TORIC IDEALS OF CHARACTERISTIC IMSETS VIA QUASI-INDEPENDENCE GLUING

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ABSTRACT. Characteristic imsets are 0-1 vectors which correspond to Markov equivalence classes of directed acyclic graphs. The study of their convex hull, named the characteristic imset polytope, has led to new and interesting geometric perspectives on the important problem of causal discovery. In this paper we begin the study of the associated toric ideal. We develop a new generalization of the toric fiber product, which we call a quasi-independence gluing, and show that under certain combinatorial homogeneity conditions, one can iteratively compute a Gröbner basis via lifting. For faces of the characteristic imset polytope associated to trees, we apply this technique to compute a Gröbner basis for the associated toric ideal. We end with a study of the characteristic ideal of the cycle and propose directions for future work.

1. Introduction

Given a directed acyclic graph (DAG) $\mathcal{G} = ([n], E)$ with vertices $[n] := \{1, \ldots, n\}$ and edges $E$ and a collection of jointly distributed random variables $(X_1, \ldots, X_n)$ with probability density function $f(x_1, \ldots, x_n)$, we say that $f(x_1, \ldots, x_n)$ is Markov to the DAG $\mathcal{G}$ if

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i | x_{\text{pa}_G(i)})$$

where $\text{pa}_G(i) = \{ j \in [n] : j \to i \in E \}$ is the set of parents of $i$ in $\mathcal{G}$.

Directed acyclic graphical models play a fundamental role in modern data science and artificial intelligence through their applications in probabilistic inference [12] and causality [17]. In such fields, the combinatorics of the graph can be used to determine complexity bounds for probabilistic inference algorithms [12, Chapter 9] and estimators for the causal effect of one variable in the system on another [17, Chapter 3]. Given these applications, it is valuable to have a DAG representation of a data-generating distribution, especially if such a representation can be learned directly from data. Unfortunately, many different DAGs may encode the same set of conditional independence (CI) statements [2, 30, 31]. Two such DAGs which encode the same set of CI statements are called Markov equivalent and so the main goal is to recover the Markov equivalence class of the DAG which generated the data. This is the basic problem of causal discovery. A wide variety of methods for doing so have been proposed and are primarily based on the combinatorics of DAGs [4, 13, 14, 20, 21, 29].

More recently, efforts were made to replace the problem of causal discovery as a linear program by embedding DAGs as 0/1-vectors in real-Euclidean space called characteristic imsets [24]. The characteristic imset vector of a DAG uniquely encodes its Markov equivalence class. The convex hull of all such vectors for DAGs with $n$ vertices is called the characteristic imset polytope CIM$_n$. Although the linear program approach that motivated the definition of CIM$_n$ remains infeasible for even reasonably small numbers of variables, due to the fact that there is no complete characterization of the facets of CIM$_n$, the geometry of CIM$_n$ appears to play a fundamental role in the general problem of causal discovery. For instance, it was recently show that the moves used by
popular search-based optimization causal discovery algorithms all correspond to edges of CIM$_n$; hence, such methods amount to walks along the edge graph of CIM$_n$ [14]. In [14], it was also shown that the polytope CIM$_G$, defined as the convex hull of all characteristic imsets $c_G$ where $G$ has adjacencies corresponding to the edges in the undirected graph $G$, is a face of CIM$_n$. In [15], the edges of CIM$_G$ for $G$ a tree are completely characterized, and it is shown that algorithms using these edges to search for an optimal Markov equivalence class with skeleton $G$ outperform classic causal discovery methods for learning directed trees.

While significant work has already been done on the polyhedral structure of CIM$_G$, there has been little attention paid to its toric geometry. There is a strong and well-established link between the algebraic properties of toric ideals and the combinatorial properties of their associated polytopes [25]. In this paper, we begin the study of the toric ideal $I_G$ which is naturally associated to CIM$_G$ with the hope that it can be used to better understand CIM$_G$. We begin by introducing a new generalization of the toric fiber product of [28] which is defined as follows.

**Definition 1.1** (Definition 3.2). Let $Q \subset [r] \times [s]$ and $I \subset \mathbb{K}[x_j \mid j \in [r]]$ and $J \subset \mathbb{K}[y_k \mid k \in [s]]$ be homogeneous ideals. The **quasi-independence gluing** of $I$ and $J$ with respect to $Q$ is

$$I \times_Q J := \phi^{-1}_Q(I + J)$$

where $\phi_Q$ is the map

$$\phi_Q : \mathbb{K}[z_{jk} \mid (j,k) \in Q] \rightarrow \mathbb{K}[x_j, y_k \mid j \in [r], k \in [s]]$$

$$z_{jk} \mapsto x_j y_k.$$

Similar to the toric fiber product, we show that the Gröbner basis of $I \times_Q J$ can be computed by lifting polynomials from the Gröbner bases of ideals $I$ and $J$ and adding in extra binomials which come from the monomial map $\phi_Q$. The Gröbner bases of the ideals $I_Q = \ker(\phi_Q)$ are completely known and have been studied in algebraic statistics [3, 5] and commutative algebra as the **toric ideals of edge rings** of bipartite graphs [8, 11, 18, 33]. Unlike the toric fiber product though, we allow for $\phi_Q$ to be any monomial homomorphism which arises as the parameterization of a **quasi-independence model** and do not require that $I$ and $J$ are homogeneous with respect to multigradings. We instead require a weaker condition which we call $Q$-homogeneity that ensures that polynomials in $I$ and $J$ can be lifted to $I \times_Q J$.

We then show that if $T$ is any tree then the ideal $I_T$ associated to the polytope CIM$_T$ can be obtained by taking iterative QIGs of the ideals of star trees, which are always the zero ideal. As a result we obtain an explicit square-free, quadratic Gröbner basis for $I_T$. These characteristic imset ideals serve as a first example of ideals which arise via iterated quasi-independence gluings, but not as codimension 0 toric fiber products (even up to change of coordinates). It would be interesting to find additional examples of ideals which arise in this way, similar to the many families of ideals which come from iterated toric fiber products [1, 6, 7, 9, 19, 26, 27].

The remainder of the paper is organized as follows. In Section 2 we provide some background on DAGs, characteristic imsets, quasi-independence ideals, and toric fiber products. In Section 3, we introduce quasi-independence gluings and show that their Gröbner bases can be computed via lifting provided the original ideals satisfy a certain technical condition. In Section 4, we apply quasi-independence gluings to characteristic imset ideals to derive the Gröbner basis of the ideal when the underlying graph is a tree. In Section 5, we discuss partial results for the characteristic imset ideal of a cycle.

2. Preliminaries

2.1. Directed Acyclic Graphs and Characteristic Imset Polytopes. Recall that given a directed acyclic graph (DAG) $\mathcal{G} = ([n], E)$ and a collection of jointly distributed random variables
2.2 states that two DAGs are Markov equivalent if and only if they are not edges of a v-structure as the pattern of DAGs as 0/1-vectors in real-Euclidean space in the following way.

Definition 2.1. Given a DAG $G = ([n], E)$, the characteristic imset of $G$ is

$$c_G : S \subseteq [n] : |S| \geq 2 \rightarrow \{0, 1\};$$

$$c_G : S \mapsto \begin{cases} 1 & \text{if there exists } i \in S \text{ such that } S \setminus \{i\} \subseteq \text{pa}_G(i), \\ 0 & \text{otherwise.} \end{cases}$$

Note that for a DAG $G$ with $n$ vertices we can realize $c_G$ as a 0/1-vector in $\mathbb{R}^{2^{n-1}-n-1}$. We then define the characteristic imset polytope to be $\text{CIM}_n = \text{conv}(c_G : G \text{ a DAG with nodes } [n])$.

An important feature of modeling with DAGs is that two different DAGs can encode the same skeleton and v-structures.

Theorem 2.2. The following are equivalent:

1. $G = ([n], E)$ and $H = ([n], E')$ are Markov equivalent,
2. $c_G = c_H$, and
3. $G$ and $H$ have the same skeleton and v-structures.

