Current conservation, screening and the magnetic moment of the Δ resonance. \(^1\)

1. Formulation without quark degrees of freedom

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Abstract

The pion-nucleon bremsstrahlung \(\pi + N \rightarrow \gamma' + \pi' + N'\) is studied in a new form of current conservation. According to this condition, the internal and external particle radiation parts of the \(\pi N\) radiation amplitude have opposite signs, i.e., they contain terms which must cancel each other. Therefore, one has a screening of the internal and external particle radiation in the \(\pi N\) bremsstrahlung. In particular, it is shown that the double \(\Delta\) exchange diagram with the \(\Delta - \gamma'\Delta'\) vertex cancel against the appropriate longitudinal part of the external particle radiation diagrams. Consequently, a model independent relation between the magnetic dipole moments of the \(\Delta^+\) and \(\Delta^{++}\) resonances and the anomalous magnetic moment of the proton \(\mu_p\) is obtained, where \(\mu_\Delta\) is expressed by \(\mu_p\) as \(\mu_\Delta^+ = \frac{M_\Delta}{m_p^2} \mu_p\) and \(\mu_{\Delta^{++}} = \frac{3}{2} \mu_{\Delta^+}\) in agreement with the values extracted from the fit for the experimental cross section of the \(\pi^+p \rightarrow \gamma'\pi^+p\) reaction.

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1. INTRODUCTION

The $\pi N$ bremsstrahlung was extensively investigated in the past in order to study the electromagnetic properties of the $\Delta$ resonances and their form factors. The main reason for the determination of the electromagnetic moments of the $\Delta$ resonances is that on one hand, the $\Delta$'s are described as a $\pi N$ resonances with the corresponding poles of the $\pi N$ amplitude and, on the other hand $\Delta$'s are often treated as independent particles in the models of strong interaction. In addition, the quark content of the proton is the same as for the $\Delta^+$, and differs from the quark content of the $\Delta^{++}$. Therefore, determination of the electromagnetic moments of the $\Delta$ resonances is important for the definition of the electromagnetic structure of the nucleons and the $\Delta$ resonances.

In contrast to nucleons, the direct experimental measurement of the electromagnetic moments of the $\Delta$'s is today impossible. Therefore, the present experimental electromagnetic moments of the $\Delta$'s are obtained using a fit to the experimental cross sections of the $\pi N$ bremsstrahlung \cite{1}-\cite{4}. The analysis of these data by different theoretical models yields different magnetic moments of the $\Delta$'s. For instance, the magnetic dipole moment of the $\Delta^{++} \mu_{\Delta^{++}}$ obtained within the framework of the low energy photon theorem \cite{5}-\cite{12} is

$$\mu_{\Delta^{++}} = 4.7 - 6.9 \mu_B \quad \text{[2]}$$

in nuclear magnetons $\mu_B = e/2m_N$, while the potential models yield $\mu_{\Delta^{++}} = 5.6 - 7.5 \mu_B$ \cite{3} or $\mu_{\Delta^{++}} = 4.5 \pm 0.95 \mu_B$ \cite{4}. The theoretical results for different models fitted to the experimental data \cite{5}-\cite{26} are shown in Table 1 in the conclusions of this paper. These results indicate substantial discrepancies between the different predicted values for $\mu_{\Delta^{++}}$ and $\mu_{\Delta^+}$.

In this paper an analytic and model-independent relation for the magnetic moments of the $\Delta$ resonances is suggested. This relation is based on a new form of current conservation for the total on mass shell and on energy shell amplitude of the $\pi N$ bremsstrahlung $A_{\pi'N'\rightarrow\pi N}^{\mu}$. The corresponding current conservation

$$k'_{\mu}A_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} = k'_{\mu}E_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} + B_{\pi'N'\rightarrow\pi N} = 0 \quad (1.1a)$$

consists of the external particle radiation amplitude $E_{\gamma'\pi'N'\rightarrow\pi N}^{\mu}$ depicted in Fig. 1 and the sum of the off shell $\pi N \rightarrow \pi'N'$ scattering amplitudes $B_{\pi'N'\rightarrow\pi N}$. This condition is obtained in the same approach as the Ward-Takahashi identities in the usual quantum field theory \cite{28, 29}. Using current conservation for the total $\pi N$ radiation amplitude

$$k'_{\mu}A_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} = k'_{\mu}E_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} + k'_{\mu}T_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} = 0 \quad (1.2)$$

for the on shell external and internal particle radiation amplitudes $E_{\gamma'\pi'N'\rightarrow\pi N}^{\mu}$ (Fig.1) and $T_{\gamma'\pi'N'\rightarrow\pi N}^{\mu}$ (Fig.2A) one can represent (1.1a) as

$$k'_{\mu}E_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} = -k'_{\mu}T_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} = -B_{\pi'N'\rightarrow\pi N} \quad (1.1b)$$

which determines an additional relation between the on shell external and internal particle radiation amplitudes.
Thus the problem of the validity of current conservation (1.1a) is reduced to the determination of the internal particle radiation amplitudes (Fig. 2A) $T_{\gamma''{\pi'}N''-\pi N}^\mu$ which satisfy the condition (1.1b).

We shall show that the $\Delta$ radiation amplitude $T_{\gamma'\pi'\pi N'-\pi N}(\Delta - \gamma\Delta)$ in Fig. 2B and the corresponding part of the external particle radiation amplitude $E_{\gamma'\pi'\pi N'-\pi N}$ (Fig. 1) denoted as $(E_{\pi N}^{3/2})_{\gamma'\pi'\pi N'-\pi N}(\Delta - \gamma\Delta)$ satisfy current conservation

$$k'_\mu\left[A^{\mu}_{\gamma'\pi'N'-\pi N}\right]_{2\Delta\text{ exchange with }\Delta-\gamma\Delta'\text{ vertex}} =$$

$$k'_\mu(E_{\pi N}^{3/2})_{\gamma'\pi'N'-\pi N}(\Delta - \gamma\Delta) + B_{\pi'N'-\pi N}^{3/2}(\Delta - \gamma\Delta) \quad (1.3a)$$

or

$$k'_\mu(E_{\pi N}^{3/2})_{\gamma'\pi'N'-\pi N}(\Delta - \gamma\Delta) = -k'_\mu T_{\gamma'\pi'N'-\pi N}(\Delta - \gamma\Delta) = -B_{\pi'N'-\pi N}^{3/2}(\Delta - \gamma\Delta), \quad (1.3b)$$

where the lower index $\pi$ and the upper index $3/2$ denote the longitudinal and the spin-isospin $(3/2, 3/2)$ part of the corresponding amplitudes. $(\Delta - \gamma\Delta)$ indicates a $\Delta$ radiation vertex with on mass shell $\Delta$'s in $(E_{\pi N}^{3/2})_{\gamma'\pi'N'-\pi N}$ and in $B_{\pi'N'-\pi N}^{3/2}$.

The intermediate $\Delta$'s in (1.3b) and in the $\Delta-\gamma\Delta$ vertex are on mass shell, i.e. the four momentum of the $\Delta$ $P_\Delta$ is determined as $P_\Delta^2 = m_\Delta^2 + P_\Delta^2$, where $m_\Delta$ denotes the effective complex mass of the $\Delta$ which is determined by the $\Delta$ pole position of the $\pi N$ amplitude. In the present approach the intermediate $\Delta$ radiation amplitude $T_{\gamma'\pi'N'-\pi N}(\Delta - \gamma\Delta)$ (Fig. 2B) is constructed unambiguously using only the on mass shell $\Delta$-pole part of the $\pi N$ amplitude. The corresponding 3D time-ordered field theoretical construction of the $\Delta$ radiation amplitude was presented by [30, 31, 32]. This approach is generalized in appendix C for any $s$ depending mass $m_\Delta(s)$. In particular, the formulation considered

![Figure 1: Diagrams describing the external particle radiation amplitude $E_{\gamma'\pi'N'-\pi N}$ in (1.1a,b). Diagrams A and C correspond to the radiation of the external nucleons. Diagrams B and D describe the emission of the photon by the external pions. The hatched circle indicates the off shell $\pi N$ elastic scattering amplitudes (2.9a,b,c,d). $N''$ and $\pi''$ denote the intermediate nucleon and pion states.](image-url)
does not use the effective Lagrangian with the Heisenberg operators of the ∆. Therefore, the off mass shell ∆ ambiguities does not appear. For the 3D time-ordered representations of the diagrams in Fig. 1 and Fig. 2B with the on mass shell intermediate pions, nucleons and ∆’s we shall use the following analytic decompositions of the amplitudes $E_{\mu}^{\gamma'\pi'N'\pi N}$ and $B_{\pi'N'\pi N}$ in (1.1a,b) in order to separate current conservation (1.3a,b):

I. Decomposition over the nucleon and antinucleon exchange parts.

II. Separation of the longitudinal and transverse parts of $E_{\mu}^{\gamma'\pi'N'\pi N}$ in (1.1a).

III. Partial wave decomposition of the off shell $\pi N$ amplitudes in $E_{\mu}^{\gamma'\pi'N'\pi N}$ and in $B_{\pi'N'\pi N}$. This procedure is necessary for separation of the $\Delta$ resonance $(3/2, 3/2)$ spin-isospin states in (1.1a,b). It also include projections on the intermediate spin $3/2$ states in the $\gamma N - N$ and $\gamma \pi - \pi$ vertices.

IV. Separation of the current conservation conditions with and without $\Delta$-pole terms in the off mass shell $\pi N$ amplitudes.

V. Reproduction of the double $\Delta$ exchange amplitudes using a sum of the $\Delta$-pole terms in $E_{\mu}^{\gamma'\pi'N'\pi N}$ and in $B_{\pi'N'\pi N}$. In the final $E_{\mu}^{\gamma'\pi'N'\pi N}(\Delta - \gamma \Delta)$ the $\Delta$ radiation vertex has the same form as the usual $\Delta - \gamma \Delta$ vertex.

Figure 2: Diagram A presents a symbolic description of the internal particle radiation amplitude $I_{\mu}^{\gamma'\pi'N'\pi N}$ in (1.1a,b). A special part of the amplitude $I_{\mu}^{\gamma'\pi'N'\pi N}$ in diagram A is given by diagram B which describes the double $\Delta$ exchange amplitude $I_{\mu}^{\gamma'\pi'N'\pi N}(\Delta - \gamma \Delta)$ with the photon emission from the intermediate $\Delta$. The $\Delta - \gamma \Delta$ vertex with on mass shell $\Delta$’s (3.7) in $I_{\mu}^{\gamma'\pi'N'\pi N}(\Delta - \gamma \Delta)$ contains the magnetic dipole moment $\mu_\Delta$ of the $\Delta$ (see appendix B). The unambiguous construction of $I_{\mu}^{\gamma'\pi'N'\pi N}(\Delta - \gamma \Delta)$ within the 3D time-ordered field theoretical approach is given in appendix C.

An important property of $I_{\mu}^{\gamma'\pi'N'\pi N}(\Delta - \gamma \Delta)$ is that it satisfies not the separate current conservation condition

$$k_{\mu}^{\gamma'\pi'N'\pi N}(\Delta - \gamma \Delta) \neq 0,$$

(1.4)
Because the $\Delta - \gamma' \Delta'$ vertex consists of the intermediate $\Delta$ four momenta $P_{\Delta}^\mu$, $P_{\Delta}'^\mu$ and $k_{\Delta}'^\mu = P_{\Delta}'^\mu - P_{\Delta}^\mu \neq k_{\Delta}^\mu$ (see appendix B).

Current conservation (1.3a,b) can be reinforced if one takes into account that only $(E_{\gamma' \pi'^{-}}^\mu \gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta))$ and $I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ have the same double $\Delta$ exchange poles and the same analytical structure of the $\Delta - \gamma \Delta$ vertex. Therefore,

$$(E_{\gamma' \pi'^{-}}^{3/2})_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)} = - I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta))}.$$

This relation allows to determine the magnetic dipole moment of the $\Delta$ resonances. Thus, from the equality of the vertex functions in $I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ and $E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ which contain $\mu_\Delta$ and $\mu_p$ correspondingly, it follows that $\mu_\Delta$ is analytically defined by $\mu_p$.

The important property of (1.5) is the equality and cancellation of the intermediate $\Delta$ radiation term $I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ (Fig. 2B) and the corresponding longitudinal part of the external particle radiation amplitudes $(E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$. This equality and cancellation are a part of the general screening of the internal particle terms via the sum of the external particle radiation amplitudes in Fig. 1 because other parts of $E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ and $I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ in (1.1b) are also equal and cancel each other.

Current conservation (1.1a,b) has the same form as in the approach based on the Low theorem (or low energy photon theorem) for the reactions with soft photons [8]-[14]. Unlike the present formulation, these approaches start from the external particle radiation amplitude $E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ which determines the full $\pi N$ bremsstrahlung amplitude in the infrared energy region of the emitted photon ($k' \to 0$). One can reproduce the double $\Delta$ exchange amplitude using the sum of the $\Delta$-pole $\pi N$ amplitudes in $E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ (Fig. 1) as it was noted in [8] and was applied in numerous other papers (see [12]) within the low energy photon approach [5]-[14]. This approach is based on a approximation $A_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)} \Rightarrow E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)} + I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ which allows to calculate the cross sections of the $\pi N$ bremsstrahlung for the extraction of the magnetic dipole moment of the $\Delta$. But in this approach the equality and cancellation of $I_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ and $E_{\gamma' \pi'^{-} \pi N (\Delta - \gamma \Delta)}$ according to (1.5) was not taken into account. Moreover, the recipe of construction of the bremsstrahlung amplitude in the low energy photon limit $k' \to 0$ is not unique due to the ambiguities of the low energy photon approximations [12].

Unlike the low energy photon approach [5]-[14], the present approach is not restricted to the infrared energy region of the emitted photon ($k' \to 0$), i.e., (1.1a,b) and (1.5) are exactly valid for any energy of the final photon. Moreover, in the present approach the electromagnetic form factors of the $\Delta$ s are determined through the $\Delta$-pole residues of the off shell $\pi N$ amplitudes. The suggested formulation can be applied to other reactions with conserved current like pion photo-production reaction, Compton scattering, processes with external vector $\rho$ or $\omega$ mesons etc.

This paper consists of four sections and three appendices. Current conservation

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2 The relationship between the external particle radiation diagrams in Fig. 1 and the double $\Delta$ exchange term in Fig. 2B were used in [12] based on the Brodsky-Brown identities [36, 37] for the diagrams in the tree approximation.
(1.1a,b) for the on shell bremsstrahlung amplitudes are derived in Section 2. In this section the equations (1.1a,b) are decomposed into independent current conservations with one nucleon and one antinucleon intermediate states. The following chain of the decompositions $\mathcal{E}_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} \implies \cdots \implies (\mathcal{E}_{\gamma'\pi'N'\rightarrow\pi N}^{3/2})^{\mu} \mathcal{B}_{\pi'N'\rightarrow\pi N}^{\nu}(\Delta - \gamma\Delta)$, $\mathcal{B}_{\pi'N'\rightarrow\pi N}^{\nu}(\Delta - \gamma\Delta)$ with the final form of current conservation (1.3a,b) is given in Section 3 and in Appendix A. The derivation of (1.5) with the extraction of the magnetic dipole moment of the $\Delta^+$ and $\Delta^{++}$ resonances is given in Sec. 3. The conclusions and the comparison with the magnetic dipole moments of other authors (Table 1) are presented in Sec. 4. In Appendix B construction of the $\Delta - \gamma'\Delta'$ vertices with the on mass shell $\Delta$'s is considered. Reproduction of the double $\Delta$ exchange diagram in Fig. 2B within the usual time-ordered field-theoretical approach for the $\pi N$ bremsstrahlung amplitude is given in Appendix C.

2. Ward-Takahashi identities for the pion-nucleon bremsstrahlung amplitude

We consider the radiative pion-nucleon scattering

$$\pi(p_\pi) + N(p_N) \implies \gamma'(k') + \pi'(p'_\pi) + N'(p'_N)$$

with the on mass shell momentum of the pions $(p_\pi = (\sqrt{p_\pi^2 + m_\pi^2}, p_\pi), p'_\pi = (\sqrt{p'_\pi^2 + m_\pi^2}, p'_\pi))$, nucleons $(p_N = (\sqrt{p_N^2 + m_N^2}, p_N), p'_N = (\sqrt{p'_N^2 + m_N^2}, p'_N))$, and final photon $(k'^2 = 0)$.

