On calculation of the interweight distribution of an equitable partition

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Abstract We derive recursive and direct formulas for the interweight distribution of an equitable partition of a hypercube. The formulas involve a three-variable generalization of the Krawtchouk polynomials.

Keywords Equitable partition · Strong distance invariance · Interweight distribution · Distance distribution · Krawtchouk polynomial

1 Introduction

We study the equitable partitions, see e.g. [10] §5.1 (also known as regular partitions, see e.g. [11] §11.1.B), partition designs, see e.g. [3], or perfect colorings, see e.g. [7]; less popular equivalent terms include coherent partitions [14], feasible colorations [4] §4.1, distributive colorings [19]), of the $n$-cubes. The goal of the paper is to derive recursive and direct formulas for the interweight distributions of an equitable partition of a hypercube. Some results are formulated in terms of the triangle distribution of the partition, which relates to the interweight distributions in the similar manner as the distance distribution relates to the weight distribution (the later two concepts are well known in coding theory, see e.g. [14]). The formulas can be used to prove the nonexistence of equitable partitions with certain parameters, which is demonstrated by examples (Proposition 1, Example 3). The results are applicable to the completely regular sets (including perfect codes, nearly perfect codes, and some kind of uniformly packed codes), as they can be represented in terms of equitable partitions (see e.g. [3]). As well, the interweight distributions have a potential in studying related objects, such as difference sets, which compose check matrices of completely regular codes, and linear two-weight codes, which are dual to completely regular codes (see e.g. [2] Corollary 4.3)).

In Section 2 we give main definitions. Section 3 contains recursive formulas for the calculation of the interweight distribution of an equitable partition of an $n$-cube. One of the formulas is proved in Section 4.
while the other are just simple corollaries of the first one. In Section 5, a direct formula for the interweight distribution is derived, in terms of polynomials in the quotient matrix of the equitable partition and their generating function. In Section 6 we observe an empirical relation with the so-called correlation-immunity bound on the quotient matrices of equitable 2-partitions of an n-cube. In Section 7 we briefly discuss a real-valued generalization of the equitable partitions. In the concluding section, we give final remarks and formulate some open questions.

2 Preliminaries

An equitable partition of a graph \( G = (V(G), E(G)) \) is an ordered partition \( C = (C_1, \ldots, C_m) \) of \( V(G) \) such that for every \( i \) and \( j \) from 1 to \( m \) and every vertex \( v \) from \( C_i \) the number \( S_{ij} \) of its neighbors from \( C_j \) depends only on \( i \) and \( j \) and does not depend on the choice of \( v \). The matrix \( S = (S_{ij})_{i,j=1}^m \) is called the quotient matrix of \( C \).

Let \( C = (C_1, \ldots, C_m) \) be a collection of vertex sets of a graph \( G \) of diameter \( d \). The weight distribution of \( C \) with respect to a vertex \( v \) of \( G \) is the collection of numbers \( (W_{r,v,j}^w)^n \) where \( W_{r,v,j}^w \) is the number of vertices of \( C_j \) at distance \( r \) from \( v \) (by the distance, we mean the natural graph distance, i.e., the length of a shortest path between two vertices).

One of well-known properties of the equitable partitions of distance regular graphs is the distance invariance. A collection \( C = (C_1, \ldots, C_m) \) of mutually disjoint vertex sets is called distance invariant if for every \( i \in \{1, \ldots, m\} \) the weight distribution of \( C \) with respect to a vertex \( v \) from \( C_i \) does not depend on the choice of \( v \) and depends only on \( i \).

**Remark 1** The concept of the distance invariance, defined as above, is applicable to partitions, as well as to single sets, \( C = (C_1) \). In the last case, \( C_1 \) is known in coding theory as a distance invariant set, or a distance invariant code [5].

The weight distribution of an equitable partition \( C \) of a distance regular graph can be calculated using recursive relations or the direct formula (see [15][12]) \( W_v^w = \Pi^{(w)}(S)V_0^w \) where \( W_v^w = (W_{v,1}^w, \ldots, W_{v,m}^w) \), \( S \) is the quotient matrix of \( C \), and \( \Pi^{(w)} \) is a polynomial related to the graph: if \( A \) is the adjacency matrix of the graph then \( A_{(w)}^w = \Pi^{(w)}(A) \) is the distance-\( w \) matrix (\( A_{(w)}^w = 1 \) if the distance between the vertices \( u, v \) equals \( w \), and \( A_{(w)}^w = 0 \) otherwise). In the current paper, we consider only n-cubes, which will be defined below; for the background on the distance regular graphs in general see, e.g., [1].

One of the known strengthenings of the distance invariance property is the strong distance invariance [17][18]. For a fixed vertex \( v \) from \( C_i \), let \( W_{r_1,r_2,r_3}^{i,j,k} \) denote the number of the pairs \((x, y)\) such that \( d(x, y) = r_2 + r_3, d(v, y) = r_1 + r_3, d(v, x) = r_1 + r_2, x \in C_j \), and \( y \in C_k \). A collection \( C = (C_1, \ldots, C_m) \) of mutually disjoint vertex sets is called strongly distance invariant if its interweight distribution \( W_i = (W_{r_1,r_2,r_3}^{i,j,k})_{r_1,r_2,r_3=0}^m \) with respect to a vertex \( v \) from \( C_i \) does not depend on the choice of \( v \) (originally, elements of the interweight distribution were indexed by the distances \( a = d(x, y), b = d(v, y), c = d(v, x) \); by the reasons that can be seen from formulas below, we reenumerate them using the indices \( r_1 = (-a + b + c)/2, r_2 = (a - b + c)/2, r_3 = (a + b - c)/2 \).

