Closed subsets in Bishop topological groups

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Abstract

We introduce the notion of a Bishop topological group i.e., a group $X$ equipped with a Bishop topology of functions $F$ such that the group operations of $X$ are Bishop morphisms with respect to $F$. A closed subset in the neighborhood structure of $X$ induced by its Bishop topology $F$ is defined in a positive way i.e., not as the complement of an open subset in $X$. The corresponding closure operator, although it is not topological, in the classical sense, does not involve sequences. As countable choice (CC) is avoided, and in agreement with Richman’s critique on the use of CC in constructive mathematics, the fundamental facts on closed subsets in Bishop topological groups shown here have a clear algorithmic content. We work within Bishop’s informal system of constructive mathematics BISH, without countable choice, equipped with inductive definitions with rules of countably many premises.

1 Introduction

The constructive non-viability of the notion of topological space is corroborated by the fact that many classical topological phenomena, like the duality between open and closed sets, are compatible only with classical logic. In a straightforward, constructive translation of general topology we cannot accept that the set-theoretic complement of a closed set is open. E.g., $\{0\}$ is a closed subset $R$, with respect to the topology on $R$ induced by its standard metric, while its complement cannot be accepted constructively as open, since that would imply the implication $\neg(x = 0) \Rightarrow (x > 0 \lor x < 0)$, which is (constructively) equivalent to the constructively unacceptable principle of Markov (see [6], p. 15). The standard use of negative definitions in classical topology does not permit a smooth translation of classical topology to a constructive framework.

In [3], chapter 3, Bishop defined a neighbourhood space $\mathcal{N} := (X, I, \nu)$, where $X, I$ are sets, and $(\nu_i)_{i \in I}$ is a family of subsets of $X$ indexed by $I$ (see [36] for an elaborate study of this notion) that satisfies the following covering (NS1) and neighborhood-condition (NS2):

(NS1) $\bigcup_{i \in I} \nu(i) = X$.

(NS2) $\forall x \in X \forall i,j \in I \left[ x \in \nu(i) \cap \nu(j) \Rightarrow \exists k \in I \left( x \in \nu(k) \land \nu(k) \subseteq \nu(i) \cap \nu(j) \right) \right]$.

A subset $O$ of $X$ is called $\nu$-open, if $\forall x \in O \forall i \in I \left( x \in \nu(i) \land \nu(i) \subseteq O \right)$. An $\nu$-closed set $C$ is not defined negatively, as the complement of a $\nu$-open set, but positively by the condition

$\forall x \in X \left( x \in \overline{C} \Rightarrow x \in C \right)$,

$x \in \overline{C} \Leftrightarrow \forall i \in I \left( x \in \nu(i) \Rightarrow \nu(i) \not\subseteq C \right)$,

where, if $A, B$ are subsets of $X$, then $A \not\subseteq B := \exists y \in X \left( y \in A \land y \not\in B \right)$. If $(Y, J, \mu)$ is a neighborhood space, a function $h : X \to Y$ is neighborhood-continuous, if $h^{-1}(\mu(j))$ is $\nu$-open, for every $j \in J$. The concept of neighborhood space was proposed as a set-theoretic alternative to the notion of topological space, and it is a formal topology in the sense of Sambin [38, 39].

In [3], chapter 3, Bishop also defined the notion of function space $F := (X, F)$, where $X$ is a set and $F$ is a subset of $\mathcal{F}(X)$, the real-valued functions on $X$, that satisfies the closure conditions of the set $\text{Bic}(R)$ of Bishop-continuous functions from $R$ to $R$. Bishop called $F$ a topology (of functions) on $X$. The set $\text{Bic}(R)$ of Bishop-continuous functions $\phi : R \to R$ is the canonical topology of functions on $R$. 
Bishop also defined inductively\(^1\) the least topology of functions on \(X\) that includes a given subset \(F_0\) of \(\mathcal{F}(X)\). The concept of function space was proposed as a function-theoretic alternative to the notion of topological space.

In [5], p. 77, Bishop and Bridges expressed in a clear way the superiority of the function-theoretic notion of function space to the set-theoretic notion of neighborhood space. As Bridges and Palmgren remark in [9], “little appears to have been done” in the theory of neighborhood spaces. Ishihara has worked in [18] (and with co-authors in [17]) on their connections to the apartness spaces of Bridges and Vîță (see [8]), and in [19] on their connections to Bishop’s function spaces, while in [20] Ishihara and Palmgren studied the notion of quotient topology in neighborhood spaces.

Bridges talked on Bishop’s function spaces at the first workshop on formal topology in 1997, and revived the subject of function spaces in [11]. Motivated by Bridges’s paper, Ishihara showed in [19] the existence of an adjunction between the category of neighbourhood spaces and the category of \(\Phi\)-closed pre-function spaces, where a pre-function space is an extension of the notion of a function space. In [27]-[35] we try to develop the theory of function spaces, or Bishop spaces, as we call them. In [33] and in [92] we also study the applications of the theory of set-indexed families of Bishop sets in the theory of Bishop spaces. In [15] connections between the theory of Bishop spaces and the theory of \(C\)-spaces of Escardó and Xu, developed in [43] and in [14], are studied.

A group \(X\) is a topological group, if there is a topology of open sets \(\mathcal{T}\) on \(X\) such that the corresponding operations \(+\) : \(X \times X \to X\) and \(-\) : \(X \to X\) are continuous functions with respect to \(\mathcal{T}\). The theory of topological groups is very well-developed, with numerous applications (see [2], [16] and [42]). We call a group \(X\), equipped with a Bishop topology of functions \(F\), a Bishop topological group, if the corresponding group operations \(+\) : \(X \times X \to X\) and \(-\) : \(X \to X\) are Bishop morphisms with respect to \(F\). A Bishop morphism between Bishop spaces is the notion of arrow in the category of Bishop spaces that was introduced by Bridges in [11] and corresponds to the notion of a continuous function between topological spaces.

Most of the concepts of the theory of Bishop spaces are function-theoretic i.e., they are determined by the Bishop topology of functions \(F\) on \(X\). Each Bishop topology \(F\) generates a canonical neighborhood structure, a family of basic open sets in \(X\), described in section 3. As explained above, a closed set \(C\) with respect to this neighborhood structure is defined positively, and independently from its set-theoretic complement. Generally we cannot show constructively that the set-theoretic complement \(X \setminus C\) of a closed set \(C\) is open. What we show in Theorem 3.2 though, is that a positive notion of complement, determined by \(F\), the \(F\)-complement \(X \setminus_F C\) of \(C\), is the largest open set included in \(X \setminus C\).

In the main core of this paper we prove some fundamental properties of the closed sets in Bishop topological groups. Using functions to describe general properties of sets, and working with the aforementioned positive notion of closed set gives us the opportunity to find constructive proofs with a clear computational content of results, which in many cases in the classical theory of topological groups depend on the use of classical negation. Moreover, our concepts and results avoid the use even of countable choice (CC). Although practitioners of Bishop-style constructive mathematics usually embrace CC, avoiding it, and using non-sequential or non-choice-based arguments instead, forces us to formulate “better” concepts and find “better” proofs. This standpoint was advocated first by Richman (see [37] and [31]).

The study of closed sets in the neighborhood structure induced by the Bishop topology of a Bishop topological group shows the fruitfulness of combining the two constructive proposals of Bishop to the classical topology of open sets. Moreover, the group-structure of a Bishop space \(X\) helps us “recover” part of the classical duality between closed and open sets. As Corollary 5.10 indicates, there are many cases of closed sets in a Bishop topological group for which we can show that their set-theoretic complement is open!

The structure of this paper is the following:

- In section 2 we include all definitions and facts on Bishop spaces that are necessary to the rest

\(^1\)This definition, together with the notion of the least algebra of Borel sets generated by a family of complemented subsets of \(X\), relative to a given set of real-valued functions on \(X\), are the main inductive definitions found in [3], both in chapter 3. The notion of the least algebra of Borel sets is avoided in [3] and [5], and the notion of the least topology is not developed neither in [3] nor in [5].
of the paper. All proofs not given here are found in [28]. For all results on constructive analysis that are used here without proof, we refer to [1].

- In section 3 we give all definitions and results on the canonical neighborhood structure of a Bishop topology that are used here. Theorem 3.2 is the result of this section that is most relevant to the study of closed sets in Bishop topological groups.

- In section 4 we introduce Bishop topological groups and we prove some of their fundamental properties.

