Modeling and experimental investigations of Lamb waves focusing in anisotropic plates

Bastien Chapuis, Nicolas Terrien, Daniel Royer

1 Département Matériaux et Structures Composites, ONERA, 29 avenue de la Division Leclerc, 92322 Châtillon Cedex, France
2 CETIM, 74 route de la Jonelière, 44326 Nantes Cedex 3, France
3 Laboratoire Ondes et Acoustique, ESPCI, Université Paris 7, CNRS UMR 7587, 10 rue Vauquelin, 75231 Paris Cedex 05, France

Bastien.Chapuis@onera.fr

Abstract. The phenomenon of Lamb waves focusing in anisotropic plates is theoretically and experimentally investigated. An analysis based on a far field approximation of the Green’s function shows that Lamb waves focusing is analog to the phonon focusing effect. In highly anisotropic structures like composite plates the focusing of $A_0$ and $S_0$ mode is strong; the energy propagates preferentially in the fibre directions, which are minima of the slowness. This has to be taken into account when developing, for example, a transducer array for structural health monitoring systems based on Lamb waves in order to avoid dead zones.

1. Introduction

Over recent years, fiber-reinforced composites have been increasingly used in industrial structures, especially in wind turbines and aircrafts. This has created the need for reliable and economic nondestructive evaluation (NDE) methods for determining the integrity and serviceability of these composite structures. In that field, elastic guided waves are of a great interest since they can propagate over a large distance [1]. Therefore, only a limited number of sensors can monitor wide zones. Structural Health Monitoring (SHM) is a strategy which consists in integrating these sensors permanently in the structure so as to detect damages as frequently as necessary [2]. Thin piezoelectric transducers (PZT) bonded to the surface or embedded between composite plies are generally used as actuators and sensors of Lamb waves for SHM of plate-like structures. A sensor array is built so that the set of acoustic paths between the PZT covers the whole surface to be inspected [3]. However, the propagation of guided waves in these multilayered and anisotropic materials is complex. We will see that the anisotropy induces a focusing of Lamb waves in some directions which can be very strong in composite materials. A precise quantification of this phenomenon is therefore useful in order to avoid any dead zone in the transducer array.

In the first part of this article, a far-field approximation of the 3D Green’s function of a multilayered anisotropic plate is introduced. This formulation is based on the analysis developed by Velichko and Wilcox in [4]. As it occurs for bulk waves [5], Rayleigh waves [6], $SH_0$ modes [7] or plate modes [8], the focusing phenomenon is related to the Maris factor. Numerical computations of
this factor are provided for a cross-ply composite plate at frequencies typically used in structural health monitoring systems.

In the second part, the field radiated by a PZT disc bonded at the surface of a composite plate is studied. This source of diameter $D$ creates an axisymmetric distribution of radial force $f$. It has been demonstrated [9] that a center of compression is equivalent to a set of three orthogonal dipole of force. In the 2D case studied, the PZT disc is assumed to be equivalent to a set of two orthogonal dipole of strength $fD$. Experiments show that Maris factor provides a good approximation of the far field radiated by the PZT disc for $S_0$ mode and is the dominant term for $A_0$ mode at the frequencies considered.

2. Lamb waves focusing in an anisotropic plate

The determination of the far-field approximation of the 3D Green’s function of a multilayered anisotropic plate is fully described in [4], in which a strategy for computing this function is also presented. Only the major steps are recalled here.

2.1. Far-field approximation of the 3D Green’s function

A multilayered anisotropic plate with Cartesian coordinates $(x_1, x_2, x_3)$ is considered with the axis $x_3$ normal to the surface. As shown in Figure 1, the acoustic source is modeled by a force distribution acting on the upper surface of the plate, at $x_3 = 0$. Using the Fourier transform method to solve the wave equation, the displacement field $u$ produced by a time-harmonic force $f(x_1,x_2)e^{-i\omega t}$ is related to the 3D Green’s function of the system $g(x_1,x_2,x_3)$ by:

$$u(x_1,x_2,x_3) = \int\int g(x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, x_3) \times f(x_1 - \tilde{x}_1, x_2 - \tilde{x}_2) d\tilde{x}_1 d\tilde{x}_2$$

