Fock space localization in the Sachdev-Ye-Kitaev model

Felipe Monteiro, Tobias Micklitz
Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180, Rio de Janeiro, Brazil

Masaki Tezuka
Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Alexander Altland
Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Cologne, Germany

(Dated: May 27, 2020)

We study the physics of many body localization in the Majorana Sachdev-Ye-Kitaev (SYK) model perturbed by a one-body Hamiltonian. Specifically, we consider the statistics of many body wave functions and spectra as the strength of the one-body term is ramped up from an ergodic phase via a phase of non-ergodic yet extended states into a (Fock space) Anderson localized phase. Our results are obtained from an effective low energy theory, derived from the microscopic model by matrix integral techniques standard in the theory of disordered electronic systems. Applicable to systems of arbitrary (yet large) particle number the analytical results produced by this formalism are in excellent agreement with numerics for systems of $O(10^3)$ particles. An important message of this study is that in systems of numerically accessible size the ‘true’ Fock space Anderson localization transition is masked by more trivial mechanisms of localization due to the trapping of states in deep potential wells. It stands to reason that such type of trapping will be operational in spatially extended systems with many body localization as well, and govern the physics of small sized systems. Only for larger particle numbers beyond $O(10^3)$ a true Anderson-Fock space transition (at much lower disorder strength than those required to induce trapping) would emerge.

PACS numbers: 05.45.Mt, 72.15.Rn, 71.30.+h

I. INTRODUCTION

Quantum mechanical wave functions in random environments often show non-ergodic or ‘localized’ behavior. Half a century after the first discovery in Ref. [1] we now distinguish three universality classes: (i) Anderson localization (AL) of single particle wave functions, (ii) Fock space localization (FSL) in the many particle Fock spaces of interacting but spatially confined quantum systems [2, 3], and (iii) many body localization (MBL) in spatially extended geometries [4, 5]. While Anderson localization is now relatively well understood, the situation in the interacting cases is less clear. The degree of uncertainty shows in that even the principal manifestations of MBL in extended geometries remain controversial [6–8]. Many body localization is a tough problem for it combines the complexity of strong interactions with that of spatial extendedness. The explosion of Fock space dimensions as the number of degrees of freedom increases makes numerical access to the phenomenon infamously hard. In this regard, the phenomenon of Fock space localization occupies an interesting middle ground: driven by correlations, it is in a universality class different from Anderson localization. However, absent spatial extension, the localizing and delocalizing tendencies of interactions are more transparent than in traditional MBL settings.

Starting with the seminal paper [2], key aspects of FSL have been addressed in numerous studies [9–17]. The most striking signature distinguishing FSL from AL is that the passage between ergodic and localized states in Fock spaces occurs via an intermediate phase where states remain extended but with non-uniform distribution. At this point, the nature of this phase of non-ergodic extended states (NEE states) is only partially understood. Absent both first principle descriptions and sufficiently high powered numerics of genuinely interacting systems, the NEE phase is described in terms of model systems mimicking the complexity of Fock spaces, such as Bethe lattices [12, 18], or random regular graphs [11, 12, 16, 19]. However, even there, the situation remains somewhat opaque, with some authors predicting NEE phases with (multi)fractal many body wave function statistics [12], and others [20, 21] less drastic scenarios of reduced Hilbert space wave function support.

In this paper, we discuss FSL for a genuinely interacting fermionic quantum system, the SYK model. The SYK Hamiltonian has become a model paradigm for strongly correlated quantum matter [22–28]. Generalized for the presence of a one-body contribution, its complex version describes spatially confined correlated quantum matter as quantum dots, heavy nuclei, complex molecules. We here focus on the real (Majorana) variant, where the lack of number conservation makes for a somewhat simpler description at essentially unaltered FSL properties. The point made in this paper is that the system is amenable to analytic solution by first principle methods of localization theory [29, 30]. For all we know, this is the only many body system which can be described analytically all the way from ergodic regimes, over an intermediate phase of non-ergodic extended states into the
phase of strong Fock space localization.
Topics addressed below include:

- the diagnostics of the FSL transition via spectral statistics,
- the description of the NEE phase via many body wave function statistics,
- the verification of the above results by parameter free comparison to exact diagonalization,
- the identification of numerical pitfalls (for finite size systems a crossover in the statistical properties of the system mimics a localization transition. The true localization transition would become visible only for sizes of $N \sim 10^4$, out of numerical reach),
- the validity of mapping FSL onto a localization problem defined on a high dimensional graph.

The work reported here builds on a previous publication [31], where three of the present authors demonstrated the existence of a phase of non-ergodic extended states in a random energy variant of the SYK model.

Model: the SYK Hamiltonian \[ \hat{H}_4 = \frac{1}{4!} \sum_{i,j,k,l=1}^{2N} J_{ijkl} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l, \] (1)
describes a system of 2N Majorana fermions, $\{ \hat{x}_i, \hat{x}_j \} = 2\delta_{ij}$, subject to all–to–all interaction, with matrix elements $\{ J_{ijkl} \}$ drawn from a Gaussian distribution of variance $\langle |J_{ijkl}|^2 \rangle = 6J_k^2/(2N)^3$. Defined in this way, it defines an ideal of a massively interacting quantum system lacking any degree of internal structure. Due to the ‘least information’ principle realized through the stochastic interaction, all single particle orbitals, $i$, stand on equal footing, and the absence of a continuous U(1)–symmetry prevents the fragmentation of the Fock space into sectors of conserved particle number. Reflecting these features, the physics of the SYK Hamiltonian at large time scales becomes equivalent to that of random matrix theory (RMT), with wave functions homogeneously distributed over the full Hilbert space.

A tendency to Fock space localization is included by adding to $\hat{H}_4$ a free particle contribution $\hat{H}_2$.

\[ \hat{H}_2 = \frac{1}{2} \sum_{j,k=1}^{2N} J_{ij} \hat{\chi}_i \hat{\chi}_j, \] (2)

with a likewise random antisymmetric matrix $J_{ij} = -J_{ji}$, with matrix elements $\{ J_{ij} \}$ drawn from a Gaussian of variance $\langle |J_{ij}|^2 \rangle = J_k^2/2N$.

We now discuss in what sense $\hat{H}_2$ induces localization. Without loss of generality, we may assume $\{ J_{ij} \}$ to be diagonalized into a form $\hat{H}_2 = \sum_i v_i \hat{\chi}_{2i-1} \hat{\chi}_{2i}$, where $\pm v_i$ are the eigenvalues of the hermitian matrix $i \{ J_{ij} \}$. We define $N$ complex fermion annihilation operators $\hat{c}_i = \frac{1}{2} (\hat{\chi}_{2i-1} + i \hat{\chi}_{2i})$ for $i = 1, \ldots, N$, satisfying $\{ \hat{c}_i, \hat{c}_j^\dagger \} = \delta_{ij}$, and number operators $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$, and $\hat{H}_2$ assumes the form $\hat{H}_2 = \sum_{i=1}^N v_i (2\hat{n}_i - 1)$. (3)

This shows that for asymptotically strong $\hat{H}_2$ the Fock space eigenstates are localized in the $2^N$ states of the Fermion occupation number basis, $|n\rangle = |n_1, n_2, ..., n_N\rangle$, $n_i = 0, 1$. Seen in this way, the Fock space becomes a hypercube defined by the $2^N$ corner sites, $\hat{H}_2$ defines an on-site localizing potential, and $\hat{H}_4$ a hopping operator connecting sites of maximal bit separation 4. (For two states $|n\rangle, |m\rangle$ we define the Hamming distance $|n - m|$ as the number of bits in which the states differ. The action of the hopping operator is limited to states of Hamming distance four and less.) [36] The nature of the ensuing random lattice system is illustrated in Fig. [1] for a Fock space of 14 Majorana fermions. The figure indicates the connectivity of the state $\{ 0, 0, 0, 1, 1, 0, 0 \}$ to illustrate the characteristics of the lattice structure — sparsity, irregularity of connections, and statistical correlation of random hopping amplitudes due to the small number of independent random coefficients in an exponentially high dimensional space.

Plan of the paper: In view of the somewhat technical core parts of our analysis, section II contains a qualitative discussion of the physics of the system and a summary of all our results. In section III we map the computation of disorder averaged correlation functions onto that of an equivalent matrix integral. In section IV a stationary phase approach is applied to reduce the matrix integral to an effective theory describing physics at large time scales. In sections V and VI we apply this representation to the discussion of wave function statistics and the localization transition, respectively. We conclude in section VII. Technical parts of our analysis are relegated to a number of Appendices.

II. SUMMARY OF RESULTS

The intuitive picture behind FSL in the SYK system is similar to that of many body localization in spatially extended systems. For asymptotically strong $\hat{H}_2$, states are localized in $|n\rangle$ with associated Poisson correlated eigenvalues [27] $v_n \equiv \sum_i v_i (2n_i - 1)$. The interaction Hamiltonian $\hat{H}_4$ induces hopping via transitions $|n\rangle \to |m\rangle$. Expressing $\hat{H}_4$ in terms of complex Feronion creation and annihilation operators, we see that $|n\rangle$ and $|m\rangle$ differ at most by four in their total occupation number $n \equiv \sum_i n_i$. Although $\hat{H}_4$ contains only an algebraically small number $\sim N^4 \sim \log(D)$ of independent matrix elements, it is efficient at inducing many body chaos, including for strengths of $\hat{H}_2$ much larger than those of $\hat{H}_4$. To see how, assume that the typical eigenvalues of $\hat{H}_2$ are dis-
The regimes of smallest and highest on site disorder, I and IV, respectively, are the easiest to understand: in I the band width of the diagonal is lesser than that of the SYK4 Hamiltonian. Wave functions are ergodic, and the spectral statistics is of Wigner-Dyson type, as in the unperturbed case. In the complementary regime IV, wave functions are localized on individual Fock space sites \( n \) with energies \( v_n \). Such sites may hybridize with neighbors \( m \) of energy \( v_m \) if the energies are resonant with the strength of the coupling matrix element, \( J/\sqrt{N} \sim |v_n - v_m| \). Typical nearest neighbor energy differences are of \( \mathcal{O}(\delta) \), but there is also a large number \( \sim N^4 \) of them. The expected number of resonant neighbors thus scales as \( N^4 J/\sqrt{N} \) which equals \( N^2/\delta \) for our chosen units. This leads to the heuristic criterion \( \delta \sim N^2 \) for delocalization, confirmed (up to corrections logarithmic in \( N \)) by the analysis below.

