Artificial magnetic field for synthetic quantum matter without breaking time reversal symmetry

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We propose an all-static method to realize an artificial magnetic field for charge neutral particles without breaking time reversal symmetry or introducing any time modulation. Our proposal consists of one dimensional tubes subject to harmonic trapping potentials with shifted centers. We show that this setup realizes an artificial magnetic field in a hybrid real-momentum space. We discuss how characteristic features of particles in a magnetic field, such as chiral edge states and quantized Hall response, can be observed in this setup. We find that the mean-field ground state of bosons in this setup in the presence of long-range interactions can have quantized vortices; such a state with vortices exhibits a supersolid structure in the physical real space. Our method can be applied to a variety of synthetic quantum matter, including ultracold atomic gases, coupled photonic cavities, coupled waveguides, and exciton-polariton lattices.

Introduction: Quantum simulation has been a central theme in the research of synthetic quantum matter based on atomic, molecular, and optical systems, such as ultracold atomic gases, trapped ions, polaritons, and photons [1–7]. In particular, simulation of topological phenomena using synthetic quantum matter has attracted considerable attention in recent years [8–13]. A prototypical example of topological phenomena is the quantum Hall effect, which was first found in a two-dimensional electron gas under a magnetic field [14–16]. In order to simulate the quantum Hall effect and other topological phenomena, one often needs to simulate physics of charged particles in a magnetic field, which is not straightforward because atoms in ultracold gases and photons in cavities and waveguides are charge-neutral.

Various methods have been employed to simulate effects of a magnetic field, namely to realize an artificial magnetic field [17]. In ultracold atomic gases, the earliest example is to rotate the system, exploiting the similarity between the Coriolis force and the Lorentz force [18–20]. Other examples include Floquet engineering [21], light-induced gauge potential [22], and synthetic dimensions [23]. All the existing proposals involve adding fast time modulation and/or exploiting different energy and time scales present in the system to explore adiabatic physics in a low energy subspace. These methods often suffer from instability and heating as a result of time modulation and spontaneous emissions. The situation is similar in photonics and optics. For light close to optical frequencies, magneto-optical effects are generally weak and thus, similar to ultracold atomic gases, one needs to realize an artificial magnetic field to explore physics of quantum Hall effects [24–26]. Static realizations of topological models in photons are in the class of quantum spin-Hall insulators where two degrees of freedom feel opposite magnetic fields and thus topological protection relies upon decoupling of different degrees of freedom [27, 28].

In this paper, we propose a completely static way to realize an artificial magnetic field, which can be applied to a variety of synthetic quantum matter. Our proposal is based on a well known fact that the Hamiltonian of a charged quantum particle in two dimensions under a uniform magnetic field is equivalent to a set of harmonic oscillators with shifted centers [29]. This fact implies, conversely, that a set of harmonic oscillators with shifted...
centers can be viewed as a charged particle in a uniform magnetic field. We pursue this analogy further and consider a set of one-dimensional systems elongated along \(x\) direction and align these one-dimensional tubes in \(y\)-direction and place them under harmonic trapping potentials with shifted centers (Fig. 1). We find that the Hamiltonian of a charged particle in a magnetic field is realized in \(x-p_y\) plane where \(p_y\) is the momentum along \(y\) direction. Even though the proposal looks deceptively simple, as we discuss in detail, we can still observe most of the phenomena characteristic of charged particles in a magnetic field, such as the chiral edge states and quantized Hall response. We note that there is no dynamical component in the proposal, and in particular, our proposal does not break the time-reversal symmetry in the physical \(x-y\) plane. Our proposal is thus expected to be more stable and free from heating compared to existing proposals which break the time-reversal symmetry.

