Mixture Gaussian Process Conditional Heteroscedasticity
Emmanouil A. Platanios and Sotirios P. Chatzis

Abstract—Generalized autoregressive conditional heteroscedasticity (GARCH) models have long been considered as one of the most successful families of approaches for volatility modeling in financial return series. In this paper, we propose an alternative approach based on methodologies widely used in the field of statistical machine learning. Specifically, we propose a novel nonparametric Bayesian mixture of Gaussian process regression models, each component of which models the noise variance process that contaminates the observed data as a separate latent Gaussian process driven by the observed data. This way, we essentially obtain a mixture Gaussian process conditional heteroscedasticity (MGPCH) model for volatility modeling in financial return series. We impose a nonparametric prior with power-law nature over the distribution of the model mixture components, namely the Pitman-Yor process prior, to allow for better capturing modeled data distributions with heavy tails and skewness. Finally, we provide a copula-based approach for obtaining a predictive posterior for the covariances over the asset returns modeled by means of a postulated MGPCH model. We evaluate the efficacy of our approach in a number of benchmark scenarios, and compare its performance to state-of-the-art methodologies.

Index Terms—Gaussian process, Pitman-Yor process, mixture model, conditional heteroscedasticity, copula, volatility modeling.

1 INTRODUCTION

Statistical modeling of asset values in financial markets requires taking into account the tendency of assets towards asymmetric temporal dependence [1]. Besides, the data generation processes of the returns of financial market indexes may be non-linear, non-stationary and/or heavy-tailed, while the marginal distributions may be asymmetric, leptokurtic and/or show conditional heteroscedasticity. Hence, there is a need to construct flexible models capable of incorporating these features. The generalized autoregressive conditional heteroscedasticity (GARCH) family of models has been used to address conditional heteroscedasticity and excess kurtosis (see, e.g., [2], [3]).

The time-dependent variance in series of returns on prices, also known as volatility, is of particular interest in finance, as it impacts the pricing of financial instruments, and it is a key concept in market regulation. GARCH approaches are commonly employed in modeling financial return series that exhibit time-varying volatility clustering, i.e., periods of swings followed by periods of relative calm, and have been shown to yield excellent performance in these applications, consistently defining the state-of-the-art in the field in the last decade. GARCH models represent the variance by a function of the past squared returns and the past variances, which facilitates model estimation and computation of the prediction errors.

Gaussian process (GP) models comprise one of the most popular Bayesian methods in the field of machine learning for regression, function approximation, and predictive density estimation [4]. Despite their significant flexibility and success in many application domains, GPs do also suffer from several limitations. In particular, GP models are faced with difficulties when dealing with tasks entailing non-stationary covariance functions, multi-modal output, or discontinuities. Several approaches that entail using ensembles of fractional GP models defined on subsets of the input space have been proposed as a means of resolving these issues (see, e.g., [5], [6], [7]).

In this work, we propose a novel GP-based approach for volatility modeling in financial time series (return) data. Our proposed approach provides a viable alternative to GARCH models, that allows for effectively capturing the clustering effects in the variability or volatility. Our approach is based on the introduction of a novel nonparametric Bayesian mixture model, the component distributions of which constitute GP regression models; the noise variance processes of the model component GPs are considered as input-dependent latent variable processes which are also modeled by imposition of appropriate GP priors. This way, our novel approach allows for learning both the observation-dependent nature of asset volatility, as well as the underlying volatility clustering mechanism, modeled as a latent model component switching procedure. We dub our approach the mixture Gaussian process conditional heteroscedasticity (MGPCH) model.

Nonparametric Bayesian modeling techniques, especially Dirichlet process mixture (DPM) models, have become very popular in statistics over the last few years, for performing nonparametric density estimation [8], [9].

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In Section 2, we provide a brief presentation of the Dirichlet process conditional heteroscedasticity (MGPCH) model, and derive efficient model inference algorithms based on the variational Bayesian framework. We also propose a Pitman-Yor process prior imposed over the component clusters and a large number of highly populated clusters. Indeed, the Pitman-Yor process prior can be viewed as a generalization of the Dirichlet process prior, and reduces to it for a specific selection of its parameter values. Consequently, the Pitman-Yor process turns out to be more promising as a means of modeling complex real-life datasets that usually comprise a high number of clusters which comprise only few data points, and a low number of clusters which are highly frequent, thus dominating the entire population.

Inspired by these advances, the component switching mechanism of our model is obtained by means of a Pitman-Yor process prior imposed over the component GP latent allocation variables of our model. We derive a computationally efficient inference algorithm for our model based on the variational Bayesian framework, and obtain the predictive density of our model using an approximation technique. We examine the efficacy of our approach considering volatility prediction in a number of financial return series.

The remainder of this paper is organized as follows: In Section 2, we provide a brief presentation of the theoretical background of the proposed method. Initially, we present the Pitman-Yor process and its function as a prior in nonparametric Bayesian models; further, we provide a brief summary of Gaussian process regression. In Section 3, we introduce the proposed mixture Gaussian process conditional heteroscedasticity (MGPCH) model, and derive efficient model inference algorithms based on the variational Bayesian framework. We also propose a copula-based method for learning the interdependencies between the returns of multiple assets jointly modeled by means of an MGPCH model. In Section 4, we conduct the experimental evaluation of our proposed model, considering a number of applications dealing with volatility modeling in financial return series. In the final section, we summarize and discuss our results.

2 Preliminaries

2.1 The Pitman-Yor process

Dirichlet process models were first introduced by Ferguson [13]. A DP is characterized by a base distribution $G_0$ and a positive scalar $\alpha$, usually referred to as the innovation parameter, and is denoted as DP($\alpha, G_0$). Essentially, a DP is a distribution placed over a distribution. Let us suppose we randomly draw a sample distribution $G$ from a DP, and, subsequently, we independently draw $M$ random variables $\{\Theta^*_{m}\}_{m=1}^{M}$ from $G$: $G|\alpha, G_0 \sim \text{DP}(\alpha, G_0)$ (1)

$$\Theta^*_m|G \sim G, \quad m = 1, \ldots M$$ (2)

