid-distortion limit (i.e., \( S \to \infty \)) the terms that scale with \( S \) are dominant and other terms are negligible in comparison. This causes the slow pressure strain correlation and the dissipation terms to be negligible. Taking the flow to be homogeneous in space causes the transport term to be negligible. In the rapid-distortion limit for homogeneous turbulent flows the transport equation for the Reynolds stress reduces to

\[
\frac{\partial}{\partial t} \overline{u_i u_j} + U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} = P_{ij} + \psi_{ij}^R
\] (4.2)

where \( \psi_{ij}^R \) is the rapid pressure strain correlation and turbulence production process is represented by \( P_{ij} \).
The linear form of the rapid pressure strain correlation model is

\[ \frac{\psi_{R}}{k} = C_2 S_{ij} + C_3 (b_{ik} S_{jk} + b_{jk} S_{ik} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij}) + C_4 (b_{ik} W_{jk} + b_{jk} W_{ik}) \]  

(4.3)

Here \( b_{ij} = \frac{\nabla \mathbf{u}_i \cdot \nabla \mathbf{u}_j}{2k} - \frac{\delta_{ij}}{3} \) is the Reynolds stress anisotropy tensor, \( S_{ij} \) is the rate of strain term for the mean velocity field and \( W_{ij} \) is the rate of rotation term for the mean velocity field. Following the notation of Speziale et al. (1991), \( C_2, C_3 \) and \( C_4 \) are the coefficients of the rapid pressure strain correlation model.

The Reynolds stress anisotropy evolution equation is derived from the Reynolds stress transport equation

\[ \frac{d b_{ij}}{d t} = 2 b_{ij} b_{mn} S_{mn} + L_2 S_{ij} + L_3 (b_{ik} S_{jk} + b_{jk} S_{ik} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij}) + L_4 (b_{ik} W_{jk} + b_{jk} W_{ik}) \]  

(4.4)

Here \( L_2 = \frac{C_2}{2} - \frac{2}{3}, L_3 = \frac{C_3}{2} - 1 \) and \( L_4 = \frac{C_4}{2} - 1 \). Once the form of the rapid pressure strain correlation model expression is fixed, the modeling problem reduces to determine the nature and values of the model coefficients \( C_2, C_3 \) and \( C_4 \) (or equivalently \( L_2, L_3 \) and \( L_4 \)).

The turbulent kinetic energy (\( k = \frac{\nabla u_i \cdot \nabla u_i}{2} \)) evolves as

\[ \frac{d k}{d t} = P - \eta \]  

(4.5)

The modeled evolution equation for the dissipation is

\[ \frac{d \eta}{d t} = C_{\eta 1} \frac{\eta}{k} P - C_{\eta 2} \eta \]  

(4.6)

Here the values of the coefficients are taken as \( C_{\eta 1} = 1.44 \) and \( C_{\eta 2} = 1.88 \).

4.3 Model formulation

The Rapid Distortion Limit is a limiting state of turbulence. Here the ratio between characteristic time scale for the turbulent flow and the characteristic time scales for the mean flow is very large, \( \frac{\delta_k}{\epsilon} \to \infty \) (Pope (2000)). The evolution of the turbulent flow is dominated by linear interactions between the mean velocity field and the turbulent velocity field. The effect of the non-linear interactions in between the fluctuating field is negligible by comparison.

In this part of the paper, we show the physical and mathematical foundation for the
formulation of the rapid pressure strain correlation model based on representation theory. The first observation is that in a general turbulent flow at the rapid-distortion limit Reynolds stress anisotropies reach stationary states after a short transient. This equilibrium represents a state where the effects of turbulence production on the Reynolds stress anisotropies is balanced by the rapid pressure strain correlation as observed by Pope (2000) and Mishra and Girimaji (2010). This is shown in Figure 1 for anisotropy evolution under different mean flows at the rapid distortion limit. The values of the Reynolds stress anisotropy $b_{ij}$ at these stationary states are designated as $b_{11}^*$, $b_{22}^*$ and $b_{12}^*$. For elliptic streamline flows there is a small degree of oscillations observed even in this equilibrium state. In such cases we focus on the mean of the anisotropy evolution. This detail is illustrated in Figure 2 (a), where the evolution of $b_{22}$ for $\beta = 0.9$ is shown. The instantaneous values are shown in the solid line and the mean (averaged over the oscillation cycle) in the dashed line. As can be seen this mean of the anisotropy component reaches a stationary value and we use this value as the proxy fixed point. Figure 2 (b) shows the evolution of the $R_{13}$ component from the DNS of Blaisdell and Shariff (1996), with the instantaneous and mean values of the Reynolds stress tensor component. As can be seen, capturing the mean while ignoring the oscillatory component may give predictions that are useful for engineering purposes. Such an approach of trying to predict the mean of the anisotropy evolution and ignoring the oscillatory component has been advocated by Blaisdell and Shariff (1996), Mishra (2014) and Mishra and Girimaji (2010). Adopting this approach we use the stationary value of the mean of the Reynolds stress anisotropy as the stationary state of $b_{ij}$ for elliptic streamline flows. We have developed and validated a Matlab code that computes the values of the Reynolds stress anisotropies at the stationary states using simulations at the rapid distortion limit based on Pope (2000).

Using representation theory the Reynolds stress anisotropy can be expressed as a polynomial function in terms of the mean rate of strain and mean rate of rotation. Based on Pope (1975), the general form of this is given by

$$b_{ij} = G_1S_{ij} + G_2(S_{ik}W_{kj} + W_{ki}S_{kj}) + G_3(S_{ik}S_{kj} + \frac{(\beta - 1)\delta_{ij}}{3})$$ (4.7)

here $\beta = \frac{W_{mn}W_{mn}}{W_{mn} + \delta_{mn}S_{mn}}$ is the ellipticity parameter introduced by Mishra and Girimaji (2013). $G_1$, $G_2$ and $G_3$ are scalars that are functions of the invariants of flow statistics. This approach can be extended to three dimensional mean flow cases (Mishra and Girimaji (2015)). In this paper, we study two dimensional mean flow cases that can be completely described using $\beta$.

Using the polynomial form from Eqs. (8), and using the Matlab code to calculate values of the Reynolds stress anisotropies at the stationary equilibrium points ($b_{11}^*$, $b_{22}^*$ and $b_{12}^*$), $G_1$, $G_2$ and $G_3$ can be expressed in terms of these stationary values of the
Reynolds stress anisotropies

\[
G_1 = \frac{b_{11} - b_{22}^*}{\sqrt{2(1-\beta)}}
\]

\[
G_2 = -\frac{b_{12}^*}{\sqrt{\beta(1-\beta)}}
\]

\[
G_3 = \frac{b_{11}^* + b_{22}^*}{(1-\beta)/3}
\] (4.8)

At the stationary states for the Reynolds stress anisotropy, the evolution equation Eqs. (5) simplifies to

\[
2b_{ij}^* b_{mn}^* S_{mn} + L_2 S_{ij} + L_3 (b_{ik}^* S_{jk} + b_{jk}^* S_{ik} - \frac{2}{3} b_{mn}^* S_{mn} \delta_{ij}) + L_4 (b_{ik}^* W_{jk} + b_{jk}^* W_{ik}) = 0
\] (4.9)

Here \(b_{ij}^*\) is the value of the Reynolds stress anisotropy at the stationary state.

Replacing \(b_{ij}^*\) in Eqs. (10) by the polynomial form from Eqs. (8), we get a equation for the coefficients \(L_2, L_3, L_4\) as functions of \(G_1, G_2, G_3\)

\[
L_2 = -2(1-\beta)G_1^2 - 4\beta(1-\beta)G_2^2 + \frac{(1-\beta)^2}{3} G_3^2
\]

\[
L_3 = -(1-\beta)G_3
\]

\[
L_4 = 2(1-\beta)G_2
\] (4.10)

Finally substituting the values of \(G_1, G_2, G_3\) computed in Eqs. (9) into the Eqs. (11), we get the values of \(L_2, L_3, L_4\).

So the steps of the formulation are as follows:

1. Using Direct Numerical Simulations at the rapid distortion limit, we find the stationary values of \(b_{ij}\) at a range of values of \(\beta\).

2. Using these values, \(b_{ij}^*\) in Eqs. (9), we find the values of \(G_1, G_2, G_3\) as functions of \(\beta\).

3. Replacing these values of \(G_1, G_2, G_3\) in Eqs. (11), at the Rapid Distortion limit, we can find the \(L_2, L_3, L_4\) as functions of \(\beta\). These are shown in Figure 2.

4. We use these values of \(L_2, L_3, L_4\) into the calculation of \(b_{ij}\) at lower strain rates as functions of \(\beta\) using equation (5), coupled with models for the slow pressure strain correlation and dissipation. Reynolds and Kassinos (1995) and Mishra and Girimaji (2017) have shown that the models developed and calibrated in rapid distortion limit are useful in modeling the behavior of turbulence away from the rapid distortion limit too.
The calculated values of the model coefficients $L_2, L_3$ and $L_4$ as functions of $\beta$ is shown in Figure 3. The form of our model expression was set to the linear form of the rapid pressure strain correlation model shown in Eqs. (4) and (5). Using this model expression with the functional relationship between $L_2, L_3$ and $L_4$ on $\beta$ finalizes the rapid pressure strain correlation model.

For testing and validation of the model we divide the DNS data available in three parts.

1. Majority of the DNS data at the rapid distortion limit is used as training data to develop the model.

2. A part of the DNS data at the rapid distortion limit is used to validate the new model at the rapid distortion limit, to see if the evolution of $b_{ij}$ is correctly predicted by the present model in the rapid distortion limit.

3. The rest of the data from DNS and experiments (not at the rapid-distortion limit) is used as a testing set to test the performance of the present rapid pressure strain correlation model.

In this paper, we use the popular models of Launder et al. (1975) and Speziale et al. (1991) for comparison of the predictions of the present model.

