Anomalous dimensions of four-quark operators in the large $N_f$ limit

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Abstract

The anomalous dimensions of four-quark operators $(\bar{q}_i q_j)_{V-A} (\bar{q}_k q_l)_{V-A}$ are calculated in the large $N_f$ limit. As expected, the result is a convergent series without renormalon ambiguities. Using the approximation of “Naive Nonabelianization”, an additional all-order contribution to the anomalous dimension matrix is obtained which is somewhat larger than the exact NLO correction itself. Possible phenomenological applications in nonleptonic $B$ decays and $B^0 - \bar{B}^0$ mixing are briefly considered.

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1 Introduction

The behaviour of the perturbative series at large orders constitutes one of the major unsolved problems of quantum field theory. Since it is in practice impossible to calculate a given physical quantity exactly to all orders of perturbation theory, the following approach for a better understanding of the a priori unknown large-order behaviour suggests itself: namely, instead of considering all possible diagrams, one tries to identify certain subclasses of diagrams which become dominant at sufficiently high orders and can be calculated exactly to all orders.

Once such a subclass of diagrams is found, it will be useful in three aspects: (i) for analyzing fundamental questions related to the summability and/or uniqueness of the perturbative series, as well as the connection of these problems to non-perturbative issues, (ii) for estimating the numerical effects of higher order contributions, which are important in many phenomenological applications, (iii) for having at one’s disposal an independent check of forthcoming exact higher-order calculations. The latter point should especially be emphasized, since such calculations will most certainly be performed in a completely computerized way. Therefore, analytical expressions for a well-defined part of the full result should be regarded as a helpful and desirable tool for the testing of these probably very extensive computer programs.

In fact, until now only one of these subclasses of Feynman graphs has been found: diagrams with an insertion of an arbitrary number of massless fermion loops (“bubbles”) in a gluon line, the so-called renormalon chains. If a physical quantity $R$ has the perturbative expansion

$$R = \sum_{n=0}^{\infty} r_n a^n,$$

where $a$ is the coupling constant of the theory, and if we perform for each coefficient $r_n$ an additional expansion in $N_f$, the number of massless fermions in the theory,

$$r_n = \sum_{m=0}^{n-1} r_n^{(m)} N_f^m, \quad n \geq 1,$$

then the coefficients $r_n^{(n-1)}$ will obviously be determined by diagrams where $n - 1$ fermion bubbles have been inserted into a gluon line of some leading-order graph.

Of course it is doubtful whether these coefficients $r_n^{(n-1)}$ correctly describe the asymptotic behaviour of the perturbative series or not. Although this will be the case in the limit $N_f \to \infty$, at first sight it looks like a gross mistake to conclude that the real world (with $N_f = 5$ in the case of $B$ physics or $e^+e^-$ annihilation) resembles at least in some aspects this fictitious limit. Yet there are some arguments for such a point of view. Above all they rely on the observation that the amplitude of an individual fermion-bubble diagram grows in general like $n!$ at high orders. If one remembers that usually not the amplitude of an individual graph but only the number of all possible graphs grows with $n!$, and if one further assumes that all diagrams yield approximately the same contribution with no preferred sign, then one can argue that the fermion-bubble diagrams should become dominant at sufficiently high orders. For recent reviews and further references see [3, 4].
In practical applications, higher order effects play a significant role only in QCD with its relative large coupling constant. However, QCD is exceptionally bad described by the limit $N_f \to \infty$, since in this limit asymptotic freedom, one of the most characteristic features of the strong interaction, is violated ($\beta_0 \sim N_f - 33/2$, the first coefficient of the QCD beta function, becomes positive for $N_f > 16$). As a generalization of the QED case, where $\beta_0 \sim N_f$, it was therefore proposed \cite{5–7} that in QCD one should expand the perturbative coefficients $r_n$ in powers of $N_f - 33/2$,

$$r_n = \sum_{m=0}^{n-1} r_n^{[m]} (N_f - 33/2)^m, \quad n \geq 1,$$

(3)

and that physical quantities are at sufficiently high orders well described by neglecting all but the coefficients $r_n^{[n-1]} = r_n^{(n-1)}$ in this expansion. This approximation is usually referred to as “Naive Nonabelianization” (NNA).

In the last two years the NNA method was applied to several observables where a comparison with existing Next-To-Leading-Order (NLO) or Next-To-Next-To-Leading-Order (NNLO) calculations was possible. Specifically, analyses were performed of the $R$ ratio and related quantities \cite{8}, of the hadronic decay width $R_\tau$ of the $\tau$ lepton \cite{8}, of semileptonic B decays \cite{9}, and of structure functions in deep inelastic scattering \cite{10–13}. In all of these, a rather good agreement between NNA and the exact calculation was claimed at third or even at second order. However, it should be stressed that a systematic reason for this agreement is still not known.

It is interesting to extend this program to some other quantities where exact NLO results are available. To this end, in this paper the anomalous dimensions of the weak current-current four-quark operators is calculated in the large $N_f$ limit. Such operators occur in the effective Hamiltonian for e.g. weak hadronic decays and particle-antiparticle-mixing involving mesons. There NLO anomalous dimension matrix was calculated in \cite{14, 15}, see also \cite{16} for a recent review on the subject of effective hamiltonians for weak decays. Clearly, those anomalous dimensions are no direct physical observables; a complete calculation of the effective hamiltonian additionally includes the matching of the effective onto the full theory at the electroweak scale. This procedure involves the calculation of finite parts of Feynman diagrams. At a given order in $\alpha_s$, the matching is therefore considerably more difficult than the computation of renormalization group functions which is performed in what follows; nevertheless a posteriori it turns out that it is presumably even more important. This point will be discussed in the last but one section.