Integrating out $G$, the joint distribution of the variables $\{\Theta^*_m\}_{m=1}^{M}$ can be shown to exhibit a clustering effect. Specifically, given the first $M-1$ samples of $G$, $\{\Theta^*_m\}_{m=1}^{M-1}$, it can be shown that a new sample $\Theta^*_M$ is either (a) drawn from the base distribution $G_0$ with probability $\frac{\alpha}{\alpha + M - 1}$, or (b) is selected from the existing draws, according to a multinomial allocation, with probabilities proportional to the number of the previous draws with the same allocation [14]. Let $\{\Theta^*_c\}_{c=1}^{C}$ be the set of distinct values taken by the variables $\{\Theta^*_m\}_{m=1}^{M-1}$. Denoting $\nu_c$ the number of values in $\{\Theta^*_m\}_{m=1}^{M-1}$ that equal to $\Theta_c$, the distribution of $\Theta^*_M$ given $\{\Theta^*_m\}_{m=1}^{M}$ can be shown to be of the form [14]

$$p(\Theta^*_M|\{\Theta^*_m\}_{m=1}^{M-1}, \alpha, G_0) = \frac{\alpha}{\alpha + M - 1} G_0 + \sum_{c=1}^{C} \frac{\nu_c^{M-1}}{\alpha + M - 1} \delta_{\Theta_c}$$ (3)

where $\delta_{\Theta_c}$ denotes the distribution concentrated at a single point $\Theta_c$. These results illustrate two key properties of the DP scheme. First, the innovation parameter $\alpha$ plays a key-role in determining the number of distinct parameter values. A larger $\alpha$ induces a higher tendency of drawing new parameters from the base distribution $G_0$; indeed, as $\alpha \to \infty$ we get $G \to G_0$. On the contrary, as $\alpha \to 0$ all $\{\Theta^*_m\}_{m=1}^{M}$ tend to cluster to a single random variable. Second, the more often a parameter is shared, the more likely it will be shared in the future.

The Pitman-Yor process (PYP) [11] functions similar to the Dirichlet process. Let us suppose we randomly draw a sample distribution $G$ from a PYP, and, subsequently, we independently draw $M$ random variables $\{\Theta^*_m\}_{m=1}^{M}$ from $G$: $G|\delta, \alpha, G_0 \sim \text{PY}(\delta, \alpha, G_0)$ (4)

with $\Theta^*_m|G \sim G, \quad m = 1, \ldots M$ (5)

where $\delta \in [0, 1)$ is the discount parameter of the Pitman-Yor process, $\alpha > -\delta$ is its innovation parameter, and $G_0$ is the base distribution. Integrating out $G$, similar to Eq. (3), we now yield

$$p(\Theta^*_M|\{\Theta^*_m\}_{m=1}^{M-1}, \delta, \alpha, G_0) = \frac{\alpha + \delta C}{\alpha + M - 1} G_0 + \sum_{c=1}^{C} \frac{\nu_c^{M-1}}{\alpha + M - 1} \delta_{\Theta_c}$$ (6)

As we observe, the PYP yields an expression for $p(\Theta^*_M|\{\Theta^*_m\}_{m=1}^{M-1}, G_0)$ quite similar to that of the DP, also possessing the rich-genericher clustering property, i.e., the more samples have been assigned to a draw from $G_0$, the more likely subsequent samples will be assigned to the same draw. Further, the more we draw from $G_0$, the more likely a new sample will again be assigned to a new draw from $G_0$. These two effects
together produce a power-law distribution where many unique $\Theta^*_{n}$ values are observed, most of them rarely [11], thus allowing for better modeling observations with heavy-tailed distributions. In particular, for $\delta > 0$, the number of unique values scales as $O(\alpha M^\delta)$, where $M$ is the total number of draws. Note also that, for $\delta = 0$, the Pitman-Yor process reduces to the Dirichlet process, in which case the number of unique values grows more slowly at $O(\alpha \log M)$ [12].

A characterization of the (unconditional) distribution of the random variable $G$ drawn from a PYP, $PY(\delta, \alpha, G_0)$, is provided by the stick-breaking construction of Sethuraman [15]. Consider two infinite collections of independent random variables $v = (v_c)_{c=1}^\infty$ and $\Theta_c$ where the $v_c$ are drawn from a Beta distribution, and the $\Theta_c$ are independently drawn from the base distribution $G_0$. The stick-breaking representation of $G$ is then given by [12]

$G = \sum_{c=1}^\infty w_c(v)\delta_{\Theta_c}$

where

$p(v_c) = \text{Beta}(1 - \delta, \alpha + \delta c)$

$w_c(v) = v_c \prod_{j=1}^{c-1} (1 - v_j) \in [0, 1]$ and

$\sum_{c=1}^\infty w_c(v) = 1$.

Under the stick-breaking representation of the Pitman-Yor process, the atoms $\Theta_c$, drawn independently from the base distribution $G_0$, can be seen as the parameters of the component distributions of a mixture model comprising an unbounded number of component densities, with mixing proportions $w_c(v)$.

### 2.2 Gaussian process models

Let us consider an observation space $\mathcal{X}$. A Gaussian process $f(x)$, $x \in \mathcal{X}$, is defined as a collection of random variables, any finite number of which have a joint Gaussian distribution [15]. A Gaussian process is completely specified by its mean function and covariance function. We define the mean function $m(x)$ and the covariance function $k(x, x')$ of a real process $f(x)$ as

$m(x) = \mathbb{E}[f(x)]$

$k(x, x') = \mathbb{E}[(f(x) - m(x))(f(x') - m(x'))]$ (11)

and we will write the Gaussian process as

$f(x) \sim \mathcal{N}(m(x), k(x, x))$ (12)

Usually, for notational simplicity, and without any loss of generality, the mean of the process is taken to be zero, $m(x) = 0$, although this is not necessary. Concerning selection of the covariance function, a large variety of kernel functions $k(x, x')$ might be employed, depending on the application considered [16]. This way, a postulated Gaussian process eventually takes the form

$f(x) \sim \mathcal{N}(0, k(x, x))$. (13)

Let us suppose a set of independent and identically distributed (i.i.d.) samples $D = \{ (x_i, y_i) \}_{i=1}^N$, with the $d$-dimensional variables $x_i$ being the observations related to a modeled phenomenon, and the scalars $y_i$ being the associated target values. The goal of a regression model is, given a new observation $x_s$, to predict the corresponding target value $y_s$, based on the information contained in the training set $D$. The basic notion behind Gaussian process regression consists in the assumption that the observable (training) target values $y$ in a considered regression problem can be expressed as the superposition of a Gaussian process over the input space $\mathcal{X}$, $f(x)$, and an independent white Gaussian noise

$y = f(x) + \epsilon$ (14)

where $f(x)$ is given by (12), and

$\epsilon \sim \mathcal{N}(0, \sigma^2)$ (15)

Under this regard, the joint normality of the training target values $y = \{y_i\}_{i=1}^N$ and some unknown target value $y_s$, approximated by the value $f_s$ of the postulated Gaussian process evaluated at the observation point $x_s$, yields [16]

$\begin{bmatrix} y \\ f_s \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} K(X, X) + \sigma^2 I_N & k(x_s) \\ k(x_s)^T & k(x_s, x_s) \end{bmatrix} \right)$. (16)

where

$k(x_s) \triangleq [k(x_1, x_s), \ldots, k(x_N, x_s)]^T$ (17)

and $X = \{x_i\}_{i=1}^N$, $I_N$ is the $N \times N$ identity matrix, and $K$ is the matrix of the covariances between the $N$ training data points (design matrix), i.e.

