The Emergence of Sparse Spanners and Greedy Well-Separated Pair Decomposition

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Abstract

A spanner graph on a set of points in $\mathbb{R}^d$ contains a shortest path between any pair of points with length at most a constant factor of their Euclidean distance. A spanner with a sparse set of edges is thus a good candidate for network backbones, as desired in many practical scenarios such as the transportation network and peer-to-peer network overlays. In this paper we investigate new models and aim to interpret why good spanners ‘emerge’ in reality, when they are clearly built in pieces by agents with their own interests and the construction is not coordinated. Our main result is to show that the following algorithm generates a $(1 + \varepsilon)$-spanner with a linear number of edges, constant average degree, and the total edge length as a small logarithmic factor of the cost of the minimum spanning tree. In our algorithm, the points build edges at an arbitrary order. When a point $p$ checks on whether the edge to a point $q$ should be built, it will build this edge only if there is no existing edge $p'q'$ with $p'$ and $q'$ at distances no more than $\frac{1}{2(1+1/\varepsilon)} \cdot |p'q'|$ from $p, q$ respectively. Eventually when all points have finished checking edges to all other points, the resulted collection of edges forms a sparse spanner as desired. This new spanner construction algorithm can be extended to a metric space with constant doubling dimension and admits a local routing scheme to find the short paths.

As a side product, we show a greedy algorithm for constructing linear-size well-separated pair decompositions that may be of interest on its own. A well-separated pair decomposition is a collection of subset pairs such that each pair of point sets is fairly far away from each other compared with their diameters and that every pair of points is ‘covered’ by at least one well-separated pair. Our greedy algorithm selects an arbitrary pair of points that have not yet been covered and puts a ‘dumb-bell’ around the pair as the well-separated pair, repeats this until all pairs of points are covered. When the algorithm finishes, we show only a linear number of pairs is generated, which is asymptotically optimal.

1 Introduction

A geometric graph $G$ defined on a set of points $\mathcal{P} \subseteq \mathbb{R}^d$ with all edges as straight line segments of weight equal to the length is called a Euclidean spanner, if for any two points $p, q \in \mathcal{P}$ the shortest path between them in $G$ has length at most $\lambda \cdot |pq|$ where $|pq|$ is the Euclidean distance. The factor $\lambda$ is called the stretch factor of $G$ and the graph $G$ is called a $\lambda$-spanner. Spanners with a sparse set of edges provide good approximations for the pairwise Euclidean distances and are good candidates for network backbones. Thus, there has been a lot of work on the construction of Euclidean spanners in both the centralized setting [17, 39] and the distributed setting [41].

In this paper we are interested in the emergence of good Euclidean spanners formed by uncoordinated agents. Many real-world networks, such as the transportation network, the Internet backbone network, the flight network, are good spanners — one can typically drive from any city to any other city in the U.S.

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with the total travel distance at most a small constant times their straight line distance; similarly, one can find connecting flights between any two towns with the total flight distance not much more than the direct flight distance, possibly by flights of different airlines. The same thing happens with the Internet backbone graph as well. However, these large networks are not owned or built by any single authority. They are often assembled with pieces built by different governments (federal, states, or county governments), different airline companies, or different ISPs. These agents have their own agenda and interests. It does not seem to be the case that they have carefully planned the construction in collaboration before hand and clearly the edges are not built all at the same time. Nevertheless altogether they provide a convenient sparse spanner for the users that is close to the best spanner one can find in a centralized coordinated setting. The work in this paper is largely motivated by this observation and we would like to interpret and understand why a good Euclidean spanner is able to ‘emerge’ from these agents incrementally.

From the application’s perspective, we are interested in the construction of nice network overlay or infrastructure topologies used in many distributed network services such as transportation network, peer-to-peer (P2P) file sharing and content distribution applications [36]. Such overlay or infrastructure networks are constructed in a distributed manner without centralized control, to achieve robustness and scalability. The agents self-organize themselves in a overlay network topology by choosing other agents to connect to directly. While prior work in overlay design has focused on system robustness and network scalability, a significant amount of work in recent years have focused on reducing routing delay [12, 43, 34, 48, 49]. Most of these work adapt the current overlay topology to respect the topology of the underlying network and only show by experiments the improvement of the average delay. The fundamental question still yet to be answered is as follows [43]: for a peer in the P2P network, what set of neighbors should it connect to, such that the shortest path routing latency on the resultant overlay is low, compared with the minimum delay in the underlying network? Obviously a spanner graph would be a good solution for the overlay construction, yet there is no centralized authority in the P2P network that supervises the spanner construction and the peers may join or leave the network frequently (the system churn rate is high). The work in this paper initiates the study of the emergence of good spanners in the setting when there is little coordination between the agents and the users only need a modest amount of incomplete information of the current overlay topology.

**Our contribution.** We consider in this paper the following model that abstracts the scenarios explained earlier. There are \( n \) points in the plane. Each point represents a separate agent and may build edges from itself to some other points by the strategy to be explained later. The edges in the final graph is the collection of edges built by all the agents. The agents may decide to build edges at different point in time. When an agent \( p \) plans on whether an edge from itself to another point \( q \) should be built or not, \( p \) checks to see whether there is already an edge from some points \( p' \) to \( q' \) such that \( |pp'| \) and \( |qq'| \) are both within \( \frac{1}{4(1+2/\varepsilon)} \cdot |p'q'| \) from \( p \) and \( q \) respectively. If not, the edge \( pq \) is built, otherwise it is not. This strategy is very intuitive — if there is already a cross-country highway from Washington D.C. to San Francisco, it does not make economical sense to build a highway from New York to Los Angeles. We assume that each agent will eventually check on each possible edge from itself to all the other points, but the order on who checks which edge can be completely arbitrary. With this strategy, the agents only make decisions with limited information and no agent has full control over how and what graph will be constructed. It is not obvious that this strategy will end up with a sparse spanner on all points. It is even not clear that the graph will be connected.

The main result in this paper is to prove that with the above strategy executed in any arbitrary order, the graph built at the end of the process is a sparse spanner graph with the following properties:

- Between any two points \( p, q \), there is a path with stretch \( 1 + \varepsilon \) and \( O(|pq|^{1/(1+2/\varepsilon)}) \) hops.
- The number of edges is \( O(n) \).
- The total edge length of the spanner is \( O(|\text{MST}| \cdot \log \alpha) \), where \( \alpha \) is the aspect ratio, i.e., the ratio of the distance between the furthest pair and the closest pair, and \( |\text{MST}| \) is the total edge length of the
minimum spanning tree of the point set. Clearly $|\text{MST}|$ is a lower bound on the total edge length of any constant stretch spanner.

- The degree of each point is $O(\log \alpha)$ in the worst case and $O(1)$ on average.

