ON DISTRIBUTION FREE SKOROKHOD-MALLIAVIN
CALCULUS

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Abstract. The starting point of the current paper is a sequence of
uncorrelated random variables. The distribution functions of these vari-
ables are assumed to be given but no assumptions on the types or the
structure of these distributions are made. The above setting constitute
the so called "distribution free" paradigm. Under these assumptions, a
version of Skorokhod-Malliavin calculus is developed and applications
to stochastic PDES are discussed.

1. Introduction

The theory and applications of Skorokhod-Malliavin calculus are well de-
veloped for Gaussian and Poisson processes, see, for example, A.V. Sko-
rokhod [15], P. Malliavin [4], [5], G. Di Nunno et al [11], D. Nualart [9], M.
Sanz-Sole [13].

In this paper we will introduce and investigate an extension of Skorokhod-
Malliavin calculus to random fields generated by an arbitrary sequence
Ξ = (ξ1, ξ2, ...) of square integrable and uncorrelated random variables on
the probability space (Ω, F, P). Let us assume that for every i, Pr (ξi < x) =
Fi (x). These distribution functions are given but their types are not speci-
fied. Of course, some or all of these distribution functions could coincide.

The above setting constitute the so called "distribution free" paradigm.
As the title suggests, our task is to develop a version of Malliavin-Skorohod
calculus in the distribution free setting. At the first glance, a rigorous im-
plementation of the distribution free Skorokhod-Malliavin calculus might
seem to be a long shot, however, it is not completely unexpected. Consider,
for example, the following quotation from P.A. Meyer [12]: "The first and
very important point is that, in the construction of multiple integrals, the
Gaussian character of the process never appears".

The main operators of Skorokhod-Malliavin calculus are Wick product
(⋄), Skorokhod integral (δ), and Malliavin derivative (D). Skorokhod in-
tegral is an anti-derivative of Malliavin derivative and Wick product is an

Date: June 20, 2014.
2000 Mathematics Subject Classification. Primary 60H05, 60H07; Secondary, 60H10,
60H15.

Key words and phrases. Distribution free Skorokhod-Malliavin calculus; linear SDEs
and SPDEs.
elementary but useful version of Skorokhod integral. In the current paper, these operators are defined and investigated in the distribution free setting under two natural assumptions (see B1 and B2 below) that hold, practically, for all standard types of random variables.

The distribution free version of Skorokhod-Malliavin calculus, developed in Sections 2-3, preserves the fundamental properties of the classic Malliavin-Skorokhod calculus. For example, the Itô-Skorokhod isometry

\[
E \left[ |\delta (u) |^2 \right] = E \| u \|^2 \tag{1.1}
\]

and

\[
E \left[ |\delta (u) |^2 \right] = E \| u \|^2 + E \langle Du, Du \rangle \tag{1.2}
\]

where (1.1) holds for functions adapted to the appropriate filtration and (1.2) holds for non-adapted random variables (see Proposition 4.12).

In the distribution free setting, it is natural to construct the driving noise \( \mathfrak{N} \) as follows:

\[
\dot{\mathfrak{N}} = \sum_k m_k \xi_k, \quad \xi_k \in \Xi,
\]

where \( \xi_k \) are uncorrelated random variables, \( E \xi_k = 0 \), \( E |\xi_k|^2 = 1 \) and \( \{m_k, k \geq 1\} \) is an orthogonal basis in some Hilbert space \( H = L_2(U, U, \mu) \), where \( (U, U, \mu) \) is a σ-finite measure space.

Let \( J \) be the set of multiindices \( \alpha = (\alpha_1, \alpha_2, ...) \), such that for every \( k \), \( \alpha_k \in \mathbb{N}_0 \), \( \mathbb{N}_0 = \{0, 1, 2, ...\} \) and \( |\alpha| = \sum_k \alpha_k \). We will construct a complete orthogonal system \( \{\mathfrak{N}_\alpha, \alpha \in J\} \) in \( L^2(\Omega, \sigma(\xi_k, k \geq 1), P) \). By construction, this basis is distribution free. The Cameron-Martin basis for Gaussian random fields is a particular case of the basis \( \{\mathfrak{N}_\alpha, \alpha \in J\} \). The set of deterministic coefficients \( \{X_\alpha = E (X \mathfrak{N}_\alpha), \alpha \in J\} \) is often referred to as the propagator of the random variable/field \( X \).

In Section 4, the distribution free Skorokhod-Malliavin "technology" is applied to the analysis of linear ordinary SDE as well as linear stochastic parabolic and elliptic SPDEs with additive or multiplicative distribution free noise. We will study these equations and their relations in adapted and non-adapted settings.

As an example, let us consider an adapted parabolic SPDE

\[
u(t) = w + \int_0^t \mathcal{L} u(s) ds + \int_0^t \int_U [u(s) G(s, v) + f(s, v)] \mathfrak{N}(ds, dv), \tag{1.3}
\]

where \( \mathcal{L} u = a^{ij}(x) u_{x_i x_j} + b^i(x) u_{x_i} \). The propagator associated with (1.3) is given by following low-triangular system of deterministic PDEs for the coefficients \( u_0(t) = w_0 \) and

\[
\begin{cases}
\partial_t u_\alpha(t) = \mathcal{L} u_\alpha + \sum_k \int_V m_k(u_\alpha G + f_\alpha) d\pi \\
u_\alpha(0) = w,
\end{cases}
\tag{1.4}
\]
where \( \alpha(k) = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \ldots) \) and \( |\alpha| > 0 \). In contrast to stochastic PDE (1.3), the related propagator is a deterministic lower-triangular system. Therefore, under quite general assumptions, system (1.4) is solvable. Therefore, one can construct a distribution free "polynomial chaos" solution of equation (1.3) in the form

\[
u = \sum_{\alpha} u_{\alpha} N_{\alpha}.
\]

Note that, due to its lower triangular structure, the propagator (1.4) can be solved sequentially. The latter bodes well to the efficiency of numerical implementation of the polynomial chaos solutions.

Similarly, under standard assumptions (e.g. positivity of the operator \( A \)), one can construct a polynomial chaos solution of the stationary equation

\[
Au + \sum_{n \geq 1} M_n u \diamond \xi_n = f.
\]

The propagator of this equation is given by the system

\[
Au_\alpha = Ef \quad \text{if} \quad |\alpha| = 0
\]

\[
Au_\alpha + \sum_{n \geq 1} M_n u_{\alpha - \varepsilon_n} = f_\gamma \quad \text{if} \quad |\alpha| > 0,
\]

where \( \varepsilon_n \) is a multiindices with \( |\varepsilon_n| = 1 \) and the only non-zero component at the \( n^{th} \) coordinate.

Again, under standard assumptions (including positivity of the operator \( A \)), one can solve the propagator of the stationary equation and reconstruct the distribution free solution of (1.5) in the polynomial chaos form.

Note that, in both settings the triangular feature of the propagator is due to the linear structure of the underlying stochastic equations.

A much more difficult and nuanced problem is the existence of "distribution free" solutions for nonlinear stochastic equations driven by arbitrary noise. In general, one should not expect a "universal" answer, because the coefficients of expansions of a nonlinear function (e.g. a product) of random variables depends on the types of this random variables. For more detail and examples, see [16], [7].

Nevertheless, there exists at least one reasonably broad class of nonlinear SPDEs that fits into the distribution free paradigm. Specifically, SPDEs with the so-called Wick-polynomial nonlinearities belong to this class. Equation

\[
Au - u^{\diamond k} + \sum_{n \geq 1} M_n u \diamond \xi_n = f
\]

is an important example of a nonlinear equation from this class.

Note that the nonlinear part of equation (1.7) is a Wick power. By this reason, it is easy to see that the propagator of equation (1.7) is again a linear lower triangular system given by

\[
Au_\alpha - \sum_{\kappa, \beta, \gamma : \kappa + \beta + \gamma = \alpha} u_{\kappa} u_{\beta} u_{\gamma} + \sum_{n \geq 1} M_n u_{\alpha - \varepsilon_n} = f_\alpha
\]

for all \( \alpha \in J \). Due to its lower-triangular this system is uniquely solvable (under reasonable assumptions on the operators).
Wick products have also been used for designing unbiased approximations of stochastic Navier-Stokes equation (see [8] and the references therein). Specifically, in this types of models, the standard nonlinear term $v \nabla v$ was approximated by the Wick product $v \diamond \nabla v$. The stochastic Navier-Stokes equation with this correction is unbiased, that is the expectation of the solution coincides with the related deterministic Navier-Stokes equation.

Note that the first examples of systems with Wick nonlinearities were introduced and investigated in Euclidean Quantum Field theory (see e.g. [14]).

The most challenging problem in the analysis of distribution free noise is development of a distribution free calculus suitable for adapted and non-adapted "arbitrary" random fields. Not surprisingly, this subject constitutes the "foundation" of the paper. It is addressed in Section 3. This Section includes construction of distribution free multiple integrals, Skorokhod integral, Malliavin derivatives, and Wick exponent, as well as Ito-Skorokhod isometry. Linear SDEs and parabolic SPDEs are discussed in Section 4.02, 4.03. Linear and Wick-nonlinear elliptic (non-adapted) SPDEs are covered in Section 4.1. The Appendix deals with Wick products of multiple integrals.

2. Driving cylindrical random fields

Let $(U,\mathcal{U},\mu)$ be a $\sigma$-finite complete measure space. Let $\mathbf{H} = L_2(U,\mathcal{U},\mu)$ and $(\Omega,\mathcal{F},P)$ be a complete probability space.

**Definition 1.** A continuous linear functional $\mathfrak{N}$ from $\mathbf{H}$ to $L_2(\Omega,\mathcal{F},P)$ such that $\mathbb{E}[\mathfrak{N}(f)^2] = |f|^2_2, f \in \mathbf{H}$, is called a driving cylindrical random field on $U$.

It is assumed that $\mathbb{E}\mathfrak{N}(f) = 0, f \in \mathbf{H}$. Clearly, $\mathfrak{N}$ is an isometric embedding of $\mathbf{H}$ into $L_2(\Omega,\mathcal{F},P)$.

**Remark 1.** If $f \in \mathbf{H}$ and $\{m_k\}$ is a complete orthonormal system in $\mathbf{H}$, then, obviously,

$$\mathfrak{N}(f) = \sum_k f_k \xi_k \text{ in } L_2(\Omega,\mathcal{F},P),$$

where $\xi_k = \mathfrak{N}(m_k)$ and $f_k = \int_{U} f m_k d\mu$ (note that $f = \sum_k f_k m_k$ in $\mathbf{H}$).

Moreover,

$$\mathbb{E}[\mathfrak{N}(f) \mathfrak{N}(g)] = \int f g d\mu, \; f, g \in \mathbf{H};$$

The noise (associated to $\mathfrak{N}$) is the formal series

$$\dot{\mathfrak{N}}(\nu) = \sum_k m_k(\nu) \xi_k, \nu \in U.$$

We write

$$\mathfrak{N}(f) = \int_{U} \dot{\mathfrak{N}}(\nu) f(\nu) \mu(d\nu) = \int_{U} f(\nu) \mathfrak{N}(d\nu) = \sum_k f_k \xi_k.$$
Remark 2. If \( (\xi_k) \) is an arbitrary sequence of centered uncorrelated r.v. in \( L^2(\Omega, \mathcal{P}) \), then for any \( L_2(U, \mu) \) the map

\[
\mathcal{M}(f) = \sum_k f_k \xi_k, \quad f = \sum_k f_k m_k \in L^2(U, \mu),
\]

is a driving cylindrical random field on \( U \), that is any sequence of centered uncorrelated r.v. can define a generalized driving random field.

We shall introduce the following assumptions about \( \mathcal{M} \).

**B1.** Let \( \xi_k = \mathcal{M}(m_k), k \geq 1 \). For each vector r.v. \( (\xi_{i_1}, \ldots, \xi_{i_n}) \), \( n \geq 1 \), the moment generating function

\[
M_{i_1\ldots i_n}(t) = M_{i_1\ldots i_n}(t_1, \ldots, t_n) = \mathbb{E} \exp \{ t_1 \xi_{i_1} + \cdots + t_n \xi_{i_n} \}
\]

exists for all \( t = (t_1, \ldots, t_n) \) in some neighborhood of \( 0 \in \mathbb{R}^n \).

This assumption implies immediately that \( \xi_k \) have all the moments (see Theorem 5a, p. 57 in [17]). Let \( J \) be the set of all multiindices \( \alpha = (\alpha_1, \ldots) \) such that \( \alpha_k \in \{0, 1, 2, \ldots\} \) and \( |\alpha| = \sum \alpha_k < \infty \). For \( \alpha = (\alpha_k) \in J \) we denote

\[
\xi^\alpha = \prod_k \xi_k^\alpha, \xi^0 = 1.
\]

The following assumption is needed as well.

**B2.** Assume we are given an orthogonalization \( \{\tilde{K}_\alpha, \alpha \in J\} \) of the system \( O = \{\xi^\alpha, \alpha \in J\} \) such that for each \( n \), \( \{\tilde{K}_p, |p| \leq n\} \) spans the same linear subspace \( H_n \) as \( \{\xi^p, |p| \leq n\} \) and for \( |\alpha| = n + 1 \),

\[
\tilde{K}_\alpha = \xi^\alpha - \text{projection}_{H_n} \xi^\alpha
\]

Set \( \mathcal{M}_\alpha = \varepsilon_{\alpha} \tilde{K}_\alpha \) so that \( \mathbb{E} [\mathcal{M}_\alpha^2] = \alpha! \). The result is the complete orthogonal system \( \{\mathcal{M}_\alpha, \alpha \in J\} \). Obviously, \( \mathcal{M}_0 = 1 \) and for \( p \in J, p = \varepsilon_k (\varepsilon_k \in J \) and has all components zeros except 1 as the \( k \)th component), \( \mathcal{M}_p = \mathcal{M}_{\varepsilon_k} = \xi_k \).

Remark 3. a) If every \( \xi_k = \mathcal{M}(m_k) \) is bounded, then **B1** obviously holds.

b) The Hilbert space \( \mathcal{H} \) can be finite dimensional. In particular, if \( U = \{1\} \) and \( \mu \) is the Dirac measure \( \delta_1 \), then \( \mathcal{H} \) is one-dimensional, \( m_1 = 1 \). If \( \xi = \mathcal{M}(1) \), then the \( \mathcal{M} \)-noise coincides with the r.v. \( \mathcal{M} = \xi \).

