Computational Code-Based Single-Server Private Information Retrieval

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Abstract—A new computational private information retrieval (PIR) scheme based on random linear codes is presented. A matrix of messages from a McEliece scheme is used to query the server with carefully chosen errors. The server responds with the sum of the scalar multiple of the rows of the query matrix and the files. The user recovers the desired file by erasure decoding the response. Contrary to code-based cryptographic systems, the scheme presented here enables to use truly random codes, not only ones disguised as such. Further, we show the relation to the so-called error subspace search problem and quotient error search problem, which we assume to be difficult, and show that the scheme is secure against attacks based on solving these problems.

I. INTRODUCTION

Private information retrieval (PIR) was first introduced in [1], enabling a user to retrieve a data item from a database without revealing the identity of the retrieved item to the system owner. A trivial solution would be to download the whole database, which is also the possibility to achieve information theoretic privacy with a single server. This solution is infeasible for modern storage systems that can contain a huge number of potentially big files. One possible solution to achieve better retrieval rates is to replicate the files and the user can recover the requested file by erasure decoding. Depending on the parameters, the achieved PIR rates are comparable to the existing computational PIR schemes. The complexity, which is the bottleneck of this construction, is the server side. Following this breakthrough, [15] gave a general construction of the scheme in [12]. Other PIR schemes based on homomorphic encryption were proposed recently in [16], [17], [18], [19]. Building on the protocol of [17], a method to significantly decrease the query size was introduced in [20].

This paper is the first to provide a computational PIR scheme based on codes, and can be seen as a counterpart to the lattice-based scheme of [12] along the same lines as code-based and lattice-based cryptography are connected in general. The query to the server can be considered as a matrix whose rows contain corrupted codewords of a secret code. The server then responds with the scalar product of the query matrix and the files and the user can recover the requested file by erasure decoding. Depending on the parameters, the achieved PIR rates are comparable to the existing computational PIR schemes of [15], [12]. The complexity, which is the bottleneck of current computational schemes, benefits from all calculations being over binary extension fields, which is advantageous for implementation.

Remark 1. This computational PIR scheme has recently been broken for all relevant parameters. For details see [21].

II. NOTATION

Let \( F_q \) denote the finite field of order \( q \) and \( \mathbb{F}_{q^s} \) its extension field of extension degree \( s \). We write \([a,b]\) for the set \( \{a, a+1, \ldots, b\} \) and if \( a = 1 \) we write \([b]\). We denote a linear code \( C \)
We begin by defining some basic functions required for the description of the PIR scheme.

**Definition 1.** Let $\mathcal{E} \subseteq [n]$ and $E \in \mathbb{F}_q^{\times |\mathcal{E}|}$. Denote by $M_{\mathcal{E}}$ the $n \times n$ identity matrix with all rows index by $[n]\setminus \mathcal{E}$ deleted. The map $\phi(\mathcal{E}, \mathcal{E})$ is given by $\phi_{\mathcal{E}}(E, \mathcal{E}) = E \cdot M_{\mathcal{E}}$.

For example, consider the mapping

$$\phi_{\mathcal{E}} \left((4 \ 2) \ , \ (2 \ 4)\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

identity matrix rows 1 and 3 deleted

In the following we need to be able to “cut” out the part of an element contained in a certain subspace.

**Definition 2.** Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ be a basis of $\mathbb{F}_q^n$, over $\mathbb{F}_q$ and $\alpha$ be an element $\alpha \in \mathbb{F}_q^n$ with $\alpha = \sum_{j=1}^{s} a_j \gamma_j$, $a_j \in \mathbb{F}_q$. For a subspace $W$ basis $W$ such that $W \subseteq \Gamma$ we define

$$\psi_W^\Gamma (\alpha) = \sum_{\gamma_j \in W} a_j \gamma_j .$$

Note that for any element $\alpha \in W$ of a subspace of $\mathbb{F}_q^n$ and element $\alpha' \in \mathbb{F}_q^n/W$ of the quotient space it holds that

$$\psi_W^\Gamma (\tilde{\alpha} \alpha + \beta \alpha') = \tilde{\alpha} \alpha \; \forall \; \beta, \beta' \in \mathbb{F}_q.$$

**IV. A CODE-BASED COMPUTATIONAL PIR SCHEME**

In a computational PIR scheme, a user generates a query $Q_i$ from a set of secret information $S$ and a set of public information $P$. For each such query the server replies with some $A_i$, which is a function of the received query $Q_i$, the $m$ files $X^1, \ldots, X^m$ stored on the server, and the public information $P$. The scheme is said to be correct if the user can recover the desired file from the replies of the servers.

