Explaining observations of rapidly rotating neutron stars in LMXBs

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We introduce a new scenario that explains the existence of rapidly rotating warm neutron stars (NSs) observed in low-mass X-ray binaries (LMXBs). The scenario takes into account the interaction between the normal (quadrupole) \( m = 2 \) r-mode and superfluid inertial modes. This interaction can only occur at some fixed ‘resonance’ stellar temperatures; it leads to formation of the ‘stability peaks’ that stabilize a star in the vicinity of these temperatures. We demonstrate that an NS in LMXB spends a substantial fraction of time on the stability peak, that is in the region of stellar temperatures and spin frequencies, that has been previously thought to be unstable with respect to excitation of r-modes. We also find that the spin frequencies of NSs are limited by the instability of normal (octupole) \( m = 3 \) r-mode rather than by \( m = 2 \) r-mode. This result agrees with the predicted value of the cut-off spin frequency \( \sim 730 \) Hz, following from the statistical analysis of the accreting millisecond X-ray pulsars. A comparison of the proposed theory with observations of rotating NSs can impose new important constraints on the properties of superdense matter.

I. INTRODUCTION

Neutron stars (NSs) are the compact rotating objects with a mass \( M \sim M_\odot \) and radius \( R \sim 10 \) km (e.g., Ref. [1]). Rotation leads to the appearance of the so-called inertial oscillation modes in NSs, whose restoring force is the Coriolis force [2]. A particular, but the most interesting class of inertial modes is r-modes for which (unlike the other inertial modes) the dominant oscillations are of toroidal type [3]. The remarkable property of r-modes is that, neglecting dissipation, they are subject to gravitational-driven (Chandrasekhar-Friedman-Schutz) instability at arbitrary spin frequency \( \nu \) of an NS [3, 6]. An account for dissipative effects stabilizes the NS to some extent resulting in the appearance of the ‘stability region’ in the \( \nu - T^\infty \) plane, where \( T^\infty \) is the red-shifted internal stellar temperature. A typical stability region is shaded in grey in Fig. 2 (see Sec. III B): r-modes cannot be spontaneously excited inside this region.

In some cases observations of rapidly rotating NSs in low-mass X-ray binaries (LMXBs) allow one to measure \( \nu \) (e.g., Refs. [4, 8] and Tab. I). It turns out that many of the rapidly rotating warm sources fall well outside the stability region, if it is plotted under realistic assumptions about the properties of superdense matter [9, 10]. In fact, calculations show that NSs in LMXBs can indeed leave the stability region for a while, but the probability to observe them there is negligibly small in most cases (see, e.g., Refs. [12, 13] and Sec. III). Thus, we face a paradox which is usually being explained following one of the two approaches.

In the first approach one tries, making some (rather artificial) assumptions, to enhance damping of r-mode oscillations due to various dissipative mechanisms. The aim is to enlarge the stability region so that it would contain all the observed sources (see, e.g., [4, 11]).

The second approach assumes that some fraction of NSs lies outside the stability region, but their spin frequency \( \nu \) and temperature \( T^\infty \) are fixed by two conditions that should be satisfied simultaneously: (i) r-mode oscillations in these NSs should reach saturation because of non-linear interaction with other inertial modes (see, e.g., [3, 11] and Sec. III B) and (ii) all the heat released due to dissipation of the ‘saturated’ r-modes should be radiated away by the neutrino emission. Unfortunately, these conditions lead to unrealistically small values of the saturation amplitude \( \alpha_{sat} \sim 10^{-9} \div 10^{-6} \) specific to each source [9, 11]. Such small \( \alpha_{sat} \) seem to contradict the results of Refs. [14, 15].

Thus, one can conclude that the existence of rapidly rotating warm NSs remains an open problem which has not yet received a satisfactory theoretical explanation [16].

This work is devoted to a possible solution to that problem. Our key idea consists in that to study evolution of NSs in LMXBs one has to correctly take into account the resonance interaction between the normal oscillation \( m = 2 \) r-mode and superfluid inertial modes, which occurs at some fixed values of \( T^\infty \) (see Sec. IV). Such resonance interaction has been completely ignored in the literature so far. However, as we will argue below, it should take place and can dramatically affect the evolution of rapidly rotating NSs.

First of all, this interaction modifies the stability region (see Sec. V) and allows us to suggest an evolution scenario (Sec. VI), that explains all the sources in LMXBs within the standard, minimal assumptions about the composition and properties of superdense matter. Moreover, as directly follows from our scenario, the NS spin frequencies \( \nu \) appear

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1 The most rapidly rotating NS observed so far is the millisecond pulsar PSR J1748-2446ad with the spin frequency \( \nu = 716 \) Hz [2].
FIG. 1: Critical temperatures of protons $T_{cp}$ (solid line) and neutrons $T_{cn}$ (dashed line) as functions of density $\rho$ in neutron star core. Vertical dotted lines indicate central densities of a star with the mass $M = 1.4M_\odot$. The left line corresponds to the relativistic star with APR EOS [21], the right line corresponds to the polytropic Newtonian star (polytropic exponent $\Gamma = 2$) with the radius $R = 10$ km.

to be bounded by the onset of the octupole $m = 3$ oscillation $r$-mode instability, which corresponds to $\nu \sim 600 - 700$ Hz at $T_{\infty} \sim 10^8$ K (see Fig. 5). The existence of an upper bound for $\nu$ can explain the sharp cut-off of the distribution function for accreting X-ray pulsars at a frequency $\nu \gtrsim 730$ Hz [17, 18]. If correct, this result presents a strong argument in favor of the idea of Refs. [19, 20] that the NS spin frequency $\nu$ is limited by the $r$-mode instability. Note, however, that in our scenario $\nu$ is limited by the octupole $m = 3$ $r$-mode rather than by quadrupole $m = 2$ $r$-mode, as it is supposed in Refs. [19, 20].

The paper is organized as follows. In Sec. II we discuss the adopted NS model and write out general equations governing the thermo-rotational evolution of an NS in LMXB with allowance for the excitation of normal $r$-modes. In Sec. III we present the summary of observations of quiescent temperatures and spin frequencies for NSs in LMXBs, and demonstrate the problem with their explanation within the scenarios available in the literature. In Sec. IV we describe and justify our model of resonance interaction between the normal and superfluid oscillation modes. In Sec. V we determine the stability region taking into account the resonance interaction of the normal $m = 2$ $r$-mode and one of the superfluid inertial modes; we also generalize the equations describing the NS dynamics to the case when a few oscillation modes are simultaneously excited in a star. These results are applied in Sec. VI to model the evolution of an accreting NS. Detailed analysis of the evolution tracks allow us to formulate an original scenario explaining all the existing data on the spin frequencies and temperatures of NSs in LMXBs. In Sec. VII we present the main conclusions.

II. PHYSICS INPUT AND GENERAL EQUATIONS

All calculations in this paper are carried out for a canonical NS with the mass $M = 1.4M_\odot$ and radius $R = 10$ km, whose core is composed of neutrons ($n$), protons ($p$) and electrons ($e$). Following Refs. [13, 22–25] we, for simplicity, consider the polytropic equation of state (EOS) with polytropic index $n = 1$ ($P \propto \rho^{\Gamma}$, where $\Gamma = 1 + 1/n = 2$; $P$ and $\rho$ are, respectively, the pressure and density of matter). We checked, that use of more realistic EOSs does not affect our main results (see also Ref. [4]).

According to numerous microscopic calculations, nucleons (neutrons and protons) in the internal layers of NSs are superfluid at temperatures $T \lesssim 10^8 \div 10^{10}$ K. Recent real-time observations of a cooling young NS in Cassiopeia A supernova remnant [26] have presented a strong evidence of this fact. They were explained [27, 28] within the so-called ‘minimal cooling scenario’, proposed in Refs. [29, 30]. In this paper we use the same models of neutron and proton superfluidity [that is the same functions $T_{ci}(\rho)$, where $T_{ci}$ is the critical temperature for transition of a nucleon species...
\[ \delta \nu = \frac{\alpha}{\sqrt{l(l+1)}} \left( \frac{r}{R} \right)^l \nabla \times (r \nabla Y_m) e^{i\omega t}, \]

where \( Y_m \) is the spherical harmonic with the multipolarity \( l \) equal to \( m \), \( l = m \); \( \alpha \) is the oscillation amplitude of \( r^o \)-mode; \( r \) is the radial coordinate. Finally, \( \omega \) is the oscillation frequency in the inertial frame, given by (also to leading order in \( \Omega \)) \[ \omega = -\frac{(l-1)(l+2)}{l+1} \Omega. \]

Below we make use of the quantity

\[ \Omega_0 \equiv \sqrt{\pi G \bar{\rho}} = 1.180 \times 10^4 \left( \frac{M}{1.4M_\odot} \right)^{1/2} \left( \frac{R}{10 \text{ km}} \right)^{-3/2} \text{s}^{-1}, \]

where \( G \) is the gravitation constant and \( \bar{\rho} = 3M/(4\pi R^3) \) is the mean stellar density. For a canonical NS \( \bar{\rho} \approx 6.646 \times 10^{14} \text{ g cm}^{-3} \).

To describe the evolution of an NS allowing for the \( r^o \)-mode instability, we follow the phenomenological approach suggested by Owen et al. \[ \text{[22]} \] and further refined in Refs. \[ \text{[24]} \text{ and [43]} \]. We mostly employ the notation of Ref. \[ \text{[24]} \]. The evolution is given by the following equations:

(i) An equation, governing the variation of canonical angular momentum \( J_c \) of \( r^o \)-mode due to radiation of gravitational waves and various dissipative effects,

\[ \frac{dJ_c}{dt} = -2J_c \left( \frac{1}{\tau_{\text{GR}}} + \frac{1}{\tau_{\text{Diss}}} \right). \]

Here \[ \text{[22, 44]} \]

\[ J_c = \frac{l}{2(\omega + l\Omega)} \int \rho \delta \nu \delta \nu^* d^3 r = -\frac{\alpha^2 l(l+1)}{4} \Omega R^{-2l+2} \int_0^R \rho r^{2l+2} dr, \]

where we apply Eqs. \[ \text{[11]} \text{ and [2]} \] in the second equality. An integral in the right-hand side of Eq. \[ \text{[5]} \] can be easily calculated if one specifies the density profile \( \rho(r) \). Obviously, the integral can generally be written in the form \( \tilde{J} M R^{2l} \), where \( \tilde{J} \) is some numerical coefficient that depends on \( \rho(r/R) \). Using this expression, \( J_c \) can be presented as

\[ J_c = \frac{l(l+1)}{4} \tilde{J} M R^2 \Omega \alpha^2. \]

2 The superscripts \( o \) and \( s \) here are the abbreviations for ‘ordinary’ and ‘superfluid’, respectively.
For the simple polytropic model with \( \Gamma = 2 \) and a given stellar mass \( M \) and radius \( R \), one has

\[
\rho(r) = \frac{M}{4\pi R^2} \sin\left(\frac{\pi r}{R}\right),
\]

which leads to \( J \approx 1.6353 \times 10^{-2} \) for \( l = m = 2 \) and \( J \approx 9.9887 \times 10^{-3} \) for \( l = m = 3 \) \( r^o \)-mode.

An intensity of gravitational radiation is determined by the current multipole; using Eq. (1) one can calculate the corresponding gravitational radiation timescale \( \tau_{GR} \),

\[
\frac{1}{\tau_{GR}} = -\frac{32\pi G \Omega^{2l+2}}{c^{2l+3}} \left(\frac{l-1}{2l+1}\right)^{2l+2} \left(\frac{l+2}{l+1}\right)^{2l+2} \int_0^R \rho r^{2l+2} dr,
\]

where \( c \) is the speed of light. For the density profile (7) this expression can be rewritten as

\[
\tau_{GR} = -\frac{\eta}{\tau_{GR0}} \left(\frac{M}{1.4M_\odot}\right)^{-1} \left(\frac{R}{10\text{km}}\right)^{-2l} \left(\frac{\nu}{1\text{kHz}}\right)^{-2l-2},
\]

where \( \tau_{GR0} \approx 46.4 \text{ s} \) and \( 1250 \text{ s} \) for \( l = m = 2 \) and \( l = m = 3 \) \( r^o \)-modes, respectively.

Further, \( 1/\tau_{Diss} \) in Eq. (4) is generally presented in the form,

\[
\frac{1}{\tau_{Diss}} = \sum_i \frac{1}{\tau_i},
\]

where the summation is assumed over all possible processes resulting in dissipation of energy and angular momentum of \( r^o \)-modes (the shear and bulk viscosities, Ekman layer dissipation, mutual friction etc. [4]). In this paper we neglect the bulk viscosity, because it is small for the range of stellar temperatures \( T < 5 \times 10^8 \text{ K} \) we are interested in (see, e.g., [38, 45–47]). One can also freely ignore the effects of mutual friction when considering \( r^o \)-modes [43, 48, 49].

On the opposite, dissipation in the Ekman layer can be a very efficient mechanism, though the corresponding damping time \( \tau_{Ek} \) is very sensitive to the chosen model of interaction between the ‘solid’ crust and liquid core of an NS [4, 50–52]. Actually, in the vicinity of the crust-core interface the crust is neither solid nor liquid, being some intermediate structure, which is called mantle. Thus, dissipation in the transition Ekman layer can be substantially lower, than it is often assumed.

Bearing this in mind, we consider dissipation due to the shear viscosity as our minimal model for the dissipation of \( r^o \)-modes. The corresponding timescale \( \tau_S \) can be calculated from the formula [22]

\[
\frac{1}{\tau_S} = (l-1)(2l+1) \int_0^R \eta r^{2l+2} dr \left(\int_0^R \rho r^{2l+2} dr\right)^{-1},
\]

that was obtained using velocity field (11). Here \( \eta \) is the shear viscosity coefficient. Estimates show that the proton shear viscosity is small in comparison to the electron one \( \eta_e \) [53], while the neutron shear viscosity is poorly known even for nonsuperfluid NS matter (its value differs for different authors by a factor of 5–10 and can be either greater [54, 55] or smaller [53] than \( \eta_e \)). In view of these facts, for \( \eta \) in this paper we take the electron shear viscosity \( \eta_e \) from Ref. [53]. Notice that, \( \eta_e \) can vary several-fold depending on a chosen EOS (or, more precisely, depending on a proton fraction predicted by an EOS; see, e.g., figure 1 in Ref. [53]). Another important ingredient, affecting \( \eta_e \), is still poorly known model of proton superfluidity [the profile \( T_{cp}(\rho) \)].