Here, the skeleton of a DAG $G = ([n], E)$ refers to its underlying undirected graph, and a v-structure refers to a triple of vertices $i, j, k$ for which $i \rightarrow j, k \rightarrow j \in E$ but $i$ and $k$ not adjacent in $G$. In the following, we will refer to the graph $P_G$ given by undirecting all edges in $G$ that are not edges of a v-structure as the pattern of $G$. An alternative graphical representation of the Markov equivalence class of $G$ is its so-called essential graph, denoted $E_G$, which is a graph with both directed and undirected edges and the same skeleton as $G$ in which the directed edges are exactly those edges that are oriented in the same direction in all members of the Markov equivalence class of $G$. Hence, Theorem 2.2 states that two DAGs are Markov equivalent if and only if they have the same characteristic imsets and the same pattern (or essential graph). See Figure 1 for an explanation of the distinction between patterns and essential graphs. Given an undirected graph $G = ([n], E)$ we let $\text{Pat}(G)$ denote the set of patterns of all DAGs that have skeleton $G$.

Due to its relevance in the problem of causal discovery, the polyhedral geometry of $\text{CIM}_n$ has been studied. In [22, 23, 34] families of facets of $\text{CIM}_n$ and certain subpolytopes of $\text{CIM}_n$ are identified. Similar results have also been found for families of edges in [14, 15]. In [14, Proposition 2.5], it is noted that for an undirected graph $G = ([n], E)$

$$\text{CIM}_G = \text{conv}(c_G : G \text{ has skeleton } G)$$

is a face of $\text{CIM}_n$. In [15], a complete characterization of the edges of $\text{CIM}_G$ for $G$ a tree is also given. The following useful fact was also observed:

Lemma 2.3. [15, Proposition 1.9] If $G = ([n], E)$ is the star graph; i.e., $E = \{j-i^* : j \in [n] \setminus \{i^*\}\}$ for some $i^* \in [n]$, then $\text{CIM}_G$ is a simplex.
Figure 1. A DAG $G$ and the associated pattern $P_G$ and essential graph $E_G$. The edge $4 \to 6$ is part of no v-structures, so it is undirected in the pattern. Since $3 \to 4 \leftarrow 5$ forms a v-structure, all DAGs in the Markov equivalence class of this DAG must have the same orientation $4 \to 6$, which appears in the essential graph.

It is also not difficult to see that $\text{CIM}_G$ is a simplex when $G$ is the path graph on four vertices. We will use these observations in the coming sections.

2.2. Gröbner bases for Toric Ideals from Bipartite Graphs and Quasi-Independence. In this section we provide some background on toric ideals. We also describe those that arise from quasi-independence models in statistics or equivalently toric ideals arising from bipartite graphs. In the following sections, we will use quasi-independence ideals to study the toric ideals associated to characteristic imset polytopes.

Quasi-independence models are log-linear models for the joint distribution of two discrete random variables $X$ and $Y$, with respective state spaces $[r]$ and $[s]$, where some subset of the states of $X$ and $Y$ are forbidden from occurring together. These forbidden states of the joint distribution are called structural zeroes of the model. These models are typically specified by a set $Q \subseteq [r] \times [s]$ which consists of the allowed states of the joint distribution. The structural zeros of the model are then all $(j, k) \notin Q$.

Definition 2.4. Let $r$ and $s$ be two positive integers and $Q \subseteq [r] \times [s]$. The quasi-independence ideal associated to the set $Q$ is the kernel of the monomial map

$$
\phi_Q : \mathbb{K}[z_{jk} \mid (j, k) \in Q] \to \mathbb{K}[x_j, y_k \mid j \in [r], k \in [s]]
$$

$$
z_{jk} \mapsto x_j y_k
$$

and is denoted by $I_Q = \ker(\phi_Q)$.

Note that in the definition of the quasi-independence model we have simply omitted coordinates which correspond to structural zeroes. Also observe that $I_Q$ is a toric ideal since it is the kernel of a monomial map. Moreover, Theorem 2.5 gives an explicit generating set for $I_Q$.

We frequently associate an $r \times s$ matrix $A_Q = (a_{jk})_{(j,k)\in[r]\times[s]}$ to the set $Q$ given by

$$
a_{jk} = \begin{cases} 
z_{jk} & \text{if } (j, k) \in Q \\ 0 & \text{otherwise} \end{cases}
$$

Such a set $Q$ also has a naturally associated bipartite graph. We define $G_Q \subseteq K_{r,s}$ to be the bipartite graph on two disjoint and independent vertex sets $[r]$ and $[s]$ with edge set $Q$. The structural zeros encode the absence of an edge in $G_Q$. This graph is usually non-planar, and so in examples and figures we will typically consider $A_Q$. However, $G_Q$ establishes a link to a different viewpoint. The quasi-independence ideals are also studied under the title of toric ideals of edge rings [33] and many combinatorial and algebraic aspects of these ideals are known [11]. Additionally the so-called edge polytopes, whose vertices are the columns of the incidence matrix of $G_Q$, are introduced in [8] and their faces are characterized in terms of certain subgraphs of $G_Q$ [18, Theorem 2.18].
Theorem 2.5. Let \( Q \subseteq [r] \times [s] \) and let \( j_1-k_1, j_2-k_1, j_2-k_2, \ldots, j_\ell-j_\ell, j_1-k_\ell \) be the edges of an induced cycle of \( G_Q \). Then
\[
\prod_{i=1}^\ell z_{j_i,k_i} - \prod_{i=1}^\ell z_{j_{i+1},k_i}
\] (where \( i \) is taken modulo \( \ell \)) is a binomial of \( I_Q \). Moreover, the set of all such binomials forms the universal Gröbner basis for \( I_Q \).

A proof of Theorem 2.5 for general directed graphs is given in [10, Proposition 4.3]. Assuming that the edges of \( G_Q \) are directed from the vertices in \([r]\) to the vertices in \([s]\), we may apply this proof. For completeness we give an elementary proof by computing the normal form of all \( S \)-polynomials.

Proof. The multisets of \( j_i \) and \( k_i \) in the two monomials are the same, and so every binomial of this form lies in \( I_Q \). Let \( f \) and \( g \) be binomials associated to cycles. We form the \( S \)-polynomial \( S(f,g) \) by multiplying \( f \) and \( g \) by monomials, so that their leading terms are the same, and subtracting to cancel the leading terms. After scaling \( f \) and \( g \), all four monomials have the same multisets of \( j_i \) and \( k_i \), and so this holds for \( S(f,g) \) as well.

We form a bipartite graph \( G' \) with vertex sets \([r]\) and \([s]\) and edges \( j-k \) for \( z_{jk} \) appearing in \( S(f,g) \) (with multiple edges allowed between two vertices). Color the edges by which monomial they appear in. Traverse the edges of the graph, alternating between the two edge colors. Because both monomials contain the same multiset of \( j_i \) and \( k_i \), each vertex is part of the same number of each color of edge. Consequently each vertex entered has an edge of the opposite color that one can exit through, so we can traverse the edges of the graph until a cycle is found. The leading term of the associated cycle binomial divides some (not necessarily leading) term of \( S(f,g) \). Note that such a cycle binomial always exists since every time you enter a vertex on an edge of one color there is always an edge of the other color that you can exit along. Hence, you can traverse the graph in this way until you return to a previously visited vertex, at which point you must have discovered such a cycle binomial.

With binomial ideals, we can perform the division algorithm even if the leading term of the cycle binomial divides the trailing term. This is because the division algorithm replaces one monomial with another when we divide by a binomial, so the number of terms never increases. Via the previous argument, at each step in the division algorithm, one can obtain a cycle in \( G' \) and use it to decrease some monomial in the term order. This process can only terminate at the normal form 0.

