COMPLEX RIEMANNIAN FOLIATIONS OF OPEN KÄHLER MANIFOLDS

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Abstract. Classification results for complex Riemannian foliations are obtained. For open subsets of irreducible Hermitian symmetric spaces of compact type, where one has explicit control over the curvature tensor, we completely classify such foliations by studying associated nearly Kähler holonomy systems. We also establish results for symmetric spaces of non-compact type and a general rigidity result for any irreducible Kähler manifold.

1. Introduction

A central theme in the study of foliations is the classification of Riemannian foliations whose leaves satisfy natural geometric conditions. This question can be traced back to Cartan, who asked if every hypersurface in a round sphere with constant principal curvatures comes from a cohomogeneity one action. Known classifications, such as the work of Gromoll-Grove [13] or Kollross [15] have mainly concentrated on two cases: classifying Riemannian foliations of space forms and classifying isometric group actions on Riemannian symmetric spaces under additional hypothesis.

In contrast, our focus is on Riemannian foliations of irreducible Kähler manifolds \((M^n, g, J)\) of complex dimension \(n = 2m \geq 2\) where the leaves are complex submanifolds (a complex Riemannian foliation). Such foliations arise in many different contexts, two natural examples being as twistor spaces of quaternionic-Kähler manifolds and in the classification of closed nearly Kähler manifolds. Our emphasis is on the local case, where existing classification results fail. This paper investigates questions which naturally arose from our previous independent work [18], [19], [20], [21].

The prototypical example of a twistor fibration comes from the two Hopf fibrations described by the diagram

\[
\begin{array}{ccc}
S^1 & \rightarrow & S^{4n+3} \\
\downarrow & & \downarrow \\
SU(2) & \rightarrow & \mathbb{C}P^{2n+1}
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{C}P^{2n+1} & \rightarrow & \mathbb{C}P^1 \\
\downarrow & & \downarrow \\
\mathbb{H}P^n & \rightarrow & \mathbb{C}P^1
\end{array}
\]

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From the inclusion $S^1 \rightarrow SU(2)$ the existence of a natural Riemannian submersion
$$
\pi : \mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n
$$
with totally geodesic leaves $SU(2)/S^1 = \mathbb{C}P^1$ is apparent. Here both spaces have their canonical symmetric space metrics. The leaves of a Riemannian submersion are well known to form a Riemannian foliation of the total space. This is the canonical twistor fibration of $\mathbb{C}P^{2n+1}$.

Recall that the O’Neill $A$ and $T$ tensors encode the fundamental information about a regular Riemannian foliation. If $A$ vanishes, $F$ is said to be polar, whereas $T$ vanishing implies $F$ is totally geodesic. Trivial complex foliations are those where every point is a leaf, or where the foliation has just one leaf. Without loss of generality, we will assume that our foliations are non-trivial throughout. The main results are as follows.

**Theorem 1.1.** Let $M$ be an open set of an irreducible Hermitian symmetric space and $F$ a complex Riemannian foliation on $M$. Then

(i) If $M$ has non-negative sectional curvature, then $F$ is the canonical twistor foliation restricted to $M^{2n+1} \subset \mathbb{C}P^{2n+1}$.

(ii) If $M$ has non-positive sectional curvature, then $F$ is polar.

Notice that our result is local, whereas all previous classification results assume completeness. Let us give some applications of the theorem. The theorem immediately implies that holomorphic foliations of locally Hermitian symmetric spaces can never be Riemannian. For example, any (singular) foliation of an open set of $\mathbb{C}P^2$ by Riemann surfaces, such as those in [4], cannot even locally be Riemannian. It is well-known that there is a large moduli space of such foliations, and they have been the focus of much study. A holomorphic foliation $F$ of $\mathbb{C}P^2$ is given by a one form $\omega = \sum_{i=0}^2 A_i dz^i$, where $[z_0, z_1, z_2]$ are homogeneous coordinates and the $A_i$ are homogeneous polynomials in $z_0, z_1, z_2$ of degree $k + 1$ such that $\sum_{i=0}^2 z_i A_i$ is identically zero. Another consequence is that we have shown that the canonical twistor fibration restricted to some open subset $M \subset \mathbb{C}P^{2n+1}$ provides the only instance when a twistor space of a positive quaternionic-Kähler manifold can be locally symmetric. Finally, combined with recent work of Di Scala–Ruiz-Hernandez [10], it completes the classification of complex Riemannian foliations of locally Hermitian symmetric spaces with non-negative sectional curvature.

Theorem 1.1 strengthens results in [18], where this result was established using different techniques for complex Riemannian foliations of open subsets of $\mathbb{C}P^n$ with maximal dimension.

Of course, it is natural to wonder if any such foliations exist in irreducible Hermitian symmetric spaces of non-compact type. As they are

\[In [9], it is claimed that the twistor space of any Wolf space is symmetric. Unfortunately this is incorrect: the twistor space of $G_2(\mathbb{C}^{m+2})$ is a counterexample. Our result shows it can never happen, even locally. This means the classification of quaternionic-Kähler manifolds with non-negative sectional curvature remains an open problem.\]
polar, we expect them to be quite rare. We also establish a rigidity result in the general case.

**Theorem 1.2.** For any complex Riemannian foliation $\mathcal{F}$ of a Kähler manifold $M$, let $\mathcal{V}$ denote the distribution tangent to the leaves. Then either $\mathcal{F}$ is totally geodesic, or else there is a non-zero integrable subdistribution $\mathcal{V}_o \subset \mathcal{V}$ defined in an open set $U$ of $M$ whose integral manifolds form a polar complex Riemannian foliation $\mathcal{F}_0$ of $U$.

