Observer-based adaptive neural network backstepping sliding mode control for switched fractional order uncertain nonlinear systems with unmeasured states

Tao Chen, Damin Cao, Jiaxin Yuan and Hui Yang

Abstract
This paper proposes an observer-based adaptive neural network backstepping sliding mode controller to ensure the stability of switched fractional order strict-feedback nonlinear systems in the presence of arbitrary switchings and unmeasured states. To avoid “explosion of complexity” and obtain fractional derivatives for virtual control functions continuously, the fractional order dynamic surface control (DSC) technology is introduced into the controller. An observer is used for states estimation of the fractional order systems. The sliding mode control technology is introduced to enhance robustness. The unknown nonlinear functions and uncertain disturbances are approximated by the radial basis function neural networks (RBFNNs). The stability of system is ensured by the constructed Lyapunov functions. The fractional adaptive laws are proposed to update uncertain parameters. The proposed controller can ensure convergence of the tracking error and all the states remain bounded in the closed-loop systems. Lastly, the feasibility of the proposed control method is proved by giving two examples.

Keywords
Fractional order nonlinear systems, adaptive backstepping control, dynamic surface control, switched systems, neural network, observer

Introduction
Fractional calculus has unique memory properties and the ability to accurately model the system, so it can be easily applied to many industries and research fields, such as research electrical circuits, signal processing, chemical processes and biological engineering. Fractional order dynamic system can reflect the system’s own conditions more truly, so as to develop new control strategies to enhance the characteristics of control loops. In recent years, many important results had been achieved in the stability analysis and control research of fractional order nonlinear systems. For example, Wang extended backstepping control scheme to fractional order system and studied problem of Mittage-Leffler stabilization of fractional order nonlinear system. Ding et al. proposed a fractional order backstepping controller for a class of fractional order nonlinear strict-feedback system with both unknown disturbance. Li et al. designed an adaptive sliding mode controller to compensate the input uncertainties for a class of fractional order nonlinear systems with unknown external disturbances and input uncertainties. Jain et al. proposed a fractional order internal model control method for non-ideal dc–dc buck and boost converter.

Adaptive backstepping control (ABC) method has been widely used to stabilize nonlinear systems in practical applications due to the excellent performance of compensating saturation. For example, Zhou et al. studied the stabilization problem for uncertain nonlinear systems and proposed an ABC algorithm. Sun et al. proposed an ABC method to handle the position tracking problem of the permanent magnet synchronous motor system. Cai et al. proposed a new ABC scheme to overcome the uncertainties for a class

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of second-order nonlinear systems with non-triangular uncertainties. At present, some control methods are combined with adaptive backstepping control methods to study classical nonlinear systems in the field of integer-order control method, such as adaptive backstepping sliding mode control (SMC),

13 adaptive neural network, or fuzzy backstepping,

14,15 adaptive dynamic surface backstepping control,16 and so on.

Traditional backstepping control requires accurate modelling information of the controlled object and cannot overcome the disturbance. To deal with the uncertain of nonlinear systems, the universal approximation theories of neural networks (NNs) and fuzzy logic systems (FLSs) can be employed to approach the unknown nonlinear functions.17 For example, the authors proposed a fuzzy double hidden layer recurrent neural network approximation technology in Refs.,18–20 which can be regarded as a combination of a fuzzy NN and a RBFNN to improve the accuracy of a nonlinear approximation, so it has the advantages of these two NNs. Aiming at the field of fractional order systems control, Liu et al.21 designed adaptive fuzzy backstepping control method is proposed for uncertain fractional order chaotic systems including unknown external disturbance and input saturation. Wang and Liang22 combined backstepping and adaptive technology, using RBFNN to approximate the system unknown nonlinear uncertainty and proposed an ABC method for a class of uncertain fractional nonlinear systems with external disturbance and input saturation. Ma and Ma23 proposed to introduce an auxiliary function to compensate the unknown external interference and the approximation error produced by the FLS approximation of the unknown function. Zhang and Li13 proposed an adaptive backstepping sliding mode controller based on NN technology for the wheel slip tracking control system. For fractional multi-agent systems with unknown uncertainty, Shahvali et al.24 designed a distributed controller based on NN control method, the NN is used to approximate the system unknown uncertainty. Based on RBFNN, an adaptive backstepping control method was proposed by Shahvali et al.25 for the consensus problem of fractional order nonlinear systems. For fractional order input saturated permanent magnet synchronous motor (PMSM), Lu and Wang26 proposed a NN method combined with command filtering technique. Sui et al.27 used NN to model uncertain fractional order systems and presented an adaptive switching dynamic surface control method for fractional order non-strict feedback nonlinear system.

In most practical applications, the system states cannot be fully accessed, and the designer cannot know the output and input of the plant. In this case, observer-based control is usually required. There have been many studies in the field of integer-order control combining observers with adaptive inversion techniques. For example, Lu and Wang28 designed a state observer to obtain the unmeasured state for a fractional chaotic PMSM with the immeasurable state and the parameter uncertainties. Tong et al.29 proposed a fuzzy state observer to estimate the unmeasurable states and designed an adaptive fuzzy output-feedback backstepping controller for uncertain strict-feedback nonlinear systems. For fractional order systems with system uncertainties and external disturbances, observers and NN approximations are rarely considered together. This is of great significance for further research on new control technologies based on neural networks and observers.

It should be recognized that the abovementioned fractional order nonlinear systems are a class of non-switched systems. It is worth pointing out that the switched system is another more complicated system. It is formed by switching signals between subsystems. Practical applications, such as hybrid vehicles and circuit systems, are all switching systems. Applying the common Lyapunov function method, Tong et al.30 solved the control problem of arbitrary switching systems. Sui et al.,27 Long and Zhao31 and Li and Tong32 studied the switching control approaches of strict feedback switched nonlinear systems through the average dwell time method.

Based on the previous discussion, this paper proposes an adaptive neural network backstepping sliding mode controller based on DSC technology for a class of the switched fractional order strict-feedback nonlinear systems with unmeasured state. It should be pointed out that the theoretical results obtained in this paper is not a simple extension from integer-order systems to fractional systems. We use some properties of the Caputo fractional derivative and the integral inequality to overcome the adverse effects from the incorporation of weakly singular kernels in fractional derivative. Compared with the current researches, the contributions of this work are list as follows:

1) In comparison with Refs.,21,22,28 this paper proposes an observer-based adaptive neural network backstepping sliding mode controller for a class of the switched fractional order strict-feedback nonlinear systems with unmeasured states. Compared with the previous works in Sui et al.,27 the state observer is introduced into the proposed method to estimate the system state.