- In section 5 the central section of our paper, we prove fundamental properties of closed sets in Bishop topological groups. As we work with functions and positively defined concepts, avoiding the use of choice, our proofs generate clear algorithms. For all algebraic notions within BISH used here, we refer to [24].

We work within Bishop’s informal system of constructive mathematics BISH, without countable choice, equipped with inductive definitions with rules of countably many premises. A set-theoretic formal framework for this system is Myhill’s CST without countable choice, or CZF, equipped with a weak form of Aczel’s regular extension axiom REA (see [1] and [26]).

2 Fundamentals of Bishop spaces

If \( a, b \in \mathbb{R} \), let \( a \lor b := \max\{a, b\} \) and \( a \land b := \min\{a, b\} \). Hence, \(|a| = a \lor (-a)\). If \( f, g \in \mathbb{F}(X) \), let \( f =_{\mathbb{F}(X)} g \iff \forall x \in X (f(x) =_{\mathbb{R}} f(y)) \). If \( f, g \in \mathbb{F}(X) \), \( \varepsilon > 0 \) and \( \Phi \subseteq \mathbb{F}(X) \), let
\[
U(g, f, \varepsilon) := \forall x \in X (|g(x) - f(x)| \leq \varepsilon),
\]
\[
U(\Phi, f) := \forall \varepsilon > 0 \exists g \in \Phi (U(g, f, \varepsilon)).
\]

A set \( X \) is inhabited, if it has an element. We denote by \( \overline{a}^X \), or simply by \( a \), the constant function on \( X \) with value \( a \in \mathbb{R} \), and by \( \text{Const}(X) \) their set.

**Definition 2.1.** A Bishop space is a pair \( \mathcal{F} := (X, F) \), where \( X \) is an inhabited set and \( F \) is an extensional subset of \( \mathbb{F}(X) \) i.e., \( \forall f, g \in \mathbb{F}(X) ([f \in F \& g =_{\mathbb{F}(X)} f] \Rightarrow g \in F) \), such that the following conditions hold:

**BS1** \( \text{Const}(X) \subseteq F \).

**BS2** If \( f, g \in F \), then \( f \lor g \in F \).

**BS3** If \( f \in F \) and \( \phi \in \text{Bic}(\mathbb{R}) \), then \( \phi \circ f \in F \).

**BS4** If \( f \in \mathbb{F}(X) \) and \( u(F, f) \), then \( f \in F \).

We call \( F \) a Bishop topology on \( X \). If \( \mathcal{G} := (Y, G) \) is a Bishop space, a Bishop morphism from \( \mathcal{F} \) to \( \mathcal{G} \) is a function \( h : X \to Y \) such that \( \forall g \in G (g \circ h \in F) \). We denote by \( \text{Mor}(\mathcal{F}, \mathcal{G}) \) the set of Bishop morphisms from \( \mathcal{F} \) to \( \mathcal{G} \). If \( h \in \text{Mor}(\mathcal{F}, \mathcal{G}) \), we say that \( h \) is open, if \( \forall f \in F \exists g \in G (f = g \circ h) \). If \( h \in \text{Mor}(\mathcal{F}, \mathcal{G}) \) is a bijection and \( h^{-1} \) is a Bishop morphism, we call \( h \) a Bishop isomorphism.

A Bishop morphism \( h \in \text{Mor}(\mathcal{F}, \mathcal{G}) \) is a “continuous” function from \( \mathcal{F} \) to \( \mathcal{G} \). If \( h \in \text{Mor}(\mathcal{F}, \mathcal{G}) \) is a bijection, then \( h^{-1} \in \text{Mor}(\mathcal{G}, \mathcal{F}) \) if and only if \( h \) is open. Let \( \mathcal{R} \) be the Bishop space of reals \((\mathbb{R}, \text{Bic}(\mathbb{R}))\). It is easy to show that if \( F \) is a topology on \( X \), then \( F = \text{Mor}(\mathcal{F}, \mathcal{R}) \) i.e., an element of \( F \) is a real-valued “continuous” function on \( X \). A Bishop topology \( F \) on \( X \) is an algebra and a lattice, where \( f \lor g \) and \( f \land g \) are defined pointwise, and \( \text{Const}(X) \subseteq F \subseteq \mathbb{F}(X) \). If \( F^*(X) \) denotes the bounded elements of \( \mathbb{F}(X) \), then \( F^* := F \cap F^*(X) \) is a Bishop topology on \( X \). If \( x =_X y \) is the given equality on \( X \), a Bishop topology \( F \) on \( X \) separates the points of \( X \), or \( F \) is separating (see [27]), if
\[
\forall x, y \in X (\forall f \in F (f(x) =_\mathbb{R} f(y)) \Rightarrow x =_X y).
\]

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2 Extensional Martin-Löf Type Theory or the theory of setoids within intensional Martin-Löf Type Theory are possible type-theoretic systems for this informal system (see [12]), although there choice, in the form of the distributivity of \( \Pi \) over \( \Sigma \), is provable.
The canonical apartness relation on $X$ induced by $F$ is defined by

$$ x \neq_F y : \Leftrightarrow \exists_{f \in F} (f(x) \neq_R f(y)). $$

An apartness relation on $X$ is a positively defined inequality on $X$. E.g., if $a, b \in \mathbb{R}$, then $a \neq_R b : \Leftrightarrow |a - b| > 0$. In Proposition 5.1.2. of [28] we show that $a \neq_R b : \Leftrightarrow a \neq_{\text{Bic}(\mathbb{R})} b$.

**Definition 2.2.** Turning the definitional clauses (BS$_1$) – (BS$_4$) into inductive rules, the least topology $\bigvee F_0$ generated by a set $F_0 \subseteq \mathcal{F}(X)$, called a subbase of $\bigvee F_0$, is defined by the following inductive rules:

$$
\begin{align*}
\frac{f_0 \in F_0}{f_0 \in \bigvee F_0}, & \quad \frac{f \in \bigvee F_0, \ g \in \mathcal{F}(X), \ g =_{\mathcal{F}(X)} f}{g \in \bigvee F_0}, & \quad \frac{a \in \mathbb{R}}{a \in \bigvee F_0}, & \quad \frac{f, g \in \bigvee F_0}{f + g \in \bigvee F_0}. \\
g_1 \in \bigvee F_0 \land U(g_1, f, \frac{f}{2}), & \quad g_2 \in \bigvee F_0 \land U(g_2, f, \frac{f}{2}), & \quad g_3 \in \bigvee F_0 \land U(g_3, f, \frac{f}{2}), & \quad \ldots .
\end{align*}
$$

The above rules induce the corresponding induction principle $\text{Ind}_{\bigvee F_0}$ on $\bigvee F_0$.

If $h : X \to Y$ and $G = \bigvee G_0$, then one can show inductively i.e., with the use of $\text{Ind}_{\bigvee G_0}$, that $h \in \text{Mor}(\mathcal{F}, \mathcal{G}) \iff \forall_{g_0 \in G_0} (g_0 \circ h \in F)$. We call this property the $\bigvee$-lifting of morphisms.

**Definition 2.3.** If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are given Bishop spaces, their product is the structure $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$, where

$$ F \times G : = \bigvee \{ f \circ \pi_1 \mid f \in F \} \cup \{ g \circ \pi_2 \mid g \in G \} : = \bigvee_{f \in F} g \circ \pi_2, $$

and $\pi_1, \pi_2$ are the projections of $X \times Y$ to $X$ and $Y$, respectively.

It is straightforward to show that $\mathcal{F} \times \mathcal{G}$ satisfies the universal property for products and that $F \times G$ is the least topology which turns the projections $\pi_1, \pi_2$ into morphisms. If $F_0$ is a subbase of $F$ and $G_0$ is a subbase of $G$, then we show inductively that

$$ \bigvee F_0 \times \bigvee G_0 = \bigvee \{ f_0 \circ \pi_1 \mid f_0 \in F_0 \} \cup \{ g_0 \circ \pi_2 \mid g_0 \in G_0 \} : = \bigvee_{f_0 \in F_0} g_0 \circ \pi_2. $$

Consequently, $\text{Bic}(\mathbb{R}) \times \text{Bic}(\mathbb{R}) = \bigvee \text{id}_R \circ \pi_1, \text{id}_R \circ \pi_2 = \bigvee \pi_1, \pi_2$.

**Corollary 2.4.** Let $\mathcal{H} = (Z, H), \mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ be Bishop spaces.

(i) If $h_1 : Z \to X$, $h_2 : Z \to Y$, the map $h_1 \times h_2 : Z \to X \times Y$, defined by $z \mapsto (h_1(z), h_2(z))$, is in $\text{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})$ if and only if $h_1 \in \text{Mor}(\mathcal{H}, \mathcal{F})$ and $h_2 \in \text{Mor}(\mathcal{H}, \mathcal{G})$.

