EXTENSIONS AND RANK-2 VECTOR BUNDLES ON IRREDUCIBLE NODAL CURVES

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Abstract. We generalize Bertram's work on rank two vector bundles to an irreducible projective nodal curve $C$. We use the natural rational map $\phi_L : \mathbb{P}(\text{Ext}_C^1(L, O_C)) \to SU_C(2, L) \subseteq SU_C(2, L)$ defined by $\phi_L([0 \to O_C \to E \to L \to 0]) = E$ to study a compactification $SU_C(2, L)$ of the moduli space $SU_C(2, L)$ of semi-stable vector bundles of rank 2 and determinant $L$ on $C$. In particular, we resolve the indeterminancy of $\phi_L$ in the case $\text{deg} L = 3, 4$ via a sequence of three blow-ups with smooth centers.

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1. Introduction

In [Ber92], Bertram used extensions of line bundles to study rank-2 vector bundles of fixed determinant on a smooth curve. We generalize his construction to an irreducible projective nodal curve $C$. The idea is to consider extensions of $L$ by $O_C$, where $L$ is a generic line bundle on $C$, and consider the ‘forgetful’ map which sends an extension to the vector bundle of rank 2 in the middle, forgetting the extension maps. This gives a rational map from $\mathbb{P}(\text{Ext}_C^1(L, O_C))$ to $SU_C(2, L)$, the moduli space of semi-stable vector bundles of rank 2 and determinant $L$. If the arithmetic genus of $C$ is $\geq 2$ and $\text{deg} L = 3$ or 4, we resolve the indeterminancy of the map by a sequence of three blow-ups with smooth centers. A nice aspect of these blow-ups is that there exists at each stage a ‘universal bundle’ which induces the rational map in a natural way.

Let $SU_C(2, L)$ be the natural compactification of $SU_C(2, L)$ via torsion-free sheaves introduced by Newstead and Seshadri (see [New78] and [Ses82]). Our main theorem is the following.
Theorem 1.1. Let $C$ be an irreducible projective nodal curve of arithmetic genus $\geq 2$, and let $L$ be a generic line bundle on $C$ of degree 3 or 4. Let $\phi_L : \mathbb{P}(\text{Ext}^1_C(L, \mathcal{O}_C)) \to \text{SU}_C(2, L) \subseteq \text{SU}_C(2, L)$ be the natural rational map defined by $\phi_L([0 \to \mathcal{O}_C \to E \to L \to 0]) = E$. There exist a sequence of three blow-ups with smooth centers
\[ \mathbb{P}_{L,3}^{\xi_3} \to \mathbb{P}_{L,2}^{\xi_2} \to \mathbb{P}_{L,1}^{\xi_1} \to \mathbb{P}(\text{Ext}^1_C(L, \mathcal{O}_C)), \]
such that
\[ \phi_L \circ \xi_1 \circ \xi_2 \circ \xi_3 : \mathbb{P}_{L,3} \to \text{SU}_C(2, L) \]
extends to a morphism $\phi_{L,3}$.

An important fact is that the fibers of $\phi_{L,3}$ are connected. As a corollary, we can give a new proof of the fact that, if $C$ is an irreducible nodal curve of arithmetic genus 2, and $\deg L$ is odd, then the normalization morphism $\text{SU}_C(2, L) \to \text{SU}_C(2, L)$ is one-to-one.

We also give the idea for a new proof of $\text{SU}_C(2, L) \cong \mathbb{P}^3$ for an irreducible nodal curve of arithmetic genus 2 when $\deg L$ is even.

If the arithmetic genus of $C$ is 1, using the morphism $\phi_L$ with $L$ of degree 1 or 2, we prove that, as in the smooth case,
\[ \text{SU}_C(2, L) = \text{SU}_C(2, L) \cong \begin{cases} \{pt\} & \text{if } \deg L \text{ is odd} \\ \mathbb{P}^1 & \text{if } \deg L \text{ is even} \end{cases}, \]
and there are no stable bundles of even degree.

In general, using the rational map $\phi_{L,3}$ with $\deg L \geq 3$, we prove that the complement of $\text{SU}_C(2, L)$ in $\text{SU}_C(2, L)$ has codimension $\geq 3$ for every irreducible nodal curve of arithmetic genus $\geq 2$. It follows, using [Bho99] and [Bho04], that $A_{3g-4}(\text{SU}_C(2, L)) \cong \mathbb{Z}$. Moreover, if $\deg L = 2g - 1$, we find open subsets $U \subseteq \mathbb{P}_{L,3}$ and $V \subseteq \text{SU}_C(2, L)$ such that $\phi_{L,3}|_U : U \to V$ is an isomorphism, and $\text{codim}(\text{SU}_C(2, L) \setminus V, \text{SU}_C(2, L)) \geq 2$. As a corollary, we prove directly that
\[ A_{3g-4}(\text{SU}_C(2, L)) = A_{3g-4}(\text{SU}_C(2, L)) \cong \mathbb{Z} \]
if $\deg L$ is odd.

Acknowledgements. A special thanks to my Ph.D. adviser E. Izadi for her constant guide and support throughout the program. I would also like to thank V. Alexeev, A. Bertram, R. Smith, V. Vologodski, and especially R. Varley, for many useful discussions.

Notation
Let $J$ be the set of nodes of $C$. For a subset $J'$ of $J$, we denote by $\pi_{J'} : C_{J'} \to C$ the partial normalization of $C$ along the nodes in $J'$. In particular, for $J' = J$ we obtain the normalization $\pi : N \to C$ of $C$. For every $p \in J$, let $p_1, p_2$ be the two points which map to $p$ under any partial normalization map $\pi_{J'}$ with $p \in J'$.

If $X$ is a projective variety, a sheaf on $X \times C$ of the form $\pi_X^* F \otimes \pi_C^* G$ (for some sheaves $F$ on $X$ and $G$ on $C$) shall be denoted by $F \boxtimes G$. If it is of the form $\pi_X^* F$ or $\pi_C^* G$, we shall sometimes just denote it by $F$ or $G$, if it is clear from the context that we are actually considering the sheaf on $X \times C$.

We shall assume throughout the paper, unless it is explicitly stated otherwise, that the arithmetic genus of $C$ is $g \geq 2$ (and we shall simply call it the genus of $C$).

Whenever we do not explicitly define a homomorphism of extension spaces throughout this paper, a natural push-forward or pull-back of extensions is understood.
2. Description of $\phi_L$

Let us start with extending to our situation some of the basic results of the smooth case. Since Ext$_{C}(L, \mathcal{O}_C) \simeq H^1(C, L^{-1}) \simeq H^0(C, L \otimes \omega_C)^*$ (see [Har77, chapter III]), the linear system $|L \otimes \omega_C|$ defines a rational map $\varphi_{L \otimes \omega_C} : C \rightarrow |L \otimes \omega_C| \simeq \mathbb{P}_L$, where we denote $\mathbb{P}(\text{Ext}_{C}(L, \mathcal{O}_C))$ by $\mathbb{P}_L$ to simplify the notation. Let $U_L \subseteq \mathbb{P}_L$ be the open locus of semi-stable extensions, i.e., the open subset where $\phi_L$ is well-defined.

**Proposition 2.1.** (1) If $\deg L < 0$, then $U_L = \emptyset$.

(2) If $0 \leq \deg L \leq 2$, then $U_L = \mathbb{P}_L$.

(3) If $3 \leq \deg L \leq 4$, then $U_L = \mathbb{P}_L \setminus \varphi_{L \otimes \omega_C}(C)$.

Proof. The same is true for a smooth curve (see [Ber92]), and the proof in our case is similar, except for two technical details that we prove in Lemmas 2.2 and 2.3.

**Lemma 2.2.** Every torsion-free sheaf of rank 1 and degree 1 on $C$ with a section is either isomorphic to $\mathcal{O}_C(q)$ for some smooth point $q \in C$ (if it is locally-free) or it is isomorphic to $(\pi_p)_*\mathcal{O}_{C_p}$ for some node $p$ (if it is not locally-free).

Proof. Every torsion-free non-locally-free coherent sheaf $F$ of rank 1 on $C$ is of the form $(\pi_{p'})_*\mathcal{F}$ for some line bundle $\mathcal{F}$ on a partial normalization $C_{p'}$ of $C$ (see [Ses82]). If it has a section, then $\mathcal{O}_C \subseteq F$ implies that $\mathcal{O}_{C_{p'}} = \pi_{p'}^*\mathcal{O}_C \subseteq \mathcal{F}$, and therefore, $\deg \mathcal{F} \geq 0$. Since $F$ has degree 1, and it is not locally-free, $J'$ must contain only one node $p$, $\mathcal{F}$ must have degree 0, and therefore be isomorphic to $\mathcal{O}_{C_p}$.

**Lemma 2.3.** If $q$ is a smooth point of $C$, then

$$\varphi_{L \otimes \omega_C}(q) = \mathbb{P}(\ker(\text{Ext}_{C}^1(L, \mathcal{O}_C) \xrightarrow{\psi_p} \text{Ext}_{C}^1(L, \mathcal{O}_C(q))))$$

If $p$ is a node of $C$, then

$$\varphi_{L \otimes \omega_C}(p) = \mathbb{P}(\ker(\text{Ext}_{C}^1(L, \mathcal{O}_C) \xrightarrow{\psi_p} \text{Ext}_{C}^1(L, (\pi_p)_*\mathcal{O}_{C_p}))$$

Proof. If $q$ is a smooth point, the proof is the same as in the smooth case. If $p$ is a node, then $\varphi_{L \otimes \omega_C}(p)$ is the hyperplane of $H^0(C, L \otimes \omega_C)$ defined by the sections vanishing at $p$. Since the sheaf generated by the regular functions vanishing at $p$ is the sheaf $((\pi_p)_*(\mathcal{O}_{C_p}(-p_1 - p_2)))$ and its dual is $(\pi_p)_*\mathcal{O}_{C_p}$, $\varphi_{L \otimes \omega_C}(p)$ corresponds to the kernel of the linear homomorphism $H^1(C, L^{-1}) \rightarrow H^1(C, L^{-1} \otimes (\pi_p)_*\mathcal{O}_{C_p})$, where we identified $H^0(C, L \otimes \omega_C)^*$ with $H^1(C, L^{-1})$. If $G$ is any coherent sheaf, we can identify $\text{Ext}_{C}^1(L, G)$ with $H^1(C, L^{-1} \otimes G)$, and the linear homomorphism above becomes $\psi_p$ as claimed.

From now on, all through Section 10 we shall restrict ourselves to the case when $\deg L$ is either 3 or 4.

**Lemma 2.4.** If $\deg L \geq 3$, then $\varphi_{L \otimes \omega_C}$ is an embedding.

Remark. Since $\varphi_{L \otimes \omega_C}$ is an isomorphism onto its image, we shall identify $C$ with $\varphi_{L \otimes \omega_C}(C) \subseteq \mathbb{P}_L$.

Proof. We need to prove that, for every $q, q' \in C$ not both equal to a node $p$, $H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) = H^1(C, L \otimes \omega_C) = 0$, where $\mathcal{I}_q$ [resp. $\mathcal{I}_{q'}$] is the ideal sheaf of the point $q$ [resp. $q'$], and that $H^1(C, L \otimes \omega_C \otimes \mathcal{I}_p^2) = H^1(C, L \otimes \omega_C) = 0$ for every node $p$ (see [Bar87]).

Case 1: $q, q'$ smooth points. Then, by Serre duality, $H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) \simeq H^0(C, L^{-1}(q + q'))^*$, which is zero because $\deg(L^{-1}(q + q')) < 0$. 

Case II: if $q = p$ node. Since $\pi^*_p \omega_C \simeq \omega_C(p_1 + p_2)$ (see [Bar87]), using the projection formula we obtain $\omega_C \otimes I_q \simeq (\pi_p)_* \omega_C$, and $L \otimes \omega_C \otimes I_q \otimes I_q \simeq L(-q) \otimes (\pi_p)_* \omega_C \simeq (\pi_p)_*(\pi^*_p(L(-q)) \otimes \omega_C)$. Therefore, $H^1(C, L \otimes \omega_C \otimes I_q \otimes I_q) \simeq H^1(C, \pi^*_p(L(-q)) \otimes \omega_C) \simeq H^0(C, \pi^*_p(L(-1)(q)))^*$, which is zero because $\deg(\pi^*_p(L(-1)(q))) < 0$.

Case III: $q = p, q' = p$ distinct nodes. Since $\pi^*_{(p,p')} \omega_C \simeq \omega_C_{(p,p')}((p_1 + p_2 + p'_1 + p'_2))$ (see [Bar87]), we obtain $\omega_C \otimes I_p \otimes I_{p'} \simeq (\pi_{(p,p')})_* \omega_C_{(p,p')}$, and $L \otimes \omega_C \otimes I_p \otimes I_{p'} \simeq L \otimes (\pi_{(p,p')})_* \omega_C_{(p,p')} \simeq (\pi_{(p,p')})_*(\pi^*_p(L \otimes \omega_C_{(p,p')}))$. Therefore, $H^1(C, L \otimes \omega_C \otimes I_p \otimes I_{p'}) \simeq H^1(C_{(p,p')}, \pi^*_p(L \otimes \omega_C_{(p,p')})) \simeq H^0(C_{(p,p')}, \pi^*_p(L^{-1}))^*$, which is zero because $\deg(\pi^*_p(L^{-1}(p_1 + p_2))) < 0$.

Case IV: $q = q' = p$ node. Then $\omega_C \otimes I_p \simeq (\pi_p)_* (\omega_C(-p_1 - p_2))$ and $L \otimes \omega_C \otimes I_p \simeq (\pi_p)_* (\pi^*_p L \otimes \omega_C(-p_1 - p_2))$. Therefore, $H^1(C, L \otimes \omega_C \otimes I_p) \simeq H^1(C, \pi^*_p L \otimes \omega_C(-p_1 - p_2)) \simeq H^0(C, \pi^*_p L^{-1}(p_1 + p_2))^*$, which is zero because $\deg(\pi^*_p L^{-1}(p_1 + p_2)) < 0$.

**Lemma 2.5.** The projective tangent plane to $C$ at a node $p$ is

\[ T_p C = \mathbb{P}(\ker(\text{Ext}^1_C(L, \mathcal{O}_C) \rightarrow \text{Ext}^1_C(L, (\pi_p)_*(\mathcal{O}_{C_p}(p_1 + p_2))))). \]

**Proof.** It is easy to see that all the kernels involved in this proof have the right dimension. The secant line between the node $p$ and a smooth point $q$ is given by

\[ \mathbb{P}(\ker(\text{Ext}^1_C(L, \mathcal{O}_C) \rightarrow \text{Ext}^1_C(L, (\pi_p)_*(\mathcal{O}_{C_p}(q)))). \]

this being a 1-dimensional linear subspace of $\mathbb{P}_L$ which contains both $p$ and $q$. If we take the limit as $q \to p$ along the branch corresponding to $p_i$ ($i = 1, 2$), we see that the projective tangent line at $p$ to that branch is $X_{p_i} := \mathbb{P}(\ker(\text{Ext}^1_C(L, \mathcal{O}_C) \rightarrow \text{Ext}^1_C(L, (\pi_p)_*(\mathcal{O}_{C_p}(p_i))))$ ($i = 1, 2$). Since $\mathbb{P}(\ker(\psi_{T_pC}))$ is a 2-dimensional linear subspace of $\mathbb{P}_L$ which contains both $X_{p_1}$ and $X_{p_2}$, it is the projective tangent plane $T_p C$ to $C$ at $p$.

We end this section with an important way to describe the rational map $\phi_L$.

**Proposition 2.6.** There exists a locally-free sheaf $\mathcal{E}_L$ on $\mathbb{P}_L \times C$ such that $\mathcal{E}_L|_{\{x\} \times C} \simeq \phi_L(x)$ for every $x \in \mathbb{P}_L \setminus C$. Moreover, $\mathcal{E}_L$ is an extension in $\text{Ext}^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L}(1))$, and for every $a \neq 0$ in $\text{Ext}^1_C(L, \mathcal{O}_C)$, if we identify $\pi^*_{(p)} C_{(p)}(1)|_{\{[a]\} \times C}$ with $\mathcal{O}_C$ using $a$, $\mathcal{E}_L$ restricts to $\{[a]\} \times C$ to the extension it is.

**Proof.** Let $\mathcal{E}_L$ be the extension corresponding to the identity homomorphism under the natural isomorphism $\text{Ext}^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L}(1)) \simeq \text{Hom}(\text{Ext}^1_C(L, \mathcal{O}_C), \text{Ext}^1_C(L, \mathcal{O}_C))$. Then, if $a \neq 0$ is an extension $0 \to \mathcal{O}_C \to E_a \to L \to 0$, $\mathcal{E}_L|_{\{[a]\} \times C}$ is $E_a$ (see [Arc04]).

**3. The first blow-up**

Since the indeterminacy locus of the rational map $\phi_L : \mathbb{P}_L \to \text{SL}_C(2, L)$ is the curve $C \subseteq \mathbb{P}_L$, to resolve the indeterminacy via a sequence of blow-ups with smooth centers, we need to begin the process with the blow-up of $\mathbb{P}_L$ at the set of nodes $J \subseteq C$. By Lemma 2.5, a node $p$ is $\mathbb{P}(\ker(\psi_p))$, where $\psi_p$ is the natural linear homomorphism $\text{Ext}^1_C(L, \mathcal{O}_C) \rightarrow \text{Ext}^1_C(L, (\pi_p)_* \mathcal{O}_{C_p})$. Therefore, the exceptional divisor $E_1$ of $\mathbb{P}_{L,1} := \mathbb{B}\mathcal{L}_J \mathbb{P}_L \rightarrow \mathbb{P}_L$ is canonically isomorphic to $\coprod_{p \in J} \mathbb{P}(\text{Ext}^1_C(L, (\pi_p)_* \mathcal{O}_{C_p})).$

**Theorem 3.1.** (a) The composition $\phi_L \circ \varepsilon_1 : \mathbb{P}_{L,1} \to \overline{SL}_C(2, L)$ extends to a rational map $\phi_{L,1}$ defined as follows: For every node $p$, a point $x \in E_1|_p$ corresponds to an extension $E'_x$ in
The natural homomorphism $\text{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p})$. Its image $\phi_{L,1}(x)$ is the torsion-free sheaf $E_x$ which is the image of $E_x'$ under the natural homomorphism

$$\text{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p}) \xrightarrow{\psi_L} \text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}).$$

(b) The indeterminacy locus of the rational map $\phi_{L,1}: \mathbb{P}_{L,1} \to SU_C(2, L)$ is the union of the strict transform $\tilde{C}_1$ of $C$ and the lines $L_p := \mathbb{P}(\ker(\psi_{L_p})) \subseteq E_1|_p$.

