Two-Loop Vacuum Diagrams in Background Field and the
Heisenberg-Euler Effective Action

Marek Krasňanský*

Department of Physics, University of Connecticut, Storrs, CT 06269-3046, USA

Abstract

We show that in arbitrary even dimensions, the two-loop scalar QED Heisenberg-Euler effective action can be reduced to simple one-loop quantities, using just algebraic manipulations, when the constant background field satisfies $F^2 = -f^2 \mathbf{1}$, which in four dimensions coincides with the condition for self-duality, or definite helicity. This result relies on new recursion relations between two-loop and one-loop diagrams, with background field propagators. It also yields an explicit form of the renormalized two-loop effective action in a general constant background field in two dimensions.

* mkras@phys.uconn.edu
I. INTRODUCTION

Great progress has been made in developing techniques of calculating multi-loop Feynman diagrams [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The "integration by parts" method, allowing to reduce higher order diagrams into a set of basic integrals, offers a powerful tool for calculating multi-loop amplitudes. This method has been applied to massless [10, 11, 12] as well as massive propagators [13, 14, 15, 16, 17]. Recently such algebraic methods proved to be very useful also for diagrams containing propagators in a constant electromagnetic background field [19, 20]. The renormalized two-loop effective action in a constant background field was derived by Ritus [21] and later by other authors [22]. The result is a rather complicated double-parameter integral. The extension of the "integration by parts" method to diagrams in background fields can dramatically simplify the computation of the renormalized two-loop effective action [18, 19, 20]. It has been shown that there is a simple diagrammatic interpretation of mass renormalization in the two-loop scalar QED effective action [20]. In the case of a self-dual background field in four dimensions, where $F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$, the field also satisfies $F^2 = - f^2 \mathbb{1}$, and the whole two-loop Heisenberg-Euler effective action acquires a very simple form [23], and moreover it can be expressed in terms of one-loop quantities [19]. This simplicity reflects the connection between helicity, self-duality and supersymmetry [24]. The self-duality condition is special to four dimensions; but in this paper we show that the simplicity of the effective action in the background field satisfying $F^2 = - f^2 \mathbb{1}$ persists in any even dimension. These results are based on certain recurrence relations, derived in this paper, for two-loop vacuum diagrams in a background field. These new relations allow us to obtain the two-loop Heisenberg-Euler effective action completely in terms of one-loop integrals in a purely algebraic way, without the need of performing any complicated proper time integrals [23]. An immediate consequence is that we obtain the fully renormalized two-loop effective action for any constant background field in two dimensions, since the condition $F^2 = - f^2 \mathbb{1}$, is satisfied by any constant field in two dimensions.
II. TWO-LOOP EFFECTIVE ACTION

Consider a scalar field in Euclidean space, in an electromagnetic background with constant field strength, such that the square of the field-strength tensor is proportional to the identity matrix:

\[ F_{\mu\alpha}F_{\alpha\nu} = -f^2 \delta_{\mu\nu}. \]  \hspace{1cm} (2.1)

Here, \( f \) denotes the strength of the field. In two dimensions this condition is satisfied for an arbitrary constant field, while in four dimensions it is satisfied by a self-dual constant electromagnetic field.

The propagator of a scalar field interacting with the background (2.1) obeys the Klein-Gordon equation

\[ (p^2 + m^2)G(p) = 1 + \frac{(ef)^2}{4} \frac{\partial^2 G(p)}{\partial p_\mu \partial p_\mu}. \]  \hspace{1cm} (2.2)

The solution of this equation in \( d \) dimensions can be written as a proper time integral:

\[ G(p) = \int_0^\infty dt \frac{e^{-m^2 t - \frac{e^2}{2} \tanh(ef t)}}{\cosh^{\frac{d}{2}}(ef t)}. \]  \hspace{1cm} (2.3)

The rotational symmetry of the propagator (2.3) in Euclidean space eliminates the complicated tensor structure of the general two-loop effective action [20] and enables us to express it in a very simple form in terms of one-loop diagrams [19, 20]. This rotational symmetry is a direct consequence of the condition (2.1). We briefly recall the form of the two-loop effective action.

The background field modifies not only the propagators but also the vertices, such that \( p_\mu \rightarrow p_\mu - i \frac{e}{2} F_{\mu\nu} \frac{\partial}{\partial p_\nu} \). Thus, the two-loop bubble diagram is

\[ \bigcirc \bigcirc = \frac{e^2}{2} \int \frac{d^dp \, d^dq}{(2\pi)^2} \frac{1}{(p-q)^2} \left\{ (p+q)^2 G(p)G(q) - e^2 f^2 \frac{\partial G(p)}{\partial p_\mu} \frac{\partial G(q)}{\partial q_\mu} \right\}. \]  \hspace{1cm} (2.4)

Here we introduce the notation that the double line denotes the scalar propagator in the background field.