(ii) If $e_1 : X \to Z$, $e_2 : Y \to Z$, then the map $e_1 \otimes e_2 : X \times Y \to Z \times Z$, defined by $(x, y) \mapsto (e_1(x), e_2(y))$, is in $\text{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H} \times \mathcal{H})$ if and only if $e_1 \in \text{Mor}(\mathcal{F}, \mathcal{H})$ and $e_2 \in \text{Mor}(\mathcal{G}, \mathcal{H})$.

**Proposition 2.5.** Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G), \mathcal{H} = (Z, H)$ are Bishop spaces, $x \in X, y \in Y, \phi : X \times Y \to \mathbb{R} \in F \times G : X \times Y \to Z \in \text{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$.

(i) $i_x : Y \to X \times Y, y \mapsto (x, y)$, and $i_y : X \to X \times Y, x \mapsto (x, y)$, are open morphisms.

(ii) $\phi_x : Y \to \mathbb{R}, y \mapsto \phi(x, y)$, and $\phi_y : X \to \mathbb{R}, x \mapsto \phi(x, y)$, are in $G$ and $F$, respectively.

(iii) $\Phi_x : Y \to Z, y \mapsto \Phi(x, y)$, and $\Phi_y : X \to Z, x \mapsto \Phi(x, y)$, are in $\text{Mor}(G, H)$ and $\text{Mor}(F, H)$, respectively.

**Proof.** (i) We show it only for $i_y$. By the $\mathcal{F}$-lifting of morphisms we have that $i_y \in \text{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G}) \iff \forall_{f \in F} ((f \circ \pi_1) \circ i_y \in F) \land \forall_{g \in G} (g \circ \pi_2) \circ i_y \in F$. If $f \in F$, then $(f \circ \pi_1) \circ i_y = f$, which shows also that $i_y$ is open, while if $g \in G$, then $(g \circ \pi_2) \circ i_y = g(y) \in F$.

(ii) We show it only for $\phi_y$. We have that $\phi_y = \phi \circ i_y$, since $(\phi \circ i_y)(x) = \phi(x, y) = \phi_y(x)$, for each $x \in X$. Since $i_y \in \text{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G})$ and $\phi \in F \times G$, we get that $\phi \circ i_y = \phi_y \in F$.

(iii) The proof is similar to the proof of (ii). Actually, (ii) is a special case of (iii).
3 The neighborhood structure of a Bishop topology

If $F$ is a Bishop topology on $X$, the neighborhood structure on $X$ induced by $F$ is the family $(U(f))_{f \in F}$, where $U(f) := \{ x \in X \mid f(x) > 0 \}$. The covering condition (NS1) follows from the equality $U(\pi X) = X$, where $a > 0$, and the neighborhood-condition (NS2) follows from the equality $U(f) \cap U(g) = U(f \wedge g)$, for every $f, g \in F$. Consequently, $O \subseteq X$ is open if
\[ \forall x \in O \exists f \in F(f(x) > 0 \& U(f) \subseteq O), \]
and $C \subseteq X$ is closed, if $\forall x \in C (x \in C \Rightarrow x \in C)$, where
\[ x \in \overline{C} \Leftrightarrow \forall f \in F(f(x) > 0 \Rightarrow \exists c \in C(f(c) > 0)). \]

**Proposition 3.1.** Let $F = (X, F), G = (Y, G)$ be Bishop spaces, $f \in F$ and $h : X \to Y$.

(i) $O$ is open in $N(\text{Bic}(F))$ if and only if it is open in the standard topology on $\mathbb{R}$.

(ii) If $h \in \text{Mor}(F, G)$, then $h$ is neighborhood-continuous.

(iii) If $h \in \text{Mor}(F, G)$, the inverse image of a closed set in $Y$ under $h$ is closed in $X$.

(iv) If $h \in \text{Mor}(F, G)$ and $A \subseteq X$, then $h(A) \subseteq h(A)$.

(v) $X \setminus U(f)$ of $U(f)$ in $X$ is closed in $X$, for every $f \in F$.

(vi) The set-theoretic complement $X \setminus U(f)$ of $U(f)$ in $X$ is closed, for every $f \in F$.

**Proof.** For (i)-(v) see [28], Proposition 4.4, and for (vi) see [28], Proposition 5.3.2. □

Clearly, $C \subseteq X$ is closed if and only if $C = \overline{C}$, and the intersection of closed sets is closed. The closure operator $A \mapsto \overline{A}$ is not topological, in the classical sense, as we cannot show constructively that the union of two closed sets is closed, in general, closed (see also [5], p. 79). If $A, B \subseteq X$ and $F$ a Bishop topology on $X$, then it is straightforward to show that (i) $A \subseteq \overline{A}$, (ii) $\overline{A} \subseteq \overline{A}$, (iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$, and (iv) $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. The inverse inclusion $\overline{A \cup B} = \overline{A} \cup \overline{B}$ cannot be shown constructively. If $F$ is a Bishop topology on $X$ and $C \subseteq X$, we define positively and through $F$ a stronger relation “$x$ is not in $C$”, where $x \in X$, by
\[ x \notin F C \Leftrightarrow \exists f \in F(f : x \notin F C), \]
\[ f : x \notin F C \Leftrightarrow f(x) > 0 \& \forall c \in C(f(c) = 0). \]

Classically one can show that $F$ is always completely regular i.e., if $C$ is closed in $X$ and $x \notin C$, then $x \notin F C$ (see [28], Proposition 3.7.6). Constructively we can show the following.

**Theorem 3.2.** Let $F$ be a Bishop topology on $X$ and $C$ closed in $X$. The $F$-complement $X \setminus F C := \{ x \in X \mid \exists f \in F(f : x \notin F C) \}$ of $C$ in $X$ is the largest open set included in $X \setminus C$.

**Proof.** We show that $X \setminus F C$ is an open set included in $X \setminus C$ and if $O$ is an open set included in $X \setminus C$, then $O \subseteq X \setminus F C$. First we show that $X \setminus F C$ is open. If $x \in X \setminus F C$, let $f \in F$ such that $f : x \notin F C$. Clearly, $x \in U(f)$, and if $y \in X$ such that $y \in U(f)$, then $f : y \notin F C$ i.e., $y \in X \setminus F C$. Next we show that if $x \in X \setminus F C$, then $x \in X \setminus C$. If $x \in C$, then $f(x) > 0$ and $f(x) = 0$, which is a contradiction. Suppose next that $O$ is an open set included in $X \setminus C$. We show that if $x \in O$, then $x \in X \setminus F C$. Since $O$ is open, there is $g \in F$ such that $g(x) > 0$ and $U(g) \subseteq O \subseteq X \setminus C$. We have that $g(c) \leq 0$, for every $c \in C$, since if $g(c) > 0$, for some $c \in C$, then $c \in U(g)$, hence $c \in X \setminus C$, which is a contradiction. By the constructively valid implication $\neg(a > 0) \Rightarrow a \leq 0$, for every $a \in \mathbb{R}$ (see [5], Lemma 2.18), we get $g(c) \leq 0$. Since $g \setminus 0^X \in F$, we conclude that $g \setminus 0^X : x \notin F C$, hence $x \in X \setminus F C$. □

Although we cannot show in general that $X \setminus C$ is open, and hence $X \subseteq C \subseteq X \setminus F C$, we can replace this computationally dubious result by the computationally meaningful fact that $X \setminus F C$ is the largest open set included in $X \setminus C$. The next result is used in the proofs of Theorem 5.6(ii) and Theorem 5.7(ii).

**Proposition 3.3.** If $F$ is a Bishop topology on $X$, the following are equivalent:

(i) $F$ separates the points of $X$.

(ii) The inequality $\neq F$ generated by $F$ is tight i.e., $\neg(x \neq F y) \Rightarrow x =_X y$, for every $x, y \in X$.

(iii) The singleton $\{ x \}$ is closed, for every $x \in X$. 


Proof. (i)⇒(ii) Let $- (x =_F y) :\Leftrightarrow \neg [\exists f \in F \{ f(x) \neq_R f(y) \}]$, for some $x, y \in X$. We show that $\forall f \in F \{ f(x) =_R f(y) \}$. Let $f \in F$ such that $f(x) \neq_R f(y)$. By our hypothesis on $x, y$ this is impossible, hence by the tightness of $\neq_R$ we conclude that $f(x) =_R f(y)$.