An example of such a spanner is shown in Figure 1 in the Appendix. The above results for Euclidean space can be extended to a metric with constant doubling dimension. We also show that the spanner can be constructed for $n$ agents with the help of a near neighbor oracle such that

- Only a total number of $O(n \log \alpha)$ messages need to be exchanged between the agents during the entire construction process.
- The spanner topology is implicitly stored on the nodes with each node’s storage cost bounded by $O(\log \alpha)$. It turns out with this representation the nearest neighbor of each node is included in the information stored on each node.
- And yet simply with the partial information stored at each node, there is a local distributed algorithm that finds a $(1+\varepsilon)$-stretch path with maximum $O(|pq|^{1/(1+2/\varepsilon)})$ hops between any two nodes.

To explain how this result is proved, we first obtain as a side product the following greedy algorithm for computing a well-separated pair decomposition of the points with optimal size. A pair of two sets of points, $(A, B)$ with $A, B$ as subsets of $\mathcal{P}$, is called $s$-well-separated if the smallest distance between any two points in $A, B$ respectively is at least $s$ times greater than the diameters of $A$ and $B$. An $s$-well-separated pair decomposition ($s$-WSPD for short) is a collection of $s$-well-separated pairs $\mathcal{W} = \{(A_i, B_i)\}$ such that for any pair of points $p, q \in \mathcal{P}$ there is a pair $(A, B) \in \mathcal{W}$ with $p \in A$ and $q \in B$. The size of an $s$-WSPD is the number of point set pairs in $\mathcal{W}$. Well-separated pair decomposition (WSPD) was first introduced by Callahan and Kosaraju [8] and they developed algorithms for computing an $s$-WSPD with linear size for points in $\mathbb{R}^d$. Since then WSPD has found many applications in computing $k$-nearest neighbors, $n$-body potential fields, geometric spanners and approximate minimum spanning trees [5, 6, 8, 7, 3, 2, 38, 35, 25, 18].

So far there are two algorithms for computing optimal size WSPD, one in the original paper [8] and one in a later paper [23]. Both of them use a hierarchical organization of the points (e.g., the fair split tree in [8] and the discrete center hierarchy in [23]) and output the well-separated pairs in a recursive way. In this paper we show the following simple greedy algorithm also outputs an $s$-WSPD with linear size. In particular, we take an arbitrary pair of points $p, q$ that is not yet covered in any existing well-separated pair, and consider the pair of subsets $(B_r(p), B_r(q))$ with $r = |pq|/(2s + 2)$ and $B_r(p)(B_r(q))$ as the set of points of $\mathcal{P}$ within distance $r$ from $p$ (q). Clearly $(B_r(p), B_r(q))$ is an $s$-well-separated pair and now all the pairs of points $(p', q')$ with $p' \in B_r(p)$ and $q' \in B_r(q)$ are covered. The algorithm continues until all pairs of points are covered. We show that, no matter in which order the pairs are selected, the greedy algorithm will always output a linear number of well-separated pairs.

The key idea in proving the linear size WSPD generated by the greedy algorithm is to show that at most a constant number of the generated well-separated pairs can be mapped to each well-separated pair generated by the deformable spanner [23], a data structure that has found many applications in proximity search with efficient update algorithm in both the kinetic [23] and dynamic settings [44, 24]. It has been shown in [23] that the deformable spanner implies a WSPD with linear size. Thus the greedy algorithm also finds a linear number of pairs. The greedy WSPD also has a number of nice properties (not necessarily carried by the WSPD constructed in [8]) as to be shown in more details later. The greedy WSPD algorithm may be of interest by itself.

Well-separated pair decomposition is deeply connected to geometric spanners. In fact, any WSPD will generate a spanner graph if one puts an edge between an arbitrary pair of points $p, q$ from each well-separated pair $(A, B) \in \mathcal{W}$ [3, 2, 38, 35]. The number of edges in the spanner equals to the size of $\mathcal{W}$. In the other direction, the deformable spanner in [23] implies a WSPD of linear size. The connection is further witnessed in this paper — in our spanner emergence algorithm, each agent $p$ constructs an edge to $q$ only when there
is no nearby edge connecting points near $p$ to points near $q$, this simple rule implies a WSPD generated in a
greedy manner. Hence our spanner construction in an uncoordinated manner inherits many nice properties of
the greedy WSPD.

Last, this paper focuses on the case when the points are distributed in the Euclidean space. The basic idea
extends naturally to metrics with constant doubling dimensions \cite{26,40,42}, as the main technique involves
essentially various forms of geometric packing arguments.

Related work. The model and the philosophy in this paper are related to the network creation game
\cite{19,13,32,1,37}. Fabrikant et al. \cite{19} was the first to introduce the network creation game, in order to understand
the evolution of network topologies by selfish agents. The model used there (and in follow-up papers) assigns
a cost function to each agent that captures the cost paid by the agents to build connections to others minus the
benefit received from the resulted network topology. The agents play a game by minimizing their individual
costs. Almost all these papers use a unit cost for each edge and they deviate in how the benefit of ‘being
connected to others’ is modeled. These papers are interested in the existence of Nash equilibria and the price
of anarchy of Nash equilibria. There are two major open questions along this direction. First, the choice of cost
functions is heuristic — often some intuitive cost functions are selected. There is little understanding on what
cost function best captures the reality yet small variation in the cost function may result in big changes in the
network topologies at Nash equilibria. It is also not easy to execute the game in practice — either because the
selfish agents may face deadlines and have to decide on building an edge or not immediately and sometimes
the edges already built cannot be removed later (e.g., in the development of the transportation network by
different local governments), or because the agents do not have the big picture and the current strategies of
all other agents (e.g., in the P2P setting). The second problem is that there is not much understanding of the
topologies at Nash equilibria. Some of the topologies at Nash equilibria in these papers are very simplistic
topologies such as trees or complete graphs (and these topologies do not show up often in real world). For
other more sophisticated topologies, there is not much understanding of their characteristics and therefore it
is not clear whether these topologies are desirable. Our model is connected to the game theoretic model in
the way that we also try to relax the requirement of a centralized authority in the graph construction, yet we
also incorporate practical considerations that may not allow the agents from playing games to reach a Nash
equilibrium. We believe such models and good algorithms under these models worth further exploration and
this paper makes a first step along this line.

