Off-shellness and inhomogeneities in the non-equilibrium dynamics of the $\lambda \phi^4$ model are studied. The non equilibrium field theory is developed in the Closed Time Path formalism, which is in turn reformulated as a kinetic field theory, in terms of a set of kinetic equations for the 2-point Green function, derived from a generalized effective action functional. We take into account all initial correlations up to the 4-point functions. It is shown that the model displays an SO(1,1) symmetry broken by interactions and initial conditions. The divergence of the corresponding Noether current is identified as a particle creation/annihilation density. The broken Ward-Takahashi relations for the SO(1,1) symmetry are derived. They constitute a set of generalized kinetic equations for the general $n$-point functions. Energy-momentum conservation is demonstrated to hold from the fully interacting transport equations, and we discuss the effect of a gradient expansion and its non-locality. As an application, we study an explicit realization of inhomogeneities and non-equilibrium conditions in the free field model, for which we give the general solution and analyze it in a particular case. The identification of the Casimir effect in our solution illustrates the intimate connection between inhomogeneities and off-shellness.

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1 Introduction

Phase transitions play an important role in the study of physical systems under extreme conditions. In
particle physics, various kinds of phase transitions are of interest and the use of equilibrium thermody-
namics may be inadequate. For example, this is so in the current descriptions of the initial stages of heavy
ion collisions, in which, heuristically, the onset of deconfinement is believed to give rise to the formation
of a quark gluon plasma [1]. Two further examples that one may speculate upon, are the electroweak
phase transition in the early universe [3], and the formation of disaligned chiral condensates (dcc) in high
energy hadronic collisions [4], such as $\pi p$ scattering.

The closed timepath (CTP) formalism, originating from the work of Schwinger [7], Keldysh [6], and
Vernon and Feynman [8], is suitable for analyzing non-equilibrium systems. It takes one beyond the more
conventional approaches used to describe small perturbations about thermal equilibrium, such as thermo
field dynamics [9], the Niemi-Semenoff methods [10] and the real time finite temperature formalism of
Dolan and Jackiw [11]. The CTP formalism is not restricted to quasi homogeneous systems close to
equilibrium, although in practice, to date, it has been difficult to implement non-equilibrium theory in
inhomogeneous systems. The full power of the CTP formalism is unfolded within the functional integral
representation [12, 15]. Recently [5, 16] an effective action principle, that leads to a hierarchy of Schwinger-
Dyson equations for the Green functions, has been formulated. These equations can be reformulated in
terms of a transport theory, or kinetic field theory, for the 2-point functions. Such a field theoretic
formalism can be shown to have a Boltzmann equation as its quasi-classical limit [3]. While the CTP
non equilibrium formalism is theoretically aesthetic and complete, it is difficult to implement practically,
and has only recently become a subject of detailed study. Transport theories in general constitute a topic
whose quantum field theoretic content is difficult to understand on the basis of quasi-classical models.
We briefly summarize some applications of transport theory that have been developed and are currently
of interest. We do so without attempting to be complete.

Non-relativistic kinetic equations have become a common tool in analyzing heavy ion collisions nu-
merically [17]. Semi-classical, but relativistic, approaches based on quantum hadron dynamics have been
put forward for the same purpose [2]. A lot of work has been done on the formulation of a quark-gluon
transport theory based on the operator formalism of quantum field theory [18]. In order to study the
restoration of chiral symmetry in relativistic heavy ion collisions, a transport theory for the Nambu-
Jona-Lasinio model, that contains quarks as its fundamental degrees of freedom, has been derived in the
framework of the CTP formalism [19]. Due to the difficulties inherent in this particular problem the
equations of motion that are obtained have however only been solved in the mean field approximation, in
order to study the influence of the phase transition on the expansion of a QGP [20], and the fluctuations
in a quark-antiquark plasma [21]. The CTP formalism has also been recently applied [22] to the O(4)
linear sigma model to investigate dcc formation. In all of the studies that have been mentioned, the sys-
tems have been required to be quasi uniform and/or the particles to be on mass-shell. These restrictions have simply been necessitated by the difficulty of the problem at hand, since they give rise to considerable simplifications in the formalism when applied. On the other hand, the lifting of these requirements may in some cases be important. The mass-shell restriction may suppress important quantum effects. This is illustrated in the calculation of Eisenberg showing that the spontaneous creation of particles from a constant background field cannot be described in this way.

In order to obtain a better understanding of the importance of off-shell behaviour and spatial inhomogeneities, it is useful to examine a simple model. To this end, we shall focus on the well-known scalar $\lambda \varphi^4$ model in this paper and study the CTP formalism and associated transport theory for this model. In doing so we shall endeavour to remove the on-shell restriction, and also investigate a scenario which is far from equilibrium. Of technical importance is the proof that energy and momentum are conserved in the presence of interactions, which we also demonstrate.

Since our analysis is performed in the setting of kinetic theory, we discuss this briefly here. If interactions are short ranged, as is the case for the $\lambda \varphi^4$ point-like interaction, one expects naively that the quantum field may, above some length-scale $\ell_f$, be conceived as a distribution of free particles, each of which temporarily goes off-shell while undergoing an interaction, during a time $\tau_c$. For this picture to have some validity, one has

$$\tau_c \ll \ell_f . \tag{1.1}$$

In that case, the time dependent particle density $n(\vec{x}, t)$ satisfies

$$n(\vec{x}, t) \ll \ell_f^{-3} \tag{1.2}$$

in describing the system. Due to particle creation and annihilation $n(\vec{x}, t)$ will, in general, have a time dependent normalization.

This separation of the system into macroscopic length scales above $\ell_f$, and microscopic length scales below $\ell_f$, is at the same time a statement about the effective interaction, and hence the effective coupling of the particles at different energy scales. It is therefore important to consider the behavior of the running coupling as we move from one length scale to another. For example, in QCD at high energies, the quark and gluon degrees of freedom are relevant, whereas at low energies these remain confined and their hadronic bound states can be treated as fundamental. In this case, the separation in length scales is accompanied by a separation of physical degrees of freedom. Such a phenomenon is difficult to implement in a two-scale transport theory. A two-scale approach is also not consistent with the running of the effective coupling in the $\lambda \varphi^4$ model. Here we are dealing with a trivial theory. If we take the quantum corrections of all scales into account, the effective coupling in the theory vanishes, implying that small scale dynamics fundamentally modify the behavior at larger scales. Imposing a momentum space cut off in the theory invalidates triviality and the theory becomes interacting. A kinetic formalism can be derived from the closed time path formalism without resorting to a two-scale interpretation. However
the resulting equations of motion that one obtains are kinetic equations for the 2 point Green functions, rather than for a quasi-classical phase-space distribution function.

A kinetic theory for the $\lambda \varphi^4$ model was originally derived via the operator formalism in the mid-seventies [14]. We study the kinetic field theory for the aforementioned model by reformulating the CTP equations of motion as a transport theory, and including up to 4-point initial correlations. A similar approach can be found in [5], where however only initial two-point correlations have been taken into account. In section two, we give a brief introduction to the CTP formalism in this context, derive the transport equations, and clarify their physical content. Our view parallels the physical interpretation put forward in the early work by Carruthers and Zachariasen [14]. Initial conditions for these equations are shown to break a symmetry underlying any CTP field theory in the third section. The Ward-Takahashi identities corresponding to this broken symmetry express direct relationships between the Green functions and the initial conditions. They also represent a set of kinetic equations generalized to the n-point Green functions. For the $\lambda \varphi^4$ model, in which the symmetry group is SO(1,1), these identities are given. In particular, the Noether current corresponding to this symmetry is interpreted physically. In addition, it is shown that the dynamical evolution of the 2-point functions guarantees energy-momentum conservation. In the fourth section we detail this explicitly by demonstrating that the divergence of the energy-momentum tensor vanishes simply by using the transport equations. This allows us to settle some of the questions concerning energy-momentum conservation and the use of gradient-expansions in both Vlasov equations, and equations that include a collision-term. By studying the off-shell behavior in the free field equations and recovering the Casimir-effect from our solutions, we illustrate the intimate connection between inhomogeneities, off-shellness and initial conditions. With this program in mind, we now give a brief exposition of the CTP formalism and the kinetic field theory that is based upon it.

2 Kinetic Field Theory

Before we turn to the $\lambda \varphi^4$ model in particular, we review the general structure of the CTP formalism and the generalized effective action. Following this, we derive the transport equations for the 2-point functions and analyse the physical meaning of the various terms in the equations and the variables we use.

2.1 The Closed Time Path Formalism

Here we will briefly recapitulate the main ideas of the CTP formalism that are relevant for our calculation. A full treatment of the CTP formalism can be found elsewhere [12]. Let $\hat{\varphi}$ denote a collection of field operators. For simplicity, we take them to be bosonic. The system is specified by a Lagrangian density $\mathcal{L}(\hat{\varphi})$, and a density operator $\hat{\rho}$, that enforces the initial conditions.
Let $|\phi_1 >$ be an eigenstate of the $\hat{\phi}$ operator satisfying

$$\hat{\phi} > |\phi_1 = \phi_1(x) |\phi_1 >$$

(2.1)

with the c-number function $\phi_1(x)$ being one of its eigenvalues. Let $|\phi_n >$ be a complete set. Then the functional

$$\rho[\phi_i(x), \phi_j(y)] = <\phi_i | \hat{\rho}(t=0) | \phi_j >$$

(2.2)

contains all the physical information stored in the density operator at $t=0$. Square brackets denote $\rho$ being a functional rather than a function. If the hamiltonian $H$ of the system is not time dependent then, in the Heisenberg picture, neither will the operator $\hat{\rho}$ be. If we now introduce an additional interaction with a source $j(\vec{x}, t)$

$$H \rightarrow H + \int d^3 \vec{x} j(\vec{x}, t) \hat{\phi}(\vec{x}, t) ,$$

(2.3)

then the time dependence of $\hat{\rho}$ is given by

$$\hat{\rho}(t) = [T \{ \exp(-i \oint_0^t dt' \int d^3 \vec{x} j(\vec{x}', t') \hat{\phi}(\vec{x}', t') \}) \] \times \hat{\rho}(t=0) \times [\tilde{T} \{ \exp(+i \oint_0^t dt'' \int d^3 \vec{x}'' j(\vec{x}'', t'') \hat{\phi}(\vec{x}'', t'') \}) \]$$

(2.4)

where $T$ is the time ordering operator, and $\tilde{T}$ is the anti time ordering operator. It is a useful bookkeeping device to introduce indices 1 and 2 for the current and field operators taken on the time ordered or anti time ordered paths respectively, that are given in Eq.(2.4). Explicitly on the time ordering path, one has $\hat{\phi}^1(\vec{x}, t)$ and $j^1$, and $\hat{\phi}^2(\vec{x}, t)$ and $j^2$ on the anti time ordering path. In a contravariant notation, the general indexed n-point Green functions are defined by

$$G^{a_1,...,a_n}(x_1,...,x_n) = \langle T_p\{\hat{\phi}^{a_1}(x_1)\dots \hat{\phi}^{a_n}(x_n)\} \rangle$$

