By numerically integrating in time the motion of vortex filaments, we study how the nonlinear interaction of Kelvin waves along vortices generates Kelvin waves of larger and larger wavenumbers (smaller and smaller wavelength). At sufficiently large wavenumbers the angular velocity of the vortices is large enough that kinetic energy is lost by sound emission. This turbulent cascade of Kelvin waves should explain why turbulence, generated in superfluid helium at very low temperature near absolute zero, quickly decays, despite the lack of any viscous dissipation.

**TURBULENT CASCADE OF KELVIN WAVES ON VORTEX FILAMENTS**

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**Summary**

By numerically integrating in time the motion of vortex filaments, we study how the nonlinear interaction of Kelvin waves along vortices generates Kelvin waves of larger and larger wavenumbers (smaller and smaller wavelength). At sufficiently large wavenumbers the angular velocity of the vortices is large enough that kinetic energy is lost by sound emission. This turbulent cascade of Kelvin waves should explain why turbulence, generated in superfluid helium at very low temperature near absolute zero, quickly decays, despite the lack of any viscous dissipation.

**KELVIN WAVES**

Turbulence can be created in superfluid helium by agitating it using propellers [1], grids [2], forks [3] or wires [4], by applying a heat flow [5] or by injecting a stream of ions [6]. Superfluid turbulence is particularly simple if the temperature is reduced to less than 1 K because thermal excitations become negligible and liquid helium can be considered a pure superfluid. In this regime it is observed that, if the forcing is not maintained, the turbulence quickly decays [7, 8], despite the lack of any viscous dissipation. Our concern is this surprising effect and the nature of the sink of turbulent kinetic energy near absolute zero.

We know that superfluid turbulence consists of a disordered tangle of vortices[9]. Because of quantum mechanical constraints on the rotational motion, all vortices have the same (fixed) circulation \(\Gamma \approx 9.97 \times 10^{-3}\)cm\(^2\)/s, and the same (fixed) vortex core radius \(a_0 \approx 10^{-8}\)cm. In a typical experiment, the vortex core is thus many orders of magnitude smaller than the average distance between vortices, typically \(\ell \approx 10^{-3}\) to \(10^{-4}\)cm. It is therefore appropriate to mathematically model quantum vortices as closed space curves of infinitesimal thickness moving in an inviscid Euler fluid according to the Biot–Savart law[10]

\[
\frac{ds}{dt} = -\frac{\Gamma}{4\pi} \int_{\mathcal{L}} \frac{(s-r)}{|s-r|^3} \times dr, \tag{1}
\]

where \(s(t)\) is the position vector along a vortex filament and \(t\) is time. The line integral extends over the entire vortex configuration \(\mathcal{L}\) and is de-singularised in a standard way [11]. Equation (1) allows the computation of the time evolution of vortex filaments suitably discretised. The reconnection of vortex filaments [12] which become close to each other is performed algorithmically[11].

Studies of superfluid vortex motion based on the Gross-Pitaevskii nonlinear Schrödinger equation have revealed that, at sufficiently short lengthscales, the kinetic energy of the vortices can be transformed into sound[13]. This effect suggests an explanation of the observed turbulence decay. The problem is that the typical lengthscale and curvature in a vortex tangle is only of the order of \(\ell\), which is far too large to account for the rate of energy decay. It is thought that the Kelvin waves cascade[14, 15] is the mechanism which generates the very short lengthscales at which sound can be efficiently radiated away.

A Kelvin wave [16] is a helical or sinusoidal displacement of a vortex filament away from it straight position at rest. In the limit of long wavelength \(\lambda\), the angular frequency of a small-amplitude Kelvin wave of wavenumber \(k = 2\pi/\lambda\) is

\[
\omega \approx \frac{\Gamma k^2}{4\pi} \ln \left( k/a_0 \right). \tag{2}
\]

The Kelvin waves cascade is the nonlinear interaction of Kelvin waves which generates waves of smaller and smaller wavenumber \(k\), until the lengthscale is so short that kinetic energy can be efficiently dissipated into the form of sound.

The Kelvin waves cascade is thus analogous to the classical Richardson-Kolmogorov cascade which shifts kinetic energy down the inertial range to a small viscous lengthscale. The main differences are that the Kelvin waves cascade is one-dimensional along individual vortex lines (rather than three-dimensional), and that the energy sink is acoustic (rather than viscous).

**KELVIN WAVES CASCADE**

The aim of our work is to investigate the Kelvin waves cascade numerically using a model based on Equation (1). We perform a number of numerical simulations of the evolution of a small number of vortex filaments, e.g. two or three
The quantity of the greatest interest is the amplitude density function of larger and larger curvature $\kappa(\xi) = |d^2s/d\xi^2|$ where $\xi$ is the arc length, which we characterise in terms of a probability density function of $\kappa$ and its exponent.

The quantity of the greatest interest is the amplitude $a(\xi)$ of the Kelvin waves and its spectrum. We introduce the concept of smoothed vortex filament: at every time $t$, we use $n$ discretization points $s_j$ along the vortex filaments as nodes of a cubic–spline interpolation to obtain a new (smoothed) filament $s_{\text{smooth}}$ from the original filament $s$. We then define the Kelvin wave amplitude as the distance between the original filament and the smoothed filament: $a(\xi) = |s - s_{\text{smooth}}|$. The amplitude spectrum $A(k)$ is defined as

$$\frac{1}{2} \int a^2(\xi) d\xi = \int_0^\infty A(k')dk', \quad (3)$$

We find that the amplitude spectrum behaves as $A(k) \sim k^\beta$ with $\beta \approx -3.1$ for large $k$, independently on the number $n$ of neighbours used to smooth the original wiggly filaments. Our result is consistent with $\beta = -3$ obtained by Vinen, Tsubota and Mitani for continuous forcing[17], and with theoretical predictions by Kozik and Svistunov ($\beta = -11/3 = -3.7$) [18] and by L’vov and Nazarenko ($\beta = -7/5 = -3.4$) [19].

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