Abstract

Approximating the sum over all gauge field configurations in the QCD partition function by a liquid of instantons, we calculate the spectrum of the Dirac operator for two and three colors and for 0, 1 and 2 flavors. We find a remarkable difference in the spectrum near zero virtuality between 2 and 3 colors, which can be explained in terms of chiral random matrix theory. For two colors the Dirac operator is real, and the appropriate random matrix ensemble has real matrix elements. For three colors the Dirac operator is complex, and the spectrum can be described by a random matrix ensemble with complex matrix elements. These results provide further evidence that the spectrum near zero virtuality is universal and is completely determined by symmetries.
1. Introduction

It is widely believed that at some critical temperature the phase of QCD changes from a chirally broken phase to a chirally symmetric phase. The order parameter is the chiral condensate, which, according to the Banks-Casher formula [1], is proportional to the thermodynamic limit of the spectral density of the Dirac operator at zero virtuality. The study of this order parameter is the central objective of this paper. In particular, we will analyze the spectrum near zero virtuality via the microscopic limit which is obtained in the thermodynamic limit $V_4 \to \infty$ by rescaling the eigenvalues by $V_4$.

A natural way to perform these studies would be via lattice QCD calculations. However, the order parameter is defined in the chiral limit which makes such calculations with dynamical fermions infeasible. Instead, we approximate the integral over all gauge field configurations by a liquid of instantons and anti-instantons. In this approach the chiral limit does not give rise to technical problems.

Our main goal is to substantiate the claim [2] that spectral correlations on the scale of one average level spacing are universal. It is well known from the study of systems that are classically chaotic [3, 4] [5, 6] [7] that correlations between eigenvalues measured in units of the average level spacing are universal in the sense that they are solely determined by the symmetries of the system. Therefore they can as well be described by random matrix theory with only the symmetries as input. It is our conjecture that the same is true for correlations of the eigenvalues of the Dirac operator measured over the ensemble of all gauge field configurations. In QCD, though, we have an important additional symmetry: the chiral symmetry. As a consequence all nonzero eigenvalues occur in pairs $\pm \lambda$. Obviously, this symmetry has to be implemented in the random matrix model. What is important is that for these types of random matrix models, which we call chiral random matrix models, the point $\lambda = 0$ is special: the spectrum has a hard edge at $\lambda = 0$ [8, 9]. Therefore, correlations on the scale of one average level spacing can be defined with respect to this point. The microscopic spectral density is exactly such correlator and is expected to be universal. There are other arguments in favor of universality. Most notably, it was recently argued by Leutwyler and Smilga [10] that the low-energy effective partition function of QCD which is solely based on symmetries leads to an infinite family
of sum rules for the inverse powers of the eigenvalues of the Dirac operator. Since all sum rules can be derived from the microscopic correlators, this strongly suggest that the spectrum at this scale is determined by symmetries as well.

For the classical random matrix ensembles we have three universality classes: the GOE, the GUE and the GSE, depending on whether the matrix elements are real, complex or quaternion real, respectively [11]. The same is true for the chiral random matrix ensembles. Recently, we succeeded the classify the $SU(N_c)$ gauge theories accordingly [12]. For $N_c = 2$ and fundamental fermions all matrix elements of the Dirac operator can be made real, for $N_c \geq 3$ gauge groups with fundamental fermions the Dirac operator is complex, whereas for adjoint fermions, the Dirac operator can be regrouped into quaternions. The corresponding random matrix ensembles will be called the chiral orthogonal ensemble (chGOE), the chiral unitary ensemble (chGUE) and the chiral symplectic ensemble (chGSE), respectively. Each of the three cases can be identified with one of the three possible scenarios of chiral symmetry breaking [14]. Up to now, the microscopic spectral density has been obtained for the chGOE [15] and the chGUE [13].

As a final note in this introduction, we want to point out that the microscopic fluctuations become universal only after they are separated from all other variations in the spectrum. This procedure, called unfolding [3], will be applied to all our numerical results. It can only be achieved if a separation of scales takes place. We want to stress that this implicit assumption is a nontrivial point, that cannot be taken for granted.

In this paper we calculate the spectrum of the Dirac operator for $SU(2)$ and $SU(3)$ color gauge groups and compare the results with the universal functions obtained from random matrix theory. The microscopic spectral density is defined in section 2. The main ingredient of the instanton liquid model are discussed in section 3. The expressions for the microscopic spectral density are given in section 4. Numerical results are presented in section 5 and concluding remarks are made in section 6.

