Fermion mass gap in the loop representation of quantum gravity

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Abstract

An essential step towards the identification of a fermion mass generation mechanism at Planck scale is to analyze massive fermions in a given quantum gravity framework. In this letter the two mass terms entering the Hamiltonian constraint for the Einstein-Majorana system are studied in the loop representation of quantum gravity and fermions. One resembles a bare mass gap because it is not zero for states with zero (fermion) kinetic energy as opposite to the other that is interpreted as ‘dressing’ the mass. The former contribution originates from (at least) triple intersections of the loop states acted on whilst the latter is traced back to every couple of coinciding end points, where fermions sit. Thus, fermion mass terms get encoded in the combinatorics of loop states. At last the possibility is discussed of relating fermion masses to the topology of space.

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The physics at Planck length $\ell_P := \sqrt{\frac{G\hbar}{c^3}} = 1.6 \times 10^{-33}\text{cm}$, where a quantum notion of spacetime is called for is acquiring deeper significance due to a number of new results. Amongst the most striking we find: i) a definition of

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a gravitational Hamiltonian \[1\] and more recently, a skein-relation interpretation of the Hamiltonian constraint \[2\], ii) a determination of area and volume spectra \[3\] and iii) an insight on the origin of black hole entropy \[4\]. All of them were obtained within a non perturbative approach to quantum gravity \[5, 6, 7\] and they present discrete and combinatorial features that seem to encode fundamental quantum aspects of spacetime at \(\ell_P\). For instance, by coupling a clock scalar field to gravity a hamiltonian was built up in \[1\] that evolves the gravitational field itself. The action of this hamiltonian on loop states is concentrated on intersection points of the loops. More recently the Hamiltonian constraint was interpreted as a skein relation when acting on the space of knots \[2\]; hereby knot polynomials satisfying the skein relation solve the full quantum Einstein equations. In \[3\] it was shown that area and volume operators can be defined at the quantum level. Their spectra are discrete and related to the way intersections occur between loops and the surface whose area should be determined or among loops inside the region whose volume is under study. This former notion of area is further exploited in relation with the horizon of a black hole in \[4\]. Hence a value for its entropy can be estimated thereby that agrees with the standard proportionality between entropy and horizon area. It should be stressed that each of these results was obtained after a suitable regularization procedure to make them well defined; the resulting operators are finite, diffeomorphisms invariant and (regularization-) background independent.

Now it is natural to wonder how compatible is the continuous picture of space we are used to with the above discreteness. It turns out that smooth space can be thought of as a large length limit of certain loopy states or weaves \[8\]. Moreover the notion of gravitons also emerges here: they are associated to embroideries on weave states \[9\]. Appealing as this idea is it
cannot describe nature as a whole; one must learn first what the notion of matter is, if any, consistently with the above discrete picture. Indeed such a consistency must be looked for in any given quantum gravity scenario.

It has been proposed in the past that wormholes at the Planck scale might behave as charged particles [10] and that quantum gravity states could have half integral angular momentum, when the space three manifold has non trivial topology [11]. Also, by following the path integral approach to quantum gravity, it was realized that one might include the contribution of non orientable spacetimes (non orientable foam) in the correspondent amplitude. This together with CP invariance could produce an effective mass for the otherwise massless fermionic fields living on such spacetimes [12]. The standard model of electroweak interactions has been studied along similar lines by taking a random Planck lattice as an effective theory coming from a Planck-scale foam spacetime [13].

In the loop representation of non perturbative quantum gravity some steps have been given to unravel the notion of matter. Coupled electromagnetic field and gravity were considered in [14] as a simple unified description of gravitational and electromagnetic interactions. Physical states were found parametrized by two loops each of which carries information on both gravity and electromagnetism – the Chern-Simons functional and Jones Polynomial playing a role in the analysis. In ref. [15] massless spin-$\frac{1}{2}$ fields and gravity were studied. A (clock-) scalar field was coupled to gravity and a Hamiltonian evolving both fermion and gravity fields was introduced. The fermionic contribution gets concentrated at the end points of the curves (where fermions sit) of the loop states acted upon. More recently in [16] a kinematical analysis was developed for the Einstein-Maxwell-Dirac theory.