In the final part of the paper, we discuss the effect of inter-particle interactions. Weakly interacting bosons under a magnetic field form vortices [18–20, 30, 31]. However, a contact interaction, common in ultracold atomic gases and photons with optical nonlinearity, translates into extremely long range interaction in \(p_y\) direction, and we find that vortices do not form under ordinary contact interactions. On the other hand, by including nearest-neighbor interaction in \(y\) direction we find that vortices can form in \(x-p_y\) plane. Such a vortex structure corresponds to supersolid structure in the physical \(x-y\) plane, where the phase coherence of a condensate is maintained and the discrete translational symmetry is broken.

**Setup:** We consider a set of uncoupled one-dimensional tubes elongated in \(x\) direction and aligned in \(y\) direction, where each tube is subject to a harmonic trapping potential. The center of the trapping potential shifts between tubes by a constant amount \(x_0\). We assume that tubes are separated by a constant amount \(y_0\) in \(y\) direction. See Fig. 1 for illustration. We use the second quantized formalism, where \(\hat{\psi}_n^\dagger(x)\) and \(\hat{\psi}_n(x)\) are the creation and annihilation operators, respectively, of a particle at coordinate \((x, ny_0)\), where \(n\) indicates that the particle is in \(n\)-th tube. The single-particle Hamiltonian is

\[
\hat{H} = \sum_n \int dx \left[ \frac{1}{2m} \left( \partial_x \hat{\psi}_n^\dagger(x) \right) \left( \partial_x \hat{\psi}_n(x) \right) + \frac{1}{2} m\omega^2 (x-nx_0)^2 \hat{\psi}_n^\dagger(x) \hat{\psi}_n(x) \right],
\]

(1)

Note that there is no motion (coupling) between different tubes in \(y\) direction. By performing Fourier transformation in \(y\) direction:

\[
\hat{\psi}_n(x) = \sqrt{\frac{y_0}{2\pi}} \int_{-\pi/y_0}^{\pi/y_0} dp_y e^{inyp_y} \hat{\psi}(x, p_y),
\]

(2)

the Hamiltonian can be rewritten as

\[
\hat{H} = \int dx \int_{-\pi/y_0}^{\pi/y_0} dp_y \left[ \frac{1}{2m} \left( \partial_x \hat{\psi}_n(x) \right) \left( \partial_x \hat{\psi}_n(x) \right) + \frac{1}{2} m\omega^2 \left( \frac{x}{\tan \theta} \right)^2 \hat{\psi}_n^\dagger(x, p_y) \hat{\psi}_n(x, p_y) \right] + \frac{1}{2} \frac{m\omega^2}{\tan^2 \theta} \left\{ \left( i\partial_{p_y} + x \tan \theta \right) \hat{\psi}_n^\dagger(x, p_y) \right\} \right\}.
\]

(3)

In the first quantized form, this Hamiltonian is

\[
H = -\frac{1}{2m} \partial_x^2 + \frac{m\omega^2}{2\tan^2 \theta} \left( -i\partial_{p_y} + x \tan \theta \right)^2.
\]

(4)

This is nothing but the Hamiltonian of a charged particle in \(x-p_y\) plane under a magnetic vector potential \(A(x, p_y) = (0, -x \tan \theta)\) in Landau gauge corresponding to the artificial magnetic field of \(B = -\tan \theta\) penetrating the plane. (We have set the charge \(e\), speed of light \(c\), and Plank constant \(\hbar\) to be unity.) We have chosen to use \(-i\partial_{p_y} = -y\) as the ‘momentum’ operator along \(p_y\) direction regarding \(p_y\) as the synthetic ‘position’. The mass is anisotropic: the mass in \(x\) direction is the original mass \(m_x = m\), whereas the mass in \(p_y\) direction is \(m_{p_y} = \tan^2 \theta/m\omega^2\). This anisotropy just leads to rescaling of units of length in two directions and does not pose any problem in the rest of the discussion. Another noticeable feature here is that \(x-p_y\) plane is a cylinder, open along \(x\) direction and periodic along \(p_y\) direction.

