New solution for dynamical symmetry breaking with top and bottom quark condensates

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Abstract

Starting from a general $SU_2 \times U_1$ invariant interaction Lagrangian $\mathcal{L}_{\text{int}}$ with four-fermion interactions between top $t$ and bottom $b$ quarks and working in the bubble approximation, we get a single Higgs as a bound state of quark pairs by allowing both $\langle \bar{t}t \rangle$ and $\langle \bar{b}b \rangle$ to be nonzero and the $t$ and $b$ states to mix. We find relations between the three four-fermion couplings $g_t, g_b,$ and $g_{tb}$ and show the new result that they may all be finite, with $g \Lambda^2 \gg 1$, where $\Lambda$ is the cutoff. Thus the dimensionless couplings $g' = g \Lambda^2$ correspond to strong interactions. Previous work with either one or more quark condensates found a fine-tuning condition giving $g \Lambda^2 \sim O(1)$. The Higgs mass $m_H$ is approximately the single-vev value $m_H \approx 2m_t$, and the quark mass ratio is $m_t/m_b \approx g_b/g_{tb}$. There is a new symmetry of $\mathcal{L}_{\text{int}}$, corresponding to a flat direction in the space of composite states. Breaking this symmetry by turning on one small eigenvalue of the coupling matrix turns on the quark masses and introduces the massive Higgs state, so that $m_b, m_t, m_H \ll \Lambda$ is natural. This symmetry takes over the role of “fine tuning” in the single-quark model or multi-quark two-doublet Higgs model. A possible regime of interest for this
solution is \( m_t \ll \Lambda \sim 2 \text{ TeV} \).

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The Higgs boson is introduced in the standard model to implement the breaking of the electroweak symmetry, where it leads to problems with the high-energy behavior. Carter and Pagels suggested using the Nambu–Jona-Lasinio mechanism to implement dynamical symmetry breaking. Bardeen, Hill, and Lindner investigated using this mechanism by treating the Higgs as a $t\bar{t}$ bound state, since the top quark is so much more massive than the other five quarks, hence closer to the massive Higgs. Many authors have carried this idea further.

The $b$ quark, as the $SU(2)_L$ partner of the $t$, can be included in the dynamical symmetry breaking in a natural way. This was pointed out by Bardeen, Hill and Lindner, and done by Suzuki, who carried his formalism to six quarks. In the standard model, the coupling of a quark to the Higgs boson is proportional to the quark’s mass, which results in the strongest coupling for the $t$ and $b$ quarks. However the inclusion of the lighter quark pairs is straightforward and makes a nice general model.

In both and , there is a fine-tuning condition needed to obtain solutions to the mass-gap equations for the bound state(s). This condition requires that all the effective four-fermion interactions be tuned to be $O(1/\Lambda^2)$, where $\Lambda$ is the cutoff in the model, so that $g\Lambda^2 \sim O(1)$.

In our work the Higgs term in the Lagrangian is replaced by three four-fermion coupling terms between both $t$ and $b$ quarks, leading ultimately to a massive bound state with the quantum numbers of the Higgs boson. We use a matrix formulation throughout, which straightforwardly generalizes to $2n$ quarks. There are massless neutral and charged bound states with the quantum numbers of Goldstone bosons of the broken $SU(2)_L \times U(1)_R$ symmetry, though the charged Goldstone boson bound state condition highlights a failure of the bubble approximation when the lines of the loop are two different particles. We give an ad hoc method for dealing with this problem. We start with no bias as to the relative strengths of the three couplings for $t\bar{t}$, $b\bar{b}$, and $b\bar{t}$. We assume that both the $t$ and $b$ quark
fields acquire nonzero vacuum expectation values, and concentrate on finding the general results of adding a second flavor and \( tb \) mixing terms to the interaction Lagrangian. One set of our solutions agrees with Suzuki’s extension, plus we have striking new results not possible with only one flavor, and extendable to any number of quark pairs.

We make the usual bubble or large \( N_c \) expansion of the Nambu–Jona-Lasinio method, and study the mass-gap equations and the bound state conditions. When there are finite quark masses as solutions to the mass-gap equations, we find a new version of the fine-tuning constraint on the couplings \( g_t, g_b, g_{tb} \). In addition to the previous solution for the \( g \)'s for which \( g \Lambda^2 \sim O(1) \), or \( g \sim 1/\Lambda^2 \), we find another solution with \( g_t g_b - g_{tb}^2 \sim 1/\Lambda^2 \). This expression is \( \text{det} \ G \), where \( G \) is the matrix of quark couplings. This condition imposes an extra slightly broken symmetry on the interaction Lagrangian \( L_{\text{int}} \), a symmetry which does not hold for the kinetic part of the Lagrangian and whose significance is not yet clear. When \( \text{det} \ G = 0 \), the symmetry holds, the quarks are massless, and the Higgs and Goldstone boson states are turned off. When the symmetry is slightly broken, the top and bottom masses are related by \( m_t/m_b \approx g_b/g_{tb} \), and the Higgs mass, in agreement with Suzuki, is given to leading order by \( m_H^2 = 4(m_t^4 + m_b^4)/(m_t^2 + m_b^2) \). Breaking the symmetry corresponds to turning on one eigenvalue \( G_- \) of \( G \) from 0 to \( 1/C \Lambda^2 \), \( C = N_c/8\pi^2 \), while the other eigenvalue \( G_+ \) remains large, \( (g/C)(1 + m_t^2/m_b^2) \), as do all the couplings. We thus do not need all couplings to be small to get \( m_t, m_b, m_H \ll \Lambda \).

The key element in our method which allows generalization to any number of quarks, and which led to recognizing the new solution, is the observation that all the divergent self-energy integrals are given in leading approximation by \( C \Lambda^2 \), and appear in the mass-gap equations multiplied by one of the couplings \( g \) times one of the quark masses \( m \) in the self-energy loop. This common factor allows us to write the leading part of the mass-gap equations in matrix form, \( (G - (1/C \Lambda^2)1)m = 0 \), where \( G \) is the matrix of couplings and \( m \) is a vector of quark masses. The usual fine-tuning condition is to take the elements of \( G \) of \( O(1/C \Lambda^2) \). Our new solution comes from expanding the couplings in series in \( 1/C \Lambda^2 \), with the leading term in the series itself finite, not infinitesimal, as \( \Lambda^2 \) approaches infinity. Thus
\( G = G_0 + O(1/C\Lambda^2) \), with \( G_0 \sim O(1) \). The new fine-tuning constraint is then \( \det G_0 = 0 \).