2) Compared with the previous works in Ma and Ma,23 the sliding mode control term is introduced to enhance robustness and the fractional order DSC technology is used to avoid “explosion of complexity” and obtain fractional derivatives for virtual control continuously. Compared with the previous works in Ding et al.,7 the uncertain disturbances are approximated by RBFNN. The stability of the closed-loop system is ensured by the constructed Lyapunov functions. And in our design, the proposed fractional update laws estimate the
unknown parameters and the upper limit of the approximation errors.

3) Under the framework of adaptive backstepping control technique, the desired tracking performance for the switched fractional order strict-feedback nonlinear systems is obtained by using the designed controller which integrates observer, RBFNN, fractional order adaptive laws, sliding mode control method, DSC technique.

The rest of the paper is organized as follows. Section 2 introduces basic theory about fractional calculus and proposes the controlled switched fractional order nonlinear system. In Section 3, we construct an observer to estimate the system states firstly, and then a controller is proposed based on the adaptive backstepping sliding mode control method. In Section 4, the effectiveness of the proposed control method is proved by giving two examples. In Section 5, we summarize the paper and give some conclusions.

Preliminaries

Fractional calculus

The Riemann-Liouville (R-L) fractional derivative definition and Caputo fractional derivative definition are commonly used in the study of fractional differential. Because Caputo fractional derivative definition is easier to give the initial value conditions of the fractional derivative equation, it is widely used in engineering applications. This paper mainly adopts the Caputo definition. The R-L fractional integral is defined as

$$\int_0^t \frac{f(\tau)}{(t-\tau)^{1-a}} d\tau$$

The R-L fractional derivative is defined as

$$\frac{d^a}{dt^a} f(t) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{n-a}} d\tau$$

where $n \in N$ and $n-1 < \alpha \leq n$, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function.

The Caputo fractional derivative is defined as

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha}} d\tau$$

Remark 1. To simplify the notation, we set $\frac{d^\alpha}{dt^\alpha} f(t) = D^\alpha f(t)$.

Definition 1. Li et al. The Mittag-Leffler function is given as

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+\gamma)}$$

where $z$ is complex number and $\alpha, \gamma > 0$. Its Laplace transform is given by

$$L(t^{-\gamma} E_{\alpha,\gamma}(z)) = \frac{s^{\alpha-\gamma}}{s^\alpha + a}$$

Lemma 1. Podlubny. For a complex number $\beta$ and two real numbers $\alpha, \gamma$ satisfying $\alpha \in (0, 1)$ and $\frac{\pi \alpha}{2} < \gamma < \min\{\pi, \pi \alpha\}$

For all integer $n \geq 1$, we can obtain

$$E_{\alpha,\gamma}(z) = \sum_{j=1}^{\infty} \frac{1}{\Gamma(\beta - aj)} + o \left( \frac{1}{|z|^\beta + 1} \right)$$

when $|z| \to \infty$, $\nu \leq |\arg(z)| \leq \pi$.

Lemma 2. Podlubny. Let $\alpha \in (0, 2)$ and $\beta$ be an arbitrary real number, and for $\forall \nu > 0$ such that $(\pi \alpha/2) < \nu \leq \min\{\pi, \pi \alpha\}$, one has

$$|E_{\alpha,\beta}(z)| \leq \frac{\mu}{1 + |\nu|}$$

where $\mu > 0$, $\nu \leq |\arg(z)| \leq \pi$, and $|z| \geq 0$.

Lemma 3. Duarte-Mermoud et al. Let $x(t) = [x_1(t), \ldots, x_n(t)] \in \mathbb{R}^n$ be a vector of continuous and differentiable function. And then, the following relationship holds

$$\frac{1}{2} D^\alpha (x^T(t) P x) \leq x^T(t) PD^\alpha x(t)$$

$$\forall \alpha \in (0, 1), \forall t > t_0$$

Lemma 4. Wang et al. For any $x, y \in \mathbb{R}^n$, the following inequality relationship holds

$$x^T y \leq \frac{c^a}{a} \|x\|^a + \frac{1}{bc(a-1)} \|y\|^b$$

where $a > 1$, $b > 1$, $c > 0$, and $(a-1)(b-1) = 1$.

Lemma 5. Li et al. In fractional-order nonlinear system, if the $n$-order derivative of Lyapunov function $V(t, x)$ satisfying

$$D^\alpha V(t, x) \leq -CV(t, x) + D$$

we can obtain

$$V(t, x) \leq V(0) E_{\alpha}(-Ct^\alpha) + \frac{D \mu}{C}, t \geq 0$$

where $0 < \alpha < 1$, $C > 0$, and $D \geq 0$. Then, $V(t, x)$ is bounded on $[0, t]$ and fractional order systems are stable, where $\mu$ is defined in Lemma 2.

System description

Many fractional order nonlinear systems can be written as a lower triangular form, and it is referred as the lower triangular system. For example, the fractional order Chua-Hartley’s system Wang et al., the fractional order Arneodo system and the fractional order...
Duffing-Holmes chaotic system. The fractional order Arneodo system:
\[
\begin{aligned}
D^a x(t) &= y(t) \\
D^a y(t) &= z(t) \\
D^a z(t) &= ax(t) - by(t) - rz(t) - x^3(t)
\end{aligned}
\]
(12)

the fractional order Chua-Hartley’s system:
\[
\begin{aligned}
D^a x_1(t) &= x_2(t) + a(x_1(t) - x_3^2(t)) \\
D^a x_2(t) &= x_3 + bx_1(t) - cx_2(t) \\
D^a x_3(t) &= dx_2(t)
\end{aligned}
\]
(13)

the fractional order Duffing-Holmes system:
\[
\begin{aligned}
D^a x_1(t) &= x_2 + f_1^{(a)}(x_1(t)) + d_1(t) \\
D^a x_2(t) &= x_3 + f_2^{(a)}(x_1(t), x_2(t), ..., x_n(t)) + d_2(t) \\
D^a x_3(t) &= u + f_3^{(a)}(x_1(t), x_2(t), ..., x_n(t)) + d_3(t)
\end{aligned}
\]
(14)

In the paper, we consider the switched fractional order strict-feedback nonlinear system with uncertain disturbances.

\[
\begin{aligned}
D^a x_1(t) &= x_2 + f_1^{(a)}(x_1(t)) + d_1(t) \\
D^a x_2(t) &= x_3 + f_2^{(a)}(x_1(t), x_2(t), ..., x_n(t)) + d_2(t) \\
D^a x_3(t) &= u + f_3^{(a)}(x_1(t), x_2(t), ..., x_n(t)) + d_3(t)
\end{aligned}
\]
(15)

where \( i = 2, ..., n - 1, \alpha \in (0, 1) \) is the order of the fractional derivative; \( Y_i = (x_1, x_2, ..., x_i)^T \in \mathbb{R}^n \) is the system states vector. \( u(t) \) is the control input of the system, \( y \) is the system output and \( f_3^{(a)}(x_1(t), x_2(t), ..., x_n(t)) \) is unknown nonlinear function; \( \sigma(t) \) is a piecewise continuous function that is used to describe the triggering conditions for switching between subsystems. It is called a switching signal, for example, if \( \sigma(t) = q \), it means that \( q - th \) subsystem is activated. \( d_i(t) \) is external disturbance and \( |d_i(t)| \leq d_i^* \), where \( d_i^* \) is an unknown constant.