The organization of this paper is as follows: In section 2, the scene is set with the basic definitions of the relevant operators and their anomalous dimensions; it is then explained how to obtain these anomalous dimensions from the calculation of truncated Green functions. In section 3, this calculation is performed and the necessary renormalization of the gluon field in the large $N_f$ limit is discussed in detail. From this an analytic large $N_f$ formula for the anomalous dimensions, summed to all orders in $\alpha_s$, can be obtained. This is done in section 4. The result is discussed in section 5, where the NNA approximation is used to estimate the perturbative contributions beyond next-to-leading order, an estimate that will be relevant for some phenomenological applications. It is also discussed how the result for
the anomalous dimensions should be seen in the context of the full large $N_f$ calculation of the effective Hamiltonian. The paper closes with a short summary in section 6. Finally, an appendix is devoted to the so-called evanescent operators, objects that occur in the course of the calculation in section 3. In particular, it is proved that these operators do not contribute to anomalous dimensions in the large $N_f$ limit.

2 Definitions

Current-current operators arise due to tree-level $W$ exchange. Their Dirac structure is given by

$$\hat{\gamma} = \bar{q}_i \gamma^\mu (1 - \gamma^5) q_j \bar{q}_k \gamma^\nu (1 - \gamma^5) q_l,$$

(4)

where $i, j, k, l$ denote flavour indices. Due to exchange of gluons between the quark legs, two different color structures can arise, in symbolic form written as

$$1 = \delta_{\alpha\beta} \delta_{\gamma\delta}, \quad \tilde{1} = \delta_{\alpha\delta} \delta_{\beta\gamma},$$

(5)

where $\alpha, \beta, \gamma, \delta$ denote the color indices of the first, second, third and fourth quark field $q$ in Eq. (4). In this compact notation, the two operators we are dealing with are given by

$$O_1 = \hat{\gamma} \tilde{1}, \quad O_2 = \hat{\gamma} 1.$$

(6)

For further purposes we also introduce the linear combinations

$$O_\pm = \frac{1}{2} (O_2 \pm O_1).$$

(7)

In this basis, the anomalous dimension matrix will be diagonal. If we now calculate some truncated Green function $\langle O_i \rangle$ with an insertion of one of these operators, the result will in general be divergent even after renormalization of fields and QCD coupling. In order to achieve a finite result, we make use of our freedom to redefine the operator basis\footnote{In general, one has to include additional “evanescent” operators in the operator basis, since they unavoidably arise while calculating loop diagrams like those of Fig. 1. Accordingly, these evanescent operators also participate in the multiplicative renormalization of Eq. (8). However, in the large $N_f$ limit they can safely be neglected, as shown in the appendix.} according to

$$O_i = Z_{ij} O_j^R.$$

(8)

If we additionally renormalize the four external fields,

$$q = Z_q^{1/2} q^R,$$

(9)

then we can write the truncated Green function as

$$\langle O_i \rangle = Z_q^{-2} Z_{ij} \langle O_j^R \rangle.$$

(10)
where the $\langle O_j^R \rangle$'s are now finite. The subtractions will be performed in the $MS$ scheme throughout, so (working in $D = 4 - 2\epsilon$ dimensions) the renormalization constants $Z_{ij}$ will have the form

$$Z_{ij} = 1 + \sum_{k=1}^{\infty} \frac{Z_{ij}^{(k)}}{\epsilon^k}. \quad (11)$$

$Z_{ij}$ is a function of the renormalized coupling and therefore depends on the renormalization scale $\mu$. The anomalous dimension matrix of the operators $O_i$ is defined as

$$\gamma_{ij} = (Z^{-1})_{ik} \mu \frac{dZ_{kj}}{d\mu}. \quad (12)$$

Introducing the quantity

$$\tilde{Z}_{ij} = Z_{ij}^{-2} Z_{ij}, \quad (13)$$

which can, according to Eq. (10), be read off directly from the divergent pieces of the truncated Green functions $\langle O_i \rangle$, we may write Eq. (12) as

$$\gamma_{ij} = \tilde{\gamma}_{ij} + 2\gamma_{\psi}, \quad (14)$$

with

$$\tilde{\gamma}_{ij} = (\tilde{Z}^{-1})_{ik} \mu \frac{d\tilde{Z}_{kj}}{d\mu}, \quad \gamma_{\psi} = Z_{\psi}^{-1} \mu \frac{dZ_{\psi}}{d\mu}. \quad (15)$$

Using the equivalents of Eq. (11) for $\tilde{Z}_{ij}$ and $Z_{\psi}$, the chain rule $d/d\mu = (dg/d\mu) d/dg$ (where $g$ is the renormalized QCD coupling) and the renormalization group equation for $g$ one obtains

$$\tilde{\gamma}_{ij} = -2g^2 \frac{d\tilde{Z}_{ij}^{(1)}}{g^2}, \quad \gamma_{\psi} = -2g^2 \frac{dZ_{\psi}^{(1)}}{dg^2}, \quad (16)$$

i.e. the anomalous dimensions are directly related to the $1/\epsilon$ pole part of the corresponding renormalization constant.

