An existence theorem on the isoperimetric ratio over scalar-flat conformal classes

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Abstract

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n\) with smooth boundary \(\partial M\), admitting a scalar-flat conformal metric. We prove that the supremum of the isoperimetric ratio over the scalar-flat conformal class is strictly larger than the best constant of the isoperimetric inequality in the Euclidean space, and consequently is achieved, if either (i) \(9 \leq n \leq 11\) and \(\partial M\) has a nonumbilic point; or (ii) \(7 \leq n \leq 9\), \(\partial M\) is umbilic and the Weyl tensor does not vanish identically on the boundary. This is a continuation of the work [12] by the second named author and Xiong.

Key Words: Conformal geometry; isoperimetric inequality.

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1 Introduction

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\) with boundary \(\partial M\). In [10], F. Hang, X. Wang and X. Yan initiated a study of the isoperimetric quotient over the scalar-flat conformal class of \(g\) on \(M\):

\[
\Theta(M, g) = \sup \left\{ \frac{\text{Vol}(M, \tilde{g})}{\text{Vol}(\partial M, \tilde{g})} : \tilde{g} \in [g] \text{ with } R_{\tilde{g}} = 0 \right\},
\]

where \([g] = \{\rho^2 g : \rho \in C^\infty(M), \rho > 0\}\) is the conformal class of \(g\), and \(R_g\) is the scalar curvature of \((M, g)\).

It was explained in [10] that the set \(\{\tilde{g} \in [g] : R_{\tilde{g}} = 0\}\) is not empty if and only if the first eigenvalue \(\lambda_1(L_g)\) of the conformal Laplacian \(L_g := -\Delta_g + \frac{n-2}{4(n-1)} R_g\) with zero Dirichlet boundary condition is positive. Note that the positivity of \(\lambda_1(L_g)\) does not depend on the choice of the metrics in \([g]\). Assuming \(\lambda_1(L_g) > 0\), they proved in [10] that

\[
\Theta(\overline{B}_1, g_{\mathbb{R}^n}) \geq \Theta(M, g) < \infty,
\]

and \(\Theta(\overline{B}_1, g_{\mathbb{R}^n})\) coincides with the best constant of the isoperimetric inequality in the Euclidean space, that is,

\[
\Theta(\overline{B}_1, g_{\mathbb{R}^n}) = n^{-\frac{1}{2}} \omega_{n-1} \left( \frac{n-1}{4(n-1)} \right).
\]

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where $\omega_{n-1}$ is the volume of the unit sphere $S^{n-1}$. They also showed in [10] that $\Theta(M, g)$ is achieved if the strict inequality

$$\Theta(B_1, g_{\mathbb{R}^n}) < \Theta(M, g) \quad (1.2)$$

holds, and made a conjecture that:

**Conjecture** ([10]). Assume $n \geq 3$, $(M, g)$ is a smooth compact Riemannian manifold of dimension $n$ with nonempty smooth boundary $\partial M$, and $\lambda_1(L_g) > 0$. If $(M, g)$ is not conformally diffeomorphic to $(B_1, g_{\mathbb{R}^n})$, then the strict inequality $(1.2)$ holds.

In the paper [12], the second named author and Xiong verified this conjecture under one of the following two conditions:

- $n \geq 12$ and $\partial M$ has a nonumbilic point;
- $n \geq 10$, $\partial M$ is umbilic and the Weyl tensor $W_g \neq 0$ at some boundary point.

At the same time, Gluck and Zhu [7] verified this conjecture when $M = \overline{B_1} \setminus B_\varepsilon$ for sufficiently small $\varepsilon > 0$ with flat metric in all dimensions.

In this paper, we reduce the dimension assumption in [12] by three.

**Theorem 1.1.** Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n$ with nonempty smooth boundary $\partial M$. Suppose that $\lambda_1(L_g) > 0$. If one of the following two conditions

(i) $9 \leq n \leq 11$ and $\partial M$ has a nonumbilic point;

(ii) $7 \leq n \leq 9$, $\partial M$ is umbilic, and the Weyl tensor $W_g \neq 0$ at some boundary point;

holds, then the strict inequality $(1.2)$ holds, and consequently, $\Theta(M, g)$ is achieved.

Throughout the paper, we will always assume that $\lambda_1(L_g) > 0$. Denote the Poisson kernel of $L_g u = 0$ with Dirichlet boundary condition by $P_g$. It was pointed out in [10] that

$$\Theta(M, g) = \sup \left\{ I[v] : v \in L^{2(n-1)/(n-2)}(\partial M), v \neq 0 \right\}, \quad (1.3)$$

where

$$I[v] = \frac{\left( \int_M |P_g v|^{\frac{2(n-1)}{n-2}} d\mu_g \right)^{\frac{n}{n-2}}}{\left( \int_{\partial M} |v|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n}{n-2}}}.$$

Therefore, to show the strict inequality $(1.2)$, we need to find a test function $v \in L^{2(n-1)/(n-2)}(\partial M)$ such that

$$I[v] > \Theta(B_1, g_{\mathbb{R}^n}). \quad (1.4)$$

Recall that it was shown in Hang-Wang-Yan [11, Theorem 1.1] that

$$\Theta(B_1, g_{\mathbb{R}^n}) = \Theta(\mathbb{R}^n_+, g_{\mathbb{R}^n}),$$

where $\mathbb{R}^n_+ = \{ x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0 \}$ is the upper half-space, and moreover, $\Theta(\mathbb{R}^n_+, g_{\mathbb{R}^n})$ defined as in $(1.3)$ is achieved by the so-called bubbles:

$$c \left( \frac{\varepsilon}{\varepsilon^2 + |x' - \xi_0|^2} \right)^{\frac{n-2}{2}}, \quad (1.5)$$

where $c \in \mathbb{R}_+$, $\varepsilon > 0$ and $\xi_0 \in \mathbb{R}^{n-1}$. The test function $v$ chosen in [12] to verify $(1.4)$ is a cut-off of the bubbles $(1.5)$ in proper coordinates on $M$ centered at a boundary point. In our proof of Theorem 1.1, we
choose the same test function, but we will give a more delicate calculation of the $L^{\frac{2n}{n-2}}$-norm of its Poisson extension $P_g v$.

One sees from the definition that $\Theta(M, g)$ depends only on the conformal class $[g]$. These results on the above variational problem (1.3) show an analogy to the Yamabe problem solved by Yamabe [19], Trudinger [17], Aubin [2] and Schoen [16], as well as to the boundary Yamabe problem (or higher dimensional Riemannian mapping problem) studied by Escobar [5, 6], Marques [13, 14], Han-Li [8, 9], Chen [3], Almaraz [1], Mayer-Ndiaye [15], Chen-Ruan-Sun [4], etc. A prescribing function problem of the isoperimetric ratio on the unit sphere, which is a Nirenberg type problem, has been studied by Xiong [18].