If \( G = G_0 \), the condition \( \det G_0 = 0 \) corresponds to having a “flat direction” in the effective interactions for composite \( q\bar{q} \) states. An eigenvector expansion shows that this is the direction which becomes the Higgs and Goldstone bosons when the symmetry is slightly broken, \( G = G_0 + O(1/C\Lambda^2) \), and the quarks acquire mass. This new solution may be particularly interesting for models in which the symmetry is broken at a scale \( \Lambda \) which is accessible, e.g., a few TeV, but is still enough greater than \( m_t \) to make \( m_t/\Lambda \) negligible. Then the dimensionless couplings \( g' = \Lambda^2 g \) would correspond to strong interactions.

In Section II we set up the matrix formulations of the Lagrangian, mass-gap equations, and bound-state conditions. In section III we show the new strong-coupling solution for the \( g \)'s, and in section IV we discuss the new symmetry of \( L_{\text{int}} \) associated with this solution. Section V is a brief summary of our results.

II. COUPLING MATRIX FORMULATION OF THE INTERACTION
LAGRANGIAN, MASS-GAP EQUATIONS, AND BOUND-STATE EQUATIONS

In this section, we find matrix expressions for all the key elements of the multi-channel (two channels in this paper) dynamical symmetry breaking problem. We write the interaction Lagrangian \( L_{\text{int}} \) for the most general SU(2)_L × U(1)_R invariant interaction between \( t \) and \( b \) quarks with three couplings \( g_t, g_b, g_{tb} \) in terms of the eigenvalues and eigenvectors of a 2 × 2 coupling matrix \( G \), and also split \( L_{\text{int}} \) into pieces made up of neutral combinations of either neutral or charged “states” (\( q\bar{q} \) bilinears).

Next we examine the mass-gap equations for this two-quark system in the bubble approximation, that is, keeping only the leading term in the \( 1/N_c \) expansion for the limit of large \( N_c \). The expressions for the quark self energies contain self-energy integrals \( I_t, I_b \) which require a cutoff mass scale \( \Lambda \), much larger than the quark masses \( m_t, m_b \). There are two mass-gap equations which must be simultaneously satisfied for the quarks to have finite masses \( m_t, m_b \); and these equations can be written in terms of the 2 × 2 coupling matrix
G and a $2 \times 2$ diagonal matrix $I$ with $I_t$ and $I_b$ on the diagonal, plus a 2-component mass vector $m$ with $m_t$ and $m_b$ the components.

Finally we derive the one-loop or bubble approximation conditions that there be three massless bound states corresponding to neutral pseudoscalar and positive and negative charged Goldstone bosons, plus one bound state which is a massive neutral scalar Higgs boson. These bound-state conditions can also be expressed in terms of $G$ and a $2 \times 2$ diagonal matrix $J$ with one-loop integrals $J_t$ and $J_b$ as the diagonal elements.

The Goldstone theorem states that when the mass-gap equations are satisfied, so that the original SU(2)$_L \times$ U(1)$_R$ is dynamically broken, the three Goldstone bosons must exist. We state the condition for satisfying the mass-gap equations as a condition on the determinant of a matrix, and find that indeed we need exactly the same condition on the same determinant to have a neutral Goldstone boson. However, as has been noted elsewhere, the bubble approximation is not quite good enough in the charged case to yield a Goldstone boson. The problem is having a massive $t$ quark in the $b$ quark channels and vice versa. This problem is neatly displayed in the matrix formulation, can be isolated, and the further necessary approximation imposed.

The same determinant condition appears as the bound-state condition for the massive scalar case, and in this case the piece of the matrix which must equal zero is the mass constraint piece expressing the mass of the Higgs $m_H$ in terms of $m_t$ and $m_b$.

The matrix formulations of $\mathcal{L}_{\text{int}}$, the mass-gap equations, and the bound-state conditions will make it very easy in the following two sections to see the physical significance of our new solution for the $g_i$ ($i = t, b, tb$) and our new symmetry of $\mathcal{L}_{\text{int}}$.

A. Lagrangian

Our interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = g_t \bar{\psi}_L t R \bar{t} R \psi_L + g_b \bar{\psi}_L b R \bar{b} R \psi_L + g_{tb} \left( \bar{\psi}_L b R \bar{t} R \tilde{c} R \psi_L + \bar{\psi}_L t R \bar{b} R \psi_L \right),$$

(1)

where all $L$'s are SU(2) doublets and all $R$'s are SU(2) singlets, and
\[
\psi_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix} \quad \text{and} \quad \tilde{\psi}_L = \begin{pmatrix} -\bar{b}_L \\ t_L^c \end{pmatrix},
\]

(2)

\[
t_L^c \equiv (t_L)^c = \frac{1 + \gamma^5}{2} t^c = \frac{1 + \gamma^5}{2} C\gamma^0(t^\dagger)^T.
\]

(3)

In each bilinear factor there is an implied summation over color indices, i.e., \(\tilde{\psi}_L t_R = \sum_a \bar{\psi}_L^a t_R^a\).

What we call \(g_t\), the coupling associated with the \(t_R^{-}t_R^{-}\) term, is the only coupling in the single (top) quark condensate model. The Lagrangian can be expressed entirely in terms of \(L, R\) spinors or in terms of \(\mathbf{1}\) and \(\gamma^5\), and can be separated into neutral-neutral and charged-charged interactions.

It is convenient to define a \(2 \times 2\) coupling matrix \(G\),

\[
G = \begin{pmatrix} g_t & -g_{tb} \\ -g_{tb} & g_b \end{pmatrix},
\]

(4)

together with state vectors \(x\) and \(x'\), respectively representing neutral and charged states,

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x^\dagger = \begin{pmatrix} x_1^\dagger, x_2^\dagger \end{pmatrix}, \quad x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \quad x'^\dagger = \begin{pmatrix} x_1'^\dagger, x_2'^\dagger \end{pmatrix},
\]

(5)

where

\[
x_1 = \bar{t}_L t_R, \quad x_2 = \bar{b}_R b_L, \quad x_1' = \bar{b}_L t_R, \quad x_2' = -\bar{b}_R t_L,
\]

\[
x_1^\dagger = \bar{t}_R t_L, \quad x_2^\dagger = \bar{b}_L b_R, \quad x_1'^\dagger = \bar{t}_R b_L, \quad x_2'^\dagger = -\bar{t}_L b_R.
\]