$K(X, X) \triangleq \begin{bmatrix} k(x_1, x_1) & k(x_1, x_2) & \ldots & k(x_1, x_N) \\ k(x_2, x_1) & k(x_2, x_2) & \ldots & k(x_2, x_N) \\ \vdots & \vdots & \ddots & \vdots \\ k(x_N, x_1) & k(x_N, x_2) & \ldots & k(x_N, x_N) \end{bmatrix}$. (18)

Then, from (16), and conditioning on the available training samples, we can derive the expression of the model predictive distribution, yielding

$p(f_s \mid x_s, D) = \mathcal{N}(f_s \mid \mu_s, \sigma_s^2)$ (19)

$\mu_s^2 = k(x_s)^T K^{-1} (X, X) + \sigma^2 I_N)^{-1} y$ (20)

$\sigma_s^2 = \sigma^2 - k(x_s)^T K^{-1} (X, X) + \sigma^2 I_N)^{-1} k(x_s) + k(x_s, x_s)$ (21)

Regarding optimization of the hyperparameters of the employed covariance function (kernel), say $\theta$, and the noise variance $\sigma^2$ of a GP model, this is usually conducted by type-II maximum likelihood, that is by maximization of the model marginal likelihood (evidence).
Using (16), it is easy to show that the evidence of the GP regression model yields

$$\log p(y|X; \theta, \sigma^2) = - \frac{N}{2} \log 2\pi - \frac{1}{2} \log |K(X, X) + \sigma^2 I_N| - \frac{1}{2} y^T (K(X, X) + \sigma^2 I_N)^{-1} y$$  

(22)

It is interesting to note that the GP regression model considers that the noise that contaminates the modeled output variables does not depend on the observations themselves, rather that it constitutes an additive white noise term with constant variance, which bears no correlation between observations, and no dependence on the values of the observations. Nevertheless, in many real-world applications, with financial return series modeling being a characteristic example, this assumption of constant noise variance is clearly implausible.

To ameliorate this issue, an heteroscedastic GP regression approach was proposed in [17], where the noise variance is considered to be a function of the observed data, similar to previously proposed heteroscedastic regression approaches applied to econometrics and statistical finance, e.g., [18], [19]. A key drawback of the approach of [17] is that their heteroscedastic regression approach does not allow for capturing the clustering effects in the variability or volatility, which is apparent in the vast majority of financial return series data, and is effectively captured by GARCH-type models. Our approach addresses these issues under a nonparametric Bayesian mixture modeling scheme, as discussed next.

3 Proposed Approach

In this section, we first define the proposed MGPCH model, considering a generic modeling problem that comprises the input variables \( x \in \mathbb{R}^p \), and the output variables \( y \in \mathbb{R}^D \). Further, we derive an efficient inference algorithm for our model under the variational Bayesian inference paradigm, and we obtain the expression of its predictive density. Finally, we show how we can obtain a predictive distribution for the covariances between the modeled output variables \( \{ y_i \}_{i=1}^D \), by utilization of the statistical tool of copulas.

3.1 Model definition

Let \( f_d(x) \) be a latent function modeling the \( d \)th output variable \( y_d \) as a function of the model input \( x \). We consider that the expression of \( y_d \) as a function of \( x \) is not uniquely described by the latent function \( f_d(x) \), but \( f_d(x) \) is only an instance of the (possibly infinite) set of possible latent functions \( f_d^c(x) \), \( c = 1, \ldots, \infty \). To determine the association between input samples and latent functions, we impose a Pitman-Yor process prior over this set of functions. In addition, we consider that each one of these latent functions \( f_d^c(x) \) has a prior distribution of the form of a Gaussian process over the whole space of input variables \( x \in \mathbb{R}^p \). At this point, we make a further key-assumption: We assume that the noise variance \( \sigma^2 \) of the postulated GPs is not a constant, but rather that it varies with the input variables \( x \in \mathbb{R}^p \). In other words, we consider the noise variance as a latent process, different for each model variable, and exhibiting a clustering effect, as described by the dynamics of the postulated PYP mixing prior.

Let us consider a set of input/output observation pairs \( \{ x_n, y_n \}_{n=1}^N \), comprising \( N \) samples. Let us also introduce the set of variables \( \{ z_{nc} \}_{n,c=1}^{N,\infty} \), with \( z_{nc} = 1 \) if the function relating \( x_n \) to \( y_n \) is considered to be expressible by the set \( \{ f_d^c(x) \}_{d=1}^D \) of postulated Gaussian processes, \( z_{nc} = 0 \) otherwise. Then, based on the previous description, we essentially postulate the following model:

$$p(y_n|x_n, z_{nc} = 1) = \prod_{d=1}^D N(y_n|f_d^c(x_n), \sigma_d^c(x_n)^2)$$  

(23)

$$p(v_c) = \text{Beta}(1-\delta, \alpha + \delta c)$$  

(27)

$$p(f_d^c|X) = \mathcal{N}(f_d^c|0, K^c(X, X))$$  

(28)

Regarding the latent processes \( \sigma_d^c(x_n)^2 \), we choose to also impose a GP prior over them. Specifically, to accommodate the fact that \( \sigma_d^c(x_n)^2 \geq 0 \) (by definition), we postulate

$$\sigma_d^c(x_n)^2 = \exp[g_d^c(x_n)]$$  

(30)

and

$$p(g_d^c|X) = \mathcal{N}(g_d^c|\bar{m}_d^c, \Lambda^c(X, X))$$  

(31)

Finally, due to the effect of the innovation parameter \( \alpha \) on the number of effective mixture components, we also impose a Gamma prior over it:

$$p(\alpha) = \mathcal{G}(\alpha|\eta_1, \eta_2)$$  

(32)

This completes the definition of our proposed MGPCH model.
3.2 Inference algorithm

Inference for nonparametric models can be conducted under a Bayesian setting, typically by means of variational Bayes (e.g., [20]), or Monte Carlo techniques (e.g., [21]). Here, we prefer a variational Bayesian approach, due to its considerably better scalability in terms of computational costs, which becomes of major importance when having to deal with large data corpora [22], [23].