The n-cube \( H^n \) is the graph \((V(H^n), E(H^n))\) whose vertex set is the set of all \( n \)-words in the alphabet \( \{0, 1\} \), two words being adjacent if and only if they differ in exactly one position.

**Theorem 1** ([18]) The equitable partitions of the n-cubes are strongly distance invariant.

The statement does not hold for distance regular graphs in general, see examples in [12].

We will show how to calculate the interweight distribution of an equitable partition of an n-cube. To formulate some of the results, we need to introduce a new notion and the corresponding concept, whose usability is briefly discussed in the beginning of the next section. Let \( T_{r_1,r_2,r_3}^{i,j,k} \) denote the number of the
triples \((v, x, y)\) such that \(d(x, y) = r_2 + r_3, d(v, y) = r_1 + r_3, d(v, x) = r_1 + r_2, v \in C_i, x \in C_j, y \in C_k.\) We will refer to the collection of \(T^{r_1, r_2, r_3}_{ijk}\) for all \(i, j, k, r_1, r_2, r_3\) as the \textit{triangle distribution} of \(C\) (which can be, in this definition, an arbitrary family of subsets of \(V(H^n)\)). Note that \(W^{r_1, r_2, r_3}_{ijk} = T^{r_1, r_2, r_3}_{ijk} / |C_i|\) holds, due to the strong distance invariance of the equitable partitions. Therefore the interweight distribution can be easily calculated from the triangle distribution. The array \(T^{r_1, r_2, r_3}\) (and, similarly, \(W^{r_1, r_2, r_3}\)) will be treated as a row-vector of length \(m^3\) whose elements \(T^{r_1, r_2, r_3}_{ijk}\) are indexed by \((i, j, k)\) and \((v, x, y)\) respectively are indexed by \(i, j, k \in \{1, \ldots, m\}\). Next, we define three \(m^3 \times m^3\) matrices \(S', S'', S'''\) such that multiplication of a row-vector to the matrix \(S' (S'', S''')\) respectively is the same as multiplication of the corresponding three-indexed array to the matrix \(S\) in the first (second, third, respectively) index. That is, for \(U = (U_{ijk})\) and \(V = (V_{ijk})\): 

\[
V = US' \iff V_{ijk} = \sum_{t=1}^{m} U_{tjk}S_{ti} \quad \text{for all } i, j, k \in \{1, \ldots, m\},
\]

\[
V = US'' \iff V_{ijk} = \sum_{t=1}^{m} U_{tik}S_{tj} \quad \text{for all } i, j, k \in \{1, \ldots, m\},
\]

\[
V = US''' \iff V_{ijk} = \sum_{t=1}^{m} U_{tij}S_{tk} \quad \text{for all } i, j, k \in \{1, \ldots, m\}.
\]

Formally, \(S' = S \otimes I \otimes I, S'' = I \otimes S \otimes I, S''' = I \otimes I \otimes S\), where \(I\) is the identity \(m \times m\) matrix and \(U = X \otimes Y \otimes Z\) denotes the \(m^3 \times m^3\) matrix with elements \(U_{ijk,i'j'k'} = X_{i,i'}Y_{j,j'}Z_{k,k'}\). We also define the diagonal matrix \(D'\) such that 

\[
T^{r_1, r_2, r_3} = W^{r_1, r_2, r_3} D'.
\] 

(1) 

Formally, \(D' = D \otimes I \otimes I\), where \(D_{ii} = |C_i|\).

\textbf{Example 1} If \(S = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}\) and the elements of a vector are arranged as 

\[
U = (U_{111}, U_{112}, U_{121}, U_{122}, U_{211}, U_{212}, U_{221}, U_{222}),
\]

then 

\[
D' = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}, \quad S' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
S'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S''' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]
Lemma 1 The matrices $S'$, $S''$, and $S'''$ commute with each other. The matrix $D'$ commutes with $S''$ and $S'''$.

Remark 2 The matrices $D'$ and $S'$ do not commute in general, see Example 1.

3 Recursive formulas

In the following theorem, we present two groups of formulas. The first three equations, (4)–(6), give recursions for $W^{r_1,r_2,r_3}$; they give more information than formulas (7)–(9), which concern to $T^{r_1,r_2,r_3}$, because $W^{r_1,r_2,r_3}$ are more refined characteristics than $T^{r_1,r_2,r_3}$. On the other hand, the system of equations (5)–(7) is symmetric with respect to all three parameters $r_1$, $r_2$, $r_3$, and we need this symmetry to derive one of the equations from the others. The symmetry is a key point of the proof and justifies the use of the triangle distribution. Without this trick (or a separate combinatorial proof of the third equation for $W^{r_1,r_2,r_3}$, which is expected to be complicated), we have only two formulas for $W^{r_1,r_2,r_3}$, which are sufficient to calculate the interweight distributions from the weight distributions recursively, but not for deriving the direct formulas (Section 5).

To understand the formulas below, it should be noted that, as it follows from definitions, the elements of $W^{r_1,r_2,r_3}$ and $T^{r_1,r_2,r_3}$ are zeros if $r_1 < 0$, $r_2 < 0$, $r_3 < 0$, or $r_1 + r_2 + r_3 > n$.