(ii)⇒(iii) Let $x, y \in X$ such that $\forall f \in F \{ f(y) > 0 \Rightarrow f(x) > 0 \}$. We show that $y =_X x$, by showing that $- (y \neq x)$. Suppose that $y \neq_F x$ and, without loss of generality, let $g \in F$ such that $g(y) = 1$ and $g(x) = 0$. By the hypothesis on $x, y$ we have that $g(y) > 0 \Rightarrow g(x) > 0$, and we get the required contradiction.

(iii)⇒(i) Let $x, y \in X$ such that $\forall f \in F \{ f(x) =_R f(y) \}$. We show that $y \in \{ x \}$, hence $y = x$. Let $f \in F$ such that $f(y) > 0$. Since $f(x) = f(y)$, we get $f(x) > 0$. □

Proposition 3.4. If $C$ is closed in $X$ and $D \subseteq Y$ is closed in $Y$, $C \times D$ is closed in $X \times Y$.

Proof. Let $(x, y) \in \overline{C \times D}$ i.e., if $h(x, y) > 0$, there is $(u, w) \in C \times D$ such that $h(u, w) > 0$, for every $h \in F \times G$. We show that $x \in \overline{C}$ and (similarly) $y \in \overline{D}$, hence $(x, y) \in C \times D$. Let $f \in F$ such that $f(x) > 0$. Since $(f \circ \pi_1)(x, y) > 0$ and $f \circ \pi_1 \in F \times G$, there is $(u, w) \in C \times D$ such that $(f \circ \pi_1)(u, w) := f(u) > 0$, hence there is $u \in C$ with $f(u) > 0$. □

4 Bishop topological groups

Definition 4.1. A Bishop topological group is a structure $\mathcal{F} := (X, +, 0, -; F)$, where $\mathcal{X} := (X, +, 0, -)$ is a group and $F$ is a Bishop space such that $+: X \times X \to X \in \text{Mor}(F \times F, F)$ and $-: X \to X \in \text{Mor}(F, F)$. If necessary, we also use the notations $+^X, 0^X$, and $-^X$ for the operations of the group $X$. If $f \in F$, let $f_+ := f \circ -$ and $f_- := f \circ -$.

By the definition of a Bishop morphism we get

$$ + \in \text{Mor}(F \times F, F) :\Leftrightarrow \forall f \in F \{ f \circ + \in F \times F \} :\Leftrightarrow \forall f \in F \{ f_+ \in F \times F \}, $$

$$ - \in \text{Mor}(F, F) :\Leftrightarrow \forall f \in F \{ f \circ - \in F \} :\Leftrightarrow \forall f \in F \{ f_- \in F \}. $$

Example 4.2 (The additive group of reals). The structure $\mathcal{R} := (\mathbb{R}, +, 0, -; \text{Bic}(\mathbb{R}))$ is a Bishop topological group. By the $\vee$-lifting of morphisms $+ \in \text{Mor}(\mathcal{R} \times \mathcal{R}, \mathcal{R}) \Leftrightarrow \text{id}_{\mathbb{R}} \circ + \in \text{Bic}(\mathbb{R}) \times \text{Bic}(\mathbb{R})$.

Example 4.3 (The trivial Bishop topological group). If $\mathcal{X} := (X, +, 0, -)$ is a group, then $\text{Const}(X)$ is the trivial Bishop topology on $X$. If $a \in \mathbb{R}$, then $\pi^X \circ + = \pi^X \times X \in \text{Const}(X \times X) = \text{Const}(X) \times \text{Const}(X)$, and $\pi^X \circ - = \pi^X \in \text{Const}(X)$.

Unless otherwise stated, from now on, $X, Y$ are Bishop topological groups with $F$ and $G$ Bishop topologies on $X$ and $Y$, respectively.

Proposition 4.4. (i) The function $- : X \to X$ is a Bishop isomorphism.

(ii) For every $x_0 \in X$ the functions $+^{x_0} : X \to X$, defined by $+^{x_0}(x) := x_0 + x$ and $-^{x_0} : X \to X$, for every $x \in X$, are Bishop morphisms.

(iii) If $f \in F$, the functions $f_+^{x_0} , f_-^{x_0} : X \to \mathbb{R}$, defined by $f_+^{x_0}(x) := f(x_0 + x)$ and $f_-^{x_0}(x) := f(x + x_0)$, for every $x \in X$, are in $F$.

(iv) For every $x_0 \in X$ the functions $+^{x_0} , -^{x_0} : X \to X$ are Bishop isomorphisms.

Proof. (i) By definition $- \in \text{Mor}(F, F)$, and it is a bijection. It is also open i.e., $\forall f \in F \exists g \in F \{ f = g_- \}$. If $f \in F$, we have that $f = (f_-)$. (ii) and (iii) If $i_{x_0} : X \to X \times X$ is defined by $i_{x_0}^1(x) := (x_0, x)$, for every $x \in X$, then $i_{x_0}^1 \in \text{Mor}(F, F \times F)$ and $+^{x_0} := \circ i_{x_0}^1 \in \text{Mor}(F, F)$ as a composition of Bishop morphisms.
Let $f_{x_0}^1 = f \circ +_{x_0}^1 \in F$, as a composition of Bishop morphisms. For $+_{x_0}^2$, we work similarly.

(iv) Clearly, $+_{x_0}^1$ is a bijection. It is also open, since for every $f \in F$ we have that $f = f_{-x_0}^{-1} \circ +_{x_0}^1$, and by (iv) $f_{-x_0}^{-1} \in F$. For $f_{x_0}^2$ we work similarly.

**Proposition 4.5.** The function $k : X \times X \to X \times X$, defined by $k(x,y) := (x,-y)$, for every $(x,y) \in X \times X$, is a Bishop isomorphism.

**Proof.** Clearly, $k$ is a bijection. By the $\vee$-lifting of morphisms we have that $k \in \text{Mor}(F \times F, F \times F)$ if and only if $(f \circ \pi_1) \circ k \in F \times F$ and $(f \circ \pi_2) \circ k \in F \times F$), for every $f \in F$. Let $f \in F$ and $(x,y) \in X \times X$.

Since $[(f \circ \pi_1) \circ k](x,y) := (f \circ \pi_1)(x,-y) := f(x) := f_{-y}(x)$, we get $(f \circ \pi_1) \circ k = f \circ \pi_1 \in F \times F$.

Moreover, $[(f \circ \pi_2) \circ k](x,y) := (f \circ \pi_2)(x,-y) := f(-y) := f_{-y}(y)$, i.e., $(f \circ \pi_2) \circ k = (f \circ \pi_2) \in F \times F$, since $f \in F$. Since $(k \circ k)(x,y) := (x,-(-y)) = (x,y)$, $k$ is its own inverse, hence $k$ is a Bishop isomorphism.

**Proposition 4.6.** Let $X := (X, +, 0, -)$ be a group, $F$ a Bishop topology on $X$, and sub : $X \times X \to X$ be defined by sub$(x,y) := x - y$, for every $(x,y) \in X \times X$. Then $F := (X, +, 0, -; F)$ is a Bishop topological group if and only if sub $\in \text{Mor}(F \times F, F)$.

**Proof.** If $F$ is a Bishop topological group, then sub $= + \circ k \in \text{Mor}(F \times F, F)$

For the converse, notice that $+ = \text{sub} \circ k \in \text{Mor}(F \times F, F)$. By Proposition 26(iii) we get $- = \text{sub}_0 \in \text{Mor}(F, F)$, where $\text{sub}_0(y) := \text{sub}(0, y) := -y$, for every $y \in X$.

**Proposition 4.7.** Let $F := (X, +, 0, -; F)$ be a Bishop topological group and $G := (Y, G)$ a Bishop space. Let the functions $+^\to : \text{Mor}(G, F) \times \text{Mor}(G, F) \to \text{Mor}(G, F)$, $-^\to : \text{Mor}(G, F) \to \text{Mor}(G, F)$, and $0^\to : Y \to X$ be defined by $(h_1 +^\to h_2)(y) := h_1(y) + h_2(y)$, $(-^\to h)(y) := -h(y)$, and $0^\to(y) := 0$, for every $y \in Y$, and $h_1, h_2, h \in \text{Mor}(G, F)$. Then $(\text{Mor}(G, F), +^\to, 0^\to, -^\to)$ is a group.