Let us comment on the time-reversal symmetry in our system. Our original Hamiltonian Eq. (1) does not break the physical time reversal symmetry. For single component (spinless) system, time reversal operation flips the sign of momentum \(\mathbf{p} \rightarrow -\mathbf{p}\). Since we want to look at the physics related to the artificial magnetic field in \(x-p_y\) plane, the relevant effective time-reversal symmetry now is to flip their reciprocal variables \(T : (p_x, -y) \rightarrow (-p_x, y)\); the effective time-reversal symmetry is simultaneous operation of flipping of \(p_x\) and the parity in \(y\). Our Hamiltonian does break the parity in \(y\) direction due to the shifted harmonic trapping potentials, which enables us to realize artificial magnetic field in \(x-p_y\) plane without breaking the physical time reversal symmetry.

Our method is related to the idea of topological charge pumping, where a momentum is replaced by an external parameter; by sweeping the parameter one obtains topological properties of the original Hamiltonian [32–34]. Our method, however, is crucially different from charge pumping in that we replace a momentum \(p_y\) by a physical real coordinate \(-y\), and in this way we succeed in recovering the full dynamics in all dimensions, whereas no dynamics occur in charge pumping in the direction replaced by an external parameter.

**Chiral edge states:** We now analyze how characteristic features of charged particles in a magnetic field can be observed in our setup in \(x-p_y\) plane. We first point out that the cyclotron frequency of the artificial magnetic field is
FIG. 2. Energy level and the Hall response. (a) The energy level of each tube at position $n$, where length of each tube is $20a_{\text{osc}}$ and we employ an open boundary condition. We choose $\tan \theta = 1$, and the energy is in units of the cyclotron frequency $\omega$. Solid lines are guide to the eye. (b) The center-of-mass motion of wavepackets when a synthetic electric field $E_x = 0.1\omega/a_{\text{osc}}$ is applied. Three solid lines show the time evolution of the center of mass in $p_y$ direction for $\tan \theta = 2/3$ (top, green), $\tan \theta = 1$ (middle, blue), and $\tan \theta = 1/2$ (bottom, red). The dashed lines going through the solid lines are theoretical prediction from Eq. (5) with $C = 1$. The unit of time is $1/\omega$, and the unit of $p_y$ is $1/y_0$.

$\omega$, as expected. This can be derived from the definition of the cyclotron frequency $B/m_\text{c}$; as the mass $m_\text{c}$, we need to use a geometric mean of anisotropic effective mass in two directions $m_\text{c} = \sqrt{m_x m_y}$. Using $B = -\tan \theta$, we then obtain $\omega = |B|/m_\text{c}$. This observation implies the well known fact that the energy level of harmonic oscillators in $x$-$y$ plane, which is $\omega(l + 1/2)$, where $l = 0, 1, 2, \cdots$ is a nonnegative integer, is nothing but the Landau level energy spectrum. High degeneracy of Landau levels comes from the many uncoupled tubes present in the system, which all have the same energy spectrum.

In the presence of edges, charged particles in two dimensions under a magnetic field should have chiral edge states. In Fig. 2(a), we plot the energy level of the system in the presence of sharp edges in $x$ direction as a function of $n$, which is the reciprocal variable of $p_y$. We consider 21 tubes and the length of each tube in $x$ direction is $20a_{\text{osc}}$, where $a_{\text{osc}} = 1/\sqrt{\pi \omega}$ is the oscillator length. We can see that at large values of $|n|$, the dispersion goes upward; these states are localized at the edges and they propagate in one direction. The existence of chiral edge states can be probed from the dynamics of wavepackets localized at the edges, as discussed in more detail in Supplemental Material.