The energy-momentum of the emitted photon is $k'_\mu = (p_N + p_\pi - p'_\pi - p'_N)_\mu$.

Following the derivation of the Ward-Takahashi identities (see e.g. ch. 8.4.1 in the book of Itzykson and Zuber[29]) we start with the on shell amplitude $A_{\gamma'\pi'N'\rightarrow\pi N}^{\mu}$

$$A_{\gamma'\pi'N'\rightarrow\pi N}^{\mu} = \pi(p'_N)(\gamma_{\nu}p'_N - p_\pi)(p'_\pi)^2 - m_\pi^2)k'_\mu \tau'^{(\gamma_{\mu}p'_N - p_\pi)(p'_\pi)^2 - m_\pi^2})u(p_N),$$

(2.1)

where the Green function $\tau^\mu$ is expressed via the photon source operator $\mathcal{J}^\mu(z)$ and the pion and nucleon field operators $\Phi(x)$ and $\Psi(y)$ as

$$k'_\mu \tau^\mu = i \int d^4zd^4yd^4xd^4y \epsilon^{k'z + yp'x' + y'p'x - yp'x - y'p'x}.$$

In this paper we use the same definition and normalization for the Dirac spinors as in [29]. In particular, $u(p_N)$ denotes the spinor of the nucleon with the three-momentum $p_N$.

We shall use the well known relation for the time-ordered product of the quantum field operators

$$\frac{\partial}{\partial z^\mu} < 0|\mathcal{T}\left(\Psi(y')\Phi(x')\mathcal{J}^\mu(z)\Psi(y)\Phi^+(x)\right)|0 > \implies 0|\mathcal{T}\left(\Psi(y')\Phi(x')(\frac{\partial}{\partial z^\mu}\mathcal{J}^\mu(z))\Psi(y)\Phi^+(x)\right)|0 >$$
\[ +\delta(z_o - x'_o) < 0|\mathcal{T}\left(\Psi(y')\left[\mathcal{J}^o(z), \Phi(x')\right]\bar{\Psi}(y)\Phi^+(x)\right)|0 > \]

\[ +\delta(z_o - y'_o) < 0|\mathcal{T}\left(\Phi(x')\left[\mathcal{J}^o(z), \Psi(y')\right]\bar{\Psi}(y)\Phi^+(x)\right)|0 > \]

\[ +\delta(z_o - x_o) < 0|\mathcal{T}\left(\Psi(y')\Phi(x')\left[\mathcal{J}^o(z), \Phi^+(x)\right]\bar{\Psi}(y)\right)|0 > \]

\[ +\delta(z_o - y_o) < 0|\mathcal{T}\left((\Psi(y')\Phi(x')\left[\mathcal{J}^o(z), \bar{\Psi}(y)\right]\Phi^+(x)\right)|0 > \]  

(2.2b)

and the equal-time commutation conditions

\[ \left[\mathcal{J}^o(z), \Psi(y')\right]\delta(z_o - y'_o) = -e_N\delta^{(4)}(z - y')\Psi(y'); \quad \left[\mathcal{J}^o(z), \bar{\Psi}(y)\right]\delta(z_o - y_o) = e_N\delta^{(4)}(z - y)\bar{\Psi}(y) \]  

(2.3a)

\[ \left[\mathcal{J}^o(z), \Phi(x')\right]\delta(z_o - x'_o) = -e_\pi\delta^{(4)}(z - x')\Phi(x'); \quad \left[\mathcal{J}^o(z), \Phi^+(x)\right]\delta(z_o - x_o) = e_\pi\delta^{(4)}(z - x)\Phi^+(x), \]  

(2.3b)

where \(e_N', e_\pi', e_N\) and \(e_\pi\) stand for the charge of the nucleons and pions in the final and initial states. In particular, \(e_N = e, 0\) for the proton and the neutron, and \(e_\pi = \pm e, 0\) for pions.

After substitution of (2.3a,b) in (2.2a) and integration over \(d^4z\) we obtain

\[ k'_{\mu} = -i \int d^4y'd^4x'd^4y d^4x e^{ip'x' + ip\nu\nu - ip_N\nu}(e_{N'}e^{i\nu'y'} + e_{\pi'}e^{i\nu'x'} - e_Ne^{i\nu'y} - e_{\pi}e^{i\nu'x}) \]

\[ < 0|\mathcal{T}\left(\Psi(y')\Phi(x')\bar{\Psi}(y)\Phi^+(x)\right)|0 > . \]  

(2.2c)

Equal-time commutators (2.3a,b) are the result of the commutation relations between the electric charge operator \(Q = \int d^3x\mathcal{J}^o(x)\) and the particle field operators with the charge \(e\). These conditions express electric charge conservation for the local fields, i.e., they represent one of the first principles in quantum field theory.

Substituting (2.2c) into (2.1) we get

\[ k'_{\mu}A'^\mu_{\nu,\pi'} = -i(2\pi)^4 \delta^{(4)}(p_N' + p'_\pi + k' - p_\pi - p_N) \]

\[ \left[\pi(p'_N)(\gamma_{\nu}p'_N - m_N)\frac{e_{N'}}{\gamma_{\nu}(p'_N + k') - m_N} < out; p'_\pi|J(0)|p_Np_N; in > \right. \]

\[ + (p'^2_\pi - m^2_\pi)\frac{e_{\pi'}}{(p'_\pi + k')^2 - m^2_\pi} < out; p'_N|j_{\pi'}(0)|p_\pi p_N; in > \]

\[ - < out; p'_\pi p'_N|J(0)|p_\pi; in > \frac{e_N}{\gamma_{\nu}(p_N - k') - m_N}(\gamma_{\nu}p_N - m_N)u(p_N) \]
where $J(x) = (i\gamma_\mu \partial_\mu - m_N)\Psi(x)$ and $j_\pi(x) = (\partial^2/\partial x^\nu \partial x_\nu + m_\pi^2)\Phi(x)$ denote the source operator of the nucleon and the pion.

For the on mass shell external particles $k' \rightarrow A_{\gamma\pi'N'N}^\mu (2.4)$ vanishes. In particular, for $k' = 0$, $k' \rightarrow A_{\gamma\pi'N'N}^\mu (2.4)$ disappears due to cancellation of the on shell $\pi N$ amplitudes in (2.4). Thus expression (2.4) presents current conservation for the on shell bremsstrahlung amplitude

$$k' \mu \left[ A_{\gamma\pi'N'N}^\mu (p_\pi', p_N', k'; p_\pi, p_N) \right]_{\text{on mass shell } \pi', N', \pi, N} = 0. \quad (2.5)$$

It is convenient to extract the full energy-momentum conservation $\delta$ function from the radiative $\pi N$ scattering amplitude $A_{\gamma\pi'N'N}^\mu$ and introduce the corresponding amplitude $<\text{out}; p_\pi' p_N' | \mathcal{J}^\mu(0) | p_\pi p_N; \text{in}>$

$$k' \mu A_{\gamma\pi'N'N}^\mu = -i(2\pi)^4 \delta^4(p_N' + k' + p_\pi - p_N)k'_\mu <\text{out}; p_\pi' p_N' | \mathcal{J}^\mu(0) | p_\pi p_N; \text{in}>. \quad (2.6)$$

Afterwards, using the identity $a/(a+b) \equiv 1 - b/(a+b)$ in (2.4) (i.e. $(\gamma_\nu p'_{\pi'} - m_\pi)(\gamma_\mu (p_N' + k') - m_\pi) = 1 - \gamma_\mu k'_\mu/((p_\pi' - m_\pi)^2 - m_\pi^2)$ etc.) we obtain

$$k' \mu <\text{out}; p_\pi' p_N' | \mathcal{J}^\mu(0) | p_\pi p_N; \text{in}> = \mathcal{B}_{\pi'N'N} + k' \mu \mathcal{E}_{\gamma\pi'N'N}^\mu = 0, \quad (2.7)$$

where $p_N + p_\pi - p_N' - p_\pi' - k' = 0$ and

$$\mathcal{B}_{\pi'N'N} = e_N \tau(p_N') <\text{out}; p_\pi' | J(0) | p_\pi p_N; \text{in}> + e_\pi <\text{out}; p_\pi' | j_\pi(0) | p_\pi p_N; \text{in}> - e_N <\text{out}; p_\pi' p_N' | \mathcal{J}(0) | p_\pi; \text{in}> > u(p_N) - e_\pi <\text{out}; p_\pi' p_N' | j_\pi(0) | p_\pi; \text{in}>. \quad (2.8a)$$

$$\mathcal{E}_{\gamma\pi'N'N}^\mu = -\left[ \tau(p_N') \gamma_\mu (p_\pi' + k') + m_\pi \right] e_N <\text{out}; p_\pi' | J(0) | p_\pi p_N; \text{in}> + (2p_{\pi'} + k') \mu e_\pi <\text{out}; p_\pi' | j_\pi(0) | p_\pi p_N; \text{in}> - e_N <\text{out}; p_\pi' p_N' | \mathcal{J}(0) | p_\pi; \text{in}> > \gamma_\mu (p_N - k') + m_\pi <\text{out}; p_\pi' p_N' | j_\pi(0) | p_\pi; \text{in}> > \frac{e_\pi}{2p_{\pi'}k'} (2p_{\pi'} - k') \mu. \quad (2.8b)$$
The identity (2.7) is derived for the on shell total $\pi N$ bremsstrahlung amplitude (2.6). This identity consists of $\mathcal{E}_{\gamma'\pi'N'\rightarrow\pi N}$ (2.8b), which has the form of the external particle radiation diagrams in Fig. 1, and $\mathcal{B}_{\pi'N'\rightarrow\pi N}$ (2.6a), which consists of the sum of the different off mass shell $\pi N$ amplitudes. (2.7) is derived using the same technique as for the well-known Ward-Takahashi identity [29]. But the usual Ward-Takahashi identity connects the off mass shell $\pi N$ different off mass shell amplitudes. (2.7) is derived using the same technique as for the on shell total bremsstrahlung amplitude, i.e., in the low energy photon theorem [5]. Various applications of this method are given in [6, 8, 12, 13, 14]. The external particle radiation amplitudes in Fig. 1 are responsible for the infrared behavior of the total bremsstrahlung amplitude, i.e., in the low energy photon limit they represent the leading diagrams. The present derivation of current conservation (2.7) based on the general condition (2.2c), i.e., (2.7) is not restricted by the limit $k' = |k'\gamma| \rightarrow 0.$

The off shell $\pi N$ amplitudes in (2.8a,b) are functions of three on mass shell moments from which one can construct only three independent Lorentz-invariant (Mandelstam) variables. Therefore, we have

$$\mathbf{u}(\mathbf{p}'_N) <\text{out}; \mathbf{p}'_\pi | J(0) | \mathbf{p}_\pi \mathbf{p}_N; \text{in}> = \mathcal{T}_{\pi'} \left( (p'_\pi - p_\pi - p_N)^2; s, t_\pi \right) = \mathcal{T}_{\pi'} \left( m_\pi^2 + 2k'p'_N; s, t_\pi \right)$$

(2.9a)

$$<\text{out}; \mathbf{p}'_N | J_\pi(0) | \mathbf{p}_\pi \mathbf{p}_N; \text{in}> = \mathcal{T}_{\pi'} \left( (p'_N - p_\pi - p_N)^2; s, t_N \right) = \mathcal{T}_{\pi'} \left( m_\pi^2 + 2k'p'_\pi; s, t_N \right)$$

(2.9b)

$$<\text{out}; \mathbf{p}'_\pi \mathbf{p}'_N | \bar{J}(0) | \mathbf{p}_\pi; \text{in}> = \mathcal{T}_{\pi} \left( s', t_\pi; (p'_\pi + p'_N - p_\pi)^2 \right) = \mathcal{T}_{\pi} \left( s', t_\pi; m_{\pi}^2 - 2k'p_N \right)$$

(2.9c)

$$<\text{out}; \mathbf{p}'_\pi \mathbf{p}'_N | J_\pi(0) | \mathbf{p}_N; \text{in}> = \mathcal{T}_{\pi} \left( s', t_N; (p'_\pi + p'_N - p_\pi)^2 \right) = \mathcal{T}_{\pi} \left( s', t_N; m_{\pi}^2 - 2k'p_\pi \right).$$

(2.9d)

The four moments of the fourth off mass shell particle in the $\pi N$ amplitudes (2.9a,b,c,d) are determined via the energy-momentum conservation for the bremsstrahlung amplitude
the particle and antiparticle contributions in the intermediate state are separated. Using antinucleon part of (2.8a,b) we have corresponding part of \( B \) intermediate one nucleon state, we obtain

\[
\langle \text{out}; p'_N p'_\pi | J^\mu(0) | p_\pi p_N; \text{in} > = (2.6) \text{ with } p'_N + p'_\pi + k' = p_\pi + p_N, \text{ i.e., the off shell behavior of these } \pi N \text{ amplitudes is defined by } k'_\pi. \text{ The related invariant variables are}
\]

\[
s' = (p'_N + p'_\pi)^2, \quad s = (p_N + p_\pi)^2 = (p'_N + p'_\pi + k')^2 = s' + 2k'(p'_N + p'_\pi) = s' + 2k'(p_N + p_\pi)
\]

\[(2.10a)\]

\[
t_N = (p'_N - p_N)^2; \quad t_\pi = (p'_\pi - p_\pi)^2
\]

\[(2.10b)\]

with the following relations between them:

\[
t_\pi + (p'_\pi - p_N)^2 + s = m^2_\pi + 2m^2_N + (p'_\pi - p_\pi - p_N)^2,
\]

\[(2.11a)\]

\[
t_N + (p'_N - p_\pi)^2 + s = m^2_N + 2m^2_\pi + (p'_N - p_\pi - p_N)^2,
\]

\[(2.11b)\]

\[
t_\pi + (p_\pi - p'_N)^2 + s' = m^2_\pi + 2m^2_N + (p_\pi - p'_\pi - p'_N)^2,
\]

\[(2.11c)\]

\[
t_N + (p_\pi - p'_N)^2 + s' = m^2_N + 2m^2_\pi + (p_\pi - p'_\pi - p'_N)^2.
\]

\[(2.11d)\]

Next we rewrite expressions (2.8a,b) in the time-ordered three-dimensional form, where the particle and antiparticle contributions in the intermediate states are separated. Using the completeness conditions \( u(p'_N + k')\pi(p_N + k') + v(p'_N - k')\pi(p_N - k') = 1 \) for the intermediate one nucleon state, we obtain

\[
B_{\pi' N' - \pi N}(N) = e_N \pi(p'_N)u(p'_N + k')\pi(p'_N + k') < \text{out}; p'_\pi | J(0) | p_\pi p_N; \text{in} > + e_{\pi'} < \text{out}; p'_N | j_\pi'(0) | p_\pi p_N; \text{in} > - e_N < \text{out}; p'_\pi p'_N | J(0) | p_\pi p_N; \text{in} > u(p_N - k')\pi(p_N - k')u(p_N) - e_\pi < \text{out}; p'_\pi p'_N | j_\pi'(0) | p_\pi p_N; \text{in} >, \quad (2.12a)
\]

\[
\mathcal{E}^\mu_{\gamma' \pi' N' - \pi N}(N) = - \left[ (2p'_\pi + k')^\mu \frac{e_\pi'}{2p'_\pi k'} < \text{out}; p'_N | j_\pi'(0) | p_\pi p_N; \text{in} > \right. \\
\left. + \frac{\pi(p'_N)}{2p'_N k'}((2p'_N + k')^\mu - i\sigma^{uv}k'_u)u(p'_N + k')\pi(p'_N + k')e_{\pi'} < \text{out}; p'_\pi | J(0) | p_\pi p_N; \text{in} > \right.
\]

\[
- e_N < \text{out}; p'_\pi p'_N | J(0) | p_\pi p_N; \text{in} > u(p_N - k')\pi(p_N - k') \frac{(2p_N - k')^\mu - i\sigma^{uv}k'_v}{2p_N k'} u(p_N) \\
- < \text{out}; p'_\pi p'_N | j_\pi'(0) | p_\pi p_N; \text{in} > \frac{e_\pi}{2p_\pi k'}(2p_\pi - k')^\mu \right], \quad (2.12b)
\]

where \((N)\) indicates the part of \( \mathcal{E}^\mu_{\gamma' \pi' N' - \pi N} \) with the one-nucleon propagator. The corresponding part of \( B_{\pi' N' - \pi N} \) is denoted as \( B_{\pi' N' - \pi N}(N) \). Similarly, for the intermediate antinucleon part of (2.8a,b) we have
\[ B_{\pi'N'N}(N) = -e_{N'} \pi(p'_N) \nu(p_N + k') \nu(p'_N + k') + e_N < out; p'_\pi p'_N | J(0) | p_{\pi}; in > v(p_N - k') \nu(p_N - k') \nu(p_N), \] 

\[ \mathcal{E}_{\gamma'\pi'\pi N}(\bar{N}) = \frac{\pi(p'_N) [(2p'_N + k') - i \sigma^{\mu\nu} k'_\nu]}{2p'_N k'} \nu(p'_N + k') e_{N'} < out; p'_\pi | J(0) | p_{\pi} p_N; in > v(p_N - k') \nu(p_N - k') \nu(p_N) \] 

For the derivation of (2.12a,b) and (2.13a,b) the simple relations of the Dirac spinors \( \pi(p'_N) \gamma^\mu (\gamma^\nu p'_N + m_N) = \pi(p'_N) 2p'_N \gamma^\mu \), \( \pi(p'_N) \gamma^\mu k'_\nu = k'^\mu \pi(p'_N) - i \pi(p'_N) \sigma^{\mu\nu} k'_\nu \) were used.

