The Cauchy problem for the Klein-Gordon equation under the quartic potential in the de Sitter spacetime

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Abstract

The Cauchy problem for the Klein-Gordon equation under the quartic potential is considered in the de Sitter spacetime. The existence of the global solution is shown based on the mechanism of the spontaneous symmetry breaking for the small positive Hubble constant. The effects of the spatial expansion and contraction on the problem are considered.

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1 Introduction

We consider the Cauchy problem for the Klein-Gordon equation under the quartic potential in the de Sitter spacetime. The de Sitter spacetime is the solution of the Einstein equations with the cosmological constant in the vacuum under the cosmological principle. We use the following convention. Let \( n \geq 1 \). The Greek letters \( \alpha, \beta, \gamma, \cdots \) run from 0 to \( n \), and the Latin letters \( j, k, \ell, \cdots \) run from 1 to \( n \). We use the Einstein rule for the sum of indices, namely, the sum is taken for the same upper and lower repeated indices, for example, \( T^\alpha_\alpha := \sum_{\alpha=0}^{n} T^\alpha_\alpha \) and \( T^i_j := \sum_{j=1}^{n} T^i_j \) for any tensor \( T^\alpha_\beta \). Let \( H \in \mathbb{R} \) be the Hubble constant, \( c > 0 \) be the speed of the light. The de Sitter spacetime that we consider in this paper is the spacetime with the metric \( \{ g_{\alpha\beta} \} \) given by

\[-c^2(d\tau)^2 = g_{\alpha\beta}dx^\alpha dx^\beta := -c^2(dx^0)^2 + e^{2Hx^0} \sum_{j=1}^{n}(dx^j)^2, \tag{1.1}\]

where we have put the spatial curvature as 0, the variable \( \tau \) denotes the proper time, \( x^0 = t \) is the time-variable (see e.g., [8, 9]). When \( H = 0 \), the spacetime with (1.1) reduces to the Minkowski spacetime.

For the imaginary number \( i \) with \( i^2 = -1 \), let \( m_* \in \mathbb{R} \cup i\mathbb{R} \) denote the real mass when \( m_* \in \mathbb{R} \), the imaginary mass when \( m_* \in i\mathbb{R} \). Let \( h > 0 \) denote the

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reduced Planck constant. The Lagrangian density $L$ of the Klein-Gordon field with a potential $V$ given by

$$L(\phi) := -\frac{1}{2} \overline{\theta^a} \partial_a \phi - V(\phi), \quad V(\phi) := \frac{1}{2} \left( \frac{m_* c}{h} \right)^2 |\phi|^2 + \frac{\lambda}{p + 1} |\phi|^{p+1},$$

yields the semilinear Klein-Gordon equation

$$\left( \frac{1}{c^2} \partial_t^2 \phi + \frac{nH}{c^2} \partial_t \phi - e^{-2Ht} \Delta \phi + \left( \frac{m_* c}{h} \right)^2 \phi + \lambda |\phi|^{p-1} \phi \right) (t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

for $(t, x) \in [0, T] \times \mathbb{R}^n$, where $0 < T \leq \infty$, $1 < p < \infty$, $\lambda \in \mathbb{R}$, $\Delta$ denotes the Laplacian defined by $\Delta := \sum_{j=1}^n (\partial / \partial x_j)^2$, $\partial^\alpha := g^{\alpha \beta} \partial_\beta$, and the matrix $(g^{\alpha \beta})$ denotes the inverse matrix $(g_{\alpha \beta})$. The equation (1.3) is rewritten as

$$c^{-2} \partial_t^2 u - e^{-2Ht} \Delta u + \left( \frac{m_* c}{h} \right)^2 - \left( \frac{nH}{2c} \right)^2 u + \lambda e^{-n(p-1)Ht/2} |u|^{p-1} u = 0 \quad (1.4)$$

by the change of $\phi$ to $u := e^{nHt/2} \phi$.

Let us consider the real mass $m_* \in \mathbb{R}$ on the Cauchy problem of (1.3) for data $\phi_0(\cdot) := \phi(0, \cdot)$ and $\phi_1(\cdot) := \partial_t \phi(0, \cdot)$. Yagdjian [29] has shown small global solutions for (1.3), provided that the norm of initial data $||\phi_0||_{H^s(\mathbb{R}^n)} + ||\phi_1||_{H^s(\mathbb{R}^n)}$ is sufficiently small for $s > n/2 \geq 1$ (see also [30] for the system of the equations), where $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order $s$. Galstian and Yagdjian [14, 32] have extended this result to the case of the Riemann metric space for each time slices. In [20], the energy solutions for $\phi_0 \in H^1(\mathbb{R}^n)$ and $\phi_1 \in L^2(\mathbb{R}^n)$ have been shown, which was extended to the case of general Friedmann-Lemaître-Robertson-Walker spacetime in [13]. Baskin [4] has shown small global solution for the equation $(\Box g + \lambda) \phi + f(\phi) = 0$ when $f(\phi)$ is a type of $|\phi|^{p-1} \phi$, $p = 1 + 4/(n - 1)$, $\lambda > n^2/4$, $\phi_0 \in H^1$ and $\phi_1 \in L^2$, where $g$ denotes the metric of the asymptotic de Sitter spacetime and $\Box g$ denotes the d’Alembertian on $g$ (see also [3] in the cases $\lambda = (n^2 - 1)/4$, $p = 5$ with $n = 3$, $p = 3$ with $n = 4$). This result was further investigated on the semilinear term including the derivatives of the solution by Hintz and Vasy [6]. We refer to [26] on numerical computations for the semilinear Klein-Gordon equation, and [21, 23] on the Cauchy problem for the semilinear Schrödinger equation and the semilinear Proca equation in the de Sitter spacetime.

On the blowing-up solution of (1.4) with the gauge variant semilinear term, Yagdjian [27, Theorem 1.1] considered the equation

$$\partial_t^2 u(t, \cdot) - e^{-2t} \Delta u(t, \cdot) + \left( m_*^2 - \left( \frac{n}{2} \right)^2 \right) u(t, \cdot) - \Gamma(t) \left( \int_{\mathbb{R}^n} |u(t, y)|^p dy \right)^\beta |u(t, \cdot)|^p = 0$$

under the normalization of $H = c = h = 1$, and has shown that $\int_{\mathbb{R}^n} u \, dx$ blows up in finite time for some small data when $0 \leq m_* \leq n/2$ and the non-decreasing or non-increasing function $\Gamma \in C^1((0, \infty))$ satisfies

$$\Gamma(t) \geq \begin{cases} C e^{-\sqrt{(n/2)^2 - m_*^2 (p(\beta + 1) - 1) t^2 + \epsilon}} & \text{if } m_* < \frac{n}{2}, \\ C t^{-1 - p(\beta + 1)} & \text{if } m_* = \frac{n}{2}. \end{cases} \quad (1.5)$$
for $t \geq 0$, where $p > 1$, $\beta > 1/p - 1$, $\epsilon > 0$ and $C > 0$. We remark that when $\beta = 0$, the weighted function $\Gamma(t) = e^{-n(p-1)t/2}$ in (1.4) can be taken for $m_*^2 < 0$, namely, for the purely imaginary number $m_* \in i\mathbb{R}$ with $m \neq 0$. He has also shown the estimate of the existence time of the solution from below in the Sobolev space $H^s(\mathbb{R}^n)$ under the conditions $s > n/2 \geq 1$ in [22] Theorem 0.1 when $H = 1$, and in [15, Theorem 0.2] when $H = -1$ with Galstian (see also the references in the summary [31]). We refer to [2] for the numerical simulation.

Firstly, we give the following result on the blowing-up solution in finite time under some conditions on data for the imaginary mass $m_* \in i\mathbb{R}$. We extend it to the case $H < 0$ and $m_* = 0$.

**Proposition 1.1.** Let $n \geq 1$, $1 < p < \infty$, $m_* \in i\mathbb{R}$. Let $H > 0$ or $H < 0$. Let $u_0, u_1 \in L^1(\mathbb{R}^n)$ be the functions which have compact supports and satisfy $w_0 := \int_{\mathbb{R}^n} u_0(x)dx > 0$, $w_1 := \int_{\mathbb{R}^n} u_1(x)dx > 0$, and

$$w_1 \geq cw_0 \left\{ -\left(\frac{m_* c}{h}\right)^2 + \left(\frac{nH}{2c}\right)^2 \right\}^{1/2}.$$

Let $w_0 > 0$ when $m_* = 0$. Then the solution of the Cauchy problem

$$\begin{cases}
e^{-2\partial_t^2 u - e^{-2Ht} \Delta u + \left\{ \left(\frac{m_* c}{h}\right)^2 - \left(\frac{nH}{2c}\right)^2 \right\} u - e^{-n(p-1)Ht/2}|u|^p = 0, \\
u(0, \cdot) = u_0(\cdot), \quad \partial_t u(0, \cdot) = u_1(\cdot)
\end{cases}$$

(1.6)

blows up in finite time in the space $L^1(\mathbb{R}^n)$. Namely, $\int_{\mathbb{R}^n} u(t, x)dx$ blows up as $t \to T$ for some positive number $0 < T < \infty$.

We have shown the blowing-up solution for the gauge variant semilinear term with the negative sign in (1.6). Next, we consider the gauge invariant case with the positive sign $\lambda > 0$, and we show the global solution for small data when $H > 0$ is small. Put $m_* = im$ in (1.2) with $m \in \mathbb{R}$, $p = 3$ and $\lambda > 0$. The potential $V$ is rewritten as

$$V(\phi) = -\frac{1}{2} \left(\frac{mc}{h}\right)^2 |\phi|^2 + \frac{\lambda}{4} |\phi|^4.$$

This potential is known as the double well potential or the Mexican hat potential, and it has the minimum when

$$|\phi| = r_0 := \frac{|m|c}{\sqrt{\lambda h}},$$

(1.7)

while $\phi = 0$ gives the local maximum for $m \neq 0$. It is expected that the solution around $\phi = 0$ is unstable, and it is stable around $|\phi| = r_0$, which causes the spontaneous symmetry breaking from $\phi = 0$ to $|\phi| = r_0$. In this paper, we characterize this breaking from the viewpoint of the Cauchy problem of (1.3) which is rewritten as

$$\begin{cases}
\frac{1}{c^2} \partial_t^2 \phi + \frac{nH}{c^2} \partial_t \phi - e^{-2Ht} \Delta \phi - \left(\frac{mc}{h}\right)^2 \phi + \lambda |\phi|^2 \phi = 0
\end{cases}$$

(1.8)
which has the potential $V(\phi) = e^{-2Ht}\Delta \phi + \lambda \{ |\phi|^2 + r_0(2\phi \Re \phi + |\phi|^2) + 2r_0^2 \Re \phi \}$

by the shift $\phi \to \phi + r_0$ (see Lemma 2.2 below). This equation is rewritten as

$$c^{-2} \partial_t^2 u - e^{-2Ht} \Delta u - \left( \frac{nh}{2c} \right)^2 u + 2 \left( \frac{mc}{\hbar} \right)^2 \Re u + h(u) = 0$$

(1.9)

by the change of the function $u = e^{\eta_4 Ht/2} \phi$ (see Lemma 2.3), where we have put

$$h(u) := \lambda |u|^2 u e^{-\eta_4 Ht} + \lambda r_0(2u \Re u + |u|^2) e^{-\eta_4 Ht/2}.$$  

(1.10)

We consider the Cauchy problem of (1.9) for the initial data given by $u(0, \cdot) = u_0(\cdot)$ and $\partial_t u(0, \cdot) = u_1(\cdot)$, and we show the problem is well-posed. We observe how the Hubble constant affects on the problem.