Theorem 2 Let $C = (C_1, \ldots, C_m)$ be an equitable partition of an $n$-cube with quotient matrix $S$. Then the vectors $W^{r_1,r_2,r_3}$ satisfy the following equations:

\begin{align*}
W^{r_1,r_2,r_3} &= (r_1 + 1)W^{r_1+1,r_2-1,r_3} + (r_2 + 1)W^{r_1-1,r_2+1,r_3} + (n - r_1 - r_2 - r_3 + 1)W^{r_1,r_2+1,r_3-1} + (r_3 + 1)W^{r_1,r_2,r_3+1}. \quad (2) \\
W^{r_1,r_2,r_3} &= (r_1 + 1)W^{r_1+1,r_2,r_3-1} + (r_3 + 1)W^{r_1,r_2+1,r_3} + (n - r_1 - r_2 - r_3 + 1)W^{r_1,r_2-1,r_3} + (r_2 + 1)W^{r_1,r_2,r_3+1}. \quad (3) \\
W^{r_1,r_2,r_3} &= (r_2 + 1)W^{r_1+1,r_2+1,r_3-1} + (r_3 + 1)W^{r_1,r_2-1,r_3} + (n - r_1 - r_2 - r_3 + 1)W^{r_1-1,r_2+1,r_3} + (r_1 + 1)W^{r_1,r_2,r_3+1}. \quad (4)
\end{align*}

The triangle distribution satisfies the following equations:

\begin{align*}
T^{r_1,r_2,r_3} &= (r_1 + 1)T^{r_1+1,r_2-1,r_3} + (r_2 + 1)T^{r_1-1,r_2+1,r_3} + (n - r_1 - r_2 - r_3 + 1)T^{r_1,r_2+1,r_3-1} + (r_3 + 1)T^{r_1,r_2,r_3+1}. \quad (5) \\
T^{r_1,r_2,r_3} &= (r_1 + 1)T^{r_1+1,r_2,r_3-1} + (r_3 + 1)T^{r_1-1,r_2+1,r_3} + (n - r_1 - r_2 - r_3 + 1)T^{r_1,r_2-1,r_3} + (r_2 + 1)T^{r_1,r_2+1,r_3}. \quad (6) \\
T^{r_1,r_2,r_3} &= (r_2 + 1)T^{r_1+1,r_2+1,r_3-1} + (r_3 + 1)T^{r_1-1,r_2-1,r_3} + (n - r_1 - r_2 - r_3 + 1)T^{r_1-1,r_2+1,r_3} + (r_1 + 1)T^{r_1+1,r_2,r_3}. \quad (7)
\end{align*}

Proof We will prove equation (2) separately, in Section 4. Equation (3) can be obtained in the same manner. Since $W^{r_1,r_2,r_3}D' = T^{r_1,r_2,r_3}$ and $D'$ commutes with $S''$ and $S'''$ (Lemma 1), we see that ...
and (3) are straightforward from (2) and (3). By analogy, (7) holds too (indeed, \( T^{r_1,r_2,r_3}_{ijk} = T^{r_2,r_1,r_3}_{ijk} \), by definition). Using again \( W^{r_1,r_2,r_3} D' = T^{r_1,r_2,r_3} \) and noting that \( D' S' D'^{-1} = S'T \) (the last follows from \( D S D^{-1} = S'T \)), we derive (4) from (7). □

The collection \( (W^{r_1,r_2,r_3}[^d_{i,j,k}])_{r_1,r_2,r_3=0} = ((W^{r_1,r_2,r_3}[^d_{i,j,k}])_{l,j,k=1}[^d_{r_1,r_2,r_3=0}} \) can be treated as \( m \) interweight distributions, accordingly with different values of \( i \). For a vertex \( v \), the interweight distribution of \( C \) with respect to \( v \) is given by the values \( W^{r_1,r_2,r_3}[^d_{i,j,k}] \) with fixed \( i \) such that \( v \in C_i \). By the definition, \( W^{0,0,0}_{iii} = 1, i = 1, \ldots, m \), and the other entries of \( W^{0,0,0} \) are zeros. Formulas (2)–(4) express \( W^{r_1,l_2,l_3} \) (in the underlined parts of formulas) as a combination of arrays with smaller index sum \( l_1 + l_2 + l_3 \); so, all the values are calculated recursively. The situation with the triangle distribution is similar with the only difference in the initial values: \( T^{0,0,0}_{iii} = |C_i|, i = 1, \ldots, m \). (Note also that, because of the obvious relations \( |C_i| S_{j,i} = |C_j| S_{j,i} \) and \( |C_1| + \ldots + |C_m| = 2^n \), the values \( |C_i| \), \( i = 1, \ldots, m \), are derived from the quotient matrix \( S \).)

4 A proof of the recursion

Before proving (2), we define two auxiliary concepts.

Given a collection \( C = (C_1, \ldots, C_m) \) of subsets of the vertex set of a graph, the spectrum of a vertex set \( X \) with respect to \( C \) is the \( k \)-tuple \( \text{Sp}_C(X) = (x_1, \ldots, x_m) \), where \( x_i = |X \cap C_i| \) (intuitively, we can treat \( C_1, \ldots, C_m \) as colors and think about the color spectrum). If \( X \) is a multiset, then \( x_i \) is defined as the sum over \( C_i \) of the multiplicities in \( X \). The multi-neighborhood \( [x] \) of a vertex set \( X \) is a multiset of vertices of the graph, where the multiplicity of a vertex is calculated as the number of its neighbors from \( X \). In other words, \( [x] = \{x \in X | x \} \) where \( [x] \) is the neighborhood of the vertex \( x \) and \( \{x \} \) is the multiset union.