Rewriting system (15):
\[
D^a X = AX + Ky + \sum_{i=1}^{n} B_i f_i^*(X_i) + Bu
\]
\[
y = CX
\]
(16)

Where
\[
A = \begin{bmatrix}
k_{11} & 0 & \cdots & 0 \\
0 & k_{12} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & k_{1n}
\end{bmatrix},
K = \begin{bmatrix}
k_{11} \\
k_{12} \\
\vdots \\
k_{1n}
\end{bmatrix},
B = \begin{bmatrix}
0_{n-1} \\
0 \\
n-1 \\
1
\end{bmatrix}^T
\]
\[
B_i = \begin{bmatrix}
0_{n-1} \\
\vdots \\
0 \\
1 \cdots \cdots \cdots 0
\end{bmatrix}^T
\]
\[
C = \begin{bmatrix}
1 \\
0 \cdots \cdots \cdots 0 \\
\vdots \\
0_{n-1}
\end{bmatrix}
\]
and given a positive matrix \( Q^T = Q \), there exists a positive matrix \( P^T = P \) satisfying
\[
ATP + PAT = -2Q
\]
(17)

Control objectives: Let \( y_d \) be a reference signal and the tracking error as \( S_1 = y - y_d \), design fractional order controller based on observer, and choose suitable parameters adaptive laws such that all the signals remain bounded and the tracking error \( S_1 \) is as small as possible in the closed-loop system.

Main results

Observer design

Assumption 1. The unknown functions \( f_i(X), i = 1, ..., n \) can be expressed as
\[
f_i(X_i|\theta_i) = \theta_i^T \phi_i(X_i), 1 \leq i \leq n
\]
(18)

where \( \theta_i \) is the ideal constant vector, \( \phi_i(X_i) \) is the basis functions vector, and Gaussian basis functions are used in this paper. In this section, assuming that the state variables of the system (15) are not available. In this case, the system states need to be estimated by an observer, and the observer is designed as
\[
D^a \hat{X} = AX + Ky + \sum_{i=1}^{n} B_i f_i^*(\hat{X}_i|\theta_i) + Bu
\]
\[
\hat{y} = CX
\]
(19)

\[\hat{X}_i = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)^T \]
Let \( e = X - \hat{X} \) be state observation errors of the system, then according to equations (16) and (19), we have
\[
D^a e = Ae + \sum_{i=1}^{n} B_i \left[ f_i^*(X_i) - f_i^*(\hat{X}_i|\theta_i) + \Delta f_i^* + d_i^* \right]
\]
(20)

where \( \Delta f_i^* = f_i^*(X_i) - f_i^*(\hat{X}_i) \).

By Assumption 1, we can obtain
\[
\Delta f_i^* (\hat{X}_i|\theta_i) = \theta_i^T \phi_i(\hat{X}_i), 1 \leq i \leq n
\]
(21)

Define the vectors of optimal parameters as
\[
\theta_i^* = \arg \min_{\theta_i \in \Omega_i} \left[ \sup_{\hat{X}_i \in U_i} \left| f_i^*(\hat{X}_i|\theta_i) \right| - f_i^*(\hat{X}_i) \right]
\]
(22)

where \( 1 \leq i \leq n, \Omega_i \) and \( U_i \) are compact regions for \( \theta_i, X_i \) and \( \hat{X}_i \).

Define errors of the optimal approximation and parameters estimation as
\[
e_i^* = f_i^*(\hat{X}_i) - \hat{f}_i^*(\hat{X}_i|\theta_i^*)
\]
\[
\dot{\theta}_i = \theta_i^* - \theta_i, i = 1, 2, ..., n
\]
(23)

Assumption 2. The optimal approximation error and the ideal constant weight vector remain bounded, there exists positive constants \( e_0, \theta_i^* \) satisfying \( |e_i^*| \leq e_0, |\theta_i| \leq \theta_i^* \).

Assumption 3. There exists a set of known constants \( m_i \), the following relationship holds
\[
|f_i(X_1) - f_i(X_2)| \leq m_i \| X_1 - X_2 \|
\]
(24)
By equations (20) and (23), we have
\[
D^\alpha e = Ae + \sum_{i=1}^{n} B_i \left( \dot{f}_i(X_i) - \ddot{f}_i(X_i) \theta_i \right) + \Delta f_i + d_i
\]
\[
= Ae + \sum_{i=1}^{n} B_i \left[ \dot{\theta}_i^T \varphi_i(X_i) \right] + \Delta f_i + d_i
\]
\[
= Ae + \Delta f + \kappa + \sum_{i=1}^{n} B_i \left[ \dot{\theta}_i^T \varphi_i(X_i) \right]
\]
(25)
where \( \kappa = [e_1 + d_1, \ldots, e_n + d_n]^T \), \( \Delta f = [\Delta f_1, \ldots, \Delta f_n]^T \).