The uncertainties, described above, and possible contribution of the Ekman layer into dissipation, can effectively increase \( \eta \) by a factor of few. For octupole (\( l = m = 3 \)) \( r^o \)-mode the situation is even more uncertain, because this mode becomes unstable (and thus important for the NS evolution, see Sec. [13] at rather high values of \( \Omega \). This means that the approximations of slowly rotating NSs, assumed in derivation of Eqs. (8) and (11), can lead to larger errors for the octupole \( r^o \)-mode [56, 57]. Taking this into account, when modeling the octupole \( r^o \)-mode (but not the quadrupole \( r^o \)-mode!), for \( \eta \) we take (somewhat arbitrary) \( \eta_e \) from Ref. [53], multiplied by a factor of 5, that is we set \( \eta = 5\eta_e \).

Using the results of Ref. [53], we approximate the electron shear viscosity \( \eta_e \) by the following fitting formula,

\[
\eta_e = 6 \times 10^{18} \left(\frac{\rho}{10^{15} \text{ g cm}^{-3}}\right)^2 \left(\frac{T}{10^9 \text{ K}}\right)^{-2} \left(\frac{T_{cp}}{2 \times 10^9 \text{ K}}\right)^{1/3} \text{ g cm s}^{-1},
\]

which particularly well describes \( \eta_e \) for APR EOS [21] (more precisely, for the parametrization [58] of APR EOS). Notice that this formula is valid only if protons are superfluid and \( T < 0.2 T_{cp} \). Notice also that, without the last
multiplier, the formula [12] coincides with the well known and widely used fit [59] of old calculations of Flowers and Itoh [60]. This is an accidental and surprising coincidence, because the physics input used in Refs. [53] and [60] is essentially different (in particular, unlike the Ref. [53], \( \eta \) from the paper by Flowers and Itoh was derived assuming no proton superfluidity and, what is more important, accounting incorrectly for the effects of transverse plasma screening on the processes of electron-electron scattering). In addition, the fitting formula of Ref. [59] was obtained for absolutely different EOS.

For our model of the proton superfluidity the last multiplier in Eq. [12] is of the order of unity in the most part of the star, \( (T_{cp}(\rho)/(2 \times 10^9 K))^{1/3} \sim 1 \). In view of the uncertainties in the value of \( \eta \), we ignore this multiplier in what follows. Using Eq. [12] and integrating (11) over \( r \), we obtain

\[
\tau_S = \tau_{S,0} \left( \frac{R}{10 \text{ km}} \right)^5 \left( \frac{M}{1.4M_\odot} \right)^{-1} (T_{S,0}^\infty)^2,
\]

where \( T_{S,0}^\infty \equiv T^\infty/(10^8 K) \); \( \tau_{S,0} \approx 2.2 \times 10^5 \text{ s} \) for \( l = m = 2 \) \( r^o \)-mode and \( \tau_{S,0} \approx 2.4 \times 10^4 \text{ s} \) for \( l = m = 3 \) \( r^o \)-mode (we remind that in the latter case we take \( \eta = 5\eta_c \)). In Eq. (13), instead of \( T \), we introduced the redshifted internal temperature \( T^\infty \equiv T e^{\nu(r)/2} \), where \( \nu(r) \) is the corresponding metric coefficient [61]. Let us remind that in the nonrelativistic approximation which has been used in derivation of this equation, \( T = T^\infty \), so that such replacement is justified. Moreover, the temperature \( T^\infty \), which is constant over the star, is a more appropriate parameter than \( T \) for the description of NS thermal evolution [see Eq. (16) below] and, especially, for the analysis of observational data (Sec. III A).

(ii) Equation, describing the change in the total angular momentum \( J_c + I\Omega \) of an NS,

\[
d(J_c + I\Omega) = \frac{2}{\tau_{GR}} J_c + \dot{J}_{acc},
\]

due to gravitational wave radiation (the first term) and accretion from the low-mass companion (the second term \( \dot{J}_{acc} \)). For simplicity, we ignore possible magneto-dipole torque in this paper (but see Sec. VII). In Eq. (14) \( I = I MR^2 \) is the stellar moment of inertia; for polytropic EOS (\( \Gamma = 2 \)) \( I \approx 0.261 \). There is a number of accretion models, leading to somewhat different estimates for \( \dot{J}_{acc} \) (see, e.g., [62,64]), however, they do not agree well with observations (see, e.g., [8]). Thus, for definiteness, we make use of the simplest estimate,

\[
\dot{J}_{acc} = p M \sqrt{GMR},
\]

which is traditionally applied in modeling of the NS evolution in binary systems. Here \( \dot{M} \) is the mass of accreted matter per unit time; \( p \) is a factor of the order of 1, that depends on the accretion radius and spin frequency \( \Omega \) of an NS (e.g., [12]). Below we analyze the large time-scale evolution of NSs, hence we assume that the quantities \( \dot{J}_{acc} \) and \( \dot{M} \) are averaged over the active and quiescent phases of accretion. Since \( \dot{J}_{acc} \propto \dot{M} \) in Eq. (15), one can use that expression for the averaged values as well. In what follows we set \( p = 1 \) and \( \dot{M} = 3.0 \times 10^{-10} M_\odot \text{ yr}^{-1} \). The chosen value of \( \dot{M} \) is close to the estimates of the accretion rates for the sources SAX J1750.8-2900 and 4U 1608-522 (see below).

(iii) Equation describing the thermal evolution of an oscillating star,

\[
C_{tot} \frac{dT^\infty}{dt} = W_{Diss} - L_{cool} + K_n \dot{M} c^2,
\]

where \( W_{Diss} \) is the energy dissipated per unit time due to the \( r^o \)-mode damping. It is presented as (e.g., Ref. [12])

\[
W_{Diss} = \frac{2E_c}{\tau_{Diss}} = \frac{\dot{J} MR^2 \Omega^2 \alpha^2}{\tau_{Diss}},
\]

where \( E_c \) is the canonical energy of \( r^o \)-mode (with arbitrary \( m \)) in a reference frame, rotating with the star. As it was shown in Refs. [44, 65], \( E_c \) is related to the canonical angular momentum \( J_c \) [see Eq. (6)] by

\[
E_c = - \frac{(\omega + m \Omega)}{m} J_c.
\]

This relation is valid for any inertial modes (not only for \( r^o \)-modes). Further, \( C_{tot}(T^\infty) \) in Eq. (16) is the total heat capacity of an NS; \( L_{cool}(T^\infty) \) is its luminosity, that is the energy carried away from the star per unit time in the form of neutrino and electromagnetic radiation from its surface. Since oscillation amplitudes of \( r^o \)-modes, analyzed
in this paper, are small ($\alpha \leq 10^{-4}$, see below), $L_{\text{cool}}$ is given by the same equation as for a non-oscillating star [66]. To determine the quantities $C_{\text{tot}}$ and $L_{\text{cool}}$ as accurate as we can, we calculate them with the relativistic cooling code, described in detail in Refs. [24, 31, 67] (we used essentially the same microphysics input as that employed in Ref. [29]). In particular, we used the parameterization [58] of APR EOS [21] and considered a star with the mass $M = 1.4M_\odot$. Although this approach is somewhat inconsistent (other equations neglect relativistic effects and employ the polytropic EOS), it allows us to use the realistic values for $C_{\text{tot}}$ and $L_{\text{cool}}$ in our simplified model. The calculations of $C_{\text{tot}}$ and $L_{\text{cool}}$ have been roughly approximated as functions of internal (redshifted) stellar temperature $T_\infty$ and are presented in Appendix A. Since the photon luminosity is not important in the temperature range of interest to us ($T_\infty > 10^8$ K), we fit only the neutrino luminosity in Appendix A. Note that for lower $T_\infty$ the photon luminosity rapidly becomes the main cooling agent and hence cannot be ignored [67]. We have checked, that the results for $r^\alpha$-mode evolution obtained using the fitting formulas from the Appendix A practically do not differ from those obtained using the exact values for $C_{\text{tot}}$ and $L_{\text{cool}}$.

Finally, the last term in Eq. (16) describes the stellar heating due to accretion (deep crustal heating, see, e.g., Ref. [68]). Under the pressure of accreted material, the matter in the stellar envelope compresses and eventually undergoes a set of exothermal nuclear transformations (pycnonuclear reactions and reactions of beta-capture, accompanying by the neutron emission). The heat released in these reactions is mostly accumulated by the core due to high thermal conductivity of the internal layers of NSs. The parameter $K_n$ characterizes the efficiency of this heating; following Refs. [14, 69] we adopt $K_n = 10^{-3}$ as a fiducial value. For a chosen NS model the heating (in the absence of $r^\alpha$-mode) is completely compensated by the cooling ($L_{\text{cool}} = K_n M_c e^2$) at $T_{\text{crit}} \approx 1.078 \times 10^8$ K.

Equations (4), (14), and (16) fully describe the evolution of non-saturated $r^\alpha$-modes. Using Eqs. (4) and (14) one can express the quantities $d\alpha/dt$ and $d\Omega/dt$,

$$
\frac{d\alpha}{dt} = -\alpha \left( \frac{1}{\tau_{GR}} + \frac{1}{\tau_{\text{Diss}}} \right),
$$

$$
\frac{d\Omega}{dt} = -2Q \alpha^2 \frac{\Omega}{\tau_{\text{Diss}}} + \dot{\Omega}_{\text{acc}},
$$

where

$$
\dot{\Omega}_{\text{acc}} \equiv \frac{\dot{j}_{\text{acc}}}{I} = p \dot{M} \frac{\sqrt{GMR}}{I}
$$

$$
\approx 3.73 \times 10^{-6} \ p \ M_{-10} \ T_{0.261}^{-1} \left( \frac{M}{1.4M_\odot} \right)^{-1/2} \left( \frac{R}{10 \text{ km}} \right)^{-3/2} \text{s}^{-1} \text{yr}^{-1},
$$

and $\dot{M}_{-10} = \dot{M}/(10^{-10} M_\odot \text{yr}^{-1})$, $\bar{T}_{0.261} = \bar{T}/0.261$. In deriving Eq. (19) we neglected the term $\propto \alpha^3$, assuming that $\alpha \ll 1$. In addition, because $\dot{\Omega}_{\text{acc}}/\Omega \ll 1/\tau_{GR}$ we also neglected the term proportional to $\dot{\Omega}_{\text{acc}}/\Omega$ in Eq. (19). Let us note that the explicit dependence of the accretion torque on an accretion regime and its parameters ($\dot{M}$, magnetic field etc.) is not important for the final equations, because they only depend on the accretion torque $\dot{\Omega}_{\text{acc}}$, averaged over a large period of time, containing both the active and quiescent phases. In principle, $\dot{\Omega}_{\text{acc}}$ can be obtained directly from observations of the spin frequency change during the active phase.

The resulting Eqs. (16), (19), and (20) correctly describe the NS evolution only until a growing oscillation mode enters the nonlinear saturation regime, where it will interact nonlinearly with other inertial modes. Under some simplifying assumptions the nonlinear regime was studied in Refs. [14, 15, 71, 74]. In particular, in the recent papers by Bondarescu et al. [14, 15] it has been shown that the saturation amplitude $\alpha_{\text{sat}}$ for $r^\alpha$-mode can be rather small, $\alpha_{\text{sat}} \approx 10^{-4} \div 10^{-1}$. Unless otherwise is stated, we, following Ref. [14], assume that $\alpha_{\text{sat}} = 10^{-4}$ for all modes considered in this paper.3

We also assume, as in Ref. [22], that in the saturation regime (when $\alpha$ reaches the value $\alpha_{\text{sat}} = 10^{-4}$) the oscillation amplitude stops to grow, so that the energy, pumped into $r^\alpha$-mode by gravitational radiation, redistributes among the other modes through the nonlinear interactions, and eventually dissipates into heat. Mathematically this can be

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3 We note that the $r^\alpha$-mode amplitude $C_R$ of Bondarescu et al. is related to our amplitude $\alpha$ by $C_R = (\bar{J}/2)^{1/2} \alpha \approx 0.1 \alpha$, see the footnote 1 in Ref. [15].
emphasized that it would be more self-consistent to replace \( \tau \) did not replace the quantity \( \tau \). [43].

spin frequencies \( \nu \) 20 neutron stars in LMXBs. The source names are given in the first column. The second column presents the NS adopt the value of \( d\alpha/dt \) that

radii were also fixed at the fiducial value \( M = 1.4M_\odot \). In Ref. [75] the apparent emission area radius \( r_e \) for the source

(qualitatively) described by introducing in Eq. (19) the effective dissipation time \( \tau_{\text{Diss}}^{\text{eff}} \) instead of \( \tau_{\text{Diss}} \), and requiring that \( d\alpha/dt = 0 \),

\[
\frac{d\alpha}{dt} = 0 = -\alpha \left( \frac{1}{\tau_{\text{GR}}} + \frac{1}{\tau_{\text{Diss}}^{\text{eff}}} \right),
\]

which leads to

\[
\tau_{\text{Diss}}^{\text{eff}} = -\tau_{\text{GR}}.
\]

Concluding, in the saturation regime we (i) fix the amplitude of \( r^\alpha \)-mode \( \alpha = \alpha_{\text{sat}} = 10^{-4} \), (ii) replace \( \tau_{\text{Diss}} \) with \( \tau_{\text{Diss}}^{\text{eff}} = -\tau_{\text{GR}} \) in Eqs. (16) and (20). Let us notice that, when modeling the saturated oscillations, Owen et al. [22] did not replace the quantity \( \tau_{\text{Diss}} \) in the thermal evolution equation (19) [but replaced it in Eq. (20)]. The first who emphasized that it would be more self-consistent to replace \( \tau_{\text{Diss}} \) with \( -\tau_{\text{GR}} \) also in Eq. (16) were the authors of Ref. [43].