This result is optimal: see the Example in Section 4, where we outline an algorithm to construct complex, non-polar and non-totally geodesic Riemannian foliations of certain Kähler manifolds. We emphasize that there is no compactness assumption on the ambient manifold, so this parallels existing results for closed Kähler manifolds [20]. There the proof hinged on a transversal version of the $\mathcal{O}$-lemma for vector bundle valued one-forms. We have to develop different arguments to deal with the open case. We also note that our proof does not recover the main result of [20] if $M$ is closed.

## 2. Preliminaries

Throughout, all manifolds are connected and smooth. We will define the real dimension of a leaf to be the real dimension of the foliation. A submanifold $X \subset M$ is said to be complex if it respects the ambient Kähler structure $J$; that is $JT_p(X) \subset T_p(X)$ for all $p \in M$. A foliation $\mathcal{F}$ of $M$ is said to be complex if every leaf $L$ is a complex submanifold. As is standard, write $\mathcal{V}$ for the leaf tangent distribution and $\mathcal{H}$ for its complement. $\mathcal{F}$ is Riemannian if it respects the metric: that is that the distance between leaves is constant. This can be written as $L_{\mathcal{V}}g(X^\mathcal{H}, Y^\mathcal{H}) = 0$ for all basic vector fields $X^\mathcal{H}, Y^\mathcal{H}$, where $L$ denotes the Lie derivative.

In any Kähler manifold there are two trivial examples of complex Riemannian foliations; the foliation with only one leaf, and the foliation where each leaf is a point. We will assume our foliation is non-trivial throughout. To prove Theorem 1.1, a first approach would be to consider a complex Riemannian foliation of $\mathbb{C}P^n$. Then [21] tells us either $\mathcal{F}$ is totally geodesic or is polar. One then can show there are never any polar regular foliations for irreducible compact Hermitian symmetric spaces. This can be seen by an short argument in [18]. Here the dual foliations of Wilking are used to show that if there were a polar regular foliation then the metric would split as a product, a contradiction. Then one is left with the case of totally geodesic Riemannian foliations. For $\mathbb{C}P^n$, these were classified by Escobales [11]. For more complicated Hermitian symmetric spaces the problem is more difficult. The structure of totally geodesic complex submanifolds of Hermitian symmetric spaces are not completely understood in general. Our approach allows us to completely answer this question, even in the local case.

For the proof of these results we must perform explicit calculations on the curvature tensor. To begin therefore, we must choose some additional notation. Assume that the foliation is complex, that is $J\mathcal{V} = \mathcal{V}$ and so
As our constructions are local we have a natural Riemannian submersion \( \pi : M \rightarrow M/F \). Throughout this paper we will denote by \( U, V, W \), sections of \( \mathcal{V} \) and by \( X, Y, Z \) etc. sections of \( \mathcal{H} \). Generic sections of \( T_pM \) will be denoted by \( E, F \). Let \( \nabla \) be the Levi-Civita connection of \( g \). Then

\[
\nabla E F = (\nabla E F)_V + (\nabla E F)_H
\]
defines a metric connection with torsion on \( M \) (here the subscripts indicate orthogonal projection on the subspace). The main property of this Bott connection is that it preserves the distributions \( \mathcal{V} \) and \( \mathcal{H} \). If \( T \) and \( A \) are the O’Neill tensors of the foliation then \( \nabla \) and \( \overline{\nabla} \) are related by

\[
\begin{align*}
\nabla X Y &= \overline{\nabla} X Y + A X Y, \\
\nabla V X &= \overline{\nabla} V X + T V X, \\
\nabla V W &= \overline{\nabla} V W.
\end{align*}
\]

Recall that \( A \) is skew-symmetric on \( \mathcal{H} \) since the foliation is Riemannian whereas \( T \) is symmetric on \( \mathcal{V} \), since the latter is integrable.

As \( \nabla J = 0 \), it follows that \( \overline{\nabla} J = 0 \). This yields information about the complex type of the tensors \( A \) and \( T \), namely

\[
\begin{align*}
A_X (JY) &= J(A_X Y), \\
A_{JX} V &= \overline{A_X (JY)} = -A_X (JY) = -A_X (JY), \\
T_{JY, W} &= J(T_{V, W}), \\
T_{JY, X} &= -J(T_{V, X}) = -T_{V, X}.
\end{align*}
\]

We also have \( A_{JX} JY = -A_X Y \) and \( T_{JY, JW} = -T_{V, W} \).

We will use now the Kähler structure on \( M \), together with suitable curvature identities to collect some geometric information about the tensors \( A \) and \( T \). Throughout, we implicitly assume our calculations are in a normal frame at the point \( p \) at which we calculate.

Of central importance is the following result.

**Lemma 2.1.** Let \( X, Y, Z \) be in \( \mathcal{H} \) and \( V, W \) in \( \mathcal{V} \). Then

\[
\begin{align*}
(i) & \quad (\overline{\nabla} X A)(Y, Z) = 0 \\
(ii) & \quad (A_X Y, T_V Z) = 0 \\
(iii) & \quad (\overline{\nabla} V A)(X, Y), W) = (\overline{\nabla} W A)(X, Y), V) \\
(iv) & \quad 2(\overline{\nabla} V A)(X, Y), W) = (\overline{\nabla} Y T(V, W), X) - (\overline{\nabla} X T(V, W), Y) \\
(v) & \quad \overline{\nabla} A_X Y A = 0.
\end{align*}
\]

**Proof.** Parts (i) to (iv) have been proved in [21]. To prove (v) we differentiate (ii) with respect to \( \overline{\nabla} \), in direction of \( Z_1 \) in \( \mathcal{H} \). Then, by taking (i) into account

\[
\langle A_X Y, (\overline{\nabla}_{Z_1} T_V Z) \rangle = 0.
\]

Now applying (iv), skew symmetrisation in \( Z_1 \) and \( Z \) yields

\[
\langle A_X Y, (\overline{\nabla} V A) Z Z_1 \rangle = 0.
\]

