Energy Cascade Rate in Compressible Fast and Slow Solar Wind Turbulence

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Abstract

Estimation of the energy cascade rate in the inertial range of solar wind turbulence has been done so far mostly within incompressible magnetohydrodynamics (MHD) theory. Here, we go beyond that approximation to include plasma compressibility using a reduced form of a recently derived exact law for compressible, isothermal MHD turbulence. Using in situ data from the THEMIS/ARTÉMIS spacecraft in the fast and slow solar wind, we investigate in detail the role of the compressible fluctuations in modifying the energy cascade rate with respect to the prediction of the incompressible MHD model. In particular, we found that the energy cascade rate (1) is amplified particularly in the slow solar wind; (2) exhibits weaker fluctuations in spatial scales, which leads to a broader inertial range than the previous reported ones; (3) has a power-law scaling with the turbulent Mach number; (4) has a lower level of spatial anisotropy. Other features of solar wind turbulence are discussed along with their comparison with previous studies that used incompressible or heuristic (nonexact) compressible MHD models.

Key words: magnetohydrodynamics (MHD) – plasmas – solar wind – turbulence

1. Introduction

A longstanding problem in the solar wind is its nonadiabatic cooling. This is manifested by the observations that the solar wind proton temperature decreases slowly as a function of the radial distance from the Sun, in comparison to the prediction of the adiabatic expansion model of the solar wind (Marsch et al. 1982; Vasquez et al. 2007). Several scenarios have been proposed to explain these observations, for example, that ions are picked up (Matthaeus et al. 1999; Smith et al. 2001; Isenberg et al. 2003; Marsch 2006). The candidate that has driven much effort is certainly the local heating of the solar wind plasma via turbulence (Bruno & Carbone 2005; Galtier 2006; Smith et al. 2006). Large-scale magnetohydrodynamics (MHD) turbulence can indeed serve as a reservoir of energy that cascades down to the small (kinetic) scales where it can be dissipated by some kinetic effects, which remain to be elucidated (Goldstein et al. 1994; Leamon et al. 1998; Sahraoui et al. 2009, 2010). The underlying assumption is that all of the energy that is injected at some large scale in the solar wind will cascade within the inertial range without dissipation, until it reaches the ion scale, where it is eventually converted into the thermal (heating) or kinetic (acceleration) energy of the plasma particles. This has led to intensive research work aimed at estimating the energy cascade rate in the solar wind using in situ spacecraft data. Direct evidence of the presence of an inertial energy cascade in the solar wind was obtained using the so-called Yaglom law (MacBride et al. 2005, 2008; Sorriso-Valvo et al. 2007; Marino et al. 2008, 2011). It is a universal law derived analytically from the incompressible MHD equations (Politano & Pouquet 1998; hereafter PP98) under the assumptions of homogeneity, stationarity, isotropy of the turbulent fluctuations, and being in the asymptotic limit of large kinetic and magnetic Reynolds numbers. Another fundamental assumption in those works is that compressible fluctuations play a minor role in the turbulent cascade. A first attempt to include the compressibility in estimating the energy cascade rate was made in Carbone et al. (2009; hereafter C09) using heuristic arguments. Indeed, a modified form of the Elsässer variables was introduced that considered the local (instead of the mean) plasma density and a new density-weighted velocity

\[ w^{\pm} = \rho^{1/3} \left( v \pm B \sqrt{\mu_0} \right), \]

where \( \rho \) is the density, \( v \) the velocity, \( B \) the magnetic field, and \( \mu_0 \) the permeability of free space. This form was inspired by the work of Kritsuk et al. (2007; see also Schmidt et al. 2008), who showed numerically, in the context of supersonic interstellar turbulence, that the density-weighted velocity offers a better understanding of isothermal compressible hydrodynamic turbulence. The application of C09 to the fast solar wind data showed a better scaling relation of the energy flux than with PP98 (Carbone et al. 2009). Furthermore, a significant increase of the turbulent cascade rate was evidenced and was shown to be sufficient to account for the local heating of the nonadiabatic solar wind expansion.

A first attempt to include the compressible fluctuations of the solar wind in the turbulence cascade using a more rigorous approach has been made recently (Banerjee et al. 2016). In that paper, an exact law derived for compressible isothermal turbulence by Banerjee & Galtier (2013; hereafter BG13) was used as well as the in situ fast solar wind data measured by the THEMIS B/ARTEMIS P1 spacecraft (Auster et al. 2009; McFadden et al. 2009). Two important improvements in the estimation of the energy cascade rate using the BG13 model were obtained: (1) a broader inertial range that extended for more than two decades of scales, and (2) a higher energy cascade rate (up to 3–4 times) than the estimation from the PP98 model. However, two discrepancies with the results of Carbone et al. (2009) were found. First, the amplification of the cascade rate is smaller than that obtained in C09. Second, the origin of that enhancement is due to the new compressible terms in the BG13 model and not to the compressible Yaglom
term (Banerjee et al. 2016). In the present study, we extend the
application of the BG13 model to a larger statistical sample and
address new questions related to the differences between the slow and fast solar wind (known to have different levels of
compressibility and different correlations between the magnetic
and the velocity fields), the nature (direct versus inverse) of the
turbulent cascade and the role of the cross-helicity, and the
effect of the turbulent Mach number and the plasma
compressibility on the spatial anisotropy of the cascade rate.
Throughout the paper, systematic comparisons with the
incompressible model are made to highlight the role of the
plasma compressibility. Discrepancies with the C09 model will be
eventually discussed.

