EXTENSIONS OF WEAK-TYPE MULTIPLIERS

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Abstract. In this paper we prove that if \( \Lambda \in M_p(\mathbb{R}^N) \) and has compact support then \( \Lambda \) is a weak summability kernel for \( 1 < p < \infty \), where \( M_p(\mathbb{R}^N) \) is the space of multipliers of \( L^p(\mathbb{R}^N) \).

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1. Introduction

Let $G$ be a locally compact abelian group, with Haar measure $\mu$ and let $\hat{G}$ be its dual. We call an operator $T : L^p(G) \to L^{p,\infty}(G)$, $1 \leq p < \infty$, a multiplier of weak type $(p, p)$, if it is bounded and translation invariant i.e. $\tau_x T = T \tau_x \forall x \in G$, and there exists a constant $C > 0$ such that

$$\mu\{x \in G : |Tf(x)| > t\} \leq \frac{Cp}{tp} \|f\|_p^p$$

for all $f \in L^p(G)$ and $t > 0$. (Here $L^{p,\infty}$ denotes the standard weak $L^p$ spaces.) Asmar, Berkson and Gillespie in [3] proved that for all such operators $T$ there exists a $\phi \in L^\infty(\hat{G})$ such that $(T^\wedge f) = \phi \hat{f}$ for all $f \in L^2 \cap L^p(G)$. We will also call such $\phi$’s to be multipliers of weak type $(p, p)$. Let $M_p^{(u)}(\hat{G})$ denote the space of multipliers of weak type $(p, p)$ for $1 \leq p < \infty$, and let $N_p^{(w)}(\phi)$ be the smallest constant $C$ such that inequality (1.1) holds.

In this paper we are concerned with extensions of weak type multipliers from $\mathbb{Z}^N$ to $\mathbb{R}^N$ through summability kernels. For similar results on strong type multipliers, see [4], [6]. Here we identify $\mathbb{T}^N$ with $[0,1)^N$ and for $f \in L^1(\mathbb{R}^N)$ we define its Fourier transform as $\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \xi \cdot x} dx$ for $\xi \in \mathbb{R}^N$. Let us define summability kernels for weak type multipliers as follows

**Definition 1.1.** A bounded measurable function $\Lambda : \mathbb{R}^N \to \mathbb{C}$ is called a weak summability kernel for $M_p^{(u)}(\mathbb{R}^N)$ if for $\phi \in M_p^{(u)}(\mathbb{Z}^N)$ the function $W_{\phi, \Lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi(n) \Lambda(\xi - n)$ is defined and belongs to $M_p^{(u)}(\mathbb{R}^N)$.

This definition is just the weak type analogue of summability kernel for strong type multipliers [4]. We first cite two important results
regarding the summability kernels of strong type multipliers from the
work of Jodeit [6] and of Berkson, Paluszynski and Weiss [4]:

**Theorem 1.1.** [6] Let $S \in L^1(\mathbb{R}^N)$ and supp $S \subseteq \left[\frac{1}{4}, \frac{3}{4}\right]^N$ with $\tau = \sum_{n \in \mathbb{Z}^N} |\hat{s}(n)| < \infty$, where $s$ is the 1-periodic extension of $S$, then the function defined by $W_{\phi,\hat{S}}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi(n)\hat{S}(\xi - n)$ belongs to $M_p(\mathbb{R}^N)$, for $1 \leq p < \infty$ with $\|W_{\phi,\hat{S}}\|_{M_p(\mathbb{R}^N)} \leq C_p \tau \|\phi\|_{M_p(\mathbb{Z}^N)}$

**Theorem 1.2.** [4] For $1 \leq p < \infty$, let $\Lambda \in M_p(\mathbb{R}^N)$ and supp $\Lambda \subseteq \left[\frac{1}{4}, \frac{3}{4}\right]^N$. For $\phi \in M_p(\mathbb{Z}^N)$ define $W_{\phi,\Lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi(n)\Lambda(\xi - n)$ on $\mathbb{R}^N$. Then $W_{\phi,\Lambda} \in M_p(\mathbb{R}^N)$ and $\|W_{\phi,\Lambda}\|_{M_p(\mathbb{R}^N)} \leq C_p \|\Lambda\|_{M_p(\mathbb{R}^N)} \|\phi\|_{M_p(\mathbb{Z}^N)}$ where $C_p$ is a constant. (Further, if $\Lambda$ has arbitrary compact support the same result holds except that the constant $C_p$ necessarily depends on the support of $\Lambda$, as shown in [4].