3 Calculation of Green functions with bubble insertions

We now proceed to calculate the truncated Green functions $\langle O_1 \rangle$ and $\langle O_2 \rangle$ with an insertion of $n$ fermion bubbles into the gluon line, i.e. the diagrams shown in Fig. 1. In order to determine the quark wave function anomalous dimension $\gamma_{\psi}$, we will additionally calculate the truncated two point function depicted in Fig. 2. To this end, we need as a first step an expression for the gluon propagator with an insertion of $n \geq 0$ fermion bubbles (but no counterterms). A straight forward calculation in $D = 4 - 2\epsilon$ dimension yields

$$\Pi_{\mu\nu}^{(n)}(k) = \frac{-i}{k^2} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \left[ -\Pi(k^2) \right]^n. \quad (17)$$
The function $\Pi(k^2)$ stems from the one loop calculation of the fermion bubble,

$$\Pi(k^2) = \frac{\alpha_s}{\pi} N_f \left( \frac{4\pi \mu^2}{-k^2} \right) \Gamma(\epsilon) B(2 - \epsilon, 2 - \epsilon), \quad (18)$$

and $k$ is the momentum flowing through the propagator. It should be stressed that Eq. (17) is valid in any gauge for $n \geq 1$. Only for $n = 0$ it is the specific result of the Landau gauge. So the fermion bubble contribution is for itself a gauge independent quantity, as of course must be the case.

Now one can insert the effective propagator (17) into the diagrams of Fig. 1 and perform the loop integrations via introduction of Feynman parameters as usual. Adding the contributions from all six diagrams, we find

$$\begin{pmatrix} \langle O_1 \rangle_{\text{bare}}^{\text{bare}} \\ \langle O_2 \rangle_{\text{bare}}^{\text{bare}} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^n \frac{(-1)^n}{(n + 1)\epsilon^{n+1}} F(\epsilon, (n + 1)\epsilon) M_c \end{pmatrix} \begin{pmatrix} \langle O_1 \rangle_{\text{tree}}^{\text{tree}} \\ \langle O_2 \rangle_{\text{tree}}^{\text{tree}} \end{pmatrix}, \quad (19)$$

with

$$F(\epsilon, z) = \left( \frac{4\pi \mu^2}{-\lambda^2} \right)^z \Gamma(1 + z) \Gamma(1 - z) \left[ 6\epsilon \Gamma(\epsilon) B(2 - \epsilon, 2 - \epsilon) \right]^{z/\epsilon - 1} \frac{6 - 4\epsilon}{\Gamma(3 - \epsilon)}, \quad (20)$$
\[ M_c = \left( \begin{array}{c} -\frac{1}{N} + \epsilon \left( C_F - \frac{1}{N} \right) \\ 1 + \epsilon \\ -\frac{1}{N} + \epsilon \left( C_F - \frac{1}{N} \right) \end{array} \right) \]  

(21)

and \( C_F = (N^2 - 1)/(2N), \) \( N = 3. \) \( \langle O_i \rangle^{\text{tree}} \) denote the trivial tree-level matrix elements of the operators \( O_i, \) and the auxiliary function \( F(\epsilon, z) \) has been defined in such a way such \( F(0,0), F(\epsilon,0) \) and \( F(0,z) \) are finite. From a calculational point of view, Eqs. (19) – (21) constitute already the main result of this paper, so several remarks are in order:

(i) The index “bare” indicates that until now no renormalization has been performed. In the next step we will renormalize the gluon field so that the transition \( \langle O_i \rangle^{\text{bare}} \rightarrow \langle O_i \rangle \) is achieved. The final operator renormalization \( \langle O_i \rangle \rightarrow \langle O_i^R \rangle, \) cf. Eq. (8), is the subject of the following section.

(ii) In calculating Eq. (19) we set all external momenta to zero which is allowed since the renormalization constants we are interested in do not depend on the external states. Infrared divergent integrals were regulated by introducing a finite gluon mass \( \lambda \) that appears as the only scale (apart from the renormalization scale \( \mu \)) in Eq. (20).

(iii) In order to obtain Eq. (19) we had to project complicated Dirac structures, generically of the form \( \Gamma \otimes \tilde{\Gamma}, \) onto the simpler \( V - A \otimes V - A \) structure of the operators \( O_i, \) i.e. we used identities like

\[ \gamma^\mu \gamma^\nu \gamma^\lambda (1 - \gamma_5) \otimes \gamma^\mu \gamma^\nu \gamma^\lambda (1 - \gamma_5) = 4 (4 - \epsilon - \epsilon^2) \gamma^\mu (1 - \gamma_5) \otimes \gamma^\mu (1 - \gamma_5). \]  

(22)

A general method for obtaining such relations is explained e.g. in [18]. However, such a projection is a very peculiar procedure since it has to be done in \( D = 4 - 2\epsilon \) dimensions where, strictly speaking, no complete finite basis \{\( \gamma^{(i)} \otimes \tilde{\gamma}^{(i)} \)} of Dirac structures on which to project can be given. Eq. (22) should therefore be understood as the projection on a certain subset of such a basis, namely, the one that forms the complete basis in \( D = 4 \) dimensions. Only in this sense the coefficient \( 4 (4 - \epsilon - \epsilon^2) \) in Eq. (22) is a well defined quantity. Rigorously, the discrepancy between the l.h.s. and the r.h.s. of Eq. (22) defines an evanescent operator (vanishing in the limit \( \epsilon \rightarrow 0 \)) which has to be treated correctly in higher order calculations [18]. However, it is shown in the appendix that such operators are irrelevant for the purpose of this paper.