One can show that the group $(\text{Mor}(G, F), +^\to, 0^\to, -^\to)$, equipped with the pointwise exponential Bishop topology (see 23, section 4.3), is a Bishop topological group. Notice that if $Y$ has also a group structure compatible with $G$, and if $h_1, h_2, h$ are group homomorphisms, then $h_1 +^\to h_2, -^\to h$ and $0^\to$ are also group homomorphisms.

**Definition 4.8.** Let $F := (X, +^X, 0^X, -^X; F)$ and $G := (Y, +^Y, 0^Y, -^Y; G)$ be Bishop topological groups. If $h \in \text{Mor}(G, F)$ such that $h$ is a $(X, Y)$-group homomorphism, then we call $h$ a Bishop group homomorphism, or simpler, a Bishop homomorphism. We denote by $\text{Mor}(F, G)$ the set of all Bishop homomorphisms from $F$ to $G$. Let $\text{BTopGrp}$ be the category of Bishop topological groups with Bishop group homomorphisms.

**Proposition 4.9.** Let $a \in \mathbb{R}$ and let $h_a : \mathbb{R} \to \mathbb{R}$ be defined by $h_a(x) = ax$, for every $x \in \mathbb{R}$. Then $h_a \in \text{Mor}(\mathcal{R}, \mathcal{R})$. Conversely, if $h \in \text{Mor}(\mathcal{R}, \mathcal{R})$, there is $a \in \mathbb{R}$ such that $h = h_a$. 

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Proof. $h_a$ is a group homomorphism, and by the $\bigvee$-lifting of morphisms $h_a \in \text{Mor}(\mathbb{R}, \mathbb{R}) \iff \text{id}_R \circ h_a := h_a \in \text{Bic}(\mathbb{R})$, which holds, since $h_a := a \cdot \text{id}_R \in \text{Bic}(\mathbb{R})$. If $h \in \text{Mor}(\mathbb{R}, \mathbb{R})$, let its restriction $h_{\{Q\}}$, where $h_{\{Q\}} : Q \to Q$ is a group homomorphism given by $(h_{\{Q\}})(g) = h(1)g$, for every $g \in Q$. Since $Q$ is metrically dense in $\mathbb{R}$, by Proposition 4.15. in [28] we have that $\text{Bic}(Q) = \text{Bic}(\mathbb{R})_{\{Q\}} = \{ \phi_{\{Q\}} \mid \phi \in \text{Bic}(\mathbb{R}) \}$. Hence $h_{\{Q\}} \in \text{Bic}(\mathbb{Q})$. By Lemma 4.7.13. in [28] there is a unique (up to equality) extension of $h_{\{Q\}}$ in $\text{Bic}(\mathbb{R})$. Hence $h = h_{h(1)}$. □

Proposition 4.10. Let $\mathcal{F} := (X, +^X, 0^X, -^X; F)$ and $\mathcal{G} := (Y, +^Y, 0^Y, -^Y; G)$ be Bishop topological groups. Let the functions $+^{X \times Y} : (X \times Y) \times (X \times Y) \to X \times Y$, $-^{X \times Y} : X \times Y \to X \times Y$, and $0^{X \times Y} \in X \times Y$ be defined by $(x, y) +^{X \times Y} (x', y') := (x + x', y + y')$, $-^{X \times Y}(x, y) := (\overline{x}, \overline{y})$, and $0^{X \times Y} := (0, 0)$. Then $\mathcal{F} \times \mathcal{G} := (X \times Y, +^{X \times Y}, 0^{X \times Y}, -^{X \times Y}; F \times G)$ is a Bishop topological group.

Proof. The proof that $X \times Y$ is a group is omitted as trivial. By the $\bigvee$-lifting of morphisms $+^{X \times Y} \in \text{Mor}(\mathbb{F} \times \mathcal{G}) \times (\mathbb{F} \times \mathcal{G}), \mathcal{F} \times \mathcal{G})$ if and only if

$$\forall_{f \in F} \forall_{g \in G} \left( (f \circ \pi^X) \circ +^{X \times Y} \in [F \times G] \times [F \times G] \& (g \circ \pi^Y) \circ +^{X \times Y} \in [F \times G] \times [F \times G] \right),$$

where by the $\bigvee$-lifting of the product Bishop topology

$$[F \times G] \times [F \times G] = \bigvee_{f \in F} \left( (f \circ \pi^X) \circ \pi^X_1, (g \circ \pi^Y) \circ \pi^Y_1, (f \circ \pi^X) \circ \pi^X_2, (g \circ \pi^Y) \circ \pi^Y_2 \right).$$

If $f \in F$, $x, x' \in X$, and $y, y' \in Y$, and if $z := ((x, y), (x', y'))$, then

$$([(f \circ \pi^X) \circ +^{X \times Y}](z)) := (f \circ \pi^X)(x + x', y + y')$$

$$:= f(x + x')$$

$$:= (f \circ +^X)(x, x')$$

$$:= (f \circ +^X)(\pi^X(x, y), \pi^X(x', y'))$$

$$:= (f \circ +^X)(\pi^X(\pi^X_1, \pi^X_2)(z), [\pi^X \circ \pi^X_2](z))$$

$$:= [(f \circ +^X \circ h)](z),$$

where by Corollary 2.24 the function $h : [(X \times Y) \times (X \times Y)] \to X \times X$, where $h := [\pi^X \circ \pi^X_1] \times [\pi^X \circ \pi^X_2]$ is a Bishop morphism. Since $(f \circ \pi^X) \circ +^{X \times Y} = (f \circ +^X) \circ h$

$$(X \times Y) \times (X \times Y) \xrightarrow{h} X \times X \xrightarrow{+^X} X \xrightarrow{f} \mathbb{R},$$

we get $(f \circ \pi^X) \circ +^{X \times Y} \in \text{Mor}([F \times \mathcal{G}] \times [F \times \mathcal{G}], \mathcal{R})$ as a composition of Bishop morphisms, hence $(f \circ \pi^X) \circ +^{X \times Y} \in [F \times G] \times [F \times G]$. Working similarly, we get $(g \circ \pi^Y) \circ +^{X \times Y} \in [F \times G] \times [F \times G]$. By the $\bigvee$-lifting of morphisms we also have that

$$-^{X \times Y} \in \text{Mor}([F \times \mathcal{G}] \times [F \times \mathcal{G}], \mathcal{R}) \iff \forall_{f \in F} \forall_{g \in G} \left( (f \circ \pi^X) \circ -^{X \times Y} \in F \times G \& (g \circ \pi^Y) \circ -^{X \times Y} \in F \times G \right).$$

If $f \in F$, $x \in X$, and $y \in Y$, then

$$[((f \circ \pi^X) \circ -^{X \times Y}](x, y) := ((f \circ \pi^X)(-x, -y) := f(-x) := f_-(x) := (f_\circ \pi^X)(x, y)$$

i.e., $(f \circ \pi^X) \circ -^{X \times Y} = f_- \circ \pi^X \in F \times G$, since $f_- \in F$. Similarly, $(g \circ \pi^Y) \circ -^{X \times Y} \in F \times G$. □

Since the projections $\pi^X, \pi^Y$ are homomorphisms, they are Bishop homomorphisms. By the universal property of the product Bishop topology, $\mathcal{F} \times \mathcal{G}$ is the product in $\text{BTopGrp}$. 
5 Closed subsets in Bishop topological groups

Proposition 5.1. Let $C \subseteq X$ and $x_0 \in X$.
(i) If $C$ is closed, then $-C := \{-c \mid c \in C\}$ is closed.
(ii) $-C = -C$.
(iii) If $C$ is closed, then $x_0 + C := \{x_0 + c \mid c \in C\}$ is closed.
(iv) $x_0 + C = x_0 + C$.

Proof. (i) We suppose that $u \in -C$ i.e., if $f(u) > 0$, there is $w \in -C$ such that $f(w) > 0$, for every $f \in F$, and we show that $u \in -C$ i.e., $-u \in C$. Since $C$ is closed, it suffices to show that $-u \in C$.

(ii) Since $C \subseteq -C$, we get $-C \subseteq -C$. Since $-C$ is closed, by (i) $-C$ is also closed, hence $-C \subseteq -C = -C$.

(iii) We suppose that $y \in x_0 + C$ i.e., if $f(y) > 0$, there is $c \in C$ such that $f(x_0 + c) > 0$, for every $f \in F$, and we show that $y \in x_0 + C$ by showing that $-x_0 + y \in C$. As $C$ is closed, it suffices to show that $-x_0 + y \in C$. Let $f \in F$ such that $f(-x_0 + y) > 0 \iff f_1^{x_0}(y) > 0$. We show that there is $c \in C$ such that $f(c) > 0$. By our hypothesis on $y$, there is $c \in C$ such that $f_1^{x_0}(x_0 + c) := f(-x_0 + x + c) = f(c) > 0$.

