ISOMONODROMIC DEFORMATIONS OF LOGARITHMIC CONNECTIONS AND STABILITY

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Abstract. Let $X_0$ be a compact connected Riemann surface of genus $g$ with $D_0 \subset X_0$ an ordered subset of cardinality $n$, and let $E_G$ be a holomorphic principal $G$–bundle on $X_0$, where $G$ is a reductive affine algebraic group defined over $\mathbb{C}$, that is equipped with a logarithmic connection $\nabla_0$ with polar divisor $D_0$. Let $(E_G, \nabla)$ be the universal isomonodromic deformation of $(E_G, \nabla_0)$ over the universal Teichmüller curve $(X, D) \to \text{Teich}_{g,n}$, where Teich$_{g,n}$ is the Teichmüller space for genus $g$ Riemann surfaces with $n$–marked points. We prove the following (see Section 5):

1. Assume that $g \geq 2$ and $n = 0$. Then there is a closed complex analytic subset $Y \subset \text{Teich}_{g,n}$, of codimension at least $g$, such that for any $t \in \text{Teich}_{g,n} \setminus Y$, the principal $G$–bundle $E_G|_{X_t}$ is semistable, where $X_t$ is the compact Riemann surface over $t$.

2. Assume that $g \geq 1$, and if $g = 1$, then $n > 0$. Also, assume that the monodromy representation for $\nabla_0$ does not factor through some proper parabolic subgroup of $G$. Then there is a closed complex analytic subset $Y' \subset \text{Teich}_{g,n}$, of codimension at least $g$, such that for any $t \in \text{Teich}_{g,n} \setminus Y'$, the principal $G$–bundle $E_G|_{X_t}$ is semistable.

3. Assume that $g \geq 2$. Assume that the monodromy representation for $\nabla_0$ does not factor through some proper parabolic subgroup of $G$. Then there is a closed complex analytic subset $Y'' \subset \text{Teich}_{g,n}$, of codimension at least $g - 1$, such that for any $t \in \text{Teich}_{g,n} \setminus Y''$, the principal $G$–bundle $E_G|_{X_t}$ is stable.

In [He1], the second–named author proved the above results for $G = \text{GL}(2, \mathbb{C})$.

1. Introduction

Take any quadruple of the form $(E \to X, D, \nabla)$, where $E$ is a holomorphic vector bundle over a smooth connected complex variety $X$, and $\nabla$ is an integrable logarithmic connection on $E$ singular over a simple normal crossing divisor $D \subset X$. The monodromy functor associates to it a representation $\rho_{\nabla} : \pi_1(X \setminus D, x_0) \to \text{GL}(E_{x_0})$, where $x_0 \in X \setminus D$. Altering the connection by a holomorphic automorphism $A$ of $E$ leads to a representation conjugated by $A(x_0)$. The monodromy functor produces an equivalence between the category of logarithmic connections $(E, \nabla)$ on $(X, D)$ such that the real parts of the residues lie in $[0, 1)$ and the category of equivalence classes of linear representations of $\pi_1(X \setminus D, x_0)$. Given a monodromy representation $\rho$, one can consider the set of all logarithmic connections $(E \to X, D, \nabla)$ (with no condition on the residues) up to holomorphic isomorphisms that produce the same monodromy representation $\rho = \rho_{\nabla}$ up to conjugation. All these connections are conjugated to each other by meromorphic gauge transformations with possible poles over $D$ (see for example [Sa]).

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The classical Riemann–Hilbert problem can be formulated as follows:

*Given a representation \( \rho : \pi_1(\mathbb{P}_C^1 \setminus D_0, x) \to \text{GL}(r, \mathbb{C}) \), is there a logarithmic connection \((E \to \mathbb{P}_C^1 \setminus D_0, \nabla)\) such that \( \rho = \rho_\nabla \) and \( E \) is holomorphically trivial?*

The answer to this problem is

1. positive if \( \text{rank } r = 2 \) \([\mathbb{P}], [\mathbb{D}]\),
2. negative in general \((r \geq 3) \([\mathbb{A}], [\mathbb{B}]\)
3. positive if \( \rho \) is irreducible \([\mathbb{B}], [\mathbb{O}]\).

On the other hand, the fundamental group \( \pi_1(\mathbb{P}_C^1 \setminus D_0, x) \) depends only on the topological and not the complex structure of \( \mathbb{P}_C^1 \setminus D_0 \). So given an integrable connection on \( \mathbb{P}_C^1 \setminus D_0 \), one can consider variations of the complex structure without changing the monodromy representation. More precisely, consider the Teichmüller space \( \text{Teich}_{0,n} \) of the \( n \)-pointed Riemann sphere together with the corresponding universal Teichmüller curve

\[
\tau : (X = \mathbb{P}_C^1 \times \text{Teich}_{0,n}, \mathcal{D}) \to \text{Teich}_{0,n},
\]

where \( n = \text{degree}(D_0) \). Since \( \text{Teich}_{0,n} \) is contractible, the inclusion

\[
(\mathbb{P}_C^1, D_t) := \tau^{-1}(t) \hookrightarrow (X, \mathcal{D}), \quad t \in \text{Teich}_{0,n},
\]

induces an isomorphism \( \pi_1(\mathbb{P}_C^1 \setminus D_t, x_t) \simeq \pi_1(X \setminus \mathcal{D}, x_t) \). Hence by the Riemann–Hilbert correspondence, we can associate to any logarithmic connection \((E_0, \nabla_0)\) on \( \mathbb{P}_C^1 \), with polar divisor \( D_0 \), its universal isomonodromic deformation: a flat logarithmic connection \((\mathcal{E}, \nabla)\) over \( X \) with polar divisor \( \mathcal{D} \) that extends the connection \((E_0, \nabla_0)\). The conjugacy class of the monodromy representation for \( \nabla|_{r^{-1}(t)} \) does not change as \( t \) moves over \( \text{Teich}_{0,n} \) (see for example \([\mathbb{H}]\)).

We are led to another Riemann–Hilbert problem:

*Given a logarithmic connection \((E_0, \nabla_0)\) on \( \mathbb{P}_C^1 \) with polar divisor \( D_0 \) of degree \( n \), is there a point \( t \in \text{Teich}_{0,n} \) such that the holomorphic vector bundle \( E_t = \mathcal{E}|_{\mathbb{P}_C^1 \times \{t\}} \) underlying the universal isomonodromic deformation \((\mathcal{E}, \nabla)\) is trivial?*

A partial answer to this question is given by the following theorem of Bolibruch:

**Theorem 1.1 ([Bol2]).** Let \((E_0, \nabla_0)\) be an irreducible trace-free logarithmic rank two connection with \( n \geq 4 \) poles on \( \mathbb{P}_C^1 \) such that each singularity is resonant. There is a proper closed complex analytic subset \( \mathcal{Y} \subset \text{Teich}_{0,n} \) such that for all \( t \in \text{Teich}_{0,n} \setminus \mathcal{Y} \), the holomorphic vector bundle \( E_t = \mathcal{E}|_{\mathbb{P}_C^1 \times \{t\}} \) underlying the universal isomonodromic deformation \((\mathcal{E}, \nabla)\) of \((E_0, \nabla_0)\) is trivial.

It should be mentioned that the condition in Theorem 1.1 that each singularity is resonant, can actually be removed \([\mathbb{H}]\).

