A Note on Extended Binomial Coefficients

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Abstract We study the distribution of the extended binomial coefficients by deriving a complete asymptotic expansion with uniform error terms. We obtain the expansion from a local central limit theorem and we state all coefficients explicitly as sums of Hermite polynomials and Bernoulli numbers.

Keywords extended binomial coefficient, composition, complete asymptotic expansion, local central limit theorem, normal approximation, Hermite polynomial, Bernoulli number

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1 Introduction

The extended binomial coefficients, occasionally called polynomial coefficients (e.g., [5, p. 77]), are defined as the coefficients in the expansion

\[ \sum_{k=0}^{\infty} \binom{n}{k}^{(q)} x^k = (1 + x + x^2 + \cdots + x^q)^n, \quad n, q \in \mathbb{N} = \{1, 2, \ldots\}. \quad (1.1) \]

In written form, they presumably appeared for the first time in works by De Moivre [6, p. 41] and later they also were addressed by Euler [9]. Since then, the extended binomial coefficients played a role mainly in the theory of compositions of integers as the number \( c(k, n, q) \) of compositions of \( k \) with \( n \) parts not exceeding \( q \) is given by

\[ c(k, n, q) = \binom{n}{k - n}^{(q-1)}. \]

Thus, the extended binomial coefficients and their modifications have been studied in various papers and from different perspectives [1, 2, 3, 4, 7, 8, 10, 12, 13, 15], and among

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the properties their distribution is of particular interest. Recently, Eger [8] showed (using a slightly different notation) that
\[
\binom{n}{nq/2}^{(q)} \sim \frac{(q + 1)^n}{\sqrt{2\pi n^{2q+2}}},
\]
as \(n \to \infty\), meaning that the quotient of both sides tends to unity. Moreover, based upon numerical simulations [8] the question arises how well those coefficients can be approximated by “normal approximations” in general. It is the aim of this note to give a precise and comprehensive answer to this question by establishing a complete asymptotic expansion for the extended binomial coefficients with error terms holding uniformly with respect to all integer \(k\). More precisely, we show the following.

**Theorem 1.1.** For all integers \(N \geq 2\) we have
\[
\sqrt{\frac{q(q + 2)n}{12}} \frac{1}{(1 + q)^n} \binom{n}{k}^{(q)} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\nu=1}^{[(N-2)/2]} \frac{q_{2\nu}(x)}{n^\nu} + o\left(\frac{1}{n^{(N-2)/2}}\right),
\]
as \(n \to \infty\), uniformly with respect to all \(k \in \mathbb{Z}\), with
\[
x = \frac{\sqrt{12}}{\sqrt{q(q + 2)n}} \left(k - \frac{q}{2} n\right),
\]
where the functions \(q_{2\nu}(x)\) are given explicitly as sums of Hermite polynomials and Bernoulli numbers (see Theorem 2.2 below for the exact formulae). Although we only deal with the very basic situation of the extended binomial coefficients in (1.1) here, the presented approach is a general one, which admits the derivation of (complete) asymptotic expansions in many applications. However, it is not always possible to obtain the involved quantities in a very explicit form, which is an instance making the case of the extended binomial coefficients further interesting and worth to be presented.

## 2 Proof of the main result

First of all we fix some notations following Petrov [14]. For a (real) random variable \(X\) we denote its characteristic function by
\[
\varphi_X(t) = E e^{itX}, \quad t \in \mathbb{R},
\]
where, as usual, \(E\) means the mathematical expectation with respect to the underlying probability distribution. If \(X\) has finite moments up to \(k\)-th order, then \(\varphi_X\) is \(k\) times continuously differentiable on \(\mathbb{R}\) and we have
\[
\frac{d^k}{dt^k} \varphi_X(t) \bigg|_{t=0} = \frac{1}{k!} EX^k.
\]
Moreover, in this case we define the cumulants of order \( k \) by
\[
\gamma_k = \frac{1}{i^k} \left. \frac{d^k}{dt^k} \log \varphi_X(t) \right|_{t=0},
\]
where the logarithm takes its principal branch. Now, let \((X_n)\) be a sequence of independent integer-valued random variables having a common distribution and suppose that for all positive integer values of \( k \) we have
\[
E|X_1|^k < \infty
\]
and
\[
EX_1 = \mu, \quad \text{Var}X_1 = \sigma^2 > 0.
\]
Thus, for the sum given by
\[
S_n = \sum_{\nu=1}^{n} X_\nu
\]
we obtain
\[
ES_n = n\mu, \quad \text{Var}S_n = n\sigma^2,
\]
and for integer \( k \) we define the probabilities
\[
p_n(k) = P(S_n = k).
\]
Furthermore, we introduce the Hermite polynomials (in the probabilist’s version)
\[
H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2},
\]
and for positive integers \( \nu \) we define the functions
\[
q_\nu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{k_1, \ldots, k_\nu \geq 0 \\ k_1 + 2k_2 + \cdots + \nu k_\nu = \nu}} H_{\nu+2s}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)! \sigma^{m+2}} \right)^{k_m}, \tag{2.1}
\]
where \( s = k_1 + \cdots + k_\nu \) and \( \gamma_{m+2} \) denotes the cumulant of order \( m+2 \) of \( X_1 \).

