Group calibration is a byproduct of unconstrained learning

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Abstract

Much recent work on fairness in machine learning has focused on how well a score function is calibrated in different groups within a given population, where each group is defined by restricting one or more sensitive attributes.

We investigate to which extent group calibration follows from unconstrained empirical risk minimization on its own, without the need for any explicit intervention. We show that under reasonable conditions, the deviation from satisfying group calibration is bounded by the excess loss of the empirical risk minimizer relative to the Bayes optimal score function. As a corollary, it follows that empirical risk minimization can simultaneously achieve calibration for many groups, a task that prior work deferred to highly complex algorithms. We complement our results with a lower bound, and a range of experimental findings.

Our results challenge the view that group calibration necessitates an active intervention, suggesting that often we ought to think of it as a byproduct of unconstrained machine learning.

1 Introduction

The scholarly debate around fairness in machine learning happened to place much weight on a formal criterion known as group calibration. A risk score is calibrated for a group, if the risk score obviates the need to solicit group membership for the purpose of predicting an outcome variable of interest. More formally, group membership is conditionally independent of the outcome variable given the score. A violation of group calibration therefore corresponds to the possibly undesirable scenario where the same risk score represents different outcome distributions for the group and its complement.

The concept of calibration has a venerable history in statistics and machine learning [8, 25, 12, 13, 27, 33, 26]. The somewhat surprising appearance of calibration as a widely adopted and discussed “fairness criterion” largely resulted from a recent debate around fairness in recidivism prediction and pre-trial detention. After journalists at ProPublica pointed out that a popular recidivism risk score had a disparity in false positive rates between white defendants and black defendants [2], the organization that produced these scores countered that this disparity was a mere consequence of the fact that their scores were calibrated by race [14]. Formal trade-offs dating back the 1970s confirm the observed tension between group calibration and other involved classification criteria [11, 5, 23, 4].

Implicit in this debate is the view that calibration is a constraint worth enforcing as a means of promoting fairness. Consequently, many papers have come up with new methods of achieving group calibration in different settings [20, 22].

In this work, we show that group calibration is a routine consequence of unconstrained machine learning. As a result, it cannot address whatever concerns we have with existing practices that

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are largely instances of unconstrained machine learning. Indeed, many have criticized the ways in which unconstrained machine learning can have harmful societal impact [9, 3, 10]. However, our results strongly suggest that calibration alone cannot and will not alleviate these concerns.

The flip side of our findings is that calibration is fairly easy to achieve and typically does not necessitate an active intervention. In fact, we don’t even need to know exactly what groups we’re concerned with ahead of time. Provided that our model family can recognize group membership given the available features, group calibration will follow from empirical risk minimization.

Finally, we perform an extensive experimental verification of our theoretical findings on a standard machine learning dataset, demonstrating the strong connection between excess risk and group calibration, as well as effectiveness of empirical risk minimization in achieving approximate multi-calibration.

1.1 Our results

We begin with a simplified presentation of our results. As is common in supervised learning, consider a pair of random variables \((X, Y)\) where \(X\) models available features, and \(Y\) is a target variable that we try to predict from \(X\). Choose any discrete random variable \(A\) in the same probability space to model group membership. For example, \(A\) could represent gender, or race. In particular, our results do not require that \(X\) perfectly encodes the attribute \(A\).

A score function \(f\) maps the random variable \(X\) to a real number. We say that the score function \(f\) is calibrated by group \(A\) if we have 
\[
E[R(Y | f(X))] = E[R(Y | f(X), A)]
\]

In words, conditioning on \(A\) provides no additional information about \(Y\) beyond what was revealed by \(f(X)\). This definition leads to a natural notion of approximate group calibration:
\[
\Delta_f(A) = E[E[R(Y | f(X))] - E[R(Y | f(X), A)]],
\]

which measures the expected deviation from satisfying calibration over a random draw of \((X, A)\).

Denote by \(L(f) = E[\ell(f, Y)]\), the population risk of the score function \(f\). Think of the loss function \(\ell\) as either the square loss or the logistic loss, although our results apply more generally.

**Theorem 1.1** (Informal). For a broad class of loss functions that includes the square loss and logistic loss, we have
\[
\Delta_f(A) \leq O \left( \sqrt{L(f) - L^*} \right).
\]

Here, \(L^*\) is the calibrated Bayes risk, i.e., the loss of the score function \(f^B(x, a) = E[Y | X = x, A = a]\).

The theorem shows that if we manage to find a score function with small excess risk over the calibrated Bayes risk, then the score function will also be reasonably well calibrated with respect to the attribute \(A\). In particular, in light of our theorem, a natural strategy for achieving group calibration is plain empirical risk minimization as is the most common supervised learning routine.

Now, supervised learning may not succeed in achieving small excess risk, \(L(f) - L^*\). After all, it is important to note that we define the calibrated Bayes score \(f^B\) as one that has access to both \(X\) and \(A\). Since the calibrated Bayes score is just the expectation of \(Y\) conditional on both \(X\) and \(A\), it satisfies calibration by group \(A\). In cases where the available features \(X\) do not encode \(A\), but \(A\) is relevant to the prediction task, the excess risk may be large. In other cases, the excess risk may be large simply because the function class over which we can feasibly optimize provides only poor approximations to the calibrated Bayes score.

The constant in front of the square root in our theorem depends on properties of the loss function, and is typically small, e.g., bounded by \(2\sqrt{2}\) for both the squared loss and the logistic
loss. The more significant question is if the square root is necessary. We answer this question in the affirmative.

**Theorem 1.2** (Informal). There is a triple of random variables \((X, A, Y)\) such that the empirical risk minimizer \(\hat{f}_n\) trained on \(n\) samples drawn i.i.d. from \((X, Y)\) satisfies \(\Delta_{\hat{f}_n}(A) \geq \Omega(1/\sqrt{n})\) and \(\mathcal{L}(\hat{f}_n) - \mathcal{L}^* \leq O(1/n)\) with probability \(\Omega(1)\).

In other words, our upper bound gives a sharp characterization of the relationship between excess risk and calibration. Moreover, our lower bound applies not such to the empirical risk minimizer \(\hat{f}_n\), but to any score learned from data which is a linear function of the features \(X\). Although group calibration is a natural consequence of unconstrained learning, it is in general untrue that group calibration implies a good predictor. Indeed, one can often find highly pathological score functions which nevertheless satisfy group calibration (see e.g. [20]).

**Experimental evaluation.** We explore how well-calibrated the result of empirical risk minimization is via comprehensive experiments on the UCI Adult dataset [15]. For various choices of group attributes, we observe that the empirical risk minimizer is fairly close to being calibrated. This is true even if we look at combinations of features so long as these correspond to large enough groups in the population.

### 1.2 Related work

Group calibration as a fairness criteria (Condition 1) was first introduced as early as the 1960s in the education testing literature. Calibration was formalized by the Cleary criterion [6, 7], which compares the slope of regression lines between the test score and the outcome in different groups. More recently, machine learning and data mining communities have rediscovered calibration, and examined the inherent tradeoffs between calibration and other fairness constraints [5, 23]. Chouldechova ’16 and Kleinberg et al. ’16 independently demonstrate that exact group calibration is incompatible with *equalized odds* (equal true positive and false positive rates), except under highly restrictive situations such as perfect predictions. Such impossibility results have been further explored in [28] by considering relaxed equalized odds.

There are multiple post-processing procedures which achieve calibration, [see e.g. 26, and references therein]. Notably, Platt scaling [27] learns calibrated probabilities for a given score function by logistic regression. Recently, Hebert-Johnson et al. ’18 [20] proposed a polynomial time learning algorithm that achieves both low prediction error, and *multi-calibration*, or simultaneous group calibration with respect to multiple, possibly overlapping, groups. Building upon the finding in [20], this work shows that low prediction error often implies multi-calibration with no additional computational cost, under very general conditions. Unlike [20], our results apply even when the attribute \(A\) may not be a function of the available classifier features \(X\).

In addition to group calibration, a variety of other fairness criteria have been proposed to address concerns of fairness with respect to a sensitive attribute. These are typically group parity constraints on the score function, including, among others, *demographic parity* (as known as *statistical parity*) [see e.g. 17], *equalized odds* [e.g. 18] (also known as *error-rate balance* [e.g. 5]), and *group calibration* (also known as *sufficiency* [e.g. 4] and *calibration* [e.g. 23, 28]), which is the main topic of investigation in this work. Beyond parity constraints, recent works have also studied dynamic aspects of fairness, such as the delayed impact of model predictions on welfare [24] and population proportion [19].
2 Formal setup and results

We consider the problem of finding a score function \( \hat{f} \) which encodes the probability of a binary outcome \( Y \in \{0, 1\} \), given access to a features \( X \in \mathcal{X} \). We assume that individuals’ features and outcomes \((X, Y)\) are random variables whose law is governed by a probability measure \( D \) over a probability space \( \Omega \), and will view functions \( f \) as maps \( \Omega \rightarrow [0, 1] \). We use \( \Pr_D[\cdot], \Pr[\cdot] \) to denote the probability of events under \( D \), and \( \mathbb{E}_D[\cdot], \mathbb{E}[\cdot] \) to denote expectation taken with respect to \( D \).

We also consider a \( D \)-measurable protected attribute \( A \in \mathcal{A} \), with respect to which we want to ensure group calibration, which states that

\[
\forall r \in [0, 1], a \in \mathcal{A}, \quad \Pr[Y = 1 \mid f = r, A = a] = \Pr[Y = 1 \mid f = r].
\]

For example, \( A \) may be the attribute that picks out the race, gender, or socioeconomic status of an individual. See Section 1.2 for a list of terms that have been used to refer to Condition 1 in the literature. In this work, we will use the terms “calibration” and “group calibration” interchangeably.