From the Birkhoff–Grothendieck classification of holomorphic vector bundles on \( \mathbb{P}_C^1 \) it follows immediately that the only semistable holomorphic vector bundle of degree zero and rank \( r \) on \( \mathbb{P}_C^1 \) is the trivial bundle \( \mathcal{O}_{\mathbb{P}_C^1}^{\oplus r} \). This leads to the following more general question:
Given a representation \( \rho : \pi_1(X \setminus D, x) \rightarrow \text{GL}_r(\mathbb{C}) \), where \( X \) is a compact connected Riemann surface, is there a logarithmic connection \( (E \xrightarrow{\nabla} X, D, \nabla) \) such that \( \rho = \rho_\nabla \) and \( E \) is semistable of degree zero?

The answer to this problem is

1. negative in general [EH],
2. positive if \( \rho \) is irreducible [EV].

Let \( \tau : (\mathcal{X}, D) \rightarrow \text{Teich}_{g,n} \) be the universal Teichmüller curve. In view of the above, it is natural to ask the following:

Given a logarithmic connection \( (E_0, \nabla_0) \), with polar divisor \( D_0 \) of degree \( n \) on a compact connected Riemann surface \( X_0 \) of genus \( g \), is there an element \( t \in \text{Teich}_{g,n} \) such that the holomorphic vector bundle \( E_t = \mathcal{E}|_{\mathcal{X}_t} \xrightarrow{} \mathcal{X}_t = \tau^{-1}(t) \) underlying the universal isomonodromic deformation \( (\mathcal{E}, \nabla) \) of \( (E_0, \nabla_0) \) is semistable?

Note that we necessarily have degree\( (E_t) = \text{degree}(E_0) \). Again, Theorem 1.1 can be generalized as follows.

**Theorem 1.2 ([HeI]).** Let \( (E_0, \nabla_0) \) be an irreducible logarithmic rank two connection with polar divisor \( D_0 \) of degree \( n \) on a compact connected Riemann surface \( X_0 \) of genus \( g \) such that \( 3g - 3 + n > 0 \). Consider its universal isomonodromic deformation \( (\mathcal{E}, \nabla) \) over \( \tau : (\mathcal{X}, D) \rightarrow \text{Teich}_{g,n} \). There is a closed complex analytic subset \( \mathcal{Y} \subset \text{Teich}_{g,n} \) (respectively, \( \mathcal{Y}_s \subset \text{Teich}_{g,n} \)) of codimension at least \( g \) (respectively, \( g - 1 \)) such that for any \( t \in \text{Teich}_{g,n} \setminus \mathcal{Y} \), the vector bundle \( E_t = \mathcal{E}|_{\mathcal{X}_t} \), where \( (\mathcal{X}_t, D_t) = \tau^{-1}(t) \), is semistable (respectively, stable).

Our aim here is to prove an analog of Theorem 1.2 in the more general context of logarithmic connections on principal \( G \)-bundles over a compact connected Riemann surface (see [Boa] for logarithmic connections on principal \( G \)-bundles).

Let \( X_0 \) be a compact connected Riemann surface of genus \( g \), and let \( D_0 \subset X_0 \) be an ordered subset of it of cardinality \( n \). Let \( G \) be a reductive affine algebraic group defined over \( \mathbb{C} \). Let \( E_G \) be a holomorphic principal \( G \)-bundle on \( X_0 \) and \( \nabla_0 \) a logarithmic connection on \( E_G \) with polar divisor \( D_0 \). Let \( (\mathcal{E}_G, \nabla) \) be the universal isomonodromic deformation of \( (E_G, \nabla_0) \) over the universal Teichmüller curve \( \tau : (\mathcal{X}, D) \rightarrow \text{Teich}_{g,n} \). For any point \( t \in \text{Teich}_{g,n} \), the restriction \( \mathcal{E}_G|_{\tau^{-1}(t)} \rightarrow \mathcal{X}_t := \tau^{-1}(t) \) will be denoted by \( \mathcal{E}_G|_{\tau^{-1}(t)} \).

We prove the following (see Section 5):

**Theorem 1.3.**

1. Assume that \( g \geq 2 \) and \( n = 0 \). Then there is a closed complex analytic subset \( \mathcal{Y} \subset \text{Teich}_{g,n} \) of codimension at least \( g \) such that for any \( t \in \text{Teich}_{g,n} \setminus \mathcal{Y} \), the holomorphic principal \( G \)-bundle \( \mathcal{E}_G|_{\tau^{-1}(t)} \rightarrow \mathcal{X}_t \) is semistable.
2. Assume that \( g \geq 1 \), and if \( g = 1 \), then \( n > 0 \). Also, assume that the monodromy representation for \( \nabla_0 \) does not factor through some proper parabolic subgroup of \( G \). Then there is a closed complex analytic subset \( \mathcal{Y}' \subset \text{Teich}_{g,n} \) of codimension at least \( g \) such that for any \( t \in \text{Teich}_{g,n} \setminus \mathcal{Y}' \), the holomorphic principal \( G \)-bundle \( \mathcal{E}_G|_{\tau^{-1}(t)} \) is semistable.
(3) Assume that \( g \geq 2 \). Assume that the monodromy representation for \( \nabla_0 \) does not factor through some proper parabolic subgroup of \( G \). Then there is a closed complex analytic subset \( \mathcal{Y}' \subset \text{Teich}_{g,n} \) of codimension at least \( g - 1 \) such that for any \( t \in \text{Teich}_{g,n} \setminus \mathcal{Y}' \), the holomorphic principal \( G \)-bundle \( \mathcal{E}_G^t \) is stable.

It is known that if a holomorphic principal \( G \)-bundle \( E_G \) over a complex elliptic curve admits a holomorphic connection, then \( E_G \) is semistable. Therefore, a stronger version of Theorem 1.3(1) is valid when \( g = 1 \).

2. Infinitesimal deformations

We first recall some classical results in deformation theory, and in the process setting up our notation.

2.1. Deformations of a \( n \)-pointed curve. Let \( X_0 \) be an irreducible smooth complex projective curve of genus \( g \), with \( g > 0 \), and let

\[
D_0 := \{x_1, \cdots, x_n\} \subset X_0
\]

be an ordered subset of cardinality \( n \) (it may be zero). We assume that \( n > 0 \) if \( g = 1 \). This condition implies that the pair \((X_0, D_0)\) does not have any infinitesimal automorphism, equivalently, the automorphism group of \((X_0, D_0)\) is finite.

Let \( B := \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \) be the spectrum of the local ring. An infinitesimal deformation of \((X_0, D_0)\) is given by a quadruple

\[
(\mathcal{X}, q, \mathcal{D}, f),
\]

(2.1)

where

- \( q : \mathcal{X} \longrightarrow B \) is a smooth proper holomorphic morphism of relative dimension one,
- \( \mathcal{D} = (\mathcal{D}_1, \cdots, \mathcal{D}_n) \) is a collection of \( n \) ordered disjoint sections of \( q \), and
- \( f : X_0 \longrightarrow \mathcal{X} \) is a holomorphic morphism such that

\[
f(X_0) \subset X_0 := q^{-1}(0) \quad \text{with} \quad f(x_i) = D_i(0) \quad \forall \quad 1 \leq i \leq n,
\]

and the morphism \( X_0 \xrightarrow{f} X_0 \) is an isomorphism.

