Algorithmic Randomness in Continuous-Time Markov Chains

Xiang Huang$^1$  
Jack H. Lutz$^2$  
Andrei N. Migunov$^2$

xhuan5@uis.edu  
lutz@iastate.edu  
amigunov@iastate.edu

$^1$University of Illinois Springfield, Springfield, IL 62703, USA  
$^2$Iowa State University, Ames, IA 50011 USA

December 20, 2021

Abstract

In this paper we develop the elements of the theory of algorithmic randomness in continuous-time Markov chains (CTMCs). Our main contribution is a rigorous, useful notion of what it means for an individual trajectory of a CTMC to be random. CTMCs have discrete state spaces and operate in continuous time. This, together with the fact that trajectories may or may not halt, presents challenges not encountered in more conventional developments of algorithmic randomness. Although we formulate algorithmic randomness in the general context of CTMCs, we are primarily interested in the computational power of stochastic chemical reaction networks, which are special cases of CTMCs. This leads us to embrace situations in which the long-term behavior of a network depends essentially on its initial state and hence to eschew assumptions that are frequently made in Markov chain theory to avoid such dependencies.

After defining the randomness of trajectories in terms of a new kind of martingale (algorithmic betting strategy), we prove equivalent characterizations in terms of algorithmic measure theory and Kolmogorov complexity. As a preliminary application we prove that, in any stochastic chemical reaction network, every random trajectory with bounded molecular counts has the non-Zeno property that infinitely many reactions do not occur in any finite interval of time.

Keywords: algorithmic randomness, continuous-time Markov chain, chemical reaction network

1 Introduction

Stochastic chemical reaction networks are used in molecular programming, DNA nanotechnology, and synthetic biology to model and specify the behaviors of natural and engineered molecular systems. Stochastic chemical reaction networks
are known to be Turing universal [27], hence capable of extremely complex dynamic behavior.

Briefly and roughly (deferring details until later in the paper), a stochastic chemical reaction network $N$ is a mathematical model of a chemical process in a volume $V$ of solution. A state of $N$ consists of the nonnegative integer populations of each of its finitely many species (types of molecules) at a given time. The state space is thus countable and discrete. The network stays in a state for a positive, real-valued sojourn time after which one of the finitely many reactions that $N$ allows to occur among its species produces an instantaneous jump transition to a different state. Both the sojourn time and the choice of the reaction are probabilistic, with the network behaving as a certain kind of continuous-time Markov chain given by the parameters of $N$. Hence, given an initial state at time $t = 0$, there are in general uncountably many trajectories (sequences of states and sojourn times) that $N$ can traverse. Some of these trajectories are finite (because $N$ reaches a state in which none of its reactions can occur), and some are infinite.

In this paper we develop the elements of the theory of algorithmic randomness in continuous-time Markov chains (CTMCs). Specifically, our main contribution is a rigorous, useful notion of what it means for an individual trajectory (also called a single orbit in dynamical systems theory) of a CTMC $C$ to be random with respect to $C$ and an initial state—or probability distribution of initial states—of $C$. This is a first step toward carrying out Kolmogorov’s program of replacing probabilistic laws stating that almost every trajectory has a given property with stronger randomness laws stating that every random trajectory has the property. More generally, we are initiating an algorithmic “single orbit” approach (in the sense of Weiss [34]) to the dynamics of CTMCs. In a variety of contexts ranging from Bernoulli processes to ergodic theory, Brownian motion, and algorithmic learning, this algorithmic single-orbit approach has led to improved understanding of known results [13, 4, 20, 25, 31, 19, 5, 8, 1, 6, 29, 24, 7, 30]. In the context of fractal geometry, this approach has even led to recent solutions of classical open problems whose statements did not involve algorithms or single orbits [15, 16, 18, 17, 26, 14].

The fact that CTMCs have discrete state spaces and operate in continuous time, together with the fact that trajectories may or may not halt, presents challenges not encountered in more conventional developments of algorithmic randomness. Our formulation of randomness is nevertheless general. Because we are interested in the computational power of stochastic chemical reaction networks, we embrace situations in which the long-term behavior of a network depends essentially on its initial state. Our development thus does not make assumptions that are frequently used in Markov chain theory to avoid such dependencies.

Our approach is also general in another sense, one involving Kolmogorov’s program, mentioned above. Once one has succeeded in replacing an “almost every” probabilistic law with an “every random” law, a natural next question is, How much randomness is sufficient for the latter? Saying that an individual object is random is saying that it “appears random” to a class of computations.
Roughly speaking, an object is algorithmically random (or Martin-Löf random) if it appears random to all computably enumerable sets. But weaker notions of randomness such as computable randomness, polynomial-space randomness, polynomial-time randomness, and finite-state randomness, have also been extensively investigated. Three examples of answers to the “how much randomness suffices” question in the context of infinite binary sequences are that (i) every algorithmically random sequence satisfies Birkhoff’s ergodic theorem [31]; (ii) every polynomial-time random sequence satisfies the Khinchin-Kolmogorov law of the iterated logarithm [32]; and (iii) every finite-state random sequence satisfies the strong law of large numbers [23].

Although we are primarily concerned with algorithmic randomness in the present paper, we want our randomness notion to be general enough to extend easily to other computational “levels” of randomness, so that “how much randomness” questions can be formulated and hopefully answered. For this reason, we define algorithmic randomness in CTMCs using the martingale (betting strategy) approach of Schnorr [21]. This approach extends to other levels of randomness in a straightforward manner, while our present state (i.e., lack) of knowledge in computational complexity theory does not allow us to extend other approaches (e.g., Martin-Löf tests or Kolmogorov complexity, which are known to be equivalent to the martingale approach at the algorithmic level [13, 4, 20, 25]) to time-bounded complexity classes.

We develop our algorithmic randomness theory in stages. In section 2 we develop the underlying qualitative structure of Boolean transition systems, defined so that (i) state transitions are nontrivial, i.e., not from a state to itself, and (ii) trajectories may or may not terminate. We then show how to use these transition systems to model rate-free chemical reaction networks.

In section 3 we add probabilities, thereby defining probabilistic transition systems. For each probabilistic transition system $\mathcal{Q}$ and each initialization $\sigma$ of $\mathcal{Q}$ we then define $(\mathcal{Q}, \sigma)$-martingales, which are strategies for betting on the successive entries in a sequence of states of $(\mathcal{Q}, \sigma)$. Following the approach of Schnorr [21], we then define a maximal state sequence $q$ of $(\mathcal{Q}, \sigma)$ to be random if there is no lower semicomputable $(\mathcal{Q}, \sigma)$-martingale that succeeds on $q$, i.e., makes unbounded money betting along $q$. This notion of randomness closely resembles the well-understood theory of random sequences over a finite alphabet [13, 4, 20, 25], except that here the state set may be countably infinite; transitions from a state to itself are forbidden; and a positive-probability state sequence may terminate, in which case it is random.

Section 4 is where we confront the main challenge of algorithmic randomness in CTMCs, the fact that they operate in continuous, rather than discrete, time. There we develop the algorithmic randomness of sequences $t = (t_0, t_1, \ldots)$ of sojourn times $t_i$ relative to corresponding sequences $\lambda = (\lambda_0, \lambda_1, \ldots)$ of nonnegative real-valued rates $\lambda_i$. Each $\lambda_i$ in such a sequence is regarded as defining an exponential probability distribution function $F_{\lambda_i}$, and the sojourn times $t_i$ are to be independently random relative to these. We use a careful binary encoding of sojourn times to define $\lambda$-martingales that bet along sequences of sojourn times, and we again follow the Schnorr approach, defining a sequence $t$ of so-
jour times to be \( \lambda \)-random if there is no lower semicomputable \( \lambda \)-martingale that succeeds in it.

In section 5 we put the developments of sections 3 and 4 together. A trajectory of a continuous-time Markov chain \( C \) is a sequence \( \tau \) of ordered pairs \((q_n, t_n)\), where \( q_n \) is a state of \( C \) and \( t_n \) is the sojourn time that \( C \) spends in state \( q_n \) before jumping to state \( q_{n+1} \). For each continuous-time Markov chain \( C \), we define the notion of a \( C \)-martingale. Following Schnorr once again, we define a trajectory \( \tau \) of \( C \) to be random if no lower semicomputable martingale succeeds on it. We also give a Kolmogorov complexity characterization of the randomness of trajectories of continuous-time Markov chains. As an example application we prove that, in any stochastic chemical reaction network, every random trajectory \( \tau \) with bounded molecular counts has the non-Zeno property that infinitely many reactions do not occur in any finite interval of time.

2 Boolean transition systems

Before developing algorithmic randomness for sequences of states with respect to computable, probabilistic transition systems, we develop the underlying qualitative (not probabilistic) structure by considering transition systems that are Boolean. Some care must be taken to accommodate the fact that, in cases of interest, a sequence of states may either be infinite or end in a terminal state.

Formally, we define a Boolean transition system to be an ordered pair \( \mathcal{L} = (Q, \delta) \) where \( Q \) is a nonempty, countable set of states, and \( \delta : Q \times Q \to \{0, 1\} \) is a Boolean state transition matrix satisfying \( \delta(q, q) = 0 \) for all \( q \in Q \).

Intuitively, a Boolean transition system \( \mathcal{L} = (Q, \delta) \) is a nondeterministic structure that may be initialized to any nonempty set of states in \( Q \). For \( q, r \in Q \), the entry \( \delta(q, r) \) in the Boolean transition matrix \( \delta \) is the Boolean value (0 = false; 1 = true ) of the condition that \( r \) is reachable from \( q \) in one “step” of \( \mathcal{L} \). The irreflexivity requirement that every \( \delta(q, q) = 0 \) (i.e., that \( \delta \) have a zero diagonal) reflects the fact that, in all cases of interest in this paper, transitions are nontrivial changes of state. We formalize this intuition, because the formalism will be useful here.

We write \( Q^{<\omega} \) for the set of all finite sequences of states in \( Q \), \( Q^\omega \) for the set of all infinite sequences of states in \( Q \), and \( Q^{\leq \omega} = Q^{<\omega} \cup Q^\omega \). The length of a sequence \( q \in Q^{\leq \omega} \) is

\[
|q| = \begin{cases} 
\omega & \text{if } q = (q_0, q_1, ..., q_{\omega-1}) \in Q^{<\omega} \\
\omega & \text{if } q \in Q^{\omega}
\end{cases}
\]

A sequence \( q \in Q^{\leq \omega} \) can thus be written as \( q = (q_i | i < |q|) \) in any case. We write () for the empty sequence (sequence of length 0).

For \( q, r = (r_i | i < |r|) \in Q^{<\omega} \), we say that \( q \) is a prefix of \( r \), and we write \( q \subseteq r \), if \( |q| \leq |r| \) and \( q = (r_i | i < |q|) \). It is easy to see that \( \subseteq \) is a partial ordering of \( Q^{\leq \omega} \).

An initialization of a Boolean transition system \( \mathcal{L} = (Q, \delta) \) is a Boolean-valued function \( \sigma : Q \to \{0, 1\} \) whose support \( \text{supp}(\sigma) = \{q \in Q | \sigma(q) \neq 0\} \) is
nonempty.

A Boolean transition system \( \mathcal{Q} = (Q, \delta) \) admits a sequence \( q = (q_i | i < |q|) \in Q^{\leq \omega} \) with an initialization \( \sigma \), and we say that \( q \) is \( \mathcal{Q} \)-admissible from \( \sigma \), if the following conditions hold for all \( 0 \leq i < |q| \).

(i) If \( i = 0 \), then \( \sigma(q_0) = 1 \).
(ii) If \( i + 1 < |q| \), then \( \delta(q_i, q_{i+1}) = 1 \).

A sequence \( q \in Q^{\leq \omega} \) that is \( \mathcal{Q} \)-admissible from \( \sigma \) is maximal if, for every sequence \( r \in Q^{\leq \omega} \) that is \( \mathcal{Q} \)-admissible from \( \sigma \), \( q \subseteq r \implies q = r \).