For energies between the extremes, \( N^{-1/2} \leq \delta \leq N^2 \), wave functions are neither localized nor ergodically extended. A key quantity to the description of these regimes is the local band center spectral density averaged over realizations of \( \hat{H}_4 \), \( \nu_n = -\frac{1}{2} \text{Im} (n|0^- - \hat{H}_2 - \hat{H}_4)^{-1}|n\rangle \). Below, we show that this quantity is given by

\[
\nu_n = \frac{1}{\pi} \text{Im} \frac{1}{v_n - i\kappa_n},
\]

\[
\kappa_n \equiv \pi (\mathcal{P}\nu)_n \equiv \pi \sum_m \mathcal{P}_{n-m}|\nu_m|,
\]

where the first line states that the average local density of states (DoS) is defined by states broadened by a scale \( \kappa_n \), and the second line expresses the broadening as a sum over SYK4 hybridization processes governed by the nearest neighbor energy denominators \( \pi \nu_m \) and a hopping operator \( \mathcal{P} \). The latter is unit normalized as \( \sum_m \mathcal{P}_{n-m} = 1 \) and limited to state separations \( |n-m| \leq 4 \) (see Eq. [31] for the precise definition), while \( \nu \) (here and in the following) denotes a diagonal matrix in occupation basis with diagonal elements \( \nu_n \). Eq. [5] is a self consistent equation for the local density of states, \( \nu_n \), and the level hybridization, \( \kappa_n \). What simplifies its solution is the effectively random distribution of site energies, \( v_m \), affecting the DoS at \( n \). The largeness \( \mathcal{O}(N^4) \) of these contributions justifies a statistical approach, which leads to the solution

\[
\kappa_n \simeq \kappa \Theta (C - |v_n|),
\]

\[
(\kappa, C) \equiv \begin{cases} (1, 1), \quad \delta < 1 \quad (I, II), \\ (\delta^{-1}, \delta), \quad \delta > 1 \quad (III, IV). \end{cases}
\]

This result states that (for center energy \( E = 0 \)) only sites with energy lower than a threshold \( C \) contribute to the DoS. The value of this threshold depends on the spread of the site energy spectrum, where in the weak disorder regimes, I, II, sites within the SYK4 band width all contribute, \( C = 1 \), and the levels are broadened over the full band width. For stronger disorder, III, IV, the spectral support is limited to energies \( |v_n| < \delta \). With \( v_n \)

FIG. 1: Hypercubical Fock space of an \( 2N = 14 \) Majorana system. The numbers indicate the bit depth of states in the computational fermion basis, and the lines are a qualitative representation of the connectivity of the reference state \( |0, 0, 0, 1, 1, 0, 0\rangle \). For large values of \( N \), the pattern of connections becomes sparse. However, there remain exponentially many \( \propto D \) connections, statistically correlated due to the small number of \( \sim N^3 \) of independent random amplitudes.
distributed over a range $\sim N^{1/2}\delta$, the ratio of active sites to the total number of sites is $\sim N^{-1/2}$, which depends on $N$ but not on the energy spreading itself. Within this restricted set, there is a still smaller set of energies $|v_n| \lesssim \delta^{-1} \sim \kappa_n$ defining a class of resonant sites for which the real part of the energy denominator, $v_n$, is comparable to the broadening, $\kappa$. This hierarchy makes regime III different from II where all active sites are automatically resonant. Active and resonant sites define the support of many-body wave functions in regimes I-III. However, to describe how this materializes in concrete ways, we need information beyond the average spectral density.

In the rest of the paper, we consider the spectral two point correlation function at the band center

$$K(\omega) = \frac{1}{\nu} \langle \nu(\frac{\omega}{2}) \nu(-\frac{\omega}{2}) \rangle_\epsilon,$$  \hspace{1cm} (7)

and the moments of wave functions $|\psi\rangle$ of zero energy, $\epsilon_\psi$,

$$I_q = \frac{1}{\nu} \sum_n \langle (|\psi| n)^q \delta(\epsilon_\psi) \rangle_J,$$  \hspace{1cm} (8)

as principal diagnostic quantities. In these expressions, $\nu = \nu(E \approx 0)$, $\nu(E) = \sum_\psi \langle \delta(E - \epsilon_\psi) \rangle_J$ is the average many-body density of states at zero energy $E \approx 0$, with $\langle \cdots \rangle_J$ indicating the average over the realization of $\{J_{ijkl}\}$ and with a fixed realization of $\{J_{ij}\}$. The subscript c refers to the connected correlation function, $\langle \nu(\frac{\omega}{2}) \nu(-\frac{\omega}{2}) \rangle_J - \langle \nu(\frac{\omega}{2}) \rangle_J \langle \nu(-\frac{\omega}{2}) \rangle_J$. Spectral statistics: In regimes I-III wave functions are extended and their eigenenergies are correlated, with Wigner-Dyson statistics. Assuming an odd number $N$ of complex fermions, for which the SYK model sits in the unitary symmetry class for simplicity, this reflects in the spectral statistics of the Gaussian unitary ensemble,

$$\tilde{K}(s) = 1 - \frac{\sin^2 s}{s^2} + \delta(s/\pi), \hspace{1cm} s = \pi \omega \nu,$$  \hspace{1cm} (9)

where $\nu = \sum_n \nu_n$ is the average density of states. Differences between the regimes I-III show in the value of this scale, which in essence counts the number of active sites according to Eq. (6). To obtain a more explicit expression, we assume that the summation over $n$ sites can be replaced by an average over the distribution of site energies, $v_n$. This leads to

$$\nu = \sum_n \nu_n = cD \left\{ \frac{1}{\sqrt{N \delta}} \right\}_{I, \text{II, III}},$$  \hspace{1cm} (10)

where, here and throughout, $c = \mathcal{O}(1)$ represents numerical constants, and

$$D = 2^{N-1}$$  \hspace{1cm} (11)

is the dimension of Fock space projected onto sites of definite (even, for concreteness) occupation number parity. (Both $\hat{H}_{4,2}$ conserve parity and we focus on a definite parity sector.) The second line states that in the regimes of intermediate disorder strength, only a fraction $D/\sqrt{N\delta}$ of active sites contributes to the spectral support of wave functions.

Wave function statistics: In the weak disorder regime, I, wave functions are ergodically distributed over the full Fock space, with moments given by those of the Porter-Thomas distribution,

$$I_q = q!D^{1-q}, \hspace{1cm} I.$$

In essence, this states that the statistics of the complex amplitudes $\langle n|\psi\rangle$ is that of independent Gaussian random variables.

In regimes II and above the wave functions no longer ergodically occupy the full Fock space. Their support is limited to the subset of active sites, and within this subset they are extended over the subset of resonant sites. We find that the corresponding set of moments is given by

$$I_q = c^q \left( \frac{D}{\sqrt{N\delta}} \right)^{1-q} \frac{2(2q - 3)!!}{\kappa q^{-1}}, \hspace{1cm} \text{II, III},$$  \hspace{1cm} (13)

where $c = \mathcal{O}(1)$. To make the connection of this expression to Eq. (12) more transparent, consider the case of large $q$, where

$$q \gg 1 : \hspace{1cm} I_q = cq!D_{\text{res}}^{1-q},$$

$$D_{\text{res}} = D \left\{ \frac{\sqrt{N\delta}}{\sqrt{N\delta^2}} \right\}_{\text{II, III}}.$$  \hspace{1cm} (14)

This results states that for large $q$ the moments become that of a Gaussian distribution defined relative to the number $D_{\text{res}}$ of resonant sites in Fock space. Noting that $\delta \sim N^\alpha$, $\alpha < 2$, the dependence of $D_{\text{res}}$ on $D$ is approximated as

$$D_{\text{res}} = D / \log D^\beta, \hspace{1cm} \beta = \begin{cases} \alpha + 1/2 & \text{II}, \\ 2\alpha + 1/2 & \text{III}. \end{cases}$$  \hspace{1cm} (15)

This suggests an interpretation of a weak fractal whose dimension differs from the naive dimension by a factor $D/\log D^\beta \sim D/D^0$, rather than the usual fractal $D/D^\gamma$ with some $\gamma > 0$.

Finally, note that the support of wave functions in regimes II and III is different (as indicated by the different value of $D_{\text{res}}$), while the DoS Eq. (10) assumes the same value. The reason for this is that, in regime II, there is no distinction between active and resonant sites: there are $\sim D/\sqrt{N\delta}$ active sites contributing with unit weight to the DoS. By contrast, in III, the dominant contribution to the DoS comes from the smaller number of $D_{\text{res}} \sim D/\sqrt{N\delta}$ resonant sites, with sharply peaked spectral weight $\sim \delta$, $\nu \sim D_{\text{res}} \delta \sim D/\sqrt{N\delta}$.

Figures 2 and 3 show a comparison of our analytical predictions to numerical simulations for $2N = 22, 26, 30$.
In both figures we used a fixed value of $\delta = 3$. See also discussion in main text.

FIG. 2: Verification of the scaling of our analytical prediction Eq. (16) in $q$ and $N$, respectively. Here $\delta = 3$. Majorana fermions. Error bars are smaller than symbol size and omitted for clarity. Numerically accessible system sizes are limited and outside $1/\sqrt{N} \ll 1$, required by regime II. We, therefore, concentrate on regime III where, keeping track of all numerical factors $c \sim \mathcal{O}(1)$, we predict (see Appendix E)

$$I_q = \frac{q(2q - 3)!!}{\delta^{2(1-q)}} \left( \frac{\pi D}{4\sqrt{N}} \right)^{-q}.$$

(16)

The upper panel of Fig. 2 verifies the scaling of Eq. (16) in $q$. Noting that $I_q \equiv I_q/(q(2q - 3)!!) = f_{N,\delta}^q$ we predict a collapse of all curves $\log(I_q)/\log(I_2)$ for different $N$ on the line $q - 1$, in agreement with the numerical simulation. The lower panel of Fig. 2 confirms the validity of the predicted $N$-dependence of (16), here for $q = 2, 3, 4$. In both figures we used a fixed value $\delta = 3$ of the diagonal distribution, deep within regime III. Fig. 3 compares the scaling of $\log(I_q)$ in $N$ for $N = 15$, where the dashed lines mark the boundaries between different regimes. The inset indicates the change of spectral statistics in terms of the Kullback-Leibler entropy of the observed level distribution relative to that of the Wigner-Dyson and Poisson distribution, respectively. More specifically, we show $D(P||Q) = \sum_k p_k \log \left( \frac{p_k}{q_k} \right)$, where $p_k$ is the spectral statistics from numerical data and $q_k$ the respective analytical distribution. In order to avoid level unfolding, we follow Ref. [10] and study the statistics of ratios of energy denominators, $r_n = \min(s_{n+1}, s_{n-1})$, where $s_n = e_{n+1} - e_n$ is the nearest neighbor spacing of the eigen energies $\{e_n\}$. $q_k$ are then the Wigner-Dyson and Poisson distributions for the variable $r_n$, given by Ref. [41]. In all figures we averaged over at least 1000 samples, eigenvectors (resp. eigenstates) at the band center [42], and both parity sectors. In computing the Kullback-Leibler entropy, the distribution of $r_n$ is obtained by splitting [0, 1] into 50 bins of equal widths. Finally, we emphasize that all comparisons are done without use of any fitting parameter.

**Fock space localization:** Turning to the regime of strong disorder, the phenomenon of localization is often addressed phenomenologically by describing Fock space as a synthetic lattice such as a Bethe lattice [12, 18] or a random regular graph [11, 12, 16, 19]. On these lattices uniform hopping amplitudes between sites of randomly chosen energy mimic the hybridization between Fock space sites which are resonant according to our classification above. A competition between the high graph connectivity, and the impedance of transport due to randomly varying energy denominators then induces an Anderson transition at a critical combination of hopping strength, disorder, and graph coordination number [30].

We here follow a somewhat different approach and work directly in Fock space. We avoid assumptions on uniform site connectivity, and the early restriction to the subset of resonant sites. In this way, we arrive at a model description where the much larger set of active sites contributes to the problem via lattice hopping amplitude showing large variations due to strong variations in the energy denominators. What helps to keep this more com-
plicated problem under control is the huge effective lattice coordination number of $\mathcal{O}(N^4)$, and a simplification known as the ‘effective medium approximation’. This approximation describes transport in Fock space as a process avoiding local loops (see Fig. 4), while multiple link traversals (top right) are included. The rational is that at any given order in hopping perturbation theory, amplitudes with the lowest number of statistically independent energy denominators contribute the strongest. (Note that in the phenomenological description via Bethe type lattices this condition is hard wired from the beginning due to the absence of loops on such graphs.) The effective medium approximation sums over all transition amplitudes avoiding loops, but including multiple traversal of links (Fig. 4 bottom). The result of this analysis is that the criterion determining whether a system with site energies $\{v_m\}$ will be in a localized phase reads,

$$1 \simeq \sum_{n-m=4} \nu_m c_N \log \left( \frac{1}{\sqrt{\nu_m c_N}} \right), \quad c_N = \frac{\sqrt{6\pi}}{N^2},$$

(17)

where the sum extends over the $\mathcal{O}(N^4)$ sites $m$ reachable from a given resonant site $n$ by SYK$_4$ hopping. While the criterion is articulated for specific sites $n$, the large number of terms in the sum implies self averaging and independence of the reference site. Replacing the sum by an average over the distribution of energies $v_m$, we find that the equation reduces to

$$\delta_c = \frac{3}{\sqrt{2}} N^2 \log N^2,$$

(18)

Physically, the equations above are in line with the naive argument for localization stated in the introduction: the factor $c_N \sim N^{-2}$ represent the strength of the interaction matrix element. The number of $n$-neighbors, $m$, whose energy is comparable to that scale is given by the product $\sim \frac{1}{\sqrt{\nu}} \sum_m \nu_m$, and the criterion asks when this product becomes of $\mathcal{O}(1)$. Finally, the logarithmic correction extends the first order perturbation theory argument to account for hopping processes of higher order.