(6)

\(\mathcal{L}_{\text{int}}\) can then be written in terms of \(G\) as

\[
\mathcal{L}_{\text{int}} = x^\dagger G x + x'^\dagger G x'.
\]

(7)

In this form, the interaction Lagrangian lends itself to diagonalization in terms of the eigenvalues and eigenvectors of \(G\). Let the eigenvalues be \(G_+\) and \(G_-\),

\[
G_\pm = \frac{g_t + g_b}{2} \pm \frac{1}{2} \sqrt{(g_t + g_b)^2 - 4 (g_t g_b - g_{tb}^2)},
\]

(8)

with eigenvectors \(\xi_+\) and \(\xi_-\).
\[ \xi_+ = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = N \begin{pmatrix} 1 \\ -g_t b/ (g_t - G_-) \end{pmatrix} = N \begin{pmatrix} 1 \\ a \end{pmatrix}, \]
\[ \xi_- = \begin{pmatrix} a_- \\ b_- \end{pmatrix} = N \begin{pmatrix} -g_t b/ (g_t - G_-) \\ -1 \end{pmatrix} = N \begin{pmatrix} a \\ -1 \end{pmatrix}, \]
\[ N = \frac{1}{\sqrt{1 + g_t^2 b^2/ (g_t - G_-)^2}} = \frac{1}{\sqrt{1 + a^2}}, \]

We take the matrix \( G \) to be positive, semi-definite, so that the eigenvalues are either zero or positive, and the interaction is attractive. If \( G \) should be negative semi-definite, we can simply switch \( G_\pm \). The interaction Lagrangian can be written using the eigenvalues and eigenvectors as a sum of two terms which involve only the neutral states \( x \) and two which involve only the charged states \( x' \),
\[ L_{\text{int}} = G_+ \left( x^\dagger \xi_+ \right) \left( \xi_+^\dagger x \right) + G_- \left( x^\dagger \xi_- \right) \left( \xi_-^\dagger x \right) + G_+ \left( x^\dagger \xi_+ \right) \left( \xi_+^\dagger x' \right) + G_- \left( x^\dagger \xi_- \right) \left( \xi_-^\dagger x' \right). \]

Some simple identities, which will be useful when we return to this form of \( L_{\text{int}} \) later, with particular interest in the case that the eigenvalue \( G_- = 0 \), are
\[ G_+ G_- = g_t g_b - g_t^2 b \det G, \quad G_+ + G_- = g_t + g_b, \quad g_t - G_- = G_+ - g_b, \]
\[ \xi_+^\dagger \xi_+ = a_+^2 + b_+^2 = 1, \quad \xi_+^\dagger \xi_- = a_+ a_- + b_+ b_- = 0. \]

**B. Mass-gap equations**

The Lagrangian of Eqs. (1), (7) or (10) describes massless fermions unless certain mass-gap equations are satisfied. These come from equating the self-energy \( \Sigma_q \) of each quark to its mass \( m_q \), and, if satisfied, lead to a dynamical breaking of the \( \mathrm{SU}(2)_L \times \mathrm{U}(1)_R \) symmetry.

We will treat the mass-gap equations for this two-quark system by looking at the bubble approximation, or leading term in the \( 1/N_c \) expansion for the limit of large \( N_c \). It is sufficient to sketch the calculation of \( \Sigma_b \), the self-energy of the \( b \) quark; for example, the \( g_b \) terms in the Lagrangian are
\[ \mathcal{L}^{(g_b)}_{\text{int}} = g_b \left( \bar{t}_L b_R \bar{b}_R t_L + \bar{b}_L b_R \bar{b}_L b_L \right) \]

\[ = 1 \frac{1}{4} g_b \left( \bar{t} b b t - \bar{t} \gamma^5 b b \gamma^5 t + b b b b - \bar{b} b \gamma^5 \bar{b} b \gamma^5 b + \bar{b} t \gamma^5 b - \bar{t} b b \gamma^5 t \right). \quad (12) \]

The only contribution to \( \Sigma_b \) from the \( g_b \) terms in the bubble approximation is

\[ \Sigma_b^{(g_b)} = g_b m_b I_b \]

\[ = g_b m_b \frac{2N_c}{(2\pi)^4} \int_{\Lambda} \frac{d^4 p}{p^2 - m_b^2} \]

\[ = g_b m_b \frac{2N_c}{(2\pi)^4} \int_{0}^{\Lambda} \frac{d^4 p_E}{p_E^2 + m_b^2}, \quad (13) \]

with \( m_b \) the physical mass of the \( b \) quark and \( p_E \) a Euclidean 4-momentum. The integral \( I_b \) is typical of integrals in self-energy expressions,

\[ I_t = \frac{2N_c}{(2\pi)^4} \int_{0}^{\Lambda} \frac{d^4 p_E}{p_E^2 + m_t^2}, \quad (14) \]

all defined with the same cutoff \( \Lambda \), which is assumed to be much larger than the quarks’ masses. Similarly to the \( g_b \) case, the \( g_t \) term gives no contribution to \( \Sigma_b \) and the \( g_{tb} \) contribution is

\[ \Sigma_b^{(g_{tb})} = -g_{tb} m_t I_t \]

\[ = -g_{tb} m_t \frac{2N_c}{(2\pi)^4} \int_{0}^{\Lambda} \frac{d^4 p_E}{p_E^2 + m_t^2}, \quad (15) \]

with \( m_t \) the physical mass of the \( t \) quark.

The total contribution to \( \Sigma_b \) is \( \Sigma_b^{(g_b)} + \Sigma_b^{(g_{tb})} \). The physical mass \( m_b \) of the \( b \) quark is the solution of the mass-gap equation

\[ m_b = \left[ \Sigma_b^{(g_b)} \left( p^2 \right) + \Sigma_b^{(g_{tb})} \left( p^2 \right) \right]_{p^2 = m_b^2} \]

\[ = g_b m_b I_b - g_{tb} m_t I_t. \quad (16) \]

The expression for the \( t \) quark physical mass \( m_t \) is derived in a similar way and is the solution of the mass-gap equation,

\[ m_t = g_t m_t I_t - g_{tb} m_b I_b. \quad (17) \]
The mass-gap Equations (16) and (17) must be satisfied simultaneously for the $b$ and $t$ quarks to have finite mass. The two equations can be written in terms of a $2 \times 2$ matrix and a 2-component mass vector $\mathbf{m} = \begin{pmatrix} m_t \\ m_b \end{pmatrix}$ as

$$
\begin{pmatrix}
  g_t I_t - 1 & -g_{tb} I_b \\
  -g_{tb} I_t & g_b I_b - 1
\end{pmatrix}
\begin{pmatrix} m_t \\ m_b \end{pmatrix} = 0.
$$

(18)

However, the dependence of the $I$'s on the $m$'s makes these nonlinear equations in the masses.

