TOPICAL REVIEW

Gravitational self-force in extreme mass-ratio inspirals

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Abstract

This review is concerned with the gravitational self-force acting on a mass particle in orbit around a large black hole. Renewed interest in this old problem is driven by the prospects of detecting gravitational waves from strongly gravitating binaries with extreme mass ratios. We begin here with a summary of recent advances in the theory of gravitational self-interaction in curved spacetime, and proceed to survey some of the ideas and computational strategies devised for implementing this theory in the case of a particle orbiting a Kerr black hole. We review in detail two of these methods: (i) the standard mode-sum method, in which the metric perturbation is regularized mode-by-mode in a multipole decomposition, and (ii) m-mode regularization, whereby individual azimuthal modes of the metric perturbation are regularized in 2+1 dimensions. We discuss several practical issues that arise, including the choice of gauge, the numerical representation of the particle singularity, and how high-frequency contributions near the particle are dealt with in frequency-domain calculations.

As an example of a full end-to-end implementation of the mode-sum method, we discuss the computation of the gravitational self-force for eccentric geodesic orbits in Schwarzschild, using a direct integration of the Lorenz-gauge perturbation equations in the time domain. With the computational framework now in place, researchers have recently turned to explore the physical consequences of the gravitational self-force; we will describe some preliminary results in this area. An appendix to this review presents, for the first time, a detailed derivation of the 'regularization parameters' necessary for implementing the mode-sum method in Kerr spacetime.

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1. Introduction

Context and motivation. Within Newtonian theory, the gravitational two-body problem, in its basic form, is readily solvable. An isolated system of two gravitationally-bound point masses admits two conserved integrals—the energy and angular momentum—and the resulting
motion is precisely periodic. The corresponding general-relativistic problem is radically—and notoriously—more difficult. In General Relativity (GR), the orbits in a bound binary are never periodic: gravitational radiation continually removes energy and angular momentum from the system, and the radiation back-reaction gradually drives the two objects tighter together until they eventually merge. Furthermore, in GR one cannot usually work consistently with pointlike mass particles (as we discuss later), and the simplest and most universal two-body problem becomes that of a binary black hole system.

In general, the description of the nonlinear radiative dynamics of a black hole binary entails a full Numerical-Relativistic (NR) treatment. There has been a much celebrated advance in NR research in the past few years \[9, 45, 138\], with NR codes now capable of tracking the complicated nonlinear evolution of a binary black hole spacetime during the final stages of the merger. However, NR methods become less efficient when the two black holes are far apart, or when one of the components is much heavier than the other. Each of these two regimes features two greatly different lengthscales, which is not easily accommodated in a NR framework \[76\]. Fortunately, the occurrence of two separate lengthscales also restores a sense of ‘pointlikeness’ (at least approximately), which allows for simpler, perturbative treatments.

In the first regime—at sufficiently large separations—the dynamics is best analyzed using post-Newtonian (PN) methods \[36\], wherein GR corrections to the Newtonian dynamics (accounting for radiation reaction, the objects’ internal structure, etc) are incorporated into the equations of motion order by order in the binary separation.

The second, so-called extreme mass-ratio regime—which will concern us in this review—is most naturally explored within the framework of black hole perturbation theory. Here the ‘zeroth-order’ configuration is that of a test particle (the lighter black hole) moving along a geodesic of the fixed background spacetime of the large black hole. This can then serve as a basis for a systematic expansion, wherein corrections due to the finite mass of the small object (and due possibly also to its internal structure) are included order by order in the small mass ratio. At first order in the mass ratio, the gravitational field of the small object is a linear perturbation of the background black hole geometry. The back-reaction from this perturbation gives rise to a gravitational self-force (SF) which gradually diverts the small object from its geodesic motion. In this picture, it is the SF that is responsible (in particular) for the radiative decay of the orbit. In principle, knowledge of the SF (along with the metric perturbation) forms a complete picture of the orbital dynamics at linear order in the small mass ratio. One therefore hopes that a calculation of the SF would facilitate a faithful, if only approximate description of the orbital dynamics in binary systems with small mass ratios.

How to determine the SF in curved spacetime is an old problem in mathematical relativity, made very relevant following recent developments in gravitational-wave research. Interest in this problem was renewed in the mid-1990s, when it was first proposed that the planned space-based gravitational wave detector LISA (the Laser Interferometer Space Antenna \[2\]) could observe signals from the inspiral of compact objects (white dwarfs, neutron stars, or stellar-mass black holes) into massive black holes in galactic nuclei. It is now believed that LISA should be able to detect tens to thousands such events \[70\], out to cosmological distances \((z \sim 1)\) \[68\]—depending on the astrophysical rates (inspirals per galaxy per year), which are still highly uncertain\(^1\). Dubbed EMRIs (extreme mass ratio inspirals)—these potential sources became key targets for LISA due to their unique facility as precision probes of strong-field gravity. A typical LISA EMRI will spend the last few years of inspiral in a very tight orbit around the massive hole, emitting some \(10^5–10^6\) gravitational wavecycles wholly within the

\(^{1}\) In fact, if inspiral rates are near the higher end of their estimated range, the unresolvable stochastic background from thousands weak inspirals could dominate the noise budget of LISA at its most sensitive frequency band \[15\].
LISA frequency band. These complicated waveforms carry extremely accurate information about the physical parameters of the inspiral system, as well as a detailed map of the spacetime geometry around the massive hole. The potential scientific implications are broad and far-reaching, ranging from astrophysics to cosmology and to fundamental theory [3, 14, 16, 50, 71, 139, 148].

The odds are that even the strongest detected EMRI signals will be buried deeply inside LISA’s instrumental noise. However, since an EMRI signal is detectable over a long time, it can be extracted with significant signal-to-noise ratio (of order $\sim 100$ for the strongest sources) using matched filtering [14]. This, however, requires precise theoretical templates of EMRI waveforms across the relevant parameter space of inspirals—indeed, the detection statistics cited above assumes that such templates were at hand by the time LISA flies. It is likely that other, non-template-based data analysis techniques could be used to detect some of the brightest EMRIs [8, 72]. Nonetheless, it remains true that an accurate parameter extraction, crucial for exploiting the full scientific value of the EMRI signal, will rely on (or be restricted by) the availability of accurate and faithful theoretical templates of the inspiral waveforms.

The theoretical challenge is derived from the astrophysical specifications of the LISA-relevant EMRIs. With its peak sensitivity at a few mHz, LISA will observe inspirals into galactic (likely Kerr) black holes with masses in the range $\sim 5 \times 10^5$–$5 \times 10^7 M_\odot$; the inspiraling objects (compact stars or stellar-mass black holes) may have masses in the range $0.5$–$50 M_\odot$, giving mass ratios of $10^{-5}$–$10^{-8}$—well within the ‘extreme mass ratio’ domain of the two-body problem. Inspirals are not expected to have any preferred orientation with respect to the central hole’s spin direction, and orbits may remain quite eccentric throughout the inspiral [14]. It is not likely that interaction with a possible accretion disk around the massive hole would play a dominant role in the orbital dynamics [31]. In a typical LISA-band EMRI—a $10 M_\odot/10^6 M_\odot$ system—gravitational radiation reaction drives the orbital decay over a timescale of months. More importantly, it affects the phasing of the inspiral orbit over mere hours. It is therefore clear that a useful model of the long-term orbital phase evolution in a LISA-relevant system ought to incorporate properly the effect of radiation reaction.

The above translates, at first approximation, to a very clean problem in black hole perturbation theory: a point mass is set in a generic (eccentric, inclined) strong-field orbit around a Kerr black hole of a much larger mass, and one wishes to calculate the gravitational waveforms emitted as radiation reaction drives the gradual inspiral up until the eventual plunge through the event horizon. While it remains important to quantify the effect of higher-order corrections to this picture (e.g. due to the spin of the inspiralling object), one expects that, thanks to the extreme mass ratio, the above simple setup should provide a good model for astrophysical inspirals. Indeed, from the perspective of GR theorists working in the field, much of the appeal of the SF problem comes from its unusual dual nature as both an elementary theoretical problem in GR and an exciting problem in contemporary astrophysics.

Scope and relation to other reviews. Our main aim here is to review the challenges and main developments in the program to calculate the gravitational SF in Kerr spacetime. Our discussion will be focused mostly on work concerned with the evaluation of the SF along a specified, non-evolving orbit (normally taken to be a geodesic of the background spacetime); we will not consider here the important question of how orbits evolve under the effect of the SF. The strategic approach envisaged here is one in which the complete analysis of the orbital evolution is carried out in two separate, consequent steps. In the first, preparatory step, one calculates the SF across the entire relevant phase space (i.e. obtain the value of the SF as a function of location and velocity, perhaps through interpolation of numerical results). In the
second step one then uses the SF information to calculate the inspiral orbits of particles with given initial conditions. Here we will be concerned only with the first, most crucial step.

In parallel to the work on SFs described in this review, there is already a substantial research effort aimed to formulate a reliable scheme for the orbital evolution, assuming that the SF has been calculated [87, 137]. This work incorporates techniques from multiple-scale perturbation theory. Another parallel effort aims to obtain an approximate model of the orbital evolution and emitted waveforms without resorting to the local SF [63, 64, 73, 92, 113, 146]. This work is largely based on a strategy proposed by Mino [110–112, 145], in which a time-average measure of the rate of change of the geodesic ‘constants of motion’ is calculated from a certain ‘radiative’ solution of the perturbation equations (to be described later in this review), which accounts for the long-term radiative aspects of the dynamics but neglects some of the conservative effects. This ‘inspiral without SF’ approach is reviewed by Mino in [111], and also by Tanaka in [154]. It is well possible that this method will prove sufficiently accurate for LISA applications. However, ultimately, its performance and accuracy could only be assessed against actual calculations of the full SF.

The fundamental formulation of the SF in curved spacetime is described in a comprehensive Living Review article by Poisson [131]. This is an excellent treatise which is both self-contained and pedagogical, and it makes an essential reading for anyone wishing to introduce oneself to the field. (The less endurant reader would find a slightly abridged version of this review in [132]; there is also a concise introduction in [133].) It would not be useful to review here at any great detail the basic SF theory already covered in [131]. Instead, we merely give a succinct summary of the essential formal results, and move on to describe, in the rest of our review, how the general formalism is implemented in actual calculations of the gravitational SF in Kerr. In this respect, our review starts where Poisson’s review ends. We will, however, briefly survey a few recent developments in SF theory not covered in [131] (last updated in 2004).

A good 2005 snapshot of the activity surrounding SF calculations is offered by a collection of review articles published in a special issue of Classical and Quantum Gravity [104]. Drasco’s review from 2006 [62] explores the utility of the various computation methods within the context of the LISA EMRI problem. The website of the 12th Capra Meeting on Radiation Reaction in Relativity (Bloomington, IN, 2009) [1] is an excellent resource of information on current research in the field, with links to all talks given in the meeting. The website also includes links to the websites of previous meetings in the series (1998–2008). There are several good reviews of EMRI astrophysics and the science potential of EMRI detections, including ones by Hopman [89], Amaro-Seoane et al [3] and Miller et al [109]. Finally, an abridged version of some of the material in our current review is to appear in [12].

Structure. The overall structure of our presentation will follow the logical route of development in SF research: from basic theory (section 2) through to practical calculation schemes (sections 3–4) and to numerical implementation in Schwarzschild (sections 5–6) and Kerr (section 7), concluding with a discussion of some physical consequences (section 8).

We begin in section 2 with a brief introduction to the general theory of the gravitational SF in curved spacetime, highlighting from the outset the role of gauge dependence in this theory. We then summarize (rather than reproduce) the main theoretical developments that over the past 12 years helped establish a robust formal framework for SF calculations. Section 3 is a brief survey of the main strategies that have been proposed for implementing this formal framework in actual calculations, particularly for orbits in Kerr (or Schwarzschild) spacetimes. This section also provides a quick-reference catalog (tables 1–3) of actual computations of the SF carried out so far. Section 4 is a self-contained introduction to
the mode-sum method, one of the leading techniques for SF calculations. The basic idea is presented through an elementary example, followed by a formulation of the method as applied to generic orbits in Kerr. In an accompanying appendix we provide, for the first time, a full derivation of the regularization parameters necessary for implementing the mode sum scheme in Kerr. (The values of these parameters were published in the past [24] without a detailed derivation.)

An essential preliminary step in almost all calculations of the SF involves the numerical integration of the relevant perturbation equations sourced by the point particle. In section 5 we discuss the practicalities of such calculations, and review some of the numerical strategies that were proposed for dealing with the particle singularity in both the frequency and time domains. Section 6 then focuses on a particular implementation strategy, based on a direct time-domain integration of the Lorenz-gauge metric perturbation equations. This approach recently led to a first computation of the gravitational SF for eccentric orbits in Schwarzschild, and we show some results from this calculation. In section 7 we discuss higher-dimensional alternatives to standard mode-sum, focusing on the recently proposed $m$-mode regularization, which may offer a more efficient treatment in the Kerr case. Section 8 reviews initial work aimed to understand and quantify the physical, gauge-invariant effects of the gravitational SF. Finally, in section 9 we reflect on recent advances and comment on future directions.

Notation. We follow here the notation conventions of Misner, Thorne and Wheeler [117]. Hence, the metric signature is $(-+++)$, the connection coefficients and Riemann tensor are $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$ and $R^{\alpha}_{\lambda\mu\nu} = \Gamma^\alpha_{\lambda,\mu\nu} - \Gamma^\alpha_{\lambda,\mu\nu} + \Gamma^\alpha_{\lambda,\nu\mu} - \Gamma^\alpha_{\lambda,\nu\mu}$, the Ricci tensor and scalar are $R^{a}_{b} = R^{\alpha}_{a\alpha b}$ and $R = R^{a}_{a}$, and the Einstein equations read $G^{a}_{b} = R^{a}_{b} - \frac{1}{2} g^{a}_{b} R = 8\pi T^{a}_{b}$. We use Greek letters for spacetime indices, and adopt standard geometrized units (with $c = G = 1$) throughout.

2. Essential theory

2.1. Gravitational forces

The concept of a gravitational force is, of course, fundamentally strange to GR: in a purely gravitational system one expects no acceleration and hence no ‘forces’ in the ordinary sense. Since the gravitational SF is an example of a force of a purely gravitational origin, we begin by explaining what is generally meant by ‘gravitational forces’ in our context.

Consider a smooth region of spacetime with metric $g_{ab}$ (which may be, for example, the stationary vacuum exterior of a Kerr black hole), and a smooth weak gravitational perturbation $h_{ab}$ of that spacetime (which may represent, for example, an incident gravitational wave). Now consider a test particle of mass $\mu$ which is moving freely in the perturbed spacetime. Neglecting SF effects, the particle’s trajectory will be a geodesic of the perturbed spacetime, described, in a given coordinate system $x^\mu$, by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\tau} \frac{dx^\nu}{d\tau} \frac{dx^\tau}{d\tau} = 0,$$

where $\tau$ is an affine parameter along the trajectory and $\Gamma^\mu_{\nu\tau}$ are the connection coefficients associated with the perturbed metric $g + h$.

In some occasions, however, it is practically useful to reinterpret the particle’s motion in terms of a trajectory in the background spacetime $g$. Under this interpretation, the trajectory (in $g$) is no longer geodesic; rather, the particle experiences ‘an external gravitational force’,
which is exerted by the perturbation \( h \). This (fictitious) force is defined through Newton’s second law as

\[
F^\alpha_{\text{grav}} = \mu \left( \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right),
\]

where \( \tau \) is an affine parameter in the background metric \( g \), and \( \Gamma^\alpha_{\mu\nu} \) are the connection coefficients associated with \( g \). Note that, in this non-covariant description, \( h \) and \( \Gamma^\alpha_{\mu\nu} \) are treated as tensor fields in \( g \), and similarly the force \( F^\alpha_{\text{grav}} \) and four-velocity \( u^\alpha \equiv \frac{dx^\alpha}{d\tau} \) are defined as vectors in \( g \). The tensorial indices of all these quantities are raised and lowered using the background metric \( g \).

To express \( F^\alpha_{\text{grav}} \) in terms of the linear perturbation field \( h_{\alpha\beta} \), we use \( \frac{d\tau'}{d\tau} = \left( \frac{d\tau'}{d\tau} \right) \frac{d\tau}{d\tau'} \) in equation (2) and introduce \( \Delta \Gamma^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu} \). By virtue of equation (1) this gives

\[
F^\alpha_{\text{grav}} = -\mu \Delta \Gamma^\alpha_{\mu\nu} u^\mu u^\nu + \zeta u^\alpha,
\]

with \( \zeta \equiv \mu \left( \frac{d\tau'}{d\tau} \right) \left( \frac{d^2 \tau'}{d\tau^2} \right) \). From its definition in equation (2), the force \( F^\alpha_{\text{grav}} \) must be perpendicular to the four-velocity \( u^\alpha \). Hence, projecting \( F^\alpha_{\text{grav}} \) orthogonally to \( u^\alpha \) keeps it unchanged, and we may use this fact to dispose of the term \( \zeta u^\alpha \) in equation (3):

\[
F^\alpha_{\text{grav}} = -\mu \left( g^\alpha_\lambda u^\lambda + u^\alpha u^\lambda \right) \Delta \Gamma^\lambda_{\mu\nu} u^\mu u^\nu.
\]

Finally, expressing \( \Delta \Gamma^\alpha_{\mu\nu} \) in terms of \( h \) (keeping only terms that are linear in \( h \)) we obtain

\[
F^\alpha_{\text{grav}} = -\frac{1}{2} \mu \left( g^{\alpha\lambda} u^\lambda + u^\alpha u^\lambda \right) \nabla_\lambda \nabla_\mu h^\lambda_{\nu\lambda} + \nabla_\lambda h^\lambda_{\mu\nu} u^\alpha u^\lambda - \nabla_\lambda h^\lambda_{\mu\nu} u^\alpha u^\lambda \right) u^\nu \nabla_\nu.
\]

Later we will often work with the trace-reversed metric perturbation,

\[
\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} h_{\mu\nu}.
\]

In terms of \( \bar{h}_{\alpha\beta} \), equation (5) becomes

\[
F^\alpha_{\text{grav}} = \mu \bar{\nabla}^\mu \bar{h}_{\mu\beta} u^\beta,
\]

with

\[
\bar{\nabla}^\mu \bar{h}_{\mu\beta} = \frac{1}{2} \left( 2 g^{\alpha\beta} u^\alpha u^\nu - 4 g^{\alpha\beta} u^\alpha u^\nu - 2 u^\alpha u^\beta u^\lambda u^\lambda + u^\alpha u^\beta u^\lambda u^\lambda + g^{\alpha\beta} g^{\gamma\delta} \right) \nabla_\gamma.
\]

Formally, the above interpretation of the motion requires a suitable procedure for mapping the physical trajectory from the full spacetime \( g + h \) onto the background spacetime \( g \). In the above discussion we adopted the following procedure: first, we assume that \( g + h \) and \( g \) were covered with two coordinate meshes that are ‘similar’ in the sense that, at the limit \( h \to 0 \), any given physical event would attain the same coordinate values in both spacetimes. Then, we identify each event \( x^\alpha \) in \( g + h \) with an event having the same coordinate value \( x^\alpha \) in \( g \). This, in particular, results in a projection of the trajectory in \( g + h \) onto one in \( g \).

Of course, there is not just one way of specifying the coordinate systems in the two spacetimes, which leads to an ambiguity in the projected trajectory: different coordinate choices would, in general, give rise to different projected trajectories in \( g \), showing different accelerations and hence interpreted as being under the influence of different gravitational forces. This, of course, is nothing but an example of the usual gauge ambiguity intrinsic to perturbation theory in GR. The gravitational force, just like the metric perturbation itself, is gauge dependent.
It is straightforward to write down a gauge transformation law for $F_{\text{grav}}^\mu$. Consider a small gauge displacement

$$x^\mu \to x^\mu - \xi^\mu,$$

where the magnitude of $\xi^\mu$ is assumed to scale like that of the external perturbation $h_{\alpha\beta}$.

Under $\xi^\mu$, the perturbation transforms as $h_{\alpha\beta} \to h_{\alpha\beta} + \delta(\xi) h_{\alpha\beta}$, where

$$\delta(\xi) h_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha.$$

From equation (5), this will induce a change in the gravitational force, given by

$$\delta(\xi) F_{\text{grav}}^\alpha = \mu (g_{\alpha\lambda} + u^\alpha u^\lambda) \ddot{\xi}_\lambda + R^{\mu}_{\mu\lambda\nu} u^\mu \xi_\lambda u_\nu,$$

where overdots denote covariant derivatives with respect to the affine parameter $\tau$ along the background trajectory. This is the general gauge transformation formula for external gravitation forces.

2.2. Gravitational self-force

To devise a theory of the gravitational SF, one might be tempted to simply interpret it as an example of a gravitational force of the type discussed above, with the source of the metric perturbation now being the particle itself. This naive interpretation would be problematic, for several reasons. First, the physical perturbation due to the particle (a retarded solution of the linearized Einstein equations, $h_{\text{ret}}^{\alpha\beta}$) is singular at the location of the particle, and the statement that the particle follows a geodesic of $g + h_{\text{ret}}$ is therefore physically meaningless. Obviously, trying to apply equation (5) (or (8)) with the external perturbation replaced with the self-perturbation $h_{\text{ret}}^{\alpha\beta}$ would yield a singular, and hence meaningless result.

Second (and relatedly), since we are now considering the self-gravity of the particle (it is no longer a test particle), we must make a mathematical sense of its being ‘pointlike’. This is not a trivial matter to address in curved spacetime. Mathematically, the usual delta-function representation of a point particle stress–energy is known to be inconsistent with the nonlinearity of the full Einstein equations [74, 161]. A familiar physical manifestation of this is that one cannot squeeze a finite amount of mass to a point without creating a black hole (of a finite size). The mathematical consistency of a delta-function source is restored in the linear theory, but it remains a challenging task to understand how the notion of a point particle might emerge (rather than be pre-assumed) from a suitable limiting procedure.

Third, the SF is conceptually different from the external forces discussed above, in that the latter are, in truth, just fictitious forces resulting from our insistence to artificially split the physical spacetime into a background and a perturbation. The SF, in contrast, must be viewed as a genuine physical effect (even if a delicate one, as the SF too is gauge dependent—see below). There indeed exists an interpretation of the motion—we will discuss it later—wherein the particle moves freely on a geodesic of a certain smooth, perturbed spacetime, subject to

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2 One might note that equation (13) reproduces the geodesic deviation equation if one sets the left-hand side to zero and reinterprets $\xi^\mu$ as the displacement vector connecting two adjacent geodesics in $g$. Indeed, a vanishing $\delta(\xi) F_{\text{grav}}^\alpha$ implies that the two adjacent trajectories (the original projected trajectory and its gauge-transformed counterpart) have the same acceleration in $g$, in which case the displacement vector between them is known to satisfy the geodesic deviation equation.
no SF. However, in this description the smooth geometry is not the physical spacetime of
the background + particle system (the metric of this geometry is not a retarded solution of
the linearized Einstein equations), and so this effective spacetime cannot be said to represent
'physical reality' any better than the SF itself.

Thanks to work commencing around 1997 and continuing to these days, we now have
a rather satisfactory theory of the gravitational SF in curved spacetime, which, in particular,
addresses all of the above issues. Initial work was strongly inspired by the classical analyses of
the analogous electromagnetic self-force problem in flat (Dirac 1938 [61]) and curved (DeWitt
and Brehme 1960 [58]) spacetimes. DeWitt and Brehme's work, especially, provided much
of the mathematical framework—that of covariant bi-tensors in curved spacetime—needed
for analyzing the gravitational problem too. In 1997 two independent groups published three
independent derivations of the gravitational SF. Quinn and Wald [141] used an axiomatic
approach, in which the physical self-acceleration is deduced, essentially, by comparing the
(divergent) self-field of the particle in question with that of a particle in a suitably constructed
tangent flat space. Independently, Mino, Sasaki and Tanaka (MST) [115] obtained the same
result using a local energy–momentum conservation argument—a direct application of DeWitt
and Brehme's method to the gravitational case. Both these methods pre-assume a notion of
a point mass particle, and neither seeks to make a consistent sense of this notion. However,
MST's paper also reported a second, independent derivation of the SF, using an approach
which, for the first time, offered a fully GR-consistent treatment of the problem.