\( \square \)

Example 2.6. Let \( r = s = 5 \) and consider \( Q = ([1,2] \times [1,4]) \cup ([2,5] \times [4,5]) \). The associated matrix \( A_Q \) is
\[
\begin{bmatrix}
  z_{11} & z_{12} & z_{13} & z_{14} & 0 \\
  z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\
  0 & 0 & 0 & z_{34} & z_{35} \\
  0 & 0 & 0 & z_{44} & z_{45} \\
  0 & 0 & 0 & z_{54} & z_{55}
\end{bmatrix}
\]

The graph \( G_Q \) has 12 cycles of length 4, which correspond to the \( 2 \times 2 \) submatrices with no structural zeroes. There are no larger cycles in \( G_Q \), so \( I_Q \) is generated by 12 quadratic binomials.

Next, we state when the quasi-independence ideal is generated by quadratic square-free binomials. A bipartite graph \( G \) is called chordal bipartite if it contains no induced cycles with at least 6 vertices. The graph \( G_Q \) above is an example of such a graph. We have the following special case of [16, Theorem 1.2].

Theorem 2.7. Let \( Q \subseteq [r] \times [s] \). Then \( I_Q \) is generated by quadratic square-free binomials if and only if \( G_Q \) is chordal bipartite.
In Section 3 we define a notion of iterated gluing by factoring through quasi-independence maps. Factoring through maps associated to chordal bipartite graphs preserves the property that the ideal is generated by square-free quadratics.

2.3. Toric Fiber Products. In this section we provide a brief outline of the toric fiber product as the contraction of an ideal under a monomial homomorphism. Unlike the usual description of the toric fiber product, we reframe the theory slightly into the language of quasi-independence maps.

Let $r$ be a positive integer and for each $i \in [r]$, let $s_i$ and $t_i$ be positive integers. Let \( \mathcal{A} = \{a_1, \ldots, a_r\} \subset \mathbb{Z}^d \) for some $d$. We define two graded polynomial rings:

\[
\mathbb{K}[x] := \mathbb{K}[x^i_j | i \in [r], j \in [s_i]] \quad \text{and} \quad \mathbb{K}[y] := \mathbb{K}[y^k_i | i \in [r], k \in [t_i]]
\]

where \( \deg(x^i_j) = \deg(y^k_i) = a_i \). Throughout this subsection let \( I \subseteq \mathbb{K}[x] \) and \( J \subseteq \mathbb{K}[y] \) be homogeneous ideals with respect to the multigrading. We define a quasi-independence map associated to the multigrading. Let

\[
Q(\mathcal{A}) = \bigsqcup_{i=1}^{r} (\{i\} \times [s_i]) \times (\{i\} \times [t_i])
\]

Let \( \mathbb{K}[z] = \mathbb{K}[z^j_{ik} : ((i, j), (i, k)) \in Q(\mathcal{A})] \). The quasi-independence map \( \phi_{Q(\mathcal{A})} : \mathbb{K}[z] \to \mathbb{K}[x, y] \) is

\[
z^j_{ik} \mapsto x^i_j y^k_i.
\]

Morally, we glue \( x^i_j \) and \( y^k_i \) together when they have the same degree in the multigrading \( \mathcal{A} \). Note that each subset \( (\{i\} \times [s_i]) \times (\{i\} \times [t_i]) \subseteq Q(\mathcal{A}) \) is the edge set of a complete bipartite subgraph of \( G_Q \). Up to reordering the rows and columns of \( A_Q \), we see that \( A_Q \) is block diagonal with blocks indexed by \( i \in [r] \). In this subsection we will only consider block diagonal \( A_Q \), but in Section 3 we extend the following definitions and results to more general quasi-independence maps.

**Definition 2.8.** The toric fiber product of the ideals \( I \) and \( J \) is

\[
I \times_{\mathcal{A}} J = \phi_{Q(\mathcal{A})}^{-1}(I + J).
\]

It is frequently difficult to compute a Gröbner basis of \( I \times_{\mathcal{A}} J \), even with Gröbner bases for \( I \) and \( J \). However, if \( \mathcal{A} \) is a set of linearly independent vectors, a Gröbner basis can be computed. Let

\[
f = \sum_{\ell} c_{\ell} x^{i_{1,1}}_{j_{\ell,1}} x^{i_{2,2}}_{j_{\ell,2}} \cdots x^{i_{d,d}}_{j_{\ell,d}} \in \mathbb{K}[x].
\]

If \( f \) is homogeneous with respect to a linearly independent multigrading, then the multiset of upper indices in each monomial must be the same. After reindexing the variables in each monomial, we can suppress \( \ell \) from the upper index.

\[
f = \sum_{\ell} c_{\ell} x^{i_{1,1}}_{j_{1,1}} x^{i_{2,2}}_{j_{1,2}} \cdots x^{i_{d,d}}_{j_{1,d}}
\]

Let \( k = (k_1, \ldots, k_d) \in [t_{i_1}] \times [t_{i_2}] \times \cdots \times [t_{i_d}] \). The lift of \( f \) by \( k \) is

\[
f_k = \sum_{\ell} c_{\ell} z^{i_{1,1}}_{j_{\ell,1},k_1} z^{i_{2,2}}_{j_{\ell,2},k_2} \cdots z^{i_{d,d}}_{j_{\ell,d},k_d} \in \mathbb{K}[z].
\]

The set of all lifts of \( f \) is denoted \( \text{Lift}(f) \). For a set of polynomials \( F \subseteq I \), we denote the set of all lifts of elements in \( F \) by \( \text{Lift}(F) \). Similarly we define lifts of elements in \( \mathbb{K}[y] \).

Let \( H_Q(\mathcal{A}) \) denote the universal Gröbner basis of the quasi-independence ideal defined in Theorem 2.5. Let \( B_Q(\mathcal{A}) \) denote the matrix of the exponent vectors of \( \phi_{Q(\mathcal{A})} \). For an ideal \( I \subseteq \mathbb{K}[x] \), we say a finite set \( G \subseteq I \) is a pseudo-Gröbner basis for \( I \) with respect to weight \( \omega \) if \( \langle \text{in}_\omega(G) \rangle = \text{in}_\omega(I) \). Note that this differs from a Gröbner basis in that here \( \text{in}_\omega(G) \) may not be a set monomials. Sullivant proved the following theorem.
Theorem 2.9. [28, Theorem 12] Let $A$ be a linearly independent multigrading. Let $F$ and $G$ be Gröbner bases for $I$ and $J$ with respect to weight order $\omega_1$ and $\omega_2$ respectively. Then

$$\text{Lift}(F) \cup \text{Lift}(G) \cup H_{Q(A)}$$

forms a pseudo-Gröbner basis of $I \times_A J$ with respect to the weight order $(\omega_1^T, \omega_2^T)B_{Q(A)}$.

After perturbing the term order, this set forms a Gröbner basis. In algebraic statistics, the toric fiber product is a commonly used tool for computing a Gröbner basis of an ideal with an associated combinatorial structure [1, 6, 7, 9, 19, 26, 27]. Typically the informal gluing of variables in the description of the toric fiber product corresponds to a gluing of combinatorial structures such as trees or simplicial complexes. The toric fiber product allows us to iteratively compute a Gröbner basis as we iteratively glue combinatorial structures.

3. Quasi-Independence Gluings

In this section we introduce a new generalization of the toric fiber product which we call a quasi-independence gluing (QIG). Similar to the toric fiber product, QIGs are the contraction of an ideal under a monomial homomorphism, so many of the techniques used to prove results for the toric fiber product can be naturally extended to prove similar results for QIGs. In particular, we describe the Gröbner basis (or generating set) of the QIG of two ideals by lifting the Gröbner basis as we iteratively glue combinatorial structures.