We assume that \( f = f(X) \) for all \( f \in \mathcal{F} \), but will compare the performance of \( f \) to the benchmark that we call the calibrated Bayes score

\[
f^B(x, a) := \mathbb{E}[Y \mid X = x, A = a], \tag{2}
\]

which depends on both the feature \( x \) and the attribute \( a \). As a consequence, \( f^B \notin \mathcal{F} \), except possibly whenever \( Y \perp X \mid A \). Nevertheless, \( f^B \) is well defined as a map \( \Omega \rightarrow [0, 1] \) and it always satisfies (1), as the following result shows:

**Proposition 2.1.** \( f^B \) is group calibrated, that is, satisfies (1). Moreover, if \( \Phi : \mathcal{X} \rightarrow \mathcal{X}' \) is any map, then the classifier \( f_{\Phi}(X) := \mathbb{E}[Y \mid \Phi(X), A] \) is group calibrated.

Proposition 2.1 is a simple consequence of the tower property (proof in Appendix A.1). In general, there are many challenges to learning perfectly group calibrated scores. As mentioned above, \( f^B \) depends on information about \( A \) which is not necessarily accessible to scores \( f \in \mathcal{F} \). Moreover, even in the setting where \( A = A(X) \), it may still be the case that \( \mathcal{F} \) is a restricted class of score, and \( f^B \notin \mathcal{F} \). Lastly, if \( \hat{f} \) is estimated from data, perfect calibration may only be possible with an infinite number of samples. To this end, we introduce the following approximate notion of calibration:

**Definition 1.** Given a \( D \)-measurable attribute \( A \in \mathcal{A} \), we define the **calibration error** of \( f \) with respect to \( A \) as

\[
\Delta_f(A) = \mathbb{E}_D \left[ \mathbb{E}[Y \mid f(X)] - \mathbb{E}[Y \mid f(X), A] \right]. \tag{3}
\]

Of particular interest are scores \( f \) which are obtained by standard learning procedures, notably empirical risk minimization (ERM). In ERM, one introduces a loss function \( \ell : [0, 1] \times \{0, 1\} \rightarrow \mathbb{R} \), and, given a finite set of examples \( S^n := \{(X_i, Y_i)\}_{i=1}^n \) and set of scores \( \mathcal{F} \), returns a score function

\[
\hat{f} \in \arg \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \ell(f(X_i), Y_i). \tag{4}
\]

It is well known that, under very general conditions, \( \mathbb{E}[\ell(\hat{f}(X), Y)] \overset{\text{prob}}{\rightarrow} \min_{f \in \mathcal{F}} \mathbb{E}[\ell(f(X), Y)] \); that is, \( \mathcal{L}(\hat{f}) \) converges in probability to the least expected loss of any score function \( f \in \mathcal{F} \). One may also wonder if an analogue is true for calibration—that the more data one collects, the more well calibrated \( \hat{f} \) is. The main motivation behind the present work is:

**Under what conditions can one guarantee that a score function \( f \) is close to being calibrated, in the sense that \( \Delta_f \) is small? In particular, when is the output of empirical risk minimization close to being group calibrated?**
2.1 Upper bounds

We now state our main results, which show that the calibration error of a function \(f\) can be controlled by its loss, relative to the calibrated Bayes score \(f^B\). All proofs are deferred to Section 3.1. Throughout, we let \(\mathcal{F}\) denote a class of score functions \(f : \mathcal{X} \to [0,1]\). For a loss function \(\ell : [0,1] \times \{0,1\} \to \mathbb{R}\) and any \(\mathcal{D}\)-measurable \(f : \Omega \to [0,1]\), recall the population risk

\[
L(f) := \mathbb{E}[\ell(f,Y)].
\]

Note that for \(f \in \mathcal{F}\), \(L(f) = \mathbb{E}[\ell(f(X),Y)]\), whereas for the calibrated Bayes score \(f^B\), we denote its population risk as \(L^* := L(f^B) = \mathbb{E}[\ell(f^B(X,A),Y)]\). We further assume that our losses satisfy the following regularity condition:

**Assumption 1.** Given a probability measure \(\mathcal{D}\), we assume that \(\ell(\cdot, \cdot)\) is (a) \(\kappa\)-strongly convex: \(\ell(z,y) \geq \kappa(z-y)^2\), (b) there exists a differentiable map \(g : \mathbb{R} \to \mathbb{R}\) such that \(\ell(z,y) = g(z) - g(y) - g'(z)(z-y)\) (that is, \(\ell\) is a Bregman Divergence), and (c) the calibrated Bayes score is a critical point of the population risk, that is

\[
\mathbb{E} \left[ \frac{\partial}{\partial z} \ell(z,Y) \bigg| z = f^B \right] = 0.
\]

Assumption 1 is satisfied by common choices for the loss function, such as the square loss \(\ell(z,y) = (z-y)^2\) with \(\kappa = 1\), and the logistic loss, as shown by the following lemma, proved in Appendix A.2.

**Lemma 2.2 (Logistic Loss).** The logistic loss \(\ell(f,Y) = -(Y \log f + (1-Y) \log(1-f))\) satisfies Assumption 1 with \(\kappa = 2/\log 2\).

We are now ready to state our main theorem, which provides a simple bound on the calibration error \(\Delta_f(A)\) in terms of the excess risk \(L(f) - L^*\):

**Theorem 2.3 (Calibration is Upper Bounded by Excess Risk).** Suppose the loss function \(\ell(\cdot, \cdot)\) satisfies Assumption 1 with parameter \(\kappa > 0\). Then, for any score \(f \in \mathcal{F}\) and any attribute \(A\),

\[
\Delta_f(A) \leq 2 \sqrt{\frac{L(f) - L^*}{\kappa}}.
\]

Theorem 2.3 applies to any \(f \in \mathcal{F}\), regardless of how \(f\) is obtained. In particular, we immediately conclude the following corollary for any score computed by the ERM algorithm:

**Corollary 2.4 (Calibration of ERM).** Let \(\hat{f}\) be the output of any learning algorithm (e.g. ERM) trained on a sample \(S^n \sim \mathcal{D}^n\), and let \(L(f)\) be as in Theorem 2.3. Then, if \(\hat{f}\) satisfies the guarantee

\[
\Pr_{S^n \sim \mathcal{D}^n} \left[ L(\hat{f}) - \min_{f \in \mathcal{F}} L(f) \geq \epsilon \right] \leq \delta,
\]

and if \(\ell\) satisfies Assumption 1 with parameter \(\kappa > 0\), then it holds that

\[
\Pr_{S^n \sim \mathcal{D}^n} \left[ \Delta_f(A) \geq 2 \sqrt{\frac{\epsilon + \min_{f \in \mathcal{F}} L(f) - L^*}{\kappa}} \right] \leq \delta.
\]
The above corollary states that if there exists score in the function class \( \mathcal{F} \) whose population risk \( \mathcal{L}(f) \) is close to that of the calibrated Bayes optimal \( \mathcal{L}^* \), then ERM succeeds in finding a well-calibrate score.

In order to apply Corollary 2.4, one must know when the gap between the best-in-class risk and Bayes risk, \( \min_{f \in \mathcal{F}} \mathcal{L}(f) - \mathcal{L}^* \), is small. In the full information setting where \( A = A(X) \) (that is, the group attribute is available to the score function), \( \mathcal{L}^* \) corresponds to the Bayes risk and \( \min_{f \in \mathcal{F}} \mathcal{L}(f) - \mathcal{L}^* \) corresponds to the approximation error for the class \( \mathcal{F} \). But when \( X \) may not contain all the information about \( A \), however, \( \min_{f \in \mathcal{F}} \mathcal{L}(f) - \mathcal{L}^* \) depends not only on the class \( \mathcal{F} \) but also on how well \( A \) can be encoded by \( X \) given the class \( \mathcal{F} \), and possibly additional regularity conditions.

Here we provide three examples of conditions under which we can meaningfully bound the excess risk (or the calibration error directly) in the incomplete information setting. As a benchmark, we introduce the uncalibrated Bayes optimal score

\[
 f^U(x) := \mathbb{E}[Y | X = x],
\]

which is minimizes empirical over all \( X \) measurable functions, and is necessarily in \( \mathcal{F} \). Our first example gives a decomposition of \( \mathcal{L}(f) - \mathcal{L}^* \) when \( \ell \) is the square loss.

**Example 2.1.** Let \( \ell(z, y) := (z - y)^2 \) denote the squared loss. Then,

\[
 \mathcal{L}(\hat{f}) - \mathcal{L}^* = \left( \mathcal{L}(\hat{f}) - \inf_{f \in \mathcal{F}}(f) \right) + \left( \inf_{f \in \mathcal{F}} \mathcal{L}(f) - \mathcal{L}(f^U) \right) + \mathbb{E} \left[ \text{Var}_A[f^B | X] \right],
\]

(5)

where \( \text{Var}_A[f^B | X] = \mathbb{E}_X[(f^B - \mathbb{E}_A[f^B | X])^2] \) denotes the conditional variance of \( f^B \) given \( X \).

The decomposition in Example 2.1 follows immediately from the fact that the excess risk of \( f^U \) over \( f^B, \mathcal{L}(f^U) - \mathcal{L}^* \), is precisely \( \text{Var}_A[f^B | X] \) when \( \ell \) is the square loss. Examining (5), (i) represents the excess risk of \( f \) over the best score in \( \mathcal{F} \), which tends to zero if \( f \) is obtained by ERM, (ii) Captures the richness of the function class, for as \( \mathcal{F} \) contains a close approximation to \( f^U \). If \( \hat{f} \) is obtained by a consistent non-parametric learning procedure, and \( f^U \) has small complexity, then both (i) and (ii) tend to zero in the limit of infinite samples. Lastly, (iii) captures the additional information about \( A \) contained in \( X \). Note that in the full information zero, this term is zero.