Quantized Hall response: Since Landau levels are topological, Hall response should be quantized, which can be a signature of nontrivial topology obtained from the bulk property of the system. The topological Chern number $C$ of Landau levels with $B < 0$ is one, $C = 1$; we now investigate if this unit Chern number can be observed in our setup. In ultracold gases, quantized Hall response can be observed through the center-of-mass response upon application of a force (synthetic electric field) $E = (E_x, E_{p_y})$ [35–37]. When a synthetic electric field $E_x$ is applied along $i$-direction, the center-of-mass velocity is $[37] v_{\text{C.M.}}^{E_x} = -\epsilon_{ij} \frac{\partial x_j}{\partial E_x} v_{\text{osc}}$, where $A_{\text{BZ}}$ is the area of the Brillouin zone and $\epsilon_{y0} = -\epsilon_{p_y} x = 1$. For our setup of a uniform magnetic field in $x$-$p_y$ plane, there is no translational symmetry in $x$ direction, and therefore we need to consider a magnetic unit cell and the magnetic Brillouin zone. A magnetic unit cell is an arbitrary rectangle in $x$-$p_y$ plane which encloses one magnetic flux quantum. Since the strength of the artificial magnetic field is $|B| = \tan \theta$, the area of a magnetic unit cell is $2\pi/\tan \theta$. Then the area of the magnetic Brillouin zone is $(2\pi)^2$ divided by the area of a magnetic unit cell and thus $A_{\text{BZ}} = 2\pi \tan \theta$. Therefore, the center-of-mass velocity should follow

$$v_{\text{C.M.}}^{E_x} = -\epsilon_{ij} E_j / \tan \theta = \epsilon_{ij} E_j / B.$$  (5)

We should then be able to determine the Chern number of the Landau levels in $x$-$p_y$ plane by monitoring the center-of-mass motion. As we see below, the quantized response upon adding $E_{p_y}$ turns out to be nontrivial, whereas the response upon $E_x$ can provide observable signatures of the quantized Hall response.

We first consider the Hall response upon adding $E_{p_y}$, which can be applied by making the artificial magnetic vector potential time dependent $A = (0, -x \tan \theta - E_{p_y} t)$. Such a time dependence can be introduced by changing the position of the minima of the harmonic trapping potentials. Changing the location of the trap minima from $nx_0$ to $nx_0 + x_c$, in a $y$-independent manner, the magnetic vector potential in $x$-$p_y$ plane becomes $A = (0, -x \tan \theta + x_c \tan \theta)$. This implies that the desired magnetic vector potential can be created by moving $x_c$ as $x_c = -E_{p_y} t / \tan \theta$. Now, the quantized Hall response through the center-of-mass velocity has a clear physical meaning. When the motion of the trapping potential is slow enough, the center of mass should follow the motion of the trap minima $x_c(t)$. Therefore, the center-of-mass velocity in $x$ direction is $v_{\text{C.M.}}^{E_x} = \partial x_c(t) / \partial t = -E_{p_y} / \tan \theta$ from Eq. (5), this implies the correct relation $C = 1$.

The quantized Hall response upon adding $E_{p_y}$, $v_{\text{C.M.}}^{E_{p_y}} = E_x / \tan \theta$ can provide a more nontrivial observable signature of topology. A synthetic electric field $E_x$ can be applied by adding a linear potential gradient $-E_x x$ to the Hamiltonian. We numerically simulate the time evolution of a wavepacket located at the center, projected onto the lowest Landau level, under the field $E_x$. In Fig. 2(b), we plot the center-of-mass position of the wavepacket in $p_y$ direction under time evolution for different values of $B = -\tan \theta$. We also plot the theoretical prediction from Eq. (5) assuming $C = 1$, which agrees well with the numerical simulation of $\langle p_y \rangle$. The numerical simulation also exhibits oscillation around the theoretical line with oscillation period $2\pi/\omega$, which is a detailed structure of the cyclotron motion. Thus, the quantized Hall response can be observed from the center-of-mass motion in $p_y$ direction under the application of $E_x$, and the Chern number
can be estimated from its slope. Interaction and vortices: We now turn to many-body cases. An important feature is that a typical short range interaction in $x$-$y$ plane translates into long-ranged and non-local interaction in $p_y$ direction. In particular, formation of vortices, which is a characteristic feature of weakly interacting bosons in a magnetic field [18–20, 30, 31], cannot be observed in $x$-$p_y$ plane with a contact interaction in $x$-$y$ plane. Nonetheless, we find that, by adopting long range interaction in $y$ direction, vortices can still form in our setup as we discuss below. We now focus on bosons with mean-field interaction and vortices can still form in $x$-$p_y$ plane. We use description in terms of the Gross-Pitaevskii equation with the condensate wavefunction $\Psi_n(x, t)$.

