Black Hole Entropy and Gravity Cutoff

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Abstract

We study the black hole entropy as entanglement entropy and propose a resolution to the species puzzle. This resolution comes out naturally due to the fact that in the presence of $N$ species the universal gravitational cutoff is $\Lambda = M_{\text{Planck}}/\sqrt{N}$, as opposed to $M_{\text{Planck}}$. We demonstrate consistency of our solution by showing the equality of the two entropies in explicit examples in which the relation between $M_{\text{Planck}}$ and $\Lambda$ is known from the fundamental theory.
1 Introduction

One of the most important unresolved theoretical problems is the existence of black holes which quantum mechanically possess entropy even though there are no obvious degrees of freedom in the theory that would constitute this entropy statistical mechanically. Despite the numerous attempts, including successful ones proposed in string theory, it remains a mystery what produces the huge Bekenstein-Hawking (BH) entropy \[1\], \[2\]

\[ S_{BH} = M_{\text{Planck}}^2 A(\Sigma), \]  

where \( M_{\text{Planck}}^2 = 1/(4G_N) \), of an uncharged macroscopic 4-dimensional black hole. Among others, the most intriguing approach to attack this problem is to use entanglement entropy. Obvious advantage of this approach is its universality, the mechanism of generation of the entropy is the same for all possible types of black holes, and its geometrical nature, the fact that the BH entropy is geometric finds a simple explanation. The identification of the BH entropy with entanglement entropy, however, has a well-known difficulty: the black hole entropy is universal finite quantity while the entanglement entropy is quadratically-sensitive to the cutoff of the theory (\( \Lambda \)) and grows with the number of particle species \( N \). In this note we suggest that the species problem results from the use of the wrong cutoff. Namely, the mistake is in ignoring the \( N \)-dependence of the cutoff of the theory. The latter dependence follows from completely independent consistency arguments, which show that there is a strict relation between \( \Lambda \), \( N \) and \( M_{\text{Planck}} \), so that with growing number of species, \( \Lambda \) decreases relative to the Planck mass by \( \sqrt{N} \). We show, that when this is taken into the account, the puzzle of species disappears and the two entropies match to leading order.

2 Entanglement Entropy of Black Holes

Entanglement entropy \[3\] is defined for a system divided by surface \( \Sigma \) into two subsystems \( A \) and \( B \). Typically, the total system is considered to be in a pure state (for instance, in a ground state) described by wave function \( |\Psi\rangle \) so that one defines \( \rho(A, B) = |\Psi\rangle\langle\Psi| \). If one does not have access to the modes in one of the subsystems, say \( B \), then one should trace over these modes and this situation is described by the reduced density matrix \( \rho_A = \text{Tr}_B \rho(A, B) \). Thus, provided we have access only to a part of the system then entanglement entropy defined as \( S_A = -\text{Tr}_A \rho_A \log \rho_A \) gives us a measure for the lack of information about the state of the total system. We could have traced over modes in subsystem A and get density matrix \( \rho_B \) and respective entropy \( S_B \). If the total system was in a pure state then \( S_A = S_B \). This property of entanglement entropy means that if the entropy is non-zero then it should not depend on
extensive quantities (such as volume) which characterize \( A \) or \( B \) but only on the surface \( \Sigma \) that separates the two subsystems.

On the other hand, this entropy is non-vanishing because there are correlations between the subsystems. Entanglement entropy, thus, can be also viewed as some measure for these correlations. In local theories the correlations are short-distance and, hence, the entropy is expected to be determined by the geometry of the boundary dividing the two subsystems. It is clear that in order to regularize these correlations one has to introduce a short-distance cut-off \( \epsilon = 1/\Lambda \) so that the entropy essentially depends on the cut-off.

