History and Physics of The Klein Paradox

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The early papers by Klein, Sauter and Hund which investigate scattering off a high step potential in the context of the Dirac equation are discussed to derive the 'paradox' first obtained by Klein. The explanation of this effect in terms of electron-positron production is reassessed. It is shown that a potential well or barrier in the Dirac equation can become supercritical and emit positrons or electrons spontaneously if the potential is strong enough. If the well or barrier is wide enough, a seemingly constant current is emitted. This phenomenon is transient whereas the tunnelling first calculated by Klein is time-independent. It is shown that tunnelling without exponential suppression occurs when an electron is incident on a high barrier, even when the barrier is not high enough to radiate. Klein tunnelling is therefore a property of relativistic wave equations and is not necessarily connected to particle emission. The Coulomb potential is investigated and it is shown that a heavy nucleus of sufficiently large Z will bind positrons. Correspondingly, as Z increases the Coulomb barrier should become increasingly transparent to positrons. This is an example of Klein tunnelling. Phenomena akin to supercritical positron emission may be studied experimentally in superfluid $^3$He.
I. SOME HISTORY

A. Introduction to the Klein Paradox(es)

Seventy years ago Klein [1] published a paper where he calculated the reflection and transmission coefficients for electrons of energy $E$, mass $m$ and momentum $k$ incident on the potential step (Fig. 1)

$$V(x) = V, \ x > 0; \ V(x) = 0, \ x < 0$$

(1)

within the context of the new relativistic equation which had just been published by Dirac [2]. He found (see Section 2 below) that the reflection and transmission coefficients $R_S, T_S$ if $V$ was large were given by

$$R_S = \left( \frac{1 - \kappa}{1 + \kappa} \right)^2 \quad T_S = \frac{4\kappa}{(1 + \kappa)^2}$$

(2)

where $\kappa$ is the kinematic factor

$$\kappa = \frac{p}{k} \frac{E + m}{E + m - V}$$

(3)

and $p$ is the momentum of the transmitted particle for $x > 0$. It is easily seen from Eq. (3) that when $E < V - m$, $\kappa$ seems to be negative with the paradoxical result that the reflection coefficient $R_S > 1$ while $T_S < 0$. So more particles are reflected by the step than are incident on it. This is what many articles and books call the Klein Paradox. It is not, however, what Klein wrote down.

Klein noted that Pauli had pointed out to him that for $x > 0$, the particle momentum is given by $p^2 = (V - E)^2 - m^2$ while the group velocity $v_g$ was given by

$$v_g = dE/dp = p/(E - V)$$

(4)

So if the transmitted particle moved from left to right, $v_g$ was positive implying that $p$ had to be assigned its negative value.
\[ p = -\sqrt{(V - E)^2 - m^2} \] (5)

With this choice of \( p \)

\[ \kappa = \frac{(V - E + m)(E + m)}{(V - E - m)(E - m)} \] (6)

and \( \kappa \geq 1 \) ensuring that both \( R_S \) and \( T_S \) are positive or zero and satisfy \( R_S + T_S = 1 \) for \( m \leq E \leq V - m \). Is there still a paradox? The general consensus both now and for the authors who followed Klein and did the calculation correctly is that there is. Let the potential step \( V \to \infty \) for fixed \( E \) then from Eq. (6) \( \kappa \) tends to a finite limit and hence \( T_S \) tends to a non-zero limit. The physical essence of this paradox thus lies in the prediction that according to the Dirac equation, fermions can pass through strong repulsive potentials without the exponential damping expected in quantum tunnelling processes. We have called this process Klein tunnelling [3].

We begin with a summary of the Dirac equation in one dimension in the presence of a potential \( V(x) \) and show how Klein’s original result for \( R_S \) and \( T_S \) is obtained. We go on to the papers of Sauter in 1931, who replaced Klein’s potential step with a barrier with a finite slope, and then to Hund in 1940 who realised that the Klein potential step gives rise to the production of pairs of charged particles when the potential strength is sufficiently strong. This result although not well known is a precursor of the famous results of modern quantum field theory of Schwinger [3] and Hawking [7] which show that particles are spontaneously produced in the presence of strong electric and gravitational fields. In Part II we turn to the underlying physics of the Klein paradox and show that particle production and Klein tunnelling arise naturally in the Dirac equation: when a potential well is deep enough it becomes supercritical (defined as the potential strength for which the bound state energy \( E = -m \)) and positrons will be spontaneously produced. Supercriticality is well-understood [8], [9] and can occur in the Coulomb potential with finite nuclear size when the nuclear charge \( Z > 137 \). Positron production via this mechanism has been the subject of
We then show that if a potential well is wide enough, a steady but transient current will flow when the potential becomes supercritical. In order to analyse these processes it is necessary to introduce the concept of vacuum charge. We consider the implications of these concepts for the Coulomb potential and for other physical phenomena and we end by pointing out that Klein was unfortunate in that the example he chose to calculate was pathological.

