Single-Source Dilation-Bounded Minimum Spanning Trees*

Otfried Cheong** Changryeol Lee**

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Abstract

Given a set \( S \) of points in the plane, a geometric network for \( S \) is a graph \( G \) with vertex set \( S \) and straight edges. We consider a broadcasting situation, where one point \( r \in S \) is a designated source. Given a dilation factor \( \delta \), we ask for a geometric network \( G \) such that for every point \( v \in S \) there is a path from \( r \) to \( v \) in \( G \) of length at most \( \delta |rv| \), and such that the total edge length is minimized. We show that finding such a network of minimum total edge length is NP-hard, and give an approximation algorithm.

Keywords: geometric network, spanner, minimum spanning tree, dilation, single source dilation

1 Introduction

Given a set \( S \) of points in the plane, a geometric network for \( S \) is an edge-weighted graph \( G \) with vertex set \( S \) and straight edges. The weight of an edge \((u,v)\) is the length of the segment \( uv \), that is, the Euclidean distance \( |uv| \) of the two points.

Various types of networks, such as communication networks, road networks, or telephone networks, have been modeled as geometric networks. One important parameter of a geometric network is its total cost \( \ell(G) \): the sum of all edge lengths. The network that minimizes the cost while connecting all points in \( S \) is the Euclidean minimum spanning tree of \( S \). Another well-studied parameter is the dilation of a network. For two points \( u,v \in S \), the dilation of the pair \((u,v)\) is defined to be the ratio

\[ \Delta_G(u,v) := \frac{d_G(u,v)}{|uv|} \]

of the length of the shortest path between \( u \) and \( v \) in \( G \) and the distance \( |uv| \). The dilation of a network is commonly defined as the maximum of \( \Delta_G(u,v) \) over all pairs \( u,v \in S \). The network minimizing the dilation is the complete graph, which has dilation 1.

The complete graph has prohibitively large cost, while the minimum spanning tree may have large dilation. Balancing these two parameters has been the subject of much research in the literature, we refer to the book by Narasimhan and Smid [4] for an overview.

In this paper, we consider a broadcasting situation, where one point \( r \in S \) is a designated source, and the purpose of the network is to broadcast information from \( r \) to all the other nodes. A node \( v \) receives the information with delay \( d_G(r,v) \), and we are interested in the relative delay, which is the dilation

\[ \Delta_G(r,v) = \frac{d_G(r,v)}{|rv|} \].

We define the (relative) delay of \( G \) to be the maximum

\[ \Delta(G) = \max_{v \in S \setminus \{r\}} \Delta_G(r,v) = \max_{v \in S \setminus \{r\}} \frac{d_G(r,v)}{|rv|} \]

Note that we can assume our network \( G \) to be a tree, as the shortest-path tree with source at \( r \) will have the same delay as \( G \) itself.

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**Department of Computer Science, KAIST, Daejeon, South Korea. (otfried,pittzzang)@kaist.ac.kr
The network minimizing $\Delta(G)$ is the star consisting of all edges $(r, v)$, for $v \in S \setminus \{r\}$. It has $\Delta(G) = 1$, but its cost $\ell(G)$ may be prohibitively large. The minimum spanning tree, which minimizes $\ell(G)$, may have large delay. In practice, a trade-off between both factors needs to be achieved. Related notions have been studied in the literature. For instance, computing the tree $T$ that minimizes $\ell(T)$ while bounding the distance of all nodes from the source in the graph is NP-complete [4]. Similarly, minimizing $\ell(T)$ while keeping all nodes at most $k$ hops from the source is NP-complete [4].

Our problem, which we term the single-source dilation-bounded minimum spanning tree problem, takes as input the point set $S$, the designated source point $r \in S$, and a delay bound $\delta \geq 1$. We ask for a spanning tree $T$ of smallest possible cost $\ell(T)$ such that $\Delta(T) \leq \delta$, or, in other words, such that for every $v \in S \setminus \{r\}$, we have $d_T(r, v) \leq \delta |rv|$. When we set $\delta = 1$, then the answer is simply the star connecting $r$ with all other points. When we set $\delta \geq n - 1$, then the answer is the minimum spanning tree of $S$, and so our problem interpolates between these two networks.