Asmar, Berkson and Gillespie proved a weak type analogue of Theorem 1.1 in [3]. In this same paper they also proved that $\Lambda$ defined by $\Lambda(\xi) = \prod_{j=1}^N \max(1 - |\xi_j|, 0)$ for $\xi = (\xi_1, ..., \xi_N)$ is a weak type summability kernel. In this paper, we prove the weak type analoge of Theorem 1.2 in §2, for $1 < p < \infty$. In §3 we relax the hypothesis that supp $\Lambda \subseteq \left[\frac{1}{4}, \frac{3}{4}\right]^N$. For the proof of our main result , as in [4], we will obtain the weak type inequalities by applying the technique of transference couples due to Berkson, Paluszyński, and Weiss [4].

**Definition 1.2.** For a locally compact group $G$, a transference couple is a pair $(S, T) = (\{S_u\}, \{T_u\})$, $u \in G$, of strongly continuous mappings defined on $G$ with values in $\mathcal{B}(X)$, where $X$ is a Banach space, satisfying
(i) \( C_S = \sup \{ \| S_u \| : u \in G \} < \infty \)

(ii) \( C_T = \sup \{ \| T_u \| : u \in G \} < \infty \)

(iii) \( S_v T_u = T_{vu} \quad \forall u, v \in G \)

In §4, as an application of our result, we prove a weak type analogue of an extension theorem by de Leeuw.

2. Weak-Type Inequality for Transference Couples and The Main Theorem

Let \( \Lambda \in L^\infty(\mathbb{R}^N) \) and \( \text{supp} \ \Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N \). Consider the following transference couple \((S, T)\) used by Berkson, Paluszyński, and Weiss in [4]. For \( u \in \mathbb{T}^N \) the family \( T = \{ T_u \} \) is given by

\[
(T_u f) \hat{\lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \Lambda(\xi - n) e^{2\pi i u \cdot n} \hat{f}(\xi), \quad \text{for } f \in L^p(\mathbb{R}^N) 
\]

and the family \( S = \{ S_u \} \) is defined by

\[
(S_u f) \hat{\lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} b(\xi - n) e^{2\pi i u \cdot n} \hat{f}(\xi), \quad \text{for } f \in L^p(\mathbb{R}^N), \quad (2.3)
\]

where \( b(\xi) = \prod_{i=1}^N b_i(\xi_i) \) for \( \xi = (\xi_1, \ldots, \xi_N) \) and for each \( i \), \( b_i \) is the continuous function defined on \( \mathbb{R} \) as \( b_i(x) = 1 \) if \( x \in [\frac{1}{4}, \frac{3}{4}] \), linear in \( [0, \frac{1}{4}] \cup (\frac{3}{4}, 1] \) and 0 otherwise. It is easy to see that

\[
S_u f(x) = \sum_{l \in \mathbb{Z}^N} \tilde{\beta}_u(l) f(x + u - l) \quad \text{a.e.,} \quad (2.4)
\]

where \( \tilde{\beta}_u \) is the inverse Fourier transform of the function \( \beta_u(\xi) = b(\xi) e^{2\pi i u \cdot \xi} \), given explicitly by

\[
\tilde{\beta}_u(\xi) = \prod_{i=1}^N \tilde{\beta}_u(\xi_i),
\]
where

\[ \tilde{\beta}_{ui}(\xi_i) = \begin{cases} 2e^{2\pi i (\xi_i - u_i/2)}(\cos \frac{\pi}{2}(\xi_i - u_i) - \cos \pi(\xi_i - u_i)) & \text{if } \xi_i \neq u_i \\ \frac{e^{2\pi i (\xi_i + u_i)/4}}{2} & \text{if } \xi_i = u_i. \end{cases} \]

Then by a straightforward calculation using Eqn. (2.5) we have

\[ \sum_{l \in \mathbb{Z}^N} |\tilde{\beta}_{u}(l)| \leq \sum_{l \in \mathbb{Z}^N} \beta(l) = C < \infty, \]

where \( \beta(l) = \prod_{i=1}^N \beta_i(l_i) \) and \( \beta_i(l_i) = \begin{cases} \frac{1}{(l_i-1)^2} & \text{if } l_i > 1 \\ \frac{1}{(l_i+1)^2} & \text{if } l_i < 1 \\ \|b_i\|_1 & \text{otherwise.} \end{cases} \)

