Analysis of Krylov Subspace Solutions of Regularized Nonconvex Quadratic Problems

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Abstract

We provide convergence rates for Krylov subspace solutions to the trust-region and cubic-regularized (nonconvex) quadratic problems. Such solutions may be efficiently computed by the Lanczos method and have long been used in practice. We prove error bounds of the form $1/t^2$ and $e^{-4t/\sqrt{\kappa}}$, where $\kappa$ is a condition number for the problem, and $t$ is the Krylov subspace order (number of Lanczos iterations). We also provide lower bounds showing that our analysis is sharp.

1 Introduction

Consider the potentially nonconvex quadratic function

$$f_{A,b}(x) \triangleq \frac{1}{2} x^T Ax + b^T x,$$

where $A \in \mathbb{R}^{d \times d}$ and $b \in \mathbb{R}^d$. We wish to solve regularized minimization problems of the form

$$\min_x f_{A,b}(x) \text{ subject to } \|x\| \leq R \quad \text{and} \quad \min_x f_{A,b}(x) + \frac{\rho}{3} \|x\|^3,$$

(1)

where $R$ and $\rho \geq 0$ are regularization parameters. These problems arise primarily in the family of trust-region and cubic-regularized Newton methods for general nonlinear optimization problems [9, 25, 14, 8], which optimize a smooth function $g$ by sequentially minimizing local models of the form

$$g(x_i + \Delta) \approx g(x_i) + \nabla g(x_i)^T \Delta + \frac{1}{2} \Delta^T \nabla^2 g(x_i) \Delta = g(x_i) + f_{\nabla^2 g(x_i), \nabla g(x_i)}(\Delta),$$

where $x_i$ is the current iterate and $\Delta \in \mathbb{R}^d$ is the search direction. Such models tend to be unreliable for large $\|\Delta\|$, particularly when $\nabla^2 g(x_i) \neq 0$. Trust-region and cubic regularization methods address this by constraining and regularizing the direction $\Delta$, respectively.

Both classes of methods and their associated subproblems are the subject of substantial ongoing research [15, 17, 4, 1, 21]. In the machine learning community, there is growing interest in using these methods for minimizing (often nonconvex) training losses, handling the large finite-sum structure of learning problems by means of sub-sampling [28, 19, 2, 33, 31].

The problems (1) are challenging to solve in high-dimensional settings when $d$ is large, where direct decomposition (or even storage) of the matrix $A$ is infeasible. In some scenarios, however, computing matrix-vector products $v \mapsto Av$ is feasible. Such is the case when $A$ is the Hessian of a neural network, where $d$ may be in the millions and $A$ is dense, and yet we can compute Hessian-vector products efficiently on batches of training data [27, 29].
In this paper we consider a scalable approach for approximately solving (1), which consists of minimizing the objective in the \textit{Krylov subspace} of order \( t \),
\[
\mathcal{K}_t(A, b) \triangleq \text{span}\{b, Ab, \ldots, A^{t-1}b\}.
\] (2)

This requires only \( t \) matrix-vector products, and the Lanczos method allows one to efficiently find the solution to problems (1) over \( \mathcal{K}_t(A, b) \) (see, e.g. \[13, 8, \text{Sec. .2}\]). Krylov subspace methods are of course familiar in numerous large-scale numerical problems, including conjugate gradient methods, eigenvector problems, or solving linear systems \[16, 22, 30, 11\].

It is well-known that, with exact arithmetic, the order \( d \) subspace \( \mathcal{K}_d(A, b) \) generically contains the global solutions to (1). However, until recently the literature contained no guarantees on the rate at which the suboptimality of the solution approaches zero as the subspace dimension \( t \) grows. This is in contrast to the two predominant Krylov subspace method use-cases, convex quadratic optimization \[11, 23, 24\] and eigenvector finding \[20\], where such rates of convergence have been known for decades. Zhang et al. \[34\] make substantial progress on this gap, establishing bounds implying a linear rate of convergence for the trust-region variant of problem (1).

In this work we complete the picture, proving that the suboptimality of the order \( t \) Krylov subspace solution to either of the problems (1) is bounded by both \( e^{-t/\sqrt{\kappa}} \) and \( t^{-2} \log^2(\|b\|/\|u_{\min}^T b\|) \).

Here \( \kappa \) is a condition number for the problem that naturally generalizes the classical condition number of the matrix \( A \), and \( u_{\min} \) is an eigenvector of \( A \) corresponding to its smallest eigenvalue. With a small random perturbation to \( b \) we may replace \( |u_{\min}^T b| \) with a term proportional to \( 1/\sqrt{d} \), circumventing the well-known “hard case” of the problem (1) (see Section 2.5). Our analysis both leverages and unifies the known results for convex quadratic and eigenvector problems, which constitute special cases of the trust-region and cubic-regularization formulations.

\textbf{Related work} \( \) Zhang et al. \[34\] show that the error of certain polynomial approximation problems bounds the suboptimality of Krylov subspace solutions to the trust region-variant of the problems (1), implying convergence at a rate exponential in \(-t/\sqrt{\kappa}\). Based on these bounds, the authors propose novel stopping criteria for subproblem solutions in the trust-region optimization method, showing good empirical results. However, the bounds of \[34\] become weak for large \( \kappa \) and vacuous in the hard case where \( \kappa = \infty \). A number of prior works develop algorithms for solving (1) with convergence guarantees that hold in the hard case. Hazan and Koren \[15\], Ho-Nguyen and Kilinc-Karzan \[17\], and Agarwal et al. \[1\] propose algorithms that obtain error roughly \( t^{-2} \) after computing \( t \) matrix-vector products. The different algorithms proposed in these works all essentially reduce the problems (1) to a sequence of eigenvector and convex quadratic problems to which well-understood algorithms apply. In previous work \[4\], we analyze gradient descent—a direct, local method—for the cubic-regularized problem. There, we show a rate of convergence roughly \( t^{-1} \), reflecting the well-known complexity gap between gradient descent (respectively, the power method) and conjugate gradient (respectively, Lanczos) methods \[30, 11\].

Our development differs in the following ways from prior work.

1. We analyze a practical approach, implemented in efficient optimization libraries \[12, 21\], with essentially no tuning parameters. Previous algorithms \[15, 17, 1\] are convenient for theoretical analysis but less conducive to efficient implementation; each has several parameters that require tuning, and we are unaware of numerical experiments with any of the approaches.

2. We provide both linear \((e^{-t/\sqrt{\kappa}})\) and sublinear \((t^{-2})\) convergence guarantees. In contrast, the papers \[15, 17, 1\] provide only a sublinear rate; Zhang et al. \[34\] provide only the linear rate.
3. Our analysis applies to both the trust-region and cubic regularization variants in (1), while [15, 17, 34] consider only the trust-region problem, and [34, 4] consider only cubic regularization.

4. We provide lower bounds—for adversarially constructed problem instances—showing our convergence guarantees are tight to within numerical constants. By a resisting oracle argument [23], these bounds apply to any deterministic algorithm that accesses $A$ via matrix-vector products.

5. Our arguments are simple and transparent, and we leverage established results on convex optimization and the eigenvector problem to give short proofs of our main results.

Paper organization In Section 2 we state and prove our convergence rate guarantees for the trust-region problem. Then, in Section 3 we quickly transfer those results to the cubic-regularized problem by showing that it always has a smaller optimality gap. Section 4 gives our lower bounds, stated for cubic regularization but immediately applicable to the trust-region problem by the same optimality gap bound. Finally, in Section 5 we illustrate our analysis with some numerical experiments.

Notation For a symmetric matrix $A \in \mathbb{R}^{d \times d}$ and vector $b$ we let $f_{A,b}(x) \triangleq \frac{1}{2} x^T A x + b^T x$. We let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of $A$, and let $u_{\min}(A), u_{\max}(A)$ denote their corresponding (unit) eigenvectors, dropping the argument $A$ when clear from context. For integer $t \geq 1$ we let $P_t \triangleq \{ c_0 + c_1 x + \cdots + c_{t-1} x^{t-1} | c_i \in \mathbb{R} \}$ be the polynomials of degree at most $t - 1$, so that the Krylov subspace (2) is $K_t(A, b) = \{ p(A) b | p \in P_t \}$. We use $\| \cdot \|$ to denote Euclidean norm on $\mathbb{R}^d$ and $\ell_2$-operator norm on $\mathbb{R}^{d \times d}$. Finally, we denote $(z)_+ \triangleq \max\{z, 0\}$ and $(z)_- \triangleq \min\{z, 0\}$.

2 The trust-region problem

Fixing a symmetric matrix $A \in \mathbb{R}^{d \times d}$, vector $b \in \mathbb{R}^d$ and trust-region radius $R > 0$, we let

$$s_t^* \in \argmin_{x \in \mathbb{R}^d, \|x\| \leq R} f_{A,b}(x) = \frac{1}{2} x^T A x + b^T x$$

denote a solution (global minimizer) of the trust region problem. Letting $\lambda_{\min}$, $\lambda_{\max}$ denote the extremal eigenvalues of $A$, $s_t^*$ admits the following characterization [9, Ch. 7]: $s_t^*$ solves problem (1) if and only if there exists $\lambda_*$ such that

$$\lambda_* \geq \lambda_{\min}, \quad (A + \lambda_* I) s_t^* = -b, \quad \lambda_* \lambda_{\min} > \lambda_{\max}.$$ (3)

The optimal Lagrange multiplier $\lambda_*$ always exists and is unique, and if $\lambda_* > -\lambda_{\min}$ the solution $s_t^*$ is unique and satisfies $s_t^* = -(A + \lambda_* I)^{-1} b$. Letting $u_{\min}$ denote the eigenvector of $A$ corresponding to $\lambda_{\min}$, the characterization (3) shows that $u_{\min}^T b \neq 0$ implies $\lambda_* > -\lambda_{\min}$.

Now, consider the Krylov subspace solutions, and for $t > 0$, let

$$s_t^r \in \argmin_{x \in K_t(A, b), \|x\| \leq R} f_{A,b}(x) = \frac{1}{2} x^T A x + b^T x$$

denote a minimizer of the trust-region problem in the Krylov subspace of order $t$. Gould et al. [13] show how to compute the Krylov subspace solution $s_t^r$ in time dominated by the cost of computing $t$ matrix-vector products using the Lanczos method (see also Section A of the supplement).
2.1 Main result

With the notation established above, our main result follows.

**Theorem 1.** For every $t > 0$,

$$f_{A,b}(s_t^\text{lr}) - f_{A,b}(s_0^\text{lr}) \leq 36 \left[ f_{A,b}(0) - f_{A,b}(s_0^\text{lr}) \right] \exp \left\{ -4t \sqrt{\frac{\lambda_{\text{min}} + \lambda_*}{\lambda_{\text{max}} + \lambda_*}} \right\},$$

(4)

and

$$f_{A,b}(s_t^\text{lr}) - f_{A,b}(s_0^\text{lr}) \leq \frac{(\lambda_{\text{max}} - \lambda_{\text{min}}) \|s_t^\text{lr}\|^2}{(t - \frac{1}{2})^2} \left[ 4 + \frac{\|s_t^\text{lr}\|^2}{8} \log^2 \left( \frac{4 \|b\|^2}{(u_{\text{min}}^T b)^2} \right) \right].$$

(5)

Theorem 1 characterizes two convergence regimes: linear (4) and sublinear (5). In the former regime, the error decays exponentially and falls beneath $\epsilon$ in roughly $\sqrt{\kappa} \log \frac{1}{\epsilon}$ Lanczos iterations, where $\kappa = \frac{\lambda_{\text{max}} + \lambda_*}{\lambda_{\text{min}} + \lambda_*} \geq 1$ is the condition number for the problem. In the latter regime, the error decays polynomially and falls beneath $\epsilon$ in roughly $\frac{1}{\sqrt{\kappa}}$ iterations. For worst-case problem instances this characterization is tight to constant factors, as we show in Section 4.