(2.5)

where $T_p$ orders the field-operators according to the rule

$$T_p\{\prod_n \hat{\phi}^{a_n}(x_n)\} = \tilde{T}\{\prod_{a_n=2} \hat{\phi}^{a_n}(x_n)\} T\{\prod_{a_k=1} \hat{\phi}^{a_k}(x_k)\} .$$

(2.6)

The action $S[\phi] = \int d^4x \mathcal{L}$ enters in the functional integral representation of the trace of the density matrix, that is obtained by writing the time-evolution operators in Eq.(2.4) as path-integrals, and using the functional Eq.(2.2)

$$Z[j, \rho] = Tr\{\hat{\rho}(t = \infty)\}$$

$$= \int D\phi^1 D\phi^2 \exp i\{S[\phi^1] + j^1 \circ \phi^1 - S[\phi^2] - j^2 \circ \phi^2 \} \rho[\phi^1, \phi^2] ,$$

(2.7)

using the abbreviation $j \circ \phi = \int d^4x j(x) \phi(x)$. 
It is convenient to replace the fields $\phi^1$ and $\phi^2$ by the doublet $\varphi^a = (\phi^1, \phi^2)$, and introduce the metric

$$c_{ab} = c^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

on this internal space. For example, the source term can then be written as

$$j_1 \circ \phi_1 - j_2 \circ \phi_2 = c_{ab} j^a \circ \varphi^b = j_a \circ \varphi^a \tag{2.9}$$

by contracting the fields $\phi$ with the metric $c^{ab}$. We will refer to these indices $a, b$ as CTP indices. We define a CTP action $S_{CTP} = S[\phi_1] - S[\phi_2]$ and write for the partition function $Z$

$$Z[j_a, \rho] = \int D\varphi^b \exp \{ i \{ S_{CTP}[\varphi^b] + j_a \circ \varphi^a \} \rho[\varphi^b] \}. \tag{2.10}$$

The general Green functions $G_{abc...}$ are computed now by taking derivatives of $Z$ with respect to the source $j_a(x)$.

By expanding the density functional in Eq.(2.2), in powers of the field $\varphi^a$

$$\rho[\varphi^a] = \exp \{ i \sum_{n=0}^{\infty} \frac{1}{n!} K_{a_1...a_n} \circ \varphi^{a_1} \circ ... \circ \varphi^{a_n} \} , \tag{2.11}$$

one obtains the expression for the generating functional of the general n-point Green functions $G^{a_1...a_n}$ to be,

$$Z[j_a, K_{ab}, ...] = \int D\varphi^b \exp \{ i \{ S_{CTP}[\varphi^b] + j_a \circ \varphi^a + \sum_{n=0}^{\infty} \frac{1}{n!} K_{a_0...a_n} \circ \varphi^{a_0} \circ ... \circ \varphi_{a_n} \} \}. \tag{2.12}$$

From Eq.(2.12), one sees that the non-local sources $K_{abc...}$ act as sources for the full Green functions $G_{abc...}$. In this way they prescribe the initial values of these Green functions. In a diagrammatic expansion of Eq.(2.12), the non-vanishing K’s are part of the tree level contributions to the non-equilibrium vertex functions, although they contain contributions from all loop orders.

The generating functional $\mathcal{W}$ of the connected Green functions $G_{abc...}(x, y, z, ...)$ is the logarithm of $Z$

$$\mathcal{W}[j_a, K_{ab}, ...] = -i \log \{ Z[j_a, K_{ab}, ...] \} . \tag{2.13}$$

and the connected Green functions can be derived from this in the standard fashion. An alternative method of obtaining the connected Green functions is as follows. One derives the full Green functions $G_{abc...}$ via differentiation of $\mathcal{W}$ with respect to the non-local sources

$$G^{a_1...a_n}(x_1, ..., x_n) = n! \frac{\delta \mathcal{W}}{\delta K_{a_1...a_n}} [j_b, K_{bc}, ...] , \tag{2.14}$$

and obtains the connected Green functions by subtracting out the disconnected contributions. This procedure is most simply convenient in deriving the hierarchy of coupled equations of motion for the (connected) Green functions in a direct fashion. In order to obtain these equations for the $G_{abc...}$, and
in particular, we construct an appropriate action functional by means of a Legendre transform with respect to the non-local sources. Let us denote the mean-field $G^a(x)$ as $\tilde{\varphi}^a$. Defining $\Gamma$ as the Legendre transform of $W$,

$$\Gamma[\tilde{\varphi}^a(x), G^{ab}(x, y), ...] = W[j_a, K_{ab}, ...] - j_a \circ \tilde{\varphi}^a - \frac{1}{2} K_{ab} \circ \{ G^{ab} + \tilde{\varphi}^a \tilde{\varphi}^b \} - ...$$  (2.15)

the mean-field $\tilde{\varphi}^a$ and the connected 2-point and n-point functions satisfy the equations

$$\frac{\delta \Gamma}{\delta \tilde{\varphi}^a}[\tilde{\varphi}^c, G^{cd}, ...] = -j_a - K_{ab} \circ \tilde{\varphi}^b - ...$$  (2.16)

$$\frac{\delta \Gamma}{\delta G^{ab}}[\tilde{\varphi}^c, G^{cd}, ...] = - \sum_{n=2}^{\infty} \frac{1}{n!} K_{a_1...a_n} \circ \frac{\delta}{\delta G^{a_1...a_n}} G^{a_1...a_n}$$  (2.17)

$$\frac{\delta \Gamma}{\delta G^{b_1...b_m}}[\tilde{\varphi}^c, G^{cd}, ...] = - \sum_{n=m}^{\infty} \frac{1}{n!} K_{a_1...a_n} \circ \frac{\delta}{\delta G^{a_1...a_n}} G^{a_1...a_n}$$  (2.18)

The general Green functions $G^{abc...}$ can be expressed in terms of the connected Green functions $G^{abc...}$ yielding the formal equations of motion. They are formal in the sense that $\Gamma$ has not been computed explicitly. For our purpose this will not be necessary. The physical implications of such a Legendre transform were treated, for example, in [17] and [26]. At this point we leave the formal development of the CTP-formalism and turn to its application to the $\lambda \varphi^4$ model.

We consider the Lagrangian density

$$L = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4.$$  (2.19)

The corresponding CTP action is then

$$S_{CTP}[\varphi^a] = \int d^4x \frac{1}{2} c_{ab} \{ \partial_\mu \varphi^a \partial^\mu \varphi^b - m^2 \varphi^a \varphi^b \} - \frac{\lambda}{4!} h_{abcd} \varphi^a \varphi^b \varphi^c \varphi^d$$  (2.20)

where we have defined

$$h_{abcd} = \begin{cases} 1 & \text{when } a = b = c = d = 1 \\ -1 & \text{when } a = b = c = d = 2 \end{cases}.$$  (2.21)

In the next subsection we focus on the equations of motion for the 2-point connected Green functions, since they lie at the heart of the kinetic formulation of the non-equilibrium theory.

### 2.2 The Kinetic Formulation

In this subsection we derive the equations of motion specifically for the 2-point functions $G^{ab}(x, y)$ and write them as kinetic equations. In particular, a hierarchy of coupled equations for the $G^{abc...}$ is obtained in which the two-point function $G^{ab}$ is embedded. For practical calculations, this hierarchy of equations for the $G^{abc...}$ must be truncated. Calzetta and Hu [1], Kandrup and Hu [27] and, in a general context,
Hu have argued that this introduces dissipative phenomena in the model. On the other hand, if the field is initially free and the density operator is diagonal in momentum space, an explicit calculation of the nonlocal sources \( K \) is feasible, and shows that the series truncates after the two point source. Consequently if one is willing to accept limitations on the range of initial conditions, the series of the \( K \)'s can be truncated. It is in this spirit that we will truncate the expansion of \( \rho[\varphi^a] \) after \( K_{abcd}(u, v, w, x) \).

### 2.2.1 The Effective Action for the \( \lambda \varphi^4 \) Model

Truncating \( \Gamma \) after the fourth term, leads to the explicit form

\[
\Gamma[\varphi^a, G^{ab}, G^{abc}, G^{abcd}] = W[j_a, K_{ab}, K_{abc}, K_{abcd}] - j_a \circ \varphi^a - \frac{1}{2} K_{ab} \circ G^{ab} - \frac{1}{3!} K_{abc} \circ G^{abc} - \frac{1}{4!} K_{abcd} \circ G^{abcd}
\]

implying that \( \varphi^a \), \( G^{ab}(x, y) \), \( G^{abc}(x, y, z) \) and \( G^{abcd}(w, x, y, z) \) can be chosen as independent variables. A more convenient choice are the 2-point connected Green functions \( G^{ac}(x, y) \)

\[
G^{ab}(x, y) = 2G^{ab}(x, y) - \varphi^a(x) \varphi^b(y),
\]

and the 3- and 4- irreducible vertices of the theory, \( \alpha^3_{abc}(x, y, z) \) and \( \alpha^4_{abcd}(w, x, y, z) \) respectively, defined by

\[
\iota \alpha^3_{abc} \circ G^{aa'G^{bb'}G^{cc'}} = 3!G^{a'b'c'} - G^{a'b'} \varphi^{c'} - G^{a'c'} \varphi^b - G^{b'c'} \varphi^a - \varphi^a' \varphi^b' \varphi^c'
\]

and,

\[
\iota \alpha^4_{abcd} \circ G^{aa'G^{bb'}G^{cc'}G^{dd'}} + G^{aa'}G^{bb'} \circ \iota \alpha^3_{abmn} \circ G^{mm'} \circ \iota \alpha^3_{m'cd} \circ G^{cc'}G^{dd'} + \text{permutations}
\]

\[
= 4!G^{a'b'c'd'} - \text{disconnected terms ,}
\]

and which are represented diagrammatically in fig.1. As shown by Weldon, in the non-equilibrium formulation these vertices are related to the equilibration rate of the distribution of quasi-particles over the available energy-momentum space, and not to the decay-rate for a single particle.