Construct the first Lyapunov function:
\[
V_0 = \frac{1}{2} e^T P e
\]
(26)
According to Lemma 3 and (17), we can obtain
\[
D^\alpha V_0 \leq \frac{1}{2} e^T (P A^T + A P) e + e^T P (\kappa + \Delta f)
\]
\[
+ \sum_{i=1}^{n} e^T P B_i \left[ \dot{\theta}_i^T \varphi_i(X_i) \right]
\]
\[
\leq -e^T Q e + e^T P (\kappa + \Delta f)
\]
\[
+ e^T P \sum_{i=1}^{n} B_i \dot{\theta}_i^T \varphi_i(X_i)
\]
(27)
By Lemma 4 and Assumption 3, we can obtain
\[
e^T P (\kappa + \Delta f) \leq \|e\|^2 + \frac{1}{2} \|P\|^2 + \frac{1}{2} \|e\|^2 + \frac{1}{2} \|P\|^2 \|\Delta f\|^2
\]
\[
\leq \|e\|^2 + \frac{1}{2} \|P\|^2 \|\theta\|^2 + \frac{1}{2} \|P\|^2 \sum_{i=1}^{n} \|\Delta f_i\|^2
\]
\[
\leq \|e\|^2 + \frac{1}{2} \|P\|^2 \|\theta\|^2 + \frac{1}{2} \|P\|^2 \sum_{i=1}^{n} \|m_i\|^2
\]
\[
\leq \|e\|^2 + \frac{1}{2} \|P\|^2 \sum_{i=1}^{n} \|m_i\|^2
\]
and
\[
e^T P \sum_{i=1}^{n} B_i \dot{\theta}_i^T \varphi_i(X_i)
\]
\[
\leq \tau e^T P e + \frac{1}{2} \sum_{i=1}^{n} \|\dot{\theta}_i^T \varphi_i(X_i) \|^2
\]
\[
\leq \tau \lambda_{\text{max}}^2 (P) \|e\|^2 + \frac{1}{2} \sum_{i=1}^{n} \|\dot{\theta}_i\|^2
\]
(28)
where \( \kappa = [e_0 + d_1, \ldots, e_n + d_n], \tau > 0 \).

By equations (27)–(29), we obtain
\[
D^\alpha V_0 \leq -q_0 \|e\|^2 + \frac{1}{2} \|P \theta\|^2 + \frac{1}{2} \sum_{i=1}^{n} \|\dot{\theta}_i\|^2
\]
(30)
where \( q_0 = \lambda_{\text{min}} (Q) - \left( 1 + \frac{1}{2} \|P\|^2 \sum_{i=1}^{n} \|m_i\|^2 + \frac{1}{2} \tau \lambda_{\text{max}}^2 (P) \right) \).

**Controller design**

**Theorem 1.** For fractional order nonlinear system (15), where Assumption 2 holds, the following designs can ensure that the tracking error tends to a small region of the origin and the closed-loop system is semi-global stability.

the error surfaces
\[
\left\{ \begin{array}{l}
S_1 = y - y_d \\
S_i = \hat{x}_i - x_i, i = 2, \ldots, n
\end{array} \right.
\]
(31)
the intermediate control functions
\[
\left\{ \begin{array}{l}
\beta_1 = - \left[ k_1 S_1 + \theta_1^T \varphi_1 (\hat{x}_1) + \text{sign}(S_1) \delta_1 - D^\alpha y_d \right] \\
\beta_i = - \left[ k_i S_i + S_{i-1} + \theta_i^T \varphi_i + \text{sign}(S_i) \delta_i \right]
\end{array} \right.
\]
(32)
the parameters update laws
\[
\left\{ \begin{array}{l}
D^\alpha \theta_i = \sigma_i \varphi_i (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) S_i - \rho_i \beta_i \\
D^\alpha \delta_i = r_i S_i - \eta_i \delta_i
\end{array} \right.
\]
(33)
the control input
\[
u = - \left( \left( S_{n-1} + k_n S_n + \theta_n^T \varphi_n + \text{sign}(S_n) \delta_n \right) \right)
\]
(34)
where \( \rho_i > 0, \eta_i > 0 \).

**Proof.** Step 1. the first tracking error surface is defined as
\[
S_1 = y - y_d
\]
(35)
According to (2) \( \dot{x}_2 = \dot{x}_2 + \epsilon_2 \), we can obtain \( \dot{x}_2 = \dot{x}_2 + \epsilon_2 \), and we have
\[
D^\alpha S_1 = D^\alpha x_1 - D^\alpha y_d
\]
\[
= x_2 + f_1 (\dot{x}_1) + d_1 - D^\alpha y_d
\]
\[
= \dot{x}_2 + f_1 (\dot{x}_1) + d_1 - D^\alpha y_d + \epsilon_2
\]
(36)
Define the second error surface \( S_2 \) and the output error of a fractional order filter
\[
S_2 = \dot{x}_2 - \epsilon_2
\]
(37)
Then we obtain
\[
D^\alpha S_2 = S_2 + \epsilon_2 - \beta_1
\]
(38)
Construct the second Lyapunov function:
\[ V_i = V_0 + \frac{1}{2} S_i^2 + \frac{1}{2 \sigma_1} \delta_i^2 \delta_1 + \frac{1}{2 r_1} \delta_i^2 \delta_1 \]  

(39)

From (30) and (38), we have

\[ D^a V_i \leq D^a V_0 + S_i D^a S_i + \frac{1}{\sigma_1} \tilde{\delta}_i^2 D^a \delta_i + \frac{1}{r_1} \delta_i D^a \delta_i \]

\[ \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ + S_i (S_2 + w_2 + \beta_1 + \theta_1^T \varphi(\dot{x}_i) + \delta_T \varphi(\dot{x}_i) - D^a y_d) \]

\[ + S_i e_2 + S_i (\tilde{e}_i^2 + \Delta \tilde{d}_i + \tilde{d}_i^2) - \frac{1}{\sigma_1} \tilde{\delta}_i^2 D^a \delta_i - \frac{1}{r_1} \delta_i D^a \delta_i \]

\[ \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ + S_i (S_2 + w_2 + \beta_1 + \theta_1^T \varphi(\dot{x}_i) - D^a y_d) + S_i e_2 \]

\[ + S_i (\tilde{e}_i^2 + \Delta \tilde{d}_i + \tilde{d}_i^2) + \tilde{\delta}_i^2 (\varphi_1 S_1 - \frac{1}{\sigma_1} D^a \theta_1) \]

\[ - \frac{1}{r_1} \delta_1 D^a \delta_1 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]  

(40)

Design the first intermediate control function \( \beta_1 \) with update laws

\[ \beta_1 = -k_1 S_1 + \theta_1^T \varphi_1 (\dot{x}_i) + \text{sign}(S_1 \delta_1) - D^a y_d \]

(41)

\[ D^a \beta_1 = \sigma_1 \varphi_1 (\dot{x}_i) S_1 - \rho_1 \theta_1 \]  

(42)

\[ D^a \delta_1 = r_1 |S_1| - \eta_1 \delta_1 \]  

(43)

According to Lemma 4, we have

\[ S_i e_2 \leq \frac{1}{2} |S_i|^2 + \frac{1}{2 \sigma_1} \tilde{e}_i^2 \delta_i \]  

(44)

\[ S_i w_2 \leq \frac{1}{2} |S_i|^2 + \frac{1}{2 \sigma_1} \tilde{e}_i^2 \delta_i \]  

(45)

