Nonlinear Modulation of Travelling Rolls in Magnetoconvection

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Abstract

Modulational dynamics of oscillatory travelling rolls in magnetoconvection is studied near the onset of a Hopf bifurcation. Using weakly nonlinear analysis, we derive an envelope equation of oscillatory travelling rolls in the plane perpendicular to an ambient vertical magnetic field. The envelope equation is the Davey-Stewartson (DS) equation with complex coefficients, from which we obtain criteria for the modulational (Benjamin-Feir) instability of oscillatory travelling rolls.

1 Introduction

A variety of spatially and temporally periodic patterns is found in weakly nonlinear Boussinesq convection in an imposed vertical magnetic field [1]. However, nonlinear modulation of the periodic patterns in the horizontal plane has thus far received less attention. Two kinds of bifurcations are known to convective patterns: one is to steady patterns and the other is to oscillatory patterns (a Hopf bifurcation) [1]. Near the onset of the former bifurcation, modulational dynamics of travelling rolls may be described by the Newell-Whitehead-Segel (NWS) equation in terms of the Newell-Whitehead (NW) orderings [2]. Near the onset of a Hopf bifurcation, the same orderings may yield a more complicated envelope equation [3], which includes various terms of different orders.

In this paper, introducing consistent orderings (different from the NW orderings) to an envelope equation near the onset of a Hopf bifurcation, we derive the DS equation with complex coefficients, where horizontal incompressible flows couple to oscillatory travelling rolls in magnetoconvection. The analysis of a instability of its spatially uniform solution yields criteria for the modulational (Benjamin-Feir) instability of oscillatory travelling rolls.
2 Hopf Bifurcation

Boussinesq convection in an imposed vertical magnetic field is described by the equations [1]

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla P + \sigma R \Theta z + \sigma \zeta Q (\nabla \times B) \times B + \sigma \Delta u, \\
\partial_t \Theta + (u \cdot \nabla) \Theta &= w + \Delta \Theta, \\
\partial_t B + (u \cdot \nabla) B &= (B \cdot \nabla) u + \zeta \Delta B,
\end{align*}
\]

(2.1) together with

\[
\nabla \cdot u = \nabla \cdot B = 0.
\]

(2.4)

Here \( u \equiv (u, v, w) \) is the dimensionless velocity in (x,y,z) coordinates, \( z \) is the unit vector in the vertical (\( z \)) direction, \( \Theta \) is the dimensionless temperature deviation from the conduction state and \( B \) is the dimensionless magnetic field. The parameters are the Prandtl number \( \sigma \), the Rayleigh number \( R \), the Chandrasekhar number \( Q \) and the ratio of ohmic to thermal diffusivity denoted by \( \zeta \).

The magnetic field is given by

\[
B = z + b,
\]

where \( b \equiv (a, b, c) \). The boundary conditions are

\[
\partial_z u = \partial_z v = w = \Theta = a = b = 0 \quad \text{at} \quad z = 0, 1.
\]

The linear stability analysis of the conduction state \( u = \Theta = b = 0 \) shows that oscillatory convection sets in, for \( \zeta < 1 \), at

\[
R = R_0 = \left( \frac{\pi^2 + k^2}{k^2} \right)^3 \left( \frac{\sigma + \zeta}{\sigma} \right) (1 + \zeta) + \frac{\pi^2 + k^2 \zeta (\sigma + \zeta)}{k^2} \left( \frac{1}{1 + \sigma} \right) \pi^2 Q.
\]

Here \( k \) is the horizontal wavenumber determined by minimizing the critical Rayleigh number \( R_0 \). Thus \( k = k_0 \), where

\[
(\pi^2 + k_0^2)^3 - \frac{3}{2} \pi^2 (\pi^2 + k_0^2)^2 = \frac{\sigma \zeta}{2(1 + \sigma)(1 + \zeta)} \pi^4 Q.
\]