In order to renormalize the gluon field, the quite ingenious method described by Beneke and Braun in the appendix of [19] is used. One first needs the counterterm appropriate to cancel the divergence of one fermion bubble. From Eq. (15) it can be read off to be \( \frac{\alpha_s}{4\pi} N_f \frac{1}{\epsilon} \) in the \( MS \) scheme. Since there are \( n!/(k! (n-k)!) \) possibilities to replace \( k \) out of \( n \) fermion bubbles with this counterterm, and since all these possibilities have to be added, one can infer from Eq. (19) the following expression for the (partially) renormalized Green functions:

\[ \left( \frac{\langle O_1 \rangle}{\langle O_2 \rangle} \right) = \left\{ 1 - \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^n \frac{1}{\epsilon^{n+1}} \sum_{k=0}^n \left[ \binom{n}{k} \frac{(-1)^k}{k+1} F(\epsilon, (k+1)\epsilon) \right] M_c \right\} \left( \frac{\langle O_1 \rangle^{\text{tree}}}{\langle O_2 \rangle^{\text{tree}}} \right). \]  

(23)

We called this the partially renormalized Green functions, because operator and wave function renormalization are still to be performed.
It is gratifying that the finite sum in Eq. (23) can be solved. Expanding $F(\epsilon, z)$ in powers of $z$,

$$F(\epsilon, z) = \sum_{j=0}^{\infty} f_j(\epsilon) z^j,$$

we have

$$\sum_{k=0}^{n} \left[ \binom{n}{k} \frac{(-1)^k}{k+1} F(\epsilon, (k+1)\epsilon) \right] = \sum_{j=0}^{n+1} f_j(\epsilon) \epsilon^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1}(k+1)^{j-1}. \quad (25)$$

The sum over $j$ stops at $j = n + 1$, since the remaining terms do not contribute in the limit $\epsilon \to 0$ (note that Eq. (25) is still to be multiplied with an overall factor $1/(n+1)!$). Using appropriate formulas from [20, 21], the sum over $k$ in Eq. (25) can easily be performed. It is different from zero only in the cases $j = 0$ and $j = n + 1$,

$$\sum_{k=0}^{n} \left[ \binom{n}{k} \frac{(-1)^k}{k+1} F(\epsilon, (k+1)\epsilon) \right] = f_0(\epsilon) \frac{1}{n+1} + f_{n+1}(\epsilon) \epsilon^{n+1} (-1)^n n! \quad (26)$$

The term proportional to $\epsilon^{n+1}$ can be discarded since it yields only a finite contribution in the limit $\epsilon \to 0$ which is irrelevant for the anomalous dimensions. With

$$f_0(\epsilon) = \frac{3 - 2\epsilon}{3\Gamma(1+\epsilon)\Gamma(2-\epsilon, 2-\epsilon)\Gamma(3-\epsilon)} \quad (27)$$

from Eq. (20), one finally obtains

$$\left( \frac{\langle O_1 \rangle}{\langle O_2 \rangle} \right) = \left\{ 1 - \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^n \left[ \frac{1}{(n+1)\epsilon^{n+1}} + \frac{3 - 2\epsilon}{3\Gamma(1+\epsilon)\Gamma(2-\epsilon, 2-\epsilon)\Gamma(3-\epsilon)} + \text{finite} \right] M_c \right\} \left( \frac{\langle O_1 \rangle_{\text{tree}}}{\langle O_2 \rangle_{\text{tree}}} \right). \quad (28)$$

From this formula the renormalization constants $\tilde{Z}_{ij}$ can immediately be read off.

A completely analogous calculation can be done for the fermion self energy with an insertion of $n$ fermion bubbles, see Fig. 2. This calculation yields

$$i\Sigma^{(n)}(p) = i\phi \left\{ 1 + \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^n \frac{1}{(n+1)\epsilon^{n+1}} \frac{\epsilon(3 - 2\epsilon)\Gamma(4 - 2\epsilon)}{6\Gamma(1+\epsilon)\Gamma(2-\epsilon)^2\Gamma(3-\epsilon)} + \text{finite} \right\} C_F \quad (29)$$

from which the wave function renormalization constant $Z_\psi$ can be read off. Eq. (28) is again gauge independent for $n \geq 1$ and specific to Landau gauge in the case $n = 0$.

### 4 Extraction of anomalous dimensions

Summarizing the results of the previous section, renormalization constants in the large $N_f$ limit may in general be written as

$$Z = 1 - \sum_{k=0}^{n} \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^k \frac{f_0(\epsilon)}{(k+1)\epsilon^{k+1}} \quad (30)$$
at any fixed order $n+1$ in perturbation theory. Here, $f_0(\epsilon)$ is some function of $\epsilon$ which may also contain color factors. However, we prefer to have at our disposal an all-order result for the corresponding anomalous dimension. This can be achieved by expanding $f_0(\epsilon)$ in powers of $\epsilon$,

$$f_0(\epsilon) = \sum_{j=0}^{\infty} \tilde{f}_j \epsilon^j.$$  

(31)

Inserting this into Eq. (30), we obtain for the $1/\epsilon$ part of the renormalization constant, in the notation of Eq. (11),

$$Z^{(1)} = -\sum_{k=0}^{n} \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^k \tilde{f}_k \frac{1}{k+1},$$

(32)

which can formally be summed up to all orders by setting $n$ to infinity,

$$Z^{(1)}_{all\text{-}orders} = -\sum_{k=0}^{\infty} \frac{\alpha_s}{4\pi} \left( \frac{\alpha_s N_f}{6\pi} \right)^k \tilde{f}_k \frac{1}{k+1}.$$  

(33)

From this the corresponding anomalous dimension is obtained according to Eq. (16) as

$$\gamma_{all\text{-}orders} = -2\alpha_s \frac{dZ^{(1)}_{all\text{-}orders}}{d\alpha_s}$$

$$= \frac{\alpha_s}{2\pi} \sum_{k=0}^{\infty} \left( \frac{\alpha_s N_f}{6\pi} \right)^k \tilde{f}_k$$

$$= \frac{\alpha_s}{2\pi} f_0 \left( \frac{\alpha_s N_f}{6\pi} \right),$$

(34)

where in the last step Eq. (31) was employed.