Crucial to the notion of matter is the understanding of the origin of
fermion masses given the unsatisfactory status in this regard of the standard model of electroweak interactions [13]. To do so a compulsory step is to find the analogue of the term $m\bar{\Psi}(x)\Psi(x)$ that reveals the fermion mass in the lagrangian form of field theory. In this letter an analysis is given of the mass of a spin-$\frac{1}{2}$ field of Majorana type coupled to gravity, using the loop representation for canonical quantum fermions and gravity of [15]. Specifically, we study the features inherited from the discreteness and combinatorial aspects appearing in such an approach.

To begin with let’s recall the outcome of the canonical analysis for the Einstein-Majorana (EM) system using Ashtekar variables. There are three first class constraints namely the Gauss, vector and Hamiltonian ones [17]

$$G_{AB} := -\mathcal{D}_a \bar{\sigma}^a_{AB} - \eta(A) \bar{\theta}_B, \quad \mathcal{V}_a := \bar{\sigma}^b_{AB} F_{ab} A - \bar{\theta}_A \mathcal{D}_a \eta^A \quad \text{and}$$

$$\mathcal{H} := -\frac{1}{2} \bar{\sigma}^a_{A C} \bar{\sigma}^b_{B C} F_{ab} A - \bar{\sigma}^a_{A B} \bar{\theta}_B \mathcal{D}_a \eta^A + m \left( (\bar{\sigma})^2 \eta_A \eta^A - \frac{1}{4} \bar{\theta}^A \bar{\theta}_A \right) \equiv \mathcal{H}_{\text{Einstein}} + \mathcal{H}_{\text{Weyl}} + \mathcal{M}_1 + \mathcal{M}_2,$$

where $A_a^{AB}(x), \eta^A(x)$ being the configuration variables and $\bar{\sigma}^a_{AB}(x), \bar{\theta}_A(x)$ the corresponding canonical momenta\(^4\). Here the Majorana spin-$\frac{1}{2}$ field contribution to $\mathcal{H}$ consists of the last three terms in (1) of which $\mathcal{M}_1$ and $\mathcal{M}_2$ are related to mass. To proceed to the quantum theory one has to solve the problem of constructing their (regularized) quantum operator version. This is achieved by adopting loop variables, where the Gauss law is automatically fulfilled, following [1, 15]. For spin-$\frac{1}{2}$ and gravity loop variables were built up in [15] as

$$X[\alpha] := \psi^A(\alpha_i) U_A^{B[\alpha]} \psi_B(\alpha_f), \quad Y[\alpha] := \bar{\pi}^A(\alpha_i) U_A^{B[\alpha]} \psi_B(\alpha_f),$$

\(^4\)Notice that the two fermionic mass terms are non vanishing because the two-spinor field $\eta^A(x)$ is Grassmann valued (e.g. $\eta_A \eta^A = -2\eta^0 \eta^1$); as opposed to the incorrect remark in [15].
\[ Y^a[\alpha](s) := \bar{\pi}^A(\alpha_i)U_A B[\alpha](0, s)\bar{\sigma}_B^a C(\alpha(s))U_C D[\alpha](s, 1)\psi_D(\alpha_f). \] (2)

Among their properties it is worth mentioning the fermionic (Grassmann) identity: if \((\alpha, \beta, \gamma)\) are open curves \(\alpha_i = \beta_i = \gamma_i\) then \(X[\alpha] X[\beta] X[\gamma] = 0\). No three fermions can coincide at the same point simultaneously. The loop variable \(Y^a[\alpha](s)\) was used to define the kinetic fermion term of the Hamiltonian constraint of the Einstein-Weyl theory [15]. Next the first mass term \(\mathcal{M}_1\) is translated into loop variables. Consider a closed loop \(\gamma\) with three gravitational hands inserted in it and an open loop \(\alpha\) with the fermion field \(\eta(x)\) placed at its ends. That is to say

\[ V^{abc}[\gamma, \alpha] := T^{abc}[\gamma](s, t, r) X[\alpha], \] (3)