**A. System Model**

We consider a single server storing $m$ files, i.e., in total we store $X \in \mathbb{F}_q^{n \times m(s-v)(n-k)}$, where each file $X^j \in \mathbb{F}_q^{L(s-v)(n-k)}$ is given by a submatrix of $(s-v)(n-k)$ columns (compare Figure 1). We denote $\delta := (s-v)(n-k)$, this parameter can be considered the required level of subpacketization. We assume that the indices of the files are known to the user.

**B. Query**

The user chooses a random $[n, k]_{q'}$ code $C$. Let $D \in \mathbb{F}_q^{\delta \times n}$ be a matrix where each row $D_{ik}$ is chosen uniformly at random from $C$. Let $\mathcal{I} \subset [n]$ with $|\mathcal{I}| = k$ be a randomly chosen information set of $C$ and denote its complement by $\mathcal{E} = [n]\setminus \mathcal{I}$. Further, the user chooses a random basis $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ of $\mathbb{F}_q^n$ over $\mathbb{F}_q$. We denote by $V$ the $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$ of dimension $s$ spanned by $V = \{\gamma_1, \ldots, \gamma_s\}$, where $v < s$, and by $W$ be the $s-v$ dimensional subspace spanned by $W = \{\gamma_{v+1}, \ldots, \gamma_s\}$, i.e., the quotient space $\mathbb{F}_q^n/V$. The user chooses a matrix $\hat{E} \in \mathbb{F}_q^{m \times n}$ i.i.d. at random. We denote $E = \phi_{\mathcal{E}}(\hat{E}, \mathcal{E})$.

Let $\Delta \in W(s-v)(n-k)$ be chosen i.i.d. random from matrices of full row-rank over $\mathbb{F}_q$, i.e., with $rk_q(\Delta) = (s-v)(n-k)$, and denote $\Delta = \phi(\Delta, \mathcal{E})$.

The query for file $X^i$ is given by

$$Q_i = D + E + \Delta \otimes e_i^m ,$$

where $e_i^m \in \mathbb{F}_2^{m \times 1}$ denotes the $i$-th unit vector and $\otimes$ denotes the Kronecker product. An illustration of the query matrix $Q_i$ is given in Figure 2.

**C. Response**

The server receives the query $Q_i \in \mathbb{F}_q^{m \times n}$ and responds with

$$A_i = X \cdot Q_i \in \mathbb{F}_q^{L \times n} ,$$

i.e., with a matrix where each row is an $\mathbb{F}_q$-linear combination of the rows of $Q_i$ with coefficients given by the respective row of $X$.

**D. Decoding**

Denote the $j$-th unit vector of length $n$ by $e_j^n$. The user receives a matrix where the $z$-th row is given by

$$A^j_{z} \cdot \Delta = \sum_{i=1}^{m} X^i_{z} \cdot (D_{(l-1)\delta + 1:i \delta}, \ E_{(l-1)\delta + 1:i \delta}) + X^i_{z} \cdot \Delta$$

for $j = z, \ldots, n$.

Let

$$e_j^m \in \mathbb{F}_2^{m \times 1}$$

be chosen i.i.d. at random. We denote $E = \phi_{\mathcal{E}}(\hat{E}, \mathcal{E})$.

Let $\Delta \in W(s-v)(n-k)$ be chosen i.i.d. random from matrices of full row-rank over $\mathbb{F}_q$, i.e., with $rk_q(\Delta) = (s-v)(n-k)$, and denote $\Delta = \phi(\Delta, \mathcal{E})$.