Lemma 2 Let \( C \) be an equitable partition (of an arbitrary graph) with quotient matrix \( S \). For every vertex set \( X \),

\[ \text{Sp}_C([X]) = \text{Sp}_C(X) \cdot S. \]

Proof

By the definitions of an equitable partition, the multi-neighborhood, and the spectrum, we have \( \text{Sp}_C([x]) = \text{Sp}_C(\{x\}) \cdot S \) for every vertex \( x \). Then,

\[
\text{Sp}_C([X]) = \text{Sp}_C(\{x \in X | x \}) = \sum_{x \in X} \text{Sp}_C([x]) = \sum_{x \in X} \text{Sp}_C(\{x\}) \cdot S = \text{Sp}_C(X) \cdot S.
\]

Now, we are ready to prove (2). For fixed vertices \( v \) and \( x \) of the \( n \)-cube, denote by \( H^{r_1,r_2,r_3}_{v,x} \) the set of vertices \( y \) such that \( d(v, y) = r_1 + r_3, d(x, y) = r_2 + r_3, d(v, x) = r_1 + r_2 \) (we do not restrict the values of the parameters \( r_1, r_2, r_3 \), but note that by the definition \( H^{r_1,r_2,r_3}_{v,x} \) is nonempty only for nonnegative \( r_1, r_2, r_3 \) satisfying \( r_1 + r_2 = d(v, x) \) and \( r_1 + r_2 + r_3 \leq n \)).

Every vertex of \( H^{r_1,r_2,r_3}_{v,x} \) has \( r_1 \) neighbors from \( H^{r_1-1,r_2+1,r_3}_{v,x} \), \( r_2 \) neighbors from \( H^{r_1+1,r_2-1,r_3}_{v,x} \), \( r_3 \) neighbors from \( H^{r_1,r_2-1,r_3}_{v,x} \), \( n - r_1 - r_2 - r_3 \) neighbors from \( H^{r_1,r_2,r_3+1}_{v,x} \), and no other neighbors (to see this, we can consider without loss of generality that \( v = 0^n, x = 1^{n-1}r_1+1r_20^{n-r_1-r_2}, y = 0^21^{r_1+r_3}0^{n-r_1-r_2-r_3} \)).

By the definition of the multi-neighborhood, we have

\[
[H^{r_1,r_2,r_3}_{v,x}] = l_1 H^{r_1+1,r_2-1,r_3}_{v,x} \cup l_2 H^{r_1-1,r_2+1,r_3}_{v,x} \cup l_3 H^{r_1,r_2-1,r_3+1}_{v,x} \cup l_0 H^{r_1,r_2,r_3-1}_{v,x}
\]

where \( l_1 = r_1 + 1, l_2 = r_2 + 1, l_3 = r_3 + 1, l_0 = n - r_1 - r_2 - r_3 + 1 \). Considering the spectrum of each side of the equation, applying Lemma 2 and denoting \( S^{r_1,r_2,r_3}_{v,x} = \text{Sp}_C(H^{r_1,r_2,r_3}_{v,x}) \), we get the following:

\[
S^{r_1,r_2,r_3}_{v,x} \cdot S = l_1 S^{r_1+1,r_2-1,r_3}_{v,x} + l_2 S^{r_1-1,r_2+1,r_3}_{v,x} + l_3 S^{r_1,r_2-1,r_3+1}_{v,x} + l_0 S^{r_1,r_2,r_3-1}_{v,x}.
\]
Summarizing the last equation over all $x$ from $C_j$, we find

$$
W_{v,j}^{r_1,r_2,r_3} : S = l_1 W_{v,j}^{r_1+1,r_2-1,r_3} + l_2 W_{v,j}^{r_1-1,r_2+1,r_3} + l_3 W_{v,j}^{r_1,r_2,r_3+1} + l_0 W_{v,j}^{r_1,r_2,r_3-1}
$$

(8)

where $W_{v,j}^{r_1,r_2,r_3} = (W_{v,j}^{r_1,r_2,r_3}, \ldots, W_{v,j}^{r_1,r_2,r_3})$ and $W_{v,j}^{r_1,r_2,r_3}$ denotes the number of the pairs $(x, y)$ of vertices such that $d(v, y) = r_1 + r_3$, $d(x, y) = r_2 + r_3$, $d(v, x) = r_1 + r_2$, $x \in C_j$, $y \in C_k$. Because of the strong distance invariance, $W_{v,j}^{r_1,r_2,r_3}$ depends on $i$ such that $v \in C_i$ and does not depend on the choice of $v$ from $C_i$. That is, $W_{v,j}^{r_1,r_2,r_3} = W_{v,j}^{r_1,r_2,r_3}$, and (8) coincides with (2). The proof is over.

Remark 3 (an alternative proof of Theorem 1) Formula (8) allows to express the values $W_{v,j}^{r_1,r_2,r_3+1}$ through $W_{v,j}^{l_1,l_2,l_3}$ with different $l_1, l_2, l_3$ satisfying $l_1 + l_2 + l_3 < r_1 + r_2 + r_3 + 1$. Since $W_{v,j}^{r_1,r_2,r_3} = W_{v,j}^{r_1,r_3,r_2}$, we can say the same about $W_{v,j}^{r_1,r_2+1,r_3}$. As a result, we can calculate $W_{v,j}^{r_1,r_2,r_3}$ recursively, starting from $W_{v,j}^{0,0,0}$, $l_1 = 0, \ldots, n$, i.e., from the weight distribution of the partition with respect to $v$. But the weight distribution depends only on $C_i$ that contains $v$ and does not depend on the choice of $v$. We conclude that the same is true for the interweight distribution. This gives another proof of Theorem 1 and makes our theory self-contained (well, we still use the distance invariance, but it is clear that formulas for the weight distribution can be obtained using the technique of Section 4).