The guarantees of Theorem 1 closely resemble the well-known guarantees for the conjugate gradient method [30], including them as the special case $R = \infty$ and $\lambda_{\text{min}} \geq 0$. For convex problems, the radius constraint $\|x\| \leq R$ always improves the conditioning of the problem, as $\lambda_{\text{max}} \lambda_{\text{min}} \geq \lambda_{\text{max}} + \lambda_*$; the smaller $R$ is, the better conditioned the problem becomes. For non-convex problems, the sublinear rate features an additional logarithmic term that captures the role of the eigenvector $u_{\text{min}}$. The first rate (4) is similar to those of Zhang et al. [34, Thm. 4.11], though with somewhat more explicit dependence on $t$.

In the “hard case,” which corresponds to $u_{\text{min}}^T b = 0$ and $\lambda_{\text{min}} + \lambda_* = 0$ (cf. [9, Ch. 7]), both the bounds in Theorem 1 become vacuous, and indeed $s_t^\text{lr}$ may not converge to the global minimizer in this case. However, as the bound (5) depends only logarithmically on $u_{\text{min}}^T b$, it remains valid even extremely close to the hard case. In Section 2.5 we describe a simple randomization technique with a convergence guarantee that is valid in the hard case as well.

2.2 Proof sketch

Our analysis reposes on two elementary observations. First, we note that Krylov subspaces are invariant to shifts by scalar matrices, i.e. $K_t(A,b) = K_t(A\lambda, b)$ for any $A, b, t$ where $\lambda \in \mathbb{R}$, and

$$A\lambda \triangleq A + \lambda I.$$ Second, we observe that for every point $x$ and $\lambda \in \mathbb{R}$

$$f_{A,b}(x) - f_{A,b}(s_0^\text{lr}) = f_{A\lambda,b}(x) - f_{A\lambda,b}(s_0^\text{lr}) + \frac{\lambda}{2} (\|s_0^\text{lr}\|^2 - \|x\|^2)$$

(6)

Our strategy then is choose $\lambda$ such that $A\lambda \succeq 0$, and then use known results to find $y_t \in K_t(A\lambda, b) = K_t(A, b)$ that rapidly reduces the “convex error” term $f_{A\lambda,b}(y_t) - f_{A\lambda,b}(s_0^\text{lr})$. We then adjust $y_t$ to obtain a feasible point $x_t$ such that the norm error term $\frac{\lambda}{2} (\|s_0^\text{lr}\|^2 - \|x_t\|^2)$ is small. To establish linear convergence we take $\lambda = \lambda_*$ and adjust the norm of $y_t$ by taking $x_t = (1 - \alpha)y_t$ for some small $\alpha$ that guarantees $x_t$ is feasible and that the “norm error” term is small. To establish sublinear convergence we set $\lambda = -\lambda_{\text{min}}$ and take $x_t = y_t + \alpha \cdot z_t$, where $z_t$ is an approximation for $u_{\text{min}}$ within $K_t(A, b)$, and $\alpha$ is chosen to make $\|x_t\| = \|s_0^\text{lr}\|$. This means the “norm error” vanishes, while the “convex error” cannot increase too much, as $A_{-\lambda_{\text{min}}} z_t \approx A_{-\lambda_{\text{min}}} u_{\text{min}} = 0$. 

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Our approach for proving the sublinear rate of convergence is inspired by Ho-Nguyen and Kilinc-Karzan [17], who also rely on Nesterov’s method in conjunction with Lanczos-based eigenvector approximation. The analysis in [17] uses an algorithmic reduction, proposing to apply the Lanczos method (with a random vector instead of b) to approximate \( u_{\min} \) and \( \lambda_{\min} \), then run Nesterov’s method on an approximate version of the “convex error” term, and then use the approximated eigenvector to adjust the norm of the result. We instead argue that all the ingredients for this reduction already exist in the Krylov subspace \( K_t(A,b) \), obviating the need for explicit eigenvector estimation or actual application of accelerated gradient descent.

### 2.3 Building blocks

Our proof uses the following classical results.

**Lemma 1** (Approximate matrix inverse). Let \( \alpha, \beta \) satisfy \( 0 < \alpha \leq \beta \), and let \( \kappa = \beta/\alpha \). For any \( t \geq 1 \) there exist a polynomial \( p \) of degree at most \( t - 1 \), such that for every \( M \) satisfying \( \alpha I \preceq M \preceq \beta I \),

\[
\| I -Mp(M) \| \leq 2e^{-2t/\sqrt{\kappa}}.
\]

**Lemma 2** (Convex trust-region problem). Let \( t \geq 1, M \succeq 0, v \in \mathbb{R}^d \) and \( r \geq 0 \), and let \( f_{M,v}(x) = \frac{1}{2} x^T Mx + v^T x \). There exists \( x_t \in K_t(M,v) \) such that

\[
\| x_t \| \leq r \quad \text{and} \quad f_{M,v}(x_t) - \min_{\| y \| \leq r} f_{M,v}(x) \leq \frac{4\lambda_{\max}(M) \cdot r^2}{(t+1)^2}.
\]

**Lemma 3** (Finding eigenvectors, [20, Theorem 4.2]). Let \( M \succeq 0 \) be such that \( u^T Mu = 0 \) for some unit vector \( u \in \mathbb{R}^d \), and let \( v \in \mathbb{R}^d \). For every \( t \geq 1 \) there exists \( z_t \in K_t(M,v) \) such that

\[
\| z_t \| = 1 \quad \text{and} \quad z_t^T M z_t \leq \frac{\| M \|}{16(t - \frac{1}{2})^2 \log^2 \left( -2 + \frac{2r^2}{(u^T v)^2} \right)}.
\]

While these lemmas are standard, their explicit forms are useful, and we prove them in Section C.1 in the supplement. Lemmas 1 and 3 are consequences of uniform polynomial approximation results (cf. supplement, Sec. B). To prove Lemma 2 we invoke Tseng’s results on a variant of Nesterov’s accelerated gradient method [32], arguing that its iterates lie in the Krylov subspace.

### 2.4 Proof of Theorem 1

**Linear convergence** Recalling the notation \( A_{\lambda_*} = A + \lambda_* I \), let \( y_t = -p(A_{\lambda_*})b = p(A_{\lambda_*})A_{\lambda_*} s_*^v \),

for the \( p \in \mathcal{P}_t \) which Lemma 1 guarantees to satisfy \( \| p(A_{\lambda_*}) - I \| \leq 2e^{-2t/\sqrt{\kappa(A_{\lambda_*})}} \). Let

\[
x_t = (1 - \alpha)y_t, \quad \alpha = \frac{\| y_t \| - \| s_*^v \|}{\max\{\| s_*^v \|, \| y_t \|\}},
\]

so that we are guaranteed \( \| x_t \| \leq \| s_*^v \| \) for any value of \( \| y_t \| \). Moreover

\[
|\alpha| = \frac{\| y_t \| - \| s_*^v \|}{\max\{\| s_*^v \|, \| y_t \|\}} \leq \frac{\| y_t - s_*^v \|}{\| s_*^v \|} = \frac{\| (p(A_{\lambda_*})A_{\lambda_*} - I)s_*^v \|}{\| s_*^v \|} \leq 2e^{-2t/\sqrt{\kappa(A_{\lambda_*})}},
\]

where the last transition used \( \| p(A_{\lambda_*})A_{\lambda_*} - I \| \leq 2e^{-2t/\sqrt{\kappa(A_{\lambda_*})}} \).
Since $b = -A\lambda_0 s^r_\ast$, we have $f_{A_\ast,b}(x) = f_{A_\ast,b}(s^r_\ast) + \frac{1}{2}\|A^{1/2}_{\lambda_0}(x - s^r_\ast)\|^2$. The equality (6) with $\lambda = \lambda_\ast$ and $\|x_t\| \leq \|s^r_\ast\|$ therefore implies

$$f_{A_\ast,b}(x_t) - f_{A_\ast,b}(s^r_\ast) \leq \frac{1}{2} \left(\|A^{1/2}_{\lambda_\ast}(x_t - s^r_\ast)\|^2 + \lambda_\ast \|s^r_\ast\|^2\right) + \lambda_\ast \|s^r_\ast\| (\|s^r_\ast\| - \|x_t\|).$$

(7)

When $\|y_t\| \geq \|s^r_\ast\|$ we have $\|x_t\| = \|s^r_\ast\|$ and the second term vanishes. When $\|y_t\| < \|s^r_\ast\|$, we have

$$\|s^r_\ast\| - \|x_t\| = \|s^r_\ast\| - \|y_t\| - \frac{\|y_t\|}{\|s^r_\ast\|} \times (\|s^r_\ast\| - \|y_t\|) = \|s^r_\ast\| \alpha^2 \leq 4 e^{-4t/\sqrt{\kappa(A_\lambda_\ast)}} \|s^r_\ast\|.$$

(8)

We also have,

$$\left\|A^{1/2}_{\lambda_\ast}(x_t - s^r_\ast)\right\| = \left\|[(1-\alpha)p(A_\lambda_\ast)A_\lambda_\ast - I] A^{1/2}_{\lambda_\ast} s^r_\ast\right\| \leq (1 + |\alpha|) \left\|p(A_\lambda_\ast)A_\lambda_\ast - I\right\| A^{1/2}_{\lambda_\ast} s^r_\ast + |\alpha| \left\|A^{1/2}_{\lambda_\ast} s^r_\ast\right\| \leq 6 \left\|A^{1/2}_{\lambda_\ast} s^r_\ast\right\| e^{-2t/\sqrt{\kappa(A_\lambda_\ast)}},$$

(9)

where in the final transition we used our upper bounds on $\alpha$ and $\|p(A_\lambda_\ast)A_\lambda_\ast - I\|$, as well as $|\alpha| \leq 1$. Substituting the bounds (8) and (9) into inequality (7), we have

$$f_{A_\ast,b}(x_t) - f_{A_\ast,b}(s^r_\ast) \leq \left(18s^r_\ast^T A_\lambda_\ast s^r_\ast + 4\lambda_\ast \|s^r_\ast\|^2\right) e^{-4t/\sqrt{\kappa(A_\lambda_\ast)}},$$

(10)

and the final bound follows from recalling that $f_{A_\ast,b}(0) - f_{A_\ast,b}(s^r_\ast) = \frac{1}{2}s^r_\ast^T A_\lambda_\ast s^r_\ast + \frac{\lambda_\ast}{2} \|s^r_\ast\|^2$ and substituting $\kappa(A_\lambda_\ast) = (\lambda_{\text{max}} + \lambda_\ast)/(\lambda_{\text{min}} + \lambda_\ast)$. To conclude the proof we note that $(1-\alpha)p(A_\lambda_\ast) = (1-\alpha)p(A + \lambda_\ast I) = p(A)$ for some $p \in P_t$, so that $x_t \in K_t(A,b)$ and $\|x_t\| \leq R$, and therefore $f_{A_\ast,b}(s^r_\ast) \leq f_{A_\ast,b}(x_t)$.