The natural (if not the only one possible) way to prevent the access to a part of the system inside a closed surface \( \Sigma \) is to put the system into a black hole space-time so that \( \Sigma \) would be a black hole horizon. In this situation the access to the inside region of \( \Sigma \) is impossible in principle, so that the notion of entanglement entropy is well suited to describe the lack of information in black hole. In order to calculate the entropy then we have to start with a pure state described by a wave function of black hole (for a construction of this function see [10]) \( \Psi(\phi_{in}, \phi_{out}) \) that is a functional of the modes inside horizon \( \Sigma \) and modes \( \phi_{out} \) outside \( \Sigma \). No observer has an access to the complete set of modes. By tracing over modes \( \phi_{in} \) we end up with a density matrix and the corresponding entropy \( S_{\text{ent}} \) depends on the geometry of \( \Sigma \). Again the short-distance correlations are present in the system that should be cured by an UV cutoff \( \epsilon = 1/\Lambda \). As far as the technicalities are concerned [11], the calculation of entanglement entropy of black hole horizon goes along the same steps as in the flat space-time. The only essential difference is that the surface in the flat spacetime may have non-trivial extrinsic geometry while the extrinsic curvature vanishes for black hole horizons. This restricts the geometric structures which may appear in the entropy. To leading order entanglement entropy is proportional to the area \( A(\Sigma) \) of \( \Sigma \). If there are \( N \) fields available, then each of them equally contributes to the entropy so that the entanglement entropy of a black hole in four dimensions is given by

\[
S_{\text{ent}} = N\Lambda^2 A(\Sigma) .
\]

The fact that the entanglement entropy is proportional to the area similarly to the Bekenstein-Hawking (BH) entropy makes it an interesting candidate for the statistical origin of the BH entropy. A would be natural identification \( \Lambda \sim M_{\text{Planck}} \) however faces a well-known problem of species (see [12] for a review): entanglement entropy depends on the number \( N \) of the species while the BH entropy is obviously universal and should not depend on \( N \). We now wish to show, that in order to resolve this puzzle we have to invoke the correct gravity cutoff in the presence of \( N \) species [4].
3 Gravity Cutoff

It was proposed in [4] that in a theory with \( N \) species the self-consistency of large-distance black hole physics implies the existence of the following new scale

\[
\Lambda = \frac{M_{\text{Planck}}}{\sqrt{N}}. \tag{3}
\]

The physical meaning of this scale is twofold. First [4], it sets the bound on the masses of the particle species. The same scale sets the bound on the gravitational cutoff of the theory [5]. We shall now briefly reproduce some of the key consistency arguments from the black hole physics.

Following [4], let us consider a theory with \( N \) particle species, \( \Phi_j \quad j = 1, 2, \ldots N \). For simplicity, we shall assume the species to be stable, and to be having equal masses, which we denote by \( M \). It is useful to monitor the “personalities” of the individual species by prescribing them parities under some gauged \( Z_2 \)-symmetries, one per each species. Under a given \( Z_2^{(j)} \)-symmetry, only a corresponding species changes the sign, \( \Phi_j \rightarrow -\Phi_j \), whereas the remaining \( N - 1 \) species stay invariant. We then perform the following thought experiment. Imagine an arbitrarily large classical black hole, and let us endow it with a maximal possible discrete charge. This can be done by throwing in the black hole one particle per each species. In this way, the resulting black hole will carry \( N \) units of different \( Z_2^{(j)} \)-charges. Since the symmetries are gauged, this charge cannot be lost and must be returned back after the black hole evaporation. (The discrete gauge charges can be continuously monitored at infinity, by Aharonov-Bohm type experiments [6]). However, irrespective of the original size, the black hole can only start giving back the charge after its Hawking temperature becomes comparable to the particle masses, \( T_H \sim M \). (Emission of particles before this moment is Boltzmann-suppressed and can only correct our results by factor \( \sim \log N \)). At this point, the black hole mass is \( M_{BH} \sim M_{\text{Planck}}^2/M \).

After this moment, we can use the conservation of energy and immediately derive that the maximal number of quanta that can be emitted by the black hole is \( N_{\text{max}} \lesssim M_{\text{Planck}}^2/M^2 \). This proves that the bound on particle masses is set by \( \Lambda \), defined by (3).

Having this bound on the particle masses already indicates that the cutoff of the theory also should be somewhere around the same scale \( \Lambda \). For example, assume that the species are self-interacting scalars. If cutoff were much higher than the scale \( \Lambda \), it would be hard to reconcile the lightness of the species versus the cutoff, in the view of quantum corrections to their masses that are quadratically sensitive to the cutoff. And indeed, the cutoff is at the scale \( \Lambda \).

