TRANSITIVITY OF NORMAL SUBGROUPS OF THE MAPPING CLASS GROUPS ON CHARACTER VARIETIES

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Abstract. We prove that the action of any non-trivial normal subgroup of the mapping class group of a surface of genus \( g \geq 2 \) is almost minimal on the character variety \( X(\pi_1 \Sigma_g, \text{SU}_2) \): the orbit of almost every point is dense.

1. Introduction

For every \( g \geq 2 \), let \( \pi_1 \Sigma_g \) denote a fundamental group of a compact, connected, orientable surface of genus \( g \), and \( \text{Mod}(\Sigma_g) \) its mapping class group. In [6], Goldman proved that \( \text{Mod}(\Sigma_g) \) acts ergodically on the character variety \( X(\pi_1 \Sigma_g, \text{SU}_2) \), and subsequently, Previte and Xia [12] proved that for every conjugacy class of representation \( \rho : \pi_1 \Sigma_g \to \text{SU}_2 \) with dense image, the orbit \( \text{Mod}(\Sigma_g) \cdot [\rho] \) is dense in \( X(\pi_1 \Sigma_g, \text{SU}_2) \).

Goldman then raised (see [7]) the question of whether smaller subgroups of \( \text{Mod}(\Sigma_g) \) still act ergodically on \( X(\pi_1 \Sigma_g, \text{SU}_2) \), and with Xia he proved [8] that when \( \Sigma \) is a twice punctured torus, the Torelli group acts ergodically in the relative \( \text{SU}_2 \) character varieties. This question was addressed by Funar and Marché [4], who proved that the Johnson subgroup, generated by the Dehn twists along separating curves, acts ergodically on this character variety. Provided \( g \geq 3 \), Bouilly (see [1]) gave a simpler proof that the Torelli group acts ergodically on this character variety, and in fact on the topological components of the character variety \( X(\pi_1 \Sigma_g, G) \) for any compact Lie group \( G \).

In this note, when a group \( \Gamma \) acts on a topological space \( X \) endowed with a Radon measure \( \mu \), we will say that the action is almost minimal if the orbit of almost every point is dense. We say the action is minimal if every orbit is dense, and ergodic if for every measurable \( \Gamma \)-invariant set \( U \), either \( U \) or its complement has measure 0. These two latter properties are independent in general, while both imply almost minimality.

The main result of this note is the following.

Theorem 1. Suppose \( g \geq 2 \). Let \( \Gamma \) be a non central, normal subgroup of \( \text{Mod}(\Sigma_g) \). Then the action of \( \Gamma \) on \( X(\pi_1 \Sigma_g, \text{SU}_2) \) is almost minimal.

When \( g \geq 3 \), the centre of \( \text{Mod}(\Sigma_g) \) is trivial, while if \( g = 2 \) this centre is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), and generated by the hyperelliptic involution. The hypothesis “non central” simply rules out the cases when \( \Gamma \) is trivial or equal to this central \( \mathbb{Z}/2\mathbb{Z} \) subgroup. Thus Theorem 1 applies, for example, to every term of the lower central series of \( \text{Mod}(\Sigma_g) \).

The mapping class group \( \text{Mod}(\Sigma_g) \) is generated by Dehn twists, while the Torelli group is generated by products of the form \( \tau_{\gamma} \tau_{\delta}^{-1} \) where \( (\gamma, \delta) \) is a pair of cobordant simple curves. Bouilly’s approach to the ergodicity of the Torelli group uses the idea that, for almost every conjugacy class of representation \( [\rho] \) and for every bounding pair \( (\gamma, \delta) \), the product \( \tau_{\gamma} \tau_{\delta}^{-1} \) acts as a totally irrational rotation along a torus
immerged in the character variety $X(\pi_1 \Sigma_g, SU_2)$. Thus, for an appropriate sequence of powers, $\tau_\gamma^n\tau_{-\delta}^{-n}$ approximates the effect of the Dehn twist $\tau_\gamma$. This reduces the ergodicity properties of the Torelli group to those of the whole mapping class group, and these are well understood.

