MULTIPLIER IDEALS OF SUFFICIENTLY GENERAL POLYNOMIALS

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ABSTRACT. It is well known that the multiplier ideal $J(r \cdot a)$ of an ideal $a$ determines in a straightforward way the multiplier ideal $J(r \cdot f)$ of a sufficiently general element $f$ of $a$. We give an explicit condition on a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ which guarantees that it is a sufficiently general element of the most natural associated monomial ideal, the ideal generated by its terms. This allows us to directly calculate the multiplier ideal $J(r \cdot f)$ (for all $r$) of “most” polynomials $f$.

1. INTRODUCTION

The multiplier ideal of a divisor $D$ or ideal sheaf $a$ tells us a great deal about the singularities of $D$, or of a general section of $a$. It is also a main ingredient of the Nadel vanishing theorem, which of course has found numerous important applications. The multiplier ideal is a powerful tool because it provides “invariants with vanishing.” In this paper we will compute the multiplier ideals of a large class of divisors in $\mathbb{A}^n$, those defined by “nondegenerate” polynomials. This nondegeneracy condition is fairly weak: It is satisfied, for example, by functions of the form $c_1 x_1^{a_1} + c_2 x_2^{a_2} + \ldots + c_n x_n^{a_n}$.

Notation 1. We write $J(r \cdot a)$ for the multiplier ideal of an ideal $a$, with constant $r$. We write $J(r \cdot D)$ for the multiplier ideal of a divisor $D$. For $f \in \mathbb{C}[x_1, \ldots, x_n]$, we abbreviate $J(r \cdot \text{Div}(f))$ by “$J(r \cdot f)$,” although “$J(f^r)$” would be more suggestive.

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1 see [Arnol’d et al., 1985] for earlier uses of nondegeneracy.
We will not develop the basic properties of multiplier ideals. We refer the reader instead to [Lazarsfeld, 2001] for a comprehensive algebraic development.

The algebraic definition of the multiplier ideal refers to an embedded resolution of $D$ or $a$, making it difficult to calculate directly in specific examples. In a previous paper [Howald, 2001b], we calculated the multiplier ideal of an arbitrary monomial ideal $a$ in a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. It happens that if $a \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal and $f \in a$ is a generic section of $a$, then the multiplier ideals of $\text{Div}(f)$ and $a$ are essentially the same:

**Theorem 2.** Let $a \subset \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal, and let $f \in a$ be a generic section. If $r < 1$ then

$$\mathcal{J}(r \cdot (f)) = J(r \cdot a).$$

See [Lazarsfeld, 2001, Proposition 9.2.29] for a proof. If $r \geq 1$, the relationship is only slightly more complicated.

This begs the question: When is a given function a “generic enough” element of some related monomial ideal? We give a sufficient condition which guarantees that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a sufficiently generic element of the (monomial) ideal $\tau(f)$ generated by its terms. This allows us to calculate the multiplier ideal of $f$.

Roughly, this condition requires that for every face of the Newton polyhedron of $f$, the function obtained by summing just those terms of $f$ which lie on that face should have nonvanishing derivative on the torus $T^n \subset \mathbb{A}^n$. (For each face $\sigma$ of the Newton polyhedron, we call this “$\sigma$-nondegeneracy.”) We prove that this guarantees equality of $\mathcal{J}(r \cdot f)$ and $\mathcal{J}(r \cdot \tau(f))$ in four steps:

1. There is a toric log resolution $\mu : X' \to X$, which principalizes $\tau(f)$.
2. By toric geometry, $X'$ is locally isomorphic to $\mathbb{A}^n$, so we may regard $\mu^*(f) = f'$ as a polynomial. We prove that it inherits nondegeneracy from $f$.
3. Since $f'$ is nondegenerate and $\tau(f')$ is principal, $\text{Div}(f')$ is a normal crossing divisor, and $\mu$ is a simultaneous toric log resolution for both $\tau(f)$ and $\text{Div}(f)$.
4. Direct comparison of the terms $r\mu^*(\text{Div}(f))$ and $r\mu^{-1}(\tau(f))$ appearing in the definition of the multiplier ideal shows them to be the same.

meaning a generic $\mathbb{C}$-linear combination of generators $\{g_1, \ldots, g_k\}$ of $a$. 

2. **Main Theorem and Examples**

The nondegeneracy condition which we will use describes the relationship of a polynomial \( f \) with the faces of its Newton polyhedron. We must first introduce some definitions and notation for dealing with such structures.