In the following theorem we shall show that the operator transferred by \( T \) (of the transference couple \((S, T)\) defined in Eqn. (2.2) and Eqn. (2.3)) given by

\[ H_k f(.) = \int_{\mathbb{T}^N} k(u)T_{u-1} f(.) du, \]

where \( k \in L^1(\mathbb{T}^N) \) and \( f \in L^p(\mathbb{R}^N) \), satisfies a weak \((p, p)\) inequality.

**Theorem 2.1.** Let \((S, T)\) be the transference couple as defined in Eqn. (2.2) and Eqn. (2.3). Then for \(1 < p < \infty\) and \( t > 0\)

\[ \lambda\{x \in \mathbb{R}^N : |H_k f(x)| > t\} \leq \left( \frac{C_p}{t} C_T N_p^{(w)}(k) \|f\|_p \right)^p, \]

where \( \lambda \) denotes the Lebesgue measure of \( \mathbb{R}^N \), \( C = \sum_{l \in \mathbb{Z}^N} \beta(l) \) as in Eqn. (2.6), \( C_T \) is the uniform bound for the family \( T = \{T_u\} \), and \( C_p = \frac{p}{p-1}. \)
Proof: Assume $f \in S(\mathbb{R}^N)$. For $t > 0$ define $E_t = \{x : |H_k f(x)| > t\}$. Notice that

$$H_k f(x) = S_{v^{-1}} S_v H_k f(x) = \sum_{l \in \mathbb{Z}^N} \tilde{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(x - v - l) du > t.$$ Let $F_t = \{(v, x) \in \mathbb{T}^N \times \mathbb{R}^N : |\sum_{l \in \mathbb{Z}^N} \tilde{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(x - l) du| > t\}$. Then, using translation invariant of Lebesgue measure

$$\lambda(E_t) = \lambda\{x \in \mathbb{R}^N : |S_{v^{-1}} \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(x) du| > t\}$$

$$= \lambda\{x \in \mathbb{R}^N : |\sum_{l \in \mathbb{Z}^N} \tilde{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(x - l) du| > t\}$$

$$= \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} \chi_{F_t}(v, x) dx dv$$

$$= \int_{\mathbb{R}^N} |\{v : \sum_{l \in \mathbb{Z}^N} \tilde{\beta}_{v^{-1}}(l) \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(x - l) du| > t\}| dx,$$ where $|E|$ denotes the measure of the subset $E \subseteq \mathbb{T}^N$. Thus

$$\lambda(E_t) \leq \int_{\mathbb{R}^N} |\{v : \sum_{l \in \mathbb{Z}^N} \beta(l)| \int_{\mathbb{T}^N} k(u) T_{u^{-1}} f(x - l) du| > t\}| dx$$

$$= \int_{\mathbb{R}^N} |\{v : \sum_{l \in \mathbb{Z}^N} \beta(l)|k \ast F(v, x - l)(v)| > t\}| dx,$$ where $F(v, x) = T_v f(x)$ a.e.

We know that $\sup_{t > 0} t \lambda_f(t)^\frac{1}{p} = \|f\|_{L^{p, \infty}}$ for $f \in L^{p, \infty}$. Also, since $p > 1$, $\|\cdot\|_{p, \infty}$ is equivalent to a norm $\|\cdot\|_{p, \infty}^*$ (\textsection 3), using triangle inequality for norms

we have

$$\lambda(E_t) \leq \int_{\mathbb{R}^N} \frac{1}{t^p} \| \sum_{l \in \mathbb{Z}^N} \beta(l)|k \ast F(., x - l)|^p_{L^{p, \infty}(\mathbb{T}^N)} dx$$

$$\leq C_p \int_{\mathbb{R}^N} \frac{1}{t^p} \left( \sum_{l \in \mathbb{Z}^N} \beta(l) \|k \ast F(., x - l)|^*_{L^{p, \infty}(\mathbb{T}^N)} \right)^p dx, \quad \text{where} \quad C_p = \frac{p}{p - 1}$$

