THERMODYNAMIC FORMALISM OF NON-AUTONOMOUS ITERATED FUNCTION SYSTEMS

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Abstract. We derive results in the thermodynamic formalism of non-autonomous iterated function systems (or NAIFSs for short) with countable infinite alphabet. NAIFSs differ from the usual (autonomous) iterated function systems, they are given [41] by a sequence of collections of continuous maps on a compact topological space, where maps are allowed to vary between iterations. The topological pressure and topological entropy are generalized to NAIFSs and several of their basic properties are provided. Especially, we generalize the classical Bowen’s result to NAIFSs, i.e., we show that the topological entropy is concentrated on the set of nonwandering points. Then, we define a notion of specification property under which, the NAIFS has positive topological entropy and all points are entropy points. In particular, each NAIFS with the specification property is topologically chaotic. Additionally, the ∗-expansive property for NAIFSs is introduced. We will finally prove that the topological pressure of any continuous potential can be computed as a limit at a definite size scale whenever the NAIFS satisfies the ∗-expansive property.

1. Introduction

The time dependent systems so-called non-autonomous, yield very flexible models than autonomous cases for the study and description of real world processes. They may be used to describe the evolution of a wider class of phenomena, including systems which are forced or driven. Non-autonomous discrete dynamical systems are strongly motivated from applications, e.g., in population biology [40] as well as applications to numerical approximations, switching systems [28] and synchronization [2, 27]. Here, we deal with non-autonomous iterated function systems (or NAIFSs for short) which differ from the usual (autonomous) iterated function systems. It is natural, and frequently necessary in applications, to consider the non-autonomous version of iterated function systems, where the system is allowed to vary at each time. (In the case where all maps are affine similarities, the resulting system is also called a Moran set construction [41]).

Generalized Cantor sets that studied by Robinson and Sharples [42] are examples of attractors of NAIFSs. Olson et al. [36] illustrate examples of pullback attractors. A pullback attractor serves as non-autonomous counterpart to the global attractor. Henderson et al. [23], extended the regularity results of [36] to a natural class of attractors of both autonomous and non-autonomous iterated functions systems of contracting similarities, and studied the Assouad, box-counting, Hausdorff and packing dimensions for the attractors of these class of dynamical systems. These regularity results are useful as pullback attractors can exhibit dimensionally different behaviour at different length scales. Rempe-Gillen and Urbański [41],

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studied the Hausdorff dimension of the limit set of NAIFSs. Under a suitable restriction on
the growth of the number of contractions used at each step, they showed that the Hausdorff
dimension of the limit set is determined by an equation known as Bowens formula. Also, they
proved Bowens formula for a class of infinite alphabet systems and deal with Hausdorff mea-
sures for finite systems, as well as continuity of topological pressure and Hausdorff dimension
for both finite and infinite systems. In particular they strengthened the existing continuity
results for infinite autonomous systems.

In this paper, we discuss NAIFSs and develop a thermodynamic formalism for them.
Thermodynamic formalism, i.e., the formalism of equilibrium statistical physics, was adapted
to the theory of dynamical systems in the classical works of Sinai, Ruelle and Bowen [13, 14, 45, 46, 48]. Topological pressure and topological entropy are two fundamental notions in
thermodynamic formalism. Topological pressure is the main tool in studying dimension of
invariant sets and measures for dynamical systems in dimension theory. On the other hand,
the notion of entropy is one of the most important objects in dynamical systems, either as
a topological invariant or as a measure of the chaoticity of dynamical systems. Hence, there
were several attempts to find their generalization for other systems in an attempt to describe
their dynamical characteristics, see, for instance, [25, 24, 29, 30, 32, 50, 58, 59, 60].

Recently, there have been major efforts in establishing a general theory of NAIFSs [23, 41],
but a global theory is still out of reach. Our main goal in this paper is to describe the
topological aspects of thermodynamic formalism for NAIFSs.

The concept of topological entropy of a map plays a central role in topological dynamics.
There are two standard definitions of topological entropy for a continuous self-map of a
compact metric space [22]. The first definition was given by Adler, Konhelm and McAndrew
[1], based on open covers, can be applied to continuous maps of any compact topological
space. In 1971, Bowen [11] and Dinaburg [17] gave other definitions, based on the dispersion
of orbits, for uniformly continuous maps in metric spaces. When the metric space is compact,
these definitions yield the same quantity, which is an invariant of topological conjugacy. Also,
Bowen [12] gave a characterization of dimension type for topological entropy of non-compact
and non-invariant sets. Topological entropy has close relationships with many important
dynamical properties, such as chaos, Lyapunove exponents, the growth of the number of
periodic points and so on. Moreover, positive topological entropy have remarkable role in the
characterization of the dynamical behaviors, for instance, Downarowicz proved that positive
topological entropy implies chaos DC2 [18]. Thus, a lot of attention has been focused on
computations and estimations of topological entropy of an autonomous dynamical system
and many good results have been obtained [8, 9, 11, 10, 21, 20, 26, 33, 34].

Beyond autonomous dynamical systems, several authors provided conditions for computa-
tions and estimations of topological entropy, for instance, Shao et al. [47] have given an
estimation of lower bound of topological entropy for coupled-expanding systems associated
with transition matrices in compact Hausdorff spaces. Some knowledge of topological entropy
of semigroup actions is also available in [4, 6, 7]. Rodrigues and Varandas [43] proved that
any finitely generated continuous semigroup action on a compact metric space with the strong
orbital specification property has positive topological entropy; moreover, every point is an en-
tropy point. Roughly speaking, entropy points are those that their local neighborhoods reflect
the complexity of the entire dynamical system from the viewpoint of entropy theory. Also,
these results extended to non-autonomous discrete dynamical systems by Nazarian Sarkooh and Ghane [35]. In the current paper, some of these results are generalized to NAIFSs.

The notion of topological pressure, based on separated sets, was brought to the theory of dynamical systems by Ruelle [44], later other definitions of topological pressure, based on open covers and spanning sets, were given by Walters [55, 56] and it was further developed by Pesin and Pitskel [39]. Pesin [38] used the dimension approach to the notion of topological pressure, which is based on the Caratheodory structure [38, 16]. Recently, there were several attempts to find suitable generalizations for other systems, see, for instance, [24] for non-autonomous discrete dynamical systems and [31, 32, 43] for semigroup actions. In this paper, we define and study the topological pressure for NAIFSs.

It is well-known that the topological pressure can be computed as the limiting complexity of the dynamical system as the size scale approaches to zero. Thus, several authors provided conditions so that the topological pressure of a dynamical system can be computed as a limit at a definite size scale. For instance, Rodrigues and Varandas [43] showed that the topological pressure of any continuous potential that satisfies the bounded distortion condition can be computed as a limit at a definite size scale for any finitely generated continuous semigroup action on a compact metric space with some kind of expansive property. Also, this result extended to non-autonomous discrete dynamical systems by Nazarian Sarkooh and Ghane [35]. Here, this result is extended to NAIFSs.

This is how the paper is organized: In Section 2, we give the precise definition of an NAIFS and present an overview of the main concepts and introduce notations that will study throughout this paper. We define and study the topological entropy for NAIFSs in Section 3. Especially, we generalize for the case of NAIFSs the classical Bowens result [10] saying that the topological entropy is concentrated on the set of nonwandering points. Then, in Section 4, we generalize the concept of specification to NAIFSs and characterize the entropy points for NAIFSs with the specification property and show that any NAIFS of surjective maps with the specification property has positive topological entropy and all points are entropy point. In particular, each NAIFS with the specification property is topologically chaotic. Finally, in Section 5 we define and study the topological pressure for NAIFSs. Also, we introduce the notion of ∗-expansive NAIFS and show that the topological pressure of any continuous potential can be computed as a limit at a definite size scale for every NAIFS with the ∗-expansive property.

2. Preliminaries

Following [41], a non-autonomous iterated function system (or NAIFS for short) is a pair (X, Φ) in which X is a set and Φ consists of a sequence \{Φ(j)\}_{j≥1} of collections of maps, where Φ(j) = {ϕ_i(j) : X → X}_{i ∈ I(j)} and I(j) is a nonempty finite index set for all j ≥ 1. By (X, Φ_k), we denote the pair of X and shifted sequence \{Φ(j)\}_{j≥k} and we use analogous notation for other sequences of objects related to an NAIFS. If the set X is a compact topological space and all the ϕ_i(j) are continuous, we speak of a topological NAIFS. Note that in the case where all ϕ_i(j) are contraction affine similarities, this is also referred to as a Moran set construction. For simplicity, we define the following symbolic spaces for positive integers m, n ≥ 1:

\[ I^{m,n} := \prod_{j=0}^{n-1} I^{m+j}, \quad I^{m,∞} := \prod_{j=m}^{∞} I^{(j)} . \]
Elements of $I^{1,n}$ are called initial $n$-words, while those of $I^{m,n}$ with $m > 1$ are called non-initial $n$-words. If there is no confusion, we use the term $n$-words for these two cases without further characterization.

A word $w$ is called finite if $w \in I^{m,n}$ for some $m, n \geq 1$, in this case its length is $n$ and denoted by $|w| := n$. While, each word $w \in I^{m,\infty}$ is called an infinite word and its length is infinity and denoted by $|w| := \infty$. For finite (infinite) word $w = w_mw_{m+1} \cdots w_{m+n-1}(w = w_mw_{m+1} \cdots ) \in I^{m,n}(I^{m,\infty})$ and $1 \leq k \leq |w|(1 \leq k < \infty)$ we define $|w|_k = w_{m+1} \cdots w_{m+k-1}$ and $|w|^k = w_mw_{m+1} \cdots w_{m+k-1}$ and $|w|^k = w_{m+k} \cdots w_{m+n-1}(w)k = w_{m+1}w_{m+2} \cdots w_{m+k}w_{m+k+1} \cdots )$.

The time evolution of the system is defined by composing the maps $\varphi_i^{(j)}$ in the obvious way. In general, for finite (infinite) word $w = w_mw_{m+1} \cdots w_{m+n-1}(w = w_mw_{m+1} \cdots ) \in I^{m,n}(I^{m,\infty})$ and $1 \leq k \leq |w|(1 \leq k < \infty)$ we define

$$\varphi^{m,k}_w := \varphi^{(m+k-1)}_{w_{m+k-1}} \circ \cdots \circ \varphi^{(m+1)}_{w_{m+1}} \circ \varphi^{(m)}_{w_m} \quad \text{and} \quad \varphi^{m,0}_w := id_X.\tag{1}$$

We put $\varphi^{m,k}_w := (\varphi^{m,k}_w)^{-1}$, which will be applied to sets, because we do not assume that the maps $\varphi_i^{(j)}$ are invertible. The orbit (trajectory) of a point $x \in X$ is the set $\{\varphi^{1,k}_w(x) : k \geq 0 \text{ and } w \in I^{1,\infty}\}$. Also, for $w \in I^{1,\infty}$, the $w$-orbit of a point $x \in X$ is the sequence $\{\varphi^{1,k}_w(x)\}_{k \geq 0}$.

Let NAIFS $(X, \Phi)$ and $n \geq 1$ be given. Denote by $(X, \Phi^n)$ the NAIFS defined by the sequence $\{\Phi^{(j,n)}\}_{j \geq 1}$, where $\Phi^{(j,n)}$ is the collection $\{\varphi^{(j,n)}_{w_j}\}_{w_j \in I^{(j,n)}}$, $I^{(j,n)} := \{w_j^{(*)} \in I^{(j-1)n+1,n}\}$ (note that $I^{(j,n)} = I^{(j-1)n+1,n}$) and $\varphi^{(j,n)}_{w_j} := \varphi^{(jn)}_{w_{jn}} \circ \varphi^{((j-1)n+2)}_{w_{(j-1)n+2}} \circ \varphi^{((j-1)n+1)}_{w_{(j-1)n+1}}$ for $w_j^{(*)} = w_{(j-1)n+1}w_{(j-1)n+2} \cdots w_{jn}$. Take $I^{m,k}_* := \prod_{j=0}^{k-1} I^{(m+j,n)}$, then $\#(I^{m,k}_*) = \#(I^{m,m})$, where $\#(A)$ is the cardinal number of the set $A$. For $w = w_1w_2 \cdots w_{mn} \in I^{1,mn}$ and $1 \leq j \leq m$, denote $w_{(j-1)n+1}w_{(j-1)n+2} \cdots w_{jn}$ by $w_j^{(*)} \in I^{(j,n)}$, then $w = w_1^{(*)}w_2^{(*)} \cdots w_m^{(*)} \in I^{m,m}_*$. For simplicity, we denote elements in $I^{m,m}_*$ by $w^{(*)}$ and use analogous notation for other sequences of objects related to an NAIFS.

Throughout this paper we work with topological NAIFSs; otherwise, we express them with the details.

3. Topological entropy of NAIFSs

In this section we deal with the topological entropy for NAIFSs. First, we extend the classical definition of topological entropy to NAIFSs via open covers. Then we give the Bowen-like definitions of topological entropy for NAIFSs and show that these different definitions coincide. We will also establish some basic properties for topological entropy of NAIFSs. Especially, we generalize the classical Bowen’s result to NAIFSs ensures that the topological entropy is concentrated on the set of nonwandering points.