Thus we get from Eq. (29) for the anomalous dimension of the quark field

$$\gamma_{\psi} = \frac{\alpha_s}{2\pi} C_F \psi \left( \frac{\alpha_s N_f}{6\pi} \right),$$

(35)

with

$$f_\psi(x) = \frac{1}{6} x (3 - 2x) \frac{\Gamma(4 - 2x)}{\Gamma(1 + x) \Gamma(2 - x)^2 \Gamma(3 - x)},$$

(36)

in agreement with a result that Gracey [22] derived some years ago using a completely different method. Having found this, we finally obtain from Eqs. (14) and (28) the following formula for the anomalous dimensions of the operators $O_1$ and $O_2$ in the large $N_f$ limit:

$$\gamma_{ij} = \tilde{\gamma}_{ij} + 2\gamma_{\psi} = \frac{\alpha_s}{2\pi} f_{\psi} \left( \frac{\alpha_s N_f}{6\pi} \right) \begin{pmatrix} -1/N & 1 \\ 1 & -1/N \end{pmatrix},$$

(37)

with

$$f_{\psi}(x) = \frac{1}{3} (1 + x)(3 - 2x) \frac{\Gamma(4 - 2x)}{\Gamma(1 + x) \Gamma(2 - x)^2 \Gamma(3 - x)},$$

(38)
This can alternatively be written in the diagonal operator basis (7), where one has
\[ \gamma_{\pm} = \pm \frac{\alpha_s}{2\pi} f_\gamma \left( \frac{\alpha_s N_f}{6\pi} \right) \frac{N \mp 1}{N}. \] (39)

Reexpanding in powers of \( \alpha_s \), our result (39) agrees with the \( \alpha_s^2 N_f \) part of the exact NLO result as given in Eq. (5.2) of [15]. One should mention that although our findings were derived in the \( \overline{\text{MS}} \) scheme, they are of course also valid in the \( \text{MS} \) scheme (with the replacement \( \alpha_s^{\overline{\text{MS}}} \to \alpha_s^{\text{MS}} \) understood). This is because the \( \text{MS} \) scheme is related to the latter simply by a multiplicative scale redefinition \( \mu \to (e^{\gamma_E}/4\pi)^{1/2}\mu \), so that the functional dependencies of renormalization group functions on \( \alpha_s \) do not change. On the other hand, it should be emphasized that the result for the anomalous dimensions \( \gamma_{\pm} \) in general as well as in the large \( N_f \) limit depends on the regularization scheme used for \( \gamma_5 \) in \( D \) dimensions: all our results were obtained in the “Naive Dimensional Regularization” (NDR) scheme with a naively anticommuting \( \gamma_5 \).

5 Discussion of results

As explained in the introduction, the large \( N_f \) limit of QCD unfortunately corresponds by no means to the physical world. Nevertheless it is widely conjectured that one can still make some contact to reality, if one uses the results of the large \( N_f \) limit as a starting point for the so-called NNA approximation, i.e. if one replaces \( N_f \) by \( N_f - 33/2 \) in these results. This procedure, motivated by the analogy to QED where \( N_f \) is essentially the first coefficient of the QED beta function, may also be called the large \( \beta_0 \) approximation. With \( \beta_0 = 2N_f/3 - 11 \), it amounts to the replacement
\[ \frac{\alpha_s N_f}{6\pi} \to \frac{\alpha_s \beta_0}{4\pi} \] (40)
in Eq. (8). To the knowledge of the author, the only justification for this procedure is a certain phenomenological success; no further comments on its validity will therefore be found in this paper. Instead, the substitution (40) will be used as an admittedly crude and naive prescription for estimating higher-order effects.

Before doing so it is interesting to investigate how well the NNA approximation works already at NLO, where a comparison with the exact result is possible. Expanding
\[ \gamma_{\pm} = \gamma_{\pm}^{(0)} \frac{\alpha_s}{4\pi} + \gamma_{\pm}^{(1)} \left( \frac{\alpha_s}{4\pi} \right)^2 + \ldots, \] (41)
the exact result [15] in the NDR scheme reads
\[ \gamma_+^{(1)} = \frac{2}{3} \beta_0 + \frac{1}{3}, \quad \gamma_-^{(1)} = -\frac{4}{3} \beta_0 - \frac{86}{3}. \] (42)
In the NNA approach, the constant terms are neglected. With \( N_f = 5 \) or \( \beta_0 = -23/3 \), this is obviously a very good (accurate to 7%) approximation for \( \gamma_+^{(1)} \), but also a very bad (wrong sign) approximation for \( \gamma_-^{(1)} \). So one cannot claim from this comparison that the NNA approximation generally works well at low orders.\(^{2}\)

\(^{2}\)See, however, the remark at the end of this section.
Despite this somewhat discouraging observation, let us now ask for the contributions of all orders beyond NLO in the NNA approach. Evaluating the anomalous dimensions at a scale $\mu \simeq 5\text{ GeV}$ or $\alpha_s(\mu) \simeq 0.21$, we have
\begin{align*}
\gamma_+ &= +0.0670 \text{ (LO)} - 0.0014 \text{ (exact NLO)} - 0.0021, \\
\gamma_- &= -0.1350 \text{ (LO)} - 0.0053 \text{ (exact NLO)} + 0.0043.
\end{align*}
Here, the first two numbers are exact result and the third number indicates the summed contribution from all orders beyond NLO in the NNA approximation (i.e. the terms $\alpha_s^3\beta_0^2$, $\alpha_s^4\beta_0^3$, ...), obtained from Eq. (39). We notice that this contribution is of the same order of magnitude as the exact NLO contribution itself.

