Luttinger liquid in a non-equilibrium steady state

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Abstract

We propose and investigate an exactly solvable model of non-equilibrium Luttinger liquid on a star graph, modeling a multi-terminal quantum wire junction. The boundary condition at the junction is fixed by an orthogonal matrix \( S \), which describes the splitting of the electric current among the leads. The system is driven away from equilibrium by connecting the leads to heat baths at different temperatures and chemical potentials. The associated non-equilibrium steady state depends on \( S \) and is explicitly constructed. In this context, we develop a non-equilibrium bosonization procedure and compute some basic correlation functions. Luttinger liquids with general anyon statistics are considered. The relative momentum distribution away from equilibrium turns out to be the convolution of equilibrium anyon distributions at different temperatures. Both the charge and heat transport are studied. The exact current–current correlation function is derived and the zero-frequency noise power is determined.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The universal features of a large class of one-dimensional quantum models, exhibiting gapless excitations with linear spectrum, are successfully described \cite{1, 2} by Tomonaga–Luttinger (TL) liquid theory \cite{3–6}. This theory\textsuperscript{3} applies to various systems, including nanowire junctions and carbon nanotubes, which are available nowadays in experiment \cite{10–13}. For this reason, the study of non-equilibrium phenomena in the TL liquid phase has attracted much attention recently \cite{14–18}.

A typical non-equilibrium setup, considered in the literature, is the junction of two or more semi-infinite leads with electrons at different temperatures and/or chemical potentials.

\textsuperscript{3} For some more recent reviews, we refer to \cite{7–9}.
The junction is an interval of finite length $L$, where the electrons injected from the leads interact among themselves. This interaction drives the system away from equilibrium. Different from the equilibrium TL liquid on the line, the non-equilibrium model defined in this way is not exactly solvable. Nevertheless, it is extensively studied \cite{14–18} by various methods, including linear response theory, bosonization combined with the non-equilibrium Keldish formalism and perturbation theory.

One of the main goals of this paper is to explore the possibility of constructing and analyzing an alternative exactly solvable model for a non-equilibrium TL junction. Since the universal features of such a system are expected to manifest themselves in the critical (scale invariant) limit, it is natural to shrink the domain of the non-equilibrium interaction to a point, taking $L \to 0$. For a complete description of the critical regime, it is essential to take into account all point-like interactions, which ensure unitary time evolution of the system. These interactions can be parametrized by a scattering matrix $S$ localized at the junction point, as shown in the multi-terminal setup displayed in figure 1. Each lead contains a TL liquid, which at infinity is in contact with a heat reservoir with (inverse) temperature $\beta_i$ and chemical potential $\mu_i$. Our first step below is to show that there exists a non-equilibrium steady state (NESS), which describes the TL configuration in figure 1. This state is characterized by non-trivial time-independent electric and heat currents, flowing in the leads. The scattering matrix $S$ is implemented by imposing specific boundary conditions at the junction. It turns out that the boundary conditions, which describe the splitting of the electric steady current at the junction, lead to an exactly solvable problem. In fact, we establish the operator solution in this case and investigate the relative non-equilibrium correlation functions in the NESS representation.

The TL theory was introduced originally \cite{3–6} for describing fermion systems. It was understood later on \cite{19–21} that the fermion TL liquid is actually an element of a more general family of anyon TL liquids, which obey Abelian braid statistics. In this paper, we explore the general anyon TL liquid, obtaining the conventional fermionic and bosonic ones as special cases.

From the two-point anyon correlation functions, we extract the NESS distribution of the TL anyon excitations. In momentum space, this non-equilibrium distribution is a nested convolution of equilibrium distributions at different temperatures and chemical potentials. As expected, the convolution depends on the scattering matrix $S$, which drives the system away from equilibrium. We also investigate the NESS correlators of the electric and energy

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A junction with scattering matrix $S$ and $n$ semi-infinite leads, connected at infinity to thermal reservoirs with temperatures $\beta_i$ and chemical potentials $\mu_i$.}
\end{figure}
currents, describing in detail the charge and heat transport at the junction. The zero-frequency noise power is deduced from the two-point current–current correlation function, whose exact expression in terms of hypergeometric functions is established. We prove the breakdown of time reversal invariance as well.

The paper has the following structure. In the following section, we construct non-equilibrium chiral fields in an NESS on a star graph modeling the junction. We derive here the non-equilibrium Casimir energy and the heat current and compare the latter with the conformal field theory (CFT) result. In section 3, we develop a non-equilibrium finite temperature operator bosonization procedure. We also establish the operator solution, subject to the current splitting boundary condition at the junction. We show that this condition covers two different physical situations, corresponding to a junction with and without charge dissipation. The non-equilibrium correlation functions are investigated in section 4, where the anyon NESS distributions are derived. The charge and heat transport as well as the noise are also studied there. Section 5 provides a concise outlook of the paper and contains some general observations. The appendix collects some results about the asymptotic properties of the anyon NESS correlators.

2. Non-equilibrium chiral fields on a star graph

The fundamental building blocks of bosonization away from equilibrium are the free massless scalar field $\phi$ and its dual $\tilde{\phi}$. The fields $\phi$ and $\tilde{\phi}$ propagate on a star graph $\Gamma$, which is shown in figure 2 and models the quantum wire junction. The edges $E_i$ are half-lines and each point $P$ in the bulk $\Gamma \setminus V$ of $\Gamma$ is uniquely determined by its coordinates $(x, i)$, where $x > 0$ is the distance to the vertex $V$ and $i = 1, \ldots, n$ labels the edge. Besides the massless Klein–Gordon equation, the fields $\phi$ and $\tilde{\phi}$ satisfy the duality relations

$$\partial_t \tilde{\phi}(t, x, i) = -\partial_x \phi(t, x, i), \quad \partial_x \tilde{\phi}(t, x, i) = -\partial_t \phi(t, x, i). \quad (2.1)$$

The initial conditions are fixed by the equal-time canonical commutation relations

$$[\phi(t, x, i), \phi(t, y, j)]_- = [\tilde{\phi}(t, x, i), \tilde{\phi}(t, y, j)]_- = 0, \quad (2.2)$$

$$[\partial_t \phi(t, x, i), \phi(t, y, j)]_- = [(\partial_t \tilde{\phi})(t, x, i), \tilde{\phi}(t, y, j)]_- = -i \delta_{ij} \delta(x - y). \quad (2.3)$$

In order to determine the dynamics completely, one must impose some boundary conditions at the vertex $x = 0$. These conditions are conveniently formulated in terms of the combinations

$$\phi_{+, R}(t - x) = \phi(t, x, i) + \tilde{\phi}(t, x, i), \quad \phi_{+, L}(t + x) = \phi(t, x, i) - \tilde{\phi}(t, x, i), \quad \phi_{-, L}(t - x) = \phi(t, x, i) + \tilde{\phi}(t, x, i), \quad \phi_{-, R}(t + x) = \phi(t, x, i) - \tilde{\phi}(t, x, i), \quad (2.4)$$

Figure 2. A star graph $\Gamma$ with $n$ edges modeling the junction of $n$ quantum wires.
which depend on \( t - x \) and \( t + x \) respectively and define right and left chiral fields \( \varphi_{i,\beta,\mu} \) on \( \Gamma \). The most general scale invariant boundary conditions, generating a unitary time evolution of \( \varphi \) and \( \tilde{\varphi} \), are parametrized by the orthogonal group \( O(n) \) and read \([24–27]\)

\[
\varphi_{i,R}(\xi) = \sum_{j=1}^{n} S_{ij} \varphi_{j,L}(\xi), \quad \mathbb{S} \in O(n).
\]  

(2.5)

These simple conditions capture the universal features of the system and \( \mathbb{S} \) has a straightforward physical interpretation: the vertex \( V \) of \( \Gamma \) represents a scale invariant point-like defect, with \( \mathbb{S} \) being the associated scattering matrix.

### 2.1. The non-equilibrium steady state \( \Omega_{\beta,\mu_\beta} \)

Our next step is to construct a steady state \( \Omega_{\beta,\mu_\beta} \), which captures the evolution of the chiral fields \( \varphi_{i,\beta,\mu} \) on \( \Gamma \), whose edges are attached at infinity to thermal reservoirs at inverse temperatures \( \beta_i \), as shown in figure 1. In the boson case, we take all chemical potentials to be equal\(^5\), setting \( \mu_{i\beta} = \mu_\beta \) in all reservoirs. The system is away from equilibrium if \( \mathbb{S} \) contains at least one non-trivial transmission coefficient among reservoirs with different temperature. The construction of \( \Omega_{\beta,\mu_\beta} \), described below, follows the scheme developed in \([28]\) and is based on scattering theory. It adapts some modern ideas about NESS to the case under consideration \([29–33]\). The framework is purely algebraic and generalizes the definition \([34]\) of an equilibrium Gibbs state over the algebra of canonical commutation relations (CCR).