### 2.2.2 The Equations of Motion

It is possible to extract all information needed to construct the equations of motion for \( G^{ab} \) from the dynamics of the mean field \( \varphi^a \) alone. For the sake of convenience we will always assume that the sources \( K_{abc} \) and \( K_{abcd} \) include the tree-level vertices. We proceed as follows. The equation of motion for \( \varphi^a \) is found by solving the identity

\[
0 = \int D\varphi^b \frac{\delta}{\delta \varphi^a} \exp \{ S_0 \ C T P[\varphi^b] + j_a \circ \varphi^a + \sum_{n=2}^4 \frac{1}{n!} K_{a_0...a_n} \circ \varphi^{a_0} \circ ... \circ \varphi^{a_n} \},
\]

(2.26)
where the free CTP action $S_{0\ CT\ P}$, given by

$$S_{0\ CT\ P} = \int d^4x \frac{c_{ab}}{2} \{ \partial_{\nu}\phi^a \partial^\nu\phi^b + m^2\phi^a\phi^b \},$$

appears due to our choice of absorbing the interaction vertices in the K’s. Working this out yields

$$G^{-1}_{(0)} \circ \varphi^b = j_a + \sum_{n=2}^{4} \frac{1}{n!} K_{a_{a_2}...a_n} \circ G^{a_2...a_n},$$

where we have denoted as $G^{-1}_{(0)}_{ab}$ the inverse of the free propagator

$$G^{-1}_{(0)}_{ab} = c_{ab} \{ \Box^2 - m^2 \}.$$  

From Eq.\[2.15\], it follows formally that

$$\frac{\delta \Gamma}{\delta \varphi^a} = -j_a - K_{ab} \circ \varphi^b - \frac{1}{2} K_{abc} \circ \frac{\delta G_{bcd}}{\delta \varphi^d} - \frac{1}{3!} K_{abcd} \circ \frac{\delta G_{bcd}}{\delta \varphi^d}. \quad (2.30)$$

Comparing Eq.\[2.28\] and Eq.\[2.30\], and using

$$\frac{\delta G^{a_2...a_n}}{\delta \varphi^a} = nG^{a_2...a_n}, \quad (2.31)$$

we can identify the l.h.s. of Eq.\[2.30\] to be

$$\frac{\delta \Gamma}{\delta \varphi^a} = G^{-1}_{(0)}_{ab} [G^{ab} \circ \varphi^b]. \quad (2.32)$$

$G^{-1}_{(0)}_{ab}$ is now a functional of the full propagator. The explicit connection is given by the usual Dyson equation. As in the zero-temperature formalism, Eq.\[2.32\] can be formally integrated with respect to $\varphi^a$ yielding

$$\Gamma[ \varphi^a, G^{ab}, \alpha_{abc}^3, \alpha_{abcd}^4] = \Gamma_0[ \varphi^a, G^{ab}] + \Gamma_{int}[G^{ab}, \alpha_{abc}^3, \alpha_{abcd}^4]. \quad (2.33)$$

$\Gamma_{int}$ can now be constructed from vacuum diagrams alone, in terms of $G^{ab}$, $\alpha_{abc}^3$ and $\alpha_{abcd}^4$, which simplifies the calculation considerably. To see this, consider, for example, a diagram D pertaining to $\Gamma_{int}$ with $n_G$ propagators, $n_3$ 3-point vertices and $n_4$ 4-point vertices. It has no external lines, since they would end in an $\varphi^a$, so

$$n_G = \frac{1}{2} \{ 3n_3 + 4n_4 \}. \quad (2.34)$$

Let $\Gamma_{int}(D)$ be the Feynman-amplitude of the diagram D contributing to the functional $\Gamma_{int}$. Cutting through one propagator line in D represents the action of a functional derivative, with respect to $G^{ab}$, on $\Gamma_{int}(D)$. The operator $G^{ab} \circ \frac{\delta}{\delta G^{ab}}$ thus counts the number of propagators in every diagram. Using the appropriate counting operators for the vertices, and using the fact that Eq.\[2.34\] holds for all diagrams contributing to $\Gamma_{int}$, one finds

$$G^{ab} \circ \frac{\delta}{\delta G^{ab}} \Gamma_{int} = \frac{3}{2} G_{abc} \circ \frac{\delta}{\delta G_{abc}} \Gamma_{int} + 2 \alpha_{abcd}^4 \circ \frac{\delta}{\delta \alpha_{abcd}^4} \Gamma_{int}. \quad (2.35)$$
The formal equation of motion for $G^{ab}$, after truncating the series of the $K_{abc...}$, is

$$\frac{\delta \Gamma}{\delta G^{ef}} [\varphi^c, G^{cd}, \alpha^3_{abc}, \alpha^4_{abcd}] = -\frac{1}{2} K_{ef} - \frac{1}{3!} K_{abc} \circ \frac{\delta G_{abc}}{\delta G^{ef}} - \frac{1}{4!} K_{abcd} \circ \frac{\delta G_{abcd}}{\delta G^{ef}}. \tag{2.36}$$

The l.h.s. of this equation can be expressed as

$$\frac{\delta \Gamma}{\delta G^{ef}} [\varphi^c, G^{cd}, \alpha^3_{abc}, \alpha^4_{abcd}] = \frac{\delta \Gamma_0}{\delta G^{ef}} [\varphi^c, G^{cd}, \alpha^3_{abc}, \alpha^4_{abcd}] + \frac{\delta \Gamma_{int}}{\delta G^{ef}} [\varphi^c, G^{cd}, \alpha^3_{abc}, \alpha^4_{abcd}]. \tag{2.37}$$

The first term here can be explicitly calculated from the functional integral for $Z$

$$\Gamma_0[\varphi^a, G^{ab}, \alpha^3_{abc}, \alpha^4_{abcd}] = S_0 \ctp [\varphi^a] - \frac{i}{2} \log \{G^{ab}\} + \frac{i}{2} G^{-1}_{(0) ab} \circ G^{ab}. \tag{2.38}$$

The last term can be written out using the formal equations of motion of the vertices $\alpha^3_{abc}$ and $\alpha^4_{abcd}$,

$$\frac{\delta \Gamma}{\delta \alpha_{abc}^3} = -\frac{i}{3!} G^{ad} G^{be} G^{cf} \circ \{K_{def} - K_{defg} \varphi^g\} - \frac{1}{4} G^{ad} G^{be} \circ K_{defg} \circ G^{ijh} G^{gij} \circ \alpha_{hij}^3 \circ G^{j^c}, \tag{2.39}$$

and $\alpha_{abcd}^4$,

$$\frac{\delta \Gamma}{\delta \alpha_{abcd}^4} = \frac{i}{4!} G^{ae} G^{bf} G^{cg} G^{dh} \circ K_{efgh}. \tag{2.40}$$

obtained from Eq. (2.13). After some tedious algebra, we find the equation for $G^{ab}$ to be

$$iG^{ab}_{-1} = iG^{-1}_{(0) ab} - \frac{\lambda}{2} h_{abcd} \circ \{\varphi^c \varphi^d + G^{cd}\} - \frac{i\lambda}{3!} h_{bece} \circ \alpha_{afgh}^4 \circ G^{fe} G^{pm} G^{he} \tag{2.41}$$
$$- \frac{i\lambda}{3!} h_{beef} \circ \varphi^g \alpha_{abcd}^3 \circ G^{ce} G^{df}$$
$$- \frac{\lambda}{3!} h_{bcde} \circ \alpha_{afgh}^3 \circ G^{fe} G^{pm} G^{gh} G^{he}$$
$$- \frac{\lambda}{3!} h_{cdef} \circ \alpha_{afgh}^3 \circ G^{fe} G^{pm} G^{gh} G^{he}$$

The solution of this equation requires knowledge of both vertices $\alpha^3_{abc}$ and $\alpha^4_{abcd}$. It describes the exact 2-point Green function for the non-equilibrium system, with initial conditions constrained by the truncation of the $K$’s after $K_{abcd}$. At this point, it makes no explicit reference to the dynamical role of inhomogeneities and off-shellness. The kinetic representation of Eq. (2.41) however allows one to address these questions directly. Since we aim at understanding the simplest effects of inhomogeneities in the field theoretic setting, there is no apparent need for allowing spontaneous symmetry breaking. This allows one to make an appreciable technical simplification of Eq. (2.41). We thus set the mean-field and the 3-vertex to zero, so that the last three terms vanish.

We progress toward the kinetic representation of the non-equilibrium dynamics by multiplying Eq. (2.41) once from the right by $G^{bc}$ and contracting over the index $b$ yielding

$$i\delta^c_a = iG^{-1}_{(0) ab} \circ G^{bc} - \frac{\lambda}{2} h_{abef} \circ G^{cd} G^{be} - \frac{i\lambda}{3!} h_{bece} \circ \alpha_{afgh} \circ G^{fe} G^{pm} G^{he} G^{bc}. \tag{2.42}$$

Here the differential operator $G^{-1}_{(0) ab}$ acts on $G^{ab}$. Multiplication of Eq. (2.41) from the left by $G^{ca}$ and contracting over the index $a$ yields

$$i\delta^a_b = iG^{ca} \circ G^{-1}_{(0) ab} - \frac{\lambda}{2} h_{abef} \circ G^{ca} G^{cf} - \frac{i\lambda}{3!} G^{ca} h_{bece} \circ \alpha_{afgh} \circ G^{fe} G^{pm} G^{he}. \tag{2.43}$$
In doing so we have formally inverted the operator \( G^{-1}_{ab}(x, y) \). In order to do so properly it is necessay to specify its kernel, i.e. the set of functions \( f(x) \) for which

\[
G^{-1}_{ab}(x, y) \circ f(y, z) = 0,
\]

\[
G^{-1}_{ab}(x, y) \circ f(x, z) = 0.
\]

(2.44)

This coincides with specifying boundary conditions on Eq. (2.42) and Eq. (2.43). Allowing inhomogeneities in the boundary conditions entails the need for both Eq. (2.42) and Eq. (2.43) instead of simply Eq. (2.41).