The key lemma in the proof of Theorem 1 is Lemma 7 below. It consists in extending Bouilly’s trick to the case when $\gamma$ and $\delta$ are no longer disjoint. We manage to control the action of $\tau_\gamma^n\tau_{-\delta}^{-n}$ for some sequences of integers $n$ dictated by classical theorems in Diophantine approximation theory.

2. Proof of Theorem 1

We first set up some notation.

2.1. Notation and reminders. The space $\text{Hom}(\pi_1 \Sigma_g, SU_2)$ of morphisms from $\pi_1 \Sigma_g$ to $SU_2$ is naturally endowed with the product topology and the character variety $X(\pi_1 \Sigma_g, SU_2)$ is the quotient of this representation space by the conjugation action of $SU_2$. From now on we will denote it simply by $X$.

The mapping class group $\text{Mod}(\Sigma_g) = \pi_0(\text{Diff}_+(\Sigma_g))$ is, by the Dehn-Nielsen-Baer theorem, isomorphic to an index two subgroup of $\text{Out}(\pi_1 \Sigma_g)$. It acts naturally on $X$, by setting, for $\phi \in \text{Aut}(\Sigma_g)$ and $[\rho] \in X$, $\phi \cdot [\rho] = [\rho \circ \phi^{-1}]$: this descends to an action of $\text{Out}(\pi_1 \Sigma_g)$.

The mapping class group is generated by the Dehn twists: when $\gamma \subset \Sigma$ is a simple closed curve, we denote by $\tau_\gamma$ the Dehn twist along $\gamma$; see e.g. [3, Chapter 3] for a definition, and numerous properties. Given such a curve, we may choose a representant in $\pi_1 \Sigma_g$: such a representant is well defined up to conjugacy and up to passing to the inverse. Yet, we will often use the same notation, $\gamma$ for the corresponding elements of $\pi_1 \Sigma_g$.

For every element $A \in SU_2$, we will write $\theta(A) = \frac{1}{2} \arccos(\frac{1}{2} \text{tr}(A)) \in [0, 1]$. Note that this is also invariant by conjugation and by taking the inverse. Thus, when $\gamma$ is an unoriented closed curve, or an element of $\pi_1 \Sigma_g$, we also define $\theta_\gamma : X \to [0, 1]$ by $\theta_\gamma([\rho]) = \theta(\rho(\gamma))$. This function is continuous, and smooth on $\theta_\gamma^{-1}((0, 1))$.

It is well-known that the subspace of irreducible representations in $X$ forms a Zariski open subset $X^{irr}$, which is the smooth part of $X$. Moreover, there is a $\text{Mod}(\Sigma_g)$-invariant symplectic form on $X^{irr}$ and the Hamiltonian flow of $\theta_\gamma$ on $X^{irr} \cap \theta_\gamma^{-1}((0, 1))$, denoted by $\Phi_\gamma^t$, is 1-periodic. This flow can be extended to $\theta_\gamma^{-1}((0, 1))$ and it satisfies the crucial identity $\tau_\gamma([\rho]) = \Phi_\gamma^t([\rho])$ for all $[\rho] \in \theta_\gamma^{-1}((0, 1))$.

We refer to [5] for all these facts.

2.2. Simultaneous Diophantine approximation. In the following definition, and subsequently in this note, for all $x \in \mathbb{R}/\mathbb{Z}$ we will denote by $|x|$ its distance to 0 in $\mathbb{R}/\mathbb{Z}$.

Definition 2. A pair $(x, y)$ of irrational elements of $\mathbb{R}/\mathbb{Z}$ will be said appropriately approximable if there exists a strictly increasing sequence $(q_n)$ of integers such that $q_n x$ converges to 0 faster than $\frac{1}{q_n}$ (i.e., $|q_n x| = o\left(\frac{1}{q_n}\right)$) and $q_n y$ converges to $y$ in $\mathbb{R}/\mathbb{Z}$.

