WEYL-IN Variant QUANTIZATION OF
LIOUVILLE FIELD THEORY

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Abstract

Liouville field theory is quantized by means of a Wilsonian effective action and its associated exact renormalization group equation. For $c < 1$, an approximate solution of this equation is obtained by truncating the space of all action functionals. The Ward identities resulting from the Weyl invariance of the theory are used in order to select a specific universality class for the renormalization group trajectory. It is found to connect two conformal field theories with central charges $25 - c$ and $26 - c$, respectively.

1 Introduction

Liouville field theory $[^1, 2, 3]$ is not only an interesting topic in its own right it is also at the heart of 2-dimensional quantum gravity and of noncritical string theory $[^4, 5]$. In the following we shall discuss the quantization of Liouville theory within the framework of the exact renormalization group approach $[^6]$. More precisely, we are going to use a formulation in terms of the effective average action $[^7] - [^11]$ which is a modification of the standard effective action $\Gamma$ with a built-in infrared cutoff $k$. It is the effective action appropriate for fields which have been averaged over spacetime volumes of size $k^{-1}$. Stated differently, $\Gamma_k$ obtains from the classical action $S$ by integrating out only the field modes with momenta larger than $k$, but excluding the modes with momenta smaller than $k$. In the limit $k \to 0$, $\Gamma_k$ approaches the standard effective action, $\Gamma_{k \to 0} = \Gamma$. Here $\Gamma$ is the generating functional for the 1PI Green’s functions and includes all quantum fluctuations. The dependence of $\Gamma_k$ on the scale $k$ is described by an exact renormalization group (RG) equation $[^8, 9]$. The quantization of a theory in terms of the effective average action proceeds in two steps: (i) specify the “short-distance” or “microscopic” action $\Gamma_\Lambda$ as the initial value of the RG evolution at some large scale $k = \Lambda$, and (ii) solve the RG equation for the trajectory $\Gamma_k, k \in [0, \Lambda]$. (For a general discussion of the effective average action we refer to a recent review $[^12]$ and to a more detailed account $[^3]$ of the material presented here $[^1]$.)

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To start with, let us briefly recall some basic facts about Liouville theory which we shall need later on. We start from an arbitrary conformal field theory of central charge $c$ coupled to gravity. Integrating out the matter field fluctuations leads to the induced gravity action \[ \Gamma_{\text{ind}}[g] = \frac{c}{96\pi} I[g] + \lambda \int d^2x \sqrt{g} \right. \]

In a second step one has to integrate over the metric $g_{\mu\nu}$. We shall do this in the conformal gauge by picking a reference metric $\hat{g}_{\mu\nu}$ and writing $g_{\mu\nu}(x) = e^{2\phi(x)} \hat{g}_{\mu\nu}(x)$. Inserting this into (1) and performing the integration over the ghost fields one is led to

$$S[\phi; \hat{g}] = -\frac{\kappa^2}{32\pi} I[\hat{g}] + S_L[\phi; \hat{g}]$$

with the Liouville action

$$S_L[\phi; \hat{g}] = \frac{\kappa^2}{8\pi} \int d^2x \sqrt{\hat{g}} \left\{ \hat{D}_\mu \phi \hat{D}^\mu \phi + \hat{R} \phi + \frac{m^2}{2} e^{2\phi} \right\}$$

and $3\kappa^2 \equiv 26 - c$. Here $\hat{D}_\mu$ and $\hat{R}$ are constructed from $\hat{g}_{\mu\nu}$. The action $S[\phi; \hat{g}]$ equals $\Gamma_{\text{ind}}[e^{2\phi} \hat{g}]$ with $c$ replaced by $c - 26$. This substitution takes care of the Faddeev-Popov determinant related to the conformal gauge fixing. We shall assume that $m^2 \equiv 16\pi\lambda/\kappa^2 > 0$ and $\kappa^2 > 0$.