As was shown [19, 20], one can express difference between (2.4) and the corresponding diagram without the background field by purely algebraic manipulations as

\[ \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right] = \frac{e^2}{2} \left( \frac{d - 1}{d - 3} \right) \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right]^2 + \left[ \bigcirc \bigcirc \right]_{p^2 = -m^2} \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right] \]
\[-2e^2 m^2 \text{[0x0]} - e^2 \left(\frac{d-2}{d-3}\right) \text{[0x0]}^2 \tag{2.5}\]

\[+ \int \frac{d^d p \, d^d q}{(2\pi)^{2d}} \frac{2e^2}{(p - q)^2} \left\{ (p^2 + m^2) + (q^2 + m^2) \right\} G(p)G(q). \]

The dashed line in the first diagram in the second line denotes a free massless scalar, with no contributions from the vertices. The second term in the first line is the mass renormalization diagram:

\[\left[ \text{[0x0]}^{\text{[0x0]}} \right]_{p^2 = -m^2} = e^2 \left(\frac{d-1}{d-3}\right) \text{[0x0]}. \tag{2.6}\]

Applying the Klein-Gordon equation (2.2) to the last term in (2.5) and integrating by parts we obtain

\[2e^2 \int \frac{d^d p \, d^d q}{(2\pi)^{2d}} \frac{p^2 + m^2 + q^2 + m^2}{(p - q)^2} G(p)G(q) = \]

\[= e^4 f^2 \int \frac{d^d p \, d^d q}{(2\pi)^{2d}} \left( \frac{\partial^2}{\partial p_\mu \partial p_\mu} - \frac{1}{(p - q)^2} \right) G(p)G(q) \tag{2.7}\]

\[= 2e^4 f^2 \left(4 - d\right) \text{[0x0]}. \]

The diagram on the RHS contains no contribution from the vertices and the free massless scalar propagator is raised to the second power. Equation (2.7), together with the recurrence formula (C1), allows us to write the two-loop effective action in the self-dual background field in the following form

\[\left[ \text{[0x0]} - \text{[0x0]} \right] = \frac{e^2}{2} \left(\frac{d-1}{d-3}\right) \text{[0x0]} - \text{[0x0]}^2 + \left[ \text{[0x0]}^{\text{[0x0]}} \right]_{p^2 = -m^2} \left[ \text{[0x0]} - \text{[0x0]} \right] - e^4 f^2 \frac{(d - 4)(d - 2)}{(d - 3)} \text{[0x0]}. \tag{2.8}\]

All terms in the bare effective action (2.8) are one-loop except the last one. It has been shown that this diagram can be reduced in four dimensions to one loop diagrams [19].

\[\textbf{III. ANALYSIS OF TWO-LOOP DIAGRAMS}\]

In this section we show how the diagrams of the form

\[= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(p - q)^{2n}} G(p)G(q) \tag{3.1}\]

\[\text{can be expressed in terms of one-loop diagrams for any } n, \text{ and that the procedure depends on the dimensionality of space } 2N. \text{ When the power of the free massless scalar propagator} \]
is half of the number of dimensions, \( n = N \), the finite part of (3.1) is proportional to a one-loop diagram. For any other power of the free massless scalar propagator, (3.1) can be written as a sum of the previous one and a one-loop diagram.

The first part comes from the fact that a convergent integral of functions \( f(p), g(q) \) and \( N \)-th power of the free massless scalar propagator in \( 2N \) dimensions can be calculated when the regularization is removed. In \( d \) dimensions we can write

\[
\int \frac{d^d p \, d^d q}{(2\pi)^{2d}} \frac{2N - d}{(p - q)^{2N}} f(p) g(q) = \frac{1}{2(N - 1)} \int \frac{d^d p \, d^d q}{(2\pi)^{2d}} \left( \frac{\partial^2}{\partial p^\mu \partial q^\mu} \frac{1}{(p - q)^{2(N - 1)}} \right) f(p) g(q) .
\]

(3.2)

If the integral converges, we can set \( d = 2N \), then the derivative of the free massless scalar propagator gives \( -\frac{4(N-1)\pi^N}{\Gamma(N)} \delta^{2N}(p - q) \) and the integral reduces to

\[
- \frac{1}{2^{2N-1}\pi^N \Gamma(N)} \int \frac{d^{2N} p}{(2\pi)^{2N}} f(p) g(p) .
\]

(3.3)

Applying this to the two-loop diagram (3.1) with \( n = N \) implies

\[
(2N - d) \begin{array}{c}
\bullet \\
\circ
\end{array} - g_1 \left[ \begin{array}{c}
\bullet \\
\circ
\end{array} \right]^2 = - \frac{1}{2^{2N-1}\pi^N \Gamma(N)} \begin{array}{c}
\bullet \\
\circ
\end{array}_2 - g_2 \begin{array}{c}
\bullet \\
\circ
\end{array} + O(\varepsilon)
\]

(3.4)

where \( 2\varepsilon = 2N - d \), and \( g_1 \) and \( g_2 \) are known functions of \( N, d, m^2 \) and \((ef)\). The double-pole divergence of the two-loop vacuum diagram on the LHS and a single-pole divergence of the one-loop vacuum diagram on the RHS are subtracted, making both sides of the equation finite. Explicit examples of (3.4) for \( N = 1, 2, 3, 4 \) are given in appendix A.

