On a total function which overtakes all total recursive functions.*

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November 1, 2018

Abstract
We discuss here a function that is part of the folklore around the $P =? NP$ problem. This function $g^*$ is defined over all time-polynomial Turing machines; we trivially change it into a $g$ which is defined over all Turing machines, and show that $g$ overtakes all total recursive functions. We then show that $g$ is dominated by a generalized Busy Beaver function.

*Partially supported by FAPESP and CNPq. Alternative address for F. A. Doria: Research Center for Mathematical Theories of Communication and Program IDEA, School of Communications, Federal University at Rio de Janeiro, Av. Pasteur, 250. 22295–900 Rio RJ Brazil.
1 Introduction

Mathematics is sometimes full of unexpected turns, like a good mystery novel. The Busy Beaver game looks naïve enough, and yet it leads us to an incredibly fast-growing, uncomputable function \cite{12}. Also, who would expect that questions about finite functions would depend \cite{7} on the existence of large cardinals?

Sometimes mathematics behaves in a rather predictable, reasonable way. When one goes from real numbers to complex numbers to quaternions and then to the Cayley numbers, each step in that succession is marked (as it should be) by a slight modification of previously known properties. Yet in many cases the generalization of properties leads us into the unexpected: a contemporary example can be found in the mathematics of gauge fields. Given an Abelian gauge field, that field is always (locally, at least) derived from a gauge potential (a connection form) which is unique modulo gauge transformations. However when we deal with non-Abelian gauge fields, we see that the same field can be derived from physically different gauge potentials, that is, potentials that aren’t related by a gauge transformation, not even locally \cite{3, 4, 14}. Just by looking at the Abelian case we would never expect such a novel phenomenon to creep up in the non-Abelian case. Moreover this precise phenomenon is quite surprisingly related to a kind of transformation first used by Einstein in his theory of the asymmetric field \cite{3, 6}.

Can we say that we are dealing here with singularities in a kind of space of mathematical theories? We have in fact recently proposed this picture: we described a coding of formal systems by dynamical systems that bifurcate: for instance, if a given (undecidable) sentence holds, the system bifurcates; if not, it doesn’t. Thus undecidable sentences are coded by singularities in those systems \cite{2}.

(However we are not sure whether the formal picture that opposes singular points to regular points is an apt characterization of our intuitions about the unexpected in mathematics; the picture may be still more complex.)

The $P = \not= NP$ question

Let’s consider a final example, which will lead us into this paper’s main topic. That example stems from the $P = \not= NP$ question.

The $P = \not= NP$ question starts from a very simple, rather obvious question that we can formulate as a short tale:

Mrs. H. is a gentle and able lady who has long been the secretary of a large university department. Every semester Mrs. H. is confronted with the following problem: there are courses to be taught, professors to be distributed among different classes of students, large and small classes, and a shortage of classrooms. She fixes a minimum acceptable level of overlap among classes and students and sets down in a tentative way to get the best possible schedule given that minimum desired overlap. It’s a tiresome task, and in most cases, when there
are many new professors or when the dean changes the classroom allocation system, Mrs. H. quite frequently has to check nearly all conceivable schedulings before she is able to reach a conclusion. In despair she asks a professor whom she knows has a degree in math: 
“tell me, can’t you find in your math a fast way of scheduling our classes with a minimum level of overlap among them?”

Mrs. H. unknowingly asks about the $P = ?NP$ question. She is able to understand its basic contours: there are questions for which it is easy to check whether a given arrangement of data fits in as a solution; however for the general case there are no known shortcuts in order to reach a solution.

If you can guess the answer, it is easy to verify it. If you have to work it out, you’ll most likely have to sweat your shirt until you reach an adequate answer. Most likely: for one always wonders whether there is some general way to get a quick solution. Who knows?

Are there general shortcuts available? This is the whole point—in a nutshell; and this is the main query in the $P = ?NP$ problem.

$P = NP$ is, roughly, “there is a time-polynomial Turing machine that inputs each instance of a problem in a NP class and that correctly guesses (outputs) a solution for that instance.” The negation $P < NP$ is, “for each polynomial machine of Gödel number $m$ there is an instance $x$ such that machine $m$ guesses wrongly about $x$.”