In the following we consider the classical action of a field $\phi$ in a fixed background geometry $\hat{g}_{\mu\nu}$, and we try to quantize $\phi$ using the exact evolution equation. Because the decomposition of the metric is invariant under the Weyl transformation $\phi'(x) = \phi(x) - \sigma(x)$, $\hat{g}_{\mu\nu}(x) = e^{2\sigma(x)} \hat{g}_{\mu\nu}(x)$, the quantization should respect this “background split symmetry”. This means that the functional integral over $\phi$ has to be performed with the Weyl invariant measure based upon the distance function $ds_{\text{Weyl}}^2 = \int d^2x \sqrt{\hat{g}} e^{2\sigma(x)} \delta\phi(x)^2$. This measure is different from the familiar translation-invariant measure which is associated to $ds_{\text{trans}}^2 = \int d^2x \sqrt{\hat{g}} \delta\phi(x)^2$.

### 2 The Weyl – Ward Identities

The derivation of the exact RG equation starts from the scale-dependent generating functional

$$e^{W_k[g;\phi]} = \int D_\phi \chi \exp \left[-S_L[\chi; g] + \int d^2x \sqrt{\hat{g}} J(x)\chi(x) \right. \]

$$

$$- \frac{1}{2} \int d^2x \sqrt{\hat{g}}(x) R_k(-D^2)\chi(x) \right]$$

(4)
where $\chi(x^\mu)$ is the “microscopic” Liouville field. Here $\mathcal{D}_g\chi$ stands for either the Weyl or the translation invariant measure. (We omit all hats from now on.) The last term in the square bracket of (4) is a diffeomorphism-invariant infrared cutoff. The function $R_k(p^2)$ vanishes if the eigenvalues $p^2$ of $-D^2 = -D_\mu D^\mu$ are much larger than the cutoff $k$, and it becomes a constant proportional to $k^2$ for $p^2 \ll k^2$. The precise shape of the function $R_k(p^2)$ is not important, except that it has to interpolate monotonically between $R_k(\infty) = 0$ and $R_k(0) = Z_k k^2$ for some constant $Z_k$. Expanding $\chi(x)$ in terms of eigenfunctions of $D^2$, we see that in $W_k$ of (4) the high-frequency modes with covariant momenta $p^2 \gg k^2$ are integrated out without any suppression whereas the low-frequency modes with $p^2 \ll k^2$ are suppressed by a smooth, mass-type cutoff term $\sim k^2 \phi^2$. It will be convenient to write the cutoff as $R_k(-D^2) = Z_k k^2 C(-D^2/k^2)$ where $C$ is a dimensionless function with $C(0) = 1$ and $C(\infty) = 0$. The effective average action $\Gamma_k[\phi; g]$ is defined as the Legendre transform of $W_k[\phi; g]$ with the infrared cutoff subtracted [7, 8]:

$$\Gamma_k[\phi; g] = \int d^2x \sqrt{g} \phi(x) J(x) - W_k[J; g] - \frac{1}{2} \int d^2x \sqrt{g} \phi(x) R_k(-D^2) \phi(x)$$

(5)

Here $J = J[\phi]$ has to be obtained by inverting the relation

$$\phi(x) \equiv \langle \chi(x) \rangle = [g(x)]^{-1/2} \frac{\delta W_k[J; g]}{\delta J(x)}$$

(6)

Obviously $\Gamma_{k=0} = \Gamma$ is the usual effective action because for $k = 0$ the function $R_k(p^2)$ vanishes for all $p^2$. The limit $\lim_{k \to \infty} \Gamma_k[\phi; g]$ is more subtle. If $R_k$ is consistent with all symmetries of the theory one can show that $\Gamma_{k \to \infty} = S$ holds true modulo a change of the bare parameters contained in $S$, which is of no importance usually. In Liouville theory the situation is more involved because the cutoff is not Weyl invariant. In order to arrive at a quantum theory which is in the right universality class, namely that of conformal field theories with central charge $26 - c$, some parameters in $\Gamma_{k \to \infty}$ have to be fine-tuned to a particular value. The starting point of the evolution, $\Gamma_\Lambda$, is the only place where the difference between the Weyl and the translation invariant measure enters. In fact, for both the Weyl and the translation invariant measure eqs. (3) and (3) give rise to the same functional RG equation (“exact evolution equation”):

$$\frac{\partial}{\partial t} \Gamma_k[\phi; g] = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)}[\phi; g] + R_k(-D^2) \right)^{-1} \frac{\partial}{\partial t} R_k(-D^2) \right]$$

Here $t \equiv \ln k$ is the renormalization group “time” and $\Gamma_k^{(2)}$ is essentially the Hessian of $\Gamma_k$ with respect to $\phi$.