$$\leq C_p \int_{\mathbb{R}^N} \frac{1}{t^p} \left( \sum_{l \in \mathbb{Z}^N} \beta(l) N_p^*(l) \|F(., x - l)|^p_{L^{p}(\mathbb{T}^N)} \right)^p dx,$$
where \( N_p^{(w)}(k) \) is the weak-type norm of the convolution operator \( f \mapsto k * f \) for \( f \in L^p(\mathbb{T}^N) \). Thus,

\[
\lambda(E_t) \leq C_p \frac{1}{tp} \sum_{l \in \mathbb{Z}^N} \beta(l) N_p^{(w)}(k) \left( \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} |T_v f(x - l)|^p \, dx \, dv \right)^{\frac{1}{p}}
\]

\[
= C_p \frac{1}{tp} \left( \sum_{l \in \mathbb{Z}^N} \beta(l) N_p^{(w)}(k) \left( \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} |T_v f(x - l)|^p \, dx \, dv \right)^{\frac{1}{p}} \right)
\]

\[
\leq \left( \frac{CC_p C_T}{t^p} N_p^{(w)}(k) \|f\|_p \right)^p.
\]

Hence, \( H_kf \) satisfies a weak \((p,p)\) inequality.

In order to prove the weak-type analogue of Theorem 1.2 we need the following Lemma proved by Asmar, Berkson, and Gillespie in [1].

**Lemma 2.1.** Suppose that \( 1 \leq p < \infty \), \( \{\phi_j\} \subseteq M_p^{(w)}(\hat{G}) \); \( \sup \{|\phi_j(\gamma)| : j \in \mathbb{N}, \gamma \in \hat{G}\} < \infty \) and suppose \( \phi_j \) converges pointwise a.e. on \( \hat{G} \) to a function \( \phi \). If \( \lim \inf j N_p^{(w)}(\phi_j) < \infty \) then \( \phi \in M_p^{(w)}(\hat{G}) \) and \( N_p^{(w)}(\phi) \leq \lim \inf j N_p^{(w)}(\phi_j) \).

In the following theorem, we use the family of operators \( \{T_u\} \) defined in (2.2) with \( \Lambda \in M_p(\mathbb{R}^N) \) and \( \text{supp} \Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N \). In this case, by [3] we have \( C_T \leq c_p \|\Lambda\|_{M_p(\mathbb{R}^N)} \), where \( c_p \) is a constant.

**Theorem 2.2.** Suppose \( 1 < p < \infty \) and \( \Lambda \in M_p(\mathbb{R}^N) \) is supported in the set \([\frac{1}{4}, \frac{3}{4}]^N \). For \( \phi \in M_p^{(w)}(\mathbb{Z}^N) \) define

\[
W_{\phi, \Lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi(n) \Lambda(\xi - n) \quad \text{on } \mathbb{R}^N.
\]

Then \( W_{\phi, \Lambda} \in M_p^{(w)}(\mathbb{R}^N) \) and \( N_p^{(w)}(W_{\phi, \Lambda}) \leq CN_p^{(w)}(\phi) \|\Lambda\|_{M_p(\mathbb{R}^N)} \).

**Proof:** Using Lemma 2.1 we first show that it is enough to prove the theorem for \( \phi \in M_p^{(w)}(\mathbb{Z}^N) \) having finite support. Suppose the theorem
is true for finitely supported \( \phi \). Then for arbitrary \( \phi \in M_p^w(\mathbb{Z}^N) \),
define \( \phi_j = \hat{k}_j \phi \), where \( k_j \) is the j-th Féjer kernel. Then for each \( j \),
\( \phi_j \)'s have finite support and \((T_{\phi_j} f)^w(n) = \phi_j(n) \hat{f}(n) = (T_{\phi_k} (k_j \ast f))^w(n)\).
So \( \phi_j \in M_p^w(\mathbb{Z}^N) \) for each \( j \) and \( N_p^w(\phi_j) \leq N_p^w(\phi) \). Define
\( W_{\phi_j, \Lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi_j(n) \Lambda(\xi - n) \).
Now \( \liminf W_{\phi_j, \Lambda}(\xi) = W_{\phi, \Lambda}(\xi) \).
Also, by our assumption
\[ N_p^w(W_{\phi_j, \Lambda}) \leq CN_p^w(\phi) \| \Lambda \|_{M_p(\mathbb{R}^N)} \]
and \( |W_{\phi_j, \Lambda}| \leq 2 \| \Lambda \|_{\infty} \| \phi_j \|_{\infty} \leq 2 \| \Lambda \|_{\infty} \| \phi \|_{\infty} \).
Thus by Lemma 2.1, applied to \( W_{\phi_j, \Lambda} \)'s, we conclude that \( W_{\phi, \Lambda} \in M_p^w(\mathbb{R}^N) \). Hence it is
enough to assume that \( \phi \in M_p^w(\mathbb{Z}^N) \) has finite support.

