Degree-distribution Stability of Evolving Networks

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Abstract: In this paper, we study a class of stochastic processes, called evolving network Markov chains, in evolving networks. Our approach is to transform the degree distribution problem of an evolving network to a corresponding problem of evolving network Markov chains. We investigate the evolving network Markov chains, thereby obtaining some exact formulas as well as a precise criterion for determining whether the steady degree distribution of the evolving network is a power-law or not. With this new method, we finally obtain a rigorous, exact and unified solution of the steady degree distribution of the evolving network.

Key words: Evolving network; Markov chain; Scale-free network; Degree distribution

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1 Introduction

Barabási and Albert (BA) found\(^1\) that for many real-world networks, the fraction of nodes with degree \(k\) is proportional over a large range to a power-law tail, i.e., \(P(k) \sim k^{-\gamma}\), where \(\gamma\) is a constant independent of the size of the network. For the purpose of establishing a mechanism to produce scale-free properties, they proposed the now-well-known BA model based on growth and preferential attachment. However, many real networks are not purely growing (as BA model), instead they are evolving networks with adding and also removing

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links and nodes throughout the developing process. A typical example is the protein-protein network, which has gene duplication, divergence, deletion, and heterodimerization.

There are many empirical and simulation studies on evolving networks\(^{[2,3]}\), but analytical models are rare. To name one, Shi et al.\(^{[4]}\) proposed a birth-and-death processing method to compute the degree distribution of an evolving network. In this paper, we study a class of stochastic processes, called evolving network Markov chains, on evolving networks. Our approach is to transform the degree distribution problem of an evolving network to a corresponding problem of evolving network Markov chains. We investigate the evolving network Markov chains, thereby obtaining some exact formulas as well as a precise criterion for determining whether the steady degree distribution of the evolving network is a power-law or not. With this new method, we finally obtain a rigorous, exact and unified solution of the steady degree distribution of the evolving network. In another recent work\(^{[5]}\), we have carried out the same, but for growing networks instead.

### 2 Main Results

For any \(i = 1, 2, \cdots\), let \(k_i(t)\) (\(t = i, i+1, \cdots\)) be a Markov chain taking values in \(\{0, 1, 2, \cdots\}\), with initial distribution \(P\{k_i(i) = k\} = d_{k,i}\) and the transition probability

\[
P\{k_i(t + 1) = l | k_i(t) = k\} = \begin{cases} f_i^+(k), & l = k + 1 \\ f_i^-(k), & l = k - 1 \\ 1 - f_i^+(k) - f_i^-(k), & l = k \\ 0, & \text{else} \end{cases} \tag{2.1}
\]

where \(0 < f_i^+(k), 0 < f_i^-(k), f_i^+(k) + f_i^-(k) \leq 1\), and \(f_i^-(0) := 0\).

Denote \(P(k, i, t) := P\{k_i(t) = k\} (t = i, i + 1, \cdots)\) and \(P(k, t) := \frac{1}{t} \sum_{i=1}^{t} P(k, i, t)\).

**Definition 2.1** The Markov chain \(\{k_i(t)\}_{t=i,i+1,\cdots} (i = 1, 2, \cdots)\) is called a series of evolving network Markov chains, or simply, evolving network Markov chains, if the limit \(P(k) := \lim_{t \to \infty} P(k, t)\) exists and

\[
P(k) \geq 0, \quad \sum_{k=0}^{\infty} P(k) = 1 \tag{2.2}
\]

In this case, it is said that the degree distribution of evolving network Markov chains exists, and \(P(k)\) is called the steady degree distribution of \(\{k_i(t)\}\). Further, if \(P(k)\) is power-law, i.e.,

\[
P(k) \sim k^{-\gamma} \quad (\gamma > 1) \tag{2.3}
\]

then \(\{k_i(t)\}\) are called scale-free evolving network Markov chains.
Assumptions

(I) The limits \( \lim_{t \to \infty} tf^+_t(k) \) and \( \lim_{t \to \infty} tf^-_t(k) \) exist, denoted by \( F^+(k) \) and \( F^-(k) \), respectively.

(II) The limits \( \lim_{t \to \infty} P(k,t), k = 0, 1, 2, \cdots \) exist.

Note. Assumption (I) is always satisfied for all the existing network models. Assumption (II) is also always satisfied for growing networks.

Theorem 2.2 If \( d_k := \lim_{t \to \infty} d_{k,t} \) exists and satisfies \( \sum_{k=0}^{\infty} d_k = 1 \), then the following relations are satisfied for \( P(k), k = 0, 1, 2, \cdots \):

\[
P(k) = \begin{cases} 
\frac{F^-(k)}{1+F^+(0)}P(k+1) + \frac{d_0}{1+F^+(0)}, & k = 0 \\
\frac{F^-(k-1)}{1+F^+(1)+F^-(k)}P(k-1) + \frac{F^-(k)}{1+F^+(1)+F^-(k)}P(k+1) + \frac{d_k}{1+F^+(1)+F^-(k)}, & k > 0
\end{cases}
\] (2.4)

Further, if there are constants \( A, B, \bar{A}, \bar{B} \), satisfying \( F^+(k) = Ak + B, F^-(k) = \bar{A}k + \bar{B} \), then

\[
P(k) = \begin{cases} 
\frac{\bar{A}+\bar{B}}{1+B}P(k+1) + \frac{d_0}{1+B}, & k = 0 \\
\frac{\bar{A}(k+1)+\bar{B}}{1+Ak+B+Ak+B}P(k-1) + \frac{\bar{A}(k+1)+\bar{B}}{1+Ak+B+Ak+B}P(k+1) + \frac{d_k}{1+Ak+B+Ak+B}, & k > 0
\end{cases}
\] (2.5)

Note. Due to the preferential attachment, one has \( A \geq 0 \) and \( \bar{A} \geq 0 \), and moreover \( A \) and \( \bar{A} \) are not both 0. In addition, \( B \geq 0 \) since there is a possibility for a node to receive new links. And the probability that a node of degree 0 loses a link is 0. Thus, \( f^-_t(k) = 0 \), and moreover \( F^-(0) = 0 \). Also, the probability that a node of degree 1 loses a link is non-negative, therefore \( \bar{A} + \bar{B} \geq 0 \).