The divisor \( \sum_{i=1}^n D_i(B) \) on \( \mathcal{X} \) will also be denoted by \( \mathcal{D} \). For a vector bundle \( \mathcal{V} \) on \( \mathcal{X} \), the vector bundle \( \mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \) will be denoted by \( \mathcal{V}(-\mathcal{D}) \).

The differential of \( f \)

\[
df : TX_0 \longrightarrow f^*T\mathcal{X}
\]

produces a homomorphism

\[
TX_0(-D_0) := TX_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0}(-D_0) \longrightarrow f^*T\mathcal{X}(-\mathcal{D})
\]

which will also be denoted by \( df \). Consider the following short exact sequence of coherent sheaves on \( X_0 \):

\[
0 \longrightarrow TX_0(-D_0) \xrightarrow{df} f^*T\mathcal{X}(-\mathcal{D}) \xrightarrow{h} \mathcal{O}_{X_0} \longrightarrow 0; \quad (2.2)
\]
note that the normal bundle to \( X_0 \subset X \) is the pullback of \( T_0 B \) by \( q_{|X_0} \), and hence this normal bundle is identified with \( O_{X_0} \). Consider the connecting homomorphism

\[
\mathcal{C} = H^0(X_0, O_{X_0}) \xrightarrow{\phi} H^1(X_0, TX_0(-D_0))
\]  

(2.3)
in the long exact sequence of cohomologies associated to the short exact sequence in (2.2). Let \( 1_{X_0} \) denote the constant function 1 on \( X_0 \). The cohomology element

\[
\phi(1_{X_0}) \in H^1(X_0, TX_0(-D_0))
\]

(2.4)
where \( \phi \) is the homomorphism in (2.3), is the Kodaira–Spencer infinitesimal deformation class for the family in (2.1). If this infinitesimal deformation class is zero, then the family \( (X, D) \rightarrow B \) is isomorphic to the trivial family \( (X_0 \times B, D_0 \times B) \rightarrow B \).

2.2. Deformations of a curve together with a principal bundle. Take \((X_0, D_0)\) as before. Let \( G \) be a connected reductive affine algebraic group defined over \( \mathbb{C} \). The Lie algebra of \( G \) will be denoted by \( g \). Let

\[
p : E_G \rightarrow X_0
\]

(2.5)
be a holomorphic principal \( G \)-bundle on \( X_0 \). The infinitesimal deformations of the triple

\[
(X_0, D_0, E_G)
\]

(2.6)
are guided by the Atiyah bundle \( \text{At}(E_G) \rightarrow X_0 \), the construction of which we shall briefly recall (see [At] for a more detailed exposition). Consider the direct image \( p_*TE_G \rightarrow X_0 \), where \( TE_G \) is the holomorphic tangent bundle of \( E_G \), and \( p \) is the projection in (2.5). It is a quasicoherent sheaf equipped with an action of \( G \) given by the action of \( G \) on \( E_G \). The invariant part

\[
\text{At}(E_G) := (p_*TE_G)^G \subset (p_*TE_G)
\]

is a vector bundle on \( X_0 \) of rank \( 1 + \dim G \) which is known as the Atiyah bundle of \( E_G \). Consequently, we have \( \text{At}(E_G) = (TE_G)/G \). Let

\[
\text{ad}(E_G) := E_G \times^G \mathfrak{g} \rightarrow X_0
\]

be the adjoint vector bundle associated to \( E_G \) for the adjoint action of \( G \) on its Lie algebra \( \mathfrak{g} \). So the fibers of \( \text{ad}(E_G) \) are Lie algebras isomorphic to \( \mathfrak{g} \). Let

\[
dp : TE_G \rightarrow p^*TX_0
\]
be the differential of the map \( p \) in (2.5). Being \( G \)-equivariant it produces a homomorphism \( \text{At}(E_G) \rightarrow TX_0 \) which will also be denoted by \( dp \). Now, the action of \( G \) on \( E_G \) produces an isomorphism \( E_G \times \mathfrak{g} \rightarrow \text{kernel}(dp) \). Therefore, we have \( \text{kernel}(dp)/G = \text{ad}(E_G) \). In other words, the above isomorphism \( E_G \times \mathfrak{g} \rightarrow \text{kernel}(dp) \) descends to an isomorphism

\[
\text{ad}(E_G) \sim (p_*(\text{kernel}(dp)))^G
\]
that preserves the Lie algebra structure on the fibers of the two vector bundles (the Lie algebra structure on the fibers of \( (p_*(\text{kernel}(dp)))^G \) is given by the Lie bracket of \( G \)-invariant vertical vector fields). Therefore, \( \text{At}(E_G) \) fits in the following short exact sequence of vector bundles on \( X_0 \)

\[
0 \rightarrow \text{ad}(E_G) \rightarrow \text{At}(E_G) \xrightarrow{dp} TX_0 \rightarrow 0,
\]

(2.7)
which is known as the Atiyah exact sequence for $E_G$. The logarithmic Atiyah bundle $At(E_G, D_0)$ is defined by

$$At(E_G, D_0) := (dp)^{-1}(TX_0(-D_0)) \subset At(E_G).$$

From (2.7) we have the short exact sequence of vector bundles on $X_0$

$$0 \rightarrow \text{ad}(E_G) \rightarrow At(E_G, D_0) \xrightarrow{\sigma} TX_0(-D_0) \rightarrow 0,$$

which is called the logarithmic Atiyah exact sequence. The above homomorphism $\sigma$ is the restriction of $dp$ to $At(E_G, D_0) \subset At(E_G)$.

An infinitesimal deformation of the triple $(X_0, D_0, E_G)$ in (2.6) is a 6–tuple

$$(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi),$$

where

- $(\mathcal{X}, q, \mathcal{D}, f)$ in an infinitesimal deformation of the $n$–pointed curve $(X_0, D_0)$ as in (2.1),
- $\mathcal{E}_G \rightarrow \mathcal{X}$ is a holomorphic principal $G$–bundle, and
- $\psi$ is a holomorphic isomorphism

$$\psi : E_G \rightarrow f^*E_G$$

of principal $G$–bundles.

The logarithmic Atiyah bundle

$$At(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{X}$$

for $(\mathcal{E}_G, \mathcal{D})$ is the inverse image, in $At(\mathcal{E}_G)$, of the subsheaf $T\mathcal{X}(-\mathcal{D})) \subset T\mathcal{X}$. We have the following short exact sequence of sheaves on $X_0$:

$$0 \rightarrow At(E_G, D_0) \rightarrow f^*At(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{O}_{X_0} \rightarrow 0$$

(2.11)

given by $\psi$ in (2.10). Let

$$\mathbb{C} = H^0(X_0, \mathcal{O}_{X_0}) \xrightarrow{\phi} H^1(X_0, At(E_G, D_0))$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to (2.11). The cohomology element

$$\tilde{\phi}(1_{X_0}) \in H^1(X_0, At(E_G, D_0))$$

(2.12)

is the cohomology class of the infinitesimal deformation of the triple $(X_0, D_0, E_G)$ given by (2.9). Let

$$\sigma_* : H^1(X_0, At(E_G, D_0)) \rightarrow H^1(X_0, TX_0(-D_0))$$

be the homomorphism given by the projection $\sigma$ in (2.8). From the commutativity of the diagram

$$0 \rightarrow At(E_G, D_0) \rightarrow f^*At(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{O}_{X_0} \rightarrow 0$$

$$0 \rightarrow TX_0(-D_0) \rightarrow f^*T\mathcal{X}(-\mathcal{D}) \rightarrow \mathcal{O}_{X_0} \rightarrow 0$$

where the top and bottom rows are as in (2.11) and (2.2) respectively, it follows that

$$\sigma_*(\tilde{\phi}(1_{X_0})) = \phi(1_{X_0}),$$
where \( \tilde{\phi}(1_{X_0}) \) and \( \phi(1_{X_0}) \) are constructed in (2.12) and (2.4) respectively. We note that \( \sigma \) is the forgetful map that sends an infinitesimal deformation of \((X_0, D_0, E_G)\) to the underlying infinitesimal deformation of \((X_0, D_0)\) forgetting the principal \( G \)–bundle.