This approach—a new implementation of the old idea of matched asymptotic expansions—is based on the assumption that there can be identified two separate lengthscales in
the problem: one associated with the particle's mass $\mu$, and another, much larger, associated
with the typical radius of curvature of the geometry in which the particle is moving. In the
strong-field EMRI problem, the second lengthscale is provided by the mass of the central
black hole, $M \gg \mu$. MST's construction further assumes that the 'particle' is actually a
Schwarzschild black hole of mass $\mu$. The two separate scales in the setup define a 'near
zone', $r \ll M$, and a 'far zone', $r \gg M$ (where $r$ is a suitable measure of distance from the
small hole). In the near zone, the geometry is approximately that of the small Schwarzschild
hole, with small tidal-type corrections from the background geometry. As we zoom away
from the small object and enter the far zone, the effect of the small object's detailed structure
becomes gradually less important, and at the far zone limit the geometry becomes that of
the background spacetime, weakly perturbed by what is now a distant 'point particle'—it is
indeed the far-zone limit through which a notion of point mass can be defined in a consistent
way. In situations where $M \gg m$ one would have a 'buffer zone' where $m \ll r \ll M$ and
both 'near zone' and 'far zone' descriptions of the geometry are valid. MST showed that
matching the near zone and far zone metrics (expressed as asymptotic expansions in $r/M$
and $m/r$, respectively) constrains the motion of the particle (from a far-zone point of view) and
thus yields an expression for the SF. This expression agreed with those obtained by Quinn
and Wald and by using DeWitt and Brehme's method; it was later coined the MiSaTaQuWa
formula, an acronym based on the names of the five contributing authors. A self-contained
and pedagogical review of the MiSaTaQuWa formula, including an elegant reproduction of
previous derivations, can be found in Poisson's [131].

More recently, Gralla and Wald [77] (see also [78] for a more concise presentation)
developed a new procedure for deriving the gravitational SF, which offers improved
mathematical rigor as well as a generalization of the MiSaTaQuWa formula. Rather than
relying on two separate asymptotic expansions of the metric as in MST's original method,
Gralla and Wald introduce a single one-parameter family of metrics, which, through two
different limiting procedures, can produce both near and far zone metrics in a natural way.
This allows us to define more robustly the criteria for the existence of the two zones, and enables a more elegant buffer-zone matching. The analysis proves that in the far-zone limit the particle is described precisely by the usual delta-function distribution, and that at the very limit $\mu \to 0$ this particle moves on a geodesic of the background. Furthermore, the analysis relaxes all assumptions about the nature of the small object: it no longer need to be a Schwarzschild black hole, but can assume the form of any sufficiently small black hole or a blob of ordinary matter. This allows, in particular, for a spin-force term to appear in the resulting, generalized version of the MiSaTaQuWa formula.

The main end product of the above theoretical developments is a firmly-established general formula for the SF in a class of background spacetimes including Kerr. It should be stressed that the SF formula stems in a deterministic way from nothing else than the Einstein equations with the usual conservation laws; it does not rely—and should not rely, as a matter of principle—on any form of ambiguous ‘regularization’ or ‘subtraction of infinities’. This idea is implemented explicitly in the matched asymptotic expansions approach, and even more so in the Gralla–Wald analysis.

2.3. MiSaTaQuWa equation

We now state the MiSaTaQuWa formula [115, 131, 141]—equation (16) below. (For simplicity we ignore spin-force terms and focus on the self-interaction part of the formula.) This will serve as a starting point for the rest of this review.

Consider a timelike geodesic $\Gamma_1$ in a background spacetime with metric $g_{\alpha\beta}$. For concreteness, let us think of $\Gamma_1$ as a test particle orbit outside a Kerr black hole, so $g_{\alpha\beta}$ is the Kerr metric. Let $\tau$ be the proper time along $\Gamma_1$, and let $x^\alpha = z^\alpha(\tau)$ describe $\Gamma_1$ in some smooth coordinate system and $u^\alpha \equiv dz^\alpha/d\tau$ be the four-velocity of the test particle. Denote by $h_{\alpha\beta}^\text{ret}$ the physical, retarded metric perturbation from a particle of mass $\mu$ whose worldline is $\Gamma_1$. Assume $h_{\alpha\beta}^\text{ret}$ is given in the Lorenz gauge:

\begin{equation}
\nabla^\beta h_{\alpha\beta}^\text{ret} = 0,
\end{equation}

where $h_{\alpha\beta}^\text{ret}$, recall, is the trace-reversed version of $h_{\alpha\beta}^\text{ret}$ (see equation (7)). Remember that throughout our discussion indices are raised and lowered using the background metric $g_{\alpha\beta}$, and covariant derivatives are taken with respect to that metric.

At any spacetime point $x$, the retarded perturbation can be written as a sum of two pieces,

\begin{equation}
h_{\alpha\beta}^\text{ret} = h_{\alpha\beta}^\text{dir} + h_{\alpha\beta}^\text{tail},
\end{equation}

the former being the ‘direct’ contribution coming from the intersection of the past light-cone of $x$ with $\Gamma_1$, and the latter being the ‘tail’ contribution arising from the part of $\Gamma_1$ inside this light cone (see figure 1). The occurrence of a tail term is a well-known feature of the wave equation in 3+1D curved spacetime, and it can be interpreted physically as arising from the effect of waves being scattered off spacetime curvature (‘failure of the Huygens principle’). Both $h_{\alpha\beta}^\text{ret}$ and $h_{\alpha\beta}^\text{dir}$ obviously diverge when evaluated on $\Gamma_1$; however, $h_{\alpha\beta}^\text{tail}$ is continuous and differentiable everywhere, including on the worldline. Notably, though, the tail field is not a smooth function on the worldline, and is not a vacuum solution of the linearized Einstein equations.

The MiSaTaQuWa formula states that the gravitational SF at a given point $z$ along $\Gamma_1$ results simply from the back-reaction of the tail field:

\begin{equation}
F_{\text{self}}(z) = \mu \nabla^\alpha \nabla_\beta h_{\alpha\beta}^\text{tail}(z).
\end{equation}
$z(\tau)$ is a point on the timelike worldline $\Gamma$ (thick solid line) and $x$ is a field point close to $z$, shown with a portion of its past light cone. $\epsilon$ is the spatial geodesic distance from $x$ to $\Gamma$ and $\delta x^\alpha = x^\alpha - z^\alpha$. The metric perturbation at $x$ consists of a direct and a tail contributions, illustrated by the thick dashed and dash-dot lines, respectively. (Graphics reproduced from [12].)

Here $\bar{\nabla}^{\alpha\beta\gamma}$ is the usual ‘force operator’, given in equation (9), which, recall, is dependent upon the four-velocity $u^\alpha$ and the background metric $g_{\alpha\beta}$ at point $z^\beta$.

### 2.4. Detweiler–Whiting reformulation

In a 2003 paper [56] (see also a preliminary discussion in [52]) Detweiler and Whiting (DW) proposed an alternative formulation, which offers an interesting re-interpretation of the perturbed motion. DW replaced the direct/tail decomposition of the retarded perturbation (equation (15)) with a new decomposition,

$$\bar{h}_{\alpha\beta}^{\text{ret}} = \bar{h}_{\alpha\beta}^{\text{S}} + \bar{h}_{\alpha\beta}^{\text{R}},$$

(17)

where the $R$-field $\bar{h}_{\alpha\beta}^{\text{R}}$ (R for Regular), unlike the tail field, is a certain smooth, vacuum solution of the perturbation equations, which, nonetheless, gives rise to the same physical SF as the tail field:

$$F_{\text{self}}(z) = \mu \bar{\nabla}^{\alpha\beta\gamma} \bar{h}_{\beta\gamma}^{\text{R}}(z).$$

(18)

The $S$-field $\bar{h}_{\alpha\beta}^{\text{S}}$ (S for Singular), which precisely mimics the singular behavior of the retarded field near the particle, exerts no SF and does not affect the motion of the particle. The precise formal prescription for constructing the $R$- and $S$-fields is described nicely in Poisson’s review [131].

DW’s discovery that the MiSaTaQuWa SF can also be expressed as the back-reaction force from a smooth vacuum perturbation leads to an interesting re-interpretation of the gravitational SF effect: the particle effectively moves freely along a geodesic of a smooth perturbed spacetime with metric $g_{\alpha\beta} + \bar{h}_{\alpha\beta}^{\text{R}}$. In this alternative picture—more in the spirit of GR’s equivalence principle—the notion of a SF becomes artificial (and obsolete) in much the same way that the notion of an external gravitational force is artificial.

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$^3$ In the original MiSaTaQuWa formulation, the SF is not expressed directly in terms of the gradient of $\bar{h}_{\alpha\beta}^{\text{tail}}$, but rather as an integral over the gradient of the relevant retarded Green’s function along the portion of the worldline to the past of $z$ (cf equation (1.9.6) of Poisson [131]). The commutation of the derivative and the worldline integral produces local terms at $z$, which, however, vanish upon contraction with $\bar{\nabla}^{\alpha\beta\gamma}$. This leads to the equivalent formulation shown here in equation (16).
It should be understood, however, that the $R$-field does not represent the actual physical perturbation from the particle (the physical perturbation field is of course $\bar{h}^{\text{ret}}_{\alpha\beta}$). The $R$-field has peculiar causal properties which make it problematic as a candidate for what we may call a ‘physical field’: the value of the $R$-field at an event $x$ depends not only on events in the causal past of $x$ but also on events outside the light-cone of $x$ [131]. Rather than an entity of physical substance, the $R$-field should be viewed as an effective field that allows us to describe the dynamics in terms of geodesic motion.

The two descriptions of the perturbed motion—self-accelerated motion in $g$ versus geodesic motion in $g + h_R$—are alternative (equivalent) interpretations of the same (genuine) physical effect. The two points of view are not contradictory but rather they are complementary in their perspective on the problem. Workers in the field often find it useful to invoke both descriptions alternately in order to get a fuller picture of the physics in question. Sago et al [147] recently demonstrated the equivalence of the two approaches with an explicit calculation of a certain gauge-invariant physical SF effect in a particular example; we shall return to discuss this work in section 8.

2.5. Singular field

Equations (16) and (18) prescribe the correct regularization of the gravitational SF and form the fundamental basis for all modern SF calculations. In later sections we will discuss practical methods for implementing these formulas. We will then need some more information on the properties of the direct (or singular) field, which we now give.

The fields $\bar{h}^\text{dir}_{\alpha\beta}$ and $\bar{h}^S_{\alpha\beta}$ share the same leading-order singular structure near the particle’s worldline $\Gamma$, but they differ in their sub-dominant singular behavior. More precisely (referring again to figure 1), consider a particular point $z$ on $\Gamma$ and a nearby off-$\Gamma$ field point $x$. The form of both the direct and $S$-fields is then given, in the Lorenz gauge, by [23, 116, 121]

$$\bar{h}^S_{\alpha\beta}(x; z) = \frac{4}{\epsilon(x; z)} \mu \hat{u}^S_{\alpha}(x; z) \hat{u}^S_{\beta}(x; z) + c^S_{\alpha\beta},$$

where $\hat{u}^S_{\beta}$ is the four-velocity vector parallelly propagated from $z$ to $x$, $\epsilon$ is the spatial geodesic distance from $x$ to $\Gamma$ (i.e. the proper length of the short normal geodesic section connecting $x$ to $\Gamma$), $u^S_{\alpha\beta}$ are smooth functions of $x$ (and $z$) which vanish at $x \rightarrow z$ at least quadratically in the coordinate differences $x^\alpha - z^\alpha$, and $c^S_{\alpha\beta}$ are constants (dependent on $z$ but not on $x$). The $S$ and direct fields differ only in the explicit form of $w^S_{\alpha\beta}$ and (possibly) in the value of $c^S_{\alpha\beta}$, but neither of these will be important for us in this review.

It is, however, important to emphasize that the singular form (19) is generally gauge dependent. The expression given here is specific to the Lorenz gauge, and is not guaranteed to keep its form if one makes other gauge choices for the perturbation. Indeed, it has been demonstrated [21] that some common gauge choices can ‘distort’ the local rest-frame isotropy of the Lorenz-gauge singularity manifest (at leading order) in equation (19).

2.6. Gauge dependence

Earlier we emphasized some important differences between the concepts of a general (external) gravitational force and that of the SF. The two notions, however, share a basic common feature: they are both defined via a mapping of the physical trajectory from a ‘perturbed’ spacetime to a ‘background’ spacetime. In both cases, such a mapping procedure gives rise to a gauge ambiguity. A thorough analysis of the gauge dependence of the gravitational SF was presented in [21], and a gauge transformation law for the SF was derived. Consider again the infinitesimal
gauge transformation of equation (10), where now the gauge displacement vector $\xi^\alpha$ is assumed to scale like the particle’s mass $\mu$. The change this induces on the physical SF was found [21] to be given by

$$\delta(\xi) F_{\text{self}}^\alpha = -\mu \left[ (g^{\alpha \beta} + u^\alpha u^\beta) \xi_\beta + R^\alpha_{\mu \lambda \gamma} u^\mu \xi_\lambda u^\gamma \right].$$

This has the same form as the gauge transformation law for the external gravitational force (recall equation (13)), which is not surprising given the similar geometrical origin of the gauge freedom in both cases.

We note, however, that the derivation of the transformation rule (20) in [21] (which predates the DW analysis) could not—and did not—follow the procedure we implemented above in deriving equation (13), simply replacing the external perturbation $h_{\alpha \beta}$ with the physical perturbation from the particle, $h_{\alpha \beta}^\text{ret}$. That is because $h_{\alpha \beta}^\text{ret}$, unlike the smooth external field $h_{\alpha \beta}$, is singular at the particle, and so expressions such as (11) or (12) (with $h_{\alpha \beta} \rightarrow h_{\alpha \beta}^\text{ret}$) would make no sense when evaluated along the particle’s worldline. DW’s later $R$-field interpretation suggests an alternative derivation of equation (20): since the $R$-field can be viewed, effectively, as a smooth external perturbation (even if one with peculiar causal properties), and since the SF is the force exerted by this perturbation, the derivation leading to equation (13) can be repeated in full with $h_{\alpha \beta} \rightarrow h_{\alpha \beta}^R$, and equation (20) follows immediately. In effect, the $R$-field interpretation allows us here to treat the SF on an equal footing with the gravitational force from an external perturbation.

The gauge dependence of the SF by no means implies that there is something ‘unphysical’ about it—the SF is as physical as the metric perturbation itself, which is also gauge dependent. The gauge dependence does mean, however, that one needs to exercise some care in decoding the physical content of the SF. One cannot expect to be able to describe the physical effect of the SF based on the value of the SF alone (to put this to extreme: one can always make a gauge choice that nullifies the SF anywhere along the orbit!). Instead, a meaningful description of the physical effect must involve both the SF and the gauge information associated with it (in the form of the metric perturbation, for example).

Another crucial point to have in mind is that the MiSaTaQuWa formula (16) is guaranteed to hold true only if $h_{\alpha \beta}^\text{ret}$ satisfies the Lorenz-gauge condition (14). Strictly speaking, the SF in a given non-Lorenz gauge only makes sense if, for $\xi^\alpha$ relating that gauge to the Lorenz gauge, the expression on the right-hand side of equation (20) is well defined. This is not at all an obvious condition; some simple counter-examples are analyzed in [21]. In gauges for which the right-hand side of (20) does not have a well-defined (i.e. finite and direction independent) particle limit, one might still devise a useful notion of the SF by averaging over angular directions, or by taking a directional limit in a consistent fashion. Such possibilities are discussed in [21, 77].

2.7. Equations of motion

Given the SF, the particle’s equation of motion becomes

$$\mu u^\alpha \nabla_\alpha u^\mu = F_{\text{self}}^\mu,$$

where on the left-hand side we have the usual four-acceleration (times $\mu$) along the background trajectory. This equation, along with the MiSaTaQuWa equation (16), describes the dynamics of the particle given the metric perturbation (and assuming one has a way of extracting the tail piece out of the full perturbation). To close the system of equations we need to know how the
The metric perturbation is determined from the particle’s trajectory. This, of course, is provided by the linearized Einstein equation, which takes the Lorenz-gauge form
\[
\nabla^\gamma \nabla_\gamma h^{\text{ret}}_{\alpha \beta} + 2 R^\mu_{\alpha \beta} \nabla^\mu h^{\text{ret}}_{\alpha \nu} = -16\pi \mu \int_{-\infty}^{\infty} (-g)^{-1/2} \delta^4 \left[ x^\mu - z^\mu(\tau) \right] u_\alpha u_\beta \, d\tau,
\]
where \( g \) is the determinant of \( g_{\alpha \beta} \) and \( z^\mu(\tau) \), recall, describes the particle’s worldline. The source on the right-hand side is the usual distributional representation of the point particle’s energy–momentum. The field equation (22) is to be supplemented by the gauge condition (14) and by suitable boundary conditions.

The set of equations (16), (21), (22) and (14) (together with a method to obtain \( h^{\text{tail}} \) out of \( h^{\text{ret}} \)) should in principle determine the dynamics of the orbit at linear order in perturbation theory. However, as pointed out by Gralla and Wald recently [77], the field equation (22) is only consistent with the Lorenz gauge condition (14) if the particle is moving strictly along a geodesic—which would then be inconsistent with the equation of motion (21). To resolve this inconsistency while allowing for orbital evolution, Gralla and Wald suggested a ‘Lorenz-gauge relaxation’ approach, wherein one relaxes the gauge condition (14) and considers solutions of the set (16), (21), (22). One then expects that, in situations where the orbit is very nearly geodesic (as is usually the case with LISA-relevant astrophysical inspirals), such solutions would give a faithful, albeit approximate description of the actual orbit. This approach is yet to be implemented and tested in actual calculations of the orbital evolution.

The proposal to use self-consistent solutions of the set (16), (21), (22) for modeling the slow orbital evolution at linear order in perturbation theory was put forward by Gralla and Wald in [77]. A different mathematical framework, based on techniques from multi-scale perturbation theory, was developed by Hinderer and Flanagan in [87]. Pound and Poisson [136, 137] performed first actual calculations of the orbital evolution under the full SF effect, using multi-scale analysis, within a weak-field, PN framework. Thus far there are no calculations of the full orbital evolution in strong-field scenarios.

In the rest of this review we will not consider any further the question of orbital evolution, but rather focus on the calculation of the SF (via equations (16) or (18)) along a pre-determined, non-evolving orbit. We consider this a preliminary step, whose output (e.g. the value of the SF at sufficiently many points across the relevant phase space) can later be incorporated into whichever evolution scheme one chooses to apply.

3. Overview of implementation frameworks and calculations to date

Starting in the late 1990s, work began to translate MiSaTaQuWa’s SF formalism into practical working schemes and to implement it in actual calculations. While the ‘holy grail’ of this program has from the outset been—and still remains—the calculation of the gravitational SF for generic orbits around Kerr black holes, much of the initial effort has focused on the simpler toy problem of the scalar-field SF, and on simple classes of orbits (radial, circular) in Schwarzschild spacetime. The last few years, however, saw first calculations of the gravitational (and electromagnetic) SFs for generic orbits in Schwarzschild—and initial work on Kerr is now underway. In this section we give a broad overview of the methods that have been suggested for implementing MiSaTaQuWa’s formalism, and we survey the actual calculations performed so far.

Our starting point are the formal expressions (16) and (18) for the gravitational SF. To facilitate the following discussion, we first use equations (15) and (17) to recast these expressions in the more practical form
\[
F^\alpha(z) = \mu \lim_{x \to z} \left[ \nabla^\alpha \nabla^\beta h^{\text{ret}}_{\beta \gamma}(x) - \nabla^\alpha \nabla^\beta h^{\text{dir}}_{\beta \gamma}(x) \right].
\]

referring once again to figure 1 for notation. This describes a ‘regularization’ procedure, which one can perform using either \( \bar{h}_{\beta\gamma}^{\text{dir}} \) or \( \bar{h}_{\beta\gamma}^{S} \)—both producing the same final value for the SF. The limit procedure is necessary here because the individual fields \( \hat{h}_{\beta\gamma}^{\text{ret}} \) and \( \bar{h}_{\beta\gamma}^{S} \), unlike their difference \( \hat{h}_{\beta\gamma}^{\text{ret}}/R \), are each singular at \( x \to z \), and so are their derivatives. Since the operator \( \bar{\nabla}_{\alpha\beta\gamma} \) is only defined along the particle’s worldline (as it involves the four-velocity—recall equation (9)), in equation (23) we needed to introduce an extension of this operator off the worldline—denoted by \( \bar{\nabla}_{\alpha\beta\gamma} \). For all \( x \) in the neighborhood of a given worldline point \( z \), the operator \( \bar{\nabla}_{\alpha\beta\gamma} \) is given by equation (9), where \( g^{\alpha\beta} \) and \( u^{\alpha} \) take the same values they have at \( z \), and \( \nabla_{\alpha} \) is the standard covariant derivative at point \( x \). Here, and throughout this review, we use this ‘fixed contravariant components’ extension of \( \bar{\nabla}_{\alpha\beta\gamma} \) exclusively, although other natural extensions are possible [23]. Of course, the final value of the SF in equation (23), after the limit \( x \to z \) is taken, is not sensitive to the choice of extension. Note that the definition of \( \bar{\nabla}_{\alpha\beta\gamma} \) is, of course, coordinate dependent, and it becomes well posed only in reference to a specific coordinate system. Also note that the operator \( \bar{\nabla}_{\alpha\beta\gamma} \) is defined with respect to a given point \( z \).

The most basic technical challenge one faces in preparing to implement the MiSaTaQuWa formula is the so-called subtraction problem: How does one go about extracting the tail (or \( R \)) piece from the full retarded perturbation in practice? Equation (23) suggests applying the subtraction \( \hat{h}_{\alpha\beta}^{\text{ret}} - \hat{h}_{\alpha\beta}^{\text{dir}/S} \), using approximate analytic expressions for the direct/\( S \)-fields, such as the one in equation (19). However, this involves the removal of one divergent quantity from another, which is not easily tractable in actual numerical calculations. Several strategies have been proposed to address this problem, and in the main part of this review we shall describe a few of them in some detail. Here we proceed with a brief overview of the main avenues of approach to this problem.

**Quasi-local/matched expansions calculations [4–7, 126, 127].** This approach tackles the calculation of the tail contribution directly, by analytically evaluating the Hadamard expansion of the relevant Green’s function. Such calculations capture the ‘near’ part of the tail, which, one might hope, represents the dominant contribution in problems of interest. Quasi-local calculations can be supplemented by a numerical computation of the ‘far’ part of the tail, a strategy referred to as ‘matched expansions’ (not to be confused with matched asymptotic expansions). The applicability of this idea was demonstrated very recently [48] with a full calculation in Nariai spacetime (a simple toy spacetime featuring many of the characteristics of Schwarzschild).

**Weak-field analysis [59, 129, 135–137].** The tail formula can be evaluated analytically for certain weak-field configurations, within a Newtonian or a post-Newtonian (PN) framework. Such work has provided important insight into the nature and properties of the SF. PN techniques have also been implemented in combination with the mode-sum method discussed below [85, 86, 119–121].

**Radiation-gauge regularization [96].** This approach proposes a reformulation of the MiSaTaQuWa regularization, in which one reconstructs the \( R \)-part of the metric perturbation in a radiation gauge (rather than in the Lorenz gauge) from a suitably regularized Newman–Penrose curvature scalar (\( \psi_{0} \) or \( \psi_{4} \)). The main advantage of this method is that it reduces the numerical component of the calculation to a solution of a single scalar-like (Teukolsky’s) equation. However, some of the technical complexity is relegated to the metric reconstruction step. This technique was implemented so far only for a particle held static in Schwarzschild,
but more interesting cases are currently being studied [67, 95]. See the discussion in section 5.1 for more details.

**Mode-sum method** [11, 20, 22–24, 27, 57, 83, 85, 86, 102, 116]. An approach whereby one evaluates the tail contribution mode by mode in a multipole expansion. The subtraction \( \text{`ret`--`dir'} \) (or \( \text{`ret`--`S'} \)) in equation (23) is performed mode by mode, avoiding the need to deal with divergent quantities. The method exploits the separability of the field equations in Kerr into multipole harmonics. The mode-sum method has provided the framework for the bulk of work on SF calculations over the last decade. We will discuss it in detail in section 4.

**Puncture methods** [17, 28, 95, 105, 158]. A set of recently-proposed methods custom-built for time-domain numerical implementation in 2+1 or 3+1 dimensions. Common to these methods is the idea to utilize as a variable for the numerical time evolution a ‘punctured’ field, constructed from the full (retarded) field by removing a suitable singular piece, given analytically. The piece removed approximates the correct \( S \)-field sufficiently well that the resulting ‘residual’ field is guaranteed to yield the correct MiSaTaQuWa SF. In the 2+1D version of this approach the regularization is done mode by mode in the azimuthal \( (m \text{-mode}) \) expansion of the full field. This procedure offers significant simplification; we shall review it in detail in section 7.

