Lie and Leibniz algebras of lower-degree conservation laws

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Received 2 August 2021
Accepted for publication 31 December 2021
Published 21 January 2022

Abstract
A relationship between the asymptotic and lower-degree conservation laws in (non-) linear gauge theories is considered. We show that the true algebraic structure underlying asymptotic charges is that of Leibniz rather than Lie. The Leibniz product is defined through the derived bracket construction for the natural Poisson brackets and the BRST differential. Only in particular, though not rare, cases that the Poisson brackets of lower-degree conservation laws vanish modulo central charges, the corresponding Leibniz algebra degenerates into a Lie one. The general construction is illustrated by two standard examples: Yang–Mills theory and Einstein’s gravity.

Keywords: symmetries and conservation laws, lower-degree conservation laws, asymptotic charges, Leibniz algebras

1. Introduction
Although exact symmetries seldom if ever occur in nature, they do play a significant role in our understanding of physical laws. The famous Noether’s theorem, for example, establishes a correspondence between global symmetries and conservation laws in Lagrangian theories. Local or gauge symmetries, in turn, govern the structure of fundamental interactions in the standard model of particle physics and general relativity. Unlike global symmetries, gauge invariance never leads to nontrivial conserved currents, as was found by Emmy Noether herself. Yet, in some cases, it is possible to associate certain conserved charges with gauge symmetries as well. These are given by integrals over lower-dimensional surfaces rather than entire physical space. Therefore one calls them the lower-degree conservation laws. For a detailed account of the subject we refer the reader to [1–5] and references therein.

It is not rare in field theory to consider a situation where the interaction of fields is presumably localized in a compact region of space, outside of which the fields behave as almost free.

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Such a supposition is a departing point of the scattering problem and an indispensable element of the multi-particle interpretation of quantum field states. In that case, one can compute the lower-degree conserved charges by integrating over surfaces lying entirely outside the interaction region. As the fields under the integral sign are supposed to be almost free, one may disregard much of the nonlinearities in the Lagrangian and charges. Not only does this choice of integration surface facilitate the actual computation of charges, but it also leads to a further generalization of the very notion of a conservation law. Indeed, if a nonlinear gauge theory admits a reasonable linearization, it may happen that the linear theory enjoys lower-degree conservation laws that are not present in the full nonlinear theory. Then one may attribute the corresponding lower-degree charges to the nonlinear theory itself or, more precisely, to those its solutions that differ arbitrary little from solutions of the linear theory outside a compact region of space (the interaction region). In the physical literature, one usually refers to the charges assigned in such a way to a nonlinear gauge theory as asymptotic charges. The ADM energy in general relativity and color charges in Yang–Mills (YM) theory are prime examples of asymptotic charges defined through the integration over two-sphere at space infinity. For an extended discussion of the above approach to asymptotic conservation laws see [3,6].

In Lagrangian theories, both the conventional and lower-degree conservation laws come equipped with a Lie algebra structure w.r.t. some natural Poisson brackets (canonical or covariant) [7–9]. Modulo central extension, this algebra is known to be isomorphic to the Lie algebra of infinitesimal symmetries. If asymptotic charges are now defined through the linearization procedure above and do not survive in the full nonlinear theory, one runs into a problem: linearization usually implies abelization of the gauge symmetry algebra, so that the corresponding asymptotic charges may form either an abelian Lie algebra or a central extension thereof, whatever a nonlinear gauge theory. This conclusion, however, contradicts to some other approaches to asymptotic conservation laws that do not appeal directly to linearization [10–13]. In the general case one expects the nonabelian gauge symmetries to generate nonabelian Lie algebras of surface charges modulo central extension. We thus face the following dilemma: a nonabelian gauge theory per se may have no nontrivial conservation laws of lower degree, while the lower-degree conservation laws resulting from its linearization cannot form a nonabelian Lie algebra w.r.t. Poisson brackets.

In this paper, we resolve this puzzle by introducing a new product on the space of lower-degree conservation laws of a linearized gauge theory. The product is constructed as a derived bracket [14] and involves the original Poisson brackets together with the classical BRST differential. It is the dependence of interaction through the BRST differential that restores a nonabelian algebra structure on conserved charges. Generally the product we introduce is not skew-symmetric but satisfies the axioms of a Leibniz algebra. This suggests that a ‘genuine’ algebraic structure underlying asymptotic symmetries and conservation laws is that of Leibniz rather than Lie1. In order to test and illustrate our construction we re-derive the well-known algebras of surface charges in Einstein’s gravity and YM theory.

Compared to other approaches our method is purely algebraic: we are not concerned with the fall-off of fields at infinity or suitable boundary conditions, nor do we make any assumption about the differential structure of field equations and/or gauge generators. What we actually use is the separation of field equations into a free part and interaction. The separation is clearly ambiguous and we regard it as part of the definition of a classical field theory. Another advantage of our construction is that it applies equally well to non-Lagrangian equations of motion

1 An instructive discussion of the origin of Leibniz algebras in the context of gauge symmetries can be found in [15, section 1].
endowed with a compatible presymplectic structure. The presence of a presymplectic structure is known to be much less restrictive for dynamics than the existence of a Lagrangian.

2. Variational tricomplex of a gauge system

This section provides a brief glossary on local gauge systems in the formalism of variational tricomplex. For a more coherent exposition of these concepts we refer the reader to [4, 7, 16, 17]. Throughout the paper, we systematically use the notation and terminology of [5].

**Classical fields.** In modern language classical fields are sections of a locally trivial fiber bundle \( \pi : E \to M \) over a spacetime manifold \( M \). The typical fiber of \( E \) is called the target space of fields. The space of all field configurations is thus identified with the space of smooth sections \( \Gamma(E) \). In this paper, we restrict ourselves to the case where \( E \) is a \( \mathbb{Z} \)-graded vector bundle over \( M \). As is customary in the physical literature, we will refer to this \( \mathbb{Z} \)-grading as the ghost number and denote the degree of a homogeneous object \( A \) by \( gh(A) \). The Grassmann parity of fields (which governs the signs) is given by the ghost number modulo two. In physical terms this means that we restrict ourselves to gauge theories without fermionic degrees of freedom. The extension of our results to general theories with bosonic and fermionic fields is straightforward.

**Variational bicomplex.** According to the principle of spacetime locality, the classical dynamics of fields are governed by partial differential equations. The jet-bundle formalism offers then a natural geometric framework for formulating and studying local field theories. A relevant jet-bundle for our considerations is the bundle of infinite jets \( \pi : J^\infty E \to M \) associated with the vector bundle \( \pi : E \to M \). Each section \( \varphi \) of \( E \) induces a section \( j^\infty \varphi \) of \( J^\infty E \) by the following rule. If \( E|_U = U \times \mathbb{R}^n \) is a trivializing chart with local coordinates \((x^i, \phi^a)\), then \((x^i, \phi^a, \phi^a_1, \phi^a_2, \ldots)\) are local coordinates in \( J^\infty E|_U \) and the induced section \( j^\infty \varphi : M \to J^\infty E \) is defined by

\[
x \mapsto (x, \varphi^a(x), \partial_i \varphi^a(x), \partial_i \partial_j \varphi^a(x), \ldots).
\]

The section \( j^\infty \varphi \) is called the \( \infty \)-jet prolongation of \( \varphi \).

The totalspace of \( J^\infty E \), defined through the inverse limit \( \lim_{\to} J^k E \) of finite-dimensional jet-bundles, inherits the structure of a \( \mathbb{Z} \)-graded manifold. Let \( \Lambda(J^\infty E) = \lim_{\to} \Lambda(J^k E) \) denote the algebra of differential forms on \( J^\infty E \). It is known that the de Rham complex of \( \Lambda(J^\infty E) \) splits naturally into a bicomplex for the vertical differential \( \delta \) and horizontal differential \( d \), so that

\[
\delta^2 = 0, \quad d^2 = 0, \quad \delta d + d \delta = 0,
\]

with \( \delta \) being the exterior differential in \( \Lambda(J^\infty E) \). In the adapted coordinates above, every form on \( J^\infty E \) can be written as a finite sum of homogeneous forms

\[
f \delta \phi^a_{i_1} \wedge \cdots \wedge \delta \phi^a_{i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} \in \Lambda^{p|q}(J^\infty E),
\]

where \( f \) is a smooth function on \( J^\infty E|_U \) and \( I = i_1 i_2 \ldots i_p \) denotes the multi-index of order \(|I|\). The numbers \( p \) and \( q \) are called the vertical and horizontal degree of a form, respectively. Since the vertical differential \( \delta : \Lambda^{p|q}(J^\infty E) \to \Lambda^{p+1|q}(J^\infty E) \) implements the action of the variational derivative on fields, one refers to the bicomplex \( \Lambda^{*,*}(J^\infty E; \delta, d) \) as variational.
**Classical BRST differential.** An evolutionary vector field\(^2\) \(Q\) on \(J^\infty E\) is called *homological* if
\[
[Q, Q] = 0, \quad \text{gh}(Q) = 1. \tag{2.4}
\]
In what follows we will use the symbol \(\delta_Q\) to denote the Lie derivative along \(Q\). Clearly, \(\delta_Q^2 = 0\). Moreover, the operator \(\delta_Q\) anticommutes with the differentials \(d\) and \(\delta\):
\[
\delta_Q d + d\delta_Q = 0, \quad \delta_Q \delta + \delta \delta_Q = 0. \tag{2.5}
\]
The equalities follow from Cartan’s formula for the Lie derivative and the fact that the vector field \(Q\) is evolutionary. This yields the *variational tricomplex* \(\Lambda^{\bullet,\bullet}(J^\infty E; \delta, d, \delta_Q)\), where the third differential
\[
\delta_Q : \Lambda^{p,q,d}(J^\infty E) \to \Lambda^{p,q+1,d}(J^\infty E) \tag{2.6}
\]
increases the ghost number by one. Under certain ‘properness’ conditions (see e.g. [16]) the operator \(\delta_Q\) is called the *classical BRST differential*. The corresponding homological vector field \(Q\) carries all the information about a classical gauge system. In particular, the true field configurations are determined by the stationary points of \(Q\). Under the standard regularity assumptions the latter form a graded submanifold \(\Sigma^\infty \subset J^\infty E\) defined through the inverse limit. By definition, a field \(\varphi \in \Gamma(E)\) satisfies the classical equations of motion iff \(j^\kappa \varphi \in \Sigma^\infty\).

In physics, the submanifold \(\Sigma^\infty \subset J^\infty E\) is usually referred to as the *shell*. By definition, the shell is invariant under the action of the homological vector field \(Q\). The restriction of the variational bicomplex \(\Lambda^{\bullet,\bullet}(J^\infty E; \delta, d)\) on \(\Sigma^\infty\) gives the *on-shell bicomplex* \(\Lambda^{\bullet,\bullet}(\Sigma^\infty; \delta, d)\). The latter inherits the additional grading by the ghost number. In general, the on-shell bicomplex is not \(d\)-exact even locally and this gives rise to interesting invariants associated with gauge dynamics. The most notable among them are the cohomology groups \(H^{p,q}(\Sigma^\infty; d)\) in ghost number zero. These are known as the *characteristic cohomology* of a gauge system [1–4, 13, 18]. The study of natural algebraic structures on characteristic cohomology is the main subject of the present paper.