5 The polynomials

In this section, we derive a direct formula and the enumerator for $T_{r_1,r_2,r_3}$. Utilizing (1) or the similarity between the systems of equations (2)–(4) and (5)–(7), one can easily see that corresponding formulas for $W_{r_1,r_2,r_3}$ are obtained by replacing $T$ by $W$ and $S'$ by $S''$.

In further considerations, we will use the following degenerated but important case of equitable partitions. By the singleton partition of a graph $G = (V(G), E(G))$, we will mean the partition $(\{x\})_{x \in V(G)}$ of the vertex set into sets of cardinality one. The singleton partition is obviously equitable, and its quotient matrix $S$ coincides with the graph adjacency matrix.

Lemma 3 For every nonnegative integers $n$, $r_1$, $r_2$, $r_3$ meeting $r_1 + r_2 + r_3 \leq n$, there is a unique polynomial $P_{r_1,r_2,r_3}(x,y,z)$ of degree at most $r_1 + r_2 + r_3$ such that the equation

$$
T_{r_1,r_2,r_3} = T_{0,0,0}^{r_1,r_2,r_3} P_{r_1,r_2,r_3}(S', S'', S''')
$$

(9)

holds for every equitable partition of the $n$-cube and its quotient matrix $S$.

Proof Since the matrices $S'$, $S''$, $S'''$ commute with each other (Lemma 1), the existence of the polynomial follows by induction from (5)–(7). It remains to prove the uniqueness, which is not straightforward; indeed, the recursion is three-parametric, and some values can be obtained in more than one way.

Since (9) must hold for every equitable partition, it is sufficient to prove the uniqueness for a fixed one. Let us consider the singleton partition; that is, $S$ is a graph adjacency matrix.

(I) We first note that the dimension of the vector space of all polynomials of degree at most $n$ in three variables equals $\binom{n+3}{3}$. This is the number of monomials of type $x^r y^r z^r$ of degree at most $n$, which form a basis.

(II) Then, we see that all $T_{r_1,r_2,r_3}$, $r_1 \geq 0$, $r_2 \geq 0$, $r_3 \geq 0$, $r_1 + r_2 + r_3 \leq n$, are linearly independent. Indeed, for every vertices $i$, $j$, $k$, there is exactly one vector $T_{r_1,r_2,r_3}$ (namely, such that the distances between the vertices $i$ and $j$, $j$ and $k$, $i$ and $k$ equal $r_1 + r_2$, $r_2 + r_3$, $r_1 + r_3$, respectively) with $T_{ij,k} = 0$. The number of different non-zero $T_{r_1,r_2,r_3}$ is $\binom{n+3}{3}$ again.
We can conclude from (I) and (II) that the linear map \( P \to T^{0,0,0}P(S', S'', S'''') \), from the space of all polynomials of degree at most \( n \) in three variables to the vector space generated by all \( T^{r_1,r_2,r_3} \), is nonsingular. Indeed, the dimensions of both spaces coincide, and the image contains a basis from \( T^{r_1,r_2,r_3} = T^{0,0,0}P^{r_1,r_2,r_3}(S', S'', S''') \). Hence, every \( T^{r_1,r_2,r_3} \) is represented as \( T^{0,0,0}P(S', S'', S'''') \), where the degree of \( P \) is not greater than \( n \), in only one way.

**Theorem 3** The generating function

\[
f(X, Y, Z) = \sum_{r_1,r_2,r_3} P^{r_1,r_2,r_3}(x, y, z) X^{r_1} Y^{r_2} Z^{r_3}
\]

of the polynomials \( P^{r_1,r_2,r_3} \) satisfying (9) for every equitable partition has the form

\[
f(X, Y, Z) = (1 + X + Y + Z)^{n+1} (1 + X - Y - Z)^{n+1} 
\times (1 - X + Y - Z)^{n+1}.
\]

**Proof** From (7) and Lemma 3, the polynomials \( P^{r_1,r_2,r_3} \) satisfy

\[
x^{r_1,r_2,r_3}(x, y, z) = (r_1 + 1) P^{r_1+1,r_2,r_3-1}(x, y, z) 
+ (r_2 + 1) P^{r_1,r_2+1,r_3-1}(x, y, z) 
+ (r_3 + 1) P^{r_1+1,r_2,r_3-1}(x, y, z) 
+ (n - r_1 - r_2 - r_3 + 1) P^{r_1-1,r_2,r_3}(x, y, z) 
+ (r_1 + 1) P^{r_1+1,r_2,r_3}(x, y, z).
\]

(10)

Multiplying by \( X^{r_1} Y^{r_2} Z^{r_3} \) and summing over all \( r_1, r_2, r_3 \) from 0 to \( \infty \), we get

\[
x \sum_{r_1,r_2,r_3} P^{r_1,r_2,r_3} X^{r_1} Y^{r_2} Z^{r_3} = Z \sum_{r_1,r_2,r_3} (r_2 + 1) P^{r_1+1,r_2+1,r_3-1} X^{r_1} Y^{r_2} Z^{r_3-1} 
+ Y \sum_{r_1,r_2,r_3} (r_3 + 1) P^{r_1+1,r_2-1,r_3+1} X^{r_1} Y^{r_2} Z^{r_3} 
+ nX \sum_{r_1,r_2,r_3} P^{r_1-1,r_2,r_3} X^{r_1-1} Y^{r_2} Z^{r_3} 
- X^2 \sum_{r_1,r_2,r_3} (r_1 - 1) P^{r_1-1,r_2+1,r_3-1} X^{r_1-1} Y^{r_2} Z^{r_3} 
- XY \sum_{r_1,r_2,r_3} r_2 P^{r_1-1,r_2,r_3} X^{r_1-1} Y^{r_2-1} Z^{r_3} 
- XZ \sum_{r_1,r_2,r_3} r_3 P^{r_1-1,r_2,r_3} X^{r_1-1} Y^{r_2} Z^{r_3-1} 
+ \sum_{r_1,r_2,r_3} (r_1 + 1) P^{r_1+1,r_2,r_3} X^{r_1} Y^{r_2} Z^{r_3}.
\]