### 3.1. SF calculations to date

As we have mentioned already, the program to calculate the SF for black hole orbits has been progressing gradually, through the study of a set of simplified model problems. Some of the necessary computational techniques were first tested within the simpler framework of a scalar-field toy model before being applied to the electromagnetic (EM) and gravitational problems. The authors have considered special classes of orbits (static, radial, circular) before attempting more generic cases, and much of the work so far has focused on Schwarzschild orbits. The state of the art in the field is numerical codes to compute the scalar, EM and gravitational SFs along any given (geodesic) orbit outside a Schwarzschild black hole. It is reasonable to expect that attention will now be increasingly drawn to the Kerr problem.

The information in tables 1–3 is meant to provide a quick reference to work done so far. It covers actual evaluations of the local SF that are based on the MiSaTaQuWa formulation (or the analogous scalar-field and EM formulations of [140] and [58, 88, 141], respectively), either directly or through one of the aforementioned implementation methods. We have included weak-field and PN implementations, but have not included work based on the radiative field approach. Some of the techniques referred to under ‘strategy’ will be discussed in the following sections.

### 4. Mode-sum method

Let us write the subtraction formula (23) using the more compact notation

\[
F_{\text{self}}^a (z) = \lim_{x \to z} \left[ F_{\text{ret}}^a (x) - F_{\text{S}}^a (x) \right],
\]

(24)

where we have introduced the fields \((\propto \mu^2)\)

\[
F_{\text{ret}}^a (x) = \mu \nabla_x^{a\beta\gamma} h_{\beta\gamma}^\text{ret} (x), \quad F_{\text{S}}^a (x) = \mu \nabla_x^{a\beta\gamma} h_{\beta\gamma}^\text{S} (x).
\]

(25)
Table 1. Calculations of the scalar-field SF as a toy model for the gravitational problem. In this table, as well as in tables 2 and 3, 'direct' implies explicit evaluation of the tail contribution to the SF. ‘Mode sum’ and ‘puncture’ refer to the two implementation schemes described, respectively, in sections 4 and 7 of this review. The method of ‘matched expansions’ (different from ‘matched asymptotic expansions’) is described briefly in sections 3.

| Case                        | Author(s)          | Strategy                      |
|-----------------------------|--------------------|-------------------------------|
| Newtonian potential         | Pfenning and Poisson [129] | Direct, analytic             |
| Spherical mass shell:       | Burko et al [43]   | Mode sum, analytic            |
| static particle             |                     |                               |
| Isotropic cosmology:        | Burko et al [44]   | Direct, analytic              |
| static particle             |                     |                               |
| Isotropic cosmology:        | Haas and Poisson [82] | Direct, analytic             |
| slow motion                 |                     |                               |
| Nariai spacetime:           | Casals et al [48]  | Matched expansions            |
| static particle             |                     |                               |
| Schwarzschild:              | Burko [39]         | Mode sum, analytic            |
| static particle             | Wiseman [163]      | Direct, analytic              |
| Schwarzschild:              | Barack and Burko [13] | Mode sum, numerical          |
| radial geodesics            |                     | (1+1D evolution)              |
| Schwarzschild:              | Nakano et al [120] | Post-Newtonian, analytic      |
| circular geodesics          | Hikida et al [86]  |                               |
|                             | Burko [40]         | Mode sum, numerical           |
|                             | Detweiler et al [57, 60] | (frequency domain)          |
|                             | Vega and Detweiler [158] | Puncture, numerical        |
|                             |                     | (1+1D evolution)              |
|                             | Vega et al [159]   | Puncture, numerical           |
|                             |                     | (3+1D evolution)              |
| Schwarzschild:              | Haas [80]          | Mode sum, numerical           |
| eccentric geodesics         |                     | (1+1D evolution)              |
| Kerr–Newman:                | Burko and Liu [42] | Mode sum, analytic            |
| static particle             |                     |                               |
| Kerr: circular-             | Warburton and Barack [162] | Mode sum, numerical          |
| equatorial geodesics        |                     | (frequency domain)            |

For concreteness and simplicity we adopt here the $S$-field subtraction, noting that the entire discussion in this section would not be altered upon replacing $S \rightarrow$ direct and $R \rightarrow$ tail. The fields $F_{\text{ret}}$ and $F_{\text{S}}$ inherit the extension ambiguity of $\nabla \alpha_{\beta\gamma} x$, but here we shall always use

4 The only exception is that statements referring to the smoothness of the $R$-field would need to be formulated more carefully to reflect the irregularity in the higher derivatives of the tail field. This irregularity, however, would have little practical impact on the discussion in this section.
Table 2. Recent calculations of the electromagnetic self-force.

| Case                        | Author(s)               | Strategy                    |
|-----------------------------|-------------------------|----------------------------|
| Newtonian potential         | Pfenning and Poisson [129] | Direct, analytic           |
| Isotropic cosmology: slow motion | Haas and Poisson [82]                          | direct, analytic           |
| Schwarzschild: static particle | Burko [39]                         | Mode sum, analytic         |
|                             | Keidl et al [96]                       | Radiation-gauge regularization, analytic |
| Schwarzschild: eccentric geodesics | Haas [81]                         | Mode sum, numerical       |
|                             |                                       | (1+1D evolution)           |

Table 3. Calculations of the gravitational self-force.

| Case                        | Author(s)               | Strategy                                      |
|-----------------------------|-------------------------|-----------------------------------------------|
| Newtonian potential         | Pfenning and Poisson [129] | Direct, analytic                             |
| Schwarzschild: radial geodesics | Barack and Lousto [18]                                  | Mode sum, 1+1D evolution in Regge–Wheeler gauge |
| Schwarzschild: static particle | Keidl et al [96]                        | Radiation-gauge regularization, analytic     |
| Schwarzschild: circular geodesics | Barack and Sago [25]                              | Mode sum, 1+1D evolution in Lorenz gauge |
|                             | Detweiler [54]                              | Mode sum, frequency domain in Regge–Wheeler gauge |
| Schwarzschild: eccentric geodesics | Barack and Sago [26, 143]                      | Mode sum, 1+1D evolution in Lorenz gauge     |

the (coordinate dependent) ‘fixed’ extension described above. Both $F_{\text{ret}}^{\alpha}$ and $F_{S}^{\alpha}$, of course, diverge at the particle, but their difference is a smooth (analytic) function of $x$ even at the particle.

In the mode-sum method we formally decompose each vectorial component of both $F_{\text{ret}}^{\alpha}$ and $F_{S}^{\alpha}$ into spherical harmonics. These harmonics are defined in the Kerr/Schwarzschild background based on the Boyer–Lindquist/Schwarzschild coordinates $(t, r, \theta, \varphi)$ in the standard way, i.e. through a projection onto an orthogonal basis of angular functions defined on surfaces of constant $t$ and $r$. Let us denote by $F_{\text{ret}}^{\alpha l}(x)$ and $F_{S}^{\alpha l}(x)$ the $l$-mode contributions to $F_{\text{ret}}^{\alpha}$ and $F_{S}^{\alpha}$, respectively (summed over $m$). A key observation is that each of these $l$-mode fields is finite even at the particle. This suggests a natural regularization procedure, which, essentially, amounts to performing the subtraction $F_{\text{ret}}^{\alpha l} - F_{S}^{\alpha l}$ mode by mode. The idea is best developed through an elementary example, as follows.
4.1. An elementary example

Consider a pointlike particle of mass $\mu$ at rest in flat space. The location of the particle is $\vec{x} = \vec{x}_p$ in a given Cartesian system. In this simple static configuration the perturbed Einstein equations (22) read

$$\nabla^2 \bar{\hat{h}}_{tt}^{\text{ret}} = -16\pi \mu \delta^3(\vec{x} - \vec{x}_p),$$

where $\nabla^2$ is the 3D Laplacian, and with all other components of $\bar{\hat{h}}_{ab}^{\text{ret}}$ vanishing\(^5\). The static perturbation $\bar{\hat{h}}_{tt}^{\text{ret}}$ automatically satisfies the Lorenz-gauge condition (14). Of course, in this simple case we can immediately write down the exact physical (Coulomb-like) static solution, where

$$\bar{\hat{h}}_{tt}^{\text{ret}} = 4\mu/|\vec{x} - \vec{x}_p|,$$

and we also trivially have $F_{tt}^{\text{ret}} = 0$. However, for the sake of our discussion, let us proceed by considering the multipole expansion of the perturbation.

To this end, introduce polar coordinates $(r, \theta, \phi)$, such that our particle is located at $\vec{x}_p = (r_0 \neq 0, \theta_0, \phi_0)$, and expand the physical solution $\bar{\hat{h}}_{tt}$ in spherical harmonics on the spheres $r = \text{const}$, in the form

$$\bar{\hat{h}}_{tt}^{\text{ret}} = \sum_{l=0}^{\infty} \tilde{h}_l^0(r, \theta), \quad \text{where} \quad \tilde{h}_l^0(r, \theta) = \sum_{m=-l}^{l} \tilde{h}_{l,m}^0(r)Y_{lm}(\theta, \phi).$$

This expansion separates the field equation (26) into radial and angular parts, the former reading (for each $l, m$)

$$\tilde{h}_{lm}^0(r) + \frac{2}{r} \tilde{h}_{lm}^0(r, \theta) + \frac{l(l+1)}{r^2} \tilde{h}_{lm}^0(r) = -\frac{16\pi \mu}{r_0^2} Y_{lm}^*(\theta_0, \phi_0)\delta(r - r_0),$$

where an asterisk denotes complex conjugation and a comma denotes partial differentiation. The unique physical $l, m$-mode solution, continuous everywhere and regular at both $r = 0$ and $r \to \infty$, reads

$$\tilde{h}_{lm}^0(r) = \frac{16\pi \mu}{(2l+1)r_0} Y_{lm}^*(\theta_0, \phi_0) \times \begin{cases} (r/r_0)^{-l-1}, & r \geq r_0, \\ (r/r_0)^l, & r \leq r_0, \end{cases}$$

(29)

giving

$$h_{lm}^0(r, \theta) = \frac{4\mu}{r_0} P_l(\cos \gamma) \times \begin{cases} (r/r_0)^{-l-1}, & r \geq r_0, \\ (r/r_0)^l, & r \leq r_0, \end{cases}$$

(30)

where $P_l$ is the Legendre polynomial and $\gamma$ is the angle subtended by the two radius vectors to $\vec{x}$ and $\vec{x}_p$ (see figure 2).

We now construct the force field $F_{\text{ret}}^{\mu}$ as it is defined in equation (25). We find $F^r_{\text{ret}} = 0$, and the spatial components are $F^\mu_{\text{ret}} = \mu \nabla^\mu \bar{\hat{h}}_{tt} = (\mu/4)\bar{\hat{h}}_{tt}^{\text{ret}}$. Focus now on the $r$ component. The $l$ mode of $F_{\text{ret}}^r$ is given simply as $F_{\text{ret}}^r = (\mu/4)\tilde{h}_{lm}^0$. Using equation (30) and evaluating $F_{\text{ret}}^r$ at the particle (taking $\gamma \to 0$ followed by $r \to r_0^-$), we obtain

$$F_{\text{ret}}^r(\vec{x}_p) = \mp L \frac{\mu^2}{r_0^2} - \frac{\mu^2}{2r_0^2}, \quad \text{where} \quad L \equiv l + \frac{1}{2}.$$  

(31)

Here the subscripts $\pm$ indicate the two (different) values obtained by taking the particle limit from ‘outside’ ($r \to r^+$) and ‘inside’ ($r \to r^-$).

Let us note the following features manifest in the above simple analysis.

- The individual $l$ modes of the metric perturbation, $\bar{\hat{h}}_{ab}^{\text{ret}}$, are each continuous at the particle’s location, although their derivatives are discontinuous there.

\(^5\) Here the label ‘ret’ is less appropriate, but we shall retain it to adhere to our general notation. As in the general case, $\bar{\hat{h}}_{tt}^{\text{ret}}$ represents the unique physical perturbation—here the unique regular static solution of equation (26).
The individual $l$ modes $F_{\text{ret}}^{\alpha l}$ have finite one-sided values at the particle.

At large $l$, each of the one-sided values of $F_{\text{ret}}^{\alpha l}$ at the particle is dominated by a term $\propto l$.

(The mode sum obviously diverges at the particle, reflecting the divergence of the full force $F_{\text{ret}}^{\alpha}$ there.)

It turns out (as we shall see later) that all above features are quite generic, and they carry over intact to the much more general problem of a particle moving in Kerr spacetime. Specifically, we find that, at any point along the particle’s trajectory, the (one-sided values of the) modes $F_{\text{ret}}^{\alpha l}$ always admit the large-$l$ form

$$F_{\text{ret}}^{\alpha l} = \pm LA^{l} + B^{l} + C^{l}/L + O(L^{-2}).$$

In our elementary problem the power series in $1/L$ truncates at the $L^{0}$ term, but in general the series can be infinite. The $l$-independent coefficients $A_{l}$, $B_{l}$, and $C_{l}$, whose values depend on the background geometry as well on the particle’s location and four-velocity, are characteristic of the local structure of the particle singularity at large $l$. These coefficients, called regularization parameters, play a crucial role in the mode-sum regularization procedure, as we describe next.

4.2. The mode-sum formula

Consider a mass particle moving on a geodesic trajectory in Kerr, and suppose we are interested in the value of the SF at a point $z$ along the trajectory, with Boyer–Lindquist coordinates $(t_{0}, r_{0}, \theta_{0}, \phi_{0})$. Starting with equation (24), let us formally expand $F_{\text{ret}}^{\alpha l}(x)$ and $F_{\text{S}}^{\alpha l}(x)$ in spherical harmonics on the surfaces $t, r = \text{const}$. Here we ignore the vectorial nature of $F_{\text{ret}}^{\alpha l}$ and $F_{\text{S}}^{\alpha l}$ and, for mathematical simplicity, treat each of their Boyer–Lindquist components as a scalar function (see [83] for a more sophisticated, covariant treatment). Denoting the respective $l$-mode contributions (summed over $m$) by $F_{\text{ret}}^{\alpha l}(x)$ and $F_{\text{S}}^{\alpha l}(x)$, we write

$$F^{\alpha \text{self}}_{\text{ret}}(z) = \lim_{x \to z} \sum_{l=0}^{\infty} \left[ F_{\text{ret}}^{\alpha l}(x) - F_{\text{S}}^{\alpha l}(x) \right].$$

We recall that the form of $F_{\text{ret}}^{\alpha l}$ and $F_{\text{S}}^{\alpha l}$ will depend on the specific off-worldline extension chosen for $\nabla_{\alpha l}$; however, this ambiguity disappears upon taking the limit $x \to z$, and the final SF is, of course, insensitive to the choice of extension.
Since \( F^\alpha_{\text{ret}}(x) - F^\alpha_S(x) \) is a smooth function for all \( x \), the mode sum in equation (33) is guaranteed to converge exponentially for all \( x \) (this is a general mathematical property of the multipole expansion). In particular, the sum converges uniformly at \( x = z \), and we are allowed to change the order of limit and summation. We expect, however, that the particle limit of the individual terms \( F^\alpha_{\text{ret}} \) and \( F^\alpha_S \) is only defined in a one-sided sense, as in the elementary example studied above. Hence, we write

\[
F^\alpha_{\text{self}}(z) = \sum_{l=0}^{\infty} \left[ F^{\alpha l}_{\text{ret}}(z) - F^{\alpha l}_S(z) \right],
\]

(34)

where \( \pm \) indicates the values obtained by first taking the limits \( t \to t_0, \theta \to \theta_0 \) and \( \phi \to \phi_0 \), and then taking \( r \to r^\pm_0 \). Of course, the difference \( F^{\alpha l}_{\text{ret}}(z) - F^{\alpha l}_S(z) \) does not depend on the direction from which the radial limit is taken: as the difference \( F^{\alpha l}_{\text{ret}}(x) - F^{\alpha l}_S(x) \) is the \( l \) mode of a smooth function, it is itself smooth for all \( x \).

Furthermore, since the mode sum in equation (34) converges faster than any power of \( 1/L \), we expect \( F^\alpha_{\text{ret}} \) and \( F^\alpha_S \) to share the same large-\( l \) power expansion (32), with the same expansion coefficients. This motivates us to re-express equation (34) in the form

\[
F^\alpha_{\text{self}}(z) = \sum_{l=0}^{\infty} \left[ F^{\alpha l}_{\text{ret}}(z) \mp LA^\alpha - B^\alpha - C^\alpha / L \right] - \sum_{l=0}^{\infty} \left[ F^{\alpha l}_S(z) \mp LA^\alpha - B^\alpha - C^\alpha / L \right].
\]

(35)

Here, each of the terms in square brackets falls off at least as \( \sim l^{-2} \) as \( l \to \infty \), and so each of the two sums converges at least as \( \sim 1/l \). We hence arrive at the following mode-sum reformulation of the MiSaTaQuWa equation:

\[
F^\alpha_{\text{self}}(z) = \sum_{l=0}^{\infty} \left[ F^{\alpha l}_{\text{ret}}(z) \mp LA^\alpha - B^\alpha - C^\alpha / L \right] - \sum_{l=0}^{\infty} \left[ F^{\alpha l}_S(z) \mp LA^\alpha - B^\alpha - C^\alpha / L \right] - D^\alpha.
\]

(36)

with

\[
D^\alpha \equiv \sum_{l=0}^{\infty} \left[ F^{\alpha l}_S(z) \mp LA^\alpha - B^\alpha - C^\alpha / L \right].
\]

(37)

The mode-sum formula (36), first proposed in [20] (scalar-field case) and [11] (gravitational case) provides a practical way of calculating the SF, once the regularization parameters \( A^\alpha, B^\alpha, C^\alpha \) and \( D^\alpha \) are known. It is practical because (i) it involves no subtraction of divergent quantities, and (ii) it builds naturally on the fact that in black hole perturbation theory one usually calculates the metric perturbation mode by mode in a multipole decomposition.

The values of the regularization parameters are obtained analytically via a local analysis of the singular (or direct) field near the particle. Early calculations of these parameters [10, 11, 20], which were restricted to specific orbits, were based on a local analysis of the Green’s function near coincidence \( (x \to z) \) at large \( l \). Later, after the local form of the direct field had been derived explicitly to sufficient accuracy [116], workers have been able to derive general expressions for the regularization parameters, valid for generic (geodesic) orbits—first in Schwarzschild [22, 23, 27], and then in Kerr [24]. Later improvements (in the scalar case) are due to [57], where higher-order terms in the \( 1/L \) expansion were derived analytically in order to accelerate the convergence of the mode sum, and [83], where the regularization parameters were redefined as scalar quantities using a covariant-form projection of the singular field onto a null tetrad based on the worldline.

With the regularization parameters given in analytic form, the calculation of the SF within the mode-sum scheme follows this procedure.
(i) For a given geodesic orbit, calculate sufficiently many multipole modes of the physical, retarded metric perturbation in the Lorenz gauge. (Here ‘multipole modes’ may refer to tensor harmonics, Fourier-tensor harmonics, or spheroidal harmonics, depending on the case considered and on the perturbation framework used for tackling the field equations.) How this might be (and is being) done in practice—usually using numerical methods—will be discussed in section 5.

(ii) Construct the $l$-modes $F^{al}_{\text{ret} \pm}$ at the particle. An example of how this is done in practice will be provided in section 6.

(iii) Use the modes $F^{al}_{\text{ret} \pm}(z)$ as input for the mode-sum formula (36).

The mode-sum procedure has the very useful ability to automatically test itself: if either the values of the regularization parameters $A^\alpha$, $B^\alpha$ and $C^\alpha$, or the values of the $l$-modes $F^{al}_{\text{ret} \pm}$ (usually computed numerically) are wrongly calculated, the $l$-mode sum in equation (36) will very likely fail to converge. Conversely, if one finds that the calculated $l$-term in the sum has the expected $\sim 1/L^2$ fall-off at large $L$, that provides an excellent validation test for the entire calculation. So far, the analytic values of the parameters $A^\alpha$, $B^\alpha$ and $C^\alpha$ have been successfully ‘tested’ (in the above sense) against numerical calculations for generic orbits in Schwarzschild, and also (in the scalar case) for circular equatorial orbits in Kerr.

In what follows we state the values of the regularization parameters in the gravitational case, for generic geodesics in Kerr, as derived in [24]. In the appendix, we provide a full derivation of these parameters (not included in [24]).

In passing, we recall that the mode-sum formula (in the gravitational case) is formulated in the Lorenz gauge, just like the MiSaTaQuWa formula on which it relies. The values of the regularization parameters; the form of the large-$l$ expansion in equation (32); or even the very definiteness of the SF—all of these are gauge dependent. It has been shown, however, that the regularization parameters remain invariant under gauge transformations (from the Lorenz gauge) that are sufficiently regular [21].

4.3. Regularization parameters for generic orbits in Kerr

We state here the values of the regularization parameters for the gravitational SF at an arbitrary point $z$ along an arbitrary geodesic in Kerr spacetime. We assume the point $z$ has Boyer–Lindquist coordinates $(t_0, r_0, \theta_0, \phi_0)$. The regularization parameters $C^\alpha$ and $D^\alpha$ are always zero:

$$C^\alpha = D^\alpha = 0.$$  \hfill (38)

The Boyer–Lindquist components of the regularization parameter $A^\alpha$ at $z$ are given by

$$A^t = -\frac{\mu^2}{V} \left( \frac{\sin^2 \theta_0}{g_{rr} g_{\theta\theta} g_{\phi\phi}} \right)^{1/2} \left( V + \frac{u_r^2}{g_{rr}} \right)^{1/2},$$

$$A^\theta = -\frac{u_r}{u_t} A^t, \quad A^\phi = A^\theta = 0,$$  \hfill (39)

where

$$V \equiv 1 + u_\theta^2/g_{\theta\theta} + u_\phi^2/g_{\phi\phi}.$$  \hfill (40)

Here $u^\alpha$ is the particle’s four-velocity and $g_{\alpha\beta}$ is the background Kerr metric—both at point $z$.

