On The Quantum Theory of Hall Effect

F.GHABOUSSI
Department of Physics, University of Konstanz
P.O. Box 5560, D 78434 Konstanz, Germany
E-mail: ghabousi@kaluza.physik.uni-konstanz.de

Abstract
We discuss a model of both classical and integer quantum Hall-effect which is based on a semi-classical Schroedinger-Chern-Simons-action, where the Ohm-equations result as equations of motion. The quantization of the classical Chern-Simons-part of action under typical quantum Hall conditions results in the quantized Hall conductivity. We show further that the classical Hall-effect is described by a theory which arises as the classical limit of a theory of quantum Hall-effect. The model explains also the preference and the domain of the edge currents on the boundary of samples.
Introduction and summary

Recently, we discussed a model of the integer quantum Hall-effect (IQHE) according to which the quantization of Hall-conductivity should result from the quantum electrodynamics in 2+1-dimensions. In this semi classical Schroedinger-Chern-Simons-model the Hall-conductivity $\sigma_H$ appears as the normalization parameter of the Chern-Simons-action. Furthermore, we assumed there according to the experimental results of QHE a vanishing longitudinal conductivity $\sigma_L$. Thereafter, the Ohm-equations of IQHE with quantized $\sigma_H$ are obtained as the equations of motion from the Schroedinger-Chern-Simons-action with quantized electromagnetic potentials.

Here we discuss a more general model for both classical Hall-effect (CHE) and IQHE, where the related Ohm-equations result as equations of motion also from a Schroedinger-Chern-Simons-action functional. Thereafter, the quantum Hall conditions cause the transition of the Hall-system into the quantum regime, where the necessary quantization of electromagnetic potentials results in the quantized $\sigma_H$ in the absence of $\sigma_L$. It is a model of non-intercating charge carriers for IQHE with a semi-classical Schroedinger-Chern-Simons-action functional, hence not the Schroedinger-term which represents the charge carriers system but only the Chern-Simons-term which represents the dynamics of the almost pure gauge potentials is quantized. Thus, a second quantization of the Schroedinger-term in our model which corresponds to the interacting particle system should result after solution of question of the ground state, in a FQHE model similar to the known models.

Our model is based on the following stand point on the theory of Hall-effects that because there are both CHE and QHE (IQHE and FQHE), thus the theory of the QHE must be the quantization of the "classical" theory of the CHE. Furthermore a rigorous quantization of a system requires the knowledge of its action functional. Accordingly, we have to construct first a "classical" action for the CHE, wherefrom the resulting equations of motion must explain the CHE behaviour. On the other hand the "classical" Ohm-equations are the only equations which describe the CHE. Thus, the action which should describe the CHE has to result in the Ohm-equations as its equations of motion. This interpretation of the Ohm-equations as the equations of motion which must result directly from an action functional is a new element of our stand point. In all other models the Ohm-equations are considered as a given relation in the sense of "material" or "phenomenological" equations.
On the other hand, in view of the well known fact that these Ohm-equations are semi-classical relations with Schroedinger-typ current densities for electrons, the desired action for CHE should be also of the semi-classical typ as it is performed in our model. Then, the canonical quatization of the classical part of this action for the case of non-interacting electrons must result in the quantum theory of the IQHE and also in the quantized Hall-conductivity according to the IQHE.

To investigate the relation between QHE and CHE, let us analyse first the Ohm-equations for QHE and CHE [1]. These are given by:

\[ j_m = \sigma_H \epsilon_{nm} E_n, \quad \epsilon_{mn} = -\epsilon_{nm} = 1 ; m, n = 1, 2 \quad , \tag{1} \]

for QHE, where \( \sigma_H = \frac{en}{B} \) becomes quantized in the units of \( \frac{e^2}{h} \). Here \( n \) is the global surface density of the charge carriers ("electrons") which we call electrons and \( B := B_3 \) is the applied magnetic field [6].