Next, denoting \( f(X, Y, Z) = \sum_{r_1,r_2,r_3=0}^{\infty} P^{r_1,r_2,r_3}(x, y, z) X^{r_1} Y^{r_2} Z^{r_3} \) and implying \( P^{r_1,r_2,r_3}(x, y, z) = 0 \) whenever \( r_1 < 0, r_2 < 0, \) or \( r_3 < 0, \) we obtain

\[
(x - nX) f(X, Y, Z) = (1 - X^2) \frac{\partial}{\partial X} f(X, Y, Z) 
+ (Z - XY) \frac{\partial}{\partial Y} f(X, Y, Z) + (Y - XZ) \frac{\partial}{\partial Z} f(X, Y, Z).
\]

The formula from the statement of the theorem satisfies this differential equation (we omit the straightforward but bulky check, but note that the most complicated part of the check is comparing polynomials,
which can be verified by computer, for example, using GAP [16]. In particular, this means that its Taylor coefficients satisfy the recursion (10). Similarly, they satisfy the recursions derived from (5) and (6). It remains to note that the coefficient at $X^0Y^0Z^0$ is 1, as desired.

**Corollary 1** For all integer $r_1 \geq 0$, $r_2 \geq 0$, $r_3 \geq 0$,

$$P_{r_1,r_2,r_3}(x,y,z) = \sum_{i,j,k} (-1)^{i_2+i_3+j_1+j_3+k_1+k_2} \times \left(\frac{n+x+y+z}{4} \right) \left(\frac{r_1-i_1-j_1-k_1}{4} \right) \left(\frac{r_2-i_2-j_2-k_2}{4} \right) \left(\frac{r_3-i_3-j_3-k_3}{4} \right)$$

where

$$\left(\begin{array}{c}
\delta \\
\alpha, \beta, \gamma, \cdots
\end{array}\right) = \frac{\delta(\delta-1)\cdots(\delta-\alpha-\beta-\gamma+1)}{\alpha! \beta! \gamma!}.$$

**Remark 4** In the partial case $r_2 = r_3 = 0$, we have

$$P_{r,r,0}(x,y,z) = K_r(K_1^{-1}(x), \text{ where } K_r(x) = \sum_i (-1)^i \binom{x}{i} \binom{n-x}{r-i}$$

is the well-known Krawtchouk polynomial, see e.g. [14 §5.2]. So, the polynomial

$$K_{r_1,r_2,r_3}(x,y,z) = P_{r_1,r_2,r_3}(K_1(x), K_1(y), K_1(z))$$

can be seen as a generalization of the Krawtchouk polynomial.

**Corollary 2** Given an equitable partition $C$ of the $n$-cube with quotient matrix $S$, the enumerator $\sum_{r_1,r_2,r_3} T_{r_1,r_2,r_3} X^{r_1} Y^{r_2} Z^{r_3}$ is equal to

$$T_{0,0,0}(1 + X + Y + Z) \frac{n+s'+s''+s'''}{4} (1 + X - Y - Z) \frac{n+s'-s''-s'''}{4} \times (1 - X + Y - Z) \frac{n-s'+s''-s'''}{4} (1 - X - Y + Z) \frac{n-s'-s''+s'''}{4}.$$

**Remark 5** The generating function $f$ in Theorem 3 is a polynomial (of degree $n$) if and only if all four powers are nonnegative integers. But the quotient matrix $S$, in general, can have eigenvalues that make the powers negative or non-integer being substituted for $x$, $y$, $z$. This means that $P_{r_1,r_2,r_3}(S', S'', S''')$ is not necessarily the zero matrix when $r_1 + r_2 + r_3 > n$. Nevertheless, $T_{0,0,0} P_{r_1,r_2,r_3}(S', S'', S''')$ will be zero in this case, and the enumerator in Corollary 2 is a degree-$n$ polynomial. This can be explained by the fact that the dimension of the matrix algebra $S$ generated by $S'$, $S''$, and $S'''$ is higher than the dimension of its restriction by the action on the vector space $T$ generated by $T_{r_1,r_2,r_3}$, $r_1, r_2, r_3 \geq 0$. For example, for the singleton partition, $S$ generates an algebra with a basis $(A^w)^n_{w=0}$, such that $A^w$ and $A^{w'}$ have no common non-zero entries provided $w \neq w'$ (this algebra is known as the Bose–Mesner algebra, and the matrices $A^w$ were already mentioned in Section 2). It follows that $(A^w \otimes A^{w'})^{n}_{w,w',w''=0}$ is a basis of $S$ as a vector space. Hence, $S$ has dimension $(n+1)^3$. On the other hand, for any element $S_0$ of $S$, the vector $T_{0,0,0} S_0$ is a linear combination of $T_{0,0,0} P_{l_1,l_2,l_3}(S', S'', S''')$ with $l_1 + l_2 + l_3 \leq n$. Because of the commutativity, the same linear relation will be valid if we replace $T_{0,0,0}$ by any of $T_{r_1,r_2,r_3}$ or by a linear combination of them. This means that the action of any $S_0$ from $S$ on $T$ is a linear combination of the actions of $P_{l_1,l_2,l_3}(S', S'', S''')$ with $l_1 + l_2 + l_3 \leq n$. That is, the space of such linear transformations of $T$ has the dimension $\binom{n+3}{3} = \frac{(n+3)(n+2)(n+1)}{6}$, which is smaller than $(n+1)^3$. 

