The Poisson Geometry of SU(1,1)

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Abstract. We study the natural Poisson structure on the group SU(1,1) and related questions. In particular, we give an explicit description of the Ginzburg-Weinstein isomorphism for the sets of admissible elements. We also establish an analogue of Thompson’s conjecture for this group.

1. Introduction

The group SU(1,1) is the group of complex linear transformations of $\mathbb{C}^2$, which preserves the pseudo-Hermitian form:

$$\langle z, w \rangle = z_1 \bar{w}_1 - z_2 \bar{w}_2 .$$

It is a subgroup of SL(2, C), transversal to AN, consisting of upper-triangular matrices with positive real diagonal entries. In the context of Poisson Lie groups, these two Lie groups can be naturally viewed as dual to each other, and thus many questions arise with regards to the induced Poisson structures.

In the present paper, we give explicit formulas in coordinates for those Poisson structures, from which one can see the symplectic leaves. We also describe the dressing action and certain natural identifications between subspaces of admissible elements in AN and $\mathfrak{su}(1,1)^*$. In short, an element is called admissible if it has real spectrum, and its eigenvalue corresponding to the timelike part is greater than the one for the spacelike part of $\mathbb{C}^2$. This definition generalizes to all quasi-Hermitian Lie groups [3], and beyond [10].

In Sections 4 and 5 of the paper, we give two explicit approaches to a Poisson isomorphism between the set of admissible elements in AN with their natural quadratic Poisson structure and the corresponding set in $\mathfrak{su}(1,1)^*$ with the Lie-Poisson structure. This can be considered as the first step in generalizing the Ginzburg-Weinstein theorem [6] to the non-compact setup. One of our approaches follows the original path by Ginzburg and Weinstein, and the other follows the idea of Flaschka and Ratiu [2], based on the Gelfand-Tsetlin coordinates. In fact, their conjecture was proven by Alekseev and Meinrenken [1] for the general SU(n) case. However, in all of those approaches, compactness was used.

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quite heavily, so that a direct generalization to a pseudo-unitary setup is not possible. This is one of the reasons that we need to be explicit in our constructions.

In the last section, we establish an analogue of Thompson’s conjecture in the pseudo-unitary setup associated to the group SU(1, 1). Thompson’s conjecture concerns the equality of spectra in the linear and non-linear situations, for the sums and products of admissible elements, respectively. In our case, the singular spectrum is replaced by the so-called *admissible spectrum*, related to the decomposition of an open subset of admissible elements in a complex reductive group $G_C$ as the product $HA_{adm}H$, where $H$ is a quasi-Hermitian real form of $G_C$.

2. Basic Facts

The group $G = SU(1, 1)$ is realized as the group of $2 \times 2$ complex matrices

$$
\begin{pmatrix}
  u & v \\
  \bar{v} & \bar{u}
\end{pmatrix}
$$

of determinant equal to 1, i.e. satisfying $|u|^2 - |v|^2 = 1$. This group is isomorphic to the group SL(2, $\mathbb{R}$) as a real Lie group. The Lie algebra $\mathfrak{g}$ of $G$ is

$$
\mathfrak{g} = \left\{ \begin{pmatrix}
  ir & \eta \\
  \bar{\eta} & -ir
\end{pmatrix} : r \in \mathbb{R}, \eta \in \mathbb{C} \right\}.
$$

A convenient basis for $\mathfrak{g}$ consists of the elements $X$, $Y$, and $H$, where

$$
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
$$

The Lie algebra structure on $\mathfrak{g}$ as well as the Poisson structure on $\mathfrak{g}^*$ is generated by the Lie bracket relations:

$$
[X, Y] = 2H, \quad [X, H] = -2Y, \quad [Y, H] = 2X.
$$

The dual vector space $\mathfrak{g}^*$ can be identified with the subspace of $\mathfrak{sl}(2, \mathbb{C})$ of the form

$$
\mathfrak{g}^* = \left\{ \begin{pmatrix}
  z & x + iy \\
  -x + iy & -z
\end{pmatrix} : x, y, z \in \mathbb{R} \right\},
$$

consisting of pseudo-Hermitian matrices of signature $(1, 1)$. The natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by the non-degenerate form

$$
\langle A, B \rangle = \Re(\text{Tr}(AB)).
$$

The linear Poisson structure on $\mathfrak{g}^*$ is then given by

$$
\pi_0 = -z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
$$

A Casimir for $\pi_0$ is $z^2 - x^2 - y^2$. 
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Let $a$ and $n$ be the Lie subalgebras of $\mathfrak{g}_c = \mathfrak{sl}(2, \mathbb{C})$ of the form

$$a = \left\{ \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} : r \in \mathbb{R} \right\}, \quad n = \left\{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} : n \in \mathbb{C} \right\}.$$  

