$N = (4,0)$ Super-Liouville Theory on the Coadjoint Orbit and PSU(1,1|2)

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Abstract

An $N = (4,0)$ supersymmetric Liouville theory is formulated by the coadjoint orbit method. It is discovered that the action has a hidden symmetry under PSU(1,1|2).

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1 Introduction

In the past few years there has been much interest in a duality between the SYK model and the D=2 effective gravity. The Schwarzian theory is considered as playing a role of a mediator between the two. The differential geometrical aspects of the Schwarzian theory got clarified when it was reformulated by the coadjoint orbit method in [1]. Supersymmetric extensions of the theory have been also discussed[2], but without a proper account on the differential geometry. Recently the extension to the $N = 4$ case has been undertaken by means of the coadjoint orbit method in [3] by following the work [1]. The obtained theory has not only the $N = 4$ superconformal symmetry, but also a hidden symmetry under PSU$(1,1|2)$ as well. The arguments there can be straightforwardly applied for the lower symmetric Schwarzian theories.

The aim of this letter is to formulate the $N = (4,0)$ super-Liouville theory in 1+1 dimensions by similarly working out the coadjoint orbit method. The Liouville theory is the simplest one for the D=2 effective gravity. After Polyakov’s work it was extensively studied by various methods. (See [4, 5] for an overview of the studies at the early stage.) Among them the coadjoint orbit method, which was proposed by Alekseev and Shatashvili[6], is the most geometrical[6]. After this work the coadjoint orbit method was generalized to get the $(1,0)$ and $(2,0)$ supersymmetric Liouville theories in [7] and [8] respectively. The left-moving sectors of the respective theories were extended so as to admit the $(1,0)$ and $(2,0)$ superconformal symmetry. On the other hand in the right-moving sector the conformal symmetry remained non-supersymmetric, but the hidden symmetry SL$(2)$ got promoted to SL$(2|1)$ and OSp$(2|2)$ for the respective theories[9, 10]. A further extension of the coadjoint orbit method to the $N = (4,0)$ case has not been discussed.

The letter is organized as follows. In Section 2 we give a short summary of the $N = (4,0)$ superconformal diffeomorphism and the $N = 4$ super-Schwarzian derivative, following [3]. In Section 3 we then work out the coadjoint orbit method to get the $N = (4,0)$ super-Liouville theory as (3.22). In contrast with the lower symmetric theories it is given in a non-local form like the WZWN model. The author believes that there is no way to find a local expression as long as we stick to the supercovariant formalism using the superfields. A local expression of the action is found as (3.21) by expanding it in components. We then check that the purely bosonic part of the action coincides with the known non-supersymmetric Liouville theory. After this check we calculate the energy-momentum tensor of the theory by using the action of the non-local form (3.22). It is found to be given by the $N = (4,0)$ super-Schwarzian derivative. In Section 4 we show that action (3.22) has a hidden symmetry under PSU$(1,1|2)$. It is realized on the coset space PSU$(1,1|2)/\{\text{SU}(2) \otimes \text{U}(1)\}$ of which holomorphic coordinates are superfields $f, \varphi_a, \varphi^a$ representing the $N = (4,0)$ superconformal diffeomorphism. The same symmetry has been discovered for the $N = 4$ super-Schwarzian theory in [3].

Appendix A is devoted to a summary on the supersymmetric extension of the Liouville theory. In Appendix B we prove some formulae which are assumed in the main body of this letter. At important steps of the arguments we have recourse to expansion of $N = (4,0)$ superfields in components, as in [3]. But the calculations are much more complicated.
here. The details of them are presented in a separate note [11].

2 The \((4,0)\) superconformal diffeomorphism

The \(N = (4,0)\) superconformal group is a group of which elements are superdiffeomorphism in the \(N = (4,0)\) superspace. The \(N = (4,0)\) superspace is described by the supercoordinates

\[
(x, \theta_1, \theta_2, \theta^1, \theta^2, t) \equiv (x, \theta, t). \tag{2.1}
\]

Here \(x\) is a real coordinate. \(\theta_a, a = 1, 2\), are fermionic ones, while \(\theta^a, a = 1, 2\), their complex conjugates. They consist of the coordinates of the supersymmetric sector, while \(t\) a real coordinate of the non-supersymmetric sector. The respective sectors are called the left- and right- moving sectors. The details of the superconformal diffeomorphism in the left-moving has been discussed in [3]. Namely we consider superdiffeomorphism in the left-moving sector

\[
x \rightarrow f(x, \theta), \quad \theta_a \rightarrow \varphi_a(x, \theta), \quad \theta^a \rightarrow \varphi^a(x, \theta). \tag{2.2}
\]