4.4 Validation of model at the Rapid Distortion Limit

Here we compare the predictions of the present model against DNS data at the rapid distortion limit to validate that the evolution of $b_{ij}$ is correctly predicted by the present model in the rapid distortion limit. We have done these comparisons at many different flows but present only 2 sets of cases for brevity, homogeneous shear mean flow and a set of elliptic mean flow. We have taken care to ensure that these flow cases were not part of the training data used to develop the model coefficients.

In Figure 4 the evolution of the Reynolds stress anisotropies for homogeneous shear mean flow is shown. The RDT simulation is shown in a solid line, the present model predictions in a dashed line. The predictions of the linearized form of the SSG and the LRR model are shown in dash-dot and dotted line. After the initial transient the present model shows improvements over earlier rapid pressure strain models for both the diagonal components of the Reynolds stress anisotropy tensor $b_{11}$ and $b_{22}$. Significant improvement in the model prediction of the non-diagonal component $b_{12}$ is also shown by the present model predictions.

Figure 5 shows the performance of various models at predicting the $b_{22}$ component of the Reynolds stress anisotropy for elliptic flows with $\beta = 0.6, 0.7$ and 0.8. At $\beta = 0.6$ (Figure 5 (a)), the present model and LRR give acceptable predictions for the $b_{22}$
component, while SSG gives unsatisfactory predictions. As \( \beta \) increases to 0.7 (Figure 5 (b)), the predictions of the LRR model become not as accurate as the present model. At \( \beta = 0.8 \) (Figure 5 (c)), the predictions of the present model are significantly better than both LRR and SSG. Figure 5 shows that as the value of \( \beta \) progressively increases (or the influence of mean rotation on the flow increases), the predictions of LRR and SSG become less accurate. The present model consistently gives predictions for the mean of the \( b_{22} \) component that are reasonably accurate and can give useful guidance for engineering purposes.

In Figure 6 we show the performance of various models at predicting the turbulent kinetic energy \( k \) for elliptic flows with \( \beta = 0.6, 0.7 \) and 0.8. At \( \beta = 0.6 \) (Figure 6 (a)), the present model and LRR are able to capture the growth of the turbulent kinetic energy. The SSG model is unable to capture this trend in the turbulent kinetic energy evolution. At \( \beta = 0.7 \) (Figure 6 (b)), the prediction of the present model are significantly more accurate than the LRR model. At \( \beta = 0.8 \) (Figure 6 (c)), the LRR and SSG models do not predict growth in the turbulent kinetic energy. The present model is able to predict the growth in turbulent kinetic energy and the rate of growth predicted by the present model is quiet close to the RDT data. Similar to what was seen for the Reynolds stress anisotropy (Figure 5), even for the turbulent kinetic energy, as the influence of mean rotation on the flow evolution increases the predictions of LRR and SSG become less accurate. The present model consistently gives accurate predictions for the growth of the turbulent kinetic energy and the predicted rate of growth is in reasonable agreement with the RDT data.

4.5 Validation of model in additional flows

Previous investigations like citelee1985, Mishra and Girimaji (2016) and Johansson and Hallbäck (1994) have shown that models developed using the rapid distortion theory (coupled with models for the slow pressure strain and dissipation) can be applied to general flows far from the rapid distortion limit. In this section we compare the predictions of the new model against DNS data for such general turbulent flows. For the return to isotropy model, we use the linear model of Rotta (1951)

\[
\frac{db_{ij}}{dt} = \frac{\eta}{k}(1 - C_R)b_{ij}
\]  

Here the value of the Rotta constant is \( C_R = 1.5 \). The complete forms of LRR and SSG models are used for comparison. We have done these comparisons at many different flows but present a selected numbers of flow cases for brevity. These include plane strain mean flow [Lee and Reynolds (1985b)] and a range of homogeneous turbulent cases under elliptic mean streamline flows simulated by citebns.
In Figure 7, the evolution of the Reynolds stress anisotropy component $b_{11}$ and the turbulent kinetic energy $k$ in a plane strain mean flow with initial $\frac{Sk}{\eta} = 0.5$. The data from the numerical simulation of Lee and Reynolds Lee and Reynolds (1985) is included for comparison. The present model predictions are shown in a solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The predictions of the present model are of comparable accuracy to the models of LRR and SSG. This is to be expected as earlier pressure strain correlation models give satisfactory predictions for strain dominated mean flows.

Next, we consider model performance in homogeneous shear flows. Homogeneous shear flows are benchmark flows for testing turbulence models and are important because of their engineering applications too. In Figure 8 (a) we outline the $b_{12}$ model predictions against the LES results of Isaza and Collins (2009) at $\frac{Sk}{\epsilon} = 27$. The predictions of the present model show significant improvement over the models of LRR and SSG. This is also seen in Figure 8 (b) for the turbulent kinetic energy evolution in the LES simulations of Isaza and Collins (2009) at $\frac{Sk}{\epsilon} = 27$. The present model predictions are capturing the trend of the evolution much better than LRR and SSG. This improvement is also seen for comparison against the DNS of Bardina et al. (1983) plotted in Figure 8 (c). The present model predictions are more accurate as compared to the models of LRR and SSG. The present model is able to capture the trend of turbulent kinetic energy evolution much better. In analyzing the performance of the present model for general homogeneous shear turbulent flows the present model is able to show improvement over the LRR and SSG models for both the Reynolds stress anisotropy and the turbulent kinetic energy evolution. The present model predictions have better agreement with the high fidelity simulations qualitatively and quantitatively.

It is documented that the LRR and SSG models do not give satisfactory performance in many elliptic streamline flows (Speziale et al. (1990), Mishra and Girimaji (2010), Mishra and Girimaji (2013) and Blaisdell and Shariff (1996)).

\[
\frac{\partial U_i}{\partial x_j} = \begin{bmatrix}
0 & 0 & -\gamma - e \\
0 & 0 & 0 \\
\gamma - e & 0 & 0
\end{bmatrix}
\] (4.12)

where $e = \sqrt{\frac{1-\beta}{2}}$ and $\gamma = \sqrt{\frac{\beta}{2}}$. For $e > \gamma$ the mean flow has elliptic streamlines of aspect ratio $E = \sqrt{\gamma + e}(\gamma - e)$. We use this data from 3 simulations with mean flows having aspect ratios E=3.2 and 1.5. The turbulent velocity field is initially isotropic and the initial $\frac{\eta}{Sk} = 0.167$.

The turbulent kinetic energy evolution from DNS is compared to the Reynolds stress models predictions in Figure 9. For case E=1.5 the LLR and SSG models do predict tur-
buent kinetic energy growth but at a rate much smaller than DNS. As the mean rotation effect on the flow increases, the performance of LLR and SSG becomes less satisfactory. For the case E=3 the LLR and SSG models predict turbulent kinetic energy decay but the DNS predicts turbulent kinetic energy growth. For all 3 cases the present model predicts turbulent kinetic energy growth at a rate similar to the DNS. The agreement between the predictions of the present model and the evolution of turbulent kinetic energy from DNS is much improved over earlier models.

The Reynolds stress anisotropy evolution from DNS for the $b_{13}$ component is compared to the Reynolds stress models predictions in Figure 10. The present model captures the equilibrium behavior of the anisotropy component well. The present model shows improvement in agreement with DNS over earlier models. As the mean strain effect on the flow decreases, the anisotropy prediction of LLR and SSG becomes less satisfactory. The present model shows agreement with DNS data for all the 3 cases of elliptic streamline flow.
Fig. 4.1 Anisotropy evolution to a stationary state for (a) plane strain mean flow, (b) elliptic mean flow and (c) axisymmetric expansion mean flow at the rapid distortion limit.
Fig. 4.2 Identification of stationary states for elliptic flows: (a) schematic outlining the methodology, (b) Comparison of methodology with DNS data from Blaisdell and Shariff (1996)
Fig. 4.3 Calculated values of the model coefficients (a) $L_2$, (b) $L_3$ and (c) $L_4$ as functions of $\beta$. 
Fig. 4.4 Evolution of the $b_{ij}$ in homogeneous shear mean flow. RDT evolution is shown by the solid line, new model predictions by the dashed line. LRR and SSG are shown by the dotted and dash-dot lines. (a) $b_{11}$, (b) $b_{22}$ and (c) $b_{12}$. 
Fig. 4.5 Evolution of $b_{22}$ in elliptic mean flows at $\beta = 0.6$, 0.7 and 0.8. RDT evolution is shown by the solid line, new model predictions by the dashed line. LRR and SSG are shown by the dotted and dash-dot lines. (a) $\beta = 0.6$ (b) $\beta = 0.7$ (c) $\beta = 0.8$
Fig. 4.6 Evolution of turbulent kinetic energy $k$ in elliptic mean flows at $\beta = 0.6, 0.7$ and 0.8. RDT evolution is shown by the solid line, new model predictions by the dashed line. LRR and SSG are shown by the dotted and dash-dot lines.
Fig. 4.7 Evolution of a) the Reynolds stress anisotropy $b_{11}$ b) turbulent kinetic energy for plane strain mean flow. The predictions of the present model are shown by the solid line. LRR and SSG are shown by the dotted and dash-dot lines. The data from the direct numerical simulation of Lee and Reynolds (1985) is included for comparison.
Fig. 4.8 Predictions for homogeneous shear flow a) $b_{12}$ evolution for the LES of Isaza and Collins (2009) with $\frac{Sk}{\epsilon} = 27$, b) tke evolution for the LES of Isaza and Collins (2009) with $\frac{Sk}{\epsilon} = 27$, c) tke evolution for the DNS of Bardina et al. (1983).
Fig. 4.9 Turbulent kinetic energy evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff (1996) is included for comparison.
Fig. 4.10 Reynolds stress anisotropy $b_{13}$ evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff (1996) is included for comparison.
4.6 Summary

In this chapter, we develop a new rapid pressure strain correlation model based on representation theory applied to equilibrium states in the Reynolds stress anisotropy evolution at the rapid distortion limit. The present model shows improved agreement with DNS results and significant improvements over these earlier pressure strain correlation models. This improvement is seen in the evolution of the turbulent kinetic energy and in the Reynolds stress anisotropy evolution. We are coding this rapid pressure strain correlation model into commercial CFD solvers to test it for inhomogeneous turbulent flows.
CHAPTER 5

A Pressure Strain Correlation Model Employing Extended Tensor Bases

5.1 General introduction

In academic and industrial applications, most investigations into turbulent flow problems use turbulence models. Turbulence models are simplified relations that express quantities that are difficult to compute in terms of simpler flow parameters. They relate higher-order unknown correlations to lower-order quantities. These unknown correlations represent the actions of viscous dissipation, pressure-velocity interactions, etc. For example pressure strain correlation is a non-local phenomenon and is difficult to compute. Using models for pressure strain correlation, it is expressed as a function of Reynolds stresses, dissipation and mean velocity gradients which are local quantities. This enables us to estimate the pressure strain correlation and its effects on flow evolution in a simpler manner that is computationally inexpensive. Turbulence models are an essential component of all computational fluid dynamics software and are used in almost all simulations into real life fluid flows of engineering importance.