In the vast amount of prior literature on geometric spanners, there are three main ideas: $\Theta$-graphs, the
greedy spanners, and the WSPD-induced spanners \cite{39}. We will review two spanner construction ideas that
are most related to our approach. The first idea is the path-greedy spanner construction \cite{11,14,15,16}. All
pairwise edges are ordered with non-decreasing lengths and checked in that order. An edge is included in the
spanner if the shortest path in the current graph is longer than $\lambda$ times the Euclidean distance, and is discarded
otherwise. Variants of this idea generate spanners with constant degree and total weight $O(|\text{MST}|)$. This idea
cannot be applied in our setting as edges constructed in practice may not be in non-decreasing order of their
lengths and in a P2P network with high churn rate it is too much overhead to compute the shortest path length
in the current overlay network (while checking the distance between two nodes, i.e., the path length in the
underlying network topology, can be done easily with a \textsc{Traceroute} command). The second idea is to use the
gap property \cite{11} — the sources and sinks of any two edges in an edge set are separated by a distance
at least proportional to the length of the shorter of the two edges and their directions are differed no more than
a given angle. The gap-greedy algorithm \cite{4} considers pairs of points, again, in order of non-decreasing
distances, and includes an edge in the spanner if and only if it does not violate the gap property. The spanner
generated this way has constant degree and total weight $O(|\text{MST}|)$. Compared with our algorithm, our strategy
is a relaxation of the gap property in the way that the edges in our spanner may have one of their endpoints
arbitrarily close (or at the same points) and we have no restriction on the direction of the edges. The proof
techniques are also quite different. The proof for the gap greedy algorithm requires heavily plane geometry
tools and our proof technique only uses packing argument and can be extended to the general metric setting as
long as a similar packing argument holds. To get these benefit our algorithm has slightly worse upper bounds on the spanner weight by a logarithmic factor.

Spanner construction for metric space of constant doubling dimension has been proposed before [9, 10, 29]. These algorithms are centralized.

**Organization.** In the rest of the paper we first elaborate the spanner construction in an uncoordinated manner and then show the connection of the spanner with the greedy WSPD. We then show the nice properties of both the greedy WSPD and our spanner. At the end, we describe how to apply the spanner in a decentralized setting to support low-storage spanner representation and efficient local low-stretch routing.

## 2 Uncoordinated spanner construction and a greedy algorithm for WSPD

Assuming $n$ points in $\mathbb{R}^d$, each point represents an agent. We consider the following algorithm for constructing a sparse spanner with stretch factor $s$ in an uncoordinated way. For any point $p$, denote by $B_r(p)$ the collection of points that are within distance $r$ from point $p$, i.e., inside the ball with radius $r$ centered at $p$.

**Uncoordinated spanner construction.** Each point/agent $p$ will check to see whether an edge from itself to another point $q$ should be constructed or not. At this point there might be some edges already constructed by other agents. The order of which agent checks on which edge is completely arbitrary. Specifically, $p$ performs the following operation:

Check where there is already an edge $p'q'$ such that $p$ and $q$ are within distance $\frac{|p'q'|}{2(s+1)}$ from $p'$, $q'$ respectively. If so, $p$ does not build the edge to $q$. Otherwise, $p$ will build an edge to $q$.

This incremental construction of edges is executed by different agents in a completely uncoordinated manner. We assume that no two agents perform the above strategy at exactly the same time. Thus when any agent conducts the above process, the decision is based on the current network already constructed. The algorithm terminates when all agents finish checking the edges from themselves to all other points. In this paper we first study the properties of the constructed graph $G$ by these uncoordinated behaviors. We will discuss later in Section 5 a proper complexity model for the uncoordinated construction in a distributed environment and also bound the computing cost of this spanner.

Before we proceed with our proof, we first realize the following invariant is maintained by the graph $G$. The proof follows immediately from the construction of $G$.

**Lemma 2.1.**

1. For any edge $pq$ that is not in $G$, there is another edge $p'q'$ in $G$ such that $|pp'| \leq |p'q'|/(2s+2)$, $|qq'| \leq |p'q'|/(2s+2)$.
2. For any two edges $pq$, $p'q'$ in the constructed graph $G$, suppose that $pq$ is built before $p'q'$, then one of the following is true: $|pp'| > |pq|/(2s+2)$ or $|qq'| > |pq|/(2s+2)$.

To show that the algorithm eventually outputs a good spanner, we first show the connection of $G$ with the notion well-separated pair decomposition.

**Definition 2.2 (Well-separated pair).** Let $s > 0$ be a constant, and a pair of sets of points $A$, $B$ is $s$-separated, if $d(A, B) \geq s \cdot \max(\text{diam}(A), \text{diam}(B))$, where $\text{diam}(A)$ is the diameter of the point set $A$, $\text{diam}(A) = \max_{p,q \in A} |pq|$, and $d(A, B) = \min_{p \in A, q \in B} |pq|$.

**Definition 2.3 (Well-separated pair decomposition).** Let $s > 0$ be a constant, and $\mathcal{P}$ be a point set. An $s$-well-separated pair decomposition (WSPD) of $\mathcal{P}$ is a set of pairs $\mathcal{W} = \{(A_1, B_1), \ldots, (A_m, B_m)\}$, s.t.

1. $A_i, B_i \subseteq \mathcal{P}$, and the pair sets $A_i$ and $B_i$ are $s$-separated for every $i$.
2. For any two points $p, q \in \mathcal{P}$, there is at least one pair $(A_i, B_i)$ such that $p \in A_i$ and $q \in B_i$. 


Here \( m \) is called the size of the WSPD.

In our construction of \( G \), it is not so hard to see that a well-separated pair decomposition is actually implied.

**Theorem 2.4.** From the uncoordinated construction of the graph \( G \), we can build the following \( s \)-well-separated pair decomposition \( \mathcal{W} \): for each edge \( pq \) in \( G \), include in \( \mathcal{W} \) the pair \((B_r(p), B_r(q))\), with \( r = |pq|/(2s + 2) \). The size of the WSPD is the number of edges in \( G \).

**Proof:** First each pair \((B_r(p), B_r(q))\) is an \( s \)-well-separated pair. Obviously, \( d(B_r(p), B_r(q)) \geq |pq| - 2r \), and \( \text{diam}(B_r(p)), \text{diam}(B_r(q)) \leq 2r \). One can then verify that \( d(B_r(p), B_r(q)) \geq s \cdot \max(\text{diam}(B_r(p)), \text{diam}(B_r(q))) \).

We now show that any point \( p, q \) is included in one well-separated pair. If the edge \( pq \) is in the graph the claim is true obviously. Otherwise, there is an edge \( p'q' \) in \( G \) such that \( |pp'| \leq |p'q'|/(2s + 2) \), \( |qq'| \leq |p'q'|/(2s + 2) \), by Lemma 2.1. This means that \( p \in B_r(p') \) and \( q \in B_r(q') \) with \( r' = |p'q'|/(2s + 2) \). This finishes the proof. \( \square \)

**A greedy algorithm for well-separated pair decomposition.** The above theorem shows the connection of the uncoordinated graph \( G \) with a WSPD \( \mathcal{W} \). In fact, the way to compute the WSPD \( \mathcal{W} \) via the construction of \( G \) is equivalent to the following algorithm that computes an \( s \)-WSPD, in a greedy fashion, with \( s > 1 \).

1. Choose an arbitrary pair \((p, q)\), not yet covered by existing well-separated pairs in \( \mathcal{W} \).
2. Include the pair of point sets \( B_r(p) \) and \( B_r(q) \) in the WSPD \( \mathcal{W} \), with \( r = |pq|/(2 + 2s) \).
3. Label every point pair \((p_i, q_i)\) with \( p_i \in B_r(p) \) and \( q_i \in B_r(q) \) as being covered.
4. Repeat the above steps until every pair of points is covered.