The value of the parameter $B^\alpha$ is more complicated. It can be expressed in a compact form as

$$B^\alpha = \mu^2 (2\pi)^{-1} P^\alpha_{abcd} F^{abcd},$$  \hfill (41)

where $P^\alpha_{abcd}$ is a four-index projector.
where hereafter roman indices \((a, b, c, \ldots)\) run over the two Boyer–Lindquist angular coordinates \(\theta, \phi\). The coefficients \(P^a_{a'b'cd}\) are given by

\[
P^a_{a'b'cd} = \frac{1}{2} \left[ P^a_\alpha (3P_{a'b} + 2P_{a'b'}) - P^a_{\alpha'b'} (2P_{a'b} + P_{a'b'}) \right] + (3P^a_\alpha P_{c'd} - P^c_a P_{a'b'}) C^\alpha_{c'd},
\]

(42)

where

\[
P_\alpha \equiv u^\alpha u^\mu g_{\mu\rho\sigma}, \quad P_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta, \quad P_{\alpha\beta\gamma} \equiv \Gamma^\lambda_{\alpha\beta} P_{\lambda\gamma},
\]

(43)

and

\[
C^\alpha_{\beta\gamma} = \frac{1}{2} \sin \theta_0 \cos \theta_0, \quad C^\alpha_{\beta\phi} = C^\alpha_{\phi\beta} = -\frac{1}{2} \cot \theta_0,
\]

(44)

with all other coefficients \(C^\alpha_{\beta\phi}\) vanishing, and with \(\Gamma^\lambda_{\alpha\beta}\) being the background connection coefficients at \(z\). The quantities \(P_{abcd}\) are

\[
I_{abcd} = (\sin \theta_0)^{-N} \int_0^{\frac{2\pi}{\alpha}} G(\chi)^{-5/2} (\sin \chi)^N (\cos \chi)^{4-N} d\chi,
\]

(45)

where

\[
G(\chi) = P_{\phi\theta} \cos^2 \chi + 2P_{\phi\phi} \sin \chi \cos \chi / \sin \theta_0 + P_{\phi\phi} \sin^2 \chi / \sin^2 \theta_0,
\]

(46)

and \(N = N(a b c d)\) is the number of times the index \(\phi\) occurs in the combination \((a, b, c, d)\), namely

\[
N = \delta^a_b + \delta^a_c + \delta^a_d + \delta^b_c + \delta^b_d + \delta^c_d + \delta^c_a + \delta^d_a + \delta^d_b + \delta^d_c + \delta^d_b.
\]

(47)

We may write \(P_{abcd}\) explicitly in terms of standard complete Elliptic integrals. Introducing the short-hand notation

\[
\alpha \equiv \sin^2 \theta_0 P_{\phi\theta} / P_{\phi\phi} - 1, \quad \beta \equiv 2 \sin \theta_0 P_{\phi\phi} / P_{\phi\phi},
\]

(48)

we have

\[
I_{abcd} = \frac{(\sin \theta_0)^{-N}}{d} \left[ Q I^{(N)}_K \hat{K}(w) + I^{(N)}_E \hat{E}(w) \right],
\]

(49)

where

\[
Q = \alpha + 2 - (\alpha^2 + \beta^2)^{1/2},
\]

(50)

\[
d = 3P^{5/2}_{\phi\phi}(\sin \theta_0)^{-5}(\alpha^2 + \beta^2)^2 (4\alpha + 4 - \beta^2)^{3/2}(Q/2)^{1/2},
\]

(51)

\[
\hat{K}(w) \equiv \int_0^{\pi/2} (1 - w \sin^2 x)^{-1/2} dx \quad \text{and} \quad \hat{E}(w) \equiv \int_0^{\pi/2} (1 - w \sin^2 x)^{1/2} dx \quad \text{are complete Elliptic integrals of the first and second kinds, respectively, and the argument is}
\]

\[
w = \frac{2(\alpha^2 + \beta^2)^{1/2}}{\alpha + 2 + (\alpha^2 + \beta^2)^{1/2}}.
\]

(52)

The ten coefficients \(I^{(N)}_K, I^{(N)}_E\) in equation (49) are given by

\[
I^{(0)}_K = 4[12\alpha^3 + \alpha^2(8 - 3\beta^2) - 4\alpha\beta^2 + \beta^2(\beta^2 - 8)],
\]

(53)

\[
I^{(0)}_E = -16[8\alpha^3 + \alpha^2(4 - 7\beta^2) + \alpha\beta^2(\beta^2 - 4) - \beta^2(\beta^2 + 4)],
\]

\[
I^{(1)}_K = 8\beta[9\alpha^2 - 2\alpha(\beta^2 - 4) + \beta^2],
\]

(54)

\[
I^{(1)}_E = -4\beta[12\alpha^3 - \alpha^2(\beta^2 - 52) + \alpha(32 - 12\beta^2) + \beta^2(3\beta^2 + 4)],
\]

\[
I^{(2)}_K = -4[8\alpha^3 - \alpha^2(\beta^2 - 8) - 8\alpha\beta^2 + \beta^2(3\beta^2 - 8)],
\]

(55)

\[
I^{(2)}_E = 8[4\alpha^4 + \alpha^3(\beta^2 + 12) - 2\alpha^2(\beta^2 - 4) + 3\alpha\beta^2(\beta^2 - 4) + 2\beta^2(3\beta^2 - 4)],
\]
$I^{(3)}_K = 8\beta [\alpha^3 - 7\alpha^2 + \alpha (3\beta^2 - 8) + \beta^2],$

$\begin{align*}
I^{(3)}_E &= -4\beta [8\alpha^4 - 4\alpha^3 + \alpha^2 (15\beta^2 - 44) + 4\alpha (5\beta^2 - 8) + \beta^2 (3\beta^2 + 4)], \\
I^{(4)}_K &= -4\beta [4\alpha^4 - 4\alpha^3 + \alpha^2 (7\beta^2 - 8) + 12\alpha\beta^2 - \beta^2 (\beta^2 - 8)], \\
I^{(4)}_E &= 16\beta [4\alpha^5 + 4\alpha^4 + \alpha^3 (7\beta^2 - 4) + \alpha^2 (11\beta^2 - 4) + (2\alpha + 1)\beta^2 (\beta^2 + 4)].
\end{align*}$

The sharp-eyed reader may observe that the expression for $P^{\alpha}_{abcd}$ in equation (42) differs somewhat from that given in [24]. The reason for this discrepancy is that in [24] we have made a different choice of extension for $\nabla^{\alpha\beta\gamma\delta}$ (one in which the metric functions $g^{\alpha\beta}$ in equation (9) take their actual value at $x$, and $s^{\alpha}$ is parallely-propagated from the worldline to $x$ along a normal geodesic). This affects the value of the parameter $B^{\alpha}$ only. We have opted here to give the parameter values corresponding to the ‘fixed’ extension because that extension is more easily implemented in actual numerical calculations. To obtain the value of $B^{\alpha}$ in the extension of [24], all one needs to do is replace the expression in equation (42) with

$$P^{\alpha}_{abcd}[PP] = \frac{1}{2} [3P^a_{dab} + P_{ab}^{\alpha}] P_{cd} + (3P^a_{dab} - P^d_{dab} - P^{\alpha}_{dab}) C^{\alpha}_{cd},$$

where ‘PP’ denotes the extension of [24] (which involves a parallel propagation of the four-velocity). Interestingly, the regularization parameters for the scalar-field SF are precisely the same as the gravitational-field ones, with the parameter $B^{\alpha}$ calculated using $P^{\alpha}_{abcd}[PP]$ (and with the obvious replacement of the mass $\mu$ with the scalar charge) [24].

5. Numerical implementation strategies

There are two broad (somewhat related) issues that need to be addressed in preparing to implement the mode-sum formula in practice (these issues arise, in some form, whether one uses the mode sum scheme or any of the other implementation methods based on the MiSaTaQuWa formalism). The first practical issue has often been referred to as the gauge problem: the mode-sum method (as the MiSaTaQuWa formalism underpinning it) is formulated in terms of the metric perturbation in the Lorenz gauge, while standard methods in black hole perturbation theory are formulated in other gauges. The second practical issue concerns the numerical treatment of the point-particle singularity; the problem takes a different form in the frequency domain and in the time domain, and we shall discuss these two frameworks separately below. We start, however, with a discussion of the gauge problem and the methods developed to address it.

5.1. Overcoming the gauge problem

The calculation of the gravitational SF requires direct information about the local metric perturbation near (and at) the particle. More specifically, one needs to be able to construct the metric perturbation, along with its derivatives, in the Lorenz gauge, at the particle’s location. The mode-sum approach allows us to do so without encountering infinities by considering individual multipoles of the perturbation, but the problem remains how to obtain these multipoles in the desired Lorenz gauge. Unfortunately, standard formulations of black hole perturbations employ other gauges, favored for their algebraic simplicity. Such gauges are simple because they reflect faithfully the global symmetries of the underlying black hole spacetimes—unlike the Lorenz gauge, which is suitable for describing the locally-isotropic particle singularity, but complies less well with the global symmetry of the background.

A common gauge choice for perturbation studies in Schwarzschild is the one introduced many years ago by Regge and Wheeler [142], and further developed by Zerilli [164].
and Moncrief [118]. In this gauge, certain projections of the metric perturbation onto a tensor-harmonic basis are taken to vanish, which results in a significant simplification of the perturbation equations. Another such useful ‘algebraic’ gauge is the radiation gauge introduced by Chrzanowski [49], in which one sets to zero the projection of the perturbation along a principle null direction of the background black hole geometry. Perturbations of the Kerr geometry have been studied almost exclusively using the powerful formalism by Teukolsky [155], in which the perturbation is formulated in terms of the Newman–Penrose gauge-invariant scalars, rather than the metric. A reconstruction procedure for the metric perturbation out of the Teukolsky variables (in vacuum) was prescribed by Chrzanowski [49] (with later supplements by Wald [160] and Ori [123]), but only in the radiation gauge. It is not known how to similarly reconstruct the metric in the Lorenz gauge.

Several strategies have been proposed for dealing with this gauge-related difficulty. Some involve a deviation from the original MiSaTaQuWa notion of SF, while others seek to tackle the calculation of the Lorenz-gauge perturbation directly. Here is a survey of the main strategies.

Self-force in a ‘hybrid’ gauge. Equation (20) in section 2 describes the gauge transformation of the SF. Let us refer here to a certain gauge as ‘regular’ if the transformation from the Lorenz gauge yields a well-defined SF in that gauge (this requires that the expression in square brackets in equation (20) admits a definite and finite particle limit). It has been shown [21] that the mode-sum formula maintains its form (36), with the same regularization parameters, for any such regular gauges. Namely, for the SF in a specific regular gauge ‘reg’ we have the mode-sum formula

$$F_{\alpha}^{\text{self}}[\text{reg}] = \sum_{l=0}^{\infty} \left[ F_{\alpha}^{\text{rad}}[\text{reg}] \mp L A^\alpha - B^\alpha \right],$$  \hspace{1cm} (59)$$

where the force modes $F_{\alpha}^{\text{rad}}[\text{reg}]$ are those constructed from the retarded metric perturbation in the ‘reg’ gauge (we have already set $C^\alpha = D^\alpha = 0$ here).

Now recall the radiative inspiral problem which motivates us here: although the momentary SF is gauge dependent, the long-term radiative evolution of the orbit (as expressed, for example, through the drift of the constants of motion) has gauge-invariant characteristics that should be accessible from the SF in whatever regular gauge. And so, insofar as the physical inspiral problem is concerned, one might have hoped to circumvent the gauge problem by simply evaluating the SF in any gauge which is both regular and practical, using equation (59). Unfortunately, while the Lorenz gauge itself is ‘regular but not practical’, both the Regge–Wheeler gauge and the radiation gauge are generally ‘practical but not regular’, as demonstrated in [21].

However, a practical solution now suggests itself: devise a gauge which is regular in the above sense, and yet practical in that it relates to one of the ‘practical’ gauges—say, the radiation gauge—through a simple, explicit gauge transformation (unlike the Lorenz gauge itself). Heuristically, one may picture such a ‘hybrid’ gauge (also referred to as ‘intermediate’ gauge [24]) as one in which the metric perturbation retains its isotropic Lorenz-like form near the particle, while away from the particle it deforms so as to resemble the radiation-gauge perturbation. The SF in such a hybrid gauge would have the mode-sum formula

$$F_{\alpha}^{\text{self}}[\text{hyb}] = \sum_{l=0}^{\infty} \left[ F_{\alpha}^{\text{rad}}[\text{reg}] \mp L A^\alpha - B^\alpha \right] - \delta D^\alpha,$$  \hspace{1cm} (60)$$

where $F_{\alpha}^{\text{rad}}[\text{reg}]$ are the $l$ modes of the full force in the radiation gauge, and the ‘counter term’ $\delta D^\alpha$ is the difference $F_{\alpha}^{\text{rad}}[\text{rad}] - F_{\alpha}^{\text{hyb}}[\text{hyb}]$, summed over $l$ and evaluated at the particle. With a
suitable choice of the hybrid gauge, the term $\delta D^\mu$ can be calculated analytically, and equation (60) then prescribes a practical way of constructing the SF in a useful, regular gauge, out of the numerically calculated modes of the perturbation in the radiation gauge.

Different variants of this idea were studied by several authors [21, 24, 114, 121, 144], but it has not been implemented in full so far.

Generalized SF and gauge invariants. Another idea (set out in [21] and further developed in [77]) involves the generalization of the SF notion through the introduction of a suitable averaging over angular directions. In some gauges which are not strictly regular in the aforementioned sense, the SF could still be defined in a directional sense. Such is the case in which the expression in square brackets in equation (20) has a finite yet direction-dependent particle limit (upon transforming from the Lorenz gauge), and the resulting ‘directional’ SF is bounded for any chosen direction. (This seems to be the situation in the Regge–Wheeler gauge, but not in the radiation gauge—in the latter, the metric perturbation from a point particle develops a one-dimensional string-like singularity [21].) In such cases, a suitable averaging over angular directions introduces a well-defined notion of an ‘average’ SF, which generalized the original MiSaTaQuWa SF (it represents a generalization since, obviously, the average SF coincides with the standard MiSaTaQuWa SF for all regular gauges, including Lorenz’s). The notion of an average SF could be a useful one if it can be used in a simple way to construct gauge-invariant quantities which describe the radiative motion. This is yet to be demonstrated.

A related method invokes the directional SF itself as an agent for constructing the desired gauge invariants. In this approach, one defines (for example) a ‘Regge–Wheeler’ SF by taking a particular directional limit consistently throughout the calculation, and then using the value of this SF to construct the gauge invariants. This approach has been applied successfully, in combination with the mode-sum method, by Detweiler and others [53, 54].

Radiation-gauge regularization. Friedman and collaborators [67, 95, 96] proposed the following construction: starting with the Lorenz-gauge $S$-field, construct the associated gauge-invariant Newman–Penrose scalar $\psi_0^S$, and decompose it into spin-weighted spheroidal harmonics. Then obtain the harmonics of the retarded field $\psi_0^{\text{ret}}$ by solving the Teukolsky equation with suitable boundary conditions, and (for each harmonic) define the $R$ part through $\psi_0^R \equiv \psi_0^{\text{ret}} - \psi_0^S$. If $\psi_0^S$ is known precisely, then $\psi_0^R$ is a vacuum solution of the Teukolsky equation. To this solution, then, apply Chrzanowski’s reconstruction procedure to obtain a smooth radiation-gauge metric perturbation $h_{0ab}^{\text{rad}}$, and use that to construct a ‘radiation gauge’ SF (via, e.g., the mode-sum method). The relation between this definition of the radiation-gauge SF and the one obtained by applying the gauge transformation formula (20) to the standard Lorenz-gauge MiSaTaQuWa SF is yet to be investigated.

In reality, the $S$-field is usually known only approximately, resulting in that $\psi_0^S$ retains some non-smoothness. How to apply Chrzanowski’s reconstruction to non-smooth potentials is a matter of appreciable technical challenge. Also, the above procedure cannot account for the contribution to the SF from the two non-radiative modes, $l = 0, 1$, which then need to be treated separately, using other methods. Nonetheless, the technique offers a natural way of circumventing the gauge problem, and has much potential promise.

Direct Lorenz-gauge implementation. In 2005, Barack and Lousto [19] succeeded in solving the full set of Lorenz-gauge perturbation equations in Schwarzschild, using numerical evolution in 1+1D. (An alternative formulation was later developed and implemented by Berndtson [33].) This development opened the door for a direct implementation of the mode-sum formula in the Lorenz gauge. It later facilitated the first calculations of the gravitational SF for bound orbits in Schwarzschild [25, 26, 143].
In section 6 we shall review this approach in some detail. Here we just point to a few of its advantages: (i) this direct approach obviously circumvents the gauge problem. The entire calculation is done within the Lorenz gauge, and the mode-sum formula can be implemented directly, in its original form. (ii) The Lorenz-gauge perturbation equations take a fully hyperbolic form, making them particularly suitable for numerical implementation in the time domain. Conveniently, the supplementary gauge conditions (which take the form of elliptic ‘constraint’ equations) can be made to hold automatically, as we explain in section 6. (iii) In this approach one solves directly for the metric perturbation components, without having to resort to complicated reconstruction procedures. This is an important advantage because metric reconstruction involves differentiation of the field variables, which inevitably results in loss of numerical accuracy. (iv) Working with the Lorenz-gauge perturbation components as field variables is also advantageous in that these behave more regularly near point particles than do Teukolsky’s or Moncrief’s variables. This has a simple manifestation, for example, within the 1+1D treatment in Schwarzschild: the individual multipole modes of the Lorenz-gauge perturbation are always continuous at the particle, just like in the simple example of section 4.1; on the other hand, the multipole modes of Teukolsky’s or Moncrief’s variables are discontinuous at the particle, and so are, in general, the modes of the metric perturbation in the Regge–Wheeler gauge. Obviously, the continuity of the Lorenz-gauge modes makes them easier to deal with as numerical variables.

5.2. Numerical representation of the point particle

Common to all numerical implementation methods is the basic preliminary task of solving the field equations (in whatever formulation) for the full (retarded) perturbation from a point particle in a specified orbit. This immediately brings about the practical issue of the numerical representation of the particle singularity. The particulars of the challenge depend on the methodological framework: in time-domain methods one faces the problem of dealing with the irregularity of the field variables near the worldline; in frequency-domain (spectral) treatments, such irregularity manifests itself in a problematic high-frequency behavior. We now survey some of the relevant methods.

5.2.1. Particle representation in the time domain. Extended-body representations. In the context of fully nonlinear Numerical Relativity, the problem of a binary black hole with a small mass ratio remains a difficult challenge because of the large span of lengthscales intrinsic in this problem. (Current NR technology can handle mass ratios as small as 1:10 [76]—still nothing near the $\sim 1:10^{8}$ ratios needed for LISA EMRI applications.) Bishop et al [34] attempted a NR treatment in which the particle is modeled by a quasi-rigid widely-extended body whose ‘center’ follows a geodesic. Comparison with perturbation results did not show sufficient accuracy, and the method requires further development.

An extended-body approach has also been implemented in perturbative studies. Khanna et al [41, 97, 101] solved the Teukolsky equation in the time domain (i.e. in 2+1D, for each azimuthal $m$ mode) with a source ‘particle’ represented by a narrow Gaussian distribution. This crude technique was much improved recently by Sundararajan et al [152, 153] using a ‘finite impulse representation’, whereby the source is modeled by a series of spikes whose relative magnitude is carefully controlled so as to assure that the source has integral properties similar to that of a delta function. Such methods were demonstrated to reproduce wave-zone solutions with great accuracy (indeed, that is what they are designed to do), but they are likely to remain less useful for computing the accurate local perturbation near the particle as required in SF calculations.
Extended-body representations suffer from the inevitable tradeoff between smoothness and localization: one can only smoothen the solution by making it less localized, and one can better localize it only by making it less smooth. In what follows we concentrate on methods in which the source particle is precisely localized on the orbit: the energy–momentum of the particle in represented by a delta-function source term (as in equation (22)), and the delta distribution is treated analytically within the numerical scheme, in an exact manner.

**Delta-function representation in 1+1D.** In full 3+1D spacetime, the full (retarded) metric perturbation obviously diverges on the particle (at any given time). The divergence is asymptotically Coulomb-like in the Lorenz gauge (and can take a different form in other gauges). In spherically-symmetric spacetimes one can decompose the perturbation into tensor harmonics and solve a separated version of the field equations in 1+1D (time+radius) for each harmonic separately. In the particle problem this becomes beneficial not only thanks to the obvious dimensional reduction, but also because it mitigates the problems introduced by the particle’s singularity: the angular integration involved in constructing the individual \( l \) modes effectively ‘smears’ the Coulomb-like singularity across the surface of a 2-sphere, and the resulting \( l \) modes are finite even at the location of the particle. Furthermore, in the case of the metric perturbation in the Lorenz gauge, the individual \( l \) modes are also continuous at the particle (cf equation (30) in section 4). The corresponding \( l \) modes of the Teukolsky or Moncrief gauge-invariant variables are generally not continuous at the particle, and generally neither are the modes of the metric perturbation in non-Lorenz gauges.

The boundedness of the \( l \) modes is, of course, a crucial feature of the \( l \)-mode regularization scheme, as we have already discussed. That same feature also greatly simplifies the 1+1D numerical treatment of the particle. Lousto and Price [106] formulated a general method for incorporating a delta-function source in a finite-difference treatment of the field equations in 1+1D. In this method, the finite-difference approximation at a numerical grid cell (in \( t-r \) space) traversed by the particle’s worldline is obtained, essentially, by integrating the field equation ‘by hand’ over the grid cell at the required accuracy. The original (1D) delta function present in the source term thereby integrates out to contribute a finite term at each time step. The original Lousto–Price scheme (corrected for a typo by Martel and Poisson [108]) was formulated with a 2nd-order global numerical convergence, and later improved by Lousto [103] to achieve a fourth-order convergence. This simple but powerful idea is at the core of many of the 1+1D finite-difference implementations presented in the last few years [25, 80, 107], including the work discussed in section 6 below.

Despite such advances, 1+1D particles remain numerically expensive to handle, because the nonsmoothness associated with them introduces a large scale variance in the solutions: the \( l \)-mode field gradients grow sharply near the particle, and, moreover, become increasingly more difficult to resolve with larger \( l \) (recall the \( l \)-mode gradient is \( \propto l \) at large \( l \)). The mode-sum formula, recall, converges rather slowly (like \( \sim 1/l \)), and so requires one to compute a considerably large number of modes (typically \( \sim 20 \) with even a moderate accuracy goal). This proves to stretch the limit of what can be achieved today using finite differentiation on a fixed mesh.

Several methods have been proposed to address this problem in the current context. Sopuerta and collaborators [46, 47, 149, 150] explored the use of finite-element discretization. This technique is particularly powerful in dealing with multi-scale problems, and, being quasi-spectral, it benefits from an exponential numerical convergence. So far it was applied successfully for generic orbits in Schwarzschild, and higher-dimensional implementations (for Kerr studies) are currently being considered. A related quasi-spectral scheme was recently
suggested by Field et al [65]. Finally, Thornburg [157] very recently developed an adaptive mesh-refinement algorithm for Lousto–Price’s finite differences scheme (with a global fourth-order convergence). This was successfully implemented for a scalar charge in a circular orbit in Schwarzschild, and generalizations are being considered.

**Puncture methods.** In Kerr spacetime, one no longer benefits from a 1+1D separability. The Lorenz-gauge perturbation equations are only separable into azimuthal \( m \)-modes, each a function of \( t, r, \theta \) in a 2+1D space. The \( m \) modes are not finite on the worldline, but rather they diverge there logarithmically (see the discussion in section II.C of [17]). Since the 2+1D numerical solutions are truly divergent, a direct finite-difference treatment becomes problematic. However, since the singular behavior of the perturbation can be approximated analytically, a simple remedy to this problem suggests itself.

The idea, which has recently been studied independently by several groups [17, 105, 158, 159], is to utilize a new perturbation variable for the numerical time evolution, which we shall call here the *residual* field. This field is constructed from the full (retarded) perturbation by subtracting a suitable function—the ‘puncture’ field, given analytically—which approximates the singular part of the perturbation well enough that the residual field is (at least) bounded and differentiable at the particle. The perturbation equations are then recast with the residual field as their independent variable, and with a new source term (depending on the puncture field and its derivatives) which now extends off the worldline but contains no delta function. The equations are then solved for the residual field in the time domain, using, e.g., standard finite differentiation.

Several variants of this method have been studied and tested with scalar-field codes in 1+1D [158] and 2+1D [17, 105], and also proposed for use in full 3+1D [95]. The various schemes differ primarily in the way they handle the puncture function far from the particle: Barack and Golbourn [17] introduce a puncture with a strictly compact support around the particle, Detweiler and Vega [158] truncate it with a smooth attenuation function, and in Lousto and Nakano [105] the puncture is not truncated at all. We will discuss the puncture method in more detail in section 7.

To obtain the necessary input for the SF mode-sum formula, the 2+1D (or 3+1D) numerical solutions need to be decomposed into \( l \) modes, in what then becomes a somewhat awkward procedure (we decompose the field into separate \( l \) modes just to add these modes all up again after regularization). Fortunately, there is a more direct alternative: Barack et al [28] showed how the SF can be constructed directly, in a simple way, from the 2+1D \( m \)-modes of the residual field (assuming only that these mode are differentiable at the particle, which is achieved by designing a suitable puncture). This direct ‘\( m \)-mode regularization’ scheme, too, will be described in section 7. It is hoped that this technique could provide a natural framework for calculations in Kerr. It is yet to be applied in practice.

### 5.2.2. Particle representation in the frequency domain: the high-frequency problem and its resolution.

The \( l \) modes required as input for the SF mode-sum formula can also be obtained using a spectral treatment of the field equations. This has the obvious advantage that one then only deals with ordinary differential equations, although constructing the \( l \) modes involves the additional step of summing over sufficiently many frequency modes. (See [32] for a recent analysis of the relative computational efficiencies of frequency- versus time-domain treatments.) As with the time-domain methods discussed above, the representation of the particle in the frequency domain too brings about technical complications, but these now take a different form.
To illustrate the problem, consider the toy model of a scalar charge in Schwarzschild, allowing the particle to move on some bound (eccentric) geodesic of the background, with radial location given as a function of time by $r = r_p(t)$. Decompose the scalar field in spherical harmonics, and denote the multipolar mode functions by $\phi_{lm}(t, r)$. The time-domain modes $\phi_{lm}$ are continuous along $r = r_p(t)$ for each $l,m$. However, the derivatives $\phi_{lm,r}$ and $\phi_{lm,t}$ will generally suffer a finite jump discontinuity across $r = r_p(t)$ (recall our elementary example in section 4.1), which reflects the presence of a source ‘shell’ representing the $l,m$ mode of the scalar charge. In particular, if the orbit is eccentric, the derivatives of $\phi_{lm}$ will generally be discontinuous functions of $t$ at a fixed value of $r$ along the orbit.