The following statement is almost obvious.

**Proposition 1.** Assume **B1**, **B2** hold. Let \( \mathcal{F}^0 = \sigma(\xi_k, k \geq 1) = \sigma(\mathcal{M}(f), f \in \mathcal{H}). \)

Then \( \{\mathcal{M}_\alpha = \xi^\alpha, \alpha \in J\} \) is a complete orthogonal system of \( L_2(\Omega, \mathcal{F}^0, \mathcal{P}) \):

for each \( \eta \in L_2(\Omega, \mathcal{F}^0, \mathcal{P}) \),

\[
\eta = \sum_\alpha \eta_\alpha \mathcal{M}_\alpha,
\]

where

\[
\eta_\alpha = \frac{\mathbb{E} [\eta \mathcal{M}_\alpha]}{\alpha!}.
\]

Note that \( \sum_\alpha \eta_\alpha^2 \alpha! = \mathbb{E} [\eta^2] < \infty \).
Proof. By Lemma [6] B1 implies that any \( \eta \in L^2(\mathcal{F}^0, \mathcal{P}) \) can be approximated by a sequence of polynomials in \( \xi^\alpha, \alpha \in J \). Therefore the orthogonalization described in B2 defines a complete orthogonal system and \( \{ \mathcal{N}_\alpha/\sqrt{\alpha!}, \alpha \in J \} \) is a CONS in \( L^2(\mathcal{F}^0, \mathcal{P}) \). \( \square \)

2.1. Examples.

2.1.1. Driving cylindrical random fields and processes generated by independent r.v.’s.

Example 1. In the case of a single r.v. \( \xi \) as in Remark [3] \( J = \{0, 1, 2, \ldots \} \), the complete orthogonal system \( \{ \mathcal{N}_n, n \geq 0 \} \) of \( L_2(\sigma(\xi), \mathcal{P}) \) must coincide with the one obtained by Gram-Schmidt orthogonalization procedure. We set \( \mathcal{N}_0 = 1, \mathcal{N}_1 = \xi \). If \( H_n \) be the subspace generated by \( \{ \mathcal{N}_l, l \leq n \} \), then we take

\[ \tilde{K}_{n+1} = \xi^{n+1} - \text{projection}_{H_n} (\xi^{n+1}) \]

and set \( \mathcal{N}_{n+1} = c_{n+1} \tilde{K}_{n+1} \) so that \( E[\mathcal{N}_{n+1}^2] = (n+1)! \).

Example 2. Let \( \xi_k \) be a sequence of independent centered r.v. whose moment generating function exists in a neighborhood of zero. Assume \( E(\xi_k^2) = 1 \).

As already discussed in Remark [3], for any \( L_2(U, \mathcal{U}, \mu) \) the map

\[ \mathcal{N}(f) = \sum_k f_k \xi_k, \quad f = \sum_k f_k m_k \in L_2(U, \mu), \]

is a driving cylindrical random field on \( U \). Obviously B1 holds. For every \( k \) we apply orthogonalization procedure of Example 1 and construct \( \mathcal{N}_l^k \) with \( E[(\mathcal{N}_l^k)^2] = l! \). For a multiindex \( \alpha \in J \), we set

\[ \mathcal{N}_\alpha = \prod_k \mathcal{N}_{\alpha_k}^k. \]

Note that \( E[\mathcal{N}_\alpha^2] = \alpha! \) For any \( \alpha \in J \) with \( |\alpha| = n+1 \), \( \mathcal{N}_\alpha \) is orthogonal to \( H_n \) and of the form \( \xi^\alpha - l_\alpha \), where \( l_\alpha \) is a linear combination of vectors in \( H_n \), i.e. \( l_\alpha \) is the orthogonal projection of \( \xi^\alpha \) on \( H_n \). Therefore B1 is satisfied as well and \( \{ \mathcal{N}_\alpha, \alpha \in J \} \) is a CONS in \( L^2(\Omega, \sigma(\xi_k, k \geq 1), \mathcal{P}) \).

Example 3. Let \( \xi_k \) be a sequence independent centered r.v. that have all the moments and \( E(\xi_k^2) = 1 \). Let \( (U, \mathcal{U}, \mu) \) be a \( \sigma \)-finite measure space. Let \( \{ m_k, k \geq 1 \} \) be a CONS in \( \mathcal{H} = L_2(U, \mathcal{U}, \mu) \). We can define \( \mathcal{N} : L_2(U, \mathcal{U}, \mu) \to L_2(\Omega, \mathcal{P}) \) as

\[ \mathcal{N}(f) = \sum_k f_k \xi_k, \quad f = \sum_k f_k m_k \in \mathcal{H}, \]

and the complete orthogonal system \( \{ \mathcal{N}_\alpha, \alpha \in J \} \) as in Example 2. The \( \mathcal{N} \)-noise is

\[ \mathcal{N}(x) = \sum_k m_k(x) \xi_k. \]
In particular, if \( U = L^2 ([0, T]) \) and \( \{ m_k, k \geq 1 \} \) is a CONS on \( L^2 ([0, T]) \), then we can regard \( \mathfrak{N} \) in (2.2) as a stochastic process

\[
\mathfrak{N}_t = \mathfrak{N} (\chi_{[0,t]}) = \sum_k \int_0^t m_k (s) \, ds \xi_k, 0 \leq t \leq T.
\]

It has uncorrelated increments, \( \mathbf{E} (\mathfrak{N}_t^2) = t, \mathbf{E} (\mathfrak{N}_t \mathfrak{N}_s) = t \wedge s, \mathbf{E} (\mathfrak{N}_t) = 0, t \geq 0 \). For any continuous deterministic \( f(t) \) on \( [0, T] \):

\[
\mathfrak{N} (f) = \lim_{n} \sum_{i=1}^{n} f (t_i) [\mathfrak{N}_{t_{i+1}} - \mathfrak{N}_{t_i}] \quad \text{in} \quad L^2 (\Omega, \mathbf{P}),
\]

where \( t_i = t^n_i, 0 \leq i \leq n \), is a partition of \([0, T]\) into \( n \) disjoint subintervals whose maximal size converges to zero as \( n \to \infty \).

Some specific examples related to Example 3:

- For a standard normal \( \xi \sim N (0, 1) \), the sequence \( \mathfrak{N}_n \) in Example 1 are Hermite polynomials: \( \mathfrak{N}_n = \frac{d^n}{dz^n} p (z) \big|_{z=0} \) with

\[
p (z) = \exp \left\{ z \xi - \frac{1}{2} z^2 \right\}, z \in \mathbb{R}.
\]

With a sequence \( \xi_k, k \geq 1 \), of independent standard normals,

\[
W_t = \sum_k \int_0^t m_k (s) \, ds \xi_k, 0 \leq t \leq T,
\]

is a standard Wiener process (see [1]).

- Let \( \xi_k \) be i.i.d. such that \( \mathbf{P} (\xi_k = 1) = \mathbf{P} (\xi_k = -1) = 1/2 \). Then the driving process \( \mathfrak{N}_t \) in (2.3) is not Gaussian: for each \( n > 1, u \in \mathbb{R} \),

\[
\mathbf{E} \exp \left\{ \iota u \sum_{k=1}^{n} \int_0^t m_k (s) \, ds \xi_k \right\} = \prod_{k=1}^{n} \cos \left\{ u \int_0^t m_k (s) \, ds \right\}.
\]

It is straightforward to find \( (\xi_k^2 = 1 \ a.s.) \) that for

\[
\mathbf{E} \left[ (W^n_t - W^n_s)^4 \right] \leq C |t - s|^2, \ s, t \in [0, T],
\]

where \( W^n_t = \sum_{k=1}^{n} \int_0^t m_k (s) \, ds \xi_k, 0 \leq t \leq T \). Therefore, by the same arguments as in the Gaussian case (see [1]), \( W_t \) has a continuous in \( t \) modification.

- Let \( \xi_k \) be uniform i.i.d. on \([-\sqrt{3}, \sqrt{3}]\). Again, we have in (2.3) a non-Gaussian driving process with continuous paths:

\[
\mathbf{E} \exp \left\{ \iota u \sum_{k=1}^{n} \int_0^t m_k (s) \, ds \xi_k \right\} = \prod_{k=1}^{n} \frac{\sin \left( u \int_0^t m_k (s) \, ds \right)}{u \int_0^t m_k (s) \, ds}.
\]
The orthogonal basis consists of Legendre polynomials in this case. Let us consider the set of Legendre polynomials $L_k$ on $[-1, 1]$ defined by the Rodrigues formula

\begin{equation}
L_n(\eta) = \frac{(-1)^n}{2^n n!} \frac{d^n}{d\eta^n} [(1 - \eta^2)^n].
\end{equation}

For any $n, m$,

\begin{equation}
\frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} L_n(x) L_m(x) dx = \frac{1}{2} \int_{-1}^{1} L_n(\eta) L_m(\eta) d\eta = \frac{1}{(2n + 1)} \delta_{nm}.
\end{equation}

Let $\tilde{B}_n = L_n(x/\sqrt{3})$. To make it "standard" ($E(B_n^2) = n!$), we set $B_0 = 1, B_n = \sqrt{n!(2n + 1)} \tilde{B}_n$.

In this case $\mathfrak{M}_k^n = \xi_k^n = B_n(\xi_k), k \geq 1, n \geq 0$ and we get the whole system by (2.1).

A sequence of independent driving random fields can be used to construct a new one.

2.1.2. Poisson random fields. Let $N$ be Poisson random measure on $U$ with $\mu$ as its Levy measure and $\tilde{N} = N - \mu$. We have isometry:

\[ E[\tilde{N}(\varphi)^2] = \int_U \varphi^2 d\mu, \varphi \in L_2(U, \mu). \]

Let $m_k(x)$ be a CONS in $L_2(U, \nu)$ such that all $m_k$ and $\int_U |m_k(\nu)| d\mu$ are bounded. In general, $\xi_k = \tilde{N}(m_k)$ are not independent.

Let $Z$ be the set of all real-valued sequences $z = (z_k)$ such that only the finite number of $z_k$ is not zero. For $z \in Z$ set, $m = m_z = \sum_k z_k m_k$. Under the assumptions above, the moment generating function

\[ M(z) = E \exp \left\{ \tilde{N}(m_z) \right\} = \exp \left\{ \int_U \left[ e^{m_z(\nu)} - 1 - m_z(\nu) \right] \mu(\nu) \right\} \]

exists and Assumption B1 is satisfied.

Charlier polynomials

For small $z$, let

\[ p(z) = \exp \left\{ \int_U \ln[1 + m_z(\nu)] N(\nu) - \int_U m_z(\nu) \mu(\nu) \right\}. \]

For $\alpha \in J$, we define Charlier polynomials as

\[ C_{\alpha} = \frac{\partial^\alpha}{\partial z^\alpha} p(z)|_{z=0}, \alpha \in J. \]
For example, if $|\alpha| = 1$, $\alpha = \varepsilon_k$, then $C_\alpha = \tilde{N}(m_k)$; If $|\alpha| = 2$, $\alpha = \varepsilon_{k_1} + \varepsilon_{k_2}$, then

$$C_\alpha = N(m_{k_1})N(m_{k_2}) - N(m_{k_1}m_{k_2}) - N(m_{k_1})\mu(m_{k_2}) - N(m_{k_2})\mu(m_{k_1}) + \mu(m_{k_1})\mu(m_{k_2}),$$

where $N(\phi) = \int_U \phi(x) N(dx), \mu(\phi) = \int_U \phi(x) \mu(dx)$. Recall $\varepsilon_k = (\alpha_i) \in J$ and has all components zero except $\alpha_k = 1$.

It is a standard fact that Charlier polynomials are orthogonal: $EC_\alpha C_{\alpha'} = \alpha!$ if $\alpha = \alpha'$ and zero otherwise. In general and contrary to the Hermite polynomials for the Gaussian random variables, they are not polynomials of Poisson random variables.

We define driving Poisson noise on $U$ as $\mathcal{N}(f) = \tilde{N}(f), f \in L^2(U,d\mu)$. From our general considerations above, it satisfies $B_1, B_2$ with $\mathcal{N}_\alpha = C_\alpha, \alpha \in J$.

2.2. Generalized random variables and processes. Let $E$ be a topological vector space and

$$D(E) = \left\{ v = \sum_\alpha v_\alpha \mathcal{N}_\alpha : v_\alpha \in E \text{ and only finite number of } v_\alpha \text{ are not zero} \right\}.$$

**Definition 2.** A generalized $D$-random variable with values in $E$ with Borel $\sigma$-algebra is a formal series $u = \sum_\alpha u_\alpha \xi_\alpha$, where $u_\alpha \in E$.

Denote the vector space of all generalized $D$-random variables by $D' = D'(E)$. If $E = \mathbb{R}$ we write simply $D, D'$. The elements of $D$ are the test random variables for $D'$. We define the action of a generalized random variable $u \in D'(E)$ on the test random variable $v \in D$ by $\langle u, v \rangle = \sum_\alpha v_\alpha u_\alpha$. If $Y$ is Hilbert and $u \in D'(Y), v \in D(E)$, then $\langle u, v \rangle = \sum_\alpha (v_\alpha, u_\alpha)_Y$.

For a sequence $u^n \in D'$ and $u \in D'$, we say that $u^n \to u$, if for every $v \in D$, $\langle u, v^n \rangle \to \langle u, v \rangle$. This implies that $u^n = \sum_\alpha u^n_\alpha \mathcal{N}_\alpha \to u = \sum_\alpha u_\alpha \mathcal{N}_\alpha$ if and only if $u^n_\alpha \to u_\alpha$ as $n \to \infty$ for all $\alpha$.

**Remark 4.** If $u = \sum_\alpha u_\alpha \mathcal{N}_\alpha \in D'(E), F$ is a vector space and $f : E \to F$ is a linear map, then we define

$$f(u) = \sum_\alpha f(u_\alpha) \mathcal{N}_\alpha \in D'(F).$$

**Definition 3.** An $E$-valued generalized $D$-field on a measurable space $(B,B)$ is a $D'(E)$-valued function on $B$ such that for each $x \in B$,

$$u(x) = \sum_\alpha u_\alpha(x) \mathcal{N}_\alpha \in D'(E),$$

where $u_\alpha(x)$ are deterministic measurable $E$-valued functions on $B$.