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In the rest of this section, we list the polynomials $P_{r_1 r_2 r_3}$ of degree at most 4.

$p_{0,0,0} = 1,$

$p_{0,0,1} = z,$

$p_{0,0,2} = (z^2 - n) / 2,$

$p_{0,1,1} = xyz - x,$

$p_{0,0,3} = (z^3 + (2 - 3n)z) / 6,$

$p_{0,1,2} = \frac{yz^2 - 2xz + (2 - n)y}{2},$

$p_{1,1,1} = \frac{xyz - x^2 - y^2 - z^2 + 2n}{2},$

$p_{0,2,2} = \frac{yz^2 - 4xyz + 2x^2 + (4 - n)y^2 + (4 - n)z^2 + (n^2 - 6n)}{4},$

$p_{1,1,2} = \frac{xyz^2 - 2x^2z - 2y^2z - z^3 + (6 - n)xy + (5n - 6)z}{2}.$

$n$:=Indeterminate(Rationals,1); SetName(n,"n");

$x$:=Indeterminate(Rationals,2); SetName(x,"x");

$y$:=Indeterminate(Rationals,3); SetName(y,"y");

$z$:=Indeterminate(Rationals,4); SetName(z,"z");

$I$:=Indeterminate(Rationals,5); SetName(I,"I"); # for Id. matrix

$P$:=function(a,b,c) # calculates $P^{a,b,c}(x,y,z)$
if (a<0)or(b<0)or(c<0) then return 0*I;
elif (a+b+c=0) then return I;
elif (a>0) then return ( x*P(a-1,b,c)-(n-a-b-c+2)*I*I*P(a-2,b,c)
-\((b+1)*I*P(a-1,b+1,c-1)
-(c+1)*I*P(a-1,b-1,c+1) )/a;
else return Value(P(b,c,a),[x,y,z],[y,z,x]); # using symmetry
fi;
end;

Print( Value(P(1,2,3),[n,I],[6,1]), "\n" ); # example

$S$:=\{[0,5], [3,2]\}; N:=Sum(S[I]);

SI :=KroneckerProduct( S,S^0); IIS:=KroneckerProduct(SI^0,S);

ISI:=KroneckerProduct(SI^0,S); SII:=KroneckerProduct(SI,S^0);

$W_0$:=\{[1,0,0,0,0,0,0,1]\};

Print(W0*Value(P(1,2,3),[n,x,y,z,I],

[5,TransposedMat(SII),ISI,IIS,SI^0]),"\n");

# Negative values mean that an equitable partition
# with quotient matrix $S$ does not exist

6 Connection with the correlation-immunity bound

The theory of the equitable 2-partitions of $n$-cubes was developed, in its current state, by D. Fon-Der-Flaass in [7,6,8]. At the moment, there are three known general necessary conditions (12), (13), (14) for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be the quotient matrix of some equitable partition (and only for one matrix $S = \begin{pmatrix} 1 & 9 \\ 1 & 3 \end{pmatrix}$ satisfying these three conditions it is known that $S$ is not the quotient matrix of an equitable partition [8]). Two conditions are rather simple:

\begin{align*}
    n = a + b = c + d; \\
    (b + c)/\gcd(b, c) \text{ divides } 2^n.
\end{align*}
Again, a perfect structure is given by a partition of its vertex set into classes such that every two adjacent vertices are in different classes. The following holds because of the connection with the corresponding bound for Boolean functions [4].

**Theorem 4** ([6]) Assume that there exists an equitable partition of an $n$-cube with quotient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $b \neq c$; then

$$c - a \leq n/3.$$  

(14)

Without loss of generality, we may assume $b \geq c$ (otherwise we can satisfy this condition by renumbering the partition elements). The following computational result connects the bound (14) with the evident condition that the elements of the interweight distribution of a partition must be nonnegative.

**Proposition 1** For every integer $n$ from 1 to 100 and every integer $a$, $c > 0$, $b > c$, $d$ satisfying (15), (16) and missing (14), there are $r_2$, $r_3$ such that $T_{111}^{0,r_2,r_3} < 0$, where $T_{\ldots}$ are formally calculated using [5]–[7].

7 The real-valued case

In this section, we will briefly discuss a generalization of the equitable partitions, where the spectrum of a single vertex can possess an arbitrary vector value over the real numbers $R$. Let $C : V(H^n) \to R^m$ be a vector function whose values are $m$-tuples over $R$. By the spectrum $C(M)$ of a set $M \subseteq V(H^n)$ we will mean the sum of values of $C$ over $M$. The function $C$ is a perfect structure with parameter $m \times m$ matrix $S$ if for every vertex $x$ the spectrum of the neighborhood of $x$ equals $C(x)S$. The concept of perfect structures generalizes the equitable partitions, whose characteristic vector functions are perfect structures. As an example, to show that this generalization can give something interesting, we refer to [13], where it is shown that the optimal binary 1-error-correcting codes of length $2^m - 4$ are related with perfect structures, but not with equitable partitions in general. Also note that the eigenfunctions of a graph are the perfect structures with $m = 1$. For this important partial case, our theory makes sense as well.