The mechanism of the spontaneous symmetry breaking is used in the study of phase transitions (see [18]). Faccioli and Salasnich [12] considered it for the Gross-Pitaevskii equation and also for the cubic nonlinear Klein-Gordon equation, and studied the spectrum of the superfluid phase of bosonic gases. Honda and Choptuik [17] considered the monotonically increasingly boosted coordinates with $n = 3$ in (1.8) to study localized and unstable solutions like oscillon. The equation (1.8) reduces to the equation

$$\partial_t^2 \phi - \partial_x^2 \phi - \phi + \phi^3 = 0,$$

which has the potential $V(\phi) = -\phi^2/2 + \phi^4/4$, when $H = 0$ and $n = m = c = h = \lambda = 1$ for the real-valued function $\phi$. This equation has the kink solution $\phi = \tanh(x/\sqrt{2})$ and appears in the $\phi^4$-theory, which has been considered in the statistical mechanics, the condensed-matter physics, and the topological quantum field theory (see [11] [17][25]).

We denote the Lebesgue space by $L^q(I)$ for an interval $I \subset \mathbb{R}$ and $1 \leq q \leq \infty$ with the norm

$$\|u\|_{L^q(I)} := \left\{ \int_I |u(t)|^q dt \right\}^{1/q} \text{ if } 1 \leq q < \infty,$$

$$\text{ess.sup}_{t \in I} |u(t)| \text{ if } q = \infty.$$  

We use the Sobolev space $H^\mu(\mathbb{R}^n)$, the homogeneous Sobolev space $\dot{H}^\mu(\mathbb{R}^n)$, and the homogeneous Besov space $\dot{B}^\mu_r(\mathbb{R}^n)$ of order $\mu \geq 0$ for $1 \leq r, s \leq \infty$ (see [6] for their definitions).

We consider the case $H \geq 0$. For $\mu \geq 0$, $0 < T \leq \infty$ and $Q \geq 0$, define the function space $X^\mu$ given by

$$\|u\|_{\dot{X}^\mu(T)} := c^{-1} \|\partial_t u\|_{L^{\infty}((0,T),H^\mu(\mathbb{R}^n))} + \|e^{-Ht} \nabla u\|_{L^{\infty}((0,T),\dot{H}^\mu(\mathbb{R}^n))}$$

$$+ \sqrt{Q} \|u\|_{L^{\infty}((0,T),H^\mu(\mathbb{R}^n))} + \sqrt{H} \|e^{-Ht} \nabla u\|_{L^{2}((0,T),\dot{H}^\mu(\mathbb{R}^n))},$$

$$X^\mu(T) := \{ u; \|u\|_{\dot{X}^\mu} < \infty \text{ for } \nu = 0, \mu_0, \mu \},$$

$$\dot{D}^\mu := c^{-1} \|u_1\|_{\dot{H}^\mu(\mathbb{R}^n)} + \|\nabla u_0\|_{\dot{H}^\mu(\mathbb{R}^n)} + \sqrt{Q} \|u_0\|_{\dot{H}^\mu(\mathbb{R}^n)}.$$  

We define the metric in $X^\mu(T)$ by $d(u,v) := \|u - v\|_{\dot{X}^\mu(T)}$ for $u, v \in X^\mu(T)$. We have the following results on the existence of local and global solutions, and the asymptotic behaviors of global solutions.
Theorem 1.2. Let \( n \geq 1, m \in \mathbb{R}, 0 \leq H < 2\sqrt{2}|m|e^2/nh \). Put
\[
Q := 2 \left( \frac{mc}{h} \right)^2 - \left( \frac{nH}{2c} \right)^2. \tag{1.11}
\]
Let \( \lambda > 0 \), and let \( r_0 \) be defined by (1.7). Let \( \mu_0 \) and \( \mu \) satisfy
\[
\max \left\{ 0, \frac{n - 3}{2} \right\} \leq \mu_0 < n/2, \quad \mu_0 \leq \mu < \infty. \tag{1.12}
\]
Then the following results hold.
(1) For any real-valued functions \( u_0 \in H^{\mu+1}(\mathbb{R}^n) \) and \( v_1 \in H^{\mu}(\mathbb{R}^n) \), there exist \( 0 < T < \infty \) and a unique solution \( u \in C([0, T), H^{\mu+1}(\mathbb{R}^n)) \cap C^1([0, T), H^{\mu}(\mathbb{R}^n)) \cap X^\mu(T) \) of (1.9).
(2) The solution \( u \) obtained in (1) is continuously dependent on the data. Namely, \( d(u, v) \to 0 \) as \( v_0 \to u_0 \) in \( H^1(\mathbb{R}^n) \) and \( v_1 \to u_1 \) in \( L^2(\mathbb{R}^n) \) for the solution \( v \) obtained in (1) for the data \( v_0 := v(0, \cdot) \) and \( v_1 := \partial_v v(0, \cdot) \).
(3) Assume that the following (i) or (ii) holds. Then the solution \( u \) obtained in (1) is the global solution if \( \hat{D}^{\mu_0} \) is sufficiently small. Namely, \( T = \infty \).
(i) \( H > 0, \mu_0 = 0, n \geq 4 \).
(ii) \( H > 0, \mu_0 > 0, n \geq 1 \).
(4) For the global solution \( u \) obtained in (3), put
\[
\begin{align*}
&u_{+0} := u_0 + e^2 \int_0^\infty K_1(s)h(u)(s)ds, \\
&u_{+1} := u_1 - e^2 \int_0^\infty K_0(s)h(u)(s)ds, \\
&u_+ := K_0(t)u_{+0} + K_1(t)u_{+1},
\end{align*}
\tag{1.13}
\]
where \( K_0 \) and \( K_1 \) are the operators defined by (2.13), below. Then \( u_{+0} \in H^{\mu}(\mathbb{R}^n), u_{+1} \in H^{\mu-1}(\mathbb{R}^n) \), and \( u \) has the asymptotic behaviors given by
\[
\|u(t) - u_+(t)\|_{H^{\mu-1}(\mathbb{R}^n)} \to 0, \quad \|\partial_t (u(t) - u_+(t))\|_{H^{\mu-1}(\mathbb{R}^n)} \to 0.
\]
(5) The solution \( u \) obtained in (1) is the global solution when \( H = 0 \) and \( \mu_0 = 0 \). Namely, \( T = \infty \).

Next, we consider the case \( H < 0 \). For \( \mu \geq 0, T > 0 \) and \( Q \geq 0 \), define the function space \( X^{\mu}(T) \) by
\[
\|u\|_{X^{\mu}(T)} := c^{-1}\|e^{HT}\partial_t u\|_{L^\infty((0, T), H^{\mu}(\mathbb{R}^n))} + \|\nabla u\|_{L^\infty((0, T), H^{\mu}(\mathbb{R}^n))}
\quad + \sqrt{Q}\|e^{HT}u\|_{L^\infty((0, T), H^{\mu}(\mathbb{R}^n))} + c^{-1}\sqrt{-H}\|e^{HT}\partial_t u\|_{L^2((0, T), H^{\mu}(\mathbb{R}^n))}
\quad + \sqrt{-HQ}\|e^{HT}u\|_{L^2((0, T), H^{\mu}(\mathbb{R}^n))}, \tag{1.14}
\]
\[
X^{\mu}(T) := \{u; \|u\|_{X^{\mu}} < \infty \text{ for } \nu = 0, \mu_0, \mu\},
\]
\[
\hat{D}^{\mu} := c^{-1}\|u_1\|_{H^{\mu}(\mathbb{R}^n)} + \|\nabla u_0\|_{H^{\mu}(\mathbb{R}^n)} + \sqrt{Q}\|u_0\|_{H^{\mu}(\mathbb{R}^n)}.
\]
We define the metric in \( X^{\mu}(T) \) by \( d(u, v) := \|u - v\|_{X^{\mu}(T)} \) for \( u, v \in X^{\mu}(T) \).
Theorem 1.3. Let \( n \geq 1, -2\sqrt{2}|m|c^2/nh < H < 0 \). Let \( Q \) be defined by (1.11). Let \( \lambda > 0 \), and let \( t_0 \) be defined by (1.7). Let \( \mu_0 \) and \( \mu \) satisfy (1.12).

(1) For any real-valued functions \( u_0 \in H^{\mu+1}(\mathbb{R}^n) \) and \( v_1 \in H^\mu(\mathbb{R}^n) \), there exist \( 0 < T < \infty \) and a unique solution \( u \in C([0,T), H^{\mu+1}(\mathbb{R}^n)) \cap C^1([0,T), H^\mu(\mathbb{R}^n)) \cap X^\mu(T) \) of (1.2). Here, \( T > 0 \) can be taken as the number which satisfies

\[
C\lambda c(-H)^{-1} \left\{ \left( \frac{e^{-4(1+\mu_0)HT} - 1}{4(1 + \mu_0)} \right)^{1/2} Q^{(n-3-2\mu_0)/2}C_0 \tilde{D} \mu_0 \right. \\
+ t_0 \left( \frac{e^{-2(1+\mu_0)HT} - 1}{2(1 + \mu_0)} \right)^{1/2} Q^{(n-4-2\mu_0)/4} \left\} \leq \frac{1}{2} \tag{1.15}
\]

for some universal constant \( C_0 > 0 \) and \( C > 0 \).

(2) The solution \( u \) obtained in (1) is continuously dependent on the data. Namely, \( d(u,v) \to 0 \) as \( v_0 \to 0 \) in \( H^1(\mathbb{R}^n) \) and \( v_1 \to 0 \) in \( L^2(\mathbb{R}^n) \) for the solution \( v \) obtained in (1) for the data \( v_0 := v(0, \cdot) \) and \( v_1 := \partial_t v(0, \cdot) \).