Its infinitesimal form is given by

\[
x \rightarrow x + \delta_v f|_{(f, \varphi) = (x, \theta)}, \quad \theta_a \rightarrow \theta_a + \delta_v \varphi_a|_{(f, \varphi) = (x, \theta)}, \quad \theta^a \rightarrow \theta^a + \delta_v \varphi^a|_{(f, \varphi) = (x, \theta)}.
\]

A superconformal field with weight \((w, 0)\) is defined as transforming by these diffeomorphisms as

\[
\delta_v \Psi_{(w,0)} = \left[ v \partial_x + \frac{1}{2} D_{\theta c} v D_{\theta}^c + \frac{1}{2} D_{\theta}^c \xi D_{\theta c} + w \partial_x v \right] \Psi_{(w,0)}, \tag{2.3}
\]

with

\[
v = \delta f|_{(f, \varphi) = (x, \theta)} + \theta_c \delta \varphi^c|_{(f, \varphi) = (x, \theta)} + \theta^c \delta \varphi_c|_{(f, \varphi) = (x, \theta)}.
\]

The superfields \(f, \varphi_a, \varphi^a\) describing the the superconformal diffeomorphisms (2.2) may be given by superfields with weight \((0,0)\). They satisfy the superconformal conditions. Moreover \(\varphi_a\) and \(\varphi^a\) satisfy the chirality conditions. That is,

\[
\delta_v f = \left[ v \partial_x + \frac{1}{2} D_{\theta c} v D_{\theta}^c + \frac{1}{2} D_{\theta}^c \xi D_{\theta c} \right] f, \tag{2.4}
\]

\[
\delta_v \varphi_a = \left[ v \partial_x + \frac{1}{2} D_{\theta c} v D_{\theta}^c \right] \varphi_a, \tag{2.5}
\]

\[
\delta_v \varphi^a = \left[ v \partial_x + \frac{1}{2} D_{\theta}^c v D_{\theta c} \right] \varphi^a. \tag{2.6}
\]

As in [3] the super-Schwarzian derivative for the \(N = (4,0)\) case is given by

\[
S(f, \varphi; x, \theta, t) = \log \det[D_{\theta a} \varphi^b]. \tag{2.7}
\]

By the superdiffeomorphisms (2.3)–(2.6) it transforms anomalously

\[
\delta_v S(f, \varphi; x, \theta, t) = \left[ v \partial_x + \frac{1}{2} D_{\theta c} v D_{\theta}^c + \frac{1}{2} D_{\theta}^c v D_{\theta c} \right] S(f, \varphi; x, \theta, t) + \partial_x v. \tag{2.8}
\]
The $(4, 0)$ super Liouville theory

Now we construct an $N = 4$ super-Liouville theory by applying the coadjoint orbit method for the superconformal group so far reviewed. The superconformal algebra $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$ are centrally extended. Their elements are given by

$$(u, k) \in \mathfrak{g}, \quad (b, c) \in \mathfrak{g}^*.$$ 

Here $k$ and $c$ are central elements. $u$ and $b$ are bosonic superfields, obeying the superconformal transformations of $\Psi(-1,0)$ and $\Psi(0,0)$ given by (2.3) respectively. But the latter transformation becomes possibly anomalous. The volume element $dx d^4 \theta$ for the left-moving sector of the $N = (4, 0)$ superspace has weight 1, so that the invariant quadratic form is defined by

$$< (b, c), (u, k) > = \int dx d^4 \theta \, bu + ck. \quad (3.1)$$

The centrally extended superconformal algebra $\mathfrak{g}$ is given by the infinitesimal adjoint action $\text{ad}(v,l)$ on $(u, k) \in \mathfrak{g}$

$$\text{ad}(v,l)(u,k) = \left( v \partial_x u - u \partial_x v + \frac{1}{2} D_\theta v D_\theta^c u + \frac{1}{2} D_\theta^c v D_\theta u, \int dx d^4 \theta \, v \partial_x u \right)$$

$$\equiv [(u,k),(v,l)]. \quad (3.2)$$

Using the relation

$$< \text{ad}^*(v,l)(b,c), (u,k) > = - < (b,c), \text{ad}(v,l)(u,k) >,$$

we then find the corresponding coadjoint action $\text{ad}^*(v,l)$ on $(b, c) \in \mathfrak{g}^*$

$$\text{ad}^*(v,l)(b,c) = \left( [v \partial_x + \frac{1}{2} D_\theta v D_\theta^c + \frac{1}{2} D_\theta^c v D_\theta] b + c \partial_x v, 0 \right), \quad (3.3)$$

which is also centrally extended. We think of a coadjoint orbit $O_{(b,c)}$, whose initial point is $(b,c) \in \mathfrak{g}^*$. The finite form of (3.3) is generated on the coadjoint orbit by the superdiffeomorphism (2.7) as

$$\text{Ad}^*(f, \varphi)(b,c) \equiv \left( b(f, \varphi, t) + c \mathcal{S}(f, \varphi; x, \theta, t), c \right). \quad (3.4)$$