We show that solving the problem exactly is NP-hard by reduction from Knapsack in Section 3. This leads us to consider an approximation algorithm. Since this algorithm is rather simple, we present it first in Section 2.

2 Approximation Algorithm

An approximate solution to our problem can be computed using known algorithms for $\delta$-spanners. We introduce these results first.

Given a point set $S$ in the plane, a $\delta$-spanner for $S$ is a geometric network on $S$ such that $d_G(u, v) \leq \delta |uv|$ for each pair of points $u, v \in S$, where $\delta \geq 1$. Many algorithms to compute a $\delta$-spanner for a given point set have been given in the literature, see again the book by Narasimhan and Smid [4] for an overview. We will make use of a result by Guzmundsson, Levcopoulos, and Narasimhan [3], who showed the following:

For any real number $\delta > 1$, a $\delta$-spanner of a point set $S$ in $\mathbb{R}^d$ can be constructed in $O(n \log n)$ time such that the spanner has $c_1 n$ edges, maximum degree $c_2$, and cost $c_3 \ell(M)$, where $M$ is the minimum spanning tree of $S$, and $c_1, c_2, c_3$ are constants depending on $\delta$ and the dimension $d$.

We can now explain our simple approximation algorithm:

**Theorem 1.** Given a set of $n$ points $S$ in the plane, a designated source $r \in S$, and a real constant $\delta > 1$, we can construct in time $O(n \log n)$ a tree $T$ with vertex set $S$ such that the delay $\Delta(T) \leq \delta$, and $\ell(T) \leq c_3 \ell(M)$, where $M$ is the minimum spanning tree of $S$ and $c_3$ is a constant depending on $\delta$.

**Proof.** We first run the algorithm by Gudmundsson et al. [3] to obtain a $\delta$-spanner $G$ for $S$, with total cost $\ell(G) \leq c_3 \ell(M)$, where $c_3$ is a constant that depends on $\delta$.

We then compute the shortest path tree $T$ with source $r$ in $G$. Clearly we have $\ell(T) \leq \ell(G) \leq c_3 \ell(M)$. For any $v \in S \setminus \{r\}$, we have $d_T(r, v) = d_G(r, v) \leq \delta |rv|$ since $G$ is a $\delta$-spanner, and so the delay $\Delta(T) \leq \delta$.

$G$ can be computed in time $O(n \log n)$ [3], and the shortest-path tree $T$ can be computed in time $O(n \log n)$, for instance using Dijkstra’s algorithm, using the fact that $G$ has $O(n)$ edges.

3 Single-Source Dilation-Bounded Minimum Spanning Tree is NP-hard

In this section, we show that the decision version of our problem is NP-hard.

**Theorem 2.** Given a set $S$ of points in the plane, a designated source $r \in S$, a delay bound $\delta \geq 1$, and a cost bound $K \geq 0$, it is NP-hard to decide if there exists a tree $T$ for $S$ with delay $\Delta(T) \leq \delta$ and cost $\ell(T) \leq K$.

The proof is by reduction from Knapsack, which we define first. Knapsack is well known to be NP-complete [2].

**Knapsack:** Given a set of $n$ items, each having an integer profit $p_i > 0$ and an integer weight $w_i > 0$, as well as a profit bound $P > 0$ and a weight bound $W > 0$. Does there exist a subset of items with total weight at most $W$ and total profit at least $P$?
Our reduction takes as input an instance of KNAPSACK, and produces an instance of single-source dilation-bounded minimum spanning tree. For a set of \( n \) items with profits \( p_i \) and weight \( w_i \), we construct a set \( S \) of \( 3n + 4 \) points. For each item \( i \), we construct three points \( a_i, b_i, c_i \). In addition, we create three extra points \( d_0, d_1, d_2 \), as well as a source \( r \).

