A Unified and Generalized Approach to Quantum Error Correction

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We present a unified approach to quantum error correction, called operator quantum error correction. This scheme relies on a generalized notion of noiseless subsystems that is not restricted to the commutant of the interaction algebra. We arrive at the unified approach, which incorporates the known techniques — i.e. the standard error correction model, the method of decoherence-free subspaces, and the noiseless subsystem method — as special cases, by combining active error correction with this generalized noiseless subsystem method. Moreover, we demonstrate that the quantum error correction condition from the standard model is a necessary condition for all known methods of quantum error correction.

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The possibility of protecting quantum information against undesirable noise has been a major breakthrough for the field of quantum computing, opening the path to potential practical applications. In this paper, we show that the various techniques used to protect quantum information all fall under the same unified umbrella. First, we will review the standard model for quantum error correction and the passive error prevention methods of “decoherence-free subspaces” and “noiseless subsystems”. We shall then demonstrate how the latter scheme admits a natural generalization, and study the necessary and sufficient conditions leading to such generalized noiseless subsystems. This generalized method in turn motivates a unified approach — called operator quantum error correction — that incorporates all aforementioned techniques as special cases. We describe this approach and discuss testable conditions that characterize when error correction is possible given a noise model. Moreover, we show that the standard error correction condition is a prerequisite for any of the known forms of error correction/prevention to be feasible.

The Standard Model — What could be called the “standard model” for quantum error correction consists of a triple \((\mathcal{R}, \mathcal{E}, \mathcal{C})\) where \(\mathcal{C}\) is a subspace, a quantum code, of a Hilbert space \(\mathcal{H}\) associated with a given quantum system. The error \(\mathcal{E}\) and recovery \(\mathcal{R}\) are quantum operations on \(\mathcal{B}(\mathcal{H})\), the set of operators on \(\mathcal{H}\), such that \(\mathcal{R}\) undoes the effects of \(\mathcal{E}\) on \(\mathcal{C}\) in the following sense:

\[
(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma \quad \text{for all} \quad \sigma = \mathcal{P}_C \sigma \mathcal{P}_C,
\]

where \(\mathcal{P}_C\) is the projector of \(\mathcal{H}\) onto \(\mathcal{C}\). As a prelude to what follows below, let us note that instead of focusing on the subspace \(\mathcal{C}\), we could just as easily work with the set of operators \(\mathcal{B}(\mathcal{C})\) which act on \(\mathcal{C}\).

When there exists such an \(\mathcal{R}\) for a given pair \(\mathcal{E}, \mathcal{C}\), the subspace \(\mathcal{C}\) is said to be correctable for \(\mathcal{E}\). The action of the noise operation \(\mathcal{E}\) can be described in an operator-sum representation as \(\mathcal{E}(\sigma) = \sum_{\alpha} \mathcal{E}_\alpha \sigma \mathcal{E}_\alpha^\dagger\). While this representation is not unique, all representations of a given map \(\mathcal{E}\) are linearly related: if \(\mathcal{E}(\sigma) = \sum_{\alpha} F_\alpha \sigma F_\alpha^\dagger\), then there exists scalars \(u_{ab}\) such that \(F_\alpha = \sum_{a} u_{ab} \mathcal{E}_a\). We shall identify the map \(\mathcal{E}\) with any of its error operators \(\mathcal{E} = \{E_a\}\). The existence of a recovery operation \(\mathcal{R}\) of \(\mathcal{E}\) on \(\mathcal{C}\) may be cleanly phrased in terms of the \(\{E_a\}\) as follows [3, 4]:

\[
P_C E_a \mathcal{E}_b P_C = \lambda_{ab} P_C \quad \text{for all} \quad a, b
\]

for some scalars \(\lambda_{ab}\). Clearly, this condition is independent of the operator-sum representation of \(\mathcal{E}\).

Noiseless Subsystems & Decoherence-Free Subspaces — Let \(\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\) be a quantum operation with errors \(\{E_a\}\). The algebra \(A\) generated by the set \(\{E_a, E_a^\dagger\}\) is a \(\mathcal{A}\)-algebra [12], called the interaction algebra, and as such it is unitarily equivalent to a direct sum of (possibly “ampliated”) full matrix algebras: \(A \cong \bigotimes_j M_{m_j} \otimes \mathbb{I}_{n_j}\).