3 Envelope Equation

In this section, we derive the equation which describes the nonlinear evolution of a slowly varying envelope of oscillatory travelling rolls near the critical Rayleigh number \( R_0 \). For values of \( R \) close to \( R_0 \),

\[
R = R_0 + \epsilon^2 R^{(2)} \quad (\epsilon \ll 1),
\]
we investigate the weakly nonlinear evolution of the wavepacket centered at the critical wavenumber \( k_0 \) and the corresponding frequency \( \omega_0 = \omega(k_0) \) by approximating \( u, b, \Theta \) and \( P \) as

\[
\begin{align*}
\mathbf{u} &= \varepsilon u_1(\xi, \eta, \tau) E \begin{pmatrix} \cos(\pi z) \\ \sin(\pi z) \end{pmatrix} + (\text{c.c.}) + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \cdots, \\
\mathbf{b} &= \varepsilon b_1(\xi, \eta, \tau) E \begin{pmatrix} \sin(\pi z) \\ \cos(\pi z) \end{pmatrix} + (\text{c.c.}) + \varepsilon^2 b^{(2)} + \varepsilon^3 b^{(3)} + \cdots, \\
\Theta &= \varepsilon \Theta_1(\xi, \eta, \tau) E \sin(\pi z) + (\text{c.c.}) + \varepsilon^2 \Theta^{(2)} + \varepsilon^3 \Theta^{(3)} + \cdots, \\
P &= P_1(\xi, \eta, \tau) E \cos(\pi z) + (\text{c.c.}) + \varepsilon^2 P^{(2)} + \varepsilon^3 P^{(3)} + \cdots, \\
E &= \exp[i(k_0x - \omega_0t)],
\end{align*}
\]

where (c.c.) denotes the complex conjugate of the previous term, and

\[
\xi = \varepsilon(x - \lambda t), \quad \eta = \varepsilon y, \quad \tau = \varepsilon^2 t.
\]

The present ordering (3.3) of the stretched variables \( \xi, \eta, \tau \) is the same as one introduced in the derivation of the DS equation [5]. Substituting the expansions (3.1)-(3.4) into Eqs (2.1)-(2.4) and using the method of multiple scales [4], we obtain from the leading order equations(linearized equations)

\[
\begin{align*}
\Theta_1 &= L_{\theta} w_1, \quad L_{\theta} = (\kappa^2 - i\omega_0)^{-1}, \\
c_1 &= \pi L_c w_1, \quad L_c = (\kappa^2 - i\omega_0)^{-1}, \\
P_1 &= \pi L_p w_1, \quad L_p = -\sigma(R_0 L_{\theta}/\kappa^2 + \zeta Q L_c), \\
u_1 &= i \pi/k_0 w_1, \quad a_1 = -i \pi/k_0 c_1, \\
b_1 &= v_1 = 0,
\end{align*}
\]

where \( \kappa^2 = k_0^2 + \pi^2 \) and the linear dispersion relation becomes

\[
f \equiv \sigma \kappa^2 - i\omega_0 - \pi^2 L_p - \sigma R_0 L_{\theta} = 0.
\]

The second order field variables are expressed as follows

\[
\begin{align*}
w^{(2)} &= w_1^{(2)} E \sin(\pi z), \\
\Theta^{(2)} &= \Theta_1^{(2)} E \sin(\pi z) + (\text{c.c.}) + \Theta_0^{(2)} \sin(2\pi z), \\
c^{(2)} &= [c_1^{(2)} \cos(\pi z) + c_0^{(2)} E] E + (\text{c.c.}), \\
P^{(2)} &= [P_1^{(2)} \cos(\pi z) + \{P_0^{(2)} \cos(2\pi z) + P_2^{(2)} \} E] E + (\text{c.c.}) + P_0^{(2)} \cos(2\pi z) + P_2^{(2)}, \\
u^{(2)} &= u_1^{(2)} E \cos(\pi z) + (\text{c.c.}) + u_0^{(2)} \cos(2\pi z) + \bar{u}_0^{(2)}, \\
a^{(2)} &= a_1^{(2)} E \sin(\pi z) + (\text{c.c.}) + a_0^{(2)} \sin(2\pi z), \\
v^{(2)} &= v_1^{(2)} E \cos(\pi z) + \bar{v}_1^{(2)}, \quad b^{(2)} = b_1^{(2)} E \sin(\pi z),
\end{align*}
\]
where $\bar{u}_0^{(2)}$ and $\bar{v}_0^{(2)}$ are determined by

\[
(\partial_\xi^2 + \partial_\eta^2)\bar{u}_0^{(2)} = \frac{\pi^2}{k_0^2\lambda}(1 - \sigma\zeta Q\pi^2|L_c|^2)\partial_\eta^2|w_1|^2, \tag{3.12}
\]

\[
(\partial_\xi^2 + \partial_\eta^2)\bar{v}_0^{(2)} = -\frac{\pi^2}{k_0^2\lambda}(1 - \sigma\zeta Q\pi^2|L_c|^2)\partial_\eta\partial_\xi|w_1|^2.
\]

