Some New Applications of Weakly $\mathcal{H}$-Embedded Subgroups of Finite Groups

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Abstract: A subgroup $H$ of a finite group $G$ is said to be weakly $\mathcal{H}$-embedded in $G$ if there exists a normal subgroup $T$ of $G$ such that $H^G = HT$ and $H \cap T \in \mathcal{H}(G)$, where $H^G$ is the normal closure of $H$ in $G$, and $\mathcal{H}(G)$ is the set of all $\mathcal{H}$-subgroups of $G$. In the recent research, Asaad, Ramadan and Heliel gave new characterization of $p$-nilpotence: Let $p$ be the smallest prime dividing $|G|$, and $P$ a non-cyclic Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if and only if there exists a $p$-power $d$ with $1 < d < |P|$ such that all subgroups of $P$ of order $d$ and $pd$ are weakly $\mathcal{H}$-embedded in $G$. As new applications of weakly $\mathcal{H}$-embedded subgroups, in this paper, (1) we generalize this result for general prime $p$ and get a new criterion for $p$-supersolubility; (2) adding the condition “$N_G(P)$ is $p$-nilpotent”, here $N_G(P) = \{g \in G|p^0 = P\}$ is the normalizer of $P$ in $G$, we obtain $p$-nilpotence for general prime $p$. Moreover, our tool is the weakly $\mathcal{H}$-embedded subgroup. However, instead of the normality of $H^G = HT$, we just need $HT$ is $S$-quasinormal in $G$, which means that $HT$ permutes with every Sylow subgroup of $G$.

Keywords: finite groups; weakly $\mathcal{H}$-embedded subgroups; $p$-supersolubility; $p$-nilpotence

1. Introduction

Throughout this paper, “$G$ is a group” always means that “$G$ is a finite group”. For convenience, one can refer to [1–4] for the definitions and notions in the paper.

The $T$-groups are defined as the groups $G$ in which normality is a transitive relation, that is, if $H \leq K \leq G$, then $H \leq G$. In 2000, Bianchi Gillio Berta Mauri, Herzog and Verardi [5] proved a characterization of soluble $T$-groups by means of $\mathcal{H}$-subgroup: a subgroup $H$ of a group $G$ is called an $\mathcal{H}$-subgroup in $G$ if $N_G(H) \cap H^G \leq H$, for every element $g \in G$, where $N_G(H) = \{x \in G|H^x = H\}$ is the normalizer of $H$ in $G$. They proved that a group $G$ is a supersolvable $T$-group if and only if every subgroup of $G$ is an $\mathcal{H}$-subgroup of $G$. Later, except for the exploration of $T$-groups, $\mathcal{H}$-subgroups were widely used to character finite groups. Csörgő and Herzog [6] obtained that a group $G$ is supersolvable if every cyclic subgroup of $G$ of prime order or order 4 is an $\mathcal{H}$-subgroup. Asaad [7] proved that a group $G$ is supersolvable if every maximal subgroup of every Sylow subgroup of $G$ is an $\mathcal{H}$-subgroup. The set of all $\mathcal{H}$-subgroups of a group $G$ is denoted by $\mathcal{H}(G)$. Moreover, Guo and Wei [8] gave new characterization of $p$-nilpotent or supersolvable by assuming some subgroups of $G$ of the same order all belong to $\mathcal{H}(G)$, which provide a unified version of the results mentioned above if the order of $G$ is odd. Moreover, Li, Zhao and Xu [9] considered the case when $G$ is of even order.

Recently, Asaad et al. [10] introduced a new subgroup embedding property called weakly $\mathcal{H}$-subgroup, which generalizes both $c$-normality and $\mathcal{H}$-subgroup, called weakly $\mathcal{H}$-subgroup. Soon after, Asaad and Ramadan [11] gave the definition of weakly $\mathcal{H}$-embedded subgroup. Please note
that a subgroup $H$ of $G$ is said to be a weakly $H$-embedded subgroup (weakly $H$-subgroup) of $G$ if there exists a normal subgroup $T$ of $G$ such that $H^G = HT$ ($G = HT$) and $H \cap T \in \mathcal{H}(G)$, where $H^G$ is the normal closure of $H$ in $G$. Clearly, $c$-normal subgroups, $H$-subgroups and weakly $H$-subgroups imply weakly $H$-embedded subgroups. However, the converse does not hold in general, see [11] (Examples 1.3, 1.4 and 1.5).

In fact, these subgroups were widely used to investigate the structure of finite groups. As a result, many interesting results have been subsequently obtained, such as [7,10–13].

In the recent research about $H$-subgroups, Asaad, Ramadan, and Heliel gave a new characterization of $p$-nilpotency.

**Theorem 1.** ([12] Theorem A) Let $p$ be the smallest prime dividing $|G|$, and $P$ a non-cyclic Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if and only if there exists a $p$-power $d$ with $1 < d < |P|$ such that all subgroups of $P$ of order $d$ and $pd$ are weakly $H$-embedded in $G$.

However, according to this result, some natural questions arise:

**Problem 1.** (1) If delete the condition “$p$ is the smallest prime dividing $|G|$”, can we claim that $G$ is $p$-supersoluble?
(2) Does there exist another condition to obtain $p$-nilpotence rather than “$p$ is the smallest prime dividing $|G|$”?
(3) As we know, the condition that $HT$ is the smallest normal subgroup of $G$ containing $H$, is too strict. Can we replace it by a weaker embedding subgroup property?

