Review and comparative analysis of computational algorithms for modeling the dynamics of elastically supported structural systems

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Abstract. The work is devoted to the comparative analysis of computational algorithms for modeling the dynamics of elastically supported structural systems with intermediate and limiting supports in the presence of point forces applied to the superstructure. The considered algorithms are based on the model formulation in the form of a system of partial differential equations for generalized functions. This approach avoids the need to subordinate the basic functions to the boundary conditions and include the presence of sources of the forces, applied at a certain point, in the model.

1. Introduction

The construction of computational algorithms for calculating the dynamics of elastically supported load-bearing structures (structural systems) is associated with a number of specific difficulties. These difficulties are caused by the fact that the mathematical model of these structures’ behavior contains the derivatives with respect to spatial variables of the fourth-fifth orders and higher. Structures can contain one-way connections, on and off switching connections, mechanisms with gaps, structures with elastic connections, which are pre-compressed by their own weight, structures with restraining arms, etc. The mathematical models of such systems, as a rule, have the form of partial differential equations’ system with uneven nonlinear conditions at the internal and external boundaries. Quite often, the situations when an elastically supported structure is under the influence of localized forces, i.e. the forces, where the contact area with the superstructure, is negligible less than the size of the supporting structure, are considered. Such situations lead to the need to use mathematical models with point sources of acting forces.

In [1] - [4], a technique for constructing an approximate solution, in which the original problem is associated with an equation for generalized functions, the solution of which will also be a solution to the original problem, is proposed. The problem representation in the form of an equation for generalized functions makes it possible to formulate the problem and construct an approximate solution in the presence of point sources. Since the boundary conditions are taken into account in the equations for generalized functions (in a weak form), then when constructing an approximate solution there is no need to subject the basis functions to the boundary conditions. Later, for the hyperbolic systems of linear
differential equations of the first order [5] - [8], this approach was developed for the case when it is possible to construct a fundamental solution of the problem operator for generalized functions. In this work, using the example of the elastic thin-walled rod transverse vibrations equations, a technique for applying the fundamental solution of the problem operator to construct an approximate solution of the initial-boundary value problem of this equation is presented.

2. Problem statement
The methods analysis for numerical modeling of the elastically supported structural systems’ dynamics will be carried out using the example of the span structure’s constructively nonlinear oscillations of a floating bridge with restrictive rigid supports. We take the model of an elastic bar with free ends of length $2l$ and linear density $\mu$ and full weight $M$ as a mechanical model under the action of the forces combination from the dynamic pressures of car tires $R(t,x)$, the interaction forces with floatation jackets $S(t,x) = \frac{Mg}{N}$, transmitted by the means of elastic restraints ($\eta_\gamma$ - stiffness of this jacket, $u(t,x)$ - superstructure profile) and interaction forces with the transitional parts of the bridge $P_p(t,x)$. The additional forces from the side of the restraint supports start acting when the gaps are closed on the elastic rod, $Q^* = -\theta(u^* - u(t,x))\eta_\gamma(u(t,x) - u^*)$; these forces are modeled by the elastic constraints with stiffness coefficients $\eta_\gamma \gg \eta_\gamma$.

Let us introduce the vector of variables $u = \begin{bmatrix} u_1 & \frac{\partial u}{\partial t} & \frac{\partial^2 u}{\partial x^2} \end{bmatrix}^T$, and the notation $\gamma(t) = u(t,x = -l)$, $\gamma^*(t) = \frac{\partial u}{\partial x}(t,x = -l)$, $\beta(t) = u(t,x = l)$, $\beta^*(t) = \frac{\partial u}{\partial x}(t,x = l)$. Then, following [9] and [6], the model of flexural-torsional vibrations of a thin-walled elastic rod can be written in the form of an initial-boundary value problem for the system of partial differential equations. This model is formulated for the distributed forces applied to an elastic thin-walled bar. To formulate this model in terms of the forces applied at a point, it is necessary to resort to the technique of the generalized functions [10].

