Thermal quantum field theory and the Casimir interaction between dielectrics

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Abstract

The Casimir and van der Waals interaction between two dissimilar thick dielectric plates is reconsidered on the basis of thermal quantum field theory in Matsubara formulation. We briefly review two main derivations of the Lifshitz formula in the framework of thermal quantum field theory without use of the fluctuation-dissipation theorem. A set of special conditions is formulated under which these derivations remain valid in the presence of dissipation. The low-temperature behavior of the Casimir and van der Waals interactions between dissimilar dielectrics is found analytically from the Lifshitz theory for both an idealized model of dilute dielectrics and for real dielectrics with finite static dielectric permittivities. The free energy, pressure and entropy of the Casimir and van der Waals interactions at low temperatures demonstrate the same universal dependence on the temperature as was previously discovered for ideal metals. The entropy vanishes when temperature goes to zero proving the validity of the Nernst heat theorem. This solves the long-standing problem on the consistency of the Lifshitz theory with thermodynamics in the case of dielectric plates. The obtained asymptotic expressions are compared with numerical computations for both dissimilar and similar real dielectrics and found to be in excellent agreement. The role of the zero-frequency term in Matsubara sum is investigated in the case of dielectric plates. It is shown that the inclusion of conductivity in the model of dielectric response leads to the violation of the Nernst heat theorem. The applications of this result to the topical problems of noncontact atomic friction and the Casimir interaction between real metals are discussed.

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I. INTRODUCTION

Both the Casimir and van der Waals interactions are the quantum phenomena caused by fluctuating electromagnetic fields. In the framework of quantum field theory, these interactions can be described through the alteration by the material boundaries of the zero-point electromagnetic energy that pervades all of space. Thus, the Casimir and van der Waals forces are found to be closely connected with the fundamental properties of quantum vacuum (see the original Ref. [1] and monographs [2, 3, 4, 5]).

In the last few years the Casimir interaction has been actively investigated in connection with topical applications in extra-dimensional physics (where it provides an effective mechanism for spontaneous compactification of extra spatial dimensions) and in the bag model of hadrons [3, 6]. Several measurements of the Casimir and van der Waals forces between metal macrobodies were performed with increased precision using modern laboratory techniques [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The experimental results were used to obtain stronger constraints on the Yukawa-type long-range interaction, predicted by many extensions to the Standard Model, in the micrometer interaction range [15, 16, 17, 19, 20, 21, 22, 23, 24, 25]. Concurrent with the fundamental applications in elementary particle physics, the use of the Casimir and van der Waals forces in nanotechnology [26, 27, 28], in quantum reflection and Bose-Einstein condensation [29, 30] and in noncontact atomic friction [31, 32, 33] was begun.

The numerous applications of the Casimir and van der Waals forces and the extensive experimental work impose severe demands on the accuracy of theoretical predictions. The basic theory of these forces acting between real materials was developed by E. M. Lifshitz and his collaborators [34, 35, 36]. It expresses the free energy and force acting between two macrobodies in terms of their frequency-dependent dielectric permittivities. The original derivation of the Lifshitz formulas for the free energy and pressure between two thick dielectric plates is based on the concept of a fluctuating electromagnetic field and uses the fluctuation-dissipation theorem. If the temperature of plates is not equal to zero, this concept includes both zero-point oscillations and thermal fluctuations of the field. In general, the Lifshitz derivation is applicable to both transparent and absorbing media. At present time, there are derivations of the Lifshitz formula based on the thermal quantum field theory in the Matsubara formulation (see Ref. [6] for a review). They do not employ the
fluctuation-dissipation theorem but lead to results identical to those obtained with the use of this theorem. Thermal quantum field theory is immediately applicable to only transparent bodies, but, under certain conditions, dissipation can also be included.

Quite independently from the Lifshitz formula, the Matsubara thermal quantum field theory was applied [37, 38] for the investigation of the Casimir force acting between two plane parallel plates made of ideal metal (at zero temperature this case was considered already in Ref. [1]). This is an exactly solvable problem of field quantization with the Dirichlet boundary conditions on the surface of the plates for the tangential component of electric field and with the identification condition in the Euclidean time variable. The obtained analytic expressions for the Casimir pressure, free energy and entropy were found to be in perfect agreement with thermodynamics. In particular, the magnitudes of the free energy and pressure were found to be monotonously increasing functions with the increase of the temperature. Entropy is nonnegative and goes to zero when temperature vanishes; this is in accordance with the third law of thermodynamics (the Nernst heat theorem) [39]. For the first time, the Lifshitz formula was considered to be in contradiction with the quantum field theoretical approach to the case of ideal metals since it leads to a different result in the limit of infinitely large dielectric permittivity. The situation was clarified by the so called Schwinger’s prescription [5, 40] whereby the limit of infinite dielectric permittivity in the Lifshitz formula must be taken prior to putting the Matsubara frequency equal to zero. Once that prescription is followed, the results obtained for ideal metals from the Lifshitz formula match those obtained from thermal quantum field theory with the Dirichlet boundary conditions.

Currently the Lifshitz formula leads to problems when applied to the case of two plates made of real metals at nonzero temperature. In Refs. [41, 42, 43, 44] metallic plates were characterized with the dielectric permittivity of the Drude model. In the limit of ideal metals this approach comes into conflict with the values of the Casimir free energy and pressure obtained using thermal quantum field theory with the Dirichlet boundary conditions. As was also proved in Refs. [45, 46], the substitution of the Drude dielectric function in the Lifshitz formula leads to a violation of the third law of thermodynamics in the case of perfect crystal lattice with no defects and impurities (as discussed above, the quantum field theory approach obeys this law in the case of ideal metals).

Another way to describe the realistic properties of a metal is to use the dielectric per-
mittivity of the free electron plasma model \[40, 47, 48\] or to impose the surface-impedance boundary conditions \[46, 49\]. In the case of ideal metals, both these approaches are in agreement with the results obtained from using thermal quantum field theory. They also obey the third law of thermodynamics. Recent experiments permit to test different theoretical approaches to the thermal Casimir force. It is significant that the first modern measurement of the Casimir force \[7\] was found \[50, 51\] to be in disagreement with Refs. \[41, 42, 43, 44\] and consistent with Ref. \[49\]. The most precise and accurate experiments of Refs. \[15, 16, 17\] exclude the theoretical approach of Refs. \[41, 42, 43, 44\] at 99% confidence and are consistent with Refs. \[46, 47, 48, 49\]. (Note that there is some controversy in the literature on the agreement of different approaches with thermodynamics and experiment which is reflected in recent Refs. \[52, 53\].)

As is evident from the foregoing, there are many questions in the Lifshitz theory of the Casimir and van der Waals interactions that remain to be answered. On the one hand, the consistency of the Lifshitz theory with thermal quantum field theory is not completely understood, and, on the other hand, the agreement of the Lifshitz theory with thermodynamics is called into question. An important point is that some thermodynamic aspects of the Lifshitz theory are still unknown even in the case of two dielectric plates for which consensus in the literature is achieved. One of the major problems realized 50 years back, but not resolved up to the present, is the elucidation of the low-temperature behavior of the Casimir (van der Waals) free energy, pressure and entropy between two dielectric plates. Without a resolution of this fundamental issue it would be impossible to settle the more complicated problems related to real metals. Even the correct way of practical computations for comparison of experimental results with theory would be uncertain.

In the present paper we reconsider the thermal van der Waals and Casimir interactions between two thick dissimilar dielectric plates. We start with a brief analysis of two derivations of the Lifshitz formula in the framework of thermal quantum field theory in the Matsubara formulation. Special attention is paid to the conditions under which the quantum field theoretical derivations are applicable not only to transparent media but also to media with dissipation. We next consider the Casimir effect between two plates made of dissimilar dilute dielectrics with constant dielectric permittivities \(\varepsilon^{(k)} (k = 1, 2)\). In this case it is possible to develop the perturbation theory in the small parameters \(\varepsilon^{(k)} - 1\) and to obtain the explicit analytic expressions for the Casimir free energy and pressure which are...
exact in the separation between the plates and their temperature. The simple asymptotic
behavior of the free energy, pressure and entropy at both low and high temperatures (short
and large separations, respectively) is also found. The leading terms of the Casimir entropy
at low temperature are shown to be of the second and third power in temperature, thus,
demonstrating the agreement of the Lifshitz formula with thermodynamics in the case of
dilute dielectrics.

As a next step, we derive analytically the low-temperature behavior of the Casimir and
van der Waals forces between dissimilar dielectrics using the frequency-dependent dielec-
tric permittivities. For this purpose, perturbation theory in a small parameter which is
proportional to the product of the separation distance between the plates and their tem-
perature is developed. It is proved that the thermal corrections to the van der Waals and
Casimir energy and pressure have an universal form and their asymptotic behavior can be
calculated in terms of the static dielectric permittivities of the plate materials. Once again,
the leading contributions to the entropy are shown to be of the second and third power in
temperature which proves the agreement of the Lifshitz formula with thermodynamics in the
case of dielectric plates with finite static dielectric permittivities. In the limit $\varepsilon^{(k)} - 1 \ll 1$
the above results for dilute dielectrics are again obtained. Results for the special case of
similar dielectrics are also provided. This means that the long-standing problem on the
low-temperature properties of the Lifshitz theory in the case of dielectric plates is resolved.

In what follows we perform the comparison between the obtained analytic results and
numerical computations for dielectric plates made of silicon and vitreous silica. The cases of
both dissimilar and similar plates are considered. The dielectric permittivities of silicon and
vitreous silica along the imaginary frequency axis are found by means of the dispersion rela-
tion using the tabulated optical data for both materials. In all cases the excellent agreement
between the analytic and numerical results is observed below some definite temperature
(separation) values.

Finally we discuss the role of the zero-frequency term in the Matsubara sum for dielectrics.
To calculate the van der Waals friction, Refs. 32, 54 use the Lifshitz-type formula and
describe the dielectric permittivity of the dielectric substrate by means of the Drude model
with appropriately low conductivity. Formally, this leads to infinitely high permittivity of a
dielectric at zero frequency and to the modification of the zero-frequency term of the Lifshitz
formula. We demonstrate that such a modification results in the violation of the Nernst heat
theorem. Hence, it follows that the dc conductivity of dielectrics is irrelevant to the van der Waals and Casimir forces and must not be included in the model of dielectric response. This conclusion leads to important consequences for both the problem of atomic friction and for the Casimir and van der Waals interactions between real metals.

The paper is organized as follows. In Sec. II, two main derivations of the Lifshitz formula for the free energy in the framework of thermal quantum field theory are briefly discussed. Special attention is paid to the restrictions imposed on the type of relaxation. Sec. III is devoted to the case of dissimilar dilute dielectrics. In Sec. IV, we derive the low-temperature behavior of the van der Waals and Casimir interactions between dissimilar dielectrics with frequency-dependent dielectric permittivities. Here, the consistency of the Lifshitz theory with thermodynamic requirements is proved. In Sec. V the obtained analytic results are compared with the results of numerical computations performed for some real dielectrics. Sec. VI demonstrates that the inclusion of the conductivity of dielectrics in the model of dielectric response leads to the violation of the Nernst heat theorem. In Sec. VII we present our conclusions and discussion touching on the topical problems of the noncontact atomic friction and the Casimir interaction between real metals. Appendices A, B and C contain the mathematical proofs of some statements used in Secs. IV and VI.

II. DERIVATIONS OF THE LIFSHITZ FORMULA IN MATSUBARA QUANTUM FIELD THEORY

We consider two thick dielectric plates (semispaces) with the frequency-dependent dielectric permittivities $\varepsilon^{(1)}(\omega)$ and $\varepsilon^{(2)}(\omega)$, restricted by the two parallel planes $z = \pm a/2$ with separation $a$ between them, in thermal equilibrium at temperature $T$.

The original Lifshitz derivation of his formula [34] was based on the assumption that the dielectric materials are characterized by randomly fluctuating sources of long wavelength electromagnetic fields. This concept includes not only the thermal fluctuations but also the zero-point oscillations of the field [36]. Lifshitz based his derivation on the fluctuation-dissipation theorem using the properties of electromagnetic fluctuations.

