Infinitely many solutions for a class of fractional Orlicz-Sobolev Schrödinger equations

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Abstract

In the present paper, we deal with a new compact embedding theorem for a subspace of the new fractional Orlicz-Sobolev spaces. We also establish some useful inequalities which yields to apply the variational methods. Using these abstract results, we study the existence of infinitely many nontrivial solutions for a class of fractional Orlicz-Sobolev Schrödinger equations whose simplest prototype is

$(-\triangle)_s^m u + V(x)m(u)u = f(x, u), \quad x \in \mathbb{R}^N,$

where $0 < s < 1$, $N \geq 2$, $(-\triangle)_s^m$ is fractional $M$-Laplace operator and the nonlinearity $f$ is sublinear as $|u| \to \infty$. The proof is based on the variant Fountain theorem established by Zou.

Keywords: Fractional Orlicz-Sobolev space, Compact embedding theorem, Fractional $M$–Laplacian, Fountain Theorem.

1 Introduction and main result

In this paper, we are concerned with the study of the nonlinear fractional $M$-Laplacian equation:

$(-\triangle)_s^m u + V(x)m(u)u = f(x, u), \quad x \in \mathbb{R}^N,$

(1.1)

where $0 < s < 1$, $N \geq 2$ and $M(t) = \int_0^{|t|} m(s)ds$.

In the last years, problem (1.1) has received a special attention for the case where $M(t) = \frac{1}{2}|t|^2$, that is, when it is of the form

$-\triangle u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N.$

(1.2)

We do not intend to review the huge bibliography of equations like (1.2), we just emphasize that the most famous conditions on the potential $V : \mathbb{R}^N \to \mathbb{R}$ are the following:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$.

(V2) There exists $\nu > 0$ such that

$$\lim_{|y| \to +\infty} \text{meas}(\{x \in \mathbb{R}^N : |x - y| \leq \nu, \ V(x) \leq L\}) = 0, \quad \forall L > 0,$$

where $\text{meas}(.)$ denotes the Lebesgue measure in $\mathbb{R}^N$. We quote here [6, 26, 27] where the existence of infinitely many nontrivial solutions for the equation (1.2) have been obtained in connection with the geometry of the function $V$.

For the case where $s = 1$, problem (1.1) becomes

$-\triangle_m u + V(x)m(u)u = f(x, u), \quad x \in \mathbb{R}^N,$

(1.1)
where the operator $\triangle_m u = \text{div}(m(\nabla u)\nabla u)$ named $M$-Laplacian. The reader can find more details involving this subject in [11, 12, 20, 21] and their references.

Notice that when $0 < s < 1$ and $M(t) = \frac{1}{p}|t|^p$ where $p > 1$ the problem (1.1) gives back the fractional Schrödinger equation
\[(−\triangle)^s_p u + V(x)|u|^{p−2}u = f(x, u), \quad x ∈ \mathbb{R}^N,\]  
where $(-\triangle)^s_p$ is the non-local fractional $p$-Laplacian operator. Concerning the equation (1.3), in the last decade, many several existence and multiplicity results have been obtained by using different variational methods. In [8], the authors studied the existence of multiple ground state solutions for the problem (1.3), when the nonlinear term $f$ is assumed to have a superlinear behaviour at the origin and a sublinear decay at infinity. Ambrosio [3] established an existence of infinitely solutions for the problem (1.3), when $f$ is $p$-superlinear and $V(x)$ can change sign. Moreover, fractional Schrödinger-type problems have been considered in some interesting papers [4, 16, 25]. The literature on non-local operators and on their applications is very interesting and, up to now, quite large. After the seminal papers by Caffarelli et al. [11, 12, 13], a large amount of papers were written on problems involving the fractional diffusion operator $(-\Delta)^s (0 < s < 1)$.

We can quote [7, 14, 15, 23, 24] and the references therein. We also refer to the recent monographs [14, 22] for a thorough variational approach of non-local problems.

Contrary to the classical fractional Laplacian case that is widely investigated, the situation seems to be in a developing state when the new fractional $M$-Laplacian is present. In this context, the natural setting for studying problem (1.1) are fractional Orlicz-Sobolev spaces. Currently, as far as we know, the only results for fractional Orlicz-Sobolev spaces and fractional $M$-Laplacian operator are obtained in [2, 5, 10]. In particular, in [10], the authors define the fractional order Orlicz-Sobolev space associated to an $N$-function $M$ and a fractional parameter $0 < s < 1$ as
\[W^{s,M}(\Omega) = \left\{ u ∈ L^M(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M\left(\frac{|u(x)−u(y)|}{|x−y|^s}\right) dx dy < \infty \right\}.\]

The previous definition creates problems in the calculus and in the embedding results, for example, the Borel measure defined as $d\mu = \frac{dx dy}{|x−y|^N}$ is not finish in the neighbourhood of the origin, that’s why, in [2], the authors introduced another definition of the fractional Orlicz-Sobolev space, i.e.,
\[W^{s,M}(\Omega) = \left\{ u ∈ L^M(\Omega) : \exists \lambda > 0 / \int_{\Omega} \int_{\Omega} M\left(\frac{\lambda(u(x)−u(y))}{|x−y|^sM^{-1}(|x−y|^N)}\right) dx dy < \infty \right\}.
\]

The authors in [5] gave some further basic properties both on this function space and the related nonlocal operator.

Motivated by the above papers, under the suitable conditions ($V_1$) and ($V_2$) on the potential $V$ and exploiting the variant Fountain theorem, we aim to study the multiplicity of nontrivial weak solutions to (1.1) where the new fractional $M$-Laplacian is present. In this spirit, we deal with a new compact embedding theorem, also, we establish some useful inequalities which yields to apply the variational methods. As far as we know, all these results are new.