Now imagine trying to reconstruct $\phi_{lm}(t, r)$ (for some fixed $r$ along the orbit) as a sum over its Fourier components:

$$\phi_{lm}(t, r) = \sum_\omega R_{lm\omega}(r) e^{-i\omega t}. \quad (61)$$

Since, for an eccentric orbit, $\phi_{lm}(t, r)$ is only a $C^0$ function of $t$ at the particle’s worldline, it follows from standard Fourier theory \[94\] that the Fourier sum in equation (61) will only converge there like $\sim 1/\omega$. The actual situation is even worse, because for SF calculations we need not only $\phi_{lm}$ but also its derivatives. Since, e.g., $\phi_{lm,r}$ is a discontinuous function of $t$, we will inevitably face here the well-known ‘Gibbs phenomenon’: the Fourier sum will fail to converge to the correct value at $r \rightarrow r_p(t)$. Of course, the problematic behavior of the Fourier sum is simply a consequence of our attempt to reconstruct a discontinuous function as a sum over smooth harmonics.

From a practical point of view this would mean that (i) at the coincidence limit, $r \rightarrow r_p(t)$, the sum over $\omega$ modes would fail to yield the correct one-sided values of $\phi_{lm,r}$ however many $\omega$ modes are included in the sum; and (ii) if we reconstruct $\phi_{lm,r}$ at a point $r = r_0$ off the worldline, then the Fourier series should indeed formally converge—however, the number of $\omega$ modes required for achieving a prescribed precision would grow unboundedly as $r_0$ approaches $r_p(t)$, making it extremely difficult to evaluate $\phi_{lm,r}$ at the coincidence limit.

This technical difficulty is rather generic, and will also show in calculations of the local gravitational field. The situation is no different in the Kerr case, because there too the mode-sum formula requires as input the spherical-harmonic modes of the perturbation field, and for each such mode the source is represented by a $\delta$-distribution on a thin shell, which renders the field derivatives discontinuous across that shell. The problem becomes even more severe when considering gravitational perturbations via the Teukolsky formalism: here, the $l,m$ modes of the perturbation field (now the Newman–Penrose curvature scalar) are not even continuous at the particle’s orbit—a consequence of the fact that the source term for Teukolsky’s equation involves (second) derivatives of the energy–momentum tensor. Again, a naive attempt to construct these multipoles as a sum over their $\omega$ modes will be hampered by the Gibbs phenomenon.

A simple way around the problem was proposed recently in \[29\]. It was shown how the desired values of the field and its derivatives at the particle can be constructed from a sum over properly weighted homogeneous (source-free) radial functions $R_{lm\omega}(r)$, instead of the actual inhomogeneous solutions of the frequency-domain equation. The Fourier sum of such homogeneous radial functions, which are smooth everywhere, converges exponentially fast. The Fourier sum of the derivatives, which are also smooth, is likewise exponentially convergent. The validity of the method (and the exponential convergence) was demonstrated in \[29\] with an explicit numerical calculation in the scalar-field monopole case ($l = 0$). It was later implemented in a frequency-domain calculation of the monopole and dipole modes of the Lorenz gauge metric perturbation for eccentric orbits in Schwarzschild.
The same method should be applicable for any of the other problems mentioned above, including the calculation of EM and gravitational perturbations using Teukolsky’s equation.

The method of [29] (dubbed method of extended homogeneous solutions) completely circumvents the problem of slow convergence (or the lack thereof) in frequency-domain calculations involving point sources. It makes the frequency-domain approach an attractive method-of-choice for some SF calculations. The method is now being implemented in first calculations of the scalar-field SF for Kerr orbits [162].

6. An example: gravitational self-force in Schwarzschild via 1+1D evolution in Lorenz gauge

As an example of a fully worked out calculation of the SF, we review here the work by Barack and Sago on eccentric geodesic in Schwarzschild [25, 143]. This work represents a direct implementation of the mode-sum formula in its original form (36). The decomposed Lorenz-gauge metric perturbation equations are integrated directly using numerical evolution in 1+1D. The numerical algorithm employs a straightforward fourth-order-convergent finite-difference scheme (a variant of the Lousto–Price method) on a fixed staggered numerical mesh based on characteristic coordinates. Below we briefly describe the perturbation formalism, discuss the numerical implementation in some more detail, and display some results.

6.1. Lorenz-gauge perturbation formalism

The linearized Einstein equations in the Lorenz gauge are given in equation (22). On the right-hand side of these equations is a distributional representation of the energy–momentum of a point particle of mass $\mu$, which is moving on a timelike geodesic $z^\mu(\tau)$ of the background spacetime. Mathematically, the linear set (22) is a diagonal hyperbolic system, which admits a well-posed initial-value formulation on a spacelike Cauchy hypersurface (see, e.g., theorem 10.1.2 of [161]). Furthermore, if the Lorenz gauge conditions of equation (14) are satisfied on the initial Cauchy surface, then they are guaranteed to hold everywhere—assuming that the field equations are satisfied everywhere and that the energy–momentum satisfies $\nabla^\beta T_{\alpha \beta} = 0$, as in the case of a geodesic particle. We recall that the gauge conditions (14) do not fully specify the gauge: there is a residual gauge freedom within the family of Lorenz gauges, $h_{\alpha \beta} \rightarrow h_{\alpha \beta} + \nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta$, with any $\xi_\mu$ satisfying $\nabla^\alpha \nabla_\alpha \xi_\mu = 0$. It is easy to verify that both equations (14) and (22) remain invariant under such gauge transformations.

Here we specialize to eccentric geodesics around a Schwarzschild black hole, and employ a Schwarzschild coordinate system $(t, r, \theta, \phi)$ in which the orbit is equatorial ($z^\theta = \pi/2$). Such orbits constitute a two-parameter family; we may characterize each orbit by the radial ‘turning points’ $r_{\text{min}}$ and $r_{\text{max}}$, or alternatively by the ‘semi-latus rectum’ $p \equiv 2(r_{\text{max}} r_{\text{min}})/(r_{\text{max}} + r_{\text{min}})$ and ‘eccentricity’ $e \equiv (r_{\text{max}} - r_{\text{min}})/(r_{\text{max}} + r_{\text{min}})$.

Barack and Lousto [19] decomposed the metric perturbation into tensor harmonics, in the form

$$h_{\alpha \beta}^{\text{ret}} = \frac{\mu}{r} \sum_{l,m} \sum_{i=1}^{10} h^{(i)lm}(r, t) Y_{i}^{l}Y_{i}^{lm}(\theta, \phi),$$

and similarly for the source $T_{\alpha \beta}$. The harmonics $Y_{i}^{l}Y_{i}^{lm}(\theta, \phi)$ (whose components are constructed from ordinary spherical harmonics and their first and second derivatives) form a complete orthogonal basis for second-rank covariant tensors on a 2-sphere (see appendix A of [19]). The time-radial functions $h^{(i)lm}$ ($i = 1, \ldots, 10$) form our basic set of perturbation
fields, and serve as variables for the numerical evolution. The tensor-harmonic decomposition decouples equation (22) with respect to $l, m$, although not with respect to $i$: for each $l, m$, the variables $\bar{h}^{(i)lm}$ satisfy a coupled set of hyperbolic (in a 1+1D sense) scalar-like equations, which may be written in the form

$$\Box^{(2)} \bar{h}^{(i)lm} + \mathcal{M}^{(i)l(j)} \bar{h}^{(j)lm} = S^{(i)lm} \quad (i = 1, \ldots, 10).$$

(63)

Here $\Box^{(2)}$ is the 1+1D scalar-field wave operator $\partial^2 + V(r)$, where $v$ and $u$ are the standard Eddington–Finkelstein null coordinates, and $V(r) = \frac{1}{4} (1 - 2M/r) [2M/r^3 + l(l+1)/r^2]$ is an effective potential. The ‘coupling’ terms $\mathcal{M}^{(i)l(j)} \bar{h}^{(j)lm}$ involve first derivatives of the $\bar{h}^{(j)lm}$’s at most (no second derivatives), so that, conveniently, the set (63) decouples at its principal part.

The decoupled source terms $S^{(i)lm}$ are each $\propto \delta [r - r_p(\tau)]$ (no derivatives of the $\delta$ function) and, as a result, the physical solutions $\bar{h}^{(j)lm}$ are continuous even at the particle. Explicit expressions for the coupling terms and the source terms in equation (63) can be found in [19].

In addition to the evolution equations (63), the functions $\bar{h}^{(i)lm}$ also satisfy four first-order elliptic equations, which arise from the separation of the gauge conditions (14) into $l, m$ modes. In the continuum initial-value problem, the solutions $\bar{h}^{(i)lm}$ satisfy these ‘constraints’ automatically if only they satisfy them on the initial Cauchy surface. This is more difficult to guarantee in a numerical finite-differences treatment, where (i) it is often impossible to prescribe exact initial data that satisfy the constraints, and (ii) discretization errors may amplify constraint violations during the numerical evolution. Inspired by a remedy proposed for a similar problem in the context of nonlinear Numerical Relativity [19, 79] proposed the inclusion of ‘divergence dissipation’ terms, $\propto \nabla \mathcal{B} \bar{h}^{\text{ret}}$, in the original set (22), so designed to guarantee that any violations of the Lorenz gauge conditions are efficiently damped during the evolution. These damping terms modify only the explicit form of the $\mathcal{M}$ terms in equation (63) as shown in [19].

6.2. Numerical implementation

The code developed by Barack and Sago [25, 143] solves the coupled set (63) (with constraint dissipation terms incorporated in the $\mathcal{M}$ terms) via time evolution. The numerical domain, covering a portion of the external Schwarzschild geometry, is depicted in figure 3. The numerical grid is based on Eddington–Finkelstein null coordinates $v, u$, and initial data (the values of the ten fields $\bar{h}^{(i)lm}$ for each $l, m$) are specified on two characteristic initial surfaces $v = \text{const}$ and $u = \text{const}$. Equations (63) are then discretized on this grid using a finite-difference algorithm which is globally fourth-order convergent. The numerical integrator solves for the various $\bar{h}^{(i)lm}$’s along consecutive $u = \text{const}$ rays. A particularly convenient feature of this setup is that no boundary conditions need be specified (the characteristic grid has no causal boundaries). Moreover, one need not be at all concerned with the determination of faithful initial conditions: it is sufficient to set $\bar{h}^{(i)lm} = 0$ on the initial surfaces and simply let the resulting spurious radiation (which emanate from the intersection of the particle’s worldline with the initial surface) dissipate away over the evolution time. The early part of the evolution, which is typically dominated by such spurious radiation, is simply discarded.

The conservative modes $l = 0$ and $l = 1$ (respectively the monopole and dipole) require a separate treatment, as they do not evolve stably using the above numerical scheme. (A naive attempt to evolve these modes leads to numerical instabilities which, so far, could

6 To simplify the appearance of equation (62) we have used here a normalization of $\bar{h}^{(i)lm}$ which is slightly different from that of [19].
not be cured.) Fortunately, the set (63) simplifies considerably for these modes, and solutions can be obtained in a semi-analytic manner based on physical considerations. Detweiler and Poisson [55] worked out the $l = 0, 1$ Lorenz-gauge solutions for circular orbits, and their work is generalized to eccentric orbits in [30], relying on the aforementioned method of extended homogeneous solutions. The calculation of [30] yields the values of the fields $\bar{h}^{(i)lm}$ and their derivatives for $l = 0, 1$.

To construct the $l$ modes $F^{\alpha l}_{\mathrm{ret}}$—the necessary input for the mode-sum formula (36)—one first substitutes the expansion (62) in the expression for the retarded force $F^{\alpha}_{\mathrm{ret}}$ (left-hand side expression in equation (25)), and then expands the result in spherical harmonics. The outcome is a formula for $F^{\alpha l}_{\mathrm{ret}}$ given in terms of the various fields $\bar{h}^{(i)lm}$ and their derivatives (evaluated at the particle and summed over $m$). The number of $l'$-modes $\bar{h}^{(i)lm}$ contributing to each $l$-mode $F^{\alpha l}_{\mathrm{ret}}$ depends on the off-worldline extension chosen for $\nabla_{\alpha\beta\gamma}$. In [25, 143] we used a convenient extension in which (referring to equation (9) and figure 1) the metric $g^{\alpha\beta}$ takes its value at the field point $x$ while the components $u^\alpha$ (in Schwarzschild coordinates) take the fixed value they have at $z$. In this extension we find that the only contributing $l'$-modes, for given $l$, are $l' - 3 \leq l' \leq l + 3$. Explicit construction formulas for $F^{\alpha l}_{\mathrm{ret}}$ can be found in [25, 143]. One then uses the numerical values of the fields $\bar{h}^{(i)lm}$ and their derivatives, as calculated along the orbit using the above code, to construct $F^{\alpha l}_{\mathrm{ret}}$ for sufficiently many $l$ modes. The mode-sum formula (36) then gives the SF. Figure 4 shows the final result for an eccentric geodesic with $p = 7M$ and $e = 0.2$.

The calculation we have just described represents a milestone in the SF program: we now finally have a code able of tackling the generic EMRI-relevant SF problem in Schwarzschild. (Indeed, to address the fully generic physical problem would require a final generalization to the Kerr case!) Using this code, and similar codes developed independently by others [54], we can now begin to explore the physical consequences of the gravitational SF—particularly those effects associated with its conservative piece. Work done so far includes (i) study of gauge invariant SF effects on circular orbits [54]; (ii) comparison of SF calculations and
results from PN theory in the weak field regime [54, 147]; (iii) comparison of SF results from different calculation schemes and using different gauges [147]; and (iv) a calculation of the shift in the location and frequency of the innermost stable circular orbit (ISCO) due to the conservative piece of the SF [26]. Work is in progress to calculate SF-related precession effects for eccentric orbits. In section 8 we will describe some of the recent work to explore the physical consequences of the gravitational SF.

7. Toward self-force calculations in Kerr: the puncture method and m-mode regularization

The time-domain Lorenz-gauge treatment of section 6 relies crucially on the separability of the field equations into (tensorial) spherical harmonics, which is no longer possible in Kerr. In the Kerr problem one can at best separate the metric perturbation equations into azimuthal $m$-modes, using the substitution

$$\bar{h}_{ab} = \sum_{m=-\infty}^{\infty} \bar{h}^m_{ab}(t, r, \theta) e^{imp}. \quad (64)$$

Then one faces solving the coupled set for the 2+1D variables $\bar{h}^m_{ab}$. Insofar as vacuum perturbations are considered, this computational task is nowadays well within reach of even

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A full separation of variables in Kerr is possible within Teukolsky’s formalism, which, alas, brings about the metric reconstruction and gauge-related difficulties discussed in previous sections. A full separation of the metric perturbation equations themselves, in Kerr, has not been formulated yet, to the best of our knowledge.
modest desktop computers. Indeed, over the past decade, 2+1D numerical evolution has been a method of choice in many studies of Kerr perturbations [17, 97–99, 101, 105, 128, 152]. The main challenge, rather, has to do with the inclusion of a \(\delta\)-function source in the 2+1D numerical domain. This is problematic, because the 2+1D variables \(\bar{h}_{\alpha\beta}^\delta\) suffer a singularity at the location of the particle. The puncture method, which we have mentioned briefly in section 5.2.1, overcomes this technical difficulty. In this section, we review this method (as implemented by Barack and Golbourn [17]) in some more detail. We also describe a new, ad hoc, regularization method—the m-mode regularization scheme—which enables a straightforward construction of the SF directly from the 2+1D numerical solutions, without resorting to a multipole decomposition.

### 7.1. Puncture method in 2+1D

In section 2, we have split the (trace-reversed) metric perturbation as \(\bar{h}_{\alpha\beta}^\text{ret} = \bar{h}_{\alpha\beta}^S + \bar{h}_{\alpha\beta}^R\), with the gravitational SF then obtained from the smooth field \(\bar{h}_{\alpha\beta}^R\) as prescribed in equation (18). Here we introduce a new splitting (defined below),

\[
\bar{h}_{\alpha\beta}^\text{ret} = \bar{h}_{\alpha\beta}^S + \bar{h}_{\alpha\beta}^R = \bar{h}_{\alpha\beta}^\text{punc} + \bar{h}_{\alpha\beta}^\text{res},
\]

so that

\[
\bar{h}_{\alpha\beta}^R = \bar{h}_{\alpha\beta}^\text{punc} - \bar{h}_{\alpha\beta}^S + \bar{h}_{\alpha\beta}^\text{res}.
\]

We denote the \(m\) modes of these various quantities, defined as in equation (64), by \(\bar{h}_{\alpha\beta}^{R,m}\), \(\bar{h}_{\alpha\beta}^{S,m}\), \(\bar{h}_{\alpha\beta}^{\text{punc},m}\), etc. The new splitting is defined (in a non-unique way) by introducing a puncture field \(\bar{h}_{\alpha\beta}^\text{punc}\), given analytically, which approximates \(\bar{h}_{\alpha\beta}^S\) near the particle well enough that the \(m\) modes of the resulting residual field \(\bar{h}_{\alpha\beta}^{\text{res},m}\) are continuous and differentiable on the particle’s worldline (and elsewhere). A particular such puncture is prescribed below (equation (76)). The form of the puncture function away from the particle can be chosen as convenient—e.g., in such a way that it can be decomposed into \(m\) modes explicitly, in analytic form.

The Lorenz-gauge perturbation equations (22) are now written in the form

\[
\nabla^\gamma \nabla_\gamma \bar{h}_{\alpha\beta}^\text{res} + 2 R^\mu_{\alpha}{}^\nu_\beta \bar{h}_{\alpha\beta}^{\text{res}} = S_{\alpha\beta}^{\text{res}},
\]

with

\[
S_{\alpha\beta}^{\text{res}} = -16\pi T_{\alpha\beta} - \nabla^\gamma \nabla_\gamma \bar{h}_{\alpha\beta}^{\text{punc}} - 2 R^\mu_{\alpha}{}^\nu_\beta \bar{h}_{\alpha\beta}^{\text{punc}},
\]

where \(-16\pi T_{\alpha\beta}\) is the original (distributional) source term appearing on the right-hand side of equation (22). The support of the source \(S_{\alpha\beta}^{\text{res}}\) now extends beyond the particle’s worldline, but it contains no \(\delta\) function on the worldline itself\(^8\). The equations are separated into \(m\) modes, with the \(m\)-mode source given by

\[
S_{\alpha\beta}^{\text{res},m}(t, r, \theta) \equiv \frac{1}{2\pi} \int_0^{2\pi} S_{\alpha\beta}^{\text{res}}(t, r, \theta, \varphi) e^{-im\varphi} d\varphi.
\]

If the puncture is sufficiently simple, the source \(S_{\alpha\beta}^{\text{res},m}\) can be evaluated in closed form (as in [17]). The \(m\)-mode field equations for the variables \(\bar{h}_{\alpha\beta}^{\text{res},m}(t, r, \theta)\), which are everywhere continuous and differentiable, can now be integrated numerically using straightforward finite differentiation on a 2+1D grid.

To ease the imposition of boundary conditions for \(\bar{h}_{\alpha\beta}^{\text{res},m}\), it is convenient to suppress the support of the puncture \(\bar{h}_{\alpha\beta}^{\text{punc}}\) away from the particle, so that the physical boundary conditions

\(^8\) This can be shown by integrating \(S_{\alpha\beta}^{\text{res}}\) over a small 3-ball containing the particle (at a given time), and then inspecting the limit as the radius of the ball tends to zero [17].
for $h_{\alpha\beta}^{\text{res}}$ become practically identical to that of the full retarded field $h_{\alpha\beta}$. In [17] this is achieved in a simple way by introducing an auxiliary ‘worldtube’ around the particle’s worldline (in the 2+1D space): inside this worldtube one solves the ‘punctured’ $m$-mode equations for $h_{\alpha\beta}^{\text{res}}$, while outside the worldtube one uses the original, retarded-field $m$-modes $h_{\alpha\beta}^{\text{ret,m}}$ as evolution variables; the value of the evolution variables is adjusted on the boundary of the worldtube using $h_{\alpha\beta}^{\text{ret,m}} = h_{\alpha\beta}^{\text{punc,m}} + h_{\alpha\beta}^{\text{ret,m}}$.

Two very similar variants of the puncture scheme have been developed and implemented independently by two groups [17, 105]—both for the toy model of a scalar charge on a circular orbit around a Schwarzschild black hole (refraining from a spherical harmonics decomposition and instead working in 2+1D). The method is yet to be applied in Kerr and for gravitational perturbations.

### 7.2. $m$-mode regularization

The 2+1D numerical solutions obtained using the puncture method can be used to construct the input modes $F_{\mu}^{\text{punc}}$ for the mode-sum formula (36), but this would require the additional step of a decomposition in spherical harmonics. It appears, however, that there is a simple way to construct the SF directly from the residual modes $h_{\alpha\beta}^{\text{res}}$. We describe it here as applied to the gravitational SF.

In analogy with equation (25), let us define the ‘force’ fields

$$F_{\alpha}^{\text{res}}(x) = \mu \tilde{\nabla}^{\alpha\beta} h_{\beta \gamma}^{\text{res}}, \quad F_{\alpha}^{\text{punc}}(x) = \mu \tilde{\nabla}^{\alpha\beta} h_{\beta \gamma}^{\text{punc}}.$$  

(70)

Then, recalling equations (18) and (66), we may write the SF at a point $z$ along the orbit as

$$F_{\alpha}^{\text{self}}(z) = \lim_{x \to z} \left[ F_{\alpha}^{\text{res}}(x) + \left( F_{\alpha}^{\text{punc}}(x) - F_{\alpha}^{\text{punc}}(x) \right) \right].$$  

(71)

Recall that the expression in square brackets ($=\mu \tilde{\nabla}^{\alpha\beta} h_{\beta \gamma}^{\text{punc}}$) is a smooth (analytic) function of $x$, and so the limit $x \to z$ is well defined. We proceed by prescribing a suitable puncture function $h_{\alpha\beta}^{\text{punc}}$.

As in previous sections, let us parameterize the particle’s (bound, timelike) geodesic orbit by proper time $\tau$, and let $z^{\alpha}(\tau)$ describe the orbit in Boyer–Lindquist coordinates. For an arbitrary spacetime point $x^{\alpha} = (t, r, \theta, \phi)$ outside the black hole, let $\Sigma_t$ be the spatial hypersurface $t = \text{const}$ containing $x$, let $\tilde{z}(t)$ be the value of $\tau$ at which $\Sigma_t$ intersects the particle’s worldline, and denote $\tilde{z}(t) \equiv [z(t)]$ and $\tilde{u}^{a} \equiv u^{a}(\tilde{z})$. This procedure assigns to any point $x$ outside the black hole a unique point $\tilde{z}$ on the worldline, with four-velocity $\tilde{u}^{a}$. We denote the coordinate distance between a given point $x$ and the point $\tilde{z}$ associated with it by $\delta x^{a} \equiv x^{a} - \bar{z}^{a}(t)$, with Boyer–Lindquist components $\delta x^{a} = (0, \delta r, \delta \theta, \delta \phi)$.

