Lattice Points, Dedekind-Rademacher Sums and a Conjecture of Kronheimer and Mrowka

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Abstract

We express the number of lattice points inside a certain simplex with vertices in \( \mathbb{Q}^3 \) or \( \mathbb{Q}^4 \) in terms of Dedekind-Rademacher sums. As an application we prove a conjecture of Kronheimer and Mrowka in the special case of Brieskorn homology spheres \( \Sigma(a_1, \ldots, a_n), n \leq 4 \). This conjecture relates the Euler characteristic of the Seiberg-Witten-Floer homology to the Casson invariant.

Introduction

Since the very beginning it was apparent that the Seiberg-Witten analogue of the instanton Floer homology of a homology 3-sphere is no longer a topological invariant since it can vary with the metric.

Recently, M. Marcolli explained in \[7\] the metric dependence of the Euler characteristic of the SWF (= Seiberg-Witten-Floer) homology. More precisely, if \( g_i (i = 0, 1) \) are two generic Riemann metrics on a homology 3-sphere \( N \) and \( \chi_{SW}(N, g_i) \) is the Euler characteristic of the SWF homology of \( (N, g_i) \) the results of \[7\] imply that

\[
\chi_{SW}(N, g_1) - \chi_{SW}(N, g_0) = \frac{1}{8} F(g_1) - \frac{1}{8} F(g_0)
\]

where

\[
F(g) = 4\eta_{\text{dir}}(g) + \eta_{\text{sign}}(g)
\]

\( \eta_{\text{dir}}(g) \) denotes the eta invariant of the Dirac operator of \( (N, g) \) while \( \eta_{\text{sign}}(g) \) denotes the eta invariant of the odd signature operator on \( (N, g) \). In particular, the above equality shows that the quantity

\[
\alpha(N) = -\chi_{SW}(N, g) + \frac{1}{8} F(g)
\]

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is independent of $g$ and thus is a topological quantity.

In 1996, Kronheimer and Mrowka conjectured (see [6]) that this quantity coincides (up to a sign) with the Casson invariant of $N$. W. Chen ([1],[2]) has established many interesting properties of this invariant.

The topological goal of this paper is to provide a proof of this conjecture, in the special case of Brieskorn homology sphere $\Sigma(a_1,\cdots,a_n)$, $n \leq 4$. Our proof is arithmetic in nature and is based on a count of lattice points inside a certain $n$-simplex in $\mathbb{R}^n$ with vertices in $\mathbb{Q}^n$, $n = 3,4$.

More precisely, recent work ([10], [13]) shows that for a certain natural metric $g_0$ on $\Sigma(a,b,c)$ (realizing the Thurston geometry of this Seifert manifold) we have

$$-\chi_{SW}(\Sigma(a_1,\cdots,a_n),g_0) = 2C_{a_1,\cdots,a_n}, \quad n = 3,4$$

where $C_{a_1,\cdots,a_n}$ is the number of lattice points in the simplex

$$\Delta(a_1,\cdots,a_n) := \left\{ (x_1,\cdots,x_n) \in \mathbb{R}^n ; x_i \geq 0, \sum_{i=1}^{n} \frac{x_i}{a_i} < \frac{1}{2} \left( n - 2 - \sum_{i=1}^{n} \frac{1}{a_i} \right) \right\}.$$

The arithmetic goal of this paper is the determination of $C_{a_1,\cdots,a_n}$ when $n = 3,4$. This is a problem of independent interest since the vertices of the simplex $\Delta(a_1,\cdots,a_n)$ are not lattice points and none of the counting techniques using Riemann-Roch on toric manifolds seem to apply. We will use instead a variation of a trick of Mordell; see [9] or [15].

R. Fintushel and R. Stern have shown in [3] that the Casson invariant of the Brieskorn sphere $\Sigma(a,b,c)$ is $\frac{1}{8}\sigma(a,b,c)$ where $\sigma(a,b,c)$ denotes the signature of the Milnor fiber of $\Sigma(a,b,c)$. This result was extended to arbitrary $\Sigma(a_1,\cdots,a_n)$ by Neuwmann-Wahl in [11]. The Kronheimer-Mrowka conjecture in this case is equivalent to

$$-2C_{a_1,\cdots,a_n} - \frac{1}{8}F(g_0) = \frac{1}{8}\sigma(a_1,\cdots,a_n). \quad (0.1)$$

The key objects in this paper will be the so called Dedekind-Rademacher sums defined for every coprime positive integers $h, k$ and any real numbers $x, y$ by

$$s(h,k;x,y) = \sum_{\mu=0}^{k-1} \left( \frac{\mu + y}{k} \right) \left( \frac{h(\mu + y)}{k} + x \right)$$

where for any $r \in \mathbb{R}$ we set

$$((r)) = \begin{cases} 0 & r \in \mathbb{Z} \\ \{q\} - \frac{1}{2} & r \in \mathbb{R} \setminus \mathbb{Z} \end{cases} \quad \{r\} := r - [r].$$

Despite their apparent complexity these sums are computationally very friendly due mainly to the reciprocity law they satisfy (see [14] or the Appendix). The sums
s(h, k; 0, 0) are precisely the Dedekind sums s(h, k) discussed in great detail in [1] or [15].

