On Some Additivity Problems in Quantum Information Theory

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1 Introduction

Quantum information theory is not merely a theoretical basis for physics of information and computation. It is also a source of challenging mathematical problems, often having elementary formulation but still resisting solution. It appears that surprisingly little is known about what may be called the combinatorial geometry of tensor products of Hilbert spaces, even in finite dimensions. One group of open problems concerns the additivity properties of various quantities characterizing quantum channels, notably the capacity for classical information, and the “maximal output purity”, defined below. All known results, including extensive numerical work in the IBM group, the Quantum Information group in the Technical University of Braunschweig, and elsewhere, are consistent with the conjecture that

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these quantities are indeed additive (resp. multiplicative) with respect to tensor products of channels. A proof of this conjecture would have important consequences in quantum information theory: in particular, according to this conjecture, the classical capacity or the maximal purity of outputs cannot be increased by using entangled inputs of the channel.

In this paper we state the additivity/multiplicativity problems, give some relations between them, and prove some new partial results, which also support the conjecture.

2 Statement of the problem

Let us give precise formulation of the additivity problem for the classical capacity (see [2], [3]). Let $\mathcal{B}(\mathcal{H})$ be the $\ast$-algebra of all operators in a finite dimensional unitary space $\mathcal{H}$. We denote the set of states, i.e. positive unit trace operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{S}(\mathcal{H})$, the set of all $m$-dimensional projections by $\mathcal{P}_m(\mathcal{H})$ and the set of all projections by $\mathcal{P}(\mathcal{H})$. A quantum channel $\Phi$ is a completely positive trace preserving linear map of $\mathcal{B}(\mathcal{H})$ (we are in the finite dimensional case and we use the Schrödinger picture). These are the maps admitting the Kraus decomposition (see e. g. [7], [5])

$$\Phi(\rho) = \sum_k A_k \rho A_k^*, \quad (1)$$

where $A_k$ are operators satisfying $\sum_k A_k^* A_k = I$.

Let $H(\rho) = -\text{Tr} \rho \log \rho$ denote the von Neumann entropy of the state $\rho$ and define

$$C(\Phi) = \max_{p_i, \rho_i} [H(\sum_i p_i \Phi(\rho_i)) - \sum_i p_i H(\Phi(\rho_i))],$$

where the maximum is taken over all finite probability distributions $\{p_i\}$ on $\mathcal{S}(\mathcal{H})$, ascribing probabilities $p_i$ to (arbitrary) states $\rho_i$. The quantity $C(\Phi)$ appears as the “one-step classical capacity” of the quantum channel $\Phi$ or the capacity with unentangled input states (we refer to [4] for a detailed information-theoretic discussion and the proof of the corresponding coding theorem). A thorough discussion of the properties of $C(\Phi)$ is given in [8].
The additivity problem can be formulated as follows: let $\Phi_1, \ldots, \Phi_n$ be channels in the algebras $\mathcal{B}(\mathcal{H}_1), \ldots, \mathcal{B}(\mathcal{H}_n)$ and let $\Phi_1 \otimes \ldots \otimes \Phi_n$ be their tensor product in $\mathcal{B}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)$. Is it true that

$$C(\Phi_1 \otimes \ldots \otimes \Phi_n) = \sum_{i=1}^n C(\Phi_i) \ ?$$

(2)

This obviously holds for reversible unitary channels; in [5] the additivity was established for the so called classical-quantum and quantum-classical channels, which map from or into an Abelian subalgebra of $\mathcal{B}(\mathcal{H})$.

Another closely related problem is the additivity of a quantity, which can be read as the “maximal output purity” of a channel. In fact, there are several quantities of this kind, depending on the way we measure “purity”. If we just take the von Neumann entropy as a measure of purity, we arrive at the question [4], [6] whether or not

$$\min_{\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)} H((\Phi_1 \otimes \ldots \otimes \Phi_n)(\rho)) = \sum_{i=1}^n \min_{\rho \in \mathcal{S}(\mathcal{H}_i)} H(\Phi_i(\rho)) \ ?$$

(3)

For a particular class of channels this property implies (2) (see the Lemma in Section 5 below).