Related to functions $M$ and $f$, our hypotheses are the following:

**Conditions on $m$ and $M$:**

The function $m : \mathbb{R}_+ → \mathbb{R}_+$ is a $C^1$-function satisfying

\[m(t), (m(t)t)’ > 0 \text{ for all } t > 0.\]

\[t \leq r < l^* = \frac{NI}{N−l} \quad \text{and} \quad l \leq \frac{m(t)t^2}{M(t)} \leq r, \quad \forall t ≠ 0,\]
where

\[ M(t) = \int_0^{|t|} m(s) ds. \]

Moreover, \( m_*(t) t \) is such that the Sobolev conjugate function \( M_* \) of \( M \) is its primitive; that is,

\[ M_*(t) = \int_0^{|t|} m_*(s) ds. \]

(M1) There exists a positive constant \( C \) such that

\[ C |t|^\mu \leq M_*(at), \quad \forall a, t \geq 0, \quad \forall 1 < \mu \leq r, \]

where \( M_* \) is the Sobolev conjugate of \( M \).

(M2) \( \lim_{|t| \to +\infty} \frac{C |t|^\mu}{M(t)} = 0, \quad \forall 1 < \mu \leq r \).

(M3) The function \( t \to M(\sqrt{t}), \ t \in [0, \infty] \) is convex.

Conditions on \( f \):

(\( f_1 \)) \( f(x, u) = p\xi(x)|u|^{p-2}u \), where \( 1 < p < l \) is a constant and \( \xi : \mathbb{R}^N \to \mathbb{R} \) is a positive continuous function such that \( \xi \in L^\infty_\mathrm{loc}(\mathbb{R}^N) \).

We mention some examples of functions \( M \), whose function \( m(t) \) satisfies the conditions (\( m_1 \))-(\( m_2 \)). The examples are the following:

1. \( M(t) = |t|^p \) for \( 1 < p < N \).
2. \( M(t) = |t|^p + |t|^q \) for \( 1 < p < q < N \) and \( q \in ]p, p^* [ \) with \( p^* = \frac{Np}{N-p} \).
3. \( M(t) = (1 + |t|^2)^\gamma - 1 \) for \( 1 < \gamma < \frac{N}{N-2} \).

Using the above hypotheses, we are able to state our main result.

**Theorem 1.1.** Suppose that (\( m_1 \)) - (\( m_2 \)), (\( M_1 \)) - (\( M_3 \)), (\( V_1 \)) - (\( V_2 \)) and (\( f_1 \)) hold. Then, problem (1.1) possesses infinitely many nontrivial solutions.

This paper is organized as follows. In Section 2, we give some definitions and fundamental properties of the spaces \( L^M(\Omega) \) and \( W^{s,M}(\Omega) \). In Section 3, we prove some basic properties of the fractional Orlicz-Sobolev space and we show a compact embedding type theorem. Finally, in Section 4, using a variant Fountain theorem, we prove our main result.

### 2 Preliminaries

In this preliminary section, for the reader’s convenience, we make a brief overview on the fractional Orlicz-Sobolev spaces studied in [2], and the associated fractional \( M \)-laplacian operator.

Let \( M : \mathbb{R} \to \mathbb{R}_+ \) be an \( N \)-function, i.e,

1. \( M \) is even, continuous, convex, with \( M(t) > t \) for \( t > 0 \),
2. \( \frac{M(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{M(t)}{t} \to +\infty \) as \( t \to +\infty \).

Equivalently, \( M \) admits the representation:

\[ M(t) = \int_0^{|t|} m(s) ds, \]
where \( m : \mathbb{R}_+ \to \mathbb{R}_+ \) is non-decreasing, right continuous, with \( m(0) = 0 \), \( m(t) > 0 \ \forall t > 0 \) and \( m(t) \to \infty \) as \( t \to \infty \). The conjugate \( N \)-function of \( M \) is defined by

\[
\overline{M}(t) = \int_0^{|t|} \overline{m}(s)ds,
\]

where \( \overline{m} : \mathbb{R}_+ \to \mathbb{R}_+ \) is given by \( \overline{m}(t) = \sup\{s : m(s) \leq t\} \). Evidently we have

\[
st \leq M(s) + \overline{M}(t),
\]

which is known as the Young inequality. Equality holds if and only if either \( t = m(s) \) or \( s = \overline{m}(t) \).

In what follows, we say that an \( N \)-function \( M \) verifies the \( \Delta_2 \) condition

\[
M(2t) \leq KM(t), \ \forall t \geq 0,
\]

for some constant \( K > 0 \). This condition can be rewritten in the following way: For each \( s > 0 \), there exists \( K_s > 0 \) such that

\[
M(st) \leq K_s M(t), \ \forall t \geq 0.
\]

If \( A \) and \( B \) are two \( N \)-functions, we say that \( A \) is stronger than \( B \) if

\[
B(x) \leq A(ax), \ x \geq x_0 \geq 0,
\]

for each \( a > 0 \) and \( x_0 \) (depending on \( a \)), \( B \prec \prec A \) in symbols. This is the case if and only if for every positive constant \( k \)

\[
\lim_{t \to +\infty} \frac{B(kt)}{A(t)} = 0.
\]

The Orlicz class \( K^M(\Omega) \) (resp. the Orlicz space \( L^M(\Omega) \)) is defined as the set of (equivalence classes of) real-valued measurable functions \( u \) on \( \Omega \) such that

\[
\rho(u; M) = \int_{\Omega} M(u(x))dx < \infty \quad \text{(resp.} \quad \int_{\Omega} M(\lambda u(x))dx < \infty \quad \text{for some } \lambda > 0)\).
\]

\( L^M(\Omega) \) is a Banach space under the Luxemburg norm

\[
\|u\|_{L^M(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u}{\lambda}\right) \leq 1 \right\},
\]

whose norm is equivalent to the Orlicz norm

\[
\|u\|_{L^M(\Omega)} = \sup_{\rho(u; M) \leq 1} \int_{\Omega} |u(x)||v(x)|dx.
\]

The next lemma and their proof can be found in [17].