For a homogeneous system, \( G^{ab}(x, y) \) depends only on the relative coordinate \( \bar{x} = x - y \). It is therefore natural to consider \( G^{ab} \) as a function of \( \bar{x} \) and the centre of mass coordinate \( X = \frac{1}{2} (x + y) \). Its 4-Fourier transform \( G^{ab}(P, \bar{\rho}) \)

\[
G^{ab}(P, \bar{\rho}) = \int d^4X d^4\bar{x} \ e^{iPX+i\bar{x}\bar{\rho}} G^{ab}(X, \bar{x})
\]

(2.45)

is related to the Wigner transform \( G^{ab}(X, \bar{\rho}) \) by

\[
G^{ab}(X, \bar{\rho}) = \int \frac{d^4P}{(2\pi)^4} e^{-iPX} G^{ab}(P, \bar{\rho}) .
\]

(2.46)

Then Eq. (2.42) and Eq. (2.43) in terms of \( G^{ab}(P, \bar{\rho}) \) become

\[
i(\bar{\rho} + \frac{P}{2})^2 G^{ab}(P, \bar{\rho}) = \lambda \int \frac{d^4P'}{(2\pi)^4} \frac{d^4\bar{\rho}'}{(2\pi)^4} G^{ab}(P', \bar{\rho}') \exp\left(\frac{1}{2} P'.\partial \bar{\rho}\right) G^{ab}(P - P', \bar{\rho})
\]

\[
= \frac{i\lambda}{12} \int \frac{d^4P'}{(2\pi)^4} \Sigma^a(P - P', \bar{\rho}) \Delta^- G^{cb}(P', \bar{\rho}),
\]

(2.47)

and

\[
i(\bar{\rho} - \frac{P}{2})^2 G^{ab}(P, \bar{\rho}) = \lambda \int \frac{d^4P'}{(2\pi)^4} \frac{d^4\bar{\rho}'}{(2\pi)^4} G^{ab}(P', \bar{\rho}') \exp\{-\frac{1}{2} P'.\partial \bar{\rho}\} G^{ab}(P - P', \bar{\rho})
\]

\[
= \frac{i\lambda}{12} \int \frac{d^4P'}{(2\pi)^4} \Sigma^a(P - P', \bar{\rho}) \Delta^+ G^{cb}(P', \bar{\rho}),
\]

(2.48)

where the 2-loop self-energy, \( \Sigma_{ab} \), is given by

\[
\Sigma_{ab} = h_{bcde} \circ \alpha_{ac'}^{d'c'} \circ G_{cc'} G_{d'd''} G_{e'e''},
\]

(2.49)

and we have introduced the gradient operators

\[
\Delta^\pm = \exp\{\pm \frac{1}{2}(P - P').\partial \bar{\rho} \}.\]

(2.50)

In this expression, the arrows indicate the direction in which the differentiation acts. By subtracting Eq. (2.43) from Eq. (2.42) we obtain the kinetic equation

\[
i \bar{\rho}.P G^{ab}(P, \bar{\rho}) = \lambda \int \frac{d^4P'}{(2\pi)^4} \frac{d^4\bar{\rho}'}{(2\pi)^4} G^{ab}(P', \bar{\rho}') \sinh\left(\frac{1}{2} P'.\partial \bar{\rho}\right) G^{ab}(P - P', \bar{\rho})
\]

\[
= \frac{i\lambda}{12} \int \frac{d^4P'}{(2\pi)^4} \left(\Sigma^a \Delta^- G^{cb} - \Sigma^b \Delta^+ G^{ac}\right)
\]

(2.51)
and, by averaging the equations, Eq.\((2.43)\) and Eq.\((2.42)\), the constraint equation

\[
\iota\left\{\bar{p}^2 + \frac{1}{4}P^2 - m^2\right\}G^{ab}(P, \bar{p}) = \frac{\lambda}{2} \int \frac{d^4P'}{(2\pi)^4} \frac{d^4\bar{p}'}{(2\pi)^4} G^{aa}(P', \bar{p}') \cosh\left(\frac{1}{2}P' \cdot \partial \bar{p}\right) G^{ab}(P - P', \bar{p}) - \frac{\iota\lambda}{12} \int d^4P' \{\Sigma^a_{\Delta} \Delta^a G^c + \Sigma^b_{\Delta} \Delta^a G^{ac}\} = \iota c^{ab}(2\pi)^4 \delta^4(P),
\]

\[(2.52)\]

is obtained. The so-called gradient expansion for the Green functions can now easily be obtained by a truncation of the series expansions of the operators in Eq.\((2.50)\) or alternatively the sinh and cosh occurring in the Eqs.\((2.51)\) and \((2.52)\). It is a well-defined approximation whenever \(|P' \partial \bar{p}|\) is small. Since \(P'\) is integrated over, this suggests that the \(P'\)-integration is regulated at an ultraviolet limit \(\Lambda\) and the gradient is restricted similarly, i.e. \(|\partial \bar{p}| < \Lambda^{-1}\). Such a cutoff necessarily introduces a length scale. Furthermore it also alters the nature of the \(\lambda \phi^4\) model substantially. Both topics have been discussed in the introduction. If the operators are to act on functions containing resonance-like singularities in \(\bar{p}\), then the approximation is no longer valid close to the pole for any value of \(P\).

We close this section with a discussion of the physical content of the various terms in the kinetic and constraint equations.

### 2.2.3 Physical Interpretation

Under homogeneous boundary conditions, such as global thermal equilibrium (T.E.), we have

\[
G^{ab}(P, \bar{p}) \propto \delta^4(P),
\]

\[(2.53)\]

which, when substituted into Eq.\((2.51)\), makes the kinetic equation void. The constraint equation reduces to the Dyson equation. The variable \(\bar{p}\) can then be identified with the momentum of the propagating boson, and, for example, \(G^{21}(\bar{p})\) can be written in terms of a Bose-Einstein distribution function \(f(\bar{p})\) as

\[
G^{21}(\bar{p}) = \{\theta(\bar{p}^0) + f(\bar{p})\} \delta(\bar{p}^2 - m^2).
\]

\[(2.54)\]

Similarly, one can identify \(\bar{p} \cdot P \ G^{ab}(P, \bar{p})\) as the flow-term of the kinetic equation. The terms containing the sinh function in Eq.\((2.51)\) describe the effect of the interactions with a background of spontaneously created (off-shell) particles. Due to the analogy with the Vlasov interaction in plasma physics, we refer to it as the Vlasov term. The Vlasov term of Eq.\((2.51)\) describes a strictly local interaction of a single particle with the density of other particles. There is no long range component to this. This situation may change however when a mean field \(\tilde{\phi}^a\) is present that contains also non-local information.

The term in the r.h.s. of the kinetic equation is the collision term. On-shell, it describes the effect of two-particle interactions on the one-particle distribution function \(f\). The relation between the two-loop
self-energy on the one hand and the decay of the bosons is found from the cutting equations implementing unitarity \[32\]. The Boltzmann equation for a Bose-gas is obtained from the kinetic equation by inserting the form of the \(G^{ab}\) from Eq.\[2.54\] into Eq.\[2.51\], generalizing \(f(\vec{p})\) to \(f(P, \vec{p})\) and neglecting all gradients in \(\vec{p}\). This equation then describes on-shell bosons. The previous remarks on the validity of the gradient expansion apply here. On-shellness is not compatible with neglecting gradients. The problem can be softened by assuming that the boson self-energy has an imaginary part, so that \[28\] the distribution \(f\) is not in equilibrium. In that case, the \(\delta(\vec{p}^2 - m^2)\) would be replaced by a Lorentzian \[23\], thus allowing \(|P'.\partial\vec{p}|\) to stay bounded for finite \(P'\). Such an imaginary parts does not arise as long as the collision term is neglected.

The physical meaning of the variable \(P\) can be clarified by analyzing the transition amplitude between two states, \(<1, p^\mu | \text{and} | q^\mu, 2\rangle\), that contain at least one, reducible, on-shell particle with initial incoming momentum \(p\), and final outgoing momentum \(q\). In our convention the momenta are directed towards the interaction vertex. By definition

\[
p^\mu - \{-q^\mu\} = P^\mu. \tag{2.55}
\]

So \(P\) is the momentum loss due to interactions with a non-trivial background. The fact that for homogeneous systems \(P = 0\) indicates that the interpretation is of a statistical nature. In homogeneous systems the particles will interact with the background, but the effects will average out. It is easy to show that

\[
p^2 = q^2 = \vec{p}^2 = m^2 \implies p' = q'. \tag{2.56}
\]

If \(G^{ab}(P, \vec{p})\) is restricted to being on-shell, with \(\vec{p}\) on-shell as well, then it contains no dynamical information on the inhomogeneities.

Thus far, we have derived the kinetic field theory for self interacting bosons. In the homogeneous limit the functions \(G^{ab}(P, \vec{p})\) describe the distribution of on-shell bosons over the available energy and momentum states. In this limit, \(\vec{p}\) is the momentum of the propagating bosons. When we only require the asymptotic states to be on-shell \(P\) can be interpreted as the loss of momentum due to interactions with inhomogeneities in the system.

In the next section, we will discuss a symmetry property of non-equilibrium field theories. It generates identities among Green functions that, among other things, express the impact of initial conditions on these Green functions. It serves to yield a satisfying interpretation of the 2-point functions and their arguments beyond the limit of quasi-homogeneity.
3 The Broken SO(1,1) Symmetry of Non-Equilibrium Field Theory

3.1 SO(1,1) Symmetry

In this section, we focus on a symmetry underlying the non-equilibrium dynamics of the scalar field. But first let us make some general remarks. The transition from standard quantum field theory to the CTP formalism gives rise to a doubling of the number of fields. Every physical field becomes a CTP doublet, and on the internal space of these CTP doublets, we have defined the degenerate metric $c_{ab}$. Now consider a theory containing an n-tuple of fermions (and/or bosons) symmetric under SU(n) transformations. The free CTP action will then, through doubling, be symmetric under SU(n,n). As expected, this symmetry is broken, although an $SU(n) \otimes SU(n)$ subgroup remains. Now SU(n,n) has $(2n)^2 - 1$ generators, whereas $SU(n) \otimes SU(n)$ only has $2n^2 - 2$ generators. Consequently the Noether theorem will yield $2n^2 + 1$ currents whose non-conservation can be computed exactly. For a broken symmetry, Ward identities can be derived and considered useful if, either the symmetry breaking terms are small or, symmetry violation is calculable. Here we will consider the symmetry relevant to the $\lambda \varphi^4$ model. The real scalar field has a trivial SO(1) symmetry, which under CTP doubling becomes a broken SO(1,1) symmetry. To our knowledge this has gone unnoticed, although there is a relation to the SU(1,1) lie algebra used in quantum optics [29]. This section focuses on the SO(1,1) symmetry of the free real scalar field. We will give the Noether current, the Ward-Takahashi identities, discuss its breaking by interactions and initial conditions.