Expressions (2.13a,b) contain the intermediate \( \pi \to \pi'N\bar{N} \) transition amplitude. Using the identity \( k'_\mu \left[ \left( P + P'^\mu \right) - i \sigma^{\mu\nu} k'_\nu \right] = s - s' \) it is easy to obtain that

\[ k'_\mu \mathcal{E}_{\gamma'\pi'\pi'N'}(\bar{N}) + B_{\pi'N'\pi N}(\bar{N}) = 0. \] 

Consequently, instead of the full Ward-Takahashi identity (2.7) we get

\[ k'_\mu < out; p'_N p_{\pi}| J^\mu(0) | p_{\pi} p_N; in > = B_{\pi'N'\pi N}(N) + k'_\mu \mathcal{E}_{\gamma'\pi'\pi'N'}(\bar{N}) = 0. \] 

The gauge terms proportional to \( k'_\mu \) in (2.12b) do not contribute to the \( \pi N \) bremsstrahlung amplitude because the product of polarization vector \( e^\mu(k', \lambda) \) of the final photon and \( k'_\mu \) vanishes \( e^\mu(k', \lambda) k'_\mu = 0 \). The terms proportional to \( k'_\mu \) modify the Green function \( \tau^\mu \) in (2.1), but for the on shell amplitude they can be ignored. In addition, due to \( k'_\mu k'^\mu \equiv k'^2 = 0 \) the terms proportional to \( k'_\mu \) in (2.12b) can also be omitted in the Ward-Takahashi identity (2.14b).

Current conservation (2.7) and (2.14b) are written for the longitudinal part of the total \( \gamma'\pi'N' - \pi N \) amplitude. In particular, the set of the diagrams which form the first term of the right side in (2.2b) with \( \partial/\partial z^\mu J^\mu(z) = 0 \) are not included in (2.7). Therefore, the modified Ward-Takahashi identity (2.7) presents a necessary condition of current conservation which contains only a longitudinal part of the external particle radiation amplitudes.

One can use the transverse part of the total \( \pi N \) bremsstrahlung amplitude in order to complete \( \mathcal{E}_{\gamma'\pi'\pi'N'}(N) \) (2.12b) up to the external particle radiation amplitude with the anomalous magnetic moment of the external nucleons. For this aim one can pick out the related transverse terms with \( \sigma^{\mu\nu} k'_\nu \) which generate the following redefinitions of \( \mu_{N'} = 1 \).
and \( \mu_N = 1 \) with the corresponding anomalous magnetic moments of the nucleon\(^3\). This procedure implies taking into account the loop corrections of the \( \gamma NN \) vertex. These corrections reproduces the anomalous magnetic moments within the minimal electromagnetic coupling scheme (see [27] consideration of (10.81)). Then we obtain\(^4\)

\[
E_{\gamma' \pi' \pi' \rightarrow \pi N}(N) = - \frac{(2p'_{\pi} + k')^\mu}{2p'_p k'} e_{\pi'} < \text{out}; p'_{\pi} [j_{\pi'}(0)] \sigma_{\pi} p_N; \text{in} >
\]

\[
+ \frac{\pi(p'_N)}{2p'_N k'} \left( (2p'_{\pi} + k')^\mu - i\mu_N \sigma_{\pi' \pi'} k_{\nu}' \right) u(p'_N + k') \sigma_{\pi} p_N; \text{in} >
\]

\[
- e_N < \text{out}; p'_{\pi} [J(0)] \sigma_{\pi} p_N; \text{in} > u(p_N - k') \left( (2p_N - k')^\mu - i\mu_N \sigma_{\pi' \pi'} k_{\nu}' \right) u(p_N)
\]

\[
- < \text{out}; p'_{\pi} p'_N [j_{\pi}(0)] \sigma_{\pi} p_N; \text{in} > \frac{e_{\pi}}{2p_{\pi} k'} (2p_{\pi} - k')^\mu, \quad (2.15)
\]

The external particle \( \pi N \) radiation amplitude (2.15) have the fixed transverse terms \( i\mu_N \sigma_{\pi' \pi'} k_{\nu}' \) in the vertex functions of the external nucleons. The full \( \gamma NN \) vertices are necessary for the realistic calculations of the \( \pi N \) radiation reactions. In particular, expression (2.15) automatically satisfies the low energy photon theorem [5]-[14].

Relation (2.14b) represents the modified Ward-Takahashi identity in the three-dimensional time-ordered form. This identity establish a relationship between the external particle \( \pi N \) bremsstrahlung amplitude (2.15) and the off mass shell elastic \( \pi N \) scattering amplitudes (2.9a,b,c,d). In order to satisfy current conservation (2.14b) it is necessary to find the internal particle radiation diagrams whose four divergence reproduces \( B_{\pi' \pi' \rightarrow \pi N}(N) \). The

\(^3\)Keeping the identity (2.14b) one can also reproduce the full electromagnetic form factors of the nucleon \( F_1(t) \) and \( F_2(t) \) in \( E_{\gamma' \pi \pi' \rightarrow \pi N}(N) \) (2.12b), where \( t \) denotes the corresponding four momentum transfer. Thus, if one picks out the terms \( \pi(p'_N) \left( (2p'_{\pi} + k')^\mu - i\mu_N \sigma_{\pi' \pi'} k_{\nu}' \right) F_2(t') F_{-1}(t') u(p'_N + k') \sigma_{\pi} T_N' \) and \( T_N \pi(p_N - k') \left( (2p_N - k')^\mu - i\mu_N \sigma_{\pi' \pi'} k_{\nu}' \right) F_2(t) F_{-1}(t) u(p_N) \) from the transverse part of the full \( \pi N \) radiation amplitude, one obtains the full \( \gamma NN \) vertices in \( E_{\gamma' \pi' \pi' \rightarrow \pi N}(N) \) (2.12b) with the redefined \( \pi N \) amplitudes (2.9a,c) \( T_N' \Rightarrow F_{-1}(t') T_N' \) and \( T_N \Rightarrow F_{-1}(t) T_N \).

Another way to take into account the full electromagnetic form factors of the external nucleons is to use the modified complete set of the intermediate Dirac spinors \( u(p'_N + k') \pi(p'_N + k') : u(p_N - k') \pi(p_N - k') \pi(p_N - k') \), where \( m_N \) is replaced by \( s'_N = \left( \sqrt{m_N^2 + (p_N^2)^2 + k'} \right)^2 - (p'_N + k')^2 \text{ or } s_N = \left( \sqrt{m_N^2 + (p_N^2)^2 - k'} \right)^2 - (p'_N - k')^2 \). Then we obtain the \( \gamma' \pi N \) vertex between the one nucleon states with the four moments \( p_{\pi'} + k' \rightarrow p_{\pi'} \) or \( p_N - k' \rightarrow p_N \). In this formulation \( t = t' = k'^2 = 0 \) and only the threshold values of the electromagnetic form factors for the external nucleons appear.

The large four momentum transfer of the external nucleons is not important for the determination of the electromagnetic moments of the \( \Delta \)'s. Therefore, we do not include them in the following text.

\(^4\)The anomalous magnetic moment of the \( \Delta \) appears in the \( -\gamma \Delta \) vertex at \( \sigma_{\mu\nu} k_{\Delta}^\nu \) with \( k_{\Delta}^\nu \neq k_{\pi'}^\nu \), i.e., the diagram in Fig. 2B with the anomalous magnetic moment of \( \Delta \) is not included in the transverse part of the \( \pi N \) bremsstrahlung amplitude.
special case of this problem for the (3/2,3/2) partial \( \pi N \) amplitudes is considered in the next Section.

3. Internal and external particle radiation parts of the \( \pi N \) bremsstrahlung amplitude.

In this Section we show, that the Ward-Takahashi identity (2.14b) after decompositions of \( (E)_{\gamma' \pi' N'-\pi N}(N) \) (2.15) and \( B_{\pi' N'-\pi N}(N) \) (2.12a) reduces to a special identity for the double \( \Delta \) exchange amplitude which has the same structure as on mass shell \( \Delta \) radiation diagram in Fig. 2B. For this aim we separate the longitudinal part of the external particle radiation amplitude and isolate the \( \Delta \) resonance parts of the off shell \( \pi N \) amplitudes (2.9a,b,c,d). The symbolic representation of this procedure is given by the chain of transformations

\[
(E)_{\gamma' \pi' N'-\pi N}(N) \Longrightarrow (E_L)_{\gamma' \pi' N'-\pi N} \Longrightarrow (E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta-pole) \tag{3.1a}
\]

and

\[
B_{\pi' N'-\pi N}(N) \Longrightarrow (B^{3/2})_{\pi' N'-\pi N}(\Delta-pole), \tag{3.1b}
\]

where the lower index \( L \) denotes the longitudinal part of the amplitude \( (E)_{\gamma' \pi' N'-\pi N} \), the upper index 3/2 corresponds to the resonance spin-isospin 3/2 state of the \( \pi N \) amplitudes. The argument \( (\Delta-pole) \) indicates the \( \Delta \)-pole part of the \( \pi N \) amplitudes (2.9a,b,c,d).

Afterwards, using the sum of the \( \Delta \)-pole terms in the different off shell \( \pi N \) amplitudes in \( (E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta-pole) \) and \( (B^{3/2})_{\pi' N'-\pi N}(\Delta-pole) \), one can separate the double \( \Delta \) exchange Ward-Takahashi identities for the double \( \Delta \) exchange amplitudes \( (E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta \Delta) \) and \( (B^{3/2})_{\pi' N'-\pi N}(\Delta \Delta) \). Moreover, after an algebraic transformation of the Ward-Takahashi identity for the double \( \Delta \) exchange amplitudes one obtains an independent identity for the amplitude which has the same structure as the internal \( \Delta \) radiation amplitude in Fig. 2B. These decompositions form the following chain of transformations

\[
(E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta-pole) \Longrightarrow (E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta \Delta) \Longrightarrow (E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta-\gamma \Delta) \tag{3.1c}
\]

\[
(B^{3/2})_{\pi' N'-\pi N}(\Delta-pole) \Longrightarrow (B^{3/2})_{\pi' N'-\pi N}(\Delta \Delta) \Longrightarrow B^{3/2}_{\pi' N'-\pi N}(\Delta-\gamma \Delta) \tag{3.1d}
\]

The algebraic decompositions (3.1a,b,c,d) of the Ward-Takahashi identity (2.14b) are detailed in Appendix A. The resulting identity is

\[
k_{\mu}' \left[ \frac{\text{Projection on spin 3/2 particle states}}{\text{2}\Delta \text{ exchange with a } \Delta-\gamma' \Delta' \text{ vertex}} \right]_{\text{out}; \ p'_{\pi} P'_{\pi} | J^\mu(0) | p_\pi P_{\pi}; \text{in} >} = k_{\mu}' (E_L^{3/2})_{\gamma' \pi' N'-\pi N}(\Delta-\gamma \Delta) + B^{3/2}_{\pi' N'-\pi N}(\Delta-\gamma \Delta) = 0, \tag{3.2}
\]
where

\[
(E^3 E_\pi^\nu)_{\gamma'\pi'N'\pi N}(\Delta - \gamma\Delta) = \frac{\langle P_N', P_{\pi'}^\nu | g_{\pi'N'\Delta}\rangle |P'_\Delta \rangle}{p^\nu_{\pi'} + p^\nu_N - P^\nu_\Delta(s')},
\]

\[
\left\{ \bar{u}'(P'_\Delta) g_{bc} [(P_\Delta + P'_\Delta)^\mu V_E - i\sigma^{\mu\nu} k'_\Delta^\nu V_H] u'(P_\Delta) \right\} \frac{\langle P_\Delta | g_{\Delta-N\pi'} | P_N, P_{\pi} \rangle}{p^\mu_{\pi} + p^\nu_N - P^\nu_\Delta(s')},
\]

and

\[
\left\{ \bar{u}'(P'_\Delta) g_{bc} [k'_\mu (P_\Delta + P'_\Delta)^\nu V_E - i k'_\mu \sigma^{\nu\rho} k'_\Delta^\rho V_H] u'(P_\Delta) \right\} \frac{\langle P_\Delta | g_{\Delta-N\pi'} | P_N, P_{\pi} \rangle}{p^\mu_{\pi} + p^\nu_N - P^\nu_\Delta(s')},
\]

where \( P = p_\pi + p_N \) and \( P' = p'_{\pi'} + p'_{N} \) are the four moments of the \( \pi N \) system in the initial and final states, \( P_\Delta \) and \( P'_\Delta \) denote the four moments of the \( \Delta \)

\[
P_\Delta \equiv \left( P^\nu_\Delta(s), P_\Delta \right) = \left( \sqrt{(M_\Delta(s) - \frac{i\Gamma_\Delta(s)}{2})^2 + P^2_\Delta}, P_\Delta \right); \quad P_\Delta = P = p_N + p_\pi \quad (3.4a)
\]

\[
P'_\Delta \equiv \left( P'^\nu_\Delta(s'), P'_\Delta \right) = \left( \sqrt{(M_\Delta(s') - \frac{i\Gamma_\Delta(s')}{2})^2 + P'^2_\Delta}, P'_\Delta \right); \quad P'_\Delta = P' = p'_N + p'_{\pi}. \quad (3.4b)
\]

We shall use two models of the \( \Delta \) mass \([31, 32, 30]\) \( m_\Delta(s) = M_\Delta(s) - i/2\Gamma_\Delta(s) \):

1. A model with the fixed mass of the intermediate \( \Delta \) resonance

\[
m_\Delta = M_\Delta - \frac{i}{2} \Gamma_\Delta = 1232 MeV - \frac{i}{2} 120 MeV, \quad (3.4c)
\]

2. and a more general model with an \( s \)-dependent mass \( m_\Delta(s) \)

\[
m_\Delta(s) = M_\Delta(s) - \frac{i}{2} \Gamma_\Delta(s), \quad (3.4d)
\]

where \( m_\Delta(s = M^2_\Delta) = m_\Delta \).

\( P_\Delta \) and \( P'_\Delta \) are on mass shell four moments because \( P^2_\Delta = m^2_\Delta(s) \) and \( P'^2_\Delta = m^2_\Delta(s') \).

\( u'(P_\Delta) \) denotes the Rarita-Schwinger spinor of the free spin 3/2 particle with the complex mass \( m_\Delta(s) \).