In the next example, we examine calibration when \( \hat{f} \) is precisely the uncalibrated Bayes score \( f^U \). The following lemma establishes an upper bound on the calibration error of the uncalibrated Bayes score in terms of the conditional mutual information between \( Y \) and \( A \), conditioning on \( X \). It is proved in Appendix A.3 as a simple consequence of Tao’s inequality.

**Example 2.2** (Calibration bound for uncalibrated Bayes score). Suppose \( X \) and \( A \) are discrete \( \mathcal{D} \)-measurable random variables, and \( \mathcal{F} \) is the set of all functions \( f : X \rightarrow [0, 1] \). Denote \( f^* = \arg\min_{f \in \mathcal{F}} \mathcal{L}(f) \). Then, under Assumption 1, \( f^* = \mathbb{E}[Y | X] \) and

\[
 \Delta_{f^*}(A) \leq \sqrt{2\log 2I(Y; A | X)}.
\]

Lastly, we consider an example when the attribute \( A \) is continuous, and there exists a function \( g \) which approximately predicts \( A \) from \( X \). A formal statement and proof is given in Appendix A.4.

**Example 2.3** (Calibration bound for continuous group attribute \( A \)). Suppose there exists \( g : \mathcal{X} \rightarrow A \) such that \( \mathbb{E}[|A - g(X)|] \leq \delta_1 \). Let \( \tilde{\mathcal{F}} = \{ f : \mathcal{X} \times A \rightarrow [0, 1] \text{s.t. } f(x, g(x)) \in \mathcal{F} \} \). Denote \( f^* = \arg\inf_{f \in \mathcal{F}} \mathcal{L}(f) \) and \( \hat{f} = \arg\inf_{f \in \tilde{\mathcal{F}}} \mathcal{L}(f) \). Then, under regularity conditions, there exists a constant \( c \) such that

\[
 \Delta_{f^*}(A) \leq \min_{f \in \mathcal{F}} \mathcal{L}(f) - \mathcal{L}(f^B) + c\delta_1.
\]
The above result shows that in the incomplete information setting we may be able to bound the calibration error of the population risk minimizer of class $\mathcal{F}$ by the approximation error for an auxiliary class $\tilde{\mathcal{F}}$ up to an additional error term that accounts for how well $g(X)$ predicts $A$. In other words, a score obtained by ERM will have low calibration error with respect to any group attribute $A$ that is sufficiently encoded in the features that are used by the score, $X$, up to the flexibility allowed by the chosen class $\mathcal{F}$. Thus, our theoretical results also suggest that ERM can achieve approximate multi-calibration, with respect to all such group attributes.

### 2.2 Lower bound

We now present a lower bound which verifies that the upper bounds described in Theorem 2.3 and Corollary 2.4 correctly depict the relationship between group calibration and excess risk. We now describe an instance which witnesses our lower bound. Let $S^1 := \{w \in \mathbb{R}^2 : \|w\|_2 = 1\}$ be the circle in $\mathbb{R}^2$. For each $w \in \mathbb{R}^2$, we consider the following affine score functions $f_w : \mathbb{R}^2 \to \mathbb{R}$ and attributes $A_w \in \{-1, 0, 1\}$:

$$ f_w(X) := \frac{1}{2} + \frac{(w, X)}{4} \quad \text{and} \quad A_w := \text{sign}((X, w)). $$

We note that $f_w(X) \in [\frac{1}{4}, \frac{3}{4}]$ whenever $w, X \in S^1$, and that our attributes are functions of our features. Lastly, we let $\mathcal{F} := \{f_w : w \in \mathbb{R}^2\}$, and for $w_\ast \in S^1$, we let $\mathcal{D}^{w_\ast}$ denote the joint distribution where

$$ \mathcal{D}^{w_\ast} := X \overset{\text{unif}}{\sim} S^1 \text{ and } Y \mid X = \text{Bernoulli}(f_{w_\ast}(X)). $$

Observe that since $A_w = A_w(X)$, we see that the calibrated Bayes score under the distribution $\mathcal{D}^{w_\ast}$ is just $f^B(X, A) = \mathbb{E}_{\mathcal{D}^{w_\ast}}[Y \mid X, A] = \mathbb{E}_{\mathcal{D}^{w_\ast}}[Y \mid X] = f_{w_\ast}(X)$, and thus $f^B \in \mathcal{F}$.

Lastly, we shall let $\Delta_{f;\mathcal{D}^{w_\ast}}(A_w)$ denote the calibration error of $f$ with respect to $\mathcal{D}^{w_\ast}$. The formal statement of our lower bound is as follows.

**Theorem 2.5 (Lower bound).** Let $f_w, \mathcal{F}, \mathcal{D}^{w_\ast}, A_w$, and $\Delta_{f;\mathcal{D}^{w_\ast}}(A_w)$ be as above. Then,

(a) For any classifier $\hat{f} \in \mathcal{F}$ trained on a sample $S^n := \{(X_i, Y_i)\}_{i=1}^n$, any $\delta_1 \in (0, 1)$,

$$ \mathbb{E}_{w_\ast, w \overset{\text{unif}}{\sim} S^1} \Pr_{S^n \sim \mathcal{D}^{w_\ast}} \left[ \Delta_{\hat{f};\mathcal{D}^{w_\ast}}(A_w) \leq \frac{1}{4\pi} \min \left\{ 1, \sqrt{\frac{3\log(1/\delta_1)}{2n}} \right\} \right] \leq 1 - \frac{\delta_1}{4}. \tag{6} $$

(b) Let $\hat{f}_n$ denote the ERM under the square loss

$$ \hat{f}_n := \arg \min_{f \in \mathcal{F}} \sum_{(X_i, Y_i) \in S^n} (f(X_i) - Y_i)^2. $$

Then (6) holds even when $\mathbb{E}_{w_\ast, w \overset{\text{unif}}{\sim} S^1}$ is replaced by a supremum $\sup_{w_\ast, w \in S^1}$. Moreover, for any $\delta_2 \in (0, 1)$ and $w_\ast \in S^1$,

$$ \Pr_{S^n \sim \mathcal{D}^{w_\ast}} \left[ \mathcal{L}_{\mathcal{D}^{w_\ast}}(\hat{f}_n) - \mathcal{L}^* \leq \frac{8 + 6\log(1/\delta_2)}{n} \right] \geq 1 - \delta_2 - 2e^{-\frac{n^2}{8 + (4/3)\delta_2 n}}, $$

where $\mathcal{L}_{\mathcal{D}^{w_\ast}}(f) = \mathbb{E}_{(X, Y) \sim \mathcal{D}^{w_\ast}}[(f(X) - Y)^2]$ denotes population risk under the square loss, and $\mathcal{L}^* = \mathcal{L}_{\mathcal{D}^{w_\ast}}(f_{w_\ast})$ denotes the (calibrated) Bayes risk.
Let’s unpack the implications of Theorem 2.5. First, by choosing $\delta_1 = 1/2$ and $\delta_2 = 1/16$, we see that there are constants $c_1, c_2 > 0$ such that

$$\Pr \left( \left\{ \mathcal{L}_{D^*}(\hat{f}_n) - \mathcal{L}^* \leq \frac{c_1}{n} \right\} \cap \left\{ \Delta_{\hat{f};D^*}(A_w) \geq \frac{c_2}{\sqrt{n}} \right\} \right) \geq \frac{1}{16} - o(n).$$

Stated otherwise, we see that as $n \to \infty$,

$$\frac{\Delta_{\hat{f};D^*}(A_w)}{\sqrt{\mathcal{L}_{D^*}(\hat{f}_n) - \mathcal{L}^*}} = \Omega(1) \text{ with constant probability.}$$

On the other hand, by our upper bound, Theorem 2.3, we have

$$\frac{\Delta_{\hat{f};D^*}(A_w)}{\sqrt{\mathcal{L}_{D^*}(\hat{f}_n) - \mathcal{L}^*}} = O(1) \text{ with probability 1,}$$

matching our lower bound.

Moreover, our lower bound shows that for any proper score estimator $\hat{f}$, one needs $\Omega(\log(1/\delta_1))$ samples to ensure that $\Delta_{\hat{f};D^*}(A_w) \leq \epsilon$, a rate similar to those obtained in agnostic PAC learning. We emphasize that this lower bound applies to any proper learning procedure, even ones that do not directly aim to minimize the empirical risk.

At a high level, our lower bound proceeds by reduction to binary hypothesis testing. We provide a proof sketch of Theorem 2.5 in Section 3.3 and defer the rest of the details to Appendix B.