**B. The Dirac Equation in One Dimension**

In one-dimension it is unnecessary to use four-component Dirac spinors. It is much easier to use two-component Pauli spinors instead. We adopt the convention $\gamma_0 = \sigma_z$, $\gamma_1 = i\sigma_x$. The above choice agrees with $\gamma_i \gamma_j + \gamma_j \gamma_i = 2g_{ij}$. The free Dirac Hamiltonian in one dimension is

$$H_0 = -\sigma_y p + \sigma_z m$$

and so the Dirac equation takes the form

$$\left(\sigma_x \frac{\partial}{\partial x} - E\sigma_z + m\right)\psi = 0 \quad (7)$$

In what follows $k$ stands for the wavevector, $k$ for its magnitude and $\varepsilon = |E| = +\sqrt{k^2 + m^2}$. We try a plane wave of the form

$$\begin{pmatrix} A \\ B \end{pmatrix} e^{ikx-iEt} \quad (8)$$

and substitute in (7). The equation is satisfied by $A = ik, B = E - m$ where $E = \pm \varepsilon$. The positive energy (or particle) solutions have the form

$$N_+(\varepsilon) \begin{pmatrix} ik \\ \varepsilon - m \end{pmatrix} e^{ikx - i\varepsilon t} \quad (9)$$

and the negative energy (or hole) solutions are
\[ N_+(\varepsilon) \left( \begin{array}{c} i k \\ E - m \end{array} \right) e^{ikx + i\varepsilon t} \]

where \( N_{\pm}(\varepsilon) \) are appropriate normalization factors. If we take the particle to be in a box of length \( 2L \) with periodic boundary conditions at \( x = -L \) and \( x = L \) we obtain

\[ N_+(\varepsilon) = \frac{1}{\sqrt{2L}\sqrt{2\varepsilon(\varepsilon - m)}}, \quad N_-(\varepsilon) = \frac{1}{\sqrt{2L}\sqrt{2\varepsilon(\varepsilon + m)}} \]

Alternatively we can use continuum states and energy normalisation; then

\[ N_+(\varepsilon) = \frac{1}{\sqrt{2\pi}\sqrt{2\varepsilon(\varepsilon - m)}}, \quad N_-(\varepsilon) = \frac{1}{\sqrt{2\pi}\sqrt{2\varepsilon(\varepsilon + m)}} \]

C. The Klein Result

In the presence of the Klein step, the Hamiltonian is

\[ H_0 = -\sigma_y p + V(x) + \sigma_z m \]

where \( V(x) \) is now given by Eq. (1). The Dirac equation reads

\[ (\sigma_z \frac{\partial}{\partial x} - (E - V(x))\sigma_z + m)\psi = 0 \]

Consider an electron incident from the left. The corresponding wavefunction is

\[ \left( \begin{array}{c} ik \\ E - m \end{array} \right) e^{ikx} + \left( \begin{array}{c} -ik \\ E - m \end{array} \right) e^{-ikx} \]

for \( x < 0 \), and

\[ F \left( \begin{array}{c} -ip \\ V - E - m \end{array} \right) e^{-ipx} \]

for \( x > 0 \) since that state is a hole state (see Fig. 2). It is easy to see from Eqs. (14, 15) that for continuity at \( x = 0 \) we require
\[ ik(1 - B) = -ipF \]  
\[ (E - m)(1 + B) = (V - E - m)F \]
giving
\[ \frac{1 - B}{1 + B} = \frac{-p}{k} \frac{E - m}{V - E - m} = \frac{1}{\kappa} \]
in terms of the quantity \( \kappa \) defined by Eq. (3). This gives the expression for \( R_S = |B|^2 \) of Eq. (2) above while that for \( T_S \) follows from \( R_S + T_S = 1 \).

**D. Sauter’s Contribution**

Klein’s surprising result was widely discussed by theoretical physicists at the time. Bohr thought that the large transmission coefficient that Klein found was because the Klein step was so abrupt. He discussed this with Heisenberg and Sommerfeld and as a result Sommerfeld’s assistant Sauter [4] in Munich calculated the transmission coefficient for a potential of the form

\[ V(x) = vx \quad 0 < x < L \]  

with \( V(x) = 0 \) for \( x < 0 \) and \( V(x) = vL \) for \( x > L \) (Fig. 3). In order to obtain negative energy states (holes) to propagate through the barrier as in the Klein problem, we require \( vL > 2m \). Sauter’s potential thus should reduce to the Klein step if \( v \) were very large. Sauter’s potential is of course more physical than Klein’s: it simply represents a constant electric field \( E = -v \) in a finite region of space. Klein tunnelling in this case would imply that low energy electrons could pass through a repulsive constant electric field without exponential damping. Bohr conjectured that the Klein result would only be reproduced if the Sauter field were so strong that the potential difference \( \Delta V > 2m \) would be attained at distances of the order of the Compton wavelength of the electron; that is to say that the electric field strength \( |E| = |v| > 2m^2 \).
After a lengthy calculation involving the appropriate hypergeometric functions, Sauter obtained the result he was seeking: he obtained an expression for the reflection and transmission coefficients $R$ and $T$ which reduced to the Klein values $R_S$ and $T_S$ for $|v| \sim m^2$; nevertheless but for weaker fields he obtained