On the other hand, the Ohm-equations for CHE are given by:

\[ j_m = \sigma_H \epsilon_{nm} E_n + \sigma_L E_m \quad \tag{2} \]

with \( \sigma_L = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \) and \( \sigma_H = \frac{\sigma_0 (\omega_c \tau)}{1 + (\omega_c \tau)^2} \), where \( \sigma_0 = \frac{e^2 n \tau}{\mu}, \omega_c := \frac{eB}{\mu}, \tau \) and \( \mu \) are the mean free time and the mass of electrons [1] [7].

The key observation is that according to quantum mechanics [8] the current density of electrons in a magnetic field without spin term and with \( C = 1 \) is given by (a): \( j_m := \frac{i e \hbar}{2 \mu} \left[ (\partial_m \psi^*) \psi - \psi^* (\partial_m \psi) \right] - \frac{e^2}{\mu} A_m \psi^* \psi \), whereas the current density of electrons in the limit \( B \to 0 \), i. e. for \( \omega_c \tau \ll 1 \) should be given by (b): \( j_m := \frac{i e \hbar}{2 \mu} \left[ (\partial_m \psi^*) \psi - \psi^* (\partial_m \psi) \right] \). Both obeying the continuity equation [8].

We deduce that the relation (a) is valid in the integer quantum Hall-regime \( (\omega_c \tau \gg 1) \) where the external magnetic field is large, whereas the relation (b) is valid in the classical Hall-regime \( (\omega_c \tau \ll 1) \) where the same external field is small or absent.

The semi-classical Schroedinger-Chern-simons-action functional in 2 + 1-dimensions is the only action from which we can obtain the mentioned Ohm-equations (1) and (2) as the equations of motion (see
below) \([8]\), where the \(\sigma_H\) plays the role of normalization parameter of the classical Chern-Simons-action.

To see the relation of the quantization of Hall-system with the empirical quantum behavior under the typical Quantum Hall-conditions \([4]\) let us recall that in a strong magnetic field the Hall-conductivity \(\sigma_H\) becomes small according to its definition which is given above \([1]\). Precisely, in the quantum Hall-limit, i.e. \(\omega_c \tau \gg 1\) the \(\sigma_H\) and \(\sigma_L\) should be considered according to their definitions which is given above of the orders \((\omega_c \tau)^{-1}\) and \((\omega_c \tau)^{-2}\) respectively, i.e. \(\sigma_H \ll 1\) and \(\sigma_L \ll \sigma_H\) or \(\sigma_L \to 0\). Moreover, in this limit the Hall-conductivity is given by \(\sigma_H = \frac{ne}{B}\) so that for small \(n\) and large \(B_{\text{external}}\) the \(\sigma_H\) becomes considerably small. Thus, if we consider in our model, \(\sigma_H\) as the normalization parameter of the Chern-Simons-action \(S_{C-S}\) \([3]\) and quantize this action according to the Schroedinger representation \([2]\):

\[
\Psi_{(C-S)}(A) \propto e^{i \frac{\sigma_H S_{C-S}}{\hbar}},
\]

the \(\sigma_H S_{C-S}\) becomes also small for relevant \(S_{C-S}\) actions in view of the above mentioned smallness of \(\sigma_H\). Therefore, for small \(\sigma_H S_{C-S}\), i.e. precisely for those \(\sigma_H S_{C-S}\), which are comparable with \(\hbar\), the quantum behaviour of action becomes dominant \([10]\) and we meet the integer quantum Hall-regime manifested by IQHE. Moreover, in this quantum limit the \(\sigma_L\) becomes, as mentioned above, very small tending to zero as it is expected in the QHE.