Finally, we demand (for convenience) that the maximal span of the distribution of \( X_1 \) is equal to one. This means that there are no numbers \( a \) and \( h > 1 \) such that the values taken on by \( X_1 \) with probability one can be expressed in the form \( a + hk \) \((k \in \mathbb{Z})\). Under all these assumptions we have the following complete asymptotic expansion in the sense of a local central limit theorem [14, p. 205].

**Theorem 2.1.** For all integers \( N \geq 2 \) we have
\[
\sigma \sqrt{np_n}(k) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\nu=1}^{N-2} \frac{q_\nu(x)}{n^{\nu/2}} + o \left( \frac{1}{n^{(N-2)/2}} \right), \tag{2.2}
\]
as \( n \to \infty \), uniformly with respect to all \( k \in \mathbb{Z} \), where we have
\[
x = \frac{k - n\mu}{\sigma \sqrt{n}}.
\]
In the following we choose \(X_1\) to take the integer values \(\{0, \ldots, q\}\) with
\[
P(X_1 = k) = \frac{1}{q + 1}, \quad k \in \{0, \ldots, q\}.
\]
Hence, we obtain
\[
p_n(k) = P(S_n = k) = \frac{1}{(1 + q)^n} \binom{n}{k} (q), \quad k \in \mathbb{Z}.
\]  \hspace{1cm} (2.3)

It is our aim to apply Theorem 2.1 in full generality and we want compute all cumulants as explicit as possible.

**Lemma 2.1**. For the \(k\)-th order cumulant \(\gamma_k\) of \(X_1\) we have
\[
\gamma_k = \begin{cases} 
\frac{q}{2}, & \text{if } k = 1; \\
0, & \text{if } k \text{ odd and } k > 1; \\
\frac{B_{2l}}{2l} ((q + 1)^{2l} - 1), & \text{if } k = 2l, l \geq 1,
\end{cases}
\]  \hspace{1cm} (2.4)

where \(B_{\nu}, \nu \geq 0\), denote the Bernoulli numbers, e.g., [11, p. 22].

**Proof.** First, we observe that the characteristic function of \(X_1\) is given by
\[
\varphi_{X_1}(t) = \frac{1 + e^{it} + \cdots + e^{qit}(q + 1)}{1 + q}.
\]

According to the definition of the cumulants we obtain for a positive integer \(k\)
\[
\gamma_k = \frac{1}{i^k} \frac{d^k}{dt^k} \log \varphi_{X_1}(t) \bigg|_{t=0} = \frac{1}{i^k} \frac{d^k}{dt^k} \left\{ \log \left(1 + e^{it} + \cdots + e^{qit}(q + 1)\right) - \log(1 + q) \right\} \bigg|_{t=0}
\]
\[
= \frac{1}{i^k} \frac{d^k}{dt^k} \log \left(\frac{e^{(q+1)it} - 1}{e^{it} - 1}\right) |_{t=0}
\]
\[
= \frac{1}{i^k} \frac{d^k}{dt^k} \left\{ \frac{q}{2} it + \log \left(\frac{\sin \frac{q+1}{2} t}{\sin \frac{1}{2} t}\right) \right\} |_{t=0}
\]
\[
= \frac{q}{2} \delta_{k,1} + \frac{1}{i^k} \frac{d^k}{dt^k} \left\{ \log \left(\frac{\sin \frac{q+1}{2} t}{\sin \frac{1}{2} t}\right) - \log \left(\frac{\sin \frac{1}{2} t}{\frac{q+1}{2} t}\right) \right\} |_{t=0},
\]

where \(\delta_{k,1}\) denotes the Kronecker delta. Using
\[
\frac{d}{dz} \log \left(\frac{\sin \frac{z}{2}}{z}\right) = \cotan z - \frac{1}{z}
\]

yields
\[
\gamma_k = \frac{q}{2} \delta_{k,1} + \frac{1}{i^k} \frac{d^{k-1}}{dt^{k-1}} \left\{ \frac{q+1}{2} \left(\cotan \frac{q+1}{2} t - \frac{2}{(q+1)t}\right) - \frac{1}{2} \left(\cotan \frac{t}{2} - \frac{2}{t}\right) \right\} \bigg|_{t=0}.
\]
Now, making use of the following expansion (see, e.g., [11, p. 35])

\[
\cotan z - \frac{1}{z} = \sum_{m=1}^{\infty} (-1)^m \frac{4^m}{(2m)!} B_{2m} z^{2m-1}, \quad 0 < |z| < \pi,
\]

after some algebra we obtain

\[
\gamma_k = \frac{q}{2} \delta_{k,1} + \frac{1}{i^k} \frac{d^{k-1}}{dt^{k-1}} \sum_{m=1}^{\infty} (-1)^m \frac{B_{2m}}{(2m)!} ((q + 1)^{2m} - 1) t^{2m-1} \bigg|_{t=0}.
\]

Carrying out the differentiation under the summation sign immediately gives us 2.4.

