Second Comment on “Contextuality within quantum mechanics manifested in subensemble mean values” [Phys. Lett. A 373 (2009) 3430]

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Abstract

I examine Pan and Home’s reply to my Comment on their proposal for testing noncontextual models. I show that the Kochen-Specker model for a qubit does explain all outcomes of a test based on such a proposal, so that it would be inconclusive about the untenability of realistic, noncontextual models.

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Pan and Home (PH) introduced recently [1] what they claimed to be a new type of contextuality between the spin and path degrees of freedom of a particle. Their goal was to show that the formalism of Quantum Mechanics (QM) embodies this kind of contextuality, in the sense that subensemble mean values of spin measurements are “contingent upon what choice is made of measuring a suitably defined comeasurable (commuting) ‘path’ observable” [1]. PH discussed a variant of a Mach-Zehnder interferometer, by means of which the alleged contextuality could be exhibited. In a Comment to PH’s paper [2] I tried to show that such a setup would be essentially equivalent to a standard Stern-Gerlach (SG) array, in the sense that any experiment performed with PH’s setup could be replicated with a SG array. As all outcomes of the latter
could be explained with the Kochen-Specker model for a qubit, so would also be the case for a test based on PH’s proposal. In a reply to my Comment Pan and Home argue [3] that this is not so, because their arrangement allows path-measurements that the SG setup does not. Such path-measurements could be effectively performed in PH’s array by recording which one of two SG devices has detected a spin-1/2 particle. Now, Pan and Home reproduce in their Reply [3] the setup I proposed; but without including a SG device which serves to prepare the spin-states I have considered. It is this missing SG device the one which carries the information that PH ascribe to their adjustable beam splitter. Last one allows to fix the different path degrees of freedom. Hence, the apparently missing stage in the array I considered, and that could serve to effectuate a “path-measurement”, was actually there. Indeed, the SG device at the entrance of this array does select one of two paths. The selected path carries then the same information as in PH’s array. This is reflected in the subensemble mean values of spin measurements that can be recorded at the two arrays – PH’s and mine – which are the same.

My aim has been to show that PH’s setup is physically equivalent to a SG setup for the sake of testing realistic models. As a consequence, the Kochen-Specker (KS) model for a qubit [4] could be invoked to explain all possible outcomes of an experiment implementing PH’s proposal. Now, it is certainly unnecessary to make the detour of showing that one setup is physically equivalent to the other, if our aim is simply to show that the KS model does explain all outcomes of some given setup. The aim of this note is to directly show how the KS model can explain the outcomes of an experiment performed according to PH’s proposal.

Let us then assume that an experimental group has implemented PH’s proposal, obtaining results that are in full agreement with the corresponding quantum-mechanical predictions. The purpose of such an experiment is to rule out some class of realistic theories. In the present case this class is constituted by noncontextual realistic theories, or models. In other words, no such a model should serve to explain the experimental outcomes. But, as we shall see, there is a model, the KS model, that would be capable of explaining all these outcomes. This is so because the KS model is capable of reproducing all quantum-mechanical predictions concerning a two-state Hilbert space, which is the only one involved in the measurements of PH’s proposal. Let us remind that observables in such a space are of the form $\hat{A} = a_0 I + \vec{a} \cdot \vec{\sigma}$, where $\vec{\sigma}$ represents...
the triple of Pauli matrices and $I$ the identity matrix.

Briefly, the experiment that realizes PH’s proposal consists in submitting to the action of a Mach-Zehnder-like array a beam of neutrons whose spins are polarized along the $+\hat{z}$-axis, i.e., neutrons being prepared in the spin-up state $|\uparrow\rangle_z$. The Mach-Zehnder-like array consists of two beam splitters, some mirrors, a spin-flipper, two SG devices and four detectors (see Fig.1 in Ref.[1]). The experimental outcomes refer to subensemble spin mean values $\langle \hat{\sigma}_\theta \rangle_{SG}$ that are drawn from the detectors set at the output of the SG devices. These SG devices can be freely oriented, $\theta$ being an angle fixing the orientation. One of the two beam splitters in the Mach-Zehnder-like array is a 50 : 50 beam splitter while the other is one of adjustable reflectivity/transmissivity, whose action can be represented by

$$A_\gamma = \begin{pmatrix} (\gamma^2 - \delta^2) & -2i\gamma\delta \\ 2i\gamma\delta & (\gamma^2 - \delta^2) \end{pmatrix},$$  

with $\gamma$ and $\delta$ being reflection and transmission (real) coefficients satisfying $\gamma^2 + \delta^2 = 1$. The matrix representation of $A_\gamma$ refers to a path-space basis $\{|\psi_1\rangle, |\psi_2\rangle\}$ constituted by the path states associated to the two input ports of the adjustable beam splitter. Its output states are $|\psi_3\rangle = -i\gamma |\psi_1\rangle + \delta |\psi_2\rangle$ and $|\psi_4\rangle = \delta |\psi_1\rangle - i\gamma |\psi_2\rangle$, in terms of which $A_\gamma$ is defined as $A_\gamma \equiv |\psi_3\rangle \langle \psi_3| - |\psi_4\rangle \langle \psi_4|.$