The subalgebra $a + n$ of $\mathfrak{g}_c$ can also be viewed as the dual vector space for $\mathfrak{g}$ and the natural pairing is given by

$$\langle A, B \rangle = 2 \Im (\text{Tr}(AB)) .$$

Thus, we have a Manin triple, i.e., a pair of transversal, Lagrangian subalgebras, $\mathfrak{g}$ and $a + n$ of $\mathfrak{sl}(2, \mathbb{C})$, together with the above non-degenerate pairing. We refer to [9] for preliminaries on Poisson Lie groups. We therefore have induced Poisson Lie group structures $\pi_G$ and $\pi_{AN}$ on $G$ and $AN = \exp(a + n)$ respectively.

The Poisson structure $\pi_G$ can be expressed as follows. Consider the element $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ given by $\Lambda = \frac{1}{2} X \wedge Y$. Then for any $g \in G$ we have

$$\pi_G(g) = (r_g)_* \Lambda - (l_g)_* \Lambda ,$$

where $r_g$ and $l_g$ are the right and left translations respectively by $g$ on $G$.

In terms of the matrix elements, for $g = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$, this bracket is quadratic and given by:

$$\{u, \bar{u}\} = -2i|v|^2 \quad \{u, v\} = -iuv \quad \{u, \bar{v}\} = -i\bar{u}v$$

$$\{\bar{u}, v\} = iuv \quad \{\bar{u}, \bar{v}\} = i\bar{u}\bar{v} \quad \{v, \bar{v}\} = 0$$

The matrix formula for the Poisson structure $\pi_{AN}$ on

$$AN = \left\{ \begin{pmatrix} \exp(\frac{x}{2}) & x + iy \\ 0 & \exp(-\frac{x}{2}) \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

is as follows:

$$\pi_{AN} = -\sinh(z) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$  

A Casimir function for this Poisson structure is

$$\xi(x, y, z) = 2 \cosh(z) - x^2 - y^2 .$$

The linearization of $\pi_{AN}$ is denoted by $\pi_0$. This should not lead to confusion because it is given by the exact same formula, (2.1).

Consider the complex anti-linear involution $\dagger$ on the Lie algebra $\mathfrak{g}_c = \mathfrak{sl}(2, \mathbb{C})$, given by

$$M^\dagger = J\bar{M}^T J, \quad \text{where} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Let $q$ denote the fixed point subspace of this involution. (Note that $\mathfrak{g} = \mathfrak{su}(1, 1)$ is the $(-1)$-eigenspace of $\dagger$) Earlier, we have identified $q$ with $\mathfrak{g}^\ast$. The same formula defines an
involution on the matrix Lie group \( \text{SL}(2, \mathbb{C}) \), and its fixed point set there is denoted by
\[
Q = \left\{ \begin{pmatrix} c & \beta \\ -\beta & d \end{pmatrix}, \quad c, d \in \mathbb{R}, \quad \beta \in \mathbb{C}, \quad cd + |\beta|^2 = 1 \right\}.
\]

### 3. Admissible loci

For an open subset \( Q' \) of \( Q \), on which \( c \neq 0 \), one can define coordinates:
\[
Q' = \left\{ \begin{pmatrix} c & a + ib \\ -a + ib & 1 - a^2 - b^2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{R} \right\}.
\]

Consider the following symmetrization map:
\[
\text{Sym} : AN \to Q, \quad M \mapsto M^\dagger M,
\]
\[
a = xe^{z/2}, \quad b = ye^{z/2}, \quad c = e^z
\]
in coordinates. Under this map, the Poisson tensor \( \pi_{AN} \) pushes down to
\[
\pi_Q = \frac{1}{2}(1 - a^2 - b^2 - c^2) \frac{\partial}{\partial a} \wedge \frac{\partial}{\partial b} + bc \frac{\partial}{\partial c} \wedge \frac{\partial}{\partial a} + ac \frac{\partial}{\partial b} \wedge \frac{\partial}{\partial c},
\]
with Casimir
\[
F(a, b, c) = \frac{1 + c^2 - a^2 - b^2}{c},
\]
which is simply the trace. Note that the dressing action of \( G \) on \( AN \) converts to conjugation on \( Q \).