Substituting (41), (42) and (43) into (40), we obtain

\[ D^a V_i \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ + S_i (S_2 - k_1 S_1 - \theta_1^T \varphi_1 - \text{sign}(S_1) \delta_1 + D^a y_d \]

\[ + \theta_1^T \varphi_1 - D^a y_d) + S_i w_2 + S_i e_2 + S_i (\tilde{e}_i^2 + \Delta \tilde{d}_i + \tilde{d}_i^2) \]

\[ + \tilde{\delta}_i^2 (\varphi_1 S_1 - \frac{1}{\sigma_1} D^a \theta_1) \]

\[ - \frac{1}{r_1} \delta_1 (r_1 |S_1| - \eta_1 \delta_1) \]

\[ \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ - k_1 S_1^2 + S_i S_2 - |S_1| \delta_1 + S_i w_2 + S_i e_2 \]

\[ + S_i (\tilde{e}_i^2 + \Delta \tilde{d}_i + \tilde{d}_i^2) + \frac{\rho_1}{\sigma_1} \tilde{\delta}_i^2 \delta_1 \]

\[ - \frac{\eta_1}{r_1} |S_1| \delta_1 \]  

(46)

note that \( \tilde{e}_i^2 + \Delta \tilde{d}_i + \tilde{d}_i^2 = \Delta_1 \), we can obtain \( S_i |\Delta_1| \leq |S_i| |\Delta_1| \leq |S_i| \delta_1 \) and \( |S_i| \delta_1 = |S_i| (|\delta_1| + |\delta_1|) \).

and we have

\[ D^a V_i \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ - k_1 S_1^2 + S_i S_2 - |S_1| \delta_1 + S_i w_2 + S_i e_2 \]

\[ + \frac{\rho_1}{\sigma_1} \tilde{\delta}_i^2 \delta_1 - |S_1| \delta_1 + \frac{\eta_1}{r_1} |\delta_1| \delta_1 \]

\[ \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ - k_1 S_1^2 + S_i S_2 - |S_1| \delta_1 + S_i w_2 + S_i e_2 + S_i |\delta_1| \delta_1 \]

\[ + \frac{\rho_1}{\sigma_1} \tilde{\delta}_i^2 \delta_1 - |S_1| \delta_1 + \frac{\eta_1}{r_1} |\delta_1| \delta_1 \]  

(47)

According to (44) and (45), we have

\[ D^a V_i \leq -q_0 \| e \|^2 + \frac{1}{2} \| P \delta^* \|^2 + \frac{1}{r_1} \sum_{i=1}^{n} \tilde{\delta}_i^2 \delta_i \]

\[ - \rho_1 \tilde{\delta}_i^2 \delta_1 - |S_1| \delta_1 + \frac{\eta_1}{r_1} |\delta_1| \delta_1 + |S_1|^2 + \frac{1}{2} |w_2|^2 \]  

(48)

where \( q_1 = q_0 - 1/2 \).

By using DSC technique, we can obtain the state variable \( v_2 \) as

\[ \lambda_2 D^a v_2 + v_2 = \beta_1, v_2(0) = \beta_1(0) \]  

(49)

according to (49), we have

\[ D^a w_2 = D^a v_2 - D^a \beta_1 \]

\[ = -w_2 - \beta_1 - D^a \beta_1 \]

\[ = -w_2 - \lambda_2 + B_2 \]  

(50)

where \( B_2 \) is a continuous function of variables \( S_1, S_2, w_2, \beta_1, \delta_1, \delta_2, w_3, y_d, D^a y_d, D^a (D^a y_d) \).

**Step 2.** Define the error surface \( S_3 \) and the output error of the fractional order filter

\[ S_3 = \dot{x}_3 - v_3 \]

\[ w_3 = v_3 - \beta_2 \]  

(51)

Further, we can obtain

\[ D^a S_3 = \dot{x}_3 + \theta_3^T \varphi_3 + \theta_3^T \varphi_3 + k_2 \dot{e}_1 \]

\[ + (\dot{e}_1 + \Delta \tilde{d}_1 + \tilde{d}_1 - D^a \theta_1) \]

\[ = S_3 + w_3 + \beta_2 + \theta_3^T \varphi_3 + \theta_3^T \varphi_3 + k_2 \dot{e}_1 \]

\[ + (\dot{e}_1 + \Delta \tilde{d}_1 + \tilde{d}_1 - D^a \theta_1) \]  

(52)

Construct the third Lyapunov function:

\[ V_2 = V_1 + \frac{1}{2} S_2^2 + \frac{1}{2 \sigma_2} \tilde{\theta}_2^2 + \frac{1}{2 r_2} \tilde{\delta}_2^2 + \frac{1}{2} w_2^2 \]  

(53)
we have

\[ D^a V_2 \leq D^a V_1 + S_2 D^a S_2 + \frac{1}{\alpha_2} \tilde{\theta}_2 D^a \tilde{\theta}_2 + \frac{1}{\eta_2} \delta_2 D^a \delta_2 \]

\[ + w_2 D^a w_2 \]

\[ \leq -q_2 ||e||^2 + \frac{1}{2} ||P \kappa^*||^2 - k_1 S_1 + S_1 \]

\[ + S_2(S_1 + w_3 + \beta_2 + \theta_2^T \varphi_2 + \tilde{\theta}_2^T \varphi_2 - D^a v_2) \]

\[ + k_1 S_2 \epsilon_1 + S_2(e_2^T + \Delta e_2^T + d_2^T) + \frac{p_1}{\sigma_1} \tilde{\theta}_2^T \theta_1 \]

\[ + \frac{\eta_1}{\epsilon_1} \delta_1 \delta_1 + |S_1|^2 + \frac{1}{2} |w_2|^2 + \frac{1}{2} \sum_{i=1}^n \tilde{\theta}_i^T \theta_i \]

\[ + \frac{\eta_1}{\sigma_1} \tilde{\theta}_2^T \theta_2 + \frac{1}{\eta_2} \delta_2 \delta_2 \]

\[ \leq -q_2 ||e||^2 + \frac{1}{2} ||P \kappa^*||^2 + \frac{1}{\eta_2} \delta_2 \delta_2 - k_1 S_1 \]

\[ + S_2(S_1 + S_3 + \beta_2 + \theta_2^T \varphi_2 + \tilde{\theta}_2^T \varphi_2 - D^a v_2) \]

\[ + S_2 \Delta_2 - \frac{1}{\alpha_2} \tilde{\theta}_2 D^a \theta_2 - \frac{1}{\eta_2} \delta_2 D^a \delta_2 \]

\[ - \frac{w_3^2}{\lambda_2} + B_2 w_2 + \frac{p_1}{\sigma_1} \tilde{\theta}_2^T \theta_1 + \frac{\eta_1}{\epsilon_1} \delta_1 \delta_1 + |S_1|^2 + |S_2|^2 \]

\[ + \frac{1}{2} |w_2|^2 + \frac{1}{2} |w_3|^2 \]

(54)

where \( q_2 = q_1 - k_2^2/2 \), and \( k_1, k_2, \epsilon_1 \leq |S_2|^2/2 |e|^2/2 \).