**Sublinear convergence** Let $A_0 \triangleq A - \lambda_{\min} I \geq 0$ and apply Lemma 2 with $M = A_0$, $v = b$ and $r = \|s^r_\ast\|$ to obtain $y_t \in K_t(A_0,b) = K_t(A,b)$ such that

$$\|y_t\| \leq \|s^r_\ast\| \text{ and } f_{A_0,b}(y_t) - f_{A_0,b}(s^r_\ast) \leq f_{A_0,b}(y_t) - \min_{\|x\| \leq \|s^r_\ast\|} f_{A_0,b}(x) \leq \frac{4\|A_0\| \|s^r_\ast\|^2}{(t+1)^2}. \tag{11}$$

If $\lambda_{\min} \geq 0$, equality (6) with $\lambda = -\lambda_{\min}$ along with (11) means we are done, recalling that $\|A_0\| = \lambda_{\text{max}} - \lambda_{\min}$. For $\lambda_{\min} < 0$, apply Lemma 3 with $M = A_0$ and $v = b$ to obtain $z_t \in K_t(A,b)$ such that

$$\|z_t\| = 1 \text{ and } z_t^T A_0 z_t \leq \frac{\|A_0\|}{16(t - \frac{1}{2})^2} \log^2 \left(\frac{4\|b\|^2}{(u_{\text{min}}^T b)^2}\right). \tag{12}$$

We form the vector

$$x_t = y_t + \alpha \cdot z_t \in K_t(A,b),$$

and choose $\alpha$ to satisfy

$$\|x_t\| = \|s^r_\ast\| \text{ and } \alpha \cdot z_t^T (A_0 y_t + b) = \alpha \cdot z_t^T \nabla f_{A_0,b}(y_t) \leq 0.$$

We may always choose such $\alpha$ because $\|y_t\| \leq \|s^r_\ast\|$ and therefore $\|y_t + \alpha z_t\| = \|s^r_\ast\|$ has both a non-positive and a non-negative solution in $\alpha$. Moreover because $\|z_t\| = 1$ we have that $|\alpha| \leq 2 \|s^r_\ast\|$. The property $\alpha \cdot z_t^T \nabla f_{A_0,b}(y_t) \leq 0$ of our construction of $\alpha$ along with $\nabla^2 f_{A_0,b} = A_0$, gives us,

$$f_{A_0,b}(x_t) = f_{A_0,b}(y_t) + \alpha \cdot z_t^T \nabla f_{A_0,b}(y_t) + \frac{\alpha^2}{2} z_t^T A_0 z_t \leq f_{A_0,b}(y_t) + \frac{\alpha^2}{2} z_t^T A_0 z_t.$$

Substituting this bound along with $\|x_t\| = \|s^r_\ast\|$ and $\alpha^2 \leq 4 \|s^r_\ast\|^2$ into (6) with $\lambda = -\lambda_{\min}$ gives

$$f_{A_\ast,b}(x_t) - f_{A_\ast,b}(s^r_\ast) \leq f_{A_\ast,b}(y_t) - f_{A_\ast,b}(s^r_\ast) + 2 \|s^r_\ast\|^2 z_t^T A_0 z_t.$$

Substituting in the bounds (11) and (12) concludes the proof for the case $\lambda_{\min} < 0$. 

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2.5 Randomizing away the hard case

Krylov subspace solutions may fail to converge to global solution when both \( \lambda_* = -\lambda_{\min} \) and \( u_{\min}^T b = 0 \), the so-called hard case [9, 26]. Yet as with eigenvector methods [20, 11], simple randomization approaches allow us to handle the hard case with high probability, at the modest cost of introducing to the error bounds a logarithmic dependence on \( d \). Here we follow the proposal of [4] and add a small random perturbation of the linear term \( b \), denoted \( \tilde{b} \). The following corollary of Theorem 1 shows that solving the problem instance \((A, \tilde{b}, R)\) in the Krylov subspace \( K_t(A, \tilde{b}) \) produces a good approximate solution to the original problem.

**Corollary 2.** Let \( v \) be uniformly distributed on the unit sphere in \( \mathbb{R}^d \), let \( \sigma > 0 \) and let

\[
\tilde{b} = b + \sigma \cdot v.
\]

Let \( s_t^{\text{tr}} \in \text{argmin}_{x \in K_t(A, \tilde{b}), \|x\| \leq R} f_{A, \tilde{b}}(x) \triangleq \frac{1}{2} x^T Ax + \tilde{b}^T x \). For any \( \delta > 0 \),

\[
f_{A, \tilde{b}}(s_t^{\text{tr}}) - f_{A, \tilde{b}}(s_t^{\text{cr}}) \leq (\lambda_{\max} - \lambda_{\min}) R^2 \left[ 4 + \frac{\| \lambda_{\min} < 0 \|}{t^{-\frac{1}{2}}} \log^2 \left( \frac{2 \|b\| \sqrt{d}}{\sigma \delta} \right) \right] + 2\sigma R.
\]

with probability at least \( 1 - \delta \) with respect to the random choice of \( v \).

See section C.2 in the supplement for a short proof, which consists of arguing that \( f_{A, b} \) and \( f_{A, \tilde{b}} \) deviate by at most \( \sigma R \) at any feasible point, and applying a probabilistic lower bound on \( \|u_{\min}^T b\| \).

For any desired accuracy \( \epsilon \), using Corollary 2 with \( \sigma = \epsilon/(4R) \) shows we can achieve this accuracy, with constant probability, in a number of Lanczos iterations that scales as \( \frac{1}{\sqrt{\epsilon}} \log \frac{\sqrt{d}}{\epsilon} \).

3 The cubic-regularized problem

We now consider the cubic-regularized problem

\[
\min_{x \in \mathbb{R}^d} f_{A, b, \rho}(x) = f_{A, b}(x) + \frac{\rho}{3} \|x\|^3 = \frac{1}{2} x^T Ax + b^T x + \frac{\rho}{3} \|x\|^3.
\]

Any global minimizer of \( f_{A, b, \rho} \), denoted \( s_\rho^{\text{cr}} \), admits the characterization [8, Theorem 3.1]

\[
\nabla f_{A, b, \rho}(s_\rho^{\text{cr}}) = (A + \rho \|s_\rho^{\text{cr}}\| I) s_\rho^{\text{cr}} + b = 0 \quad \text{and} \quad \rho \|s_\rho^{\text{cr}}\| \geq -\lambda_{\min}.
\]

Comparing this characterization to its counterpart (3) for the trust-region problem, we see that any instance \((A, b, \rho)\) of cubic regularization has an equivalent trust-region instance \((A, b, R)\), with \( R = \|s_\rho^{\text{cr}}\| \). These instances are equivalent in that they have the same set of global minimizers. Evidently, the equivalent trust-region instance has optimal Lagrange multiplier \( \lambda_* = \rho \|s_\rho^{\text{cr}}\| \). Moreover, at any trust-region feasible point \( x \) (satisfying \( \|x\| \leq R = \|s_\rho^{\text{cr}}\| = \|s_\rho^{\text{cr}}\| \)), the cubic-regularization optimality gap is smaller than its trust-region equivalent,

\[
\hat{f}_{A, b, \rho}(x) - \hat{f}_{A, b, \rho}(s_t^{\text{tr}}) = f_{A, b}(x) - f_{A, b}(s_t^{\text{cr}}) + \frac{\rho}{3} \left( \|x\|^3 - \|s_t^{\text{cr}}\|^3 \right) \leq f_{A, b}(x) - f_{A, b}(s_t^{\text{tr}}).
\]

Letting \( s_t^{\text{cr}} \) denote the minimizer of \( \hat{f}_{A, b, \rho} \) in \( K_t(A, b) \) and letting \( s_t^{\text{tr}} \) denote the Krylov subspace solution of the equivalent trust-region problem, we conclude that

\[
\hat{f}_{A, b, \rho}(s_t^{\text{tr}}) - \hat{f}_{A, b, \rho}(s_t^{\text{cr}}) \leq \hat{f}_{A, b, \rho}(s_t^{\text{tr}}) - \hat{f}_{A, b, \rho}(s_t^{\text{tr}}) \leq f_{A, b}(s_t^{\text{tr}}) - f_{A, b}(s_t^{\text{tr}});
\]

cubic regularization Krylov subspace solutions always have a smaller optimality gap than their trust-region equivalents. The guarantees of Theorem 1 therefore apply to \( \hat{f}_{A, b, \rho}(s_t^{\text{tr}}) - \hat{f}_{A, b, \rho}(s_t^{\text{cr}}) \) as well, and we arrive at the following.
Corollary 3. For every $t > 0$,

$$
\hat{f}_{A,b,\rho}(s_t^\tau) - \hat{f}_{A,b,\rho}(s_\star^\tau) \leq 36 \left[ \hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s_\star^\tau) \right] \exp \left\{ -4t \sqrt{\frac{\lambda_{\min} + \rho \|s_\star^\tau\|^2}{\lambda_{\max} + \rho \|s_\star^\tau\|^2}} \right\},
$$

(16)

and

$$
\hat{f}_{A,b,\rho}(s_t^\tau) - \hat{f}_{A,b,\rho}(s_\star^\tau) \leq \frac{(\lambda_{\min} - \lambda_{\min}) \|s_\star^\tau\|^2}{(t - \frac{1}{2})^2} \left[ 4 + \frac{\|\lambda_{\min} < 0\|}{8} \log^2 \left( \frac{4 \|b\|^2}{(u_{\min}^T b)^2} \right) \right].
$$

(17)

Proof. Use the slightly stronger bound (10) derived in the proof of Theorem 1 with the inequality $18s_t^T A_s \lambda_{\star} s_t^\tau + 4\lambda_s \|s_t^\tau\|^2 = 18s_\star^T A_s \lambda_{\star} s_\star^\tau + 22\rho \|s_\star^\tau\|^3 \leq 36[\hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s_\star^\tau)]$. \hfill \qed

Here too it is possible to randomly perturb $b$ and obtain a guarantee for cubic regularization that applies in the hard case. In [4] we carry out such analysis for gradient descent, and show that perturbations to $b$ with norm $\sigma$ can increase $\|s_\star^\tau\|^2$ by at most $2\sigma / \rho$ [4, Lemma 4.6]. Thus the cubic-regularization equivalent of Corollary 2 amounts to replacing $R^2$ with $\|s_\star^\tau\|^2 + 2\sigma / \rho$ in (13).

We note briefly—without giving a full analysis—that Corollary 3 shows that the practically successful Adaptive Regularization using Cubics (ARC) method [8] can find $\epsilon$-stationary points in roughly $\epsilon^{-7/4}$ Hessian-vector product operations (with proper randomization and subproblem stopping criteria). Researchers have given such guarantees for a number of algorithms that are mainly theoretical [1, 7], as well as variants of accelerated gradient descent [5, 18], while more practical still require careful parameter tuning. In contrast, ARC requires very little tuning and it is encouraging that it may also exhibit the enhanced Hessian-vector product complexity $\epsilon^{-7/4}$, which appears to be at least near-optimal [6].

4 Lower bounds

We now show that the guarantees in Theorem 1 and Corollary 3 are tight up to numerical constants for adversarially constructed problems. We state the result for the cubic-regularization problem; corresponding lower bounds for the trust-region problem are immediate from the optimality gap relation (15).