### 3 Proofs of main results

#### 3.1 Proofs of upper bounds

Throughout, we consider a fixed distribution $D$ and attribute $A$. We shall also use the shorthand $f = f(X)$ and $f^B = f^B(X, A)$. We begin by proving the following lemma, which establishes Theorem 2.3 in the case where $f$ is the squared loss:

**Lemma 3.1.** Let $f^B$ be the Bayes classifier, and let $f$ denote any function. Then,

$$\Delta_f \leq 2\sqrt{\mathbb{E}_{X,A}[(f - f^B)^2]} \quad (7)$$

To conclude the proof of Theorem 2.3, we first show that $\mathbb{E}[\ell(f, f^B)] = \mathbb{E}[\ell(f, Y)] - \mathbb{E}[\ell(f^B, Y)]$. Indeed Since $\ell$ is a Bregman divergence and calibrated at $f^B$,

$$\mathbb{E}[\ell(f, f^B)] = \mathbb{E}[\ell(f, Y)] - \mathbb{E}[\ell(f^B, Y)] + \mathbb{E}[(g'(Y) - g'(f^B)) \cdot (f - f^B)]$$

$$= \mathbb{E}[\ell(f, Y)] - \mathbb{E}[\ell(f^B, Y)] - \mathbb{E}[\mathbb{E}[\nabla_f \ell(f, Y) | f^B, X, A] \cdot (f - f^B)]$$

$$= \mathbb{E}[\ell(f, Y)] - \mathbb{E}[\ell(f^B, Y)]$$

Moreover, by strong convexity, we have that $\mathcal{L}(f) \geq \frac{1}{\kappa} \mathbb{E}[(f - Y)^2]$. Thus,

$$\kappa \mathbb{E}[(f - f^B)^2] \leq \mathbb{E}[\ell(f, f^B)] = \mathbb{E}[\ell(f, Y)] - \mathbb{E}[\ell(f^B, Y)] = \mathcal{L}(f) - L(f^B).$$

Applying Lemma 3.1 concludes the proof.
3.2 Proof of Lemma 3.1

First, we bound the $L_2$ difference of the conditional expectations. Note that since $f = f(X)$,

$$
\mathbb{E}[Y | f, A] = \mathbb{E}[\mathbb{E}[Y | X, A, f] | f, A] = \mathbb{E}[\mathbb{E}[Y | X, A] | f, A] = \mathbb{E}[f_B | f, A].
$$

Moreover, by the definition of $f^B$

$$
\mathbb{E}[Y | f^B, A] = \mathbb{E}[\mathbb{E}[Y | A, X, f^B], f^B, A] = \mathbb{E}[\mathbb{E}[Y | A, X, \mathbb{E}[Y | A, X]]
$$

and thus, by (8) and (9), we have

$$
\mathbb{E}_{X,A}[ (\mathbb{E}[Y | f, A] - \mathbb{E}[Y | f^B, A])^2 ] = \mathbb{E}_{X,A}[ (\mathbb{E}[f_B | f, A] - f_B)^2 ] \quad \text{by (8) and (9)}
$$

$$
= \mathbb{E}_{X,A}[ (\mathbb{E}[f_B - f | f, A] + f - f_B)^2 ]
$$

$$
\leq 2\mathbb{E}_{X,A}[ (\mathbb{E}[f_B - f | f, A])^2 + (f - f_B)^2 ]
$$

$$
\leq 2\mathbb{E}_{X,A}[ (f_B - f)^2 | f, A] + (f - f_B)^2 ] \quad \text{Jensen's inequality}
$$

$$
= 4\mathbb{E}_{X,A}[ (f - f_B)^2 ] .
$$

Now, define the cross term

$$
K := \mathbb{E}_{X,A} \left[ (\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B]) \cdot \mathbb{E}[Y | f, A] - \mathbb{E}[Y | f^B, A] \right]
$$

$$
= \mathbb{E}_{X} \left[ (\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B]) \cdot \mathbb{E}_{A}[\mathbb{E}[Y | f, A] - \mathbb{E}[Y | f^B, A]] \right]
$$

$$
= \mathbb{E}_{X} \left[ (\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B])^2 \right] .
$$

(11)

We then find that

$$
\Delta_f = \Delta_f - \Delta_f^B
$$

$$
= \mathbb{E}_{X,A}[|\mathbb{E}[Y | f] - \mathbb{E}[Y | f, A]| - |\mathbb{E}[Y | f^B] - \mathbb{E}[Y | f^B, A]|]
$$

$$
= \mathbb{E}_{X,A}[|\mathbb{E}[Y | f] - \mathbb{E}[Y | f, A]| - |\mathbb{E}[Y | f^B] - \mathbb{E}[Y | f^B, A]|]
$$

$$
= \mathbb{E}_{X,A}\left[ \sqrt{(\mathbb{E}[Y | f] - \mathbb{E}[Y | f, A] - (\mathbb{E}[Y | f^B] - \mathbb{E}[Y | f^B, A]))^2} \right]
$$

$$
\leq \sqrt{\mathbb{E}_{X,A}[(\mathbb{E}[Y | f] - \mathbb{E}[Y | f, A] - (\mathbb{E}[Y | f^B] - \mathbb{E}[Y | f^B, A]))^2]}
$$

$$
= \sqrt{\mathbb{E}_{X,A}[ (\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B)]^2 + \mathbb{E}_{X,A}[(\mathbb{E}[Y | f, A] - \mathbb{E}[Y | f^B, A])] - 2K .}
$$

Using the fact that $K = \mathbb{E}_{X}[(\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B])^2]$ (see (11)), we conclude

$$
\Delta_f = \sqrt{\mathbb{E}_{X,A}[(\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B)]^2 + \mathbb{E}_{X,A}[(\mathbb{E}[Y | f, A] - \mathbb{E}[Y | f^B, A])] - 2\mathbb{E}_{X}[(\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B])^2]}
$$

$$
= \sqrt{\mathbb{E}_{X,A}[(\mathbb{E}[Y | f, A] - \mathbb{E}[Y | f^B, A])] - \mathbb{E}_{X}[(\mathbb{E}[Y | f] - \mathbb{E}[Y | f^B)]^2]}
$$

$$
\leq 2\sqrt{\mathbb{E}_{X,A}[(f - f_B)^2]}
$$

(by (10))

3.3 Proof sketch of lower bound

In this section, we shall sketch a proof of the lower bound (6) in Theorem 2.5. The properties of the least squares estimator are standard, and deferred to Appendix B.1. Since $\hat{f} \in \mathcal{F}$, we shall write $\hat{f} = f_{\hat{w}}$ for some $\hat{w} \in \mathbb{R}^2$. We begin by given a precise characterization of the calibration error:
Lemma 3.2. Let \( w, w \in S^1 \), and suppose that either \( \hat{w} = 0 \), or \( \text{span}(\hat{w}, w) = \mathbb{R}^2 \). Then, for \((X, Y) \sim D^{w^*}\),

\[
\Delta_{f_{\hat{w}}, D^{w^*}}(A_w) = \frac{\sqrt{\Phi(\hat{w}; w)}}{2\pi}, \quad \text{where} \quad \Phi(\hat{w}; w^*) = \begin{cases} 
1 - \langle w^*, \hat{w} \rangle^2 & \hat{w} \neq 0 \\
1 & \hat{w} = 0
\end{cases}
\]

At the heart of Lemma 3.2 is noting that when \( \hat{w} \) and \( w \) are linearly independent, then \( f_{\hat{w}}(X) \) and \( A_w(X) = \text{sign}(\langle X, w \rangle) \) uniquely determine \( X \in S^1 \). Hence, \( \mathbb{E}[Y|f_{\hat{w}}(X), A_w(X)] = \mathbb{E}[Y|X] = f_{w^*}(X) \). In the proof of Lemma 3.2, we show that a similar simplification occurs in the case that \( \hat{w} = 0 \). Because the attribute \( A_w \) is independent of the distribution \( D^{w^*} \), and because \( w \sim S^1 \), we have that, for any \( w^* \),

\[
\Pr_{w, S^n \sim D^{w^*}}[\{\text{span}(w, \hat{w}) = \mathbb{R}^2\} \cup \{\hat{w} = 0\}] = 1
\]

so that the conditions of Lemma 3.2 hold with probability one.

Next, we observe that \( \Phi(\hat{w}; w^*) \) corresponds to the square norm of the projection of \( w^* \) onto a direction perpendicular to \( \hat{w} \), or equivalently, the square of the sign of the angle between \( \hat{w} \) and \( w^* \). Note that calibration can occur when the angle between \( \hat{w} \) and \( w^* \) is either close to zero, or close to \( \pi \)-radians; this is in contrast to prediction, where a small loss implies that the angle between \( \hat{w} \) and \( w^* \) is necessarily close to zero. Nevertheless, we can still prove an information theoretic lower bound on the probability that \( \Phi(\hat{w}; w^*) \) is small by a reduction to binary hypothesis testing. This is achieved in the next proposition:

**Proposition 3.3.** For any \( n \geq 1 \), \( \delta \in (0, 1) \), and any estimator \( \hat{w} \),

\[
\mathbb{E}_{w \sim S^1} \Pr_{S^n \sim D^{w^*}} \left[ \Phi(\hat{w}; w^*) \leq \min \left\{ \frac{1}{2}, \frac{3\log(1/\delta)}{n} \right\} \right] \leq 1 - \frac{\delta}{4}.
\]

The first part of Theorem 2.5, Equation (6), now follows immediately from combining the bound in Proposition 3.3, (12), and the computation of \( \Delta_{f_{\hat{w}}, D^{w^*}}(A_w) \) in Lemma 3.2. The proof of Lemma 3.2 and Proposition 3.3 are deferred to Sections B.3 and B.2, respectively.

### 4 Experiments

In this section, we show that the Adult dataset from the UCI Machine Learning Repository [15] corroborates our theoretical findings that small excess risk yields well-calibrated score functions. The Adult dataset contains 14 demographic features for 48842 individuals, with the goal of predicting whether one’s annual income is greater than $50,000. We first examine group calibration with respect to two sensitive attributes, gender and race. Then, in Section 4.3 we present results for multi-calibration, where a score is calibrated with respect to multiple sensitive attributes \((A_1, A_2, \ldots)\) simultaneously. Implementation details are described in Section 4.5.

#### 4.1 Calibration Metrics

We shall use two descriptions of group calibration. First, we measure group calibration by an empirical estimate of \( \Delta_f(A) \), described in further detail in Section 4.5. In addition, we give a more fine-grained visualization of group calibration using a so-called calibration plot [see e.g. 5], which plots observed positive outcome rates against score deciles. The shaded regions indicate 95% confidence intervals for the rate of positive outcomes under a binomial model.
In Figure 1 we show that our group calibration metric $\Delta_f(A)$ is consistent with the calibration levels depicted by a calibration plot: models with higher $\Delta_f$ correspond to calibration plots where the curves for different groups are far apart (Figure 1). In contrast, models with lower $\Delta_f$ correspond to calibration plots where the rate of positive outcomes for different groups are closer together.