Now let \( \phi \in M_p^w(\mathbb{Z}^N) \) be finitely supported. Define \( k(u) = \sum_{n \in \mathbb{Z}^N} \phi(n) e^{-2\pi i u \cdot n} \)
then \( k \in L^1(\mathbb{T}^N) \) and \( \hat{k}(n) = \phi(n) \). For this particular \( k \) and the transference couple \((S,T)\) defined above. We have
\[ (H_k f)^w(\xi) = (T_{W_{\phi, \Lambda}} f)^w(\xi). \]
Thus \( T_{W_{\phi, \Lambda}} f = H_k f \). Hence from Theorem 2.1 and since \( C_T \leq c_p \| \Lambda \|_{M_p(\mathbb{R}^N)} \),
we have
\[ \lambda\{x \in \mathbb{R}^N : |T_{W_{\phi, \Lambda}} f(x)| > t\} \leq \left( \frac{C}{t} N_p^w(\phi) \| \Lambda \|_{M_p(\mathbb{R}^N)} \| f \|_p \right)^p. \]

3. Lattice Preserving Linear Transformations and Multipliers

We shall now relax the hypothesis that \( \text{supp} \ \Lambda \subseteq [\frac{1}{4}, \frac{3}{4}]^N \) to allow \( \Lambda \) to have arbitrary compact support. In fact this can be done by a
partition of identity argument as in \([4]\). Here we give a different method by proving Lemma 3.2 below. Particular cases of this lemma occur in
\([3]\) and in \([2]\). Suppose \( \text{supp} \ \Lambda \subseteq [-M, M]^N \); define \( \Lambda_M(\xi) = \Lambda_1(4M\xi) \),
where $\Lambda_1(\xi) = \Lambda(\xi - \frac{1}{2})$. So $\text{supp } \Lambda_M \subseteq \left[ \frac{1}{4}, \frac{3}{4} \right]^N$. Thus if we define a non-singular transformation $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $Ax = 4Mx$ then $\Lambda_M = \Lambda_1 \circ A$. In order to replace the support condition we need to prove $\Lambda_M \circ A^{-1}$ is a summability kernel. In the work of Jodeit and of Asmar, Berkson and Gillespie they assume $A$ in Lemma 3.2 to be multiplication by 2. We have combined some of the results proved by Gröchenig and Madych [5] in the following lemma which will help us to prove Lemma 3.2. In the proof of Theorem 3.1, we only use the case of a diagonal linear transform, but the more general results proved below are of some interest in their own right.

**Lemma 3.1.** [5] Let $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a non-singular linear transformation which preserves the lattice $\mathbb{Z}^N$ (i.e. $A(\mathbb{Z}^N) \subseteq \mathbb{Z}^N$). Then the following are true.

(i) The number of distinct coset representatives of $\mathbb{Z}^N/A\mathbb{Z}^N$ is equal to $q = |\det A|$.

(ii) If $Q_0 = [0, 1)^N$ and $k_1, \ldots, k_q$ are the distinct coset representatives of $\mathbb{Z}^N/A\mathbb{Z}^N$ then the sets $A^{-1}(Q_0 + k_i)$ are mutually disjoint.

(iii) Let $Q = \bigcup_{i=1}^q A^{-1}(Q_0 + k_i)$, then $\lambda(Q) = 1$ and $\cup_{k \in \mathbb{Z}^N}(Q + k) \simeq \mathbb{R}^N$.

(iv) $AQ \simeq \bigcup_{i=1}^q (Q_0 + k_i)$.

Where $E \simeq F$ if $\lambda(F \triangle E) = 0$.