### III. OBSERVATIONAL DATA AND STABILITY OF RAPIDLY ROTATING NSS

#### A. Observational data

Observational data on spin frequencies, quiescent temperatures, and accretion rates are summarized in Tab. I for 20 neutron stars in LMXBs. The source names are given in the first column. The second column presents the NS spin frequencies \( \nu \) which are mainly taken from Ref. [4]. An exception is the source IGR J17498-2921, for which we adopt the value of \( \nu \) from the review [8]. The third column summarizes observational data on NS redshifted effective temperatures \( T_{\text{eff}}^{\infty} \) in the quiescent state. The corresponding values are taken from the papers quoted in the fourth column. In those papers the thermal component was fitted by the hydrogen atmosphere models with the fiducial value of the NS mass \( M = 1.4M_\odot \). Except for the sources EXO 0748-676 and 4U 1608-522, the NS circumferential radii were also fixed at the fiducial value \( R = 10 \) km. In Ref. [72] the apparent emission area radius \( r_e \) for the source

### TABLE I: Observational data and internal temperatures on NSs in LMXBs

| Source          | \( \nu \) [Hz] | \( T_{\text{eff}}^{\infty} \) Ref. | \( T_{\text{fe}}^{\infty} \) | \( T_{\text{ad}}^{\infty} \) | \( T_{\text{fe}} \) | \( T_{\text{ad}} \) | \( M \) [\( M_\odot \)] | \( \tau_{\text{Diss}}^{\text{eff}} \) | \( \tau_{\text{Diss}} \) | \( \tau_{\text{GR}} \) |
|-----------------|----------------|----------------------------------|---------------------------|--------------------------|-----------------|-----------------|--------------------------|--------------------------|--------------------------|
| 4U 1608-522     | 620            | 1.51 [75]                        | 0.93                      | 1.90                     | 2.47            | 3.6 \times 10^{-10} [76] |
| SAX J1750.8-2900| 601            | 1.72 [77]                        | 1.18                      | 2.57                     | 3.11            | 2 \times 10^{-10} [77] |
| IGR J00291-5934 | 599            | 0.63 [78]                        | 0.21                      | 0.24                     | 0.52            | 2.5 \times 10^{-12} [78] |
| MXB 1659-298    | 556            | 0.63 [79]                        | 0.21                      | 0.24                     | 0.52            | 1.7 \times 10^{-12} [76] |
| EXO 0748-676    | 552            | 1.26 [80]                        | 0.68                      | 1.20                     | 1.79            | 4 \times 10^{-10} [76] |
| Aql X-1         | 550            | 1.26 [81]                        | 0.68                      | 1.20                     | 1.79            | 4 \times 10^{-10} [76] |
| KS 1731-260     | 526            | 0.73 [82]                        | 0.27                      | 0.32                     | 0.67            | < 1.5 \times 10^{-9} [76] |
| SWIFT J1749.4-2807 | 518        | 1.16 [83]                        | 0.59                      | 0.96                     | 1.54            |                  |
| SAX J1748.9-2021 | 442            | 1.04 [84]                        | 0.49                      | 0.72                     | 1.27            | 1.8 \times 10^{-12} [76] |
| XTE J1751-305   | 435            | 0.63 [78]                        | 0.21                      | 0.24                     | 0.52            | 6 \times 10^{-12} [78] |
| SAX J1808.4-3658 | 401            | 0.27 [78]                        | 0.05                      | 0.05                     | 0.11            | 9 \times 10^{-12} [78] |
| IGR J17498-2921 | 401            | < 0.93 [83]                      | 0.41                      | 0.55                     | 1.04            |                  |
| HETE J1900.1-2455 | 377            | < 0.65 [10]                      | 0.22                      | 0.25                     | 0.55            |                  |
| XTE J1814-338   | 314            | < 0.61 [78]                      | 0.20                      | 0.22                     | 0.49            | 3 \times 10^{-12} [78] |
| IGR J17191-2821 | 294            | 0.86 [10]                        | 0.36                      | 0.45                     | 0.90            |                  |
| IGR J17511-3057 | 245            | < 1.1 [10]                       | 0.54                      | 0.84                     | 1.40            |                  |
| NGC 6440 X-2    | 205            | < 0.37 [10]                      | 0.09                      | 0.09                     | 0.20            | 1.3 \times 10^{-12} [85] |
| XTE J1807-294   | 190            | < 0.45 [78]                      | 0.12                      | 0.13                     | 0.28            | < 8 \times 10^{-12} [78] |
| XTE J0929-314   | 185            | < 0.58 [86]                      | 0.19                      | 0.20                     | 0.45            | < 2 \times 10^{-11} [78] |
| Swift J1756-2508 | 182            | < 0.96 [10]                      | 0.43                      | 0.59                     | 1.10            |                  |

aWe treat the effective temperature from the table 2 of Ref. [78] as a local one to reproduce the thermal luminosity from that reference. bThe radius of this source was fixed at 15.6 km in spectral fits of Ref. [80].
4U 1608-522 was treated as a free parameter, and the value \( r_e = 9.4^{+4.3}_{-2.7} \) km, extracted from the spectral fitting, is compatible with the fiducial value \( R = 10 \) km. At the same time, the spectral fitting for EXO 0748-676 with the canonical mass \( M = 1.4M_\odot \) and radius \( R = 10 \) km leads to unrealistic estimates of the distance and/or hydrogen column density \( N_H \) \cite{80}, which made the authors of that reference to fix the radius at the best-fit value \( R = 15.6 \) km. Let us also note that we treat the values of the effective temperatures shown in Table 2 of Ref. \cite{78} as the local (non-redshifted) ones to reproduce the objects’ thermal luminosities, calculated in the same paper\footnote{For the source XTE J1751-305 we reproduce an upper limit of \( 2 \times 10^{32} \) erg s\(^{-1}\) for the thermal luminosity obtained in Ref. \cite{86}, rather than the value \( 4 \times 10^{32} \) erg s\(^{-1}\) shown in the table 2 of Ref. \cite{78}.}. It is interesting, that the parameters of the sources EXO 0748-676 and Aql X-1 almost coincide in Tab. \cite{1}. For each \( T_{\text{eff}}^\infty \) we calculate the internal redshifted temperature \( T^\infty \) by employing the analytical fitting formulas from Ref. \cite{51} (see Appendix A3 of that reference), and assuming canonical values of mass and radius for each source (including EXO 0748-676). The relation between \( T_{\text{eff}}^\infty \) and \( T^\infty \) depends on the amount of material accreted onto the NS surface. To get an impression about uncertainty in the value of \( T^\infty \) at a fixed effective temperature \( T_{\text{eff}}^\infty \) we, following Ref. \cite{10}, consider three models of envelope composition, (i) fully accreted envelope (the corresponding internal temperature \( T_{\text{acc}}^\infty \) is given in the fifth column of Tab. \cite{1}; (ii) partially accreted envelope with a layer of accreted light elements down to a column depth of \( P/g = 10^9 \) g cm\(^{-2}\) (the corresponding ‘fiducial’ temperature \( T_{\text{fid}}^\infty \) is presented in the sixth column; \( P \) is the pressure at the bottom of the accreted column, \( g \) is the gravitational acceleration at the stellar surface; the same fiducial value of \( P/g \) has been considered in Refs. \cite{10, 88}); (iii) pure iron envelope (the corresponding temperature \( T_{\text{Fe}}^\infty \) is given in the seventh column). For all sources \( T_{\text{acc}}^\infty < T_{\text{fid}}^\infty < T_{\text{Fe}}^\infty \), because the thermal conductivity of pure iron envelope is lower than that of the envelope with admixture of light elements (iron envelope is better heat insulator). Note, however, that this inequality (and its explanation) is only justified at not too low temperatures \( T_{\text{eff}}^\infty > 10^5 \) K \cite{87}. Finally, the eighth column presents estimates of the averaged accretion rates \( \dot{M} \) onto NSs and the corresponding references. The averaging is performed over a long period of time, which includes both active and quiescent phases. Unfortunately, we have not found estimates of \( \dot{M} \) for some sources.

### B. Observational data vs stability of rapidly rotating NSs

The region of typical temperatures and spin frequencies for NSs in LMXBs is shown in Fig. \ref{fig:stability2}. The small filled circles demonstrate the fiducial temperatures \( T_{\text{fid}}^\infty \) of the sources from Tab. \cite{1}, corresponding to the column depth of light elements \( P/g = 10^9 \) g cm\(^{-2}\). The error bars indicate uncertainties in the internal temperature, which can vary from \( T_{\text{acc}}^\infty \) (fully accreted envelope) to \( T_{\text{Fe}}^\infty \) (iron envelope), see Tab. \cite{1}. If only an upper limit for the effective temperature is known for a source, then the left error bar ends with arrow and the values of \( T_{\text{fid}}^\infty \), \( T_{\text{acc}}^\infty \), and \( T_{\text{Fe}}^\infty \) are calculated for that upper limit. Note that, because \( \nu \) and \( T_{\text{eff}}^\infty \) for the sources EXO 0748-676 and Aql X-1 are very close to one another, the corresponding error bars almost merge in Fig. \ref{fig:stability2}.

By dashes we plot the ‘instability curve’ for the quadrupole \( m = 2 \) \( r^2 \)-mode, which is determined by the condition 
\[ 1/\tau_{\text{GR}} + 1/\tau_{\text{Diss}} = 0. \]
Above this curve \( 1/\tau_{\text{GR}} + 1/\tau_{\text{Diss}} < 0 \), and, as follows from Eq. \eqref{eq:19}, a star becomes unstable with respect to excitation of \( r^2 \)-mode (\( da/dt > 0 \)). This region is often referred to as the instability window for \( r \)-modes \cite{4}. The region filled with \( r^2 \) in the figure is the stability region for \( m = 2 \) \( r^2 \)-mode. One can observe that a number of NSs appears well beyond the stability region.

As it was first shown by Levin \cite{12} (see also Ref. \cite{13}), NSs in LMXBs can undergo a cyclic evolution. This results in a closed track in the \( \nu - T^\infty \) plane with a part of the track belonging to the instability region. For the NS model described in Sec. \ref{sec:2} and the \( r^2 \)-mode saturation amplitude \( \alpha_{\text{sat}} = 10^{-4} \) such a track \( A - B - C - D - A \) is shown in Fig. \ref{fig:stability2} by the thick solid line; medium and thin solid lines demonstrate similar tracks for \( \alpha_{\text{sat}} = 5 \times 10^{-3} \) and \( \alpha_{\text{sat}} = 10^{-1} \), respectively. It is worth noting that, qualitatively, the shape of these tracks does not depend on the details of microphysics input adopted in Sec. \ref{sec:2}.

The evolution tracks in Fig. \ref{fig:stability2} consist of four main stages. Let us describe them briefly, taking the \( A - B - C - D - A \) track as an example (a detailed discussion with a number of useful estimates can be found in the Appendix \cite{13}):

(i) Spin up of the star in the stability region at a temperature \( T_A^\infty = T_{\text{eq}}^\infty \) (the stage \( A - B \)).

The star stays in the stability region and \( r^2 \)-modes are not excited (\( \alpha = 0 \)). In accordance with Eq. \eqref{eq:20}, the spin frequency increases linearly with time due to accretion of matter onto the NS, while the stellar temperature \( T^\infty \), governed by Eq. \eqref{eq:10}, stays constant. This stage lasts \( \tau_{AB} \approx 4 \times 10^9 \) yrs and ends by crossing the instability curve. (ii) Runaway heating of the star in the instability region (the stage \( B - C \)).
This stage starts when the star leaves the $m = 2$ $r^p$-mode stability region due to accretion-driven spin up. The corresponding oscillation amplitude $\alpha$ begins to increase rapidly from the initial value determined by fluctuations (for example, the thermal fluctuations or those, related with accretion). Even at very low initial amplitude $\alpha = 10^{-30}$ it takes $\Delta t_{\text{torq}} \approx 4500$ yrs for the torque associated with viscous damping of $r^p$-mode to become equal to the accretion torque $[d\Omega/dt = 0$, see Eq. (20)]. In the next $\approx 4$ yr $r^p$-mode reaches saturation ($\alpha = \alpha_{\text{sat}}$). During these two periods of time $T^\infty$ and $\Omega$ remain almost unchanged (the shift of the star in Fig. 2 is smaller than the width of the evolution track line).

Having reached saturation, the amplitude of $m = 2$ $r^p$-mode stops to grow and the star (within the time $\Delta t_T \approx 3000$ yrs) warms up to the temperature, at which the neutrino emission exactly compensates the heating caused by the dissipation of the saturated oscillation mode [see Eq. (16)],

$$- \frac{\tilde{J} M R^2 \Omega^2 \alpha_{\text{sat}}^2}{\tau_{\text{GR}}} - L_{\text{cool}} + K_n \dot{M} c^2 = 0.$$  \hspace{1cm} (25)

The temperatures, that satisfy this condition, strongly depend on the stellar spin frequency and the saturation amplitude. We will refer to the corresponding curves in the $\nu - T^\infty$ plane as the Cooling=Heating curves; they are shown in Fig. 2 for $\alpha_{\text{sat}} = 10^{-4}$, $5 \times 10^{-3}$, and $10^{-1}$ by the dotted lines. These lines constrain the region of temperatures and frequencies accessible for NSs in LMXBs; the star can not intersect the Cooling=Heating curve during its runaway, since this requires a more intensive heating than the dissipation of the saturated mode can provide. Note that the frequency keeps almost unchanged during the $B - C$ stage.

(iii) Spin down of the star along the Cooling=Heating curve in the instability region (the stage $C - D$).
Having reached the point $C$, the star starts to move along the Cooling=Heating curve, that is, its temperature is determined by the balance of neutrino luminosity and cooling due to dissipation of the saturated mode. As the rate of the angular momentum loss associated with the emission of gravitational waves is larger than the accretion torque in our NS model, the star starts to spin down\(^5\). Eventually, the star returns into the stability region. This stage lasts $\Delta t_{CD} \approx 8 \times 10^8$ yrs.

(iv) Cooling of the star in the stability region (the stage $D \rightarrow A$).

Having entered into the stability region, the $r^o$-mode amplitude vanishes rapidly (in $\sim 400$ yrs), and after that a cooling of the star down to the temperature $T_{eq}^\infty$ (the point $A$) takes place. The cooling lasts $\sim 10^9$ yrs, then the cycle repeats. The spin frequency does not change noticeably during the $D \rightarrow A$ stage.

Summarizing, the star spends most of the time in the stage (i) and only rarely gets into the instability region. Furthermore, in the instability region the star spends in the stage (ii) a few orders of magnitude less time than in the stage (iii).

Obviously, none of the observed NSs in LMXBs evolve along the tracks in Fig. 2. Various modifications of the standard scenario described above, for example, decreasing of $T_{eq}^\infty$ (with the aim to increase $\Omega_B$) and increasing of the saturation amplitude $\alpha_{sat}$, can allow one to interpret the observed sources as moving along the horizontal part of the evolution track, that corresponds to the stage (ii) – runaway heating of a star in the instability region. However, such modifications would make the detection of any source in this stage even more unlikely since they would further decrease the fraction of time spent there by the star\(^6\). In addition, this interpretation of observations would also suggest that a significant number of NSs in LMXBs should be located in the stage (iii) (on the Cooling=Heating curve), since the duration of this stage is a few orders of magnitude larger than that of the stage (ii) (see Appendix B). As follows from Fig. 2, the Cooling=Heating curves (the dotted lines) correspond to very high temperatures ($T^\infty \sim 4 \times 10^8$ K), so such stars should have been observed. Nevertheless, none of the NSs detected in LMXBs have a redshifted effective temperature larger than $T_{eq}^\infty \gtrsim 2 \times 10^6$ K (which corresponds to $T_{eq}^\infty \gtrsim 4 \times 10^8$ K for the canonical NS model).

In other words, the NS temperatures and frequencies inferred from the LMXB observations cannot be explained within the standard scenario. Therefore, to explain the sources from Fig. 2 one usually follows a different approach, trying to rise the instability curves so that all the sources would be contained inside the stability region. To this aim one needs to enhance dramatically the dissipation of the unstable oscillation mode. Since the mode frequencies are non-linear functions of the angular momentum loss associated with the emission of gravitational waves is larger than the accretion torque, the spin frequency does not change noticeably during the

\(^5\) For lower saturation amplitudes the latter condition may be violated. In that case the star moves to the stationary point at the Cooling=Heating curve, where the accretion torque is balanced by the angular momentum loss due to emission of gravitational waves from the unstable oscillation mode.