3.1. Topological entropy of NAIFSs via open covers. In this subsection we are going to extend the definition of topological entropy to NAIFSs via open covers, which is a natural generalization of the definition of topological entropy for autonomous dynamical systems [56], non-autonomous discrete dynamical systems [29] and semigroup actions [49]. In fact, if $\#(I^{(j)}) = 1$ and $\Phi^{(j)} = \{\varphi^{(j)}_1\}$ for every $j \geq 1$, then we get the definition of topological entropy for non-autonomous discrete dynamical system $(X, \varphi_{1,\infty})$, where $\varphi_{1,\infty}$ is the sequence $\{\varphi^{(j)}_1\}_{j=1}^{\infty}$. Additionally, if $\varphi^{(j)}_1 = \varphi$ for every $j \geq 1$, then we get the classical definition of topological entropy for autonomous dynamical system $(X, \varphi)$. Moreover, in the case that
\(\Phi^{(i)} = \Phi^{(j)}\) for all \(i, j \geq 1\), then we get the definition of topological entropy for semigroup action \((X,G)\) with generator set \(\{\varphi^{(i)}_g : i \in I^{(1)}\}\).

Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact topological space \(X\). We define its topological entropy as follows. A family \(\mathcal{A}\) of subsets of \(X\) is called a cover (of \(X\)) if their union is all of \(X\). For open covers \(A_1, A_2, \cdots, A_n\) of \(X\) we denote

\[
\bigvee_{i=1}^{n} A_i = A_1 \vee A_2 \vee \cdots \vee A_n = \{A_1 \cap A_2 \cap \cdots \cap A_n : A_i \in A_i \text{ for } 1 \leq i \leq n\}.
\]

Note that \(\bigvee_{i=1}^{n} A_i\) is also an open cover of \(X\). For open covers \(A\), finite word \(w = w_m w_{m+1} \cdots w_{m+n-1} \in I^{m,n}\) and \(0 \leq j \leq n\) we denote \(\varphi^{m-j}_w(A) = \{\varphi^{m-j}_w(A) : A \in A\}\) and \(A^{m,n}_w := \bigvee_{j=0}^{n} \varphi^{m-j}_w(A)\). For each \(0 \leq j \leq n\), \(\varphi^{m-j}_w(A)\) is an open cover, so \(A^{m,n}_w\) is also an open cover. Next, we denote by \(\mathcal{N}(\mathcal{A})\) the minimal possible cardinality of a subcover chosen from \(\mathcal{A}\). Then

\[
h(X, \Phi; A) := \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \mathcal{N}(A^{1,n}_w) \right)
\]

is said to be the topological entropy of NAIFS \((X, \Phi)\) on the cover \(A\), where \(\#(I^{1,n})\) is the cardinality of the set \(I^{1,n}\). The topological entropy of NAIFS \((X, \Phi)\) is defined by

\[
h_{\text{top}}(X, \Phi) := \sup \{h(X, \Phi; A) : A \text{ is an open cover of } X\}.
\]

For open covers \(A, B\) of \(X\), continuous map \(g : X \to X\) and finite word \(w \in I^{m,n}\), the following inequalities hold:

\[
(2) \quad \mathcal{N}(A \vee B) \leq \mathcal{N}(A) \cdot \mathcal{N}(B),
\]

\[
(3) \quad \mathcal{N}(\varphi^{m-n}_w(A)) \leq \mathcal{N}(A),
\]

\[
(4) \quad g^{-1}(A \vee B) = g^{-1}(A) \vee g^{-1}(B).
\]

We say that a cover \(A\) is finer than a cover \(B\), and write \(A > B\), when each element of \(A\) is contained in some element of \(B\). Clearly, we have

\[
(5) \quad \text{if } A > B \text{ then } \mathcal{N}(A) \geq \mathcal{N}(B).
\]

and for each \(w \in I^{1,n}\)

\[
(6) \quad \text{if } A > B \text{ then } A^{1,n}_w > B^{1,n}_w.
\]

Hence,

\[
(6) \quad \text{if } A > B \text{ then } h(X, \Phi; A) \geq h(X, \Phi; B).
\]

Since \(X\) is compact, in the definition of \(h_{\text{top}}(X, \Phi)\) it is sufficient to take the supremum only over all open finite covers. If \(A\) is an open finite cover of \(X\) and \(w \in I^{1,n}\) then the cardinality of \(A^{1,n}_w\) is at most \((\#(A))^n\). Therefore, \(h(X, \Phi; A) \leq \log(\#(A))\) and so \(0 \leq h(X, \Phi; A) < \infty\). But, it can be \(h_{\text{top}}(X, \Phi) = \infty\).

Now, we extend the definition of topological entropy of an NAIFS to not necessarily compact and not necessarily invariant subsets of a compact topological space. Note that the idea of defining the topological entropy for non-compact and non-invariant sets is not new. See [12] and [37], where Bowen and Pesin introduce the dimension definition of topological entropy for autonomous dynamical systems, that applied to not necessarily compact and not necessarily invariant subsets of a topological space. Let \((X, \Phi)\) be an NAIFS of continuous
maps on a compact topological space $X$ and $Y$ be a nonempty subset of $X$. The set $Y$ may not be compact and may not exhibit any kind of invariance with respect to $\Phi$. If $\mathcal{A}$ is a cover of $X$ we denote by $\mathcal{A}|_Y$ the cover $\{A \cap Y: A \in \mathcal{A}\}$ of the set $Y$. Then we define the topological entropy of NAIFS $(X,\Phi)$ on the set $Y$ by

$$h_{\text{top}}(Y,\Phi) := \sup\{h(Y,\Phi;\mathcal{A}): \mathcal{A} \text{ is an open cover of } X\},$$

where

$$h(Y,\Phi;\mathcal{A}) := \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} \mathcal{N}(\mathcal{A}_{w,n}|_Y) \right).$$

Again, it is sufficient to take the supremum only over all open finite covers of $X$.

### 3.2. Equivalent Bowen-like definitions of topological entropy.

Let $(X,\Phi)$ be an NAIFS of continuous maps on a compact metric space $(X,d)$. For finite (infinite) word $w = w_m w_{m+1} \cdots w_{m+n-1}(w = w_m w_{m+1} \cdots) \in I^{m,n}(I^{m,\infty})$ and $1 \leq k \leq |w|(1 \leq k < \infty)$ we introduce on $X$ the Bowen-metrics

$$d_{w,k}(x,y) := \max_{0 \leq j \leq k} d(\varphi_w^j(x),\varphi_w^j(y)). \tag{7}$$

Also, for finite (infinite) word $w = w_m w_{m+1} \cdots w_{m+n-1}(w = w_m w_{m+1} \cdots) \in I^{m,n}(I^{m,\infty})$, $1 \leq k \leq |w|(1 \leq k < \infty)$, $x \in X$ and $\epsilon > 0$, we define

$$B(x;w,k,\epsilon) := \{y \in X: d_{w,k}(x,y) < \epsilon\}, \tag{8}$$

which is called the dynamical $(k+1)$-ball with radius $\epsilon$ relative to word $w$ around $x$.

Fix $w \in I^{1,n}$ for some $n \geq 1$. A subset $E$ of the space $X$ is called $(n,w,\epsilon;\Phi)$-separated, if for any two distinct points $x,y \in E$, $d_{w,n}(x,y) > \epsilon$ (note that $|w| = n$). Also, a subset $F$ of the space $X$, $(n,w,\epsilon;\Phi)$-spans another subset $K \subseteq X$, if for each $x \in K$ there is a $y \in F$ such that $d_{w,n}(x,y) \leq \epsilon$. For subset $Y$ of $X$ we define $s_n(Y;w,\epsilon,\Phi)$, as the maximal cardinality of an $(n,w,\epsilon;\Phi)$-separated set in $Y$ and $r_n(Y;w,\epsilon,\Phi)$ as the minimal cardinality of a set in $Y$ which $(n,w,\epsilon;\Phi)$-spans $Y$. If $Y = X$ we sometime suppress $Y$ and shortly write $s_n(w,\epsilon,\Phi)$ and $r_n(w,\epsilon,\Phi)$.

#### Lemma 3.1

Let $(X,\Phi)$ be an NAIFS of continuous maps on a compact metric space $(X,d)$ and $Y$ be a nonempty subset of $X$. Then,

$$h_{\text{top}}(Y,\Phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_n(Y;\epsilon,\Phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R_n(Y;\epsilon,\Phi),$$

where

$$S_n(Y;\epsilon,\Phi) := \frac{1}{\#(I^n)} \sum_{w \in I^n} s_n(Y;w,\epsilon,\Phi)$$

and

$$R_n(Y;\epsilon,\Phi) := \frac{1}{\#(I^n)} \sum_{w \in I^n} r_n(Y;w,\epsilon,\Phi).$$

Note that the limits can be replaced by $\sup_{\epsilon > 0}$, because for $\epsilon_2 < \epsilon_1$ and $w \in I^{1,n}$ we have

$$r_n(Y;w,\epsilon_2,\Phi) \geq r_n(Y;w,\epsilon_1,\Phi) \quad \text{and} \quad s_n(Y;w,\epsilon_2,\Phi) \geq s_n(Y;w,\epsilon_1,\Phi).$$
Proof. First we prove the second equality. Fix $\epsilon > 0$. It is enough to show that

$$R_n(Y; \epsilon, \Phi) \leq S_n(Y; \epsilon, \Phi) \leq R_n(Y; \frac{\epsilon}{2}, \Phi).$$

To do this it is enough to prove

$$r_n(Y; w, \epsilon, \Phi) \leq s_n(Y; w, \epsilon, \Phi) \leq r_n(Y; w, \frac{\epsilon}{2}, \Phi)$$

for every $w \in I^{1,n}$. Fix $w \in I^{1,n}$. It is obvious that any maximal $(n, w, \epsilon; \Phi)$-separated subset of $Y$ is an $(n, w, \epsilon; \Phi)$-spanning set for $Y$. Therefore $r_n(Y; w, \epsilon, \Phi) \leq s_n(Y; w, \epsilon, \Phi)$. To show the other inequality of (10) suppose $E$ is an $(n, w, \epsilon; \Phi)$-separated subset of $Y$ and $F \subset X$ is an $(n, w, \frac{\epsilon}{2}; \Phi)$-spanning set of $Y$. Define $\psi : E \to F$ by choosing, for each $x \in E$, some point $\psi(x) \in F$ with $d_{w,n}(x, \psi(x)) \leq \frac{\epsilon}{2}$. The point $\psi(x) \in F$ satisfying this condition is unique. Thus $\psi$ is injective and therefore the cardinality of $E$ is not greater than that of $F$. Hence, $s_n(Y; w, \epsilon, \Phi) \leq r_n(Y; w, \frac{\epsilon}{2}, \Phi)$. This completes the proof of relations (9) and (10), thus the proof of the second equality is completed.

To prove the first equality, let $\epsilon > 0$ and $w \in I^{1,n}$ be given. Let $E$ be an $(n, w, \epsilon; \Phi)$-separated subset of $Y$ and $A$ be an open cover of $X$ by sets of diameter less than $\epsilon$. Then by definition of $(n, w, \epsilon; \Phi)$-separated sets two distinct point of $E$ cannot lie in the same element of $A \vee \varphi_w^{1-n}(A) \vee \varphi_w^{1-2}(A) \vee \cdots \vee \varphi_w^{1-n}(A)$. Therefore $s_n(Y; w, \epsilon, \Phi) \leq N(A_{w,n}^{1,n}|Y)$. Hence, by the definition of topological entropy, it follows that

$$h_{top}(Y, \Phi) \geq \lim \sup_{n \to \infty} \frac{1}{n} \log s_n(Y; \epsilon, \Phi).$$

To show the inverse of relation (11), let $A$ be an open cover of $X$ and $\lambda > 0$ be a Lebesgue number for $A$. Then, for every $x \in X$ and $\epsilon < \frac{\lambda}{2}$, the closed $\epsilon$-ball $B_\epsilon(x)$ lies inside some element $A_\alpha \in A$. Fix $w \in I^{1,n}$. Let $F$ be an $(n, w, \epsilon; \Phi)$-spanning set of $Y$ with minimal cardinality that gives $r_n(Y; w, \epsilon, \Phi)$. For each $z \in F$ and each $0 \leq k \leq n$ (note that $|w| = n$), let $A_k(z)$ be some element of $A$ containing $B_\epsilon(\varphi_w^{1-k}(z))$. On the other hand, as $F$ is an $(n, w, \epsilon; \Phi)$-spanning set of $Y$ with minimal cardinality, for any $y \in Y$ there is a $z \in F$ such that $\varphi_w^{1-k}(y) \in B_\epsilon(\varphi_w^{1-k}(z))$ for $0 \leq k \leq n$. Thus, $\varphi_w^{1-k}(y) \in A_k(z)$ for $0 \leq k \leq n$, and the family

$$\{ A_0(z) \cap \varphi_w^{1-1}(A_1(z)) \cap \cdots \cap \varphi_w^{1-n}(A_n(z)) : z \in F \}$$

is a subcover of the cover $A_{w,n}^{1,n}|Y$ of $Y$. Hence, $N(A_{w,n}^{1,n}|Y) \leq #(F) = r_n(Y; w, \epsilon, \Phi)$. Now, by the definition of topological entropy, it follows that

$$h_{top}(Y, \Phi) \leq \lim \sup_{n \to \infty} \frac{1}{n} \log R_n(Y; \epsilon, \Phi) = \lim \sup_{n \to \infty} \frac{1}{n} \log S_n(Y; \epsilon, \Phi).$$

By relations (11) and (12) we have

$$h_{top}(Y, \Phi) = \lim \sup_{n \to \infty} \frac{1}{n} \log S_n(Y; \epsilon, \Phi),$$

and the proof is completed.