![Calibration Graphs](image)

Figure 1: Less calibrated model with empirical $\Delta_f = 0.01039$ (left), well-calibrated model with empirical $\Delta_f = 0.00681$ (right)

### 4.2 Calibration improves with model accuracy and model flexibility

Our theoretical results suggest that risk of a score function is positively correlated with its calibration error. In general, it impossible to determine the excess risk of a given classifier with respect to the Bayes risk $\mathcal{L}^*$ from experimental data. Instead, we test whether the excess calibration error of a score trained by logistic regression decreases as the risk of that score decreases.

Specifically, we explore the effects of decreased risk on calibration error due to (a) increased number of training examples (Figure 2) and (b) increased expressiveness of the class $\mathcal{F}$ of score functions (Figure 3). As the number of training samples increases, the gap between the ERM and least-risk score function in a given class $\mathcal{F}$ decreases. On the other hand, as the number of model parameters grows, the class $\mathcal{F}$ becomes more expressive, and $\min_{f \in \mathcal{F}} \mathcal{L}(f)$ may become closer to the Bayes risk $\mathcal{L}^*$.

Figures 2 and 3 display, for each experiment, the calibration error and logistic loss on a test set averaged over 10 random trials, each using a randomly chosen training set. The shaded region in the figures indicates two standard deviations from the average value. In Figure 2, as the number of training examples increase, the logistic loss of a classifier decreases, and so does the calibration error. As expected given our theoretical analysis, the loss appears to track the calibration error, for both gender and race group attributes.

In Figure 3 (right), we gradually restrict the model class by reducing the number of features used in logistic regression. As the number of features decreases, the logistic loss increases and so
does the calibration error. In Figure 3 (left), we implicitly restrict the model class by varying the
regularization parameter. In this case, a smaller regularization parameter corresponds to more
severe regularization, which constrains the learned weights to be inside a smaller L1 ball. As we in-
crease regularization, the logistic loss increases and so does the calibration error. Both experiments
show that the calibration error is reduced when the model class is enlarged, again demonstrating
its tight connection to the excess risk.

4.3 Multicalibration

We observe that empirical risk minimization with logistic regression already achieves approximate
multicalibration, that is simultaneous calibration with respect to any group attributes defined on
the basis of the given features, not only the protected attributes—gender and race. In Figure 4, we
show the calibration of the ERM model with respect to Age, Education-Num, Workclass, and Hours per week. In Figure 5, we show that for combinations of two features. In each case, the confidence intervals for the rate of positive outcomes for all groups overlap at all, if not most, score deciles. In particular, Figure 5 (right) shows that the ERM model is close to group-calibrated even for a newly defined group attribute that is the intersectional combination of race and gender. The calibration plots for other features, as well as implementation details, can be found in Appendix D.1.

Figure 4: Calibration with respect to other group attributes
4.4 Training with group information has modest effects on calibration

Calibration can hold approximately even when the score is not a function of the group attribute. In Figures 6 and 7, we compare calibration when the group attribute is included as a feature for regression with that when it is not. Again, the shaded regions indicate 95% confidence intervals for the rate of positive outcomes under a binomial model. Without the group variable, the ERM model is only slightly less calibrated; the confidence intervals for both groups still overlap at every score decile.

Figure 6: Calibration with group as a feature
4.5 Implementation

We remark that our implementation is partially adapted from the code example in Chapter 2 of [4]. Score functions are trained with logistic regression on a training set that is 80% of the original dataset, unless otherwise stated. The calibration error $\Delta_f$ is estimated using a test set that is 20% of the original data.

**Empirical estimate of $\Delta_f$** We estimate $\Delta_f$ from the test set and the scores for this test set, that is $\{x_i, y_i, a_i, f(x_i)\}_{i=1}^{n}$. We divide the scores into deciles, that is, $B = 10$ equally spaced intervals on $[0, 1]$. For any score value $f \in [0, 1]$, let $d(f)$ denote the corresponding decile for $f$. We estimate $E[Y|f]$ as the average rate of positive outcomes in the corresponding decile of the score $f$, i.e.

$$\hat{g}(f) = \frac{1}{N} \sum_{i=1}^{n} y_i 1\{f(x_i) \in d(f)\},$$

where $N = \sum_{i=1}^{n} 1\{f(x_i) \in d(f)\}$. For any group $a$ and score $f$, we estimate $\approx E[Y|f, A = a]$ as the average rate of positive outcomes in the corresponding decile of the score $f$ in group $a$, i.e.

$$\hat{g}(f, a) = \frac{1}{M} \sum_{i=1}^{n} y_i 1\{f(x_i) \in d(f) \cap a_i = a\},$$

where $M = \sum_{i=1}^{n} 1\{f(x_i) \in d(f) \cap a_i = a\}$. $\hat{\Delta}_f$ is then computed as the sample average $\frac{1}{n} \sum_{i=1}^{n} |\hat{g}(f(x_i), a_i) - \hat{g}(f(x_i))|$. In general, the value of the estimate does vary with the chosen number of intervals $B$; we find that on the chosen dataset, $B = 10$ gives us an estimate that adequately captures the expected calibration error for experimental purposes. We leave the statistical properties of this estimator, including the ramifications of different $B$, to future work.

5 Conclusion and future work

In summary, our results show that group calibration follows from closeness to the Bayes optimal score function. Consequently, empirical risk minimization is a simple and efficient recipe for achiev-
ing group calibration, provided that (1) the function class is sufficiently rich, (2) there are enough training samples, and (3) the group attribute can be approximately predicted from the available features. Under these general conditions, active intervention is unnecessary to ensure calibration.

Importantly, our work suggests that group calibration does not and cannot solve fairness concerns that pertain to the Bayes optimal score function, such as disparate true (false) positive rates and disparate delayed impact, among others.

In addition, we have highlighted the role of the function class in ensuring group calibration and indeed, low excess risk relative to the calibrated Bayes score. Whereas restricted function class may yield lower sample complexity, richer function classes may yield better approximation to the Bayes score. Better understanding this tradeoff in the context of group fairness is an exciting direction for future work.

Another direction is to extend the current work to measures of calibration that enforce the calibration error to be uniform across groups. In this case, function class restriction may give rise to tradeoffs in accuracy between different groups, as discussed in [16].

References

[1] Rudolf Ahlswede. The final form of tao’s inequality relating conditional expectation and conditional mutual information’. Advances in Mathematics of Communications, 1:239, 2007.

[2] Julia Angwin, Jeff Larson, Surya Mattu, and Lauren Kirchner. Machine bias. ProPublica, May 2016. URL https://www.propublica.org/article/machine-bias-risk-assessments-in-criminal-sentencing.

[3] Solon Barocas and Andrew D Selbst. Big data’s disparate impact. UCLA Law Review, 2016.

[4] Solon Barocas, Moritz Hardt, and Arvind Narayanan. Fairness and Machine Learning. fairmlbook.org, 2018. http://www.fairmlbook.org.

[5] A. Chouldechova. Fair prediction with disparate impact: A study of bias in recidivism prediction instruments. Big Data, 5, 2017.

[6] T. Anne Cleary. Test bias: Validity of the scholastic aptitude test for negro and white students in integrated colleges. ETS Research Bulletin Series, 1966(2):i–23.

[7] T. Anne Cleary. Test bias: Prediction of grades of negro and white students in integrated colleges. Journal of Educational Measurement, 5(2):115–124, 1968.

[8] David R. Cox. Two further applications of a model for binary regression. Biometrika, 45(3-4):562–565, 1958.

[9] Kate Crawford. The hidden biases in big data. Harvard Business Review, 1, 2013.

[10] Kate Crawford. The trouble with bias. NIPS Keynote https://www.youtube.com/watch?v=fMym_BkWqzk, 2017.

[11] Richard B Darlington. Another look at “cultural fairness”. Journal of Educational Measurement, 8(2):71–82, 1971.

[12] A. P. Dawid. The well-calibrated bayesian. Journal of the American Statistical Association, 77(379):605–610, 1982.
[13] Morris H. DeGroot and Stephen E. Fienberg. The comparison and evaluation of forecasters. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 32(1/2):12–22, 1983.

[14] William Dieterich, Christina Mendoza, and Tim Brennan. Compas risk scales: Demonstrating accuracy equity and predictive parity, 2016. URL https://www.documentcloud.org/documents/2998391-ProPublica-Commentary-Final-070616.html.

[15] Dheeru Dua and Efi Karra Taniskidou. UCI machine learning repository, 2017. URL http://archive.ics.uci.edu/ml.

[16] Cynthia Dwork, Nicole Immorlica, Adam Tauman Kalai, and Max Leiserson. Decoupled classifiers for group-fair and efficient machine learning. In Sorelle A. Friedler and Christo Wilson, editors, *Proceedings of the 1st Conference on Fairness, Accountability and Transparency*, volume 81 of *Proceedings of Machine Learning Research*, pages 119–133, New York, NY, USA, 23–24 Feb 2018. PMLR.

[17] M. Feldman, S. Friedler, J. Moeller, C. Scheidegger, and S. Venkatasubramanian. Certifying and removing disparate impact. In *International Conference on Knowledge Discovery and Data Mining (KDD)*, pages 259–268, 2015.

[18] M. Hardt, E. Price, and N. Srebro. Equality of opportunity in supervised learning. In *Advances in Neural Information Processing Systems (NIPS)*, pages 3315–3323, 2016.