\(^6\) In the very recent paper \(^9\) it is argued that the minimum saturation amplitude $|C_R|_{\nu IT, \min}$ can be as low as $\approx 10^{-7}$ (hence $\alpha_{sat} \approx 10^{-6}$, see the footnote \(^8\) for fiducial values of the stellar parameters $\nu = 500$ Hz, $T = 10^8$ K, $R = 10$ km, $K_4 = 1$, and $k_D = 1$ (see equations (16) or (35) of Ref. \(^8\)). The latter two parameters $K_4$ and $k_D \leq 1$ refer to, respectively, the shear viscosity $\eta$ and the three-mode coupling. This value of $|C_R|_{\nu IT, \min}$ is obtained assuming that the two daughter modes have the principal mode numbers $n_D \sim 100$ and the inverse damping timescale $\tau_D = 1/\gamma_D \approx 5000$ s is only $\approx 50$ times smaller than the shear viscosity damping timescale $\tau_\eta \approx 2.2 \times 10^8$ s of $m = 2$ mode, following from our Eq. (15). This value of $\tau_D$ does not look realistic since one would expect $\tau_D/\tau_\eta \sim 1/n^2_D = 10^{-8}$ for $n_D = 100$.
IV. SUPERFLUID AND NORMAL MODES

A. Two main assumptions

In this section we formulate and discuss two main assumptions which are made in order to explain observations. Strictly speaking, these two types are clearly distinct only if one sets to zero the so-called coupling parameter $s$ [36–38]. In the absence of other mechanisms of mode decoupling (see the end of this section), $s = s_{EOS}$, where the parameter $s_{EOS}$ depends only on EOS of superdense matter and is given by [37]

$$s_{EOS} \equiv \frac{n_e}{n_b} \frac{\partial P(n_b, n_e)/\partial n_e}{\partial P(n_b, n_e)/\partial n_b}.$$  

Here $n_b$ and $n_e$ are, respectively, the baryon and electron number densities. As it was shown in Refs. [36, 37], when $s$ vanishes, equations governing superfluid and normal modes decouple into two independent systems of equations. In this approximation a system, that describes the normal modes, can be written in the exactly the same form as for a nonsuperfluid star. Hence, the spectrum and eigenfunctions of normal modes coincide with the corresponding quantities of a normal star, and oscillation frequencies $\omega$ are independent of NS temperature $T_\infty$. Superfluid inertial modes, in turn, do not have a counterpart in normal stars; unlike the normal modes, $\omega$ for superfluid modes is a strong function of $T_\infty$.

In reality, the actual coupling parameter $s$ is although small but finite (for example, for APR EOS $s_{EOS} \sim 0.01 \div 0.03$ [37]). This leads to a strong interaction of normal and superfluid modes when their frequencies become close to one another. As a result, instead of crossings of these modes in the $\omega - T_\infty$ plane, one has avoided crossings: At $T_\infty > T_\infty^0$ the mode II starts to behave as a superfluid mode while the mode I becomes normal-like. In contrast, assuming $s = 0$, one would obtain crossing of instead of avoided crossing (see the dashed lines in the figure); in that case superfluid and normal modes would not ‘feel’ each other.

The qualitative behavior of oscillation modes in superfluid NSs described above has been confirmed by direct calculation of radial oscillation modes [38, 47]. The concept of weakly interacting superfluid and normal modes has also been used in Refs. [38, 90] for a detailed analysis of nonradial oscillation spectra of nonrotating NSs and damping of these oscillations.

Unfortunately, self-consistent calculations of oscillations of rotating superfluid NSs at finite temperatures are still unavailable in the literature. However, it seems natural that the behavior of inertial modes (in particular, $r$-modes)
in superfluid NSs should be quite similar. The results of Refs. [33 35 48] provide indirect independent confirmation of this assumption (see below).

Thus, our first main assumption is:
1. *An oscillation mode of a superfluid rotating NS, which behaves, at some \( T^\infty \), as a normal quadrupole \( m = 2 \) \( r^s \)-mode (\( i^s \)-mode) can, as the temperature gradually changes, transform into a superfluid-like inertial mode (\( i^s \)-mode).

Our second main assumption is:
2. *Dissipative damping of an NS oscillation mode in the regime when it mimics the \( m = 2 \) \( r^s \)-mode is much smaller than damping of this mode in the superfluid-like (\( i^s \)-mode) regime* [see Fig. 5(b,c)], which shows a qualitative dependence of the damping timescale \( \tau_{\text{Diss}} \) and its inverse \( \frac{1}{\tau_{\text{Diss}}} \) on \( T^\infty \) for the same two modes as in Fig. 5(a)].

What is the second assumption based on?

First, on the analysis of \( \tau_{\text{Diss}} \) for nonradial oscillations of a nonrotating NS [38], damping of oscillation modes due to the shear viscosity in the superfluid-like regime occurs approximately 10 times faster than their damping in the normal-like regime. The reasons of that are discussed in detail in section 7.4 of Ref. [38] and should be applicable to \( r \)-modes. This is also in line with the results of Refs. [33 34], where it was found that \( \tau_2 \) for the zero-temperature \( i^s \)-modes is generally more than one order of magnitude smaller than for normal \( r^s \)-modes (compare the table 1 of Ref. [33] and the table 2 of Ref. [34]).

But the main dissipation mechanism, that leads to a drastic difference (by orders of magnitude) of \( \tau_{\text{Diss}} \) in superfluid- and normal-like regimes, is the mutual friction between the superfluid and normal matter components [39 40 91]. The friction occurs because of electron scattering off the magnetic field of Feynman-Onsager vortices. The corresponding magnetic field is generated because of entrainment [92] of superconducting protons by the motion of superfluid neutrons.

This mechanism tends to equalize the velocities of normal and superfluid components; it does not noticeably affect dissipation of the normal modes, since for normal modes these velocities approximately coincide (co-moving motion). On the opposite, mutual friction is extremely effective for superfluid modes, because in that case the difference between the normal and superfluid velocities is large (counter-moving motion). In application to \( r \)-modes the effects of mutual friction were studied in detail in Refs. [33 38 49 53]. In particular, the damping timescale for normal \( r \)-modes (\( r^s \)-modes) due to mutual friction was shown to be

\[
\frac{1}{\tau_{\text{MF norm}}} = \frac{1}{\tau_{\text{MF norm} 0}} \left( \frac{\Omega}{\Omega_0} \right)^5,
\]

where \( \tau_{\text{MF norm} 0} \sim 10^3 \div 10^4 \) s [33 48]. Superfluid \( r \)-modes (\( r^s \)-modes) and superfluid inertial modes (\( i^s \)-modes) were studied, for the first time, in Refs. [33 and 34], respectively; for the damping timescale of these modes due to mutual friction they obtain

\[
\frac{1}{\tau_{\text{MF eff}}} = \frac{1}{\tau_{\text{MF eff} 0}} \left( \frac{\Omega}{\Omega_0} \right),
\]

where \( \tau_{\text{MF eff} 0} \sim 0.1 \) s (see the table 1 in Ref. [33] and the table 2 in Ref. [34]). It is interesting that \( i^s \)-modes were also presumably found in Ref. [48] (see the resonances in their figure 6 and the corresponding discussion in that reference).

The results obtained by Lee and Yoshida [33 34] indirectly confirm our main assumptions 1 and 2. These authors employed the zero temperature approximation (\( T^\infty = 0 \)) and varied the so-called ‘entrainment’ parameter \( \eta \) (\( \eta \) in their paper), that parameterizes interaction between the superfluid neutrons and superconducting protons. It follows from the microphysics calculations [94 96] that \( \eta \) is a function of \( T^\infty \). Hence, its variation is analogous to a variation of stellar temperature. In other words, the eigenfrequencies and eigenfunctions for the superfluid oscillation modes should depend on \( \eta \), while these for the normal modes should be almost insensitive to this parameter. Thus, all the discussed above peculiarities in the behavior of oscillation modes with changing \( T^\infty \) should also be observed in calculations of Refs. [33 34], where \( \eta \) is varied. (In particular, Fig. 5 should still be applicable, provided that one replaces \( T^\infty \) with \( \eta \) there.)

And indeed, Lee and Yoshida [33 34] found numerous avoided crossings of superfluid and normal inertial modes (see their figures 5–8 in Ref. [34]). Concerning \( r \)-modes, in Ref. [33] they found avoided crossing between \( m = 2 \) \( r^s \)-mode and one of the normal inertial \( i^s \)-modes (see their figure 7) and crossings of \( m = 2 \) \( r^s \)-mode with two superfluid inertial modes (see their figure 8). In the latter case Lee and Yoshida emphasized on p. 409 that: ‘it is quite difficult to numerically discern whether the mode crossings result in avoided crossings or degeneracy of the mode frequencies at the crossing point’. If our interpretation is correct there should be avoided crossings.

This point of view is supported by the figure 12 of the same Ref. [33]. The figure shows the timescale \( \tau_{\text{MF norm}} \) [corresponding to our time \( \tau_{\text{MF norm} 0} \), introduced in Eq. (27)] for \( m = 2 \) \( r^s \)-mode as a function of \( \eta \) for the same stellar parameters as in figure 8 of that reference. One can see that \( \tau_{\text{MF norm}} \) in figure 12 sharply decreases (by few orders of magnitude) at the values of \( \eta \) at which one observes crossing of \( r^s \)- and \( i^s \)-mode in figure 8. This is exactly what
one would expect if our assumptions 1 and 2 were correct. Near the crossing of modes (which is avoided crossing in reality) \( m = 2 \) \( r^s \)-mode start to transform into \( i^s \)-mode, and hence \( \tau_{MF,0} \) drops down rapidly. Moving away from the avoided crossing (by decreasing or increasing \( \tilde{\eta} \)) the solution found by Lee and Yoshida resembles more and more \( m = 2 \) \( r^s \)-mode. Consequently, \( \tau_{MF,0} \) grows on the both sides of the resonance, approaching the asymptote value corresponding to the pure (with no admixture) \( m = 2 \) \( r^s \)-mode. The results obtained in figure 12 of Ref. \[33\] are shown qualitatively by filled circles in our Fig. 3(c).

The fact that Lee and Yoshida \[33\] fail to discriminate between crossing and avoided crossing of modes in their figure 8 indicates that the real coupling parameter \( s \) responsible for the interaction of \( m = 2 \) \( r^s \) and \( i^s \) is actually much smaller than the parameter \( s_{EOS} \) given by Eq. (26). The reason is the stellar matter only weakly deviates from the beta-equilibrium state in the course of \( m = 2 \) \( r^s \)-mode oscillations [the deviation \( \delta \mu \sim (\Omega/\Omega_0)^4 \) \[48\] is small since \( \Omega \ll \Omega_0 \)]. It can be shown [30, 33] that in that case the superfluid degrees of freedom decouple from the normal ones especially well. According to our preliminary estimates, the real coupling parameter can be of the order \( s \sim s_{EOS} (\Omega/\Omega_0)^2 \). If this estimate is correct then for \( s_{EOS} = 0.01 \) and \( \Omega/\Omega_0 = 0.1 \) one has \( s \sim 10^{-4} \). However, in view of the existing uncertainties, in this paper we adopt the larger value, \( s = 0.001 \). We checked that the variation of \( s \) within the very wide range (by orders of magnitude) does not affect our principal results.

### B. Mixing the modes

Obviously the fact that the real oscillation modes of superfluid NSs demonstrate, depending on \( T^\infty \), either normal- or superfluid-like behavior should have a major effect on the stability region discussed in Sec. IIIB. To describe this effect it is necessary to understand how the timescales \( \tau_s \), \( \tau_{MF} \), and \( \tau_{GR} \) are modified during the transformation of the mode from the normal-like to superfluid-like regime (see Fig. 4). Since there are no accurate calculations of these timescales in the literature, below we develop a simple phenomenological model evoked by the perturbation theory of quantum mechanics.

Assume for a moment that the coupling parameter \( s = 0 \), so that the systems of equations describing the superfluid and normal oscillation modes are completely decoupled. The solution to these systems of equations describes two types of independent modes, the superfluid and normal ones. Let us present the eigenfunctions of normal modes in the form of a column vector \( \Psi_{norm} \), and those of superfluid modes – in the form of a column vector \( \Psi_{sfl} \). Assume further that \( \Psi_{norm} \) and \( \Psi_{sfl} \) are normalized by the same oscillation energy \( E_c \) and that the timescale \( \tau_s \) of damping/excitation of oscillations due to some dissipation mechanism [e.g., shear viscosity (\( X = S \)), mutual friction (\( X = MF \)), or gravitational radiation (\( X = GR \))] is given by the general formula of the form

\[
\frac{1}{\tau_X} = - \frac{1}{2E_c} \frac{dE_c}{dt} = - \frac{1}{2E_c} (\Psi, \dot{\Psi}),
\]

where \( \dot{A} \) is a matrix differential operator and \((\Psi_1, \Psi_2)\) is a scalar product, both specified by the actual mechanism of dissipation. For example, for \( X = S \) or \( MF \) the scalar product is defined as (e.g., Ref. \[34\])

\[
(\Psi_1, \Psi_2) \equiv \int_{star} \Psi_1^\dagger \Psi_2 dV,
\]

where the integration is performed over the NS volume \( V \). To determine the timescale \( \tau_{norm}^s \) for normal modes one should set \( \Psi \equiv \Psi_{norm} \) in Eq. (29); similarly, to determine the timescale \( \tau_{sfl}^s \) for superfluid modes one should assume \( \Psi \equiv \Psi_{sfl} \). Note that for normal \( \tau \)-modes the timescales \( \tau_{GR}^s \) and \( \tau_{S}^s \) have been already calculated in Sec. III and are given by, respectively, Eqs. (13) and (15).