\( V_E \) and \( V_H \) in (3.3a,b) are defined through the \( \Delta \)-pole residues \( R_{N'}, R_{\pi'}, R_N \) and \( R_{\pi} \) of the off shell \( \pi N \) amplitudes in equations (A.9a,b,c,d), (A.15a,b) and (A.17a,b) of Appendix A. The \( \Delta - \pi N \) and \( \pi N - \Delta \) vertices \( g_{\pi'N'\Delta} \) and \( g_{\Delta-N\pi} \) are defined as

\[
\langle P_N', P_{\pi'} | g_{\pi'N'\Delta} | P'_\Delta \rangle = g_{\pi'N'\Delta}(s', k') \pi(p'_N) i\gamma_5 \left( \frac{p'_N}{|p'_N|} \right) u'(P'_\Delta), \quad (3.5a)
\]
\[ < \mathbf{P}_\Delta | g_{\Delta - \pi N} | \mathbf{p}_N, \mathbf{p}_\pi > = g_{\Delta - \pi N}(s) \pi'(\mathbf{P}_\Delta) \frac{(\mathbf{p}_N)_i}{|\mathbf{p}_N|} i\gamma_5 u(\mathbf{p}_N). \] (3.5b)

The longitudinal part of the external particle radiation amplitude with the \( \Delta \) intermediate states \((\mathcal{E}_3 \pi N)^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta)\) (3.3a) has the same form as the internal \( \Delta \) radiation amplitude \( \mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) \) in Fig. 2B.

\[
\mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) = -\frac{< \mathbf{P}'_{\Delta}, \mathbf{m}_\Delta(s')|\mathcal{J}^\mu(0)|\mathbf{P}_\Delta, \mathbf{m}_\Delta(s) >}{p'^\mu_{\pi} + p'^2_N - p'^2_\Delta(s')},
\]

(3.6)

where the details of the \( \Delta - \gamma \Delta \) vertex \(< \mathbf{P}'_{\Delta}, \mathbf{m}_\Delta(s')|\mathcal{J}^\mu(0)|\mathbf{P}_\Delta, \mathbf{m}_\Delta(s) >\) with on mass shell \( \Delta \)'s are given in Appendix B. In particular, for the low energy photons we have

\[
< \mathbf{P}'_{\Delta}, \mathbf{m}_\Delta(s')|\mathcal{J}^\mu(0)|\mathbf{P}_\Delta, \mathbf{m}_\Delta(s) > =
\]

\[
\bar{u} (\mathbf{P}'_{\Delta}) g_{\rho\sigma} \left[ \frac{(\mathbf{P}_\Delta + \mathbf{P}'_{\pi})_\mu}{2M_\Delta} G_C(k'^2, s, s') - i\sigma_\mu k'_\Delta \frac{G_{M1}(k'^2, s, s')}{2M_\Delta} \right] u(\mathbf{P}_\Delta)
\]

(3.7)

where \( G_C \) and \( G_{M1} \) denote the electric and magnetic dipole form factors of the \( \Delta \)'s.

The unambiguous construction of the \( \Delta \) radiation amplitude \( \mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) \) (3.6) is given in Appendix C following our previous papers \([31, 32]\). Thus \( \mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) \) in Fig. 2B can be determined as a projection of the complete internal particle radiation amplitude

\[
\mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) = \mathcal{T}^\mu_{\gamma' \pi' N' - \pi N} \bigg|_{\text{Projection on spin 3/2 particle states}} \bigg|_\text{\(2\Delta\) exchange with a \(\Delta-\gamma'\Delta'\) vertex}
\]

Consequently, expressions (3.3a) and (3.6) determine the complete projection of the \( \pi N \) radiation amplitude, and we have

\[
k'_\mu \left[ < \text{out}; \mathbf{p}'_{\pi} | \mathcal{J}^\mu(0) | \mathbf{p}_{\pi} \mathbf{p}_N; \text{in} > \right] \bigg|_{\text{2\(\Delta\) exchange with a \(\Delta-\gamma'\Delta'\) vertex}} =
\]

\[
k'_\mu \left( (\mathcal{E}_3 \pi N)^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) + \mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) \right) = 0.
\]

(3.8)

Combining (3.2) and (3.8) we obtain

\[
k'_\mu (\mathcal{E}_3 \pi N)^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) = -k'_\mu \mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta) = -B^{3/2}_{\pi' N' - \pi N}(\Delta - \gamma \Delta)
\]

(3.9)

Conditions (3.2) and (3.9) present the four-divergence of the amplitudes \((\mathcal{E}_3 \pi N)^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta)\) (3.3a) and \(\mathcal{T}^\mu_{\gamma' \pi' N' - \pi N}(\Delta - \gamma \Delta)\) (3.6) with the same double \( \Delta \) poles and the corresponding \( \Delta - \gamma \Delta \) vertex. There are no other amplitudes with the same analytical structure. Moreover, all gauge terms \( A_\mu \) with the separate current conservation
condition $k'_\mu A^\mu = 0$ are included in the transverse part of the external particle radiation amplitude $(E_{TR})_{\gamma'\pi'N'\rightarrow\pi N} (A.4b)$ and other transverse parts of the total amplitude which corresponds to the first term in (2.2b) with $\partial/\partial z^\mu J_\mu(z)$. Therefore, current conservation (3.2) and (3.9) are fulfilled if

\[
(E_L^{3/2})_{\gamma'\pi'N'\rightarrow\pi N}(\Delta - \gamma \Delta) = -T_{\gamma'\pi'N'\rightarrow\pi N}(\Delta - \gamma \Delta)
\]  

(3.10)

This equation coincides with the final equation (1.5) and allows to determinate the connection between the form factors of the $\Delta - \gamma \Delta$ vertices in $T_{\gamma'\pi'N'\rightarrow\pi N}(\Delta - \gamma \Delta)$ (3.6) and the analogical form factors in $(E_L^{3/2})_{\gamma'\pi'N'\rightarrow\pi N}(\Delta - \gamma \Delta)$ (3.3a).

Using the condition (3.10) one easily gets

\[
k'_\mu \left[ (E_L^{3/2})_{\gamma'\pi'N'\rightarrow\pi N}(\Delta - \gamma \Delta) + T_{\gamma'\pi'N'\rightarrow\pi N}(\Delta - \gamma \Delta) \right] = 0
\]  

(3.11)

which immediately gives

\[
G_{C0}(k'_\Delta, s, s') = -2M_\Delta V_E(s'P'_\Delta; sP_\Delta)
\]  

(3.12a)

Combining this equation with (3.24) we obtain

\[
G_{M1}(k'_\Delta, s, s') = -2M_\Delta V_H(s'P'_\Delta; sP_\Delta).
\]  

(3.12b)

Equations (3.12a,b) determine $G_{C0}(k'_\Delta, s, s')$ and $G_{M1}(k'_\Delta, s, s')$ via the residues of the $\pi N$ amplitudes $R$ (A.9a,b,c,d) which yield $V_E$ and $V_H$ in (A.17a,b) and $V_E^{(+)}$ and $V_H^{(+)}$ in (A.15a,b).

The threshold values of (3.12a,b) give a relations between $e_\Delta$, $\mu_\Delta$ and $V_E^{k'=0}$, $V_H^{k'=0}$ respectively

\[
e_\Delta = G_{C0}(k'_\Delta = 0, s' = M^2_\Delta, s = M^2_\Delta) = \left[ 2M_\Delta V_E \right]_{k'=0}^{\sqrt{s'}/\sqrt{s} = M_\Delta}
\]  

(3.13a)

and

\[
\mu_\Delta = G_{M1}(k'_\Delta = 0, s' = M^2_\Delta, s = M^2_\Delta) = \left[ 2M_\Delta V_H \right]_{k'=0}^{\sqrt{s'}/\sqrt{s} = M_\Delta}
\]  

(3.13b)

These conditions determine the relations between $e_\Delta$, $\mu_\Delta$ and residues of the $\pi N$ amplitudes $R$ (A.9a,b,c,d).

\[
e_\Delta = -\left[ N(s) \left[ g_{\pi'N'\rightarrow\Delta}(s', k') \right]^{-1} \left( e_N \frac{R_{N'}}{2} + e_\pi \frac{R_{\pi'}}{2} + e_\pi' \frac{R_{\pi}}{2} \right) \right]_{\sqrt{s'}/\sqrt{s} = M_\Delta}
\]  

(3.14a)

where $N(s) = 1/(d\sqrt{s}/dk') - dP_\Delta^2(s)/d\sqrt{s}$ and
\[
\mu_{\Delta} = - \left[ \mathcal{N}(s) \left[ g_{\pi'N'-\Delta'}(s', k') \right]^{-1} \left( \mu_N \frac{R_{N'} + R_N}{2} \right) \left[ g_{\Delta - \pi N}(s) \right]^{-1} \right]_{k'=0}^{s'=\sqrt{s}=M_\Delta}.
\] (3.14b)

The similarity of the conditions (3.14a) and (3.14b) allows to determine \( \mu_\Delta \) using (3.14a) as a normalization condition. For this aim we consider (3.14b) separately for the \( \pi^+ n \rightarrow \gamma' \pi'^+ n' \) and \( \pi^0 p \rightarrow \gamma' \pi'' p' \) reactions. For the \( \pi^+ n \rightarrow \gamma' \pi'^+ n' \) reaction (3.14a) generates the independent normalization condition

\[
1 = - \left[ \mathcal{N}(s) \left[ g_{\pi'N'-\Delta'}(s', k') \right]^{-1} \left( \frac{R_{p'} + R_p}{2} \right) \left[ g_{\Delta - \pi N}(s) \right]^{-1} \right]_{s'=\sqrt{s}=M_\Delta}^{k'=0}.
\] (3.15)

and for the \( \pi^0 p \rightarrow \gamma' \pi'' p' \) reaction we get

\[
1 = - \left[ \mathcal{N}(s) \left[ g_{\pi'N'-\Delta'}(s', k') \right]^{-1} \left( \frac{R_{p'} + R_p}{2} \right) \left[ g_{\Delta - \pi N}(s) \right]^{-1} \right]_{s'=\sqrt{s}=M_\Delta}^{k'=0}.
\] (3.16)

Expressions (3.15) and (3.16) are the normalization conditions for the residues \( R_\pi \) and \( R_p \) (A.9a,b,c,d) of the \( \pi N \) matrices at the \( \Delta \) resonance pole position. They show the dependence of \( R_\pi \) and \( R_p \) (A.9a,b,c,d) on the \( \Delta \) mass \( m_\Delta(s) \) (3.4c,d). Therefore, \( g_{\Delta - \pi N}(s) \) and \( g_{\pi'N'-\Delta'}(s', k') \) form factors must also include a dependence on a \( m_\Delta(s) \).

The right side of (3.16) differs from the right side of (3.14b) only by the factor \( \mu_N \). Therefore, substituting (3.16) into (3.14b) we obtain

\[
\mu_{\Delta^+} = \mu_p \frac{M_\Delta}{m_p},
\] (3.17)

where the factor \( M_\Delta/m_p \) arises because of the different units for \( \mu_{\Delta^+} \) and \( \mu_p \).

The magnetic dipole moment of \( \Delta^{++} \) can be determined from the relationship between \( \mu_\Delta \) and \( G_{M1}(k'_\Delta^2 = 0, s' = M_{\Delta^2}, s = M_{\Delta^2}^2) \) (3.13b) and (3.14b). The difference between \( (R_{N'} + R_N)/2 \) in (3.16) for the \( \pi^0 p \rightarrow \gamma' \pi'' p' \) and \( \pi^+ p \rightarrow \gamma' \pi'^+ p' \) reactions is in the isospin factors of the corresponding \( \pi N \) amplitudes. Using the isospin symmetry between the \( \pi^0 p \rightarrow \pi^0 p \) and \( \pi^+ p \rightarrow \pi^+ p \) amplitudes we get

\[
\mu_{\Delta^{++}} = \frac{3}{2} \mu_{\Delta^+} = \frac{3}{2} \mu_p \frac{M_\Delta}{m_p}.
\] (3.18)

One cannot use directly the \( \pi^0 n \rightarrow \gamma' \pi'^0 n' \) and \( \pi^- n \rightarrow \gamma' \pi'^- n' \) reactions for determination of the magnetic moments of \( \Delta^0 \) and \( \Delta^- \), because the equal-time commutator (2.3a) is zero in this case. This problem is considered in the next part of the present paper.

The modified Ward-Takahashi identity (3.9) requires equality and cancellation of the internal \( \Delta \) radiation amplitude in Fig. 2B and the corresponding part of the external particle radiation amplitude according to relation (3.10). This cancellation is the result
of current conservation for the $\pi N$ radiation amplitude and the special sum of the off shell $\pi N$ amplitudes in $(E_L^{3/2})_\pi' N'\pi N (\Delta - \gamma \Delta)$ (3.3a), which have the same analytical structure as the internal $\Delta$ radiation amplitude (3.6) in Fig. 2B. Therefore, the amplitude (3.6) in Fig. 2B cancels exactly with $(E_L^{3/2})_\pi' N'\pi N (\Delta - \gamma \Delta)$ (3.3a). Consequently, the internal $\Delta$ radiation diagram in Fig. 2B is screened by the appropriate part of the external particle radiation diagrams. Generally, screening is built into the initial Ward-Takahashi identity (2.7), where $B_{\pi' N'\pi N}$ must be compensated by the internal particle radiation diagrams. In other words, the screening corresponds to equality and cancellation of special parts of the internal and external particle radiation terms in the total $\pi N$ bremsstrahlung amplitude.

4. Conclusion

In the present paper $\Delta$'s are considered as resonances of the $\pi N$ system which generate appropriate $\Delta$-poles in the off mass shell $\pi N$ amplitudes (A.9a,b,c,d). The sums of the corresponding residues determine the $\Delta$ form factors $G_C$ and $G_{M1}$ in (3.14a,b). Thus current conservation (3.2) makes it possible to determine $G_C$ and $G_{M1}$ only using the dynamical information about the residues of the off shell $\pi N$ amplitudes (2.9a,b,c,d). The threshold values of $G_C$ and $G_{M1}$ define the magnetic dipole moments of $\Delta^+$ and $\Delta^{++}$ via the anomalous magnetic moment of the proton. The difference between $\mu_\Delta$ and $\mu_p$ is formed by different units in the $\Delta - \gamma \Delta$ and $\pi - \gamma \pi$ electromagnetic vertex functions. Another dynamical input for reproduction of the magnetic dipole moment of $\Delta^+$ and $\Delta^{++}$ is the anomalous magnetic moment of the proton, which requires loop corrections for the $\gamma NN$ vertices and the corresponding redefinition of the external particle radiation amplitude (2.12b) by expression (2.15). This redefinition does not change the initial current conservation (2.7) and is necessary for reproduction of the realistic results for the $\pi N$ bremsstrahlung reactions.

The present investigation of the $\pi N$ radiation is based on current conservation for the on shell amplitudes (2.7). From the general point of view only the sum of the external and internal particle radiation parts of the full bremsstrahlung amplitude satisfies current conservation. The modified Ward-Takahashi identity (2.7) specifies this statement for the special form of the external particle radiation amplitude $E_{\gamma' \pi' N'\pi N}^\mu$ and the appropriate sum of the off shell $\pi N$ scattering amplitudes $B_{\pi' N'\pi N}$. In particular, $E_{\gamma' \pi' N'\pi N}^\mu$ contains only the diagrams which are responsible for the infrared behavior of the $\pi N$ radiation amplitude. The sums of the $\Delta$-pole $\pi N$ amplitudes in the longitudinal part of $E_{\gamma' \pi' N'\pi N}^\mu$ reproduce the double $\Delta$ exchange poles.

The model-independent properties of current conservation (2.7) can be generalized for any amplitude of an arbitrary reaction $a + b \rightarrow \gamma' + f_1 + \ldots + f_n$ ($n = 1, 2, \ldots$). Current conservation requires the existence of the internal particle radiation amplitude $T_{\gamma' f_1 \ldots f_n - ab}^\mu$ which satisfies the relation $k'_\mu T_{\gamma' f_1 \ldots f_n - ab}^\mu = -k'_\mu E_{\gamma' f_1 \ldots f_n - ab}^\mu = B_{f_1 \ldots f_n - ab}$, where $E_{\gamma' f_1 \ldots f_n - ab}^\mu$ is the external particle radiation amplitude. Therefore, the appropriate parts of $E_{\gamma' f_1 \ldots f_n - ab}^\mu$ and $T_{\gamma' f_1 \ldots f_n - ab}^\mu$ have a different sign and they must be subtracted from each other. Consequently, we have a screening of the internal particle radiation amplitudes.
by the external one-particle radiation terms. In the limit \( k' \to 0 \) our approach exactly reproduces the low energy photon theorems for the bremsstrahlung reactions.