We define a diagonal $2 \times 2$ matrix $\mathbf{I}$,

$$
\mathbf{I} = \begin{pmatrix} I_t & 0 \\ 0 & I_b \\ \end{pmatrix},
$$

(19)

so that the mass-gap equations are

$$
(\mathbf{G} \mathbf{I} - 1) \mathbf{m} = 0,
$$

(20)

and the condition for a finite solution $m_t, m_b$ is

$$
\text{det} (\mathbf{G} \mathbf{I} - 1) = 0
$$

(21)

when $\mathbf{I}$ is evaluated in terms of the physical masses.

The integrals $I_t$ and $I_b$ are both quadratically divergent, and as long as $\Lambda^2/m^2 \gg 1$,

$$
I_t = C \left( \Lambda^2 - m_t^2 \ln \frac{\Lambda^2}{m_t^2} + O \left( \frac{m_t^4}{\Lambda^2} \right) \right),
$$

$$
I_b = C \left( \Lambda^2 - m_b^2 \ln \frac{\Lambda^2}{m_b^2} + O \left( \frac{m_b^4}{\Lambda^2} \right) \right),
$$

(22)

where $C = N_c/8\pi^2$. For many purposes the slight difference in magnitude between $I_t$ and $I_b$ can be ignored relative to their overall magnitude, in which case

$$
I_t \simeq I_b \simeq C \Lambda^2.
$$

(23)

To this order, the mass-gap Eqs. (18) then become linear, homogeneous equations in the masses,
\[
\begin{pmatrix}
g_t - \frac{1}{CA^2} & -g_{tb} \\
-g_{tb} & g_b - \frac{1}{CA^2}
\end{pmatrix}
\begin{pmatrix}
m_t \\
m_b
\end{pmatrix} = 0,
\]
or
\[
\left( G - \frac{1}{CA^2} I \right) \mathbf{m} = 0.
\] (24)

One way to satisfy this (leading order) condition is for one of the eigenvalues of \( G \) to be equal to \( 1/CA^2 \), while at the same time the mass vector \( \mathbf{m} \) is proportional to the corresponding eigenvector. We will show in Sec. III, see Eqs. (62) and (57), that this is a natural solution for the eigenvalue we called \( G_- \) in Eq. (8),
\[
G_- = \frac{1}{CA^2} \quad \text{and} \quad \mathbf{m} \propto \xi_-.
\] (25)

C. Bound-state conditions

We now set up the (dynamical symmetry-breaking) bound-state conditions for three massless bound states corresponding to neutral pseudoscalar and positive and negative charged Goldstone bosons, plus a bound state which is a massive neutral scalar Higgs boson. In the simple sum of one-loop or bubble graphs we are considering, there is only an \( s \)-channel, no \( t \)- or \( u \)-channels, and eigenstates scatter to eigenstates, \( 1 \rightarrow 1, \gamma^5 \rightarrow \gamma^5, LR \rightarrow LR \). Diagonalizing the interaction to find the eigenstates amounts to diagonalizing the \( s \)-channel scattering matrix.

The scattering matrix can be written as
\[
\frac{1}{2} \left[ G + GJG + GJGJG + \cdots \right] = \frac{1}{2} G \frac{1}{1 - JG} = \frac{1}{2} G \frac{1}{G - GJG} = \frac{1}{2} G \frac{[\text{cofac} (1 - JG)]^T \frac{1}{\text{det} (1 - JG)}}, \] (26)

where
\[ \mathbf{J} = \begin{pmatrix} J_t & 0 \\ 0 & J_b \end{pmatrix}, \]  

and \( J_i \) is a one-loop function (integral). The condition for a bound state is that the scattering matrix have a pole, \( i.e. \), that the denominator in the last form of Eq. (20) vanish,

\[ \det (\mathbf{1} - \mathbf{JG}) = 0. \]  

(28)

Since \( \mathbf{G} \) is symmetric and \( \mathbf{J} \) is diagonal, we have

\[ \det (\mathbf{GJ} - \mathbf{1}) = 0. \]  

(29)

We will show that the one-loop integrals \( J_i \) can be expressed in terms of the self-energy integrals \( I_i \) as

\[ J_i = I_i + \Delta J_i, \]  

(30)

so the bound-state condition becomes

\[ \det (\mathbf{GI} + \mathbf{G} \Delta \mathbf{J} - \mathbf{1}) = 0. \]  

(31)

If the mass-gap equations are satisfied, then Eq. (21), \( \det (\mathbf{GI} - \mathbf{1}) = 0 \), already holds, and we can use straightforward matrix manipulations to find the conditions for bound states, which must exist by the Goldstone Theorem and Nambu–Jona-Lasinio construction. Our procedure follows, and is easily generalized to \( 2n \times 2n \), not only \( 2 \times 2 \), matrices.

When \( \det (\mathbf{GI} - \mathbf{1}) = 0 \), in its diagonal form the matrix \( (\mathbf{GI} - \mathbf{1}) \) must have one zero eigenvalue. Other eigenvalues must be of leading order \( GCA^2 \), where \( G \) is a “large” eigenvalue of \( \mathbf{G} \). For our \( 2 \times 2 \) case, we choose the “minus” eigenvalue \( G_- \) of \( (\mathbf{GI} - \mathbf{1}) \) to be zero, and call the corresponding eigenstate \( \xi'_- \).

Now we write the extra piece \( \mathbf{G} \Delta \mathbf{J} \) of \( (\mathbf{GJ} - \mathbf{1}) \) in the \(-,+\) basis, so the matrix elements are labeled \(- -, -+, +-, ++\). The full matrix \( (\mathbf{GJ} - \mathbf{1}) \) is thus these four matrix elements, plus the nonzero eigenvalue of \( (\mathbf{GI-1}) \) in the ++ position,
\[(GJ - 1) = \begin{pmatrix} x_i^T G \Delta J x_i' - G \Delta J x_i' & x_i^T G \Delta J x_i' \\ x_i'^T G \Delta J x_i - x_i'^T G \Delta J x_i' + G_+ \Lambda^2 - 1 \end{pmatrix} \]  

(32)

Clearly for the determinant to be zero in leading approximation, \(x_i'^T G \Delta J x_i'\) must be zero, since \(G_+ \Lambda^2\) is so large. (In the \(2n \times 2n\) case, there would be \(2n - 1\) large eigenvalues of \(G I - 1\) on the diagonal.) All other terms, as we will show explicitly later, are at most \(O(\ln \Lambda^2)\).