From (49) we can obtain

\[ D^a V_2 \leq -q_2 ||e||^2 + \frac{1}{2} ||P \kappa^*||^2 + \frac{1}{\eta_2} \delta_2 \delta_2 - k_1 S_1 \]

\[ + S_2 \left( S_1 + S_3 + \beta_2 + \theta_2^T \varphi_2 + \tilde{\theta}_2^T \varphi_2 - \frac{\beta_1}{\alpha_2} \right) \]

\[ + |S_2| \left( \delta_2 + \delta_2 - \frac{1}{\alpha_2} \tilde{\theta}_2 D^a \theta_2 - \frac{1}{\eta_2} \delta_2 D^a \delta_2 \right) \]

\[ - \frac{w_3^2}{\lambda_2} + B_2 w_2 + \frac{p_1}{\sigma_1} \tilde{\theta}_2^T \theta_1 + \frac{\eta_1}{\epsilon_1} \delta_1 \delta_1 + |S_1|^2 \]

\[ + |S_2|^2 + \frac{1}{2} |w_2|^2 + \frac{1}{2} |w_3|^2 \]

(55)

where \( e_2^T + \Delta e_2^T + d_2^T = \Delta_2 \), \( S_2 \Delta_2 \leq |S_2| \Delta_2 \leq \|S_2||\Delta_2| \), and

\[ |S_2| \Delta_2 \leq |S_2| \delta_2 \leq |S_2| (\delta_2 + \delta_2) \]

Choose the first intermediate control function \( \beta_2 \) with update laws,

\[ \beta_2 = - \left[ k_2 S_2 + S_1 + \theta_2^T \varphi_2 + \text{sign}(S_2) \delta_2 \right] \]

\[ D^a \theta_2 = \sigma_2 \sigma_2^T (\hat{\xi}_1, \hat{\xi}_2) S_2 - \rho_2 \theta_2 \]

\[ D^a \delta_2 = \epsilon_2 S_2 - \eta_2 \delta_2 \]

(56)

(57)

(58)

Substituting equations (56)–(58) into (55), we obtain

\[ D^a V_2 \]

\[ \leq -q_2 ||e||^2 + \frac{1}{2} ||P \kappa^*||^2 + \frac{1}{\eta_2} \delta_2 \delta_2 - k_1 S_1 \]

\[ + S_2(S_1 + S_3 - k_2 S_2 - \theta_2^T \varphi_2 - \text{sign}(S_2) \delta_2 \]

\[ + \frac{\beta_1}{\alpha_2} + \theta_2^T \varphi_2 + \tilde{\theta}_2^T \varphi_2 - \frac{\beta_1 - v_2}{\alpha_2} + |S_2| (\delta_2 + \delta_2) \]

\[ - \frac{1}{\alpha_2} \tilde{\theta}_2^T (\sigma_2 \varphi_2 S_2 - \rho_2 \theta_2) - \frac{1}{\eta_2} \delta_2 (\sigma_2 \varphi_2 S_2 - \eta_2 \delta_2) \]

\[ + \frac{p_1}{\sigma_1} \tilde{\theta}_2^T \theta_1 + \frac{\eta_1}{\epsilon_1} \delta_1 \delta_1 + |S_1|^2 + |S_2|^2 + \frac{1}{2} |w_2|^2 \]

\[ + \frac{1}{2} |w_3|^2 - \frac{w_3^2}{\lambda_2} + B_2 w_2 \]

(59)

By employing Young’s inequality, we have

\[ w_2 B_2 \leq \frac{w_2^2 B_2^2}{2 \mu} + 2 \mu \].

Then, we have

\[ D^a V_2 \]

\[ \leq -q_2 ||e||^2 + \frac{1}{2} ||P \kappa^*||^2 + \frac{1}{\eta_2} \delta_2 \delta_2 - k_1 S_1 \]

\[ + S_2(S_1 + S_3 - k_2 S_2 - \theta_2^T \varphi_2 - \text{sign}(S_2) \delta_2 \]

\[ + \frac{\beta_1}{\alpha_2} + \theta_2^T \varphi_2 + \tilde{\theta}_2^T \varphi_2 - \frac{\beta_1 - v_2}{\alpha_2} + |S_2| (\delta_2 + \delta_2) \]

\[ + \frac{p_1}{\sigma_1} \tilde{\theta}_2^T \theta_1 + \frac{\eta_1}{\epsilon_1} \delta_1 \delta_1 + |S_1|^2 + |S_2|^2 + \frac{1}{2} |w_2|^2 \]

\[ + \frac{1}{2} |w_3|^2 - \frac{w_3^2}{\lambda_2} + B_2 w_2 \]

(60)

Step 1. According to Theorem 1, design the error surface \( S_{i+1} \) and the output error of a fractional order filter.

\[ S_{i+1} = \hat{\xi}_{i+1} - v_{i+1} \]

\[ w_{i+1} = v_{i+1} - \beta_i \]

(61)

The intermediate control function \( \beta_i \) and the update laws are designed as

\[ \beta_i = - \left[ \frac{S_{i-1} + k_1 S_i + \theta_i^T \varphi_i + \text{sign}(S_i) \delta_i}{\beta_{i-1} - v_i} \right] \]

(62)
Employing DSC technique, \( v_i \) can be obtain as
\[
\lambda_i D^\alpha v_i + v_i = \beta_{j_i-1}, \quad v_i(0) = \beta_{j_i-1}(0)
\]
by (65), we have
\[
D^\alpha w_i = \frac{w_i}{\lambda_i} + B_i
\]
where \( B_i = -D^\alpha \alpha_{j_i-1} \).

