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Integrability properties and Limit Theorems for the exit time from a cone of planar Brownian motion

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Abstract

We obtain some integrability properties and some limit Theorems for the exit time from a cone of a planar Brownian motion, and we check that our computations are correct via Bougerol’s identity.

Key words: Bougerol’s identity, planar Brownian motion, skew-product representation, exit time from a cone.

MSC Classification (2010): 60J65, 60F05.

1 Introduction

We consider a standard planar Brownian motion§ (Z_t = X_t + iY_t, t ≥ 0), starting from x_0 + i0, x_0 > 0, where (X_t, t ≥ 0) and (Y_t, t ≥ 0) are two independent linear Brownian motions, starting respectively from x_0 and 0.

As is well known [ItMK65], since x_0 ≠ 0, (Z_t, t ≥ 0) does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process θ_t = Im(∫_0^t dZ_s), t ≥ 0 is well defined. A scaling argument shows that we may assume x_0 = 1, without loss of generality, since, with obvious notation:

\[
\left( Z_t^{(x_0)}, t ≥ 0 \right) \overset{(law)}{=} \left( x_0 Z_{(t/x_0^2)}, t ≥ 0 \right).
\]

§When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.
Thus, from now on, we shall take $x_0 = 1$.

Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \bigg|_{u=H_t}$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i0 = 0$.

Thus, the Bessel clock $H$ plays a key role in many aspects of the study of the winding number process $(\theta_t, t \geq 0)$ (see e.g. [Yor80]).

Rewriting (2) as:

$$\log |Z_t| = \beta H_t; \quad \theta_t = \gamma H_t,$$

we easily obtain that the two $\sigma$-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

We shall also use Bougerol’s celebrated identity in law [Bou83, ADY97] and [Yor01] (p. 200), which may be written as:

$$\sinh(\beta_t) \stackrel{(law)}{=} \hat{\beta}_A_t(\beta)$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$ and $(\hat{\beta}_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$. For the random times $T_c^{[\theta]} \equiv \inf\{t : |\theta_t| = c\}$, and $T_c^{[\gamma]} \equiv \inf\{t : |\gamma_t| = c\}$, $(c > 0)$ by using the skew-product representation (3) of planar Brownian motion [ReY99], we obtain:

$$T_c^{[\theta]} = A_{\hat{T}_c^{[\gamma]}}(\beta) \equiv \int_0^{T_c^{[\gamma]}} ds \exp(2\beta_s) = H_u^{-1} \bigg|_{u=\hat{T}_c^{[\gamma]}}.$$

Moreover, it has been recently shown that, Bougerol’s identity applied with the random time $T_c^{[\theta]}$ instead of $t$ in (4) yields the following [Vak11]:

**Proposition 1.1** The distribution of $T_c^{[\theta]}$ is characterized by its Gauss-Laplace transform:

$$E \left[ \sqrt{\frac{2c^2}{\pi T_c^{[\theta]}}} \exp \left( -\frac{x^2}{2T_c^{[\theta]}} \right) \right] = \frac{1}{\sqrt{1+x}} \varphi_m(x),$$

for every $x \geq 0$, with $m = \frac{\pi}{2c}$, and:

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad G_{\pm}(x) = \sqrt{1 + x} \pm \sqrt{x}.$$  

The remainder of this article is organized as follows: in Section 2 we study some integrability properties for the exit times from a cone; more precisely we obtain some new results concerning the negative moments of $T_c^{[\theta]}$ and of $T_c^{\theta} \equiv \inf\{t : \theta_t = c\}$. In Section 3 we state and prove some limit Theorems for these random times for $c \to 0$ and for $c \to \infty$ followed by several generalizations (for extensions of these works to more general planar processes, see e.g. [DoV12]). We use these results in order to obtain (see Remark 3.4) a
new simple non-computational proof of Spitzer’s celebrated asymptotic Theorem [Spi58],
which states that:

$$\frac{2}{\log t} \theta_t \xrightarrow{\text{law}} C_1,$$

with $C_1$ denoting a standard Cauchy variable (for other proofs, see also e.g. [Wil74, 
Dur82, MeY82, BeW94, Yor97, Vak11]). Finally, in Section 4 we use the Gauss-Laplace
transform (6) which is equivalent to Bougerol’s identity (4) in order to check our results.