Definition 3.1. Let $\omega \in \mathbb{R}^r$ be a weight. Let $Q \subseteq [r] \times [s]$ and $f \subseteq \mathbb{K}[x]$ be a polynomial which is homogeneous with respect to total degree. Let $\text{in}_\omega(f) = x_{j_1}x_{j_2} \ldots x_{j_d}$. We say $f$ is weakly $Q$-homogeneous with respect to $\omega$ if for every monomial $m' = x_{j'_1}x_{j'_2} \ldots x_{j'_d}$ of $f$ there exists a permutation $\sigma$ of the elements of $[d]$ such that

$$(k_1, \ldots, k_d) \in [s]^d \quad \text{and} \quad (j_{\ell}, k_\ell) \in Q, 1 \leq \ell \leq d \quad \text{and} \quad ((k_1, \ldots, k_d) \in [s]^d \quad \text{and} \quad (j_{\ell}', k_\ell) \in Q, 1 \leq \ell \leq d.$$ 

We typically omit the $\sigma$ when describing this reordering since the specific permutation is not important. Later we define and motivate a notion of strong $Q$-homogeneity. Note that while weak $Q$-homogeneity does require homogeneity with respect to total degree, it does not necessarily correspond to homogeneity with respect to a multigrading.

Definition 3.2. Let $Q \subseteq [r] \times [s]$ and $I \subseteq \mathbb{K}[x]$ and $J \subseteq \mathbb{K}[y]$ be homogeneous ideals. The quasi-independence gluing of $I$ and $J$ with respect to the set $Q$ is

$$I \times_Q J := \phi_Q^{-1}(I + J).$$

Recall that defining the toric fiber product required a set $Q(A)$, defined in terms of a multigrading $A$, and that $A_{Q(A)}$ is block diagonal. So QIGs include toric fiber products but allow for more general structures.

In this section, we show that when $I$ and $J$ are weakly $Q$-homogeneous, we can construct a Gröbner basis for $I \times_Q J$ from Gröbner bases of $I$ and $J$. Our approach is essentially the same as that used to determine Gröbner bases for toric fiber products. Because of this we try to use the same notation and present our argument in the same format whenever possible. We begin with the following two lemmas from [28].
Let \( \omega : \mathbb{K}[z_1, \ldots, z_n] \to \mathbb{K}[t_1, \ldots, t_d] \) be a monomial map with a matrix of exponent vectors \( B \). If \( I \) is an ideal in \( \mathbb{K}[t] \) and \( \omega \in \mathbb{Z}_{\geq 0}^d \) is a weight vector on \( \mathbb{K}[t] \), then
\[
\text{in}_{\omega}x_B(\phi^{-1}(I)) \subseteq \phi^{-1}(\text{in}_{\omega}(I)).
\]

**Lemma 3.4** ([28, Lemma 2.3]). Let \( M = \langle m_1, \ldots, m_r \rangle \subset \mathbb{K}[t] \) be a monomial ideal. Then
\[
\phi^{-1}(M) = \phi^{-1}(\langle m_1 \rangle) + \ldots + \phi^{-1}(\langle m_r \rangle).
\]
Furthermore, \( \phi^{-1}(M) = M' + \ker(\phi) \) where \( M' \) is a monomial ideal.

As explained in [28], these two lemmas suggest a potential strategy for determining \( \phi^{-1}(I) \):

1. Determine \( \phi_B^{-1}(\text{in}_{\omega}(I)) \).
2. Find a candidate Gröbner basis \( G \subseteq \phi^{-1}(I) \), so that by Lemma 3.3 we have
   \[
   \text{in}_{\omega}x_B((G)) \subseteq \text{in}_{\omega}x_B(\phi^{-1}(I)) \subseteq \phi^{-1}(\text{in}_{\omega}(I)).
   \]
3. Show that \( \text{in}_{\omega}x_B((G)) = \phi^{-1}(\text{in}_{\omega}(I)) \).
4. Deduce that \( G \) is a Gröbner basis for \( \phi^{-1}(I) \) since
   \[
   \text{in}_{\omega}x_B((G)) = \text{in}_{\omega}x_B(\phi^{-1}(I)) = \phi^{-1}(\text{in}_{\omega}(I)).
   \]

We will follow this strategy in order to determine a Gröbner basis for \( I \times_Q J \). We begin by proving the following lemma which is analogous to [28, Lemma 2.5] for QIGs and has the exact same proof.

**Lemma 3.5.** Let \( Q \subseteq [r] \times [s] \) and let \( m = x_{j_1}x_{j_2} \ldots x_{j_d} \) be a monomial in \( \mathbb{K}[x, y] \). Then
\[
\phi_Q^{-1}(\langle m \rangle) = \langle z_{j_1k_1}z_{j_2k_2} \ldots z_{j dk_d} | k_1, \ldots, k_d \in [s], (j_1, k_1) \in Q \rangle + I_Q.
\]
Similarly, if \( n = y_{k_1}y_{k_2} \ldots y_{k_d} \) is a monomial in \( \mathbb{K}[x, y] \). Then
\[
\phi_Q^{-1}(\langle n \rangle) = \langle z_{j_1k_1}z_{j_2k_2} \ldots z_{j dk_d} | j_1, \ldots, j_d \in [r], (j_1, k_1) \in Q \rangle + I_Q.
\]

**Proof.** By symmetry it suffices to prove the first statement. By Lemma 3.4 we have that \( \phi_Q^{-1}(\langle m \rangle) = M' + I_Q \) for some monomial ideal \( M' \) which we claim is the ideal
\[
M' = \langle z_{j_1k_1}z_{j_2k_2} \ldots z_{j_dk_d} | k_1, \ldots, k_d \in [s], (j_1, k_1) \in Q \rangle.
\]
Consider a monomial \( m' = z_{j_1} \ldots z_{j_d} \) and note that \( \phi_Q(m') = x_{j_1}y_{k_1} \ldots x_{j_d}y_{k_d} \) which belongs to \( \langle m \rangle \) if and only if \( x_{j_1} \ldots x_{j_d} \) belongs to \( \langle m \rangle \). If \( x_{j_1} \ldots x_{j_d} \in \langle m \rangle \) then \( x_{j_1}x_{j_2} \ldots x_{j_d} \) divides \( x_{j_1} \ldots x_{j_d} \) but this immediately implies that there exists a monomial in \( M' \) dividing \( m' \).

We now discuss how to lift Gröbner bases for the ideals \( I \) and \( J \) to a Gröbner basis for \( I \times_Q J \). This once again is similar to the lifting presented in [28] (or subsection 2.3), but the lifting procedure is determined by \( Q \) instead of a multigrading.

Let \( \omega \) be a weight and consider a weakly \( Q \)-homogeneous polynomial \( f \in I \) of the form
\[
f = \sum_{\ell} c_{\ell}x_{j_{\ell,1}}x_{j_{\ell,2}} \ldots x_{j_{\ell,d}}.
\]
Let \( \text{in}_{\omega}(f) = c_as_{j_{1,1}}s_{j_{2,2}} \ldots s_{j_{d,d}} \) and let \( k = (k_1, \ldots, k_d) \in [s]^d \) such that the \( k_i \) satisfy \( (j_{s,i}, k_i) \in Q \). Note that since \( f \) is weakly \( Q \)-homogeneous, we have that \( (j_{s,i}, k_i) \in Q \) for all \( \ell \) after rearranging each monomial \( x_{j_{\ell,1}}x_{j_{\ell,2}} \ldots x_{j_{\ell,d}} \) appropriately.

**Definition 3.6.** Let \( Q \subseteq [r] \times [s] \) A lift of \( f \) by lower indices \( k = (k_1, \ldots, k_d) \) is a homogeneous polynomial \( f_k \) of the form
\[
f_k = \sum_{\ell} c_{\ell}z_{j_{\ell,1}k_1}z_{j_{\ell,2}k_2} \ldots z_{j_{\ell,d}k_d}.
\]
Furthermore, we denote the set of all possible lifts of \( f \) by \( \text{Lift}(f) \) which is the set
\[
\text{Lift}(f) = \{ f_k | k \in [s]^d, (j_{s,i}, k_i) \in Q, 1 \leq i \leq d \}\]
Observe that for any polynomial $f_k \in \text{Lift}(f)$ it holds that $f_k \in I \times_Q J$ since

$$\phi_Q(f_k) = y_{k_1}y_{k_2} \ldots y_{k_d}f \in I.$$ 

Now given a collection $F \subseteq I$ of weakly $Q$-homogeneous polynomials, let

$$\text{Lift}(F) = \bigcup_{f \in F} \text{Lift}(f)$$

and for a collection $G \subseteq J$ let Lift$(G)$ be the analogous collection. In terms of lifting, weak $Q$-homogeneity means that whenever the leading monomial of $f$ lifts by $k$, there exists lifts of all non-leading monomials of $f$ by $k$ as well.