Our variational Bayesian inference algorithm for the MGPCH model comprises derivation of a family of variational posterior distributions \( q(\cdot) \) which approximate the true posterior distribution over the infinite sets \( Z, v = (v_c)_{c=1}^{\infty} \), \( \{f_c\}_{c=1}^{\infty} \), and \( \{g_c\}_{c=1}^{\infty} \), and the innovation parameter \( \alpha \). Apparently, Bayesian inference is not tractable under this setting, since we are dealing with an infinite number of parameters.

For this reason, we employ a common strategy in the literature of Bayesian nonparametrics, formulated on the basis of a truncated stick-breaking representation of the PYP [20]. That is, we fix a value \( C \) and we let the variational posterior over the \( v_c \) have the property \( q(v_C = 1) = 1 \). In other words, we set \( \pi_c(v) \) equal to zero for \( c > C \). Note that, under this setting, the treated MGPCH model involves a full PYP prior; truncation is not imposed on the model itself, but only on the variational distribution to allow for tractable inference. Hence, the truncation level \( C \) is a variational parameter which can be freely set, and not part of the prior model specification.

Let \( W = \{v, \alpha, Z, \{f_c\}_{c=1}^{C}, \{g_c\}_{c=1}^{C}\} \) be the set of all the parameters of the MGPCH model over which a prior distribution has been imposed, and \( \Xi \) be the set of the hyperparameters of the model priors and kernel functions. Variational Bayesian inference introduces an arbitrary distribution \( q(W) \) to approximate the actual posterior \( p(W|\Xi, X, Y) \) which is computationally intractable, yielding [24]

\[
\log p(X, Y) = \mathcal{L}(q) + \text{KL}(q||p) \tag{33}
\]

where

\[
\mathcal{L}(q) = \int dW q(W) \log \frac{p(X, Y, W|\Xi)}{q(W)} \tag{34}
\]

and \( \text{KL}(q||p) \) stands for the Kullback-Leibler (KL) divergence between the (approximate) variational posterior, \( q(W) \), and the actual posterior, \( p(W|\Xi, X, Y) \). Since KL divergence is nonnegative, \( \mathcal{L}(q) \) forms a strict lower bound of the log evidence, and would become exact if \( q(W) = p(W|\Xi, X, Y) \). Hence, by maximizing this lower bound \( \mathcal{L}(q) \) (variational free energy) so that it becomes as tight as possible, not only do we minimize the KL-divergence between the true and the variational posterior, but we also implicitly integrate out the unknowns \( W \).

Due to the considered conjugate exponential prior configuration of the MGPCH model, the variational posterior \( q(W) \) is expected to take the same functional form as the prior, \( p(W) \) [25]:

\[
q(W) = q(Z) q(\alpha) \left( \prod_{c=1}^{C-1} q(v_c) \right) \prod_{c=1}^{C} \prod_{d=1}^{D} q(f_c^d) q(g_c^d) \tag{35}
\]

with

\[
q(Z) = \prod_{n=1}^{N} \prod_{c=1}^{C} q(z_{nc} = 1) \tag{36}
\]

Then, the variational free energy of the model reads (ignoring constant terms)

\[
\mathcal{L}(q) = \sum_{c=1}^{C} \sum_{d=1}^{D} \int d\mathbf{f}_d^c q(\mathbf{f}_d^c) \log \frac{p(\mathbf{f}_d^c|\mathbf{0}, \mathbf{K}(X, X))}{q(\mathbf{f}_d^c)}
\]

\[
+ \sum_{c=1}^{C} \sum_{d=1}^{D} \int d\mathbf{g}_d^c q(\mathbf{g}_d^c) \log \frac{p(\mathbf{g}_d^c|\tilde{\nu}^c_1, \Lambda^c(X, X))}{q(\mathbf{g}_d^c)}
\]

\[
+ \int d\alpha q(\alpha) \left\{ \frac{\log p(\hat{\alpha}|\hat{n}_1, \hat{n}_2)}{q(\alpha)} + \sum_{c=1}^{C-1} \int dv_c q(\nu_c) \log \frac{p(\nu_c|\alpha)}{q(\nu_c)} \right\}
\]

\[
+ \sum_{c=1}^{C} \sum_{n=1}^{N} q(z_{nc} = 1) \left\{ \int dv q(v) \log \frac{p(z_{nc} = 1|v)}{q(z_{nc} = 1)} \right\}
\]

\[
+ \sum_{d=1}^{D} \int d\mathbf{f}_d^c d\mathbf{g}_d^c q(\mathbf{f}_d^c, \mathbf{g}_d^c) \log p(y_{nd}|\mathbf{f}_d^c(x_n), \sigma_d^2(x_n)^2)
\}
\tag{37}
\]

Derivation of the variational posterior distribution \( q(W) \) involves maximization of the variational free energy \( \mathcal{L}(q) \) over each one of the factors of \( q(W) \) in turn, holding the others fixed, in an iterative manner [26]. By construction, this iterative, consecutive updating of the variational posterior distribution is guaranteed to monotonically and maximally increase the free energy \( \mathcal{L}(q) \) [25].

Let us denote as \( \langle \cdot \rangle \) the posterior expectation of a quantity. From (37), we obtain the following variational (approximate) posteriors over the parameters of our model:

1. Regarding the PYP stick-breaking variables \( v_c \), we have

\[
q(v_c) = \text{Beta}(v_c|\beta_{c,1}, \beta_{c,2}) \tag{38}
\]

where

\[
\beta_{c,1} = 1 - \delta + \sum_{n=1}^{N} q(z_{nc} = 1) \tag{39}
\]

\[
\beta_{c,2} = (\alpha) + c\delta + \sum_{c' = c+1}^{C} \sum_{n=1}^{N} q(z_{nc'} = 1) \tag{40}
\]

2. The innovation parameter \( \alpha \) yields

\[
q(\alpha) = \mathcal{G}(\alpha|\hat{\eta}_1, \hat{\eta}_2) \tag{41}
\]

where

\[
\hat{\eta}_1 = \eta_1 + C - 1 \tag{42}
\]
\[ \eta_2 = \eta_1 - \sum_{c=1}^{C-1} [\psi(\beta_{c,2}) - \psi(\beta_{c,1} + \beta_{c,2})] \quad (43) \]

\[ \psi(\cdot) \text{ denotes the Digamma function, and} \]

\[ \langle \alpha \rangle = \frac{\eta_1}{\eta_2} \quad (44) \]

3. Regarding the posteriors over the latent functions \( f_\theta \), we have

\[ q(f_\theta) = \mathcal{N}(f_\theta | \mu_\theta, \Sigma_\theta) \quad (45) \]

where

\[ \Sigma_\theta = \left( (K^c(X,X))^{-1} + B_\theta \right)^{-1} \quad (46) \]

\[ \mu_\theta = \Sigma_\theta B_\theta^T y_d \]

\[ B_\theta \triangleq \text{diag} \left( \frac{1}{\langle \sigma_\theta^2(x_n) \rangle} \right) \quad (48) \]

and \( y_d \triangleq [y_{nd}]_{n=1}^N \).

