A simple derivation of the Henon’s isochrone potentials

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By using well-known techniques of one-dimensional motion, we get all isochrone central potentials as exact solutions of simple quadratic equations. Moreover, we show that all potentials with Keplerian radial period \( T^2 \propto |E|^{-3} \) are necessarily isochrone. Bertrand’s theorem, which states that only Newtonian and harmonic central potentials have closed orbits for arbitrary initial conditions, is a direct consequence of our main result.

I. INTRODUCTION

The central-force problem is one of the most important and inspiring topics in Classical Mechanics, with its origin dating back to Newton’s Principia [1]. The problem of a test particle moving under the action of a central potential field \( V(r) \) is known to admit at least two conserved quantities, the total energy \( E \) and the angular momentum \( \ell = \vec{r} \times \vec{p} \). The conservation of \( \ell \) implies that all orbits under the action of a central-force field lie in the plane perpendicular to \( \ell \), so that the motion can be completely described in polar coordinates \((r, \theta)\) on this plane. The equations of motion are then obtained from these conserved quantities

\[
\frac{m}{2} \ddot{r}^2 + U(r) = E
\]

and

\[
m r^2 \dot{\theta} = \ell,
\]

where \( \ell = ||\vec{\ell}|| \) and \( U(r) \) stands for the effective central potential

\[
U(r) = V(r) + \frac{\ell^2}{2mr^2},
\]

with the typical centrifugal barrier \( 1/r^2 \) term. From \([1]\) and \([2]\), one can obtain the so-called apsidal angle \( \Theta \), which is the angular variation of a bounded trajectory between the points of smaller (periapsis) and greater (apoapsis) approximation of the center

\[
\Theta(E, \ell) = \int_{r_{\min}}^{r_{\max}} \frac{\ell}{r^2} \frac{1}{\sqrt{2m[E - U(r)]}} \, dr.
\]

Note that, if an orbit is closed, then \( \Theta = \pi p/q \), where \( p \) and \( q \) are integers forming an irreducible fraction. Similarly, from \([1]\) one can obtain the time interval between the start and first return to a given apsis, which expresses the radial period \( T \) of an orbit

\[
T(E, \ell) = 2\sqrt{m} \int_{r_{\min}}^{r_{\max}} \frac{1}{\sqrt{E - U(r)}} \, dr.
\]

A celebrated result in the central-force problem is the Bertrand’s theorem \([2, 3]\), which roots can also be traced back to the Newton’s Principia \([1]\) (proposition 45, Book 1). Bertrand’s theorem states that there are only two types of central potentials \( V(r) \) for which all (non-colliding) bounded orbits are also closed, independent of \( E \) and \( \ell \) and, hence, all orbits will correspond to the same kind of closed periodic curve in the plane of motion. These potentials are the well known Newtonian and isotropic harmonic potentials

\[
V_{Ne}(r) = -\frac{k}{r}, \quad V_{ha}(r) = \frac{k}{2} r^2.
\]

A somewhat more general problem encompassing Bertrand’s theorem was put forward by Michel Hénon in the fifties \([4, 5]\) (see \([6]\) for a brief review of the subject), who sought generic central potentials for which the radial periods of bounded orbits would not depend on angular momentum \( \ell \). He dubbed this kind of potential as \textit{isochrone}, and showed that it could provide good descriptions of globular clusters \([6]\). Hénon’s problem encompasses the Bertrand’s theorem in the sense that the two central potentials leading to closed orbits are trivially isochrone, but there are isochrone potentials for which generic bounded orbits are not closed. A dynamical mechanism called resonant relaxation was proposed to explain how the mass distribution of a globular cluster could evolve towards an isochrone model \([7]\). Hénon’s work has attracted considerable attention in recent years and his original approach has been increasingly elucidated and deepened, see \([7, 12]\). Although the modern understanding is that realistic globular clusters do require a more sophisticated description, the isochrone models are still inspiring and are being actively used in astrophysical applications \([13, 14]\). In summary, the isochrone potentials are, besides the Bertrand’s cases \([6]\), the so-called Hénon potential

\[
V_{He}(r) = -\frac{k}{b + \sqrt{b^2 + r^2}},
\]
and, respectively, the bounded and hollowed potentials

\[ V_{bo}(r) = \frac{k}{b + \sqrt{b^2 - r^2}}, \quad V_{ho}(r) = -\frac{k}{r^2}\sqrt{r^2 - b^2}. \]  