We will also consider this problem for other measures of purity, based on the noncommutative $\ell_p$-norms

$$\|A\|_p = (\text{Tr}|A|^p)^{\frac{1}{p}},$$

defined for $p \geq 1$, and $A \in \mathcal{B}(\mathcal{H})$, with the operator norm $\|A\|$ corresponding naturally to the case $p = \infty$. For an arbitrary quantum channel $\Phi$ let us introduce the following notations for the “highest purity” of outputs of a channel

$$\nu_H(\Phi) = \min_{\rho} H(\Phi(\rho)), \quad (4)$$

$$\nu_p(\Phi) = \max_{\rho} \|\Phi(\rho)\|_p, \quad (5)$$

$$\nu_{-\infty}(\Phi) = \min_{\rho} \|\Phi(\rho)^{-1}\|^{-1}, \quad (6)$$

where the extrema are taken with respect to all input density matrices $\rho$. By convexity of the norms, the extrema in the above definitions are attained.
on pure states (in the first (resp. last) case the operator convexity of the function \( x \mapsto x \log x \) (resp. \( x \mapsto x^{-1} \)) is also relevant).

Then the additivity/multiplicativity inequalities

\[
\nu_p(\Phi_1 \otimes \Phi_2) \geq \nu_p(\Phi_1)\nu_p(\Phi_2) \quad (7)
\]

\[
\nu_H(\Phi_1 \otimes \Phi_2) \leq \nu_H(\Phi_1) + \nu_H(\Phi_1) \quad (8)
\]

are clear from inserting product density operators into the defining variational expressions. The standing conjecture is that equality always holds in these inequalities, i.e. that choosing entangled input states is never helpful for getting purer output states.

Before proceeding to show some new partial results on this problem, it is helpful to establish the relation between \( \nu_H(\Phi) \), and \( \nu_p(\Phi) \) for \( p \) close to one. Of course, some relationship is expected, as the von Neumann entropy \( H(\rho) \) can be computed in terms of the derivative of \( \|\rho\|_p \) at \( p = 1^+ \). Here we find that if the equality holds in (7) for \( p \) arbitrarily close to 1, then it holds also in (8).

**Proof.** We shall use the fact that for every \( 0 < x \leq 1 \)

\[
\frac{1 - x^p}{p - 1} \uparrow -x \frac{\log x}{\log e}
\]

if \( p \downarrow 1 \). Thus \( \frac{1 - \text{Tr}(\Phi(\rho))^p}{p - 1} \) is a monotonely increasing family of continuous functions of the variable \( \rho \) which varies in the compact set \( \mathcal{S}(\mathcal{H}) \), converging pointwise to the continuous function \( H(\Phi(\rho)) \). By Dini’s Theorem, the convergence is uniform, and

\[
\min_\rho H(\Phi(\rho)) = \lim_{p \downarrow 1} \frac{1 - \max_\rho \text{Tr}(\Phi(\rho))^p}{p - 1}.
\]

Therefore, if the equality holds in (7) for \( p \) close to 1,

\[
\min_\rho H((\Phi_1 \otimes \Phi_2)(\rho)) = \lim_{p \downarrow 1} \frac{1 - \max_\rho \text{Tr}((\Phi_1 \otimes \Phi_2)(\rho))^p}{p - 1}
\]

\[
= \lim_{p \downarrow 1} \frac{1 - \max_\rho \text{Tr}(\Phi_1(\rho))^p \max_\rho \text{Tr}(\Phi_2(\rho))^p}{p - 1} = \min_\rho H(\Phi_1) + \min_\rho H(\Phi_2) + \min_\rho H(\Phi_2). \square
\]
3 Tensoring with an ideal channel

The first natural step is to establish the multiplicativity property when one factor is the identity channel.

Lemma. For \( * = p, H, -\infty \)

\[
\nu_*(\Phi \otimes \text{Id}) = \nu_*(\Phi).
\] (9)

Since \( \nu_p(\text{Id}) = 1 \), and \( \nu_H(\text{Id}) = 0 \), this is indeed an instance of the additivity/multiplicativity conjecture.

Proof. We shall restrict to the case \( * = p, 1 \leq p \leq \infty \). The argument in the case \( * = -\infty \) is similar and the case \( * = H \) follows by the argument given above.