There are two derivations of the Lifshitz formula at nonzero temperature in the framework of thermal quantum field theory without use of the fluctuation-dissipation theorem. The first of them is due to M. Bordag [6] and is based on the scattering approach [55]. An
electromagnetic wave $e^{ikz}$ coming from the negative part of $z$-axis in the dielectric with the permittivity $\varepsilon^{(1)}(\omega)$ will be scattered on the empty gap between the two semispaces and there occurs a transmitted and a reflected wave,

$$\varphi_1(z) \sim e^{ikz} + s_{12}e^{-ikz},$$
$$\varphi_1(z) \sim s_{11}e^{ikz}. \quad (1)$$

A similar situation holds for the linear independent wave $e^{-ikz}$ coming from the positive part of $z$-axis

$$\varphi_2(z) \sim s_{22}e^{-ikz},$$
$$\varphi_2(z) \sim e^{-ikz} + s_{21}e^{ikz}. \quad (2)$$

The matrix $\{s_{ij}\}$ composed of the coefficients in Eqs. (1) and (2) is unitary.

For the evaluation of the free energy one has to use the Euclidean version of the field theory obtained by Wick rotation $x_0 \to ix_4$ with the electromagnetic field periodic in the Euclidean time variable with a period $\beta = \hbar/(k_BT)$, where $k_B$ is the Boltzmann constant. Starting from the usual representation of the free energy $F$ in thermal quantum field theory at the one loop level, we arrive at the result (see Ref. [6] for details)

$$F(a, T) = -\frac{\hbar}{\beta} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\delta_{l0}\right) \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} \times \left[\ln s^\parallel_{11}(i\xi_l, k_\perp) + \ln s^\perp_{11}(i\xi_l, k_\perp)\right]. \quad (3)$$

Here $s^\parallel,^\perp_{11}$ are the elements of the scattering matrix for the two independent polarizations of electromagnetic field, $k_\perp = |k_\perp|$ is the magnitude of the wave vector in the plane of plates, and $\xi_l = 2\pi k_BTl/\hbar$ are the Matsubara frequencies.

To obtain the explicit expressions for the elements of the scattering matrix, we assume that the electric field satisfy the Maxwell equations and the corresponding boundary conditions on the boundary surfaces $z = \pm a/2$ (i.e., that the normal components of $D^{(k)} = \varepsilon^{(k)}E^{(k)}$ and tangential components of $E^{(k)}$ are continuous). By solving the scattering problem with the demand that for infinitely remote plates both the free energy and pressure are equal to
zero, one obtains

\[
s_{ll}^{\parallel}(i\xi_l, k_\perp) = \left(1 - \frac{\varepsilon_l^{(1)} q_l - k_l^{(1)}}{\varepsilon_l^{(1)} q_l + k_l^{(1)}} \frac{\varepsilon_l^{(2)} q_l - k_l^{(2)}}{\varepsilon_l^{(2)} q_l + k_l^{(2)}} e^{-2a q_l} \right)^{-1},
\]

\[
s_{ll}^{\perp}(i\xi_l, k_\perp) = \left(1 - \frac{k_l^{(1)} - q_l}{k_l^{(1)} + q_l} \frac{k_l^{(2)} - q_l}{k_l^{(2)} + q_l} e^{-2a q_l} \right)^{-1},
\]

where

\[
q_l = \sqrt{\varepsilon_l^2 + k_\perp^2}, \quad k_l^{(k)} = \sqrt{\varepsilon_l^{(k)}(i\xi_l) \frac{\varepsilon_l^2}{c^2} + k_\perp^2}.
\]

An important point is that the Maxwell equations supplemented by the corresponding boundary conditions lead to a definite solution of the scattering problem, given by Eqs. (4) and (5), only for \(l \geq 1\). For \(l = 0\), however, \(q_0 = k_0^{(k)} = k_\perp\) and, as a result, the respective system of linear algebraic equations has infinitely many solutions, i.e., it is satisfied for any \(s_{ll}^{\perp}(0, k_\perp)\) [56]. The definite value of \(s_{ll}^{\perp}(0, k_\perp)\) is obtained by the use of the unitarity condition resulting in \(|s_{ll}^{\perp}(0, k_\perp)| = 1\) and dispersion relation leading to \(s_{ll}^{\perp}(0, k_\perp) = 1\). We will return to this point below when discussing the role of dissipation.

Substituting Eq. (4) in Eq. (3), we arrive at the Lifshitz formula for the free energy of the van der Waals and Casimir interaction

\[
\mathcal{F}(a, T) = \frac{k_B T}{2\pi} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2} \delta_{l0} \right) \int_0^{\infty} k_\perp dk_\perp \times \left\{ \ln \left[ 1 - r_{ll}^{(1)}(\xi_l, k_\perp) r_{ll}^{(2)}(\xi_l, k_\perp) e^{-2a q_l} \right] + \ln \left[ 1 - r_{\perp l}^{(1)}(\xi_l, k_\perp) r_{\perp l}^{(2)}(\xi_l, k_\perp) e^{-2a q_l} \right] \right\},
\]

where the reflection coefficients are defined by

\[
r_{ll}^{(k)}(\xi_l, k_\perp) = \frac{\varepsilon_l^{(k)} q_l - k_l^{(k)}}{\varepsilon_l^{(k)} q_l + k_l^{(k)}},
\]

\[
r_{\perp l}^{(k)}(\xi_l, k_\perp) = \frac{k_l^{(k)} - q_l}{k_l^{(k)} + q_l}.
\]

The second field theoretical approach to the derivation of the Lifshitz formula at nonzero temperature without recourse to the fluctuation-dissipation theorem is based on the direct summation of the free energies of all photon oscillator modes [49] (at zero temperature this method was proposed in Refs. [57, 58]; see also the generalizations in Refs. [59, 60, 61]).

Equations for the determination of the frequencies of oscillator modes between two dissimilar plates are obtained from the Maxwell equations with the corresponding boundary
conditions on the boundary planes $z = \pm a/2$. They are given by\cite{60, 61}

$$\Delta_\parallel (\omega, k_\perp) \equiv \left[ k^{(1)} + \varepsilon^{(1)}(\omega)q \right] \left[ k^{(2)} + \varepsilon^{(2)}(\omega)q \right] e^{aq}$$

$$- \left[ k^{(1)} - \varepsilon^{(1)}(\omega)q \right] \left[ k^{(2)} - \varepsilon^{(2)}(\omega)q \right] e^{-aq} = 0,$$

$$\Delta_\perp (\omega, k_\perp) \equiv (k^{(1)} + q) (k^{(2)} + q) e^{aq}$$

$$- (k^{(1)} - q) (k^{(2)} - q) e^{-aq} = 0,$$

(8)

where

$$k^{(k)} = \sqrt{k_\perp^2 - \varepsilon^{(k)}(\omega)\omega^2/c^2}, \quad q = \sqrt{k_\perp^2 - \omega^2/c^2}.$$  

(9)

The solutions of Eq. (8) can be denoted as $\omega_{k_\perp, n}^\parallel$ and $\omega_{k_\perp, n}^\perp$. The free energy of one oscillator mode is given by

$$F_{k_\perp, n}^\parallel (a, T) = \frac{\hbar \omega_{k_\perp, n}^\parallel}{2} + k_B T \ln \left( 1 - e^{\frac{-\hbar \omega_{k_\perp, n}^\parallel}{k_B T}} \right)$$

$$= k_B T \ln \left( 2 \sinh \frac{\hbar \omega_{k_\perp, n}^\parallel}{2k_B T} \right).$$  

(10)

After summation over all quantum numbers, we obtain

$$F(a, T) = \frac{k_B T}{2\pi} \sum_n \int_0^\infty k_\perp dk_\perp \left[ \ln \left( 2 \sinh \frac{\hbar \omega_{k_\perp, n}^\parallel}{2k_B T} \right) \right.$$  

$$+ \ln \left( 2 \sinh \frac{\hbar \omega_{k_\perp, n}^\perp}{2k_B T} \right) \right].$$  

(11)

Eq. (11) can be identically represented by the use of the argument theorem in terms of the quantities $\Delta_\parallel (\omega, k_\perp)$ and $\Delta_\perp (\omega, k_\perp)$ from Eq. (8) having their roots at the frequencies of the oscillator modes. After transformation of the branch points into poles by means of integration by parts and calculation of the residues at all imaginary Matsubara frequencies $i\xi_l$, we rewrite Eq. (11) in the form (see Ref. \cite{49} for details)

$$F(a, T) = \frac{k_B T}{2\pi} \sum_{l=0}^\infty \left( 1 - \frac{1}{2} \delta_{l0} \right) \int_0^\infty k_\perp dk_\perp \times \left[ \ln \Delta_\parallel (i\xi_l, k_\perp) + \ln \Delta_\perp (i\xi_l, k_\perp) \right].$$  

(12)

Expression (12) is infinite. To remove the divergences, we subtract from the right-hand side of Eq. (12) the free energy in the case of infinitely remote plates ($a \to \infty$). Then Eq. (6) with the notation (7) is reobtained.
Both quantum field theoretical derivations of the Lifshitz formula discussed above are immediately applicable for dielectrics described by real dielectric permittivities $\varepsilon^{(k)}(\omega)$. In this case the unitarity condition in the first derivation is valid and the photon oscillator modes in the second derivation are real quantities. Both derivations, however, under certain conditions, can be generalized to the case of media with dissipation [3, 62]. In so doing a medium under the influence of electromagnetic oscillations can be represented as a set of oscillators

$$\frac{d^2x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t),$$

(13)

where $\gamma$ is a damping parameter and $f(t) = f_\omega \exp(-i\omega t)$ is the harmonically changing external force. Eq. (13) has the solution $x(t) = x_\omega \exp(-i\omega t)$ where $x_\omega = \chi(\omega)f_\omega$, and

$$\chi(\omega) = \frac{1}{\omega_0^2 - i\gamma\omega - \omega^2}$$

(14)

is the susceptibility of the system. The latter is connected with the dielectric permittivity of a medium

$$\varepsilon(\omega) = 1 + \frac{2g}{\pi} \chi(\omega),$$

(15)

where $g$ is the oscillator strength.

The dielectric permittivity (15) is complex and describes an absorption band (or bands if one considers several oscillators with different parameters) of finite width and amplitude at nonzero frequency. Note that in thermal equilibrium net heat losses are absent on average [62, 63]. One can conclude that for such media all processes of absorption are balanced by the respective processes of emission, and the global unitarity condition remains valid.

Regarding the second field theoretical approach described above, the oscillator frequency, determined from the equation $1/\chi(\omega) = 0$, becomes complex when $\gamma \neq 0$. In this case the free energy is not given by Eq. (11) (which is already clear from the complexity of the right-hand side of this equation). For the complex dielectric permittivity given by Eq. (15) the correct expression for the free energy is obtained from the auxiliary electrodynamic problem and leads to Eqs. (12) and (6) (see Ref. [62] for details). The qualitative reason for the validity of this statement is that the free energy depends only on the values of the dielectric permittivity along the imaginary frequency axis [i.e., only on $\varepsilon(i\xi)$] which are always real. Note that these considerations are not applicable to real metals described by the Drude dielectric function because in this case the proper frequency $\omega_0$ of the respective oscillator
in Eq. (13) turns into zero and the absorption band is shifted to zero frequency and achieves an infinite amplitude. As a result, dissipation leads to heating of a metal and the unitarity condition is violated (see Refs. 49, 56).

III. CASIMIR EFFECT FOR TWO PLATES MADE OF DISSIMILAR DILUTE DIELECTRICS

In this section we anticipate that the dielectric permittivities of both plates \((k = 1, 2)\) are constant and equal to \(\varepsilon^{(k)} = 1 + \eta_k\), where \(\eta_k \ll 1\). The assumption that the dielectric permittivity does not depend on frequency implies that the separation distance \(a\) is rather large (in fact larger than the characteristic absorption wavelength) and, consequently, relativistic effects are essential [36]. Thus, in this case we are dealing with the Casimir effect.