It is clear that the bounded and hollowed potentials are not defined for all \( r \), and that the Newtonian potential arises in the \( b = 0 \) limits of the Hénon \([7]\) and the hollowed \([8]\) potentials. Despite being a rather basic result, the derivation of the isochrone potentials are quite involved, and the relevant details can be found, for instance, in \([7]–[12]\).

In the present work, we obtain all the isochrone potentials by using a well-known technique from the inverse problem for one-dimensional motion \([10]\). Our alternative derivation is quite enlightening since it is at the same time rigorous and elementary, involving only simple algebraic expressions and a classic result, with physical relevance \([17]\), on the inversion of integrals. Our approach also allows to show that all potentials with Keplerian evanescence \([17]\), on the inversion of integrals. Our approach

\[ \beta \]

then

\[ g(v) = \frac{1}{\pi} \frac{d}{dv} \int_{u_0}^{v} \frac{f(u)}{\sqrt{v - u}} du. \]  

For further details and references on the Abel integral equation, as well as some of its applications, see \([17]\).

From \([12] \) and \([13]\) we have

\[ \frac{1}{r_1} - \frac{1}{r_2} = \frac{2m}{\pi \ell} \int_{U_0}^{U} \frac{\Theta(E, \ell)}{\sqrt{U - E}} dE, \]  

\[ r_2 - r_1 = \frac{1}{\pi \sqrt{2m}} \int_{U_0}^{U} \frac{T(E, \ell)}{\sqrt{U - E}} dE. \]  

Notice that equation \([14]\) was originally used in \([18]\) for deriving an alternative proof of Bertrand’s theorem, whereas equation \([15]\) is presented in Landau and Lifshitz’s classic textbook \([10]\).

III. ISOCHRONE SOLUTIONS

The isochronism condition, \( i.e. \), the requirement that the radial period \( T \) does not depend on the angular momentum \( \ell \), is fully equivalent to the condition that the apsidal \( \Theta \) angle does not depend on the energy \( E \). This can be seen form the identity

\[ \frac{\partial T}{\partial \ell} = -2 \frac{\partial v}{\partial E}, \]  

which can be proved directly evoking the so-called radial action

\[ A_r(E, \ell) = \sqrt{2m} \int_{r_{min}}^{r_{max}} \sqrt{E - U(r, \ell)} dr. \]  

By differentiating under the integral sign and using that the integrand vanishes at both \( r_{min} \) and \( r_{max} \), we have

\[ T = \frac{\partial A_r}{\partial E}, \quad \Theta = -\frac{\partial A_r}{\partial \ell}, \]  

from where \([16]\) follows straightforwardly. Hence, assuming the isochronism condition

\[ \Theta(\ell) = \pi \lambda(\ell), \]  

equation \([14]\) can be integrated directly as

\[ \frac{1}{r_1} - \frac{1}{r_2} = \beta \sqrt{U - U_0}, \]  

where \( \beta \ell = 2\sqrt{2m \lambda(\ell) / \ell} \). For potentials admitting closed orbits for all values of \( \ell \), \( i.e. \), for the case of Bertrand’s theorem, \( \lambda \) is a rational number independent of \( \ell \).