3. Obstruction to extension of a reduction of structure group to \( P \)

Given a holomorphic reduction of structure group of \( E_G \) to a parabolic subgroup of \( G \), our aim in this section is to compute the obstruction for this reduction to extend to a reduction of an infinitesimal deformation \( \mathcal{E}_G \rightarrow \mathcal{X} \) as in (2.9) (see the paragraph after the proof of Lemma 3.1).

A parabolic subgroup of \( G \) is a connected Zariski closed subgroup \( P \) such that \( G/P \) is a complete variety. Fix a parabolic subgroup \( P \subset G \). The Lie algebra of \( P \) will be denoted by \( p \). As before, \( E_G \) is a holomorphic principal \( G \)–bundle on \( X_0 \). Let

\[
E_P \subset E_G
\]

be a holomorphic reduction of structure group of \( E_G \) to the subgroup \( P \subset G \). Let

\[
\text{ad}(E_P) := E_P \times^P p \rightarrow X_0
\]

be the adjoint vector bundle associated to \( E_P \) for the adjoint action of \( P \) on its Lie algebra \( p \). The vector bundle over \( X_0 \) associated to the principal \( P \)–bundle \( E_P \) for the adjoint action of \( P \) on the quotient \( g/p \) will be denoted by \( E_P(g/p) \). So, \( E_P(g/p) = \text{ad}(E_G)/\text{ad}(E_P) \). The logarithmic Atiyah bundle for \((E_P, D_0)\) will be denoted by \( \text{At}(E_P, D_0) \). We have

\[
\text{ad}(E_P) \subset \text{ad}(E_G) \quad \text{and} \quad \text{At}(E_P, D_0) \subset \text{At}(E_G, D_0);
\]

both the quotient bundles above are identified with \( E_P(g/p) \). In other words, we have the following commutative diagram of vector bundles on \( X_0 \)

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{ad}(E_P) & \rightarrow & \text{At}(E_P, D_0) & \rightarrow & TX_0(-D_0) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{ad}(E_G) & \rightarrow & \text{At}(E_G, D_0) & \rightarrow & TX_0(-D_0) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
E_P(g/p) & = & E_P(g/p) & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & & &
\end{array}
\]

(3.2)

where \( \sigma \) is the homomorphism in (2.8). Let

\[
\tilde{\xi} : H^1(X_0, \text{At}(E_P, D_0)) \rightarrow H^1(X_0, \text{At}(E_G, D_0))
\]

be the homomorphism induced by the canonical injection \( \xi \) in (3.2).

Take any \((\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)\) as in (2.9). Assume that the reduction \( E_P \subset E_G \) in (3.1) extends to a holomorphic reduction of structure group

\[
\mathcal{E}_P \subset \mathcal{E}_G
\]
to $P \subset G$ on $\mathcal{X}$. Consider the short exact sequence on $X_0$

$$0 \rightarrow \text{At}(E_P, D_0) \rightarrow f^*\text{At}(\mathcal{E}_P, \mathcal{D}) \rightarrow \mathcal{O}_{X_0} \rightarrow 0,$$

(3.4)

where $\text{At}(\mathcal{E}_P, \mathcal{D}) \rightarrow \mathcal{X}$ is the logarithmic Atiyah bundle associated to the principal $P$–bundle $\mathcal{E}_P$, and $f$ is the map in (2.1). Let

$$\theta \in H^1(X_0, \text{At}(E_P, D_0))$$

(3.5)

be the image of the constant function $1_{X_0}$ by the homomorphism

$$H^0(X_0, \mathcal{O}_{X_0}) \rightarrow H^1(X_0, \text{At}(E_P, D_0))$$

in the long exact sequence of cohomologies associated to (3.4).

**Lemma 3.1.** The cohomology class $\theta$ in (3.5) satisfies the equation

$$\tilde{\xi}(\theta) = \tilde{\phi}(1_{X_0}),$$

where $\tilde{\xi}$ and $\tilde{\phi}(1_{X_0})$ are constructed in (3.3) and (2.12) respectively.

**Proof.** Consider the commutative diagram of vector bundles on $X_0$

$$\begin{array}{ccc}
0 & \rightarrow & \text{At}(E_P, D_0) \\
| & | & | \\
0 & \rightarrow & \text{At}(E_G, D_0) \\
\downarrow{\xi} & & \downarrow{\xi} \\
0 & \rightarrow & f^*\text{At}(\mathcal{E}_P, \mathcal{D}) & \rightarrow & \mathcal{O}_{X_0} \rightarrow 0 \\
& & & & \\
& & & & \\
& & & & \\
\end{array}$$

(3.6)

where the top and bottom rows are as in (3.4) and (2.11) respectively, and $\xi$ is the homomorphism in (3.2); the above homomorphism $f^*\text{At}(\mathcal{E}_P, \mathcal{D}) \rightarrow f^*\text{At}(\mathcal{E}_G, \mathcal{D})$ is the pullback of the natural homomorphism $\text{At}(\mathcal{E}_P, \mathcal{D}) \rightarrow \text{At}(\mathcal{E}_G, \mathcal{D})$. In view of (3.6), the lemma follows by comparing the constructions of $\theta$ and $\tilde{\phi}(1_{X_0})$.

Consider the homomorphism $\mu_* : H^1(X_0, \text{At}(E_G, D_0)) \rightarrow H^1(X_0, E_P(\mathfrak{g}/\mathfrak{p}))$ induced by the homomorphism $\mu$ in (3.2). From Lemma 3.1 we conclude that $\mu_*(\tilde{\phi}(1_{X_0})) = 0$; to see this consider the long exact sequence of cohomologies associated to the right vertical exact sequence in (3.2). Therefore, $\mu_*(\tilde{\phi}(1_{X_0}))$ is an obstruction for the reduction $E_P \subset E_G$ to extend to a reduction of $\mathcal{E}_G$ to $P$.

4. **Logarithmic connections and the second fundamental form**

In this section we characterize those infinitesimal deformations of the principal bundle $E_G$ on the $n$–pointed curve that arise from the isomonodromic deformations.