**Lemma 2.1.** Assume that \((m_1)\) and \((m_2)\) hold and let \(\xi_0(t) = \min\{t^l, t^r\}, \xi_1(t) = \max\{t^l, t^r\}, \) for all \( t \geq 0 \). Then,

\[
\xi_0(\rho)M(t) \leq M(\rho t) \leq \xi_1(\rho)M(t) \ \text{for} \ \rho, t \geq 0
\]

and

\[
\xi_0(\|u\|_{L^M(\Omega)}) \leq \int_{\mathbb{R}^N} M(|u|)dx \leq \xi_1(\|u\|_{L^M(\Omega)}) \ \text{for} \ u \in L^M(\mathbb{R}^N).
\]

**Definition 2.2.** Let \( M \) be an \( N \)-function. For a given domain \( \Omega \) in \( \mathbb{R}^N \) and \( 0 < s < 1 \), we define the fractional Orlicz-Sobolev space \( W^{s,M}(\Omega) \) as follows,

\[
W^{s,M}(\Omega) = \left\{ u \in L^M(\Omega) : \exists \lambda > 0 / \int_{\Omega} \int_{\Omega} M\left(\frac{\lambda (u(x) - u(y))}{|x - y|^s M^{-1}(|x - y|^N)}\right)dxdy < \infty \right\}.
\]

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Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and let $s \in ]0,1[.$ Let $M$ be an $N$-function. Then there exists a positive constant $\mu$ such that,

$$\|u\|_{M} \leq \mu [u]_{s,M}, \quad \forall \ u \in W^{s,M}_0(\Omega).$$

Therefore, if $\Omega$ is bounded and $M$ be an $N$-function, then $[u]_{s,M}$ is a norm of $W^{s,M}_0(\Omega)$ equivalent to $\|u\|_{s,M}$.

Let $M$ be a given $N$-function, satisfying the following conditions:

$$\int_0^1 \frac{M^{-1}(\tau)}{\tau^{\frac{N}{s}}}d\tau < \infty \quad (2.11)$$

and

$$\int_1^{+\infty} \frac{M^{-1}(\tau)}{\tau^{\frac{N}{s}}}d\tau = \infty. \quad (2.12)$$

If $(2.12)$ is satisfied, we define the inverse Sobolev conjugate $N$-function of $M$ as follows,

$$M_{-1}^{-1}(t) = \int_0^t \frac{M^{-1}(\tau)}{\tau^{\frac{N}{s}}}d\tau. \quad (2.13)$$

**Theorem 2.4.** Let $M$ be an $N$-function and $s \in ]0,1[.$ Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If $(2.11)$ and $(2.12)$ hold, then

$$W^{s,M}(\Omega) \hookrightarrow L^{M}(\Omega). \quad (2.14)$$

Moreover,

$$W^{s,M}(\Omega) \hookrightarrow L^{B}(\Omega). \quad (2.15)$$

is compact for all $B \prec \prec M_s.$

The fractional $M$-Laplacian operator is defined as

$$(-\Delta)^s_{M} u(x) = 2P.V \int_{\mathbb{R}^N} m\left(\frac{u(x) - u(y)}{|x - y|^sM^{-1}(|x - y|^N)}\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{dy}{|x - y|^sM^{-1}(|x - y|^N)}, \quad (2.16)$$

where $P.V$ is the principal value.

This operator is well defined between $W^{s,M}(\mathbb{R}^N)$ and its dual space $W^{-s,M}(\mathbb{R}^N).$ In fact, in $[2,$ lemma 3.5$]$ the following representation formula is provided

$$\langle (-\Delta)^s_{M} u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m\left(\frac{u(x) - u(y)}{|x - y|^sM^{-1}(|x - y|^N)}\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{v(x) - v(y)}{|x - y|^sM^{-1}(|x - y|^N)} dxdy, \quad (2.17)$$

for all $v \in W^{s,M}(\mathbb{R}^N).$
3 Variational setting and some useful tools

In this section, we will first introduce the variational setting for problem (1.1). In view of the presence of potential \( V(x) \), our working space is

\[
E = \left\{ u \in W^{s,M}(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x)M(u)dx < \infty \right\},
\]
equipped with the following norm

\[
\|u\| = \|u\|_{(s,M)} + \|u\|_{(V,M)}
\]

where

\[
\|u\|_{(V,M)} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} V(x)M\left(\frac{u}{\lambda}\right)dx \leq 1 \right\}.
\]

We define the functional \( G : E \to \mathbb{R} \) by

\[
G(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M(h_u(x,y))dxdy,
\]

where \( h_u(x,y) = \frac{u(x) - u(y)}{|x - y|^{s}M^{-1}(|x - y|^N)} \).

After integrating, we obtain from \((f_1)\) that for any \((x, t) \in \mathbb{R}^N \times \mathbb{R}\)

\[
F(x, t) = \int_{0}^{t} f(x,s)ds = \xi(x)|t|^p.
\]

In order to prove Theorem 1.1 we will consider the following family of functionals

\[
I_\lambda(u) = G(u) + \Psi(u) - \lambda B(u)
\]

with \( \lambda \in [1, 2], u \in E \) and

\[
\Psi(u) = \int_{\mathbb{R}^N} V(x)M(u)dx, \quad B(u) = \int_{\mathbb{R}^N} F(x,u)dx.
\]

We will show that \( I_\lambda \) satisfies the assumptions of the following variant of fountain Theorem due to Zou [28].

**Theorem 3.1.** Let \((E, \|\|)\) be a Banach space and \( E = \bigoplus_{j \in \mathbb{N}} X_j \) with \( \dim X_j < \infty \) for any \( j \in \mathbb{N} \). Set \( Y_k = \bigoplus_{j=1}^{k} X_j \) and \( Z_k = \bigoplus_{j=k}^{\infty} X_j \). Consider the following \( C^1 \)-functional \( I_\lambda : E \to \mathbb{R} \) defined by

\[
I_\lambda(u) = A(u) - \lambda B(u), \; \lambda \in [1, 2].
\]

Assume that \( I_\lambda \) satisfies the following assumptions:

(i) \( I_\lambda \) maps bounded sets to bounded sets for \( \lambda \in [1, 2] \) and \( I_\lambda(-u) = I_\lambda(u) \) for all \( (\lambda, u) \in [1, 2] \times E \).