In this paper, we further explore weakly $H$-embedded subgroups and pay attention to Problem 1. However, instead of the normality of $HT$, we just consider $HT$ is $S$-quasinormal in $G$. As we know, a subgroup $K$ is $S$-quasinormal in $G$, means that $K$ permutes with every Sylow subgroup $P$ of $G$, that is $KP = PK$. However, for convenience, we also called it a weakly $H$-embedded subgroup, that is:

**Definition 1.** A subgroup $H$ of a group $G$ is said to be weakly $H$-embedded in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \in H(G)$.

As an application of these subgroups, we give a positive answer to Problem 1 in the class of $p$-soluble groups, for detail:

**Theorem 2.** Let $E$ be a $p$-soluble normal subgroup of a group $G$ such that $G/E$ is $p$-supersoluble, where $p$ is a prime divisor of $|E|$. Let $P$ be a Sylow $p$-subgroup of $E$. Suppose that $P$ has a subgroup $D$ with $1 \leq |D| < |P|$ such that all subgroups of $P$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $G$. When $|D| = 1$ and $P$ is a non-abelian 2-group, we further assume that all cyclic subgroups of $P$ of order 4 are weakly $H$-embedded in $G$. Then $G$ is $p$-supersoluble.

Moreover, to avoid the condition “$p$ is the smallest prime dividing $|G|$” of Theorem 1, we further prove that the conclusion holds if this condition is replaced by “$N_G(P)$ is $p$-nilpotent”. Consequently, we give an answer to Problem 1.

**Theorem 3.** Let $E$ be a normal subgroup of $G$ such that $G/E$ is $p$-nilpotent, and $P$ be a non-cyclic Sylow $p$-subgroup of $E$, where $p$ is a prime dividing $|E|$. Assume that $N_G(P)$ is $p$-nilpotent and $P$ has a subgroup $D$ with order $1 < |D| < |P|$ such that all subgroups of $P$ of order $|D|$ and order $p|D|$ are weakly $H$-embedded in $G$. Then $G$ is $p$-nilpotent.

In the second section, we list some lemmas which will be useful for the proofs of the above results. The proofs of Theorems 2 and 3 are put in the third section. Some previously known results are generalized by our theorems, and we list some in the fourth section.
2. Preliminaries

Lemma 1. (see ([1], Chapter 1) or ([3], Chapter 1, Lemmas 5.34 and 5.35)) Assume that $H, E$ are subgroups of $G$ and $N 	rianglelefteq G$.

1. If $H$ is $S$-quasinormal in $G$, then $H \cap E$ is $S$-quasinormal in $E$, and $HN/N$ is $S$-quasinormal in $G/N$.
2. Assume that $H$ is a $p$-group. Then $H$ is $S$-quasinormal in $G$ if and only if $O_p^r(G) \leq N_G(H)$.
3. The set of $S$-quasinormal subgroups of $G$ is a sublattice of the subnormal subgroup lattice of $G$.
4. If $H$ is a $p$-group and $H$ is subnormal in $G$, then $H \leq O_p(G)$.

Lemma 2. ([11] Lemma 2.1) Let $H, N$ be subgroups of $G$ satisfying $H \in \mathcal{H}(G)$ and $N \trianglelefteq G$. Then:

1. If $E$ is a subgroup of $G$ containing $H$, then $H \in \mathcal{H}(E)$;
2. If $H$ is subnormal in $G$, then $H$ is normal in $G$;
3. Assume that $N \leq N_G(H)$. Then $NH \in \mathcal{H}(G)$;
4. If $E$ is a subgroup of $G$ satisfying $N \leq E$, then $E \in \mathcal{H}(G)$ if and only if $E/N \in \mathcal{H}(G/N)$;
5. If $H$ is a $p$-group and $p \nmid |N|$, then $NH \in \mathcal{H}(G)$ and $HN/N \in \mathcal{H}(G/N)$.

Lemma 3. Let $H$ be a weakly $\mathcal{H}$-embedded subgroup of a group $G$.

1. Assume that $E$ is a subgroup of $G$ containing $H$, then $H$ is weakly $\mathcal{H}$-embedded in $E$.
2. Assume that $N$ is a normal subgroup of $G$ satisfying $N \leq H$, then $H/N$ is weakly $\mathcal{H}$-embedded in $G/N$.
3. Assume that $H$ is a $p$-group and $N$ a normal $p'$-subgroup of $G$. Then $HN/N$ is weakly $\mathcal{H}$-embedded in $G/N$.

Proof. By the hypothesis, $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \in \mathcal{H}(G)$.

1. Clearly, $T \cap E$ is a normal subgroup of $E$ such that $H(T \cap E) = HT \cap E$ is $S$-quasinormal in $E$ and $H \cap (T \cap E) = H \cap T \in \mathcal{H}(E)$ (see Lemmas 1(1) and 2(1)). This shows that $H$ is weakly $\mathcal{H}$-embedded in $E$.