Let us denote by \( \mathcal{H}_1, \mathcal{H}_2 \) the Hilbert spaces of the first and the second system, respectively. Let \( \phi_{12} \) be a unit vector in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), and write \( \rho_{12} = |\phi_{12}\rangle\langle\phi_{12}| \) and \( \rho_1 = \text{Tr}_2 \rho_{12} \) for the partial state in \( \mathcal{H}_1 \). If \( \Phi \) is the channel in \( \mathcal{H}_1 \), we denote \( \rho'_{12} = (\Phi \otimes \text{Id})(\rho_{12}) \). Let us dilate the channel \( \Phi \) to a unitary evolution \( U_{13} \) with the environment \( \mathcal{H}_3 \), initially in a pure state \( \rho_3 = |\phi_3\rangle\langle\phi_3| \). The final state of the environment is

\[
\rho'_3 = \text{Tr}_1 U_{13} (\rho_1 \otimes |\phi_3\rangle\langle\phi_3|) U_{13}^* \equiv \Psi(\rho_1).
\]

Since the state of the composite system \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \) remains pure after the unitary evolution, its partial states \( \rho'_{12}, \rho'_3 \) are isometric \([7]\). Therefore

\[
|| (\Phi \otimes \text{Id})(\rho_{12}) ||_p = || \rho'_{12} ||_p = || \rho'_3 ||_p = || \Psi(\rho_1) ||_p.
\]

Now the map \( \rho_1 \to \Psi(\rho_1) \) is affine and the norm is convex, therefore the maximum of the quantity above is attained on pure \( \rho_1 \), whence \( \rho_{12} = \rho_1 \otimes \rho_2 \), and the statement follows. \( \Box \)

We shall specifically need this Lemma in the case \( p = \infty \). It is instructive to see an alternative direct proof in this case.

Proof. In what follows we take \( \rho = |\phi\rangle\langle\phi| \). We compute the operator norm of the Hermitian operator \( \Phi(\rho) \) as \( ||\Phi(\rho)|| = \sup_{\psi} \langle \psi, \Phi(\rho)\psi \rangle \), and take \( \Phi \) to
be given in the Kraus decomposition (\[1\]). Then
\[
\nu_\infty(\Phi) = \sup_{\phi, \psi} \sum_k \langle \psi, A_k \phi \rangle \langle \phi, A_k^* \psi \rangle,
\]
where the supremum is over all unit vectors in the appropriate spaces. The expression under the supremum can be read as the $\ell^2$-norm of a vector with components $\langle \psi, A_k^* \psi \rangle$. We write this norm also as “the largest scalar product with a unit vector” $\chi$, i.e.,
\[
\nu_\infty(\Phi) = \left( \sup_{\psi, \chi} \sum_k \overline{\chi}_k \langle \phi, A_k^* \psi \rangle \right)^2 = \left( \sup_{\psi, \chi} \| \sum_k \overline{\chi}_k A_k^* \psi \| \right)^2 = \sup_{\chi} \| \sum_k \overline{\chi}_k A_k \| ^2 = \nu_\infty(\Phi),
\]
where all suprema are over unit vectors. Obviously, the Kraus operators for $\Phi \otimes \text{Id}$ are $A_k \otimes I$, so
\[
\nu_\infty(\Phi \otimes \text{Id}) = \sup_{\chi} \| \sum_k \chi_k (A_k \otimes I) \| ^2 = \nu_\infty(\Phi).
\]

We will also need the analogous result for a quantity in which the two vectors $\phi, \psi$ in the above proof are fixed to be the same: for any channel $\Phi$, let
\[
\nu_\flat(\Phi) = \sup_\psi \langle \psi, \Phi(\psi) \rangle,
\]
where the supremum is again over all unit vectors. Note that this expression only makes sense, if the channel does not change the type of system, i.e., input and output algebra are the same. Then
\[
\nu_\flat(\Phi \otimes \text{Id}) = \nu_\flat(\Phi).
\]