One can go one step further and examine to what extent the Wilson coefficients $C_i(\mu)$ of the operators $O_i$ are affected by these additional fermion-bubble contributions to the anomalous dimensions. We will investigate here two cases that are of some phenomenological interest: nonleptonic $B$ decays and $B^0 - \bar{B}^0$ mixing.

(i) Nonleptonic $B$ decays

Only decays of the $b$ quark into three different flavours, e.g. $b \rightarrow c\bar{u}d$, are considered. For these decay modes, no penguin operators can occur and the complete effective Hamiltonian is given by
\begin{equation}
H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{CKM} [C_1(\mu)O_1 + C_2(\mu)O_2],
\end{equation}
with the four-quark operators $O_i$ as defined in Eq. (6). The renormalization group analysis is most easily be done in the diagonal basis $O_\pm$ with the corresponding Wilson coefficients $z_\pm = C_2 \pm C_1$. These Wilson coefficients obey the renormalization group equation
\begin{equation}
\mu \frac{d}{d\mu} z_\pm(\mu) = \gamma_\pm z_\pm(\mu)
\end{equation}
which can be solved numerically using the anomalous dimension (39) and the initial condition
\begin{equation}
z_+(M_W) = 1 + \frac{11}{3} \frac{\alpha_s(M_W)}{4\pi}, \quad z_-(M_W) = 1 - \frac{22}{3} \frac{\alpha_s(M_W)}{4\pi}.
\end{equation}
In this way we obtain (after transforming back to the basis $O_{1,2}$ which is more commonly used) at $\mu = m_b \simeq 5\text{ GeV}:
\begin{align*}
C_1(m_b) &= +1.0952 \text{ (LO)} - 0.0250 \text{ (exact NLO)} - 0.0017, \\
C_2(m_b) &= -0.2250 \text{ (LO)} + 0.0575 \text{ (exact NLO)} + 0.0043.
\end{align*}
These equations should be read in analogy to Eq. (42): the first two numbers are the exact LO and NLO results, the last number indicates the summed contribution beyond NLO from the anomalous dimensions in the NNA approximation\textsuperscript{3}). We conclude that this contribution

\textsuperscript{3)In the NNA approximation (essentially corresponding to the large $\beta_0$ limit of QCD) one should for consistency always use the LO running coupling $\alpha_s(\mu)$, neglecting higher order corrections to the QCD beta function. Both of the last numbers in Eq. (47) were nevertheless obtained by solving Eq. (46) with the NLO running coupling (and the anomalous dimensions exact up to NLO), since we wished to estimate the sum of all contributions beyond NLO in the NNA approximation.
is completely negligible as compared to the NLO correction. The reason for this is that the
NLO contribution itself is dominated by the large matching correction \[10\] so that the small
modifications due to \[13\] in the running become unimportant.

(ii) $B^0 - \bar{B}^0$ mixing

The effective Hamiltonian for $B^0 - \bar{B}^0$ - mixing is \[23\]

$$
H_{eff} = \frac{G_F^2}{16\pi^2} M_W^2 |V_{td}^* V_{tb}|^2 z_{B\bar{B}}^+(\mu) \mathcal{O}_+.
$$

(48)

It contains only the operator $\mathcal{O}_+$ for which the NNA approximation worked very well even
at NLO. This leads to the conjecture that NNA estimates of higher orders might be to some
extent reliable here. The initial condition for the Wilson coefficient $z_{B\bar{B}}^+(\mu)$ is a complicated
function of the top mass which can be found in the appendix of \[23\]. Evaluating this function
at $m_t(M_W) \simeq 177$ GeV, we obtain

$$
z_{B\bar{B}}^+(M_W) = 2.624 + 2.798 \frac{\alpha_s(M_W)}{4\pi}.
$$

(49)

The evolution down to the low scale with Eq. (45) then yields

$$
z_{B\bar{B}}^+(m_b) = 2.284 \text{ (LO)} + 0.028 \text{ (exact NLO)} + 0.006.
$$

(50)

Again, the summed contributions beyond NLO are minute and can safely be neglected.

From Eqs. (47) and (50) we conclude that the higher-order fermion-bubble contributions
to the anomalous dimensions $\gamma_\pm$ are irrelevant in phenomenological applications. The reason
for this is not the generic smallness of these additional contributions, cf. Eq. (13), but the
smallness of the running in the current-current sector in general. Our analysis indicates that
the termination of the perturbative series at NLO is in this case a very good approximation,
with an associated theoretical error of almost zero.

However, a word of caution is in order here. We examined in this paper only the large
$N_f$ contribution to the anomalous dimension. The result Eq. (39) was a perfectly convergent
series, essentially a product of gamma functions with $\alpha_s N_f/6\pi$ as argument, without any
renormalon ambiguity. This is in fact the general structure of renormalization group functions
in the large $N_f$ limit \[13, 19, 22\]. Conversely, this implies that there is no $n!$ growth of the perturbative coefficients and so there is no longer a good reason to believe that at sufficiently
high orders the large $N_f$ contributions become dominant. In the best case we can consider
them as a guess of what may happen at higher orders.