## 2 Integrability Properties

Concerning the moments of $T_{\varepsilon}^{[\theta]}$, we have the following (a more extended discussion is
found in e.g. [MaY05]):

**Theorem 2.1** For every $c > 0$, $T_{\varepsilon}^{[\theta]}$ enjoys the following integrability properties:

(i) for $p > 0$, $E \left[ \left( T_{\varepsilon}^{[\theta]} \right)^p \right] < \infty$, if and only if $p < \frac{\pi}{4c}$.

(ii) for any $p < 0$, $E \left[ \left( T_{\varepsilon}^{[\theta]} \right)^p \right] < \infty$.

**Corollary 2.2** For $0 < c < d$, the random times $T_{\varepsilon,d,c} \equiv \inf \{ t : \theta_t \notin (-d,c) \}$, $T_{\varepsilon}^{[\theta]}$ and $T_{\varepsilon}^{\theta}$ satisfy the inequality:

$$T_{\varepsilon}^{\theta} \geq T_{\varepsilon,d,c} \geq T_{\varepsilon}^{[\theta]}.$$

Thus, their negative moments satisfy:

$$E \left[ \frac{1}{(T_{\varepsilon}^{[\theta]})^p} \right] \leq E \left[ \frac{1}{(T_{\varepsilon,d,c})^p} \right] \leq E \left[ \frac{1}{(T_{\varepsilon}^{\theta})^p} \right] < \infty.$$

**Proofs of Theorem 2.1 and of Corollary 2.2**

(i) The original proof is given by Spitzer [Spi58], followed later by many authors [Wil74, 
Bur77, MeY82, Dur82, Yor85]. See also [ReY99] Ex. 2.21/page 196.

(ii) In order to obtain this result, we might use the representation $T_{\varepsilon}^{[\theta]} = A_{T_{\varepsilon}^{[\gamma]}}$ together
with a recurrence formula for the negative moments of $A_t$ [Duf00], Theorem 4.2, p. 417
(in fact, Dufresne also considers $A_t^{(\mu)} = \int_0^t ds \exp(2\beta s + 2\mu s)$, but we only need to take
$\mu = 0$ for our purpose, and we note $A_t \equiv A_t^{(0)}$) [Vakth11]. However, we can also obtain
this result by simply remarking that the RHS of the Gauss-Laplace transform (6) in
Proposition 1.1 is an infinitely differentiable function in 0 (see also [VaY11]), thus:

$$E \left[ \frac{1}{(T_{\varepsilon}^{[\theta]})^p} \right] < \infty, \text{ for every } p > 0. \quad (11)$$

Now, Corollary 2.2 follows immediately from Theorem 2.1 (ii).
3 Limit Theorems for $T_c^{[\theta]}$

3.1 Limit Theorems for $T_c^{[\theta]}$, as $c \to 0$ and $c \to \infty$

The skew-product representation of planar Brownian motion allows to prove the three following asymptotic results for $T_c^{[\theta]}$.

