Comonotonic risk measures in a world without risk-free assets

Pablo Koch-Medina\textsuperscript{1}, Cosimo Munari\textsuperscript{2}

\textit{Center for Finance and Insurance, University of Zurich, Switzerland}

Gregor Svindland\textsuperscript{3}

\textit{Mathematics Institute, LMU Munich, Germany}

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Abstract

We study comonotonicity of risk measures in terms of the primitives of the theory: acceptance sets and eligible assets. We show that comonotonicity cannot be characterized by the properties of the acceptance set alone and heavily depends on the choice of the eligible asset. In fact, in many important cases, comonotonicity is only compatible with risk-free eligible assets. The incompatibility with risky eligible assets is systematic whenever the acceptability criterion is based on Value at Risk or any convex distortion risk measures such as Expected Shortfall. These findings show the limitations of the concept of comonotonicity in a world without risk-free assets and raise questions about the meaning and the role of comonotonicity within a capital adequacy framework. We also point out some potential traps when using comonotonicity for “discounted” capital positions.

Keywords: comonotonicity, risk measures, acceptance sets, eligible assets

1 Introduction

The theory of acceptance sets and risk measures occupies an important place in current debates about solvency regimes in both the insurance and the banking world. A variety of theoretical properties of risk measures have been studied since the seminal publication by Artzner et al. (1999), among which the property of comonotonicity has received considerable attention. The connections between risk measures and comonotonicity were first established by Kusuoka (2001) and Delbaen (2002) in the mathematical finance literature and by Dhaene et al. (2002) in the actuarial

\textsuperscript{1}Email: pablo.koch@bf.uzh.ch
\textsuperscript{2}Email: cosimo.munari@bf.uzh.ch
\textsuperscript{3}Email: svindla@mathematik.uni-muenchen.de
Capital requirements and the rationale for comonotonicity

To best highlight the message of our paper, we start by introducing the concept of comonotonicity for risk measures in a rather informal way. Consider a one-period economy and assume that capital positions—assets net of liabilities—of financial institutions at the terminal date are represented by random variables belonging to a suitable ordered vector space which, in line with much of the risk measure literature, we take to be the space $L^\infty$ of bounded random variables over a given probability space. In a capital adequacy context, risk measures are interpreted as capital requirement rules and are used to determine the amount of capital a company has to hold as a regulatory buffer against unexpected future losses (we use the term “regulatory” in a loose sense to encompass any externally or internally imposed requirement). Mathematically, a risk measure can be represented by a decreasing map $\rho : L^\infty \to \mathbb{R}$.

We say that a risk measure $\rho$ is comonotonic if it satisfies

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

whenever $X$ and $Y$ are comonotone random variables in $L^\infty$, i.e. random variables that, being increasing functions of a common risk driver, are perfectly positively dependent. The main idea behind comonotonic risk measures is that merging two comonotone positions $X$ and $Y$ will not lead to diversification and, hence, the amount of risk capital required for the aggregated position $X + Y$ should correspond to the sum of the individual capital requirements. This interpretation is particularly appealing if one requires $\rho$ to be subadditive so that the sum of the individual capital requirements always constitutes an upper bound for the capital requirement of a diversified position. This is the standard argument put forward in the literature to argue that comonotonicity might be a natural and desirable normative requirement in a capital adequacy framework, see e.g. the section on comonotonic risk measures in Föllmer and Schied (2011).

The actuarial literature on capital adequacy has mainly focused on comonotonicity, especially in conjunction with subadditivity, to derive approximations and bounds for capital requirements of aggregated positions. We refer to Dhaene et al. (2002) and to the survey article by Dhaene et al. (2006) for a detailed presentation of such techniques. Note that, as already pointed out in Embrechts et al. (2002), the highest capital requirement under a comonotonic risk measure for an aggregated position with given marginals may not be attained by the comonotonic copula unless the risk measure is subadditive. We refer to Embrechts et al. (2013) and the references therein for a study of bounds under Value-at-Risk.

1As said, the focus of this paper will be on comonotonic risk measures in a capital adequacy framework. It is worth mentioning that comonotonicity has a wide spectrum of applications that we do not cover here. In particular, comonotonicity has been extensively investigated in the context of optimal risk sharing, decision theory and insurance pricing. In fact, comonotonicity was originally studied, under different names, precisely in the context of Pareto optimal allocations. The earliest reference in this respect seems to be Borch (1962). We refer to Landsberger and Meilijson (1994) and to the more recent contributions by Ludkowski and Rüschendorf (2008) and Carlier et al. (2012) for more information about this line of research. In this framework, comonotonicity arises as a natural property of “optimal” allocations in the presence of convex risk measures, see e.g. Jouini et al. (2008) and Filipović and Svindland (2008). Since the fundamental works by Quiggin (1982), Yaari (1987) and Schmeidler (1989), comonotonicity has become a key ingredient of decision theory beyond the classical paradigm of expected utility. This line of research has inspired a variety of applications to insurance pricing as documented in Denneberg (1990), Wang (1996), Wang et al. (1997) and Wang (2000).
The framework of Artzner, Delbaen, Eber, and Heath

The objective of this paper is to investigate the property of comonotonicity for the prominent class of risk measures introduced in the paper by Artzner et al. (1999). The fundamental idea in that paper was to provide an operational definition of risk measures by emphasizing the role of two basic primitive objects:

- the acceptance set $\mathcal{A} \subset L^\infty$, representing the set of capital positions that are deemed acceptable from a regulatory perspective, and
- the eligible asset $S$ with price $S_0 > 0$ and payoff $S_1 \in L^\infty_+$, representing a liquid financial asset used to reach acceptability.

The acceptance set plays the role of a capital adequacy test to discriminate between financial institutions that are adequately capitalized from a regulatory perspective, i.e. whose capital position belongs to $\mathcal{A}$, and those that are inadequately capitalized, i.e. whose capital position does not belong to $\mathcal{A}$. If a company does not pass the capital adequacy test, then its management needs to implement a pre-specified remedial action, namely raising capital and investing it into the eligible asset, to become acceptable.

The risk measure associated with $\mathcal{A}$ and $S$ was defined in the aforementioned paper by setting

$$\rho_{\mathcal{A},S}(X) := \inf\{m \in \mathbb{R} \mid X + \frac{m}{S_0} S_1 \in \mathcal{A}\}, \quad X \in L^\infty.$$

If $S$ is the cash asset so that $S_0 = S_1 = 1$, we simply write

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}, \quad X \in L^\infty.$$

When finite, the quantity $\rho_{\mathcal{A},S}(X)$ has a very clear operational interpretation. If positive, $\rho_{\mathcal{A},S}(X)$ is the “minimum” amount of capital that, if raised and invested in the eligible asset, makes the position $X$ acceptable. If negative, $-\rho_{\mathcal{A},S}(X)$ is the maximum amount of capital that can be returned to the owners while retaining acceptability. As discussed in Section 2 in this paper we rule out the situations where $\rho_{\mathcal{A},S}$ is not finitely valued by making suitable assumptions on the payoff $S_1$.

Note that risk measures of the form $\rho_{\mathcal{A},S}$ satisfy the property of $S$-additivity, i.e.

$$\rho_{\mathcal{A},S}(X + \lambda S_1) = \rho_{\mathcal{A},S}(X) - \lambda S_0$$

for every position $X \in L^\infty$ and $\lambda \in \mathbb{R}$. If $S$ is the cash asset, then we speak of cash-additivity.

The central question

As stated above, the central question of this paper is:

**How to characterize comonotonicity for risk measures of the form $\rho_{\mathcal{A},S}$?**

This question has been addressed in the literature only for the special case of a risk-free eligible asset, e.g. cash, but not for other more realistic choices. As established in Theorem 2.4, for a risk measure of the form $\rho_{\mathcal{A},S}$ to be comonotonic it is necessary (and sufficient) that the corresponding cash-additive risk measure $\rho_{\mathcal{A}}$ is itself comonotonic and coincides with $\rho_{\mathcal{A},S}$ (up to a multiple). Hence, we are in fact asking whether a comonotonic cash-additive risk measure can be additive with respect to a risky eligible asset. Since, as illustrated by our examples and not surprisingly, comonotonicity of $\rho_{\mathcal{A}}$ is not sufficient for $\rho_{\mathcal{A},S}$ to be comonotonic, the question is not settled by the existing literature.
Comonotonicity beyond risk-free assets

The main contribution of our paper is to provide a full picture on comonotonic risk measures when the eligible asset is not risk-free. In this sense, our results will clarify the role of comonotonicity in a world in which no risk-free asset may exist.