We use the following notations.

\[ \text{Adm}_\mathcal{Q}(\sigma) = \{x \in Q^{<\omega} | x \text{ is } \mathcal{Q} \text{-admissible from } \sigma \}. \]

\[ \mathcal{A}[\mathcal{Q}](\sigma) = \{q \in Q^{\leq \omega} | q \text{ is a maximal } \mathcal{Q} \text{-admissible sequence from } \sigma \}. \]

When \( \mathcal{Q} \) is obvious from the context, we omit it from the notation and write these sets as \( \text{Adm}(\sigma) \) and \( \mathcal{A}(\sigma) \). Note that elements of \( \text{Adm}_\mathcal{Q}(\sigma) \) are required to be finite sequences.

Intuitively, \( \mathcal{A}[\mathcal{Q}](\sigma) \) is the set of all possible “behaviors” of the Boolean transition system \( \mathcal{Q} = (Q, \delta) \) with the state initialization \( \sigma : Q \to \{0, 1\} \). The fact that \( \delta \) is irreflexive implies that \( q_i \neq q_{i+1} \) holds for all \( i \in \mathbb{N} \) such that \( i + 1 < |q| \) in every admissible sequence \( q = (q_i | i < |q|) \in \mathcal{A}[\mathcal{Q}](\sigma) \). In this paper we do not regard the indices \( i = 0, 1, \ldots \) in a state sequence \( q = (q_0, q_1, \ldots) \) as successive instants in discrete time. In our main applications, the amount of time spent in state \( q_i \) varies randomly and continuously, so it is more useful to think of the indices \( i = 0, 1, \ldots \) as finite ordinal numbers, i.e., to think of \( q_i \) as merely the \( i \)th state in the sequence \( q \).

Each \( x \in \text{Adm}_\mathcal{Q}(\sigma) \) is the name of the \( \mathcal{Q} \)-cylinder

\[ \mathcal{A}_x(\sigma) = \{q \in \mathcal{A}[\mathcal{Q}](\sigma) | x \subseteq q \}. \]

Each \( x \in \text{Adm}(\sigma) \) is a finite—and typically partial—specification of each sequence \( q \in \mathcal{A}_x(\sigma) \). The collection

\[ \mathcal{A}(\sigma) = \mathcal{A}[\mathcal{Q}](\sigma) = \{\mathcal{A}_x(\sigma) | x \in \text{Adm}_\mathcal{Q}(\sigma)\} \]

is a basis for a topology on \( \mathcal{A}(\sigma) \). The open sets in this topology are simply the sets that are unions of (finitely or infinitely many) cylinders in \( \mathcal{A}(\sigma) \). The metric (in fact, ultrametric) \( d \) on \( Q^{\leq \omega} \) defined by

\[ d(q, r) = 2^{-|p|}, \]

where \( p \) is the longest common prefix of \( q \) and \( r \) (and \( 2^{-\infty} = 0 \)), induces this same topology on \( \mathcal{A}[\mathcal{Q}](\sigma) \) for each Boolean transition system \( \mathcal{Q} = (Q, \delta) \) and each state initialization \( \sigma : Q \to [0, 1] \). With this topology, \( \mathcal{A}[\mathcal{Q}](\sigma) \) is a Polish space (a complete, separable metric space). The isolated points in \( \mathcal{A}[\mathcal{Q}](\sigma) \) are (when they exist) the sequences in \( \mathcal{A}[\mathcal{Q}](\sigma) \) that are finite, i.e., the sequences
\( x \in Q^{\omega} \cap A[\mathcal{D}](\sigma) \). Such sequences \( x \) are said to halt, or terminate, in \( \mathcal{D} \) from \( \sigma \).

A Boolean transition system \( \mathcal{D} = (Q, \delta) \) is computable if the elements of \( Q \) are naturally represented in such a way that (i) the Boolean-valued function \( \delta \) is computable, and (ii) the set of terminal states (i.e., states \( q \in Q \) such that \( \delta(q, r) = 0 \) for all \( r \in Q \)) is decidable. An initialization \( \sigma : Q \to \{0, 1\} \) is computable if its support is decidable.

An important class of examples of Boolean transition systems consists of those that model rate-free chemical reaction networks. Formally, let \( S = \{X_0, X_1, X_2, \ldots\} \) be a countable set of distinct species \( X_n \), each of which we regard as an abstract type of molecule. A rate-free chemical reaction network (or rate-free CRN) is an ordered pair \( N = (S, R) \), where \( S \subseteq S \) is a finite set of species, and \( R \) is a finite set of (rate-free) reactions on \( S \), each of which is formally an ordered pair \( \rho = (r, p) \) of distinct vectors \( r, p \in \mathbb{N}^S \) (equivalently, functions \( r, p : S \to \mathbb{N} \)). Informally, we write species in notations convenient for specific problems (\( X, Y, Z, \hat{X}, \hat{Y} \), etc.) rather than as subscripted elements of \( S \), and we write reactions in a notation more suggestive of chemical reactions. For example,

\[
X + Z \to 2Y + Z \tag{2.1}
\]

is a rate-free reaction on the set \( S = \{X, Y, Z\} \). If we consider the elements of \( S \) to be ordered as written, then the left-hand side of (2.1) is formally the reactant vector \( r = (1, 0, 1) \), and the right-hand side of (2.1) is the product vector \( p = (0, 2, 1) \). A species \( Y \in S \) is called a reactant of a reaction \( \rho = (r, p) \) if \( r(Y) > 0 \) and a product of \( \rho \) if \( p(Y) > 0 \).

Intuitively, the reaction \( \rho \) in (2.1) means that, if a molecule of species \( X \) encounters a molecule of species \( Z \), then the reaction \( \rho \) may occur, in which case the reactants \( X \) and \( Z \) disappear and the products – two molecules of species \( Y \) and a molecule of species \( Z \) – appear in their place. Accordingly, the net effect of a reaction \( \rho = (r, p) \) is the vector \( \Delta \rho \in \mathbb{Z}^S \) defined by

\[
\Delta \rho(Y) = p(Y) - r(Y)
\]

for all \( Y \in S \). Since we have required \( r \) and \( p \) to be distinct, \( \Delta \rho \) is never the zero-vector in \( \mathbb{Z}^S \).

In this paper, a state of a chemical reaction network \( N = (S, R) \) is a vector \( q \in \mathbb{N}^S \). Intuitively, \( N \) is modeling chemical processes in a solution, and the state \( q \) denotes a situation in which, for each \( Y \in S \), exactly \( q(Y) \) molecules of species \( Y \) are present in the solution.

A reaction \( \rho = (r, p) \in R \) of a chemical reaction network \( N = (S, R) \) can occur (or is enabled) in a state \( q \in \mathbb{N}^S \) if

\[
q(Y) \geq r(Y)
\]

holds for every \( Y \in S \), i.e. if the reactants of \( \rho \) are present in \( q \). If this reaction \( \rho \) does occur in state \( q \), then it transforms \( q \) to the new state \( q + \Delta \rho \).

The behavior of a rate-free chemical reaction network \( N = (S, R) \) clearly coincides with that of the Boolean transition system \( \mathcal{D}_N = (\mathbb{N}^S, \delta) \), where
δ : N^S × N^S → [0, 1] is defined by setting each δ(q, q') to be the Boolean value of the condition that some reaction ρ ∈ R transforms the state q to the state q'. Boolean transition systems of this form are clearly computable and have other special properties. As one example, for each q ∈ N^S, there only exist finitely many q' ∈ N^S for which δ(q, q') = 1.

Rate-free chemical reaction networks, and Boolean transition systems more generally, raise significant and deep problems in distributed computing [10, 3], but our focus here is on randomness, which we begin in the following section.

3 Random state sequences

This section develops the elements of algorithmic randomness for sequences of states with respect to computable, probabilistic transition rules.

Formally, we define a probabilistic transition system to be an ordered pair \( \mathcal{Q} = (Q, \pi) \), where Q is a countable set of states, and \( \pi : Q \times Q \rightarrow [0, 1] \) is a probabilistic transition matrix, by which we mean that \( \pi \) satisfies the following two conditions for each state \( q \in Q \).

1. \( \pi(q, q) = 0 \).
2. The sum \( \pi(q) = \sum_{r \in Q} \pi(q, r) \) is either 0 or 1.

If the sum \( \pi(q) \) in condition 2 is 0, then \( q \) is a terminal state. If \( \pi(q) \) is 1, then \( q \) is a nonterminal state.

If \( \mathcal{Q} = (Q, \pi) \) is a probabilistic transition system, and we define \( \delta : Q \times Q \rightarrow \{0, 1\} \) by

\[
\delta(q, r) = sgn(\pi(q, r))
\]

for all \( q, r \in Q \), where \( sgn : [0, \infty) \rightarrow \{0, 1\} \) is the signum function

\[
sgn(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}
\]

then \( \mathcal{Q}_B = (Q, \delta) \) is the Boolean transition system corresponding to \( \mathcal{Q} \). The essential difference between \( \mathcal{Q}_B \) and \( \mathcal{Q} \) is that, while \( \delta(q, r) \) merely says whether it is possible for \( \mathcal{Q}_B \) (or \( \mathcal{Q} \)) to transition from \( q \) to \( r \) in one step, \( \pi(q, r) \) is the quantitative probability of doing so.

An initialization of a probabilistic transition system \( \mathcal{Q} = (Q, \pi) \) is a discrete probability measure \( \sigma \) on \( Q \), i.e., a function \( \sigma : Q \rightarrow [0, 1] \) satisfying \( \sum_{q \in Q} \sigma(q) = 1 \). The Boolean version of such an initialization \( \sigma \) is the function \( \sigma_B : Q \rightarrow \{0, 1\} \) defined by

\[
\sigma_B(q) = sgn(\sigma(q))
\]

for each \( q \in Q \). It is clear that \( \sigma_B \) is an initialization of \( \mathcal{Q}_B \).

Given a probabilistic transition system \( \mathcal{Q} = (Q, \pi) \) and an initialization \( \sigma \) of \( \mathcal{Q} \), we define the sets
\[ \text{Adm}(\sigma) = \text{Adm}_{2}(\sigma) = \text{Adm}_{2B}(\sigma_B), \]

\[ \text{X}(\sigma) = X[2](\sigma) = X_{2B}(\sigma_B), \]

by the fact that the right-hand sets were defined in section 2. The notations and terminology in section 2 leading up to these definitions are similarly extended to probabilistic transition systems, as are the definitions of the \(2\)-cylinders \(X_x(\sigma)\) and the basis \(\mathcal{A}(\sigma)\) for the topology \(X(\sigma)\).

What we can do here that we could not do for Boolean transition systems is define a Borel probability measure on each set \(X_{2}(\sigma)\). Specifically, for each probabilistic transition system \(2 = (Q, \pi)\) and each initialization \(\sigma\) of \(2\), define the function

\[ \mu_{2,\sigma} : \text{Adm}_{2}(\sigma) \to [0, 1] \]

as follows. Let \(x = (x_i | i < |x|) \in \text{Adm}_{2}(\sigma)\). If \(|x| = 0\), then \(\mu_{2,\sigma}(x) = 1\). If \(|x| > 0\), then

\[ \mu_{2,\sigma}(x) = \sigma(x_0) \prod_{i=0}^{|x|-2} \pi(x_i, x_{i+1}). \]

Since \(x\) is a name of the cylinder \(X_x[2](\sigma)\), each \(\mu_{2,\sigma}(x)\) here should be understood as an abbreviation of \(\mu_{2,\sigma}(X_x(\sigma))\), which is intuitively the probability that an element of \(X_{2}(\sigma)\) begins with the finite sequence \(x\).

**Observation 1.** If a sequence \(x \in \text{Adm}_{2}(\sigma)\) does not terminate, then

\[ \mu_{2,\sigma}(x) = \sum_{x \sqsubseteq y \in \text{Adm}_{2}(\sigma), |y| = |x| + 1} \mu_{2,\sigma}(y). \]

The above observation implies that \(\mu_{2,\sigma}\) can, by standard techniques, be extended to a Borel probability measure on \(X[2](\sigma)\), i.e., to a function \(\mu_{2,\sigma}\) that assigns probability \(\mu_{2,\sigma}(E)\) to every Borel set \(E \subseteq X[2](\sigma)\).