Remark 3.2. Note that, $r_n(Y; w, \epsilon, \Phi)$ is defined for $w \in I^{1,n}$ as the minimal cardinality of a set in $Y$ which $(n, w, \epsilon; \Phi)$-spans $Y$. If we take $r_n^X(Y; w, \epsilon, \Phi)$ for $w \in I^{1,n}$ as the minimal cardinality of a set in $X$ which $(n, w, \epsilon; \Phi)$-spans $Y$, again we have

$$h_{top}(Y, \Phi) = \lim \sup_{n \to \infty} \frac{1}{n} \log R_n^X(Y; \epsilon, \Phi).$$
where
\[ R^X_n(Y; \epsilon, \Phi) := \frac{1}{\#(I^n)} \sum_{w \in I^n} r^X_n(Y; w, \epsilon, \Phi). \]

Hence, it is not important that we take \( r^X_n(Y; w, \epsilon, \Phi) \) for \( w \in I^n \) as the minimal cardinality of a set in \( Y \) which \((n, w, \epsilon, \Phi)\)-spans \( Y \) or as the minimal cardinality of a set in \( X \) which \((n, w, \epsilon, \Phi)\)-spans \( Y \).

### 3.3. Basic properties of topological entropy

In this section we want to prove some basic properties of topological entropy of NAIFSs. First, we give the following auxiliary lemma that is an extension of [3, Lemma 4.1.9] and we use it in the proof of Proposition 3.4.

**Lemma 3.3.** Let for \( 1 \leq i \leq k \), \( n = 1, 2, \ldots \) and \( w \in I^n \) in which \( I^n \) is a nonempty finite set, \( a_{n,w,i} \)'s be non-negative numbers. Then
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right) = \max_{1 \leq i \leq k} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right).
\]

**Proof.** For every \( 1 \leq j \leq k \) and every \( w \in I^n \) we have \( \sum_{i=1}^k a_{n,w,i} \geq a_{n,w,j} \), thus by taking summation over \( w \in I^n \) and dividing to \( \#(I^n) \) we have
\[
\frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \geq \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,j}
\]

Hence
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right) \geq \max_{1 \leq i \leq k} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right).
\]

On the other hand
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right) \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{k}{\#(I^n)} \max_{1 \leq i \leq k} \sum_{w \in I^n} a_{n,w,i} \right)
\]
\[
= \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \max_{1 \leq i \leq k} \sum_{w \in I^n} a_{n,w,i} \right)
\]
\[
= \limsup_{n \to \infty} \max_{1 \leq i \leq k} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right)
\]
\[
\leq \max_{1 \leq i \leq k} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^n)} \sum_{w \in I^n} a_{n,w,i} \right).
\]

The proof is completed. \( \square \)

**Proposition 3.4.** Let \( (X, \Phi) \) be an NAIFS of continuous maps on a compact topological space \( X \). If \( X = \bigcup_{i=1}^k X_i \) in which each \( X_i \) is an arbitrary nonempty subset of \( X \), then
\[
h_{\text{top}}(X, \Phi) = \max_{1 \leq i \leq k} h_{\text{top}}(X_i, \Phi).
\]

Note that, we don’t need to assume that the sets \( X_i \) are closed or invariant (invariant in the sense that they contain the trajectories of all points), because we have defined the topological entropy of NAIFS \((X, \Phi)\) on every subset of \( X \).
Proof. By the definition of topological entropy we have \( h_{\text{top}}(X, \Phi) \geq \max_{1 \leq i \leq k} h_{\text{top}}(X_i, \Phi) \). Hence, it is enough to prove the converse inequality. To do this, consider \( w \in I^{1,n} \) and open cover \( \mathcal{A} \) of \( X \). Let \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k \) be subcovers chosen from the covers \( \mathcal{A}_{w}^{1,n} |_{X_1}, \mathcal{A}_{w}^{1,n} |_{X_2}, \ldots, \mathcal{A}_{w}^{1,n} |_{X_k} \), respectively. Then each element of \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k \) is contained in some element of \( \mathcal{A}_{w}^{1,n} \) and \( \mathcal{B} \) is an open cover of \( X \). Thus,

\[
\mathcal{N}(\mathcal{A}^{1,n}_w) \leq \sum_{i=1}^{k} \mathcal{N}(\mathcal{A}^{1,n}_{w_i} |_{X_i}) \text{ for each } w \in I^{1,n} \text{ and } n \geq 1.
\]

Now, by Lemma 3.3, we have

\[
h(X, \Phi; \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \mathcal{N}(\mathcal{A}^{1,n}_w) \right) \\
\leq \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \mathcal{N}(\mathcal{A}^{1,n}_{w_i} |_{X_i}) \right) \\
= \max_{1 \leq i \leq k} \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \mathcal{N}(\mathcal{A}^{1,n}_{w_i} |_{X_i}) \right) \\
= \max_{1 \leq i \leq k} h(X_i, \Phi; \mathcal{A}) \\
\leq \max_{1 \leq i \leq k} h_{\text{top}}(X_i, \Phi).
\]

Since open cover \( \mathcal{A} \) of \( X \) was arbitrary, we obtain \( h_{\text{top}}(X, \Phi) \leq \max_{1 \leq i \leq k} h_{\text{top}}(X_i, \Phi) \), that completes the proof. \( \square \)

Now, we extend an analogue of the well known property \( h_{\text{top}}(\varphi^n) = n \cdot h_{\text{top}}(\varphi) \) of the topological entropy of autonomous dynamical systems to NAIFSs and we use it in the proof of Theorem 3.15. To do this we begin with the following lemma.

Lemma 3.5. Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact topological space \( X \). Then \( h_{\text{top}}(X, \Phi^n) \leq n \cdot h_{\text{top}}(X, \Phi) \) for every \( n \geq 1 \).

Proof. Fix \( n \geq 1 \) and consider NAIFS \((X, \Phi^n)\) which induced by NAIFS \((X, \Phi)\). For finite word \( w^* = w^*_m w^*_{m+1} \ldots w^*_m s - 1 \in I^m_s \) and \( 1 \leq k \leq s \), similar to equation (1) we define

\[
\varphi^{m,k}_{w^*} := \varphi^{(m+k-1,n)}_{w^*_m+k-1} \circ \cdots \circ \varphi^{(m+1,n)}_{w^*_{m+1}} \circ \varphi^{(m,n)}_{w^*_m} \text{ and } \varphi^{m,0}_{w^*} := \text{id}_X,
\]

where \( \varphi^{(r,n)}_{w^*} = \varphi^{(r,n)}_{w^*_m} \circ \cdots \circ \varphi^{((r-1)n+2)}_{w^*_m s - 1} \circ \varphi^{((r-1)n+1)}_{w^*_m s - 1} \) for \( w^*_m = w_{(r-1)n+1} w_{(r-1)n+2} \cdots w_{rn} \). Also, we put \( \varphi^{m,-k}_{w^*} := (\varphi^{m,k}_{w^*})^{-1} \). Then for any open cover \( \mathcal{A} \) of \( X \) by relation (5) we have
\[ h(X, \Phi; \mathcal{A}) = \limsup_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(I_{1,m})} \sum_{w \in I_{1,m}} \mathcal{N}(A_{w}^{1,m}) \right) \]
\[ \geq \limsup_{m \to \infty} \frac{1}{mn} \log \left( \frac{1}{\#(I_{1,mn})} \sum_{w \in I_{1,mn}} \mathcal{N}(A_{w}^{1,mn}) \right) \]
\[ = \limsup_{m \to \infty} \frac{1}{mn} \log \left( \frac{1}{\#(I_{1,mn})} \sum_{w \in I_{1,mn}} \mathcal{N}\left( \bigvee_{j=0}^{m} \varphi_{w,j}^{1}(A) \right) \right) \]
\[ \geq \limsup_{m \to \infty} \frac{1}{mn} \log \left( \frac{1}{\#(I_{1,mn})} \sum_{w \in I_{1,mn}} \mathcal{N}\left( \bigvee_{j=0}^{m} \varphi_{w,j}^{1}(A) \right) \right) \]
\[ = \frac{1}{n} \limsup_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(I_{1,m})} \sum_{w^{n} \in I_{1,m}} \mathcal{N}(A_{w^{n}}^{1,m}) \right) \]
\[ = \frac{1}{n} \limsup_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(I_{1,m})} \sum_{w^{n} \in I_{1,m}} \mathcal{N}(A_{w^{n}}^{1,m}) \right) \]
\[ = \frac{1}{n} \cdot h(X, \Phi^{n}; \mathcal{A}). \]

Thus, \( h_{\text{top}}(X, \Phi^{n}) \leq n \cdot h_{\text{top}}(X, \Phi) \), and the proof is completed.\( \square \)

**Remark 3.6.** In a similar way one can prove that for every subset \( Y \) of \( X \) and every \( n \geq 1 \), \( h_{\text{top}}(Y, \Phi^{n}) \leq n \cdot h_{\text{top}}(Y, \Phi) \). Also, in general we cannot claim that \( h_{\text{top}}(X, \Phi^{n}) = n \cdot h_{\text{top}}(X, \Phi) \) (see the comment after Lemma 4.2 in [29], where \( \#(I^{j}) = 1 \) for every \( j \in \mathbb{N} \). Note that the results in [29] are about non-autonomous discrete dynamical systems which are a special case of NAIFSs.

In Theorem 3.8, we give some sufficient conditions to have equality in Lemma 3.5. Let us begin with the following definition.

**Definition 3.7.** An NAIFS \( (X, \Phi) \) of continuous maps on a compact metric space \( (X, d) \) is said to be equicontinuous, if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that the implication \( d(x, y) < \delta \Rightarrow d(\varphi_{i}^{j}(x), \varphi_{i}^{j}(y)) < \epsilon \) holds for every \( x, y \in X, j \geq 1 \) and \( i \in I^{j} \).

**Theorem 3.8.** Let \( (X, \Phi) \) be an equicontinuous NAIFS on a compact metric space \( (X, d) \). Then for every \( n \geq 1 \), \( h_{\text{top}}(X, \Phi^{n}) = n \cdot h_{\text{top}}(X, \Phi) \).

**Proof.** For \( n = 1 \) everything is obvious. For \( n \geq 2 \), by Lemma 3.5 it suffices to show that \( h_{\text{top}}(X, \Phi^{n}) \geq n \cdot h_{\text{top}}(X, \Phi) \). By equicontinuity, for every \( \epsilon > 0 \) take \( \Delta(\epsilon) \geq \epsilon \) such that \( \Delta(\epsilon) \to 0 \) if \( \epsilon \to 0 \), and \( d(\varphi_{w^{n,k}}^{m,k}(x), \varphi_{w^{n,k}}^{m,k}(y)) \leq \Delta(\epsilon) \) whenever \( m \geq 1, k \in \{1, \ldots, n\}, w \in I^{m,k} \) and \( d(x, y) \leq \epsilon \). Note that one can put

\[ \Delta(\epsilon) = \epsilon + \sup_{m \geq 1} \max_{1 \leq k \leq n} \{ d(\varphi_{w^{m,k}}^{m,k}(x), \varphi_{w^{m,k}}^{m,k}(y)) : x, y \in X, d(x, y) \leq \epsilon \}. \]

Let \( m \) be a positive integer and \( w = w_{1} w_{2} \cdots w_{mn} \in I_{1,mn}^{1,mn} \), then \( w = w_{*} = w_{1}^{*} w_{2}^{*} \cdots w_{m}^{*} \in I_{*}^{1,m} \) in which \( w_{i}^{*} = w_{(i-1)n+1} w_{(i-1)n+2} \cdots w_{in} \) for \( 1 \leq i \leq m \). By definition of \( \Delta(\epsilon) \), for
\( \epsilon > 0 \) any \((mn, w, \Delta(\epsilon); \Phi)\)-separated set is \((m, w^*, \epsilon; \Phi^n)\)-separated. Thus, \( s_{mn}(w, \Delta(\epsilon), \Phi) \leq s_m(w^*, \epsilon, \Phi^n) \). Further, by the definition of separated sets

\[
s_{(m-1)n+r}(w|_{(m-1)n+r}, \Delta(\epsilon), \Phi) \leq s_{mn}(w, \Delta(\epsilon), \Phi) \quad \text{for } 1 \leq r \leq n \text{ and } w \in \mathcal{I}^{1,mn}.
\]

Hence, by equation (13) for every \( 1 \leq r \leq n \),

\[
h_{\text{top}}(X, \Phi^n) = \lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(\mathcal{I}^{1,m})} \sum_{w^* \in \mathcal{I}^{1,m}} s_m(w^*, \epsilon, \Phi^n) \right)
\]

\[
\geq \lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(\mathcal{I}^{1,mn})} \sum_{w \in \mathcal{I}^{1,mn}} s_{mn}(w, \Delta(\epsilon), \Phi) \right)
\]

\[
\geq \lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(\mathcal{I}^{1,mn})} \sum_{w \in \mathcal{I}^{1,mn}} s_{(m-1)n+r}(w|_{(m-1)n+r}, \Delta(\epsilon), \Phi) \right)
\]

\[
= \lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{\#(\mathcal{I}^{1,mn})} \sum_{w \in \mathcal{I}^{1,mn}} s_{(m-1)n+r}(w, \Delta(\epsilon), \Phi) \right)
\]

\[
\geq n \cdot \lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{(m-1)n + r} \log \left( \frac{1}{\#(\mathcal{I}^{1,mn})} \sum_{w \in \mathcal{I}^{1,mn}} s_{(m-1)n+r}(w, \Delta(\epsilon), \Phi) \right)
\]

Since this is true for every \( 1 \leq r \leq n \) and \( \Delta(\epsilon) \to 0 \) when \( \epsilon \to 0 \), then

\[
h_{\text{top}}(X, \Phi^n) \geq n \cdot \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{1}{\#(\mathcal{I}^{1,t})} \sum_{w \in \mathcal{I}^{1,t}} s_t(w, \Delta(\epsilon), \Phi) \right)
\]

\[
= n \cdot h_{\text{top}}(X, \Phi).
\]

The proof is completed. \( \Box \)

**Remark 3.9.** In the same way under the assumption of Theorem 3.8 one can prove that for any subset \( Y \) of \( X \) and \( n \geq 1 \), \( h_{\text{top}}(Y, \Phi^n) = n \cdot h_{\text{top}}(Y, \Phi) \).