**Definition 3.** Let \( X = \mathbb{C}^n \). In questions of membership in monomial ideals, the coefficient of a monomial \( m \in \mathbb{C}[x_1, \ldots, x_n] \) is irrelevant. Discarding this information, we will refer to the lattice \( L_X \cong \mathbb{Z}^n \) of rational monomials on \( X \), whose nonnegative orthant contains the monomials. Let \( a \subset \mathcal{O}_X \) be a monomial ideal. We may identify \( a \) with a subset of \( L_X \). The *Newton polyhedron* \( P(a) \) of \( a \) is the convex hull of this set, an unbounded closed subset of \( L_X \otimes \mathbb{R} \). By a *face* \( \sigma \) of the Newton polyhedron, we mean a face of arbitrary codimension (even the codimension-0 face). Because the polyhedron is unbounded, faces need not be compact.

**Definition 4.** Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial. The *term ideal* \( \tau(f) \) of \( f \) is the monomial ideal generated by the terms of \( f \). We write \( P(f) \) for \( P(\tau(f)) \), though this is not the usual definition of the Newton polyhedron of a polynomial.

We can now introduce the main nondegeneracy condition which will guarantee that \( f \) and \( \tau(f) \) have similar multiplier ideals.

**Definition 5.** Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \), and let \( \sigma \) be a face of \( P(f) \). We write \( f_\sigma \) for the polynomial composed of the terms of \( f \) which lie in \( \sigma \). Note that \( f_\sigma \) remembers the coefficient of each term. If the 1-form \( df_\sigma \) is nonvanishing on the torus \( T_n = (\mathbb{C} \setminus 0)^n \), then we say that \( f \) is *nondegenerate* for \( \sigma \). If \( f \) is nondegenerate for every face of its Newton polyhedron, then we call \( f \) simply *nondegenerate*. If \( f \) is nondegenerate for just its compact faces, then we say \( f \) has *nondegenerate principal part*.

This last condition follows [Arnol’d et al., 1985, 6.2.2], in which it is used (from the analytical point of view) in the calculation of the log canonical threshold of \( f \). Notice that if \( \sigma \) is the origin in \( L_X \), then \( f_\sigma \) is a nonzero constant. In this special case, we call \( f \) nondegenerate for \( \sigma \) despite the vanishing of \( df_\sigma \). At the other extreme, \( \sigma \) may be the entire Newton polyhedron, in which case nondegeneracy for \( \sigma \) implies that \( \text{Div}(f|_{T^n}) \) is nonsingular. (If this condition seems too strong, remember that we will allow ourselves only toric resolutions, and these are isomorphisms on \( T^n \).)
We will soon see that nondegeneracy of $f$ implies convenient global properties for $\text{Div}(f)$, whereas nondegeneracy of the principal part tells us that $\text{Div}(f)$ is “nice” only near the origin.

**Definition 6.** Let $X = \mathbb{C}^n$, and let $\mathfrak{a}$ be an ideal sheaf on $X$. By a *toric log resolution* of $\mathfrak{a}$, we mean a log resolution $\mu : X' \to X$ which is obtained by a sequence of blowings-up along orbits of the torus action. Equivalently, it is a log resolution for which $X'$ and $\mu$ belong to the toric category. We define a toric log resolution of a divisor similarly.

Locally on $X'$, such a map is determined by a monomial ring map $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[y_1, \ldots, y_n]$. A toric log resolution of $\mathfrak{a}$ can also be described in terms of refinements of the dual polyhedron of $P(\mathfrak{a})$. See [Howald, 2001b] for more on this perspective.

**Proposition 7.** Every monomial ideal has a toric log resolution.

See [Howald, 2001a] for a sketch of the proof, or [Fulton, 1993] for the basic theory of toric varieties.

**Example 8.** The functions $f = y^2 - y(x - 1)^2$, $g = (xy - 1)^9$, and $h = (x + y)^2 - (x - y)^5$ on $\mathbb{C}^2$ are all degenerate. Although $df$ is nonzero on the torus, there is a proper face $\sigma \subset P(f)$ for which $f_\sigma = y(x - 1)^2$. Clearly $df_\sigma = 0$ at the point $(1,0)$. $\text{Div}(f)$ has a singularity at $(1,0)$, but toric resolutions up can only blow up the origin so the singularity cannot be corrected by such maps. Nevertheless, $f$ has nondegenerate principal part.