4. Similarly, regarding the posteriors over the latent noise variance processes \( g_\theta \), we have

\[ q(g_\theta) = \mathcal{N}(g_\theta | m_\theta, S_\theta) \quad (49) \]

where

\[ S_\theta = \left( (\Lambda^c(X,X))^{-1} + Q_\theta \right)^{-1} \quad (50) \]

\[ m_\theta = \Lambda^c(X,X) \left( Q_\theta - \frac{1}{2} \text{diag} [q(z_{nc} = 1)] \right) + \frac{1}{2} m_\theta 1 \]

and \( Q_\theta \) is a positive semi-definite diagonal matrix, whose components comprise variational parameters that can be freely set. Note that, from this result, it follows

\[ \langle \sigma_\theta^2(x_n) \rangle = \exp \left( m_\theta^2 - \frac{1}{2} S_\theta^2 \right) \quad (52) \]

5. Finally, the posteriors over the latent variables \( Z \) yield

\[ q(z_{nc} = 1) \propto \exp \langle \log \pi_c(v) \rangle \exp \langle r_{nc} \rangle \quad (53) \]

where

\[ \langle \log \pi_c(v) \rangle = \sum_{c=1}^{C-1} \langle \log (1 - v_c) \rangle + \langle \log v_c \rangle \quad (54) \]

and

\[ r_{nc} = -\frac{1}{2} \sum_{d=1}^D \left\{ \frac{1}{\langle \sigma_\theta^2(x_n) \rangle^2} \left[ (y_{nd} - [\mu_\theta^2])^2 + [\Sigma_\theta^2]_{nn} \right] + [m_\theta^2]_{nn} \right\} \quad (55) \]

where \( [\xi]_n \) is the \( n \)th element of vector \( \xi \), \( [\Sigma_\theta^2]_{nn} \) is the \( (n,n) \)th element of \( \Sigma_\theta \), and it holds

\[ \langle \log v_c \rangle = \psi(\beta_{c,2}) - \psi(\beta_{c,1} + \beta_{c,2}) \]

\[ \langle \log (1 - v_c) \rangle = \psi(\beta_{c,1}) - \psi(\beta_{c,1} + \beta_{c,2}) \]

As a final note, estimates of the values of the model hyperparameters set \( \Xi \), which comprises the hyperparameters of the model priors and the kernel functions \( k(\cdot, \cdot) \) and \( \lambda(\cdot, \cdot) \), are obtained by maximization of the model variational free energy \( \mathcal{L}(q) \) over each one of them. For this purpose, in this paper we resort to utilization of the limited memory variant of the BFGS algorithm (L-BFGS) [27].

### 3.3 Predictive density

Let us consider the predictive distribution of the \( d \)th model output variable corresponding to \( x_* \). To obtain it, we begin by deriving the predictive posterior distribution over the latent variables \( f \). Following the relevant derivations of Section 2.2, we have

\[ q(f_\star) = \sum_{c=1}^C \langle \pi_c(v) \rangle \prod_{d=1}^D \mathcal{N}(f_\star^c | a_{\star,d}^c, (\sigma_{\star,d}^c)^2) \quad (58) \]

where

\[ a_{\star,d}^c = k^c(x_\star)^T \left( K^c(X,X) + (B_\theta^c)^{-1} \right) y_d \]

\[ (\sigma_{\star,d}^c)^2 = -k^c(x_\star)^T \left( K^c(X,X) + (B_\theta^c)^{-1} \right) k^c(x_\star) \quad (60) \]

and \( k(x_\star) = [k(x_1, x_\star), ..., k(x_N, x_\star)]^T \quad (63) \)

Further, we proceed to the predictive posterior distribution over the latent variables \( g \); we yield

\[ q(g_\star) = \mathcal{N}(g_\star | g_{\star,d}, \varphi_{\star,d}) \quad (64) \]

where

\[ r_{\star,d}^c = \lambda^c(x_\star)^T \left( Q_\theta - \frac{1}{2} \right) 1 + m_\theta^c \quad (65) \]

\[ \varphi_{\star,d}^c = \lambda^c(x_\star)^T - \lambda^c(x_\star)^T \left( \Lambda_\theta^c + (Q_\theta^c)^{-1} \right) \lambda^c(x_\star) \quad (66) \]

and

\[ \lambda(x_\star) = [\lambda(x_1, x_\star), ..., \lambda(x_N, x_\star)]^T \quad (67) \]

Based on these results, the predictive posterior of our model output variables yields

\[ q(y_\star) = \int \mathcal{N}(y_\star | \sum_{c=1}^C \langle \pi_c(v) \rangle a_{\star,d}^c, C \sum_{c=1}^C \langle \pi_c(v) \rangle^2 (\sigma_{\star,d}^c)^2 + \exp(g_{\star,d}^c) ) \]
We note that this expression does not yield a Gaussian predictive posterior. However, it is rather straightforward to compute the predictive means and variances of $y_{sd}$. It holds

$$
\hat{y}_{sd} = \mathbb{E}[y_{sd} | x_s; D] = \sum_{c=1}^{C} (\omega_c(v)) a^c_{sd}
$$

and

$$
\forall y_{sd} | x_s; D] = \sum_{c=1}^{C} (\pi_c(v))^2 \left[ (\sigma^c_{sd})^2 + \psi^c_{sd} \right]
$$

where

$$
\psi^c_{sd} \equiv \mathbb{E}[\exp(g^c_{sd}) | x_s; D] = \exp \left( \tau^c_{sd} + \frac{1}{2} \varphi^c_{sd} \right)
$$

### 3.4 Learning the covariances between the modeled output variables

As one can observe from (23), a characteristic of our proposed MGIPCH model is its assumption that the distribution of the modeled output vectors $y \in \mathbb{R}^D$ factorizes over their component variables $\{y_d\}_{d=1}^{D}$. Indeed, this type of modeling is largely the norm in Gaussian process-based modeling approaches [16]. This construction in essence implies that, under our approach, the modeled output variables are considered independent, i.e., their covariance is always assumed to be zero. However, when jointly modeling the return series of various assets, the modeled output variables (asset returns) are rather strongly correlated, and it is desired to be capable of predicting the values of their covariances for any given input value.