The second important fact is that the two-loop diagram (3.1) with any power \( n \) of the free massless scalar propagator can be written as a sum of the same diagram with \( N \)-th power of the free massless scalar propagator and the square of the one-loop diagram:

\[
\begin{array}{c}
\bullet \\
\circ
\end{array}_n = h_1 \begin{array}{c}
\bullet \\
\circ
\end{array} + h_2 \left[ \begin{array}{c}
\bullet \\
\circ
\end{array} \right]^2 .
\]

(3.5)

Here, \( h_1 \) and \( h_2 \) are simple functions of \( n, N, d, m^2 \) and \((ef)\). If \( h_1 \) is proportional to \( 2N - d \), we can use (3.4) and write the diagram (3.1) completely in terms of one-loop diagrams plus a term vanishing as \( d \to 2N \).

This statement is based on the following recurrence relation proven in appendix B

\[
(ef)^2 n \left( 2(n + 1) - d \right)^2 \begin{array}{c}
\bullet \\
\circ
\end{array}_n - 2m^2 (2n - d + 1) \begin{array}{c}
\bullet \\
\circ
\end{array} - (n - d + 1) \begin{array}{c}
\bullet \\
\circ
\end{array}_2 = 0 .
\]

(3.6)
The identity (3.6) implies that for any \( n \) bigger than 1, by successive application of (3.6), the diagram (3.1) can be written as

\[(2n - d)^2 \square = f_1 \square + f_2 \square^2, \tag{3.7}\]

where \( f_1 \) and \( f_2 \) are some functions of \( n, d, m^2 \) and \( ef \). Relevant examples are given in appendix C. If the power of the free massless scalar propagator is zero, (3.1) turns into the square of the one-loop diagram, producing the second term in (3.7). Because a particular form of (3.7) can be found for any \( n \), and always contains the two diagrams \( \square^2 \) and \( \square \), one can use such equations for \( n \) and \( N \) to eliminate \( \square \) and obtain (3.5). In the case of a vanishing background field the recurrence formula (3.6) turns into a relation of the vacuum loop diagrams with the free propagators:

\[2m^2(2n - d + 1) \square + (n - d + 1) \square = 0. \tag{3.8}\]

This relation gives us a formula equivalent to (3.7) for the free propagators:

\[\square = \frac{1}{m^{2n}} \frac{\Gamma(1 - \frac{d}{2} + n)\Gamma(d - 1 - 2n)}{\Gamma(1 - \frac{d}{2})\Gamma(d - 1 - n)} \square^2. \tag{3.9}\]

By evaluating the one-loop diagram on the RHS of (3.9), one can obtain a special case of Vladimirov’s formula [9].

**IV. TWO-LOOP EFFECTIVE ACTION IN 2N DIMENSIONS**

The two-loop Heisenberg-Euler effective action (2.8) in the background field (2.1) in four dimensions has been already extensively discussed [19, 20, 23]. In this section we focus on the last two-loop diagram in the effective action (2.8) in different dimensions:

\[- e^4 f^2 \frac{(d - 4)(d - 2)}{(d - 3)} \square . \tag{4.1}\]

We show that this diagram can be expressed in terms of single-loop diagrams, using the identities from the previous section. The particular form of the result depends on the dimensionality of space.

Splitting the two-loop vacuum diagram (3.1) into one-loop parts can be done only for its convergent part and only if \( n = N \), the power of the photon propagator \( n \) is one half of the number of dimensions \( 2N \). Therefore first we use the identity (3.6) to write (4.1) as a sum of
the second power of the one-loop diagram and $\Box^N$. From this diagram we have to subtract its divergent part that can be expressed in terms of diagrams without the background field (these can always be reduced into one-loop diagrams) and the remaining convergent part can be written as a one-loop diagram in the background field. We describe this procedure in 2, 4 and 6 dimensions and outline how it can be done in any even dimension.