Our tool for the determination of $\Gamma_\Lambda$ are the Ward identities related to the Weyl transformations. To derive them, we apply the transformation
\( J' = e^{-2\sigma} J \), \( g'_{\mu\nu} = e^{+2\sigma} g_{\mu\nu} \), \( \chi' = \chi - \sigma \) to the functional integral (4) and use that

\[
S_L[\chi'; g'] = S_L[\chi; g] - \frac{\kappa^2}{8\pi} \int d^2 x \sqrt{g} \{ D_\mu \sigma D^\mu \sigma + R \sigma \}
\]

(8)

The measure responds according to

\[
D g' \chi' = D g \chi \exp \left[ \frac{\tau}{24\pi} \int d^2 x \sqrt{g} \{ D_\mu \sigma D^\mu \sigma + R \sigma \} \right]
\]

(9)

where \( \tau = 0 \) (\( \tau = 1 \)) for the Weyl (translation) invariant measure. The resulting Ward identity for \( W_k \) can be rewritten as a statement about \( \Gamma_k \).

It reads

\[
\mathcal{L} \Gamma_k[\phi; g, \psi_i] = -\frac{26 - c + \tau}{24\pi} R(x) + A_k(x)
\]

(10)

where

\[
\mathcal{L} = 2 \frac{g_{\mu\nu}(x)}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} - \frac{1}{\sqrt{g}} \frac{\delta}{\delta \phi(x)} - 2 \sum_i \Delta_i \frac{\psi_i(x)}{\sqrt{g}} \frac{\delta}{\delta \psi_i(x)}
\]

(11)

and

\[
A_k(x) \equiv \langle x | R_k(\Gamma_k(2)^2 + R_k)^{-1} | x \rangle + \text{Tr} \left[ \hat{R}_k(x)(\Gamma_k(2)^2 + R_k)^{-1} \right]
\]

(12)

with

\[
\hat{R}_k(y) \equiv \frac{g_{\mu\nu}(y)}{\sqrt{g}(y)} \frac{\delta}{\delta g_{\mu\nu}(y)} R_k(-D^2[g_{\mu\nu}(\cdot)]).
\]

(13)

We have not yet explained the meaning of the external fields \( \psi_i(x) \). In the derivation of the Ward identity we actually replaced \( S_L \) in (4) by \( S_L + S_\psi \) with

\[
S_\psi[\chi; g, \psi_i] = \sum_i \int d^2 x \sqrt{g} \psi_i(x) \exp[2(1 - \Delta_i^0)\chi]
\]

(14)

The action \( S_\psi \) is Weyl invariant provided \( \psi'_i = e^{-2\Delta_i^0} \psi_i \). The \( \psi_i \)'s could be some spin-0 primary fields of the underlying conformal theory of matter and ghost fields. Their scaling dimensions in absence of gravity are \((\Delta_i^0, \Delta_i^0)\). We shall investigate how these dimensions change when the system is coupled to quantized gravity ("gravitational dressing" [5, 2]).

The Ward identity is an equation for the trace of the energy momentum tensor derived from \( \Gamma_k \). When an arbitrary conformal field theory with action \( S \) and central charge \( c[S] \) is coupled to gravity the trace of its energy momentum tensor \( T^{\mu\nu}[S] \equiv 2g^{-1/2} \delta S/\delta g_{\mu\nu} \) is (at least on shell) given by

\[
T^{\mu\nu}[S] = -c[S] R/24\pi + \text{const}.
\]

The classical Liouville action (3) satisfies this condition with the central charge \( c[S_L] = 3\kappa^2 = 26 - c \). Coming back to the Ward identity, \( R_k \) vanishes for \( k \to 0 \) so that \( A_k = 0 \) in this limit.
Thus (10) with $\phi$ on shell and $\psi_i \equiv 0$ tells us that the evolution ends at a theory which is conformally invariant and has central charge

$$c[\Gamma_{k \to 0}] = 26 - c + \tau$$

(15)

For the Weyl-invariant measure ($\tau = 0$), this is the correct value.