As has been mentioned above, in reality the parameter \( s \) is small but finite. This means that the eigenfunctions \( \Psi_{norm} \) and \( \Psi_{sfl} \) approximate well the exact solution far from the avoided crossings of neighboring modes (\( \Psi_{norm} \) describes well the exact solution in the normal-like regime, while \( \Psi_{sfl} \) – in superfluid-like regime). However, in the vicinity of an avoided crossing the eigenfunctions of the exact solution should be presented as a linear superposition of \( \Psi_{norm} \) and \( \Psi_{sfl} \). In particular, in Fig. 4 avoided crossing occurs between the modes I and II. Denoting the corresponding eigenfunctions as \( \Psi_1 \) and \( \Psi_{II} \), one can write

\[
\Psi_1 = -\sin(\theta(x)) \Psi_{norm} + \cos(\theta(x)) \Psi_{sfl},
\]

\[
\Psi_{II} = \cos(\theta(x)) \Psi_{norm} + \sin(\theta(x)) \Psi_{sfl},
\]

\[\text{[34]}\]

The definition of scalar product for \( X = GR \) follows, e.g., from the equations (36) and (37) of Ref. \[34\].
where \( \cos \theta(x) \) and \( \sin \theta(x) \) guarantee the correct normalization of the eigenfunctions \( \Psi_1 \) and \( \Psi_{11} \) by the oscillation energy \( E_c \), while the function \( \theta(x) \) determines how the normal mode transforms into superfluid one (and vice versa). This function depends on the parameter \( x \equiv (T^\infty - T_0^\infty) / \Delta T^\infty \) [see Fig. 4(a)] and ranges from 0 to 1 on a temperature scale specified by the characteristic width \( \Delta T^\infty \) of the avoided crossing, \( \Delta T^\infty \sim s T_0^\infty \). The exact form of the function \( \theta(x) \) can be found only by direct solution to the coupled oscillation equations. However, using as the analogy the problem of intersection of electron terms in molecules (see, e.g., Ref. [97], §79), one can immediately write down an approximate expression for \( \theta(x) \) that correctly reproduces its main properties,

\[
\theta(x) = \frac{1}{2} \left[ \frac{\pi}{2} + \arctg(x) \right].
\]

(33)

Consider, for example, the mode II. At \( x \to -\infty \) one has \( \theta(x) \to 0 \), and it follows from Eq. (32) that the mode II is in normal-like regime (\( \Psi_{11} = \Psi_{\text{norm}} \)); at \( x \to +\infty \) one obtains \( \theta(x) \to \pi/2 \), which corresponds to superfluid-like behavior of the mode II (\( \Psi_{11} = \Psi_{\text{sfl}} \)).

Now, substituting Eqs. (31) and (32) into (29) and neglecting the interfering terms of the form\(^8\)

\[
- \frac{1}{2E_c} \cos \theta(x) \sin \theta(x) \left( \Psi_{\text{norm}}^* A \Psi_{\text{sfl}} + \Psi_{\text{sfl}}^* A \Psi_{\text{norm}} \right),
\]

(34)

one gets

\[
\frac{1}{\tau_X} \approx \frac{1}{\tau_X^{\text{norm}}} \sin^2 \theta(x) + \frac{1}{\tau_X^{\text{sfl}}} \cos^2 \theta(x)
\]

(35)

for the mode I and

\[
\frac{1}{\tau_X} \approx \frac{1}{\tau_X^{\text{norm}}} \cos^2 \theta(x) + \frac{1}{\tau_X^{\text{sfl}}} \sin^2 \theta(x)
\]

(36)

for the mode II. These are the main formulas of our approximate model. Their use for \( X = S, \text{MF} \), GR enables us to plot the instability windows for the real oscillation modes (similar to the modes I and II shown in Fig. 8).

V. REALISTIC INSTABILITY WINDOWS AND THREE-MODE REGIME

A. Realistic instability windows

Let us assume, that a certain oscillation mode of a rotating superfluid NS (by analogy with previous section we will refer to it as the mode II) behaves like \( m = 2 r^o \)-mode at low temperatures, and that at \( T^\infty = T_0^\infty \) it experiences an avoided crossing with another mode (with the same \( m = 2 \), let us call it the mode I), which behaves like a superfluid inertial mode (\( i^s \)-mode) at low \( T^\infty \) (exactly as on the scheme in Fig. 3). After avoided crossing the mode I starts to behave as \( m = 2 r^o \)-mode, while the mode II – as \( i^s \)-mode. Let us determine the instability windows for these modes.

The instability windows are defined by the following inequality (see also Sec. III B above)

\[
\frac{1}{\tau_{\text{GR}}} + \frac{1}{\tau_S} + \frac{1}{\tau_{\text{MF}}} < 0.
\]

(37)

Each of these timescales can be calculated using Eq. (36) for the mode II and Eq. (35) for the mode I. One only needs to specify the values for \( \tau_X^{\text{norm}} \) and \( \tau_X^{\text{sfl}} \), which will be employed in each case.

(i) Shear Viscosity (\( X = S \)). The damping timescale \( \tau_S^{\text{norm}} \) for \( m = 2 r^o \)-mode is determined by Eq. (13). According to the discussion in Sec. III A, \( \tau_S^{\text{MF}} \) for \( i^s \)-mode is taken to be

\[
\tau_S^{\text{MF}} = c_S \tau_S^{\text{norm}},
\]

(38)

where \( c_S = 0.1 \). Since the mutual friction dissipation dominates for superfluid \( i^s \)-mode [see item (ii) below and compare Eqs. (28) and (38)], the specific value of the coefficient \( c_S \) is not important for our scenario, one can take 1 or 0.01 instead of 0.1, the main results will not change.

\(^8\) The contribution of these terms can be neglected since the timescales \( \tau_X^{\text{norm}} \) and \( \tau_X^{\text{sfl}} \) differ by at least one order of magnitude (see Sec. IIIA for details).
FIG. 4: Instability curves for superfluid NS oscillations. The red and blue solid curves correspond to \( m = 2 \) modes I and II, which experience avoided crossing at \( T_0^\infty = 1.5 \times 10^8 \) K. Coupling parameter was chosen to be \( s = 0.001 \). The dashed red and blue curves correspond to \( m = 2 \) \( r^o \)- and \( i^o \)-mode plotted under assumption that they are completely decoupled (\( s = 0 \)). The grey line is the instability curve for \( m = 3 \) \( r^o \)-mode, plotted ignoring the resonance coupling with the superfluid modes. The temperature \( T_0^\infty \) is shown by the vertical dotted line. Similar to Fig. 2, the panel (b) shows temperatures and frequencies of the sources from Tab. II Only the fastest source 4U 1608-522 is shown in the panel (a). See text for details.

(ii) Mutual friction (X = MF). The damping timescale \( \tau_{\text{norm}}^{\text{MF}} \) is given by Eq. (27) with \( \tau_{\text{norm}}^{\text{MF}} = 10^4 \) s; the time \( \tau_{\text{MF}}^{\text{eff}} \) is determined from Eq. (28) with \( \tau_{\text{MF}}^{\text{eff}} = 2.5 \) s. Our scenario is insensitive to the actual choice of \( \tau_{\text{MF}}^{\text{eff}} \) because the mutual friction is not a dominating dissipative process for normal modes. However, it is crucial that \( \tau_{\text{MF}}^{\text{eff}} \) be sufficiently small, \( \tau_{\text{MF}}^{\text{eff}} \lesssim 100 \) s.

(iii) Gravitational radiation (X = GR). The timescale \( \tau_{\text{GR}}^{\text{eff}} \) is given by Eq. (39); \( \tau_{\text{GR}}^{\text{eff}} \) is taken to be

\[
\tau_{\text{GR}}^{\text{eff}} = c_{\text{GR}} \tau_{\text{GR}}^{\text{norm}},
\]

where \( c_{\text{GR}} = 100 \). Such expression for gravitational radiation timescale for \( i^o \)-mode agrees qualitatively with the results of Refs. [33, 34] (see equation (44) and table 2 of Ref. [34]), where even longer timescales were obtained, corresponding to \( c_{\text{GR}} > 10^4 \). Thus, our fiducial value \( c_{\text{GR}} \) underestimates the \( \tau_{\text{GR}}^{\text{eff}} \) for \( m = 2 \) \( i^o \)-mode significantly. Increasing of \( c_{\text{GR}} \) and (even further decreasing of \( c_{\text{GR}} \) down to \( \sim 1 \)) does not affect the scenario suggested in this paper.

Instability curves for the modes I (solid red line) and II (solid blue line) are shown in Fig. 4(a,b). The curves are obtained by making use of Eqs. (35)–(39) with the coupling parameter \( s = 0.001 \). The panel (b) is a version of panel (a), but plotted in a different scale. The dotted line in Fig. 4(a,b) corresponds to the temperature \( T_0^\infty = 1.5 \times 10^8 \) K, at which the modes I and II experience avoided crossing. In addition, Fig. 4(a,b) shows the instability curves for: (i) octupole \( m = 3 \) \( r^o \)-mode (grey solid line; to plot it we take the characteristic timescales \( \tau_3 \) and \( \tau_{\text{GR}} \) from Sec. III and ignore the mutual friction, \( \tau_{\text{MF}} \equiv \infty \)); (ii) \( m = 2 \) \( r^o \)-mode (blue dashed line); (iii) superfluid \( i^o \)-mode with \( m = 2 \) (red dashed line). The latter curves (i)–(iii) are obtained using the approximation \( s = 0 \) (neglecting the interaction between the superfluid and normal modes).

As one would expect, far from the avoided crossing point the solid (modes I and II) and dashed (\( r^o \)- and \( i^o \)-modes) lines almost coincide. The region, where \( m = 2 \) modes I, II, and the octupole \( m = 3 \) \( r^o \)-mode are simultaneously stable, is filled with grey in Fig. 4(a,b). The presence of the ‘stability peak’ at \( T^\infty \approx T_0^\infty \) is an important characteristic feature of this region. The height of the peak is determined by the lowest-frequency intersection of the mode II instability curve with the other instability curves. The instability curves for the modes I and II intersect at a very high frequency \( \nu \approx 1580 \) Hz; hence, the lowest-frequency intersection corresponds to that with the octupole \( m = 3 \) \( r^o \)-mode and occurs at \( \nu \approx 625 \) Hz. As a result, at \( T^\infty = T_0^\infty \) the most unstable mode is \( m = 3 \) \( r^o \)-mode, and the
height of the stability peak is $\nu \approx 625 \text{ Hz}$.

As follows from Fig. 4 the evolution of an NS with such complicated structure of the instability windows can be accompanied by excitation of each of the three oscillation modes. Therefore, prior to discussing the evolution tracks one should formulate the equations describing an oscillating star in a three-mode regime.

### B. Three-mode regime

The equations governing the evolution of an NS and allowing for possible excitation of the three modes (I, II and $m = 3 \ r^o$-mode), can be derived in much the same fashion as it was done in Sec. II (see the one-mode equations (16), (19), and (20) in that section). If all the modes are non-saturated, they can be written as

$$\frac{d\alpha_i}{dt} = -\alpha_i \left( \frac{1}{\tau_{GR_i}} + \frac{1}{\tau_{Diss_i}} \right),$$

$$\frac{d\Omega}{dt} = -\sum_i 2Q_i \alpha_i^2 \Omega + \Omega_{acc},$$

$$C_{tot} \frac{d{T^\infty}}{dt} = \sum_i W_{Diss_i} - L_{cool} + KnMc^2,$$

where we neglect the terms $\propto \alpha_i^3$. The index $i$ in Eqs. (40)–(42) runs over the mode types, and

$$W_{Diss_i} = \frac{2E_{c_i}}{\tau_{Diss_i}},$$

$$\frac{1}{\tau_{Diss_i}} = \frac{1}{\tau_{SI_i}} + \frac{1}{\tau_{MF_i}},$$

where $\tau_{SI_i}$ and $\tau_{MF_i}$ for the modes I and II are calculated as it is described in Sec. VA, while for octupole $m = 3 \ r^o$-mode – as is described in Sec. II (we neglect the effects of mutual friction on damping of the octupole $r^o$-mode).

Thus, only the quantities $E_{c_i}$ and $Q_i$ in Eqs. (41) and (42) are left to be determined. The corresponding Eqs. (18) and (22) for the octupole $r^o$-mode are presented in Sec. II. In the case of the modes I and II one can argue as follows.

First, let us discuss the mode II. At low $T^\infty$ (before the avoided crossing) it behaves like $m = 2 \ r^o$-mode. Accordingly, its canonical angular momentum $J_{c,II}$ is given by Eq. (6), where the coefficient $\tilde{J} \approx 1.6353 \times 10^{-2}$. At the avoided crossing point the behavior of the mode changes and it turns into the $i^o$-mode. However, since the canonical angular momentum is an adiabatic invariant [24, 25, 65], $J_{c,II}$ is conserved (neglecting dissipative processes) and stays the same even after passing the avoided crossing. Without any loss of generality, one can assume it to be still related to the oscillation amplitude $\alpha_{II}$ by exactly the same Eq. (6) (with the same $\tilde{J} = 1.6353 \times 10^{-2}$), as before the avoided crossing. This assumption, which should be treated as the definition of the amplitude $\alpha_{II}$ in the superfluid-like regime, has already been implicitly employed when deriving the system of Eqs. (40)–(42). It ensures that $\alpha_{II}$ is continuous throughout the avoided crossing region.

The same reasoning also holds true for the mode I. For a given $J_{c,i}$ the quantities $Q_i$ and $E_{c,i}$ can be found from Eqs. (18) and (22). The problem, however, consists in that the mode energy $E_{c,i}$ depends on the oscillation frequency $\omega$, which is only known for the modes I and II in the normal-like regime [in that case it is given by Eq. (2)]. In the superfluid-like regime $\omega$ depends not only on $\Omega$, but also on $T^\infty$; unfortunately, the function $\omega(\Omega, T^\infty)$ has not yet been calculated. Below, for simplicity, we assume that the frequency $\omega$ is determined by the same Eq. (2) even in the superfluid-like regime. This assumption does not influence our main conclusions and is well justified because the range of $T^\infty$, which is of interest in our scenario (see Sec. VI), is located near avoided crossings of modes. In that region $\omega$ for both modes can indeed be estimated from Eq. (2). Beyond this region any mode in the superfluid-like regime is stable, unexcited, and, correspondingly, not important for NS evolution.

Equations (40)–(42) are satisfied if the oscillation amplitudes $\alpha_i$ are less than the correspondent saturation amplitudes $\alpha_{\text{sat },i}$. In the following, the saturation amplitudes for all the modes are taken to be $\alpha_{\text{sat },i} = 10^{-4}$. Note

---

9 The octupole $m = 3 \ r^o$-mode can also experience a resonant coupling with the superfluid $m = 3$ oscillation modes. However, the correspondent resonance temperatures are unlikely to be close to those for $m = 2 \ r^o$-mode. Therefore, at $T^\infty \approx T^\infty_0$ the instability curve for $m = 3 \ r^o$-mode will hardly be essentially affected by coupling with superfluid modes.
FIG. 5: Evolution of the spin frequency $\nu$ and temperature $T_\infty^8$ for a superfluid NS in LMXB allowing for the avoided crossing of $m=2$ modes I and II. The corresponding track $A-B-C-D-E-F-A$ is shown by thick solid line. The dotted line shows the Cooling=Heating curve (see text for details). Other notations are the same as in Fig. 4.

that our main results are insensitive to the actual value of $\alpha_{\text{sat}}$. If one or more modes are saturated, the evolution equations can be derived in a similar way as it was done in Sec. II.