To state the result, we require a bit more notation. Let $\mathcal{L}$ map cubic-regularization problem instances of the form $(A, b, \rho)$ to the quadruple $(\lambda_{\min}, \lambda_{\max}, \lambda_s, \Delta) = \mathcal{L}(A, b, \rho)$ such that $\lambda_{\min}, \lambda_{\max}$ are the extremal eigenvalues of $A$ and the solution $s_\star^\tau = \arg \min_x \hat{f}_{A,b,\rho}(x)$ satisfies $\rho \|s_\star^\tau\|^2 = \lambda_s$, and $\hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s_\star^\tau) = \Delta$. Similarly let $\mathcal{L}'$ map an instance $(A, b, \rho)$ to the quadruple $(\lambda_{\min}, \lambda_{\max}, \tau, R)$ where now $\|s_\star^\tau\|^2 = R$ and $\|b\|/|u_{\min}^T b| = \tau$, with $u_{\min}$ an eigenvector of $A$ corresponding to eigenvalue $\lambda_{\min}$.

With this notation in hand, we state our lower bounds. (See supplemental section D for a proof.)

Theorem 4. Let $d, t \in \mathbb{N}$ with $t < d$ and $\lambda_{\min}, \lambda_{\max}, \lambda_s, \Delta$ be such that $\lambda_{\min} \leq \lambda_{\max}$, $\lambda_s > (\lambda_{\min})_+$, and $\Delta > 0$. There exists $(A, b, \rho)$ such that $\mathcal{L}(A, b, \rho) = (\lambda_{\min}, \lambda_{\max}, \lambda_s, \Delta)$ and for all $s \in K_t(A, b)$,

$$
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s_\star^\tau) > \frac{1}{K} \left[ \hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s_\star^\tau) \right] \exp \left\{ -\frac{4t}{\sqrt{\kappa - 1}} \right\},
$$

(18)

where $K = 1 + \frac{\rho \|s_\star^\tau\|^2}{3(\rho \|s_\star^\tau\|^2 + \lambda_{\min})}$ and $\kappa = \frac{\rho \|s_\star^\tau\|^2 + \lambda_{\max}}{\rho \|s_\star^\tau\|^2 + \lambda_{\min}}$. Alternatively, for any $\tau, R > 0$, there exists $(A, b, \rho)$ such that $\mathcal{L}'(A, b, \rho) = (\lambda_{\min}, \lambda_{\max}, \tau, R)$ and for $s \in K_t(A, b)$,

$$
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s_\star^\tau) > \min \left\{ (\lambda_{\max})_+ - \lambda_{\min}, \frac{\lambda_{\max} - \lambda_{\min}}{16(t - \frac{1}{2})^2} \log^2 \left( \frac{4 \|b\|^2}{(u_{\min}^T b)^2} \right) \right\} \frac{\|s_\star^\tau\|^2}{32},
$$

(19)
\begin{align}
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s_{cr}^\star) &> \frac{(\lambda_{\text{max}} - \lambda_{\text{min}}) \| s_{cr}^\star \|^2}{16(t + \frac{1}{2})^2}.
\end{align}

The lower bounds (18) matches the linear convergence guarantee (16) to within a numerical constant, as we may choose \( \lambda_{\text{max}}, \lambda_{\text{min}} \) and \( \lambda \) so that \( \kappa \) is arbitrary and \( K < 2 \). Similarly, lower bounds (19) and (20) match the sublinear convergence rate (17) for \( \lambda_{\text{min}} < 0 \) and \( \lambda_{\text{min}} \geq 0 \) respectively. Our proof flows naturally from minimax characterizations of uniform polynomial approximations (Lemmas 4 and 5 in the supplement), which also play a crucial role in proving our upper bounds.

One consequence of the lower bound (18) is the existence of extremely badly conditioned instances, say with \( \kappa = (100d)^2 \) and \( K = 3/2 \), such that in the first \( d - 1 \) iterations it is impossible to decrease the initial error by more than a factor of 2 (the initial error may be chosen arbitrarily large as well). However, since these instances have finite condition number we have \( s_{cr}^\star \in K_d(A,b) \), and so the error supposedly drops to 0 at the \( d \)th iteration. This seeming discontinuity stems from the fact that in this case \( s_{cr}^\star \) depends on the Lanczos basis of \( K_d(A,b) \) through a very badly conditioned linear system and cannot be recovered with finite-precision arithmetic. Indeed, it is well-known that running Krylov subspace methods for \( d \) iterations with inexact arithmetic often results in solutions that are very far from exact, while guarantees of the form (16) are more robust to roundoff errors [3, 10, 30].

While we state the lower bounds in Theorem 4 for points in the Krylov subspace \( K_t(A,b) \), a classical “resisting oracle” construction due to Nemirovski and Yudin [23, Chapter 7.2] (see also [22, §10.2.3]) shows that (for \( d > 2t \)) these lower bounds hold also for any deterministic method that accesses \( A \) only through matrix-vector products, and computes a single matrix-vector product per iteration. The randomization we employ in Corollary 2 breaks the lower bound (19) when \( \lambda_{\text{min}} < 0 \) and \( \| b \|/\| u_{\text{min}}^T b \| \) is very large, so there is some substantial power from randomization in this case. Whether randomization can break the lower bounds in the convex case (\( \lambda_{\text{min}} \geq 0 \)) is a longstanding open question, even for symmetric positive definite systems with \( \rho = 0 \).

5 Numerical experiment

To see whether our analysis applies to non-worst case problem instances, we generate 5,000 random cubic-regularization problem instances with \( d = 10^6 \) and controlled condition number \( \kappa = (\lambda_{\text{max}} + \rho \| s_{cr}^\star \|)/(\lambda_{\text{min}} + \rho \| s_{cr}^\star \|) \) (see Section E in the supplement for more details). We repeat the experiment three times with different values of \( \kappa \) and summarize the results in Figure 1.

As seen in the figure, about 20 Lanczos iterations suffice to solve even the worst-conditioned instances to about 10% accuracy, and 100 iterations give accuracy better than 1%. Moreover, for \( t \gtrsim \sqrt{\kappa} \), the approximation error decays exponentially with precisely the rate \( 4/\sqrt{\kappa} \) predicted by our analysis, for almost all the generated problems. For \( t \ll \sqrt{\kappa} \), the error decays approximately as \( t^{-2} \). Thus, Theorem 1 seems to describe the performance of Krylov subspace solutions well beyond the worst case.
Figure 1: Optimality gap of Krylov subspace solutions to random cubic-regularization problems, as a function of the subspace dimension $t$. Different columns show ensembles with different condition numbers $\kappa$, and the rows differ by scaling of $t$. Thin lines indicate results for individual instances, and bold lines indicate ensemble median and maximum suboptimality. The dashed lines are proportional to our linear and sublinear convergence rates, for the top and bottom row respectively.
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Supplementary material

A Computing Krylov subspace solutions

Generic instances of the trust-region and cubic-regularized problems can be globally optimized by solving the one-dimensional equations

\[ \|A^{-1}\lambda b\| = R, \ \lambda > \max\{-\lambda_{\text{min}}, 0\}. \]

and

\[ \|A^{-1}\lambda b\| = \lambda/\rho, \ \lambda \geq -\lambda_{\text{min}}, \]

respectively. However, when \(d\) is very large, even a single exact evaluation of \(\|A^{-1}\lambda b\|\) (which requires a direct linear system solution) can become prohibitively expensive.

In this case, a general approach to obtaining approximate solutions is to constrain the domain to a linear subspace \(Q_t \subset \mathbb{R}^d\) of dimension \(t \ll d\). Let \(Q_t \in \mathbb{R}^{d \times t}\) be an orthogonal basis for \(Q_t\) \((Q_t^T Q_t = I)\). Finding the global minimizer in \(Q_t\) is equivalent to re-parameterizing \(x\) as \(x = Q_t \tilde{x}\) and solving for \(\tilde{x} \in \mathbb{R}^t\), which is also equivalent to solving a \(t\)-dimensional problem instance with \(\tilde{A} = Q_t^T AQ_t\) and \(\tilde{b} = Q_t^T b\). For sufficiently large \(d\) the time to solve such problems will be dominated by the \(t\) matrix-vector products required to construct \(\tilde{A}\).

In this paper we focus on the choice \(Q_t = K_t(A,b)\) the Krylov subspace of order \(t\). This choice offers a significant efficiency boost: we can efficiently construct a basis \(Q_t\) for which \(Q_t^T AQ_t\) is tridiagonal, using the Lanczos process, which consists of the following recursion, starting with \(q_1 = b/\|b\|, q'_1 = 0\),

\[ \alpha_t = q_t^T A q_t, \ \beta_t = \|q'_t\|, \ q_{t+1} = A q_t - \alpha_t q_t - \beta_t q_{t-1}, \ q_{t+1} = q'_{t+1}/\|q'_{t+1}\|. \]

The vectors \(q_1, \ldots, q_t\) give the columns of \(Q_t\) while \(\alpha_1, \ldots, \alpha_t\) and \(\beta_2, \ldots, \beta_t\) respectively give the diagonal and off-diagonal elements of the symmetric tridiagonal matrix \(\tilde{A} = Q_t^T AQ_t\). The tridiagonal structure of \(\tilde{A}\) allows us to solve \(\tilde{A}_{\lambda} x = z\) in time linear in \(t\), which combined with Newton steps on \(\lambda\) [9, 8] allows us to solve equations (21) and (22) in time essentially linear in \(t\). It is also possible to avoid keeping \(Q_t\) in memory (when \(t \cdot d\) storage is too demanding) by running the Lanczos process twice, once for evaluating \(\tilde{x}\) and again to obtain \(x = Q_t \tilde{x}\).

The Lanczos process produces the same result as Gram-Schmidt orthonormalization of the vectors \([b, Ab, \ldots, A^{t-1}b]\) but uses the special structure of that matrix to avoid computing inner products that are known in advance to be zero. When run for many iterations, the Lanczos process has well-documented numerical stability issues [30]. However, in our setting we usually seek low to moderate accuracy solutions and will usually stop at \(t < 100\), for which Lanczos is reasonably stable with floating point arithmetic even when \(d\) is quite large. The application of the Lanczos process—which is typically used for eigenvector computation—in the context of regularized quadratic optimization is sometimes referred to as the generalized Lanczos process [13].