where \(T^{abc}[\gamma](s, t, r) := \text{Tr} \left\{ \bar{\sigma}^a(\gamma(s))U_\gamma(s, t)\bar{\sigma}^b(\gamma(t))U_\gamma(t, r)\bar{\sigma}^c(\gamma(r))U_\gamma(r, s) \right\} \) is the loop variable used in the construction of the volume operator [3] and \(X[\alpha]\) was given above. It is straightforward to show that when the two loops shrink down to a common point \(x\) we have the local quantity

\[ \mathcal{M}_1 = \left( \frac{m}{3\sqrt{2}} \right) \lim_{\gamma, \alpha \to x} \eta_{abc} V^{abc}[\gamma, \alpha] = m(\sigma)^2 \eta_A \eta^A. \] (4)

The construction becomes more transparent if Eq.(3) is rewritten as follows. Then use is made of the fundamental spinor identity \(\epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} = \epsilon_{AD}\epsilon_{CB}\) inserted at the intersection point of the two loops. The result is the difference of further loop variables

\[ T^{abc}[\gamma](s, t, r) X[\alpha] = N^{abc}[\alpha \cdot \gamma](s^*, t^*, r^*) \]
\[ \quad - N^{cba}[\alpha \cdot \gamma^{-1}](1 - s^*, 1 - t^*, 1 - r^*) \] (5)

\[ N^{abc}[\alpha \cdot \gamma](s^*, t^*, r^*) := \text{Tr} \left\{ \eta(\alpha_i)U[\alpha](0, p)U[\gamma](0, s)\bar{\sigma}^a(\gamma(s))U[\gamma](s, t)\bar{\sigma}^b(\gamma(t)) \right. \]
\[ \left. \quad U[\gamma](t, r)\bar{\sigma}^c(\gamma(r))U[\gamma](r, 1)U[\alpha](p, 1)\eta(\alpha_f) \right\}, \] (6)
with $\alpha(p) = \gamma(0) = \gamma(1) = x$ and $s^*, t^*, r^*$ are the values of the parameter of $\alpha \cdot \gamma$, where the gravitational hands are inserted. Here $\text{Tr}\{\psi \mathcal{O}^{(1)} \ldots \mathcal{O}^{(n)} \psi\} := \psi_{A_1} \mathcal{O}^{(1)}_{A_1} \ldots \mathcal{O}^{(n)}_{A_n} \psi_{A_n+1}$. 

Regarding $\mathcal{M}_2$, it can be expressed as

$$\mathcal{M}_2 := -\frac{m^4}{4} \lim_{\alpha \to x} Z[\alpha] \quad \text{with} \quad Z[\alpha] := \tilde{\theta}^A(\alpha_i) U_A^B[\alpha] \tilde{\theta}_B(\alpha_f),$$

(7)

when $\alpha$ shrinks down to the point $x$.

Based on the loop transform of [15] the action of the (non regularized) operators (3) and (7) can be defined on loop states as

$$\hat{Z}[\alpha] \Psi[\beta] = \delta^3(\alpha_f, \beta_i) \delta^3(\alpha_i, \beta_f) \Psi[\alpha \cdot \beta]$$

$$+ \delta^3(\alpha_f, \beta_f) \delta^3(\alpha_i, \beta_i) \Psi[\alpha \cdot \beta^{-1}],$$

(8)

$$\hat{V}^{abc}[\gamma, \alpha] \Psi[\beta] = \sum_{\mu=\pm1} \sum_{j=1}^8 \sum_{i=1}^6 \left( \frac{1}{\sqrt{2}} \right)^3 \cdot$$

$$\left( \Delta^a[\alpha \cdot \gamma^\mu(s^*), \beta] \Delta^b[\alpha \cdot \gamma^\mu(t^*), \beta] \Delta^c[\alpha \cdot \gamma^\mu(r^*), \beta] \right)$$

$$\cdot (-1)^{r_{ij}} c_{ij} (-1)^{(1-\mu)} \Psi[(\alpha \cdot \gamma^\mu \cdot \beta)_{ij}],$$