### 3 Truncation and Initial Value

While the evolution equation is exact, it is usually not possible to find exact solutions. Approximate solutions can be found by the method of truncations which is of a nonperturbative nature and does not require any small expansion parameter. The idea is to project the renormalization group flow on a subspace of the space of all action functionals. In our case this subspace has finite dimension and is coordinatized by a finite number of generalized couplings. We make the ansatz

$$\Gamma_k[\phi; g, \psi] = \Gamma^L_k[\phi; g] + \Gamma^\psi_k[\phi; g, \psi] - \tilde{\kappa}_k^2 I[g]/32\pi$$

(16)

where

$$\Gamma^L_k[\phi; g] = \frac{\kappa_k^2}{8\pi} \int d^2 x \sqrt{g} \left\{ \zeta_k(\phi) D_\mu \phi D^\mu \phi + \omega_k(\phi) R + v_k(\phi) \right\}$$

(17)

$$\Gamma^\psi_k[\phi; g, \psi] = \frac{1}{16\pi} \sum_{i} \left( \frac{m_{ik}}{\alpha_{ik}} \right)^2 \int d^2 x \sqrt{g} \psi_i \exp[2\alpha_{ik}\kappa_k\phi]$$

(18)

The running parameter $\kappa_k$ is defined as the coefficient of the $\phi R$-term by the convention $(\partial \omega_k/\partial \phi)(0) = 1$. For the main part of our analysis in ref. [13] we assumed the more restrictive ansatz

$$v_k(\phi) = \frac{m_k^2}{2\alpha_k \kappa_k} \exp[2\alpha_{ik}\kappa_k \phi], \quad \omega_k(\phi) = \phi, \quad \zeta_k(\phi) = \zeta_k$$

(19)

In this case $\Gamma^L_k$ is parametrized by four functions of $k$, namely $\kappa_k, \zeta_k, m_k$ and $\alpha_k$, which have to be determined from the evolution equation along with $m_{ik}, \alpha_{ik}$ and $\tilde{\kappa}_k$.

Before we can embark on solving the projected RG equation we must analyze the constraints which the Ward identities impose on the allowed initial points $\Gamma_\Lambda$. By inserting the above ansatz into (13) and extracting the relevant field monomials from $A_k$ in the limit $k = \Lambda \to \infty$, one finds

$$\zeta_\Lambda = 1, \quad 3\kappa_\Lambda^2 = 25 - c + \tau, \quad 3\tilde{\kappa}_\Lambda^2 = 26 - c$$

(20)

In the classical action $S_L$ the coefficient corresponding to $3\kappa_\Lambda^2$ is $c[S_L] = 3\kappa^2 = 26 - c$. For the translation invariant quantization ($\tau = 1$) these values coincide, but for the Weyl invariant case ($\tau = 0$) we have to start
from a different value, $3\kappa_\Lambda^2 = 25 - c$. Therefore $\Gamma^\tau_\Lambda$ cannot coincide with $S_L$. We also obtain a relation which determines the initial value $\alpha_\Lambda$ in terms of $\kappa_\Lambda$, $\alpha_\Lambda \kappa_\Lambda = 1 + 2\alpha_\Lambda^2$. With $\tau = 0$ for the Weyl-invariant measure we have the two solutions

$$\alpha_\Lambda = \frac{1}{4\sqrt{3}} \left[ \sqrt{25 - c} \pm \sqrt{1 - c} \right]$$

(21)

with the perturbative branch corresponding to the minus sign. They are real only for $c \leq 1$ and $c \geq 25$. In the following we restrict ourselves to $c \leq 1$. There is a similar equation for $\alpha_i \Lambda$ which is most conveniently expressed in terms of the gravitationally dressed scaling dimension relative to the area operator, $\Delta_i(\kappa) \equiv 1 - \alpha_{ik}/\alpha_k$. For the perturbative branch and for $\tau = 0$ it reads

$$\Delta_i(\Lambda) = \frac{\sqrt{1 - c} + 24\Delta^i_0 - \sqrt{1 - c}}{\sqrt{25 - c} - \sqrt{1 - c}}$$