We denote the linear space of all such fields by $D'(B;E)$. If $E$ is a topological vector space and a generalized $D$-field $u(x)$ is continuous on $B$ we
write \( u \in C \mathcal{D}'(B; E) \) (note that \( u(x) \) is continuous if and only if all coefficient functions \( u_{\alpha} \) on \( B \) are continuous). In particular, if \( B = [0, T] \), we say \( u(t) \) is a generalized \( \mathcal{D} \)-process. If there is no room for confusion, we will often say \( \mathcal{D} \)-process (\( \mathcal{D} \)-random variable) instead of generalized \( \mathcal{D} \)-process (generalized \( \mathcal{D} \)-random variable).

If \((B, \mathcal{B}, \kappa)\) is a measure space and \( E \) is a normed vector space, we denote

\[
L_2(\mathcal{D}'(B; E), \kappa) = \{ u(x) = \sum_{\alpha} u_{\alpha} \mathcal{N}_{\alpha} \in \mathcal{D}'(B; E) : \int_B |u_{\alpha}(x)|^2 d\kappa < \infty, \alpha \in J \}.
\]

For \( u(t) = \sum_{\alpha} u_{\alpha}(t) \mathcal{N}_{\alpha} \in L_1(\mathcal{D}'([0, T], E)) \) we define \( \int_0^t u(s) ds, 0 \leq t \leq T, \) in \( \mathcal{D}'([0, T]; E) \) by

\[
\int_0^t u(s) ds = \sum_{\alpha} \left( \int_0^t u_{\alpha}(s) ds \right) \mathcal{N}_{\alpha}, 0 \leq t \leq T.
\]

If \( u(t) = \sum_{\alpha} u_{\alpha}(t) \mathcal{N}_{\alpha} \in \mathcal{D}'([0, T]; E) \), then \( u(t) \) is differentiable in \( t \) if and only if \( u_{\alpha}(t) \) are differentiable in \( t \). In that case,

\[
\frac{d}{dt} u(t) = \dot{u}(t) = \sum_{\alpha} i u_{\alpha}(t) \mathcal{N}_{\alpha} \in \mathcal{D}'([0, T], E).
\]

3. Distribution Free Skorokhod-Malliavin Calculus

Let \( \mathbf{H}^\otimes n = L_2(U^n, \mathcal{U}^\otimes n, \mu_n) \), where \( U^n = U \times \ldots \times U \) n-times, \( \mu_n = \mu^\otimes n \).

Let \( \mathbf{H}^\hat{\otimes} n \) be the symmetric part of \( \mathbf{H}^\otimes n \) : it is the set of all symmetric \( \mu_n \)-square integrable functions on \( U^n \). For \( \alpha \in J \) with \( |\alpha| = n \) we define its characteristic set \( K_{\alpha} = \{ k_1, \ldots, k_n \} \) with each \( k \) represented in it by \( \alpha_k \) copies. Let \( \mathcal{G}_n \) be the group of permutations of \( \{1, \ldots, n\} \) and

\[
E_{\alpha} = \sum_{\sigma \in \mathcal{G}_n} m_{k_{\sigma(1)}} \otimes \ldots \otimes m_{k_{\sigma(n)}}.
\]

Then

\[
e_{\alpha} = \frac{E_{\alpha}}{\sqrt{\alpha!|\alpha|!}}, \alpha \in J, |\alpha| = n,
\]

is a CONS of the symmetric part of \( \mathbf{H}^\hat{\otimes} n \). If \( |p| = 1 \) with kth component non zero, then \( E_p = e_p = m_k \).

3.1. Multiple integrals, Wick product and Skorokhod integral. Multiple integrals

Now we construct multiple integrals with respect to \( \mathcal{N} \) on the symmetric part \( \mathbf{H}^\otimes n \) (\( \mathbf{H}^\hat{\otimes} 0 = \mathbf{R} \)). Set for \( |\alpha| = n \geq 1, \)

\[
I_n(E_{\alpha}) = n! \mathcal{N}_{\alpha}.
\]
Let $Y$ be a Hilbert space and denote $\mathcal{H}^{\hat{n}}(Y)$ the space of all $Y$-valued symmetric functions $v = \sum_{|\alpha|=n} v_{\alpha}E_{\alpha}$ on $U^n$ such that

$$|v|^2_{\mathcal{H}^{\hat{n}}(Y)} = \int_{U^n} |v(r)|^2_{Y} \, d\mu_n = \sum_\alpha |v_\alpha|^2_{Y} |\alpha|!|\alpha|! < \infty.$$  

For $v = \sum_{|\alpha|=n} v_{\alpha}E_{\alpha} \in \mathcal{H}^{\hat{n}}(Y)$, we define its $n$-tuple integral as

$$I_n(v) = \sum_{|\alpha|=n} v_{\alpha}I_n(E_{\alpha}) = \sum_{|\alpha|=n} v_{\alpha}n!\alpha!,$$

and $I_0(c) = c, c \in \mathbb{R}$. Note that

$$\int_{U^n} \left( \sum_{|\alpha|=n} v_{\alpha}E_{\alpha} \right)^2 \, d\mu_n = \sum_{|\alpha|=n} v_{\alpha}^2n!\alpha!$$

and

$$\mathbb{E}\left[|I_n(v)|^2_{Y}\right] = n!^2 \sum_{|\alpha|=n} |v_\alpha|^2_{Y}|\alpha|! = n!|v|^2_{\mathcal{H}^{\hat{n}}(Y)}.$$

Let $\mathcal{S}(Y)$ be the space of all finite linear combinations $\sum_k \frac{I_k(F_k)}{k!}$ with $F_k \in \mathcal{H}^{\hat{k}}(Y)$.

**Definition 4.** A generalized $\mathcal{S}$-random variable is a formal sum

$$u = \sum_k \frac{I_k(F_k)}{k!} \text{ with } F_k \in \mathcal{H}^{\hat{k}}(Y).$$

We denote the set of all generalized $\mathcal{S}$-random variables by $\mathcal{S}'(Y)$.

The action of $u \in \mathcal{S}'(Y)$ on $v \in \mathcal{S}(Y)$ is defined as

$$\langle u, v \rangle = \sum_k \int_{U^k} (u_k, v_k)_Y \, d\mu,$$

where $u = \sum_k I_k(u_k)/k!, v = \sum_k I_k(v_k)/k!$ with $u_k, v_k \in \mathcal{H}^{\hat{k}}(Y)$.

**Definition 5.** A generalized $\mathcal{S}$-field on a measurable space $(B, \mathcal{B})$ is a $\mathcal{S}'(Y)$-valued function on $B$ such that for each $x \in B$,

$$u(x) = \sum_n \frac{I_n(F_n(x))}{n!} \in \mathcal{S}'(Y),$$

where $F_n(x) = F_n(x; v_1, \ldots, v_n)$ are deterministic measurable $\mathcal{H}^{\hat{n}}(Y)$-valued functions on $B$.

We denote the linear space of all such fields by $\mathcal{S}'(B; E)$. If a generalized $\mathcal{S}$-field $u(x)$ is continuous on $B$ we write $u \in C\mathcal{S}'(B; E)$ (note that $u(x)$ is continuous if and only if all coefficient functions $F_n$ on $B$ (as $\mathcal{H}^{\hat{k}}(Y)$-valued functions) are continuous). In particular, if $B = [0, T]$, we say $u(t)$ is a generalized $\mathcal{S}$-process.
Remark 5. Obviously, \( \mathcal{D} \subseteq \mathcal{S}(\mathbb{R}) \) and \( \mathcal{S}'(Y) \subseteq \mathcal{D}'(Y) \). In fact,

\[
\mathcal{S}'(Y) = \left\{ u = \sum_{\alpha} u_\alpha \mathfrak{N}_\alpha \in \mathcal{D}'(Y) : \sum_{|\alpha|=n} |u_\alpha|_Y^2 \alpha! < \infty \ \forall n \geq 1 \right\}.
\]

Indeed, if \( u = \sum_{\alpha} u_\alpha \mathfrak{N}_\alpha \) with \( \sum_{|\alpha|=n} |u_\alpha|_Y^2 \alpha! < \infty, n \geq 1 \), then

\[
u_n = \sum_{|\alpha|=n} u_\alpha E_\alpha \in \mathcal{H}^\otimes n(Y)
\]

and

\[
u = \sum_{\alpha} u_\alpha \mathfrak{N}_\alpha = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} u_\alpha \mathfrak{N}_\alpha = \sum_{n=0}^{\infty} \frac{I_n(u_n)}{n!} \in \mathcal{S}'(Y).
\]

Note that for \( |\alpha| = n \),

\[
u_\alpha = \frac{1}{\alpha!n!} \int_{U^n} u_n(v) E_\alpha(v) d\mu_n, n \geq 1.
\]

For \( n \geq 0 \), let \( \mathcal{E} \mathcal{H}^\otimes n \) be the space of all finite linear combinations of \( E_\alpha, |\alpha| = n \). The following statement provides some insight about the transition from \( n \)-tuple integral to an integral on \( U^{n+1} \).

Proposition 2. Let \( f \in \mathcal{E} \mathcal{H}^\otimes n, g \in \mathcal{E} \mathcal{H} \), \( f \otimes g = f(z) g(v), z \in U^n, v \in U \), and let \( f \otimes g \) be the standard symmetrization of \( f \otimes g \). Then

\[
I_{n+1}(f \otimes g) = I_n(f) I_1(g) - \text{projection}_{H_n} [I_n(f) I_1(g)]
\]

Proof. Let

\[
f = \sum_{|p|=n} f_p E_p, g = \sum_{|p|=1} g_p E_p.
\]

Then

\[
f g = \sum_{p,p'} f_p g_{p'} E_p E_{p'},
\]

\[
\tilde{f} \otimes g = \sum_{p,p'} f_p g_{p'} \tilde{E}_p \tilde{E}_{p'} = \frac{1}{n+1} \sum_{p,p'} f_p g_{p'} E_{p+p'}
\]

and

\[
I_{n+1}(f \otimes g) = \frac{1}{n+1} \sum_{p,p'} f_p g_{p'} I_{n+1}(E_{p+p'}) = n! \sum_{p,p'} f_p g_{p'} \mathfrak{N}_{p+p'},
\]

\[
I_n(f) = \sum_{p} f_p \mathfrak{N}_p, I_1(g) = \sum_{p'} g_{p'} \mathfrak{N}_{p'}
\]

Since

\[
\mathfrak{N}_{p+p'} = \mathfrak{N}_p \mathfrak{N}_{p'} - \text{projection}_{H_n} [\mathfrak{N}_p \mathfrak{N}_{p'}],
\]
it follows that
\[ I_{n+1}(f \otimes g) = n! \sum_{p,p'} f_{p} g_{p'}[\mathcal{H}_{p} \otimes \mathcal{H}_{p'} - \text{projection}_{H_{n}}(\mathcal{H}_{p} \otimes \mathcal{H}_{p'})] \]
\[ = I_{n}(f) I_{1}(g) - \text{projection}_{H_{n}}[I_{n}(f) I_{1}(g)]. \]

\[ \square \]

**Remark 6.** If \( U = [0, T], d\mu = dt \) and \( m_1 = \chi_{(a,b)} \) (an interval of unit length), then according to Proposition 2, the "measure" of the square
\[ I_{2} \left( \chi_{(a,b)}^{\otimes 2} \right) = I_{1} \left( \chi_{(a,b)} \right)^{2} - \text{projection}_{H_{1}} \left[ I_{1} \left( \chi_{(a,b)} \right)^{2} \right]. \]

**Wick product and Skorohod integral.** We define Wick product
\[ N_{\alpha} \diamond N_{\beta} = N_{\alpha + \beta}, \quad 1 \diamond N_{\alpha} = N_{\alpha}, \quad \alpha, \beta \in J. \]
For \( u = \sum_{\alpha} u_{\alpha} N_{\alpha}, v = \sum_{\alpha} v_{\alpha} N_{\alpha} \in \mathcal{D}'(E) \) (\( E \) is Hilbert),
\[ u \diamond v = \sum_{\alpha} (u_{\beta}, v_{\alpha-\beta}) E N_{\alpha}. \]
For a generalized random field on \( u = \sum_{\alpha} u_{\alpha} N_{\alpha} \in \mathcal{D}'(H(Y)) \), we define its Skorokhod integrals:
\[ \delta_{p}(u) = \int_{U} u(v) \diamond N_{p} (dv) = \sum_{\alpha} \int_{U} a_{\alpha}(v) E_{p}(v) d\mu_{\alpha+p}, \quad |p| = 1, \]
and
\[ \delta(u) = \int_{U} u(v) \diamond \mathcal{M} (dv) = \sum_{|p| = 1} \delta_{p}(u) \]
\[ = \sum_{\alpha} \sum_{|p| = 1} \int_{U} a_{\alpha}(x) E_{p}(x) d\mu_{\alpha+p} = \sum_{|\alpha| \geq 1} \sum_{|p| = 1} \int_{U} a_{\alpha-p}(x) E_{p}(x) d\mu_{\alpha}. \]
Note that for a deterministic \( u = u \mathcal{M}_{0} \in H(Y) \),
\[ \delta_{\epsilon}(u) = \int_{U} u(v) m_{k}(v) d\mu \mathcal{M}_{\epsilon_{k}}, \quad \delta(u) = \sum_{k} \int_{U} u(v) m_{k}(v) d\mu \mathcal{M}_{\epsilon_{k}} = \mathcal{M}(u). \]
Now, we describe Skorokhod integral in terms of the multiple integrals \( I_{n} \). We show that \( \delta : S \rightarrow S \) and \( \delta : S' \rightarrow S' \).

**Proposition 3.** Let
\[ u = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n}(u_{n}) \in S'(L^{2}(U,d\mu)), \]
i.e.,
\[ u_{n} = u_{n}(v;v_{1}, \ldots, v_{n}) = \sum_{|\alpha| = n} u_{\alpha}(v) E_{\alpha}(v_{1}, \ldots, v_{n}), \]
with
\[ \sum_{|\alpha|=n} \int |u_{\alpha}(v)|^2 \, d\mu_\alpha! < \infty \quad \forall n. \]

Then
\[ \delta(u) = \int u(v) \diamond \mathfrak{M}(dv) = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\tilde{F}_n), \]
where \( \tilde{F}_n \) is the standard symmetrization of \( F_n \) on \( U^{n+1} \).