On Theorems 1.2 and 1.3, we remark that the estimate of the lifespan of time-local solutions from below has been shown in \cite{32} (iii) in Theorem 0.1] when \( H > 0 \), and in \cite{15} Theorem 0.2] when \( H < 0 \), for data with high regularity such as \( \mu > n/2 \geq 1 \). Theorem 1.2 shows the existence of global solutions for small rough data, and that the asymptotic behaviors are given by the free solutions defined by (1.13) when \( H = 0 \) or \( H > 0 \) is small, while Theorem 1.3 gives more explicit estimate of the lifespan of time-local solutions from below for rough data.

We denote the inequality \( A \leq CB \) for some constant \( C > 0 \) which is not essential for the argument by \( A \lesssim B \). This paper is organized as follows. In Section 2 we collect several results on the derivation of the first equation in (1.9) as the Euler-Lagrange equation from a Lagrangian in the de Sitter spacetime, the energy estimate of the equation, which are required to prove Proposition 1.1, Theorems 1.2 and 1.3 in Sections 3, 4 and 5 respectively.

2 Preliminaries

In this section, we collect several results to prove the results in the previous section. We introduce the following fundamental result with its proof since Lemma 2.2 is based on it.

Lemma 2.1. Let \( m \in \mathbb{R} \). Let \( \lambda \in \mathbb{R} \). Let \( H \in \mathbb{R} \), and put \( (g_{\alpha\beta}) := \text{diag}(-e^2, e^{2Ht}, \cdots, e^{2Ht}) \). Let \( L \) be the Lagrangian density defined by

\[
L(\phi) := -\frac{1}{2} \overline{\partial^\alpha \phi} \partial_\alpha \phi + \frac{1}{2} \left( \frac{mc}{h} \right)^2 |\phi|^2 - \frac{\lambda}{4} |\phi|^4.
\]

Let \( g \) denote the determinant of the matrix \( (g_{\alpha\beta}) \). Then the Euler-Lagrange equation of the action

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} L(\phi) \sqrt{-g} dx dt
\]

is given by (1.8).
Proof. Since we have
\[
\delta(\partial^2 \phi \partial_\alpha \phi) = 2 \partial_\alpha (\text{Re}(\delta \phi \partial^2 \phi)) - 2 \text{Re}(\delta \phi \partial_\alpha \partial^2 \phi),
\]
\[
\delta|\phi|^2 = 2 \text{Re}(\delta \phi \phi), \quad \delta|\phi|^4 = 4|\phi|^2 \text{Re}(\delta \phi \phi),
\]
we obtain
\[
\delta L(\phi) = -\partial_\alpha (\text{Re}(\delta \phi \partial_\alpha \phi)) + \text{Re}(\delta \phi E),
\]
where we have put
\[
E := \partial_\alpha \partial^2 \phi + (\frac{mc}{\hbar})^2 \phi - \lambda |\phi|^2 \phi.
\]
So that, we have
\[
\delta \int_{\mathbb{R}} \int_{\mathbb{R}^n} L(\phi) \sqrt{-g} dx dt = -\int_{\mathbb{R}} \int_{\mathbb{R}^n} \partial_\alpha (\text{Re}(\delta \phi \partial^2 \phi)) \sqrt{-g} dx dt
\]
\[
+ \int_{\mathbb{R}} \int_{\mathbb{R}^n} \text{Re}(\delta \phi F) \sqrt{-g} dx dt,
\]
where \( F \) is defined by
\[
F := E + \partial^2 \phi \frac{\partial_\alpha \sqrt{-g}}{\sqrt{-g}} = E - \frac{nH}{c^2} \partial_0 \phi,
\]
which yields the required equation \( F = 0 \) as the Euler-Lagrange equation. \( \square \)

**Lemma 2.2.** Let \( \lambda > 0 \). Under the assumptions in Lemma 2.1, put
\[
V(\phi) := -\frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 r^2 + \frac{\lambda}{4} r^4.
\]
Let \( r_0 \) be the number defined by (1.7). Then the Euler-Lagrange equation of the action \( \int_{\mathbb{R}} \int_{\mathbb{R}^n} L(\phi + r_0) \sqrt{-g} dx dt \) is given by
\[
\frac{1}{c^2} \partial^2_t \phi - e^{-2Ht} \Delta \phi + \frac{nH}{c^2} \partial_t \phi + J = 0, \tag{2.1}
\]
where
\[
J := \lambda \left( |\phi|^2 \phi + r_0 (2 \phi \text{Re} \phi + |\phi|^2) + 2r_0^2 \text{Re} \phi \right). \tag{2.2}
\]
Proof. We note that \( r_0 \) gives the minimum of \( V \) by \( V'(r) = \lambda r(r + r_0)(r - r_0) \). We have
\[
V(\phi + r_0) = \frac{\lambda}{4} (|\phi|^4 + 4r_0 |\phi|^2 \text{Re} \phi + 4r_0^2 (\text{Re} \phi)^2 - r_0^4)
\]
and \( \delta V(\phi + r_0) = \text{Re}(J \delta \phi) \) and \( \delta L(\phi + r_0) = -\text{Re}(\partial_0 \delta \phi \partial^2 \phi) - \text{Re}(J \delta \phi) \) by direct calculations, where we have used
\[
\delta|\phi|^2 = 2 \text{Re}(\delta \phi \phi), \quad \delta|\phi|^4 = 4 \text{Re}(|\phi|^2 \delta \phi \phi), \quad \delta(\text{Re} \phi)^2 = 2 \text{Re}(\text{Re} \phi \delta \phi),
\]
\[
\delta(|\phi|^2 \text{Re} \phi) = \text{Re} ((2 \phi \text{Re} \phi + |\phi|^2) \delta \phi), \quad \delta(\partial^2 \phi \partial_\alpha \phi) = 2 \text{Re}(\partial_\alpha \delta \phi \partial^2 \phi).
\]
Thus, we obtain

\[
\delta \int \int_{\mathbb{R}^n} L(\phi + r_0) \sqrt{-g} dx dt = -\text{Re} \int \int_{\mathbb{R}^n} \partial_\alpha (\delta \phi \partial^\alpha \phi \sqrt{-g}) dx dt + \text{Re} \int \int_{\mathbb{R}^n} K \delta \phi \sqrt{-g} dx dt,
\]

where we have put \( K := \partial_\alpha (\partial^\alpha \phi \sqrt{-g}) / \sqrt{-g} - J \). So that, the Euler-Lagrange equation is given by \( K = 0 \), which yields the required equation (2.1). \( \square \)

**Lemma 2.3.** Under the assumptions in Lemma 2.2, put \( u := e^{nHt/2} \phi \). Then the equation (2.1) is rewritten as (1.9), where \( h(u) \) is defined by (1.10).

**Proof.** Let \( K \) be defined by the left hand side in (2.1). Put \( \eta := -nH/2 \). Since we have

\[
J e^{-\eta t} = \lambda |u|^2 e^{2\eta t} + \lambda r_0 (2u \text{Re} u + |u|^2)e^{\eta t} + 2\lambda r_0^2 \text{Re} u,
\]

we obtain

\[
-Ke^{-\eta t} = \frac{1}{c^2} \partial_t^2 u - e^{-2Ht} \Delta u - \left( \frac{nH}{2c} \right)^2 u + h(u) + 2 \left( \frac{mc}{\eta} \right)^2 \text{Re} u,
\]

which shows \( K = 0 \) gives the required equation (1.9). \( \square \)

**Lemma 2.4.** Let \( H \geq 0 \), \( \lambda \in \mathbb{R} \) and \( r_0 \in \mathbb{R} \). Consider the Cauchy problem

\[
\begin{aligned}
&\frac{1}{c^2} \partial_t^2 u - e^{-2Ht} \Delta u - \left( \frac{nH}{2c} \right)^2 u + 2\lambda r_0^2 \text{Re} u + h = 0, \\
&u(0, \cdot) = u_0(\cdot), \quad \partial_t u(0, \cdot) = u_1(\cdot)
\end{aligned}
\]

(2.3)

for any given function \( h \) which decay rapidly at spatial infinity. Then the following results hold.

1. \( \partial_\alpha e^\alpha + e^{n+1} + e^{n+2} = 0 \), where

\[
e^0 := \frac{1}{2c^2} |\partial_t u|^2 - \frac{1}{2} \left( \frac{nH}{2c} \right)^2 |u|^2 + \lambda r_0^2 (\text{Re} u)^2 + \frac{e^{-2Ht}}{2} |\nabla u|^2,
\]

\[
e^j := -e^{-2Ht} \text{Re} \left( \partial_i u_0 \partial_j u \right),
\]

\[
e^{n+1} := H e^{-2Ht} |\nabla u|^2, \quad e^{n+2} := \text{Re} \left( \partial_t uh \right).
\]

2. If \( u \) is real-valued and \( Q := 2\lambda r_0^2 - (nH/2c)^2 \geq 0 \), then the following estimate holds;

\[
\frac{1}{c} \| \partial_t u \|_{L^\infty((0,\infty), L^2(\mathbb{R}^n))} + \sqrt{Q} \| u \|_{L^\infty((0,\infty), L^2(\mathbb{R}^n))} + \| e^{-Ht} \nabla u \|_{L^\infty((0,\infty), L^2(\mathbb{R}^n))}) + \| e^{-Ht} \nabla u \|_{L^2((\infty,0) \times \mathbb{R}^n)}
\]

\[
\leq \frac{1}{c} \| u_0 \|_{L^2(\mathbb{R}^n)} + \sqrt{Q} \| u_0 \|_{L^2(\mathbb{R}^n)} + \| \nabla u_0 \|_{L^2(\mathbb{R}^n)} + c \| h \|_{L^1((\infty,0), L^2(\mathbb{R}^n))}.
\]

3. If \( h = h(u) \) is given by (1.10), then \( \partial_0 e^0 + \partial_j e^j + e^{n+1} = 0 \) holds, where

\[
e^0 := e^0 + \frac{\lambda}{4} |u|^4 e^{-nHt} + \lambda r_0 |u|^2 \text{Re} u e^{-nHt/2}
\]

\[
e^{n+1} := e^{n+1} + \frac{\lambda nH}{4} |u|^4 e^{-nHt} + \frac{\lambda r_0 nH}{2} |u|^2 \text{Re} u e^{-nHt/2},
\]

and \( e^\alpha \) is defined in (1) for \( 0 \leq \alpha \leq n + 1 \).
Proof. (1) Multiplying $\partial_t u$ to the first equation in (2.3) and taking its real part, we obtain the required equation by

$$\text{Re} \left( \partial_t u \partial_t^2 u \right) = \partial_t \left( \frac{1}{2} |\partial_t u|^2 \right),$$

$$e^{-2Ht} \text{Re} \left( \partial_t u \Delta u \right) = \nabla \{ e^{-2Ht} \text{Re} \left( \partial_t u \nabla u \right) \} - \partial_t \left( \frac{e^{-2Ht}}{2} |\nabla u|^2 \right) - He^{-2Ht} |\nabla u|^2,$$

$$\text{Re} \left( \partial_t uu \right) = \partial_t \left( \frac{|u|^2}{2} \right), \quad \text{Re} \left( \partial_t u Re u \right) = \partial_t \left( \frac{|Re u|^2}{2} \right).$$

(2) Integrating the both sides in $\partial_\alpha e^\alpha + e^{n+1} + e^{n+2} = 0$ in (1), we have

$$\int_{\mathbb{R}^n} e^0(t)dt + H\|e^{-Hs\nabla u}\|^2_{L^2((0,t),L^2)} = \int_{\mathbb{R}^n} e^0(0)dt - \int_{t}^{\infty} \int_{\mathbb{R}^n} e^{n+2}dxdt$$

for $t > 0$, where we have used

$$\int_{\mathbb{R}^n} e^0(t)dt = \frac{1}{2c^2} \|\partial_t u(t)\|^2_2 + \frac{Q}{2} \|u(t)\|^2_2 + \frac{e^{-2Ht}}{2} \|\nabla u(t)\|^2_2$$

since $u$ is real-valued. By the Hölder inequality $\int_{t}^{\infty} \int_{\mathbb{R}^n} |e^{n+2}dxdt \leq \|\partial_t u\|_{L^\infty L^2} \|h\|_{L^1 L^2}$, we obtain the required result.