(ii) \( B(u) \geq 0, B(u) \to +\infty \) as \( \|u\| \to +\infty \) on any finite dimensional subspace of \( E \).

(iii) There exists \( r_k > \rho_k \) such that

\[
a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0 \quad \text{and} \quad b_k(\lambda) := \max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u), \; \forall \lambda \in [1, 2],
\]

and

\[
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \; \text{as} \; k \to \infty \; \text{uniformly for} \; \lambda \in [1, 2].
\]

Then there exist \( \lambda_n \to 1, u_{\lambda_n} \in Y_n \) such that

\[
I'_{\lambda_n}(u_{\lambda_n}) \to 0 \quad \text{as} \quad k \to \infty.
\]

Particularly, if \((u_{\lambda_n})\) has a convergent subsequence for every \( k \), then \( I_1 \) has infinitely many nontrivial critical points \( \{u_k\} \in E \setminus \{0\} \) satisfying \( I_1(u_k) \to 0^- \) as \( k \to \infty \).

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Now we give the definition of weak solution for the problem \(1.1\). We define the functional \(I_1\) on \(E\) by
\[
I_1(u) = A(u) - B(u),
\]
where
\[
A(u) = G(u) + \Psi(u).
\]

**Definition 3.2.** We say that \(u \in E\) is a weak solution to \(1.1\) if \(u\) satisfies
\[
\langle I_1'(u), v \rangle = \langle A'(u), v \rangle - \langle B'(u), v \rangle
\]
for all \(v \in E\).

The functional \(I_\lambda\) is well defined on \(E\) moreover \(I_\lambda \in C^1(E, \mathbb{R})\) and
\[
\langle I_\lambda'(u), v \rangle = \langle A'(u), v \rangle - \lambda \langle B'(u), v \rangle \quad \forall v \in E.
\]

Then the critical points of \(I_1\) are weak solutions to \(1.1\).

Now, we introduce some important inequalities that show that the functional \(I_\lambda\) satisfies the hypothesis of Theorem 3.1.

**Lemma 3.3.** we assume that \((m_1), (m_2)\) and \((V_1)\) are satisfied. Then, the following properties hold true:

(i) \(\xi_0([u]_{(s,M)}) \leq G(u) \leq \xi_1([u]_{(s,M)}) \forall u \in E\),

(ii) \(\xi_0(\|u\|_{(V,M)}) \leq \int_{\mathbb{R}^N} V(x)M(u)dx \leq \xi_1(\|u\|_{(V,M)}) \forall u \in E\).

**Proof.** (i) By Lemma 2.1, we know that
\[
\xi_0(\|u\|_{(M)}) \leq \int_{\mathbb{R}^N} M(u)dx \leq \xi_1(\|u\|_{(M)}) \forall u \in L^M(\mathbb{R}^N).
\]

It follows that
\[
\xi_0([h_u]_{(M,\mathbb{R}^{2N})}) \leq G(u) \leq \xi_1([h_u]_{(M,\mathbb{R}^{2N})}) \forall u \in L^M(\mathbb{R}^N).
\]

Having in mind that, \(\|h_u\|_{L^M(\mathbb{R}^{2N})} = [u]_{(s,M)}\), we obtain
\[
\xi_0([u]_{(s,M)}) \leq G(u) \leq \xi_1([u]_{(s,M)}) \forall u \in E,
\]

(ii) Using Lemma 2.1 and Choosing \(\rho = \|u\|_{(V,M)}\), we have
\[
M(u) \leq \xi_1(\|u\|_{(V,M)})M\left(\frac{u}{\|u\|_{(V,M)}}\right),
\]
then,
\[
V(t)M(u) \leq \xi_1(\|u\|_{(V,M)})V(t)M\left(\frac{u}{\|u\|_{(V,M)}}\right) \text{ for } t \in \mathbb{R}^N.
\]

From the definition of the norm \(2.7\), we obtain,
\[
\int_{\mathbb{R}^N} V(t)M(u)dt \leq \xi_1(\|u\|_{(V,M)}) \int_{\mathbb{R}^N} V(t)M\left(\frac{u}{\|u\|_{(V,M)}}\right)dt \leq \xi_1(\|u\|_{(V,M)}).
\]

Using the similar reasoning with \(\rho = \|u\|_{(V,M)} - \epsilon\) and \(\epsilon > 0\), we get
\[
\xi_0(\|u\|_{(V,M)} - \epsilon)V(t)M\left(\frac{u}{\|u\|_{(V,M)} - \epsilon}\right) \leq V(t)M(u),
\]

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then
\[ \int_{\mathbb{R}^N} V(t)M(u)dt \geq \xi_0(\|u\|_{(V,M)} - \epsilon) \int_{\mathbb{R}^N} V(t)M\left(\frac{u}{\|u\|_{(V,M)} - \epsilon}\right)dt \geq \xi_0(\|u\|_{(V,M)} - \epsilon). \]

Letting \( \epsilon \to 0 \) in the above inequality, we obtain
\[ \xi_0(\|u\|_{(V,M)}) \leq \int_{\mathbb{R}^N} V(t)M(u)dt. \]

The proof of Lemma 3.3 is complete. \( \square \)

Now we show that the following compactness result holds.

**Lemma 3.4.** We suppose that \((m_1)\) and \((m_2)\) are satisfied. Let \( \Phi \) be an \( N \)-function satisfying the \( \Delta_2 \) condition, \( \Phi \prec \prec M \), and
\[
\lim_{|t| \to +\infty} \frac{\Phi(t)}{M(t)} = 0. \tag{3.21}
\]

Under the assumption \((V_1)\) and \((V_2)\), the embedding from \( E \) into \( L^{\Phi}(\mathbb{R}^N) \) is compact.