The manuscript is structured as follows. In Section 2 we
recall the basic equations and assumptions of the different
theoretical models used in this work. In Section 3 we describe
the procedure we used to select our data samples. In Section 4
we present the main results of the study along with their
comparisons to previous works, and in Section 5 we discuss the
origin of the discrepancies with the results reported in Carbone
et al. (2009) and other caveats related to the theoretical models
used and the data selection. In Section 6 we provide a summary of
the results.

2. Theoretical Models

We briefly recall the basic equations of the three theoretical
models, namely PP98, C09, and BG13, that will be used
throughout the paper. These models are based on MHD, which
is a relevant model for most of the astrophysical plasmas
(Galtier 2016).

Incompressible Model: The PP98 law is written in terms of the
Elsässer variables $\xi = v \pm v_h$, where $v$ is the plasma flow
velocity, $v_h = B/\sqrt{\mu_0 \rho}$ is the magnetic field normalized to a
velocity, and $\rho_0 = <\rho>$ is the mean plasma density. It reads in the
isotropic case as

$$-\frac{4}{3} \bar{\varepsilon}_T \ell = F_{C+0}(\ell), \quad (4)$$

where the general definition of an increment of a variable $\psi$ is
used: $\delta \psi \equiv \psi(x + \ell) - \psi(x)$. The longitudinal components
are denoted by the index $\ell$ with $\ell \equiv |\ell|$, $\langle \cdot \rangle$ stands for the
statistical average, and $\bar{\varepsilon}_T$ is the dissipation rate of the total
energy. Note that in S.I. units we have the relation $\rho_0 = 1.673 \times 10^{-21} (n_p)$.

Heuristic Compressible Model: The heuristic C09 law is
based from Equation (2). The Elsässer variables are simply
replaced by cube-root density-weighted compressible Elsässer
variables $\mathbf{w}^\pm \equiv \rho^{1/3} \xi^\pm$. Then, the isotropic law becomes

$$-\frac{4}{3} \bar{\varepsilon}_W \ell = F_{W}(\ell), \quad (3)$$

where $\bar{\varepsilon}_W$ is the dissipation rate of the total compressible energy
(following the notations introduced by Carbone et al. 2009, that
means $2 \bar{\varepsilon}_W = \varepsilon^+ + \varepsilon^-$. Note that the renormalization
proposed is inspired directly by studies of supersonic hydro-
dynamic interstellar turbulence (Kritsuk et al. 2007; Schmidt
et al. 2008).

Compressible Model: Following the approach used in
Banerjee et al. (2016), the original equations of the BG13
model can be reduced to the following compact form in the
isotropic case:

$$-\frac{4}{3} \varepsilon_c \ell = F_{C+0}(\ell), \quad (4)$$

where

$$F_{C+0}(\ell) = F_1(\ell) + F_2(\ell) + F_3(\ell), \quad (5)$$

and

$$F_1(\ell) = \left(\frac{1}{2} \delta(\ell^2) \cdot \delta(z^+) \delta(z^-) + \frac{1}{2} \delta(\ell^2) \cdot \delta(z^+) \delta(z^-)\right),$$

$$F_2(\ell) = 2\delta \rho e \delta v,,$$

$$F_3(\ell) = 2\delta \left(1 + \frac{1}{\beta} \right) e + \frac{v_2^2}{2} \delta(\rho_1 v),$$

$$e = c_s^2 \ln(\rho/\rho_0)$$

is the internal energy, with $c_s$ the constant
isothermal sound speed and $\rho$ the local plasma density
($\rho = \rho_0 + \rho_1$), and $\beta = 2c_s^2 / \nu_3$ is the local ratio of the total
thermal to magnetic pressure ($\beta = \beta_2 + \beta_3$). We recall that,
contrary to incompressible MHD theory, the BG13 compressible
model yields an energy cascade rate that is not simply
related to third-order expressions of different turbulent
fluctuations, but rather involves more complex combinations of the
turbulent fields in the new flux and source terms. However,
the model still assumes the existence of an inertial range
(similarly to incompressible MHD theory) in which the
forcing and the dissipation terms are negligible. If significant
dissipation of the compressible fluctuations occurs in the
inertial range, then the applicability of the BG13 model should
be questioned. Note finally that the (large-scale) forcing term,
generally considered in turbulence equations, must be
distinguished from the source terms included in the BG13 model that
act within the inertial range.

To obtain Equations (4)–(6), several assumptions have been used
(see details in Banerjee et al. 2016). First, the source terms have
been neglected based on the argument they are probably
important only in supersonic turbulence, whereas solar wind
 turbulence is subsonic (Galtier & Banerjee 2011; Kritsuk
et al. 2013), and on a preliminary estimation using numerical
simulations of isothermal MHD turbulence (S. Servidio 2015,
private communication). Note that the source terms cannot be
estimated reliably using single-spacecraft data in this work
because of the local spatial divergence involved in those terms.
Second, the plasma $\beta$ is assumed to be nearly stationary, which
is a stringent requirement in selecting the data to use in the
present study. To these assumptions are added the classical
generally used to derive similar equations in turbulence
theories, namely statistical homogeneity and stationarity of the
turbulent fluctuations. The statistical isotropic assumption is
further made to obtain the reduced form given by Equations (4)–(6) (this point will be discussed in more detail
in Section 4.5). In this work, it is these equations that will be
evaluated using spacecraft data in the fast and slow solar wind.