**Theorem 3.7.** Let $F \subseteq I$ be a weakly $Q$-homogeneous Gröbner basis for $I$ with respect to the weight $\omega_1$ and $G \subseteq J$ be a weakly $Q$-homogeneous Gröbner basis for $J$ with respect to the weight $\omega_2$. Then

$$\text{Lift}(F) \cup \text{Lift}(G) \cup H_Q$$

is a pseudo-Gröbner basis for $I \times_Q J$ with respect to the weight $\omega^T B_Q$ where $B_Q \in \mathbb{Z}^{(r+s) \times \#Q}$ is the matrix of exponents of the map $\phi_Q$ and $\omega = (\omega_1, \omega_2)$.

**Proof.** First recall that by Lemma 3.3, we have that

$$\text{in}_{\omega^T B_Q}(\phi_Q^{-1}(I + J)) \subseteq \phi_Q^{-1}(\text{in}_\omega(I + J)).$$

We have already shown that Lift$(F) \cup$ Lift$(G) \cup H_Q \subseteq I \times_Q J = \phi_Q^{-1}(I + J)$ so if we show that the leading terms of Lift$(F) \cup$ Lift$(G) \cup H_Q$ form a pseudo-Gröbner basis for $\phi_Q^{-1}(\text{in}_{\omega_1, \omega_2}(I + J))$ then we have that

$$\text{in}_{\omega^T B_Q}(\phi_Q^{-1}(I + J)) = \phi_Q^{-1}(\text{in}_\omega(I + J))$$

and thus Lift$(F) \cup$ Lift$(G) \cup H_Q$ is a pseudo-Gröbner basis for $\phi_Q^{-1}(I + J) = I \times_Q J$.

Let $M = \text{in}_\omega(I + J)$ and recall that by Lemma 3.4 we only need to show that any monomial in $\phi_Q^{-1}(M)$ is divisible by the $\omega^T B_Q$-leading term of a polynomial in Lift$(F) \cup$ Lift$(G)$. So suppose $m' = z_{j_1}k_1z_{j_2}k_2 \ldots z_{j_d}k_d$ is a monomial in $\phi_Q^{-1}(M)$. Now observe that $F \cup G$ is a Gröbner basis for $I + J$ with respect to $\omega$ since $I$ and $J$ are in disjoint sets of variables and that any minimal generator of $M$ is either in $\mathbb{K}[x]$ or $\mathbb{K}[y]$ for the same reason. So by Lemma 3.5 $m'$ is divisible by a monomial $m = z_{j_1}k_1z_{j_2}k_2 \ldots z_{j_d}k_d$ and there exists some $f \in F$ or some $g \in G$ such that $m \in \text{Lift}(\text{in}_{\omega}(f))$ or $m \in \text{Lift}(\text{in}_{\omega}(g))$.

We now suppose that $m \in \text{Lift}(\text{in}_{\omega}(f))$ since the proof of the other case is the same. So $m$ is the lift of in$_w(f)$ for some valid tuple $k \in [s]^d$. But then $f_k \in \text{Lift}(f)$ since $f$ is weakly $Q$-homogeneous and of course $m = \text{in}_{\omega^T B_Q}(f_k)$. Thus we have that $m'$ is divisible by the $\omega^T B_Q$-leading term of a polynomial in Lift$(F)$ or Lift$(G)$ which completes the proof.

**Corollary 3.8.** With the same assumptions as Theorem 3.7. Let $\omega$ be a weight vector such that $H_Q$ is a Gröbner basis for $I_Q$. Then there exists $\epsilon > 0$ such that

$$\text{Lift}(F) \cup \text{Lift}(G) \cup H_Q$$

is a Gröbner basis for $I \times_Q J$ with respect to the weight $\omega' = (\omega_1, \omega_2)^T B_Q + \epsilon \omega$.

**Proof.** By Theorem 2.5, there exists $\omega$ such that $H_Q$ is a Gröbner basis for $I_Q$. For sufficiently small $\epsilon$, $\omega'$ specifies the same leading terms of Lift$(F)$ and Lift$(G)$ while additionally determining leading terms of polynomials in $H_Q$. 

The following example illustrates how one can use Theorem 3.7 to find a generating set for a quasi-independence gluing.
Example 3.9. Consider the following monomial map:

\[ z_{ijk} \mapsto \alpha_i \beta_j \gamma_k \]

for \((i, j, k)\) in the set \(Q = \{(1, 0, 0), (2, 0, 0), (0, 0, 1), (2, 0, 1), (0, 1, 0), (1, 1, 0), (0, 1, 1)\}\). This monomial map factors through the quasi-independence map \(z_{ijk} \mapsto x_i y_j k\) associated to the following matrix:

\[
\begin{bmatrix}
  y_{00} & y_{01} & y_{10} & y_{11} \\
  x_0 & 0 & z_{01} & z_{010} & z_{011} \\
  x_1 & z_{100} & 0 & z_{110} & 0 \\
  x_2 & z_{200} & z_{201} & 0 & 0
\end{bmatrix}
\]

By Theorem 2.5, the kernel of this quasi-independence map is generated by the following cubic:

\[ z_{001} z_{110} z_{200} - z_{010} z_{100} z_{201}. \]

By Theorem 3.7, it remains to compute the lifts of the generators of the following two maps:

\[ x_i \mapsto \alpha_i \quad y_j k \mapsto \beta_j \gamma_k. \]

The kernel of the former is the trivial ideal, so there are no lifts. Choose any weight order with weight \(\omega_1\) on this trivial ideal. The kernel of the latter is generated by \(y_{00} y_{11} - y_{01} y_{10}\). We have the following sets of lifts for each monomial:

\[
\begin{align*}
\text{Lift}(y_{00} y_{11}) &= \{z_{100} z_{011}, z_{200} z_{011}\} & \text{Lift}(y_{01} y_{10}) &= \{z_{001} z_{010}, z_{001} z_{110}, z_{201} z_{010}, z_{201} z_{110}\}
\end{align*}
\]

Note that for each monomial in \(\text{Lift}(y_{00} y_{11})\), the multiset of first coordinates also appears as a multiset of first coordinates for some monomial in \(\text{Lift}(y_{01} y_{10})\) (this inclusion is depicted by a coloring of the monomials). Consequently, for any weight \(\omega_2\) such that the weight order chooses leading term \(y_{00} y_{11}\), we can complete all lifts of \(y_{00} y_{11}\) to lifts of \(y_{00} y_{11} - y_{01} y_{10}\). We obtain the following lifts:

\[ z_{100} z_{011} - z_{001} z_{110} \quad \text{and} \quad z_{200} z_{011} - z_{201} z_{010}. \]

Together with the generator from the quasi-independence ideal, these lifts form a pseudo-Gröbner basis with respect to the weight order induced by \((\omega_1, \omega_2)^T B_Q\) where \(Q\). In contrast to the toric fiber product, the quasi-independence gluing of two ideals which are quadratically generated can still yield an ideal which is not quadratically generated.

As we saw in Example 3.9, one must know the leading term of a polynomial to determine that it is weakly \(Q\)-homogeneous and construct the set of lifts. Consequently for this more general construction, one must keep track of the weight order through each iterative gluing. This differs from the toric fiber product, where one can obtain a Gröbner basis with respect to some weight order without ever computing the weight. This motivates the following definition.