Proposition 3.1 a) For $c \to 0$, we have:

$$
\frac{1}{c^2} T_c^{[\theta]} \xrightarrow{\text{law}} T_{1}^{[\gamma]}.
$$

b) For $c \to \infty$, we have:

$$
\frac{1}{c} \log\left( T_c^{[\theta]} \right) \xrightarrow{\text{law}} 2|\beta| T_{1}^{[\gamma]}.
$$

c) For $\varepsilon \to 0$, we have:

$$
\frac{1}{\varepsilon^2} \left( T_{c+\varepsilon}^{[\theta]} - T_c^{[\theta]} \right) \xrightarrow{\text{law}} \exp\left( 2\beta T_{1}^{[\gamma]} \right) T_{1}^{[\gamma]},
$$

where $\gamma'$ stands for a real Brownian motion, independent from $\gamma$, and $T_{1}^{[\gamma]} = \inf\{t : \gamma'_t = 1\}$

Proof of Proposition 3.1:

We rely upon (5) for the three proofs. By using the scaling property of BM, we obtain:

$$
T_c^{[\theta]} = A_{T_{1}^{[\gamma]}}(\beta) \xrightarrow{\text{law}} A_u(\beta) \bigg|_{u = c^2 T_{1}^{[\gamma]}},
$$

thus:

$$
\frac{1}{c^2} T_c^{[\theta]} \xrightarrow{\text{law}} \int_0^{T_{1}^{[\gamma]}} dv \exp(2c\beta_v).
$$

a) For $c \to 0$, the RHS of (15) converges to $T_{1}^{[\gamma]}$, thus we obtain part a) of the Proposition.

b) For $c \to \infty$, taking logarithms on both sides of (15) and dividing by $c$, on the LHS we obtain $\frac{1}{c} \log\left( T_c^{[\theta]} \right) - \frac{2}{c} \log c$ and on the RHS:

$$
\frac{1}{c} \log\left( \int_0^{T_{1}^{[\gamma]}} dv \exp(2c\beta_v) \right) = \log\left( \int_0^{T_{1}^{[\gamma]}} dv \exp(2c\beta_v) \right)^{1/c},
$$

which, from the classical Laplace argument: $\|f\|_p \xrightarrow{p \to \infty} \|f\|_\infty$, converges for $c \to \infty$, towards:

$$
2 \sup_{v \leq T_{1}^{[\gamma]}} (\beta_v) \xrightarrow{\text{law}} 2|\beta| T_{1}^{[\gamma]}.
$$
This proves part \( b \) of the Proposition.

c)  
\[
T_{c+\varepsilon}^{[\theta]} - T_{c}^{[\theta]} = \int_{T_{c}^{[\gamma]}}^{T_{c+\varepsilon}^{[\gamma]}} du \exp \left( 2\beta u \right) = \int_{0}^{T_{c+\varepsilon}^{[\gamma]} - T_{c}^{[\gamma]}} dv \exp \left( 2\beta_T^{[\gamma]} \right) \exp \left( 2 \left( \beta_{T^{[\gamma]} - \beta_T^{[\gamma]} - \varepsilon} \right) \right)
\]
\[
= \exp \left( \left( \beta_T^{[\gamma]} \right) \right) \int_{0}^{T_{c+\varepsilon}^{[\gamma]} - T_{c}^{[\gamma]}} dv \exp \left( 2B_v \right),
\]  
(16)

where \( \left( B_s \equiv \beta_{s+T^{[\gamma]} - \beta_T^{[\gamma]}, s \geq 0} \right) \) is a BM independent of \( T_{c}^{[\gamma]} \).

We study now \( T_{c+\varepsilon}^{[\gamma]} \equiv T_{c+\varepsilon}^{[\gamma]} - T_{c}^{[\gamma]} \), the first hitting time of the level \( c+\varepsilon \) from \( |\gamma| \), starting from \( c \). Thus, we define: \( \rho_u \equiv |\gamma_u| \), starting also from \( c \). Thus, \( \rho_u = c + \delta_u + L_u \), where \( (\delta_u, s \geq 0) \) is a BM and \( (L_u, s \geq 0) \) is the local time of \( \rho \) at 0. Thus:

\[
T_{c+\varepsilon}^{[\gamma]} = \inf \{ u \geq 0 : \rho_u = c + \varepsilon \} = \inf \{ u \geq 0 : \delta_u + L_u = \varepsilon \}
\]
\[
= \varepsilon^2 \inf \left\{ v \geq 0 : \frac{1}{\varepsilon} \delta_{v\varepsilon} + \frac{1}{\varepsilon} L_{v\varepsilon} = 1 \right\}.
\]  
(17)