4.1. **Canonical extension of a logarithmic connection.** As before, let $p : E_G \rightarrow X_0$ be a holomorphic principal $G$–bundle. A logarithmic connection on $E_G$ with polar divisor $D_0$ is a holomorphic splitting of the logarithmic Atiyah exact sequence in (2.8). In other words, a logarithmic connection is a homomorphism

$$\delta : T_{X_0}(-D_0) \rightarrow \text{At}(E_G, D_0)$$

(4.1)

such that $\sigma \circ \delta = \text{Id}_{T_{X_0}(-D_0)}$, where $\sigma$ is the homomorphism in (2.8). Note that given such a $\delta$, there is a unique homomorphism

$$\delta'' : \text{At}(E_G, D_0) \rightarrow \text{ad}(E_G)$$

(4.2)
such that $\delta'' \circ \delta = 0$, and the composition

$$\text{ad}(E_G) \hookrightarrow \text{At}(E_G, D_0) \xrightarrow{\delta''} \text{ad}(E_G)$$

(see \(2.8\)) is the identity map of $\text{ad}(E_G)$. As there are no nonzero \((2,0)\)-forms on $X_0$, all logarithmic connections on $E_G$ are automatically integrable.

At the level of first order infinitesimal deformations, given a principal $G$–bundle

$$E_G \xrightarrow{q} B,$$

a logarithmic connection on $E_G$ with polar divisor $D$ is a homomorphism $\text{At}(E_G, D) \rightarrow \text{ad}(E_G)$ that splits the logarithmic Atiyah exact sequence for $E_G$. We note that a connection on $E_G$ need not be integrable, as we have added an (infinitesimal) extra dimension. However, the Riemann–Hilbert correspondence for principal $G$–bundles yields the following:

**Lemma 4.1.** Let $(\mathcal{X}, q, D, f)$ be an infinitesimal deformation of $(X_0, D_0)$ as in \(2.1\). Let $\delta$ be a logarithmic connection on a holomorphic principal $G$–bundle $E_G$ on $X_0$ with polar divisor $D_0$. Then there exists a unique pair $(E_G, \nabla)$, where

- $E_G$ is a holomorphic principal $G$–bundle on $\mathcal{X}$, and
- $\nabla$ is an integrable logarithmic connection on $E_G$ with polar divisor $D$,

such that $(f^* E_G, f^* \nabla) = (E_G, \delta)$.

Let us recall a few elements of the proof of this (classical) result. Choose a covering $\mathcal{U}$ of $X_0 \setminus D_0$ by complex discs and a small neighborhood $U_i$ for each $x_i \in D_0$. Since $\delta$ is integrable, we can choose local charts for $E_G$ over $\mathcal{U}$ such that all transition functions are constants. Now if the curve fits into an analytic family $\mathcal{X} \rightarrow B$, one can, restricting $B$ if necessary, cover $\mathcal{X}$ by open subsets of the form $V_j \times B$, where $V_j$ are the open subsets of $X_0$ in the collection $\mathcal{U} \cup \{U_i\}_{i=1}^n$. The isomonodromic deformation is then given by simply extending the transition maps by keeping them to be constant in deformation parameters.

The logarithmic connection $\delta$ gives a splitting of the logarithmic Atiyah bundle

$$\text{At}(E_G, D_0) = \text{ad}(E_G) \oplus TX_0(-D_0).$$

The corresponding cohomological decomposition

$$H^1(X_0, \text{At}(E_G, D_0)) = H^1(X_0, \text{ad}(E_G)) \oplus H^1(X_0, TX_0(-D_0))$$

gives a splitting of the infinitesimal deformations of $(X_0, D_0, E_G)$ into the infinitesimal deformations of $(X_0, D_0)$ and the infinitesimal deformations of $E_G$ (keeping $(X_0, D_0)$ fixed). In other words, let

$$\delta' : H^1(X_0, TX_0(-D_0)) \rightarrow H^1(X_0, \text{At}(E_G, D_0))$$

be the homomorphism induced by the homomorphism

$$\delta : TX_0(-D_0) \rightarrow \text{At}(E_G, D_0)$$

(4.3)

in Lemma \(4.1\) defining the logarithmic connection on $E_G$. Given an infinitesimal deformation $(\mathcal{X}, q, D, f)$ of $(X_0, D_0)$, the above homomorphism $\delta'$ produces an infinitesimal deformation
Lemma 4.2. Given \((\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)\) of \((X_0, D_0, E_G)\). As explained above, this holomorphic principal \(G\)-bundle \(\mathcal{E}_G\) on \(\mathcal{X}\) coincides with the holomorphic principal \(G\)-bundle on \(\mathcal{X}\) produced by the isomonodromic deformation in Lemma 4.1.

We will now construct the exact sequence in (2.1) corresponding to the above infinitesimal deformation \((\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)\). Consider the projection \(\mathcal{X}_0(-D_0) \to \text{At}(E_G, D_0) \oplus f^*\mathcal{T}\mathcal{X}(-\mathcal{D})\), \(v \mapsto (\delta(v), -(df)(v))\), where \(df\) is the differential in (2.2) and \(\delta\) is the homomorphism in (4.3). The corresponding cokernel \(\text{At}^\delta(E_G, D_0) := (\text{At}(E_G, D_0) \oplus f^*\mathcal{T}\mathcal{X}(-\mathcal{D}))/\text{(TX}_0(-D_0))\) possesses a canonical projection \(\hat{\delta} : \text{At}^\delta(E_G, D_0) \to \mathcal{O}_{X_0}\), \((v, w) \mapsto h(w)\), (4.4)

where \(h\) is the homomorphism in (2.2); note that the above homomorphism \(\hat{\delta}\) is well-defined because \(h\) vanishes on the image of \(\text{TX}_0(-D_0)\) in \(\text{At}(E_G, D_0) \oplus f^*\mathcal{T}\mathcal{X}(-\mathcal{D})\). The kernel of \(\hat{\delta}\) is identified with \(\text{At}(E_G, D_0)\) by sending any \(z \in \text{At}(E_G, D_0)\) to the image in \(\text{At}^\delta(E_G, D_0)\) of \((z, 0) \in \text{At}(E_G, D_0) \oplus f^*\mathcal{T}\mathcal{X}(-\mathcal{D})\). Therefore, we obtain the following exact sequence of vector bundles over \(X_0\):

\[0 \to \text{At}(E_G, D_0) \to \text{At}^\delta(E_G, D_0) \overset{\hat{\delta}}{\to} \mathcal{O}_{X_0} \to 0,\]

This exact sequence coincides with the one in (2.1). In particular, we have \(\text{At}^\delta(E_G, D_0) = f^*\text{At}(\mathcal{E}_G)\).

Consider the projection

\[\text{At}(E_G, D_0) \oplus f^*\mathcal{T}\mathcal{X}(-\mathcal{D}) \to \text{ad}(E_G), (z_1, z_2) \mapsto \delta''(z_1),\]

where \(\delta''\) is constructed in (1.2) from \(\delta\). It vanishes on the image of \(\text{TX}_0(-D_0)\), yielding a projection \(\lambda : \text{At}^\delta(E_G, D_0) \to \text{ad}(E_G)\).

Let \(\nabla'' : \text{At}(\mathcal{E}_G, \mathcal{D}) \to \text{ad}(\mathcal{E}_G)\) be the homomorphism given by the logarithmic connection \(\nabla\) in Lemma 4.1. The homomorphism in (4.5) fits in the commutative diagram

\[
\begin{array}{ccc}
\text{At}^\delta(E_G, D_0) & \xrightarrow{\lambda} & \text{ad}(E_G) \\
\| & & \| \\
f^*\text{At}(\mathcal{E}_G, \mathcal{D}) & \xrightarrow{f^*\nabla''} & f^*\text{ad}(\mathcal{E}_G)
\end{array}
\]

(the vertical identifications are evident).