Our proof of (0.1) is based on the following facts.
• According to Zagier (see [1] or [11]) the signature σ(a_1, ⋯, a_n) can be expressed in terms of Dedekind sums.
• According to the computations in [13] the quantity \( \frac{1}{2}η_{dir}(g_0) + \frac{1}{8}η_{sign}(g_0) \) can be expressed in terms of Dedekind-Rademacher sums.
• For \( n = 3, 4 \) the number \( C_{a_1, ⋯, a_n} \) can be expressed in terms of Dedekind-Rademacher sums.

As an arithmetic byproduct of the proof we obtain a divisibility result for certain expressions involving Dedekind-Rademacher sums (see Remark 1.2).

The present paper consists of three sections and an appendix. In the first section we use the results of [13] to express \( F \) in terms of Dedekind-Rademacher sums and to reduce the computation of \( χ_{SW} \) to a lattice point count. In the next section we describe a variation of the Mordell trick which reduces the lattice point count to a certain arithmetic expression. The third section describes this arithmetic expression in terms of Dedekind-Rademacher sums and completes the proof of (1.13). For the reader’s convenience we have included a brief appendix containing the basic properties of Dedekind-Rademacher sums used in this paper.

Note After this paper was completed we found out that the Kronheimer-Mrowka conjecture was proved independently and in its entire generality by Lim and Carey-Marcolli-Wang. We believe the nice arithmetic behind the Brieskorn homology spheres is interesting enough to warrant a separate treatment.

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1 Geometric preliminaries

For pairwise coprime integers $a_1, \cdots, a_n \geq 2$, $n \geq 3$ we denote by $\Sigma(\vec{a})$ the Brieskorn homology sphere $\Sigma(a_1, \cdots, a_n)$ with $n$ singular fibers (see [3] for a precise definition). We orient $\Sigma(\vec{a})$ as the boundary of a complex manifold. $\Sigma(\vec{a})$ is a Seifert manifold. With respect to the above orientation it is a singular $S^1$ fibration over the orbi-sphere $S^2(\vec{a})$ which has $n$ cone points of isotropies $\mathbb{Z}_{a_i}$, $1 \leq i \leq n$. This fibration has rational degree

$$\ell = -\frac{1}{A}, \quad A := a_1 a_2 \cdots a_n.$$ 

Set $b_i := A/a_i$. The Seifert invariants $\vec{\beta} = (\beta_1, \cdots, \beta_n)$ (normalized as in [13]) are defined by

$$\beta_i b_i \equiv -1 \pmod{a_i}, \quad 0 \leq \beta_i < a_i$$

Set $q_i = \beta_i^{-1} = -b_i \mod a_i$, $i = 1, \ldots, n$. The canonical line bundle of $S^2(\vec{a})$ has rational degree

$$\kappa = (n-2) - \sum_{i=1}^{n} \frac{1}{a_i}.$$ 

The universal covering space of $\Sigma(\vec{a})$ is a Lie group $G = G(\vec{a})$ determined by

$$G = \begin{cases} 
SU(2) & \kappa < 0 \\
\tilde{PSL}_2(\mathbb{R}) & \kappa > 0 
\end{cases}$$

where $\tilde{PSL}_2(\mathbb{R})$ denotes the universal cover of $PSL_2(\mathbb{R})$. Moreover $\Sigma(\vec{a}) \cong \Gamma/G(\vec{a})$ where $\Gamma$ is a discrete subgroup of $G$. The natural left invariant metrics on $G$ (see [17]) induce a natural metric $g_0$ on $\Sigma(\vec{a})$. All the geometric quantities discussed in the sequel are computed with respect to this metric and for simplicity we will omit $g_0$ from the various notations. Thus $F(\vec{a})$ is $F(g_0)$.

Set

$$\rho := \left\{ \begin{array}{ll}
\kappa / 2\ell \\
\frac{1}{2} & A \text{ even} \\
0 & A \text{ odd} 
\end{array} \right.$$ 

and define $\vec{\gamma} = (\gamma_1, \cdots, \gamma_n)$ by the equalities

$$\gamma_i = m\beta_i \pmod{a_i}, \quad 1 \leq i \leq n$$

where $m$ is the integer

$$m := \frac{\kappa - \rho}{2\ell} - \rho = \frac{u - A - 2\rho}{2}, \quad u := \sum_{i} b_i.$$ 

In [13] we have proved the following.