For convenience in analytic and numerical calculations, we introduce the dimensionless variables \(\zeta\) and \(y\) given by

\[
\zeta_l = \frac{\xi_l}{\xi_c} = \frac{2a \xi_l}{c} = \tau l, \quad y = 2q_l a,
\]

where \(\xi_c = c/(2a)\) is the so called characteristic frequency, \(\tau = 4\pi k_B a T/\hbar c\), and \(q_l\) was defined in Eq. (5). Then the Lifshitz formula (6) takes the form

\[
\mathcal{F}(a, T) = \frac{\hbar c \tau}{32\pi^2 a^3} \sum_{l=0}^{\infty} \left( 1 - \frac{1}{2} \delta_{l0} \right) \int_{\zeta_l}^{\infty} y dy \times \left\{ \ln \left[ 1 - r^{(1)} (\zeta_l, y) r^{(2)} (\zeta_l, y) e^{-y} \right] + \ln \left[ 1 - r^{(1)}_\perp (\zeta_l, y) r^{(2)}_\perp (\zeta_l, y) e^{-y} \right] \right\}.
\]

(17)

Here the reflection coefficients (7) are expressed in terms of new variables (16) as follows:

\[
\begin{align*}
  r^{(k)} (\zeta_l, y) &= \frac{\varepsilon^{(k)}_l y - \sqrt{y^2 + \zeta^2 (\varepsilon^{(k)}_l - 1)}}{\varepsilon^{(k)}_l y + \sqrt{y^2 + \zeta^2 (\varepsilon^{(k)}_l - 1)}} , \\
  r^{(k)}_\perp (\zeta_l, y) &= \frac{\sqrt{y^2 + \zeta^2 (\varepsilon^{(k)}_l - 1) - y}}{\sqrt{y^2 + \zeta^2 (\varepsilon^{(k)}_l - 1) + y}}.
\end{align*}
\]

(18)

Note that \(\varepsilon^{(k)}_l \equiv \varepsilon^{(k)}(i\xi_l) = \varepsilon^{(k)}(i\zeta_l \xi_c)\). In the case of constant dielectric permittivities, which are under consideration in this section, \(\varepsilon^{(k)}_l = \varepsilon^{(k)}\), i.e., being independent on \(l\).
Substitution of $\varepsilon^{(k)} = 1 + \eta_k$ in Eq. (17) and expansion of the logarithms in powers of the small parameters $\eta_1, \eta_2$ (preserving all powers up to order 3 inclusive) leads to

\[
\mathcal{F}(a, T) = -\frac{\hbar c T \eta_1 \eta_2}{256 \pi^2 a^3} \sum_{l=0}^{\infty} \left( 1 - \frac{1}{2} \delta_0 \right) \left[ I_1(\zeta_l) - \frac{\eta_1 + \eta_2}{2} I_2(\zeta_l) \right],
\]

(19)

where

\[
I_1(\zeta_l) \equiv \int_{\zeta_l}^{\infty} dy \frac{e^{-y}}{y^3} \left( 2y^4 - 2\zeta_l^2 y^2 + \zeta_l^4 \right)
\]

\[
= e^{-\zeta_l} \left( 2 + 2\zeta_l - \frac{\zeta_l^2}{2} - \frac{\zeta_l^3}{2} \right) + \zeta_l^2 \left( 2 - \frac{\zeta_l^2}{2} \right) \text{Ei}(-\zeta_l),
\]

\[
I_2(\zeta_l) \equiv \int_{\zeta_l}^{\infty} dy \frac{e^{-y}}{y^5} \left( 2y^6 - \zeta_l^2 y^4 - \zeta_l^4 y^2 + \zeta_l^6 \right)
\]

\[
= e^{-\zeta_l} \left( 2 + 2\zeta_l - \frac{\zeta_l^2}{4} - \frac{5\zeta_l^3}{12} + \frac{\zeta_l^4}{24} + \frac{\zeta_l^5}{24} \right)
\]

\[
+ \zeta_l^2 \left( 1 - \frac{\zeta_l^2}{2} - \frac{\zeta_l^4}{24} \right) \text{Ei}(-\zeta_l),
\]

and $\text{Ei}(z)$ is the exponential integral function.

Summation in Eq. (19) can be performed in the following way:

\[
f_1(\tau) \equiv \sum_{l=0}^{\infty} \left( 1 - \frac{1}{2} \delta_0 \right) I_1(\zeta_l)
\]

\[
= 1 + 2 \frac{(1 + \tau) e^\tau - 1}{(e^\tau - 1)^2} + \frac{\tau^2 e^\tau [(1 - \tau) e^{2\tau} - 4\tau e^\tau - \tau - 1]}{2(e^\tau - 1)^4}
\]

\[
+ \sum_{l=1}^{\infty} F_1(\zeta_l),
\]

\[
f_2(\tau) \equiv \sum_{l=0}^{\infty} \left( 1 - \frac{1}{2} \delta_0 \right) I_2(\zeta_l)
\]

\[
= 1 + 2 \frac{(1 + \tau) e^\tau - 1}{(e^\tau - 1)^2} + \frac{\tau^2 e^\tau [(5\tau - 3) e^{2\tau} + 20\tau e^\tau + 5\tau + 3]}{12(e^\tau - 1)^4}
\]

\[
- \frac{\tau^3 e^\tau [(\tau - 1) e^{4\tau} + 2(13\tau - 5) e^{3\tau} + 66\tau e^{2\tau} + 2(13\tau + 5) e^\tau + \tau + 1]}{24(e^\tau - 1)^6}
\]

\[
+ \sum_{l=1}^{\infty} F_2(\zeta_l),
\]
where
\[
F_1(\zeta_l) = \zeta_l^2 \left( 2 - \frac{\zeta_l^2}{2} \right) \text{Ei}(-\zeta_l),
\]
\[
F_2(\zeta_l) = \zeta_l^2 \left( 1 + \frac{\zeta_l^2}{2} - \frac{\zeta_l^4}{24} \right) \text{Ei}(-\zeta_l).
\]

Thus, the exact (in \(\tau\)) expression for the Casimir free energy for the configuration of two dissimilar dilute dielectric plates is given by
\[
\mathcal{F}(a, T) = -\frac{\hbar c \tau \eta_1 \eta_2}{256 \pi^2 a^3} \left[ f_1(\tau) - \frac{\eta_1 + \eta_2}{2} f_2(\tau) \right],
\]
where \(f_{1,2}(\tau)\) are defined in Eqs. (21)–(23).

It is not difficult to find the asymptotic behavior of Eq. (24) at \(\tau \ll 1\). For this purpose the remaining sums on the right-hand side of Eqs. (21), (22) are calculated with the help of the Abel-Plana formula
\[
\sum_{l=0}^{\infty} \left( 1 - \frac{1}{2} \delta_{l0} \right) F(l) = \int_0^\infty F(t) dt + i \int_0^\infty dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1},
\]
where \(F(z)\) is an analytic function in the right half-plane. Taking into account that \(F_{1,2}(0) = 0\), these sums are equal to
\[
\sum_{l=1}^{\infty} F_{1,2}(\zeta_l) = \int_0^\infty F_{1,2}(\tau t) dt + i \int_0^\infty dt \frac{F_{1,2}(i\tau t) - F_{1,2}(-i\tau t)}{e^{2\pi t} - 1}.
\]

Using Eq. (23) one obtains
\[
\int_0^\infty F_1(\tau t) dt = \frac{16}{15\tau}, \quad \int_0^\infty F_2(\tau t) dt = \frac{128}{105\tau}.
\]

The second integral on the right-hand side of Eq. (26) can be calculated perturbatively. From Eq. (23) it follows:
\[
F_1(i\tau t) - F_1(-i\tau t) = -2i\pi^2 t^2 + 4i\tau^3 t^3 + O(\tau^4),
\]
\[
F_2(i\tau t) - F_2(-i\tau t) = -i\pi^2 t^2 + 2i\tau^3 t^3 + O(\tau^4).
\]
Substituting Eqs. (27) and (28) into Eq. (26), we arrive at

\[ \sum_{l=1}^{\infty} F_1(\zeta_l) = \frac{16}{15\tau} + \frac{\zeta(3)}{2\pi^2} \tau^2 - \frac{\tau^3}{60} + O(\tau^4), \]  

\[ \sum_{l=1}^{\infty} F_2(\zeta_l) = \frac{128}{105\tau} + \frac{\zeta(3)}{4\pi^2} \tau^2 - \frac{\tau^3}{120} + O(\tau^4), \]  

where \( \zeta(z) \) is the Riemann zeta-function. After expansion of the remaining terms on the right-hand sides of Eqs. (21), (22) in powers of \( \tau \), one obtains

\[ f_1(\tau) = \frac{46}{15\tau} + \frac{\zeta(3)}{2\pi^2} \tau^2 - \frac{7\tau^3}{360} + O(\tau^4), \]  

\[ f_2(\tau) = \frac{338}{105\tau} + \frac{\zeta(3)}{4\pi^2} \tau^2 + \frac{\tau^3}{360} + O(\tau^4). \]  

The substitution of Eq. (30) in Eq. (24) leads to the final result

\[ \mathcal{F}(a, T) = -\frac{\hbar c \eta_1 \eta_2}{256\pi^2 a^3} \left[ \frac{46}{15} + \frac{\zeta(3)}{2\pi^2} \tau^3 - \frac{7\tau^4}{360} \right], \]  

The asymptotic behavior of the Casimir free energy at \( \tau \gg 1 \) (high temperatures or large separations) also can be obtained from Eq. (24). With the proviso that \( \tau \gg 1 \), Eqs. (21) and (22) lead to \( f_1(\tau) = f_2(\tau) = 1 \) up to exponentially small corrections. As a result

\[ \mathcal{F}(a, T) = -\frac{k_B T \eta_1 \eta_2}{64\pi a^2} \left( 1 - \frac{\eta_1 + \eta_2}{2} \right). \]  

The same expression follows from the zero-frequency contribution to the free energy in Eq. (17)

\[ \mathcal{F}(a, T) = \frac{\hbar c T}{64\pi a^3} \int_{y_0}^{\infty} y dy \ln \left[ 1 - \frac{\eta_1 \eta_2}{(2 + \eta_1)(2 + \eta_2)} e^{-y} \right] \]  

after integration with respect to \( y \) and expansion in powers of \( \eta_1 \) and \( \eta_2 \). Note that at zero frequency only \( r^{(k)}_{||}(0, y) = 0 \).

Now let us consider the Casimir pressure and entropy for the configuration of two dissimilar dielectric plates. The Lifshitz formula for the Casimir pressure presented in terms of dimensionless variables (16) has the form

\[ P(a, T) = -\frac{\hbar c T}{32\pi^2 a^4} \sum_{l=0}^{\infty} \left( 1 - \frac{1}{2} \delta_{l0} \right) \int_{\zeta_l}^{\infty} y^2 dy \times \left[ \frac{r^{(1)}_{||}(\zeta_l, y) r^{(2)}_{||}(\zeta_l, y)}{e^y - r^{(1)}_{||}(\zeta_l, y) r^{(2)}_{||}(\zeta_l, y)} + \frac{r^{(1)}_{\perp}(\zeta_l, y) r^{(2)}_{\perp}(\zeta_l, y)}{e^y - r^{(1)}_{\perp}(\zeta_l, y) r^{(2)}_{\perp}(\zeta_l, y)} \right]. \]  

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Expanding in powers of small parameters $\eta_1$, $\eta_2$ by the same procedure as in the case of the free energy, we obtain the expression exact in $\tau$

$$ P(a, T) = -\frac{\hbar c \tau \eta_1 \eta_2}{256 \pi^2 a^4} \left[ p_1(\tau) - \frac{\eta_1 + \eta_2}{2} p_2(\tau) \right], \quad (35) $$

where $p_{1,2}(\tau)$ are given by

$$ p_1(\tau) = 2 + 4 \left( 1 + \tau \right) e^\tau - 1 \left( e^\tau - 1 \right)^2 + \frac{\tau^4 e^\tau (e^{2\tau} + 4 e^\tau + 1)}{(e^\tau - 1)^4}$$

$$ + \tau^4 \sum_{l=1}^{\infty} l^4 \text{Ei}(-\tau l), \quad (36) $$

$$ p_2(\tau) = 2 + 4 \left( 1 + \tau \right) e^\tau - 1 \left( e^\tau - 1 \right)^2 - \frac{\tau^2 e^\tau (2\tau - 3)e^{2\tau} + 8\tau e^\tau + 2\tau + 3}{3(e^\tau - 1)^4}$$

$$ + \frac{\tau^4 e^\tau [(\tau - 1)e^{4\tau} + 2(13\tau - 5)e^{3\tau} + 66\tau e^{2\tau} + 2(13\tau + 5)e^\tau + \tau + 1]}{6(e^\tau - 1)^6}$$

$$ - \tau^4 \sum_{l=1}^{\infty} l^4 \left( 1 - \frac{\tau^2 l^2}{6} \right) \text{Ei}(-\tau l). $$

In the limiting case $\tau \ll 1$, Eqs. (35), (36) lead to

$$ P(a, T) = -\frac{\hbar c \eta_1 \eta_2}{256 \pi^2 a^4} \left[ \frac{46}{5} + \frac{7\tau^4}{360} \right]$$

$$ - \frac{\eta_1 + \eta_2}{2} \left( \frac{338}{35} - \frac{\tau^4}{360} \right) + O(\tau^5). \quad (37) $$

This expression can also be obtained as

$$ P(a, T) = -\frac{\partial F(a, T)}{\partial a}, \quad (38) $$

where $F(a, T)$ is given by Eq. (31).