A majority of industrial applications use simple two-equation turbulence models like the $k-\epsilon$ and $k-\omega$ models. However recent emphasis in the scientific community has shifted to Reynolds stress models (Hanjalić and Launder (2011), Durbin (2017), Klifi and Lili (2013), Mishra and Girimaji (2014), Jakirlić and Maduta (2015), Manceau (2015), Eisfeld et al. (2016), Schwarzkopf et al. (2016), Moosaie and Manhart (2016), Lee et al. (2016), Mishra and Girimaji (2017), Sun et al. (2017)). Reynolds stress models have the potential to provide better predictions than turbulent viscosity based models at a computational expense significantly lower than DNS studies. They may be able to model the directional effects of the Reynolds stresses and additional complex interactions in turbulent flows (Johansson and Hallbäck (1994)). They have the ability to accurately model the return to isotropy of decaying turbulence and the behavior of turbulence in the rapid distortion limit (Pope (2000)). The pressure strain correlations are the main building block of the Reynolds stress models. Due to its importance, there have been many attempts to develop closure models for the pressure strain correlation.
Chou (1945) established the formulation for the second moment closure approach and introduced the pressure strain correlation term. Rotta (1951) developed a linear closure for the slow pressure strain correlation term using a modeling expression that was linear in the Reynolds stresses. Launder et al. (1975) developed a model for the complete pressure strain correlation. They developed a novel closure for the rapid pressure term and incorporated the model of Rotta (1951) for the slow pressure strain correlation. Jones and Musonge (1984) attempted to develop pressure strain correlation models that could be applicable for complex recirculating flows. Their model expression was similar to Launder et al. (1975) but the closure coefficients were calibrated to different values determined by the best agreement with their data for high Reynolds number homogeneous flows. Sarkar and Speziale (1990) developed a nonlinear extension for the slow pressure strain correlation for high Reynolds number flows. This model was able to show improved agreement with the non-linear trends in the return to isotropy behavior. This was extended to a fully non-linear quadratic model for the complete pressure strain correlation in Speziale et al. (1991). Johansson and Hallbäck (1994) formulated a non-linear model for the rapid pressure strain correlation with quartic terms. This model showed improved agreement for some homogeneous turbulent flows.

In spite of these modeling developments, there remain deficiencies in the performance of established models for the pressure strain correlation. These deficiencies are two-fold: limitations in accuracy and limitations in realizability.

Established pressure strain correlation models have unsatisfactory accuracy in some important classes of flows. For example in vorticity dominated flows their predictions may not be satisfactory. For these flows linear stability theory, experiments and DNS show growth in the turbulent kinetic energy. However established models predict that turbulence is decaying in these cases Blaisdell and Shariff (1996). Similarly the predictions of available pressure strain correlation models are often inadequate in non-equilibrium turbulent flows, flows with swirl and re-circulation, etc Mishra and Girimaji (2015).

Established pressure strain correlation models suffer from realizability issues. Realizability conditions ensure that the predictions of the turbulence model are consistent with a random stochastic process. The pressure strain correlation models available presently lead to realizability violations at or in the neighborhood of the two-component limit of turbulence. While the two-component limit is termed as a limiting state for the Reynolds stresses, it is found in many engineering flows. For example in near wall turbulence the state of the Reynolds stress tensor is extremely close to the two-component limit with the wall normal component of the Reynolds stresses being negligible. Such realizability violations in important flows limit the applicability of pressure strain correlation models.
Most classical pressure strain correlation models have focused on the closure modeling expression and the values of the closure coefficients to improve the performance of models. With respect to the model expression there has been a trend toward more complex terms that are non-linear in the Reynolds stress tensor \( \text{Speziale et al. (1991)} \). For example while the model of \( \text{Launder et al. (1975)} \) was linear in the Reynolds stress tensor, the model of \( \text{Speziale et al. (1991)} \) is quadratic and the model of \( \text{Johansson and Hallbäck (1994)} \) is quartic. With respect to the closure coefficients, investigations have tried to calibrate them to more specialized data sets from experiments and DNS. Investigators have also made the closure coefficients functions of the invariants of the Reynolds stress tensor. This allows additional degrees of freedom in the modeling expression and enables better agreement with additional data sets. However the improvements due to such steps have been incremental. The central issues of unsatisfactory accuracy in specific important classes of flow or the issues with realizability are still present and important.

Investigators have raised questions about the inadequacy of the modeling basis used in pressure strain correlation closures. The modeling basis is composed of the set of tensors used in the modeled constitutive equation for the pressure strain correlation. In classical one-point closure modeling these are one-point tensors including the Reynolds stress anisotropies, the turbulent kinetic energy and the dissipation. The set of tensors used in the modeling basis determines the type and extent of information about the turbulent flow field that is available in the model formulation. In an incompressible flow pressure is governed by the Poisson equation. Due to the elliptic nature of this governing equation the pressure strain correlation is not a one-point tensor and attempts to model it using one-point tensors may be limited. \( \text{Reynolds and Kassinos (1995)} \) have claimed that in rotation-dominated turbulent flows, the modeling basis for the pressure strain correlation is limited. They introduced additional non-local tensors to the modeling basis like stropholysis, circulicity, etc. \( \text{Cambon et al. (1992)} \) have also claimed that additional tensors may be needed to model the pressure strain correlation in rotation-dominated flows. However both these models use non-local tensors that may not be available in an engineering application. \( \text{Mishra and Girimaji (2010) and Mishra and Girimaji (2013)} \) have carried out a spectral analysis to outline the manner in which the modeling basis is limited and the manner in which it affects the ability of the model to replicate specific features of turbulent flows.

If some of the limitations in the pressure strain correlation models are due to limited modeling basis, there are three important questions to be answered:

1. What tensors need to be added to the modeling basis to have additional information that is relevant for modeling.

2. While many different correlations and turbulent statistics may be added to the
modeling basis and may offer different degrees of benefit, we must identify the optimal tensors to be added.

3. Finally with these added tensors, how much improvement can we show in the performance of single-point pressure strain correlation models.

In this chapter we address these questions in order. Using physics based arguments we outline a set of tensors to be added to the modeling basis for the slow pressure strain correlation and separately for the rapid pressure strain correlation. We show that these tensors add missing information to the modeling effort that is important to improve the potential performance of pressure strain correlation closures. We outline a complete model for the pressure strain correlation using this extended modeling basis.

This model is tested for a range of mean flows while compared to DNS results. In this investigation, we use the popular models of Launder et al. (1975) and Speziale et al. (1991) for comparison. The present model shows improved agreement with DNS results and significant improvements over these earlier pressure strain correlation models.

An important feature of our investigation is the approach used for testing the pressure strain correlation model. Turbulence modeling has a common practice of testing the slow and rapid rapid pressure strain models separately. This is an attractive approach as it allows the investigator to gauge the performance of the rapid and slow models separately and in isolation. It has been shown that this may be an unsound practice especially in uniformly strained turbulent flows Speziale et al. (1992), Lasher and Taulbee (1994), Gatski (1996). This happens because of the ambiguity of the decomposition of the pressure strain correlation into the rapid and slow components for these flow cases. Speziale et al. (1992) have proved that such a procedure can lead to counter-intuitive conclusions being drawn where the model with the superior performance for both the rapid and slow pressure strain correlations in isolation may give the inferior performance for the pressure strain correlation model as a whole for the same testing data set. At present there are many rapid and slow pressure strain correlation models that have been developed and tested only for the limiting cases of turbulence (that is: rapid distortion limit or decaying turbulence). In engineering practice however the behavior in a general turbulent flow is important and often the investigators have to rely of individual models for the rapid and slow pressure strain correlations that might have been developed in separate investigations and have not been tested together as a unit. Based on the analysis of Speziale et al. (1992) this approach may have pitfalls.

Because of the importance of the benchmark case of uniformly strained turbulence, the fact that the final engineering application of the model is for general turbulent flows (and not limiting cases of turbulence) and the observations regarding the unsoundness of testing model components in isolation we perform the testing for this rapid pressure strain correlation model in conjunction with a slow pressure strain correlation model.
The specific slow pressure strain correlation model is chosen as the methodology of its formulation can be outlined in a similar manner as the rapid model where additional tensors are investigated to account for missing physics.

The overall objective of this investigation is to offer a complete pressure strain correlation model (that is the rapid and slow components together) that has been tested as a unit for a range of different turbulent flows and shows improvement over established models. To this end the key contributions of this article include

1. We introduce a physical methodology for turbulence modeling that identifies key information missing in modeling expressions and determines tensors that may be able to provide this information. These are added to the modeling basis to attempt to improve the model.

2. We outline a model for the linear component of the pressure strain correlation that uses such additional tensors based on physical arguments. Similar arguments are outlined for the non-linear model for completeness.

3. This entire pressure strain correlation model is tested as a single unit for multiple examples of homogeneous shear flow, elliptic flow, besides other case. These testing cases are chosen so that they are far off the limiting states of turbulence so as to act as robust tests for the model. Such an approach is shown to be more reliable by other investigations [Speziale et al. (1992), Lasher and Taulbee (1994), Gatski (1996)].

4. The present model shows improved agreement with DNS and LES results. In comparison to the models of LRR and SSG it shows significant improvements.