**Presymplectic structure.** Another geometric entity present in most gauge theories is called a *presymplectic structure*. This is given by a form \(\omega \in \Lambda^2(J^\infty E)\) obeying the condition
\[
\delta \omega \simeq 0. \tag{2.7}
\]
Hereinafter the sign \(\simeq\) means equality modulo \(d\)-exact forms. Notice that the horizontal degree of a presymplectic form may take any value in the interval \(0 \leq m \leq \dim M\). We also impose no restriction on the ghost number of \(\omega\). Two presymplectic forms are considered equivalent if they differ by a \(d\)-exact form. By abuse of notation, we will not distinguish between a presymplectic form \(\omega\) and its equivalence class in \(\Lambda^2(J^\infty E)/d\Lambda^{2,m-1}(J^\infty E)\). As we are dealing with a vector bundle \(E\), the relative \(\delta\) modulo \(d\) cohomology appears to be trivial in positive vertical degree [7, section 19.3.9]. Among other things this means that each presymplectic structure has a \(\delta\)-exact representative, i.e. there exists \(\theta \in \Lambda^{1,m}(J^\infty E)\) such that \(\omega \simeq \delta \theta\). The form \(\theta\) is called a *presymplectic potential*. In what follows we will always work with \(\delta\)-exact representatives \(\omega = \delta \theta\) of presymplectic forms, so that \(\delta \omega = 0\).

\(^2\) Recall that a vertical vector field \(X\) is called evolutionary if \(i_X d + (-1)^{\dim X} d i_X = 0\), where \(i_X\) is the operation of contraction of \(X\) with differential forms. The Lie algebra of evolutionary vector fields will be denoted by \(\mathcal{X}_e(J^\infty E)\).
An evolutionary vector field $X$ is called Hamiltonian relative to $\omega$ if $L_X \omega \simeq 0$. Again, the triviality of relative $\delta$-cohomology implies that
\[ i_X \omega \simeq \delta \alpha \tag{2.8} \]
for some $\alpha \in \Lambda^{0,1}(J^\infty E)$. It is natural to refer to the form $\alpha$ as Hamiltonian or as a Hamiltonian of the vector field $X$. Notice that equation (2.8) defines $\alpha$ up to a $d$-exact form; hence, two Hamiltonians $\alpha$ and $\alpha'$ are considered equivalent if $\alpha \simeq \alpha'$. The space of all Hamiltonian $m$-forms is a graded Lie algebra with the bracket
\[ \{ \alpha, \beta \} = (-1)^{gh(\alpha) \cdot gh(\beta)} i_{X_\alpha} i_{X_\beta} \omega \tag{2.9} \]
of degree $-gh(\omega)$. Here $X_\alpha$ and $X_\beta$ are Hamiltonian vector fields with the Hamiltonians $\alpha$ and $\beta$. The bracket enjoys the symmetry property
\[ \{ \alpha, \beta \} \simeq -(-1)^{gh(\alpha) \cdot gh(\beta)} \{ \beta, \alpha \} \tag{2.10} \]
and obeys the Jacobi identity
\[ \{ \gamma, \{ \alpha, \beta \} \} \simeq \{ \{ \gamma, \alpha \}, \beta \} + (-1)^{gh(\gamma) \cdot gh(\beta)} \{ \{ \gamma, \beta \}, \alpha \} \tag{2.11} \]
For more details see [9, proposition 2.1].

**Gauge systems and their descendants.** By a gauge system on $J^\infty E$ we mean a pair $(Q, \omega)$ composed of a homological vector field $Q$ and a $Q$-invariant presymplectic form $\omega$ of type $(2, m)$. In other words, the vector field $Q$ is supposed to be Hamiltonian relative to $\omega$, so that $\delta Q \omega \simeq 0$. The last relation is equivalent to
\[ \delta Q \omega = d \omega_1 \tag{2.12} \]
for some $\omega_1 \in \Lambda^{2,m-1}(J^\infty E)$. We will refer to the horizontal degree of the form $\omega$ as the degree of a gauge system $(Q, \omega)$. Applying the differentials $\delta$ and $\delta Q$ to both sides of (2.12) and using the acyclicity of $d$ in positive vertical degree, we find that
\[ \delta \omega_1 \simeq 0, \quad \delta Q \omega_1 \simeq 0. \]
Hence, $\omega_1$ is a $Q$-invariant presymplectic form of type $(2, m - 1)$. Again, without loss of generality we may assume $\omega_1 = \delta \theta_1$ for some presymplectic potential $\theta_1 \in \Lambda^{1,m-1}(J^\infty E)$. We call the pair $(Q, \omega_1)$ the descendant gauge system. Iterating the above construction once and again, one can produce a sequence of gauge systems $(Q_k, \omega_k)$ where the $k$th presymplectic structure $\omega_k \in \Lambda^{2,m-k}(J^\infty E)$ is the descendant of $\omega_{k-1}$. The minimal $k$ for which $\omega_k \simeq 0$ is called the length of a gauge system.

**Symmetries.** An evolutionary vector field $X$ is called a symmetry of a gauge system $(Q, \omega)$ if it commutes with $Q$, i.e. $[X, Q] = 0$. Therefore the flow generated by $X$ preserves the shell $\Sigma^\infty \subset J^\infty E$ mapping solutions to solutions.

A symmetry $X$ is called trivial or gauge symmetry if their exists an evolutionary vector field $Y$ such that $X = [Q, Y]$. Clearly, the gauge symmetries form an ideal in the Lie algebra of all symmetries, so that one may regard the corresponding quotient algebra, denoted by Sym$(Q)$, as the Lie algebra of nontrivial symmetries. Alternatively, we can identify the nontrivial symmetries Sym$(Q)$ with the cohomology of the differential graded Lie algebra $(X_{ev}(J^\infty E), \delta_Q)$. 

\[ \text{J. Phys. A: Math. Theor. 55 (2022) 065201} \]

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A symmetry $X$ is called Hamiltonian if $X$ is a Hamiltonian vector field, that is, $L_X\omega \simeq 0$. Finally, we say that a symmetry $X$ is on-shell Hamiltonian if

$$i_X\omega|_{\Sigma^\infty} \simeq \delta \alpha|_{\Sigma^\infty}$$

(2.13)

for some $\alpha$. Read from right to left, this relation defines an on-shell Hamiltonian form $\alpha$. Writing $\text{Sym}(Q, \omega)$ and $\text{Sym}_\Sigma(Q, \omega)$ for the Lie algebras of Hamiltonian and on-shell Hamiltonian symmetries, respectively, we get the following sequence of subalgebras in the Lie algebra of vector fields:

$$\text{Sym}(Q, \omega) \subset \text{Sym}_\Sigma(Q, \omega) \subset \text{Sym}(Q) \subset X_{\text{ev}}(J^\infty E) \subset X(J^\infty E).$$

When dealing with on-shell Hamiltonian symmetries and forms it is convenient to introduce the following equivalence relation on the space of differential forms $\Lambda(J^\infty E)$:

$$\alpha \approx \alpha' \iff \alpha|_{\Sigma^\infty} \simeq \alpha'|_{\Sigma^\infty}.$$  

(2.14)

Then equation (2.13) takes the form $i_X\omega \approx \delta \alpha$. Although the last equation implies $L_X\omega \approx 0$, the converse is not always true as the on-shell bicomplex may not be $\delta$-exact even for vector bundles. Notice that equation (2.13) defines $\alpha$ only modulo $d$-exact and on-shell vanishing forms. Therefore, it makes sense to consider the equivalence classes of on-shell Hamiltonian forms defined by relation (2.14). Due to the regularity condition, this implies the existence of forms $\beta$ and $\gamma$ such that $\alpha - \alpha' = i_Q\beta + d\gamma$. It is significant that the equivalence classes of on-shell Hamiltonian forms constitute a Lie algebra, which we denote by $\Lambda^H(Q, \omega)$, for the same Lie bracket (2.9). The last fact is quite obvious as we can identify the on-shell Hamiltonian forms with the Hamiltonian forms on $\Sigma^\infty$ endowed with the induced presymplectic structure $\omega|_{\Sigma^\infty}$.

**Conservation laws.** A form $\alpha \in \Lambda^{0,m}(J^\infty E)$ is said to define a conservation law of degree $m$ if

$$d\alpha|_{\Sigma^\infty} = 0.$$  

(2.15)

A conservation law is called trivial if $\alpha \approx 0$. In other words, the space of nontrivial conservation laws of degree $m$ is identified with the cohomology group $H^{0,m}(\Sigma^\infty; d)$ of the on-shell bicomplex. Due to the standard regularity conditions on $Q$, equation (2.15) implies the existence of a form $\chi \in \Lambda^{1,m+1}(J^\infty E)$ such that

$$d\alpha = i_Q\chi.$$  

(2.16)

The form $\chi$ is called the characteristic of a conservation law $\alpha$.

Let $C \subset M$ be an $m$-cycle, $\alpha$ conservation law of degree $m$, and $\varphi \in \Gamma(E)$ a solution to the field equations, then the functional

$$I[\varphi] = \int_C (j^\infty \varphi)^*(\alpha)$$  

(2.17)

depends only on the homotopy class of $C$ in $M$ and is called the conserved charge.

**Flat gauge systems.** We say that a homological vector $Q$ field on $J^\infty E$ is flat if $i_Q\alpha|_M = 0$ for all $\alpha \in \Lambda(J^\infty E)$. Geometrically, this means that the submanifold $M \subset J^\infty E$, identified with the zero section, belongs to the zero locus of $Q$. Similarly, a form $\alpha \in \Lambda(J^\infty E)$ is called flat if $\alpha|_M = 0$. The flat forms constitute an ideal in the exterior algebra $\Lambda(J^\infty E)$ and a subcomplex in the variational tricomplex whenever $Q$ is flat. Let us denote the latter by
Λ_{flat}^{* \ast}(J^{\infty}E; \delta, d, \delta Q). The horizontal, vertical, and relative cohomology of the corresponding flat bicomplex Λ_{flat}^{* \ast}(J^{\infty}E; \delta) are given by the groups

\[ H_{flat}^{p,q}(J^{\infty}E; d) = 0, \quad p < n, \quad H_{flat}^{0,0}(J^{\infty}E; d) \simeq \Lambda_{flat}^{0,0}(J^{\infty}E)/d\Lambda_{flat}^{0,0-1}(J^{\infty}E), \]

(2.18)

n being the dimension of the base manifold M. The proof can be found in [7, chapter 19]. Amongst the elements of H_{flat}^{0,0}(J^{\infty}E; d) are the equivalence classes of Lagrangians. We will say that (Q, \omega) is a flat gauge system if Q is flat. From the physical viewpoint, flat gauge systems correspond to field theory models without external sources.