For the perfect structures, we can define the interweight distribution as follows. For two (similarly, for three) $m$-tuples $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$, we define their tensor product $a \otimes b$ as

$$(c_{jk})_{j,k=1}^m = (c_{11}, \ldots, c_{1m}, c_{21}, \ldots, c_{mm}), \quad \text{where} \quad c_{jk} = a_j b_k.$$  

For a vertex $v$, let $W_v^{r_1,r_2,r_3}$ denote the sum of $C(x) \otimes C(y)$ over all pairs of vertices $(x, y)$ such that $d(v, x) = r_2 + r_3$, $d(v, y) = r_1 + r_3$, $d(v, x) = r_1 + r_2$. The value $W_v^{r_1,r_2,r_3}$ can be still be treated as “the number of the triangles $(v, x \in C_j, y \in C_k)$ such that $d(v, x) = r_2 + r_3$, $d(v, y) = r_1 + r_3$, $d(v, x) = r_1 + r_2$” if the values of $C$ are treated as the multiplicities of a collection $(C_1, \ldots, C_m)$ of multisets. Similarly, let $T_{r_1,r_2,r_3}$ denote the sum of $C(x) \otimes C(x) \otimes C(y)$ over all triples of vertices $(v, x, y)$ such that $d(v, x) = r_2 + r_3$, $d(v, y) = r_1 + r_3$, $d(v, x) = r_1 + r_2$. In the case of perfect structures, the equations similar to (2) and (3) hold for $W_v^{r_1,r_2,r_3}$, and $T_{r_1,r_2,r_3}$ satisfy equations (4)–(7).

**Corollary 3** Given a perfect structure $C$ and a vertex $v$, all the values $W_v^{r_1,r_2,r_3}$, $r_1 + r_2 + r_3 \leq n$ can be calculated from $W_v^{r_1,0,0}$, $r_1 = 0, 1, \ldots, n$ and the parameter matrix $S$.

But in general, in contrast to the case of equitable partitions, the values $W_v^{r_1,0,0}$ cannot be calculated from $C(v)$.

**Corollary 4** Given a perfect structure $C$, all the values $T_{r_1,r_2,r_3}$, $r_1 + r_2 + r_3 \leq n$ can be calculated from $T_{0,0,0}$ and the parameter matrix $S$.

Again, $T_{0,0,0}$ is not invariant over all perfect structures with the same parameter matrix.
Example 2 Let \( n = m = 2 \), \( C(00) = C(01) = (2, 0) \), \( C(10) = C(11) = (0, 2) \), \( C'(00) = (2, 0) \), \( C'(01) = (1, 1) \), \( C'(11) = (0, 2) \). Then \( C \) and \( C' \) are perfect structures with the same parameter matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). Also, \( C(v) = C'(v) \) for \( v = 00 \). Nevertheless, it is easy to see that the values \( W^1_{1,0,0} \) and \( T^0_{0,0,0} \) differ for \( C \) and \( C' \).

So, we can see that the strong distance invariance, in contrast to the distance invariance, cannot be generalized to the perfect structures, while the recursive relations (2)–(3), (5)–(7) are still valid.

8 Open problems

For a partition of the vertex set of the \( n \)-cube, the distance invariance and the strong distance invariance are equivalent. Indeed, the equitability of the partition follows, by definition, from each of these properties. In its turn, the equitability implies the distance invariance and the strong distance invariance. The things are not so easy if we consider collections \( C \) of subsets that are not partitions. In particular, if \( C \) consists of only one set.

**Problem 1** Does there exist a distance invariant set of vertices of an \( n \)-cube that is not strongly distance invariant?

Clearly, any set that is a cell of some equitable partition is out of consideration because it is strongly distance invariant. To formulate the next question, recall that the distance distribution of a set is the multiset of mutual distances between the set elements.

**Problem 2** Do there exist two distance invariant sets of vertices of an \( n \)-cube with the same distance distributions but different triangle distributions?

The question can be treated as follows: is the triangle distribution the function of the distance distribution, for the distance invariant sets in \( n \)-cubes? The following example in the 4-cube shows that it is not the case for arbitrary sets: \( \{0000, 0001, 0111, 1111\} \), \( \{0000, 0001, 0111, 1111\} \). The distances in both cases are 1, 1, 2, 3, 3, 4, and the triangles, in terms of \( r_1, r_2, r_3 \), are \( (0, 1, 2), (0, 1, 2), (0, 1, 3), (0, 1, 3) \) and \( (0, 1, 3), (0, 1, 3), (1, 1, 1), (1, 1, 2) \), respectively.

Another question arises from the computational results in Section 6.

**Problem 3** Explain Proposition 1 theoretically; prove it for an arbitrary \( n \).

In the case on \( m \geq 3 \), the interweight distributions can be used to obtain new nonexistence results.

**Example 3** Consider the matrix \( S = \begin{pmatrix} 0 & 2 & 0 \\ 5 & 6 & 11 \\ 0 & 10 & 12 \end{pmatrix} \). The eigenvalues \(-10, 6, 22\) of the matrix are eigenvalues of the 22-cube too. Calculating the weight distributions of a hypothetical equitable partition with the quotient matrix \( S \), we do not find a contradiction as all the elements found are nonnegative integers. However, calculating the triangle distribution gives a negative value for \( T^0_{111,8,9} \), for example. Hence, equitable partitions of the 22-cube with the quotient matrix \( S \) do not exist.

Another theoretical question is to find connections between the interweight distributions and the Terwilliger algebra of the \( n \)-cube [9]. One of the relations was occasionally found by the author of the current paper during a search in his local database of papers: it is the subword “terw”, which occurs in both notations. There should be deeper connections, as the Terwilliger algebra is related to distance triangles and its dimension coincides with \( \frac{n+3}{3} \), the dimension of the algebra of linear transformations of \( \mathcal{T} \) considered in Section 5 (Remark 5). But the last algebra is commutative, while the Terwilliger algebra is not.

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