VI. OUR RESONANCE UP-LIFT SCENARIO

Using the results of the preceding sections, we can examine quantitatively how the resonance coupling of superfluid and normal modes modifies the standard scenario discussed in Sec. III B (see also Fig. 2).

A typical NS evolution track $A-B-C-D-E-F-A$ is shown in Fig. 5 by the thick solid line, calculated for exactly the same model as the instability curves in Sec. V A (see Fig. 4). Other notations coincide with those in Fig. 4. As in Sec. V A we suppose that the mode I experiences an avoided crossing with the mode II at $T_\infty^\infty = T_\infty^0 = 1.5 \times 10^8$ K.

To plot the Cooling=Heating curve (shown by the dotted line in Fig. 5) we use Eq. (42) with $dT_\infty^\infty/dt = 0$. When doing this we assume that all the modes, which are unstable at a given temperature and frequency, are saturated, while the stable modes have vanishing oscillation amplitudes. This means that in each point of the Cooling=Heating curve the neutrino luminosity is exactly compensated by the stellar heating due to nonlinear damping of saturated modes. Let us note that in the stability region (the grey filled area in the figure) we do not use this definition, but instead, by analogy with Fig. 2 we continue the Cooling=Heating curve according to Eq. (25)\textsuperscript{11}. A break of the Cooling=Heating curve the neutrino luminosity is exactly compensated by the stellar heating due to nonlinear damping of saturated modes.

\textsuperscript{10} In particular, the choice of $\alpha_{\text{sat}}$ for $m=3$ $\nu^3$-mode appears to be insignificant and does not even affect the position of the Cooling=Heating curve (see Sec. VI).

\textsuperscript{11} The point is that the Cooling=Heating curve in the instability region is almost indistinguishable from the curve given by Eq. (25), see the following discussion herein.
curvature at the intersection point with the instability curve for \( m = 3 \) \( r^o \)-mode is imperceptible, because the contribution of the octupole mode to stellar heating can be neglected owing to a longer gravitational radiation timescale for this mode [see Eq. (9)]. Therefore, along the whole Cooling=Heating curve the nonlinear damping of the mode I, behaving as the saturated \( m = 2 \) \( r^o \)-mode, is the dominating heating mechanism. This means that the Cooling=Heating curve, obtained allowing for the resonance coupling of modes, is practically indistinguishable from that given by Eq. (25) (see Sec. [11B] and Fig. [2]).

During the \( A \) \( - \) \( B \) stage, an NS stays inside the stability region and gradually spins up by accretion. This stage is completely analogous to \( A \) \( - \) \( B \) stage of the standard scenario shown in Fig. [2]. In the point \( B \) the star becomes unstable with respect to excitation of the mode II, which behaves there as \( m = 2 \) \( r^o \)-mode. In the next stage \( B \) \( - \) \( C \) the amplitude of the mode II increases and rapidly reaches saturation \((\alpha_{\text{sat}} = 10^{-5})\). After that, the star heats up without any significant variation of the spin frequency \( \nu \). This stage ends by reaching the stability peak at point \( C \).

The next stage \( C \) \( - \) \( D \) is the most interesting and is absent in the standard scenario described in Sec. [11B]. Owing to accretion, the star is spinning up along the boundary of the stability peak produced by the avoided crossing of modes I and II. This stage is discussed in details below. In the point \( D \) the star, for the first time, becomes unstable with respect to excitation of the octupole \( m = 3 \) \( r^o \)-mode. The amplitude of this mode increases rapidly and hits saturation, which leads to heating up of the star. As a result, it leaves the stability peak, becomes unstable also with respect to excitation of the mode I, and quickly moves to the point \( E \). Thus, the \( D \) \( - \) \( E \) stage is quite similar to the \( B \) \( - \) \( C \) stage of the standard scenario with the only difference that two modes \((m = 3 \) \( r^o \)-mode and the mode II) are excited (and saturated) in this stage instead of one. The spin frequency is almost constant during this stage. In the point \( E \) the star approaches the Cooling=Heating curve and then spins down along this curve until it enters the stability region in the point \( F \) (the stage \( E \) \( - \) \( F \)). All the oscillation modes vanish in the very beginning of the subsequent stage \( F \) \( - \) \( A \) and the star cools down to the equilibrium temperature \( T_{\text{eq}}^{\infty} \) without noticeable variation of the spin frequency. The stages \( E \) \( - \) \( F \) and \( F \) \( - \) \( A \) are close analogues of, respectively, the stages \( C \) \( - \) \( D \) and \( D \) \( - \) \( A \) of the standard evolution scenario (see Fig. [2]).

Let us return to the almost vertical stage \( C \) \( - \) \( D \) in Fig. [5] and discuss it in more detail. During this stage, the NS moves along the instability curve for the mode II; only the mode II is excited, the amplitudes of other modes are all equal to zero. Since in the stage \( C \) \( - \) \( D \) the stellar temperature \( T^{\infty} > T_{\text{eq}}^{\infty} \), the star requires an additional heating to maintain its thermal balance. This heating is provided by the damping of the mode II. A required power is determined from Eq. (12) by setting \( dT^{\infty}/dt \approx 0 \),

\[
W_{\text{Diss II}} \approx L_{\text{cool}} - K_n \dot{M} c^2. \tag{45}
\]

Using Eqs. (13) and (15), together with Eqs. (6) and (18), one can determine the corresponding equilibrium oscillation amplitude

\[
\alpha_{\text{II}}^{(\text{eq})} = \sqrt{\frac{(L_{\text{cool}} - K_n \dot{M} c^2) \tau_{\text{Diss II}}}{J M R^2 \Omega^2}}, \tag{46}
\]

where \( J \approx 1.6353 \times 10^{-2} \) for the mode II. Since \( \tau_{\text{Diss II}} = -\tau_{\text{GR II}} \) on the instability curve, one can use \( -\tau_{\text{GR II}} \) instead of \( \tau_{\text{Diss II}} \) in this equation\(^{13}\). For example, taking the point on the instability curve with coordinates \( \nu = 400 \) Hz and \( T^{\infty} \approx 1.48 \times 10^8 \) K, we obtain \( L_{\text{cool}} \approx 3.3 \times 10^{34} \) erg s\(^{-1} \), \( \tau_{\text{Diss II}} = -\tau_{\text{GR II}} \approx 1.13 \times 10^4 \) s, and, as follows from Eq. (46), \( \alpha_{\text{II}}^{(\text{eq})} \approx 8 \times 10^{-7} \ll \alpha_{\text{sat II}} = 10^{-4} \).

It is possible for a star to maintain finite, but not saturated, oscillation amplitude for a long time, because it penetrates into the instability region with decreasing \( T^{\infty} \). Indeed, if, for some reason, the mode II has a lower amplitude than that required by Eq. (46), then the star starts to cool down and becomes unstable with respect to excitation of the mode II. This immediately leads to increasing of the amplitude \( \alpha_{\text{II}} \) and to accelerated heating of the

\(^{12}\) Due to this fact, it is easy to understand an impact that the instability curve for \( m = 2 \) \( r^o \)-mode has on the stellar evolution track \( A \) \( - \) \( B \) \( - \) \( C \) \( - \) \( D \) \( - \) \( E \) \( - \) \( F \) \( - \) \( A \) (see its description in the text). The point \( F \) is determined by the intersection of the Cooling=Heating curve, given by Eq. (29), with the instability curve; its frequency fixes the frequency of the point \( A \). The point \( B \) lies on the instability curve at \( T^{\infty} = T_{\text{eq}}^{\infty} \), and specifies the frequency of the point \( C \). Points \( D \) and \( E \) do not depend on the position of \( m = 2 \) \( r^o \)-mode instability curve.

\(^{13}\) It is convenient to use \( -\tau_{\text{GR II}} \) instead of \( \tau_{\text{Diss II}} \) in Eq. (45), since \( \tau_{\text{Diss II}} \) is a strong function of \( T^{\infty} \) in the vicinity of the stability peak. The reason is the increasing role of the mutual friction dissipation owing to admixture of the superfluid mode to the real solution near avoided crossing (see Sec. IVB). Thus, one cannot estimate \( \tau_{\text{Diss II}} \) directly from Eq. (45). On the opposite, the simple Eq. (9) provides an accurate estimate for \( \tau_{\text{GR II}} \) because the gravitational radiation timescale for the normal mode is smaller than for the superfluid one. Hence, an admixture of the superfluid mode has almost no effect on the gravitational timescale for the real NS mode II [see Eq. (36)].
star. As a result, the star moves toward the stability region, where \( \alpha_{II} \) decreases rapidly, the heating becomes less and less efficient and, eventually, is replaced by cooling. The process of modulation of \( \alpha_{II} \) may occur repeatedly, but the correspondent variation of \( T^\infty \) is very small. The characteristic modulation period varies from a few months to years.

It can be shown that the modulation magnitude may decrease or increase in time depending on the parameters of the model. In the first case, during the NS motion along the peak, the amplitude of the mode II adjusts itself to the equilibrium value \( \alpha_{II} \approx \alpha_{II}^{eq} \), and does not experience modulation. In the second case, the maximum value of \( \alpha_{II} \) is typically limited by the saturation amplitude \( (\alpha_{II} = \alpha_{sat II}) \), thus limiting the modulation magnitude. However, even in this case the temperature oscillations accompanying the modulation are very small, less than the thickness of the line in Fig. 5 and can hardly be observed\(^{14} \). At the same time, strong modulation of the oscillation amplitude \( \alpha_{II} \) is also accompanied by the modulation of \( d\Omega/dt \), which is, in principle, observable\(^{15} \). The effects of \( \alpha_{II} \) modulation described above will be discussed in detail in our subsequent publication.

Let us estimate the duration of the spin up stage \( C - D \). Using Eqs. (11) and (10), we get

\[
\frac{d\Omega}{dt} = -\frac{2Q_{II} (L_{\text{cool}} - K_n M c^2)}{J M R^2 \Omega} + \dot{\Omega}_{\text{acc}},
\]  

(47)

where \( Q_{II} \approx 0.094 \) [see Eq. (22)]. First of all, taking into account Eq. (21) one can determine from this formula the minimal NS accretion rate \( M_{\text{min}} \) required to spin up the star,

\[
\dot{M}_{\text{min}} = \frac{3L_{\text{cool}}}{3 K_n c^2 + p \Omega \sqrt{GM^2 R}} \approx 3 \times 10^{-12} \left( \frac{L_{\text{cool}}}{10^{34} \text{erg s}^{-1}} \right) \left( \frac{\Omega}{\Omega_0} \right)^{-1} \left( \frac{M}{1.4 M_\odot} \right)^{-1} \left( \frac{R}{10 \text{km}} \right) \frac{M_\odot}{\text{yr}}.
\]

(48)

At point \( C \) one has \( \Omega_C \approx 1500 \text{ s}^{-1} (\nu_C \approx 239 \text{ Hz}) \), \( T_C^\infty \approx 1.37 \times 10^8 \text{ K} \), \( L_{\text{cool}} \approx 3 \times 10^{34} \text{ erg s}^{-1} \), and it follows from Eq. (48) that \( \dot{M}_{\text{min}} \approx 6.2 \times 10^{-11} \text{M}_\odot/\text{yr} \). If \( \dot{M} > \dot{M}_{\text{min}} \), then the duration of the \( C - D \) stage can be estimated by noticing that the first term in Eq. (47) is smaller than the second one at \( \Omega \gtrsim \Omega_C \). Because \( \Omega_D \approx 3930 \text{ s}^{-1} (\nu_D \approx 625 \text{ Hz}) \), we find

\[
\Delta t_{CD} \approx \frac{\Omega_D - \Omega_C}{\dot{\Omega}_{\text{acc}}} \approx 2.2 \times 10^8 \text{ yrs},
\]

(49)

where we make use of Eq. (21) with our fiducial accretion rate \( \dot{M} = 3 \times 10^{-10} \text{M}_\odot/\text{yr} \). An accurate calculation, which is done without any additional simplifications, gives a close value \( \Delta t_{CD} \approx 2.3 \times 10^8 \text{ yrs} \). This time constitutes approximately 82\% of the period of the A – B - C – D - E – F - A cycle. For comparison, the A – B and E – F stages constitute, respectively, 15\% and 3\% of the cycle; the contribution of all other stages is negligible. Note that the time \( \Delta t_{CD} \) can be even longer, if the magneto-dipole torque is sufficiently large. The corresponding term of the form

\[
\dot{\Omega}_B = -\frac{B^2 R^6}{6 c^3 I} \Omega^3
\]

(50)

should then be added to the right-hand side of Eq. (47). In particular, for a strong enough dipolar magnetic field \( B \) a star can stop spinning up at a frequency at which \( \dot{\Omega}_{\text{acc}} + \dot{\Omega}_B \approx 0 \). For example, this will happen at \( \nu = 600 \text{ Hz} \) (for \( \dot{M} = 3 \times 10^{-10} \text{M}_\odot/\text{yr} \)) if the magnetic field at the poles is \( B \approx 8.8 \times 10^8 \text{ G} \).

Four conclusions can be drawn from the analysis of Fig. 5 and the estimates presented above.

(i) The high spin frequencies of the sources 4U 1608-522, SAX J1750.8-2900, EXO 0748-676, Aq1 X-1, and SWIFT J1749.4-2807 can be explained assuming that these stars are climbing up the peak in the \( C - D \) stage;

(ii) The probability to find these stars with the observed (high) frequencies is not small, since they spend a substantial amount of time in the high frequency region;

(iii) The maximum NS spin frequency is limited by the \( m = 3 \) \( r^\text{a} \)-mode instability curve within our scenario;

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\(^{14} \) The thermal relaxation of an NS crust can also smoothen the temperature oscillations.

\(^{15} \) Note that only the period of modulation and its magnitude depend on the shape of the instability curve; in contrast, the fact that the star stays attached to this curve is purely due to the onset of gravitational instability with decrease of \( T^\infty \). Consequently, the exact form of the instability curve and the function \( \theta(x) \), which determines it [see Eq. (23)], are insignificant for our model.
FIG. 6: An example of the stability curves in case of two avoided crossings of $m = 2$ oscillation modes of a superfluid NS. As in Fig. 4 the solid red and blue lines are plotted for the modes I and II experiencing an avoided crossing at $T^\infty = 1.5 \times 10^8$ K (the coupling parameter $s = 0.001$). An additional violet solid line corresponds to the mode III, which exhibit an avoided crossing with the mode II at $T^\infty = 4.5 \times 10^7$ K. This avoided crossing is drawn for $s = 0.01$. Other notations are the same as in Fig. 4.