In order to see this, we can again use as a tool the well-known properties of the black holes [5]. Let us assume the opposite, that the gravity cutoff is much higher than the scale \( \Lambda \).
We shall now see that with this assumption, we shall inevitably run into an inconsistency with the well-established macro black hole physics. If the true gravitational cutoff is much higher than the scale $\Lambda$, then the black holes of size $\Lambda^{-1}$ must behave as normal Schwarzschild black holes. Thus, consider such a black hole. In the normal case, with only few species, the lifetime of a black hole of size $\Lambda^{-1}$ would be

$$\tau_{BH} \sim \frac{M_{\text{Planck}}^2}{\Lambda^3}. \quad (4)$$

For $\Lambda \ll M_{\text{Planck}}$, this lifetime would be perfectly consistent with our assumption that such a black hole is a quasi-classical object with well-defined (slowly-changing) Hawking temperature, $T_H \sim \Lambda$. However, because of $N$ species, the evaporation rate is enhanced by factor of $N$. Taking into the account (3), this enhancement reduces the black hole lifetime to

$$\tau_{BH} \sim \Lambda^{-1}. \quad (5)$$

Thus, the black hole has a lifetime of order of its inverse temperature! This is a clear indication that such a black hole cannot be regarded as a quasi-classical state, with a well-defined temperature. Thus, we are inevitably lead into the contradiction with our initial assumption that black holes of size $\Lambda^{-1}$ are normal Schwarzschild black holes. The only resolution of this inconsistency is that the gravity cutoff is $\Lambda$.

Finally, let us note that the above presented non-perturbative arguments exactly match the perturbative ones [5, 7, 8], which also indicate that in theory with $N$ species gravitational cutoff must be set by $\Lambda$, and this is the scale above which the low energy perturbation theory breaks down.[1]

Thus, there are number of different perturbative and non-perturbative arguments for the validity of the relation (3), all of which will not be reproduced here. For more detailed discussions the reader is referred to the above mentioned papers. The most relevant point for our present subject is the fact that the scale $\Lambda$ is the gravitational cutoff.

Given the relation (3), our strategy for the resolution of the species puzzle is clear. Since the scale (3) is the right UV cutoff, we should use $\Lambda$ and not $M_{\text{Planck}}$ in the entanglement entropy calculation. Then the disturbing dependence of the entanglement entropy on the number of species disappears and the entropy (2) precisely agrees with the BH entropy (1).

[1] In [7] this fact was used to explain largeness of the four-dimensional Planck scale relative to the fundamental high-dimensional one, in theory where $N$ four-dimensional particle species are localized on a 3-brane embedded in extra space.
4 Explicit Examples

4.1 Entropy of High-Dimensional Black Holes

We now wish to demonstrate the consistency of our solution on explicit examples, in which the relation (3) is fixed by the fundamental theory. We then demonstrate that in such cases, the two entropies are automatically equal, despite the presence of the arbitrarily large number of species.

As such example, consider $4+n$ dimensional theory with $n$ space dimensions compactified on $n$-torus, and 4 non-compact dimensions forming Minkowskian geometry. Without affecting any of our results, for simplicity, we shall set all the radii being equal to $R$. We shall choose the radius $R$ to be much larger than the fundamental Planck length $\Lambda^{-1}$, and otherwise keep it as a free parameter, which can be consistently taken to infinity. In this limit one recovers a $4+n$ dimensional Minkowski space. The fundamental high-dimensional theory has one dimensionfull parameter, the $4+n$-dimensional Planck mass $\Lambda$, which is the cutoff of the theory. We shall assume that the only species in the theory is a $4+n$-dimensional graviton. From the four-dimensional point of view it is a theory of the tower of spin-2 Kaluza-Klein species. For any $R$, the relation between the four-dimensional Planck mass and the cutoff of the theory is

$$M_{\text{Planck}}^2 = \Lambda^2 (RA)^n$$

(6)

Notice that the factor $(RA)^n$ measures the number of KK species

$$N = (RA)^n.$$  

(7)

Thus, as already noted in [4, 5], the relation (6) is a particular example of relation (3) in which $N$ has to be understood as the number of KK species.

Let us now prove that a well known Bekenstein-Hawking entropy of a high-dimensional black hole, is correctly reproduced by the entanglement entropy of $N$ KK species. For this consider a high dimensional black hole of gravitational radius $r_g$.