We define the subset of admissible elements \( q_{\text{adm}} \subset q \) as the set of elements conjugate to the diagonal matrices \( \text{diag}(\lambda, -\lambda) \) with \( \lambda > 0 \). The set of admissible elements forms an open cone in \( q \) defined by
\[
z^2 - x^2 - y^2 > 0, \quad \text{and} \quad z > 0.
\]

Denote also \( Q_{\text{adm}} = \exp(q_{\text{adm}}) \). The exponential map is easily checked to be invertible and we denote its inverse
\[
\log : \quad Q_{\text{adm}} \to q_{\text{adm}}.
\]

Actually, if one denotes \( A_{\text{adm}} = \text{diag}(e^{z/2}, e^{-z/2}) \), and \( (AN)_{\text{adm}} = G.A_{\text{adm}} \), where “.” denotes the dressing action, then the image of \( (AN)_{\text{adm}} \) under the symmetrization map is exactly \( Q_{\text{adm}} \). On the set of admissible elements, \( (AN)_{\text{adm}} \), the right dressing action is globally defined. (Which is not true for the left dressing action.) This is also the case for a general pseudo-unitary groups \( SU(p, q) \). In our case, we have explicitly:
\[
\begin{pmatrix}
e^{z/2} & 0 \\
0 & e^{-z/2}
\end{pmatrix} \cdot \begin{pmatrix} u & v \\
v & \bar{u}
\end{pmatrix} = \begin{pmatrix} u' & v' \\
v' & \bar{u'}
\end{pmatrix} \cdot \begin{pmatrix} \rho & m \\
0 & \rho^{-1}
\end{pmatrix},
\]
where
\[
\rho = \sqrt{|u|^2 e^z - |v|^2 e^{-z}}, \quad u' = \frac{ue^{z/2}}{\rho}, \quad v' = \frac{ve^{-z/2}}{\rho}, \quad m = 2\bar{uv} \sinh(z)\rho.
\]
Note that $\rho$ is well-defined, since $z > 0$. It is also easy to see that the symmetrization map is a diffeomorphism on the set of admissible elements.

For an element $B \in Q_{\text{adm}}$, denote by $(e^\lambda, e^{-\lambda})$ the set of its eigenvalues with $\lambda > 0$. Then the log map from $Q_{\text{adm}}$ to $q_{\text{adm}}$,

$$
\log : \left( \begin{array}{ccc}
c & a + ib \\
-a + ib & \frac{1}{-a^2 - b^2}
\end{array} \right) \mapsto \left( \begin{array}{ccc}
z & x + iy \\
x - iy & -z
\end{array} \right)
$$

is given by

$$(a, b, c) \mapsto (a \frac{\lambda}{\sinh(\lambda)}, b \frac{\lambda}{\sinh(\lambda)}, c \frac{\lambda}{\sinh(\lambda)} - \lambda \coth(\lambda)) = (x, y, z) .$$

Under the composition of these two diffeomorphisms on the sets of admissible elements, the Poisson structure $\pi_{AN}$ pushes down to

$$
\pi := \log_*(\text{Sym}_*(\pi_{AN}))
$$

$$
= -z(\lambda \coth \lambda + z) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y(\lambda \coth \lambda + z) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x(\lambda \coth \lambda + z) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}
$$

$$
= (\lambda \coth \lambda + z)\pi_0 .
$$

The natural Casimir function for this Poisson structure is clearly $x^2 + y^2 - z^2 = -\lambda^2$, which is the determinant. Note that the symplectic leaves of those two structures are actually the same, and are hyperboloids. One can view $\pi$ as a family of Poisson structures on a single hyperboloid, depending on $\lambda$, and identify it diffeomorphically with the lower hemisphere, and show that it extends to the whole sphere. In terms of a holomorphic coordinate $w$ on the sphere, this one-parameter family of Poisson structures can also be written as

$$
\pi(\tau) = i(1 - |w|^2)|w|^2 \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}} + \tau \cdot i(1 - |w|^2)^2 \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}},
$$

for $\tau \in \mathbb{R}$, where the first term is the so-called $\Pi_v$ structure from [5], and the second is an $SU(1, 1)$-invariant Poisson structure on $S^2$.