(22)

The r.h.s. of this expression is precisely what appears in the famous KPZ-formula [5] for the scaling dimension of $\psi_i$ in presence of quantized gravity [2]. However, in the renormalization group framework the properties of the quantum theory are obtained for $k \to 0$ rather than at $k = \Lambda$. Therefore (22) is not the KPZ-formula yet. It obtains only provided we can show that $\Delta_i(0) = \Delta_i(\Lambda)$.

4 The RG Evolution

The next step is to derive the coupled system of ordinary differential equations for the functions $\alpha_k, m_k, \ldots$. Upon inserting the truncation (16) into the renormalization group equation (7) we have to evaluate

$$\partial_t \Gamma_k = \text{Tr} \left[ \left( (1 - \eta_k/2)k^2C + D^2C' \right) \left( -D^2 + k^2C + E_k \right)^{-1} \right]$$

(23)

Here $\eta_k \equiv -\partial_k \ln(\kappa_k^2 \zeta_k)$, $C \equiv -C(-D^2/k^2)$ and $E_k = \tilde{m}_k^2 \exp[2\alpha_k \kappa_k \phi] + \cdots$ with $\tilde{m}_k^2 \equiv m_k^2/\zeta_k$. By appropriate derivative and curvature expansions of the functional trace in (23) one can extract its contribution proportional to the field monomials present in the truncation ansatz. By comparing the coefficients of these monomials on both sides of the RG equation one finds the set of ordinary differential equations for the coefficient functions (generalized couplings). From the term $\int d^2x \sqrt{g} R \omega_k(\phi)$, say, one obtains (omitting the subscripts $k$)

$$\partial_t [k^2 \omega(\phi)] = \frac{1}{3} \frac{k^2}{k^2 + \tilde{m}^2 \exp(2\alpha_k \phi)}$$

(24)

This equation shows a rather typical feature which occurs for the other couplings as well and which is crucial for the understanding of the RG flow.
in Liouville theory: depending on the relative magnitude of the two terms in the denominator of \( \frac{2}{k^2} \) either \( k^2 \) or the \( \phi \)-dependent Liouville mass term acts as the relevant infrared cutoff. In the latter case the couplings do not run any longer as a function of \( k \). Let us define the \( k \)-dependent critical value \( \phi_c \) by the condition that these two terms are equal at \( \phi = \phi_c \):

\[
\phi_c(k) \equiv \frac{1}{2\alpha k^2} - \frac{\ln k^2}{m_0^2} \tag{25}
\]

We see that \( \kappa^2 \omega(\phi) \) changes at a constant rate \( \partial_t [\kappa^2 \omega] = \frac{1}{3} \) for \( \phi \ll \phi_c \) (\( k \) large) whereas the evolution stops for \( \phi \gg \phi_c \) (\( k \) small) due to an exponential suppression factor:

\[
\partial_t [\kappa^2 \omega(\phi \gg \phi_c)] = \frac{1}{3} \exp[-2\alpha k(\phi - \phi_c)] \tag{26}
\]

The subtle aspect of this decoupling phenomenon is that in different regions of field space the threshold separating the two regimes is crossed at different values of \( k \).

For a detailed discussion of the evolution equations for the various couplings we have to refer to [13] where also the approximations are described, which went into their solution. Here we only quote the result for \( \Gamma_k \) in the physical limit \( k \to 0 \). One finds the \( k \)-independent functional

\[
\Gamma_0 = \frac{26 - c}{24\pi} \int d^2x \sqrt{g} \left\{ D_{\mu} \phi D^{\mu} \phi + R \phi + \frac{\bar{m}_i^2}{2} \psi_i \exp[2(1 - \Delta_0^i) \phi] \right\} - \frac{26 - c}{96\pi} I[g] \tag{27}
\]