We show that risk measures of the form $\rho_{A,S}$ can be comonotonic although the eligible asset is not risk-free, so that our question is meaningful. However, as illustrated by Theorem 2.6 and its corollaries, comonotonicity turns out to be compatible only with a very limited range of eligible assets. When applied to concrete examples, the characterization established there shows that comonotonicity is typically compatible only with eligible assets that are close to being risk-free, i.e. with payoffs that are constant with sufficiently high probability. As discussed in Corollary 2.8 when the underlying acceptance set is pointed, the situation becomes more extreme: comonotonicity is compatible only with risk-free eligible assets. Note that the pointedness condition is far from being pathological and is satisfied, for example, by acceptance sets based on distortion risk measures such as Expected Shortfall.

Comonotonicity in a world of “discounted” positions

As observed above, the literature on comonotonic risk measures has focused exclusively on the cash-additive case. The reason for this focus goes back to the following “discounting” argument (see the remark after Definition 2.1 in Delbaen (2002) or Remark 2.2 in Föllmer and Schied (2002)), which suggests that the cash-additive case encompasses more general risk measures. Assume $S_1$ is bounded away from zero. If we use $S$ as a new numéraire, then, expressed in the new numéraire, a capital position $X \in L^\infty$ becomes $X' = X/S_1$, which is often referred to as the “discounted” version of $X$. After the change of numéraire, the eligible asset “formally” has the look and feel of the cash asset. Moreover, the acceptance set for “discounted” positions is given by

$$A' = \{X/S_1 \mid X \in A\}.$$ 

We can now express $\rho_{A,S}$ — applied to “undiscounted” positions — in terms of the cash-additive risk measure $\rho_{A'}$ — applied to “discounted” positions — as follows:

$$\rho_{A,S}(X) = S_0 \rho_{A'}(X'), \quad X \in L^\infty.$$ 

Note that $\rho_{A,S}(X)$ represents a number of units of the original numéraire and $\rho_{A'}(X')$ a number of units of $S$.

Cash-additive risk measures therefore appear as risk measures either with respect to the cash asset or with respect to a more general eligible asset but using the eligible asset as a numéraire. The study of comonotonic cash-additive risk measures can apply to either of these situations. While for the former comonotonicity may have a financially meaningful interpretation, this does not seem to be the case for the latter. The critical issue in this respect is that it is not clear how to interpret comonotonicity in the “discounted” world once we move back to the “undiscounted” world. Indeed, note that $\rho_{A'}$ is comonotonic if, and only if, we have

$$\rho_{A,S}(X + Y) = \rho_{A,S}(X) + \rho_{A,S}(Y)$$

whenever $X' = X/S_1$ and $Y' = Y/S_1$ are comonotone. However, it is easy to verify that $X'$ and $Y'$ will generally lose their comonotonic relationship once expressed in the original numéraire. This is because, intuitively speaking, multiplication with a (nonconstant) random variable, e.g.
S_1$, reshuffles the outcomes of $X'$ and $Y'$ in a way that will typically disrupt their perfect positive dependence. Not only can $X'$ and $Y'$ be comonotone while $X$ and $Y$ may be far from being comonotone, but also $X'$ and $Y'$ may fail to be comonotone while $X$ and $Y$ are. Moreover, regardless of what $S_1$ might have been, $S'_1 = 1$ becomes comonotone with any random variable in the “discounted” world. Thus, attaching a financial interpretation to the comonotonicity of a “discounted” risk measure in terms of the original numéraire seems to be an impossible undertaking.

The above discussion implies that the comonotonicity of $\rho_A$ is neither implied nor implies the comonotonicity of $\rho_{A,S}$ and highlights, from a different angle, that our results cannot be obtained from the cash-additive theory by a change of numéraire. This conclusion reinforces the general concerns about the cash-additive reduction pointed out in Farkas et al. (2014b). We refer to the monograph by Vecer (2011) and to the recent work by Herdegen (2014) for a discussion about the conceptual importance of a numéraire-independent approach to mathematical finance.

**Structure of the paper**

In Section 2 we describe the underlying framework and state our main results. In Section 3 we discuss a variety of examples. Section 4 concludes. All the proofs are relegated to a final Appendix.

## 2 Risk measures and comonotonicity

In this section we provide a comprehensive study of comonotonicity for the class of risk measures introduced by Artzner et al. (1999). We refer to Appendix A for a brief review of the main mathematical notions used below.

### Introducing risk measures

We consider a one-period economy with dates $t = 0$ and $t = 1$ in which future uncertainty is modelled by a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The capital of a financial institution at time 1, i.e. the value of the company’s assets net of liabilities, is represented by a random variable $X \in L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. We refer to $X$ as being a capital position.

A set $A \subset L^\infty$ is said to be an acceptance set if it is nonempty, strictly contained in $L^\infty$, and monotone, i.e.

$$X \in A, \ Y \geq X \implies Y \in A$$

for every $X, Y \in L^\infty$. The acceptance set $A$ is used to model a capital adequacy test prescribed by (internal or external) regulators: A company with capital position $X$ is deemed adequately capitalized from a regulatory perspective if, and only if, $X$ belongs to $A$. In this sense, $A$ divides the world of financial institutions into the class of acceptable institutions, i.e. those institutions that hold enough capital, and unacceptable ones, i.e. those institutions that have to raise capital to become adequately capitalized.

To make unacceptable capital positions acceptable we allow financial institutions to raise capital and to invest it in a pre-specified traded asset, the so-called eligible asset. We assume the eligible asset to be liquid and frictionless so that it can be bought or sold in any quantity and its initial price is a linear function of the volume traded. In this case, the eligible asset can be represented by a couple

$$S = (S_0, S_1) \in \mathbb{R}_+ \times L^\infty_+,$$
where $S_0$ is its \textit{initial price} and $S_1$ its \textit{terminal payoff}. We assume throughout that $S_0 > 0$ and $S_1 \geq \varepsilon$ for some $\varepsilon > 0$. We say that $S$ is \textit{risk-free} whenever $S_1$ is constant. Otherwise, we speak of a \textit{risky} eligible asset.

In this setting, the rule to compute the amount of required capital can be therefore modelled by the map $\rho_{A,S} : L^\infty \to \mathbb{R}$ defined by setting

$$\rho_{A,S}(X) := \inf\{m \in \mathbb{R} \mid X + \frac{m}{S_0}S_1 \in A\}, \quad X \in L^\infty.$$ 

The requirement that $S_1$ be bounded away from zero ensures that $\rho_{A,S}$ is indeed finitely valued, see Farkas et al. (2014a). When the eligible asset is \textit{cash}, i.e. when $S = (1,1)$, we simply write $\rho_A$ so that

$$\rho_A(X) := \inf\{m \in \mathbb{R} \mid X + m \in A\}, \quad X \in L^\infty.$$ 

Following Artzner et al. (1999), the functional $\rho_{A,S}$ will be called the \textit{risk measure} associated to $A$ and $S$.

As a result of the monotonicity of the acceptance set and the linearity of the pricing rule, the risk measure $\rho_{A,S}$ is easily seen to enjoy the following fundamental properties, which will be freely used in the sequel; see Artzner et al. (1999) and, for the present setting, Farkas et al. (2014a). Here, we say that $\rho : L^\infty \to \mathbb{R}$ is \textit{S-additive} whenever

$$\rho_{A,S}(X + \lambda S_1) = \rho_{A,S}(X) - \lambda S_0$$

for all $X \in L^\infty$ and $\lambda \in \mathbb{R}$. If $S$ is the cash asset, then we speak of \textit{cash-additivity}. Moreover, we adopt the notation

$$A(\rho) := \{X \in L^\infty \mid \rho(X) \leq 0\}.$$ 

**Proposition 2.1.** The risk measure $\rho_{A,S}$ satisfies the following properties:

(i) $\rho_{A,S}$ is \textit{S-additive}.