Here $\mathcal{S}(f, \varphi; x, \theta, t)$ is the super-Schwarzian derivative given by (2.7). Now we consider the right-moving sector in an enlarged space with coordinates $(t_1, t_2, \ldots, t_n)$, called $\mathcal{O}_n$. That is, the supercoordinates (2.1) describing the $N = (4, 0)$ superspace are extended as

$$(x, \theta_1, \theta_2, \theta^1, \theta^2, t_1, t_2, \ldots, t_n) \equiv (x, \theta, t). \quad (3.5)$$

1 Here the arguments of $b$ have been explicitly written as $b(f, \varphi, t)$. As in [3] our convention is that superfields always depend on $(x, \theta, t)$ if any argument is not written.
The Kirillov-Kostant 2-form is given by

\[ \Omega_{(b,c)} = \frac{1}{2} < \text{Ad}^*(f, \varphi)(b,c), [(y,0), (y,0)] >, \]  

(3.6)
on the coadjoint orbit \( O_{(b,c)} \) in \( O_n \). Here \( y \) is a \( g \)-valued 1-form in \( O_n \), while \( f \) and \( \varphi \) are 0-forms. They are superfields with the coordinates \( x, \theta, t_1, t_2, \cdots, t_n \). It is determined so that its infinitesimal coadjoint action on \((b, c) \in g^*\) is the exterior derivative of \((b, c) \in g^*\) on the orbit \( O_{(b,c)} \) as

\[ d(b(f, \varphi, t), c) = \text{ad}^*(y, 0))(b(f, \varphi, t), c). \]  

(3.7)
Keep in mind that the exterior derivative acts only on the coordinates \( t_1, t_2, \cdots \). In \( 3 \) \( y \) is found to be a solution to this equation such that

\[ y = \frac{1}{\Delta}(df + \varphi_c d\varphi^c + \varphi^c d\varphi_c), \]  

(3.8)
with

\[ \Delta = \partial_x f + \varphi_c \partial_x \varphi^c + \varphi^c \partial_x \varphi_c = \det[D_{\theta \alpha} \varphi^b]. \]  

(3.9)
The centrally extended commutator in (3.6) becomes

\[ [(y,0), (y,0)] = \left(2y \partial_x y + D_{\theta c} y D_{\theta}^c y, \int dxd^4 \theta \ y \partial_x y \right), \]  

(3.10)
from (3.2). By the definition \( d\Omega_{(b,c)} = 0 \) so that it may be locally expressed such that

\[ \Omega_{(b,c)} = d\alpha. \]  

(3.11)
Integrating this 1-form on the orbit \( O_1 \) gives an \( N = (4,0) \) supersymmetric action

\[ I = \int_{O_1} \alpha. \]  

(3.12)
We propose that this is the \( (4,0) \) super-Liouville theory.

We put the Kirillov-Kostant 2-form (3.6) in an explicit form as

\[ 2\Omega_{(b,c)} = \int dxd^4 \theta \left[ \left(b(f, \varphi, t) + cS(f, \varphi; x, \theta, t)\right)(2y \partial_x y + D_{\theta c} y D_{\theta}^c y) + cy \partial_x y \right], \]  

by (3.1) with (3.4) and (3.10). Choose the initial point \((b, c)\) of the coadjoint orbit to be constant, say \((b_0, c_0)\). Then by integration by part it becomes

\[ 2\Omega_{(b_0, c_0)} = c_0 \int dxd^4 \theta \left[ S(f, \varphi; x, \theta, t)(2y \partial_x y + D_{\theta c} y D_{\theta}^c y) + y \partial_x y \right]. \]  

(3.13)
To proceed with our argument we need the following formulae

\[
dS(f, \varphi; x, \theta, t) = y\partial_x y + \frac{1}{2}D_\theta y D_\theta y\]

\[
dy = y\partial_x y + \frac{1}{2}D_\theta y D_\theta y,
\]