Existing approaches for resolving these issues of GP-based models are based on the introduction of an additional kernel-based modeling mechanism that allows for capturing this latent covariance structure [28, 29, 30, 31, 32]. For example, in [28] the authors propose utilization of a convolution structure to induce correlations between two output components. In [31], a generalization of the previous method is proposed for the case of more than two modeled outputs combined under a convolved kernel. Along the same lines, multitask learning approaches for resolving these issues are presented in [29] and [30], where separate GPs are postulated for each output, and are considered to share the same prior in the context of a multitask learning framework.

A drawback of the aforementioned existing approaches is that, in all cases, learning entails employing a tedious optimization procedure to estimate a large number of hyperparameters of the used kernel functions. As expected, such a procedure is, indeed, highly prone to getting trapped to bad local optima, a fact that might severely undermine model performance.

In this work, to avoid being confronted with such optimization issues, and inspired by the financial research literature, we devise a novel way of capturing the interdependencies between the modeled output variables $\{y_d\}_{d=1}^{D}$, expressed in the form of their covariances:

specifically, we use the statistical tool of copulas [33]. The copula, introduced in the seminal work of Sklar [33], is a model of statistical dependence between random variables. A copula is defined as a multivariate distribution with standard uniform marginal distributions, or, alternatively, as a function (with some restrictions mentioned for example in [34]) that maps values from the unit hypercube to values in the unit interval.

#### 3.4.1 Copulas: An introduction

Let $y = [y_d]_{d=1}^{D}$ be a $D$-dimensional random variable with joint cumulative distribution function (cdf) $F\left([y_d]_{d=1}^{D}\right)$, and marginal cdf’s $F_d(y_d), d = 1, \ldots, D$, respectively. Then, according to Sklar’s theorem, there exists a $D$-variate copula $C(\cdot, \ldots, \cdot)$ on $[0, 1]^D$ such that

$$
F(y_1, \ldots, y_D) = C(F_1(y_1), \ldots, F_D(y_D))
$$

for any $y \in \mathbb{R}^D$. Additionally, if the marginals $F_d(\cdot), d = 1, \ldots, D$, are continuous, then the $D$-variate copula $C(\cdot, \ldots, \cdot)$ satisfying (72) is unique. Conversely, if $C(\cdot, \ldots, \cdot)$ is a $D$-dimensional copula and $F_i(\cdot), i = 1, \ldots, D$, are univariate cdf’s, it holds

$$
C(u_1, \ldots, u_D) = F(F_1^{-1}(u_1), \ldots, F_D^{-1}(u_D))
$$

where $F_d^{-1}(\cdot)$ denotes the inverse of the cdf of the $d$th marginal distribution $F_d(\cdot)$, i.e., the quantile function of the $d$th modeled variable $y_d$. It is easy to show that the corresponding probability density function of the copula model, widely known as the copula density function, is given by

$$
c(u_1, \ldots, u_D) = \frac{\partial^D}{\partial u_1 \ldots \partial u_D} C(u_1, \ldots, u_D)
$$

$$
= \frac{\partial^D}{\partial u_1 \ldots \partial u_D} F(F_1^{-1}(u_1), \ldots, F_D^{-1}(u_D))
$$

$$
= \frac{p(F_1^{-1}(u_1), \ldots, F_D^{-1}(u_D))}{\prod_{i=1}^{D} p_i(F_i^{-1}(u_i))}
$$

where $p_i(\cdot)$ is the probability density function of the $i$th component variable $y_i$.

Let us now assume a parametric class for the copula $C(\cdot, \ldots, \cdot)$ and the marginal cdf’s $F_i(\cdot), i = 1, \ldots, D$, respectively. In particular, let $\zeta$ denote the (trainable) parameter (or set of parameters) of the postulated copula. Then, the joint probability density of the modeled variables $y = [y_i]_{i=1}^{D}$ yields

$$
p(y_1, \ldots, y_D | \zeta) = \prod_{i=1}^{D} p_i(y_i) c(F_1(y_1), \ldots, F_D(y_D) | \zeta)
$$

Since the emergence of the concept of copula, several copula families have been constructed, e.g., Gaussian, Clayton, Frank, Gumbel, Joe, etc, that enable capturing of any form of dependence structure. By coupling different marginal distributions with different copula functions, copula-based models are able to model a wide variety
of marginal behaviors (such as skewness and fat tails), and dependence properties (such as clusters, positive or negative tail dependence) [34]. Selection of the best-fit copula has been a topic of rigorous research efforts during the last years, and motivating results have already been achieved [35] (for excellent and detailed discussions on copulas, c.f. [34, 36]).

3.4.2 Proposed Approach

In this work, to capture the interdependencies (covariances) between the MGPCPCH-modeled output variables, we propose a conditional copula-based dependence modeling framework. Specifically, for the considered $D$-dimensional output vectors $y = [y_d]_{d=1}^D$, we postulate pairwise parametric conditional models for each output pair $(y_i, y_j)$ with $i, j = 1, \ldots, D$, respectively.

$$F(y_i, y_j | x) = C(F_i(y_i | x), F_j(y_j | x) | x)$$

(76)

where the marginals $F_i(y_i | x)$ are the cdf’s that correspond to the predictive posteriors $q(y_{i,d})$ given by (68), and the used input-conditional copulas are defined under a parametric construction as

$$C(u_i, u_j | x) \equiv C(u_i, u_j | \zeta_{ij}(x))$$

(77)

and we consider that the $\zeta_{ij}(x)$ are given by

$$\zeta_{ij}(x) = \xi(\gamma_{ij}(x))$$

(78)

where the $\gamma_{ij}(x)$ are trainable real-valued models, and $\xi(\cdot)$ is a link function ensuring that the values of $\zeta_{ij}(x)$ will always be within the range allowed by the copula model employed each time. For instance, if a Clayton copula $C(\cdot)$ is employed, it is required that its parameter be positive, i.e., $\zeta_{ij}(x) > 0$ [34]; in such a case, $\xi(\cdot)$ may be defined as the exponential function, i.e., $\xi(\alpha) = \exp(\alpha)$.

Note that the predictive posteriors $q(y_{i,d})$ are difficult to compute analytically, since (68) does not yield a Gaussian distribution. For this reason, and in order to facilitate efficient training of the postulated pairwise conditional copula models, in the following we approximate (68) as a Gaussian with mean and variance given by (69) and (70), respectively.