With the \( s \)-WSPD \( \mathcal{W} \), the uncoordinated construction of the graph \( G \) is in fact by taking an edge from each and every well-separated pair in \( \mathcal{W} \) — the simple rule in Lemma 2.1 prevented two edges from the same well-separated pair in \( \mathcal{W} \) to be constructed. It is already known that for any well-separated pair decomposition, if one edge is taken from each well-separated pair, then the edges will become a spanner on the original point set \([3, 2, 38, 35] \). For our specific greedy \( s \)-WSPD, we are able to get a slightly better stretch as shown in the theorem below.

**Theorem 2.5.** Graph \( G \) constructed from the greedy \( s \)-WSPD is a spanner with stretch factor \((s + 1)/(s - 1)\).

**Proof:** Denote by \( \pi(p, q) \) the shortest path length between \( p, q \) in the graph \( G \). We show that \( \pi(p, q) \leq \beta \cdot |pq| \) for any \( p, q \in P \), with \( \beta = (s + 1)/(s - 1) \). We prove this claim by induction on the distance between two points \( p, q \). Take \( p, q \) as the closest pair of \( P \). Then any \( s \)-WSPD will have to use a singleton pair \((p, q)\) to cover the pair \((p, q)\) if \( s > 1 \). If otherwise, say \((P, Q)\) is an \( s \)-well-separated pair that covers \((p, q)\), and \(|P| > 1 \). Then \( \text{diam}(P) > |pq| \), and \( d(P, Q) = |pq| \). This contradicts with the fact that \( d(P, Q) \geq s \cdot \text{diam}(P) \).

Therefore the edge \( pq \) is included in \( G \) for sure and \( \pi(p, q) = |pq| \).

Now suppose that for all pairs of nodes \( x, y \) with Euclidean distance \(|xy| \leq \ell \), we have \( \pi(x, y) \leq \beta \cdot |xy| \). Now we consider the pair of nodes \( p, q \) with the smallest distance (among all remaining pairs) that is still greater than \( \ell \). \((p, q)\) is covered by an \( s \)-well-separated pair \((P, Q) \in \mathcal{W} \), where \( P = B_r(p') \) and \( Q = B_r(q') \) with \( r = |p'q'|/(2s + 2) \) and \( p'q' \) an edge in \( G \). Now we argue that \(|pp'| \leq \ell \). If otherwise, \(|pp'| \geq d(P, Q) \geq s \cdot \text{diam}(P) \geq s \cdot |p'q'| > |pp'| \). So we should have selected the pair \((p, p')\) instead of \((p, q)\). Similarly, \(|qq'| \leq \ell \). Thus by induction hypothesis \( \pi(p, p') \leq \beta \cdot |pp'| \), \( \pi(q, q') \leq \beta \cdot |qq'| \).

By triangle inequality, we have \( \pi(p, q) \leq \pi(p, p') + |p'q'| + \pi(q, q') \leq \beta \cdot (|pp'| + |qq'|) + |p'q'| \leq 2 \beta \cdot r + |p'q'| \). On the other hand, we know by triangle inequality that \( |pq| \geq |p'q'| - 2r = \frac{s + 1}{s} \cdot |p'q'| \). Combining everything we get that \( \pi(p, q) \leq \left(\frac{s}{s + 1} + 1\right) \cdot \frac{s + 1}{s} \cdot |pq| = \beta \cdot |pq| \), with \( \beta = (s + 1)/(s - 1) \). This finishes the proof. \( \square \)

To make the stretch factor as \( 1 + \varepsilon \), we just take \( s = 1 + 2/\varepsilon \) in our spanner construction. We also want to show that the spanner is sparse and has some other nice properties useful for our applications. For that we will first show that the greedy WSPD algorithm will output a linear number of well-separated pairs.
Figure 1. An example of the uncoordinated spanner for 100 points with aspect ratio $\alpha = 223$, the average degree is 6.5, and the stretch is 3.4.
3 The uncoordinated spanner has linear size

To show that the WSPD by the greedy algorithm has a linear number of pairs, we actually show the connection of this WSPD with a specific WSPD constructed by the deformable spanner \([23]\), in the way that at most a constant number of pairs in \(W\) is mapped to each well-separated pair constructed by the deformable spanner. To be consistent, in the following description, the greedy WSPD is denoted by \(W\) and the WSPD constructed by the deformable spanner is denoted by \(\hat{W}\).

3.1 Deformable spanner and WSPD

In this section, we review the basic definition of the deformable spanner and some related properties, which will be used in our own algorithm analysis in the next subsection.

Given a set of points \(P\) in the plane, a set of discrete centers with radius \(r\) is defined to be the maximal set \(S \subseteq P\) that satisfies the covering property and the separation property: any point \(p \in P\) is within distance \(r\) to some point \(p' \in S\); and every two points in \(S\) are of distance at least \(r\) away from each other. In other words, all the points in \(P\) can be covered by balls with radius \(r\), whose centers are exactly those points in the discrete center set \(S\). And these balls do not cover other discrete centers.

We now define a hierarchy of discrete centers in a recursive way. \(S_0\) is the original point set \(P\). \(S_i\) is the discrete center set of \(S_{i-1}\) with radius \(2^i\). Without loss of generality we assume that the closest pair has distance 1 (as we can scale the point set and do not change the combinatorial structure of the discrete center hierarchy). Thus the number of levels of the discrete center hierarchy is \(\lg \alpha\), where \(\alpha\) is the aspect ratio of the point set \(P\), defined as the ratio of the maximum pairwise distance to the minimum pairwise distance, that is, \(\alpha = \max_{u,v \in P} |uv| / \min_{u,v \in P} |uv|\). Since a point \(p\) may stay in multiple consecutive levels and correspond to multiple nodes in the discrete center hierarchy, we denote by \(P^{(i)}(p)\) the existence of \(p\) at level \(i\). For each point \(p^{(i-1)} \in S_{i-1}\) on level \(i-1\), it is within distance \(2^i\) from at least one other point on level \(i\). Thus we assign to \(p^{(i-1)}\) a parent \(q^{(i)}\) in \(S_i\) such that \(|p^{(i-1)}q^{(i)}| \leq 2^i\). When there are multiple points in \(S_i\) that cover \(p^{(i-1)}\), we choose one as its parent arbitrarily. We denote by \(P(p^{(i-1)})\) the parent of \(p^{(i-1)}\) on level \(i\). We denote by \(P(i)(p) = P(P(i-1)(p))\) the ancestor of \(p\) at level \(i\).

The deformable spanner is based on the hierarchy, with all edges between two points \(u\) and \(v\) in \(S_i\) if \(|uv| \leq c \cdot 2^i\), where \(c\) is a constant equal to \(4 + 16/\varepsilon\).