At high temperatures (large separations) it holds $\tau \gg 1$ and from Eqs. (35), (36) it follows:

$$ P(a, T) = -\frac{k_B T \eta_1 \eta_2}{32 \pi a^3} \left( 1 - \frac{\eta_1 + \eta_2}{2} \right). \quad (39) $$

Using the above procedure, one can obtain the Casimir entropy for the configuration of two dissimilar dilute dielectric plates. It can be found also as

$$ S(a, T) = -\frac{\partial F(a, T)}{\partial T}, \quad (40) $$

where $F(a, T)$ is given by Eq. (31). In the limit $\tau \ll 1$ the result is

$$ S(a, T) = \frac{3k_B \zeta(3) \eta_1 \eta_2 \tau^2}{128 \pi a^3} \left[ 1 - \frac{\eta_1 + \eta_2}{4} \right]$$

$$ - \left( 1 + \frac{\eta_1 + \eta_2}{14} \right) \frac{7\pi^2 \tau}{135 \zeta(3)} + O(\tau^2). \quad (41) $$
As is seen from Eq. (41), \( S(a,0) = 0 \), i.e., the Nernst heat theorem is satisfied.

The obtained results \( 31 \), \( 37 \), and \( 11 \) are used as the tests in the next section where the low-temperature behavior the Casimir and van der Waals interactions between plates with arbitrary (not dilute) dielectric permittivities is studied.

**IV. LOW-TEMPERATURE BEHAVIOR OF THE VAN DER WAALS AND CASIMIR INTERACTIONS BETWEEN DISSIMILAR DIELECTRICS: ANALYTICAL RESULTS**

We will now look at the configuration of two parallel plates made of real dissimilar dielectrics described by frequency-dependent dielectric permittivities. A distinguishing feature of dielectrics is the finite value of their static dielectric permittivity \( \varepsilon_0^{(k)} = \varepsilon^{(k)}(0) \). It is common for dielectrics that the dielectric permittivity is practically equal to its static value within some frequency region \([0, \omega_k]\). At higher frequencies \( \varepsilon^{(k)}(\omega) \) is smaller than \( \varepsilon_0^{(k)} \).

We start with the Lifshitz formula \( 17 \), \( 18 \) which is applicable to any dielectric plates at any separation (note that separations \( a \) should be large enough so one can ignore the atomic structure of the plates, i.e., \( a \) should be larger than 2 or 3 nm).

Applying the Abel-Plana formula \( 25 \), we can identically rearrange Eq. (17) to the form

\[
\mathcal{F}(a, T) = E(a) + \Delta \mathcal{F}(a, T).
\]

Here \( E(a) \) is the energy of the van der Waals or Casimir interaction at zero temperature given by

\[
E(a) = \frac{hc}{32\pi^2 a^3} \int_0^\infty d\zeta \int_\zeta^\infty dy f(\zeta, y),
\]

where we introduce the notation

\[
f(\zeta, y) = y \left\{ \ln \left[ 1 - r^{(1)}(\zeta, y)r^{(2)}(\zeta, y)e^{-y} \right] + \ln \left[ 1 - r^{(1)}_\perp(\zeta, y)r^{(2)}_\perp(\zeta, y)e^{-y} \right] \right\}.
\]

The second contribution on the right-hand side in Eq. (42) is the thermal correction to the energy

\[
\Delta \mathcal{F}(a, T) = \frac{ihc\tau}{32\pi^2 a^3} \int_0^\infty dt \frac{F(i\tau t) - F(-i\tau t)}{e^{2\pi t} - 1},
\]
where the function $F(x)$ is defined by

$$F(x) = \int_x^\infty dy f(x, y).$$

(46)

The behavior of the energy (43) at both short separations (the limit of the van der Waals forces) and large separations (the limit of the Casimir forces) is well-studied [36]. At short separations ($a \ll \lambda_0$) the dependence of the dielectric permittivities on frequency is important and the asymptotic behavior of the energy is given by $E(a) = -H/(12\pi a^2)$, where $H$ is the Hamaker constant [6, 36]. At large separations ($a \gg \lambda_0$) $\varepsilon^{(k)}(\omega)$ can be approximated by their static values $\varepsilon_0^{(k)}$. As a result, the energy takes the form $E(a) = -\Psi(\varepsilon_0^{(1)}, \varepsilon_0^{(2)})/a^3$, where $\Psi$ is some tabulated function of two variables [6, 36].

Importantly, the low-temperature behavior of the thermal correction [45] under the condition $\tau \ll 1$ has a universal form valid at both short and large separations. The qualitative explanation for this fact, proven below in detail, is that for dielectrics at sufficiently low temperatures the Matsubara frequencies contributing to the thermal correction belong to the region where $\varepsilon^{(k)}$ practically do not depend on the frequency and are equal to their static values $\varepsilon_0^{(k)}$.

To prove the universal low-temperature character of the thermal correction, we use the Ninham-Parsegian representation for the dielectric permittivities along the imaginary frequency axis [4, 64, 65],

$$\varepsilon^{(k)}(i\xi) = 1 + \sum_j \frac{C_j^{(k)}}{1 + i\xi^2 / \omega_j^{(k)}},$$

(47)

Here, the parameters $C_j^{(k)}$ are the absorption strengths for different dielectrics ($k = 1, 2$) satisfying the condition,

$$\sum_j C_j^{(k)} = \varepsilon_0^{(k)} - 1,$$

(48)

and $\omega_j^{(k)}$ are the characteristic absorption frequencies. Although Eq. (47) is the approximate one, it gives a very accurate description for many dielectrics [65].

To obtain the asymptotic behavior of the thermal correction to the energy given by Eq. (45) at $\tau \ll 1$, we substitute Eq. (47) in Eq. (44) and expand the function $f(x, y)$ in powers of $x = \tau t$. The subsequent integration of this expansion with respect to $y$ from $x$ to
infinity (see Appendix A) leads to
\[ F(ix) - F(-ix) = i\pi \varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)} \frac{(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} x^2 \]
\[ - i\alpha x^3 + O(x^4). \]  

(49)

On the right-hand side of Eq. (49) we separated the third-order contribution with a real coefficient \(\alpha\). It cannot be determined at this stage of our calculations because all terms in the expansion of \(f(x, y)\) in powers of \(x\) contribute to this coefficient [the value of \(\alpha\) is found below in Eq. (61)]. As is seen from Eq. (49), only the static dielectric permittivities contribute to \(F(ix) - F(-ix)\) in leading order and, consequently, to the thermal correction.

Substituting Eq. (49) in Eqs. (45) and (42) and performing the integration, we arrive at the result
\[ \mathcal{F}(a, T) = E(a) - \frac{\hbar c}{32\pi^2 a^3} \left[ \zeta(3) \left( \varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)} \right) \frac{(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{8\pi^2(\varepsilon_0^{(1)} + \varepsilon_0^{(2)})} \tau^3 - C_4 \tau^4 + O(\tau^5) \right], \]

where \(C_4 \equiv \alpha/240\).

So far we have considered the free energy. The pressure can be obtained, using Eqs. (48) and (50), as
\[ P(a, T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} \left[ C_4 \tau^4 + O(\tau^5) \right], \]

(51)

where \(P_0(a) = -\partial E(a)/\partial a\) is the pressure at zero temperature. Our aim is to determine the value of \(C_4\). To attain this, the pressure is expressed directly by the Lifshitz formula (34). Applying the Abel-Plana formula (25) in Eq. (34), we represent the pressure in the form
\[ P(a, T) = P_0(a) + \Delta P(a, T), \]

(52)

where the thermal correction to the pressure is
\[ \Delta P(a, T) = \frac{i\hbar c\tau}{32\pi^2 a^4} \int_0^\infty dt \frac{\Phi(i\tau t) - \Phi(-i\tau t)}{e^{2\pi t} - 1} \]

(53)

and the function \(\Phi(x) \equiv \Phi_\parallel(x) + \Phi_\perp(x)\) is given by
\[ \Phi_\parallel,\perp(x) = \int_x^\infty y^2 dy \frac{r^{(1)}_\parallel(x, y)r^{(2)}_\parallel(x, y)}{e^y - r^{(1)}_\parallel(x, y)r^{(2)}_\parallel(x, y)}. \]

(54)
First, let us determine the leading term of the expansion of $\Phi_\parallel(x)$ in powers of $x$. By introducing the new variable $v = y/x$ one arrives at

$$
\Phi_\perp(x) = x^3 \int_1^\infty \frac{v^2 dv r^{(1)}_\perp(x,v)r^{(2)}_\perp(x,v)}{e^{uv} - r^{(1)}_\perp(x,v)r^{(2)}_\perp(x,v)}. \tag{55}
$$

Note that the reflection coefficients $r^{(k)}_\perp(x,v)$ depend on $x$ only through the frequency dependence of $\varepsilon^{(k)}$ according to Eq. (17). Expanding in powers of $x$ in Eq. (55), we obtain

$$
\Phi_\perp(x) = x^3 \int_1^\infty \frac{v^2 dv r^{(1)}_\perp(0,v)r^{(2)}_\perp(0,v)}{1 - r^{(1)}_\perp(0,v)r^{(2)}_\perp(0,v)} + O(x^4), \tag{56}
$$

where, according to Eq. (18),

$$
r^{(k)}_\perp(v,0) = \frac{\sqrt{v^2 + \varepsilon^{(k)}_0} - 1 - v}{\sqrt{v^2 + \varepsilon^{(k)}_0} - 1 + v}. \tag{57}
$$

Integration of Eq. (56) with account of Eq. (57) results in

$$
\Phi_\perp(x) = \left(1 - \frac{\varepsilon^{(1)}_0 + \varepsilon^{(2)}_0 + \sqrt{\varepsilon^{(1)}_0 \varepsilon^{(2)}_0 - \varepsilon^{(1)}_0 \varepsilon^{(2)}_0}}{\sqrt{\varepsilon^{(1)}_0} + \sqrt{\varepsilon^{(2)}_0}}\right) \frac{x^3}{6} + O(x^4). \tag{58}
$$

From Eq. (58) it follows

$$
\Phi_\perp(ix) - \Phi_\perp(-ix) = -i \left(1 - \frac{\varepsilon^{(1)}_0 + \varepsilon^{(2)}_0 + \sqrt{\varepsilon^{(1)}_0 \varepsilon^{(2)}_0 - \varepsilon^{(1)}_0 \varepsilon^{(2)}_0}}{\sqrt{\varepsilon^{(1)}_0} + \sqrt{\varepsilon^{(2)}_0}}\right) \frac{x^3}{3} + O(x^5). \tag{59}
$$