[19] T. B. Hashimoto, M. Srivastava, H. Namkoong, and P. Liang. Fairness without demographics in repeated loss minimization. In *International Conference on Machine Learning (ICML)*, 2018.

[20] Ursula Hebert-Johnson, Michael Kim, Omer Reingold, and Guy Rothblum. Multicalibration: Calibration for the (Computationally-identifiable) masses. In *Proceedings of the 35th International Conference on Machine Learning*, pages 1944–1953, Stockholm, Sweden, 2018.

[21] Daniel Hsu, Sham M Kakade, and Tong Zhang. An analysis of random design linear regression. Citeseer, 2011.

[22] Michael J. Kearns, Seth Neel, Aaron Roth, and Zhiwei Steven Wu. Preventing fairness gerrymandering: Auditing and learning for subgroup fairness. *CoRR*, abs/1711.05144, 2017. URL http://arxiv.org/abs/1711.05144.

[23] Jon M. Kleinberg, Sendhil Mullainathan, and Manish Raghavan. Inherent trade-offs in the fair determination of risk scores. *Proc. 8th ITCS*, 2017.

[24] Lydia T. Liu, Sarah Dean, Esther Rolf, Max Simchowitz, and Moritz Hardt. Delayed impact of fair machine learning. In *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 3156–3164, Stockholm, Sweden, 2018.

[25] Allan H. Murphy and Robert L. Winkler. Reliability of subjective probability forecasts of precipitation and temperature. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 26(1):41–47, 1977.

[26] Alexandru Niculescu-Mizil and Rich Caruana. Predicting good probabilities with supervised learning. In *Proceedings of the 22Nd International Conference on Machine Learning*, ICML
[27] John C. Platt. Probabilistic outputs for support vector machines and comparisons to regularized likelihood methods. In *Advances in Large Margin Classifiers*, pages 61–74. MIT Press, 1999.

[28] Geoff Pleiss, Manish Raghavan, Felix Wu, Jon Kleinberg, and Kilian Q Weinberger. On fairness and calibration. In *Advances in Neural Information Processing Systems 30*, pages 5684–5693, 2017.

[29] Max Simchowitz, Kevin Jamieson, and Benjamin Recht. Best-of-k-bandits. In *Conference on Learning Theory*, pages 1440–1489, 2016.

[30] Terence Tao. Szemerédi’s regularity lemma revisited. *Contributions to Discrete Mathematics*, 1, 2006.

[31] Joel A. Tropp. An introduction to matrix concentration inequalities. *Foundations and Trends® in Machine Learning*, 8(1-2):1–230, 2015.

[32] Alexandre B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer Publishing Company, Incorporated, 1st edition, 2008. ISBN 0387790519, 9780387790510.

[33] Bianca Zadrozny and Charles Elkan. Obtaining calibrated probability estimates from decision trees and naive bayesian classifiers. In *Proceedings of the Eighteenth International Conference on Machine Learning*, ICML ’01, pages 609–616, San Francisco, CA, USA, 2001. Morgan Kaufmann Publishers Inc. ISBN 1-55860-778-1. URL http://dl.acm.org/citation.cfm?id=645530.655658.
A  Additional Proofs

In this section we prove the claims and examples given in Section 2.

A.1 Proof of Proposition 2.1

Consider the full information setting $A = A(X)$. Then,

$$f_B(x,a) := \mathbb{E}[Y \mid X = x, A = a] = \mathbb{E}[Y \mid X = x]$$

which shows that $f_B$ depends only on the features $X$, and is therefore valid. Now, by the tower rule for conditional expectation,

$$\Pr[Y = 1 \mid f_B(X,A)] = \mathbb{E}[Y \mid f_B(X,A)] = \mathbb{E}[\mathbb{E}[Y \mid f_B(X,A), A] \mid f_B(X,A)] = \mathbb{E}[f_B(X,A) \mid f_B(X,A)] = f_B(X,A),$$

and, $\Pr[Y = 1 \mid f_B(X,A)] = \mathbb{E}[Y \mid f_B(X,A)] = \mathbb{E}[\mathbb{E}[Y \mid f_B(X,A), A] \mid f_B(X,A)] = f_B(X,A)$. Therefore, the calibrated Bayes score $f_B(X)$ is well-calibrated.

A.2 Proof of Lemma 2.2

To see that this is true, first note that $\ell(f, y)$ is a Bregman divergence. We can easily check that

$$\mathbb{E}[\nabla f \ell(f(x,a), Y)] = \mathbb{E}[\frac{Y}{f} - \frac{1-Y}{1-f}] = 0.$$  

Finally, $\kappa$-strong convexity follows from Pinkser's inequality for Bernoulli random variables:

$$(f - f')^2 \leq \frac{\log 2}{2} \left( f' \log \frac{f'}{f} + (1 - f') \log \frac{1 - f'}{1 - f} \right) = \frac{\log 2}{2} \ell(f, f').$$

A.3 Proof of Example 2.2

For $f = \mathbb{E}[Y \mid X]$, we have the following identity for $\Delta_f(A)$ by the tower rule:

$$\Delta_f(A) = \mathbb{E}[\mathbb{E}[Y \mid f] - \mathbb{E}[Y \mid f, A]] = \mathbb{E}[\mathbb{E}[Y \mid X] - \mathbb{E}[Y \mid X, A]].$$

By applying Tao's inequality [30, 1], we have that

$$\mathbb{E}[\mathbb{E}[Y \mid X] - \mathbb{E}[Y \mid X, A]] \leq \sqrt{2\log 2 I(Y; A \mid X)}$$

Note that $I(Y; (X,A) \mid X) = I(Y; A \mid X)$ and the result follows.

A.4 Restatement and proof of Example 2.3

We first restate Example 2.3 more formally in the following lemma:
Observe that $E$ where we let $∥\hat{x}∥$. We readily see that the distribution of $\tilde{w}$ depend on $Hence, the conclusion of (12) holds for any fixed $w \sim \text{unif } S^1$, so

The last inequality follows from the fact that $\forall y, \ell(\cdot, y)$ is $\frac{1}{\min \{\delta_2, 1 - \delta_3\}}$-Lipschitz on $[\delta_2, \delta_3]$, and that $\tilde{f}$ is only supported on $[\delta_2, \delta_3]$.  

\section{Supplementary Material for the Proof of Theorem 2.5}

\subsection{Analysis of $\hat{f}_n$}

We can write $\hat{f}_n = f_{\tilde{w}_LS}$, where

$$\tilde{w}_LS := \arg \min_w \sum_{(X_i,Y_i) \in S^n} (f_w(X_i) - Y_i)^2$$

We readily see that the distribution of $\tilde{w}_LS$ marginalized over $w_\ast \sim \text{unif } S^1$ is radially symmetric. Hence, the conclusion of (12) holds for any fixed $w \in S^1$.

Moreover, since $\Delta_{f_{\tilde{w}_LS}, D^w}(A_w) = \sqrt{\Phi(\tilde{w}_LS: w_\ast)}$, and both the least-squares algorithm and the error $\Phi(\cdot, \cdot)$ are radially symmetric, we see that for any $t$, $\Pr_{S^\ast \sim D^w} [\Delta_{f_{\tilde{w}_LS}, D^w}(A_w) \leq t]$ does not depend on $w_\ast \in S^1$ either.

It now suffices to prove an upper bound for least squares. We have that

$$\mathcal{L}(\hat{f}_n) - \mathcal{L}^* = \mathbb{E} \left[ (f_{\tilde{w}_LS}(A)(X) - f_{w_\ast}(X))^2 \right]$$

$$= \mathbb{E}[(X, \tilde{w}_LS - w_\ast)^2]$$

$$= \|\tilde{w}_LS - w_\ast\|^2_{\frac{1}{n} \Sigma[X]} T,$$

where we let $\|x\|_2^2 := x^\top \Sigma x$. Now, conditioning on $\{X_1, \ldots, X_n\}$, and let $\hat{\Sigma} := \frac{1}{n} \sum_{i=1}^n X_i X_i^\top$. Observe that $\mathbb{E}[Y \mid X_i] = \langle w_\ast, \frac{X_i}{n} \rangle$, and $Y_i - \mathbb{E}[Y \mid X_i]$ are independent random variables in $[0, 2)$, so
are 1-subgaussian by Hoeffding’s inequality. Hence, Hsu et al. [21, Proposition 1] with \( \sigma^2 = 1 \) and \( d = 2 \) implies that

\[
\Pr \left[ \left\| \hat{w}_{LS} - \hat{w}_{LS} \right\|^2 \leq \frac{4 + 3 \log(1/\delta)}{n} \mid \{X_1, \ldots, X_n\} \right] \\
\geq \Pr \left[ \left\| \hat{w}_{LS} - \hat{w}_{LS} \right\|^2 \leq \frac{2 + 2 \sqrt{2 \log(1/\delta) + 2 \log(1/\delta)}}{n} \mid \{X_1, \ldots, X_n\} \right] \geq 1 - \delta.
\]

where \((i)\) uses the elementary inequality \( ab \leq \frac{a^2 + b^2}{2} \). Lastly, we note that \( E[XX^T] = \frac{1}{2} I \), so on the event \( \lambda_{\min}(\Sigma) \geq \frac{1}{4} \), we have \( \left\| \hat{w}_{LS} - w_* \right\|^2 \leq \frac{1}{2} \left\| \hat{w}_{LS} - w_* \right\|^2 \). To this end, define \( M_i = E[XX^T] - X_iX_i^T = \frac{1}{2} I - X_iX_i^T \). Note that \( \lambda_{\max}(M_i) \leq \frac{1}{2} \) and \( E[M_i^2] = \frac{1}{4} I \). Hence, by the Matrix Bernstein inequality Tropp [31, Theorem 6.6.1], we have

\[
\Pr \left[ \lambda_{\min}(\Sigma) \leq \frac{1}{4} \right] = \Pr \left[ \lambda_{\max}(\sum_{i=1}^{n} M_i) \geq \frac{n}{4} \right] \leq 2e^{-\frac{\epsilon^2}{(7/2)\delta(7/2)}} \mid_{\epsilon = \frac{\sqrt{n^2}}{\delta}} = 2e^{-\frac{n^2}{8 \pi (1/3)n}}.
\]