B Polynomial approximation results

In this section we state (and prove for ease of reference) two classical results on uniform polynomial approximation (cf. [20, 22]) that stand at the core of the technical development in this work.
Lemma 4. Let \( n \geq 1 \) and \( 0 < \alpha \leq \beta \), and let \( \kappa = \beta / \alpha \). Then

\[
\min_{p \in \mathcal{P}_n} \max_{x \in [\alpha, \beta]} |1 - xp(x)| = \Xi_n(\kappa) = 2 \left( \left( \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \right)^n + \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^n \right)^{-1}
\]

and

\[
2 \left( e^{2n/(\sqrt{\kappa} - 1)} + 1 \right)^{-1} \leq \Xi_n(\kappa) \leq 2 e^{-2n/\sqrt{\kappa}}.
\]

Moreover, there exist \( x_0, x_1, \ldots, x_n \in [\alpha, \beta] \) and probability distribution \( \pi_0, \pi_1, \ldots, \pi_n \) such that

\[
\min_{p \in \mathcal{P}_n} \sum_{k=0}^{n} \pi_k (1 - x_k p(x_k))^2 = [\Xi_n(\kappa)]^2.
\]

Proof. Let

\[
T_n(x) = \begin{cases} 
\cos(n \arccos(x)) & |x| \leq 1 \\
\frac{1}{2} \left( (x - \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right) & |x| \geq 1
\end{cases}
\]

denote the order \( n \) Chebyshev polynomial of the first kind. We claim that \( p^* \in \mathcal{P}_n \) that solves the minimax problem \( \min_{p \in \mathcal{P}_n} \max_{x \in [\alpha, \beta]} |1 - xp(x)| \) is given by

\[
1 - xp^*(x) = \Xi_n(\kappa) \cdot T_n \left( \frac{\kappa + 1 - 2x/\alpha}{\kappa - 1} \right),
\]

where \( \Xi_n(\kappa) = \left[ T_n \left( \frac{\pi + 1}{\kappa - 1} \right) \right]^{-1} \) guarantees that the RHS has value 1 at \( x = 0 \) and therefore \( p^* \) is well defined. Since clearly \( |T_n(y)| \leq 1 \) for every \( y \in [-1, 1] \), we have that \( \max_{x \in [\alpha, \beta]} |1 - xp^*(x)| = \Xi_n(\kappa) \).

We argue that \( p^* \) is optimal using the classical alternating signs argument, sometimes also referred to as Chebyshev’s theorem. First, note that \( T_n(y) \) has \( n+1 \) extrema in \( [-1, 1] \) (at \( y_k = \cos(k\pi/n) \) for \( k = 0, \ldots, n \)) and that their values alternate between \(-1\) and \(1\) (i.e. \( T_n(y_k) = (-1)^k \)). Therefore, there exist \( n+1 \) distinct points \( x_0, x_1, \ldots, x_n \in [\alpha, \beta] \) for which \( 1 - x_k p^*(x_k) = (-1)^k \Xi_n(\kappa) \). Let \( q \in \mathcal{P}_n \) satisfy \( \max_{x \in [\alpha, \beta]} |1 - xq(x)| = \Xi_n(\kappa) \). Then,

\[
p^*(x_k) - q(x_k) = \frac{[1 - x_k q(x_k)] - [1 - x_k p^*(x_k)]}{x_k}
\]

must be non-positive for even \( k \) and non-negative for odd \( k \), and therefore \( p^* - q \) must have at least \( n \) roots in \([\alpha, \beta]\). However, \( p^* - q \) is a polynomial of degree at most \( n - 1 \) and can have \( n \) roots only if it is identically 0, so we have that \( q = p^* \), proving that \( p^* \) is the unique solution of the minimax problem.

To see the upper and lower bounds on \( \Xi_n(\kappa) \), note that \( \Xi_n(\kappa) = 1 / \cosh(\log(1 + \frac{2}{\sqrt{\kappa} - 1})) \), that \( \frac{1}{2} e^y \leq \cosh(y) \leq \frac{1}{2} (e^y + 1) \), and that

\[
\frac{2}{z} \leq \log \left( 1 + \frac{2}{z} \right) \leq \frac{2}{z - 1},
\]

where the lower bound above can be seen by comparing derivatives.

To see the final part of the lemma, let \( x_0, x_1, \ldots, x_n \in [\alpha, \beta] \) be the points constructed in the optimality argument above, and note that this argument continues to hold if the inner maximization is restricted to these points. Therefore,

\[
\min_{p \in \mathcal{P}_n} \max_{0 \leq k \leq n} (1 - x_k p(x_k))^2 = \left[ \min_{p \in \mathcal{P}_n} \max_{0 \leq k \leq n} (1 - x_k p(x_k)) \right]^2 = [\Xi_n(\kappa)]^2.
\]
Letting $\Delta_{n+1}$ denote the probability simplex with $n + 1$ variables, we may write

$$\max_{0 \leq k \leq n} (1 - x_k p(x_k))^2 = \max_{\mu \in \Delta_{n+1}} \sum_{k=0}^{n} \mu_k (1 - x_k p(x_k))^2.$$

Finally, noting that the objective $\sum_{k=0}^{n} \mu_k (1 - x_k p(x_k))^2$ is linear (and hence concave) in $\mu$ and convex in (the coefficients of) $p$, we may use Von-Neumann’s lemma and swap the min and max above, writing

$$\max_{\mu \in \Delta_{n+1}} \min_{p \in \mathcal{P}_n} \sum_{k=0}^{n} \mu_k (1 - x_k p(x_k))^2 = \min_{p \in \mathcal{P}_n} \max_{\mu \in \Delta_{n+1}} \sum_{k=0}^{n} \mu_k (1 - x_k p(x_k))^2 = [\Xi_n(\kappa)]^2.$$

Letting $\pi$ denote the distribution attaining the outer maximum, we get the desired result. We remark in passing that $\pi$ may be constructed explicitly using the orthogonality principle of least squares estimation and orthogonality relations of Chebyshev polynomials.

**Lemma 5.** Let $n \geq 1$ and $0 < \alpha \leq \beta$, let $\kappa = \beta/\alpha$ and define $w(x) \triangleq \sqrt{x - \alpha}$. Then

$$\min_{p \in \mathcal{P}_n} \max_{x \in [\alpha, \beta]} w(x) |1 - xp(x)| = \Upsilon_n(\kappa) \triangleq 2 \sqrt{\alpha} \left( (\frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1})^{n+\frac{1}{2}} - (\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1})^{n+\frac{1}{2}} \right)^{-1}$$

and

$$2 \sqrt{\alpha} \left( e^{2(2n+1)/\sqrt{\kappa}} - 1 \right)^{-\frac{1}{2}} \leq \Upsilon_n(\kappa) \leq 2 \sqrt{\alpha} \left( e^{2(2n+1)/\sqrt{\kappa}} - 2 \right)^{-\frac{1}{2}}.$$

Moreover, there exist $x_0, x_1, \ldots, x_n \in [\alpha, \beta]$ and probability distribution $\pi_0, \pi_1, \ldots, \pi_n$ such that

$$\min_{p \in \mathcal{P}_n} \sum_{k=0}^{n} \pi_k w^2(x_k)(1 - x_k p(x_k))^2 = [\Upsilon_n(\kappa)]^2.$$

**Proof.** Let

$$U_n(x) = \begin{cases} \frac{1}{\sqrt{1-x}} \sin((n+1) \arccos(x)) & |x| \leq 1 \\ \frac{1}{2\sqrt{x^2-1}} \left( (x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n-1} \right) & |x| \geq 1 \end{cases}$$

denote the order $n$ Chebyshev polynomial of the second kind. We claim that $p^* \in \mathcal{P}_n$ that solves the minimax problem $\min_{p \in \mathcal{P}_n} \max_{x \in [\alpha, \beta]} (x - \alpha)^{1/2} |1 - xp(x)|$ is given by

$$1 - xp^*(x) = \frac{\Upsilon_n(\kappa)}{w(\beta)} \cdot U_{2n} \left( \sqrt{\frac{\kappa - x/\alpha}{\kappa - 1}} \right),$$

where $\Upsilon_n(\kappa) = w(\beta) \left[ U_{2n} \left( \sqrt{\frac{\kappa - 1}{\kappa - 1}} \right) \right]^{-1}$ guarantees that the RHS has value 1 at $x = 0$ and therefore $p^*$ is well defined (note that $U_{2n}(\cdot)$ is an even polynomial and therefore $U_{2n}(\sqrt{\cdot})$ is a polynomial of degree $n$). For $x \in [\alpha, \beta]$, we have by the definition of $p^*$ and the expression for $U_{2n},$

$$w(x)(1 - xp^*(x)) = \Upsilon_n(\kappa) \cdot \sin \left( (2n + 1) \arccos \left( \sqrt{\frac{\kappa - x/\alpha}{\kappa - 1}} \right) \right).$$
Therefore, we have that \( w(x)|1 - xp^*(x)| \leq \mathcal{U}_n(\kappa) \) for every \( x \in [\alpha, \beta] \), and moreover we have that \( w(x_k)(1 - xp^*(x_k)) = (-1)^k \cdot \mathcal{U}_n(\kappa) \), for the points \( x_0, \ldots x_n \in [\alpha, \beta] \) satisfying

\[
\sqrt{\frac{\kappa - x_k/\alpha}{\kappa - 1}} = \cos \left( \frac{\pi}{2} \cdot \frac{2k + 1}{2n + 1} \right).
\]

Hence, the alternating signs argument from the proof of Lemma 4 holds here as well and we have that \( p^* \) is optimal and that \( \min_{p \in \mathbb{P}_n} \max_{x \in [\alpha, \beta]} w(x)|1 - xp(x)| = \mathcal{U}_n(\kappa) \).

To see the upper and lower bounds on \( \mathcal{U}_n(\kappa) \), note that \( \mathcal{U}_n(\kappa) = \sqrt{\alpha} / \sinh(n + \frac{1}{2}) \log(1 + 2^{-\frac{1}{2} + 1}) \), that for \( y \geq 0, \sinh(y) = \frac{1}{\sqrt{2}} \sqrt{\cosh(2y) - 1} \) gives \( \frac{1}{2} \sqrt{e^{2y} - 2} \leq \sinh(y) \leq \frac{1}{2} \sqrt{e^{2y} - 1} \), and that (as in Lemma 4) \( \frac{2}{z} \leq \log \left( 1 + \frac{2}{z-1} \right) \leq \frac{2}{z-1} \).

The final part of the lemma follows exactly as in Lemma 4.

\[\square\]

C Proofs from Section 2

C.1 Proof of auxiliary lemmas

Lemma 1 (Approximate matrix inverse). Let \( \alpha, \beta \) satisfy \( 0 < \alpha \leq \beta \), and let \( \kappa = \beta/\alpha \). For any \( t \geq 1 \) there exist a polynomial \( p \) of degree at most \( t - 1 \), such that for every \( M \) satisfying \( \alpha I \preceq M \preceq \beta I \),

\[
\|I - Mp(M)\| \leq 2e^{-2t/\sqrt{\kappa}}.
\]

Proof. This is an immediate consequence of Lemma 4, as

\[
\min_{p \in \mathbb{P}_n} \max_{\alpha I \preceq M \preceq \beta I} \|I - Mp(M)\| = \min_{p \in \mathbb{P}_n} \max_{\lambda \in [\alpha, \beta]} |1 - \lambda \cdot p(\lambda)| = \mathcal{U}_t(\kappa).
\]

\[\square\]

Lemma 2 (Convex trust-region problem). Let \( t \geq 1, M \succeq 0, v \in \mathbb{R}^d \) and \( r \geq 0 \), and let \( f_{M,v}(x) = \frac{1}{2} x^T M x + v^T x \). There exists \( x_t \in K_t(M, v) \) such that

\[
\|x_t\| \leq r \text{ and } f_{M,v}(x_t) - \min_{\|x\| \leq r} f_{M,v}(x) \leq \frac{4\lambda_{\max}(M) \cdot y^2}{(t+1)^2}.
\]