We first define

\[
\alpha_i := p_i + w_i > 0 \\
\beta_i := 2p_i + w_i = \alpha_i + p_i > \alpha_i \\
\gamma_i := 3p_i + w_i = \beta_i + p_i > \beta_i.
\]

We thus have

\[
\alpha_i + \beta_i = 3p_i + 2w_i = \gamma_i + w_i > \gamma_i,
\]

and so \( 0 < \alpha_i < \beta_i < \gamma_i < \alpha_i + \beta_i \). This implies that the three values \( \alpha_i, \beta_i, \gamma_i \) satisfy the triangle inequality. We also set

\[
m := \max_{1 \leq i \leq n} \gamma_i, \quad L := \sum_{1}^{n} (\gamma_i + m),
\]

**Construction of \( S \).** Our construction starts by placing the source \( r \) at the origin. We then place the \( 2n \) points \( a_i \) and \( b_i \) on the negative \( y \)-axis as follows (see Fig. 1):

\[
a_i := (0, -4L - \sum_{j=1}^{i-1} (\gamma_j + m)) \\
b_i := a_i - (0, \gamma_i)
\]
We have $|a_ib_i| = \gamma_i$ and $|b_ia_{i+1}| = m$. We can now place $c_i$, for $1 \leq i \leq n$ on the right side of the $y$-axis such that $|a_ic_i| = \beta_i$ and $|c_ib_i| = \alpha_i$. Finally, we define $d_0$, $d_1$, and $d_2$ as follows:

$$d_0 := (0, -5L)$$
$$d_1 := (0, -8L)$$
$$d_2 := (-6L, -8L)$$

**Regular trees.** We classify the edges connecting points of $S$ into regular and irregular edges. Regular edges are the edges $r\alpha_i, b_i\beta_i, d_0d_1, d_1d_2$, the edges $a_ib_i, b_i\alpha_i, a_i\beta_i$, for $1 \leq i \leq n$, and the edges $b_ia_{i+1}$ for $1 \leq i < n$. A regular tree is a spanning tree on $S$ which contains only regular edges. A tree containing irregular edges is irregular.

In a regular tree $T$, the dilation of the pair $(r, d_2)$ is much larger than the dilation of any other pair $(r, v)$, and so we have

**Lemma 3.** The delay of a regular tree $T$ is $\Delta(T) = \Delta_T(r, d_2)$.

**Proof.** Let $L_i = |a_i\alpha_i| = \sum_{j=1}^{i-1} (\gamma_j + m)$. Since $T$ is a tree, there is a unique path from $b_i$ to $a_{i+1}$ and we have $d_T(a_i, b_i) \leq |a_ic_i| + |c_ib_i|$. Since $\alpha_i < \beta_i < \gamma_i \leq m$, we have

$$d_T(r, b_i) \leq |r, a_1| + \sum_{j=1}^{i} (|a_jc_j| + |c_jb_j|) + \sum_{j=1}^{i-1} |b_ja_{j+1}|$$

$$= 4L + \sum_{j=1}^{i} (\beta_j + \alpha_j) + \sum_{j=1}^{i-1} m < 4L + L_i = 5L + L_i$$

$$d_T(r, c_i) \leq |r, a_1| + \sum_{j=1}^{i-1} (|a_jc_j| + |c_jb_j| + |b_ja_{j+1}|) + |a_ib_i| + |b_ic_i|$$

$$= 4L + \sum_{j=1}^{i-1} (\beta_j + m) + \sum_{j=1}^{i-1} \alpha_j < 4L + L_i + L = 5L + L_i$$

Note that $d_T(r, b_i) = d_T(r, a_i) + d_T(a_i, b_i) > d_T(r, a_i)$. Thus, we get

$$\Delta_T(r, a_i) = \frac{d_T(r, a_i)}{|ra_i|} < \frac{d_T(r, b_i)}{|ra_i|} = \frac{5L + L_i}{4L + L_i} \leq 1.25$$

$$\Delta_T(r, b_i) = \frac{d_T(r, b_i)}{|rb_i|} < \frac{d_T(r, b_i)}{|ra_i|} \leq 1.25.$$