**Proof.** According to Remark 5, for \( |\alpha| = n \),
\[ u_{\alpha}(v) = \frac{1}{\alpha! n!} \int_{U^n} u_n(v; \upsilon') E_\alpha(\upsilon') \, \mu_n(\upsilon') \]
and
\[ u = \sum_{\alpha} u_{\alpha}(v) \mathfrak{M}_\alpha. \]

Note that
\[ \int_{U \times U^n} u_n(v; \upsilon')^2 \, d\mu_{n+1} < \infty \]
and
\[ \int_U u_{\alpha}(v) E_p(v) \, d\mu = \frac{1}{\alpha! n!} \int_U \int_{U^n} u_n(r; \upsilon') E_p(r) (dr) E_\alpha(\upsilon') \, \mu_n(\upsilon') \]
\[ = \frac{1}{\sqrt{\alpha! n!}} \int_U \int_{U^n} u_n(r; \upsilon') E_p(r) (dr) e_\alpha(\upsilon') \, \mu_n(\upsilon') \]
and
\[ u_n(v, \upsilon') = \sum_{|\alpha|=n, |\beta|=1} \int_U u_{\alpha}(r) E_p(r) \, d\mu E_p(v) E_\alpha(\upsilon'). \]

Therefore, the standard symmetrization of \( u_n(v; \upsilon'), v \in U, \upsilon' = (v_1, \ldots, v_n) \in U^n \), is
\[ \bar{u}_n(v, \upsilon') = \sum_{|\alpha|=n+1, |\beta|=1} \int_U u_{\alpha-p}(r) E_p(r) \, d\mu \frac{n!}{(n+1)!} E_\alpha(v, \upsilon'). \]

By definition of the Skorokhod integral,
\[ \delta(u) = \sum_{\alpha} \sum_{|\beta|=1} \int_U a_{\alpha}(r) E_p(r) \, d\mu \mathfrak{M}_{\alpha+p} \]
\[ = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \sum_{|\beta|=1} \int_U a_{\alpha}(r) E_p(r) \, d\mu \frac{I_{n+1}(E_{\alpha+p})}{(n+1)!} \]
\[ = \sum_{n=0}^{\infty} \sum_{|\alpha|=n+1} \sum_{|\beta|=1} \int_U a_{\alpha-p}(r) E_p(r) \, d\mu \frac{I_{n+1}(E_{\alpha})}{(n+1)!} \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\tilde{F}_n), \]
and the statement follows. \( \square \)
Multiple Skorokhod integrals

For symmetric $u = \sum_{\alpha} u_\alpha \mathcal{N}_\alpha \in \mathcal{D}'(H^\otimes n)$, with $u_\alpha \in H^\otimes n$, and $|p| = n$, we define

$$
\delta_p (u) = \sum_{\alpha} \int_{U^n} u_\alpha \frac{E_p}{p!} d\mu_n \mathcal{N}_{\alpha + p},
$$

$$
\delta^n (u) = \sum_{|p|=n} \delta_p (u) = \sum_{|p|=n} \int_{U^n} u_\alpha \frac{E_p}{p!} d\mu_n \mathcal{N}_{\alpha + p}.
$$

Let $\delta^0 (u) = u_0$. Note that for a deterministic $u(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)\mathcal{N}_0$ in $H^\otimes n$,

$$
\delta^n (u) = \sum_{|p|=n} \int_{U^n} u \frac{E_p}{p!} d\mu_n \mathcal{N}_p = \sum_{|p|=n} \int_{U^n} u \frac{E_p}{p!} d\mu_n I_n (E_p) / n!.
$$

It is easy to show that for $u = \sum_{\alpha} u_\alpha \mathcal{N}_\alpha \in \mathcal{D}'(H^\otimes n), n \geq 1$, (with $a_\alpha \in H^\otimes n$), $\delta^n (u) = \delta (\delta^{n-1} (\tilde{u} (v)))$, where

$$
\tilde{u} (v) = \tilde{u} (v; v_1, \ldots, v_{n-1}) = u(v, v_1, \ldots, v_{n-1}), v, v_i \in U.
$$

**Remark 7.** In the framework of a single r.v. $\xi$ (see Remark 3),

$$
\delta^k (\mathcal{N}_n) = \mathcal{N}_{n+k}, k \geq 1.
$$

**Lemma 1.** For a deterministic $u = u\mathcal{N}_0$ in $H^\otimes n$,

$$
\mathbb{E} \left[ \delta^n (u)^2 \right] = n! \int_{U^n} |u|^2 d\mu_n.
$$

**Proof.** Indeed,

$$
\delta^n (u) = \sum_{|p|=n} \int_{U^n} u \frac{E_p}{p!} d\mu_n \mathcal{N}_p,
$$

$$
\mathbb{E} \left[ \delta^n (u)^2 \right] = \sum_{|p|=n} \left( \int_{U^n} u \frac{E_p}{p!} d\mu_n \right)^2 p! = n! \int_{U^n} |u|^2 d\mu_n.
$$

**Remark 8.** We can rewrite Proposition 7 using multiple integrals. For each $\eta \in L_2 (\Omega, \mathcal{F}_0, \mathbb{P})$,

$$
\eta = \sum_{\alpha} \eta_\alpha \mathcal{N}_\alpha = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|\alpha|=n} \eta_\alpha I_n (E_\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n (\eta_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n (\eta_n)
$$

with

$$
\eta_\alpha = \frac{\mathbb{E} [\eta \mathcal{N}_\alpha]}{\alpha!}.
$$
and

\[ \eta_n = \eta_n(v) = \sum_{|\alpha|=n} \eta_\alpha E_\alpha(v). \]

Note that

\[ \eta_\alpha = \frac{1}{\alpha! n!} \int_{U^n} F_n(v) E_\alpha(v) \, d\mu_n, \alpha \in J. \]

3.2. Malliavin derivative. We define

\[ \mathbb{D} \mathcal{N}_\alpha = \sum_{|p|=1}^{\alpha} \frac{\alpha!}{(\alpha - p)!} \mathcal{N}_{\alpha-p} E_p(v) = \sum_{\gamma} \sum_{|p|=1, \gamma + p = \alpha} \frac{(\gamma + p)!}{\gamma!} E_p(v) \mathcal{N}_\gamma. \]

For \( u = \sum_{\alpha} a_\alpha \mathcal{N}_\alpha \in \mathcal{D}, \)

\[ \mathbb{D}_k u = \sum_{|\alpha| \geq 1} \alpha_k u_\alpha \mathcal{N}_{\alpha(k)} m_k(v), \mathbb{D}_p u = \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha - p)!} u_\alpha \mathcal{N}_{\alpha-p} E_p(v), |p| = 1, \]

\[ \mathbb{D} u = \sum_{|p|=1} \mathbb{D}_p u = \sum_{\alpha \geq p} \sum_{|p|=1} \frac{\alpha!}{(\alpha - p)!} u_\alpha \mathcal{N}_{\alpha-p} E_p(v) \]

\[ = \sum_{\alpha} \sum_{|p|=1} \frac{(\alpha + p)!}{\alpha!} u_{\alpha+p} E_p(v) \mathcal{N}_\alpha. \]

In a standard way we define the higher order Malliavin derivatives: for \( u = \sum_{\alpha} a_\alpha \mathcal{N}_\alpha \in \mathcal{D}, \)

\[ \mathbb{D}^n_{p} u = \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha - p)!} u_\alpha \mathcal{N}_{\alpha-p} \frac{E_p(v_1, \ldots, v_n)}{p!}, |p| = n, \]

\[ \mathbb{D}^n u = \sum_{|p|=n} \mathbb{D}^n_{p} u = \sum_{|p|=n} \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha - p)! p!} u_\alpha \mathcal{N}_{\alpha-p} E_p(v_1, \ldots, v_n) \]

\[ = \sum_{\alpha} \sum_{|p|=n} \frac{(\alpha + p)!}{\alpha! p!} u_{\alpha+p} E_p(v_1, \ldots, v_n) \mathcal{N}_\alpha. \]

We define Malliavin derivative for multiple integrals as well.

Proposition 4. Let \( v = \sum_{|\alpha|=n} v_\alpha E_\alpha \in H_\wedge^n(Y) \) has only finite number of \( v_\alpha \neq 0. \) Then

\[ \mathbb{D} I_n(u) = n I_{n-1}(u(\cdot, t)), t \in U. \]

Proof. By definition,

\[ I_n(v) = \sum_{|\alpha|=n} v_\alpha I_n(E_\alpha) = \sum_{|\alpha|=n} v_\alpha n! \mathcal{N}_\alpha \in \mathcal{D}. \]

Since

\[ v_\alpha = \frac{1}{\alpha! n!} \int_{U^n} v E_\alpha d\mu_n, \]
and for \( t, v_1, \ldots, v_n \in U \),

\[
v(t, v_1, \ldots, v_{n-1}) = \sum_{|p|=1} \int_U v(t', v_1, \ldots, v_{n-1}) E_p(t') \, d\mu E_p(t)
\]

(it is a finite sum), its Malliavin derivative (\( \{(\alpha, p) : |\alpha + p| = n, |p| = 1\} = \{(\alpha, p) : |\alpha| = n - 1, |p| = 1\} \)) is

\[
\mathbb{D} I_n(v) = \sum_{|\alpha|=1} \sum_{|\alpha+p|=n} \frac{(\alpha+p)!}{\alpha!} v_{\alpha+p} n! E_p(v) \mathcal{N}_\alpha = \sum_{|\alpha|=n-1} \sum_{|p|=1} \frac{(\alpha+p)!}{\alpha!} v_{\alpha+p} n! E_p(v) \mathcal{N}_\alpha
\]

\[
= \sum_{|\alpha|=n-1} \sum_{|p|=1} \frac{1}{\alpha!} \int v E_{\alpha+p} d\mu_n E_p(v) \mathcal{N}_\alpha = \sum_{|\alpha|=n-1} \sum_{|p|=1} n \int v E_{\alpha} E_p d\mu_n E_p(t) \mathcal{N}_\alpha
\]

\[
= \sum_{|\alpha|=n-1} n(n-1)! \int v(t, \cdot) \frac{E_\alpha}{\alpha!(n-1)!} d\mu_{n-1} \mathcal{N}_\alpha = \sum_{|\alpha|=n-1} n(n-1)! v_\alpha(t, \cdot) \mathcal{N}_\alpha
\]

\[
= n I_{n-1}(v(t, t)).
\]

We used here that

\[
\overline{E_\alpha E_p} = \frac{|\alpha|!}{|\alpha+p'!} E_{\alpha+p} = \frac{1}{n} E_{\alpha+p},
\]

where \( \overline{f} \) is the symmetrization of \( f \).

As suggested by Proposition 4, for an arbitrary \( v = \sum_{|\alpha|=n} v_\alpha E_\alpha \in \mathcal{H}^{\hat{n}}(Y) \) we define

\[
\mathbb{D} I_n(v) = n I_{n-1}(v(y, \cdot)), y \in U,
\]

\( I_0(c) = c, c \in \mathbb{R} \). For \( u = \sum_n \frac{I_n(u_n)}{n!} \in \mathcal{S}'(Y) \) we define

\[
\mathbb{D} u(y) = \sum_n \frac{\mathbb{D} I_n(u_n)}{n!} = \sum_n \frac{I_{n-1}(u_n(y, \cdot))}{(n-1)!}
\]

We see that \( \mathbb{D} \) maps \( \mathcal{S}(Y) \) into \( \mathcal{S}'(Y) \).

**Remark 9.** In the framework of a single r.v. \( \xi \) (see Remark 4), \( \mathbb{D}^k(\xi^{\otimes n}) = \frac{n!}{(n-k)!} \xi^{\otimes(n-k)} \), \( k \geq 1 \).

### 3.3. Adapted stochastic processes.

In this subsection we assume that \( U = [0, T] \times \mathcal{V}, \mathcal{U} = \mathcal{B}([0, T]) \times \mathcal{V}, d\mu = dt d\pi \). Let

\[
(3.1) \quad u = \sum_n \frac{I_n(u_n)}{n!} \in \mathcal{S}'(Y),
\]

with \( u_n \in \mathcal{H}^{\hat{n}}(Y), n \geq 0 \) (\( u_n = u_n(t_1, v_1, \ldots, t_n, v_n) \), \( t_i, v_i \in U, i = 1, \ldots, n \)).

For \( t \in [0, T] \), let \( Q^n_t = ([0, t] \times \mathcal{V})^n \).
Definition 6. Let \( t_0 \in [0, T] \). A random variable \( u \) defined by (3.1) is called \( \mathcal{F}_{t_0} \)-measurable if for each \( n \), \( \text{supp}(u_n) \subseteq Q_{t_0}^n \), i.e. \( \mu_{n-1} \)-a.e.

\[
u_n(t_1, v_1, \ldots, t_n, v_n) = 0 \text{ if } t_i > t_0 \text{ for some } i
\]

Proposition 5. A random variable \( u \in S(Y) \) defined by (3.1) is \( \mathcal{F}_{t_0} \)-measurable iff

\[
\mathbb{D}u(t, v) = 0 \text{ if } t > t_0.
\]

Proof. For \( u \in S(Y) \) defined by (3.1),

\[
\mathbb{D}u = \sum_{n \geq 1} \frac{1}{(n - 1)!} I_{n-1}(u_n(t, v, \cdot), (t, v)) \in U.
\]

and

\[
\mathbb{E}\left[ \mathbb{D}u(t, v)^2 \right] = \sum_n \frac{1}{(n - 1)!} \mathbb{E}[I_{n-1}(u_n(t, v, \cdot))^2] = \sum_n \int_{U^{n-1}} u_n(t, v, \cdot)^2 d\mu_{n-1} = 0
\]

for \( t > t_0 \) iff \( u_n(t, v, \cdot) = 0 \) \( \mu_{n-1} \)-a.e. for \( t > t_0 \). \( \square \)

Now, we will introduce a notion of an adapted random process. Consider \( u \in S(U; Y) \), i.e.,

\[
u(t, v) = \sum_n I_n(u_n(t, v, \cdot)) \frac{1}{n!}
\]

with \( u_n(t, v, \cdot) \in H^\otimes n(Y) \) for all \((t, v) \in U:

\[
u(t, v, \cdot) = \sum_{|\alpha|=n} u_{\alpha}(t, v) E_{\alpha}(\cdot), (t, v) \in U.
\]

Definition 7. A random field \( u(t, v) \) on \( U \) defined by (3.3) is called adapted if \( \text{supp}(u_n(t, v, \cdot)) \subseteq Q_t^n, v \in V, \) for every \( t \in [0, T] \).