The expansion of $\Phi_\parallel(x)$ from Eq. (54) in powers of $x$ is somewhat more cumbersome. It is presented in detail in Appendix B leading to the result

$$
\Phi_\parallel(ix) - \Phi_\parallel(-ix) = -i \left\{1 + \frac{1}{\left(\sqrt{\varepsilon^{(1)}_0} + \sqrt{\varepsilon^{(2)}_0}\right)^2} \left[-\left(\varepsilon^{(1)}_0 + \varepsilon^{(2)}_0\right)^3 \right.ight.
$$

$$
+ \varepsilon^{(1)}_0 \varepsilon^{(2)}_0 \left(5\varepsilon^{(1)}_0 \varepsilon^{(2)}_0 - 3\varepsilon^{(1)}_0 - 3\varepsilon^{(2)}_0 + 1\right)
$$

$$
+ \sqrt{\varepsilon^{(1)}_0} \varepsilon^{(2)}_0 \sqrt{\varepsilon^{(1)}_0} \varepsilon^{(2)}_0 \left(\varepsilon^{(1)}_0 \varepsilon^{(2)}_0 - \varepsilon^{(1)}_0 - \varepsilon^{(2)}_0\right)
$$

$$
+ 3\varepsilon^{(1)}_0 \varepsilon^{(2)}_0 \left(\varepsilon^{(1)}_0 - 1\right) \left(\varepsilon^{(2)}_0 - 1\right) \text{Artanh} \left(\varepsilon^{(1)}_0 + \varepsilon^{(2)}_0 \sqrt{\varepsilon^{(1)}_0 - \varepsilon^{(2)}_0} \right) \right\} \frac{x^3}{3} + O(x^4). \tag{60}
$$
Adding Eqs. (59) and (60) and integrating the obtained result according to Eq. (53), we find the coefficient $C_4$ in the thermal correction to the energy (50) and pressure (51)

$$C_4 = \frac{1}{720} \left\{ 2 + \frac{1}{\left( \sqrt{\varepsilon_0^{(1)}} + \sqrt{\varepsilon_0^{(2)}} \right) \left( \sqrt{\varepsilon_0^{(1)}} + \sqrt{\varepsilon_0^{(2)}} \right)^2} \left[ - \left( \varepsilon_0^{(1)} + \varepsilon_0^{(2)} \right)^2 \right] \right. $$

$$\times \left( 2\varepsilon_0^{(1)} + 2\varepsilon_0^{(2)} + \sqrt{\varepsilon_0^{(1)}} \varepsilon_0^{(2)} - \varepsilon_0^{(1)} \varepsilon_0^{(2)} \right) + \varepsilon_0^{(1)} \varepsilon_0^{(2)} \sqrt{\varepsilon_0^{(1)}} \varepsilon_0^{(2)} \left( 5\varepsilon_0^{(1)} \varepsilon_0^{(2)} - 3\varepsilon_0^{(1)} - 3\varepsilon_0^{(2)} + 1 \right)$$

$$+ \sqrt{\varepsilon_0^{(1)}} \varepsilon_0^{(2)} \left( \varepsilon_0^{(1)} \varepsilon_0^{(2)} \sqrt{\varepsilon_0^{(1)}} \varepsilon_0^{(2)} - \varepsilon_0^{(1)} \varepsilon_0^{(2)} - \varepsilon_0^{(1)} - \varepsilon_0^{(2)} \right)$$

$$- \frac{3\varepsilon_0^{(1)} \varepsilon_0^{(2)} \left( \varepsilon_0^{(1)} - 1 \right) \left( \varepsilon_0^{(2)} - 1 \right)}{\left( \sqrt{\varepsilon_0^{(1)}} - \sqrt{\varepsilon_0^{(2)}} \right) \sqrt{\varepsilon_0^{(1)}} + \varepsilon_0^{(2)}} \text{Artanh} \left( \varepsilon_0^{(1)} + \varepsilon_0^{(2)} \right) \right\} \right.$$  

This expression gets much more simplified for the case of two plates made of one and the same dielectric ($\varepsilon_0^{(1)} = \varepsilon_0^{(2)} \equiv \varepsilon_0$):

$$C_4 = \frac{1}{720} \left( \sqrt{\varepsilon_0} - 1 \right) \left( \varepsilon_0^2 + \varepsilon_0 \sqrt{\varepsilon_0} - 2 \right).$$  

Thus, for two similar dielectrics the free energy takes the form

$$F(a, T) = E(a) - \frac{hc}{32\pi^2 a^3} \left[ \frac{\zeta(3)(\varepsilon_0 - 1)^2}{8\pi^2(\varepsilon_0 + 1)} \tau^3 - \frac{1}{720} \left( \sqrt{\varepsilon_0} - 1 \right) \left( \varepsilon_0^2 + \varepsilon_0 \sqrt{\varepsilon_0} - 2 \right) \tau^4 + O(\tau^5) \right].$$

Another limiting case is given by dilute dielectrics. Putting $\varepsilon_0^{(k)} = 1 + \eta_k$ in Eqs. (50), (51) and (61) and expanding in powers of the small parameters $\eta_1$, $\eta_2$, one arrives at precisely the same thermal corrections (terms proportional to $\tau^3$ and $\tau^4$) as were obtained for dilute dielectrics in Eqs. (31) and (37). Note that Eqs. (31) and (37) also contain terms independent of $\tau$ having a physical meaning of the energy and pressure at zero temperature. These terms coincide with $E(a)$ and $P_0(a)$ from Eqs. (50) and (51) only in the relativistic limit of large separations. This is because Eqs. (31) and (37) were obtained under the condition that the dielectric permittivities are constant. However, as indicated above, the thermal corrections obtained in such a way have a universal character and are valid at any separation with the constraint $\tau \ll 1$.

Eqs. (50), (61) and (63) solve the vital issue about the thermodynamic consistency of the Lifshitz formula for the case of two dielectric plates. Using Eqs. (50) and (51), the entropy
of the interaction between plates takes the form

\[ S(a, T) = \frac{3k_B \zeta(3)(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{64\pi^3 a^2(\varepsilon_0^{(1)} + 1)} \tau^2 \left[ \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)}\varepsilon_0^{(2)}}{2(\varepsilon_0^{(2)} + 1)(\varepsilon_0^{(1)} + \varepsilon_0^{(2)})} \right]
\]

\[ - \frac{32\pi^2(\varepsilon_0^{(1)} + 1)C_4}{3\zeta(3)(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)} \tau + O(\tau^2) \],

\( (64) \)

where the coefficient \( C_4 \) is determined in Eq. \( (61) \). As is seen from Eq. \( (64) \), the entropy vanishes when the temperature goes to zero as it must be in accordance with the third law of thermodynamics (the Nernst heat theorem).

In the limiting case of two similar dielectrics Eq. \( (64) \) can be rearranged as follows:

\[ S(a, T) = \frac{3k_B \zeta(3)(\varepsilon_0 - 1)^2}{64\pi^3 a^2(\varepsilon_0 + 1)} \tau^2 \left[ 1 - \frac{2\pi^2(\varepsilon_0 + 1)(\varepsilon_0\sqrt{\varepsilon_0} + 2\varepsilon_0 + 2\sqrt{\varepsilon_0} + 2)}{135\zeta(3)(\sqrt{\varepsilon_0} + 1)^2} \tau + O(\tau^2) \right] . \]

\( (65) \)

In the limiting case of two dilute dielectrics the expansion of Eq. \( (64) \) in powers of \( \eta_1 \) and \( \eta_2 \) coincides with Eq. \( (61) \). As is seen from Eqs. \( (64) \) and \( (65) \), in the limit \( \tau \to 0 \) (\( T \to 0 \)) the lower order contributions to the entropy are of the second and third powers in the small parameter \( \tau \). In this manner for dielectrics at low temperatures the entropy obeys the same universal law which was previously found for ideal \([38, 39, 45, 66]\) and real \([45, 48]\) metals. Recall that ideal metals (i.e., two plates with the Dirichlet boundary conditions \( E_t = 0 \)) lead to an exactly solvable model in Matsubara quantum field theory \([37, 38]\). As was shown in Refs. \([39, 45]\), in the case of plates made of ideal metal the entropy at low temperatures is given by

\[ S(a, T) = \frac{3k_B \zeta(3)}{32\pi^3 a^2} \tau^2 \left[ 1 - \frac{2\pi^2}{135\zeta(3)} \tau + O(\tau^2) \right] . \]

\( (66) \)

This demonstrates that the thermal quantum field theory approach is in perfect agreement with thermodynamics for ideal metals. It should be pointed out that the expansion coefficients of the free energy, pressure and entropy in powers of \( \tau \) in the case of ideal metals cannot be obtained as a straightforward limit \( \varepsilon_0^{(k)} \to \infty \) in Eqs. \( (50) \), \( (51) \) and \( (64) \). The mathematical reason is that it is impermissible to interchange the limits \( \tau \to 0 \) and \( \varepsilon \to \infty \) in the power expansion of functions depending on \( \varepsilon \) as a parameter.

In the case of large separations (high temperatures) \( \tau \gg 1 \) and the approximation of static dielectric permittivities is applicable. With this condition the major contribution is
given by the zero-frequency term of the Lifshitz formula (17)

\[ F(a, T) = \frac{\hbar c T}{64\pi^2 a^3} \int_0^\infty y dy \ln \left[ 1 - \frac{\varepsilon_0^{(1)} - 1}{\varepsilon_0^{(1)} + 1} \frac{\varepsilon_0^{(2)} - 1}{\varepsilon_0^{(2)} + 1} e^{-y} \right]. \]  \hspace{1cm} (67)

(the other terms being exponentially small). Integration in Eq. (67) leads to

\[ F(a, T) = -\frac{k_B T}{16\pi a^2} \text{Li}_3 \left( \frac{\varepsilon_0^{(1)} - 1}{\varepsilon_0^{(1)} + 1} \frac{\varepsilon_0^{(2)} - 1}{\varepsilon_0^{(2)} + 1} \right), \]  \hspace{1cm} (68)

where \( \text{Li}_n(z) \) is the polylogarithm function. In a similar manner at \( \tau \gg 1 \) the pressure is given by

\[ P(a, T) = -\frac{k_B T}{8\pi a^3} \text{Li}_3 \left( \frac{\varepsilon_0^{(1)} - 1}{\varepsilon_0^{(1)} + 1} \frac{\varepsilon_0^{(2)} - 1}{\varepsilon_0^{(2)} + 1} \right). \]  \hspace{1cm} (69)

In the limiting case of dilute dielectrics Eqs. (68) and (69) coincide with Eqs. (32) and (39), respectively.

V. COMPARISON BETWEEN ANALYTIC AND NUMERICAL RESULTS

In this section we compare our analytic results in Eqs. (50) and (51) for the thermal corrections to the free energy and pressure with computations performed for both dissimilar and similar dielectrics using the complete tabulated optical data and the Lifshitz formulas (17), (34). This enables us to illustrate the applicability regions of the above asymptotic expressions for different plate materials and to gain an impression on the temperature dependence of the free energy and pressure within a wide range of parameters. As an example we consider two dielectrics (Si and vitreous SiO\(_2\)) which differ in their dielectric properties.

The dielectric permittivity of real dielectrics along the imaginary frequency axis can be obtained through the dispersion relation

\[ \varepsilon(i\xi) = 1 + \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega \text{Im}\varepsilon(\omega)}{\omega^2 + \xi^2}. \]  \hspace{1cm} (70)

Here \( \text{Im}\varepsilon(\omega) = 2n_1(\omega)n_2(\omega) \), where \( n_1(\omega) = \text{Re} n(\omega) \), \( n_2(\omega) = \text{Im} n(\omega) \), and \( n(\omega) \) is the complex refractive index tabulated in Ref. [67]. For the dielectric Si (of high resistivity \( \rho_0 = 1000 \Omega \text{ cm} \)) and vitreous SiO\(_2\) the dielectric permittivities were computed by the use of Eq. (70) in Ref. [68]. The obtained results are presented in Fig. 1 for Si (line 1) and for SiO\(_2\) (line 2). The flat steps in both lines should be extended for all frequencies \( 0 \leq \xi \leq 10^{10} \text{ rad/s} \).
(the lowest frequency indicated in Fig. 1). As a result, the static dielectric permittivities are equal to \( \varepsilon_0^{(1)} = 11.66 \) (for Si) and \( \varepsilon_0^{(2)} = 3.84 \) (for SiO\(_2\)).

In Fig. 2, we compare the computational results for the thermal corrections to the energy \( \Delta F(a, T) = F(a, T) - E(a) \) (a) and pressure \( \Delta P(a, T) = P(a, T) - P_0(a) \) (b) in the configuration of two plates one made of Si and another one of SiO\(_2\) at separation \( a = 400 \) nm as a function of the temperature. Short-dashed lines are obtained by the use of the Lifshitz formulas (17), (34), (43) with the static dielectric permittivities \( \varepsilon_0^{(k)} \). Solid lines are computed using the same Lifshitz formulas with the complete frequency-dependent dielectric permittivities of Si and SiO\(_2\) presented in Fig. 1. Long-dashed lines show our asymptotic expressions for \( \Delta F(a, T) \) and \( \Delta P(a, T) \) on the right-hand sides of Eqs. (50) and (51), respectively, with a coefficient \( C_4 \) defined in Eq. (61).