The claim follows now from the symmetry property for \( \overline{\nabla} A \) in (iii). \( \square \)

In particular some of the components of the Riemann curvature tensor become algebraic in \( A \) and \( T \), as recorded below. Here \( R \) is the curvature tensor of the connection \( \nabla \). Our convention is that \( R(X, Y) = \nabla^2_{Y, X} - \nabla^2_{X, Y} \) for all \( X, Y \) in \( TM \).
Corollary 2.2. We have the following.
\[ R(X, Y, Z, V) = 0, \]
\[ R(V, W, X, Y) = \langle A_X V, A_Y W \rangle - \langle A_X W, A_Y V \rangle - \langle T_Y X, T_W Y \rangle + \langle T_Y X, T_W Y \rangle, \]
\[ R(V, X, W, Y) = \langle (\nabla X)T(V, W), Y \rangle + \langle (\nabla V)A(X, Y), W \rangle - \langle T_Y X, T_W Y \rangle + \langle A_X V, A_Y W \rangle. \]

Proof. These follow from Lemma 2.1 and the O’Neill formulae in [3] applied to curvature terms of the type listed above.

3. Reduction results

Our goal here is to obtain a geometric interpretation of Lemma 2.1.

It is standard theory that a nearly Kähler metric \( g^{NK} \) is induced by taking the canonical variation of \( g \) with respect to the Riemannian submersion \((M, g) \to (M/\mathcal{F}, \hat{g}) = (N, \hat{g})\) (see [3], Chapter 9 for further details) with parameter \( t = \frac{1}{2} \). In particular,

\[ g^{NK}|_{\mathcal{V}} = \frac{1}{2} g|_{\mathcal{V}}, \quad g^{NK}|_{\mathcal{H}} = g|_{\mathcal{H}}, \quad g^{NK}(\mathcal{H}, \mathcal{V}) = 0. \]

The corresponding compatible almost complex structure \( \mathcal{J} \) is given by \( \mathcal{J}|_{\mathcal{V}} = -J \) and \( \mathcal{J}|_{\mathcal{H}} = J \). Moreover it may easily be checked that the canonical Hermitian connection associated with \( g^{NK} \) coincides precisely with the Bott connection \( \nabla \).

For an element \( V \in \mathcal{V} \) define \( \gamma_V : \mathcal{H} \to \mathcal{H} \) by

\[ \gamma_V X = A_X V. \]

Then \( \gamma_V \) is skew-symmetric and moreover

\[ \gamma_J V = -\gamma_V J = J \gamma_V. \]

We also have that \( \nabla \gamma_V = 0 \). This follows from general theory in [22]. Following the notation there, this fact is due to the correspondence between \( \gamma_V \) and \( d^3 \omega(V, \cdot, \cdot) = \psi^+_V \) coupled with Kirchenko’s Theorem, which states that \( \nabla \psi^+ = 0 \).

Lemma 3.1. If \( \mathcal{F} \) is a totally geodesic foliation of an irreducible Kähler manifold, then

(i) \( \gamma_V \neq 0 \) for all \( v \in \mathcal{V} \), and

(ii) \( \mathcal{V} = A_{\mathcal{H}} \mathcal{H} \).

Proof. For the first part, consider

\[ \beta(V, W) = \sum_{e_i \in \mathcal{H}} (\gamma_V(e_i), \gamma_W(e_i)), \]

where the sum is over an orthonormal basis of \( \mathcal{H} \). The fact that \( \nabla \beta = 0 \) coupled with Schur’s Lemma yields

\[ \beta(V, W) = kg(V, W) \]
for some constant $k$. Suppose now, to the contrary, that $\gamma_V(H) = 0$ for some $V \in \mathcal{V}$. Then
\[ k_1\|V\|^2 = g(V, V) = \beta(V, V) = 0 \]
which implies
\[ \beta(V, W) = 0 \]
for all $V, W \in \mathcal{V}$. But this implies $\gamma_V^2 = 0$ for all $V \in \mathcal{V}$, which means that $\gamma_V = 0$ for all $V \in \mathcal{V}$. This forces $A = 0$, which would mean that the Kähler metric $g$ splits. This is a contradiction, so the first part is established. For the second part, since $A_H H$ is a linear subspace of $\mathcal{V}$, the statement must hold unless there is an element $V \in \mathcal{V}$ such that $\langle V, A_X Y \rangle = 0$ for all $X, Y \in H$. This implies $\gamma_V = 0$, which contradicts the first statement of the lemma. \[ \square \]

From this result, combined with Lemma 2.1, it follows that if $\mathcal{F}$ is a totally geodesic foliation of a Kähler metric $g$, then $\nabla A = 0$. The converse to Lemma 3.1 (2) also holds:

**Lemma 3.2.** If $\mathcal{V} = A_H H$, then $\mathcal{F}$ is totally geodesic.

*Proof.* $\mathcal{F}$ is totally geodesic precisely when
\[ \langle \nabla_V W, X \rangle = \langle \nabla_{A_{E_1} E_2 A_{E_3} E_4} X \rangle = \langle (\nabla_{A_{E_1} E_2} A)_{E_3} E_4, X \rangle = 0 \]
by Lemma 2.1 (v).

Now consider the locally defined distributions
\[ \mathcal{V}_1 = A_H H, \text{ and } \mathcal{V}_0 = \mathcal{V}_1^\perp. \]
Both are complex. Moreover, we have the following general result.

**Proposition 3.3.** Let $(M, g, J)$ be a Kähler manifold equipped with a complex Riemannian foliation $\mathcal{F}$. Then either $\mathcal{F}$ is totally geodesic, or there is an open subset $U \subset M$ such that $\mathcal{V}_0$ is a polar Riemannian complex foliation of $U$.