(3) Put $e^\alpha_\ast := e^0 - e^\alpha$, $e^{n+1}_\ast := e^{n+1} - e^{n+1}$. We have $\text{Re} \left( \partial_t uh \right) = \partial_t e^\alpha_\ast + e^{n+1}_\ast$ by

$$\text{Re} \left( \partial_t |u|^2 u \right) e^{-nHt} = \partial_t \left( \frac{1}{4} |u|^4 e^{-nHt} \right) + \frac{nH}{4} |u|^4 e^{-nHt},$$

$$\text{Re} \left\{ \partial_t \left( 2u Re u + |u|^2 \right) \right\} e^{-nHt/2} = \partial_t \left( |u|^2 Re u e^{-nHt/2} \right) + \frac{nH}{2} |u|^2 Re u e^{-nHt/2}.$$

So that, we obtain $0 = \partial_\alpha e^\alpha + e^{n+1} + e^{n+2} = \partial_0 e^0 + \partial_j e^j + e^{n+1}_\ast$ by (1), which is the required result. \hfill \Box

Lemma 2.5. Let $H < 0$, $\lambda \in \mathbb{R}$ and $r_0 \in \mathbb{R}$. Consider the problem (2.3). Then the following results hold.

(1) $\partial_\alpha e^\alpha + e^{n+1} + e^{n+2} = 0$, where

$$e^0 := \frac{e^{2Ht}}{2c^2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 - \frac{e^{2Ht}}{2} \left( \frac{nH}{2c} \right)^2 |u|^2 + \lambda r_0^2 e^{2Ht} |Re u|^2,$$

$$e^j := - \text{Re} \left( \partial_t u \partial_j u \right),$$

$$e^{n+1} := - \frac{He^{2Ht}}{c^2} |\partial_t u|^2 + \left( \frac{nH}{2c} \right)^2 He^{2Ht} |u|^2 - 2\lambda r_0^2 He^{2Ht} |Re u|^2,$$

$$e^{n+2} := e^{2Ht} \text{Re} \left( \partial_t uh \right).$$

(2) Let $q$ be any number with $2 \leq q \leq \infty$, and let $q'$ be the conjugate number with $1/q + 1/q' = 1$. If $u$ is real-valued and $Q := 2\lambda r_0^2 - (nH/2c)^2 \geq 0$, then the
following estimate hold;

\[
\frac{1}{c} e^{\lambda t} \partial_t u \| L^\infty((0,\infty),L^2(\mathbb{R}^n)) + \sqrt{Q} e^{\lambda t} u \| L^\infty((0,\infty),L^2(\mathbb{R}^n)) + \| \nabla u \| L^\infty((0,\infty),L^2(\mathbb{R}^n)) \\
+ \frac{\sqrt{-H}}{c} e^{\lambda t} \partial_t u \| L^2((0,\infty) \times \mathbb{R}^n) + \sqrt{Q} e^{\lambda t} u \| L^2((0,\infty) \times \mathbb{R}^n) \\
\leq \frac{1}{c} \| u_1 \| L^2(\mathbb{R}^n) + \sqrt{Q} \| u_0 \| L^2(\mathbb{R}^n) + \| \nabla u_0 \| L^2(\mathbb{R}^n) + \frac{c}{(-H)^{1/4}} \| e^{\lambda t} h \| L^\infty((0,\infty),L^2(\mathbb{R}^n)) .
\]

(3) If \( h = h(u) \) is given by \((1.10)\), then \( \partial_0 e^0 + \partial_j e^{2} + \partial_{n+1} e = 0 \) holds, where

\[
e^0 := e^0 + \frac{\lambda}{4} |u|^4 e^{-(n-2)H} + \lambda \Re \| u \| 2 \Re u e^{-(n-4)H/2}
\]
\[
e^{n+1} := e^{n+1} + \frac{\lambda(n-2)H}{4} |u|^4 e^{-(n-2)H} + \lambda \Re \| u \| 2 \Re u e^{-(n-4)H/2},
\]

and \( e^n \) is defined in (1) for \( 0 \leq \alpha \leq n + 1 \).

Proof. (1) Multiplying \( e^{2Ht} \partial_t u \) to the first equation in \((2.3)\) and taking its real part, we obtain the required equation by

\[
e^{2Ht} \Re (\partial_t u \partial_t u) = \partial_t \left( \frac{e^{2Ht}}{2} |\partial_t u|^2 \right) - He^{2Ht} |\partial_t u|^2,
\]
\[
\Re (\partial_t u \Delta u) = \nabla \Re (\partial_t u \nabla u) - \partial_t \left( \frac{1}{2} |\nabla u|^2 \right),
\]
\[
e^{2Ht} \Re (\partial_t uu) = \partial_t \left( \frac{e^{2Ht}}{2} |u|^2 \right) - He^{2Ht} |u|^2,
\]
\[
e^{2Ht} \Re (\partial_t u u) = \partial_t \left( \frac{e^{2Ht} |\Re u|^2}{2} \right) - He^{2Ht} |\Re u|^2.
\]

(2) Integrating the both sides in \( \partial_0 e^n + e^{n+1} + e^{n+2} = 0 \) in (1), we have

\[
\int_{\mathbb{R}^n} e^0(t) dx + \int_0^t \int_{\mathbb{R}^n} e^{n+1} dx ds = \int_{\mathbb{R}^n} e^0(0) dx - \int_0^t \int_{\mathbb{R}^n} e^{n+2} dx dt
\]

for \( t > 0 \), where we note

\[
\int_{\mathbb{R}^n} e^0(t) dx = \frac{1}{2c^2} \| e^{Ht} \partial_t u(t) \|_{L^2}^2 + \frac{1}{2} \| \nabla u(t) \|_{L^2}^2 + \frac{Q}{2} \| e^{Ht} u(t) \|_{L^2}^2,
\]

and

\[
\int_0^t \int_{\mathbb{R}^n} e^{n+1} dx ds = - \frac{H}{c^2} \| e^{Hs} \partial_t u(t) \|_{L^2}^2 - HQ \| e^{Hs} u(t) \|_{L^2}^2
\]

since \( u \) is real-valued. We estimate the last term by the H"older inequality

\[
\int_0^t \int_{\mathbb{R}^n} e^{n+2} dx dt \leq \| e^{Hs} \partial_t u \|_{L^2} \| e^{Hs} h \|_{L^2} \]
\[
\leq \frac{\varepsilon^2 (-H)^{2/4}}{4c^2} \| e^{Hs} \partial_t u \|_{L^2}^2 + \frac{c^2}{\varepsilon^2 (-H)^{2/4}} \| e^{Hs} h \|_{L^2}^2
\]
for any number \( \varepsilon > 0 \). So that, the required inequality follows from the interpolation inequality
\[
\| e^{H_s t} \partial_t u \|_{L^q L^r} \leq \| e^{H_s t} \partial_t u \|_{L^{1-q/2} L^{r/2}} \| e^{H_s t} \partial_t u \|_{L^2 L^2} \text{ with } \varepsilon > 0 \text{ taken sufficiently small.}
\]

(3) Put \( e^0 := \tilde{e}^0 - e^0, e^{n+1} := \tilde{e}^{n+1} - e^{n+1} \). We have
\[
e^{2Ht} \Re (\partial_t u) = \partial_t e^0 + e^{n+1}
\]
by
\[
\Re \left( \partial_t u |u|^2 u \right) e^{-(n-2)Ht} = \partial_t \left( \frac{1}{4} |u|^4 e^{-(n-2)Ht} \right) + \frac{(n-2)H}{4} |u|^4 e^{-(n-2)Ht},
\]
and
\[
\Re \left\{ \partial_t u \left( 2u \Re u + |u|^2 \right) \right\} e^{-(n-4)Ht/2} = \partial_t \left( |u|^2 \Re u e^{-(n-4)Ht/2} \right) + \frac{(n-4)H}{2} |u|^2 \Re u e^{-(n-4)Ht/2}.
\]
So that, we obtain
\[
0 = \partial_\alpha e^0 + e^{n+1} + e^{n+2} = \partial_\alpha e^0 + \partial_\beta e^1 + \tilde{e}^{n+1}
\]
from (1) as required.

We confirm that the Euler-Lagrange equation (2.1) with (2.2) is obtained from (1.8) by the shift of the function \( \phi \) as follows.

**Lemma 2.6.** For \( \lambda > 0 \) and \( r_0 \) defined by (1.7), the equation (2.1) with (2.2) is obtained from (1.8) with \( \phi \) replaced by \( \phi + r_0 \).

**Proof.** The result follows from a direct calculation by
\[
\lambda |\phi + r_0|^2 (\phi + r_0) = \lambda \left\{ |\phi|^2 \phi + r_0(2\phi \Re \phi + |\phi|^2) + 2r_0^2 \Re \phi \right\} + \lambda r_0^2 (\phi + r_0)
\]
and
\[
\lambda r_0^2 = (mc/h)^2.
\]

To express the solution of the differential equation as the integral equation, we recall the following fundamental results for ordinary differential equations (see, e.g., [19]). Put \( D_t := \partial/\partial t \).

**Lemma 2.7.** For any fixed nonnegative function \( \tilde{a} \in C([0,T)) \) for \( T > 0 \), let \( \rho_0 \) and \( \rho_1 \) be the solutions of the Cauchy problem
\[
\begin{cases}
(D_t^2 + \tilde{a}(t)) \rho_j(t) = 0 \text{ for } t \in [0,T], \\
\rho_j(0) = \delta_{0j}, \quad D_t \rho_j(0) = \delta_{1j}
\end{cases}
\]
(2.4)
for \( j = 0, 1 \), where \( \delta_{00} = \delta_{11} = 1 \) and \( \delta_{01} = \delta_{10} = 0 \). Let \( b \in L^1((0,T)) \), and let \( \rho \) be the solution of the equation
\[
(D_t^2 + \tilde{a}(t)) \rho(t) = b(t)
\]
(2.5)
for \( t \in [0,T) \). Put
\[
\Phi = \begin{pmatrix}
\rho_0 & \rho_1 \\
D_t \rho_0 & D_t \rho_1
\end{pmatrix}.
\]
Then the following results hold.