Definition 3.10. Let \(Q \subseteq [r] \times [s]\) and \(f \in \mathbb{K}[x_1, \ldots, x_r]\) be a polynomial which is homogeneous with respect to total degree. Then \(f\) is strongly \(Q\)-homogeneous if for every two monomials \(m = x_{j_1} x_{j_2} \ldots x_{j_d}\) and \(m' = x'_{j_1} x'_{j_2} \ldots x'_{j_d}\) of \(f\) there exists a permutation \(\sigma\) of the elements of \([d]\) such that

\[
\{(k_1, \ldots, k_d) \mid (j_\ell, k_\ell) \in Q, 1 \leq \ell \leq d\} = \{(k_1, \ldots, k_d) \mid (j'_{\sigma(\ell)}, k_\ell) \in Q, 1 \leq \ell \leq d\}.
\]

When \(f\) is strongly \(Q\)-homogeneous, the subset relation in the definition of weakly \(Q\)-homogeneous holds regardless of which monomial is the leading term. Therefore it is unnecessary to keep track of the term order because it is not needed to compute the lifts of each generator. In this way, the strong version of \(Q\)-homogeneity is similar to the toric fiber product. In Section 4, we use strong \(Q\)-homogeneity to compute a Gröbner basis for toric ideals associated to the face CIM \(T\) of the characteristic imset polytope for \(T\) a tree.

While the preceding results hold for ideals \(I\) and \(J\) which are not toric, there is a nice interpretation of the construction of \(I \times_Q J\) when \(I\) and \(J\) are toric.
Corollary 3.11. Let $I_A$ and $I_{A'}$ be toric ideals associated to integer matrices $A$ and $A'$. Then $I_A \times_Q I_{A'}$ is the toric ideal of the matrix $A \times_Q A'$ whose columns are all vectors of the form $(a_j, a'_k)^T$ such that $(j, k) \in Q$.

Proof. Let $\psi_A$, $\psi_{A'}$, and $\psi_{A \times_Q A'}$ be the monomial maps with associated integer matrices $A$, $A'$, and $A \times_Q A'$. The following diagram commutes

$$
\begin{array}{c}
\mathbb{K}[z] \xrightarrow{\psi_{A \times_Q A'}} \mathbb{K}[t] \\
\phi_Q \downarrow \quad \quad \quad \quad \quad \quad \downarrow \psi_{A, A'} \\
\mathbb{K}[x, y]
\end{array}
$$

where

$$
\psi_{A, A'} : \mathbb{K}[x, y] \to \mathbb{K}[t]
$$

$$
x_j \mapsto \psi_A(x_j) \\
y_k \mapsto \psi_{A'}(y_k).
$$

Hence we have the factorization $\psi(z_{jk}) = \psi_{A, A'}(x_j y_k) = \psi_A(x_j) \psi_{A'}(y_k)$, whenever $(j, k) \in Q$, which concludes the proof. \hfill \square

Example 3.12. In Example 3.9, we consider the quasi-independence gluing of the trivial ideal with the ideal for the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the given $Q$. Thus we obtain the following integer matrix:

$$
A \times_Q A' =
\begin{bmatrix}
z_{001} & z_{010} & z_{011} & z_{100} & z_{110} & z_{200} & z_{201} \\
\alpha_0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & 1 & 1 & 0 \\
\alpha_2 & 0 & 0 & 0 & 0 & 1 & 1 \\
\beta_0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\beta_1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\gamma_0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\gamma_1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}.
$$

4. Characteristic Imset Ideals via Quasi-Independence

In this section we define the toric ideal of $\text{CIM}_G$ and show that when $G$ is a tree, the ideal is an iterated quasi-independence gluing. We then use this to show that these ideals have square-free quadratic Gröbner bases. We use $\mathcal{P}$ to denote a general pattern without reference to a specific DAG.

The vertices of $\text{CIM}_G$ correspond to the Markov equivalence classes with underlying skeleton $G$. Consequently, we can equivalently represent the characteristic imset $c_G$ for $\mathcal{G}$ with the pattern $\mathcal{P}$ of the Markov equivalence class of $G$. We do so in the following. Given a pattern $\mathcal{P}$, we let $\mathcal{G}(\mathcal{P})$ denote any (fixed) DAG in the Markov equivalence class represented by $\mathcal{P}$.

Definition 4.1. Let $G = ([p], E)$ be a graph and $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be patterns which represent the Markov equivalence classes with skeleton $G$. Then the characteristic imset ideal of $G$ is the toric ideal associated to $\text{CIM}_G$, which is the kernel of the map

$$
\psi_G : \mathbb{K}[z_{\mathcal{P}_1}, \ldots, z_{\mathcal{P}_n}] \to \mathbb{K}[t_S \mid S \subseteq [p], |S| \geq 2] \\
z_{\mathcal{P}_i} \mapsto t^{c_{\mathcal{G}(\mathcal{P}_i)}}.
$$

We denote this ideal by $I_G = \ker(\psi_G)$. 
To prove this we extend the main technique introduced in Section 3 to show that ideals are toric fiber products to QIGs.

Let $\mathcal{T} \in \text{Pat}(T)$ and $\text{part}(\mathcal{T}, e) = (\mathcal{T}_u, \mathcal{T}_v)$. Let the ambient rings of $I_{\mathcal{T}}$, $I_{\mathcal{T}_u}$ and $I_{\mathcal{T}_v}$ be $\mathbb{K}[z] = \mathbb{K}[z_T \mid T \in \text{Pat}(T)]$, $\mathbb{K}[x] = \mathbb{K}[x_{\mathcal{T}_u} \mid \mathcal{T}_u \in \text{Pat}(T_u)]$, and $\mathbb{K}[y] = \mathbb{K}[y_{\mathcal{T}_v} \mid \mathcal{T}_v \in \text{Pat}(T_v)]$ respectively. If we square the parameter $t_{u,v}$ everywhere it appears in $\psi_T$ then

$$\psi_T(z_T) = \psi_{T_u}(x_{\mathcal{T}_u})\psi_{T_v}(y_{\mathcal{T}_v}).$$

(1)

This follows because if $S \subseteq [p]$ is the vertex set of a star subgraph of $T$ (so that $t_S$ appears in $\psi_T(z_T)$ or $\psi_{T_u}(x_{\mathcal{T}_u})\psi_{T_v}(y_{\mathcal{T}_v})$ and $S \neq \{u,v\}$, then $S$ is a subset of the vertices of $T_u$ or $T_v$ and thus either $T|_S = T_u|_S$ or $T|_S = T_v|_S$. So then the parameter $t_S$ appears in $\psi_T(z_T)$ if and only if
$t_S$ it appears in either $\psi_{T_u}(x_{T_u})$ or $\psi_{T_v}(y_{T_v})$. Note that squaring the parameter $t_{\{u,v\}}$ everywhere it appears in $\psi_T$ does not change the kernel of the map whatsoever. The factorization in Equation 1 implies that the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{K}[z] & \xrightarrow{\psi_T} & \mathbb{K}[t_S \mid S \subseteq [p], |S| \geq 2] \\
\downarrow{\phi_Q} & & \downarrow{\psi_{T_u,T_v}} \\
\mathbb{K}[x,y] & & \\
\end{array}
$$

where the map $\psi_{T_u,T_v}$ is given by

$$
\psi_{T_u,T_v} : \mathbb{K}[x,y] \rightarrow \mathbb{K}[t_S]
$$

$$
x_{T_u} \mapsto \psi_{T_u}(x_{T_u})
$$

$$
y_{T_v} \mapsto \psi_{T_v}(y_{T_v}).
$$

But recall that $I_T = \ker(\psi_T) = \psi_T^{-1}(0)$. Since the above diagram commutes, we have

$$
\psi_T^{-1}(0) = \phi_Q^{-1}\left(\psi_T^{-1}(0)\right) = \phi_Q^{-1}(I_{T_u} + I_{T_v}) = I_{T_u} \times Q I_{T_v}
$$

which completes the proof. \hfill \Box

**Example 4.5 (Quartet Tree).** In general, to compute a Gröbner basis for a QIG one must show that the defining monomial map factors through a quasi-independence map $\phi_Q$ and that the two ideals that we wish to glue together are weakly $Q$-homogeneous. Let $T$ be the quartet graph, with vertices labeled by elements of $[6]$ and edges $\{1-3, 2-3, 3-4, 4-5, 4-6\}$. Let $Q \subseteq \text{Pat}(T|_{1234}) \times \text{Pat}(T|_{3456})$ be the set of all partings of patterns with skeleton $T$.