Using this approach we have attempted to develop and offer a complete model for the pressure strain correlation. The model formulation is clearly outlined with physical rationale for the addition of specific tensors to the basis. The improvements shown by the model may make it an more accurate alternative to popular models. As the components of this model have been tested together extensively this may increase the confidence in its application in a general engineering problem.

5.2 Theoretical and mathematical details

In this section we outline our procedure to select specific tensors to the modeling basis for the pressure strain correlation. During this process, physical arguments for the choice of specific tensors and the particular benefits that they offer, with respect to the modeling of definite features of the pressure velocity interaction term. We demarcate this procedure sequentially, first for the slow pressure strain correlation model and then
for the rapid pressure strain correlation model. During this selection, we try to consider
tensors that are still single point and are available in the engineering single point mod-
eling methodology. Following this selection, we develop the individual slow and rapid
pressure strain correlation models with this expanded modeling basis.

5.2.1 Slow pressure strain correlation modeling basis

Considering the slow pressure strain correlation model, we commence with the details
of the rate of dissipation tensor. The rate of dissipation tensor can be decomposed into
its deviatoric and isotropic components:

$$
\epsilon_{ij} = D_{ij} + \frac{2}{3} \epsilon \delta_{ij}
$$

(5.1)

Here, $\epsilon = \epsilon_{ii}$ and $D_{ij}$ is the deviatoric component of the rate of dissipation tensor.

Traditionally, The deviatoric component of the rate of dissipation tensor is com-
bined with the pressure strain correlation mechanism and the two are modeled together
Lumley and Newman (1977)

$$
\phi_{ij} = D_{ij} + \phi'_{ij}
$$

(5.2)

In flows where the Reynolds number is large dissipation can be assumed to be
isotropic $D_{ij} = 0$. In most Reynolds Stress Modeling investigations this assumption
is adopted and it is assumed that the rate of dissipation tensor is nearly isotropic. For all
practical modeling purposes, $\phi_{ij}$ is the slow pressure strain correlation only. However
recent direct numerical simulation studies suggest that this assumption is inadequate
Kim et al. (1987), Lee and Reynolds (1985a). For example in near wall turbulence this
assumption is unsatisfactory Lee and Reynolds (1985a). In fact Yeung and Brasseur
(1991) have proved that if the large scale structures in a turbulent flow are anisotropic
the small scale turbulent motions will have a significant level of anisotropy. Due to
these arguments the assumption of the isotropy of the rate of dissipation is a significant
shortcoming and causes deficiencies in the slow pressure strain correlation model. To
address the shortcomings due to this assumption we introduce the dissipation anisotropy
tensor ($e_{ij}$) in the modeling basis:

$$
e_{ij} = \frac{\epsilon_{ij}}{\epsilon} - \frac{2}{3} \delta_{ij}
$$

(5.3)

This tensor allows the model to have information about the anisotropy in the rate of
dissipation mechanism and should improve the predictions of the models especially in
the inhomogeneous flows.

A considerable amount of information required for the closure modeling of the
terms in the Reynolds Stress Models is contained in the two-point correlation tensor,
The two-point correlation contains significant information about the dissipation and the pressure strain correlation, both of which can be expressed as functionals of the two-point correlation. The two-point correlation also has important information about the turbulent length scales. As the two-point correlation is non-local it is not used in the single-point modeling basis. This causes another significant shortcoming in classical Reynolds Stress Models that is the assumption of a single integral length scale. This is markedly true in flows where the geometry of the flow domain or body forces lead to a co-ordinate direction in the flow being decidedly preferred. For example axisymmetric expansion and axisymmetric contraction mean flows. In many anisotropic turbulent flows, the characteristic length scale is observed to be varying in different directions [Panda et al. (2017), Breuer and Peters (2005)]. At the most basic level, we must try to include this anisotropy in the length scale in the modeling basis for the pressure strain correlation. We introduce the length scale anisotropy tensor ($l_{ij}$) in the modeling basis and derive it as follows. The length scale tensor ($L_{ij}$) is defined as in equation 3.5:

5.2.2 Rapid pressure strain correlation modeling basis

Considering the rapid (or linear) pressure strain correlation term, we address the level of information used to characterize the state of the turbulent flow field. One of the key shortcomings in the Reynolds Stress Modeling approach to pressure strain correlation closures is the use of only the Reynolds stress tensor to describe the state of the turbulent flow field. This leads to a coarse grained description that limits the potential accuracy of the rapid pressure strain correlation model. [Mishra (2014), Mishra and Girimaji (2010, 2013, 2014, 2015)] have made important insights about the specific limitations due to this level of characterization of the turbulent flow field. They have shown that turbulent flow fields with the same Reynolds stresses can have very different internal structuring and lead to very different evolution [Mishra and Girimaji (2010, 2013)]. Using spectral analysis, they have established a universal evolution (termed the statistically most likely behavior) that is dependent on the mean velocity gradient. This behavior was shown to be highly dependent on the mean velocity gradient of the flow [Mishra and Girimaji (2013, 2014)]. At the primary level, including information about the local mean velocity gradients may be a good substitute for detailed multi-point information about the internal structuring of the turbulent flow field. For information about the mean velocity gradient, we introduce the invariants of the mean velocity gradient in the modeling basis. In this paper we restrict ourselves to planar mean velocity gradients. Here the information about the mean velocity field can be included using the ellipticity...
5.2.3 Integration of additional tensors into model expressions

To integrate these three tensors into the model expression we adopt a practical recourse. For the slow pressure strain correlation, the addition of the tensors requires that the model expression be extended. On experimenting with variants where the coefficients of the closure expression were made functions of $e_{ij}$ and $l_{ij}$, we found the final model to not perform well. However the general expression for the rapid pressure strain correlation closure is retained and the closure coefficients are made functions of these three tensors. On experimenting with variations (where additional terms involving these tensors were included in the model expression) we have found that this does not negatively affect the performance of the new model. Additionally we hope that retaining the established closure expression and only changing the nature of the closure coefficients will encourage the scientific community to incorporate this model into their proprietary codes.

We have outlined the additional tensors to be added to the modeling basis for the pressure strain correlation and the specific reasons for their addition in chapter 3 and 4. The slow and rapid pressure strain correlation models with these additional tensors have been formulated. In the next section we use these two model expressions together to simulate the evolution of general turbulent flows that are far off the limiting states of turbulence. This methodology follows the procedure counseled by Speziale et al. (1992), where they have warned against testing models of the rapid and slow pressure strain correlation in isolation.

During the test cases, the turbulent kinetic energy ($k = \overline{uu''}$) evolves as

$$\frac{dk}{dt} = P - \epsilon$$  \hspace{1cm} (5.5) 

The modeled evolution equation for the dissipation is

$$\frac{d\eta}{dt} = C_{\eta_1} \frac{\eta}{k} P - C_{\eta_2} \frac{\eta^2}{k}$$  \hspace{1cm} (5.6) 

Here the values of the coefficients are taken as $C_{\eta_1} = 1.44$ and $C_{\eta_2} = 1.88$.

5.3 Results and discussion

In this section, the present pressure strain correlation model is tested for a wide variety of general turbulent flows. We ensure that these flows are general in the sense that they
are not at the limiting states of decaying turbulence or the rapid distortion limit. We use the predictions of established models by Launder et al. (1975) and Speziale et al. (1991) as yard sticks to compare the performance of the present model.

In Figure 1 the evolution of Reynolds stress anisotropy and turbulent kinetic energy is shown for plane strain mean flow. The present model predictions are shown in a solid line, the SSG and the LRR model are shown in dash-dot and dotted lines respectively. DNS data from Lee and Reynolds (1985b) is shown using unfilled circles in the figure. The predictions of the present model for both the components of the Reynolds stress anisotropy and the evolution of the turbulent kinetic energy show agreement with the DNS data. The present model is able to show some improvement in comparison to the predictions of popular models like those by Launder et al. (1975) and Speziale et al. (1991).

It is documented that the LRR and SSG models may not give satisfactory performance in many elliptic streamline flows. Blaisdell and Shariff (1996) have simulated homogeneous turbulence subjected to elliptic mean flows:

\[
\frac{\partial U_i}{\partial x_j} = \begin{bmatrix}
0 & 0 & -\gamma - e \\
0 & 0 & 0 \\
\gamma - e & 0 & 0 \\
\end{bmatrix}
\]  

(5.7)

where \( e = \sqrt{1-\beta^2} \) and \( \gamma = \sqrt{\frac{\beta}{2}} \). For \( e > \gamma \) the mean flow has elliptic streamlines of aspect ratio \( E = \sqrt{(\gamma + e)(\gamma - e)} \). We use this data from 3 simulations with mean flows having aspect ratios E=3, 2 and 1.5. The turbulent velocity field is initially isotropic and the initial \( \frac{u^3}{\nu} = 0.167 \).

In Figure 2 and Figure 3 we use the data from the direct numerical simulations of Blaisdell and Shariff (1996) in elliptic streamline mean flows. Figure 2 represents the time evolution of turbulence kinetic energy for elliptic mean flow with different values of aspect ratio. For case E=1.5 in Figure 2 (a) the LLR and SSG models predict turbulent kinetic energy growth but at a rate much lower than the DNS of Blaisdell and Shariff (1996). As the relative strength of mean rotation effect increases, in Figure 2 (b) and (c), the performance of LLR and SSG becomes less satisfactory. For the case E=3 the LLR and SSG models predict turbulent kinetic energy decay but the DNS predicts turbulent kinetic energy growth. For all 3 cases the predictions of the present model are in agreement with the DNS data qualitatively and quantitatively. Unlike LRR and SSG models, the present model predicts growth of turbulent kinetic energy for all three cases of elliptic streamline mean flow. The rate of growth of turbulent kinetic energy predicted by the present model is able to show quantitative agreement with the DNS data also.

In Figure 3, the evolution of Reynolds stress anisotropy (\( b_{13} \) component) is shown.
For all three values of aspect ratio the new model predictions shows improvement agreement with the DNS data of Blaisdell and Shariff (1996). Testing across a variety of elliptic streamline flows seems to suggest that the present model is able to show significant improvement in predictions of both the Reynolds stress anisotropy and the turbulent kinetic energy evolution.