We finally address a question which may be of high relevance to the study of localization phenomena in many body systems at large:

**Will the Fock space transition ever be numerically observable?** To understand what is meant by this question, suppose we are at disorder concentrations below the localization threshold Eq. (18). The inverse participation ratio $I_2$ will then be given by Eq. (13), and with (Eq. (6)) $\kappa = \delta^{-1}$, we obtain

$$I_2 \simeq c \sqrt{N} \delta^2 \frac{1}{D}.$$

For $\delta \equiv \delta_c' \sim \sqrt{D/\sqrt{N}}$ the inverse participation ratio as described by this formula becomes of $\mathcal{O}(1)$, signifying localization. However, this trivial mechanism of ‘localization’ by trapping is different from that of the Fock space transition above. It simply reflects the fact that the potential fluctuations become so large that the support of wave functions is limited to only few sites with finite density of states. By contrast, the true FSL transition occurs within a larger set of sites with finite density of states and separates an extended phase from one where wave functions are localized by quantum interference. To illustrate this point on a conceptionally similar phenomenon, consider a disordered two-dimensional single particle problem with $v$ being the energy scale of the disorder. Localization of particles with energy $E \sim E_F$ can be induced by strong potential fluctuations, $v \sim E$ (trivial, wave functions running out of support), or by genuine Anderson localization for arbitrarily weak disorder of strength at exponentially large scales $\sim \exp (E/v)$ times the lattice constant. For finite size systems, increasing the disorder may drive the first mechanism of localization before traces of the second are seen. (In two dimensions, arbitrarily weak disorder induces localization, different from the present, effectively high dimensional system. However, for the sake of the argument, the analogy carries.) Such masking of an Anderson transition happens in SYK systems below a certain finite size threshold.

More specifically, the wave functions in regime III are ergodically extended over a set of $D/(\sqrt{N} \delta^2)$ resonant sites (which makes them non-ergodic relative to the full Fock space). Ergodicity implies that the system is described by a low energy theory which is structureless in Fock-space, much as the real space ergodic structureless theories describing disordered single particle systems be-
low the localization threshold. In more concrete terms, this means that the system is in the universality class of the Rosenzweig-Porter model: a random matrix ($H_{nm}$) of dimension $D$ (mimicking the extended nature of wave functions) subject to randomness, $v$, on the diagonal (assuming the role of the energies $v_n$). Beyond the scale $\delta_c$, the site randomness is strong enough to concentrate states on individual sites, effecting the trivial localization mechanism by trapping, and Poissonian spectral statistics.

For the SYK system in the thermodynamic limit, $N \to \infty$, the above localization crossover $\delta_c$ is never an issue. Long before $\delta'_c \sim 2^{N/2}$ is reached, we hit the FSL transition at $\delta_c$, where the equivalence of the model to the RP model breaks down. At the FSL threshold, the inverse participation ratio has reached values $I_2 \sim \frac{\sqrt{N}v^2}{D} \sim \frac{N^{3/2}}{2} \ll 1$. This means that the transition shows via a sharp upturn of $I_2$ (through a critical region whose description is beyond the scope of this paper) towards the saturation value $I_2 = 1$ at $\delta > \delta_c$. By contrast, the trivial localization mechanism by trapping simply means that the above power law Eq. \([14]\) reaches values of $O(1)$.

### III. MATRIX MODEL

We start the derivation of the results summarized above by constructing an exact matrix integral representation of the above correlation functions. The physics behind the matrix representation and the stationary phase analysis of the theory will be discussed in the next section.

All information on spectra and wave functions of the system is contained in the Fock space matrix elements of resolvent operators,

$$G_{nm}^{\pm} = \langle n | (z_{\pm} - \hat{H})^{-1} | m \rangle,$$

where $z_{\pm} = \pm (\frac{\hbar}{2} + i\nu)$ and, here and throughout, $\nu$ is infinitesimal (with a limit $\nu \to 0$ to be taken in the final step of all calculations). Specifically, the correlation functions above are obtained as

$$I_q = \frac{(2i\nu)^{q-1}}{2i\nu^q} \sum_n \langle G_{nn}^{+}(q) G_{nn}^{-} \rangle,$$

$$K(\omega) = \frac{1}{2\pi^2 \nu^2} \sum_{nm} \text{Re} \langle G_{nn}^+ G_{mn}^- \rangle,$$

where $I_q$ is computed at $\omega = 0$, and $\langle \cdots \rangle$ denotes the average over matrix elements $\{J_{ijk}\}$

Construction of the matrix integral. — Following standard protocols, we raise the Green functions to an exponential representation before performing the Gaussian average. The basic auxiliary formula in this context is $M_{nn}^{-1} = \int \hat{D}(\hat{\psi}, \bar{\psi}) e^{-\psi^\dagger M \psi_{mn}^\dagger \psi_{nm}^\dagger}$, where $M$ is a general $L \times L$ matrix and the $2L$ dimensional ‘graded’ vector $\psi = (\psi^b, \psi^f)^T$ contains $L$-commuting components $\psi^b$, and an equal number of Grassmann components $\psi^f$. The double integral over these variables cancels unwanted determinants $\text{det}(M)$, while the pre-exponential factors, either commuting or anti-commuting, $\sigma = b, f$, isolate the inverse matrix element. With the identification $M = \text{diag}(-i[G^+], i[G^-]) = -i\sigma_3 (E + z - \hat{H})$, we are led to consider the generating function

$$\mathcal{Z}[j] = \int \hat{D}(\hat{\psi}, \bar{\psi}) \langle e^{-\bar{\psi}^\dagger (E + z - \hat{h} - j) \psi} \rangle_{\hat{H}_4}. \quad \text{(21)}$$

Here, $z \equiv (\frac{\hbar}{2} + i\nu) \sigma_3$, contains the energy arguments of the Green functions and $\sigma_3$ is a Pauli matrix distinguishing between advanced and retarded components. The matrix $j$ acts as a source for the generation of the required moments of Green function matrix elements. Specifically, we define

$$j_K(\alpha, \beta) = \alpha \pi^b \otimes \pi^+ + \beta \pi^f \otimes \pi^-,$$

$$j_{1,n}(\alpha, \beta) = \int_k (\alpha, \beta) \otimes |n\rangle \langle n|,$$

where $\pi^{b,f}$ is a projector onto commuting and anticommuting-variables, respectively, $\pi^+ \pi^- \psi = \bar{\psi}^\dagger \psi$, and $\pi^\pm$ projects in causal space, $\bar{\psi}^\dagger \pi^\pm \psi = \psi^\pi \psi$, $s = \pm$. With these definitions, an elementary computation shows that

$$K(\omega) = \frac{1}{2\pi^2 \nu^2} \text{Re} \sum_{\alpha, \beta} \partial_{\alpha \beta}^2 \mathcal{Z}[j_K] |_{\alpha, \beta = 0}, \quad \text{(24)}$$

$$I_q = c_q (2i\nu)^{q-1} \sum_n \partial_{\alpha} \partial_{\beta}^n \mathcal{Z}[j_{1,n}] |_{\alpha, \beta = 0}, \quad \text{(25)}$$

with $c_q \equiv 1/(2i\nu (q - 1)!$. In the following, we consider the sources absorbed in a redefined energy matrix, $z \to z - j$, and remember their presence only when needed.
At this point, the averaging over $\hat{H}_4$ can be performed, and it generates a quartic term

$$Z = \int D(\bar{\psi}, \psi) e^{-\bar{\psi}\hat{G}^{-1}\psi + \frac{i}{2} \sum_a (\bar{\psi}\hat{X}_a \psi)^2},$$

(26)

where we defined $w^2 = 6J^2/(2N)^3 \equiv \frac{1}{2} N^{-4}$ for the scaled variance of the SYK Hamiltonian, $G \equiv (E + z - \hat{H}_2)^{-1}$,

$$\hat{X}_a \equiv \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l,$$

(27)

and $a = (i, j, k, l)$ with $i < j < k < l$. We next perform an innocuous but physically meaningful (see next section) rearrangement $(\bar{\psi}\hat{X}_a \psi)^2 = \text{STr}((\bar{\psi}\hat{X}_a \psi)^2)$, where the supertrace $\text{STr}(X) \equiv \text{tr}(X^{bb}) - \text{tr}(X^{ff})$ accounts for the minus sign caught when exchanging anti-commuting variables. The next step is a Hubbard-Stratonovich transformation decoupling the matrices $\psi \hat{X}_a \sim A_a$ in terms of $(2N)^4/4!$ auxiliary matrix fields $A_a$. Referring for details of the procedure to Appendix A we note that after the decoupling the integral over $\psi$-variables has become Gaussian and can be carried out. A more interesting statement is that of the $\rho \equiv (2N)^2$ Hubbard-Stratonovich fields $A_a$, all but one can be removed, too, by straightforward Gaussian integration. Upon restricting to $E = 0$ this leaves us with a single integration,

$$Z = \int D Y e^{-S[Y]},$$

(28)

$$S[Y] = -\frac{1}{2} \text{STr}(Y^{PPY}) + \text{STr} \log (z - \hat{H}_2 + iPY),$$

over a $2 \times 2 \times D$ dimensional matrix $Y = \{Y_{nm}^{\sigma\sigma',ss'}\}$ carrying indices in causal space, super-space, and Fock space. The information on the SYK system now sits in the site-diagonal one-body term, $\hat{H}_2$, and the hopping operator $\hat{P}$ which represents the interaction and acts on matrices $Z = \{Z_{nm}\}$ in Fock space as

$$\hat{P}Z \equiv \frac{1}{\rho} \sum_a X_a Z X_a^\dagger.$$

(29)

Finally, $\gamma = w\rho^{1/2} = 1$ represents the SYK$_4$ band width, which we have set to unity. To simplify formulas, we will consider energies $\hat{H}_2 \rightarrow \gamma \hat{H}_2$, $\omega \rightarrow \gamma \omega$ scaled by this parameter, and suppress it throughout.

