Quantization of Chern-Simons Coefficient

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Abstract
The relation between the Dirac quantization condition of magnetic charge and the quantization of the Chern-Simons coefficient is obtained. It implies that in a (2+1)-dimensional QED with the Chern-Simons topological mass term and the existence of a magnetic monopole with magnetic charge $g$, the Chern-Simons coefficient must be also quantized, just as in the non-Abelian case.

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Recently, the quantization of Chern-Simons coefficient has attracted considerable attention [1]. The action with an added Chern-Simons topological mass term to the usual Yang-Mills gauge theory in a (2+1)-dimensional space-time remains invariant under small gauge transformations, but, to ensure invariance of the exponentiated action under large gauge transformations, the coefficient of Chern-Simons topological mass term has to be quantized [2].

In the Abelian case, however, in general, the Chern-Simons coefficient is not quantized in the absence of a topological charge, but, the Chern-Simons coefficient must be also quantized in the presence of a topological charge, for example, a magnetic pole, just as in the non-Abelian case.

In the letter we will demonstrate it. By the two-loop radiative corrections perturbatively for the fermionic current vector in QED with a Chern-Simons topological mass term in a (2+1)-dimensional space-time, we look for the relation between the Dirac quantization condition of a magnetic charge and the quantization of the Chern-Simons coefficient, thus the Dirac quantization...
condition of a magnetic charge leads to the quantization of the Chern-Simons coefficient.

Let us consider the following Lagrangian in the (2+1)-dimensional QED with a Chern-Simons mass term,

\[ 
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{m}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{\lambda}{2} (\partial_\mu A^\mu)^2 + \bar{\Psi} [i \gamma_\mu (i \partial^\mu - e A^\mu) - m_f] \Psi 
\]  

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \); \( m, \lambda \) and \( m_f \) are the photon topological mass, a parameter of the gauge-fixing term and the fermion mass respectively. The space-time is Minkowski with signature (+, −, −). The \( \epsilon^{\mu\nu\lambda} \) is a three-dimensional antisymmetric tensor. Our purpose is to find out the relation between the quantization of the Chern-Simons coefficient and the Dirac quantization condition of a magnetic charge in the presence of a magnetic pole in the Abelian case, therefore we can start with the electron current vector \( J_\mu(x) \) in an external electromagnetic field \( A_\mu(x) \) with the Chern-Simons topological mass term in 2+1 dimensions.

The ground state current vector of fermion fields in a (2+1)-dimensional space-time is

\[ 
\langle 0 | J_\mu(x) | 0 \rangle = \langle 0 | \frac{1}{2} [\bar{\Psi}(x), \gamma_\mu \Psi(x)]_{-} | 0 \rangle = -Tr[\gamma_\mu G(x,x')]|_{x' \rightarrow x} 
\]  

where \( G(x,x') \) is the fermion propagator in interaction with an external electromagnetic field \( A_\mu(x) \) with the Chern-Simons mass term and it satisfies the following equation

\[ 
(\not{D} - m_f)G(x,x') = \delta^4(x-x')
\]  

with

\[ 
\not{D} = i \partial^\mu - e A_\mu = \gamma_\mu [i \partial_\mu - eA_\mu(x)].
\]  

What are considered are the effects which are brought by the Chern-Simons mass term, so the photon propagator \( D_{\mu\nu} \) is taken into the following form which is produced by the pure Chern-Simons term, i.e., the second term of the Lagrangian in eq. (1),

\[ 
D_{\mu\nu} = \frac{\epsilon_{\mu\nu\lambda} k^\lambda}{mk^2}
\]
where it is in momentum space and we adopt the Landau gauge, $\lambda = 0$, in eq. (1) in order to avoid the infrared divergences. In the (2+1)-dimensional perturbative QED, the pure Chern-Simons effects brought by the fermion vector current in eq. (2) are in the two-loop corrections in the lowest order, so we have to consider the two-loop Feynman diagrams for the ground state current vector of the fermion fields, $\langle 0 \mid J_\mu \mid 0 \rangle$, in a (2+1)-dimensional space-time in eq. (2). These two-loop Feynman diagrams in momentum space are the diagrams (a), (b), (c) and (d) in Fig. 1.