2. Consider the normal subgroup $TN/N$ of $G/N$. Please note that $N \leq N_G(H \cap T)$, so $(H \cap T)N \in \mathcal{H}(G)$ by Lemma 2(3). Furthermore, we have that $(H/N)(TN/N) = HT/N$ is $S$-quasinormal in $G/N$ and

$$(H/N) \cap (TN/N) = (H \cap T)N/N \in \mathcal{H}(G/N)$$

(see Lemmas 1(1) and 2(4)). By the definition, $H/N$ is weakly $\mathcal{H}$-embedded in $G/N$.

3. By Lemma 1(1), the normal subgroup $TN/N$ of $G/N$ such that $(HN/N)(TN/N) = HTN/N$ is $S$-quasinormal in $G/N$. Please note that

$$(|HN \cap T : H \cap T|,|HN \cap T : N \cap T|) = (|N \cap HT|,|H \cap NT|) = 1,$$

so $HN \cap T = (H \cap T)(N \cap T)$. Combining with Lemma 2(5),

$$(HN/N) \cap (TN/N) = (HN \cap T)N/N = (H \cap T)N/N \in \mathcal{H}(G/N).$$

Hence $HN/N$ is weakly $\mathcal{H}$-embedded in $G/N$. □

Recall that a class of groups $\mathcal{F}$ is called a formation if for every group $G$, every homomorphic image of $G/G^\mathcal{F}$ belongs to $\mathcal{F}$, where $G^\mathcal{F} = \bigcap\{N \trianglelefteq G|G/N \in \mathcal{F}\}$. Furthermore, a formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. The intersection of all formations containing the set $\{G/O^r_{p',p}(G)|G \in \mathcal{F}\}$ is denoted by $\mathcal{F}(p)$, and $F(p)$ denotes the class of all groups $G$ such that $G^\mathcal{F}(p)$ is a $p$-group. Associated with a saturated formation $\mathcal{F}$, there is a function $f$ of the form $f : P \to \{\text{group formations}\}$, where $f(p) = F(p)$ for any prime $p$, which divides $|G|$ for some $G \in \mathcal{F}$, and $f(p) = \emptyset$ otherwise. The function $f$ is called the canonical local satellite of $\mathcal{F}$. For more detail, please turn to ([3] P. 3) or ([2] Chap. IV, Theorem 3.7 and Definitions 3.9). Now we recall the subgroup $Z_\mathcal{F}(G)$
of $G$, which is called the $\mathfrak{N}$-hypercenter of $G$. In fact, $\mathcal{Z}_\mathfrak{N}(G)$ the product of all such normal subgroups $N$ of $G$ whose $G$-chief factors $H/K$ satisfying $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{N}$.

**Lemma 4.** Let $\mathfrak{N}$ be a saturated formation and $f$ the canonical local satellite of $\mathfrak{N}$. Let $P$ be a normal $p$-subgroup of $G$. Then $P \leq \mathcal{Z}_{\mathfrak{N}}(G)$ if and only if one of the following holds:

1. $G/C_G(P) \in f(p)$ ([3] Chap. 1, Lemma 2.26) or ([14] Lemma 2.14);
2. $P/\Phi(P) \leq \mathcal{Z}_{\mathfrak{N}}(G/\Phi(P))$ ([15] Lemma 2.8).

**Lemma 5.** ([1] Lemma 2.1.6) If $G$ is $p$-supersoluble and $O_p'(G) = 1$, then $G$ has the unique Sylow $p$-subgroup.

**Lemma 6.** ([2] Chap. A, Lemma 8.4) Let $N$ be a nilpotent normal subgroup of $G$ and $M$ a maximal subgroup of $G$ such that $N \nsubseteq M$. Then $N \cap M$ is a normal subgroup of $G$.

### 3. Proofs of Main Results

The following proposition plays an important role in the proof of Theorem 2.

**Proposition 1.** Let $P$ be a normal $p$-subgroup of a group $G$. Assume that $P$ has a subgroup $D$ satisfying $1 \leq |D| < |P|$, such that all subgroups of $P$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $G$. When $|D| = 1$ and $P$ is a non-abelian 2-group, we further assume that all cyclic subgroups of $P$ of order 4 are weakly $H$-embedded in $G$. Then $P \leq \mathcal{Z}_{\mathfrak{M}}(G)$.

**Proof.** Assume by contradiction that $(G, P)$ is a counterexample of minimal order $|G| + |P|$. We proceed via the following steps.

1. $P$ is not a minimal normal subgroup of $G$.

Assume that $P$ is minimal normal in $G$. Let $H$ be a subgroup of $P$ of order $|D|$ or $p|D|$, which is normal in some Sylow subgroup of $G$. By the hypothesis, $H$ is weakly $H$-embedded in $G$. So $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \in \mathcal{H}(G)$. Please note that $P \cap T$ is normal in $G$, so $P \cap T = 1$ or $P \cap T = P$ by the minimality of $P$. If $P \cap T = 1$, then $H = H(P \cap T) = P \cap HT$ is $S$-quasinormal in $G$. However, by the choice of $H$ and $G$, $H \leq G$, a contradiction. So $P \leq T$. In this case, $H = H \cap T \in \mathcal{H}(G)$ and then $H \leq G$ by the relationship $H \leq P \leq G$ and Lemma 2(2), which is impossible. Thus, $P$ is not a minimal normal of $G$.