They are shown in Appendix B. By means of these formulae the Kirillov-Kostant 2-form \(3.13\) becomes

\[
2\Omega_{(b_0, c_0)} = c_0 \int dxd^4\theta \left[ - d\left(2yS(f, \varphi; x, \theta, t)\right) - y\partial_x y \right].
\]

\(\Omega_{(b_0, c_0)}\) is closed so that

\[
d \int dxd^4\theta \ y\partial_x y = 0.
\]

Then the anomaly term takes an exact form such that

\[
\int dxd^4\theta \ y\partial_x y = d\gamma,
\]

with a 1-form \(\gamma\). To study \(\gamma\) it is helpful to remember the arguments for the lower supersymmetric cases. The Kirillov-Kostant 2-form takes the form as \(3.16\), in which \(y\), \(S\), the anomaly term and the integration measure should be replaced by those for the lower supersymmetries, given in Appendix A. The anomaly terms

\[
- \int dx \ y\partial^2 y, \quad \text{for } N = (0, 0),
\]

\[
- \int dx d\theta \ y D_\theta \partial^2 y, \quad \text{for } N = (1, 0),
\]

\[
- \int dx d^2\theta \ y \partial_x [D_{\theta+} D_{\theta-}] y, \quad \text{for } N = (2, 0),
\]

are also closed. For these we can find local expressions for \(\gamma\) as

\[
\gamma = -d\int dx \left[ y \left( S + \frac{1}{2} (\partial^2 f)^2 \right) \right], \quad \text{for } N = (0, 0),
\]

\[
\gamma = -d\int dx d\theta \left[ y \left( S + \left( \frac{D^3 \varphi}{D_\theta \varphi} \frac{D^2 \varphi}{D_\theta \varphi} \right) \right) \right], \quad \text{for } N = (1, 0),
\]

\[
\gamma = -d\int dx d^2\theta \left[ y \left( S - \frac{\partial_x \varphi^+}{D_{\theta+} \varphi^+} \frac{\partial_x \varphi^-}{D_{\theta-} \varphi^-} \right) \right], \quad \text{for } N = (2, 0).
\]

\(3.17\) is obvious by the definition of \(\Omega_{(b_0, c_0)}\), but it may be directly checked by \(3.15\).
with $S$ the Schwarzian derivative for the respective supersymmetric case. Owing to these formulae the Kirillov-Kostant 2-form takes the exact form (3.11), in which the 1-form $\alpha$ is explicitly given. As the result the action (3.12) gives the known Liouville actions for the lower supersymmetric cases, summarized in Appendix A. However for the case of $N = (4,0)$ case we could not find a local expression for $\gamma$. Presumably it would not be possible at all in the supercovariant formulation with the superfields. Hopefully it would be possible if the anomaly term $y \partial y$ is expanded in components. To see this we have done rather massive calculations, using the expansion formulae of $f, \varphi$, $\psi$ discussed in Appendix A in [3]. We have indeed found it in an exact form as (3.18). The details of the calculations were exposed in [11]. We quote here only the result:

$$\gamma = \int dx \left\{ dh \left[ \partial_2^2(-\frac{1}{\rho \xi}) + \partial_x \log(\rho/\xi)\partial_x(-\frac{1}{\rho \xi}) \right] + \frac{4}{(\rho \xi)^2} \left[ \left( \partial_2^2(\rho \eta) \cdot \partial_x(\xi \eta) \right) - \left( \partial_x(\rho \eta) \cdot \partial_2^2(\xi \eta) \right) \right] - \frac{2}{\rho \xi} \left[ \left( \partial_2(\rho \eta) \cdot \partial_x(\xi \eta) \right) - \left( \partial_x(\rho \eta) \cdot d\partial_2(\xi \eta) \right) \right] \right.$$

$$\left. - \frac{1}{\rho \xi} \partial_x \left[ \left( \partial_2(\rho \eta) \cdot d(\xi \eta) \right) - \left( d(\rho \eta) \cdot \partial_x(\xi \eta) \right) \right] \right. + \frac{1}{2} \left[ \left( \rho \eta \cdot d(\xi \eta) \right) - \left( d(\rho \eta) \cdot \xi \eta \right) \right] \left[ \rho \xi \left( \partial_x \left( \frac{1}{\rho \xi} \right) \right)^2 + \partial_2^2(-\frac{1}{\rho \xi}) \right]$$

$$\left. \left. - \frac{2}{(\rho \xi)^2} \left[ \left( \rho \eta \cdot d(\xi \eta) \right) - \left( d(\rho \eta) \cdot \xi \eta \right) \right] \left[ \left( \partial_2(\rho \eta) \cdot \partial^2(\xi \eta) \right) - \left( \partial_2^2(\rho \eta) \cdot \partial_2(\xi \eta) \right) \right] \right. \right. + \frac{8}{(\rho \xi)^3} \left[ \left( \rho \eta \cdot d(\xi \eta) \right) - \left( d(\rho \eta) \cdot \xi \eta \right) \right] \left( \partial_x(\rho \eta) \cdot \partial_x(\xi \eta) \right)^2 \right\},$$