4. POISSON ISOMORPHISM: THE GINZBURG-WEINSTEIN APPROACH

The Ginzburg-Weinstein approach in the compact situation to finding a Poisson isomorphism between $\mathfrak{t}^*$ and $K^*$ was to prove the existence of a vector field whose flow would connect $\pi_0$ and $\pi$. We will construct such a vector field explicitly for the $SU(1, 1)$ case.

Following the Ginzburg-Weinstein argument, define a bivector field $\pi_t$ on $\mathfrak{q}$ by

$$
\pi_t(\tilde{v}) := \frac{\pi(t\tilde{v})}{t},
$$

where the expression on the right-hand side is identified with an element of $\wedge^2 (T_{\tilde{v}}\mathfrak{q})$ by translation. Now set

$$
\dot{\pi}_t := \frac{d}{dt} \pi_t,
$$

for $t \in \mathbb{R}$. The vector field $\dot{\pi}_t$ is then a Poisson isomorphism $\pi_t : (\mathfrak{q}, \pi_t) \to (\mathfrak{t}^*, \pi_0)$.
and
\[ \dot{\pi} := \left. \frac{d}{dt} \right|_{t=1} \pi_t. \]

In coordinates,
\[
\dot{\pi} = -z \left( \lambda \coth \lambda + z - \frac{\lambda^2}{\sinh^2 \lambda} \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \\
y \left( \lambda \coth \lambda + z - \frac{\lambda^2}{\sinh^2 \lambda} \right) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \\
x \left( \lambda \coth \lambda + z - \frac{\lambda^2}{\sinh^2 \lambda} \right) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

**Proposition 4.1.** There exists a vector field $X$ on $\mathfrak{q}_{adm} \cong (\mathfrak{a} + \mathfrak{n})_{adm}$ such that

1. $[X, \pi] = \dot{\pi}$,
2. $X$ has the zero linearization at the origin,
3. $X$ is tangent to the symplectic leaves of $\pi$ (and $\pi_0$), and
4. $X$ is complete.

Note that in the compact case considered by Ginzburg and Weinstein, completeness simply follows from the fact that $X$ is tangent to the symplectic leaves of $\pi$.

**Proof.** It will be convenient to convert to hyperbolic coordinates $(\lambda, \phi, s)$. The relations between rectangular and hyperbolic coordinates are
\[
x = \lambda (\sinh s) (\cos \phi) \\
y = \lambda (\sinh s) (\sin \phi) \\
z = \lambda (\cosh s)
\]
\[ \lambda = \sqrt{z^2 - x^2 - y^2} \]
\[ \phi = \arctan(y/x) \]
\[ s = \cosh^{-1}(z/\lambda). \]

In these coordinates,
\[ \pi = \frac{1}{\sinh s} \left( \coth \lambda + \cosh s \right) \frac{\partial}{\partial \phi} \wedge \frac{\partial}{\partial s}, \]
and
\[ \dot{\pi} = \frac{1}{\sinh s} \left( \coth \lambda + \cosh s - \frac{\lambda}{\sinh^2 \lambda} \right) \frac{\partial}{\partial \phi} \wedge \frac{\partial}{\partial s}. \]

Set
\[ g(s) := \frac{1}{\sinh s} \left( \coth \lambda + \cosh s \right), \]
\[ h(s) := \frac{1}{\sinh s} \left( \coth \lambda + \cosh s - \frac{\lambda}{\sinh^2 \lambda} \right). \]

The action of the diagonal torus $T \subset G$ on $\mathfrak{a} + \mathfrak{n}$ corresponds to rotation about the $z$-axis in $\mathfrak{q}$. Since $\pi$, and hence $\dot{\pi}$, are invariant under the torus action, we may assume that $X$
has the form \( f(s) \frac{\partial}{\partial s} \), where \( f(s) \) does not depend on \( \phi \). Then the equation \([X, \pi] = \dot{\pi}\) reduces to the ODE

\[
f \frac{\partial g}{\partial s} - g \frac{\partial f}{\partial s} = h.
\]