(9)

$$\Delta^a[\alpha \cdot \gamma^\mu(s^*), \beta] = \frac{1}{2} \int_0^1 du \tilde{\beta}^a(u) \delta^3(\alpha \cdot \gamma^\mu(s^*), \beta(u)).$$

$\hat{V}^{abc}[\gamma, \alpha]$ produces sixteen multiloop states\(^5\) (indexes $\mu$ and $j$) for each exclusive configuration labeled by $i$. In other words, $(\Delta^a[\alpha \cdot \gamma^\mu(s), \beta] \Delta^b[\alpha \cdot \gamma^\mu(t), \beta] \Delta^c[\alpha \cdot \gamma^\mu(r), \beta])_i$ represent the six different ways in which the open loop $\beta$ is attached to the open loop $\alpha \cdot \gamma^\mu$. $\Psi[(\alpha \cdot \gamma^\mu \cdot \beta)_{ij}]$ denote the multiloop states resulting from rerouting $\alpha \cdot \gamma^\mu$ and $\beta$. $r_{ij}$ is the number of orientation-reversed segments of any $\alpha \cdot \gamma^\mu$ or $\beta$-loop segments between intersections to get a consistent overall orientation while the parameter $c_{ij}$ is such that

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\(^5\)Of these sixteen contributions, eight come from attaching the open loop $\beta$ to the loop $\alpha \cdot \gamma$ and the other eight by attaching the open loop $\beta$ to $\alpha \cdot \gamma^{-1}$. 
$c_{ij} = -1$ if the multiloop $\Psi[(\alpha \cdot \gamma \cdot \beta)]_{ij}$ has open component starting at $\alpha_i$ and ending at $\beta_f$; otherwise $c_{ij} = +1$.

To regularize the mass terms $\mathcal{M}_1$ and $\mathcal{M}_2$ amounts to regularize (8) and (9). To this aim use is made of an auxiliary background flat metric and a prefered set of coordinates in which this metric is Euclidean. Lets take the partition of the 3-dimensional fictitious space in cubes of sides $L$. The regularised quantum version of the mass operators is readily performed by introducing

$$\hat{M} := \lim_{L, \xi \to 0} \sum_I L^3 \sqrt{-\hat{M}_1^L - \hat{M}_2^L}. \quad (10)$$

Lets describe $\hat{M}_1^L$ ($\hat{M}_2^L$ will be analysed below.)

$$\hat{M}_1^L := \frac{\sqrt{2}m}{8 \cdot 3! L^6} \int_{\partial I} d\sigma^2 \int_{\partial I} d\tau^2 \int_{\partial I} d\rho^2 (-1)^{r_a + r_b + r_c} \eta^{abc} n_a(\sigma) n_b(\tau) n_c(\rho) \nabla_{abc}[\gamma, \alpha],$$

$$= \frac{\sqrt{2}m}{8 \cdot 3! L^6} \int_{\partial I} d\sigma^2 \int_{\partial I} d\tau^2 \int_{\partial I} d\rho^2 (-1)^{r_a + r_b + r_c} \eta^{abc} n_a(\sigma) n_b(\tau) n_c(\rho) \cdot \{\hat{N}^{abc} - \hat{N}^{cba}\}, \quad (11)$$

