Efficient Universal Leakage Elimination for Physical and Encoded Qubits

L.-A. Wu, M.S. Byrd, and D.A. Lidar

Chemical Physics Theory Group, University of Toronto,
80 St. George St., Toronto, Ontario M5S 3H6, Canada

(Dated: March 31, 2022)

Decoherence-induced leakage errors can couple a physical or encoded qubit to other levels, thus potentially damaging the qubit. They can therefore be very detrimental in quantum computation and require special attention. Here we present a general method for removing such errors by using simple decoupling and recoupling pulse sequences. The proposed gates are experimentally accessible in a variety of promising quantum computing proposals.

PACS numbers: 03.67.-a,03.67.Lx,03.65.Yz

The unit of quantum information is the qubit: an idealized two-level system consisting of a pair of orthonormal quantum states. However, this idealization neglects other levels which are typically present and can mix with those defining the qubit. Such mixing, the prevention of which is the subject of this work, is known as “leakage”. Leakage may be the result of the application of logical operations, or induced by system-bath coupling. In the former case, a rather general solution was proposed in [1]. Here we are interested in decoherence-induced leakage. E.g., in the ion-trap QC proposal the two-level approximation may break down and spontaneous transitions may leak population out of those levels that represent the qubit in the ion [2]. This is part of a more general problem: quantum computation (QC) depends on reliable components and a high degree of isolation from a noisy environment. When these conditions are satisfied, it is known that it is possible to stabilize a quantum computer using an encoding of a “logical qubit” into several physical qubits. Methods which profitably exploit such an encoding are, e.g., (closed-loop) quantum error correcting codes (QECC) [3,4] and (open-loop) decoherence-free subspaces or subsystems (DFS) [5,6,7,8]. The logical qubits of these codes can also undergo leakage errors, which are particularly serious: by mixing states from within the code and outside the code space, leakage completely invalidates the encoding. A simple procedure to detect and correct leakage, which can be incorporated into a fault-tolerant QECC circuit, was given in [3]. This scheme is, however, not necessarily compatible with all encodings [9]. Here we present a universal, open-loop solution to leakage elimination, which makes use of fast and strong “bang-bang” (BB) pulses [10,11]. We first give a general scheme for protecting qubits (whether encoded or physical) from leakage errors using an efficient pulse sequence. Then we illustrate the general result with examples taken from a variety of promising QC proposals. Particularly important is the fact that our scheme is experimentally feasible in these examples, in the sense that we only make use of the naturally available interactions.

Universal leakage elimination operator. — Here we give a general, existential argument for eliminating all leakage errors on encoded or physical qubits. Suppose that $n$ two-level systems (e.g., electron spins in quantum dots [12]) are used to encode one logical qubit, or that a $N$-level Hilbert space $\mathcal{H}_N$ supports a two-dimensional physical qubit subspace (e.g., hyperfine energy levels of an ion [2]). Let us arrange the basis vectors $\{ |n\rangle \}_{n=0}^{N-1}$ of $\mathcal{H}_N$ so that $|0\rangle$ and $|1\rangle$ represent the (physical or encoded) qubit states ($N = 2^n$ for the encoded case). We refer to this as the “ordered basis”. In this basis we can classify all system operators as follows:

$$
E = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad E^\perp = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \quad L = \begin{pmatrix} 0 & D \\ F & 0 \end{pmatrix},
$$

where $B$ and $C$ are $2 \times 2$ and $(N-2) \times (N-2)$ blocks respectively, and $D, F$ are $2 \times (N-2)$, $(N-2) \times 2$ blocks. Operators of type $E$ represent logical operations, i.e., they act entirely outside the qubit subspace. $E^\perp$ operators, on the other hand, have no effect on the qubit as they act entirely outside the qubit subspace. Finally, $L$ represents the leakage operators. The total system-bath Hamiltonian can be written as $H_{SB} = H_E + H_{E^\perp} + H_L$, where $H_E$ ($H_{E^\perp}$, $H_L$) is a linear combination of elements of the set $E$ ($E^\perp$, $L$), tensored with bath operators. Now consider

$$
R_L = e^{i\phi} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},
$$

where the blocks have the same dimensions as in Eq. (1) and $\phi$ is an overall phase. This operator satisfies $[R_L, L] = 0$, while $[R_L, E] = [R_L, E^\perp] = 0$. Using a BB parity-kick sequence [10] it follows that $R_L$ is a leakage-elimination operator (LEO):

$$
\lim_{n \to \infty} (e^{-iH_{SB}t/n} R_L e^{-iH_{SB}t/n} R_L)^n = e^{-iH_{E} t} e^{-iH_{E^\perp} t}
$$