Rewriting the left-hand side using the quotient rule gives

\[
- \frac{\partial}{\partial s} \left( f g \right) \cdot g^2 = h,
\]

or, equivalently,

\[
(4.1) \quad f = -g \cdot \int \frac{h}{g^2} \, ds.
\]

Integrating, we obtain

\[
(4.2) \quad \int \frac{h}{g^2} \, ds = \ln(\coth \lambda + \cosh s) + \frac{\lambda}{\sinh^2 \lambda} \left( \frac{1}{\coth \lambda + \cosh s} \right) + C,
\]

where \( C \) is constant with respect to \( s \) and \( \phi \). Note that \( g \to \infty \) as \( s \to 0 \). Therefore, to ensure smoothness when \( s = 0 \), set

\[
C = -\ln(\coth \lambda + 1) - \frac{\lambda}{\sinh^2 \lambda} \left( \frac{1}{\coth \lambda + 1} \right).
\]

Thus, we obtain the vector field

\[
X = \left( \frac{\coth \lambda + \cosh s}{\sinh s} \right) \left[ \ln \left( \frac{\coth \lambda + \cosh s}{\coth \lambda + 1} \right) + \frac{\lambda}{\sinh^2 \lambda} \left( \frac{1}{\coth \lambda + \cosh s} - \frac{1}{\coth \lambda + 1} \right) \right] \frac{\partial}{\partial s},
\]

which extends smoothly to the positive \( z \)-axis (where it vanishes). This vector field is smooth on the open cone \( z > \sqrt{x^2 + y^2} \), extends continuously to the boundary \( z = \sqrt{x^2 + y^2} \), and satisfies \([X, \pi] = \dot{\pi}\).

It is easy to check that \( X \) has zero linearization at the origin. Since when \( s \to \infty \), we have \( X \sim s \frac{\partial}{\partial s} \), the restriction \( X_\lambda \) of \( X \) to any hyperboloid \( \lambda = \sqrt{z^2 - x^2 - y^2} \) extends continuously to the boundary, which we have identified with the unit circle in the plane. Since the closed unit disk is compact, it follows that \( X_\lambda \) is complete for every \( \lambda \), which implies that \( X \) is complete. This completes the proof. \( \square \)

Given the vector field \( X \) from Proposition 4.1, the Ginzburg-Weinstein argument goes through as in the compact case. Defining \( X_t \) by

\[
X_t(\tp) := \frac{X(t \tp)}{t^2},
\]

the corresponding flow \( \phi_t \) pushes \( \pi_0 \) forward to \( \pi_t \), and in particular, \( (\varphi_1)_*(\pi_0) = \pi \). Thus, \( \phi_1 \) is the desired Poisson isomorphism.
5. Poisson Isomorphism: The Flaschka-Ratiu Approach

In this section, we apply the procedure used by Flaschka and Ratiu in [2] for the SU(2) case to our pseudo-unitary situation. The idea is to use the Gelfand-Tsetlin coordinates, which are given by the eigenvalues of the principal minors of a matrix. The Gelfand-Tsetlin coordinates were developed in [7] for the unitary case and extended in [3] to the pseudo-unitary case.

The elements of $q_{\text{adm}}$ with eigenvalues $(\lambda, -\lambda)$ for $\lambda > 0$ can be parameterized by the matrices

$$\begin{pmatrix}
  z \\
  -\sqrt{z^2 - \lambda^2} \cdot e^{-i\theta} \\
  \sqrt{z^2 - \lambda^2} \cdot e^{i\theta}
\end{pmatrix},
$$

with $z \geq \lambda$ and $0 \leq \theta < 2\pi$. Define coordinates on $q$ by identifying (5.1) with $(z, \lambda, \theta) \in \mathbb{R}^3$. The coordinates $z$ and $\lambda$ are the eigenvalues of the upper left $1 \times 1$ and $2 \times 2$ minors. In these coordinates,

