GRAVITATIONAL FREQUENCY SHIFT OF LIGHT SIGNALS IN A PULSATING DARK MATTER HALO

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We study the gravitational frequency shift of light signals from the center of a spherically symmetric nonstatic matter distribution. In the case of oscillating scalar dark matter with a logarithmic potential, we obtain explicit formulas for the ratio of the emitted and received frequencies.

Keywords: dark matter halo, oscillating scalar field, nonstatic metric, red shift, blue shift

DOI: 10.1134/S0040577919120109

Modern observations show that approximately a quarter of the total mass of the Universe is dark matter, which forms galactic halos and, possibly, separate self-gravitating lumps [1]. These results are based on studying the motion of visible matter and the propagation of photons in the gravitational field created by various spatial distributions of dark matter. If the distribution of dark matter is nonstatic, then the corresponding space–time metric is also nonstatic, generally speaking. Here, we investigate the gravitational frequency shift of a light signal propagating in a spherically symmetric pulsating lump of scalar dark matter. The corresponding metric is

\[ ds^2 = B(t, r) \, dt^2 - A(t, r) \, dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\phi^2). \]  

(1)

The general formula for the frequency shift of a photon propagating in a gravitational field is well known (see, e.g., [2]). Namely, the ratio of the source proper frequency at the photon emission point to the photon frequency at the observation point is equal to the ratio of the scalar product of the photon four-momentum at the emission point and the source four-velocity to the scalar product of the photon four-momentum at the observation point and the observer four-velocity. The four-momentum of the photon satisfies the null-geodesic equation, which in the case of a nonstatic metric reduces to a rather complicated system of nonlinear differential equations with the affine parameter as an independent variable. Analytic solutions of this system can be obtained only in certain particular cases, for example, for metrics describing some expanding isotropic space–times [2], plane gravitational waves [3], [4], and small fluctuations of the gravitational potential [5], [6]. In the case of spherically symmetric metric (1), the geodesic equation for the photon four-momentum was solved numerically in [7].

We note that a simple formula for the frequency shift can be obtained in the case where the source is located at the center of a spherically symmetric distribution and the observer is at rest at a point \( r = R \). Indeed, setting \( ds = d\vartheta = d\phi = 0 \) in (1), we obtain the equation for the photon trajectory in the form

\[ \frac{dt}{dr} = \sqrt{\frac{A(t, r)}{B(t, r)}}. \]  

(2)

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Prepared from an English manuscript submitted by the authors; for the Russian version, see Teoreticheskaya i Matematicheskaya Fizika, Vol. 201, No. 3, pp. 440–445, December, 2019. Received June 27, 2019. Revised June 27, 2019. Accepted July 25, 2019.
We now consider two light pulses that are successively emitted at the point \( r = 0 \) through a small time interval \( \Delta t(0) \). The corresponding close trajectories \( t(r) \) and \( t(r) + \Delta t(r) \) satisfy Eq. (2) on the interval \( 0 \leq r \leq R \). Substituting \( t(r) + \Delta t(r) \) in (2) and expanding the right-hand side near the trajectory \( t(r) \), we obtain the ratio \( \Delta t(r)/\Delta t(0) \). As a result, for the ratio of the source proper frequency \( \nu_0 \) to the observed frequency \( \nu_s \), we obtain

\[
\frac{\nu_0}{\nu_s} = \sqrt{\frac{B(t(R), R)}{B(t(0), 0)}} \exp \int_0^R \left[ \frac{\partial}{\partial t} \sqrt{\frac{A(t, r)}{B(t, r)}} \right]_{t=t(r)} \, dr. \tag{3}
\]

We note that this formula also holds in the comoving coordinates: in this case, we must set \( B = 1 \) in (3). For example, it immediately follows from (3) that the ratio \( \nu_0/\nu_s \) for the Friedmann–Robertson–Walker metric is equal to the ratio of the scale factors taken at the respective instants \( t(R) \) and \( t(0) \).

It is easy to verify that formula (3) can be equivalently represented in the form

\[
\frac{\nu_0}{\nu_s} = \sqrt{\frac{A(t(R), R)}{A(t(0), 0)}} \exp \int_{t(0)}^{t(R)} \left[ \frac{\partial}{\partial r} \sqrt{\frac{B(t, r)}{A(t, r)}} \right]_{r=r(t)} \, dt, \tag{4}
\]

whence follows the formula obtained in [8] in the post-Newtonian approximation.