(1)

A 2D spatial transform of this function gives:

$$g(x_1,x_2,x_3) = \frac{1}{(2\pi)^2} \int\int G(k_1,k_2,x_3) e^{i(k_1x_1 + k_2x_2)} dk_1 dk_2 ,$$

(2)

where $G(k_1,k_2,x_3)$ stands for the Green’s function for plane waves propagating in the direction given by the wave vector $k(k_1 = k \cos \theta, k_2 = k \sin \theta, 0)$, with $\theta$ the angle between the direction of the phase velocity and the $x_1$ axis.

In the following, polar coordinates are used both for displacements and forces. For a given source/receiver configuration defined by the distance $r = OP$ and the angle $\varphi$, $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$, $x_3 = z$. The Green’s function can be written as:
\[ g(r, \varphi, z) = \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \left( \int_{\varphi - \pi/2}^{\varphi + \pi/2} G(k \cos \theta, k \sin \theta, z) e^{ikr \cos \psi} dk \right) d\theta \]  

where \( \Gamma \) is the real positive half axis and \( \psi = \varphi - \theta \).

The residue theorem is then used to evaluate the far field contribution as \( r \to \infty \). The contribution of complex poles in the far field is neglected as these modes are localized near the source and cannot propagate.

The Green’s function for propagating modes \( M \) is so given by:

\[ g(r, \varphi, z) = \frac{1}{2\pi} \sum_M G_M(\theta, z) \left| k_M(\theta) \right| e^{ik_Mr \cos \psi_M} dk d\theta \]  

However, in the far-field, for each Lamb mode \( M \) and for a given direction of observation \( \varphi \), only angles \( \theta \) for which the phase \( \Phi_M(\theta) = k_Mr \cos \psi \) of the oscillatory term varies slowly give a significant contribution. This is an application of the stationary phase method. The first derivative of the phase \( \Phi_M \) vanishes for values \( \theta_M \) such that:

\[ \frac{dk_M}{d\theta} = -k_M \tan \psi_M, \quad \text{with} \quad \psi = \varphi - \theta. \]  

As shown in Figure 2, equation (5) indicates that the propagation direction for which the phase is stationary is such that the observation direction is parallel to the energy velocity \( V_e \), i.e. normal to the slowness curve \( s(\theta) = k/\omega \). In the following, we assume that there will be only one stationary phase point. In this case, the slowness curve is convex for all \( \theta \) and equation (5) defines a one-to-one relation between angles \( \varphi \) and \( \theta \). The application of the stationary phase method gives the following far-field approximation of the 3D Green’s function [10]:

\[ g(r, \varphi, z) = \frac{1}{2\pi} \sum_M G_M(\theta_M, z) e^{\Phi_M + \text{sign} \Phi_M} \left| A_M k_M \cos \psi_M \right|, \]  

where the factor \( A_M \) is defined for each Lamb mode \( M \) by the relation:

\[ A = \frac{d\theta}{d\varphi}. \]  

The factor \( A \) measures the acoustic ray density anisotropy shown in Figure 3. It is called Maris factor since it has been postulated by Maris to explain the phonon focusing effect [11]. This factor is related to the slowness curve of the mode plotted at the angular frequency \( \omega \) by [8]:

\[ A = \left[ k^2 + (ds/d\omega)^2 \right]^{1/2} \left| K_s \right|^{-1}, \]  

where \( K_s \) is the curvature of the slowness curve. Equation 8 gives a geometrical interpretation of the limitation of the approach developed here. When \( K_s \to 0 \), i.e. at transition points for which the slowness curve changes from a convex shape to a concave one, Maris factor presents singularities. This is due to the presence of caustics, for which the second derivative of the phase in equation (4) also vanishes.
Maris factor has already been used to describe the directivity pattern of bulk [5] and Rayleigh waves [6]. It has also been highlighted for plate modes [7] and $S_{0}$ mode [8] at low frequency-thickness product. The formulation given in this paper shows that this factor plays a role for Lamb waves too.