(iv) Since $C$ is closed, and $x_0 + C \subseteq x_0 + C$, we get $x_0 + C \subseteq x_0 + C = x_0 + C$. For the converse inclusion, let $x := x_0 + y$ with $y \in C$. We show that $x \in x_0 + C$. Let $f \in F$ such that $f(x_0 + y) > 0 \iff f_1^{x_0}(y) > 0$. We find $c \in C$ such that $f(x_0 + c) > 0$. By our hypothesis on $y$ though, there is $c \in C$ such that $f_1^{x_0}(c) := f(x_0 + c) > 0$.

Corollary 5.2. $F$ separates the points of $X$ if and only if $\{0^X\}$ is closed.

Proof. If $\{0^X\}$ is closed, then by Proposition 5.1 (iii) $\{x\} = x + \{0^X\}$ is closed, for every $x \in X$. By Proposition 3.3 $F$ is separating. The converse follows immediately from Proposition 3.3.

Proposition 5.3. If $C$ is an open subgroup of $X$, then $C$ is closed in $X$.

Proof. Let $x \in C$ i.e., if $f(x) > 0$, there is $u \in C$ such that $f(u) > 0$, for every $h \in F$. We show that $x \in C$. Since $C$ is a subgroup of $X$, we have that $0 \in C$. Since $C$ is open in $X$, there is $g \in F$ such that $g(0) > 0$ and $U(g) \subseteq C$. Since $g_1^{x_0} \in F$ and

$$g_1^{x_0}(x) := g(-x + x) = g(0) > 0,$$

by our hypothesis on $x$ there is $u \in C$ such that $g_1^{x_0}(u) := g(-x + u) > 0$. Since $U(g) \subseteq C$, we get $-x + u \in C$, and since $C$ is a subgroup of $X$, we get $x \in C$.

Classically, $C$ is closed, since its complement in $X$ is the open set $\bigcup \{x + C \mid x \notin C\}$, where $x + C$ is open, for every $x \in X$, as $C$ is open (this holds also constructively). The double use of negation in the classical proof is replaced here by the clear algorithm of the previous proof.

Lemma 5.4. The commutator map $\text{abel} : X \times X \to X$ is defined by $\text{abel}(x, y) := x + y - x - y$, for every $(x, y) \in X \times X$.
(i) $\text{abel} \in \text{Mor}(F \times F, F)$.
(ii) If $x, y \in X$, then $x + y = x + y \iff \text{abel}(x, y) = 0^X$.
(iii) If $x \in X$, the mapping $\text{abel}_x : X \to X$, where $\text{abel}_x(y) := \text{abel}(x, y)$, for every $y \in X$, is in $\text{Mor}(F, F)$, and for every $f \in F$ the composition $f \circ \text{abel}_x \in F$.
Proof. (i) By the definition of the product Bishop topology \( \pi_1, \pi_2 \in \text{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F}) \). Since \( \text{abel} := \pi_1 + \pi_2 - \pi_1 - \pi_2 \), by Proposition 4.7 we get \( \text{abel} \in \text{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F}) \).

(ii) and (iii) The proof for (ii) is immediate. By (i) and Proposition 2.3(iii) \( \text{abel}_x \in \text{Mor}(\mathcal{F}, \mathcal{F}) \), hence by the definition of a Bishop morphism \( f \circ \text{abel}_x \in F \), for every \( f \in F \).

(iv) is trivial and (v) follows immediately from (iv).

Lemma 5.5. Let \( x \in X \) and \( H \subseteq X \). The maps \( \text{normal}_x : X \to X \) and \( \text{Normal}_x : X \to X \) are defined, for every \( x \in X \), respectively, by \( \text{normal}_x(y) := x + y - x \) and \( \text{Normal}_x(y) := y + x - y \). Let \( \text{normal}^H_x, \text{Normal}^H_x : H \to X \) be the restrictions of \( \text{normal}_x \) and \( \text{Normal}_x \) to \( H \), respectively.

(i) If \( x, y \in X \), then \( \text{normal}_x(y) = \text{Normal}_y(x) \).

(ii) \( \text{normal}_x \in \text{Mor}(\mathcal{F}, \mathcal{F}) \) and \( \text{Normal}_x \in \text{Mor}(\mathcal{F}, \mathcal{F}) \).

(iii) If \( H \leq X \), then \( H \) is normal if and only if \( \text{normal}^H_x : H \to H \), for every \( x \in X \).

(iv) If \( f \in F \), the compositions \( f \circ \text{normal}_x \in F \) and \( f \circ \text{Normal}_x \in F \).

(v) If \( H \) is normal, then \( \text{normal}^H_x \in \text{Mor}(\mathcal{F}_H, \mathcal{F}_H) \).

(vi) If \( \text{Normal}^H_x : H \to H \), then \( \text{Normal}^H_x \in \text{Mor}(\mathcal{F}_H, \mathcal{F}_H) \).

Proof. (i) The proof is immediate. For the proof of (ii), the function \( c_x : X \to X \), defined by \( c_x(y) := x \), for every \( y \in X \), is in \( \text{Mor}(\mathcal{F}, \mathcal{F}) \); if \( f \in F \), then \( (f \circ c_x) := f(x) \), for every \( y \in X \), hence \( f \circ c_x = f(x) \in F \). The identity map \( \text{id}_X \) on \( X \) is also in \( \text{Mor}(\mathcal{F}, \mathcal{F}) \). Since \( \text{normal}_x := c_x + \text{id}_X - c_x \), by Proposition 4.7 \( \text{normal}_x \in \text{Mor}(\mathcal{F}, \mathcal{F}) \). Since \( \text{Normal}_x := \text{id}_X + c_x - \text{id}_X \), we get \( \text{Normal}_x \in \text{Mor}(\mathcal{F}, \mathcal{F}) \).

(iii) and (iv) are immediate to show. For the proof of (v), by the \( \forall \)-lifting of morphisms we have that \( \text{normal}^H_x \in \text{Mor}(\mathcal{F}_H, \mathcal{F}_H) \) if and only if \( \forall f \in F \) \( f(x) \in F \). If \( f, v \in H \), then \( (f_H \circ \text{normal}^H_x)(v) := f_H(x + v - x) := f(x + v - x) := (f \circ \text{normal}_x)(v) := (f \circ \text{normal}_x)_H(v) \) i.e., \( f_H \circ \text{normal}^H_x = (f \circ \text{normal}_x)_H \in F_H \), since \( f \circ \text{normal}_x \in F \). For the proof of (vi), we proceed as in the proof of (v).

Theorem 5.6. Let \( H \) be a subgroup of \( X \).

(i) The closure \( \overline{H} \) of \( H \) is also a subgroup of \( X \).

(ii) If \( F \) is a separating Bishop topology and \( H \) is abelian, then \( \overline{H} \) is abelian.

(iii) If \( H \) is normal, then \( \overline{H} \) is normal.

Proof. (i) Since \( H \subseteq \overline{H} \) and \( 0 \in H \), we get \( 0 \in \overline{H} \). Let \( x, y \in \overline{H} \), where by definition

\[
\begin{align*}
x \in \overline{H} & \iff \forall f \in F(f(x) > 0 \Rightarrow \exists v \in H(f(v) > 0)), \\
y \in \overline{H} & \iff \forall f \in F(f(y) > 0 \Rightarrow \exists v \in H(f(v) > 0)).
\end{align*}
\]

We show that \( x + y \in \overline{H} \), i.e., \( \forall f \in F(f(x + y) > 0 \Rightarrow \exists v \in H(f(v) > 0)) \). Let \( f \in F \) such that \( f(x + y) > 0 \).

By Proposition 4.2(iii) the map \( f^2_0 : X \to \mathcal{F} \in F \), where \( f^2_0(u) := f(u + y) \), for every \( u \in X \). By hypothesis, \( f^2_0(x) > 0 \). Since \( x \in \overline{H} \), there is \( z \in H \) such that

\[
f^2_0(z) > 0 \Rightarrow f(z + y) > 0 \Rightarrow f^1_2(y) > 0,
\]

where by Proposition 4.2(iii) the map \( f^1_2 : X \to \mathcal{F} \in F \), where \( f^1_2(u) := f(z + u) \), for every \( u \in X \), is in \( F \). Since \( y \in \overline{H} \), there is \( w \in H \) such that \( f^1_2(w) > 0 \Rightarrow f(z + w) > 0 \). Since \( H \leq X \), we get \( z + w \in H \), which is what we need to show. Next we show that \( -x \in \overline{H} \) i.e.,

\[
\forall f \in F(f(-x) > 0 \Rightarrow \exists v \in H(f(v) > 0)).
\]
Let \( f \in F \) such that \( f(-x) > 0 \Leftrightarrow f_-(x) > 0 \). Since \( f_- \in F \) and \( x \in \overline{H} \), there is \( v \in H \) such that \( f_-(v) := f(-v) > 0 \). Since \( H \leq X \), we get \( -v \in H \), and our proof is completed.