We remark that Krauthgamer and Lee \([33]\) independently proposed a very similar hierarchical structure for proximity search in metrics with doubling dimension \([26]\).

Now we will restate some important properties of the deformable spanner that will be useful in our algorithm analysis.

**Lemma 3.1 (Packing Lemma \([23]\)).** In a point set \(S \subseteq R^d\), if every two points are at least distance \(r\) away from each other, then there can be at most \((2R/r + 1)^d\) points in \(S\) within any ball with radius \(R\).

**Lemma 3.2 (Deformable spanner properties \([23]\)).** For a set of \(n\) points in \(R^d\) with aspect ratio \(\alpha\),

1. For any point \(p \in S_0\), its ancestor \(P(i)(p) \in S_i\) is of distance at most \(2^{i+1}\) away from \(p\).
2. Any point \(p \in S_i\) has at most \((1 + 2c)^d - 1\) edges with other points of \(S_i\).
3. The deformable spanner \(\hat{G}\) is a \((1+\varepsilon)\)-spanner with \(O(n/\varepsilon^d)\) edges.
4. \(\hat{G}\) has total weight \(O(|\text{MST}| \cdot \lg \alpha/\varepsilon^{d+1})\), where \(|\text{MST}|\) is the weight of the minimal spanning tree of the point set \(S\).

As shown in \([23]\), the deformable spanner implies a well-separated pair decomposition \(\hat{W}\) by taking all the ‘cousin pairs’. Specifically, for a node \(p^{(i)}\) on level \(i\), we denote by \(P_i\) the collection of points that are
decendents of \( p(i) \) (including \( p(i) \) itself). Now we take the pair \((P_i, Q_i)\), the sets of decendents of a cousin pair \( p(i) \) and \( q(i) \), i.e., \( p(i) \) and \( q(i) \) are not neighbors in level \( i \) but their parents are neighbors in level \( i + 1 \). This collection of pairs constitutes a \( \frac{2}{\epsilon} \)-well-separated pair decomposition. The size of \( \mathcal{W} \) is bounded by the number of cousin pairs and is shown to be in the order of \( O(n / \epsilon^d) \).

### 3.2 Greedy well-separated pair decomposition has linear size

With the WSPD \( \mathcal{W} \) constructed by the deformable spanner, we now prove that the greedy WSPD \( \mathcal{W} \) has linear size as well. The basic idea is to map the pairs in \( \mathcal{W} \) to the pairs in \( \mathcal{W} \) and show that at most a constant number of pairs in \( \mathcal{W} \) map to the same pair in \( \mathcal{W} \).

**Theorem 3.3.** The greedy \( s \)-WSPD \( \mathcal{W} \) has size \( O(ns^d) \).

**Proof:** Suppose that we have constructed a deformable spanner \( DS \) with \( c = 4(s + 1) \) and obtained an \( s \)-well-separated pair decomposition (WSPD) of it, call it \( \mathcal{W} \), where \( s = \frac{c}{4} - 1 \). The size of \( \mathcal{W} \) is \( O(ns^d) \). Now we will construct a map that takes each pair in \( \mathcal{W} \) and map it to a pair in \( \mathcal{W} \).

Each pair \((P, Q)\) in \( \mathcal{W} \) is created by considering the points inside the balls \( B_r(p), B_r(q) \) with radius \( r = |pq|/(2 + 2s) \) around \( p, q \). Now we consider the ancestors of \( p, q \) in the spanner \( DS \) respectively. There is a unique level \( i \) such that the ancestor \( u_i = P(i)(p) \) and \( v_i = P(i)(q) \) do not have an edge in between but the ancestor \( u_{i+1} = P(i+1)(p) \) and \( v_{i+1} = P(i+1)(q) \) have an edge in between. The pair \( u_i, v_i \) is a cousin pair by definition and thus their decendents correspond to an \( s \)-well-separated pair \((P, Q)\) in \( \mathcal{W} \). We say that the pair \((B_r(p), B_r(q)) \in \mathcal{W} \) maps to the descendant pair \((P_i, Q_i) \in \mathcal{W} \).

By the discrete center hierarchy (Lemma 3.2) and that \( u_i, v_i \) do not have an edge in the spanner, i.e., \( |u_i v_i| > c \cdot 2^i \), we show that,

\[
|pq| \geq |u_i v_i| - |pu_i| - |qv_i| \geq |u_i v_i| - 2 \cdot 2^{i+1} \geq (c - 4) \cdot 2^i.
\]

Also, since \( u_{i+1}, v_{i+1} \) have an edge in the spanner, \( |u_{i+1} v_{i+1}| \leq c \cdot 2^{i+1} \),

\[
|pq| \leq |pu_{i+1}| + |u_{i+1} v_{i+1}| + |qv_{i+1}| \leq 2 \cdot 2^{i+2} + c \cdot 2^{i+1} = 2(c + 4) \cdot 2^i.
\]

Similarly, we have

\[
c \cdot 2^i < |u_i v_i| \leq |u_i u_{i+1}| + |u_{i+1} v_{i+1}| + |v_i v_{i+1}| \leq 2 \cdot 2^{i+1} + c \cdot 2^{i+1} = 2(c + 2) \cdot 2^i.
\]

Therefore the distance between \( p \) and \( q \) is \( c' \cdot |u_i v_i| \), where \((c - 4)/(2c + 4) \leq c' \leq (2c + 8)/c\).

Now suppose two pairs \((B_{r_1}(p_1), B_{r_1}(q_1)), (B_{r_2}(p_2), B_{r_2}(q_2))\) in \( \mathcal{W} \) map to the same pair \( u_i \) and \( v_i \) by the above process. Without loss of generality suppose that \( p_1, q_1 \) are selected before \( p_2, q_2 \) in our greedy algorithm. Here are some observations:

1. \( |p_1 q_1| = c_1' \cdot |u_1 v_1|, |p_2 q_2| = c_2' \cdot |u_1 v_1|, r_1 = |p_1 q_1|/(2 + 2s) = c_1' \cdot |u_1 v_1|/(2 + 2s), r_2 = c_2' \cdot |u_1 v_1|/(2 + 2s), \) where \( (c-4)/(2c+4) \leq c_1', c_2' \leq (2c+8)/c \), and \( r_1, r_2 \) are the radii of the balls for the two pairs respectively.

2. The reason that \((p_2, q_2)\) can be selected in our greedy algorithm is that at least one of \( p_2 \) or \( q_2 \) is outside the balls \( B(p_1), B(q_1) \), by Lemma 2.1. This says that at least one of \( p_2 \) or \( q_2 \) is of distance \( r_1 \) away from \( p_1, q_1 \).