Our bound-state condition is now reduced to setting one matrix element equal to zero, and we now make some approximations, all good to \(O(\ln \Lambda^2/\Lambda^2)\). First, we consider the relation of \(x_i'\), the eigenvectors of \((G I - 1)\), to \(x_i\), the eigenvectors of \(G\). The matrix \(I\) is diagonal,

\[I = \begin{pmatrix} CA^2 - m_t^2 \Lambda^2 + O\left(\frac{m_t^4}{\Lambda^2}\right) & 0 \\ 0 & CA^2 - m_b^2 \ln \frac{\Lambda^2}{m_b} + O\left(\frac{m_b^4}{\Lambda^2}\right) \end{pmatrix},\]

(33)

and to leading order is just \(I \approx CA^2 I\). Thus to leading order the same eigenvectors \(x_i\) diagonalize \(G I\) as diagonalize \(G\) alone. Since \(\Delta J\) is diagonal, our bound-state condition is now

\[x_i'^T \Delta J G x_i = x_i'^T \Delta J G x_i = 0.\]

(34)

The \(G\) factors out, and our generic condition for a bound state in the one-loop approximation, when the mass-gap equations are satisfied, in terms of the components of \(x_i\), is

\[a_-^2 \Delta J_t + b_-^2 \Delta J_b = 0,\]

(35)

where \(a_-\) and \(b_-\) are defined in Eq. (9). Next we show the specific cases.

The neutral scalar channel terms in \(L_{\text{int}}\) which contribute to the bubble expansion scattering matrix are

\[L_{\text{int}}^{\text{neu}} = \frac{1}{4} g_t t t t t + \frac{1}{4} g_b b b b b - \frac{1}{2} g_{t b} t b b.\]

(36)
The one-loop integral \( J_i \) is \((i = t, b)\),

\[
J_i = -\frac{i}{2} \frac{N_c}{(2\pi)^4} \int d^4k \, Tr \frac{i}{(p + k) - m_i} \frac{i}{k - m_i}
\]

\[
= i \frac{2N_c}{(2\pi)^4} \int d^4k \, \frac{k \cdot (p + k) + m_i^2}{[(p + k)^2 - m_i^2][k^2 - m_i^2]}
\]

\[
= i \frac{2N_c}{(2\pi)^4} \left\{ \int d^4k \frac{1}{k^2 - m_i^2} - \frac{1}{2} (p^2 - 4m_i^2) \int d^4k \frac{1}{[(p + k)^2 - m_i^2][k^2 - m_i^2]} \right\}.
\]

(37)

We recognize the first integral as \( I_i \), see Eq. (14), so that we can write \( J_i \) as

\[
J_i = I_i + \Delta J_i.
\]

(38)

We neglect the difference between \( \ln(\Lambda^2/m_i^2) \) and \( \ln(\Lambda^2/m_b^2) \), and find that to have a neutral scalar bound state, we must have

\[
a_+^2 (p^2 - 4m_i^2) + b_+^2 (p^2 - 4m_b^2) = 0.
\]

(39)

In the next section, we will find the values of \( G_{\pm} \), \( x_{i\pm} \) which allow the mass-gap equation to hold. For now we invoke those solutions, where \( a_+ = m_t^2/(m_t^2 + m_b^2) \) and \( b_+ = m_b^2(m_t^2 + m_b^2) \).

This gives us the (leading order) constraint on the neutral scalar bound state, identified as the Higgs boson,

\[
p^2 = m_H^2 = \frac{4(m_t^4 + m_b^4)}{(m_t^2 + m_b^2)},
\]

(40)

in agreement with Suzuki [4]. Because we know that \( m_t^2 \gtrsim 1200 m_b^2 \), see Ref. [12], this places the mass of the Higgs very near \( m_H = 2m_t \),

\[
m_H = 2m_t \left[ 1 - \frac{1}{2} \frac{m_b^2}{m_t^2} + O \left( \frac{m_b^4}{m_t^4} \right) \right].
\]

(41)

Thus \( m_H \) is barely changed from the result of Bardeen et al. [3], \( m_H = 2m_t \), by the inclusion of a second quark vacuum expectation value (vev).

The pseudoscalar neutral channel terms in \( L_{\text{int}} \) which contribute to the scattering matrix are
\[
\mathcal{L}_{\text{int}}^{\text{neu ps}} = -\frac{1}{4} g_t \bar{t} \gamma^5 t \Gamma^5 t - \frac{1}{4} g_b \bar{b} \gamma^5 b \Gamma^5 b - \frac{1}{2} g_t \bar{t} \gamma^5 t \bar{b} \gamma^5 b. \quad (42)
\]

The one-loop integral \( J_i \) is
\[
J_i = -\frac{i}{2} \frac{N_c}{(2\pi)^4} \int d^4 k \text{Tr} \frac{i}{k - m_i} \gamma^5 \frac{i}{\bar{p} + \bar{k} - m_i} \Gamma^5
= -\frac{i}{2} \frac{2N_c}{(2\pi)^4} \int d^4 k \frac{k \cdot (p + k) - m_i^2}{[(p + k)^2 - m_i^2] [k^2 - m_i^2]}, \quad (43)
\]
or
\[
J_i = -I_i - \frac{C}{2} p^2 \left[ \ln \frac{\Lambda^2}{m_i^2} + \cdots \right]
= -I_i - \Delta J_i. \quad (44)
\]

The signs of the \( g_t, g_b \) couplings in Eq. (42) are opposite to Eq. (36), which means that the coupling matrix is not \( G \), but \( \tilde{G} \),
\[
\tilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (45)
\]

It is not obvious that \( \det(\tilde{G}I + \tilde{G}\Delta J - 1) = 0 \). However, it is, and furthermore we find after some manipulations that the bound-state condition we have reduces to \( \det(GI + G\Delta J - 1) = 0 \), thanks to the commutativity of diagonal matrices and the properties of determinants,
\[
\det(\tilde{G}I + \tilde{G}\Delta J - 1) = \det \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (GI + G\Delta J - 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = \det(GI + G\Delta J - 1) = 0. \quad (46)
\]

This construction generalizes to the \( 2n \times 2n \) matrices encountered with \( n \) generations of quarks.