- If $A$ is even then

$$F(\vec{a}) = 1 - 4 \sum_{i} s(\beta_i, a_i) - 4 \sum_{i} \left( \left( \frac{q_i \gamma_i + \rho}{a_i} \right) + 2s(\beta_i, a_i; \gamma_i + \beta_i \rho, -\rho) \right) \quad (1.2)$$
The above expression can be further simplified using the identities

\[ s(\beta_i, a_i) = -s(b_i, a_i) \]  \hspace{1cm} (1.3)

\[ s(\beta_i, a_i; \gamma_i + \beta_i/2, -1/2) = -s(b_i, a_i, 1/2, 1/2) - \frac{1}{2} \left( \left( \frac{\gamma_i + 1/2}{a_i} \right) \right) \]  \hspace{1cm} (1.4)

The identity (1.3) is elementary and can be safely left to the reader. The identity (1.4) is proved in the Appendix. Putting the above together we deduce that when \( A \) is even we have

\[ F(\vec{a}) = 1 + 4 \sum_i s(b_i, a_i) + 8 \sum_i s(b_i, a_i; 1/2, 1/2). \]  \hspace{1cm} (1.5)

- If \( A \) is odd then

\[ F(\vec{a}) = 1 - \frac{1}{A} - 4 \sum_i s(\beta_i, a_i) - 4 \sum_{i=1}^n \left( 2s(\beta_i, a_i; \gamma_i + \beta_i, -\rho) + \left( \left( \frac{\gamma_i + \rho}{a_i} \right) \right) \right). \]  \hspace{1cm} (1.6)

Similarly, we have an identity

\[ s(\beta_i, a_i; \gamma_i, 0) + \frac{1}{2} \left( \left( \frac{\gamma_i}{a_i} \right) \right) = -s(b_i, a_i; 1/2, 1/2) \]  \hspace{1cm} (1.7)

and we deduce

\[ F(\vec{a}) = 1 - \frac{1}{A} + 4 \sum_i s(b_i, a_i) + 8 \sum_i s(b_i, a_i; 1/2, 1/2). \]  \hspace{1cm} (1.8)

The signature \( \sigma(\vec{a}) \) of the Milnor fiber of \( \Sigma(\vec{a}) \) can be expressed in terms of Dedekind sums as well (see [11], Sect.1) and we have

\[ \sigma(\vec{a}) = -1 - \frac{(n - 2)A}{3} + \frac{1}{3A} + \frac{1}{3} \sum_i b_i a_i - 4 \sum_i s(b_i, a_i). \]  \hspace{1cm} (1.9)

We deduce that

\[ F(\vec{a}) + \sigma(\vec{a}) = -\frac{(n - 2)A}{3} + \frac{\varepsilon}{3A} + \frac{1}{3} \sum_i b_i a_i + 8 \sum_i s(b_i, a_i; 1/2, 1/2) \]  \hspace{1cm} (1.10)

where

\[ \varepsilon = \begin{cases} 1 & \text{if } A \text{ even} \\ -2 & \text{if } A \text{ odd} \end{cases} \]

The description of \( \chi_{SW} \) requires a bit more work. Introduce the simplex

\[ \Delta(\vec{a}) = \{ \vec{x} \in \mathbb{Z}^n ; x_i \geq 0, \sum_i \frac{x_i}{a_i} < \kappa/2 \}. \]
For each \( \vec{x} \in \Delta(\vec{a}) \) set
\[
d(\vec{x}) = \sum_i \left[ \frac{x_i}{a_i} \right]
\]
and \( S_{\vec{x}} \) = symmetric product of \( d(\vec{x}) \) copies of \( S^2 \). Note that if \( n = 3, 4 \) then \( d(\vec{x}) = 0 \) for all \( \vec{x} \in \Delta(\vec{a}) \).

The irreducible part of the adiabatic Seibert-Witten equations on \( \Sigma(\vec{a}) \) (studied in \cite{10} and \cite{12}) can be described as
\[
\mathcal{M}_{\vec{a}} = \bigcup_{\vec{x}} \mathcal{M}_{\vec{x}}
\]
where
\[
\mathcal{M}_{\vec{x}} = \mathcal{M}_{\vec{x}}^+ \cup \mathcal{M}_{\vec{x}}^- \quad \text{and} \quad \mathcal{M}_{\vec{x}}^\pm \cong S_{\vec{x}}.
\]
Moreover, the virtual dimensions of the spaces of finite energy gradient flows originating at the unique reducible solution and ending at one of the \( \mathcal{M}_{\vec{x}} \) are all odd. Using the adiabatic argument in §3.3 of \cite{13} we deduce that if all \( d(\vec{x}) \) are zero the Seiberg-Witten-Floer homology obtained using the usual Seiberg-Witten equations is isomorphic with the Seiberg-Witten-Floer homology obtained using the adiabatic equations. Moreover, all the even dimensional Betti numbers of the Seiberg-Witten-Floer homology are zero and we deduce
\[
\chi_{SW}(\vec{a}) = -2C_{\vec{a}} := -2\#\Delta(\vec{a}). \tag{1.11}
\]

In 1996 Kronheimer and Mrowka conjectured that
\[
\chi_{SW}(N, g) - \frac{1}{8}\mathcal{F}(N, g) = \lambda(N)
\]
for any homology sphere \( N \) and any generic metric \( g \) on \( N \). Above, \( \lambda(N) \) denotes the Casson invariant of \( N \). It was shown in \cite{11} that for the Brieskorn homology spheres \( \Sigma(\vec{a}) \)
\[
\lambda(\Sigma(\vec{a})) = \frac{1}{8}\sigma(\vec{a}). \tag{1.12}
\]

The main result of this paper is the following.