We claim $\psi_T$ factors. Consider the pattern $T$ with directed edges $1 \rightarrow 3$, $2 \rightarrow 3$, $3 \rightarrow 4$, and $6 \rightarrow 4$. Then under $\psi_T$ we have

$$
z_T \mapsto t_{13}t_{23}t_{13}t_{34}t_{45}t_{46}t_{346}.
$$

After squaring $t_{34}$ (which does not change the kernel), we have the following factorization:

$$
z_T \mapsto x_{T_3}y_{T_4}, \quad x_{T_3} \mapsto t_{13}t_{23}t_{34}t_{123} \quad y_{T_4} \mapsto t_{34}t_{45}t_{46}t_{346}
$$

where $T_3$ and $T_4$ are defined as in Definition 4.3. Similarly we see this factorization for all other patterns.

For the quartet (and more generally for all bistars) the two trees we glue together are star graphs. Consequently their characteristic imset ideals are trivial and are strongly $Q$-homogeneous. It follows that the QIG is $\phi_Q^{-1}(0)$, in particular it is the quasi-independence ideal of the bipartite graph $G_Q$. By Theorem 2.7, the kernel of $\phi_Q$ is given by the $2 \times 2$ minors of the matrix in Figure 4 that contain no structural zeros, which are the binomials appearing in Example 4.2.

Recall that toric fiber products are QIGs associated to block diagonal matrices. In Figure 4 we have two overlapping blocks, one for each possible orientation of the gluing edge. These blocks overlap in the pattern with no $v$-structures, and so this characteristic imset ideal fails to be a toric fiber product.

We have shown that the ideal $I_T = I_{T_u} \times_Q I_{T_v}$, but to use Theorem 3.7 to find a Gröbner basis for $I_T$ we will show that $I_{T_u}$ and $I_{T_v}$ are actually strongly $Q$-homogeneous.

**Lemma 4.6.** Let $T = ([p], E)$ be a tree, $e = u - v$ be a non-leaf edge of $T$ and $T_u, T_v$ be the undirected trees obtained from parting $T$ at $e$. Let $Q = \{\text{part}(T, e) \mid T \in \text{Pat}(T)\}$. Then $I_{T_u}$ and $I_{T_v}$ are strongly $Q$-homogeneous.
Figure 4. The matrix of patterns associated to the QIG defining the characteristic imset polytope of the bistar. Structural zeros correspond to pairs of essential graphs indexing rows and columns where the gluing edge in both essential graphs has forced opposing orientations.

Proof. We will show that $I_{T_u}$ is strongly $Q$-homogeneous since the proof is the same for $I_{T_v}$. For each pattern with skeleton $T_u$, we have one of the following 3 orientations of $e$:

$u \to v$, $v \to u$, or $u - v$.

To order the rows and columns of $A_Q$, we consider the essential graph associated to each pattern. Order the rows and columns such that all essential graphs with $e$ oriented $u \to v$ are first, then all essential graphs with $u - v$, and lastly the essential graphs with $v \to u$. The nonzero entries of this matrix are row and column convex. Therefore $G_Q$ is chordal bipartite and all generators from the quasi-independence map are quadratic. Since lifting does not change the degree of a generator, we may assume by induction that $I_{T_u}$ is quadratically generated. Furthermore, the coefficients of any generator are $\pm1$ since all generators are liftings of $2 \times 2$ determinants. Suppose $f = x_{P_1}x_{P_2} - x_{P_3}x_{P_4}$ is a generator of $I_{T_u}$ and without loss of generality, let $(P_1, P_1'), (P_2, P_2') \in Q$. Consequently we must have either $(P_3, P_1'), (P_4, P_2') \in Q$ or $(P_3, P_2'), (P_4, P_1') \in Q$. Hence $f$ is strongly $Q$-homogeneous. □

We know by Lemma 2.3 that the CIM ideal of a star graph (and of a path graph up to 4 vertices) is the trivial ideal. Thus, one can construct the CIM ideal of $I_T$ of any tree $T$ as an iterated QIG of trivial ideals. By Corollary 3.8, we have the following corollary.

Corollary 4.7. Let $T$ be a tree. Then there exists a weight vector $\omega$ such that the reduced Gröbner basis of $I_T$ with respect to $\omega$ consists of square-free quadratics. Moreover, these quadratics can be explicitly constructed via iterated quasi-independence gluing.

We close this section with an example involving the path graph. Let $P_n$ denote the path with $n$ vertices and label the vertices by $\{1, \ldots, n\}$ in the order of the path. The $v$-structures on $P_n$ have edges from $i - 1$ and $i + 1$ to $i$, where the nonleaf vertex is $i \in \{2, \ldots, n - 1\}$. We cannot have both $i$ and $i + 1$ being nonleaf vertices of $v$-structures in a pattern since the edge containing $i$ and $i + 1$ would require two opposing orientations. Consequently we may index variables by sets $S \subseteq \{2, \ldots, n - 1\}$ of nonconsecutive integers in place of patterns.

Example 4.8 (6 Vertex Path). For this example we consider $I_{P_3} \subseteq \mathbb{K}[x]$, $I_{P_5} \subseteq \mathbb{K}[y]$, and $I_{P_6} \subseteq \mathbb{K}[z]$. We glue a path with 3 vertices to $P_5$ to form $P_6$, as in Figure 5. This gluing yields 3 generators from quasi-independence:

$z_0z_25 - z_2z_5, z_0z_235 - z_3z_5, z_2z_35 - z_3z_25$. 
Since $P_3$ is a star graph, $I_{P_3} = 0$, so we only consider lifts of Gröbner basis elements in $I_{P_3}$. One can show that the Gröbner basis for $I_{P_3}$ consists of just the polynomial $y_0y_{24} - y_2y_4$, since it can be seen as the QIG of $I_{P_3}$ with the ideal of the star graph $I_{P_3}$ which are both trivial ideals. Both monomials can lift by the unordered tuples of lower indices $(\emptyset, \emptyset)$ and $(\emptyset, 5)$, which result in the following lifts of the binomial:

$$z_0z_{24} - z_2z_{24}, z_5z_{24} - z_{25}z_{24}.$$  

The 5 binomials we have constructed form a Gröbner basis for $I_{P_6}$.

5. **Weak Q-homogeneity, Cycles, and Future Directions**

In this section we examine the characteristic imset ideal of a cycle. We show that the ideal can be obtained by taking the quasi-independence gluing of two paths but the Gröbner basis of the path constructed via QIG is not weakly Q-homogeneous. We end by exploring possible directions for finding a weakly Q-homogeneous Gröbner basis for the path.

We begin with a general gluing of two path graphs, of the form pictured in Figure 6. This is similar to the partings for trees, except we part at two edges. Since the end points of paths are not glued together and since we glue along two edges, each path must have at least four vertices.

Label $P_n$ as in the previous section, and consider $I_{P_n}$ in variables $y_S$ for $S \subseteq \{2, \ldots, n-1\}$ a set of pairwise non-consecutive integers. Let $P'_m$ denote the path on $m$ vertices, which we label \{n-1, n, n+1, \ldots, n+m-4, 1, 2\} in the order of the path. Similar to $P_n$ we index variable $x_S$ by sets $S \subseteq \{n-1, n, \ldots, n+m-4, 1, 2\}$ containing no two integers which are consecutive in the written order. We now define a gluing rule $Q_{m,n}$ to be the set of $(P', P) \in \text{Pat}(P'_m) \times \text{Pat}(P_n)$ such that both of the following hold

1. for all v-structures $i \rightarrow k \leftarrow j$ and $i' \rightarrow k' \leftarrow j'$ in $P$ or $P'$ the numbers $k$ and $k'$ are not cyclically consecutive,
2. either $P'$ or $P$ contains a v-structure.