(3.19)

with $(\mu \cdot \nu) \equiv \mu_a \nu^a$ the inner product of SU(2) doublets. Here $\rho \xi$ is constrained by

$$\rho \xi = \partial_x h + \left( \rho \eta \cdot \partial_x \xi \eta \right) - \left( \partial_x(\rho \eta) \cdot \xi \eta \right).$$

(3.20)

Consequently $\gamma$ is a function of $h, \rho \eta_a, \xi \eta^a$, which are the lowest componnet of the superfields $f, \varphi_a, \varphi^a$ respectively. With this $\gamma$ put in (3.16) the $N = (4,0)$ super-Liouville action (3.12) gets a local expression in 1+1 dimensions as

$$I = -c_0 \int_{\Omega_1} \int dx \, d^4 \theta \, y S(f, \varphi; x, \theta, t) - \frac{c_0}{2} \int_{\Omega_1} \gamma.$$

(3.21)

It is important to discuss non-supersymmetric approximation of the theory. The purely bosonic parts of $y$ and $S$ are given by

$$S(f, \varphi; x, \theta, t) = \log \rho \xi + \frac{1}{2}(\theta \cdot \theta)^2 \left[ - \frac{\partial_2^2 \rho \xi}{\xi} - \frac{\partial_2^2 \rho}{\rho} + \frac{1}{2} \frac{\partial_2 \xi \partial_x \rho \eta}{\rho} \right] + O(\eta),$$

$$y = \frac{dh}{\rho \xi} + (\theta \cdot \theta) \left( \frac{d \xi}{\xi} - \frac{d \rho}{\rho} \right) + \frac{1}{2} (\theta \cdot \theta)^2 \left( d \left[ \frac{h \partial_2^2(-\frac{1}{\rho \xi})}{\rho \xi} \right] - \partial_x \left[ h d \partial_2 \left(-\frac{1}{\rho \xi} \right) \right] \right) + O(\eta).$$

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3Our convention is that $\int d^4 \theta (\theta \cdot \theta)^2 = 1$
Put them into (3.21) as well as \( \gamma \) given in (3.19). Calculate the \( yS \) as \([y]_{\theta'}[S]_{\theta'} + [y]_{\theta'}[S]_{\theta'} \). In the action (3.21) the purely bosonic part of the second term is cancelled by the one of \( \frac{1}{2}\gamma \) owing to \( \rho \xi = \partial_y h \) which is (3.20) in the non-supersymmetric approximation. Then we get

\[
I = -\frac{c_0}{2} \int_{\mathcal{O}_1} \int dx \, dh \left[ -\frac{\partial^3_y h}{\partial_x h} + 2(\frac{\partial^2_y h}{\partial_x h})^2 + O(\eta) \right],
\]

by using again (3.20) in the non-supersymmetric approximation. This is the ordinary non-supersymmetric Liouville theory.

Thus we have assured ourselves that our arguments are going on a right track. However to study symmetries of the theory furthermore it is not convenient to go with the local form of the action (3.21). Instead we prefer the non-local form without using (3.18)

\[
I = -\frac{c_0}{2} \int_{\mathcal{O}_1} \int dx d^4\theta \, yS(f, \varphi; x, \theta) - \frac{c_0}{2} \int_{\mathcal{O}_2} \int dx d^4\theta \, y \partial_y x.
\]

With this we check the energy-momentum tensor to be given by the \( N = (4, 0) \) Schwarzian derivative. To this end we need the formula (2.8) for \( \delta_v \mathcal{S} \) and also the following ones

\[
\delta_v \int dx d^4\theta \, y \partial_y x = d \int dx d^4\theta \, 2v \partial_x y,
\]

which are shown in Appendix B. By using these formulae we find that

\[
\delta_v I = -c_0 \int_{\mathcal{O}_1} \int dx d^4\theta \, dv S(f, \varphi; x, \theta).
\]

It may be written in the form

\[
\delta_v I = c_0 \int_{\mathcal{O}_1} dt \int dx d^4\theta \, v \frac{d}{dt} S(f, \varphi; x, \theta).
\]

Thus the Schwarzian derivative is the energy-momentum tensor of the theory in the left-moving sector. When \( dv/dt = 0 \) it is conserved. This was the hallmark of the Liouville theory for the lower supersymmetric cases.