Since $|rc_i| > |ra_i|$, we also have

$$\Delta_T(r, c_i) = \frac{d_T(r, c_i)}{|rc_i|} < \frac{d_T(r, c_i)}{|ra_i|} = \frac{5L + L_i}{4L + L_i} \leq 1.25$$

$$\Delta_T(r, d_0) = \frac{d_T(r, d_0)}{|rd_0|} = \frac{d_T(r, b_i) + m}{5L} < \frac{5L + L_i + m}{5L} < \frac{6L}{5L} < 1.25$$

$$\Delta_T(r, d_1) = \frac{d_T(r, d_1)}{|rd_1|} = \frac{d_T(r, d_0) + |d_0d_1|}{5L + |d_0d_1|} < \frac{6L + |d_0d_1|}{5L + |d_0d_1|} < 1.25.$$  

On the other hand,

$$\Delta_T(r, d_2) = \frac{d_T(r, d_2)}{|rd_2|} \geq \frac{|rd_1| + |d_1d_2|}{10L} = \frac{14L}{10L} = 1.4$$

Hence, the delay of $T$ is determined by $d_2$:

$$\Delta(T) = \max_{v \in S(r)} \Delta_T(r, v) = \Delta_T(r, d_2)$$

We define a special regular tree, the base tree $T_0$, which contains all regular edges except for the edges $a_ic_i$. Delay and cost of the base tree are as follows.
Lemma 4. The total edge length of the base tree is $\ell(T_0) < 14.5L$, and its delay is $\Delta(T_0) = 1.4$.

Proof. The total edge length and the delay of the base tree $T_0$ are

$$\ell(T_0) = |rd_1| + \sum_{i=1}^{n} (|b_i c_i|) + |d_1 d_2|$$

$$= 8L + \frac{1}{2} \sum_{i=1}^{n} 2\alpha_i + 6L < 14L + \frac{1}{2} \sum_{i=1}^{n} (\gamma_i + m) = 14.5L$$

$$\Delta(T_0) = \Delta_{T_0}(r, d_2) = \frac{|rd_1| + |d_1 d_2|}{|rd_2|} = \frac{8L + 6L}{10L} = 1.4$$

Connecting $d_2$ to any point other than $d_1$ will always produce trees with higher cost than the base tree:

Lemma 5. If $T$ is an irregular tree that contains an edge $vd_2$ for $v \neq d_1$, then $\ell(T) > \ell(T_0)$.

Proof. Assume $T$ contains an edge $vd_2$, for $v \neq d_1$, and consider the path $Q$ in $T$ connecting $r$ and $d_1$.

There are two possible cases. First, assume that $Q$ passes through $d_2$. Then we have

$$\ell(T) \geq \ell(Q) \geq |rd_2| + |d_2 d_1| = 10L + 6L = 16L > \ell(T_0).$$

In the second case, $Q$ does not pass through $d_2$. Since $|vd_2| \geq |d_2d_1|$ and $\ell(Q) \geq |rd_1|$, we have

$$\ell(T) \geq \ell(Q) + |vd_2| \geq |rd_1| + |d_2d_2| = 8L + 3\sqrt{5}L \approx 14.7082L > \ell(T_0).$$

We can now show that for $\delta \geq 1.4$, regular trees are better than irregular trees.

Lemma 6. For every irregular tree $T_1$ with $\Delta(T_1) \geq 1.4$, there exists a regular tree $T^*$ such that $\ell(T_1) \geq \ell(T^*)$ and $\Delta(T_1) \geq \Delta(T^*)$.

Proof. If $T_1$ includes an edge $vd_2$ with $v \neq d_1$, then Lemma 5 implies the lemma with $T^* = T_0$. We can therefore assume that $T_1$ contains the edge $d_1 d_2$, and no other edge incident to $d_2$.

For a spanning tree $T$ of $S$, let $\pi(T)$ denote the path connecting $r$ and $d_1$ in $T$. We define $J(T)$ to be the set of indices $i \in \{1, \ldots, n\}$ such that $\pi(T)$ contains all three vertices $a_i, b_i, c_i$, but does not contain both edge $a_i c_i$ and $b_i c_i$.

Let $T$ denote the set of spanning trees of $T$ such that $d_2$ is adjacent only to $d_1$ in $T$, $\ell(T) \leq \ell(T_1)$, and $\ell(\pi(T)) \leq \ell(\pi(T_1))$. We have $T_1 \in T$, so $T \neq \emptyset$.