2. The microscopic spectral density

For $N_f$ flavors with masses $m_f \ (m_f \to 0)$ the QCD partition function in the sector
with \( \nu \) zero modes is defined by

\[
Z_{\nu}^{\text{QCD}}(m) = \langle \prod_{f=1}^{N_f} \prod_{\lambda_n > 0} (\lambda_n^2 + m_f^2)m_f^{\nu} \rangle_{S_\nu(A)},
\]

where the average is over gauge field configurations with \( \nu \) fermionic zero modes weighted by the gauge field action \( S_\nu(A) \). The product is over all eigenvalues of the Dirac operator. The spectral density of the QCD Dirac operator for a given field configuration is defined by

\[
\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n).
\]

The average spectral density \( \langle \rho(\lambda) \rangle \) is defined with respect to the partition function (1). According to the Banks-Casher formula [1], the chiral order parameter can be expressed as

\[
\langle \bar{\psi} \psi \rangle = \lim_{V_4 \to \infty} \frac{1}{V_4^4} \rho(0),
\]

where it is understood that the limit \( V_4 \to \infty \) is taken before the chiral limit \( m \to 0 \). Its existence implies that the eigenvalues near zero scale as \( 1/V_4 \). Therefore, the order parameter can be analyzed in much greater detail via the microscopic limit of the spectral density defined as

\[
\rho_S(z) = \lim_{V_4 \to \infty} \frac{1}{V_4^4} \langle \rho(zV_4) \rangle_{\nu}.
\]

Instead of a single number, our new order parameter is a function that not only gives the thermodynamic limit of \( \langle \bar{\psi} \psi \rangle \), but also provides information on \textit{how} the thermodynamic limit is approached.

3. Instanton liquid model

The sum over all gauge field configurations in (1) has only been performed exactly in lattice gauge theory. However, in calculations with dynamical fermions, the quark masses cannot be taken equal to zero, which makes it difficult to study the chiral limit of the microscopic spectral density. This problem is not present in our approach, where we...
approximate the sum over all gauge field configuration in (1) by a sum over instantons and anti-instantons.

In this approximation, the non-zero mode quantum fluctuations about the classical configurations are taken into account to one-loop order under the assumption that contributions from different instantons factorize. This is justified when the ensemble of instantons is sufficiently dilute, which according to phenomenological arguments \[16\], is indeed the case. The integral over the zero modes, *i.e.* the size, the position and the orientation of each of the pseudoparticles, is done exactly via a Monte Carlo simulation. Also the integral over the fermionic zero modes is performed exactly. The latter amounts to evaluating the fermion determinant in the space of fermionic zero modes under the assumption that it factorizes from the contribution of the non-zero modes. Again, this can be justified on the basis that the instanton ensemble is sufficiently dilute. The resulting Dirac operator is given by the \((N_I + N_A) \times (N_I + N_A)\) matrix

\[
\begin{pmatrix}
im_f & T \\
T^\dagger & \nim_f
\end{pmatrix},
\]

where the overlap matrix elements are given by

\[
T_{IA} = \int d^4x \psi_0^I(x) i \hat{D} \psi_0^A(x) = \frac{1}{2(\rho_I \rho_A)^{3/2}} \text{Tr} \left( \tau_\mu^+ \hat{R}_\mu^{IA} U_I^{-1} U_A \right) F(\lambda),
\]

Here, \(R_I, R_A\) are the positions of the instanton and the anti-instanton, \(\rho_I, \rho_A\), their sizes and \(U_I\) and \(U_A\) their orientations in color space. The vector \(\hat{R}_\mu^{IA}\) is a unit vector in the direction \((R_I - R_A)_\mu\), and we have introduced the notation \(\tau_\mu^+ = (\vec{\tau}, i)_\mu\) with \(\vec{\tau}\) are the Pauli spin matrices. The scalar function \(F(\lambda)\) depends on the specific ansatz for the gauge field configuration. We use the so called streamline field configurations \[17, 18\], which are optimal in the sense that the first derivative with respect to directions perpendicular to the collective coordinates vanishes. In this case \(\lambda\) is given by a conformal invariant combination of \(R_{IA}, \rho_I\) and \(\rho_A\). Asymptotically, we have

\[
F(\lambda) \sim 4 \frac{(\rho_I \rho_A)^{3/2}}{R_{IA}^3}
\]

