A NOTE ON PENNER’S COCYCLE
ON THE FATGRAPH COMPLEX

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Abstract. We study a 1-cocycle on the fatgraph complex of a punctured surface introduced by Penner. We present an explicit cobounding cochain for this cocycle, whose formula involves a summation over trivalent vertices of a trivalent fatgraph spine. In a similar fashion, we express the symplectic form of the underlying surface of a given fatgraph spine.

1. Introduction

The fatgraph (or ribbon graph) complex \( \hat{G} = \hat{G}(\Sigma) \) of a punctured oriented surface \( \Sigma \) serves as a combinatorial model for the Teichmüller space of \( \Sigma \) and the action of the mapping class group \( \mathcal{M} = \mathcal{M}(\Sigma) \) on it. The cells of \( \hat{G} \) are indexed by isotopy classes of fatgraph spines, which are graphs embedded in the surface satisfying certain conditions, and its face relations are described by contracting non-loop edges in fatgraph spines.

The complex \( \hat{G} \) has been used to study the cohomology of the mapping class group and the moduli space of Riemann surfaces. Among others, Morita and Penner \([9]\) studied twisted first cohomology of the mapping class group by using \( \hat{G} \). More concretely, they considered a once punctured (or bordered) surface and “lifted” the extended first Johnson homomorphism

\[
\tilde{k} \in H^1(\mathcal{M}; \frac{1}{2} \wedge^3 H)
\]

to a 1-cocycle on \( \hat{G} \). Here, \( H \) is the first integral homology group of \( \Sigma \) and \( \wedge^3 H \) is its third exterior power. The key feature of \([9]\) is that their cocycle

\[
j \in Z^1(\hat{G}; \wedge^3 H)
\]

takes a very explicit and simple form for flips (or Whitehead moves), which are local deformations of a trivalent fatgraph spine and which correspond to oriented 1-cells of \( \hat{G} \). In this sense, the Morita-Penner construction is “canonical”.

After the work of Morita and Penner, there have been known several variations and generalizations in other cohomological or topological objects related to the mapping class group; see \([1, 2, 3, 5, 6]\). In each of them one can find interesting constructions making use of the combinatorics of fatgraph spines.

In this paper, based on the same line of study as above, we study invariants of trivalent fatgraph spines obtained by summation over trivalent vertices. More concretely, we focus on a 1-cocycle introduced by Penner in

\[
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\]
his textbook on decorated Teichmüller theory \cite{10},
\[ s \in Z^1(\hat{G}; S^2(\wedge^2 H)), \]
where \( S^2(\wedge^2 H) \) is the second symmetric power of the second exterior power of \( H \). The coefficient of \( s \) comes from a description of the target of the second Johnson homomorphism given in \cite{7, 8}. As was shown in \cite{5}, the corresponding twisted cohomology class in \( H^1(\mathcal{M}; S^2(\wedge^2 H)) \) is trivial over the rational coefficients, but this is simply because the cohomology group itself is trivial. We will prove that this vanishing of the cohomology class more directly and over the integral coefficients, by giving an explicit 0-cochain
\[ \xi \in C^0(\hat{G}; S^2(\wedge^2 H)) \]
which cobounds \( s \) and is \( \mathcal{M} \)-equivariant. Since 0-cells of \( \hat{G} \) correspond to trivalent fatgraph spines, \( \xi \) is nothing but an assignment
\[ \xi_G \in S^2(\wedge^2 H) \]
to each trivalent fatgraph spine \( G \subset \Sigma \). The formula for \( \xi_G \) is of the form
\[ \xi_G = \frac{1}{2} \sum_v \eta_v \cdot \eta_v, \]
where the sum is taken over all trivalent vertices of \( G \), and \( \eta_v \in \wedge^2 H \) is defined by using the homology classes dual to the oriented edges pointing toward \( v \).

As a by-product of our study, we find a simple combinatorial formula for the symplectic form
\[ \omega \in \wedge^2 H \]
of the underlying surface of a given fatgraph spine \( G \). The element \( \omega \) is dual to the intersection form on \( H \). Our formula is of the form
\[ \omega = \frac{1}{2} \sum_v \eta_v. \]

The results above will be stated more precisely and proved in \S 3 (Theorem 3.2 and Theorem 3.9).