We now pick a spanning tree $T_2$ in $T$ such that $J(T_2)$ is minimal under inclusion. Let us suppose first that $J(T_2) \neq \emptyset$. Then there is an $i \in J(T_2)$ such that $\pi(T_2)$ contains $\{a_i, b_i, c_i\}$, but does not contain both $a_i c_i$ and $b_i c_i$.

Let $u$ be the first of the three vertices encountered by $\pi(T_2)$, let $v$ be the second one, and let $w$ be the last one. One of the two edges incident to $v$ in $\pi(T_2)$ is different from $a_i c_i$ and $b_i c_i$. Denote this edge by $e$. Then $|uv| \geq \gamma_i \geq |uw|$. We obtain a new spanning tree $T_3$ from $T_2$ by adding $uw$ and removing $e$. Then $\ell(T_3) = \ell(T_2) + |uw| - |e| \leq \ell(T_2) < \ell(T_1)$. Clearly $\ell(\pi(T_3)) \leq \ell(\pi(T_2))$, and so $T_3 \in T$. Furthermore, $J(T_3) \subseteq J(T_2)$ as $i \notin J(T_3)$, a contradiction to the choice of $T_2$.

It follows that $J(T_2) = \emptyset$. Let us define the set $I \subset \{1, \ldots, n\}$ of indices $i$ such that $\pi(T_2)$ contains both edge $a_i c_i$ and $c_i b_i$. We define $T^*$ as the tree consisting of all regular edges, except that we remove $a_i b_i$ when $i \in I$, and that we remove $a_i c_i$ when $i \notin I$. Fig. 2 shows an example of the regular tree we construct. Let $E_0$ denote the set of edges $a_i c_i$ and $c_i b_i$ for all $i \in I$. By definition of $I$, $E_0 \subseteq \pi(T_2)$, and we define $E_1 = \pi(T_2) \setminus E_0$. Let $E_0^*$ and $E_1^*$ be the projection of $E_0$ and $E_1$ on the $y$-axis. Then $E_0^* \cup E_1^*$ must be equal to the segment $r d_1$, and so $\ell(E_0) \geq \ell(E_1) \geq |rd_1| - \ell(E_0)$. It follows that

$$\ell(\pi(T^*)) = |rd_1| - \ell(E_0) \leq \ell(E_0) + \ell(E_0) = \ell(\pi(T_2)) \leq \ell(\pi(T_1)).$$

This implies that $\Delta(T_1) \geq \Delta_{T_1}(r, d_2) \geq \Delta_{T_2}(r, d_2) = \Delta(T^*)$ by Lemma 5.

For each point $v \in S \setminus \{r\}$, let $p(v)$ be the second vertex on the path from $v$ to $r$ in $T_2$. We have $\ell(T_2) = \sum_{v \in S \setminus \{r\}} |vp(v)|$. For the vertices on $\pi(T_2)$, we have $\sum_{v \in \pi(T_2) \setminus \{r\}} |vp(v)| = \ell(\pi(T_2))$. Since
such that \( a \) the three edges \( d \) include an edge incident to
\( \text{Lemma 7.} \) tree of small delay and small cost if and only if the original
\( \ell \) and the claim follows.
It remains to show that the constructed point set \( S \) has a spanning
tree of small delay and small cost if and only if the original \textsc{Knapsack} instance had a positive answer.
\textbf{Lemma 7.} The \textsc{Knapsack} instance has a positive answer if and only if there is a spanning tree \( T \) for \( S \) with delay \( \Delta(T) \leq 1.4 + (W/10L) \) and cost \( \ell(T) \leq \ell(T_0) - P \).

\textbf{Proof.} We first assume that the \textsc{Knapsack} instance has a positive answer. Let \( I \subseteq \{1, 2, \ldots, n\} \) be a set of indices such that \( \sum_{i \in I} p_i \geq P \) and \( \sum_{i \in I} w_i \leq W \). Let \( T \) be the tree consisting of all regular edges, except that we exclude \( a_ib_i \) for \( i \in I \), and exclude \( a_ic_i \) for \( i \not\in I \).