The CTP action for the free field

$$S_{CTP}[\varphi^a] = \int d^4x \frac{1}{2} c_{ab} \left\{ \partial_\mu \varphi^a \partial^{\mu} \varphi^b - m^2 \varphi^a \varphi^b \right\}$$

is invariant under global SO(1,1) transformations. The generator of these is

$$\tau^b_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and an infinitesimal transformation applied to the fields $\varphi^a$ yields the change

$$\delta \varphi^a = \alpha \tau^a_b \varphi^b, \quad |\alpha| << 1,$$

The associated SO(1,1) Noether current is

$$j^\nu = \epsilon_{ab} \varphi^a \partial^\nu \varphi^b.$$
Its zero’th component is related to the canonical commutator by

\[ j^0(\vec{x}, t) = \lim_{\vec{y} \to \vec{x}} \langle [ \hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t) ] \rangle, \] (3.6)

so that we have

\[ j^0(P) = \int \frac{d^3 \vec{p}}{(2\pi)^4} \delta^4(P) \] (3.7)

from canonical quantization. It implies that \( j^0(X) \) is an infinite constant. To see the meaning of this, consider the expectation value of the commutator of two fields

\[ \langle [ \hat{\phi}(x), \hat{\phi}(y) ] \rangle = G^{21}(x, y) - G^{12}(x, y). \] (3.8)

It represents a pulse arriving at the point \( x = y \) from the backward lightcone, and subsequently going out into the forward lightcone \[30\]. The temporal charge density is the amplitude corresponding to this, summed over all possible momenta. Put differently, \( j^0(x) \) is the amplitude density for absorbing a particle at \( x \) and subsequently emitting it. That it is infinite is due to the fact that it is obtained from

\[ \frac{\partial}{\partial \vec{x}^0} \{ G^{21}(X, \vec{x}) - G^{12}(X, \vec{x}) \}, \] (3.9)

evaluated at \( \vec{x}^0 = 0 \). It represents the amplitude density for absorbing a particle and reemmiting it at a relative distance \( \vec{x} \). Physically, at \( \vec{x}^0 = 0 \) the only contribution can come from \( \vec{x} = 0 \), yielding the delta function. The infinity is a result of the expansion of these amplitudes in terms of plane waves. The use of extended wavepackets on the other hand would yield finite expressions. We see that the propagation of particles is represented by subsequent absorptions and reemissions. In the free field vacuum this amplitude is conserved. The temporal charge density is a creation/annihilation density, whose free field vacuum value describes the free propagation of particles. It is sensible to make a vacuum subtraction setting the free field temporal charge equal to zero, which corresponds to normal ordering.

Analyzing the eigenstates of the SO(1,1) generator sheds light on the interpretation of \( j^0(P) \) in terms of particles. The fields

\[ \hat{\phi}^\pm = \frac{1}{\sqrt{2}} \{ \hat{\phi}^1 \pm \hat{\phi}^2 \} \] (3.10)

are normalised eigenvectors of the SO(1,1) generator. These are the generators of the Keldysh base i.e. the physical representation \[12\]. We assign the eigenvalues \( q_{\pm} = \pm 1 \), or so-called temporal charge, to these. With respect to particle/anti-particle exchange, the states generated by \( \hat{\phi}^+ \) are symmetric and those generated by \( \hat{\phi}^- \) are anti-symmetric. Since, for neutral bosons, particles and anti-particles are indistinguishable, the 1-particle, negative temporal charge states are redundant. In the physical limit, the field carrying negative temporal charge \( < \hat{\phi}^- > \) vanishes. The field-operator \( \hat{\phi}^+ \) creates a symmetric state out of the vacuum, representing a physical particle.

Let us see if the propagator is consistent with the demand that physical particles always move into the forward light-cone. The propagator is given by \( T_p \{ \hat{\phi}^- \hat{\phi}^+ \} \). By considering the retarded and advanced
2-point Green functions, given by

\[
\begin{align*}
G_{\text{ret}}(x, y) &= \langle \varphi^+(x) \varphi^-(y) \rangle \\
G_{\text{adv}}(x, y) &= \langle \varphi^+(y) \varphi^-(x) \rangle 
\end{align*}
\]  
(3.11)

we can identify the propagators for the particles with + and - temporal charge respectively. They represent respectively the outgoing and the incoming particles. In terms of these functions we can write

\[
j^\nu(P) = \int \frac{d^4 \bar{p}}{(2\pi)^4} \bar{p}^\nu \{ G_{\text{ret}}(P, \bar{p}) - G_{\text{adv}}(P, \bar{p}) \}
\]  
(3.12)

showing that the temporal current is directly related to the absorptive part of the 2-point Green function.

Thus we find that

\[
\{ G^{21}(P, \bar{p}) - G^{12}(P, \bar{p}) \}
\]  
(3.13)

is the creation/annihilation density, in \((P, \bar{p})\)-space, corresponding to momentum-loss \(P\) and the momentum-flow \(\bar{p}\). Since we have lost all coordinate space information, we cannot say in what region of coordinate space the momentum flows and where it is lost. However such questions can be addressed by constructing appropriate wave-packets. The commutation relation Eq. (3.6) expresses that if one integrates out the energy-flow, then \(P = 0\). Put differently, the momentum is lost to other particles with different energies but it is not destroyed. The temporal current is the total flow of the creation/annihilation density in terms of either a location, \(j^\nu(X)\), or the energy-momentum dissipation \(j^\nu(P)\). In detailed balance, the vacuum subtracted flow will be zero. The total temporal charge

\[
Q(t) = \int d^3 x \, \epsilon_{ab} \varphi^a \partial_0 \varphi^b ,
\]  
(3.14)

at time \(t\) is the total creation/annihilation amplitude. Its Fourier transform is

\[
Q(e) = \int dt e^{iet} Q(t)
\]  
(3.15)

the amplitude associated with the energy-loss \(e\). The divergence of the temporal current is thus the amplitude for the energy-momentum loss \(P\) due to all interactions.

The divergence of the temporal current is related to the integrated kinetic equations for \(G^{21}\) and \(G^{12}\) since

\[
P^\nu j^\nu(P) = \int \frac{d^4 \bar{p}}{(2\pi)^4} \bar{p}^\nu \{ G^{21}(P, \bar{p}) - G^{12}(P, \bar{p}) \} .
\]  
(3.16)

As expected, and as can be seen from Eq. (2.51), the global SO(1,1) symmetry is broken by the \(\lambda \varphi^4\) self-interaction. There is, however, another source of SO(1,1) symmetry breaking. Consider the free-field, vacuum, 2-point functions. If SO(1,1) symmetry were realized then \(G^{ab}\) would be expressible as

\[
G_{\text{sym}}^{ab}(\bar{p}) = g_1 c^{ab} + g_2 \epsilon^{ab} .
\]  
(3.17)
with \( g_1 \) and \( g_2 \) functions of the momentum \( \vec{p} \). The actual Green function can be decomposed however to display the form

\[
G_{\text{vac}}^{ab} = (\vec{p}^2 - m^2 - i\alpha)^{-1} \epsilon^{ab} - 2\pi i \delta(\vec{p}^2 - m^2) \epsilon^{ab} - 2\pi i \delta(\vec{p}^2 - m^2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]  

(3.18)

where one sees that the third term explicitly breaks SO(1,1) symmetry. A proper definition of the non-equilibrium theory, i.e. fixing the initial conditions, also requires the symmetry to be broken.

### 3.2 Initial Conditions Breaking SO(1,1)

For the sake of simplicity, we examine the symmetry breaking by a 2-point kernel \( K_{ab}(x, y) \). This covers both equilibrium and non-equilibrium initial conditions, that describe an inhomogeneous system of free bosons. If \( \lambda = 0 \), then the divergence of the temporal current is

\[
\partial_{\nu} j^{\nu}(x) = \int d^4 y \ K_{ab}(x, y) \varphi^a(y) \tau^b \varphi^c(x).
\]  

(3.19)

If this is integrated over \( x \), after vacuum subtraction, we obtain the total particle production amplitude,

\[
N_{\text{tot}} = K_{ab} \circ \{ G_{ab} - G_{\text{vac}}^{ab} \}.
\]  

(3.20)

These particles are created to form the initial state. Taking the most general form for the symmetry-breaking kernel to be

\[
K_{ab}(x, y) = k_1(x, y) \delta_{ab} + k_2 \tilde{\tau}_{ab}
\]  

(3.21)

with

\[
\tilde{\tau}_{ab} = \tau^a_b.
\]  

(3.22)

and using

\[
G^{21} + G^{12} = G^{11} + G^{22} = G_{\text{corr}},
\]  

(3.23)

we see the divergence of the temporal current is directly related to the 2-point correlation function

\[
\partial_{\nu} j^{\nu}(x) = \int d^4 y \left\{ k_1(x, y) + k_2(x, y) \right\} G_{\text{corr}}(x, y) \}
\]  

(3.24)

In terms of temporal charge fields we have

\[
G_{\text{corr}}(x, y) = \langle \phi^+(x) \phi^+(y) \rangle,
\]  

(3.25)

representing the correlation among the particle production at \( x \) and at \( y \). So far we have discussed the expectation values of the SO(1,1) current operator. However in the context of Quantum Field Theory symmetries, even explicitly broken symmetries, generate a whole hierarchy of identities among Green functions.
By applying the usual techniques to the functional integral, assuming the absence of anomalous breaking of the SO(1,1) symmetry, we can derive Ward-Takahashi identities for the broken symmetry. In the case of initial conditions symmetry-breaking (ICSB) discussed above we find

\[
\frac{\partial}{\partial z^\nu} \langle j^\nu(z) \prod_{j=1}^n \phi^{a_j}(x_j) \rangle = \sum_{j=1}^n (\delta^4(z - x_j) \tau_e^{a_j} \phi^e(z) \prod_{i \neq j}^n \phi^{a_i}(x_i)) \\
+ \int d^4y \, K_{ab}(z, y) \langle \phi^a(z) \phi^b(y) \prod_{j=1}^n \phi^{a_j}(x_j) \rangle,
\]

(3.26)

where the braces indicate functional integral averages. For \( n = 0 \) we recover Eq. (3.19). For \( n > 0 \) we expect the first term to become small as \( z \to \infty \) and the application of Gauss theorem to the l.h.s. of Eq. (3.26) leads to the relation

\[
\sum_{j=1}^n \tau^{a_j}_e \langle \phi^e(x_j) \prod_{i \neq j}^n \phi^{a_i}(x_i) \rangle = -K_{ab} \circ \langle \phi^a \phi^b \prod_{j=1}^n \phi^{a_j}(x_j) \rangle.
\]

(3.27)

For \( n = 1, 2 \) we have explicitly

\[
\tilde{\varphi}^a(x) = -K_{bc}(y, z) \circ G^{bca}(y, z, x),
\]

(3.28)

and

\[
G_{cor}(x, y) = -K_{bc}(z, z') \circ G^{bc11}(z, z', x, y),
\]

(3.29)

representing the contribution of the initial conditions to particle distributions at later times.

We have seen how initial conditions break SO(1,1) symmetry in order to prepare the initial state, and we have calculated the effect this has on the CTP Green functions. The self-interaction also breaks the symmetry and we will deal with that next.

### 3.3 Interactions Breaking SO(1,1)

The divergence in the temporal current due to interactions is

\[
\partial_\nu j^\nu(x) = -\frac{\lambda}{3!} h_{abcd} \tau^d_e \varphi^a(x) \varphi^b(x) \varphi^c(x) \varphi^e(x).
\]

(3.30)

This explicit symmetry breaking (ESB) is local and therefore persistent, unlike the case discussed above. A simple example of the violation of particle number by interactions is the decay of one boson into three bosons. Note that Eq. (3.30) is just the \( d^4 \, \tilde{\rho} \) integrated kinetic equation for \( G^{21} - G^{12} \). The integrated Vlasov- and collision terms can all be recovered from Eq. (3.30), and interpreted along the lines of (off-shell) particle creation and annihilation in the vacuum of the interacting theory.