The counterexample function $f(m)$ is the function that enumerates each first instance $x$ where a polynomial algorithm fails; it is recursive on the set of all polynomial machines and $P < NP$ holds if and only if $f$ is total.

So, one of the possible approaches to this major problem is to focus on the counterexample function $f$. How are we to understand its properties? Perhaps by analogy: we use a tamer function $g$ that more or less looks like $f$, and try to infer the properties of $f$ from those of $g$. The chief property we want to consider is whether $f$ can be proved to be total in Peano Arithmetic (PA), for in that case:

If the counterexample function $f$ is bounded by a prescribed total recursive function—for Peano Arithmetic, the fast-growing function $F_{\epsilon_0}$—then $P < NP$ will be provable in Peano Arithmetic.

For an adequately conceived $g$, it should be easy to check the desired property for $g$, and it should now be just a small step from $g$ to $f$.

(For $F_{\epsilon_0}$ see [15] and references therein; for the counterexample function see [4].)

Then our main query is: is $g$ bounded by such a function $F_{\epsilon_0}$? If not, where do the properties of $g$ will lead us concerning $f$?

**Functions $g_0$ and $g^*$**

**Remark 1.1** Let’s motivate $g^*$. Consider the following:

$$g_0(m) = \mu_x(x^m < 2^x)$$

(1)
for non-negative integers $m$. This function is trivially recursive and total, and bounded by the exponential $2^x$ itself.

Now suppose that we have defined some (necessarily nonrecursive) enumeration for all time-polynomial Turing machines (the poly machines). This enumeration is noted, $P_m, m \in \omega$; we note $\mathcal{P}$ the set of all polymachines. The following function:

$$g^*(m) = \mu_x[P_m(x) < 2^x]$$

looks very similar to (1). In fact we have only changed the monomial $x^m$ for the output $P_m(x)$ of a poly machine, and that output is necessarily bounded in its length by some polynomial, that is, there is a positive integer $k$ so that

$$|P_m(x)| < |x|^k.$$ 

Therefore $g^*$ is also total.

$g^*$ is usually presented as a kind of analogue of $g_0$ in eq. (1), and it is argued that, since $g_0$ is bounded by an exponential, the same should be true of $g^*$ in eq. (2). Yes, a moment’s thought shows that this should be the case...

Wait a moment—is it really so?

Let’s ponder it: the relation between exponent $k$ and machine index $m$ is possibly quite complicated—and index $m$ roams over a nonrecursive set.

The analogy starts to break down here.

The analogy breaks down right at the beginning. Can we pass over those differences and still have a rather “tame” behavior for $g^*$ as the one exhibited by $g_0$?

We show in this paper how wildly different is the behavior of $g^*$ when compared to $g_0$, despite the fact that the two functions are apparently very similar. The whole point, as we will see, is the relation between machine index $m$ and the exponent $k$ for the bounding polynomial.

**Analogies between $f$ and $g^*$**

The main analogy between the counterexample function $f$ and $g^*$ is: both are defined over the set of all poly machines $\mathcal{P}$ and their values are given by the application of the $\mu$ operator to a recursive predicate on $\mathcal{P}$ that depends on the index $m$ of a poly machine $P_m$ (for the case of $f$ see [5]). Of course the main difference between both functions is that $g^*$ is intuitively total, while the whole point of the $P=\uparrow NP$ question is whether $f$ is total or not.

**Summary of the paper**

$g^*$ is part of the folklore around the $P=\uparrow NP$ question. As noticed, it is sometimes used as a simile for the behavior of the counterexample function to the $P=NP$ (no question–mark here!) hypothesis. We consider its behavior in detail in the present paper in order to show how wildly different is the behavior of $g^*$ or its modification $g$ (see below) when compared to that of $g_0$. 
This paper shows that this function \( g^* \) isn’t well–behaved at all. In fact, given a natural version of it defined over all Turing machines, the modified \( g^* \) (noted \( g \)) will oscillate in a wild way, and in its “ups” it will tower over all total recursive functions, so that the only immediate bounds we can find for it are in the realm of those functions which bound all total recursive functions.

One of those upper bounds is a generalized Busy Beaver function, as we show in Section 4 of this paper.

**Preliminary comments**

**Remark 1.2** If \( M \) is the set of all Turing machines given by their Gödel numbers, the subset of all poly machines, \( P \subset M \), isn’t recursive.