The contribution of Fig. 1(a) to the ground state current vector of the fermion fields, $\langle 0 \mid J_\mu \mid 0 \rangle$, has the power of $e^3$, but the contributions of Fig. 1(b), (c), (d) to $\langle 0 \mid J_\mu \mid 0 \rangle$ have the power of $e^4$. After calculating the contribution of Fig. 1(a) to the fermion current, we have discovered that it vanishes. Therefore, in the perturbative QED of 2+1 dimensions, the lowest order of the nonvanishing contributions with the effect of Chern-Simons term to the ground state current vector of the fermion fields is in $O(e^4)$, and the corresponding Feynman diagrams are the three diagrams (b), (c), and (d) in Fig. 1.

We now consider the contributions of the three diagrams Fig. 1(b), (c) and (d). Let us first take a look at the subdiagram in Fig. 1(b), the electron self-energy. To its contribution in momentum space

$$\Sigma(p) = \frac{e^2}{4\pi m} \left\{ (p^2 - m_f^2) \int_0^1 \alpha^\frac{1}{2} [(1 - \alpha)p^2 + m_f^2]^{-\frac{1}{2}} d\alpha + 2m_f \right\}$$

where the Landau gauge has been adopted and the photon propagator has been taken as the form in eq. (5) due to the pure Chern-Simons theory, and the dimensional regularization has been also used. We compute the contribution $\Pi_{\mu\nu}^{(b)}(k) A_\mu(k)$ of Fig.1(b) in terms of the electron self-energy $\Sigma(p)$ of eq.(6). Because we concern with the pure Chern-Simons effects in $\Pi_{\mu\nu}^{(b)}(k)$, for simplicity, the electron mass $m_f$ in $\Pi_{\mu\nu}(k)$ can be omitted, i.e., assuming the gauge boson mass $m \gg m_f$. We have got for $\Pi_{\mu\nu}^{(b)}(k)$ after a tedious calculation

$$\Pi_{\mu\nu}^{(b)}(k) = \frac{e^4}{48\pi^2 m} \left\{ \frac{2}{\varepsilon} - \ln \frac{k^2}{\mu^2} - \gamma + O(\varepsilon) \right\} \epsilon_{\mu\nu\lambda} k^\lambda$$

where $\varepsilon = 3 - d$, the dimensional regularization has been used and the dimension $d \rightarrow 3$, and $\gamma$ is Euler constant and $\mu$ is an arbitrary mass scale.
The two diagrams (b) and (c) in Fig. 1 obviously give equal contributions. We now turn to the diagram (d) in Fig. 1. Similarly, we proceed to the cumbersome computation of the contribution $\Pi^{(d)}_{\mu\nu}(k)A_\nu(k)$ of Fig. 1(d) to the ground state vector current of the electron fields. We have also omitted the electron mass $m_f$ in $\Pi^{(d)}_{\mu\nu}(k)$. We compute the contribution $\Pi^{(d)}_{\mu\nu}(k)$ in terms of the Ward-Takahashi identities between the electron self-energy and the vertex function which is in the subdiagram of Fig. 1(d). After a tedious computation for $\Pi^{(d)}_{\mu\nu}(k)$ we have obtained

$$
\Pi^{(d)}_{\mu\nu}(k) = \frac{e^4}{12\pi^2 m} \left[ -\frac{2}{\varepsilon} - \frac{\ln k^2}{\mu^2} - \gamma + O(\varepsilon) \right] \epsilon_{\mu\nu\lambda} k^\lambda
$$