In two dimensions, $N = 1$ and $2\varepsilon = 2 - d$. The identity (3.6) for $n = 1$ implies

$$- e^4 f^2 \frac{(d-4)(d-2)}{(d-3)(d-4)} = e^2 \frac{d-2}{(d-3)(d-4)} \left\{ 2m^2(d-3) + (d-2)\left[ \Box^2 \right]^2 \right\}. \quad (4.2)$$

Despite the fact that all the diagrams are divergent, they are multiplied by an appropriate power of $d - 2$, and thus each term of this equation is finite. The second term on the RHS is already one-loop. In two dimensions its finite part is the same as for the free propagator:

$$e^2 \frac{(d-2)^2}{(d-3)(d-4)} \left[ \Box^2 \right]^2 = \frac{e^2}{2} (d-2)^2 \left[ \Box^2 \right]^2 + O(\varepsilon),$$

and can be manipulated using the identity (3.6) for $n = 1$ without a background field:

$$\frac{d-2}{d-3} \left[ \Box^2 \right]^2 = -2m^2 \Box^2,$$

into the same form as the first term on the RHS of (4.2):

$$e^2 m^2 (d-2) \Box^2 + O(\varepsilon)$$

Now we can use the equation (4.1) and an identical equation for a vanishing background field to obtain:

$$- e^4 f^2 \frac{(d-4)(d-2)}{(d-3)} \Box^2 = -e^2 m^2 (d-2) \left[ \Box^2 - \Box^2 \right] + O(\varepsilon)$$

$$= -e^2 m^2 \frac{2\pi}{2\pi} \left[ \Box^2 - \Box^2 \right] + O(\varepsilon). \quad (4.3)$$

In two dimensions (4.1) does not contain any divergence and after we drop off the mass renormalization from (2.8), the renormalized two-loop effective action has the following form

$$\left[ \Box^2 - \Box^2 \right]_{\text{ren.}} = -\frac{e^2}{2} \left[ \Box^2 - \Box^2 \right]^2 - \frac{e^2 m^2}{2\pi} \left[ \Box^2 - \Box^2 \right]. \quad (4.4)$$

This is the finite fully renormalized two-loop effective action in a general constant background field in two dimensions.

In four dimensions, $N = 2$ and $2\varepsilon = 4 - d$, and the part of the effective action containing charge renormalization (4.1) has already the appropriate form and we do not have to use
the identity (3.6). In this case we can directly subtract the divergent part and use (A2) to obtain
\[- e^4 f^2 \frac{(d-4)(d-2)}{(d-3)} = - e^4 f^2 \frac{d-2}{8\pi^2 d-3} \left[ \begin{array}{c}
\end{array} \right] - e^4 f^2 \frac{(d-4)(d-2)}{(d-3)} \left[ \begin{array}{c}
\right] + O(\varepsilon). \]

The term
\[- e^4 f^2 \frac{(d-4)(d-2)}{(d-3)} \left[ \begin{array}{c}
\right] = - e^4 f^2 \frac{(d-2)^2 (d-4)}{4m^4 (d-3)(d-5)} \left[ \begin{array}{c}
\right] \]

(4.5)
corresponds to charge renormalization and so the renormalized two-loop effective action in four dimensions has the form:
\[
\left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right]_{\text{ren.}} = \frac{3e^2}{2} \left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right]^2 - \frac{e^4 f^2}{4\pi^2} \left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right], \tag{4.7}
\]
as derived in [19].

Notice the similarity between this four dimensional result and the previous two dimensional result (4.4). The fully renormalized effective action in each case, is a linear combination of the same two one-loop terms, but with different coefficients. Moreover these one-loop terms are closely related. Using the propagator (2.3) in the background field (2.1), we find
\[
\left[ \begin{array}{c}
\end{array} \right] - \left[ \begin{array}{c}
\end{array} \right] = \frac{(ef)^{\frac{d-1}{2}}}{(4\pi)^{\frac{d}{2}}} \int_0^\infty dt e^{-2\kappa t} \left( \frac{1}{\sinh \frac{d}{2} t} - \frac{1}{t^\frac{d}{2}} \right) \tag{4.8}
\]
\[
\left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right]_2 = \frac{(ef)^{\frac{d-2}{2}}}{(4\pi)^{\frac{d}{2}}} \int_0^\infty dt t e^{-2\kappa t} \left( \frac{1}{\sinh \frac{d}{2} t} - \frac{1}{t^\frac{d}{2}} \right), \tag{4.9}
\]
where \( \kappa = \frac{m^2}{2ef} \). From these two equations we see that
\[
\left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right]_2 = - \frac{1}{2ef d\kappa} \left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right]. \tag{4.10}
\]
Thus, (4.9) is derivative of (4.8) with respect to \( \kappa \). In four dimensions, we define [19, 23]:
\[
\xi_{4D}(\kappa) \equiv - \frac{(4\pi)^2}{m^2} \kappa \left[ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} \right]_{d=4} = - \kappa \left( \psi(\kappa) - \ln(\kappa) + \frac{1}{2\kappa} \right), \tag{4.11}
\]
where \( \psi(\kappa) = \frac{d}{d\kappa} \ln \Gamma(\kappa) \) is the Euler digamma function [25]. The fully renormalized two-loop effective action in a self-dual constant background in four dimensions has the form:
\[
S^{(2)}_{d=4} = \alpha \frac{m^4}{(4\pi)^3 \kappa^2} \left( \frac{3}{2} \xi_{4D} - \xi_{4D}' \right), \tag{4.12}
\]
as was shown in [19, 23]. Similarly, in two dimensions, we define