3. Lie algebra of conservation laws

The presence of the zero section j^{\infty}(0) : M \rightarrow J^{\infty}E together with the canonical projection π^{\infty} : J^{\infty}E \rightarrow M allows us to split the complex of purely horizontal forms into the direct sum

\[ \Lambda^{0, \ast}(J^{\infty}E; d) = \Lambda_{flat}^{0, \ast}(J^{\infty}E; d) \bigoplus \pi^{\ast}_{\infty} \Lambda^{\ast}(M; d). \]

(3.1)

For flat gauge systems, this results in the natural spitting of the cohomology groups

\[ H^{0, \ast}(\Sigma^{\infty}; d) = H_{flat}^{0, \ast}(\Sigma^{\infty}; d) \bigoplus H^{\ast}(M; d) \]

(3.2)

associated with the equivalence classes of conservation laws. The second summand in (3.2) owes its existence to the topology of M rather than field dynamics. Therefore, in the sequel, we will ignore it, focusing on the subspace of flat conservation laws H_{flat}^{0, \ast}(\Sigma^{\infty}; d). By definition, the elements of the latter subspace vanish identically when evaluated on the zero field configuration.

Let us denote by \hat{\alpha} the projection of a purely horizontal form \alpha on the first summand in (3.1). The elements of the second summand in (3.1), being in the kernel of the differential \delta, belong to the center of the Lie algebra \hat{\Lambda}^{H}(Q, \omega) of on-shell Hamiltonian forms. This gives the short exact sequence of Lie algebras

\[ 0 \rightarrow \pi^{\ast}_{\infty} \Lambda(M) \rightarrow \Lambda^{H}(Q, \omega) \rightarrow \hat{\Lambda}^{H}(Q, \omega) \rightarrow 0, \]

(3.3)

where \textit{i} is the natural embedding and \textit{p} is the canonical projection induced by the assignment \alpha \mapsto \hat{\alpha}. The exact sequence does not split in general, in which case \Lambda^{H}(Q, \omega) is a nontrivial central extension of \hat{\Lambda}^{H}(Q, \omega).

With the definitions above we are ready to formulate a generalization of Noether’s first theorem to gauge systems.

**Theorem 3.1.** ([5]). Let (Q, \omega) be a flat gauge system of degree m. Then each Hamiltonian symmetry \textit{X} gives rise to a uniquely defined sequence \alpha_{1}, \alpha_{2}, \ldots, \alpha_{m} of flat conservation laws. Trivial symmetries generate trivial conservation laws.

**Proof.** Let \alpha_{0} denote the Hamiltonian of the symmetry \textit{X}, then

\[ i_{\textit{X}}\omega \simeq \delta \alpha_{0}. \]

(3.4)

Applying \delta_{Q} to this equality yields \delta_{Q} \alpha_{0} \simeq 0. Since the form \delta_{Q} \alpha_{0} is flat, we can write it as

\[ \delta_{Q} \alpha_{0} = d \alpha_{1} \]

(3.5)
for some $\alpha_1 \in \Lambda^{0,m-1}(J^\infty E)$. By definition, $\alpha_1$ is a conservation law with characteristic $\delta \alpha_0$. It follows from equation (3.5) that $d\delta_Q \alpha_1 = 0$. Since $\delta_Q \alpha_1$ is flat, the triviality of the group $H^0_{\text{flat}}(J^\infty E; d)$ implies the existence of a form $\alpha_2$ such that $\delta_Q \alpha_1 = d\alpha_2$. Hence, $\alpha_2$ is a conservation law with characteristic $\delta \alpha_1$. Iterating this construction once and again, we get the desired sequence of conservation laws $\alpha_1, \alpha_2, \ldots, \alpha_m$. These are not claimed to be all nontrivial; quite the opposite: if the $k$th conservation law happens to be trivial, $\alpha_k \approx 0$, then so are all its descendants $\alpha_{k+1}, \ldots, \alpha_m$.

If now $X = [Q, Y]$ for some Hamiltonian vector field $Y$, then

$$i_X \omega = \delta \alpha_0 \simeq i_X \omega \simeq \delta_Q i_Y \omega \simeq - \delta \delta_Q \beta,$$

where $\beta$ is the Hamiltonian of $Y$. The group $H^0_{\text{flat}}(J^\infty E; \delta/d)$ being trivial, $\alpha_0 = - \delta \delta_Q \beta + d\gamma$ for some $\gamma \in \Lambda^{0,m-1}(J^\infty E)$. Equation (3.5) then gives $d(d\delta_Q \gamma - \alpha_1) = 0$. Since the group $H^0_{\text{flat}}(J^\infty E; d)$ is zero, we conclude that $\alpha_1 \approx 0$ and the conservation law $\alpha_1$ is trivial together with all its descendants.

It is worth noting that the assumption of flatness is rather technical; in many cases, one can replace it with other mild conditions that ‘kill’ the de Rham cohomology of the base manifold $M$. In particular, the above theorem holds for any nonflat $Q$ whenever $M$ is contractible.

**Theorem 3.2. ([5]).** Let $\{\alpha_k\}$ be a sequence of conservation laws associated with a Hamiltonian symmetry $X$. Then the form $\alpha_k$ is on-shell Hamiltonian relative to the $k$th descendent presymplectic structure $\omega_k$.

**Proof.** As above, we let $\alpha_0$ denote the Hamiltonian of $X$. Then equation (3.4) implies that

$$i_X \omega = \delta \alpha_0 \simeq d\alpha_0'$$

for some $\alpha_0' \in \Lambda^{1,m-1}(J^\infty E)$. Acting by $\delta_Q$ on both sides of the equality, we find

$$(-1)^{gh(X)h} i_X \delta_Q \omega = \delta \delta_Q \alpha_0 \simeq d\delta_Q \alpha_0'.$$

(3.7)

Combining this with equations (2.12) and (3.5), we get $d(i_X \omega_1 = d\delta \alpha_1 \simeq d\delta_Q \alpha_0'$. The acyclicity of $d$ in positive vertical degree implies that

$$i_X \omega_1 = \delta \alpha_1 \simeq d\delta_Q \alpha_0'$$

for some $\alpha_1' \in \Lambda^{1,m-2}(J^\infty E)$. Therefore, $i_X \omega_1 \approx \delta \alpha_1$ and the form $\alpha_1$ is on-shell Hamiltonian relative to the descendent presymplectic structure $\omega_1$. Applying the operator $\delta_Q$ once again, we obtain from (3.8)

$$(-1)^{gh(X)h} i_X \delta_Q \omega_1 = \delta \delta_Q \alpha_1 \simeq d\delta_Q \alpha_0,'$$

(3.9)

This equation coincides in form with (3.7). Hence, there is a form $\alpha_2'$ such that

$$i_X \omega_2 = \delta \alpha_2 \simeq d\alpha_2'$$

(3.10)

and the conservation law $\alpha_2$ is on-shell Hamiltonian w.r.t. $\omega_2$. Proceeding in such a way, we obtain the sequence of relations $i_X \omega_k = \delta \alpha_k \simeq \delta \delta_Q \alpha_{k-1} \simeq d\alpha_k'$, which imply that all forms $\alpha_k$ are on-shell Hamiltonian.

**Theorem 3.3.** Let $\{\alpha_k\}$ be the sequence of flat conservation laws associated with a Hamiltonian symmetry $X$, then the assignment $X \mapsto \alpha_k$ defines a Lie algebra homomorphism

$$h_k: \text{Sym}(Q, \omega) \to \hat{\Lambda}^H(Q, \omega_k).$$

(3.11)
Proof. For every pair of Hamiltonian symmetries \( X, Y \in \text{Sym}(Q, \omega) \) there corresponds a Hamiltonian symmetry \([X, Y] \) satisfying the equation

\[
i_{[X, Y]} \omega \simeq \delta \{ \alpha_0, \beta_0 \}.
\]

Here \( \alpha_0, \beta_0 \), and \( \{ \alpha_0, \beta_0 \} = (-1)^{gh(X)}i_Xi_Y \omega \) are the Hamiltonians of \( X, Y \), and \([X, Y]\), respectively. Let \( \{ \alpha_k \} \) and \( \{ \beta_k \} \) denote the sequences of conservation laws associated with \( X \) and \( Y \). By definition,

\[
\delta_0 \alpha_k = d\alpha_{k+1}, \quad \delta_0 \beta_k = d\beta_{k+1}, \quad k = 0, 1, \ldots, m.
\]

Applying the Lie derivative \( L_X \) to the second sequence of equations, we get

\[
\delta_0 L_X \beta_k = dL_X \beta_{k+1} \Rightarrow \delta_0 \{ \alpha_k, \beta_k \} = d\{ \alpha_{k+1}, \beta_{k+1} \}, \tag{3.12}
\]

where \( \{ \alpha_k, \beta_k \} = (-1)^{gh(X)}i_Xi_Y \omega_k \). For \( k > 0 \), this yields a uniquely defined sequence \( \{ \alpha_k, \beta_k \} \) of flat conservation laws associated with the Hamiltonian symmetry \([X, Y]\). \( \square \)

The homomorphism (3.11) defines a Lie subalgebra \( \text{Im} h_1 \subset \hat{\Lambda}^H(Q, \omega) \). By the \( k \)-th Lie algebra of conservation laws, denoted by \( \text{CL}_k(Q, \omega) \), we will understand the preimage of \( \text{Im} h_k \) in \( \Lambda^H(Q, \omega) \); that is, \( \text{CL}_k(Q, \omega) = p^{-1}(\text{Im} h_k) \). In view of (3.3), the Lie algebra \( \text{CL}_k(Q, \omega) \) is nothing more than a central extension of \( \text{Im} h_k \). A necessary condition for the existence of a nontrivial central extension of \( \text{Im} h_k \) is that \( H^{m-k}(M; \text{d}) \neq 0 \).

One more version of Noether’s first theorem is given by the next statement.

Proposition 3.4. Let \((Q, \omega)\) be a regular gauge system. Suppose the following two conditions are satisfied:

(a) \( d\omega|_{\Sigma^\omega} = 0 \),
(b) \( H^1_{\text{hor}}(\Sigma^\omega; \delta) = 0 \);

then every on-shell Hamiltonian form is a conservation law and equation (2.13) establishes a homomorphism \( h \colon \text{Sym}_{\Sigma}(Q, \omega) \to \hat{\Lambda}^H(Q, \omega) \) of the Lie algebras.