If \(2\) is a probabilistic transition system and \(\sigma\) is an initialization of \(2\), then a \((2, \sigma)\)-martingale is a function \(d : \text{Adm}_{2}(\sigma) \to [0, \infty)\) such that, for every non-terminating sequence \(x \in \text{Adm}_{2}(\sigma)\),

\[ d(x)\mu(x) = \sum_{x \sqsubseteq y \in \text{Adm}_{2}(\sigma), |y| = |x| + 1} d(y)\mu(y), \quad (3.1) \]

where \(\mu = \mu_{2,\sigma}\).

Intuitively, a \((2, \sigma)\)-martingale \(d\) is a gambler that bets on the successive states in a sequence \(q = (q_i | i < |q|) \in X[2](\sigma)\). The gambler’s initial capital is \(d(\langle\rangle)\), and its capital after betting on a prefix \(x \in \text{Adm}_{2}(\sigma)\) of \(q\) is \(d(x)\). The condition (3.1) says that the payoffs are fair with respect to the probability measure \(\mu = \mu_{2,\sigma}\) in the sense that the conditional expectation of the gambler’s capital after betting on the state following \(x\) in \(q\), given that \(x \sqsubseteq q\), is exactly the gambler’s capital before placing this bet.
A \((\mathcal{Q}, \sigma)\)-martingale \(d\) succeeds on a sequence \(q \in \mathbb{A}[\mathcal{Q}](\sigma)\) if the set
\[
\{d(x) | x \in \text{Adm}_{\mathcal{Q}}(\sigma) \text{ and } x \sqsubseteq q\}
\]
is unbounded. The success set of a \((\mathcal{Q}, \sigma)\)-martingale \(d\) is
\[
S^\infty[d] = \{q \in \mathbb{A}[\mathcal{Q}](\sigma) | d \text{ succeeds on } q\}.
\]

Following standard practice, we develop randomness by imposing computability conditions on martingales. Recall that, if \(D\) is a discrete domain, then a function \(f : D \rightarrow \mathbb{R}\) is computable if there is a computable function \(\hat{f} : D \times \mathbb{N} \rightarrow \mathbb{Q}\) such that, for all \(x \in D\) and \(r \in \mathbb{N}\),
\[
|\hat{f}(x, r) - f(x)| \leq 2^{-r}.
\]
The parameter \(r\) here is called a precision parameter.

A function \(f : D \rightarrow \mathbb{R}\) is lower semi-computable if there is a computable function \(\hat{f} : D \times \mathbb{N} \rightarrow \mathbb{Q}\) such that the following two conditions hold for all \(x \in D\).

(i) For all \(s \in \mathbb{N}\),
\[
\hat{f}(x, s) \leq \hat{f}(x, s+1) < f(x).
\]
(ii) \(\lim_{s \rightarrow \infty} \hat{f}(x, s) = f(x)\).

The parameter \(s\) is sometimes called a patience parameter, because the convergence in (ii) can be very slow.

A probabilistic transition system \(\mathcal{Q} = (Q, \pi)\) is computable if the elements of \(Q\) are naturally represented in such a way that (i) the probability transition matrix \(\pi : Q \times Q \rightarrow [0, 1]\) is computable in the above sense, and (ii) the support of \(\pi\) and the set of terminal states are decidable. (It is well known ([9], [33]) that (ii) does not follow from (i). Fortunately, (ii) does hold in many cases of interest, including chemical reaction networks).

Similarly, an initialization \(\sigma\) of a probabilistic transition system \(\mathcal{Q} = (Q, \pi)\) is computable if (i) the function \(\sigma : Q \rightarrow [0, 1]\) is computable, and (ii) the support of \(\sigma\) is decidable.

Let \(\mathcal{Q}\) be a probabilistic transition system that is computable, and let \(\sigma\) be an initialization of \(\mathcal{Q}\) that is also computable. A state sequence \(q \in \mathbb{A}[\mathcal{Q}](\sigma)\) is (algorithmically) random if there is no lower semi-computable \((\mathcal{Q}, \sigma)\)-martingale that succeeds on \(q\).

This notion of random sequences in \(\mathbb{A}[\mathcal{Q}](\sigma)\) closely resembles the well-understood theory of random sequences on a finite alphabet [35, 22]. The main differences are that here the state set may be countably infinite; transitions from a state to itself are forbidden; and a positive-probability state sequence may terminate, in which case it is clearly random. The following analog of Ville’s theorem holds for probabilistic transition sequences. This theorem, as well as an algorithmic version, follow from a stronger theorem proven in Section 5.
**Theorem 2** (Ville [28]). Let $\mathcal{D}$ be a probabilistic transition system, let $\sigma$ be an initialization of $\mathcal{D}$, and let $\mu = \mu_{\mathcal{D}, \sigma}$. For every set $E \subseteq A_{\mathcal{D}}(\sigma)$, the following two conditions are equivalent.

1. $\mu(E) = 0$.
2. There is a $(\mathcal{D}, \sigma)$-martingale $d$ such that $E \subseteq S^\infty[d]$.

### 4 Random sequences of sojourn times

The “sojourn time” that a continuous-time Markov chain spends in a state before jumping to a new state may be any element of $(0, \infty]$, i.e., any duration $t$ that is either a (strictly) positive real number or $\infty$. This section thus develops the elements of algorithmic randomness for sequences of durations $t \in (0, \infty]$ with respect to sequences of probability measures that occur in continuous-time Markov chains.

A *rate* in this paper is a non-negative real number $\lambda \in [0, \infty)$. We rely on context to distinguish this standard use of $\lambda$ from the equally standard use of $\lambda$ to denote the empty string. We interpret each rate $\lambda > 0$ as a name of the exponential probability measure with rate $\lambda$, i.e., the probability measure on $(0, \infty]$ whose cumulative distribution function $F_\lambda : (0, \infty] \to [0, 1]$ is given by

$$F_\lambda(t) = 1 - e^{-\lambda t}$$

for all $t \in (0, \infty]$, where $e^{-\infty} = 0$. We interpret the rate $\lambda = 0$ as a name of the point-mass probability on $(0, \infty]$ that concentrates all the probability at $\infty$. This has the cumulative distribution function $F_0 : (0, \infty] \to [0, 1]$ given by

$$F_0(t) = \begin{cases} 
0 & \text{if } t \in (0, \infty) \\
1 & \text{if } t = \infty
\end{cases}.$$

We associate each string $w \in \{0, 1\}^*$ with the interval $I_w \subseteq [0, 1]$ defined as follows. Let $w$ be the lexicographically $i^{\text{th}}$ ($0 \leq i < 2^{|w|}$) element of $\{0, 1\}^{|w|}$ where $0^{|w|}$ is the $0^\text{th}$ element and $1^{|w|}$ is the $(2^{|w|} - 1)^\text{st}$ element. Then

$$I_w = (2^{-|w|}, 2^{-|w|}(i + 1]].$$

Note that, for each $w \in \{0, 1\}^*$ and $l \in \mathbb{N}$, the intervals $I_{wu}$, for $u \in \{0, 1\}^l$, form a left-to-right partition of $I_w$, i.e., a partition of $I_w$ in which $I_{wu}$ lies to the left of $I_{wv}$ if and only if $u$ lexicographically precedes $v$.

For each rate $\lambda \in [0, \infty)$ and each string $w \in \{0, 1\}^*$, define the interval

$$D_\lambda(w) = F_\lambda^{-1}(I_w) \subseteq (0, \infty].$$

**Example 3.** If $\lambda > 0$, then

$$D_\lambda(00) = (0, a_1], \quad D_\lambda(01) = (a_1, a_2],$$
$$D_\lambda(10) = (a_2, a_3], \quad D_\lambda(11) = (a_3, \infty].$$
where \( a_1 = \frac{2\ln 2 - \ln 3}{\lambda}, a_2 = \frac{\ln 2}{\lambda}, \text{ and } a_3 = \frac{2\ln 2}{\lambda}. \) On the other hand, \( D_0(00) = (0, \infty), D_0(01) = D_0(10) = \emptyset, \text{ and } D_0(11) = \{ \infty \} \).

**Observation 4.** If \( \lambda > 0, \) then, for each \( l \in \mathbb{N}, \) the intervals \( D_\lambda(w) \), for \( w \in \{0, 1\}^l \), form a left-to-right partition of \((0, \infty]\) into intervals that are equiprobable with respect to \( F_\lambda \).

Example 3 shows that the assumption \( \lambda > 0 \) is essential here.

For each rate \( \lambda \in [0, \infty], \) each duration \( t \in (0, \infty], \) and each \( w \in \{0, 1\}^*, \) we call \( w \) a \( \lambda \)-approximation (or a partial \( \lambda \)-specification) of \( t, \) and we write \( w \sqsubseteq_\lambda t, \) if \( t \in D_\lambda(w) \).

A rate sequence is a nonempty sequence \( \lambda = (\lambda_i \mid 0 \leq i < |\lambda|) \in [0, \infty)^{\leq \omega} \) with the property that, for each \( 0 \leq i < |\lambda|, \)

\[
i + 1 < |\lambda| \iff \lambda_i > 0.
\]

(That is, either \( \lambda \) is finite with a single 0 entry, occurring at the end, or \( \lambda \) is infinite with no 0 entries.)

If \( \lambda = (\lambda_i \mid 0 \leq i < |\lambda|) \) is a rate sequence, then a \( \lambda \)-duration sequence is a sequence

\[
t = (t_i \mid 0 \leq i < |\lambda|) \in (0, \infty)^{\leq \omega}
\]

such that, for each \( 0 \leq i < |\lambda|, \)

\[
t_i < \infty \iff \lambda_i > 0.
\]

We write \( D_\lambda \) for the set of all \( \lambda \)-duration sequences. Note that

\[
D_\lambda = \begin{cases} 
(0, \infty)^{|\lambda|} \times \{ \infty \}, & \text{if } |\lambda| < \omega, \\
(0, \infty)^\omega, & \text{if } |\lambda| = \omega.
\end{cases}
\]

depends only on the length of \( \lambda, \) not on the components of \( \lambda. \)

If \( \lambda = (\lambda_i \mid 0 \leq i < |\lambda|) \) is a rate sequence, \( t = (t_i \mid 0 \leq i < |\lambda|) \in D_\lambda \) is a \( \lambda \)-duration sequence, and \( w = (w_i \mid 0 \leq i < |w|) \in (\{0, 1\}^*)^{< \omega} \) is a finite sequence of binary strings with \( |w| \leq |\lambda|, \) then we call \( w \) a \( \lambda \)-approximation (or a partial \( \lambda \)-specification) of \( t, \) and we write \( w \sqsubseteq_\lambda t, \) if \( w_i \sqsubseteq_\lambda t_i \) holds for all \( 0 \leq i < |w|. \)

If \( \lambda \) is a rate sequence and \( w \in (\{0, 1\}^*)^{< \omega} \) is a finite sequence of binary strings with \( |w| \leq |\lambda|, \) then the \( \lambda \)-cylinder generated by \( w \) is the set

\[
D_\lambda(w) = \{ t \in D_\lambda \mid w \subseteq_\lambda t \}
\]

of \( \lambda \)-duration sequences.

It is routine to verify that, for each rate sequence \( \lambda, \) the collection

\[
\mathcal{D}_\lambda = \{ D_\lambda(w) \mid w \in (\{0, 1\}^*)^{< \omega} \text{ and } |w| \leq |\lambda| \}
\]

is a semi-algebra of subsets of \( D_\lambda \) that generates the \( \sigma \)-algebra \( \mathcal{B}_\lambda \) of all Borel subsets of \( D_\lambda \). If we define

\[
\mu_\lambda : \mathcal{D}_\lambda \to [0, 1]
\]
by
\[ \mu_\lambda(\mathcal{B}_\lambda(w)) = 2^{-\sum_{i=0}^{\|w\|-1} |w_i|} \]
for all \( w = (w_i \mid i < \|w\|) \in (\{0,1\})^{<\omega} \) with \( |w| \leq |\lambda| \), then it follows by standard techniques that \( \mu_\lambda \) extends uniquely to a probability measure
\[ \mu_\lambda : \mathcal{B}_\lambda \to [0,1]. \]

Note that \( \mathcal{B}_\lambda \) only depends on the length of \( \lambda \), but \( \mu_\lambda \) also depends on the components of \( \lambda \). When convenient, we use the abbreviation
\[ \mu_\lambda(w) = \mu_\lambda(D_{\lambda}(w)). \]

If \( \lambda = (\lambda_i \mid 0 \leq i < |\lambda|) \) is a rate sequence, then a \( \lambda \)-martingale is a function
\[ d : (\{0,1\})^{<|\lambda|} \to [0,\infty) \]
that satisfies the following two conditions for all \( w = (w_0, ..., w_{n-1}) \in (\{0,1\})^{<|\lambda|} \).