$$i \partial_t \Psi_n(x, t) = H_{\text{hop}} + H_{\text{int}} +$$

$$\left\{ -\frac{\partial^2}{2m} + \frac{1}{2} m \omega^2 (x - nx_0)^2 + \frac{1}{2} m \tilde{\omega}^2 y^2 \right\} \Psi_n(x, t). \quad (6)$$

$$H_{\text{hop}} = J_y (\Psi_{n-1}(x, t) + \Psi_{n+1}(x, t)) \quad (7)$$

$$H_{\text{int}} = g_s |\Psi_n(x, t)|^2 \Psi_n(x, t)$$

$$+ g_1 (|\Psi_{n-1}(x, t)|^2 + |\Psi_{n+1}(x, t)|^2) \Psi_n(x, t) \quad (8)$$

We additionally included an overall weak harmonic potential with the oscillator frequency $\tilde{\omega}$ which confines the particles. We take $\tilde{\omega}$ to be much smaller than $\omega$ so that its only effect is to prevent particles from flying away. We also added a small inter-tube hopping term, $H_{\text{hop}}$, which allows exchange of particles between different tubes. This hopping term allows the system to establish an overall phase coherence. In the numerical calculations below, we take $a_{\text{osc}} \equiv 1/\sqrt{m \tilde{\omega}} = 10 a_{\text{osc}}$ and $J_y = -0.01 \omega$. The interaction term $H_{\text{int}}$ contains the contact interaction with strength $g_s$ and also a nearest-neighbor (long-range) interaction between tubes with strength $g_1$.

We first consider the ground state when only the contact interaction $g_s$ is present. In Fig. 3(a,b), we plot the ground state, obtained by imaginary-time propagation of the Gross-Pitaevskii equation (6), when $g_s = a_{\text{osc}} \omega$ and $g_1 = 0$, assuming the normalization $\int dx |\Psi_n(x, t)|^2 = 1$. Fig. 3(a) shows $|\Psi_n(x)|$ in $x$-$y$ plane, whereas Fig. 3(b) shows $|\Psi(x, p_y)|$, where $\Psi(x, p_y)$ is the Fourier transform of $\Psi_n(x)$ in $y$ direction. We see that the particles are clumped at the center of the system, and no vortex-like structure is seen. This is because, in order to obtain vortices in $x$-$p_y$ plane, we need a short range repulsive interaction in $x$-$p_y$, but the interaction due to $g_s$ is infinite-ranged in $p_y$ direction.

In order to obtain vortices in $x$-$p_y$ plane, we include the nearest-neighbor interaction $g_1$, which introduces a shorter-range interaction in $x$-$p_y$ plane. Such an anisotropic interaction in $x$-$y$ plane may be realized in ultracold atomic gases using dipole-dipole interaction with proper orientation of the dipole field [38]. In Fig. 3(c, d), we plot the ground state when $g_s = a_{\text{osc}} \omega$ and $g_1 = 2 a_{\text{osc}} \omega$. We see that $|\Psi(x, p_y)|$ has six dips. By looking at its phase profile in Fig. 3(f), we confirm that these dips are the vortices formed in $x$-$p_y$ plane. In $y$-$p_y$ plane, the state has density peaks at every other tube, as seen in the integrated density for each tube plotted in Fig. 3(e). Vortices on a cylinder are known to have dynamical properties different from those on a plane [39]; our proposal provides a unique setup where such phenomena can be studied. Since the Hamiltonian has a discrete translational symmetry along the angle $\theta$ from the $x$-axis, the state breaks this translational symmetry, thus forming a supersolid state in $x$-$y$ plane. If this supersolid structure persists in the absence of the confining potential $\tilde{\omega}$ in a thermodynamic limit is a question to be addressed in a future work. More details of the vortices are discussed in the Supplemental Material.