We start by observing that the massless Klein–Gordon equation and the relations (2.1) lead to the following representation:

\[
\varphi_{i,R}(\xi) = \int_{0}^{\infty} \frac{dk}{\pi \sqrt{2}} \sqrt{\Delta_i(k)} \left[ a_i^+(k) e^{i\xi} + a_i(k) e^{-i\xi} \right],
\]  

(6)

\[
\varphi_{i,L}(\xi) = \int_{0}^{\infty} \frac{dk}{\pi \sqrt{2}} \sqrt{\Delta_i(k)} \left[ a_i^-(k) e^{i\xi} + a_i(-k) e^{-i\xi} \right],
\]  

(7)

with \( \Delta_i \) being some distribution to be fixed below. Using the fact that \( \mathbb{S} \) is a real matrix, the boundary condition (2.5) implies the constraints

\[
a_i(k) = \sum_{j=1}^{n} S_{ij}(k) a_j(-k), \quad a_i^+(k) = \sum_{j=1}^{n} S_{ij}(k) a_j^+(-k),
\]  

(8)

where

\[
S(k) = \theta(-k)\mathbb{S}^T + \theta(k)\mathbb{S},
\]  

(9)

\( \theta \) is the Heaviside step function and \( \mathbb{S}^T \) indicates the transpose of \( \mathbb{S} \). From the equal-time commutation relations (2.2) and (2.3), one infers that the elements \( \left\{ a_i(k), a_i^+(k) : k \in \mathbb{R}, i = 1, \ldots, n \right\} \) generate the following deformation \( \mathcal{A} \) of the algebra of CCR:

\[
[a_i(k), a_j(p)] = [a_i^+(k), a_j^+(p)] = 0,
\]  

(10)

\[
[a_i(k), a_j^+(p)] = 2\pi [\delta(k - p)\delta_{ij} + S_{ij}(k)\delta(k + p)].
\]  

(11)

Moreover, (2.2) and (2.3) imply that

\[
|k|\Delta_i(k) = 1.
\]  

(12)

\(^5\) This choice will not prevent us to deal in the fermion case below with arbitrary \( \mu_i \).
There exists a one-parameter family of tempered distributions, which solve this equation in $\mathbb{R}$. A convenient representation of this family is given by \[35\]

$$
\Delta_\lambda(k) = \frac{d}{dk} \left[ \theta(k) \ln \frac{k}{\lambda} \right],
$$

(2.13)

where $\lambda > 0$ is a free parameter with dimension of mass, having well-known infrared origin.

The above structure is very general and equations (2.6) and (2.7) apply to any representation of the algebra $A$, which is a simplified version of the so-called reflection–transmission (RT) algebra \[36–38\], describing factorized scattering in integrable models with point-like defects in one dimension. The Fock and the Gibbs states over $A$ describe equilibrium physics and have been largely explored. We will investigate here the NESS $\Omega_{\beta,\mu_b}$, which describes the physical situation shown in figure 1. For this purpose, we first observe that the sub-algebras $A_{\text{in}}$ and $A_{\text{out}}$, generated by the elements $\{a_i(k), a_i^*(k) : k < 0\}$ and $\{a_i(k), a_i^*(k) : k > 0\}$ respectively, parametrize the asymptotic incoming and outgoing fields. Accordingly, both $A_{\text{in}}$ and $A_{\text{out}}$ are conventional CCR algebras; in fact, the $\delta(k + p)$ term in (2.11) vanishes if both momenta are negative or positive. It is worth stressing that (2.8) relate $A_{\text{in}}$ with $A_{\text{out}}$ and that the whole RT algebra $A$ can be generated via (2.8) either by $A_{\text{in}}$ or by $A_{\text{out}}$. The main idea for constructing $\Omega_{\beta,\mu_b}$ is based on this kind of asymptotic completeness property. Starting with an equilibrium state on $A_{\text{in}}$, we will extend it by means of (2.8) to a non-equilibrium state on the whole algebra $A$. For this purpose, we introduce the edge Hamiltonian and number operators

$$
h_i = \int_{-\infty}^{0} \frac{dk}{2\pi} [k|a_i^*(k)a_i(k), \qquad n_i = \int_{-\infty}^{0} \frac{dk}{2\pi} a_i^*(k)a_i(k),
$$

(2.14)

which describe the asymptotic dynamics at $t = -\infty$ (i.e. before the interaction) in terms of $A_{\text{in}}$. Defining

$$
K = \sum_{i=1}^{n} \beta_i(h_i - \mu_b n_i), \qquad \beta_i \geq 0,
$$

(2.15)

we introduce the equilibrium Gibbs state over $A_{\text{in}}$ in the standard way \[34\]. For any polynomial $P$ over $A_{\text{in}}$, we set

$$
(\Omega_{\beta,\mu_b}, P(a_i^*(k_i), a_j(p_j))\Omega_{\beta,\mu_b}) \equiv \langle P(a_i^*(k_i), a_j(p_j)) \rangle_{\beta,\mu_b} = \frac{1}{Z} \text{Tr}[e^{-\beta P(a_i^*(k_i), a_j(p_j))}],
$$

(2.16)

where $k_i < 0$, $p_j < 0$ and $Z = \text{Tr}(e^{-\beta K})$. All the expectation values (2.16) can be computed \[34\] by purely algebraic manipulations and can be expressed in terms of the two-point functions, which are written in terms of the familiar Bose distribution

$$
h_i(k) = \frac{e^{-\beta|k| - \mu_b}}{1 - e^{-\beta|k| - \mu_b}}
$$

(2.17)

in the following way:

$$
\langle a_i^*(p) a_i(k) \rangle_{\beta,\mu_b} = b_i(k) \delta_{ij} 2\pi \delta(k - p),
$$

(2.18)

$$
\langle a_i(k) a_i^*(p) \rangle_{\beta,\mu_b} = [1 + b_i(k)] \delta_{ij} 2\pi \delta(k - p).
$$

(2.19)

We stress that (2.18) and (2.19) hold on $A_{\text{in}}$, i.e. only for negative momenta. The chemical potential common for all reservoirs, $\mu_b < 0$, allows us to avoid in (2.18) and (2.19) the infrared singularity at $k = 0$. We anticipate that $\mu_b$ has nothing to do with the fermion chemical potentials, appearing in the non-equilibrium bosonization procedure described in the following section, where the limit $\mu_b \to 0^-$ exists and will be performed.
The next step is to extend (2.16)–(2.19) to the whole RT algebra $\mathcal{A}$, namely to positive momenta. Employing (2.8), one finds
\[
\langle a_\ast^\alpha(p) a_\beta(k) \rangle_{\beta,\mu} = 2\pi \left[ \theta(-k) b_\beta(k) \delta_{ij} + \theta(k) \sum_{j=1}^{n} S_{ij} b_j(k) S_{ij}^\ast \right] \delta(k - p) + [\theta(-k) b_\beta(k) S_{ij}^\ast + \theta(k) S_{ij} b_j(k)] \delta(k + p) \quad (2.20)
\]

The expression for $\langle a_\beta(k) a_\ast^\alpha(p) \rangle_{\beta,\mu}$ is obtained from (2.20) by the substitution
\[
b_\beta(k) \mapsto 1 + b_\beta(k) = \frac{1}{1 - e^{-\delta(|k| - \mu)}} \quad (2.21)
\]

The final step is to compute a generic correlation function. By means of the commutation relations (2.10) and (2.11), this problem is reduced to the evaluation of correlators of the form
\[
\left\langle \prod_{m=1}^{M} a_{\alpha_m}(k_{\alpha_m}) \prod_{n=1}^{N} a_{\beta_n}^\ast(p_{\beta_n}) \right\rangle_{\beta,\mu} \quad (2.22)
\]

We would like to mention in conclusion that the use of the RT algebra $\mathcal{A}$ in the construction of $\Omega_{\beta,\mu}$ represents only a convenient choice of coordinates, which has a simple physical interpretation in terms of scattering data and applies to a variety of systems [28, 39] with point-like defects.

### 2.2. Energy density and energy transport in $\Omega_{\beta,\mu}$

In order to illustrate the physical properties of $\Omega_{\beta,\mu}$, it is instructive to investigate the non-equilibrium energy density and transport associated with the scalar field $\phi$. The equations of motion imply the conservation
\[
\partial_t \theta_j(t, x, i) - \partial_i \theta_j(t, x, i) = 0, \quad (2.23)
\]
of the energy–momentum tensor
\[
\theta_j(t, x, i) = \frac{1}{2} : [\partial_j \phi(t) \partial_i \phi(t) - \phi^2(t) \partial_j \phi(t)] : (t, x, i), \quad (2.24)
\]

where $\cdot \cdot \cdot$ denotes the normal product in the algebra $\mathcal{A}$. The boundary condition (2.5) implies the Kirchhoff rule
\[
\sum_{i=1}^{n} \theta_j(t, 0, i) = 0, \quad (2.26)
\]

which, combined with (2.23), ensures energy conservation.

The derivation of $\theta_j(t, x, i)_{\beta,\mu}$ and $\theta_j(t, x, i)_{\beta,\mu}$ is based on the expectation value
\[
\langle \phi(t_1, x_1, i_1) \phi(t_2, x_2, i_2) \rangle_{\beta,\mu} = \int_0^\infty \frac{dk}{2\pi} \Delta_k(k) \left\{ \delta_{ij} b_i(k) [k(t_{12} + x_{12})] + S_{ij} b_j(k) \cos[k(t_{12} - x_{12})] \right. \\
+ b_i(k) S_{ij}^\ast \cos[k(t_{12} - x_{12})] + \sum_{j=1}^{n} S_{ij} b_j(k) S_{ij}^\ast \cos[k(t_{12} - x_{12})] \right\} \quad (2.27)
\]
where $x_{12} = x_1 + x_2$. Plugging (2.27) in the definitions (2.24) and (2.25), one obtains

\begin{equation}
\mathcal{E}_i(\beta, \mu_b) \equiv \langle \theta_{il}(t, x, i) \rangle_{\beta, \mu_b} = S_{ii} \int_0^\infty \frac{dk}{\pi} k \cos(2kx)b_i(k) + \sum_{j=1}^n \left( \delta_{ij} + S_{ij}^2 \right) \frac{1}{2\pi \beta_j^2} \operatorname{Li}_2(e^{\beta_i \mu_b}),
\end{equation}