3. Data Selection

We used the THEMIS B/ARTEMIS P1 spacecraft data
during time intervals when it was traveling in the free-
streaming solar wind. The magnetic field data and plasma moments (density, velocity, and temperature) were measured respectively by the Flux Gate Magnetometer (FGM) and the Electrostatic Analyzer (ESA). All data have 3 s time resolution (i.e., spin period). A large survey of the THEMIS/ARTEMIS data from the period 2008–2011 has been performed that covered both the fast and slow solar wind. Fast winds are those having an average velocity \( V > 450 \text{ km s}^{-1} \). In selecting the data, we have tried to avoid intervals that contained significant disturbances or large-scale gradients (e.g., coronal mass ejection or interplanetary shocks). As mentioned above, a limiting criterion in choosing the data is the condition of having a stationary plasma \( \beta \), which has been checked for each case separately in this work. Another parameter that has been checked is the uniformity of \( \Theta_{VB} \), the angle between the local solar wind speed \( \mathbf{V} \) and the magnetic field \( \mathbf{B} \). Indeed, when using the Taylor hypothesis on single-spacecraft measurements, the time sampling of the data is converted into a 1D spatial sampling of the turbulent fluctuations along the flow direction. Therefore, the stationarity of the angle \( \Theta_{VB} \) is required to guarantee that the spacecraft is sampling nearly the same direction in space with respect to the local magnetic field, which would ensure a better convergence in estimating the cascade rate. This point will be further developed in Section 5.3.

The obtained intervals that fulfilled all of the previous criteria were divided into a series of samples of equal duration of \( \sim 35 \) minutes, which corresponds to a number of data points \( N \sim 700 \) with a 3 s time resolution. The duration of 35 minutes ensures having at least one correlation time of the turbulent fluctuations estimated to vary in the range \( \sim 20–30 \) minutes. Eventually, the data selection yielded 148 samples (\( \sim 1 \times 10^5 \) data points) in the fast solar wind and 182 (\( \sim 1.3 \times 10^5 \) data points) in the slow solar wind. An example of the analyzed time intervals is shown in Figure 1 in the slow solar wind.

The average solar wind speed and plasma \( \beta \) for all of the statistical samples are shown in Figure 2. Note that most of the values of \( \beta \) are larger than one.

4. Observational Results in the Fast and Slow Solar Wind

4.1. Cascade Rate versus Plasma Compressibility and Turbulent Mach Number

For all of the selected time intervals, we computed the energy cascade rates \( \varepsilon_f \) and \( \varepsilon_C \) from the PP98 and the BG13 models using respectively Equations (2) and (4)–(6). To this end, we had constructed temporal structure functions of the different turbulent fields involved in those equations, namely \( \mathbf{B} \), \( n \), and \( v \), at different time lags \( \tau \). In order to probe into the scales of the inertial range, known to lie within the frequency range \( \sim [10^{-4}, 1] \text{ Hz} \) (based on the observation of the Kolmogorov-like \( -5/3 \) magnetic energy spectrum; Bruno & Carbone 2005), we vary the time lag \( \tau \) from 10 to 1000 s, thereby being well inside the targeted frequency range. Note that this range of scales is slightly shifted toward small scales in comparison to previous studies that used ACE data (MacBride et al. 2008).

A detailed comparison of the different fluxes of the BG13 model is given in Figure 3(a). We see that the pure compressible flux \( \mathcal{F}_c \) dominates the other fluxes for most of the timescales \( \tau \). For comparison, we also show the incompressible flux \( \mathcal{F}_i \) given by the PP98 model, which is clearly much lower than \( \mathcal{F}_c \). In Figure 3(b) are plotted the cascade rates deduced from the flux analysis. The estimate from BG13 gives a flat cascade rate over two decades of scales, whereas the estimate from the PP98 exhibits hollows, which are the manifestation of a change of sign. This difference is a generic behavior found in many other cases (see inset). The sign change of the incompressible cascade rate does not always occur at the same time lag \( \tau \), as can be seen in Figure 3. This rules out the possible role of minor heavy ions, e.g., \( H^+ \), whose characteristic scales would belong to the range of scales analyzed in this work. This was confirmed by a visual check of the power spectra of the magnetic fluctuations that did not show any significant enhancement of power in the frequency range \( [10^{-3}, 10^{-1}] \text{ Hz} \), which would be caused by an energy injection via a kinetic plasma instability of heavy ions.

Examples of the obtained cascade rates computed in the fast and slow solar wind are shown in Figure 4. A first observation is that both the compressible and incompressible cascade rates \( \langle |\varepsilon_c| \rangle \) and \( \langle |\varepsilon_i| \rangle \) are larger in the fast wind than in the slow wind, as indicated by the histogram and the average (absolute) values. This confirms the previous finding regarding the incompressible cascade rate \( \langle |\varepsilon_i| \rangle \) (MacBride et al. 2008; Stawarz et al. 2009; Coburn et al. 2012) and shows that compressibility does not change that trend.

![Figure 1](image-url)  
Figure 1. From top to bottom: the solar wind magnetic field components (nT), ion velocity (km s\(^{-1}\)), ion number density, \( \Theta_{VB} \) angle, and total plasma beta \( (\beta = \beta_i + \beta_e) \) measured by the FGM and ESA experiments on board the THEMIS B spacecraft on 2009 November 20 from 03:33 to 04:08.