**Theorem 1.1** The Kronheimer-Mrowka conjecture is true for Brieskorn homology spheres \( \Sigma(a_1, \cdots, a_n) \), \( n = 3, 4 \). According to (1.11), (1.12) and (1.10) this is equivalent to
\[
-16C_{\vec{a}} = \mathcal{F}(\vec{a}) + \sigma(\vec{a})
\]
\[
= -\frac{(n-2)A}{3} + \frac{\varepsilon}{3A} + \frac{1}{3} \sum_i b_i a_i + 8 \sum_i s(b_i, a_i; 1/2, 1/2). \tag{1.13}
\]

**Remark 1.2** As indicated in \cite{13}, Rohlin’s theorem implies that the term \( \mathcal{F}(\vec{a}) \) is divisible by 8. The results of \cite{11} show the signature \( \sigma(\vec{a}) \) is also divisible by 8. Thus the right-hand-side of (1.10) is an integer divisible by 8. The above theorem shows that \( \mathcal{F}(\vec{a}) + \sigma(\vec{a}) \) is in effect divisible by 16!!!
2 The Mordell trick

Let $\vec{a} \in \mathbb{Z}^n$ be as in the previous section. Denote by $\mathcal{P} = \mathcal{P}_\vec{a}$ the parallelopiped

$$\mathcal{P} := ([0, a_1 - 1] \times \cdots \times [0, a_n - 1]) \cap \mathbb{Z}^n.$$  

When $n = 3$ we will use the notation $\vec{a} = (a, b, c)$. Define $q : \mathcal{P} \to \mathbb{R}$ by

$$q(\vec{x}) = \sum_i \frac{x_i + 1/2}{a_i} = \sum_i \frac{x_i}{a_i} + \frac{u}{2A}.$$  

Remark 2.1 (a) Suppose $n = 3$, $\vec{a} = (a, b, c)$. Note that $q(p) \in \frac{1}{2}\mathbb{Z}$ for some $p \in \mathcal{P}$ if and only if $abc$ is odd and

$$p = p_0 = (\frac{a - 1}{2}, \frac{b - 1}{2}, \frac{c - 1}{2}).$$  

In this case $q(p_0) = \frac{3}{2}$.

(b) Suppose $n = 4$, $\vec{a} = (a_1, \cdots, a_4)$. Then $q(p) \in \mathbb{Z}$ for some $p \in \mathcal{P}$ if and only if $A$ is odd and

$$p = p_0 = (\frac{a_1 - 1}{2}, \cdots, \frac{a_4 - 1}{2})$$  

in which case $q(p_0) = 2$.

For every interval $I \subset \mathbb{R}$ we put

$$N_I := \#q^{-1}(I).$$  

Note that

$$C_\vec{a} = N_{(0,(n-2)/2)}.$$  

In particular, if $n = 3$

$$C_{a,b,c} = N_{(0,1/2)}.$$  

while if $n = 4$

$$C_{a_1,\cdots,a_4} = N_{(0,1)}.$$  

For every $r \in \mathbb{R}$ define $\|r\| = [r + 1/2]$ where $[\cdot]$ denotes the integer part function. Note that $\|r\|$ is the integer closest to $r$. We now discuss separately the two cases, $n = 3$ and $n = 4$

- The case $n = 3$, $\vec{a} = (a, b, c)$. Inimitating Mordell (see [9] or [15]) we introduce the quantity

$$\alpha := \sum_{I} (\|q\| - 1)(\|q\| - 2).$$  

Observe that

$$\alpha = 2N_{[0,1/2]} + 2N_{[5/2,3]} = 2N_{(0,1/2)} + 2N_{(5/2,3)}.$$  

The importance of the last equality follows from the following elementary result.
Lemma 2.2

\[ N_{(0,1/2)} = N_{(5/2,3)} \]

**Proof**  Consider the involution

\[ \omega : \mathcal{P} \rightarrow \mathcal{P}, \quad (x,y,z) \mapsto (a-1-x, b-1-y, c-1-z). \]

It has the property

\[ q(\omega(p)) = 3 - q(p) \]

from which the lemma follows immediately. \( \square \)

Using the lemma and the equalities (2.1), (2.3) we deduce

\[ 4C_{a,b,c} = \sum_{P} (\|q\| - 1)(\|q\| - 2). \quad (2.4) \]

- **The case** \( n = 4, \vec{a} = (a_1, \cdots, a_4) \). Arguing exactly as above we deduce

\[ 4C_{\vec{a}} = N_{(0,1)} = \sum_{P} ([q] - 1)([q] - 2). \quad (2.5) \]

The proof of Theorem 1.1 will be completed by providing an expression for the above sums in terms of Dedekind-Rademacher sums. This will be achieved in the next section following the strategy of [9] (see also [15]).