Proof. The second part of the statement is obvious. To prove the first consider an on-shell Hamiltonian vector field \( X \) with Hamiltonian \( \alpha \in \Lambda^H(Q, \omega) \). Applying the horizontal differential \( \text{d} \) to both sides of equation (2.13), we get

\[
\delta \text{d} \alpha|_{\Sigma^\omega} = (-1)^{gh(X)}i_X \omega|_{\Sigma^\omega}.
\]

Regularity implies that \( d\omega = i_0 \beta + \delta_0 \gamma \) for some forms \( \beta \) and \( \gamma \), whence

\[
i_X d\omega = i_0 i_X \beta - (-1)^{gh(X)} \delta_0 i_X \gamma. \tag{3.14}
\]

Here we used the equality \([Q, X] = 0\). Therefore, \( \delta(\text{d} \alpha)|_{\Sigma^\omega} = 0 \). Condition (b) then means that \( d\alpha|_{\Sigma^\omega} = 0 \), i.e. \( \alpha \) is a flat conservation law. \( \square \)

Notice that all descendent presymplectic structures \( \omega_k \) meet condition (a), see equation (2.12). Condition (b) is fulfilled e.g. for homogeneous linear systems of field equations; in that case, the components of the homological vector field \( Q \) are homogeneous linear functions of the vertical coordinates \( \phi^a_{\Sigma^\omega} \).

Example 3.5. All the notions above are best illustrated by the example of Chern–Simons theory. Let us consider a \( U(1) \)-vector bundle over a three-dimensional manifold \( M \). For simplicity, we assume that the bundle is trivial. Then the affine space of all \( U(1) \)-connections...
is naturally isomorphic to the space of one-forms $\Lambda^1(M)$, that is, the sections of the cotangent bundle $T^*M$. In the Batalin–Vilkovisky (BV) formalism, the field spectrum of abelian Chern–Simons theory includes the connection one-form $A \in \Lambda^1(M)$, ghost field $C \in \Lambda^0(M)$ and their antifields $A^* \in \Lambda^2(M)$ and $C^* \in \Lambda^3(M)$. By definition,

$$\text{gh}(C) = 1, \quad \text{gh}(A) = 0, \quad \text{gh}(A^*) = -1, \quad \text{gh}(C^*) = -2. \tag{3.15}$$

The space of fields and antifields carries the canonical symplectic structure

$$\omega = \delta A \wedge \delta A^* + \delta C \wedge \delta C^*, \quad \text{gh}(\omega) = -1, \tag{3.16}$$

of top horizontal degree. Here $\delta$ stands for the usual variational differential on fields, whose properties are identical to those of the vertical differential above, hence the notation. The action of the classical BRST differential is given by

$$\delta_Q C = 0, \quad \delta_Q A = dC, \quad \delta_Q A^* = dA, \quad \delta_Q C^* = dA^*. \tag{3.17}$$

As is seen, only flat connections belong to the stationary surface of $Q$. The zero-curvature equation $dA = 0$ enjoys the standard gauge symmetry $\delta A = d\epsilon$, which manifests itself through the second equation in (3.17). The Hamiltonian of the homological vector field $Q$ is given by the BV master Lagrangian of Chern–Simons theory:

$$i_Q \omega \simeq \delta L, \quad L = \frac{1}{2} A \wedge dA + dC \wedge A^*. \tag{3.18}$$

Thus, equations (3.16) and (3.17) define a gauge system of degree three.

The canonical symplectic structure (3.16) generates the full sequence of descendants:

$$\delta_Q \omega = d\omega_1, \quad \omega_1 = \frac{1}{2} \delta A \wedge \delta A + \delta C \wedge \delta A^*, \quad \text{gh}(\omega_1) = 0,$$

$$\delta_Q \omega_1 = d\omega_2, \quad \omega_2 = \delta C \wedge \delta A, \quad \text{gh}(\omega_2) = 1, \tag{3.19}$$

$$\delta_Q \omega_2 = d\omega_3, \quad \omega_3 = \frac{1}{2} \delta C \wedge \delta C, \quad \text{gh}(\omega_3) = 2.$$

Hence, the length of Chern–Simons theory is four. Notice that $\omega_1$ coincides with the canonical presymplectic structure associated with the Lagrangian (3.18). The classical BRST differential enjoys a nontrivial global symmetry $Y$ in ghost number $-1$, whose action on fields is given by

$$L_Y C = 1, \quad L_Y A = 0, \quad L_Y A^* = 0, \quad L_Y C^* = 0. \tag{3.20}$$

The variational vector field $Y$ is Hamiltonian, $i_Y \omega = \delta C^*$, and gives rise to the sequence of conservation laws $A^*$, $A$, and $C$. The latter fact is readily seen from (3.17). The corresponding Lie brackets read

$$\{A^*, A\}_1 = 0, \quad \{A, A\}_2 = 0, \quad \{C, C\}_3 = 1. \tag{3.21}$$

The last equality exemplifies the phenomenon of central extension: the abelian subalgebra $[Y, Y] = 0$ of global symmetries gets a nontrivial central extension $1 \in H^0(M; d)$ when evaluated on the corresponding conservation laws. Among the conservation laws above only the

$^3$ aka field-antifield formalism, see e.g. [4].
one-form $A$ has ghost number zero. Integrating it over a loop $\gamma \subset M$ yields the conserved charge $I[A] = \oint_{\gamma} A$, which is nothing but the holonomy invariant of the flat connection $A$.

4. Leibniz algebras of symmetries and conservation laws

Most of the gauge systems encountered in physics depend on some numerical parameters such as masses of particles, coupling constants, etc. This motivates us to consider families of gauge systems rather than isolated systems. For the sake of simplicity we will restrict ourselves to one-parameter families $(Q_t, \omega_t)$. Also, it is convenient to be a bit sloppy about the class of functions of $t$. The homological vector field $Q_t$, for instance, may be a smooth function of $t$ or a formal power series

$$Q_t = Q_0 + tQ_1 + t^2Q_2 + \cdots .$$

Such formal expansions are at the heart of perturbation theory: if one regards $t$ as a coupling constant, then the leading term $Q_0$ defines a ‘free gauge system’ $(Q_0, \omega_0)$, while the higher order terms $Q_1, Q_2, \ldots$ describe a ‘consistent interaction’. In many instances, the corresponding presymplectic form $\omega_t$ does not depend on $t$ at all.

Even though the gauge system $(Q_t, \omega_t)$ varies ‘smoothly’ with $t$, this may not be the case with the corresponding cohomology groups. For example, the inclusion of interaction may violate some nontrivial symmetries of a free gauge system. It is the differences in cohomology for different $t$’s that give rise to interesting algebraic constructions that we consider below.

Symmetries. Differentiating the defining equality $[Q_t, Q_\tau] = 0$, we obtain

$$[Q_t, \dot{Q}_t] = 0, \quad [\dot{Q}_t, \dot{Q}_\tau] = -[Q_t, \ddot{Q}_\tau],$$

where the overdots stand for the $t$-derivatives. The first equation says that the evolutionary vector field $\dot{Q}_t$ is $Q_t$-invariant; hence, it defines an odd symmetry of the gauge system $(Q_t, \omega_t)$. According to the second equation, the symmetry $\dot{Q}_t$ always squares to a trivial symmetry. Thus, our first observation is that any one-parameter family of gauge systems possesses a canonical (perhaps trivial) symmetry generated by $\dot{Q}_t$. Furthermore, this symmetry makes $\text{Sym}(Q_t)$ into a differential graded Lie algebra with the deferential

$$\partial_t \hat{X} = [\hat{Q}_t, \hat{X}], \quad \partial_t^2 = 0.$$

Here $\hat{X}$ denotes the equivalence class of the symmetry generated by a $Q$-invariant vector field $X \in \mathfrak{X}(\mathcal{E})$.

The main purpose of this section is to equip the space of symmetries $\text{Sym}(Q_t)$ with the structure of a graded Leibniz algebra. The latter is defined as follows.

Definition 4.1. A graded Leibniz algebra is a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ together with a bilinear product

$$\circ : V_n \otimes V_m \to V_{n+m}$$

satisfying the Leibniz identity

$$a \circ (b \circ c) = (a \circ b) \circ c + (-1)^{|a||b|} b \circ (a \circ c)$$

for all $a, b, c \in V$; here $|a|$ denotes the degree of a homogeneous element $a \in V$. 

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It is convenient to split the $\circ$-product into its graded symmetric and skew-symmetric parts:\(^4\)

$$
[[a, b]] = \frac{1}{2} \left( a \circ b - (-1)^{|a||b|} b \circ a \right),
\quad \{a, b\} = \frac{1}{2} \left( a \circ b + (-1)^{|a||b|} b \circ a \right).
$$

Then it follows from the Leibniz identity (4.5) that the subspace $I \subset V$ spanned by all elements of the form $[[a, b]]$ is an ideal of the Leibniz algebra $(V, \circ)$. Moreover, $I \circ V = 0$. The condition $I = 0$ is clearly equivalent to the skew-symmetry of the $\circ$-product. In that case, relation (4.5) boils down to the Jacobi identity for the Lie bracket $[[a, b]] = a \circ b$. Therefore, a graded Leibniz algebra with skew-symmetric product is the same as a graded Lie algebra and vice versa. In general, the skew-symmetric part of the $\circ$-product does not satisfy the Jacobi identity, but it always induces a Lie bracket in the quotient space $V/I$.

Central to our construction is a functor

$$
F : dgLie \to gLeib
$$

(4.7)
from the category of differential graded Lie algebras to that of graded Leibniz algebras. It is defined in terms of the so-called derived bracket [14]. Given a differential graded Lie algebra $L = \bigoplus_{n \in \mathbb{Z}} L_n$ with bracket $[-, -]$ of degree $n$ and differential $\partial$ of degree $1$, we write

$$
a \circ b = (-1)^{|a||\partial|} \partial [a, b].
$$

(4.8)

It is straightforward to check the $\circ$-product makes the graded vector space $L[-n-1]$ into a Leibniz algebra\(^5\). One can also see that the symmetric part of the $\circ$-product (4.8) is in the image of the differential:

$$
\{a, b\} = \frac{1}{2} (-1)^{|a||\partial|} \partial [[a, b]].
$$

(4.9)

As a consequence, the skew-symmetric part of $\circ$ induces a Lie bracket in the quotient space $L/\partial L$. Furthermore, the differential $\partial$ defines a homomorphism of the Leibniz algebras $(L[-n-1], \circ)$ and $(L, [-, -])$.

Applying the construction of the derived bracket (4.8) to our situation, we can turn the differential graded Lie algebra of symmetries $(\text{Sym}(Q), \partial)$ into a graded Leibniz algebra w.r.t. the product

$$
\tilde{X} \circ \tilde{Y} = (-1)^{|\partial|X-1} [[\tilde{Q}, X], Y] = [[X, \tilde{Q}], Y],
$$

(4.10)

$X$ and $Y$ being $Q$-invariant evolutionary vector fields. We will denote this Leibniz algebra by $\text{Sym}(Q)$; as a graded vector space $\text{Sym}(Q) = \text{Sym}(Q)[-n-1]$. In general, the product (4.10) is not skew-symmetric and $\text{Sym}(Q)$ is not a Lie algebra. An important particular situation when it does degenerate to a Lie algebra structure is the following. Let $\mathcal{L} \subset \text{Sym}(Q)$ be a commutative subalgebra in the Lie algebra of symmetries such that $[\partial \mathcal{L}, \mathcal{L}] \subset \mathcal{L}$. Then $F \mathcal{L}$ is again a Lie algebra. Moreover, the derived Lie algebra $F \mathcal{L}$ may well be noncommutative.