The other second order amplitudes such as $\Theta_1^{(2)}, \Theta_0^{(2)}, c_1^{(2)}$ and so on are given in Appendix and give

\[
\lambda = \left[\partial_k\omega(k)\right]_{k=k_0} \equiv \partial_k\omega_0.
\]

As shown in Appendix, the solvability condition for the third order variable $w_1^{(3)}$ yields the following equation of the envelope of the first order vertical fluid velocity $w_1$:

\[
i\partial_\tau w_1 + \frac{\partial_k^2\omega_0}{2}\partial_\xi^2 w_1 + \frac{\partial_k\omega_0}{2k_0}\partial_\eta w_1 + \left(\frac{q}{\partial_{\omega_0} f}|w_1|^2 - k_0\bar{u}_0^{(2)}\right)w_1 + \frac{\partial_{R_0} f}{\partial_{\omega_0} f} R_0^{(2)} w_1 = 0, \tag{3.13}
\]

where $\partial_k\omega_0$ is real, while $\partial_k^2\omega_0 \equiv (\partial_k^2\omega)_{k=k_0}$ has a complex value in general and

\[
q = \sigma R_0 \frac{k_0^2}{2\kappa^2} L_\theta L_\theta' + \sigma Q \zeta^4 \frac{2\kappa^4 - \pi^2}{2\kappa^2 - \pi^2} \left[3k_0^2 - \pi^2|L_c|^2 - L_c^2\right]
\]

\[
- \frac{8i\sigma Q\pi^6}{\kappa^2(Q + 4\pi^2)} L_c'' L_c
\]

\[
+ \frac{iQ\pi^2}{Q + 4\pi^2} \left(\pi^2\sigma Q L_c^2 + \sigma R_0 \frac{k_0^2}{\kappa^2} L_\theta^2 + \frac{k_0^2 - 3\pi^2}{\kappa^2} L_c^2\right) L_c''.
\tag{3.14}
\]

A coupled system of equations (3.13) and (3.12) is the DS equation with complex coefficients, in which the real field $\bar{u}_0^{(2)}$ represents $z$ independent incompressive horizontal flows varying slowly.

## 4 Modulational Instability of Oscillatory Travelling Rolls

For later conveniences, the DS equation with complex coefficients (a coupled system of equations (3.13) and (3.12) ) is rewritten as follows.

\[
i\partial_\tau \Psi + \alpha \partial_\xi^2 \Psi + \beta \partial_\eta^2 \Psi + (\gamma|\Psi|^2 + su)\Psi = ir\Psi, \tag{4.1}
\]

\[
(\partial_\xi^2 + a\partial_\eta^2)u = (b\partial_\xi^2 + c\partial_\eta^2)|\Psi|^2, \tag{4.2}
\]
where $\alpha, \beta$ and $\gamma$ are complex constants ($\alpha = \alpha' + i\alpha'', \beta = \beta' + i\beta'', \gamma = \gamma' + i\gamma''$), while $s, r, a, b$ and $c$ are real constants ($a = 1, b = 0, \beta'' = 0$ in the present case). A spatially uniform solution of Eqs. (4.1) and (4.2) is given by
\[ \Psi = \Psi_0 \equiv \psi_0 \exp(-i\Omega t), \quad u = 0, \]
where $|\psi_0|^2 = r/\gamma''$ and $\Omega = -\gamma'|\psi_0|^2$. Setting
\[ \Psi = \Psi_0 + \psi_1(t) \exp[i(k \cdot r - \Omega_1 t)] + \psi_2(t) \exp[i(-k \cdot r - \Omega_2 t)], \]
\[ u = u_1(t) \exp[i\{k \cdot r - (\Omega_1 - \Omega)t\}] + u_2(t) \exp[i\{-k \cdot r - (\Omega_2 - \Omega)t\}] + (c.c.), \]
where $k = (k_x, k_y), r = (x, y), 2\Omega = \Omega_1 + \Omega_2$ and linearizing Eqs. (4.1) and (4.2) with respect to $\psi_1$ and $\psi_2$, we have
\[ \frac{d\psi_1}{dt} = [r - i\{\alpha k_x^2 + \beta k_y^2 - \Omega_1 - (\gamma + \hat{\gamma})|\psi_0|^2\}]\psi_1 + i\hat{\gamma}\psi_0^2\psi_2^*, \quad (4.3) \]
\[ \frac{d\psi_2^*}{dt} = [r + i\{\alpha^* k_x^2 + \beta^* k_y^2 - \Omega_2 - (\gamma^* + \hat{\gamma}^*)|\psi_0|^2\}]\psi_2^* - i\hat{\gamma}^*\psi_0^2\psi_1^*, \quad (4.4) \]
where $\hat{\gamma} = \gamma + s(bk_x^2 + ck_y^2)/(k_x^2 + ak_y^2)$ and $^*$ denotes the complex conjugate. The linear equations (4.3) and (4.4) have exponentially growing solutions if the following condition is satisfied.
\[ (\alpha'\gamma' + \alpha''\hat{\gamma}'')k_x^2 + (\beta'\gamma' + \beta''\hat{\gamma}'')k_y^2 > 0. \quad (4.5) \]
Since $a > 0$ in the present case, Eq. (4.3) yields the following instability criteria.
\[ \hat{\alpha} \equiv \alpha'(\gamma' + sb) + \alpha''\gamma'' > 0, \quad (4.6) \]
or
\[ \hat{\beta} \equiv \beta'(\gamma' + sc/a) + \beta''\gamma'' > 0, \quad (4.7) \]
or
\[ (\hat{\alpha} - \hat{\beta}/a)^2 + \hat{s}[2(\hat{\alpha} + \hat{\beta}/a) + \hat{s}] > 0, \quad \text{and} \quad \hat{\alpha} + \hat{\beta}/a + \hat{s} > 0, \quad (4.8) \]
where $\hat{s} = s(c/a - b)(\alpha' - \beta'\alpha) > 0$. The criterion (4.6) or (4.7) is essentially the same as the case of the two-dimensional complex Ginzburg-Landau equation (Eq. (4.1) with $s = 0$). A new criterion (4.8) comes from the coupling between convective rolls and horizontal incompressible flows. If $a < 0$, although this is not the case in magnetoconvection, a spatially uniform solution of Eqs. (4.1) and (4.2) is shown to be always modulatally unstable.
5 Concluding Remarks