(ii) **Case I:** \( x \in H \) and \( y \in \overline{H} \).

We show that \( \text{abel}_x(y) = x \ 0^X \). Suppose that \( \text{abel}_x(y) \neq x \ 0^X \). By Proposition 5.2.5 in [28] there is \( f \in F \) such that \( f(\text{abel}_x(y)) = 1 \) and \( f(0^X) = 0 \). Since \( y \in \overline{H} \), and \( (f \circ \text{abel}_x)(y) = 1 > 0 \), and \( f \circ \text{abel}_x \in F \), there is \( v \in H \) such that \((f \circ \text{abel}_x)(v) > 0 \). Since \( x, v \in H \) and \( v \in H \), we have that \( \text{abel}_x(v) = 0^X \), hence 0 = \( f(0^X) = (f \circ \text{abel}_x)(v) > 0 \), which is a contradiction. Since \( F \) is separating, the canonical apartness relation \( \neq_F \) of \( F \) is tight (see Proposition 5.1.3 in [28]), hence the negation of \( \text{abel}_x(y) \neq x \ 0^X \) implies that \( \text{abel}_x(y) = x \ 0^X \).

**Case II:** \( x \in \overline{H} \) and \( y \in \overline{H} \).

We show that \( \text{abel}_x(y) = x \ 0^X \). Suppose that \( \text{abel}_x(y) \neq x \ 0^X \). As in the previous case, there is \( f \in F \) such that \( f(\text{abel}_x(y)) = 1 \) and \( f(0^X) = 0 \). Since \( y \in \overline{H} \), and \((f \circ \text{abel}_x)(y) = 1 > 0 \), there is \( v \in H \) such that \((f \circ \text{abel}_x)(v) > 0 \). By case I we have that \( \text{abel}_v(x) = 0^X \), hence by Lemma 5.4(v) we get \( \text{abel}_x(v) = 0^X \), hence 0 = \( f(0^X) = (f \circ \text{abel}_x)(v) > 0 \), which is a contradiction. Since \( F \) is separating, we conclude, similarly to Case I, that \( \text{abel}_x(y) = x \ 0^X \).

(iii) By Lemma 5.5(ii) it suffices to show that \( \text{normal}_x : H \rightarrow \overline{H} \), for every \( x \in X \). If \( x \in X \) and \( y \in \overline{H} \), we show that
\[
\text{normal}_x(y) := x + y - x \in \overline{H}.
\]
Let \( f \in F \) such that \( f(x + y - x) > 0 \Leftrightarrow (f \circ \text{normal}_x)(y) > 0 \). We need to find \( u \in H \) such that \( f(u) > 0 \). Since \( y \in \overline{H} \) and \( f \circ \text{normal}_x \in F \), there is \( v \in H \) with
\[
(f \circ \text{normal}_x)(v) := f(x + v - x) > 0.
\]
Since \( H \) is normal and \( v \in H \), we have that \( u := x + v - x \in H \) and \( f(u) > 0 \).

As we explain in [28], section 5.8, by the Stone-Čech theorem for Bishop spaces we have that the separating hypothesis on \( F \) in case (ii) of the previous proposition is not a serious restriction on \( F \). Note that classically, if \( X \) is a Hausdorff topological group, then if \( H \) is an abelian subgroup, then \( \overline{H} \) is abelian. The \( F \)-version of a Hausdorff topology is the following (see [28], section 5.2): if \( \neq \) is a given apartness relation on \( X \), then \( F \) is \( \neq \)-Hausdorff, if \( \neq \subseteq \neq_F \). In Proposition 5.2.3. of [28] we show that this is equivalent to the positive \( \neq \)-version of the induced neighborhood structure being Hausdorff. If \( F \) is \( \neq \)-Hausdorff, \( n \)-many pairwise \( \neq \)-apart points of \( X \) are mapped to given \( n \)-many real numbers. This is essential to the proof we gave above. So, we capture computationally the requirement of a Hausdorff topology for \( \overline{H} \) being abelian. By Proposition 5.3 the tightness of \( \neq_F \) in a Bishop topological group implies that the induced topology is Hausdorff in the classical sense (classically, a topological group is Hausdorff if and only if there is a closed singleton). In the next proposition the subset \( H \) of \( X \) is extensional i.e., \( H \) is closed under the given equality \( =_X \) on \( X \), so that the defining property of \( \text{Normal}_X(H) \) in the use of the separation scheme is also extensional.

**Theorem 5.7.** Let \( H \) be an extensional subset of \( X \). The normalizer \( \text{Normal}_X(H) \) and the center \( \text{Center}_X(H) \) of \( H \) in \( X \), are defined by
\[
\text{Normal}_X(H) := \{ x \in X \mid \text{Normal}_x^H : H \rightarrow H \},
\]
\[
\text{Center}_X(H) := \{ x \in X \mid \forall v \in H (\text{abel}_x(v) = 0^X) \}.
\]
(i) If \( H \) is closed, then \( \text{Normal}_X(H) \) is closed.
(ii) If \( F \) is separating, then \( \text{Center}_X(H) \) is closed.

**Proof.** (i) We suppose that \( H \) is closed i.e.,
\[
\text{(Hyp)} \quad \forall x \in X \left( \forall f \in F (f(x) > 0 \Rightarrow \exists v \in H (f(v) > 0)) \Rightarrow x \in H \right),
\]
and we show that \( \text{Normal}_X(H) \) is closed i.e.,
\[
\text{(Goal)} \quad \forall x \in X \left( \forall f \in F (f(x) > 0 \Rightarrow \exists u \in \text{Normal}_X(H) (f(u) > 0)) \Rightarrow x \in \text{Normal}_X(H) \right).
\]
For that we fix some $x \in X$ and we suppose that 

\[(\text{Hyp}_2) \quad \forall f \in F (f(x) > 0 \Rightarrow \exists u \in \text{Normal}_X(H)(f(u) > 0)),\]

and we show that $(\text{Goal}_2) \quad x \in \text{Normal}_X(H) := \text{Normal}^u_x : H \rightarrow H$. Let $v \in H$ be fixed, and we show

\[(\text{Goal}_3) \quad \text{Normal}^u_x(v) := v + x - v \in H.\]

By Hyp$_1$ it suffices to show the following

\[(\text{Goal}_4) \quad \forall f \in F \left(f(\text{Normal}^u_x(v)) > 0 \Rightarrow \exists w \in H(f(w) > 0)\right).\]

If we fix $f \in F$, we suppose that

\[(\text{Hyp}_3) \quad f(\text{Normal}^u_x(v)) > 0 \iff f(\text{normal}_v(x)) > 0 \iff (f \circ \text{normal}_v)(x) > 0,\]

and we show $(\text{Goal}_5) \quad \exists w \in H(f(w) > 0)$. Since $f \circ \text{normal}_v \in F$, by $(\text{Hyp}_2)$ there is $u \in \text{Normal}_X(H)$ with $(f \circ \text{normal}_v)(u) := f(v + u - v) > 0$. Since $u \in \text{Normal}_X(H)$, $\text{Normal}^u_x : H \rightarrow H$, and since $v \in H$, we get $\text{Normal}^u_x(v) := v + u - v \in H$. Hence, $w := v + u - v \in H$ and $f(w) > 0$.