As an example of the screening the identity and cancellation of the double \( \Delta \) exchange amplitude is demonstrated in (3.10). This cancellation allows to determine the magnetic dipole moments

\[
\mu_{\Delta^+} = G_{M1}(0) = \frac{M_\Delta}{m_\Delta} \mu_p \quad \text{and} \quad \mu_{\Delta^{++}} = \frac{3}{2} \mu_{\Delta^+} = 5.46 \mu_B \quad \text{or} \quad \mu_{\Delta^{++}}/\mu_p \sim 1.95 \quad \text{of the} \quad \Delta^+ \quad \text{and} \quad \Delta^{++} \quad \text{resonances.}
\]

Our result for \( \mu_{\Delta^{++}} \) roughly agrees with the prediction of the naive \( SU(6) \) quark model \[15, 16\], with the nonrelativistic potential model \[4\] \( \mu_{\Delta^{++}} = 4.52 \pm 0.95 \mu_B \) and with extraction of \( \mu_{\Delta^{++}} \) from the experimental \( \pi^+p \to \gamma \pi^+p \) cross section within the low energy photon approach \( \mu_{\Delta^{++}} = 3.6 \pm 2.0 \mu_B \) \[9\], \( \mu_{\Delta^{++}} = 5.6 \pm 2.1 \mu_B \) \[10\] and \( \mu_{\Delta^{++}} = 4.7 - 6.9 \mu_B \) \[2\]. Our result is larger than the predictions of the modified \( SU(6) \) models \[18, 17\] and the low energy photon approximation \( \mu_{\Delta^{++}} = 3.7 \sim 4.9 \mu_B \) \[14\]. On the other hand, our result is smaller than the values obtained within the effective meson-nucleon Lagrangian \( \mu_{\Delta^{++}} = 6.1 \pm 0.5 \mu_B \) \[25\], in the effective quark model \( \mu_{\Delta^{++}} = 6.17 \mu_B \) \[26\], in the modified bag model \( \mu_{\Delta^{++}} = 6.54 \mu_B \) \[20\], and in the constituent quark model \[21\].

The resulting magnetic dipole moments of \( \Delta \)'s obtained in various theoretical models differ. Moreover, the results obtained using the same Low theorem approach for soft photons also differ. This difference can be explained with the various recipes for the construction of the bremsstrahlung amplitude in the low energy photon limit \( k' \to 0 \). These ambiguities are listed in \[12\]. Our formulation is free off these ambiguities.

**Table 1. Magnetic moments of \( \Delta^+ \) and \( \Delta^{++} \) in units of the nuclear magneton \( \mu_B = e/2m_N \). The upper index * at the reference indicates the theoretical model which is used to fit of the experimental data and to extract the magnetic moment \( \mu_{\Delta} \).**

| Model | This work | \( \mu_{\Delta^+} \) | \( \mu_{\Delta^{++}} \) | Potential and K-matr. app. | Skyrme | Low ener. phot. theorem | Eff. \( \pi N \) Lagran. | quark |
|-------|-----------|------------------------|------------------------|-----------------------------|--------|--------------------------|-----------------------------|--------|
| \( \mu_{\Delta^+} \) | 3.66 | 2.79 \[15, 16\] | 2.13 \[17\] | 2.20-2.45 \[18\] | 3.27 \[20\] | 2.0-3.0 \[24\] | 3.49 \[21\] | 2.85 \[22\] | 2.3-2.7 \[23\] | 2.79 \[26\] |
| \( \mu_{\Delta^{++}} \) | 5.49 | 5.58 \[15, 16\] | 4.25 \[17\] | 4.41-4.89 \[18\] | 6.54 \[20\] | 6.9-9.7 \[19\]* | 4.52\pm0.95 \[4\]* | 4.2-7.4 \[24\] | 3.6\pm2.0 \[9\]* | 5.6\pm2.1 \[10\]* | 4.7-6.9 \[2\]* | 3.7-4.9 \[14\]* | 6.1\pm0.5 \[25\]* | 6.98 \[21\] | 5.33 \[22\] | 5.1-5.4 \[23\] | 6.17 \[26\] |

The numerical values of the magnetic moments of the \( \Delta^+ \) and \( \Delta^{++} \) resonances are given in Table 1. In a number of approaches the magnetic moment of \( \Delta \) is treated as an adjustable parameter. The corresponding results obtained from the experimental cross sections of the \( \pi^+p \to \gamma \pi^+p \) reaction are indicated in Table 1 with the upper index *. It must be emphasized that only our approach and the naive \( SU(6) \) quark model give an
analytical form for $\mu_{\Delta^+}$ and $\mu_{\Delta^{++}}$. But our result for $\mu_{\Delta^+}$ is $M_{\Delta}/m_p \sim 1.31$ times larger as $\mu_{\Delta^+} = \mu_p = 2.79 \mu_B$ in refs. [15, 26].

The $SU(6)$ models [15, 16, 17] and their bag model modifications require proportionality between the charge and the magnetic dipole moment $\mu_\Delta = e_\Delta \mu_p$ of the $\Delta$ resonance. Therefore, $\mu_{\Delta^+} = 1/2 \mu_{\Delta^{++}}$ in [15, 16, 17, 18, 20]. This property is preserved in the constituent quark model [21]. But it is broken in the Skyrme model [24], chiral quark model [23], chiral quark-soliton model [22], and effective quark model [26]. Our result for the ratio $\mu_{\Delta^{++}}/\mu_{\Delta^+}$ is determined by the isospin factors of the $\pi^+p$ and $\pi^0p$ elastic scattering amplitudes. In addition, we take into account the difference between units of $\mu_\Delta$ and $\mu_p$ in the $\Delta-\gamma\Delta$ and $\gamma NN$ vertices. This difference generates the factor $M_{\Delta}/m_N$. Therefore the present value $\mu_{\Delta^+} = 3.64 \mu_B$ is larger than other predictions.

Our approach is based on usual local quantum field theory [28, 29]. This approach is not dependent on the form of the Lagrangian. Moreover, we have not used a special representation of the $\pi N$ amplitude and the $\Delta$ propagator. Therefore, the suggested relations between $\mu_\Delta$ and the anomalous magnetic moment of the proton are model independent. But the present field-theoretical formulation does not include the quark degrees of freedom. The generalization of the present formulation based on the field-theoretical approach with the quark-gluon degrees of freedom will be given in the following paper.

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Appendix A: Projections on the intermediate $\Delta$ states

In this section a set of transformations (3.1a,c) of the external particle radiation amplitude $(E)_{\gamma^* \pi'N' \rightarrow \pi N}(N)(2.15)$ and the corresponding Ward-Takahashi identity (2.14b) is performed. The resulting condition (3.2) as well as other intermediate current conservation conditions are obtained on the basis of the algebraic identity

$$k'_\mu [(P + P')^\mu - i\sigma^{\mu\nu}k'_\nu] = s - s'. \quad (A.1)$$

The decompositions (3.1a,b,c,d) of current conservation are detailed in the following subsections.

A. Decomposition over the transverse and longitudinal parts of the external particle radiation amplitude (2.15), i.e., $(E)_{\gamma^* \pi'N' \rightarrow \pi N}(N) \rightarrow (E_L)_{\gamma^* \pi'N' \rightarrow \pi N}(N)$.

In order to separate the transverse part of $E_{\gamma^* \pi'N' \rightarrow \pi N}(N)$ (2.15) (Fig. 1) it is convenient to introduce the total and relative moments

$$P = p_\pi + p_N; \quad p = \frac{\alpha_\pi p_N - \alpha_N p_\pi}{\alpha_\pi + \alpha_N}; \quad p_N = \frac{\alpha_N P}{\alpha_\pi + \alpha_N} + p, \quad p_\pi = \frac{\alpha_\pi P}{\alpha_\pi + \alpha_N} - p, \quad (A.2a)$$
\[ P' = p'_\pi + p'_N; \quad p' = \frac{\alpha'_N p'_N - \alpha'_N p'_H}{\alpha'_N + \alpha'_N}; \quad p'_N = \frac{\alpha'_N P'}{\alpha'_N + \alpha'_N} + p'; \quad p'_\pi = \frac{\alpha'_N P'}{\alpha'_N + \alpha'_N} - p', \quad (A.2b) \]

where

\[ \alpha_N = k'_\nu p'_N, \quad \alpha'_N = k'_\nu p'_N, \quad \alpha'_N = k'_\nu p'_N. \quad (A.2c) \]

The relative moments \( p \) and \( p' \) are transverse to \( k'_\mu \)

\[ k'_\nu p'_N = 0; \quad k'_\nu p'_N = 0. \quad (A.2d) \]

Now one can separate the transverse part \( (E_{TR})_{\gamma' \pi' N' - \pi N}^\mu \) from \( E_{\gamma' \pi' N' - \pi N}^\mu(N) \) (2.15) as

\[ (E_{TR})_{\gamma' \pi' N' - \pi N}^\mu(N) = (E_L)_{\gamma' \pi' N' - \pi N}^\mu + (E_{TR})_{\gamma' \pi' N' - \pi N}^\mu, \quad (A.3) \]

where \( k'_\mu(E_L)_{\gamma' \pi' N' - \pi N}^\mu = -B_{\pi' N' - \pi N}(N), \quad k'_\mu(E_{TR})_{\gamma' \pi' N' - \pi N}^\mu = 0 \) and

\[ (E_L)_{\gamma' \pi' N' - \pi N}^\mu = -\frac{1}{s - s'} \left[ \pi(p'_N) \left[ 2e_N P'^\mu - i \mu_N \sigma_{\mu\nu} k'_\nu \right] u(p'_N + k') \pi(p'_N) \right] < out; p'_\pi|J(0)|p_N; in > 
- \left[ 2e_N P'^\mu < out; p'_N|J(0)|p_N; in > -2e_N P'^\mu < out; p'_\pi|J(0)|p_N; in > \right] \quad (A.4a) \]

\[ (E_{TR})_{\gamma' \pi' N' - \pi N}^\mu = -\left[ e_N \frac{p'^\mu}{\alpha'_N} \pi(p'_N) u(p'_N + k') \pi(p'_N) < out; p'_\pi|J(0)|p_N; in > 
- e_N \frac{p'^\mu}{\alpha'_N} < out; p'_N|J(0)|p_N; in > + e_N \frac{p'^\mu}{\alpha'_N} < out; p'_\pi|J(0)|p_N; in > 
- e_N \frac{p'^\mu}{\alpha'_N} < out; p'_\pi|J(0)|p_N; in > u(p_N - k') \pi(p_N - k') u(p_N) \right] \quad (A.4b) \]

Other non-longitudinal terms can be obtained using new total four moments \( P_\pm \)

\[ P_\pm = \frac{1}{2}(P \pm P'), \quad \text{where} \ P = P_+ + P_-, \quad P' = P_+ - P_- \quad \text{and} \quad P_\pm^\mu = \frac{1}{2} k'_\mu. \quad (A.5) \]

This allows to separate of the term \( K_{\gamma' \pi' N' - \pi N}^\mu \) proportional to \( k'_\mu \)
\[
\left( E_L \right)^\mu_{\gamma^*\pi N' - \pi N} = \left( E_L \right)^\mu_{\gamma^*\pi N' - \pi N} + K^\mu_{\gamma^*\pi N' - \pi N} \]  

where

\[
\left( E_L \right)^\mu_{\gamma^*\pi N' - \pi N} = -\frac{1}{s - s'} \left[ \bar{\nu}(p_N) \left[ \epsilon_{N'}(P + P') - i\mu_N'\sigma^{\mu
u}k'_\mu \right] u(p_N + k') \right] <\text{ out }; p'_\pi | J(0) | p_\pi p_N; \text{ in } > - <\text{ out }; p'_\pi p'_{N'} | J(0) | p_\pi; \text{ in } > u(p_N - k') \bar{\nu}(p_N - k') \left[ \epsilon_{N'}(P + P') - i\mu_N'\sigma^{\mu
u}k'_\mu \right] u(p_{N'}) \]  

\[
- \frac{1}{s - s'} \left[ \epsilon_{N'}(P + P') <\text{ out }; p'_N | j_{\pi'}(0) | p_{\pi'} p_{N'}; \text{ in } > - \epsilon_{\pi}(P + P') <\text{ out }; p'_\pi p'_{N'} | j_{\pi}(0) | p_{\pi'} p_{N'}; \text{ in } > \right], \]  

\[
K^\mu_{\gamma^*\pi N' - \pi N} = \frac{k'^\mu}{s - s'} \left[ \bar{\nu}(p'_{N'}) u(p'_N + k') \bar{\nu}(p'_N + k') \epsilon_{N'} <\text{ out }; p'_\pi | J(0) | p_\pi p_N; \text{ in } > + \epsilon_{\pi'} <\text{ out }; p'_N | j_{\pi'}(0) | p_\pi p_N; \text{ in } > u(p_N - k') \bar{\nu}(p_N - k') u(p_{N'}) + \epsilon_{\pi} <\text{ out }; p'_\pi p'_{N'} | j_{\pi}(0) | p_{\pi'} p_{N'}; \text{ in } > \right]. \]  

(A.7a)

(A.7b)

It is easy to see that \( k'^\mu \mathcal{E}^\mu_{\gamma^*\pi N' - \pi N}(N) = k'^\mu \mathcal{E}^\mu_{\gamma^*\pi N' - \pi N}(N) = -\mathcal{B}_{\pi N' - \pi N}(N). \) The resulting expression \( \mathcal{E}^\mu_{\gamma^*\pi N' - \pi N} \) (3.7a) differs from \( \mathcal{E}^\mu_{\gamma^*\pi N' - \pi N}(N) \) (2.15) by the \( \gamma N N \) and \( \gamma \pi \pi \) vertices which have unified factors \( (P + P')^\mu - i\mu_N(N)\sigma^{\mu
u}k'_\nu \) and \( (P + P')^\mu. \)

**B. Projection on the \( \Delta \)-pole terms** \( \mathcal{E}^\mu_{\gamma^*\pi N' - \pi N}(N) \Rightarrow \mathcal{E}^{3/2\mu}_{\gamma^*\pi N' - \pi N}(\Delta - \text{pole}) \) and \( \mathcal{B}_{\pi N' - \pi N}(N) \Rightarrow \mathcal{B}^{3/2}_{\pi N' - \pi N}(\Delta - \text{pole}). \)

In order to separate the \( \Delta \)-pole parts in \( \mathcal{E}^\mu_{\gamma^*\pi N' - \pi N} \) (A.7a) we shall use a projection of the \( \gamma N - N \) and \( \gamma \pi - \pi \) vertex on the spin 3/2 intermediate states

\[
(p'_N \cdot p_N) \left[ \bar{\nu}(p'_N) \gamma_5 \left[ \epsilon_{N'}(P + P')^\mu - i\mu_N'\sigma^{\mu
u}k'_\nu \right] \gamma_5 u(p'_N + k') \right]^{\text{Projection on spin 3/2 states}} = \]  

\[
\left( \bar{\nu}(p'_N) \gamma_5 \bar{p'}_{N_a} \right)^a(P') \left( \bar{\nu}(P') \gamma_5 \bar{p}_N \right) \left[ \epsilon_{N'}(P + P')^\mu - i\mu_N'\sigma^{\mu
u}k'_\nu \right] u_c(P) \frac{\bar{\nu}(P)(p_N) d \gamma_5 u(p_N)}{\bar{\nu}(p_N) u(p'_N + k')}, \]  

\[
(p'_N \cdot p_N) \left[ \epsilon_{\pi'}(P + P')^\mu \right]^{\text{Projection on spin 3/2 states}} = \left( \bar{\nu}(p'_N) \gamma_5 \bar{p'}_{N_a} \right)^a(P') \]  

22
\[
[\bar{\pi}'(P') g_{bc} e_\pi(P + P')^\mu u^c(P)] \bar{\pi}'(P)(p_N) a\gamma_5 u(p_N) \bar{\pi}(p_N) u(p'_N). \tag{A.8b}
\]

where \( (p'_N, p_N) = (p'_N, a(p_N))^a \) and we omit the spin index \( S = \pm 1/2, \pm 3/2 \) of the Rarita-Schwinger spinor \( u^a(P) \) with the mass \( m_{D_{3/2}}^2 = P_o^2 - P^2 \). The matrix element (A.8a) corresponds to the transitions \( \pi N \to D_{3/2} \to \gamma' D'_{3/2} \to \pi' N' \) with the intermediate spin \( 3/2 \) particles \( D_{3/2} \) and \( D'_{3/2} \).