As shown in [6], $u(t,x)$, the considered generalized function from $S'$, meets the condition of the partial differential equations system.

$$\frac{\partial u}{\partial t} + A \frac{\partial^2 u}{\partial x^2} + Bu = f + \alpha \delta(t) + b$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1^2 & 0 \\ 0 & -1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad f = -g(\theta(x+l) - \theta(x-l)) + \frac{1}{\mu} \sum \psi_i \delta(x-x_i) \quad (1)$$

$$b = \begin{bmatrix} 0 \\ +a_1^2 \gamma_1(t)\delta(x+l) + a_1^2 \gamma_1(t)\delta'_i(x+l) - a_1^2 \beta_1(t)\delta(x-l) - a_1^2 \beta_1(t)\delta'_i(x-l) \\ -a_1^2 \gamma_1(t)\delta(x+l) - a_1^2 \gamma_1(t)\delta'_i(x+l) + a_1^2 \beta_1(t)\delta(x-l) + a_1^2 \beta_1(t)\delta'_i(x-l) \end{bmatrix}$$

3 Computational algorithms

3.1 Algorithm based on global basis functions.
We define the system of linearly independent basis functions $H_p(x)$, for example, Legendre polynomials or Chebyshev polynomials on the segment $-l, l$. We will seek an approximate solution in the form $u(t, x) = \sum_{p=1}^{P} H_p(x) u^p(t)$. The functions of the form $\varphi^p(t, x) \in S'$, kind, indexed $p' = 1 \div P$ will be taken as test functions. $\chi(t)$ - is an arbitrary finite infinitely differentiable function. Following the stated in [6] to determine $u^p(t)$, we obtain the Cauchy problem for the ordinary differential equations system (ODE)

$$
D_p^p \frac{d u^p}{dt} + B_p^p u^p = f^p
$$

(2)

Matrices $D_p^p$ are not unitary and densely filled, since the basic functions are not orthonormal and their support coincides with the entire segment $-l, l$. Having solved this Cauchy problem for the ODE system, we find an approximate solution to the original problem.

3.2 Algorithm Based on Finite Piecewise Linear Functions.

This algorithm differs from the one described above only in the choice of basic functions. Let us plot the mesh of nodes $x_p$, $p = 1 \div P$ on the segment $-l, l$ so, that $x_1 = -l, x_p = l$. This grid splits the segment $-l, l$ at intervals. Let us connect the function $H_p(x)$ with each node $x_p$, which is equal at the point $x_p$, equal at all other grid points and linear at each interval. Matrices $D_p^p$ will also not be single, but unlike the previous algorithm, they will be strongly decompressed, due to the use of finite functions as a basis.

3.3 Algorithm based on the problem operator fundamental solution construction.

3.3.1 Fundamental solution.

Let us plot a fundamental solution to the operator of the problem (1). The fundamental solution to the operator of the problem (1) is called the generalized matrix function $G(t, x) \in S'$, which satisfies the equation

$$
\frac{\partial G(t, x)}{\partial t} + A \frac{\partial^2 G(t, x)}{\partial x^2} + B G(t, x) = I \delta(t, x)
$$

(3)

Let us denote the Fourier transformation $G(t, x)$ by the spatial variables by $V(t, \xi) = F_x[G](t, \xi)$. Let us perform the Fourier equations transformation (3) in spatial variables

$$
\frac{\partial V(t, \xi)}{\partial t} - (\xi^2 A - B) V(t, \xi) = I \delta(t)
$$

(4)

The solution of the problem (4) exists, is unique and has the form $V(t, \xi) = \theta(t) e^{(\xi^2 A - B)t}$.
Matrix $\xi^2 A - B$ is hyperbolic, i.e. has a complete set of linearly independent eigenvectors and therefore can be represented in the form $\xi^2 A - B = \Omega^{-1} \Lambda \Omega$. Here $\Lambda$ is a diagonal matrix of the matrix eigenvalues, ordered in non-decreasing order, $\Omega$ is the matrix whose rows are the left eigenvectors of the matrix corresponding to the eigenvalues $\Lambda$. Then $e^{t(\xi^2 A - B)} \Omega^{-1} e^t \Omega$ and accurate to $O(\Delta t^2)$.

$$V(t, \xi) = \theta(t) \begin{pmatrix} 1 & \frac{\sin a\xi t}{a\xi} & 0 \\ 0 & \frac{\partial}{\partial t} \frac{\sin a\xi t}{a\xi} & -a^2 (i\xi)^2 \frac{\sin a\xi t}{a\xi} \\ 0 & (i\xi)^2 \frac{\sin a\xi t}{a\xi} & \frac{\partial}{\partial t} \frac{\sin a\xi t}{a\xi} \end{pmatrix}$$ (5)

Performing the inverse Fourier transformation, we obtain a fundamental solution to the operator of the problem (1) or Green’s matrix function

$$G(t, x) = \theta(t) \begin{pmatrix} \delta(x) & \frac{\theta(\Delta t - |x|)}{2a} & 0 \\ 0 & \frac{\partial}{\partial t} \frac{\theta(\Delta t - |x|)}{2a} & -a^2 \frac{\partial^2}{\partial x^2} \frac{\theta(\Delta t - |x|)}{2a} \\ 0 & \frac{\partial^2}{\partial x^2} \frac{\theta(\Delta t - |x|)}{2a} & \frac{\partial}{\partial t} \frac{\theta(\Delta t - |x|)}{2a} \end{pmatrix}$$ (6)