If \(\mathcal{E}\) is a unital quantum operation, by which we mean that the maximally mixed state \(I\) remains unaffected by \(\mathcal{E}\) (i.e., \(\mathcal{E}(I) = I\)), then the fundamental noiseless subsystem (NS) method of quantum error correction [13, 14] may be applied. This method makes use of the operator algebra structure of the “noise commutant”:

\[
\mathcal{A}' = \{\sigma \in \mathcal{B}(\mathcal{H}) : E \sigma = \sigma E \forall E \in \{E_a, E_a^\dagger\}\},
\]

to encode states that are immune to the errors of \(\mathcal{E}\). As such, it is in effect a method of error prevention. Notice that with the structure of \(A\) given above, the noise commutant is unitarily equivalent to \(\mathcal{A}' \cong \bigoplus_j M_{m_j} \otimes \mathbb{I}_{n_j}\).

In [13, 14] it was proved that for unital \(\mathcal{E}\), the noise commutant coincides with the fixed point set for \(\mathcal{E}\); i.e.,

\[
\mathcal{A}' = \text{Fix}(\mathcal{E}) = \{\sigma \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\sigma) = \sigma\}.
\]

This is precisely the reason that \(\mathcal{A}'\) may be used to produce NS for unital \(\mathcal{E}\). We note that while many of the physical noise models satisfy the unital constraint, there...
are important non-unital models as well. Below we show how shifting the focus from $\mathcal{A}'$ to $\text{Fix}(\mathcal{E})$ (and related sets) quite naturally leads to a generalized notion of NS that applies to non-unital quantum operations as well.

Note that the structure of $\mathcal{A}$ given above induces a natural decomposition of the Hilbert space

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j^A \otimes \mathcal{H}_j^B,$$

where the “noisy subsystems” $\mathcal{H}_j^A$ have dimension $m_j$ and the “noiseless subsystems” $\mathcal{H}_j^B$ have dimension $n_j$. For brevity, we focus on the case where information is encoded in a single noiseless sector of $\mathcal{B}(\mathcal{H})$, so

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \otimes \mathcal{K},$$

with $\dim(\mathcal{H}^A) = m$, $\dim(\mathcal{H}^B) = n$ and $\dim \mathcal{K} = \dim \mathcal{H} - mn$. The generalization to multiple $J$’s is straightforward. We shall write $\sigma^A$ for operators in $\mathcal{B}(\mathcal{H}^A)$ and $\sigma^B$ for operators in $\mathcal{B}(\mathcal{H}^B)$. Thus the restriction of the noise commutant $\mathcal{A}'$ to $\mathcal{H}^A \otimes \mathcal{H}^B$ consists of the operators of the form $\sigma^{AB} = 1^A \otimes \sigma^B$ where $1^A$ is the identity element of $\mathcal{B}(\mathcal{H}^A)$. It is easy to see that such states are immune to noise in the unital case.

For notational purposes, assume that ordered orthonormal bases have been chosen for $\mathcal{H}^A = \text{span}\{|\alpha_i\rangle\}_{i=1}^m$ and $\mathcal{H}^B = \text{span}\{|\beta_j\rangle\}_{j=1}^n$ that yield the matrix representation of the corresponding subalgebra of $\mathcal{A}'$ as $1_l \otimes \mathcal{M}_n$. We let

$$\{P_{kl} = |\alpha_k\rangle\langle\alpha_l| \otimes 1_l : 1 \leq k, l \leq m \}$$

denote the corresponding family of “matrix units” associated with this decomposition. In terms of these matrix units, the minimal reducing projectors for $\mathcal{A}'$ are given by $P_k = |\alpha_k\rangle\langle\alpha_k| \otimes 1_l = P_{kk} \in \mathcal{A}$. The following equalities are readily verified and in fact are the defining properties for a family of matrix units.

$$P_{kl} = P_k P_{kl} P_l \quad \forall 1 \leq k, l \leq m$$

$$P^l_{kl} = P_{lk} \quad \forall 1 \leq k, l \leq m$$

$$P_{kl} P_{l' k'} = \begin{cases} P_{k k'} & \text{if } l = l' \\ 0 & \text{if } l \neq l' \end{cases}.$$ 

With these properties in hand, the following useful result may be easily proved.