Theorem 2.3 Suppose that (i) when \( F^+(k) = Ak + B, F^-(k) = \bar{A}k + \bar{B} \), and there are \( 0 \leq m \leq M < \infty \) such that \( d_k = 0 \) when \( k < m \) or \( k > M \), the degree distribution of evolving network Markov chains satisfies

\[
P(k) = \begin{cases} 
\lim_{\varepsilon \to 0} \frac{-\int_{\varepsilon}^{1} \int_{1}^{\infty} b_2(s) e^{-\int_{s}^{A} a(u) du} ds}{\int_{\varepsilon}^{1} \int_{1}^{\infty} b_1(s) e^{-\int_{s}^{A} a(u) du} ds} > 0, & k = 0 \\
\frac{1}{g} \sum_{i=1}^{M} z_i & 1 \leq k \leq M - 1
\end{cases}
\] (2.6)

(ii) when \( k \geq M \), the degree distribution of evolving network Markov chains satisfies

\[
P(k) = \begin{cases} 
C \int_{0}^{1} z^{k-1+\frac{\mu}{\lambda}} (1 - z)^{\frac{1}{\lambda} - \frac{1}{\mu}} e^{-\frac{z}{\lambda}} \frac{\mu}{\lambda} dz, & A = 0, \bar{A} \neq 0 \\
C \int_{0}^{1} z^{k-1+\frac{\mu}{\lambda}} (1 - z)^{\frac{1}{\lambda} - \frac{1}{\mu}} e^{-\frac{z}{\lambda}} \frac{\mu}{\lambda} dz, & \bar{A} = 0, A \neq 0 \\
C \int_{0}^{1} z^{k-1+\frac{\mu}{\lambda}} (1 - z)^{\frac{1}{\lambda} - \frac{1}{\mu}} e^{-\frac{z}{\lambda}} \frac{1}{\lambda} dz, & \bar{A} = A \neq 0 \\
C \int_{0}^{1} z^{k-1+\frac{\mu}{\lambda}} (1 - z)^{\frac{1}{\lambda} - \frac{1}{\mu}} |z - \frac{A}{\lambda}|^{\frac{1}{\lambda} - \frac{1}{\mu}} e^{-\frac{z}{\lambda}} \frac{1}{\lambda} dz, & \text{else}
\end{cases}
\] (2.7)
where $\varepsilon > 0$ is small, and

$$h = \begin{cases} 1, & A \leq \overline{A} \\ \frac{\varepsilon^2}{4}, & A > \overline{A} \text{ and } \frac{B}{A} + \frac{1}{A^2 - \overline{A}} - \frac{B}{\overline{A}} > 0 \\ \varepsilon^2, & A > \overline{A} \text{ and } \frac{B}{A} + \frac{1}{A^2 - \overline{A}} - \frac{B}{\overline{A}} \leq 0 \end{cases} \quad (2.8)$$

$$a(z) = -\frac{1}{A} Bz^2 - (1 + B + \overline{B})z + \overline{B} \frac{(\overline{A} - z)(A - z)}{z(1 - z)} \quad (2.9)$$

$$b_1(z) = \frac{1}{A} \frac{1}{z(\overline{A} - z)} \quad (2.10)$$

$$b_2(z) = -\frac{1}{A} \frac{\sum_{k=m}^{M} d_k z^{k+1}}{z(1 - z)(\overline{A} - z)} \quad (2.11)$$

$$e_i = -\frac{i\overline{A} + \overline{B}}{(i - 1)(A + \overline{A}) + 1 + B + \overline{B}}, \quad (i = 1, 2, \ldots, M) \quad (2.13)$$

$$f_i = -\frac{(i - 1)A + B}{i(A + \overline{A}) + 1 + B + \overline{B}}, \quad (i = 1, 2, \ldots, M - 1) \quad (2.14)$$

$$D = \begin{pmatrix} 1 & 0 \\ 1 & e_1 \\ f_1 & 1 & e_2 \\ \vdots & & \ddots \\ f_{i-1} & 1 & e_i \\ \vdots & & \ddots \\ f_{M-2} & 1 & e_{M-1} \\ f_{M-1} & 1 & e_M \end{pmatrix} \quad (2.15)$$

$$\overline{D} = \begin{pmatrix} \lim_{\varepsilon \to 0} \frac{-\int_{0}^{\varepsilon} b_2(s)e^{-\int_{s}^{0} a(\theta)d\theta}ds}{1 + B\int_{0}^{\varepsilon} b_1(\theta)e^{-\int_{\theta}^{0} a(\theta)d\theta}d\theta} \\ d_0 \\ \vdots \\ d_M \end{pmatrix} \quad (2.16)$$

$$g = \begin{cases} \int_{0}^{1} z^{M-1+\frac{B}{\overline{A}}}(1 - z)^{-\frac{1}{A}}e^{\frac{B}{\overline{A}}z}dz, & A = 0, \overline{A} \neq 0 \\ \int_{0}^{1} z^{M-1+\frac{B}{\overline{A}}}(1 - z)^{\frac{1}{A}}e^{\frac{B}{\overline{A}}z}dz, & \overline{A} = 0, A \neq 0 \\ \int_{0}^{1} z^{M-1+\frac{B}{\overline{A}}}(1 - z)^{\frac{1}{A}}e^{\frac{B}{\overline{A}}z}dz, & \overline{A} = A \neq 0 \\ \int_{0}^{1} z^{M-1+\frac{B}{\overline{A}}}(1 - z)^{\frac{1}{A}}|z - \frac{A}{\overline{A}}|^{\frac{B}{\overline{A}}}e^{\frac{B}{\overline{A}}(z - \frac{A}{\overline{A}})}dz, & \text{else} \end{cases} \quad (2.17)$$
in which $D_j$ is a matrix obtained by replacing the $j$th column with $D$ in the matrix $D$, and $|D_j|$ is the determinant of $D_j$. Then,

$$C = \frac{\det D_{M+1}}{g \sum_{i=1}^{M} c_i} \quad (2.18)$$

**Theorem 2.4** When $F^+(k) = Ak + B, F^-(k) = \overline{Ak} + \overline{B}$, one has:

(I)

$$P(k) \geq 0, \quad \sum_{k=0}^{\infty} P(k) = 1 \quad (2.19)$$

(II) When $A > \overline{A}$, the network is scale-free with the scaling exponent $1 + \frac{1}{A-\overline{A}}$. However, when $A \leq \overline{A}$, the network is not scale-free.