The solution of the problem (1) is defined as the convolution of the fundamental solution with the right-hand side. Let us consider the point $(t + \Delta t, x_i)$, corresponding to the location of one of the intermediate supports. Then

$$u_i(t + \Delta t, x_i) = -g \frac{\Delta t^2}{2} + \frac{1}{2a\mu} \int_{\tau_i}^{t+\Delta t} \left( \frac{Mg}{N} - \eta \mu u_i(\tau, x_i) \right) d\tau + u_i(t, x_i) + \frac{1}{2a} \int_{x_i+a\Delta t}^{x_i} u_\tau(t, \xi) d\xi$$

$$u_\tau(t + \Delta t, x_i) = -g \Delta t + \frac{Mg}{2a\mu N} - \frac{\eta}{2a\mu} u_i(t + \Delta t, x_i) + \sum_r \left( 1 + 2\theta(x_i - x_r) - 1 \right) \frac{\partial^2}{\partial x^2} \frac{\theta(\Delta t - |x_i - x_r|)}{2a\mu} \int_{x_i+a\Delta t}^{x_i} u_\tau(t, \xi) d\xi$$

$$u_j(t + \Delta t, x_i) = -\frac{\eta}{2a\mu} u_j(t + \Delta t, x_i) + \sum_r \frac{\theta(\Delta t - |x_i - x_r|)}{2a^3 \mu} \frac{\partial^2}{\partial t} \frac{\partial u_j}{\partial x} (t + \Delta t - |x_i - x_r|)$$

$$+ \frac{\partial^2}{\partial x} \frac{\partial u_j}{\partial t} (t, x_i - a\Delta t) + \frac{\partial^2}{\partial x} \frac{\partial u_j}{\partial t} (t, x_i + a\Delta t)$$

$$+ \frac{2}{a} \frac{\partial u_j}{\partial x} (t, x_i - a\Delta t) + \frac{\partial u_j}{\partial x} (t, x_i + a\Delta t)$$

$$+ \frac{2}{a} \frac{\partial u_j}{\partial x} (t, x_i - a\Delta t) + \frac{\partial u_j}{\partial x} (t, x_i + a\Delta t)$$

$$+ \frac{u_j(t, x_i - a\Delta t) + u_j(t, x_i + a\Delta t)}{2}$$ (7)
At the points \((t + \Delta t, x_q)\), corresponding to location of one of the limiting supports, an expression follows for the total force transmitted to the superstructure from the side of the \(s\) th intermediate support, \(S_s\), replaced by \(Q_q\) - total force transferred to the superstructure from the side of the \(q\) th limiting support. At the points \((t + \Delta t, x)\), that do not coincide with either intermediate or limiting supports, in all formulas the terms responsible for the total force transferred to the superstructure from the side of the intermediate support should be equated to zero.

Considering the points \((t + \Delta t, x = \mp l)\), coinciding with the left or right boundary, we obtain

\[
\begin{align*}
\frac{1}{\Delta t} [u_1(t + \Delta t, l) - u_1(t, l)] = & -\frac{\Delta t^2}{4} + \frac{\int_{t}^{t+\Delta t} u_2 dq}{2a} \quad \text{for } l < \gamma < 0

\frac{1}{\Delta t} [u_2(t + \Delta t, l) - u_2(t, l)] = & -\frac{\Delta t}{2} + \frac{1}{2} u_2(t, l - a\Delta t) \quad \text{for } l < \gamma < 0

\frac{1}{\Delta t} [u_3(t + \Delta t, l) - u_3(t, l)] = & 0 \quad \text{for } l < \gamma < 0
\end{align*}
\]

3.3.2 Computational Algorithm and Numerical Experiments

Let us split the segment \([-l, l]\) with the nodes \(x_n\), \(n = 1: N - 1\) on \(N - 1\) intervals. In this case, we require that the position of each intermediate and each restraint support coincides with some of the nodes \(x_n\).

Let us denote the minimum length \(h_n = x_{n+1} - x_n\), \(n = 1: N - 1\) by \(h\) in these intervals. Also on the time interval \(0, T\) we introduce a uniform grid \(t_m\), \(m = 1: M\) with spacing \(\Delta t = t_{m+1} - t_m\).