**Discussion of the matrix integral.** — This is now a good point to discuss the meaning of the above Hubbard-Stratonovich transformation and of the matrix-representation. The two-fermion vertices $\bar{\psi}\hat{X}_a \psi$ entering the theory after disorder averaging describe the scattering of Fock space states off the four-Majorana operators contained in the Hamiltonian, and in this way introduce the lattice connectivity indicated in Fig. I. While a direct analysis of individual Fock space amplitudes seems hopeless, progress can be made if the propagators are paired to two-amplitudes composites as indicated in Fig. II. For two reasons, the pair amplitudes $Y_{nn',\sigma\sigma'} = \psi_{n\sigma} \psi_{n'}^{\sigma'}$ are more convenient degrees of freedom: First, the pair action $Y \rightarrow \sum_a \hat{X}_a Y \hat{X}_a = \rho \gamma Y$ governing scattering in the two-state channel (cf. the structure of the action $S[Y]$) is relatively easy to describe, see below. Second, the advanced/retarded combinations $Y_{nn',\sigma\sigma'} = \psi_{n\sigma} \psi_{n'}^{\sigma'}$ appear as terminal vertices in the computation of Green functions $G^{+}_{ss'} G^{-}_{ss'}$, where the dots stand for the unspecified final points of the correlation function. With the exact identity $\langle G^+ \rangle^{-1} - \langle G^- \rangle^{-1} = \omega \tau \equiv \omega + 2i0$, we have $\langle G^{+}\rangle_{nm} \langle G^{-}\rangle_{nm} = \langle \text{tr}(G^+ G^-) \rangle = \frac{1}{2\pi} \text{tr}(G^+ [(G^+)^{-1} - (G^-)^{-1}]) = \frac{1}{4\pi} (\text{tr}(G^+ - G^-)) \sim \frac{2}{\pi \nu}$, where $\nu$ is the density of states at the band center. The way to read this (Ward) identity is that the product of Green functions contains a singularity, provided $\text{tr}(G^- - G^+) \sim \nu$ is a structureless quantity. (The latter condition does not hold in systems with localization, where the isolated eigenstates support a point spectrum with poles rather than a uniform cut.) This argument indicates that the ‘soft mode’ $G^+ G^- \sim \omega^{-1}$ is key to the understanding of observables probing spectrum and eigenfunctions of the system.

In the matrix integral framework, the above singularity shows in the presence of a soft mode in the integration over the variables $Y_{nn',\sigma\sigma'}$. To isolate this mode, we note that Eq. (28) has an approximate symmetry

$$Y \rightarrow TY T^{-1}, \quad T = \{T^{ss',\sigma\sigma'}\}$$

(30)

under rotations homogeneous in Fock space. The set of these transformations defines $\text{GL}(2/2)$, i.e. the group of invertible $4 \times 4$ matrices with anti-commuting entries. Invariance under this symmetry is weakly broken only by the frequency/source matrix $z$, which, ignoring the infinitesimal sources, transforms as $z \rightarrow z T^{-1} \sigma_3 T$. This reduces the symmetry down to the transformations diagonal in advanced-retarded $(s$-indices) space, $\text{GL}(1/1) \times \text{GL}(1/1)$.

The essential question now is whether the above weak explicit symmetry breaking is spontaneously broken in the matrix integral (much as a weak explicit symmetry breaking by a finite magnetic field gets upgraded to spontaneous symmetry breaking in a ferromagnetic phase.) In the latter case, we expect a soft Goldstone mode whose ‘mass’ is set by the symmetry breaking parameter, $\omega$, and $\omega^{-1}$ singularities in line with the observation above. To investigate this question and the consequences in the observables $K(\omega)$ and $I_{4\omega}$, we next subject the theory to a stationary phase analysis.

**IV. EFFECTIVE THEORY**

In this section, we map the exact theory Eq. (28) to an approximate but more manageable effective theory. We have already established the presence of an exact (in the limit $\omega \rightarrow 0$) rotational soft mode isotropic in Fock space. Since much of the analysis below will focus on strong $\hat{H}_2$ with eigenvalues $v_n$ of $\hat{H}_2$ compara-
Causality requires $\text{sgn} Y = \text{sgn} \text{Im} z$, i.e. the sign of the self energy is dictated by that of the imaginary part contained in the energy arguments. Otherwise the saddle point equation is rotationally invariant in the internal indices of the theory. This motivates an ansatz,

$$\text{Re} Y = \sum_n (\text{Re} Y)_n \langle n | n \rangle \equiv \sum_n \pi \nu_n | n \rangle \otimes \sigma_3 \otimes 1_{\text{bf}}$$

with real coefficients $\nu_n$. Inspection of Eq. (32) shows that these coefficients afford an interpretation as mean field local density of states.

Substituting this expression into the equation and temporarily ignoring the small energy argument, $z$, as small compared to both $H_2$ and $Y$, we obtain the variational equation (5). The structure of this equation contains the key to its solution: For $v_n = 0$, the normalization $\sum_m P_{[n-m]} = 1$ implies that it is solved by $\kappa_n = 1$. In the chosen units, this is ($\pi \times$) the density of states at the SYK band center. For finite $v_n$, the summation over $m$ implements an effective average over the connected states, which now carry random energy. In Appendix C we show that the average stabilizes the solution (6), where $\sim$ stands for equality up to corrections exponentially small in $\exp(- (v_n/\delta)^2)$). We interpret this result as the spectral density of sites with energy $v_n$ and decay rate $\kappa_n$ into neighboring sites. The latter is finite for states below a threshold $|v_n| < C$. For $\delta > 1$, the rate is given by the energy denominator $\kappa \sim \delta^{-1}$ of neighboring sites. In the opposite regime, $\delta < 1$, the energy denominators of states $v_n \sim 1$ in resonance with the SYK band width are of $O(1)$, leading to the second line in Eq. (6).

The saddle point solutions discussed thus far are distinguished for their diagonality in all matrix indices. However, we now recall that the $z = 0$ action is invariant under Fock space uniform rotations Eq. (30), implying that uniformly rotated saddle point configurations $Y_n \rightarrow T_n Y_n T_n^{-1}$ are solutions, too. (Technically, this follows from the cyclic invariance of the trace.) Next to this uniform Goldstone mode, configurations $Y_n \rightarrow T_n Y_n T_n^{-1}$ with site-diagonal rotations commutative with $H_2$ are expected to cost the least amount of action. With $Y_n = \pi \nu_n \sigma_3$, this makes $Y_n \rightarrow \pi \nu_n Q_n$, $Q_n = T_n \sigma_3 T_n^{-1}$ the effective degrees of freedom of the theory, and substitution into Eq. (28) defines the Goldstone mode integral,

$$Z = \int DQ e^{-S[Q]},$$

$$S = -\frac{\pi^2}{2} \text{Str}((\nu \tilde{Q})P(\nu \tilde{Q})) + \text{Str} \log \left( z - H_2 + i \pi P(\nu \tilde{Q}) \right).$$

In the next two sections, we investigate what this integral has to say about wave function statistics and Fock space localization, respectively.
V. SPECTRAL AND WAVE FUNCTION STATISTICS

In this section, we explore the spectral and wave function statistics in regimes I-III. The presumption is that wave functions are not yet localized and correlated with each other. This should lead to Wigner-Dyson spectral statistics and wave function moments reflecting the extended nature on the subsets of Fock space corresponding to active or resonant sites.

To test these hypothesis it is sufficient to consider the integral \[ \mathcal{I} \] in the presence of effectively infinitesimal explicit symmetry breaking \( z \); besides the sources, \( j \), this parameter contains a frequency argument \( \omega \sim D^{-1} \) of the order of the exponentially small inverse many body level spacing in the case of spectral statistics, Eq. (24), or the infinitesimal parameter \( \eta \) in the case of wave function statistics, Eq. (25). On general grounds, we expect the smallness in the ‘explicit’ symmetry breaking in a Goldstone mode integral to lead to singular contributions \( \sim z^{-n} \) proportional in the inverse of the that parameter after integration. (Inspection of the prefactors, \( \eta^{n-1} \) in the definition of the wave function statistics shows that such singularities are actually required to obtain non-vanishing results.) These most singular contributions to the integral must come from the Goldstone mode fluctuations of least action, which are fluctuations homogeneous in Fock space, \( \mathcal{Q}_n = T_n \sigma_3 T_n \to T \sigma_3 T^{-1} = Q \).

With \( [T, \mathcal{P}] = 0 \), the substitution \( \nu \mathcal{Q} \to \nu Q \) into the action Eq. (33) leads to

\[
S_0[Q, j] = \text{STr} \log \left( z - j - \hat{H}_2 + i\kappa Q \right),
\]

where we made the dependence \( z \to z - j \) of the action on the sources \( j \equiv j_{n, \eta} \) required to calculate moments via Eq. (23) explicit again, and we noted \( \pi \mathcal{P} \nu = \kappa \).

Before proceeding, we note that the structure of this action is identical to that describing the Rosenzweig-Porter model — a single random matrix of dimension \( D \) containing Gaussian distributed disorder on the matrix diagonal \[ \mathcal{H}_0 \]. An important difference is, however, that the diagonal disorder in the latter is uncorrelated, while the Fock-space diagonal disorder induced by \( \mathcal{H}_2 \) is highly correlated. As a consequence, the effective action for the Rosenzweig-Porter model only allows for homogenous saddle point solutions \[ \mathcal{Q}_n \mathcal{H}_1 \mathcal{Q}_n \], while here we encounter solutions that become inhomogeneous in Fock space once on-site disorder exceeds the SYK4 band width. The inhomogeneity accounts for a site-dependent broadening \( \kappa_n \), induced by correlations in the disorder amplitudes, and also manifests in a separation into regimes II/III of the phase of non ergodic extended states. In the following, we discuss what this reduction of the model has to say about spectral and wave function statistics.

Spectral statistics: To obtain a prediction for spectral correlations based on the representation Eq. (33) with Fock space zero mode, we consider the correlation function \( \langle \mathcal{Q}_n \rangle \), represented through matrix integral represented Green functions as in Eq. (20) and Eq. (24). To compute these quantities from the effective theory, we need to expand the action Eq. (35) to lowest order in the parameter \( \omega / \kappa \sim 1 / D \), and to second order in the sources. The straightforward \( \omega \)-expansion yields (cf. Eq. (33))

\[
S_w[Q] \equiv -i\frac{\pi \nu (\omega + i\eta)}{2} \text{STr}(Q \sigma_3),
\]

where \( \nu \) is the zero energy density of states, Eq. (10). What remains, is the source differentiation and the integration over the matrix \( Q \). To get some intuition for the integral, notice that the non-linear degree of freedom \( Q = T \sigma_3 T^{-1} \) affords a representation, \( Q = U \sigma_3 U^{-1} \), where \( U \) contains various compact angular variables (cf. Appendix E), and

\[
Q_0 = \left( \begin{array}{cc} \cos \theta & i \sin \theta \\ -i \sin \theta & \cos \theta \end{array} \right),
\]

a rotation matrix in causal space. Diagonal in superspace, this matrix is parameterized in terms of the two ‘Bogolubov’ angles \( \theta = (i \theta_3, \theta_1)^T \), where \( \theta_1 \in [0, \pi] \) is a compact rotation variable, and \( \theta_0 \in \mathbb{R}^+ \) a non-compact real variable. This representation reveals the geometry of the integration manifold as the product of a sphere \( \theta_1 \) and a hyperboloid \( \theta_0 \) (coupled by variables contained in \( U \)). Where the physics of non-perturbative structures in spectral and wave function statistics, and localization is concerned, the most important player is the non-compact variable, \( \theta_0 \), as only this one has the capacity to produce singular results. Heuristically, one may think of the model reduced to its dependence on this variable as a non-compact version of a Heisenberg-model, containing hyperboloidal, rather than compact spins as degrees of freedom.

Referring for details of the source differentiation and the subsequent integration over the matrix \( Q \) to Appendix E the above reduction of the model yields the GUE spectral correlation function \( \langle \mathcal{Q}_n \rangle \) for the spectral statistics on scales of the many body level spacing in regimes I-III. With increasing energies, the assumption of homogeneity of fluctuations in Fock space breaks down (cf. the next section) beyond a ‘Thouless energy’ whose value depends on the specific observable under consideration. However, the detailed investigation of Thouless thresholds for the present model is beyond the scope of the paper.