If we consider a function \( g \) given by:
- \( g(m) = \mu_x[M_m(x) < 2^x] \), for \( m \) in \( P \),
- \( g(m) = 0 \), otherwise,
we intuitively see that \( g \) is total. \( \blacksquare \)

We consider \( g \) instead of \( g^* \) as it is clear that if there is some total recursive \( f \) that bounds \( g \), then we can obtain an adequately modified \( h^* \) that will bound \( g^* \).

**Goals**

Our goal in the present paper is to discuss some properties of \( g \). We will try to understand its relation to the set of all total recursive functions; everything proceeds within an intuitive framework, where we will be able to compare it to a generalization of the Busy Beaver function [12].

As noticed above, functions \( g^* \) and \( g \) are part of the folklore around the \( P=\text{NP} \) question; \( g^* \) was suggested to the authors by F. Cucker and, independently, by W. Mitchell.

**2 Conventions on Turing and \( \ell \)-machines**

We will use here what we call \( \ell \)-machines, for “labeled” or “parametric” Turing machines; they are defined in this section. They are defined out of the usual Turing machines, which we now describe. They are simply Turing machines of which we keep track with the help of a tag given by a parameter.

**Remark 2.1** Suppose given the canonical enumeration of binary words

\[ \emptyset, 0, 1, 00, 01, 10, 11, 000, 001, \ldots \]

which code the empty word and the integers; they correspond to

\[ 0, 1, 2, \ldots \]

We take this correspondence to be fixed for the rest of this paper. \( \blacksquare \)
Turing machines

Remark 2.2 We describe the behavior of the Turing machines we deal here to avoid ambiguities:

1. Turing machines are defined over the set $A^*_2$ of finite words on the binary alphabet $A_2 = \{0, 1\}$.

2. Each machine has $n \geq 0$ states $s_0, s_1, \ldots, s_{n-1}$, where $s_0$ is the final state. (The machine stops when it moves to $s_0$.)

We allow for a machine with 0 states and an empty table; it is discussed below in Remark 2.6.

3. The machine’s head roams over a two–sided infinite tape.

4. Machines input a single binary word and either never stop or stop when the tape has a finite, and possibly empty set of binary words on it.

5. The machine’s output word will be the one over which the head rests if and when $s_0$ is reached. (If the head lies on a blank square, then we take the output word to be the empty word, that is, 0.)

Remark 2.3 The Turing machine inputs a binary string $\lfloor x \rfloor$ and (if it stops over $\lfloor x \rfloor$) outputs a binary string $\lfloor y \rfloor$. The corresponding recursive function inputs the numeral $x$ and outputs the numeral $y$. Whenever it is clear from context, we write $x$ for both the binary sequence and the numeral.

Remark 2.4 We will use upper case and lower case sans serif letters (such as $M, \ldots$) for Turing machines (and also for $\ell$–machines). If $M_n$ is a Turing machine of Gödel number $n$, its input–output relation is noted $M_n(x) = y$.

Remark 2.5 Turing machines are given by tables. We can write the tables as code lines $\xi, \xi', \ldots$, separated by blanks $\sqcup$, such as $\xi \sqcup \xi' \sqcup \ldots \sqcup \xi''$.

Let $\Xi$ be one such set of code lines separated by blanks. Let $\Xi'$ be obtained out of $\Xi$ by a permutation of the lines $\xi, \xi', \ldots$. Both $\Xi$ and $\Xi'$ are seen as different machines that compute the same algorithmic function, that is, in this case, if $f_\Xi (f_{\Xi'})$ is computed by $\Xi$ ($\Xi'$), then $f_\Xi = f_{\Xi'}$.

Remark 2.6 We define the empty or trivial machine to be the Turing machine with an empty table; we take it to be the simplest example of the identity machine, again by definition. (See also Remark 2.10.)
**Busy Beaver**

**ℓ–machines**

**Remark 2.7** We describe here the ℓ–machines. Consider a two–tape Turing machine where the input is written over tape 1, while tape 2 comes with a possibly empty binary string \( n \). \( n \) is the machine’s label, or parameter.

- One easily sees that each Turing machine \( M_m \) is simulated by infinitely many ℓ–machines: write an arbitrary \( n \) on tape 2 and write all instructions for \( M_m \) solely for tape 1.