Putting pieces together, we conclude that

\[
\Pr \left[ E \left[ (f_{\hat{w}_{LS}}(A)(X) - f_{w_*}(X))^2 \right] \leq \frac{8 + 6 \log(1/\delta)}{n} \right] \geq 1 - \delta - 2e^{-\frac{n^2}{8 \pi (1/3)n}}.
\]

**B.2 Proof of Information Theoretic Bound, Proposition 3.3**

Let \( R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denote the linear operator corresponding to rotation by \( \theta \). Our strategy will be to show that for any \( w \in \mathcal{S}^1 \), for

\[
\epsilon(n) = \sqrt{\min \left\{ \frac{1}{2}, \frac{3 \log(1/\delta)}{n} \right\}}, \quad (13)
\]

and some \( \theta = \theta(n) \), we have

\[
\frac{1}{2} \left( \Pr[\Phi(\hat{w}; w) \leq \epsilon(n)^2] + \Pr[\Phi(\hat{w}; R_\theta w) \leq \epsilon(n)^2] \right) \leq 1 - \frac{\delta}{4} \quad (14)
\]

Now, we can express \( w = R_\phi e_1 \) where \( e_1 = (1, 0) \) and \( \phi \in [0, 2\pi] \). Thus, taking an expectation over \( \phi \sim_{\text{unif}} [0, 2\pi] \), we observe that

\[
E_{\phi \sim_{\text{unif}} [0, 2\pi]} \Pr[\Phi(\hat{w}; R_\phi e_1) \leq \epsilon(n)^2] = E_{w_* \sim \mathcal{S}^1} \Pr[\Phi(\hat{w}; w_*) \leq \epsilon(n)^2],
\]

and similarly

\[
E_{\phi \sim_{\text{unif}} [0, 2\pi]} \Pr[\Phi(\hat{w}; R_\phi R_\phi e_1) \leq \epsilon(n)] = E_{\phi \sim_{\text{unif}} [0, 2\pi]} \Pr[\Phi(\hat{w}; R_\phi \phi e_1) \leq \epsilon(n)]
\]

\[
= E_{\phi \sim_{\text{unif}} [0, 2\pi]} \Pr[\Phi(\hat{w}; R_\phi w) \leq \epsilon(n)^2].
\]

Hence,

\[
1 - \frac{\delta}{4} \geq \frac{1}{2} E_{\phi \sim_{\text{unif}} [0, 2\pi]} \left[ \Pr[\Phi(\hat{w}; R_\phi w) \leq \epsilon(n)^2] + \Pr[\Phi(\hat{w}; R_\phi R_\phi e_1) \leq \epsilon(n)^2] \right]
\]

\[
= \frac{1}{2} \cdot 2E_{w_* \sim \mathcal{S}^1} \Pr[\Phi(\hat{w}; w_*) \leq \epsilon(n)^2], \quad \text{as needed.}
\]
We now turn to proving (14). By rotation invariance argument, it suffices to prove the inequality for \(w = e_1 = (1, 0)\). We now fix an \(\epsilon = \epsilon(n)\) as in (13) to be chosen later, and choose \(\theta = \arccos(1 - 2\epsilon^2)\). Note that \(\epsilon \in (0, \frac{1}{2}]\) implies \(\theta \in (0, \pi/2]\).

We construct two alternative instances \(w^{(1)} = e_1\), and let \(w^{(2)} = e_1 \cos \theta + e_2 \sin \theta\). We will establishe lower bound on the problem of testing between \(w = w^{(1)}\) and \(w = w^{(2)}\), and then translate this into a bound on \(\Phi(\tilde{w}; \cdot)\). The first step is a KL-divergence computation established in Section B.2.1:

**Lemma B.1.** There exists a constant \(K > 0\) such that, if \((D^{(w)}) \otimes n\) denote the distribution of \(n\) i.i.d. samples from \(D^{(w)}\), then

\[
KL((D^{(w)}) \otimes n, (D^{w'}) \otimes n) \leq \frac{n}{12} \|w - w'\|^2.
\]

In our setting, we see that

\[
\|w^{(1)} - w^{(2)}\|^2 = (1 - \cos \theta)^2 + \sin^2 \theta = 1 + \cos^2 \theta + \sin^2 \theta - 2 \cos \theta = 2(1 - \cos \theta) = 4\epsilon^2.
\]

Hence, we have that \(KL((D^{w^{(1)}) \otimes n}, (D^{w^{(2)})} \otimes n) \leq n\epsilon^2 \cdot \frac{3}{4}\). Therefore, given any estimator \(\hat{i}\) of \(i\), the proof of [Theorem 2.2.iii in [32]] reveals that

\[
\frac{1}{2} \sum_{i \in \{1, 2\}} \Pr_{(D^{w^{(i)})} \otimes n} \left[ \{\hat{i} \neq i\} \right] \geq \frac{1}{4} e^{- \frac{n\epsilon^2}{4}}.
\]

In particular, since \(\epsilon = \epsilon(n)^2 \leq \frac{3\log(1/\delta)}{n}\), as in (13), and considering the complement of \(\{\hat{i} \neq i\}\), we have

\[
\frac{1}{2} \sum_{i \in \{1, 2\}} \Pr_{(D^{w^{(i)})} \otimes n} \left[ \{\hat{i} = i\} \right] \leq 1 - \frac{1}{4} e^{- \log(1/\delta)} = 1 - \frac{\delta}{4}.
\]

(15)

Lastly, we show how a small value of \(\Phi(\tilde{w}; w^{(i)})\) yields an accurate estimator of \(\hat{i}\). Given an estimator \(\hat{w}\), we define the estimator of \(w^{(i)}\) give \(\hat{w}\), where \(\hat{i}\) is given by:

\[
\hat{i} \in \arg \min_{i \in \{1, 2\}} \Phi(w^{(i)}; w^{(i)}),
\]

where we arbitrarily choose \(\hat{i} = 1\) if both values of \(i\) attain the same value in the display above. The following lemma, proved in Section B.2.2, shows that gives a reduction from estimating \(i\) to obtaining a small value of \(\Phi^{(w^{(i)}, \tilde{w})}\):

**Lemma B.2.** For \(\theta \in [0, \frac{\pi}{2}]\), \(\Phi(w^{(i)}, \tilde{w}) < \sin^2 \frac{\theta}{2}\) implies that \(\hat{i} = i\).

Hence, combining Lemma B.2 with (15), we have

\[
\frac{1}{2} \sum_{i \in \{1, 2\}} \Pr[\Phi(w^{(i)}, \tilde{w}) < \sin \frac{\theta}{2}] \leq 1 - \frac{\delta}{4}.
\]

Lastly, we find that as \(\theta \in (0, \frac{\pi}{2})\), \(\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{2\epsilon^2}{2}} = \epsilon\), thereby concluding the proof.
B.2.1 Proof of Lemma B.1

By the tensorization of the KL-divergence,
\[ KL \left( (\mathcal{D}^w)^\otimes n, (\mathcal{D}^{w'})^\otimes n \right) = n KL (\mathcal{D}^w, \mathcal{D}^{w'}) \]
\[ = n \mathbb{E}_{X \sim S_1} KL (\text{Bernoulli}(f_w(X)), \text{Bernoulli}(f_{w'}(X))) , \]
Now we use a standard Taylor-expansion upper bound on the KL-divergence between two Bernoulli random variables (see, e.g. Lemma E.1 in [29]):

**Lemma B.3.** Let \( p, q \in (0, 1) \). Then,
\[ KL (\text{Bernoulli}(p), \text{Bernoulli}(q)) \leq \frac{(p - q)^2}{2 \min\{p(1 - p), q(1 - q)\}}. \]

In our setting, \( f_w(X) = \frac{1}{2} + \langle w, X \rangle / 4 \) \( \in \left[ \frac{1}{4}, \frac{3}{4} \right] \) because \( |\langle X, w \rangle| \leq \|X\| \|w\| / 4 = \frac{1}{4} \).

Hence, since \( 2 \min_{p \in [1/4, 3/4]} p(1 - p) = 2 \cdot \frac{1}{4} \cdot \frac{3}{4} = 3/8, \)
\[ KL (\text{Bernoulli}(f_w(X)), \text{Bernoulli}(f_{w'}(X))) \leq \frac{8 \|f_w(X) - f_{w'}(X)\|_2^2}{3} \]
Hence,
\[ KL \left( (\mathcal{D}^w)^\otimes n, (\mathcal{D}^{w'})^\otimes n \right) \leq n \cdot \frac{8}{3} \mathbb{E}_{X \sim S_1} \|f_w(X) - f_{w'}(X)\|_2^2 \\
= \frac{8n}{3} \mathbb{E}_{X \sim S_1} \left\| \left\langle \frac{w - w'}{4}, X \right\rangle \right\|_2^2 \\
= \frac{n}{6} (w - w')^\top \mathbb{E}_{X \sim S_1} [XX^\top] (w - w') = \frac{n}{12} \|w - w'\|_2^2. \]

B.2.2 Proof of Lemma B.2

Since \( \theta \leq \frac{\pi}{2}, \sin^2 \frac{\theta}{2} < 1 \), so \( \Phi(\hat{w}; w^{(i)}) < 1 \), ruling out the case \( \hat{w} = 0 \). Thus, we may write can write
\[ \frac{\hat{w}}{\|\hat{w}\|} = e_1 \cos \phi + e_2 \sin \phi \]
for some \( \phi \in [-\pi, \pi] \). Since \( \hat{w} \neq 0 \), \( \Phi(\hat{w}; w^{(i)}) \) corresponds to the square norm of the projection of \( w^{(i)} \) onto a direction perpendicular to \( \hat{w} \). Therefore,
\[ \Phi(\hat{w}; w^{(1)}) = \sin^2 \phi \] and \( \Phi(\hat{w}; w^{(2)}) = \sin^2(\phi - \theta) \).