The common factor \( \left( \overline{\pi}(p_N) \gamma_5 p'_{N a} u^a(P') \right) \left[ \bar{\pi}'(P') g_{bc}(\ldots)^\mu u^c(P) \right] \bar{\pi}(p_N) a\gamma_5 u(p_N) \) in (A.8a,b) generates the following redefinition of (A.7a)

\[
[\bar{\pi}'(P') g_{bc} e_\pi(P + P')^\mu u^c(P)] \bar{\pi}'(P)(p_N) a \gamma_5 u(p_N) \bar{\pi}(p_N) u(p'_N).
\]

\[
(\mathcal{E}_L)^{\mu}_{\gamma' \pi' \pi' - \pi N} \equiv (\mathcal{E}_L^{3/2})^{\mu}_{\gamma' \pi' \pi' - \pi N} = \frac{(p'_N) a (p_N) d (P + P')^\mu}{|p'_N||p_N|(s - s')} \overline{\pi}(p'_N) i\gamma_5 u^a(P') \left\{ \bar{\pi}'(P') g_{bc} e'_c(P) \right\} \bar{\pi}'(P) i\gamma_5 u(p_N)
\]

\[
\left[ \frac{|p'_N||p_N|}{|p'_N||p_N|} \left( \overline{\pi}(p'_N) u(p'_N) + k') \overline{\pi}(p'_N) + k') e_{\pi'} < \text{out}; p'_\pi |J(0)| p_\pi p_N; \text{in} > \right.
\]

\[
- \mu_N < \text{out}; p'_\pi p'_N |J(0)| p_\pi; \text{in} > u(p_N - k') \overline{\pi}(p_N - k') u(p'_N)
\]

\[
+ \overline{\pi}(p_N) u(p_N') e_{\pi'} < \text{out}; p'_\pi |j_{\pi'}(0)| p_\pi p_N; \text{in} >
\]

\[
- e_{\pi} < \text{out}; p'_\pi p'_N |j_{\pi'}(0)| p_\pi; \text{in} > \overline{\pi}(p_N) u(p'_N) \right]^{\text{Projection on spin 3/2 states}}
\]

\[
+ \frac{(p'_N) a (p_N) d}{|p'_N||p_N|(s - s')} \overline{\pi}(p'_N) i\gamma_5 u^a(P') \left\{ \bar{\pi}'(P') (-i\sigma^\mu\kappa'_a) u^c(P) \right\} \bar{\pi}'(P) i\gamma_5 u(p_N)
\]

\[
\left[ \frac{|p'_N||p_N|}{|p'_N||p_N|} \left( \overline{\pi}(p'_N) u(p'_N) + k') \overline{\pi}(p'_N) + k') \mu_{N'} < \text{out}; p'_\pi |J(0)| p_\pi p_N; \text{in} > \right.
\]

\[
- \mu_N < \text{out}; p'_\pi p'_N |J(0)| p_\pi; \text{in} > u(p_N - k') \overline{\pi}(p_N - k') u(p'_N) \right]^{\text{Projection on spin 3/2 states}}.
\]

The \( \pi N \) amplitudes (2.9a,b,c,d) consist of the resonant and non-resonant parts \(^7\)

\(^6\)Using the completeness conditions of the spin 3/2 functions \( u^a(P, S) \) [38, 39, 40] and \( u^a(P, S) \) of the free particle and antiparticle states

\[
\sum_{s = -3/2}^{3/2} \left( u^a(P, S) \overline{u}^b(P, S) + \frac{\gamma_5 \sigma_{pq}^a + s_{1/2}^a}{2s_{1/2}^b} \left\{ \text{projections on spin 1/2 states} \right\}^{ab} \right) +
\]

\[
\sum_{s = -3/2}^{3/2} \left( v^a(P, S) \overline{v}^b(P, S) + \frac{-\gamma_5 \sigma_{pq}^a + s_{1/2}^a}{2s_{1/2}^b} \left\{ \text{projections on spin 1/2 states} \right\}^{ab} \right) = g^{ab}
\]

one can rewrite the \( \gamma N - N \) vertex as
Takahashi identities. Then for the final nucleon radiation term we obtain
\[
\left[ \frac{\left| \mathbf{P}'_N \right| \mathbf{P}_N}{\left( \mathbf{p}'_N \cdot \mathbf{p}_N \right)} \bar{\pi}(\mathbf{p}_N) \left( \mathbf{p}'_N + k' \right) \gamma^\sigma + m_N \right] \mathbf{p}(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| J(0) \right| \mathbf{p}_N; \text{in} > = \frac{\mathcal{R}_N(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta)}{p'_\pi + p'_N - p_\Delta(s)} + \mathcal{R}_{N'}(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta) / \pi(\mathbf{p}_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| j'_\pi(0) \right| \mathbf{p}_N; \text{in} > \]
\[= \mathcal{R}_N(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta) / \pi(\mathbf{p}_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| J(0) \right| \mathbf{p}_N; \text{in} > \]
\[= \mathcal{R}_N(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta) / \pi(\mathbf{p}_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| J(0) \right| \mathbf{p}_N; \text{in} > \]

In the ∆ resonance region one can take into account only the spin 3/2 intermediate states. In addition, other degrees of freedom with antiparticle and spin 1/2 intermediate states form the independent Ward-Takahashi identities. Then for the final nucleon radiation term we obtain
\[
\left[ \frac{\left| \mathbf{P}'_N \right| \mathbf{P}_N}{\left( \mathbf{p}'_N \cdot \mathbf{p}_N \right)} \bar{\pi}(\mathbf{p}_N) \left( \mathbf{p}'_N + k' \right) \gamma^\sigma + m_N \right] \mathbf{p}(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| J(0) \right| \mathbf{p}_N; \text{in} > = \frac{\mathcal{R}_N(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta)}{p'_\pi + p'_N - p_\Delta(s)} + \mathcal{R}_{N'}(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta) / \pi(\mathbf{p}_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| j'_\pi(0) \right| \mathbf{p}_N; \text{in} > \]
\[= \mathcal{R}_N(s' \mathbf{P}'_\Delta; s \mathbf{P}_\Delta) / \pi(\mathbf{p}_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| j'_\pi(0) \right| \mathbf{p}_N; \text{in} > \]

In the same way for the final pion radiation term we get
\[
< \text{out}; \mathbf{p}'_N \left| j'_\pi(0) \right| \mathbf{p}_N; \text{in} > = \frac{1}{(\mathbf{p}'_N \cdot \mathbf{p}_N)} \left[ \bar{\pi}(\mathbf{p}'_N) \gamma^\sigma \mathbf{p}(\mathbf{p}'_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| j'_\pi(0) \right| \mathbf{p}_N; \text{in} > \right] \]
\[= \bar{\pi}(\mathbf{p}'_N) \gamma^\sigma \mathbf{p}(\mathbf{p}'_N) u(\mathbf{p}'_N) < \text{out}; \mathbf{p}'_N \left| j'_\pi(0) \right| \mathbf{p}_N; \text{in} > \]

Hereafter the spin indexes S and S' are omitted for the sake of simplicity.
one-particle Heisenberg operator of $\Delta$ requires an additional assumption. Therefore, the introduction of the effective intermediate one-particle $\Delta$'s are not equivalent to real particle degrees of freedom which have appropriate $r$ functions and the $\Delta$ states which decay into the asymptotic quantum mechanical $\Delta$ propagator.  

Projection on the spin $3/2$ states and separation of the $\Delta$ pole terms modifies Ward-Takahashi identity (A.7c) as

$$
\left[ k'_\mu < out; p'_N p'_\pi | J^\mu(0) | p_N p_N; in > \right]_{\text{projection on spin } 3/2 \text{ states}} = k'_\mu (E^{3/2})_{\gamma' \gamma' N' N - \pi N} + B^{3/2}_{\gamma' \gamma' N' N - \pi N} + R^{\gamma'}_{\gamma}(s'P'_\Delta; sP_\Delta).
$$

(A.9d)

The linear relativistic $\Delta$ propagator in (A.9a,b,c,d) is most similar to non-relativistic quantum mechanical $\Delta$ propagator.

With identity (A.1), it is easy to see, that the $\Delta$-pole and non-pole parts of $(E^{3/2})_{\gamma' \gamma' N' N - \pi N}$ separately satisfy the independent Ward-Takahashi identities. In particular,

$$
\left[ k'_\mu < out; p'_N p'_\pi | J^\mu(0) | p_N p_N; in > \right]_{\Delta \text{ - pole}} = k'_\mu (E^{3/2})_{\gamma' \gamma' N' N - \pi N} + B^{3/2}_{\gamma' \gamma' N' N - \pi N}(\Delta - \text{pole}) = 0.
$$

(A.10b)

For the on energy shell $\pi N$ and $\pi' N'$ states with $s = s'$, i.e., $k' = 0$, the operator

$$
Q(p'_N, p_N, P'_\Delta, P_\Delta) = \frac{1}{(p'_N - p_N)} \pi(p'_N) i\gamma_5 (p'_N) a u^a (P'_\Delta) \pi^i(P_\Delta)(p_N) e^{i\gamma_5 u(p_N)}
$$

is transformed into the projection operator on the $\pi N$ state with the orbital momentum $L = 1$ and the total momentum $J = 3/2$ $P^{3/2} = (p'_N, p_N)$ [32]

$$
P^{3/2} (p'_N, p_N) = \frac{6 m_N}{4 \pi p' (m_N + \sqrt{m_N^2 + p'^2})} \pi(p'_N) i\gamma_5 p'_N a u^a (P'_\Delta) \pi^i(P_\Delta)(p_N) e^{i\gamma_5 u(p_N)}.
$$

Therefore, we have

$$
\left[ Q(p'_N, p_N, P'_\Delta, P_\Delta) \right]_{|k'|=0} = \frac{4 \pi p' (m_N + \sqrt{m_N^2 + p'^2})}{6 m_N (p'_N - p_N)} P^{3/2} (p'_N, p_N)
$$

(A.11)

Equations (A.9a,b,c,d) are also valid in the models, where the $\Delta$'s are considered as the intermediate one-particle states which decays into the asymptotic $\pi N$ states (see for example ref. [42]). The $R$ and $r$ functions and the $\Delta - \pi N$ vertices in this case are defined in the one-particle approach. Nevertheless, intermediate one-particle $\Delta$'s are not equivalent to real particle degrees of freedom which have appropriate one-particle asymptotic states. Therefore, the introduction of the effective $\pi N \Delta$ Lagrangian with the one-particle Heisenberg operator of $\Delta$ requires an additional assumption.
where

$$(E_L^{3/2})^{\mu}_{\gamma^\prime N' \to N}(\Delta - \text{pole}) = \frac{1}{|P_N||P'_N|} \overline{\pi}(P'_N)(P'_N)_{\alpha} i\gamma_5 u^\alpha(P')$$

$$\left\{ \overline{\pi'}(P') g_{bc} \left[ (P + P')\mu \nu V_E - i\sigma^{\mu\nu} k'_\nu V_H \right] u^\nu(P) \right\} \overline{\pi}(P)(p_N)_{\alpha} i\gamma_5 u(P_N),$$

(A.10c)

$$B_{\pi N' \to N}^{3/2}(\Delta - \text{pole}) = \frac{(p'_N)_{\alpha}(p_N)_{\alpha} d |P'_N||P_N|}{|P'_N||P_N|} \overline{\pi}(P'_N) i\gamma_5 u^\alpha(P') \left\{ \overline{u}^b(P') g_{bc} u^\nu(P) \right\} \overline{u}^d(P) i\gamma_5 u(P_N)$$

$$\left[ R_{N'}(s'P_{\Delta}'sP_{\Delta}) + \frac{R_{\pi'}(s'P'_{\Delta}'sP_{\Delta})}{p^\alpha_{\pi} + p^\alpha_{N} - P^\alpha_{\Delta}(s)} \right] - \frac{R_{N}(s'P_{\Delta}'sP_{\Delta}) - R_{\pi}(s'P'_{\Delta}'sP_{\Delta})}{p^\alpha_{\pi} + p^\alpha_{N} - P^\alpha_{\Delta}(s')}.$$

(A.10d)

where

$$V_E = \frac{e_{N'} R_{N'}}{(s - s')(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s))} + \frac{e_{\pi'} R_{\pi'}}{(s - s')(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s'))}$$

$$- \frac{e_{N} R_{N}}{(s - s')(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s'))} - \frac{e_{\pi} R_{\pi}}{(s - s')(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s))},$$

(A.11a)

$$V_H = \frac{\mu_{N'} R_{N'}}{(s - s')(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s))} - \frac{\mu_{N} R_{N}}{(s - s')(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s'))},$$

(A.11b)

Hereafter we omit the variables of $R$ functions for the sake of simplicity.

The resulting expressions (A.10c,d) have the common factors $\overline{\pi}(P'_N)(p'_N)_{\alpha} i\gamma_5 u^\alpha(P')$, $\overline{\pi}(P')(p_N)_{\alpha} i\gamma_5 u(P_N)$, $\overline{\pi}(P') g_{bc}(P + P')\mu \nu u^\nu(P)$, $\overline{\pi}(P') g_{bc} [-i\sigma^{\mu\nu} k'_\nu] u^\nu(P)$ and $\overline{\pi}(P') g_{bc} u^\nu(P)$ which are needed for the $\Delta - \gamma\Delta$-type vertex in (3.6) depicted in Fig. 2B.

C. The double $\Delta$ exchange amplitude and transitions $(E_L^{3/2})^{\mu}_{\gamma^\prime N' \to -N}(\Delta - \text{pole}) \Rightarrow (E_L^{3/2})^{\mu}_{\gamma N' \to N}(\Delta\Delta)$ and $B_{\pi' N' \to N}^{3/2}(\Delta - \text{pole}) \Rightarrow B_{\pi N' \to N}^{3/2}(\Delta\Delta)$.

Next we have to extract from $E_L^{3/2})^{\mu}_{\gamma^\prime N' \to N}(\Delta - \text{pole})$ (A10b) the amplitude which has the same analytical properties as the double $\Delta$ exchange term in Fig. 2B. Using a simple algebra we rewrite (A.11b) as

$$V_H = \frac{1}{(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s'))(p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s))} \left\{ R_+ \left[ \frac{|k|}{s - s'} \frac{s - P^\alpha_{\Delta}(s) - P^\alpha_{\Delta}(s')} {s - s'} \right] 
\right\}$$

$$+ \frac{R_-}{s - s'} \left[ \frac{p_{\pi}^\alpha + p_{N}^\alpha + p_{\pi}^\alpha + p_{N}^\alpha - P^\alpha_{\Delta}(s) - P^\alpha_{\Delta}(s')} {s - s'} \right],$$

(A.12a)
where

\[ R_\pm = \frac{1}{2} \left[ \mu_{N'} R_{N'} \pm \mu_N R_N \right] \tag{A.12b} \]

and we use the identities \( 1/a \pm 1/b = 1/(a \pm b) \) with \( a = \left( p_\pi^o + p_N^o - P_\Delta^o(s') \right) \) and \( b = \left( p_\pi^o + p_N^o - P_\Delta^o(s) \right) \). These transformations play a central role in connection between the amplitudes in Fig. 1 and Fig. 2B.

The first part in (A.12a) is regular at \( |k'| = 0 \), where \( s = s' \), because \( (P_\Delta^o(s) - P_\Delta^o(s'))/(s - s') \) is finite at \( s = s' \).

The second part of \( \mathcal{V}_H \) (A.12a) can describe only one \( \Delta \) exchange because \( p_\pi^o + p_N^o + p_\pi^o + p_N^{o'} - P_\Delta^o(s) - P_\Delta^o(s') \) cancels out one of the \( \Delta \) propagators. This expression has different behavior in two different cases:

1. For charge exchange reactions \( R_-/(s - s') \) can be singular at the threshold \( |k'| = 0 \). This case needs a special investigation which is out of the scope of this article.