![Figure 2](image-url)  
Figure 2. The average solar wind speed (a) and the total plasma \( \beta \) (b) for all of the data intervals used.

![Figure 3](image-url)  
4.1. Cascade Rate versus Plasma Compressibility and Turbulent Mach Number

![Figure 4](image-url)  
4. Observational Results in the Fast and Slow Solar Wind

4.1. Cascade Rate versus Plasma Compressibility and Turbulent Mach Number
In Figure 5 we compare the ratio between the compressible and the incompressible cascade rate $R = \langle | \varepsilon_C | \rangle / \langle | \varepsilon_I | \rangle$ in the fast and slow winds. Here we use the average value of the cascade rate over all of the time lags $\tau$ within the range 10–1000 s. As will be discussed in Section 5.4, the cascade rate may change its sign within a single time interval for two (or more) different values of $\tau$ in the inertial range, which may make the choice of a single value of $\varepsilon$ at a given value of $\tau$ questionable. This motivated a new criterion applied to further narrow down the selection of our time intervals: we kept only those samples for which the compressible cascade rate shows a constant (negative or positive) sign for all of the time lags in the range 10–1000 s.

A first feature that can be seen in Figure 5, and already reported in Banerjee et al. (2016) regarding the fast solar wind, is that the plasma compressibility, which on average may not modify significantly the cascade rate (since the bulk of the distribution of the ratio $R$ is centered around 1), in some cases does nevertheless amplify it by a factor of 3–4. This trend is enhanced in the slow wind where the (blue) histogram of $R$ in Figure 5 is found to shift to higher values (up to 7–8) and for a larger number of events than in the fast wind. Note, however, that these amplification values remain smaller than those reported in Carbone et al. (2009), as will be discussed in Section 5.1.

To evidence the role of the density fluctuations $\sqrt{\langle \rho^2 \rangle - \langle \rho \rangle^2} / \langle \rho \rangle$ in enhancing the cascade rate $\langle | \varepsilon_C | \rangle$ with respect to the incompressible one $\langle | \varepsilon_I | \rangle$, we plotted in Figure 6 $\langle | \varepsilon_C | \rangle$ as a function of the wind speed and the density fluctuation.
fluctuations. First, one can find the property discussed above that, overall, the fast wind has a higher $\langle |\varepsilon_c| \rangle$ than the slow wind. Moreover, one can see an increase in the cascade rate as compressibility increases in particular in the case of the slow wind. This trend is less evident in the case of the fast solar wind, possibly because the spread in the compressibility values is smaller (~3%–15%) than in the case of the slow wind (~1%–20%).

The correlation is better seen with the estimated turbulent sonic Mach number defined as $M_{\text{rms}} = \sqrt{\nu_1^2/C_s^2}$ ($\nu_1$ being the average fluctuating plasma flow), as shown in Figure 7. The slow wind shows a clear power law in

$$\varepsilon_C \sim M_{\text{rms}}^{2.67},$$

while the fast wind exhibits more spread around an approximate power law

$$\varepsilon_C \sim M_{\text{rms}}^{1.5}.$$  

4.2. Cascade Rate versus the Energy of the Turbulent Fluctuations

Another interesting feature that can be analyzed is the dependence of the cascade rate $\varepsilon_C$ on the energy of the compressible turbulent fluctuations $E_{i,\text{comp}}$ and the possible existence of a scaling law relating each of the energy components to $\varepsilon_C$. Indeed, unlike in the incompressible model PP98, the total energy of the fluctuations is not given simply by

$$E_{i,\text{inc}} = \rho_0 \left( \frac{1}{2} v_1^2 + \frac{1}{2} B_1^2 \right),$$

where $v_1$ and $B_1$ are the fluctuating velocity and magnetic fields, but includes the fluctuating internal energy $U_1$:

$$E_{i,\text{comp}} = \rho_0 \left( \frac{1}{2} v_1^2 + \frac{1}{2} B_1^2 + U_1 \right),$$

where, to the lowest order of $\rho_i/\rho_0$, $U_1$ can be written as

$$U_1 = \rho_0 \varepsilon_i^2 \ln(1 + \rho_1/\rho_0).$$

First, we plotted in Figure 8 the variation of $\langle |\varepsilon_c| \rangle$ as a function of compressibility at $\varepsilon_C$ for all of the statistical samples analyzed in the fast and slow solar wind. One can see clearly that the higher the amplitude of the fluctuation, the larger the cascade rate $\langle |\varepsilon_c| \rangle$. This observation is valid in both the fast and slow wind and is consistent with previous observations (Smith et al. 2006; MacBride et al. 2008). Note that there is no significant variation of $\langle |\varepsilon_c| \rangle$ as a function of compressibility at a fixed value of the energy of the turbulent fluctuations.

Figure 9 shows the three components of the total energy of the fluctuations $E_{i,\text{comp}}$ as a function of the estimated compressible cascade rate $\langle |\varepsilon_c| \rangle$ for all of the statistical samples analyzed in the fast and slow solar wind. First, one can see that, statistically, the magnetic energy dominates over the kinetic and internal energies, the latter being the smallest, which confirms the results of Podesta et al. (2007). Second, a relatively clear power-law scaling between $E_{i,i}$ ($i = K, M, I$ for kinetic, magnetic, and internal energies) and $|\varepsilon_c|$ can be evidenced with nearly the same slope in the fast and slow winds:

$$E_{iK} \sim \varepsilon_C^{0.57},$$

and

$$E_{IM} \sim \varepsilon_C^{0.60}.$$
The scaling of the internal energy is shallower and is different for the two types of wind:

\[ E_{II} \sim \varepsilon_C^{0.42} \]  \hspace{1cm} (14)

for the fast wind, and

\[ E_{II} \sim \varepsilon_C^{0.32} \]  \hspace{1cm} (15)

for the slow wind. While the scaling of the magnetic and kinetic energies with the cascade rate is very close to the theoretical prediction from the Kolmogorov theory (Frisch 1995), \( E_l \sim \varepsilon_l^{2/3} \), to the best of our knowledge, no theoretical prediction exists so far to help in interpreting the empirical laws of Equations (14)–(15).