**Lemma 1** The map $\Gamma = \{P_{kl}\}$ from $\mathcal{B}(\mathcal{H})$ to itself satisfies the following two properties

$$\Gamma(\sigma) = \sum_{k,l} P_{kl} \sigma P^l_{kl} \in \mathcal{A}'$$

$$\Gamma(\sigma^A \otimes \sigma^B) \propto 1^A \otimes \sigma^B.$$ 

for all operators $\sigma^A$, $\sigma^B$ and $\sigma \in \mathcal{B}(\mathcal{H})$.

We note that the NS method contains the method of decoherence-free subspaces (DFS) as a special case. Specifically, if we are given an error operation $\mathcal{E}$, then the DFS method encodes information in a subspace of the system’s Hilbert space that is immune to the evolution. However, instead of working at the level of vectors, we could work at the level of operators. In particular, as in the standard model, we may identify a given Hilbert space $\mathcal{H}$ with the full algebra $\mathcal{B}(\mathcal{H})$ of operators acting on $\mathcal{H}$. In doing so, the DFS method may be regarded as a special case of the NS method in the sense that the DFS method in effect makes use of the “unampliated” summands, $1_{m_j} \otimes \mathcal{M}_{n_j}$ where $m_j = 1$, inside the noise commutant $\mathcal{A}'$ for encoding information.

**Generalized Noiseless Subsystems** — We now describe a generalized notion of noiseless subsystems that serves as a building block for the unified approach to error correction discussed below and applies equally well to non-unital maps. In the standard NS method, the quantum information is encoded in $\sigma^B$; i.e., the state of the noiseless subsystem. Hence, it is not necessary for the noisy subsystem to remain in the maximally mixed state $1^A$ under $\mathcal{E}$, it could in principle get mapped to any other state.

In order to formalize this idea, define for a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \otimes \mathcal{K}$ the set of operators

$$\mathfrak{A} = \{\sigma \in \mathcal{B}(\mathcal{H}) : \sigma = \sigma^A \otimes \sigma^B, \text{ for some } \sigma^A \text{ and } \sigma^B\}.$$ 

Notice that this set has the structure of a semigroup and includes operator algebras such as $1^A \otimes \mathcal{B}(\mathcal{H}^B)$. For notational purposes, we assume that bases have been chosen and define the matrix units $P_{kl}$ as above, so that $P_k = P_{kk}$, $P_\mathfrak{A} = P_1 + \ldots + P_m$, $P_\mathfrak{A} \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, $P_\mathfrak{A}^\perp = 1^A \otimes P_\mathfrak{A}$ and $P_\mathfrak{A} \mathcal{H} = \mathcal{K}$. We also define a map $P_\mathfrak{A}$ by the action $P_\mathfrak{A}(|\psi\rangle) = P_\mathfrak{A}(|\psi\rangle) P_\mathfrak{A}$. The following result leads to our generalized definition of NS.

**Lemma 2** Given a fixed decomposition $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{K}$ and a map $\mathcal{E}$, the following three conditions are equivalent:

1. $\forall \sigma^A \forall \sigma^B$, $\exists \tau^A : \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
2. $\forall \sigma^B$, $\forall \tau^A : \mathcal{E}(1^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
3. $\forall \sigma \in \mathfrak{A} : (\text{Tr}_A \circ P_\mathfrak{A} \circ \mathcal{E})(\sigma) = \text{Tr}_A(\sigma)$.