This paper is organized as follows. In the next section, we will review the proof in [12]. In Section 3, we will first set up our objectives on how to reduce the dimension assumption in [12], and carry out our detailed calculations afterwards.

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2 An overview

The purpose of this section is to summarize the proof of [12, Theorem 1.2], on which our calculations are based.

Let $P \in \partial M$ be a non-umbilic point in case (i) of Theorem 1.1, or a point at which the Weyl tensor of $M$ does not vanish in case (ii). Since $\Theta(M, g)$ is a conformal invariant, we can choose conformal Fermi coordinates (see Marques [13]) $x = (x', x_n)$ centered at $P$ to simplify the computations.

For any fixed $1 \gg \rho \gg \varepsilon > 0$, denote by $\chi_\rho(t)$ a smooth cut-off function supported on $[0, 2\rho]$ such that $\chi_\rho(t) = 1$ in $[0, \rho]$ and $0 < \chi_\rho(t) < 1$ in $(\rho, 2\rho)$. Let

$$v_\varepsilon(x') = \left(\frac{\varepsilon}{\varepsilon^2 + |x'|^2}\right)^{\frac{n-2}{2}} \chi_\rho(|x'|)$$

in the above coordinates. Let $u_\varepsilon$ be the $L_g$-harmonic extension of $v_\varepsilon$ in $M$, or equivalently, the solution to

$$L_g u_\varepsilon = 0 \quad \text{in} \quad M, \quad u_\varepsilon = v_\varepsilon \quad \text{on} \quad \partial M.$$

Roughly, we can regard $u_\varepsilon$ in $(M, g)$ as a small perturbation of the harmonic extension of $\left(\frac{\varepsilon}{\varepsilon^2 + |x'|^2}\right)^{\frac{n-2}{2}}$ in the Euclidean upper half-space $\mathbb{R}^n_+$, which is

$$\overline{U}_\varepsilon(x) = \left(\frac{\varepsilon}{(x_n + \varepsilon)^2 + |x'|^2}\right)^{\frac{n-2}{2}}.$$

Thus, it is natural to consider the error term $W_\varepsilon := u_\varepsilon - \overline{U}_\varepsilon$, which satisfies

\[
\begin{cases}
L_g W_\varepsilon = -L_g \overline{U}_\varepsilon =: F[\overline{U}_\varepsilon] & \text{in} \quad \Omega, \\
W_\varepsilon(x', 0) = 0 & \text{on} \quad \partial \Omega \cap \partial M, \\
W_\varepsilon = u_\varepsilon - \overline{U}_\varepsilon & \text{on} \quad \partial \Omega \setminus \partial M.
\end{cases}
\]

Here $\Omega$ is a smooth domain in $\mathbb{R}^n_+$ such that $B_\rho^+(0) \subset \Omega \subset B_{2\rho}^+(0)$ and radially symmetric with respect to $x'$. Denote by $G(f)$ the solution to

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.$$
Decompose $W_\varepsilon$ into three parts:

$$W_\varepsilon^{(1)} = \mathcal{G}(F|U_\varepsilon|), \quad W_\varepsilon^{(2)} = \mathcal{G}(F|W_\varepsilon^{(1)}|) \quad \text{and} \quad W_\varepsilon^{(3)} = W_\varepsilon - W_\varepsilon^{(1)} - W_\varepsilon^{(2)}.$$  \hspace{1cm} (2.1)

It has been obtained in [12] that

$$\int_M |u_\varepsilon|^{\frac{2n}{n-2}} d\mu_g = \frac{1}{n^{2n}} \omega_{n-1} + \frac{2n}{n-2} \int_\Omega \frac{n+2}{\varepsilon} W_\varepsilon^{(1)} dx + \frac{2n}{n-2} \int_\Omega \frac{n+2}{\varepsilon} W_\varepsilon^{(2)} dx + \frac{2n}{n-2} \int_\Omega \frac{n+2}{\varepsilon} W_\varepsilon^{(3)} dx$$

$$+ \frac{n(n+2)}{(n-2)^2} \int_\Omega \frac{n-2}{\varepsilon} W_\varepsilon^2 dx + h.o.t.,$$

\hspace{1cm} (2.2)

and

$$\int_{\partial M} v_\varepsilon^{\frac{n-1}{n-2}} d\sigma_g = 2^{1-n} \omega_{n-1} + O(\varepsilon^{n-1}).$$

\hspace{1cm} (2.3)

**Case 1.** If $\partial M$ admits a non-umbilic point $P$, then it follows from [12, (19), (22), (24)] that for $n \geq 5$ and $s = 1, 2, 3, 4$,

$$|W_\varepsilon^{(1)}| + |x + \varepsilon e_n|^s |\nabla^s W_\varepsilon^{(1)}| \leq C\varepsilon^{\frac{2s-2}{s}} |x + \varepsilon e_n|^{3-n},$$  \hspace{1cm} (2.4.1)

$$|W_\varepsilon^{(2)}| + |x + \varepsilon e_n|^s |\nabla^s W_\varepsilon^{(2)}| \leq C\varepsilon^{\frac{2s-2}{s}} |x + \varepsilon e_n|^{4-n},$$

$$|W_\varepsilon^{(3)}| \leq \begin{cases} C\varepsilon^{\frac{2s-2}{s}} |x + \varepsilon e_n|^{5-n} & \text{if } n \geq 6, \\ C\varepsilon^{\frac{n-2}{2}} |\log \varepsilon| & \text{if } n = 5, \end{cases}$$

and from [12, (30), (37), (38)] that

$$\frac{2n}{n-2} \int_\Omega \frac{n+2}{\varepsilon} W_\varepsilon^{(1)} dx = \frac{\omega_{n-2}(n-12)B(\frac{n-1}{2}, \frac{n+1}{2})}{4(n-1)(n-2)(n-3)} |h|^2 \varepsilon^2 + O(\varepsilon^3),$$

$$\frac{2n}{n-2} \int_\Omega \frac{n+2}{\varepsilon} W_\varepsilon^{(2)} dx \approx \frac{8n^2(n+2)}{3} |h|^2 \varepsilon^2 \int_{\mathbb{R}_+} x_1^2 |x + \varepsilon e_n|^{-(n+4)} V(|x'|, x_n)x_1 x_4 dx + O(\varepsilon^3),$$

$$\frac{2n}{n-2} \int_\Omega \frac{n+2}{\varepsilon} W_\varepsilon^{(3)} dx \approx O(\varepsilon^3),$$

$$\frac{n(n+2)}{(n-2)^2} \int_\Omega \frac{n-2}{\varepsilon} W_\varepsilon^2 dx \approx \frac{8n^3(n+2)}{3} |h|^2 \varepsilon^2 \int_{\mathbb{R}_+} |x + \varepsilon e_n|^{-4} V^2(|x'|, x_n)x_1^2 dx + O(\varepsilon^3),$$

where $h$ is the second fundamental form at $P$ with respect to the outward unit normal vector and $V$ is a positive function. Moreover, if we define

$$\widetilde{V}(x) = V(|x'|, x_{n+4})$$

($x \in \mathbb{R}_{n+4}^+, x' = (x_1, \cdots, x_{n+3})$) as a function in $\mathbb{R}_{n+4}^+$, then $\widetilde{V}$ satisfies (see [12, (35)])

$$\begin{cases} -\Delta \widetilde{V} = x_{n+4}|x + e_{n+4}|^{-n-2}, & \text{in } \mathbb{R}_{n+4}^+, \\ \widetilde{V} = 0, & \text{on } \partial \mathbb{R}_{n+4}^+. \end{cases}$$