Proof: Again we use Kraus decomposition (\[1\]). For $\psi$ we use the Schmidt decomposition $\psi = \sum \sqrt{c_\mu} e_\mu \otimes e'_\mu$, where the $e_\mu$ and $e'_\mu$ are orthonormal systems. Then the expression to maximized on the left-hand side becomes
\[
\sum_{\mu \nu \alpha \beta k} (c_\mu c_\nu c_\alpha c_\beta)^{1/2} \langle e_\mu \otimes e'_\mu, (A_k \otimes I) e_\nu \otimes e'_\nu \rangle \langle e_\alpha \otimes e'_\alpha, (A_k^* \otimes I) e_\beta \otimes e'_\beta \rangle
\]
\[
\begin{align*}
\sum_{\mu\nu\alpha\beta k} (c_\mu c_\nu c_\alpha c_\beta)^{1/2} & \langle e_\mu, A_k e_\nu \rangle \langle e_\alpha, A_k^* e_\beta \rangle \delta_{\mu\nu} \delta_{\alpha\beta} \\
= \sum_{\mu \alpha k} c_\mu c_\alpha & \langle e_\mu, A_k e_\mu \rangle \langle e_\alpha, A_k^* e_\alpha \rangle = \sum_k \text{tr}(\rho_1 A_k)\text{tr}(\rho_1 A_k^*),
\end{align*}
\]

where \( \rho_1 = \sum_\mu c_\mu |e_\mu\rangle \langle e_\mu| \) is the reduced density matrix belonging to \( \psi \). Since the function \( \rho_1 \mapsto |\text{tr}(\rho_1 A_k)|^2 \) is convex, this expression attains its maximum with respect to \( \psi \) when \( \rho_1 \) is pure, i.e., when \( \psi \) is a product. \( \square \)

\section{Weak Noise}

One testing ground for the multiplicativity/additivity conjecture are channels close to the identity. For such channels the purity parameters can be evaluated in lowest order in the deviation from the identity. Doing this for each subchannel and for their tensor product, one can explicitly check the conjecture. As the following result shows, this test supports the conjecture.

Consider a channel with weak noise, i.e. choose some channel \( \Phi \) on \( \mathcal{B}(\mathcal{H}) \), and set

\[
\Phi(\epsilon) = (1 - \epsilon)\text{Id} + \epsilon \Phi.
\]

For small \( \epsilon \) this is a weak noise channel, which has the property that for any pure input the output will be nearly pure.

\textbf{Theorem.} The multiplicativity hypothesis for the quantities \( \nu_p(\Phi) \) with \( 1 \leq p \leq \infty \) and the additivity hypothesis for the quantity \( \nu_H(\Phi) \) hold true approximately in the leading order in \( \epsilon \).

\textbf{Proof.} In order to estimate these quantities for the weak noise channels, we need to estimate entropy, and the \( p \)-norms near a pure state. Let \( \rho \) be a density operator on a \( d \)-dimensional Hilbert space, and suppose that \( \|\rho\| = 1 - \epsilon + o(\epsilon) \). Then the leading order of the other norms is determined completely by \( \epsilon \):

\[
\begin{align*}
\|\rho\|_p &= 1 - \epsilon + o(\epsilon) \quad \text{for } p > 1 \\
H(\rho) &= -\epsilon \log \epsilon + o(\epsilon \log \epsilon),
\end{align*}
\]

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where as usual $o(\epsilon)$ stands for terms going to zero faster than $\epsilon$ as $\epsilon \to 0$. In this case we can say more: in first line we have $0 \leq \text{remainder} \leq C\epsilon$, for $\epsilon < 1/2$, where $C$ is a constant depending only on the dimension. Similarly, the estimates in the second line are independent of the details of $\rho$. Hence in leading order all the variational expressions are equivalent: each one amounts to maximizing $\epsilon$.

Let us go back to the weak noise channel (14). To get high fidelity we need to maximize the leading term, so we can take $\eta = \xi$ in the following computation:

$$
\nu_\infty(\Phi^{(\epsilon)}) = \sup_{\xi,\eta} \langle \xi, \Phi^{(\epsilon)}(\eta) \langle \eta \rangle \xi \rangle
$$

$$
= \sup_{\xi,\eta} \left( (1 - \epsilon)|\langle \xi, \eta \rangle|^2 + \epsilon \langle \xi, \Phi(\eta) \langle \eta \rangle \xi \rangle \right)
$$

$$
= 1 - \epsilon + \epsilon \sup_{\xi} \langle \xi, \Phi(\langle \xi \rangle \langle \xi \rangle) \rangle + o(\epsilon)
$$

$$
= 1 - \epsilon + \epsilon \nu_\delta(\Phi) + o(\epsilon). \quad (17)
$$

Note that in all these estimates the remainder estimates can be done uniformly for all channels, depending only on dimension.