Wave function statistics: In the same manner, we may consider the local moments of wave functions Eq. (31), represented via Green functions Eq. (20), and obtained from the matrix integral through Eq. (25). A key feature of this expression is that it contains a limit \( \lim_{\eta \to 0} \eta^{n-1} (\ldots) \) the factor \( \eta^{n-1} \) must thus be compensated for by an equally strong singularity \( \eta^{1-q} \) from the integral, where \( \eta \) couples through \( z = i\eta \sigma_3 \). Setting \( \omega = 0 \) in Eq. (36) and integrating over the functional differentiated in sources.
VI. ERGODIC-TO-LOCALIZATION TRANSITION

In regimes, II, III, the dominant contribution to the matrix integral at the lowest energies comes from homogeneous contributions \( Q \). Upon approaching the localization threshold III/IV, inhomogeneous fluctuations \( Q \to \tilde{Q} = \{ \nu_n \} \) gain in importance and eventually destabilize the mean field theory. To describe this physics, we need an effective action generalized for inhomogeneous fluctuations, and more manageable than Eq. (33). We derive it in Appendix E under the assumption that the sum over a large number of fluctuating terms represented by the term \( P(\tilde{Q}) \) is largely self-averaging. An expansion to lowest order in fluctuations around the homogeneous average than leads to the effective hopping action

\[
S[Q] = S_P[Q] + S_\omega[Q],
\]

\[
S_P[Q] = \frac{\pi^2}{2} \sum_{n,m} \nu_n \nu_m P_{n,m} \text{str}(Q_n Q_m), \quad (38)
\]

\[
S_\omega[Q] = -i\pi \sum_n \nu_n \text{str}(z Q_n), \quad (39)
\]

with \( Q_n = T_n^{-1} \sigma_3 T_n \). Eqs. (38) and (39) are the main result of this section. Depending on the value of \( \kappa_n \), Eq. (37), this action describes the entire range from vanishing to large deformations \( H_2 \). We next discuss what this action has to say on the ergodic-to-localization transition.

The key player in this problem is the hopping term (38) where \( Q \) matrices at SYK-1-neighboring sites are coupled, subject to a weight which contains the local spectral densities. In analytic approaches to localization on high dimensional lattices, it is common to set these weights to unity. However, in view of the massive site-to-site fluctuations of \( \nu_n \) we prefer not to make this assumption and work with a given realization \( \{ \nu_n \} \) for as long as possible. Approaching the transition from the localized side where the integration over \( Q \)’s is subject to only small damping \( \nu_n \), the essential degrees of freedom are once again the non-compact variables, \( \theta_n \) contained in \( Q_n \), Eq. (37).

To better understand the significance of this structure, we write \( Q_n Q_m = (Q_n - \sigma_3)(Q_m - \sigma_3) + \sigma_3 Q_n + \sigma_3 Q_m - 1 \) to represent the hopping part of the action as

\[
S_P[Q] = \frac{\pi^2}{2} \sum_n \Gamma_n \text{str}(Q_n \sigma_3)
\]

\[
+ \frac{\pi^2}{2} \sum_{n,m} \nu_n \nu_m P_{n,m} \text{str}((Q_n - \sigma_3)(Q_m - \sigma_3)),
\]

where \( \Gamma_n \equiv \nu_n \sum_m P_{n,m} \nu_m \). Consider a situation where the accumulate hopping weights \( \Gamma_n \) out of site \( n \) are small. In this case, large fluctuations of the non-compact angles, \( \lambda_{b,n} \equiv \cosh(\theta_n b) \) dominate the functional integral. To understand the consequences, we note that the measure of the \( Q \)-integration in the angular representation is given by [30]

\[
\int dQ = \int dU \int_{-1}^{1} d\lambda_1 \int_{1}^{\infty} d\lambda_2 \frac{1}{(\lambda_b - \lambda_1)^2}.
\]

where \( \lambda_1 = \cos(\theta_1) \). For small typical values \( \Gamma \sim \Gamma_n \ll 1 \), the exponential weights effectively cut off the integration over \( \lambda_b \) at \( \sim \Gamma^{-1} \gg 1 \). Individual terms in the second line of the above representation of \( S_P \) are smaller than the accumulated weights in the first line, and so the integral can be approached by perturbative expansion in the hopping terms. As an example, consider the sixth order expansion indicated via the highlighted links in Fig. 4. Retaining only the information on the non-compact integrations \( \lambda \equiv \lambda_b \), the contribution with a loop (left) and that with doubly occurring links evaluate to

\[
\text{loop: } \int_{1}^{\Gamma^{-1}} \frac{d\lambda_1}{\lambda_1^2} \frac{d\lambda_2}{\lambda_2^2} \frac{d\lambda_3}{\lambda_3^2} \lambda_1^2 \lambda_2^2 \lambda_3^2 \sim \Gamma^{-6},
\]

\[
\text{no loop: } \int_{1}^{\Gamma^{-1}} \frac{d\lambda_1}{\lambda_1^2} \frac{d\lambda_2}{\lambda_2^2} \frac{d\lambda_3}{\lambda_3^2} \lambda_1^2 \lambda_2^2 \lambda_3^2 \sim \Gamma^{-7},
\]

where the indices refer to the participating \( Q \)-matrices, \( Q_1...4 \). This estimate shows that the contribution of loops in the perturbation expansion is suppressed. At the same time, the largeness of the individual contributions signals that infinite order summations are required. The effective medium approximation achieves this summation, loops excluded. The approximation is called ‘effective medium’ because from the perspective of individual sites in Fock space the contribution of all hopping processes terminating at that site adds up to the influence of an effectively homogeneous background medium, transmissive or not depending on the strength of the couplings.

To see how this comes about, consider a site \( n \) with local configuration \( Q_n \) and let \( \Psi_{n,m}(Q_n) = \int_{\text{corr}_{m,Q}} DQ e^{-S[Q]} \), be the contribution to the functional integrated over all links connected to \( n \) via the
neighbor $m$, through the loopless coral like structure indicated in Fig. [7]. The essence of the approximation is the recursion relation,

$$
\Psi_{nm}(Q) = \int dQ' N_{w_{nm}}(Q, Q') e^{-S_0(Q')} \prod_n \Psi_{mo}(Q'),
$$

$$
N_w(Q, Q') = e^{w \str(QQ')},
$$

where the product extends over all sites, $o$, connected to $m$ by hopping, $S_0(Q) \equiv S_{w \rightarrow i}(Q)$ acts as a convergence generating factor, and we defined

$$
w_{nm} \equiv \frac{\pi^2}{2} \nu_n \nu_m \mathcal{P}_{n,m}, \quad (41)
$$

for the coupling constants weighting the hopping kernel. If we now take the product $\Psi_n(Q) \equiv \prod_m \Psi_{n,m}(Q)$ (assuming self averaging in the sense that the fully integrated amplitude, $\Psi_n$ depends on the terminal site, $n$, but not on the detailed values of the $O(N^4)$ neighbor amplitudes), the equation assumes the form,

$$
\Psi_n(Q) = \prod_m \int dQ' N_{w_{nm}}(Q, Q') \Psi_m(Q'),
$$

where the presence of the convergence generator $\exp(-S_0)$ is left implicit. In the deeply localized regime, $N_{n,m} \approx 1$, the integral decouples, and $\Psi_n = 1$ is a solution by supersymmetry (i.e. the unit normalization of all source-less integrals in the present formalism). This suggests [30] a linearization, $\Psi_n(Q) = 1 - \Phi_n(Q)$, where the emergence of a non-trivial solution $\Phi_n$ is taken as a criterion for the localization transition. Substituting this ansatz into the equation, and again using supersymmetry, $\prod_m \int dQ' N_w(Q, Q') = 1$, we obtain

$$
\Phi_n(Q) = \sum_m \int dQ' N_{w_{nm}}(Q, Q') \Phi_m(Q'). \quad (42)
$$

This is a linear integral equation governed by a random lattice structure in Fock space via the couplings $w_{nm}$ and an internal structure encoding the randomness of the SYK$_4$ system via the $Q'$-integrals. Although the integral equation may look helpless complicated, progress is possible recalling our previous observation: we again have a situation where the $Q$ integrations extend over wide parameter intervals such that the leading non-compact variable is the key player. Assuming that the solutions depend on the non-compact variable as $\Phi(Q) \rightarrow \Phi(t), t \equiv \log(\lambda_1/\delta)$, and referring to Ref. [10] for details of the integration over remaining variables, the reduction of Eq. (42) to the regime of interest, $t \ll 0$, $w_{nm} \ll 1$ reads

$$
\Phi_n(t) = \sum_m \int dt' L_{w_{nm}}(t-t') \Phi_m(t'),
$$

$$
L_w(t) = \left( \frac{w}{2\pi} \right)^{1/2} e^{-w \cosh t} + \frac{1}{2} \left( w \cosh t + \frac{1}{2} \right).
$$

Ref. [46] contains a pedagogical discussion of the solution of (the homogeneous variant $w_{nm} = \text{const.}$ of this equation, including the somewhat subtle issue of boundary conditions. It turns out that the key to the stability of the localized solution, $\Psi = 1$ lies in the spectrum of the linear kernel $\{L_{w_{nm}}(t-t')\}$: a spectrum with lower bound $\epsilon > 0$ means that perturbations $\delta \psi$ will grow under the application of the linearized kernel, signifying destabilization of the null solution $\psi = 1$. We thus declare the existence of a minimal eigenvalue $\epsilon = 1$ as a delocalization criterion. Due to translational invariance in $t-t'$ eigenstates are of the form $e^{\theta(t-t')} \Phi_n$, where the coefficients are determined by the reduced equation, $\Phi_n = L_{\theta,nm} \Phi_m$, with

$$
L_{\theta,nm} = \int_{-\infty}^{\infty} dt L_{w_{nm}}(t) e^{-\theta t}.
$$

Substitution of the kernel in Eq. (43) followed by differentiation in $\theta$ shows that the positive matrix $L_{\theta,nm}$ assumes its smallest values at $\theta = 1/2$, and the straightforward integration at that value defines the matrix,

$$
L_{nm} = L_{1/2,nm} = \left( \frac{w_{nm}}{2\pi} \right)^{1/2} \int dt e^{-w_{nm} \cosh t} \left( w_{nm} \cosh t + \frac{1}{2} \right)
$$

$$
\approx \left( \frac{w_{nm}}{2\pi} \right)^{1/2} \log \left( \frac{2}{w_{nm}} \right).
$$

In an approximation that neglects site-to-site fluctuations of the logarithm, the factorization (Eq. (41)) $w_{nm} = \sum_{\nu = |n-m|=4} \nu_\nu \nu_m \simeq \frac{12\pi^2}{\nu_\nu \nu_m}$ implies that the matrix has the structure of a dyadic product. The criterion for the dominant eigenvalue thus assumes the form of Eq. (17), where the sum extends over the Hamming distance four neighborhood of a reference site $n$. Due to the large number of contributing terms, the dependence on $n$ is negligible and replacing the sum by an average we obtain Eq. (18) as the final result for the critical disorder strength.

**VII. DISCUSSION**

In this paper, we have presented a first principle analysis of Fock space localization in the Majorana SYK$_{4+2}$ model. Our work builds on a previous publication [31], where three of the present authors demonstrated the existence of a phase of non-ergodic extended states in a random energy variant of the SYK model. The present work focuses on the SYK model perturbed by a two Majorana potential, and gives a complete description of the ergodic-to-Fock space localization transition, including an intermediary phase of non-ergodic extended states. To the best of our knowledge, this is the only genuine many body system for which microscopic and analytical studies of many body localization are possible at this stage. Many of the theoretical tools applied above to
describe the localization phenomenon were originally developed for synthetic systems — such as Bethe lattices subject to disorder, or modified variants of random matrix theory — meant to mimic true Fock spaces. From a methodological perspective, the main contribution of the present paper is a controlled reduction of the microscopic SYK model to a form where the above concepts become applicable. While the physics of interactions in the $N$ body Fock space of the SYK system required a somewhat more realistic variant of the above synthetic SYK model to a form where the above concepts are finally extended systems, and to what extent our present approach can be generalized in this direction.