Conversely, if the magnetic field is not strong, i.e. for \(\omega_c \tau \ll 1\), \(\sigma_H\) and \(\sigma_H S_{C-S}\) become large or \(\sigma_H S_{C-S} \gg \hbar\) and we meet the classical regime, where the quantum fluctuations of the action are compensated \([10]\) and the original quantum theory reduces to its classical limit which is the theory of CHE. In this classical limit \(\sigma_L \approx \sigma_0\), thus both typ of conductivities are no more small but of considerable magnitudes, since they are also present in the Ohm-equations of the CHE \((2)\). We avoid to discuss here the typical FQH-conditions including the high mobility of particles in view of the fact that we consider only the IQHE.

On the other hand, it is known that if one considers currents involved in the IQHE only as the boundary currents, then most of experimental data can be understood in a satisfactory manner \([3]\). It is a favour of the Chern-Simons-ansatz in a manifold with a spatial boundary that the boundary currents are the
only allowed ones according to the constraints of the theory under the typical quantum Hall-conditions \[4\]. Therefore, for construction of a theoretical model for both CHE and IQHE one is left with the Schroedinger-Chern-Simons-action, from which we know already that it results, at least, in the Ohm-equations for CHE and IQHE as the equations of motion \[2\].

The Chern-Simons-Action for Classical and Quantum Hall-Effect

The general action from which the Ohm-equations of CHE and IQHE ((2) and (1)) can be obtained as the equations of motion is the following Schroedinger-Chern-Simons-action defined on the 2+1-dimensional manifold \( M = \Sigma \times \mathbb{R} \).

\[
S = \frac{1}{8\pi} \int dt \int_{\Sigma} \psi^* \left[ i\hbar \partial_t - \frac{1}{2\mu} (-i\hbar \partial_m - eA_m)^2 - eA_0 \right] \psi + \text{h.c.} - \frac{\sigma_H}{8\pi} \int_M \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma ,
\]

where \( A_\alpha(x_m, t) \) is still the classical electromagnetic potential which remains classical in the classical Hall-regime but must be quantized in the quantum Hall-regime. Furthermore, \( \{\alpha, \beta, \gamma\} = \{0, 1, 2\} \) and everywhere is \( C = 1, \partial_m = \frac{\partial}{\partial x_m} \) and we consider (in accordance with the experimental arrangements of the QHE) that the \( \Sigma \) has a boundary. Furthermore, as already mentioned the Schroedinger-term represents the mechanics of the non-interacting particle system, whereas the Chern-Simons-term represents the dynamics of the coupled electromagnetic potentials.

Obviously, we use the \( \sigma_H \) as the locally constant normalization parameter of the Chern-Simons-action. It is justified to do so, because \( \sigma_H \) can be considered as a dimensionless and locally constant quantity in 2 + 1-dimensions also in view of its well known topological or global character \[1\] \[3\] \[11\]. Moreover, we suppressed the spin term within the usual Schroedinger-action for "electron" in a magnetic field in view of the well known fact about QHE that the spin degeneracy is not essential for the IQHE \[1\].

In view of the gauge freedom of \( A_m \) we choose the gauge fixing condition \( A_0 = 0 \) to retain the true degrees of freedom of the electromagnetic fields in the action (4). Thereafter, the action reduces to the following one:

\[
\frac{1}{8\pi} \int dt \int_{\Sigma} \psi^* \left[ i\hbar \partial_t - \frac{1}{2\mu} (-i\hbar \partial_m - eA_m)^2 \right] \psi + \text{h.c.} - \frac{\sigma_H}{8\pi} \int \epsilon^{mn} A_m A_n ,
\]

(5)
The equations of motion for classical $A_m$ potentials which result from this action are

$$j_m - \frac{e^2\hbar}{\mu} A_m = \sigma_H e^{nm} \dot{A}_n,$$

(6)

where we used according to $\omega_c \tau \ll 1$ in the classical regime the corresponding definition $j_m := \frac{i e \hbar}{2\mu} [(\partial_m \psi^*) \psi - \psi^*(\partial_m \psi)]$.