The above result is essentially contained in [5].

**Lemma 3.2.** Let $A$ be as in Lemma 3.1. Denote $A^t = B$, where $A^t$ is the transpose of $A$. For $\phi \in l_\infty(\mathbb{Z}^N)$ define

$$\psi(n) = \phi(Bn)$$

and
\( \eta(n) = \begin{cases} 
\phi(B^{-1}n) & n \in B\mathbb{Z}^N \\
0 & \text{otherwise.} 
\end{cases} \)

(i) If \( \phi \in M_p(\mathbb{Z}^N) \) then \( \psi, \eta \in M_p(\mathbb{Z}^N) \) with multiplier norms not exceeding the multiplier norm of \( \phi \).

(ii) If \( \phi \in M_p^{(w)}(\mathbb{Z}^N) \) then \( \psi, \eta \in M_p^{(w)}(\mathbb{Z}^N) \) with weak multiplier norms not exceeding the weak multiplier norm of \( \phi \).

**Proof:** (i) For \( f \in L^p(Q_0) \), we let \( f \) again denote the periodic extension to \( \mathbb{R}^N \). Define \( Sf(x) = f(Ax) \), then \( Sf \) is also periodic and

\[
\int_{Q_0} |Sf(x)|^p dx = \int_{Q_0} |Sf(x)|^p \sum_j \chi_Q(x - j) dx \\
= \sum_j \int_{Q_0 + j} |Sf(x)|^p \chi_Q(x) dx \\
= \int_Q |Sf(x)|^p dx \\
= \frac{1}{|\det A|} \int_{AQ} |f(x)|^p dx \\
= \frac{1}{q} \sum_{i=1}^q \int_{Q_0 + k_i} |f(x)|^p dx \quad ((iv) \text{ of Lemma 3.1}) \\
= \int_{Q_0} |f(x)|^p dx.
\]

Thus \( S \) is an isometry, i.e., \( \|Sf\|_{L^p(Q_0)} = \|f\|_{L^p(Q_0)} \). Further, from the orthogonality relations of the characters (Lemma 1, [7]) we have

\[
(Sf)\wedge(n) = \begin{cases} 
\hat{f}(B^{-1}n) & \text{if } n \in B\mathbb{Z}^N \\
0 & \text{otherwise.} 
\end{cases}
\]
For $f \in L^p(Q_0)$ we define an operator $W$ on $L^p(Q_0)$ given by $Wf(x) = \frac{1}{q} \sum_{i=1}^{q} f(A^{-1}(x + k_i))$, where $k_1, \ldots, k_q$ are distinct cosets representations of $\mathbb{Z}^N/A\mathbb{Z}^N$. Then for a trigonometric polynomial $f$,

$$(Wf)^\wedge(n) = \hat{f}(Bn),$$

and so

$$\left( \int_{Q_0} |Wf(x)|^p dx \right)^{\frac{1}{p}} = \left( \int_{Q_0} \frac{1}{q} \sum_{i=1}^{q} |f(A^{-1}(x + k_i))|^p dx \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{q} \sum_{i=1}^{q} \left( \int_{Q_0} |f(A^{-1}(x + k_i))|^p dx \right)^{\frac{1}{p}}$$

$$= \frac{q^{1/p}}{q} \sum_{i=1}^{q} \left( \int_{A^{-1}(Q_0 + k_i)} |f(x)|^p dx \right)^{\frac{1}{p}}.$$ 

Therefore $\|Wf\|_{L^p(Q_0)} \leq q^{1-p/p} \|f\|_{L^p(Q_0)}$, since

$$\int_{Q_0} |f(x)|^p dx = \int_{Q} |f(x)|^p dx$$

as above. It is easy to see that

$$ST\phi W = T_\eta \tag{3.7}$$

and

$$WT_\psi S = T_\psi \tag{3.8}$$

It follows that, if $\phi \in M_p(\mathbb{Z}^N)$ then $\|T_\psi f\| \leq C_p \|\phi\|_{M_p(\mathbb{Z}^N)} \|f\|_{L^p(Q_0)}$. Also $\|T_\eta f\|_{L^p(Q_0)} \leq C_p \|\phi\|_{M_p(\mathbb{Z}^N)} \|f\|_{L^p(Q_0)}$. Hence $\psi, \eta \in M_p(\mathbb{Z}^N)$.