As shown in equation (6) many terms are responsible for the angular dependence of the displacement. However the focusing factor $\sqrt{A}$ (which is the relevant quantity for the displacement) is generally found to dominate the others in highly anisotropic materials [5]. The term $G_{M}(\theta_{M},x_{y})$ is related to the way that Lamb waves are excited and detected. It can be calculated by the procedure given in [4].

The description given here is for a monochromatic wave, the only dispersion involved in the problem is the angular dispersion of Lamb waves. In this case and as noted in [12], it is the direction of the group velocity that is important and not its modulus. However, to fully describe the focusing phenomenon, a study at different frequencies is necessary. The frequency dispersion is expected to soften the phenomenon [13] since the focusing patterns of the spectral contributions in a wave packet are different.

2.2. Computation of the focusing factor in composite plates

The computation of the Maris factor (or of the focusing factor) is easily derived from the dispersion curves. In this paper the dispersion curves have been determined using the SAFE method [14]. The plate considered here is a $[0^\circ/90^\circ]_{s}$-T700GC/M21 cross-ply composite fiber reinforced polymer (CFRP) whose characteristics are given in Table 1. Dispersion curves show that below 500 kHz and in any direction of the plate only three modes can propagate: the two fundamental Lamb modes $A_{0}$ and $S_{0}$ and the first shear horizontal mode $S H_{0}$. However it should be noted that, except for $0^\circ$ and $90^\circ$ directions, the polarizations of $S_{0}$ and $S H_{0}$ modes are never purely radial nor orthoradial. In the following $S_{0}$ denotes the quasi-longitudinal mode and $S H_{0}$ the quasi-shear mode. In the next part we will show that $S_{0}$ and $S H_{0}$ modes are in fact coupled in some direction by the axisymmetric source used in the experiments (see also [4] for the case of an out-of-plane source).

Table 1. Parameters of the CFRP T700GC/M21 (Hexcel Composites).

| Parameter          | Value     |
|--------------------|-----------|
| $C_{11}$ (GPa)     | 123.44    |
| $C_{22} = C_{33}$ (GPa) | 11.54    |
| $C_{12} = C_{13}$ (GPa) | 5.55     |
| $C_{23}$ (GPa)     | 6.35      |
| $C_{44}$ (GPa)     | 2.6       |
| $C_{55} = C_{66}$ (GPa) | 4.5      |
| Mass density (kg/m$^3$) | 1.6×10$^3$ |
| Ply thickness (mm) | 0.25    |
Slowness curves and associated focusing factors for A_0 mode at different frequencies are plotted in Figure 4. At 40 kHz, the focusing factor is maximal in the direction 0°, but a second local maximum exists in the direction ϕ = 90°. The amplitude ratio between the direction ϕ = 90° and ϕ = 0° is 47%, the minimum occurs in the direction ϕ = 50° where the ratio is only 23.4% compared to the direction ϕ = 0°. When the frequency increases, the shape of the focusing factor becomes more and more complex.

![Figure 4](image1.png)

**Figure 4.** (a) Slowness curves in s.km\(^{-1}\) in a cross-ply [0°/90°]_S CFRP T700GC/M21 for the A_0 mode (b) Corresponding focusing factor \(\sqrt{A}\).

Slowness curve and associated focusing factor for S_0 mode at 300 kHz are presented in Figure 5. Since this mode is non dispersive at low frequency-thickness products, the curves obtained for the frequencies below 300 kHz are superimposed. The focusing occurs at ϕ = 0° and 90°, and the minimum is in the direction ϕ = 45°. The focusing is very acute in fiber direction since the ratio between the minimum and the maximum of the focusing factor \(\sqrt{A}\) is 8%. In the frequency range [40 kHz-300 kHz], S_0 mode focusing is much more important than A_0 mode focusing.