\[ \xi_{2D}(\kappa) \equiv 4\pi \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right]_{d=2} = - \left( \psi(\kappa + \frac{1}{2}) - \ln(\kappa) \right), \tag{4.13} \]

and the renormalized two-loop effective action in a general constant background field in two dimensions is

\[ S_{d=2}^{(2)} = - \frac{e^2}{32\pi^2} \left( \xi_{2D}^2 - 4\kappa \xi'_{2D} \right). \tag{4.14} \]

In six dimensions, \( N = 3 \) and \( 2\varepsilon = 6 - d \), and we have to express (4.11) in terms of \( \bigcirc \bigcirc \). Its convergent part then can be reduced to a one-loop diagram. To achieve this, we use (A3) and (C2), and thus for (4.11) we obtain:

\[- e^4 f^2 \frac{(d - 4)(d - 2)}{(d - 3)} \bigcirc \bigcirc = \frac{4e^4 f^2 m^2}{C_6} \frac{(d - 2)(d - 6)^2}{(d - 3)} \bigcirc \bigcirc - \frac{e^2 (d - 2)^2}{C_6 (d - 3)} \bigcirc \bigcirc \bigcirc \bigcirc \]

where \( C_6 = \left( \frac{2m^2}{ef} \right)^2 \left( \frac{5 - d}{4 - d} \right) + (4 - d) \). \tag{4.15}

The first diagram on the RHS can be written by using (A3) as

\[ (d - 6) \bigcirc \bigcirc = \frac{1}{(4\pi)^2} \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right] + (d - 6) \bigcirc \bigcirc + O(\varepsilon) \tag{4.16} \]

and (4.15) becomes

\[- e^4 f^2 \frac{(d - 4)(d - 2)}{(d - 3)} \bigcirc \bigcirc = \frac{e^4 f^2 m^2}{2(2\pi)^2 C_6} \frac{(d - 2)(d - 6)}{(d - 3)} \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right] - \frac{4e^4 f^2 m^2}{C_6} \frac{(d - 2)(d - 6)^2}{(d - 3)} \bigcirc \bigcirc - \frac{e^2 (d - 2)^2}{C_6 (d - 3)} \bigcirc \bigcirc \bigcirc \bigcirc + O(\varepsilon). \tag{4.17} \]

The difference of the two integrals in the first line is already finite and thus when multiplied by \( (d - 6) \), it vanishes in the limit \( d \to 6 \). The free diagram in the second term has a double-pole divergence and gives a finite result when multiplied by \( (d - 6)^2 \). By virtue of (3.9) it can be reduced to a one-loop diagram. Thus the two-loop effective action (2.8) in six dimensions has the following form

\[ \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right] = \frac{e^2}{2} \left( \frac{d - 1}{d - 3} \right) \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right]^2 + e^2 \left( \frac{d - 1}{d - 3} \right) \bigcirc \bigcirc \left[ \bigcirc \bigcirc - \bigcirc \bigcirc \right] - \frac{e^2 (d - 2)^2}{C_6 (d - 3)} \bigcirc \bigcirc \bigcirc \bigcirc + \frac{16e^4 f^2}{3m^4 C_6} (d - 6)^2 \bigcirc \bigcirc \bigcirc \bigcirc + O(\varepsilon) \tag{4.18} \]

The action (4.18) still contains divergences, but this is because QED in six dimensions is nonrenormalizable. Nevertheless, the two-loop effective action in six dimensions is expressed in (4.18) in terms of one-loop diagrams.
In this way we could continue to higher dimensions. In $2N$ dimensions, $2\varepsilon = 2N - d$, the last two-loop term (4.1) of the effective action can be written using the recurrence formula (3.6) as a sum of $\bigcirc$ and $\bigcirc^2$. For $N > 2$, the first diagram will be always multiplied by $(2N - d)^2$, as can be seen from (3.6). In such a way there always will be the factor $2N - d$ in front of $\bigcirc$ allowing us to reduce the diagram into a one-loop as described in III. Moreover, because the factor in front of it contains the factor $(2N - d)^2$, the finite part of $(2N - d)^2 \bigcirc$ will vanish as $d \to 2N$, and its divergent part will give a finite contribution to the effective action. Thus the effective action will always be of the form:

$$\bigcirc + \bigcirc^2 = \frac{e^2}{2} \left( \frac{d-1}{d-3} \right) \bigcirc \bigcirc^2 + e^2 \left( \frac{d-1}{d-3} \right) \bigcirc \bigcirc \bigcirc \bigcirc^2,$$

$$- f_1 \bigcirc \bigcirc^2 + f_2 (d-2N)^2 \bigcirc \bigcirc^2 + O(\varepsilon) \quad (4.19)$$

where $f_1$ and $f_2$ are certain known functions of $N, d, m$ and $(ef)^2$.