First we consider a small black hole for which $R \gg r_g \gg \Lambda^{-1}$. In this regime, on one hand black hole is classical, and on the other hand the effects of compactification can be ignored on the near-horizon geometry. The black hole horizon is $(2+n)$-sphere of radius $r_g$ in this case. We can then use the well known generalization of Beckenstein-Hawking entropy for a high dimensional black hole [13]. Not surprisingly, this entropy is given by the black hole area in fundamental Planck units

$$S_{\text{BH}} = (\Lambda r_g)^{2+n}.$$  

(8)

Note that this is entropy of the black hole in the $(n+2)$-dimensional theory. Let us now show that, from 4-dimensional perspective, this equation can be understood as entanglement entropy...
of KK species. We can do this in two ways, working with species that are either coordinate
or momentum eigenstates in extra dimensions. Conventional KK expansion in the flat space is
done in eigenstates of high-dimensional momentum operator, which coincides with eigenstates
of four-dimensional mass operator. In accordance with the usual uncertainty relation, the wave-
functions of KK states are plane waves $\Psi_\vec{p} = e^{i\vec{p}_{extra}\vec{Y}_{extra}}$ in the extra coordinate, forming a complete set of functions. Cutting of this infinite tower by $P_{extra} \leq \Lambda$, we get $N$ KK, species

By forming appropriate orthogonal superpositions of the momentum eigenstates, we could
form a complete set of coordinate eigenstates, each being localized at one particular point in
extra dimensions, $\psi_{\vec{Y}} = \delta^n(\vec{Y} - \vec{Y}_0)$ However, since we are limited by the cutoff in momentum space, the delta functions in position space will be smeared over a distance $\sim \Lambda^{-1}$, and again we get $N$ coordinate eigenstates, localized at $\Lambda^{-1}$ distance apart. For each of these localized states we can compute the entanglement entropy. For each state what will matter is the radial (in extra coordinate) distance from its localization site to the center of the black hole. In the other words, for each state localized on a 3-dimensional surface in $3 + n$ dimensional space, the black hole horizon will cut out a 2-dimensional sphere of radius $r$, where $0 \leq r \leq r_g$. Suppressing the factors of order one, the entanglement entropy for each such state will be

$$S_{ind} = (r\Lambda)^2. \quad (9)$$

Integrating this over all possible localization sites, we obviously get

$$S_{ent} = (r_g\Lambda)^2 N(r_g) = (r_g\Lambda)^{n+2}, \quad (10)$$

where $N(r_g) = (r_gN)^n$ is the number of distinct species localized in the extra $n$ dimensions
that see the black hole horizon of radius $r_g$. This equation correctly reproduces $[3]$. Despite
the existence of arbitrarily large number of species, no puzzle appears, since $N$ dependence is automatically taken care of by the relation (6). Had we incorrectly used $M_{\text{Planck}}$ as a cutoff we would end up with the species puzzle.

We can do the same computation using momentum eigenstates. Each KK species sees the in
four-dimensions the cut-out sphere of surface $r_g^2$, which is simply a four-dimensional projection
of the high-dimensional black hole horizon. However, because the wave-function of each KK is
spread out in extra dimension, we have to take into the account the intersection probability.
Since the species are the uniform plane waves in $n$ extra dimensions, whereas the cross-section
of the black hole is $n$ dimensional sphere of volume $r_g^n$, the intersection probability is equal to

$$\frac{r_g^n}{R^n}. \quad (11)$$

Thus the individual contributions to the entanglement entropies are equal to

$$S_{KK} = (r_g\Lambda)^2 \left(\frac{r_g}{R}\right)^n. \quad (11)$$
Summing this over all the KK states, with total number $N$ given by (7), we get exactly (8)

$$S_{\text{ent}} = \sum_{KK} S_{KK} = NS_{KK} = S_{BH}. \quad (12)$$

Consider now the case of large black hole $r_g \gg R$. In this case the black hole horizon is a product of 2-sphere of radius $r_g$ and $n$-dimensional torus of size $R$. From the point of view of 4-dimensional gravity this black hole has entropy

$$S_{BH}^{(4)} = (M_{\text{Planck}} r_g)^2 \quad (13)$$

which, by means of relation (6), is identical to the Bekenstein-Hawking entropy in $(4+n)$-dimensional theory

$$S_{BH}^{(4+n)} = \Lambda^{n+2} r_g^2 R^n, \quad (14)$$

where $4\pi r_g^2 R^n$ is the area of $(4+n)$-dimensional horizon.