(2.28)

\begin{equation}
\mathcal{T}_i(\beta, \mu_b) \equiv \langle \theta_{il}(t, x, i) \rangle_{\beta, \mu_b} = \sum_{j=1}^n \left( \delta_{ij} - S_{ij}^2 \right) \frac{1}{2\pi \beta_j^2} \operatorname{Li}_2(e^{\beta_i \mu_b}),
\end{equation}

(2.29)

where $\operatorname{Li}_2$ is the dilogarithm function. Equation (2.28) determines the Casimir energy, whereas (2.29) describes the energy (heat) transport. Both are time independent, thus confirming that we are dealing with a steady state. Note also that the energy density is $x$-dependent, which reflects the breaking of translation invariance by the junction and is consistent with the conservation law (2.23). Since $S$ is an orthogonal matrix, $\mathcal{T}_i(\beta, \mu_b)$ obviously satisfies the Kirchhoff rule (2.26). The energy density $\mathcal{E}_i(\beta, \mu_b)$ can be written in the equivalent form as

\begin{equation}
\mathcal{E}_i(\beta, \mu_b) = S_{ii} \int_0^\infty \frac{dk}{\pi} k \cos(2kx)b_i(k) + \frac{1}{\pi \beta_i} \operatorname{Li}_2(e^{\beta_i \mu_b}) - \sum_{j=1}^n \left( \delta_{ij} - S_{ij}^2 \right) \frac{1}{2\pi \beta_j^2} \operatorname{Li}_2(e^{\beta_i \mu_b}),
\end{equation}

(2.30)

where the last term vanishes at equilibrium and describes therefore the non-equilibrium contribution to the Casimir energy.

For a junction with $n = 2$ wires, there are two one-parameter families,

\begin{equation}
S^+ = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad S^- = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \quad \theta \in [0, 2\pi),
\end{equation}

(2.31)

of scattering matrices with $\det(S^+) = \pm 1$. For both families, one finds

\begin{equation}
\mathcal{T}_i(\beta, \mu_b) = -\mathcal{T}_2(\beta, \mu_b) = \frac{\sin^2 \theta}{2\pi} \left[ \frac{1}{\beta_i^2} \operatorname{Li}_2(e^{\beta_i \mu_b}) - \frac{1}{\beta_j^2} \operatorname{Li}_2(e^{\beta_j \mu_b}) \right],
\end{equation}

(2.32)

which deserves a comment. Heat transport has been investigated recently in the framework of CFT for generic central charge $c$ in [40]. Equation (2.32) confirms the result of [40] for $c = 1$ and extends this result in two directions: imperfect junction with transmission probability $\sin^2(\theta)$ and $\mu_b \neq 0$. We observe in this respect that the derivation of the dilogarithm terms in (2.32) is problematic in the CFT context, because both $\beta_1 \mu_b$ and $\beta_2 \mu_b$ are non-trivial dimensionless parameters.

In the limit $\mu_b \to 0^-$, the $x$-dependent integral in (2.28) can be evaluated explicitly, and one finds

\begin{equation}
\mathcal{E}_i(\beta, 0) = \frac{\pi}{12} \sum_{j=1}^n \left( \delta_{ij} + S_{ij}^2 \right) \frac{1}{\beta_j^2} + S_{ii} \frac{1}{8\pi x^2} - S_{ii} \frac{\pi}{2\beta_i^2 \left[ \sinh \left( 2\pi \frac{x}{\beta_i} \right) \right]^2},
\end{equation}

(2.33)

\begin{equation}
\mathcal{T}_i(\beta, 0) = \frac{\pi}{12} \sum_{j=1}^n \left( \delta_{ij} - S_{ij}^2 \right) \frac{1}{\beta_j^2},
\end{equation}

(2.34)
2.3. Chiral NESS correlators

In the non-equilibrium bosonization procedure, developed below, we will need the correlation functions of the chiral fields (2.6) and (2.7) in the NESS $\Omega_{\beta,\mu_b}$. It is easily seen that all of them can be expressed in terms of the distribution
\[
\omega(\xi, \beta; \lambda, \mu_b) = \frac{1}{\pi} \int_0^\infty \frac{dk}{\Delta_1(\lambda)} \left[ \frac{e^{-\beta|k-\mu_b|}}{1 - e^{-\beta|k-\mu_b|}} e^{ik\xi} + \frac{1}{1 - e^{-\beta|k-\mu_b|}} e^{-i k\xi} \right].
\] (2.35)

The full $\lambda$-dependence and the singularity at $\mu_b = 0$ of (2.35) are captured by [19]
\[
\omega(\xi, \beta; \lambda, \mu_b) = \frac{1}{\pi} \left\{ \frac{2}{\beta|\mu_b|} \ln \frac{|\mu_b|}{\lambda} - \ln \left[ 2i \sinh \left( \frac{\pi}{\beta} \xi - i \epsilon \right) \right] \right\} + o(\mu_b),
\] (2.36)
where $o(\mu_b)$ stands for $\lambda$-independent terms, which vanish in the limit $\mu_b \to 0^-$. It is convenient at this point to relate the (up to now free) infrared regularization parameter $\lambda$ to $\mu_b$ by means of
\[
\lambda = |\mu_b|.
\] (2.37)

The limit $\mu_b \to 0^-$ in (2.35) now exists and gives the distribution
\[
\omega(\xi, \beta) \equiv \lim_{\mu_b \to 0^-} \omega(\xi, \beta; \lambda = |\mu_b|, \mu_b) = -\frac{1}{\pi} \ln \left[ 2i \sinh \left( \frac{\pi}{\beta} \xi - i \epsilon \right) \right],
\] (2.38)
which is the fundamental block of the NESS chiral correlation functions. In fact, for the two-point correlators one obtains
\[
\langle \psi_{i,L}(\xi_1)\psi_{i,L}(\xi_2) \rangle_{\beta} = \delta_{i_{12}} w(\xi_{12}, \beta_i),
\] (2.39)
\[
\langle \psi_{i,L}(\xi_1)\psi_{i,R}(\xi_2) \rangle_{\beta} = w(\xi_{12}, \beta_i) S_{i_{12}},
\] (2.40)
\[
\langle \psi_{i,R}(\xi_1)\psi_{i,L}(\xi_2) \rangle_{\beta} = S_{i_{12}} w(\xi_{12}, \beta_i),
\] (2.41)
\[
\langle \psi_{i,R}(\xi_1)\psi_{i,R}(\xi_2) \rangle_{\beta} = \sum_{j=1}^{n} S_{i_{1j}} w(\xi_{12}, \beta_j) S_{j_{2i}}^*,
\] (2.42)
where $\xi_{12} = \xi_1 - \xi_2$. As expected, the point-like interaction at the vertex of $\Gamma$ induces a non-trivial left–right mixing described by (2.40) and (2.41).

3. Bosonization away from equilibrium

The possibility of expressing fermions in terms of bosons in (1+1)-dimensional spacetime was discovered long ago by Jordan and Wigner [41]. The bosonization technique in the Fock representation of the fields $\varphi$ and $\tilde{\varphi}$ has been applied for solving the TL model in [1–6]. The framework was extended later [19, 42] to the finite temperature Gibbs representation of $\varphi$ and $\tilde{\varphi}$. Both the Fock and Gibbs representations describe equilibrium physics. Our goal in what follows will be to apply the NESS representation, constructed in the previous section, for investigating the non-equilibrium TL liquid in the multi-terminal configuration shown in figure 1.

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6 The $i\epsilon$ prescription, adopted throughout the paper, indicates as usual the weak limit $\epsilon \to 0^+$. 
8
3.1. The Tomonaga–Luttinger model on $\Gamma$

The bulk dynamics is governed by the TL Lagrangian density

$$\mathcal{L} = i\psi_1^* (\partial_t - v_F \partial_x) \psi_1 + i\psi_2^* (\partial_t + v_F \partial_x) \psi_2 - g_+ (\psi_1^* \psi_1 + \psi_2^* \psi_2)^2 - g_- (\psi_1^* - \psi_2^* \psi_2)^2,$$

(3.1)

where $[\psi_\alpha(t, x, i) : \alpha = 1, 2]$ are complex fermion fields, $v_F > 0$ is the Fermi velocity and $g_\pm \in \mathbb{R}$ are the coupling constants.

The bulk theory has an obvious $U_L(1) \otimes U_R(1)$ symmetry. In fact, the Lagrangian density (3.1) is left invariant by the two independent phase transformations

$$\psi_\alpha \rightarrow e^{is_\alpha} \psi_\alpha, \quad \psi_\alpha^* \rightarrow e^{-is_\alpha} \psi_\alpha, \quad s_\alpha \in \mathbb{R}, \quad \alpha = 1, 2,$$

(3.2)

implying the current conservation laws

$$\partial_t \rho_Z(t, x, i) - v_F \partial_x j_Z(t, x, i) = 0,$$

(3.3)

where the charge and current densities are given by

$$\rho_Z(t, x, i) = \begin{cases} \{\psi_1^* \psi_1]\{t, x, i\}, & Z = L, \\ \{\psi_2^* \psi_2]\{t, x, i\}, & Z = R, \end{cases} \quad j_Z(t, x, i) = \begin{cases} [\psi_1^* \psi_1]\{t, x, i\}, & Z = L, \\ -[\psi_2^* \psi_2]\{t, x, i\}, & Z = R. \end{cases}$$

(3.4)

The currents $j_Z(t, x, i)$ have simple physical meaning: $j_L(t, x, i)$ and $j_R(t, x, i)$ represent the particle excitations moving along the edge $E_i$ toward and away from the vertex $V$, respectively. Interpreting the vertex as a defect, which can be characterized by some scattering matrix, the currents $j_L$ and $j_R$ describe therefore the incoming and outgoing flows, respectively.