The strict transform $\tilde{C}_1$ of $C$ is isomorphic to $N$ and, for each node $p$, it intersects $E_1|_p$ at the two points $p_1, p_2$ lying on $p$. The following lemma describes the lines $L_p$.

**Lemma 3.2.** The points on $L_p$ correspond to the directions tangent to $p$ in $T_pC$, the projective tangent plane to $C$ at $p$. In particular, $L_p$ is the line through $p_1$ and $p_2$ in $E_1|_p$.

**Proof.** It suffices to show that $L_p$ contains $p_1$ and $p_2$. It is easy to see that, for $i = 1, 2$, $p_i = \mathbb{P}(\ker(\psi_i)) \subseteq \mathbb{P}(\text{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p})) \simeq E_1|_p$, where $\psi_i$ is the natural map $\text{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p}) \to \text{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p}(p_i))$. To prove that $p_1, p_2 \in L_p$, we need to show that $\ker(\psi_i) \subseteq \ker(\psi_{L,p})$ for $i = 1, 2$. A non-trivial extension $E$ in $\text{Ext}^1_C(L, (\pi_p)_*\mathcal{O}_{C_p})$ is in the kernel of $\psi_i$ if and only if there exists a surjective map $E \to (\pi_p)_*\mathcal{O}_{C_p}(p_i)$. The kernel of this map is $L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*$, and $E$ is therefore also in the kernel of $\psi_{L,p}$. □

It can be shown that, for every node $p$, the image of the natural linear homomorphisms $\psi_{L,p}$ is isomorphic to the space of extensions $\text{Ext}^1_C((\pi_p)_*L(-p_1 - p_2), \mathcal{O}_{C_p})$ via the homomorphism $(\pi_p)_*$. In particular, no torsion-free sheaf in the image of $\phi_{L,1}|_{E_1}$ is locally-free, being a push-forward from a partial normalization of $C$.

**Corollary 3.3.** The image $\phi_{L,1}(\mathbb{!p}_{L,1} \setminus (\bigsqcup_{p \in J} L_p \cup \tilde{C}_1))$ of $\phi_{L,1}$ in $SU_C(2, L)$ is given by

$$\phi_L(\mathbb{P}_L \setminus C) \cup \{E \in SU_C(2, L) | E = (\pi_p)_*\mathcal{E} \text{ for some } p \in J \text{ and } \mathcal{E} \in \text{Ext}^1_C((\pi_p)_*L(-p_1 - p_2), \mathcal{O}_{C_p})\}.$$

Before we prove Theorem 3.1, we need the following lemma.

**Lemma 3.4.** For every node $p$, all non-trivial extensions $0 \to (\pi_p)_*\mathcal{O}_{C_p} \to E \to L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \to 0$ in $\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})$ are semi-stable.

**Proof.** Assume that $E$ is not semi-stable. Then there exists a torsion-free quotient $F$ of $E$ of rank 1 and degree $\leq 1$. Consider the composite map $(\pi_p)_*\mathcal{O}_{C_p} \hookrightarrow E \to F$. If it is the zero-map, then the morphism $E \to F$ factors through $L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*$, and this is not possible since $\deg(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*) > 1 \geq \deg F$. If it is not the zero-map, then it is an inclusion because $(\pi_p)_*\mathcal{O}_{C_p}$ is torsion-free, and this implies that $\deg F = 1$ and $F \simeq (\pi_p)_*\mathcal{O}_{C_p}$. But this can happen only if the extension we started with is trivial. □

We saw in Proposition 2.4 that there exists a locally-free sheaf $\mathcal{E}_L$ on $\mathbb{P}_L \times C$ such that $\mathcal{E}_L|_{x \times C} \simeq \phi_L(x)$ for every $x \in \mathbb{P}_L \setminus C$. To prove Theorem 3.1, we introduce a torsion-free sheaf $\mathcal{E}_{L,1}$ on $\mathbb{P}_{L,1} \times C$ which induces the rational map $\phi_{L,1}$. It is defined by

$$\mathcal{E}_{L,1} := \ker \left( (\varepsilon_1, 1)^*\mathcal{E}_L \to \bigoplus_{p \in J} \mathcal{O}_{E_1|_p} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \right).$$

Note that, since $((\pi_p)_*\mathcal{O}_{C_p})^* \simeq (\pi_p)_*\mathcal{O}_{C_p}(-p_1 - p_2)$, the push-forward of $\pi_p^*L(-p_1 - p_2)$ from $C_p$ to $C$ is isomorphic to $L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*$ by the projection formula.
Note that the map is surjective because, for every node \( p \), \( E_L|_p \times C \) is isomorphic to \( \phi_L(p) \), which surjects onto \( (\pi_p)_*O_C \) by Lemma 2.3. Moreover, the sheaf \( \bigoplus_{p \in J}O_{E_1|_p} \otimes (\pi_p)_*O_{C_p} \) is supported on \( E_1 \times C \), and so \( E_{L,1} \) defines the same map as \( E_L \) on \( \mathbb{P}_L \setminus E_1 \cong \mathbb{P}_L \setminus J \), i.e., \( E_{L,1}|_{(x) \times C} \cong E_L|_{(x) \times C} \cong \phi_L(x) = \phi_L(e_1(x)) \) for every \( x \in \mathbb{P}_L \setminus E_1 \). If \( x \in E_1|_p \), we have an exact sequence \( E_{L,1}|_{(x) \times C} \rightarrow (e_1, 1)^*E_L|_{(x) \times C} \rightarrow (\pi_p)_*O_{C_p} \rightarrow 0 \), which completes to an exact sequence on \( C \)

\[
0 \rightarrow \mathcal{T} \rightarrow E_{L,1}|_{(x) \times C} \rightarrow E_L|_{(x) \times C} \rightarrow (\pi_p)_*O_{C_p} \rightarrow 0,
\]

where \( \mathcal{T} \) is the torsion sheaf \( Tor^1_{\mathbb{P}_L \times C}(O_{E_1|_p} \otimes (\pi_p)_*O_{C_p}, O_{(x) \times C}) \), that we shall see to be isomorphic to \((\pi_p)_*O_{C_p}\). Since the kernel of \( E_{L,1}|_{(x) \times C} \rightarrow (\pi_p)_*O_{C_p} \) is \( L \otimes ((\pi_p)_*O_{C_p})^* \), \( E_{L,1}|_{(x) \times C} \) is an extension of \( L \otimes ((\pi_p)_*O_{C_p})^* \) by (\( \pi_p \))\( O_{C_p} \). Therefore, by Lemma 3.4, \( E_{L,1}|_{(x) \times C} \) is semi-stable if and only if it does not split as an extension. To prove Theorem 3.1, we need to show that \( E_{L,1}|_{E_1 \times C} \) induces the rational map \( \phi_{L,1}|_{E_1} \), i.e., that \( E_{L,1}|_{(x) \times C} \cong E_x \) for every \( x \in E_1 \). This will be proved in Proposition 4.1.2.

4. Description of \( E_{L,1} \)

The main goal of this section is to prove that \( E_{L,1} \) induces the rational map \( \phi_{L,1} \), and we start by analyzing \( E_{L,1} \). Since \( E_L \) fits into a short exact sequence \( 0 \rightarrow O_{F_L}(1) \rightarrow E_L \rightarrow L \rightarrow 0 \) on \( \mathbb{P}_L \times C \), and the image of the composite map \( \varepsilon^*_1O_{F_L}(1) \hookrightarrow (e_1, 1)^*E_L \rightarrow \bigoplus_{p \in J}O_{E_1|_p} \otimes (\pi_p)_*O_{C_p} \) is \( O_{E_1 \times C} \), we obtain the following commutative diagram on \( \mathbb{P}_L \times C \)

\[
\begin{array}{cccccc}
0 & \rightarrow & A_1 & \rightarrow & E_{L,1} & \rightarrow & B_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\varepsilon^*_1O_{E_1(1)} & \rightarrow & (e_1, 1)^*E_L & \rightarrow & L & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & O_{E_1 \times C} & \rightarrow & \bigoplus_{p \in J}O_{E_1|_p} \otimes (\pi_p)_*O_{C_p} & \rightarrow & \bigoplus_{p \in J}O_{E_1 \times \{p\}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

where \( A_1, E_{L,1}, \) and \( B_1 \) are defined by the exactness of the vertical exact sequences. In particular, \( A_1 \cong \pi^*_E(\varepsilon^*_1O_{F_L}(1) \otimes O_{F_L}(-E_1)) \).

This shows that \( E_{L,1} \) fits in a short exact sequence \( 0 \rightarrow \varepsilon^*_1O_{E_1}(1) \otimes O_{E_1|_p}(-E_1) \rightarrow E_{L,1} \rightarrow B_1 \rightarrow 0 \) on \( \mathbb{P}_{L,1} \times C \) which restricts to a short exact sequence \( 0 \rightarrow O_{E_1}(1) \rightarrow E_{L,1}|_{E_1 \times C} \rightarrow B_1|_{E_1 \times C} \rightarrow 0 \) on \( E_1 \times C \). The restriction stays exact because \( O_{E_1}(1) \) is locally-free, and the map \( O_{E_1}(1) \rightarrow E_{L,1}|_{E_1 \times C} \) is generically injective. Therefore, the image of any \( Tor \) sheaf which would appear is 0.

Remark. We shall use this fact several times when restricting diagrams or short exact sequences. When no comments are made about a sequence staying exact after a restriction, the reason shall be the same as here, i.e., the first sheaf is locally-free, and the first map is generically injective.

Lemma 4.1. For each node \( p \in J \), there exists a short exact sequence

\[
0 \rightarrow O_{E_1|_p}(1) \otimes (\pi_p)_*O_{C_p} \rightarrow E_{L,1}|_{E_1 \times C} \rightarrow L \otimes ((\pi_p)_*O_{C_p})^* \rightarrow 0
\]
on $E_1|_p \times C$.

**Proof.** If we restrict the diagram (1) to $E_1|_p \times C$, we obtain

\[
\begin{array}{cccccc}
0 & \to & O_{E_1|_p}(1) & \to & O_{E_1|_p}(1) \boxtimes (\pi_p)_* O_{C_p} & \to & O_{E_1|_p \times \{p\}}(1) & \to & 0 \\
& & \downarrow \cong & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & O_{E_1|_p}(1) & \to & E_{L,1}|_{E_1|_p \times C} & \to & B_1|_{E_1|_p \times C} & \to & 0 \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & O_{E_1|_p \times C} & \to & \pi_C^* E_{L}|_{\{p\} \times C} & \to & L & \to & 0 \\
& & \downarrow \cong & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & O_{E_1|_p \times C} & \to & O_{E_1|_p} \boxtimes (\pi_p)_* O_{C_p} & \to & O_{E_1|_p \times \{p\}} & \to & 0 \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

It follows from the commutativity of the diagram that

\[
\ker(\pi_C^* E_{L}|_{\{p\} \times C} \to \pi_C^*((\pi_p)_* O_{C_p})) \cong \ker(\pi_C^* L \to O_{E_1|_p \times \{p\}}) \cong \pi_C^*(L \otimes ((\pi_p)_* O_{C_p})),
\]

which implies our statement. \qed

**Proposition 4.2.** The sheaf $E_{L,1}$ on $\mathbb{P}_{L,1} \times C$ induces the rational map $\phi_{L,1}$.

**Proof.** Since we already saw that $E_{L,1}$ defines the rational map $\phi_{L,1}$ on $\mathbb{P}_{L,1} \setminus E_1$, it suffices to show that, for every node $p$, $E_{L,1}|_{E_1|_p \times C}$ induces the rational map $\phi_{L,1}|_{E_1|_p}$. Fix a node $p \in J$.

If we pull-back the extension $E_L$ to $\mathbb{P}_{L,1} \times C$, and then push it forward via the inclusion $\pi_{E_{L,1}}^*(\varepsilon_1^* O_{E_{L}}(1)) \to \varepsilon_1^* O_{E_{L,1}}(1) \boxtimes (\pi_p)_* O_{C_p}$

\[
\begin{array}{cccccc}
0 & \to & \varepsilon_1^* O_{E_{L}}(1) & \to & (\varepsilon_1, 1)^* E_{L} & \to & L & \to & 0 \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & \varepsilon_1^* O_{E_{L,1}}(1) \boxtimes (\pi_p)_* O_{C_p} & \to & E'_{0} & \to & L & \to & 0
\end{array}
\]

we obtain an extension $E'_0$ which splits when restricted to $E_1|_p \times C$. Indeed, $\varepsilon_1^* O_{E_{L,1}}(1)|_{E_1|_p} \cong O_{E_1|_p}$, and since $\text{Ext}^1_{E_1|_p \times C}(L, (\pi_p)_* O_{C_p}) \cong H^0(E_1|_p, O_{E_1|_p}) \otimes \text{Ext}^1_{C}(L, (\pi_p)_* O_{C_p})$ (see [Arc04]), we see that $E'_0|_{E_1|_p \times C}$ splits as long as $E'_0|_{\{x\} \times C}$ splits for some $x \in E_1|_p$. Restricting the diagram (2) above to $\{x\} \times C$ for any $x \in E_1|_p$, we see that $E'_0|_{\{x\} \times C}$ is the trivial extension $\psi_p(E_{L}|_{\{p\} \times C})$.

Therefore, there exists a surjective map $E'_0 \to O_{E_1|_p} \boxtimes (\pi_p)_* O_{C_p}$: Define $E'_1$ to be its kernel.
There exists a commutative diagram on $\mathbb{P}_{L,1} \times C$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & A'_1 & \longrightarrow & \mathcal{E}'_1 & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \varepsilon'_1 \mathcal{O}_{\mathbb{P}_L(1)} \boxtimes (\pi_p)_* \mathcal{O}_C & \longrightarrow & \mathcal{E}'_0 & \longrightarrow & L & \longrightarrow & 0,
\end{array}
$$

with $A'_1 \simeq (\varepsilon'_1 \mathcal{O}_{\mathbb{P}_L(1)} \otimes \mathcal{O}_{\mathbb{P}_L(-E_1|_p)}) \boxtimes (\pi_p)_* \mathcal{O}_C$. Moreover, if we restrict $i_1: \mathcal{E}_{L,1} \hookrightarrow (\varepsilon_1, 1)^* \mathcal{E}_L$ and $i'_1: \mathcal{E}'_1 \hookrightarrow \mathcal{E}'_0$ to $E_1|_p \times C$, we obtain the following commutative diagram, where the first row is the exact sequence described in Lemma 11.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_* \mathcal{O}_C & \longrightarrow & \mathcal{E}_{L,1}|_{E_1|_p \times C} & \stackrel{i_1|_{E_1|_p \times C}}{\longrightarrow} & L \otimes ((\pi_p)_* \mathcal{O}_C)^* & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_* \mathcal{O}_C & \longrightarrow & \mathcal{E}'_1|_{E_1|_p \times C} & \stackrel{i'_1|_{E_1|_p \times C}}{\longrightarrow} & L & \longrightarrow & 0
\end{array}
$$

This shows that $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ is the pull-back of $\mathcal{E}'_1|_{E_1|_p \times C}$ via the inclusion $L \otimes ((\pi_p)_* \mathcal{O}_C) \hookrightarrow L$ pulled-back from $C$ to $E_1|_p \times C$.

This is a summary of the steps we took in the construction of $\mathcal{E}'_1|_{E_1|_p \times C}$:

$$
\begin{align*}
\mathcal{E}_L & \in \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ Ext^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L(1)}) \\
(\varepsilon_1, 1)^* \mathcal{E}_L & \in \ \ Ext^1_{\mathbb{P}_{L,1} \times C}(L, \varepsilon_1^* \mathcal{O}_{\mathbb{P}_L(1)}) \\
\mathcal{E}'_0 & \in \ \ Ext^1_{\mathbb{P}_{L,1} \times C}(L, \varepsilon_1^* \mathcal{O}_{\mathbb{P}_L(1)} \boxtimes (\pi_p)_* \mathcal{O}_C) \\
\mathcal{E}'_1 & \in \ Ext^1_{\mathbb{P}_{L,1} \times C}(L, (\varepsilon_1^* \mathcal{O}_{\mathbb{P}_L(1)} \otimes \mathcal{O}_{\mathbb{P}_L(-E_1|_p)}) \boxtimes (\pi_p)_* \mathcal{O}_C) \\
\mathcal{E}'_1|_{E_1|_p \times C} & \in \ Ext^1_{E_1|_p \times C}(L, \mathcal{O}_{E_1|_p}(1) \boxtimes (\pi_p)_* \mathcal{O}_C)
\end{align*}
$$

Using the natural isomorphisms $Ext^1_{Y \times C}(L, F \boxtimes G) \simeq H^0(Y, F) \otimes Ext^1_C(L, G)$ (see [Arc04]), we can understand what extension $\mathcal{E}'_1|_{E_1|_p \times C}$ is by tracking the corresponding elements in these spaces. Let $v_0, \ldots, v_n$ be a basis of $Ext^1_C(L, \mathcal{O}_C)$, with $\text{Span} \{v_0\} = \langle p \rangle$, and let $v_0^*, \ldots, v_n^*$ be the corresponding dual basis in $Ext^1_{\mathbb{P}_L \times C}(L, \mathcal{O}_{\mathbb{P}_L(1)})$. Then $\mathcal{E}_L$ corresponds to the element $\sum_{i=0}^n v_i^* \otimes v_i \in H^0(\mathbb{P}_L, \mathcal{O}_{\mathbb{P}_L(1)}) \otimes Ext^1_C(L, \mathcal{O}_C)$, and $\mathcal{E}'_1|_{E_1|_p \times C}$ corresponds to the element $\sum_{i=1}^n \psi_p(v_i^*)^* \otimes \psi_p(v_i) \in H^0(E_1|_p, \mathcal{O}_{E_1|_p}(1)) \otimes Ext^1(C, (\pi_p)_* \mathcal{O}_C)$. Therefore, since $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ is the pull-back of $\mathcal{E}'_1|_{E_1|_p \times C}$ via $L \otimes ((\pi_p)_* \mathcal{O}_C)^* \hookrightarrow L$, $\mathcal{E}_{L,1}|_{E_1|_p \times C}$ corresponds to $\psi_{L_p}$ itself. This proves that, for any $a \in Ext^1_C(L, (\pi_p)_* \mathcal{O}_C)$, $a \neq 0$, $\mathcal{E}_{L,1}|_{\{a\} \times C}$ is $\psi_{L_p}(a)$ as extensions of $L \otimes ((\pi_p)_* \mathcal{O}_C)^*$ by $(\pi_p)_* \mathcal{O}_C$. \qed
5. The Second Blow-up

We now blow-up $\mathbb{P}_{L,1}$ along the lines $L_p \subseteq E_1|_p$ ($p \in J$). Let

$$\mathbb{P}_{L,2} := BL_{|_{p \in J}} L_p \mathbb{P}_{L,1} \overset{\varepsilon_2}{\longrightarrow} \mathbb{P}_{L,1} \overset{\varepsilon_1}{\longrightarrow} \mathbb{P}_L,$$

and let $E_2 \subseteq \mathbb{P}_{L,2}$ be the exceptional divisor, which is the disjoint union of projective bundles $E_{2,p} \rightarrow L_p$ ($p \in J$).