![Figure 5](image2.png)

**Figure 5.** (a) Slowness curves in s.km–1 in a cross-ply [0°/90°]_S CFRP T700GC/M21 for S_0 and SH_0 modes at 300 kHz (b) Focusing factor \(\sqrt{A}\) for the S_0 mode.
Slowness curve of SH₀ mode is also presented in Figure 5. This curve is an example of situation for which the slowness curve is not always convex. Due to the presence of two caustics at $\varphi = 7^\circ$ and $\varphi = 83^\circ$, the focusing factor becomes singular and the model previously developed is no more valid.

3. Acoustic field radiated by a PZT disc

The radiated field from a PZT disc is very different from one mode to the other. The effect of the anisotropy on the propagation of the guided waves has been observed with an air-coupled transducer mounted in a XY-translational stage, which permits to observe the wave front during the propagation in the plate. Figure 6 shows a cartography at a fixed time in the propagation of $S₀$ and $A₀$ modes excited by a thin PZT disc of diameter $D = 10$ mm bonded at the center of a $[0^\circ/90^\circ]_\text{s}$-T700GC/M21 cross-ply composite of thickness $d = 1$ mm. The PZT disc is excited with a three cycles tone burst of amplitude $10 \text{ V}_{pp}$ and a repetition rate of $10 \text{ ms}$, the detected signal is averaged 200 times at each measurement point. The PZT disc imposes an axisymmetric force at the surface of the plate along its periphery [15]. Lamb waves of wavelength $\lambda = 1.7 \times D$, i.e. 17 mm, having a large radial displacement are preferentially launched [16]. Taking into account the dispersion curves, the wave number-thickness product $kd = 0.37$ corresponds to frequencies equal to 40 kHz and 300 kHz for $A₀$ and $S₀$ modes, respectively.

Due to the anisotropy of the material, the $S₀$ mode travels faster in the directions $\varphi = 0^\circ$ and $90^\circ$. The $A₀$ mode is less sensitive to the anisotropy. However, for both modes an increase in the amplitude signal is detected at $0^\circ$ and $90^\circ$ (less visible for $A₀$ mode in the second case).

![Figure 6](image)

**Figure 6.** Cartography of (b) $S₀$ mode at 300 kHz and (c) $A₀$ mode at 40 kHz propagating in a cross-ply CFRP plate obtained by scanning the plate with an air-coupled transducer. Lamb waves are generated with a thin PZT disc bonded at the center of the plate.

Due to their respective polarization, $S₀$ and SH₀ modes are coupled and it is not possible to excite them separately. However, as discussed in [9], the generation by the PZT and the detection techniques used are much less efficient for SH₀ mode than for $S₀$ mode. Therefore, due to the shape of the slowness curves presented in Figure 5, the contribution of the shear mode should be significant only in the direction of the two caustics, for $\theta = 35^\circ$ and $\theta = 55^\circ$, i.e. for observation directions $\varphi = 7^\circ$ and $\varphi = 83^\circ$.
Green’s function represents the response of the plate to a point-like source. In practice, the lateral dimensions of the source must be smaller than the wavelength. With \( D \equiv \lambda/2 \), this condition is nearly fulfilled and due to the symmetry of the source, the finite dimension of the PZT disc introduces only a frequency filtering. Therefore, the Green’s function calculated in the first part is a good approximation of the far-field radiated by the PZT disc. To comply with our model hypothesis, the normal displacement at surface of the plate is measured at \( r = 10 \text{ cm} \) from the center of the PZT disc. Since \( kr \approx 40 \), the far-field assumption is fulfilled. Since the Green’s function is calculated at a given frequency, the spectral contribution at the desired frequency is extracted by a Fast Fourier Transform of the incident wave packet measured in each direction.

Results for \( S_0 \) and \( SH_0 \) modes at 300 kHz are presented in Figure 7 (a). The measurements have been achieved with an air-coupled transducer and also with a BMI heterodyne interferometer. In this case, amplitude fluctuations can be ascribed to the surface roughness of the plate. The air-coupled transducer is not affected by the roughness, but its spot size tends to enlarge the central lobe. The strong focusing in the direction 0° for \( S_0 \) mode is however clearly observed and fits well with the prediction. \( \sqrt{A} \) is a very good approximation of the field radiated by the PZT disc. The possible contribution of \( SH_0 \) mode at \( \phi = \pm 7^\circ \) is blurred in the central peak at 0° and consequently cannot be highlighted.