For every proper face $\sigma$ of $P(g)$, $g_\sigma$ is nondegenerate. But $dg$ itself vanishes at points in the torus, so $g$ is degenerate. Although $\text{Div}(g)$ has a toric log resolution, the high multiplicity of $g$ along the proper transform of $\text{Div}(g)$ complicates the calculation of the multiplier ideal.

For properly chosen $\sigma$, $h_\sigma$ is a perfect square $(x + y)^2$, and is clearly degenerate. Despite being singular only at the origin, $h$ has no toric log resolution. The multiplier ideal of $\text{Div}(h)$ is not monomial, even for some $r < 1$. This function has degenerate principal part.

**Example 9.** Let $f = c_1x_1^{a_1} + \ldots + c_nx_n^{a_n}$, with $c_i \neq 0$ and $a_i > 0$. Then $f$ is nondegenerate, as the reader may check. Furthermore, if the simplex spanned by $\{a_ie_i\}_{i \in \{1 \ldots n\}}$ (the unique maximal compact face of $P(f)$) contains no lattice points other than its vertices, then any polynomial with the same Newton polyhedron as $f$ has nondegenerate principal part.

**Notation 10.** We write $J(r \cdot \mathfrak{a})$ or $J(r \cdot D)$ for the multiplier ideal of an ideal or divisor, with rational coefficient $r$. We also write $\lfloor r \rfloor$ and $\{r\}$ for the round-down and fractional part of a rational number $r$. We
will write \([D]\) for the round-down of a \(\mathbb{Q}\)-divisor \(D\), and \([D]\) for its round-up.

The multiplier ideal of a monomial ideal can be calculated combinatorially:

**Theorem 11.** Let \(a \subset \mathcal{O}_{\mathbb{A}^n}\) be a monomial ideal. Then \(\mathcal{J}(r \cdot a)\) is a monomial ideal, and contains exactly the following monomials:

\[
\{ m : mx_1x_2 \ldots x_n \in \text{Interior}(rP(a)) \}.
\]

**Proof.** See [Howald, 2001a] or [Howald, 2001b]. □

For small values of \(r\), we can hope for the equality \(\mathcal{J}(r \cdot \text{Div}(f)) = \mathcal{J}(r \cdot \tau(f))\), provided that \(f\) is a sufficiently generic element of \(\tau(f)\). This requires a Bertini argument which shows that a toric log resolution \(\mu\) of \(\tau(f)\) is in fact a log resolution of a general section of \(\tau(f)\). The divisors \(\mu^*(\text{Div}(f))\) and \(\mu^{-1}(a)\) differ only by the proper transform of \(\text{Div}(f)\), which disappears on rounding if \(r < 1\).

So a claim that \(f\) is sufficiently general primarily means that any toric log resolution of \(\tau(f)\) also resolves \(\text{Div}(f)\). Our result is in essence an effective substitute for the use of Bertini’s theorem to guarantee this. Basic facts about multiplier ideals will take care of the rest of the calculation. Here then is our main theorem.

**Theorem 12** (Main Theorem). Let \(f \in \mathbb{C}[x_1, \ldots, x_n]\). Let \(\mu : X' \to \mathbb{C}^n\) be a toric log resolution of \(\tau(f)\). If \(f\) is nondegenerate, then \(\mu\) also log-resolves \(\text{Div}(f)\). If \(f\) has nondegenerate principal part, then there is a Zariski neighborhood \(U\) of the origin over which \(\mu\) is a log resolution of \(f|_U\).

We will prove this theorem in the next section.

**Corollary 13.** Let \(f \in \mathbb{C}[x_1, \ldots, x_n]\) be nondegenerate, and let \(r < 1\). Then

\[
\mathcal{J}(r \cdot \text{Div}(f)) = \mathcal{J}(r \cdot \tau(f)).
\]

**Proof** (12 ⇒ 13). Since we have a toric log resolution \(\mu : X' \to X = \mathbb{C}^n\) of both \(f\) and \(\tau(f)\), we may use it to calculate each of these multiplier ideals. Let \(K_{X'/X}\) be the relative canonical bundle, and consider the divisors

\[
F_1 = K_{X'/X} - [r\mu^*\text{Div}(f)] \quad \text{and} \quad F_2 = K_{X'/X} - [r\mu^{-1}(\tau(f))],
\]

which when pushed forward via \(\mu_*\) calculate the two multiplier ideals. We claim that these divisors are equal. Because the resolution is toric, each exceptional divisor \(E\) of \(\mu\) corresponds to a weighted blowup. The order along \(E\) of \(\mu^*\text{Div}(f)\) is the minimal weight of the terms of \(f\), and
the order along $E$ of $\mu^{-1}(\tau(f))$ is the minimal weight of monomials in $\tau(f)$. These are obviously the same, so $F_1$ and $F_2$ have the same order along each exceptional divisor. Clearly they also agree on the proper transforms of the coordinate axes.