From Skorokhod’s Lemma [ReY99]:

\[
L_u = \sup_{y \leq u} \left( (-c - \delta_y) \lor 0 \right)
\]

we deduce:

\[
\frac{1}{\varepsilon} L_{v\varepsilon} = \sup_{y \leq v\varepsilon} \left( (-c - \delta_y) \lor 0 \right) = \sup_{y \leq \varepsilon^2 \sigma} \left( (-c - \frac{1}{\varepsilon} \delta_{\varepsilon^2 \sigma}) \lor 0 \right) = 0.
\]  
(18)

Hence, with \( \gamma' \) denoting a new BM independent from \( \gamma \), (16) writes:

\[
T_{c+\varepsilon}^{[\theta]} - T_{c}^{[\theta]} = \exp \left( \left( \beta_T^{[\gamma]} \right) \right) \int_{0}^{e^{2T_{T}^{\gamma'}}} dv \exp \left( 2B_v \right).
\]  
(19)

Thus, dividing both sides of (19) by \( \varepsilon^2 \) and making \( \varepsilon \to 0 \), we obtain part \( c \) of the Proposition. 

\[\blacksquare\]

**Remark 3.2** The asymptotic result \( c \) in Proposition 3.1 may also be obtained in a straightforward manner from (16) by analytic computations. Indeed, using the Laplace transform of the first hitting time of a fixed level by the absolute value of a linear Brownian motion \( E \left[ e^{-\frac{\lambda^2}{2} (T_{c}^{[\gamma]} - T_{c}^{[\gamma]})} \right] = \frac{1}{\cosh(\lambda b)} \) (see e.g. Proposition 3.7, p 71 in Revuz and Yor [ReY99]), we have that for \( 0 < c < b \), and \( \lambda \geq 0 \):

\[
E \left[ e^{-\frac{\lambda^2}{2} (T_{c}^{[\gamma]} - T_{c}^{[\gamma]})} \right] = \frac{\cosh(\lambda c)}{\cosh(\lambda b)}
\]  
(20)

Using now \( b = c + \varepsilon \), for every \( \varepsilon > 0 \), the latter equals:

\[
\frac{\cosh(\lambda c)}{\cosh \left( \frac{\lambda}{2} (c + \varepsilon) \right)} \xrightarrow{\varepsilon \to 0} e^{-\lambda}.
\]

The result follows now by remarking that \( e^{-\lambda} \) is the Laplace transform (for the argument \( \lambda^2/2 \)) of the first hitting time of 1 by a linear Brownian motion \( \gamma' \), independent from \( \gamma \).
3.2 Generalizations

Obviously we can obtain several variants of Proposition 3.1, by studying $T^\theta_{bc,ac}$, $0 < a, b \leq \infty$, for $c \to 0$ or $c \to \infty$, and $a, b$ fixed. We define $T_{-c} \equiv \inf\{t : \gamma_t \not\in (-d, c)\}$ and we have:

- $\frac{1}{c^2} T^\theta_{bc,ac} \xrightarrow{\text{law}} T_{-b,a}^\gamma$.

- $\frac{1}{c} \log (T^\theta_{bc,ac}) \xrightarrow{\text{law}} 2|\beta|T_{-b,a}^\gamma$.

In particular, we can take $b = \infty$, hence:

**Corollary 3.3**

a) For $c \to 0$, we have:

$$\frac{1}{c^2} T^\theta_{ac} \xrightarrow{\text{law}} T_{a}^\gamma.$$ (21)

b) For $c \to \infty$, we have:

$$\frac{1}{c} \log (T^\theta_{ac}) \xrightarrow{\text{law}} 2|\beta|T_{a}^\gamma = 2|C_a|,$$ (22)

where $(C_a, a \geq 0)$ is a standard Cauchy process.