**Remark 4.2.** The functor (4.7) is not the only one that can be attributed to the category of differential graded Lie algebras. In [19], Getzler constructed a functor from $dgLie$ to the category of $L_{\infty}$-algebras. A remarkable property of Getzler’s functor is that it extends the

\(^4\)In the following, we will often omit the word ‘graded’.

\(^5\)For every $\mathbb{Z}$-graded vector space $V = \bigoplus V_k$ and $k \in \mathbb{Z}, V[k]$ is a graded vector space with $V[k]_n = V_{n+k}$.
skew-symmetric part of the derived bracket (4.8), restricted to the subspace $L_{-1}$, to the full $L_{\infty}$-structure. The discussion of other interesting functors from the category of Leibniz algebras to that of $L_{\infty}$-algebras can be found in the recent paper [20].

**Remark 4.3.** Differentiating the identity $\delta_Q \omega_t = 0$ w.r.t. the parameter $t$, one can see that the symmetry $\dot{Q}_t$ is on-shell Hamiltonian. If, as often happens, $\omega_t$ does not depend on $t$, the symmetry $\dot{Q}_t$ is Hamiltonian and the Leibniz product (4.10) restricts consistently onto the subspace of Hamiltonian symmetries $\text{Sym}(Q, \omega)$. In that case, one can speak of the Leibniz algebras of Hamiltonian symmetries and the corresponding conservation laws.

**From Lie to Leibniz and back.** Let us return to the interpretation of the expansion (4.1) as a formal deformation of the free gauge system $(Q_0, \omega_0)$ by interaction. Evaluating (4.2) at $t = 0$, we find

$$[Q_0, Q_1] = 0, \quad [Q_1, Q_1] = -2 [Q_0, Q_2].$$

These equalities say that the first-order interaction $Q_1$ defines an odd symmetry whose square is a trivial symmetry of the free system. Hence, $\dot{Q}_1 \in \text{Sym}(Q_0)$. Formula (4.10) makes then the space of free symmetries into the Leibniz algebra with product

$$\tilde{X} \circ \tilde{Y} = [[X, \dot{Q}_1], Y].$$

We denote this Leibniz algebra by $\text{Sym}(Q_0, Q_1)$ to emphasise that the product (4.12) on the symmetries of the free system $(Q_0, \omega_0)$ depends on the first-order interaction $Q_1$. At the level of graded vector spaces we have the isomorphism $\text{Sym}(Q_0, Q_1) = \text{Sym}(Q_0)[-1]$. As is often the case, the inclusion of interaction breaks some symmetries of the free system, and those that survive form a subalgebra $L_{\text{int}}$ in the Lie algebra $\text{Sym}(Q_0)$. The quotient $L_{\text{br}} = \text{Sym}(Q_0)/L_{\text{int}}$, identified with the space of broken symmetries, carries no natural Lie algebra structure unless $L_{\text{int}}$ is an ideal in $\text{Sym}(Q_0)$.

From the viewpoint of Leibniz algebras, the unbroken symmetries constitute a central ideal in $\text{Sym}(Q_0, Q_1)$, that is,

$$\text{Sym}(Q_0, Q_1) \circ L_{\text{int}} \subset L_{\text{int}}, \quad L_{\text{int}} \circ \text{Sym}(Q_0, Q_1) = 0.$$  

In view of relation (4.9), this includes the ideal $I \subset L_{\text{int}}$ spanned by the unbroken symmetries of the form

$$\{\tilde{X}, \tilde{Y}\} = \frac{1}{2} (-1)^{gh(X)} [\dot{Q}_1, [X, Y]]$$

$X$ and $Y$ being arbitrary $Q_0$-invariant evolutionary vector fields. As a result, the Leibniz product (4.12) canonically induces a Lie bracket on the vector space $L_{\text{br}}[-1]$, the suspended space of broken symmetries.

Let us summarize the results of this subsection by the following thesis. Given a consistent first-order interaction $Q_1$, we can split (noncanonically) the symmetries of the corresponding free system $(Q_0, \omega_0)$ into two groups—those that preserve $\dot{Q}_1$ and those that do not—and make them both into graded Lie algebras. We emphasize that the Lie brackets in $L_{\text{int}}$ and $L_{\text{br}}[-1]$ are essentially different: the former is just the restriction of that in $\text{Sym}(Q_0)$, while the latter, being induced by the Leibniz product (4.12), involves the interaction $\dot{Q}_1$. For free gauge systems, the Lie algebra of symmetries $\text{Sym}(Q_0)$ is normally commutative in nonzero ghost number [13, section 3.9] and so is its subalgebra $L_{\text{int}}$. By contrast, the Lie algebra of broken symmetries $L_{\text{br}}[-1]$ may well be nonabelian due to the interaction $Q_1$. Notice that
whenever Sym($Q_0$) is abelian the corresponding Leibniz algebra Sym($Q_0, Q_1$) is a Lie algebra with $L_{\text{int}}[-1]$ belonging to its centre.

**Conservation laws.** The same construction of the derived bracket (4.8) allows one to make the Lie algebra of on-shell Hamiltonian forms $\Lambda^H (Q_0, \omega_0)$ into a Leibniz algebra. We can proceed from the simple observation that each first-order interaction $Q_1$ defines an on-shell Hamiltonian symmetry. Indeed, evaluating the equation $i_{Q_1} \omega_0 \simeq \delta H_1$ at first order in $t$, we readily find

$$i_{Q_1} \omega_0 \simeq \delta H_1 - i_{Q_0} \omega_0 \approx \delta H_1.$$  \hspace{1cm} (4.15)

Hence, $Q_1$ is an on-shell Hamiltonian symmetry of the free gauge system ($Q_0, \omega_0$). It follows from the second relation in (4.11) that the Lie bracket (2.9) of the on-shell Hamiltonian form $H_1$ with itself is equivalent to zero:

$$\{H_1, H_1\} \simeq -2 \{H_0, H_2\} = 2i_{Q_0} i_{Q_1} \omega_0 \approx 0.$$  \hspace{1cm} (4.16)

Notice that the degree of the bracket is opposite to the ghost number of the presymplectic structure. Considering now the adjoint action of $H_1$, we turn $\Lambda^H (Q_0, \omega_0)$ into a differential graded Lie algebra with the differential $\partial$ defined by the relation

$$\partial \alpha = \{H_1, \alpha\}, \quad \forall \alpha \in \Lambda^H (Q_0, \omega_0).$$  \hspace{1cm} (4.17)

Then the functor (4.7) gives us immediately the Leibniz product

$$\alpha \circ \beta = (-1)^{gh(\alpha)+gh(\beta)} \cdot \{\{H_1, \alpha\}, \beta\} = \{\{\alpha, H_1\}, \beta\}.$$  \hspace{1cm} (4.18)

for all $\alpha, \beta \in \Lambda^H (Q_0, \omega_0)$. With the definition of an on-shell Hamiltonian form (2.13) we can rewrite this product in several equivalent ways:

$$\alpha \circ \beta \approx (-1)^{gh(\alpha)+gh(\beta)} \{L_{Q_1} \alpha, \beta\} \approx i_{Q_1} X \cdot \gamma \omega \approx L_{Q_1} X \beta.$$  \hspace{1cm} (4.19)

Here $X$ and $Y$ are the on-shell Hamiltonian symmetries associated with $\alpha$ and $\beta$. We will denote this Leibniz algebra by $\Lambda^H (Q_0, Q_1, \omega_0)$. As usual the passage from the Lie algebra $\Lambda^H (Q_0, \omega_0)$ to the Leibniz algebra $\Lambda^H (Q_0, Q_1, \omega_0)$ implies the shift in degree of on-shell Hamiltonian forms.

The above considerations apply then to all the descendent gauge systems $(Q_0, \omega_0^k)$ associated with the free gauge system $(Q_0, \omega_0)$. Indeed, replacing $\omega_0$ and $\omega_1$ in (4.15) with $\omega_0^k$ and $\omega_1^k$, respectively, we readily conclude that the vector field $Q_1$ is on-shell Hamiltonian relative to $(Q_0, \omega_0^k)$, that is,

$$i_{Q_1} \omega_0^k \simeq \delta H_1^k - i_{Q_0} \omega_1^k \approx \delta H_1^k,$$  \hspace{1cm} (4.20)

$H_1^k$ being the Hamiltonian. The derived bracket construction gives then the sequence of Leibniz products

$$\alpha \circ_k \beta \equiv \{\{\alpha, H_1^k\}, \beta\}, \quad k = 1, 2, \ldots, m,$$  \hspace{1cm} (4.21)

for all $\alpha, \beta \in \Lambda^H (Q_0, \omega_0^k)$. We denote the corresponding Leibniz algebras by $\Lambda^H (Q_0, Q_1, \omega_0^k)$. If the assumption is made that $H_1^0 \Sigma \equiv (\Sigma^0; \delta) = 0$, the elements of these algebras become conservation laws of the free gauge system, see proposition 3.4.

In special, but not rare, instances where the presymplectic structure of the family $(Q_t, \omega_t)$ does not depend on $t$, the vector field $Q_1$ is Hamiltonian relative to $\omega_0 = \omega_t$. Being a symmetry, it generates the sequence of conservation laws $H_1^k \subset \text{CL}_{\text{int}} (Q_0, \omega_0), k = 1, \ldots, m$. In that case the product (4.21) restricts onto the subspace $\text{CL}_{\text{int}} (Q_0, \omega_0) \subset \Lambda^H (Q_0, \omega_0^k)$ making it into a Leibniz algebra, which we denote by $\text{CL}_{\text{int}} (Q_0, Q_1, \omega_0)$. 


5. Applications

In this section, we exemplify the general approach developed above by two fundamental physical models: YM theory and Einstein’s gravity without matter. Both the theories enjoy asymptotic conservation laws with nontrivial Leibniz algebras. The technical simplicity of these models combined with the BV formalism enables us to exhibit explicitly all the relevant presymplectic structures, conserved currents and their algebras. As the third example we consider the gravity field subject to the so-called unimodularity condition. This model, being highly nonlinear, is intended to demonstrate that lower-degree conservation laws are not prerogative of free gauge theories alone.