(ii) $\rho_{A,S}$ is decreasing.

(iii) $A(\rho_{A,S}) = A$ whenever $A$ is closed.

The first property tells us that the worse the capital position, the higher the amount of required capital. The second property shows that adding the payoff of the eligible asset to a given capital position has a linear impact on the corresponding capital requirement. The last property establishes a useful relation between the underlying acceptance set and the risk measure under the assumption of closedness, which is satisfied in all the relevant examples and will be systematically required in the sequel.

**Characterizing comonotonicity**

We start our study of comonotonic risk measures by recalling the notion of comonotonicity. The terminology was introduced in Schmeidler (1986) and the characterization in terms of a common risk driver can be found in Denneberg (1994). We refer to the introduction for a financial interpretation in the context of capital adequacy.
Definition 2.2. We say that $X, Y \in L^\infty$ are comonotone whenever there is a $\mathbb{P} \otimes \mathbb{P}$-null set $N \subset \mathcal{F} \otimes \mathcal{F}$ such that

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \text{ for all } (\omega, \omega') \in \Omega \times \Omega \setminus N$$

This is equivalent to the existence of $Z \in L^\infty$ and of two increasing functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfying

$$X = f(Z) \text{ and } Y = g(Z).$$

A set $C \subset L^\infty$ is said to be comonotonic if $X$ and $Y$ are comonotone for any choice of $X, Y \in C$. A functional $\rho : L^\infty \to \mathbb{R}$ is called comonotonic whenever

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

for all comonotone $X, Y \in L^\infty$.

We start our investigation of comonotonic risk measures of the form $\rho_{A,S}$ by highlighting that any comonotonic decreasing functional can be expressed as the risk measure associated to a closed conic acceptance set and to a risk-free asset.

Lemma 2.3. Let $\rho : L^\infty \to \mathbb{R}$ be a nonzero comonotonic decreasing map. Then, $\rho(1) < 0$ and we have

$$\rho(X) = \rho_{A(\rho), R}(X)$$

for every $X \in L^\infty$, where $A(\rho)$ is a closed conic acceptance set and $R = (-\rho(1), 1)$.

Armed with this result we can now provide necessary and sufficient conditions for a risk measure $\rho_{A,S}$ to be comonotonic in terms of properties of the acceptance set and of the eligible asset. First of all, we show that $\rho_{A,S}$ cannot be comonotonic unless the cash-additive risk measure $\rho_A$ is itself comonotonic.

Proposition 2.4. Assume $A$ is closed and $\rho_{A,S}$ is comonotonic. Then, $\rho_A$ is comonotonic as well and we have

$$\rho_{A,S}(X) = \rho_{A,R}(X) = -\rho_{A,S}(1)\rho_A(X)$$

for every $X \in L^\infty$, where $R = (-\rho_{A,S}(1), 1)$.

It follows from from the above result that a necessary condition for the comonotonicity of $\rho_{A,S}$ is that, besides being additive with respect to the eligible asset $S$, the risk measure $\rho_{A,S}$ is also additive with respect to a particular risk-free asset. In particular, we are naturally led to investigate the equality of two risk measures $\rho_{A,S}$ and $\rho_{A,R}$ based on the same acceptance set but different eligible assets. The following lemma establishes a necessary and sufficient condition for this equality to hold. Here, we denote by $\text{span}(X)$ the linear space generated by $X \in L^\infty$.

Lemma 2.5. Assume $A$ is closed and consider two eligible assets $S = (S_0, S_1)$ and $R = (R_0, R_1)$. Then, the following statements are equivalent:

(a) $\rho_{A,S}(X) = \rho_{A,R}(X)$ for every $X \in L^\infty$.

(b) $A + \text{span} \left( \frac{S_1}{S_0} - \frac{R_1}{R_0} \right) \subset A$. 

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We are now in a position to state our first characterization of comonotonicity for risk measures \( \rho_{A,S} \). This result identifies which combinations of acceptance sets and eligible assets give rise to comonotonicity. In particular, it shows that comonotonicity cannot be characterized by the properties of the acceptance set alone: Given the same acceptance set, a risk measure \( \rho_{A,S} \) will be comonotonic only for special choices of the eligible asset \( S \).

**Theorem 2.6.** Assume \( A \) is closed and \( \rho_A \) is comonotonic. Then, the following statements are equivalent:

(a) \( \rho_{A,S} \) is comonotonic.

(b) \( A \pm (1 + \frac{\rho_{A,S}(1)}{S_0} S_1) \subset A \).

In particular, \( \rho_{A,S} \) is comonotonic only if \( S_1 + \frac{S_0}{\rho_{A,S}(1)} \in A \cap (-A) \).

To best interpret condition (b) in the above theorem assume that \( \rho_{A,S}(1) = -1 \) so that the condition boils down to

\[
A \pm \left( 1 - \frac{S_1}{S_0} \right) \subset A.
\]

This means that acceptability is not compromised once we add to any acceptable position the payoff of a *fully-leveraged portfolio* obtained by borrowing at the risk-free rate and going long the eligible asset or, similarly, by shorting the eligible asset and going long the risk-free asset (assuming the risk free asset is available in the market). In particular, since \( 0 \in A \), we must have

\[
1 - \frac{S_1}{S_0} \in A,
\]

showing that the payoffs of these fully-leveraged portfolios are themselves acceptable. This highlights that comonotonicity for \( \rho_{A,S} \) corresponds to a property of acceptability, namely the “blindness” of the acceptance set with respect to aggregations of fully-leveraged portfolios, that is far from being desirable from a regulatory perspective.

We specify the preceding theorem to convex acceptance sets, in which case comonotonicity can be characterized by means of a simpler condition as follows.

**Corollary 2.7.** Assume \( A \) is closed and convex and \( \rho_A \) is comonotonic. Then, the following statements are equivalent:

(a) \( \rho_{A,S} \) is comonotonic.

(b) \( \pm (1 + \frac{\rho_{A,S}(1)}{S_0} S_1) \in A \).

(c) \( S_1 + \frac{S_0}{\rho_{A,S}(1)} \in A \cap (-A) \).

In the setting of the above corollary the set \( A \cap (-A) \) is a linear space whose elements might be called *risk invariants* in that the risk measure \( \rho_A \) is not affected once we add them to a given capital position, i.e.

\[
A \cap (-A) = \{ X \in L^\infty \mid \rho_A(X + Y) = \rho_A(Y), \ Y \in L^\infty \}.
\]
To see this, note first that \( A \cap (-A) \) is indeed a linear space since \( A \) is a convex cone. Then, for any \( X \in A \cap (-A) \) and \( Y \in L^\infty \) we have
\[
\rho_A(Y) = \rho_A(Y + X - X) \leq \rho_A(X + Y) + \rho_A(-X) \leq \rho_A(X + Y) \leq \rho_A(X) + \rho_A(Y) \leq \rho_A(Y)
\]
by subadditivity of \( \rho_A \), showing the above representation of \( A \cap (-A) \). In this sense, the preceding result tells us that \( \rho_{A,S} \) is comonotonic if, and only if, the payoff of the “fully-leveraged” position \( S_1 + \frac{S_0}{\rho_{A,S}(1)} \) is a risk invariant.

As a consequence of the preceding corollary we see that, if the acceptance set \( A \) satisfies the pointedness condition
\[
A \cap (-A) = \{0\},
\]
i.e. if there exists no nonzero risk invariant, then a risk measure \( \rho_{A,S} \) fails to be comonotonic unless the eligible asset \( S \) is risk-free. As will be illustrated by our examples in the next section, this situation is far from being exceptional. In fact, the vast majority of acceptance sets used in applications satisfy the above pointedness condition.

**Corollary 2.8.** Assume \( A \) is closed and satisfies \( A \cap (-A) = \{0\} \). Then, \( \rho_{A,S} \) is comonotonic only if
\[
S_1 = -\frac{S_0}{\rho_{A,S}(1)}.
\]
In particular, \( \rho_{A,S} \) is never comonotonic if \( S \) is a risky asset.