V. CONCLUSIONS

In this paper we have further developed the algebraic rules for vacuum diagrams with scalar propagators in a constant electromagnetic field [19, 20]. Such rules are generalizations of the "integration by parts" technique for free propagators [10, 11, 12, 13, 14, 15, 16, 17], a powerful method to perform multi-loop calculations. The simplicity of the background field approach opens a way to calculations of higher order loop diagrams in a background field. We have shown that the reduction of the two-loop Heisenberg-Euler effective action into one-loop diagrams and possible further subtraction of divergences can be done in any even number of dimensions. This extends previously obtained results in four dimensions [19, 20, 21, 22, 23]. We also derived the fully renormalized effective action in a constant background field in two dimensions.

Acknowledgments

I thank G. Dunne for discussions, and the US DOE for support through grant DE-FG02-92ER40716.
APPENDIX A: EXAMPLES OF SPLITTING TWO-LOOP DIAGRAM (3.4) INTO ONE-LOOP DIAGRAMS.

In this section we present some examples of (3.4), the way to write diagram \( N \) in terms of one-loop diagrams in a 2\( N \) dimensional space as was described in [III]. In the dimensional regularization 2\( \varepsilon = 2N - d \).

In two dimensions, \( N = 1 \), the integral on the LHS of (3.2), with the propagators \( G \) (2.3) in a constant background field as functions \( f(p) \) and \( g(q) \), is convergent. Therefore we obtain

\[
(d - 2) \quad \begin{array}{c} \bigcirc \bigcirc \end{array} = \frac{1}{2\pi} \quad \begin{array}{c} \bigcirc \bigcirc \end{array} + O(\varepsilon) \quad \text{(A1)}
\]

without any need of subtracting divergences. Such subtraction is necessary in higher dimensions. The identity (A1) holds also for zero background field.

In four dimensions, \( N = 2 \), we have to subtract the divergent part of the two-loop diagram. Then the formula (3.4) has the following form:

\[
(d - 4) \left[ \begin{array}{c} \bigcirc \bigcirc - \bigcirc \bigcirc \end{array} \right] = \frac{1}{8\pi^2} \left[ \begin{array}{c} \bigcirc \bigcirc - \bigcirc \bigcirc \end{array} \right] + O(\varepsilon) \quad \text{(A2)}
\]

This formula can be brought to a form identical to (3.4) by using well known identities (3.9) and (A6). The procedure can be repeated in the same way in six dimensions, \( N = 3 \), producing:

\[
(d - 6) \left[ \begin{array}{c} \bigcirc \bigcirc - \bigcirc \bigcirc \end{array} \right] = \frac{1}{(4\pi)^3} \left[ \begin{array}{c} \bigcirc \bigcirc - \bigcirc \bigcirc \end{array} \right] + O(\varepsilon) \quad \text{(A3)}
\]

In eight dimensions, \( N = 4 \), we have to subtract another divergence to obtain the finite result:

\[
(d - 8) \left[ \begin{array}{c} \bigcirc \bigcirc - \bigcirc \bigcirc - (ef)^2 (4 - d) \bigcirc \bigcirc \end{array} \right] = \frac{1}{3(4\pi)^4} \left[ \begin{array}{c} \bigcirc \bigcirc - \bigcirc \bigcirc - (ef)^2 (4 - d) \bigcirc \bigcirc \end{array} \right] + O(\varepsilon) \quad \text{(A4)}
\]

All the above expressions (A2), (A3) and (A4) can be written in the same form as (3.4) by using the equation (3.9) for the loop integrals of the free propagators which can be obtained by integration by parts and

\[
\begin{array}{c} \bigcirc \bigcirc \end{array} = \frac{1}{2m^2} \frac{(6 - \frac{d}{2}) (5 - \frac{d}{2}) (4 - \frac{d}{2}) (3 - \frac{d}{2}) (2 - \frac{d}{2}) (1 - \frac{d}{2})}{(d - 8)(d - 9)(d - 10)(d - 11)} \left[ \begin{array}{c} \bigcirc \bigcirc \end{array} \right]^2 \quad \text{(A5)}
\]
together with:

$$\frac{1}{n} = \frac{1}{m^{2(n-1)}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n) \Gamma(1 - \frac{d}{2})} \cdot$$

\[\text{(A6)}\]