Consider Lyapunov function:
\[
V_i = V_{i-1} + \frac{1}{2} S_i^2 + \frac{1}{2} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
From (54) and (57), we obtain
\[
D^\alpha V_i \leq -q_i \|e\|^2 - \sum_{i=1}^{n} k_i S_i^2 + S_{i-1}S_i + \frac{1}{\tau} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
where \( q_i = k_i - k_{i-1}^2/2 \) and \( w_i B_i \leq \frac{w_i^2}{2} \mu + 2 \mu \)

Step n. Define the error surface \( S_n \) and the output error of a fractional order filter
\[
S_n = \hat{x}_n - x_n, \quad w_n = x_n - \beta_{n-1}
\]
The Lyapunov function is chosen as
\[
V_n = V_{n-1} + \frac{1}{2} S_n^2 + \frac{1}{2} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
According to (68), we obtain
\[
D^\alpha V_n \leq -q_n \|e\|^2 - \sum_{i=1}^{n-1} k_i S_i^2 + S_{n-1} S_n + \frac{1}{\tau} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
Design control law \( u(t) \) with update laws,
\[
u = - \left( \frac{S_{n-1} + k_n S_n + \delta_n^T \phi_n + \text{sign}(S_n) \delta_n}{\lambda_n} - \frac{\beta_n - v_n}{\lambda_n} \right)
\]
where \( q_n = q_n - k_{i-1}^2/2 \).

From (71) - (74), we obtain
\[
D^\alpha V_n \leq -q_n \|e\|^2 - \sum_{i=1}^{n} k_i S_i^2 + S_{n-1} S_n + \frac{1}{\tau} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
where
\[
D^\alpha \theta_n = \sigma_n \phi_n S_n - \rho_n \theta_n
\]
\[
D^\alpha \delta_n = r_n |S_n| - \eta_n \delta_n
\]
From (71) - (74), we obtain
\[
D^\alpha V_n \leq -q_n \|e\|^2 - \sum_{i=1}^{n} k_i S_i^2 + S_{n-1} S_n + \frac{1}{\tau} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
where
\[
D^\alpha \theta_n = \sigma_n \phi_n S_n - \rho_n \theta_n
\]
\[
D^\alpha \delta_n = r_n |S_n| - \eta_n \delta_n
\]
From (71) - (74), we obtain
\[
D^\alpha V_n \leq -q_n \|e\|^2 - \sum_{i=1}^{n} k_i S_i^2 + S_{n-1} S_n + \frac{1}{\tau} \sum_{i=1}^{n} \delta_i^2 + \frac{1}{2} \sum_{i=1}^{n} w_i^2
\]
By using DSC technique, we can obtain
\[
\lambda_n D^n v_n + v_n = \beta_{n-1}, \quad v_n(0) = \beta_{n-1}(0)
\] (76)
and
\[
D^n w_n = D^n v_n - D^n \beta_{n-1}
= - \frac{v_n - \beta_{n-1}}{\lambda_n} - D^n \beta_{n-1}
= - \frac{w_n}{\lambda_n} + B_n
\] (77)
where \(B_n = - D^n \alpha_{n-1}\).

According to \(w_n B_n \leq \frac{w_2 B_2^2}{2\mu} + 2\mu\), we can have
\[
D^n V_n \leq - q_n \|e\|^2 - \sum_{i=1}^n k_i S_i^2 + \frac{1}{\tau} \sum_{i=1}^n \tilde{\theta}^T \theta_i + \sum_{i=1}^n \frac{\eta_i}{\tau} \tilde{\theta}^T \theta_i
+ \sum_{i=1}^n \frac{\eta_i}{\tau} \delta_{i} \theta_{i} + \sum_{i=1}^n |S_i|^2 + \frac{1}{2} \|P \kappa^*\|^2
+ w_n \left(- \frac{w_n}{\lambda_n} + B_n\right) + \sum_{i=1}^{n-1} \left(\frac{B_i^2}{\lambda_n} - \frac{1}{\lambda_i} + \frac{1}{2}\right) w_i^2
+ \frac{1}{2} w_n^2 + 2\mu(n-2)
\leq - q_n \|e\|^2 - \sum_{i=1}^n k_i S_i^2 + \frac{1}{\tau} \sum_{i=1}^n \tilde{\theta}^T \theta_i + \sum_{i=1}^n \frac{\eta_i}{\tau} \tilde{\theta}^T \theta_i
+ \sum_{i=1}^n \frac{\eta_i}{\tau} \delta_{i} \theta_{i} + \sum_{i=1}^n |S_i|^2 + \frac{1}{2} \|P \kappa^*\|^2
+ \sum_{i=1}^n \left(\frac{B_i^2}{\lambda_n} - \frac{1}{\lambda_i} + \frac{1}{2}\right) w_i^2 + 2\mu(n-1)
\] (78)

According to Lemma 4, we have
\[
\tilde{\theta}^T \theta_i \leq - \frac{1}{2} \tilde{\theta}^T \theta_i + \frac{1}{2} \tilde{\theta}^T \theta_i
\] (79)
Substituting (79) into (78), we obtain
\[
D^n V_n \leq - q_n \|e\|^2 - \sum_{i=1}^n k_i S_i^2 - \frac{1}{2} \sum_{i=1}^n \tilde{\theta}^T \theta_i
+ \frac{1}{2} \sum_{i=1}^n \frac{\eta_i}{\tau} \tilde{\theta}^T \theta_i
+ \sum_{i=1}^n \frac{\eta_i}{\tau} \delta_{i} \theta_{i} + \sum_{i=1}^n |S_i|^2 + \frac{1}{2} \|P \kappa^*\|^2
+ \sum_{i=1}^n \left(\frac{B_i^2}{\lambda_n} - \frac{1}{\lambda_i} + \frac{1}{2}\right) w_i^2 + 2\mu(n-1)
\leq - q_n \|e\|^2 - \sum_{i=1}^n k_i S_i^2 + \frac{1}{2} \sum_{i=1}^n \frac{\rho_i}{\sigma_i} \tilde{\theta}^T \theta_i
+ \sum_{i=1}^n \frac{\eta_i}{\tau} \delta_{i} \theta_{i} + \sum_{i=1}^n |S_i|^2 + \frac{1}{2} \|P \kappa^*\|^2
+ \sum_{i=1}^n \left(\frac{B_i^2}{\lambda_n} - \frac{1}{\lambda_i} + \frac{1}{2}\right) w_i^2 + 2\mu(n-1) + \frac{\eta_i}{\tau} \delta_{i} \theta_{i}
\] (80)

Denote
\[
\xi = \frac{1}{2} \sum_{i=1}^n \left(\frac{\rho_i}{\sigma_i} \tilde{\theta}^T \theta_i + \frac{\eta_i}{\tau} \delta_{i} \theta_{i} + 2|S_i|^2\right)
+ 2\mu(n-1) + \frac{1}{2} \|P \kappa^*\|^2
\]
\[
C = \min \left\{2q_n - 1, \lambda_{\min}(P), \lambda_{\max}(P), 2k_i, \eta_i, \left(\frac{\rho_i}{\lambda_i} - \frac{B_i^2}{2\mu} - \frac{1}{2}\right)\right\}
\]

Then (80) becomes
\[
D^n V_n \leq - CV_n + \xi
\] (83)
According to (83), we can obtain
\[
D^n V_n + M(t) = - CV_n + \xi
\] (84)
where \(M(t) \equiv 0\).