(1) \( \det \Phi = 1 \).
(2) \( \rho \) is given by
\[
\begin{pmatrix}
\rho(t) \\
D_t \rho(t)
\end{pmatrix} = \Phi(t) \begin{pmatrix}
\rho(0) \\
D_t \rho(0)
\end{pmatrix} + \int_0^t \Phi(t) \Phi(s)^{-1} \begin{pmatrix}
0 \\
b(s)
\end{pmatrix} ds,
\]
which is rewritten as
\[
\begin{align*}
\rho(t) &= \rho_0(t)\rho(0) + \rho_1(t)D_t\rho(0) + \int_0^t \rho_{12}(t, s)b(s)ds, \\
D_t \rho(t) &= D_t \rho_0(t)\rho(0) + D_t \rho_1(t)D_t \rho(0) + \int_0^t \rho_{22}(t, s)b(s)ds,
\end{align*}
\]
where \( \rho_{12} \) and \( \rho_{22} \) are defined by
\[
\begin{align*}
\rho_{12}(t, s) &= -\rho_0(t)\rho_1(s) + \rho_1(t)\rho_0(s), \\
\rho_{22}(t, s) &= -D_t \rho_0(t)\rho_1(s) + D_t \rho_1(t)\rho_0(s).
\end{align*}
\]

(3) If \( D_t \tilde{a} \leq 0 \), then
\[
|\rho_0(\cdot)| \leq \sqrt{\tilde{a}(0)} \sqrt{\tilde{a}(\cdot)}, \quad |D_t \rho_0(\cdot)| \leq \sqrt{\tilde{a}(0)}, \quad |\rho_1(\cdot)| \leq \frac{1}{\sqrt{\tilde{a}(\cdot)}}, \quad |D_t \rho_1(\cdot)| \leq 1.
\]

(4) If \( D_t \tilde{a} \geq 0 \), then
\[
|\rho_0(\cdot)| \leq 1, \quad |D_t \rho_0(\cdot)| \leq \sqrt{\tilde{a}(\cdot)}, \quad |\rho_1(\cdot)| \leq \frac{1}{\sqrt{\tilde{a}(0)}}, \quad |D_t \rho_1(\cdot)| \leq \frac{\tilde{a}(\cdot)}{\tilde{a}(0)}.
\]

(5) \( \rho \in C([0, T)) \). If \( \tilde{a} \in C([0, T)) \) and \( b \in C([0, T)) \), then \( \rho \in C^2([0, T)) \).

For \( H \in \mathbb{R} \) and \( Q \in \mathbb{R} \), let \( \rho_0 = \rho_0(t, \xi) \) and \( \rho_1 = \rho_1(t, \xi) \) be the functions obtained by Lemma 2.7 putting
\[
\tilde{a} = \tilde{a}(t, \xi) := \frac{c^2}{e^{2ht}} \sum_{j=1}^n (\xi_j)^2 + c^2 Q
\]
for \( \xi = (\xi_1, \cdots, \xi_n) \). Put
\[
\begin{align*}
K_0(t) &= F^{-1} \rho_0(t, \cdot)F, \quad K_1(t) &= F^{-1} \rho_1(t, \cdot)F, \\
K(t, s) &= c^2 \{ -K_0(t)K_1(s) + K_1(t)K_0(s) \}
\end{align*}
\]
for \( t, s \in \mathbb{R} \), where \( F \) and \( F^{-1} \) denote the Fourier transform and its inverse for \( (x^1, \cdots, x^n) \). Then the Cauchy problem
\[
\frac{1}{c^2} \partial_t^2 u - e^{-2Ht} \Delta u + Qu + h = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1
\]
for a given function \( h \) on \( \mathbb{R}^{1+n} \) can be regarded as the solution of the integral equation
\[
u(t) = K_0(t)u_0 + K_1(t)u_1 - \int_0^t K(t, s)h(s)ds.
\]

By the estimates 2.10, 2.11 and the Plancherel theorem for the Fourier transform, we obtain the following results (see e.g., 23, Lemma 4.4).
Lemma 2.8. Let \( \mu \in \mathbb{R} \), and let \( h \in L^1((0, \infty), H^\mu(\mathbb{R}^n)) \). Let \( K_0 \) and \( K_1 \) be the operators defined by (2.13). Put

\[
\begin{align*}
    u_{+0} &:= u_0 + c^2 \int_0^\infty K_1(s)h(s)ds, \\
u_{+1} &:= u_1 - c^2 \int_0^\infty K_0(s)h(s)ds, \\
    u_+(t) &:= K_0(t)u_{+0} + K_1(t)u_{+1}
\end{align*}
\]

for \( t \geq 0 \). Let \( u \) be the solution of (2.15). Then the following estimates hold.

1. \( ||u_{+0}||_{H^\mu(\mathbb{R}^n)} \lesssim ||u_0||_{H^\mu(\mathbb{R}^n)} + c \int_0^\infty ||h(s)||_{H^\mu(\mathbb{R}^n)} ds \)
2. \( ||u_{+1}||_{H^{\mu-1}(\mathbb{R}^n)} \lesssim ||u_1||_{H^{\mu-1}(\mathbb{R}^n)} + c^2 \int_0^\infty ||h(s)||_{H^\mu(\mathbb{R}^n)} ds \)
3. \( ||u_+||_{H^{\mu-1}(\mathbb{R}^n)} \lesssim ||u_0||_{H^\mu(\mathbb{R}^n)} + \frac{1}{c} ||u_1||_{H^{\mu-1}(\mathbb{R}^n)} + c \int_0^\infty ||h(s)||_{H^\mu(\mathbb{R}^n)} ds \)
4. \( ||\partial_t u_+||_{H^{\mu-1}(\mathbb{R}^n)} \lesssim c ||u_0||_{H^\mu(\mathbb{R}^n)} + ||u_1||_{H^{\mu-1}(\mathbb{R}^n)} + c^2 \int_0^\infty ||h(s)||_{H^\mu(\mathbb{R}^n)} ds \)
5. \( ||u(t) - u_+(t)||_{H^{\mu-1}(\mathbb{R}^n)} \lesssim \frac{1}{c} \int_t^\infty ||h(s)||_{H^\mu(\mathbb{R}^n)} ds \)
6. \( ||\partial_t (u(t) - u_+(t))||_{H^{\mu-1}(\mathbb{R}^n)} \lesssim c^2 \int_t^\infty ||h(s)||_{H^\mu(\mathbb{R}^n)} ds \)

3 Proof of Proposition 1.1

Put

\[
Q := \left( \frac{m^* c}{h} \right)^2 - \left( \frac{nH}{2c} \right)^2 \quad \text{and} \quad M := \sqrt{-Q}
\]

which satisfies \( M \geq 0 \) by \( Q \leq 0 \). Put \( w(t) := \int_{\mathbb{R}^n} u(t, x) dx \) for \( t \geq 0 \). It suffices to show that \( w \) blows up in finite time. Integrating the first equation in (1.6), we have

\[
\frac{1}{c^2} \partial_t^2 w + Qw = h,
\]

(3.1)

where \( h := e^{-n(p-1)Ht/2} \int_{\mathbb{R}^n} |u|^p dx \) and we have used the divergence theorem. So that, \( w \) is written as

\[
w(t) = (\cosh cMt)w_0 + \frac{\sinh cMt}{cM} w_1 + c^2 \int_0^t \frac{\sinh cM(t-s)}{cM} h(s)ds,
\]

(3.2)

and \( w \) satisfies \( w(t) \geq 0 \) and \( \partial_t w(t) \geq 0 \) for \( t \geq 0 \) by \( w_0 \geq 0, \ w_1 \geq 0, \ h \geq 0 \). By the finite speed of the propagation, we may assume that the support of \( u(t, \cdot) \) is
in the ball of the radius \( r(t) := r_0 + c(1 - e^{-Ht})/H \) for some number \( r_0 > 0 \) with \( \text{supp} u_0 \cup \text{supp} u_1 \subset \{ x \in \mathbb{R}^n; |x| \leq r_0 \} \). By this support condition and the Hölder inequality, we have

\[
w(t)^p \leq \{\omega_n r(t)^n\}^{p-1} \int_{\mathbb{R}^n} |u(t, x)|^p dx,
\]

which yields

\[
h(t) \geq b(t)|w(t)|^p,
\]

where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \), and we have put

\[
b(t) := e^{-n(p-1)Ht/2}\{\omega_n r(t)^n\}^{-p+1}.
\]

From this, \( w \) satisfies the differential inequality

\[
\partial_t^2 w(t) + c^2 Qw(t) - c^2 b(t)w^p \geq 0,
\]

which yields

\[
\partial_t^2 w(t) + c^2 Qw(t) \geq 0
\]

by \( b(t)w(t)^p \geq 0 \). Multiplying \( \partial_t w \) to this inequality, integrating it, and using the assumptions \( w_0 \geq 0, w_1 \geq 0 \) and \( w_1 \geq cM w_0 \), we have \( (\partial_t w)^2 + c^2 Qw^2 \geq 0 \). This inequality yields \( \partial_t w - cM w \geq 0 \) by \( w \geq 0 \) and \( \partial_t w \geq 0 \), by which we obtain

\[
w(t) \geq w_0 e^{cMt}.
\]

We have \( r(t) \leq r_0 + c/H \) for \( t \geq 0 \) when \( H > 0 \), and we also have \( r(t) \leq 2ce^{-Ht}/|H| \) for sufficiently large \( t \) when \( H < 0 \) by the definition of \( r(t) \). So that, \( b(t) \) is bounded as

\[
b(t) \geq B e^{-n(p-1)|H|t/2}
\]

for \( t \gg 1 \), where \( B \) is a constant defined by

\[
B := \omega_n^{-p+1} \left\{ \begin{array}{ll}
(r_0 + c/H)^{-n(p-1)} & \text{if } H > 0 \text{ and } t \geq 0, \\
\left(\frac{2c}{|H|}\right)^{-n(p-1)} & \text{if } H < 0 \text{ and } t \gg 1.
\end{array} \right.
\]