In order to simplify the exposition we restrict ourselves to the case of a once punctured surface, but the general case may be addressed similarly.

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2. Fatgraph complex and Penner’s cocycle

We collect basic materials about the fatgraph complex of a punctured surface, and recall the definition of Penner’s cocycle. For more details, see \cite{10, 5}.

Throughout the paper, we fix a positive integer \( g \).
2.1. Fatgraph spines. Let $\Sigma = \Sigma^1_g$ be a closed oriented surface of genus $g$ together with a distinguished basepoint $\ast$. We call $\Sigma$ a once punctured surface of genus $g$.

By a graph, we mean a finite CW complex of dimension one. Each edge of a graph admits two orientations. An oriented edge of a graph is an edge of the graph equipped with an orientation. If $e$ is an oriented edge of a graph, we denote by $\overline{e}$ the oriented edge that is obtained by reversing the orientation of $e$.

**Definition 2.1.** A fatgraph spine of $\Sigma$ is a graph $G$ embedded in $\Sigma \setminus \{\ast\}$ such that the inclusion map $G \hookrightarrow \Sigma \setminus \{\ast\}$ is a homotopy equivalence and the valency of every vertex of $G$ is at least three.

Figure 1 shows a fatgraph spine of a surface of genus one. The condition that $G \hookrightarrow \Sigma \setminus \{\ast\}$ is a homotopy equivalence implies that if we cut $\Sigma$ along $G$ we obtain a polygon with a puncture in its interior which corresponds to $\ast \in \Sigma$. We give the polygon an orientation which matches that of $\Sigma$. Going around the boundary of the polygon in the clockwise manner, we traverse each oriented edge of $G$ exactly once. This gives the set of oriented edges of $G$ a cyclic ordering, which we denote by $\prec$. For instance, the polygon obtained from the fatgraph spine in Figure 1 is depicted in Figure 2, and we have $e_1 \prec e_2 \prec e_3 \prec \overline{e}_1 \prec \overline{e}_2 \prec \overline{e}_3 \prec e_1$.

For each vertex of a fatgraph spine of $\Sigma$, the set of oriented edges pointing toward the vertex is endowed with a cyclic ordering induced from the orientation of the tangent space of $\Sigma$ at the vertex.

2.2. Fatgraph complex and flips. The fatgraph complex of $\Sigma$, denoted by $\hat{\mathcal{G}} = \hat{\mathcal{G}}(\Sigma)$, is a CW complex of dimension $4g - 3$ whose cells are in one to one correspondence with isotopy classes of fatgraph spines of $\Sigma$. Each 0-cell of $\hat{\mathcal{G}}$ corresponds to a trivalent fatgraph spine, i.e. a fatgraph spine with all the vertices being trivalent. In general, $n$-dimensional cells of $\hat{\mathcal{G}}$ correspond to fatgraph spines which can be obtained from a trivalent one by collapsing $n$ edges.

Each oriented 1-cell of $\hat{\mathcal{G}}$ corresponds to a flip (or a Whitehead move) between two trivalent fatgraph spines; if $e$ is an edge of a trivalent fatgraph spine $G$, a flip along $e$, denoted by $W_e$, is to deform $G$ by collapsing and
expanding $e$ in the natural way to obtain another trivalent fatgraph spine $G'$. We use a shorthand notation

$$G \xrightarrow{W_e} G'$$

for the flip $W_e$. See Figure 3, where the orientation of the surface matches the trigonometric orientation of the plan. Here, since there is a canonical bijective correspondence between the four oriented edges in $G$ adjacent to $e$ and the four oriented edges in $G'$ adjacent to $e'$, we use the same letter for each pair of corresponding edges. Note that it can happen for instance that $a$ and $c$ have the same underlying edge.