4 Hidden PSU(1,1|2) symmetry

We show that the action (3.22) has a hidden symmetry under PSU(1,1|2). Following [3] such a symmetry is non-linearly realized on a supercoset space \( \text{PSU}(1,1|2)/\{\text{SU}(2) \otimes \text{U}(1)\} \) for which the generators of PSU(1,1|2) are decomposed as

\[
\{T^A\} = \left\{ L, F_a, \overline{L}, \overline{F}_a, \overline{F}^a, L^0, R^a \right\}. \]

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8
The local coordinates are the so far discussed superdiffeomorphism $f, \varphi, \varphi^a$ and their complex conjugate $\overline{f}, \overline{\varphi}, \overline{\varphi}^a$. They correspond to the coset generators $L, F_a, F^a$ and $\mathcal{L}, \mathcal{F}_a, \mathcal{F}^a$. The fermionic coordinates $\varphi_a$ and $\varphi^a$ are doublets of the subgroup SU(2).

We can calculate the Killing vectors on this coset space following the general method developed in [13]. They were worked out in [14] and given by

$$\delta_\epsilon f \equiv -i \epsilon A R^A$$
$$= \epsilon L + f \epsilon L^0 + (\varphi_c \epsilon F_c + \varphi^c \epsilon F^c) + \left( f^2 \epsilon \mathcal{T} + f (\varphi_c \epsilon \mathcal{F}_c + \varphi^c \epsilon \mathcal{F}^c) \right)$$
$$+ (\varphi_b \epsilon b)(\varphi_c \epsilon F^c - \varphi^c \epsilon \mathcal{F}_c) + (\varphi_c \varphi^c)^2 \epsilon \mathcal{T}, \tag{4.1}$$

$$\delta_\epsilon \varphi_a \equiv -i \epsilon A R^A_a$$
$$= \epsilon F_a + f \epsilon F^a + \frac{1}{2} \varphi_a \epsilon L^0 - \varphi^c \epsilon R^c_a$$
$$+ (f \varphi_a \epsilon \mathcal{T} + (\varphi_c \varphi^c \epsilon \mathcal{F}_a + 2 \varphi_c \epsilon \mathcal{F}^c_\varphi_a) + \varphi_c \varphi^c \varphi_a \epsilon \mathcal{T}, \tag{4.2}$$

$$\delta_\epsilon \varphi^a \equiv -i \epsilon A R^{Aa}$$
$$= \epsilon F^a + f \epsilon F_a + \frac{1}{2} \varphi^a \epsilon L^0 - \varphi^c \epsilon R^c a$$
$$+ (f \varphi^a \epsilon \mathcal{T} - (\varphi_c \varphi^c \epsilon \mathcal{F}_a - 2 \varphi_c \epsilon \mathcal{F}^c \varphi^0) - \varphi_c \varphi^c \varphi^0 \epsilon \mathcal{T}. \tag{4.3}$$

Here $\epsilon A$ are infinitesimal transformation parameters of PSU(1,1|2). In [3] they have been shown to satisfy the chirality transformation conditions as well as the superconformal conditions. We then found a remarkable transformation law as

$$\delta_\epsilon \Delta = \left( \epsilon L^0 + 2 f \epsilon \mathcal{T} + 2 \varphi^c \epsilon \mathcal{F}_c + 2 \varphi^c \epsilon \mathcal{F}^c \right) \Delta,$$

for $\Delta$ given by [3,9]. As $d\epsilon A = 0$ and $\partial_x \epsilon A = 0$, the quantity $\delta_\epsilon (df + \varphi_a d\varphi^a + \varphi^a d\varphi_a)$ obeys the same transformation law as $\Delta$. Consequently the 1-form $y$ is invariant under the transformations (4.1)~(4.3). Using these transformation properties we find the action transform as

$$\delta_\epsilon I = -c \int_\mathcal{O}_1 \int dxd^4 \theta \ y \left( \epsilon L^0 + 2 f \epsilon \mathcal{T} + 2 \varphi^c \epsilon \mathcal{F}_c + 2 \varphi^c \epsilon \mathcal{F}^c \right). \tag{4.4}$$

But here we remember that the purely bosonic part of the action is identical with that of the non-supersymmetric Liouville theory. The latter is invariant under SU(1,1)(\cong SU(2)), which is a subgroup of PSU(1,1|2). It does not admit such a transformation as [4,4] at all. This observation suggests us that the integration of (4.4) is vanishing. It can be hardly seen in the supercovariant form as it is. So we have expanded the integrand in components to examine this. Here also we were involved in massive calculations. Finally we have found that the quantities $y, yf, y\varphi, y\varphi^c$ are all of the form $\partial_x (\cdots)$. The details are reported in [11]. Thus the $N = (4,0)$ theory has turned out to be invariant under the hidden symmetry group PSU(1,1|2).