3 The proof of Theorem 1.1

We will consider separately the two cases \( n = 3 \) and \( n = 4 \).

§3.1  **The case** \( n = 3 \)  Set \( \vec{a} = (a, b, c) \) so that \( A = abc, u = ab + bc + ca \). We will distinguish two cases: \( A \) is even and \( A \) is odd.

- **A is even**  In this case \( q(p) + \frac{1}{2} \notin \mathbb{Z} \) so that

\[ \|q\| = q + 1/2 - \{q + 1/2\} = q - ((q + 1/2)). \]

The sum can be rewritten as

\[
\sum_{P} q - ((q + 1/2)) - 1)(q - ((q + 1/2)) - 2) = \sum_{P} q^2 - 3 \sum_{P} q + 2 \sum_{P} 1 - 2 \sum_{P} q((q + 1/2)) + \sum_{P} ((q + 1/2))^2 + 3 \sum_{P} ((q + 1/2)).
\]
We compute each of these 6 sums separately.

\[
\sum_{p} 1 = \#\mathcal{P} = abc.
\]

\[
\sum_{p} q = \sum_{p} \frac{x + 1/2}{a} + \sum_{p} \frac{y + 1/2}{b} + \sum_{p} \frac{z + 1/2}{c}
\]

\[
= \frac{bc}{2a} \sum_{x=0}^{a-1} (2x + 1) + \frac{ca}{2b} \sum_{y=0}^{b-1} (2y + 1) + \frac{ab}{2c} \sum_{z=0}^{c-1} (2z + 1)
\]

\[
= \frac{3}{2} abc.
\]

\[
\sum_{p} q^2 = \sum_{p} \left( \frac{x + 1/2}{a} \right)^2 + \sum_{p} \left( \frac{y + 1/2}{b} \right)^2 + \sum_{p} \left( \frac{z + 1/2}{c} \right)^2
\]

\[
+2c \left( \sum_{x=0}^{a-1} \frac{x + 1/2}{a} \right) \left( \sum_{y=0}^{b-1} \frac{y + 1/2}{b} \right) + 2b \left( \sum_{x=0}^{a-1} \frac{x + 1/2}{a} \right) \left( \sum_{z=0}^{c-1} \frac{z + 1/2}{c} \right)
\]

\[
+2a \left( \sum_{z=0}^{c-1} \frac{z + 1/2}{c} \right) \left( \sum_{y=0}^{b-1} \frac{y + 1/2}{b} \right)
\]

Using basic properties of Bernoulli polynomials (see [16]) we deduce

\[
\sum_{x=0}^{a-1} \left( \frac{x + 1/2}{a} \right)^2 = \frac{1}{3a^2} \left( B_3(a + \frac{1}{2}) - B_3(\frac{1}{2}) \right)
\]

where

\[
B_3(t) = \frac{t(2t-1)(t-1)}{2}
\]

is the third Bernoulli polynomial. Note that \( B_3(1/2) = 0 \) and

\[ B_3(t + 1/2) = t(t^2 - 1/4). \]

Using the identity

\[
\frac{1}{n} \sum_{k=0}^{n-1} k + 1/2 = \frac{n}{2}
\]

we conclude

\[
\sum_{p} q^2 = \frac{5}{2} abc - \frac{1}{12} \left( \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right).
\]

Next

\[
\sum_{p} ((q + 1/2)) = \sum_{p} \left( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u}{2abc} \right)
\]
\[
= \sum_{k=0}^{abc-1} \left( \left( \frac{k}{abc} + \frac{u}{2abc} \right) \right)
\]

According to the Kubert identity [A.4] in the Appendix the last sum is equal to \( ((u/2)) \) which is zero. Thus
\[
\sum_{p} (\left( q + \frac{1}{2} \right)) = 0.
\]

The sum \( \sum q((q + 1)/2) \) requires a bit more work. Note first that it decomposes as
\[
\sum_{x=0}^{a-1} \frac{x + 1/2}{a} \sum_{y,z} \left( \left( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u + abc}{2abc} \right) \right)
\]
\[
+ \sum_{y=0}^{b-1} \frac{y + 1/2}{b} \sum_{z,x} \left( \left( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u + abc}{2abc} \right) \right)
\]
\[
+ \sum_{z=0}^{c-1} \frac{z + 1/2}{c} \sum_{x,y} \left( \left( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u + abc}{2abc} \right) \right)
\]
\[
= S_1 + S_2 + S_3.
\]

We describe in detail the computation of \( S_1 \). The other two sums are entirely similar.