The case of the integral \([15]\) is different, we cannot integrate it explicitly with the isochronism condition since the function \( T = T(E) \) is unknown. However, we can
write the right-handed side of (15) in a convenient functional form, without loss of generality, as

\[ r_2 - r_1 = \frac{\sqrt{U - U_0}}{h(U, U_0)}, \quad (21) \]

where \( h(U, U_0) \) is an undetermined arbitrary function. For the sake of notation, we will denote this function simply as \( h(U) \). It is important to stress that (21) is merely a definition for the function \( h(U) \), there is absolutely no loss of generality in this choice, whose main motivation comes from the fact that \( U \) must have a local minimum at \( r = r_0 \) in order to guarantee the existence of bounded orbits. A simple Taylor series expansion of \( U \) gives us \( U - U_0 = \left[U''(r_0)/2\right](r_2 - r_1)^2 \) as \( r_{1,2} \to r_0 \), so that (21) can be locally verified, with \( h(U_0) = \sqrt{U''(r_0)/8} \). Thus, (21) captures the essence of behavior of \( U(r) \), which must have a local minimum, avoiding non-rigorous perturbative calculations. Moreover, the convenience of this choice for the right-handed side of (15) will become clear by solving the equations (20) and (21) for the two branches \( r_{1,2}(U) \), leading to

\[ \sqrt{U - U_0} = r_2 h(U) - \frac{1}{\beta r_2} - \left[r_1 h(U) - \frac{1}{\beta r_1}\right], \quad (22) \]

from where we have

\[ U - U_0 = \left[r h(U) - \frac{1}{\beta r}\right]^2. \quad (23) \]

As we can see, the choice (21) allows us to obtain symmetrical expressions in (22) for both branches \( r_{1,2}(U) \) and, consequently, a unique expression (23) valid for all \( r \).

The isochronism condition is now equivalent to the existence of solutions of (23) for the effective central potential \( U \) and, rather surprisingly, this is sufficient to constrain the unknown function \( h(U) \)! The \( r^2 h^2 \) term from the right-handed side of (23) implies that \( h \) must be a linear function in \( U \) since any other dependence on \( U \) would give origin to \( \ell \)-depending terms different than the centrifugal barrier, which is the only \( \ell \)-depending term allowed in \( U \), see (19). The general linear form \( h = \alpha U + \gamma \) reduces to two cases: \( \alpha = 0 \) or \( \gamma = 0 \). If the two coefficients are considered non-zero, we can rewrite \( h = \alpha (U + \gamma/\alpha) \), and then \( \gamma/\alpha \) can be disregarded, without loss of generality, otherwise it would mean adding a constant to the potential \( V \). In fact, if \( V(r) \) is a isochrone potential, then \( V(r) + \varepsilon + \Lambda/\ell^2 \) will be also isochrone since the extra terms only imply some shifts in the conserved quantities \( E \) and \( \ell \). These extra terms are called \((\varepsilon, \Lambda)\)-gauges in this context since they do not alter any dynamical property of the system.

The simplest case here is the constant \( h(U) = \sqrt{k/2} \), which gives us the isotropic harmonic potential \( V_{\text{iso}}(r) \) of (8), together with the closed orbit condition \( \Lambda = 1/2 \), and therefore a Bertrand’s solution. Notice that, in this case, from (23), we also have \( U_0 = \ell \sqrt{k/m} \), as expected for this harmonic potential.

In the second case, \( h(U) = \alpha U \), which is a bit more involved. The substitution of \( U \) given by (3) in (23) results in the quadratic equation

\[ r^2 V^2 - \left(1 - \frac{\varepsilon^2}{m} + \frac{\ell}{\alpha \sqrt{2m}}\right) V + \left(\frac{U_0}{\alpha^2} + \frac{c}{\ell^2}\right) = 0, \quad (24) \]

where we conveniently set

\[ -\frac{\varepsilon^2}{m} + \frac{\ell}{\alpha \sqrt{2m}} = 2bk, \quad -\frac{U_0}{\alpha^2} = \pm k^2, \quad (25) \]