We introduce the gauge $A_m = E_m \tau$ in (6) which is more appropriate for the case of low magnetic fields, i.e. precisely it is appropriate for the classical Hall-regime with $\omega_c \tau \ll 1$. It is equivalent to the relaxation time approximation which is the usual approach in this case. Substituting $A_m = E_m \tau$ in (6) we obtain the desired Ohm-equations for CHE

$$j_m = \sigma_L E_m + \sigma_H e^{nm} E_n,$$

(7)

where we used $\sigma_L \approx \sigma_0$ according to $\omega_c \tau \ll 1$.

Thus, we obtained the Ohm-equations of the CHE as the equations of motion from the action (4) in the classical Hall-regime, consistently, according to $\omega_c \tau \ll 1$.

The quantization of the action (5) under the typical IQH-conditions, i.e. in the limit $\omega_c \tau \gg 1$ results then in the action which is responsible for the Ohm-equations of IQHE, where one must use obviously the definition (a) for the current density in the quantum Hall-regime according to $\omega_c \tau \gg 1$.

Recalling our previous analysis we like to mention that the quantum regime of Hall-effect is related in double sense to the strong exterior magnetic field which is applied on the two dimensional electronic systems: In the limit $\omega_c \tau \gg 1$ the $\sigma_H$ and $\sigma_L$ should be considered theoretically of the other $(\omega_c \tau)^{-1}$ and $(\omega_c \tau)^{-2}$ respectively, i.e. $\sigma_H$ becomes small and $\sigma_L$ tends to zero, as it is confirmed by experiments. On the other hand, under typical quantum Hall conditions where the number or the density of electrons is small the $\sigma_H$ and $\sigma_H S_{C-S}$ become more smaller and so the last one becomes comparable with $\hbar$ which results in the integer quantization of $\sigma_H$ as it is also confirmed by experiments.
In other words, the \( \omega_c \tau \gg 1 \) limit together with small \( n \) corresponds with the quantum regime \(^3\) where \( \sigma_H S_{C-S} \) becomes comparable with \( \hbar \), whereas the \( \omega_c \tau \ll 1 \) limit together with \( n \) around the usual electronic density in metals corresponds with the classical limit where the action \( \sigma_H S_{C-S} \gg \hbar \).

Therefore, for large magnetic fields and small density of electrons which are the typical quantum Hall conditions the two dimensional Hall-system is in the IQHE-regime \(^4\) which is described by the same action (4) or (5) after gauge fixing:

\[
\frac{1}{8\pi} \int dt \int_\Sigma \psi^* \left[ i\hbar \partial_t - \frac{1}{2\mu}(-i\hbar \partial_m - eA_m)^2 \right] \psi + h.c. - \frac{\sigma_H}{8\pi} \int dt \int_\Sigma \epsilon^{mn} \dot{A}_m A_n , \tag{8}
\]

but in view of \( \sigma_H S_{C-S} \approx \hbar \) with \( A_m \) potentials now obeying the usual quantization algebra \(^4\)

\[
\left[ \dot{A}_m(x_l,t), \dot{A}_n(y_l,t) \right] = \frac{4\pi i\hbar}{\sigma_H} \epsilon_{mn} \delta^2(X - Y) ; X,Y \in \Sigma , \tag{9}
\]

which can be read off directly from the Chern-Simons-action in (8). It means that \( \dot{A}_m := \frac{i\partial}{\partial A_m} \) which is the usual polarization of the \( \{ A_m \} \) phase space.

However, for practical use it is convenient to introduce the Schrödinger representation \( \Psi(A) \propto e^{i\sigma_H S_{C-S}/\hbar} \) of the Chern-Simons-action

\[
- \frac{\sigma_H}{8\pi} \int dt \int_\Sigma \epsilon^{mn} \dot{A}_m A_n , \tag{10}
\]

after its quantization according to (9), hence \( \Psi(A) \) must fulfill the relation (9) in the sense of its eigen functions.