*Proof.* Let us start by proving that $\mathcal{V}_0$ is locally integrable. Choose $V_0, W_0$ in $\mathcal{V}_0$. By definition $A_H H$ and $\mathcal{V}_0$ are orthogonal, and so
\[ \langle (\nabla_{V_0} A) X, W_0 \rangle = \langle \nabla_{V_0} (A_X Y), W_0 \rangle = -\langle A_X Y, \nabla_{V_0} W_0 \rangle. \]
This expression is symmetric in $V_0$ and $W_0$ by Lemma 2.1 (ii). The integrability of $\mathcal{V}_0$ follows.

The next step is to prove $(\mathcal{V}_1 \oplus H)$ is locally integrable and totally geodesic. For $X, Y$ in $H$ we have $\nabla_X Y = \nabla_X Y + A_X Y$ in $H \oplus \mathcal{V}_1$. Lemma 2.1 (ii) shows that
\[ T(\mathcal{V}_1, \mathcal{V}) = 0. \]
At the same time, Lemma 3.1 (v) guarantees that $\nabla_{\mathcal{V}_1} A = 0$ for $\mathcal{V}_1$ in $\mathcal{V}_1$. In particular $\mathcal{V}_1$ is totally geodesic with respect to $\nabla$. Therefore
\[ \nabla_{\mathcal{V}_1} W_1 = \nabla_{\mathcal{V}_1} W_1 \in \mathcal{V}_1. \]
If \((X,V_1)\) is in \(H \times V_1\) we have \(\nabla_{V_1} X = \nabla_{V_1} X\) in \(\mathcal{H}\) again by using 
\(T(V_1, \cdot) = 0\). Finally, note that \(V_1\) is parallel in direction of \(\mathcal{H}\), with respect to \(\nabla\). This is a consequence of Lemma 2.1 (i). Thus for a pair \((V_1, X)\) as above \(\nabla_X V_1 = \nabla_X V_1 + A_X V_1\) belongs to \(V_1 \oplus \mathcal{H}\). These calculations were local. An open subset \(U \subset M\) can then be chosen where \(\text{dim}(A_{\mathcal{H}} \mathcal{H})\) is constant by smoothness arguments, and our claim follows from combining these facts.

This establishes Theorem 1.2 because \(V_0\) vanishes when the foliation is totally geodesic. The result parallels the situation for closed Kähler manifolds, where \(\mathcal{F}\) must be either totally geodesic or polar. The natural question now is whether the above result could be improved to show the second author’s result also holds in the non-compact case. Certainly the techniques in [20] will not work here. In fact the above theorem is optimal, as we will demonstrate by showing how to construct a wide family of examples \((M, \mathcal{F})\) with \(\mathcal{F}\) a complex Riemannian foliation that is neither polar nor totally geodesic.

To this end, we follow a construction in [7]. Let \(N\) be an arbitrary Kähler manifold and \(U\) the unit ball in \(\mathbb{C}\). Take an arbitrary holomorphic function \(f : U \rightarrow N\). Set

\[
\Phi = \begin{pmatrix} \text{Re}(f) & \text{Im}(f) \\ \text{Im}(f) & -\text{Re}(f) \end{pmatrix}.
\]

Then \(\Phi \in S^2(TU)\) with respect to the standard basis. Now consider the product bundle with projection map \(\pi : M = U \times N \rightarrow N\). Fix a Kähler metric \(g_N\) on \(N\), and choose the standard complex structure on \(TU\), called \(J_0\). Take the metric

\[
g = g_0((1 + \Phi)^{-1}(1 - \Phi), \cdot) + g_N
\]
together with the compatible complex structure

\[
J = (1 - \Phi)^{-1} J_0 (1 + \Phi) + J_N,
\]

Then \((M, g, J)\) is Kähler, and

\[
TM = \mathbb{C} \oplus TN = V_0 + \mathcal{H}_0
\]
is a polar foliation. The key point is that there is no restriction on \(N\), so we can choose it to be a Kähler manifold equipped with a totally geodesic Riemannian foliation, inducing a splitting \(\mathcal{H}_0 = TN = V_1 \oplus \mathcal{H}\). Taking now the splitting \(\mathcal{V} \oplus \mathcal{H}\) of \(T(U \times N)\), with \(\mathcal{V}\) equal to \(V_0 \oplus V_1\), it is easy to check that \(\mathcal{V}\) is a complex Riemannian foliation which is not polar unless \(V_1\) splits off. Of course one also sees that \(V_0\) induces a polar foliation as the above theorem states. Concrete examples are given by taking \(N\) to be a twistor space of a quaternionic Kähler manifold with positive scalar curvature.

Our next result reduces the study of complex Riemannian foliations to a special case for Hermitian symmetric spaces of compact type.
Proposition 3.4. Let $M$ be an open set of an irreducible Hermitian symmetric space and $\mathcal{F}$ a complex Riemannian foliation on $M$. If $M$ has non-negative sectional curvature, then $\mathcal{F}$ is totally geodesic.

Proof. Consider the splitting

$$TM = V_0 \oplus (V_1 \oplus \mathcal{H})$$

over $M$. The integrability of both factors is equivalent to saying that $(g, \tilde{J})$ is almost-Kähler, where

$$\tilde{J}|_{V_0} = -J, \quad \tilde{J}|_{V_1 \oplus \mathcal{H}} = J.$$

That is $\tilde{J}$ is $g$-orthogonal and the Kähler form $g(\cdot, \cdot)$ is closed. Since $(M, g)$ is symmetric of compact type it follows that the isotropic sectional curvatures are non-negative. It follows from [1], proposition 1, (iii) that $\nabla \tilde{J} = 0$. Equivalently the distributions $V_0$ and $V_1 \oplus \mathcal{H}$ are parallel with respect to the Levi-Civita connection of $g$. Since the latter is irreducible and real analytic either $V_0 = 0$, corresponding to the totally geodesic case or $V_1 \oplus \mathcal{H} = 0$, which happens when the foliation is trivial. $\square$

4. Proof of Theorem 1.1(i)

4.1. Initial Computations. Without loss of generality, $\mathcal{F}$ may be assumed to be a totally geodesic complex Riemannian foliation of $M$, an open subset of a Hermitian symmetric space with non-negative sectional curvature by Proposition 3.4. As has already been demonstrated, the condition that $T = 0$ implies that $\nabla A = 0$.