**Proof.** Let \( (u_n) \subset E \) be a sequence verifying \( u_n \to 0 \) in \( E \). We have to show that \( u_n \to 0 \) in \( L^{\Phi}(\mathbb{R}^N) \). By using Theorem 1.1, we know that \( u_n \to 0 \) in \( L^{\Phi}(\mathbb{R}^N) \). Thus it suffices to show that, for any \( \epsilon > 0 \), there exists \( R > 0 \) such that
\[ \int_{B^c_R(0)} \Phi(u_n)dx < \epsilon; \]
here \( B^c_R(0) = \mathbb{R}^N \setminus B_R(0) \).

Choose \( (y_i)_{i \in \mathbb{N}} \subset \mathbb{R}^N \) such that \( \mathbb{R}^N \subset \bigcup_{i \in \mathbb{N}} B_{\nu}(y_i) \) and each point \( x \in \mathbb{R}^N \) is contained in at most \( 2^N \) such balls \( B_{\nu}(y_i) \). Let
\[ A_{R,L} = \{ x \in B^c_R : V(x) \geq L \} \quad \text{and} \quad B_{R,L} = \{ x \in B^c_R : V(x) < L \}. \]

The fact that \( (u_n) \) converges weakly to \( u \) in \( E \) implies that \( \|u_n\|_{E} \leq T, \forall n \in \mathbb{N} \) with \( T > 0 \). From the \( \Delta_2 \) condition, there is \( K > 0 \) such that
\[ M(T \frac{u_n}{T}) \leq KM\left(\frac{u_n}{T}\right), \forall n \in \mathbb{N}. \tag{3.22} \]

Given \( \epsilon > 0 \), by (3.21), there is \( R > 0 \) such that
\[ \Phi(t) \leq \epsilon M(t), \text{ if } |t| > R. \tag{3.23} \]

Combining (3.22) and (3.23), we get
\[ \int_{A_{R,L}} \Phi(u_n)dx \leq \frac{\epsilon}{L} \int_{\mathbb{R}^N} V(x)M(u_n)dx \leq \frac{\epsilon K}{L} \int_{\mathbb{R}^N} V(x)M\left(\frac{u_n}{T}\right)dx \leq \frac{\epsilon K}{L} \int_{\mathbb{R}^N} V(x)M\left(\frac{u_n}{\|u_n\|_{(V,M)}}\right)dx \leq \frac{\epsilon K}{L} \]
and this can be made arbitrarily small by choosing \( L \) large.

Take \( C \) an \( N \)-function such that \( C \circ \Phi \prec \prec M \) and let \( C' \) be the conjugate of \( C \). By Theorem 4.17.4 in [18], there exist \( K' > 0 \) such that
\[ \|u\|_{C \circ \Phi} \leq K' \|u\|_{(M)} , \forall u \in L^M(\mathbb{R}^N). \tag{3.25} \]
Proof. Let $\Phi(\cdot)$ be compactly embedded in $L^r$ where $\epsilon(3.24)$ and (3.26) we get our desired result.

Indeed, using Lemma 3.7, we get

\[
\xi_1(\|u_n\|_M) \leq \xi_1 \left( \frac{1}{V_0} \int_{\mathbb{R}^N} M(u_n) dx + 1 \right)
\]

\[
\leq \xi_1 \left( \frac{1}{V_0} \int_{\mathbb{R}^N} V(x) M(u_n) dx + 1 \right)
\]

\[
\leq \xi_1 \left( \frac{1}{V_0} \xi_1(\|u_n\|(V,M)) + 1 \right)
\]

\[
\leq \xi_1 \left( \frac{1}{V_0} \xi_1(\|u_n\|_E) + 1 \right).
\]

We fix $L > 0$. Combining (3.25), claim 1 and Lemma 2.1 and applying the Hölder inequality, we infer that

\[
\int_{B_{R,L}} \Phi(u_n) dx \leq \sum_{i \in \mathbb{N}} \int_{B_{R,L} \cap B_i(y_i)} \Phi(u_n) dx
\]

\[
\leq \sum_{i \in \mathbb{N}} \|\Phi(u_n)\|_{L^C(B_i(y_i))} \|\chi_{B_{R,L} \cap B_i(y_i)}\|_{L^\infty(B_i(y_i))}
\]

\[
\leq \epsilon R \sum_{i \in \mathbb{N}} \|\Phi(u_n)\|_{L^C(B_i(y_i))} \leq \epsilon R^{2N} \|\Phi(u_n)\|_{L^C(\mathbb{R}^N)}
\]

\[
\leq \epsilon R^{2N} \left\{ \int_{\mathbb{R}^N} C(\Phi(u_n)) dx + 1 \right\}
\]

\[
\leq \epsilon R^{2N} \{ \xi_1(\|u_n\|_{C^0\Phi}) + 1 \}
\]

\[
\leq \epsilon R^{2N} \{ \xi'' \xi_1(\|u_n\|_{(M)} + 1) \}
\]

\[
\leq \epsilon R^{2N} \left\{ K'' \xi_1 \left[ \frac{1}{V_0} \xi_1(T) + 1 \right] + 1 \right\}
\]

where $\epsilon_R = \sup_{y_i} \|\chi_{B_{R,L} \cap B_i(y_i)}\|_{L^\infty(B_i(y_i))}$ and $K'' > 0$. By assumption $(V_2)$ and Proposition 4.6.9 in [18] we can infer that $\epsilon_R \to 0$ as $R \to \infty$. Thus we may make this term small by choosing $R$ large. Combining (5.24) and (5.20) we get our desired result.

**Claim 1:** $\xi_1(\|u_n\|_{(M)}) \leq \xi_1 \left( \frac{1}{V_0} \xi_1(\|u_n\|_E) + 1 \right)$

**Corollary 3.5.** Under $(M_1)$ and $(M_2)$, the embedding from $E$ into $L^\mu(\mathbb{R}^N)$ is compact for all $1 < \mu \leq r$.