At this moment we want to point out that for \(SU(2)\) all matrix elements of \(T\) are *real*, whereas for \(SU(3)\) and all larger gauge groups they are *complex*. As we will see below, this distinction has important consequences for the spectral correlations near zero virtuality.
Summarizing, the QCD partition function is approximated by

\[ Z = \int \prod_{i=1}^N dU_i dz_i d\rho_i \mu(\rho_i \Lambda) \prod_f \det(T^\dagger T + m_f^2) \exp(-\beta(\rho \Lambda) \sum_{i<j} S_{ij}^{int}), \]  

(8)

where \( \mu(\rho \Lambda) \) is size distribution as induced by the quantum fluctuations (\( \Lambda \) is the QCD scale parameter). The total density of instantons \( N/V_4 \) is kept fixed \( (N = N_I + N_A) \), and we always work in the sector of zero total topological charge. For the interaction of the instantons we use the streamline action supplemented by a hard core. The latter has been introduced in order to be able to simulate a stable liquid of instantons. In this way, instantons are simulated as they occur in the full QCD partition function as given by lattice QCD simulations [19]. For a more detailed discussion of the instanton liquid model described above we refer to refs. [20, 21, 22].

4. Random matrix theory

Our hypothesis is that the fluctuations of the eigenvalues of the Dirac operator near zero virtuality are universal and can be described by chiral random matrix theory. Using the instanton partition function as an inspiration [23, 24, 25] it is clear that the appropriate random matrix model in the sector with \( \nu \) fermionic zero modes is defined by [12]

\[ Z_{\beta,\nu}(m) = \int \mathcal{D}T P_\beta(T) \prod_f \det \left( \begin{array}{cc} m_f & iT \\ iT & m_f \end{array} \right), \]  

(9)

where \( T \) has the symmetries of the corresponding Dirac operator and the masses are in the chiral limit \( (m_f \to 0) \). As discussed above, depending on the universality class the matrix \( T \) is real \( (\beta = 1, \text{chGOE}) \), complex \( (\beta = 2, \text{chGUE}) \) or quaternion real \( (\beta = 4, \text{chGSE}) \). In the latter case the square root of the fermion determinant appears in (9). The matrix \( T \) is a rectangular \( n \times m \) matrix with \( |n - m| = \nu \) (for definiteness we take \( m > n \)), so that the 'Dirac operator' in (9) has exactly \( \nu \) zero modes. The function \( P(T) \) is chosen gaussian:

\[ P_\beta(T) = \exp \left( -\frac{\Sigma^2 \beta n}{2} \sum_{k=1}^n \lambda_k^2 \right). \]  

(10)
In this normalization, the chiral condensate (as given by (3)) equals $\Sigma$ in each of the three random matrix ensembles. The total number of modes is $N \equiv m + n$. The latter quantity is identified with the volume of space time.

The spectral density can be obtained by transforming to new integration variables in which $T$ is diagonal. For $\beta = 2$ the corresponding Jacobian can be rewritten in terms of generalized Laguerre polynomials. After some manipulations, which are well-known from random matrix theory [6], one obtains a formula for the spectral density in terms of these polynomials. From their asymptotic properties it follows that the microscopic limit is given by

$$\rho_S(z) = \frac{\Sigma^2 z}{2} (J_{N_f+\nu}(\Sigma z) - J_{N_f+\nu+1}(\Sigma z) J_{N_f+\nu-1}(\Sigma z)).$$

(11)

In a more modern language, this limit is a double scaling limit, which exists as a consequence of the fact that the Laguerre ensemble has a hard edge at zero [8, 9].

The situation for the chGOE is much more complicated. Because of the structure of the Jacobian, the standard orthogonal polynomial method no longer works. However, in this case it is possible to express the spectral density in terms of skew-orthogonal polynomials [25]. Remarkably, using a number of tricks from [26] it is also possible in this case to obtain an analytical result for the microscopic spectral density [15]. The result can be expressed as an integral over the Bessel kernel

$$\rho_S(z) = \frac{\Sigma}{4} J_{2a+1}(z\Sigma) + \frac{\Sigma}{2} \int_0^\infty dw(zw)^{2a+1} (z - w) \left( \frac{1}{w} \frac{d}{dw} - \frac{1}{z} \frac{d}{dz} \right)$$

$$\times \frac{w J_{2a}(z\Sigma) J_{2a-1}(w\Sigma) - z J_{2a-1}(z\Sigma) J_{2a}(w\Sigma)}{(zw)^{2a}(z^2 - w^2)}. \tag{12}$$

The parameter $a$ is defined by

$$a = N_f - \frac{1}{2} + \frac{\nu}{2}. \tag{13}$$

Recently, the Bessel kernel got a great deal of attention in the literature. For details and further references we refer to [8, 9].