2. If every maximal subgroup of $P$ is weakly $H$-embedded in $G$, then $P \leq \mathcal{Z}_{\mathfrak{M}}(G)$.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$. By Lemma 3(2), $(G/N, P/N)$ satisfies the hypothesis. So, the choice of $(G, P)$ implies that: (i) $P/N \leq \mathcal{Z}_{\mathfrak{M}}(G/N)$; (ii) $N$ is non-cyclic; (iii) $N$ is the unique minimal normal subgroup of $G$ contained in $P$. Now assume that $\Phi(P) = 1$. In this case, $P$ is elementary abelian and $P = N \times B$, where $B$ is a complement of $N$. Let $N_1$ be a maximal subgroup of $N$ such that $N_1$ is normal in some Sylow $p$-subgroup $G_p$ of $G$. Then $P_1 = N_1 B$ is a maximal subgroup of $P$. By the hypothesis, $G$ has a normal subgroup $T$ such that $P_1 T$ is $S$-quasinormal in $G$ and $P_1 \cap T \in \mathcal{H}(G)$. Please note that $P \cap T$ is a normal subgroup of $G$ contained in $P$, so $N \leq P \cap T$ or $P \cap T = 1$ by (iii). First, assume that $N \leq T$. Then $1 < N_1 \leq P_1 \cap T$. However, $P_1 \cap T \leq G$ by the relationship $P_1 \cap T \leq P \leq G$ and Lemma 2(2). Thus, the uniqueness of $N$ deduces that $N \leq P_1 \cap T \leq P_1$, a contradiction. Secondly, if $P \cap T = 1$, then $P_1 = P_1 (P \cap T) = P \cap P_1 T$ is $S$-quasinormal in $G$, moreover $P_1 \cap N = N_1$ is $S$-quasinormal in $G$ by Lemma 1(3). Hence Lemma 1(2) and the choice of $N_1$ imply that $N_1 \leq G$, a contradiction. The above shows that $\Phi(P) = 1$ and consequently, $N \leq \Phi(P)$. Furthermore, $P/\Phi(P) \leq \mathcal{Z}_{\mathfrak{M}}(G/\Phi(P))$. However, we have $P \leq \mathcal{Z}_{\mathfrak{M}}(G)$ by Lemma 4. This contradiction shows that (2) holds.

3. If every cyclic subgroup of $P$ of order $p$ or 4 (when $P$ is a non-abelian 2-group) is weakly $H$-embedded in $G$, then $P \leq \mathcal{Z}_{\mathfrak{M}}(G)$.

If $P$ is not a non-abelian 2-group, then we use $\Omega$ to denote the subgroup $\Omega_1(P)$ of $P$. Otherwise, $\Omega = \Omega_2(P)$. 
Let $R$ be a normal subgroup of $G$ such that $P/R$ is a $G$-chief factor. Obviously, $R$ satisfies the hypothesis. So $R \leq Z_{\Omega}(G)$ and $P/R$ is non-cyclic by the choice of $(G,P)$. Moreover, for any normal subgroup $L$ of $G$ satisfying $L < P$, we have $L \leq R$. In fact, if $L \not\leq R$, then similarly $L \leq Z_{\Omega}(G)$, and $P = RL \leq Z_{\Omega}(G)$, a contradiction. Now, assume that $\Omega \leq R$. Then $\Omega \leq Z_{\Omega}(G)$. From Lemma 4 and ([16] Lemma 2.4), it follows that $G/C_{G}(\Omega) \in F(p)$ and $C_{G}(\Omega)/C_{G}(P) \in \mathcal{N}_p$, where $F$ is the canonical local satellite of $\Omega$ and $\mathcal{N}_p$ is the class of $p$-groups. Consequently, $G/C_{G}(P) \in \mathcal{N}_p F(p) = F(p)$, and thereby $P \leq Z_{\Omega}(G)$ by Lemma 4 again. This contradiction shows that $\Omega = P$.

Let $L/R$ be a minimal subgroup of $Z(G/R) \cap P/R$ and $x \in L \setminus R$, where $G_p$ is a Sylow $p$-subgroup of $G$. Then $H = \langle x \rangle$ has order $p$ or 4 and $L = HR$. By the hypothesis, $H$ is weakly $\mathcal{H}$-embedded in $G$, so $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \in \mathcal{H}(G)$. Please note that $P \cap T \leq G$. Combining with the above result, we have $P \cap T = P$ or $P \cap T \leq R$. If $P \cap T = P$, that is, $P \leq T$, then $H = H \cap T \in \mathcal{H}(G)$. Moreover, the relationship $H \leq \mathcal{F} \leq G$ and Lemma 2(2) deduce $H \leq G$. By the choice of $H$, we have $P/R = L/R$ is cyclic, which is a contradiction. Now assume that $P \cap T \leq R$. Then

$$L/R = HR/R = H(P \cap T)R/R = P/R \cap HTR/R$$

is $S$-quasinormal in $G/R$ by Lemma 1(3). From Lemma 1(2) and the choice of $L/R$, it follows that $L/R \leq G/R$, which also shows that $P/R = L/R$, a contradiction. This completes the proof of (3).

(4) $p < |D| < |P^F|$ (it follows directly from (2) and (3)).