Proof. Let \( g : \mathbb{R}^d \to \mathbb{R} \) have \( L \)-Lipschitz gradient and let \( Q \subseteq \mathbb{R}^d \) be a convex set containing the point 0. Consider Nesterov’s accelerated gradient method for minimization of \( g \), which comprises the following recursion [24, Scheme (2.2.17)],

\[
x_{k+1} = \min_{x \in Q} \left\{ x^T \nabla g(y_k) + \frac{L}{2} \|x - y_k\|^2 \right\} = \Pi_Q \left( y_k - \frac{1}{L} \nabla g(y_k) \right)
\]

\[
\alpha_{k+1}^2/(1 - \alpha_{k+1}) = \alpha_k^2 \Rightarrow \alpha_{k+1} = -\frac{\alpha_k^2}{2} + \frac{\alpha_k^2}{2} \sqrt{1 + \frac{4}{\alpha_k^2}}
\]

\[
y_{k+1} = x_{k+1} + \alpha_{k+1}(\alpha_k^{-1} - 1)(y_{k+1} - y_k),
\]

where \( \Pi_Q(\cdot) \) is the Euclidean projection to \( Q \). Letting \( \alpha_0 = 1 \) and \( y_0 = x_0 = 0 \), and letting \( x^* \) denote any minimizer of \( g \) in \( Q \), the analysis of Tseng [32, Corollary 2(b)] gives\(^1\),

\[
g(x_t) - g(x^*) \leq \frac{4L \max_{z \in Q} \|z\|^2}{(t+1)^2}.
\]

\(^1\) translating to the notation of [32], take \( \phi(x, v) = g(x) \) and \( P(x) \) to be the indicator of \( Q \), so that \( q^P(\cdot) = g(x^*) \), note that \( \theta_k (\alpha_k \text{ in our notation}) \) satisfies \( \theta_k \leq 2/(2+k) \).
Taking \( g = f_{M,v} \) and \( Q = B_r = \{ x \mid \|x\| \leq r \} \), we note that \( f_{M,v} \) has \( L \triangleq \lambda_{\max}(M) \)-Lipschitz gradient, and that the projection step guarantees that \( \|x_t\| \leq r \) for every \( t \). Therefore, to establish the lemma it remains only to argue that \( x_t \) as defined above is in \( K_t(M,v) \); we shall see this by simple induction, whose basis is \( y_0, x_0 \in K_0(M,v) = \{0\} \). Assume now that \( y_k, x_k \in K_k(M,v) \) for some \( k \geq 0 \). This implies

\[
y_k - \frac{1}{L} \nabla g(y_k) = y_k - \frac{1}{L} A y_k - \frac{1}{L} v \in K_{k+1}(M,v).
\]

Further, note that projection to the Euclidean ball \( B_r \) is simply scaling:

\[
\Pi_Q(z) = \Pi_{B_r}(z) = \frac{r}{\max\{r, \|z\|\}} \cdot z,
\]

and therefore \( x_{k+1} \in K_{k+1}(M,v) \). Finally, \( y_{k+1} \) is simply a linear combination of \( x_{k+1} \) and \( x_k \) and therefore is also in \( K_{k+1}(M,v) \), concluding the induction and the proof. \( \square \)

**Lemma 3** (Finding eigenvectors, [20, Theorem 4.2]). Let \( M \succeq 0 \) be such that \( u^T M u = 0 \) for some unit vector \( u \in \mathbb{R}^d \), and let \( v \in \mathbb{R}^d \). For every \( t \geq 1 \) there exists \( z_t \in K_t(M,v) \) such that

\[
\|z_t\| = 1 \quad \text{and} \quad z_t^T M z_t \leq \frac{\|M\|}{16(t - \frac{q}{2})^2 \log^2 \left( -2 + \frac{4 \|v\|^2}{(u^T v)^2} \right)}.
\]

**Proof.** Let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d \) denote the eigenvalues of \( M \) and let \( u_1, u_2, \ldots, u_d \) denote their corresponding (orthonormal) eigenvectors. By our assumption \( \lambda_1 = 0 \) and we have also \( \lambda_d = \|M\| \). We let

\[
v(i) \triangleq u_i^T v
\]

denote the component of \( v \) in the eigenbasis of \( M \). Define

\[
\text{err}_t \triangleq \min_{p \in \mathcal{P}_t} \frac{(p(M)v)^T M p(M)v}{\|p(M)v\|^2} = \min_{p \in \mathcal{P}_t} \frac{\sum_{i=1}^d v_i^2 p^2(\lambda_i) \lambda_i}{\sum_{i=1}^d v_i^2 p^2(\lambda_i)},
\]

and let \( q \in \mathcal{P}_t \) attain the minimum above. Setting \( z_t = q(M)v/\|q(M)v\| \), we see that

\[
\text{err}_t = z_t^T M z_t = \frac{\sum_{i=1}^d v_i^2 q^2(\lambda_i) \lambda_i}{\sum_{i=1}^d v_i^2 q^2(\lambda_i)},
\]

and so our proof comprises of bounding \( \text{err}_t \) from above.

We invoke Lemma 5 with \( n = t - 1, \alpha = \text{err}_t \) and \( \beta = \lambda_d = \|M\| \); let \( \tilde{q}(x) = 1 - x p^*(x) \in \mathcal{P}_t \) be the polynomial for which the Lemma guarantees

\[
\max_{x \in \text{err}_t, \lambda_d} \left( x - \text{err}_t \right)^{1/2} |\tilde{q}(x)| = \Omega_{t-1}(\kappa).
\]

By the optimality of \( q \), we have that

\[
\text{err}_t \leq \frac{\sum_{i=1}^d v_i^2 q^2(\lambda_i) \lambda_i}{\sum_{i=1}^d v_i^2 q^2(\lambda_i)}.
\]
Rearranging and noting that \( \tilde{q}(\lambda(1)) = \tilde{q}(0) = 1 \), we obtain
\[
\text{err}_t \leq \sum_{i=2}^{d} \frac{v_i^2(\lambda) - \text{err}_t)}{v_1(\lambda)} q^2(\lambda(i)) \leq \frac{\|v\|^2 - v^2(1)}{v_1^2} \max_{\lambda \in [\text{err}_t, \lambda(d)]} (\lambda - \text{err}_t) q^2(\lambda) = \left( \frac{\|v\|^2}{v_1^2} - 1 \right) [\mathcal{U}_{t-1}(\kappa)]^2.
\]

Lemma 5 provides the bound
\[
[\mathcal{U}_{t-1}(\kappa)]^2 \leq \frac{4 \text{err}_t}{e^{2(2t-1)\sqrt{\text{err}_t} \|M\|} - 2}.
\]

Substituting the upper bound into \( \text{err}_t \leq \left( \frac{\|v\|^2}{v_1^2} - 1 \right) [\mathcal{U}_{t-1}(\kappa)]^2 \) and rearranging gives the result. \( \square \)

### C.2 Proof of Corollary 2

**Corollary 2.** Let \( v \) be uniformly distributed on the unit sphere in \( \mathbb{R}^d \), let \( \sigma > 0 \) and let
\[
\tilde{b} = b + \sigma \cdot v.
\]
Let \( \tilde{s}_t^\text{tr} \in \arg\min_{x \in \mathcal{K}(\tilde{A}, \tilde{b})} \| x \| \leq R \hat{f}_{\tilde{A}, \tilde{b}}(x) \triangleq \frac{1}{2} x^T A x + \tilde{b}^T x \). For any \( \delta > 0 \),
\[
f_{\tilde{A}, \tilde{b}}(\tilde{s}_t^\text{tr}) - f_{\tilde{A}, \tilde{b}}(s_t^\text{tr}) \leq \frac{(\lambda_{\max} - \lambda_{\min}) R^2}{(t - \frac{1}{2})^2} \left[ 4 + \frac{\mathbb{I}_{\{\lambda_{\min} < 0\}}}{2} \log^2 \left( \frac{2 \|\tilde{b}\| \sqrt{\delta}}{\sigma \delta} \right) \right] + 2 \sigma R. \tag{13}
\]

with probability at least \( 1 - \delta \) with respect to the random choice of \( v \).

**Proof.** Let \( \tilde{x}_t^* \in \arg\min_{x \in \mathcal{K}(\tilde{A}, \tilde{b}), \| x \| \leq R} \hat{f}_{\tilde{A}, \tilde{b}}(x) \) be a solution to the perturbed problem. Since \( v \) is a unit vector, for any feasible \( x \) we have
\[
f_{\tilde{A}, \tilde{b}}(x) - f_{\tilde{A}, \tilde{b}}(s_t^\text{tr}) = f_{\tilde{A}, \tilde{b}}(x) - f_{\tilde{A}, \tilde{b}}(s_t^\text{tr}) + \sigma \cdot v^T (s_t^\text{tr} - x) \leq f_{\tilde{A}, \tilde{b}}(x) - f_{\tilde{A}, \tilde{b}}(s_t^\text{tr}) + 2 \sigma R
\]
and so it suffices to argue about the perturbed optimality gap \( f_{\tilde{A}, \tilde{b}}(s_t^\text{tr}) - f_{\tilde{A}, \tilde{b}}(s_t^\text{tr}) \).

Applying the bound (5) on the perturbed problem gives us
\[
f_{\tilde{A}, \tilde{b}}(\tilde{s}_t^\text{tr}) - f_{\tilde{A}, \tilde{b}}(\tilde{x}_t^*) \leq \frac{(\lambda_{\max} - \lambda_{\min}) R^2}{(t - \frac{1}{2})^2} \left[ 4 + \frac{\mathbb{I}_{\{\lambda_{\min} < 0\}}}{2} \log^2 \left( \frac{2 \|\tilde{b}\| \sqrt{\delta}}{\sigma \delta} \right) \right], \tag{24}
\]
and a simple argument on the density of \( u_{\text{min}}^T \tilde{b} \) (cf. [4, Lemma 4.6]) shows that
\[
\|u_{\text{min}}^T \tilde{b}\| \geq \frac{\sigma \cdot \delta}{\sqrt{d}} \quad \text{with probability at least} \quad 1 - \delta. \tag{25}
\]
Combining the bounds (23), (24) and (25) gives the result (13). \( \square \)

### D Proof of lower bounds

In what follows, we break Theorem 4 into two parts, one for the linear convergence lower bound (18) and one for the sublinear lower bounds (19) and (20). We restate each sub-theorem in a way that clearly shows our control over problem-dependent parameters when constructing the hard problem instances. In our proofs we will make use of the following expression for the optimality gap in the cubic-regularization problem,
\[
\hat{f}_{\tilde{A}, \tilde{b}, \rho} (x) - \hat{f}_{\tilde{A}, \tilde{b}, \rho} (s_t^\text{cr}) = \frac{1}{2} (x - s_t^\text{cr})^T A_\rho ||s_t^\text{cr}|| (x - s_t^\text{cr}) + \frac{\rho}{6} (||s_t^\text{cr}|| - ||x||)^2 (||s_t^\text{cr}|| + 2 ||x||), \tag{26}
\]
where \( A_\rho ||s_t^\text{cr}|| = A + \rho ||s_t^\text{cr}|| I. \)
D.1 Proof of linear convergence lower bound

**Theorem 4, part I.** Let \( \lambda_{\text{min}}, \lambda_{\text{max}}, \lambda_*, \Delta \in \mathbb{R} \) such that \( \lambda_{\text{min}} \leq \lambda_{\text{max}}, \lambda_* > \max\{-\lambda_{\text{max}}, 0\} \) and \( R, \Delta > 0 \). For every \( t \geq 1 \) and every \( d > t \) there exists \( A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d \) and \( \rho > 0 \) such that

- all eigenvalues of \( A \) are in \([\lambda_{\text{min}}, \lambda_{\text{max}}]\),
- the solution \( s^* \in \mathbb{R}^d \) satisfies \( \rho \| s^* \| = \lambda_* \),
- \( \hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s^*) = \Delta \), and

\[
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s^* \| s^* \| > \left( 1 + \frac{\rho \| s^* \|}{3(\rho \| s^* \| + \lambda_{\text{min}})} \right)^{-1} \left[ \hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s^*) \right] \exp \left\{ -\frac{4t}{\sqrt{\rho \| s^* \| + \lambda_{\text{min}}} - 1} \right\}
\]

for every \( s \in \mathcal{K}_t(A, b) \).