**Theorem 5.1.** (a) The composition $\phi_{L,1} \circ \varepsilon_2: \mathbb{P}_{L,2} \longrightarrow SU_C(E, L)$ extends to a rational map $\phi_{L,2}$ with the following property. For each $l \in L_p$, the rational map $\phi_{L,2}|_{E_2(l)}: E_2(l) \longrightarrow SU_C(E, L)$ is the projectivization of a linear homomorphism $N_{L,1}/E_{L,1}|_l \rightarrow H'$, where $H'$ is the closure of the locus of vector bundles of determinant $L$ in $Ext^1_C(L \otimes ((\pi_p)_* O_{C_p}), (\pi_p)_* O_{C_p})$. This linear homomorphism is an isomorphism if $l \neq p_1, p_2$, and it maps $N_{L,1}/E_{L,1}|_{p_1}$ (for $i = 1, 2$) surjectively onto the hyperplane $\text{Im} \psi_{L,p} \subseteq H'$.

(b) The indeterminacy locus of $\phi_{L,2}: \mathbb{P}_{L,2} \rightarrow SU_C(E, L)$ is the strict transform $\tilde{C}_2$ of $\tilde{C}_1$.

**Corollary 5.2.** The image $\phi_{L,2}(\mathbb{P}_{L,2} \setminus \tilde{C}_2)$ of $\phi_{L,2}$ in $SU_C(E, L)$ is given by

$$\phi_L(\mathbb{P}_L \setminus C) \cup \bigcup_{p \in J} \mathbb{P}(H'_p).$$

**Remark.** Here $\mathbb{P}(H'_p)$ actually stands for its image in $SU_C(E, L)$ via the natural ‘forgetful’ map, which is a morphism by Lemma 3.4.

We shall prove that this morphism is injective if $g > \text{deg } L$.

Note that the strict transform $\tilde{C}_2$ of $\tilde{C}_1$ is isomorphic to $\tilde{C}_1$ and $N$, and, for each node $p$, it intersects $E_{2,p}$ at two points $\tilde{p}_1$ and $\tilde{p}_2$ lying over $p_1$ and $p_2$, respectively.

The first step in the proof of Theorem 5.1 is the analysis of the exceptional divisor $E_2$. For each node $p$, $E_{2,p}$ is canonically isomorphic to the projective bundle $\mathbb{P}(N_{L,p}/E_{L,1})$ over $L_p$. Since $N_{L,p}/E_{L,1}$ is the normal bundle to $L_p$ in $\mathbb{P}_{L,1}$, it contains the normal bundle to $L_p$ in $E_1$, and we obtain short exact sequences $0 \rightarrow N_{L_p}/E_1 \rightarrow N_{L_p}/E_{L,1} \rightarrow N_{E_1}/E_{L,1}|_l \rightarrow 0$ of vector bundles on $L_p$.

**Lemma 5.3.** For each node $p$, the sequence $0 \rightarrow N_{L_p}/E_1 \rightarrow N_{L_p}/E_{L,1} \rightarrow N_{E_1}/E_{L,1}|_l \rightarrow 0$ splits. If $N = \dim \mathbb{P}_L$, then $N_{L,p}/E_1 \simeq O_{L_p}(1)^{\oplus N-2}$, and $N_{L_p}/E_{L,1}|_l \simeq O_{L_p}(-1)$. Moreover, $\mathbb{P}(N_{E_1}/E_{L,1}|_l)$ maps isomorphically to $\mathbb{P}(C \cap E_2)$ via $\varepsilon_2 \circ \varepsilon_1$, where $\mathbb{P}(C)$ is the strict transform of $T_pC$ in $\mathbb{P}_{L,2}$.

We shall denote $\mathbb{P}(N_{E_1}/E_{L,1}|_l)$ by $L_{2,p}$. It is isomorphic to $L_p$ via $\varepsilon_2|_{L_{2,p}}$, and it corresponds to a section of $E_{2,p} \rightarrow L_p$.

**Proof.** The short exact sequence $0 \rightarrow T_{L_p}/T_{L_p}^C \rightarrow \Omega_{E_1}|_l \rightarrow \Omega_{L_p} \rightarrow 0$, together with the standard short exact sequence for $\Omega_{P^n}$ (see [Har77, II.8.13]), proves that $N_{L_p}/E_1 \simeq O_{L_p}(1)^{\oplus N-2}$. It is a standard fact about blow-ups that $N_{E_1}/E_{L,1}|_l \simeq O_{L_p}(-1)$. Finally, the short exact sequence $0 \rightarrow N_{L_p}/E_1 \rightarrow N_{L_p}/E_{L,1} \rightarrow N_{E_1}/E_{L,1}|_l \rightarrow 0$ splits because every extension of $O_{L_p}(-1)$ by $O_{L_p}(1)^{\oplus N-2}$ splits. Indeed, $\text{Ext}^1_{C}(O_{L_p}(-1), O_{L_p}(1)^{\oplus N-2})$ is isomorphic to $\oplus^{N-2} H^1(L, O_{L_p}(2)) = 0$.

The proof of the last statement of the lemma, note that, for each $l \in L_p$, if we let $X_l$ be the projective line in $\mathbb{P}_L$ which passes through $p$ and corresponds to $l$, we obtain the following canonical isomorphisms:

$$N_{E_1}/E_{L,1}|_l \simeq T_{l|_E} \mathbb{P}_{L,1} \overset{T}{\longrightarrow} T_p X_l \simeq \frac{\langle X_l \rangle}{(p)} \simeq \frac{\langle l \rangle}{\cap} \cap \frac{\text{Ext}^1_{C}(L, O_{C})}{(p)} \simeq \text{Ext}^1_{C}(L, (\pi_p)_* O_{C_p})$$
where, if $V$ is a vector space, and $S$ is a linear subspace of $\mathbb{P}(V)$, we denote by $\langle S \rangle$ the linear subspace of $V$ corresponding to $S$.

Before we proceed to the proof of Theorem 5.1, it is important to study the following situation: Fix a node $p$ of $C$ (throughout this section), let $T_p C$ be the projective tangent plane to $C$ at $p$, and let $X$ be a projective line in $T_p C$ passing through $p$. As we saw in Lemma 3.2, such lines are parametrized by $L_p$. Any such line $X_l (l \in L_p)$ intersects $C$ at $p$ (and possibly at other points, but always a finite number), and there exists a rational map $\phi_l|_{X_l} : X_l \rightarrow \mathbb{SU}(2, L)$, which extends uniquely to a morphism $\psi_l$ defined on the whole $X_l$. We are interested in finding $\psi_l(p)$. The points of $L_p$ are in one-to-one correspondence with torsion-free sheaves $M_l (l \in L_p)$ of rank 1 and degree 2 containing $(\pi_p)_* \mathcal{O}_{C_p}$, and

$$X_l = \mathbb{P}(\ker(\text{Ext}^1_C(L, \mathcal{O}_C) \rightarrow \text{Ext}^1_C(L, M_l))).$$

Note that, if $l \neq p_1, p_2$, then $M_l$ is a line bundle, and if $l = p_i (i = 1, 2)$, then $M_{p_i}$ is $(\pi_p)_* \mathcal{O}_{C_p}(p_i)$.

**Lemma 5.4.** Let $l \in L_p$, $l \neq p_1, p_2$. Then $\psi_l(p)$ is the unique (up to isomorphisms) torsion-free sheaf $E_l$ which can be written both as an extension

$$0 \rightarrow (\pi_p)_* \mathcal{O}_{C_p} \rightarrow E_l \rightarrow L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* \rightarrow 0$$

and an extension

$$0 \rightarrow L \otimes M_l^* \rightarrow E_l \rightarrow M_l \rightarrow 0.$$

In particular, it is locally-free.

**Proof.** Since for every $x \in X_l \setminus \{p\}$, $\psi_l(x)$ maps onto $M_l$, the same is true for $\psi_l(p)$. Indeed, it cannot surject onto something of smaller degree, or it would not be semi-stable. Since $\psi_l(p)$ is in $\mathbb{SU}(2, L)$, the kernel of $\psi_l(p) \rightarrow M_l$ must then be $L \otimes M_l^*$, and we have a short exact sequence

$$0 \rightarrow L \otimes M_l^* \rightarrow \psi_l(p) \rightarrow M_l \rightarrow 0.$$

Since $\psi_l(x)$ surjects onto $L$ for every $x \in X_l \setminus \{p\}$, $\psi_l(p)$ surjects onto some torsion-free sheaf $F \subseteq L$, which must have deg $F \geq 2$ because $\psi_l(p)$ is semi-stable. Moreover, $F \neq L$ because in that case $\psi_l(p)$ would be an extension of $L$ by $\mathcal{O}_C$, but we know that the limit in $\mathbb{P}_L$ is $E_L|_{\{p\} \times C}$, which is not semi-stable. Since $\psi_l(p)$ is an extension of $M_l$ by $L \otimes M_l^*$, and every map from $M_l$ to $F$ is zero because $M_l \neq F$, the composite map $L \otimes M_l^* \rightarrow \psi_l(p) \rightarrow F$ is non-zero, and therefore $L \otimes M_l^* \subseteq F \subseteq L$, which implies that $F \simeq L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*$.

Therefore, $\psi_l(p)$ is both an extension of $M_l$ by $L \otimes M_l^*$ and of $L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*$ by $(\pi_p)_* \mathcal{O}_{C_p}$, as claimed. Any such sheaf is in the kernel of the natural linear homomorphism $\text{Ext}^1_C(M_l, L \otimes M_l^*) \rightarrow \text{Ext}^1_C(M_l, L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*)$ which is one-dimensional, being isomorphic to $\text{Hom}_C(M_l, \mathcal{C}_p)$.

We shall prove in Proposition 3.2 that the points of $L_{2, p} = \widehat{T_p C} \cap E_2 \simeq L_p$ map to these vector bundles $E_l$. The following lemma describes their geometry.

**Lemma 5.5.** The torsion-free sheaves $E_l (l \in L_p)$ form a conic in a quadric $Q$ in

$$\mathbb{P}^3 \simeq \mathbb{P}(\ker(\text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \rightarrow \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}(p_1 + p_2))).$$

**Proof.** For every $l \in L_p$, let

$$X_{l,1} := \mathbb{P}(\ker(\text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \rightarrow \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}(p_1 + p_2))))$$

$$X_{l,2} := \mathbb{P}(\ker(\text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \rightarrow \text{Ext}^1_C(L \otimes M_l^*, (\pi_p)_* \mathcal{O}_{C_p}))).$$

For each $l \in L_p$, the lines $X_{l,1}$ and $X_{l,2}$ span the plane.

$^2$For the dimension of the kernels involved in this proof, see [Arc04].
and the union of all these lines is a quadric \( Q \). We shall show in Lemma 5.7 that \( H' \) is a hyperplane in \( \text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \). Similarly, \( \mathbb{P}(H'_p) \cap \mathbb{P}^3 \) is a hyperplane in \( \mathbb{P}^3 \), and the intersection of \( \mathbb{P}(H'_p) \) with \( Q \) is the conic in the lemma, since we know that each \( E_l \) is contained in both.

To prove Theorem 5.1, we shall first construct a torsion-free sheaf \( \mathcal{E}_{L,2} \) on \( \mathbb{P} \times C \), and then show that it induces the rational map \( \phi_{L,2} \). We can construct \( \mathcal{E}_{L,2} \), starting with the torsion-free sheaf \( \mathcal{E}_{L,1} \) corresponding to the rational map \( \phi_{L,1} \), as follows\(^3\)

\[
\mathcal{E}_{L,2} := \ker \left( (\varepsilon_2, 1)^*\mathcal{E}_{L,1} \rightarrow \bigoplus_{p \in J} \varepsilon_2^*\mathcal{O}_{L_p}(1) \otimes (\pi_p)_*\mathcal{O}_{C_p} \right).
\]

Moreover, the sheaf \( \bigoplus_{p \in J} \varepsilon_2^*\mathcal{O}_{L_p}(1) \otimes (\pi_p)_*\mathcal{O}_{C_p} \) is supported on \( E_2 \times C \), and so \( \mathcal{E}_{L,2} \) defines the same map as \( \mathcal{E}_{L,1} \) on \( \mathbb{P} \times C \) \( \setminus E_2 \sim \mathbb{P} \setminus \coprod_{p \in J} L_p \). The situation is very similar to the one in the first blow-up, and it is easy to see that, for every node \( p \), every \( l \in L_p \) and every \( x \in E_2|_l \), we have an exact sequence

\[
0 \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \rightarrow \mathcal{E}_{L,2}|_{x \times C} \rightarrow \mathcal{E}_{L,1}|_{x \times C} \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \rightarrow 0.
\]

Since \( \mathcal{E}_{L,1}|_{x \times C} \cong (\pi_p)_*\mathcal{O}_{C_p} \otimes (L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*) \) for every \( l \in L_p \), we obtain that, for every \( x \in E_2, \mathcal{E}_{L,2}|_{x \times C} \) is an extension of the following type:

\[
0 \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \rightarrow \mathcal{E}_{L,2}|_{x \times C} \rightarrow L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \rightarrow 0.
\]

We shall prove in Proposition 6.2 that the sheaf \( \mathcal{E}_{L,2} \) induces the rational map \( \phi_{L,2} \). In particular, for every \( l \in L_p \), the restriction of \( \phi_{L,2} \) to \( E_2|_l \) is a rational map

\[
E_2|_l \rightarrow \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))
\]

that we want to prove to be linear, and to be a morphism for \( l \neq p_1, p_2 \).

**Lemma 5.6.** For every node \( p \), and every \( l \in L_p \), the rational map

\[
\phi_{L,2}|_{E_2|_l} : E_2|_l \rightarrow \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))
\]

is linear.

**Proof.** We give a direct proof of this lemma, but it also follows from the fact, that we shall prove in Lemma 6.1 that \( \mathcal{E}_{L,2}|_{E_2|_l \times C} \in \text{Ext}^1_{E_2|_l \times C}(E_2, (\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \) (see [Arc04]).

From the proof of Lemma 5.3 it is clear that there exists a commutative diagram:

\[
\begin{array}{ccc}
E_2|_l & \xrightarrow{\phi_{L,2}|_{E_2|_l}} & \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})) \\
\cup & \xrightarrow{=} & \mathbb{P}(\text{Im } \psi_{L_p}) \\
\mathbb{P}(\mathcal{N}_{L_p/E_1|_l}) & \xrightarrow{=} & \mathbb{P}(\text{Im } \psi_{L_p})
\end{array}
\]

Since the morphism \( \mathbb{P}(\mathcal{N}_{L_p/E_1|_l}) \rightarrow \mathbb{P}(\text{Im } \psi_{L_p}) \) is a linear isomorphism, the map itself is linear. \( \square \)

Let us now show that \( H'_p \) is a hyperplane in \( \text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \).

**Lemma 5.7.** The closure \( H'_p \) of the locus \( \{ E \in \text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \mid \det E \simeq L \} \) in \( \text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \) is a vector subspace of codimension 1.

\(^3\)For the existence of the map in the definition of \( \mathcal{E}_{L,2} \), for a proof of its surjectivity, and for a more in depth analysis of the sheaf, see Section 6.
Proof. It is enough to show that the closure of
\[ \mathbb{P} \left( \{ E \in \text{Ext}^1_C(\mathcal{L} \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \mid \det E \simeq L \} \right) \]
in \( \mathbb{P}(\text{Ext}^1_C(\mathcal{L} \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})) \) is a linear hyperplane. Let \( E \) be a vector bundle in \( \text{Ext}^1_C(\mathcal{L} \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \): What are the possible values for \( \det E \)?

Let \( l \in L_p \setminus \{ p_1, p_2 \} \), and consider the lines \( X_{l,1} \) and \( X_{l,2} \) that we defined in Lemma 5.5. All vector bundles \( E \) in \( X_{l,1} \) are of the form \( 0 \to L \otimes M_i^* \to E \to M_i \to 0 \) for \( l' \in L_p \setminus \{ p_1, p_2 \} \), and their determinant is of the form \( L \otimes M_i \otimes M_i^* \). Similarly, all of the vector bundles \( E \) in \( X_{l,2} \) are of the form \( 0 \to L \otimes M_i^* \to E \to M_i \to 0 \), and so their determinant is of the same form as above. Consider the rational map
\[ \det : \mathbb{P}(\text{Ext}^1_C(\mathcal{L} \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})) \to \{ L' \in \text{Pic}^d(C) \mid \pi_p^*L' \simeq \pi_p^*L \} \simeq \mathbb{P}^1. \]

It is defined on the locus of locally-free sheaves, and it extends to the locus of the extensions which are not push-forwards of extensions from \( C_p \) (see [Bho92]). Since it is an isomorphism on each line \( X_{l,i} \) with \( l \neq p_1, p_2 \) and \( i \in \{ 1, 2 \} \), it is a surjective linear map. \( \square \)

**Proposition 5.8.** For every node \( p \), and every \( l \in L_p, l \neq p_1, p_2 \), the rational map
\[ E_{2|l} \to \mathbb{P}(\text{Ext}^1_C(\mathcal{L} \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p})) \]
is an isomorphism onto its image \( \mathbb{P}(H'_p) \). If \( l = p_i \) \((i = 1, 2)\), then it maps \( E_{2|l} \) onto \( \mathbb{P}(\text{Im} \psi_{L_p}) \). In particular, if \( l \neq p_1, p_2 \), then \( \phi_{2|E_{2|l}} \) is a morphism.

**Proof.** We already saw in Lemma 5.6 that the map is linear for every \( l \in L_p \). Let \( x_i \) be the point of intersection between the strict transform of the projective line \( X_l \) with \( E_2 \). We shall prove in Proposition \( \Box \) that, if \( l \neq p_1, p_2 \), \( \mathcal{E}_{2,2}|_{(x_i)x(C)} \) is isomorphic to the vector bundle \( E_l \) of Lemma 5.3. Therefore, in this case, the image of \( E_{2|l} \) contains \( \mathbb{P}(\text{Im} \psi_{L_p}) \) and \( E_l \). Since \( E_l \notin \mathbb{P}(\text{Im} \psi_{L_p}) \), the image of \( E_{2|l} \) is the hyperplane \( H'_p \).