**Proof.** The implications 1. $\Rightarrow$ 2. and 1. $\Rightarrow$ 3. are trivial. To prove 2. $\Rightarrow$ 1., observe that $\sum_{k=1}^m |\alpha_k\rangle\langle\alpha_k| = 1^A$, so condition 2. implies that for any $|\psi\rangle \in \mathcal{H}^B$,

$$\sum_{k=1}^m \mathcal{E}(|\alpha_k\rangle\langle\alpha_k| \otimes |\psi\rangle\langle\psi|) = \tau^A \otimes |\psi\rangle\langle\psi|$$

for some $\tau^A \in \mathcal{B}(\mathcal{H}^A)$. Since $\mathcal{E}$ is a quantum operation, $\sigma_{\phi,k} = \mathcal{E}(|\alpha_k\rangle\langle\alpha_k| \otimes |\phi\rangle\langle\phi|)$ are positive for $k = 1, \ldots, m$. 
Equation 3 implies that \( \tau^A \otimes |\psi\rangle\langle\psi| \) is a convex combination of the operators \( \sigma_{\psi,k} \), which is only possible if \( \sigma_{\psi,k} = \sigma_{\psi,k}^A \otimes |\psi\rangle\langle\psi| \) for some positive \( \sigma_{\psi,k}^A \). Through an application of the Stinespring dilation theorem \( \text{(1)} \) and a linearity argument, it follows that \( \sigma_{\psi,k}^A \) does not depend on \( \psi \). Since the basis \( \{ |a_k\rangle \} \) and the state \( |\psi\rangle \) were chosen arbitrarily, the result now follows from the linearity of \( \mathcal{E} \).

To prove \( \Rightarrow \), note that since \( \mathcal{E} \) and \( \text{Tr}_B \) are trace preserving, \( \Rightarrow \) implies that \( (P_A \circ \mathcal{E}) (\sigma) = \mathcal{E} (\sigma) \) for all \( \sigma \in \mathfrak{A} \). By setting \( \sigma = \mathbb{1}^A \otimes |\psi\rangle\langle\psi| \) as above, we conclude from \( \Rightarrow \) that \( \mathcal{E} (\sigma) = \tau^A \otimes |\psi\rangle\langle\psi| \) for some \( \tau^A \). The rest follows from linearity.

The subsystem \( \mathcal{H}^B \) is said to be noiseless when it satisfies one — and hence all — of the conditions in Lemma 2. It is clear from the third condition that the fate of the noisy subsystem \( \mathcal{H}^A \) has no importance: only the information stored in the noiseless subsystem \( \mathcal{H}^B \) must be preserved by \( \mathcal{E} \). Note that the generalized definition of NS coincides with the standard definition when \( \dim(\mathcal{H}^A) = 1 \). Hence, the notion of DFS is not altered by this generalization.

Given this new notion of a NS, the crucial question is to determine what are the necessary and sufficient conditions for a map \( \mathcal{E} = \{ E_a \} \) to admit a NS described by a semigroup \( \mathfrak{A} \). Recall that the condition expressed by Eq. 2 gives an answer to standard error correction. The following Theorem provides an answer to this question in the general noiseless subsystem setting.

**Theorem 1** Let \( \mathcal{E} = \{ E_a \} \) be a quantum operation on \( \mathcal{B}(\mathcal{H}) \) and let \( \mathfrak{A} \) be a semigroup in \( \mathcal{B}(\mathcal{H}) \) as above. Then \( \mathfrak{A} \) encodes a noiseless subsystem (decoherence-free subspace in the case \( m = 1 \)) — as defined by any of the three conditions of Lemma 2 — if and only if the following two conditions hold:

\[
P_k E_a P_l = \lambda_{akl} P_{kl} \quad \text{for all} \quad a, k, l \quad \text{(4)}
\]

for some set of scalars \( \{ \lambda_{akl} \} \) and

\[
P_A^+ E_a P_A \equiv 0 \quad \text{for all} \quad a \quad \text{(5)}
\]

**Proof.** To prove the necessity of Eqs. 4 and 5, note that Lemma 1 and Lemma 2 imply

\[
(\Gamma \circ \mathcal{E} \circ \Gamma)(\sigma) \propto \Gamma(\sigma) \quad \text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}). \quad \text{(6)}
\]

By linearity, the proportionality factor cannot depend on \( \sigma \), so the sets of operators \( \{ P_{kl} E_a P_{jl} \} \) and \( \{ \lambda P_{kk'} \} \) define the same map for some scalar \( \lambda \). We may thus find a set of scalars \( \mu_{kiajl,k'l'} \) such that

\[
P_{kl} E_a P_{jl} = \sum_{k'l'} \mu_{kiajl,k'l'} P_{k'l'}. \quad \text{(7)}
\]