Note first that
\[
\sum_{y,z} \left( \left( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u + abc}{2abc} \right) \right) = \sum_{y,z} \left( \left( \frac{y}{b} + \frac{z}{c} + \frac{x}{a} + \frac{u + abc}{2abc} \right) \right)
\]
\[
= \sum_{k=0}^{bc-1} \left( \left( \frac{k}{bc} + \frac{x}{a} + \frac{u + abc}{2abc} \right) \right)
\]

(use the Kubert identity [A.4])
\[
= \left( \left( \frac{bcx}{a} + \frac{u + abc}{2a} \right) \right).
\]
\[
= \left( \left( \frac{bc(x+1/2)}{a} + \frac{bc + b + c}{2} \right) \right) = \left( \left( \frac{bc(x+1/2)}{a} + \frac{1}{2} \right) \right).
\]

We conclude
\[
S_1 = \sum_{x=0}^{a-1} \left( \left( \frac{bc(x+1/2)}{a} + \frac{1}{2} \right) \right)
\]
\[
= \sum_{x=0}^{a-1} \left( \left( \frac{x+1/2}{a} \right) \right) \left( \left( \frac{bc(x+1/2)}{a} + \frac{1}{2} \right) \right) - \frac{1}{2} \sum_{x=0}^{a-1} \left( \left( \frac{bc(x+1/2)}{a} + \frac{1}{2} \right) \right)
\]
(use the Kubert identity)

\[
\sum_{x=0}^{a-1} \left( \left( \frac{x + 1/2}{a} \right) \left( \frac{bc(x + 1/2)}{a} + \frac{1}{2} \right) \right) = s(bc, a; 1/2, 1/2).
\]

Hence

\[
\sum q((q + 1/2)) = s(bc, a; 1/2, 1/2) + s(ca, b; 1/2, 1/2) + s(ab, c; 1/2, 1/2).
\]

Finally

\[
\sum \left( (q + 1/2)^2 \right) = \sum \left( \left( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u + abc}{2abc} \right) \right)^2 = \sum_{k=0}^{abc-1} \left( k + \frac{u + abc}{2abc} \right)^2
\]

( use the fact that \( u + abc \) is odd in this case)

\[
= \sum_{k=0}^{abc-1} \left( \frac{k + 1/2}{abc} \right)^2 = s(1, abc; 0, 1/2)
\]

( use (A.1)

\[
= \frac{abc}{12} - \frac{1}{12abc}.
\]

Putting together the above information we deduce that if \( abc \) is even then

\[
4C_{a,b,c} = \frac{abc}{12} - \frac{1}{12abc} - \frac{1}{12} \left( \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right)
\]

\[
- 2(s(bc, a; 1/2, 1/2) + s(ca, b; 1/2, 1/2) + s(ab, c; 1/2, 1/2)).
\]

The identity (1.13) is now obvious.

- \( A \) is odd In this case, using Remark 2.1 we deduce

\[
\|q(p)\| = \begin{cases} 
q - ((q + 1/2)) & p \neq p_0 \\
q - (q + 1/2) + \frac{1}{2} & p = p_0
\end{cases}.
\]

Thus

\[
(\|q\| - 1)(\|q\| - 2) = \begin{cases} 
(q - ((q + 1/2)) - 1)(q - ((q + 1/2)) - 2) & p \neq p_0 \\
(q - ((q + 1/2)) - 1/2)(q - ((q + 1/2)) - 3/2) & p = p_0
\end{cases}
\]

Hence

\[
4C_{a,b,c} = \sum_{p} (q - ((q + 1/2)) - 1)(q - ((q + 1/2)) - 2)
\]
\[+ (q - ((q + \frac{1}{2})) - \frac{1}{2})(q - ((q + \frac{1}{2})) - \frac{3}{2})|_{p_0} -(q - ((q + \frac{1}{2})) - 1)(q - ((q + \frac{1}{2})) - 2)|_{p_0}\]
\[= \sum_{p} (q - ((q + 1/2)) - 1)(q - ((q + 1/2)) - 2) + \frac{1}{4}. \tag{3.2}\]

The above sum can be computed exactly as in the even case with one notable difference namely
\[
\sum((q + 1/2))^2 = \sum_{k=0}^{abc-1} \left( \left( \frac{k}{abc} \right) \right)
\]
(u + abc is even)
\[
= \sum_{k=0}^{abc-1} \left( \left( \frac{k}{abc} \right) \right) = s(1, abc; 0, 0) = \frac{abc}{12} + \frac{1}{6abc} - \frac{1}{4}.
\]

Thus, when abc is odd we have
\[
4C_{a,b,c} = \frac{abc}{12} + \frac{1}{6abc} - \frac{1}{12} \left( \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right) - 2(s(bc, a; 1/2, 1/2) + s(ca, b; 1/2, 1/2) + s(ab, c; 1/2, 1/2)).
\]

This completes the proof of Theorem 1.1 when \(n = 3\)

§3.2 The case \(n = 4\)  We follow a similar strategy with some obvious modifications. Set \(\vec{a} = (a_1, \cdots, a_4)\), \(A = 4\), \(u = b_1 + \cdots + b_4\) and
\[S_{\vec{a}} = \sum_{p_\vec{a}} ([q] - 1)([q] - 2).\]

As in the previous subsection will distinguish two situations.