![Figure 3. Flip.](image)

Flips can be composed as morphisms of the fundamental path groupoid of the CW complex $\hat{G}$. We read composition of flips from right to left. There are three types of relations satisfied by compositions of flips:

- **Involutivity relation:** $W_e W_e = \text{id}$ in the notation of Figure 3.
- **Commutativity relation:** $W_{e_1} W_{e_2} = W_{e_2} W_{e_1}$ if $e_1$ and $e_2$ are edges sharing no vertices in a trivalent fatgraph spine.
- **Pentagon relation:** $W_{f_4} W_{g_3} W_{f_2} W_{g_1} W_f = \text{id}$ in the notation of Figure 4.

The involutivity relation comes from the fact that $W_{e'}$ is the oriented 1-cell of $\hat{G}$ obtained by reversing the orientation of $W_e$. The commutativity and pentagon relations come from the boundary of 2-cells of $\hat{G}$.

There is a homotopy equivalence between the Teichmüller space of the punctured surface $\Sigma$ and $\hat{G}$ which is equivariant under the action of the mapping class group of $\Sigma$. In particular, the space $\hat{G}$ is contractible. This implies the following basic facts:

- For any trivalent fatgraph spines $G$ and $G'$, there is a finite sequence of flips connecting $G$ to $G'$:
  $$G = G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} \cdots \xrightarrow{W_m} G_m = G'$$

- Any two sequences of flips connecting $G$ to $G'$ are related by a finite application of the three types of relations above.

2.3. **Homology marking.** We denote by $H$ the first integral homology group of $\Sigma$. Let $G$ be a fatgraph spine of $\Sigma$ and let $e$ be an oriented edge of $G$. Then there is an oriented simple loop $\hat{e}$ in $\Sigma$ such that $\hat{e}$ and $G$ have only one intersection which lies in the interior of $e$, and the ordered pair $(\hat{e}, e)$
matches the positive frame for the tangent space of $\Sigma$ at the intersection; see Figure 5. The homotopy class of $\hat{e}$ is well defined. We denote by

$$\mu(e) \in H$$

the homology class of $\hat{e}$ and call it the homology marking of $e$.

The homology marking satisfies the following properties:

1. For any oriented edge $e$ of $G$, we have $\mu(\overline{e}) = -\mu(e)$.
2. If $v$ is a vertex of $G$ and $e_1, \ldots, e_n$ are the oriented edges of $G$ pointing toward $v$, then $\sum_{i=1}^{n} \mu(e_i) = 0$.

**Example 2.2.**

1. If $v$ is a trivalent vertex of $G$ and the three oriented edges pointing toward $v$ are arranged as in Figure 6, we have

$$\mu(a_v) + \mu(b_v) + \mu(c_v) = 0.$$

2. In the notation of Figure 3, we have

$$\mu(a) + \mu(b) + \mu(c) + \mu(d) = 0.$$

**Figure 4.** Pentagon relation.

**Figure 5.** Homology marking: the loop $\hat{e}_1$ is dual to $e_1$. 
2.4. Penner’s cocycle. We keep the notation in Figure 3. For the flip $W_e$, we set

$$s(W_e) := (\mu(a) \wedge \mu(c)) \cdot (\mu(b) \wedge \mu(d)) \in S^2(\wedge^2 H).$$

Here, $\wedge^2 H$ is the second exterior power of $H$, and $S^2(\wedge^2 H)$ is the second symmetric power of $\wedge^2 H$. It is easy to see that $s(W_e) = -s(W_e)$. Since flips correspond to oriented 1-cells in $\hat{G}$, we can regard $s$ as a 1-cochain of $\hat{G}$ with values in $S^2(\wedge^2 H)$. It is in fact a 1-cocycle:

$$s \in Z^1(\hat{G}; S^2(\wedge^2 H)).$$

This means that $s$ respects the three types of relations among flips. The only nontrivial relation one has to check is the pentagon relation, and we must have

$$s(W_f) + s(W_{g_1}) + s(W_{g_2}) + s(W_{g_3}) + s(W_{g_4}) = 0$$

where we use the notation in Figure 4. A proof of this can be found in [5, §3, Theorem 5].

The cocycle $s$ was introduced by Penner in [10, Chapter 6, Remark 2.8].