A tensor product of weak noise channels (14) is again of the same form:

$$
\Phi^{(\epsilon)} = \Phi_1^{(\epsilon)} \otimes \cdots \otimes \Phi_n^{(\epsilon)}
$$

$$
= (1 - \epsilon)^n \text{Id} + \epsilon (1 - \epsilon)^{n-1} (\Phi_1 \otimes \text{Id}_{2\ldots n} + \cdots + \text{Id}_{1\ldots n-1} \otimes \Phi_n) + o(\epsilon)
$$

$$
= (1 - n\epsilon) \text{Id} + n\epsilon \delta\Phi + o(\epsilon),
$$

where $\delta\Phi$ is the average of the $n$ channels $\text{Id}_{1\ldots k-1} \otimes \Phi_k \otimes \text{Id}_{k+1\ldots n}$. Hence, in order to compute the leading order of $\nu_\infty(\Phi^{(\epsilon)})$ by formula (17) we have to determine $\nu_\delta(\delta\Phi)$. We have

$$
\frac{1}{n} \sum_{k=1}^m \nu_\delta(\Phi_k) \leq \nu_\delta(\delta\Phi)
$$

$$
\leq \frac{1}{n} \sum_{k=1}^m \nu_\delta(\Phi_k \otimes \text{Id})
$$

$$
= \frac{1}{n} \sum_{k=1}^m \nu_\delta(\Phi_k)
$$

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where we have used in turn: insertion of product states into the supremum defining \( \nu_\flat(\delta \Phi) \), convexity of \( \nu_\flat \) as a supremum of affine functionals, and finally the restricted additivity result (13). Hence equality holds, which means that in the leading order in \( \epsilon \) all the variational expressions for the purity quantities \( \nu_\ast() \), with \( \ast = p, H, \flat \) are attained at product states. \( \Box \)

5 Depolarizing Channels

A channel is called bistochastic if \( \Phi(I) = I \), where \( I \) is the unit operator in \( \mathcal{B}(\mathcal{H}) \). An important example is the depolarizing channel [2]

\[
\Phi(\rho) = (1 - p)\rho + \frac{p}{d}(\text{Tr}\rho)I, \quad \rho \in \mathcal{B}(\mathcal{H}), \quad 0 < p < 1,
\]

where \( d = \text{dim}\mathcal{H} \). A channel is called binary if \( d = 2 \).

Lemma. Let \( \Phi \) be binary bistochastic channel, then

\[
C(\Phi) = \log 2 - \min_{\rho \in \mathcal{S}(\mathcal{H})} H(\Phi(\rho)). \tag{18}
\]

If \( \Phi_i \) are binary bistochastic channels, then (3) implies (2).

Proof. The \( \leq \) part of (18) is obvious from the fact that for any channel

\[
C(\Phi) \leq \log \text{dim}\mathcal{H} - \min_{\rho \in \mathcal{S}(\mathcal{H})} H(\Phi(\rho)), \tag{19}
\]

so we need to prove only \( \geq \) part. Since the entropy is convex, the minimum is achieved at the set of extreme points of \( \mathcal{S}(\mathcal{H}) \) which is \( \mathcal{P}_1(\mathcal{H}) \). Let \( \rho \) be the minimum point, then taking equiprobably \( \rho_0 = \rho, \rho_1 = I - \rho \), we obtain

\[
C(\Phi) \geq H\left(\frac{1}{2}\Phi(I)\right) - \frac{1}{2}\left[H(\Phi(\rho)) + H(\Phi(I - \rho))\right].
\]

Since the channel is bistochastic, this is equal to \( H\left(\frac{1}{2}I\right) - \frac{1}{2}\left[H(\Phi(\rho)) + H(I - \Phi(\rho))\right], \) and since it is binary, this is equal to the right-hand side of (18).