Conclusion: We showed that a set of tubes with harmonic trapping potentials whose centers are shifted is
equivalent to charged particles in a magnetic field in $x$-$p_y$ plane. Our proposal can be realized in various platforms of synthetic quantum matter; the quantized Hall response and the phenomena involving interactions are directly relevant in ultracold atomic gases [35–37], whereas edge related physics are more easily accessible in photon-related platforms [24, 27, 40]. The underlying idea of our proposal is to replace a momentum in one direction by position to realize a simpler Hamiltonian, which simulates the original Hamiltonian in a hybrid real-momentum space. This method is quite versatile; for example, with a similar mechanism, a collection of one-dimensional Aubry-André model [41] (the Harper model) with shifted modulation phases simulates a two-dimensional lattice with a magnetic field, the Harper-Hofstadter model [42, 43], in a real-momentum space, without breaking the physical time reversal symmetry. Our proposal provides a simple and powerful method applicable to realize various Hamiltonians, opening a new possibility for quantum simulation in synthetic quantum matter.

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Supplemental Material for “Artificial magnetic field for synthetic quantum matter without breaking time reversal symmetry”

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I. WAVEPACKET DYNAMICS OF CHIRAL EDGE MODES

We show that the chiral edge mode in $x$-$p_y$ plane can be observed through wavepacket dynamics. We choose the strength of the artificial magnetic field to be $B = -\tan \theta = -1$ for the numerical simulation. As an initial state, we consider three wavepackets, located at the left edge, center, and the right edge of the system. To consider physics relevant to a single band or band gap, we project the wavepacket onto the lowest level of each tube. Experimentally, in ultracold gases, such an initial state can be prepared by creating a wavepacket out of single particle ground states of each tube by starting from Bose-Einstein condensates in each tube. In photonics, where light can be injected from outside, we can insert light with frequency corresponding to the lowest Landau level to prepare a similar initial condition. The wavefunction of the initial state is plotted in Fig. S1(a,b), in both $x$-$y$ and $x$-$p_y$ planes. We then let the wavepacket evolve freely. The wavefunction in $x$-$p_y$ plane after time $t = 1/\omega$, $t = 2/\omega$, and $t = 3/\omega$ are plotted in Fig. S1(c,d,e). We observe that the wavepacket at the left edge goes upward, whereas the wavepacket at the right edge goes downward, showing that the edge state goes clockwise in a chiral manner. This motion of the at the edge is consistent with the artificial magnetic field in $x$-$p_y$ plane; the strength of the magnetic field is $B = -\tan \theta < 0$, which implies that the cyclotron motion inside the bulk of the system is counter-clockwise, and the edge state goes clockwise. We note that the absolute value of the wavefunction plotted in $x$-$y$ plane looks identical during time evolution. This is because our initial state is projected onto the lowest Landau level, and thus the wavefunction in each tube is an eigenstate of the Hamiltonian for each tube, and thus the only time evolution appears in the phase of the wavefunction. The evolution of the phase in $x$-$y$ plane appears as chiral edge motion in $x$-$p_y$ plane.

![Fig. S1. Dynamics of a wavepacket and chiral edge modes.](image)

(a) The absolute value of the initial wavefunction is plotted in $x$-$y$ plane, where each $n$ represents a tube located at the position $y = n\gamma_0$. (b) The initial state in $x$-$p_y$ plane. The lower panel shows the wavefunction in $x$-$p_y$ plane after time evolution for duration (c) $t = 1/\omega$, (d) $t = 2/\omega$, and (e) $t = 3/\omega$. Chiral edge motion is clearly visible.
We should also comment on the direction of the edge mode with respect to the energy spectrum, Fig. 2(a) of the main text. One sees that, in the energy spectrum, the energy goes up at large values of $n$. These states at large values of $n > 0$ are localized at the right edge. Since the energy goes up, one might want to conclude that the group velocity is positive at the right edge. However, as one sees from Fig. S1, this is not the case; the group velocity is negative at the right edge. This apparent discrepancy is because one needs to consider $-n$ as the conjugate variable of $p_y$. Namely, the horizontal axis of Fig. 2(a) should be flipped when one wants to find the correct group velocity. By flipping the horizontal axis of Fig. 2(a), one obtains the correct group velocity which is negative at the right edge.