3.2. The current splitting boundary condition

It is well known [1–6] that the TL model (3.1) is exactly solvable on the line $\mathbb{R}$. On the graph $\Gamma$, the situation is more involved, because one should take into account the boundary conditions at the vertex $V$. The conditions

$$\psi_1(t, 0, i) = \sum_{j=1}^n U_j \psi_2(t, 0, j), \quad U \in U(n),$$

(3.5)

which work in the free case $g_- = g_+ = 0$, do not lead [43] to an exactly solvable problem after switching on the TL interactions$^7$. The fact that on $\mathbb{R}$ the quartic bulk interactions in (3.1) are solved exactly via bosonization suggest that we try boundary conditions which, differently from (3.5), are formulated in terms of real bosonic fields. In this spirit and according to our previous comments on the chiral currents (3.4), it is quite natural to consider

$$j_R(t, 0, i) = \sum_{k=1}^n jk \cdot j_k(t, 0, k), \quad J \in O(n),$$

(3.6)

which has been proposed and explored first in the two-terminal case in [44]. An advantage of (3.6) is the direct interpretation in terms of gauge invariant physical observables, which represent the basic building blocks of algebraic quantum field theory (see e.g. [45]). In fact, (3.6) describes the splitting in the vertex $V$ of the outgoing current $j_R(t, 0, i)$ along the edge $E_i$ into incoming currents $j_k(t, 0, k)$ along the edges $E_k$. For this reason, we refer to $J$ as the current splitting matrix and show in the following subsection that $J$ actually coincides with the boson scattering matrix $S$.

$^7$ We recall that this fact is a consequence of the exponential boundary interactions of $\psi_1, \psi_2$, generated by (3.5) after bosonization.
It is instructive to compare at this stage the boundary conditions (3.5) and (3.6). First of all we observe that (3.5) is linear, whereas according to (3.4), the condition (3.6) is quadratic in the fields $\psi_\alpha$. Moreover, from the physical point of view, (3.5) and (3.6) describe the transport at the junction at two different levels. Equation (3.5) provides a microscopic description, based on the hopping of the TL excitations among different edges. Equation (3.6) gives instead a collective description in terms of the currents, parametrizing the incoming and outgoing flows. In this sense, (3.5) is more fundamental and in theory can be used to derive the current splitting matrix $J$ as a function of the hopping matrix $\mathbb{U}$. The current splitting boundary condition (3.6) allows us to bypass this difficult and yet unsolved problem, the price to pay being that the dependence of $J$ on the microscopic parameters remains unknown. Nevertheless, imposing (3.6) instead of (3.5) is mathematically perfectly consistent and, as shown below, allows us to solve the TL model on $\Gamma$ exactly. In our opinion, the solution is not only of conceptual but also of practical interest because in principle the current splitting at the junction can be measured directly.

### 3.3. Operator solution of the TL model on $\Gamma$

Referring for the details to [46], we recall here the anyon operator solution of the TL model on a star graph $\Gamma$. The solution provides a unified description of all anyon Luttinger liquids and is expressed in terms of the chiral fields (2.6), (2.7) and the parameters $\sigma$, $\tau \in \mathbb{R}$ and the sound velocity $v \in \mathbb{R}$ as follows:

$$
\psi_1(t, x, i) = \eta_i : e^{i\sqrt{\pi} [\sigma_\psi R(x t - i) + \tau_\psi L(x t + 1)]} :,
$$

$$
\psi_2(t, x, i) = \eta_i : e^{i\sqrt{\pi} [\sigma_\psi R(x t - i) + \tau_\psi L(x t + 1)]} :.
$$

(3.7)

(3.8)

Here, $:\ldots :$ denotes the normal product in the RT algebra $\mathcal{A}$ and $\eta_i$ are some Klein factors, controlling the statistics of $\psi_\alpha$. In this respect, we impose the general anyon exchange relation

$$
\psi_\alpha^x(t, x_1, i_1) \psi_\alpha(t, x_2, i_2) = e^{i\epsilon(x_1, x_2)} \psi_\alpha(t, x_2, i_2) \psi_\alpha^x(t, x_1, i_1), \quad x_1 \neq x_2,
$$

(3.9)

where $\epsilon(x)$ is the sign function and $\kappa > 0$ is the so-called statistical parameter which interpolates between bosons ($\kappa$-even integer) and fermions ($\kappa$-odd integer). A simple realization of the Klein factors is

$$
\eta_i = \frac{1}{\sqrt{2\pi}} e^{i2\pi(\gamma_i + \gamma_i^*)},
$$

(3.10)

where $[\gamma_i, \gamma_j^*] : i = 1, \ldots, n$ generate the auxiliary algebra

$$
[y_i, y_{j}^*] = [\gamma_i^*, \gamma_j^*] = 0, \quad [y_i, \gamma_j^*] = i\frac{\kappa}{2} \epsilon_{ij},
$$

(3.11)

with $\epsilon_{ij} = -1$ for $i < j$, $\epsilon_{ii} = 0$ and $\epsilon_{ij} = 1$ for $i > j$.

In order to fix the solution (3.7) and (3.8) completely, one should determine the parameters $\sigma$, $\tau$ and $v$ in terms of coupling constants $g_{\pm}$ and the statistical parameter $\kappa$. Using a standard short distance expansion and (3.7), (3.8), one obtains the charge and current densities$^8$

$$
\rho_\pm(t, x, i) \equiv (\psi_1^* \psi_1 : \psi_2^* \psi_2 :)(t, x, i) = \frac{-1}{2\sqrt{\pi} \xi_\pm}[(\partial_\psi R)(vt - x) \pm (\partial_\psi L)(vt + x)],
$$

(3.12)

$$
\mathbf{j}_\pm(t, x, i) = \frac{v}{2\sqrt{\pi} v_F \xi_\pm}[(\partial_\psi R)(vt - x) \mp (\partial_\psi L)(vt + x)],
$$

(3.13)

$^8$ Without loss of generality, we assume in what follows $\tau \geq 0$ and $\tau \neq \pm \sigma$. 

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where for convenience the variables
\[ \zeta_\pm = r \pm \sigma \] (3.14)
have been introduced. The normalization of (3.12) is fixed [46] by the Ward identities associated with the electric charge \( Q_+ \) and the helicity \( Q_- \) defined by
\[ Q_\pm = \sum_{i=1}^{n} \int_{0}^{\infty} dx \rho_\pm(t, x, i). \] (3.15)
The normalization of (3.13) in turn is determined by the conservation law
\[ \partial_t \rho_\pm(t, x, i) - \nu \partial_i j_\pm(t, x, i) = 0. \] (3.16)
Plugging (3.7), (3.8) and (3.12) in the quantum equations of motion
\[ i(\partial_t + (-1)^i \nu \partial_i) \psi_\alpha(t, x, i) = 2[g_+ : \rho_+(t, x, i) \psi_\alpha : (t, x, i)] - (-1)^i g_- : \rho_-(t, x, i) \psi_\alpha : (t, x, i)], \] (3.17)
one finds
\[ \nu \zeta_+^2 = \nu \kappa + \frac{2}{\pi} g_+, \] (3.18)
\[ \nu \zeta_-^2 = \nu \kappa + \frac{2}{\pi} g_- \] (3.19)
Moreover, the exchange relation (3.9) implies
\[ \zeta_+ \zeta_- = \kappa. \] (3.20)
Equations (3.20), (3.18) and (3.19) provide a system for determining \( \nu \) and \( \zeta_\pm \) (or equivalently \( \sigma \) and \( r \)) in terms of \( \kappa \) and \( g_\pm \), respectively. The solution is
\[ \zeta_\pm^2 = \kappa \left( \frac{\pi \kappa \nu_F + 2g_+}{\pi \kappa \nu_F + 2g_-} \right)^{\pm 1/2}, \] (3.21)
\[ \nu = \frac{\sqrt{(\pi \kappa \nu_F + 2g_-)(\pi \kappa \nu_F + 2g_+)}}{\pi \kappa}, \] (3.22)
where the positive roots are taken on the right-hand side. Relations (3.21) and (3.22) represent the anyon generalization [46] of the well-known result for canonical fermions \( \kappa = 1 \), where an alternative and frequently used notation [7] is
\[ g_2 = 2(g_+ - g_-), \quad g_4 = 2(g_+ + g_-), \quad K = \zeta_-^2 = \zeta_+^{-2}. \] (3.23)
Considering the general anyon solution (3.21) and (3.22), we assume in what follows that the parameters \( \kappa, \zeta_\pm \) belong to the domain
\[ \mathcal{D} = \{ \kappa > 0, \ 2g_\pm > -\pi \kappa \nu_F \}, \] (3.24)
which ensures that \( \sigma \), \( r \) and \( \nu \) are real and finite.
Let us discuss finally the current splitting boundary condition (3.6) and establish the relation between \( \mathcal{J} \) and \( \mathbb{S} \). Expressing the chiral currents \( j_\mathcal{E} \) in terms of the chiral fields \( \psi_{iZ} \), one finds
\[ j_\mathcal{E}(t, x, i) = \frac{1}{2} (\zeta_- j_- + \zeta_+ j_+) (t, x, i) = \frac{\nu}{2 \sqrt{\pi} \nu_F} \partial \psi_{iZ}(\nu t - x), \] (3.25)
\[ j_\mathcal{L}(t, x, i) = \frac{1}{2} (\zeta_- j_- - \zeta_+ j_+) (t, x, i) = \frac{\nu}{2 \sqrt{\pi} \nu_F} \partial \psi_{iZ}(\nu t + x), \] (3.26)
which, according to (2.5), satisfy the current splitting boundary condition (3.6), provided that
\[ \mathcal{J} = S \in O(n). \] (3.27)

The symmetry content of the TL junction is strongly influenced by (3.27). The point is that in the presence of a defect the continuity equation (3.16) alone is not enough to ensure the electric charge conservation. A direct computation shows indeed that
\[ \partial_t Q_+ = \frac{v}{2\sqrt{\pi}\zeta} \sum_{k=1}^{n} \left( 1 - \sum_{i=1}^{n} S_{ik} \right) (\partial\psi_{i,L})(vt). \] (3.28)

The independence of \( \psi_{i,L} \) implies that the electric charge \( Q_+ \) is conserved if and only if
\[ \sum_{i=1}^{n} S_{ik} = 1, \quad \forall \, k = 1, \ldots, n, \] (3.29)

which, as expected, is equivalent to the Kirchhoff rule
\[ \sum_{i=1}^{n} j_+(t, 0, i) = 0. \] (3.30)

Since \( S \in O(n) \), one infers from (3.29) that \( S' \) satisfies (3.29) as well. Therefore, the electric charge \( Q_+ \) is conserved for those \( S \), whose entries along each column (line) sum up to 1. In geometric terms, these scattering matrices belong to the stability subgroup \( O_\kappa \subset O(n) \) of the \( n \)-vector \( v = (1, 1, \ldots, 1) \). An explicit parametrization of \( O_\kappa \) in terms of angular variables is given in [47].