(ii) For $\phi \in M_p^{(w)}(\mathbb{Z}^N)$, we need to calculate the distribution function of $Sf$ and $Wf$. Denote $E_t = \{x \in Q_0 : |Sf(x)| > t \}$, then

$$|E_t| = \int_{Q_0} \chi_{E_t}(x) dx$$

$$= \int_{Q_0} \chi_{\mathbb{R}^+}(|f(Ax)| - t) dx$$

$$= \frac{1}{q} \int_{AQ} \chi_{\mathbb{R}^+}(|f(x)| - t) dx.$$
\[ \frac{1}{q} \sum_{i=1}^{q} \int_{Q_0 + k_i} \chi_{\mathbb{R}^+}(|f(x)| - t) \, dx \]
\[ = |\{x : |f(x)| > t\}|. \]

Therefore,

\[ \{x \in Q_0 : |Sf(x)| > t\} = \{x \in Q_0 : |f(x)| > t\} \]

(3.9)

Also

\[ \{x \in Q_0 : |Wf(x)| > t\} = \{x \in Q_0 : \left| \sum_{i=1}^{q} f(A^{-1}(x + k_i)) \right| > tq\} \]
\[ \leq \{x \in Q_0 : \sum_{i=1}^{q} |f(A^{-1}(x + k_i))| > tq\} \]
\[ = \sum_{i=1}^{q} \int_{Q_0} \chi_{\mathbb{R}^+}(|f(A^{-1}(x + k_i))| - t) \, dx \]
\[ = \sum_{i=1}^{q} \int_{A^{-1}(Q_0 + k_i)} \chi_{\mathbb{R}^+}(|f(x)| - t) \, dx. \]

Thus

\[ \{x \in Q_0 : |Wf(x)| > t\} \leq q \{x \in Q_0 : |f(x)| > t\}. \]

(3.10)

From the relations (3.7) - (3.10), we conclude that \( \psi, \eta \in M_p^{(w)}(\mathbb{Z}^N) \) whenever \( \phi \in M_p^{(w)}(\mathbb{Z}^N) \). Also \( N_p^{(w)}(\psi) \leq CN_p^{(w)}(\phi) \) and \( N_p^{(w)}(\eta) \leq CN_p^{(w)}(\phi) \).

As an application of this Lemma we get the following result regarding weak summability kernels.

**Lemma 3.3.** Let \( A \) be as in Lemma 3.1. Suppose \( \Lambda \) is a weak (strong) summability kernel then \( \Lambda \circ B \) and \( \Lambda \circ B^{-1} \) are also weak (strong) summability kernels.
Proof: Define $W_{\phi,\Lambda \circ B}$ on $\mathbb{R}^N$ for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$.

$$W_{\phi,\Lambda \circ B}(x) = \sum_{n \in \mathbb{Z}^N} \phi(n) \Lambda \circ B(x - n)$$

$$= \sum_{n \in \mathbb{Z}^N} \eta(n) \Lambda(Bx - n)$$

$$= W_{\eta,\Lambda}(Bx).$$

As $\eta \in M_p^{(w)}(\mathbb{Z}^N)$ (by Lemma 3.2) and since $\Lambda$ is a summability kernel we have $W_{\eta,\Lambda} \in M_p^{(w)}(\mathbb{R}^N)$. Hence $W_{\phi,\Lambda \circ B} \in M_p^{(w)}(\mathbb{R}^N)$. Similarly

$$W_{\phi,\Lambda \circ B^{-1}}(x) = \sum_{n \in \mathbb{Z}^N} \phi(n) \Lambda(B^{-1}x - B^{-1}n)$$

$$= \sum_{j=1}^{q} \sum_{n \in B\mathbb{Z}^N + p_j} \phi(n) \Lambda(B^{-1}x - B^{-1}n)$$

where $p_1,...,p_q$ are distinct coset representatives of $B\mathbb{Z}^N/\mathbb{Z}^N$ ($p_1 = 0$).

$$W_{\phi,\Lambda \circ B^{-1}}(x) = \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}^N} \phi(Bn + p_j) \Lambda(B^{-1}x + B^{-1}p_j - n)$$

$$= W_{\psi,\Lambda}(B^{-1}x) + \ldots + W_{\psi_{q-1},\Lambda}(B^{-1}x - B^{-1}p_q)$$

where $\psi_{p_i}(l) = \phi(Bl + p_j)$, $i = 1, 2, ..., q$. As $\psi \in M_p^{(w)}(\mathbb{Z}^N)$ and $\Lambda$ is a summability kernel we conclude that $W_{\phi,\Lambda \circ B^{-1}} \in M_p^{(w)}(\mathbb{R}^N)$.