Then we have
\[
\ell(T) = \ell(T_0) - \sum_{i \in I} (\gamma_i - \beta_i) = \ell(T_0) - \sum_{i \in I} p_i \leq \ell(T_0) - P,
\]
\[
d_T(r, d_2) = |rd_1| + |d_1d_2| + \sum_{i \in I} (\alpha_i + \beta_i - \gamma_i) = 14L + \sum_{i \in I} w_i \leq 14L + W
\]
\[
\Delta(T) = \Delta_T(r, d_2) \leq \frac{14L + W}{10L} = 1.4 + (W/10L),
\]
and the claim follows.
Assume now that \( T \) is a spanning tree for \( S \) with the given bounds. If \( \Delta(T) < 1.4 \), then \( T \) must include an edge incident to \( d_2 \) other than \( d_1d_2 \) and is not regular. But then Lemma \[5\] implies that \( \ell(T) > \ell(T_0) \), a contradiction. So \( \Delta(T) \geq 1.4 \), and by Lemma \[6\] we can assume that \( T \) is regular.

Since \( T \) is a spanning tree, it must include all regular edges, except that for each \( 1 \leq i \leq n \), one of the three edges \( a_ib_i, a_ic_i, \) or \( b_ic_i \) must be missing. We define \( I \subseteq \{1, 2, \ldots, n\} \) to be the set of indices \( i \) such that \( T \) does not include the edge \( a_ib_i \).
We have 

\[ d_T(r, d_2) = |rd_1| + |d_1d_2| + \sum_{i \in I}(\alpha_i + \beta_i - \gamma_i) = 14L + \sum_{i \in I} w_i. \]

Since \( \Delta(T) = \Delta_T(r, d_2) = d_T(r, d_2)/10L \), we have

\[ \sum_{i \in I} w_i = d_T(r, d_2) - 14L = 10L\Delta(T) - 14L \leq 14L(1.4 + (W/10L)) - 14L = W. \]

The cost of \( T \) is

\[ \ell (T) \geq \ell(T_0) - \sum_{i \in I}(\gamma_i - \beta_i) = \ell(T_0) - \sum_{i \in I} p_i, \]

and so

\[ \sum_{i \in I} p_i \geq \ell(T_0) - \ell(T) \geq \ell(T_0) - (\ell(T_0) - P) = P. \]

It follows that the Knapsack instance has a positive answer. \( \Box \)

**Reduction with integer coordinates.** To complete our proof of Theorem 2 we need to construct a set of points with integer coordinates, such that the total number of bits is polynomial in the size of Knapsack instance. The construction given so far does not achieve this yet, since the points \( c_i \), defined as the solution of a quadratic equation. We will therefore compute approximations \( \tilde{c}_i \) with \( |c_i - \tilde{c}_i| < \varepsilon \), for an \( \varepsilon \) to be determined later. The set of points obtained in that way will be denoted by \( \tilde{S} \), which is the set of points \( r, a_1, b, \tilde{c}_i, d_0, d_1, d_2 \). In the following lemma, we bound by how much this approximation can change the tree cost and delay.

**Lemma 8.** If \( T \) is a spanning tree on \( S \) and \( \tilde{T} \) is the corresponding tree on \( \tilde{S} \), then \( |\ell(T) - \ell(\tilde{T})| < 12n\varepsilon \), and \( |\Delta(T) - \Delta(\tilde{T})| < 20n\varepsilon \).

*Proof.* Let \( u, v \) be a pair of points in \( S \), with \( \tilde{u}, \tilde{v} \) denoting the corresponding points in \( \tilde{S} \). Since \( |uv| \leq \varepsilon \) and \( |\tilde{u}\tilde{v}| \leq \varepsilon \), we have \( |uv| - |\tilde{u}\tilde{v}| \leq 2\varepsilon \). The tree \( T \) has \( 3n + 3 \) edges, and so \( |\ell(T) - \ell(\tilde{T})| < (6n + 6)\varepsilon \leq 12\varepsilon \).