Let us connect the system of difference equations with each node \(x_n\). At the internal nodes, we obtain these equations by approximating the equalities (7). The functions included in these equalities are approximated on each interval of the spatial grid by the linear functions plotted from the values at the nearest nodes of the spatial grid. We replace the integrals with finite-difference relations, for example, by the formula of rectangles or by the formula of trapeziums. At the nodes corresponding to the boundary points, the system of difference equations is obtained by approximating the equations (8) respectively. As a result, we obtain an explicit difference scheme for calculating the solution values at the next time layer.

A number of numerical experiments were carried out with the constructed computational model, including those with a moving active load at various parameters of this load. In these experiments, a computational scheme with the parameters described in detail in [11] and further used in [6], [8], [12], [13] was used. These experiments have shown, even in a single-processor implementation, high performance of a computational model based on this algorithm. The computation time is significantly less than the time required to calculate similar scenarios by other methods. For example, when compared with probably the most effective of the known methods - the modified Galerkin method, on piecewise-linear basis the functions developed by the authors in earlier works [8], the performance increase in the same scenarios reached 6-8 times.

Since the algorithm is explicit and the solution value at each node on the next time layer is calculated as a linear combination of the solution values at the nodes on the previous time layer and this linear combination involves the solution values only in several nodes of the previous time layer, then the solution at each node can be computed independently, which allows the algorithm to efficiently parallelize. And the potential for increasing the algorithm speed due to parallelization on multiprocessor or multicore systems is obviously great. This aspect of the proposed algorithm should and will be investigated in further works on its improvement.
Summary
In this work, using the example of an initial boundary value problem for an equation describing constructively nonlinear oscillations of the floating bridge span structure with intermediate supports and restrictive rigid supports, a review and comparative analysis of algorithms for finding an approximate solution is carried out. All considered algorithms are based on the original initial-boundary value problem’s reformulation in the form of a partial differential equations system for generalized functions. This approach in a weak form includes boundary and initial conditions in the equations for generalized functions. This, in turn, leads to the fact that there is no need to subordinate the basic functions, according to which the solution is expanded to the boundary and initial conditions. This aspect is very important, especially in case of problems with many spatial variables. The use of the generalized functions technique makes it possible to pose and solve the problems with point and even instant sources of forces acting on the system.

The algorithms in which the solution was approximated by the global basic functions, i.e. the functions which support coincided with the entire domain of the solution search, the algorithms based on finite piecewise linear basic functions as well as the algorithms based on the construction of a fundamental solution to the problem or the Green’s matrix function have been investigated.

The algorithms on the global basic functions, for example, on Legendre polynomials or Chebyshev polynomials reduce the original problem to the Cauchy problem for the ODE system dynamics in time of the coefficients of expansion of the sought solution in terms of basic functions. The resulting algorithm is implicit, since the matrix at the derivatives is not unit. Such algorithms make it possible to find only rather rough approximations of the initial-boundary value problems. The attempts to achieve higher accuracy by increasing the number of basic functions lead to a deterioration in the conditionality of the problem and, as a consequence, to the emergence of computational instability. They are also quite slow due to the implicit algorithm.

The algorithms in which the solution is approximated by finite piecewise linear basis functions also reduce the original problem to the Cauchy problem for the ODE system, they are also implicit, since the matrix at the derivatives is not unit, however, they make it possible to achieve a significantly higher accuracy of the approximate solution by increasing the number of basic functions. An increase in the number of basic functions does not lead to computational instability. However, these algorithms, due to the fact that they are implicitly not fast enough, especially in comparison with the explicit algorithms based on the problem operator fundamental solution plotting.

The algorithms based on the problem operator fundamental solution plotting reduce the finding of the solution at any point on the next time level to explicit formulas for recalculating the linear combination of the solution at the previous time level in a small number of nearby nodes of the spatial grid. Traditionally, numerical algorithms, including those described above, are based on one or another approximation of the differential operators of the problem. The algorithm based on the fundamental solution of the operator of the problem is based on the approximation of explicit formulas expressing the solution of the initial initial-boundary value problem for the system of partial differential equations on the next time layer through the solution on the previous layer. The algorithm is explicit, it gives an opportunity to achieve the required accuracy by increasing the number of nodes of the spatial grid and its performance on the identical scenarios is multiples of the performance of the other algorithms presented. Also, the algorithm has a high potential for increasing performance due to parallelization.

The study makes it possible to conclude that for calculating the dynamics of the elastically supported structural systems’ behavior with intermediate and restraining supports, especially in computing systems for mass industrial calculations, it is most expedient to use the algorithms based on plotting a fundamental solution to the operator of the problem.

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