$$R \simeq 1 \quad T = e^{-\pi m^2/v} = e^{-\left(\pi m^2 L/\Delta V\right)}$$

(18)

a non-paradoxical result since it shows the exponentially-suppressed tunnelling typical of quantum phenomena. What no one realised at the time is that Sauter had anticipated Schwinger’s [6] result of quantum electrodynamics by twenty years (see next Section). Note also that Eq. (18) shows explicitly that Bohr’s conjecture is correct: in order to violate the rule that tunnelling in quantum mechanics is exponentially suppressed we require electric fields of field strength $|E| = |v| \sim \pi m^2$.

E. Hund’s Contribution

The next major contribution to the subject came ten years later. Hund [5] looked again at the Klein step potential but from the viewpoint of quantum field theory, not just the one particle Dirac equation. He concentrated on charged scalar fields rather than spinor fields. He considered both the Klein step potential and a sequence of step potentials. His result was as surprising as Klein’s original result. Hund found that provided $\Delta V > 2m$ where $\Delta V = V(\infty) - V(-\infty)$, then a non-zero constant electric current $j$ had to be present where the current was given by an integral over the transmission coefficient $T(E)$ with respect to energy $E$. The current had to be interpreted as spontaneous production out of the vacuum of a pair of oppositely charged particles. Hund attempted to derive the same result for a spinor field but was unsuccessful: it was left to Hansen and Ravndal [11] forty years later to generalise this result to spinors (for a good discussion of the difference between scalar and spinor fields incident on a Klein step see Manogue [12]). We show in the Appendix for a Klein step or more general step potential such as those considered by Hund and Sauter.
in the Dirac equation that there is indeed a spontaneous current of electron-positron pairs produced given by
\[ \langle 0| j |0 \rangle = -\frac{1}{2\pi} \int dET(E) \] (19)
in agreement with Hund’s result for scalars. Eq. (19) is very powerful: it is a sort of optical theorem. If spontaneous pair production occurs at a constant rate, then the time-independent reflection and transmission coefficients must incorporate this process. If Sauter had known of Eq. (19), he would have been able to predict Schwinger’s [6] result on spontaneous pair production by a constant electric field simply by using the value of the transmission coefficient he had calculated in Eq. (18).

II. THE UNDERLYING PHYSICS

A. Scattering by a Square Barrier

We now investigate the underlying physics behind these phenomena. Why is it that electrons can tunnel so easily through a high potential barrier? Why are particles produced in strong potentials? Are these two questions the same question; that is to say is the result that particles are produced by a Klein step or other strong field the reason for Klein tunnelling. To answer these questions we turn our attention to a potential barrier which is not the Klein step but is similar and has better-defined properties. This is the square barrier (Fig. 4)
\[ V(x) = V, |x| < a; V(x) = 0, |x| > a. \] (20)
Electrons incident from the left would not be expected to be able to distinguish between a wide barrier (i.e. \( ma >> 1 \)) and a Klein step. The results are in fact not identical but they do display the same characteristics.

It is easy to show that the reflection and transmission coefficients are given for a square barrier by \[ 13 \]
\[ R = \frac{(1 - \kappa^2)^2 \sin^2(2pa)}{4\kappa^2 + (1 - \kappa^2)^2 \sin^2(2pa)} \]  
(21)

\[ T = \frac{4\kappa^2}{4\kappa^2 + (1 - \kappa^2)^2 \sin^2(2pa)} \]  
(22)

Note that tunnelling is easier for a barrier than a step: if

\[ 2pa = N\pi \]  
(23)

corresponding to \( E_N = V - \sqrt{m^2 + N^2\pi^2/4a^2} \) then the electron passes right through the barrier with no reflection: this is called a transmission resonance \([14]\).

As \( a \) becomes very large for fixed \( m, E \) and \( V \), \( pa \) becomes very large and \( \sin(pa) \) oscillates very rapidly. In those circumstances we can average over the phase angle \( pa \) using \( \sin^2(pa) = \cos^2(pa) = \frac{1}{2} \) to find the limit

\[ R_\infty = \frac{(1 - \kappa^2)^2}{8\kappa^2 + (1 - \kappa^2)^2} \quad T_\infty = \frac{8\kappa^2}{8\kappa^2 + (1 - \kappa^2)^2} \]  
(24)

It may seem unphysical that \( R_\infty \) and \( T_\infty \) are not the same as \( R_S \) and \( T_S \) but it is not: it is well known in electromagnetic wave theory \([13]\) that reflection off a transparent barrier of large but finite width (with 2 sides) is different from reflection off a transparent step (with 1 side). The square barrier thus demonstrates Klein tunnelling but it now arises in a more physical problem than the Klein step. The zero of potential is properly defined for a barrier whereas it is arbitrary for a step and the energy spectrum of a barrier (which attracts positrons) or well (which attracts electrons) is easily calculable. Particle emission from a barrier or well is described by supercriticality: the condition when the ground state energy of the system overlaps with the continuum \( (E = m \) for a barrier; \( E = -m \) for a well) and so any connection between particle emission and the time-independent scattering coefficients \( R \) and \( T \) can be investigated.
B. Fermionic Emission from a Narrow Well