The $n!$ growth is indeed not present (at least in dimensional regularization) in the divergent
parts of Feynman diagrams but only in the finite parts - precisely the pieces we neglected in
this calculation. This somewhat surprising statement can be verified directly from Eq. (26).
The finite parts manifest themselves in corrections to the matching of the effective onto the full
theory, i.e. they contribute to the initial conditions $C_i(M_W)$ of the Wilson coefficients at the
electroweak scale. So this initial condition is the place where one really expects renormalon
singularities in the Borel plane. Hence, a complete large $N_f$ analysis should certainly include
these matching corrections – a task which remains to be done in the future. If the general
perturbative expansion of a Wilson coefficient is written as

\[ C(\mu) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{nm} \alpha_s^n \left( \ln \frac{\mu}{M_W} \right)^m, \]  

(51)

with expansion coefficients that are by themselves power series in \( N_f \),

\[ C_{nm} = \sum_{k=0}^{n-1} C_{nm}^{(k)} N_f^k, \]

(52)

then the calculation of the anomalous dimension provides us, in principle, with all the coefficients \( C_{nm}^{n-1} \) with \( m > 0 \), i.e. with all the logarithmic pieces. The matching calculation on
the other hand gives us the coefficients \( C_{n0}^{n-1} \) which will grow as \( (n - 1)! \) and will therefore
presumably dominate the whole perturbative expansion at sufficiently high orders. Strictly
speaking, this means that the whole renormalization group evolution from \( \mu \approx M_W \) down to
\( \mu \approx m_b \), i.e. the summation of large logarithms, becomes meaningless in the large \( N_f \) limit,
since the perturbative series in this limit is (again: at sufficiently high orders) dominated by
the constant and not by the logarithmic pieces, even if the latter are large. More work needs
to be done in order to clarify these questions completely.

6 Summary

In this paper the anomalous dimensions of the current-current four-quark operators \( O_1 \) and
\( O_2 \) were calculated in the large \( N_f \) limit in the \( \overline{MS} \) (or \( MS \)) scheme. The calculation
was done via direct computation of Feynman diagrams in dimensional regularization, with
a special emphasis on the role of evanescent operators. Our findings agree with the \( \alpha_s^2 N_f \)
part of the exact NLO result and may furthermore serve as an independent check for future
NNLO calculations. Using the NNA approximation, we estimated with our result the total
contribution of all orders beyond NLO to the anomalous dimension. This contribution was
found to be of the same order of magnitude as the exact NLO correction itself. We also
investigated to which extent this contribution affects physical quantities. As an example, we
considered the Wilson coefficients relevant for nonleptonic \( B \) decays and \( B^0 - \bar{B}^0 \) mixing.
In these two cases, the effects were negligibly small and we concluded that the truncation of
the perturbative series at NLO is a very good approximation in the current-current sector.
Finally, future steps were discussed that will be necessary for a complete large \( N_f \) analysis of
effective weak Hamiltonians.

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Appendix: Evanescent operators

As we observed in Sec. 3, evanescent structures as e.g.
\[ E_1 = \gamma^\mu \gamma^\nu \gamma^\lambda (1 - \gamma_5) \otimes \gamma_\mu \gamma_\nu \gamma_\lambda (1 - \gamma_5) - 4 (4 - \epsilon - \epsilon^2) \gamma^\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5) \] (53)
unavoidably arise while calculating matrix elements of weak four-quark operators\footnote{It is essential for the following argument to define the evanescent operators in such a way that \( E_i \perp \mathcal{O} \) in \( D = 4 - 2\epsilon \) dimensions so that the coefficients \( c_i \) in Eq. (55) start at \( O(\alpha_s) \) and not at \( O(\alpha_s^0) \). This condition fixes the coefficient \( 4 (4 - \epsilon - \epsilon^2) \) in Eq. (53) uniquely. In general, however, the \( O(\epsilon) \) part of this coefficient can be chosen arbitrarily (implying different definitions of the evanescent operators) without affecting the physical results. For a detailed discussion on the subject of evanescent operators we refer to the work of Herrlich and Nierste [18].} For a consistent calculation these operators must be included from the very beginning, i.e. one should start such a calculation from a “true” effective Hamiltonian
\[ \mathcal{H}_{\text{eff}} = C\mathcal{O} + \sum_{j=1}^{\infty} C_{E_j} E_j, \] (54)
where the second term constitutes an infinite tower of evanescent operators vanishing in the limit \( \epsilon \to 0 \). To avoid an embarrassing amount of indices, here and in the following two simplifying assumptions are made: (i) we are dealing with only one physical operator \( \mathcal{O} \), (ii) only one new evanescent operator arises at each loop, i.e. the \( n \) loop matrix elements can be written as
\[ \langle \mathcal{O} \rangle^{(n)} = a \langle \mathcal{O} \rangle^{(0)} + \sum_{j=1}^{n} b_j \langle E_j \rangle^{(0)}, \]
\[ \langle E_i \rangle^{(n)} = c_i \langle \mathcal{O} \rangle^{(0)} + \sum_{j=1}^{\infty} d_{ij} \langle E_j \rangle^{(0)}, \] (55)
where the sum in the first equation stops at \( n \) and not at infinity, and with some coefficients \( a, b_i, c_i, d_{ij} \) which are computed by explicit calculation. Of course both assumptions can easily be relaxed.