Further, we consider the functions $\gamma_{ij}(x)$ to be linear basis functions models. Specifically, we postulate

$$\gamma_{ij}(x) = w_{ij}^T h(x)$$

(79)

where the $w_{ij}$ are trainable model parameters, and the basis functions $h(x)$ are defined using a small set of basis input observations $\{x_i\}_{i=1}^L$, and an appropriate kernel function $k$:

$$h(x) \equiv [k(x, x_i)]_{i=1}^L$$

(80)

Training for the postulated pairwise conditional copula models can be performed by optimizing the logarithm of the copula density function that corresponds to the parametric conditional model (77), given a set of training data $\mathcal{D} = (x_n, y_n)_{n=1}^N$, which yields

$$\mathcal{P}_{ij} = \sum_{n=1}^N \log c(F_i(y_{ni} | x_n), F_j(y_{nj} | x_n) | \xi(w_{ij}^T h(x_n)))$$

(81)

with respect to the parameter vectors $w_{ij}$. To effect this procedure, in this paper we resort to the L-BFGS algorithm [27].

After training the postulated pairwise models $C(u_i, u_j | \zeta_{ij}(x))$ for $i \neq j$, computation of the predictive covariance $\mathbb{V}[y_{i*}, y_{j*} | x, \mathcal{D}]$ between the $i$th and the $j$th model output given the input observation $x$, can be conducted using the corresponding conditional copula model and marginal predictive densities. Specifically, from Hoeffding’s lemma [37, 38, 39], we directly obtain [Eq. (82)]; this latter integral can be approximated by means of numerical analysis methods.

4 Experimental Evaluation

In this section, we elaborate on the application of our MGPCPCH approach to volatility modeling for financial return series data. We perform an experimental evaluation of its performance in volatility modeling, and examine how it compares to state-of-the-art competitors. We also assess the efficacy of the proposed copula-based approach for learning the predictive covariances between the modeled output variables of the MGPCPCH model.

For this purpose, we consider modeling the daily return series of various financial indices, including currency exchange rates, global large-cap equity indices, and Euribor rates. We note that, in this work, asset return $r(t)$ is defined as the difference between the logarithm of the prices $p(t)$ in two subsequent time points, i.e., $r(t) \triangleq \log p(t) - \log p(t-1)$. All our source codes were developed in MATLAB R2012a.

4.1 Volatility prediction using the MGPCPCH model

In this set of experiments, we consider three application scenarios:

- In the first scenario, we model the return series pertaining to the following currency exchange rates, over the period December 31, 1979 to December 31, 1998 (daily closing prices):
  1. (AUD) Australian Dollar / US $.
  2. (GBP) UK Pound / US $.
  3. (CAD) Canadian Dollar / US $.
  4. (DKK) Danish Krone / US $.
  5. (FRF) French Franc / US $.
  6. (DEM) German Mark / US $.
  7. (JPY) Japanese Yen / US $.
  8. (CHF) Swiss Franc / US $.

- In the second scenario, we model the return series pertaining to the following global large-cap equity indices, for the business days over the period April
27, 1993 to July 14, 2003 (daily closing prices):
1. (TSX) Canadian TSX Composite
2. (CAC) French CAC 40
3. (DAX) German DAX
4. (NIK) Japanese Nikkei 225
5. (FTSE) UK FTSE 100
6. (SP) US S&P 500
7. (EB3M) Three-month Euribor rate.

These series have become standard benchmarks for assessing the performance of volatility prediction algorithms [40], [41], [42].

In all the considered scenarios, the proposed MGPCH model is trained using as input data, $x(t)$, vectors containing the daily returns of all the assets considered in each scenario. The corresponding training output data $y(t)$ essentially comprise the same series of input vectors shifted one-step ahead. In other words, the output series are defined as $y(t) \equiv r(t+1)$, $t > 0$, and the input series as $x(t) \equiv r(t)$, $t < T$, where $T$ is the total duration of the modeled return series, and the vectors $r(t)$ contain the return values of all the considered indices at time $t$.

In our experiments, we evaluate the MGPCH model using zero kernels for the mean process, i.e. $k^c(x, x') = 0 \forall c$; this construction allows for our model to remain consistent with the existing literature, where it is typically considered that the modeled return series constitute a zero-mean noise-only process, i.e. $f^c_\gamma(x) = 0 \forall d, c$. Note though that our approach can seamlessly deal with learning the mean process $f^c_\gamma(x)$, if a model for its covariance is available. Further, we consider autoregressive kernels of order one for the noise variance process of the model, of the form

$$\lambda^c(x, x') = \frac{\sigma^2_0}{1 - \phi^2} \phi ||x - x'||$$  \hspace{1cm} (83)

where the $\phi$ and $\sigma^2_0$ are model hyperparameters, estimated by means of free energy optimization (using the L-BFGS algorithm).

To obtain some comparative results, we also evaluate: (i) a common baseline approach from the field of financial engineering and econometrics, namely the GARCH(1,1) model [3], that is a GARCH model with volatility terms of order one and residual terms of order one; and (ii) the recently proposed VHGP approach of [17]. Both these approaches have been shown to be very competitive in the task of volatility prediction in financial return series [43], [17]. Note that the GARCH(1,1) model uses as input the time variable, while the VHGP model is trained similar to MGPCH.

In our experiments, similar to [41], all the evaluated methods are trained using a rolling window of the previous 120 days of returns to make 1, 7, and 30 days ahead volatility forecasts; we retrain the models every 7 days. We use two performance metrics to evaluate the considered algorithms: The first one is the mean squared error (MSE) between the model-estimated volatilities and the squared returns of the modeled return series. The second one is the MSE between the generated predictions and the historical volatilities computed over rolling windows of 10 contiguous return values (days). As discussed in [44], these two groundtruth measurements (squared returns and historical volatilities) constitute two of the few consistent ways of volatility measuring.

In Tables 1-3, we provide the obtained results for the three considered scenarios. These results are means of the obtained MSEs over all the assets modeled in each scenario. As we observe, our approach yields a clear advantage and a significant improvement over its competitors, of at least one order of magnitude, in all the considered scenarios, in terms of both the employed evaluation metrics.

4.2 Copula-based modeling of the covariances between asset returns

Here, we evaluate the performance of the proposed copula-based approach for learning a predictive model of the covariances between the MGPCH-modeled asset returns. For this purpose, we repeat the previous experimental scenarios, with the goal now being to obtain predictions regarding the covariances between the assets modeled each time.

In our experiments, we consider application of three popular Archimedean copula types, namely Clayton, Frank, and Gumbel copulas [34]. The employed MGPCH models are trained similar to the previous experiments. The postulated conditional-copula pairwise models use a basis set of input observations (to compute the $h(x)$ in (80)) that comprises the 10% of the available training data points, i.e. 12 data points sampled at regular time intervals (one sample every 10 days).