We discussed the field theoretic treatment of this topic in a previous paper [14] which we refer to as CDI. We quickly review the argument of that paper. Spontaneous fermionic emission is a non-static process and in the case of a seemingly static potential, it is necessary to ask how the potential was switched on from zero. We follow CDI in turning on the potential adiabatically. We will consider the square well

\[ V(x) = -V, |x| < a; V(x) = 0, |x| > a \]  \hspace{1cm} (25)

but it is easiest to begin with the very narrow potential \( V(x) = -\lambda \delta(x) \) which is the limit of a square well with \( \lambda = 2Va \). The bound states are then very simple: for a given value of \( \lambda \) there is just one bound state corresponding to either the even (e) or odd (o) wave functions [14] with energy given by

\[ E = m \cos \lambda \hspace{1cm} (e) \hspace{1cm} E = -m \cos \lambda \hspace{1cm} (o) \]  \hspace{1cm} (26)

When the potential is initially turned on and \( \lambda \) is small the bound state is even and its energy \( E \) is just below \( E = m \). As \( \lambda \) increases, \( E \) decreases and at \( \lambda = \pi/2 \), \( E \) reaches zero. For \( \lambda > \pi/2 \), \( E \) becomes negative. Assuming that we started in the vacuum state and therefore that the well was originally vacant, we now have for \( \lambda > \pi/2 \) the absence of a negative energy state which must be interpreted as the presence of a (bound) positron according to Dirac’s hole theory. Let \( \lambda \) increase further and \( E \) decreases further until at \( \lambda = \pi \), \( E = -m \) which is the supercriticality condition. So for \( \lambda > \pi \), the bound positron acquires sufficient energy to escape from the well. This is the phenomenon of spontaneous positron production as described originally by Gershtein and Zeldovich [8] and Pieper and Greiner [9]. Note that this picture requires that positrons (as well as electrons) are bound by potential wells when the potential strength is large enough: we return to this point later when we discuss the Coulomb potential.
C. Digression on Vacuum Charge

How is it possible to conserve charge and produce positrons out of the vacuum? This question has been a fruitful ground for theorists in recent years. The key point is that the definition of the vacuum state of the system (and of the other states) depends on the background potential: this leads to the concept of vacuum charge [16], [17]. At this point a single particle interpretation of a potential in the Dirac equation is insufficient and field theory becomes necessary (as is also seen in the discussion of radiation from the Klein step in the Appendix). But nevertheless it turns out that once the concept of vacuum charge is introduced, first quantisation is all that is necessary to determine its value. We shall refer the reader to CDI for a proper treatment of vacuum charge; we just write down the essential equations here.

The total charge is defined by (according to our conventions the electron charge is $-1$)

$$ Q(t) = \int dx \rho(x, t) = -\frac{1}{2} \int dx \left[ \psi^\dagger(x, t), \psi(x, t) \right] $$

Writing the wave function $\psi(x, t)$ in terms of creation and annihilation operators we eventually find that

$$ Q = Q_p + Q_0 $$

where the particle charge $Q_p$ is an operator which counts the number of electrons in a state minus the number of positrons while the vacuum charge $Q_0$ is just a number which is defined by the difference in the number of positive energy and negative energy states of the system:

$$ Q_0 = \frac{1}{2} \left\{ \sum_k \text{(states with } E > 0) - \sum_k \text{(states with } E < 0) \right\} $$

Given the definition of the vacuum we immediately get

$$ \langle 0 | Q | 0 \rangle = Q_0 $$
We illustrate the use of the vacuum charge by returning to the delta function potential \( V(x) = -\lambda \delta(x) \). For \( \lambda \) just larger than \( \pi/2 \), \( Q_p = +1 \) because a positron has been created, but now the vacuum charge \( Q_0 = -1 \) because the number of positive energy states has decreased by one while the number of negative energy states has increased by one. So the total charge \( Q \) is in fact conserved. As the potential is increased further, \( \lambda \) will reach \( \pi \), where \( E = -m \) and the bound positron reaches the continuum and becomes free. Note that at supercriticality, there is no change in vacuum charge; the change occurs when \( E \) crosses the zero of energy. Note also that at supercriticality the even bound state disappears and the first odd state appears.