1. \( d(w) = \frac{d(w_0, ..., w_{n-1}0) + d(w_0, ..., w_{n-1}1)}{2} \).
2. If \( n + 1 < |\lambda| \), then
   \[ d(w_0, ..., w_{n-1}, \lambda) = d(w_0, ..., w_{n-1}). \]
   (Note that the \( \lambda \) entry on the left-hand side is the empty string.)

Intuitively, a \( \lambda \)-martingale \( d \) is a strategy that a gambler may use for betting on approximations \( w_i \) of the durations \( t_i \) in a \( \lambda \)-duration sequence \( t = (t_i \mid i < |t|) \). The gambler’s initial amount of money is the value \( d(()) \) of \( d \) at the empty sequence ( ) of binary strings. If \( w = (w_0, ..., w_{n-1}) \subseteq_{\lambda} t \), then \( d(w) \) is the amount of money that the gambler has after betting on \( w \). This condition \( w \subseteq_{\lambda} t \) means that each \( t_i \) is in the interval \( D_{\lambda}(w_i) \subseteq [0,\infty) \). If the gambler then chooses to bet on which of the subintervals \( D_{\lambda_{n-1}}(w_{n-1}0) \) and \( D_{\lambda_{n-1}}(w_{n-1}1) \) \( t_{n-1} \) lies in, condition 1 above says that the payoffs of these bets are fair with respect to the exponential probability measure with rate \( \lambda_{n-1} \). (Note that \( D_{\lambda_{n-1}}(w_{n-1}0) \) and \( D_{\lambda_{n-1}}(w_{n-1}1) \) partition \( D_{\lambda_{n-1}}(w_{n-1}) \) into equiprobable subintervals, but these subintervals may have very different lengths.) Condition 2 above says that the extension from \( (w_0, ..., w_{n-1}) \) to \( (w_0, ..., w_{n-1}, \lambda) \), does not involve a bet. The martingale has values \( d(w) \) for all \( w \in (\{0,1\})^{<|\lambda|} \), but our intuitive gambler may place bets in many different orders. For example, the gambler may place a finite number of bets on approximations of \( t_{11} \), then a finite number of bets on approximations of \( t_2 \), etc., but this ordering of bets is an intuitive fancy, not part of the definition of the \( \lambda \)-martingale \( d \).

A \( \lambda \)-martingale \( d \) succeeds on a \( \lambda \)-duration sequence \( t \) if the set
\[ \{d(w) \mid w \subseteq_{\lambda} t\} \]
is unbounded. The success set of a $\lambda$-martingale $d$ is

$$S^\infty[d] = \{ t \in D_\lambda \mid d \text{ succeeds on } t \}.$$  

The following analog of Ville’s theorem holds for $\lambda$–martingales.

**Theorem 5.** If $\lambda$ is a rate sequence, then, for each set $E \subseteq D_\lambda$, the following two conditions are equivalent.

1. $\mu_\lambda(E) = 0$.
2. There is a $\lambda$-martingale $d$ such that $E \subseteq S^\infty[d]$.

The proof of Theorem 5 is a slight simplification of the proof of Theorem 8, and the proof of the algorithmic version of this theorem is a slight simplification of the proof of Theorem 9.

## 5 Random CTMC trajectories

We now develop the theory of randomness for sequences of state-time pairs, representing trajectories of continuous-time Markov chains.

### 5.1 Continuous-time Markov chains

A CTMC is an ordered triple

$$C = (Q, \lambda, \sigma),$$

where $Q$ is a countable set of states, $\lambda : Q \times Q \to [0, \infty)$ is the rate matrix satisfying $\lambda(q, q) = 0$ for every $q \in Q$, and $\sigma$ is the state initialization as described in section 3. Let $C = (Q, \lambda, \sigma)$ be a CTMC. At each time $t \in [0, \infty)$ $C$ is probabilistically in some state. At time $t = 0$, this state is chosen according to $\sigma$. For each state $q \in Q$, the real number

$$\lambda_q = \sum_{r \in Q} \lambda(q, r)$$

is the rate out of state $q$. If $\lambda_q = 0$, then $q$ is a terminal state, meaning that, if $C$ ever enters state $q$, then $C$ remains in state $q$ forever. If a state $q$ is nonterminal, i.e., $\lambda_q > 0$ and $C$ enters $q$ at some time $t$, then the sojourn time for which $C$ remains in state $q$ before moving to a new state is a random variable that has the exponential distribution with rate $\lambda_q$. Hence the expected sojourn time of $C$ in state $q$ is $\frac{1}{\lambda_q}$. When $C$ does move to a new state, it moves to state $r \in Q$ with probability

$$p(q, r) = \frac{\lambda(q, r)}{\lambda_q}.$$

Note that the CTMC model uses “continuous time” (times ranging over $(0, \infty)$) but a “discrete state space”. Accordingly, its state transitions, called
jump transitions, are instantaneous. Mathematically, if $C$ jumps from state $q$ to state $r$ at time $t$, we say that $q$ is in the “new” state $r$ at time $t$, having been in the “old” state $q$ throughout some time interval $[s, t)$ where $s < t$.

A trajectory of a CTMC $C = (Q, \lambda, \sigma)$ is a sequence $\tau$ of the form

$$\tau = ((q_n, t_n) \mid n \in \mathbb{N}) \in (Q \times (0, \infty))^\infty.$$  

Intuitively, such a trajectory $\tau$ denotes the turn of events in which $q_0, q_1, ...$ are the successive states of $C$ and $t_0, t_1, ...$ are the successive sojourn times of $C$ in these states. Accordingly, we write

$$\text{state}_\tau(n) = q_n, \text{ soj}_\tau(n) = t_n$$

for each $n \in \mathbb{N}$. When convenient we write $\tau$ as an ordered pair $\tau = (q, t)$, where

$$q = (q_n \mid n \in \mathbb{N}), t = (t_n \mid n \in \mathbb{N}).$$

There are two ways in which a trajectory $(q, t)$ may fail to represent a “true trajectory” of the CTMC $C$ in the above intuitive sense. First, it may be the case that $p(q_n, q_{n+1}) = 0$ (i.e. $\lambda(q_n, q_{n+1}) = 0$) for some $n \in \mathbb{N}$. This presents no real difficulty, since it merely says that the event “state$_\tau(n) = q_n$ and state$_\tau(n + 1) = q_{n+1}$” has probability 0. The second way in which $(q, t)$ may fail to represent a “true trajectory” is for some $q_n$ to be a terminal state of $C$. We deal with this by defining the length of a trajectory $\tau = (q, t)$ to be

$$||\tau|| = \min\{n \in \mathbb{N} \mid q_n \text{ is terminal} \},$$

where $\min \emptyset = \infty$. We then intuitively interpret a trajectory $\tau = (q, t)$ with $||\tau|| < \infty$ as the finite sequence

$$\tau' = ((q_n, t'_n) \mid n \leq ||\tau||),$$

where each

$$t'_n = \begin{cases} t_n & \text{if } n < ||\tau|| \\ \infty & \text{if } n = ||\tau|| \end{cases}.$$  

We write

$$\Omega = \Omega[C] = (Q \times (0, \infty))^\infty$$

for the set of all trajectories of a CTMC, $C$.

Elements of $(Q \times \{0, 1\}^*)^*$ are called approximations or partial specifications of trajectories. The cylinder generated by $w = (q_0, u_0), (q_1, u_1), ..., (q_{n-1}, u_{n-1}) \in (Q \times \{0, 1\}^*)^*$ is the set $\Omega_w$ of trajectories defined as follows: If $q_i$ is terminal for some $0 \leq i < n - 1$ then $\Omega_w = \emptyset$. If $q_i$ is nonterminal for all $0 \leq i < n - 1$ and $q_{n-1}$ is terminal, then

$$\Omega_w = \{\tau \in \Omega \mid (\forall 0 \leq i < n)\text{state}_\tau(i) = q_i \}.$$
and \((\forall 0 \leq i < n - 1) soj_{\tau}(i) \in D_{\lambda_i}(u_i)\). 

If \(q_i\) is nonterminal for all \(0 \leq i < n\) then 
\[
\Omega_w = \{ \tau \in \Omega \mid (\forall 0 \leq i < n) [\text{state}_{\tau}(i) = q_i \\
\text{and } soj_{\tau}(i) \in D_{\lambda_i}(u_i)]\}.
\]

For \(w, x \in (Q \times \{0, 1\}^*)^*\) and \(\tau \in \Omega\), we write \(w \subseteq \tau\) to indicate that \(\tau \in \Omega_w\) and \(w \subseteq x\) to indicate that \(\Omega_x \subseteq \Omega_w\). It is easily verified that the latter condition is decidable.

The probability \(\mu_C(\Omega_w)\), usually written \(\mu_C(w)\), of a cylinder \(\Omega_w\), is defined as follows: If \(n = 0\) (i.e. \(w = \lambda\)), then \(\mu_C(\Omega_w) = 1\). If \(q_i\) is terminal for some \(0 \leq i < n - 1\), then \(\mu_C(\Omega_w) = 0\). If \(q_i\) is nonterminal for all \(0 \leq i < n - 1\) and \(q_{n-1}\) is terminal, then 
\[
\mu_C(\Omega_w) = \sigma(q_0) \prod_{i=0}^{n-2} [p(q_i, q_{i+1})2^{-|u_i|}] .
\]
If \(n > 0\) and \(q_i\) is nonterminal for all \(0 \leq i < n\), then 
\[
\mu_C(\Omega_w) = \sigma(q_0) \prod_{i=0}^{n-2} p(q_i, q_{i+1}) \prod_{i=0}^{n-1} 2^{-|u_i|} .
\]

A set \(X \subseteq \Omega\) has probability 0, and we write \(\mu_C(X) = 0\), if, for \(\epsilon > 0\), there is a set \(A \subseteq (Q \times \{0, 1\}^*)^*\) such that 
\[
X \subseteq \bigcup_{w \in A} \Omega_w
\]
and 
\[
\sum_{w \in A} \mu_C(\Omega_w) \leq \epsilon .
\]

From now on we assume that the states \(q \in Q\) have canonical representations, so that it is clear what it means for function \(f : Q \to Q\), etc., to be computable.

A set \(X \in \Omega\) has algorithmic probability 0 (or is an algorithmic null set), and we write \(\mu_{C,\text{alg}}(X) = 0\), if there is a computable function 
\[
g : \mathbb{N} \times \mathbb{N} \to (Q \times \{0, 1\}^*)^*
\]
such that, for every \(k \in \mathbb{N}\), 
\[
X \subseteq \bigcup_{l=0}^{\infty} \Omega_{g(k,l)}
\]
and 
\[
\sum_{l=0}^{\infty} \mu_C(\Omega_{g(k,l)}) \leq 2^{-k} .
\]

A set \(X \subseteq \Omega\) has algorithmic probability 1, and we write 
\[
\mu_{C,\text{alg}}(X) = 1 ,
\]
if \(\mu_{C,\text{alg}}(\Omega \setminus X) = 0\).
5.2 CTMC martingales

In place of $\mu_\Lambda, \mu_C,$ and $\mu_{\mathcal{E},\sigma}$ we will simply write $\mu, \mu_{\text{alg}}.$ It should be clear from context which measure is being used.

If $C = (Q, \lambda, \pi)$ is a CTMC, then a $C$-martingale is a function

$$d : (Q \times \{0, 1\}^*)^* \to [0, \infty)$$

with the following two properties.