We summarize the above constructions in the following lemma:

**Lemma 4.2.** Given \((\mathcal{X}, q, \mathcal{D}, f)\) as in (2.1), and also given a logarithmic connection \(\delta\) on a holomorphic principal \(G\)-bundle \(E_G \to X_0\), the exact sequence in (2.1) corresponding to the infinitesimal deformation of \((X_0, D_0, E_G)\) in Lemma 4.1 is

\[0 \to \text{At}(E_G, D_0) \to \text{At}^\delta(E_G, D_0) \overset{\hat{\delta}}{\to} \mathcal{O}_{X_0} \to 0,\]

where \(\hat{\delta}\) is constructed in (4.4).
4.2. The second fundamental form. Fix a logarithmic connection $\delta$ on $(E_G, D_0)$ as in (4.1). Take a holomorphic reduction of structure group $E_P \subset E_G$ to $P$ as in (3.1). The composition

$$S(\delta) := \mu \circ \delta : TX_0(-D_0) \longrightarrow E_P(\mathfrak{g}/\mathfrak{p}) \quad (4.7)$$

where $\mu$ is constructed in (3.2), is called the second fundamental form of $E_P$ for the connection $\delta$. We note that $\delta$ is induced by a logarithmic connection on the holomorphic principal $P$–bundle $E_P$ if and only if we have $S(\delta) = 0$.

Assume that $E_P$ satisfies the condition that $S(\delta) \neq 0$. Let $L \subset E_P(\mathfrak{g}/\mathfrak{p}) (4.8)$ be the holomorphic line subbundle generated by the image of the homomorphism $S(\delta)$ in (4.7). More precisely, $L$ is the inverse image, in $E_P(\mathfrak{g}/\mathfrak{p})$, of the torsion part of the quotient $E_P(\mathfrak{g}/\mathfrak{p})/(S(\delta)(TX_0(-D_0)))$. Now consider the homomorphism

$$S'(\delta) : TX_0(-D_0) \longrightarrow L \quad (4.9)$$

given by the second fundamental form $S(\delta)$. Let

$$S' : H^1(X_0, TX_0(-D_0)) \longrightarrow H^1(X_0, L) \quad (4.10)$$

be the homomorphism of cohomologies induced by $S'(\delta)$ in (4.9).

**Proposition 4.3.** As before, let $\delta$ be a logarithmic connection on $E_G \longrightarrow X_0$ with polar divisor $D_0$, and let $(\mathcal{X}, q, \mathcal{D}, f)$ be an infinitesimal deformation of $(X_0, D_0)$. Let $E_G \longrightarrow \mathcal{X}$ be the isomonodromic deformation of $\delta$ along $(\mathcal{X}, \mathcal{D})$ obtained in Lemma 4.1. Let $E_P \subset E_G$ be a holomorphic reduction of structure group to $P$ over $X_0$ that extends to a holomorphic reduction of structure group $E_P \subset E_G$ to $P$ over $\mathcal{X}$. Then

$$S'(\phi(1_{X_0})) = 0 ,$$

where $\phi(1_{X_0})$ is the cohomology class constructed in (2.4) corresponding to $(\mathcal{X}, q, \mathcal{D}, f)$, and $S'$ is constructed in (4.10).

**Proof.** Consider the inverse images

$${\text{At}}_P(E_G, D_0) := \mu^{-1}(L) \subset \text{At}(E_G, D_0) \quad \text{and} \quad \text{ad}_P(E_G) := \mu_1^{-1}(L) \subset \text{ad}(E_G),$$

where $\mu$ and $\mu_1$ are the quotient maps in (3.2), and $L$ is constructed in (4.8). These two vector bundles fit in the following commutative diagram produced from (3.2):

$$
\begin{array}{c}
0 \longrightarrow \text{ad}(E_P) \longrightarrow \text{At}(E_P, D_0) \longrightarrow TX_0(-D_0) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \xi \quad \parallel \\
0 \longrightarrow \text{ad}_P(E_G) \longrightarrow \text{At}_P(E_G, D_0) \longrightarrow TX_0(-D_0) \longrightarrow 0 \\
\downarrow \mu_1 \quad \downarrow \mu \quad \downarrow \\
L \quad = \quad L \\
\downarrow \quad \downarrow \\
0 \quad 0
\end{array}
$$

(4.11)
By the construction of \( \text{At}_P(E_G, D_0) \), the connection homomorphism \( \delta \) in (4.3) factors through a homomorphism

\[
\delta^1 : TX_0(-D_0) \to \text{At}_P(E_G, D_0).
\]

Consider the homomorphism

\[
\delta^1_* : H^1(X_0, TX_0(-D_0)) \to H^1(X_0, \text{At}_P(E_G, D_0))
\]

induced by the above homomorphism \( \delta^1 \), and let

\[
\Phi := \delta^1_*(\phi(1_{X_0})) \in H^1(X_0, \text{At}_P(E_G, D_0))
\]

be the image of the cohomology class \( \phi(1_{X_0}) \) that characterizes the deformation \((X, D)\) as in (2.4).

As in the statement of the proposition, let \( \mathcal{E}_P \to X \) be a holomorphic extension of the reduction \( E_P \). Note that from (3.6), (3.2) and Lemma 4.2 we have

\[
\text{At}(E_G, D_0)/\text{At}(E_P, D_0) = E_P(g/p) = (f^*\text{At}(E_G, D))/f^*\text{At}(E_P, D)
\]

\[
= \text{At}^\delta(E_G, D_0)/f^*\text{At}(E_P, D).
\]

Let \( \mu_2 : \text{At}^\delta(E_G, D_0) \to E_P(g/p) \) be the above quotient map. Define

\[
\text{At}^\delta_P(E_G, D_0) := \mu_2^{-1}(L) \subset \text{At}^\delta(E_G, D_0),
\]

where \( L \) is constructed in (4.8).

Now we have the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \to & TX_0(-D_0) & \to & f^*TX(-D) & \to & \mathcal{O}_{X_0} & \to & 0 \\
0 & \to & \text{At}_P(E_G, D_0) & \to & \text{At}^\delta_P(E_G, D_0) & \to & \mathcal{O}_{X_0} & \to & 0
\end{array}
\]

\[
\begin{array}{ccccccc}
\delta & \downarrow & f^*\nabla & \parallel & \\
\end{array}
\]

(4.14)

where the bottom exact sequence is obtained from (3.6), and the top exact sequence is as in (2.2) (see also (4.6)). Let

\[
\nu : H^0(X_0, \mathcal{O}_{X_0}) \to H^1(X_0, \text{At}_P(E_G, D_0))
\]

be the connecting homomorphism in the long exact sequence of cohomologies associated to the bottom exact sequence in (4.14). From the commutativity of (4.14) and the construction of \( \phi(1_{X_0}) \) (see (2.4)) it follows that

\[
\nu(1_{X_0}) = \delta^1_*(\phi(1_{X_0})) = \Phi,
\]

where \( \nu \) and \( \delta^1_* \) are the homomorphisms constructed in (4.15) and (4.12) respectively, and \( \Phi \) is the cohomology class in (4.13).
The diagram in (3.6) produces the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{At}(E_P, D_0) \\
\downarrow_{\xi} & & \downarrow \\
0 & \rightarrow & \text{At}_P(E_P, D_0) \\
\downarrow_{\mu} & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

where \( \xi \) and \( \mu \) are the homomorphisms in (4.11). Using this diagram we can check that

\[
\Phi = \xi_*(\theta),
\]

where \( \theta \) is the cohomology classes in (3.5), and

\[
\xi_* : H^1(X_0, \text{At}(E_P, D_0)) \rightarrow H^1(X_0, \text{At}_P(E_P, D_0))
\]

is the homomorphism induced by \( \xi \) in (4.17). Indeed, (4.18) follows from (4.16), the commutativity of (4.17) and the construction of \( \theta \).