**A weaker form of comonotonicity**

What drives the strong structural results about comonotonic risk measures is the fact that the constant random variable 1 is comonotone with any random variable. As a result, whenever \( X \) and \( Y \) are comonotonic, the set \( \{X, Y, 1\} \) is a comonotonic set. One could then wonder whether it is possible to obtain a wider spectrum of results by weakening the notion of a comonotonic risk measure and only require that
\[
\rho_{A,S}(X + Y) = \rho_{A,S}(X) + \rho_{A,S}(Y)
\]
whenever there is a comonotonic set \( C \) such that \( X, Y, S_1 \in C \). In this case, besides being comonotone, \( X \) and \( Y \) must also be comonotone with \( S_1 \). This property could be called \( S \text{-comonotonicity} \).

The above question has, in fact, a purely mathematical interest. Indeed, being expressed in terms of a given eligible asset, the above weaker notion of comonotonicity is tailored to risk measures with respect to that eligible asset. However, the eligible asset is nothing but an instrument to reach acceptability and there seems to be no way to establish what is the “right” eligible asset to consider in this context. Anyway, from a purely mathematical perspective, it is not difficult to see that we arrive at similar constraints on the acceptance set \( A \) and the eligible asset \( S \) as in the standard comonotone case. Indeed, an inspection of our results show that they can be localized to comonotonic sets containing \( S_1 \). For instance, keeping in mind that 1 is comonotonic with \( S_1 \), it follows as in Lemma 2.3 that
\[
\rho_{A,S}(X + \lambda) = \rho_{A,S}(X) - \lambda \rho_{A,S}(-1)
\]
for all \( \lambda \in \mathbb{R} \) whenever \( X \) is comonotone with \( S_1 \). Moreover, \( \rho_{A,S} \) must be positively homogeneous on the cone
\[
U = \{X \in L^\infty \mid X \text{ is comonotone with } S_1\}.
\]
A straightforward generalization of the preceding results thus shows that, if \( \mathcal{A} \) is closed, then we must have
\[
\mathcal{A} \cap \mathcal{U} \pm \left( 1 + \frac{\rho_{A,S}(1)}{S_0} S_1 \right) \subset \mathcal{A}.
\]
Thus, the condition \( \rho_{A,S}(X + Y) = \rho_{A,S}(X) + \rho_{A,S}(Y) \) whenever there is a comonotonic set \( \mathcal{C} \) such that \( X, Y, S_1 \in \mathcal{C} \) only holds if
\[
S_1 + \frac{S_0}{\rho_{A,S}(1)} \in \mathcal{A} \cap (-\mathcal{A}).
\]
This latter condition is the same as in Theorem \ref{thm:main} and implies that \( S \) must be risk-free once \( \mathcal{A} \) is pointed.

3 Examples

In this final section we illustrate our results by focusing on a variety of explicit acceptance sets. These examples will make clear that comonotonicity is the exception rather than the rule as soon as the eligible asset is not risk-free.

Capital adequacy based on Value-at-Risk

The Value-at-Risk (VaR) of a capital position \( X \in L^\infty \) at the level \( \alpha \in (0, 1) \) is defined by
\[
\text{VaR}_\alpha(X) := \inf \{ m \in \mathbb{R} \mid \mathbb{P}(X + m < 0) \leq \alpha \}.
\]
Note that \( \text{VaR}_\alpha(X) \) is, up to a sign, the upper \( \alpha \)-quantile of \( X \). The corresponding acceptance set is the closed cone given by
\[
\mathcal{A}_{\text{VaR}}(\alpha) := \{ X \in L^\infty \mid \text{VaR}_\alpha(X) \leq 0 \} = \{ X \in L^\infty \mid \mathbb{P}(X < 0) \leq \alpha \}.
\]
We are interested in studying the comonotonicity of the risk measure \( S\text{-VaR}_\alpha : L^\infty \to \mathbb{R} \) given by
\[
S\text{-VaR}_\alpha(X) := \rho_{A_{\text{VaR}}(\alpha),S}(X) = \inf \{ m \in \mathbb{R} \mid \mathbb{P}(X + \frac{m}{S_0} S_1 < 0) \leq \alpha \}, \quad X \in L^\infty.
\]
Since \( \text{VaR}_\alpha \) is well-known to be comonotonic, one easily sees that \( S\text{-VaR}_\alpha \) will be automatically comonotonic whenever \( S \) is risk-free. In this case, \( S\text{-VaR}_\alpha \) will be, in fact, just a multiple of \( \text{VaR}_\alpha \).

At the same time, it is not difficult to verify that the acceptance set \( \mathcal{A}_{\text{VaR}}(\alpha) \) is not pointed in general and, therefore, Corollary \ref{cor:main} does not apply to risk measures based on VaR-acceptability.

The first result is derived by applying Theorem \ref{thm:main} to VaR-acceptability and shows that \( S\text{-VaR}_\alpha \) can be comonotonic only if the payoff \( S_1 \) is constant with sufficiently high probability. Indeed, as values of \( \alpha \) close to 0 are the interesting ones from a practical perspective, the bound given in (3.1) is close to 1.

Proposition 3.1. Assume \( S\text{-VaR}_\alpha \) is comonotonic. Then, we have
\[
\mathbb{P}\left( S_1 = \frac{1}{\text{VaR}_\alpha(1/S_1)} \right) \geq 1 - 2\alpha.
\]

(3.1)
Remark 3.2. Condition (3.1) is generally not sufficient for comonotonicity. To see this, let \( \{ A, B, C \} \) be a measurable partition of \( \Omega \) such that \( \mathbb{P}(A) = \mathbb{P}(B) = \alpha \) and \( \mathbb{P}(C) = 1 - 2\alpha \). Consider an eligible asset \( S \) with \( S_0 > 0 \) and

\[
S_1 = \begin{cases} 
S_0 & \text{on } A \cup C, \\
2S_0 & \text{on } B.
\end{cases}
\]

It is easy to verify that \( S - \text{VaR}_\alpha(1) = -1 \) and, thus, (3.1) is satisfied. However, since \(-1_A \in \mathcal{A}_{\text{VaR}}(\alpha) \) but

\[-1_A + \left( 1 - \frac{S_1}{S_0} \right) = -1_{A \cup B} \notin \mathcal{A}_{\text{VaR}}(\alpha),\]

then condition \((b)\) in Theorem 2.6 is violated and, hence, \( S - \text{VaR}_\alpha \) is not comonotonic.

In view of the previous result, it is natural to wonder whether comonotonicity is at all compatible with risky eligible assets. The next proposition characterizes all the underlying probabilistic models where \( S - \text{VaR}_\alpha \) is comonotonic for some risky eligible asset.

**Lemma 3.3.** The following statements are equivalent:

(a) There exists a risky eligible asset \( S \) such that \( S - \text{VaR}_\alpha \) is comonotonic.

(b) There exists \( A \in \mathcal{F} \) such that \( 0 < \mathbb{P}(A) \leq \alpha \) and for every \( B \in \mathcal{F} \) we have

\[\mathbb{P}(B) \leq \alpha \implies \mathbb{P}(A) + \mathbb{P}(A^c \cap B) \leq \alpha.\]

The preceding result has the following remarkable consequence when specified to the common setting of a nonatomic probability space, i.e. a probability space that supports random variables with any prescribed distribution: Risk measures based on VaR-acceptability are never comonotonic unless the eligible asset is risk-free.

**Proposition 3.4.** Assume \( (\Omega, \mathcal{F}, \mathbb{P}) \) is nonatomic. Then, \( S - \text{VaR}_\alpha \) is comonotonic if and only if \( S \) is risk-free.