**APPENDIX B: PROOF OF THE RECURRENCE RELATION (3.6).**

A two-loop vacuum diagram of the form (3.1) in the background field (2.1) is equal to the following parametric integral

$$\frac{1}{n} \cdot$$

\[\text{(B1)}\]

For $d < n + 1$, the value of the following integral is equal to zero

$$K_n \int_0^\infty dy \ dy \left[ e^{-2\kappa y} (\sinh y)^{n-d+1} \right]$$

\[\text{(B2)}\]

By performing the derivative and using the decomposition $1 = \frac{n}{2n-d+1} + \frac{(n-d+1)}{2n-d+1}$ we obtain:

$$0 = -2\kappa \cdot$$

\[\text{(B3)}\]

The final step follows from two identities for hypergeometric functions [25]:

\[\text{(B4)}\]
and the observation that $K_n$ can be written as

$$K_n = -2ef(2n - d + 3) K_{n+1}$$

$$K_n = -\frac{1}{2ef(2n - d + 1)} K_{n-1}$$

From the resulting integrals one can quickly recover the two-loop integrals (B1) with $(n+1)$-th and $(n-1)$-th power of the free massless scalar propagator and obtain the formula (3.6). By analytic continuation we can extend region of validity of (3.6) beyond $d < n + 1$.

**APPENDIX C: EXAMPLES OF THE RECURRENCE FORMULA (3.6).**

In this section we give some examples of the relation (3.6) for $n = 1, 2, 3$ and also the way (3.7) of writing the two-loop diagram \(\begin{array}{c}
\end{array}\) in terms of the diagrams \(\begin{array}{c}
\end{array}\) and \(\begin{array}{c}
\end{array}\).

For $n = 1$, the identity (3.6) acquires the following form:

$$(ef)^2 (d - 4)^2 \begin{array}{c}
\end{array} - 2m^2(3 - d) \begin{array}{c}
\end{array} - (2 - d) \begin{array}{c}
\end{array}^2 = 0 \quad \text{(C1)}$$

For $n = 2$, the identity (3.6) has the following form:

$$2(ef)^2 (d - 6)^2 \begin{array}{c}
\end{array} - 2m^2(5 - d) \begin{array}{c}
\end{array} - (3 - d) \begin{array}{c}
\end{array} = 0 \quad \text{(C2)}$$

We can use (C1) to write (C2) as

$$\begin{aligned}
\begin{array}{c}
\end{array} &= \frac{1}{(ef)^2 2 (d - 6)^2} \left\{ \left( \frac{2m^2}{ef} \right)^2 \frac{(5 - d)(3 - d)}{(d - 4)^2} + (3 - d) \right\} \begin{array}{c}
\end{array} + \frac{2m^2 (5 - d)(2 - d)}{(ef)^2 (d - 4)^2} \begin{array}{c}
\end{array}^2 \right\}.
\end{aligned} \quad \text{(C3)}$$

For $n = 3$, the identity (3.6) has the following form:

$$3(ef)^2 (d - 8)^2 \begin{array}{c}
\end{array} - 2m^2(7 - d) \begin{array}{c}
\end{array} - (4 - d) \begin{array}{c}
\end{array} = 0 \quad \text{(C4)}$$

Now we use (C1) and (C3) to write (C4) as

$$\begin{aligned}
\begin{array}{c}
\end{array} &= \frac{1}{(ef)^2 3 (d - 8)^2} \left\{ \frac{1}{2ef} \left[ \left( \frac{2m^2}{ef} \right)^3 \frac{(7 - d)(5 - d)(3 - d)}{(d - 6)^2 (d - 4)^2} + \frac{2m^2 (7 - d)(4 - d)(3 - d) + 2(6 - d)^2 (3 - d)}{(d - 6)^2 (4 - d)} \right] \begin{array}{c}
\end{array} + \frac{1}{(ef)^2} \left[ \left( \frac{2m^2}{ef} \right)^2 \frac{(7 - d)(5 - d)(2 - d)}{2 (d - 6)^2 (d - 4)^2} + \frac{2 - d}{4 - d} \right] \begin{array}{c}
\end{array}^2 \right\}.
\end{aligned} \quad \text{(C5)}$$

13
In this way we can continue and express the two-loop diagram of the form $\{(3.1)\}$ for any $n$ as a linear combination of the diagram $\bigcirc\bigotimes$ and the square of the diagram \(\bigcirc\).
[13] L. V. Avdeev, “Recurrence Relations For Three-Loop Prototypes Of Bubble Diagrams With A Mass,” Comput. Phys. Commun. 98, 15 (1996) [arXiv:hep-ph/9512442]. L. V. Avdeev, J. Fleischer, M. Y. Kalmykov and M. N. Tentyukov, “Towards automatic analytic evaluation of diagrams with masses,” Comput. Phys. Commun. 107, 155 (1997) [arXiv:hep-ph/9710222].