To obtain \( \Psi(A) \) we use the method introduced in a previous work on IQHE \(^2\). It is based on the representation of the state functions \( \Psi(A) \) in terms of the eigen states of the quantum orbital angular momentum. For equivalent quantization of \( S_{C-S} \) and its Schrödinger representations see \(^1\).

Introducing polar coordinates in the phase space described by the action (10), the quantum orbital angular momentum becomes \( \dot{L} = -i\hbar \partial_\phi \). Thereafter, \( \Psi(A) \) is given as the eigen states of the operator \( \dot{L} \) by:
\( \Psi(A) = F(R) e^{\frac{i}{\hbar} \sigma_H l \phi} \),

(11)

Here \( F(R) \) is an arbitrary function of \( R \) and \( l = R^2 \) is the value of angular momentum of the system which is a constant of motion according to the \( SO(2) \) symmetry of the system. We normalize the constant \( l = 1 \).

Thus, the necessary single-valuedness of \( \Psi(A) \) forces the \( \sigma_H \) to be

\[ \sigma_H = 0, 1, 2, ... , N, ... ; N \in \mathbb{Z}_+ \],

(12)

where we restricted us to the positive values [15].

Recall that the normalization parameter of the \( \Psi_{CS} \) becomes allways quantized as integers in view of the single valuedness of \( \Psi_{CS} \) in its first quantization no matter what kind of quantization is performed [14].

Empirically it is the mentioned typical IQH-conditions [4] which prepares the electrons, according to their density and mobility and the strength of the exterior magnetic fields, to be in IQHE situation (see also the conclusion).

The equations of motion for \( A_m \) potentials which result from the quantized action (8) for the non-interacting system of charge carriers, according to (11)-(12) and using the corresponding definition (a) for the current density in magnetic fields, are:

\[ j_m = \sigma_H \epsilon_{nm} E_n \],

(13)

which are the desired Ohm-equations with quantized \( \sigma_H \).

It is obvious from the comparison between the quantized Chern-Simons-action in units of \( \hbar \), i.e. \( \frac{\sigma_H S}{\hbar} \) and the Schroedinger-action in (8) that in the atomic units the \( \sigma_H \) should be considered in units of \( \frac{e^2}{\hbar} \), which is equivalent to a redefinition of the quantized \( A_m \)-potentials absorbing the coupling constant \( e \).
Thus, we obtained the quantized Ohm-equations of IQHE as the equations of motion from the quantized Schroedinger-Chern-Simons-action.

To summarize the quantum and classical behavior in this model let us recapitulate the analysis of the integer quantum and classical Hall conditions:

If the Hall-system is prepared with $\omega_c \tau \gg 1$ and with small $n$, then the quantum modes of its action become dominant, but if it is prepared with $\omega_c \tau \ll 1$ and with $n$ around the density of CHE-samples then its classical modes become dominant.

The theoretical description of this situation is according to our model so that the general semi-classical action functional for both cases should be given by (4) where the Schroedinger-term remains the same in both cases in view of the non-interacting particles in IQHE. Then, the action (4) with quantized Chern-Simons-term describes the integer quantum Hall-regime, whereas the action (4) with classical Chern-Simons-term describes the classical Hall-regime.

In the first case the typical quantum Hall conditions, i.e. $\omega_c \tau \gg 1$, and small $n$ cause the smallness of $\sigma_H$ so that $\sigma_H S_{C-S}$ becomes comparable with $\hbar$. Thus the quantum modes of the action $\sigma_H S_{C-S}$ which are represented by $\Psi(A)$ become dominant requiring the quantization of $\sigma_H$. Since the total quantum action results in the "quantum" Ohm-equations with integrally quantized $\sigma_H$ and vanishing $\sigma_L$ as it is shown above.

In the second case the action is of the order $\sigma_H S_{C-S} \gg \hbar$, therefore the classical limit of Chern-Simons-action, i.e. the classical Chern-Simons-action becomes dominant. Then the total action reduces to the semi-classical Chern-Simons-Schroedinger-action which describe the semi-classical theory of the CHE. Since it results in the "classical" Ohm-equations as it is shown above.