For ESB we also give a Ward-Takahashi identity, assuming no ICSB one finds

\[
\frac{\partial}{\partial z^\nu} \langle j^\nu(z) \prod_{j=1}^n \phi^{a_j}(x_j) \rangle = \sum_{j=1}^n (\delta^4(z - x_j) \tau^{a_j}_e \phi^e(z) \prod_{i \neq j}^n \phi^{a_i}(x_i)) \\
- \frac{\lambda}{3!} h_{abcd} \tau^d_e \langle \phi^a(z) \phi^b(z) \phi^c(z) \phi^e(z) \prod_{j=1}^n \phi^{a_j}(x_j) \rangle.
\]

(3.31)
Once again for $n > 0$, we expect an integrated identity to hold. We find

$$ \sum_{j=1}^{n} \tau_{c}^{a_{j}} G^{a_{1}...a_{j-1}c_{a_{j+1}}...a_{n}} = \frac{\lambda}{3!} h_{abcd} \circ \tau_{c}^{d} G^{abc_{a_{1}}...a_{n}}. $$

(3.32)

Again we give to explicit examples, for $n = 1$

$$ \varphi^{a} = \frac{\lambda}{3!} h_{bcde} \circ \tau_{f}^{a} \tau_{g}^{c} G^{bc_{d}gf}, $$

(3.33)

and for $n = 2$, we find an equation for the correlation function

$$ G_{corr}(x, y) = \frac{\lambda}{3!} h_{abcd} \tau_{c}^{d} \circ G^{abc_{1}11}. $$

(3.34)

In any realistic case, both mechanisms of SO(1,1) breaking contribute. The broken SO(1,1) symmetry generates a hierarchy of equations for the Green functions. The generalization of Eq. (3.26) and Eq. (3.31) to higher-order kernels $K_{abc...}$ and other polynomial interactions is straightforward.
4 Energy-Momentum Conservation

In this section, we show that the divergence of the canonical energy-momentum tensor vanishes due to the kinetic and constraint equations. It will turn out that the true difference between Vlasov- and collision-terms lies in the (non-)locality of the interactions that they represent.

The energy-momentum tensor in the $\lambda\varphi^4$ model is given by

$$T_{\mu\nu} = \partial_{\mu}\varphi\partial_{\nu}\varphi - \eta_{\mu\nu}\frac{1}{2}(\partial_{\alpha}\varphi\partial^\alpha\varphi - m^2\varphi^2 - 2\frac{\lambda}{4!}\varphi^4)$$

$$= T_{\mu\nu}^0 + \frac{\lambda}{4!}T_{\mu\nu}^{int}, \quad (4.1)$$

where $T_{\mu\nu}^0$ refers to the free field energy-momentum tensor and $T_{\mu\nu}^{int}$ to the interacting part. In terms of 2-point functions this can be written as

$$\langle T_{\mu\nu}^0 \rangle = \frac{1}{2}\int \frac{d^4\vec{p}}{(2\pi)^4}\left\{ -\frac{1}{4}P^\mu P^\nu + \vec{p}^\mu\vec{p}^\nu - \frac{1}{2}\eta^{\mu\nu}(\vec{p}^2 - \frac{1}{4}P^2 - m^2) \right\} G_{corr}(P, \vec{p}), \quad (4.2)$$

where $G_{corr}$ has been defined in Eq.(3.23). In deriving this expression, linear terms in $\vec{p}^\nu$ have been dropped, because they do not contribute to the integral since $G_{corr}(P, \vec{p})$ is symmetric in $\vec{p}^\nu$. The interaction contribution to the energy momentum tensor is connected with the coincidence limit of the four point Green functions. One possible choice is

$$\langle T_{\mu\nu}^{int} \rangle = \frac{3}{2}\eta^{\mu\nu}\{G^{11}(x,x)G^{12}(x,x) + G^{12}(x,x)G^{21}(x,x)\}. \quad (4.3)$$

It is important to recognize that the interaction part also contains the contribution of the independent variable $\alpha_{abcd}^4$. Energy-momentum conservation requires

$$P_\mu\langle T_{\mu\nu}^0 \rangle = \frac{\lambda}{4!}P_\mu(T_{\mu\nu}^{int}). \quad (4.4)$$

We will investigate this relation for three cases. First we will consider the free field situation, followed by a verification of Eq.(4.4) taking into account the Vlasov interaction and, finally, treating the full theory. Our procedure will be to use the kinetic and constraint equations to calculate the l.h.s. of Eq.(4.4) from Eq.(4.2) and conclude the equality with the r.h.s. of Eq.(4.4).

For the free field case, $\lambda = 0$, the kinetic and constraint equations read

$$\begin{cases} \vec{p}.P G^{ab}(P, \vec{p}) = 0 \\
\{ \vec{p}^2 + \frac{1}{4}P^2 - m^2 \}G^{ab}(P, \vec{p}) = \epsilon^{ab}(2\pi)^4\delta^4(P) \end{cases}. \quad (4.5)$$

Applying $P_\mu$ to Eq.(4.2), it is straightforward to see that the l.h.s. of Eq.(4.4) vanishes. In the Vlasov approximation, we neglect the collision terms, i.e. we treat the 4-point Green function as a disconnected product of 2-point functions. So the interaction term is

$$\langle T_{\mu\nu}^{int} \rangle = \frac{3}{2}\eta^{\mu\nu}\{G^{11}(x,x)G^{12}(x,x) + G^{22}(x,x)G^{21}(x,x)\}. \quad (4.6)$$
The factor 3 counts the possible ways of writing a 4-point Green function as a disconnected product of 2-point Green functions. The kinetic and constraint equations are now given as

\[
\begin{align*}
&\iota\,P\,G^{ab}(P, \bar{\rho}) - \frac{1}{2} \int \frac{d^4 P'}{(2\pi)^4} \frac{d^4 \bar{\rho}'}{(2\pi)^4} G^{aa}(P', \bar{\rho}') \sinh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}ight) G^{ab}(P - P', \bar{\rho}) = 0 \\
&\iota\{ \bar{\rho}^2 + \frac{1}{2} P_\mu^2 - m^2\} G^{ab}(P, \bar{\rho}) - \frac{1}{2} \int \frac{d^4 P'}{(2\pi)^4} \frac{d^4 \bar{\rho}'}{(2\pi)^4} G^{aa}(P', \bar{\rho}') \cosh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{ab}(P - P', \bar{\rho}) \\
&= \iota e^{ab}(2\pi)^4 \delta^4(P)
\end{align*}
\]  

(4.7)

Applying $P_\mu$ to Eq. (4.2) and using the Vlasov equations yields

\[
P_\mu(T_{0}^{\mu\nu})(P) = \frac{1}{2} \int \frac{d^4 \bar{\rho}}{(2\pi)^4} \int \frac{d^4 P'}{(2\pi)^4} \frac{d^4 \bar{\rho}'}{(2\pi)^4} \bar{\rho}' \left( G^{22}(P', \bar{\rho}') \sinh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{21}(P - P', \bar{\rho}) + G^{11}(P', \bar{\rho}') \sinh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{12}(P - P', \bar{\rho}) \right) + \frac{P_\nu}{2} \left( G^{22}(P', \bar{\rho}') \cosh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{21}(P - P', \bar{\rho}) + G^{11}(P', \bar{\rho}') \cosh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{12}(P - P', \bar{\rho}) \right).
\]  

(4.8)

The $d^4 \bar{\rho}$-integration in the terms containing the sinh and cosh functions can be performed. For the sinh terms this yields

\[
\int \frac{d^4 \bar{\rho}}{(2\pi)^4} \bar{\rho}' G^{aa}(P', \bar{\rho}') \sinh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{ab}(P - P', \bar{\rho}) = -\frac{P_\nu}{2} G^{aa}(P', \bar{\rho}') \int \frac{d^4 \bar{\rho}}{(2\pi)^4} G^{ab}(P - P', \bar{\rho}),
\]  

(4.9)

to be integrated over $P'$ and $\bar{\rho}'$. For the cosh terms one can use

\[
\int \frac{d^4 \bar{\rho}}{(2\pi)^4} \frac{P_\nu}{2} G^{aa}(P', \bar{\rho}') \cosh\left(\frac{1}{2} P' \cdot \partial P' \bar{\rho}\right) G^{ab}(P - P', \bar{\rho}) = -\frac{P_\nu}{2} G^{aa}(P', \bar{\rho}') \int \frac{d^4 \bar{\rho}}{(2\pi)^4} G^{ab}(P - P', \bar{\rho}).
\]  

(4.10)

These terms generate the r.h.s. of Eq. (4.4). Note that, since the Vlasov term has gradient operators between a $\bar{\rho}$-independent factor and $G^{ab}(P, \bar{\rho})$, the gradients play no role in energy-momentum conservation.

In the fully interacting case, the confirmation of Eq. (4.4) becomes difficult as long as we stay in $(P, \bar{\rho})$-space. Substitution of Eq. (2.51) and Eq. (2.52) into the l.h.s. of Eq. (4.4) now yields

\[
P_\mu(T_{0}^{\mu\nu}(P)) = \frac{1}{2} \int \frac{d^4 \bar{\rho}}{(2\pi)^4} \frac{d^4 P'}{(2\pi)^4} \bar{\rho}' \left\{ \Sigma^2_c(P - P', \bar{\rho}) \Delta^{-} G^{c1}(P', \bar{\rho}) + \Sigma^1_c(P - P', \bar{\rho}) \Delta^{-} G^{c2}(P', \bar{\rho}) - G^{22}(P - P', \bar{\rho}) \Delta^{-} G^{21}(P', \bar{\rho}) \right\} + \frac{P_\nu}{2} \left\{ \Sigma^2_c(P - P', \bar{\rho}) \Delta^{-} G^{c1}(P', \bar{\rho}) + \Sigma^1_c(P - P', \bar{\rho}) \Delta^{-} G^{c2}(P', \bar{\rho}) + G^{22}(P - P', \bar{\rho}) \Delta^{-} G^{21}(P', \bar{\rho}) \right\}
\]  

(4.11)

where we have already used the fact that the free field and Vlasov terms either vanish or are taken care of by corresponding terms in the interaction part. The $d^4 \bar{\rho}$-integration in Eq. (4.11) is not trivial, as it
was for the Vlasov term. All the gradients must be included. We do this by transforming Eq.(4.11) back to coordinate-space. There we have

\[
\lim_{y \to x} \frac{1}{2} \frac{\lambda}{4!} \int d^4 z \left\{ \frac{\partial}{\partial x^\nu} \left[ \Sigma^2(x, z) G^{c_1}(z, y) + \Sigma^1(x, z) G^{c_1}(z, y) \right] + \frac{\partial}{\partial y^\nu} \left[ G^2(y, z) \Sigma^{c_1}(z, x) + G^1(y, z) \Sigma^{c_1}(z, x) \right] \right\}, \tag{4.12}
\]

which we can identify as the point-split implementation of the product rule by using the symmetry of the Green functions under the coherent exchange of CTP indices and the corresponding space-time arguments. Consequently we have the divergence of the free field energy-momentum tensor expressed in terms of the connected 4-point Green function

\[
\partial_\mu \langle T^{\mu \nu}_0 \rangle = \frac{\lambda}{4!} \frac{\lambda}{2} \left\{ G^{2221}(x, x, x, x) + G^{1112}(x, x, x, x) \right\}. \tag{4.13}
\]

Using Eq.(4.6) we see that Eq.(4.4) is satisfied, so that in general energy and momentum are conserved. Contrary to energy-momentum conservation for the Vlasov terms we required all the gradients. The physical reason for this lies in the nature of the interactions that these terms represent. The Vlasov-term results from \textit{local} interactions with (off-shell) particles from the background. The interaction vertex is the point-like \(\lambda \phi^4\) interaction. The collision terms contain \textit{non-local} interactions, like, for example, the two-boson exchange interaction. The gradients probe the locality of these interactions. Neglecting gradients thus violates energy-momentum conservation, unless specific non-local interactions are added to the Lagrangian density.