In figure 4 and 5, we perform a exhaustive validation for the case of homogeneous sheared mean flow. This flow case is of great importance theoretically and from the point of view of the engineering applicability of the model. We use the data from Isaza and Collins (2009) where the evolution of the Reynolds stress anisotropies and the turbulent kinetic energy was collected for a range of different shear parameter $S^*$. This is important as it tests the performance of the slow and rapid pressure strain correlation models when used in conjunction with each other. This issue is emphasized in Speziale et al. (1992) where the authors comment that testing the rapid and slow pressure strain correlation models in isolation can lead to unsound and misleading results. Testing the complete pressure strain correlation model, for a range of in this manner acts as an exhaustive test of entire pressure strain correlation model as a unit where the rapid and slow models work in conjunction with each other. We select three specific cases of the shear parameter from Isaza and Collins (2009), a) $S^* = 3$ b) $S^* = 15$ c) $S^* = 27$. At $S^* = 3$, the nonlinear behavior is dominant in the flow physics and the performance depends more on the accuracy of slow pressure strain model. At $S^* = 27$, the linear behavior is dominant and the performance depends more on the accuracy of rapid pressure strain model. Finally, at $S^* = 15$, both linear and non-linear physics are of equal importance in the turbulence evolution. This case tests how well the entire pressure strain correlation model works as a unit. The present model predictions matches well with the DNS data for all three values of the shear parameter. There is a significant improvement over the predictions of both the LLR and SSG models.

In Figure 6, the present model prediction of the evolution of turbulence kinetic energy is compared with the large eddy simulation data of Bardina et al. (1983) for purely sheared flows. The predictions of the present model are in reasonable agreement with the LES data and show accuracy at par with the models of LLR and SSG.

In testing across these flows we find that the present model is able to show some improvements in accuracy for strain dominated flows like multiple examples of homogeneous shear flow Bardina et al. (1983), Isaza and Collins (2009) and plane strain flow Lee and Reynolds (1985b). For rotation dominated flows like those investigated by Blaisdell and Shariff (1996) the present model shows much improvement over the established models of SSG (Speziale et al. (1991)) and LRR (Launder et al. (1975)).
Fig. 5.1 Evolution of a) the Reynolds stress anisotropy $b_{11}$ b) turbulent kinetic energy $k$ for plane strain mean flow. The predictions of the present model are shown by the solid line. SSG and LRR model are shown by the dashed and dash-dot lines. The data from the direct numerical simulation of Lee and Reynolds (1985b) is included for comparison.
Fig. 5.2 Turbulent kinetic energy evolution for elliptic flows a) E=1.5 b) E=2 c) E=3. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff [Blaisdell and Shariff (1996)] is included for comparison.
Fig. 5.3 Reynolds stress anisotropy $b_{13}$ evolution for elliptic flows a) $E=1.5$ b) $E=2$ c) $E=3$. The present model predictions are in the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Blaisdell and Shariff [Blaisdell and Shariff (1996)] is included for comparison.
Fig. 5.4 Turbulence kinetic evolution for purely sheared flows a) $S^*=3$ b) $S^*=15$ c) $S^*=27$. The predictions of the present model are shown by the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Isaza and Collins [Isaza and Collins (2009)] is included for comparison.
Fig. 5.5 Reynolds stress anisotropy $b_{12}$ for purely sheared flows a) $S^*=3$ b) $S^*=15$ c) $S^*=27$. The predictions of the present model are shown by the solid line, the SSG and the LRR model are shown in dash-dot and dotted lines. The data from the direct numerical simulation of Isaza and Collins [Isaza and Collins (2009)] is included for comparison.
Fig. 5.6 Evolution of turbulent kinetic energy for the purely sheared flow. The predictions of the present model are shown by the solid line. SSG and LRR model are shown by the dashed and dash-dot lines. The data from the direct numerical simulation of Bardina et al. Bardina et al. (1983) is included for comparison.
5.4 Summary

It is accepted in the turbulence modeling community that the pressure strain correlation model is a critical component for the success of the Reynolds Stress Modeling approach. Pressure strain correlation models try to capture the effects of the interaction of fluctuating pressure with the fluctuating rate of strain tensor. Such models try to express the effects of the pressure strain correlation using a tensor basis of local tensors like Reynolds stresses, dissipation and mean velocity gradients. The physics that the pressure strain correlation model tries to capture is non-local due to the non-local nature of pressure. Using a limited set of local tensors to capture this physics leads to limitations in model performance. In this investigation we extend the tensor basis used for pressure strain correlation modeling. This set of additional tensors sequentially justified based on physics based arguments. We formulate a model using this extended modeling basis. The present model is tested for a wide variety of turbulent flows and contrasted against the predictions of other popular models like those by Launder et al. (1975) and Speziale et al. (1991). It is shown that the new model provides significant improvement in predictive accuracy. We are currently testing this pressure strain correlation model for inhomogeneous turbulent flows where the effects of boundaries and walls are important. This chapter aims to communicate the promising performance of this model in homogeneous turbulent flows to the turbulence modeling community.
CHAPTER 6

Experimental and Numerical Analysis of Grid Generated Turbulence With and Without Mean Strain

6.1 Introduction

At present there are no analytical solutions to predict the evolution of complex engineering turbulent flows. Studies of turbulence have to use turbulence models that characterize the statistical evolution of turbulence. Industrial studies use simple eddy viscosity based turbulence models like the $k - \epsilon$ and $k - \omega$ models. Recent emphasis in the scientific research community has shifted to Reynolds stress models [Gerolymos et al. (2010), Klifi and Lili (2013), Mishra and Girimaji (2014), Jakirić and Maduta (2015), Manceau (2015), Moosaie and Manhart (2016), Sun et al. (2017), Bois (2017)].

Reynolds stress models have the potential to give better predictions than turbulent viscosity based models at a reasonable computational expense. They may be able to model the directional effects of Reynolds stresses and complex interactions in turbulent flows [Mishra and Girimaji (2013), Hanjalić and Launder (2011)]. They have the potential to accurately model the return to isotropy of decaying turbulence and evolution in the rapid distortion limit [Mishra and Girimaji (2017), Durbin (2017)]. Reynolds stress models are used to develop improved simplified eddy viscosity based $k - \epsilon - \alpha$ models for variable density flows [Schwarzkopf et al. (2016)], better algebraic closures and more accurate sub-grid scale models.

The fruition of this potential of Reynolds stress models depends on the quality of the closures for the individual turbulence processes in the Reynolds Stress Modeling approach. Along with progress in modeling, this requires accurate, varied and detailed data from experimental investigations. Experimental studies have a symbiotic relationship with turbulence modeling. Data from such experiments can guide the development and testing of models. For example the experiments of [Lumley and Newman (1977)] pointed to a non-linear return to isotropy phenomenon in decaying turbulence. This led to the formulation of advanced slow pressure strain correlation models like [Sarkar and Speziale (1990)]. The shortcomings in models also guide the organization of new experiments. For example the drawbacks of turbulence models in rotation dominated mean
flows led to the investigations of Blaisdell and Shariff (1996), Bardina et al. (1985). While established models are available for the evolution of turbulence processes there remain many questions about the model expressions and the closure coefficients. For example the closure coefficient values used in the rate of dissipation evolution model are varied between different studies in literature. Most of these studies use closure coefficient values that are well outside the range established by theoretical guidelines and experimental investigations. Similarly the form of the model expression used in pressure strain correlation models is also not universally accepted. The model of Rotta (1951) is linear in the Reynolds stress anisotropies, but the model of Sarkar and Speziale (1990) is non-linear with coefficients that are functions of the Reynolds stress invariants. While the models of Rotta (1951) and Sarkar and Speziale (1990) use a modeling basis consisting of the Reynolds stress anisotropies, the model of Panda et al. (2017) uses additional tensors in the modeling expression. Using the experimental data from this study we evaluate these variabilities and make recommendations for improvement.

In this investigation we study the canonical cases of Homogeneous Isotropic Turbulence (HIT) and Homogeneous Anisotropic Turbulence (HAT). HIT conditions are well replicated in experimental grid generated turbulence. Such tests are conducted in wind tunnels or water tanks where the grid is placed at the beginning of the test section. The rods in the grid interact with the flow through them leading to wakes. Just downstream of the grid, the wakes from individual rods interact with each other producing turbulence. If there is no externally imposed forcing downstream of the grid this turbulent kinetic energy is viscously dissipated at small scales leading to a decay in the velocity fluctuations. This turbulent velocity field becomes statistically isotropic at a distance of the order of 10-20 mesh lengths from the grid Lumley and Newman (1977). Beyond this length this turbulent flow is statistically stationary with variation along the stream wise direction as the turbulence decays. The rate of energy decay is approximately equal to the viscous dissipation rate. Many authors have explored grid generated turbulence Warhaft and Lumley (1978), Le Penven et al. (1985), Comte-Bellot and Corrsin (1966), Choi and Lumley (2001). In addition to the insight into the decay of turbulence such studies provide data for benchmarking and calibrating turbulence models.

HAT conditions are imposed by using passage of the turbulent flow through an area change in the flow duct. Axisymmetric contraction increases the turbulent velocity fluctuations along the transverse directions. Choi and Lumley (2001) have studied wind-tunnel turbulence experimentally and explored plane distortion, axisymmetric expansion and contraction to introduce anisotropy in grid turbulence. Murzyn and Bélorgey (2005) investigated the grid generated turbulence experimentally using a water tank and have studied the evolution of turbulence kinetic energy, dissipation rate and other flow parameters. Ayyalasomayajula and Warhaft (2006) experimentally investigated grid-
generated turbulence subjected to axisymmetric strain and indicated that single-point turbulence models may not be adequate to describe the relaxation of the turbulence towards an isotropic state. In a very important investigation Torrano et al. (2015) studied the properties of turbulence downstream of a passive grid. This experimental data was used to evaluate the accuracy of eddy viscosity models and suggest optimal values of model coefficients. Such studies provide essential guidance for the limitations and development of improved turbulence models. In spite of these investigations few researchers have investigated the detailed evolution of Reynolds stress anisotropies near the grid at a large range of grid Reynolds numbers.