**Remark 3.4** *(Yet another proof of Spitzer’s Theorem)*

Taking $a = 1$, from Corollary 3.3(b), we can obtain yet another proof of Spitzer’s celebrated asymptotic Theorem stated in (8). Indeed, (22) can be equivalently stated as:

$$P (\log T^\theta_{ac} < cx) \xrightarrow{\text{law}} P (2|C_1| < x).$$ (23)

Now, the LHS of (23) equals:

$$P (\log T^\theta_{ac} < cx) \equiv P (T^\theta_{ac} < \exp(cx)) \equiv P \left( \sup_{u \leq \exp(cx)} \theta_u > c \right)$$

$$= P \left( |\theta_{\exp(cx)}| > c \right) = P \left( |\theta_t| > \frac{\log t}{x} \right),$$ (24)

with $t = \exp(cx)$. Thus, because $|C_1| \xrightarrow{\text{law}} |C_1|^{-1}$, (23) now writes:

for every $x > 0$ given, $P \left( |\theta_t| > \frac{\log t}{x} \right) \xrightarrow{\text{law}} P \left( |C_1| > \frac{2}{x} \right),$ (25)

which yields precisely Spitzer’s Theorem (8).
3.3 Speed of convergence

We can easily improve upon Proposition 3.1 by studying the speed of convergence of the distribution of $\frac{1}{c^2}T_\theta^\theta$ towards that of $T_1^\gamma$, i.e.:

**Proposition 3.5** For any function $\varphi \in C^2$, with compact support,

$$\frac{1}{c^2} \left( E \left[ \varphi \left( \frac{1}{c^2}T_\theta^\theta \right) \right] - E \left[ \varphi \left( T_1^\gamma \right) \right] \right) \to E \left[ \varphi' \left( T_1^\gamma \right) T_1^\gamma \right]^2 + \frac{2}{3} \varphi'' \left( T_1^\gamma \right) \left( T_1^\gamma \right)^3.$$

(26)

**Proof of Proposition 3.5:**

We develop $\exp (2c\beta_v)$, for $c \to 0$, up to the second order term, i.e.:

$$e^{2c\beta_v} = 1 + 2c\beta_v + 2c^2\beta_v^2 + \ldots.$$

More precisely, we develop up to the second order term, and we obtain:

$$E \left[ \varphi \left( \frac{1}{c^2}T_\theta^\theta \right) \right] = E \left[ \varphi \left( \int_0^{T_1^\gamma} dv \exp (2c\beta_v) \right) \right] = E \left[ \varphi \left( T_1^\gamma \right) + \varphi' \left( T_1^\gamma \right) \int_0^{T_1^\gamma} \left( 2c\beta_v + 2c^2\beta_v^2 \right) dv \right] + \frac{1}{2} E \left[ \varphi'' \left( T_1^\gamma \right) \left( \int_0^{T_1^\gamma} \beta_v dv \right)^2 \right] + c^2 o(c).$$

We then remark that $E \left[ \int_0^t \beta_v dv \right] = 0$, $E \left[ \int_0^t \beta_v^2 dv \right] = t^2/2$ and $E \left[ \left( \int_0^t \beta_v dv \right)^2 \right] = t^3/3$, thus we obtain (26).