where one particular box has been considered (in the fictitious metric). $\partial I$ indicate its boundary, namely the union of the six faces of the cube oriented outwards and $\gamma$ and $\alpha$ are closed and open, respectively, with a common point of intersection. The closed loop $\gamma$ has three gravitational hands inserted lying on the boundary of the box at the point $\sigma, \tau$, and $\rho$. The loop $\gamma$ is the triangle formed by three segments that connect the three points $\sigma, \tau$, and $\rho$. The loop $\gamma$ has three gravitational hands inserted lying on the boundary of the box at the point $\sigma, \tau$, and $\rho$. The loop $\gamma$ is the triangle formed by three segments that connect the three points $\sigma, \tau$, and $\rho$. The open loop has two fermions at its ends, $\eta(\alpha_i)$ and $\eta(\alpha_f)$, respectively. These fermions are in the box; $n_a$ is the normal one-form to the box’s boundary. No sum convention is applied to $r_a$ and $\eta^{abc}$; $r_a = 0$ at the front and $r_a = 1$ at the back of the boundary.
The action of $\hat{\mathcal{M}}_{L_1}$ on the loop states is as follows. The three surface integrals on the boundary of the box $I$ and the three line integrals along the loop $\beta$ that parametrize the loop state combine to give three numbers related to the intersections of the open loop with the boundary of the cube. The non-vanishing contribution can be traced back to the intersection of the open loop $\beta$ simultaneously with three different faces of the cube. That is to say when the open loop $\beta$ has at least a triple point of intersection in the cube. The cube shrinks down to that point in the limit $L \to 0$. Hereby the gravitational hands are smeared on the boundary of the cube, for each permutation of them there are $8 \cdot 16$ terms which correspond to all the possibilities in which the hands can lie on the faces of the cube. Each one of these terms has, in general, different weight because of two reasons: the first one is due to the orientation of the open loop $\beta$ when it intersects one of the three faces of the cube. The second one comes from the factor $+1$ or $-1$ depending on whether the hands lie at the front or the back faces. Note that of these $8 \cdot 16$ terms only one particular permutation of the hands enters once: 16 of them contribute for a specific loop $\beta$ since each hand intersects the open loop once. By taking into account the six permutations of the hands there are, for a specific loop, $16 \cdot 3!$ terms and for a general situation $8 \cdot 16 \cdot 3!$ terms.

It is important to mention that the prescription given above depends in a way on the open loop $\beta$ labeling the loop state. More precisely, one needs to know some topological information (intersections and end points) of $\beta$ in order to calculate its specific contribution. Observe that the prefactor in (11) is finite in the $L \to 0$ limit because the surface integrals produce a factor $L^6$ that cancels out the one in the denominator. Hence one gets a finite action of the operator. This is analogous to the case of the volume operator [3].

Due to the fact that $\hat{\mathcal{M}}_{L_1}$ has only gravitational hands it can even “see”
loop states without fermionic excitations, namely loop states parametrized by closed loops with at least a triple point of intersection. This is precisely the difference with the kinetic fermion term [15] (i.e. $H_{\text{Weyl}}$ above) which is “blind” to this type of loop states (it yields zero on such states.) Hence the interpretation here proposed for $\hat{\mathcal{M}}_1$ is as forming the gap fermion mass. This is the major result presented here.

For the case of $\hat{\mathcal{M}}_2$ and hence $\hat{Z}[\alpha]$ consider $(\alpha_{\vec{x},\vec{y}})(s)$ a straight line (in the background metric) that starts at $\vec{x}$ an points in the $\vec{y}$ direction

$$(\alpha_{\vec{x},\vec{y}})(s) = \vec{x} + s\vec{y}, \quad (\alpha_{\vec{x},\vec{y}})(0) = \vec{x}, \quad (\alpha_{\vec{x},\vec{y}})(1) = \vec{x} + \vec{y}. \quad (12)$$

Then define the following

$$\hat{\mathcal{M}}_2^\xi := \frac{1}{L^3} \int_I d^3x \hat{\mathcal{M}}_2^\xi(\vec{x}), \quad (13)$$

$$\hat{\mathcal{M}}_2^\xi(\vec{x}) := \frac{-m}{3\pi^3} \int d^3y \theta(\xi - |\vec{y}|) \hat{Z}[\alpha_{\vec{x},\vec{y}}], \quad (14)$$

where $\xi < L$ and $\theta$ is the step function.