3. Results

First we fix some conventions in §3.1. Then we state and prove our results.

3.1. Mapping class group and tensor modules of $H$. Let $M_{g,*}$ be the mapping class group of the punctured surface $\Sigma$. Namely, $M_{g,*}$ is the group of orientation preserving homeomorphisms of $\Sigma$ that preserve $*$ modulo isotopies that fix $*$. The group $M_{g,*}$ acts naturally as cellular automorphisms on the fatgraph complex $\hat{G}$. It acts also on $H = H_1(\Sigma; \mathbb{Z})$ and hence on tensor modules of $H$ such as $\wedge^2 H$ and $S^2(\wedge^2 H)$.

The cocycle $s$ is $M_{g,*}$-equivariant in the sense that

$$\varphi \cdot s(W_e) = s(W_{\varphi(e)})$$

for any flip $W_e$ and $\varphi \in M_{g,*}$.

We regard $\wedge^2 H$ as a submodule of $H \otimes H$ defined as the image of the homomorphism

$$H \otimes H \longrightarrow H \otimes H, \quad x \otimes y \mapsto x \wedge y := x \otimes y - y \otimes x,$$

and $S^2(\wedge^2 H)$ as a submodule of $\wedge^2 H \otimes \wedge^2 H$ defined as the image of the homomorphism

$$\wedge^2 H \otimes \wedge^2 H \longrightarrow \wedge^2 H \otimes \wedge^2 H, \quad u \otimes v \mapsto u \cdot v := u \otimes v + v \otimes u.$$
Hence we can think of $S^2(\wedge^2 H)$ as a submodule of $H^\otimes 2 \otimes H^\otimes 2 = H^\otimes 4$.

The first homology group $H = H_1(\Sigma; \mathbb{Z})$ is equipped with a non-degenerate skew-symmetric bilinear form

$$H^\otimes 2 \longrightarrow \mathbb{Z}, \quad x \otimes y \mapsto (x \cdot y)$$

called the intersection form. Note that it sends

$$x \wedge y \mapsto 2(x \cdot y)$$

for any $x, y \in H$.

### 3.2. A cobounding cochain for Penner’s cocycle.

Let $G$ be a trivalent fatgraph spine of $\Sigma$ and let $v$ be a trivalent vertex of $G$. In the notation in Figure 6, it holds that

$$\mu(a_v) \wedge \mu(b_v) = \mu(b_v) \wedge \mu(c_v) = \mu(c_v) \wedge \mu(a_v)$$

since $\mu(a_v) + \mu(b_v) + \mu(c_v) = 0$ as explained in Example 2.2 (1). We put

$$\eta_v := \mu(a_v) \wedge \mu(b_v) \in \wedge^2 H.$$

**Definition 3.1.**

$$\xi_G := \frac{1}{2} \sum_v \eta_v \cdot \eta_v = \sum_v \eta_v \otimes \eta_v.$$ 

Here, the sum is taken over all trivalent vertices of $G$.

A priori, $\xi_G$ is an element of $(1/2)S^2(\wedge^2 H)$. The collection $\xi = \{\xi_G\}_G$ can be regarded as a 0-cochain of $\widehat{G}$ with values in $(1/2)S^2(\wedge^2 H)$.

**Theorem 3.2.** For any fatgraph spine $G$ of $\Sigma$, we have $\xi_G \in S^2(\wedge^2 H)$. The 0-cochain $\xi \in C^0(\widehat{G}; S^2(\wedge^2 H))$ cobounds the 1-cocycle $s$ and is $M_{g, s}$-equivariant in the sense that $\varphi \cdot \xi_G = \xi_{\varphi(G)}$ for any $G$ and $\varphi \in M_{g, s}$.