**Proof.** From Lemma 4 with \( \alpha = \lambda_* + \lambda_{\text{min}}, \beta = \lambda_* + \lambda_{\text{max}} \) and \( n = \), we have that there exist \( \xi_0, \ldots, \xi_t \in [\alpha, \beta] \) and probability distribution \( \pi_0, \ldots, \pi_{t+1} \) such that

\[
\min_{p \in \mathcal{P}_t} \sum_{k=0}^n \pi_k (1 - \xi_k p(\xi_k))^2 \geq e^{-4t/(\sqrt{\kappa} - 1)},
\]

where \( \kappa = \beta/\alpha = (\lambda_{\text{max}} + \lambda_*)/(\lambda_{\text{min}} + \lambda_*) \). We let \( \xi \) and \( \sqrt{\kappa} \) denote vectors with entries \( \xi_0, \ldots, \xi_t \) and \( \sqrt{\kappa_0}, \ldots, \sqrt{\kappa_t} \) respectively.

To construct the problem instance \((A, b, \rho)\) we assume without loss of generality \( d = t + 1 \) as otherwise we may zero-pad \( A \) and \( b \) to make the problem effectively \((t + 1)\)-dimensional. We set

\[
A = \text{diag}(\xi - \lambda_*), \quad b = \mu A^{-1/2} \sqrt{\kappa}, \quad \rho = \| A^{-1} b \| / \lambda_*,
\]

where we will choose \( \mu > 0 \) to set the value of \( \hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s^*) \). First, we note that for any value of \( \mu \) our choice of \( \rho \) guarantees that \( \| A^{-1} b \| = \rho \lambda_* \), making \( s^* = -A^{-1} b \) the unique global minimizer of \( \hat{f}_{A,b,\rho} \). We therefore have by equation (26)

\[
\hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s^*) = \frac{1}{2} A_{\lambda_*} s^* + \frac{\rho \| s^* \|}{6} \| s^* \|^2 = \frac{\mu^2}{2} \left( 1 + \frac{\lambda_*}{3(\lambda_* + \lambda_{\text{min}})} \right)
\]

so for every \( \Delta > 0 \) there is \( \mu \) for which \( \hat{f}_{A,b,\rho}(0) - \hat{f}_{A,b,\rho}(s^*) = \Delta \). Noting that \( \sqrt{\kappa} A^{-1}_\lambda \sqrt{\kappa} \leq (\lambda_* + \lambda_{\text{min}})^{-1} \| \sqrt{\kappa} \|^2 = (\lambda_* + \lambda_{\text{min}})^{-1} \), we also have

\[
\frac{\mu^2}{2} \geq \left( 1 + \frac{3(\lambda_* + \lambda_{\text{min}})}{\lambda_*} \right)^{-1}.
\]

Now, every \( s \in \mathcal{K}_t(A, b) \) is of the form \( s = -p(A_{\lambda_*}) b \) for \( p \in \mathcal{P}_t \), and using equation (26) again we have

\[
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s^*) \geq \frac{1}{2} \left\| A_{\lambda_*}^{1/2} (s - s^*) \right\|^2 \geq \frac{1}{2} \left\| (I - A_{\lambda_*} p(A_{\lambda_*})) A^{-1/2} b \right\|^2 \geq \frac{\mu^2}{2} \left( \sum_{k=0}^n \pi_k (1 - \xi_k p(\xi_k))^2 \right) \geq \frac{\mu^2}{2} e^{-4t/(\sqrt{\kappa} - 1)},
\]

where in (a) we substituted \( s = -p(A_{\lambda_*}) b \) and \( s^* = -A^{-1} b \), in (b) we used our construction of \( A \) and \( b \), and in (c) we used the guarantee from Lemma 4. The result follows from substituting our lower bound on \( \mu^2 \) and recalling that \( \lambda_* = \rho \| s^* \|. \)  \( \square \)
D.2 A lower bound for finding eigenvectors

The “non-convex” lower bound is in its heart a statement about the difficulty of approximating an extremal eigenvector in a Krylov subspace, which we state explicitly here. The proof of the lemma consists of applying “in reverse” the same polynomial approximation result (Lemma 5) that Kuczynski and Woźniakowski [20] use for proving upper bounds on finding eigenvector with the Lanczos method (which we state as Lemma 3).

**Lemma 6** (Finding eigenvectors: lower bound). For every $d > 0$, vector $v \in \mathbb{R}^d$, unit vector $u \in \mathbb{R}^d$ and $t < d$, there exists matrix $M \in \mathbb{R}^{d \times d}$ such that $M \succeq 0$, $Mu = 0$, and for every $z \in K_t(M,v)$,

$$
\frac{z^T M z_t}{\|M\| \|z_t\|^2} \geq \min \left\{ \frac{1}{4}, \frac{1}{64(t - \frac{1}{2})^2} \log \left( -3 + 4 \frac{\|v\|^2}{(u^T v)^2} \right) \right\}.
$$

**Proof.** We take $\|M\| = 1$ without loss of generality; results for arbitrary norms of $M$ follow by scaling the construction below. Define

$$
err_t \triangleq \min \left\{ \frac{1}{4}, \frac{1}{64(t - \frac{1}{2})^2} \log \left( -3 + 4 \frac{\|v\|^2}{(u^T v)^2} \right) \right\}.
$$

We apply Lemma 5 with $n = t - 1$, $\alpha = err_t$ and $\beta = 1$, to obtain $\xi_1, \ldots, \xi_t \in [err_t, 1]$ and probability distribution $\pi_1, \ldots, \pi_t$ such that

$$
\min_{p \in P_{t-1}} \sum_{k=1}^t \pi_k (\xi_k - err_t) (1 - \xi_k p(\xi_k)) \geq \frac{4err_t}{e^{2\log(\sqrt{\frac{4}{e}})} - 1}.
$$

We assume without loss of generality that $d = t + 1$ (otherwise we zero-pad), and construct $M$ as follows. First, we take the eigenvalues of $M$ to be $0, \xi_1, \ldots, \xi_t$, satisfying $0 \leq M \leq I$. Next, we let $u$ be the eigenvector of $M$ corresponding to eigenvalue $0$, satisfying $Mu = 0$. Finally, for $i = 1, \ldots, t$ we choose the eigenvector $u_i$ corresponding to eigenvalue $\xi_i$ such that $(u_i^T v)^2 = \pi_i (\|v\|^2 - (u^T v)^2)$.

Assume by contradiction

$$
\min_{z \in K_t(M,v)} \frac{z^T M z}{\|z\|^2} < err_t,
$$

and let $q \in P_t$ be such that

$$
\frac{\sum_{i=1}^t \xi_i q^2(\xi_i) (u_i^T v)^2}{\sum_{i=1}^t q^2(\xi_i) (u_i^T v)^2} = \frac{(q(M)v)^T M q(M)v}{\|q(M)v\|^2} = \min_{z \in K_t(M,v)} \frac{z^T M z}{\|z\|^2} < err_t.
$$

Rearranging and using $(u_i^T v)^2 = \pi_i (\|v\|^2 - (u^T v)^2)$ and letting $\bar{q}(x) = q(x)/q(0)$, we have that

$$
err_t > \left( \frac{\|v\|^2}{(u^T v)^2} - 1 \right) \sum_{i=1}^t \pi_i (\xi_i - err_t) \bar{q}^2(\xi_i) \geq \left( \frac{\|v\|^2}{(u^T v)^2} - 1 \right) \frac{4err_t}{e^{2\log(\sqrt{\frac{4}{e}})} - 1}.
$$

where in the last transition we used that $\bar{q}(0) = 1$ and therefore it is of the form $1 - xp(x)$ for some $p \in P_{t-1}$, so the lower bound (28) applies. Rearranging gives

$$
err_t > h \left( \frac{1}{16(t - \frac{1}{2})^2} \log \left( -3 + 4 \frac{\|v\|^2}{(u^T v)^2} \right) \right), \quad h(x) = \frac{x}{(1 + \sqrt{x})^2}.
$$

Using $h(x) \geq \frac{1}{4} \min \{ 1, x \}$ and the definition (27) of $err_t$, we see that the above bound gives the contradiction $err_t > err_t$ and therefore assumption (29) must be false and we have the desired result

$$
\min_{z \in K_t(M,v)} \frac{z^T M z}{\|z\|^2} \geq err_t.
$$

□
D.3 Proof of sublinear convergence lower bound

**Theorem 4, part II.** Let \( \lambda_{\min}, \lambda_{\max}, R, \tau \in \mathbb{R} \) such that \( \lambda_{\min} \leq \lambda_{\max}, \tau \geq 1 \) and \( R > 0 \). For every \( t \geq 1 \) and every \( d > t \) there exists \( A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d \) and \( \rho > 0 \) such that

- all eigenvalues of \( A \) are in \( [\lambda_{\min}, \lambda_{\max}] \),
- the solution \( s^*_{cr} = \arg\min_{x \in \mathbb{R}^d} \hat{f}_{A,b,\rho}(x) \) satisfies \( \|s^*_{cr}\| = R \),
- there exists unit eigenvector \( u_{\min} \) such that \( u_{\min}^T A u_{\min} = \lambda_{\min} \) and \( \frac{\|b\|}{\|u_{\min}\|} = \tau \), and

\[
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s^*_{cr}) > \min \left\{ \lambda_{\max}^2 - \lambda_{\min}^2, \frac{\lambda_{\max} - \lambda_{\min}}{16(t - \frac{1}{2})^2} \log^2 \left( \frac{\|b\|^2}{(u_{\min}^T b)^2} \right) \right\} \frac{\|s^*_{cr}\|^2}{32},
\]

where \( \lambda_{\max} = \min \{\lambda_{\max}, 0\} \), and

\[
\hat{f}_{A,b,\rho}(s) - \hat{f}_{A,b,\rho}(s^*_{cr}) > \frac{(\lambda_{\max} - \lambda_{\min}) \|s^*_{cr}\|^2}{16(t + \frac{1}{2})^2}.
\]

for every \( s \in \mathcal{K}_t(A,b) \).