A straightforward consequence of Proposition 5 is the following claim.

Corollary 1. A random field \( u \in S(U; Y) \) is adapted iff for each \( t \in [0, T] \), the Malliavin derivative \( \mathbb{D}u(t, v; s_1, v_1, \ldots, s_n, v_n) = 0, v \in V, \) if \( s_i > t \) for some \( i \).

Given a random field \( u(t, v) \) on \( U = [0, T] \times V \), consider its Skorokhod integral

\[
\delta(u)_t = \delta(\chi_{[0,t]} u) = \int_0^t u(r, y) \circ \mathfrak{N}(dr, dy), 0 \leq t \leq T.
\]

Proposition 6. Consider a random field \( u \in S(H(Y); Y) \), i.e. (3.3) holds with

\[
\int_{U^n+1} |u_n|_Y^2 d\mu_{n+1} < \infty \forall n.
\]

If it is adapted, then \( \delta(u)_t, 0 \leq t \leq T, \) is adapted as well.
Proof. Since \( u(t, v) = \sum_n I_n(u_n(t, v)) \) is adapted with \( u_n \) satisfying Proposition \( 3 \), \( \text{supp}(u_n(t, v, \cdot)) \subseteq Q_n \), \( v \in V \), for all \( n \) and \( t \in [0, T] \), i.e. \( u_n(t, v) = u_n(t, v)\chi_{Q_n} \). By Proposition \( 3 \)

\[
\delta(u) = \delta(\chi_{[0,t]}u) = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\chi_{[0,t]}u_n),
\]

where \( \chi_{[0,t]}u_n \) is the standard symmetrization of \( \chi_{[0,t]}u_n = u_n\chi_{Q_n+1} \). Since its support is obviously a subset of \( Q_n+1 \), the statement follows. \( \square \)

3.4. Ito-Skorokhod isometry. Now we estimate the \( L_2 \)-norm of the Skorokhod integral.

Proposition 7. Let \( u = u(v) = \sum \alpha u_{\alpha}(v) \mathfrak{N}_\alpha \in L_2(D(U; \mathbb{Y}), d\mu) \), i.e. \( u_\alpha \in H(Y) \):

\[
\int |u_{\alpha}(v)|^2 \, d\mu < \infty, \alpha \in J,
\]

with a finite number of \( u_\alpha \neq 0 \). Then

\[
E[|\delta(u)|^2] = E\left[ \int_U |u(v)|^2 \, d\mu \right] + E\left[ \int_{U^2} (\mathbb{Y}u(v; v'), \mathbb{Y}u(v'; v)) \mu(dv) \mu(dv') \right],
\]

where

\[
\mathbb{Y}u(v; v') = \sum_{\alpha} \sum_{|\alpha|=1} \frac{(\alpha + p)!}{\alpha!} u_{\alpha+p}(v) E_p(v') \mathfrak{N}_\alpha.
\]

Proof. By definition,

\[
\delta(u) = \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_U u_{\alpha-p}(x) E_p(x) d\mu \mathfrak{N}_\alpha = \sum_{\alpha} \sum_{|p|=1} \int_U u_{\alpha}(r) E_p(r) d\mu \mathfrak{N}_{\alpha+p}.
\]

Hence

\[
E[|\delta(u)|^2] = \sum_{|\alpha| \geq 1} \left| \sum_{|p|=1} \int_U u_{\alpha-p}(x) E_p(x) d\mu \right|^2 \frac{\alpha!}{Y}.
\]

\[
= \sum_{|p|=1} \sum_{|p'|=1} \sum_{|\alpha| \geq 1} \int_{U^2} (u_{\alpha-p}(x), u_{\alpha-p'}(x')) Y E_p(x) E_{p'}(x') \mu(dx) \mu(dx') \frac{\alpha!}{\alpha', \alpha''}.
\]

\[
= \sum_{p=p'} \sum_{|p|=1} A + \sum_{p \neq p'} \sum_{|p|=1} A = A + B.
\]
Now
\[
A = \sum_{|p|=1} \sum_{\alpha \geq p} | \int u_{\alpha-p}(x) E_p(x) d\mu(x)^2 \alpha! = \sum_{|p|=1} \sum_{\alpha \geq p} | \int u_{\gamma}(x) E_p(x) d\mu(x)^2 \gamma! (\gamma + p)!
\]
\[
= \sum_{|p|=1} \sum_{\gamma \geq p} | \int u_{\gamma}(x) E_p(x) d\mu(x)^2 \gamma! + \sum_{|p|=1} \sum_{\gamma \geq p} | \int u_{\gamma}(x) E_p(x) d\mu(x)^2 \gamma! ((\gamma + p)! - \gamma!)
\]
\[
= \sum_{|p|=1} \int \gamma! |u_{\gamma}(x)\|^2 \mu + \sum_{\gamma \geq p} | \int u_{\gamma}(x) E_p(x) d\mu(x)^2 (\gamma! + p)! - \gamma!]
\]

Also,
\[
B = \sum_{p \neq p'} \sum_{|p|=1} \sum_{\alpha \geq p+p'} \int_{U^2} (\alpha + p)(u_{\alpha+p}(x), u_{\alpha+p}(x')) Y E_p(x) E_p'(x') \mu(dx) \mu(dx') \alpha!
\]
\[
= \sum_{p \neq p'} \sum_{|p|=1} \int_{U^2} (u_{\beta+p}(x), u_{\beta+p}(x')) Y \times
\]
\[
E_p(x) E_p'(x') \mu(dx) \mu(dx') (\beta + p + p')!
\]

On the other hand,
\[
E \left[ \mathbb{D}_u(x; x'), \mathbb{D}_x u(x'; x) \right]_Y = \sum_{\alpha} \sum_{|p|=|p'|=1} \frac{(\alpha + p)!}{\alpha!} (u_{\alpha+p}(x), u_{\alpha+p}(x')) Y \frac{(\alpha + p')!}{\alpha!} E_p(x) E_p'(x) \alpha!
\]
\[
= \sum_{\alpha} \sum_{p=p',|p|=1} ... + \sum_{\alpha} \sum_{p \neq p',|p|=1} ... = C + D.
\]

Obviously,
\[
C = \sum_{|p|=1} \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha - p)!} (u_{\alpha}(x), u_{\alpha}(x')) Y E_p(x) E_p(x') \alpha!.
\]

Comparing
\[
D = \sum_{\alpha} \sum_{p \neq p',|p|=1} \frac{(\alpha + p)!}{\alpha!} (a_{\alpha+p}(x) E_p(x'), a_{\alpha+p}(x') E_p'(x)) Y \frac{(\alpha + p')!}{\alpha!} \alpha!,
\]
\[
\int_{U^2} C d\mu_2, \int_{U^2} D d\mu_2 \text{ and } A, B, \text{ the statement follows.} \]

\textbf{Corollary 2.} Let \( u \in S(H(Y), Y) \), i.e. (iii) holds with
\[
\int_{U^{n+1}} |u_n|_Y^2 d\mu_{n+1} < \infty \forall n.
\]

Then the statement of Proposition \( \square \) holds for \( \delta(u) \).

\textbf{Proof.} It is enough to prove the statement for \( u(v) = I_n(u_n(v)) \), where
\[
\begin{align*}
   u_n(v, \cdot) &= \sum_{|\alpha|=n} u_{\alpha}(v) E_\alpha(\cdot) \\
\end{align*}
\]
with a finite number nonzero \( u_\alpha \in H (Y) : \)
\[
\int_U |u_\alpha|^2_Y d\mu < \infty.
\]
In this case, \( u = n! \sum_{|\alpha|=n} u_\alpha (v) N_\alpha \in L(\mathbb{D} (U; Y) , d\mu) \) and Proposition 7 applies. We obtain the general case by linearity and passing to the limit. □

**Remark 10.** In the framework of a single r.v. \( \xi \) (see Remark 3), for \( u = \sum_n u_n \xi^n \) we have
\[
E [\delta (u)^2] = E [|u|^2] + E [(\mathcal{D} u)^2].
\]

For an adapted random field on \( U = [0, T] \times V, d\mu = dt d\pi \), the standard isometry holds. It is an obvious consequence of Corollary 2.

**Corollary 3.** Let \( H = L^2 ([0, T] \times V, dtd\pi) \). Assume \( u \in \mathcal{S} (H (Y), Y) \) is an adapted random field on \( U = [0, T] \times V \). Then
\[
E [|\delta (u)|^2_Y] = E \left[ \int_U |u(t, v)|^2_Y d\mu \right].
\]

**Duality between \( \delta \) and \( \mathcal{D} \)**

**Proposition 8.** Let \( u = u (x) = \sum_\alpha u_\alpha (x) N_\alpha \in L(\mathbb{D} (U; \mathbb{R}) , d\mu) \), i.e.
\[
\int |u_\alpha (x)|^2 d\mu < \infty, \alpha \in J,
\]
with a finite number of \( u_\alpha \neq 0 \). Let \( v = \sum_\alpha v_\alpha N_\alpha \in \mathcal{D} \). Then
\[
E[\delta (u) v] = E \left[ \int_U u(x) \mathcal{D} v(x) d\mu \right].
\]

**Proof.** Indeed,
\[
E[\delta (u) v]
\begin{align*}
&= \sum_{|\alpha| \geq 1} \sum_{|p| = 1} \int_U u_{\alpha - p}(x) E_p(x) d\mu v_\alpha! \\
&= \sum_{|p| = 1} \sum_{\alpha \geq p} \int_U u_{\alpha - p}(x) E_p(x) d\mu v_\alpha! \\
&= \sum_{|p| = 1} \sum_{\gamma} \int_U u_\gamma(x) E_p(x) d\mu v_{\gamma + p} \frac{(\gamma + p)!}{\gamma!} = E \left[ \int_U u(x) \mathcal{D} v(x) d\mu \right].
\end{align*}
\]

\[
\Box
\]

4. **Stochastic Differential Equations**

4.0.1. **Wick exponent.** We start with the definition of Wick exponent.
Let \( f = \sum_k f_km_k \in L^2(U, d\mu) \). For \( n \geq 1 \) and \( \mathfrak{N}(f) = \sum_k f_k \xi_k \) we have (denoting \( f^\alpha = \prod_k f^\alpha_k \))

\[
\mathfrak{N}(f)^\circ_n = \mathfrak{N}(f) \circ \ldots \circ \mathfrak{N}(f) \text{ (n times)} = \sum_{|\alpha|=n} \frac{n!}{\alpha!} f^\alpha \mathfrak{N}_\alpha.
\]

Note that \( \mathfrak{N}(f)^\circ_n \in L^2(\Omega) \):

\[
\mathbb{E} \left( \frac{1}{n!} \mathfrak{N}(f)^\circ_n \right)^2 = \sum_{|\alpha|=n} \frac{z^{2\alpha}}{\alpha!} = \frac{1}{n!} \sum_{|\alpha|=n} \frac{n! z^{2\alpha}}{\alpha!} = \frac{1}{n!} \left( \sum_i z_i^2 \right)^n
\]

\[
= \frac{1}{n!} |f|^2_{L^2(\mu)} < \infty.
\]

Let \( Z \) be the set of all number sequences \( z = (z_k) \) with finite number of nonzero terms. The following statement holds.

**Proposition 9.** a) Let \( f = \sum_k f_km_k \in L^2(\mu) \). Then

\[
\mathfrak{N}(f)^\circ_n = I_n \left( f^{\otimes n} \right)
\]

and

\[
\exp \{ \mathfrak{N}(f) \} := \sum_{n=0}^{\infty} \frac{\mathfrak{N}(f)^\circ_n}{n!} = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} \mathfrak{N}_\alpha = \sum_{\alpha} \frac{f^\alpha}{\alpha!} \mathfrak{N}_\alpha \in L^2(\Omega)
\]

with

\[
\frac{f^\alpha}{\alpha!} = \frac{1}{n!\alpha!} \int f^{\otimes n} E_\alpha d\mu_n.
\]

Moreover,

(4.1) \[
\mathbb{E} \left( \exp \{ \mathfrak{N}(f) \} \right)^2 = \exp \left( |f|^2_{L^2(\mu)} \right).
\]

b) Let \( z = (z_k) \in Z \). Then \( \mathbb{P} \)-a.s.

\[
p(z) = \exp \left\{ \mathfrak{N} \left( \sum_k z_km_k \right) \right\} = \sum_{\alpha} \frac{z^\alpha}{\alpha!} \mathfrak{N}_\alpha, z = (z_k) \in Z,
\]

is analytic in \( z \) and

\[
\frac{\partial^{|\alpha|} p(z)}{\partial z^\alpha} \big|_{z=0} = \mathfrak{N}_\alpha.
\]

**Proof.** a) In terms of multiple integrals we have

\[
\mathfrak{N}(f)^\circ_n = \sum_{|\alpha|=n} \frac{n! f^\alpha}{\alpha!} \mathfrak{N}_\alpha = \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} I_n (E_\alpha) =
\]

\[
\sum_{|\alpha|=n} \frac{n! f^\alpha}{\alpha!} I_n \left( E_{\alpha_1} \ldots E_{\alpha_n} \right) =
\]

\[
I_n \left( f^{\otimes n} \right) = I_n \left( \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} E_\alpha \right),
\]
with
\[ f^\otimes n = \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} E^\alpha, \quad f^\alpha = \frac{1}{n!} \int f^\otimes n E^\alpha d\mu_n. \]

In addition,
\[
\mathbb{E} \left( \left( \frac{\mathcal{M}(f)^\otimes n}{n!} \right)^2 \right) = \sum_{|\alpha|=n} \frac{f^{2\alpha}}{\alpha!} = \frac{1}{n!} \sum_{|\alpha|=n} n! f^{2\alpha} = \frac{1}{n!} \left( \sum_i f_i^2 \right)^n
\]
(note also using multiple integrals: \( \mathbb{E} \left( \frac{1}{(n!)^2} I_n (f^\otimes n)^2 \right) = \frac{1}{n!} |f|^{2n}_{L^2(\mu)} = \frac{1}{n!} |f|^{2n}_{L^2(\mu)} \). Moreover,
\[
\mathbb{E} \left( \sum_n \left( \frac{\mathcal{M}(f)^\otimes n}{n!} \right)^2 \right) = \sum_n \frac{1}{n!} |f|^{2n}_{L^2(\mu)} = \exp \left\{ |f|^{2n}_{L^2(\mu)} \right\}.
\]

b) Let \( z = (z_k) \in \mathbb{Z} \). Then
\[
\mathbb{E} |p(z)| \leq \sum_{\alpha} \frac{|z^\alpha|}{\sqrt{\alpha!}} \mathbb{E} \left| \frac{\mathcal{M}_\alpha}{\sqrt{\alpha!}} \right| \leq \sum_{\alpha} \frac{|z^\alpha|}{\sqrt{\alpha!}} \leq \prod_k \sum_n \frac{|z_k|^n}{\sqrt{n!}}
\]
and the statement follows.