(5) $\Phi(P) = 1$.

Suppose that $\Phi(P) > 1$. We compare the order of $\Phi(P)$ with $|D|$. First, assume that $|\Phi(P)| \geq |D|$. In this case, we have $\Phi(P) \leq Z_{\Omega}(G)$ by the hypothesis and the choice of $P$. Let $N$ be a minimal normal subgroup of $G$ contained in $\Phi(P)$. Clearly, $|N| = p$ and by (4), $P/N$ satisfies the hypothesis. Thus, $P/N \leq Z_{\Omega}(G/N)$ and consequently $P \leq Z_{\Omega}(G)$, a contradiction. So $|\Phi(P)| \leq |D|$. Please note that $P/\Phi(P)$ is elementary abelian, so we can easily prove that $P/\Phi(P)$ satisfies the hypothesis. Therefore, $P(\Phi(P)) \leq Z_{\Omega}(G/\Phi(P))$ and by Lemma 4, we further have $P \leq Z_{\Omega}(G)$. This contradiction shows that $\Phi(P) = 1$.

(6) Final contradiction.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$. Clearly, $N < P$. Compare the order of $N$ with $|D|$. If $|D| < |N|$, then $N$ satisfies the hypothesis and the choice of $P$ implies that $N \leq Z_{\Omega}(G)$. Consequently, $|N| = p$ and then $|D| = 1$, which contradicts (4). Thus, $|D| \geq |N|$. By (5), $P$ is elementary abelian, and all subgroups of $P/N$ of order $|D|/|N|$ and $|p|D|/|N|$ are weakly $\mathcal{H}$-embedded in $G$ (see Lemma 3(2)). Therefore $P/N \leq Z_{\Omega}(G/N)$ by the choice of $P$. Please note that $|P/N| \geq |P|/|D| > p^2$. So there exists a normal subgroup $E$ of $G$ contained in $P$ satisfying $N \leq E \leq P$ and $|P/E| = p$. Consider the subgroup $E$. Then $E \leq Z_{\Omega}(G)$ by the hypothesis and the choice of $P$, which implies $|N| = p$. Combining with $P/N \leq Z_{\Omega}(G/N)$, we finally obtain $P \leq Z_{\Omega}(G)$, which is a contradiction. The final contradiction completes the proof of the proposition. ∎

Now we give the proof of Theorem 2:

Proof. Suppose that the assertion is false and consider a counterexample $(G, E)$ with minimal $|G| + |E|$. We proceed via the following steps.

(1) $O_{p'}(E) = 1$.

Clearly, $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis by Lemma 3(3). If $O_{p'}(E) > 1$, then the choice of $G$ implies that $G/O_{p'}(E)$ is $p$-supersoluble. Furthermore, $G$ is $p$-supersoluble, which is a contradiction. Thus, $O_{p'}(E) = 1$.

(2) $E = G$.

Suppose that $E < G$. Please note that Lemma 3(1) shows that $(E, E)$ satisfies the hypothesis, so $E$ is $p$-supersoluble. Combining (1) with Lemma 5, we have $P \leq E$ and consequently, $P \leq G$. From the
hypothesis and Proposition 1, it follows that \( P \leq Z_\Delta(G) \). This result implies \( E \leq Z_\rho\Delta(G) \) and then \( G \) is \( p \)-supersoluble, which is a contradiction. Thus, \( E = G \).

(3) If every maximal subgroup of \( P \) is weakly \( \Delta \)-embedded in \( G \), then \( G \) is \( p \)-supersoluble.

Let \( N \) be a minimal normal subgroup of \( G \). Since \( G \) is \( p \)-soluble and \( O_{p'}(G) = 1, N \leq O_p(G) \). By Lemma 3(2), \( G/N \) satisfies the hypothesis, so: (i) \( G/N \) is \( p \)-supersoluble; (ii) \( |N| > p \); (iii) \( N \) is the unique minimal normal subgroup of \( G \). Obviously, \( N \leq Z(G) \), so there exists a maximal subgroup \( M \) of \( G \) such that \( G = N \times M \). By Lemma 6, \( O_p(G) \cap M \leq G \). So \( O_p(G) \cap M = 1 \) by the uniqueness of \( N \), and then
\[
O_p(G) = N(O_p(G) \cap M) = N.
\]

On one hand, \( O_p(G) \leq C_G(O_p(G)) \) by the minimality of \( O_p(G) \). On the other hand, since \( G \) is \( p \)-soluble and \( O_{p'}(G) = 1 \),
\[
C_G(O_p(G)) = C_G(F(G)) \leq F(G) = O_p(G).
\]

In general, \( C_G(O_p(G)) = O_p(G) \). Now we show that \( O_p(G) < P \). In fact, if \( P \leq G \), then \( P \leq Z_\Delta(G) \) by Proposition 1. Similar to step (2), it is impossible.