• A is even  Note that for every \(p \in P\) we have \(q(p) \notin \mathbb{Z}\) so that
\[[q] = q - ((q)) - 1/2.\]

Thus
\[S_{\vec{a}} = \sum_{p} (q - ((q)) - 3/2)(q - ((q)) - 5/2) = \sum_{q} (q^2 - 4q + 15/4) - 2 \sum_{p} q((q)) + \sum_{p} ((q))^2 + 4 \sum_{p} ((q)).\]

The computation of the above terms follows the same pattern as in the previous subsection.
\[\sum_{p} ((q)) = 0.\]
\[
\sum_p 15/4 = 15\#P/4 = 15A/4.
\]

\[
\sum_p q = \sum_{i=1}^4 b_i \sum_{x_i=0}^{a_i-1} \frac{x_i + 1/2}{a_i} = \sum_{i=1}^4 b_i a_i = 2A.
\]

\[
\sum_p q^2 = \sum_{i=1}^4 b_i \sum_{x_i=0}^{a_i-1} \left(\frac{x_i + 1/2}{a_i}\right)^2
+ 2 \sum_{i<j} A \frac{a_i}{a_ia_j} \left(\sum_{x_i=0}^{a_i-1} \frac{x_i + 1/2}{a_i}\right) \left(\sum_{x_j=0}^{a_j-1} \frac{x_j + 1/2}{a_j}\right)
= \sum_{i=1}^4 b_i a_i (a_i^2 - 1/4) + \sum_{1 \leq i < j \leq 4} \frac{A}{2}
= \sum_{i=1}^4 \left(\frac{A}{3} - \frac{b_i}{12a_i}\right) + 3A = \frac{13A}{3} - \frac{1}{12} \sum_i b_i / a_i.
\]

When \(A\) is even \(u\) is odd and we have
\[
\sum_p ((q))^2 = \sum_{k=0}^{A-1} \left(\frac{k + u/2}{A}\right)^2 = s(1, A; 0, 1/2) = \frac{A}{12} + \frac{1}{2A}.
\]

Finally,
\[
\sum_p q((q)) = S_1 + \cdots + S_4
\]
where
\[
S_1 = \sum_{x_1=0}^{a_1-1} \frac{x_1 + 1/2}{a_1} \cdot \sum_{x_2,x_3,x_4} \left(\frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_1}{a_1} + \frac{u}{2A}\right).
\]

\(S_2, S_3, S_4\) are defined similarly. To compute \(S_1\) note that
\[
\sum_{x_2,x_3,x_4} \left(\frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_1}{a_1} + \frac{u}{2A}\right)
= \sum_{k=0}^{b_1-1} \left(\frac{k + x_1}{b_1} + \frac{u}{2A}\right)
= \left(\frac{b_1 x_1}{a_1} + \frac{u}{2A}\right) = \left(\frac{b_1(x_1 + 1/2)}{a_1} + \frac{u - b_1}{2a_1}\right)
\]
\((u - b_1)/a_1\) is odd
\[
= \left(\frac{b_1(x_1 + 1/2)}{a_1} + \frac{1}{2}\right).
\]
Thus

\[ S_1 = \sum_{x_1} a_1 - 1 \cdot \frac{x_1 + 1/2}{a_1} \left( \left( \frac{b_1(x_1 + 1/2)}{a_1} + \frac{1}{2} \right) \right) \]

and we deduce as in the previous subsection that

\[ S_1 = s(b_1, a_1; 1/2, 1/2). \]

By adding all the above together we deduce that if \( A \) is odd then

\[ 4C \vec{a} = S \vec{a} = \frac{A}{6} - \frac{1}{12A} - \frac{1}{12} \sum_i b_i a_i - 8 \sum_i s(b_i, a_1; 1/2, 1/2). \]

The identity (1.13) is now obvious.

**A is odd** In this case \( u \) is even. Arguing as in the previous subsection we deduce

\[ S \vec{a} = \sum_p (q - ((q)) - 3/2)(q - ((q)) - 5/2) + \frac{1}{4}. \]

The only term in the previous computations which is influenced by the parity of \( A \) is

\[ \sum_p ((q))^2 = \sum_{k=0}^{A-1} \left( \frac{k + u/2}{A} \right)^2 = s(1, A) \]

\[ = \frac{A}{12} + \frac{1}{6A} - \frac{1}{4}. \]

Putting together all the terms we obtain again the identity (1.13). The Theorem 1.1 is proved. \( \square \)

A Basic facts concerning Dedekind-Rademacher sums

In [14] Rademacher consider for every pair of coprime integers \( h, k \) and any real numbers \( x, y \) the following generalization of the classical Dedekind sums

\[ s(h, k; x, y) = \sum_{\mu=0}^{k-1} \left( \frac{\mu + y}{k} \right) \left( \frac{h(\mu + y)}{k} + x \right). \]

A simple computations shows that \( s(h, k; x, y) \) depends only on \( x, y \) mod 1. When \( h = 1 \) and \( x = 0 \) one can prove (see [14])

\[ s(1, k; 0, y) = \begin{cases} \frac{k}{12} + \frac{1}{6k} - \frac{1}{4} & y \in \mathbb{Z} \\ \frac{k}{12} + \frac{1}{k} B_2(\{y\}) & y \in \mathbb{R} \setminus \mathbb{Z} \end{cases} \quad (A.1) \]

where \( B_2(t) = t^2 - t + 1/6 \) is the second Bernoulli polynomial.
Perhaps the most important property of these Dedekind-Rademacher sums is their reciprocity law which makes them computationally very friendly. To formulate it we must distinguish two cases.