Using these results, we calculate the gravitational frequency shift of the light signal from the center of a pulsating self-gravitating lump of a real scalar field \( \phi(t, r) \) with the potential

\[
U(\phi) = \frac{1}{2} m^2 \phi^2 \left( 1 - \log \frac{\phi^2}{\sigma^2} \right), \tag{5}
\]

where \( \sigma \) is the characteristic magnitude of the scalar field and \( m \) is the mass (in units \( h = c = 1 \)). Such potentials arise in inflationary cosmology [9] and also in some supersymmetric extensions of the Standard Model [10]. The corresponding solution of the Einstein–Klein–Gordon system describing the self-gravitating pulson was found in [11] in the weak-field approximation. It has the form

\[
\frac{\phi(t, r)}{\sigma} = [a(\theta) + \Omega(\varphi) + O(\varphi^2)] e^{(3-\rho^2)/2}, \tag{6}
\]

\[
A(t, r) = \left( 1 - \frac{\rho_g}{\rho} \right)^{-1}, \quad B(t, r) = \left( 1 - \frac{\rho_g}{\rho} \right) e^{-s}, \tag{7}
\]

\[
\rho_g(\tau, \rho) = \varphi \left[ V_{\text{max}} \frac{\sqrt{3}}{2} e^{\rho^2} \text{erf} \rho - \rho^3 \right] e^{3-\rho^2} + O(\varphi^2), \tag{8}
\]

\[
s(\tau, \rho) = \varphi \left[ 2V_{\text{max}} + a^2 \log a^2 + a^2 \rho^2 \right] e^{3-\rho^2} + O(\varphi^2), \tag{9}
\]

where \( a(\theta(\tau)) \) satisfies the nonlinear oscillator equation

\[
a_{\theta \theta} = -\frac{dV}{da}, \quad V(a) = \frac{a^2}{2} \left( 1 - \log a^2 \right), \tag{10}
\]

\( \tau = mt, \rho = mr, \varphi = 4\pi G\sigma^2 \ll 1 \) (\( G \) is the gravitational constant), \( V_{\text{max}} = V(a_{\text{max}}) \), \( \theta_r = 1 + \varphi \Omega + O(\varphi^2) \), and \( \varphi \Omega \) is the pulson frequency correction. The first term in formula (6) describes the anharmonic oscillations \( -a_{\text{max}} \leq a(\theta) \leq a_{\text{max}} \) in the symmetric potential \( V(a) \). The function \( Q(\theta, \rho) \) is a series in Hermite polynomials whose coefficients are periodic (in \( \theta \)) solutions of the nonhomogeneous Hill equations. In [11], initial conditions and the correction \( \varphi \Omega \) were found for which such solutions exist. The stability of
these solutions depends essentially on the oscillation amplitude $a_{\text{max}}$. It turned out that in some intervals of $a_{\text{max}}$ values, solutions retain their periodicity with high accuracy, making hundreds of oscillations. Similar quasistability intervals were also found when studying the effect of external perturbations on a pulson in the absence of gravity [12], [13]. Because we only need to know the metric coefficients $A(t, r)$ and $B(t, r)$ to calculate the gravitational frequency shift, we do not discuss the structure of the function $Q(\theta, \rho)$ in more detail here; its influence on the metric is manifested in the next orders in $\kappa$. We merely assume that $a_{\text{max}}$ belongs to one of the quasistability intervals.

Let the function $a(\theta)$ be an even periodic solution of Eq. (10) with the period

$$T(a_{\text{max}}) = \frac{2\pi}{\omega} = 4 \int_{0}^{1} \frac{d\xi}{\sqrt{(1 - \log a_{\text{max}}^{2}(1 - \xi^{2})) + \xi^{2}\log \xi^{2}}}. \quad (11)$$

Then

$$a^{2} = \frac{A_{0}}{2} + \sum_{n=1}^{\infty} A_{n} \cos 2n\omega\theta, \quad (12)$$

$$a^{2} \log a^{2} = \frac{C_{0}}{2} + \sum_{n=1}^{\infty} C_{n} \cos 2n\omega\theta. \quad (13)$$

Using (10), we can easily show that the Fourier coefficients $A_{n}$ and $C_{n}$ are related by

$$C_{0} = \frac{A_{0}}{2} - 2V_{\text{max}}, \quad C_{n} = \frac{A_{n}}{2} [1 - 2(n\omega)^{2}], \quad (14)$$

where

$$A_{0} = \frac{8a_{\text{max}}^{2}}{T(a_{\text{max}})} \int_{0}^{1} \frac{n\omega \rho d\rho}{\sqrt{(1 - \log a_{\text{max}}^{2})(1 - \xi^{2}) + \xi^{2}\log \xi^{2}}}, \quad (15)$$

$$A_{n} = \frac{8}{T} \int_{0}^{T/4} a^{2}(\theta) \cos 2n\omega\theta d\theta. \quad (16)$$

