Second-order efficiency of fully sequential designs for estimating the product of two means with application in reliability estimation

Abstract

For estimating the product of two means from the general one parameter exponential family, we consider a fully Bayesian approach with conjugate priors. We derive a sharp lower bound for the Bayes Risk. We also propose a fully sequential design with an incurred Bayes Risk near the second order lower bound. An application to reliability estimation is performed analytically and through Monte Carlo simulation.

Keywords: Bayesian estimation, sequential designs, second-order optimality, one-parameter exponential family, reliability estimation

Introduction

Solving estimation problems using a fully Bayesian approach for the one-parameter exponential family with conjugate priors for sequential designs was first investigated by Shapiro and Wardrop and developed for linear combinations of means thoroughly by Rekab et al. (2013) and many others.13

For estimating the product of means of two independent Bernoulli populations with Beta priors which has an application in estimating the reliability of a system, Rekab & Song1 have derived the first order lower bound for the Bayes Risk and proposed a sequential design that achieves it.

The first-order efficiency for any sequential procedure, \( \rho \), is

\[
R(\rho) \geq \frac{E\left[ \sum_{i=1}^{k} \sqrt{T(\eta)} \prod_{i=1}^{k} \psi(\eta) \right]^2}{T} + o\left( \frac{1}{T} \right)
\]

In this article, we will derive a sharper lower bound for the Bayes Risk and propose a sequential design that will achieve at least asymptotically. Such a design will be referred to as second order efficient design.

The problem of estimating system reliability is the same as estimating the product of means of independent Bernoulli populations. We use independent Beta priors for the means and propose a sequential design that is second order efficient, it converges faster to the optimal ratio than the first order designs. Second order sequential designs are sought and show the optimality of the fully sequential design through an application of reliability estimation using Monte Carlo simulation.

Second-order efficiency

The density function of one-parameter exponential family

\[
f_{\eta}(x) = \exp \left\{ \eta x - \varphi(\eta) \right\} \mathcal{d}\eta \wedge (x), -\infty < x < \infty, \eta \in \Omega, \quad (2.1)
\]

where \( \eta = \eta(\theta) \) is real-valued function, \( \varphi \) is a continuously differentiable, real-valued function, \( \wedge \) is a non-degenerate sigma-finite measure and \( \Omega = [\eta_l, \eta_u], -\infty < \eta_l < \eta_u < \infty \), is a non-empty open interval, \( i=1,2 \). Using the square error loss, and by adopting the Bayesian approach, the conjugate prior \( \pi(\theta) \) and the posterior density \( \pi(\theta|X_1, X_2) \) are both the one-parameter exponential family. The prior density as derived by Diaconis & Ylvisaker.4

\[
\pi(\theta) = \frac{\exp \{ r_{i0} \mu - r_{i0}' \theta \}}{C(r_{i0}, \mu_0)}, \mu_i \in \Omega, \quad (2.2)
\]

where

\[
C(r_{i0}, \mu_0) = \int \exp \{ r_{i0} \mu - r_{i0}' \theta \} d\theta \quad (2.3)
\]

Consider the problem of allocating a fixed total number of observations from the independent populations, where \( T = M_1 + M_2 \), \( M_i \) is the sample size of component \( i=1,2 \).

The Bayes Risk for estimating the product of two means

\[
\rho(\theta) = \frac{E\left[ \sum_{i=1}^{k} \sqrt{T(\eta)} \prod_{i=1}^{k} \psi(\eta) \right]^2}{T} + o\left( \frac{1}{T} \right)
\]

where \( C \) is the sigma algebra generated by a total of \( T \) observations.

To ease the notation, let \( D_i = E\left[ \psi^2(\eta) \right] \), \( C_i = E\left[ \psi(\eta) \right], i=1,2, \)

Next, we derive the second order lower bound for the Bayes Risk.

\[
R(\rho) = E\left[ \frac{E\left[ \psi^2(\eta) | \mathcal{F}_T \right] E\left[ \psi(\eta) | \mathcal{F}_T \right]}{M_1} + E\left[ \psi^2(\eta) | \mathcal{F}_T \right] E\left[ \psi(\eta) | \mathcal{F}_T \right] \right] + \frac{E\left[ \psi^2(\eta) | \mathcal{F}_T \right]}{M_2} \left( E\left[ \psi(\eta) | \mathcal{F}_T \right] + \frac{E\left[ \psi(\eta) | \mathcal{F}_T \right]}{M_2} \right)
\]

where \( \mathcal{F}_T \) is the sigma algebra generated by a total of \( T \) observations.

Theorem 2.1: For any sequential procedure, \( \mathcal{F}_T \) that satisfies the following conditions:

\[
M_i \rightarrow \infty \text{ in probability, as } T \rightarrow \infty,
\]

\[
\frac{M_i}{T} \rightarrow \frac{\sum_{i=1}^{k} \sqrt{T(\eta)} \psi(\eta)}{\sum_{i=1}^{k} \sqrt{T(\eta)} \psi(\eta)} \text{ in probability, as } T \rightarrow \infty
\]

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Let the speed be defined as \( \Delta = R(p) - R(p_1) \).

The data used to support the findings of this study have been produced by Monte Carlo simulation from Bernoulli trials with 5000 replications.

Table 1 Fully sequential sampling design with uniform priors

| \( T \) | 50 | 70 | 100 | 200 | 400 | 600 | 800 | 1000 | 3000 |
|------|----|----|-----|-----|-----|-----|-----|------|------|
| \( E[m_1] \) | 26 | 36 | 51 | 101 | 200 | 301 | 400 | 498 | 1504 |
| \( E[m_2] \) | 24 | 34 | 49 | 99 | 200 | 299 | 400 | 502 | 1496 |

Table 2 \( T^\beta \Delta < C \) as \( t \to \infty \), where \( \Delta = R(p) - R(p_1), 0 \leq \beta < 2, C \in \mathbb{R} \)

| \( T \) | 50 | 70 | 100 | 200 | 400 | 600 | 800 | 1000 | 3000 |
|------|----|----|-----|-----|-----|-----|-----|------|------|
| \( T^\Delta \) | 0.02 | 0.019 | 0.026 | 0.029 | 0.033 | 0.035 | 0.034 | 0.031 | 0.032 |
| \( T_1^\Delta \) | 0.138 | 0.159 | 0.262 | 0.415 | 0.662 | 0.619 | 0.955 | 0.969 | 1.776 |
| \( T_2^\Delta \) | 0.976 | 1.329 | 2.62 | 5.863 | 13.231 | 15.169 | 27.001 | 30.627 | 97.281 |

Conclusion

The proposed sequential design enables the experimenter to determine the optimal allocation of units to be tested from each component when component is functional independently. Rekab used a classical approach to estimate the reliability of a series system, and proposed a sequential design that was shown to be first-order optimal, that is \( T^\Delta \to 0 \). The fully Bayesian approach enables us to investigate a design with a converge rate of the order of \( T^{\beta}, 0 \leq \beta < 2 \).

Data availability

The data used to support the findings of this study have been produced by Monte Carlo simulation from Bernoulli trials with 5000 replications.

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Figure 1 $\Delta = R(p) - R(p_2)$ is bounded by $T^{-\beta}$ as $T \to \infty$, where $0 \leq \beta < 2$.

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Conflicts of interest

Authors declare that there are no conflicts of interest.

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