To obtain some comparative results, we also evaluate the performance of two state-of-the-art methods used
Squared Returns

Table 1

| Evaluation Metric | Squared Returns | Historical Volatility |
|-------------------|----------------|-----------------------|
|                   | 1-step | 7-step | 30-step | Average | 1-step | 7-step | 30-step | Average |
| GARCH             | 4.99  | 1.89   | 6.11    | 5.03    | 4.98   | 5.01   | 5.08    | 5.02    |
| VHGP              | 2.15  | 2.16   | 2.15    | 2.15    | 1.63   | 1.63   | 1.62    | 1.63    |
| MGFC              | 1.46  | 1.46   | 1.48    | 1.47    | 1.03   | 1.02   | 1.02    | 1.03    |

Table 2

| Evaluation Metric | Squared Returns | Historical Volatility |
|-------------------|----------------|-----------------------|
|                   | 1-step | 7-step | 30-step | Average | 1-step | 7-step | 30-step | Average |
| GARCH             | 9.56  | 9.96   | 9.66    | 9.99    | 1.00   | 1.03   | 1.01    | 1.01    |
| VHGP              | 3.42  | 3.53   | 3.45    | 4.10    | 4.06   | 3.98   | 4.05    | 4.05    |
| MGFC              | 1.39  | 1.45   | 1.41    | 3.52    | 3.52   | 3.47   | 3.50    | 3.50    |

Table 3

| Evaluation Metric | Squared Returns | Historical Volatility |
|-------------------|----------------|-----------------------|
|                   | 1-step | 7-step | 30-step | Average | 1-step | 7-step | 30-step | Average |
| GARCH             | 4.14  | 4.94   | 4.33    | 4.50    | 4.61   | 5.06   | 4.72    | 4.72    |
| VHGP              | 4.03  | 4.26   | 4.08    | 4.87    | 4.89   | 4.91   | 4.88    | 4.88    |
| MGFC              | 1.45  | 1.52   | 1.47    | 4.36    | 4.42   | 4.42   | 4.40    | 4.40    |

for modeling dynamic covariance matrices (multivariate volatility) for high-dimensional vector-valued observations; specifically, we consider the CCC-MVGARCH(1,1) approach of [45], and the GARCH-BEKK(1,1) method of [46]. As our evaluation metric, we use the products of the returns of the corresponding asset pairs at each time point. Our obtained results are depicted in Tables 4-6. We observe that our approach yields a very competitive result: specifically, in two out of the three considered scenarios, the yielded improvement was equal to or exceeded one order of magnitude, while, in one case, all methods yielded comparable results. We also observe that switching the employed Archimedean copula type had only marginal effects on model performance, in all our experiments.

5 Conclusions

In this paper, we proposed a novel nonparametric Bayesian approach for modeling conditional heteroscedasticity in financial return series. Our approach consists in the postulation of a mixture of Gaussian process regression models, each component of which models the noise variance process that contaminates the observed data as a separate latent Gaussian process driven by the observed data. We imposed a nonparametric prior with power-law nature over the distribution of the model mixture components, namely the Pitman-Yor process prior, to allow for better capturing modeled data distributions with heavy tails and skewness. In addition, in order to provide a predictive posterior for the covariances over the modeled asset returns, we devised a copula-based covariance modeling procedure built on top of our model. To assess the efficacy of our approach, we applied it to several asset return series, and compared its performance to several state-of-the-art methods in the field, on the grounds of standard evaluation metrics. As we observed, our approach yields a clear performance improvement over its competitors in all the considered scenarios.

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Table 4
First Scenario: Obtained MSEs considering comparison against the asset pair return products.

| Evaluation Metric | Squared Returns | | | |
|-------------------|-----------------|-----------------|-----------------|-----------------|
| Prediction Horizon | 1-step | 7-step | 30-step | Average |
| CCC-MVGARCH | 1.19×10⁻⁴ | 1.17×10⁻⁴ | 1.20×10⁻⁴ | 1.20×10⁻⁴ |
| BEKK | 1.16×10⁻⁴ | 1.16×10⁻⁴ | 1.17×10⁻⁴ | 1.16×10⁻⁴ |
| MGPC: Clayton | 1.18×10⁻⁴ | 1.17×10⁻⁴ | 1.17×10⁻⁴ | 1.17×10⁻⁴ |
| MGPC: Frank | 1.16×10⁻⁴ | 1.16×10⁻⁴ | 1.17×10⁻⁴ | 1.16×10⁻⁴ |
| MGPC: Gumbel | 1.16×10⁻⁴ | 1.16×10⁻⁴ | 1.16×10⁻⁴ | 1.16×10⁻⁴ |

Table 5
Second Scenario: Obtained MSEs considering comparison against the asset pair return products.

| Evaluation Metric | Squared Returns | | | |
|-------------------|-----------------|-----------------|-----------------|-----------------|
| Prediction Horizon | 1-step | 7-step | 30-step | Average |
| CCC-MVGARCH | 1.4×10⁻⁶ | 1.4×10⁻⁶ | 1.4×10⁻⁶ | 1.4×10⁻⁶ |
| BEKK | 1.7×10⁻⁶ | 1.7×10⁻⁶ | 1.7×10⁻⁶ | 1.7×10⁻⁶ |
| MGPC: Clayton | 3.1×10⁻⁷ | 3.1×10⁻⁷ | 3.1×10⁻⁷ | 3.1×10⁻⁷ |
| MGPC: Frank | 3.2×10⁻⁷ | 3.2×10⁻⁷ | 3.2×10⁻⁷ | 3.2×10⁻⁷ |
| MGPC: Gumbel | 3.1×10⁻⁷ | 3.1×10⁻⁷ | 3.1×10⁻⁷ | 3.1×10⁻⁷ |

Table 6
Third Scenario: Obtained MSEs considering comparison against the asset pair return products.

| Evaluation Metric | Squared Returns | | | |
|-------------------|-----------------|-----------------|-----------------|-----------------|
| Prediction Horizon | 1-step | 7-step | 30-step | Average |
| CCC-MVGARCH | 0.0044 | 0.0044 | 0.0044 | 0.0044 |
| BEKK | 0.84 | 0.85 | 0.85 | 0.85 |
| MGPC: Clayton | 9.8×10⁻⁶ | 9.8×10⁻⁶ | 9.8×10⁻⁶ | 9.8×10⁻⁶ |
| MGPC: Frank | 9.8×10⁻⁶ | 9.8×10⁻⁶ | 9.8×10⁻⁶ | 9.8×10⁻⁶ |
| MGPC: Gumbel | 9.8×10⁻⁶ | 9.8×10⁻⁶ | 9.8×10⁻⁶ | 9.8×10⁻⁶ |

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