**Proof.** Let $\Phi(t) = C|t|^\mu$. By condition $(M_1)$, $(M_2)$ and applying Lemma 3.4 we can deduce that $E$ is compactly embedded in $L^\mu(\mathbb{R}^N)$ for all $1 < \mu \leq r$.

**Lemma 3.6.** The functional $A$ is weakly lower semi-continuous.

**Proof.** By Lemma 3.3 in [5], it is enough to show that $\Psi$ is weakly lower semi-continuous. Let $(u_n) \subset E$ be a sequence which converges weakly to $u$ in $E$. Since $E$ is compactly embedded in $L^r(\mathbb{R}^N)$ it follows that $(u_n)$ converges strongly to $u$ in $L^r(\mathbb{R}^N)$. Then, up to a subsequence, we obtain

$u_n(x) \to u(x)$, a.e in $\mathbb{R}^N$.

This along with Fatou’s lemma yield

$\Psi(u) \leq \liminf_{n \to \infty} \Psi(u_n)$.

Therefore, $A$ is weakly lower semi-continuous. The proof of Lemma 3.6 is complete.

**Lemma 3.7.** If $u_n \rightharpoonup u$ in $E$ and

\[
\langle A'(u_n), u_n - u \rangle \to 0,
\]

then $u_n \to u$ in $E$. 

Proof. Since \((u_n)\) converges weakly to \(u\) in \(E\) implies that \((|u_n|)_{(E,M)}\) and \(\left(\|u_n\|_{(V,M)}\right)\) are a bounded sequences of real numbers. That fact and relations \((i)\) and \((ii)\) from lemma \([3.3]\) imply that the sequences \((G(u_n))\) and \((\Psi(u_n))\) are bounded, it means that the sequence \((A(u_n))\) is bounded. Then, up to a subsequence, we deduce that \(A(u_n) \to c\). Furthermore, Lemma \([3.4]\) implies
\[
A(u) \leq \liminf_{n \to \infty} A(u_n) = c.
\] (3.28)

On the other hand, since \(A\) is convex, we have
\[
A(u) \geq A(u_n) + \langle A'(u_n), u - u_n \rangle.
\] (3.29)

Therefore, combing \([3.28]\) and \([3.29]\) and the hypothesis \([3.27]\), we conclude that \(A(u) = c\).

Taking into account that \(\frac{u_n + u}{2}\) converges weakly to \(u\) in \(E\) and using again the weak lower semi-continuity of \(A\) we find
\[
c = A(u) \leq \liminf_{n \to \infty} A\left(\frac{u_n + u}{2}\right).
\] (3.30)

We assume by contradiction that \((u_n)\) does not converge to \(u\) in \(E\). Then by \((i)\) and \((ii)\) in lemma \([3.3]\) it follows that there exist \(\epsilon > 0\) and a subsequence \((u_{n_m})\) of \((u_n)\) such that
\[
A\left(\frac{u_{n_m} - u}{2}\right) \geq \epsilon, \ \forall m \in \mathbb{N}.
\] (3.31)

On the other hand, relations \([2.5]\) and \((M_3)\) enable us to apply \([19],\) theorem 2.1\) in order to obtain
\[
\frac{1}{2}A(u) + \frac{1}{2}A(u_{n_m}) - A\left(\frac{u_{n_m} + u}{2}\right) \geq A\left(\frac{u_{n_m} - u}{2}\right) \geq \epsilon, \ \forall m \in \mathbb{N}.
\] (3.32)

Letting \(m \to \infty\) in the above inequality we obtain
\[
c - \epsilon \geq \limsup_{m \to \infty} A\left(\frac{u_{n_m} + u}{2}\right).
\] (3.33)

and that is a contradiction with \([3.30]\). It follows that \((u_n)\) converges strongly to \(u\) in \(E\) and lemma \([3.7]\) is proved.

4 Proof of Theorem 1.1

We further need the following lemmas.

Lemma 4.1. Let \((V_1), (V_2), (M_1), (M_2)\) and \((f_1)\) be satisfied. Then \(B(v) \geq 0\). Furthermore, \(B(v) \to \infty\) as \(\|v\| \to \infty\) on any finite dimensional subspace of \(E\).

Proof. Evidently \(B(u) \geq 0\) follows by \((f_1)\). We claim that for any finite dimensional subspace \(F \subset E\) there exists a constant \(\epsilon > 0\) such that
\[
\text{meas}\{x \in \mathbb{R}^N : \xi(x)|u(x)|^p \geq \epsilon\|u\|^p\} \geq \epsilon, \ \forall u \in F \setminus \{0\}.
\] (4.34)

We argue by contradiction and we suppose that for any \(n \in \mathbb{N}\) there exists \(0 \neq u_n \in F\) such that
\[
\text{meas}\{x \in \mathbb{R}^N : \xi(x)|u_n(x)|^p \geq \frac{1}{n}\|u_n\|^p\} \leq \frac{1}{n}, \ \forall n \in \mathbb{N}.
\]

For each \(n \in \mathbb{N}\), let \(v_n = \frac{u_n}{\|u_n\|} \in F\). Then \(\|v_n\| = 1\) for all \(n \in \mathbb{N}\), and
\[
\text{meas}\{x \in \mathbb{R}^N : \xi(x)|v_n(x)|^p \geq \frac{1}{n}\} \leq \frac{1}{n}, \ \forall n \in \mathbb{N}.
\] (4.35)
Up to a subsequence, we may assume that $v_n \rightarrow v$ in $E$ for some $v \in F$ since $F$ is a finite dimensional space. Clearly $\|v\| = 1$. Consequently, there exists a constant $\delta_0 > 0$ such that

$$\text{meas}\{(x \in \mathbb{R}^N : \xi(x)|v(x)|^p \geq \delta_0)\} \geq \delta_0. \quad (4.36)$$

In fact, if not, then we have

$$\text{meas}\{(x \in \mathbb{R}^N : \xi(x)|v(x)|^p \geq \frac{1}{n}) = 0 \ \forall n \in \mathbb{N},$$

which implies that

$$0 \leq \int_{\mathbb{R}^N} \xi(x)|v|^p+\delta dx \leq \frac{\|v\|_{L^p(\mathbb{R}^N)}}{n} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

This together (4.35) yields $v = 0$, which is in contradiction to $\|v\| = 1$.