Using the above symbol, \( G = O_p(G) \times M \) and then \( P = O_p(G) \times (P \cap M) \). Let \( P_1 \) be a maximal subgroup of \( P \) containing \( P \cap M \). Then \( P_1 \cap O_p(G) > 1 \) and it is not normal in \( G \). In fact, if \( P_1 \cap O_p(G) \leq G \), then \( O_p(G) \leq P_1 \cap O_p(G) \leq P_1 \) by the minimality of \( O_p(G) \) and consequently, \( P = P_1 \), a contradiction. By the hypothesis, \( P_1 \) is weakly \( \Delta \)-embedded in \( G \). So \( G \) has a normal subgroup \( T \) such that \( P_1T \) is \( S \)-quasinormal in \( G \) and \( P_1 \cap T \in H(G) \). If \( T = 1 \), then \( P_1 \) is \( S \)-quasinormal in \( G \), which implies that \( P_1 \leq O_p(G) \) by Lemma 3(2)-4 and then \( O_p(G) = P \). However, it contradicts the above result. So, the uniqueness of \( O_p(G) \) implies that \( O_p(G) \leq T \). Next, we prove that
\[
P_1 \cap O_p(G) \in H(G).
\]

First, we show that \( N_G(P_1 \cap O_p(G)) = N_G(P_1 \cap T) \). On one hand, note that
\[
P_1 \cap O_p(G) = (P_1 \cap T) \cap O_p(G),
\]
so
\[
P \leq N_G(P_1 \cap T) \leq N_G(P_1 \cap O_p(G)) < G.
\]

On the other hand, \( N_G(P_1 \cap O_p(G)) \) is \( p \)-supersoluble by Lemma 3(1) and the relation
\[
N_G(P_1 \cap O_p(G)) \leq G.
\]

Please note that \( C_G(O_p(G)) = O_p(G) \), so it is rather clear that \( O_{p'}(N_G(P_1 \cap O_p(G))) = 1 \). Thus, \( P \) is normal in \( N_G(P_1 \cap O_p(G)) \) by Lemma 5. At this moment, we have
\[
P_1 \cap T \leq P \leq N_G(P_1 \cap O_p(G)),
\]
and by Lemma 2(1),
\[
P_1 \cap T \in H(N_G(P_1 \cap O_p(G))).
\]
Consequently, \( P_1 \cap T \leq N_G(P_1 \cap O_p(G)) \) by Lemma 2(2), that is, \( N_G(P_1 \cap O_p(G)) \leq N_G(P_1 \cap T) \). Together with the above proof, we finally obtain \( N_G(P_1 \cap O_p(G)) = N_G(P_1 \cap T) \). Please note that \( P_1 \cap T \in H(G) \). So, for any element \( g \in G \),
\[
(P_1 \cap O_p(G)) \leq N_G(P_1 \cap O_p(G)) = (P_1 \cap T)^g \cap O_p(G) \leq N_G(P_1 \cap T) \leq P_1 \cap T \cap O_p(G) = P_1 \cap O_p(G).
\]

This shows that \( P_1 \cap O_p(G) \in H(G) \). By Lemma 2(2), we further have \( P_1 \cap O_p(G) \leq G \), a contradiction. This completes the proof of (3).
(4) If every cyclic subgroup of $P$ of order $p$ or 4 (when $P$ is a non-abelian 2-group) is weakly $H$-embedded in $G$, then $G$ is $p$-supersoluble.

Let $M$ be any proper subgroup of $G$ and $M_p$ a Sylow $p$-subgroup of $M$. Clearly, $(M_p)^g \leq P$ for some element $g \in G$. Then consider $M^g$, which has a Sylow $p$-subgroup $(M_p)^g$ contained in $P$. So, without loss of generality, assume that $M_p \leq P$. By Lemma 3(1), $M$ satisfies the hypothesis, so the choice of $G$ implies that $M$ is $p$-supersoluble. As a result, $G$ is a minimal non-$p$-supersoluble group.

By [17] Theorem 1, $G^{\Phi}(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$, where $\Phi$ is the class of all $p$-supersoluble groups. Clearly, $p \mid |G^{\Phi}(G)/\Phi(G)|$, so $G^{\Phi}(G)/\Phi(G)$ is a $p$-group and $G^{\Phi}$ is solvable. From (18) Theorem 3.4.2), it follows that $G^{\Phi}$ is a $p$-group of exponent $p$ or 4 (when $G^{\Phi}$ is a non-abelian 2-group). By the hypothesis, every cyclic subgroup of $G^{\Phi}$ of order $p$ is weakly $H$-embedded in $G$. When $G^{\Phi}$ is a non-abelian 2-group, clearly, $P$ is also a non-abelian 2-group, so every cyclic subgroup of $G^{\Phi}$ of order 4 is also weakly $H$-embedded in $G$ in this case. Hence, we have $G^{\Phi} \leq Z_{U}(G)$ by Proposition 1, and then $G$ is $p$-supersoluble, a contradiction. So (4) holds.

(5) $p < |D| < |P|$ (It follows directly from (3) and (4)).

(6) $p > 2$ (It follows directly from (2), (5) and Theorem 1).

(7) $O_p(G)$ is the unique minimal normal subgroup of $G$ and $G/O_p(G)$ is $p$-supersoluble.