**Proof.** That $\xi$ is $M_{g, s}$-equivariant is clear from construction.

Next we prove that $d \xi = s$. We need to show that $\xi_{G'} - \xi_G = s(W_e)$, where we use the notation in Figure 3. For simplicity, we write $\mu(a) = a$, etc. Note that there is a canonical bijection between the set of vertices of $G$ that are different from the endpoints of $e$ and the set defined in the same way for $G'$, and this bijection preserves the value of $\eta$. Thus we have

$$2(\xi_{G'} - \xi_G) = \eta \cdot \eta + \eta \cdot \eta = (b \wedge c)^2 - (a \wedge b)^2 - (c \wedge d)^2,$$

where $(b \wedge c)^2 = (b \wedge c) \cdot (b \wedge c)$, etc. The right hand side is equal to

$$(b \wedge c + a \wedge b) \cdot (b \wedge c) + (d \wedge a + c \wedge d) \cdot (d \wedge a - c \wedge d).$$

By Example 2.2 (2), we have $a + b + c + d = 0$. Hence we have

$$b \wedge c + a \wedge b = b \wedge (a + c) = -b \wedge (b + d) = -b \wedge d,$$

and similarly $d \wedge a - c \wedge d = b \wedge d$. Therefore,

$$2(\xi_{G'} - \xi_G) = (b \wedge c - a \wedge b + d \wedge a + c \wedge d) \cdot (b \wedge d)$$

$$= (-a \wedge (b + d) + (b + d) \wedge c) \cdot (b \wedge d)$$

$$= (a \wedge (a + c) + (a + c) \wedge c) \cdot (b \wedge d)$$

$$= 2(a \wedge c) \cdot (b \wedge d)$$

$$= 2s(W_e).$$
In the third line, we have used again the relation $a + b + c + d = 0$. Since $S^2(\wedge^2 H)$ is torsion free, we obtain $\xi_G - \xi_G = s(W_e)$.

Finally we prove that $\xi_G \in S^2(\wedge^2 H)$ for any $G$. Consider the trivalent fatgraph spine $G_0$ in Example 3.3 below. Then, as we will see, it holds that $\xi_{G_0} \in S^2(\wedge^2 H)$. Let $G$ be any trivalent fatgraph spine of $\Sigma$. Take a sequence of flips $G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} \cdots \xrightarrow{W_m} G_m = G$ connecting $G_0$ to $G$. Then from $\delta \xi = s$ we obtain $\xi_G = \sum_{j=1}^m s(W_j)$, which shows that $\xi_G \in S^2(\wedge^2 H)$ as well.

Example 3.3. Let $G_0$ be the trivalent fatgraph spine as in the left part of Figure 7 and fix a symplectic basis $\{a_i, b_i\}_{i=1}^g$ for $H$ as shown in the right part of the same figure. Then, we have $\mu(e_i) = -b_i$, $\mu(e'_i) = a_i$, and $\mu(e''_i) = b_i$. Thus $\eta_{v_1} = \mu(e_i) \wedge \mu(e'_i) = a_i \wedge b_i$ and $\eta_{v_2} = \mu(e''_i) \wedge \mu(e'_i) = a_i \wedge b_i$. The value of $\eta$ on the other vertices are zero, and hence

$$\xi_{G_0} = \sum_{i=1}^g (a_i \wedge b_i) \cdot (a_i \wedge b_i).$$

In particular, we see that $\xi_{G_0} \in S^2(\wedge^2 H)$.

![Figure 7. The fatgraph spine $G_0$ and a symplectic basis $(g = 2)$.](image)

Remark 3.4. (1) Since the module of $\mathcal{M}_{g,*}$-invariant tensors in $S^2(\wedge^2 H)$ is non-trivial (it is of rank 2 if $g$ is at least 4; see [8, Lemma 4.1] for example), the $\mathcal{M}_{g,*}$-equivariant cobounding cochains for $s$ are not unique.

(2) The cochain $\xi$ can be considered as a secondary invariant associated with the vanishing of the cohomology class $[s_G]$ which will be explained in the next subsection. In [4], another secondary invariant associated with the vanishing of a certain cohomology class in $H^1(\mathcal{M}_{g,*}; H)$ was studied. In that case, the uniqueness of the secondary invariant holds since there are no non-trivial $\mathcal{M}_{g,*}$-invariant elements in $H$.