$$\pi_0 = \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial z}.$$

The symplectic structure induced by $\pi_0$ on any symplectic leaf

$$\Theta_\lambda = \left\{(x, y, z) : \sqrt{z^2 - y^2} - x^2 = \lambda\right\}$$

is then given by:

$$\omega_0 := d\theta \wedge dz.$$

Similarly, the elements of $Q$ with eigenvalues $(e^\lambda, e^{-\lambda})$ can be parameterized by the matrices

$$\begin{pmatrix}
  e^w \\
  -\sqrt{(e^w - e^\lambda)(e^w - e^{-\lambda})} \cdot e^{-i\theta} \\
  \sqrt{(e^w - e^\lambda)(e^w - e^{-\lambda})} \cdot e^{i\theta} / 2 \cosh(\lambda - e^w)
\end{pmatrix}.
$$

Define coordinates on $Q$ by identifying (5.1) with $(w, \lambda, \theta) \in \mathbb{R}^3$. In these coordinates:

$$\pi_Q = \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial w}.$$

The symplectic structure induced by $\pi_Q$ on any symplectic leaf

$$\Psi_\lambda = \left\{(a, b, c) : \frac{1 + c^2 - a^2 - b^2}{c} = 2(\cosh \lambda)\right\}$$

is then given by:

$$\omega_Q := d\theta \wedge dw.$$

Given these simple expressions for $\omega_0$ and $\omega_1$, for each $\lambda$, we can define a symplectomorphism from $\Theta_\lambda$ to $\Psi_\lambda$ by identifying the matrix (5.1) with the matrix (5.2). A Poisson isomorphism $f$ from $(q_{\text{adm}}, \pi_0)$ to $(Q_{\text{adm}}, \pi_Q)$ is obtained by allowing $\lambda$ to vary over the interval $(0, \infty)$. Equivalently, $f$ sends $(z, \lambda, \theta)$ to $(w, \lambda, \theta)$. In terms of the coordinates

$$\begin{pmatrix}
  z \\
  -x + iy \\
  -z
\end{pmatrix} \leftrightarrow (x, y, z)$$

we have

$$f: q \rightarrow Q, \quad \lambda \rightarrow \lambda.$$
on \( q \) and
\[
\begin{pmatrix}
  c & a + ib \\
  -a + ib & \frac{1-a^2-b^2}{c}
\end{pmatrix} \leftrightarrow (a, b, c)
\]
on \( Q \), \( f \) is given by
\[
a &= \sqrt{e^{2z} - 2e^z \cosh(\sqrt{z^2 - x^2 - y^2}) + 1} \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \\
b &= \sqrt{e^{2z} - 2e^z \cosh(\sqrt{z^2 - x^2 - y^2}) + 1} \left( \frac{y}{\sqrt{x^2 + y^2}} \right) \\
c &= e^z.
\]
The map \( f \) is neither one-to-one nor onto, but it is actually a diffeomorphism when restricted to \( q_{\text{adm}} \).

6. Thompson’s conjecture

In this section we establish an analogue of Thompson’s conjecture for the \( SU(1, 1) \) case. The original proof in the unitary case is due to Klyachko [3]. First, we say that an element \( g \in G_C = \text{SL}(2, \mathbb{C}) \) is admissible if it can be decomposed as a product \( g = hb \), with \( h \in G = SU(1, 1) \) and \( b \in (AN)_{\text{adm}} \). In this case, the admissible spectrum of \( g \) is the number \( \gamma > 0 \) such that the pair \((e^\gamma, e^{-\gamma})\) is the spectrum of \( g^\dagger g = b^\dagger b \).

This is equivalent to saying that \( g \in G_C \) is admissible if it lies in the open subset \( GA_{\text{adm}}G \) of \( G_C \), in which case its admissible spectrum can be read from the middle term of this decomposition.

For example, if
\[
b = \begin{pmatrix}
  e^{z/2} & x + iy \\
  0 & e^{-z/2}
\end{pmatrix} \in (AN)_{\text{adm}},
\]
then it is admissible if and only if \( z > 0 \) and \( \Delta := e^z + e^{-z} - (x^2 + y^2) > 2 \), in which case the admissible spectrum is given by
\[
\gamma = \log \left( \frac{\Delta + \sqrt{\Delta^2 - 4}}{2} \right).
\]

Lemma 6.1. If \( g_1, g_2 \in G_C \) are admissible, then their product \( g_1g_2 \) is also admissible.