Now if the end points of the open loop $\beta$, $\beta_i$ and $\beta_f$, coincide with the end points of the open loop $\alpha$ inside the ball centered at $\vec{x}$ and with radius $\xi$ one has

$$\lim_{L,\xi \to 0} \frac{L^3}{L} \sqrt{-\hat{M}_2^\xi} \Psi[\beta] = \lim_{L \to 0, \xi \to 0} \sqrt{-\frac{m}{3\pi^3} L^3 (\hat{F}_\beta + \hat{F}_\beta')} \Psi[\beta], \quad (15)$$

$$\hat{F}_e \Psi[\beta] = \begin{cases} \Psi[\alpha_{\vec{x},\beta_i - \vec{x}} \cdot \beta] & \text{if } e = \beta_f \\ \Psi[\alpha_{\vec{x},\beta_f - \vec{x}} \cdot \beta^{-1}] & \text{if } e = \beta_i. \end{cases} \quad (16)$$

The regularization parameters can be chosen as $L(\epsilon) = b\epsilon$, and $\xi(\epsilon) = b \sin \epsilon$, where $b$ is an arbitrary length. Remarkably the prefactor in (15) is

6This election corrects the one in [15].
finite and given by
\[ \lim_{\epsilon \to 0} C(L(\epsilon), \xi(\epsilon)) = \sqrt{\frac{m}{\pi L^4}} = \sqrt{\frac{3m}{8\pi}}. \] (17)

In this way, as opposed to \( \hat{\mathcal{M}}_1 \), only loops with pairs of coinciding point-like fermion excitations have contribution to \( \hat{\mathcal{M}}_2 \), which is quadratic in the fermion momentum variables. In this respect is rather similar to the kinetic energy fermion contribution to the Hamiltonian constraint \( \mathcal{H}_{\text{Weyl}} \). However since anyhow it modifies the fermion mass a “dressing” interpretation seems more appropriate.

In summary the Majorana type mass for fermions has been studied in the loop representation of non perturbative quantum gravity and fermions. There are two contributions one of which resembles a mass gap whereas the other seems to dress the corresponding mass. The former is non zero for loop states even lacking fermion excitations and containing at least triple intersections. The latter requires the presence of coinciding pairs of end points characterising the loop states. Setting the Einstein and kinetic fermion terms to zero, the Majorana mass operator turns out to be

\[ \hat{\mathcal{M}}_{\text{Majorana}} = \sum_{i,e} \sqrt{\hat{\mathcal{M}}_1^{(i)} + \hat{\mathcal{M}}_2^{(e)}} \] (18)

where the sum runs over (at least triple) intersections i of the loop states (with and without fermionic excitations) and end points e for open loops with pairs of coinciding point-like fermionic excitations. The limit in which the regularization parameters go to zero is understood on the r.h.s. of Eq.(18).

Some further comments are in order.

Topology of space. Recently Smolin put forward the equivalence between minimalist quantum wormholes (i.e. identifying pairs of space points) without matter and quantum Einstein-Weyl theory expressed in loop variables
In this picture, the fermionic character of the Weyl field being associated to the antisymmetrization of the mouths of the wormholes. In this way the fermionic matter gets encoded in the topological properties of the space. Also, non minimalist wormholes (that is smooth manifolds) could be considered having the results of the minimalist ones as their low energy limit [23]. This in turn suggests a scenario of the kind a Weyl field living on a space foam as the one considered by Friedman et al [12] that yields an effective theory in which the fermion field becomes massive! Nevertheless, the analysis of [12] relies on a perturbative approach which it is better to avoid in the loop representation. A way out consists in following Smolin’s strategy of studying the equivalence of the Einstein-Majorana theory, as given in the present work, to the non minimalist quantum wormholes. Further work is needed to settle down the issue of generating fermion masses out of the topology of the space in the context of non perturbative quantum gravity.

A mechanism of mass generation would allow to calculate the values of the masses of course but realistic values, i.e. related to nature, might well only come from the incorporation with the other non gravitational fundamental interactions. This possibility was left open but some steps are in progress [24] along the lines presented here combined with those of [13]. Also, a massive Dirac field is currently under study.

**Reality conditions and spin networks.** Relying on Ashtekar approach for gravity and spin-$\frac{1}{2}$ fields involves two complex local degrees of freedom for the gravitational field unless reality conditions are supplemented [20]. Of course the question remains open that such reality conditions will single out the correct inner product at the quantum level. Nevertheless, the present analysis is expected to be robust enough to encompass real variables along the lines of [21]. This will be possible after extending the spin-network framework to
include spin-$\frac{1}{2}$ fields, as in [22].

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