Let $N$ be a minimal normal subgroup of $G$. Clearly, $N \leq O_p(G)$. If $|N| > |D|$, then $N \leq Z_{U}(G)$ by Proposition 1, which shows that $|N| = p$ and $|D| = 1$, a contradiction. So, we have $|N| \leq |D|$. Please note that $p > 2$, so it is easy to show that $(G/N, P/N)$ satisfies the hypothesis. Thus, the choice of $G$ implies that $G/N$ is $p$-supersoluble; $|N| > p$; $N$ is the unique minimal normal subgroup of $G$. Since $N \not\leq \Phi(G)$, there exists a maximal subgroup $M$ of $G$ such that $G = N \cdot M$. By Lemma 6, $O_p(G) \cap M \leq G, O_p(G) \cap M = 1$ and $O_p(G) = N(O_p(G) \cap M) = N$. Thus, (7) holds.

(8) Final contradiction.

Let $R$ be a normal subgroup of $G$ such that $O_p(G) \leq R \leq G$ and $G/R$ is a G-chief factor. Please note that $G/O_p(G)$ is $p$-supersoluble. So $|G/R| = p$ or $p \mid |G/R|$. First, assume that $|G/R| = p$. Then $|P : P \cap R| = p$ and by (6), $R$ satisfies the hypothesis of the theorem. So $R$ is $p$-supersoluble. Please note that $O_p(R, G) = 1$. Together with Lemma 5, $R$ has the unique Sylow $p$-subgroup $P \cap R$, and furthermore, $P \cap R \leq G$. By (6), $P \cap R$ satisfies the hypothesis of Proposition 1. Thus, $P \cap R \leq Z_{U}(G)$, that is, $R \leq Z_{p}(G)$, which deduces that $G$ is $p$-supersoluble, a contradiction. Then assume that $p \mid |G/R|$, that is $P \leq R$. In this case, $R$ satisfies the hypothesis and so $R$ is $p$-supersoluble by the choice of $G$. Similarly, we have $O_p(R) = 1$ and by Lemma 5, $P \leq R$, which implies that $P \leq G$. By Proposition 1, $P \leq Z_{U}(G)$ and consequently, $G$ is $p$-supersoluble, a contradiction. The final contradiction completes the proof of the theorem.

Next we give the proof of Theorem 3:

Proof. Suppose that the assertion is false and consider a counterexample $G$ of minimal order. According to Theorem 1, we only need to consider that $p$ is odd. We proceed via the following steps.

(1) $O_{p'}(E) = 1$.

If $O_{p'}(E) > 1$, then it is normal in $G$. Consider $\overline{G} = G/O_{p'}(E)$. Please note that $\overline{G}$ a Sylow $p$-subgroup of $\overline{E}$ and $N_{\overline{G}}(\overline{P}) = N_{\overline{G}}(P)$ is $p$-nilpotent. Moreover, by hypothesis and Lemma 3(3), all subgroups of $\overline{P}$ of order $|D|$ and order $|D|$ are weakly $H$-embedded in $\overline{G}$, that is $G$ satisfies the hypothesis for $G$. Thus, the choice of $G$ implies that $\overline{G}$ is $p$-nilpotent. Consequently, $G$ is $p$-nilpotent, a contradiction. So $O_{p'}(E) = 1$.

(2) $E = G$.

By Lemma 3(1), all subgroups of $P$ of order $|D|$ and order $p|D|$ are weakly $H$-embedded in $E$. Since $N_{\overline{E}}(\overline{P}) = N_{\overline{G}}(P) \cap E, N_{\overline{E}}(P)$ is $p$-nilpotent. Then $E$ satisfies the hypothesis. If $E < G$, then $E$ is $p$-nilpotent by the choice of $G$. Let $E_{p'}$ be the normal $p'$-Hall subgroup of $E$. Clearly, $E_{p'} \leq G$. So, by (1), $E_{p'} = 1$, that is, $E = P$. In this case, $G = N_{\overline{G}}(P)$ is $p$-nilpotent. This contradiction shows that $E = G$.

(3) $O_{p'}(G) > 1$. 


Let $J(P)$ be the Thompson subgroup of $P$. Then clearly, $Z(J(P)) > 1$, $P \leq N_{G}(Z(J(P)))$ and $N_{G}(Z(J(P)))$ is $p$-nilpotent. Assume that $N_{G}(Z(J(P))) \leq G$. Please note that $N_{G}(Z(J(P)))$ satisfies the hypothesis by Lemma 3(1). So, the choice of $G$ implies that $N_{G}(Z(J(P)))$ is $p$-nilpotent. However, it contradicts ([19] Theorem 8.3.1). Thus, $N_{G}(Z(J(P))) = G$, that is $Z(J(P)) \leq G$, which shows that (3) holds.

(4) $G$ is not $p$-soluble.

Suppose that $G$ is $p$-soluble. Then $G$ is $p$-supersoluble by the Theorem 2. Please note that $O_{p'}(G) = 1$. So $P \leq G$ by Lemma 5, which shows that $N_{G}(P) = G$ is $p$-nilpotent, a contradiction. Thus, (4) holds.