In this paper, we have derived an envelope equation of oscillatory travelling rolls near a Hopf bifurcation, which is not a type of the NWS equation [3] but the DS equation with complex coefficients. The NWS type equation has a defect if the group velocity at the critical (bifurcation) point does not vanish. That is, it consists of different order terms: the main term is linear and proportional to the group velocity, while the other interesting terms such as nonlinear terms are of higher order. Although our derivation is based on weakly nonlinear analysis with multiple scales similar to [3], the present ordering of stretched variables (3.5) is different from the NW ordering [2] and yields the DS equation with complex coefficients which consists of the same order terms.

Analyzing the modulational instability of a spatially independent oscillatory solution of the DS equation with complex coefficients, we obtain criteria of the modulational instability, which include not only the known criterion for the complex Ginzburg-Landau equation, which was first given in [6], but also a new criterion due to the coupling between convective rolls and horizontal incompressible flows.

Appendix

A Higher Order Amplitudes

In terms of the first order amplitudes and \( w_1^{(2)} \), the second order amplitudes defined in Eqs. (3.6)-(3.11) are given by

\[
\begin{align*}
\Theta_1^{(2)} & = \theta_0 w_1^{(2)} - i \dot{\theta}_0 \partial_k w_1, \\
P_1^{(2)} & = \pi L_p w_1^{(2)} - i \pi \dot{L}_p \partial_k w_1, \\
a_1^{(2)} & = -i \frac{\pi}{k_0} (c_1^{(2)} + \frac{i}{k_0} \partial_k c_1), \\
\dot{L}_\theta & = \lambda \partial_{\omega_0} L_\theta + \partial_{k_0} L_\theta, \text{ etc.},
\end{align*}
\]

and

\[
\begin{align*}
\Theta_0^{(2)} & = - \frac{L_\theta}{2\pi} |w_1|^2, \\
u_0^{(2)} & = -2Q \frac{\pi^2}{k_0 (Q + 4\pi^2)} L''_c |w_1|^2, \\
a_0^{(2)} & = \frac{2\pi}{Q\zeta} u_0^{(2)}, \\
P_0^{(2)} & = \left(1 + \frac{\sigma R_0 L'_\theta}{4\pi^2}\right) |w_1|^2 + \frac{\sigma \zeta Q k^2}{2k_0^2} |c_1|^2, \\
\bar{P}_0^{(2)} & = \lambda u_0^{(2)} \frac{\pi^2}{k_0^2} |w_1|^2 - \frac{\sigma \zeta Q}{2} \left(1 - \frac{\pi^2}{k_0^2}\right) |c_1|^2,
\end{align*}
\]