Let us now calculate first-order information on the structure of the curvature tensor of $g$, using the fact that $\nabla R = 0$. In what follows $[,]$ denotes the commutator on operators.

Lemma 4.1. The following equations hold:

(i) $\gamma_R(V_1, V_2) V_3 = [\gamma V_1, \gamma V_2], \gamma V_3]$

(ii) $\gamma R(X_1, X_2) = [\gamma X_1, \gamma X_2] \gamma V - \gamma [\gamma X_1, \gamma X_2] V$.

Proof. (i) is a consequence of the differential Bianchi identity and holds independently of the assumption $\nabla R = 0$. See [20] for details. (ii) is a consequence of the fact our metric is locally symmetric:

$$(\nabla_{X_3} R)(X_1, X_2, X_3, V) = 0.$$ 

Expanding and using the fact that $R(X_1, X_2, X_3, V) = 0$ yields

$$0 = R(A_{X_4} X_1, X_2, X_3, V) + R(X_1, A_{X_4} X_2, X_3, V) + R(X_1, X_2, A_{X_4} X_3, V) + R(X_1, X_2, X_3, A_{X_4} V).$$

Expressing the first three terms above by means of Corollary 2.2 a short algebraic computation establishes the second claim. $\square$
Let $\mathcal{R}$ refer to the curvature tensor associated to the canonical Hermitian connection $\nabla$ of $g^{NK}$. It is easy to see that the nearly Kähler metric $g^{NK}$ is irreducible, as otherwise the metric $g$ would split and this would contradict the assumption of irreducibility. The rest of the proof involves establishing the following two statements:

**Statement 1** $\dim(\mathcal{F}) = \dim(\mathcal{V}) = 2$.

**Statement 2** $g$ has constant holomorphic sectional curvature.

To do this, we will appeal to classification results of the second author [21] throughout the proof. Take the Lie subalgebra $\mathfrak{hol}(\nabla)$ of the unitary algebra $u(T_pM)$ given by

$$\mathfrak{hol}(\nabla) = \text{Lie}\{R_{EF} : E,F \in T_pM\}.$$ 

The classification proceeds by studying the various representations of $\mathfrak{hol}(\nabla)$.

For a nearly Kähler metric arising from a totally geodesic complex Riemannian foliation $\mathcal{F}$, the 3 form $\psi^+$ is defined by the equation

$$\psi^+_E F = (\nabla_E \mathcal{J}) F$$

where $\mathcal{J}$ is the nearly Kähler almost-complex structure (see [20], [22]). The tensor $\psi^- = \psi^+(J\cdot\cdot\cdot)$ is precisely the torsion of $\nabla$.

5. **Proof of Statement 1.**

5.1. **Local Homogeneity.** We begin by expressing $\mathcal{R}_{XY}$ in terms of $R_{XY}$, where $X,Y \in \mathcal{H}$. In what follows, denote the Levi-Civita connection associated to the nearly Kähler metric by $\nabla^{NK}$. An easy calculation using standard identities yields that

$$R^{NK}_{XY} = R_{XY} - \gamma_{AXY} + \frac{1}{2}[A_X, A_Y]$$

for $X,Y \in \mathcal{H}$.

**Lemma 5.1.** For a complex totally geodesic Riemannian foliation $(M,g,\mathcal{F})$, one has

$$\psi^+_X Y = -\gamma_{AXY}.$$  

**Proof.** This follows from calculating

$$\langle \psi^+_X Z_1, Z_2 \rangle = \langle \psi^+_X Y, \psi^+_Z Z_2 \rangle.$$ 

Here $X,Y,Z_1,Z_2 \in \mathcal{H}$. But $\psi^+_X Y = (\nabla_X \mathcal{J}) Y = A_X JY$, and similarly for $\psi^+_Z Z_2$. So this equation is just

$$\langle A_{Z_1} (J Z_2), A_X JY \rangle = -\langle \gamma_{AXY} Z_1, Z_2 \rangle.$$ 

From here there is a short calculation to get the last equality. \qed
A short algebraic calculation using Equation (3.2) of [22] shows that
\[ [\psi^-, X, \psi^-, Y] = [\psi^+, X, \psi^+, Y] \]
and furthermore
\[ \tilde{R}_{XY} = R^{NK}_{XY} + \frac{1}{4} \left( [\psi^-, X, \psi^-] - 2\psi^- \psi^- Y \right) \]
\[ = R^{NK}_{XY} + \frac{1}{2}[A_X, A_Y] + \gamma_{AXY} \]
\[ = R_{XY} - \gamma_{AXY} + \frac{1}{2}[A_X, A_Y] + \frac{1}{2}[A_X, A_Y] + \gamma_{AXY} \]
\[ = R_{XY} + \frac{1}{2}[A_X, A_Y]. \]

Recall that a metric connection is said to be locally homogeneous if the torsion and curvature are parallel with respect to the metric. We are now in a position to prove the following result.