### 3 Proofs of the Main Results

**Lemma 3.1** If $d_k := \lim_{t \to \infty} d_{k,t}$ exists and satisfies $\sum_{k=0}^{\infty} d_k = 1$, then the following relations are satisfied for $P(k), k = 0, 1, 2, \cdots$:

$$P(k) = \begin{cases} \frac{F^-(0)}{1+F^+(0)} P(k+1) + \frac{d_0}{1+F^+(0)}, & k = 0 \\ \frac{F^+(k-1)}{1+F^+(k)+F^-(k)} P(k-1) + \frac{F^-(k+1)}{1+F^+(k)+F^-(k)} P(k+1) + \frac{d_k}{1+F^+(k)+F^-(k)}, & k > 0 \end{cases} \quad (3.1)$$

**Proof.** It follows from the Markovian properties that

$$P(0, i, t + 1) = P(0, i, t)[1 - f^+_t(0)] + P(1, i, t)f^-_t(1) \quad (3.2)$$

Then, by the definitions of $P(k, t)$ and $P(0, i, i) = d_{0,i}$, one obtains

$$P(0, t + 1) = \frac{t}{t+1} P(0, t)[1 - f^+_t(0)] + \frac{t}{t+1} P(1, t)f^-_t(1) + \frac{1}{t + 1} d_{0,t+1} \quad (3.3)$$

The above difference equation has the following solution:

$$P(0, t) = \frac{1}{t} \prod_{i=1}^{t-1} [1 - f^+_i(0)] \times \left\{ P(0, 1) + \sum_{l=1}^{t-1} \frac{P(1, l)f^-_l(1) + d_{0,l+1}}{\prod_{j=1}^{l-1} [1 - f^+_j(0)]} \right\} \quad (3.4)$$
Let

\[ x_t = P(0, 1) + \sum_{i=1}^{t-1} \frac{P(1, l) f_t^-(1) + d_{0,t+1}}{\prod_{j=1}^{t}[1 - f_j^+(0)]} \]  

(3.5)

\[ y_t = \prod_{i=1}^{t}[1 - f_i^+(0)]^{-1} \]  

(3.6)

Then, one can easily get

\[ x_{t+1} - x_t = \frac{P(1,t) t f_t^- (1) + d_{0,t+1}}{\prod_{j=1}^{t} [1 - f_j^+(0)]} \]  

(3.7)

\[ y_{t+1} - y_t = \frac{[1 + t f_t^+(0)] \prod_{i=1}^{t} [1 - f_i^+(0)]^{-1}}{1 + F^+(0)} \]  

(3.8)

With the given condition, one has

\[ \frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \frac{P(1,t) t f_t^- (1) + d_{0,t+1}}{1 + t f_t^+(0)} \rightarrow \frac{F^-(1) P(1) + d_0}{1 + F^+(0)} \]  

(3.9)

With \( P(0, t) = \frac{a_t}{y_t} \) and by the Stolz Theorem [6], one obtains

\[ P(0) = \frac{F^-(1)}{1 + F^+(0)} P(1) + \frac{d_0}{1 + F^+(0)} \]  

(3.10)

When \( k > 0 \), one has

\[ P(k, i, t + 1) = P(k, i, t) [1 - f_i^+(k) - f_i^-(k)] + P(k + 1, i, t) f_i^- (k + 1) + P(k - 1, i, t) f_i^+(k - 1) \]  

(3.11)

Similar to the above, one has

\[ P(k) = \frac{F^+(k - 1)}{1 + F^+(k) + F^-(k)} P(k - 1) + \frac{F^-(k + 1)}{1 + F^+(k) + F^-(k)} P(k + 1) + \frac{d_k}{1 + F^+(k) + F^-(k)} \]  

(3.12)

Thus, the Lemma is proved. □

**Lemma 3.2** If there are constants \( A, B, A \) and \( B \), such that \( F^+(k) = Ak + B \) and \( F^-(k) = \overline{A}k + \overline{B} \), then

\[ P(k) = \begin{cases} \frac{\overline{A} + \overline{B}}{1 + \overline{A}} P(k + 1) + \frac{d_0}{1 + B}, & k = 0 \\ \frac{A(k-1) + B}{1 + Ak + B} P(k - 1) + \frac{A(k+1) + \overline{B}}{1 + Ak + B} P(k + 1) + \frac{d_k}{1 + Ak + B + Ak + B}, & k > 0 \end{cases} \]  

(3.13)

**Proof.** It follows immediately from Lemma 3.1 □
Proof of Theorem 2.2

Proof. The theorem follows easily from Lemmas 3.1 and 3.2. □

Lemma 3.3 Suppose that \( f^+_t(k) = a_t k + b_t + o(t) \) and \( f^-_t(k) = \bar{a}_t k + \bar{b}_t + o(t) \). Then, \( \lim_{t \to \infty} tf^+_t(k) = Ak + B \) and \( \lim_{t \to \infty} tf^-_t(k) = \bar{A}k + \bar{B} \) if and only if \( \lim_{t \to \infty} ta_t = A \), \( \lim_{t \to \infty} tb_t = B \), \( \lim_{t \to \infty} t\bar{a}_t = \bar{A} \), and \( \lim_{t \to \infty} t\bar{b}_t = \bar{B} \).