To prove the second statement, it is sufficient to prove the \( \leq \) part of (2), since \( \geq \) part follows from the definitions. But this follows from (19) and (18). \( \Box \)
In the paper [3] the relation (2) was proven for the two binary depolarizing
channels $\Phi_1, \Phi_2$. The proof heavily uses Schmidt decomposition and as such
does not generalizes to the case $n > 2$. The main difficulty is evaluating the
entropy of the product channel. However, it appears to be possible to check
the additivity in the limiting cases of “weak” and “strong” depolarization
in the leading order. Let us consider a collection of depolarizing channe ls
$\{\Phi_i\}$ in the Hilbert spaces $\mathcal{H}_i$, with parameters $p_i, d_i$, $i = 1, 2, \ldots, n$, and denote
$\Phi = \otimes_{i=1}^n \Phi_i$, $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$, $d = \prod_{i=1}^n d_i$. In the following we shall use symbols
$I_i$ and $I = \otimes_{i=1}^n I_i$ for the identity operators in
$\mathcal{H}_i$ and $\mathcal{H}$ respectively. Let $\epsilon_L$ be the tensor product
$\phi_1 \otimes \ldots \otimes \phi_n$, where $\phi_i(\rho) = \frac{1}{d_i} \text{Tr}(\rho) I_i$, $i \in L \subset \{1, 2, \ldots, n\}$, and $\phi_i(\rho) = \rho$ otherwise, $\rho \in S(\mathcal{H}_i)$. Then $\epsilon_L$ is a conditional
expectation onto the subalgebra $\mathcal{M}_L$, generated by operators of the form
$A_1 \otimes \ldots \otimes A_n$, where $A_i = I_i$ for $i \in L$, and is normalized partial trace with
respect to its commutant. It has the property $\min_{P \in \mathcal{P}(\mathcal{M}_L)} \text{dim} P = \prod_{i=1}^n \theta_L(i)$,
where $\theta_L(i) = 1$ if $i \in L$ and $\theta_L(i) = 0$ otherwise. Here we denoted by
$\mathcal{P}(\mathcal{M}_L)$ the set of all orthogonal projections in $\mathcal{M}_L$. Notice that the inclusion
$\mathcal{M}_{L_1} \subset \mathcal{M}_{L_2}$ holds if $L_2 \subset L_1$. So $\epsilon_{L_1} \epsilon_{L_2} = \epsilon_{L_1 \vee L_2}$.

We shall use the expansion

$$\Phi = \sum_{L} \prod_{i=1}^n \theta_L(i) (1 - p_i)^{1 - \theta_L(i)} \epsilon_L.$$  \hfill (20)

Weak (strong) depolarization corresponds to the case where all $p_i$ (respec-
tively, $1 - p_i$) are small parameters, which we assume to be of the same
order.

**Proposition.** The relation (2) holds in the cases of weak and strong
depolarization approximately in the leading order.

**Proof.** In the case of weak depolarization the statement follows from the
Theorem in Section [4].

In the case of strong depolarization we have to retain all the terms up to
the second order in $q_i = 1 - p_i$. Then the leading terms are

$$\Phi(P) \sim d^{-1} \left[ I + \sum_{i=1}^n q_i (d_i P_i - I) + \sum_{1 \leq i < j \leq n} q_i q_j (1 - d_i P_i - d_j P_j + d_i d_j P_{ij}) \right],$$

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where we denoted $d = \prod_{i=1}^{n} d_i$, $P_i$ is the partial state of $P$ in the $i$-th Hilbert space, multiplied by the unit operator in the tensor product of the remaining Hilbert spaces, and similarly $P_{ij}$. Denoting the first (second) sum in the squared brackets $A_1$ (respectively $A_2$) one easily sees that both are traceless operators. Moreover, up to the second order,

$$\Phi(P) \log \Phi(P) \sim d^{-1} \left[ (1 + A_1 + A_2) \log d^{-1} + \left( A_1 + A_2 + \frac{A_1^2}{2} \right) \right]$$

and

$$H(\Phi(P)) \sim \log d - \frac{\text{Tr}A_1^2}{2d}.$$  

But

$$\text{Tr}A_1^2 = d \sum_{i=1}^{n} q_i^2 (d_i \text{Tr} \rho_i^2 - 1),$$

where $\rho_i$ is the partial state of $P$ in the $i$-th Hilbert space, which is maximized if and only if $\rho_i$ is one-dimensional projection, i.e. $P = \rho_1 \otimes \ldots \otimes \rho_n$. $\square$

Partial answers to the multiplicativity hypothesis are given by the following Theorem. In fact, multiplicativity of $\nu_{\infty}(\Phi)$ for binary bistochastic maps follows from a more general result in [6].