2. For the \( \pi N \) bremsstrahlung reactions without charge exchange (e.g. \( \pi^\pm p \rightarrow \gamma \pi^\pm p \) or \( \pi^o p \rightarrow \gamma \pi^o p \) we have \( e_{N'} = e_N \) and \( \mu_{N'} = \mu_N \). In this case \( R_-/(s - s') \) is finite at the threshold \( |k'| = 0 \) and this part corresponds to the one \( \Delta \) exchange diagrams of the \( \pi N \rightarrow \gamma' \pi' N' \) reaction. Using identity (A.1) one can separate one and double \( \Delta \) exchange contributions through the different current conservation conditions, i.e., one can split \( (\mathcal{E}_L)^{3/2}_{\gamma' \pi'; \pi' N' \rightarrow \pi N}(\Delta - pole) \) and \( B_{\pi' N' \rightarrow \pi N}(\Delta - pole) \) into two parts

\[
(\mathcal{E}_L)^{3/2}_\gamma \gamma' \pi' N' \rightarrow \pi N(\Delta - pole) = (\mathcal{E}_L)^{3/2}_\gamma \gamma' \pi' N' \rightarrow \pi N(\Delta \Delta) + (\mathcal{E}_L)^{3/2}_\gamma \gamma' \pi' N' \rightarrow \pi N(\Delta), \tag{A.13}
\]

where

\[
(\mathcal{E}_L)^{3/2}_\gamma \gamma' \pi' N' \rightarrow \pi N(\Delta \Delta) = \frac{1}{|p'_N||p_N|} \overline{u}(p'_N)(p'_N)_{a} \gamma_{5} u(a)(p'),
\]

\[
\left\{ \overline{u}(p')_{\gamma} \left[ (P + P')_{\mu} \gamma_{\mu} [1 - i \sigma_{\mu} k'_\nu \gamma_{\nu}] u(a) \right] \overline{u}(p')(p_N)_{a} \gamma_{5} u(a), \right. \tag{A.14a}
\]

\[
(\mathcal{E}_L)^{3/2}_\gamma \gamma' \pi' N' \rightarrow \pi N(\Delta) = \frac{1}{|p'_N||p_N|} \overline{u}(p'_N)(p'_N)_{a} \gamma_{5} u(a)(p'),
\]

\[
\left\{ \overline{u}(p')_{\gamma} \left[ (P + P')_{\mu} \gamma_{\mu} [1 - i \sigma_{\mu} k'_\nu \gamma_{\nu}] u(a) \right] \overline{u}(p')(p_N)_{a} \gamma_{5} u(a), \right. \tag{A.14b}
\]

\[
B_{\pi' N' \rightarrow \pi N}(\Delta \Delta) = \frac{s - s'}{|p'_N||p_N|} \overline{u}(p'_N)(p'_N)_{a} \gamma_{5} u(a)(p'),
\]

\[
\left\{ \overline{u}(p')_{\gamma} \left[ (P + P')_{\mu} \gamma_{\mu} [1 - i \sigma_{\mu} k'_\nu \gamma_{\nu}] u(a) \right] \overline{u}(p')(p_N)_{a} \gamma_{5} u(a), \right. \tag{A.14c}
\]
\[ \mathcal{B}_{\pi'N'\pi N}(\Delta) = \frac{s - s'}{|p'_N||p_N|} \pi(p'_N)(p'_N)_{a'i}^a \gamma_5 u^a(p') \left\{ \pi'(P') g_{bc} \gamma_5 u^c(P') \right\} \pi'(P)(p_N)_{a'i}^a \gamma_5 u(p_N), \]

where

\[ \mathcal{V}^{(+)}_E = \frac{\epsilon_N(R_{N'} + R_N) + \epsilon_{\pi}(R_{\pi'} + R_{\pi})}{2(s - s')} \left| k' \right| - \left( P'_\Delta(s) - P'_{D}(s') \right) \left( p'_\pi + p'_N - P'_{\Delta}(s') \right) \left( p'_\pi + p'_N - P'_{\Delta}(s) \right) \]

\[ \mathcal{V}^{(+)}_H = \frac{\mu_N(R_{N'} + R_N)}{2(s - s')} \left| k' \right| - \left( P'_\Delta(s) - P'_{D}(s') \right) \left( p'_\pi + p'_N - P'_{\Delta}(s') \right) \left( p'_\pi + p'_N - P'_{\Delta}(s) \right) \]

\[ \mathcal{V}^{(-)}_E = \frac{\epsilon_N(R_{N'} - R_N) + \epsilon_{\pi}(R_{\pi'} - R_{\pi})}{2(s - s')} \left( p'_\pi + p'_N - P'_{\Delta}(s') \right) + p'_\pi + p'_N - P'_{\Delta}(s) \]

\[ \mathcal{V}^{(-)}_H = \frac{\mu_N(R_{N'} - R_N)}{2(s - s')} \left( p'_\pi + p'_N - P'_{\Delta}(s') \right) + p'_\pi + p'_N - P'_{\Delta}(s) \]

With (A.1) it easy to check that

\[ k'_\mu (\mathcal{E}_{L}^{3/2})^\mu_{\gamma'\pi'N'\pi N}(\Delta) = -\mathcal{B}_{\pi'N'\pi N}(\Delta); \quad k'_\mu (\mathcal{E}_{L}^{3/2})^\mu_{\gamma'\pi'N'\pi N}(\Delta) = -\mathcal{B}_{\pi'N'\pi N}(\Delta). \]

Thus we have extracted the double \( \Delta \) exchange part from the \( \Delta - \text{pole} \) amplitudes (A.10b,c). Afterwards, the Ward-Takahashi identity (A.10a) is divided into two independent identities (A.16). The resulting Ward-Takahashi identity (A.16) contains the double \( \Delta \) exchange terms.

D. An alternative form of the double \( \Delta \) exchange amplitude \( (\mathcal{E}_{L}^{3/2})^\mu_{\gamma'\pi'N'\pi N}(\Delta) \).

Hereafter it is convenient to represent \( (\mathcal{E}_{L}^{3/2})^\mu_{\gamma'\pi'N'\pi N}(\Delta) \) (A.14a) and \( \mathcal{B}_{\pi'N'\pi N}(\Delta) \) (A.14c) through the \( \pi N \to \Delta, \Delta \to \gamma'\Delta', \Delta' \to \pi'N' \) vertices (3.5a,b) and the intermediate \( \Delta \) propagators. Therefore we rewrite (A.15a,b) as

\[ \mathcal{V}^{(+)}_E = \frac{1}{\left( p'_\pi + p'_N - P'_{\Delta}(s') \right) \left( p'_\pi + p'_N - P'_{\Delta}(s) \right)} g_{\pi'N'\Delta}(s', k') \mathcal{V}_E g_{\Delta - \pi N}(s), \]

\[ \mathcal{V}^{(+)}_H = \frac{1}{\left( p'_\pi + p'_N - P'_{\Delta}(s') \right) \left( p'_\pi + p'_N - P'_{\Delta}(s) \right)} g_{\pi'N'\Delta}(s', k') \mathcal{V}_H g_{\Delta - \pi N}(s). \]
Relations (A.17a,b) allows to rewrite (A.14a,c) as

\[
(E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta \Delta) = \frac{\langle P_N', P'_\pi | g_{\pi' \pi' - \Delta} | P'_\Delta >}{p_{\pi'}^0 + p_N'^0 - P_N'^0(s')}
\]

\[
\left\{ \bar{\psi}(P'_\Delta) g_{bc} \left[ (P + P')^\mu V_E - i\sigma^{\mu \nu} k'^\nu V_H \right] u^c(P_\Delta) \right\} \frac{\langle P \Delta | g_{\Delta - \pi N} | P_N, P_\pi >}{p_\pi^0 + p_N^0 - P_N^0(s)}.
\tag{3.15d}
\]

The amplitude (A.18a) has the form of the usual \( \Delta \) radiation diagram in Fig. 2B with the \( \Delta - \gamma \Delta \) vertex function \( (P + P')^\mu V_E - i\sigma^{\mu \nu} k'^\nu V_H \) instead of the \( \Delta - \gamma \Delta \) vertex function \( (B.3a,b) \) in Appendix B.

\[
E. \text{ Transitions } (E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta \Delta) \implies (E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta - \gamma \Delta) \text{ and }
\]

\[
B_{\pi' \pi' \pi - \pi N}(\Delta \Delta) \implies B_{\pi' \pi' \pi - \pi N}(\Delta - \gamma \Delta)
\]

It is important to note that \( \mathcal{Z}_{\gamma' \pi' \pi - \pi N}(\Delta - \gamma \Delta) \) (3.6) and \( \langle E_L^{3/2} \rangle_{\gamma' \pi' \pi - \pi N}(\Delta \Delta) \) (A.18a) contain different \( \Delta - \gamma \Delta \) vertices. In order to unify these vertex functions we extract from \( (E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta \Delta) \) (3.15d) the part with the \( \Delta - \gamma \Delta \) vertex from (3.6)

\[
(E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta \Delta) = (E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta - \gamma \Delta) + (E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta - \gamma \Delta) 
\]

\[
(B_{\pi' \pi' \pi - \pi N}(\Delta \Delta) = (B_{\pi' \pi' \pi - \pi N}(\Delta - \gamma \Delta) + (B_{\pi' \pi' \pi - \pi N}(\Delta - \gamma \Delta).
\tag{3.19b}
\]

where \( (E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta - \gamma \Delta) \) and \( (B_{\pi' \pi' \pi - \pi N}(\Delta - \gamma \Delta) \) are defined by (3.3a,b) and

\[
(E_L^{3/2})_{\gamma' \pi' \pi - \pi N}(\Delta) = \frac{\langle P_N', P'_\pi | g_{\pi' \pi' - \Delta} | P'_\Delta >}{p_{\pi'}^0 + p_N'^0 - P_N'^0(s')}
\]

\[
\left\{ \bar{\psi}(P'_\Delta) g_{bc} \left[ (P + P' - P_\Delta - P'_\Delta)^\mu V_E - i\sigma^{\mu \nu} (k' - k'_\Delta)^\nu V_H \right] u^c(P_\Delta) \right\} \frac{\langle P \Delta | g_{\Delta - \pi N} | P_N, P_\pi >}{p_\pi^0 + p_N^0 - P_N^0(s)}.
\tag{3.6}
\]

\[
B_{\pi' \pi' \pi - \pi N}(\Delta) = \frac{\langle P_N', P'_\pi | g_{\pi' \pi' - \Delta} | P'_\Delta >}{p_{\pi'}^0 + p_N'^0 - P_N'^0(s')}
\]

\[
\left\{ \bar{\psi}(P'_\Delta) g_{bc} \left[ k'_o (P + P' - P_\Delta - P'_\Delta)^o V_E - i k'_o (k' - k'_\Delta)^o V_H \right] u^c(P_\Delta) \right\} \frac{\langle P \Delta | g_{\Delta - \pi N} | P_N, P_\pi >}{p_\pi^0 + p_N^0 - P_N^0(s)}.
\tag{3.20b}
\]

29
The difference between \( (\mathcal{E}_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta \Delta) \) (A.18a) and \( (\mathcal{E}_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta - \gamma \Delta) \) (3.3a) makes the zero components of the kinematic factors \((P + P')_{\mu} - (P_\Delta + P_\Delta')_{\mu} = \delta \nu \left[ P_\rho + P_\rho' - P_\Delta - P_\Delta' \right] \) cancel out one of the \( \Delta \) propagators \(1/(P_\rho^\nu - P_\rho'^\nu)\) or \(1/(P_\Delta - P_\Delta')\) in \( (\mathcal{E}_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta) \) (3.20c). Therefore \( (\mathcal{E}_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta) \) (A.20b) corresponds to the one \( \Delta \) exchange term.

Modification of \( (\mathcal{E}_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta \Delta) \) (3.3a) and \( B_{\pi'/N'-\pi N}(\Delta \Delta) \) (3.3b) according to (3.19a,b) generates two new Ward-Takahashi identities: identity (3.2) for the amplitudes (3.3a,b) and \( k_{\mu}^\prime (E_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta) = -B_{\pi'/N'-\pi N}(\Delta) \) for the amplitudes (A.20a,b).

An additional expression like \( (E_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta) \) (A.20a) was also used in other papers [8, 19] in order to ensure current conservation for the total \( \pi N \) bremsstrahlung amplitude. Unlike the case in those papers, \( (E_L^{3/2})_{\gamma'}^{\mu N'-\pi N}(\Delta) \) corresponds to the one-\( \Delta \) exchange amplitude which satisfies the independent Ward-Takahashi identity.

**Appendix B: \( \gamma' \Delta' - \Delta \) vertex function with on mass shell \( \Delta \)'s**

The \( \Delta - \gamma \Delta \) vertices can be constructed using the analytical continuation of the spin 3/2 particle electromagnetic vertex function \( < \text{out}; P' | J^\mu(0) | P; \text{in} > \) in the complex region. The electromagnetic vertex of the spin 3/2 particles with the real mass \( m_{3/2} \) is

\[
< \text{out}; P' | J^\mu(0) | P; \text{in} > = (P + P')_{\mu} \left( \pi' (P') \left[ g^{\rho\sigma} G_1(k'^2) + \frac{k'^\rho k'^\sigma}{m_{3/2}^2} G_3(k'^2) \right] \right) u^\rho(P)
\]

\[
+ \pi' (P') \left( -i \sigma^{\mu\nu} k'^\nu \left[ g^{\rho\sigma} G_2(k'^2) + \frac{k'^\rho k'^\sigma}{m_{3/2}^2} G_4(k'^2) \right] \right) u^\rho(P), \quad (B.1)
\]

where \( k'^\mu = (P - P')_{\mu} \) denotes the four momentum of the emitted photon, \( P = P_N + P_\pi = P_\Delta \), \( P'^\mu = \sqrt{m_{3/2}^2 + P'^2} \); \( P' = P'_N + P'_\pi = P'_{\Delta} \), \( P'^\rho = \sqrt{m_{3/2}^2 + P'^2} \) are the four moments of spin 3/2 particles with a mass \( m_{3/2} \) in the initial and final states.

Expression (B.1) can be analytically continued for the unequal masses of the particles in the “in” \( (m_{3/2}(\text{in}) = m_{3/2}) \) and in the ”out” \( (m_{3/2}(\text{out}) = m'_{3/2}) \) states

\[
< \text{out}; P' m'_{3/2} | J^\mu(0) | P m_{3/2}; \text{in} > = (P + P')_{\mu} \left( \pi' (P') \left[ g^{\rho\sigma} G_1(k'^2, m_{3/2}^2, m'_{3/2}^2) + \frac{k'^\rho k'^\sigma}{m_{3/2}^2 m'_{3/2}^2} G_3(k'^2, m_{3/2}^2, m'_{3/2}^2) \right] \right) u^\rho(P)
\]

\[
+ \pi' (P') \left( -i \sigma^{\mu\nu} k'^\nu \left[ g^{\rho\sigma} G_2(k'^2, m_{3/2}^2, m'_{3/2}^2) + \frac{k'^\rho k'^\sigma}{m_{3/2}^2 m'_{3/2}^2} G_4(k'^2, m_{3/2}^2, m'_{3/2}^2) \right] \right) u^\rho(P), \quad (B.2)
\]

where for \( m'_{3/2} = m_{3/2} \) expression (B.2) coincides with (B.1).

The extension of (B.2) in the complex energy and mass region of the \( \Delta \) resonance implies the replacements \( m_{3/2} \rightarrow m_\Delta(s) \) and \( m'_{3/2} \rightarrow m_\Delta(s') \), where \( m_\Delta \) is given in
(3.9b). Correspondingly, we obtain the complex energies (3.9c,d) and the complex zero component of the four-vector of the four momentum transfer $k' = k'_\Delta$, $k'_o \rightarrow (k'_\Delta)_o = \sqrt{m^2(s) + P^2_\Delta} - \sqrt{m^2(s') + P'^2_\Delta}$.