The first group of solutions for isochrone potentials comes from \( c = 0 \), after introducing the parameters

\[ \alpha = \pm \sqrt{\frac{m}{\ell}} \frac{1}{\sqrt{1 + 2bkm/\ell^2 + \sqrt{1 + 4bkm/\ell^2}}} \quad \frac{1}{\sqrt{1 + 4bkm/\ell^2}} \quad (28) \]

\[ \lambda = \frac{1}{2} \left(1 \pm \frac{1}{\sqrt{1 + 4bkm/\ell^2}}\right). \quad (29) \]

The second type of solution arises after setting the parameters

\[ \frac{1}{\alpha^2} - \frac{\varepsilon^2}{m} + \frac{\ell}{\alpha \sqrt{2m}} = 0, \quad (30) \]

\[ c = b^2 k^2, \quad -\frac{U_0}{\alpha^2} = k^2, \quad (31) \]

where \( k > 0 \) and \( b \geq 0 \), resulting in the attractive potential

\[ V(r) = -k \frac{r^2 - b^2}{r^2}, \quad (32) \]

which is the remaining isochrone potential \( V_{\text{iso}}(r) \), see (8). The parameters \( \alpha \) and \( \lambda \) for this case are

\[ \alpha = \frac{1}{\sqrt{2}} \sqrt{\frac{m}{\ell}} \frac{1}{\sqrt{1 + \sqrt{1 + (2bkm/\ell^2)^2}}} \quad (33) \]

\[ \lambda = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1 + \sqrt{1 + (2bkm/\ell^2)^2}}}. \quad (34) \]
Evidently, Newtonian potential is a particular case of the Hénon potential in (27) and the hollowed potential (32) for \( b = 0 \), with the closed orbit condition \( \lambda = 1 \), as also expected from the Bertrand’s theorem.

It is important to stress that the \( U_0 \) constants for all cases discussed above do indeed correspond to the respective minima of the effective potential \( U(r) \), corroborating the consistence of the approach based in the equation (26).

### IV. ISOCHRONE POTENTIALS ARE KEPLERIAN

Our approach allows the exact determination of the radial period for all isochrone potentials. Notice that equations (13) and (21) imply that

\[
\frac{\sqrt{U - U_0}}{h(U)} = \frac{1}{\pi \sqrt{2m}} \int_{U_0}^{U} \frac{T(E)}{\sqrt{U - E}} dE.
\]

We can determine \( T(E) \) by means of the function \( h(U) \) by inverting equation (35) again using Abel’s integrals (12) and (13),

\[
T(E) = \sqrt{2m} \frac{dE}{dE} \int_{U_0}^{E} \frac{\sqrt{U - U_0}}{\sqrt{E - U}} \frac{1}{h(U)} dU.
\]

For the isotropic harmonic potential \( V_{\text{iso}}(r) \) in (10), we saw that \( h(U) = \sqrt{k/2} \), and therefore

\[
T(E) = \pi \sqrt{\frac{m}{k}}.
\]

The other group of isochrone potentials are more interesting. By using only \( h(U) = aU \) and the common condition for the minimum \( |U_0| = \alpha^2 k^2 \), Eq. (69) yields

\[
T(E) = \pi k \sqrt{\frac{m}{|2E|}}.
\]

which is exactly a Keplerian equation, independent of \( b \). Moreover, the radial period (68) implies \( h = aU \) with \( |U_0| = \alpha^2 k^2 \), i.e., all potentials with Keplerian radial period are necessarily isochrone potentials.

We can also obtain a Kepler’s third law \( T^2 \propto a^3 \) for these models. In order to do it, we have to find orbital characteristic lengths \( a \) that are inversely proportional to the energy \( E \). The starting point is the equation that gives the solution for both periapsis and apoapsis, i.e., \( V(r) + \ell^2/2mr^2 = E \). For the isochrone potentials (27) we define \( \xi \pm = \sqrt{b^2 \pm r^2} \), which then satisfies

\[
|E|\xi_\pm^2 - k\xi \pm + \left( kb + \frac{\ell^2}{2m} - |E|b^2 \right) = 0,
\]

and therefore

\[
a_\pm = \frac{\sqrt{b^2 + r_{\text{max}}^2} + \sqrt{b^2 + r_{\text{min}}^2}}{2} = \frac{k}{2|E|}.
\]