For the \( A_0 \) mode at 40 kHz, measurements have been performed with a BMI heterodyne interferometer, an air-coupled transducer and a Polytech velocimeter. Results are presented in Figure 7 (b). Small discrepancies exist between the three curves, however they clearly indicates that the focusing occurs mainly in the direction \( \phi = 0^\circ \), and with less intensity in the 90°-direction. The agreement between theoretical and experimental curves is rather good. Conversely to \( S_0 \) mode case, \( \sqrt{A} \) is not sufficient to fully describe the spectral displacement field radiated by the PZT disc for \( A_0 \) mode at 40 kHz. However, this factor is still the dominant term and indicates the directions of focusing.

![Figure 7](image_url)

**Figure 7.** (a) \( S_0 \) Lamb mode at 300 kHz. Normalized displacement measured at the surface of the cross-ply composite plate with an air-coupled transducer (dashed line) and an heterodyne interferometer (solid line) compared with the theoretical value (cross) and focusing factor \( \sqrt{A} \) (circles). (b) \( A_0 \) Lamb mode at 40 kHz. Normalized displacement measured at the surface of the cross-ply composite plate with a velocimeter (dots), an air-coupled transducer (dashed line) and an heterodyne interferometer (solid line) compared with the theoretical value (cross) and focusing factor \( \sqrt{A} \) (circles).
4. Conclusion
An approximation of the Green’s function giving a general description of the generation of guided waves in a thin anisotropic plate was presented and the role of the Maris factor in the phenomenon of Lamb mode focusing was emphasized. The far-field radiation of PZT discs bonded at the surface of composite plates can be described by this Green’s function. It was experimentally and numerically shown that in cross-ply composite plates, focusing occurs mainly in fiber directions and can be very strong, especially for $S_0$ mode. The investigation presented here provides useful information for the development of structural health monitoring systems based on PZT discs array in anisotropic structures.

References
[1] Alleyne D and Cawley P 1992 IEEE Trans. Ultrasonics, Ferroelec. and Freq. Control 39 381–397
[2] Boller C Structural Health Monitoring - An Introduction and Definitions (Encyclopedia of Structural Health Monitoring vol 1) eds C Boller, F-K Chang and Y Fujino, (Chichester: John Wiley & Sons Ltd), chapter 1 p 3–25
[3] Su Z, Ye L and Lu Y 2006 J. Sound Vib. 295 753–780
[4] Velichko A and Wilcox P D 2007 J. Acoust. Soc. Am. 121 60–69
[5] Royer D 2001 Ultrasonics 39 345–354
[6] Maznev A, Lomonosov A, Hess P and Kolomenskii A 2003 Eur. Phys. J. B 35 429–439
[7] Chen H-Y and Man C-S 1994 Appl. Phys. Lett. 64 966–968
[8] Maznev A and Every A 1995 Acta Acustica 3 387–391
[9] Achenbach J D 1980 Wave Propagation in Elastic Solids (Amsterdam: North-Holland) p 101
[10] Chapuis B, Terrien N and Royer D 2010 J. Acoust. Soc. Am. 127 198–203
[11] Maris H 1971 J. Acoust. Soc. Am. 50 812–818
[12] Potel C, Baly S, de Bellevial J-F, Lowe M and Gatignol P 2005 IEEE Trans. Ultrasonics, Ferroelec. and Freq. Control 52 987–1001
[13] Neau G 2003 PhD dissertation Université Bordeaux 1
[14] Liu G, Tani J, Ohyoshi T and Watanabe K 1991 J. Vib. Acoust. 113 230–234
[15] Giurgiu V 2005 J. Intell. Mater. Syst. Struct. 16 291–305
[16] Rose L R F and Wang C H 2004 J. Acoust. Soc. Am. 116 154–171