By using Corollary 3.5 and the fact that all norms are equivalent on $F$, we deduce that

$|v_n - v|^r_{L^p(\mathbb{R}^N)} \rightarrow 0 \text{ as } n \rightarrow \infty.$

By the Hölder inequality, it holds that

$$\int_{\mathbb{R}^N} \xi(x)|v_n - v|^p dx \leq \|\|v\|_{L^p(\mathbb{R}^N)}\|^\frac{p}{r} \left(\int_{\mathbb{R}^N} |v_n - v|^r dx\right)^\frac{p}{r} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.37)$$

Set

$$\Lambda_0 := \{x \in \mathbb{R}^N : \xi(x)|v(x)|^p \geq \delta_0\}$$

and for all $n \in \mathbb{N}$,

$$\Lambda_n := \{x \in \mathbb{R}^N : \xi(x)|v_n(x)|^p < \frac{1}{n}\}, \ \Lambda^c_n := \mathbb{R}^N \setminus \Lambda_n.$$

Taking into account (4.35) and (4.36), we get

$$\text{meas}(\Lambda_n \cap \Lambda_0) \geq \text{meas}(\Lambda_0) - \text{meas}(\Lambda^c_n) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}$$

for $n$ large enough. Therefore we obtain

$$\int_{\mathbb{R}^N} \xi(x)|v_n - v|^p dx \geq \int_{\Lambda_n \cap \Lambda_0} \xi(x)|v_n - v|^p dx \geq \frac{1}{2p} \int_{\Lambda_n \cap \Lambda_0} \xi(x)|v|^p - \int_{\Lambda_n \cap \Lambda_0} \xi(x)|v_n|^p dx \geq \left(\frac{\delta_0}{2p} - \frac{1}{n}\right) \text{meas}(\Lambda_n \cap \Lambda_0) \geq \left(\frac{\delta_0}{2p+2}\right) > 0$$

which contradicts (4.37). For the $\epsilon$ given in (4.34), let

$$\Lambda_u = \{x \in \mathbb{R}^N : \xi(x)|u(x)|^p \geq \epsilon \|u\|^p\}, \ \forall u \in F \setminus \{0\}.$$

Then by (4.34),

$$\text{meas}(\Lambda_u) \geq \epsilon, \ \forall u \in F \setminus \{0\}.$$

Therefore

$$B(u) = \int_{\mathbb{R}^N} \xi(x)|u|^p dx \geq \int_{\Lambda_u} \epsilon \|u\|^p dx \geq \epsilon \|u\|^p \text{meas}(\Lambda_u) = \epsilon^2 \|u\|^p.$$ 

This implies that $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of $E$. The proof is complete. \qed
Lemma 4.2. Assume that \((m_1) - (m_2), (M_1) - (M_2), (V_1) - (V_2)\) and \((f_1)\) are satisfied. Then there exists a sequence \(\rho_k \to 0^+ \) as \(k \to \infty\) such that

\[
a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) > 0, \ \forall k \in \mathbb{N},
\]

and

\[
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \quad \text{as} \quad k \to \infty \quad \text{uniformly for} \ \lambda \in [1, 2].
\]

Proof. Using Lemma 3.3 for any \(u \in Z_k\) and \(\lambda \in [1, 2]\), we can see that

\[
I_\lambda(u) \geq \xi_0([u]_{(s,M)}) + \xi_0([u]_{(V,M)}) - 2 \int_{\mathbb{R}^N} \xi(x) |u|^p dx
\]

\[
\geq \xi_0([u]_{(s,M)}) + \xi_0([u]_{(V,M)}) - 2\|\xi\|_{\frac{p}{p-1}} \|u\|_{L^p(\mathbb{R}^N)}^p.
\]

Let

\[
l_k = \sup_{u \in Z_k, \|u\| = 1} \|u\|_{L^p(\mathbb{R}^N)}, \ \forall k \in \mathbb{N}.
\]

By the next lemma, there hold

\[
l_k \to 0 \quad \text{as} \quad k \to +\infty.
\]

Combining (4.38) and (4.39), we have

\[
I_\lambda(u) \geq \xi_0([u]_{(s,M)}) + \xi_0([u]_{(V,M)}) - 2\|\xi\|_{\frac{p}{p-1}} \|u\|_{L^p(\mathbb{R}^N)}^p, \ \forall k \in \mathbb{N} \quad \text{and} \quad (\lambda, u) \in [1, 2] \times Z_k.
\]

For each \(k \in \mathbb{N}\), choose

\[
\rho_k = (4\|\xi\|_{\frac{p}{p-1}} L_k^p)^{\frac{1}{p}}.
\]

Since \(1 < p < r\), then by (4.40), we have

\[
\rho_k \to 0 \quad \text{as} \quad k \to +\infty,
\]

and so, for \(k\) large enough, we have \(\rho_k \leq 1\). Then, by Lemma 2.1

\[
\xi_0([u]_{(s,M)}) + \xi_0([u]_{(V,M)}) \geq \|u\|^r, \quad \text{when} \ \|u\| = \rho_k.
\]

By (4.41), (4.42) and (4.44), direct computation shows

\[
a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq \frac{1}{2} \rho_k^r > 0, \ \forall k \in \mathbb{N}.
\]