The quantum-mechanical predictions that the experiment should confirm are given by

$$\langle \hat{\sigma}_\theta \rangle_{SG1} = \frac{1}{2}\langle \downarrow |\hat{\sigma}_\theta |\downarrow \rangle_\varphi = -\frac{1}{2}\cos (2(\vartheta - \theta)),$$

$$\langle \hat{\sigma}_\theta \rangle_{SG2} = \frac{1}{2}\langle \uparrow |\hat{\sigma}_\theta |\uparrow \rangle_\varphi = +\frac{1}{2}\cos (2(\vartheta - \theta)).$$

Here, I have set $\gamma = \sin \vartheta$ and $\delta = \cos \vartheta$. The mean values refer to states $|\uparrow\rangle_\varphi = \sin \vartheta |\downarrow\rangle_z + \cos \vartheta |\uparrow\rangle_z$ and $|\downarrow\rangle_\varphi = \cos \vartheta |\downarrow\rangle_z - \sin \vartheta |\uparrow\rangle_z$. These states are prepared by choosing an appropriate value of $\vartheta$, viz., of $\gamma$ (or $\delta$).

Let us now turn to a KS model that explains the above results. Being a realistic model, the KS model assumes that any quantum-mechanical state $|\psi\rangle$ conveys incomplete information about a physical system, and this should be the reason why we cannot predict with certainty the results of
measurements performed on the system. This means that $|\psi\rangle$ represents in fact a whole family of systems, whose members could be in principle distinguished from one another by a series of supplementary parameters $\lambda$, so-called “hidden variables”. In the case of a two-level system we can generally write $|\psi\rangle = \cos(\theta_\psi/2) e^{-i\varphi_\psi/2} |+\rangle + \sin(\theta_\psi/2) e^{i\varphi_\psi/2} |\rangle$, with $|\pm\rangle$ being the eigenvectors of $\sigma_z$. Within the formalism of QM the states $|\psi\rangle$ and $e^{i\alpha} |\psi\rangle$ represent one and the same physical state. This state is thus more properly represented by an equivalence class, $\{e^{i\alpha} |\psi\rangle \sim |\psi\rangle\}$, or by the projector $|\psi\rangle \langle \psi| = (I + \hat{n}_\lambda \cdot \vec{\sigma})/2$, say, the KS model assigns to it the following probability for its occurrence:

$$\rho_a(\lambda) d\lambda = \frac{\hat{n}_\lambda \cdot \hat{n}_a}{\pi} \Theta (\hat{n}_\lambda \cdot \hat{n}_a) d\lambda,$$

(4)

with $\Theta$ meaning the Heaviside’s step-function and $d\lambda$ the uniform measure on the sphere ($d\lambda = \sin \theta_\lambda d\theta_\lambda d\varphi_\lambda$). It can then be proved [2] that for any $\hat{A} = a_0 I + \vec{a} \cdot \vec{\sigma}$ and $|\psi\rangle$ there is a function $A(\lambda)$ such that $\langle \psi| \hat{A} |\psi\rangle = \int \rho_\psi(\lambda) A(\lambda) d\lambda$.

Focusing on PH’s proposal, a KS model tailored to explain its outcomes can be as follows. Neutrons entering PH’s array are described by a probability distribution like that of Eq.(4) with $\hat{n}_a = \hat{z}$. The effect that the Mach-Zehnder part of PH’s array has on neutrons – so the model’s prescription – is to flip $\hat{z}$. Those neutrons exiting the array through port 3 of the adjustable beam-splitter (see Fig.1 in Ref. [1]) are in a state whose probability distribution is like that of Eq.(4) with $\hat{n}_a = \hat{n}_\downarrow$. The unit vector $\hat{n}_\downarrow$ is the one corresponding to the projector $|\downarrow\rangle \langle \downarrow|$, with $|\downarrow\rangle = \cos \theta |\downarrow\rangle_z - \sin \theta |\uparrow\rangle_z = \delta |\downarrow\rangle_z - \gamma |\uparrow\rangle_z$. Analogous prescriptions hold for exit port 4: change $\hat{n}_\downarrow$ by $\hat{n}_\uparrow$, viz., change $|\downarrow\rangle$ by $|\uparrow\rangle$. In other words, the effect of the Mach-Zehnder array is to flip the vector $\hat{z}$ either to $\hat{n}_\downarrow$ or to $\hat{n}_\uparrow$, depending on the exit channel. This is a perfectly acceptable effect that such a device can have on variables like $\hat{n}_\lambda$, irrespective of the physical meaning that we might ascribe to these variables. As was shown in Ref. [2], the KS model then predicts that $\langle \hat{\sigma}_\theta \rangle_{SG1}$ and $\langle \hat{\sigma}_\theta \rangle_{SG2}$ will be given by the corresponding expressions in Eqs.(2) and (3), thereby explaining the experimental outcomes as good as QM does.
In summary, an experiment that implements PH’s proposal would produce results that could be explained by a noncontextual realistic model, the KS model. Such an experiment would therefore be inconclusive about the untenability of realistic, noncontextual models.

References

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