Multiplying both sides of this equality on the right by \( P_l \) and on the left by \( P_k \), we see that \( \mu_{kiajl,k'l'} = 0 \) when \( k \neq k' \) or \( l \neq l' \). This implies Eq. 4 with \( \lambda_{akl} = \mu_{kkall,kl} \). For the second condition, note that by definition \( P_A^+ \sigma P_A^+ = 0 \) for all \( \sigma \in \mathfrak{A} \). Together with Lemma 1 and Lemma 2 this implies \( P_A^+ \mathcal{E}(\Gamma(\sigma)) P_A^+ = 0 \) for all \( \sigma \in \mathcal{B}(\mathcal{H}) \). Equation 3 follows from this observation via a consideration of the operator-sum representation for \( \mathcal{E} \).

To prove sufficiency, we use the definitions \( ll = P_A + P_A^+ \) and \( P_A = \sum_{k=1}^m P_k \) to establish for all \( \sigma \in \mathfrak{A} \)

\[
\mathcal{E}(\sigma) = (P_A + P_A^+) \sum_a E_a P_A \sigma P_A^+ E_a^\dagger (P_A + P_A^+ ) = \sum_a P_A E_a \sigma E_a^\dagger P_A = \sum_{a,k,k'} P_k E_a \sigma E_a^\dagger P_{k'}. \quad \text{(8)}
\]

Combining this with the identity \( \tau^A \otimes \sigma^B = P_A (\tau^A \otimes \sigma^B) P_A \sum_{l,l'} P_l (\tau^A \otimes \sigma^B) P_{l'} \) implies

\[
\mathcal{E}(\tau^A \otimes \sigma^B) = \sum_{a,k,k',l,l'} \lambda_{alkl'} P_{kl} (\tau^A \otimes \sigma^B) P_{l'l'}. \quad \text{(9)}
\]

The proof now follows from the fact that the matrix units \( P_{kl} \) act trivially on the \( \mathcal{B}(\mathcal{H}^B) \) sector. □

Conditions Eqs. 1, 5 do not necessarily imply that the noiseless operators are in the commutant of the interaction algebra \( \mathfrak{A} = \{ E_a \} \) since \( P_A E_a P_A^+ \) is not necessarily equal to zero. Hence, this generalization does indeed admit new possibilities.

**The Unified Approach** — The unified scheme for quantum error correction consists of a triple \( (\mathcal{R}, \mathcal{E}, \mathfrak{A}) \) where again \( \mathcal{R} \) and \( \mathcal{E} \) are quantum operations on some \( \mathcal{B}(\mathcal{H}) \), but now \( \mathfrak{A} \) is a semigroup in \( \mathcal{B}(\mathcal{H}) \) defined as above with respect to a fixed decomposition \( \mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K} \). Given such a triple \( (\mathcal{R}, \mathcal{E}, \mathfrak{A}) \) we say that \( \mathfrak{A} \) is correctable for \( \mathcal{E} \) if

\[
(\text{Tr}_A \circ P_A \circ \mathcal{R} \circ \mathcal{E})(\sigma) = \text{Tr}_A(\sigma) \quad \text{for all} \quad \sigma \in \mathfrak{A}. \quad \text{(8)}
\]

In other words, \( (\mathcal{R}, \mathcal{E}, \mathfrak{A}) \) is a correctable triple if the \( \mathcal{H}^B \) sector of the semigroup \( \mathfrak{A} \) encodes a noiseless subsystem of the error map \( \mathcal{R} \circ \mathcal{E} \). Thus, substituting \( \mathcal{E} \) by \( \mathcal{R} \circ \mathcal{E} \) in Lemma 2 offers alternative equivalent definitions of a correctable triple. Observe that the standard model for error correction is given by the particular case in this model that occurs when \( m = 1 \). Lemma 2 shows that the generalized (and standard) NS and DFS methods are captured in this model when \( \mathcal{R} = \text{id} \) is the identity channel and, respectively, \( m \geq 1 \) and \( m = 1 \).