The contribution of this chapter are twofold: firstly concerning the experimental investigation carried out and secondly regarding the utilization of this experimental data to aid numerical investigations of grid generated turbulence.

In the experimental facet of this investigation we study the evolution of the anisotropy in the Reynolds stress tensor near the grid at multiple Reynolds numbers. This study is carried out both for cases with and without mean strain to provide a comprehensive overview of the anisotropy behavior. This represents one of the novel contributions of this paper. While grid generated turbulence is one of the simplest and best approximations of isotropic turbulence, it is accepted that grid generated turbulence is not exactly isotropic. For example it is known that longitudinal velocity fluctuations are more energetic than the lateral in such cases. This leads to many important hurdles in our understanding of turbulence and specifically the return to isotropy phenomenon. For example the existence of anisotropic structures in the flow may slow down the return to isotropy at different scales of flow. Numerous investigations (for example Grant and Nisbet (1957) and Ertunç et al. (2010)) have stated that to develop a detailed understanding of the differences between grid generated turbulence experimental realizations and the ideal case of perfectly isotropic turbulence, we need to study the anisotropy in the Reynolds stresses specifically in the region near the grid under different conditions. This necessity is addressed by the experimental work conducted in this investigation. In addition to the anisotropy, the evolution of the turbulent kinetic energy and the Reynolds stress components are plotted downstream of the grid for giving a comprehensive picture of turbulence structure near the grid.

In the numerical analysis, we use this experimental data to analyze the closure coefficients of the rate of dissipation model and the pressure strain correlation models used in Reynolds Stress Modeling simulations. This addresses important doubts in the turbulence community regarding the validity of the use of RANS models to simulate grid generated turbulence. Grid generated turbulence experiments represent a cornerstone for fundamental investigations into turbulence. Study of the decay of turbulent fluctuation in such cases represents a classical methodology to develop concepts and
theories on the kinematics and dynamics of turbulent flows. Till very recent developments wherein turbulence could be generated in a periodic cube via direct numerical simulations (for instance in Laizet and Vassilicos (2011) and Djenidi (2006)), such experimental studies represented the only recourse for studying isotropic turbulence.

However experimental studies have many limitations with respect to the possible information that can be gathered and the fidelity of such measurements. Due to experimental limitations, the entire three-dimensional structure cannot be studied using experiments. Similarly higher order spectral quantities cannot be reliably recreated using experimental methods. These limitations have lead to the rise of numerical studies of grid generated turbulence. The recourse offer the highest fidelity is Direct Numerical Simulations. For example Nagata et al. (2008) have conducted detailed dns studies of turbulent flows generated by different grids. However because of the high computational cost such numerical investigations are limited to small Reynolds numbers and for very short times. This limits such investigations from providing a complete picture of the time evolution of such turbulent flows. Due to this recent investigators have started to use Reynolds Averaged Navier Stokes models to simulate the decay of grid generated turbulence. These RANS models can handle varying initial conditions, complex geometries, different types of grid and arrangements and simulations for long time periods. However owing to limitations in the accuracy of RANS models there are questions in the scientific community about the validity of such simulations to account for fundamental turbulence features in such simulations. Most RANS models are calibrated for simple homogeneous flows and can lead to significant discrepancies for grid generated flows.

Thus a significant hurdle in gaining confidence in studies using RANS models for grid generated turbulence is calibration of the closure model coefficients with relevant datasets. A major objective of this study is to use the data set generated by our experiments to tune to closure coefficients and find the optimal values of these parameters, specifically for different cases of grid generated turbulence. The closure coefficients of the rate of dissipation model and the pressure strain correlation models used in Reynolds Stress Modeling simulations are analyzed. Based on this analysis we make recommendations for optimal values of the coefficients for studies of grid-generated turbulence.

6.2 Experimental and modeling details

The experiments for this paper were conducted in the recirculating water tank at the department of Ocean Engineering and Naval Architecture, IIT Kharagpur. Side walls of the water tank are made up of glass. The schematic of the experimental apparatus is shown in figure 6.1. The water is recirculated by a pump, the rpm of the pump is controlled by an electrical control unit. x, y and z are the main flow(streamwise), transverse...
and vertical directions respectively. \( U, V \) and \( W \) are the corresponding mean velocities and respective small letters indicate fluctuating velocities. A mean flow velocity of \( 1 \text{ m/s} \) is achievable for a water depth of 0.8 meter in the main flow direction. The water tank has width 2 meters and depth 1.5 meter. The grids were placed immediately preceding the test section through a grid holder (at the grid position, \( x=0 \)). The depth of water was 0.8 meter for all the cases of the experiments. Turbulence was generated by using a grid made up of cylindrical pipes. The diameter \( (d_b) \) of the pipes used was 0.025 meter. The mesh length of the grids \( (M) \) was 10 cm. The rigidity of the grid was calculated as 0.43 by using equation (1) as described in \cite{Comte-Bellot and Corrsin 1966}:

\[
\sigma = \frac{d_b}{M}(2 - \frac{d_b}{M})
\]  

(6.1)

Reynolds number based on the grid mesh size \cite{Nagata et al. 2008b} is calculated as:

\[
Re_M = \frac{UM}{\nu},
\]  

(6.2)

here \( M \) is the mesh size, \( U \) is the streamwise mean velocity and \( \nu \) is the kinematic viscosity of water. Experiments were conducted at three different grid Reynolds numbers, \( Re_M = 25000, 32000 \) and 39000.

Fig. 6.1 Schematic of the recirculating water tank and the wedge, all dimensions are in meter, the width of the wedge is same as width of the water tank.

A Wedge was used downstream of the grid for contracting the flow at a distance of 0.6 meter downstream of the grid. The detailed dimensions of the wedges is shown in
An Acoustic Doppler Velocimeter (NORTEK Vectrino) is used in our experiment to measure instantaneous velocity components at different downstream locations of the grid, which is mostly suitable for use in hydraulic models and laboratory flumes with sampling rate up to 200 Hz. The ADV was fixed for 5 minutes at each location. This is sufficient to obtain stable and converged results as reported in literature Voulgaris and Trowbridge (1998). The ADV used in our experiments has spatial and temporal resolution of $1\text{cm}^3$ and $200\text{Hz}$ respectively.

An ADV measures three-dimensional flow velocities using Doppler shift principle. Flow velocities can be measured at high sampling rates with small sampling volume with such systems. Main components of the instrument are a sound emitter, three sound receivers and a signal conditioning electronic module. The schematic of the ADV probe and signal conditioning module is shown in figure 6.2. An acoustic signal is generated by the sound emitter and is reflected back by the sound scattering particles, which are supposed to move at the waters velocity. The flow velocity in the beam or radial directions is calculated form the Doppler phase shift which is computed from the scattered sound signal detected by the receivers. More information on velocimeter operation is available in McLelland and Nicholas (2000), García et al. (2007).

The ADV system uses different pulse repetition rates separated by a dwell time which can be adjusted by changing the velocity range of the measurement. Each pulse is a square shaped pulse train of an acoustic signal having different frequencies. The phase shift can be calculated from the cross- and auto correlation computed for each pulse pair using pulse-to-pulse coherent Doppler techniques Lhermitte and Serafin (1984). The velocity ($\eta_i$) can be calculated from the phase shift using Doppler principle:

$$\eta_i = \frac{C}{4\pi f_{ADV}} \frac{d\phi}{dt} \tag{6.3}$$

Where, $C$ is speed of sound, $f_{ADV}$ is the sound signal frequency, $d\phi$ is the change in signal phase and $dt$ is the change in time.

The measurement errors are intrinsic to the ADV system. The probe orientation, instrument velocity range, sampling frequency and local flow properties mainly control...
the errors in the system. The three main sources of measurement errors in the ADV system are the sampling errors, noise and velocity gradients in the sampling volume. The ADV mean flow velocities are accurate within one percent, the ADV used in this experiment has a maximum obtainable sampling frequency of 200 Hz, with such high frequency, accuracy of 0.5 percent of measured value ±1 mm/s can be achieved for the water velocity.

The errors associated with Reynolds stress are within 1% of the estimated true value as reported in Voulgaris and Trowbridge (1998) for a sampling frequency of 25 Hz. Those were obtained by comparing Semi empirical model predictions of open channel flow statistics with measured values of Reynolds stresses.

6.2.1 Data Analysis

The data collected from the ADV were decomposed into mean and fluctuating velocities in all the three directions. The streamwise mean($U$) and fluctuating($u$) velocities can be calculated from the following formula:

\[ U = 1/n \sum_{i=1}^{n} U_i \]  

\[ u = \sqrt{1/n \sum_{i=1}^{n} (U_i - U)^2} \]

The mean and fluctuating velocities in other two directions were calculated by using similar formulae.

Since turbulence is considered as eddying motion of fluid, secondary stresses appear in the fluid and those stresses are known as Reynolds stresses, which is a second order tensor having nine components, out of which six are independent. Diagonal components are called as normal stresses and the off diagonal components are called as shear stresses. The turbulent kinetic energy is defined as $k = \frac{1}{2}(u^2 + v^2 + w^2)$, which is the mean kinetic energy per unit mass in the fluctuating velocity field.

6.3 Experimental Results

In the experimental results we report the Reynolds stress anisotropies and the decay of the turbulence kinetic energy downstream of the mesh for a range of different Reynolds numbers. In figure 6.3, we show the evolution of the streamwise mean velocity $U$ and the fluctuating velocity $u_{rms}$ downstream of the mesh for $Re_M = 25000$. There is a gradual increase in the streamwise mean velocity due to the development of the boundary layers along the configuration walls till it reaches its maximum value. The power law decay
The pressure distribution is expressed in terms of the pressure coefficient $C_p$. For an incompressible flow $C_p = \frac{P - P_{ref}}{\frac{1}{2} \rho U_{ref}^2}$, where $P$ is the local static pressure, $P_{ref}$ is the static pressure at the beginning of the contraction, the local free stream velocity is $U_{\infty}$, and $U_{ref}$ is the reference free stream velocity at the beginning of the contraction. In figure 6.4 the evolution of the pressure coefficient along the stream wise direction is presented, since the contraction used in the experiment produces a favorable pressure gradient, a sharp decrease in pressure coefficient is observed along the stream wise direction. The downstream distance is normalized with respect to the step height of the contraction and $x/M = 0$ represents the beginning of the contraction.