Now, we give the following theorem that will be used in the next section.

**Theorem 3.10.** Let \( (X, \Phi) \) be an NAIFS of continuous maps on a compact topological space \( X \). Then \( h(X, \Phi; \mathcal{A}) \leq h(X, \Phi_j; \mathcal{A}) \) for every \( 1 \leq i < j < \infty \) and every open cover \( \mathcal{A} \) of \( X \). In particular, \( h_{\text{top}}(X, \Phi_i) \leq h_{\text{top}}(X, \Phi_j) \).

**Proof.** It is enough to show that for every \( 1 \leq i < \infty \) and every open cover \( \mathcal{A} \) of \( X \), \( h(X, \Phi_i; \mathcal{A}) \leq h(X, \Phi_{i+1}; \mathcal{A}) \). Hence, \( h_{\text{top}}(X, \Phi_i) \leq h_{\text{top}}(X, \Phi_{i+1}) \).

Let \( \mathcal{A} \) be an open cover of \( X \), \( i \geq 1 \) and \( w = w_i w_{i+1} \cdots w_{i+n-1} \in \mathcal{I}^{i,n} \) for some \( n \geq 1 \), then \( w' := |w| = w_i w_{i+1} \cdots w_{i+n-1} \in \mathcal{I}^{i+1,n-1} \). Now, by relation (4), we have

\[
\mathcal{A}^{i,n}_w = \mathcal{A} \cup \phi_w^{i-1}(\mathcal{A}) \cup \phi_w^{i-2}(\mathcal{A}) \cup \cdots \cup \phi_w^{i-n}(\mathcal{A})
\]

\[
= \mathcal{A} \cup (\phi_w^{(i)})^{-1}(\mathcal{A}) \cup \phi_w^{i+1,n-1}(\mathcal{A}) \cup \phi_w^{i+1,2}(\mathcal{A}) \cup \cdots \cup \phi_w^{i+1,(n-1)}(\mathcal{A})
\]

\[
= \mathcal{A} \cup (\phi_w^{(i)})^{-1}(\mathcal{A}^{i+1,n-1}_w).
\]
Using relations (2) and (3), we have
\[
\begin{align*}
    h(X, \Phi_i; A) &= \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I_{1,n})} \sum_{w \in I_{1,n}} \mathcal{N}(A_{w}^{i,n}) \right) \\
    &\leq \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I_{1,n})} \sum_{w' \in I_{1,n}} \mathcal{N}(A) \cdot \mathcal{N}(A_{w'}^{i+1,n-1}) \right) \\
    &= \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{\#(I_{1,n})}{\#(I_{1,n+1})} \sum_{w' \in I_{1,n+1}} \mathcal{N}(A) \cdot \mathcal{N}(A_{w'}^{i+1,n-1}) \right) \\
    &= \lim_{n \to \infty} \frac{1}{n} \log \mathcal{N}(A) + \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I_{1,n+1})} \sum_{w' \in I_{1,n+1}} \mathcal{N}(A_{w'}^{i+1,n-1}) \right) \\
    &= \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I_{1,n})} \sum_{w \in I_{1,n+1}} \mathcal{N}(A_{w}^{i+1,n}) \right) \\
    &= h(X, \Phi_{i+1}; A).
\end{align*}
\]
Now, by taking supremum over all open covers \( A \) of \( X \) we have \( h_{\text{top}}(X, \Phi_i) \leq h_{\text{top}}(X, \Phi_{i+1}) \), and the proof is completed. \( \square \)

In general, without more assumptions, we cannot claim that \( h_{\text{top}}(X, \Phi) = h_{\text{top}}(X, \Phi_i) \) for all \( i \geq 1 \). Nevertheless, in Corollary 4.4 we will give a condition that guarantees the equality \( h_{\text{top}}(X, \Phi) = h_{\text{top}}(X, \Phi_i) \) for all \( i \geq 1 \).

**Remark 3.11.** Because, in general, the inequality \( \mathcal{N}\left( (\varphi_{w_i}^{(i)})^{-1}(A)|_{Y} \right) \leq \mathcal{N}(A|_{Y}) \) is not true, the proof of Theorem 3.10 cannot be modified to prove an analogue of the theorem for the topological entropy on the subsets \( Y \) of \( X \). Hence, it is not very surprising that such an analogue does not hold (see [29, Fig.2 and comments], where \( \#(I_{1,j}) = 1 \) for every \( j \in \mathbb{N} \)).

### 3.4. Asymptotical topological entropy and topologically chaotic NAIFSs.

As an autonomous dynamical system \((X, f)\) is usually called topologically chaotic if \( h_{\text{top}}(f) > 0 \), one could consider also an NAIFS \((X, \Phi)\) with \( h_{\text{top}}(X, \Phi) > 0 \) to be topologically chaotic. But, we give another definition which is an extension of the definition of topologically chaotic that given by Kolyada and Snoha for non-autonomous discrete dynamical systems [29].

Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact topological space \( X \) and \( A \) be an open cover of \( X \), then by Theorem 3.10 the limit
\[
    h^*(X, \Phi; A) := \lim_{n \to \infty} h(X, \Phi_n; A) = \lim_{n \to \infty} \lim_{k \to \infty} \frac{1}{k} \log \left( \frac{1}{\#(I_{n,k})} \sum_{w \in I_{n,k}} \mathcal{N}(A_{w}^{n,k}) \right)
\]
exists. The quantity \( h^*(X, \Phi; A) \) is said to be the asymptotical topological entropy of the NAIFS \((X, \Phi)\) on the cover \( A \). Put
\[
h^*(X, \Phi) := \sup_{A} h^*(X, \Phi; A)
\]
where the supremum is taken over all open covers \( \mathcal{A} \) of \( X \). By definition and the proof of Theorem 3.10 it is easy to see that

\[
h^\ast(X, \Phi) = \sup_{\mathcal{A}} h^\ast(X, \Phi; \mathcal{A}) = \sup_{\mathcal{A}} \lim_{n \to \infty} h(X, \Phi_n; \mathcal{A}) = \sup_{\mathcal{A}} \sup_{n} h(X, \Phi_n; \mathcal{A})
\]

\[
= \sup_{(A,n)} h(A,n) = \sup_{n} \lim_{A} h(X, \Phi_n; \mathcal{A}) = \lim_{n \to \infty} \sup_{\mathcal{A}} h(X, \Phi_n; \mathcal{A})
\]

\[
= \lim_{n \to \infty} h_{\text{top}}(X, \Phi_n).
\]

If \( X \) is a compact metric space, then by the definition of topological entropy via separated and spanning sets, we have

\[
h^\ast(X, \Phi) = \lim_{n \to \infty} \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{k} \log S_k(\epsilon, \Phi_n) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{k} \log S_k(\epsilon, \Phi_n)
\]

\[
= \lim_{\epsilon \to 0} \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{k} \log R_k(\epsilon, \Phi_n) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{k} \log R_k(\epsilon, \Phi_n).
\]

where

\[
S_k(\epsilon, \Phi_n) := \frac{1}{\#(I_{n,k})} \sum_{w \in I_{n,k}} s_k(w, \epsilon, \Phi_n) \quad \text{and} \quad R_k(\epsilon, \Phi_n) := \frac{1}{\#(I_{n,k})} \sum_{w \in I_{n,k}} r_k(w, \epsilon, \Phi_n).
\]

The quantity \( h^\ast(X, \Phi) \) is said to be the asymptotical topological entropy of NAIFS \((X, \Phi)\).

**Definition 3.12.** An NAIFS \((X, \Phi)\) of continuous maps on a compact topological space \( X \) is said to be topologically chaotic if it has positive asymptotical topological entropy, i.e., \( h^\ast(X, \Phi) > 0 \).

By Remark 3.11, since for a proper subset \( Y \) of \( X \) \((Y \subset X)\) we may have \( h_{\text{top}}(Y, \Phi_i) \geq h_{\text{top}}(Y, \Phi_j) \) for some \( j > i \), there is a problem with the extension of the concept of asymptotical topological entropy to a proper subset \( Y \) of \( X \). But, we can define \( h^\ast(Y, \Phi) := \limsup_{n \to \infty} h_{\text{top}}(Y, \Phi_n) \) for proper subsets \( Y \) of \( X \).

Many results that hold for the topological entropy of NAIFSs can be carried to asymptotical topological entropy of NAIFSs. Hence, it is not difficult to see that Proposition 3.4, Lemma 3.5, Theorem 3.8 and Theorem 3.10 have analogues versions for asymptotical topological entropy of NAIFSs by replacing \( h_{\text{top}} \) by \( h^\ast \).

**3.5. Entropy of NAIFSs of monotone interval maps or circle maps.** Sometimes in computing the topological entropy of a dynamical system, one may be very interested in whether it is positive or zero rather than its exact value. Also, computing the exact value may be impossible. In the theory of autonomous dynamical systems, a homeomorphism on the interval or on the circle has zero topological entropy (see, e.g., [1, 56]). Also, in [29] in the theory of non-autonomous discrete dynamical systems, Kolyada and Snoha showed that any non-autonomous discrete dynamical systems of continuous (not necessarily strictly) monotone maps on the interval or on the circle, have zero topological entropy. In the following theorem, we extend these results to NAIFSs on the interval and on the circle.

We consider the unit circle \( S^1 \) as the quotient space of the real line by the group of translations by integers \((S^1 = \mathbb{R}/\mathbb{Z})\). Let \( q : \mathbb{R} \to S^1 \) be the quotient map. In the unit circle \( S^1 \), we consider the metric (denoted by \( \rho \)) and the orientation induced from the metric and orientation of the real line via \( q \) (hence the distance between any two points is at most \( \frac{1}{2} \)). Also, we denote by \( I \) the unit interval \([0, 1]\).
Note that a homeomorphism of \( I \) or \( S^1 \) is either strictly increasing (orientation preserving) or strictly decreasing (orientation reversing). The desired result can be followed from the following theorem. In it, when we speak about an NAIFS of monotone maps we do not assume that the type of monotonicity is the same for all of them.

**Theorem 3.13.** Let \((X, \Phi)\) be an NAIFS of continuous monotone maps in which \( X \) is \( I \) or \( S^1 \). Then, the topological entropy \( h_{top}(X, \Phi) \) is zero. Thus, \( h^*(X, \Phi) = 0 \).

**Proof.** First let \( X = I \) and fix \( w \in I^{1,n} \). Let \( E := \{x_1, x_2, \ldots, x_k\} \) be a subset of \( I \) with \( x_1 < x_2 < \cdots < x_k \). Since the maps \( \varphi_{w_1}, \varphi_{w_2}, \ldots, \varphi_{w_n} \) are monotone, for every \( 0 \leq j \leq n \) either \( \varphi_{w_1}^{1,j}(x_1) \leq \varphi_{w_1}^{1,j}(x_2) \leq \cdots \leq \varphi_{w_1}^{1,j}(x_k) \) or \( \varphi_{w_1}^{1,j}(x_1) \geq \varphi_{w_1}^{1,j}(x_2) \geq \cdots \geq \varphi_{w_1}^{1,j}(x_k) \).

This implies that the set \( E \) is \( (w, n, \epsilon; \Phi) \)-separated. Since the maps \( \varphi_{w_1}, \varphi_{w_2}, \ldots, \varphi_{w_n} \) are monotone, for every \( 0 \leq j \leq n \) at most \( [1/\epsilon] \) distances from \( |\varphi_{w_1}^{1,j}(x_1) - \varphi_{w_1}^{1,j}(x_2)|, |\varphi_{w_1}^{1,j}(x_2) - \varphi_{w_1}^{1,j}(x_3)|, \ldots, |\varphi_{w_1}^{1,j}(x_k-1) - \varphi_{w_1}^{1,j}(x_k)| \) are longer than \( \epsilon \), where \( [1/\epsilon] \) is the integer part of \( 1/\epsilon \). Hence, at most \( (n+1)[1/\epsilon] \) sets of the form \( \{x_i, x_{i+1}\} \), \( 1 \leq i \leq k-1 \) are \( (w, n, \epsilon; \Phi) \)-separated. So if \( E \) is \( (w, n, \epsilon; \Phi) \)-separated then \( k-1 \leq (n+1)[1/\epsilon] \). Consequently, \( s_n(w, \epsilon, \Phi) \leq 1 + (n+1)[1/\epsilon] \).