We can continue to increase \( \lambda \) and count positrons: the total number of positrons produced for a given \( \lambda \) is the number of times \( E \) has crossed \( E = 0 \); that is

\[
Q_p = Int\left[\frac{\lambda}{\pi} + \frac{1}{2}\right]
\]

and \( Q_0 = -Q_p \) where \( Int[x] \) denotes the integer part of \( x \). For positron emission the more interesting quantity is the number of supercritical positrons \( Q_S \), that is the number of states which have crossed \( E = -m \). This is given by

\[
Q_S = Int\left[\frac{\lambda}{\pi}\right]
\]

**D. Wide Well**

We can now return to the case that we are interested in which is that of a wide well or barrier. So let us consider the general case of a square well potential of strength \( V > 2m \) and then look at a wide well for which \( ma >> 1 \) most closely corresponding to the Klein step. We follow the discussion given in our papers CDI and CD [3]. We must find first the condition for supercriticality and then the number of bound and supercritical positrons produced for a given \( V \).
The bound state spectrum for the well \( V(x) = -V, |x| < a; V(x) = 0, |x| > a \) is easily obtained: there are even and odd solutions given by the equations

\[
\tan pa = \frac{\sqrt{(m - E)(E + V + m)}}{(m + E)(E + V - m)} \quad (33)
\]

\[
\tan pa = -\frac{\sqrt{(m + E)(E + V + m)}}{(m - E)(E + V - m)} \quad (34)
\]

where now the well momentum is given by \( p^2 = (E + V)^2 - m^2 \). We have changed the sign of \( V \) so that it is now attractive to electrons rather than positrons in order to conform with other authors who have studied supercritical positron emission rather than electron emission.

From Eq (33) we see that the ground state becomes supercritical when \( pa = \pi/2 \) and therefore \( V_1^c = m + \sqrt{m^2 + \pi^2/4a^2} \). From Eq (34) the first odd state becomes supercritical when \( pa = \pi \) and \( V_2^c = m + \sqrt{m^2 + \pi^2/a^2} \). Clearly the supercritical potential corresponding to the Nth positron is

\[
V_N^c = m + \sqrt{m^2 + N^2\pi^2/4a^2} \quad (35)
\]

It follows from Eq (35) that \( V = 2m \) is an accumulation point of supercritical states as \( ma \to \infty \). Furthermore it is a threshold: a potential \( V \) is subcritical if \( V < 2m \). It is not difficult to show for a given \( V > 2m \) that the number of supercritical positrons is given by

\[
Q_S = \text{Int}\left[\left(2a/\pi\right)\sqrt{V^2 - 2mV}\right] \quad (36)
\]

The corresponding value of the total positron charge \( Q_p \) can be shown using Eqs (33,34) to satisfy

\[
Q_p - 1 \leq \text{Int}\left[\left(2a/\pi\right)\sqrt{V^2 - m^2}\right] \leq Q_p \quad (37)
\]

so for large \( a \) we have the estimates.
\[ Q_p \sim (2a/\pi)\sqrt{V^2 - m^2}; \quad Q_S \sim (2a/\pi)\sqrt{V^2 - 2mV} \] (38)

Now we can build up an overall picture of the wide square well \( ma >> 1 \). When \( V \) is turned on from zero in the vacuum state an enormous number of bound states is produced. As \( V \) crosses \( m \) a very large number \( Q_p \) of these states cross \( E = 0 \) and become bound positrons. As \( V \) crosses \( 2m \) a large number \( Q_S \) of bound states become supercritical together. This therefore gives rise to a positively charged current flowing from the well. But in this case, unlike that of the Klein step, the charge in the well is finite and therefore the particle emission process has a finite lifetime. Nevertheless, for \( ma \) large enough the transient positron current for a wide barrier is approximately constant in time for a considerable time as we shall see in the next section.

**E. Emission Dynamics**

We now restrict ourselves to the case \( V = 2m + \Delta \) with \( \Delta << m \). This is not necessary but it avoids having to calculate the dynamics of positron emission while the potential is still increasing beyond the critical value. We can assume all the positrons are produced almost instantaneously as the potential passes through \( V = 2m \). It also means that the kinematics are non-relativistic. Hence for a sufficiently wide well so that \( \Delta a \) is large, \( Q_S \sim (2a/\pi)\sqrt{2m\Delta} \). The well momentum of the Nth supercritical positron is still given by Eq \( (23) \) \( p_N a = N\pi/2 \) which corresponds to an emitted positron energy \( |E_N| = 2m + \Delta - \sqrt{p_N^2 + m^2} > m \). Note that the emitted energies have discrete values although for \( a \) large, they are closely spaced.

The lifetime \( \tau \) of the supercritical well is given by the time for the slowest positron to get out of the well. The slowest positron is the deepest lying state with \( N = 1 \) and momentum \( p_1 = \pi/2a \). Hence \( \tau \approx ma/p_1 = 2ma^2/\pi \). So the lifetime is finite but scales as \( a^2 \). But a large number of positrons will have escaped well before \( \tau \). There are \( Q_S \) supercritical positrons
initially and their average momentum $\overline{p}$ corresponds to $N = Q_s/2$; hence $\overline{p} = \sqrt{m\Delta}/2$ which is independent of $a$. Thus a transient current of positrons is produced which is effectively constant in time for a long time of order $\tau = a\sqrt{2m/\Delta}$. We thus see that the square well (or barrier) for $a$ sufficiently large behaves just like the Klein step: it emits a seemingly constant current with a seemingly continuous energy spectrum. But initially the current must build up from zero and eventually must return to zero. So the well/barrier is a time-dependent physical entity with a finite but long lifetime for emission of supercritical positrons or electrons.