Thus, the theory of CHE, i.e. its action arises as the classical limit from the quantum action of IQHE.
The Edge Currents in QHE

Obviously, the motion of system which is described by the action (8) together with the quantization relations (9)-(12) is constrained by the constraint:

$$-\sigma_H \epsilon^{mn} \partial_m A_n = e\psi^* \psi,$$

(14)

with \( e\psi^* \psi := j_0 \).

If we integrate the relation (14) over the sample surface and consider \( B := \epsilon_{nm} \partial_m A_n \) as a constant field strength, then we obtain the well known relation between the Hall-conductivity and the magnetic field, namely

$$\sigma_H = \frac{ne}{B},$$

(15)

where \( n = (a)^{-1} \int da (\psi^* \psi) \) is the global density of charge carriers and \( a \) is the area of sample.

Recall, that the relation (15) is conforme with the general definition of \( \sigma_H \) in the limit \( \omega_c \tau \gg 1 \).

However, the constraint (14) influences the motion of the IQHE-system in a way which is known from the experimental results of IQHE.

To see this let us note first some of main experimental features of IQHE reviewed from [4]:

1. Most of IQHE-data can be understood in a satisfactory manner if one reduces the involved currents to the edge currents.
2. The typical IQHE-regime is related to \( B \gg 1 \) and small \( n \).
3. Under integer quantum Hall conditions the edge of Hall-systems are characterized by the \( n \rightarrow 0 \).
4. For the large current densities the IQHE can not be simply described by the edge currents located on the boundary, whereas the low currents are transported by the edge channels.

All these features of IQHE can be understood if we take into account the constraint (14).

Recall that, in view of the Ohm-equations the currents are restricted to those regions where the \( A_m \)-potentials are allowed to exist. Thus, the question of the edge currents is related with the questions of
the regions where the $A_m$-potentials are defined. Moreover, according to the constraint (14) the potential $A_m$ becomes pure gauge potential with vanishing field strength if $n \to 0$.

This is the case if one has to do with samples with small $n$ under the large $B$ for example on the edges of quantum Hall-system. Thus under these circumstances we should replace the constraint (14) by the following one

$$\epsilon^{mn} \partial_m A_n \approx 0 \quad ,$$

for systems under quantum Hall conditions \[4\]. Thereafter, the $A_m$-potentials become pure gauge potentials, i.e. $A_m \approx ig^{-1} \partial_m g$, where $g$ is an element of the U(1)-gauge group. Recall however that this is a local relation in quantum mechanics, therefore 1.) it should be valid only within the limit of uncertainty relations and 2.) a locally pure gauge potential has the well known geometric, i.e. globally well defined and observable effects in quantum mechanics \[17\].

On the other hand, the constraint tensor $\epsilon^{mn} \partial_m A_n$ generates a gauge transformation $A_m' = A_m + \partial_m \lambda$ in the phase space of the $A_m$-potentials \[14\]. Therefore, according to the constraint (16) one must identify $A_m' = A_m$ everywhere in the phase space. Furthermore, if as in our case the $\Sigma$ possess a boundary we must choose boundary conditions for $A_m$ and $\lambda$ on the boundary. We choose free boundary conditions for $A_m$ but $\lambda = 0$ on the boundary. A reason for this choise is that the Chern-Simons-action is not invariant under gauge transformations that do not vanish on the boundary \[14\].

Accordingly, it must be required that $A_m' = A_m$ for any $\lambda$ which vanishes on the boundary $\partial \Sigma$. The only pure $A_m$ gauge potentials which obey this additional condition are those restricted to be defined only on the boundary \[14\]. In other words, the only $A_m$-potentials obeying both restrictions caused by the constraint (16) are those restricted to exists on the boundary region of $\Sigma$. Thereafter, the currents $j_m$ should be considered also to be restricted to the boundary region of $\Sigma$, i.e. to the so called edge currents. Accordingly, under quantum Hall conditions \[4\] the edge currents are the preferred ones.