Hence, by the definition of topological entropy, it follows that

\[
   h_{top}(I, \Phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{#(I^{1,n})} \sum_{w \in I^{1,n}} s_n(w, \epsilon, \Phi) \right)
\]

\[
   \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{#(I^{1,n})} \sum_{w \in I^{1,n}} (1 + (n+1)[1/\epsilon]) \right)
\]

\[
   = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log (1 + (n+1)[1/\epsilon]) = 0.
\]

Now, let \( X = S^1 \) and fix \( w \in I^{1,n} \). Let \( E := \{x_1, x_2, \ldots, x_k\} \) be a maximal \( (w, n, \epsilon; \Phi) \)-separated set in \( S^1 \) with \( x_1 < x_2 < \cdots < x_k \), i.e., \( s_n(w, \epsilon, \Phi) = k \). Then the sets \( \{x_i, x_{i+1}\} \), \( 1 \leq i \leq k-1 \) and \( \{x_k, x_1\} \) are \( (w, n, \epsilon; \Phi) \)-separated. Since for every \( 0 \leq j \leq n \) the sum of distances

\[
   \sum_{i=1}^{k-1} \rho(\varphi_{w_i}^{1,j}(x_i), \varphi_{w_i}^{1,j}(x_{i+1})) + \rho(\varphi_{w_i}^{1,j}(x_k), \varphi_{w_i}^{1,j}(x_1))
\]

equals to the length of the circle = 1, at most \( [1/\epsilon] \) of them are longer than \( \epsilon \). Hence, at most \( (n+1)[1/\epsilon] \) sets of the form \( \{x_i, x_{i+1}\} \), \( 1 \leq i \leq k-1 \) or \( \{x_k, x_1\} \) are \( (w, n, \epsilon; \Phi) \)-separated. Thus \( s_n(w, \epsilon, \Phi) = k \leq (n+1)[1/\epsilon] \), since all of these sets are \( (w, n, \epsilon; \Phi) \)-separated. Hence, by the definition of topological entropy, it follows that

\[
   h_{top}(S^1, \Phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{#(I^{1,n})} \sum_{w \in I^{1,n}} s_n(w, \epsilon, \Phi) \right)
\]

\[
   \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{#(I^{1,n})} \sum_{w \in I^{1,n}} (n+1)[1/\epsilon] \right)
\]

\[
   = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log ((n+1)[1/\epsilon]) = 0.
\]

In a similar way, for every \( n \geq 2 \) one can conclude that \( h_{top}(I, \Phi_n) = 0 = h_{top}(S^1, \Phi_n) \). Hence, \( h^*(I, \Phi) = 0 = h^*(S^1, \Phi) \), and the proof is completed. \( \square \)
3.6. Topological entropy on the set of nonwandering points. If \((X, \varphi)\) be a autonomous dynamical system in which \(\varphi\) is a continuous self-map of a compact topological space \(X\), then by \([10]\), the topological entropy of \(\varphi\) and of \(\varphi|_{\Omega(\varphi)}\) are equal. Where, \(\Omega(\varphi)\) is the set of nonwandering points of \(\varphi\). A point \(x \in X\) is said to be a nonwandering point of \(\varphi\) if for every nonempty open neighborhood \(U_x\) of \(x\) in \(X\), there exists a positive integer \(n\) such that \(\varphi^n(U_x) \cap U_x \neq \emptyset\). Also, in the context of non-autonomous discrete dynamical systems, Kolyada and Snoha \([29]\) showed that for every sequence \(\varphi_1, \infty = \{\varphi_i\}_{i=1}^{\infty}\) of equicontinuous self-maps of a compact metric space \(X\), the topological entropy of non-autonomous discrete dynamical system \((X, \varphi_1, \infty)\) is equal to the topological entropy of its restriction to the set of nonwandering points, i.e., \(h_{\text{top}}(\varphi_1, \infty) = h_{\text{top}}(\varphi_1, \infty)|_{\Omega(\varphi_1, \infty)}\). Where, \(\Omega(\varphi_1, \infty)\) is the set of nonwandering points of sequence \(\varphi_1, \infty\). In addition, Eberlein \([19]\) asserted that the topological entropy of an (abelian) finitely generated semigroup action is equal to the topological entropy of its restriction to its nonwandering set.

In the following theorem, we want to find a analogous result for NAIFSs (in the proof we will use Theorem 3.8 and therefore we restrict ourselves to equicontinuous NAIFSs of a compact metric space).

**Definition 3.14.** Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact topological space \(X\). A point \(x \in X\) is said to be nonwandering for \(\Phi\) if for every open neighborhood \(U_x\) of \(x\) there is a finite word \(w \in I^{m,n}\) for some \(m, n \geq 1\), such that \(\varphi_w^{m,n}(U_x) \cap U_x \neq \emptyset\). The set of all nonwandering points of \(\Phi\) is called the nonwandering set of \(\Phi\) and denoted by \(\Omega(\Phi)\). It is easy to see that \(\Omega(\Phi)\) is a closed subset of \(X\).

An open subset \(U \subseteq X\) is said to be wandering for \(\Phi\) if \(\varphi_w^{m,n}(U) \cap U = \emptyset\) for every finite word \(w \in I^{m,n}\) and every \(m, n \geq 1\). A point \(x \in X\) is said to be wandering for \(\Phi\) if it belongs to some wandering set \(U\). Hence, \(x\) is wandering if and only if it is not nonwandering.

In an NAIFS \((X, \Phi)\), if \(#(I^j) = 1\) and \(\Phi^j = \{\varphi^{(j)}\}\) for every \(j \geq 1\), then this definition coincides with the usual definition of nonwandering points of non-autonomous discrete dynamical system \((X, \varphi_1, \infty)\), i.e., \(\Omega(\Phi) = \Omega(\varphi_1, \infty)\), where \(\varphi_1, \infty\) is the sequence \(\{\varphi^{(j)}\}_{j=1}^{\infty}\). Additionally, if \(\varphi^{(j)} = \varphi\) for every \(j \geq 1\), then this definition coincides with the usual definition of nonwandering points of autonomous dynamical system \((X, \varphi)\), i.e., \(\Omega(\Phi) = \Omega(\varphi)\).

**Theorem 3.15.** Let \((X, \Phi)\) be an equicontinuous NAIFS of a compact metric space \((X, d)\). Then \(h_{\text{top}}(X, \Phi) = h_{\text{top}}(\Omega(\Phi), \Phi|_{\Omega(\Phi)})\).

**Proof.** By the definition of topological entropy, we have \(h_{\text{top}}(X, \Phi) \geq h_{\text{top}}(\Omega(\Phi), \Phi|_{\Omega(\Phi)})\). Hence, it is enough to prove the converse inequality. To do this we will follow the main ideas from the proof of \([29, \text{Theorem H}]\) and \([3, \text{Lemma 4.1.5}]\).

So let \(A\) be an open cover of \(X\). Fix \(n \geq 1\) and \(w \in I^{1,n}\). Let \(\zeta_w\) be an minimal subcover of \(\Omega(\Phi)\) chosen from \(A_{w,n}^1\). Since \(X\) is a compact metric space, the set \(K = X \setminus \bigcup_{B \in \zeta_w} B\) is compact and consists of wandering points. Hence, we can cover \(K\) with a finite number of wandering sets (subsets of \(X\), not necessarily of \(K\)), each of them contained in some element of \(A_{w,n}^1\). These sets, together with all elements of \(\zeta_w\), form an open cover \(\xi_w\) of \(X\), finer than \(A_{w,n}^1\).

Now, in NAIFS \((X, \Phi^n)\) for \(w^* = w^1*w^2*\cdots*w_k^* \in I^{1,k}\) with \(w^* = w = w^1*w^2*\cdots*w_{kn}\) \(\in I^{1,kn}\), consider any nonempty element of \((\xi_w)^{1,k}_w\). It is of the form \(C_0 \cap (\varphi_{w^1_1}^{(1,n)})^{-1}(C_1) \cap (\varphi_{w^2_1}^{(1,n)})^{-1} \circ (\varphi_{w^2_2}^{(2,n)})^{-1}(C_2) \cap \cdots \cap (\varphi_{w^k_1}^{(1,n)})^{-1} \circ (\varphi_{w^k_2}^{(2,n)})^{-1} \circ \cdots \circ (\varphi_{w^k_k}^{(k,n)})^{-1}(C_k)\).
that is equal to
\[ C_0 \cap \varphi_{w}^{1,-n}(C_1) \cap \varphi_{w}^{1,-n} \circ \varphi_{w}^{n+1,-n}(C_2) \cap \cdots \cap \varphi_{w}^{1,-n} \circ \varphi_{w}^{n+1,-n} \circ \cdots \circ \varphi_{w}^{(k-1)n+1,-n}(C_k), \]
where \( \varphi_{w}^{(j,n)} = \varphi_{w}^{(j-1)n+1,n} \in \Phi^{(j,n)} \) for \( 1 \leq j \leq k \) and \( C_i \in \xi_w \) for \( 0 \leq i \leq k \). Since we assume that this element is nonempty, we get that if \( C_i = C_j \) for some \( i < j \), then
\[ \varphi_{w}^{1,-n} \circ \cdots \circ \varphi_{w}^{(j-1)n+1,-n} \circ \varphi_{w}^{n+1,-n} \circ \cdots \circ \varphi_{w}^{(j-1)n+1,-n}(C_i) \]
and \( \varphi_{w}^{n+1,-n} \circ \cdots \circ \varphi_{w}^{(j-1)n+1,-n}(C_i) \neq \emptyset \), so \( \varphi_{w}^{(j-1)n+1,-n} \circ \cdots \circ \varphi_{w}^{n+1,-n}(C_i) \cap \varphi_{w}^{1,-n} \circ \cdots \circ \varphi_{w}^{(j-1)n+1,-n}(C_i) \neq \emptyset \), hence \( C_i \) cannot be wandering for \( \Phi \), this implies that \( C_i \in \zeta_w \).

In the same way as in the proof of [3, Lemma 4.1.5] one can show that the number of elements in cover \( (\xi_w)^{1,k} \) is not larger than \((m + 1)! \cdot (k + 1)^m \cdot (\#(\zeta_w))^{k+1}\), where \( m = \#(\xi_w \setminus \zeta_w) \). Thus,
\[
 h(X, \Phi^n; \xi_w) = \limsup_{k \to \infty} \frac{1}{k} \log \left( \frac{1}{\#(I_{+}^{1,k})} \sum_{w^* \in I_{+}^{1,k}} N((\xi_w)^{1,k}) \right)
\leq \limsup_{k \to \infty} \frac{1}{k} \log \left( \frac{1}{\#(I_{+}^{1,k})} \sum_{w^* \in I_{+}^{1,k}} (m + 1)! \cdot (k + 1)^m \cdot (\#(\zeta_w))^{k+1} \right)
= \limsup_{k \to \infty} \frac{1}{k} \log \left( (m + 1)! \cdot (k + 1)^m \cdot (\#(\zeta_w))^{k+1} \right)
= \log(\#(\zeta_w)).
\]

Now we are ready to finish the proof. By the definition of topological entropy it follows that for any \( \epsilon > 0 \) there is an open cover \( \mathcal{A} \) of \( X \) with \( h_{\text{top}}(X, \Phi^n) < h(X, \Phi^n; \mathcal{A}) + \epsilon \). Using this fact and by Theorem 3.8 and (6), we get that for any positive integer \( n \) and \( \epsilon > 0 \) there is an open cover \( \mathcal{A} \) of \( X \) with
\[
h_{\text{top}}(X, \Phi) = \frac{1}{n} h_{\text{top}}(X, \Phi^n) < \frac{1}{n} h(X, \Phi^n; \mathcal{A}) + \epsilon \leq \frac{1}{n} h(X, \Phi^n; \mathcal{A}_{w}) + \frac{\epsilon}{n}
\leq \frac{1}{n} h(X, \Phi^n; \xi_w) + \frac{\epsilon}{n} \leq \frac{1}{n} \log(\#(\zeta_w)) + \frac{\epsilon}{n} = \frac{1}{n} \log N(\mathcal{A}_{w}^{1,n}|_{\Omega(\Phi)}) + \frac{\epsilon}{n},
\]
where \( w \in I_{+}^{1,n} \) is arbitrary. Thus,
\[
h_{\text{top}}(X, \Phi) \leq \frac{1}{n} \log \left( \frac{1}{\#(I_{+}^{1,n})} \sum_{w^* \in I_{+}^{1,n}} N(\mathcal{A}_{w}^{1,n}|_{\Omega(\Phi)}) \right) + \frac{\epsilon}{n}.
\]
Taking the upper limit when \( n \to \infty \), we have
\[
h_{\text{top}}(X, \Phi) \leq h(\Omega(\Phi), \Phi|_{\Omega(\Phi)}; \mathcal{A}) \leq h_{\text{top}}(\Omega(\Phi), \Phi|_{\Omega(\Phi)}),
\]
that completes the proof. \( \square \)

4. Specification property for NAIFSs and its relationship with entropy

The notion of entropy is one of the most important objects in dynamical systems, either as a topological invariant or as a measure of the chaoticity of dynamical systems. Several notions of entropy have been introduced for other branches of dynamical systems in an attempt to describe their dynamical characteristics. In this section, we define entropy points for NAIFSs. Roughly speaking, entropy points are those that their local neighborhoods reflect
the complexity of the entire dynamical system in the context of topological entropy. Also, we define a notion of specification property for NAIFSs and characterize entropy points and topological entropy for NAIFSs having the specification property.