In addition, we have

\[
b'(t) \leq 0
\]

for \( t \gg 1 \) since

\[
b'(t) = -n(p-1)\omega_n^{-p+1} \left( e^{Ht/2} r(t) \right)^{-n(p-1)} \left( \frac{H}{2} + \frac{r'(t)}{r(t)} \right)
\]

and

\[
\frac{r'(t)}{r(t)} = \frac{ce^{-Ht}}{r_0 + c/H(1 - e^{-Ht})} \to \begin{cases} 
0 & \text{if } H > 0, \\
-H & \text{if } H < 0
\end{cases}
\]

as \( t \to \infty \).
Multiplying $\partial_t w$ to (3.4), which is non-negative, we have
\[ \partial_t e_0(t) + e_1(t) \geq 0 \]
for $t \geq 0$, where we have put
\[ e_0 := \frac{1}{2c^2}(\partial_t w)^2 + \frac{Q}{2} w^2 - \frac{b}{p+1} w^{p+1}, \quad e_1 := \frac{\partial_t b}{p+1} w^{p+1}. \]
Integrating the both sides of this inequality on the interval $[t_0, t]$ for sufficiently large $t_0 \gg 1$, and using $b' \leq 0$ in (3.7), we obtain
\[ (\partial_t w(t))^2 + c^2 Q w(t)^2 - \frac{2c^2 b(t)}{p+1} w(t)^{p+1} \geq 2c^2 e_0(t_0). \tag{3.8} \]
The term $b w^{p+1}$ in this inequality is estimated by
\[ b(t)w(t)^{p+1} \geq B w_0^{p-1} e^{(p-1)(cM - n|H|/2) t} w(t)^2 \]
\[ \geq B w_0^{p-1} e^{2cMt} \to \infty \]
as $t \to \infty$ by (3.5), (3.6), and $cM - n|H|/2 \geq 0$ due to $m_* \in \mathbb{R}$. Thus, the inequality (3.8) yields
\[ (\partial_t w(t))^2 \geq -c^2 Q w(t)^2 + \frac{c^2 b(t)}{p+1} w(t)^{p+1} \geq c^2 M_1^2 w(t)^2 \tag{3.10} \]
for $t \gg 1$ by (3.9), where we have put
\[ M_1 := \left( M^2 + \frac{B w_0^{p-1}}{p+1} \right)^{1/2}. \]
So that, we have $\partial_t w \geq cM_1 w$, by which we obtain
\[ w(t) \geq w(t_1)e^{cM_1(t-t_1)} \tag{3.11} \]
for $t \geq t_1 \gg 1$. For any sufficiently small number $\varepsilon > 0$, we estimate the term $b(t)w(t)^{p+1}$ in (3.10) as
\[ b(t)w(t)^{p+1} = b(t)w(t)^{(1-\varepsilon)(p-1)} w(t)^{2+\varepsilon(p-1)} \]
\[ \geq B \left( (w(t_1) e^{-cM_1 t_1})^{1-\varepsilon} e^{-n|H| t/2 + cM_1 (1-\varepsilon) t} \right)^{p-1} w(t)^{2+\varepsilon(p-1)} \]
\[ \geq B \left( w(t_1) e^{-cM_1 t_1} \right)^{(1-\varepsilon)(p-1)} w(t)^{2+\varepsilon(p-1)} \]
for $t \geq t_1 \gg 1$ by (3.6) and (3.11), where we have used $-n|H| t/2 + cM_1 (1-\varepsilon) t \geq 0$ for sufficiently small $\varepsilon > 0$ by $cM_1 \geq cM > n|H|/2$ when $m_* \neq 0$, and by $cM_1 > cM = n|H|/2$ when $m_* = 0$ and $w_0 > 0$. By this estimate, (3.10) and $Q \leq 0$, we obtain the differential inequality
\[ \partial_t w(t) \geq c \sqrt{\frac{B}{p+1}} (w(t_1) e^{-cM_1 t_1})^{(1-\varepsilon)(p-1)/2} w(t)^{1+\varepsilon(p-1)/2} \tag{3.12} \]
for $0 < \varepsilon \ll 1$ and $t \geq t_1 \gg 1$. Since $w$ is positive, and the positive solution of (3.12) must blow up in finite time, the function $w$ blows up as required.
4 Proof of Theorem 1.2

(1) Let $q = \infty$ when $H = 0$, and $0 \leq 1/q \leq \min \{1/2, 1/(n - 2\mu_0)\}$ when $H > 0$. Let $\tilde{q} = \infty$ when $H = 0$, and $0 \leq 1/\tilde{q} \leq \min \{1/2, 2/(n - 2\mu_0)\}$ when $H > 0$. Put $1/q_* := 1 - (n - 2\mu_0)/q$ and $1/\tilde{q}_* := 1 - (n - 2\mu_0)/\tilde{q}$.

Let $\rho_0, \rho_1, K_0, K_1$ and $K$ be the functions and the operators defined in Lemma 2.7 and (2.13) for the function $\tilde{a}$ in (2.12). Then the solution of (1.9) is regarded as the fixed point of the operator $\Phi$ defined by

$$
\Phi(u)(t) := K_0(t)u_0 + K_1(t)u_1 - \int_0^t K(t,s)h(u)(s)ds,
$$

where $h(u)$ is defined by (1.10). For constants $T > 0$, $R_\nu > 0$ for $\nu = 0, \mu_0, \mu$, we define the closed ball defined by

$$
X^\mu(T, R_\nu, R_{\mu_0}, R_\mu) := \{u; \|u\|_{X^\nu} \leq R_\nu \text{ for } \nu = 0, \mu_0, \mu\},
$$

and we show $\Phi$ is a contraction mapping on this space for the suitable constants. The solution is obtained as the fixed point of $\Phi$. Let $\mu$ and we show $\Phi$ is a contraction mapping on this space for the suitable constants.

Define the closed ball defined by

$$
X^\mu(T, R_0, R_{\mu_0}, R_\mu) := \{u; \|u\|_{X^\nu} \leq R_\nu \text{ for } \nu = 0, \mu_0, \mu\},
$$

and we show $\Phi$ is a contraction mapping on this space for the suitable constants.

The solution is obtained as the fixed point of $\Phi$. Let $\mu$ and we show $\Phi$ is a contraction mapping on this space for the suitable constants. We define $\theta$, $r_*$, $r_{**}$, $q_*$ by

$$
\theta := \frac{n - 2\mu_0}{3}, \quad r_* := \frac{1}{6} - \frac{\mu_0}{3n}, \quad r_{**} := \frac{1}{6} + \frac{2\mu_0}{3n}, \quad q_* := 1 - \frac{n - 2\mu_0}{q}.
$$

We note

$$
0 \leq \theta \leq 1, \quad \frac{1}{2} = \frac{2 + \frac{2}{r_*}}{r_*}, \quad 0 < \frac{1}{r_*} \leq \frac{1}{2}, \quad 0 < \frac{1}{r_{**}} \leq \frac{1}{2}
$$

hold by the definition of $\theta$, $(n - 3)/2 \leq \mu_0 < n/2$. We have

$$
\|u\|^2\|u\|_{H^\nu} \lesssim \|u\|^2_{L^{r_*} \cap \dot{B}^0_{r_{**}, 2}} \|u\|_{\dot{B}^0_{r_{**}, 2}}
$$

for $\nu \geq 0$ by the nonlinear estimate in the Besov spaces (see [24, Lemm 3.1]) and (4.4). Since we have the embeddings $H^{\mu_0+\theta} \hookrightarrow \dot{B}^0_{r_{**}, 2} \hookrightarrow L^{r_*} \cap \dot{B}^0_{r_{**}, 2}$ and $H^{\nu+\theta} \hookrightarrow \dot{B}^0_{r_{**}, 2}$ by $1/r_* = 1/r_{**} - \mu_0/n$ and $1/r_{**} = 1/2 - \theta/n$, we have

$$
\|u\|^2\|u\|_{H^\nu} \lesssim \|u\|_{H^{\mu_0+\theta}}\|u\|_{H^{\nu+\theta}} \\
\lesssim \|u\|_{H^{\nu+\theta}}^2 \|u\|_{L^{2\theta}} \|u\|_{L^{2\theta}} \|u\|_{H^{\nu+1}} \|u\|_{H^{\nu+1}},
$$

where we have used the interpolation inequalities at the last line. Thus, we obtain

$$
\|u\|^2\|ue^{-nHt}\|_{L^1H^\nu} \lesssim \|A\|_{q_*} \|u\|_{L^\infty H^{\mu_0}}^{2(1-\theta)} \|u\|_{L^\infty H^{\mu_0+1}}^{2\theta} \|u\|_{L^\infty H^{\nu+1}}^{1-\theta} \|u\|_{L^\infty H^{\nu+1}}^\theta \\
\lesssim \|A\|_{q_*} \|B\|_{q_*} \|u\|_{X^{q_*}} \|u\|_{\dot{X}^{\nu}}
$$

for $\nu = 0, \mu_0, \mu$ by the Hölder inequality and $1 = 1/q_* + 3\theta/q$, where we have put

$$
A = A(t) := e^{-nHt+3\theta Ht} \quad \text{and} \quad B := Q^{-3(1-\theta)/2}H^{-3\theta/q}.
$$
We note
\[ \|A\|_{L^{q_r}(0,T)} = \begin{cases} T^{1/q_r} & \text{if } 1 \leq q_r \leq \infty, \ H\mu_0 = 0, \\ 1 & \text{if } q_r = \infty, \ H\mu_0 > 0, \\ \left\{ 1 - e^{-2\nu_0 H\bar{q}_r} \right\}^{1/q_r} & \text{if } q_r < \infty, \ H\mu_0 > 0 \end{cases} \] 
(4.9)
by a direct calculation.