\hspace{1cm} (2.5)

Therefore, from (2.2) we have

$$\int_M |u_\varepsilon|^{\frac{2n}{n-2}} d\mu_g = \frac{1}{n^{2n}} \omega_{n-1} + C_1(n) \varepsilon^2 |h|^2 + h.o.t.,$$

\hspace{1cm} (3.4)
where

\[
C_1(n) := \frac{\omega_{n-2}(n-12)B\left(\frac{n-1}{2}, \frac{n+1}{2}\right)}{4n(n-1)(n-2)(n-3)} + \frac{8n^2(n+2)}{3} \int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-(n+4)} \chi(|x'|, x_n) x_1^4 dx \\
+ \frac{8n^3(n+2)}{3} \int_{\mathbb{R}^n_+} |x + e_n|^{-4} \chi(|x'|, x_n) x_1^4 dx,
\]

(2.6)

and \(B(\cdot, \cdot)\) is the Beta function. Since \(|h|^2 > 0\) in this case, there holds \(C_1(n) > 0\) when \(n \geq 12\), from which the result in [12] follows.

Case 2. Assume that \(\partial M\) is umbilic and the Weyl tensor \(W_g\) of \(M\) is nonzero at some boundary point \(P\), then it follows from [12, (43)] that for \(n \geq 7\) and \(s = 1, 2, 3, 4\),

\[
|W^{(1)}_e| + |x + \varepsilon e_n|^s |\nabla^s W_e^{(1)}| \leq C \varepsilon^{\frac{n-2}{2}} |x + \varepsilon e_n|^{4-n},
\]

\[
|W^{(2)}_e| + |x + \varepsilon e_n|^s |\nabla^s W_e^{(2)}| \leq C \varepsilon^{\frac{n-2}{2}} |x + \varepsilon e_n|^{6-n},
\]

\[
|W^{(3)}_e| \leq \begin{cases} 
C \varepsilon^{\frac{n-2}{2}} |x + \varepsilon e_n|^{8-n} & \text{if } n \geq 9, \\
C \varepsilon^{\frac{n-2}{2}} |\log \varepsilon| & \text{if } n = 8, \\
C \varepsilon^{\frac{n-2}{2}} & \text{if } n = 7,
\end{cases}
\]

(2.7)

and from [12, (46), (51), (52)] that

\[
\frac{2n}{n-2} \int_{\Omega} U^{\frac{n+2}{2}} W_e^{(1)} dx = \frac{3(n-10)\omega_{n-2}B\left(\frac{n-1}{2}, \frac{n+1}{2}\right)}{2n(n-1)(n-2)(n-3)(n-4)(n-5)} (R_{\text{inj}})^2 \varepsilon^4 + a(n) |\nabla W|^2 \varepsilon^4 + O(\varepsilon^5),
\]

\[
\frac{2n}{n-2} \int_{\Omega} U^{\frac{n+2}{2}} W_e^{(2)} dx = \frac{2n^2(n+2)}{3} (R_{\text{inj}})^2 \varepsilon^4 \int_{\mathbb{R}^n_+} |x + e_n|^{-n-4} \Lambda(|x'|, x_n) x_n^3 x_1^4 dx + O(\varepsilon^5),
\]

\[
\frac{2n}{n-2} \int_{\Omega} U^{\frac{n+2}{2}} W_e^{(3)} dx = O(\varepsilon^5),
\]

\[
\frac{n(n+2)}{(n-2)^2} \int_{\Omega} U^{\frac{n+4}{2}} W_2^2 dx = \frac{2n^3(n+2)}{3} (R_{\text{inj}})^2 \varepsilon^4 \int_{\mathbb{R}^n_+} |x + e_n|^{-4} x_1^4 \Lambda^2(|x'|, x_n) dx + O(\varepsilon^5),
\]

where \(a(n) > 0\) is a constant, \(\nabla\) is the Weyl tensor of \(\partial M\) with the induced metric of \(g\), and \(\Lambda\) is a positive function. Moreover, if we define

\[
\tilde{\Lambda}(x) = \Lambda(|x'|, x_{n+4})
\]

as a function in \(\mathbb{R}^{n+4}_+\), then \(\tilde{\Lambda}\) satisfies (see [12, (50)])

\[
\begin{cases} 
- \Delta \tilde{\Lambda} = x_{n+4}^2 |x + e_{n+4}|^{-n-2}, & \text{in } \mathbb{R}^{n+4}_+, \\
\tilde{\Lambda} = 0, & \text{on } \partial \mathbb{R}^{n+4}_+.
\end{cases}
\]

(2.8)

Therefore, from (2.2) we have

\[
\int_M |u_\varepsilon|^{\frac{2n}{n+2}} d\mu_g = \frac{1}{n^{2n}} \omega_{n-1} + C_2(n) \varepsilon^4 (R_{\text{inj}})^2 + a(n) |\nabla W|^2 \varepsilon^4 + h.o.t.,
\]
where
\[
C_2(n) := \frac{3(n-10)\omega_{n-2}B\left(\frac{n-1}{2}, \frac{n+1}{2}\right)}{2n(n-1)(n-2)(n-3)(n-4)(n-5)} + \frac{2n^2(n+2)}{3} \int_{\mathbb{R}^n_+} |x + e_n|^{-n-4}\Lambda(|x'|, x_n)x_n^3x_4^4dx \\
+ \frac{2n^3(n+2)}{3} \int_{\mathbb{R}^n_+} |x + e_n|^{-4}\Lambda^2(|x'|, x_n)dx,
\]
and \(B(\cdot, \cdot)\) is the Beta function. It is known from Almaraz [1, Lemma 2.5] that under conformal Fermi coordinates around \(P\), \(\bar{W}^2 + (R_{\text{minj}})^2 \neq 0\) at \(P\) is equivalent to the Weyl tensor \(|\bar{W}| \neq 0\) at \(P\). Therefore, if \(n \geq 10\), then we have \(C_2(n) > 0\), from which the result in [12] follows.

## 3 Proofs

As mentioned in [12, Remarks 3.4 and 4.2], one may reduce the dimension assumptions in [12] if one can explicitly calculate, or obtain useful lower bounds of \(V\) and \(\Lambda\), that are solutions of (2.5) and (2.8) respectively.