Now we look at all the pairs \((p_i, q_i)\) that map to the same ancestor pair \((u_i, v_i)\). The pairs are ordered in the same order as they are constructed, i.e., \( p_1, q_1 \) is the first pair selected in the greedy WSPD algorithm. Suppose \( r_{\text{min}} \) is the minimum among all radius \( r_i \), \( r_{\text{min}} = |pq|_{\text{min}}/(2 + s) \geq (c-4)/(2c+4) \cdot c \cdot 2^i/(2 + 2s) = s/(2s + 3) \cdot 2^{i+1} \). We group these pairs in the following way. The first group \( H_1 \) contains \((p_1, q_1)\) and all the
pairs \((p_\ell, q_\ell)\) that have \(p_\ell\) within distance \(r_{\text{min}}/2\) from \(p_1\). We say that \((p_1, q_1)\) is the representative pair in \(H_1\) and the other pairs in \(H_1\) are close to the pair \((p_1, q_1)\). The second group \(H_2\) contains, among all remaining pairs, the pair that was selected in the greedy algorithm the earliest, and all the pairs that are close to it. We repeat this process to group all the pairs into \(k\) groups, \(H_1, H_2, \ldots, H_k\). For all the pairs in each group \(H_j\), we have one representative pair, denoted by \((p_j, q_j)\) and the rest of the pairs in this group are close to it.

We first bound the number of pairs belonging to each group by a constant with a packing argument. With our group criteria and the above observations, all \(p_i\) in the group \(H_j\) are within radius \(r_{\text{min}}\) from each other. This means that the \(q_i\)'s must be far away — the \(q_i\)'s must be at least distance \(r_{\text{min}}\) away from each other, by Lemma 2.1. On the other hand, all the \(q_i\)'s are descendant of the node \(v_i\), so \(v_i q_\ell\) is at most \(2^{i+1}\) by Theorem 3.2. That is, all the \(q_i\)'s are within a ball of radius \(2^{i+1}\) centered at \(v_i\). By the packing Lemma 3.1, the number of such \(q_i\)'s is at most \((2 \cdot 2^{i+1}/r_{\text{min}} + 1)^d \leq (2 \cdot 2^{i+1}(2s + 3)/(s \cdot 2^{i+1} + 1))^d = (5 + 6/s)^d\). This is also the bound on the number of pairs inside each group.

Now we bound the number of different groups, i.e., the value \(k\). For the representative pairs of the \(k\) groups, \((p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)\), all the \(p_i\)'s must be at least distance \(r_{\text{min}}/2\) away from each other. Again these \(p_i\)'s are all descendant of \(u_i\) and thus are within distance \(2^{i+1}\) from \(u_i\). By a similar packing argument, the number of such \(p_i\)'s is bounded by \((4 \cdot 2^{i+1}/r_{\text{min}} + 1)^d \leq (9 + 12/s)^d\). So the total number of pairs mapped to the same ancestor pair in \(W\) will be at most \((5 + 6/s)^d \cdot (9 + 12/s)^d = (O(1 + 1/s))^d\). Thus the total number of pairs in \(W\) is at most \(O(ns^d)\). This finishes the proof.

4 Size, degree, weight and diameter of the uncoordinated spanner

With the result that the greedy WSPD has linear size in the previous section and the connection of the greedy WSPD with the uncoordinated spanner construction in Section 2, we are able to obtain the following theorems. The proofs use various of packing arguments.

\textbf{Theorem 4.1.} The uncoordinated spanner \(G\) with parameter \(s\) is a spanner with stretch factor \((s + 1)/(s - 1)\) and has \(O(ns^d)\) number of edges.

\textbf{Proof:} The number of edges in the spanner is the same as the size of the greedy WSPD \(W\) with the same parameter \(s\) constructed by selecting the same set of edges in the same order. \(\square\)

\textbf{Theorem 4.2.} \(G\) has a maximal degree of \(O(\lg \alpha \cdot s^d)\) and average degree \(O(s^d)\).

\textbf{Proof:} With the same argument as in Theorem 3.4, each pair \((p, q)\) built in the uncoordinated spanner maps to a pair of ancestors \((P^{(i)}(p), P^{(i)}(q))\) in the deformable spanner that is a cousin pair. Consider all the edges of \(p\) in \(G, (p, q_\ell)\), that map to the same ancestor pair \((P^{(i)}(p), P^{(i)}(q))\). By a similar argument, all the \(q_i\)'s must be at least distance \(r_{\text{min}}\) away from each other (since all these pairs have \(p\) as the first element in the pair).

Thus we have the number of such edges is bounded by \((5 + 6/s)^d\). The number of cousin pairs associated with \(P^{(i)}(p)\) is at most \(5^d\) times the number of adjacent edges of \(P^{(i+1)}(p)\), and is bounded by \(5^d \cdot (8s + 9)^d - 1\) (by Theorem 3.2). Since there are \(\lfloor \lg \alpha \rfloor\) levels, the total number of edges associated with the node \(p\) is at most \(\lfloor \lg \alpha \rfloor \cdot 5^d \cdot (8s + 9)^d - 1 \cdot (5 + 6/s)^d\). Then the maximal degree of the spanner is \(O(\lg \alpha \cdot s^d)\).

Since the spanner has total \(O(ns^d)\) edges, the average degree is \(O(s^d)\). \(\square\)

\textbf{Theorem 4.3.} \(G\) has total weight \(O(\lg \alpha \cdot |\text{MST}| \cdot s^{d+1})\).

\textbf{Proof:} Again we use the mapping of the uncoordinated spanner edges to the cousin pairs in the deformable spanner \(DS\), as in Theorem 3.4. We also use the same notation here. Consider all the edges \((p_\ell, q_\ell)\) that map to the same ancestor cousin pair \((u_i, v_i)\). We now map them to the edge between the parents of this cousin.
pair, i.e., edge \( u_{i+1}v_{i+1} \) in \( DS \). The pair \((u_{i+1}, v_{i+1})\) has at most \( 5^{2d} \) number of cousin pairs. Thus at most 
\((5 + 6/s)^d \cdot (9 + 12/s)^d \cdot 5^{2d} = (O(1 + 1/s))^d \) edges in \( G \) are mapped to one edge in \( DS \).

Now we will bound the length of an edge \( pq \) in \( G \) and the edge \( u_{i+1}v_{i+1} \) in \( DS \) it maps to. From the proof of Theorem 3.3, we know that \((c - 4) \cdot 2^i \leq |pq| \leq 2(c + 4) \cdot 2^i \). In addition, \(|u_{i+1}v_{i+1}| \leq 2c \cdot 2^i \) as \( u_{i+1}v_{i+1} \) is an edge in \( DS \), and \(|u_{i+1}v_{i+1}| \geq |u_iv_i| - |u_{i+1}u_i| - |v_{i+1}v_i| \geq c \cdot 2^i - 2 \cdot 2^{i+1} = (c - 4) \cdot 2^i \).

Thus, \((c - 4)/(2c) \leq |pq|/(u_{i+1}v_{i+1}) \leq 2(c + 4)/(c - 4) \).