In practice one takes $n = 1$ and makes the time $t$ very small compared to the bath correlation time $\tau_B$. Eq. (3) then holds to order $\tau_B$, and implies that one intersperses periods of free evolution for time $t$ with $R_L, R_L^\dagger$ pulses which are so strong that $H_{SB}$ is negligible during these
BB pulses. The term $e^{-iH_S t}$ in Eq. (3) has no effect on the qubit subspace. The term $e^{-iH_S t}$ may result in logical errors, which will have to be treated by other methods, e.g., concatenation with a QECC [1, 4, 3, 4], or additional BB pulses [1, 3]. Note that since $R_L$ commutes with the logical operations, they can be performed at the same time, i.e., our leakage elimination procedure is fully compatible with universal QC. We now give a procedure for generating LEOs from a controllable system Hamiltonian $H_S$ acting for a time $\tau$, i.e., $R_L = \exp(-iH_S \tau)$. From Eq. (3) it follows that $H_S$ must act as a projection operator $P$ onto the qubit subspace. Furthermore, $\tau$ must be chosen so that $R_L$ acts as $-I$ in the qubit subspace. A general choice is

$$R_L^{(1)} = \exp\left(\pm i\pi \hat{n} \cdot \vec{\sigma} P\right)$$

(4)

where $\vec{\sigma}$ denotes the vector of Pauli matrices, which we refer to as logical $X, Y, Z$ operations, and $\hat{n}$ is a real unit vector. This is a valid LEO since $\exp(-i\pi \hat{n} \cdot \vec{\sigma})$ expresses a $2\pi$ rotation about the axis $\hat{n}$ on the qubit Bloch sphere, upon which the qubit state acquires a minus sign. A useful example is $\tau = \pi$ and $H_S = |0\rangle \langle 0| + |1\rangle \langle 1|$, which is a projector onto the qubit subspace and acts as identity there. This example generalizes immediately to $d$-dimensional qubits:

$$R_L^{id(id)} = \exp(\pm i\pi \sum_{k=0}^{d-1} |k\rangle \langle k|).$$

(5)

Now let us consider leakage prevention on a code subspace of $K$ logical qubits, each supported by $n$ physical qubits. In analogy to $R_L^{(1)}$ we can construct a general LEO as follows. Let $S_i$ be a single-qubit logical (unitary) operation on the $i^{th}$ (encoded or physical) qubit, and let $P_i$ be a projection on the code subspace of that qubit. Then

$$R_L^{E(K)} = \exp(\pm i\pi \bigotimes_{i=1}^{K} S_i P_i)$$

(6)

is a valid LEO.

Proof: We can always rotate the Bloch sphere of a qubit so that each $S_i$ is independently transformed into $Z_i$: $S_i = U_i Z_i U_i^\dagger$ (where $U_i$ is an appropriate single-qubit unitary); $\bigotimes_{i=1}^{K} Z_i$ is a diagonal matrix of $\pm 1$, so $\exp(-i\pi \bigotimes_{i=1}^{K} Z_i) = -I$. Thus $R_L^{E(K)} = \exp(-i\pi \bigotimes_{i=1}^{K} U_i Z_i U_i^\dagger P_i) = (\bigotimes_{i=1}^{K} U_i) \exp(-i\pi \bigotimes_{i=1}^{K} Z_i P_i) \bigotimes_{i=1}^{K} U_i^\dagger = -I$ on the code subspace, and (because of the $P_i = I$) on the orthogonal complement. QED.

Sometimes we shall be able to construct single-qubit logical operators which are automatically projectors on that qubit’s subspace. We refer to such operators as “canonical”. We are now ready to apply these considerations to a number of promising QC proposals.