Lemma 3.4 Suppose that \( F^+(k) = Ak + B \), \( F^-(k) = \bar{A}k + \bar{B} \), and there are \( 0 \leq m < M < \infty \) such that \( d_k = 0 \) when \( k < m \) or \( M > k \). Then,

\[
P(0) = \lim_{\epsilon \to 0} \frac{- \int_\epsilon^h b_2(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds}{1 + \bar{B} \int_\epsilon^h b_1(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds}
\]

where \( h, a(z), b_1(z), b_2(z) \) are given in Theorem 2.3.

Proof. Let \( F(z) = \sum_{k=0}^\infty P(k) z^k \). Then, one has \( F(0) = P(0) \). With Eq. (2.5) and the given condition \( d_k = 0 \), when \( k < m \) and \( M > k \), one obtains

\[
Az(1 - z) \left( \frac{A}{A} - z \right) F'(z) = -[Bz^2 - (1 + B + \bar{B})z + \bar{B}]F(z) + \bar{B}P(0)(1 - z) - \sum_{k=m}^M d_k z^{k+1}
\]

Solving the above equation gives

\[
F(z) = F(\epsilon) e^{\int_\epsilon^z a(\theta)d\theta} + \bar{B}P(0) e^{\int_\epsilon^z a(\theta)d\theta} \int_\epsilon^z b_1(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds + e^{\int_\epsilon^z a(\theta)d\theta} \int_\epsilon^z b_2(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds
\]

where \( \epsilon > 0 \) is small, and

\[
\frac{1}{e^{\int_\epsilon^z a(\theta)d\theta}} F(z) = F(\epsilon) + \bar{B}P(0) \int_\epsilon^z b_1(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds + \int_\epsilon^z b_2(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds
\]

When \( z \uparrow h \), the left hand of Eq. (3.17) is 0, so

\[
F(\epsilon) + \bar{B}P(0) \int_\epsilon^h b_1(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds + \int_\epsilon^h b_2(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds = 0
\]

With \( P(0) = \lim_{\epsilon \to 0} F(\epsilon) \), and by letting \( \epsilon \downarrow 0 \), one obtains

\[
P(0) = \lim_{\epsilon \to 0} \frac{- \int_\epsilon^h b_2(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds}{1 + \bar{B} \int_\epsilon^h b_1(s) e^{-\int_\epsilon^s a(\theta)d\theta} ds}
\]

Since \( P(0) \) is uniquely determined, the solution of Eq. (2.5), i.e., \( P(k) \), \( k = 0, 1, 2, \cdots \), is unique. □
Lemma 3.5 If $A$ and $\overline{A}$ are not both 0, then when $k \geq M$, Eq. (2.3) has the following solutions:

$$
P(k) = \begin{cases} 
C \int_0^1 z^{k-1+x} \left(1 - \frac{1}{A} \right) e^{-\frac{z}{A}} dz, & A = 0, \overline{A} \neq 0 \\
C' \int_0^1 z^{k-1+x} \left(1 - \frac{1}{A} \right) e^{-\frac{z}{A}} dz, & \overline{A} = 0, A \neq 0 \\
C'' \int_0^1 z^{k-1+x} \left(1 - \frac{1}{A} \right) e^{-\frac{z}{A}} dz, & \overline{A} = A \neq 0 \\
C''' \int_0^1 z^{k-1+x} \left(1 - \frac{1}{A} \right) e^{-\frac{z}{A}} dz, & \text{else}
\end{cases}$$

(3.20)

where $C$ is a constant.

Proof. It is easily to see that (3.20) satisfies Eq. (2.5). \qed

Proof of Theorem 2.3

Proof. From Lemma 3.5 and Eqs. (2.5) and (3.20), one has

$$
P(0) = \lim_{\varepsilon \to 0} \frac{-\int_{-\varepsilon}^{\varepsilon} b(s)e^{-\int_0^{\varepsilon} a(\theta)d\theta} ds}{1 + B \int_{-\varepsilon}^{\varepsilon} b(s)e^{-\int_0^{\varepsilon} a(\theta)d\theta} ds} \\
P(0) = \frac{d_h}{1 + B} P(1) + \frac{d_o}{A} + \frac{d_h}{B} \\
P(1) = \frac{d_h}{1 + B} P(0) + \frac{2\overline{A} + B}{A + A + 1 + B + B} P(2) + \frac{d_h}{A + A + 1 + B + B} \\
\vdots
$$

(3.21)

This is a system of equations with $M + 1$ unknown variables, where $g$ is given by Eq. (2.7). Solving this system of equations, one obtains (2.6), and (2.7) is obtained by substituting $C$ into (3.20). \qed

Lemma 3.6 When $A \leq \overline{A}$, $P(k)$ is not power-law.

Proof. When $k > m$, Eq. (2.5) can be rewritten as

$$
[(A + \overline{A})k + 1 + B + \overline{B}]P(k) = [A(k - 1) + B]P(k - 1) + [\overline{A}(k + 1) + \overline{B}]P(k + 1)
$$

(3.22)