Consider now the local form of the $S$-field $\tilde{h}_{\alpha\beta}^{S}(x, \tilde{z})$, given in equation (19), with $x$ being an arbitrary point near the worldline, and $\tilde{z}$ being the worldline point $\tilde{z}$ associated with $x$. Recall in equation (19) that $\epsilon(x; \tilde{z})$ is the normal geodesic distance from $x$ to the worldline, and $\bar{u}^{a}(x; \tilde{z})$ is the four-velocity parallelly propagated from $\tilde{z}$ to $x$. The two quantities $\epsilon^{2}(x; \tilde{z})$ and $\bar{u}^{a}(x; \tilde{z})$ are well-defined and smooth functions of $x$ if $x$ is sufficiently close to the worldline (hence also close to $\tilde{z}(t)$). For $x \to \tilde{z}$ (hence $\delta x \to 0$) we have the asymptotic expansions

$$\epsilon^{2}(x, \tilde{z}) = \tilde{S}_{0} + \tilde{S}_{1} + O(\delta x^{2}), \quad \bar{u}^{a}(x, \tilde{z}) = \bar{u}^{a} + \delta u^{a} + O(\delta x^{2}),$$  

(72)

where $\tilde{S}_{0}$ and $\tilde{S}_{1}$ are, respectively, quadratic and cubic in $\delta x$, and $\delta u^{a}$ is linear in $\delta x$. Explicitly, these expansion coefficients read

$$\tilde{S}_{0} = F_{\alpha\beta} \delta u^{\alpha} \delta x^{\beta}$$  

(73)
\[ \delta \tilde{u}_\alpha = \Gamma_{\alpha \beta \gamma}^{\delta} \tilde{u}_\beta \delta x^\gamma, \]  
\[ \tilde{S}_1 = P_{\alpha \beta} \Gamma_{\beta \gamma}^{\delta} \delta x^\alpha \delta x^\beta \delta x^\gamma, \]  
\[ (74) \]

where \( \tilde{P}_{\alpha \beta} \equiv \tilde{g}_{\alpha \beta} + \tilde{u}_\alpha \tilde{u}_\beta \), \( \tilde{g}_{\alpha \beta} \equiv g_{\alpha \beta}(\xi) \) and \( \Gamma_{\beta \gamma}^{\delta} \equiv \Gamma_{\beta \gamma}^{\delta}(\xi) \) (for a derivation of \( \tilde{S}_1 \), see, for example, appendix A of \cite{22}). We now wish to regard equations (73)–(75) as definitions of the quantities \( \tilde{S}_0, \tilde{S}_1 \) and \( \delta \tilde{u}_\alpha \), taken to be valid far from the particle in order to control its global properties, but such modifications will not affect our discussion here.

The support of this function can be attenuated far from the particle in order to control its global properties, but such modifications will not affect our discussion here.

It is not difficult to show \cite{28} that, near the particle, the difference between the puncture (76) and the actual \( \tilde{S} \)-field given in equation (19) (with \( z \) taken to be \( \tilde{z} \)) has the form

\[ \tilde{h}_{\text{punc}}^{\alpha \beta}(x) = \frac{4 \mu (\tilde{u}_\alpha \tilde{u}_\beta + \delta \tilde{u}_\alpha \tilde{u}_\beta + \tilde{u}_\alpha \delta \tilde{u}_\beta)}{(S_0 + S_1)^{1/2}}, \]  
\[ (76) \]

The support of this function can be attenuated far from the particle in order to control its global properties, but such modifications will not affect our discussion here.

We now arrive at the crucial step of our discussion. In equation (71), for a given point \( z \) along the particle’s worldline, we express the (analytic) function in square brackets as a formal sum over \( m \)-modes, in the form

\[ F_{\text{self}}^a(z) = \lim_{x \to z} \sum_{m=-\infty}^{\infty} \left[ F_{\text{res}}^{a,m}(x) + (F_{\text{punc}}^{a,m}(x) - F_{\text{res}}^{a,m}(x)) \right], \]  
\[ (79) \]

where

\[ F_{\text{res}}^{a,m}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\text{res}}(t, r, \theta, \phi'; \tilde{z}) e^{im(\phi - \phi')} \, d\phi', \]  
\[ (80) \]

and similarly for \( F_{\text{punc}}^{a,m} \) and \( F_{\text{res}}^{a,m} \). Since the \( m \)-mode sum is formally a Fourier expansion, and since the Fourier expansion of an analytic function converges uniformly, we may replace the order of limit and summation in equation (79):

\[ F_{\text{self}}^a(z) = \sum_{m=-\infty}^{\infty} \lim_{x \to z} \left[ F_{\text{res}}^{a,m}(x) + (F_{\text{punc}}^{a,m}(x) - F_{\text{res}}^{a,m}(x)) \right]. \]  
\[ (81) \]

From equation (78), omitting the terms \( O(\delta x) \) as they cannot possibly affect the final value of the SF in equation (71), we have

\[ \lim_{x \to z} \left( F_{\text{punc}}^{a,m}(x) - F_{\text{res}}^{a,m}(x) \right) = \lim_{x \to z} \frac{1}{2\pi} e^{im\psi} \int_{-\pi}^{\pi} \tilde{\epsilon}_0^{-1} P_{\text{res}}^a e^{-im\phi'} \, d\phi', \]  
\[ (82) \]
where the integrand is evaluated at $\phi = \phi'$ and where we have used the fact that $x \to z$ implies also $x \to \bar{z}$. Crucially, one finds [28] that the last integral vanishes at the limit $x \to \bar{z}$, for any $m$, regardless of the explicit form of $P^{ij}_{\phi \psi}$. Hence, equation (81) reduces to

$$F^\alpha_{\text{self}}(z) = \sum_{m=-\infty}^{\infty} F^\alpha_{\text{res},m}(z).$$

Here the substitution $x = z$ is allowed since the limit $x \to z$ is known to be well defined (and, in particular, direction independent). Phrased differently, the modes $F^\alpha_{\text{res},m}$ corresponding to our puncture (76) are continuous at the particle, as desired.

It might perhaps seem suspicious that, in the above analysis, the sum over $m$-modes $F^\alpha_{\text{punc}} - F^\alpha_{\text{S}}$ (which are all zero at $x = z$) fails to recover the original function $F^\alpha_{\text{punc}} - F^\alpha_{\text{S}}$ (which is discontinuous at $x = z$ and hence indefinite there). This, however, should not come as a surprise. Recall that the formal Fourier sum at a step discontinuity (where the function itself is indefinite) is in fact convergent: it yields the two-side average value of the function at the discontinuity. Technically, this peculiarity of the formal Fourier expansion is due simply to the non-interchangeability of the sum and limit at the point of discontinuity.

Relatedly, we should emphasize that the full (4D) residual field, $F^\alpha_{\text{res}}$, does not yield the correct SF upon taking the limit $x \to z$: this field does not even have a well-defined limit $x \to z$, as we argued in the discussion below equation (78). However, the sum over the formal Fourier modes of $F^\alpha_{\text{res}}$ which indeed fails to reproduce $F^\alpha_{\text{res}}$ at the particle, does turn out to give the correct SF, as we have established in the discussion leading to equation (83).

It is possible to design an improved, higher-order-accurate puncture $\bar{h}^\alpha_{\text{punc}}$, so that the SF is indeed simply given by $F^\alpha_{\text{res}}(z)$. Such an improvement is necessary for 3+1D implementations of the puncture scheme, and is beneficial also in a 2+1D framework as it enhances the differentiability of the residual variable and improves the convergence of the $m$-mode sum in equation (83). However, such improvements entail an explicit calculation of higher-order terms in the $S$-field (including the term $w^\alpha_{\text{reg}}$ and possibly higher-order terms in equation (19)). A higher-order puncture, suitable for 3+1D implementation, was devised recently by Vega and collaborators [158, 159].

Our result (83) can be written explicitly in terms of the $m$ modes of the residual field $\bar{h}^\alpha_{\text{res},m}$ and easily so if we use the fixed off-worldline extension of $\bar{\nabla}^{\alpha\beta\gamma}$ (the choice of extension may affect the value of the individual $m$ modes, but not the eventual value of the SF). We then have

$$F^\alpha_{\text{self}} = \mu \sum_{m=-\infty}^{\infty} \hat{w}^{\alpha\beta\gamma}(\bar{h}^\alpha_{\text{res},m} \epsilon^{m\phi}),$$

where, of course, the right-hand side is evaluated at the particle. This formula prescribes a straightforward method for constructing the SF in a 2+1D framework. In the puncture scheme we effectively ‘regularize’ the field equations themselves (not their solutions, as in the standard $l$-mode regularization method), by writing them in the form (67) with a sufficiently accurate puncture function—like the one we give in (76). Once the $m$-modes of the residual perturbation have been calculated, the SF is constructed directly from these modes, via equation (84), with no need for further regularization. This ‘$m$-mode regularization scheme’, yet to be applied in actual calculations of the SF, offers a simple and efficient framework for such calculations in Kerr spacetime.

8. Physical effects of the self-force

With the development of the first gravitational SF codes in the past few years, it became possible to start exploring quantitatively the physical consequences of the SF. While the
ultimate goal of the SF program remains the description of the long-term orbital evolution, knowledge of the SF along geodesic orbits already gives access to some interesting physics associated with the $O(\mu)$ dynamics. Among the questions that one can address: How does the finite mass of the particle affect the rates of periapsis precession and orbital plane precession? How does it modify the location and frequency of the ISCO? Are there other, less familiar conservative SF effects that could manifest themselves in a characteristic way in the emitted gravitational waveforms? Can transient resonances between the radial and polar motion (for eccentric and inclined orbits in Kerr) have important observational implications [66]? Identifying and quantifying concrete SF effects that are gauge invariant and have a clear physical interpretation is also vital if one intends to cross-validate SF calculations carried out in different gauges [147], and in making a connection with results from PN theory [54]. Quantitative information about concrete SF effects (such as the $O(\mu)$ correction to the strong-field periapsis precession rate as a function of the geodesic parameters) can be incorporated 'by hand' into approximate (PN) models of EMRI orbital evolution [69, 90]. In this way, the study of SF effects for geodesic orbits can inform the improvement of EMRI models even before a reliable and workable scheme for the long-term evolution is at hand.

In this penultimate section we describe some of the recent initial investigations into the physical consequences of the gravitational SF.

8.1. Conservative and dissipative pieces of the SF

In discussing the physical consequences of the SF it is very useful to distinguish between ‘conservative’ and ‘dissipative’ effects. These are normally defined as effects arising, correspondingly, from the ‘time symmetric’ and ‘time antisymmetric’ pieces of the SF [87]. To formulate this more precisely, let us re-write equation (24) as

$$F^{\alpha}_{\text{self}(\text{ret})}(z) = \lim_{x \to z} \left[ F^{\alpha}_{\text{ret}}(x) - F^{\alpha}_{S}(x) \right],$$

where the label ‘ret’ is to remind us that the physical SF is derived from the physical, retarded metric perturbation (recall $F^{\alpha}_{\text{ret}} = \mu \bar{\nabla}^{\alpha \beta \gamma} x^{\bar{\beta}} h^{\text{ret}}_{\beta \gamma}$). Then introduce the force

$$F^{\alpha}_{\text{self}(\text{adv})}(z) = \lim_{x \to z} \left[ F^{\alpha}_{\text{adv}}(x) - F^{\alpha}_{S}(x) \right],$$

where $F^{\alpha}_{\text{adv}} = \mu \bar{\nabla}^{\alpha \beta \gamma} x^{\bar{\beta}} h^{\text{adv}}_{\beta \gamma}$ (namely, $F^{\alpha}_{\text{self}(\text{adv})}$ is obtained from $F^{\alpha}_{\text{self}(\text{ret})}$ by replacing the retarded perturbation with the advanced one). The conservative and dissipative pieces of the SF are then defined through

$$F^{\alpha}_{\text{cons}} = \frac{1}{2} \left( F^{\alpha}_{\text{self}(\text{ret})} + F^{\alpha}_{\text{self}(\text{adv})} \right), \quad F^{\alpha}_{\text{diss}} = \frac{1}{2} \left( F^{\alpha}_{\text{self}(\text{ret})} - F^{\alpha}_{\text{self}(\text{adv})} \right).$$

The physical SF is the sum of the two pieces: $F^{\alpha}_{\text{self}} = F^{\alpha}_{\text{self}(\text{ret})} = F^{\alpha}_{\text{cons}} + F^{\alpha}_{\text{diss}}$.

The dissipative force, $F^{\alpha}_{\text{diss}}$, is responsible (in particular) for the long-term secular drift in the value of the intrinsic ‘constants’ of motion—the energy $E$, azimuthal angular momentum $L_z$, and Carter constant $Q_c$, associated with the geodesic motion in Kerr. $F^{\alpha}_{\text{cons}}$, on the other hand, has no such long-term influence on the evolution of these orbital parameters. Here ‘long-term’ refers to a period of time much longer than the longest orbital period$^9$. It should be noted, however, that both $F^{\alpha}_{\text{diss}}$ and $F^{\alpha}_{\text{cons}}$ affect the momentary values of the intrinsic parameters $E$, $L_z$, and $Q_c$ (in a gauge-dependent way); in the case of $F^{\alpha}_{\text{cons}}$ this effect ‘averages out’ over many orbital periods, whereas for $F^{\alpha}_{\text{diss}}$ it accumulates. It should also be noted (what is usually less well recognized) that $F^{\alpha}_{\text{cons}}$, too, gives rise to secular, long-term effects—those associated with the evolution of the phase-type orbital elements [135–137].

$^9$ Recall that bound geodesics in Kerr are 3-periodic, with three characteristic frequencies corresponding to the azimuthal, radial and longitudinal motion.
How does one go about separating the full SF into its conservative and dissipative pieces in practice? In a frequency-domain framework this poses no difficulty: in addition to calculating the physical SF $F_{\text{self}(\text{ret})}$, one also calculates the advanced metric perturbation modes by suitably inverting the boundary conditions in the relevant ordinary differential equations, and use these modes to construct the quantity $F_{\alpha l}^{\text{adv}}$ (e.g. using the mode-sum formula (36), replacing $F_{\text{ret} l}^{\text{self}} \rightarrow F_{\text{adv} l}^{\text{self}}$). The conservative and dissipative pieces are then obtained using equation (87). This procedure was implemented, for example, in the analysis of [60]. The situation is slightly different in a time-domain framework, as one does not normally control the boundary conditions during the time evolution (cf section 6). Haas [81] proposed to obtain the necessary advanced SF data by reversing the time direction of the evolution: for a given orbit, one solves the relevant time-domain field equations once evolving forward in time as usual to obtain $F_{\text{self}(\text{ret})}$, and then again evolving backward in time (starting with initial Cauchy data specified in the future) to obtain $F_{\text{self}(\text{adv})}$.

There is a more computationally economic method for constructing $F_{\alpha}^{\text{cons/diss}}$ in the time domain, at least in the case of orbits that are either equatorial or circular. This method, first proposed and implemented in [26], makes use of the symmetries of Kerr geodesics, first noted in the current context by Mino [110]. It can be shown (see, e.g., section II.G of [87]) that, at any given point in Kerr spacetime, the following relation applies:

$$F_{\text{self}(\text{ret})} (u_t, u_r, u_{\theta}, u_{\phi}; \tau) = \epsilon(\alpha) F_{\text{self}(\text{adv})} (u_t, -u_r, -u_{\theta}, u_{\phi}; -\tau),$$

where

$$\epsilon(\alpha) = (-1, 1, 1, -1)$$

in Boyer–Lindquist coordinates (no summation over $\alpha$ on the right-hand side). Here we are treating the SF as a function of the four-velocity at the given spacetime point. Now consider a bound geodesic orbit in Kerr, which is either circular (and possibly inclined) or equatorial (and possibly eccentric). We parameterize this geodesic by the proper time $\tau$, and, if the orbit is eccentric, take $\tau = 0$ at one of the radial ‘turning points’ (i.e. where $dr/d\tau = 0$) without loss of generality. Now examine the retarded and advanced SFs $F_{\text{self}(\text{ret/adv})} (\tau)$ along this geodesic. It is clear from symmetry that

$$F_{\text{self}(\text{ret/adv})} (u_t, u_r, u_{\theta}, u_{\phi}; \tau) = \epsilon(\alpha) F_{\text{self}(\text{ret/adv})} (u_t, -u_r, -u_{\theta}, u_{\phi}; -\tau),$$

From equation (88) it then follows that

$$F_{\text{self}(\text{ret/adv})} (\tau) = \epsilon(\alpha) F_{\text{self}(\text{ret/adv})} (-\tau),$$

and equation (87) gives

$$F_{\text{cons}} (\tau) = \frac{1}{2} \left( F_{\text{self}(\text{ret})} (\tau) + \epsilon(\alpha) F_{\text{self}(\text{ret})} (-\tau) \right),$$

$$F_{\text{diss}} (\tau) = \frac{1}{2} \left( F_{\text{self}(\text{ret})} (\tau) - \epsilon(\alpha) F_{\text{self}(\text{ret})} (-\tau) \right).$$

Since in time-domain calculations the physical (retarded) SF is computed along an entire radial period (at least), equations (92) and (93) can be used to obtain the conservative and dissipative components of this force anywhere along the orbit without resorting to a calculation of the advanced perturbation.

The above trick can, of course, be implemented for any geodesic orbit in Schwarzschild (as in this case the orbit can always be taken as ‘equatorial’). However, it cannot be applied for orbits in Kerr that are both eccentric and inclined, because for such orbits the symmetry relation (90) does not hold in general. Note that, for circular orbits, equations (92) and (93) immediately yield $F_{\text{cons}} = F_{\text{cons}}' = 0$ as well as $F_{\text{diss}} = F_{\text{diss}}' = 0$, so in this case the $t$ and
\( \phi \) components of the SF are purely dissipative, while the \( r \) and \( \theta \) components are purely conservative.

Finally, it is useful to have at hand separate mode-sum formulas in the manner of (36) for the conservative and dissipative pieces. To obtain such formulas, first write a mode-sum expression for \( F_{\alpha}^{self(adv)} \) by replacing \( F_{\alpha}^{ret \pm} \rightarrow F_{\alpha}^{adv \pm} \) in equation (36). Here \( F_{\alpha}^{adv \pm} \) are derived from the modes of the advanced metric perturbation in just the same way as \( F_{\alpha}^{ret \pm} \) are derived from the modes of the retarded perturbation. (The same regularization parameters apply to both retarded and advanced forces, because the corresponding metric perturbations share the same singular piece.) Then substitute the mode-sum expressions for \( F_{\alpha}^{self(ret)} \) and \( F_{\alpha}^{self(adv)} \) in equation (87). This gives

\[
F_{\alpha}^{cons} = \sum_{l=0}^{\infty} \left( \frac{F_{\alpha}^{cons \pm} \pm LA_{\alpha}}{2} \right),
\]

\[
F_{\alpha}^{diss} = \sum_{l=0}^{\infty} F_{\alpha}^{diss \pm},
\]

where we have used \( C_{\alpha} = D_{\alpha} = 0 \) and introduced \( F^{cons \pm}_{\alpha} \equiv \left( F^{ret \pm}_{\alpha} + F^{adv \pm}_{\alpha} \right)/2 \) and \( F^{diss \pm}_{\alpha} \equiv \left( F^{ret \pm}_{\alpha} - F^{adv \pm}_{\alpha} \right)/2 \). Note that the dissipative component of the SF requires no regularization.

In fact, one can show that the mode sum in equation (95) converges exponentially fast. For that reason, the computation of the dissipative piece of the SF via the mode-sum method is technically much less challenging than that of the conservative piece.

8.2. Dissipation of energy and angular momentum

Perhaps the most familiar aspect of the self-interaction physics in the binary context is the long-term radiative decay of the orbit. In the language of the SF, we say that the work done on the particle by the dissipative component of the SF converts orbital energy (and angular momentum) into gravitational-wave energy (and angular momentum). The relation between the SF and dissipation is immediately evident from the equation of motion (21), whose (Boyer–Lindquist) \( t \) and \( \phi \) components read, respectively,

\[
\mu \frac{d \mu}{d \tau} = F_{t}^{self}, \quad \mu \frac{d \phi}{d \tau} = F_{\phi}^{self}.
\]

In the absence of SF (that is, in the test-particle limit), equations (96) tell us that \( u_t \) and \( u_\phi \) are constants of the motion, and we interpret these as the orbital energy, \( E \equiv -u_t \), and azimuthal angular momentum, \( L_\phi \equiv u_\phi \)—both per unit charge \( \mu \).\(^{10}\) Equations (96) also tell us how \( E \) and \( L_\phi \) change under the effect of the SF. This effect is not entirely dissipative: the SF includes periodic components which do not lead to a net long-term change in the values of \( E \) and \( L_\phi \). The non-periodic component of \( F_t \) and \( F_\phi \) does lead to a secular drift in the values of \( E \) and \( L_\phi \), interpreted as dissipation.

This non-periodic component is entirely contained in the dissipative piece of \( F_t^{self} \) and \( F_\phi^{self} \), as expected. This is immediately clear from equation (92) in the case of equatorial or circular orbits in Kerr (and for all orbits in Schwarzschild): the anti-symmetry of \( F_{t}^{cons}(\tau) \) and \( F_{\phi}^{cons}(\tau) \) with respect to \( \tau \rightarrow -\tau \) means that these pieces of the SF vanish upon integration over one complete (radial or longitudinal) period, and thus produce no net change in the values of \( E \).

\(^{10}\) This interpretation is motivated by the observation that \(-u_t \) and \( u_\phi \) are, in fact, projections of the four-velocity onto the Killing vectors associated with the invariance of the Kerr background under (respectively) time translations and spatial rotations.

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and $L_z$ over that period (this, of course, assumes that the orbit is very nearly geodesic over one period). The same applies generically for all orbits in Kerr, if the orbital integration is taken over sufficiently many periods (formally, infinitely many). These statements are encapsulated in the relations

$$\mu \langle \dot{E} \rangle = -\langle F_\phi / u' \rangle = -\langle F_{\phi \text{diss}} / u' \rangle, \quad \mu \langle \dot{L}_z \rangle = \langle F_\psi / u' \rangle = \langle F_{\psi \text{diss}} / u' \rangle,$$

(97)

where an overdot represents $d/dr$ (hence the factor $1/u' = dr/dt$ on the right-hand side), and $\langle \cdot \rangle$ indicates an average over sufficiently long time. One always finds $\langle F_{\phi \text{diss}} \rangle > 0$ and $\langle F_{\psi \text{diss}} \rangle < 0$ (see, e.g., [122]), so that $\langle E \rangle < 0$ and $\langle L_z \rangle < 0$, and orbital energy and angular momentum are lost over time as one expects. The time-average quantities in equation (97) are expected to be independent of the gauge even though the momentary values of $F_{\phi \text{diss}}$ and $F_{\psi \text{diss}}$ are themselves gauge dependent.

Implicit in the above discussion is the assumption that the orbit is indeed ‘available’ for the necessary time averaging, i.e. that $E$ and $L_z$ evolve very slowly over the required averaging time—otherwise the averaging procedure would be meaningless. Such ‘adiabaticity’ requirement (formulated more accurately, e.g., in [91]) is believed to hold well in LISA-relevant EMRIs throughout the inspiral and until very close to the innermost stable orbit—see, e.g., [91] for a more quantitative discussion of this point.

The loss of orbital energy and angular momentum is precisely balanced by the flux of the corresponding quantities in the gravitational waves radiated to infinity and down into the black hole. Using equation (97) we can express this balance in terms of the local SF:

$$\mu^{-1} \langle F_{\phi \text{diss}} / u' \rangle = \langle E_{\text{GW}} \rangle_{\text{eh}} + \langle E_{\text{GW}} \rangle_{\infty},$$

$$-\mu^{-1} \langle F_{\psi \text{diss}} / u' \rangle = \langle L_z^{\text{GW}} \rangle_{\text{eh}} + \langle L_z^{\text{GW}} \rangle_{\infty}.$$  

(98)

The quantities on the right-hand side are the time-averaged asymptotic fluxes down the event horizon (‘eh’) and out to infinity (‘$\infty$’), with the convention that the infinity fluxes are positive. This convention also fixes the sign of the horizon fluxes for a given orbit, but it should be noted that the latter can be either positive or—for certain orbits in Kerr—negative. Negative horizon fluxes reflect a superradiance behavior, whereby energy and angular momentum are, in effect, transferred from the ergosphere of the Kerr hole to the orbit [75, 91, 156]. The formal proof of the balance equations (98) involves the application of the Gauss theorem in the 3-volume bounded between a 2-sphere at $r \to \infty$ and another one just outside the horizon; how this is done is demonstrated, for example, in appendix D of [54] (in the special case of circular orbits in Schwarzschild).

The asymptotic fluxes $\langle E_{\text{GW}} \rangle_{\infty}$ and $\langle L_z^{\text{GW}} \rangle_{\infty}$ are calculated across a large 2-sphere $r \to \infty$ residing in the ‘wave zone’, where the background’s radius of curvature is much larger than the characteristic wavelength of the emitted gravitational waves. In this case Isaacson’s effective energy–momentum tensor formulation [93] applies, and one can use it to calculate the fluxes at infinity. This is a standard calculation in black hole perturbation theory, and since the early 1970s it has been performed many times using frequency-domain methods (e.g. [75, 91, 130, 156]) and more recently within a time-domain framework [19, 25, 107]. There is also a standard prescription for calculating the horizon fluxes $\langle E_{\text{GW}} \rangle_{\infty}$ and $\langle L_z^{\text{GW}} \rangle_{\infty}$; it is due to Teukolsky and Press [156] (based on the horizon perturbation formalism of Hawking and Hartle [84]) and is formulated in the frequency domain in terms of curvature scalars. A time-domain formulation of the horizon fluxes was more recently developed by Poisson [134].