Summarizing, the condition \( S \in O(n) \) guarantees the energy conservation at the TL-junction. Concerning the electric charge \( Q_+ \), one must distinguish two different regimes. \( Q_+ \) is conserved for \( S \in O_\kappa \). If instead \( S \) belongs to the complement \( \bar{O}_\kappa = O(n) \setminus O_\kappa \), then there is an external incoming or outgoing charge flow at the junction and \( Q_+ \) is not conserved. The possibility of describing such imperfect junctions is a remarkable feature of the current splitting boundary condition (3.6). The physical details about the charge transport in the junction are discussed in section 4.2.

3.4. NESS representation and chemical potentials

The crucial property of the operator solution (3.7), (3.8), (3.21) and (3.22) is that it is universal, meaning that it applies for any representation of the chiral field algebra generated by \( \psi_{i,Z} \). The Fock and Gibbs representations have been largely studied and describe the equilibrium properties of the TL model on \( \Gamma \). In order to explore the behavior of the Luttinger liquid away from equilibrium, we investigate below the operator solution in the NESS representation of the RT algebra \( \mathcal{A} \), constructed in section 2.2.

The first step in this direction is the introduction of the fermion chemical potentials
\[ \mu_i = k_F - V_i, \] (3.31)

where \( k_F \) defines the Fermi energy for \( \kappa = 1 \) and \( V_i \) is the external voltage applied to the thermal reservoir in the edge \( E_i \) of figure 2. In what follows, we keep \( k_F \) fixed and vary eventually the gate voltages \( V_i \). As already mentioned, the boson chemical potential \( \mu_b < 0 \) has been introduced for avoiding some infrared singularities at the boson level and has nothing to do with \( \mu_i \). In fact, in the chiral correlators (2.39)–(2.42), we already performed the limit \( \mu_b \to 0^+ \). In order to recover \( \mu_i \), following [19] we introduce the shift \( \alpha_{\mu_i} \), defined by
\[ \psi_{i,L}(\xi) \mapsto (\alpha_{\mu_i}\psi_{i,L})(\xi) = \psi_{i,L}(\xi) - \frac{\xi}{\sqrt{\pi}\zeta^+} \mu_i \] (3.32)
and, consistently with the boundary condition (2.5),
\[ \psi_{l,R}(\xi) \mapsto (\alpha_\mu \psi_{l,R})(\xi) = \psi_{l,R}(\xi) = \frac{\xi}{\sqrt{\pi}} \xi + \sum_{j=1}^n S_{ij} \mu_j. \] (3.33)

The transformations (3.31) and (3.32) extend to an automorphism \( \alpha_\mu \) on the whole algebra generated by the chiral fields \( \psi_{l,Z} \), which is directly implemented in the operator solution (3.7), (3.8), (3.12) and (3.13). At this stage, the TL correlation functions in the NESS are defined by
\[ \langle O_1[\psi_{l,Z}] \cdots O_4[\psi_{l,Z}] \rangle_{\beta,\mu} = \langle O_1[\alpha_\mu \psi_{l,Z}] \cdots O_4[\alpha_\mu \psi_{l,Z}] \rangle_{\beta}. \] (3.34)

In the rest of the paper, we focus on the correlation functions (3.34), which capture the physical properties of the Luttinger liquid with the current splitting boundary condition (3.6) away from equilibrium. We will show in particular that (3.34) satisfy the Kubo–Martin–Schwinger (KMS) condition [34, 45] at equilibrium, which justifies the introduction of the chemical potentials \( \mu_i \) by means of (3.32) and (3.33).

4. Non-equilibrium TL correlation functions

4.1. Anyon correlators

We derive here the two-point correlators of \( \psi_\mu(t, x, i) \) defined by (3.7) and (3.8) in the NESS and discuss their properties. For this purpose, we extend away from equilibrium the finite temperature results of [19]. Using (2.22), for \( \psi_1 \) one finds
\[ \langle \psi_1^\dagger(t_1, x_1, i) \psi_1(t_2, x_2, j) \rangle_{\beta,\mu} = A_{ij} B_{ij}(t_1, x_1, i; \mu) \left\{ \frac{1}{\pi} \sinh \left[ \frac{\pi}{\beta} (v t_{12} + x_{12}) - i \epsilon \right] \right\}^{\sigma T S_{ij}} \times \left\{ \frac{1}{\pi} \sinh \left[ \frac{\pi}{\beta} (v t_{12} - x_{12}) - i \epsilon \right] \right\}^{\sigma T S_{ij}} \times \prod_{k=1}^{n} \left\{ \frac{1}{\pi} \sinh \left[ \frac{\pi}{\beta} (v t_{12} - x_{12}) - i \epsilon \right] \right\}^{\sigma T S_{ij}}, \] (4.1)

where
\[ A_{ij} = e^{i \pi^2 \kappa_{ij} / 2} \left( \frac{1}{2 \pi} \right)^{(\sigma T + \tau T) S_{ij} + \sigma T S_{ij}} \] (4.2)
\[ B_{ij}(t_1, x_1, i; \mu) = e^{(v t_{11}, x_{11}, -x_{12}) \mu_i + \sigma ((v t_{11} - x_{11}) \sum_{k=1}^{n} S_{ik} \mu_k - (v t_{12} - x_{12}) \sum_{k=1}^{n} S_{ik} \mu_k) / (\sigma + \tau)}. \] (4.3)

The \( \psi_2 \)-correlator has the analogous form,
\[ \langle \psi_1^\dagger(t_1, x_1, i) \psi_2(t_2, x_2, j) \rangle_{\beta,\mu} = (4.1) \quad \text{with} \quad \sigma \leftrightarrow \tau. \] (4.4)

The TL junction involves two types of \( \psi_1-\psi_2 \) interactions. First, the Lagrangian (3.1) contains a \( \psi_1-\psi_2 \) bulk coupling proportional to \( (g_+ - g_-) \). Second, the current splitting boundary condition (3.6) provides an additional boundary interaction described by the mixed left–right correlators (2.40) and (2.41). Consequently, the mixed \( \psi_1-\psi_2 \) correlators are non-trivial and have the form
As expected, in the equilibrium limit \( \beta \to \infty \) the current splitting boundary condition (3) is reached in full detail in [47]. One obtains

\[
\langle \psi_1^* (t_1, x_1, i) \psi_2 (t_2, x_2, j) \rangle_{\mu, \rho} = \tilde{A}_{ij} \tilde{B}_{ij} (t_{1,2}, x_{1,2}; \mu)
\]

\[
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} + x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij}
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} - x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij}
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} + x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij}
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} - x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij},
\]

where

\[
\tilde{A}_{ij} = \frac{e^{i \pi \epsilon \epsilon_{ij}}}{2 \pi} \left[ \frac{1}{\beta} \right] \sigma^\tau S_{ij} + 2 \epsilon \tau \delta_{ij}
\]

\[
\tilde{B}_{ij} (t_{1,2}, x_{1,2}; \mu) = e^{i \left( \tau \left[ (v n_{1,1}) \mu_1 - (v n_{2,2}) \sum_{k=1}^{n} S_{1k} \mu_k \right] + \sigma \left[ (v n_{1,1}) \sum_{k=1}^{n} S_{1k} \mu_k - (v n_{2,2}) \mu_k \right] / (\sigma + \tau) \right)}.
\]

Finally,

\[
\langle \psi_2^* (t_1, x_1, i) \psi_1 (t_2, x_2, j) \rangle_{\mu, \rho} = (4.5) \text{ with } \sigma \leftrightarrow \tau.
\]

As expected, in the equilibrium limit \( \beta_i = \beta \) and \( \mu_i = \mu \) for all \( i \), the correlators (4.1)–(4.8) simplify and satisfy the KMS condition, which represents a non-trivial check both on the computation and on the shift (3.32) and (3.33) introducing the chemical potentials. Let us consider for instance (4.1), which in this limit takes the form

\[
\langle \psi_1^* (t_1, x_1, i) \psi_1 (t_2, x_2, j) \rangle_{\rho, \mu} = A_{ij} e^{i \left( \tau \left( (v n_{1,1}) \mu_1 - (v n_{2,2}) \sum_{k=1}^{n} S_{1k} \mu_k \right) + \sigma \left( (v n_{1,1}) \sum_{k=1}^{n} S_{1k} \mu_k - (v n_{2,2}) \mu_k \right) / (\sigma + \tau) \right)}
\]

\[
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} + x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij}
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} - x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij}
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} + x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij}
\times \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v t_{12} - x_{12}) - i \epsilon \right] \right\} \sigma^\tau S_{ij},
\]

Recalling that the KMS automorphism \( \varphi \) acts on \( \psi_\mu \) as follows,

\[
[\varphi, \psi_\mu] (t, x, i) = e^{i \mu t} \psi_\mu (t + s, x, i),
\]

one can check that the equilibrium correlator (4.9) satisfies the KMS condition

\[
\langle \psi_1^* (t_1, x_1, i) \psi_1 (t_2, x_2, j) \rangle_{\rho, \mu} = (\langle [\varphi, \psi_1] (t_2, x_2, j) \psi_1^* (t_1, x_1, i) \rangle_{\rho, \mu})
\]

for all values of the statistical parameter \( \kappa \).