The notion of entropy points was defined for finitely generated pseudogroup actions, finitely generated semigroup actions and non-autonomous discrete dynamical systems, respectively in [5], [43] and [35]. In the following definition, we extend the notion of entropy points to NAIFSs.

**Definition 4.1.** The NAIFS \((X, \Phi)\) of continuous maps on a compact topological space \(X\), admits an entropy point \(x_0 \in X\) if for every open neighbourhood \(U\) of \(x_0\) the equality \(h_{\text{top}}(X, \Phi) = h_{\text{top}}(\text{cl}(U), \Phi)\) holds.

The notion of specification was first introduced in the 1970s as a property of uniformly hyperbolic basic pieces and became a characterization of complexity in dynamical systems. Thus, several notions of specification had been introduced in an attempt to describe their dynamical characteristics for autonomous dynamical systems, non-autonomous discrete dynamical systems and semigroup actions [35, 43, 51, 52, 54, 53, 57]. In the following definition, we give a concept of specification property for NAIFSs.

**Definition 4.2.** The NAIFS \((X, \Phi)\) of continuous maps on a compact metric space \((X, d)\), is said to have the specification property if for every \(\delta > 0\) there is \(N(\delta) \in \mathbb{N}\) such that for each \(w \in I^{1,\infty}\), any \(x_1, x_2, \ldots, x_s \in X\) with \(s \geq 2\) and any sequence \(0 = j_1 \leq j_2 \leq k_2 < \cdots < j_s \leq k_s\) of integers with \(j_{n+1} - k_n \geq N(\delta)\) for \(n = 1, \ldots, s - 1\), there is a point \(x \in X\) such that for each \(1 \leq m \leq s\) and any \(j_m \leq i \leq k_m\), \(d(\varphi_{w}^{1,i}(x), \varphi_{w}^{1,i}(x_m)) \leq \delta\). In other words, an NAIFS \((X, \Phi)\) of continuous maps on a compact metric space \((X, d)\) has the specification property if we have the specification property along every branch \(w \in I^{1,\infty}\) as a non-autonomous discrete dynamical system, where \(N(\delta)\) is independent of \(w \in I^{1,\infty}\), for each \(\delta > 0\).

Rodrigues and Varandas [43] showed that for any finitely generated continuous semigroup action of local homeomorphisms on a compact Riemannian manifold with the strong orbital specification property (weak orbital specification property), every point is an entropy point. Also, they showed that any finitely generated continuous semigroup action on a compact metric space with the strong orbital specification property (weak orbital specification property under some other conditions) has positive topological entropy. Also, Nazarian Sarkooh and Ghane [35] showed that every non-autonomous discrete dynamical system of surjective maps with the specification property has positive topological entropy and all points are entropy point; in particular, it is topologically chaotic. In this section, we extend these results to NAIFSs.

4.1. Specification and entropy points. We investigate here the relation between the specification property of NAIFSs and the existence of entropy points.

**Theorem 4.3.** If \((X, \Phi)\) is an NAIFS of surjective continuous maps on a compact metric space \((X, d)\) with the specification property, then every point is an entropy point.

**Proof.** According to Theorem 3.10, \(h_{\text{top}}(X, \Phi) \leq h_{\text{top}}(X, \Phi_k)\) for every \(k \geq 1\). Also, by Lemma 3.1,

\[
h_{\text{top}}(X, \Phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_n(\epsilon, \Phi),
\]
where

\[ S_n(\epsilon, \Phi) := \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} s_n(w, \epsilon, \Phi), \]

Using these facts, we show that for every \( z \in M \) and every open neighborhood \( V \) of \( z \), \( h_{\text{top}}(X, \Phi) = h_{\text{top}}(\text{cl}(V), \Phi) \). For \( \delta > 0 \) define \( W_\delta := \{ y \in V : d(y, \partial V) > \frac{\delta}{2} \} \). Fix \( \epsilon > 0 \) such that the open set \( W_\epsilon \) is nonempty. Take \( N(\frac{\delta}{2}) \geq 1 \) given by the definition of specification property. Fix \( w = w_1 w_2 \cdots w_N(\frac{\delta}{2}) w_{N(\frac{\delta}{2})+1} \cdots w_{N(\frac{\delta}{2})+n} \in I^{1,N(\frac{\delta}{2})+n} \) and take

\[ w' := w|^{N(\frac{\delta}{2})+1} w_{N(\frac{\delta}{2})+2} \cdots w_{N(\frac{\delta}{2})+n} \in I^{N(\frac{\delta}{2})+1,n}. \]

Let

- \( E := \{ z_1, z_2, \ldots, z_l \} \subseteq X \) be a maximal \((n, w', \epsilon; \Phi_{N(\frac{\delta}{2})+1})\)-separated set,
- \( E' = \{ z'_1, z'_2, \ldots, z'_l \} \subseteq X \) be a preimage set of \( E \) under \( \varphi_w^{1,N(\frac{\delta}{2})} \), i.e., \( \varphi_w^{1,N(\frac{\delta}{2})}(z'_i) = z_i \) for \( 1 \leq i \leq l \),
- \( y \in W_\epsilon \) be an arbitrary point (note that \( W_\epsilon \neq \emptyset \)).

Let \( j_1 = k_1 = 0, j_2 = N(\frac{\delta}{2}) \) and \( k_2 = N(\frac{\delta}{2}) + n. \) By the definition of specification property, for every \( z'_i \in E' \), by taking \( x_1 = y \) and \( x_2 = z'_i \), there exists \( y_i \in B(y, \frac{\delta}{2}) \) such that \( \varphi_w^{1,N(\frac{\delta}{2})}(y_i) \in B(\varphi_w^{1,N(\frac{\delta}{2})}(z'_i); w', \frac{\delta}{2}) = B(z'_i; w', \frac{\delta}{2}). \) Since \( E := \{ z_1, z_2, \ldots, z_l \} \subseteq X \) is a maximal \((n, w', \epsilon; \Phi_{N(\frac{\delta}{2})+1})\)-separated set, the set \( \{ y_i \}_{i=1}^l \subseteq \text{cl}(V) \) is \((N(\frac{\delta}{2}) + n, w, \frac{\delta}{2}; \Phi)\)-separated. So

\[ s_{N(\frac{\delta}{2})+n}(\text{cl}(V); w, \frac{\epsilon}{2}; \Phi) \geq s_n(w', \epsilon, \Phi_{N(\frac{\delta}{2})+1}). \]

By taking summation over \( w \in I^{1,N(\frac{\delta}{2})+n} \) we have

\[ S_{N(\frac{\delta}{2})+n}(\text{cl}(V); \frac{\epsilon}{2}, \Phi) \geq \#(I^{1,N(\frac{\delta}{2})}) \cdot S_n(\epsilon, \Phi_{N(\frac{\delta}{2})+1}) \geq S_n(\epsilon, \Phi_{N(\frac{\delta}{2})+1}). \]

Thus

\[ \limsup_{n \to \infty} \frac{1}{n} \log S_n(\text{cl}(V); \frac{\epsilon}{2}, \Phi) = \limsup_{n \to \infty} \frac{1}{N(\frac{\delta}{2}) + n} \log S_{N(\frac{\delta}{2})+n}(\text{cl}(V); \frac{\epsilon}{2}, \Phi) \]

\[ \geq \limsup_{n \to \infty} \frac{1}{N(\frac{\delta}{2}) + n} \log S_n(\epsilon, \Phi_{N(\frac{\delta}{2})+1}) \]

\[ = \limsup_{n \to \infty} \frac{1}{n} \log S_n(\epsilon, \Phi_{N(\frac{\delta}{2})+1}) \]

This implies that

\[ h_{\text{top}}(X, \Phi) \geq h_{\text{top}}(\text{cl}(V), \Phi) \geq h_{\text{top}}(X, \Phi_{N(\frac{\delta}{2})+1}) \geq h_{\text{top}}(X, \Phi). \]

Hence, \( h_{\text{top}}(X, \Phi) = h_{\text{top}}(\text{cl}(V), \Phi) \), i.e., every point is an entropy point. \( \square \)

By Theorem 3.10 and the proof of Theorem 4.3, we conclude the following corollary.

**Corollary 4.4.** Let \((X, \Phi)\) be an NAIFS of surjective continuous maps on a compact metric space \((X, d)\) with the specification property. Then \( h_{\text{top}}(X, \Phi) = h_{\text{top}}(X, \Phi_i) \) for every \( i \geq 1 \).
4.2. Specification and positive topological entropy. In this subsection we show that the specification property is a sufficient condition for an NAIFS has positive topological entropy.

**Theorem 4.5.** Let $(X, \Phi)$ be an NAIFS of surjective continuous maps on a compact metric space $(X, d)$ with the specification property. Then, the NAIFS $(X, \Phi)$ has positive topological entropy, i.e., $h_{\text{top}}(X, \Phi) > 0$.

**Proof.** By Lemma 3.1, we know that

$$h_{\text{top}}(X, \Phi) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log S_n(\epsilon, \Phi),$$

where

$$S_n(\epsilon, \Phi) := \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} s_n(w, \epsilon, \Phi)$$

and the limit can be replaced by $\sup_{\epsilon > 0}$. Thus, it is enough to prove that there exists $\epsilon > 0$ small enough so that

$$\lim_{n \to \infty} \frac{1}{n} \log S_n(\epsilon, \Phi) > 0.$$ 

Let $\epsilon > 0$ be small and fixed so that there are at least two distinct $2\epsilon$-separated points $x_1, y_1 \in M$, i.e., $d(x_1, y_1) > 2\epsilon$. Let $N(\frac{\epsilon}{2}) \geq 1$ be given by the definition of specification property.

Fix $w \in I^{1,N(\frac{\epsilon}{2})}$. Take $j_1 = k_1 = 0$, $j_2 = k_2 = N(\frac{\epsilon}{2})$ and consider preimages $x_2$ of $x_1$ and $y_2$ of $y_1$ under $\varphi_{w}^{1,N(\frac{\epsilon}{2})}$, i.e., $\varphi_{w}^{1,N(\frac{\epsilon}{2})}(x_2) = x_1$ and $\varphi_{w}^{1,N(\frac{\epsilon}{2})}(y_2) = y_1$. By applying the specification property to pairs $(x_1, x_2)$, $(x_1, y_2)$, $(y_1, x_2)$ and $(y_1, y_2)$, there are $x_1, x_1, x_1, y_2 \in B(x_1, \frac{\epsilon}{2})$ and $y_1, y_1, y_1, y_1 \in B(y_1, \frac{\epsilon}{2})$ such that

$$\varphi_{w}^{1,N(\frac{\epsilon}{2})}(x_1, x_1) \in B(\varphi_{w}^{1,N(\frac{\epsilon}{2})}(x_2), \frac{\epsilon}{2}) = B(x_1, \frac{\epsilon}{2}),$$

$$\varphi_{w}^{1,N(\frac{\epsilon}{2})}(x_1, y_1) \in B(\varphi_{w}^{1,N(\frac{\epsilon}{2})}(y_2), \frac{\epsilon}{2}) = B(y_1, \frac{\epsilon}{2}),$$

$$\varphi_{w}^{1,N(\frac{\epsilon}{2})}(y_1, x_1) \in B(\varphi_{w}^{1,N(\frac{\epsilon}{2})}(x_2), \frac{\epsilon}{2}) = B(x_1, \frac{\epsilon}{2}),$$

$$\varphi_{w}^{1,N(\frac{\epsilon}{2})}(y_1, y_1) \in B(\varphi_{w}^{1,N(\frac{\epsilon}{2})}(y_2), \frac{\epsilon}{2}) = B(y_1, \frac{\epsilon}{2}).$$

It is clear that the set $\{x_1, x_1, y_1, y_1 \}$ is $(N(\frac{\epsilon}{2}), w, \epsilon; \Phi)$-separated. In particular, it follows that $s_{N(\frac{\epsilon}{2})}(w, \epsilon, \Phi) \geq 2^2$. By taking summation over $w \in I^{1,N(\frac{\epsilon}{2})}$, it follows that

$$S_{N(\frac{\epsilon}{2})}(\epsilon, \Phi) = \frac{1}{\#(I^{1,N(\frac{\epsilon}{2})})} \sum_{w \in I^{1,N(\frac{\epsilon}{2})}} s_{N(\frac{\epsilon}{2})}(w, \epsilon, \Phi) \geq \frac{1}{\#(I^{1,N(\frac{\epsilon}{2})})} \sum_{w \in I^{1,N(\frac{\epsilon}{2})}} 2^2 = 2^2.$$ 