4.3. Role of the Different Flux Terms

To gain insight into the role of the different flux terms involved in estimating the compressible energy cascade rate, we plotted in Figure 10 statistical results for the contribution of the different compressible fluxes, \( F_1, F_2, \) and \( F_3 \), relative to the incompressible (Yaglom) flux \( F_I \) for the slow and fast winds. A first observation is that most of the samples have their compressible Yaglom flux \( F_I \) of the order of the incompressible flux \( F_I \). This indicates that it is the new compressible fluxes \( F_2 \) and \( F_3 \) that contribute more to enhancing the compressible cascade rate \( \varepsilon_C \) (with respect to \( \varepsilon_l \)) rather than the compressible Yaglom term \( F_I \). This is better seen when observing that high values of \( \langle |\varepsilon_C|/|\varepsilon_l| \rangle \) (up to \( \sim 4 \) in the fast wind and up to \( 8 \) in the slow wind) are observed when \( \langle |\varepsilon_I| + |\varepsilon_l| \rangle/|\varepsilon_I| > 1 \). We recall that stronger amplification has been reported in Carbone et al. (2009), which stems from a heuristic modification of the incompressible (Yaglom) term via density fluctuations. The discrepancy between that observation and the present ones will be clarified in Section 5.1.

The second observation is that the compressible energy cascade rate (4.3. Role of the Different Flux Terms)

\[ \text{Figure 11. Histograms of the signed energy cascade rate estimated using the compressible model BG13 in the fast (b) and slow (a) solar wind.} \]

4.4. Sign of the Energy Transfer Rate and Cross-helicity

In this section we discuss the sign of the cascade rate as estimated from the incompressible (PP98) and compressible (BG13) models. We first recall that this property can be discussed only when the dependence of the energy flux on the time increments \( \tau \) are converted into the spatial ones \( \ell \) via the Taylor frozen-in flow assumption. With the positive convention of the time increments \( \tau > 0 \) used in this work, the Taylor hypothesis implies \( \ell \sim -\nu \tau \). In this convention, positive (respectively negative) values of \( \varepsilon_C \) correspond to a direct (inverse) energy cascade. The histograms of the signed compressible cascade rate are shown in Figure 11. Although the statistical sample used here is not as large as those used in previous studies based on the PP98 model (e.g., Coburn et al. 2014, 2015) for the reasons explained in Section 3, our results confirm the previously reported features of the solar wind. First, Figure 11 shows that both the histogram and the mean values (red lines) of the signed cascade rates indicate a direct cascade in the slow solar wind and an inverse cascade in the fast wind. The average cascade rates over all of the statistical samples in the slow wind, \( \sim 1.3 \times 10^{-17} \text{ J m}^{-3} \text{ s}^{-1} \sim 2.5 \times 10^{3} \text{ J (kg s)}^{-1} \), are slightly higher than those reported in, for example, MacBride et al. (2008) (\( \sim 1.9 \times 10^{3} \text{ J (kg s)}^{-1} \)).

The second observation is that the compressible fluctuations do not influence the direction of the cascade. This can be seen in Figure 12, which shows the correlations between the estimated signed incompressible and compressible cascade rates \( \varepsilon_C \) and \( \varepsilon_C \); most of the studied cases showed the same sign for the averaged incompressible and compressible energy cascade rates.

To understand the difference in the direction of the cascade in the slow and the fast wind, we investigated the role of the cross-helicity as suggested in Smith et al. (2009). The results of...
the analysis are shown in Figure 13. Several interesting features can be seen. First, we observe again the property evidenced in Section 4.1 that the fast wind has higher $$|\langle \varepsilon_c \rangle|$$ than does the slow wind (Figure 13(a)). Furthermore, we observe the known feature that the fast wind is generally characterized by higher values of cross-helicity $$|\sigma_c| \gtrsim 0.5$$ with more preference for outward-propagating waves ($$\sigma_c > 0$$; Figure 13(b)). This property is not observed in the slow solar wind, where $$\sigma_c$$ is uniformly distributed in the range $$\sim [-0.8, +0.8]$$. Our observation of the dominance of the inverse cascade in the fast solar wind (dominated by outward-propagating waves) is consistent with the finding of Smith et al. (2009), who suggested that this process could explain the survival of regions of high cross-helicity in the fast wind at large radial distances from the Sun (Roberts et al. 1987).