In this case, then, \( \Delta J \propto p^2 \), and the bound state constraint reduces to \( p^2 = 0 \). This is the massless pseudoscalar neutral Goldstone boson.

The charged channel terms which contribute to the scattering matrix, left in terms of \( L, R \) in \( \mathcal{L}_{\text{int}} \), are
Thus all the diagrams involve $b \leftrightarrow t$ loops, and the natural inclination is to assume $m_t \neq 0, m_b \neq 0, m_t \neq m_b$. We write as illustration the $J^\pm$ integral for a $\bar{b}t$ diagram,

$$J^\pm = -i \frac{N_c}{(2\pi)^4} \int d^4 k \, Tr \left[ \frac{i}{k - m_b} \frac{1 \pm \gamma^5}{2} \frac{i}{p + k - m_t} \frac{1 \mp \gamma^5}{2} \right].$$

(48)

We can express this in terms of $I_t$ and $I_b$ as

$$J^\pm = I_t + I_b - \frac{C}{2} (p^2 - m_t^2 - m_b^2) \int d^4 k \, \frac{1}{(k^2 - m_b^2)((p + k)^2 - m_t^2)},$$

(49)

where the integral is logarithmically divergent, or in terms of $I_t$ alone as

$$J^\pm = I_t + p^2 I_1 + m_b^2 I_2,$$

(50)

where $I_1$ and $I_2$ are both logarithmically divergent, and both have the denominator containing both masses.

Alarmingly, in this one-loop approximation for the multichannel case, the Goldstone Theorem fails: there should be a $p^2 = 0$ (massless) charged Goldstone boson. Clearly, dynamical symmetry breaking is more subtle than a bubble approximation. By putting in finite unequal masses we have already broken the symmetry by hand. This problem was not noted by Suzuki [4] and was noted but not discussed by Pham [5]. There is an *ad hoc* remedy: set $m_t \neq 0, m_b = 0$ in loops with coupling $g_t$; set $m_b \neq 0, m_t = 0$ in loops with coupling $g_b$; and do one and then the other separately in the two off-diagonal terms in the matrix $GJ$ which correspond to loops with coupling $g_{tb}$. We do not give these details here. The result is that $JG = IG + O(p^2)$, so as in the neutral pseudoscalar case, there is a $p^2 = 0$ massless charged bound state.
III. TWO SOLUTIONS FOR COUPLINGS WHICH GIVE FINITE QUARK MASSES

We reverse the usual approach to solving the mass-gap equations, Eq. (18). Instead of assuming that the couplings in $G$ are known and solving for allowed $m$’s, we assume that $m_t, m_b$ exist, are finite, and are unequal, $\Lambda \gg m_t > m_b$. This of course makes sense experimentally. Then we look for constraints on the $g$’s which allow solutions to Eq. (18).

We find two sets of solutions. The first set is known, see Suzuki [4], and is the generalization of the Bardeen, Hill, and Lindner fine-tuning constraint on the top quark coupling of [3]. It is found by approximating $I_i \simeq C\Lambda^2$ and solving Eq. (24). The second solution, new in this paper, is found by expanding and matching terms order-by-order in the mass-gap equations.

To get the first set of $g$’s we assume that all the couplings are of magnitude $1/C\Lambda^2$,

$$g_i = \frac{\hat{g}_i}{C\Lambda^2}, \quad G = \frac{\hat{G}}{C\Lambda^2}. \quad (51)$$

This condition makes all the $gI$ terms in Eq. (18) finite in the limit $\Lambda^2 \to \infty$. These $g$’s are a (leading order) solution of Eq. (24) provided

$$\det(\hat{G} - 1) = 0 \quad (52)$$

up to terms of $O(1/\Lambda^2)$, a constraint on the coupling matrix which has apparently not been noted before. In this solution, the “fine-tuning” is the condition $g\Lambda^2 \sim O(1)$.

The second set of $g$’s comes from allowing $g\Lambda^2 \gg 1$, which allows a natural Higgs mass $m_H \ll \Lambda$. In this case the $g_i$ are of $O(1)$, plus corrections of $O(1/\Lambda^2)$, and the condition

$$\det G = (g_t g_b - g_{tb}^2) = 0 \quad (53)$$

must hold up to terms of $O(1/\Lambda^2)$. This solution involves a new symmetry as the eigenvalue $G_- \to 0$.

We note that for both sets of solutions, corrections of $O(1/\Lambda^2)$ are essential to get non-trivial solutions to the mass-gap equations.
The solution for the $g_i$ noted in Eq. (51) is analogous to the fine-tuning solutions of [3] and [4] with the couplings set very small to give finite fermion masses,

$$
g_t = \frac{\hat{g}_t}{C\Lambda^2}, \quad g_b = \frac{\hat{g}_b}{C\Lambda^2}, \quad g_{tb} = \frac{\hat{g}_{tb}}{C\Lambda^2}.
$$

(54)

We can express the $\hat{g}_i$ which allow positive, finite masses $m_t, m_b$ in terms of a single constant $\hat{g}$ and the two masses,

$$
\hat{g}_t = \hat{g}, \quad \hat{g}_b = 1 + \frac{m_t^2}{m_b^2} (\hat{g} - 1), \quad \hat{g}_{tb} = \frac{m_t}{m_b} (\hat{g} - 1).
$$

(55)

The determinant of $G$ is

$$
\det G = \frac{1}{C^2\Lambda^4} \left[ \hat{g} + \frac{m_t^2}{m_b^2} (\hat{g} - 1) \right].
$$

(56)

The eigenvalues and eigenvectors of $G$ are

$$
G_+ = \frac{1}{C\Lambda^2} \left[ \hat{g} + \frac{m_t^2}{m_b^2} (\hat{g} - 1) \right],
$$

$$
G_- = \frac{1}{C\Lambda^2},
$$

$$
\frac{a_+}{b_+} = N \begin{pmatrix} \frac{1}{m_t/m_b} \\ -1 \end{pmatrix} = N \begin{pmatrix} \frac{1}{a} \\ -1 \end{pmatrix},
$$

$$
\frac{a_-}{b_-} = N \begin{pmatrix} \frac{m_t}{m_b} \\ -1 \end{pmatrix} = N \begin{pmatrix} \frac{a}{m_t/m_b} \\ -1 \end{pmatrix},
$$

$$
N = \frac{1}{\sqrt{1 + m_t^2/m_b^2}} = \frac{1}{\sqrt{1 + a^2}}.
$$

(57)

The mass ratio $m_t/m_b$ in terms of $\hat{g}$’s is

$$
\frac{m_t}{m_b} = \frac{\hat{g}_b - 1}{\hat{g}_{tb}}.
$$

(58)

This solution corresponds to a small perturbation by the mixing $g_{tb}$ of the solution of Ref. [3] for two independent $\langle \bar{t}t \rangle, \langle \bar{b}b \rangle$ vevs with $g_t \approx g_b \approx 1/C\Lambda^2$, $\hat{g}_t \approx \hat{g}_b \approx 1$.