1. For all $w \in (Q \times \{0, 1\}^*)^*$,
   $$d(w)\mu(w) = \sum_{q \in Q} d(w(q, \lambda))\mu(w(q, \lambda)).$$

2. For all $w \in (Q \times \{0, 1\}^*)^*$, $q \in Q,$ and $u \in \{0, 1\}^*$,
   $$d(w(q, u))\mu(w(q, u)) = \sum_{b \in \{0, 1\}} d(w(q, ub))\mu(w(q, ub)).$$

Intuitively, a $C$-martingale $d$ is a strategy for betting on successive approximations $w$ of a trajectory $\tau$ of $C.$ A gambler using $d$ starts with initial capital $d(\lambda) \in [0, \infty).$ More generally, each value $d(w)$ is the amount of money that the gambler will have after betting on $w.$ At this stage, the $C$-martingale $d$ tells the gambler how it may proceed in either of the following two ways.

(i) The gambler may “move on” to bet on the value of $\text{state}_\tau(|w|)$, which is the next state of $\tau.$ In this case condition (1) ensures that the payoffs for this bet are fair.

(ii) The gambler may “stay” with the current state, which is $\text{state}_\tau(|w| - 1)$, and bet further on the approximate value of $\text{soj}_\tau(|w| - 1).$ In this case condition (2) ensures that the payoffs for this bet are fair.

As in section 4, we emphasize that these orderings of bets are intuitive fancies, not part of the martingale definition.

A $C$-martingale $d$ succeeds on a trajectory $\tau$ if, for every real number $\alpha > 0,$ there exists $w \in (Q \times \{0, 1\}^*)^*$ such that $w \subseteq \tau$ and $d(w) > \alpha.$

The success set of a $C$-martingale $d$ is

$$S^\infty[d] = \{\tau \in \Omega[C] \mid d \text{ succeeds on } \tau\}.$$

An analog of Ville’s theorem holds for $C$-martingales. In order to prove this, we first prove two useful lemmas.

**Lemma 6** (Generalized Kraft Inequality). Let $C = (Q, \lambda, \sigma)$ be a CTMC, $d$ a $C$-martingale, and $B \subseteq (Q \times \{0, 1\}^*)^*$ a prefix set. Then,

$$d(\lambda)\mu(\lambda) = d(\lambda) \geq \sum_{w \in B} d(w)\mu(w).$$
Proof. If \( d(\lambda) = 0 \), this is immediate. Assume that \( d(\lambda) > 0 \). Let \( wb \) be shorthand for \((q_0, u_0)\ldots(q_{n-2}, u_{n-2})(q_{n-1}, u_{n-1})\). Note that \( \mu \) is a probability measure on \((Q \times \{0,1\}^{*})^{\infty}\), because it satisfies the following conditions.

1. \( \mu : (Q \times \{0,1\}^{*})^{*} \to [0,1] \).
2. \( \mu(\lambda) = 1 \).
3. If \(|w| = n\) and \( w = (q_0, u_0)\ldots(q_{n-2}, u_{n-2})(q_{n-1}, u_{n-1}) \) then
\[
\sum_{q \in Q} \mu(w(q, \lambda)) = \sum_{q \in Q} \sigma(q_0) \Pi_{i=0}^{n-2} (q_i, q_{i+1}) \Pi_{i=0}^{n-1} 2^{-|u_i|} p(q_{n-1}, q) = \mu(w).
\]
4. If \(|w| = n, u \in \{0,1\}^{*}\) and \( w = (q_0, u_0)\ldots(q_{n-2}, u_{n-2})(q_{n-1}, u) \) then
\[
\sum_{b \in \{0,1\}^{*}} \mu(wb) = \sum_{b \in \{0,1\}^{*}} \sigma(q_0) (\Pi_{i=0}^{n-2} (q_i, q_{i+1})) (\Pi_{i=0}^{n-1} 2^{-|u_i|}) 2^{-|ub|} = \mu(w).
\]

Define \( \pi : (Q \times \{0,1\}^{*})^{*} \to [0,1] \) by
\[
\sigma(w) = \frac{d(w)\mu(w)}{d(\lambda)}.
\]

It is straightforward to verify that this is a probability measure on \((Q \times \{0,1\}^{*})^{\infty}\).

For all \( w \in (Q \times \{0,1\}^{*})^{*} \), we then have
\[
d(w) = d(\lambda) \frac{\pi(w)}{\mu(w)}.
\]

Intuitively, \( \pi \) is the “strategy” of \( d \) and \( \mu \) is its “environment”.

Then choose \( \omega \in (Q \times \{0,1\}^{*})^{\infty} \) according to \( \pi \) and let \( E \) be the event that there exists \( w \in (Q \times \{0,1\}^{*})^{*} \) such that \( w \subseteq \omega \) for some \( w \in B \) in this experiment. Then
\[
1 \geq Pr(E)
= \sum_{w \in B} \pi(w)
= \frac{1}{d(\lambda)} \sum_{w \in B} d(w)\mu(w),
\]
so
\[
d(\lambda) \geq \sum_{w \in B} d(w)\mu(w).
\]
\[\square\]
Lemma 7. Let \( d_0, d_1, d_2, d_3, \ldots \) be a sequence of C-martingales (resp. \( \lambda \)-martingales or \( \mathcal{Q} \)-martingales) such that
\[
\sum_{n=0}^{\infty} d_n(\lambda) < \infty.
\]
Then the function \( d : (Q \times \{0,1\}^*)^* \rightarrow [0, \infty) \) defined by: for all \( w \in (Q \times \{0,1\}^*)^* \)
\[
d(w) = \sum_{n=0}^{\infty} d_n(w)
\]
is a C-martingale.

Proof. Let \( d_0, d_1, \ldots \) and \( d \) be as given.

For all \( w \in (Q \times \{0,1\}^*)^*, q \in Q, \) and \( u \in \{0,1\}^* \)
\[
\sum_{b \in \{0,1\}} d(w(q, ub))\mu(w(q, ub))
= \mu(w(q, u)) \sum_{b \in \{0,1\}} (\sum_{n=0}^{\infty} d_n(w(q, ub)))
= \mu(w(q, u)) \sum_{n=0}^{\infty} \sum_{b \in \{0,1\}} d_n(w(q, ub))
= \mu(w(q, u)) \sum_{n=0}^{\infty} d_n(w(q, u))
= \mu(w(q, u))d(w(q, u)),
\]
and
\[
\sum_{q \in Q} d(w(q, \lambda))\mu(w(q, \lambda)) = \mu(w) \sum_{q \in Q} (\sum_{n=0}^{\infty} d_n(w(q, \lambda)))
= \mu(w) \sum_{n=0}^{\infty} \sum_{q \in Q} d_n(w(q, \lambda))
= \mu(w) \sum_{n=0}^{\infty} d_n(w)
= \mu(w)d(w).
\]

Since \( d(\lambda) \) is finite and the martingale conditions hold, it follows by simple induction that for all \( w, d(w) \) is also finite. Thus, \( d \) is a C-martingale.

We now prove the analog of Ville’s theorem for CTMCs.

Theorem 8. For every CTMC \( C \) and every set \( X \subseteq \Omega[C] \), the following two conditions are equivalent.
(1) $\mu(X) = 0$.

(2) There is a $C$-martingale $d$ such that $X \subseteq S^\infty[d]$.

**Proof.** Suppose $\mu(X) = 0$. It suffices to show that there exists a $C$-martingale $d$ such that $X \subseteq S^\infty[d]$.

Assume the hypothesis. Then, for all $k \in \mathbb{N}$ there exists $C_k \subseteq (Q \times \{0, 1\}^*)^*$ such that

$$X \subseteq \bigcup_{w \in C_k} \Omega_w,$$

and

$$\sum_{w \in C_k} \mu(\Omega_w) \leq 2^{-k}.$$

Let $k \in \mathbb{N}$. Suppose there exists $C_k \subseteq (Q \times \{0, 1\}^*)^*$ satisfying the above conditions. Then there exists $g : \mathbb{N} \times \mathbb{N} \to (Q \times \{0, 1\}^*)^* \cup \{\emptyset\}$ such that

$$X \subseteq \bigcup_{n=0}^\infty \Omega_{g(k,n)},$$

and

$$\sum_{n=0}^\infty \mu(g(k,n)) \leq 2^{-k}.$$

Define $\wedge : (Q \times \{0, 1\}^*)^* \times (Q \times \{0, 1\}^*)^* \to (Q \times \{0, 1\}^*)^* \cup \emptyset$ by

$$x \wedge y = \begin{cases} x & \text{if } y \subseteq x \\ y & \text{if } x \subseteq y \\ \emptyset & \text{otherwise.} \end{cases}$$

We define a martingale which succeeds on every $\tau \in X \cap \Omega_{g(k,n)}$. Let $\tau \in X \cap \Omega_{g(k,n)}$. Define the function $d_k : (Q \times \{0, 1\}^*)^* \to [0, \infty)$ by

$$d_k(\lambda) = 2^{-k},$$

and

$$d_k(w) = \frac{\sum_{n=0}^\infty \mu(g(k,n) \wedge w)}{\mu(w)}.$$

Recall that $d_k$ is a $C$-martingale if it satisfies the conditions

(1A.) For all $w \in (Q \times \{0, 1\}^*)^*$,

$$d(w)\mu(w) = \sum_{q \in Q} d(w(q, \lambda))\mu(w(q, \lambda)), \text{ and}$$

(2A.) For all $w \in (Q \times \{0, 1\}^*)^*$, $q \in Q$, and $u \in \{0, 1\}^*$,

$$d(w(q, u))\mu(w(q, u)) = \sum_{b \in \{0, 1\}} d(w(q, ub))\mu(w(q, ub)).$$
Let $k \in \mathbb{N}, q \in Q, u \in \{0,1\}$, and $w \in (Q \times \{0,1\}^*)^*$. 

$$\sum_{q \in Q} d_k(w(q, \lambda))\mu(w(q, \lambda))$$

$$= \sum_{q \in Q} \sum_{n=0}^{\infty} \frac{\mu(g(k, n) \wedge (w(q, \lambda)))}{\mu(w(q, \lambda))} \mu(w(q, \lambda))$$

$$= \sum_{q \in Q} \sum_{n=0}^{\infty} \mu(g(k, n) \wedge w(q, \lambda))$$

$$= \sum_{n=0}^{\infty} \sum_{q \in Q} \mu(g(k, n) \wedge w(q, \lambda))$$

$$= \sum_{n=0}^{\infty} \mu(g(k, n) \wedge w)$$

$$= d_k(w)\mu(w),$$

satisfying (1A).

$$\sum_{b \in \{0,1\}} d_k(w(q, ub))\mu(w(q, ub))$$

$$= \sum_{b \in \{0,1\}} \sum_{n=0}^{\infty} \frac{\mu(g(k, n) \wedge (w(q, ub)))}{\mu(w(q, ub))} \mu(w(q, ub))$$

$$= \sum_{b \in \{0,1\}} \sum_{n=0}^{\infty} \mu(g(k, n) \wedge w(q, ub))$$

$$= \sum_{n=0}^{\infty} \sum_{b \in \{0,1\}} \mu(g(k, n) \wedge w(q, ub))$$

$$= \sum_{n=0}^{\infty} \mu(g(k, n) \wedge w(q, u))$$

$$= d_k(w(q, u))\mu(w(q, u)),$$

satisfying (2A). Hence, for all $k \in \mathbb{N}, d_k$ is a $C$-martingale.

Define the unitary success set of a martingale $d$ to be $S^1[d] = \{ \tau \in (Q \times (0,\infty))^\infty | \text{There exists } w \sqsubseteq \tau \text{ such that } d(w) \geq 1 \}$.