Example 1.— As a simple first example, consider physical qubits (without encoding), such as electrons on liquid helium [9], or an electron-spin qubit in quantum dots [12, 7], or a nuclear-spin qubit in donor atoms in silicon [18]. In those cases, a potential well at each site traps one fermion. Usually, the ground and first excited state are taken as a qubit for a given site: $|k\rangle = |c_k\rangle|\text{vac}\rangle$, where $|c_k\rangle$ is a fermionic creation operator for level $k = 0, 1$. Let $n_k = c_k^\dagger c_k$ be the fermion number operator. The logical operations for this qubit are $E = \{X = c_k^\dagger c_0 + c_0^\dagger c_k, Y = i(c_k^\dagger c_0 - c_0^\dagger c_k), Z = n_0 - n_1\}$ whose elements satisfy $su(2)$ commutation relations. In this case, a general linear Hamiltonian which includes hopping terms, $H_{SB} = \sum_{k,l} a_{kl} c_k^\dagger c_l$, where $a_{kl}$ includes parameters and bath operators, and $k,l$ denote all electron states, can leak the qubit states $k = 0, 1$ into any of the other states. Using parity-kicks, we can eliminate this leakage in terms of the LEO [recall Eq. (3)]: $R_L^{E(1)} = \exp[\pm i\pi(n_0 + n_1)]$. This LEO is implemented simply by controlling on-site energies.

Let us now generalize this to $K$ qubits. The states of the $i^{th}$ qubit are $|k\rangle_i = |c_k\rangle_i|\text{vac}\rangle$. States outside of the code subspace contain at least one creation operator $c_k^\dagger(i)$ with $k \geq 2$. A logical $Z$ operator on the $i^{th}$ qubit is $Z_i = n_0(i) - n_1(i)$, which is canonical. It follows from Eq. (3) that an LEO is:

$$(R_L^{E(K)})_{\text{term}} = \exp(\pm i\pi Z_1 Z_2 \cdots Z_K).$$

(7)

The term $\bigotimes_{i=1}^{K} Z_i$ involves a many-body interaction which is not naturally available. However, it can be constructed from available interactions as follows: Let us assume that the interaction between neighboring sites $i, j$ contains a controllable $Z_i Z_j$ term (in reality such control may have to be obtained indirectly, e.g., by controlling an $X_i X_j + Y_i Y_j$ term, as shown in [14]), and as discussed in more detail below). We note the following useful “conjugation by $\pi/4$” formula [14]:

$$T_A \circ e^{i\theta B} \equiv e^{i\pi A} e^{i\theta B} e^{-i\pi A} = e^{i\theta (iAB)},$$

(8)

which holds provided $[A, B] = 0$ and $A^2 = B^2 = I$. Using this we can efficiently generate long-range interactions by alternately switching interactions $A, B$ on/off. E.g.,

$$T_{Z_2} \circ [T_{Z_2} Z_3 \circ (T_{X_2} \circ e^{i\theta Z_1 Z_2})] = e^{i\theta Z_1 Z_2 Z_3},$$

(9)

which can, in turn, be used to generate $e^{i\theta Z_1 Z_2 Z_3 Z_4}$, etc. Using this recursive process, the implementation of the LEO $(R_L^{E(K)})_{\text{term}}$ takes $O(K)$ steps. Fig. 1 shows a circuit for the 4-qubit case.

Finally, we note that we can also treat bosonic systems, such as the linear optical QC proposal [15]. In this case, a qubit is encoded into two modes. The first qubit has states $|0\rangle_1 = b_1^\dagger|\text{vac}\rangle$ and $|1\rangle_1 = b_2^\dagger|\text{vac}\rangle$, and the second qubit is $|0\rangle_2 = b_3^\dagger|\text{vac}\rangle$ and $|1\rangle_2 = b_4^\dagger|\text{vac}\rangle$, where $b_i^\dagger$ are bosonic creation operators. Encoded two-qubit states are $|00\rangle = b_1^\dagger b_3^\dagger|\text{vac}\rangle$, $|01\rangle = b_1^\dagger b_4^\dagger|\text{vac}\rangle$, $|10\rangle =$
This form for \( (R_L^{E(1)})_{2-\text{DFS}} \) is an instance of Eq. (8) with \( \hat{n} = \hat{x} \). Note that, in agreement with our general comments above, \( (R_L^{E(1)})_{2-\text{DFS}} \) commutes with every element of \( E_{2-\text{DFS}} \), meaning that logical operations can be performed on the encoded subspace while eliminating leakage.