[14] P. A. Baikov, “Explicit solutions of the 3–loop vacuum integral recurrence relations,” Phys. Lett. B 385, 404 (1996) [arXiv:hep-ph/9603267]; P. A. Baikov and M. Steinhauser, “Three-loop vacuum integrals in FORM and REDUCE,” Comput. Phys. Commun. 115, 161 (1998) [arXiv:hep-ph/9802429].

[15] V. A. Smirnov and M. Steinhauser, “Solving recurrence relations for multi-loop Feynman integrals,” Nucl. Phys. B 672, 199 (2003) [arXiv:hep-ph/0307088].

[16] Y. Schroder, “Automatic Reduction Of Four-Loop Bubbles,” Nucl. Phys. Proc. Suppl. 116, 402 (2003) [arXiv:hep-ph/0211288].

[17] K. G. Chetyrkin, M. Faisst, C. Sturm and M. Tentyukov, “e-finite basis of master integrals for the integration-by-parts method,” Nucl. Phys. B 742, 208 (2006) [arXiv:hep-ph/0601165].

[18] G. V. Dunne, “Heisenberg-Euler effective Lagrangians: Basics and extensions,” in Ian Kogan Memorial Collection, From Fields to Strings: Circumnavigating Theoretical Physics, Vol. I, M. Shifman (ed.) et al, (World Scientific, 2005) [arXiv:hep-th/0406216].

[19] G. V. Dunne, “Two-loop diagrammatics in a self-dual background,” JHEP 0402, 013 (2004) [arXiv:hep-th/0311167].

[20] G. V. Dunne and M. Krasnansky, “Background field integration-by-parts’ and the connection between one-loop and two-loop Heisenberg-Euler effective actions,” JHEP 0604, 020 (2006) [arXiv:hep-th/0602216].

[21] V. I. Ritus, “Lagrangian Of An Intensive Electromagnetic Field And Quantum Electrodynamics At Short Distances,” Sov. Phys. JETP 42, 774 (1975) [Pisma Zh. Eksp. Teor. Fiz. 69, 1517 (1975)]; “On The Relation Between The Quantum Electrodynamics Of An Intense Field And The Quantum Electrodynamics At Small Distances,” Zh. Eksp. Teor. Fiz. 73, 807 (1977); “The Lagrangian Function of an Intense Electromagnetic Field”, in Proc. Lebedev Phys. Inst. Vol. 168, Issues in Intense-field Quantum Electrodynamics, V. I. Ginzburg, ed., (Nova Science Pub., NY 1987).

[22] W. Dittrich and M. Reuter, “Effective Lagrangians In Quantum Electrodynamics,” Lect. Notes Phys. 220, 1 (1985); M. Reuter, M. G. Schmidt and C. Schubert, “Constant external
fields in gauge theory and the spin 0, 1/2, 1 path integrals,” Annals Phys. 259, 313 (1997) [arXiv:hep-th/9610191]; D. Fliegner, M. Reuter, M. G. Schmidt and C. Schubert, “Two-loop Euler-Heisenberg Lagrangian in dimensional regularization,” Theor. Math. Phys. 113, 1442 (1997) [Teor. Mat. Fiz. 113, 289 (1997)] [arXiv:hep-th/9704194]; B. Kors and M. G. Schmidt, “The effective two-loop Euler-Heisenberg action for scalar and spinor QED in a general constant background field,” Eur. Phys. J. C 6, 175 (1999) [arXiv:hep-th/9803144].

[23] G. V. Dunne and C. Schubert, “Closed-form two-loop Euler-Heisenberg Lagrangian in a self-dual background,” Phys. Lett. B 526, 55 (2002) [arXiv:hep-th/0111134]; “Two-loop self-dual Euler-Heisenberg Lagrangians. I: Real part and helicity amplitudes,” JHEP 0208, 053 (2002) [arXiv:hep-th/0205004]; “Two-loop self-dual Euler-Heisenberg Lagrangians. II: Imaginary part and Borel analysis,” JHEP 0206, 042 (2002) [arXiv:hep-th/0205005].

[24] M. J. Duff and C. J. Isham, “Selfduality, Helicity, And Supersymmetry: The Scattering Of Light By Light,” Phys. Lett. B 86, 157 (1979); A. D’Adda and P. Di Vecchia, “Supersymmetry and instantons, Phys. Lett. B 73, 162 (1978); I. Bialynicki-Birula, E. T. Newman, J. Porter, J. Winicour, B. Lukacs, Z. Perjes and A. Sebestyen, “A Note On Helicity,” J. Math. Phys. 22, 2530 (1981).

[25] A. Erdélyi (ed.), *Higher Transcendental Functions, Vol. I*, (Kreiger, Florida, 1981).