3.3. Cohomology vanishing. Recall that the cocycle $s$ is $\mathcal{M}_{g,*}$-equivariant. In this situation, given a trivalent fatgraph spine $G$ of $\Sigma$ one can construct a twisted 1-cocycle

$$s_G: \mathcal{M}_{g,*} \longrightarrow S^2(\wedge^2 H).$$

Explicitly, for $\varphi \in \mathcal{M}_{g,*}$ we take a sequence of flips $G = G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} \cdots \xrightarrow{W_m} G_m = \varphi(G)$
which connects $G$ to $\varphi(G)$, and set

$$s_G(\varphi) := \sum_{i=1}^{m} s(W_i).$$

The twisted cohomology class

$$[s_G] \in H^1(M_{g,*}; S^2(\wedge^2 H))$$

is independent of the choice of $G$. A more detailed and general construction is explained in [5, the proof of Theorem 5] and in [4, §2]. This combinatorial construction of cocycles of $M_{g,*}$ from $M_{g,*}$-equivariant cocycles of $\hat{G}$ originally dates back to the work of Morita and Penner [9].

In [5, §3], it was shown that $[s_G]$ vanishes over the rational coefficients. As a consequence of Theorem 3.2, we see that the vanishing of $[s_G]$ holds over the integral coefficients.

**Corollary 3.5.** $[s_G] = 0 \in H^1(M_{g,*}; S^2(\wedge^2 H))$.

**Proof.** Let $\varphi \in M_{g,*}$. By construction and by Theorem 3.2, we have $s_G(\varphi) = \xi_{\varphi(G)} - \xi_G$. Since the 0-cochain $\xi$ is $M_{g,*}$-equivariant, the right hand side is equal to $\varphi \cdot \xi_G - \xi_G$. Hence $s_G = \delta \xi_G$ and $[s_G] = 0$. \hfill \Box

### 3.4. Contractions of Penner’s cocycle.

Using the intersection form on $H$, we introduce the contraction maps

$$\text{Cont}_{1,2}: H^{\otimes 4} \rightarrow H^{\otimes 2}, \quad x \otimes y \otimes z \otimes w \mapsto (x \cdot y) z \otimes w$$

and

$$\text{Cont}_{1,3}: H^{\otimes 4} \rightarrow H^{\otimes 2}, \quad x \otimes y \otimes z \otimes w \mapsto (x \cdot z) y \otimes w.$$ 

Their restrictions to $S^2(\wedge^2 H) \subset H^{\otimes 4}$ take values in $\wedge^2 H$. We consider 1-cocycles

$$s' := \text{Cont}_{1,2} \circ s \quad \text{and} \quad s'' := \text{Cont}_{1,3} \circ s.$$ 

Explicitly, their values on the flip $W_e$ are given as follows:

$$s'(W_e) = 2 ((a \cdot c) b \wedge d + (b \cdot d) a \wedge c),$$

$$s''(W_e) = (a \cdot b) c \wedge d + (a \cdot d) b \wedge c + (c \cdot d) a \wedge b + (b \cdot c) a \wedge d.$$ 

Here and in the rest of this section, we simply write $\mu(a) = a$, etc.

**Proposition 3.6.** We have $s' = 2s'' \in Z^1(\hat{G}; \wedge^2 H)$.

**Proof.** We work with Figure 3 and prove the equality $s'(W_e) = 2s''(W_e)$. We need to take into account possible cyclic orderings among oriented edges $a, b, c, d$. By changing the role of $a$ and $c$ and the role of $G$ and $G'$, it is sufficient to consider the following six cases:

(i) $a \prec b \prec c \prec d \prec a$;

(ii) $a \prec b \prec d \prec c \prec a$;

(iii) $a \prec c \prec b \prec d \prec a$;

(iv) $a \prec c \prec d \prec b \prec a$;

(v) $a \prec d \prec b \prec c \prec a$;

(vi) $a \prec d \prec c \prec b \prec a$.

We only consider the cases (i) and (ii), the other cases being similar.

Case (i). From Figure 8, we see that all the intersection numbers $(a \cdot c), (b \cdot d), (a \cdot b), (a \cdot d), (c \cdot d),$ and $(b \cdot c)$ are zero, and thus $s'(W_e) = s''(W_e) = 0$. 