We tested the validity of the theory in the numerically accessible regimes (I-III) by comparison with the results of direct diagonalization for systems up to $N = 15$, or $N_{\text{Majorana}} = 30$ and found excellent agreement. However, the numerical validation of the Fock space transition III/IV presents a more tricky affair:

We found that the diminishing support of the wave functions in regime III has bearings on the observability of the true FSL transition III/IV in systems of numerically manageable size. For intermediate disorder, III, wave functions are spread over a limited set of 'resonant' sites with energies $v_n \sim \delta^{-1}$. For 'exponentially strong' disorder $\delta^2 \sim D/N$, the support set becomes of $O(1)$, which can be seen as a trivial crossover into a localized regime. This trivial form of localization by trapping is different from the genuine FSL transition which occurs within the set of resonant sites and at disorder $\delta_c \sim N^2$ only polynomial in $N$. The problem is that for system sizes $N_{\text{Majorana}} \lesssim O(60)$, the exponential disorder threshold is reached before that $\delta_c \sim N^2$ of the genuine Fock space localization transition, implying that the true transition remains masked in systems of numerically realistic size. In such systems, the saturation of inverse participation ratios (and other indicators of localization, including changes in the spectral statistics) reflects the diminishing spectral support of wave functions due to large potential fluctuations, rather than a genuine MBL transition. The numerical monitoring of these effects will lead to overly conservative estimates for localization 'transitions' (i.e. the actual MBL transition in a system of sufficiently large $N$ would be driven by weaker potential fluctuations within a band of spectral support.)

Irrespective of this practical challenge to observability, the description of the FSL transition in the SYK system relies on the assumption of the irrelevance of loops in the Fock space propagation and accompanying interference of wave function amplitudes. In the present context, this assumption is stabilized by the high coordination number of the lattice. It will be interesting to explore if it requires modification in the more complex setting of spatially extended systems, and to what extent our present approach can be generalized in this direction.

Acknowledgments

Discussions with A. D. Mirlin and K. Tikhonov are gratefully acknowledged. F. M and T. M. acknowledge financial support by Brazilian agencies CNPq and FAPERJ. The work of M. T. was partially supported by JSPS KAKENHI Grants JP17K17822 and JP20K03787. Work funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 277101999 - TRR 183 (project A03). Part of the numerical computation in this work was carried out at the Supercomputer Center, ISSP, University of Tokyo.

Appendix A: Derivation of the action

We here derive the action Eq. (28) from the averaged functional [26]. We start by rewriting the quartic term as $(\hat{\psi} \hat{X}_a \psi)^2 = \text{STr}((\hat{\psi} \hat{X}_a \psi)^2)$. To decouple this nonlinearity, we multiply the functional with the unit normalized Gaussian integral $1 = \int DA \exp(-\frac{1}{2} \sum_a \text{STr}(A_a \hat{X}_a)^2)$, where $DA = \prod_a dA_a$, and $A_a = \{A_{nn',\sigma\sigma'}\}$ are 4D-dimensional matrices. A shift $A_a \rightarrow A_a + w \psi$ then removes the quartic term, and the subsequent integration over $\psi$ leads to

$$Z[j] = \int DA e^{-\frac{1}{2} \sum_a \text{STr}(A_a \hat{X}_a)^2 - \text{STr} \log(G^{-1} + w \sum_a A_a)} ,$$

where $G^{-1} = z - \hat{H}_2$. We now observe that the nonlinear part of the action couples only to the combination $\sum_a A_a$. This motivates the definition, $A_a = \frac{i}{\rho}(Y + Y_a)$, where the factor of $i$ is included for later convenience, and $\sum_a Y_a =$

| site disorder | state extension | spectral statistics |
|---------------|-----------------|---------------------|
| $\delta < N^{-1/2}$ | D | Wigner-Dyson |
| $N^{-1/2} < \delta \ll 1$ | $D/\sqrt{N}$ | Wigner-Dyson |
| $1 < \delta \ll N^2$ | $O(D^0)$ | Wigner-Dyson |
| $N^2 < \delta$ | | Poisson |

TABLE I: Summary of regimes with different wave function and spectral statistics.
0. Adding a Lagrange multiplier $\frac{1}{\rho} \sum_a \text{Str}(Y_a \Lambda)$ to enforce the constraint, we are led to consider the functional $Z[j] = \int D\!Y D\!A \exp(-S[Y, \Lambda])$, with action

$$S[Y, \Lambda] = -\frac{1}{2\rho^2} \sum_a \text{Str}((Y + Y_a)P_a(Y + Y_a)) + \frac{i}{\rho} \sum_a \text{Str}(AY_a) + \text{Str} \log(\hat{G}^{-1} + iwY),$$

where $\rho = (2N^4) \frac{1}{\rho}$ and we defined the operator $\hat{P}_a B = \hat{X}_a B \hat{X}_a$. Note that $\hat{P}_a$ is self-inverse, $\hat{P}_a^2 = \hat{X}_a^2 B \hat{X}_a^2 = B$, and hermitian in the sense that $\text{Str}(C \hat{P}_a B) = \text{Str}(\hat{P}_a CB)$. We now do the Gaussian integrals over $Y_a$ to obtain,

$$S[Y, \Lambda] = -\text{Str}(\rho \frac{1}{2} \Lambda P A + iAY) + \text{Str} \log(\hat{G}^{-1} + iwY),$$

where $\mathcal{P} = \frac{1}{\rho} \sum_a \hat{P}_a$. The Gaussian integration over $\Lambda$ may now be performed and after rescaling $Y \to \rho^{1/2} Y$, and defining $\gamma = w\rho^{1/2} = \frac{1}{2} (2N)^{1/2}$ we obtain the action

$$S[Y] = -\frac{1}{2} \text{Str}(YP^{-1}Y') + \text{Str} \log(z - \hat{H}_2 + i\gamma Y).$$

In a final step, we perform a linear transformation $P^{-1} Y \to Y$, and recall that in our units $J^2 = 2/N$ and $\gamma = 1$, to arrive at Eq. (28).

**Appendix B: The operator $\mathcal{P}$**

In this Appendix, we discuss the action of the operator $\mathcal{P}$ states $|n\rangle\langle n|$ diagonal in the occupation number basis. To this end, note that for a state $|n\rangle = |n_1, \ldots, n_i, \ldots, n_N\rangle$, the action of the Majorana operator $\hat{X}_{2i} = c_i + c_i^\dagger$ produces the state $|n_i\rangle = \hat{X}_{2i} |n\rangle = |n_1, \ldots, \bar{n}_i, \ldots, n_N\rangle$, where $\bar{n}$ is 0 for $n = 1$, and vice versa. Similarly, $\hat{X}_{2i-1} |n\rangle = i(-)^n |n\rangle$. Except for $n_i$ all other occupation numbers remain unchanged, and no superpositions of states are generated. The adjoint action thus generates $\hat{X}_{2i-1} |n\rangle\langle n| \hat{X}_{2i} = \hat{X}_{2i-1} |n\rangle\langle n| \hat{X}_{2i-1} = |n_i\rangle\langle n_i|$, which we interpret as nearest neighbor hopping in Fock space. Notice that $(\hat{X}_{2i-1} \hat{X}_{2i-1}) |n\rangle\langle n| (\hat{X}_{2i-1} \hat{X}_{2i}) = |n\rangle\langle n|$ leaves the state unchanged.

With these structures in place, it is straightforward to describe the action of $\mathcal{P} |n\rangle\langle n| = \frac{1}{\rho} \sum_a \hat{X}_a |n\rangle\langle n| \hat{X}_a$. The summation contains contributions changing the particle number $|n\rangle$ by 0, 2 and 4. With $\mathcal{P}_{n,m} = \langle m| \mathcal{P} |n\rangle\langle n|m\rangle$, the diagonal contribution, $\mathcal{P}_0$ is obtained from the $\binom{N}{2}$ terms of the structure $\hat{X}_{2i} \hat{X}_{2i+1} \hat{X}_{2j} \hat{X}_{2j+1}$. Similar counting for the contributions changing $|n\rangle$ by two and 4 gives the matrix elements stated in the main text,

$$\mathcal{P}_0 = \frac{N(N-1)}{2\rho}, \quad \mathcal{P}_2 = \frac{4(N-2)}{\rho}, \quad \mathcal{P}_4 = \frac{16}{\rho}, \quad (\text{B1})$$

and it is verified that

$$\sum_m \mathcal{P}_{m,n} = \left( \begin{array}{c} N \end{array} \right) \frac{N(N-1)}{2\rho} + \left( \begin{array}{c} N \end{array} \right) \frac{4(N-2)}{\rho} + \left( \begin{array}{c} N \end{array} \right) \frac{16}{\rho} = 1. \quad (\text{B2})$$

**Appendix C: Saddle point equations**

In this Appendix we address the solution of the saddle point equation Eq. (5). The non-trivial element in this equation is the quantity $\kappa_n = \pi \langle \mathcal{P} \rangle_n$ in the denominator. In terms of this quantity, Eq. (4) becomes the simple algebraic equation \( \kappa = 1 \). A closed yet site non-local equation for $\kappa$ is obtained by acting on Eq. (5) with the operator $\mathcal{P}$,

$$\kappa_n = \sum_m \mathcal{P}_{|n-m|} \text{Im} \left. \frac{1}{v_m - i\kappa_m} \right|_{m}$$

$$= \sum_m \mathcal{P}_{|n-m|} \text{Re} \int_0^\infty dt e^{i v_m t - \kappa_m t},$$

where in the second line switch to a temporal Fourier representation to facilitate the treatment of the argument $v_m$. The solution of this equation relies on two conceptual elements, first the ansatz Eq. (6) and second a replacement of the sum over the $\rho$ neighboring sites $m$ by a Gaussian average over energies $v_m$. Specifically, we note that up to corrections small in $N^{-1}$, the neighbor sites $m$ are separated by Hamming distance 4 from $n$ and each change in $n_i$ changes $v_n \to v_n \pm 2v_i$. This means that $v_m = v_n + v$, where we assume $v$ to be Gaussian distributed with width $\sqrt{2\delta} = 4\delta$. Substituting the ansatz $\kappa_m = \kappa \Theta(C - |v_m|)$ into the equation, and splitting the integral over $v$ into regions with $C - |v_m| = C - |v_n| + v$ smaller and larger than zero, respectively, we obtain after shifting $v \to v - v_n$,

$$\kappa_n = \frac{1}{\sqrt{2\pi} \delta} \text{Re} \int_0^\infty dt \times \left( \int_C^\infty dv e^{-\frac{(v-v_n)^2}{2\delta^2}} + \int_C^{-C} dv e^{-\frac{(v-v_n)^2}{2\delta^2}} (e^{-\kappa t} - 1) \right) e^{i vt}.$$
now explore for which configurations \((C, \kappa)\) it represents a self consistent solution.

The details of this analysis depend on whether we work with weakly (I, II) or strongly (III, IV) distributed on-site energies.

**Strong on-site disorder III, IV:** Anticipating that all solutions satisfy \(\kappa < 1\), the width of \(\delta_\kappa(v)\) is much smaller than that of the Gaussian weight, \(\delta\). The function \(\delta_\kappa\) thus collapses the integral, and we obtain

\[
\kappa_\nu = \frac{\sqrt{\pi}}{\sqrt{32\delta}} e^{-\frac{v^2}{32\delta^2}}, \quad (C2)
\]

This is consistent with our ansatz with \(C = 2\delta\) and \(\kappa \sim \delta^{-1}\).