5. Numerical results
All results have been obtained for an ensemble of 32 instantons and 32 anti-instantons confined to a box of $2.37^3 \times 4.74 \Lambda_{QCD}^{-4}$. For $N_f = 0$ averages were carried out for an ensemble of 100,000 configurations, whereas for $N_f = 1, 2$ the average was over 20,000 configurations. In the latter case, configurations are correlated due to the Monte-Carlo evolution, so that, effectively, the size of the ensemble is somewhat smaller.

As stated in the introduction, our hypothesis is that the scale of the microscopic fluctuations and the scale of the secular variations are independent. The two scales are separated via a procedure called unfolding. This procedure is not new. It has been widely used in the analysis of spectra of quantum systems in terms of random matrix ensembles [3, 4]. It is always the unfolded spectrum that has universal properties. In the present case, we first obtain the average spectral density $\bar{\rho}$, by fitting a smooth curve to the histogram of the eigenvalues of the Dirac operator. In our case we use a combination of a Gaussian and an exponential function. This function is then used to generate the unfolded spectrum $\{\lambda_n^U\}$ from the original spectrum $\{\lambda_n\}$ according to

$$\lambda_n^U = \int_0^{\lambda_n} \bar{\rho}(\lambda) d\lambda.$$  \hspace{1cm} (14)

The average spacing of the unfolded spectrum is unity.

In Figs. 1 and 2 the solid lines show histograms of the unfolded spectrum $\{\lambda_n^U\}$ for $N_c = 2$ and $N_c = 3$, respectively. These results are compared to the microscopic spectral density for the chGUE (dashed curve) and the chGOE (dotted curve). Results are given for 0, 1 and 2 flavors. In all cases, we find that the spectrum for $N_c = 2$ is described by the chGOE, whereas the spectrum for $N_c = 3$ is described by the chGUE. The only discrepancy is that the oscillations for $\beta = 2$ are not reproduced several level spacings away from zero. Since the positions of the peaks give the average location of the eigenvalues, this implies that level fluctuations are larger than expected according to random matrix theory. At this moment it is not clear whether this effect is a finite size effect that will go away in the thermodynamic limit. For more than two flavors the gap near zero becomes so big that the microscopic and the macroscopic scales can no longer be separated unambiguously.

6. Discussion
We have calculated the spectrum of the Dirac operator for an instanton liquid approximation to the QCD partition function for both two and three colors. For 0, 1 and 2 flavors, we have found that the spectrum near zero virtuality is different in both cases. In all cases the microscopic limit of the spectrum, obtained in the thermodynamic limit by rescaling the eigenvalues $\sim V_4$, can be described by chiral random matrix theory with only the symmetries of the Dirac operator as input. For two colors the Dirac operator is real and the appropriate ensemble, the chiral orthogonal ensemble, has real matrix elements. For three colors the Dirac operator is complex. Now the appropriate ensemble, the chiral unitary ensemble, has complex matrix elements. This confirms our universality hypothesis that the spectrum near zero virtuality is entirely determined by symmetries. In a sense, this is natural because this region of the spectrum carries information on the very long wavelength excitations which are determined by the global symmetries of the QCD partition function. Indeed, the microscopic correlation functions are consistent with the static limit of the QCD partition function.

As has been argued in [12], there is a third universality class, where the matrix elements of the chiral random matrix ensemble are quaternion real. This case is realized for fermions in the adjoint representation. It would of great interest to subject this case to numerical studies as well.

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Figure Captions

Fig. 1. The microscopic spectral density $\rho_S(z)$ versus the virtuality $z$ measured in units of the average level spacing. The results are for the color gauge group SU(2). The full line shows the histogram of our numerical calculations. Analytical results for the chGOE and the chGUE are represented by dotted and dashed curves, respectively. For the number of flavors we refer to the labels in the figure.

Fig. 2. The microscopic spectral density $\rho_S(z)$ versus the virtuality $z$ measured in units of the average level spacing. The results are for the color gauge group SU(3). Further explanation can be found in the caption of Fig. 1.