Case (ii). From Figure 8, we have 
\[(a \cdot c) = 1, (b \cdot d) = 0, (a \cdot b) = 0, (a \cdot d) = -1, (c \cdot d) = +1, (b \cdot c) = 0\]. Therefore, 
\[s'(W_e) = 2b \wedge d, \text{ and} \]
\[s''(W_e) = -b \wedge c + a \wedge b = -b \wedge (a + c) = b \wedge (b + d) = b \wedge d. \]
Thus 
\[s'(W_e) = 2s''(W_e), \text{ as required.} \]

\[\text{Figure 8. Case (i) and Case (ii).}\]

3.5. Cobounding cochains for \(s'\). By Theorem 3.2,
\[\xi' := \text{Cont}_{1,2} \circ \xi \in C^0(\hat{G}; \wedge^2 H)\]
is an \(\mathcal{M}_{g,*}\)-equivariant 0-cochain which cobounds the 1-cocycle \(s'\). Let \(G\) be a trivalent fatgraph spine of \(\Sigma\). To describe \(\xi'_G\) explicitly, we recall the following definition from [4, §3].

\[\text{Definition 3.7. Let } v \text{ be a trivalent vertex of } G \text{ and use the notation in Figure 6. The vertex } v \text{ is called of type 1 if } a_v < b_v < c_v < a_v; \text{ and of type 2 if } a_v < c_v < b_v < a_v. \]

Observe the following:
\[\begin{align*}
(a_v \cdot b_v) &= (b_v \cdot c_v) = (c_v \cdot a_v) = 0 \text{ if } v \text{ is of type 1; } \\
(a_v \cdot b_v) &= (b_v \cdot c_v) = (c_v \cdot a_v) = 1 \text{ if } v \text{ is of type 2.}
\end{align*}\]
By (3.1) and \(\xi_G = \sum_v \eta_v \otimes \eta_v = \sum_v (a_v \wedge b_v) \otimes (a_v \wedge b_v)\), we have
\[\xi'_G = 2 \sum_v (a_v \cdot b_v) a_v \wedge b_v = 2 \sum_{v: \text{type 2}} a_v \wedge b_v. \]

The \(\mathcal{M}_{g,*}\)-equivariant cobounding cochains for the 1-cocycle \(s'\) are not unique. Let us consider another \(\mathcal{M}_{g,*}\)-equivariant cochain \(\tilde{\xi} = \{\tilde{\xi}_G\}_G \in C^0(\hat{G}; \wedge^2 H)\) defined by
\[\tilde{\xi}_G := - \sum_{v: \text{type 1}} a_v \wedge b_v + \sum_{v: \text{type 2}} a_v \wedge b_v. \]

\[\text{Lemma 3.8. } \delta \tilde{\xi} = s'. \]
Proof. We work with Figure 3 and prove \( \tilde{\xi}_{G'} - \tilde{\xi}_G = s'(W_e) \). We consider the six cases in the proof of Proposition 3.6. We only treat the first two cases, the other cases being similar.

Case (i). All the trivalent vertices in Figure 3 are of type 1, and hence

\[
\tilde{\xi}_{G'} - \tilde{\xi}_G = -b \wedge c - d \wedge a - (a \wedge b - c \wedge d) = (a + c) \wedge (b + d) = 0.
\]

Since \( s'(W_e) = 0 \) as shown in the proof of Proposition 3.6, we obtain \( \tilde{\xi}_{G'} - \tilde{\xi}_G = s'(W_e) \).

Case (ii). Suppose \( a \prec b \prec d \prec c \prec a \). Let \( v_1 \) be the vertex of \( G \) adjacent to \( a, b, e \) and \( v_2 \) the one adjacent to \( c, d, e \). Similarly, let \( v_1' \) be the vertex of \( G' \) adjacent to \( b, c, e' \) and \( v_2' \) the one adjacent to \( a, d, e' \). Then \( v_1 \) and \( v_1' \) are of type 1, and \( v_2 \) and \( v_2' \) are of type 2. Thus we compute

\[
\tilde{\xi}_{G'} - \tilde{\xi}_G = -b \wedge c + d \wedge a - (a \wedge b + c \wedge d)
= -b \wedge (a + c) - (a + c) \wedge d
= b \wedge (b + d) + (b + d) \wedge d
= 2b \wedge d.
\]