Suppose that $P(k)$ is power-law. Then, one has $P(k) = Ck^{-\gamma}[1 + o_k(1)]$, where $\gamma > 1$ is the scaling exponent, $C$ is a constant, and $o_k(1)$ is an infinitesimal with respect to $k$. It follows that

$$
[1 + (A + \overline{A})k + B + \overline{B}]k^{-\gamma}[1 + o_k(1)] = [A(k - 1) + B](k - 1)^{-\gamma}[1 + o_{k-1}(1)]
$$

$$
+ [\overline{A}(k + 1) + \overline{B}](k + 1)^{-\gamma}[1 + o_{k+1}(1)]
$$

(3.23)
that is,
\[
\left( A + \frac{1 + B + B}{k} \right) \left( 1 - \frac{1}{k} \right)^\gamma \left( 1 + \frac{1}{k} \right)^\gamma - \left( A + \frac{B - A}{k} \right) \left( 1 + \frac{1}{k} \right)^\gamma \\
- \left( \frac{A + \frac{A + B}{k}}{1 - \frac{1}{k}} \right)^\gamma = - \left( A + \frac{A + B + B}{k} \right) \left( 1 - \frac{1}{k} \right)^\gamma \left( 1 + \frac{1}{k} \right)^\gamma o_k(1) \\
+ \left( A + \frac{B - A}{k} \right) \left( 1 + \frac{1}{k} \right)^\gamma o_{k-1}(1) + \left( \frac{A + \frac{A + B}{k}}{1 - \frac{1}{k}} \right)^\gamma o_{k+1}(1) \quad (3.24)
\]

The first term on the left of the above expansion is \([(1 + A - A) - (A - A)]_1^1\), the first term of the right is \[-(A + A)o_k(1) + Ao_{k-1}(1) + Ao_{k+1}(1)\]. These two terms must be equal after neglecting the high-order infinitesimals; that is,

\[
[(1 + A - A) - (A - A)]_1^1 = -(A + A)o_k(1) + Ao_{k-1}(1) + Ao_{k+1}(1) \quad (3.25)
\]

Thus, summing over \(k\), one obtained

\[
[(1 + A + A) - (A - A)]_1^1 \sum_{k=k_0}^{\infty} \frac{1}{k} = -A o_{k_0}(1) + Ao_{k_0-1}(1) \quad (3.26)
\]

To this end, one has \((1 + A - A) - (A - A) = 0\) since \(o_k(1)\) is a infinitesimal, so that \(\gamma = 1 + \frac{1}{A - A}\). And, since \(\gamma > 1\), one has \(A > \overline{A}\). From the assumption, the proof is complete.

\[\Box\]

**Lemma 3.7** (I) When \(A > \overline{A} = 0\),

\[
P(k) = C \int_0^1 z^{k-1+\frac{\mu}{\alpha}} (1 - z)^{\frac{1}{A - A}} e^{-\frac{\mu}{\alpha}z} dz \sim e^{-\frac{\mu}{\alpha}} k^{-(1+\frac{\mu}{\alpha})} \quad (3.27)
\]

(II) When \(A > \overline{A} > 0\),

\[
P(k) = C \int_0^1 z^{k-1+\frac{\mu}{\alpha}} (1 - z)^{\frac{1}{A - A}} \left( \frac{A}{\overline{A}} - z \right)^{\frac{1}{A - A} - \frac{\mu}{\alpha}} dz \\
\sim k^{-(1+\frac{1}{A - A})} \quad (3.28)
\]

**Proof.** (I) When \(A > \overline{A} = 0\), one has \(\lim_{k \to \infty} k^\gamma \frac{\Gamma(k + k_0)}{\Gamma(k + k_0 + \gamma)} = 1\), where \(\gamma, k_0\) are non-negative real numbers, i.e., there is a number \(K\) satisfying \(k^\gamma \frac{\Gamma(k)}{\Gamma(k + \gamma)} < 1 + \varepsilon < 2\) when \(k > K\), where \(\varepsilon > 0\) can be arbitrary.
When \( k > K \), one has

\[
\left| \frac{P(k)}{k^{-(1+\frac{1}{A})}} \right| = \left| k^{1+\frac{1}{A}} P(k) \right| = \left| k^{1+\frac{1}{A}} C \int_0^1 z^{k-1+\frac{B}{A}} (1-z)^{\frac{1}{A}} e^{-\frac{B}{A} z} dz \right|
\]

\[
= \left| C k^{1+\frac{1}{A}} \int_0^1 z^{k-1+\frac{B}{A}} (1-z)^{\frac{1}{A}} \sum_{s=0}^{\infty} \frac{(-\frac{B}{A^2})^s}{s!} dz \right|
\]

\[
= C \sum_{s=0}^{\infty} \frac{1}{s!} \left( -\frac{B}{A} \right)^s k^{1+\frac{1}{A}} \int_0^1 z^{k+s+\frac{B}{A}-1} (1-z)^{\frac{1}{A}} dz
\]

\[
= C \sum_{s=0}^{\infty} \frac{1}{s!} \left( -\frac{B}{A} \right)^s k^{1+\frac{1}{A}} \frac{\Gamma(k+s+\frac{B}{A})\Gamma(1+\frac{1}{A})}{\Gamma(k+s+\frac{B}{A}+1+\frac{1}{A})}
\]

\[
\leq C \sum_{s=0}^{\infty} \frac{1}{s!} \left( \frac{B}{A} \right)^s \left( k+s+\frac{B}{A} \right)^{1+\frac{1}{A}} \frac{\Gamma(k+s+\frac{B}{A})\Gamma(1+\frac{1}{A})}{\Gamma(k+s+\frac{B}{A}+1+\frac{1}{A})}
\]

\[
\leq C \sum_{s=0}^{\infty} \frac{1}{s!} \left( \frac{B}{A} \right)^s \left( 2\Gamma(1+\frac{1}{A}) + 1 \right)
\]

\[
= 2|C| \Gamma \left( 1 + \frac{1}{A} \right) e^{\frac{B}{A}} < +\infty
\]

(3.29)

Thus, one obtains

\[
\lim_{k \to \infty} \frac{P(k)}{k^{-(1+\frac{1}{A})}} = \lim_{k \to \infty} C \sum_{s=0}^{\infty} \frac{1}{s!} \left( -\frac{B}{A} \right)^s k^{1+\frac{1}{A}} \frac{\Gamma(k+s+\frac{B}{A})\Gamma(1+\frac{1}{A})}{\Gamma(k+s+\frac{B}{A}+1+\frac{1}{A})}
\]

\[
= C \sum_{s=0}^{\infty} \frac{1}{s!} \left( -\frac{B}{A} \right)^s \left( \lim_{k \to \infty} k^{1+\frac{1}{A}} \frac{\Gamma(k+s+\frac{B}{A})\Gamma(1+\frac{1}{A})}{\Gamma(k+s+\frac{B}{A}+1+\frac{1}{A})} \right)
\]

\[
= C \sum_{s=0}^{\infty} \frac{1}{s!} \left( -\frac{B}{A} \right)^s \Gamma \left( 1 + \frac{1}{A} \right)
\]

\[
= C \Gamma \left( 1 + \frac{1}{A} \right) e^{\frac{B}{A}}
\]