Now the crucial point is that we wish to avoid finite contributions from evanescent operators to the physical matrix element
\[ \langle \mathcal{H}_{\text{eff}}^R \rangle = C\mathcal{O} \langle \mathcal{O}^R \rangle + \sum_{j=1}^{\infty} C_{E_j} \langle E_j^R \rangle. \] (56)
In principle, this can be achieved by choosing the multiplicative operator renormalization
\[ \left( \begin{array}{c} \mathcal{O}^R \\ E_i^R \end{array} \right) = \left( \begin{array}{cc} \tilde{Z}_{\mathcal{O}\mathcal{O}} & \tilde{Z}_{\mathcal{O}E_j} \\ \tilde{Z}_{E_i\mathcal{O}} & \tilde{Z}_{E_iE_j} \end{array} \right) \left( \begin{array}{c} \mathcal{O} \\ E_j \end{array} \right) \] (57)
in such a way that the \( \langle E_i^R \rangle \) vanish. Plugging Eq. (57) into Eq. (56) and using the matrix elements (55), this condition (together with the finiteness of \( \langle \mathcal{H}_{\text{eff}}^R \rangle \)) enables us to determine the renormalization constant \( \tilde{Z} \). Note that \( \tilde{Z} \) is the inverse of the usual operator renormalization constant \( Z \).
However, this procedure involves of course a finite renormalization in the $\tilde{Z}_{\xi_i O}$ in order to remove the finite contribution $c_i\langle O \rangle^{(0)}$ of the evanescents to the matrix element (55). The general structure of the renormalization constant $\tilde{Z}$ at $n$ loops is therefore

$$\tilde{Z} = \begin{pmatrix} s & s & \ldots & s & 0 & 0 & \ldots \\ f & f & & & & & \\ \vdots & & & & & & \\ f & & & & & & s \\ \end{pmatrix},$$

(58)

where $s$ means “only singular parts” and $f$ “also finite parts”. Note that finite $O(\epsilon^0)$ contributions to the $b_i$ and $d_{ij}$ in Eq. (55) nevertheless yield no finite contribution to the matrix element due to the vanishing of the evanescents in four dimensions. This is the reason why the $\tilde{Z}_{\xi_i\xi_j}$ and $\tilde{Z}_{\xi_i E_j}$ in (58) contain only singular pieces. Furthermore, $\tilde{Z}_{\xi_i E_j} = 0$ for $j > n$ since only the first $n$ evanescent operators contribute to the $n$ loop matrix element $\langle O \rangle^{(n)}$.

However, there is a price to pay for the finite renormalization in (58). Namely, the finite parts of the $\tilde{Z}_{\xi_i O}$ will in general result in an additional finite contribution to the anomalous dimension $\gamma_{OO}$ of the physical operator $O$. Hence, the running and thereby the numerical value of the Wilson coefficient $C_O$ in Eq. (56) is modified in a calculable way although the evanescent matrix elements themselves do not contribute. This subtle effect was first observed in this context in [15].

The relation between the renormalization constant $Z = \tilde{Z}^{-1}$ and the corresponding anomalous dimension

$$\gamma = \tilde{Z}_\mu \frac{dZ}{d\mu}$$

(59)

can easily be calculated at $n$ loops. From Eq. (58) it follows for the general structure of $Z$

$$Z = \begin{pmatrix} s & s & \ldots \\ f & f & \\ \vdots & & s \\ \end{pmatrix}.$$  

(60)

Using (58), (60) and the finiteness of $\gamma$ one obtains in the usual way

$$\gamma_{OO} = -\frac{\partial Z^{(1)}_{\xi_i O}}{\partial \xi} - \frac{\partial Z^{(1)}_{\xi_i E_j}}{\partial \xi} g \frac{\partial Z^{(0)}_{\xi_j O}}{\partial \xi},$$

(61)

where we have expanded $Z$ and $\tilde{Z}$ in powers of $1/\epsilon$,

$$Z = Z^{(0)} + \frac{1}{\epsilon} Z^{(1)} + O(\epsilon^{-2}),$$

$$\tilde{Z} = \tilde{Z}^{(0)} + \frac{1}{\epsilon} \tilde{Z}^{(1)} + O(\epsilon^{-2}).$$

(62)

The second term in Eq. (61) would vanish without the finite contribution due to the evanescents, so Eq. (61) indeed makes sense as generalization of the usual formula (10).
Now one can immediately conclude from Eq. (61) that there is no contribution to $\gamma_{OO}$ from the evanescents in the large $N_f$ limit. This is for the following reasons: $N_f$ enters the matrix elements (55) and therefore the renormalization constants $Z, \tilde{Z}$ only at $O(\alpha_s^2)$, i.e.

$$
Z_{E_k} = \alpha_s A_k + \alpha_s^2 N_f B_k + \ldots,
$$

$$
\tilde{Z}_{E_k} = \alpha_s C_k + \alpha_s^2 N_f D_k + \ldots
$$

(63)

Multiplying these we see that the $N_f$ terms in the evanescent contributions to the r.h.s. of Eq. (61) start with $\alpha^3_s N_f$, so they are suppressed compared to the $\alpha^3_s N_f^2$ terms contained in $Z_{OO}^{(1)}$. In the limit $N_f \to \infty$ they can therefore be neglected. Note that this statement is only true if there are no $O(\alpha_s^0)$ terms present in Eq. (63). In order to ensure this we defined the evanescent operators as described in the footnote above.

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