Remarkably enough, up to modified mass parameters \( \bar{m} \) and \( \bar{m}_i \) this is precisely the classical action. In particular, the value of the coefficient \( 3\kappa_0^2 \) has increased from \( 25 - c \) at the initial point \( k = \Lambda \) to \( 26 - c \) at the final point \( k = 0 \). Here we encounter the slightly unusual situation that even during an infinitely long running “time” \( (\Lambda \to \infty) \) a certain coupling changes by a finite amount only \( \frac{1}{3} \). The same remark also applies to \( \alpha_k \) which changes in such a way that \( \alpha_k \kappa_0 = 1 \), i.e., for \( k \to 0 \) we recover the classical Liouville potential of [3]. The fact that the effective potential equals the classical one can be proven [13] on the basis of the rather general truncation (17); it is not necessary to make the ansatz (19). Likewise the exponentials multiplying the \( \psi_i \)’s return to their classical form as \( k \) approaches zero. One obtains \( \lim_{k \to 0} \alpha_k \kappa_k = 1 - \Delta_0^i \). This implies that the gravitationally dressed scaling dimensions equal the classical ones, \( \Delta_i(k = 0) = \Delta_0^i \). Even though it seems that our truncation is too simple

\[ ^2 \text{We set } \tau = 0 \text{ from now on.} \]

\[ ^3 \text{A similar finite renormalization was found for the Chern-Simons parameter in } d = 3 \]

\[ ^4 \text{and for the } \theta- \text{parameter in } d = 4 \].
to reproduce the KPZ formulas in the infrared, one can verify that (within
the approximations which were made in solving the evolution equations)
the resulting RG trajectory is perfectly consistent with the Ward identities
In particular, $\Gamma_0$ is easily seen to describe a conformal field theory
with total central charge zero.

5 The UV-Fixed Point

Let us look at the space of 2-dimensional field theories with one scalar
field as a whole now. Typically a critical scalar theory is governed by a
fixed point which is IR-unstable in one or several directions. The flow
starts then for $k = \Lambda$ in the immediate vicinity of this fixed point and
remains there for a long “running time”. Depending on the value of $\phi$ the
running stops at a certain point, and $\Gamma_k[\phi]$ becomes independent of $k$ for
$k \to 0$. Generically there is also a region in $\phi$ where the running never
stops. In the critical $\mathbb{Z}_2$-symmetric $\phi^4$-theory (Ising model) this region
shrinks to one point – the potential minimum – as $k \to 0$. In our case this
region is $\phi < \phi_c(k)$ and it moves to $\phi \to -\infty$ as $k \to 0$.

We search for fixed points of the renormalization group flow in the
subspace of action functionals of the form (“local potential approxima-
tion”)

$$\Gamma_k[\phi; g] = Z_k \int d^2 x \sqrt{g} \left\{ \frac{1}{2} D_\mu \phi D^\mu \phi + R f_k(\phi) + k^2 u_k(\phi) \right\}$$

(28)

Here $f_k$ and $u_k$ are arbitrary dimensionless functions. Hence the condition
for a fixed point is simply $\partial_t u_k = \partial_t f_k = \partial_t Z_k = 0$. By inserting (28) into
(7) we obtain the following equation for the fixed point potential $u_*(\phi)$:

$$8 \pi Z_* u_*(\phi) = \int_0^\infty dy \frac{C(y) - y C'(y)}{y + C(y) + u_*'(\phi)}$$

(29)

It is not difficult to see [3] that a generic solution to this equation is
a periodic function of $\phi$. Solutions of this type are not related to Liou-
ville theory and represent different universality classes. The only solution
which is not periodic is the constant one, $u_*(\phi) \equiv u_*$. We are now going
to show that Liouville theory can be understood as the perturbation of this
“Gaussian” fixed point by a relevant operator. When $k$ is lowered
from infinity down to zero, this operator drives the theory from the IR
unstable Gaussian fixed point to a different IR stable one which repre-
sents quantum Liouville theory. In order to linearize the RG flow in the
vicinity of the Gaussian fixed point we write $u_k(\phi) = u_* + \varepsilon \delta u_k(\phi)$ with
($t \equiv \ln[k/\Lambda]$)

$$\delta u_k(\phi) = e^{-\gamma t} Y(\phi) + \sum_i e^{-\gamma t} e^{2\Delta_i t} \psi_i T_i(\phi)$$