II. INTERACTION AND VORTICES

In the main text, we have shown that the mean-field ground state of bosons when there is a nearest-neighbor interaction between tubes exhibit vortices. Here we give more detailed analysis of the vortices. We first show, in Sec. II A, the wavefunction profile when a hypothetical contact interaction in $x$-$p_y$ plane is present. We see that the vortices translate into a supersolid structure in $x$-$y$ plane, confirming that the vortices we observed in the main text is indeed created by the same mechanism as the ordinary vortices in the presence of a magnetic field. We then discuss, in Sec. II B, how vortices form as one increases the nearest-neighbor interaction $g_1$.

A. Vortices under a contact interaction in x-$p_y$ plane

We first analyze the bosonic ground state if the interaction is contact in $x$-$p_y$ plane. Note that this is the situation one often encounters in $x$-$y$ plane in ultracold atomic gases in the presence of an artificial magnetic field [S1–S3], and one expects formation of vortices. We include the contact interaction in $x$-$p_y$ plane by adding a term $\int dx \int dp_y \hat{g}_s |\Psi(x, p_y)|^4$ in the energy functional, where $\Psi(x, p_y)$ is the normalized wavefunction in $x$-$p_y$ plane. We choose $\hat{g}_s = 10a_{osc}y_0\omega$, $g_s = g_1 = 0$ and we take all the other parameters to be the same as in the main text; in particular, we take $\tan \theta = 1$ for all the simulations in this section. In Fig. S2, we plot the ground state obtained by imaginary time evolution of the Gross-Piatevskii equation. The absolute values of the wavefunction in $x$-$y$ as well as $x$-$p_y$ planes are plotted in Fig. S2(a), (b), respectively. We can see two density peaks in $x$-$y$ plane, similar to the one found for middle column of Fig. S2(a), (b), respectively. We can see two density peaks in $x$-$y$ plane, similar to the structure seen in the main text when the nearest-neighbor interaction is included. The total density of each tube is plotted in Fig. S2(c), where the vertical axis denotes the integral of $|\Psi_n(x)|^2$ over $x$. The corresponding wavefunction in $x$-$p_y$ plane, as confirmed by the phase profile in Fig. S2(d). These vortices, formed by an artificial magnetic field and a contact interaction in $x$-$p_y$ plane, is the ordinary vortices which have been observed in ultracold gases in the presence of an artificial magnetic field in the real $(x,y)$ space. Comparing these results with the wavefunction found in the main text when the nearest-neighbor interaction is present, we conclude that the vortex structure found in the main text has the same origin as the conventional vortices in the presence of an artificial magnetic field.

B. Vortex formation

Now we come back to the interaction we considered in the main text: contact interaction $g_s$ and the nearest-neighbor interaction $g_1$. We analyze how vortices are formed by varying the value of $g_1$ and looking at the wavefunction profile. In Fig. S3, we plot, for three different values of $g_1$, the wavefunction in $x$-$y$ plane and $x$-$p_y$ plane, as well as the phase profile of the wavefunction in $x$-$p_y$ plane and the integrated density for each tube. We use $\tan \theta = 1$, $g_s = a_{osc}\bar{\omega}$, $\bar{\omega} = \omega/100$, and $J_y = -0.01\omega$. We can observe that vortices form as one increases $g_1$. We see that, when $g_1 = 0.8a_{osc}\bar{\omega}$, there is no phase singularity in $x$-$p_y$ plane, and the integrated density for each tube has a peak at $n = 0$ and monotonically decreases as one goes away from $n = 0$. The situation changes for $g_1 = 1.0a_{osc}\bar{\omega}$. Although it is still difficult to see vortex formation in $|\Psi(x, p_y)|$, by looking at the phase profile of $\Psi(x, p_y)$, we clearly see six phase singularities, indicating the formation of vortices. Correspondingly, density modulation starts to be seen in the integrated density for each tube. When $g_1 = 1.2a_{osc}\bar{\omega}$, the vortices become easily visible in $|\Psi(x, p_y)|$. The phase singularity and the density modulation also become more pronounced.