- For the converse, if \( M_m \) is an arbitrary Turing machine, and if (by an abuse of notation) \( \tau \) is the machine that polynomially computes the onto pairing function \( \omega \times \omega \rightarrow \omega \), then the set of all coupled machines \( M_m \circ \tau(n, x) \), all \( n, m \), represents the set of ℓ–machines.

- Moreover, to avoid a pre–fixed label or parameter, we can use the family of constant Turing machines \( i_n \) that print \( n \) over any input, and couple it to the arrangement above, that is \( M_m \circ \tau \circ i_n \) (in order to generate \( n \)), to obtain a set of emulated ℓ–machines.

The Gödel number \( k = c(m, \# \tau, n) \) (\( \# \ldots \) denotes the Gödel number of \ldots) is given by a primitive recursive \( c \).

From that we have:

**Definition 2.8** An ℓ–machine is a pair \( \langle n, M_m \rangle \), where \( n \in \omega \) and \( M_m \) is a 2–tape Turing machine of Gödel number \( m \), with \( n \) written on tape 2.

**Remark 2.9** From now on whenever we write “machine” we will mean “ℓ–machine,” unless otherwise stated.

**Remark 2.10** So, ℓ–machines are just a convenient bookkeeping device to keep track of families of Turing machines which are labeled by some parameter. However, under the guise of ℓ–machines they allow us the simple result stated in Lemma 2.13.

**Gödel numbering**

(We use here Kleene’s “method of digits,” without however “closing the gaps” to squeeze out the junk; see [10], p. 178.)

**Definition 2.11** Each ℓ–machine is coded by the pair \( \langle n, m \rangle \), where both \( n, m \) range over the whole of \( \omega \).

**Remark 2.12** We refer to \( \langle n, m \rangle \) as the Gödel number of the ℓ–machine, or ℓ–Gödel number, or ℓ–index. In order to compute some values for \( g \) as we do here one needs to know the ℓ–index for some poly machines which are given as ℓ–machines.
A lemma on Gödel numbers

In order to compute pieces of \( g \) one needs to know the Gödel numbers for some polynomial Turing machines. We use the following result:

Lemma 2.13 Let \( M_{(n,m)} \), \( n \in \omega \), \( m \) fixed, be a family of \( \ell \)-machines. We can explicitly construct an \( \ell \)-Gödel numbering for the set of all \( \ell \)-machines so that the \( \ell \)-indexes for the \( M_{(n,m)} \), \( n \in \omega \), are given by a linear function \( N(n) = an + b \).

Moreover there is a primitive recursive map from that \( \ell \)-Gödel numbering to standard Gödel numberings. Therefore, if \( k \) is such a map, the induced \( N'(n) = k \circ N(n) \) is primitive recursive.

Remark 2.14 Recall that the length \(|x|\) of a finite string \( x \) is the number of letters in \( x \).

Proof of Lemma 2.13: The idea goes as follows: 1547, 2547, 3547, \ldots are in arithmetic progression with \( r = 1000 = 10^3 = 10^{|\text{length}(547)|} \). We extend this to the proof of our lemma.

- Recall the ordering of binary words as described in Remark 2.1.
- \( \ell \)-machines are fully characterized by pairs \( \langle n, m \rangle \), where \( n \) is the parameter and \( m \) the 2-tape Turing machine’s code.
- If \( x_n \) and \( x_m \) are the corresponding binary words (numerals), code pair \( \langle n, m \rangle \) as the word \( x_n10\widehat{x_m} \), where if
  \[
  x_m = x_m^0x_m^1x_m^2\ldots x_m^k,
  \]
  then
  \[
  \widehat{x_m} = x_m^0x_m^0x_m^1x_m^1\ldots x_m^kx_m^k.
  \]
- So, given an arbitrary binary word, there is a simple recursive procedure to decide whether it codes an \( \ell \)-machine \( \langle n, m \rangle \) or not: go to the rightmost end and (by going backwards) see if we have a duplicated word that ends in the sequence 10. If so, we have a coded binary word for one of our pairs.
- Map the in-between mumbo-jumbo onto the trivial \( \ell \)-machine.
- This coding is certainly recursive and 1–1 onto the natural numbers.
- Finally, if \(|x_m|\) is the length of \( x_m \), then \(|\widehat{x_m}| = 2|x_m| + 2 \), and therefore, between \( \ell \)-machine \( \langle n, m \rangle \) and \( \ell \)-machine \( \langle n + 1, m \rangle \) there are \( 2^{2|x_m|+2} \) binary words.
Follows the lemma, and the Gödel number \(N(n)\) of the \(\ell\)-machines just described is given by an arithmetic progression \(N(n)\), of ratio \(2^{2^{|x_m|}+2}\).