(3.30)

Consequently, one has

\[
P(k) = C \int_0^1 z^{k-1+\frac{B}{A}} (1-z)^{\frac{1}{A}} e^{-\frac{B}{A} z} dz
\]

\[
\sim C \Gamma \left( 1 + \frac{1}{A} \right) e^{-\frac{B}{A}} k^{-\left(1+\frac{1}{A}\right)}
\]

(3.31)
(II) When $A > \bar{A} > 0$, one has

$$\left| \frac{P(k)}{k^{-1(1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}})}} \right| = \left| k^{1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} C \int_0^1 z^{k-1+\frac{B}{\bar{A}}}(1-z)^{\frac{1}{\bar{A}}-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \frac{\bar{A}}{A} - z \right|$$

$$= C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} k^{1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \int_0^1 z^{k-1+\frac{B}{\bar{A}}}(1-z)^{\frac{1}{\bar{A}}-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \frac{\bar{A}}{A} - z \right|$$

$$= C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} k^{1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \int_0^1 z^{k-1+\frac{B}{\bar{A}}}(1-z)^{\frac{1}{\bar{A}}-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \sum_{s=0}^{\infty} H_s \left( \frac{\bar{A}}{A} z \right)^s \right|$$

$$\leq C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \sum_{s=0}^{\infty} H_s \left( \frac{\bar{A}}{A} \right)^s \Gamma(k+s+\frac{B}{\bar{A}}) \Gamma(1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}})$$

$$= C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \sum_{s=0}^{\infty} H_s \left( \frac{\bar{A}}{A} \right)^s 2\Gamma(1+\frac{1}{A-\bar{A}})$$

$$= 2|C| \Gamma \left( 1 + \frac{1}{A-\bar{A}} \right) \left( \frac{A}{\bar{A}} + 1 \right)^{\frac{B}{A-\bar{A}}} < +\infty \quad (3.32)$$

where $H_s$ is the coefficient of $(\frac{\bar{A}}{A} z)^s$ in the expansion of $(1 - \frac{\bar{A}}{A} z)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}}$. It follows that

$$\lim_{k \to \infty} \frac{P(k)}{k^{-1(1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}})}} = \lim_{k \to \infty} k^{1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} C \int_0^1 z^{k-1+\frac{B}{\bar{A}}}(1-z)^{\frac{1}{\bar{A}}-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \frac{\bar{A}}{A} - z \right|$$

$$= \lim_{k \to \infty} C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} k^{1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \int_0^1 z^{k-1+\frac{B}{\bar{A}}}(1-z)^{\frac{1}{\bar{A}}-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \sum_{s=0}^{\infty} H_s \left( \frac{\bar{A}}{A} z \right)^s \right|$$

$$= C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \sum_{s=0}^{\infty} H_s \left( \frac{\bar{A}}{A} \right)^s \Gamma(k+s+\frac{B}{\bar{A}}) \Gamma(1+\frac{1}{\bar{A}}-\frac{B}{\bar{A}})$$

$$= C \left( \frac{A}{\bar{A}} \right)^{-\frac{1}{\bar{A}}-\frac{B}{\bar{A}}} \sum_{s=0}^{\infty} H_s \left( \frac{\bar{A}}{A} \right)^s \Gamma \left( 1 + \frac{1}{A-\bar{A}} \right)$$

$$= CT \left( 1 + \frac{1}{A-\bar{A}} \right) \left( \frac{A}{\bar{A}} + 1 \right)^{\frac{B}{A-\bar{A}}} \quad (3.33)$$
Thus,
\[
P(k) = C \int_0^1 z^{k-1} (1-z)^{\frac{1}{A-1}} (\frac{A}{A} - z)^{\frac{1}{A-1}} - \frac{B}{A} dz
\]
\sim CT \left(1 + \frac{1}{A - \tilde{A}}\right) \left(\frac{A}{A} - 1\right)^{\frac{1}{A-1}} k^{-\frac{1}{A-1}} (1 + \frac{1}{A - \tilde{A}})
\]

(3.34)

The Lemma is proved. □

Proof of Theorem 2.4:

Proof. Summing \( k \) from 0 to \( 2^n \) in Eq. (2.3), where \( n \) is an integer, gives
\[
\sum_{k=0}^{2^n} P(k) + A2^n P(2^n) + B P(2^n) = \tilde{A}(2^n + 1) P(2^n + 1) + \overline{B} P(2^n + 1) + \sum_{k=0}^{2^n} d_k
\]

(3.35)

along with \( P(k) \geq 0, \sum_{k=0}^{\infty} P(k) \leq 1 \) and Eq. (3.20), one has \( P(k) \downarrow 0 \) when \( k > M \) and \( k \uparrow \infty \), and \( \sum_{k=0}^{\infty} 2^k P(2^k) < \infty \). Moreover, one has \( (2^k + s)P(2^k + s) \to 0 \) when \( k \to \infty \), where \( s \) is an integer. Letting \( n \to \infty \) in Eq. (54) yields \( \sum_{k=0}^{\infty} P(k) = 1 \).

From Lemma 3.7, one can see that \( P(k) \) is power-law with scaling exponent \( 1 + \frac{1}{A - \tilde{A}} \) when \( A > \tilde{A} \). From Lemma 3.6, one can see that the network is not scale-free when \( A \leq \tilde{A} \). □

4 Examples

Ex 4.1 Start with a small number \( m_0 \) of nodes, which together have a total degree \( N_0 \).