Next we now show how to efficiently eliminate leakage in this case on an arbitrary number of encoded qubits. The \( m \)th logical qubit is encoded as \( |0\rangle_m = |0_{2m-1}1_{2m}\rangle \), \( |1\rangle_m = |1_{2m-1}0_{2m}\rangle \). The logical Z operator is \( Z_m = (Z_{2m-1} - Z_{2m})/2 \), and is canonical. It follows from Eq. (9) that a valid LEO is:

\[
(R_L^{E(K)})_{2-\text{DFS}} = \exp(\pm i\pi \mathbf{Z}_1 \mathbf{Z}_2 \ldots \mathbf{Z}_K).
\]

The next question is how to efficiently construct such an operator. The term \( \mathbf{Z}_1 \mathbf{Z}_2 = \frac{1}{4}(Z_1Z_3 + Z_2Z_4 - Z_1Z_4 - Z_2Z_3) \) contains next and second-next-nearest-neighbor interactions. Using “conjugation by \( \pi/4 \)”, they can all be generated using only nearest-neighbor interactions in terms of the relation

\[
e^{i\theta Z_{i+2}} = T_{X_{i+1}X_{i+2}} \circ (T_{Y_{i+1}Y_{i+2}} \circ e^{i\theta Z_{i+1}}) = T_{X_{i+1}X_{i+2}+Y_{i+1}Y_{i+2}} \circ e^{i\theta Z_{i+1}}
\]

where, in accordance with [19], we have only assumed controllability of the XY interaction term \( X_{i+1}X_{i+2} + Y_{i+1}Y_{i+2} \). At this point we can use the recursive construction of Eq. (8) again, by replacing \( X, Y, Z \) there by their encoded counterparts. Doing so takes \( \mathbf{Z}_1 \mathbf{Z}_{i+1} \) to \( \mathbf{Z}_i \mathbf{Z}_{i+1} \mathbf{Z}_{i+2} \), etc., and will again efficiently construct the LEO \( (R_L^{E(K)})_{2-\text{DFS}} \), i.e., using \( O(K) \) steps. An example of this for 4 encoded qubits is shown in Fig. 1. It is interesting to contrast the linear scaling of this leakage elimination procedure with general error elimination using BB pulses. As shown in [11], without additional symmetry assumptions restricting the order of coupling terms in the Hamiltonian Eq. (3), the BB procedure, if used to eliminate all errors, requires a number of pulses that is exponential in \( N \).

**Example 4.** — Collective decoherence is a system-bath interaction that obvies full qubit permutation symmetry: \( H_{\text{Col.Dec.}}^{SB} = \sum_{\alpha=x,y,z}(\sum_i \sigma_i^\alpha) \otimes B^\alpha \), where \( \sigma_i^\alpha \) are the Pauli matrices and \( B^\alpha \) are bath operators. This situation can be created from an arbitrary linear system-bath coupling \( H_{SB}^{(1)} = \sum_i \sigma_i \cdot \tilde{B}_i \), where \( \sigma_i = (X_i, Y_i, Z_i) \) and \( \tilde{B}_i \) are bath operators, using a BB symmetrization pulse-sequence that employs only the Heisenberg exchange interaction. The shortest DFS (or “noiseless subsystem”) encoding that protects a single logical qubit against collective decoherence uses 3 physical qubits [8]. In [23] it was shown that one can perform universal QC on this DFS, again using only the Heisenberg interaction. To explain the encoding, note that the Hilbert space of 3 spin 1/2’s has total-spin \( \vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3) \) and splits into two \( S = 1/2 \) subspaces (denoted \( \lambda = 0, 1 \)), and \( S = 3/2 \) subspace. The states can be labeled as \( |S, \lambda, S_\lambda\rangle \), and

**Example 3.** — A substantial number of promising solid-state QC proposals, e.g. [12, 13, 14, 15, 22], are governed by effective isotropic and anisotropic exchange interactions, which quite generally, can be written as

\[
H_{\text{ex}} = \sum_{i<j} J_{ij}^x X_i X_j + J_{ij}^y Y_i Y_j + J_{ij}^z Z_i Z_j.
\]