In this paper, we find a way of obtaining some useful subsolutions to (2.5) and (2.8), which serve as lower bounds of \(V\) and \(\Lambda\), respectively. These give better estimates of the constants \(C_1(n)\) and \(C_2(n)\), defined in (2.6) and (2.9), respectively, which in return reduce the dimension assumptions in [12].

### 3.1 Two calculus lemmas

The Laplacian operator in \(\mathbb{R}^{n+4}\) applying to functions that are radial in the first \(n+3\) variables (denoting \(r = |(x_1, \cdots, x_{n+3})|, s = x_{n+4}\)) is
\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{n+2}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial s^2}.
\]
All of our calculations are based on the next calculus lemma.

**Lemma 3.1.** For \(\phi(r, s) = f(s)(r^2 + (1 + s)^2)^{-\frac{n}{2}}, \alpha \in \mathbb{R}, r \geq 0, s \geq 0, \) we have
\[
-\left(\phi_r + \frac{n+2}{r}\phi_r + \phi_{ss}\right) = \left[\alpha(n+2-\alpha)f(s) + 2\alpha(1+s)f'(s) - f''(s)(r^2 + (1 + s)^2)\right](r^2 + (1 + s)^2)^{-\frac{n+2}{2}}.
\]
In particular, if \(f\) is convex, then we have
\[
-\left(\phi_r + \frac{n+2}{r}\phi_r + \phi_{ss}\right) \leq \left[\alpha(n+2-\alpha)f(s) + 2\alpha(1+s)f'(s)\right](r^2 + (1 + s)^2)^{-\frac{n+2}{2}}.
\]

The proof of this lemma is elementary, and we omit the details.

We also summarize the following calculation as a lemma, which will be frequently used as well. It is essentially just the integration by parts.

**Lemma 3.2.** Suppose \(\phi_1, \phi_2 \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)\) are two nonnegative functions with sufficiently fast decay at infinity. For \(i = 1, 2\), let \(\tilde{u}_i\) be the unique solution (decay to zero at infinity) of
\[
\begin{cases}
-\Delta \tilde{u}_i = \phi_i(|x'|, x_{n+4}), & \text{in } \mathbb{R}^{n+4}_+, \\
\tilde{u}_i = 0, & \text{on } \partial \mathbb{R}^{n+4}_+,
\end{cases}
\]

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where \( x' = (x_1, \ldots, x_{n+3}) \) for \( x \in \mathbb{R}^{n+4} \). For \( r, s \geq 0 \), let \( u_i(r, s) = \tilde{u}_i(r, 0, \ldots, 0, s) \). Then
\[
\int_{\mathbb{R}^n_+} \phi_1(|x'|, x_n) u_2(|x'|, x_n)x^4 dx = \int_{\mathbb{R}^n_+} \phi_2(|x'|, x_n) u_1(|x'|, x_n)x^4 dx.
\]

**Proof.** From the Green’s function of \(-\Delta \) in \( \mathbb{R}^{n+4}_+ \) with zero Dirichlet boundary condition, we know that \( \tilde{u}_i \) is radial in \( x' \). Using this symmetry and integration by parts, we have
\[
\int_{\mathbb{R}^n_+} \phi_1(|x'|, x_n) u_2(|x'|, x_n)x^4 dx
= \frac{3\omega_{n-2}}{(n-1)(n+1)\omega_{n+2}} \int_{\mathbb{R}^{n+4}_+} \phi_1(|x'|, x_{n+4}) u_2(|x'|, x_{n+4}) dx
= \frac{3\omega_{n-2}}{(n-1)(n+1)\omega_{n+2}} \int_{\mathbb{R}^{n+4}_+} (-\Delta \tilde{u}_1) \tilde{u}_2 dx
= \frac{3\omega_{n-2}}{(n-1)(n+1)\omega_{n+2}} \int_{\mathbb{R}^{n+4}_+} (-\Delta \tilde{u}_2) \tilde{u}_1 dx
= \int_{\mathbb{R}^n_+} \phi_2(|x'|, x_n) u_1(|x'|, x_n)x^4 dx.
\]

This finishes the proof. \( \square \)

For convenience, we did not explicitly assume the required decay rates on \( \phi_1 \) and \( \phi_2 \) in Lemma 3.2. However, when this lemma is applied in the next two sections, it will be clear that the decay rates there will be sufficient.

### 3.2 Non-umbilic boundary in dimensions \( 9 \leq n \leq 11 \)

As stated earlier, to estimate \( \tilde{V} \) defined in (2.5), we want to find a sub-solution of
\[
\begin{align*}
-\Delta \tilde{V} &= x_{n+4}|x + e_{n+4}|^{-n-2}, & & \text{in } \mathbb{R}^{n+4}_+, \\
\tilde{V} &= 0, & & \text{on } \partial \mathbb{R}^{n+4}_+.
\end{align*}
\]

According to Lemma 3.1, we will choose \( \alpha = n \), and search for the solution of
\[
\begin{align*}
\alpha(n + 2 - \alpha)f(s) + 2\alpha(1 + s)f'(s) &= s, \\
f(0) &= 0.
\end{align*}
\tag{3.1}
\]

The solution of (3.1) is
\[
\frac{s^2}{4n(1 + s)},
\]
which is a convex function. So we have
\[
-\Delta \left( \frac{x_{n+4}^2}{4n(1 + x_{n+4})} |x + e_{n+4}|^{-n} \right) = x_{n+4}|x + e_{n+4}|^{-n-2} - \frac{1}{2n(1 + x_{n+4})^3}|x + e_{n+4}|^{-n}.
\]

Set
\[
\tilde{u}_1(x) = \tilde{V}(x) - \frac{1}{4n} \frac{x_{n+4}^2}{1 + x_{n+4}} |x + e_{n+4}|^{-n}, \quad x \in \mathbb{R}^{n+4}_+.
\]
Then
\[
\begin{cases}
-\Delta \tilde{u}_1 = \frac{1}{2n(1 + x_{n+4})^3} |x + e_{n+4}|^{-n}, & \text{in } \mathbb{R}^n_{+}, \\
\tilde{u}_1 = 0, & \text{on } \partial \mathbb{R}^n_{+}. 
\end{cases}
\] (3.2)

Thus, it follows from the maximum principle that
\[
\tilde{u}_1(x) = \bar{V}(x) - \frac{1}{4n1 + x_{n+4}} |x + e_{n+4}|^{-n} \geq 0 \text{ in } \mathbb{R}^n_{+}. 
\] (3.3)

Now we are going to give a better estimate of \(C_1(n)\) defined in (2.6), which is
\[
C_1(n) = \frac{\omega_{n-2}(n-12)B\left(\frac{n-1}{2}, \frac{n+1}{2}\right)}{4n(n-1)(n-2)(n-3)}
+ \frac{8n^2(n+2)}{3} \int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-(n+4)} V(|x'|, x_n)x_1^4 dx
+ \frac{8n^3(n+2)}{3} \int_{\mathbb{R}^n_+} |x + e_n|^{-4}V^2(|x'|, x_n)x_1^4 dx.
\]