As is seen from Fig. 2a, the thermal correction to the energy can be calculated with the static dielectric permittivities of Si and SiO\(_2\) at temperatures below 100 K. At higher temperatures the solid line computed by the use of the tabulated optical data departs from the short-dashed line computed with the static dielectric permittivities. At \( T < 60 \) K the asymptotic expression for \( \Delta F(a, T) \) in Eq. (50) with the coefficient (61) leads to the same results as the original Lifshitz formula. Quite similar situation takes place for the thermal correction to the pressure (see Fig. 2b). The numerical results obtained by the use of the Lifshitz formula with static \( \varepsilon^{(k)} \) coincide with those obtained using the tabulated optical data at \( T < 100 \) K (solid and short-dashed lines). As to the asymptotic expression for the thermal correction to the pressure [Eqs. (51) and (61)], it becomes applicable at \( T < 50 \) K.

To give a more comprehensive idea on the application regions of the obtained asymptotic expressions, in Fig. 3 we present the same information, as in Fig. 2, for the case of two similar plates made of vitreous SiO\(_2\) at separation \( a = 450 \) nm. As before, in Fig. 3a the thermal correction to the energy is plotted and in Fig. 3b to the pressure versus temperature. As is seen from Fig. 3a, all three theoretical descriptions of the free energy (in terms of the tabulated data, static dielectric permittivity and the asymptotic one) become applicable at \( T < 85 \) K. If we consider the thermal correction to the pressure (Fig. 3b), different models of \( \varepsilon \) are applicable at \( T < 100 \) K whereas the asymptotic expression (61) with the coefficient (62) coincides with them at \( T < 65 \) K. The distinguishing feature of Fig. 3b is the intersection of the solid and long-dashed lines. It is explained by a greater deviation of the solid line, computed by the use of tabulated data, from the short-dashed line, computed using static
\( \varepsilon_0^{(2)} \), because of the presence of a second flat step in the frequency dependence of \( \varepsilon^{(2)}(i\xi) \) (see line 2 in Fig. 1).

In Fig. 4 we present the dependences of the thermal correction to the energy (a) and pressure (b) for two Si plates at separation \( a = 300 \text{ nm} \) computed in the framework of the same approaches. For two Si plates within the separation and temperature region under consideration the lines computed with frequency-dependent and static \( \varepsilon^{(1)} \) almost coincide (see Figs. 4a,b). The asymptotic expressions in Eqs. (53) and (54) with the coefficient (62) are applicable at \( T < 45 \text{ K} \) and \( T < 35 \text{ K} \), respectively. In Fig. 4a the asymptotic expression for the thermal correction to the free energy achieves its maximal value at about \( T \approx 105 \text{ K} \). This is explained by the relatively large static dielectric permittivity of Si.

Comparing all the above figures, one can conclude that the obtained asymptotic expressions for the free energy (containing two terms of order \( \tau^3 \) and \( \tau^4 \)) have a wider application range than the asymptotic expressions for the pressure containing only one term of order \( \tau^4 \). At the same time, by decreasing the separation distance one can widen the range of temperatures where our asymptotic expressions are applicable. The remarkable feature of Figs. 2–4 is the monotonous increase of the magnitude of all thermal corrections with the increase of temperature for real dielectrics (see solid lines overlapping with our asymptotic expressions in the applicability region of the latter). This confirms the same conclusion made in Refs. [46, 69] based on qualitative thermodynamical considerations.

VI. ROLE OF THE ZERO-FREQUENCY TERM IN THE MATSUBARA SUM FOR DIELECTRICS

According to Eq. (17), the free energy of the van der Waals and Casimir interaction is represented by the Matsubara sum from zero to infinity. The zero-frequency term in Eq. (17),

\[
F_0(a, T) = \frac{k_B T}{16\pi a^2} \int_0^{\infty} y \, dy \ln \left[ 1 - r^{(1)}(0, y) r^{(2)}(0, y) e^{-y} \right],
\]

is of special interest. For dielectrics with finite static dielectric permittivities \( \varepsilon_0^{(k)} \) from Eq. (18) it follows:

\[
r^{(k)}(0, y) = \frac{\varepsilon_0^{(k)} - 1}{\varepsilon_0^{(k)} + 1}, \quad r^{(k)}(0, y) = 0.
\]
Eqs. (71) and (72) were already used in Eqs. (33) and (67) to obtain the asymptotic expressions for the free energy at high temperatures where the contributions from all terms in the Matsubara sum with \( l \geq 1 \) are exponentially small.

Note that the second equality in Eqs. (72) is somewhat analogous with the same equality in the case of real metals described by the Drude dielectric function [41]. In reality, however, for metals the second reflection coefficient at zero frequency is \( r_\parallel(0, y) = 1 \), i.e., equal to its physical value. Together with \( r_\perp(0, y) = 0 \), this leads to the violation of the Nernst heat theorem for perfect crystal lattices with no impurities [46] and to contradictions with experiment [15, 16, 17, 50, 51]. On the contrary, for dielectrics \( r_\parallel^{(k)}(0, y) \) in Eq. (72) is larger than its physical value at normal incidence [the latter is equal to \( (\sqrt{\varepsilon_{0}^{(k)}} - 1) / (\sqrt{\varepsilon_{0}^{(k)}} + 1) \)] and, as was demonstrated in Sec. IV, the Lifshitz formula incorporating Eqs. (72) is in perfect agreement with the Nernst heat theorem.

We now turn to a problem of outstanding importance which arises when one includes the dc conductivity of dielectric materials into the model of their dielectric response. What this means is that, instead of dielectric permittivities \( \varepsilon_l^{(k)} = \varepsilon^{(k)}(i\xi_l) \), one uses \( \tilde{\varepsilon}_l^{(k)}(k) = \varepsilon^{(k)}(i\xi_l) + \frac{4\pi\sigma_0^{(k)}}{\xi_l} = \varepsilon^{(k)} + \frac{\beta^{(k)}(T)}{l} \). (73)

Here \( \sigma_0^{(k)} \) is the dc conductivity of the plate materials and \( \beta^{(k)} = 2\hbar\sigma_0^{(k)}/(k_B T) \). It is common knowledge [70] that the conductivity of dielectrics depends on the temperature as \( \sigma_0^{(k)} \sim \exp(-b^{(k)}/T) \) where \( b^{(k)} \) is determined by the energy gap \( \Delta^{(k)} \) which is different for different materials. It cannot be too highly stressed that for dielectrics the conductivity at constant current is very low. To take an example [71], for SiO\(_2\) at \( T = 300 \) K it holds \( \beta^{(2)} \sim 10^{-12} \) and, thus, negligible for all \( l \geq 1 \). From physical considerations the inclusion of the term \( \beta^{(k)}(T)/l \) into the model of the dielectric response (73) seems of dubious value since \( \beta^{(k)}(T) \) quickly decreases with decrease of \( T \) and, thus, remains negligible at any \( T \).

In spite of this, the substitution of Eq. (73) into the Lifshitz formula (17) leads to Eq. (71) with different value of one of the reflection coefficients at zero frequency than in Eq. (72):

\[ \tilde{r}_\parallel^{(k)}(0, y) = 1, \quad \tilde{r}_\parallel^{(k)}(0, y) = 0. \] (74)

The change from Eq. (72) to Eq. (74) has far-reaching consequences. To analyze them, we calculate the free energy \( \tilde{\mathcal{F}}(a, T) \) with the dielectric permittivities \( \tilde{\varepsilon}_l^{(k)} \) identically rearranging
Eq. (17) to the form

\[
\tilde{F}(a, T) = \frac{k_B T}{16\pi a^2} \int_0^\infty y dy \left\{ \ln \left( \frac{\varepsilon_0^{(1)} - 1}{\varepsilon_0^{(1)} + 1} \right) - 1 \right\}
- \ln \left[ \frac{1 - (\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{e^{-y}} \right]
+ \frac{k_B T}{16\pi a^2} \int_0^\infty y dy \ln \left[ \frac{1 - (\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{e^{-y}} \right]
+ \frac{k_B T}{16\pi a^2} \sum_{l=1}^{\infty} \int_0^\infty y dy \ln \left[ 1 - \tilde{r}_{\parallel}^{(1)}(\zeta_l, y) \tilde{r}_\perp^{(2)}(\zeta_l, y) e^{-y} \right]

+ \ln \left[ 1 - \tilde{r}_\parallel^{(1)}(\zeta_l, y) \tilde{r}_\perp^{(2)}(\zeta_l, y) e^{-y} \right],
\]

where the reflection coefficients \( \tilde{r}_{\parallel,\perp}^{(k)} \) are obtained from Eq. (18) by replacing \( \varepsilon_l^{(k)} \) with \( \tilde{\varepsilon}_l^{(k)} \).

Now we expand the last, third term on the right-hand side of Eq. (75) in powers of the small parameters \( \beta^{(k)}/l \). Combining the zero-order contribution in this expansion with the second term on the right-hand side of Eq. (75), one obtains the free energy \( F(a, T) \) calculated with the dielectric permittivities \( \varepsilon_l^{(k)} \). Calculating the first integral on the right-hand side of Eq. (75), we arrive at

\[
\tilde{F}(a, T) = F(a, T) - \frac{k_B T}{16\pi a^2} \left\{ \zeta(3) - \text{Li}_3 \left[ \frac{(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \right] \right\} + R(a, T).
\]

In this formula \( R(a, T) \) is of order \( O(\beta^{(k)}/l) \) and it stands for the first and higher-order contributions in the expansion of the third term on the right-hand side of Eq. (75) in powers of \( \beta^{(k)}/l \). The explicit expression for \( R(a, T) \) is given in Eqs. (C1) and (C2) of Appendix C. As is shown in Appendix C, \( R(a, T) \) exponentially goes to zero with the decrease of \( T \).

Eq. (76) leads to the important conclusion about the thermodynamic inconsistency of the Lifshitz formula for dielectrics if one includes the dc conductivity in the model of dielectric response. Substituting Eq. (76) into Eq. (40), we obtain the entropy for the plates with the dielectric permittivities (73) in the form

\[
\tilde{S}(a, T) = S(a, T) + \frac{k_B}{16\pi a^2} \left\{ \zeta(3) - \text{Li}_3 \left[ \frac{(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \right] \right\} - \frac{\partial R(a, T)}{\partial T},
\]

\( \Box \)
where $S(a, T)$ is the entropy for the plates with the dielectric permittivities $\varepsilon^{(k)}$ given by Eq. (65).

In the limit $T \to 0$ it follows:

$$\tilde{S}(a, 0) = \frac{k_B}{16 \pi a^2} \left\{ \zeta(3) - \text{Li}_3 \left[ \frac{(\varepsilon_0^{(1)} - 1)(\varepsilon_0^{(2)} - 1)}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \right] \right\} > 0. \quad (78)$$

Eq. (78) depends on the parameter of the system under consideration (the separation distance $a$) and implies the violation of the Nernst heat theorem. Thus, the dc conductivity of a dielectric is irrelevant to the origin of the van der Waals and Casimir forces and must not be included in the models of the dielectric response. The neglect of this rule results in the violation of thermodynamics. Physically it is amply clear that for high-frequency phenomena like the van der Waals and Casimir forces the behavior of dielectric materials at low frequencies is described by the static dielectric permittivities (the approach used by E. M. Lifshitz and his collaborators [35]). The results of this section provide the necessary theoretical background for this conclusion.

VII. CONCLUSIONS AND DISCUSSION

In the foregoing, we have reconsidered the Lifshitz theory of the Casimir and van der Waals interaction between dielectrics on the basis of thermal quantum field theory in Matsubara formulation. As was shown above, there are field theoretical derivations of the Lifshitz formula without use of the fluctuation-dissipation theorem which permit generalization for the presence of dissipation. The special analysis demonstrates, however, that only a particular dissipation is compatible with the Lifshitz formula, those without net heat losses and with balanced processes of absorption and emission.

In this paper, we have analytically solved the long-standing problem of the low-temperature (short separation) behavior of the thermal corrections to the Casimir energy and pressure between dissimilar dielectric plates and demonstrated that the Casimir (van der Waals) entropy vanishes when the temperature goes to zero. This was done, first, using the idealized model of dilute dielectrics, and, then, for real dielectrics with finite static dielectric permittivities. The free energy, pressure and entropy of both the van der Waals and Casimir interactions between dielectrics demonstrate at low temperatures the same universal temperature dependence which was previously discovered for ideal metals. This proves the
thermodynamic consistency of the Lifshitz theory in application to dielectric plates. The obtained asymptotic expressions were compared with the results of numerical computations for both similar and dissimilar real dielectrics (silicon and vitreous silica), and were found to be in excellent agreement.