(8)

where $3-d = \varepsilon$, $(k^2)^{\frac{d}{2}} = 1 + \frac{2}{\varepsilon} \ln k^2 + \cdots$ and $\Gamma(\frac{3-d}{2}) = \frac{2}{\varepsilon} - \gamma + \cdots$. Ultimately, adding the contributions of the three diagrams in Fig. 1 to the fermion vector current $J_\mu$, we have obtained the total contributions of Fig. 1(b), (c), (d) to the electron vector current $J_\mu$ in momentum space therefore

$$
J_\mu(k) = \frac{e^4}{8\pi^2 m} \left[ \frac{2}{\varepsilon} + \frac{\ln k^2}{\mu^2} + \gamma + O(\varepsilon) \right] \epsilon_{\mu\lambda\nu} k^\lambda A_\nu(k)
$$

(9)

We now transform from a momentum space into a configuration space to consider eq. (9). In configuration space a magnetic field

$$
\vec{B} = \vec{\partial} \times \vec{A}(x) = \epsilon_{ij} \partial_i A_j(x)
$$

(10)

where $\epsilon_{ij} = \epsilon_{0ij}, i, j = 1, 2$.

The magnetic flux through the surface (i.e., the two space dimensions) is

$$
\oint \vec{B} \cdot d\vec{\sigma} = \int \vec{B} \cdot d\vec{x} = \vec{g}
$$

(11)

by definition of the magnetic charge $g$ contained inside the sphere. There is a magnetic monopole with a magnetic charge $g$.

The contributions of the self-energy counterterms and the vertex counterterms corresponding to Fig. 1(b), (c) and (d) can be also readily computed. After having considered the contributions of these counterterms and renormalized for the quantities in eq. (9), we finally get the result in configuration space

$$
\int J_0(x) d\vec{x} = e_R = \frac{e^2_R}{8\pi^2 m_R} \int \vec{B} \cdot d\vec{x} = \frac{e^2_R}{8\pi^2 m_R} g
$$

(12)
where the zeroth dividual quantity $J_0(x)$ of $J_\mu(x)$ is the electric charge density, and $e_R$ and $m_R$ are a renormalized electron charge and a renormalized Chern-Simons topological mass respectively. This yields

$$\frac{1}{2\pi}(eg)_R = n, \quad (13)$$

$$4\pi\left(\frac{m}{e^2}\right)_R = n, \quad (14)$$

where $n = 0, 1, 2, 3, \cdots$. Equation (13) is called the Dirac quantization condition of magnetic charge. Equation (14) implies the quantization of the coefficient of Chern-Simons term. Thus it can be seen that we have established the relation between the Dirac quantization condition of magnetic charge and the quantization of the Chern-Simons coefficient in QED of 2+1 space-time dimensions with the Chern-Simons topological mass term in the existence of any magnetic monopole with magnetic charge $g$. It is easy to see from eq. (12) that the Dirac quantization condition of magnetic charge leads to the quantization of the Chern-Simons coefficient or conversely, the requirement of the quantization of the Chern-Simons coefficient in order to keep the nontrivial gauge invariance leads to the Dirac quantization condition of magnetic charge. Therefore, this demonstrate that the Chern-Simons coefficient must be also quantized in the Abelian case in the existence of a topological charge.

In this letter, in the (2+1)-dimensional QED with a Chern-Simons term the magnetic monopole is not put by hand, but is innate in the theory, so the quantization of the Chern-Simons coefficient is not also put by hand, but is also innate in the theory. The work of the quantization of the Chern-Simons coefficient of the (2+1)-dimensional non-Abelian gauge theory at zero temperature and finite temperature is in progress.

References

[1] G. Dunne, K. Lee, and C. Lu, Phys. Rev. Lett. 78, 3434 (1997); S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. Lett. 79, 1976 (1997); C.D. Fosco, G.L. Rossini, and F.A. Schaposnik, Phys. Rev. Lett. 79, 1980 (1997).
Fig. 1. The two-loop Feynman diagrams for \( \langle 0 \mid J_\mu \mid 0 \rangle \). The solid line stands for a fermion propagator, the wavy line stands for a gauge propagator in the pure Chern-Simons theory, \( D_{\mu\nu}(k) \) in eq. (5), the cross for \( eA \) and the black dot for \( e\gamma_\mu \).