In a time dependent case the following statement holds.

**Corollary 4.** Let \( U = [0, T] \times V, \; d\mu = dt d\pi \). Let \( G \in L^2([0, T] \times V, d\mu) \).
Consider \( M_t = \exp \left\{ \mathcal{M} \left( \chi_{[s,t]} G \right) \right\}, 0 \leq s \leq t \leq T. \)
Then
\[
M_t = \sum_{\alpha} \frac{H(s, t)^\alpha}{\alpha!} \mathcal{M}_\alpha = \sum_{n=0}^{\infty} \frac{I_n(H_n(s, t))}{n!}, \quad s \leq t \leq T,
\]
with
\[
H_k(s, t) = \int_s^t \int G(r, v) m_k(r, v) dr d\pi, \quad H(s, t)^\alpha = \prod_k H_k(s, t)^{\alpha_k},
\]
\[
\frac{H(s, t)^\alpha}{\alpha!} = \frac{1}{n! \alpha!} \int \left( \chi_{[s,t]} G \right)^{\otimes n} E^\alpha d\mu_n \text{ if } |\alpha| = n,
\]
\[
H_n(s, t) = \left( \chi_{[s,t]} G \right)^{\otimes n}.
\]
Moreover, \( M \) is adapted (for each \( t \), the \( \text{supp}(H_n(t)) \subseteq Q^n_t = ([0, t] \times V)^n. \)
4.0.2. Linear SDE. Let $U = [0, T] \times V, d\mu = dt d\pi$. Let $w = \sum_{\alpha} w_{\alpha} \mathcal{N}_{\alpha} \in \mathcal{D}'$, $f = \sum_{\alpha} f_{\alpha} (t, v) \mathcal{N}_{\alpha} \in L_{2} (\mathcal{D}' (U), d\mu)$. For $G \in L^{2} (\mu)$, consider the a non-homogeneous equation

\begin{equation}
\dot{u} (t) = \int [u (t) G (t, v) + f (t, v)] \circ \mathcal{N} (t, v) \nu (dv), u (0) = w,
\end{equation}

that is, equivalently,

\begin{equation}
u (t) = w + \int_{0}^{t} [u (s) G (s, v) + f (s, v)] \circ \mathcal{N} (ds, dv), 0 \leq t \leq T.
\end{equation}

We seek a solution to (4.3) in the form of (4.4) with continuous coefficients $\nu$. Plugging the series (4.4) into (4.3) we immediately get the solution $u$.

**Lemma 2.** Let $w = \sum_{\alpha} w_{\alpha} \mathcal{N}_{\alpha} \in \mathcal{D}'$, $f = \sum_{\alpha} f_{\alpha} (t, v) \mathcal{N}_{\alpha} \in L_{2} (\mathcal{D}' (U), d\mu)$.

Then there is a unique solution to (4.3) in $C \mathcal{D}' ([0, T]; \mathbb{R})$ (Recall $C \mathcal{D}' ([0, T]; \mathbb{R})$ is the class of all generalized processes $u = \sum_{\alpha} u_{\alpha} (t) \mathcal{N}_{\alpha}$ on $[0, T]$ such that $u_{\alpha}$ is continuous on $[0, T]$ for all $\alpha \in J$). The solution $u$ given by (4.4) has the following coefficients: $u_{0} (t) = w_{0}$,

\begin{equation}
u_{\alpha} (t) = w_{\alpha} + \sum_{|\beta| = 1} \int_{0}^{t} [u_{\alpha - \beta} (r) G (r, v) + f_{\alpha - \beta} (r, v)] E_{\beta} (r, v) d\pi dr, 0 \leq t \leq T.
\end{equation}

**Proof.** We seek the solution $u$ to (4.3) in the form of (4.4) with continuous coefficients $\nu_{\alpha}$. Plugging the series (4.4) into (4.3) we immediately get the unique solution $u_{\alpha} (t)$ for $|\alpha| \geq 1$.

Let

\begin{equation}
H_{k} (t) = \int_{0}^{t} \int G (s, v) m_{k} (s, v) ds d\pi, H_{k} (t)^{\alpha} = \prod_{k} H_{k} (t)^{\alpha_{k}}, \alpha \in J.
\end{equation}

For $w = \sum_{\alpha} w_{\alpha} \mathcal{N}_{\alpha} \in \mathcal{D}'$, let

\begin{equation}
||w||^{2} = \sum_{\alpha} |w_{\alpha}|^{2} \beta! + \sup_{t} \sum_{\alpha} \alpha! \left( \sum_{\beta \leq \alpha} w_{\alpha - \beta} \frac{H (t)^{\beta}}{\beta!} \right)^{2}
\end{equation}

**Lemma 3.** Let $f = 0, w = \sum_{\alpha} w_{\alpha} \mathcal{N}_{\alpha} \in \mathcal{D}'$.

(i) The solution to (4.3) is given by

\begin{equation}
u (t) = w \circ \exp \left\{ \mathcal{N} \left( \chi_{[0, t]} G \right) \right\} = \sum_{\alpha} \sum_{\beta \leq \alpha} w_{\alpha - \beta} \frac{H (r)^{\beta}}{\beta!} \mathcal{N}_{\alpha}, 0 \leq r \leq T.
\end{equation}

(ii) If $w \in \mathcal{S}' (\mathbb{R})$, then $u \in C \mathcal{S}' ([0, T]; \mathbb{R})$;

(iii) $\sup_{t} \mathbb{E} [u (t)^{2}] \leq ||w||^{2}$,
i.e. it is in $L^2(\Omega, \mathbb{P})$ if $||w|| < \infty$ (see Example 4 below).

(iv) If $w = w_0$ is a constant, then $u(t)$ is adapted and

$$
\sup_t \mathbb{E} [u(t)^2] = w_0^2 \exp \left\{ \int |G|^2 \, d\mu \right\}.
$$

Proof. (i) Let $M_t = \exp^\circ \{ \mathcal{M}(\chi_{[0,t]}G) \}$. By Corollary 4,

$$
v(r) = w \circ M_r = \sum_{\alpha} \sum_{\beta \leq \alpha} w_{\alpha-\beta} H(r)^{\beta} \mathcal{M}_\alpha, 0 \leq r \leq T.
$$

We will show that $v$ solves (4.3). Indeed,

$$
\delta \left( \chi_{[0,t]} G \right) = \sum_{|\alpha| \geq 1} \sum_{|\beta| = 0} \int_0^t \int_v \sum_{\beta \leq \alpha - p} w_{\alpha - p} \frac{H(r)^{\beta}}{\beta!} G(r, \nu) E_p(r, \nu) d\mu \mathcal{M}_\alpha
$$

$$
= \sum_{|\alpha| \geq 1} \sum_{|\beta| = 0} \int_u \int_v \sum_{\beta \leq \alpha - p} w_{\alpha - p} \frac{\chi_{[0,r]} G}{|\beta|! \beta!} E_{|\beta|} \chi_{[0,t]} G(r, \nu) E_p(r, \nu) d\mu \mathcal{M}_\alpha
$$

$$
= \sum_{|\alpha| \geq 1} \sum_{|\beta| = 0} \int_u \int_v \sum_{\gamma \leq \alpha, |\gamma| \geq 1} w_{\alpha - \gamma} \chi_{[0,r]} G \chi_{[0,t]} G(r, \nu) \frac{E_\gamma}{|\gamma|! \gamma!} \chi_{[0,t]} G(r, \nu) E_p(r, \nu) d\mu \mathcal{M}_\alpha
$$

$$
= \sum_{|\alpha| \geq 1} \sum_{|\beta| = 0} \int_u \int_v \sum_{\gamma \leq \alpha, |\gamma| \geq 1} w_{\alpha - \gamma} \chi_{[0,r]} G \chi_{[0,t]} G(r, \nu) \frac{E_\gamma}{|\gamma|! \gamma!} \mathcal{M}_\alpha = v(t) - w,
$$

and (4.3) holds.

(ii) follows immediately by Lemma 7 and Proposition 9. The part (iii) is a direct consequence of (4.6). Finally, (iv) follows from (4.6), Proposition 9 and Corollary 4.

Example 4. Let $F \in \mathbb{L}^2(U, d\mu)$. Taking $w = \exp^\circ \{ \mathcal{M}(F) \}$ in (4.6), we see that the solution to (4.3)

$$
u(t) = \exp^\circ \{ \mathcal{M}(F) \} \circ \exp^\circ \{ \mathcal{M}(\chi_{[0,t]} G) \}
$$

$$
= \exp^\circ \{ \mathcal{M}(F) + \mathcal{M}(\chi_{[0,t]} G) \}
$$

is clearly non-adapted in general but $\sup_t \mathbb{E} [u(t)^2] < \infty$ (Proposition 3).

Let for $s \leq t$,

$$
H_k(s, t) = \int \chi_{[s,t]} G \mu_k \, d\mu, H(s, t) = \prod_k H_k(s, t)^{\alpha_k}, \alpha \in J.
$$
For \( f = \sum_{\alpha} f_{\alpha}(t, v) \mathcal{M}_{\alpha} \in L_2(\mathcal{D}'(U), d\mu) \), let \( \|f\|_{0,T}^2 = E \int_U |f|^2 \ d\mu \) and

\[
\|f\|_{T}^2 = \sup_t \sum_{\alpha} \alpha! \left( \sum_{|p|=1} \int_0^t \int_U \sum_{\beta+p \leq \alpha, |\beta| \leq n} f_{\alpha-p-\beta}(s, v) E_p(s, v) \frac{H(s, t)^{\beta}}{\beta!} \ d\mu \right)^2 + \sum_{\alpha} \int_U |f_{\alpha}|^2 d\mu!.
\]

**Proposition 10.** Let \( f = \sum_{\alpha} f_{\alpha}(t, v) \mathcal{M}_{\alpha} \in L_2(\mathcal{D}'(U), d\mu) \), \( w = \sum_{\alpha} w_{\alpha} \mathcal{M}_{\alpha} \in \mathcal{D}' \).

(i) The unique solution to (4.3) in \( C \mathcal{D}'([0, T]; \mathbb{R}) \) can be given as

\[
(4.7) u(t) = w \ast \exp \{ \mathcal{M}(\chi_{[0,t]}G) \} + \int_0^t \int_U \exp \{ \mathcal{M}(\chi_{[s,t]}G) \} \circ f(s, v) \circ \mathcal{M}(ds, dv)
\]

\[
= \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_0^t \int_U \sum_{\beta+p \leq \alpha, |\beta| \leq n} f_{\alpha-p-\beta}(s, v) E_p(s, v) \frac{H(s, t)^{\beta}}{\beta!} d\mu \mathcal{M}_{\alpha}
\]

\[+ \sum_{\alpha} \sum_{\beta \leq \alpha} w_{\alpha-\beta} H(r)^{\beta} \beta! \mathcal{M}_{\alpha}.
\]

(ii) The solution is the limit of Picards iterations \( u^n(t) \): \( u^0(t) = w + \int_0^t \int f(s, v) \circ \mathcal{M}(ds, dv) \)

\[
(4.8) u^{n+1}(t) = w + \int_0^t \int [u^n(s)G(s, v) + f(s, v)] \circ \mathcal{M}(ds, dv) , 0 \leq t \leq T.
\]

In fact,

\[
(4.9) u^n(t) = w \circ \sum_{k=0}^n \frac{\mathcal{M}(\chi_{[0,t]}G)^{\circ k}}{k!} + \int_0^t \int \sum_{k=0}^n \frac{\mathcal{M}(\chi_{[s,t]}G)^{\circ k}}{k!} \circ f(s, v) \circ \mathcal{M}(ds, dv).
\]

If \( f \in \mathcal{S}'(H, \mathbb{R}) \) and \( w \in \mathcal{S}'(\mathbb{R}) \), then \( u^n, u \in CS'([0, T]; \mathbb{R}) \); (iii)

\[
\sup_t E \left[ u(t)^2 \right] \leq 2 \left( \|w\|^2 + \|f\|_{0,T}^2 \right).
\]

(iv) If \( w = w_0 \) is deterministic and \( f \) is adapted with \( \|f\|_{0,T}^2 < \infty \), then \( u(t) \) is adapted and square integrable:

\[
\sup_t E \left[ u(t)^2 \right] \leq C \left( w_0^2 + E \int_U |f|^2 \ d\mu \right).
\]

**Proof.** Because of Lemma 3, we assume \( w = 0 \).
(i) Let

\[ l(s, v) = f(s, v) \circ \exp^o \left\{ \mathfrak{H} \left( \chi[s, t] \right) G \right\} \]
\[ = \sum_{\alpha} \sum_{\beta \leq \alpha, |\beta| \leq n} f_{\alpha - \beta}(s, v) \frac{H(s, r)^\beta}{\beta!}, 0 \leq s \leq r \leq T \]
\[ = \sum_{\alpha} \sum_{\beta \leq \alpha, |\beta| \leq n} f_{\alpha - \beta}(s, v) \frac{1}{|\beta|!} \int \left( \chi[s, t] G \right)^{\otimes |\beta|} E_\beta d\mu_{|\beta|} \mathfrak{N}_\alpha. \]

and set

\[ v(r) = \int_0^r \int l(s, v) \circ \mathfrak{H} (ds, dv) \]
\[ = \sum_{|\alpha| \geq 1, |p| = 1, p \leq \alpha} \sum_{\beta + p \leq \alpha} \int \sum_{\beta \leq \alpha, |\beta| \leq n} f_{\alpha - (p + \beta)}(s, v) E_p(s, v) d\mu \frac{H(s, r)^\beta}{\beta!} \mathfrak{N}_\alpha, 0 \leq r \leq T. \]