5.1. Yang–Mills fields

Let \( M \) be a four-dimensional spacetime manifold, \( \mathcal{G} \) a compact Lie algebra, and \( \text{Tr} \) an invariant nondegenerate trace on the universal enveloping algebra \( \mathcal{U}(\mathcal{G}) \). Denote by \( \Lambda_{q}^{p}(M, \mathcal{G}) \) the space of differential \( p \)-forms on \( M \) with values in \( \mathcal{G} \) that have ghost number \( q \in \mathbb{Z} \). Recall that in the BV formalism the spectrum of YM theory without matter consists of the following fields and antifields:

\[ A \in \Lambda_{0}^{1}(M, \mathcal{G}), \quad C \in \Lambda_{1}^{0}(M, \mathcal{G}), \quad A^{*} \in \Lambda_{-1}^{3}(M, \mathcal{G}), \quad C^{*} \in \Lambda_{-2}^{4}(M, \mathcal{G}). \]  

(5.1)

The space of fields and antifields carries the canonical symplectic structure\(^6\)

\[ \omega = \text{Tr} \left( \delta A \wedge \delta A^{*} + \delta C \wedge \delta C^{*} \right), \quad \text{gh}(\omega) = -1. \]

(5.2)

The two-form \( \omega \) being nondegenerate, the corresponding homological vector field \( Q \) is uniquely defined by the relation \( i_{Q} \omega = \delta L \), where the Hamiltonian form \( L \) is given by the standard BV extension of the YM Lagrangian, namely,

\[ L = \text{Tr} \left( \frac{1}{2} F \wedge \tilde{F} + A^{*} \wedge DC + g C^{*} C \right). \]

(5.3)

Here \( \tilde{F} \) is the Hodge dual of the curvature two-form \( F = dA + g A \wedge A \) and \( D \) stands for the covariant derivative, e.g. \( \text{DC} = dC + g[A, C] \). Regarding the YM coupling constant \( g \) as deformation parameter, we can write

\[ L = L_{0} + g L_{1} + g^{2} L_{2}, \]

(5.4)

and similar expansion takes place for the homologous vector field \( Q = Q_{0} + g Q_{1} + g^{2} Q_{2} \). One can easily check that \( Q^{2} = 0 \) or, what is the same, \( \{ L, L \} \simeq 0 \), where the braces stand for the canonical BV bracket associated with the symplectic structure (5.2). The leading term \( L_{0} \) describes the dynamics of free YM fields, while \( L_{1} \) and \( L_{2} \) introduce a consistent interaction.

Applying the BRST differential \( \delta_{Q} \) to the symplectic structure (5.2), one can find the following descendants:

\[ \delta_{Q} \omega = d \omega_{1}, \quad \omega_{1} = \text{Tr} (\delta A \wedge \delta \tilde{F} + \delta C \wedge \delta A^{*}), \quad \omega_{1} \in \Lambda^{3}(M), \quad \text{gh}(\omega_{1}) = 0, \]

\[ \delta_{Q} \omega_{1} = d \omega_{2}, \quad \omega_{2} = \text{Tr} (\delta C \wedge \delta \tilde{F}), \quad \omega_{2} \in \Lambda^{4}(M), \quad \text{gh}(\omega_{2}) = 1. \]

(5.5)

\( ^6 \) Hereinafter the wedge product combines the exterior product of forms with the product in \( \mathcal{U}(\mathcal{G}) \).
Since $\delta Q = 0$, the length of YM theory is equal to 3. Unlike (5.2), the descendent presymplectic forms $\omega_1$ and $\omega_2$ depend on the coupling constant $g$. The first descendent is nothing but the canonical presymplectic structure associated with the Lagrangian (5.3), that is, $\omega_1 = \delta \theta$, where the presymplectic potential $\theta$ comes from the variation $\delta L = iQ \omega + d\theta$. The second descendent $\omega_2$ gives rise to the Lie and Leibniz algebra structures on lower-degree conservation laws (surface charges).

In order to define these algebraic structures we note that $Q_1$ is a Hamiltonian vector field generating a symmetry of the free Lagrangian $L_0$. By theorem 3.1, it yields a pair of conservation laws $J_1$ and $J_2$ defined by

\[
\begin{align*}
\delta Q_0 L_1 &= dJ_1, & J_1 &= \text{Tr} \left( \tilde{F}_0 \wedge [A, C] + A^* CC \right), & J_1 &\in \Lambda^3(M), & \text{gh}(J_1) &= 1, \\
\delta Q_0 J_1 &= dJ_2, & J_2 &= \text{Tr} \left( \tilde{F}_0 CC \right), & J_2 &\in \Lambda^2(M), & \text{gh}(J_2) &= 2, \\
\delta Q_0 J_2 &= 0,
\end{align*}
\]

(5.6)

$F_0 = dA$ being the strength of free YM fields. In the calculations above we used the following formulas for the action of the free BRST differential on fields and antifields:

\[
\begin{align*}
\delta Q_0 C^* &= dA^*, & \delta Q_0 A^* &= d\tilde{F}_0, & \delta Q_0 A &= dC, & \delta Q_0 C &= 0.
\end{align*}
\]

(5.7)

With the help of $J$’s we can turn the Lie algebras of on-shell Hamiltonian forms of degree three and two into a pair of Leibniz algebras by setting

\[
\alpha \circ_k \beta = \{\{\alpha, J_k\}, \beta\}_k.
\]

(5.8)

Here the braces with the subscript $k = 1, 2$ stand for the Lie brackets determined by the presymplectic structures (5.5).

The two-form $J_2$ is not the only surface current that one can attribute to free YM fields. The free Lagrangian $L_0$ obviously enjoys the shift symmetry $C \to C + \epsilon \xi$, where $\xi$ is a vector of $\mathcal{G}$ and $\epsilon$ is a constant parameter of ghost number one ($d\epsilon = 0$). By Noether’s theorem, we get immediately the conserved current

\[
J_\xi = \text{Tr}(\xi A^*) \in \Lambda^3(M), & \text{gh}(J_\xi) = -1,
\]

(5.9)

and its descendant

\[
J_\xi = \text{Tr}(\xi \tilde{F}_0) \in \Lambda^2(M), & \text{gh}(J_\xi) = 0.
\]

(5.10)

Obviously,

\[
dJ_1 \approx 0, & dJ_2 \approx 0, & \delta Q_0 J_1 = dJ_2, & \delta Q_0 J_2 = 0.
\]

(5.11)

The conserved current $J_\xi$, being of ghost number zero, admits a straightforward physical interpretation. If $S$ is a closed space-like surface in $M$, then the integral

\[
q^\xi = \int_S J_\xi
\]

(5.12)

defines the net (color) charge enclosed by the surface $S$. The surface currents (5.9) and (5.10) form the abelian Lie algebras

\[
\{J_k^\ell, J_k^\ell\}_k = 0, & k = 1, 2,
\]

(5.13)
as is usually the case for free theories. The first-order interaction $\mathcal{L}_1$ breaks the shift symmetry above, leading thus to nontrivial Leibniz products of surface currents:

$$ J_k^\xi \circ J_k^{\xi'} = J_{k}^{[\xi,\xi']}, \quad \forall \, \xi, \xi' \in \mathcal{G}, \quad k = 1, 2. \quad (5.14) $$

As is seen, both the products are skew-symmetric and define the Lie algebra structure isomorphic to $\mathcal{G}$. In such a way we are able to reproduce the color Lie algebra at the level of surface currents. We emphasize that the currents $J_k^\xi$ are conserved on the free equations of motion $Q_0 = 0$ and cannot be promoted to conservation laws of full YM theory. Nevertheless, to make them into a nonabelian Lie algebra we need the cubic interaction vertices accommodated in $\mathcal{L}_1$. These vertices are essentially responsible for the ‘nonabelian part’ of the gauge generators.

In order to present a genuine example of Leibniz algebra which is not Lie, we note that the free Lagrangian $\mathcal{L}_0$ is invariant under orthogonal transformations. Indeed, as the Lie algebra $\mathcal{G}$ is supposed to be compact, there is a basis $\{t_a\}_{a=1}^n$ in $\mathcal{G}$ such that $\text{Tr}(t_at_b) = \delta_{ab}$. Expanding now the fields in terms of this basis, e.g. $A = A^at_a$, we see that the quadratic in fields Lagrangian $\mathcal{L}_0$ is determined by the Euclidean metric $\delta_{ab}$. As a result, it appears to be invariant under the $O(n)$-rotations of fields: $A^a \to A'^a = R^a_bA^b$ and similar transformations for the other fields; here $R = (R^a_b)$ is an orthogonal matrix from $O(n)$. Applying Noether’s theorem to infinitesimal rotations gives the following conserved currents for the free YM fields:

$$ f_{1b}^a = A^a \wedge F_0^b + C^{[a}A^b], \quad f_{2b}^a \in \Lambda^2(M), \quad \text{gh}(f_{1b}^a) = 0. \quad (5.15) $$

These have descendants of the first generation:

$$ \delta_{Q_0} f_{1b}^a = d f_{2b}^a, \quad \delta_{Q_0} f_{2b}^a = C^{[a}F_0^b], \quad f_{2b}^a \in \Lambda^2(M), \quad \text{gh}(f_{2b}^a) = 1. \quad (5.16) $$

As usual the square brackets stand for skew-symmetrization of indices. By theorem 3.3, the Lie brackets of the currents reproduce the commutation relations of the Lie algebra $o(n)$:

$$ \{ f_{2b}^a, f_{2d}^c \}_k = \delta^{bc} f_{2k}^d - \delta^{ac} f_{2d}^k - \delta^{ad} f_{2k}^c + \delta^{bd} f_{2k}^c, \quad k = 1, 2. \quad (5.17) $$

The currents $f_{2b}^a$ generate the Hamiltonian action of $o(n)$ on the space of fields and antifields. Evaluating now the symmetric part of the Leibniz products $f_{2b}^a \circ f_{2d}^c$, one easily finds

$$ \{ f_{2b}^a, f_{2d}^c \}_k = -\frac{1}{2} \{ J_k, \{ f_{2b}^a, f_{2d}^c \}_k \} = \delta^{bc} J_k^d - \delta^{ac} J_k^d + \delta^{ad} J_k^c - \delta^{bd} J_k^c, \quad (5.18) $$

where $J_{ab} = -\frac{1}{2} \{ J_k, f_{2b}^a \}_k$ are new conserved currents of ghost number $k$. Generally the structure constants of the Lie algebra $\mathcal{G}$, entering the currents $J_k$, are not $O(n)$-invariant, so that the new currents $J_{ab}$ are different from zero.

Writing the currents (5.9) and (5.10) in terms of the basis above, $J_k = \delta_{ab} f_{2b}^a$, we obtain

$$ f_{2b}^a \circ f_{2d}^c = f^{ac} J_k^d - f^{bc} J_k^d + \delta^{ac} f_{2d}^k - \delta^{bc} f_{2k}^d, \quad J_k \circ f_{2b}^a = f^{bc} J_k^d - f^{ac} J_k^d, \quad k = 1, 2. \quad (5.19) $$

These Leibniz products also have nonzero symmetric parts.

5.2. Einstein gravity

In the vierbein formalism, the dynamics of the gravity field are described by ten one-form fields: the vierbein $e^a$ and the spin-connection $\omega^{ab} = -\omega^{ba}$. As usual, we use the Minkowski metric $\eta_{ab}$ for raising and lowering the Lorentz indices $a, b = 0, 1, 2, 3$. In order to have explicit
control over the gauge symmetries corresponding to the diffeomorphisms of the spacetime manifold $M$ and local Lorentz invariance, one extends the field content with the ghost fields $c_a$ and $c_{ab} = -c_{ba}$ as well as the corresponding antifields labeled by star. Table 1 collects the full spectrum of fields and antifields together with their form degrees and ghost numbers.