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for $j = z, \ldots, n$.
As the positions \([i] \setminus \mathcal{E}\) are an information set of \(C\) by definition and the set \(\mathcal{E}\) is known to the user, the entire vector \(\sum_{t=1}^m X^i_{z_t}\) can be recovered and thereby

\[
A^{i}_{z_t} = \left( \sum_{t=1}^m X^i_{z_t} \cdot D(t-1)\delta + 1: \delta \right) = \left( \sum_{t=1}^m X^i_{z_t} \cdot E(t-1)\delta + 1: \delta \right) + X^i_{z_t} \cdot \Delta .
\]

Applying the function from Definition 2 with respect to \(W\) yields

\[
\psi_W^\mathcal{V}\left( \left( \sum_{t=1}^m X^i_{z_t} \cdot E(t-1)\delta + 1: \delta \right) + X^i_{z_t} \cdot \Delta \right) = X^i_{z_t} \cdot \Delta .
\]

As \(\hat{\Delta}\) is of full row-rank over \(\mathbb{F}_q\), so is \(\Delta\). Hence, the vector \(X^i_{z_t}\) can be recovered and finally the entire file \(X^i\) by performing these steps on each row \(z \in [L]\).

E. Analysis

The upload, i.e., the size of the query, in bits is

\[
H(Q^i) = m \delta n \log_2(q^s) = m \delta n \log_2(q).
\]

The download, i.e., the size of the response, in bits is

\[
H(A^i) = L n \log_2(q^s) = L n \log_2(q).
\]

**Theorem 1.** The rate of the scheme is given by

\[
R_{\text{PIR}} = \frac{L \delta \log_2(q)}{m \delta n \log_2(q) + L n \log_2(q)} = \frac{L}{m \delta + L} \left( 1 - \frac{k + \frac{1}{2}(n - k)}{n} \right).
\]

A common assumption in literature is that the size of the file is much larger than the number of files, i.e., \(L >> \delta m\). In this case it is reasonable to neglect the upload cost in the calculation of the rate of the scheme.

**Corollary 1** (PIR Rate). For \(L >> \delta m\), the rate of the scheme is

\[
R_{\text{PIR}} \approx 1 - \frac{k + \frac{1}{2}(n - k)}{n} .
\]

V. Security Analysis

A. Subspace Attack

The security of the system is based on the idea that it is difficult for the attacker to differentiate which rows of \(D\), i.e., elements of \(C\), are corrupted by elements from a different subspace than the other rows (indicated by the green columns of \(\hat{\Delta}\) in Figure 3).

The security of our system is therefore tightly related to the following search problem.

**Problem 1** (Error Subspace Search Problem). Given a set of words in \(\mathbb{F}_q^n\) which are each the sum of a codeword of a random code \(C\) and an error vector. Find a \(v\)-dimensional subspace that contains the largest possible number of these error vectors.

Solving this general problem efficiently would break our system. Since the code of our system is unknown, it appears as a random code to an attacker. It is known that decoding a random code (i.e., explicitly finding the error vector(s)) is an NP hard problem. Problem 1 is easier than decoding as we do not want to decode all words (or many), but find the \(v\)-dimensional subspace that contains the most error vectors. However, we are not aware of how to find this subspace other than just trying all \(v\)-dimensional subspaces which results in an exponential complexity. Once this \(v\)-dimensional subspace is known, an approach to break our system is derived in the following.

The rows of the matrix \(Q^T\) are considered as the basis of a code. As the rows of \(D\) are random elements of a \(k\)-dimensional vector space, another basis of this code is given by

\[
\langle Q^T \rangle = \left\{ \left( D \cdot A, \hat{\Delta} \otimes e_q^m \right)^T \right\},
\]

for some full-rank matrix \(A \in \mathbb{F}^{n \times k}\). Recall that the elements of \(\hat{E}\) are from the space \(V\) and the elements of \(\Delta\) are from the quotient space \(\hat{V}\). It follows that all elements in \(\langle \hat{E} + \Delta \otimes e_q^m \rangle^T\) are from \(V\), except for the ones corresponding to file \(i\), which can be from the entire field \(\mathbb{F}_q^s\). Therefore, if the attacker is able to find such a basis, the index of the desired file can easily be determined. Hence we can restate the problem as: find a subspace of \(\langle Q^T \rangle\) such that all positions except for those corresponding to one file are from a subspace of dimension \(\text{dim}(V) = v\).

Once a suitable subspace \(V\) is known (or for any guessed subspace), an attacker can proceed by the following procedure:

1. Consider the \([m \delta, n]\) code spanned by \(Q^T\). Puncture the positions belonging to the file \(l\).
2. Calculate the parity-check matrix of this code. This matrix spans the dual code of dimension \((m - 1)\delta - n\).
3. Extend the parity-check matrix to the subfield. If every-thing is random, the dimension of the subfield subcode is \(\max\{(m - 1)\delta - ((m - 1)\delta - n), 0\}\) w.h.p. As \(m >> n\), this is almost certainly 0.
4. If the dimension of the subcode is zero, then \(l \neq i\). If it is non-zero, then \(l = i\) w.h.p.