It is well-known that \( \text{VaR}_\alpha \) fails to be subadditive. However, being comonotone, it satisfies

\[\text{VaR}_\alpha(X + Y) = \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)\]

for any comonotone \( X, Y \in L^\infty \). This allows to control the capital required for an aggregated position of comonotone random variables by means of the individual capital requirements. Since \( S - \text{VaR}_\alpha \) is in general not comonotone if \( S \) is a risky asset, one may wonder whether the capital required for an aggregated position of comonotone random variables can still be controlled in terms of the individual capital requirements or not. Here, we show that the undesirable situation

\[S - \text{VaR}_\alpha(X + Y) > S - \text{VaR}_\alpha(X) + S - \text{VaR}_\alpha(Y)\]

is possible also for comonotone \( X, Y \in L^\infty \), so that summing up the individual capital requirements of comonotone random variables does not help find a bound for the capital required for the aggregated position.
We provide an example in the setting of Example 3.2 above. If we consider the comonotone random variables

\[ X = \begin{cases} 
-2 & \text{on } A \\
-3 & \text{on } B \\
2 & \text{on } C 
\end{cases} \quad \text{and} \quad Y = \begin{cases} 
-4 & \text{on } A \\
-9 & \text{on } B \\
0 & \text{on } C 
\end{cases}, \]

then it is not difficult to show that

\[ S-\text{VaR}_\alpha(X + Y) = 6 > \frac{3}{2} + 4 = S-\text{VaR}_\alpha(X) + S-\text{VaR}_\alpha(Y). \]

**Capital adequacy based on Expected Shortfall**

The *Expected Shortfall* (ES) of a capital position \( X \in L^\infty \) at the level \( \alpha \in (0, 1) \) is defined by

\[ \text{ES}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) \, d\beta. \]

The corresponding acceptance set is the closed convex cone defined by

\[ \mathcal{A}_{\text{ES}}(\alpha) := \{X \in L^\infty \mid \text{ES}_\alpha(X) \leq 0\}. \]

We aim to characterize comonotonicity for the risk measure \( S-\text{ES}_\alpha : L^\infty \to \mathbb{R} \) given by

\[ S-\text{ES}_\alpha(X) := \rho_{\mathcal{A}_{\text{ES}}(\alpha), S}(X) = \inf\{m \in \mathbb{R} \mid \text{ES}_\alpha(X + mS_1) \leq 0\}, \quad X \in L^\infty. \]

We show that risk measures based on ES-acceptability are comonotonic if, and only if, the eligible asset is risk-free. This result will be a direct consequence of the following lemma.

**Lemma 3.5.** For every nonconstant \( X \in L^\infty \) we have

\[ \text{ES}_\alpha(X) > -\mathbb{E}[X]. \]

The preceding lemma implies that acceptance sets based on ES are pointed and, thus, we are in a position to apply Corollary 2.8 and conclude that risk measures based on ES-acceptability fail to be comonotonic unless the eligible asset is risk-free.

**Proposition 3.6.** The risk measure \( S-\text{ES}_\alpha \) is comonotonic if and only if \( S \) is risk-free.

**Capital adequacy based on distortion risk measures**

We denote by \( \mathcal{P}([0, 1]) \) the set of all probability measures \( \mu : [0, 1] \to [0, 1] \). The *distortion risk measure* associated to \( \mu \in \mathcal{P}([0, 1]) \) is the map \( \text{DR}_\mu : L^\infty \to \mathbb{R} \) defined by

\[ \text{DR}_\mu(X) := \int_0^1 \text{ES}_\alpha(X) \, \mu(d\alpha). \]

Here, as is commonly done, we extend ES by setting

\[ \text{ES}_0(X) := -\inf\{m \in \mathbb{R} \mid X \geq m\} \quad \text{and} \quad \text{ES}_1(X) := -\mathbb{E}[X]. \]
The corresponding acceptance set is the closed convex cone given by
\[ \mathcal{A}_{\text{DR}}(\mu) := \{ X \in L^\infty \mid \text{DR}_\mu(X) \leq 0 \} . \]

The class of acceptance sets based on distortion risk measures is huge and includes, in a nonatomic setting, all the acceptance sets that are convex, law-invariant and compatible with comonotonicity. This is made precise in the next proposition, which is a direct consequence of Theorem 4.93 in Föllmer and Schied (2011).

**Proposition 3.7.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic and \(\mathcal{A}\) is closed, convex and law-invariant. Then, the following statements are equivalent:

(a) \(\rho_A\) is comonotonic.

(b) \(\mathcal{A} = \mathcal{A}_{\text{DR}}(\mu)\) for some \(\mu \in \mathcal{P}([0, 1])\).

**Remark 3.8.** In a nonatomic setting, distortion risk measures can be equivalently identified, up to a sign, with Choquet integrals associated with concave distortion functions, see Theorem 4.70 in Föllmer and Schied (2011).

We aim to characterize comonotonicity for the risk measure \(S\)-\(\text{DR}_\mu : L^\infty \rightarrow \mathbb{R}\) given by
\[ S\text{-}\text{DR}_\mu(X) := \rho_{\mathcal{A}_{\text{DR}}(\mu), S}(X) = \inf \{ m \in \mathbb{R} \mid \text{DR}_\mu(X + \frac{m}{S_0} S_1) \leq 0 \}, \quad X \in L^\infty . \]

The next proposition shows that distortion-based risk measures are never comonotonic unless they reduce to standard expectations or the eligible asset is taken to be risk-free.

**Proposition 3.9.** The risk measure \(S\text{-}\text{DR}_\mu\) is comonotonic if and only if one of the following conditions holds:

(i) \(\mu(\{1\}) = 1\) (so that \(\text{DR}_\mu(X) = -\mathbb{E}[X]\) for all \(X \in L^\infty\)).

(ii) \(S\) is risk-free.

### 4 Conclusions

The preceding discussion and results unveil a variety of pitfalls of the notion of comonotonicity that have not been highlighted hitherto:

- **Comonotonicity typically fails in a world without risk-free assets.** Comonotonicity is compatible with a limited range of eligible assets, which typically need to be close to risk-free. In fact, for many important acceptance sets — such as those based on distortion risk measures like Expected Shortfall — comonotonicity is compatible only with risk-free eligible assets.

- **Comonotonicity cannot be expressed in terms of properties of \(\mathcal{A}\) alone.** Differently from other important properties of risk measures such as convexity, subadditivity, and positive homogeneity, our results imply that comonotonicity cannot be characterized in terms of properties of the underlying acceptance set alone but critically depends upon the choice of the eligible asset. Since the eligible asset is simply a vehicle to ensure acceptability, which is the key capital adequacy objective, it would seem that, in the context of capital adequacy, comonotonicity is more an incidental property than a fundamental one.
Comonotonicity depends on the underlying numéraire. The fact that comonotonicity is not preserved by changes of numéraire implies that requiring comonotonicity for “discounted” risk measures has no clear interpretation in the “undiscounted” setting. This seriously limits the financial interpretability of comonotonicity in a capital adequacy context.

These conclusions qualify the meaning and the role of comonotonicity within a capital adequacy context when the eligible asset is not risk-free. In particular, the critical dependence of comonotonicity on the choice of the underlying numéraire shows that the assumption of comonotonicity in a world of “discounted” capital positions is not financially plausible. This seems to suggest that requiring comonotonicity in a capital adequacy framework may not be as desirable as sometimes claimed and calls for a critical appraisal of the concept of comonotonicity within a capital adequacy framework.

A The mathematical setup

Throughout the whole paper we consider a fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and equip \(\mathbb{R}\) with its canonical Borel measurable structure. A random variable \(X : \Omega \to \mathbb{R}\) is said to be \(\mathbb{P}\)-almost surely (a.s.) bounded whenever

\[
\|X\| := \inf\{m \in \mathbb{R} \mid \mathbb{P}(|X| > m) = 0\} < \infty.
\]

The constant random variable with value \(\lambda \in \mathbb{R}\), which is still denoted by \(\lambda\), and the indicator function of an event \(E \in \mathcal{F}\), denoted by \(1_E\), are examples of bounded random variables. The equivalence classes of \(\mathbb{P}\)-a.s. bounded random variables with respect to \(\mathbb{P}\)-a.s. equality form the vector space \(L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})\). We identify, as is customary, equivalence classes in \(L^\infty\) with any of their representatives. The space \(L^\infty\) is a Banach lattice if equipped with the above norm and with the partial order

\[
X \geq Y \iff \mathbb{P}(X \geq Y) = 1.
\]

The set of positive elements of \(L^\infty\) is given by

\[
L^\infty_+ := \{X \in L^\infty \mid X \geq 0\}.
\]

For any \(X, Y \in L^\infty\) we write \(X \sim Y\) whenever \(X\) and \(Y\) have the same distribution under \(\mathbb{P}\), i.e. \(\mathbb{P}(X \leq \lambda) = \mathbb{P}(Y \leq \lambda)\) for all \(\lambda \in \mathbb{R}\).