Fix $w \in I^{1,2N(\frac{\epsilon}{2})}$. Take $j_1 = k_1 = 0$, $j_2 = k_2 = N(\frac{\epsilon}{2})$ and $j_3 = k_3 = 2N(\frac{\epsilon}{2})$. Consider preimages $x_2$ of $x_1$ and $y_2$ of $y_1$ under $\varphi_{w}^{1,2N(\frac{\epsilon}{2})}$, i.e., $\varphi_{w}^{1,2N(\frac{\epsilon}{2})}(x_2) = x_1$ and $\varphi_{w}^{1,2N(\frac{\epsilon}{2})}(y_2) = y_1$. Also, consider preimages $x_3$ of $x_1$ and $y_3$ of $y_1$ under $\varphi_{w}^{1,2N(\frac{\epsilon}{2})}$, i.e., $\varphi_{w}^{1,2N(\frac{\epsilon}{2})}(x_3) = x_1$ and $\varphi_{w}^{1,2N(\frac{\epsilon}{2})}(y_3) = y_1$. By applying the specification property to triples $(x_1, x_2, x_3)$, $(x_1, x_2, x_3)$, $(x_1, y_2, x_3)$, $(x_1, y_2, x_3)$, $(y_1, y_2, y_3)$, $(y_1, y_2, x_3)$, $(y_1, x_2, y_3)$ and $(y_1, x_2, x_3)$, there are $x_1, x_1, x_1, x_1 \in B(x_1, \frac{\epsilon}{2})$ and $y_1, y_1, y_1, y_1 \in B(y_1, \frac{\epsilon}{2})$ such that
\[ \varphi^1_{w}(x_{1,1}) \in B(x_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(x_{1,1}) \in B(x_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(x_{1,2}) \in B(x_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(x_{1,2}) \in B(y_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(x_{1,3}) \in B(y_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(x_{1,3}) \in B(x_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(x_{1,4}) \in B(y_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(x_{1,4}) \in B(y_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(y_{1,1}) \in B(y_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(y_{1,1}) \in B(y_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(y_{1,2}) \in B(y_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(y_{1,2}) \in B(x_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(y_{1,3}) \in B(x_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(y_{1,3}) \in B(y_1, \frac{c}{2}), \]
\[ \varphi^1_{w}(y_{1,4}) \in B(x_1, \frac{c}{2}) \quad \text{and} \quad \varphi^2_{w}(y_{1,4}) \in B(x_1, \frac{c}{2}). \]

It is clear that the set \( \{x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}\} \) is \((2N(\frac{c}{2}), w, \epsilon; \Phi)\)-separated. In particular, it follows that \( s_{2N(\frac{c}{2})}(w, \epsilon; \Phi) \geq 2^3 \). By taking summation over \( w \in I^{1,2N(\frac{c}{2})} \), it follows that
\[
S_{2N(\frac{c}{2})}(\epsilon, \Phi) = \frac{1}{#(I^{1,2N(\frac{c}{2})})} \sum_{w \in I^{1,2N(\frac{c}{2})}} s_{2N(\frac{c}{2})}(w, \epsilon; \Phi) \geq \frac{1}{#(I^{1,2N(\frac{c}{2})})} \sum_{w \in I^{1,2N(\frac{c}{2})}} 2^3 = 2^3.
\]

Now, fix \( w \in I^{1,dN(\frac{c}{2})} \) where \( d \in \mathbb{N} \). Taking \( j_1 = k_1 = 0, j_2 = k_2 = N(\frac{c}{2}), j_3 = k_3 = 2N(\frac{c}{2}), \ldots, j_d = k_d = (d-1)N(\frac{c}{2}), j_{d+1} = k_{d+1} = dN(\frac{c}{2}) \) and consider the preimages \( x_i \) of \( x_1 \) and \( y_i \) of \( y_1 \) under \( \varphi^1_{w}(x_{i-1}) \) for \( i = 2, \ldots, d+1 \), i.e., \( \varphi^1_{w}(x_{i-1}) = x_1 \) and \( \varphi^1_{w}(y_{i-1}) = y_1 \). By repeating the previous reasoning for \((d+1)\)-tuples in which the \( i \)th component choosing from the set \( \{x_i, y_i\} \), it follows that \( s_{dN(\frac{c}{2})}(w, \epsilon; \Phi) \geq 2^{d+1} \). By taking summation over \( w \in I^{1,dN(\frac{c}{2})} \), we have
\[
S_{dN(\frac{c}{2})}(\epsilon, \Phi) = \frac{1}{#(I^{1,dN(\frac{c}{2})})} \sum_{w \in I^{1,dN(\frac{c}{2})}} s_{dN(\frac{c}{2})}(w, \epsilon; \Phi) \geq \frac{1}{#(I^{1,dN(\frac{c}{2})})} \sum_{w \in I^{1,dN(\frac{c}{2})}} 2^{d+1} = 2^{d+1}.
\]

Hence,
\[
\limsup_{n \to \infty} \frac{1}{n} \log S_n(\epsilon, \Phi) \geq \limsup_{d \to \infty} \frac{1}{dN(\frac{c}{2})} \log S_{dN(\frac{c}{2})}(\epsilon, \Phi) \geq \limsup_{d \to \infty} \frac{1}{dN(\frac{c}{2})} \log 2^{d+1} = \frac{\log 2}{N(\frac{c}{2})}.
\]

This proves that the topological entropy is positive and finishes the proof. \( \Box \)

As a direct consequence of Theorems 4.5 and 3.10 we have the following corollary.

**Corollary 4.6.** Let \( (X, \Phi) \) be an NAIFS of surjective continuous maps on a compact metric space \( (X, d) \) with the specification property. Then, the NAIFS \( (X, \Phi) \) has positive asymptotical topological entropy. Thus, the NAIFS \( (X, \Phi) \) is topologically chaotic.
In Theorem 4.3, we show that for surjective NAIFSs with the specification property, local neighborhoods reflect the complexity of the entire dynamical system from the viewpoint of entropy theory. Also, in Theorem 4.5 we show that surjective NAIFSs with the specification property have positive topological entropy. Hence, by Theorem 4.3, local neighborhoods have positive topological entropy. More precisely, we have the following corollary.

**Corollary 4.7.** Let \((X, \Phi)\) be an NAIFS of surjective contiguous maps on a compact metric space \((X, d)\) with the specification property. Then \(h_{\text{top}}(\text{cl}(V), \Phi) > 0\) for any \(x \in X\) and any open neighborhood \(V\) of \(x\).

5. **Topological pressure of NAIFSs**

The notion of topological pressure is a generalization of topological entropy for dynamical systems [56], which is a fundamental notion in thermodynamic formalism. Topological pressure is the main tool in studying dimension of invariant sets and measures for dynamical systems in dimension theory. Our purpose in this section is to introduce and study the notion of topological pressure for NAIFSs on a compact topological space.

Consider an NAIFS \((X, \Phi)\) of continuous maps on a compact metric space \((X, d)\). Let \(C(X, \mathbb{R})\) be the space of real-valued continuous functions of \(X\). For \(\psi \in C(X, \mathbb{R})\) and finite word \(w \in I^{m,n}\) we denote \(\Sigma^n_{j=0}(\varphi_{w,j}) (x)\) by \(S_{w,n} \psi(x)\). Also, for subset \(U\) of \(X\) we put \(S_{w,n} \psi(U) = \sup_{x \in U} S_{w,n} \psi(x)\).

### 5.1. Definition of topological pressure using spanning sets

For \(\epsilon > 0, n \geq 1, w \in I^{1,n}\) and \(\psi \in C(X, \mathbb{R})\), put

\[
Q_n(\Phi; w, \psi, \epsilon) := \inf_F \left\{ \sum_{x \in F} e^{S_{w,n} \psi(x)} : F \text{ is a } (w, n, \epsilon; \Phi)\text{-spanning set for } X \right\}
\]

and taking

\[
Q_n(\Phi; \psi, \epsilon) := \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} Q_n(\Phi; w, \psi, \epsilon).
\]

**Remark 5.1.** According to the foregoing description, the following statements are true.

1. \(0 < Q_n(\Phi; w, \psi, \epsilon) \leq \| e^{S_{w,n} \psi} \| r_n(w, \epsilon, \Phi) < \infty\), where \(\| \psi \| = \max_{x \in X} |\psi(x)|\). Hence, \(0 < Q_n(\Phi; \psi, \epsilon) \leq e^{(n+1)\| \psi \|} R_n(\epsilon, \Phi) < \infty\).
2. If \(\epsilon_1 < \epsilon_2\), then \(Q_n(\Phi; w, \psi, \epsilon_1) \geq Q_n(\Phi; w, \psi, \epsilon_2)\). Hence, \(Q_n(\Phi; \psi, \epsilon_1) \geq Q_n(\Phi; \psi, \epsilon_2)\).
3. \(Q_n(\Phi; w, 0, \epsilon) = r_n(w, \epsilon, \Phi)\). Hence, \(Q_n(\Phi; 0, \epsilon) = R_n(\epsilon, \Phi)\).
4. In the definition of \(Q_n(\Phi; w, \psi, \epsilon)\), it suffices to take the infimum over those spanning sets which don’t have proper subsets that \((w, n, \epsilon; \Phi)\)-span \(X\). This is because \(e^{S_{w,n} \psi(x)} > 0\).

Set

\[
Q(\Phi; \psi, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log Q_n(\Phi; \psi, \epsilon).
\]

**Remark 5.2.** By Remark 5.1, the following two facts hold.

1. \(Q(\Phi; \psi, \epsilon) \leq \| \psi \| + \limsup_{n \to \infty} \frac{1}{n} \log R_n(\epsilon, \Phi) < \infty\).
2. If \(\epsilon_1 < \epsilon_2\), then \(Q(\Phi; \psi, \epsilon_1) \geq Q(\Phi; \psi, \epsilon_2)\), i.e., \(Q(\Phi; \psi, \epsilon)\) is monotonic with respect to \(\epsilon\).
Definition 5.3. For $\psi \in C(X, \mathbb{R})$, the topological pressure of an NAIFS $(X, \Phi)$ with respect to $\psi$ is defined as

$$P_{\text{top}}(\Phi, \psi) := \lim_{\epsilon \to 0} Q(\Phi; \psi, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q_n(\Phi; \psi, \epsilon).$$

This is a natural extension of the definition of topological pressure for autonomous dynamical systems, non-autonomous discrete dynamical systems and semigroup actions. Also, it is clear that $P_{\text{top}}(\Phi, 0) = h_{\text{top}}(X, \Phi)$.

Remark 5.4. By part (2) of Remark 5.2, $P_{\text{top}}(\Phi, \psi)$ exists, but could be $\infty$. For example, when $\#(I(j)) = 1$, $\Phi(j) = \{\varphi\}$ for all $j \geq 1$ and $\psi = 0$, we have $P_{\text{top}}(\Phi, \psi) = h_{\text{top}}(\varphi)$ which is the classical topological entropy in the sense of Bowen. Many expositions show that $h_{\text{top}}(\varphi) = \infty$, see, for instance, [15].

5.2. Definition of topological pressure using separated sets. For $\epsilon > 0$, $n \geq 1$, $w \in I_1^n$, and $\psi \in C(X, \mathbb{R})$, put

$$P_n(\Phi; w, \psi, \epsilon) := \sup \left\{ \sum_{x \in E} e^{S_{w,n}(\psi(x))} : E \text{ is a } (w, n, \epsilon; \Phi)-\text{separated set for } X \right\}$$

and taking

$$P_n(\Phi; \psi, \epsilon) := \frac{1}{\#(I_1^n)} \sum_{w \in I_1^n} P_n(\Phi; w, \psi, \epsilon).$$

Remark 5.5. According to the foregoing description, the following statements are true.

1. If $\epsilon_1 < \epsilon_2$, then $P_n(\Phi; w, \psi, \epsilon_1) \geq P_n(\Phi; w, \psi, \epsilon_2)$. Hence, $P_n(\Phi; \psi, \epsilon_1) \geq P_n(\Phi; \psi, \epsilon_2)$.
2. $P_n(\Phi; 0, \psi, \epsilon) = s_n(w, \epsilon, \Phi)$. Hence, $0 < P_n(\Phi; 0, \epsilon) = s_n(\epsilon, \Phi)$.
3. In the definition of $P_n(\Phi; w, \psi, \epsilon)$, it suffices to take the supremum over all $(w, n, \epsilon; \Phi)$-separated sets which fail to be $(w, n, \epsilon; \Phi)$-separated when any point of $X$ is added. This is because $e^{S_{w,n}(\psi(x))} > 0$.
4. $Q_n(\Phi; \psi, \epsilon) \leq P_n(\Phi; \psi, \epsilon)$.

Proof. Fix $w \in I_1^n$. Since $e^{S_{w,n}(\psi(x))} > 0$ and by the fact that each $(w, n, \epsilon; \Phi)$-separated set which cannot be enlarged to another $(w, n, \epsilon; \Phi)$-separated set must be a $(w, n, \epsilon; \Phi)$-spanning set for $X$, we have $Q_n(\Phi; w, \psi, \epsilon) \leq P_n(\Phi; w, \psi, \epsilon)$. Hence, by the definition of $Q_n(\Phi; \psi, \epsilon)$ and $P_n(\Phi; \psi, \epsilon)$, we have $Q_n(\Phi; \psi, \epsilon) \leq P_n(\Phi; \psi, \epsilon)$.

5. If $\delta = \sup \{|\psi(x) - \psi(y)| : d(x, y) < \frac{\epsilon}{2}\}$, then $P_n(\Phi; \psi, \epsilon) \leq e^{(n+1)\delta} Q_n(\Phi; \psi, \frac{\epsilon}{2})$.