4 Checks via Bougerol’s identity

So far, we have not made use of Bougerol’s identity (4), which helps us to characterize the distribution of $T_\theta^\theta$ [Vak11]. In this Subsection, we verify that writing the Gauss-Laplace transform in (6) as:

$$E \left[ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_\theta^\theta}} \exp \left( - \frac{xc^2}{2T_\theta^\theta} \right) \right] = \frac{1}{\sqrt{1 + xc^2}} \varphi_m(xc^2),$$

(27)

with $m = \pi/(2c)$, we find asymptotically for $c \to 0$ the Gauss-Laplace transform of $T_1^\gamma$. Indeed, from (27), for $c \to 0$, we obtain:

$$E \left[ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^\gamma}} \exp \left( - \frac{x}{2T_1^\gamma} \right) \right] = \lim_{c \to 0} \frac{2}{\left( \sqrt{1 + xc^2} + \sqrt{xc^2} \right)^{\pi/2c} + \left( \sqrt{1 + xc^2} - \sqrt{xc^2} \right)^{\pi/2c}}.$$
Let us now study:
\[
\left( \sqrt{1 + xc^2} + \sqrt{xc^2} \right)^{\pi/2c} = \exp \left( \frac{\pi}{2c} \log \left[ 1 + \left( \sqrt{1 + xc^2} - 1 \right) + \sqrt{xc^2} \right] \right)
\]
\[
\sim \exp \left( \frac{\pi}{2c} \left[ c \sqrt{x} + \frac{xc^2}{2} \right] \right) \rightarrow \exp \left( \frac{\pi \sqrt{x}}{2} \right).
\]

A similar calculation finally gives:
\[
E \left[ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1 \gamma}} \exp \left( -\frac{x}{2T_1 \gamma} \right) \right] = \frac{1}{\cosh \left( \frac{\pi}{2} \sqrt{x} \right)}, \quad (29)
\]

a result which is in agreement with the law of \( \beta_{T_1 \gamma} \), whose density is:
\[
E \left[ \exp \left( i \frac{\lambda}{\sqrt{T_1 \gamma}} \right) \right] = \frac{1}{\cosh(\lambda c)}. \quad (32)
\]

It is well known that \([\text{Lev}80, \text{BiY}87]\):
\[
E \left[ \exp(i \lambda \beta_{T_1 \gamma}) \right] = \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\frac{\pi \lambda \sqrt{x}}{2})} = \int_{-\infty}^{\infty} e^{i(\frac{\pi \lambda}{2})y} \frac{1}{2 \cosh(\frac{\pi y}{2c})} \ dy = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{1}{2 \cosh(\frac{\pi \sqrt{x} y}{2c})} \ dx = \int_{\infty}^{\infty} e^{i\lambda x} \frac{1}{2 \cosh(\frac{\pi \sqrt{x} y}{2c})} \ dx. \quad (31)
\]

So, the density \( h_{-c,c} \) of \( \beta_{T_1 \gamma} \) is:
\[
h_{-c,c}(y) = \left( \frac{1}{2c} \right) \frac{1}{\cosh(\frac{\pi \sqrt{x} y}{2c})} = \left( \frac{1}{c} \right) e^{\frac{\pi \sqrt{x} y}{2c}} + e^{-\frac{\pi \sqrt{x} y}{2c}},
\]
and for \( c = 1 \), we obtain (30).

We recall from Remark 3.2 that (see also \([\text{PiY}03]\), where further results concerning the infinitely divisible distributions generated by some Lévy processes associated with the hyperbolic functions \( \cosh, \sinh \) and \( \tanh \) can also be found):
\[
E \left[ \exp \left( -\frac{\lambda^2}{2c} T_1 \right) \right] = \frac{1}{\cosh(\lambda c)}, \quad (32)
\]

thus, for \( c = 1 \) and \( \lambda = \frac{\pi}{2} \sqrt{x} \), (29) now writes:
\[
E \left[ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1 \gamma}} \exp \left( -\frac{x}{2T_1 \gamma} \right) \right] = E \left[ \exp \left( -\frac{x \pi^2}{8 T_1 \gamma} \right) \right], \quad (33)
\]
a result which gives a probabilistic proof of the reciprocal relation in [BPY01] (using the notation of this article, Table 1, p.442):

\[ f_{C_1}(x) = \left( \frac{2}{\pi x} \right)^{3/2} f_{C_1} \left( \frac{4}{\pi^2 x} \right). \]

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