**Remark 2.1.** As an immediate consequence of Lemma 2.1 we obtain

\[
EX_1 = \mu = \gamma_1 = \frac{q}{2}
\]

and, as we know \(B_2 = \frac{1}{6}\),

\[
Var X_1 = \sigma^2 = \gamma_2 = \frac{B_2}{2} ((q + 1)^2 - 1) = \frac{q(q + 2)}{12}.
\]

We now are ready to state the main theorem in form of a complete asymptotic expansion with explicit coefficients for the extended binomial coefficients \( \binom{n}{k}^{(q)} \).

**Theorem 2.2.** For all integers \( N \geq 2 \) we have

\[
\sqrt{\frac{q(q + 2)n}{12}} \frac{1}{(1 + q)^n} \binom{n}{k}^{(q)} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\nu=1}^{\lfloor (N-2)/2 \rfloor} \frac{q_{2\nu}(x)}{n^{\nu}} + o \left( \frac{1}{n^{(N-2)/2}} \right),
\]

as \( n \to \infty \), uniformly with respect to all \( k \in \mathbb{Z} \), with

\[
x = \frac{\sqrt{12}}{\sqrt{q(q+2)n}} \left( k - \frac{q}{2} n \right),
\]

and

\[
q_{2\nu}(x) = \frac{1}{\sqrt{2\pi}} \left( \frac{12}{q(q+2)} \right)^{\nu} e^{-x^2/2}
\]

\[
\times \sum_{\begin{array}{c}
\nu = 0 \\
k_2, k_4, \ldots, k_{2\nu} \geq 0 \\
k_2 + 2k_4 + \cdots + \nu k_{2\nu} = \nu
\end{array}} H_{2(\nu+s)} \left( \frac{6}{q(q+2)} \right)^s \prod_{m=1}^{\nu} \frac{1}{k_{2m}!} \left( \frac{B_{2(m+1)} ((q + 1)^{2m+2} - 1)}{(2m + 2)!(m + 1)} \right)^{k_{2m}},
\]

where \( s = k_2 + k_4 + \cdots + k_{2\nu} \).
Proof. The proof is based on an application of Theorem 2.1 to the probabilities defined in (2.3). First we observe that in our situation the functions given in (2.1) vanish identically for odd indices, which turns out to be a consequence of (2.4). Indeed, if \( \nu = 2l + 1 \) for an integer \( l \geq 0 \), then in every solution \( k_1, \ldots, k_{2l+1} \geq 0 \) of the equation

\[
k_1 + 2k_2 + \cdots + (2l + 1)k_{2l+1} = 2l + 1
\]

there is at least one odd index \( i \) with \( k_i > 0 \). Consequently, using (2.4) we have

\[
\prod_{m=1}^{2l+1} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!\sigma^m} \right)^{k_m} = 0,
\]

from which follows that \( q_{2l+1}(x) \) vanishes identically. Thus, only the functions \( q_{2\nu}(x) \) appear in (2.2) and here we have

\[
q_{2\nu}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k_1, \ldots, k_{2\nu} \geq 0, \ k_1 + 2k_2 + \cdots + 2\nu k_{2\nu} = 2\nu} H_{2(\nu+s)}(x) \prod_{m=1}^{2\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!\sigma^m} \right)^{k_m},
\]

where \( s = k_1 + \cdots + k_{2\nu} \). An analogous argument as in the odd case above shows that a solution \( k_1, \ldots, k_{2\nu} \) of the equation

\[
k_1 + 2k_2 + \cdots + 2\nu k_{2\nu} = 2\nu
\]

with a positive entry at an odd index does not give any contribution to the whole sum, so that we can write

\[
q_{2\nu}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k_1, k_4, \ldots, k_{2\nu} \geq 0, k_1 + 2k_2 + \cdots + \nu k_{2\nu} = \nu} H_{2(\nu+s)}(x) \prod_{m=1}^{\nu} \frac{1}{k_{2m}!} \left( \frac{\gamma_{2m+2}}{(2m+2)!\sigma^{2m+2}} \right)^{k_{2m}},
\]

where \( s = k_2 + k_4 + \cdots + k_{2\nu} \). Now, taking the explicit form of the cumulants in (2.4) into account, after some elementary computation we obtain (2.5).

\[\square\]

Remark 2.2. As a concluding remark we state the meaning of Theorem 2.2 for \( N = 5 \) explicitly. Using the known facts

\[
H_4(x) = x^4 - 6x^2 + 3, \quad B_4 = -\frac{1}{30},
\]

we obtain

\[
\sqrt{\frac{q(q+2)n}{12}} \frac{1}{(1+q)^n} \binom{q}{k} (n^q) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left\{ 1 - \frac{(q+1)^4 - 1}{20nq^2(q+2)^2} \left( x^4 - 6x^2 + 3 \right) \right\} + o\left( \frac{1}{n^{3/2}} \right),
\]

as \( n \to \infty \), uniformly with respect to all \( k \in \mathbb{Z} \), where we have

\[
x = \frac{\sqrt{12}}{\sqrt{q(q+2)n}} \left( k - \frac{q}{2} n \right).
\]
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