**Proposition 5.2.** \((M, g^{NK}, \nabla)\) is locally homogeneous.

**Proof.** The fact that \(\nabla \psi^- = 0\) is due to Kirichenko, and well-known. From Corollary 2.2 together with the fact that \(\nabla A = 0\) we see that the only non-trivial case is
\[ \nabla R(X, Y, Z_1, Z_2) \]
for \(X, Y, Z_1, Z_2 \in \mathcal{H}\). The upshot is that we must show
\[ (\nabla R)_{XY} = \nabla R_{XY} : \mathcal{H} \to \mathcal{H} \]
vanishes. From the equation
\[ \tilde{R}_{XY} = R_{XY} + [A_X, A_Y] \]
and the fact that \(\nabla\) satisfies \(\nabla A = 0\) it is immediate that
\[ (\nabla R)_{XY} = \nabla R_{XY}. \]
Because the Kähler metric is locally symmetric, we differentiate the second equation in Lemma 4.1 with respect to \(\nabla\), using that \(\nabla \gamma = 0\), to deduce that
\[ \gamma_V(\nabla R_{XY}) = \nabla([A_X, A_Y] \gamma_V - \gamma_{[A_X, A_Y]V}) = 0. \]
Lemma 3.1 shows that \(\gamma_V\) is injective on \(\mathcal{H}\), and so one may conclude that \(\nabla R_{XY} = 0\).

\(\square\)

5.2. **Local-to-global arguments.** In the previous section the local homogeneity of \((M, g^{NK}, \nabla)\) was established. Now one meets the problem that one cannot necessarily conclude that \((M, g^{NK})\) is isometric to an open subset of a closed homogeneous space \(G/H\). In this section Theorem 5.4 is proved, which allows us to overcome this obstacle. For this we will firstly recall some background on the Nomizu construction associated to locally homogeneous connections. We refer the reader to the expositions in [9] and [25] as a reference for these calculations. This will be utilized in the proof of Theorem 5.4.
Locally homogeneous connections $\nabla$ are those which, by definition, have parallel curvature and torsion. They are also called Ambrose-Singer connections. Associated to any such connection is the infinitesimal model on a vector space $V$. This is basically given by the action of the torsion $T : V \to \text{End}(V)$ and the curvature tensor $K : V \times V \to \text{End}(V)$. The Nomizu construction associates a Lie algebra $\mathfrak{g}$ to the infinitesimal model in the following manner. Let

$$\mathfrak{h} = \{ A \in \mathfrak{so}(n) : A \cdot T = A \cdot K = 0 \}$$

be the subalgebra whose elements vanish under the action of the torsion and curvature. Then

$$\mathfrak{g} = \mathfrak{h} \oplus V,$$

with Lie brackets

$$[X, Y] = -T_X Y + K_{XY}$$
$$[A, X] = A(X),$$
$$[A, B] = AB - BA,$$

where $X, Y \in V$ and $A, B \in \mathfrak{h}$. Now take $G$ to be the connected, simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and take $H$ to be its connected subgroup corresponding to $\mathfrak{h}$. If $H$ is closed in $G$, then the model is regular. In this situation one can conclude $(M, g)$ is isometric to an open subset of $G/H$. So our goal is to prove the infinitesimal model is regular. For this, we will need the following key lemma.

**Lemma 5.3.** The action of $\mathfrak{hol}(\nabla)$ on $\mathcal{H}$ is irreducible.

**Proof.** Suppose to the contrary that the action of $\mathfrak{hol}(\nabla)$ on $\mathcal{H}$ is reducible. This implies that $\mathcal{H} = \mathcal{E}_1 \oplus \mathcal{E}_2$, with

$$\langle \overline{R}_{XY} E_1, E_2 \rangle = 0$$

for all $X, Y \in \mathcal{H}$ and $E_i \in \mathcal{E}_i$. In particular

$$0 = \langle \overline{R}_{E_1 E_2} E_1, E_2 \rangle$$
$$= \langle (R_{E_1 E_2} + [A_{E_1}, A_{E_2}]) E_1, E_2 \rangle$$
$$= \text{sec}_g(E_1, E_2) + \| A_{E_1} E_2 \|^2.$$

As $g$ has non-negative sectional curvatures, it follows that $A_{E_1} \mathcal{E}_2 = 0$. A short calculation shows that this implies that

$$\langle \overline{R}^H_{XY} E_1, E_2 \rangle = 0$$

for all $X, Y \in \mathcal{H}$ and $E_i \in \mathcal{E}_i$, where

$$\overline{R}_{XY} = \overline{R}_{XY} + \psi^+_X Y$$

is the refined curvature tensor for the $\mathfrak{hol}(\nabla)$-invariant subspace $\mathcal{H}$ (see [22], Section 5.3). Let

$$\mathfrak{h}^\mathcal{H} = \text{Lie}\{ \overline{R}^H_{XY} : X, Y \in \mathcal{H} \}. $$
The last equation implies that the metric representation \((h^H, H)\) is reducible. Then Theorem 5.2 of [22] implies \(g^{NK}\) is reducible, which is a contradiction. So one concludes the original action of \(\frak{hol}(\nabla)\) on \(H\) is irreducible. \(\square\)

We now present the main theorem of this section.

**Theorem 5.4.** Any locally homogeneous nearly Kähler manifold \((M, g^{NK}, \nabla)\) arises as an open part of a closed homogeneous nearly Kähler manifold \(G/H\).

**Proof.** Applying the Nomizu construction, let \(\frak{hol}(\nabla) = g\) and \(V = T_p M\). In our situation, a sufficient condition for the infinitesimal model to be regular is that the centers \(Z\) of the Lie algebras \(\frak{hol}(\nabla)\) and \(\frak{hol}(\nabla) \oplus T_p M\) vanish. This implies the Lie algebras are semisimple and of compact type.