We have tested equality on all divisors not intersecting the torus $T_n$. No other divisors can appear in the support of $F_2$ because $\tau(f)$ is monomial. We must similarly rule out components of $F_1$ which intersect the torus. Suppose $D'$ were one such. Since $\mu$ is an isomorphism on the torus, $\mu(D')$ would be a divisor $D$ and $\text{ord}_D(\mu^*(f)) = \text{ord}_D(f) = 1$ because $df|_{T_n}$ does not vanish. To calculate $\text{ord}_D F_1$, we would multiply this number by $r$ and round down, giving 0 because $r < 1$. □

**Corollary 14.** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be nondegenerate. Then

$$\mathcal{J}(r \cdot \text{Div}(f)) = (f^{|r|}) \mathcal{J}({|r|} \tau(f)).$$

**Proof.** The integer divisor $|r| \text{Div}(f)$ pulls outside the multiplier ideal (see [Howald, 2001a]), so we have

$$\mathcal{J}(r \cdot \text{Div}(f)) = (f^{|r|}) \mathcal{J}(|r| \text{Div}(f)).$$

Corollary 13 finishes the proof. □

**Example 15.** As in Example 9, let $f = c_1 x_1^{a_1} + \ldots + c_n x_n^{a_n}$, with $c_i \neq 0$ and $a_i > 0$. Let $v = (\frac{1}{a_1}, \ldots, \frac{1}{a_n})$, so that $m \in P(f)$ if and only if $v \cdot m \geq 1$. (Here we regard $m$ as a vector in $L_X$ and take its dot product with $v$.) Theorem 11 tells us that $\mathcal{J}(r \cdot \tau(f))$ is generated by $
omb{m : v \cdot (m x_1 x_2 \ldots x_n) > r}$.

By Corollary 14

$$\mathcal{J}(r \cdot \text{Div}(f)) \text{ is generated by } \{f^{|r|}m : v \cdot (m x_1 x_2 \ldots x_n) > |r|\}$$

3. PROOF OF MAIN THEOREM

We will require a few more definitions and notations before we can comfortably prove the main theorem. Let $X = \mathbb{C}^n$.

**Definition 16.** Let $a \subset \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal, and let $\sigma \subset P(a)$ be a face of its Newton polyhedron. Let $v_\sigma$ be any linear functional on $L_X$ which cuts out $\sigma$ in the sense that $v_\sigma(P(a)) = [b, \infty)$ and $v_\sigma^{-1}(b) \cap P(a) = \sigma$. We may regard $v$ as a hyperplane distinguishing $\sigma \subset P(a)$. Let $\mathcal{R}(\sigma)$ be the ring generated by all monomials $m$ with $v_\sigma(m) = 0$. Because $v_\sigma$ is a nonnegative vector (when written in the standard dual basis), $\mathcal{R}(\sigma)$ is of the form $\mathbb{C}[S]$ for some $S \subseteq \{x_1, \ldots, x_n\}$. Let $\mathcal{L}(\sigma) = \text{Spec}(\mathcal{R}(\sigma)) \subset \mathbb{C}^n$. Notice that $\mathcal{L}(\sigma)$ is just a coordinate linear subspace of $\mathbb{C}^n$. By the *relative interior* of $\mathcal{L}(\sigma)$, we will mean the points of $\mathcal{L}(\sigma)$ not contained in some smaller coordinate linear subspace.
The reader may check that although \( v_\sigma \) is not completely determined by \( \sigma \), \( \mathcal{R}(\sigma) \) and \( \mathcal{L}(\sigma) \) are. Readers familiar with valuations will notice that \( v_\sigma \) is a valuation on the ring \( \mathbb{C}[x_1, \ldots, x_n] \), and \( \mathcal{L}(\sigma) \) is its center.