**Narrow on-site disorder I, II:** In this regime, we test for the validity of the ansatz with \(C = 1\) and \(\kappa = 1\). First assume \(|v_n| > 1 = C \gg \delta\). In this case, the ansatz requires exponentially suppressed \(\kappa\), the \(\delta_\kappa\)-function again becomes effective, and the integral collapses to \(\kappa_\nu = \frac{\sqrt{\pi}}{\sqrt{32\delta}} \exp(-\frac{v^2}{32\delta^2})\) consistent with the assumed smallness of \(\kappa\). Conversely, for \(|v_n| < 1 = C\), the ansatz requires \(\kappa = 1\). The function \(\delta_\kappa = \delta_1\) is now much wider than the width of the Gaussian, \(\sim \delta\), and the integration boundaries can be extended to infinity. Doing the integral, we obtain \(\kappa_\nu \equiv \kappa = 1/\kappa\), or \(\kappa = 1\), consistent with Eq. (46).

**Appendix D: Effective matrix theory**

In this appendix we discuss the derivation of Eqs. (38) and (39) from Eq. (28). In Eq. (28), we substitute \(Y \to \pi \tilde{\nu} Q\) with \(Q_n = T_n \sigma_3 T_n^{-1}\). The expansion of the action in fluctuations then comprises three parts: the Gaussian weight, the expansion of the ‘Str log’ in site-to-site fluctuations, and the expansion of the ‘Str log’ in small frequency arguments, \(z\) (reflecting the non-commutativity, \([z, T_n] \neq 0\).)

**Gaussian weight:** A straightforward substitution yields

\[
\frac{-1}{2} \text{Str}(Y PY) \to -\frac{\pi^2}{2} \text{Str}(\tilde{\nu} Q \mathcal{P}(\tilde{\nu} Q)) = -\frac{\pi^2}{2} \sum_{nm} \nu_n \nu_m P_{|n-m|} \text{Str} Q_n Q_m. \quad (D1)
\]

**Fluctuation action:** Substituting the ansatz into the ‘Str log’ and temporarily neglecting the frequency arguments, \(z\), we obtain

\[
\begin{align*}
\text{Str log}(\tilde{H}_2 + i\pi \mathcal{P}(\tilde{\nu} Q)) & = \text{Str log}(\tilde{H}_2 + iT^{-1} \pi \mathcal{P}(\tilde{\nu} Q) \hat{T}) \\
& = \text{Str log}(\tilde{H}_2 + i\pi \mathcal{P}(\tilde{\nu} \sigma_3) + iT^{-1} \mathcal{P}(\tilde{\nu} Q) \hat{T} - \mathcal{P}(\tilde{\nu} \sigma_3)) \\
& \approx \text{Str log}(1 + \pi^2 \tilde{\nu} \sigma_3 T^{-1} \mathcal{P}(\tilde{\nu} Q) \hat{T} - \mathcal{P}(\tilde{\nu} \sigma_3)) \\
& \approx \pi^2 \text{Str}(\tilde{\nu} \sigma_3 T^{-1} \mathcal{P}(\tilde{\nu} Q) \hat{T} - \mathcal{P}(\tilde{\nu} \sigma_3)) \\
& = \pi^2 \text{Str}(\tilde{\nu} Q \mathcal{P}(\tilde{\nu} Q)), \quad (D2)
\end{align*}
\]

identical to \((-2\times)\) the Gaussian weight. In the second line we used the cyclic invariance \(\text{Str log}(\ldots) \text{Str log}(\hat{T}^{-1}(\ldots) \hat{T})\), and in the fourth the saddle point equation \((-\tilde{H}_2 + i\pi \mathcal{P}(\tilde{\nu} \sigma_3))^{-1} = -i\pi \tilde{\nu} \sigma_3\).

**Frequency action:** In a similar manner, we obtain

\[
\begin{align*}
\text{Str log}(\tilde{H}_2 + i\pi \mathcal{P}(\tilde{\nu} Q) + z) & \approx \text{Str log}(\tilde{T}(\tilde{\nu} \sigma_3 + T^{-1} \tilde{\nu} Q + z) \\
& = \text{Str log}(\tilde{H}_2 + i\pi \mathcal{P}(\tilde{\nu} \sigma_3) + T^{-1} z \hat{T}) \\
& \approx -i\pi \text{Str}(\tilde{\nu} \sigma_3 T^{-1} z \hat{T}) = -i\pi \text{Str}(\tilde{\nu} Q z), \quad (D3)
\end{align*}
\]

where in the first line, we neglected local fluctuations \(P(\tilde{\nu} \sigma_3 T^{-1}) \approx TP(\tilde{\nu} \sigma_3) T^{-1}\), in the third used cyclic invariance, and in the fourth the saddle point condition.

Combining terms, we obtain the effective action \((38)\).

**Appendix E: Wave function and spectral statistics from matrix model**

In this section we provide details on the computation of wave-function and spectral statistics in the deformed SYK\(_3\) model. The starting point for both statistics is Eq. (35), with sources \(j = J_k\) or \(J = J_{1,n}\), respectively, given in Eq. (22). Using the commutativity \([T, H_2] = 0\) we represent the action as

\[
S[T] = \text{Str log}(1 + \tilde{G} \Omega_T) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \text{Str}(\tilde{G} \Omega_T)^k,
\]

where \(\Omega_T \equiv T^{-1} [z - j(\alpha, \beta)] T\) is an operator in which we need to expand to the order required by the correlation function, and we have made the source contribution, \(j(\alpha, \beta)\), to the matrix \(z = \omega \nu_n \sigma_3\) explicit again. Concerning the resolvent, \(\tilde{G}^{-1} \equiv i\kappa \sigma_3 - H_2\), we notice that fluctuation variables commute through the real part of \(\tilde{G}\), and keep only \(i \text{Im} \tilde{G} = -i \kappa \tilde{\nu}\), with local components \(\nu_n\) defined in Eq. (4). Specifically, to zeroth order in the sources, and first order in an expansion in \(z \nu_n \sim \omega / \Delta\), the action assumes the form \((36)\).

For the computation of the spectral and wave function statistics, we need the expansion in sources to first order in \(\beta\) and higher orders in \(\alpha\). With the above definitions, the expansion of the action assumes the form

\[
S[T] = -\pi \sum_{k=1}^{\infty} (-i \nu_n \alpha)^k \left( \frac{1}{k} [Q_{bb}^{++}]^k + \frac{\beta}{\alpha} [Q_{bb}^{+}]^{k-1} Q_{ff}^{-} \right), \quad (E1)
\]

where in the terms \(k > 2\) we used the approximation \(Q_{bb}^{++} \approx Q_{bb}^{+} \approx Q_{ff}^{-}\) valid in the limit \(\nu_n \to 0\) implied in the calculation of wave function moments \((39)\). Doing the derivatives in the source parameters, we arrive at

\[
\begin{align*}
\partial_{\alpha} Q_{bb}^{++} \partial_{\beta} Z|_{\alpha, \beta = 0} & = (-i \nu_n \alpha)^q q! \langle [Q_{bb}^{+}]^{q-1} Q_{ff}^{-} \rangle, \quad (E2)
\end{align*}
\]

where \(\langle ... \rangle = \int dQ e^{-S[T]} Q \ldots\).
The remaining integral over the four-dimensional matrix $Q$ is conceptually straightforward but technically the hardest part of the calculation. Referring for details to Ref. [30] we here review the main steps. The starting point is a ‘polar coordinate’ representation $Q = UQ_0U^{-1}$ with $Q_0$ defined in Eq. (37), $\hat{\theta} = \text{diag}(i\bar{\theta}_1, \bar{\theta}_1)$ containing compact and non-compact angles $0 < \bar{\theta}_1 < \pi$ and $\bar{\theta}_1 > 0$, respectively [30]. The matrix $U$ is block-diagonal in causal space and contains four Grassmann variables $\eta^\pm, \bar{\eta}^\pm$, and two more commuting variables $0 \leq \phi, \chi < 2\pi$. More specifically, $U = \text{diag}(u_1, u_2, v)\eta^\pm, v_2 = \text{diag}(e^{i\phi}, e^{i\chi})\bar{u}^\pm$ and supermatrices $u_1 = e^{-2i\bar{\eta}^+}, v = e^{-2i\eta^-}$, generated by $\eta^\pm = \begin{pmatrix} 0 & \eta^- \\ -\eta^- & 0 \end{pmatrix}$. In this representation, the matrix elements entering the correlation function are given by $Q_{bb}^{\pm} = \cosh \theta_{bb}(1 - 4\eta^+\eta^+)$ and $Q_{\bar{b}b}^{\pm} = \cos \theta_{bb}(1 - 4\eta^+\bar{\eta}^-)$, and the integration measure reads $dQ = 2^{\pm}\pi^2 \sinh \bar{\theta}_1 \sin \bar{\theta}_1 d\bar{\theta}_1 d\bar{\theta}_1 d\eta^+ d\eta^- d\bar{\eta}^+ d\bar{\eta}^- [30]$. The essential advantage of the polar representation is that the action only depends on the ‘radial variables’ $S_{\text{p}}[Q] = -i2\nu v(\omega + i\eta)\cos \theta_1 - \cos \theta_1$.

Wave function statistics: In the calculation of the wave function moments, we may set $\omega = 0$. The integration over the non-compact angle is then cut by the parameter $\eta$ at values $1 \leq \cos \theta_1 \lesssim 1/\eta$, while the integration over the compact angles $\theta_1$ is free. With this simplification, the integration over all variables except the non-compact one, $\theta$, becomes elementary, and one obtains [30]

$$G_{nn}(\eta^{-1}) G_{nn}^{-1} = 2q(q - 1) (-i\pi\nu_n)^q \int_0^\infty d\theta_1 \sinh \theta_1 \cosh \theta_1 (\cosh \theta_1)^{q-2} e^{-2\pi\nu_\eta \cosh \theta_1}. \quad (E3)$$

The final integral gives $(2\pi\nu_\eta)^{-q+1} q!$ and collecting all factors we arrive at

$$I_q = q! \nu^q \sum_n \nu_n^q. \quad (E4)$$

This result expresses the $n$th moment of the local wave function amplitudes through that of the local density of states individually averaged over SYK$_{2n}$ fluctuations. The energies $\nu_n$ at each individual site are obtained as sums of $N$ random coefficients $v_i$ (cf. Eq. (3)). For large $N$, this makes the sum self averaging, and we replace $I_q \rightarrow \langle I_q \rangle_v$ by its average over single particle energies, $v_i$. Using Eq. (6), we thus obtain

$$I_q = \left( \frac{-q^{-1} q}{\pi \nu} \right) \frac{D}{(\nu)^q} \kappa^q \frac{\partial^{q-1}}{\partial \nu^q} \left[ \frac{1}{\nu^2 + \kappa^2} \right].$$

The evaluation of this expression now depends on which on-site disorder regime we are in. In regime I, $\delta < N^{-1/2}$, or $|v_i| < 1$, the mean field broadening assumes the uniform value $\kappa = 1$. In this case, the dependence of $I_q$ on site energies, $v_n$, is weak. This implies $\nu \simeq \frac{1}{\pi} \sum n = D/\pi$. Doing the $\kappa$ derivatives, we obtain

$$I_q = q! D^{1-q}, \quad \text{regime I,} \quad (E5)$$

which is the RMT result for a matrix of dimension $D$.

For larger disorder, only a fraction of sites have finite decay width. Using Eq. (9) and assuming self averaging to replace the $n$-sum to an average over a distribution of site energies of width $\delta N$, the DoS is evaluated as

$$\nu \simeq \frac{1}{\pi \sqrt{2\pi\nu N^2}} \int_C \frac{D}{v^2 + \kappa^2} \left( e^{-\frac{v^2}{2\nu^2 + \kappa^2}} \frac{D}{v^2 + \kappa^2} \right) \approx \frac{1}{\pi \sqrt{2\pi\nu N^2}} \arctan(C/\kappa), \quad (E6)$$

where in the second line we used that the distribution of energies is much wider than the tolerance window $C$. Substituting the values specified in Eq. (6), this leads to

$$\nu = c \rho D \sqrt{\frac{1}{N}}, \quad (E6)$$

where $c$ is of order unity and the suppression relative to $\nu = cD$ in regime I accounts for the improbability to find resonant sites.