The critical scaling dimensions \( d_i \) can be extracted from (4.1)–(4.8) in the limit \( \beta_i \to \infty \) and \( \mu_i \to 0 \). Because of the operator mixing, this is a subtle issue, which has been discussed in full detail in [47]. One obtains

\[
d_i = \frac{1}{2} (\sigma^2 + \tau^2) + \sigma \tau S_i, \quad i = 1, \ldots, n.
\]

where \( S_i = \pm 1 \) are the eigenvalues of \( S \). As already observed in [26], the impact of the vertex interaction is captured by the term \( \sigma \tau S_i \), which preserves unitarity in the sense of CFT because \( d_i \geq 0 \).

A remarkable special case is obtained by setting \( g_+ = g_- = g \). In this case, the bulk \( \psi_1 - \psi_2 \) coupling vanishes and one is left only with the boundary interaction induced by the current splitting boundary condition (3.6). From (3.22), one obtains

\[
u = v_F + \frac{2g}{\pi \kappa},
\]

where \( v_F \) is the Fermi velocity.
and (using (3.20) with $\kappa > 0$ and $\tau \geq 0$)
\[
\sigma = 0, \quad \tau = \sqrt{\kappa}. \tag{4.14}
\]
Inserting (4.14) in (4.1)–(4.4) and localizing the fields in the same edge (i.e. setting $i = j$), one finds that the correlation functions simplify to
\[
C_{11}(v_{12} + x_{12}, i; \beta, \mu) \equiv \langle \psi_{1}^* (t_{1}, x_{1}, i) \psi_{1}(t_{2}, x_{2}, i) \rangle_{\beta, \mu} = \frac{1}{2\pi} \left( \frac{1}{2i} \right)^{\kappa} e^{i(v_{12} + x_{12})/\beta} \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v_{12} + x_{12}) - i\epsilon \right] \right\}^{\kappa}, \tag{4.15}
\]
\[
C_{22}(v_{12} + x_{12}, i; \beta, \mu) \equiv \langle \psi_{2}^* (t_{1}, x_{1}, i) \psi_{2}(t_{2}, x_{2}, i) \rangle_{\beta, \mu} = \frac{1}{2\pi} \left( \frac{1}{2i} \right)^{\kappa} e^{i(v_{12} - x_{12})/\beta} \left\{ \frac{1}{\beta} \sinh \left[ \frac{\pi}{\beta} (v_{12} - x_{12}) - i\epsilon \right] \right\}^{\kappa}. \tag{4.16}
\]
The condition $g_{+} \neq g_{-}$ and the left–right asymmetry of the NESS construction in section 2 imply that only left moving (incoming) excitations contribute to $C_{11}$, which therefore coincides with the equilibrium correlator [19]. All the non-equilibrium features are captured by $C_{22}$, which involves only right moving (outgoing) excitations. In fact, in spite of being localized at the edge $E$, of the graph, (4.16) depends on the temperatures and chemical potentials of all $n$ edges.

It is instructive for this reason to derive and compare the Fourier transforms of (4.15) and (4.16). We will show first that they can be expressed in terms of the finite temperature TL anyon distribution discovered in [19]. Consider in fact
\[
\hat{C}_{11}(E, p, i; \beta, \mu) \equiv \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \ e^{-i(Evt + \mu x)} C_{11}(vt + x, i; \beta, \mu). \tag{4.17}
\]
Plugging (4.15) in (4.17), one obtains
\[
\hat{C}_{11}(E, p, i; \beta, \mu) = \frac{\pi}{\kappa} \delta(E - p) d(p - \mu_{i}, \beta_{i}; \kappa), \tag{4.18}
\]
where the $\delta$-function fixes the dispersion relation and $d$ is the equilibrium anyon momentum distribution [19]
\[
d(p, \beta; \kappa) = \frac{\beta^{1-\kappa} e^{-\beta p/2}}{2\pi} B \left( \frac{\kappa}{2}, \frac{\kappa}{2} \right) \left\{ \frac{1}{2\pi} \beta p \right\}^{\kappa/2} = \frac{\beta^{1-\kappa} e^{-\beta p/2}}{2\pi \Gamma(\kappa)} \Gamma \left( \frac{\kappa}{2}, \frac{\kappa}{2} \right) \Gamma \left( \frac{\kappa}{2} + \frac{\kappa}{2} \beta p \right), \quad \kappa > 0, \tag{4.19}
\]
with $B$ and $\Gamma$ being the beta and gamma functions (Euler’s integrals of first and second kind respectively). Note that for $\kappa \neq 1$ the distribution (4.19) depends not only on the dimensionless combination $\beta p$, but also on $\beta$ separately. Equation (4.19) defines a smooth function of $p \in \mathbb{R}$, which satisfies
\[
\lim_{p \to 0} d(p, \beta; \kappa) = \frac{\beta^{1-\kappa} \Gamma^{2}(\kappa/2)}{2\pi \Gamma(\kappa)}, \tag{4.20}
\]
and has the following asymptotic behavior:
\[
\lim_{p \to \infty} d(p, \beta; \kappa) = 0, \quad \forall \kappa > 0, \tag{4.21}
\]
As expected, at the fermion point $\kappa = 1/2$ where $n = 1$ of the TL liquid, one obtains the familiar Fermi distribution. In spite of the fact that the remaining boson and fermion points ($\kappa = 2n, \ldots$) were established in [19, 20] more than a decade ago, to our knowledge their physical meaning and potential applications of (4.19) have not been fully explored.

In order to give an idea about the anyon distributions in the interval $0 < \kappa < 1$, we show some of them in figure 3, where the standard Fermi distribution (continuous red curve) is given for comparison. Figure 4 displays the behavior of the anyon distribution (4.19) for fixed $\kappa = 1/4$ and different temperatures. For $0 < \kappa < 1$ and with decreasing temperature $T \sim 1/\beta$, one observes the formation of a sharp peak at $p = 0$ (in agreement with equation (4.20)), which signals a condensation-like phenomenon [19].

Concerning the Fourier transform of (4.16), it is useful to consider first the case when all the temperatures are equal ($\beta_1 = \beta$), with the system being driven away from equilibrium only by the voltages $V_i$. In this case,

$$C_{22}(v_{t_1} - x_{t_2}; i; \beta, \mu) \equiv \langle \psi_2^\dagger(t_1, x_1, i) \psi_2(t_2, x_2, i) \rangle_{\beta, \mu}$$

$$= \frac{1}{2\pi} \left( \frac{1}{2\beta} \right)^\kappa \epsilon^{(v_{t_1} - x_{t_2}) \sum_{i=1}^2 5i\mu_i} \frac{1}{\beta} \sinh \left[ \frac{1}{\beta} (v_{t_1} - x_{t_2}) - i\varepsilon \right] \right)^\kappa,$$
and therefore
\[ \hat{C}_{22}(E, p, i; \beta, \mu) = \frac{\pi^k}{v} \delta(E - p) d(p - \sum_{k=1}^n S_{ik}\mu_k, \beta; \kappa). \]  

One has still the equilibrium distribution, with the energy shifted by a linear combination of the chemical potentials \( \mu_k \), whose coefficients are the \( S \)-matrix elements.

Figure 4. The distribution \( d \) at fixed \( \kappa = 1/4 \) for different temperatures \( \beta = 0.2 \) (continuous red line), \( \beta = 0.4 \) (dashed black line) and \( \beta = 0.8 \) (dotted blue line).

Finally, in the coordinate space the general expression (4.16) is a product of \( C_{11} \)-factors with different temperatures and chemical potentials. One obtains therefore in momentum space the nested convolution formula
\[ \hat{C}_{22}(E, p, i; \beta, \mu) = \frac{\pi^k}{v} \delta(E - p) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \cdots \frac{dk_n}{2\pi} \]
\[ \times d(k_1 - S_{i1}\mu_1, \beta_1; \kappa S_{i1}^1)d(k_2 - S_{i2}\mu_2 - k_1, \beta_2; \kappa S_{i2}^2) \]
\[ \cdots d(p - S_{i\mu_\mu} - k_{\mu-1}, \beta_\mu; \kappa S_{i\mu}^\mu). \]  

Being a convolution of distributions, (4.28) is also a well-defined distribution. The NESS \( \Omega_{\beta, \mu} \) has therefore a remarkable property: the associated non-equilibrium distribution is a convolution of equilibrium distributions with different temperatures and chemical potentials.