At each time step, perform the following two operations independently.

(i) Add a new node with \( m \) (\( 1 < m \leq m_0 \)) edges that link the new node to \( m \) different nodes already present in the network. And, the preferential probability is similar to that in the BA model, i.e., the probability that the new node is connected to an old node \( i \) depends on the connectivity (degree) \( k_i \) of that node; that is,
\[
\Pi(k_i) = \frac{k_i}{\sum_j k_j}
\]

(4.1)

(ii) Delete an old edge. In so doing, select a node \( i \) with probability \( \Pi'(k_i) \) given by Eq. (4.1), and select a node \( j \) at random in the domain of \( i \); then, remove the edge \( l_{ij} \). After \( t \) steps, the model becomes a random network with \( t + m_0 \approx t \) nodes with the total degree \( 2(m - 1)t + N_0 \approx 2(m - 1)t \).
From (i), one can see that the probability \( \Pi^+_i(k_i(t)) \) for node \( i \) to increase its degree \( k_i(t) \) by one is

\[
\Pi^+_i(k_i(t)) = m \Pi(k_i(t)) = m \frac{k_i(t)}{\sum_j k_j(t)} = m \frac{k_i(t)}{2(m-1)t}
\] (4.2)

From (ii), one obtains the probability \( \Pi^-_i(k_i(t)) \) for node \( i \) to decrease its degree \( k_i(t) \) by one, which is

\[
\Pi^-_i(k_i(t)) = \Pi'(k_i(t)) + \sum_{j \in O_i} \Pi'(k_j(t)) \frac{1}{k_j(t)} = \frac{k_i(t)}{2(m-1)t} + \sum_{j \in O_i} \frac{1}{2(m-1)t} = \frac{2}{2(m-1)t} = \frac{k_i(t)}{(m-1)t}
\] (4.3)

where \( \Pi'(k_i(t)) \) is the probability of node \( i \) to be selected preferentially, and \( \sum_{j \in O_i} \Pi'(k_j(t)) \frac{1}{k_j(t)} \) is the probability of node \( i \) to be selected randomly.

Consider Eq. (4.2) and Eq. (4.3). One has the following transition probability:

\[
P\{k_i(t+1) = l | k_i(t) = k\} = \begin{cases} 
\Pi^+_i(k)[1 - \Pi^-_i(k)] = \frac{mk}{2(m-1)t} - \frac{mk^2}{2(m-1)t^2}, & l = k + 1 \\
\Pi^+_i(k)[1 - \Pi^+_i(k)] = \frac{k}{(m-1)t} - \frac{2k^2}{2(m-1)t^2}, & l = k - 1 \\
1 - \Pi^+_i(k) - \Pi^-_i(k) = 1 - \frac{(m+2)k}{2(m-1)t} + \frac{mk^2}{(m-1)t^2}, & l = k \\
0, & \text{otherwise}
\end{cases}
\] (4.4)

and \( A = \frac{m}{2(m-1)}, \bar{A} = \frac{1}{m-1}, B = B = 0, d_m = 1. \)

When \( m = 2 \) (\( A = \bar{A} \)), this network is not scale-free, and

\[
P(0) = e \int_0^1 \frac{s^2}{(1-s)^2} e^{\frac{1-s}{1-s}} ds
\] (4.5)

When \( m > 2(A > \bar{A}) \), the network is scale-free and

\[
P(0) = \int_0^\frac{m}{m-1} \frac{\frac{z^m}{2(m-1)^2} - \frac{1}{m-1}}{(z-1)(\frac{z}{2(m-1)} - \frac{1}{m-1})} e^{-\frac{\frac{z}{2(m-1)} - \frac{1}{m-1}}{2(m-1)}} dz
\] (4.6)

\[
P(k) = C \int_0^1 z^{k-1}(1-z)\frac{\frac{2(m-1)}{m-1} - \frac{z}{2} - \frac{z^2}{m-1}}{2(m-1)} \frac{1}{2(m-1)} dz \quad (k \geq m)
\] (4.7)

One can easily obtain \( P(m) \) from \( P(0) \). Further, one can obtain \( C \). It is clear that \( \sum_{k=0}^{\infty} P(k) = 1 \) is an distribution according to Theorem 2.4, and that \( P(k) \) is power-law with scaling exponent \( 3 + \frac{2}{m-2} \) according to Lemma 3.7, i.e.,

\[
P(k) \sim C \Gamma(3 + \frac{2}{m-2})(\frac{m}{2} - 1)^{-(2 + \frac{2}{m-2})} k^{-(1 + \frac{2(m-1)}{m-2})}
\] (4.8)
For instance, when $m = 3$, one has $A = \frac{3}{4}, \overline{A} = \frac{1}{2}, B = \overline{B} = 0, d_3 = 1$ and

$$P(0) = 47 - \frac{171}{4} \ln 3$$

(4.9)

It follows from Eq. (4.3) that

$$P(1) = 2P(0) = 94 - \frac{171}{2} \ln 3$$
$$P(2) = \frac{9}{2} P(0) = \frac{423}{2} - \frac{1539}{8} \ln 3$$
$$P(3) = \frac{19}{2} P(0) = \frac{19}{2} (47 - \frac{171}{4} \ln 3)$$

(4.10)

When $k \geq 3$, $P(k)$ has the following form:

$$P(k) = C \int_0^1 z^{k-1}(1-z)^4\left(\frac{3}{2} - z\right)^{-4}dz$$

(4.11)

and $C = \frac{\int_0^1 z^{k-1}(1-z)^4(\frac{3}{2} - z)^{-4}dz}{\int_0^1 z^{k-1}(1-z)^4(\frac{3}{2} - z)^{-4}dz} = \frac{171}{4}$. Furthermore, $C \Gamma(1 + \frac{1}{A-A})(\frac{A}{A} - 1)^{-\frac{1}{A-A}} = \frac{171}{4} \Gamma(5)(\frac{3}{2})^{-4} = 16416$. Therefore, the degree distribution is

$$P(k) = \begin{cases} 
47 - \frac{171}{4} \ln 3, & k = 0 \\
94 - \frac{171}{2} \ln 3, & k = 1 \\
\frac{423}{2} - \frac{1539}{8} \ln 3, & k = 2 \\
\frac{171}{4} \int_0^1 z^{k-1}(1-z)^4\left(\frac{3}{2} - z\right)^{-4}dz \sim 16416 k^{-5}, & k > 2 
\end{cases}$$