Besides, by (4.41), for each \(k \in \mathbb{N}\), we have

\[
I_\lambda(u) \geq -2\|\xi\|_{\frac{p}{p-1}} L_k^p \|u\|^p,
\]

for all \(\lambda \in [1, 2]\) and \(u \in Z_k\) with \(\|u\| \leq \rho_k\). Therefore,

\[
-2\|\xi\|_{\frac{p}{p-1}} L_k^p \|u\|^p \leq \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \leq 0, \ \forall \lambda \in [1, 2] \quad \text{and} \quad \forall k \in \mathbb{N}.
\]

Combining (4.40) and (4.43), we have

\[
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \quad \text{as} \quad k \to \infty \quad \text{uniformly for} \ \lambda \in [1, 2].
\]

The proof is complete. \(\square\)

Lemma 4.3. We have that

\[
l_k := \sup_{u \in Z_k, \|u\| = 1} \|u\|_{L^r(\mathbb{R}^N)} \to 0 \quad \text{as} \quad k \to +\infty.
\]
Proof. It is clear that $l_k$ is decreasing with respect to $k$ so there exist $l \geq 0$ such that $l_k \to 0$ as $k \to +\infty$. For any $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L^r(R^N)} \geq \frac{\lambda}{4}$. By definition of $Z_k$, $u_k \to 0$ in $E$. Lemma 3.4 implies that $u_k \to 0$ in $L^r(R^N)$. Thus we proved that $l = 0$. □

Lemma 4.4. Under the hypotheses and the sequence $\rho_k$ of in Lemma 4.2 there exists $r_k > \rho_k$ for any $k \in \mathbb{N}$ such that

$$b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) < 0.$$  

Proof. By using the fact that $Y_k$ is with finite dimensional and (4.34), we can find $\epsilon > 0$ such that

$$\text{meas}(\Lambda^k_\lambda) \geq \epsilon_k, \forall u \in Y_k \setminus 0,$$

where $\Lambda^k_\lambda := \{x \in \mathbb{R}^N : \xi(x)|u(x)|^p \geq \epsilon_k\|u\|^p\}$. By (4.45), for any $k \in \mathbb{N}$, we have

$$I_\lambda(u) \leq \xi_1([|u|]_{(\mathcal{E},M)}) + \xi_1([||u||]_{(\mathcal{E},M)}) - \int_{\mathbb{R}^N} \xi(x)|u|^p dx$$
$$\leq \xi_1([|u|]_{(\mathcal{E},M)}) + \xi_1([||u||]_{(\mathcal{E},M)}) - \int_{\Lambda^k_\lambda} \epsilon_k\|u\|^p dx$$
$$\leq \xi_1([|u|]_{(\mathcal{E},M)}) + \xi_1([||u||]_{(\mathcal{E},M)}) - \epsilon_k\|u\|^p \text{meas}(\Lambda^k_\lambda)$$
$$\leq \xi_1([|u|]_{(\mathcal{E},M)}) + \xi_1([||u||]_{(\mathcal{E},M)}) - \epsilon_k^2\|u\|^p$$
$$\leq \|u\|^t - \epsilon_k^2\|u\|^p \leq -\|u\|^t$$

for all $u \in Y_k$ with $\|u\| \leq \min\{\rho, 2^{-\frac{1}{\theta - 1}} \epsilon_k^{\frac{2}{\theta - 1}}, 1\}$. If we choose

$$r_k < \min\{\rho, 2^{-\frac{1}{\theta - 1}} \epsilon_k^{\frac{2}{\theta - 1}}, 1\}, \forall k \in \mathbb{N},$$

and using (4.46), we deduce that

$$b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) = -r_k^t < 0, \forall k \in \mathbb{N}.$$  

□

Proof of Theorem 1.1 From Lemma 3.4 we can see that $I_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Moreover, $I_\lambda$ is even. Then the condition (i) in Theorem 3.1 is satisfied. Besides, Lemma 4.4 shows that the condition (ii) holds while Lemma 4.2 together with Lemma 4.4 implies that the condition (iii) holds. Therefore, by Theorem 3.1 for each $k \in \mathbb{N}$, there exist $\lambda_n \to 1$, $u_{\lambda_n} \in Y_n$ such that

$$I'_{\lambda_n} y_n(u_{\lambda_n}) = 0, I_{\lambda_n}(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)] \text{ as } n \to \infty.$$  

Claim 2: The sequence $(u_{\lambda_n})_{n \in \mathbb{N}}$ obtained in (4.47) is bounded in $E$.

For the sake of notational simplicity, in what follows we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$. In fact, combining (4.47), Lemmas 3.3 and 3.4 and the Hölder inequality, we obtain

$$\|u_n\|^\theta \leq I_{\lambda_n}(u_n) + \lambda_n \int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p dx \leq C_0 + 2\|\xi\|_{r-\theta} \|u_n\|^p_{L^r(\mathbb{R}^N)}$$
$$\leq C_0 + 2\tau^\theta \|\xi\|_{r-\theta} \|u_n\|^p$$

for some $C_0 > 0$ and $\tau > 0$, where $\theta = r$ or $\theta = l$. Therefore, the claim above is true since $p < \theta$.

Claim 3: The sequence $(u_n)$ has a strong convergent subsequence for every $k$.

In view of Claim 2, without loss of generality, we may assume

$$u_n \to u_0 \text{ as } n \to +\infty$$  

(4.48)
for some $u_0 \in E$. Hence
\[
\langle I_{\lambda_n}'(u_n) - I_{\lambda_n}'(u_0), u_n - u_0 \rangle = \langle G'(u_n) - G'(u_0), u_n - u_0 \rangle + \int_{\mathbb{R}^N} V(x)[m(u_n)u_n - m(u_0)u_0](u_n - u_0)dx
\]
\[- \lambda_n \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_0)](u_n - u_0)dx \to 0, \ n \to +\infty.
\]
Therefore, by Lemma 3.7 we can deduce that $u_n$ converges strongly to $u_0$ in $E$.

Now from the last assertion of Theorem 3.1 we know that $I_1$ has infinitely many nontrivial critical points. Therefore, (1.1) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is complete. \qed

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