• Both \(x\) and \(y\) are integers. Then

\[
s(\beta, \alpha; x, y) + s(\alpha, \beta; y, x) = -\frac{1}{4} + \frac{\alpha^2 + \beta^2 + 1}{12\alpha\beta}.
\]

(A.2)

• \(x\) and/or \(y\) is not an integer. Then

\[
s(\beta, \alpha; x, y) + s(\alpha, \beta; y, x) = ((x)) \cdot ((y)) + \frac{\beta^2\psi_2(y) + \psi_2(\beta y + \alpha x) + \alpha^2\psi_2(x)}{2\alpha\beta}.
\]

(A.3)

where \(\psi_2(x) := B_2(\{x\})\).

An important ingredient behind the reciprocity law is the following identity (Lemma 1 in [14])

\[
\sum_{\mu=0}^{k-1} \left( \frac{\mu + w}{k} \right) = (w) \quad \forall w \in \mathbb{R}.
\]

(A.4)

Following the terminology in [8] we will call the above equality the Kubert identity.

We conclude with a proof of the identity (1.4). For simplicity we consider only the case \(n = 3\) and \(i = 1\). Set \(\vec{a} = (a, b, c)\). Thus \(A = \text{even}\), \(u = bc + ca + ab\) and \(b_1 = bc\). For arbitrary \(n\) the proof is only notationally more complicated.

The proof of (1.4) goes as follows.

\[
s(\beta_1, a; \frac{\gamma_1 + \beta_1/2}{a}, -1/2) = \sum_{x=0}^{a-1} \left( \frac{x - 1/2}{a} \right) \left( \frac{\beta_1 x + \gamma_1}{\alpha_1} \right)
\]

\((\gamma_1 = \beta_1(u - abc - 1)/2 \text{ mod } a)\)

\[
= \sum_{x=0}^{a-1} \left( \frac{x - 1/2}{a} \right) \left( \frac{\beta_1(x - abc/u + 1)}{2} \right)
\]

\((y := x - \frac{abc-u+1}{2} \text{ mod } a)\)

\[
= \sum_{y=0}^{a-1} \left( \frac{y + abc - u + 1}{2} - 1/2 \right) \left( \frac{\beta_1 y}{a} \right)
\]

( use \(z = -bcy \text{ mod } a\) and \(\beta_1 bc \equiv 1 \text{ mod } a_1\))

\[
= -\sum_{z=0}^{a-1} \left( \frac{bcz - abc - u}{2} \right) \left( \frac{z}{a} \right)
\]

\[
= -\sum_{z=0}^{a-1} \left( \frac{bc(z + 1/2) + b + c - bc}{2} \right) \left( \frac{z}{a} \right)
\]

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\[= - \sum_{z=0}^{a-1} \left( \left( \frac{bc(z + 1/2)}{a} + \frac{1}{2} \right) \left( \frac{z}{a} \right) \right).\]

At this point we use the elementary identity
\[
\left( \left( \frac{z}{a} \right) \right) = \left( \left( \frac{z + 1/2}{a} \right) \right) - \frac{1}{2a} + \frac{1}{2} \delta(z)
\]

where
\[
\delta(z) = \begin{cases} 
1 & z \equiv 0 \ (	ext{mod} \ a) \\
0 & \text{otherwise}
\end{cases}
\]

We deduce
\[
s(\beta_1, \alpha_1; \frac{\gamma_1 + \beta_1/2}{\alpha_1}, -1/2) = - \sum_{z=0}^{a-1} \left( \left( \frac{bc(z + 1/2)}{a} + \frac{1}{2} \right) \left( \frac{z + 1/2}{a} \right) \right)
\]
\[+ \frac{1}{2a} \sum_{z=0}^{a-1} \left( \left( \frac{bc(z + 1/2)}{a} + \frac{1}{2} \right) \right) - \frac{1}{2} \left( \left( \frac{bc}{2a} + \frac{1}{2} \right) \right).
\]

The Kubert identity shows that the second sum above vanishes. Also
\[
\left( \left( \frac{q_1 \gamma_1 + 1/2}{\alpha_1} \right) \right) = \left( \left( \frac{\alpha_1 - abc}{2a} \right) \right) = \left( \left( \frac{b + c - bc}{2} + \frac{bc}{2a} \right) \right)
\]
\[= \left( \left( \frac{bc}{2a} + \frac{1}{2} \right) \right).\]

The identity (1.4) is proved. The proof of (1.7) is similar and is left to the reader.

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