The calculation of the entanglement entropy remains the same. The entropy of a single KK state (in the coordinate representation) has entropy

$$S_{\text{ind}} = (r_g \Lambda)^2. \quad (15)$$

The number of states in the case when $r_g \gg R$ does not depend on $r_g$ and is equal to $N = (R \Lambda)^n$ (7), the full number of KK species. Summing over all species we get

$$S_{\text{ent}} = (r_g \Lambda)^2 N = (r_g \Lambda)^2 (R \Lambda)^n \quad (16)$$

which exactly agrees both with (13) and (14).

That the entanglement entropy calculation gives correct entropy in two limiting cases ($R \gg r_g$ and $r_g \gg R$) makes us believe that this method should work also in the intermediate regime although to do the computation one should know the concrete profile of the black hole horizon along the extra dimensions.

### 4.2 AdS/CFT Example

To further illustrate our point and provide the reader with another explicit example we consider a 3-brane in a $Z_2$ configuration in anti-de Sitter space-time. This is the Randall-Sundrum set-up, in the framework of the AdS/CFT correspondence it has a description in terms of CFT on the brane coupled to gravity at a UV cutoff [14]. If the brane is placed at the distance $\rho = \epsilon^2$ from the Anti-de Sitter boundary, one obtains that there is a dynamical gravity induced on the brane with the induced Newton constant $1/G_N = 2N^2/(\pi \epsilon^2)$. According to the AdS/CFT
dictionary, in the theory on the brane $\Lambda = 1/\sqrt{2\pi \epsilon}$ has the meaning of the UV cut-off. Thus, one finds that $\Lambda = M_{\text{Planck}}/N$, in agreement with (3). We remind that the quantum field theory defined on the brane is the super-conformal $SU(N)$ gauge theory, so that in the large $N$ limit $N^2$ represents the number of species. Thus, the 3-brane in the anti-de Sitter space-time gives us an example when the relation (3) holds automatically. Consider now a black hole on the 3-brane and compute its entanglement entropy [15] due to the CFT. To leading order one finds

$$S_{\text{ent}} = \frac{N^2}{4\pi \epsilon^2} A(\Sigma) = N^2 \Lambda^2 A(\Sigma),$$

which is exactly the Bekenstein-Hawking entropy of the black hole provided one expresses the UV cut-off in terms of the induced Planck scale $M_{\text{Planck}} = N\Lambda$.

The fact that the BH entropy is correctly reproduced in the entanglement entropy calculation, in accordance to our proposal, is not limited to the above particular examples, but is much more general and relies solely on the existence of the cutoff (3).

5 Conclusions

In this note we consider the black hole entropy as entanglement entropy and propose a resolution to the species puzzle. We suggest that the puzzle never appears provided the correct cutoff in the theory is $\sqrt{N}$ suppressed relative to the Planck mass. This suppression follows from completely independent consistency arguments given in [4]. We demonstrate the equality of two entropies in explicit examples in which the relation between the Planck mass and the cutoff is known from the fundamental theory. It would be interesting to verify our proposal directly in string theory. For this one would have to compute entanglement entropy of strings (for a recent work in this direction see [16]).

It is straightforward to generalize our proposal to arbitrary dimensions. Indeed, the BH entropy of a black hole in $d$ dimensions, $S_{\text{BH}}^{(d)} = M_{(d)}^{d-2} A(\Sigma)$, where $M_{(d)}$ is the Planck mass in $d$-dimensional gravity and $A(\Sigma)$ is the area of $(d-2)$-surface of the horizon, matches the entanglement entropy $S_{\text{ent}}^{(d)} = N\Lambda^{d-2} A(\Sigma)$ produced by $N$ species at cutoff $\Lambda$ if the relation between $M_{(d)}$, $N$ and $\Lambda$ is given by

$$\Lambda^{d-2} N = M_{(d)}^{d-2}.$$  \hspace{1cm} (18)

This is exactly the generalized gravity cutoff in $d$ dimensions found in [5]. Thus, our proposal resolves the species problem in arbitrary dimensions.
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