4.5. Spatial Anisotropy and the Energy Cascade Rate

In this section we explore the anisotropic nature of the cascade rate and the differences between the incompressible and compressible models. The anisotropy of the cascade rate has been previously explored using the PP98 model, and it has been shown that the cascade rate is more anisotropic in the fast than in the slow solar wind (MacBride et al. 2008). In the previous works, the original PP98 equations were modified to fit the limit of either 1D (slab) or 2D geometry, through the appropriate projection of the flux terms onto the two directions parallel and perpendicular to the mean magnetic field. Here, we do not use that approach for either the PP98 or the BG13 models. Instead, we simply examine the dependence of the estimated cascade rates on the angle $$\Theta_{VB}$$. As we explained above, the use of the Taylor hypothesis ($$l = V \tau$$) to convert time lags $$\tau$$ into spatial scales implies that the analysis samples only the direction along the solar wind flow. Hence, when $$\Theta_{VB} \sim 0^\circ$$ (respectively $$\Theta_{VB} \sim 90^\circ$$), the analysis yields information in the direction parallel (perpendicular) to the local mean magnetic field. It is worth recalling that the derivation of the BG13 model does not require the isotropy assumption. Therefore, estimating the cascade rate using that model as a function of the sampling direction in space given by the angle $$\Theta_{VB}$$ should allow us to gain insight into the anisotropic nature of the fluctuations. We used this approach by splitting our statistical samples (in the fast and slow winds) as a function of the angle $$\Theta_{VB}$$. The result is given in Figure 14. Two important observations can be made. First, both models, PP98 and BG13, provide a cascade rate that is strongly dependent on the angle $$\Theta_{VB}$$. This dependence is even more pronounced in the slow wind than in the fast wind. This contrasts with the finding of MacBride et al. (2008), who showed no significant anisotropic cascade in the slow wind. This discrepancy may come from the criterion of uniform angle $$\Theta_{VB}$$ used in this work, which allows us to better show the difference in the cascade rates parallel and perpendicular to the mean field. However, similar to that in MacBride et al. (2008), the heating is smaller in the parallel direction (where $$E_{\text{comp}}^{\parallel}$$ is lower) than in the perpendicular one (where $$E_{\text{comp}}^{\perp}$$ is higher) for both winds, with a lower $$\langle |\varepsilon_c| \rangle$$ for the slow compared to the fast one. Second, we can see that the compressible model BG13 slightly reduces the level of anisotropy in particular in the slow wind (by a factor of $$R \sim 2$$). This observation can easily be understood considering that, unlike the shear Alfvén mode in
the PP98 model, the BG13 model also includes the compressible MHD (slow and fast) modes, which have a parallel magnetic field component, although they are minor in the solar wind. In particular, the fast mode turbulence is shown to be isotropic from numerical simulations of MHD turbulence (Cho & Lazarian 2002). That property naturally tends to isotropize the full turbulent fluctuations, which are no longer simply guided by the mean magnetic field as in incompressible MHD theory.

5. Discussion

Before summarizing the main finding of the present statistical study, we address some important points related to the use of compressible models to estimate the energy cascade rate in the solar wind. These points are related to the subtle role of the background (mean) density and velocity of the solar wind plasma. Other caveats will be discussed, such as the role of the angle \( \Theta_{VP} \) and the statistical significance of the single (at a given value of \( \tau \)) versus the average (over all values of \( \tau \)) of the estimated cascade rates.

5.1. On the Role of Mean Flow Velocity

In the first attempt to include compressible fluctuations in solar wind turbulence studies, Carbone et al. (2009) found that the energy transfer rate \( \varepsilon_{C09} \) is around 10–15 times greater than the one given by PP98 and that amplification comes from a heuristic modification of the original (incompressible) Yaglom terms in the PP98 model. Our results showed that the compressible Yaglom term \( F_I \) does not play a significant role in enhancing \( \varepsilon_C \) with respect to the PP98 model. The amplification comes from the new flux terms \( F_2 \) and \( F_3 \) that are not included in the C09 model. This discrepancy may originate from the role of the mean flow velocity that could have been erroneously included in the modified (compressible) Elsässer variables \( \tilde{w}^\pm \) (Equation (3)) used in Carbone et al. (2009), which is much larger (by a factor of \( \sim 10 \)) than the velocity fluctuations. Indeed, when using the incompressible MHD model (PP98) with the Elsässer variables that include the full (i.e., mean and fluctuating) flow and magnetic field vectors, the mean fields are systematically suppressed while estimating the increments of the different variables, and consequently the latter depend only on the fluctuating turbulent fields. This is consistent with the theoretical derivation of the exact laws in turbulence, where a zero mean flow velocity is generally assumed. However, in the empirical compressible model of Carbone et al. (2009; hereafter C09), the difficulty arises when dealing with the density-weighted velocity given in Equation (1). Because of the density dependence of the modified Elsässer variables, the mean flow velocity will remain involved when estimating the field increments in Equation (3) of C09. In other words, the estimation of the cascade rate will involve not only the turbulent fluctuations but also the mean flow velocity, which is not relevant in turbulence studies and in particular for the estimation of the cascade rate. To test this hypothesis, we compared the energy transfer rates computed using PP98, BG13, C09, and a modified version of the C09 model that uses the fluctuating velocity \( \tilde{v}_I \) instead of the total one \( (V + \tilde{v}_I) \), namely

\[
\tilde{w}^\pm = \rho^{2/3} \left( \tilde{v}_I \pm \frac{B}{\sqrt{\rho v_0}} \right).
\]

The results are shown in Figure 15. As one can see, not only does the cascade rate \( \langle |\varepsilon| \rangle \) of C09 (blue) not give a linear scaling as does the BG13 model, but it also gives a cascade rate that is at least 10 times higher than that of the other models. However, when using the modified C09 with the variables \( \tilde{w}^\pm \), the corresponding \( \langle |\varepsilon| \rangle \) (green curve) decreases and becomes comparable to the Yaglom term of PP98 (black curve). This implies that the modified C09 model, which considers compressibility corrections to the Yaglom term in the PP98 model, does not modify significantly the energy cascade rate, in agreement with our finding using the BG13 model.