**Proof.** We begin with the first, “non-convex” bound, which is essentially a reduction to the eigenvector problem. Here we assume \( \lambda_{\min} \leq 0 \) as otherwise the lower bound is vacuous. We use Lemma 6 to construct \( M \in \mathbb{R}^{d \times d} \) and unit vectors \( u_{\min}, v \in \mathbb{R}^d \) such that \( M \succeq 0, \|M\| = \lambda_{\max} - \lambda_{\min}, M u_{\min} = 0, \|v\|/\|u_{\min}\| = \tau \) and for every \( z \in \mathcal{K}_t(M,v) \)

\[
\frac{z^T M z}{\|z\|^2} \geq \frac{\lambda_{\max} - \lambda_{\min}}{4} \min \left\{ 1, \frac{1}{16(t - \frac{1}{2})^2} \log^2 \left( -3 + 4 \frac{\|v\|^2}{(u_{\min}^T v)^2} \right) \right\}
\geq \frac{1}{4} \min \left\{ \lambda_{\max} - \lambda_{\min}, \frac{\lambda_{\max} - \lambda_{\min}}{16(t - \frac{1}{2})^2} \log^2 \left( \frac{\|v\|^2}{(u_{\min}^T v)^2} \right) \right\} \triangleq \epsilon_t,
\]

where \( \lambda_{\max} = \min \{\lambda_{\max}, 0\} \). We let \( \epsilon > 0 \) be a parameter to be specified later. We let

\[
\lambda_* = -\lambda_{\min} + \epsilon
\]

and construct the cubic regularization instance as follows

\[
A = M + \lambda_{\min} I, \quad b = \frac{R}{\|A^{-1} \lambda_* v\|} v, \quad \rho = \lambda_*/R.
\]

The solution for this instance is unique and satisfies \( s^*_{cr} = -A^{-1} \lambda_* b = -RA^{-1} \lambda_* v/\|A^{-1} \lambda_* v\| \) so that \( \|s^*_{cr}\| = R \), and moreover we note that \( \|b\| \to 0 \) as \( \epsilon \to 0 \). For every \( s \in \mathcal{K}_t(M,v) = \mathcal{K}_t(A,b) \),

\[
\hat{f}_{A,b,\rho}(s) = \frac{1}{2} s^T A s + b^T s + \rho \frac{3}{2} \|s\|^3 \geq -\|b\| \|s\| + \frac{1}{2} (\lambda_{\min} + \epsilon_t) \|s\|^2 + \rho \frac{3}{2} \|s\|^3.
\]

The RHS above is minimal for

\[
\|s\| = \tilde{R} \triangleq \frac{-\lambda_{\min} + \epsilon_t}{2\rho} + \sqrt{\left( \frac{-\lambda_{\min} + \epsilon_t}{2\rho} \right)^2 + \frac{\|b\|^2}{\rho}} - \frac{\lambda_{\min} - \epsilon_t}{\rho} + \sqrt{\frac{\|b\|^2}{\rho}},
\]

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where the bound holds since our definition of $\epsilon_t$ implies $\epsilon_t \leq -\lambda_{\text{min}}$ and so $-\lambda_{\text{min}} - \epsilon_t \geq 0$. The minimum value of the RHS satisfies

$$f_{A,b,\rho}(s) \geq -\frac{2}{3} \|b\| \tilde{R} + \frac{1}{6} (\lambda_{\text{min}} + \epsilon_t) \tilde{R}^2.$$  \hspace{1cm} (31)

Taking without loss of generality $u_{\text{min}}^T b \leq 0$ and using $\rho = \lambda_\star / R$, and $\lambda_\star = -\lambda_{\text{min}} + \epsilon$, we have

$$f_{A,b,\rho}(s_\star^t) \leq f_{A,b,\rho}(R \cdot u_{\text{min}}) \leq \frac{1}{2} \lambda_{\text{min}} R^2 + \frac{1}{3} \lambda_\star R^2 = \frac{1}{6} \lambda_{\text{min}} R^2 + \frac{\epsilon}{3} R^2. \hspace{1cm} (32)$$

Recall that $\|b\| \to 0$ as $\epsilon \to 0$, and take $\epsilon > 0$ sufficiently small so that

$$\epsilon < \epsilon_t / 24 \quad \text{and} \quad \|b\| \leq \min\{\epsilon_t R / 24, \epsilon_t^2 / \rho\},$$

which implies also

$$\tilde{R} \leq \frac{-\lambda_{\text{min}} - \epsilon_t}{\rho} + \frac{\epsilon_t}{\rho} = \frac{-\lambda_{\text{min}}}{\rho} \leq \frac{\lambda_\star}{\rho} = R.$$

Substituting the bounds on $\tilde{R}$, $\|b\|$, and $\epsilon$ into expressions (31) and (32) yields

$$f_{A,b,\rho}(s) - f_{A,b,\rho}(s_\star^t) \geq \frac{\epsilon_t}{6} R^2 - \frac{2}{3} \|b\| R - \frac{\epsilon}{3} R^2 \geq \frac{\epsilon_t}{8} R^2.$$

Recalling $\|s_\star^t\| = R$ and the definition (30) of $\epsilon_t$, we get the desired “non-convex” lower bound.

To derive the alternative, “convex”, lower bound, we again let $0 < \epsilon < \lambda_{\text{max}} - \lambda_{\text{min}}$ be a parameter to be determined, and we apply Lemma 5 with $n = t$, $\alpha = \epsilon$, $\beta = \lambda_{\text{max}} - \lambda_{\text{min}}$ to obtain points

$$\xi_0, \ldots, \xi_t \in [0, \lambda_{\text{max}} - \lambda_{\text{min}}]$$

and probability masses $\pi_0, \ldots, \pi_t$ such that

$$\min_{p \in \mathcal{P}_t} \sum_{k=0}^n \pi_k (\xi_k - \epsilon) (1 - \xi_k p(\xi_k))^2 = \left[ \mathcal{U}_t \left( \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\epsilon} \right) \right]^2.$$

To construct the hard instance we again set

$$\lambda_\star = -\lambda_{\text{min}} + \epsilon.$$

Letting $\xi$ and $\sqrt{\pi}$ denote vectors with entries $\xi_i$ and $\sqrt{\pi}_i$, we set

$$A = \text{diag}(\xi - \lambda_\star), \quad b = R \cdot A \lambda_\star \sqrt{\pi}, \quad \rho = \lambda_\star / R.$$

Again we have that $s_\star^t = -A_\lambda^{-1} b$ is the unique solution and $\|s_\star^t\| = R \|\sqrt{\pi}\| = R$. Let $s \in \mathcal{K}_t(A,b)$, then

$$s = -p(A_\lambda) b = p(A_\lambda) A_\lambda s_\star^t = Rp(A_\lambda) A_\lambda \sqrt{\pi}$$

for some $p \in \mathcal{P}_t$. By equality (26) we have

$$f_{A,b,\rho}(s) - f_{A,b,\rho}(s_\star^t) \geq \frac{1}{2} \left\| A \lambda_{\text{star}}^{1/2} (s - s_\star^t) \right\|^2 = \frac{R^2}{2} \sum_{k=0}^t \pi_k \xi_k (1 - \xi_k p(\xi_k))^2 \geq \frac{R^2}{2} \sum_{k=0}^t \pi_k (\xi_k - \epsilon) (1 - \xi_k p(\xi_k))^2 = \frac{R^2}{2} \left[ \mathcal{U}_t \left( \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\epsilon} \right) \right]^2.$$

Note that

$$\lim_{\epsilon \to 0} \mathcal{U}_t \left( \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\epsilon} \right) = \sqrt{\lambda_{\text{max}} - \lambda_{\text{min}}} \frac{2t + 1}{2t + 1}.$$

Therefore, we can choose $\epsilon$ sufficiently small so that

$$\left[ \mathcal{U}_t \left( \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\epsilon} \right) \right]^2 \geq \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{2(2t + 1)^2},$$

which gives the proof for the “convex” lower bound, as $\|s_\star^t\| = R$. \hfill \Box
E Numerical experiment details

Random problem generation We generate random cubic regularization instances \((A, b, \rho)\) as follows. We take \(\lambda_{\max} = 1\) and draw \(\lambda_{\min} \sim U[-1, -0.1]\), where \(U[a, b]\) denotes the uniform distribution on \([a, b]\). We then fix two eigenvalues of \(A\) to be \(\lambda_{\min}, \lambda_{\max}\) and draw the other \(d - 2\) eigenvalues independently from \(U[\lambda_{\min}, \lambda_{\max}]\). We then take \(A\) to be diagonal with said eigenvalues. This is without much loss of generality (as the Krylov subspace method is rotationally invariant), and it allows us to quickly compute matrix-vector products, whose computation nevertheless accounts for much of the experiment running time when using \(d = 10^6\).

For a desired condition number \(\kappa\), we let
\[
\lambda_* \triangleq \frac{\lambda_{\max} - \kappa \lambda_{\min}}{\kappa - 1}
\]
and as usual denote \(A_{\lambda_*} = A + \lambda_* I\). To generate \(b, \rho\), we draw a standard normal \(d\)-dimensional vector \(v \sim \mathcal{N}(0; I)\) and let
\[
b = \sqrt{\frac{2}{v^T A_{\lambda_*}^{-1} v + \lambda_*^2 v^T A_{\lambda_*}^{-2} v}} \cdot v, \quad \rho = \frac{\lambda_*}{\|A_{\lambda_*}^{-1} b\|}.
\]
The above choice of \(b\) and \(\rho\) guarantee that \(\rho \|A_{\lambda_*}^{-1} b\| = \lambda_*\) and therefore \(s_{\sigma}^\ast = -A_{\lambda_*}^{-1} b\) is the unique solution and the problem condition number satisfies
\[
\frac{\lambda_{\max} + \rho \|s_{\sigma}^\ast\|}{\lambda_{\min} + \rho \|s_{\sigma}^\ast\|} = \frac{\lambda_{\max} + \lambda_*}{\lambda_{\min} + \lambda_*} = \kappa
\]
as desired. Moreover, our scaling of \(b\) guarantees that
\[
\hat{f}_{A, b, \rho}(0) - \hat{f}_{A, b, \rho}(s_{\sigma}^\ast) = \frac{1}{2} (s_{\sigma}^\ast)^T A_{\lambda_*} s_{\sigma}^\ast + \frac{\rho}{6} \|s_{\sigma}^\ast\|^3 = \frac{1}{2} \left( b^T A_{\lambda_*}^{-1} b + \frac{\lambda_*}{3} b^T A_{\lambda_*}^{-2} b \right) = 1.
\]

Our technique for generating \((A, b, \rho)\) is similar to the one we used in [4] to test gradient descent for cubic regularization. The main difference is that in [4] the value of \(\rho\) is fixed and consequently there is no control over the initial optimality gap.

Hardness of generated problems It is well known that the performance of subspace methods improves dramatically when the eigenvalues of \(A\) are clustered [30]. Taking the eigenvalues of \(A\) to be uniformly distributed produces very little clustering, making the instances we draw somewhat hard. However, examining the proof of the lower bound (18) we see that the worst case eigenvalues are of the form \(\lambda_k = \lambda_{\min} + (\lambda_{\max} - \lambda_{\min}) \sin^2 \theta_k\) where \(\theta_1, \ldots, \theta_d\) are equally spaced in \([0, \pi/2]\). This is fairly different from a uniform distribution (asymptotically as \(d \to \infty\) it becomes an arcsine distribution), and consequently we think that uniformly distributing the eigenvalues makes for a challenging but not quite adversarial test case.

Computing Krylov subspace solutions We use the Lanczos process to obtain a tridiagonal representation of \(A\) as described in Section A. To obtain full optimization traces we solve equation (22) after every Lanczos iteration, warm-starting \(\lambda\) with the solution from the previous step and the minimum eigenvalue of the current tridiagonal matrix. We use the Newton method described by Cartis et al. [8, Algorithm 6.1] to solve the equation (22) in the Krylov subspace, stopping the process when \(\|A_{\lambda}^{-1} b\| - \lambda/\rho < 10^{-12}\) or after 25 tridiagonal system solves are computed.