6
\[
\begin{align*}
P^{(2)}_2 &= -\frac{\sigma \zeta Q \kappa^2}{4k_0^2} c_1^2, \\
\bar{P}^{(2)}_2 &= -\sigma \zeta Q \left[ \frac{\pi}{2\zeta k_0^2 - i\omega_0} w_1 + \frac{\kappa^2}{4k_0^2} c_1 \right] + \frac{\pi^2}{2k_0^2} u_1,
\end{align*}
\]
\[
\begin{align*}
c^{(2)}_2 &= \frac{\pi}{2\zeta k_0^2 - i\omega_0} w_1 c_1, \\
Lc'' &= \text{Im}(L_c), \\
L\theta' &= \text{Re}(L_\theta),
\end{align*}
\]

Lengthy but straightforward calculations of the third order terms in Eqs. \ref{eq:2.1}-\ref{eq:2.4} give the following third order field variables (envelopes) proportional to \( E \) in terms of \( w_1 \) and \( w^{(2)}_1 \).

\[
\begin{align*}
\Theta^{(3)}_1 &= L_\theta w_1^{(3)} - i L_\theta \partial_\xi w^{(2)}_1 - (\bar{L}_\theta/2) \partial_\tau^2 w_1 - L_\theta^2 (\partial_\tau - \partial_\eta^2) w_1 \\
&\quad - i k_0 L_\theta (\bar{u}^{(2)}_0 - u_0^{(2)} / 2) w_1 + \pi L_\theta \Theta^{(2)}_0 w_1,

c^{(3)}_1 &= \pi [L_c w_1^{(3)} - i \bar{L}_c \partial_\xi w^{(2)}_1 - (\bar{L}_c/2) \partial_\tau^2 w_1 - L_c^2 (\partial_\tau - \zeta \partial_\eta^2) w_1] \\
&\quad + i k_0 L_c [a_0^{(2)}/2 - \pi L_c (\bar{u}^{(2)}_0 + u_0^{(2)} / 2)] w_1 - \pi L_c c^{(2)}_2 w_1,

P^{(3)}_1 &= \pi [L_p w_1^{(3)} - i \bar{L}_p \partial_\xi w^{(2)}_1 - (\bar{L}_p/2) \partial_\tau^2 w_1 - 1/(2k_0) \partial_{\omega_0} L_p \partial_\eta^2 w_1 \\
&\quad + i \partial_{\omega_0} L_p \partial_\tau w_1 - (\sigma R^{(2)}/\kappa^2) L_\theta w_1] \\
&\quad + \sigma \zeta \pi [L_c - L_c^* (k_0^2 - 3\pi^2/\kappa^2) c^{(2)}_2 w_1] \\
&\quad - [(\sigma \zeta Q / 2)^2 (k_0^4 + 4k_0^2 \pi^2 - \pi^4) / \kappa^2] L_c a_0^{(2)} + k_0 \pi \partial_{\omega_0} L_p u_0^{(2)} \\
&\quad - (ik_0 \pi / 2) \{ \sigma \zeta Q L_c^2 - (\sigma R_0 / \kappa^2) L_\theta^2 - (4/\kappa^2) \} u_0^{(2)} \\
&\quad + (\sigma R_0 \pi^2 / \kappa^2) L_\theta \Theta^{(2)}_0 w_1,
\end{align*}
\]

and

\[
\begin{align*}
f w^{(3)}_1 &= i \tilde{f} \partial_\xi w^{(2)}_1 - i \partial_{\omega_0} f \partial_\tau w_1 + (1/2) (\tilde{f} \partial_\tau^2 w_1 + k_0^{-1} \partial_{\omega_0} f \partial_\eta^2 w_1) \\
&\quad + (k_0 \partial_{\omega_0} f \bar{u}^{(2)}_0 - \partial_{R_0} f R^{(2)}) w_1 - q |w_1|^2 w_1, \quad (A.1)
\end{align*}
\]

where \( q \) is given in Eq.\ref{eq:3.14}. Since

\[
\begin{align*}
f = \dot{f} = 0, \\
\frac{\partial_{\omega_0} f}{\partial_{\omega_0} \dot{f}} = -\partial_{\omega_0} \omega_0, \\
\frac{\dot{f}}{\partial_{\omega_0} \dot{f}} = -\partial_{\omega_0} \omega_0,
\end{align*}
\]

Eq.\ref{eq:A.1} gives Eq.\ref{eq:3.13}.

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