This result is confirmed by a statistical analysis of all of the events for which \( \varepsilon_{C09} \) is constant in sign. The corresponding results are shown in Figure 16, which compares the ratios \( R \) of the average energy cascade rates obtained using the original and modified C09 models to those given by the PP98 model. As one can see, \( R \) reaches values as high as \( \sim 50 \) in both the fast and slow winds (blue histograms), while this ratio drops down to \( \sim 1 \) with the modified C09 model (red histogram), in agreement with our finding using the BG13 model.
5.2. The Mean Plasma Density

Another point that deserves clarification is the influence of the mean density $\rho_0$ in the BG13 model. Indeed, the original form of $\mathcal{F}_3$ includes the total density $\rho = \rho_0 + \rho_1$, as in the following (Banerjee & Galtier 2013):

$$\nabla_t \cdot \mathcal{F}_3(t) = \nabla_t \cdot \left\{ 2 \delta \left[ \left( 1 + \frac{1}{\beta} \right) e + \frac{v_A^2}{2} \right] \delta(\rho) \right\}$$

$$= \nabla_t \cdot \left\{ 2 \delta \left[ \left( 1 + \frac{1}{\beta} \right) e + \frac{v_A^2}{2} \right] \delta(\rho_0) \right\}$$

$$+ \nabla_t \cdot \left\{ 2 \delta \left[ \left( 1 + \frac{1}{\beta} \right) e + \frac{v_A^2}{2} \right] \delta(\rho_1) \right\}. \quad (17)$$

In the incompressible limit ($\rho_1 \to 0$ and $\nabla \cdot v = 0$), the divergence of $\mathcal{F}_3(t)$ vanishes. However, since in the estimation of flux terms $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ using spacecraft data we do not explicitly apply the divergence operator $\nabla_t$, but rather $\nabla_t \to 1/\ell$, it is practically impossible to ensure that $\mathcal{F}_3$ vanishes in the incompressible limit. To guarantee the convergence of the BG13 and PP98 models in the limit of incompressibility, we kept only the second term of Equation (17), while the first term can be easily transformed into source terms since

$$2 \nabla_t \cdot (\delta X \delta(\rho_0) v) = 2 \rho_0 \nabla_t \cdot (\delta X v)$$

$$= \rho_0 \nabla_t \cdot (Xv' - Xv + X'v' - X'v)$$

$$= \rho_0 \nabla_t \cdot (Xv' - X'v)$$

$$= (\rho_0 X \nabla \cdot v') + (\rho_0 X' \nabla \cdot v), \quad (18)$$

where $X = \delta \left[ \left( 1 + \frac{1}{\beta} \right) e + \frac{v_A^2}{2} \right]$. It is easy to see that both expressions Equation (18) and the flux term $\mathcal{F}_3(t)$ of Equation (6) converge to zero in the incompressible limit (i.e., $\nabla \cdot v = 0$ and $\rho_0 = 0$).

As a final remark, it is important to note that the term $\mathcal{F}_3$ used in the BG13 model still retains the role of the mean magnetic field through the term proportional to $v_A^2$ in Equation (17). This is a significant difference from the PP98 model, which yields a third-order law that depends only on the fluctuating fields, as discussed in Section 5.1, whereas it is well known that a mean magnetic field has an impact on the nonlinear dynamics. This apparent inconsistency is solved if one extends the derivation in the incompressible case to the next order because the time evolution of the third-order correlation depends explicitly on the mean magnetic field, and so does the associated characteristic timescale. Consequently, the second-order correlation may be affected by the mean magnetic field (see, e.g., Oughton et al. 2013).

5.3. The Influence of the Angle $\Theta_{VB}$

In Section 3 we emphasized the importance of having relatively stationary angles $\Theta_{VB}$ in order to have a more reliable estimate of the energy cascade rate (both its sign and its absolute value) when dealing with single-spacecraft data, and regardless of the theoretical model used. Here we discuss two possible effects of the nonstationarity of the angle $\Theta_{VB}$ that may influence the estimation of the cascade rate.

Let us first start with the case of the presence of sharp variations (i.e., discontinuities) in the angle $\Theta_{VB}$ as in the example of Figure 17. Such discontinuities may be due to different causes, such as the crossing of strong current sheets frequently observed in the solar wind and the magnetosheath (Gosling & Szabo 2008; Chasapis et al. 2015). We estimated the energy cascade rate using BG13 from a long but nonstationary time interval (04:40–06:00) that contained two discontinuities in $\Theta_{VB}$ (at about 05:00 and 05:40) and from a shorter one (05:05–05:38) where such discontinuities were excluded. The results are shown in Figure 17 (bottom). As one can see, the long, nonstationary time interval yields a nonuniform energy cascade rate that changes its sign, whereas the shorter one where the $\Theta_{VB}$ sharp discontinuities were excluded is more uniform and has a constant sign. This result should balance the usual wisdom that argues for the use of long time intervals (i.e., a large number of data points) to guarantee the statistical convergence of the third-order moment estimates (e.g., Podesta et al. 2009): the existence of a few (i.e., statistically minor) sharp discontinuities such as those in Figure 17 can significantly influence the estimates of the cascade rate, as we showed here.