Then
\[
X_0 + V_0 \in Z(g \oplus V)
\]

means that
\[
[X_0 + V_0, X + V] = 0
\]
for all \(X \in g\) and \(V \in V\). This directly implies \(X_0 = 0\). Note that
\[
0 = [V_0, V] = \psi^+(V_0, V) - \mathcal{R}_{V_0}V \in g \otimes V
\]
for all \(V \in V\) implies, looking at the \(g\) component, that
\[
0 = \psi^+_V = \gamma_{V_0},
\]
which cannot happen by Lemma 3.1. Thus \(V_0 = 0\), and hence the center \(Z((g \oplus V) = 0\).

Next we will show \(Z(g) = 0\). Let \(\rho \in g\) be such that \([\rho, g] = 0\). As \(\rho \in g\), we have that \(\rho J\) is a symmetric endomorphism of \(V\). Note that \(\rho J(V) \subset V\) and \(\rho J(H) \subset H\). Moreover, as the action of \(g\) on both \(V \subset V\) is irreducible, we have that
\[
\rho J|_V = \lambda J|_V
\]
by Schur’s Lemma. By the preceding Lemma, the action of \(g\) on \(H\) is also irreducible in our setting. Thus we recover
\[
\rho = \lambda_1 J|_V \oplus \lambda_2 J|_H
\]
with \(\lambda_i \in \mathbb{R}\).

Recalling that \([\rho, \psi^+]\) is a 3-form defined by the equation
\[
[\rho, \psi^+](E, F, G) = \psi^+(\rho E, F, G) + \psi^+(E, \rho F, G) + \psi^+(E, F, \rho G)
\]
for \(E, F, G \in T_p M\), we see that the equation \([\rho, \psi^+] = 0\) implies that
\[
\rho \psi^+ = 0.
\]
As \(\psi^+\) is invariant under the action of \(g\), \(\rho \psi^+(V) \subset V\) and \(\rho \psi^+(H) \subset H\). Then
\[
0 = \rho \psi^+(X, Y, Z) = \left(\frac{\lambda_2}{3}\right) J(\psi^+|_H)(X, Y, Z)
\]

for $X, Y, Z \in \mathcal{H}$ which implies $\lambda_2 = 0$ by the strictness of the nearly Kähler structure. Secondly

$$0 = \rho \psi^+(X, Y, V) = \left(\frac{\lambda_1}{3}\right) J \psi^+(X, Y, V) = \lambda_1 \langle \gamma V, X, Y \rangle$$

for $X, Y \in \mathcal{H}$ and $V \in \mathcal{V}$. This implies $\lambda_1 = 0$ by Lemma 3.1. Thus $Z(g) = 0$, so we have a regular infinitesimal model and the theorem is established. □

The upshot of the last result is that a locally homogeneous nearly Kähler manifold $M$ is isometric to an open subset of a closed homogeneous space $G/H$. Shrinking $M$ to take it to be sufficiently small if necessary, and lifting $G/H$ to its universal cover, homogeneity allows us to conclude without loss of generality that $(G/H, g^{NK})$ is a simply connected homogeneous nearly Kähler metric. Here we extend our metric $g^{NK}$ on $M$ to all of $G/H$ by homogeneity. Again by homogeneity, considering the metric $g$ at $p \in M$ given by reversing the canonical variation to return to the Kähler metric, we see that $(G/H, g)$ is a simply connected Hermitian symmetric space of compact type. It is clear that the isometry group remains the same. Hence $(G/H, g)$ must be isometric to one of the following symmetric spaces:

| G       | H                                      | Restrictions |
|---------|----------------------------------------|--------------|
| $SU_n/\mathbb{Z}_n$ | $(SO_{2n-1} \times SO_2)/\mathbb{Z}_n$ | $n \geq 2$  |
| $SO_{2n+1}$     | $SO_{2n-1} \times SO_2$               | $n \geq 2$  |
| $Sp_n/\mathbb{Z}_2$ | $U_n/\mathbb{Z}_2$                   | $n \geq 3$  |
| $SO_{2n}/\mathbb{Z}_2$ | $(SO_{2n-1} \times SO_2)/\mathbb{Z}_2$ | $n \geq 4$  |
| $SO_{2n}/\mathbb{Z}_2$ | $(SO_{2n-1} \times SO_2)/\mathbb{Z}_2$ | $n \geq 4$  |
| $E_6/\mathbb{Z}_3$ | $(SO_{10} \times SO_2)/\mathbb{Z}_2$  | $-            |
| $E_7/\mathbb{Z}_3$ | $(E_6 \times T^1)/\mathbb{Z}_3$       | $-            |

Table 1. Hermitian Symmetric Spaces

Observe that the splitting $T_p M = \mathcal{V} \oplus \mathcal{H}$ must agree with the holonomy representation of the homogeneous nearly Kähler metric $g^{NK}$ on $G/H$. This now allows us to utilize the classification results in [21]. It follows that we have three possibilities to consider:

- $(G/H, g^{NK})$ is the twistor space of a quaternionic-Kähler manifold of positive scalar curvature,
- $(G/H, g^{NK})$ is six dimensional,
- $(G/H, g^{NK})$ is a 3-symmetric space of type I, II, III, or IV.

In the first case, Statement 1 automatically holds. For the second case, if we were to have a six dimensional Kähler manifold $(G/H, g)$ endowed with a complex Riemannian foliation, then we need to show $\dim(\mathcal{V}) = 2$. Otherwise it is four, but then $\dim(\mathcal{H}) = 2$. From the equation $A_{JX}JY = -A_XY$ we see this implies $A = 0$, which would imply the metric splits. Hence $\dim(\mathcal{V}) = 2$ in this case.
3-symmetric spaces of type I cannot occur by definition, because \( \nabla \) has special algebraic torsion (so the representation of \( \mathfrak{so}(\nabla) \) cannot be irreducible). To rule out 3-symmetric spaces of Type II, notice that by definition \( \mathfrak{so}(\nabla) \) induces a Lagrangian splitting \( T_pM = E \oplus JE \) into irreducible totally real subspaces. Since \( \mathfrak{so}(\nabla)(V) \subset V \), either \( V = E \) or \( V = JE \). Both cases cannot happen, as \( V \) is complex and \( E \) and \( JE \) are totally real.