Consider now \( X := d_T(r, v), \tilde{X} := d_T(r, \tilde{v}), Y := |rv|, \text{ and } \tilde{Y} := |r\tilde{v}|. \) Since \( |v\tilde{v}| \leq \varepsilon \), we have \( |Y - \tilde{Y}| \leq \varepsilon \). The path from \( r \) to \( \tilde{v} \) in \( \tilde{T} \) passes through most \( n \) approximated points, and so \( |X - \tilde{X}| \leq 2n\varepsilon \). Since the longest edge in \( T \) has length \( 10L \) and the path has at most \( 3(n + 1) \) edges, we have \( X \leq 30L(n + 1) \).

We also have that \( Y \geq 4L \) by the construction of \( S \). This means that \( X/Y < 7.5(n + 1) \). We get

\[
\frac{\tilde{X}}{\tilde{Y}} - \frac{X}{Y} = \frac{\tilde{X}Y - XY}{\tilde{Y}Y} < \frac{Y(X + 2n\varepsilon) - X(Y - \varepsilon)}{YY} = \frac{2n\varepsilon + \varepsilon}{Y} \cdot \frac{X}{Y},
\]

\[
\frac{X}{\tilde{Y}} - \frac{\tilde{X}}{\tilde{Y}} = \frac{XY - \tilde{X}Y}{\tilde{Y}Y} < \frac{X(Y + \varepsilon) - \tilde{X}(Y - 2n\varepsilon)}{YY} = \frac{2n\varepsilon + \varepsilon}{Y} \cdot \frac{X}{Y}.
\]

And since \( \tilde{Y} \geq 1 \),

\[
\frac{2n\varepsilon + \varepsilon}{Y} \cdot \frac{X}{Y} \leq 2n\varepsilon + \varepsilon \times 7.5(n + 1) < 20n\varepsilon.
\]

If the Knapsack instance has a positive answer, then \( S \) has a spanning tree \( T \) with \( \Delta(T) \leq 1.4 + (W/10L) \) and \( \ell(T) \leq \ell(T_0) - P \). On the other hand, if the instance has a negative answer, then this implies that for any subset of indices \( I \subset \{1, 2, \ldots, n\} \) we have either \( \sum_{i \in I} w_i \geq W + 1 \) or \( \sum_{i \in I} p_i \leq P - 1 \). By Lemma 7 this means that any spanning tree \( T \) for \( S \) has either delay \( \Delta(T) \geq 1.4 + ((W + 1)/10L) \) or cost \( \ell(T) \geq \ell(T_0) - P + 1 \).

Let us set \( \varepsilon = 1/600nL \). We approximate the \( c_i \) with a precision of most \( \varepsilon \), resulting in the point set \( \tilde{S} \). This set is the input to our problem, with a delay bound of \( \delta = 1.4 + (W/10L) + (1/20L) \), and a cost bound of \( K = \ell(T_0) - P + 0.5 \).

If the Knapsack instance has a positive answer, then by Lemma 8 there is a spanning tree \( \tilde{T} \) for \( \tilde{S} \) with \( \Delta(\tilde{T}) \leq 1.4 + (W/10L) + 20n\varepsilon = 1.4 + (W/10L) + (1/30L) < \delta \) and \( \ell(\tilde{T}_0) - \ell(\tilde{T}) > P - 24n\varepsilon > P - 0.5 \).

On the other hand, if the Knapsack instance has a negative answer, then by Lemma 8 every spanning tree \( \tilde{T} \) for \( \tilde{S} \) has either \( \Delta(\tilde{T}) \geq 1.4 + (W/10L) + (1/10L) - 20n\varepsilon = 1.4 + (W/10L) + (1/10L) - (1/30L) = 1.4 + (W/10L) + (1/15L) > \delta \), or we have \( \ell(\tilde{T}_0) - \ell(\tilde{T}) < P - 1 + 24n\varepsilon < P - 0.5 \).
In both cases, solving our single-source dilation-bounded minimum spanning tree problem correctly answers the Knapsack instance.

By construction, the points \( r, a_i, b_i, d_j \) have integer coordinates. We construct the points \( \tilde{c}_i \) by solving a quadratic equation with an error of at most \( 2^{-k} \), that is, with \( k \) bits after the binary point, where \( k \) is chosen such that \( 2^k > 600nL \). Clearly \( k \) is polynomial in the input size. If we multiply all point coordinates in our construction and the cost bound by \( 2^k \), then all points have integer coordinates.

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