(5) Let $N$ be a minimal normal subgroup of $G$ contained in $O_{p'}(G)$. Then $|N| > |D|$. If $|N| = |D|$, then every subgroup of $P/N$ of order $p$ is weakly $H$-embedded in $G/N$ by Lemma 3(2). Denote $\overline{G} = G/N$. Let $\overline{M}$ be a proper subgroup of $\overline{G}$ and $\overline{M}_{p}$ a Sylow $p$-subgroup of $\overline{M}$. Clearly, $\overline{M}_{p}^{\overline{P}} \leq \overline{P}$ for some $\overline{P} \in \overline{G}$. Now consider $\overline{M}_{p}$, which has a Sylow $p$-subgroup $\overline{M}_{p}^{\overline{P}}$ contained in $\overline{P}$. Without loss of generality, we can assume that the Sylow $p$-subgroup $\overline{M}_{p}$ of $\overline{M}$ contains in $\overline{P}$. By Lemma 3(1), every cyclic subgroup of $\overline{M}_{p}$ of order $p$ is weakly $H$-embedded in $\overline{M}$. Moreover, $N_{\overline{M}}(\overline{P}) = N_{\overline{M}}(P)$ is $p$-nilpotent. So $\overline{M}$ satisfies the hypothesis, and the choice of $G$ implies that $\overline{M}$ is $p$-nilpotent. Consequently, $G$ is a minimal non-$p$-nilpotent group. However, in this case, $G$ is soluble, which contradicts (4). Suppose that $|N| < |D|$. Then all subgroups of $P/N$ of order $|D|/|N|$ and $p|D|/|N|$ are weakly $H$-embedded in $G/N$ by Lemma 3(2), that is $G/N$ satisfies the hypothesis for $G$. So $\overline{G}$, from the choice of $G$, we deduce that $G/N$ is $p$-nilpotent. Similarly, $G$ is $p$-soluble in this case, a contradiction. Thus, $|N| > |D|$.

(6) Final contradiction.

By (5), all subgroups of $N$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $G$. Then $N \leq Z_{H}(G)$ by Proposition 1. From this result, we deduce that $|N| = p$ and $|D| = 1$, that is, every subgroup of $P$ of order $p$ is weakly $H$-embedded in $G$. Similarly, as the proof of (5), we can prove that in this case $G$ is soluble, a contradiction. The final contradiction completes the proof. □

4. Some Applications

In this section, we list some applications of our results.

Corollary 1. Let $E$ be a normal subgroup of $G$. For every non-cyclic Sylow subgroup $P$ of $E$, assume that $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups of $P$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $G$. Then $E \leq Z_{H}(G)$.

Proof. Assume that $p$ is the smallest prime divisor of $|E|$ and $P$ is a Sylow $p$-subgroup of $E$. If $P$ is cyclic, then $E$ is $p$-nilpotent by the famous Burnside Theorem. Otherwise, by Lemma 3(1) and the hypothesis, all subgroups of $P$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $E$. So $E$ is $p$-nilpotent by Theorem 1, and then $E$ is soluble. By Lemma 3(1) again, we have that for any prime $p$ dividing $|E|$, $E$ satisfies the hypothesis of Theorem 2. So $E$ is supersoluble. Let $q$ be the maximal prime dividing $|E|$ and $Q$ the unique Sylow $q$-subgroup of $E$. Clearly, $Q \leq G$. Note that $Q$ satisfies the hypothesis of Proposition 1, so $Q \leq Z_{H}(G)$. Now consider $E/Q$. By Lemma 3(3), $E/Q$ satisfies the hypothesis of corollary. So $E/Q \leq Z_{H}(E/G)$ by induction. Therefore, $E \leq Z_{H}(G)$. □

Corollary 2. ([12]) Assume that the Sylow subgroups of $G$ are non-cyclic for all primes $p$ dividing $|G|$. Assume further that for each such $p$ there is a $p$-power $d$ with $1 < d < |G|_{p}$ such that all subgroups of $P$ of order $d$ and $pd$ are weakly $H$-embedded in $G$, then $G$ is supersoluble.

Proof. Let $p$ be the smallest prime dividing $|G|$. By Theorem 1, $G$ is $p$-nilpotent. Consequently, $G$ is soluble. From the Theorem 2, it follows that $G$ is $q$-supersoluble, for any prime divisor $q$ of $|G|$, that is, $G$ is supersoluble. □
Corollary 3. ([10]) Let $P$ be a normal $p$-subgroup of a group $G$. If all maximal subgroups of $P$ are weakly $H$-subgroups in $G$, then $P \leq Z_U(G)$.

Corollary 4. ([10]) Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. A group $G$ lies in $\mathfrak{F}$ if and only if it has a normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and all maximal subgroups of every Sylow subgroup of $H$ (or $F^*(H)$) are weakly $H$-subgroups in $G$.

Corollary 5. $G$ is supersolvable, if one of the following holds:

1. $G$ has a normal subgroup $H$ such that $G/H$ is supersolvable and all maximal subgroups of every Sylow subgroup of $H$ belong to $H(G)$ [7];
2. all maximal subgroups of every Sylow subgroup of $F^*(G)$ belong to $H(G)$ [7];
3. all maximal subgroups of every Sylow subgroup of a group $G$ are weakly $H$-subgroups in $G$ [10].

5. Conclusions

In this paper, we further explore weakly $H$-embedded subgroups. As new applications, we generalize the characterization of $p$-nilpotent given by Asaad, Ramadan and Heliel and get a new criterion for $p$-supersolubility for general prime $p$. Moreover, adding condition “$N_G(P)$ is $p$-nilpotent”, we obtain $p$-nilpotence for general prime $p$.

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