(ii) We fix $x \in X$, we suppose that

\[(\text{Hyp}_4) \quad \forall f \in F (f(x) > 0 \Rightarrow \exists u \in \text{Center}_X(H)(f(u) > 0)),\]

and we show $(\text{Goal}_6) \quad x \in \text{Center}_X(H) := \forall v \in H(\text{abel}_x(v) = 0^X)$. Let $v \in H$ be fixed. Since $F$ is a separating Bishop topology, it suffices to prove $\neg(\text{abel}_x(v) \neq F 0^X)$. If we suppose that $\text{abel}_x(v) \neq F 0^X$, there is $f \in F$ such that

\[f(\text{abel}_x(v)) = 1 > 0 \quad \& \quad f(0^X) = 0.\]

By Lemma 5.4(iv) we get

\[1 = f(\text{abel}_x(v)) = f(\text{abel}_v(x)) := f_-(\text{abel}_v(x)) \quad \& \quad 0 = f(0^X) = f_-(0^X).\]

Since $f \in F$, we have that $f_- \circ \text{abel}_v \in F$ and $(f_- \circ \text{abel}_v)(x) > 0$. By $(\text{Hyp}_4)$ there is $u \in \text{Center}_X(H)$ such that $(f_- \circ \text{abel}_v)(u) > 0$. Since $u \in \text{Center}_X(H)$, we get $\forall v \in H(\text{abel}_u(w) = 0^X)$. Moreover, $(f_- \circ \text{abel}_v)(u) := f_-(v + u - v - u) > 0$. Since $v \in H$, we get

\[v + u - v - u := \text{abel}_u(u) = -\text{abel}_u(v) = -\text{abel}_v(v) = -0^X = 0^X,\]

hence $f_-(0^X) > 0$, which contradicts the previously established equality $f_-(0^X) = 0^X$. \hfill $\square$

**Proposition 5.8.** If $G$ is a separating Bishop topology on $Y$ and $h \in \text{Mor}(\mathcal{F}, \mathcal{G})$, the kernel $\text{Ker}(h) := \{ x \in X | h(x) = y 0^Y \}$ of $h$ is a closed set in $\mathcal{F}$.

**Proof.** Let $x \in X$ such that $\forall f \in F (f(x) > 0 \Rightarrow \exists v \in \text{Ker}(h)(f(v) > 0))$. Since $\not\exists G$ is tigh, it suffices to show that $\neg(h(x) \neq G 0^Y)$. If $h(x) \neq G 0^Y$, there is $g \in G$ such that $g(h(x)) = 1 > 0$ and $g(0^Y) = 0$. Since $h \in \text{Mor}(\mathcal{F}, \mathcal{G})$, we have that $g \circ h \in F$. As $(g \circ h)(x) > 0$, there is $v \in \text{Ker}(h)$ such that $0 = g(0^Y) = g(h(v)) = (g \circ h)(v) > 0$, which is a contradiction. \hfill $\square$

**Theorem 5.9** (Characterisation of a closed (open) subgroup). Let $C$ be a subgroup of $X$.

(i) $C$ is closed if and only if there is an open set $O \in X$ such that $O \cap C$ is inhabited and closed in $O$.

(ii) $C$ is open if and only if there is an inhabited, open set $O \in X$ such that $O \subseteq C$.

**Proof.** (i) Let $O$ be open in $X$ such that $O \cap C$ is inhabited and closed in $O$. We show that $C$ is closed. Suppose that $x \in C$, i.e., if $f(x) > 0$, there is $u \in C$ such that $f(u) > 0$, for every $h \in F$. We prove that $x \in C$. Let $c_0 \in O \cap C$. Since $O$ is open, there is $g \in F$ such that $g(c_0) > 0$ and $U(g) \subseteq O$. Since $\text{abel}_x := \text{abel}_x + c_0 \in F$ and

\[g^1_{-x+c_0}(x) := g(x - x + c_0) = g(c_0) > 0,\]

by our hypothesis on $x$ there is $c \in C$ such that

\[g^1_{-x+c_0}(c) := g(c - x + c_0) > 0.\]
As \( U(g) \subseteq O \), we get \( c - x + c_0 \in O \). The hypothesis \( \text{“} O \cap C \text{” is closed in } O \text{” means}

\[ \forall z \in O (z \in \overline{O \cap C} \Rightarrow z \in O \cap C). \]

Let \( z_0 := c - x + c_0 \in O \). We show that \( z_0 \in \overline{O \cap C} \), hence \( z_0 \in O \cap C \). Since \( C \) is a subspace, and since then \( z_0 \in C \), we get the required membership \( x \in C \). To show that \( z_0 \in \overline{O \cap C} \), let \( f \in F \) such that \( f(z_0) > 0 \). We find \( w \in O \cap C \) such that \( f(w) > 0 \). Since \( f(z_0) > 0 \) and \( g(z_0) > 0 \), we have that \( (f \wedge g)(z_0) > 0 \) (see [8], p. 57). By Theorem 5.6(i) \( C \) is a subgroup of \( X \). Since \( C \subseteq \overline{C} \), and \( c, c_0, -x \in \overline{C} \), we get \( z_0 \in \overline{C} \). Since \( f \wedge g \in F \) and \( (f \wedge g)(z_0) > 0 \), there is \( w \in C \) such that

\[ (f \wedge g)(w) > 0. \]

Since \( g(w) \geq (f \wedge g)(w) > 0 \), we get \( w \in O \), hence \( w \in O \cap C \). Since \( f(w) \geq (f \wedge g)(w) > 0 \), we conclude that \( f(w) > 0 \), as required. For the converse, if \( C \) is closed, then \( X \) is open, \( C = C \cap X \) is inhabited by 0 and it is trivially closed in \( X \).

(ii) Let \( O \) be open in \( X \) such that \( O \subseteq C \), and let \( c_0 \in O \). Suppose that \( g \in F \) with \( g(c_0) > 0 \) and \( U(g) \subseteq O \). Since also \( U(g) \subseteq C \), we get \( c_0 \in C \). Let \( c \in C \). The function \( g_{-c+c_0} \in F \) and \( g_{-c+c_0}(c) = g(c_0) > 0 \). We show that \( U(g_{-c+c_0}) \subseteq C \), hence, since \( c \) is an arbitrary element of \( C \), we conclude that \( C \) is open. Let \( u \in X \) such that \( g_{-c+c_0}(u) := g(u - c + c_0) > 0 \). As \( U(g) \subseteq O \subseteq C \), we get \( u - c + c_0 \in C \). As \( c, c_0 \in C \) and \( C \) is a subgroup of \( X \), we have that \( u \in C \). For the converse, if \( C \) is open, then \( C \) is an inhabited open set included in \( O \).

The classical proof of Theorem 5.9(i) is based on multiple use of negation (see [21]). As usual in constructive mathematics, we replaced the “non-empty intersection” of \( O \) and \( C \) in Theorem 5.9(i) with \( O \nsubseteq C \), and the “non-emptiness” of \( O \) in Theorem 5.9(ii) with the stronger inhabitedness of \( O \). Although, in general, it is not possible to show that the \( F \)-complement \( X \setminus F \) of a closed set in a Bishop space \( X \) is equal to \( X \setminus C \), there is a number of cases in the theory of Bishop topological groups where this is possible.

**Corollary 5.10.** If \( C \) is closed in \( X \), such that \( X \setminus C \) is a subgroup of \( X \) and \( X \setminus F \) is inhabited, then \( X \setminus C = X \setminus F \) and \( X \setminus C \) is clopen.

**Proof.** By Theorem 3.2 we have that \( X \setminus F \) is open in \( X \) and \( X \setminus F \subseteq X \setminus C \). Since \( X \setminus F \) is inhabited and \( X \setminus C \) is a subgroup of \( X \), by Theorem 5.9(ii) we have that \( X \setminus C \) is open. As \( X \setminus C \subseteq X \setminus C \), by Theorem 3.2 we get \( X \setminus C \subseteq X \setminus F \), hence \( X \setminus C = X \setminus F \). As \( X \setminus C \) is an open subgroup, by Proposition 5.3 we have that \( X \setminus C \) is also closed.

Notice that classically a subgroup \( C \) of a topological group is either clopen or has empty interior. By replacing the hypothesis of “non-empty interior of \( C \)” with the positive “existence of an inhabited open subset of \( C \)”, constructively we have the following corollary.

**Corollary 5.11.** Let \( C \) be a subgroup of \( X \). If \( O \) is an inhabited open set in \( X \) such that \( O \subseteq C \), then \( C \) is clopen.

**Proof.** By Theorem 5.9(ii) \( C \) is open, and by Proposition 5.3 \( C \) is also closed.

Here we have presented some very first, fundamental results in the theory of Bishop topological groups. Clearly, we can work similarly for other algebraic structures, like rings and modules, equipped with a compatible Bishop topology. There is a plethora of open questions related to Bishop topological groups. The complete regularity of a Bishop topological group, the study of “compact” subsets of Bishop topological groups, the uniform continuity of the elements of \( F \) when \( X \) is a compact Bishop topological group with \( F \), and the interpretation of local compactness in the theory of Bishop topological groups, are some of the numerous topics to which our future work could hopefully be directed.

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