Special attention was paid to the role of the zero-frequency term in Matsubara sum in the case of dielectric plates and to the extrapolation of the dielectric permittivity along the imaginary frequency axis to zero frequency. As was proved above, the inclusion of the conductivity at constant current in the model of dielectric response leads to a modification of the zero-frequency term of the Lifshitz formula and to the violation of the Nernst heat theorem. The conclusion was made that the conductivity of dielectrics at constant current (low but formally nonzero at nonzero temperature) is irrelevant to physical phenomena described by the Lifshitz theory. This conclusion leads to far-reaching consequences for both the problem of noncontact atomic friction and for Casimir interaction between real metals at nonzero temperature.

As to the problem of atomic friction, the observed effect is many orders of magnitude larger \cite{31} than the van der Waals friction between metals caused by the vacuum fluctuations and thermal photons (in the experiment of Ref. \cite{31} the gold-coated tip and substrate were used). In Refs. \cite{32,54} it has been proposed that measurements performed on metal films are strongly affected by the underlying dielectric substrate. To calculate the friction due to the substrate, Refs. \cite{32,54} use the Lifshitz-type formula including the low dc conductivity in the model of dielectric response. Then, the large value of friction coefficient is obtained in rough agreement with the experimental data of Ref. \cite{31}. According to our results, the proposition of Refs. \cite{32,54} would not be in agreement with the Nernst heat theorem. It is also qualitatively clear that the low-frequency behavior of the dielectric permittivity in the region below $\sim 600$ rad/s cannot cause the large friction effect at characteristic frequencies from $3.75 \times 10^{14}$ rad/s to $1.5 \times 10^{17}$ rad/s (in the experiment of Ref. \cite{31} separations vary from 1 to 400 nm).

Regarding the application of the obtained results to real metals, the case of two dielectric plates with including the dc conductivity turns out to be analogous to two metal plates described by the Drude dielectric function. In both cases the region of low frequencies contributes significantly to the description of high-frequency phenomena resulting in contradictions with thermodynamics (see Introduction). It is well known that in the region of
the normal skin effect, described by the Drude dielectric function, dissipation leads to the heating of a metal (see Ref. [49] for details). For dissipation of this kind, as discussed above, the field theoretical derivation of the Lifshitz formula is inapplicable. In this connection it is not surprising also that the results obtained from the Lifshitz formula combined with the Drude model are in conflict with the conclusions obtained for ideal metals using the thermal quantum field theory approach with the Dirichlet boundary conditions. The proposed measurement of the Casimir force between both metals and semiconductors at large separations [72] is aimed to resolve this contradiction experimentally. At large separations (high temperatures) the classical limit of the Casimir effect is achieved where forces acting between real and ideal metals practically coincide [73, 74]. In our opinion, this important experiment could bring the final verification of the results obtained from the thermal quantum field theory approach (see also Refs. [75, 76] where two short-separation experiments are proposed with the same aim).

It is worthwhile to note that in real experiments the most frequent configuration is not the two parallel plates but a large sphere above a plate. There is a prediction in recent literature [74] that for ideal metals in the case of open geometries the thermal correction to the Casimir force is not as suppressed as in the parallel plate geometry in the limit of zero temperature. By this reason it is of much interest to derive on fundamental grounds the low-temperature behavior of the Casimir free energy between real bodies (metallic or dielectric) for any open geometry.

To conclude, the above reconsideration of the Casimir and van der Waals interactions between dielectric semispaces not only confirms the mutual agreement between the thermal quantum field theory approach, Lifshitz formula and thermodynamics, but also suggests ways for the resolution of other complicated problems in modern applications of quantum electrodynamics.

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APPENDIX A

In this Appendix we prove Eq. (49) in Sec. IV. Let us present Eq. (44) in the form

\[ f(\zeta, y) = f_\parallel(\zeta, y) + f_\perp(\zeta, y), \]  

(A1)

where

\[ f_\parallel, \perp(\zeta, y) = y \ln \left[ 1 - r_\parallel(1)(\zeta, y)r_\perp(2)(\zeta, y)e^{-y} \right]. \]  

(A2)

It is easy to check that \( f_\perp(\zeta, y) \) does not contribute to the first term on the right-hand side of Eq. (49). Substituting Eq. (47) in \( f_\parallel(\zeta, y) \) and expanding in powers of \( x = \tau t \) one obtains

\[ f_\parallel(x, y) = y \ln \left[ 1 - r_0(1)r_0(2)e^{-y} \right] \]
\[ + \frac{\varepsilon_0(1) + \varepsilon_0(2) + 2\varepsilon_0(1)\varepsilon_0(2)}{(\varepsilon_0(1) + 1)(\varepsilon_0(2) + 1)} \left( \frac{r_0(1)r_0(2)e^{-y}}{y(1 - r_0(1)r_0(2)e^{-y})} \right) x^2 \]
\[ + 2 \left[ \frac{1}{(\varepsilon_0(1) + 1)\omega(1)^2} + \frac{1}{(\varepsilon_0(2) + 1)\omega(2)^2} \right] \frac{r_0(1)r_0(2)e^{-y}}{1 - r_0(1)r_0(2)e^{-y}} x^2 + O(x^3), \]  

(A3)

where the static values of the reflection coefficients are found from Eq. (18)

\[ r_0^{(k)} \equiv r^{(k)}(0, y) = \frac{\varepsilon^{(k)}_0 - 1}{\varepsilon^{(k)}_0 + 1} < 1. \]  

(A4)

Note that for simplicity we consider only one oscillator in Eq. (47) and omit index \( j \). The obtained results, however, are valid for any number of oscillators.

As a next step, we should integrate Eq. (A3) according to Eq. (46). Integration of the first term on the right-hand side of Eq. (A3) results in

\[ Z_1(x) \equiv \int_x^\infty ydy \ln \left( 1 - r_0(1)r_0(2)e^{-y} \right) \]
\[ = -\sum_{n=1}^{\infty} \frac{(1 + nx)e^{-nx}}{n^3} \left( r_0(1)r_0(2) \right)^n. \]  

(A5)

Expanding (A5) in powers of \( x \) and summing the obtained series, we arrive at

\[ Z_1(x) = -\text{Li}_3 \left( r_0(1)r_0(2) \right) \]
\[ - \frac{x^2}{2} \ln \left( 1 - r_0(1)r_0(2) \right) + O(x^3). \]  

(A6)
Integration of the second term on the right-hand side of Eq. (A3) contains the integral

\[ Z_2(x) = x^2 \int_x^\infty dy \frac{r_0^{(1)} r_0^{(2)} e^{-y}}{y \left(1 - r_0^{(1)} r_0^{(2)} e^{-y}\right)} \]

\[ = -x^2 \sum_{n=1}^\infty \left(r_0^{(1)} r_0^{(2)}\right)^n \text{Ei}(-nx). \quad (A7) \]

Integration of the third term on the right-hand side of Eq. (A3) leads to the integral

\[ Z_3(x) = x^2 \int_x^\infty dy \frac{r_0^{(1)} r_0^{(2)} e^{-y}}{1 - r_0^{(1)} r_0^{(2)} e^{-y}} \]

\[ = x^2 \sum_{n=1}^\infty \left(r_0^{(1)} r_0^{(2)}\right)^n \frac{(1 + nx)e^{-nx}}{n^2}. \quad (A8) \]

Expanding in powers of \( x \) in Eq. (A8) and performing the summation, we obtain

\[ Z_3(x) = x^2 \text{Li}_2 \left(r_0^{(1)} r_0^{(2)}\right) - \frac{x^4}{2} \frac{r_0^{(1)} r_0^{(2)}}{1 - r_0^{(1)} r_0^{(2)}} + O(x^5). \quad (A9) \]

Let us now calculate the different contributions to the quantity (49). According to Eq. (A6)

\[ Z_1(ix) - Z_1(-ix) = O(x^3). \quad (A10) \]

From Eq. (A7) it follows:

\[ Z_2(ix) - Z_2(-ix) = i\pi x^2 \frac{r_0^{(1)} r_0^{(2)}}{1 - r_0^{(1)} r_0^{(2)}} + O(x^3). \quad (A11) \]

Finally, Eq. (A9) leads to

\[ Z_3(ix) - Z_3(-ix) = O(x^5). \quad (A12) \]

As is seen from Eqs. (A10)–(A12), only Eq. (A11) contributes to the leading term on the right-hand side of Eq. (49). As to the frequency dependence of the dielectric permittivity in accordance to Eq. (47), it contributes only to the fifth order term [see the last term on the right-hand side of Eq. (A3) and Eq. (A12)].

Assigning the numerical coefficient to \( Z_2 \) as in the second term on the right-hand side of Eq. (A3) and using Eq. (A11), we arrive at the result

\[ F(ix) - F(-ix) = \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \frac{i\pi r_0^{(1)} r_0^{(2)}}{1 - r_0^{(1)} r_0^{(2)}} x^2 + O(x^3). \quad (A13) \]

In terms of a notation (A4), this coincides with Eq. (49).
APPENDIX B

This Appendix is devoted to the derivation of Eq. (60) in Sec. IV where the function $\Phi_{\parallel}(ix)$ was used in the calculation of pressure. As was shown in Appendix A, the dependence of the dielectric permittivity on frequency contributes to the expansion of $F(ix)$ in powers of $x$, used in the calculation of the free energy, starting from only the 5th power [i.e., to $\Phi_{\parallel}(ix)$ starting from the 4th power in $x$]. Here we are looking for the lowest (third) order expansion term of $\Phi_{\parallel}(ix)$. Because of this, it is possible to disregard the frequency dependence of $\varepsilon$ and describe the plate materials by their static dielectric permittivities.

Let us identically rearrange Eq. (54) by subtracting and adding the two first expansion terms of the function under the integral in powers of $x$

$$
\Phi_{\parallel}(x) = \int_x^\infty dy \left[ y^2 \frac{r_{\parallel}^{(1)}(x,y)r_{\parallel}^{(2)}(x,y)e^{-y}}{1 - r_{\parallel}^{(1)}(x,y)r_{\parallel}^{(2)}(x,y)e^{-y}} - y^2 \frac{r_0^{(1)}r_0^{(2)}e^{-y}}{1 - r_0^{(1)}r_0^{(2)}e^{-y}} \right] + x^2 \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)}\varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \frac{r_0^{(1)}r_0^{(2)}e^{-y}}{\left(1 - r_0^{(1)}r_0^{(2)}e^{-y}\right)^2} + \int_x^\infty y^2 dy \frac{r_0^{(1)}r_0^{(2)}e^{-y}}{1 - r_0^{(1)}r_0^{(2)}e^{-y}} - x^2 \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)}\varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \int_x^\infty dy \frac{r_0^{(1)}r_0^{(2)}e^{-y}}{\left(1 - r_0^{(1)}r_0^{(2)}e^{-y}\right)^2}.
$$

(B1)

We consider the first integral on the right-hand side of Eq. (B1) written in terms of a new variable $v = y/x$

$$
Q_1(x) = x^3 \int_1^\infty dv \left[ v^2 \frac{r_{\parallel}^{(1)}(0,v)r_{\parallel}^{(2)}(0,v)e^{-vx}}{1 - r_{\parallel}^{(1)}(0,v)r_{\parallel}^{(2)}(0,v)e^{-vx}} - v^2 \frac{r_0^{(1)}r_0^{(2)}e^{-vx}}{1 - r_0^{(1)}r_0^{(2)}e^{-vx}} \right] + x^2 \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)}\varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \frac{r_0^{(1)}r_0^{(2)}e^{-vx}}{\left(1 - r_0^{(1)}r_0^{(2)}e^{-vx}\right)^2},
$$

(B2)

where, in accordance with Eq. (B3),

$$
r_{\parallel}^{(k)}(0,v) = \frac{\varepsilon_0^{(k)}v - \sqrt{v^2 + \varepsilon_0^{(k)} - 1}}{\varepsilon_0^{(k)}v + \sqrt{v^2 + \varepsilon_0^{(k)} - 1}}.
$$

(B3)
The leading expansion order of $Q_1(x)$ from Eq. (B2) in powers of $x$ is

$$x^3 \int_1^\infty dv \left[ v^2 \frac{r_0^{(1)}(0,v) r_0^{(2)}(0,v)}{1 - r_0^{(1)}(0,v) r_0^{(2)}(0,v)} - v^2 \frac{r_0^{(1)}(0,v)}{1 - r_0^{(1)} r_0^{(2)}} \right] + \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \frac{r_0^{(1)} r_0^{(2)}}{(1 - r_0^{(1)} r_0^{(2)})^2}$$ \quad \text{(B4)}