As was shown in the proof of Proposition 3.6, we have \( s'(W_e) = 2b \wedge d \). This completes the proof.

\[\]}

3.6. A combinatorial formula for the symplectic form. Let \( \omega \in \wedge^2 H \) be the symplectic form on \( H = H_1(\Sigma; \mathbb{Z}) \). This is the element corresponding to the identity \( 1_H \in \text{Hom}_\mathbb{Z}(H,H) \) through the isomorphism

\[
H \otimes H \cong \text{Hom}_\mathbb{Z}(H,H), \quad x \otimes y \mapsto (z \mapsto (x \cdot z)y)
\]

defined by the intersection form, where \( x, y, z \in H \). Explicitly, if \( \{a_i, b_i\}_{i=1}^g \) is a symplectic basis as in Example 3.3, then

\[
\omega = \sum_{i=1}^g a_i \wedge b_i.
\]

It is a generator of the module of \( M_{g,*} \)-invariants in \( \wedge^2 H \).

Theorem 3.9. Let \( G \) be a trivalent fatgraph spine of \( \Sigma \) and keep the notations above. Then we have

\[
\omega = \frac{1}{2} \sum_{v} \eta_v = \frac{1}{2} \sum_{v} a_v \wedge b_v,
\]

where the sum is taken over all trivalent vertices of \( G \).

Proof. As we have seen in §3.5, \( \xi' \) and \( \tilde{\xi} \) are cobounding cochains for the same 1-cocycle \( s' \). Thus their difference is a 0-cocycle:

\[
\zeta := \xi' - \tilde{\xi} \in Z^0(\tilde{G}; \wedge^2 H).
\]

This means that the value \( \zeta_G \) is invariant under flips and hence is independent of \( G \). Let us denote this value by \( \zeta_0 \in \wedge^2 H \). Then \( \zeta_0 \) is \( M_{g,*} \)-invariant since \( \zeta \) is \( M_{g,*} \)-equivariant. Thus we can write \( \zeta_0 = m \omega \) for some \( m \in \mathbb{Z} \).

To determine \( m \), we use the intersection form \( H^{\otimes 2} \to H \) restricted to \( \wedge^2 H \). On the one hand, from (3.2) and (3.3) we see that for any \( G \)

\[
\zeta_0 = \xi'_G - \tilde{\xi}_G = \sum_v a_v \wedge b_v.
\]
As shown in [4, Proposition 3.3], there are $2g$ vertices of type 2. (In [4] one considers fatgraph spines with tail, but notice that the trivalent vertex adjacent to the tail is of type 1.) Therefore, the value of the intersection form for $\zeta_0$ is $2\sum_v(a_v \cdot b_v) = 2\#\{$vertices of type 2$\} = 4g$. On the other hand, the value of the intersection form for $\omega$ is $2g$. Therefore, we conclude that $m = 2$ and $\zeta_0 = 2\omega$. This completes the proof. □

Remark 3.10. (1) Our formula for $\omega$ works for a fatgraph spine with tail as well. We note that the vertex next to the tail does not contribute to the sum $(1/2)\sum_v \eta_v$ since the homology marking of the tail is trivial.

(2) In [6], Massuyeau constructed a 2-chain $Z_G$ in the normalized bar complex of $\pi := \pi_1(\Sigma_{g,1})$, where $\Sigma_{g,1}$ is a once bordered surface of genus $g$ and $G \subseteq \Sigma_{g,1}$ is a fatgraph spine with tail. By the canonical homomorphism $\pi \to H_1(\Sigma_{g,1}) \cong H$, the 2-chain $Z_G$ is mapped to a normalized 2-cocycle $Z_G'$ of $H$. Massuyeau points out that $Z_G'$ has a similar expression to our bivector $(1/2)\sum_v \eta_v$ and its homology class in $H_2(H) \cong \Lambda^2 H$ coincides with $\omega$.

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