Let
\[
u_1(r, s) = \tilde{u}_1(r, 0, \cdots, 0, s),
\]
and recall
\[
V(r, s) = \bar{V}(r, 0, \cdots, 0, s).
\]

By (3.3) we have
\[
\int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-(n+4)} V(|x'|, x_n)x_1^4 dx
= \frac{1}{4n} \int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-n-4} \frac{x_n^2}{1 + x_n} |x + e_n|^{-n}x_1^4 dx + \int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-(n+4)} u_1(|x'|, x_n)x_1^4 dx.
\] (3.4)

For the first term, we have
\[
\frac{1}{4n} \int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-2n-4} \frac{x_n^4}{1 + x_n} dx = \frac{3\omega_{n-2}}{4n(n-1)(n+1)} \int_0^\infty \frac{x_n^4 dx_n}{(1 + x_n)^{n+2}} \int_0^\infty \frac{x_n^{n+2} dx_n}{(1 + x_n^2)^{n+2}}
= \frac{9\omega_{n-2}B\left(\frac{n-1}{2}, \frac{n+1}{2}\right)}{4(n+1)^2n^3(n-1)(n-2)(n-3)}. 
\] (3.5)

For the second term, if we let \(\tilde{v}_1\) be the solution (decay to zero at infinity) of
\[
\begin{cases}
-\Delta \tilde{v}_1 = x_{n+4}^2 |x + e_{n+4}|^{-n-4}, & \text{in } \mathbb{R}^n_{+}, \\
\tilde{v}_1 = 0, & \text{on } \partial \mathbb{R}^n_{+},
\end{cases}
\]
and notice that \(\tilde{v}_1\) is radial in the \(x' = (x_1, \cdots, x_{n+3})\) variable, then it follows from Lemma 3.2 and (3.2) that
\[
\int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-(n+4)} u_1(|x'|, x_n)x_1^4 dx = \frac{1}{2n} \int_{\mathbb{R}^n_+} v_1(|x'|, x_n) \frac{1}{(1 + x_n)^4} |x + e_n|^{-n} x_1^4 dx,
\]

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where
\[ v_1(r, s) = \hat{v}_1(r, 0, \cdots, 0, s). \]

Next we give some lower bound estimates of \( \hat{v}_1 \). According to Lemma 3.1, we will choose \( \alpha = n + 2 \), and search for the solution of
\[
\begin{cases}
2\alpha(1 + s)f'(s) = s^2, \\
f(0) = 0.
\end{cases}
\tag{3.6}
\]
The solution of (3.6) is
\[
\frac{1}{2(n + 2)} \left[ \frac{1}{2} s^2 - s + \log(1 + s) \right],
\]
which is a convex function. Hence, we have in \( \mathbb{R}_+^{n+4} \) that
\[
-\Delta \left( \frac{1}{2(n + 2)} \left[ \frac{1}{2} x_{n+4}^2 - x_{n+4} + \log(1 + x_{n+4}) \right] \right) \geq x_{n+4}^2 |x + e_{n+4}|^{-n-2} \geq 0 \quad \text{in} \quad \mathbb{R}_+^{n+4}.
\tag{3.7}
\]
Therefore,
\[
\int_{\mathbb{R}_+^n} x_n^2 |x + e_n|^{-(n+4)} u_1(|x'|, x_n)x_1^4 dx
\geq \frac{1}{4n} \int_{\mathbb{R}_+^n} \left[ \frac{1}{2} x_n^2 - x_n + \log(1 + x_n) \right] |x + e_n|^{-2n-2} \frac{x_n^4}{(1 + x_n)^3} dx
= \frac{3\omega_{n-2}}{4n(n + 1)(n - 1)(n + 1)} \int_0^\infty \frac{1}{2} x_n^2 - x_n + \log(1 + x_n) dx_n \int_0^\infty \frac{r^{n+2}}{(1 + r^2)^{n+1}} dr
= \frac{3\omega_{n-2}}{4n(n + 1)(n - 1)(n + 1)^2} B \left( \frac{n - 1}{2}, \frac{n + 1}{2} \right)
= \frac{3\omega_{n-2} B \left( \frac{n - 1}{2}, \frac{n + 1}{2} \right)}{8(n + 2)(n + 1)^2 n(n - 1)^2}.
\tag{3.8}
\]
By (3.3) we obtain
\[
\int_{\mathbb{R}_+^n} |x + e_n|^{-4}V^2(|x'|, x_n)x_1^4 dx
= \frac{3\omega_{n-2}}{16n^2(n + 1)(n - 1)} \int_0^\infty \frac{x_n^4}{(1 + x_n)^n+3} \int_0^\infty \frac{r^{n+2} dr}{(1 + r^2)^{n+2}} dx
+ \int_{\mathbb{R}_+^n} |x + e_n|^{-4} u_1^2(|x'|, x_n)x_1^4 dx
= I_1 + I_2 + I_3.
\tag{3.9}
\]
For \( I_1 \), we have
\[
I_1 = \frac{3\omega_{n-2}}{16n^2(n + 1)(n - 1)} \int_0^\infty \frac{x_n^4}{(1 + x_n)^n+3} \int_0^\infty \frac{r^{n+2} dr}{(1 + r^2)^{n+2}} dx
= \frac{9\omega_{n-2} B \left( \frac{n - 1}{2}, \frac{n + 1}{2} \right)}{16(n + 2)(n + 1)^2 n(n - 1)(n - 2)}.
\tag{3.10}
\]
For $I_2$, if we let $\tilde{w}_1$ be the solution (decay to zero at infinity) of

$$
\begin{cases}
-\Delta \tilde{w}_1 = \frac{x_{n+4}^2}{1 + x_{n+4}} |x + e_{n+4}|^{-n-4}, & \text{in } \mathbb{R}_{+}^{n+4}, \\
\tilde{w}_1 = 0, & \text{on } \partial \mathbb{R}_{+}^{n+4},
\end{cases}
$$

and notice that $\tilde{w}_1$ is radial in the $x' = (x_1, \cdots, x_{n+3})$ variable, then it follows from Lemma 3.2 and (3.2) that

$$
I_2 = \frac{1}{4n^2} \int_{\mathbb{R}_+^n} w_1(|x'|, x_n)(1 + x_n)^{-3}|x + e_n|^{-n} x_1^4 \, dx,
$$

where

$$
w_1(r, s) = \tilde{w}_1(r, 0, \cdots, 0, s).
$$

We will give a lower bound estimate for $\tilde{w}_1$. According to Lemma 3.1, we will choose $\alpha = n + 2$, and search for the solution of

$$
\begin{cases}
2\alpha(1 + s)f'(s) = \frac{s^2}{1+s}, \\
f(0) = 0.
\end{cases}
$$