To get the new solution for the $g$’s, we solve Eq. (18) explicitly, with $g_{tb}$ real and $m_t, m_b$ free parameters. We find
\[ g_{tb} = \frac{g m_t I_t - m_t}{m_b I_b} = m_t \frac{g_t I_t - 1}{m_b I_b}, \]
\[ g_b = \frac{m_b + g_{tb} m_t I_t}{m_b I_b} = \frac{1}{I_b} \left( 1 + \frac{m_t^2}{m_b^2} \frac{g_t I_t - 1}{I_t} \right). \]  

(59)

Now we set \( g_t = g/C \), and using Eq. (22) for the \( I \)'s with \( g \Lambda^2 \gg 1 \), we find

\[ g_{tb} = \frac{g}{C} \frac{m_t}{m_b} \left( 1 - \frac{m_t^2}{\Lambda^2} \ln \frac{\Lambda^2}{m_t^2} + \frac{m_b^2}{\Lambda^2} \ln \frac{\Lambda^2}{m_b^2} + \cdots \right) - \frac{m_t}{m_b} \frac{1}{C \Lambda^2} + O \left( \frac{1}{\Lambda^4} \right), \]
\[ g_b = \frac{g}{C} \frac{m_t^2}{m_b^2} \left( 1 - \frac{2 m_t^2}{\Lambda^2} \ln \frac{\Lambda^2}{m_t^2} + \frac{2 m_b^2}{\Lambda^2} \ln \frac{\Lambda^2}{m_b^2} + \cdots \right) + \left( 1 - \frac{m_t^2}{m_b^2} \right) \frac{1}{C \Lambda^2} + \cdots, \]
\[ g_t = \frac{g}{C}. \]  

(60)

The determinant of \( G \) is simple, and is zero up to \( O(1/\Lambda^2) \),

\[ \text{det} G = \frac{g}{C^2 \Lambda^4} \left( 1 + \frac{m_t^2}{m_b^2} \right) + O \left( \frac{1}{\Lambda^4} \right). \]  

(61)

The eigenvalues and eigenvectors of \( G \) are

\[ G_+ = \frac{g}{C} \left( \frac{m_t^2}{m_b^2} + 1 \right) - \frac{2 g \ln \Lambda^2}{C \Lambda^2} \frac{m_t^2}{m_b^2} \left( \frac{m_t^2}{m_b^2} \right) + \frac{m_t^2}{m_b^2} \left[ -1 + 2 g \left( \frac{m_t^2}{m_b^2} \ln \frac{m_t^2}{m_b^2} \right) \right], \]
\[ G_- = \frac{1}{C \Lambda^2}; \]

\[ \xi_+ = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = N \begin{pmatrix} 1 \\ \frac{m_t}{m_b} \end{pmatrix} + O \left( \frac{1}{C \Lambda^2} \right) \approx N \begin{pmatrix} 1 \\ a \end{pmatrix}, \]
\[ \xi_- = \begin{pmatrix} a_- \\ b_- \end{pmatrix} = N \begin{pmatrix} \frac{m_t}{m_b} \\ -1 \end{pmatrix} + O \left( \frac{1}{C \Lambda^2} \right) \approx N \begin{pmatrix} a \\ -1 \end{pmatrix}, \]
\[ N = \frac{1}{\sqrt{1 + m_t^2/m_b^2}} = \frac{1}{\sqrt{1 + a^2}}. \]  

(62)

Thus the eigenvectors for the two cases are the same to leading order, and the eigenvalue \( G_- \) is the same. By inspection, \( m \propto \xi^- \) as required in Eq. (25); and we see that it is natural for the leading order mass gap equations, as expressed by Eq. (24), to be satisfied for either solution of the couplings \( g_i \).

In this case the mass ratio is (to order \( 1/\Lambda^2 \)),
\[
\frac{m_t}{m_b} = \frac{g_b}{g_{tb}} = \frac{g_{tb}}{g_t}
\]  

(63)

We note that \( m_t \gg m_b \) requires \( g_b \gg g_{tb} \gg g_t \), a surprising ordering of the couplings. This solution is not obtainable from the solution of Ref. [3], \( g_t = 1/C \Lambda^2, g_b = g_{tb} = 0 \).

The dimensionless couplings \( g' = g \Lambda^2 \) are, for the old solution \( g' \sim O(1) \) and for the new solution \( g' \gg 1 \). The new solution corresponds to strong coupling, and in the eigenvector space, the direction \( x_i^- \) which goes with the small eigenvalue \( G_- = 1/C \Lambda^2 \) is a weakly coupled direction while the \( x_i^+ \) direction which goes with the finite-sized \( G_+ \) eigenvalue is strongly coupled. States which are expressed in terms of \( x_i^- \) are very massive, while states which are expressed in terms of \( x_i^+ \) have small masses relative to \( \Lambda \).

IV. A NEW SYMMETRY OF THE INTERACTION LAGRANGIAN AND STRONG COUPLING

In the new solution for the \( g' \)'s, \( \det G \) is very small, \( \det G = O(1/\Lambda^2) + O(1/\Lambda^4) \). We can investigate what would happen if \( \det G = 0 \), which corresponds to one eigenvalue being zero. Since \( G_- = 1/C \Lambda^2 \) and \( G_+ = (g/C)(1 + m_t^2/m_b^2) \), \( G_- \) is the obvious eigenvalue to turn off. In this regime, \( \mathcal{L}_{\text{int}} \) can be written as \( \Psi^\dagger \Psi \), where \( \Psi \) is a mixed neutral and charged state, though there is nothing particular to be gained by this. We will label quantities with a zero to designate this regime, e.g., \( \mathcal{L}_{\text{int}}^0, G_0, G_+^0, x_i^0 \).