For \( r \in [0, T], v = (s_1, v_1, \ldots, s_k, v_k) \in U^k, k \geq 1, \) define

\[ \Phi(r, k, G, f) = \Phi(r, k, G, f) (s_1, v_1, \ldots, s_k, v_k) \]
\[ = \sum_{j=1}^k f(\hat{s}, v_j) \prod_{i \neq j, i=1}^k \chi[s, r] (s_i) G (s_i, v_i), \]

where \( \hat{s} = \min \{ s_i, 1 \leq i \leq k \}. \)

By Corollary 4

\[ v(r) = \sum_{|\alpha| \geq 1, |p| = 1, p \leq \alpha} \int_0^r \int \sum_{\beta + p \leq \alpha} f_{\alpha - (p + \beta)}(s, v) E_p(s, v) \frac{H(s, r)^\beta}{\beta!} \int \left( \chi[s, t] G \right)^{\otimes |\beta|} E_\beta d\mu_{|\beta|} d\mu \mathfrak{N}_\alpha \]
\[ = \sum_{\alpha} \sum_{\beta' \leq \alpha, 1 \leq |\beta'|} \int_{U^{|\beta'|+1}} \left( \chi[s, r] G \right)^{\otimes |\beta'|} - 1 f_{\alpha - \beta'}(s, v) \left( \frac{E_{\beta'}}{|\beta'|!} - 1 \right) \frac{H(s, r)^\beta}{\beta!} d\mu_{|\beta'|} \mathfrak{N}_\alpha \]
\[ = \sum_{\alpha} \sum_{\beta' \leq \alpha, 1 \leq |\beta'|} \int_{U^{|\beta'|}} \Phi(r, |\beta'|, G, f_{\alpha - \beta'}) \left( \frac{E_{\beta'}}{|\beta'|!} - 1 \right) d\mu_{|\beta'|} \mathfrak{N}_\alpha, 0 \leq r \leq T. \]
We will show that $v$ solves (4.3). Indeed,

$$
\int_0^t v(r) G(r, v) \circ \mathfrak{N} (dr, dv)
= \sum_{|\alpha| \geq 2} \sum_{|p| = 1} \sum_{\beta' + p \leq \alpha, 1 \leq |\beta'|} \int_{U^{[\beta']}} \chi_{[0,t]}(r) G(r, v) E_p (r, v) \times \\
\times \int_{U^{[\beta']}} \Phi \left( r, |\beta'|, G, f_{\alpha-(p+\beta')} \right) \frac{E_{|\beta'| \gamma} \gamma \gamma d\mu_{|\beta'| \gamma} d\mu_{\alpha}}{E_{|\gamma| \gamma} \gamma \gamma d\mu_{|\gamma| \gamma} d\mu_{\alpha}}
= \sum_{|\alpha| \geq 2} \sum_{|p| = 1} \sum_{\beta' + p \leq \alpha, 2 \leq |\gamma|} \int_{U^{[\gamma]}} \chi_{[0,t]}(r) G(r, v) \Phi \left( r, |\gamma| - 1, G, f_{\alpha-\gamma} \right) \times \\
\times \frac{E_{\gamma \gamma} \gamma \gamma d\mu_{|\gamma| \gamma} d\mu_{\alpha}}{E_{|\gamma| \gamma} \gamma \gamma d\mu_{|\gamma| \gamma} d\mu_{\alpha}}
$$

and we see that

$$
\int_0^t v(r) G(r, v) \circ \mathfrak{N} (dr, dv) = v(t) - \int_0^t \int f (r) \circ G (r, v) \mathfrak{N} (dr, dv).
$$

(ii) Consider $u^n(t)$ defined by (4.9) with $w = 0$. Then (4.10)

$$
u^n(t) = \sum_{|\alpha| \geq 1} \sum_{|p| = 1, p \leq \alpha} \int \sum_{\beta + p \leq \alpha, |\beta| \leq n} f_{\alpha-(p+\beta)}(s, v) E_p(s, v) d\mu \frac{H(s, t)^\beta}{\beta!} \mathfrak{N}_\alpha, 0 \leq r \leq T,
$$

and repeating the proof of part (i) we see that for $0 \leq t \leq T$

$$
\int_0^t \int u^n(r) G(r, v) \circ \mathfrak{N} (dr, dv) = u^{n+1}(t) - \int_0^t \int f (s, v) \circ \mathfrak{N} (ds, dv).
$$

If $f \in S'(H, R)$, then $u^0 \in CS'([0,T]; R)$. If $u^n \in CS'([0,T]; R)$, then $u^n G \in S'(H, R)$. By Proposition 3, $u^{n+1} \in CS'([0,T]; R)$ and the statement follows by comparing (4.10) and (4.7).

The part (iii) is a direct consequence of (4.7).

(iv) Since $\mathbb{E} \int |f|^2 d\mu < \infty$, it follows that $f \in S'(H, R)$, and according to part (ii) and Proposition 3, all the iterations are adapted. Therefore Ito isometry holds. Obviously,

$$
\sup_t \mathbb{E} \left[ u^0(t)^2 \right] \leq \mathbb{E} \int_U |f|^2 d\mu < \infty.
$$

Assume $\sup_t \mathbb{E} \left[ u^n(t)^2 \right] < \infty$. Using (4.8) and Ito isometry,

$$
\mathbb{E} \left[ u^{n+1}(t)^2 \right] \leq C \int_0^t \mathbb{E} \left( u^n(s)^2 \right) G(s, v)^2 ds + \mathbb{E} \int_0^t \int |f|^2 ds d\pi
$$
and by Gronwall’s lemma there is a constant \( C \) independent of \( n \) such that
\[
\sup_{n,t} \mathbb{E} \left( u^n(t)^2 \right) \leq C \mathbb{E} \int_0^T \int |f|^2 \, ds \, d\pi.
\]
Similarly, using Gronwall’s lemma, we show that
\[
\sum_n \sup_t \mathbb{E} \left( |u^{n+1}(t) - u^n(t)|^2 \right) < \infty.
\]
The statement follows. \( \square \)

4.0.3. Linear parabolic SPDEs. In this section we extend the results on the linear SDE to a simple parabolic SPDE.

Again, let \( U = [0,T] \times V, d\mu = dt \, d\pi \). We denote \( \mathbb{R}^d_T = \mathbb{R}^d \times [0,T] \) and suppose that the following measurable functions are given

\[
a : \mathbb{R}^d \to \mathbb{R}^{d^2}, \quad b : \mathbb{R}^d \to \mathbb{R}^d.
\]

The following is assumed.

**A1.** Functions \( a, b \), are infinitely differentiable and bounded with all derivatives, and the matrix \( a = (a_{ij}(x)) \) is symmetric and non-degenerate:

\[
a_{ij}(x)\xi_i\xi_j \geq \delta |\xi|^2, \quad \xi \in \mathbb{R}^d,
\]

for some \( \delta > 0 \).

Let \( H^2 = H^2(\mathbb{R}^d), s = 1, 2 \), be the Sobolev class of square-integrable functions \( v \) on \( \mathbb{R}^d \) having generalized space derivatives up to the \( s \)-order with the finite norm

\[
|v|_{s,2} = |v|_2 + |D_x^s v|_2,
\]

where \( |v|_2 = \int_{\mathbb{R}^d} |v|^2 \, dx \).

Let \( G \in L^2([0,T] \times V, d\mu) \) with \( d\mu = dt \, d\pi \). Let \( L^2 = L^2(\mathbb{R}^d \times [0,T] \times V , dx \, dt \, d\pi) \) be the space of all measurable functions \( g \) on \( \mathbb{R}^d \times [0,T] \times V \) such that

\[
||g||^2_{2,2} = \int_0^T \int_{\mathbb{R}^d} \int_V [\|g(s,x,v)\|^2 + |\nabla_x g(s,x,v)|^2] \, ds \, dx \, d\pi < \infty.
\]

Let \( w = \sum_{\alpha} w_\alpha(x) \mathfrak{H}_\alpha \in \mathcal{D}'(H^3_2(\mathbb{R}^d)) \) and \( f = \sum_\alpha f_\alpha(x,v) \mathfrak{H}_\alpha \in \mathcal{D}'(L^2) \).

The main objective of this section is to study the equation for \( u(t) = u(t,x) \),

\[
\begin{align*}
\partial_t u(x,t) &= \mathcal{L}u(x,t) \\
&\quad + \int_U (u(x,t)G(t,v) + f(x,t,v)) \circ \mathfrak{H}(t,v) \, \pi \, (dv) \\
\end{align*}
\]

where \( \mathcal{L}u = a_{ij}(x)u_{x_ix_j} + b^i(x)u_{x_i} \).

Equivalently, we understand (4.11) as

\[
\begin{align*}
u(t) &= w + \int_0^t \mathcal{L}u(s) \, ds \\
&\quad + \int_0^t \int_U [u(s)G(s,v) + f(s,v)] \circ \mathfrak{H}(ds,dv),
\end{align*}
\]
0 ≤ t ≤ T.

We will seek a solution to (4.12) in the form

\[ u(t) = \sum_{\alpha} u_{\alpha}(t) \mathcal{N}_\alpha \in C^{\alpha}([0, T]; H^2 ) \]

We start our analysis of equation (4.12) by introducing the definition of a solution in the "weak sense".

**Definition 8.** We say that a generalized \(D\)-process \( u(t) = \sum_{\alpha} u_{\alpha}(t) \xi_{\alpha} \in C^{\alpha}([0, T], H^2 ) \) is \(D\)-\(H^2\) solution of equation (4.12) in \([0, T]\), if the equality (4.12) holds in \(D(L^2(\mathbb{R}^d))\) for every \(0 ≤ t ≤ T\).

**Lemma 4.** Assume A1 holds, \( w(x) = \sum_{\alpha} w_{\alpha}(x) \mathcal{N}_\alpha \in \mathcal{D}'(H^2 ) \),

\[ g = \sum_{\alpha} g_{\alpha}(x, s, v) \mathcal{N}_\alpha \in \mathcal{D}'(L^2) \].

Then there is a unique solution to (4.12) in \(\mathcal{D}'([0, T], H^2 )\) (Recall \(\mathcal{D}'([0, T], H^2 )\) is the class of all generalized processes \( u = \sum_{\alpha} u_{\alpha}(t) \mathcal{N}_\alpha \) on \([0, T]\) such that \( u_{\alpha} \) is \(H^2\)-valued continuous on \([0, T]\) \(∀\alpha \in J\). The solution \( u \) given by (4.13) has the following coefficients: \( u_{\alpha}(t) = w_{\alpha} \).

Proof. We seek the solution \( u \) to (4.12) in the form of (4.13) with continuous coefficients \( u_{\alpha} \). Plugging the series (4.13) into (4.12) we immediately get the system (4.14). Indeed, by definition, for \( t ∈ [0, T] \),

\[ \sum_{\alpha} u_{\alpha}(x, t) \mathcal{N}_\alpha = \sum_{\alpha} w_{\alpha}(x) \mathcal{N}_\alpha + \sum_{\alpha} \int_0^t \mathcal{L}u_{\alpha}(x, s)ds \mathcal{N}_\alpha + \sum_{\alpha} \sum_{k} \int_0^t \int_V m_k[u_{\alpha(k)}G + g_{\alpha(k)}]d\pi ds \mathcal{N}_\alpha. \]

Since the system is triangular, starting with \( u_0 (t) = w_0 \) we find unique continuous \( u_{\alpha}(t) \) for \(|\alpha| ≥ 1\) (see [2]).

Denote by \( T_t f \) the solution of the problem

\[ \left\{ \begin{array}{l}
\partial_t u = \mathcal{L}u, \quad 0 ≤ t ≤ T, \\
u(0, x) = h(x), x ∈ \mathbb{R}^d.
\end{array} \right. \]

**Remark 11.** If A1 holds, then it is well known that

\[ |T_t h|_{L^2(\mathbb{R}^d)}^2 \leq e^{Ct} |h|_{L^2(\mathbb{R}^d)}^2, \]

(see [2]).

Note that for each \( \alpha \), the solution \( u_{\alpha}(\cdot) \) of (4.14) satisfies for \( t ∈ [0, T] \),

\[ u_{\alpha}(t) = T_t w_{\alpha} + \sum_{k} \int_0^t \int_V [m_k(s, v)(T_{t-s} u_{\alpha(k)}(s)G(s, v)
+ T_{t-s} g_{\alpha(k)}(s, v))]dsd\pi. \]
Lemma 5. Assume \( A_1 \) holds,
\[
w(x) = \sum_{\alpha} w_\alpha(x) \mathcal{M}_\alpha \in \mathcal{D}'(H^3_2), \quad g = \sum_{\alpha} g_\alpha(x, s, v) \mathcal{M}_\alpha \in \mathcal{D}'(L^{2,1}).
\]

Then \( u \) is the unique solution to (4.12) in \( \mathcal{C}D'([0, T], H^2_2) \) iff it is the unique solution to
\[
(4.17) \quad u(t) = \int_0^t \int_U [T_{t-s}w(s)(s, v) + T_{t-s}g(s, v)] \circ \mathcal{M}(ds, dv) + T_tw,
\]
\( 0 \leq t \leq T. \)

Proof. Since (4.16) holds, the statement is an immediate consequence of Lemma 4.

The following statement holds.

Proposition 11. Let \( A_1 \) hold and \( w = \sum_{\alpha} w_\alpha(x) \mathcal{M}_\alpha \in \mathcal{D}'(H^3_2(R^d)) \) and \( g = \sum_{\alpha} f_\alpha(x, s, v) \mathcal{M}_\alpha \in \mathcal{D}'(L^{2,1}) \).