We now bound the total weight of the spanner \( G \). We group all the edges by the spanner edge in \( DS \) that they map to. Thus we have the total weight of \( G \) is at most \( 2(c + 4)/(c - 4) \cdot (O(1 + 1/s))^d \) the weight of \( DS \). By Theorem 3.2, the weight of \( DS \) is at most \( O(|\lg \alpha \cdot |\text{MST}| \cdot s^{d+1}) \). Thus the weight of \( G \) is at most 
\( O(|\lg \alpha \cdot |\text{MST}| \cdot s^{d+1}) \).

**Theorem 4.4.** For any two points \( p \) and \( q \) in \( G \), there is a path with stretch \((s + 1)/(s - 1)\) between \( p \) and \( q \) with at most \( 2|pq|^{1/(1+\lg s)} \) hops.

**Proof:** For any two point \( p \) and \( q \), they will be covered by a pair set \((P, Q)\) with respect to edge \( pq' \) so that \( p \in P \) and \( q \in Q \). The path \( p \rightsquigarrow q \) between \( p \) and \( q \) can be found recursively by taking the path \( p \rightsquigarrow p' \), then the edge \( p'q' \), and then the path \( q' \rightsquigarrow q \). This path found recursively will have stretch \((s + 1)/(s - 1)\) according to Theorem 2.5.

Obviously \(|pp'| \leq r \leq |pq|/(2s)\). Denote by \( h(|pq|) \) the hop count of the path between \( p \) and \( q \) with stretch \((s + 1)/(s - 1)\). Thus we can get the following recurrence \( h(|pq|) = h(|pp'|) + 1 + h(|qq'|) \leq h(|pq|/(2s)) + 1 + h(|pq|/(2s)), \) that is, \( h(x) = 2h(x/(2s)) + 1 \). Solve this recurrence we get \( h(|pq|) = 2|\lg 2s| |pq| + 1 = 2|\lg |pq|/|pq|(2s) + 1 = 2|pq|^{1/(1+\lg s)}. \)

The analysis is tight as shown by Figure 2. In the example, we assume all the nodes almost lying on the line. The 2 balls with the same radius on the left side(or right side) is the pair that covers \( p \) or \( q \), and there is an edge between them(in order to see clearly, we draw curve line between them). The red bold balls are the balls that contain \( p \) or \( q \) on the left side. If we create these pairs in decreasing order on the radius, then we will have a path between \( p \) and \( q \) with exactly \( 2|pq|^{1/(1+\lg s)} \) hops. And we assume all the rest points are far away, then this is the only path between \( p \) and \( q \).

**5 Spanner construction and routing**

The spanner construction is so far only defined on points in Euclidean space. Now, we will first extend our spanner results to a more general metric, metric with constant doubling dimension. The doubling dimension of a metric space \((X, d)\) is the smallest value \( \gamma \) such that each ball of radius \( R \) can be covered by at most \( 2^\gamma \) balls of radius \( R/2 \) [20]. Metrics with constant doubling dimension appear in many different settings. For example, it has been discovered that the Internet delay metric has restricted growth rate and approximately
has a constant doubling dimension \cite{40,42}. As the main technique used in the spanner analysis is packing argument, the following Theorem is the extension of the spanner in the metric case. The proof is in the Appendix.

Theorem 5.1. For \( n \) points and a metric space defined on them with constant doubling dimension \( \gamma \), the uncoordinated spanner construction outputs a spanner \( G \) with stretch factor \((s + 1)/(s - 1)\), has total weight \(O(\gamma^9 \cdot \lg \alpha \cdot |MST| \cdot s^{O(\gamma)})\) and has \( O(\gamma^4 \cdot n \cdot s^{O(\gamma)}) \) number of edges. Also it has a maximal degree of \( O(\gamma^4 \cdot \lg \alpha \cdot s^{O(\gamma)}) \) and average degree \( O(\gamma^4 \cdot s^{O(\gamma)}) \).

Proof: The proof follows almost the same as those in the previous section. The deformable spanner can be applied for metrics with constant doubling dimension \cite{23}. Whenever we use a geometric packing argument, we replace by the property of metrics of constant doubling dimension. We just need to notice that, with the proposition from \cite{26}, the number of pairs in a group that might be mapped to the same ancestor is bounded by \( \gamma^{|\lg(2R/r)|} = \gamma^{|\lg(2 \cdot 2^{i+1}/(s/(2s+3) \cdot 2^{i+1}))|} = \gamma^{|\lg(2(2+3)/s)|} < \gamma^4 \), and the number of groups is bounded by \( \gamma^{|\lg(4 \cdot 2^{i+1}/(s/(2s+3) \cdot 2^{i+1}))|} < \gamma^5 \).

Model of computing. Now we would like to discuss the computing model as well as the construction cost of an uncoordinated spanner. Each distributed agent checks whether an edge to another agent should be built or not. The order for these operations can be arbitrary. We assume that there is already an oracle that answers near neighbors queries: return the list of nodes within distance \( r \) from any node \( x \). Such near neighbor oracle may be already available in the applications. For example, in transportation network, if we think each city as an agent, it usually stores the nearby cities and the corresponding transportation road by them. For a metric with doubling dimension, this oracle can be implemented in a number of ways \cite{42,33,30}. For example, such functions can be implemented in Tapestry, a P2P system \cite{41,50}.

Spanner construction and representation. The spanner edges are recorded in a distributed fashion so that no node has the entire picture of the spanner topology. After each edge \( p q \) in \( G \) is constructed, the peers \( p, q \) will inform their neighboring nodes (those in \( B_r(p) \) and \( B_r(q) \) with \( r = |pq|/(2s + 2) \)) that such an edge \( pq \) exists so that they will not try to connect to one another. These neighboring nodes can be obtained through a near neighbor search from both \( p \) and \( q \). We assume that these messages are delivered immediately so that when any edge is built the previous constructed edges have been informed to nodes of relevance. We remark that the nodes in \( B_r(p) \) and \( B_r(q) \) will only store the single edge \( pq \), as well as their distance to \( p \) (for those in \( B_r(p) \)) or \( q \) (for those in \( B_r(q) \)) and the value of \( |pq| \), but not all the nodes in \( B_r(p) \) or \( B_r(q) \). When a node \( x \) considers whether an edge \( xy \) should be built, \( x \) could simply consult with the edges that it has stored locally. The amount of storage at each node \( x \) is proportional to the number of well-separated pairs that include \( x \). The number of messages for this operation is bounded by \( |B_r(p)| + |B_r(q)| \). The following theorem shows that the total number of such messages during the execution of the algorithm is almost linear in \( n \). That is, on average each node sends about \( O(s^d \cdot \lg \alpha) \) messages. Also the amount of storage at each node is bounded by \( O(s^d \cdot \lg \alpha) \).

Theorem 5.2. For the uncoordinated spanner \( G \) and the corresponding greedy WSPD \( \mathcal{W} = \{(P_i, Q_i)\} \) with size \( m \), each node \( x \) is included in at most \( O(s^d \cdot \lg \alpha) \) well-separated pairs in \( \mathcal{W} \).