Similarly, for the isochrone potential (32) we now define \( \xi = \sqrt{r^2 - b^2} \), which satisfies

\[
|E|\xi^2 - k\xi + \left( \frac{\ell^2}{2m} - |E|b^2 \right) = 0,
\]

and therefore

\[
a = \frac{\sqrt{r_{\text{max}}^2 - b^2} + \sqrt{r_{\text{min}}^2 - b^2}}{2} = \frac{k}{2|E|}.
\]

For cases where \( b = 0 \) are allowed, \( a \) corresponds to the semi-major axis of Kepler’s problem.

The common Keplerian period (68) for the isochrone potentials (27) and (32) is not a mere coincidence. We can solve the equation of motion (11)

\[
t = \sqrt{\frac{m}{2}} \int \frac{1}{\sqrt{E - V(r) - \ell^2/2mr^2}} dr,
\]

so that it has a Keplerian form common to all these potentials

\[
t = \sqrt{\frac{m}{2}} \int \frac{1}{\sqrt{E_\pm + k/\xi - \xi_\pm^2/2m\xi^2}} d\xi,
\]

after performing the change of variables \( \xi = \sqrt{b^2 \pm r^2} \) for the isochrone potentials (27), with orbital parameters

\[
E_\pm = \pm E, \quad \frac{\ell^2}{2m} = \frac{\ell^2}{2m} + kb + E_\pm b^2,
\]

and the change of variable \( \xi = \sqrt{r^2 - b^2} \) for the isochrone potential (32), with orbital parameters

\[
E_\pm = E, \quad \frac{\ell^2}{2m} = \frac{\ell^2}{2m} - E_\pm b^2.
\]

The result of integral (44) can be directly borrowed from Kepler’s problem after setting the parameters below

\[
p = \frac{\ell^2}{mk}, \quad e = \sqrt{1 + \frac{2E_\pm \ell^2}{mk^2}},
\]

leading to the following parametric solutions (see (16))

\[
\xi = a(1 - e \cos \psi), \quad t = \sqrt{\frac{ma^3}{k}} (\psi - e \sin \psi),
\]

\[
\xi = \frac{p}{2}(1 + \psi^2), \quad t = \frac{1}{2} \sqrt{\frac{ma^3}{k}} \left( \psi + \frac{\psi^3}{3} \right),
\]

\[
\xi = a(e \cosh \psi - 1), \quad t = \sqrt{\frac{ma^3}{k}} (e \sinh \psi - \psi),
\]

for \( E_\pm < 0, E_\pm = 0, \) and \( E_\pm > 0 \), respectively, where we use the initial condition that \( \xi \) takes its smallest value at \( t = 0 \) for all cases. The corresponding values of \( r \) can be simply redeemed by reversing the \( \xi \)-transformations of each potential.
V. FINAL REMARKS

Despite many direct similarities with Kepler’s problem, the trajectory equations \( r(\theta) \) for the isochronous potentials \((7)\) and \((8)\) evidently do not belong to the conic sections group for \( b \neq 0 \). Anyway, the reader can also calculate the time evolution \( \theta(t) \) based on the equations \((48-50)\) solving the integral below, which comes from \((2)\).

\[
\theta = \frac{\ell}{m} \int \frac{1}{\sqrt{r^2(\xi(\psi))}} \, d\psi.
\]  

(51)

We finish by noticing that our results imply another simple proof for Bertrand’s theorem. From \((16)\), we have that any potential satisfying Bertrand’s theorem must be isochrone. By inspecting \( \lambda \) determined in the Section \([III]\) we conclude immediately that only the Newtonian and the harmonic potentials have apsidal angle \( \Theta \) independent of \( E \) and \( \ell \). For other proofs for Bertrand’s theorem, see \([18-23]\).

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