We next present a mathematical condition that characterizes correctable codes for a given channel \( \mathcal{E} \) in terms of its error operators and generalizes Eq. 2 for the standard model. Again, we assume that matrix units \( P_{kl} \) associated with the noise commutant have been defined as above.
Theorem 2 Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. If there is a quantum operation $\mathcal{R}$ on $\mathcal{B}(\mathcal{H})$ such that

$$\{\text{Tr}_A \circ \mathcal{P}_A \circ \mathcal{R} \circ \mathcal{E}\}(\sigma) = \text{Tr}_A(\sigma) \text{ for all } \sigma \in \mathfrak{A},$$

then there are scalars $\Lambda = \{\lambda_{abk l}\}$ such that

$$P_k E_a^\dagger E_b P_l = \lambda_{abk l} P_k \text{ for all } a, b, k, l.$$  \tag{10}

Proof. As noted above $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ being a correctable triple implies that $\mathfrak{A}$ encodes a generalized noiseless subsystem of the map $\mathcal{R} \circ \mathcal{E}$. Applying Theorem 1 and in particular condition Eq. (11), to the map $\mathcal{R} \circ \mathcal{E}$ implies the existence of a set of scalars $\mu_{abk l}$ for which $P_k R_k E_a P_l = \mu_{abk l} P_k$. It now follows from Eq. (10) applied to the map $\mathcal{R} \circ \mathcal{E}$ and $P_A = \sum_j P_j$ that

$$P_k E_a^\dagger E_b P_l = \sum_c P_k^c E_c^\dagger R_c E_b P_l$$

$$= \sum_{c,j} P_k^c E_c^\dagger R_c^j P_j R_c E_b P_l$$

$$= \sum_{c,j} P_c^\alpha_{c,j} \mu_{c,j} P_j^\dagger P_j$$

$$= \left( \sum_{c,j} P_c^\alpha_{c,j} \mu_{c,j} \right) P_k \text{,}$$

and this completes the proof of the Theorem. \hfill $\square$

Remark 1 The condition Eq. (11) is independent of the choice of basis $\{\ket{\alpha_i}\}$ that defines the family $P_{kl}$ and of the operator-sum representation of $\mathcal{E}$. In particular, under the changes $\ket{\alpha_i^\prime} = \sum_j u_{kl} \ket{\alpha_j}$ and $F_a = \sum_{b'} w_{ab} E_b$, the scalars $\Lambda$ change to $\lambda_{abk l} = \sum_{a',b',k' l'} u_{k' l'} u_{a a'} w_{b b'} \lambda_{abk l}$.\hfill $\square$

Equation (10) generalizes the quantum error correction condition Eq. (2) to the case where information is encoded in operators, not necessarily restricted to act on a fixed code subspace $C$. However, observe that setting $k = l$ in Eq. (10) gives the standard error correction condition Eq. (2) with $P_C = P_k$. This leads to the following result.

Theorem 3 If $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ is a correctable triple for some semigroup $\mathfrak{A}$ defined as above, then $(\mathcal{P}_k \circ \mathcal{R} \circ \mathcal{E}, \mathcal{P}_k \mathfrak{A} \mathcal{P}_k)$ is a correctable triple according to the standard definition Eq. (2), where $\mathcal{P}_k$ is any minimal reducing projector of $\mathfrak{A}$, and the map $\mathcal{P}_k$ is defined by $\mathcal{P}_k(\cdot) = \sum_l P_{kl}^\dagger P_{kl}^\prime$.\hfill $\square$

Proof. The error correction condition Eq. (3) and Lemma 2 imply that for all $\sigma^B$ there is a $\tau^A$ such that

$$(\mathcal{R} \circ \mathcal{E})(P_k(\mathbf{1}^A \otimes \sigma^B) P_k) \propto \tau^A \otimes \sigma^B.$$ \hfill $\square$

Observe that $\mathcal{P}_k(\tau^A \otimes \sigma^B) \propto \ket{\alpha_k} \bra{\alpha_k} \otimes \sigma^B$ for all $\sigma^B$ and $\tau^A$. Combining these two observations, we conclude that

$$(\mathcal{P}_k \circ \mathcal{R} \circ \mathcal{E})(P_k(\mathbf{1}^A \otimes \sigma^B) P_k) \propto P_k(\mathbf{1}^A \otimes \sigma^B) P_k,$$

completing the proof.

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