The solution of (3.11) is

$$
\frac{1}{2(n+2)} \left( 1 + s - 2 \log(1 + s) - \frac{1}{1 + s} \right),
$$

which is a convex function. Hence, we have

$$
-\Delta \left\{ \frac{1}{2(n+2)} \left[ 1 + x_{n+4} - 2 \log(1 + x_{n+4}) - \frac{1}{1 + x_{n+4}} \right] |x + e_{n+4}|^{-n-2} \right\}
\leq \frac{x_{n+4}^2}{1 + x_{n+4}} |x + e_{n+4}|^{-n-4} \text{ in } \mathbb{R}_{+}^{n+4}.
$$

So it follows from the maximum principle that

$$
\tilde{w}_1(x) \geq \frac{1}{2(n+2)} \left( 1 + x_{n+4} - 2 \log(1 + x_{n+4}) - \frac{1}{1 + x_{n+4}} \right) |x + e_{n+4}|^{-n-2} \geq 0 \text{ in } \mathbb{R}_{+}^{n+4}.
$$

Thus,

$$
I_2 \geq \frac{1}{8n^2(n+2)} \int_{\mathbb{R}_+^n} \left( 1 + x_n - 2 \log(1 + x_n) - \frac{1}{1 + x_n} \right)(1 + x_n)^{-3}|x + e_n|^{-2n-2} x_1^4 \, dx
$$

$$
= \frac{3\omega_{n-2}}{8n^2(n+2)(n-1)(n+1)} \int_0^\infty \frac{1 + x_n - 2 \log(1 + x_n) - \frac{1}{1 + x_n}}{(1 + x_n)^{n+2}} \, dx_n \int_0^\infty \frac{r^{n+2} \, dr}{(1 + r^2)^{n+1}}
$$

$$
= \frac{3\omega_{n-2}}{8n^2(n+2)(n-1)(n+1)} \frac{2}{n+1} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)
= \frac{3\omega_{n-2}}{16n^4(n+2)^2(n+1)^2(n-1)} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right).
$$

Finally,

$$
I_3 \geq 0.
$$

Therefore, putting (2.6), (3.4), (3.5), (3.8), (3.9), (3.10), (3.12) and (3.13) together, we obtain

$$
C_1(n)
$$
We first need to find a sub-solution of 

\[ \frac{\omega_{n-2}(n-12) B(\frac{n-1}{2}, \frac{n+1}{2})}{4n(n-1)(n-2)(n-3)} \]

\[ + \frac{8n^2(n+2)}{3} \left[ \frac{9\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{4(n+1)^2n^3(n-1)(n-2)(n-3)} + \frac{3\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{8(n+2)(n+1)^2n^3(n-1)^2} \right] \]

\[ + \frac{8n^3(n+2)}{3} \left[ \frac{9\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{16(n+2)(n+1)^2n^3(n-1)(n-2)} + \frac{3\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{16n^4(n+2)^2(n+1)^2(n-1)} \right] \]

\[ = \omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2}) \frac{n^2 - 8n - 5}{4(n+2)(n+1)(n-1)^2(n-3)}. \]

Hence,

\[ C_1(n) > 0 \quad \text{if} \quad n \geq 9. \]

**Remark 3.3.** Using Lemma 3.1, one can check that the function

\[ \frac{x^{n+4}}{4n} |x + e_{n+4}|^{-n}, \quad x \in \mathbb{R}^{n+4}, \]

is a supersolution of (2.5), and thus,

\[ \tilde{V}(x) \leq \frac{x^{n+4}}{4n} |x + e_{n+4}|^{-n} \quad \text{in} \quad \mathbb{R}^{n+4}. \]

Therefore,

\[ C_1(n) = \frac{\omega_{n-2}(n-12) B(\frac{n-1}{2}, \frac{n+1}{2})}{4n(n-1)(n-2)(n-3)} \]

\[ + \frac{8n^2(n+2)}{3} \int_{\mathbb{R}^n_+} x_n^2 |x + e_n|^{-(n+4)} V(|x'|, x_n) x_n^2 dx \]

\[ + \frac{8n^3(n+2)}{3} \int_{\mathbb{R}^n_+} |x + e_n|^{-4} V^2(|x'|, x_n) x_n^4 dx. \]

\[ \leq \frac{(n-12)\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{4n(n-1)(n-2)(n-3)} \]

\[ + \frac{8n^2(n+2)}{3} \frac{9\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{16(n+1)n^3(n-1)(n-2)(n-3)} \]

\[ + \frac{8n^3(n+2)}{3} \frac{3\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{64(n+1)n^4(n-1)(n-2)} \]

\[ = \frac{(3n^2 - 11n - 6)\omega_{n-2} B(\frac{n-1}{2}, \frac{n+1}{2})}{8(n+1)n(n-1)(n-2)(n-3)} < 0 \quad \text{if} \quad n \leq 4. \]

Hence, the best possible dimension one can achieve in this case under the present proof is \( n \geq 5. \)

### 3.3 Umbilic boundary in dimensions \( 7 \leq n \leq 9 \)

We first need to find a sub-solution of

\[ \begin{cases} -\Delta \tilde{\Lambda} = x_{n+4}^2 |x + e_{n+4}|^{-n-2}, & \text{in} \quad \mathbb{R}^{n+4}_+, \\ \tilde{\Lambda} = 0, & \text{on} \quad \partial \mathbb{R}^{n+4}_+. \end{cases} \]
According to Lemma 3.1, we will choose $\alpha = n$, and search for the solution of
\begin{align*}
\left\{ \begin{array}{l}
\alpha(n + 2 - \alpha)f(s) + 2\alpha(1 + s)f'(s) = s^2, \\
f(0) = 0.
\end{array} \right. 
\tag{3.14}
\end{align*}

The solution of (3.14) is
\[ s^3 \over 6n(1 + s), \]
which is a convex function. So we have
\[-\Delta \left( \frac{x_{n+4}^3}{6n(1 + x_{n+4})} |x + e_{n+4}|^{-n} \right) = x_{n+4}^2 |x + e_{n+4}|^{-n-2} - \frac{1}{3n} \left( 1 - \frac{1}{(1 + x_{n+4})^3} \right) |x + e_{n+4}|^{-n}. \]

Set
\[ \tilde{u}_2(x) := \overline{\Lambda}(x) - \frac{x_{n+4}^3}{6n(1 + x_{n+4})} |x + e_{n+4}|^{-n}, \quad x \in \mathbb{R}_{+}^{n+4}. \]

Then it becomes
\begin{align*}
\left\{ \begin{array}{l}
-\Delta \tilde{u}_2 = \frac{1}{3n} \left( 1 - \frac{1}{(1 + x_{n+4})^3} \right) |x + e_{n+4}|^{-n} \quad \text{in } \mathbb{R}_{+}^{n+4}, \\
\tilde{u}_2 = 0, \\
\end{array} \right. \tag{3.15}
\end{align*}