The second possible effect of the angle $\Theta_{VB}$ can come from its steady but significant variation in a single time interval. Indeed, as we argued in Section 3, the Taylor frozen-in flow assumption generally used on single-spacecraft data allows one to convert the time sampling of the data into a 1D spatial
sampling of the turbulent fluctuations along the flow direction. In anisotropic turbulence, the direction of the spatial sampling carries therefore a particular importance since the sampling can be either parallel ($\Theta_{VB} \sim 0^\circ$) or perpendicular ($\Theta_{VB} \sim 90^\circ$) to the mean field. These two directions, as demonstrated in Figure 14, have different values of the energy cascade rate. Therefore, if $\Theta_{VB}$ oscillates strongly between $0^\circ$ and $90^\circ$, then the analysis would mix the two cascade rates estimated along the directions parallel and perpendicular to the local magnetic field and would lead to higher uncertainty in the estimated values. This might be what explains the discrepancy in the cascade rate in the slow solar wind found between our results and those of MacBride et al. (2008).

5.4. Mean Value of Cascade Rate and Sign Change

As explained in Section 3, among the criteria that we used to select our statistical samples is the constant sign of the estimated cascade rate $\varepsilon_C$ over the time lag $\tau \in [10, 1000]$ s. This step is necessary in order to get a reliable estimate of the mean cascade rate $\langle \varepsilon_C \rangle$ averaged over all of the time lags $\tau$. Indeed, if the sign of $\varepsilon$ changes, the resulting average will yield (by cancellation) lower values of the cascade rates. Another alternative to this approach has been used in previous works based on performing statistical studies of the cascade rate obtained at a given value of the time lag $\tau$ (MacBride et al. 2008; Smith et al. 2009). The choice of the particular $\tau$ value has not been justified apart from the fact that it belongs to the inertial range. The drawback of this approach is that, since the sign of $\varepsilon$ can vary within the inertial range, as can be seen in Figure 17 and in, for example, in Sorriso-Valvo et al. (2007), the choice of the value of $\tau$ may influence the conclusion regarding the nature (direct versus inverse) of the turbulent cascade.

Figure 18 shows the histogram of $\varepsilon_\tau$ computed using PP98 at different values of $\tau$. For $\tau = 21$ s (Figure 18(a)), $\langle \varepsilon_\tau \rangle$ is positive, implying a direct cascade, whereas for $\tau = 81$ s (Figure 18(b)), $\langle \varepsilon_\tau \rangle$ is negative, indicating an inverse cascade. This result underlines the need to be cautious when interpreting statistical results for cascade rates estimated at a single value of the time lag $\tau$ within the inertial range.

6. Summary and Conclusions

In this paper we provided the first statistical study of the compressible energy cascade rate in fast and slow solar wind MHD turbulence using a large survey of the THEMIS/ARTEMIS spacecraft data. The work is based on the reduced form of the isothermal compressible MHD turbulence model recently derived in Banerjee & Galtier (2013). Several new results have been obtained, which include the amplification of the cascade rate and its slight isotropization (in particular in the slow wind) due to compressible fluctuations and a better definition of the inertial range thanks to a steadier (in value and sign) estimated compressible cascade rate over more than two decades of scales in comparison to the incompressible PP98 model. The new flux terms contained in the BG13 model were shown to play a leading role in amplifying the compressible energy cascade rate, rather than the modified compressible Yaglom term. This result disagrees with the finding of Carbone et al. (2009), who used a heuristic compressible model based on a modification of the Yaglom term in the PP98 model via density fluctuations. That discrepancy motivated a comparative study with the C09 model, which eventually showed that the origin of the cascade rate amplification found in Carbone et al. (2009) is due to the mean solar wind velocity included in that estimation through the modified (compressible) Elsässer variables. Other important results have been obtained, such as the new empirical scaling laws relating the new compressible cascade rate to the sonic turbulent Mach number, and to the different components (magnetic, kinetic, and internal) of the fluctuating energy. Interpreting those empirical laws requires further theoretical investigations. Several caveats related to the data selection and to the role the angle $\Theta_{VB}$ plays in the convergence of the energy cascade rate were highlighted.

While this work based on the new BG13 model undoubtedly sheds light on new features of solar wind turbulence, it remains however a perfectible model. Two particular aspects require improvement. The first one is related to the source terms that could not have been estimated in this work using single-spacecraft data (as they involve local divergences of the Alfvén and the plasma velocity fields). A reliable estimation of those terms can be done using multispacecraft observations. Cluster spacecraft offer that possibility, but the plasma data (density, velocity, and temperature) are available only on two (out of four) spacecraft, which does not allow us to obtain a 3D estimation of the source terms. The recently launched Magnetospheric Multiscale Mission (MMS) mission offers a more interesting alternative as both the magnetic field and plasma data are available on the four spacecraft. However, the mission in its current phase explores only the magnetopause and magnetosheath regions (with a focus on the former) and will reach into the solar wind only in 2018. Another possible shortcoming is the spacecraft separation ($\sim$10 km), which would not allow accurate estimation of the gradients at the scales of the inertial range $>$100 km (Robert et al. 1998). Other than spacecraft data, numerical simulation of isothermal compressible MHD turbulence should allow for a straightforward estimation of the source terms and their comparison to the flux terms. This task is planned for the upcoming months. On a longer run, the BG13 model needs to be extended to more general closure equations, such as the polytropic one, to go beyond the current simplified isothermal closure.

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