This attack is successful w.h.p. for all parameters that lead to a reasonable rate. However, it requires that the attacker knows that subspace \(V\) in order to determine the dimension of the corresponding subcode. Hence, to prevent this attack, the system parameters need to be chosen such that the probability of the attacker guessing the correct subspace is small.

The number of \(v\)-dimensional subspaces of an \(s\)-dimensional space (where \(v \leq s\)) is given by the Gaussian binomial coefficient, i.e.,

\[
\binom{s}{v}_q = \frac{(1 - q^s)(1 - q^{s-1}) \ldots (1 - q^{s-v+1})}{(1 - q^v)(1 - q^{v-1}) \ldots (1 - q)}.
\]

Instead of guessing the actual \(v\)-dimensional subspace \(V\), the attacker can also guess a larger subspace in the hope that it contains the correct space \(V\), as any subspace subcode can be expected to be empty if the number of files \(m\) is large (the probability approaches 1 as \(m \to \infty\)). The probability
of picking a space containing \( V \) depends on the number of possible extensions spaces, i.e., the number of higher dimensional subspaces a smaller subspace is contained in.

**Lemma 1.** Every \( v \)-dimensional subspace of \( \mathbb{F}_q^r \) is a subspace of
\[
\begin{bmatrix}
s - v \\
\vdots \\
z - v
\end{bmatrix}_{q^s}
\]
subspaces of dimension \( z \).

**Proof:** Let \( V \) be any \( v \)-dimensional subspace of \( \mathbb{F}_q^r \), and \( Z \) be a \( z \)-dimensional subspace containing it. Then there is a one-to-one mapping between the \( Z \) and the \( z - v \)-dimensional subspaces of the quotient space \( \mathbb{F}_q^r/V \).

The attack is successful if the attacker picks one of these \((s-1)\)-dimensional “superspaces”, which happens with probability
\[
\Pr\{V \subseteq Z\} = \frac{s - v}{s - 1 - v} \cdot \frac{s}{s - 1},
\]
if the space \( Z \) is chosen uniformly at random. To prevent the attack, we require the inverse of this probability to be larger than the security level of the scheme.

**B. Linear Dependency Attack**

The goal of the attacker is to determine for which \( l \in [n] \) the corresponding rows in \( Q \) differ from the other rows. In this section, we discuss an attack that aims at directly finding the file index \( i \) by comparing the probability of rows of the query matrix being independent, given that positions corresponding to \( l \) are included or not. We can therefore say that if one can efficiently solve the following problem, our system would be broken.

**Problem 2 (Quotient Error Search Problem).** Given a set of words in \( \mathbb{F}_{q^n} \), which are each the sum of a codeword of a random code \( C \) and an error vector from a subspace \( \mathbb{F}_{q^r} \), except for one, to which an additional error vector from the quotient space \( \mathbb{F}_{q^n}/\mathbb{F}_{q^r} \) is added. Find the word with the additional error vector from the quotient space.

We analyze the probability of a square submatrix of \( Q \) being of full rank if it does not contain any rows corresponding to the \( i \)-th file. This probability differs from the probability for a submatrix containing rows corresponding to the \( i \)-th file, as the probability of a matrix being full-rank decreases with the size of the subspace. For simplicity we only consider the case where \( \mathbb{F}_q \) is a subfield of \( \mathbb{F}_q^r \) and leave the generalization to arbitrary subspaces for an extended version of this work.

**Theorem 2.** Let \( v|s \). Then for any \( I \subset [m]\delta \setminus \{(i-1)\delta + 1, \ldots, i\delta\} \) with \( |S| = n \) it holds that
\[
\Pr\{\text{rk}(D_{I,:} + E_{I,:}) = n\} \geq \left( \prod_{j=n-k+1}^{n} \left(1 - \frac{1}{q^{m^j}}\right) \right) \cdot \left( \prod_{j=1}^{n-k} \left(1 - \frac{1}{q^{m^j}}\right) \right) - \left( \prod_{j=k+1}^{n} \left(1 - \frac{1}{q^{m^j}}\right) \right).
\]

**Proof:** Without loss of generality assume that \( E = [n-k] \). By slight abuse of notation we drop the index \( I \) in the following, i.e., instead of \( D_{I,:} \) and \( E_{I,:} \), we simply write \( D \) and \( E \).