Since the general form of (4.28) is quite complicated, it is instructive to consider below the case \( n = 2 \) and \( \mu_1 = \mu_2 = 0 \), focusing on
\[ D_2(p; \beta_1, \beta_2; \kappa, \theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} d(k, \beta_1; \kappa \cos^2 \theta) d(p - k, \beta_2; \kappa \sin^2 \theta), \quad \theta \in [0, \pi). \]  

Using the \( x \)-space representation, at equal temperatures one finds the relation
\[ D_2(p; \beta, \beta; \kappa, \theta) = d(p, \beta; \kappa), \quad \forall \theta \in [0, \pi). \]  

For \( \beta_1 \neq \beta_2 \), the convolution \( D_2 \) defines a new distribution. Since we were not able to determine its explicit analytic form, we give some plots which are obtained numerically. The plots in figures 5 and 6 illustrate the behavior of \( D_2 \) for different values of \( \beta_{1,2}, \kappa \) and \( \theta \). We see that even for \( \beta_1 \neq \beta_2 \) the distribution \( D_2 \) is similar to \( d \) with a kind of ‘effective’ temperature and statistical parameter depending on \( \beta_{1,2}, \kappa \) and \( \theta \).

Summarizing, we derived above the two-point TL anyon correlation functions away from equilibrium. The results (4.1) and (4.5) are expressed as products of \( S \)-dependent powers of equilibrium correlators at different temperatures. In agreement with this fact, the momentum space anyon NESS distribution is the convolution of equilibrium anyon distributions (4.28).

The equilibrium limit satisfies the KMS conditions. At criticality one is dealing with a \( c = 1 \) CFT, whose anomalous dimensions (4.12) depend not only on the coupling constants \( g_{\pm \pm} \), but also on the scattering matrix \( S \). The above technique allows us to compute higher anyon
correlation functions as well, but in order to investigate the transport properties of the system, we concentrate below on the electric and energy current correlators away from equilibrium.

4.2. Charge and heat transport

The charge transport in the NESS is described by

\[ \langle \partial_t Q^+ \rangle_{\beta,\mu} = \frac{v}{2\pi \zeta^2} \sum_{i,j=1}^{n} (S_{ij} - \delta_{ij}) \mu_j, \]  

(4.31)

\[ \langle j^+(t, x, i) \rangle_{\beta,\mu} = \frac{v}{2\pi v_F \zeta^2} \sum_{j=1}^{n} (\delta_{ij} - S_{ij}) \mu_j, \]  

(4.32)

which follow by substituting (3.13) and (3.28) in (3.34). Equation (4.31) describes the external charge flow at the junction: it is constant in time and is incoming for \( \langle \partial_t Q^+ \rangle_{\beta,\mu} > 0 \) and outgoing for \( \langle \partial_t Q^+ \rangle_{\beta,\mu} < 0 \). Equation (4.32) determines instead the value of the currents along the leads. The charge balance

\[ \langle \partial_t Q^+ \rangle_{\beta,\mu} + v_F \sum_{i=1}^{n} \langle j^+(t, x, i) \rangle_{\beta,\mu} = 0 \]  

(4.33)

is satisfied and represents a useful check. If \( S \in O_\alpha \), then the electric charge is conserved \( \langle \partial_t Q^+ \rangle_{\beta,\mu} = 0 \) and the \( k_F \)-dependence in (4.32) drops out, leading to

\[ \langle j^+(t, x, i) \rangle_{\beta,\mu} = \frac{v}{2\pi v_F \zeta^2} \sum_{j=1}^{n} (S_{ij} - \delta_{ij}) V_j. \]  

(4.34)
The current (4.34) satisfies the Kirchhoff rule (3.30) and vanishes at equilibrium ($V_i = V$ for all $i$) as it should be. The dependence on the statistical parameter $\kappa$ is explicit and deserves a comment. In the physical domain $D$, defined by (3.24), the overall coefficient in front of the sum in (4.34) is positive,

$$G(g_-, \kappa) = \frac{v}{2\pi v_F \xi^2} = \frac{\pi \kappa v_F + 2g_-}{2\pi^2 \kappa^2 v_F} > 0.$$  

(4.35)

For $g_+ > 0$, the coefficient $G$ decreases monotonically with $\kappa > 0$. For $g_- < 0$, one has that $\kappa > -2g_-/\pi v_F$ in the physical domain $D$. In this case, the coefficient $G$ increases in the interval $-2g_-/\pi v_F < \kappa < -4g_-/\pi v_F$, reaching the maximal value $-v_F/16g_-$ and decreases for $\kappa > -4g_-/\pi v_F$. This behavior is illustrated in figure 7.

The current (4.34) is proportional to the applied external voltages. Nonlinear effects are absent in the critical regime under consideration, which implies the conductance tensor

$$G_{ij} = G(g_-, \kappa) \sum_{j=1}^n (S_{ij} - \delta_{ij}).$$  

(4.36)

We see that the NESS approach, adopted in this paper, confirms the result for $G$, obtained previously for $\kappa = 1$ by different methods, including renormalization group techniques [48, 49], linear response theory [26, 27] and CFT [50, 51]. The novelty in (4.36) is the explicit dependence on the statistical parameter $\kappa$, shown in figure 7.

A similar computation gives the energy (heat) flow

$$\langle \theta(t, x, i) \rangle_{\beta, \mu} = \frac{v^2}{8\pi \xi^2} \left[ \mu_i^2 - \left( \sum_{j=1}^n (S_{ij} - \delta_{ij}) \right)^2 \right] + \frac{\pi v^2}{12} \sum_{j=1}^n \left( \delta_{ij} - S_{ij}^2 \right) \frac{1}{\beta_j^2},$$  

(4.37)

which satisfies the Kirchhoff rule (2.26) for all $S \in O(n)$. For $S \in O_+$, the expression (4.37) takes the form

$$\langle \theta(t, x, i) \rangle_{\beta, \mu} = \frac{v^2}{8\pi \xi^2} \left[ V_i^2 - 2k_F \sum_{j=1}^n (\delta_{ij} - S_{ij}) V_j - \left( \sum_{j=1}^n S_{ij} V_j \right)^2 \right]$$

$$+ \frac{\pi v^2}{12} \sum_{j=1}^n (\delta_{ij} - S_{ij}^2) \frac{1}{\beta_j^2}.$$  

(4.38)

The heat flow depends therefore not only on $k_F$ and the voltages $V_i$, but also on the temperatures $\beta_i$. 

Figure 7. Behavior of $G$ at $g_+ = 1$ (dashed black line) and $g_- = -1$ (continuous red line).
As before, we consider for illustration the case of $n = 2$ wires. Inserting the scattering the matrixes (2.31) in (4.32), one has

$$
\langle \hat{a} Q^+ \rangle_{\beta, \mu} = \begin{cases} 
G_{vF}[(\mu_1 + \mu_2)(\cos \theta - 1) - (\mu_1 - \mu_2) \sin \theta], & \det S = 1, \\
G_{vF}[(\mu_1 + \mu_2)(\sin \theta - 1) + (\mu_1 - \mu_2) \cos \theta], & \det S = -1.
\end{cases}
$$

(4.39)

Therefore,

$$
\langle \hat{a} Q^+ \rangle_{\beta, \mu} = 0 \implies \begin{cases} 
\theta = 0 \implies j_+ (t, x, 1) = j_+ (t, x, 2) = 0, & \det S = 1, \\
\theta = \pi / 2 \implies j_+ (t, x, 1) = -j_+ (t, x, 2) = G(\mu_1 - \mu_2), & \det S = -1,
\end{cases}
$$

(4.40)

corresponding respectively to full reflection (disconnected edges) and complete transmission in the junction. Finally,

$$
\langle \theta_\nu (t, x, 1) \rangle_{\beta, \mu} = -\langle \theta_\nu (t, x, 2) \rangle_{\beta, \mu}
$$

\begin{equation}
= \frac{v^2}{4 \pi \zeta_\nu} (\mu_1^2 - \mu_2^2) \sin^2 \theta - \mu_1 \mu_2 \sin 2\theta \left[ \frac{1}{\beta_1^2} - \frac{1}{\beta_2^2} \right],
\end{equation}

(4.41)

for both families in (2.31).

4.3. Quantum noise

In this section, we derive the noise power in the TL junction in figure 1. For this purpose, we need (52) the two-point connected current–current correlator

$$
\langle j_+ (t_1, x_1, i) j_+ (t_2, x_2, j) \rangle_{\beta, \mu}^{\text{conn}} = \langle j_+ (t_1, x_1, i) j_+ (t_2, x_2, j) \rangle_{\beta, \mu} - \langle j_+ (t_1, x_1, i) \rangle_{\beta, \mu} \langle j_+ (t_2, x_2, j) \rangle_{\beta, \mu}.
$$

(4.42)

After some algebra, one finds

\begin{equation}
\langle j_+ (t_1, x_1, i) j_+ (t_2, x_2, j) \rangle_{\beta, \mu}^{\text{conn}} = \left( \frac{v}{2 \nu_\nu} \right)^2 \left[ \frac{1}{\beta_1^2} \sinh^2 \left( \frac{\nu}{\beta_1} (v t_{12} + x_{12}) - i \varepsilon \right) \right] \delta_{ij}
\end{equation}

\begin{equation}
+ \sum_{i=1}^{n} \frac{1}{\beta_1^2 \sinh^2 \left( \frac{\nu}{\beta_1} (v t_{12} - x_{12}) - i \varepsilon \right)} \delta_{ij} - \frac{1}{\beta_1^2 \sinh^2 \left( \frac{\nu}{\beta_1} (v t_{12} + x_{12}) - i \varepsilon \right)} \delta_{ij} \right].
\end{equation}

(4.43)

One easily verifies that the equilibrium limit ($\beta_i \rightarrow \beta$ for all $i$) of (4.43) satisfies the KMS condition

$$
\langle j_+ (t_1, x_1, i) Q_{x_1, x_2} j_+ (t_2, x_2, j) \rangle_{\beta, \mu}^{\text{conn}} = \langle Q_{x_1, x_2} j_+ (t_2, x_2, j) j_+ (t_1, x_1, i) \rangle_{\beta, \mu}^{\text{conn}},
$$