The explicit calculation of the integral in Eq. (B4) leads to

$$Q_1(x) = \left[ 1 - \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}}{\sqrt{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}}} - \frac{3\varepsilon_0^{(1)} \varepsilon_0^{(2)} \sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)}}}{\sqrt{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}} (\varepsilon_0^{(1)} + \varepsilon_0^{(2)})} \right] \left[ \varepsilon_0^{(1)} \varepsilon_0^{(2)} \right]$$

$$+ \frac{\varepsilon_0^{(1)} \varepsilon_0^{(2)} (3 \sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)}} + \varepsilon_0^{(1)} + \varepsilon_0^{(2)})}{\sqrt{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}} (\varepsilon_0^{(1)} + \varepsilon_0^{(2)})^2} \right]$$

$$- \frac{\varepsilon_0^{(1)} \varepsilon_0^{(2)} (\sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)} - \varepsilon_0^{(1)} - \varepsilon_0^{(2)})}{\sqrt{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}} (\varepsilon_0^{(1)} + \varepsilon_0^{(2)})^2} \right]$$

$$- \frac{3\varepsilon_0^{(1)} \varepsilon_0^{(2)} (\varepsilon_0^{(1)} - 1) (\varepsilon_0^{(2)} - 1)}{(\varepsilon_0^{(1)} - \varepsilon_0^{(2)}) (\varepsilon_0^{(1)} + \varepsilon_0^{(2)})^{5/2}} \right]$$

$$+ \frac{\varepsilon_0^{(1)} \varepsilon_0^{(2)} (\sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)} - \varepsilon_0^{(1)} - \varepsilon_0^{(2)})}{\sqrt{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}} (\varepsilon_0^{(1)} + \varepsilon_0^{(2)})^2} \right]$$

$$x^3 \frac{r_0^{(1)} r_0^{(2)}}{3 \left[ 1 - r_0^{(1)} r_0^{(2)} \right]} - x^3 \frac{\varepsilon_0^{(1)} \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)} + \varepsilon_0^{(1)} \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} \right]$$

$$+ O(x^4).$$

Now we calculate the second integral on the right-hand side of Eq. (B5),

$$Q_2(x) = \int_x^\infty dy \frac{r_0^{(1)} r_0^{(2)} e^{-y}}{1 - r_0^{(1)} r_0^{(2)} e^{-y}}$$

$$= \sum_{n=1}^\infty \left( r_0^{(1)} r_0^{(2)} \right)^n \frac{(2 + 2nx + n^2 x^2) e^{-nx}}{n^3}.$$

Expanding Eq. (B6) in powers of $x$, one obtains

$$Q_2(x) = 2 \text{Li}_3 \left( r_0^{(1)} r_0^{(2)} \right) - x^3 \frac{r_0^{(1)} r_0^{(2)}}{3 \left[ 1 - r_0^{(1)} r_0^{(2)} \right]} + O(x^4).$$

The integral contained in the third term on the right-hand side of Eq. (B5) is simply
calculated,
\[ Q_3(x) = x^2 \int_x^\infty dy \frac{r_0^{(1)} r_0^{(2)} e^{-y}}{\left(1 - r_0^{(1)} r_0^{(2)} e^{-y}\right)^2} \]  
(B8)

\[ = x^2 \frac{r_0^{(1)} r_0^{(2)} e^{-x}}{1 - r_0^{(1)} r_0^{(2)} e^{-x}}. \]

It can be expanded in powers of \( x \) as follows:
\[ Q_3(x) = x^2 \frac{r_0^{(1)} r_0^{(2)}}{1 - r_0^{(1)} r_0^{(2)}} - x^3 \frac{r_0^{(1)} r_0^{(2)}}{\left(1 - r_0^{(1)} r_0^{(2)}\right)^2} + O(x^4). \]  
(B9)

Finally, according to Eq. (B1), the function under consideration is given by
\[ \Phi_\parallel(x) = Q_1(x) + Q_2(x) - \frac{\varepsilon_0^{(1)} + \varepsilon_0^{(2)} + 2\varepsilon_0^{(1)} \varepsilon_0^{(2)}}{(\varepsilon_0^{(1)} + 1)(\varepsilon_0^{(2)} + 1)} Q_3(x), \]  
(B10)

where \( Q_1(x), Q_2(x), \) and \( Q_3(x) \) are found in Eqs. (B5), (B7), and (B9), respectively. It is notable that the contributions of the third power from \( Q_2(x) \) and \( Q_3(x) \) in Eq. (B10) cancel the second and third contributions from \( Q_1(x) \).

Using Eq. (B10), one arrives at Eq. (60) after some identical rearrangements.

**APPENDIX C**

The quantity \( R(a, T) \) was introduced in Eq. (76) of Sec. VI and has the following explicit form:
\[ R(a, T) = R^{(1)}(a, T) + R^{(2)}(a, T) + O \left[ (\beta^k / l)^2 \right], \]  
(C1)

where
\[ R^{(1)}(a, T) = \frac{k_B T}{8 \pi a^2} \sum_{l=1}^{\infty} \frac{\beta^{(1)}}{l} \int_{\zeta_l}^{\infty} \frac{dy y^2 e^{-y}}{\sqrt{y^2 + \zeta_l^2 (\varepsilon_l^{(1)} - 1)}} \]
\[ \times \left[ \frac{(2 - \varepsilon_l^{(1)}) \zeta_l^2 - 2y^2}{(\varepsilon_l^{(1)} y + \sqrt{y^2 + \zeta_l^2 (\varepsilon_l^{(1)} - 1)})^2} \frac{r_\parallel^{(2)}(\zeta_l, y)}{1 - r_\parallel^{(1)}(\zeta_l, y) r_\parallel^{(2)}(\zeta_l, y) e^{-y}} \right. \]
\[ \left. - \frac{\zeta_l^2}{\sqrt{y^2 + \zeta_l^2 (\varepsilon_l^{(1)} - 1) + y}} \frac{r_\perp^{(2)}(\zeta_l, y)}{1 - r_\parallel^{(1)}(\zeta_l, y) r_\perp^{(2)}(\zeta_l, y) e^{-y}} \right] \]  
(C2)

and \( R^{(2)}(a, T) \) is obtained from \( R^{(1)}(a, T) \) by interchanging of the upper indices (1) and (2). In this Appendix we demonstrate that \( R(a, T) \) vanishes exponentially when \( T \to 0 \).
Let us consider the integral with respect to \( y \) from Eq. (C2), expand the integrated function in powers of \( \tau \) (we recall that \( \zeta_l = \tau l \)) and restrict ourselves by the main contribution to the result at \( \tau = 0 \):

\[
-2 \int_{\zeta_l}^{\infty} dy \frac{y e^{-y}}{(\varepsilon_0^{(1)})^2 + 1} \frac{r_0^{(2)}}{1 - r_0^{(1)} r_0^{(2)} e^{-y}},
\]

where \( r_0^{(k)} \) was defined in Eq. (A4). With account of Eq. (A4), Eq. (C3) can be rearranged to the form

\[
-\frac{2}{(\varepsilon_0^{(1)})^2 - 1} \int_{\zeta_l}^{\infty} dy \frac{y r_0^{(1)} r_0^{(2)} e^{-y}}{1 - r_0^{(1)} r_0^{(2)} e^{-y}} = -\frac{2}{(\varepsilon_0^{(1)})^2 - 1} \sum_{n=1}^{\infty} \left( r_0^{(1)} r_0^{(2)} \right)^n \int_{\zeta_l}^{\infty} dy \frac{y}{1 - r_0^{(1)} r_0^{(2)} e^{-y}}.
\]

Substituting Eq. (C4) in Eq. (C2), we find

\[
R^{(1)}(a, T) = -\frac{k_B T \beta^{(1)}}{4 \pi a^2 \left( \varepsilon_0^{(1)} \right)^2 - 1} \sum_{n=1}^{\infty} \left( r_0^{(1)} r_0^{(2)} \right)^n \int_{\zeta_l}^{\infty} dy \frac{y}{1 - r_0^{(1)} r_0^{(2)} e^{-y}} \times \left[ \sum_{l=1}^{\infty} e^{-n\tau l} + n\tau \sum_{l=1}^{\infty} e^{-n\tau l} \right] + T \beta^{(1)} O(\tau^0) = \\
= -\frac{k_B T \beta^{(1)}}{4 \pi a^2 \left( \varepsilon_0^{(1)} \right)^2 - 1} \sum_{n=1}^{\infty} \left( r_0^{(1)} r_0^{(2)} \right)^n \sum_{n=1}^{\infty} \frac{1 + n\zeta_l}{n^2} e^{-n\zeta_l}.
\]

It is readily seen that

\[
-\ln \left( 1 - e^{-n\tau} \right) + \frac{n\tau}{e^{n\tau} - 1} = -\ln \tau + 1 - \ln n + O(\tau^2).
\]

The substitution of the leading term on the right-hand side of Eq. (C6) in Eq. (C5) results in

\[
R^{(1)}(a, T) = -\frac{k_B T \beta^{(1)}}{4 \pi a^2 \left( \varepsilon_0^{(1)} \right)^2 - 1} \sum_{n=1}^{\infty} \left( r_0^{(1)} r_0^{(2)} \right)^n \int_{\zeta_l}^{\infty} dy \frac{y}{1 - r_0^{(1)} r_0^{(2)} e^{-y}} \times \left[ -\ln \left( 1 - e^{-n\tau} \right) + \frac{n\tau}{e^{n\tau} - 1} \right] + T \beta^{(1)} O(\tau^0) = \\
= \frac{k_B \ln_2 \left( r_0^{(1)} r_0^{(2)} \right)}{4 \pi a^2 \left( \varepsilon_0^{(1)} \right)^2 - 1} T \beta^{(1)} \ln \tau + T \beta^{(1)} O(\tau^0).
\]
Taking into account that $\beta^{(1)} \sim (1/T) \exp(-b^{(1)}/T)$ (see Sec. VI), we get the conclusion that the temperature dependence of $R^{(1)}(a, T)$ is determined by the term

$$R^{(1)}(a, T) \sim e^{-b^{(1)}/T \ln T},$$

(C8)

i.e., both $R^{(1)}(a, T)$ and its derivative go to zero as $\exp(-b^{(1)}/T)$ when the temperature goes to zero.

In perfect analogy to $R^{(1)}(a, T)$, the same conclusion is obtained for $R^{(2)}(a, T)$. The terms of the second and higher powers in $\beta^{(k)}$ [see Eq. (C1)] go to zero even faster than $R^{(k)}(a, T)$ when $T \to 0$. In this way, with account of Eq. (C1), we have proven the exponentially fast vanishing of $R(a, T)$ with decrease of the temperature.

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Figures
FIG. 1: Dielectric permittivity of Si (line 1) and vitreous SiO$_2$ (line 2) along the imaginary frequency axis as a function of the logarithm of frequency.
FIG. 2: Magnitudes of the thermal corrections to the energy (a) and pressure (b) in configuration of two plates one made of Si and another one of SiO$_2$ at a separation $a = 400$ nm as a function of temperature calculated by the use of different approaches: by the Lifshitz formula and tabulated optical data (solid lines), by the Lifshitz formula and static dielectric permittivities (short-dashed lines), by the asymptotic expressions in Eqs. (50), (51) and (61) (long-dashed lines).
FIG. 3: Magnitudes of the thermal corrections to the energy (a) and pressure (b) in configuration of two plates made of vitreous SiO$_2$ at a separation $a = 450 \text{ nm}$ as a function of temperature calculated by the use of different approaches: by the Lifshitz formula and tabulated optical data (solid lines), by the Lifshitz formula and static dielectric permittivities (short-dashed lines), by the asymptotic expressions in Eqs. (51), (62) and (63) (long-dashed lines).
FIG. 4: Magnitudes of the thermal corrections to the energy (a) and pressure (b) in configuration of two plates made of Si at a separation $a = 300 \text{ nm}$ as a function of temperature calculated by the use of different approaches: by the Lifshitz formula and tabulated optical data (solid lines), by the Lifshitz formula and static dielectric permittivities (short-dashed lines), by the asymptotic expressions in Eqs. (51), (62) and (63) (long-dashed lines).