**Example 17.** Let \( a = (x^3, y^3, z^3) \subset \mathbb{C}[x, y, z] \). The triangular face \( \sigma \) of \( P(a) \) is determined by \( v_\sigma = (1/3, 1/3, 1/3) \). For this face and in fact for all of its subfaces, \( v_\sigma \) is strictly positive, so \( \mathcal{R}(\sigma) = \mathbb{C} \) and \( \mathcal{L}(\sigma) \) is the origin. On the other hand, if we take \( \sigma \) to be the intersection of \( P(a) \) with the \( x \)-axis in the lattice of monomials, then \( \mathcal{R}(\sigma) = \mathbb{C}[x] \), and \( \mathcal{L}(\sigma) \) is the \( x \)-axis in \( \mathbb{C}^3 \).

**Proposition 18.** If, as in Example 17, \( \mathcal{L}(\sigma) \) is the origin, then \( \sigma \) is compact.

**Proof.** If \( \mathcal{L}(\sigma) \) is the origin, then \( \mathcal{R}(\sigma) = \mathbb{C} \), so \( v_\sigma \) is a strictly positive dual vector. From the above definition, \( v_\sigma^{-1}(b) \cap P(a) = \sigma \). Since \( v_\sigma^{-1}(b) \) has compact intersection with the nonnegative orthant in \( L_X \), and since \( P(a) \) is closed, \( \sigma \) itself must be compact.

**Definition 19.** Let \( X = \text{Spec} \mathbb{C}[x_1, \ldots, x_n] \), let \( X' = \text{Spec} \mathbb{C}[y_1, \ldots, y_n] \), and let \( \mu : X' \to X \) be a monomial map. If \( a \subset O_X \) is a monomial ideal, then so is \( a' =_{\text{def}} \mu^{-1} a \). Let \( \sigma' \) be a face of \( P(a') \). We write \( \mu(\sigma') \) for \( \{m \in L_X : \mu^*(m) \in \sigma' \} \). It is the preimage of \( \sigma' \) under \( \mu^* \).

**Proposition 20.** For any face \( \sigma' \) of \( P(a') \), \( \mu(\sigma') \) is a face of \( P(a) \).

**Proof.** Whereas \( \sigma' \) is the intersection of \( P(a') \) with the level set of some linear functional \( v_{\sigma'} \), \( \sigma =_{\text{def}} \mu(\sigma') \) is the intersection of \( P(a) \) with the level set of \( v =_{\text{def}} v_{\sigma'} \circ \mu^* \). This \( v \) is a linear functional because \( \mu^* \) acts linearly on the lattice of monomials. The reader may verify that \( v(\sigma) \) is the smallest value of \( v \) on \( P(a) \). This proves that \( \sigma \) is a face.

**Proposition 21.** Let \( X, X', a, a', \) and \( \mu : X' \to X \) be as in Definition 19. For any face \( \sigma' \) of \( a' \), \( \mu(\mathcal{L}(\sigma')) = \mathcal{L}(\mu(\sigma')) \).

**Proof.** Since everything is invariant under the torus action, each set is at least torus invariant, and so is defined by a monomial ideal in \( \mathbb{C}[x_1, \ldots, x_n] \). Choose \( v' = v_{\sigma'} \) cutting out \( \sigma' \). As in Proposition 20, \( \sigma \) is cut out by \( v = v' \circ \mu^* \). Unraveling the definitions involved, we find that the ideal of the left hand side is generated by \( \{m \in L_X : v'(\mu^*(m)) > 0 \} \). The ideal of the right hand side is generated by \( \{m \in L_X : (v' \circ \mu^*)(m) > 0 \} \). These are obviously the same set.

We now have the tools in hand to prove the main theorem, which we restate here:
Theorem 22 (Main Theorem). Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Let $\mu : X' \to \mathbb{C}^n$ be a toric log resolution of $\tau(f)$. If $f$ is nondegenerate, then $\mu$ also log-resolves $\text{Div}(f)$. If $f$ has nondegenerate principal part, then there is a Zariski neighborhood $U$ of the origin over which $\mu$ is a log resolution of $f|_U$.

We begin with a special case:

Lemma 23. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$, and assume that $\tau(f)$ is principal. Let $\sigma$ be a face of $P(f)$. If $f$ is nondegenerate for $\sigma$, then $\text{Div}(f)$ is a normal crossing divisor at points $p$ in the relative interior of $L(\sigma)$.