We shall show that if \( \sin^2 \phi < \sin^2 \frac{\theta}{2} \), then \( \sin^2 \phi < \sin^2(\phi - \theta) \), and thus \( \hat{i} = 1 \); the other direction is analogous. To this end, suppose that \( \sin^2 \phi < \sin^2 \frac{\theta}{2} \). We consider three cases, and show that in each case, \( \sin^2(\phi - \theta) \geq \sin^2 \frac{\theta}{2} \).

1. Case (a): \( |\phi| < \frac{\theta}{2} \). Then, we have that \( \theta - \phi \in (\frac{\theta}{2}, \frac{3\theta}{2}) \). Since \( \theta \leq \frac{\pi}{2}, \min_{\varphi \in [\frac{\theta}{2}, \frac{3\theta}{2}]} \sin^2 \varphi = \sin^2 \frac{\theta}{2} \), whence \( \sin^2 \frac{\theta}{2} < \sin^2(\phi - \theta) \).
2. Case (b.1): $\phi \in (\pi - \frac{\theta}{2}, \pi]$. Then, we have that $\phi - \theta \in (\pi - \frac{3\theta}{2}, \pi - \theta]$. Now, if $\phi - \theta \in [\frac{\pi}{2}, \pi - \theta)$, we have that $\sin^2(\phi - \theta) \in [\sin^2 \theta, 1]$, so that $\sin^2(\phi - \theta) \geq \sin^2 \frac{\theta}{2}$. On the other hand, if $\phi - \theta \in [\pi - \frac{3\theta}{2}, \frac{\pi}{2}]$, then since $\theta \leq \frac{\pi}{2}$, we have $\sin^2(\phi - \theta) \in [\frac{\pi}{4}, \pi/2]$. Thus, $\sin^2(\phi - \theta) \geq \sin^2 \frac{\pi}{4} \geq \sin^2 \frac{\theta}{2}$.

3. Case (b.2): $\phi \in [-\pi, -\pi + \frac{\theta}{2})$. Then, we have that $\phi - \theta \in [-\pi - \theta, \pi - \frac{\theta}{2})$, and the rest is similar to (b.1).

### B.3 Proof of Calibration Error Computation, Lemma 3.2

We shall show that if either $\text{span}\{w, \hat{w}\} = \mathbb{R}^2$ or $\hat{w} = 0$, then there is a unit vector $v \in S^1$ for which

$$\Delta_{f_{\hat{w}}, D_{w_*}}(A_w) = \frac{\Phi(\hat{w}; w_*)}{4} \cdot \mathbb{E}_{\mathcal{X} \sim S^1}[|\langle v, X \rangle|].$$

This is enough to conclude the proof, since

$$\mathbb{E}_{\mathcal{X} \sim S^1}[|\langle v, X \rangle|] = \frac{1}{2\pi} \int_0^{2\pi} |\sin \theta| d\theta = \frac{1}{\pi} \int_0^\pi \sin\theta d\theta = \frac{2}{\pi}.$$

First, suppose that $\hat{w} \neq 0$. Choose an orthonormal basis $\{e_1, e_2\}$ so that $\hat{w} = ||\hat{w}|| e_1$. Then, we can write

$$X = X_1 e_1 + X_2 e_2,$$

where $X_i = \langle X, e_i \rangle$. Then, letting $w_* = w_{*,1} e_1 + w_{*,2} e_2$, we see that

$$w_{*,2} = \sqrt{\Phi(\hat{w}; w_*)},$$

and we have

$$f_{w_*}(X) = \langle w_*, X \rangle + \frac{1}{2} = \frac{1}{4} X_1 w_{*,1} + \frac{1}{4} X_2 w_{*,2} + \frac{1}{2}.$$

First, suppose that $\hat{w} \neq 0$. Then, since $f_{\hat{w}}(X) = \frac{1}{2} + ||\hat{w}|| \cdot X_1$ is in bijection with $X_1$, and since

$$\mathbb{E}[X_2 | X_1] = 0 \text{ for } (X_1, X_2) \sim S^1,$$

we have

$$\mathbb{E}[f_{w_*}(X) | f_{\hat{w}}(X)] = \mathbb{E}[f_{w_*}(X) | X_1] = \frac{1}{2} + \frac{1}{4} X_1 w_{*,1} + \frac{1}{4} w_{*,2} \mathbb{E}[X_2 | X_1] = \frac{1}{2} + \frac{1}{4} X_1 w_{*,1}.$$

Moreover, if $w$ and $w = ||w|| e_1$ are linearly independent, then since $X \in S^1$, $A_w = \text{sign}(\langle w, X \rangle)$ and $X_1 = \langle e_1, X \rangle$ uniquely determine $X$. Hence,

$$\mathbb{E}[f_{w_*}(X) | f_{\hat{w}}, A_w] = \mathbb{E}[f_{w_*}(X) | X] = f_{w_*}(X) = \frac{1}{2} + \frac{1}{4} X_1 w_{*,1} + \frac{1}{4} w_{*,2} X_2.$$

Thus,

$$\mathbb{E}[f_{w_*}(X) | f_w(X), A] - \mathbb{E}[f_{w_*}(X) | f_w(X)] = \frac{w_{*,2} X_2}{4}.$$
Hence, we conclude that

$$\Delta_f(A) = \frac{|w_x|^2}{4} \cdot \mathbb{E}[X_2] = \frac{\sqrt{\Phi(\hat{w}; w)} \cdot \mathbb{E}[\langle e_2, X \rangle]}{4}.$$ 

We now address the edge-case $\hat{w} = 0$. Since $f_{\hat{w}}(X) = 0$ for all $X$, $\mathbb{E}[Y | f_{\hat{w}}] = \mathbb{E}[Y] = \frac{1}{2}$, and $\mathbb{E}[Y | f_{\hat{w}}, A_w] = \mathbb{E}[Y | A_w]$. To compute $\mathbb{E}[Y | A_w]$, let $e_1 = w$, and let $e_2$ be such that $\{e_1, e_2\}$ form an orthonormal basis, and write $X = X_1 e_1 + X_2 e_2$. Then,

$$\mathbb{E}[X | A] = \mathbb{E}[X_1 | \text{sign}(X_1)] + \mathbb{E}[X_2 | \text{sign}(X_1)]$$

(since $X_2 \perp \text{sign}(X_1)$)

$$= \text{sign}(X_1) \mathbb{E}[\text{sign}(X_1) X_1 | \text{sign}(X_1)]$$

$$= \text{sign}(X_1) \mathbb{E}[||X_1| | \text{sign}(X_1)] = \text{sign}(X_1) \mathbb{E}[||X_1||].$$

Hence, $\mathbb{E}[Y | A] = \frac{1}{2} + \frac{1}{4} (w_x, \mathbb{E}[X | A]) = \frac{\text{sign}(X_1) \mathbb{E}[||X_1||]}{4}$. Therefore,

$$\Delta_{f_{\hat{w}}, D_{w^*}}(A_w) = \mathbb{E}[\mathbb{E}[f_{w^*}(X) | f_{\hat{w}}(X), A_w] - \mathbb{E}[f_{w^*}(X) | f_{\hat{w}}(X)]]$$

$$= \mathbb{E}\left[\frac{1}{2} + \frac{\text{sign}(X_1) \mathbb{E}[||X_1||]}{4} - \frac{1}{2}\right] = \frac{1}{4} \mathbb{E}[||X_1||] = \frac{\Phi(\hat{w}; w_x) \cdot \mathbb{E}[\langle w, X \rangle]}{4},$$

where we recall the convention $\Phi(\hat{w}; w_x) = 1$ if $\hat{w} = 0$.

### C Additional result

The following corollary provides a bound on the worst possible calibration error for any one group $A = a$.

**Corollary C.1** (Single group calibration). *Denote the single group calibration error as

$$\Delta_f^a = \mathbb{E}_{X|A=a} [\mathbb{E}[Y | f(X, a)] - \mathbb{E}[Y | f(X, a), A = a]]$$

and $p_a = \text{Pr}\{A = a\}$. Under Assumption 1, we have

$$\Delta_f^a \leq \frac{C'}{p_a} \sqrt{\mathcal{L}(f) - \mathcal{L}^*}.$$* 

*Proof. Note that $\sum_{a \in A} \Delta_f^a \text{Pr}\{A = a\} = \Delta_f(A)$. Then the desired inequality follows from Theorem 2.3. □

As shown in the above result, we can use our bound on a score with small calibration error $\Delta_f(A)$ to derive a bound on single group calibration error. However, the upper bound deteriorates as the mass of the single group $\text{Pr}\{A = a\}$ decreases. In other words, Theorem 2.3 implies small single group calibration error as long as the group is large enough and the excess risk of the score is small.

### D Additional experiments

#### D.1 Multicalibration

For numerical features (e.g. Age), we split the data into 2 groups according to an arbitrary threshold (e.g. above 40 years old and below 40 years old) and compute calibration with respect to those groups. For categorical features, we only visualize the calibration of the top three most populous groups, for clarity. This is shown in Figure 8. Note that by Corollary C.1, groups that have small mass have a worse bound on calibration.
Figure 8: Group calibration with respect to other features and combinations of features