Proof. Fix $w \in I_1^n$. Let $E$ be a $(w, n, \epsilon; \Phi)$-separated set and $F$ is a $(w, n, \frac{\epsilon}{2}; \Phi)$-spanning set. Define $\phi : E \to F$ by choosing, for each $x \in E$, some point $\phi(x) \in F$ with $d_{w, n}(x, \phi(x)) < \frac{\epsilon}{2}$. The point $\phi(x) \in F$ that satisfies in this condition is unique. Then $\phi$ is injective and

$$\sum_{y \in F} e^{S_{w,n}(\psi(y))} \geq \sum_{y \in \phi(E)} e^{S_{w,n}(\psi(y))} \geq \left( \min_{x \in E} e^{S_{w,n}(\psi(\phi(x)) - S_{w,n}(\psi(x))} \right) \sum_{x \in E} e^{S_{w,n}(\psi(x))} \geq e^{-(n+1)\delta} \sum_{x \in E} e^{S_{w,n}(\psi(x))}.$$

Therefore $P_n(\Phi; w, \psi, \epsilon) \leq e^{(n+1)\delta} Q_n(\Phi; w, \psi, \frac{\epsilon}{2})$. Hence, by the definition of $Q_n(\Phi; \psi, \frac{\epsilon}{2})$ and $P_n(\Phi; \psi, \epsilon)$, we have $P_n(\Phi; \psi, \epsilon) \leq e^{(n+1)\delta} Q_n(\Phi; \psi, \frac{\epsilon}{2})$. \qed
Then, set
\[ P(\Phi; \psi, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon). \]

**Remark 5.6.** As above, the following statements are true.

1. \( Q(\Phi; \psi, \epsilon) \leq P(\Phi; \psi, \epsilon), \) by part (4) of Remark 5.5.
2. If \( \delta = \sup\{|\psi(x) - \psi(y)| : d(x, y) < \frac{\epsilon}{2}\}, \) then \( P(\Phi; \psi, \epsilon) \leq \delta + Q(\Phi; \psi, \frac{\epsilon}{2}), \) by part (5) of Remark 5.5.
3. If \( \epsilon_1 < \epsilon_2, \) then \( P(\Phi; \psi, \epsilon_1) \geq P(\Phi; \psi, \epsilon_2), \) by part (1) of Remark 5.5.

**Theorem 5.7.** If \( \psi \in C(X, \mathbb{R}) \) then \( P_{\text{top}}(\Phi, \psi) = \lim_{\epsilon \to 0} P(\Phi; \psi, \epsilon). \)

**Proof.** The limit exists by part (3) of Remark 5.6. By part (1) of Remark 5.6, we have
\[ P_{\text{top}}(\Phi, \psi) \leq \lim_{\epsilon \to 0} P(\Phi; \psi, \epsilon). \]

By part (2) of Remark 5.6, for any \( \delta > 0, \) we have
\[ \lim_{\epsilon \to 0} P(\Phi; \psi, \epsilon) \leq \delta + P_{\text{top}}(\Phi, \psi), \]
which implies
\[ \lim_{\epsilon \to 0} P(\Phi; \psi, \epsilon) \leq P_{\text{top}}(\Phi, \psi). \]

Hence, \( P_{\text{top}}(\Phi, \psi) = \lim_{\epsilon \to 0} P(\Phi; \psi, \epsilon). \) The proof is completed. \( \square \)

### 5.3. Definition of topological pressure using open covers.

In this subsection we introduce a special class of continuous potentials and provide a formula via open covers to compute the topological pressure of an NAIFS respect to this class of continuous potentials. Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact metric space \((X, d)\). Given \( \epsilon > 0 \) and \( w \in I^{m,n} \), we say that an open cover \( \mathcal{U} \) of \( X \) is a \((w, n, \epsilon)\)-cover if any open set \( U \in \mathcal{U} \) has \( d_{w,n}\)-diameter smaller than \( \epsilon \), where \( d_{w,n} \) is the Bowen-metric introduced in (7). To obtain another characterization of the topological pressure using open covers, we need continuous potentials satisfying a regularity condition. Given \( \epsilon > 0, \) \( w \in I^{m,n} \) and \( \psi \in C(X, \mathbb{R}) \) we define the variation of \( S_{w,n} \psi \) on dynamical balls of radius \( \epsilon \) (see (8)) alongside the word \( w \) by
\[ \text{Var}_{w,n}(\psi, \epsilon) := \sup_{d_{w,n}(x, y) < \epsilon} |S_{w,n} \psi(x) - S_{w,n} \psi(y)|. \]

We say that potential \( \psi \) has uniform bounded variation on dynamical balls of radius \( \epsilon \) if there exists \( C > 0 \) so that
\[ \sup_{n \geq 1, w \in I^{1,n}} \text{Var}_{w,n}(\psi, \epsilon) \leq C. \]

The potential \( \psi \) has the uniformly bounded variation property whenever there exists \( \epsilon > 0 \) so that \( \psi \) has the uniform bounded variation on dynamical balls of radius \( \epsilon \).

In the following proposition, we use open covers to provide a formula for computation the topological pressure of an NAIFS respect to this class of continuous potentials.

**Proposition 5.8.** Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact metric space \((X, d)\) and \( \psi : X \to \mathbb{R} \) be a continuous potential with the uniformly bounded variation property. Then,
\[ P_{\text{top}}(\Phi, \psi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{S_{w,n} \psi(U)} \right). \]
where the infimum is taken over all open covers \( \mathcal{U} \) of \( X \) such that \( \mathcal{U} \) is a \((w,n,\epsilon)\)-cover.

**Proof.** By Theorem 5.7 we know that

\[
P_{\text{top}}(\Phi, \psi) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon)
\]

where

\[
P_n(\Phi; \psi, \epsilon) = \frac{1}{\#(I^1,n)} \sum_{w \in I^1,n} P_n(\Phi; w, \psi, \epsilon) = \frac{1}{\#(I^1,n)} \sum_{w \in I^1,n} \sup_{x \in E} e^{S_{\psi}(x)}
\]

and the supremum is taken over all sets \( E \) that are \((w,n,\Phi)\)-separated. Now for simplicity, we denote

\[
C_n(\Phi; w, \psi, \epsilon) := \inf_{U \in \mathcal{U}} \sum_{w \in I^1,n} e^{S_{\psi}(U)} \quad \text{and} \quad C_n(\Phi; \psi, \epsilon) := \frac{1}{\#(I^1,n)} \sum_{w \in I^1,n} C_n(\Phi; w, \psi, \epsilon)
\]

where the infimum is taken over all open covers \( \mathcal{U} \) of \( X \) such that \( \mathcal{U} \) is a \((w,n,\epsilon)\)-cover.

Take \( \epsilon > 0 \) and \( w \in I^1,n \). Given a \((w,n,\epsilon; \Phi)\)-maximal separated set \( E \), it follows that \( \mathcal{U} := \{B(x; w, n, \epsilon)\}_{x \in E} \) is a \((w,n,2\epsilon)\)-cover. By the uniformly bounded variation property we have

\[
S_{w,n}(B(x; w, n, \epsilon)) = \sup_{z \in B(x; w, n, \epsilon)} S_{w,n}(z) \leq S_{w,n}(x) + C
\]

for some constant \( C > 0 \), depending only on \( \epsilon \). Consequently, we have

\[
\lim_{n \to \infty} \frac{1}{n} \log C_n(\Phi; \psi, 2\epsilon) \leq \lim_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon).
\]

On the other hand, if \( U \) is \((w,n,\epsilon)\)-cover of \( X \), then for any \((w,n,\epsilon; \Phi)\)-separated set \( E \) we have that \( N(E) \leq N(\mathcal{U}) \), since the diameter of any \( U \in \mathcal{U} \) in the metric \( d_{w,n} \) is less than \( \epsilon \). By the uniformly bounded variation property, we have

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon) \leq \lim_{n \to \infty} \frac{1}{n} \log C_n(\Phi; \psi, \epsilon).
\]

Now, combining equations (14) and (15), we get that

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon) \leq \lim_{n \to \infty} \frac{1}{n} \log C_n(\Phi; \psi, \epsilon) \leq \lim_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \frac{\epsilon}{2}),
\]

this completes the proof. \( \square \)

### 5.4. The topological pressure of \(*\)-expansive NAIFSs

In this subsection, we will be mostly interested in providing conditions to compute the topological pressure of an NAIFS as a limit at a definite size scale. Hence, we begin with the following definition.

**Definition 5.9.** Let \((X, \Phi)\) be an NAIFS of continuous maps on a compact metric space \((X,d)\). For \( \delta > 0 \), the NAIFS \((X, \Phi)\) is said to be \(\delta\)-expansive if for any \( \gamma > 0 \) and any \( x, y \in X \) with \( d(x, y) \geq \gamma \), there exists \( k_0 \geq 1 \) (depending on \( \gamma \)) such that \( d_{w,n}(x, y) > \delta \) for each \( w \in I^{m,n} \) with \( n \geq k_0 \). Also, an NAIFS is said to be \(*\)-expansive if it is \(\delta^*\)-expansive for some \( \delta > 0 \).

In the rest of this section, we prove that the topological pressure of an \(*\)-expansive NAIFS can be computed as the topological complexity that is observable at a definite size scale. More precisely, we get the next result.
Theorem 5.10. Let \((X, \Phi)\) be a \(\delta^*\)-expansive NAIFS of continuous maps on a compact metric space \((X, d)\) for some \(\delta > 0\). Then, for every continuous potential \(\psi : X \to \mathbb{R}\) and every \(0 < \epsilon < \delta\),

\[
P_{\text{top}}(\Phi; \psi) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \sup_{E} \sum_{x \in E} e^{S_{w,n,\psi}(x)} \right),
\]

where the supremum is taken over all sets \(E\) that are \((w, n, \epsilon; \Phi)\)-separated.

Proof. Since \(X\) is compact and \(\psi : X \to \mathbb{R}\) is continuous, without loss of generality, we assume that \(\psi\) is non-negative. Fix \(\gamma\) and \(\epsilon\) with \(0 < \gamma < \epsilon < \delta\). Then by part (3) of Remark 5.6 we have the following inequality

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \gamma) \geq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon).
\]

Hence, it is enough to prove the following converse inequality

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon).
\]

By the definition of \(\delta^*\)-expansivity, for any two distinct points \(x, y \in X\) with \(d(x, y) \geq \gamma\), there exists \(k_0 \geq 1\) (depending on \(\gamma\)) such that \(d_{w,n}(x, y) > \delta\) for each \(w \in I^{m,n}\) with \(n \geq k_0\). Take \(w \in I^{1,n+k}\) with \(n, k \geq k_0\). Given any \((w|n, n, \gamma; \Phi)\)-separated set \(E\), we claim that the set \(E\) is \((w, n + k, \epsilon; \Phi)\)-separated. In fact, given \(x, y \in E\) there exists a \(0 \leq j \leq n\) so that \(d(\varphi^{1,j}_w(x), \varphi^{1,j}_w(y)) > \gamma\). Using that \(n + k - j \geq k_0\) and the definition of \(\delta^*\)-expansivity, it follows that \(d_{w|n+k-j}(\varphi^{1,j}_w(x), \varphi^{1,j}_w(y)) > \delta > \epsilon\). This implies that \(d_{w,n+k}(x, y) > \epsilon\). Hence, \(E\) is \((w, n + k, \epsilon; \Phi)\)-separated, that prove the claim. Since \(\psi\) is non-negative, we have

\[
e^{S_{w,n+k,\psi}(x)} = e^{S_{w,n,\psi}(x)} e^{S_{w|n+k,\psi(\varphi^{1,n}_w(x))}} \geq e^{S_{w,n,\psi}(x)},
\]

which implies that \(P_n(\Phi; \psi, \gamma) \leq P_{n+k}(\Phi; \psi, \epsilon)\) because by relation (16) we have

\[
P_n(\Phi; \psi, \gamma) = \frac{1}{\#(I^{1,n})} \sum_{w \in I^{1,n}} \sup_{E} \sum_{x \in E} e^{S_{w,n,\psi}(x)} = \frac{\#(I^{n+1,k})}{\#(I^{1,n+k})} \sum_{w \in I^{1,n}} \sup_{E} \sum_{x \in E} e^{S_{w,n,\psi}(x)}
\]

\[
= \frac{1}{\#(I^{1,n+k})} \sum_{w \in I^{1,n+k}} \sup_{E} \sum_{x \in E} e^{S_{w,n,\psi}(x)} \leq \frac{1}{\#(I^{1,n+k})} \sum_{w \in I^{1,n+k}} \sup_{E} \sum_{x \in E} e^{S_{w,n+k,\psi}(x)} = P_{n+k}(\Phi; \psi, \epsilon).
\]

Thus,

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_{n+k}(\Phi; \psi, \epsilon)
\]

\[
= \limsup_{n \to \infty} \frac{1}{n+k} \log P_{n+k}(\Phi; \psi, \epsilon)
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi; \psi, \epsilon).
\]

This completes the proof.
Remark 5.11. We observe that in view of the previous characterization given in Proposition 5.8, the same result as Theorem 5.10 also holds if we consider open covers instead of separated sets. More precisely, let $(X, \Phi)$ be a $\delta^*$-expansive NAIFS of continuous maps on a compact metric space $(X, d)$ for some $\delta > 0$. Then, for every continuous potential $\psi : X \to \mathbb{R}$ with the uniformly bounded variation property and every $0 < \epsilon < \delta$, 

\[ P_{\text{top}}(\Phi; \psi) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\#(I_{1,n}^{1,n})} \sum_{w \in I_{1,n}^{1,n}} \inf_{U \in \mathcal{U}} e^{\sum_{U \in \mathcal{U}} S_{w,n}(\psi(U))} \right) \]

where the infimum is taken over all open covers $\mathcal{U}$ of $X$ such that $\mathcal{U}$ is a $(w, n, \epsilon)$-cover.

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