Let $n \in \mathbb{N}$, and $\tau \in \Omega_{g(k,n)}$. Then $g(k,n) \sqsubseteq \tau$ and

$$d_k(g(k,n)) \geq \frac{\mu(g(k,n) \wedge g(k,n))}{\mu(g(k,n))} = 1.$$
For each \( k \in \mathbb{N} \), define \( \hat{d}_k : (Q \times \{0,1\}^*)^* \to [0, \infty) \) by
\[
\hat{d}_k(\lambda) = d_k(\lambda),
\]
and
\[
\hat{d}_k(wa) = \begin{cases} 
    d_k(wa) & \text{if } \hat{d}_k(w) < 1 \\
    \hat{d}_k(w) & \text{if } \hat{d}_k(w) \geq 1.
\end{cases}
\]

\( \hat{d}_k \) is a \( C \)-martingale. Define \( \hat{d} : (Q \times \{0,1\}^*)^* \to [0, \infty) \) by
\[
\hat{d}(w) = \sum_{k=0}^{\infty} \hat{d}_k(w).
\]

Then there exists \( x \subseteq \tau \) such that \( d(x) \geq \alpha \). Thus, one direction is proven.

Now let \( C = (Q, \lambda, \pi) \) be a CTMC. Let \( X \subseteq \Omega[C] \). Suppose there exists a \( C \)-martingale, \( d \) such that \( X \subseteq S^{\infty}[d] \). Then, for all \( \tau \in X \) and \( \alpha > 0 \), there exists \( w \in (Q \times \{0,1\}^*)^* \) such that \( w \subseteq \tau \) and \( d(w) > \alpha \). It suffices to show that \( \mu(X) = 0 \).

We will show that there exists \( g : \mathbb{N} \times \mathbb{N} \to (Q \times \{0,1\}^*)^* \cup \{\emptyset\} \), such that

(1B.) \( X \subseteq \bigcup_{n=0}^{\infty} \Omega_{g(k,n)} \), and

(2B.) \( \sum_{n=0}^{\infty} \mu(\Omega_{g(k,n)}) \leq 2^{-k} \).

For each \( k \in \mathbb{N} \), define
\[
A_k = \{ w \in (Q \times \{0,1\}^*)^* | d(w) \geq 2^kd(\lambda) \}
\]
and
\[
B_k = \{ w \in A_k | \text{For all } v \subseteq w, v \notin A_k \}.
\]

\( B_k \) is thus the set of all partial specifications \( w \) “at which” \( d \) has accumulated \( 2^k \) value for the first time along the unique path of that \( w \).

For all \( k \in \mathbb{N} \), define \( B_k(i) \) to be the \( i \)-th element of \( B_k \) in standard enumeration of strings and define the function \( g : \mathbb{N} \times \mathbb{N} \to (Q \times \{0,1\}^*)^* \cup \emptyset \) by
\[ g(k, n) = \begin{cases} B_k(n) & \text{if } |B_k| \geq n \\ \emptyset & \text{otherwise.} \end{cases} \]

To see that (1B) is satisfied, let \( k \in \mathbb{N} \) and \( \tau \in X \), and let \( d_k \) be defined as in the previous section. Since \( \tau \in S^\infty[d_k] \), there exists \( w \in B_k \) such that \( w \subseteq \tau \). Then, there exists \( n \in \mathbb{N} \) such that \( g(k, n) \subseteq \tau \), whence

\[ \tau \in \bigcup_{n=0}^{\infty} \Omega_{g(k,n)}. \]

Thus,

\[ X \subseteq \bigcup_{n=0}^{\infty} \Omega_{g(k,n)} \]

To see that (2B) is satisfied, by Lemma 6 we have that

\[ d(\lambda) \geq \sum_{w \in B_k} d(w) \mu(w) \]

\[ \geq 2^k d(\lambda) \sum_{w \in B_k} \mu(w) \]

\[ = 2^k d(\lambda) \sum_{n=0}^{\infty} \mu(g(k,n)). \]

Thus,

\[ \sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}, \]

and we have that \( \mu(X) = 0 \). \( \square \)

We next derive from Theorem 8 a proof of Theorem 2. An algorithmic version of Theorem 2 has the same proof, using lower semicomputable martingales, algorithmically measure-zero sets, and Theorem 9 (the algorithmic version of Theorem 8) where needed.

**Proof. (Theorem 2).** Let \( C = (Q, \lambda, \pi) \) be a CTMC. Let \( d' \) be a \( \mathcal{A} \)-martingale which succeeds on some set \( S \subseteq A_Q[\pi] \). Note that for all \( q \in S \), \( |q| = \infty \). Define a \( C \)-martingale \( d \) for all \( q \in Q, u \in \{0, 1\}^*, \) and partial specifications \( w = (q_0, u_0), (q_1, u_1), \ldots(q_k, u_k) \) by

\[ d(w(q, u)) = d'(q_0, \ldots q_k, q). \]

It is straightforward to verify that \( d \) is a \( C \)-martingale which succeeds on the set \( S \times \mathbb{R}^\infty \). Then,

\[ \mu_C(S \times \mathbb{R}^\infty) = 0, \]

by Theorem 8. It follows that \( \mu_Q(S) = 0 \).
Now assume instead that \( \mu_Q(S) = 0 \). Then,

\[
\mu_C(S \times \mathbb{R}^\infty) = 0.
\]

Then, by Theorem 8 there is a \( C \)-martingale \( d \) which succeeds on \( S \times \mathbb{R}^\infty \). Then, by Lemma 10, there is a \( \mathcal{Z} \)-martingale \( d' \) which succeeds on \( S \).

### 5.3 Algorithmic CTMC martingales

**Theorem 9.** For every CTMC \( C \) and every set \( X \subseteq \Omega[C] \), the following two conditions are equivalent.

1. \( \mu_{\text{alg}, C}(X) = 0 \).
2. There is a lower semi-computable \( C \)-martingale \( d \) such that \( X \subseteq S^\infty[d] \).

**Proof.** Let \( C \) a CTMC.

Suppose \( \mu_{\text{alg}, C}(X) = 0 \). We wish to show that there exists a lower semicomputable \( C \)-martingale, \( d \), such that \( X \subseteq S^\infty[d] \).

By the hypothesis, there exists a computable function \( g : \mathbb{N} \times \mathbb{N} \to (Q \times \{0,1\}^*)^* \cup \emptyset \) with the property that for all \( k \in \mathbb{N} \),

\[
X \subseteq \bigcup_{n=0}^\infty C_{g(k,n)},
\]

and

\[
\sum_{n=0}^\infty \mu(g(k,n)) \leq 2^{-k}.
\]

We can also assume that for all \( k \in \mathbb{N} \), the elements of \( \{g(k,n) \mid n \in \mathbb{N} \} \) constitute a disjoint set of cylinders.

For each \( k \in \mathbb{N} \), define the function \( d_k : (Q \times \{0,1\}^*)^* \to [0,\infty) \) as in [4] (Theorem 6.4.3):

\[
d_k(w) = \#\{\sigma \in \{g(k,n) \mid n \in \mathbb{N} \} \mid \sigma \subseteq w\} + \sum_{\sigma \in \{g(k,n) \mid n \in \mathbb{N} \}, w \subseteq \sigma} 2^{\mid w \mid - \mid \sigma \mid}.
\]

Since \( \{g(k,n) \mid n \in \mathbb{N} \} \) is a set of partial specifications whose elements form a disjoint set of cylinders, if the first term of \( d_k(w) \) is 1 then the second term is zero, and the first term cannot be greater than 1. If the first term is zero, the second term ‘closes the gap’ to value 1 by way of the first \( \sigma \) contained in the cylinder at \( w \) “approaching” that \( \sigma \). It is straightforward to verify that each \( d_k \) is a \( C \)-martingale, and that it is lower semicomputable.

Let \( d = \sum_{k \in \mathbb{N}} d_k \). \( d \) is also a lower semicomputable \( C \)-martingale.

To see that \( X \subseteq S^\infty[d] \), let \( \tau \in X, \alpha \in \mathbb{Z}^+ \). It suffices to show that there exists \( w \subseteq \tau, d(w) \geq \alpha \).
Let $S^1[d_k] = \{ w \in (Q \times \{0,1\}^*)^* \mid d_k(w) \geq 1 \}$ be the unitary success set of $d_k$.

Let $w_0, \ldots, w_{\alpha-1}$ be elements of $S^1[d_0], \ldots, S^1[d_{\alpha-1}]$, respectively, such that $w_i \subseteq \tau$ for all $0 \leq i < \alpha$. Let $w$ be such that $w_i \subseteq w$ for all $0 \leq i < \alpha$. Then $d(w) = \sum_{k=0}^{\alpha-1} d_k(w) \geq \alpha$.

Now assume instead that there exists a lower semicomputable $C$-martingale $d$ such that $X \subseteq S^\infty[d]$. Then, for all $\tau \in X, \alpha > 0$, there exists $w \in (Q \times \{0,1\}^*)^*$ such that $w \subseteq \tau$ and $d(w) > \alpha$. We wish to show that $\mu_{\text{alg},C}(X) = 0$.

We will show that there exists a uniformly computable function $g : N \times N \rightarrow (Q \times \{0,1\}^*)^* \cup \{\emptyset\}$, satisfying

$$X \subseteq \bigcup_{n=0}^{\infty} C_{g(k,n)},$$

and

$$\sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}.$$

For each $k \in N$, define

$$A_k = \{ w \in (Q \times \{0,1\}^*)^* \mid d(w) \geq 2^kd(\lambda) \}$$

and

$$B_k = \{ w \in A_k \mid \forall v \subseteq w, v \notin A_k \}.$$

$B_k$ is thus a set of all partial specifications “by which” $d$ has accumulated $2^k$ value for the first time along some “path” to $w \in B_k$.

For all $k \in N$, define $B_k(i)$ to be the $i$th element of $B_k$ in standard enumeration of strings and define the function $g : N \times N \rightarrow (Q \times \{0,1\}^*)^* \cup \emptyset$ by

$$g(k, n) = \begin{cases} B_k(n) & \text{if } |B_k| \geq n \\ \emptyset & \text{otherwise.} \end{cases} \quad (5.1)$$

Note that since $A_k$ is enumerable, $g(k, n)$ is computable. Let $k \in N, \tau \in X$, and let $d_k$ be defined as before. Since $\tau \in S^\infty[d_k]$, there exists $w \in B_k$ such that $w \subseteq \tau$. Then, there exists $n \in N$ such that $g(k, n) \subseteq \tau$, whence

$$\tau \in \bigcup_{n=0}^{\infty} C_{g(k,n)}$$

and we have that

$$X \subseteq \bigcup_{n=0}^{\infty} C_{g(k,n)}.$$
By Lemma 6,
\[
d(\lambda) \geq \sum_{w \in B_k} d(w)\mu(w)
\geq 2^k d(\lambda) \sum_{w \in B_k} \mu(w)
= 2^k d(\lambda) \sum_{n=0}^{\infty} \mu(g(k,n))
\]
and
\[
\sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}.
\]
Thus, \(\mu_{\text{alg},C}(X) = 0\).

We use the shorthand \(wb\), where \(w = (q_0, u_0), ..., (q_k, u_k)\), and \(b \in \{0, 1\}\), to denote \((q_0, u_0) ... (q_k, u_k, b)\) and \(wq\), where \(q \in Q\), to denote \((q_0, u_0) ... (q_k, u_k, (q, \lambda))\).

**Lemma 10.** Let \(C = (Q, \lambda, \pi)\) be a CTMC. Then, if \(S \subseteq Q^{\leq \infty}\) and \(d\) is a \(C\)-martingale that succeeds on
\[
T_S = \{(q, r) \mid q \in S, r \in \mathbb{R}^{\leq \infty}, |q| = |r| \} \subset S \times \mathbb{R}^{\leq \infty},
\]
then there is a \(2\)-martingale \(d'\) that succeeds on \(S\). Moreover, if \(d\) is lower semicomputable, then so is \(d'\).

**Proof.** Assume the hypothesis. Define \(d'\) by
\[
d'(q_0, ..., q_k) = d((q_0, \lambda), (q_1, \lambda), ..., (q_k, \lambda)),
\]
for all \(k \in \mathbb{N}\) and \(q_0, ..., q_k \in Q^k\).