Similarly to the above estimate, let \( \tilde{\mu} = \infty \) when \( H = 0 \), \( 0 \leq 1/\tilde{\mu} \leq \min\{1/2, 2/(n-2\mu_0)\} \) when \( H > 0 \). We define \( \tilde{\mu}, \tilde{r}, \tilde{r}_s, \tilde{q}_s, \tilde{\mu}_0 \) by
\[ \tilde{\mu} := \frac{n - 2\mu_0}{4}, \quad \tilde{r} := \frac{1}{4}, \quad \tilde{r}_s := \frac{\mu_0}{2n}, \quad \tilde{r}_s := \frac{1}{4}, \quad \tilde{q}_s := 1 - \frac{n - 2\mu_0}{2\bar{q}}. \] 
(4.10)

We note
\[ 0 \leq \tilde{\mu} \leq 1, \quad 0 < \frac{1}{\tilde{r}} \leq 1, \quad 0 < \frac{1}{\tilde{r}_s} \leq \frac{1}{2} \] 
(4.11)
hold by the definition of \( \tilde{\mu} \), \((n - 4)/2 \leq \mu_0 < n/2\). We have
\[ \|2u \text{Re } u + |u|^2\|_{H^\nu} \lesssim \|u\|_{L^{r_s} \cap \tilde{B}^{\tilde{r}_s,2}_{\tilde{r}_s}} \|u\|_{\tilde{B}^{\tilde{r}_s,2}_{\tilde{r}_s}} \] 
(4.12)
by the nonlinear estimate and (4.11). Since we have the embeddings \( \tilde{H}^{\mu_0+\tilde{\mu}} \hookrightarrow \tilde{B}^{\tilde{r}_s,2}_{\tilde{r}_s} \hookrightarrow L^{r_s} \cap \tilde{B}^{\tilde{r}_s,2}_{\til\nu} \) and \( \tilde{H}^{\til\nu+\til\mu} \hookrightarrow \til{B}^{\til{r}_s+2}_{\til{r}_s} \) by \( 1/\til{r}_s = 1/\til{r}_s - \mu_0/n \) and \( 1/\til{r}_s = 1/2 - \til{\theta}/n \), we have
\[ \|2u \text{Re } u + |u|^2\|_{H^\nu} \lesssim \|u\|_{\til{H}^{\mu_0+\til{\mu}}} \|u\|_{\til{H}^{\til{\nu}+\til{\mu}}} \] 
\[ \lesssim \|u\|_{\til{H}^{\mu_0+1}} \|u\|_{\til{H}^{\til{\nu}+1}} \] 
(4.13)
where we have used the interpolation inequalities at the last line. Thus, we obtain
\[ \|2u \text{Re } u + |u|^2\|_{L^1 H^\nu} \lesssim \|\til{A}\|_{L^1 \til{H}^\nu} \] 
\[ \lesssim \|\til{A}\|_{L^1 \til{H}^\nu} \|u\|_{L^1 \til{H}^{\mu_0+1}} \|u\|_{L^1 \til{H}^{\til{\nu}+1}} \|e^{-Ht} u\|_{L^1 \til{H}^\nu} \] 
\[ \lesssim \|\til{A}\|_{L^1 \til{H}^\nu} \|\til{B}\| \|u\|_{\til{H}^{\mu_0+1}} \|u\|_{\til{H}^{\til{\nu}+1}} \] 
(4.14)
for \( \mu = 0, \mu_0, \mu \) by the H"older inequality and \( 1 = 1/\til{q}_s + 2\til{\theta}/\til{\mu}_0 \), where we have put \( \til{A} = \til{A}(t) := e^{-\nu Ht/2 + 2\til{\theta}Ht} \) and \( \til{B} = Q^{-(1-\til{\theta})H - 2\til{\theta}/\til{\mu}_0} \). We note
\[ \|\til{A}\|_{L^{q_r}(0,T)} = \begin{cases} T^{1/q_s} & \text{if } 1 \leq \til{q}_s \leq \infty, \ H\mu_0 = 0, \\ 1 & \text{if } \til{q}_s = \infty, \ H\mu_0 > 0, \\ \left\{ 1 - e^{-\mu_0 H\til{q}_s} \right\}^{1/q_s} & \text{if } \til{q}_s < \infty, \ H\mu_0 > 0 \end{cases} \] 
(4.15)
by a direct calculation.

By (1.7) and (4.14), we have
\[ \|h(u)\|_{L^1 H^\nu} \lesssim \lambda \|A\|_{q_s} \|B\| \|u\|_{\til{H}^{\mu_0+1}} \|u\|_{\til{H}^{\til{\nu}+1}} \] 
(4.16)
for $\nu = 0, \mu_0, \mu$, and any $u \in X^\mu(T, R_0, R_{\mu_0}, R_{\mu})$. Since we have

$$\|\Phi(u)\|_{X^\nu} \lesssim \frac{1}{c} \|u\|_{\dot{H}^\nu} + \|\nabla u_0\|_{\dot{H}^\nu} + \sqrt{Q} \|u_0\|_{\dot{H}^\nu} + c \|h(u)\|_{L^1 \dot{H}^\nu}$$

by Lemma 2.23 we have

$$\|\Phi(u)\|_{X^\nu} \leq C_0 \dot{B}^\nu + Cc\lambda \left( \|A\|_{Q, R_0 B_{\mu_0} + r_0} \|\tilde{A}\|_{\tilde{q}, \tilde{B}} \right) R_{\mu_0} R_\nu \leq R_\nu$$ (4.17)

for $\nu = 0, \mu_0, \mu$ for some constants $C_0 > 0$, $C > 0$, and any $u \in X^\mu(T, R_0, R_{\mu_0}, R_{\mu})$ by (4.16) if $R_0, R_{\mu_0}$ and $R_{\mu}$ satisfy

$$R_\nu \geq 2C_0 \dot{B}^\nu, \quad Cc\lambda \left( \|A\|_{Q, R_0 B_{\mu_0} + r_0} \|\tilde{A}\|_{\tilde{q}, \tilde{B}} \right) R_{\mu_0} \leq \frac{1}{2}$$ (4.18)

for $\nu = 0, \mu_0, \mu$.

Next, we consider the estimate for the metric. Since we have

$$\|u^2 u - |v|^2 v\|_2 \lesssim \max_{\nu = u, v} \|w\|_2 \|u - v\|_{\nu, e}$$

for any functions $u$ and $v$ similarly to (4.5) by the Hölder inequality, we obtain

$$\left\|\left(\|u^2 u - |v|^2 v\|_2 e^{-n H^1}\right)_{L^1 L^2}\right\| \lesssim \|A\|_{Q, B} \max_{\nu = u, v} \|w\|_{X_{\mu_0}} \|u - v\|_{X_0}$$

similarly to (4.7) by the same argument. Since we also have

$$\|2u \text{Re} u + |u|^2 - (2v \text{Re} v + |v|^2)\|_2 \lesssim \max_{\nu = u, v} \|w\|_{L^\nu \ast} \|u - v\|_{L^\nu \ast}$$

similarly to (4.11), we obtain

$$\left\|\left\{2u \text{Re} u + |u|^2 - (2v \text{Re} v + |v|^2)\right\} e^{-n H^1/2}\right\|_{L^1 L^2} \lesssim \|\tilde{A}\|_{\tilde{q}, \tilde{B}} \max_{\nu = u, v} \|w\|_{X_{\mu_0}} \|u - v\|_{X_0}$$

similarly to (4.13). So that, we have

$$\|h(u) - h(v)\|_{L^1 L^2} \lesssim \lambda \|A\|_{Q, B} \max_{\nu = u, v} \|w\|_{X_{\mu_0}} \|u - v\|_{X_0} + \lambda r_0 \|\tilde{A}\|_{\tilde{q}, \tilde{B}} \max_{\nu = u, v} \|w\|_{X_{\mu_0}} \|u - v\|_{X_0}$$

for any $u, v \in X^\mu(T, R_0, R_{\mu_0}, R_{\mu})$ similarly to (4.16), by which we obtain

$$d(\Phi(u), \Phi(v)) \leq Cc\lambda \left( \|A\|_{Q, R_0 B_{\mu_0} + r_0} \|\tilde{A}\|_{\tilde{q}, \tilde{B}} \right) R_{\mu_0} d(u, v) \leq \frac{1}{2} d(u, v)$$ (4.19)

by Lemma 2.23 similarly to (4.17) under the second condition in (4.18).

We take $q = \tilde{q} = \infty$, thus, $q_\ast = \tilde{q}_\ast = 1$, and $R_{\mu_0} = 2C_0 \dot{B}_{\mu_0}$. Then the second condition in (4.18) is satisfied for sufficiently small $T > 0$, and $T$ depends on the size of $\dot{B}_{\mu_0}$. So that, $\Phi$ is a contraction mapping, and we obtain the local in time solutions.
The continuity of the solution \( u \in C([0, T), H^{\mu+1}) \cap C^1([0, T), H^\mu) \) follows from the continuity of the operators \( K_0, K_1 \) and \( K \) such as \( K_0(\cdot)u_0 \in C([0, T), H^{\mu+1}), \partial_t K_0(\cdot)u_0 \in C([0, T), H^\mu) \) for \( u_0 \in H^{\mu+1}, K_1(\cdot)u_1 \in C([0, T), H^{\mu+1}), \partial_t K_1(\cdot)u_1 \in C([0, T), H^\mu) \) for \( u_1 \in H^\mu, \int_0^t K(t, s)h(s)ds \in C([0, T), H^{\mu+1}), \partial_t \int_0^t K(t, s)h(s)ds \in C([0, T), H^\mu) \) for \( h \in L^1 H^\mu \). The uniqueness of the solution in \( C([0, T), H^{\mu+1}) \cap C^1([0, T), H^\mu) \cap X^\mu(T) \) follows from the continuity of the solution, and the result that the existence time \( T \) is taken on the size of the norm of the data in our argument. See e.g., [23] for the details.

(2) Let \( v \) be the solution of the Cauchy problem for the data \( v_0 \) and \( v_1 \). Put \( \dot{D}^0(u - v) := \frac{1}{\varepsilon}\|u_1 - v_1\|^2 + \|\nabla(u_0 - v_0)\|^2 + \sqrt{Q}\|u_0 - v_0\|^2 \). By Lemma [2.4], we have

\[
d(u, v) \leq D^0(u - v) + c\|h(u) - h(v)\|_{L^1 L^2}
\]

and thus,

\[
d(u, v) \leq C_0 D^0(u - v) + Cc\lambda \left(\|A\|_q, BR_{\mu_0} + r_0\|\tilde{A}\|_{\tilde{q}, \tilde{\beta}}\right) R_{\mu_0} d(u, v).
\]

Since \( u \) is the solution in \( X^\mu(T, R_0, R_{\mu_0}, R_{\mu}) \) under the condition (4.18), \( v \) is in \( X^\mu(T, R_0 + \varepsilon, R_{\mu_0} + \varepsilon, R_{\mu} + \varepsilon) \) for sufficiently small \( \varepsilon > 0 \) when \( (v_0, v_1) \) is sufficiently close to \( (u_0, u_1) \). So that, \( Cc\lambda \left(\|A\|_q, BR_{\mu_0} + r_0\|\tilde{A}\|_{\tilde{q}, \tilde{\beta}}\right) R_{\mu_0} < 1 \) when \( (v_0, v_1) \) is sufficiently close to \( (u_0, u_1) \), which yields \( d(u, v) \to 0 \) as \( (v_0, v_1) \to (u_0, u_1) \).

(3) We take \( q_\ast = \tilde{q}_\ast = \infty \), thus, \( q = n, \tilde{q} = n/2 \) for the condition (i). We take \( q_\ast < \infty, \tilde{q}_\ast < \infty \) when for the condition (ii). Then the second condition in (4.18) is satisfied by

\[
C\lambda(BR_{\mu_0} + r_0\tilde{\beta})R_{\mu_0} \leq \frac{1}{2}
\]

for (i), where we need at least \( n \geq 4 \) to make \( \tilde{q}_\ast = \infty \) by \( \tilde{q} = n/2 \geq 2 \), or

\[
C\lambda \left( (2\mu_0 Hq_\ast)^{-1/q}, BR_{\mu_0} + r_0(\mu_0 H\tilde{q}_\ast)^{-1/\tilde{q}}, \tilde{\beta}\right) R_{\mu_0} \leq \frac{1}{2}
\]

for (ii). Since (4.20) or (4.21) holds for \( T = \infty \) and sufficiently small \( \dot{D}^\mu > 0 \), We obtain the global solutions under the condition (i) or (ii).