(iv) A star, which starts to evolve in the stability region with the temperature lower than that of the avoided crossing of modes I and II, will eventually find itself in the stage $C - D$. The other sources with lower $T^\infty$ (e.g., IGR J00291-5934) can be explained in a similar manner. First, it is obvious that the temperature $T_0^\infty$ of the avoided crossing of modes I and II depends on the NS mass. Hence, if the masses of colder sources differ from those of the hotter ones, the avoided crossing of modes I and II can occur at a different $T_0^\infty$. In particular, it can be shifted to the region of lower temperatures, which are typical for these (rather cold) stars. Second, as it was shown in calculations of non-rotating NS oscillation spectra [38, 47, 90, 98], a normal mode can experience an avoided crossing with the superfluid modes more than once. To illustrate this idea, we demonstrate in Fig. 6 the instability curves in case of two avoided crossings of oscillation modes. The first avoided crossing takes place at $T^\infty = 4.5 \times 10^7$ K between the mode III (the violet solid line), which behaves as $m = 2$ $r^o$-mode at low $T^\infty$, and the mode II (the blue solid line). For this avoided crossing the coupling parameter was chosen to be $s = 0.01$. The second avoided crossing of modes I and II is discussed above (see Fig. 5); it takes place at $T^\infty = 1.5 \times 10^8$ K. In this case the mode II behaves as $m = 2$ $r^o$-mode only at intermediate temperatures $6 \times 10^7$ K $\lesssim T^\infty \lesssim 1.3 \times 10^8$ K. At higher and at lower temperatures it transforms into superfluid modes, which are different. It is easy to demonstrate that, for low enough $T_{eq}^\infty \lesssim 4 \times 10^7$ K, the evolution track goes along the left (low-temperature) boundary of the first stability peak, corresponding to the avoided crossing of the modes II and III (i.e., along the violet line in Fig. 6). This stage is a direct analogue of the $C - D$ stage in Fig. 5 and an NS stays there for a long time. One sees that two avoided crossings are already sufficient to explain all the existing observations of frequencies and quiescent temperatures of NSs in LMXBs.

\footnote{In reality, the number of avoided crossings can be larger.}
The sources IGR J00291-5934, MXB 1659-298, KS 1731-260, and XTE J1751-305 can be interpreted as moving along the instablility curve of the mode III (the violet line in the figure). This interpretation requires that their equilibrium temperature $T_{\text{eq}} \lesssim 4 \times 10^7$ K. The parameters of the objects IGR J17498-2921 and SAX J1748.9-2021 can be explained by accretion spin up at $T^\infty = T_{\text{eq}} \sim 5 \times 10^7$ K, which takes place inside the stability region (an analogue of the $A-B$ stage in the scenarios discussed above). Finally, an explanation of the hottest stars 4U 1608-522, SAX J1750.8-2900, EXO 0748-676, Aql X-1, and SWIFT J1749.4-2807 remains the same as in the scenario with one avoided crossing (however, because of the additional avoided crossing the equilibrium temperature $T_{\text{eq}}$ should comply with the condition $5 \times 10^7$ K $\lesssim T_{\text{eq}} \lesssim 1.4 \times 10^8$ K for these sources). The rest of the stars lie in the stability region (even without accounting for the resonant coupling of modes), so they can be explained as being in the $A-B$ stage with the corresponding equilibrium temperature $T_{\text{eq}}^\infty$ (see Sec. III B).

Let us note that, to spin up the rapidly rotating sources up to the observed spin frequencies $\Omega$ during the time period shorter than the age of the Universe $t_{\text{Un}}$, one needs quite a strong accretion torque $\Omega_{\text{acc}}^{(\text{crit})} \gtrsim \Omega/t_{\text{Un}} \sim 3 \times 10^{-7}$ s$^{-1}$ yr$^{-1}$, and hence quite a high accretion rate $[M_{\text{crit}} \gtrsim 10^{-11} M_\odot$/yr if one uses Eq. (21)]. Thus, to explain the low-temperature sources (those like MXB 1659-298) a rapid NS cooling may be required (e.g., with the open direct Urca process in the central regions of the star [93]), which allows one to have a lower $T_{\text{eq}}^\infty$ at higher $M$. This can indicate that the coldest rapidly rotating NSs in LMXBs are more massive. An alternative explanation of these sources (not requiring the enhanced cooling) is also possible. It implies a more efficient accretion torque due to some peculiarities in the accretion onto these objects, that results in a large value of $\Omega_{\text{acc}}$ at a relatively small accretion rate $M$. The last hypothesis agrees well with the very low observational estimate $M \approx 2.5 \times 10^{-12}$ M$\odot$/yr $\ll M_{\text{crit}}$ for the source IGR J00291-5934 (see Tab. I).

In Fig. [6] we have considered a situation in which an additional avoided crossing appears at lower $T^\infty$ than for the modes I and II. It is also interesting to see, how the additional avoided crossing affects the NS evolution if it appears at higher $T^\infty$. This possibility is studied in Appendix C, where it is shown that the four conclusions (i)–(iv) stated above hold true even in this case.

Thus, we demonstrate that the high spin frequencies of NSs in LMXBs can be explained within our new scenario. But is this scenario consistent with the existence of millisecond pulsars? It is generally believed, that millisecond pulsars originate from LMXBs, in which the accretion has ceased for some reason. We interpret the millisecond pulsars as the stars evolving in the stage $\bar{D}$ (see Fig. [3]), or, what is more likely, in a similar stage associated with another low-temperature avoided crossing (such as that shown at $T^\infty \approx 4.5 \times 10^7$ K in Fig. [6]). Since for these NSs $\Omega_{\text{acc}} = 0$, their spin frequency will gradually decrease, while the rotation energy will be carried away by gravitational waves and neutrinos. Clearly, the interpretation of millisecond pulsars as stars climbing down in the stage $\bar{D} - C$ (or a similar stage) will be convincing only if they spin down rather slowly, so that the probability to observe them rapidly rotating would not be too small. To get an impression about the typical times of climbing down, let us estimate the time $\Delta t_{\text{DC}}$ spent by a pulsar on the way from point $D$ to point $C$. Even for a very high $T^\infty = 1.5 \times 10^8$ K (and hence $L_{\text{cool}} \approx 3.45 \times 10^{34}$ erg s$^{-1}$), Eq. (47) with $\Omega_{\text{acc}} = 0$ and $M = 0$ gives

$$\Delta t_{\text{DC}} = \frac{\bar{J} M R^2 (\Omega_D^2 - \Omega_s^2)}{4 Q_H L_{\text{cool}}} \approx 1.5 \times 10^9 \text{ yrs}. \quad (51)$$

Such a long time indicates that the proposed scenario can be applicable for millisecond pulsars as well. It is easy to demonstrate that, due to magneto-dipole losses only [see Eq. (50)], a star would spin down during the same period of time if it had the dipolar magnetic field at the poles $B \approx 7 \times 10^8$ G.

VII. CONCLUSIONS

We demonstrate that the key role in the evolution of NSs in LMXBs is played by the resonance interaction of the normal $m = 2$ oscillation $r$-mode ($r^\alpha$-mode) and the superfluid inertial modes ($i^\alpha$-modes). This result allows us to formulate a scenario that explains the existence of rapidly rotating warm NSs observed in LMXBs, and agrees with the existence of millisecond pulsars (see Sec. VII). A detailed application of our scenario to millisecond pulsars will be reported in our subsequent publication.

The conclusion about the resonance interaction of $i^\alpha$- and $r^\alpha$-modes is based on the following facts:

1. Detailed calculations [37, 58, 47, 51, 98] of the oscillation spectra of nonrotating superfluid NSs at finite temperatures $T^\infty$ reveal that: (i) The frequencies of the superfluid modes essentially depend on $T^\infty$, while those of the normal modes are almost insensitive to a temperature variation. (ii) If, at some $T^\infty$, the frequencies $\omega$ of two arbitrary (but with the same ‘quantum’ number $m$) superfluid and normal modes become equal, they start to interact resonantly. As a result of such interaction, the superfluid mode turns into the normal one and vice versa; that is, an avoided
crossing of modes is formed in the $\omega - T_0^\infty$ plane. (iii) Far from the avoided crossings superfluid and normal modes are almost non-interacting and are described by the two weakly coupled systems of equations.

2. According to computations of Lee and Yoshida \[33, 34\] performed in the $T_0^\infty = 0$ approximation, the frequencies of $i^s$-modes are sensitive to a variation of the so-called entrainment parameter $\tilde{\eta}$ (see Sec. [V A]). In particular, at some specific values of $\tilde{\eta}$ avoided crossings of superfluid and normal oscillation modes are observed (see also Sec. [V B]).

3. An account for finite $T_0^\infty$ leads to a temperature dependence of a number of parameters of superfluid hydrodynamics (including $\tilde{\eta}$).

The items 2 and 3 give us a ground to assume that the results formulated in the item 1 in application to nonrotating NSs remain valid for rotating NSs as well. Hence, the frequencies of $i^s$-modes should also depend on $T_0^\infty$. This means, in particular, that avoided crossings between $m = 2 r^s$-mode and $i^s$- modes should be formed at some values of $T_0^\infty$ (see, e.g., Fig. 3). When passing through an avoided crossing, $i^s$-mode transforms into $m = 2 r^s$-mode, while $m = 2 r^s$-mode becomes $i^s$-mode. During such a transformation the eigenfunctions of $m = 2 r^s$-mode mix intensively with those of $i^s$-mode. This leads to the enhancement of $r^s$-mode damping due to mutual friction (see Sec. [V B]). In the $\nu - T_0^\infty$ plane, this effect is manifested by the appearance of a sharp ‘stability peak’ over the standard (usually considered) stability region of fast rotating NSs (see Sec. [V A] and Fig. 3) the stability region is filled with grey there.

An analysis of evolution of an NS in LMXB taking into account the stability peak shows that the star spends a significant amount of time climbing the left side of this peak in the region, which has been previously thought to be unstable with respect to excitation of $r$-modes. To keep on the peak, the average oscillation amplitude adjusts itself so as the star heating due to dissipation of oscillations is compensated by neutrino cooling. Under such circumstances, a spin-down due to gravitational wave emission can be insufficient to oppose the accretion torque on the star. This leads to a gradual increasing of the NS spin frequency as it slowly climbs up the peak (see Sec. [VI] for details).

The spin-up stage is terminated when the star reaches the instability curve for $m = 3$ oscillation $r^s$-mode, which is the next unstable mode in normal NSs after $m = 2 r^s$-mode\[17\]. As a result, the star jumps off the peak and shortly returns to the stability region (see Sec. [VI]). Thus, the real limit on the spin frequency of NSs is set by the instability curve for the octupole $m = 3 r^s$-mode. This result allows us to explain the fast rotation of NSs in LMXBs within the minimal assumptions about the properties of superdense matter. Moreover, this result agrees with the predicted\[17, 18\] abrupt cut-off above $\sim 730$ Hz of the spin frequency distribution of accreting millisecond X-ray pulsars. Furthermore, because $\Omega = \Omega_{\sec}$ in the $A$ -- $B$ and $C$ -- $D$ stages, our scenario predicts the frequency distribution to be almost constant at 200-600 Hz, in agreement with observations (see, e.g., figure 5 of Ref. [7]).

It is important to emphasize that our scenario is almost insensitive to an actual choice of the parameters regulating the resonance interaction between the modes (resonance temperatures and width of the peaks, see Secs. [IV] and [V], and does not require any nonrealistic enhancement of the kinetic coefficients and/or additional exotic damping mechanisms.

Obviously, the new approach to the evolution of rapidly rotating NSs and interpretation of their observations, suggested in the present paper, needs further development and refinement. In particular, one needs to perform detailed calculations in order to confirm the presence of avoided crossings in the oscillation spectra of warm superfluid rotating NSs, and to study how the resonance interaction of modes affects the oscillation damping times. We expect that the corresponding resonance temperatures will depend on the NS mass and on the parameters of superfluidity. A detailed analysis of the effect of various damping processes (such as, e.g., Ekman layer dissipation\[4, 100, 101\]) on the instability curve of the octupole $m = 3 r^s$-mode will place further restrictions on the spin frequencies of NSs.

If our scenario is correct, then the observed temperatures of the most rapidly rotating NSs must coincide with the temperatures $T_0^\infty$, at which avoided crossings occur between $m = 2 r^s$-mode and the superfluid $i^s$-modes. Comparison of these temperatures $T_0^\infty$ with the results of (still not available) theoretical calculations can impose stringent constraints on the properties of superdense matter and parameters of superfluidity. Clearly, a direct observational test of our scenario is a very important task which we plan to address in the nearest future. In particular, we plan to study in detail the modulation of the NS spin frequency, appearing when the star moves along the stability peak (see Sec. [VI]), and to examine whether this effect can be confirmed observationally.

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\[17\] The possibility that, under certain circumstances, another (secular or dynamical) instability could set in at lower $\Omega$ than the instability of the octupole ($m = 3$) $r^s$-mode cannot be excluded and should be carefully analyzed.
Appendix A: The approximations for neutrino luminosity and heat capacity

The neutrino luminosity $L_{\text{cool}}$ and total heat capacity $C_{\text{tot}}$ of an NS with the mass $M = 1.4\,M_\odot$ are calculated with the relativistic cooling code, described in detail in Refs. [29, 31, 67]. We use essentially the same microphysics input as in Ref. [29]. In particular, we employ the parameterization [58] of APR EOS [21]. The results of our calculations of $L_{\text{cool}}$ and $C_{\text{tot}}$ are roughly fitted by the following formulas:

\[
\begin{align*}
L_{\text{cool}} &= 7 \times 10^{30} (T_8^\infty)^8 \left\{ \sqrt{1.25 T_8^\infty} + 140 \exp \left( -\frac{30}{T_8^\infty} \right) \right\} \text{erg s}^{-1}, \\
C_{\text{tot}} &= 2.1 \times 10^{37} T_8^\infty \left( 1 + \frac{6}{(0.18/T_8^\infty)^{3.9} + 1} \right) \text{erg K}^{-1}.
\end{align*}
\] (A1)

The last term in the expression for $L_{\text{cool}}$ corresponds to the enhancement of the neutrino luminosity due to neutron Cooper pairing.

Appendix B: NS evolution in the absence of resonant interaction with superfluid modes

Let us analyze the evolution of an NS in LMXB in the absence of resonant interaction of the normal $r^\omega$-mode with superfluid modes. A similar scenario was proposed, for the first time, in Ref. [12] (see also Ref. [13]). Here we reconsider it, employing the physics input, described in Sec. [11] and perform a number of useful estimates, supplementing the consideration of Sec. [H13] Fig. [7] presents the stellar spin frequency $\nu$ as a function of the internal red-shifted temperature $T^\infty$. The thick solid line shows the cyclic evolution track of the NS $A-B-C-D-A$ for the saturation amplitude of $r^\omega$-mode $\alpha_{\text{sat}} = 10^{-4}$. The medium and thin solid lines show similar tracks for $A-B-C-D-A$.