It is important to mention that if we consider this restrictions of the potentials and currents to the boundary or to the edge of Hall-system "quantum mechanically", then there is an uncertainty of the position of currents, or so to say there is an uncertainty of the "quantum mechanical" edge $\Delta(\partial \Sigma)$ in
view of the Heisenberg’s uncertainty relations. Thus, if we consider the uncertainty of momentum equal to $(2m\Delta E)^{1/2}$ with $\Delta E = E_{n+1} - E_n = \frac{\hbar \omega_c}{2}$ the uncertainty of the mentioned edge or the width of the current’s orbit is given by $\Delta X = \left(\frac{\hbar}{eB}\right)^{1/2}$ which is the magnetic length $l_B$. Since, the edge current is according to its empirical definition the current which flows, in the ideal case, close to the edge within the length scale of the magnetic length $l_B$. Moreover, this circumstance shows also that the constraint (16) should be fullfield within the uncertainty dictated by the energy-time uncertainty relation. Since the $\Delta E \propto \Delta B$ in the Landau-levels [8].

On the other hand, if $n >$ for large transport currents the right hand side of the constraint (14) and thereby also the field strength in (14) is obviously non-vanishing and the IQHE breaks down as manifested by early experiments [4].

**Conclusion**: This was a model of IQHE based on the non-interacting system of charge carriers coupled on an electromagnetic potential in $2 + 1$-dimensions. There are strong hints that the FQHE which is belived to be a many particle effect, i. e. of interacting particles, should results from the second quantization of the Schroedinger-field of charge carriers involved in an action similar to one which is used in this model [8]. Hence, the conformity of our model for IQHE with an erlear model of FQHE [8] is a hint about the possibility that, if one consider a proper modification of our model for the case of interacting charge carriers, then after the second quantization of the Schroedinger-term in our action for the interacting (“many particle”) system one should arrive in a theory of FQHE. However, this is possible if one can solve the problem of ground state of interacting particles in such models [8]. We discuss the second quantization of our model and the resulting fractionality elsewhere [18].

Footnotes and references

**References**

[1] For a general review on QHE and its experimental setting see:

[1a] R.E. Prange and S.M. Girvin, ed., The quantum Hall effect, Graduate Texts in Contemporary Physics (Springer, New York, 1987);
[1b] A.H. Macdonald, ed., Quantum Hall effect: A Perspective, Perspectives in Condensed Matter Physics (Kluver Academic Publishers, 1989)

[1c] G. Morandi, The role of Topology in Classical and Quantum Physics, Lecture Notes in Physics m7 (Springer, New York 1992)

[1d] M. Janssen, et al., J. Hajdu, Introduction to the Theory of the Integer Quantum Hall effect (VCH-verlag, Weinheim, New York, 1994)

[2] F. Ghaboussi, On the Integer Quantum Hall Effect, KN-UNI-preprint-95-1; A Model of the Integer Quantum Hall Effect, KN-UNI-preprint-95-2, submitted for publication.

[3] The use of $\sigma_H$ as normalization parameter of Chern-Simons-action is in accordance with the use of similar parameters in interacting system of particles which become afterwards proportional to $\sigma_H$ in FQHE models: G. W. Semenof, Phys. Rev. Lett. 61, 517, (1988); S. C. Zhang, T. H. Hanssson and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989). Recall that it is expected that non-interacting particles in quantum Hall-samples result under proper conditions in the IQHE, whereas the interacting particle systems should be responsible for the fractional QHE (FQHE). In the last case it seems that depending on the theoretical treatment of the question of the ground state one is lead to one of the above mentioned models.

[4] K. von Klitzing, Physica B 204 (1995) 111-116; R. Knott, W. Dietsche, K. von Klitzing, K. Eberl and K. Ploog, Semicond. Sci. Technol. 10 (1995) 117-126

[5] For a different model, where only the Ohm-equations of the IQHE but not that of CHE are derived as equations of motion see: J. Fröhlich, T. Kerler, Nucl. Phys. B354 (1991) 369-417.