A classical theorem of Khinchin [11] states that if $(\psi_n)$ is a decreasing sequence of real numbers and if $\sum \psi_n$ diverges, then for almost every $x$ there are infinitely many integers $q$ such that $|qx| \leq \psi_q$. In particular for example, for almost every $x$, there are infinitely many integers $q$ satisfying $|qx| \leq \frac{1}{q \ln q}$.

Now, a classical theorem of Hardy and Littlewood [9, Theorem 1.40] states that for every strictly increasing sequence of integers $(q_n)$, for almost every $y \in \mathbb{R}/\mathbb{Z}$ the
set \( \{g_n y, n \geq 0\} \) is dense in \( \mathbb{R}/\mathbb{Z} \). In particular, for almost every \( y \), the number \( y \) is an accumulation point of the sequence \( (g_n y) \).

These two theorems together imply the following observation.

**Observation 3.** The set \( \text{App} \subset (\mathbb{R}/\mathbb{Z})^2 \) of appropriately approximable pairs has full measure.

We continue with some preliminary observations concerning mapping class groups and character varieties.

2.3. Preliminary observations. In the next statements, we denote by \( P \) the set of pairs \( (\gamma, \delta) \) of isotopy classes of non-separating and non-isotopic simple curves.

**Observation 4.** Let \( \gamma \in \Sigma_g \) be an unoriented, non-separating simple closed curve. Then there exists \( \varphi \in \Gamma \) such that \( (\gamma, \varphi(\gamma)) \in P \).

**Proof.** Since \( \Gamma \) is not central, and since \( \text{Mod}(\Sigma_g) \) is generated by Dehn twists along non-separating curves, there exists a non-separating simple closed curve \( \delta \), and \( \psi \in \Gamma \) such that \( \psi \) and the Dehn twist \( \tau_\delta \) do not commute. There exists \( \phi \in \text{Mod}(\Sigma_g) \) mapping \( \delta \) to \( \gamma \), so \( \phi \tau_\delta \phi^{-1} = \tau_\gamma \). Now \( \varphi = \phi \psi \phi^{-1} \) is in \( \Gamma \) since \( \Gamma \) is normal, and \( \varphi \) does not commute with \( \tau_\gamma \); this implies the statement. \( \square \)

For every \( (\gamma, \delta) \in P \), we denote by \( \text{Ind}(\gamma, \delta) \) the subset of \( X \) consisting of those \( [\rho] \) such that \( (\theta(\rho(\gamma)), \theta(\rho(\delta))) \in \text{App} \). As we will see below, this condition gives some independence of the traces of \( \rho(\gamma)^n \) and \( \rho(\delta)^n \) for \( n \) large.

**Observation 5.** Let \( (\gamma, \delta) \in P \). Then \( \text{Ind}(\gamma, \delta) \) has full measure in \( X \).

**Proof.** Consider the map \( \Theta = (\theta_\gamma, \theta_\delta) : X \to [0, 1]^2 \). We want to show that \( \Theta^{-1}(\text{App}) \) has full measure in \( X \). If \( \Theta \) is a submersion at \([\rho]\), the implicit theorem implies that \( \Theta^{-1}(\text{App}) \) has full measure locally around \([\rho]\). Hence it suffices to show that \( \Theta \) is a submersion in a dense Zariski open subset of \( X \). Consider the Zariski open set \( U = \Theta^{-1}(0, 1)^2 \); it is well-known that \( d\theta_\gamma \) and \( d\theta_\delta \) are smooth non-vanishing forms on \( U \). If \( \gamma \) and \( \delta \) are disjoint, \( \Theta \) can be extended to a system of action-angle coordinate, which implies that \( \Theta \) is a submersion everywhere in \( U \), see for instance [10]. If \( \gamma, \delta \) do intersect, then it is known that their Poisson bracket does not vanish identically, see for instance [2] Corollary 5.2. As \( X \) is irreducible, it follows that \( d\theta_\gamma, d\theta_\delta \) are linearly independent in a Zariski-open subset of \( U \), proving the lemma. \( \square \)