In the same manner, we obtain

$$I_q \simeq \left( \frac{-q^{-1} q}{\pi \nu} \right) \frac{D}{(\nu)^q} \kappa^q \frac{\partial^{q-1}}{\partial \nu^q} \left[ \frac{1}{\nu^2 + \kappa^2} \right] \frac{1}{\nu^2 + \kappa^2} \approx \left( \frac{-q^{-1} q}{\pi \nu} \right) \frac{D}{(\nu)^q} \kappa^q \frac{\partial^{q-1}}{\partial \nu^q} \frac{1}{\kappa^2} \arctan(C/\kappa) \approx 2 \left( \frac{-q^{-1} q}{\pi \nu} \right) \frac{D}{(\nu)^q} \kappa^q \frac{\partial^{q-1}}{\partial \nu^q} \frac{1}{\kappa} \arctan(C/\kappa)$$

$$\approx \frac{1}{\pi \sqrt{2\pi N^2}} \sqrt{2\pi\nu N^2} \kappa^q \frac{\partial^{q-1}}{\partial \nu^q} \frac{1}{\kappa} \frac{1}{\kappa} \arctan(C/\kappa)$$

where ‘$\simeq$’ here means equality up to some constant $c \sim O(1)$. Insertion of Eq. (6) leads to Eq. (13). Using Eq. (6), we finally obtain

$$q \gg 1: \quad I_q = c^q q! \left( \frac{D}{\sqrt{N}} \right)^{1-q} \left( \frac{\delta^{q-1}}{\delta^{2(q-1)}} \right)^{1-q} \delta^{q-1}, \quad \text{regime II,} \quad (E7)$$

Finally, for a quantitative comparison to numerical simulations in regime III without fitting parameter we trace all constants $c \sim O(1)$ in $\nu$ and $I_q$. Noting that one in regime III we can substitute $\arctan(C/\kappa) = \pi/2$ we arrive at,

$$I_q = q! (2q - 3)!! \left( \frac{\pi D}{4\sqrt{N}} \right)^{1-q} \frac{\delta^{q-1}}{\delta^{2(q-1)}} \left( \frac{\pi D}{4\sqrt{N}} \right)^{1-q} \delta^{q-1}, \quad (E8)$$

where in the second equality we used Eq. (C2) for $\kappa$.

Level-statistics: For the level statistics we need to keep finite $\omega$, and differentiate the functional to first order in
where \( \theta_h \) and \( \theta_f \) are the non-compact bosonic and compact fermionic angle, respectively. These integrals can be carried out in closed form, and yield the two-point correlation function of the Gaussian Unitary Ensemble \([9]\).

\[ K(\omega) = \frac{1}{2} \text{Re} \int_0^\infty d\theta_h \int_{-\pi/2}^{\pi/2} d\theta_f \sinh \theta_h \sin \theta_f \delta^{\omega}(\cosh \theta_h - \cos \theta_f), \]  

(E9)

[1] P. W. Anderson, Absence of Diffusion in Certain Random Lattices, Phys. Rev. 109, 1492 (1958).

[2] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Quasiparticle Lifetime in a Finite System: A Nonperturbative Approach, Phys. Rev. Lett. 78, 2803 (1997).

[3] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Many-body delocalization transition and relaxation in a quantum dot, Phys. Rev. B 93, 125419 (2016).

[4] D. Basko, I.Aleiner, and B. Altshuler, Metalinsulator transition in a weakly interacting many-electron system with localized single-particle states, Ann. Phys. 321, 1126 (2006).

[5] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Interacting Electrons in Disorder: Anderson Localization and Low-T Transport, Phys. Rev. Lett. 95, 206603 (2005).

[6] J. Suntajs, J. Bonca, T. Prosen, and L. Vidmar, Quantum chaos challenges many-body localization, arXiv:1905.06345.

[7] D. A. Abanin, et al., Distinguishing localization from chaos: challenges in finite-size systems, arXiv:1911.04501.

[8] R. K. Panda, A. Scardicchio, M. Schulz, S. R. Taylor, and M. Žnidarič, Can we study the many-body localisation transition?, Eur. Phys. Lett. 128, 67003 (2020).

[9] A. De Luca, B. L. Altshuler, V. E. Kravtsov, and A. Scardicchio, Anderson Localization on the Bethe Lattice: Nonergodicity of Extended States, Phys. Rev. Lett. 113, 046806 (2014).

[10] V. E. Kravtsov, I. M. Khaymovich, E. Cuevas, and M. Amini, A random matrix model with localization and ergodic transitions, New Journal of Physics 17, 122002 (2015).

[11] B. L. Altshuler, E. Cuevas, L. B. Ioffe, and V. E. Kravtsov, Nonergodic Phases in Strongly Disordered Random Graphs, Phys. Rev. Lett. 117, 156601 (2016).

[12] V. E. Kravtsov, B. L. Altshuler, and L. B. Ioffe, Non-ergodic delocalized phase in Anderson model on Bethe lattice and regular graph, Annals of Physics 389, 148 (2018).

[13] K. S. Tikhonov, and A. D. Mirlin, Statistics of eigenstates near the localization transition on random regular graphs, Phys. Rev. B 99, 024202 (2019).

[14] E. J. Torres-Herrera, and L. F. Santos, Extended non-ergodic states in disordered manybody quantum systems, Ann. Phys. 529, 1600284 (2017).

[15] L. Faoro, M. Feigel’man, and L. Ioffe, Non-ergodic extended phase of the Quantum Random Energy model, Ann. of Phys. 409, 167916 (2019).

[16] G. Biroli, A. C. Ribeiro-Teixeira, and M. Tarzia, Difference between level statistics, ergodicity and localization transitions on the Bethe lattice, arXiv:1211.7334.

[17] G. Biroli, and M. Tarzia, Delocalization and ergodicity of the Anderson model on Bethe lattices, arXiv:1810.07545.

[18] G. Biroli, and M. Tarzia, Delocalized glassy dynamics and many-body localization, Phys. Rev. B 96, 201114 (2017).

[19] A. De Luca, B. L. Altshuler, V. E. Kravtsov and A. Scardicchio, Anderson Localization on the Bethe Lattice: Nonergodicity of Extended States, Phys. Rev. Lett. 113, 046806 (2014).

[20] K. S. Tikhonov, and A. D. Mirlin, Statistics of eigenstates near the localization transition on random regular graphs, Phys. Rev. B 99, 024202 (2019).

[21] K. S. Tikhonov, and A. D. Mirlin, Critical behavior at the localization transition on random regular graphs, Phys. Rev. B 99, 214202 (2019).

[22] X. Y. Song, C. M. Jian, and L. Balents, Strongly Correlated Metal Built from Sachdev-Ye-Kitaev Models, Phys. Rev. Lett. 119, 216601 (2017).

[23] S. Banerjee and E. Altman, Solvable model for a dynamical quantum phase transition from fast to slow scrambling, Phys. Rev. B 95, 134302 (2017).

[24] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography, Phys. Rev. B 95, 155131 (2017).

[25] D. Chowdhury, Y. Werman, E. Berg, and T. Senthil, Translationally Invariant Non-Fermi-Liquid Metals with Critical Fermi Surfaces: Solvable Models, Phys. Rev. X 8, 031024 (2018).

[26] A. A. Patel, J. Mc Greevy, D. P. Arovas, and S. Sachdev, Magnetotransport in a Model of a Disordered Strange Metal, Phys. Rev. X 8, 021049 (2018).

[27] A. V. Lukin, K. S. Tikhonov, and M. V. Feigel’man, Sachdev-Ye-Kitaev Model with Quadratic Perturbations: The Route to a Non-Fermi Liquid, Phys. Rev. Lett. 121, 236601 (2018).

[28] A. Altland, D. Bagrets, and A. Kamenev, Sachdev-Ye-Kitaev Non-Fermi-Liquid Correlations in Nanoscopic Quantum Transport, Phys. Rev. Lett. 123, 226801 (2019).

[29] F. Wegner, The mobility edge problem: Continuous symmetry and a conjecture, Z. Phys. B 35, 207 (1979).

[30] K. B. Efetov, Supersymmetry in Disorder and Chaos (Cambridge Univ. Press, 1999).

[31] T. Micklitz, F. Monteiro, and A. Altland, Nonergodic Extended States in the Sachdev-Ye-Kitaev Model, Phys. Rev. Lett. 123, 125701 (2019).

[32] S. Sachdev and J. Ye, Gapless spin-fluid ground state in a random quantum Heisenberg magnet, Phys. Rev. Lett. 70, 3339 (1993).

[33] A. Kitaev, http://online.kitp.ucsb.edu/online/entangled15/kitaev/ .... /kitaev2/ (Talks at KITP on April 7th and May 27th 2015).
A. M. García-García, B. Loureiro, A. Romero-Bermúdez, and M. Tezuka, Chaotic-Integrable Transition in the Sachdev-Ye-Kitaev Model, Phys. Rev. Lett. 120, 241603 (2018).

A. R. Kolovsky and D. L. Shepelyansky, Dynamical thermalization in isolated quantum dots and black holes, Eur. Phys. Lett. 117, 10003 (2017).

Recalling conservation of fermion parity, distances are four, two, or zero.

Although the eigenvalues \( \{ \pm \nu_i \} \) of \( J_{ij} \) are correlated, their sums, i.e. the eigenvalues of \( \hat{H}_2 \), become uncorrelated for large \( N \).

Here we ignore corrections of \( \mathcal{O}(1/N) \). However, for numerically accessible sizes it is important to keep in mind the full expressions for the SYK\(_{4} \) band width \( w_4 = \sqrt{\frac{3}{2N^2}} \left( \frac{2N}{4} \right) \). Then, in order to still have the SYK\(_{2} \) band width \( w_4 = \Delta \), we need to rescale \( J_2 = \delta (1 - \frac{3}{2N} + \frac{3}{2N^2})^{1/2} \).

Notice that the inverse participation ratio here, eq. (16), has not been normalized by its value at \( \delta = 0 \), as in our previous publication [31].

V. Oganesyan and D. Huse, Localization of interacting fermions at high temperature, Phys. Rev. B 75, 155111 (2007).

Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Distribution of the Ratio of Consecutive Level Spacings in Random Matrix Ensembles, Phys. Rev. Lett. 110, 084101 (2013).

We kept 1/7 of the total spectrum and verified a constant density of states, resp., that results remain unchanged when restricting to smaller energy windows.

The average is dominated by resonant sites, \( \sum_{\nu_m} f(\nu_m) = \frac{N}{\sqrt{2\pi \delta}} f(\pi\delta^{-1}) \), and we neglect a contribution \( \log(N^2\nu^{-1}) \ll \log(N^4) \) since \( \nu_n \gg \delta^{-1} \) for generic active sites.

K. Truong and A. Ossipov, Eigenvectors under a generic perturbation: Non-perturbative results from the random matrix approach, Eur. Phys. Lett 116, 37002 (2016).

Unlike with low dimensional single particle problems, the effectively high dimension of Fock space implies non-universality of the Thouless energy. For example, non-zero mode corrections to the spectral form factor (the Fourier transform of the two-point correlation function in energy) and the form factor itself, respectively, become visible at different energy scales.

M.R. Zirnbauer, Anderson Localization and Nonlinear \( \sigma \) Model With Graded Symmetry, Nucl. Phys. B [FS] 265, 375 (1986)

The inverse participation ratio at the localization transition, \( I_2(\delta_c) = \frac{8\sqrt{\pi} \nu_{\mathrm{c}}^2}{\pi^{3/2}} = 9\pi^{-1/2} N^{9/2} \log^2 N \), takes the value one for \( N \approx 32 \) complex fermions.