Note again that the transmission resonances of the time-independent scattering problem coincide with the energies of particles emitted by the well or barrier. It is therefore tempting to use the Pauli principle to explain the connection. Following Hansen and Ravndal [11], we could say that $R$ must be zero at the resonance energy because the electron state is already filled by the emitted electron with that energy. But it is easy to show that the reflection coefficient is zero for bosons as well as fermions of that energy, and no Pauli principle can work in that case. Furthermore emission ceases after time $\tau$ whereas $R = 0$ for times $t > \tau$. It follows that we must conclude that Klein tunnelling is a physical phenomenon in its own right, independent of any emission process. It seems that Klein tunnelling is indeed distinct from the particle emission process: to show this is so we return to the square barrier to show that Klein tunnelling occurs even when the barrier is subcritical.

**F. Klein Tunnelling and the Coulomb Barrier**

It is clear from Eq (21) that while the reflection coefficient $R$ for a square barrier cannot be 0, neither is the transmission coefficient $T$ exponentially small for energies $E < V$ when $V > 2m$ even though the scattering is classically forbidden. The simplest way to understand this is to consider the negative energy states under the potential barrier as corresponding to physical particles which can carry energy in exactly the same way that positrons are
described by negative energy states which can carry energy. It follows from Eq (2) that $R_S$ and $T_S$ correspond to reflection and transmission coefficients in transparent media with differing refractive indices: thus $\kappa$ is nothing more than an effective fermionic refractive index corresponding to the differing velocities of propagation by particles in the presence and absence of the potential. On this basis, tuning the momentum $p$ to obtain a transmission resonance for scattering off a square barrier is nothing more than finding the frequency for which a given slab of refractive material is transparent. This is not a new idea. In Jensen’s words ”A potential hill of sufficient height acts as a Fabry-Perot etalon for electrons, being completely transparent for some wavelengths, partly or completely reflecting for others” [18].

We can now look in more detail at Klein tunnelling: both in terms of our model square well/barrier problem and at the analogous Coulomb problem. The interesting region is where the potential is strong but subcritical so that emission dynamics play no role and sensible time independent scattering parameters can be defined. For electron scattering off the square barrier $V(x) = V$ we would thus require $V < V_1^c = m + \sqrt{m^2 + \pi^2/4a^2}$ together with $V > 2m$ so that positrons can propagate under the barrier. For the corresponding square well $V(x) = -V$ there are negative energy bound states $0 > E > -m$ provided that $V > \sqrt{m^2 + \pi^2/4a^2}$ [cf. Eq.(37)]. So when the potential well is deep enough, it will in fact bind positrons. Correspondingly, a high barrier will bind electrons. It is thus not surprising that electrons can tunnel through the barrier for strong subcritical potentials since they are attracted by those potential barriers. Another way of seeing this phenomenon is by using the concept of effective potential $V_{eff}(x)$ which is the potential which can be used in a Schrödinger equation to simulate the properties of a relativistic wave equation. For a potential $V(x)$ introduced as the time-component of a four-vector into a relativistic wave equation (Klein-Gordon or Dirac), it is easy to see that $2mV_{eff}(x) = 2EV(x) - V^2(x)$. Hence as the energy $E$ changes sign, the effective potential can change from repulsive to attractive.
For the pure Coulomb potential, it is well known that there is exponential suppression of the wave functions for a repulsive potential compared with an attractive potential. For example, if $\rho = |\psi(0)|_{\text{pos}}^2 / |\psi(0)|_{\text{el}}^2$ is the ratio of the probability of a positron penetrating a Coulomb barrier to reach the origin compared with the probability of an electron of the same energy, then if the particles are non-relativistic

$$\rho = e^{-2\pi Z\alpha E/p}$$  \hspace{1cm} (39)

where $p$ and $E$ are the particle momenta and energies and this is exponentially small as $p \to 0$ \cite{19}. But if the particles are relativistic \cite{20}

$$\rho = fe^{-2\pi Z\alpha}$$  \hspace{1cm} (40)

where $f$ is a ratio of complex gamma-functions and is approximately unity for large $Z$. So $\rho \sim e^{-2\pi Z\alpha} \approx 10^{-3}$ for $Z\alpha \sim 1$ which is not specially small although it still decreases exponentially with $Z$.

In order to demonstrate Klein tunnelling for a Coulomb potential we require first the inclusion of nuclear size effects so that the potential is not singular at $r = 0$ and second that $Z$ is large enough so that bound positron states are present. This means that $Z$ must be below its supercritical value $Z_c$ of around 170 but large enough for the $1s$ state to have $E < 0$. The calculations of references \cite{8} and \cite{9} which depend on particular models of the nuclear charge distribution give this region as $150 < Z < Z_c$ which unfortunately will be difficult to demonstrate experimentally. Nevertheless, the theory seems to be clear: in this subcritical region positrons should no longer obey a tunnelling relation which decreases exponentially with $Z$ such as that of Eq. (40). Instead the Coulomb barrier should become more transparent as $Z$ increases, at least for low energies. By analogy with the square barrier we may expect that maximal transmission for positron scattering on a Coulomb potential should occur around $Z = Z_c$ although the onset of supercriticality implies that time independent scattering quantities may no longer be well-defined. We are now carrying out further de-
tailed calculations to clarify the situation for positron scattering off nuclei with $Z$ near $Z_c$ to see if we can simulate Klein tunnelling.