We collect in the next definition a variety of properties of functionals that will be used later on. Recall that, under positive homogeneity, convexity and subadditivity are equivalent.

**Definition A.1.** A functional \(\rho : L^\infty \to \mathbb{R}\) may satisfy the following properties:

(i) \(\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)\) for all \(\lambda \in [0, 1]\) and \(X, Y \in L^\infty\) (convexity).

(ii) \(\rho(\lambda X) = \lambda \rho(X)\) for all \(\lambda \in \mathbb{R}_+\) and \(X \in L^\infty\) (positive homogeneity).

(iii) \(\rho(X + Y) \leq \rho(X) + \rho(Y)\) for all \(X, Y \in L^\infty\) (subadditivity).

(iv) \(\rho(X) \leq \rho(Y)\) for all \(X, Y \in L^\infty\) with \(X \geq Y\) (decreasing monotonicity).

(v) \(|\rho(X) - \rho(Y)| \leq c\|X - Y\|\) for all \(X, Y \in L^\infty\) and some \(c \in \mathbb{R}_+\) (Lipschitz continuity).
(vi) \( \rho(X) = \rho(Y) \) for all \( X, Y \in L^\infty \) such that \( X \sim Y \) (law-invariance).

For ease of reference we also collect a number of properties of sets of random variables. Note that, under conicity, convexity is equivalent to closedness under addition.

**Definition A.2.** A set \( \mathcal{A} \subset L^\infty \) may satisfy the following properties:

(i) \( \lambda X + (1 - \lambda)Y \in \mathcal{A} \) for all \( \lambda \in [0, 1] \) and \( X, Y \in \mathcal{A} \) (convexity).

(ii) \( \lambda X \in \mathcal{A} \) for all \( \lambda \in \mathbb{R}_+ \) and \( X \in \mathcal{A} \) (conicity).

(iii) \( X + Y \in \mathcal{A} \) for all \( X, Y \in \mathcal{A} \) (closedness under addition).

(iv) \( -X \notin \mathcal{A} \) for all nonzero \( X \in \mathcal{A} \) (pointedness).

(v) \( X \in \mathcal{A} \) for all \( X \in L^\infty \) with \( X \geq Y \) for some \( Y \in \mathcal{A} \) (increasing monotonicity).

(vi) \( X \in \mathcal{A} \) for all \( X \in L^\infty \) such that \( X_n \to X \) for some \( (X_n) \subset \mathcal{A} \) (closedness).

(vii) \( X \in \mathcal{A} \) for all \( X \in L^\infty \) such that \( X \sim Y \) for some \( Y \in \mathcal{A} \) (law-invariance).

**B Proofs**

**Proof of Lemma 2.3** We easily see that comonotonicity implies \( \rho(0) = 0 \) and, hence, \( \rho(-1) = -\rho(1) \). Moreover, we have \( \rho(1) \leq 0 \) by monotonicity. We claim that \( \rho(1) < 0 \) holds. To prove this, assume that \( \rho(1) = 0 \) and take \( X \in L^\infty \). Since 1 is comonotone with itself and \( a \geq X \geq -a \) for \( a \in \mathbb{N} \) large enough, we obtain

\[
0 = a \rho(1) = \rho(a) \leq \rho(X) \leq \rho(-a) = a \rho(-1) = 0,
\]

which is only possible if \( \rho \) is the zero functional. In conclusion, we see that \( \rho(1) < 0 \) must hold.

It follows from the monotonicity of \( \rho \) that \( \mathcal{A} \) is an acceptance set. Closedness and conicity will follow once we show that \( \rho \) is Lipschitz continuous and positively homogeneous, respectively. To this end, take first an arbitrary \( X \in L^\infty \) and two integers \( a, b \in \mathbb{N} \). Since every random variable is comonotone with itself we have

\[
\rho(\frac{a}{b} X) = a \rho(\frac{1}{b} X) = a \rho(b \frac{1}{b} X) = a \rho(X).
\]

As a result, for every rational number \( \lambda \in \mathbb{Q} \) it follows that

\[
\rho(X + \lambda) = \rho(X) + \rho(\lambda) = \rho(X) - \lambda \rho(-1),
\]

where we used that \( \lambda \) is comonotone with \( X \) and that \( \rho(-1) = -\rho(1) \).

Take now an arbitrary \( \lambda \in \mathbb{R} \) and let \( (\lambda_n^+) \) and \( (\lambda_n^-) \) be sequences of rational numbers converging to \( \lambda \) from above and below, respectively. Using the monotonicity of \( \rho \) we finally obtain

\[
\rho(X) - \lambda_n^+ \rho(-1) = \rho(X + \lambda_n^+) \leq \rho(X + \lambda) \leq \rho(X + \lambda_n^-) = \rho(X) - \lambda_n^- \rho(-1)
\]

for any \( n \in \mathbb{N} \). By letting \( n \) tend to infinity, we conclude that \( \rho(X + \lambda) = \rho(X) - \lambda \rho(-1) \) must hold. Since, setting \( S = (-\rho(1), 1) \), the condition \( X + \frac{m}{s_0} S_1 \in \mathcal{A}(\rho) \) is then easily seen to be equivalent to \( \rho(X) \leq m \) for any \( m \in \mathbb{R} \), we easily see that

\[
\rho(X) = \inf \{ m \in \mathbb{R} \mid X + \frac{m}{s_0} S_1 \in \mathcal{A}(\rho) \} = \rho_{\mathcal{A}(\rho), S}(X).
\]  

(B.1)
We have already established positive homogeneity for positive rational numbers. Take now a strictly-positive \( \lambda \in \mathbb{R}_+ \) and let \( (\lambda_n^+) \) and \( (\lambda_n^-) \) be sequences of positive rational numbers converging to \( \lambda \) from above and below, respectively. In view of (B.1) we can assume that \( X \in L_\infty^+ \) without loss of generality. Then, the monotonicity of \( \rho \) yields

\[
\lambda_n^+ \rho(X) = \rho(\lambda_n^+ X) \leq \rho(\lambda X) \leq \rho(\lambda_n^- X) = \lambda_n^- \rho(X)
\]

for every \( n \in \mathbb{N} \). Letting \( n \) go to infinity, we conclude that \( \rho(\lambda X) = \lambda \rho(X) \). This proves positive homogeneity.

Since \( Y - \|X - Y\| \leq X \leq Y + \|X - Y\| \) for any \( X, Y \in L_\infty \), it follows immediately from monotonicity and (B.1) that

\[
\rho(Y) + \|X - Y\| \rho(-1) \geq \rho(X) \geq \rho(Y) - \|X - Y\| \rho(-1),
\]

which shows that \( \rho \) is Lipschitz continuous. This concludes the proof of the proposition. \( \square \)

**Remark B.1.** An alternative way to establish the previous lemma is to rely on the representation theorem in Schmeidler (1986) and derive the above properties of \( \rho \) from the properties of Choquet integrals, see e.g. Proposition 5.1 in Denneberg (1994). The above proof has the advantage of being direct and of showing explicitly the role of each of our assumptions.

**Proof of Proposition 2.4** Recall from Proposition 2.1 that \( A(\rho_{A,S}) = A \) since \( A \) is closed. Hence, Lemma 2.3 implies that

\[
\rho_{A,S}(X) = \rho_{A,R}(X) = \inf \{ m \in \mathbb{R} \mid X - \frac{m}{\rho_{A,S}(1)} \in A \} = -\rho_{A,S}(1) \rho_A(X)
\]

for all \( X \in L_\infty \). This also shows that \( \rho_A \) is comonotonic. \( \square \)

**Proof of Lemma 2.5** To prove that \( (a) \) implies \( (b) \), assume that \( \rho_{A,S} = \rho_{A,R} \). Then, for any \( X \in L_\infty \) and \( m \in \mathbb{R} \) we have

\[
\rho_{A,S} \left( X + m \left( \frac{S_1}{S_0} - \frac{R_1}{R_0} \right) \right) = \rho_{A,S}(X),
\]

where we used that \( \rho_{A,S} \) is additive with respect to both \( S \) and \( R \) by assumption. The assertion now follows from \( A(\rho_{A,S}) = A \).