(i) The unique solution to (4.12) is given by
\[
u(t) = T_tw(x) \circ \exp\{ \mathcal{M}(\chi_{[0,t]}G) \}
\]
\[
+ \int_0^t \int \exp\{ \mathcal{M}(\chi_{[s,t]}G) \} \circ T_{t-s}(s, x, v) \circ \mathcal{M}(ds, dv)
\]
In the form of the series,
\[
u(t) = \sum_{|\alpha| \geq 1} \sum_{|p| = 1} \int_0^t \int_U T_{t-s}f_{\alpha - p - \beta}(s, v) E_p(s, v) \frac{H(s, t)^\beta}{\beta!} d\mu_\alpha
\]
\[
+ \sum_{\beta \leq \alpha} T_tw_{\alpha - \beta} \frac{H(t)^\beta}{\beta!} \mathcal{M}_\alpha.
\]

(ii) The solution is the limit of Picards iterations \( u^n(t) \): \( u^0(t) = T_tw + \int_0^t \int T_{t-s}f(s, v) \circ \mathcal{M}(ds, dv) \),
\[
u^{n+1}(t) = T_tw + \int_0^t \int [T_{t-s}u^n(s)(s, v) + T_{t-s}f(s, v)] \circ \mathcal{M}(ds, dv),
\]
\( 0 \leq t \leq T. \) In fact, for \( 0 \leq t \leq T \),
\[
u^n(t) = T_tw \circ \sum_{k=0}^n \frac{\mathcal{M}(\chi_{[0,t]}G)^{\otimes k}}{k!}
\]
\[
+ \int_0^t \int \sum_{k=0}^n \frac{\mathcal{M}(\chi_{[s,t]}G)^{\otimes k}}{k!} \circ T_{t-s}f(s, v) \circ \mathcal{M}(ds, dv).
\]

If \( f \in \mathcal{S}'(H, R) \) and \( w \in \mathcal{S}'(R) \), then \( u^n, u \in \mathcal{C}\mathcal{S}'([0, T]; R) \);

\]
(iii) If $w$ is deterministic and $g$ is adapted, then the solution $u$ is $L^2(\mathbb{R}^d)$-valued and
\[
\sup_{t} \mathbb{E} \left[ |u(t)|^2_{L^2(\mathbb{R}^d)} \right] \leq C \mathbb{E} |w|^2_{L^2(\mathbb{R}^d)} + \int_0^T \int_{\mathbb{R}^d} \int_U |g(x, s, v)|^2 ds dx d\pi.
\]

Proof. We repeat the main arguments of Proposition 10 (as in the case of linear SDE). The changes in the proof (i) are obvious. The proof of (ii)-(iii) is identical to the proof of (ii), (iv) in Proposition 10 with the use of (4.15) for the estimate of the iterations $L^2(\Omega, \mathbb{P})$-norm. $\square$

4.1. Stationary SPDEs. Let us consider a stationary (time independent) equation
\[
Au + \delta_{\mathfrak{N}}(Mu) = g
\]
where, as previously, $\mathfrak{N}$-noise is a formal series $\mathfrak{N} = \sum_k m_k \xi_k$ and $\{m_k\}$ is a CONS in a Hilbert space $H$ and $\xi_k$ are independent random variables with zero mean and variance 1.

We will consider equation (4.18) in a triple of Hilbert spaces $(V, H, V')$,
- A. $V \subset H \subset V'$ and the imbeddings $V \subset H$ and $H \subset V'$ are dense and continuous;
- B. The space $V'$ is dual to $V$ relative to the inner product in $H$;
- C. There exists a constant $C > 0$ such that $|(u, v)_H| \leq C \|u\|_V \|v\|_{V'}$ for all $u$ and $v$.

A triple of Hilbert spaces is often call normal if the assumptions A,B,C hold. A typical example of a normal triple is the Sobolev spaces
\[
\left( H^{l+\gamma}(\mathbb{R}^d), H^l(\mathbb{R}^d), H^{l-\gamma}(\mathbb{R}^d) \right) \quad \text{for } \gamma > 0.
\]

Everywhere in this section it is assumed that $A : V \to V'$ and $M : V \to V' \otimes l_2$ are bounded linear operators.

As we already know, equation (4.18) can be rewritten in the form
\[
Au + \sum_{n \geq 1} M_n u \diamond \xi_n = f,
\]
where $u = \sum_{a} u_a \mathfrak{N}_a$. Since
\[
M_n u = \sum_{a \in \mathcal{J}} M_n u_a \mathfrak{N}_a,
\]
we get
\[
\sum_{n \geq 1} M_n u \diamond \xi_n
\]
\[
= \sum_{a \in \mathcal{J}} \sum_{n \geq 1} M_n u_a \xi_n \diamond \xi_n + \sum_{a \in \mathcal{J}} \sum_{n \geq 1} M_n u_a \mathfrak{N}_{a+\varepsilon_n}
\]
\[
= \sum_{n \geq 1} \sum_{\beta \in \mathcal{J}: |\beta| \geq 1} M_n u_{\beta-\varepsilon_n} \mathfrak{N}_{\beta}.
\]
Therefore, for $\gamma \in J$ such that $|\gamma| > 0$, we have

$$\left( \sum_{n \geq 1} M_n u \circ \xi_n \right)_\gamma = \sum_{n \geq 1} M_n u_{\gamma - \varepsilon_n}$$

It is readily checked that the set $(u_\alpha, \alpha \in J)$ solves the following system of deterministic equations related to (4.19) is given by

$$A u_\alpha = Ef \quad \text{if } |\alpha| = 0$$

$$A u_\alpha + \sum_{n \geq 1} M_n u_{\alpha - \varepsilon_n} = f_\gamma \quad \text{if } |\alpha| > 0$$

Note that the propagator (4.21) is lower triangular. Therefore, if $A$ has an appropriate inverse $A^{-1}$, then the propagator can be solved sequentially. Then, a solution to equation (4.19) could be defined by the following formula

$$u = \sum_{\alpha \in J} u_\alpha \mathcal{R}_\alpha$$

where the sequence $\{u_\alpha, \alpha \in J\}$.

Of course, an appropriate question to ask is: does equation (4.19) has finite variance? The answer to this question is negative. The following simple example clarifies this issue.

**Example 5.** Consider the following simple version of equation

$$u = 1 + u \circ \xi.$$  

(4.23)

Obviously, in this setting, $J = (0, 1, 2, ...)$ and consists of one-dimensional indices $\alpha = 0, 1, 2, ...$ and $E[\mathcal{R}_n^2] = n!$.

It is easy to see that $\{u_n = E(u \mathcal{R}_n), n \geq 0\}$ solves the following system of equations:

$$u_0 = 1, \quad u_n = I_{n=0} + \sqrt{n} u_{n-1}, \quad n \geq 1$$

Obviously, $u_n = \sqrt{n}!$ and $v = 1 + \sqrt{n}! \mathcal{R}_n$, Therefore,

$$E u^2 = \sum_{n \geq 1} u_n^2 = \infty.$$  

One could also define a solution to equation (4.19) as a generalized $D$-random variable with values in $V$, such that (4.19) holds in $D(V')$.

4.1.1. **Weighted Norms.** Another popular definition of solutions based on rescaling/weighting of the coefficients $u_\alpha$ was discussed thoroughly in the literature on polynomial chaos expansion for Gaussian and Levy processes (see, for example, [11], [3], [8], [10]). This technique is also suitable for the current setting and we will describe it briefly.
Given a separable Hilbert space $X$ and sequence of positive numbers $R = \{r_\alpha, \alpha \in J\}$, we define the space $RL_2(X)$ as the collection of formal series $f = \sum f_\alpha N_\alpha$, $f_\alpha \in X$ such that

\begin{equation}
(4.24) \quad \|f\|_{RL_2(X)}^2 = \sum_\alpha \|f_\alpha\|_X^2 r_\alpha^2 < \infty
\end{equation}

If (4.24) holds, then $\sum_\alpha r_\alpha f_\alpha \xi_\alpha \in L_2(X)$.

Similarly, the space $RL_2^{-1}(X)$ corresponds to the sequence $R^{-1} = \{r^{-1}_\alpha, \alpha \in J\}$.

Important and popular examples of the space $RL_2(X)$ correspond to the following weights:

(a) $r_\alpha^2 = \prod_{k=1}^\infty q_\alpha^k$, where $\{q_\alpha, k \geq 1\}$ is a non-increasing sequence of positive numbers;

(b) Kondratiev’s spaces $(S)_{\rho,\alpha}$:

\[ r_\alpha^2 = (\alpha!)^\rho (2N)^{\alpha l} \rho \leq 0, \ l \leq 0. \]

In particular, in the setting of Example 5, $E u^2 = \|u\|_{(S)_{0,0}}^2 = \infty$, but $E \|u\|_{(S)_{\rho,0}}^2 < \infty$ for sufficiently small $\rho$.

4.1.2. Wick-Nonlinear SPDEs. Let us consider equation

\begin{equation}
(4.25) \quad A u - u^{\circ 3} + \sum_{n \geq 1} M_n u \circ \xi_n = f,
\end{equation}

where $u^{\circ 3} = u \circ u \circ u$. As in the previous section, we will look for a chaos solution of the form

\[ u = \sum_{\kappa \in J} u_\kappa N_\kappa. \]

Obviously,

\[ u^{\circ 3} = \sum_{\kappa, \beta, \gamma \in J} u_\kappa u_\beta u_\gamma N_{\kappa + \beta + \gamma}. \]

Therefore,

\[ (u^{\circ 3})_\alpha = \sum_{\kappa, \beta, \gamma, \kappa + \beta + \gamma = \alpha} u_\kappa u_\beta u_\gamma \]

and the propagator of equation (4.25) is given by

\begin{equation}
(4.26) \quad A u_\alpha - \sum_{\kappa, \beta, \gamma, \kappa + \beta + \gamma = \alpha} u_\kappa u_\beta u_\gamma + \sum_{n \geq 1} M_n u_{\alpha - \varepsilon_n} = f_\alpha
\end{equation}

for all $\alpha \in J$.

Similarly to (4.21), system (4.26) is also lower triangular and could be solved sequentially, assuming that operator $A$ has an appropriate inverse.

It is readily checked that if the Wick cubic $u^{\circ 3}$ is replaced by any Wick type polynomial then the related propagator system remains to be lower triangular.
Acknowledgement 1. We express our gratitude to D. Nualart for a useful discussion.

5. Appendix

Let \((\xi_k)\) be a sequence of r.v.. We assume that the following assumption holds.

\[ \text{G.} \] For each vector r.v. \((\xi_{i_1}, \ldots, \xi_{i_n})\) the moment generating function

\[ M_{i_1, \ldots, i_n}(t) = M_{i_1, \ldots, i_n}(t_1, \ldots, t_n) = E\exp\{t_1\xi_{i_1} + \ldots t_n\xi_{i_n}\} \]

exists for all \(t = (t_1, \ldots, t_n)\) in some neighborhood of \(0 \in \mathbb{R}^n\).

Denote \(J\) the set of all multiindices \(\alpha = (\alpha_1, \alpha_2, \ldots)\) such that \(|\alpha| < \infty\) and \(\alpha_k \in \{0, 1, 2, \ldots\}\).

\[ \text{Let } G = \sigma(\xi_k, k \geq 1). \]

Lemma 6. Let \(G\) holds. Then every \(f \in L^2(G, \mathbb{P})\) can be approximated in \(L^2(G, \mathbb{P})\) by a sequence of polynomials in \(\xi_{\alpha} = \prod_k \xi_k^{\alpha_k}, \alpha \in J\).

Proof. The assumption \(G\) implies that all the moments of \(\xi_k, k \geq 1\), exist. Assume \(f \in L^2(G, \mathbb{P})\) and

\[ E[f\xi^n] = 0 \quad \forall \alpha \in J. \]

It is enough to show that \(f = 0\) a.s. in such a case.

Let \(\xi^n = (\xi_1, \ldots, \xi_n), \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n\). Then

\[ h(r, \theta) = E[\exp\{r\theta \cdot \xi^n\} f] \]

exists for \(r \in (-\varepsilon, \varepsilon)\) for some \(\varepsilon > 0\). Since \(h(r)\) is a bilateral Laplace transform, by Theorem 5a, p. 57 in [17], it must exist and be analytic for all complex values of \(r\) in the strip \(-\varepsilon < \text{Re} r < \varepsilon\). In addition, because of (5.1),

\[ h(iu, \theta) = E[\exp\{iu \theta \cdot \xi^n\} f] = 0 \]

for all \(u \in \mathbb{R}\) (here \(i^2 = -1\)). In particular,

\[ \tilde{h}(\theta) = E[\exp\{i \theta \cdot \xi^n\} f] = 0 \]

for all \(\theta \in \mathbb{R}^n\). Therefore,

\[ E[g(\xi^n)f] = 0 \]

for any continuous periodic function on \(\mathbb{R}^n\). Then approximating with long period functions we see that (5.2) holds for a continuous function on \(\mathbb{R}^n\) with compact support, and then for any continuous bounded function \(g\) as well. Since \(n\) is arbitrary, it follows that \(f = 0\) a.s.

\[ \square \]

Lemma 7. For \(v = \sum_{|\alpha| = n} v_{\alpha} E_\alpha \in H_0(Y)\) and \(u = \sum_{|\alpha| = n} u_{\alpha} E_\alpha \in H_0(Y)\), we have

\[ I_n(v) \diamond I_m(u) = I_{n+m}(v \otimes_Y u), \]

where \(v \otimes_Y u = (v(x), u(y))_Y, x \in U^n, y \in U^m\).
Proof. Indeed, \( I_n(v) = n! \sum_{|\alpha| = n} v_{\alpha} \mathfrak{N}_\alpha \), \( I_m(u) = m! \sum_{|\alpha| = m} u_{\alpha} \mathfrak{N}_\alpha \) and

\[
I_n(v) \diamond I_m(u) = n! m! \sum_{\alpha} \sum_{\beta \leq \alpha} (v_{\beta}, u_{\alpha - \beta}) \mathfrak{N}_\alpha
\]

\[
= n! m! \sum_{|\alpha| = n+m} \sum_{\beta \leq \alpha} \int (v \otimes_Y u) \frac{E_\beta}{\beta!} \frac{E_{\alpha - \beta}}{(\alpha - \beta)!} \mathfrak{N}_\alpha
\]

\[
= \sum_{\alpha} \sum_{\beta \leq \alpha} \int (v \otimes_Y u) \frac{E_\beta}{\beta!} \frac{E_{\alpha - \beta}}{(\alpha - \beta)!} \mathfrak{N}_\alpha = \sum_{\alpha} \int (v \otimes_Y u) \frac{E_\alpha}{\alpha!} \mathfrak{N}_\alpha
\]

\[
= (n + m)! \sum_{|\alpha| = n+m} l_\alpha \mathfrak{N}_\alpha = I_{n+m}(v \otimes_Y u),
\]

because \( v \otimes_Y u = \sum_{|\alpha| = n+m} l_\alpha E_\alpha \) with

\[
l_\alpha = \frac{1}{\alpha!(n + m)!} \int (v \otimes_Y u) E_\alpha d\mu_{n+m}.
\]

\[\qed\]

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