According to Lemma 5, we can obtain
\[
V_n \leq V(0) E_{\alpha}( - C \alpha^\alpha ) + \frac{\xi \mu}{C}.
\] (85)

Then, we have
\[
\lim_{t \to \infty} |V_n(t)| \leq \frac{\xi \mu}{C}
\] (86)
Since \(\frac{1}{2} |S_1(t)|^2 \leq V_n(t)\), which yields that
\[
\lim_{t \to \infty} |S_1(t)| \leq \sqrt{\frac{2\xi \mu}{C}}
\] (87)

This means that if the parameters are suitable, \(\|S_1(t)\|\) can converge to an arbitrary small region in \(t \in (k_0, \infty)\). And it is easy to see that all signals will remain bounded in a closed-loop system.

**Simulations**

In this section, we use the following two examples to verify the validity of the proposed method. Figure 1 shows the block diagram of the designed control system.

**Example I**

Consider a fractional order nonlinear system:
\[
\begin{cases}
D^\alpha x_1 = x_2 + f_1^\alpha \left( x_1(t) \right) + d_1^\alpha(t)
D^\alpha x_2 = x_1 + f_2^\alpha \left( x_1(t) \right) + d_2^\alpha(t)
\end{cases}
\] (88)
where \(\alpha = 0.98\), \(f_1^\alpha = 0\), \(f_2^\alpha = x_1(t) - 0.25x_2(t) - x_3^2(t)\), \(d_1^\alpha = 2x_1(t) - 0.25x_2(t) - 0.5x_3^2(t)\), \(d_2^\alpha = 4x_1(t) - 0.5x_2(t) - x_3^2(t)\), \(d_1^\alpha = 0\), \(d_2^\alpha = 0.5 \sin(t)\) and \(y_d = \sin t\) is defined as the reference signal. The initial conditions are \(x(0) = [0.1, 0.1]^T\), \(\dot{x}(0) = [0.2, 0.2]^T\), \(\theta_i(0) = [0.01, ..., 0.01]^T\), \(\delta_i(0) = 0.01\) and choose design parameters as \(k_1 = 30\), \(k_2 = 20\), \(\sigma_1 = 1\), \(\rho_1 = 40\), \(\eta_1 = 1\), \(\eta_2 = 20\), select the gain matrix as \(K = [100, 4000]^T\).
Figures 2 to 7 show the simulation results of Example 1. Figure 2 displays the trajectories of the fractional order Duffing-Holmes system (88) with $\alpha = 0.98$. Figure 3 shows that the tracking trajectories of $x_1$ and the comparison trajectories with traditional backstepping method. Figure 4 gives the trajectories of estimation of $x_1$ and its estimation error. Figure 5 gives the trajectories of the $x_2$ signal tracking and its estimation. Figure 6 displays the trajectory of the control
input. Figure 7 shows the trajectories of \( f(x) \) and the switching signal \( s(t) \). It can be seen from Figure 6 that \( f(x) \) changes randomly between \( f^1(x) \). From the Example 1 simulation results, the proposed method can obtain good control performance for fractional order systems containing switched unknown functions and external disturbances.

**Example 2**

Consider the fractional order strict-feedback nonlinear system:

\[
\begin{align*}
D^\alpha x_1(t) &= x_2(t) \\
D^\alpha x_2(t) &= x_3(t) \\
D^\alpha x_3(t) &= u + f^2(X(t)) + d^k(t) \\
y &= x_1(t)
\end{align*}
\]  

where the system order is \( \alpha = 0.97 \), the unknown function is \( f^1 = 5.5x_1(t) - 3.5x_2(t) - 0.8x_3(t) - x_3^2(t) \), \( f^2 = 2x_1(t) - x_2(t) - x_3(t) - x_3^2(t) \), \( d^k(t) = 0.5\sin(t) \) is external disturbance, \( y_d = \sin t \) is defined as the reference signal.

Choose the initial conditions of the system as \( x(0) = [0.1, 0.1, 0.1]^T \), the observer initial conditions are chosen as \( \hat{x}(0) = [0.2, 0.2, 0.2]^T \) and \( \theta(0) = [0.01, ..., 0.01]^T \), \( \delta(0) = 0.1 \). Choose design parameters as \( k_1 = 20, k_2 = 20, k_3 = 40, \alpha = 5, p = 40, r = 2, \eta = 40 \) select the gain matrix as \( K = [15, 150, 1500]^T \).

Figures 8 to 14 show the simulation results of Example 2. Figure 8 displays the trajectory of the fractional order Arneodo system (89) with \( \alpha = 0.97 \). Figure 9 shows that the tracking trajectories of \( x_1 \) and the comparison trajectories with traditional backstepping methods, it can be clearly seen from the figure that the proposed method can converge faster and have a
smaller tracking error. It can be seen from Figure 10 that the designed observer has good estimation performance, and its estimation error can converge. Figure 11 gives the estimation trajectories of $x_2$ and $x_3$, it can obtain good estimation performance for $x_2$ and $x_3$. Figure 12 shows that the tracking trajectories of $x_2$ and $x_3$. Figure 13 displays the trajectory of the control input. Figure 14 shows the trajectories of $f(x)$ and the switching signal $\sigma(t)$ of example 2.

From Example 1 and Example 2, we can conclude that the proposed controller can stabilize the switched fractional order systems of the equation (15), and the control performance is satisfactory. The designed control law can ensure the convergence of the tracking errors and the boundedness of all signals in the system.

Conclusions

This paper proposes an adaptive neural network backstepping sliding mode controller based on DSC to
stabilize the fractional order strict-feedback nonlinear systems. The fractional order nonlinear system under consideration contains arbitrary switchings, unmeasured states, the unknown nonlinear functions and uncertain disturbances. The unknown nonlinear functions and uncertain disturbances are approximated by the radial basis function neural network and an observer is designed for state estimation of the fractional order systems. Fractional order DSC technology is used to avoid “explosion of complexity” and obtain fractional derivatives for virtual control laws continuously. The stability of the closed-loop system is ensured by the constructed Lyapunov functions. Examples and simulation results show that the tracking error can quickly converge to a small region of the origin by the proposed adaptive fractional order control method, and the semi-global stability can be ensured in the closed-loop system. An interesting future topic involves fractional order systems Synchronization control, multi-agent systems control and fractional order model of DC model.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

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