From the Observation 5 it follows that the set

\[
\text{Ind} = \bigcap_{(\gamma, \delta) \in P} \text{Ind}(\gamma, \delta)
\]

has full measure in \( X \). It is obviously \( \text{Mod}(\Sigma_g) \)-invariant, and for any \( [\rho] \in \text{Ind} \) and any non-separating simple curve \( \gamma \), we have \( \theta_\gamma(\rho) \in (0, 1) \); in fact \( \theta_\gamma(\rho) \) is irrational.

2.4. The proof. Since the action of \( \text{Mod}(\Sigma_g) \) on \( X \) is ergodic (by Goldman [6]), the set

\[
D = \{[\rho] : \text{Mod}(\Sigma_g) \cdot [\rho] \text{ is dense in } X\}
\]

has full measure in \( X \). In fact, this set is known explicitly from the work of Previte and Xia [12]; it is the set of those \([\rho]\) such that the image of \( \rho \) is dense in \( \text{SU}_2 \). Thus, the set \( D \cap \text{Ind} \) also has full measure, and Theorem 1 will follow from the following statement.

**Proposition 6.** For all \([\rho] \in D \cap \text{Ind} \), the set \( \Gamma \cdot [\rho] \) is dense in \( X \).

The proof resides on the following lemma.
Proof. Consider an element \( \varphi \in \Gamma \) as in Observation \[\text{4}\] and set \( \delta = \varphi(\gamma) \). We observe that for any \( n \in \mathbb{N} \), \( \tau_\gamma^n \tau_\delta^{-n} = \tau_\gamma^n \varphi \tau_\delta^{-n} \varphi^{-1} \) belongs to \( \Gamma \).

Write \( \alpha = \theta_\delta(p) \) and \( \beta = \theta_\gamma(p) \). As \( \alpha \in (0,1) \), the twist flow \( (\Phi^t_\delta)_{t \in \mathbb{R}/\mathbb{Z}} \) is well defined on \( \Phi^t_\gamma([\rho]) \) for all \( t \) in a neighborhood \( I \) of \( 0 \) in \( \mathbb{R}/\mathbb{Z} \). We set \( F(t,s) = \Phi^t_\delta \Phi^{s-t}_\delta([-\rho]) \) for \( (t,s) \in I \times \mathbb{R}/\mathbb{Z} \). From the identity \( \tau_\gamma = \Phi^{\theta_\gamma}_\delta \), we get for all \( n \) such that \( n\alpha \in I \)

\[
\tau_\gamma^n \tau_\delta^{-n} [-\rho] = F(n\alpha, n f(n\alpha))
\]

where \( f(t) = \theta_\gamma(\Phi^{s-t}_\delta([-\rho])) \). As \( \beta \in (0,1) \), the function \( \theta_\gamma \) is smooth at \( [-\rho] \) hence \( f \) is smooth at \( 0 \). To prove the lemma, it is sufficient to show that one has \( (n\alpha, n f(n\alpha)) \to (0, \beta) \) for a sequence of \( n \)'s going to infinity.

Since \( (\alpha, \beta) \in \text{App} \), there exists a sequence \( (q_n) \) of integers as in Definition \[\text{2}\]. We have \( q_n \alpha \to 0 \), so we consider the Taylor expansion of \( f \) at \( 0 \): since \( |q_n \alpha| = o\left(\frac{1}{q_n}\right) \), this gives

\[
f(q_n \alpha) = f(0) + o\left(\frac{1}{q_n}\right),
\]

so \( q_n f(q_n \alpha) = q_n \beta + o(1) \). Now, \( q_n \alpha \) tends to \( \beta \), by Definition \[\text{2}\]. □

We are ready to conclude the proof of Theorem \[\text{1}\].