Hence from Lemma 3.3 and the discussion preceding Lemma 3.1 we conclude the following theorem.

**Theorem 3.1.** Suppose $\Lambda \in M_p(\mathbb{R}^N)$ and $\text{supp }\Lambda \subseteq [-M, M]$; for $\phi \in M_p^{(w)}(\mathbb{Z}^N)$ define $W_{\phi,\Lambda}(\xi) = \sum_{n \in \mathbb{Z}^N} \phi(n) \Lambda(\xi - n)$ on $\mathbb{R}^N$, then $W_{\phi,\Lambda} \in M_p^{(w)}(\mathbb{R}^N)$ and $N_p^{(w)}(W_{\phi,\Lambda}) \leq C_{\Lambda} N_p^{(w)}(\phi) \|\Lambda\|_{M_p(\mathbb{R}^N)}$, where $C_{\Lambda}$ is a constant depending on $\Lambda$. 

4. An Application

As an application of Theorem 3.1, we prove a weak-type version of a result proved by de Leeuw [8].

**Theorem 4.1.** For \(1 < p < \infty\), and \(\epsilon > 0\); let \(\{\phi_\epsilon\} \subseteq M_p^w(\mathbb{Z})\) satisfy

(i) \(\lim_{\epsilon \to 0} \phi_\epsilon([\frac{x}{\epsilon}]) = \phi(x)\) a.e.

(ii) \(\sup_{\epsilon} N_p^w(\phi_\epsilon) = K < \infty\).

Then \(\phi \in M_p^w(\mathbb{R})\) and \(N_p^w(\phi) \leq \sup_{\epsilon} N_p^w(\phi_\epsilon)\).

**Proof:** For each \(\epsilon > 0\), define \(W_{\phi_\epsilon}^{ \epsilon} on \mathbb{R}\) by

\[
W_{\phi_\epsilon}^{ \epsilon}(x) = \sum_{n \in \mathbb{Z}} \phi_\epsilon(n) \chi_{[0,1)}(x - n).
\]

(4.11)

As \(\chi_{[0,1)} \in M_p(\mathbb{R})\) for \(1 < p < \infty\), from Theorem 3.1 we have \(W_{\phi_\epsilon}^{ \epsilon} \in M_p^w(\mathbb{R})\) and \(N_p^w(W_{\phi_\epsilon}^{ \epsilon}) \leq CN_p^w(\phi_\epsilon) \leq CK\). We define another function \(\psi_\epsilon\), for each \(\epsilon > 0\), by \(\psi_\epsilon(x) = W_{\phi_\epsilon}^{ \epsilon}(\frac{x}{\epsilon})\). Then \(\psi_\epsilon \in M_p^w(\mathbb{R})\) and

\[
N_p^w(\psi_\epsilon) \leq N_p^w(W_{\phi_\epsilon}) \leq CK.
\]

(4.12)

From (4.11) we have

\[
\psi_\epsilon(x) = W_{\phi_\epsilon}^{ \epsilon}(\frac{x}{\epsilon}) = \sum_{n \in \mathbb{Z}} \phi_\epsilon(n) \chi_{[0,1)}(\frac{x}{\epsilon} - n) = \phi_\epsilon([\frac{x}{\epsilon}]).
\]

So from our hypothesis

\[
\lim_{\epsilon \to 0} \psi_\epsilon(x) = \phi(x) a.e.
\]

(4.13)

Also we have \(|\psi_\epsilon(x)| < \infty\) (as \(\sup_{\epsilon,n} |\phi_\epsilon(n)| < \infty\)).

Hence from (4.11), (4.12) and (4.13) along with Lemma 2.1 we have \(\phi \in M_p^w(\mathbb{R})\) and \(N_p^w(\phi) \leq \lim_{\epsilon} N_p^w(\phi_\epsilon) \leq CK\).
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