If \( l = p_i \) \((i = 1, 2)\), then the image is just \( \mathbb{P}(\text{Im} \psi_{L_p}) \). Indeed, we already know that the map cannot be defined everywhere on \( E_{2|p_i} \) \((i = 1, 2)\) because it contains a point on the strict transform \( \mathcal{C}_2 \) of \( C \) which is contained in the locus of indeterminacy of \( \phi_{E_{2,2}} \). Therefore, it cannot be an isomorphism. Being a linear map, its image is contained in a hyperplane, which has to be \( \mathbb{P}(\text{Im} \psi_{L_p}) \). \( \square \)

Theorem \( \Box \) will now follow from Proposition \( \Box \).

### 6. Description of \( \mathcal{E}_{L,2} \)

Since, for every node \( p \), \( \mathcal{E}_{L,1} \mid_{L_p \times C} \) splits as \( (\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p}) \oplus (L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*) \), the map \( \mathcal{E}_{L,1} \to \bigoplus_{p \in J} \mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \) which appears in the definition of \( \mathcal{E}_{L,2} \) is surjective.

Since the image of the composite map
\[ (\varepsilon_2, 1)^*\mathcal{A}_1 \to (\varepsilon_2, 1)^*\mathcal{E}_{L,1} \to \bigoplus_{p \in J} \varepsilon_2^*\mathcal{O}_{L_p}(1) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \]
is $\pi_{E_2}^*\varepsilon_f^*\mathcal{O}_{L_p}(1)$, we obtain the following commutative diagram on $\mathbb{P}_{L,2} \times C$:

\[
\begin{array}{cccccc}
0 & \longrightarrow & A_2 & \longrightarrow & \mathcal{E}_{L,2} & \longrightarrow & B_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\varepsilon_2,1)^*A_1 & \longrightarrow & (\varepsilon_2,1)^*\mathcal{E}_{L,1} & \longrightarrow & (\varepsilon_2,1)^*B_1 & \longrightarrow & 0, \\
\end{array}
\]

where $A_2$, $\mathcal{E}_{L,2}$, and $B_2$ are defined by the vertical exact sequences, and we denoted the locally-free sheaf $\varepsilon_2^*\mathcal{O}_{L_p}(1)$ on $E_{2,p}$ by $F_p$ to simplify the notation.

We want to show that, for every node $p$, and every $l \in L_p$, $l \neq p_1, p_2$, $\mathcal{E}_{L,2}|_{E_{2,l} \times C}$ is the universal bundle associated to $H'_p$ when we identify $E_{2,l}$ with $H'_p$. Let us start with a lemma.

**Lemma 6.1.** For every node $p$, there exists a short exact sequence

\[
0 \longrightarrow (\varepsilon_2^*\mathcal{O}_{L_p}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{E_2}) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \longrightarrow \mathcal{E}_{L,2}|_{E_{2,p} \times C} \longrightarrow L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \longrightarrow 0
\]

on $E_{2,p} \times C$. Moreover, for each $l \in L_p$, there exists a short exact sequence

\[
0 \longrightarrow \mathcal{O}_{E_{2,l}(1)} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \longrightarrow \mathcal{E}_{L,2}|_{E_{2,l} \times C} \longrightarrow L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* \longrightarrow 0
\]

on $E_{2,l} \times C$.

**Proof.** The restriction of diagram $\mathbf{3}$ to $E_{2,p} \times C$ is

\[
\begin{array}{cccccc}
0 & \longrightarrow & F_p(-E_2) & \longrightarrow & F_p(-E_2) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & F_p(-E_2) \boxtimes \mathcal{C}_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{p}(-E_{2}) & \longrightarrow & \mathcal{E}_{L,2}|_{E_{2,p} \times C} & \longrightarrow & \mathcal{B}_{2}|_{E_{2,p} \times C} & \longrightarrow & 0 \\
\downarrow \overset{\simeq}{\longrightarrow} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{p} & \longrightarrow & (\varepsilon_2,1)^*\mathcal{E}_{L,1}|_{L_p \times C} & \longrightarrow & (\varepsilon_2,1)^*\mathcal{B}_1|_{L_p \times C} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{p} & \longrightarrow & F_{p} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} & \longrightarrow & F_{p} \boxtimes \mathcal{C}_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

where $F_p$ is $\varepsilon_2^*\mathcal{O}_{L_p}(1)$ as above, and $F_p(-E_2)$ is $\varepsilon_2^*\mathcal{O}_{L_p}(1) \otimes \mathcal{O}_{E_2}(-E_2)$. 


The first statement of the lemma follows directly from the diagram by looking at the middle column and observing that the kernel of the map $(\varepsilon_2, 1)^*E_{L,1}|_{L_p \times C} \to \varepsilon_2^*O_{L_p}(1) \boxtimes (\pi_p)_*O_C$ on $E_{2,p} \times C$ is $\pi_C^*(L \otimes ((\pi_p)_*O_C)^*)$.

For the second statement, the diagram shows that $B_2|_{E_{2,p} \times C}$ has torsion. Also, for every $l \in L_p$, $B_2|_{E_{2,l} \times C}$ has torsion, and $B_2|_{E_{2,l} \times C}/\text{Tors}$ is isomorphic to $\ker((\varepsilon_2, 1)^*B_0|_{\{l\} \times C} \to O_{E_{2,l} \times \{l\}})$. By the way we defined this last map, it is clear that this kernel is just the pull-back via $(\varepsilon_2, 1)$ of $B_1|_{\{l\} \times C}$ modulo torsion, which is $\pi_C^*(L \otimes ((\pi_p)_*O_C)^*)$. From the diagram, it is clear that the kernel of the map $E_0|_{E_{2,l} \times C} \to \pi_C^*(L \otimes ((\pi_p)_*O_C)^*)$ is the same as the kernel of the map $E_{2,l}|_{E_{2,l} \times C} \to (\varepsilon_2, 1)^*E_{L,1}|_{\{l\} \times C}$, which is $O_{E_{2,l}}(1) \boxtimes (\pi_p)_*O_C$.

The following proposition will conclude the proof of Theorem 5.1.

**Proposition 6.2.** The sheaf $E_{L,2}$ on $\mathbb{P}_{L,2} \times C$ induces the rational map $\phi_{L,2}$.

Let us start with a lemma.

**Lemma 6.3.** If $Y \subseteq \mathbb{P}_{L,2}$ is a smooth subvariety such that

$$\text{codim}(Y, \mathbb{P}_{L,2}) = \text{codim}(Y \cap E_2, E_2),$$

then $\text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times C}, O_Y \times C) = 0$ and $\text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times \{p\}}, O_Y \times C) = 0$.

**Proof.** Consider the short exact sequence $0 \to O_{E_2 \times C}(-E_2) \xrightarrow{f} O_{E_2 \times C} \xrightarrow{g} O_{E_2} \to 0$ on $\mathbb{P}_{L,2}$ and its pull-back to $\mathbb{P}_{L,2} \times C$. If we tensor it with $O_Y \times C$, we obtain the exact sequence

$$0 \to \text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times C}, O_Y \times C) \to O_{E_2 \times C}(-E_2)|_Y \xrightarrow{f|_Y} O_Y \times C \xrightarrow{g|_Y} O_{(Y \cap E_2) \times C} \to 0$$

on $Y \times C$, where the zero on the left occurs because $O_{\mathbb{P}_{L,2} \times C}$ is locally-free.

Since the codimension of $Y \cap E_2$ in $Y$ is 1, $g$ is zero on the dense open subset $(Y \setminus (Y \cap E_2)) \times C$, and $f$ is an isomorphism on it. Therefore, $\text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times C}, O_Y \times C)$ is supported on $(Y \cap E_2) \times C$, and it must be zero, being a subsheaf of $O_{\mathbb{P}_{L,2}}(-E_2)|_{Y \times C}$, which is a locally-free sheaf on a bigger dimensional variety.

Consider now the short exact sequence $0 \to ((\pi_p)_*O_C)^* \to O_C \to \mathbb{C}_p \to 0$ on $C$ and its pull-back $0 \to O_{E_2} \boxtimes ((\pi_p)_*O_C)^* \to O_{E_2 \times C} \to O_{E_2 \times \{p\}} \to 0$ to $E_2 \times C$. If we tensor this exact sequence with $O_Y \times C$ over $O_{\mathbb{P}_{L,2} \times C}$, we obtain the exact sequence

$$0 \to T \to O_{Y \cap E_2} \boxtimes ((\pi_p)_*O_C)^* \to O_{(Y \cap E_2) \times C} \to O_{(Y \cap E_2) \times \{p\}} \to 0$$

on $(Y \cap E_2) \times C$, where $T = \text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times \{p\}}, O_Y \times C)$ and the zero on the left is the sheaf $\text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times \{p\}}, O_Y \times C)$. Just as above, $O_{Y \cap E_2} \boxtimes ((\pi_p)_*O_C)^* \to O_{(Y \cap E_2) \times C}$ is an isomorphism on the dense open subset $(Y \cap E_2) \times (C \setminus \{p\})$, whose complement has codimension 1, and therefore $\text{Tor}_1^\mathbb{P}_{L,2}(O_{E_2 \times \{p\}}, O_Y \times C)$ must be zero, being supported on $(Y \cap E_2) \times \{p\}$ and contained in the torsion-free sheaf $O_{Y \cap E_2} \boxtimes ((\pi_p)_*O_C)^*$, which is supported on a bigger dimensional variety. 

**Proof (of Proposition 6.2).** It is clear that $E_{L,2}$ defines $\phi_{L,2}$ on $\mathbb{P}_{L,2} \setminus E_2$. On $E_2$, we shall divide the proof in two parts. We shall first show that, for every node $p$, $E_{L,2}$ defines the rational map $\phi_{L,2}$ on $\widetilde{E}_1 \cap E_{2,p} \simeq \mathbb{P}(N_{L_p/E_1}) \subseteq E_2$, where $\widetilde{E}_1$ is the strict transform of $E_1$, and then we prove that, if $l \neq p_1, p_2$, then $E_{L,2}|_{\{x\} \times C} \simeq E_l$, where $E_l$ is the vector bundle of Lemma 5.4.

Since $\phi_{L,2}$ agrees with the rational map defined by $E_{L,2}$ on a dense open subset, we have that $\phi_{L,2}(x)$ is $E_{L,2}|_{\{x\} \times C}$ whenever this is semi-stable. In particular, the proposition will follow from

---

4For another proof of $T$ being 0, note that the map $O_{Y \cap E_2} \boxtimes ((\pi_p)_*O_C)^* \to O_{(Y \cap E_2) \times C}$ is injective because it is the pull-back of the injective map $((\pi_p)_*O_C)^* \to O_C$ via the flat morphism $(Y \cap E_2) \times C \to C$. 

the fact that \( \mathcal{E}_{L_2} \) is semi-stable for every \( x \in E_2 \), except for \( x = \tilde{p}_i, i = 1, 2 \), which are the only two points on \( E_2 \) where we know that \( \phi_{L_2} \) cannot be defined.

To prove that \( \mathcal{E}_{L_2} \) defines the rational map \( \phi_{L_2} \) on \( \widetilde{E}_1 \cap E_2 \simeq \mathbb{P}(\mathcal{N}_{L_2/E_1}) \subseteq E_2 \), restrict the commutative diagram (3) to \( \widetilde{E}_1 \times C \) to obtain:

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{A}_2|_{\tilde{E}_1 \times C} & \rightarrow & \mathcal{E}_{L,2}|_{\tilde{E}_1 \times C} & \rightarrow & \mathcal{B}_2|_{\tilde{E}_1 \times C} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \sigma^*\mathcal{O}_{E_1}(1) & \rightarrow & (\sigma, 1)^*\mathcal{E}_{L,1}|_{\tilde{E}_1 \times C} & \rightarrow & (\sigma, 1)^*\mathcal{B}_1|_{\tilde{E}_1 \times C} & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \sigma^*\mathcal{O}_{L_1}(1) & \rightarrow & \sigma^*\mathcal{O}_{L_1}(1) \otimes (\pi_p)_*\mathcal{O}_{C_p} & \rightarrow & \sigma^*\mathcal{O}_{L_1}(1) \boxdot \mathbb{C}_p & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

where \( \sigma : \widetilde{E}_1 \rightarrow E_1 \) is the restriction of \( \varepsilon_2 \) to \( \widetilde{E}_1 \). The vertical columns are exact because \( \mathcal{T}_{\text{or}} \mathcal{F}_{L_2 \times C}(\varepsilon_{L,1}, \mathcal{O}_{E_1 \times C}) = \mathcal{T}_{\text{or}} \mathcal{F}_{L_2 \times C}(\varepsilon_{1}^*\mathcal{O}_{L_1}(1) \boxdot \mathbb{C}_p, \mathcal{O}_{E_1 \times C}) = 0 \). This is true because \( \varepsilon_{L,1}^*\mathcal{O}_{L_1}(1) \simeq \varepsilon_{L,1}^*\mathcal{O}_{E_1 \times C} \), and since \( \varepsilon_{L,1}^*\mathcal{O}_{E_1 \times C} \) is a locally-free sheaf, it is enough to show that \( \mathcal{T}_{\text{or}} \mathcal{F}_{L_2 \times C}(\mathcal{O}_{E_2 \times C}, \mathcal{O}_{E_1 \times C}) = \mathcal{T}_{\text{or}} \mathcal{F}_{L_2 \times C}(\mathcal{O}_{E_2 \times C}, \mathcal{O}_{E_1 \times C}) = 0 \), which was proved in Lemma 6.2.

Since \((\sigma, 1)^*\mathcal{E}_{L,1}|_{\tilde{E}_1 \times C}\) is an extension of \( \pi_{C}^*(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*) \) by \( \sigma^*\mathcal{O}_{E_1}(1) \boxdot (\pi_p)_*\mathcal{O}_{C_p} \), there exists a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{A}'_2 & \rightarrow & \mathcal{E}_{L,2}|_{\tilde{E}_1 \times C} & \rightarrow & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & \sigma^*\mathcal{O}_{E_1}(1) \boxdot (\pi_p)_*\mathcal{O}_{C_p} & \rightarrow & (\sigma, 1)^*\mathcal{E}_{L,1}|_{\tilde{E}_1 \times C} & \rightarrow & L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^* & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
\sigma^*\mathcal{O}_{L_1}(1) \boxdot (\pi_p)_*\mathcal{O}_{C_p} & = & \sigma^*\mathcal{O}_{L_1}(1) \boxdot (\pi_p)_*\mathcal{O}_{C_p} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

where \( \mathcal{A}'_2 \simeq (\sigma^*\mathcal{O}_{E_1}(1) \otimes \mathcal{O}_{\widetilde{E}_1}(\widetilde{E}_1 \cap E_2)) \boxdot (\pi_p)_*\mathcal{O}_{C_p} \).
Using the natural isomorphisms $\text{Ext}^{1}_{Y \times C}(L, F \boxtimes G) \simeq H^{0}(Y, F) \otimes \text{Ext}^{1}_{C}(L, G)$ (see [Arc04]) as we did in the proof of Proposition 4.2 we have the following diagram\footnote{Note that $\psi_{L_{p}}(w_{1}) = \psi_{L_{p}}(w_{2}) = 0$.}

$$\begin{align*}
\mathcal{E}_{L,1}|_{E_{1} \cap C} & \leftrightarrow \sum_{i=1}^{n} w_{i}^{*} \otimes \psi_{L_{p}}(w_{i}) \in H^{0}(E_{1}|_{p}, \mathcal{O}_{E_{1}|_{p}}(1)) \otimes V \\
(\sigma, 1)^{*} \mathcal{E}_{L,1}|_{E_{1} \cap C} & \leftrightarrow \sum_{i=1}^{n} w_{i}^{*} \otimes \psi_{L_{p}}(w_{i}) \in H^{0}(\tilde{E}_{1}|_{p}, \sigma^{*} \mathcal{O}_{E_{1}|_{p}}(1)) \otimes V \\
\mathcal{E}_{L,2}|_{E_{2} \cap C} & \leftrightarrow \sum_{i=3}^{n} w_{i}^{*} \otimes \psi_{L_{p}}(w_{i}) \in H^{0}(E_{2}|_{p} \cap E_{2}|_{i}, \mathcal{O}_{E_{1} \cap E_{2}|_{i}}(1)) \otimes V \\
\mathcal{E}_{L,2}|(E_{1}|_{p} \cap E_{2}|_{l}) \times C & \leftrightarrow \sum_{i=3}^{n} \psi_{L_{p}}(w_{i})^{*} \otimes \psi_{L_{p}}(w_{i}) \in H^{0}(\tilde{E}_{1}|_{p} \cap E_{2}|_{l}, \mathcal{O}_{E_{1} \cap E_{2}|_{l}}(1)) \otimes V
\end{align*}$$

where $w_{1}, \ldots, w_{n}$ is a basis of $\text{Ext}^{1}_{C}(L, (\pi_{p})_{*} \mathcal{O}_{C_{p}})$ such that $\text{Span} \{w_{1}, w_{2}\} = \ker \psi_{L_{p}}$, $w_{1}^{*}, \ldots, w_{n}^{*}$ is the corresponding dual basis of $\text{Ext}^{1}_{C}(L, (\pi_{p})_{*} \mathcal{O}_{C_{p}})^{*} \simeq H^{0}(E_{1}|_{p}, \mathcal{O}_{E_{1}|_{p}}(1))$, and we denoted $\text{Ext}^{1}_{C}(L \otimes ((\pi_{p})_{*} \mathcal{O}_{C_{p}})^{*}, (\pi_{p})_{*} \mathcal{O}_{C_{p}})$ by $V$ to simplify the diagram.

This proves that the torsion-free sheaf $\mathcal{E}_{L,2}|_{(E_{1}|_{p} \cap E_{2}|_{l}) \times C}$ corresponds to the inclusion when we identify the vector space $\text{Ext}^{1}_{C}(L \otimes ((\pi_{p})_{*} \mathcal{O}_{C_{p}})^{*}, (\pi_{p})_{*} \mathcal{O}_{C_{p}})$ with the vector space $\text{Hom}(\text{Im} \psi_{L_{p}}, \text{Ext}^{1}_{C}(L \otimes ((\pi_{p})_{*} \mathcal{O}_{C_{p}})^{*}, (\pi_{p})_{*} \mathcal{O}_{C_{p}}))$. In particular, for every $a \in \text{Im} \psi_{L_{p}}$, $a \neq 0$, $[a] \in \mathbb{P}(\text{Im} \psi_{L_{p}}) \simeq \mathbb{P}(\mathcal{N}_{L_{p}/E_{1}|_{p}}(1))) \simeq \tilde{E}_{1}|_{p} \cap E_{2}|_{l}$, and $\mathcal{E}_{L,2}|{(a)} \times C \simeq a$ as extensions of $L \otimes ((\pi_{p})_{*} \mathcal{O}_{C_{p}})^{*}$ by $(\pi_{p})_{*} \mathcal{O}_{C_{p}}$.