The balance equations (98), with (97), allow us to infer the leading-order long-term evolution of $E$ and $L_z$ without resorting to an explicit calculation of the local SF (assuming adiabaticity). To get a full description of the time-average orbital decay one must, in general, also be able to calculate the evolution of the Carter constant $Q_c$. It is not difficult to write
down a relation analogous to (97) which expresses \( \langle \dot{Q}_c \rangle \) directly in terms of the local SF \([122]\), but there is no known analog to (98), relating \( \langle \dot{Q}_c \rangle \) to the asymptotic flux of radiation. However, thanks to a breakthrough idea by Mino \([110, 111]\) and follow-up work by Sago and collaborators \([73, 146]\), there is now known a practical formula for \( \langle \dot{Q}_c \rangle \) which does not require knowledge of the SF (it does require knowledge of the local advanced and retarded modes of the metric perturbation).

In conclusion, information on the time-averaged evolution of all three intrinsic ‘constants of motion’, \(E\), \(L_z\) and \(Q_c\), is directly accessible from the dissipative piece of the SF, but there are alternative methods (likely more computationally efficient) for accessing this information without resorting to the SF and to equation (98). However, from the perspective of the SF program development, the balance equations (98) are still a very useful tool for validating SF codes. Two independent tests are possible: first, since a SF code always involves a calculation of the metric perturbation itself, one can use this perturbation to construct the asymptotic fluxes of energy and angular momentum, and then use equations (98) to test the self-consistency of the code. Second, one can test the computed values of the dissipative SF against asymptotic flux data available in the literature (e.g. \([75, 107]\)). Such a twofold test is an important, reassuring check on the validity of a SF code. Explicit calculations demonstrating the balance relations (98) have been carried out so far for circular \([25]\) and eccentric \([143]\) orbits in Schwarzschild.

8.3. Conservative effects on circular orbits in Schwarzschild

Detweiler was first to explore quantitatively the conservative effects of the SF—in the simple example of circular geodesics outside a Schwarzschild black hole \([54]\). He pointed out two gauge-invariant quantities that carry non-trivial information about the conservative SF dynamics in this setting: the orbital frequency (with respect to time \(t\)),

\[ \Omega \equiv \frac{u^\phi}{u^t}, \]

and the contravariant \(t\) component of the four-velocity, denoted by

\[ U \equiv u^t. \]

Here \(u^\alpha = dx^\alpha/d\tau\) denotes the value of the perturbed four-velocity along the circular geodesic orbit, including SF corrections. We have \(u^r = 0\) along the orbit, and we also take (without loss of generality) \(u^\theta = 0\). The said gauge invariance of \(\Omega\) and \(U\) is restricted to transformations for which the gauge displacement vector \(\xi^\alpha\) respects the helical symmetry of the perturbed spacetime system, i.e. it satisfies

\[ (\partial_t + \Omega \partial_\phi) \xi^\alpha = 0 \]

through \(O(\mu)\), at least along the particle’s worldline. It is easy to see that all contravariant components \(u^\alpha\) are invariant through \(O(\mu)\) under such transformations \([147]\). The two nontrivial components \(u^t\) and \(u^\phi\) give two independent gauge invariants, which we can recombine to form Detweiler’s gauge-invariant variables \(\Omega\) and \(U\).

While the physical significance of the frequency \(\Omega\) is clear, that of \(U\) is slightly less obvious. Detweiler \([54]\) discusses two physical interpretations of \(U\). First, it is a measure of the gravitational red-shift experienced by photons emitted by the orbiting particle and observed at a distance. Second, \(U\) is intimately related to the helical Killing vector of the perturbed spacetime (with its gauge invariance being simply the statement that this Killing vector is invariant under gauge transformations satisfying equation (99)). Unfortunately, as we explain below, the conservative SF corrections to \(\Omega\) and \(U\) are not expected to have any short-term observable imprint on the gravitational waveforms emitted from the circular orbits.
Explicit expressions for $\Omega$ and $U$, including SF terms, are easily obtained from the $r$ component of the equation of motion (21), setting $u^r = 0$ and $du^r/d\tau = 0$. One finds

$$\Omega = \Omega_0 \left[ 1 - \frac{r_0(r_0 - 3M)}{2\mu M} F_r \right],$$

(102)

$$U = U_0 \left( 1 - \frac{r_0}{2\mu} F_r \right),$$

(103)

through $O(\mu)$, where $r_0$ is the orbital radius (Schwarzschild $r$ coordinate), and $\Omega_0 = (M/r_0^3)^{1/2}$ and $U_0 \equiv (1 - 3M/r_0)^{-1/2}$ are the geodesic (unperturbed) values of $\Omega$ and $U$, respectively. Recall that the radial component of the SF, $F_r$, is purely conservative in the case of a circular orbit; hence the SF terms in equations (102) and (103) represent conservative corrections to $\Omega$ and $U$. Of course, the two quantities would also evolve dissipatively (this effect is described by the $t$ and $\phi$ components of the equation of motion), but here we ignore the dissipative piece of the SF in order to study the effect of its conservative piece in isolation. Equations (102) and (103) tell us that the effect of the conservative SF is to ‘shift’ the values of $\Omega$ and $U$ as compared with their non-perturbed geodesic values (for a given orbital radius $r_0$). The conservative SF shift in $\Omega$, $\Delta \Omega \equiv \Omega - \Omega_0$, was first calculated (numerically, as a function of $r_0$) in [25], along with the conservative shift in the orbital energy $E$ (which is gauge dependent).

It should be understood that, despite the formal gauge invariance of $\Omega$, the relation $\Delta \Omega (r_0)$ is, in fact, gauge dependent, because the radius $r_0$ itself is gauge dependent (the calculation in [25] was carried out in the Lorenz gauge). To illustrate this point, suppose that we carry out two independent calculations of the SF in two different gauges, and we wish to test our results by comparing the values of $\Omega$ using equation (102). The problem we would face is that a given value of $r_0$ would, in general, correspond to two physically distinct orbits, due to the gauge ambiguity at $O(\mu)$; there is no way of relating the coordinate values of the two physical radii in the two gauges without further information about the gauge (in the form of the local metric perturbation, for example). The conservative SF shift in $U$, $\Delta U (r_0) \equiv U - U_0$, is similarly gauge dependent, and thus it too cannot be utilized usefully as a gauge-invariant measure of the SF effect.

One might hope to get around this problem by expressing one of our gauge invariants in terms of the other. To this end, it is convenient to introduce the gauge invariant radius,

$$R \equiv (M/\Omega^2)^{1/3},$$

(104)

and then express $U$ in terms of $R$. However, one then simply finds

$$U(R) = (1 - 3M/R)^{-1/2} + O(\mu^2),$$

(105)

which contains no explicit information about the SF. This is an obvious result: an $O(\mu)$ term on the right-hand side of equation (105) could only involve $R$ and $F_r$, and since $R$ is gauge invariant while $F_r$ is not, the occurrence of such a term would necessarily violate the gauge invariance of $U$. We therefore have to maintain our conclusion that $U$ and $\Omega$ (or $R$), each on its own or even both combined, do not contain gauge-invariant information about the SF effect, unless, in addition, we have access to local gauge information. For that reason, also, we cannot expect to be able to identify the effect of the conservative SF in a detected gravitational wave from a circular EMRI merely by measuring the momentary values of $\Omega$ and/or $U$ (even if there was a way of extracting $U$ from the waveform, which is unlikely)\textsuperscript{11}.

It is less clear how the conservative piece of the SF might affect the long-term evolution of the circular orbit under the full SF (dissipation included), and what influence it may have on the emitted gravitational waveform—these questions await investigation.

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Fortunately, information about the local metric perturbation, not available to the gravitational-wave astronomer, is available to the SF theorist running a SF code. This allows the theorist to utilize $\Omega$ and $U$ usefully in quantifying the gauge-invariant content of the conservative SF, as first shown by Detweiler [53, 54] and further demonstrated in [147]. In the following we outline the method of [147].

Consider again a circular orbit of radius $r_0$ (in Schwarzschild), but now interpret this, in the manner of Detweiler and Whiting, as a geodesic of the effective perturbed spacetime with metric $g_{\alpha\beta} + h^R_{\alpha\beta}$, where $h^R_{\alpha\beta}$ is the $R$-part of the physical perturbation associated with the particle. Let $\bar{\tau}$ be proper time along this geodesic. To a given physical event along the orbit there thus correspond two proper time values, $\tau$ and $\bar{\tau}$, associated with the two interpretations. (The two values will differ, in general, even if we calibrate $\tau$ and $\bar{\tau}$ to agree with one another at some initial moment.) It is easily shown [147] that, through $O(\mu)$,

$$\frac{d\tau}{d\bar{\tau}} = 1 + H,$$

with $H \equiv \frac{1}{2} h^R_{\alpha\beta} u^\alpha u^\beta$,

(106)

where the perturbation is, of course, evaluated at the worldline point in question. (The four-velocity in $H$ can be defined interchangeably with respect to either $\tau$ or $\bar{\tau}$: $h^R_{\alpha\beta}$ is already $O(\mu)$, the difference would only affect terms of $O(\mu^2)$ which are anyway neglected in our discussion.) Next, re-define the gauge invariants $U$ and $\Omega$ in terms of the four-velocity $\bar{u}^\alpha \equiv dx^\alpha / d\bar{\tau}$:

$$\bar{\Omega} \equiv \bar{u}^\phi / \bar{u}^t = u^\phi / u^t = \Omega, \quad \bar{U} \equiv \bar{u}^t = U(1 + H),$$

(107)

where we have used equation (106). Finally, express $\bar{U}$ in terms of the gauge invariant radius $\bar{R} = R$, and construct the $O(\mu)$ difference $\Delta \bar{U}(R) \equiv \bar{U}(R) - (1 - 3M/R)^{-1/2}$. The final result is [147]

$$\Delta \bar{U}(R) = (1 - 3M/R)^{-1/2} H.$$

(108)

Evidently, the metric perturbation combination $H$ is gauge-invariant, as can be verified with an explicit calculation.

Equation (108) provides a nontrivial gauge-invariant relation which explicitly involves the $R$-part of the local metric perturbation (although it does not involve directly the SF). Two theorists working in two different gauges should be able to agree on the value of $\Delta \bar{U}(R)$, and such an agreement would constitute a nontrivial test of the calculation of $h^R_{\alpha\beta}$—and, to an extent, of the SF too.

Two such tests, based on equation (108), were carried out so far. In [147] the results of Barack–Sago’s Lorenz-gauge SF code [25] were compared with those of Detweiler’s Regge–Wheeler SF code [54]. The two codes were shown to agree on the values of $\Delta \bar{U}(R)$ to within the computational error (of merely $\sim 10^{-5}$ in fractional terms). In [54], Detweiler worked out a 2PN expression for $\Delta \bar{U}(R)$ and compared it with the numerical SF data—showing an astonishing agreement down to radii as small as $R \sim 8M$. Later, a 3PN expression derived by Le Tiec and collaborators [100] showed an even closer agreement. The 3PN expression approximates the ‘exact’ SF value of $\Delta \bar{U}$ to within mere $\sim 1\%$ at $R = 12M$ and $\sim 5\%$ at $R = 7M$.

8.4. ISCO shift

To identify conservative SF effects with truly ‘observable’ consequences (by which we mean ones that can be measured in the gravitational waveform at least in principle), we must move

12 Note our notation here for $\tau$ and $\bar{\tau}$ is reversed compared to that of [147].
away from the simplicity of circular orbits. We need not move very far, though. There is a simple effect with a clear physical significance, which is manifest already in the dynamics of orbits with infinitesimally small eccentricity: the conservative SF-induced shift in the value of the orbital frequency $\Omega_1$ at the ISCO. This frequency shift, which is gauge invariant (under transformations satisfying equation (101)), was calculated recently in [26], and we shall describe this analysis briefly below.

Strictly speaking, the ISCO is defined in a precise way only at the test particle limit, $\mu \to 0$. When $\mu$ is finite, dissipation ‘smears’ the transition from inspiral to plunge and it is no longer precisely localizable. Ori and Thorne showed [124] (for circular orbits in Kerr, a result later generalized to other orbits by O’Shaughnessy [125] and Sundararajan [151]) that the ‘width’ of the radiative transition regime (measured, e.g., in terms of the frequency bandwidth of the emitted gravitational wave) scales as a low power of the mass ratio: $\sim (\mu/M)^{2/5}$. The same scaling was discovered independently by Buonanno and Damour for a binary of nonrotating black holes with an arbitrary mass ratio [38]. Ori and Thorne’s analysis ignores the conservative effect of the SF, but the latter is expected to modify the location of the ISCO only by an amount $\sim (\mu/M)$, which for very small mass ratios we expect to be negligible compared with the width of the radiative transition. Still, there is a strong motivation to study the conservative effect of the SF on the ISCO. First, we would like to confirm the above expectation and quantify it better. Second, the conservative $O(\mu)$ shift of the ISCO frequency (dissipation ignored) is a precisely specifiable gauge-invariant quantity, and as such it can serve as a convenient ‘anchor’ point for comparison between different calculation schemes. In particular, one can envisage it being used as a strong-field benchmark for calibration of PN calculations. Indeed, the value of the conservative ISCO frequency shift, for mass ratios not necessarily small, has been utilized extensively in the past by PN theorists in assessing the performance of various PN schemes (see, e.g., [51, 35, 37]). The calculation in [26] now provides this value precisely, at $O(\mu)$, in the Schwarzschild case.

To outline the calculation, we first remind how the occurrence of an ISCO is explained from the point of view of an effective radial potential. Ignoring the SF for the moment, the radial component of the equation of motion for a timelike geodesic in Schwarzschild takes the familiar form

$$\mu \ddot{r} = -\frac{\mu}{2} \frac{\partial V_{\text{eff}}(r, L_z)}{\partial r} \equiv F_{\text{eff}}, \quad (109)$$

where $V_{\text{eff}} = \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L_z^2}{r^2}\right)$ is an effective potential for the radial motion, and throughout our current discussion an overdot will denote $d/d\tau$ (not $d/dt$ as in subsection 8.2). The quantity $F_{\text{eff}}(r, L_z)$ can be interpreted as an effective radial force acting on the geodesic test particle. A stable circular orbit is associated with the (single) minimum of $V_{\text{eff}}$ (for given $L_z$), which occurs only for $L_z > \sqrt{12M}$. The radius of the circular orbit, $r = r_0(L_z)$, decreases monotonically with decreasing $L_z$, and at the limiting value of $L_z = \sqrt{12M}$ we have $r_0 = 6M$—the innermost stable circular orbit.

To better understand the dynamical significance of the ISCO it is instructive to examine the behavior of a circular orbit under a small-eccentricity perturbation. Writing $r(\tau) = r_0 + \epsilon \delta r(\tau)$ and considering the linear variation of equation (109) with respect to the small eccentricity $\epsilon$, one readily obtains [26]

$$\ddot{\delta r}(\tau) = -\omega_0^2 \delta r(\tau), \quad (110)$$

with

$$\omega_0^2 = \frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}. \quad (111)$$
Thus the radius of the perturbed orbit is a linear oscillator with frequency \( \omega_0 \) (the subscript 0 indicates the geodesic (no-SF) value). For \( r_0 > 6M \) we have \( \omega_0^2 > 0 \), and the circular orbit is dynamically stable under small-\( e \) perturbations; the condition \( \omega_0 = 0 \) identifies the ISCO at \( r_0 = 6M \).\(^{13}\) To avoid confusion, it should be understood that the radial frequency \( \omega_0 \) is a characteristic of the circular orbit (not the perturbed orbit), which, however, does not manifest itself in the dynamics of the circular orbit itself. Mathematically, \( \omega_0 \) is associated with the curvature of the effective potential at its minimum; physically, it describes the radial frequency of an eccentric orbit as the eccentricity tends to zero. In our current discussion, we simply use the value of \( \omega_0 \) as a convenient handle on the location of the ISCO.

Now consider the effect of the SF, ignoring its dissipative piece. Equation (109) becomes

\[
\mu \ddot{r} = F_\text{eff}(r, L_z) + F_\text{cons}^r.
\]

The quantity \( L_z (= u_{\phi} \) by its definition) is no longer constant along the orbit; its (conservative) time variation is determined by the \( \phi \) component of the equation of motion, reading

\[
\mu \dot{L}_z = F_\phi^\text{cons}.
\]

To learn how the SF affects the radial frequency of slightly eccentric orbits, we need to consider the linear variation of equations (112) and (113) with respect to \( \epsilon \). Through \( O(\epsilon) \), the relevant SF components along the orbit assume the general form

\[
F_\text{cons}^r = F_0^r + e F_1^r \cos \omega_0 \tau, \quad F_\phi^\text{cons} = e \omega_0 F_0^\phi \sin \omega_0 \tau
\]

(this is shown in [26] based on equation (92)), where the coefficients \( F_0^r, F_1^r \) and \( F_0^\phi \) depend only on \( r_0 \). In these expressions we can use the unperturbed radial frequency \( \omega_0 \) instead of the perturbed frequency since the \( F \) coefficients are already \( O(\mu^2) \), and it is only the leading-order SF effect which concerns us here. The linear variation procedure once again yields an equation of the form (110) where now the radial frequency, perturbed by the conservative SF, is found to be

\[
\omega^2 = \omega_0^2 + \mu^{-1} \left( \alpha F_0^r + \beta F_1^r + \gamma F_0^\phi \right),
\]

with \( \alpha = -3r_0^{-1}(r_0 - 4M)/(r_0 - 3M), \beta = r_0^{-1} \) and \( \gamma = -2r_0^{-3}[M(r_0 - 3M)]^{1/2} \). This formula describes the \( O(\mu) \) conservative shift in the radial frequency off its geodesic value. Note that it requires knowledge of the SF through \( O(\epsilon) \).

The perturbed ISCO radius, \( r_0 = 6M + \Delta r_{\text{isco}} \), is now obtained from the condition \( \omega(r_0) = 0 \). Recalling equations (111), this gives

\[
\Delta r_{\text{isco}} = \mu^{-1} \left( r_0^3/M \right) (3M - r_0)(\alpha F_0^r + \beta F_1^r + \gamma F_0^\phi) \bigg|_{r_0 = 6M} = \mu^{-1} \left( 216M^2 F_0^r - 108M F_1^r + 3F_0^\phi \right) \bigg|_{r_0 = 6M},
\]

where the substitution \( r_0 = 6M \) is allowed since this only introduces an error of \( O(\mu^2) \) on the right-hand side. This equation describes the \( O(\mu) \) shift in the location of the ISCO due to the effect of the conservative SF. This shift is well defined albeit gauge dependent. However, the value of the azimuthal orbital frequency \( \Omega \) at the (shifted) location of the ISCO is gauge invariant. To obtain the shift in \( \Omega \) we simply substitute \( r_0 = 6M + \Delta r_{\text{isco}} \) in equation (102), writing \( \Omega = \Omega_{0,\text{isco}} + \Delta \Omega_{\text{isco}} \) where \( \Omega_{0,\text{isco}} \equiv \Omega_0(6M) = (6^{3/2} M)^{-1} \). We find, through \( O(\mu) \),

\[
\frac{\Delta \Omega_{\text{isco}}}{\Omega_{0,\text{isco}}} = -\frac{1}{4} \frac{\Delta r_{\text{isco}}}{M} - \frac{27}{2} M \mu^{-1} F_1^r (6M).
\]

\(^{13}\) The fact that equation (111) gives \( \omega_0^2 > 0 \) also in the range \( r_0 < 3M \) is irrelevant here, since there are no timelike circular orbits in that range.
In [26] the SF coefficients $F_0^r$, $F_1^r$ and $F_1^{\psi}$ are extracted numerically using the Lorenz-gauge SF code reviewed here in section 6. With these numerical results, equations (116) and (117) give the final values

$$\Delta r_{\text{isco}} = -3.269 \mu,$$

$$\frac{\Delta \Omega_{\text{isco}}}{\Omega_{\text{isco}}} = 0.4870 \mu/M.$$ (118)

We recall that the value of $\Delta r_{\text{isco}}$ is specific to the Lorenz gauge, but that for $\Delta \Omega_{\text{isco}}$ is invariant under all gauge transformations related to the Lorenz gauge through a transformation satisfying equation (101).

It is interesting to compare the conservative shift $\Delta \Omega_{\text{isco}}$ with the frequency bandwidth of the dissipative transition across the ISCO, calculated by Ori and Thorne in [124]. Denoting the latter (in the Schwarschild case) by $\Delta \Omega_{\text{diss}}$, one finds $\Delta \Omega_{\text{diss}}/\Delta \Omega_{\text{isco}} \sim 9 (\mu/M)^{-3/5}$, giving, for example, $\sim 35,830$, $9,000$ and $2,261$ for mass ratios $\mu/M = 10^{-6}$, $10^{-5}$ and $10^{-4}$, respectively. This confirms our expectation that the conservative shift in the ISCO is practically negligible from the observational point of view. The main practical value of the results in equation (118) remains that they provide an accurate strong-field benchmark to inform the development of approximate methods.

9. Reflections and prospects

We have attempted here to capture a snapshot of the activity surrounding the development of reliable, efficient and accurate computational methods for the gravitational SF in black hole spacetimes. The problem still attracts considerable attention (more than half the items on our bibliographic list date $\geq$ 2005), with a multitude of different approaches being studied by different groups. This multitude offers the opportunity for cross-validation of techniques and results—a particularly important prospect given the intricate nature of the numerics involved and the fact that SF calculations explore a new territory in black hole physics, yet uncharted neither by PN theory nor by Numerical Relativity.

Indeed, the field has by now matured sufficiently that such cross-validation exercises are becoming possible. As we described in section 8, the last year has seen first quantitative comparisons between results from different calculations carried out in different gauges and using different numerical methods. We are now able to use SF codes to explore, for the first time, the conservative post-geodesic dynamics of strong-field orbits around a Schwarzschild black hole [26, 53, 54, 147]. We can compute physical gauge-invariant SF effects and test them directly against results from PN theory in the weak-field regime [53, 54, 100, 147]. Indeed, we can now start to use strong-field SF results in order to calibrate PN methods and assess their performance [26]. The exciting prospects for synergistic interaction between SF and PN theories are beginning to materialize, with much scope for important developments in the coming years.

At present, the state of the art in SF calculations is a code that can calculate the gravitational SF along any bound geodesic of the Schwarzschild geometry (currently at substantial computational cost, which future developments in the numerical technology may help reduce). This code, as many others mentioned in our review, is an implementation of the mode-sum regularization method (section 4), which has proven a useful framework for calculations in Schwarzschild.

The Kerr problem, on the other hand, has hardly been tackled so far, and it represents the next significant challenge. Although the standard mode-sum method is in principle applicable to the gravitational SF in Kerr spacetime, the details of its numerical implementation in this case are yet to be worked out. It is possible that higher-dimensional techniques (like the
m-mode scheme discussed in section 7) could provide an attractive alternative to standard mode-sum in the Kerr case.

In the short term, activity is likely to focus on the following tasks: (i) continue to improve the computational efficiency of SF calculations using advanced numerical techniques (mesh refinement, spectral methods, etc); (ii) tackle the Kerr problem; (iii) use SF codes to study post-geodesic physical effects (such as the finite-mass correction to the orbital precession rate), and in particular assess the relative importance of conservative SF effects in the EMRI problem; (iv) explore what can be learned from a comparison between SF and PN results.

Within the wider context of the LISA EMRI problem, the computation of the SF on momentarily-geodesic particles is only one essential ingredient. There is still much more to understand before a faithful model of an astrophysical inspiral can be developed. Most crucially, a reliable and workable method must be devised for calculating the long-term evolution of the inspiral orbit. Work to address this problem has started recently [77, 87, 136, 137] but much further development is required.

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Appendix. Derivation of the regularization parameters

We describe here the derivation of the regularization parameters for the gravitational SF in Kerr. The values of these parameters (stated in section 4, equations (38), (39) and (41)) were first published in [24], but the detailed derivation has not appeared in print so far. We reproduce it here using the original method of [24].

Let the arbitrary timelike geodesic $\Gamma_1$ be the worldline of a particle with mass $\mu$ in a Kerr geometry with mass $M \gg \mu$ and arbitrary spin $aM$. We wish to calculate the regularization parameters for the gravitational SF acting on the particle at an arbitrary point $z$ along $\Gamma_1$ with Boyer–Lindquist coordinates $(t_0, r_0, \theta_0, \phi_0)$. Without loss of generality we shall take $\phi_0 = 0$.