(4.12)

**Ex 4.2** Start with a small number $m_0$ of nodes, which together have a total degree $N_0$.

At each time step, add a new node with $m$ ($1 < m \leq m_0$) edges that link the new node to $m$ different nodes already present in the network. The probability that the new node is connected to $m$ old nodes is the group preferential attachment$[7]$, i.e., the probability for an old node $i$ to receive one edge is

$$\Pi^+(k_i(t)) = \frac{m_0 + t - m}{m_0 + t - 1} \sum_j k_j(t) + \frac{m - 1}{m_0 + t - 1}$$

(4.13)

At the same time, remove an old edge. To do so, select a node $i$ with probability $\frac{k_i(t)}{\sum_j k_j(t)}$, and select a node $j$ at random in the domain of $i$; then, remove the edge $l_{ij}$, i.e., the probability for the old node $i$ to remove one edge is

$$\Pi^-(k_i(t)) = \frac{k_i(t)}{\sum_j k_j(t)} + \sum_j \frac{k_j(t)}{\sum_j k_i(t) k_j(t)} 1$$

$$= \frac{k_i(t)}{2(m-1)t} + \sum_{j \in O_i} \frac{1}{2(m-1)t}$$

$$= 2 \frac{k_i(t)}{2(m-1)t} = \frac{k_i(t)}{(m-1)t}.$$
After $t$ steps, the model becomes a random network with $t + m_0 \approx t$ nodes having the total degree $2(m-1)t + N_0 \approx 2(m-1)t$.

The probability for a node with degree $k$ to increase its degree by one or to decrease its degree by one, denoted by $f^+_t(k)$ or $f^-_t(k)$ respectively, is given by

\[
\begin{align*}
 f^+_t(k) &= \left(\frac{t-m}{t-1} \frac{k}{2(m-1)t} + \frac{m-1}{t-1}\right) \left(1 - \frac{k}{(m-1)t}\right) \\
 f^-_t(k) &= \frac{k}{(m-1)t} \left(1 - \frac{t-m}{t-1} \frac{k}{2(m-1)t} - \frac{m-1}{t-1}\right)
\end{align*}
\] (4.15) (4.16)

one thus has $A = \frac{1}{2(m-1)}$, $B = m - 1$, $\overline{A} = \frac{1}{m-1}$, $\overline{B} = 0$, $B = \overline{B} = 0$, and $d_m = 1$.

It follows that $A < \overline{A}$ when $m > 1$, and

\[
P(0) = 2(m-1) \int_0^1 s^m (1-s)^{2m-3} (2-s)^{2(m-1)(m-2)-1} ds
\] (4.17)

One can see that this network is not scale-free by Lemma [3.6].

**Ex 4.3** This model is a revised model proposed by Albert et al.\cite{8}.

Start with $m_0$ isolated nodes. At each time step, add a new node and perform one of the following three operations:

(i) With probability $p$, add $m$ ($m \leq m_0$) new edges: In so doing, randomly select one node as the starting point of the new edge, and the other end of the edge is selected with probability

\[
\Pi(k_i) = \frac{k_i + 1}{\sum_j (k_j + 1)}
\] (4.18)

Taking into account the fact that new edges preferentially point to popular nodes with large numbers of connections. This process is repeated $m$ times.

(ii) With probability $q$, rewire $m$ edges: In so doing, randomly select a node $i$ and an edge $l_{ij}$ connected to it. Then, remove this edge and replace it with a new edge $l_{ij'}$ that connects to $i$ with node $j'$ chosen, with probability $\Pi(k'_j)$ given by (4.18). This process is repeated $m$ times.

(iii) With probability $1 - p - q$, the new node with $m$ new edges are connected to $i$ nodes already present in the network, with probability $\Pi(k_i)$.

In this model, one has $d_0 = p + q, d_m = 1 - p - q$ and the probability $f^+_t(k)$ for a node with degree $k$ to increase its degree by one is

\[
f^+_t(k) = pm \left(\frac{1}{N} + \frac{k + 1}{\sum_j (k_j + 1)}\right) + qm \left(1 - \frac{1}{N} \sum_j \frac{k + 1}{(k_j + 1)}\right) + (1 - p - q) \frac{k + 1}{\sum_j (k_j + 1)}
\]
\[
A = \frac{m}{1 + (1 - q)2m + o(1)} + pm \\
B = \frac{m}{1 + (1 - q)2m + o(1)} + pm.
\]

One thus obtains \(A = \frac{m}{1 - q)2m + 1}\), \(B = \frac{m}{1 - q)2m + 1}\). 

The probability \(f_t^-(k)\) for a node with degree \(k\) to decrease its degree by one is

\[
f_t^-(k) = \frac{qm}{N} \left(1 - \frac{k + 1}{\sum_j (k_j + 1)}\right)
\]

\[
= \frac{qm}{N} - \frac{q m}{N} \frac{k + 1}{N + \sum_i k_i(t)}
\]

\[
= \frac{1}{t} - \frac{q m}{t^2} 1 + (1 - q)2m + o(1)
\]

(4.20)

One thus has \(\overline{A} = 0\) and \(\overline{B} = qm\). Since \(P(k)\) is power-law, by Lemma 3.7 and

\[
P(k) \sim Ck^{-(3-2q+\frac{1}{m})}
\]

(4.21)

the scaling exponent is \(3 - 2q + \frac{1}{m}\). So, the network is scale-free.

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