(30)

8
Here $\Upsilon(\phi)$ and $\Upsilon_i(\phi)$ are the eigenvectors of the linearized evolution operator with the eigenvalues $\gamma$ and $\gamma_i$, respectively. By expanding the evolution equation to first order in $\varepsilon$ and solving the resulting linear equation one obtains for the potential near the fixed point:

$$k^2 \delta u_k(\phi) \propto m^2 \left[ (\Lambda/k)^{4\alpha^2} e^{2\alpha \kappa \phi} + \sum_i \psi_i (\Lambda/k)^{4\alpha_i^2} e^{2\alpha_i \kappa_i \phi} \right]$$  \hspace{1cm} (31)

This result is universal in the sense that it does not depend on the form of the cutoff function $C(\cdot)$. Now we understand what is so special about the exponential interaction potentials: they are precisely the eigenvectors of the linearized renormalization group flow at the Gaussian fixed point. For $\alpha, \alpha_i$ real, the perturbation grows for decreasing $k$ and this fixed point is IR-unstable therefore. Hence we can identify $\alpha, \alpha_i$ and $\kappa$ with the UV parameters $\alpha_\Lambda, \alpha_i \Lambda$ and $\kappa_\Lambda$ whose values are fixed by the Ward identities. So far we discussed only the potential term. If we also take the fixed point condition for $f_k$ into account we find that the fixed point action is of the Feigin-Fuks type:

$$\Gamma_* = \frac{25 - c}{24\pi} \int d^2x \sqrt{g}(D_{\mu} \phi D^{\mu} \phi + R\phi + \text{const})$$  \hspace{1cm} (32)

This theory is conformally invariant with $c[\Gamma_*] = 25 - c$.

Note that the area operator is required to be an $(1, 1)$ operator in the quantum theory, i.e., for $k \to 0$, but not necessarily at $k = \Lambda$. In the UV it is not marginal and this is what drives the system away from the UV-fixed point. In fact, the area operator responds to a conformal reparametrization $z \to z'(z)$:

$$s' = |dz'/dz|^2 s, \phi' = \phi - \frac{1}{2} \ln |dz'/dz|^2$$

according to

$$(\sqrt{g} e^{2\alpha \kappa \phi})' = |dz'/dz|^{-4\alpha^2 \Lambda} \sqrt{g} e^{2\alpha \kappa \phi}$$  \hspace{1cm} (33)

This is a direct consequence of the Ward identity $\alpha \Lambda \kappa_\Lambda = 1 + 2\alpha^2 \Lambda$.

6 Conclusion

Let us summarize the results obtained from the Ward identities, the evolution equation and the fixed point equation. The overall picture which emerges is that of a crossover phenomenon from one fixed point to another. The renormalization group trajectory starts for $k = \Lambda$ at a conformal theory with central charge $3\kappa^2_\Lambda = 25 - c$. This initial point, approached for $\Lambda \to \infty$, is a Feigin-Fuks free field action $\Gamma_*$. The perturbations at this fixed point are governed by eigenvectors of the linearized RG flow, $\Upsilon(\phi)$ and $\Upsilon_i(\phi)$, which have an exponential dependence on $\phi$. Adding such a perturbation to $\Gamma_*$ we obtain the Liouville action. The Weyl–Ward identities fix the free parameters in this action (except for $m_\Lambda$). For these
initial values the perturbation is a relevant operator which drives the system away from the IR-unstable fixed point. For $k \to 0$, $\Gamma_k^L$ approaches an IR-stable fixed point. The Ward identity guarantees that $\Gamma_k^L \to 0$ is a conformal field theory with $c[\Gamma_k^\to 0] = 26 - c$. Within our truncation we find that $\Gamma_0^L$ equals the classical Liouville action. We expect that this is a very good approximation (and perhaps even exact) for the low momentum part of the effective action. An improved truncation should, for instance, reproduce the KPZ dimensions at the final point $k = 0$. It remains slightly mysterious why the present truncation yields precisely the KPZ scaling dimensions at the initial point of its RG trajectory.

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