In figure 6.5 the turbulence kinetic energy evolution for three different grid Reynolds number is presented. It is observed in the absence of mean strain there is a sharp decay of the fluctuating velocity is shown in the second sub-figure. The exponent of decay for this case was calculated from the experimental data to be 1.35.
Fig. 6.4 Distribution of pressure coefficient along the streamwise direction for $Re_M = 25000$, the reference pressure was taken at the beginning of the contraction, the downstream distance is non-dimensionalized by the wedge step height.

of turbulence kinetic energy and with increase in grid Reynolds number turbulence kinetic energy increases. However in presence of mean strain, an increase in turbulence kinetic energy just after the contraction is observed. Because of the favorable pressure gradient the turbulence kinetic energy near the grid increased five times on an average at all three grid Reynolds numbers.

In figure 6.6 and 6.7 the evolution of normal components of Reynolds stresses are shown. It is observed that the imposed strain has no effect on the distribution of Reynolds stresses along the transverse direction, but in longitudinal direction, there is an increase in Reynolds stress components towards the end of the contraction.

The comparison of the evolution of Reynolds stresses at $Re_M = 32,000$ is shown in figure 6.8. It is shown in the figure that the imposed strain enhances the magnitude of Reynolds stress only in the longitudinal direction.

The downstream evolution of Reynolds stress anisotropies for grid with contraction case is shown in figure 6.9. The imposed contraction leads to increase in the anisotropy of the turbulent flow field.

6.4 Numerical results and analysis

6.4.1 Analysis of the rate of dissipation model

Experimental investigations of the decay of grid generated turbulence are essentially important, but there is a marked trend to investigate such flow cases using numerical
simulations. Numerical simulations can provide idealized conditions for the experiment (that may only be met approximately in a real experiment). Numerical simulations also provide a large amount of detailed data which is free of measurement errors. Most such numerical simulations into the decay of turbulence use the Reynolds stress modeling approach.

An important shortcoming for the Reynolds Stress Modeling approach is the approximate nature of the rate of dissipation equation. While the model equations for the evolution of the Reynolds stress anisotropy components are exact and based on the Reynolds stress transport equation the evolution equation for the rate of dissipation is empirically derived \( \text{Pope (2000)} \). This model expression is

\[
\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_i} \left( \nu + \nu_t \right) \frac{\partial \epsilon}{\partial x_i} + C_{\epsilon 1} \frac{P\epsilon}{k} - C_{\epsilon 2} \frac{\epsilon^2}{k} \tag{6.6}
\]

The first term on the right hand side represents the diffusive transport of \( \epsilon \). The second and third terms on the right side represent the generation of \( \epsilon \) due to vortex stretching and the destruction of \( \epsilon \) by viscous action. The standard values for the closure coefficients are given by \( \sigma_\epsilon = 1.3, C_{\epsilon 1} = 1.44 \) and \( C_{\epsilon 1} = 1.92 \), based on the constants determined by \text{Launder and Spalding (1974)}. The value of the \( C_{\epsilon 2} \) coefficient is calibrated to be in agreement with the power law decay observed in decaying turbulence. Here the decay exponent corresponds to the power law decay observed as \( k(t) = k(t_0)(t/t_0)^{-n} \) and \( \epsilon(t) = \epsilon(t_0)(t/t_0)^{-n-1} \). In terms of the decay exponent \( n \) this is given by

\[
n = \frac{1}{C_{\epsilon 2} - 1} \tag{6.7}
\]

Most experimental investigations have found the decay exponent to lie in the range of \( 1.15 - 1.45 \). This obligates the value of \( C_{\epsilon 2} \) to approximately lie in the range \( 1.69 - 1.87 \). However the values used in different models often lies well outside this bound. Based on \text{Batchelor and Townsend (1948), Hanjalic and Launder (1972)} chose \( C_{\epsilon 2} = 2.0 \) to make the turbulent kinetic energy vary inversely with distance from the origin. Both the investigations of \text{Launder et al. (1975)} and \text{Sarkar and Speziale (1990)} changed it to \( C_{\epsilon 2} = 1.9 \) so as to get faster rate of decay for their model simulations. \text{Speziale et al. (1991)} chose to adopt the value of \( C_{\epsilon 2} = 1.92 \) for better calibration of their model. Since then, different modeling investigations have used different values for the coefficient ranging from 1.90 to 2.0. All these chosen values lie outside the range prescribed by experimental investigations and are often varying from one numerical investigation to another. In this context the value of the \( C_{\epsilon 2} \) is important to ensure correct simulations and its determination may represent a hurdle for the Reynolds stress modeling approach.

The values of the \( C_{\epsilon 1} \) is chosen to match the steady state parameters in homoge-
neous turbulent shear flow. The form is given by \( \frac{P}{\epsilon} = \frac{C_{\epsilon 2}}{C_{\epsilon 1}} \). It can be seen from this relationship that the choice of the value of the coefficient \( C_{\epsilon 2} \) also in turn affects the value of the \( C_{\epsilon 1} \) coefficient. Any errors in the values of \( C_{\epsilon 2} \) will have a cascading effect and will affect the accuracy of the entire model.

In this section, we vary the value of \( C_{\epsilon 2} \) while using different established Reynolds Stress Models to find the optimal value for this coefficient. The values of \( C_{\epsilon 1} \) and \( \sigma_{\epsilon} \) (\( \sigma_{\epsilon} = \frac{\kappa^2}{\sqrt{C_{\epsilon 1}(C_{\epsilon 2} - C_{\epsilon 1})}} \)) where the Von Karman constant is \( \kappa = 0.41 \)) are determined by their relationship with \( C_{\epsilon 2} \). While we have done this investigation for a large number of \( C_{\epsilon 2} \) values, we show the results for four values \( C_{\epsilon 2} = 1.75, 1.80, 1.85, 1.90 \). Only the first three values are in the range allowed by experimental data. Mesh independence studies were carried out for this case and are reported in Panda et al. (2017).

In figure 6.11 we show the the decay of turbulent kinetic energy downstream of the mesh predicted by the model of Speziale et al. (1991). The rate of decay of turbulent kinetic energy is captured well by the model predictions. At \( C_{\epsilon 2} = 1.75 \) (corresponding to the decay exponent calculated from experimental data) the model over predicts the value of the turbulent kinetic energy. Increasing the value of \( C_{\epsilon 2} \) leads to improvement in the prediction of the turbulent kinetic energy downstream of the mesh. At \( C_{\epsilon 2} = 1.90 \) we get the best agreement with the experimental data. This value of \( C_{\epsilon 2} \) is not in agreement with the decay exponent calculated from experimental data and is outside the range prescribed by experimental investigations in literature.

In figure 6.12 we show the the decay of turbulent kinetic energy downstream of the mesh predicted by the model of Launder et al. (1975). In this case we see that at \( C_{\epsilon 2} = 1.90 \) the dissipation downstream of the mesh is over-predicted. At \( C_{\epsilon 2} = 1.85 \) the agreement between the experimental data and the model predictions is much better.

Such variability in the coefficient values for turbulence models arises due to the empirical nature of the modeling expression and the modeling methodology where the coefficient values are calibrated using limited data from select experiments. To highlight such variability Cheung et al. (2011) have developed different models calibrated against experimental data that show significant variation in the values of the coefficients. Edeling, Cinnella, Dwight and Bijl (2014) and Edeling, Cinnella and Dwight (2014) have studied the parameter variability across flows for the \( k - \epsilon \) model. With respect to Reynolds stress modeling Mishra et al. (2015) and Mishra et al. (2016) have shown that in homogeneous turbulence there can be significant variation in turbulence evolution if the Reynolds stresses are assumed to completely describe the state of the turbulent flow field.

This disagreement between theoretical analysis with experimental data and the numerical results against experimental data observed in this paper may be arising due to the empirical nature of the rate of dissipation evolution equation. In this case we did
not vary the coefficient values of the model of Speziale et al. (1991) and Launder et al. (1975). These model coefficients are calibrated for homogeneous shear flows. Variation in these values may be advantageous because it would sample from all the degrees of freedom in the system of equations.

From this analysis it is clear that the optimal values of the closure coefficients of the rate of dissipation equation depend on the flow being simulated and also on the pressure strain correlation model used. This represents a hurdle in numerical simulations as the rate of dissipation equation should be ideally independent of other models. As a compromise, we recommend the value of $C_{\epsilon 2} = 1.87$. This determines the values of $C_{\epsilon 1} = 1.42$ and $\sigma_\epsilon = 1.25$. For these values both the models of Speziale et al. (1991) or Launder et al. (1975) give acceptable agreement with experimental data. This value is also contained inside the range recommended by most experimental investigations in literature.

6.4.2 Analysis of the Pressure strain correlation model

In our analysis of the pressure strain correlation model we focus on the very popular LRR model of Launder et al. (1975). The form of this model is given by

$$
\phi_{ij} = -(C_1^0 + C_1^1 P) b_{ij} + C_2 k S_{ij} + C_3 k (b_{ik} S_{jk} + b_{jk} S_{ik} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij}) + C_4 k (b_{ik} W_{jk} + b_{jk} W_{ik})
$$

(6.8)

The closure coefficients are given as $C_1^0 = 3$, $C_1^1 = 0$, $C_2 = 0.8$, $C_3 = 1.75$ and $C_4 = 1.31$. This model can be thought of as the summation of the slow pressure strain correlation model of Rotta (1951) with a rapid pressure strain correlation model developed by Launder et al. (1975).