The instability curve for the quadrupole $m = 2$ $r^\omega$-mode, given by the condition $1/\tau_{\text{GR}} + 1/\tau_{\text{Diss}} = 0$, is shown by thick dashed line. In the region over the curve one has $1/\tau_{\text{GR}} + 1/\tau_{\text{Diss}} < 0$ and, as follows from Eq. (19), the star is unstable with respect to excitation of the $r^\omega$-mode ($d\alpha/dt > 0$). In the figure the stability region for $m = 2$ $r^\omega$-mode is filled with grey.

Let us discuss in more detail the stellar evolution along the track $A-B-C-D-A$.

(i) The stage $A-B$.

The star has initial equilibrium temperature $T_8^\infty = T_8^\text{eq} \approx 1.078 \times 10^8$ K (see Sec. [H11]), the amplitude of $r^\omega$-mode $\alpha = 0$ and the spin frequency $\nu_A \approx 164$ Hz ($\Omega_A = 2\pi \nu_A \approx 1030$ s$^{-1}$). The stellar spin frequency grows linearly due to accretion of matter onto the NS according to Eq. (20) with $\alpha = 0$. The stellar temperature $T^\infty$ remains unchanged. The star reaches the boundary of the stability region in the point $B$. In this point $\nu_B \approx 238$ Hz ($\Omega_B \approx 1495$ s$^{-1}$), so that the time spent by the star in the stage $A-B$, equals: $\Delta t_{AB} = (\Omega_B - \Omega_A)/\dot{\Omega}_{\text{acc}} \approx 4 \times 10^7$ yrs.

(ii) The stage $B-C$.

In the point $B$ the star is located on the instability curve of $m = 2$ $r^\omega$-mode. Further increasing of the stellar spin frequency makes it unstable. However, if the $r^\omega$-mode amplitude is strictly zero, then, as follows from Eq. (19), $d\alpha/dt$ remains to be zero even in the instability region. In reality, any fluctuation of the amplitude $\alpha$ (for instance, the thermal fluctuation or a fluctuation, related to accretion onto an NS) will lead to instability growth. In numerical calculations we modeled this effect by specifying the initial condition $\alpha_B = 10^{-30}$ for the oscillation amplitude in the point $B$. Naturally, the subsequent NS evolution is not sensitive to the actual value of the initial amplitude.

Getting unstable, the amplitude of $m = 2$ $r^\omega$-mode grows rapidly, so that after $\Delta t_{\text{torq}} \approx 4500$ yrs the torque acting on the NS due to $r^\omega$-mode dissipation becomes equal to the accretion torque $|d\Omega/dt| = 0$, see Eq. (20)]. This happens at $\alpha_0 \approx 1.8 \times 10^{-5}$. Approximately 4 yr later the $r^\omega$-mode reaches saturation ($\alpha = \alpha_{\text{sat}} = 10^{-4}$). During these evolution phases $\Omega$ and $T^\infty$ almost do not change. The spin frequency $\Omega$ does not change because a typical time-scale of its variation is much greater than $\Delta t_{\text{torq}}$ (see below). The temperature $T^\infty$ does not change, because its typical time-scale is $\propto \alpha^{-2}$ [see Eqs. (16) and (17)] and is also much greater than $\Delta t_{\text{torq}}$ most of the time.
Using the fact that \( \Omega \) and \( T^\infty \) are almost constant, \( \alpha_0 \) can be derived from Eq. (20) if we fix \( \Omega = \Omega_B \) and \( T^\infty = T^\infty_{\text{eq}} \), and vanish its left-hand side,

\[
\alpha_0 = \sqrt{\frac{\Omega_{\text{acc}} \tau_{\text{Diss}}(T^\infty_{\text{eq}})}{2 Q \Omega_B}} \approx 1.8 \times 10^{-5}.
\]  

(B1)

The time \( \Delta t_{\text{torq}} \) can also be roughly estimated if one keeps in mind that, in the initial stage of the instability, the amplitude \( \alpha \) stays small. In that case, the first term in the right-hand side of Eq. (20) can be neglected, so that one obtains

\[
\Omega - \Omega_B \approx \Omega_{\text{acc}}(t - t_B).
\]  

(B2)

Using this equation and expanding into Tailor series the right-hand side of Eq. (19) around the point \( B \) (at fixed \( T^\infty = T^\infty_{\text{eq}} \)), one gets

\[
\frac{d\alpha}{dt} \approx \alpha \frac{|\tau_{\text{GR}}(\Omega_B)|}{\tau_{\text{GR}}(\Omega_B)} \Omega_{\text{acc}}(t - t_B),
\]  

(B3)

or, after trivial integration,

\[
\alpha = \alpha_B e^{(t-t_B)^2/\tau_}\alpha^2}, \quad \text{where} \quad \tau_\alpha = \sqrt{\frac{2\tau_{\text{GR}}^2(\Omega_B)}{|\tau_{\text{GR}}(\Omega_B)| \Omega_{\text{acc}}}} \approx 600 \text{ yrs.}
\]  

(B4)
Substituting now $\alpha = \alpha_0$ into this equation, one finds

$$\Delta t_{\text{torq}} \approx \tau_\alpha \sqrt{\ln \left( \frac{\alpha_0}{\alpha_B} \right)} \approx 4600 \text{ yrs.} \quad (B5)$$

This result is just a little bit larger than the exact value $\Delta t_{\text{torq}} \approx 4500 \text{ yrs}$. As shown in Sec. [11] the NS evolution with the saturated $r^\alpha$-mode is governed by the simpler equations. In particular, instead of Eq. (20) one will have

$$\frac{d\Omega}{dt} = \frac{2Q\alpha_{\text{sat}}^2\Omega}{\tau_{\text{GR}}} + \dot{\Omega}_{\text{acc}}, \quad (B6)$$

which can be integrated independently. Neglecting the term $\dot{\Omega}_{\text{acc}}$, which is small in our case [\dot{\Omega}_{\text{acc}} becomes comparable to the first term in the right-hand side of Eq. (B6)] at a small $\Omega \approx 915 \text{ s}^{-1} (\nu \approx 146 \text{ Hz})$, and can be omitted for a rough estimate, we get

$$\Omega = \frac{\Omega_B}{(1 + t/\tau_\Omega)^{1/6}}, \quad \text{with} \quad \tau_\Omega = \frac{1}{12} \frac{\tau_{\text{GR}}(\Omega_B)}{Q\alpha_{\text{sat}}^2} \approx \frac{3 \times 10^{-8}}{\alpha_{\text{sat}}^2} \left( \frac{\Omega_0}{\Omega_B} \right)^6 \text{ yrs} \approx 7 \times 10^5 \text{ yrs}, \quad (B7)$$

where the time is counted from the moment when $r^\alpha$-mode reaches saturation. In practice, this formula describes the $\Omega(t)$ dependence on the whole interval $B - C - D$ sufficiently well.

Let us now estimate the time $\Delta t_T$ required to heat up the star from the moment of $r^\alpha$-mode saturation to the point $C$. The point $C$ lies on the Cooling=Heating curve, given by the condition

$$-\frac{\tilde{J}MR^2\Omega B\alpha_{\text{sat}}^2}{\tau_{\text{GR}}} - L_{\text{cool}} + K_n M c^2 = 0, \quad (B8)$$

which means that at this curve the stellar heating due to dissipation of the saturated $r^\alpha$-mode is exactly compensated by the neutrino cooling. After reaching the curve, the star moves along it, until it enters the stability region. As we will see from the estimate, $\Delta t_T$ is much smaller than $\tau_\Omega$, thus in the further derivation one can set $\Omega = \Omega_B (= \Omega_C)$ in Eq. (B6). Bearing in mind that the mode is saturated (that is $\alpha = \alpha_{\text{sat}}$ and instead of $\tau_{\text{Diss}}$ one should write $-\tau_{\text{GR}}$), Eq. (B6) can be rewritten as

$$C_{\text{tot}} \frac{d\tau_C^\infty}{dt} = -\frac{\tilde{J}MR^2\Omega B\alpha_{\text{sat}}^2}{\tau_{\text{GR}}} - L_{\text{cool}} + K_n M c^2. \quad (B9)$$

Vanishing the left-hand side of this equation, one finds the stellar temperature in the point $C$, $\tau_C^\infty \approx 4.6 \times 10^8 \text{ K}$. Equation (B9) can now be integrated in quadratures. However (since we are only interested in the order-of-magnitude estimate for $\Delta t_T$), we additionally simplify it by neglecting the last two terms in the right-hand side of this equation. In addition, we make use of the fact that in the range of temperatures under consideration the heat capacity $C_{\text{tot}} \approx \gamma T^2$, where $\gamma \approx 1.5 \times 10^{30} \text{ erg K}^{-2}$. Integrating now Eq. (B9), we obtain

$$\Delta t_T = \frac{\gamma \tau_{\text{GR}}(\tau_C^\infty - \tau_C^2)}{2\tilde{J}M R^2 \Omega B \alpha_{\text{sat}}^2} \approx 1200 \text{ yrs.} \quad (B10)$$

Because we ignored the star luminosity $L_{\text{cool}}$ in Eq. (B9), our rough estimate is smaller than the real time $\Delta t_T \approx 3300 \text{ yrs}$. In reality, $L_{\text{cool}}$ becomes important and slows down the NS heating only in the very vicinity of the curve Cooling=Heating. According to our estimate, the first $\sim 1300 \text{ yrs}$ the star rapidly heats up and reaches the boundary of the circle, showing the point $C$ in the figure. During subsequent $\sim 2000 \text{ yrs}$ the star heating proceeds very slowly and its position in Fig. 7 almost does not change. As we expected, $\Delta t_T \ll \tau_\Omega$.

(iii) The stage $C - D$.

This is the longest stage in the instability region. During it the star moves along the Cooling=Heating curve. The time spent on the horizontal stage $B - C$ is several orders of magnitude smaller. The Cooling=Heating curve crosses the instability curve in the point $D$, $\Omega_D \approx \Omega_A$. The traveling time along $C - D$ can be easily estimated from Eq. (B7),

$$\Delta t_{CD} \approx \tau_\Omega \left( \frac{\Omega_0}{\Omega_D^6} - 1 \right) \approx \frac{3 \times 10^{-8}}{\alpha_{\text{sat}}^2} \left( \frac{\Omega_0}{\Omega_D^6} - \frac{\Omega_0}{\Omega_C^6} \right) \text{ yrs} \approx 6 \times 10^6 \text{ yrs.} \quad (B11)$$

The exact calculation shows that $\Delta t_{CD} \approx 8 \times 10^6 \text{ yrs}$. The discrepancy is due to neglect of the term $\dot{\Omega}_{\text{acc}}$ in the derivation of Eq. (B7).
FIG. 8: Similar to Fig. 5 but with additional avoided crossing of the modes I and III at $T^\infty = 5 \times 10^8$ K. An instability curve for the mode III is shown by the violet solid line; the coupling parameter parameterizing interaction between the modes I and III is $s = 0.001$. The evolution track $A - B - C - D - E - X_1 - X_2 - X_3 - F - A$ of a star is shown by the black solid line. Other notations (and input parameters) coincide with those of Fig. 5.

(iv) The stage $D - A$.

Just after the star reaches the stability region, the amplitude of $r^\circ$-mode rapidly (during $\sim 400$ yrs) decreases to negligible values; then the star cools down to the temperature $T^\infty_{eq}$ (the point $A$). The cooling takes $\sim 10^5$ yrs, after that the cycle repeats.

The main conclusion, that can be drawn from the discussion of the evolution tracks is following: in the stability region the star spends most of the time in the stage $A - B$, while in the instability region – in the stage $C - D$. The ratio of the time spent in the instability region (without accounting for the time $\Delta t_{torq} \approx 4500$ yr, during which the star ‘sits’ in the point $B$, see Fig. 7) to the period of the cycle, equals $k \approx 0.16$ for the model with $\alpha_{sat} = 10^{-4}$. This ratio drops rapidly with increasing $\alpha_{sat}$ [13], because the typical time $\tau_\Omega$ of $\Omega$ variation during the $C - D$ stage is $\tau_\Omega \propto \alpha_{sat}^{-2}$, see Eq. (B7). For $\alpha_{sat} = 5 \times 10^{-3}$ we have $k \approx 1.7 \times 10^{-4}$, whereas for $\alpha_{sat} = 10^{-1}$ we obtain $k \approx 10^{-6}$. Let us note that, since in the saturation regime $W_{Diss} \propto \alpha_{sat}^2$, the higher is $\alpha_{sat}$ the farther the NS gets into the region of high temperatures (the more horizontally elongated is the track $A - B - C - D - A$, see Fig. 7).

Appendix C: NS evolution in the case of two avoided crossings of oscillation modes

Assume that, besides the avoided crossing of modes I and II, there is one more avoided crossing of modes I and III at $T^\infty = 5 \times 10^8$ K such that the mode III becomes $m = 2$ $r^\circ$-mode at $T^\infty > 5 \times 10^8$ K (see Fig. 8). Solid black line in Fig. 8 shows the typical evolution track $A - B - C - D - E - X_1 - X_2 - X_3 - F - A$ of an NS in this case. The main difference between this track and the one discussed in Sec. VII (see Fig. 5) is the stage $X_1 - X_2 - X_3$, in which
the star evolves in the region of avoided crossing of the modes I and III. Let us discuss this stage in more detail. In the point \( X_1 \) the only excited mode is the oscillation mode III, which is saturated (i.e., its amplitude equals \( 10^{-4} \)). At the stage \( X_1 - X_2 \) the star enters the stability region, where the amplitude of the mode III rapidly vanishes, and the star cools down to the point \( X_2 \) during \( \sim 130 \) yr. At point \( X_2 \) the star becomes unstable with respect to excitation of the mode I; similar to the case of the mode II at the stage \( C - D \) (see Fig. 8), its equilibrium amplitude \( \alpha_{I}^{(eq)} \) is then defined by the thermal equilibrium condition (46). Since the stage \( X_2 - X_3 \) is close to the curve Cooling=Heating, intensive heating is required to maintain the temperature, and the mode I appears to be close to saturation. Such a high oscillation amplitude means that the spin down of the NS due to viscous dissipation of the mode I will dominate the accretion torque [see Eq. (47)]. As a result, the stellar spin frequency will decrease. Finally, in \( 4.6 \times 10^5 \) yr after leaving the point \( X_2 \) (this time constitutes \( \sim 0.16\% \) of the full period of the cycle), the star again reaches the Cooling=Heating curve in the point \( X_3 \). To continue spinning down along the instability curve of the mode I, it needs a more intensive heating than the saturated mode can provide. Thus, the further evolution of the star (the stage \( X_3 - F \)) goes along the Cooling=Heating curve, as in the scenario shown in Fig. 5.

We arrive at the conclusion, that the existence of additional avoided crossings of oscillation modes does not affect noticeably the scenario, proposed in Sec. VI, and does not change the main results of the paper.

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18 As in Fig. 5 the curve Cooling=Heating, shown by dots in Fig. 8 is given by Eq. (50) in the stability region (see also the discussion of this curve in Sec. VI).
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