Energy-momentum conservation is guaranteed by the kinetic- and constraint equations. This points to the fact that any inhomogeneities occurring in such a field theoretic model come from a spontaneous breaking of translation invariance. A gradient-expansion violates locality if used in the collision term. A Vlasov-approximation is restricted to the local response of the system to inhomogeneities. The temporal current non-conservation is a differential equation for the difference \(G^{21} - G^{12}\) in the coincidence limit and energy-momentum conservation is a differential equation for the sum \(G^{21} + G^{12}\) in the coincidence limit. The two equations, given appropriate boundary conditions, consequently determine the coincidence limit of the 2-point Green functions. Put differently, energy-momentum conservation and temporal current non-conservation determine the U.V.behaviour of the dressed propagators. In the next section we analyse a free field case in the presence of inhomogeneities.

### 5 Free field

In the previous sections we have seen that energy-momentum conservation and temporal current non-conservation equations determine the coincidence limits of \(G^{21}(x, y)\) and \(G^{12}(x, y)\). Inhomogeneities have played a major role in establishing the conservation since they reflect the energy-momentum exchange
between the particles. Inhomogeneities on the other hand make their appearance in the gradient expansions of the interaction terms in the kinetic and constraint equations, but also they appear quadratically in the free field equations. In this section we will concentrate on the free field equations

\[
\begin{cases}
\bar{p}.P G^{ab}(P, \bar{p}) = 0 \\
\{ \bar{p}^2 + \frac{1}{4}P^2 - m^2 \} G^{ab}(P, \bar{p}) = c^{ab}(2\pi)^4 \delta^4(P)
\end{cases}
\]  

(5.1)

To find the solution to Eq.(5.1) for arbitrary initial data, we exploit the linearity of these equations. It is used to discuss the effect of inhomogeneities and off-shellness. Following this we give an explicit example of an inhomogeneous non-equilibrium system.

### 5.1 General solution

By inspection of Eq.(5.1) one sees that the decomposition of \( G^{ab} \) into a vacuum contribution, \( G_{\text{vac}}^{ab} \), and a medium term, \( \tilde{G}^{ab} \),

\[
G^{ab}(P, \bar{p}) = G_{\text{vac}}^{ab}(P, \bar{p}) + \tilde{G}^{ab}(P, \bar{p})
\]  

(5.2)

solves the free field equations, where \( G_{\text{vac}}^{ab} \) is

\[
G_{\text{vac}}^{ab}(P, \bar{p}) = (2\pi)^4 c^{ab} \frac{\delta^4(P^\nu)}{\bar{p}^2 + \frac{1}{4}P^2 - m^2}
\]  

(5.3)

and \( \tilde{G}^{ab}(P, \bar{p}) \) satisfies the set of homogeneous equations

\[
\begin{cases}
\bar{p}.P \tilde{G}^{ab}(P, \bar{p}) = 0 \\
\{ \bar{p}^2 + \frac{1}{4}P^2 - m^2 \} \tilde{G}^{ab}(P, \bar{p}) = 0
\end{cases}
\]  

(5.4)

Since \( G_{\text{vac}}^{ab} \) describes the propagation of free particles, their \( P \)-dependence is a simple \( \delta^4(P) \) as expected from the absence of interactions. All the momentum-loss information is included in \( \tilde{G}^{ab} \). It is the momentum-loss suffered by the particles during the preparation of the initial condition. Particle propagation at later times will not couple to these initial inhomogeneities and thus not dissipate them.

Let us consider that \( G^{ab}(X, \bar{x}) \) is given by an initial value at \( X^0 = 0 \) and \( \bar{x}^0 = 0 \) and denote it as \( G_0^{ab}(X^i, \bar{x}^j) \). Since we work in momentum space, it is useful to introduce the Fourier transform of the initial conditions \( G_0^{ab}(P^i, \bar{p}^j) \)

\[
\tilde{G}_0^{ab}(P^i, \bar{p}^j) = \int d^3 X \int d^3 \bar{x} e^{-iP^\mu X^\mu} \bar{p}_\nu \delta(\bar{p}^\nu P^\nu) G_0^{ab}(X^i, \bar{x}^j).
\]  

(5.5)

This Fourier transform can also be decomposed in the fashion of Eq.(5.2) as \( G_0^{ab} = G_{0\text{vac}}^{ab} + \tilde{G}_0^{ab} \). The linearity of the equations implies the solution for \( \tilde{G}^{ab} \) has the form

\[
\tilde{G}^{ab}(P, \bar{p}) = \tilde{G}_0^{ab}(P^\nu, \bar{p}^\nu) \delta(\bar{p}^\nu P^\nu) \delta(\bar{p}^2 + \frac{1}{4}P^2 - m^2).
\]  

(5.6)

Due to the symmetry, \( G^{21}(P, \bar{p}) = G^{12}(P, -\bar{p}) \), of the off-diagonal 2-point functions and the relation

\[
G^{11}(X, \bar{x}) = \theta(\bar{x}^0)G^{21}(X, \bar{x}) + \theta(-\bar{x}^0)G^{12}(X, \bar{x}),
\]  

(5.7)
specifying the diagonal elements in terms of the off-diagonal elements, the four functions $\tilde{G}^{ab}$ contain only one unknown function $g(P, \bar{p})$, satisfying Eq. (5.4). In thermal equilibrium, it is well-known that $g(P, \bar{p})$ is given as

$$g(P, \bar{p}) = g_{BE}(p^0)\delta(\bar{p}^2 - m^2)\delta^4(P),$$

(5.8)

where $g_{BE}(p^0)$ is the Bose-Einstein distribution function.

Consider the Wigner representation of the general solution

$$g(X, \bar{p}) = \int \frac{d^4P}{(2\pi)^4} e^{ip\cdot X} \delta(\bar{p}_\mu P^\mu)\delta(\bar{p}^2 + \frac{1}{4}P^2 - m^2)g(P, \bar{p})$$

(5.9)

Since we deal with massive particles $\bar{p}^0 \neq 0$ and two of the four $P$ integrations, say over $P^0$ and the component parallel to $\bar{p}^i$,

$$P^i = \frac{P^0}{\bar{p}^0} \bar{p}^i,$$

(5.10)

can be performed using

$$\begin{cases}
\delta(\bar{p}_\mu P^\mu) = \frac{1}{|\bar{p}|} \delta(P_0 - \bar{p}_0/P^0) \\
\delta(\bar{p}^2 + \frac{1}{4}P^2 - m^2) = \frac{1}{|\gamma\alpha|} \{\delta(P^0 - 2\gamma\alpha) + \delta(P^0 + 2\gamma\alpha)\}
\end{cases}$$

(5.11)

where we have defined the measure of off-shellness $\alpha$ by

$$\alpha = \sqrt{\bar{p}^2 - m^2 - \frac{1}{4}P^2},$$

(5.12)

and

$$\gamma = \frac{1}{\sqrt{1 - \frac{\bar{p}^0}{P^0}}},$$

(5.13)

and used the first identity in Eq. (5.11) to obtain the second. In these coordinates $\bar{p}$ has only two non-vanishing elements: $\bar{p}^0$ and $\bar{p}^i$. Thus $\gamma$ is a Lorentz boost factor and $\alpha$ is not a Lorentz-invariant quantity, but rather a frequency. After performing the integrals in $P^0$ and $P^0$, the integrand will have the general form

$$g(X, \bar{p}, \bar{p}^0) = \int \frac{d^2P_\perp}{(2\pi)^2} W(\bar{p}, \bar{p}^0, \gamma, \alpha, X_\perp)e^{i2\gamma\alpha|X_\perp - \bar{p}/P^0X_0|} + W(\bar{p}, \bar{p}^0, \gamma, \alpha, X_\perp)e^{-i2\gamma\alpha|X_\perp - \bar{p}/P^0X_0|}.$$  

(5.14)

Eq. (5.14) is a superposition of waves with frequencies $\gamma\alpha$ moving along the $X_\parallel$-axis. The Lorentz boost factor relates these waves to spatial oscillations in the rest-frame defined by $\bar{p} = 0$.

Although there is no interaction among the particles, there are collective off-shell effects due to the preparation of the initial condition. In a free field theory, particles with a pure momentum are described by plane waves. Eq. (5.14) tells us that a particle cannot be in a pure momentum state if there are transversal inhomogeneities (T.I.’s). Clearly they would disturb any plane wave into a more complex wave. From the reality condition on $\alpha$, we see that T.I.’s must vanish for $\bar{p}$ to be on-shell. The constraint $\bar{p}_\mu P^\mu = 0$ implies that the rest-mass of the particles is unaffected by inhomogeneities. The energy-loss is solely due
to longitudinal inhomogeneities (L.I.’s), in contrast to the T.I.’s, responsible for removing the particles off
the mass shell. By going to the rest frame \( \bar{p} = 0 \) of the particles moving along \( X_\parallel \), one sees that \( \bar{p}_0 > m \)
for non-vanishing T.I.’s. This implies that the particles will not reach asymptotic on-shell states unless
they were not deflected in the first place. One of the two remaining integrations over the components of
\( P_\perp \) can be replaced by an integration over the off-shellness. Integrating out the \( \alpha \) represents a summing
over quantum fluctuations dressing the particles.