The fields and antifields are canonically conjugate to each other w.r.t. the BV symplectic structure

$$\omega = \delta e^a \wedge \delta e^a + \delta w^a_{ab} \wedge \delta w^{ab} + \delta c^a \wedge \delta c^a + \delta c^a_{ab} \wedge \delta c^{ab}. \quad (5.20)$$

The BV master Lagrangian of Einstein’s gravity now reads

$$L = \frac{1}{2} \epsilon_{abcd} e^c \wedge e^d \wedge R^{cd} + e^a \wedge D e^a + w^a_{ab} \wedge D w^{ab} + (e^a \wedge \epsilon^{cd} \wedge b^c - \epsilon_{ab} \wedge \epsilon^{cd} b^d) + \frac{1}{2} \epsilon_{abca} c^a c^b. \quad (5.21)$$

Here $D = d + w$ is the Lorentz covariant differential with the curvature two-form $R^{ab} = d w^{ab} + w^a_c \wedge w^b_c$. The homological vector field $Q$ underlying Einstein’s gravity is just the Hamiltonian vector field generated by the master Lagrangian (5.21) and the symplectic structure (5.20). As is well known, see e.g. [21], the characteristic cohomology of pure gravity is empty. Thenontrivial conservation laws arise only upon linearization of the Lagrangian (5.21) about a suitable geometric background. Consider, for simplicity, the flat background geometry with a vierbein $h^a$ obeying $dh^a = 0$. Then we can put $e^a = h^a + \tilde{e}^a$, where the one-forms $\tilde{e}^a$ describe fluctuations over the flat background. On substituting this decomposition into (5.21), we get\(^7\)

$$L \simeq L_0 + L_1 + L_2, \quad (5.22)$$

where

$$L_0 = \epsilon_{abcd} d w^{ab} \wedge \tilde{e}^c \wedge h^d + \frac{1}{2} \epsilon_{abcd} d e^a \wedge w^{ab} \wedge h^c \wedge \bar{h}^d + e^a \wedge (D e^a + c^{ab} h_b)$$

$$+ w^a_{ab} \wedge \epsilon^{cd} b^c,$$

$$L_1 = \frac{1}{2} \epsilon_{abcd} d w^{ab} \wedge \tilde{e}^c \wedge \tilde{e}^d + \epsilon_{abcd} d e^a \wedge w^{ab} \wedge \tilde{e}^c \wedge h^d + e^a \wedge (u^a \epsilon^c + c^{ab} \bar{e}_b)$$

$$+ w^a_{ab} \wedge u^a \epsilon e^b + \epsilon^a c^{ab} b^b \epsilon + \frac{1}{2} c_{abca} c^a c^b,$$

$$L_2 = \frac{1}{2} \epsilon_{abcd} d e^a \wedge u^{ab} \wedge \tilde{e}^c \wedge \tilde{e}^d.$$

\(^7\) Rescaling all the fields and antifields by $g$, while multiplying $L$ by $g^{-2}$, we can bring the Lagrangian (5.22) into the form (5.4) with $g$ playing the role of coupling constant.
It is convenient to introduce the following background one- and two-forms:

\[ h_{abc} = \epsilon_{abc} h^d, \quad H_{ab} = \epsilon_{abcd} h^c \wedge h^d. \]  

(5.23)

Then the action of the free BRST differential is given by

\[
\begin{align*}
\delta Q_0 e^a &= de^a + e^b h^b, \\
\delta Q_0 w_{ab} &= dw_{ab} \wedge h_{abc}, \\
\delta Q_0 w_{ab}^* &= de^a \wedge h_{abc} + H_{abc} \wedge w_{ab} - H_{ab} \wedge w_{bc}, \\
\delta Q_0 e^a &= 0, \\
\delta Q_0 e^a_* &= de^a_* \wedge h_{abc}, \\
\delta Q_0 \epsilon_{ab} &= 0, \\
\delta Q_0 \epsilon_{ab}^* &= dw_{ab} - e^a_\cdot h_b + e^a_b \wedge h_a. 
\end{align*}
\]  

(5.24)

The BV symplectic structure (5.20) has the following descendants:

\[
\begin{align*}
\delta \omega_1 &= d\omega_1, \\
\delta \omega_2 &= d\omega_2, \\
\delta \omega_3 &= d\omega_3, \\
\delta \omega_a &= d\omega_a, \\
\delta \omega_{ab} &= d\omega_{ab}. 
\end{align*}
\]  

(5.25)

The free Lagrangian \( L_0 \) is invariant under the shifts

\[
\begin{align*}
e^a &\rightarrow e^a + \xi^a, \\
e^a_{ab} &\rightarrow e^a_{ab} + \xi^a_{ab},
\end{align*}
\]  

(5.26)

where the transformation parameters \( \xi = (\xi^a, \xi^{ab}) \) obey the conditions

\[
\begin{align*}
\xi_{ab} &= -\xi^{ba}, \\
d\xi_{ab} &= 0, \\
d\xi^a &= \xi^a h_b.
\end{align*}
\]  

(5.27)

In local coordinates where \( h^d = dx^d \), we can solve these equations as \( \xi^a = \xi^a + \xi^{ab} x_a \) and \( \xi^{ab} = \xi^{ab} \), with \( \xi^a \) and \( \xi^{ab} = -\xi^{ba} \) being arbitrary constant parameters. The symmetry (5.26) is Hamiltonian relative to the BV symplectic structure (5.20) and is generated by the Hamiltonian form \( H = \xi^a e^a + \xi^{ab} e^a_{ab} \). By Noether’s first theorem, we obtain the ten-parameter family of conserved currents. We find

\[
\delta Q_0 H = dJ_1^a, \quad J_1^a = \xi^a e^a_{ab} + \xi^{ab} w_{ab} = \zeta^a_{1a} + \zeta_{ab} M_1^{ab}.
\]  

(5.28)

Applying the BRST differential (5.24) yields the following descendants:

\[
\begin{align*}
\delta Q_0 J_1^a &= dJ_2^a, \quad J_2^a = \zeta^a_{2a} + \zeta_{ab} M_2^{ab} = (\xi^{ab} e^a_{ab} + \xi^{ab} w_{ab}) \wedge h_{abc}, \\
\delta Q_0 J_1^a &= dJ_3^a, \quad J_3^a = \zeta^a_{3a} + \zeta_{ab} M_3^{ab} = (\xi^{ab} e^a_{ab} + \xi^{ab} e^a_{bc}) \wedge h_{abc},
\end{align*}
\]  

(5.29)

and \( \delta Q_0 J_3^a = 0 \). The conserved currents \( J_2^a \) being two-forms of ghost number zero, are used to define the total energy-momentum \( P \) and the angular momentum \( M \) of an asymptotically flat universe. These are given by the charges

\[
\begin{align*}
P^a &= \int_{S^2} P^a, \\
M_{ab} &= \int_{S^2} M_{ab},
\end{align*}
\]  

(5.30)

where the integrals are over a closed space-like surface \( S \subset M \) at infinity. In particular, \( P^0 \) gives the ADM energy [3, 13]. From the physical viewpoint, it is quite natural to identify the above charges with the generators of the Poincaré group—the isometry group of flat spacetime—and
to expect them to form the Poincare algebra w.r.t. the Lie brackets. However, this is not the case. A straightforward calculation yields

\[ \{ J^\xi_1, J^{\xi'}_1 \} = 0, \quad \{ J^\xi_2, J^{\xi'}_2 \} = 0, \quad \{ J^\xi_3, J^{\xi'}_3 \} = - (\xi^a \xi^{bc} + \xi'^a \xi^bc) h_{abc}. \] (5.31)

As is seen, the abelian Lie algebra of the symmetry transformations (5.26) gets a central extension when evaluated at the level of the one-form currents \( J^\xi_k \). This is the phenomenon of central extension that we discussed in section 3.

In order to reproduce the commutation relations of the Poincare algebra the cubic interaction of gravitons needs to be taken into account. As usual, the cubic part of the Lagrangian (5.22) leads to a sequence of conserved currents of the corresponding free theory. Explicitly,

\[ \delta Q_0 L^1 = dJ^1, \quad J^1 = \left( \frac{1}{2} e^a w^b_d \wedge w^d_c - w^a_d \wedge e^c b_d \right) \wedge h_{abc} + e^a e^{b c} e = \frac{1}{2} w^{a b} e^a c b + \frac{1}{2} e^a e^{b c} e^c b, \]

\[ \delta Q_0 J^1 = dJ^2, \quad J^2 = \left( e^a w^b_d \wedge e^c b_d \right) \wedge h_{abc}, \]

\[ \delta Q_0 J^2 = dJ^3, \quad J^3 = \left( \frac{1}{2} e^a e^{b c} d e^c b \right) h_{abc}. \] (5.32)

By formulas (4.18) and (4.21), these currents define Leibniz algebra structures on the spaces of on-shell Hamiltonian forms of degree 3, 2, and 1. Since the currents \( J^\xi_k \) commute to each other modulo central elements, the symmetric part of the corresponding Leibniz products

\[ \{ J^\xi_k, J^\xi'_k \} = \{ \{ J^\xi_k, J^\xi_k \}, J^\xi'_k \}, \quad k = 1, 2, 3, \] (5.33)

vanishes identically, while the skew-symmetric one reproduces the structure relations of the Poincare algebra:

\[ \{ M^{ab}_{k}, M^{ab'}_{k} \} = \eta^{id} M^{id}_{k} - \eta^{bd} M^{db}_{k} + \eta^{bd} M^{ad}_{k} - \eta^{ab} M^{bd}_{k}, \]

\[ \{ P^b_{k}, M^{bc}_{k} \} = \eta^{bc} P^b_{k} - \eta^{ab} P^c_{k}, \]

\[ \{ P^b_{k}, P^b_{k} \} = 0. \] (5.34)

It should be recognized that the construction of this Lie algebra involves essentially the cubic part \( L^1 \) of the Lagrangian, and not just the conserved currents of linearized gravity.