To prove that, for every $l \in L_{p}$, $l \neq \rho_{1}, \rho_{2}$, $\mathcal{E}_{L,2}|_{(x_{l}) \times C} \simeq \tilde{E}_{l}$, and conclude the proof of the proposition, it suffices to show that $\mathcal{E}_{L,2}|_{(x_{l}) \times C}$ is semi-stable. This is done by tracking the restrictions of $\mathcal{E}$, $\mathcal{E}_{1}$, and $\mathcal{E}_{2}$ to the product of $X_{l}$ and its strict transforms with the curve $C$. Restricting the diagrams defining $\mathcal{E}$ and $\mathcal{E}_{2}$ to $\tilde{X}_{l} \times C$, we obtain a short exact sequence

$$0 \rightarrow L \otimes M_{l}^{*} \rightarrow \mathcal{E}_{2}|_{\tilde{X}_{l} \times C} \rightarrow \mathcal{O}_{\tilde{X}_{l}}(-1) \otimes M_{l} \rightarrow 0.$$  

Therefore, $\mathcal{E}_{L,2}|_{(x_{l}) \times C}$ cannot split, being an extension of $M_{l}$ by $L \otimes M_{l}^{*}$ and an extension of $L \otimes ((\pi_{p})_{*} \mathcal{O}_{C_{p}})^{*}$ by $(\pi_{p})_{*} \mathcal{O}_{C_{p}}$, and therefore it is semi-stable. By continuity, it must be isomorphic to $E_{l}$, that we proved to be the limit of $\psi_{l}(x)$ as $x \rightarrow p$.

7. The third blow-up

To resolve the indeterminacy of $\phi_{L,2}$, we now blow-up $\mathbb{P}_{L,2}$ along $\tilde{C}_{2}$. Let

$$\mathbb{P}_{L,2} := \mathcal{B} \mathcal{L}_{\tilde{C}_{2}} \mathbb{P}_{L,2} \xrightarrow{\varepsilon_{2}} \mathbb{P}_{L,2} \xrightarrow{\varepsilon_{2}} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_{2}} \mathbb{P}_{L},$$

and let $E_{3} \subseteq \mathbb{P}_{L,3}$ be the exceptional divisor. For each node $p$, let $\tilde{p}_{1}, \tilde{p}_{2}$ be the points in $\tilde{C}_{2}$ which map to $p_{1}, p_{2} \in \tilde{C}_{1}$, respectively.

**Theorem 7.1.** The composition $\phi_{L,2} \circ \varepsilon_{3} \colon \mathbb{P}_{L,3} \rightarrow SU_{C}(2, \tilde{L})$ extends to a morphism $\phi_{L,3}$ such that for each $q \in \tilde{C}_{2}$ not lying above a node of $C$, the restriction of $\phi_{L,3}$ to $E_{3}|_{q}$ maps $E_{3}|_{q}$
isomorphically onto $\mathbb{P}(\text{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$, and for each node $p \in C$, its restriction to $E_3|_{\tilde{p}_i}$ sends $E_3|_{\tilde{p}_i}$ isomorphically onto $\mathbb{P}(H_p')$ ($i = 1, 2$).

**Corollary 7.2.** The image of $\phi_{L,3}$ in $\overline{SU_C(2, L)}$ is given by\(^7\)

$$
\phi_{L}(\mathbb{P}_L \setminus C) \cup \bigcup_{p \in J} \mathbb{P}(H_p') \cup \bigcup_{q \in J} \mathbb{P}(\text{Ext}^1_C(L(-q),\mathcal{O}_C(q))).
$$

We shall prove Theorem 7.1 in the next section, after we study the exceptional divisor $E_3$ in this section. We know that $E_3$ is canonically isomorphic to $N_{\tilde{C}_2/P_{L,2}}$.

Let $q$ be a point of $\tilde{C}_2$ not lying above a node of $C$. Then we have the canonical isomorphisms

$$N_{\tilde{C}_2/P_{L,2}}|_q \simeq \frac{T_q\mathbb{P}_{L,2}}{T_q\tilde{C}_2} \simeq \frac{T_q\mathbb{P}_L}{T_qC} \simeq \frac{\text{Ext}^1_C(L,\mathcal{O}_C)}{\langle q \rangle},$$

and so, to prove that $E_3|_q \simeq \mathbb{P}(\text{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$, it is necessary to prove that

$$T_qC \simeq \frac{\ker(\text{Ext}^1_C(L,\mathcal{O}_C) \rightarrow \text{Ext}^1_C(L(-q),\mathcal{O}_C(q)))}{\langle q \rangle}.$$ 

Since, as in the proof of Lemma 2.3, the secant line joining two smooth points $q, q'$ of $C$ is $\mathbb{P}(\ker(\text{Ext}^1_C(L,\mathcal{O}_C) \rightarrow \text{Ext}^1_C(L,\mathcal{O}_C(q + q'))))$, and $\text{Ext}^1_C(L,\mathcal{O}_C(q + q')) \simeq \text{Ext}^1_C(L(-q'),\mathcal{O}_C(q))$, this follows when taking the limit as $q' \rightarrow q$.

Let now $p$ be a node of $C$, and $q = \tilde{p}_i$ with $i \in \{1, 2\}$. Then

$$N_{\tilde{C}_2/P_{L,2}}|_{\tilde{p}_i} \simeq \frac{T_{\tilde{p}_i}\mathbb{P}_{L,2}}{T_{\tilde{p}_i}\tilde{C}_2} \simeq T_{\tilde{p}_i}E_2.$$ 

This contains the canonical hyperplane $T_{\tilde{p}_i}(E_2|_{\tilde{p}_i})$ which maps isomorphically to $\text{Im} \psi_{L_p}$. Indeed, using Lemma 5.3 and the fact that $E_2 \simeq \mathbb{P}(\mathcal{N}_{L_p/E_1})$,

$$T_{\tilde{p}_i}(E_2|_{\tilde{p}_i}) \simeq \frac{\mathcal{N}_{L_p/E_1}|_{\tilde{p}_i}}{\langle \tilde{p}_i \rangle} \simeq \frac{\mathcal{N}_{L_p/E_1}|_{\tilde{p}_i} \oplus \mathcal{O}_{L_p}(-1)|_{\tilde{p}_i}}{\mathcal{O}_{L_p}(-1)|_{\tilde{p}_i}} \simeq \mathcal{N}_{L_p/E_1}|_{\tilde{p}_i},$$

that we already saw to be canonically isomorphic to $\text{Im} \psi_{L_p}$. We shall see in Proposition 9.3 that the morphism $\mathbb{P}(T_{\tilde{p}_i}(E_2|_{\tilde{p}_i})) \rightarrow \overline{SU_C(2, L)}$ factors through this canonical isomorphism, i.e., there exists a commutative diagram

$$
\begin{array}{ccc}
E_3|_{\tilde{p}_i} & \simeq & \mathbb{P}(T_{\tilde{p}_i}E_2) \\
& \cup & \\
& \uparrow & \\
\mathbb{P}(T_{\tilde{p}_i}(E_2|_{\tilde{p}_i})) & \longrightarrow & \phi_{L,3}(E_3|_{\tilde{p}_i}) \subseteq \overline{SU_C(2, L)} \\
\end{array}
$$

(4)

We shall then show that the top map factors through an isomorphism $E_3|_{\tilde{p}_i} \simeq \mathbb{P}(H')$.

As for the other blow-ups, to prove Theorem 7.1 the strategy is to construct a universal sheaf $\mathcal{E}_{L,3}$ on $\mathbb{P}_{L,3} \times C$, and then prove that $\mathcal{E}_{L,3}$ induces the correct rational map. In this case, we also want to prove that $\mathcal{E}_{L,3}$ induces a morphism, i.e., that $\mathcal{E}_{L,3}|_{\{x\} \times C}$ is semi-stable for every $x \in \mathbb{P}_{L,3}$. The definition of $\mathcal{E}_{L,3}$ is not as evident as in the other two blow-ups, and we postpone it to the

---

\(^6\)We identify here a point $q$ on $\tilde{C}_2$, $q \neq \tilde{p}_1, \tilde{p}_2$, with its image $q$ on $C$.

\(^7\)Note that by $\mathbb{P}(H_p')$ [resp. $\mathbb{P}(\text{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$] we actually mean its image into $\overline{SU_C(2, L)}$ by the morphism described in the remark after Corollary 5.2 [resp. in Corollary 7.2].
induces the isomorphism of \( E \) that, if \( \phi \) next section. By construction, \( E \) shall agree with \( E \) on \( \mathbb{P}_L \setminus E = \mathbb{P}_L \setminus \tilde{C}_2 \), and we shall show in Propositions \( 9.1 \) and \( 9.2 \) that, if \( q \) does not lie over a node of \( C \), then \( E \mid_{E_q \times C} \) induces the isomorphism of \( E \) with \( \mathbb{P}(\text{Ext}^1_C(L(-q), \mathcal{O}_C(q))) \) described above. To prove that this induces a morphism from \( E_q \) to \( \mathcal{S}U_C(2, L) \), we need to prove the following result.

**Lemma 7.3.** All non-trivial extensions in \( \text{Ext}^1_C(L(-q), \mathcal{O}_C(q)) \) are semi-stable.

**Proof.** This proof is identical to the one of Lemma \( 3.4 \). \( \square \)

**Corollary 7.4.** The natural ‘forgetful’ map \( \mathbb{P}(\text{Ext}^1_C(L(-q), \mathcal{O}_C(q))) \to \mathcal{S}U_C(2, L) \) which sends an extension \( 0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0 \) to \( E \) is a morphism.

Fix now a node \( p \) in \( C \). From the definition of \( E \), it will be clear that, as in the case of the first two blow-ups, \( \phi_L(E_q|_{\tilde{p}_i}) \subseteq \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_C)|, (\pi_p)_*\mathcal{O}_C)) \), and therefore, using diagram \( \square \), we can prove the following linearity result.

**Lemma 7.5.** For \( i = 1, 2 \), the rational map \( \phi_{L,p} : E_q|_{\tilde{p}_i} \to \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_C)|, (\pi_p)_*\mathcal{O}_C)) \) is linear.

**Proof.** The proof is the same as the one of Lemma \( 3.6 \). \( \square \)

For each \( i \in \{1, 2\} \), since the map is linear, and we know it to send the hyperplane \( \mathbb{P}(\text{Ext}^1_C(L, \mathcal{O}_C(q))) \) isomorphically onto \( \mathbb{P}(\text{Im} \psi_{L,p}) \), to prove that \( E_q|_{\tilde{p}_i} \) isomorphically onto \( \mathbb{P}(\text{Im} \psi_{L,p}) \), it suffices to show that there exists a point \( y \in E_q|_{\tilde{p}_i} \) which maps to some point \( x \in \mathbb{P}(\text{Im} \psi_{L,p}) \).

For each point \( x \in \mathbb{P}(\text{Im} \psi_{L,p}) \), there exists a section \( s_x \) of \( E_p \to L_p \) defined as follows: If \( l \neq p_1, p_2 \), \( s_x(l) \) is the unique point of \( E_p|_{\tilde{p}_i} \) which maps to \( x \in \mathbb{P}(\text{Im} \psi_{L,p}) \). This defines a section on \( L_p \setminus \{p_1, p_2\} \), which can be completed to a section of \( E_p \to L_p \) by taking its closure. Note that its closure must satisfy \( s_x(p_i) = \tilde{p}_i \) for \( i = 2, 1 \), because \( \tilde{p}_i \) is the only point on \( E_p|_{\tilde{p}_i} \) which does not map to \( \mathbb{P}(\text{Im} \psi_{L,p}) \), and \( x \notin \mathbb{P}(\text{Im} \psi_{L,p}) \).

We shall prove in Proposition \( 9.4 \) that, for every \( l \in L_p \), \( l \neq p_1, p_2 \), the point \( y_{l,i} \) defined as the intersection of the strict transform of \( s_{E_l}(L_p) \) with \( E_q|_{\tilde{p}_i} \), maps to \( E_l \) for \( i = 1, 2 \), and this shall complete the proof of Theorem \( 7.3 \).

**8. Definition of \( E \)**

We shall define \( E \) as the kernel of a map \( (\varepsilon_3, 1)^*E \to (\varepsilon_3, 1)^*(A_2|_{\tilde{C}_2 \times C} \otimes F_2) \), with \( F_2 \) a sheaf on \( \tilde{C}_2 \times C \) such that \( F_2|_{(q) \times C} \simeq \mathcal{O}_C(q) \) if \( q \neq \tilde{p}_1, \tilde{p}_2 \), and \( F_2|_{(\tilde{p}_i) \times C} \simeq (\pi_p)_*\mathcal{O}_C \) for \( i = 1, 2 \). The map corresponds to a map \( E_L \to \pi_{\mathcal{F} \times \mathcal{F} \times \mathcal{F} \times \mathcal{F}} \mathcal{F}_x|_{\mathcal{F}_x \times \mathcal{F}_x} \times \mathcal{F} \), where \( \mathcal{F} := \mathcal{I}_\Delta, \mathcal{I}_\Delta \) being the ideal sheaf of the diagonal \( \Delta \) in \( C \times C \).

Before we define the map, let us study \( F \) in more detail.

**Lemma 8.1.** There exists a short exact sequence

\[
0 \to \mathcal{O}_C \to \mathcal{F} \to \omega^{-1}_\Delta \to 0.
\]

Moreover, \( F|_{(q) \times C} \simeq \mathcal{O}_C(q) \) if \( q \neq p \), and \( F|_{(p) \times C} \simeq (\pi_p)_*\mathcal{O}_C \).

**Proof.** Starting with the short exact sequence \( 0 \to \mathcal{I}_\Delta \to \mathcal{O}_C \to \mathcal{O}_\Delta \to 0 \), and applying the functor \( \mathcal{H}om_{C \times C}(-, \mathcal{O}_C) \), we obtain the short exact sequence\(^8\)

\[
0 \to \mathcal{O}_C \to \mathcal{F} \to \mathcal{E}xt_1^{C \times C}(\mathcal{O}_\Delta, \mathcal{O}_C) \to 0.
\]

\(^8\)The sequence starts with \( \mathcal{H}om_{C \times C}(\mathcal{O}_\Delta, \mathcal{O}_C) \) which is zero because \( \mathcal{O}_C \) is torsion-free, and ends with \( \mathcal{E}xt_1^{C \times C}(\mathcal{O}_C, \mathcal{O}_C \times C) \) which is also zero (see [Har77, III.6.3]).
Moreover, $\mathcal{E}_{\text{I}C_{\times C}}(\mathcal{O}_{\Delta}, \mathcal{O}_{C_{\times C}}) \cong \mathcal{E}_{\text{I}C_{\times C}}(\mathcal{O}_{\Delta}, \omega_{C_{\times C}}) \cong \mathcal{O}_{\Delta} \otimes \omega_{C_{\times C}}^{-1} \cong \omega_{C_{\times C}}^{-1}$ (see [Har77, III.6.7]) since $\omega_{\Delta} \cong \mathcal{E}_{\text{I}C_{\times C}}(\mathcal{O}_{\Delta}, \omega_{C_{\times C}})$ (see [Eis95, 21.15]), and $\omega_{C_{\times C}}^{-1} \cong \omega_{\Delta}^{-1}$.

Now, for any $q \in C$, restricting the short exact sequence to $\{q\} \times C$, we obtain a short exact sequence $0 \to \mathcal{O}_{C} \to \mathcal{F}_{|\{q\} \times C} \to \mathbb{C}_{q} \to 0$ which does not split.

We know that $\mathcal{E}_{L}$ is the extension of $\pi_{L}^{*}L$ by $\pi_{L}^{*}\mathcal{O}_{\mathbb{P}_{L}}(1)$ which corresponds to the identity in $\text{Hom}(\mathcal{E}_{L}|_{C \times C}, \mathcal{O}_{C})$. Since $\pi_{L}^{*}\mathcal{O}_{\mathbb{P}_{L}}(1)|_{C \times C} \cong \pi_{1}^{*}(L \otimes \omega_{C})$, where $\pi_{1}$ is the first projection $C \times C \to C$, we obtain the following short exact sequence on $C \times C$:

$$0 \to \pi_{1}^{*}(L \otimes \omega_{C}) \to \mathcal{E}_{L}|_{C \times C} \to \pi_{2}^{*}L \to 0.$$  

The map $\pi_{1}^{*}(L \otimes \omega_{C}) \hookrightarrow \pi_{1}^{*}(L \otimes \omega_{C}) \otimes \mathcal{F}$ extends to a map $\mathcal{E}_{L}|_{C \times C} \to \pi_{1}^{*}(L \otimes \omega_{C}) \otimes \mathcal{F}$ if $\mathcal{E}_{L}|_{C \times C}$ is in the kernel of the natural linear homomorphism

$$\text{Ext}_{C \times C}^{1}(\pi_{2}^{*}L, \pi_{1}^{*}(L \otimes \omega_{C})) \to \text{Ext}_{C \times C}^{1}(\pi_{2}^{*}L, \pi_{1}^{*}(L \otimes \omega_{C}) \otimes \mathcal{F}),$$

i.e., if $\mathcal{E}_{L}|_{C \times C}$ is in the image of the natural linear homomorphism

$$\text{Hom}_{C \times C}(\pi_{2}^{*}L, \pi_{1}^{*}(L \otimes \omega_{C}) \otimes \omega_{L}^{-1}) \to \text{Ext}_{C \times C}^{1}(\pi_{2}^{*}L, \pi_{1}^{*}(L \otimes \omega_{C})).$$

Let us prove that this is the case. Since $\pi_{1}^{*}(L \otimes \omega_{C}) \otimes \omega_{L}^{-1}$ is isomorphic to $L$ on $\Delta \cong C$, $\text{Hom}_{C \times C}(\pi_{2}^{*}L, \pi_{1}^{*}(L \otimes \omega_{C}) \otimes \omega_{L}^{-1}) \cong H^{0}(\Delta, \mathcal{O}_{\Delta}) \cong \mathbb{C}$. Moreover,

$$\text{Ext}_{C \times C}^{1}(\pi_{2}^{*}L, \pi_{1}^{*}(L \otimes \omega_{C})) \cong \text{Ext}_{C}(\mathcal{O}_{C}^{*}) \otimes \text{Ext}_{C}^{1}(L, \mathcal{O}_{C}),$$

which has the canonical identity element corresponding to $\mathcal{E}_{L}|_{C \times C}$. The constant section $1$ of $\mathcal{O}_{\Delta}$ maps to the identity, and our claim is proved, i.e., there exists a map $\mathcal{E}_{L}|_{C \times C} \to \pi_{L}^{*}\mathcal{O}_{\mathbb{P}_{L}}(1)|_{C \times C} \otimes \mathcal{F}$ as claimed at the beginning of the section.