So it follows from the maximum principle that
\[ \tilde{u}_2(x) = \overline{\Lambda}(x) - \frac{x_{n+4}^3}{6n(1 + x_{n+4})} |x + e_{n+4}|^{-n} \geq 0 \quad \text{in } \mathbb{R}_{+}^{n+4}. \tag{3.16} \]

Now we are going to give a better estimate of $C_2(n)$ defined in (2.9), which is
\begin{align*}
C_2(n) := \frac{3(n - 10)\omega_n - 2B \left( \frac{n - 1}{2}, \frac{n + 1}{2} \right)}{2n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)} & + \frac{2n^2(n + 2)}{3} \int_{\mathbb{R}_{+}^n} |x + e_n|^{-n-4} \Lambda(|x'|, x_n)x_n^3 x_n^4 dx \\
& + \frac{2n^3(n + 2)}{3} \int_{\mathbb{R}_{+}^n} |x + e_n|^{-n-4} x_n^4 \Lambda^2(|x'|, x_n) dx.
\end{align*}

Let
\[ u_1(r, s) = \tilde{u}_1(r, 0, \ldots, 0, s), \]
and recall
\[ \Lambda(r, s) = \overline{\Lambda}(r, 0, \ldots, 0, s). \]

We have
\begin{align*}
\int_{\mathbb{R}_{+}^n} |x + e_n|^{-n-4} \Lambda(|x'|, x_n)x_n^3 x_n^4 dx = \frac{1}{6n} \int_{\mathbb{R}_{+}^n} |x + e_n|^{-2n-4} \frac{x_n^6}{1 + x_n} x_n^4 dx + \int_{\mathbb{R}_{+}^n} |x + e_n|^{-n-4} x_n^3 x_n^4 u_2(|x'|, x_n) dx. \tag{3.17}
\end{align*}

On one hand,
\begin{align*}
\frac{1}{6n} \int_{\mathbb{R}_{+}^n} |x + e_n|^{-2n-4} \frac{x_n^6}{1 + x_n} x_n^4 dx &= \omega_{n-2} \int_{0}^{\infty} \frac{x_n^6 dx_n}{(1 + x_n)^{n+\frac{3}{2}}} \int_{0}^{\infty} \frac{r^{n+2} dr}{(1 + r^2)^{n+\frac{3}{2}}}.
\end{align*}
\[
\frac{45\omega_{n-2}B\left(\frac{n-1}{2}, \frac{n+1}{2}\right)}{(n+1)^2n^3(n-1)(n-2)(n-3)(n-4)(n-5)}. \tag{3.18}
\]

On the other hand, if we let \( \tilde{v}_2 \) be the solution (decay to zero at infinity) of
\[
\begin{cases}
-\Delta \tilde{v}_2 = x_{n+4}^3|x + e_{n+4}|^{-n-4}, & \text{in } \mathbb{R}_+^{n+4}, \\
\tilde{v}_2 = 0, & \text{on } \partial \mathbb{R}_+^{n+4},
\end{cases}
\]
and notice that \( \tilde{v}_2 \) is radial in the \( x' = (x_1, \cdots, x_{n+3}) \) variable, then it follows from Lemma 3.2 and (3.15) that
\[
\int_{\mathbb{R}_+^n} x_n^3|x + e_n|^{-n-4}u_2(|x'|, x_n)x_n^4dx = \frac{1}{3n} \int_{\mathbb{R}_+^n} v_2(|x'|, x_n) \left(1 - \frac{1}{(1 + x_n)^3}\right)|x + e_n|^{-n}x_n^3dx,
\]
where
\[v_2(r, s) = \tilde{v}_2(r, 0, \cdots, 0, s).\]

We will give a lower bound on \( \tilde{v}_2 \). According to Lemma 3.1, we will choose \( \alpha = n + 2 \), and search for the solution of
\[
\begin{cases}
2\alpha(1 + s)f'(s) = s^3, \\
f(0) = 0.
\end{cases} \tag{3.19}
\]
The solution of (3.19) is
\[
\frac{1}{2(n+2)} \left[ \frac{(1+s)^3}{3} - \frac{3(1+s)^2}{2} + 3(1+s) - \log(1+s) - \frac{11}{6} \right],
\]
which is a convex function. Hence, we have
\[-\Delta \left[ \frac{1}{2(n+2)} h_1(x_{n+4})|x + e_{n+4}|^{-n-2}\right] \leq x_{n+4}^3|x + e_{n+4}|^{-n-4} \quad \text{in } \mathbb{R}_+^{n+4},
\]
where
\[h_1(s) = \frac{(1+s)^3}{3} - \frac{3(1+s)^2}{2} + 3(1+s) - \log(1+s) - \frac{11}{6}.
\]
Therefore, we have
\[\tilde{v}_2(x) \geq \frac{1}{2(n+2)} h_1(x_{n+4})|x + e_{n+4}|^{-n-2} \quad \text{in } \mathbb{R}_+^{n+4}.
\]
Hence,
\[
\int_{\mathbb{R}_+^n} x_n^3|x + e_n|^{-n-4}u_2(|x'|, x_n)x_n^4dx
\]
\[= \frac{1}{3n} \int_{\mathbb{R}_+^n} v_2(|x'|, x_n) \left(1 - \frac{1}{(1 + x_n)^3}\right)|x + e_n|^{-n}x_n^3dx
\]
\[\geq \frac{1}{6n(n+2)} \int_{\mathbb{R}_+^n} h_1(x_n) \left(1 - \frac{1}{(1 + x_n)^3}\right)|x + e_n|^{-2n-2}x_n^4dx
\]
\[\geq \frac{1}{6n(n+2)} \frac{3\omega_{n-2}}{(n-1)(n+1)} \int_0^\infty h_1(x_n) \left(1 - \frac{1}{(1 + x_n)^3}\right) (1 + x_n)^{-n+1}dx_n \int_0^\infty \frac{r^{n+2}}{(1 + r^2)^{n+2}}dr
\]
\[= \frac{1}{6n(n+2)} \frac{3\omega_{n-2}}{(n-1)(n+1)} \frac{n + 1}{4n} B\left(\frac{n-1}{2}, \frac{n+1}{2}\right) \int_0^\infty h_1(x_n) \left(1 - \frac{1}{(1 + x_n)^3}\right) (1 + x_n)^{-n+1}dx_n.
\]
It is elementary to calculate that
\[ \int_0^\infty h_1(x_n) \left( 1 - \frac{1}{(1 + x_n)^3} \right) (1 + x_n)^{-(n+1)} \, dx_n = \frac{18(5n^3 - 24n^2 + 51n - 40)}{(n-5)(n-4)(n-3)(n-2)^2(n-1)n(n+1)^2}. \]

Hence,
\[ \int_{\mathbb{R}^n_+} x^3_n |x + e_n|^{-n-4} u_2(|x'|, x_n) x_1^4 \, dx \geq \frac{\omega_{n-2} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{4n^2(n+2)(n-1)} \frac{9(5n^3 - 24n^2 + 51n - 40)}{(n-5)(n-4)(n-3)(n-2)^2(n-1)n(n+1)^2}. \quad (3.20) \]