Let \( B \in \mathbb{F}_q^{n \times n} \) be chosen uniformly at random from all full-rank matrices with
\[
B \cdot (D + E) = (B_1 \\ B_2) \cdot \left( \begin{bmatrix} D_{1,1} \\ D_{2,1} \end{bmatrix} & D_{1,2} \\ D_{2,2} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right) = \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} \Rightarrow \begin{bmatrix} 0_{n-k \times k} \\ 0_{n-k \times k} \end{bmatrix}
\]
(3)

Note that such a matrix always exists since the rows of \( D \) are taken from a \( k \)-dimensional subspace and \( E \) is only supported on \( \mathcal{E} \).

The matrix \( B \cdot (D + E) \) is of full rank if and only if \( D_2' \) and \( E_2' \) are of full rank, therefore
\[
\Pr\{\text{rk}(D_2 + E_2) = n\} = \Pr\{\text{rk}(D_2') = k \land \text{rk}(E_2') = n - k\}
\]
\[
= \Pr\{\text{rk}(D_2') = k\} \cdot \Pr\{\text{rk}(E_2') = n - k | \text{rk}(D_2') = k\}
\]
Since \( [n]\setminus\mathcal{E} \) is an information set of \( C \) by definition, the matrix \( D_2' \) is of full rank if and only if the matrix \( D \) contains a basis of the code \( C \). Let \( G_s \) be a generator matrix of the code \( C \), then there is an \( U \in \mathbb{F}_{q^n} \) such that
\[
U \cdot G_s = D.
\]

The codewords in \( D \) are chosen uniformly at random, which is equivalent to \( U \sim \text{unif}(\mathbb{F}_{q^n}^{m^k}) \). Since the generator matrix \( G_s \) is full-rank by definition, the multiplication is rank preserving. Hence, it holds that \( \text{rk}(D) = k \), i.e., the matrix \( D \) contains a basis of \( C \), if and only if \( \text{rk}(U) = k \), which is well-known to be
\[
\Pr\{\text{rk}(D_2') = k\} = \Pr\{\text{rk}(U) = k\} = \prod_{j=n-k+1}^{n} \left(1 - \frac{1}{q^{m^j}}\right).
\]

Now consider the bottom part of the matrix. From (3) we get
\[
0_{n-k \times k} = B_2 \cdot \begin{bmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix} = B_2 \cdot \left( \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \cdot G_s \right) \Rightarrow B_2 \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0_{n-k \times k}.
\]

Since \( \text{rk}(U) = k \) and \( \text{rk}(B) = n \) by assumption, it follows that \( B_2 \) is a basis of the dual space of \( U \). As \( U \) is chosen uniformly at random, every full rank \( U \) is equally likely and therefore also any \( B_2 \). From (3) we further get
\[
E_2' = B_2 \cdot \left( \frac{E_1}{E_2} \right) = B_2 \cdot \hat{E},
\]
where \( B_2 \in \mathbb{F}_q^{n \times k} \) and \( \hat{E}_1, \hat{E}_2 \in \mathbb{F}_q^m \). Let \( M \sim \text{unif}(\mathbb{F}_{q^{n-k \times n}}) \).
We are interested in the probability
\[
\Pr\{\text{rk}(B_2 \cdot \hat{E}) = n - k | \text{rk}(D'_2) = k\}
\]
= \Pr\{\text{rk}(B_2 \cdot \hat{E}) = n - k | \text{rk}(U) = k\}
\stackrel{(a)}{=} \Pr\{\text{rk}(B_2 \cdot \hat{E}) = n - k\}
= \Pr\{\text{rk}(M \cdot \hat{E}) = n - k \land \text{rk}(M) = n - k\}
= \frac{\Pr\{\text{rk}(M \cdot \hat{E}) = n - k \land \text{rk}(M) = n - k\}}{\Pr\{\text{rk}(M) = n - k\}}
\geq \Pr\{\text{rk}(M \cdot \hat{E}) = n - k \land \text{rk}(M) = n - k\}
= 1 - \Pr\{\text{rk}(M \cdot \hat{E}) < n - k \lor \text{rk}(M) < n - k\}
\geq 1 - (\Pr\{\text{rk}(M \cdot \hat{E}) < n - k\} + \Pr\{\text{rk}(M) < n - k\})
= 1 - (1 - \Pr\{\text{rk}(M \cdot \hat{E}) = n - k\} - \Pr\{\text{rk}(M) < n - k\})
= \Pr\{\text{rk}(M \cdot \hat{E}) = n - k\} - \Pr\{\text{rk}(M) < n - k\},
\]
where (a) holds because \(\hat{E}\) is independent of \(U\) and \(B_2\) is uniformly distributed if \(U\) is uniformly distributed over all full rank matrices. To obtain the first probability, we fix a basis of \(\mathbb{F}_q\) over \(\mathbb{F}_q\) and consider the extension of \(M \in \mathbb{F}^{n-k}_{q^{n-k}}\) to \(\hat{M} \in \mathbb{F}^{n-k}_{q^{n-k}}\) obtained by representing every element in this basis. As \(M\) is random over \(\mathbb{F}^q\), the matrix \(M\) is random over \(\mathbb{F}^q\). The multiplication of two random matrices is again a random matrix, hence we get \(\hat{M} \cdot \hat{E} \sim \text{unif}(\mathbb{F}^{n-k}_{q^{n-k}})\) and equivalently, when mapping back to \(\mathbb{F}^q\), we get \(M \cdot \hat{E} \sim \text{unif}(\mathbb{F}^{n-k}_{q^{n-k}})\).