(4) The required results follow directly from Lemma [2.5] and \( h(u) \in L^1((0, \infty), H^\mu) \) as we have shown in (4.16).

(5) The local in time solution is obtained in (1) by setting \( \mu_0 = \mu = 0 \). Integrating the both sides in the equation \( \partial_\theta \tilde{e}^{0} + \partial_\beta \tilde{e}^{1} + \tilde{e}^{\mu+1} = 0 \) in (3) in Lemma [2.4], we have

\[
\int_{\mathbb{R}^n} \tilde{e}^{0}(t)dx = \frac{1}{2c^2}\left|\partial_\theta u(t)\right|^2 + \frac{1}{2}\left|\nabla u(t)\right|^2 + \lambda \left| r_0 \text{Re} u(t) + \frac{\left|u(t)\right|^2}{2}\right|_{L^2}^2 = \int_{\mathbb{R}^n} \tilde{e}^{0}(0)dx
\]

by the divergence theorem, where we have used

\[
\tilde{e}^{0} = \frac{1}{2c^2}\left|\partial_\theta u\right|^2 + \frac{1}{2}\left|\nabla u\right|^2 + \lambda \left( r_0 \text{Re} u + \frac{1}{2}\left|u\right|^2\right)^2
\]
and $\tilde{e}^{n+1} = 0$ by $H = 0$. So that, $\|\partial_t u(t)\|_2$ and $\|\nabla u(t)\|_2$ are uniformly bounded. In addition, $\|u(t)\|_2$ does not blow up since $\|\partial_t u(t)\|_2$ is bounded by

$$\|u(t)\|_2 \leq \|u(0)\|_2 + \int_0^t \|\partial_t u(s)\|_2 ds.$$ 

Since the existence time of our solutions obtained in (1) is taken by the size of the norm of the data $\tilde{D}^0$, we are able to show the existence of the global solution connecting the local solution.

5 Proof of Theorem 1.3

The proof of Theorem 1.3 follows analogously to that of Theorem 1.2. We only focus on the essential parts to prove (1).

(1) We consider the operator $\Phi$ defined by (4.1), and we show that $\Phi$ is a contraction mapping on the closed ball defined by (4.2) for some $T > 0$, $R_0 > 0$, $\nu = 0$, $\mu_0, \mu$, where $\|\cdot\|_{X^\nu}$ is defined by (4.1). We define $\theta, r_*, r_{**}$ by (4.2), and we have the property (4.1). We obtain the estimates (4.3) and (4.6) by the same argument. For any $q_0$ with $2 \leq q_0 < \infty$, assume $3 - n + 2\mu_0 \leq q_0$ when $\mu_0 > (n - 3)/2$. For any $q$ with

$$0 \leq \frac{1}{q} \leq \min \left\{ \frac{3 - n + 2\mu_0}{q_0}, \frac{1}{\tilde{q}_*} - \frac{3 - n + 2\mu_0}{q_0} \right\},$$

we define $q_*$ by

$$\frac{1}{q_*} := \frac{1}{q'} - \frac{3 - n + 2\mu_0}{q_0}. \quad (5.1)$$

We note that $q_*$ satisfies $1/q' = 1/q_* + 3(1 - \theta)/q_0$, and $q' = q_* \leq q_0$ holds by the conditions on $q_0$ and $q$. Thus, we have

$$c(-H)^{-1/q}e^{-(n-1)\mu_0 + 1} \leq B \|A\|_{q_*} \left\{ \langle -H \rangle^{1/q_0} \|u\|_{2q_0} \right\}^{\theta} \|u\|_{L^\infty \mu_0 + 1}$$

$$\leq B \|A\|_{q_*} \left\{ \langle -H \rangle^{1/q_0} \|u\|_{2q_0} \right\}^{\theta} \|u\|_{L^\infty \mu_0 + 1}$$

$$\leq B \|A\|_{q_*} \|u\|_{X^{q_0}} \|u\|_{X^*}$$

$$\leq B \|A\|_{q_*} R_{\mu_0}^2 R_{\nu} \quad (5.2)$$

for $\nu = 0, \mu_0, \mu$, and any $u \in X^{\mu}(T, R_0, R_{\mu_0}, R_{\mu})$, where we have put

$$A = A(t) := e^{-(1+\mu_0)HT} \quad \text{and} \quad B := c(-H)^{-1/q} Q^{(n-2\mu_0)/2}. \quad (5.3)$$

We note

$$\|A\|_{L^{2\nu}(0,T)} = \begin{cases} e^{-(1+\mu_0)HT} & \text{if } q_* = \infty, \\ e^{-(1+\mu_0)HT} \left( \frac{1}{2(1+\mu_0)H q_* - 1} \right) & \text{if } q_* < \infty. \end{cases}$$

We define $\tilde{\theta}$, $\tilde{r}_*$, $\tilde{r}_{**}$ by (4.10). Since we have the property (4.11), we obtain the estimates (4.12) and (4.13) by the same argument. For any $\tilde{q}_0$ with $2 \leq \tilde{q}_0 \leq \infty,$
assume \((4 - n + 2\mu_0)/2 \leq \tilde{q}_0\) when \(\mu_0 > (n - 4)/2\). For any \(\tilde{q}\) with
\[
0 \leq \frac{1}{\tilde{q}} \leq \min \left\{ \frac{1}{2}, 1 - \frac{4 - n + 2\mu_0}{2\tilde{q}_0} \right\},
\]
we define \(\tilde{q}_*\) by
\[
\frac{1}{\tilde{q}_*} := \frac{1}{\tilde{q}'} - \frac{4 - n + 2\mu_0}{2\tilde{q}_0}. \tag{5.4}
\]
We note that \(\tilde{q}_*\) satisfies \(1/\tilde{q}' = 1/\tilde{q}_* + 2(1 - \tilde{\theta})/\tilde{q}_0\), and \(\tilde{q}' \leq \tilde{q}_* \leq \infty\) holds by the conditions on \(\tilde{q}_0\) and \(\tilde{q}\). Thus, we have
\[
e^{(-H)^{-1/\tilde{q}}} e^{-(n-2)Ht/2} (2u \operatorname{Re} u + |u|^2) L_{\tilde{q}'} H\nu
\leq \tilde{B} \|\tilde{A}\| \tilde{q}_* \|u\| \hat{\chi}_{\tilde{q}_0} \|u\| \hat{\chi}_\nu
\leq \tilde{B} \|\tilde{A}\| \tilde{q}_* R_{\mu_0} R_{\nu}
\]
for \(\nu = 0, \mu_0, \mu\), and any \(u \in X^\mu(T, R_0, R_{\mu_0}, R_{\mu})\), where we have put
\[
\tilde{A} = \tilde{A}(t) := e^{-(1+\mu_0)Ht} \text{ and } \tilde{B} := e^{-H}_{-1/\tilde{q}_*} Q^{(n-4-2\mu_0)/4}.
\tag{5.6}
\]
We note
\[
\|\tilde{A}\|_{L_{\tilde{q}_*}((0,T))} = \begin{cases} e^{-(1+\mu_0)HT} & \text{if } \tilde{q}_* = \infty, \\ \left\{ -e^{-(1+\mu_0)HT} \frac{e^{-(n-1)Ht}}{\tilde{q}_0} \right\}^{1/\tilde{q}_*} & \text{if } \tilde{q}_* < \infty. \end{cases}
\]
Since we have
\[
\|\Phi(u)\|_{\tilde{\chi}'} \lesssim \frac{1}{c} \|u_1\|_{H^\nu} + \|\nabla u_0\|_{H^\nu} + \sqrt{Q} \|u_0\|_{H^\nu}
+ \lambda c(-H)^{-1/\tilde{q}} e^{-(n-1)Ht} |u|^2 \|u\|_{L_{\tilde{q}_*}^\nu H^\nu}
+ \lambda r_0 c(-H)^{-1/\tilde{q}_*} e^{-(n-2)Ht/2} (2u \operatorname{Re} u + |u|^2) \|L_{\tilde{q}_*} H\nu
\]
by Lemma 2.5, we have
\[
\|\Phi(u)\|_{\tilde{\chi}'} \leq C_0 \tilde{D}' + C \lambda \left( \|A\| q_*, B R_{\mu_0} + r_0 \|\tilde{A}\| \tilde{q}_* \tilde{B} \right) R_{\mu_0} R_{\nu} \leq R_{\nu} \tag{5.7}
\]
for \(\nu = 0, \mu_0, \mu\) for some constants \(C_0 > 0, C > 0\), and any \(u \in X^\mu(T, R_0, R_{\mu_0}, R_{\mu})\) by (5.2) and (5.5) if \(R_0, R_{\mu_0}\) and \(R_{\mu}\) satisfy
\[
R_{\nu} \geq 2C_0 \tilde{D}', \quad C \lambda \left( \|A\| q_*, B R_{\mu_0} + r_0 \|\tilde{A}\| \tilde{q}_* \tilde{B} \right) R_{\mu_0} \leq \frac{1}{2} \tag{5.8}
\]
for \(\nu = 0, \mu_0, \mu\). On the metric, we are able to obtain
\[
d(\Phi(u), \Phi(v)) \leq C \lambda \left( \|A\| q_*, B R_{\mu_0} + r_0 \|\tilde{A}\| \tilde{q}_* \tilde{B} \right) R_{\mu_0} d(u, v) \leq \frac{1}{2} d(u, v) \tag{5.9}
\]
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for any \( u, v \in X(T, R_0, R_{\mu_0}, R_{\mu}) \) analogously to (4.19), provided the second condition in (5.8). So that, \( \Phi \) is a contraction mapping on \( X(T, R_0, R_{\mu_0}, R_{\mu}) \) under (5.8). Especially, (5.8) holds if \( T > 0 \) is sufficiently small such that

\[
C\lambda c(-H)^{-1} \left\{ \frac{e^{-4(1+\mu_0)HT}-1}{4(1+\mu_0)} \right\}^{1/2} Q^{(n-3-2\mu_0)/2} R_{\mu_0} + r_0 \left\{ \frac{e^{-2(1+\mu_0)HT}-1}{2(1+\mu_0)} \right\}^{1/2} Q^{(n-4-2\mu_0)/4} \leq \frac{1}{2}
\]

when \( q_0 = \tilde{q}_0 = \infty \) and \( q = \tilde{q} = q_s = \tilde{q}_s = 2 \), and \( R_\nu = 2C_0 D^\nu \) for \( \nu = 0, \mu_0, \mu \).

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