\footnote{Precisely speaking the coset space PSU(2|2)/[SU(2)\otimes U(1)] was studied in ([14]). The result was adapted for PSU(1,1|2)/[SU(2)\otimes U(1)] in [3].}

\footnote{This is the same as (5.8) in [3].}
5 Conclusions

Thus in this letter we have formulated the $N = (4,0)$ super-Liouville theory by the coadjoint orbit method and have shown that it has all the properties which are characteristic in the lower symmetric Liouville theory, except for one. Namely the hidden symmetry under PSU(1,1|2) has been shown only as a global symmetry above, while for the lower supersymmetric cases it was also a local one in the right-moving sector. But when promoted the constant parameters $\epsilon^A$ as $d \epsilon \neq 0, \partial_x \epsilon = 0$ it is extremely hard to check the symmetry with either the non-local form of the action (3.22) or the local one (3.21). The author hopes to be able to answer the question in a future work.

To conclude this letter we would like to comment on a prospect for a farther development of the work. PSU(1,1|2), being a subgroup of PSU(2,2|4), is the key symmetry group for the string/QCD duality which was intensively studied in the last decade[15]. On the string side it is a symmetry for spin-chains, while on the QCD side it is a symmetry for the $D = 4, N = 4$ YM supermultiplets[16]. In [12] it was proposed to interpret the spin-chain system by a non-linear $\sigma$-model on a coset space realizing PSU(1,1|2) as a subgroup. But it is worth revisiting the issue of the string/QCD duality by means of the $N = (4,0)$ super-Liouville theory, since it has the root of the string theory more than the non-linear $\sigma$-model. The study is in progress.

A The lower supersymmetric Liouville theories

$N = (0,0)$: [6]

- Action:

$$I = \int dx \; y \left[ S - \frac{1}{2} \left( \frac{\partial_x^2 f}{\partial_x f} \right)^2 \right],$$

with

$$y = \frac{df}{\partial_x f} = dt \frac{\partial t f}{\partial_x f}, \quad S = \frac{\partial_x^3 f}{\partial_x f} - \frac{3}{2} \left( \frac{\partial_x^2 f}{\partial_x f} \right)^2.$$

- Conformal transformation:

$$\delta_v I = -2 \int dx dt \; v \partial_t S,$$

$$\delta_v y = dv + [v \partial_x - \partial_x v] y, \quad \delta_v S = [v \partial_x + 2 \partial_x v] S + \partial_x^3 v,$$

with $v = \delta_v x$.

$N = (1,0)$: [4]

- $\{D_\theta, D_\theta\} = 2 \partial_x$.

- Constraint: $D_\theta f = \varphi D_\theta \varphi$. 

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\begin{itemize}
  \item Action:
  \[ I = \int dx d\theta \ y \left[ S - \frac{D^3_\theta \varphi \ D^3_\theta \varphi}{D_\theta \varphi \ D_\varphi \varphi} \right] = -2 \int dx d\theta \ d\varphi \ \frac{D^3_\theta \varphi}{(D_\theta \varphi \ D_\varphi \varphi)^2}, \]
  with
  \[ y = \frac{df + \varphi d\varphi}{(D_\theta \varphi)^2} = dt \frac{\partial_t f + \varphi \partial_t \varphi}{(D_\theta \varphi)^2}, \quad S = \frac{D^4_\theta \varphi}{D_\theta \varphi} - 2 \frac{D^3_\theta \varphi \ D^2_\varphi \varphi}{D_\theta \varphi \ D_\varphi \varphi}. \]

  \item Superconformal transformation:
  \[ \delta_v I = -2 \int dx d\theta dt \ v \partial_t S, \]
  \[ \delta_v y = dv + [v \partial_x + \frac{1}{2} D_\theta v D_\theta - \partial_x v] y, \]
  \[ \delta_v S = [v \partial_x + \frac{1}{2} D_\theta v D_\theta + \frac{3}{2} \partial_x v] S + \frac{1}{2} D_\theta \partial^2_x v, \]
  with \( v = \delta_v x + \theta \delta_v \theta. \)