We now return to Eqs. (2) and (3). We note that because $\kappa$ is small, the right-hand side of (2) is close to unity and its time derivative in (3) is of the order $O(\kappa)$. Therefore, calculating the integral in (3), we can take the photon trajectory in the form $t(r) = r + t(0)$. Hence, substituting (7)–(9) in (3) and using (12)–(14), we can set $\theta \approx \tau(\rho) = \rho + \tau(0)$, where $\tau(0) = \tau(R) - R, R = mR$. As a result, we finally obtain

$$\nu_{0} = \frac{\nu_{\kappa}}{\nu_{\kappa}} = 1 + \kappa \approx 1 + \kappa \frac{e^{3}}{2} \left( V_{\text{max}} \left[ 1 - \frac{\sqrt{\pi} \text{erf} R}{R} \right] + \frac{A_{0}}{4} (1 - e^{-R^{2}}) + \right.$$}

$$+ \sum_{n=1}^{\infty} \left[ C_{n} (\cos 2n\omega \tau(0) - e^{-R^{2}} \cos 2n\omega \tau(R)) - 2n\omega P_{n}(\tau(0), R) \right] \right), \quad (17)$$

where

$$P_{n}(\tau(0), R) = [C_{n} I_{1}(n\omega, R) - A_{n} I_{3}(n\omega, R)] \cos 2n\omega \tau(0) +$$

$$+ [C_{n} I_{2}(n\omega, R) - A_{n} I_{4}(n\omega, R)] \sin 2n\omega \tau(0), \quad (18)$$

$$I_{1}(n\omega, R) = \int_{0}^{R} e^{-\rho^{2}} \sin 2n\omega \rho d\rho, \quad I_{2}(n\omega, R) = \int_{0}^{R} e^{-\rho^{2}} \cos 2n\omega \rho d\rho,$$

$$I_{3}(n\omega, R) = \int_{0}^{R} \rho^{2} e^{-\rho^{2}} \sin 2n\omega \rho d\rho, \quad I_{4}(n\omega, R) = \int_{0}^{R} \rho^{2} e^{-\rho^{2}} \cos 2n\omega \rho d\rho.$$
Integrating by parts, we can easily show that

\[
I_3(n\omega, R) = \frac{n\omega}{2} + \frac{1}{2} [1 - 2(n\omega)^2] I_1(n\omega, R) - \frac{1}{2} e^{-R^2} (n\omega \cos 2n\omega R + R \sin 2n\omega R),
\]

\[
I_4(n\omega, R) = \frac{1}{2} [1 - 2(n\omega)^2] I_2(n\omega, R) + \frac{1}{2} e^{-R^2} (n\omega \sin 2n\omega R - R \cos 2n\omega R).
\]

At long distances from the source, we have

\[
I_1(n\omega, \infty) = n\omega e^{-(n\omega)^2} F_1\left(\frac{1}{2}, \frac{3}{2}; (n\omega)^2\right), \quad I_2(n\omega, \infty) = \frac{\sqrt{\pi}}{2} e^{-(n\omega)^2},
\]

\[
I_3(n\omega, \infty) = \frac{n\omega}{2} + \frac{n\omega}{2} [1 - 2(n\omega)^2] e^{-(n\omega)^2} F_1\left(\frac{1}{2}, \frac{3}{2}; (n\omega)^2\right),
\]

\[
I_4(n\omega, \infty) = \frac{\sqrt{\pi}}{4} [1 - 2(n\omega)^2] e^{-(n\omega)^2}.
\]

The second square bracket in the function \(P_n\) given by (18) with (14) taken into account then vanishes, and therefore

\[
P_n(\tau(0), \infty) = -\frac{n\omega}{2} A_n \cos 2n\omega \tau(0).
\]

As a result, from (17), we obtain

\[
\left(\frac{\nu_0}{\nu_n}\right)_{R \to \infty} \approx 1 + \frac{e^3}{2} \left\{ V_{\text{max}} + \frac{A_0}{4} + \frac{1}{2} \sum_{n=1}^{\infty} A_n \cos 2n\omega \tau(0) \right\}.
\]

Figures 1 and 2 show the dependence of the frequency shift \(z = (\nu_0 - \nu_n)/\nu_n\) on the coordinate time at the observation point, calculated using formula (17) for different amplitudes of the halo oscillations \(a_{\text{max}}\) and for different distances \(R\) from the source. It turns out that because the series rapidly converges, it suffices to take only the first eight Fourier harmonics into account when summing in (17).

It can be seen from the plots that the shift is modulated by the doubled halo oscillation frequency. The magnitude of the modulations increases as \(a_{\text{max}}\) increases. We note that for some amplitudes and at some distances, a blue shift \((z < 0)\) is observed at certain time intervals instead of a redshift \((z > 0)\).
Fig. 2. Dependence of the frequency shift on time for $a_{max} = 0.24$ at different distances from the source: (1) $R = 0.5$, (2) $R = 1$, (3) $R = 1.5$, (4) $R = 3$, and (5) $R = 15$.

Conflicts of interest. The authors declare no conflicts of interest.

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