We recall that the parameters $\pm A^\alpha, B^\alpha$ and $C^\alpha$ are the leading-order coefficients in the formal expansion of the $l$-modes $F_{\pm}^{a\pm} (z)$ at large $l$ (recall equation (32)). We also recall that the difference $F_{\text{ret}}^{a\pm} (z) - F_{\text{reg}}^{a\pm} (z)$ is expected to die off at large $l$ faster than any power of $1/l$ (recall the discussion surrounding equation (35)). Therefore, the values of $A^\alpha, B^\alpha$ and $C^\alpha$ can also—more conveniently—be deduced from the large-$l$ asymptotics of the $S$-field modes:

$$F_{\text{reg}}^{a\pm} (z) = \pm A^\alpha L + B^\alpha + C^\alpha / L + O(L^{-2}).$$  \hspace{1cm} (A.1)

Once $A^\alpha, B^\alpha$ and $C^\alpha$ are known, the parameter $D^\alpha$ is given as the residual quantity (equation (37))

$$D^\alpha \equiv \sum_{l=0}^{\infty} \left[ F_{\text{reg}}^{a\pm} (z) \mp L A^\alpha - B^\alpha - C^\alpha \right].$$  \hspace{1cm} (A.2)
Our starting point will be the local expression for the $S$-field $\hat{h}^S_{\alpha\beta}$, given in equation (19). Referring back to figure 1, we denote the Boyer–Lindquist coordinate difference between point $x$ (an arbitrary point in the immediate vicinity of $z$) and point $z$ by $\delta x^a \equiv x^a - z^a$. In equation (19) the quantities $\epsilon^2(x; z)$ and $\tilde{u}_a(x; z)$ are smooth functions of $\delta x$, and we may expand them in the form

$$\epsilon^2 = S_0 + S_1 + O(\delta x^4), \quad \tilde{u}_a = u_a + \delta u_a + O(\delta x^2), \quad (A.3)$$

where $S_0$ and $S_1$ are, respectively, quadratic and cubic in $\delta x$, and $\delta u_a$ is linear in $\delta x$. Explicitly, these expansion terms read

$$S_0 = (g_{\alpha\beta} + u_\alpha u_\beta)\delta x^a \delta x^\beta = P_{\alpha\beta} \delta x^a \delta x^\beta, \quad (A.4)$$

$$S_1 = \Gamma^\alpha_{\gamma\beta} P_{\gamma\beta} \delta x^a \delta x^\beta \delta x^\gamma = P_{\alpha\beta\gamma} \delta x^a \delta x^\beta \delta x^\gamma, \quad (A.5)$$

$$\delta u_a = \Gamma^a_{\beta\gamma} u_\beta \delta x^\gamma, \quad (A.6)$$

where the background metric and connections are evaluated at $z$, and the coefficients $P_{\alpha\beta}$ and $P_{\alpha\beta\gamma}$ are those defined in equation (43) (for a derivation of $S_1$ see, for example, appendix A of [22]).

We now substitute the above expansions (A.3) in equation (19), and subsequently construct the field $F^\mu_a(x)$ $= \mu \nabla^a_{\alpha\beta} \hat{h}^S_{\alpha\beta}(x)$, where, recall, the operator $\nabla^a_{\alpha\beta}$ is the one defined immediately below equation (23). The resulting expression for $F^\mu_a(x)$ can be written down as a sum of terms sorted according to how they scale with $\delta x$:

$$\mu^{-2} F^\mu_a(x) = \frac{P_{\alpha(1)}^a (\delta x)}{\epsilon^3_0} + \frac{P_{\alpha(4)}^a (\delta x)}{\epsilon^5_0} + \frac{P_{\alpha(7)}^a (\delta x)}{\epsilon^7_0} + O(\delta x). \quad (A.7)$$

Here $\epsilon_0 \equiv S_0^{1/2}$, and $P_{\alpha}^{(n)}$ denotes a certain multilinear function of the coordinate differences $\delta x^a$, of homogeneous order $n$ in $\delta x$. Note that the first term on the right-hand side scales as $\delta x^{-2}$, the second as $\delta x^{-1}$ and the third as $\delta x^0$. The $O(\delta x)$ remainder disappears at the limit $x \rightarrow z$ and cannot affect the value of the final SF; it is therefore safe to ignore it. The explicit form of $P_{\alpha}^{(7)}$ will not be needed in our analysis. The two other coefficients read

$$P_{\alpha(1)}^a = -2 u_\beta u_\gamma \nabla^a_{\alpha\beta\gamma} S_0 = -P_{\rho}^a \delta x^\rho \quad (A.8)$$

and

$$P_{\alpha(4)}^a = u_\beta u_\gamma (3S_1 \nabla^a_{\alpha\beta\gamma} S_0 - 2S_0 \nabla^a_{\alpha\beta\gamma} S_1) - 2(\delta u_\beta u_\gamma + u_\beta \delta u_\gamma)S_0 \nabla^a_{\alpha\beta\gamma} S_0 \quad (A.9)$$

The second equality in each of (A.8) and (A.9) was evaluated with the help of the following identities, which are easily confirmed:

$$u_\beta u_\gamma \nabla^a_{\alpha\beta\gamma} = \frac{1}{4} P^{a\delta} \nabla_\delta, \quad P^{a\delta} P_{\gamma\delta} = P_{\gamma}^a, \quad (A.10)$$

$$\nabla_\delta S_0 = 2 P_{\gamma\delta} \delta x^\gamma, \quad \nabla_\delta S_1 = (2P_{\alpha\gamma\delta} + P_{\alpha\delta\gamma}) \delta x^a \delta x^\beta, \quad (A.11)$$

$$\delta u_\beta u_\gamma + u_\beta \delta u_\gamma \nabla^a_{\alpha\beta\gamma} S_0 = -\delta u_\gamma u_\beta P^a_{\rho} \delta x^\rho = -\frac{1}{2} P^a_{\rho} P_{\gamma\rho} \delta x^\gamma, \quad (A.12)$$

where $P_{\rho}$ is given in equation (43).

To obtain the regularization parameters, we need to consider the decomposition of $F^\mu_a(x)$ in spherical harmonics (and then take $x \rightarrow z$). Notwithstanding the spheroidicity of the Kerr background, the spherical harmonics $Y_{lm}(\theta, \phi)$ are defined as usual on surfaces of constant $r$ and $t$, where $(t, r, \theta, \phi)$ are Boyer–Lindquist coordinates. To make this $l$-mode
decomposition easier, we introduce new coordinates: In general (i.e. for general \( \theta_0 \)), the multipole decomposition will involve all azimuthal numbers \(|m| \leq l\) for each \( l \). However, if we choose a new set of angular coordinates, \((\theta', \varphi') \rightarrow (\theta^0, \varphi^0)\), such that in the new coordinates \( z \) is positioned at the pole (i.e. \( \theta^0_0 = 0 \)), then the only contribution to each \( l \) mode in the new system would come from the axially-symmetric \( m = 0 \) mode alone. Yet, due to the invariance of the Legendre functions under rotations on the 2-sphere, the overall \( l \) mode contribution (summed over \( m \)) would be the same in both systems. A transformation \((\theta, \varphi) \rightarrow (\theta', \varphi')\) is given explicitly by

\[
\cos \theta' = \cos \theta \sin \theta_0 + \cos \theta_0 \cos \theta_0, \\
\tan(\varphi' - \varphi_0') = \frac{\sin \varphi \sin \theta}{\cos \varphi \sin \theta \cot \theta_0 - \cos \theta \sin \theta_0},
\]

where \( \varphi'_0 \) indicates the azimuthal direction of the particle’s velocity in the new polar system (i.e. in the new system the velocity is momentarily tangent to the longitude line \( \varphi' = \text{const} = \varphi'_0 \) on the 2-sphere). The angle \( \varphi'_0 \) depends on \( u^\theta \) and \( u^\varphi \), but the explicit relation will not be needed here.

Since the new angular coordinates are not well behaved at \( z \), we also introduce Cartesian-like local coordinates based around \( z \):

\[
x \equiv \rho(\theta') \cos(\varphi' - \varphi'_0), \quad y \equiv \rho(\theta') \sin(\varphi' - \varphi'_0),
\]

where we require \( \rho = \theta' + O(\theta'^3) \) near \( z \). Note that at \( z \) we have \( x = y = 0 \) as well as \( u^\varphi = 0 \). For later convenience, we make here the concrete choice

\[
\rho = 2 \sin(\theta'/2),
\]

giving

\[
\cos \theta' = 1 - \frac{1}{2}(x^2 + y^2).
\]

In the following we will need the transformation \((\theta, \varphi) \leftrightarrow (x, y)\) only in the immediate neighborhood of \( z \). To sufficient order, we find

\[
\delta \theta = x + (1/2) \cot \theta_0 y^2 + O(\delta x^3),
\]

\[
\delta \varphi = (\sin \theta_0)^{-1} (y - \cot \theta_0 x y) + O(\delta x^3),
\]

where, recall, \( \delta \theta = \theta - \theta_0 \) and \( \delta \varphi = \varphi - \varphi_0 = \varphi \) are the Boyer–Lindquist coordinate differences, and \( O(\delta x^3) \) represents terms at least cubic in \( x \) and \( y \). We denote

\[
X^a = (x^0, x^1, x^2, x^3) \equiv (\delta t, \delta r, x, y / \sin \theta_0),
\]

noting \( X^a = 0 \) at the particle. The direct transformation from \( X^a \) to the Boyer–Lindquist coordinate differences \( \delta x^a \) can be written to sufficient order in the compact form

\[
\delta x^a = X^a + C^a_{\mu \nu} X^\mu X^\nu,
\]

where the constant coefficients \( C^a_{\mu \nu} \) are those defined in equation (44).

Our next step is to express the S-field force \( F^a_S(x) \) in terms of the \( X^a \) coordinates (note, however, that we do not consider here the \( X^a \) components of \( F^a_S(x) \), but rather the Boyer–Lindquist components; the \( X^a \) coordinates are merely introduced to assist with the Legendre integration below). We substitute equation (A.19) in equation (A.7) and then re-expand in the coordinate differences \( X^a \). The result takes the form

\[
\mu^{-2} F^a_S(x) = F^a_{\xi \mu} + F^a_\xi + F^a_{\mu} + O(\delta x),
\]
where

\[ F^\alpha_A = -\frac{P^\alpha_{\mu} X^\mu}{\epsilon^3_{0X}}, \quad (A.21) \]

\[ F^\alpha_B = \frac{P^\alpha_{\rho \beta \gamma \delta} X^\rho X^\beta X^\gamma X^\delta}{\epsilon^5_{0X}}, \quad (A.22) \]

\[ F^\alpha_C = \frac{P^\alpha_{\mu \nu \lambda \sigma \beta \gamma \delta} X^\mu X^\nu X^\lambda X^\sigma X^\beta X^\gamma X^\delta}{\epsilon^7_{0X}}. \quad (A.23) \]

Here

\[ \epsilon^2_{0X} = P_{\alpha \beta} X^\alpha X^\beta, \quad (A.24) \]

\[ P^\alpha_{\mu \nu \lambda \sigma \beta \gamma \delta} \] are constant coefficients whose explicit values will not be needed, and

\[ P^\alpha_{\rho \beta \gamma \delta} = \frac{1}{2} \left[ P^\alpha_4 (3 P_{\rho \beta \gamma} + 2 P_{\rho \beta} P_{\gamma}) - P^\alpha_2 (2 P_{\rho \beta} + P_{\rho \beta \lambda}) P_{\gamma \delta} \right] + \left( 3 P^\alpha_{\rho \beta} P_{\rho \beta} - P^\alpha_{\rho \beta \lambda} P_{\rho \beta \lambda} \right) C_{\gamma \delta}. \quad (A.25) \]

(In the last expression, the first line is simply the coefficient of \( P^\alpha_4 \) from equation (A.9), and the second line arises from expanding the term \( \epsilon_0^{-2} P^\alpha_{\alpha \beta} \) in equation (A.7) in powers of \( X^\alpha \).) Note in equation (A.20) that the terms \( F^\alpha_A, F^\alpha_B \) and \( F^\alpha_C \) scale as \( \sim X^{-2}, X^{-1} \) and \( X^{-0} \), respectively. The terms included in \( O(\delta x) \) vanish at \( x \to z \), and are therefore irrelevant for calculating the regularization parameters—we can hence safely ignore them in what follows.

To obtain the regularization parameters, via equations (A.1) and (A.2), we now need to consider the spherical-harmonic \( l \) modes \( F_{\delta \pm \beta}^\alpha(z) \). These are constructed, by definition, through

\[ F_{\delta \pm \beta}^\alpha = \lim_{\delta r \to 0^\pm} \frac{L}{2\pi} \int_1^{r^{-1}} \frac{1}{d \cos \theta} \int_0^{2\pi} d \varphi' P_l (\cos \theta') F_{\delta \pm}^\alpha(r, \theta', \varphi'; z), \quad (A.26) \]

where \( P_l (\cos \theta') \) is the Legendre polynomial and, recall, \( L = l + 1/2 \). Here we have already taken the limits \( t \to t_0, \theta \to \theta_0 \) and \( \varphi \to \varphi_0 \), thereby choosing to approach the particle along the radial direction, recalling that we expect two different one-side values (hence the label ‘\( \pm \)’). Let us write

\[ \mu^{-2} F_{\delta \pm}^\alpha = F_{\delta \pm}^\alpha + F_{\delta \pm}^\alpha + F_{\delta \pm}^\alpha, \quad (A.27) \]

where the three terms on the right-hand side represent the respective contributions to \( F_{\delta \pm}^\alpha \) from the three terms \( F_{A,B,C}^\alpha \) in equation (A.20). As explained in [22], the second and third terms (scaling near the particle as \( \sim \delta x^{-1} \) and \( \sim \delta x^0 \), respectively) are sufficiently regular to allow interchanging the order of the limit and integration in equation (A.26). For the same reason, the radial limit is well defined and the two-side ambiguity does not occur. As we discuss below, such ambiguity does appear when considering \( F_{A,B,C}^\alpha \)—hence the label \( \pm \). In what follows we consider each of the three contributions to \( F_{\delta \pm}^\alpha \) in turn.

We start with \( F_{\delta \pm}^\alpha \). This term is obtained by replacing \( F_{S}^\alpha \) in equation (A.26) with \( F_{C}^\alpha \) from equation (A.23), and we are allowed to pull the limit \( \delta r \to 0^\pm \) over the integrals. The outcome has the form

\[ F_{\delta \pm}^\alpha = -\frac{L}{2\pi} \int_1^{r^{-1}} d \cos \theta \int_0^{2\pi} d \varphi' P_l (\cos \theta') \tilde{F}_{\delta \pm}^\alpha(x, y), \quad (A.28) \]

where \( \tilde{F}_{\delta \pm}^\alpha(x, y) \) is a certain polynomial of homogeneous order 7 in the two ‘angular’ coordinates \( x, y = (x^2, X^3 \sin \theta_0) \), and

\[ \tilde{\epsilon}_{0X} \equiv \epsilon_{0X}(\delta r \to 0, \delta t \to 0) = \left[ \left( g_{xx}(z) + u_z^2 \right) x^2 + g_{yy}(z) y^2 \right]^{1/2}. \quad (A.29) \]
In the last equality we used equation (A.24) with \( P_{\alpha \beta} = g_{\alpha \beta} + u_{\alpha} u_{\beta} \), recalling \( u_{\gamma} = 0 \) along with \( g_{\alpha \gamma}(z) = 0 \). We observe that \( \hat{e}_{\alpha \beta} \) is an even function of both \( x \) and \( y \), and, recalling equation (A.16), so is \( \cos \theta' \). However, each of the terms in \( P_{\beta}(x, y) \) (such as \( \alpha x^2 y^2, \alpha x^4 y^3, \) etc) is odd in either \( x \) or \( y \). It then immediately follows from symmetry that the integral in equation (A.28) vanishes:

\[
F_{\alpha l} = 0. \tag{A.30}
\]

Next consider \( F_{\alpha l}^{C} \). First, taking the limits \( \delta r \to 0 \) and \( \delta t \to 0 \), we have from equation (A.22)

\[
F_{\alpha l}^{C}(\delta r, \delta t) \to 0 \Rightarrow \rho_{w}^a \rho_{w} b \rho_{w} c \rho_{w} d,
\]

where roman indices run over the angular coordinates \((X^2, X^3)\) only. Then, using \( (X^2, X^3) = (x, y/\sin \theta_0) = \rho(\cos \varphi', \sin \varphi'/\sin \theta_0) \equiv \rho u^a \), we write \( X^a = \rho u^a \) and \( \rho_{w}^a = \rho^2 P_{\alpha \beta}^{ab} u^a u^b \).

This allows us to write the Legendre integral for \( F_{\alpha l}^{C} \) in a factorized form:

\[
F_{\alpha l}^{C} = \left[ \int_{-\pi}^{\pi} \frac{d \cos \theta'}{\rho(\theta')} \int_{0}^{2\pi} \frac{d \varphi'}{(\rho u^a u^b)^{3/2}} \right]. \tag{A.32}
\]

Here the \( \theta' \) integral is elementary: the entire term in the first set of square brackets reads simply \((2\pi)^{-1}\) (for any \( l \)). The term in the second set of square brackets is \( P_{\alpha \beta}^{ab} I_{\alpha \beta}^{abcd} \), where \( I_{\alpha \beta}^{abcd} \) are the (\( l \)-independent) integrals given in equation (45). These integrals, recall, are not elementary, but they can be expressed explicitly in terms of complete elliptic integrals, as in equation (49). In conclusion, we find

\[
F_{\alpha l}^{C} = (2\pi)^{-1} P_{\alpha \beta}^{ab} I_{\alpha \beta}^{abcd}. \tag{A.33}
\]

Importantly, the term \( F_{\alpha l}^{C} \) contributes to each \( l \)-mode \( F_{a l}^{C} \) a constant amount, independent of \( l \).

We have one more contribution to \( F_{a l}^{C} \) to consider: that of \( F_{a l}^{C} \). Recalling equation (A.21), we have

\[
F_{a l}^{C} = \lim_{\delta r \to 0} \int_{-\pi}^{\pi} \frac{d \cos \theta'}{\rho(\theta')} P_{\beta}(\cos \theta') \epsilon_{0\alpha}^{\beta} \mu^a X^a,
\]

where the integral is taken over the 2-sphere, we have used the Jacobian \( \hat{\beta}(\cos \theta', \varphi')/\beta(x, y) = -1 \), and the integrand is understood to be already evaluated at \( t = t_0 \). The Legendre polynomial can be written in the form \( P_{\beta}(\cos \theta') = 1 + \rho^2 H(\rho) \), where \( H(\rho) \) admits a regular Taylor expansion in \( \rho^2 \). Consider first the contribution of the term \( \propto \rho^2 H(\rho) \) to \( F_{a l}^{C} \). the corresponding integrand in equation (A.34) has the form \( \epsilon_{0\alpha}^{\beta} H(\rho) \bar{P}_{\beta}(\delta r, x, y) \) (recalling \( \rho^2 = x^2 + y^2 \)), where \( \bar{P}_{\beta} \) is a polynomial of homogeneous order 3 in \( \delta r, x, y \). This contribution to the integrand is therefore bounded, and we are allowed to swap the limit and integral just as we did with \( F_{\alpha l}^{C} \). The resulting integral then vanishes by virtue of the same symmetry argument applied in the case of \( F_{a l}^{C} \). Both functions \( \epsilon_{0\alpha}^{\beta} \) and \( H(\rho) \) are even in each of \( x \) and \( y \), while all the possible terms in \( P_{\beta}(0, x, y) \) are odd in either \( x \) or \( y \).

We are thus left with the contribution

\[
F_{a l}^{C} = \lim_{\delta r \to 0} \int_{-\pi}^{\pi} \frac{d \cos \theta'}{\rho(\theta')} \epsilon_{0\alpha}^{\beta} P_{\beta}^{\mu} X^\mu. \tag{A.35}
\]

To evaluate this integral, we divide the integration domain (the 2-sphere) into two regions: let \( D_{in} \) denote the square \(-h < x, y < h\) for some particular \( 0 < h < 1 \) (say, \( h = 1/10 \)), and let \( D_{out} \) denote the remaining integration area. Consider first the contribution to \( F_{a l}^{C} \) from \( D_{in} \): since in this domain the integrand is regular (the only singularity is at \( x = y = 0 \), which is in \( D_{in} \)), we are allowed to interchange the limit and integration. As a result, the integrand takes
the form $\tilde{\delta}^{-3} P^{\mu}_{\alpha} X^{\mu}$, and the integral over $D_{\text{out}}$ vanishes by virtue of the odd symmetry. The remaining piece of the integral, over $D_{\text{in}}$, is

$$F^{\mu l}_{A k} = \lim_{\delta r \to 0} \frac{L}{2\pi} \int_{-h}^{h} \int_{-h}^{h} dx dy \, \epsilon_{0x}^{-3} P^{\mu}_{\alpha} X^{\mu}. \quad (A.36)$$

This integral takes a simpler form if we express it in terms of the rescaled coordinates $\tilde{x} \equiv x/\delta r$, $\tilde{y} \equiv y/\delta r$ and $\tilde{X}^{\mu} \equiv X^{\mu}/\delta r = (0, 1, \tilde{x}, \tilde{y}/\sin \theta_{01})$ (where we have already taken $\delta t \to 0$).

For given $\delta r \neq 0$ we have $e_{0x} = (P_{\alpha\beta} X^{\alpha} X^{\beta})^{1/2} = \pm \delta r (P_{\alpha\beta} \tilde{X}^{\alpha} \tilde{X}^{\beta})^{1/2}$, where the sign corresponds to the sign of $\delta r$. Hence we obtain

$$F^{\mu l}_{A k} = \pm \lim_{\delta r \to 0} \frac{L}{2\pi} \int_{-h/\delta r}^{h/\delta r} \int_{-h/\delta r}^{h/\delta r} d\tilde{x} d\tilde{y} \, \frac{P^{\mu}_{\beta} \tilde{X}^{\beta}}{(P_{\alpha\beta} \tilde{X}^{\alpha} \tilde{X}^{\beta})^{3/2}},$$

$$= \pm \frac{L}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tilde{x} d\tilde{y} \, \frac{P^{\mu}_{\beta} \tilde{X}^{\beta}}{(P_{\alpha\beta} \tilde{X}^{\alpha} \tilde{X}^{\beta})^{3/2}}, \quad (A.37)$$

where the last equality holds because the integrand, expressed in terms of the tilde variables, no longer depends on $\delta r$. This integral is now elementary, and evaluating it gives

$$F^{\mu l}_{A k} = \pm L A^{\alpha}, \quad (A.38)$$

where the signs correspond to $\delta r \to 0^{+}$, and where the various Boyer–Lindquist components of $A^{\alpha}$ are given in equation (39). (Section V.D of [22] explains in detail how the integral in equation (A.37) is evaluated in the special case of Schwarzschild, and the method is directly applicable to Kerr.) Note that $F^{\mu l}_{A k}$ is found to depend on $l$ only through the prefactor $L$.

In summary, collecting the results (A.20), (A.30), (A.33) and (A.38), we find that the $l$ modes $F^{\mu l}_{A k}$ are given precisely by

$$F^{\mu l}_{A k} = \pm L A^{\alpha} + B^{\alpha}, \quad (A.39)$$

where $A^{\alpha}$ and $B^{\alpha} \equiv (2\pi)^{-1} F^{\alpha}_{abcd} I^{abcd}$ are $l$-independent. Comparing with equation (A.1), we identify $A^{\alpha}$ and $B^{\alpha}$ as the first two of the regularization parameters. This comparison also implies $C^{\alpha} = 0$. Furthermore, we find that each of the individual terms in the sum over $l$ in equation (A.2) vanishes, giving also $D^{\alpha} = 0$. This completes the derivation of all regularization parameters.

We comment on a potentially confusing aspect of the above analysis: we have discarded in equation (A.20) the $O(\delta x)$ terms of $F^{\mu l}_{A k}(x)$, which vanish at $x \to z$. Clearly, the multipole expansion of these neglected terms could contribute to $F^{\mu l}_{A k}$ (e.g. they may well add a term $\propto L^{-2}$ in equation (A.39)). Such terms, however, must add up to zero upon summation over $l$ (when evaluated at the particle). They hence affect neither the value of $D^{\alpha}$ in equation (A.2), nor the value of the final SF in equation (36).

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