  \item \( N = (2, 0): \]

  \begin{itemize}
    \item \( \{ D_{\theta^+}, D_{\theta^-} \} = 2 \partial_x, \quad \{ D_{\theta^\pm}, D_{\theta^\mp} \} = 0. \)
    \item Constraints: \( D_{\theta^+} \varphi^+ = 0, \quad D_{\theta^+} f = \varphi^- D_{\theta^+} \varphi^+, \quad D_{\theta^-} f = \varphi^+ D_{\theta^-} \varphi^- \)
    \item Action:
      \[ I = \int dx d^2 \theta \ y \left[ S + 2 \frac{\partial_x \varphi^+}{D_{\theta^+} \varphi^+} \frac{\partial_x \varphi^-}{D_{\theta^-} \varphi^-} \right] = 2 \int dx d^2 \theta \ d(\log D_{\theta^+} \varphi^+) \log D_{\theta^-} \varphi^-, \]
      with
      \[ y = \frac{df + \varphi^+ d\varphi^- + \varphi^- d\varphi^+}{D_{\theta^+} \varphi^+ D_{\theta^-} \varphi^-} = dt \frac{\partial_t f + \varphi^+ \partial_t \varphi^- + \varphi^- \partial_t \varphi^+}{D_{\theta^+} \varphi^+ D_{\theta^-} \varphi^-}, \]
      \[ S = \partial_x (\log D_{\theta^+} \varphi^+ - \log D_{\theta^-} \varphi^-) + 2 \frac{\partial_x \varphi^+}{D_{\theta^+} \varphi^+} \frac{\partial_x \varphi^-}{D_{\theta^-} \varphi^-}. \]
  \end{itemize}

  \item Superconformal transformation:
    \[ \delta_v I = -2 \int dx d^2 \theta dt \ v \partial_t S, \]
    \[ \delta_v y = dv + [v \partial_x + \frac{1}{2} D_{\theta^+} v D_{\theta^-} + \frac{1}{2} D_{\theta^-} v D_{\theta^+} - \partial_x v] y, \]
    \[ \delta_v S = [v \partial_x + \frac{1}{2} D_{\theta^+} v D_{\theta^-} + \frac{1}{2} D_{\theta^-} v D_{\theta^+} + \partial_x v] S + \frac{1}{2} \partial_x [D_{\theta^+}, D_{\theta^-}] v, \]
    with \( v = \delta_v x + \theta^+ \delta_v \theta^- + \theta^- \delta_v \theta^+. \)

  Here \( S \) is the Schwarzian derivative for the relevant type of the supersymmetry.
\end{itemize}
B Proofs of some formulae

We shall prove (3.14), (3.15), (3.23) and (3.24). The first two formulae may be shown by taking the exterior derivative directly as has been done in Appendix 3, while the last two by the superconformal transformation (2.5) and (2.6). Here we prefer to do it by using the language of the differential geometry. Namely the requirement (3.7) is equivalent to saying that

\[ i_v df = \delta_v f, \quad i_v d\varphi_a = \delta_v \varphi_a, \quad i_v d\varphi^a = \delta_v \varphi^a. \]  

(B.1)

Here \( i_v \) is the known operation of the differential geometry, called anti-derivative, while the superconformal transformation \( \delta_v \) is known as the Lie-derivative. Applying (B.1) for \( y \) gives

\[ i_v y = v. \]  

(B.2)

To find the energy-momentum tensor we have intentionally assumed \( dv \neq 0 \). Otherwise we may set \( dv = 0 \) as usual. \((y, 0)\) in (3.10) is an element of the superconformal algebra \( g \) such as \((v, l)\) appearing in (3.2). \( \delta_v y \) should be the superconformal transformation given by (2.3) with weight \((-1, 0)\). Thus we obtain (3.23). \( \delta_v y \) may be alternatively obtained

\[ \delta_v y = dv + i_v dy, \]  

(B.3)

by the identity of the differential geometry

\[ \delta_v = di_v + i_v d. \]  

(B.4)

Comparing (B.3) with (3.23) we know that \( dy \) in (B.3) takes the form (3.15) owing to (B.2). Finally (3.14) and (3.24) can be also shown by means of the identity (B.4). Namely we have

\[ i_v dS = \delta_v S, \quad di_v \int dxd^4\theta \ y\partial xy = \delta_v \int dxd^4\theta \ y\partial xy, \]  

(B.5)

with

\[ i_v S = 0, \quad d \int dxd^4\theta \ y\partial xy = 0. \]

By applying (B.2) in the l.h.s.s of the equations in (B.5) it is confirmed that (3.14) and (3.24) are right.

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