Proof. The proof is omitted, as it is a short computational affair, which uses the fact that the dressing action on \( (AN)_{\text{adm}} \) does not change the admissible spectrum, and therefore we can assume one of the two elements in \( (AN)_{\text{adm}} \) is diagonal.

It is easy to see that the possible admissible spectrum of an element \( b \) given by (6.1) lies in the interval \([z, \infty)\).
Next, one can readily establish that for two elements $M_1, M_2 \in q_{\text{adm}} = \mathfrak{su}(1,1)^*_{\text{adm}}$ with respective eigenvalues $(\lambda_1, -\lambda_1)$ and $(\lambda_2, -\lambda_2)$ such that $\lambda_1, \lambda_2 > 0$, the possible spectrum $(\lambda, -\lambda)$ of $M_1 + M_2$ satisfies $\lambda \geq \lambda_1 + \lambda_2$, which is equivalent to the reversed triangle inequality in Minkowski space [4].

Thompson’s conjecture in our particular case now is equivalent to the following

**Proposition 6.2.** For two admissible elements $g_1$ and $g_2$ from $G_C$ with admissible spectra $\lambda_1$ and $\lambda_2$ respectively, the admissible spectrum of their product $g_1 g_2$ lies in the interval $[\lambda_1 + \lambda_2, \infty)$. 

**Proof.** Clearly, we can assume $g_1 = b_1 \in (AN)_{\text{adm}}$ and $g_2 = b_2 \in (AN)_{\text{adm}}$ as well. Also, using the dressing action, one can assume that one of those elements, say $b_2$, is diagonal: $b_2 = a_2 \in A_{\text{adm}}$, where $a_2 = \text{diag}(\rho, \rho^{-1})$. Let $a_1 = \text{diag}(r, r^{-1})$ and $g \in SU(1,1)$ be such that $b_1 = a_1 g$, with respect to the dressing action. The element $g$ is given by 

$$g = \begin{pmatrix} u & v \\ \bar{u} & \bar{v} \end{pmatrix},$$

with $|u|^2 - |v|^2 = 1$. The admissible spectra of $b_1$ and $b_2$ are $\lambda_1 = 2 \log(r)$ and $\lambda_2 = 2 \log(\rho)$ respectively. Now consider the product $b = b_1 b_2$ and compute:

$$b^\dagger b = b_2^\dagger b_1^\dagger b_1 b_2 = a_2^\dagger (a_1 g)^\dagger (a_1 g) a_2 = a_2 g^{-1} a_1^2 g a_2.$$

In terms of matrices, we have

$$\text{Tr}(b^\dagger b) = r^2 \rho^2 |u|^2 - r^{-2} \rho^2 |v|^2 + r^{-2} \rho^2 |u|^2 - r^2 \rho^{-2} |v|^2.$$ 

Thus, if $\mu$ is the greatest root of the quadratic equation

$$\mu + \frac{1}{\mu} = \text{Tr}(b^\dagger b),$$

then the admissible spectrum of $\lambda$ of $b = b_1 b_2$ is given by $\lambda = \log(\mu)$. It follows that Thompson’s conjecture in our case is equivalent to proving

$$\mu \geq r^2 \rho^2.$$

Now, if we substitute $|u|^2 = 1 + |v|^2$, then we obtain:

$$\mu + \frac{1}{\mu} = |v|^2 \left( r^2 \rho^2 - \frac{\rho^2}{r^2} + \frac{1}{r^2 \rho^2} - \frac{\rho^2}{\rho^2} \right) + r^2 \rho^2 + \frac{1}{r^2 \rho^2} =$$

$$= |v|^2 \left( \rho^2 - \frac{1}{\rho^2} \right) \left( r^2 - \frac{1}{r^2} \right) + r^2 \rho^2 + \frac{1}{r^2 \rho^2} \geq r^2 \rho^2 + \frac{1}{r^2 \rho^2}.$$

Since $\mu$ is the greater root of this equation, and $\xi(x) = x + \frac{1}{x}$ is an increasing function of $x$, for $x > 1$, we conclude that $\mu \geq r^2 \rho^2$ as desired. \qed
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