III. CONCLUSIONS

It seems that Klein was very unfortunate in that the potential step he considered is pathological and therefore a misleading guide to the underlying physics. Klein’s step represents a limit in which time-dependent emission processes become time-independent and therefore a relationship between the emitted current and the transmission coefficient exists, as we show in the Appendix. In general no direct relationship would exist between the transient current emitted and the time-independent transmission coefficient. The physics of the Dirac equation which underlies Klein’s result is rich: it includes spontaneous fermionic production by strong potentials and the separate phenomenon of Klein tunnelling by means of the negative energy states characteristic of relativistic wave equations, similar to interband tunnelling in semiconductors [21]. Spontaneous positron production due to supercriticality has not yet been unambiguously demonstrated experimentally in heavy ion collisions but experiments on superfluid $^3$He-B [22], [23] have displayed anomalous effects when the velocity of a body moving in the fluid exceeds the critical Landau velocity $v_L$. These experiments have now been interpreted in the same way as supercritical positron production [24]. It may well be that fermionic many-body systems can be used to demonstrate the fundamental quantum processes which Klein unearthed seventy years ago.

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IV. APPENDIX: PAIR PRODUCTION BY A STEP POTENTIAL

Consider the Klein step of Eq. (1) for $V > 2m$. We will show that the expectation value of the current in the vacuum state in the presence of the step is non-zero which means that
the Klein step produces electron-positron pairs out of the vacuum at a constant rate. The derivation hinges on a careful definition of the vacuum state. We use the derivation of CD2 [3].

**A. The normal modes in the presence of the Klein step.**

An energy-normalised positive energy or particle solution to the Dirac equation can be written from eq. (12)

\[
\sqrt{\frac{\varepsilon + m}{2k}} \begin{pmatrix} \frac{i}{k} \\ \frac{E + m}{E + m} \end{pmatrix} e^{ikx}
\]  

A negative energy or hole solution reads

\[
\sqrt{\frac{\varepsilon - m}{2k}} \begin{pmatrix} \frac{i}{k} \\ \frac{E + m}{E + m} \end{pmatrix} e^{ikx}
\]

Scattering is usually described by a solution describing a wave incident (say from the left) plus a reflected wave (from the right) plus a transmitted wave (to the right). It is convenient here to use waves of different form either describing a wave (subscript \( L \)) incident from the left with no reflected wave or describing a wave (subscript \( R \)) incident from the right with no reflected wave. Particle and hole wavefunctions will be denoted by \( u \) and \( v \) respectively. It is clear that the nontrivial result we are seeking arises from the overlap of the hole continuum \( E < V - m \) on the right with the particle continuum \( E > m \) on the left. We are thus concerned with wavefunctions with energies in the range \( m < E < V - m \). The expressions for \( u_L, u_R \) in this energy range are given below.
\[ \sqrt{2\pi} u_L(E, x) = \frac{\sqrt{2}\kappa}{\kappa + 1} \left \{ \begin{array}{c} \frac{1}{E + m} \left( \frac{i}{k} \right) e^{ikx} \theta(-x) + \\
\frac{V - E - m}{2|p|} \left( \frac{i}{|p|} \right) e^{ip|x|} + \sqrt{\frac{V - E - m}{2|p|}} \left( \frac{i}{E + m - V} \right) e^{-ip|x|} \end{array} \right \} \theta(x) \]

\[ \sqrt{2\pi} u_R(E, x) = \left \{ \begin{array}{c} 1 - \frac{\kappa}{1 + \kappa} \frac{E + m}{2k} \left( \frac{i}{k} \right) e^{ikx} + \sqrt{\frac{E + m}{2k}} \left( \frac{i}{E + m} \right) e^{-ikx} \\
\frac{\sqrt{2}\kappa}{\kappa + 1} \frac{V - E - m}{|p|} \left( \frac{i}{|p|} \right) e^{ip|x|} \theta(x) \end{array} \right \} \theta(-x) + \]

We write \(|p|\) rather than \(p\) in these equations since the group velocity is negative for \(x > 0\) (cf. Eq. (15)).