So the remaining case to analyze are when \( (G/H, g^{NK}) \) is a 3-symmetric space of type III or IV. A complete list of all such 3-symmetric spaces together is given in [6], Tables 2 and 3. Symmetric spaces of Type IV are immediately ruled out, as it has been shown that \( g \) must act on \( H \) irreducibly. The list of 3-symmetric spaces of Type III with \( \text{dim}(V) \neq 2 \) is as follows:

| G          | H                                      | \( \text{dim}(V) \) | Restrictions       |
|------------|----------------------------------------|----------------------|--------------------|
| \( SO_{2n+1} \) | \( SO_{(2n-i)+1} \times U_i \)        | \( i(i - 1) \)      | \( n > 2 \) and \( i > 2 \) |
| \( Sp_n/Z_2 \) | \( (U_i \times Sp_{n-1})/Z_2 \)         | \( i(i + 1) \)      | \( n > 2 \) and \( 1 < i < n \) |
| \( SO_{2n}/Z_2 \) | \( (SO_{2(n-i)} \times U_i)/Z_2 \) | \( i(i - 1) \)      | \( n > 2 \) and \( 2 < i < n - 1 \) |
| \( F_4 \) | \( Spin_7 \times T^1 \)                  | 14                   | –                  |
| \( E_6/Z_3 \) | \( (S(U_5 \times U_1 \times SU_2))/Z_2 \) | 10                   | –                  |
| \( E_7/Z_2 \) | \( (SU_2 \times (SO_{10} \times SO_2))/Z_2 \) | 20                   | –                  |
| \( E_7/Z_2 \) | \( S(U_7 \times U_1)/Z_4 \)              | 14                   | –                  |
| \( E_8 \) | \( SO_{14} \times SO_2 \)               | 28                   | –                  |

Table 2. 3-symmetric spaces of type III with \( \text{dim}(V) \neq 2 \)

It is easy to compute on a case-by-case basis that there is no overlap between Tables 1 and 2. Since we know the isometry group \( G \) has to remain the same, one only has to check that the dimensions of the corresponding isotropy groups are different. Hence the only possibility is \( \text{dim}(V) = 2 \).

5.3. Conclusion. Combining the above arguments, we see that Statement 1 is established.

6. Proof of Statement 2.

Since \( V \) is two-dimensional,

\[
R(V,W)U = \alpha(V,W)U
\]

and via Schur’s Lemma we see that

\[
\alpha(V,W) = \Omega g(JV,W)J
\]

for some constant \( \Omega \), because \( \gamma \) (and hence \( \alpha \)) are \( \nabla \)-parallel. Here we utilize Lemma 4.1(i).
Now, 
\[ [\gamma_V, \gamma_JV] = 2\gamma_V^2 J = \Omega g(V, V) J, \]
in particular \( \Omega \geq 0 \). We employ algebraic polarization and calculate 
\[ 2\gamma_V \gamma_W = [\gamma_V, \gamma_W] + J[\gamma_V \gamma_JW] \]
It follows that 
\[ \gamma_V \gamma_W = \frac{\Omega}{2} \left( g(JV, W) J - g(V, W) Id \right) \].
In particular
\[(6.1) \quad A_X \gamma_V Y = \frac{\Omega}{2} (g(X, Y)V - g(JX, Y)JV) \]
To conclude the proof we will calculate the holomorphic sectional curvatures of \((M, g)\). From Lemma (4.1)
\[ \gamma_V R(X, JX)X = [A_X, A_JX] \gamma_V X - \gamma_{[A_X, AJX]} X = 4JA_X (A_X \gamma_V X) \]
for all \((V, X)\) in \( V \times H \), by using the invariance properties of \( A \). By Lemma (4.1) we get
\[ \gamma_V R(X, JX)X = 2\Omega g(X, X) \gamma_V (JX), \]
so applying injectivity of \( \gamma_V \) one sees that 
\[ R(X, JX)X = 2\Omega g(X, X)JX \]
for \( X \) in \( H \). The rest of the curvature terms are handled similarly by using Corollary 3.1 whilst taking Equation (6.1) into account. Eventually \( g \) turns out to have constant holomorphic sectional curvature \( 2\Omega \), and it follows that \((M, g)\) is isometric to a complex projective space.

To see that \( M/F = N \) is isometric to an open subset of \( \mathbb{H}P^n \), observe that a local geodesic reflection in \( p \in M/F \) of a curve \( c \) can be obtained by taking a horizontal lift of the curve to \( M \), reflecting it in the fibre over \( p \) and projecting back to the quotient space. Hence \( N \) is a locally symmetric space, and by the standard O'Neill formula for sectional curvatures under a Riemannian submersion we see \( N \) is locally isometric to a rank one symmetric space. \( N \) is an open piece of \( \mathbb{H}P^n \): the quaternionic structure is given by projecting 
\[ \mathbb{R}J \oplus \{\gamma_V : V \in V\} \]
down to \( N \). This concludes the proof of Theorem 1.1(i).

7. **Proof of Theorem 1.1 (ii)**

**Proof.** We have the splitting 
\[ TM = V_0 \oplus (V_1 \oplus H). \]
Since \( M \) is locally symmetric with non-positive sectional curvatures and \((V_1 \oplus H)\) is totally geodesic, the sectional curvatures of the integral manifolds are non-positive with respect to the induced metric. Each integral manifold carries a totally geodesic foliation corresponding to the leaves of \( V_1 \) since
$\mathcal{F}$ is polar. □
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