### 5.3. Unimodular gravity

Unimodular gravity provides an example of a nonlinear gauge theory with lower-degree conservation laws, see e.g. [22–24]. This time it is convenient to work in the metric instead of the vierbein formalism we used above. So, let \( M \) be a four-dimensional spacetime manifold endowed with a pseudo-Riemanian metric \( g_{\mu \nu} \) and let \( \sqrt{-g} d^4 x \) denote the canonical volume form on \((M, g)\). The main idea of unimodular gravity, which goes back to Einstein, is to impose the algebraic constraint

\[ \sqrt{-g} = 1 \] (5.35)

on the metric tensor. Physically, one may regard this as the partial fixing of a reference frame. The unimodularity condition (5.35) breaks the group of spacetime diffeomorphisms—the
Table 2. Fields and antifields of unimodular gravity.

| Field Index | \( C_{\mu\lambda} \) | \( C_{\lambda\mu} \) | \( C_{\mu\nu} \) | \( g_{\mu\nu}^* \) | \( g^{\mu\nu} \) | \( C^{\lambda\mu} \) | \( C^{\sigma\lambda\mu} \) |
|-------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| gh          | -4               | -3               | -2               | -1               | 0                | 1                | 2                | 3                |

gauge group of general relativity—to the subgroup of volume preserving diffeomorphisms. If \( \xi \) is a vector field generating an infinitesimal gauge transformation

\[
\delta \xi g_{\mu\nu} = L_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad (5.36)
\]

then the unimodularity condition requires that

\[
\delta \xi \sqrt{-g} = L_\xi \sqrt{-g} = \sqrt{-g} \nabla_\mu \xi^\mu = 0 \quad \Rightarrow \quad \nabla_\mu \xi^\mu = 0. \quad (5.37)
\]

Hence, the gauge parameter \( \xi \) appears to be constrained by a differential equation. One can solve this equation for \( \xi^\mu \) in terms of an arbitrary bivector \( \xi_{\mu\nu} = -\xi_{\nu\mu} \) as \( \xi^\mu = \nabla_\nu \xi_{\nu\mu} \). Then the volume preserving gauge transformations take the form

\[
\delta \xi g_{\mu\nu} = \nabla_\mu \nabla^\lambda \xi_{\lambda\nu} + \nabla_\nu \nabla^\lambda \xi_{\lambda\mu}. \quad (5.38)
\]

Unlike (5.36), these transformations appear to be reducible. Indeed, using the symmetry properties of the Riemann and Ricci tensors, one can see that the shift

\[
\xi^\mu_{\nu} \rightarrow \xi^\mu_{\nu} + \nabla_\sigma \xi^\sigma_{\lambda\mu\nu}, \quad (5.39)
\]

does not affect the rhs of equation (5.38) for an arbitrary three-vector \( \xi_{\lambda\mu\nu} \); hence, one may regard (5.39) as gauge symmetry for gauge symmetry. The latter transformation, in its turn, is invariant under similar redefinitions

\[
\xi_{\lambda\mu\nu} \rightarrow \xi_{\lambda\mu\nu} + \nabla_\sigma \xi^\sigma_{\lambda\mu\nu}, \quad (5.40)
\]

with \( \xi^\sigma_{\lambda\mu\nu} \) being an arbitrary four-vector. At this step the sequence of reducibility relations stops by the reason of dimension. Now, to put unimodular gravity into the standard BV formalism, one just promotes the gauge parameters \( \xi^\mu_{\nu} \) to the corresponding ghost fields \( C_{\mu\nu} \) and introduces the conjugate antifields\(^8\). The full field content is represented in table 2 below.

Notice that all the ghost and antighost fields are totally skew-symmetric tensors and the antifield \( g_{\mu\nu}^* = g_{\nu\mu}^* \) is supposed to be traceless, i.e.

\[
g^*_{\mu\nu} g^{*\mu\nu} = 0. \quad (5.41)
\]

A relevant presymplectic structure on the fields and antifields is obtained by restricting the canonical symplectic form

\[
\omega = \left( \delta g^{*}_{\mu\nu} \land \delta g_{\mu\nu} + \delta C^*_{\mu\nu} \land \delta C_{\mu\nu} + \delta C^*_{\lambda\mu\nu} \land \delta C_{\lambda\mu\nu} + \delta C^*_{\sigma\lambda\mu\nu} \land \delta C_{\sigma\lambda\mu\nu} \right) \sqrt{-g} \, \mathrm{d}^4 x
\]

onto the subspace of fields obeying the constraints (5.35) and (5.41). Since the bracket of constraints defined by \( \omega \) is clearly nonzero,

\[
\left\{ \mathrm{d}^4 x (\sqrt{-g} - 1), \, \mathrm{d}^4 x \sqrt{-g} g_{\mu\nu}^* g^{*\mu\nu} \right\} = 2 \sqrt{-g} \, \mathrm{d}^4 x > 0,
\]

\(^8\) For an alternative BV formulation of unimodular gravity we refer to [25].
the restricted form is likewise symplectic.

Now, the (minimal) master action for unimodular gravity can be written as

\[ S = \int d^4x \sqrt{-g} \left[ R + g^*_{\mu\nu} \nabla^\lambda \nabla_{\lambda} C^{\lambda\mu\nu} + C^*_{\mu\nu} \nabla_{\lambda} C^{\lambda\mu\nu} + C^*_{\mu\nu\lambda} \nabla_{\sigma} C^{\sigma\mu\nu\lambda} \right]. \] (5.44)

Although the first term looks like the conventional Einstein–Hilbert action, one should keep in mind the algebraic constraint (5.35) enforced on the metric. Evaluating the variation of the action (5.44) under the variation of metric subject to the condition \( g_{\mu\nu} \delta g_{\mu\nu} = 0 \), one can see that the corresponding equations of motion are fully equivalent to the Einstein equations with cosmological constant:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{4} \Lambda g_{\mu\nu}. \] (5.45)

The only difference is that the constant \( \Lambda \) is now a constant of integration rather than a parameter in the action functional. It follows from equation (5.45) that \( R = \Lambda \), i.e. the scalar curvature \( R \) represents a zero-degree conservation law. One may wonder about a global symmetry this conservation law comes from. The answer is almost obvious: the action is invariant under the transformations

\[ C^{\mu\nu\lambda\sigma} \rightarrow C^{\mu\nu\lambda\sigma} + \kappa \sqrt{-g} \epsilon^{\mu\nu\lambda\sigma}, \] (5.46)

where \( \epsilon^{\mu\nu\lambda\sigma} \) is the Levi-Civita symbol and \( \kappa \) is a constant parameter of ghost number 3. By the Noether theorem this gives immediately the conserved current

\[ J_0 = C^*_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda, \quad \text{gh}(J_0) = -3. \] (5.47)

Although this current on its own has no physical interpretation, it gives rise to a sequence of lower-degree conservation laws. Indeed, applying the BRST differential generated by the master Lagrangian (5.44) and the symplectic structure (5.42), we find

\[ \delta Q J_0 = \delta J_1, \quad J_1 = C^*_{\mu\nu} dx^\mu \wedge dx^\nu, \quad \text{gh}(J_1) = -2, \]
\[ \delta Q J_1 = \delta J_2, \quad J_2 = \nabla^\mu g^*_{\mu\nu} dx^\nu, \quad \text{gh}(J_2) = -1, \]
\[ \delta Q J_2 = \delta J_3, \quad J_3 = -\frac{1}{4} R, \quad \text{gh}(J_3) = 0. \] (5.48)

The last nontrivial current of ghost number zero is proportional to the scalar curvature we have discussed above.

As one more source of examples of nonlinear gauge theories with lower-degree conservation laws, we would like to mention higher spin gravity. As shown in the recent paper [26], the higher-spin extension of four-dimensional gravity enjoys an infinite number of nontrivial conservation laws of degrees zero and two.

**Acknowledgments**

The second author is grateful to Glenn Barnich for useful discussions. The results of sections 2–4 were obtained under support of the Foundation for the Advancement of Theoretical Physics and Mathematics ‘BASIS’; the study of section 5 was supported by the Tomsk State University Development Programme (Priority-2030).
Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Tsujishita T 1991 Homological method of computing invariants of systems of differential equations  Differ. Geom. Appl. 1 3–34
[2] Bryant R L and Griffiths P A 1995 Characteristic cohomology of differential systems: I. General theory  J. Am. Math. Soc. 8 507–96
[3] Anderson I M and Torre C G 1996 Asymptotic conservation laws in classical field theory  Phys. Rev. Lett. 77 4109–13
[4] Barnich G, Brandt F and Henneaux M 2000 Local BRST cohomology in gauge theories  Phys. Rep. 338 439–569
[5] Sharapov A A 2016 Variational tricomplex, global symmetries and conservation laws of gauge systems  Symmetry, Integrability Geometry Methods Appl. 12 098
[6] Torre C G 1997 Local cohomology in field theory (with applications to the Einstein equations) 2nd Mexican School on Gravitation and Mathematical Physics (arXiv:hep-th/9706092)
[7] Dickey L A 2003 Soliton Equations and Hamiltonian Systems 2nd edn (Singapore: World Scientific)
[8] Barnich G and Henneaux M 1996 Isomorphisms between the Batalin–Vilkovisky antibracket and the Poisson bracket  J. Math. Phys. 37 5273–96
[9] Sharapov A A 2015 Variational tricomplex of a local gauge system, Lagrange structure and weak Poisson bracket  Int. J. Mod. Phys. A 30 1550152
[10] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity  Ann. Phys., NY 88 286–318
[11] Brown J D and Henneaux M 1986 Central charges in the canonical realization of asymptotic symmetries: an example from three dimensional gravity  Commun. Math. Phys. 104 207–26
[12] Silva S 1999 On superpotentials and charge algebras of gauge theories  Nucl. Phys. B 558 391–415
[13] Barnich G and Brandt F 2002 Covariant theory of asymptotic symmetries, conservation laws and central charges  Nucl. Phys. B 633 3–82
[14] Kosmann-Schwarzbach Y 2004 Derived brackets  Lett. Math. Phys. 69 61–87
[15] Bonezzi R and Hohm O 2020 Leibniz gauge theories and infinity structures  Commun. Math. Phys. 377 2027–77
[16] Kaparulin D S, Lyakhovich S L and Sharapov A A 2011 Local BRST cohomology in (non-) Lagrangian field theory  J. High Energy Phys. JHEP09(2011)006
[17] Anderson I M 1992 Introduction to the variational bicomplex  Contemp. Math. 132 51–73
[18] Verbovetsky A 1998 Notes on the horizontal cohomology  Contemp. Math. 219 211–31
[19] Getzler E 2010 Higher derived brackets (arXiv:1010.5859 [math-ph])
[20] Lavau S and Stasheff J 2020 L∞-algebra extensions of Leibniz algebras (arXiv:2003.07838 [math-ph])
[21] Torre C G 1977 Spinors, jets, and the Einstein equations  The Sixth Canadian Conf. General Relativity and Relativistic Astrophysics (Fields Institute Communications vol 15) ed S P Braham, J D Gegenberg and R J McKellar (Providence, RI: American Mathematical Society) pp 125–36
[22] Unruh W G 1989 Unimodular theory of canonical quantum gravity  Phys. Rev. D 40 1048–52
[23] Henneaux M and Teitelboim C 1989 The cosmological constant and general covariance  Phys. Lett. B 222 195–9
[24] Percacci R 2018 Unimodular quantum gravity and the cosmological constant  Found. Phys. 48 1364–79
[25] Kaparulin D S and Lyakhovich S L 2019 Unfree gauge symmetry in the BV formalism *Eur. Phys. J. C* 79 718

[26] Sharapov A and Skvortsov E 2020 Characteristic cohomology and observables in higher spin gravity *J. High Energy Phys.* JHEP12(2020)190