\(\ddagger\)From Remark 2.7 one immediately sees that there are primitive recursive maps that lead from that coding to more usual Gödel numberings, such as the original one used by Gödel, or Kleene’s “method of digits,” so that \(N'(n)\) is also primitive recursive. \(\square\)

**Remark 2.15** So, it is enough to know that the Gödel numbers \(N\) for the families of machines we will be using here are related to the machine’s “tag” or parameter by \(N = q(n)\), where \(q\) is primitive recursive, that is, Ackermann’s function \(F_\omega\) dominates \(q\). \(\square\)

**Quasi-trivial machines**

We use \(\ell\)-machines; therefore here “machine” stands for \(\ell\)-machine.

**Remark 2.16** Again the trivial \(\ell\)-machine is the one with the empty table; more formally, \(\ell\)-machine \(\langle 0, 0 \rangle\). \(\square\)

We will now consider a set of, say, nearly trivial machines. They are all polynomial. Recall that the operation time of a Turing machine is given as follows: if \(M\) stops over an input \(x\), then the operation time over \(x\),

\[ t_M = |x| + \text{number of cycles of the machine until it stops}. \]

**Example 2.17**

- **First nearly trivial machine.** Note it \(O\). \(O\) inputs \(x\) and stops.

  \[ t_O = |x| + \text{moves to halting state + stops}. \]

  So, operation time of \(O\) has a linear bound.

- **Second nearly trivial machine.** Call it \(O'\). \(O'\) inputs \(x\), always outputs 0 (zero) and stops.

  Again operation time of \(O'\) has a linear bound.

- **Quasi-trivial machines.** A quasi-trivial machine \(Q\) operates as follows: for \(x \leq x_0\), \(x_0\) a constant value, \(Q = R\), \(R\) an arbitrary total machine. For \(x > x_0\), \(Q = O\) or \(O'\).

  This machine has also a linear bound. \(\square\)

We will use several \(\ell\)-families of quasi-trivial machines. Please allow for some abuse of language here; we need it in order to avoid cumbersome notations. (For instance, the machines defined below depend on the Gödel numbers of the machines that appear as their subroutines.)
Remark 2.18 Now let $H$ be any fast–growing, superexponential total machine. Let $H'$ be another such machine. Form the following family $Q^{H(n)}$ of quasi–trivial $\ell$–machines with subroutines $H$ and $H'$:

1. $H(n) = k(n)$, all $n$, is the way we introduce the parameter in the family.
2. If $x \leq k(n)$, $Q^{H,n}(x) = H'(x)$;
3. If $x > k(n)$, $Q^{H,n}(x) = 0$. □

For function $g$ in Remark 1.2:

Proposition 2.19 If $N(n)$ is the Gödel number of an $\ell$–machine as in Remark 2.18, then $g(N(n)) = k(n) + 1 = H(n) + 1$.

Proof: Use Lemma 2.13. Recall that $N(n) = an + b$, $a$ and $b$ constants that depend on the actual algorithms (i.e., in the Gödel numbers) for $H, H'$ and $Q$. □

3 A domination lemma

Recall:

Definition 3.1 For $f, g : \omega \rightarrow \omega$,

\[ f \text{ dominates } g \leftrightarrow_{\text{Def}} \exists y \forall x (x > y \rightarrow f(x) \geq g(x)). \]

We write $f \succ g$ for $f$ dominates $g$. □

Our goal here is to prove the following result:

Proposition 3.2 For no total recursive function $h$ does $h \succ g$.

Proof: Suppose that there is a total recursive function $h$ such that $h \succ g$.