We need to evaluate the currents corresponding to the solutions of Eqs (42,43). According to our conventions \(\alpha_x = \gamma_0 \gamma_x = -\sigma_y\) so

\[ j_L \equiv -u_L^\dagger(E, x) \sigma_y u_L(E, x) = -\frac{2\kappa/\pi}{(\kappa + 1)^2} \] (44)

\[ j_R \equiv -u_R^\dagger(E, x) \sigma_y u_R(E, x) = -\frac{2\kappa/\pi}{(\kappa + 1)^2} \] (45)

B. The definition of the vacuum and the vacuum expectation value of the current.

Now expand the wave function \(\psi\) in terms of creation and annihilation operators which refer to our left- and right-travelling solutions:
\[
\psi(x,t) = \int dE \{ a_L(E)u_L(E,x)e^{-iEt} + a_R(E)u_R(E,x)e^{-iEt} + \\
+ b_L^\dagger(E)v_L(E,x)e^{iEt} + b_R^\dagger(E)v_R(E,x)e^{iEt} \} 
\]  

(46)

with \(\psi^\dagger\) given by the Hermitian conjugate expansion. We must now determine the appropriate vacuum state in the presence of the step. States described by wavefunctions \(u_L(E,x)\) and \(v_L(E,x)\) correspond to (positive energy) electrons and positrons respectively coming from the left. Hence with respect to an observer to the left (of the step) such states should be absent from the vacuum state, so

\[
a_L(E)|0\rangle = 0, \quad b_L(E)|0\rangle = 0 
\]  

(47)

Wavefunctions \(u_R(E,x)\) for \(E > m + V\) describe for an observer to the right, electrons incident from the right. These are not present in the vacuum state hence

\[
a_R(E)|0\rangle = 0 \text{ for } E > m + V 
\]  

(48)

Wavefunctions \(v_R(E,x)\) describe, again with respect to an observer to the right, positrons incident from the right; again

\[
b_R(E)|0\rangle = 0 
\]  

(49)

The wavefunctions that play the crucial role in the Klein problem belong to the set \(u_R(E,x)\) for \(m < E < V - m\). For an observer to the right these states are positive energy positrons and hence they should be filled in the vacuum state, i.e.

\[
a_R^\dagger(E)a_R(E')|0\rangle = \delta(E - E')|0\rangle, \quad m < E < V - m 
\]  

(50)

Having specified the vacuum the next and final step is the calculation of the vacuum expectation value of the current:

\[
\langle 0 | j | 0 \rangle = \frac{1}{2} \left( -\langle 0 | \psi^\dagger \sigma_y \psi | 0 \rangle + \langle 0 | \psi \sigma_y \psi^\dagger | 0 \rangle \right) 
\]  

(51)

Substituting (46) in (51) and noticing that all terms involving \(v_L\) and \(v_R\) can be dropped since the corresponding energies lie outside the interesting range \(m < E < V - m\) we end up with
\[ \langle 0 | j | 0 \rangle = -\frac{1}{2} \int dE dE' \{ \langle 0 | a_L^\dagger(E) a_L(E') | 0 \rangle u_L^\dagger(E, x) \sigma_y u_L(E', x) + \\
+ \langle 0 | a_L(E) a_L^\dagger(E') | 0 \rangle u_L^\dagger(E', x) \sigma_y u_L(E, x) - \langle 0 | a_R^\dagger(E) a_R(E') | 0 \rangle u_R^\dagger(E, x) \sigma_y u_R(E', x) + \\
+ \langle 0 | a_R(E) a_R^\dagger(E') | 0 \rangle u_R^\dagger(E', x) \sigma_y u_R(E, x) \} \]

The first term in (52) vanishes due to (47). The second term becomes

\[ u_L^\dagger(E', x) \sigma_y u_L(E, x) \delta(E - E') \]

if we use the anticommutation relations and (47). The third term yields \(-u_R^\dagger(E, x) \sigma_y u_R(E, x) \delta(E - E')\) using (50) and the fourth term vanishes using the anticommutation relations (i.e. the exclusion principle; the state \(|0\rangle\) already contains an electron in the state \(u_R\) hence we get zero when we operate on it with \(a_R^\dagger\)). One energy integration is performed immediately using the \(\delta\) function. We obtain

\[ \langle 0 | j | 0 \rangle = \frac{1}{2} \int dE ( - j_L + j_R ) = -\frac{1}{2\pi} \int dE \frac{4\kappa(E)}{(\kappa(E) + 1)^2} = -\frac{1}{2\pi} \int dE T_S(E) \]

(53)

where the energy integration is over the Klein range \(m < E < V - m\).

It is now straightforward to generalise Eq (53) to any step potential for which \(V(x < 0) = V_1\); \(V(x > L) = V_2\) and \(V_2 - V_1 > 2m\) such as those considered by Sauter and Hund to obtain Eq (19) linking the pair production current with the transmission coefficient.

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FIG. 1. The potential $V(x)$ of the Klein step
FIG. 2. An electron of energy $E$ scattering off a Klein step of height $V > 2m$. The electrons are shown with solid arrowheads; the hole state has a hollow arrowhead. The particle continuum (slant background) and the hole continuum (shaded background) overlap when $m < E < V - m$. 
FIG. 3. A potential $V(x)$ of the Sauter form representing constant electric field in the region $0 < x < L$. 
FIG. 4. A potential $V(x)$ representing a square barrier of height $V$ in the region $-a < x < a$. 