The encoding \( |0\rangle_L = |01\rangle, |1\rangle_L = |10\rangle \) (using two physical qubits per logical qubit) is highly compatible with \( H_{\text{ex}} \), in the sense that universal QC can be performed by controlling the single parameter \( J_{ij}^x \) in the Heisenberg (\( J_{ij}^y = J_{ij}^z = 0 \)), XXZ (\( J_{ij}^x \neq J_{ij}^y \neq J_{ij}^z \)), and XY (\( J_{ij}^x = J_{ij}^y \neq J_{ij}^z \)) instances of \( H_{\text{ex}} \), provided there is a Zeeman splitting that distinguishes single-qubit \( Z_i \) terms. This is done using the “encoded selective recoupling” method [19]. Furthermore, the \( \{|01\rangle, |10\rangle\} \) encoding is a DFS for collective dephasing (where the bath couples only to system \( Z_{2i-1} + Z_{2i} \) operators) [14, 24, 25]. A set of logical operations on the code is \( E = \{X_1 = (X_1X_2 + Y_1Y_2)/2, Y_1 = (X_2Y_1 - Y_2X_1)/2, Z_1 = (Z_1 - Z_2)/2\} \). Only the \( X_1 \) term is assumed to be directly controllable (by manipulation of \( J_{ij}^x \)), while the \( Z_1 \) term can be turned on/off using recoupling [19]. The \( Y_1 \) term can then be reached in a few steps: \( e^{-i\phi Y_1} = T_{X_1}^\phi e^{-i\phi Z_1} \). The leakage errors are due to system-bath interactions where the system terms include any of \( X_1, Y_1, X_2, Y_2 \), and \( Z_1, Z_2 \), since as is easily seen, such terms do not preserve the \( \{|01\rangle, |10\rangle\} \) code subspace. As pointed out first in [18], the LEO can be expressed as \( (R_L^{E(1)})_{2-\text{DFS}} = \exp(i\pi X_1) = Z_1Z_2 \), which means that it is implementable using just the controllable \( J_{ij}^x \) parameter in the instances of \( H_{\text{ex}} \) mentioned above. This form for \( (R_L^{E(1)})_{2-\text{DFS}} \) is an instance of Eq. (8).
the DFS qubit is \( |0_L⟩ = α |1/2, 0, 1/2⟩ + β |1/2, 0, −1/2⟩, \)
\( |1_L⟩ = α |1/2, 1, 1/2⟩ + β |1/2, 1, −1/2⟩, \)
\( |α|^2 + |β|^2 = 1, \)
i.e., the encoding is into the degeneracy of the two 
\( S = 1/2 \) subspaces [8, 22]. Collective errors can change the \( α, β \) coefficients, but have the same effect on the \( |0_L⟩, |1_L⟩ \) states, which is why this encoding is a DFS. If, however, we also consider bilinear system-bath coupling 
\[ H_\text{SB} = \sum_{i,j} g_{ij}^{αβ} σ_i^α σ_j^β \otimes B_i^β, \]
where \( g_{ij}^{αβ} \) is a rank-2 tensor, then the symmetrization procedure of [23], that prepares collective decoherence conditions, will not work. In this case we must consider the possibility of leakage. The bilinear term \( g_{ij}^{αβ} σ_i^α σ_j^β \) can be decomposed into (i) a scalar \( g_δ δ_1 δ_2 \), which has the effect of logical errors \( E \); (ii) two rank-1 tensors \( \tilde{β} \cdot (δ_1 \times δ_2) \) and \( (δ_1 \cdot \tilde{γ}) (δ_2 \cdot \tilde{γ}) \), which can couple between \( S = 1/2 \) states, and can couple them to \( S = 3/2 \) states. Note that this also applies to imperfect symmetrization at the level of a linear system-bath Hamiltonian \( H_\text{SB} \). Thus we see that the \( S = 3/2 \) subspace acts as a source for leakage, and that there is also the possibility of (non-collective) errors [both from (ii)] which do not have the same effect on the \( |0_L⟩, |1_L⟩ \) states. We defer an analysis of the latter "\( S = 1/2 \to 1/2" \) errors to a separate publication [24], but we note that they can be suppressed using techniques similar to those we discuss next.

An open-loop leakage correction circuit for this DFS, that once more uses only the Heisenberg interaction, was given in [23]. There the DFS qubit was defined to be \( |0_L⟩ = |1/2, 0, 1/2⟩, |1_L⟩ = |1/2, 1, 1/2⟩ \) and transitions to any of the other 6 states were considered as leakage (this includes errors caused by collective decoherence, which are normally avoided by a DFS encoding). Here we add another element to this picture of the Heisenberg interaction as an enabler of universal, fault-tolerant QC, by showing that it can also provide an LEO. The importance of Heisenberg-only QC, as pointed out in [23], is in the relative ease of manipulating this interaction in a number of the most promising solid-state QC proposals [12, 13]. Now, as shown in [22],
\[ \bar{X} = \frac{1}{\sqrt{3}} δ_1 δ_3 − δ_2 δ_3, \]
acts as a logical X on the DFS qubit defined above, and annihilates the \( S = 3/2 \) states (i.e., it is canonical). Therefore, using Eq. (4), \( (R_L^{(1)})_{3-\text{DFS}} = \exp(±iπ \bar{X}) \) is a Heisenberg-only LEO for a single 3-qubit DFS which eliminates transitions to the \( S = 3/2 \) subspace. An LEO for the K-qubit case is then, from Eq. (4):

\[ (R_L^{(K)})_{3-\text{DFS}} = \exp(±iπ \bar{X}_1 \bar{X}_2 \cdots \bar{X}_K). \]