Let \( \mu_* : H^1(X_0, \text{At}_P(E_P, D_0)) \rightarrow H^1(X_0, \mathcal{L}) \) be the homomorphism induced by the homomorphism \( \mu \) in (4.17). It is straightforward to check that

\[
\mu_*(\Phi) = S'(\phi(1_{X_0}))
\]

(see (4.10), (2.4) and (4.13) for \( S' \), \( \phi(1_{X_0}) \) and \( \Phi \) respectively). Indeed, this follows from (4.16) and the definition of \( S(\delta) \) in (4.7). Combining this with (4.18), we have

\[
S'(\phi(1_{X_0})) = \mu_*(\Phi) = \mu_*(\xi_*(\theta)).
\]

Since \( \mu \circ \xi = 0 \) (see (4.17)), it now follows that \( S'(\phi(1_{X_0})) = 0 \).

5. Logarithmic connections and semistability of the underlying principal bundle

Let \( \mathcal{T}_{g,n} \) be the Teichmüller space of genus \( g \) compact Riemann surfaces with \( n \) ordered marked points. As before, we assume that \( g > 0 \), and if \( g = 1 \), then \( n > 0 \). Let

\[
\tau : \mathcal{C} \rightarrow \mathcal{T}_{g,n}
\]

be the universal Teichmüller curve with \( n \) ordered sections \( \Sigma \). The fiber of \( \mathcal{C} \) over any point \( t \in \mathcal{T}_{g,n} \) will be denoted by \( \mathcal{C}_t \). The ordered subset \( \mathcal{C}_t \cap \Sigma \subset \mathcal{C}_t \) will be denoted by \( \Sigma_t \).

Take a \( n \)-pointed Riemann surface \( (C_0, \Sigma_0) \) of genus \( g \), which is represented by a point of \( \mathcal{T}_{g,n} \). Let

\[
\nabla_0
\]

be a logarithmic connection on a holomorphic principal \( G \)-bundle \( F_G \rightarrow C_0 \) with polar divisor \( \Sigma_0 \). By the Riemann–Hilbert correspondence, the connection \( \nabla_0 \) produces a flat
(isomonodromic) logarithmic connection $\nabla$ on a holomorphic principal $G$–bundle $\mathcal{F}_G \to \mathcal{C}$ with polar divisor $\Sigma$.

The following lemma is a special case of the main theorem in [GN] (see also [Sh]). Although the families of $G$–bundles considered in [GN] are algebraic, all arguments there go through in the analytic case of our interest with obvious modifications.

**Lemma 5.1** ([GN]). Let $\mathcal{F}_G \to \mathcal{C} \to T_{g,n}$ be as above. For each Harder–Narasimhan type $\kappa$, the set

$$Y_\kappa := \{ t \in T_{g,n} \mid \mathcal{F}_G|_{\mathcal{C}_t} \text{ is of type } \kappa \}$$

is a (possibly empty) locally closed complex analytic subspace of $T_{g,n}$. More precisely, for each Harder–Narasimhan type $\kappa$, the union $Y_{\leq \kappa} := \bigcup_{\kappa' \leq \kappa} Y_{\kappa'}$ is a closed complex analytic subset of $T_{g,n}$. Moreover, the principal $G$–bundle $\mathcal{F}_G|_{\tau^{-1}(Y_\kappa)} \to \tau^{-1}(Y_\kappa)$ possesses a canonical holomorphic reduction of structure group inducing the Harder–Narasimhan reduction of $\mathcal{F}_G|_{\mathcal{C}_t}$ for every $t \in Y_\kappa$.

In the following two Sections 5.1 and 5.2 we will see that under certain assumptions, the only Harder-Narasimhan stratum $Y_\kappa$ of maximal dimension $\dim(T_{g,n}) = 3g - 3 + n$ is the trivial one, in the sense that the Harder–Narasimhan parabolic subgroup is $G$ itself. In other words, if the principal $G$–bundle $F_G$ is not semistable, and therefore has a non-trivial Harder–Narasimhan reduction $E_P \subset E_G$ to a certain parabolic subgroup $P \subsetneq G$, then there is always an isomonodromic deformation in which direction the reduction $E_P$ is obstructed meaning it does not extend.

5.1. The case of $n = 0$. In this subsection we assume that $n = 0$. So, we have $g > 1$.

**Theorem 5.2.** There is a closed complex analytic subset $\mathcal{Y} \subset T_{g,0}$ of codimension at least $g$ such that for every $t \in T_{g,0} \setminus \mathcal{Y}$, the holomorphic principal $G$–bundle $\mathcal{F}_G|_{\mathcal{C}_t} \to \mathcal{C}_t$ is semistable.

**Proof.** Let $\mathcal{Y} \subset T_{g,0}$ denote the (finite) union of all Harder-Narasimhan strata $Y_\kappa$ as in Lemma 5.1 with nontrivial Harder-Narasimhan type $\kappa$. From Lemma 5.1 we know that $\mathcal{Y}$ is a closed complex analytic subset of $T_{g,0}$.

Take any $t \in Y_\kappa \subset \mathcal{Y}$. Let $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$ be the holomorphic principal $G$–bundle on $X_0 := \mathcal{C}_t$. The holomorphic connection on $E_G$ obtained by restricting $\nabla$ will be denoted by $\delta$. Since $E_G$ is not semistable, there is a proper parabolic subgroup $P \subsetneq G$ and a holomorphic reduction of structure group $E_P \subset E_G$ to $P$, such that $E_P$ is the Harder–Narasimhan reduction [Be], [AAB]; the type of this Harder–Narasimhan reduction is $\kappa$. From Lemma 5.1 we know that $E_P$ extends to a holomorphic reduction of structure group of the principal $G$–bundle $\mathcal{F}_G|_{\tau^{-1}(Y_\kappa)}$ to the subgroup $P$.

Let $\text{ad}(E_P)$ and $\text{ad}(E_G)$ be the adjoint vector bundles of $E_P$ and $E_G$ respectively. Consider the vector bundle

$$\text{ad}(E_G)/\text{ad}(E_P) = E_P(g/p)$$
(see (3.2)). We know that

$$\mu_{\text{max}}(E_P(g/p)) < 0 \quad (5.2)$$

[ABB p. 705] (see sixth line from bottom). In particular

$$\text{degree}(E_P(g/p)) < 0. \quad (5.3)$$

A holomorphic connection on $E_G$ induces a holomorphic connection on $\text{ad}(E_G)$, hence $\text{degree}(\text{ad}(E_G)) = 0$. Combining this with (5.3) it follows that $\text{degree}(\text{ad}(E_P)) > 0$, because $\text{ad}(E_G)/\text{ad}(E_P) = E_P(g/p)$. Since $\text{degree}(\text{ad}(E_P)) \neq 0$, the holomorphic vector bundle $\text{ad}(E_P)$ does not admit any holomorphic connection, hence the principal $P$–bundle $E_P$ does not admit a holomorphic connection. Consequently, the second fundamental form $S(\delta)$ of $E_P$ for $\delta$ (see (4.7)) is nonzero.