To prove the converse implication, take \( X \in L_\infty \) and assume that \( (b) \) holds. If \( X + \frac{m}{S_0} S_1 \in A \) for some \( m \in \mathbb{R} \), then we easily see that

\[
X + \frac{m}{R_0} R_1 = X + \frac{m}{S_0} S_1 - m \left( \frac{S_1}{S_0} - \frac{R_1}{R_0} \right) \in A
\]

as well. This implies that \( \rho_{A,R}(X) \leq \rho_{A,S}(X) \). Since the argument is symmetric in \( S \) and \( R \), we conclude that \( \rho_{A,S} = \rho_{A,R} \). \( \square \)

**Remark B.2.** The above lemma could be derived from Proposition 5.1 in Farkas et al. (2014a), see also Proposition 1-a in Artzner et al. (2009) in a convex setting. We have provided a short proof for the sake of completeness.
Proof of Theorem 2.6 First, assume that (a) holds so that \( \rho_{A,S} \) is comonotonic. Then, by Proposition 2.4 we have
\[
\rho_{A,S}(X) = -\rho_{A,S}(1)\rho_A(X) = \rho_{A,R}(X)
\]
for every \( X \in L^\infty \), where \( R = (-\rho_{A,S}(1),1) \). Hence, Lemma 2.5 immediately yields (b).

Conversely, assume that (b) holds. As \( \rho_A \) is comonotonic, \( A \) must be conic by Lemma 2.3. Moreover, note that \( \rho_{A,S}(1) < 0 \) by virtue of the same lemma. Since condition (b) is then easily seen to be equivalent to
\[
A + \text{span}(\frac{S_1}{S_0} - \frac{1}{-\rho_{A,S}(1)}) \subset A,
\]
we can rely on Lemma 2.5 to conclude that \( \rho_{A,S} \) coincides, up to the constant \(-\rho_{A,S}(1)\), with the comonotonic risk measure \( \rho_A \) and is therefore itself comonotonic. This establishes (a) and concludes the proof of the equivalence. The last assertion follows immediately because 0 \( \in A \) by conicity.

Proof of Corollary 2.7 Since 0 \( \in A \) by conicity, it follows directly from Theorem 2.6 that (a) implies (b). To prove the converse implication, assume that (b) holds and take an arbitrary \( X \in A \). Since, being convex and conic, \( A \) is closed under addition, we infer from (b) that \( X \pm (1 + \frac{\rho_{A,S}(1)}{S_0}S_1) \in A \). That \( \rho_{A,S} \) is comonotonic now follows from Theorem 2.6. The equivalence between (b) and (c) is a direct consequence of conicity.

Proof of Proposition 3.1 Since 0 \( \in A_{\text{VaR}}(\alpha) \), it follows from Theorem 2.6 that \( S-\text{VaR}_\alpha \) can be comonotonic only if
\[
\mathbb{P}(1 + \frac{S-\text{VaR}_\alpha(1)}{S_0}S_1 < 0) \leq \alpha \quad \text{and} \quad \mathbb{P}(-1 - \frac{S-\text{VaR}_\alpha(1)}{S_0}S_1 < 0) \leq \alpha .
\] (B.2)

By rearranging, the above inequalities are easily seen to imply
\[
\mathbb{P}(S_1 = -\frac{1}{\text{VaR}_\alpha(1/S_1)}) = 1 - \mathbb{P}(S_1 < -\frac{1}{\text{VaR}_\alpha(1/S_1)}) - \mathbb{P}(S_1 > -\frac{1}{\text{VaR}_\alpha(1/S_1)}) \geq 1 - 2\alpha ,
\]
where we used that \( S-\text{VaR}_\alpha(1) = S_0 \text{VaR}_\alpha(1/S_1) \).

Proof of Lemma 3.3 To prove that (a) implies (b), assume that \( S-\text{VaR}_\alpha \) is comonotonic for some eligible asset \( S \) with nonconstant payoff \( S_1 \). Consider the random variable
\[
Z = 1 + \frac{S-\text{VaR}_\alpha(1)}{S_0}S_1
\]
and note that, by (B.2), it satisfies
\[
\max\{\mathbb{P}(Z < 0), \mathbb{P}(Z > 0)\} \leq \alpha .
\]
Since \( Z \) is nonconstant, one of the above probabilities must be strictly positive. Without loss of generality, assume that \( \mathbb{P}(Z < 0) > 0 \) and set
\[
A = \{Z < 0\} \in \mathcal{F}.
\]
Now, take any \( B \in \mathcal{F} \) satisfying \( \mathbb{P}(B) \leq \alpha \). Since \(-1_B \in A_{\text{VaR}}(\alpha) \), Theorem 2.6 implies that \( Z - 1_B \) must belong to \( A_{\text{VaR}}(\alpha) \) so that \( \mathbb{P}(Z < 1_B) \leq \alpha \). Note that \( \mathbb{P}(Z < 1) = 1 \) because \( S-\text{VaR}_\alpha(1) < 0 \). Then, it is easy to see that
\[
\mathbb{P}(A) + \mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup (A^c \cap B)) \leq \mathbb{P}(Z < 1_B) \leq \alpha .
\]
Hence, (a) implies (b).

To prove the converse implication, assume that (b) holds and define the eligible asset $S$ by setting $S_0 = 1$ and

$$S_1 = \begin{cases} 1 & \text{on } A^c, \\
2 & \text{on } A. \end{cases}$$

Moreover, consider the random variable

$$Z = 1 + \frac{S - \text{VaR}_\alpha(1)}{S_0} S_1.$$ 

It is easy to verify that $S - \text{VaR}_\alpha(1) = -1$ so that $Z = -1_\mathbb{A}$.

By applying (b) to $B = \{X < 0\}$ we obtain

$$\mathbb{P}(X + Z < 0) = \mathbb{P}(A \cap \{X < 1_A\}) + \mathbb{P}(A^c \cap \{X < 1_A\}) \leq \mathbb{P}(A) + \mathbb{P}(A^c \cap \{X < 0\}) \leq \alpha.$$ 

In addition, we easily see that

$$\mathbb{P}(X - Z < 0) = \mathbb{P}(X < -1_A) \leq \mathbb{P}(X < 0) \leq \alpha.$$ 

As a result, Theorem 2.6 implies that $S - \text{VaR}_\alpha$ is comonotonic and we conclude that (b) implies (a). This completes the proof of the equivalence.

Proof of Proposition 3.4 The “if” implication is always true. To prove the “only if” implication, it suffices to note that condition (b) in Lemma 3.3 is never satisfied in a nonatomic setting. Indeed, if $\mathbb{P}(A) \leq \alpha$, then we can always find $B \in \mathcal{F}$ such that $B \subset A^c$ and $\alpha - \mathbb{P}(A) < \mathbb{P}(B) < \alpha$ by nonatomicity. Since we easily have

$$\mathbb{P}(A) + \mathbb{P}(A^c \cap B) = \mathbb{P}(A) + \mathbb{P}(B) > \alpha,$$

Lemma 3.3 tells us that $S - \text{VaR}_\alpha$ can be comonotonic only if $S_1$ is constant.

Proof of Lemma 3.5 Since, for given $X \in L^\infty$, we have that $\text{VaR}_\beta(X)$ is decreasing as a function of $\beta$, it follows that

$$\mathbb{E}[\alpha](X) - \int_0^\alpha \text{VaR}_\beta(X) \, d\beta = \frac{1 - \alpha}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) \, d\beta \geq (1 - \alpha) \text{VaR}_\alpha(X) \geq \int_\alpha^1 \text{VaR}_\beta(X) \, d\beta,$$

with equality between the left- and the right-hand side if and only if $\text{VaR}_\beta(X) = \text{VaR}_\alpha(X)$ for all $\beta \in (0,1)$ or, equivalently, if and only if $X$ is constant. Hence, by rearranging, we obtain

$$\mathbb{E}[\alpha](X) \geq \int_0^\alpha \text{VaR}_\beta(X) \, d\beta + \int_\alpha^1 \text{VaR}_\beta(X) \, d\beta = \int_0^1 \text{VaR}_\beta(X) \, d\beta = -\mathbb{E}[X],$$

with equality if and only if $X$ is constant.