This map is surjective because its restriction to $\{q\} \times C$, $q \neq p$, [resp. to $\{p\} \times C$] is the surjective map $\mathcal{E}_{L}|_{\{q\} \times C} \to \mathcal{O}_{C}(q)$ [resp. $\mathcal{E}_{L}|_{\{p\} \times C} \to (\pi_{p})_{*}\mathcal{O}_{C_{p}}$] which makes $\mathcal{E}_{L}|_{\{q\} \times C}$ [resp. $\mathcal{E}_{L}|_{\{p\} \times C}$] not semi-stable.

There exists a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}_{L}|_{\tilde{\mathcal{C}}_{1} \times C} & \longrightarrow & \mathcal{A}_{1}|_{\tilde{\mathcal{C}}_{1} \times C} \otimes \mathcal{F}_{1} \\
\downarrow & & \downarrow \\
(\sigma, 1)^{*}(\mathcal{E}_{L}|_{C \times C}) & \longrightarrow & \sigma^{*}\mathcal{O}_{\mathbb{P}_{L}}(1)|_{\tilde{\mathcal{C}}_{1} \times C} \otimes (\sigma, 1)^{*}\mathcal{F} \longrightarrow 0
\end{array}
$$

where $\sigma : \tilde{\mathcal{C}}_{1} \to C$ is the restriction of $\varepsilon$ to $\tilde{\mathcal{C}}_{1}$, and $\mathcal{F}_{1}$ is defined by the first row being exact. Since the restriction of the short exact sequence defining $\mathcal{E}_{L,1}$ to $\tilde{\mathcal{C}}_{1} \times C$ stays exact (see Lemma [6.3]), the cokernel of the vertical map on the left is $\mathcal{O}_{\{p_{i}, p_{2}\}} \boxtimes (\pi_{p})_{*}\mathcal{O}_{C_{p}}$. Since the fiber of ker $g$ at $\{p_{i}\} \times C$ has degree $2$ for $i = 1, 2$, it maps to zero into $(\pi_{p})_{*}\mathcal{O}_{C_{p}}$, and we obtain the following
If we restrict to \( \{q\} \times C \), for \( q \in \tilde{C}_1 \), \( q \neq p_1, p_2 \), then \( \mathcal{F}_1|_{\{q\} \times C} \simeq \mathcal{F}|_{\{q\} \times C} \simeq \mathcal{O}_C(q) \). Let \( i \in \{1, 2\} \).

If we restrict the right column to \( \{p_i\} \times C \), we obtain

\[
0 \rightarrow T \rightarrow \mathcal{F}_1|_{\{p_i\} \times C} \rightarrow \mathcal{F}|_{\{p\} \times C} \xrightarrow{\simeq} (\pi_p)_*\mathcal{O}_{C_p} \rightarrow 0,
\]

where \( T = \mathcal{T}or_1^{\tilde{C}_1 \times C}(\mathcal{O}_{\{p_1, p_2\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p}, \mathcal{O}_{\{p_i\} \times C}) \).

To calculate this sheaf, consider \( 0 \rightarrow \mathcal{O}_{\tilde{C}_1}(-p_i) \rightarrow \mathcal{O}_{\tilde{C}_1} \rightarrow \mathcal{O}_{\{p_i\}} \rightarrow 0 \) on \( \tilde{C}_1 \) and its pull-back to \( \tilde{C}_1 \times C \). We want to tensor it with \( \mathcal{O}_{\{p_1, p_2\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \), and we do it in two steps. We first tensor it with \( \pi_{C}^!((\pi_p)_*\mathcal{O}_{C_p}) \) to obtain

\[
0 \rightarrow \mathcal{O}_{\tilde{C}_1}(-p_i) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \rightarrow 0,
\]

where the map \( \mathcal{O}_{\tilde{C}_1}(-p_i) \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \) is injective because it is an isomorphism on the dense open subset \((\tilde{C}_1 \setminus \{p_i\}) \times C \) whose complement has codimension 1, and therefore the image of any torsion sheaf appearing on the left will be zero, being a subsheaf of a torsion-free sheaf supported on a codimension 1 subvariety. Then we tensor the short exact sequence with \( \pi_{\tilde{C}_1}^*\mathcal{O}_{\{p_1, p_2\}} \) to obtain

\[
0 \rightarrow T \rightarrow \mathcal{O}_{\tilde{C}_1}(-p_i)|_{\{p_1, p_2\}} \boxtimes \pi_\sharp\mathcal{O}_N \rightarrow \mathcal{O}_{\{p_1, p_2\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \rightarrow (\pi_p)_*\mathcal{O}_{C_p} \rightarrow 0,
\]

from which is clear that \( T \simeq \mathcal{O}_{\{p_i\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \), and therefore \( \mathcal{F}_1|_{\{p_i\} \times C} \simeq (\pi_p)_*\mathcal{O}_{C_p} \).

An identical process defines a sheaf \( \mathcal{F}_2 \) on \( \tilde{C}_2 \times C \) such that

\[
\mathcal{E}_{L, 2}|_{\tilde{C}_2 \times C} \longrightarrow \mathcal{A}_2|_{\tilde{C}_2 \times C} \boxtimes \mathcal{F}_2 \longrightarrow 0
\]

where we identify \( \tilde{C}_2 \) and \( \tilde{C}_1 \) via the isomorphism \( \varepsilon_2|_{\tilde{C}_2} \). Since the cokernel of the vertical maps is again \( \mathcal{O}_{\{\tilde{p}_1, \tilde{p}_2\}} \boxtimes (\pi_p)_*\mathcal{O}_{C_p} \), the exact same proof as above shows that \( \mathcal{F}_2|_{\{q\} \times C} \simeq \mathcal{O}_C(q) \) if \( q \in \tilde{C}_2 \), \( q \neq \tilde{p}_1, \tilde{p}_2 \), and \( \mathcal{F}_2|_{\{\tilde{p}_i\} \times C} \simeq (\pi_p)_*\mathcal{O}_{C_p} \) for \( i = 1, 2 \).

We define \( \mathcal{E}_{L, 3} \) to be the kernel of the map \( \varepsilon_3|_{\tilde{C}_2}^* \mathcal{E}_{L, 2} \rightarrow (\varepsilon_3, 1)^* (\mathcal{A}_2|_{\tilde{C}_2 \times C} \boxtimes \mathcal{F}_2) \).
9. Relation between $\mathcal{E}_{L,3}$ and $\phi_{L,3}$

**Proposition 9.1.** For every $\phi \in \tilde{C}_2$ mapping to a smooth point $q \in C$, $\phi_{L,3}|_{E_3 q}$ is a morphism, and it maps $E_3 q$ isomorphically to $\mathbb{P}(\text{Ext}^1_C(L(q), \mathcal{O}_C(q)))$.

**Proof.** We proved in Section 7 that $E_3 q \cong \mathbb{P}(\text{Ext}^1_C(L(q), \mathcal{O}_C(q)))$, and therefore we need to show that under this identification, $\phi_{L,3}$ is the identity map. The proposition follows from Proposition 9.2. \qed

**Proposition 9.2.** The restriction of the torsion-free sheaf $\mathcal{E}_{L,3}$ to $E_3 q \times C$ is the element of the vector space $\text{Ext}^1_{E_3 q \times C}(\pi_C^* L(q), \mathcal{O}_{E_3 q}(1) \otimes \mathcal{O}_C(q))$ which corresponds to the identity under the identification of this extension space with $\text{Hom}(\text{Ext}^1_C(L(q), \mathcal{O}_C(q)), \text{Ext}^1_C(L(q), \mathcal{O}_C(q)))$.

**Proof.** This proof is very similar to the proof of Proposition 4.2. Let $H \subseteq \mathbb{P}_L$ be a linear hyperplane which contains $q$, does not contain any node $p$ of $C$, and is transverse to the curve $C$ at $q$ (i.e., $H$ does not contain $T_q C$). Then $H$ is isomorphic to its strict transform in $\mathbb{P}_L$, that we shall still denote by $H$. It is clear that $\mathcal{E}_{L,2}|_{H \times C}$ is $\mathcal{E}_L|_{H \times C}$, and therefore it is an extension $0 \rightarrow \pi^*_H \mathcal{O}_H(1) \rightarrow \mathcal{E}_{L,2}|_{H \times C} \rightarrow \pi^*_H L \rightarrow 0$ on $H \times C$.

Let $\sigma : \tilde{H} \rightarrow H$ be the blow-up of $H$ at $q$, and let $E' \subseteq \tilde{H}$ be the exceptional divisor. Then there exists a commutative diagram on $\tilde{H} \times C$:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \sigma^* \mathcal{O}_H(1) \otimes \mathcal{O}_{\tilde{H}}(-E') & \rightarrow & \mathcal{E}_{L,3}|_{\tilde{H} \times C} & \rightarrow & \mathcal{B}_3|_{\tilde{H} \times C} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \sigma^* \mathcal{O}_H(1) & \rightarrow & (\sigma, 1)^* \mathcal{E}_{L,2}|_{H \times C} & \rightarrow & L & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_{E' \times C} & \rightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & \rightarrow & \mathcal{O}_{E' \times \{q\}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

where $\mathcal{B}_3$ is defined exactly in the same way we defined $\mathcal{B}_1$ and $\mathcal{B}_2$, and the columns are exact (see Lemma 6.3).

Let $\mathcal{E}'_H$ be the push-forward of $(\sigma, 1)^* \mathcal{E}_{L,2}|_{H \times C}$ via $\sigma^* \mathcal{O}_H(1) \hookrightarrow \sigma^* \mathcal{O}_H(1) \otimes \mathcal{O}_C(q)$:

\[
\begin{array}{cccc}
0 & \rightarrow & \sigma^* \mathcal{O}_H(1) & \rightarrow & (\sigma, 1)^* \mathcal{E}_{L,2}|_{H \times C} & \rightarrow & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \sigma^* \mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q) & \rightarrow & \mathcal{E}'_H & \rightarrow & L & \rightarrow & 0
\end{array}
\]

(5)

Then the restriction of $\mathcal{E}'_H$ to $E' \times C$ splits. Indeed, $\sigma^* \mathcal{O}_H(1)|_{E'} \cong \mathcal{O}_{E'}$, and via the identification $\text{Ext}^1_{E' \times C}(L, \mathcal{O}_C(q)) \cong H^0(E', \mathcal{O}_{E'}) \otimes \text{Ext}^1_C(L, \mathcal{O}_C(q))$ (see [Arc04]), we see that $\mathcal{E}'_H|_{E' \times C}$ splits as long as $\mathcal{E}'_H|_{\{x\} \times C}$ splits for any $x \in E'$. Restricting the diagram (3) above to $\{x\} \times C$ for any $x \in E'$, we see that $\mathcal{E}'_H|_{\{x\} \times C}$ is the trivial extension $\mathcal{E}_L(q)$. Therefore, there exists a surjective map $\mathcal{E}'_H \rightarrow \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q)$, and we can define $\mathcal{E}'_{H,1}$ to be its kernel: $0 \rightarrow \mathcal{E}'_{H,1} \rightarrow \mathcal{E}'_H \rightarrow \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \rightarrow 0$. Then there exists the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{E}'_{H,1} & \rightarrow & \mathcal{E}'_H & \rightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & \rightarrow & 0
\end{array}
\]
on \( \widetilde{H} \times C \):

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \sigma^* \mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
\mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & = & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

Moreover, we have the following commutative diagram on \( \widetilde{H} \times C \) which relates \( \mathcal{E}'_H \) and \( \mathcal{E}'_{H,1} \) to \( \mathcal{E}_{L,2}|_{\widetilde{H} \times C} \) and \( \mathcal{E}_{L,3}|_{\widetilde{H} \times C} \):

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & (\sigma, 1)^* \mathcal{E}_{L,2}|_{\widetilde{H} \times C} \\
\downarrow & & \downarrow \\
\mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & \rightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
\sigma^* \mathcal{O}_H(1) \boxtimes \mathbb{C}_q & = & \sigma^* \mathcal{O}_H(1) \boxtimes \mathbb{C}_q \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

When we restrict the first two rows of this diagram to \( E' \times C \), and we look at the image of the restrictions of \( i_1 \) and \( i'_1 \) to \( E' \times C \), we obtain the following diagram, where the first row shows that the restriction of \( \mathcal{E}_{L,3} \) to \( E_3|_q \times C \simeq E' \times C \) is an extension of \( \pi^*_C L(-q) \) by \( \mathcal{O}_{E_3|_q} (1) \boxtimes \mathcal{O}_C(q) \):

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
\mathcal{E}'_{L,3}|_{E' \times C} & \rightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
\sigma^* \mathcal{O}_H(1) \boxtimes \mathbb{C}_q & \rightarrow & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
\mathcal{E}'_{H,1}|_{E' \times C} & \rightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
\mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & \rightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]
This shows that $\mathcal{E}_{L,3}|_{E' \times C}$ is the pull-back of $\mathcal{E}'_{H,1}|_{E' \times C}$ via the pull-back of the inclusion $L(-q) \hookrightarrow L$ from $C$ to $E' \times C$. Here is a summary of how to construct $\mathcal{E}_{L,3}|_{E' \times C}$:

$$
\begin{align*}
\mathcal{E}_{L,2}|_{E' \times C} & \in \quad \text{Ext}^1_{C}(L, \mathcal{O}_C(1)) \\
\rightarrow & \\
\mathcal{E}'_H & \in \quad \text{Ext}^1_{E' \times C}(L, \sigma^* \mathcal{O}_C(1) \boxtimes \mathcal{O}_C(q)) \\
\rightarrow & \\
\mathcal{E}'_{H,1} & \in \quad \text{Ext}^1_{E' \times C}(L, (\sigma^* \mathcal{O}_C(1) \boxtimes \mathcal{O}_C(-E')) \boxtimes \mathcal{O}_C(q)) \\
\rightarrow & \\
\mathcal{E}_{L,3}|_{E' \times C} & \in \quad \text{Ext}^1_{E' \times C}(L(-q), \mathcal{O}_C(1) \boxtimes \mathcal{O}_C(q))
\end{align*}
$$

Using the isomorphisms $\text{Ext}^1_{E' \times C}(L, F \boxtimes G) \simeq H^0(Y, F) \otimes \text{Ext}^1_C(L, G)$ (see [Arc04]), we can understand what extension $\mathcal{E}_{L,3}|_{E' \times C}$ is by tracking the corresponding elements in these spaces. Let $v_0, \ldots, v_n$ be a basis of $\text{Ext}^1_C(L, \mathcal{O}_C)$ with Span $\{v_1, \ldots, v_n\} = \langle H \rangle$, and Span $\{v_0, v_1\} = \langle T_qC \rangle$. Let $v_0^*, \ldots, v_n^*$ be the corresponding dual basis in $\text{Ext}^1_C(L, \mathcal{O}_C)^*$. Then $v_1^*, \ldots, v_n^*$ is a basis of $\langle H \rangle^* \simeq H^0(H, \mathcal{O}_C(1))$, and $\mathcal{E}_{L,2}|_{E' \times C}$ corresponds to the element $\sum_{i=1}^n v_i^* \otimes v_i \in H^0(H, \mathcal{O}_C(1) \otimes \text{Ext}^1_C(L, \mathcal{O}_C))$. Let $\psi_q: \text{Ext}^1_C(L, \mathcal{O}_C(q)) \to \text{Ext}^1_C(L(-q), \mathcal{O}_C(q))$ be the natural linear homomorphism. Since $\psi_q(v_1) = 0$ and $\ker(\psi_q \circ \psi_q) = \text{Span} \{v_0, v_1\}$, we can calculate that $\mathcal{E}_{L,3}|_{E' \times C}$ corresponds to the element $\sum_{i=2}^n w_i^* \otimes v_i \in H^0(E', \mathcal{O}_C(1)) \otimes \text{Ext}^1_C(L(-q), \mathcal{O}_C(q))$, where, for $2 \leq i \leq n$, $w_i = \psi_q(v_i)$. Therefore, $\mathcal{E}_{L,3}$ corresponds to the identity in the vector space $\text{Hom}(\text{Ext}^1_C(L(-q), \mathcal{O}_C(q)), \text{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$, as claimed.

We now prove that, for every node $p$ of $C$, and for every $i \in \{1, 2\}$, $\phi_{L,3}|_{\text{PT}_p(E_2|_p)}$ factors through the canonical isomorphism $\mathbb{P}(T_{\tilde{p}_i}/E_2|_p)) \to \mathbb{P}(\text{Im} \psi_{Lp})$ described in Section 7.

**Proposition 9.3.** For every node $p$ in $C$, and every $i \in \{1, 2\}$, the extension

$$
\mathcal{E}_{L,3}|_{\mathbb{P}(\mathcal{N}_{(p_i)/E_2|_p})} \subseteq \text{Ext}^1_{\mathbb{P}(\mathcal{N}_{(p_i)/E_2|_p})} (\mathbb{L} \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p})
$$

corresponds to the inclusion in $\text{Hom}(\text{Im} \psi_{Lp}, \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}))$ under the canonical identification $\mathcal{N}_{(\tilde{p}_i)/E_2|_p} \cong \text{Im} \psi_{Lp}$.

**Proof.** Let $i \in \{1, 2\}$. From our description of $\mathcal{E}_{L,2}$ and $\phi_{L,2}$, it is clear that $\mathcal{E}_{L,2}|_{E_2|_p \times C}$ induces the linear map given by projection from $\tilde{p}_i$, i.e., it corresponds to a linear homomorphism $\mathcal{N}_{Lp}/\mathcal{L}_1|_{p_i} \to \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p})$ of kernel $\langle \tilde{p}_i \rangle$ and image $\text{Im} \psi_{Lp}$. Therefore, we can find a basis $w_1, \ldots, w_n$ of $\mathcal{N}_{Lp}/\mathcal{L}_1|_{p_i}$ and a basis $v_0, \ldots, v_n$ of $\text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p})$ such that $\langle \tilde{p}_i \rangle = \text{Span} \{w_1\}$, $\text{Im} \psi_{Lp} = \text{Span} \{v_2, \ldots, v_n\}$, and $\text{Span} \{w_1, v_j\}$ maps to $\text{Span} \{v_j\}$ for every $2 \leq j \leq n$ under the homomorphism corresponding to $\mathcal{E}_{L,2}|_{E_2|_p \times C}$. In particular, this sheaf corresponds to $\sum_{j=2}^n w_j^* \otimes v_j$ in $H^0(E_2|_p, \mathcal{O}_{E_2|_p}(1)) \otimes \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p})$.