The vortices are fragile against increase of the inter-tube hopping amplitude $J_y$. If we use $J_y = -0.02\omega$, the density dip in $x$-$y$ plane, and hence the vortex structure in $x$-$p_y$ plane, similar to the one found for middle column of Fig. S3 will not appear until we increase $g_1$ until around $g_1 \approx 1.3a_{osc}\bar{\omega}$. Thus, a stronger interaction is necessary to form vortices. This tendency can be understood in the following way. Inter-tube hopping $J_y$ introduces a sinusoidal potential energy in $x$-$p_y$ plane along $p_y$ direction. In order to create vortices in $x$-$p_y$ plane, the interaction effect should be strong enough to overcome the sinusoidal potential created by $J_y$; this is an intuitive picture of why larger values of $g_1$ is required to realize vortices when $J_y$ becomes larger. We note that the vortices can in principle form
FIG. S2. The mean-field ground state when we consider a contact interaction in $x$-$p_y$ plane. The unit of length in $x$ direction is $a_{osc}$, and the unit of $p_y$ is $1/y_0$. We take the strength of the contact interaction in $x$-$p_y$ plane to be $\tilde{g}_s = 10a_{osc}y_0\omega$, and no other inter-particle interaction exists. Other parameters are $\tan = \theta = 1$, $\tilde{\omega} = \omega/100$, and $J_y = -0.01\omega$. (a) The absolute value of the wavefunction in $x$-$y$ plane, where two density peaks are seen. (b) The wavefunction in $x$-$p_y$ plane, where two dips of the wavefunction are seen. (c) The integrated wavefunction of each tube, $\int dx |\Psi_n(x)|^2$, from which one can see the density peaks more clearly. (d) The phase profile of the wavefunction in $x$-$p_y$ space; the two blue circles indicate the position of the phase singularity, namely the vortices.

also when there is no inter-tube hopping, $J_y = 0$. In such a case, however, there is no reason for the entire system to develop a phase coherence, and thus the resulting structure in $x$-$y$ plane is not a supersolid anymore; it would rather be a phase incoherent density wave.

[S1] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Vortex Formation in a Stirred Bose-Einstein Condensate, Phys. Rev. Lett. 84, 806 (2000).
[S2] N. R. Cooper, Rapidly rotating atomic gases, Advances in Physics 57, 539 (2008).
[S3] A. L. Fetter, Rotating trapped Bose-Einstein condensates, Rev. Mod. Phys. 81, 647 (2009).
FIG. S3. Formation of vortices in $x$-$p_y$ plane. Three columns represent the ground states for $g_1 = 0.8a_{osc}\omega$, $g_1 = 1.0a_{osc}\omega$, and $g_1 = 1.2a_{osc}\omega$, respectively, from left to right. In each column, the top row shows $|\Psi_n(x)|$ of the ground state, the second row shows $\Psi(x, p_y)$, the third row shows the phase profile of $\Psi(x, p_y)$, and the bottom row shows $\int dx |\Psi_n(x)|^2$ as a function of $n$. In the phase profile panels, we marked phase singularities at $p_y > 0$ by blue circles. (There are also the same number of phase singularities at $p_y < 0$, which are not encircled.) The unit of length in $x$ direction is $a_{osc}$, and the unit of $p_y$ is $1/y_0$. 