To see that \(d'\) succeeds on every \(q \in S\), for each \(i \in \mathbb{N}\) and \(q_0, ..., q_i \in Q^{i+1}\), define \(t_i \in \{0, 1\}^\infty\) by
\[
t_i[j] = 1 \text{ if } d((q_0, \lambda)(q_1, \lambda)...(q_i, t_i[0...j-1]1)) \leq d((q_0, \lambda)(q_1, \lambda)...(q_i, t_i[0...j-1]0))
\]
and \(t_i[j] = 0\) otherwise. In other words, \(t_i\) is an ‘artificially’ constructed ‘minimal-payoff’ time for \(q_i\). Lastly, note that for all \(u_i \subseteq t_i\) and \(u_0...u_{i-1} \in \{0, 1\}^*\),
\[
d((q_0, u_0)(q_1, u_1)...(q_i, u_i)) \leq d((q_0, u_0)(q_1, u_1)...(q_i, \lambda))
\]
Since \(d\) succeeds on \(T_S\), \(d'\) succeeds on \(S\). If \(d\) is lower semicomputable, so is \(d'\).
5.4 Random CTMC trajectories

As in the Cantor space setting, we define a trajectory $\tau$ of a CTMC to be algorithmically random (a.k.a. Martin-Löf random) if $\tau$ does not have algorithmic measure 0.

**Lemma 11.** Let $C$ be a CTMC and $\tau = (q_0, t_0)(q_1, t_1)\ldots \in \Omega[C]$ be random. Then, the subsequence consisting of all states in $\tau$, $q = q_0, q_1, q_2, \ldots \in Q^\infty$ is random with respect to $(Q, \sigma)$.

**Proof.** Let $\tau, q$ be as described. Suppose there exists a lower semicomputable $(Q, \sigma)$-martingale $d : Q^* \to [0, \infty)$ which succeeds on $q$ (that is, $q$ is not random).

Define the $C$-martingale $\hat{d} : (Q \times \{0, 1\}^*)^* \to [0, \infty)$ as follows:

If $q \in Q$

$$\hat{d}(wq) = d(q_0, \ldots, q_{n-1}, q).$$

If $b \in \{0, 1\}$

$$\hat{d}(wb) = \hat{d}(w).$$

That is, $\hat{d}$ only bets on states (and bets on them according to $d$’s strategy), while hedging its bets on times. To see that $\hat{d}$ is in fact a $C$-martingale:

For all $w \in (Q \times \{0, 1\}^*)^*, q \in Q, u \in \{0, 1\}^*$

$$\sum_{b \in \{0, 1\}} \hat{d}(w(q, ub))\mu(w(q, ub)) = \hat{d}(w(q, u)) \sum_{b \in \{0, 1\}} \mu(w(q, ub))$$

$$= \hat{d}(w(q, u))\mu(w(q, u))$$

and for all $w \in (Q \times \{0, 1\}^*)^*, |w| = n$

$$\sum_{q \in Q} \hat{d}(w(q, \lambda))\mu(w(q, \lambda)) = \sum_{q \in Q} d(q_0, \ldots, q_{n-1}, q)\mu(w(q, \lambda))$$

$$= d(q_0, \ldots, q_{n-1}) \sum_{q \in Q} \mu(w(q, \lambda))$$

$$= \hat{d}(w) \sum_{q \in Q} \mu(w(q, \lambda))$$

$$= \hat{d}(w)\mu(w).$$

To see that $\hat{d}$ succeeds on $\tau$, let $\alpha > 0$. Since $d$ succeeds on $q$, there exists $n \in \mathbb{N}$ and $w_n \subseteq q$ such that $d(w_n) > \alpha$. Then, since $\hat{d}$ does not bet on sojourn times and bets on states according to $d$,

$$\hat{d}(q_0, u_0)(q_1, u_1)\ldots(q_{n-1}, u_{n-1}) > \alpha$$

To see that $\hat{d}$ is lower semicomputable, let $d' : Q^* \times \mathbb{N} \to Q$ be a function testifying to the fact that $d$ is lower semicomputable. Define $\hat{d}'$ as $\hat{d}$ is defined above, replacing instances of $d$ with instances of $d'$.
Lemma 12. Let $\tau = (q_0, t_0)(q_1, t_1)\ldots \in \Omega[C]$ where $C$ is some CTMC. Suppose there exists $m \in \mathbb{N}$ such that $t_m$ is not random. Then, $\tau$ is not random.

Proof. Assume the hypothesis and let $d$ be a lower semi-computable martingale which succeeds on some $t_m$. For every $n \in \mathbb{N}$ define a C-martingale $\hat{d}_n$ which doesn’t bet on states and bets according to $d$ on only the $n$th sojourn time $t_n$. That is,

$$\hat{d}_n(\lambda) = 2^{-n},$$
$$\hat{d}_n(w(q, \lambda)) = \hat{d}_n(w),$$

if $|w| = n, w = (q_0, u_0)(q_1, u_1)\ldots(q_{n-1}, u), u \in \{0, 1\}^*$, and $b \in \{0, 1\}$ then

$$\hat{d}_n(w[0\ldots n - 2](q_{n-1}, ub)) = d(ub),$$

and if $|w| = k \neq n, w = (q_0, u_0)(q_1, u_1)\ldots(q_{k-1}, u), u \in \{0, 1\}^*$, and $b \in \{0, 1\}$ then

$$\hat{d}_n(w[0\ldots k - 2](q_{k-1}, ub)) = \hat{d}_n(w[0\ldots k - 2](q_{k-1}, u)).$$

Let $n \in \mathbb{N}$. We must prove $\hat{d}_n$ is indeed a martingale. If $q \in Q$ then

$$\sum_{q \in Q} \hat{d}_n(w(q, \lambda))\mu(w(q, \lambda)) = \sum_{q \in Q} \hat{d}_n(w)\mu(w(q, \lambda))$$
$$= \hat{d}_n(w) \sum_{q \in Q} \mu(w(q, \lambda))$$
$$= \hat{d}_n(w)\mu(w).$$

If $|w| = k \neq n$ then

$$\sum_{b \in \{0, 1\}} d(ub)\mu(w[0\ldots n - 2](q, ub))$$
$$= \sum_{b \in \{0, 1\}} d(ub)\mu(w[0\ldots n - 2](q, ub))$$
$$= d(u)\mu(w[0\ldots n - 2](q, u))$$
$$= \hat{d}_n(w[0\ldots n - 2](q, u))\mu(w[0\ldots n - 2](q, u)).$$

If $|w| = n$ then

$$\sum_{b \in \{0, 1\}} d(ub)\mu(w[0\ldots n - 2](q, ub))$$
$$= \sum_{b \in \{0, 1\}} d(ub)\mu(w[0\ldots n - 2](q, ub))$$
$$= d(u)\mu(w[0\ldots n - 2](q, u))$$
$$= \hat{d}_n(w[0\ldots n - 2](q, u))\mu(w[0\ldots n - 2](q, u)).$$

27
Define $\hat{d}$ to be a $C$-martingale obtained by applying Lemma 7:

$$\hat{d} = \sum_{n=0}^{\infty} \hat{d}_n.$$ 

$\hat{d}$ succeeds on $\tau$. Since $d$ is lower semicomputable, let $d'$ testify to this. Substituting $d'$ in the above construction shows that $\hat{d}_n$ is lower semicomputable for all $n$, and thus that $\hat{d}$ is also lower semicomputable. Thus, $\tau$ is not random.

Theorem 13. Let $\tau \in \Omega[C]$ be a trajectory in a CTMC, C. If $\tau$ is random, then all sojourn times $t_1, t_2, \ldots$ in $\tau$ are independently random.

Proof. Let $\tau \in \Omega[C]$ and suppose there exists $n \in \mathbb{N}$ such that $t_1, \ldots, t_n$ are not independently random. Then, there exists a lower semicomputable martingale $d : \{0, 1\}^*_n \rightarrow [0, \infty)$ (where $\{0, 1\}^*_n$ denotes the set of all $n$-tuples of strings of the same length) such that for all $w \in \{0, 1\}^*_n$,

$$d(w)\mu(w) = \sum_{a \in \{0, 1\}^*_n} d(wa)\mu(wa),$$

and

$$\limsup_{k \to \infty} d((t_1, \ldots, t_n) \mid k) = \infty,$$

where $\mu$ refers to the probability measure on $\{0, 1\}^*_n$ defined by

$$\mu((w_1, \ldots, w_n)) = \Pi_{i=1}^{n} \mu_i(w_i).$$

Define the martingale $\hat{d} : \{0, 1\}^* \rightarrow [0, \infty)$ by

$$\hat{d}(w) = d(w, t_2 \mid |w|, \ldots, t_n \mid |w|).$$

It is straightforward to verify that $\hat{d}$ is a lower semicomputable martingale which succeeds on $t_1$, from which it follows by Lemma 12 that $\tau$ cannot be random.

Example. There exists a rate sequence $\lambda$ and a sequence $R = (t_0, t_1, \ldots)$ of $\lambda$-durations such that $t_0, t_1, \ldots$ are independently random but $R$ is not random with respect to $\mu_\lambda$.

Proof. Let $\lambda$ be a rate sequence and let $S_0, S_1, \ldots$ be a sequence of elements of $\{0, 1\}^\infty$ representing times $t_0, t_1, \ldots$ each of which are random with respect to the rates $\lambda_0, \lambda_1, \ldots$. Then, the times (as binary sequences) in the $\lambda$-duration sequence $(0S_0, 0S_1, 0S_2, \ldots)$ are not independently random since a lower-semicomputable $\lambda$-martingale exists which can bet only on the first bit of each sequence and hedge on all other bits.
5.5 Kolmogorov complexity and CTMC randomness

Random trajectories can also be characterized using Kolmogorov complexity. First, we briefly review this notion in the classical setting. We fix a universal self-delimiting Turing machine (see [13]), \( U \). The Kolmogorov complexity, \( K \), of a (finite) string \( x \) in \( \{0, 1\}^* \) is the length of a shortest program for a self-delimiting Turing machine which prints \( x \). That is, \( K : \{0, 1\}^* \to \mathbb{N} \) is defined by
\[
K(x) = \min\{|\pi| \mid U(\pi) = x \ \text{and} \ \pi \in \{0, 1\}^*\}.
\]

When \( x \) is not a binary string, but some other finite object, \( K(x) \) is defined from the above by routine coding.

The profile of a cylinder \( \Omega_w \) of a CTMC is
\[
\text{prof}(w) = (|u_1|, ..., |u_n|),
\]
where \( w = ((q_1, u_1), ..., (q_n, u_n)) \).

**Observation 14.** For each CTMC \( C \) and each profile \( p \),
\[
\sum_{\{w : \text{prof}(w) = p\}} \mu_C(w) = 1.
\]

The following two lemmas are analogous to standard results used in the Kolmogorov complexity characterization of algorithmically random sequences.

**Lemma 15.** For every cylinder, \( \Omega_w \) of a CTMC \( C \),
\[
K(w) \leq l(w) + K(\text{prof}(w)) + O(1),
\]
where \( l(w) = \log \frac{1}{\mu_C(w)} \) is the “self-information” of \( w \).