To simplify the notation, let us denote $E_2|_p$ by $X$ and its blow-up at $\tilde{p}_i$ by $\sigma : \tilde{X} \to X$, with $E'$ the exceptional divisor. Then there exists a short exact sequence

$$
0 \to \mathcal{E}_{L,3}|_{\tilde{X} \times C} \to (\sigma, 1)^* \mathcal{E}_{L,2}|_{X \times C} \to \mathcal{O}_{E'} \boxtimes (\pi_p)_* \mathcal{O}_{C_p} \to 0,
$$

obtained by restricting the short exact sequence defining $\mathcal{E}_{L,3}$ to $\tilde{X} \times C$. It stays exact because of Lemma 6.3.
There exists the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & 
\rightarrow & \mathcal{A} \otimes (\pi_p)_* \mathcal{O}_{C_p} & 
\rightarrow & \mathcal{E}_{L,3}|_{\tilde{X} \times C} & 
\rightarrow & L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow \\
0 & \rightarrow & \sigma^* \mathcal{O}_X(1) \otimes (\pi_p)_* \mathcal{O}_{C_p} & 
\rightarrow & (\sigma, 1)^* \mathcal{E}_{L,2}|_{X \times C} & 
\rightarrow & L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* & 
\rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{E'} \otimes (\pi_p)_* \mathcal{O}_{C_p} & = & \mathcal{O}_{E'} \otimes (\pi_p)_* \mathcal{O}_{C_p} & 
\rightarrow & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & & 0 & & 0 & & 0
\end{array}
\]

where \( \mathcal{A} = \sigma^* \mathcal{O}_X(1) \otimes \mathcal{O}_{\tilde{X}}(-E') \). If we restrict the first row to \( E' \times C \), we obtain a short exact sequence

\[
0 \rightarrow \mathcal{O}_{E'}(1) \otimes (\pi_p)_* \mathcal{O}_{C_p} \rightarrow \mathcal{E}_{L,3}|_{E' \times C} \rightarrow L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* \rightarrow 0.
\]

Remember that \( E' \) is \( \mathbb{P}(\mathcal{N}_{(\tilde{p}_i)/E_2|p}) \). The following diagram, where, to simplify the notation, we denoted \( \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \) by \( V \), illustrates the steps we took in finding \( \mathcal{E}_{L,3}|_{E' \times C} \):

\[
\begin{array}{ccc}
\mathcal{E}_{L,2}|_{X \times C} & \leftarrow & \sum_{j=2}^n w_j^* \otimes v_j \in H^0(X, \mathcal{O}_X(1)) \otimes V \\
& \downarrow & \\
(\sigma, 1)^* \mathcal{E}_{L,2}|_{X \times C} & \leftarrow & \sum_{j=2}^n w_j^* \otimes v_j \in H^0(\tilde{X}, \sigma^* \mathcal{O}_X(1)) \otimes V \\
& \uparrow & \\
\mathcal{E}_{L,3}|_{\tilde{X} \times C} & \leftarrow & \sum_{j=2}^n w_j^* \otimes v_j \in H^0(\tilde{X}, \mathcal{A}) \otimes V \\
& \downarrow & \\
\mathcal{E}_{L,3}|_{E' \times C} & \leftarrow & \sum_{j=2}^n v_j^* \otimes v_j \in H^0(E', \mathcal{O}_{E'}(1)) \otimes V
\end{array}
\]

Therefore, \( \mathcal{E}_{L,3}|_{E' \times C} \) corresponds to the inclusion \( \text{Im} \psi_{L,p} \hookrightarrow \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \), as claimed.

Let \( p \) be a node of \( C \), let \( l \in L_p \), \( l \neq p_1, p_2 \), and let \( Y_l := s_{E_l}(L_p) \), where \( s_{E_l} \) is the section of \( E_{2,p} \rightarrow L_p \) defined in Section 7. Remember that we denoted by \( y_{l,i} \) the only point of intersection of the strict transform \( \tilde{Y}_l \) of \( Y_l \) with \( E_{3|\tilde{p}_i} \ (i = 1, 2) \).

**Proposition 9.4.** The restriction of \( \mathcal{E}_{L,3} \) to \( \tilde{Y}_l \times C \) is a non-zero element in the vector space \( \text{Ext}^1_{Y_l \times C}(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \cong H^0(\tilde{Y}_l, \mathcal{O}_{\tilde{Y}_l}) \otimes \text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*, (\pi_p)_* \mathcal{O}_{C_p}) \). In particular \( \mathcal{E}_{L,3}|_{Y_l \times C} \cong E_l \) for \( i = 1, 2 \).

For the proof, we need the following result.

**Lemma 9.5.** The restriction of \( \mathcal{O}_{E2}(-E_2) \) to \( L_{2,p} \cong \mathbb{P}^1 \) is \( \mathcal{O}_{L_{2,p}}(1) \).
Proof. Remember that $L_{2,p} = \tilde{T}_p C \cap E_2$. It is isomorphic to $L_p = \tilde{T}_p C \cap E_1$ via $\varepsilon_2$, and it is therefore the exceptional divisor of the blow-up of $T_p C$ at $p$. Therefore,
\[
\mathcal{O}_{\tilde{T}_p,2}(-E_2)|_{L_{2,p}} = (\mathcal{O}_{\tilde{T}_p,2}(-E_2)|_{\tilde{T}_p C})|_{L_{2,p}} = \mathcal{O}_{\tilde{T}_p C}(-L_{2,p})|_{L_{2,p}} = \mathcal{O}_{L_{2,p}}(1).
\]

Proof (of Proposition 9.2). We saw in Lemma 6.1 that there exists a short exact sequence
\[
0 \to (\varepsilon_2^* \mathcal{O}_{L_p}(1) \otimes \mathcal{O}_{E_2}(-E_2)) \otimes (\pi_p)_* \mathcal{O}_{C_p} \to \mathcal{E}_{L,2}|_{E_2 \times C} \to L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* \to 0
\]
on $E_2 \times C$. If we restrict it to $Y_l \times C$, we obtain the short exact sequence
\[
0 \to \mathcal{O}_{Y_l}(2) \otimes (\pi_p)_* \mathcal{O}_{C_p} \to \mathcal{E}_{L,2}|_{Y_l \times C} \to L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* \to 0,
\]
because $\mathcal{O}_{E_2}(-E_2)|_{Y_l} = \mathcal{O}_{Y_l}(1)$ since $Y_l$ and $L_2$ are in the same linear system. Therefore, there exists the following commutative diagram on $\tilde{Y}_l \times C$:
\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{Y_l}(2) \otimes (\pi_p)_* \mathcal{O}_{C_p} & \to & \mathcal{E}_{L,2}|_{Y_l \times C} & \to & L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* & \to & 0 \\
0 & \to & \mathcal{O}_{Y_l \times C}(2) \otimes (\pi_p)_* \mathcal{O}_{C_p} & \to & \mathcal{E}_{L,2}|_{Y_l \times C} & \to & L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* & \to & 0,
\end{array}
\]
and $\mathcal{E}_{L,2}|_{Y_l \times C}$ is an element of Ext$_1^{\mathcal{Y}_l \times C}(L \otimes ((\pi_p)_* \mathcal{O}_{C_p}), (\pi_p)_* \mathcal{O}_{C_p})$. Since this is isomorphic to $H^0(\tilde{Y}_l, \mathcal{O}_{\tilde{Y}_l}) \otimes \text{Ext}_1^C(L \otimes ((\pi_p)_* \mathcal{O}_{C_p}), (\pi_p)_* \mathcal{O}_{C_p})$ (see [Arc04]), $\mathcal{E}_{L,3}|_{Y_l \times C}$ is non-zero because we know that $\mathcal{E}_{L,3}|_{\{y\} \times C}$ does not split for every $y \in \tilde{Y}_l$, $y \neq y_{l,1}, y_{l,2}$.

10. Fibers of $\phi_{L,3}$

We prove in this section that the fibers of $\phi_{L,3}$ are connected. Let us start with characterizing the image.

Lemma 10.1. An element $E \in \overline{SU}_C(2, L)$ is in the image of $\phi_{L,3}$ if and only if $H^0(E) \neq 0$.

Proof. By our description of $\phi_{L,3}$ it is clear that if $E$ is in its image, then there exists a non-zero map $\mathcal{O}_C \to E$, and therefore $E$ has a non-zero section. Conversely, if $H^0(E) \neq 0$, there exists a non-zero section $\mathcal{O}_C \to E$. If $E$ is stable, then $s$ can vanish at at most one point, and therefore $E$ fits in at least one of the following exact sequences:
\[
\begin{align*}
&0 \to \mathcal{O}_C \to E \to L \to 0, \\
&0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0 \quad \text{for some smooth point } q \in C, \\
&0 \to (\pi_p)_* \mathcal{O}_{C_p} \to E \to L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^* \to 0 \quad \text{for some node } p \in C,
\end{align*}
\]
and it is therefore in the image of $\phi_{L,3}$. This proves the lemma, since $\phi_{L,3}$ is proper and the locus of stable bundles is dense in $\{E \in \overline{SU}_C(2, L) \mid H^0(E) \neq 0\}$.

As in the smooth case, we have the following result.
Proposition 10.2. For every stable $E \in \mathcal{SU}_C(2, L)$ there exists a morphism

$$\psi_E : \mathbb{P}(H^0(E)) \longrightarrow \mathbb{P}_L$$

such that, for every $x \in \mathbb{P}_L \setminus C$,

$$\phi_L(x) = E \iff x \in \text{Im}(\psi_E).$$

Proof. For every $s \in H^0(E)$, define $\psi_E([s]) := \mathbb{P}(\ker(\text{Ext}^1_C(L, \mathcal{O}_C) \to \text{Ext}^1_C(L, E)))$. To prove that $\psi_E$ is a morphism, it suffices to show that, for every $s \in H^0(E)$, the kernel in the definition of $\psi_E([s])$ is one-dimensional. Note that, since $E$ is stable of degree $\leq 4$, every torsion-free subsheaf of $E$ of rank 1 has degree $\leq 1$, and therefore $s$ can vanish at at most one point.

Case I: $s$ is no-where vanishing. There exists a short exact sequence $0 \to \mathcal{O}_C \to E \to L \to 0$, and applying to it the functor $\text{Hom}_C(L, -)$ we obtain

$$0 \to \text{Hom}_C(L, L) \to \text{Ext}^1_C(L, \mathcal{O}_C) \to \text{Ext}^1_C(L, E) \to \cdots,$$

where the sequence starts with $\text{Hom}_C(L, E)$, which is zero because $E$ is stable. This proves that the kernel of $\text{Ext}^1_C(L, \mathcal{O}_C) \to \text{Ext}^1_C(L, E)$ is isomorphic to $\text{Hom}_C(L, L)$, and it is therefore one-dimensional. Moreover, the image of the identity element of $\text{Hom}_C(L, L)$ in $\text{Ext}^1_C(L, \mathcal{O}_C)$ is the extension associated to $\mathcal{O}_C \to E$, and therefore $\phi_L(\psi_E([s])) = E$.

Case II: $s$ vanishes at exactly one point. Let $F$ be $\mathcal{O}_C(q)$ if $s$ vanishes at a smooth point $q \in C$ or let $F$ be $(\pi_p)_*\mathcal{O}_{C_p}$ if $s$ vanishes at a node $p$ of $C$. There exists a short exact sequence

$$0 \to F \to E \to L \otimes F^* \to 0,$$

and $\text{Ext}^1_C(L, \mathcal{O}_C) \to \text{Ext}^1_C(L, E)$ factors through $\text{Ext}^1_C(L, F) \to \text{Ext}^1_C(L, E)$. Since the kernel of $\text{Ext}^1_C(L, \mathcal{O}_C) \to \text{Ext}^1_C(L, F)$ is one-dimensional, to conclude the proof it suffices to show that $\text{Ext}^1_C(L, F) \to \text{Ext}^1_C(L, E)$ is injective. Applying the functor $\text{Hom}_C(L, -)$ to (6), we see that this is the case, because $\text{Hom}_C(L, L \otimes F^*) = 0$. \hfill \Box

In what follows, we also need the following result, which is similar to the proposition above.

Lemma 10.3. Let $E \in \mathcal{SU}_C(2, L)$ be stable.

(a) If every section $s \in H^0(E)$ vanishes at a smooth point $q \in C$, then there exists a morphism

$$\psi_{E,q} : \mathbb{P}(H^0(E)) \longrightarrow E_{3,q} \subset \mathbb{P}(\text{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$$

such that

$$\phi_{L,3}^{-1}(E) = \text{Im} \psi_{E,q}.$$

(b) If every section $s \in H^0(E)$ vanishes at a node $p \in C$, then there exists a morphism

$$\psi_{E,p} : \mathbb{P}(H^0(E)) \longrightarrow \mathbb{P}(H_p^*) \subset \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}))$$

such that $x$ is in the image of $\psi_{E,p}$ if and only if $x$ maps to $E$ under the natural forgetful map $\mathbb{P}(H_p^*) \to \mathcal{SU}_C(2, L)$, which is a morphism by Lemma 3.4.

Proof. Define $\psi_{E,q}([s]) := \mathbb{P}(\ker(\text{Ext}^1_C(L(-q), \mathcal{O}_C(q)) \to \text{Ext}^1_C(L(-q), E)))$ in part (a), and $\psi_{E,p}([s]) := \mathbb{P}(\ker(\text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, (\pi_p)_*\mathcal{O}_{C_p}) \to \text{Ext}^1_C(L \otimes ((\pi_p)_*\mathcal{O}_{C_p})^*, E)))$ in part (b). If we let $F = \mathcal{O}_C(q)$ for part (a) and $F = (\pi_p)_*\mathcal{O}_{C_p}$ for part (b), the proof is the same as the proof of Case I in Proposition 10.2, where we now use the unique extension $0 \to F \to E \to L \otimes F^* \to 0$ associated to $s$. \hfill \Box

To simplify the notation, for the rest of this section, we shall say that an element $E \in \mathcal{SU}_C(2, L)$ is of type $Q$ if there exists a non-zero map $\mathcal{O}_C(q) \to E$ for some smooth point $q \in C$, and is of type $P$ if there exists a non-zero map $(\pi_p)_*\mathcal{O}_{C_p} \to E$ for some node $p \in C$. 

Proposition 10.4. The fibers of $\phi_{L,3}$ are connected.

We shall divide the proof of this proposition into several lemmas analyzing various different cases.

Lemma 10.5. If $g > \deg L$, and $E \in \text{Im} \phi_{L,3}$ is not of type $P$, then $\phi_{L,3}^{-1}(E)$ is just a point.

Proof. Since $E$ is not of type $P$, there are two possibilities:

Case I: There exists an $x \in \mathbb{P}_L \setminus C$ such that $\phi_{L}(x) = E$. Then there exists a short exact sequence $0 \to \mathcal{O}_C \to E \to L \to 0$, and $h^0(E) = h^0(C, \mathcal{O}_C) = 1$, since $H^0(L) = 0$ because $g > \deg L$ and $L$ is generic. This proves that there is only one way to write $E$ as an extension of $L$ by $\mathcal{O}_C$, and $\mathcal{O}_C$ must be a maximal subbundle of $E$ (i.e., $E$ is not in the image of the natural morphisms $\prod \text{Ext}^1_C(L(-q), \mathcal{O}_C(q)) \to \mathcal{SU}_C(2, L)$ and $\prod H^0 \to \mathcal{SU}_C(2, L)$). Therefore $\phi_{L,3}^{-1}(E) = \{pt\}$.

Case II: There exists a smooth point $q \in C$ and an $x \in E_3|_q$ such that $\phi_{L,3}(x) = E$. Then there exists a short exact sequence $0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0$, and again $h^0(E) = h^0(C, \mathcal{O}_C(q)) = 1$ and $\phi_{L,3}^{-1}(E) = \{pt\}$. □

Lemma 10.6. If $g > \deg L$, and $E \in \text{Im} \phi_{L,3}$ is a non-locally-free sheaf of type $P$, then $\phi_{L,3}^{-1}(E)$ is the union of (the strict transforms of) a plane and two lines intersecting the plane.

Proof. If $E$ is of type $P$, then $E$ is in the image of a point of $\widetilde{E}_1$, $\widetilde{E}_2$, or $E_3|_p$, for some node $p$ of $C$ and $i \in \{1, 2\}$. There exists a short exact sequence $0 \to (\pi_p)_* \mathcal{O}_{C_p} \to E \to L \otimes ((\pi_p)_* \mathcal{O}_{C_p^*})^* \to 0$. Since $L$ is generic, $H^0(L) = 0$, and this implies that $H^0(L \otimes ((\pi_p)_* \mathcal{O}_{C_p})^*) = 0$. Then $h^0(E) = h^0((\pi_p)_* \mathcal{O}_{C_p}) = 1$, and there is only one way to write $E$ as such an extension (and $E$ cannot be written as an extension of $L$ by $\mathcal{O}_C$ or $L(-q)$ by $\mathcal{O}_C$ for some smooth $q \in C$ or $L(-q)$ by $\mathcal{O}_C$ for some other node $p'$).

Since $E$ is not locally-free, it is in the image of $\psi_{L_p}$, and by Theorem 3.1 $\phi_{L,1}^{-1}(E)$ is a plane in the projective space $E_1|_p$ containing the line $L_p$ (except for the line itself, where $\phi_{L,1}$ is not defined). Then, by Theorem 5.1 $\phi_{L,2}^{-1}(E)$ is the union of the strict transform $\phi_{L,1}^{-1}(E)$ of the plane $\phi_{L,1}^{-1}(E)$ and two lines which are contained in $E_2|_{p_1}$ and $E_2|_{p_2}$, respectively (except for the points $\tilde{p}_1$ and $\tilde{p}_2$, which are on the lines, where $\phi_{L,2}$ is not defined). The lines intersect the plane at the points $\phi_{L,1}^{-1}(E) \cap E_2|_{p_1}$ and $\phi_{L,1}^{-1}(E) \cap E_2|_{p_1}$, respectively. The last blow-up just adds the two missing points, and $\phi_{L,3}^{-1}(E)$ is the union of a plane and two lines which intersect it, as claimed. □

Lemma 10.7. If $g > \deg L$, and $E \in \text{Im} \phi_{L,3}$ is a locally-free sheaf of type $P$, then $\phi_{L,3}^{-1}(E)$ is (the strict transform of) a line.