To generate this LEO from available interactions we use a procedure similar to Eq. (4). First, note that
\[ T_{\bar{X}_3} \circ T_{\bar{X}_2} \circ T_{\bar{X}_1} \circ e^{iθ\bar{X}_1} = e^{iθ\bar{X}_3 \bar{X}_2 \bar{X}_1}. \]
Efficient schemes for generating \( \bar{X}_1 \bar{X}_2 \bar{X}_3 \) were given in [25], while \( \bar{X}_1, \bar{X}_2 \) are directly obtainable from the Heisenberg interaction [23]. The recursive construction of \( (R_L^{(K)})_{3-\text{DFS}} \) then proceeds using \( T_{\bar{X}_3} \circ T_{\bar{X}_2} \circ T_{\bar{X}_1} \circ e^{iθ\bar{X}_1} = e^{iθ\bar{X}_3 \bar{X}_2 \bar{X}_1} \), etc., which again is a procedure that scales as \( O(K) \).

Conclusions.— Decoherence-induced leakage from the logical space of (physical or encoded) qubits is a severe source of errors for quantum computation. We have shown how to efficiently and universally eliminate such errors using sequences of “bang-bang” pulses. These pulses can be applied at the same time as logical operations, so that leakage elimination can be performed in conjunction with universal quantum computation. Applications to a variety of promising quantum computing proposals were discussed, and leakage elimination methods were presented that are directly applicable using only experimentally available interactions.

The present study was sponsored by the DARPA-QuIST program (managed by AFOSR under agreement No. F49620-01-1-0468), by PRO, PREA, and the Connaught Fund (to D.A.L.). We thank Dr. S. Schneider for helpful discussions.

[1] L. Tian and S. Lloyd, Phys. Rev. A 52, 050301 (2000).
[2] J.I. Cirac and P. Zoller, Phys. Rev. Lett. 74, 4019 (1995); M. Plenio and P. Knight, Proc. Roy. Soc. London Ser. A 453, 2017 (1997).
[3] J. Preskill, Proc. Roy. Soc. London Ser. A 454, 385 (1998).
[4] E. Knill et al., Science 279, 342 (1998); A.M. Steane, Nature 399, 124 (1999); D. Aharonov and M. Ben-Or, eprint quant-ph/9906129.
[5] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997).
[6] L.-M Duan and G.-C. Guo, Phys. Rev. A 57, 737 (1998).
[7] D.A. Lidar et al., Phys. Rev. Lett. 81, 2594 (1998).
[8] E. Knill et al., Phys. Rev. Lett. 84, 2525 (2000).
[9] J. Kempe et al., eprint quant-ph/0112013.
[10] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998); D. Vitali and P. Tombesi, Phys. Rev. A 59, 4178 (1999); P. Zanardi, Phys. Rev. A 63, 012301 (2001).
[11] L. Viola et al., Phys. Rev. Lett. 82, 2417 (1999).
[12] D. Loss and D.P. DiVincenzo, Phys. Rev. A 57, 120 (1998); J. Levy, Phys. Rev. A 64, 052306 (2001).
[13] D.A. Lidar et al., Phys. Rev. Lett. 82, 4556 (1999).
[14] D.A. Lidar et al., Phys. Rev. A 63, 022307 (2001).
[15] M.S. Byrd and D.A. Lidar, eprint quant-ph/0112054.
[16] P.M. Platzman and M.I. Dykman, Science 284, 1967 (1999).
[17] A. Imamoglu et al., Phys. Rev. Lett. 83, 4204 (1999); E. Pazzy et al., eprint cond-mat/0109033.
[18] B.E. Kane, Nature 393, 133 (1998); R. Vrijen et al., Phys. Rev. A 62, 042307 (2000).
[19] D.A. Lidar and L.-A. Wu, Phys. Rev. Lett. 88, 017905 (2002).
[20] E. Knill et al., Nature 409, 46 (2001).
[21] D. Mozyrsky et al., Phys. Rev. Lett. 86, 5112 (2001).
[22] J. Kempe et al., Phys. Rev. A 63, 042307 (2001).
[23] L.-A. Wu and D.A. Lidar, eprint quant-ph/0112144.
[24] M.S. Byrd, L.-A. Wu and D.A. Lidar, in preparation.
[25] D.P. DiVincenzo et al., Nature 408, 339 (2000).