**Proof.** In the following proof, we let \( p \) range over all profiles, and assume there is some natural encoding (enumerating process) between natural numbers and profiles, and also between natural numbers and cylinders.
\[
\Omega = \sum_p 2^{-K(p)},
\]
\[
= \sum_p \left( 2^{-K(p)} \sum_{\text{prof}(w) = p} 2^{-l(w)} \right), \text{ note that the second summation is 1.}
\]
\[
= \sum_w 2^{-(K(\text{prof}(w)) + l(w))} < \infty.
\]

Then, by the minimality of \( K \) and the coding relation between cylinders and natural numbers, we have
\[
K(w) \leq l(w) + K(\text{prof}(w)) + O(1).
\]

\( \square \)
Lemma 16. There is a constant $c \in \mathbb{N}$ such that, for every profile $p$ of a CTMC $C$ and every $k \in \mathbb{N},$

$$\mu_C\left( \bigcup_{w \text{ prof}(w)=p \atop K(w)<l(w)+K(p)-k} \Omega_w \right) < 2^{c-k}.$$ 

Substituting $k + K(\text{prof}(w))$ for $k$ here gives

$$\mu_C\left( \bigcup_{w \text{ prof}(w)=p \atop K(w)<l(w)-k} \Omega_w \right) < 2^{c-k-K(p)}.$$

Proof. We only need to note that

$$\sum_{p} \sum_{\text{prof}(w)=p} 2^{-K(w)} = \sum_{w} 2^{-K(w)} < \infty.$$

Then by the minimality [2] of $K,$ we have,

$$2^{-K(p)+c} \geq \sum_{\text{prof}(w)=p} 2^{-K(w)}$$

$$= \sum_{\text{prof}(w)=p} \mu(w) \frac{1}{\mu(w)} 2^{-K(w)}$$

$$= \sum_{\text{prof}(w)=p} \mu(w) 2^{\log \frac{1}{\mu(w)}} 2^{-K(w)}$$

$$= \sum_{\text{prof}(w)=p} \mu(w) 2^{l(w)-K(w)}$$

$$= \mathbb{E}_\mu[2^{l(w)-K(w)}]$$

Therefore,

$$\mu \left\{ w \mid K(w) < l(w) + K(\text{prof}(w)) - k \right\}$$

$$= \mu \left\{ w \mid l(w) - K(w) > k - K(\text{prof}(w)) \right\}$$

$$= \mu \left\{ w \mid 2^{l(w)-K(w)} > 2^{k-K(\text{prof}(w))} \right\}$$

$$< \frac{\mathbb{E}_\mu[2^{l(w)-K(w)}]}{2^{k-K(\text{prof}(w))}} \leq \frac{2^{-K(p)+c}}{2^{k-K(\text{prof}(w))}} = 2^{c-k}.$$ 

The first inequality in the last row follows by the Markov inequality.

With these lemmas, we can establish the Kolmogorov complexity characterization of randomness for trajectory objects, which is exactly analogous to a well-known characterization of the algorithmic randomness of sequences over finite alphabets [35, 22].
Theorem 17. A trajectory $\tau$ is Martin-Löf random if and only if there exists $k \in \mathbb{N}$, such that for every $w \sqsubseteq \tau$, $K(w) \geq l(w) - k$.

Proof. “Only if”: We prove the contrapositive. Suppose that for every $k$, there is at least one $w \sqsubseteq \tau$, such that $K(w) < l(w) - k$. We let

$$U_k = \{w \mid K(w) < l(w) - k\}.$$ 

Note that $w$ ranges over all cylinders in the above definition. Therefore, it is clear that $\tau$ is covered by the $U_k$.

Next, we are going to estimate the measure of $U_k$. First we consider the $p$-slice of $U_k$, $U^p_k$, defined as:

$$U^p_k = \{w \mid \text{prof}(w) = p \text{ and } w \in U_k\}.$$ 

Note that by Lemma 16, we have $\mu[U^p_k] < 2^{c-k-K(p)}$, therefore

$$\mu[U_k] = \sum_p \mu[U^p_k] \leq \sum_t 2^{c-k-K(p)} \leq 2^{c-k} \Omega \leq 2^{c-k}.$$ 

Also note that each $U_k$ is recursively enumerable, and $V_k = U_{c+k}$ is a Martin-Löf test.

“If”: Again by contrapositive: Assume $\tau$ is not Martin-Löf random, and let $\{U_k\}$ be a Martin-Löf test. We construct the following (output, size-of-program) requirement pairs as follows:

$$\{(w, l(w) - k) \mid w \in U_k, k \geq 2\}$$

It can be checked this requirement satisfies Kraft’s inequality, since the measure of the size-of-program is bounded from above by

$$\sum_{k \geq 2} 2^{-(k^2-k)} = 1/2^2 + 1/2^6 + 1/2^{12} \cdots < 1$$

Then by Levin’s coding lemma [11, 12], this requirement can be fulfilled.

Note that $\tau$ can be covered by $U_{k^2}$, and therefore for each $k \geq 2$ there are prefixes $w$ of $\tau$ for which $K(w) \leq l(w) - k < l(w) - (k - 1)$.

That is, for every $k' = k - 1 > 0$, there is some $w \sqsubseteq \tau$, such that $K(w) < l(w) - k'$. Hence $\tau$ is not random in the Kolmogorov sense.

\[ \square \]

5.6 CTMC randomness and Zeno phenomena

Like the rate-free chemical reaction networks described in section 2, a chemical reaction network (CRN) is an ordered pair, $N = (S, R)$ where $S$ is a finite set of species, $Q$ is a countable set of states $q \in \mathbb{N}^S$ representing integer quantities of each species, and $R = \{\rho_1, \rho_2, \ldots, \rho_r\}$ is a finite set of reactions. However, the reactions in a CRN are triples $\rho = (r, p, k)$, where $r$ and $p$ are as before,
and the nonnegative real number $k$ is a rate constant governing, along with the reactants $r$ and state $q$, the rate $\lambda_q$ of reaction $\rho$ at state $q$. As before, for $\rho$ to be applicable in state $q$, it must be that $r \leq q$, otherwise the reaction cannot occur at $q$.

**Theorem 18** (Non-Zeno property). Let $C$ be a CRN. Then, if $\tau = (q_0, t_0), (q_1, t_1), ... \in \Omega[C]$ is random and has bounded molecular counts, then $\tau$ satisfies the non-Zeno property that

$$\sum_{i=0}^{\infty} t_i = \infty.$$ 

**Proof.** Let $C$ be a CRN and $\tau \in \Omega[C]$ a trajectory with bounded molecular counts. Since $\tau$ has bounded molecular counts, there exists a constant $M \in \mathbb{R}$ which is the maximum reaction rate along $\tau$. Since $\tau$ has the Zeno property, there must exist $i \in \mathbb{N}$ such that for all $k \geq i, t_k \in D_{\lambda_q}(0)$. Define a $C$-martingale which only bets on the first bit of each sojourn time $t_i, t_{i+1}, ...$ as follows:

$$d_i(\lambda) = 1$$
$$d_i(w(q, \lambda)) = d_i(w)$$
$$d_i(w(q, w)) = \begin{cases} (2d_i(w(q, u)) & \text{if } |w| = i, u = \lambda, \text{ and } b = 0 \\ d_i(w(q, w)) & \text{if } |w| < i \\ 0 & \text{if } |w| = i, \text{ and } b \neq 0 \end{cases}$$

Intuitively, $d_i$ does not begin to bet until it reaches the $i$-th sojourn time, and $d_i$ only bets on the first bit of each sojourn time after the $i$th. Thus, $d_i$ succeeds on $\tau$. $d_i$ is clearly lower semicomputable. Thus, $\tau$ cannot be random. 

**Acknowledgments.** This research was supported in part by National Science Foundation grants 1247051, 1545028, and 1900716. A preliminary version of a portion of this work was presented at the 2019 Allerton Conference on Communication, Control, and Computing. We thank Elvira Mayordomo and Neil Lutz for useful discussions. We also thank anonymous reviewers for the 2019 Allerton Conference for helpful comments.

**References**

[1] Kelty Allen, Laurent Bienvenu, and Theodore Slaman. On zeros of Martin-Löf random Brownian motion. *Journal of Logic and Analysis*, 6, 2015.

[2] Gregory J. Chaitin. Incompleteness theorems for random reals. *Advances in Applied Mathematics*, 8(2):119–146, 1987.

[3] Wojciech Czerwiński, Sławomir Lasota, Ranko Lazić, Jérôme Leroux, and Filip Mazowiecki. The reachability problem for petri nets is not elementary. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 24–33, 2019.
[4] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer Science & Business Media, 2010.

[5] Willem L. Fouché. Fractals generated by algorithmically random Brownian motion. In *Conference on Computability in Europe*, pages 208–217. Springer, 2009.

[6] Willem L. Fouché. Kolmogorov complexity and the geometry of Brownian motion. *Mathematical Structures in Computer Science*, 25(7):1590–1606, 2015.

[7] Mrinalkanti Ghosh and Satyadev Nandakumar. Predictive complexity and generalized entropy rate of stationary ergodic processes. In *International Conference on Algorithmic Learning Theory*, pages 365–379. Springer, 2012.

[8] Bjørn Kjos-Hanssen and Anil Nerode. Effective dimension of points visited by Brownian motion. *Theoretical Computer Science*, 410(4-5):347–354, 2009.

[9] Ker-I Ko. *Complexity Theory of Real Functions*. Birkhäuser, 1991.

[10] Jérôme Leroux and Sylvain Schmitz. Demystifying reachability in vector addition systems. In *LICS 2015*, pages 56–67. IEEE, 2015.

[11] Leonid A. Levin. On the notion of a random sequence. *Soviet Mathematics Doklady*, 14(5), 1973.

[12] Leonid A. Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. *Problemy Peredachi Informatsii*, 10(3):30–35, 1974.

[13] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications (Fourth Edition)*. Springer, 2019.

[14] Jack H Lutz. The point-to-set principle, the continuum hypothesis, and the dimensions of Hamel bases. *Technical report 2109.10981, arxiv.org*, 2021.

[15] Jack H Lutz and Neil Lutz. Who asked us? How the theory of computing answers questions about analysis. In *Complexity and Approximation*, pages 48–56. Springer, 2020.

[16] Neil Lutz. Fractal intersections and products via algorithmic dimension. *ACM Transactions on Computation Theory (TOCT)*, 13(3):1–15, 2021.

[17] Neil Lutz and Donald M Stull. Projection theorems using effective dimension. In *43rd International Symposium on Mathematical Foundations of Computer Science (MFCS 2018)*, 2018.

[18] Neil Lutz and Donald M Stull. Bounding the dimension of points on a line. *Information and Computation*, 275:104601, 2020.
[19] Satyadev Nandakumar. An effective ergodic theorem and some applications. In Proceedings of the fortieth annual ACM symposium on Theory of computing, pages 39–44. ACM, 2008.

[20] André Nies. Computability and Randomness, volume 51. OUP Oxford, 2009.

[21] Claus-Peter Schnorr. A unified approach to the definition of random sequences. Mathematical Systems Theory, 5(3):246–258, 1971.

[22] Claus-Peter Schnorr. A survey of the theory of random sequences. In Basic Problems in Methodology and Linguistics, pages 193–211. Springer, 1977.

[23] Claus-Peter Schnorr and Hermann Stimm. Endliche Automaten und Zufallsfolgen. Acta Informatica, 1(4):345–359, 1972.

[24] Glenn Shafer and Vladimir Vovk. A tutorial on conformal prediction. Journal of Machine Learning Research, 9(Mar):371–421, 2008.

[25] Alexander Shen, Vladimir A. Uspensky, and Nikolay Vereshchagin. Kolmogorov Complexity and Algorithmic Randomness, volume 220. American Mathematical Society, 2017.

[26] Theodore Slaman. Kolmogorov complexity and capacitability of dimension. Report No. 21/2021. Mathematisches Forschungsinstitut Oberwolfach, 2021.

[27] David Soloveichik, Matthew Cook, Erik Winfree, and Jehoshua Bruck. Computation with finite stochastic chemical reaction networks. Natural Computing, 7(4):615–633, 2008.

[28] Jean Ville. Etude critique de la notion de collectif. Gauthier-Villars Paris, 1939.

[29] Vladimir Vovk, Alex Gammerman, and Glenn Shafer. Algorithmic Learning in a Random World. Springer Science & Business Media, 2005.

[30] Volodya Vovk, Alexander Gammerman, and Craig Saunders. Machine-learning applications of algorithmic randomness. In Proceedings of the Sixteenth International Conference on Machine Learning, pages 444–453. Morgan Kaufmann Publishers Inc., 1999.

[31] Vladimir V. V’yugin. Ergodic theorems for individual random sequences. Theoretical Computer Science, 207(2):343–361, 1998.

[32] Yongge Wang. Randomness and Complexity. PhD thesis, University of Heidelberg, 1996.

[33] Klaus Weihrauch. Computable Analysis: An Introduction. Springer, 2000.
[34] Benjamin Weiss. *Single Orbit Dynamics*. Number 95 in Regional Conference Series in Mathematics. American Mathematical Society, 2000.

[35] Alexander K. Zvonkin and Leonid A. Levin. The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. *Russian Mathematical Surveys*, 25(6):83, 1970.