**Theorem.**

(i) $\nu_2(\Phi) = \prod_{i=1}^{n} \nu_2(\Phi_i) = \prod_{i=1}^{n} \left( \frac{d_i-1}{d_i} (1-p_i)^2 + \frac{1}{d_i} \right)^{1/2}$,

(ii) $\nu_{\infty}(\Phi) = \prod_{i=1}^{n} \nu_{\infty}(\Phi_i) = \prod_{i=1}^{n} \left( 1 - \frac{p_i(d_i-1)}{d_i} \right)$,

(iii) $\nu_{-\infty}(\Phi) = \prod_{i=1}^{n} \nu_{-\infty}(\Phi_i) = \prod_{i=1}^{n} \frac{p_i}{d_i}$.

**Proof.** (i) It follows from the relation $\min_{P \in P(M_L)} \dim P = \prod_{i=1}^{n} d_i^{\min_L(i)}$ that

$$\text{Tr}(\epsilon_{L_1}(P)\epsilon_{L_2}(P)) = \text{Tr}(P\epsilon_{L_1 \vee L_2}(P)) = \text{Tr}(\epsilon_{L_1 \vee L_2}(P)^2) \leq \prod_{i=1}^{n} d_i^{-\max_L(i)}$$

for arbitrary $P \in P_1(H)$ and equality holds only for factorizable projections. Hence

$$\text{Tr}((\Phi(P))^2) = \text{Tr}(\sum_{L} \prod_{i=1}^{n} P_{i,\epsilon_{L}}(1 - p_i)^{1-\epsilon_{L}(i)}\epsilon_{L}(P))^2)$$

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We have

\[ \sum = (1 - p_i)^2 - \theta_1(i) - \theta_2(i) d_i^{\max(\theta_1(i), \theta_2(i))} = \]

\[ = \prod_{i=1}^{n} \sum_{\theta_1, \theta_2 = 0, 1} p_i^{\theta_1 + \theta_2} (1 - p_i)^{2 - \theta_1 - \theta_2} d_i^{\max(\theta_1, \theta_2)} = \prod_{i=1}^{n} \left( \frac{d_i - 1}{d_i} (1 - p_i)^2 + \frac{1}{d_i} \right). \]

(ii) Let us estimate

\[ ||\Phi(P)|| \leq \sum_{i=1}^{n} L \prod_{i=1}^{n} p_i^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} ||\epsilon_L(P)|| \]

\[ \leq \sum_{L} \prod_{i=1}^{n} \left( \frac{p_i}{d_i} \right)^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} = \prod_{i=1}^{n} \left( 1 - \frac{p_i(d_i - 1)}{d_i} \right), \]

which proves the second statement.

(iii) We have

\[ ||\Phi(P)||^{-1} \| \leq \sum_{k=0}^{+\infty} ||\Phi(I - P)||^k. \]  \hspace{1cm} (21)

Let us calculate

\[ \max_{P \in P_t(B(\mathcal{H}))} ||\Phi(I - P)|| \leq \max_{Q \in P_{d-1}(B(\mathcal{H}))} ||\Phi(Q)||. \]

We have

\[ ||\Phi(Q)|| \leq \prod_{i=1}^{n} p_i ||\epsilon_{1,2,...,n}(Q)|| \]

\[ + \sum_{L \neq \{1,2,...,n\}} \prod_{i=1}^{n} p_i^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} ||\epsilon_L(Q)|| \leq \]

\[ (1 - d^{-1}) \prod_{i=1}^{n} p_i + \sum_{L \neq \{1,2,...,n\}} \prod_{i=1}^{n} p_i^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} = 1 - \prod_{i=1}^{n} \frac{p_i}{d_i}. \] \hspace{1cm} (22)

Here we have used the equality \( \sum_{L} \prod_{i=1}^{n} p_i^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} = 1. \) Substituting (22) into (21), we get the last statement. \( \square \)

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