Proof of Proposition 3.6 The “if” implication is clear since $S - \mathbb{E}[\alpha]$ is simply a multiple of $\mathbb{E}[\alpha]$ in this case and $\mathbb{E}[\alpha]$ is well known to be comonotonic. To prove the “only if” implication, note that

$$\mathbb{E}[\alpha](X) + \mathbb{E}[\alpha](-X) > -\mathbb{E}[X] + \mathbb{E}[X] = 0$$

(B.3)
holds for every nonconstant \( X \in L^\infty \) by Lemma \[3.5\]. As a result, it follows that the acceptance set \( A_{ES}(\alpha) \) satisfies the pointedness condition

\[
A_{ES}(\alpha) \cap (-A_{ES}(\alpha)) = \{0\}.
\]

To see this, assume that \( X \in A_{ES}(\alpha) \cap (-A_{ES}(\alpha)) \) so that \( ES_\alpha(X) \leq 0 \) and \( ES_\alpha(-X) \leq 0 \) both hold. Since \( ES_\alpha(X) + ES_\alpha(-X) \leq 0 \), it follows from \[3.3\] that \( X \) must be constant. However, the only constant random variable belonging to \( A_{ES}(\alpha) \cap (-A_{ES}(\alpha)) \) is clearly the zero random variable. In conclusion, \( A_{ES}(\alpha) \) is pointed and Corollary \[2.8\] implies that \( S-ES_\alpha \) is comonotonic only if \( S \) is risk-free.

\[\square\]

**Proof of Proposition \[3.9\]** The “if” implication is clear since under any of the two conditions the risk measure \( S-DR_\mu \) is simply a multiple of \( DR_\mu \), which is comonotonic. In particular, if \( \mu(\{1\}) = 1 \), then we clearly have \( DR_\mu(X) = -E[X] \) for any \( X \in L^\infty \) so that

\[
S-DR_\mu(X) = \inf\{m \in \mathbb{R} \mid E[X + \frac{m}{E[S_1]}S_1] \geq 0\} = -\frac{S_0}{E[S_1]}E[X] = \frac{S_0}{E[S_1]}DR_\mu(X)
\]

for all \( X \in L^\infty \). To prove the “only if” implication, assume that \( S-DR_\mu \) is comonotonic but \( \mu(\{1\}) < 1 \). Then, it follows from Lemma \[3.5\] that any nonconstant \( X \in L^\infty \) satisfies \( DR_\mu(X) > -E[X] \) and therefore

\[
DR_\mu(X) + DR_\mu(-X) > -E[X] + E[X] = 0.
\]

As in the proof of Proposition \[3.6\] this yields the pointedness condition

\[
A_{DR}(\mu) \cap (-A_{DR}(\mu)) = \{0\}.
\]

Hence, we can apply Corollary \[2.8\] and conclude that \( S \) must be risk-free. This completes the proof of the “only if” implication.

\[\square\]

**References**

[1] Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk. *Mathematical Finance* 9, 203–228 (1999)

[2] Artzner, Ph., Delbaen, F., Koch-Medina, P.: Risk measures and efficient use of capital, *ASTIN Bulletin*, 39(1), 101–116 (2009)

[3] Borch, K.: Equilibrium in a reinsurance market, *Econometrica*, 30(3), 424–444 (1962)

[4] Carlier, G., Dana, R.-A., Galichon, A.: Pareto efficiency for the concave order and multivariate comonotonicity, *Journal of Economic Theory*, 147(1), 207–229 (2012)

[5] Delbaen, F.: Coherent risk measures on general probability spaces, In: Sandmann, K., Schönbucher, P.J. (eds.), *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann*, pp. 1–37, Springer (2002)

[6] Denneberg, D.: Distorted probabilities and insurance premium, *Methods of Operations Research*, 52, 21–42 (1990)

[7] Denneberg, D.: *Non-Additive Measure and Integral*, Kluwer (1994)

[8] Demui, M., Dhaene, J., Goovaerts, M.J., Kaas, R.: *Actuarial Theory for Dependent Risks: Measures, Orders and Models*, Wiley (2005)

[9] Dhaene, J., Demui, M., Goovaerts, M.J., Kaas, R., Vyncke, D.: The concept of comonotonicity in actuarial science and finance: theory, *Insurance: Mathematics and Economics*, 31(1), 3–33 (2002)
[10] Dhaene, J., Vanduffel, S., Goovaerts, M.J., Kaas, R., Tang, Q., Vyncke, D.: Risk measures and comonotonicity: A review, *Stochastic Models*, 22, 573–606 (2006)

[11] Embrechts, P., McNeil, A.J., Straumann, A.: Correlation and dependence in risk management: properties and pitfalls, In: Dempster, M.A.H. (ed.), *Risk Management: Value at Risk and Beyond*, pp. 176–223, Cambridge University Press (2002)

[12] Embrechts, P., Puccetti, G., Rüschendorf, L.: Model uncertainty and VaR aggregation, *Journal of Banking & Finance*, 37(8), 2750–2764 (2013)

[13] Föllmer, H., Schied, A.: Robust preferences and convex measures of risk, In: Sandmann, K., Schönbucher, P.J. (eds.), *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann*, pp. 39–56, Springer (2002)

[14] Föllmer, H., Schied, A.: Stochastic Finance. An Introduction in Discrete Time. Berlin: De Gruyter (2011)

[15] Farkas, W., Koch-Medina, P., Munari, C.: Capital requirements with defaultable securities, *Insurance: Mathematics and Economics*, 55, 58–67 (2014a)

[16] Farkas, W., Koch-Medina P., Munari, C.: Beyond cash-additive capital requirements: when changing the numéraire fails, *Finance and Stochastics*, 18(1), 145–173 (2014b)

[17] Filipović, D., Svindland, G.: Optimal capital and risk allocations for law- and cash-invariant convex functions, *Finance and Stochastics*, 12, 423–439 (2008)

[18] Herdegen, M.: No-arbitrage in a numéraire-independent modeling framework, *Mathematical Finance*, forthcoming (2014)

[19] Jouini, E., Schachermayer, W., Touzi, N.: Optimal risk sharing for law invariant monetary utility functions, *Mathematical Finance*, 18(2), 269–292 (2008)

[20] Kusuoka, S.: On law invariant coherent risk measures, *Advances in Mathematical Economics*, 3, 83–95 (2001)

[21] Landsberger, M., Meilijson, I.: Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion, *Annals of Operations Research*, 52(2), 97–106 (1994)

[22] Ludkovski, M., Rüschendorf, L.: On comonotonicity of Pareto optimal risk sharing, *Statistics & Probability Letters*, 78, 1181–1188 (2008)

[23] McNeil, A.J., Frey, R., Embrechts, P.: *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press (2015)

[24] Quiggin, J.: A theory of anticipated utility, *Journal of Economic Behavior & Organization*, 3(4), 323–343 (1982)

[25] Schmeidler, D.: Integral representation without additivity, *Proceedings of the American Mathematical Society*, 97(2), 255–261 (1986)

[26] Schmeidler, D.: Subjective probability and expected utility without additivity, *Econometrica*, 57(3), 571–587 (1989)

[27] Vecer, J.: *Stochastic Finance: A Numéraire Approach*, Chapman & Hall (2011)

[28] Wang, S.: Premium calculation by transforming the layer premium density, *ASTIN Bulletin*, 26, 71–92 (1996)

[29] Wang, S.: A class of distortion operators for pricing financial and insurance risks, *Journal of Risk and Insurance*, 67, 15–36 (2000)

[30] Wang, S., Young, V., Panjer, H.: Axiomatic characterization of insurance prices, *Insurance: Mathematics and Economics*, 21, 173–183 (1997)

[31] Yaari, M.E.: The dual theory of choice under risk, *Econometrica*, 55(1), 95–115 (1987)