Proof. Let $m$ be the (unique) monomial generator of $\tau(f)$. By reordering the coordinates, we may assume $\sigma = \{mx_1^{a_1} \cdots x_k^{a_k} : a_i \geq 0\}$ for some $k$, so $L(\sigma) = \text{Spec}(\mathbb{C}[x_1, \ldots, x_k]) = \mathbb{V}(x_{k+1}, \ldots, x_n)$. We may write $g = f/m$, where $m$ is a monomial and $g$ has nonzero constant term. Let us also write $g_{\sigma} = f_{\sigma}/m$. Notice that $g = g_{\sigma}$ on $L(\sigma)$, and $g_{\sigma} \in \mathbb{C}[x_1, \ldots, x_k]$. (See the figure below.)

Let $p$ be in the relative interior of $L(\sigma)$. If $g(p) \neq 0$, the normal crossing conclusion is trivial, so we assume $g(p) = 0$. We must show that the relevant 1-forms $\{dx_{k+1}, \ldots, dx_n, dg\}$ corresponding to the components of $\text{Div}(f)$ which meet at $p$ are independent at $p$. For this, it suffices to show that $d(g|_{L(\sigma)})$ is nonzero at $p$. Let $q = p + (0, \ldots, 0, 1, \ldots, 1)$ (k zeroes, n-k ones). Notice that $g_{\sigma}(q) = g_{\sigma}(p) = g(p) = 0$. Since $q$ is in the torus, we have at $q$,

$$0 \neq df_{\sigma} = d(mg_{\sigma}) = mdg_{\sigma} + g_{\sigma}dm = mdg_{\sigma}.$$ 

So $0 \neq dg_{\sigma}$ at $q$. Since $g_{\sigma} \in \mathbb{C}[x_1, \ldots, x_k]$,

$$0 \neq dg_{\sigma}|_q = dg_{\sigma}|_p = d(g_{\sigma}|_{L(\sigma)})|_p = d(g|_{L(\sigma)})|_p.$$
as desired.

Remark 24. Notice that every point \( p \in \mathbb{C}^n \) is in the relative interior of some coordinate linear space \( L \), and such a space can always be written \( L = L(\sigma) \) for some \( \sigma \).

Proof of Theorem 12. Let \( f' = \mu^*(f) \). We must show that \( \text{Div}(f') \) has normal crossing support. Since this property is local on \( X' \), and since \( \mu \) is a toric log resolution of \( \tau(f) \), we may take \( X' = \text{Spec} \mathbb{C}[y_1, \ldots, y_n] \), with \( \mu : X' \to X \) a monomial map. Note that \( \tau(f') \) is principal, because \( \mu \) resolves \( \tau(f) \).

First we discuss the sense in which \( f' \) inherits nondegeneracy from \( f \). Let \( \sigma' \) be a face of \( P(f') \), and let \( \sigma = \mu(\sigma') \) be the associated face of \( P(f) \). Clearly \( \mu^*f_\sigma = f'_{\sigma'} \). Because \( \mu \) is an isomorphism on the torus, \( d_{f'}_{\sigma'}|_{\tau} = d_{f_\sigma}|_{\tau} \). This shows that \( f' \) is \( \sigma' \)-nondegenerate as long as \( f \) is \( \sigma \)-nondegenerate, where \( \sigma = \mu(\sigma') \).

If we assume that \( f \) is nondegenerate for every face then we can conclude the same for \( f' \), so \( \text{Div}(f') \) is a normal crossing divisor (by Lemma 23) and we are done.

If however we assume only that \( f \) is nondegenerate for its compact faces, then we know only that \( f' \) is nondegenerate for each \( \sigma' \) with \( \mu(\sigma') \) compact. In particular (by Proposition 18), \( f' \) is nondegenerate for each \( \sigma' \) with \( L(\mu(\sigma')) \) equal to the origin. Equivalently, (by Proposition 21), for each \( \sigma' \) with \( L(\mu(\sigma')) \) the origin. We may conclude (again by Lemma 23) that \( f' \) has normal crossing support near points \( p' \in X' \) which map to the origin. Since the locus of point \( p' \) which fail the normal crossing condition is a closed subset of \( X' \), and since \( \mu \) is proper, there is a neighborhood of the origin above which \( \mu \) is a log resolution of \( \text{Div}(f) \). Therefore if \( f \) has nondegenerate principal part, then \( \mu \) is indeed a log resolution of \( f \), over a neighborhood of the origin. □

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