Hence
\[
\Pr\{\text{rk}(M \cdot \hat{E}) = n - k\} = \prod_{j=1}^{n-k} \frac{1}{1 - \frac{1}{q^j}}.
\]

It follows that
\[
\Pr\{\text{rk}(B_2 \cdot \hat{E}) = n - k\}
\geq \prod_{j=1}^{n-k} \left(1 - \frac{1}{q^j}\right) - \prod_{j=k+1}^{n} \left(1 - \frac{1}{q^j}\right)
\]

and the theorem statement follows.

### VI. PARAMETER CHOICES

Table I shows the achieved PIR rate for different choices of parameters together with lower bounds on the complexity of the respective attacks, as derived in Section V-A and V-B. Note that rate of the presented scheme depends greatly on the chosen parameters. Increasing \(q\) and/or \(s\) increases the security and therefore allows for increasing the rate of the scheme by adapting \(v\) and/or \(k\). However, increasing the values of \(q\) or \(s\) increases the complexity of the scheme, as the server is required to perform multiplications over the respective fields. As the computational complexity is regarded as the bottleneck for computational PIR [11], we present parameters resulting in a low rate, but relatively good complexity. Due to a lack of space a detailed comparison of the complexity compared to the existing schemes of [12, 13] is left as future work. Instead we provide some intuition on why the scheme can perform favorably compared to these schemes in terms of complexity. Although the field size resulting from the parameters given in Table I appear to be large from a coding-theoretic point of view, the majority of the more complex operations, i.e., multiplications, is not over these fields, but instead between elements of the field and elements of a subfield. Especially, since the files are only from \(\mathbb{F}_q\), all multiplications performed on the server side, the number of which depends on the (generally large) number of files and their size, are of the form \(\alpha \beta\) with \(\alpha \in \mathbb{F}_{q^r}\) and \(\beta \in \mathbb{F}_q\). Each element \(\alpha \in \mathbb{F}_{q^r}\) can be represented as a polynomial of degree \(s - 1\) over \(\mathbb{F}_q\), so the complexity of this multiplication is just the complexity of multiplying the \(s\) coefficients of this polynomial by \(\beta\). Assuming a complexity of \((\log(q))^2\) for the multiplication of elements from a field \(\mathbb{F}_q\), this gives a complexity of \(s(\log(q))^2 = (\log(q^r))^2\), which is equivalent to performing multiplications over a field \(\mathbb{F}_{q^r}\). For example, for \(q = 32\) and \(s = 32\) this is approximately equivalent to the complexity of multiplications over \(\mathbb{F}_{2}\). As a comparison, the parameters proposed in [12, Section IV] require the multiplication of matrices of similar size to our scheme on the server side, but over the integer field \(\mathbb{F}_{2^{60+32}}\). This is not only significantly larger than the "equivalent field" in our construction, but additionally does not provide the hardware advantages that extension fields of \(\mathbb{F}_2\) provide, namely the possibility of implementation based on shifts and XORs.
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