The first condition guarantees that we do not glue edges together with forced opposing orientations while the second condition prevents an additional illegal gluing. Indeed, if we direct the edges of the cycle, the only way that no v-structures can appear is if the edges form a directed cycle, which is not a DAG. Then $I_{C_{n+m-4}}$ is an ideal in the polynomial ring with variables $z_S$ for $S \subseteq \{1, \ldots, n+m-4\}$ containing no cyclically consecutive integers modulo $n+m-4$. 

![Figure 5. A QIG yielding the characteristic imset ideal of the path with 6 vertices.](image)
**Theorem 5.1.** Let $m, n \geq 4$. Then $I_{C_{n+4}} = I_{P'_m} \times Q_{m,n} I_{P_n}$.

**Proof.** The proof is analogous to the proof of Theorem 4.4. The main difference is that we must square both $t_1$ and $t_{n-1,n}$, since we glue along two edges. □

The previous theorem shows that the cycle is a quasi-independence gluing but to compute a Gröbner basis with Theorem 2.5 we also need that there are Gröbner bases for both $I_{P'_m}$ and $I_{P_n}$ which are weakly $Q_{m,n}$-homogeneous with respect to some term order. For the remainder of this section we investigate combinatorial and polyhedral restrictions on the weight order that are imposed by weak $Q_{m,n}$-homogeneity.

**Lemma 5.2.** Let $I_{P_n} \subseteq \mathbb{K}[y_S]$, $\omega$ be a weight order on $\mathbb{K}[y]$, and $F$ be a Gröbner basis for $I_{P_n}$ with respect to $\omega$. If $I_{P_n}$ is weakly $Q_{m,n}$-homogeneous with respect to $\omega$ then for every binomial in $F$ of the form $y_0y_{S_2} \cdots y_{S_k} - y_{T_1}y_{T_2} \cdots y_{T_k}$ where $T_i \neq \emptyset$, it holds that

$$y_{T_1}y_{T_2} \cdots y_{T_k} \leq_w y_0y_{S_2} \cdots y_{S_k}.$$  

**Proof.** First, note that the only variable which does not lift by $\emptyset$. Thus if every $T_i \neq \emptyset$ then the monomial $y_{T_1}y_{T_2} \cdots y_{T_k}$ lifts by indices $(\emptyset, \emptyset, \ldots, \emptyset)$ but the monomial $y_0y_{S_2} \cdots y_{S_k}$ does not. Thus $F$ can only be weakly $Q_{m,n}$-homogeneous if $y_0y_{S_2} \cdots y_{S_k}$ is the leading term. □

Lemma 5.2 implies the following restriction on the sizes of the paths that we glue together.

**Proposition 5.3.** Suppose that $m \geq 5$ and $n \geq 5$. Then there is no term order such that $I_{P_n}$ is weakly $Q_{m,n}$-homogeneous.

**Proof.** Since $n \geq 5$, $I_{P_n} \neq 0$ and we have the following generator of $I_{P_n}$:

$$y_0y_{2n-1} - y_2y_{n-1}.$$  

Observe that every Gröbner basis of $I_{P_n}$ must contain this binomial since no other binomial in the ideal contains either of these monomials. By Lemma 5.2 the underlined term must be the leading term.

But if $m \geq 5$, then a pattern on $P'_m$ could have $v$-structures at both end points $\{1, n\}$ of $P_n$. We can then lift $y_0y_{2n-1}$ by $\{(1, n), \emptyset\}$ to $z_{1,n}z_{2,n-1}$. However, both 2 and $n-1$ are adjacent to some element of $\{1, n\}$, and so the other monomial does not lift. Since each monomial lifts by an indexing set that the other does not lift by, this polynomial cannot be weakly $Q_{m,n}$-homogeneous with respect to any term order. □

We consequently restrict our focus to $m = 4$. We suppress $m$ from the notation, instead considering the path $P'_m$ of length 4 and weak $Q_n$-homogeneity. Note that CIM$_P$ is a simplex, and so $I_{P'} = 0$. So we need only investigate weak $Q_n$-homogeneity for a Gröbner basis of $I_{P_n}$. We now show that any term order producing a weakly $Q_n$-homogeneous Gröbner basis cannot be constructed via the iterated QIG in Section 4.

**Lemma 5.4.** For $n \geq 6$, there does not exist a Gröbner basis of $I_{P_n}$ produced by iterated QIG that is also weakly $Q_n$-homogeneous.

**Proof.** Observe that every Gröbner basis of $I_{P_n}$ produced by iterative QIG contains the polynomial $y_0y_{24} - y_{24}y_4$. Let $\omega$ be any weight such that $y_{24}y_4 \leq_w y_0y_{24}$ since by Lemma 5.2 we know $\omega$ must select monomials with $y_0$ as the leading term. Then QIG produces the following two lifts in $I_{P_n}$:

$$z_0z_{24} - z_{24}z_4 \text{ and } z_{n-1}z_{24} - z_{2,n-1}z_4.$$  

Lifting does not change the leading term (regardless of the term order chosen on $I_{P'}$), and so the underlined terms are leading terms.

Now we consider lifting the polynomial $z_{n-1}z_{24} - z_{2,n-1}z_4 \in I_{P_n}$ to $I_{C_n}$. The leading monomial $z_{n-1}z_{24}$ lifts by indices $\{(\emptyset, \emptyset), (\emptyset, n), (1, \emptyset), (1, n)\}$ while the trailing monomial $z_{2,n-1}z_4$ lifts by indices $\{(\emptyset, \emptyset), (\emptyset, n), (1, \emptyset)\}$. Consequently $z_{n-1}z_{24} - z_{2,n-1}z_4$ is not weakly $Q_n$-homogeneous. □
Figure 7. A QIG that yields the characteristic imset ideal of the cycle with 6 vertices. One can visually check that there is no term order constructed by QIG such that $z_0z_{24} - y_2y_4$ and $z_5z_{24} - z_{25}z_{24}$ are not both weakly $Q_6$-homogeneous.

Figure 7 illustrates Lemma 5.2. The monomial map of the characteristic imset ideal factors according to $Q_n$, but weak $Q_n$-homogeneity does not hold for the Gröbner basis of the path that we constructed in Section 4.

The previous lemma shows that it is impossible for iterative quasi-independence gluing to produce a Gröbner basis for $I_{P_n}$ which can then be used to compute $I_{C_n}$. However, our computations for $n = 6, 7$ suggest that there do exist Gröbner bases for $I_{P_n}$ which are weakly $Q_n$-homogeneous.

**Problem 5.5.** Find a term order $\omega$ on $I_{P_n}$ such that a Gröbner basis $F$ of $I_{P_n}$ with respect to $\omega$ is weakly $Q_n$-homogeneous.

A solution to this problem would allow one to construct the Gröbner basis for the cycle ideal $I_{C_n}$ using Theorem 3.7. One potential first step is to begin with a Gröbner basis $G$ produced by QIG and then construct a new term order $\omega$, which does not come from QIG, such that $G$ is a weakly $Q_n$-homogeneous generating set with respect to $\omega$. Since weak $Q_n$-homogeneity corresponds to homogeneous linear inequalities on $\omega$, finding a term order $\omega$ just amounts to finding an interior point in the corresponding cone. The existence of such a term order would immediately allow one to find a generating set for $I_{C_n}$ using QIG. To find a Gröbner basis one would instead need to compute a Gröbner basis for $I_{P_n}$ with respect to the term order $\omega$ and certify that any new polynomials which arise in the generating set are also weakly $Q_n$-homogeneous with respect to $\omega$.

Another direction for further work is finding other classes of ideals which arise via quasi-independence gluing. It would be particularly interesting to find ideals which arise via weakly $Q_n$-homogeneous quasi-independence gluing. This is summarized in the following question.

**Question 5.6.** Are there other interesting families of ideals which can be constructed via iterated quasi-independence gluing?

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