Lastly,
\[ \int_{\mathbb{R}^n_+} |x + e_n|^{-4} x_1^4 \Lambda^2(|x'|, x_n) \, dx \]
\[ = \int_{\mathbb{R}^n_+} |x + e_n|^{-4} x_1^4 \left( \frac{x_3^3}{6n^2 + 3} \right) |x + e_n|^{-n} + u_2(|x'|, x_n) \, dx \]
\[ = \int_{\mathbb{R}^n_+} |x + e_n|^{-4} x_1^4 \left( \frac{x_3^3}{6n^2 + 3} \right) |x + e_n|^{-n} \, dx \]
\[ \quad + \frac{1}{3n} \int_{\mathbb{R}^n_+} |x + e_n|^{-n-4} x_1^4 \frac{x_3^3}{1 + x_n} u_2(|x'|, x_n) \, dx + \int_{\mathbb{R}^n_+} |x + e_n|^{-4} x_1^4 u_2^2(|x'|, x_n) \, dx \]
\[ = II_1 + II_2 + II_3. \quad (3.21) \]

For \( II_1 \), we have
\[ II_1 = \frac{\omega_{n-2}}{12(n+1)n^2(n-1)} \int_0^\infty x_n^6 \, dx_n \int_0^\infty \frac{r^{n+2}}{(1 + r^2)^{n+2}} \, dr \]
\[ = \frac{15\omega_{n-2} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{2(n+2)(n+1)^2 n^4(n-1)(n-2)(n-3)(n-4)}. \quad (3.22) \]

For \( II_2 \), if we let \( \tilde{w}_2 \) be the solution (decay to zero at infinity) of
\[ \left\{ \begin{array}{ll} -\Delta \tilde{w}_2 = \frac{x_3^3}{1 + x_{n+4}} |x + e_{n+4}|^{-n-4}, & \text{in } \mathbb{R}^{n+4}_+, \\
\tilde{w}_2 = 0, & \text{on } \partial \mathbb{R}^{n+4}_+, 
\end{array} \right. \]
and notice that \( \tilde{w}_2 \) is radial in the \( x' = (x_1, \cdots, x_{n+3}) \) variable, then it follows from Lemma 3.2 and (3.15) that
\[ II_2 = \frac{1}{9n^2} \int_{\mathbb{R}^n_+} w_2(|x'|, x_n) \left( 1 - \frac{1}{(1 + x_n)^3} \right) |x + e_n|^{-n} x_1^4 \, dx, \]
where
\[ w_2(r, s) = \tilde{w}_2(r, 0, \cdots, 0, s). \]

We will give a lower bound estimate for \( w_2 \). According to Lemma 3.1, we will choose \( \alpha = n + 2 \), and search for the solution of
\[ \left\{ \begin{array}{ll} 2\alpha(1 + s) f'(s) = \frac{s^3}{1+s}, \\
f(0) = 0. \end{array} \right. \quad (3.23) \]
The solution of (3.23) is
\[
\frac{1}{2(n+2)} \left[ (s+1)^2 - 3(s+1) + 3 \log(s+1) + \frac{1}{s+1} + \frac{3}{2} \right],
\]
which is a convex function. Hence, we have
\[
-\Delta \left\{ \frac{1}{2(n+2)} h_2(x_{n+4}) |x + e_{n+4}|^{-n-2} \right\} \leq \frac{x_{n+4}^3}{1 + x_{n+4}} |x + e_{n+4}|^{-n-4} \quad \text{in } \mathbb{R}^+_n,
\]
where
\[
h_2(s) := \frac{(s+1)^2}{2} - 3(s+1) + 3 \log(s+1) + \frac{1}{s+1} + \frac{3}{2}.
\]
By the maximum principle, we have
\[
\bar{w}_2(x) \geq \frac{1}{2(n+2)} h_2(x_{n+4}) |x + e_{n+4}|^{-n-2} \quad \text{in } \mathbb{R}^+_n.
\]
Thus,
\[
II_2 \geq \frac{1}{18n^2(n+2)} \int_{\mathbb{R}^+_n} |x + e_n|^{-2n-2} x_n^4 h_2(x_n) \left( 1 - \frac{1}{(1 + x_n)^3} \right) \, dx
\]
\[
\geq \frac{3\omega_{n-2}}{18n^2(n+2)(n-1)(n+1)} \frac{n+1}{4n} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)
\cdot \int_0^\infty h_2(x_n)(1 + x_n)^{-n+1}(1 - (1 + x_n)^{-3}) \, dx_n
\]
\[
= \frac{3(5n^3 - 13n^2 + 26n - 16)\omega_{n-2} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{4(n-4)(n-3)(n-2)^2(n-1)^2n^4(n+1)^2(n+2)^2}.
\] 
(3.24)
Finally,
\[
II_3 \geq 0.
\] 
(3.25)
Therefore, putting (2.9), (3.17), (3.18), (3.20), (3.21), (3.22), (3.24) and (3.25) together, we obtain
\[
C_2(n) > \frac{3(n-10)\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{2n(n-1)(n-2)(n-3)(n-4)(n-5)}
+ \frac{30(n+2)\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{(n+1)^2n(n-1)(n-2)(n-3)(n-4)(n-5)}
+ \frac{3(5n^3 - 24n^2 + 51n - 40)\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{2(n-5)(n-4)(n-3)(n-2)^2(n-1)^2n(n+1)^2}
+ \frac{10\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{(5n^3 - 13n^2 + 26n - 16)\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}
+ \frac{2(n+1)^2n(n-1)(n-2)(n-3)(n-4)}{2(n-4)(n-3)(n-2)^2(n-1)^2n(n+1)^2(n+2)}
+ \frac{(3n^5 - 33n^4 + 106n^3 - 119n^2 + 59n - 40)\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{2(n+1)^2n(n-1)^2(n-2)^2(n-3)(n-4)(n-5)}
+ \frac{(5n^3 - 13n^2 + 26n - 16)\omega_{n-2}B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{2(n-4)(n-3)(n-2)^2(n-1)^2n(n+1)^2(n+2)}
\]
\[
\frac{(3n^3 - 24n^2 + 27n + 34)\omega_{n-2} B \left( \frac{n-1}{2}, \frac{n+1}{2} \right)}{2(n-5)(n-4)(n-3)(n-2)(n-1)^2(n+1)(n+2)}.
\]

Hence,

\[C_2(n) > 0, \quad \text{if} \quad n \geq 7.\]

**Remark 3.4.** According to (2.7), if \(n \leq 6\), then the dominant error term in (2.2) will be of order \(\varepsilon^4 |\log \varepsilon|\), and thus, the right hand side of (2.2) needs more delicate expansion in \(\varepsilon\). Therefore, in this umbilic case, \(n = 7\) is the best one can do under the present proof.

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