Inspite of its popularity and widespread use there are many questions raised in literature regarding the values of its coefficients. The closure coefficient values of Launder et al. (1975) were calibrated using experimental data using simple turbulent flows. In Launder et al. (1975) the experiments used were restricted to the low shear experiments. Different investigators like Jones and Musonge (1984) used data from other turbulent flows to re-calibrate the coefficients of this model and determined different values of the closure coefficients. Mishra and Girimaji (2010) have analyzed this form of the pressure strain correlation model expression and have recommended that the closure coefficients be explicit function of the mean rate of strain and mean rate of rotation tensors. This would make the values of the closure coefficients vary across different flows for the same model.

In our analysis of the pressure strain correlation model of Launder et al. (1975) we analyze the model for the slow pressure strain correlation in isolation first. This is given
by $\phi_{ij} = -C_1^0 \epsilon b_{ij}$ and is equal to the return to isotropy model of Rotta (1951). For assessment of this pressure strain correlation models, the downstream distance relative to the start point of contraction was measured by the transit time of the turbulence advection from the beginning of the contraction to a given stream wise position, $x$ Hearst and Lavoie (2014):

$$t = \int_{0}^{x} \frac{1}{U(x)} \, dx$$

(6.9)

where $x$ is the dummy integration variable and $U(x)$ is the local mean velocity at a position $z$. The experimentally calculated value for the initial value of the $S^* = \frac{Sk_0}{\epsilon_0} = 1.43$ is used for the simulations.

As can be seen in figure 6.13 there is very good agreement between experimental data and model prediction when the linear interactions between the mean velocity field and the fluctuating velocity field are absent. This indicates that the slow pressure strain correlation model is adequate for simulation of grid generated decaying turbulence.

Considering the rapid pressure strain correlation model given by the form

$$\phi^R_{ij} = C_2 k S_{ij} + C_3 k (b_{ik} S_{jk} + b_{jk} S_{ik} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij}) + C_4 k (b_{ik} W_{jk} + b_{jk} W_{ik})$$

(6.10)

There are 3 closure coefficients representing three potential degrees of freedom. However the value of the $C_2$ coefficient is fixed by the analytical Crow Constraint Crow (1968). The other two coefficients are related to each other as $\frac{C_3}{6} + \frac{3C_4}{7} = \frac{8}{7}$ to maintain symmetry conditions on the $M_{ijkl}$ tensor Pope (2000). Because of this there is just one degree of freedom in the coefficient values of the rapid pressure strain correlation model of Launder et al. (1975). We choose to vary this degree of freedom to explore the optimal value for grid generated turbulence with and without the effects of mean straining.

In figure 6.14 we show the predictions of LRR model variants for the evolution of turbulent kinetic energy. The model coefficients can be expressed in terms of a closure coefficient of the $M_{ijkl}$ tensor, $\alpha$ as $C_3 = -6\alpha$ and $C_4 = \frac{2}{3} (4 + 7\alpha)$. We vary the value of $\alpha$ from $-0.29$ (corresponding to the model coefficient values in Launder et al. (1975)) to $-0.45$. As can be seen in the figure the decrease in value of $\alpha$ leads to improved predictions till $\alpha = -0.42$. Based on our analysis we recommend the value of $\alpha = -0.42$ (or $C_3 = 2.52, C_4 = 0.71$) for investigations grid generated turbulence undergoing mean straining effects.
Fig. 6.5 Downstream evolution of turbulence kinetic energy, a) without mean strain b) with mean strain, solid lines corresponds to $Re_M$ of 39000, dashed lines 32000 and dashed dot lines 25000.
Fig. 6.6 Downstream evolution of Reynolds stress, a) without mean strain b) with mean strain, solid lines corresponds to $Re_M$ of 39000, dashed lines 32000 and dashed dot lines 25000.
Fig. 6.7 Downstream evolution of Reynolds stress, a) without mean strain b) with mean strain, solid lines corresponds to $Re_M$ of 39000, dashed lines 32000 and dashed dot lines 25000.
Fig. 6.8 Effect of mean strain on free stream turbulence, a) represents evolution of free stream turbulence without mean strain b) with mean strain. solid line represents $R_{11}$, dashed lines $R_{22}$ and dashed dot lines $R_{33}$ corresponding to $Re_M$ of 32000.
Fig. 6.9 $b_{ij}$ evolution under mean strain at $Re_M = a)32000$ b)39000. Solid lines show the longitudinal component, dashed lines the transverse normal component.
Fig. 6.10 Contrasting the relationship between $C_{ε2}$ and $n$ based on experimental studies (solid black line) and the values used in Reynolds Stress Modeling investigations.

Fig. 6.11 Turbulent kinetic energy downstream of the mesh and the decay predicted by the model [Speziale et al. (1991)] with different values of the coefficients for the rate of dissipation equation.
Fig. 6.12 Turbulent kinetic energy downstream of the mesh and the decay predicted by the model of Launder et al. (1975) with different values of the coefficients for the rate of dissipation equation.

Fig. 6.13 Turbulence kinetic energy evolution for decaying grid generated turbulence, $S^* = 1.43$. Predictions of Rotta (1951) are shown by solid line, results at $Re_M = 25000$ by unfilled circles.
Fig. 6.14 Turbulence kinetic evolution for grid generated turbulence under axisymmetric contraction. Results at $Re_M = 25000$ are shown as unfilled circles.
6.5 Summary

The focus of this investigation is grid generated turbulence under a variety of different conditions. We carried out experimental and numerical analysis of grid generated turbulence with and without mean strain for a range of Reynolds numbers. In the experimental phase data of turbulence statistics including Reynolds stress anisotropies is collected and analyzed for grid generated turbulence. The experimental data was used to calibrate the coefficients of the rate of dissipation model and the pressure strain correlation models used in Reynolds Stress Modeling. For both models we recommend values of coefficients that should be used for experimental studies of grid generated turbulence.

Some of the main results of this investigation include the details of the experimental decay of turbulence and the numerical calibration of model coefficients. For all the Reynolds numbers that we conducted experiments the decay of turbulent kinetic energy is mainly concentrated in the near grid region \((x/M < 20)\). The rate of decay of the turbulent kinetic energy is more rapid at higher Reynolds numbers. In the absence of any mean strain the decay of the individual components of the Reynolds stress tensor are uniform. This is due to the conditions of the experiment representing a good approximation of isotropy. Application of mean strain leads to changes in the Reynolds stress anisotropy. However this is predominantly focused in the longitudinal component of the Reynolds stresses. The transverse component of the Reynolds stress tensor is relatively unaffected. This causes the longitudinal component of the Reynolds stress anisotropy to dominate the transverse component.

In the application of this experimental data to calibrate the closure coefficients of the the rate of dissipation model and the pressure strain correlation model shortcomings of these models were observed. It was exhibited that the values of the closure coefficients for the rate of dissipation model may not be optimal for grid generated decaying turbulence simulations. Further it was observed that there are inconsistencies between the optimal values of these closure coefficients based on calibration with experimental data and the theoretical bounds on the coefficients. This may arise due to the empirical nature of the modeled evolution equation for the rate of dissipation equation. As a best possible compromise we recommend closure coefficient values of \(C_{\epsilon 2} = 1.87\), \(C_{\epsilon 1} = 1.42\) and \(\sigma_{\epsilon} = 1.25\). A similar analysis was carried out for the pressure strain correlation closure models and we recommend closure coefficient values of \(C_3 = 2.52\), \(C_4 = 0.71\).

It is re-iterated that these recommendations are limited to CFD studies replicating grid generated turbulence in the presence and absence of mean strain and should not be extrapolated to general turbulent flows. Further there may be additional dependencies of the decay of the Reynolds stress anisotropies and also the closure coefficient values on additional parameters of grid generated turbulence experiments such as \(\frac{S_k}{\tau}, Re_M\).
$S$, $W$ and different initial conditions for the turbulent velocity field. Additionally the nature of the grid used may also lead to variations.

In ensuing future work we are using active grids to generate more varied data sets that cover a wider range of such parameters. This data will be used to generate probability distribution functions for the values of the closure coefficients that may be useful for Bayesian investigations into the variability of the values of these model coefficients.
CHAPTER 7

Closure

7.1 Summary

The field of turbulence modeling provides a topic of intense research activity in the field of fluid flow simulations. Although in industrial applications the eddy viscosity models are widely used, the popularity of Reynolds stress models are growing extensively because of the increased computational power. It is accepted in the turbulence modeling community that the pressure strain correlation model is a critical component for the success of the Reynolds Stress Modeling approach. Pressure strain correlation models try to capture the effects of the interaction of fluctuating pressure with the fluctuating rate of strain tensor. Such models try to express the effects of the pressure strain correlation using a tensor basis of local tensors like Reynolds stresses, dissipation and mean velocity gradients. The physics that the pressure strain correlation model tries to capture is non-local due to the non-local nature of pressure. Using a limited set of local tensors to capture this physics leads to limitations in model performance. In this investigation we extend the tensor basis used for pressure strain correlation modeling. This set of additional tensors sequentially justified based on physics based arguments. We formulate a model using this extended modeling basis. The present model is tested for a wide variety of turbulent flows and contrasted against the predictions of other popular models. It is shown that the new model provides significant improvement in predictive accuracy. Experiments were also carried out for decaying grid generated turbulence and grid turbulence with mean strain, to calibrate the model constants for the pressure strain correlation and the dissipation rate equation for free surface flows.

7.2 Scope of future work

In the present Section, future research scopes in the light of the work pursued in the present thesis are highlighted point-wise as below:

- The newly developed pressure strain correlation model can be implemented in standard computational fluid dynamics codes.
• The model can be utilized to develop algebraic Reynolds stress models.

• The predictive capability of the model can be tested against other available dissipation rate equations.

• The model can be extended towards compressible turbulence.

• We are currently testing this pressure strain correlation model for inhomogeneous turbulent flows where the effects of boundaries and walls are important.

• The experimental data sets presented can be used to develop new turbulence models for free surface flow.

• In related future active grids can be used to generate more varied data sets that cover a wider range of parameters including $\frac{2k}{\epsilon}$, $Re_M$, $S$, $W$ and different initial conditions for the turbulent velocity field.
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