Proof. The proof of this lemma follows exactly the proof of the previous lemma up to the description of $\phi_{L,1}^{-1}(E)$. In our case now, since $E$ is locally-free, $\phi_{L,1}^{-1}(E)$ is empty. Then, by Theorem 5.1 $\phi_{L,2}^{-1}(E)$ is a section of $E_{2,p} \to L_p$ which passes through the points $\tilde{p}_1$ and $\tilde{p}_2$, where $\phi_{L,2}$ is not defined. The last blow-up just adds the two missing points, and therefore $\phi_{L,3}^{-1}(E)$ is isomorphic to the line $L_p$, as claimed. □

The previous lemmas prove that the fibers of $\phi_{L,3}$ are connected if $g > \deg L$. To prove that the fibers are always connected, we need to still study the cases of $g = 2$, $g = 3$ and the case $g = \deg L = 4$. We shall now prove the case $g = 2$ and $\deg L = 3$ (since it is the case for which we have an application), and leave the other cases as an exercise for the reader.

Lemma 10.8. If $g = 2$ and $\deg L = 3$, then the fibers of $\phi_{L,3}$ are connected.
Proof. For every $E \in \overline{SU_C(2, L)}$, $E$ is stable, and $H^0(E) \neq 0$ because $\chi(E) = 1$. Therefore, $\phi_{L,3}$ is surjective by Lemma 10.11 and it is a birational morphism. If $h^0(E) = 1$, then the fiber $\phi_{L,3}^{-1}(E)$ is the same as the fibers described in Lemmas 10.5, 10.6, and 10.7 above. Suppose that $h^0(E) \geq 2$. Since $E$ is stable, every section $s \in H^0(E)$ satisfies $\deg(Z(s)) \leq 1$. Moreover, we have the morphism $\psi_\E: \mathbb{P}(H^0(E)) \to \mathbb{P}_L$ described in Proposition 10.2 and we saw that $\phi_{L}^{-1}(E) = \text{Im} \psi_\E \setminus \text{C}.$

Case I: $\text{Im} \psi_\E \subset \text{C}$. If $\text{Im} \psi_\E$ does not intersect $\text{C}$, then $\phi_{L,3}^{-1}(E) = \overline{\text{Im} \psi_\E}$ is connected. If $\text{Im} \psi_\E$ intersects $\text{C}$ at a smooth point $q$, then there exists a unique section $s \in H^0(E)$ such that $Z(s) = \{q\}$, and $E$ can be written as an extension of $L(-q)$ by $\mathcal{O}_C(q)$. By continuity, this extension must be the point in $E_{3|q} \simeq \mathbb{P}(\text{Ext}^1_C(\mathcal{O}_C(q), L(-q)))$ in the strict transform of the closure of $\text{Im} \psi_\E$. Similarly, if $\text{Im} \psi_\E$ intersects $\text{C}$ at a node $p$, then $E$ can be written as an extension of $L \otimes ((\pi_p)_* \mathcal{O}_C)_p^*$ by $(\pi_p)_* \mathcal{O}_C_p$ in a unique way. Since $\text{Im} \psi_\E \subset \text{C}$, $E$ is locally-free. Therefore, $E$ is not in the image of $E_1$, and its preimage in $\overline{E}_2$ is a line, which, by continuity, must intersect the strict transform of the closure of $\text{Im} \psi_\E$.

Case II: $\text{Im} \psi_\E \subset \text{C}$. In this case, $\text{Im} \psi_\E$ must be just a point. Indeed, $\psi_\E(s) = x \in \mathbb{C}$ if and only if $Z(s) = \{x\}$, and if $\text{Im} \psi_\E = C$, then there would exist two distinct sections in $H^0(E)$ mapping to each node of $\text{C}$. If $\text{Im} \psi_\E$ is a smooth point $q$ of $\text{C}$, then every section of $E$ vanishes at $q$, and by Lemma 10.3 $\phi_{L,3}^{-1}(E) = \text{Im} \psi_\E_q$ is connected. If $\text{Im} \psi_\E$ is a node $p$ of $\text{C}$, then every section of $E$ vanishes at $p$, and by Lemma 10.3 there exists a morphism

$$\psi_{E,p} : \mathbb{P}(H^0(E)) \to \mathbb{P}(H'_p) \subseteq \mathbb{P}(\text{Ext}^1_C(L \otimes ((\pi_p)_* \mathcal{O}_C)_p^*, (\pi_p)_* \mathcal{O}_C)_p))$$

such that $x$ is in the image of $\psi_{E,p}$ if and only if $x$ maps to $E$ under the natural forgetful morphism $\mathbb{P}(H'_p) \to \overline{SU_C(2, L)}$. For each point $x$ of $\mathbb{P}(H'_p)$, the space of all points in $\overline{E}_{2|p} \cup E_{3|p_1} \cup E_{3|p_2}$ which map to $x$ under the maps to $\mathbb{P}(H'_p)$ of Theorems 5.1 and 7.1 is connected by Lemmas 10.6 and 10.7. Since the image of $\psi_{E,p}$ in $\mathbb{P}(H'_p)$ is connected, this proves that $\phi_{L,3}^{-1}(E)$ is also connected. \hfill $\square$

Remark. The proof shows that $\phi_{L,3}^{-1}(E)$ is connected for every stable $E$ in the image of $\phi_{L,3}$.

11. The case $\deg L > 4$

If $\deg L > 4$, the rational map $\mathbb{P}_{L,3} \to \overline{SU_C(2, L)}$ is not a morphism, but we can still prove the following.

Proposition 11.1. Let $\deg L = 2g - 1$, and let

$$V = \{E \in \overline{SU_C(2, L)} \mid H^0(E) \simeq \mathbb{C} \cdot s \text{ and } Z(s) = \emptyset \text{ or } \{q\}, \text{ with } q \text{ a smooth point of } \text{C}\}.$$

Then the codimension of $\overline{SU_C(2, L)} \setminus V$ in $\overline{SU_C(2, L)}$ is $\geq 2$, and there exists an open subset $U$ of $\mathbb{P}_{L,3}$ such that $\phi_{L,3}|_U : U \to V$ is an isomorphism.

Proof. Let $E \in V$. If $Z(s) = \emptyset$ [resp. $Z(s) = \{q\}$], then $E$ is an extension of $L$ by $\mathcal{O}_C$ [resp. of $L(-q)$ by $\mathcal{O}_C(q)$], and by Lemma 10.2 [resp. 10.3] there exists a unique point $x$ of $\mathbb{P}_L$ [resp. $\mathbb{P}_{L,3}$] such that $\phi_L(x) = E$ [resp. $\phi_{L,3}(x) = E$]. No other point of $\mathbb{P}_{L,3}$ can map to $E$, and therefore $E$ has a unique preimage under $\phi_{L,3}$.

To prove the claim about the codimension of $\overline{SU_C(2, L)} \setminus V$ in $\overline{SU_C(2, L)}$, let us first study $U$. If we identify $\mathbb{P}_L \setminus \text{C}$ with its isomorphic image in $\mathbb{P}_{L,3}$, then

$$U \cap (\mathbb{P}_L \setminus \text{C}) = \{E \in \mathbb{P}_L \mid E \text{ is semi-stable and } h^0(E) = 1\}.$$
As in the smooth case, let
\[ \Gamma_L := \{ E \in \mathbb{P}_L \mid h^0(E) > 1 \}. \]
It is a hypersurface of degree \( g \) in \( \mathbb{P}_L \) (see [Ber92]). Then
\[ \mathbb{P}_L = (U \cap (\mathbb{P}_L \setminus C)) \cup \Gamma_L \cup B, \]
where \( B \) is the base locus of \( \phi_L \), which has codimension \( \geq 2 \) in \( \mathbb{P}_L \). Moreover, \( U \) does not intersect the exceptional divisors \( E_1 \) and \( E_2 \), and \( U \cap E_3 \) is a dense open subset of \( E_3 \). Therefore, the complement of \( U \) in \( \mathbb{P}_{L,3} \) is the union of \( \tilde{\Gamma}_L \), \( E_1 \), \( E_2 \), and a locus of codimension \( \geq 2 \) in \( \mathbb{P}_{L,3} \). Since the map \( \phi_{L,3} \) restricted to \( \tilde{\Gamma}_L \cup E_1 \cup E_2 \) has positive dimensional fibers, the image of \( \tilde{\Gamma}_L \cup E_1 \cup E_2 \) in \( \mathcal{S}U_C(2, L) \) has codimension 2, and therefore \( \text{codim}(\mathcal{S}U_C(2, L) \setminus V, \mathcal{S}U_C(2, L)) \geq 2. \) \( \square \)

12. Applications

Before we give direct applications of our construction, let us point out how the rational map \( \phi_L \) can be used to describe \( \mathcal{S}U_C(2, L) \) on an irreducible nodal curve of genus 1. In this case, the normalization \( N \) of \( C \) is isomorphic to \( \mathbb{P}^1 \), and \( C \) has only one node.

Proposition 12.1. Let \( C \) be an irreducible projective curve of arithmetic genus 1 with one node \( p \) as singularity.

1. Let \( L \) be any line bundle of degree 1. Then
   \[ \phi_L : \mathbb{P}(\text{Ext}^1_C(L, \mathcal{O}_C)) \to \mathcal{S}U_C(2, L) \subseteq \mathcal{S}U_C(2, L) \]
   is an isomorphism, and therefore \( \mathcal{S}U_C(2, L) = \mathcal{S}U_C(2, L) \simeq \mathbb{P}^0 \) as in the smooth case (see [Tu93]).

2. Let \( L \) be any line bundle of degree 2. Then
   \[ \phi_L : \mathbb{P}(\text{Ext}^1_C(L, \mathcal{O}_C)) \to \mathcal{S}U_C(2, L) \subseteq \mathcal{S}U_C(2, L) \]
   is an isomorphism, and therefore \( \mathcal{S}U_C(2, L) = \mathcal{S}U_C(2, L) \simeq \mathbb{P}^1 \) as in the smooth case (see [Tu93]).

Proof. The map \( \phi_L \) is a morphism by Proposition 24.1. Note that every \( E \in \mathcal{S}U_C(2, L) \) has at least one section because \( h^0(E) \geq \chi(E) = \deg L \geq 1. \)

If \( \deg L = 1 \), since every element of \( \mathcal{S}U_C(2, L) \) is stable, the sections of \( E \) cannot vanish at any point, therefore \( E \) is an extension of \( L \) by \( \mathcal{O}_C \), and \( \phi_L \) is surjective.

If \( \deg L = 2 \), then \( h^0(L) = 2 \). We have that \( \text{Ext}^1_C(L, \mathcal{O}_C) \simeq H^0(L)^* \), and therefore \( \mathbb{P}_L \simeq |L|^* \), which, being a \( \mathbb{P}^1 \), is canonically equal to \( |L| \). The morphism \( \phi_L \) is defined by \( \phi_L(q_1 + q_2) = \mathcal{O}_C(q_1) \oplus \mathcal{O}_C(q_2) \) for \( q_1 + q_2 \in |L| \simeq \mathbb{P}(\text{Ext}^1_C(L, \mathcal{O}_C)) \). It is clearly injective. To prove that \( \phi_L \) is surjective, note that, if \( E \in \mathcal{S}U_C(2, L) \) is not stable, then it must \( S \)-equivalent to \( \mathcal{O}_C(q_1) \oplus \mathcal{O}_C(q_2) \) with \( q_1 + q_2 \in |L| \), and if it is stable, then the sections of \( E \) cannot vanish at any point, and \( E \) is in the image of \( \phi_L \). \( \square \)

From the direct description of \( \phi_L \) in the proof, we deduce the following fact, which is also true in the smooth case (see [Tu93]).

Corollary 12.2. If \( C \) is an irreducible projective curve of arithmetic genus 1 with one node \( p \) as singularity, and \( L \) is a line bundle of even degree, then every vector bundle \( E \in \mathcal{S}U_C(2, L) \) is semi-stable but not stable.

For an irreducible nodal curve of genus 2, we have the following application as a corollary of Proposition 10.4. This fact is already known (see [BhoNew90]).

Corollary 12.3. If \( g = 2 \) and \( \deg L \) is odd, the normalization morphism \( \mathcal{S}U_C(2, L) \to \mathcal{S}U_C(2, L) \) is one-to-one.
\textbf{Proof.} This follows from the fact that $\phi_{L,3}$ is a birational morphism with connected fibers for $g = 2$ and $\deg L = 3$. \hfill \Box

\textbf{Remark.} If $g = 2$ and $\deg L = 4$, then $\phi_L: \mathbb{P}_L \to \SU_C(2, L)$ is a rational map defined by sections of $[\mathcal{I}_C(2)]$. It should be possible to prove that $h^0(\mathcal{I}_C(2)) = 4$, and obtain as a corollary the known fact that, if $\deg L$ is even, $\SU_C(2, L) \simeq \mathbb{P}^3$ for an irreducible nodal curve of genus 2 (see [Bho98]). We can prove that $h^0(\mathcal{I}_C(2))$ is indeed 4 for a generic such curve with one node and for a generic such curve with two nodes.

For curves of genus $\geq 2$, as a corollary of Proposition \ref{11.4} we can prove the following results.

\textbf{Corollary 12.4.} If $\deg L$ is odd, then $A_{3g-4}(\SU_C(2, L)) \simeq \mathbb{Z}$.

\textbf{Proof.} Let $\deg L = 2g - 1$. The isomorphism $\phi_{L,3}[U]: U \to V$ of Proposition \ref{11.1} induces an isomorphism $(\phi_{L,3}|_U)^*: A_{3g-4}(U) \to A_{3g-4}(V)$, and $A_{3g-4}(V) \simeq A_{3g-4}(\SU_C(2, L))$ because the complement of $V$ has codimension $\geq 2$ in $\SU_C(2, L)$. Recall that $U$ is an open subset of $\mathbb{P}_L$, whose complement has codimension one, and therefore there exists an exact sequence (see [Ful84, 1.8])

$$A_{3g-4}(\mathbb{P}_L,3 \setminus U) \longrightarrow A_{3g-4}(\mathbb{P}_L,3) \longrightarrow A_{3g-4}(U) \longrightarrow 0,$$

where

$$A_{3g-4}(\mathbb{P}_L,3) \simeq \mathcal{Z}H \oplus_{p \in J} \mathcal{Z}E_1|_p \oplus_{p \in J} \mathcal{Z}E_2,p \oplus \mathcal{Z}E_3,$$

with $H$ the pull-back of a hyperplane class from $\mathbb{P}_L$. It follows from our description of $U$ in the proof of Proposition \ref{11.4} that

$$A_{3g-4}(\mathbb{P}_L,3 \setminus U) \simeq \mathcal{Z}\tilde{H}_L \oplus_{p \in J} \mathcal{Z}\tilde{E}_1|_p \oplus_{p \in J} \mathcal{Z}\tilde{E}_2,p,$$

and therefore

$$A_{3g-4}(U) \simeq \frac{\mathcal{Z}H \oplus \mathcal{Z}E_3}{\mathcal{Z}\tilde{H}_L}.$$

Since, as in the smooth case, $\mathcal{Z}\tilde{H}_L \sim gH - (g - 1)E_3$, we obtain

$$A_{3g-4}(\SU_C(2, L)) \simeq \frac{\mathcal{Z}H \oplus \mathcal{Z}E_3}{\mathcal{Z}(gH - (g - 1)E_3)} \simeq \mathbb{Z}. \hfill \Box$$

We now study the complement of $\SU_C(2, L)$ in $\SU_C(2, L)$.

\textbf{Proposition 12.5.} For every irreducible nodal curve $C$ of genus $\geq 2$,

$$\text{codim}(\SU_C(2, L) \setminus \SU_C(2, L), \SU_C(2, L)) \geq 3.$$

\textbf{Proof.} If suffices to prove this in the case when $\deg L = d \gg 0$. The generic element of $\SU_C(2, L) \setminus \SU_C(2, L)$ is a torsion-free non-locally-free sheaf such that every section vanishes at a node. Therefore, it is a non-locally-free extension of the form

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \otimes (\mathcal{O}_C)^* \longrightarrow 0$$

for some node $p$ in $C$, i.e., the push-forward of an extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \pi_p^*L(-p_1 - p_2) \longrightarrow 0$$

via the partial normalization $\pi_p: C_p \to C$. A generic such extension $\mathcal{E}$ is in $\SU_{C_p}(2, \pi_p^*L(-p_1 - p_2))$, and there exists a morphism

$$\SU_{C_p}(2, \pi_p^*L(-p_1 - p_2)) \longrightarrow \SU_C(2, L)$$
defined by $E \mapsto (\pi_p)_*E$. It is a morphism because if $F \subseteq (\pi_p)_*E$ is a rank-1 torsion-free subsheaf of $(\pi_p)_*E$, then $\pi^*F/Tors \subseteq E$. Since $E$ is semi-stable, the degree of $\pi^*F/Tors$ is $\leq \deg E/2 = (d - 2)/2$. Therefore $\deg F \leq (d - 2)/2 + 1 = d/2$, and $(\pi_p)_*E$ is semi-stable.

The dimension of $SU_C(2, L) \setminus SU_C(2, L)$ is therefore less than or equal to the dimension of $SU_C(2, \pi_p Tors C(-p_1 - p_2))$, which is $3g - 6$ unless $g = 2$ and $\deg L$ is even when it is $1 = 3g - 5$. But if $g = 2$ and $\deg L$ is even then $SU_C(2, L) \cong SU_C(2, L) \simeq \mathbb{P}^3$ (see [Bho98]). Therefore,

$$\text{codim} (SU_C(2, L) \setminus SU_C(2, L), SU_C(2, L)) \geq (3g - 3) - (3g - 6) = 3.$$ 

\[\square\]

**Remark.** The proof of the proposition shows that, except for the case $g = 2$ and $\deg L$ even, $SU_C(2, L) \setminus SU_C(2, L)$ is the union of $|J|$ varieties of dimension $3g - 6$, the images of the morphisms $(\pi_p)_*$ described above, one for each node.

An immediate consequence of Corollary 12.4 and Proposition 12.5 is the following result, which was already proved by Bhosle (see [Bho99] and [Bho04]).

**Corollary 12.6.** If $C$ is an irreducible nodal curve of genus $\geq 2$ and $\deg L$ is odd, then $A_{3g-4}(SU_C(2, L)) \simeq \mathbb{Z}$.

Our last application is the following corollary.

**Corollary 12.7.** If $C$ is an irreducible nodal curve of genus $\geq 2$, then $A_{3g-4}(SU_C(2, L)) \simeq \mathbb{Z}$.

**Proof.** Bhosle proved in [Bho99] and [Bho04] that $A_{3g-4}(SU_C(2, L)) \simeq \mathbb{Z}$. The result follows from the exact sequence (see [Ful84, 1.8])

$$A_{3g-4}(SU_C(2, L) \setminus SU_C(2, L)) \rightarrow A_{3g-4}(SU_C(2, L)) \rightarrow A_{3g-4}(SU_C(2, L)) \rightarrow 0$$

and Proposition 12.5. \[\square\]

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