Using the second fundamental form $S(\delta)$, construct the holomorphic line subbundle

$$L \subset E_P(g/p)$$

as done in (4.8). From (5.2) we have

$$\text{degree}(L) < 0. \quad (5.4)$$

Consider the short exact sequence of sheaves on $X_0$

$$0 \longrightarrow TX_0 \xrightarrow{S(\delta)'} L \longrightarrow T^\delta := L/(S(\delta)(TX_0)) \longrightarrow 0, \quad (5.5)$$

where $S(\delta)'$ is constructed in (4.9); note that $T^\delta$ is a torsion sheaf because $S(\delta)' \neq 0$ (recall that $S(\delta) \neq 0$). From (5.4) it follows that

$$\text{degree}(T^\delta) = \text{degree}(L) - \text{degree}(TX_0) < -\text{degree}(TX_0) = 2g - 2.$$

So, we have

$$\dim H^0(X_0, T^\delta) = \text{degree}(T^\delta) < 2g - 2 = \dim H^1(X_0, TX_0) + 1 - g.$$ 

This implies that

$$\dim H^1(X_0, TX_0) - \dim H^0(X_0, T^\delta) \geq g. \quad (5.6)$$

Consider the long exact sequence of cohomologies

$$H^0(X_0, T^\delta) \longrightarrow H^1(X_0, TX_0) \xrightarrow{\zeta} H^1(X_0, L)$$

associated to the short exact sequence of sheaves in (5.5). From (5.6) it follows that

$$\dim \zeta(H^1(X_0, TX_0)) \geq g. \quad (5.7)$$

Since the reduction $E_P$ extends to a holomorphic reduction of structure group of the principal $G$–bundle $F_G|_{\tau^{-1}(Y_\kappa)}$ to the subgroup $P$, combining (5.7) and Proposition 4.3 we conclude that the codimension of the complex analytic subset $Y_\kappa \subset T_{g,0}$ is at least $g$. This completes the proof of the theorem. \qed
5.2. When $n$ is arbitrary. Now we assume that $n > 0$.

A logarithmic connection $\eta$ on a holomorphic principal $G$–bundle $F_G \to X_0$ is called reducible if there is pair $(P, F_P)$, where $P \subset G$ is a parabolic subgroup and $F_P \subset F_G$ is a holomorphic reduction of structure group of $F_G$ to $P$, such that $\eta$ is induced by a connection on $F_P$. Note that $\eta$ is induced by a connection on $F_P$ if and only if the second fundamental form of $F_P$ for $\eta$ vanishes identically. A connection is called irreducible if it is not reducible or, equivalently, if the monodromy representation of the corresponding flat principal $G$–bundle does not factor through any proper parabolic subgroup of $G$.

**Proposition 5.3.** Assume that the logarithmic connection $\nabla_0$ in (5.1) is irreducible. Then there is a closed complex analytic subset $\mathcal{Y} \subset T_{g,n}$ of codimension at least $g$ such that for every $t \in T_{g,n} \setminus \mathcal{Y}$, the holomorphic principal $G$–bundle $\mathcal{F}_G|_{C_t}$ is semistable.

**Proof.** The proof of Theorem 5.2 goes through after some obvious modifications. As before, let $\mathcal{Y} \subset T_{g,n}$ be the locus of all points $t$ such that the principal $G$–bundle $\mathcal{F}_G|_{C_t}$ is not semistable. Take any point $t \in \mathcal{Y} \subset T_{g,n}$. Let $E_G = \mathcal{F}_G|_{C_t}$ be the holomorphic principal $G$–bundle on $X_0 := C_t$. The logarithmic connection on $E_G$ with polar divisor $D_0 := \Sigma_t$ obtained by restricting $\nabla$ will be denoted by $\delta$.

Let $E_P \subset E_G$ be the Harder–Narasimhan reduction; its type is $\kappa$. Since $\nabla_0$ is irreducible, the second fundamental form $S(\delta)$ of $E_P$ for $\delta$ (see (4.7)) is nonzero. We note that for the monodromy of a logarithmic connection, the property of being irreducible is preserved under isomodromic deformations. Consider the short exact sequence of sheaves on $X_0$

$$0 \to TX_0(-D_0) \xrightarrow{S(\delta)'} \mathcal{L} \to T^\delta := \mathcal{L}/(S(\delta)(TX_0(-D_0))) \to 0,$$

where $S(\delta)'$ is constructed in (4.9). As before, $\deg(\mathcal{L}) < 0$, because $\mu_{\max}(E_P(g/p)) < 0$. So,

$$\deg(T^\delta) = \deg(\mathcal{L}) - \deg(TX_0(-D_0)) < -\deg(TX_0(-D_0)) = 2g - 2 + n.$$

We now have

$$\dim H^0(X_0, T^\delta) = \deg(T^\delta) < 2g - 2 + n = \dim H^1(X_0, TX_0(-D_0)) + 1 - g.$$

Hence the dimension of the image of the homomorphism

$$H^1(X_0, TX_0(-D_0)) \to H^1(X_0, \mathcal{L})$$

in the long exact sequence of cohomologies associated to (5.8) is at least $g$. Since the reduction $E_P$ extends to a holomorphic reduction of $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\nu)}$ to $P$, and the dimension of the image of the homomorphism in (5.9) is at least $g$, from Proposition 4.3 we conclude that the codimension of $\mathcal{Y}_\nu \subset T_{g,0}$ is at least $g$. \qed

5.3. Stability of underlying principal bundle. We now assume that $g \geq 2$.

**Proposition 5.4.** Assume that the logarithmic connection $\nabla_0$ in (5.1) is irreducible. There is a closed analytic subset $\mathcal{Y}' \subset T$ of codimension at least $g - 1$ such that for every $t \in T_{g, \mathcal{Y}}$, the holomorphic principal $G$–bundle $\mathcal{F}_G|_{C_t}$ is stable.
Proof. The proof is identical to the proof of Proposition 5.3. If \( E_G = \mathcal{F}_G|_C \) is not stable, there is a maximal parabolic subgroup \( P \subsetneq G \) and a holomorphic reduction of structure group \( E_P \subset E_G \) to \( P \), such that the quotient bundle
\[
\text{ad}(E_G)/\text{ad}(E_P) = E_P(g/p)
\]
is semistable of degree zero. Therefore, we have \( \text{degree}(\mathcal{L}) \leq 0 \). This implies that
\[
\text{degree}(T^\delta) \leq 2g - 2 - n.
\]
Hence the dimension of the image of the homomorphism \( H^1(X_0, TX_0(-D_0)) \to H^1(X_0, \mathcal{L}) \) in the long exact sequence of cohomologies associated to (5.8) is at least \( g - 1 \). \( \square \)

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