Remark 3.3 Given such a function $h$, obtain another total recursive function $h'$ which satisfies:

1. $h'$ is strictly increasing.
2. For $n > n_0$, $h'(n) > h(\frac{an}{b} + \frac{1}{b})$, for integer values of the argument of $g$.

That is, $h'(an + b) > h(n)$. □

Constants $a, b$ are from the Gödel numbers of the quasi–trivial machines described in Remark 2.18.

Lemma 3.4 Given a total recursive $h$, there is a total recursive $h'$ that satisfies the conditions in Remark 3.3.
**Proof**: Given $h$, obtain out of that total recursive function a strictly increasing total recursive $h^*$. Then if, for instance, $F^\omega$ is Ackermann’s function, $h' = h^* \circ F^\omega$ will do. □

From Lemma 2.13, we have that the Gödel numbers $\#[Q^{h',K,n}]$ of the $Q^{h',K,n}$ are given by $\#[Q^{h',K,n}] = an + b$, $a, b \in \omega$.

Therefore, $g(an + b) = h'(n) + 1$. From Lemma 2.13, Remark 3.3 and Lemma 3.4 we conclude our argument. If we make explicit the computations, for $q(n) = an + b$ (as the argument holds for any strictly increasing primitive recursive $q$):

$$g(q(n)) = h'(n) + 1 = h^*(F^\omega(n)) + 1,$$

and

$$h^*(F^\omega(n)) > h^*(q(n)).$$

For $N = q(n)$ (see Remark 2.11),

$$g(N) > h^*(N) \geq h(N), \text{ all } N.$$

Therefore no such $h$ can dominate $g$. □

**Remark 3.5** No need to emphasize that we could have used ordinary Turing machines, as at the end of Remark 2.7; then $N(n)$ would be primitive recursive on $n$. □

### 4 An extended Busy Beaver function

So $g$ overtakes all total recursive functions. Can we put a ceiling to it? Yes: and that ceiling is a kind of generalized Busy Beaver function (which depends on $g$ itself).

We now sketch an argument that shows that this generalized Busy Beaver function dominates $g$.

The Busy Beaver function $B$ can be easily and intuitively defined. If $N$ is the number of states of a given Turing machine, then $B(N)$ is the biggest number of 1s a $N$–state Turing machine prints over a tape filled with 0s as its input.

**Remark 4.1** We are going to generalize that function to $B'$ as follows:

- First, notice that there is a simple relation between a Turing machine’s table and the number $\#[S]$ of its states.

More precisely, from that reference we see that a $N$–state Turing machine with a binary alphabet has a table given by a $N' = (3N + 1) \times 4$ matrix.

So we can define a primitive recursive relation $f$ from the Gödel numbers to the set of states of Turing machines so that if Gödel numbers $m' > m$, then states $f(m') \geq f(m)$.
• We can therefore see the Busy Beaver function $B$ as a monotonic nondecreasing function of Turing machine Gödel numbers.

• Now for each Gödel number $m$ for a polynomial Turing machine, $g(m)$ is effectively computed.

• We will define $B'(m)$ as: the maximum number of 1s printed by a $N(m)$–state Turing machine over a tape with only 0s and over a tape with all inputs $< g(m) + 1$.

• Clearly $B'(N) \geq B(N)$ and $B'(N(m)) = B(N(m))$ if the Turing machine with Gödel number $m$ isn’t a polynomial machine. □

**Proposition 4.2** For all $m$, $B'(m) \geq B(m)$. □

**Proposition 4.3** $B'$ dominates $g$. □

**Remark 4.4** We conjecture that $g$ is of the same order of growth as the Busy Beaver function itself. □

We intend to apply those ideas to the counterexample function $f$ in a follow–up paper.

5 Acknowledgments

The present paper is part of the Research Project on Complexity and the Foundations of Computation, at the Institute for Advanced Studies, University of São Paulo (IEA–USP). The authors thank the Institute for support, especially its Director Prof. A. Bosi, as well as F. Katumi and S. Sedini. FAD also wishes to thank Professors S. Amoedo de Barros and A. Cintra at the Federal University of Rio de Janeiro, as well its Rector José Vilhena.

Portions of the main questions dealt with in the present paper were discussed and considered in the newsgroup THEORY–EDGE@YAHOOGROUPS.COM from November to December 2000; please search its files for details. We heartily thank its participants for criticisms and suggestions.

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