The general double $\Delta$ exchange term $I^\mu_{\gamma'\pi'N-N\gamma\Delta}(\Delta - \gamma\Delta)$ (3.18) contains the following full $\Delta - \gamma'\Delta'$ vertex function $< P'_{\Delta}, s'|J_\mu(0)|P_{\Delta}, s >$ with on mass shell $\Delta$'s

$$< P'_{\Delta}, m_{\Delta}(s')|J^\mu(0)|P_{\Delta}, m_{\Delta}(s) > = (P_{\Delta} + P'_{\Delta})^\mu \left( \bar{\tau}'(P'_{\Delta}) [g_{\rho\sigma} G_1(k'^2_{\Delta}, s, s') + \frac{k'_{\Delta}\sigma k'_{\Delta}\rho}{M^2_{\Delta}} G_3(k'^2_{\Delta}, s, s')] u^\rho(P_{\Delta}) \right)$$

$$+ \bar{\tau}'(P'_{\Delta}) \left( -i \sigma^{\mu\nu} k'_{\Delta\nu} [g_{\rho\sigma} G_2(k'^2_{\Delta}, s, s') + \frac{k'_{\Delta}\sigma k'_{\Delta}\rho}{M^2_{\Delta}} G_4(k'^2_{\Delta}, s, s')] \right) u^\rho(P_{\Delta}),$$

(B.3)

where we introduced the auxiliary four-vector $k'_\Delta = (P_{\Delta} - P'_{\Delta})_\mu$ for the momentum transfer and $g_{\mu\nu}$ is the metric tensor. An additional dependence of the form factors $G_i(k'^2_{\Delta}, s, s')$ on the variables $s$ and $s'$ is generated by $m_{\Delta}(s)$ and $m_{\Delta}(s')$ (3.4d).

The form factors $G_i(k'^2_{\Delta}, s, s')$ are simply connected with the charge monopole $G_{C_0}$, magnetic dipole $G_{M_1}$, electric quadrupole $G_{E_2}$ and magnetic octupole $G_{M_3}$ form factors of the $\Delta$ resonance.

The terms $\sim k'^2_{\Delta}/4M^2_{\Delta}$ in the $\Delta - \gamma'\Delta'$ vertex for the low energy photons can be neglected and only terms $\sim 1/M_{\Delta}$ can be taken into account. Then (B.3) reduces to (3.7).

An important property of the electromagnetic $\Delta$ vertices (B.3) and (3.7) is that at the threshold ($k' = 0$ and $k'_\Delta = 0$) they coincide with $G_i(0, m^2_{\frac{1}{2}}, m^2_{\frac{3}{2}})$ in (B.2). But the form factors $G_i(k'^2, m^2_{\frac{1}{2}}, m^2_{\frac{3}{2}})$ in (B.2) are real according to the $C, P, T$ invariance. Consequently, the form factors $G_i(k'^2_{\Delta}, s, s')$ at the threshold are also real. Therefore, $G_{C_0}(k'^2_{\Delta}, s, s')$ and $G_{M_1}(k'^2_{\Delta}, s, s')$ satisfy the following normalization conditions

$$G_{C_0}(k'^2_{\Delta} = 0, s, s) = \epsilon_{\Delta}; \quad G_{M_1}(k'^2_{\Delta} = 0, s, s) = \mu_{\Delta},$$

(B.4a)

where $\epsilon_{\Delta}$ and $\mu_{\Delta}$ denote the charge and magnetic dipole moment of the $\Delta$'s.

The exact form of vertex functions (B.3a,b) ensure the validity of the special one-body current conservation condition for the $\Delta$ vertex function

$$k'^\mu_{\Delta} < P'_{\Delta}, m_{\Delta}(s')|J_\mu(0)|P_{\Delta}, m_{\Delta}(s) > =$$

$$\left\{ \begin{array}{ll}
0 & \text{for constant } m_{\Delta} \text{ (3.4c)} \\
\frac{m^2(s) - m^2(s')}{2m_{\Delta}} G_{C_0}(k'^2_{\Delta}, s, s')\bar{\tau}'(P'_{\Delta}) g_{\rho\sigma} u^\rho(P_{\Delta}) & \text{for } m_{\Delta}(s) \text{ (3.4d)}
\end{array} \right.$$

(B.4b)

\footnote{Another double $\Delta$ exchange term contains the $\Delta - \pi'\Delta'$ vertex function. But this term is negligible small\cite{8}.}

\footnote{Other choices of $G_i$ form factors are considered in ref. \cite{35}}
Equation (B.4b) expresses the analytical continuation of usual current conservation for the real spin 3/2 particle vertex function in the complex energy-mass region of the Δ’s. Certainly, this “one-body intermediate Δ current” conservation is not necessary for real current conservation of the full πN radiation amplitude.

It must be emphasized that the present formulation of the Δ degrees of freedom does not use a Heisenberg local field operator of the Δ resonance or a Lagrangian with the local Δ field operators. This simplifies the formulation because it is not possible to construct a Fock space for a “free” resonance state with a complex mass and a complex energy. A renormalization procedure for real spin 3/2 states can generate intermediate Δ complex states. This renormalization is equivalent to the extension of the vertex functions (B.1) or (B.2) into the complex Δ vertex (B.3). In the present approach we use the vertices only with the on mass shell Δ’s. Therefore, ambiguities generated by unphysical gauge transformations of the Δ-particle field operator \( \Psi^a_\Delta \rightarrow \Psi^a_\Delta + C \gamma^a \gamma_b \Psi^b_\Delta \) [40] with an arbitrary parameter \( C \) do not appear in the present formulation. Sensitivity of the \( \gamma p \rightarrow \gamma' \pi' p' \) observable to the choice of the form of the intermediate Δ propagator is demonstrated in [32].

In the off mass shell region, where \( P_{\Delta}^2 \neq m_{\Delta}^2 \) and \( P'_{\Delta}^2 \neq m_{\Delta}^2 \), the Δ − γ′Δ′ vertex is a function of two independent four momenta of each Δ. Therefore, for the off mass shell Δ’s (B.3) and (3.7) take a much more complicated forms with the increasing number of the form factors \( G_i \), because each of the conditions \( P_{\Delta}^2 \neq m_{\Delta}^2 \) and \( (i \gamma_\sigma P^2_\Delta \neq m_{\Delta}^2) \) reduplicates the number of the form factors. Therefore, instead of two form factors in (3.7) we get 8 form factors for the off mass shell Δ − γ′Δ′ vertices. The role of these six additional form factors is as important as the contribution of the off shell effects like the mass and charge renormalization. In addition, these form factors of the Δ − γ′Δ′ vertex with off mass shell Δ’s depend on three complex variables \( k_{\Delta}^2, P_{\Delta}^2 \) and \( P'_{\Delta}^2 \). Therefore, the use of the off mass shell Δ propagators together with the on mass shell Δ − γ′Δ′, as is done in refs. [33, 34, 35], is inconsistent.

Appendix C: On mass shell Δ degrees of freedom and construction of the double Δ exchange term in Fig. 2B.

The on mass shell intermediate Δ states are usually introduced via the Δ resonance pole position in the πN amplitude. We shall shortly consider the corresponding formulation within the time-ordered field-theoretical approach [31, 32, 30]. In this formulation the off mass shell πN amplitudes (2.9a,b,c,d) are simply connected with the πN \( t \)-matrix \( T(p_{\pi}p_{\Delta}^s, p_{\pi}p_{\Delta}^t; E) \), which satisfies the relativistic Lippmann-Schwinger-type equation in the c.m. frame

\[
T(p', p; E_p) = U(p', p; E_p) - \int U(p', q; E_p) \frac{d^3q}{E_p - E_q - i\epsilon} T(q, p; E_p), \tag{C.1}
\]

where \( E_p \equiv p_o = \sqrt{p^2 + m_{\pi}^2} + \sqrt{p'^2 + m_{\Delta}^2} \) and \( p \) are the πN energy and the relative three-momentum in c.m. frame. Equation (C.1) can be symbolically represented as
\[ T(E_p) = U(E_p) + U(E_p)G_0(E_p)T(E_p) = U(E_p) + U(E_p)G_{\pi N}(E_p)U(E_p), \quad (C.2) \]

where \( G_0(E_p) \) and \( G_{\pi N}(E_p) \) are the free and total Green functions of the \( \pi N \) system and \( U(E) \equiv U(p', p; E) = A(p', p) + EB(p', p) \) is the linear energy depending on the field-theoretical potential with a Hermitian \( A(p', p) \) and \( B(p', p) \) matrices. The full \( \pi N \) Green function satisfies the completeness condition

\[ G_{\pi N}(E) = \sum_{\pi N} |\Psi_{\pi N}(q)|^2 d^3q < \Psi_{\pi N}(q)|(1 - B)u(E - E_{\pi N} - i\epsilon), \quad (C.3) \]

where \( \Psi_{\pi N}(q) \) denotes the \( \pi N \) wave function which can be determined via the solution of (C.1).

The \( \Delta \) resonance pole in the complex energy region generates the following representation of the full \( \pi N \) wave function

\[ G_{\pi N}(E) = \sum_{\Delta} |\Psi_{\Delta} > < \Psi_{\Delta}|(1 - B)u(E - E_{\Delta}) + \text{non-pole part}, \quad (C.4a) \]

where \( m_{\Delta} = 1232 MeV - \frac{1}{2}120 MeV \) and \( E_{\Delta} \equiv p_{\Delta}^2 = \sqrt{m_{\Delta}^2 + P^2} \) according to (3.9a) and (3.9c). \( m_{\Delta} \) indicates the \( \Delta \)-resonance pole position of the full \( \pi N \) Green function or the total \( \pi N \) amplitude.

Using (C.4) and (C.2) one can extract the \( \Delta \) exchange part of the \( \pi N \) t-matrix

\[ \left[ T(E) \right]_{\text{one } \Delta \text{ exchange part}} = \sum_{\Delta} \frac{U(E)|\Psi_{\Delta} > < \Psi_{\Delta}|(1 - B)U(E)}{E - E_{\Delta}}, \quad (C.4b) \]

This expression can be reproduced in the separable model of the \( \pi N \) t-matrix [31, 32]

\[ T(p', p; E) = \lambda g(p')g(p)(p_{\Delta} - p'_{\pi N}) u^\alpha(p_{\Delta})\pi^{\beta}(p_{\Delta}) d_{\Delta}(E) \quad (C.4c) \]

where

\[ d_{\Delta}(E) = 1 - \lambda \int \frac{d^3q}{(2\pi)^3} \frac{m_N}{2E_{q_{\pi N}}E_{q_N}} \frac{q^2g^2(q)}{E + i\epsilon - E_{q_{\pi N}} - E_{q_N}} \quad (C.4d) \]

in the usual separable potential model and

\[ d_{\Delta}(E) = E - E_{\Delta}(\text{bare}) - \Sigma_{\pi N}(E) \quad (C.4e) \]

in the more complicated microscopic models with the bare energy \( E_{\Delta}(\text{bare}) \) and the mass operator of the \( \Delta \) resonance \( \Sigma_{\pi N}(E) \).

Using the normalization condition [19], we get

\[ d_{\Delta}(E = \sqrt{m_{\Delta}^2 + P_{\Delta}^2}) = 0. \quad (C.4f) \]
Equations (C.4c) and (C.4d) can be represented in the form of (C.4b) with the corresponding redefinition of the form factors of the ∆ resonances

$$|g_\Delta(E) > \equiv \left( \frac{d_\Delta(E)}{E - E_\Delta} \right)^{\frac{1}{2}} U(E)|\Psi_\Delta > \quad \text{(C.4g)}$$

In this way the expression $E - E_\Delta(s)$ can be replaced by the propagator $d_\Delta(E)$ which is constructed in the separable model.

$$E - E_\Delta(s) \mapsto d_\Delta(E). \quad \text{(C.4h)}$$

This allows to separate the ∆ pole and non-pole parts of the πN amplitude in accordance with the (A.9a,b,c,d).

The double ∆ exchange term with the $\Delta - \gamma'\Delta'$ vertex (3.21c) (Fig. 2B) can be extracted from the πN bremsstrahlung amplitude $< \text{out: } \mathbf{p}'_N\mathbf{p}'_\pi | \mathcal{J}^\mu(0) | \mathbf{p}_\pi\mathbf{p}_N; \text{in} >$ (2.6) in the same way as in our previous papers [31, 32, 30]. Thus the s-channel part of the full πN bremsstrahlung amplitude with the double πN intermediate states is

$$< \text{out: } \mathbf{p}'_N\mathbf{p}'_\pi | \mathcal{J}^\mu(0) | \mathbf{p}_\pi\mathbf{p}_N; \text{in} > \mapsto \sum_{\pi''''N'', \pi'''N'''} \int d^4x' e^{i\mathbf{p}'_\pi x' - i\mathbf{p}_N x} < \text{out: } \mathbf{p}'_N | j_\pi(x') > |\pi'''N'''; \text{out} >$$

$$\theta(x'_o < \text{out: } \pi''''N'' | \mathcal{J}^\mu(0) | \pi'''N'''; \text{in} > \theta(-x'_o) < \text{in: } \pi''N'' | j_\pi(x) | \mathbf{p}_N; \text{in} > \quad \text{(C.5a)}$$

which after integration is transformed to

$$< \text{out: } \mathbf{p}'_N\mathbf{p}'_\pi | \mathcal{J}^\mu(0) | \mathbf{p}_\pi\mathbf{p}_N; \text{in} > \mapsto (2\pi)^6 \sum_{\pi''''N'', \pi'''N'''} \frac{< \text{out: } \mathbf{p}'_N | U(E_{\mathbf{p}'}) | \Psi_{\pi''''N'''}(\mathbf{p}'') >}{p''_N + p''_\pi - p'''_N + p'''_\pi - i\epsilon}$$

$$< \text{out: } \pi''''N'' | \mathcal{J}^\mu(0) | \pi'''N'''; \text{in} > \frac{< \Psi_{\pi''''N''}(\mathbf{p}'') | U(E_{\mathbf{p}'}) | \mathbf{p}_N; \text{in} >}{p''_N + p''_\pi - p'''_N + p'''_\pi - i\epsilon} \quad \text{(C.5b)}$$

where we used a connection between the πN amplitude and the πN wave function [31, 30]

$$< \text{out: } \mathbf{p}'_N\mathbf{p}'_\pi | j_\pi(0) | \mathbf{p}_N; \text{in} > =< \text{out: } \mathbf{p}'_N | U(E_{\mathbf{p}'}) | \Psi_{\pi_N}(\mathbf{p}) >, \quad \text{(C.6)}$$

Next we separate the πN irreducible part $< \text{out: } \pi''''N'' | \mathcal{J}^\mu(0) | \pi'''N'''; \text{in} >$ of the full πN bremsstrahlung amplitude $< \text{out: } \pi'N' | \mathcal{J}^\mu(0) | \pi N; \text{in} >$ as

$$< \text{out: } \pi'N' | \mathcal{J}^\mu(0) | \pi N; \text{in} > = \sum_{\pi''''N'', \pi'''N'''} < \Psi_{\pi'N'} | (1 - B) | \pi'''N'''; \text{out} >$$

$$< \text{out: } \pi''''N'' | \mathcal{J}^\mu(0) | \pi'''N'''; \text{in} > $$

$$\left[< \text{out: } \pi''''N'' | \mathcal{J}^\mu(0) | \pi'''N'''; \text{in} > \right]_{\pi N \text{ irreducible}} < \text{in: } \pi''N'' | (1 - B) | \Psi_{\pi_N} >, \quad \text{(C.7)}$$

Substituting (C.7) into (C.5b) and using (C.3) and (C.4a) we get (3.6), where
\[< P_\Delta', m_\Delta(s') | J^\mu(0) | P_\Delta, m_\Delta(s) > = \sum_{\pi''N''} < \Psi_\Delta' | (1 - B) | \pi''N''; \text{out} > \]

\[\left[ < \text{out}; \pi''N'' | J^\mu(0) | \pi''N''; \text{in} > \right]_{i \pi N \text{ irreducible}} < \text{in}; \pi''N'' | (1 - B) | \Psi_\Delta >, \quad (C.8)\]

and

\[< p_\pi' g_{\pi'N'-\Delta'} | P_\Delta' > = < p_\pi' | U(E_{p'}) | \Psi_{\Delta'}(P') >, \quad (C.9a)\]

\[< P_\Delta | g_{\Delta-\pi N} | p_N, p_\pi > = < \Psi_\Delta(\mathbf{P}) | U(E_p) | p_N, p_\pi > \quad (C.9b)\]

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