5.2 Free expansion of a Bose-Einstein gas

In this subsection, we give an example of a system as discussed above. The free expansion of a Bose-
Einstein gas will be treated. Consider \((3 + 1)\) dimensional Minkowski space divided into two half-spaces,
\( M_+ \) and \( M_- \), by a Dirichlet-wall at \( x^3 = 0 \). The gas in \( M_- \) is assumed to be in thermal equilibrium at a
temperature \( T \), while in \( M_+ \) a vacuum persists. The total-space propagator \( G(x, y) \) is defined as a sum
of two half-space propagators

\[
G(x, y) = G_vac(x; y) + G_T(P^0, \bar{p}^0) \theta(-x^3) \theta(-y^3) \frac{1}{2} P^3
\]

where \( G_+ \) is a vacuum propagator and \( G_- \) is a thermal propagator. These can be computed from total-
space vacuum and thermal propagators by the method of images. This procedure can be justified by an
eigenfunction expansion in terms of solutions of the half-space Klein-Gordon equation. For the half-space
vacuum, one can express the associated Green function in terms of the total-space vacuum propagator

\[
G_vac(x, y) = G_vac(x; y) - G_vac(x^i, -x^3; y) \text{.} \tag{5.15}
\]

By using the fact that the reflection \((x^3, y^3) \rightarrow (-x^3, y^3)\), is equivalent to \((X^3, \bar{x}^3) \rightarrow (-\frac{1}{2} X^3, -2X^3)\),
in the centre of mass (c.m.) and relative coordinates, we can write down the relation in the Fourier
representation as

\[
G_+(P; \bar{p}) = G_vac(P; \bar{p}) - G_vac(P^i, -2 \bar{p}^3; \bar{p}^i, -\frac{1}{2} P^3) \text{.} \tag{5.16}
\]

In the thermal half-space on the other hand we have, using the thermal total-space propagator \( G_T(P, \bar{p}) \),

\[
G_-(P; \bar{p}) = G_T(P; \bar{p}) - G_T(P^i, -2 \bar{p}^3; \bar{p}^i, -\frac{1}{2} P^3) \text{.} \tag{5.17}
\]

These propagators describe a stationary state which at \( X^0 = \bar{x}^0 = 0 \) coincides with our initial condition.
In momentum space the initial conditions in each of the half-spaces is now represented by the apropriate
propagator integrated over \( P_0 \) and \( \bar{p}_0 \). Thus we compute the initial condition for each half-space

\[
G_{\pm 0}(P^i, \bar{p}^i) = \int \frac{dP^0}{2\pi} \frac{d\bar{p}^0}{2\pi} G_{\pm}(P, \bar{p}) \text{.} \tag{5.18}
\]

The total space initial condition is calculated from the functions \( G_{\pm 0} \) according to Eq.\( (5.15) \). The
product with the step functions in coordinate space becomes a convolution in momentum space. The
Fourier transform of the product of step functions is found by noting that
\[ \theta(\pm x^3)\theta(\pm y^3) = \theta(\pm X^3)\{\theta(\mp 2X^3 - x^3) - \theta(\mp 2X^3 - x^3)\}, \tag{5.20} \]
and Fourier transforming this yields
\[ \hat{\theta}_\pm(P, \bar{p}) \equiv \pm 4 \bar{p}^3 \frac{\theta(\pm 2 \bar{p}^3 + \iota \epsilon)}{|(P^3)^2 - 4(\bar{p}^3)^2 \pm 2P^3 \epsilon - \epsilon^2|}, \tag{5.21} \]
where the limit \( \epsilon, \iota \to 0 \) is understood. Thus the required full-space initial conditions are given by
\[ G_0(P_i, \bar{p}_j) = \int \frac{dP'_i}{2\pi} \frac{d\bar{p}'_j}{2\pi} \hat{\theta}_\pm(P', \bar{p}') G_\pm(P_i - P'_i, \bar{p}_j - \bar{p}'_j) \tag{5.22} \]
Having now determined the initial condition Green function, we focus on the calculation of the medium. Using the linearity of the free field equations and the Fourier transform we can subtract all vacuum contributions from \( G_0(P_i, \bar{p}_j) \) to obtain \( \tilde{G}_0(P_i, \bar{p}_j) \). By making a high-temperature expansion
\[ g_{BE}(\omega) \approx \frac{1}{\beta \omega}, \quad (\beta \omega) \ll 1, \tag{5.23} \]
these integrals can be calculated analytically. Repeating the steps leading from Eq.(5.9) to Eq.(5.14) yields the general solution. In the case under consideration inhomogeneities only exist in the \( X^3 \) direction, so the dependence on \( P_\perp \) is simply a \( \delta(P_\perp) \). The measure of off-shellness now takes the form
\[ \alpha = \sqrt{\bar{p}^2 - m^2}, \tag{5.24} \]
and the boost factor is
\[ \gamma = \sqrt{1 - \frac{\bar{p}^2}{P_0^2}}. \tag{5.25} \]
The solution thus has a simpler form
\[ g(X, \bar{p}) = \frac{1}{\gamma \alpha \bar{p}^3} \{(A) \sin \gamma \alpha (X^3 - \frac{\bar{p}^3}{\bar{p}^0} X^0) + (B) \cos \gamma \alpha (X^3 - \frac{\bar{p}^3}{\bar{p}^0} X^0)\}, \tag{5.26} \]
where the A and B are given by
\[
A = -\pi T \frac{\bar{p}_0^2 - \frac{1}{4} \gamma^2 \alpha^2}{\bar{p}_0^2 - \gamma^2 \alpha^2 \bar{p}_3^2}, \\
B = -\pi T \frac{\sqrt{\bar{p}_0^2 - \bar{p}_3^2 - \frac{1}{4} \gamma^2 \alpha^2}}{\bar{p}_0^2 - \gamma^2 \alpha^2 \bar{p}_3^2}. \tag{5.27}
\]
If one averages Eq.(5.26) over small segments in \( X^3 \), or in \( \bar{p}^3 \), the \( \alpha = 0 \) poles become stronger by a factor \( \alpha^{-2} \). This suggests that the semi-classical limit may be uncovered not by setting \( \alpha = 0 \), but rather by some kind of averaging. Since the propagation of the waves appears to be dominated by the pole at \( \alpha = 0 \), this suggests averaging over the off-shellness. In the general case this could replace an
integration over $P_\perp$, but here there is no non-trivial $P_\perp$ dependence. Consequently we will integrate over the momenta perpendicular to the $\bar{x}^3$ axis when averaging over $\alpha$. The integration over $\alpha$ is restricted to the range

$$0 \leq \alpha \leq \sqrt{\bar{p}_0^2 - \bar{p}_3^2 - m^2}$$

(5.28)
due to the reality condition on $\alpha$. Using the freedom to make a Lorentz transform to the $\bar{p}^3$ frame, one obtains

$$g(X_3, \bar{p}_0) = 4\pi \frac{T}{\bar{p}_0} \{ \frac{\pi}{2} - Si(2\sqrt{\bar{p}_0^2 - m^2}X_3) \}
+ \frac{1}{4 \bar{p}_0 X_3} [2\sqrt{\bar{p}_0^2 - m^2} \cos(2\sqrt{\bar{p}_0^2 - m^2}X_3) + \sin(2\sqrt{\bar{p}_0^2 - m^2}X_3)]
+ \frac{2j_{-1}(\sqrt{\bar{p}_0^2 - m^2}, \bar{p}_0 X_3)}{X_3 \bar{p}_0^2} \} ,$$

(5.29)

where we have defined the function $j_{-\nu}(x, y)$ by

$$j_{-\nu}(x, y) = \int_0^x dz \left( \frac{1}{2} y \right)^{\nu} (\sqrt{1 - z^2})^{\nu - \frac{1}{2}} \cos(yz) .$$

(5.30)

with $\nu = 1$. The function $g(X_3, \bar{p}_0)$ is related to the non-equilibrium generalization $f(X_3, \bar{p}_0)$ of the Bose-Einstein distribution $g_{BE}$ by

$$f(X_3, \bar{p}_0) = \bar{p}_0^2 g(X_3, \bar{p}_0) ,$$

(5.31)

except for an overall normalization. The distribution function $f(X_3, \bar{p}_0)$ is represented graphically in fig.2 as a function of $X_3$ for fixed $\bar{p}_0$, and in fig.3 as a function of $\bar{p}_0$ for fixed $X_3$. Two interesting limiting cases that one may consider are $\bar{p}_0 \to \infty$, and $X_3 \to 0$. At increasing energy, $f(X_3, \bar{p}_0)$, behaves more and more like a step function being zero in $M_+$ and thermal in $M_-$. This is the result that one expects classically. The second limit, $X_3 \to 0$, represents a purely field theoretical effect. The divergence near the Dirichlet wall is nothing but the one-plate Casimir effect. In the presence of such a wall, quantum fluctuation near it will be modified. The modification of vacuum fluctuations gives rise to a $1/X_3^2$ behaviour [31]. On dimensional grounds one expects the finite temperature contribution to be proportional to $T/X_3$, which is the behaviour we find. An explicit calculation verifies this interpretation.

As the wall is instantaneously removed at $t = 0$, this leads to particle creation. It is exactly this type of effect that the SO(1,1) Ward identities for ICSB adress.

### 6 Conclusions

In principle a kinetic field theory of non-equilibrium physics is viable beyond a two-scale approach. It is the ideal starting point for treating inhomogeneous systems out of equilibrium. In particular we have studied a transport theory for the scalar $\lambda \phi^4$ field theory, constructed using the closed time path formalism.
In this case we find that energy momentum conservation is guaranteed by the dynamics of the 2-point functions. Gradient expansions occurring naturally within the formalism cannot be truncated without either violating energy-momentum conservation, or introducing non-locality. It has been shown how the broken SO(1,1) symmetry of non-equilibrium field theory is useful in establishing a physical interpretation for $G^{ab}$. Temporal current non-conservation combined with energy-momentum conservation determines the ultra-violet behaviour of the dressed propagators. The free field equations have been analyzed on their non-equilibrium content. Inhomogeneity and off-shellness are intimately related in this context. This is not expected to change in the interacting theory. By integrating out the off-shell contributions one arrives at quasi classical distribution functions.

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Figure Captions

• Fig. 1a and 1b: The diagrammatical representations of Eq.(2.24) and Eq.(2.25) are given. The shaded triangle denotes the 1PI 3-vertex $\alpha_{abc}^3$, an open triangle refers to the general 3-point Green function $G^{abc}$, thin solid lines represent outgoing lines, single blobs correspond to the mean fields $\tilde{\varphi}^a$, and lines with a blob along the line represent the full propagator $G^{ab}$. Additionally in 1b the shaded disc is the 1PI 4-vertex $\alpha_{abcd}^4$ and the open disc the general Green function $G^{abcd}$.

• Figures 2 a-d: The distribution function $f(x, e)$ as a function of the energy $e = \tilde{p}_0$ at $x = -5, -1, 1, 5$ fm/c. Here the temperature $T = 2$ GeV and the mass of the bosons was set $m = 0.2$ GeV.

• Figures 3 a-d: The function $f(x, e)$ as a function of $x$ at $e = 0.2, 0.3, 1, 2$ GeV.
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