Proof of Proposition \[\text{2}\]. Recall that \( D \cap \text{Ind} \) is \( \text{Mod}(\Sigma_g) \)-invariant. Let \( [\rho] \in D \cap \text{Ind} \) and let \( \gamma_1, \gamma_2 \) be two non-separating simple closed curves. By Lemma \[\text{7}\] there exists a sequence \( (\varphi_n) \) of elements of \( \Gamma \) such that \( \varphi_n \cdot [\rho] \to \tau_{\gamma_2} \cdot [\rho] \). For all \( n \), we may apply Lemma \[\text{7}\] to \( \varphi_n \cdot [\rho] \), and now we can apply a diagonal argument to show that \( \tau_{\gamma_1} \tau_{\gamma_2} \cdot [\rho] \) is in the closure of \( \Gamma \cdot [\rho] \). We proceed by induction: for all \( [\rho] \in D \cap \text{Ind} \), and all curves \( \gamma_1, \ldots, \gamma_n \), the representation \( \tau_{\gamma_1} \cdots \tau_{\gamma_n} \cdot [\rho] \) is in the closure of \( \Gamma \cdot [\rho] \).

We notice that Lemma \[\text{7}\] works equally well for \( \tau_{\gamma}^{-1} \) instead of \( \tau_{\gamma} \), hence we deduce that the whole orbit \( \text{Mod}(\Sigma_g) \cdot [\rho] \) (and hence, also its closure) is contained in the closure of \( \Gamma \cdot [\rho] \). As \( [\rho] \in D \), this implies that \( \Gamma \cdot [\rho] \) is dense in \( X \). □

References

[1] Yohann Bouilly. Ergodic actions of Torelli groups on character varieties and pure modular groups on relative character varieties and topological dynamics of modular groups. PhD thesis, Université de Strasbourg, 2021.
[2] Laurent Charles and Julien Marché. Multicurves and regular functions on the representation variety of a surface in \( SU(2) \). Comment. Math. Helv., 87(2):409–431, 2012.
[3] Benson Farb and Dan Margalit. A primer on mapping class groups, volume 49 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012.
[4] Louis Funar and Julien Marché. The first Johnson subgroups act ergodically on relative character varieties and topological dynamics of modular groups. J. Differential Geom., 95(3):407–418, 2013.
[5] William M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math., 85(2):263–302, 1986.
[6] William M. Goldman. Ergodic theory on moduli spaces. Ann. of Math. (2), 146(3):475–507, 1997.
[7] William M. Goldman. Mapping class group dynamics on surface group representations. In Problems on mapping class groups and related topics, volume 74 of Proc. Sympos. Pure Math., pages 189–214. Amer. Math. Soc., Providence, RI, 2006.
[8] William M. Goldman and Eugene Z. Xia. Action of the Johnson-Torelli group on representation varieties. Proc. Amer. Math. Soc., 140(4):1449–1457, 2012.

\[\text{1}\] Alternatively, we may use the beautiful fact that \( \text{Mod}(\Sigma_g) \) is positively generated by Dehn twists, see \[\text{3}\] Paragraph 5.1.4.]
[9] G. H. Hardy and J. E. Littlewood. Some problems of diophantine approximation. *Acta Math.*, 37(1):155–191, 1914.

[10] L. C. Jeffrey and J. Weitsman. Toric structures on the moduli space of flat connections on a Riemann surface: volumes and the moment map. *Adv. Math.*, 106(2):151–168, 1994.

[11] A. Khintchine. Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.*, 92(1-2):115–125, 1924.

[12] Joseph P. Previte and Eugene Z. Xia. Topological dynamics on moduli spaces. II. *Trans. Amer. Math. Soc.*, 354(6):2475–2494, 2002.

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