When \( G_- \) is turned off, the direction \( x_i^0 \) is “flat”. States of the Lagrangian proportional to \( x_i^- \) do not couple, scatter, or bind in the bubble approximation. Bound states which still may exist, those proportional to \( x_i^+ \), would have very large masses \( \sim \Lambda \). We think of setting \( G_- \), expressed in terms of \( g_t, g_b, \) and \( g_{tb} \), to zero, not of taking \( \Lambda \rightarrow \infty \). Then when the couplings are changed slightly so that \( \det G \approx g_t g_b - g_{tb}^2 \neq 0 \), i.e., \( G \neq G_0 \), the coupling eigenvalue \( G_- \) is turned on with value \( 1/C \Lambda^2 \), and the quark masses and boson bound states turn on at a mass scale \( \ll \Lambda \). The old solution with all couplings \( O(1/\Lambda^2) \) used the fine-tuning condition \( g \sim O(1/\Lambda^2) \) to get small enough masses, while the new solution
has a different “fine-tuning” condition $\det G \sim O(1/\Lambda^2)$, or $\det G_0 = 0$, corresponding to the existence of the flat direction, and naturally small masses.

We actually discovered the new solution by studying the expressions for the eigenvalues $G_\pm$ and asking what would happen if one eigenvalue were zero, $G_-$ being the obvious choice. That implied $\det G = 0$, and we realized that in principle there was no restriction on the size of $G_+$, hence on the sizes of all the $g$’s, as long as $G_-$ and $\det G$ were small, $O(1/\Lambda^2)$.

The interaction Lagrangian is reduced to

$$\mathcal{L}^0_{\text{int}} = G^0_+ \left[ (x_i^0 x_i^0) (x_i^0 x_i^0) + (x_i^0 x_i^0) (x_i^0 x_i^0) \right].$$

(64)

The symmetry transformation which leaves $\mathcal{L}^0_{\text{int}}$ invariant is to change any vector by a multiple of $x_i_-,$

$$x \to x + \delta x,$$

$$\delta x = c(x_i_0 x_i_0 x_i_-)$$

$$\delta x^\dagger = c(x_i_0 x_i_0 x_i_-),$$

$$\mathcal{L}^0_{\text{int}} \to \mathcal{L}^0.$$ (65)

This is not a symmetry of the kinetic part of the Lagrangian, only of the interaction.

We know that the solution to the mass-gap equations requires $\det G \neq 0$, so $m_t = m_b = 0$ initially, and quantities reduce to simpler forms. Since $g^2_{tb} = g_t g_b$, the component $a$ in $x_i^0$, see Eq. (65), is now

$$a^0 = \sqrt{g_b/g_t}. \quad (66)$$

The full set of eigenstates $\Phi$ of the Lagrangian, see Eq. (10), is

$$\Phi^{\text{neu}}_{+} = \left( \xi_{+}^x \right)^s = \frac{tt + \bar{a}b}{(1 + a^2)^{1/2}},$$

$$\Phi^{\text{neu}}_{-} = \left( \xi_{-}^x \right)^s = \frac{at - \bar{b}b}{(1 + a^2)^{1/2}},$$

$$\Phi^{\text{neu}}_{+} = \left( \xi_{+}^{x'} \right)^s = \frac{2(-a \bar{b}_R t_L + \bar{b}_L t_R)}{(1 + a^2)^{1/2}},$$

$$\Phi^{\text{neu}}_{-} = \left( \xi_{-}^{x'} \right)^s = \frac{2(\bar{b}_R t_L + a \bar{b}_L t_R)}{(1 + a^2)^{1/2}}.$$

(67)
The $\Phi$ states are the ones corresponding to composite bound state Higgs ($\Phi_{\text{neu}}$), neutral Goldstone ($\Phi_{\text{neu ps}}$), and charged Goldstone ($\Phi_{\text{chg}}$) bosons. One way to view the breaking of the new symmetry is that as $G_-$ turns on from 0 to $1/C\Lambda^2$, the $\alpha$ in these states changes slightly from $-g_{tb}/g_t = -\sqrt{g_b/g_t}$ to $-g_{tb}/(g_t - G_-)$.

We do not yet understand this new symmetry in any fundamental way.

V. SUMMARY OF RESULTS

Every two-quark result presented in this paper generalizes straightforwardly to $2n$ quarks. For example, there would be $(2n - 1)$ large eigenvalues and one small one for $G$, $\det(GI - 1) = 0$, and $\det(GJ - 1) = 0$. We have checked the charged Goldstone boson bound-state condition and find the same type of ad hoc rules to impose for $2n$ quarks as for two quarks. In the $G_- = 0$ regime, there would be $(2n - 1)$ times as many high-mass bound states, but the same "flat" states.

Our most important new result is the new solution for the couplings $g_i$ which allow the mass-gap equations to be satisfied, presented in Section III. These couplings correspond to dimensionless couplings $g'_i = g_i\Lambda^2 \gg 1$, instead of $g'_i \sim O(1)$ as in the previous solutions with fine-tuning condition $g_i \sim O(1/\Lambda^2)$. Despite the strong coupling, the existence of a nearly flat direction in the interaction gives naturally small masses $m_t, m_b, m_H \ll \Lambda$.

The new solution grew out of the matrix formulation of Section II. In that section, we pointed out some details not previously discussed in the literature. Even for the old solution with $g_i \sim O(1/\Lambda^2)$, the next-to-leading terms are important for getting exact solutions to the mass-gap equations and bound-state conditions. Since $G_- = 1/C\Lambda^2$ in both solutions for the $g_i$, our determinant arguments hold for both solutions. The non-vanishing mass in the charged Goldstone boson case is a consequence of the one-loop approximation and can be fixed by setting $m_t$ or $m_b = 0$ in loops, depending on which coupling and which connecting states are involved.

The new solution led to the realization of a new symmetry of $L_{\text{int}}$, discussed in Section
V. Breaking this symmetry appears to have the same result as breaking SU(2)$_L \times$ U(1)$_R$, yet it is not a symmetry of the whole Lagrangian. When the eigenvalue $G_-$ is zero, the relevant bound states are in the flat $\mathbf{x}_i$ direction. When $G_-$ is turned on with value $1/C\Lambda^2$, these states appear as the Higgs and three Goldstone bosons, and the quarks acquire masses.

We believe that our solution should be of interest in the mass condition $m_t \ll \Lambda \sim 2$ TeV, which corresponds to one model (technicolor).

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