[6] Precisely, the total magnetic field acting on the Hall system described by the Schroedinger-Chern-Simons-action (9) is given by $B_{total} := B_{external} + B(A_m)$ with $B_{external} \gg B(A_m)$, where $B_{external}$ is the external homogenous strong magnetic field applied on the system. The $B(A_m)$ is the magnetic field arised from the dynamics of $A_m$-potentials which is also responsible for the electric fields $E_m$. The $B(A_m)$ is usually so small that $\omega_c \tau \ll 1$ and so its influence on the conductivity is contained already in what is known under the classical Hall-effect. Since, to achieve magnetic influence of the
quantum Hall-type one needs strong magnetic fields as those used in QHE-experiments (see refs. [1] and [3]).

[7] Recall also that relation (2) can be obtained from relation (1) by an infinitesimal SO(2)-transformation in the $E_m$- or in the $A_m$-space. The infinitesimal angle $\delta \chi = \frac{\sigma_L}{\sigma_H}$ becomes almost zero in the quantum Hall-regime.

[8] See L.D.Landau, E.M.Lifschitz, III Vol.

[9] See also the Ref. [2].

[10] R. P. Feynman and A. R. Hibbs, Quantum mechanics and Path Integrals (McGraw-Hill 1965)

[11] This means that $d\sigma_H = 0$. Furthermore, recall also that both charge carrier density $n$ in two dimensions and the $B$-field are of dimension $L^{-2}$. Thus, the $\sigma_H = \frac{en}{B}$ becomes dimensionless. For further arguments in favour of the local constancy of $\sigma_H$ see the paper quoted in Ref. [5].

[12] Recall that in presence of magnetic fields the well known Landau-gauge is given by $A_m = Bx_\epsilon m$ (see ref. [8]).

[13] J. Callaway, Energy band Theory (Academic Press 1964). Recall also that the relaxation time $\tau$ is indeed introduced in this approximation to calculate the electric conductivity from the Ohm-equations. One can show that the usual relaxation time approximation results in the approximation $\Delta A_m = E_m \Delta t \approx E_m \tau$, where the defining vanishing average velocity $\tilde{V} = 0$ is given according to the operator $\tilde{V}_m = \hat{P}_m - e\hat{A}_m$.

[14] E. Witten, Cumm. Math. Phys. 121 (1989) 351-399; R. Jakiw, in: Physics, Geometry, and Topology, Nato ASI Series, ed., H.C. Lee (Plenum Press, New York, 1990); G.V. Dunne, R. Jackiw, and C Treugenerberger, Ann. Phys. 194 (1989) 197-223; G.V.Dunne and C. Treugenerberger, Mod. Phys. Lett. A Vol4 (1989) 1635-1644

[15] See for other methods of quantizations Ref. [14]. The fractional quantization of the normalization parameter should be a result of multivaluedness of the wave function of the electrons in the Schroedinger-term in its second quantization which is related with the interacting electrons (see
the models quoted in Ref. [3]). It is well known that the mentioned properties of electrons like the mobility and also the strength of exterior magnetic field has different values for the FQHE-samples [4].

[16] Recall also that $\sigma_H$ becomes $\sigma_H = \frac{ne}{B}$ only in the quantum Hall-limit, whereas in the classical Hall-limit it is given by $\sigma_H = \sigma_0 \omega_c \tau$.

[17] It is this discrepancy between the local and global properties of pure gauge potential which makes a classical or a either semi-classical understanding of QHE difficult. The quantum mechanical, i.e. the global or invariant character of pure gauge potential is given by its line integral which is the phase of wave function and results in the flux quantization. Furthermore we mean here always a pure gauge potential of the electromagnetic or U(1)-type in a multiply connected region.

[18] Under preparation.