(4.44)

where the KMS automorphism $Q$ acts on $j_+$ as follows,

$$
[Q, j_+](t, x, i) = j_+ (t + s/v, x, i).
$$

(4.45)

The explicit expression (4.43) contains fundamental physical information about the NESS. First of all, since

$$
\langle j_+ (t_1, x_1, i_1) j_+ (t_2, x_2, i_2) \rangle_{\beta, \mu}^{\text{conn}} \neq \langle j_+ (-t_1, x_1, i_1) j_+ (-t_2, x_2, i_2) \rangle_{\beta, \mu}^{\text{conn}},
$$

(4.46)

The bar indicates complex conjugation.
the NESS breaks down time reversal invariance, even if the junction interaction preserves it, i.e. if $S \equiv S'$ [47]. Nevertheless, time translation invariance is preserved, which allows one to use the conventional definition [52] of noise power

$$P_{ij}(\beta; x_1, x_2; \omega) \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} \langle j_i(t, x_1, i) j_j(0, x_2, j) \rangle_{\beta, \mu}^{\text{conn}}. \quad (4.47)$$

Equation (4.47) defines a complex matrix whose entries can be expressed [53] in terms of the hypergeometric function $_2F_1$, namely

$$P_{ij}(\beta; x_1, x_2; \omega) = \left(\frac{v}{2v_F \zeta_+}\right)^2 \left(\frac{\zeta_+}{\zeta_-}\right)^{\delta_{ij}} \left\{ \sum_{l=1}^{n} \zeta_l \beta_l^{-1} \left[ F_-(\omega, \beta, x_12) - F_+(\omega, \beta, x_12) \right] \delta_{ij} \right. $$

$$+ \sum_{l=1}^{n} \zeta_l \beta_l^{-1} \left[ F_-(\omega, \beta, -x_12) - F_+(\omega, \beta, -x_12) \right] \delta_{ij} $$

$$- \sum_{l=1}^{n} \zeta_l \beta_l^{-1} \left[ F_-(\omega, \beta, -\bar{x}_12) - F_+(\omega, \beta, -\bar{x}_12) \right] $$

$$- \left[ F_-(\omega, \beta, \bar{x}_12) - F_+(\omega, \beta, \bar{x}_12) \right] \delta_{ij} \}, \quad (4.48)$$

with

$$F_\pm(\omega, \beta, x) = \frac{e^{\pm i \pi v \beta}}{i \omega \beta \pm 2 \pi v} \ 2F_1 \left(2, 1 \pm \frac{i \omega \beta}{2 \pi v}, 2 \pm \frac{i \omega \beta}{2 \pi v}, e^{\pm i \pi v / \beta}\right). \quad (4.49)$$

From (4.48) and (4.49), one can deduce the zero-frequency limit (zero-frequency noise power)

$$P_{ij}(\beta) \equiv \lim_{\omega \to 0^+} P_{ij}(\beta; x_1, x_2; \omega). \quad (4.50)$$

Using

$$\lim_{\omega \to 0^+} [F_-(\omega; \beta, x) - F_+(\omega; \beta, x)] = \frac{1}{2 \pi v}, \quad (4.51)$$

one obtains

$$P_{ij}(\beta) = \frac{G(g, \kappa)}{v_F} \left(\frac{\zeta_+}{\zeta_-}\right)^{\delta_{ij}} \left( \beta_i^{-1} \delta_{ij} - \zeta_j \beta_j^{-1} - \beta_i^{-1} \zeta_j \delta_{ij} + \sum_{l=1}^{n} \zeta_l \beta_l^{-1} \delta_{ij} \right), \quad (4.52)$$

where $G$, defined by (4.35), captures the dependence (see figure 7) of the noise on the statistical parameter $\kappa$. As expected, $P_{ij}(\beta)$ turns out to be an $x_{1,2}$-independent real symmetric matrix. If the electric charge is conserved ($S \subset O_\phi$), then the noise power (4.52) satisfies in addition the Kirchhoff rule

$$\sum_{j=1}^{n} P_{ij}(\beta) = 0. \quad (4.53)$$

Expression (4.52) admits the typical Johnson–Nyquist $\beta^{-1}$ behavior and shows the non-trivial interplay between the different temperatures and the scattering matrix. For example, in the two-terminal case with $S = S^+$ one finds

$$P^+ = \frac{G}{v_F k_B} \left( T_1(1 - \cos \theta)^2 + T_2 \sin^2 \theta \ (T_1 - T_2)(1 - \cos \theta) \sin \theta \right) \quad (4.54)$$

where $T = (k_B \beta)^{-1}$ is the absolute temperature and $k_B$ is the Boltzmann constant. The eigenvalues of $P^+$

$$\rho_i^+ = \frac{2G}{v_F k_B} (1 - \cos \theta) T_i \geq 0, \quad i = 1, 2 \quad (4.55)$$

are non-negative in agreement with the positivity of the two-point function (4.42). An analogous result holds for $P^-$ corresponding to $S^-$.
5. Outlook and conclusions

In this paper, we constructed and investigated an exactly solvable model of a non-equilibrium Luttinger junction. The basic points of our approach are:

(i) a scale invariant point-like interaction, which is described by a scattering matrix $S$ and drives the system away from equilibrium;

(ii) a representation generated by a NESS $\Omega_{\beta,\mu}$, which encodes the point-like interaction in the chiral fields $\psi_{i,Z}$;

(iii) an exact operator solution of the TL model (in terms of $\psi_{i,Z}$) on a star graph with the current splitting boundary condition at the vertex;

(iv) an extension of the conventional fermion Luttinger liquid to anyon statistics.

Combining these ingredients, we derived the basic correlation functions in the state $\Omega_{\beta,\mu}$. The essential characteristic features of these functions are:

(a) the non-equilibrium two-point anyon correlations are products of $S$-dependent powers of equilibrium correlations at the temperatures and chemical potentials of the heat baths, connected to the leads;

(b) accordingly, the corresponding momentum space distribution is the convolution of equilibrium anyon distributions at different temperatures and chemical potentials;

(c) the Fourier transform of the leading terms in the large distance expansion of the anyon correlations gives Cauchy–Lorentz distributions, which after convolution reproduce themselves with appropriate width and median;

(d) in the critical limit, one has a $c = 1$ conformal field theory with $S$-dependent anomalous dimensions, which are explicitly derived;

(e) the expected breakdown of time reversal invariance is manifest in the current–current correlator.

We investigated in detail the energy and charge transport in the junction for all values of the statistical parameter. The energy is conserved for $S \in O(n)$, which covers both possibilities of a junction without and with electric charge dissipation. In the latter case, we determined the exact expression for the charge flow leaving or entering the junction. The connected current–current correlation is a linear combination of hypergeometric functions. The associated zero-frequency noise power depends linearly on the temperatures.

Our investigation above has been focused essentially on the critical properties of anyon Luttinger liquids away from equilibrium. It will be interesting to study the noncritical aspects as well. The generalization of the results of this paper beyond the Luttinger liquid paradigm, when the nonlinearity of the dispersion relation becomes essential, is also a challenging open problem.

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Appendix. Large space separation asymptotics

The behavior of the correlators (4.15) and (4.16) at large space separation \(|x_{12}| \gg \beta_i\) is encoded in

\[
C_{11}(x, i; \beta, \mu) = \left( \frac{1}{2i} \right)^\chi \left( \frac{2\pi}{\beta_i} \right)^\chi L \left( x; \frac{\pi\kappa}{\beta_i}, -\mu_i \right) + \cdots, \tag{A.1}
\]

\[
C_{22}(x, i; \beta, \mu) = \left( \frac{1}{2i} \right)^\chi \prod_{k=1}^n \left( \frac{2\pi}{\beta_k} \right)^\chi L \left( x; \frac{\pi\kappa S_{ik}^2}{\beta_k}, -S_{ik} \mu_k \right) + \cdots, \tag{A.2}
\]

where \(x \gg \beta_i > 0\), the dots stand for sub-leading contributions and

\[
L(x; \gamma, \mu) \equiv \frac{1}{2\pi} e^{-iux-\gamma x}, \quad x > 0. \tag{A.3}
\]

The Fourier transform

\[
\hat{L}(p; \gamma, \mu) \equiv \int_{-\infty}^{\infty} dx e^{ipx} L(x; \gamma, \mu) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (p - \mu)^2} \tag{A.4}
\]

is the familiar Cauchy–Lorentz distribution\(^{10}\), where \(\gamma\) is the half-width at half-maximum and \(\mu\) is the statistical median. Using the fact that the class of Cauchy–Lorentz distributions is closed under convolution, one finds

\[
\hat{C}_{11}(p, i; \beta, \mu) \equiv \int_{-\infty}^{\infty} dx e^{ipx} C_{11}(x, i; \beta, \mu) \sim \hat{L} \left( p; \frac{\pi\kappa}{\beta_i}, -\mu_i \right) + \cdots, \tag{A.5}
\]

\[
\hat{C}_{22}(p, i; \beta, \mu) \equiv \int_{-\infty}^{\infty} dx e^{ipx} C_{22}(x, i; \beta, \mu) \sim \hat{L} \left( p; \pi\kappa \sum_{j=1}^n S_{ij}^2 \beta_j, -\sum_{j=1}^n S_{ij} \mu_j \right) + \cdots. \tag{A.6}
\]

Summarizing, the Fourier transform of the leading term in the long distance expansion of both (4.15) and (4.16) is a Cauchy–Lorentz distribution. Note that the width and the median of (A.6) depend on the temperatures and chemical potentials of all heat baths, as well as on \(\vec{S}\).

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\(^{10}\) Known also as non-relativistic Breit–Wigner distribution.
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