Proof: For each point \( x \), each well-separated pair that contains \( x \) can be mapped to some ancestor pair in some level in the corresponding deformable spanner. Let’s consider the set pair \( (P, Q) \) (with respect to edge \( pq \)) that is mapped to the ancestor pair \( (P^{(i)}(p), P^{(i)}(q)) = (u_i, v_i) \) at level \( i \) with \( x \in P \). Now we map \( x \) to the node \( u_i \) and we count how many such nodes \( u_i \) on level \( i \) that a node \( x \) may map to. The corresponding radius with \( (p, q) \) is \( r = |pq|/(2s + 2s) \leq (1 + 1/(s + 1))2^{i+2} \). In this case, \( |xp_i| \leq |pu_i| + |xp| \leq 2^{i+1} + r \leq (3 + 2/(s + 1))2^{i+1} \). According to Lemma \ref{3.2} different nodes in level \( i \) must be at least \( 2^i \) away from each
other. With a packing argument, there can be at most \[\left\lfloor \frac{(2 \cdot 3 + 2/(s+1)) \cdot 2^i + 1}{2^i + 1} \right\rfloor = (13 + 8/(s+1))^d\]
different such nodes \(p_i\) on level \(i\) that \(x\) may map to. So the number of well-separated pairs that cover \(x\) is at most

\[
\sum_{i=1}^{\log_2 n} \sum_{p_i} \text{cousin pairs with } p_i \cdot |\text{pairs in } \mathcal{W} \text{ mapping to a cousin pair with } p_i| \\
\leq \log_2 n \cdot (13 + 8/(s+1))^d \cdot 5^d(8s + 9)^d - 1 \cdot (O(1 + 1/s)^d)) \\
= O(s^d \log_2 n).
\]

\[\square\]

**Theorem 5.3.** For the uncoordinated spanner \(G\) and the corresponding greedy WSPD \(\mathcal{W} = \{(P_i, Q_i)\}\) with size \(m\), \[\sum_{i=1}^{m} (|P_i| + |Q_i|) = O(ns^d \cdot \log_2 n)\]. Thus the total messages involved in the spanner construction algorithm is \(O(ns^d \cdot \log_2 n)\).

**Proof:** This follows immediately from Theorem 5.2 as

\[\sum_{i=1}^{m} (|P_i| + |Q_i|) \leq \sum_x |\text{pairs that include } x| = O(ns^d \cdot \log_2 n).
\]

\[\square\]

**Distributed low-stretch routing on spanner.** Although the spanner topology is implicitly stored on the nodes with each node only knows some piece of it, we are actually able to do a distributed and local routing on the spanner with only information available at the nodes such that the path discovered has maximum stretch \((s+1)/(s-1)\). In particular, for any node \(p\) who has a message to send to node \(q\), it is guaranteed that \((p, q)\) is covered by a well-separated pair \((B_r(p'), B_r(q'))\) with \(p \in B_r(p')\) and \(q \in B_r(q')\). By the construction algorithm, the edge \(p'q'\), after constructed, is informed to all nodes in \(B_r(p') \cup B_r(q')\), including \(p\). Thus \(p\) includes in the packet a partial route with \((p \leadsto p', p'q', q' \leadsto q)\). The notation \(p \leadsto p'\) means that \(p\) will need to first find out the low-stretch path from \(p\) to the node \(p'\) (inductively), from where the edge \(p'q'\) can be taken, such that with another low-stretch path to be found out from \(q'\) to \(q\), the message can be delivered to \(q\). This way of routing with partial routing information stored in the packet is similar to the idea of source routing [47] except that we do not include the full routing path at the source node. By the same induction as used in the proof of spanner stretch (Theorem 2.3), the final path is going to have stretch at most \((s+1)/(s-1)\) and at most \(2|pq|/(1+\log_2 s)\) hops.

**Support for nearest neighbor search.** The constructed spanner can be used to look for the nearest peer in the P2P network. Since we let each point \(x\) keep all the edges \((p, q)\) that cover \(x\), among all these \(p\)'s, one of them must be the nearest neighbor of \(x\). If otherwise, suppose \(y\) is the nearest neighbor of \(x\), and \(y\) is not one of \(p\). But in the WSPD \(\mathcal{W}\), \((x, y)\) will belong to one of the pair set \((P, Q)\), which corresponds to a spanner edge \((p', q')\) that covers \(x\). Then there is a contradiction, as \(|xp'| \leq \text{diam}(P) \leq d(P, Q)/s < d(P, Q) = |xy|\) implies that \(y\) is not the nearest neighbor of \(x\). According to Theorem 5.2, \(x\) will belong to at most \(O(s^d \log_2 n)\) different pair sets. So the nearest neighbor search can be done in \(O(s^d \log_2 n)\) time, with only the information stored on \(x\).

**Node insertion and deletion.** The uncoordinated spanner construction supports node insertion and deletion. When a peer \(x\) joins the network, it will check with each other peer whether or not a nearby edge exists as specified in our greedy algorithm. When a peer \(y\) leaves the network, \(p\) will notify the nodes that are covered by \(p\)'s edges, i.e., for each edge \(pq\), \(p\) will notify \(q\), and all the nodes within \(|pq|/(2s+2)\) from \(p\) and \(q\). Then the notified nodes will check and possibly build new edges to restore the spanner property. In this way, the spanner and the good properties can be maintained.
6 Conclusion and future work

This paper aims to explain the emergence of good spanners from the behaviors of agents with their own interests. The results can be immediately applied to the construction of good network overlays by distributed peers with incomplete information. For our future work we would like to explore incentive-based overlay construction [20]. One problem faced in the current P2P system design is to reward peers that contribute to the network maintenance or service quality and punish the peers that try to take free rides [22][27][28][21][45][46]. We would like to extend the results in this paper and come up with a spanner construction with different quality of service for different peers to achieve fairness — those who build more edges should have a smaller stretch to all other nodes and those who do not build many edges are punished accordingly by making the distances to others slightly longer.

References

[1] S. Albers, S. Eilts, E. Even-Dar, Y. Mansour, and L. Roditty. On nash equilibria for a network creation game. In SODA ’06: Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm, pages 89–98, New York, NY, USA, 2006. ACM.

[2] S. Arya, G. Das, D. M. Mount, J. S. Salowe, and M. Smid. Euclidean spanners: short, thin, and lanky. In Proc. 27th ACM Symposium on Theory Computing, pages 489–498, 1995.

[3] S. Arya, D. M. Mount, and M. Smid. Randomized and deterministic algorithms for geometric spanners of small diameter. In Proc. 35th IEEE Symposium on Foundations of Computer Science, pages 703–712, 1994.

[4] S. Arya and M. Smid. Efficient construction of a bounded-degree spanner with low weight. Algorithmica, 17:33–54, 1997.

[5] Callahan and Kosaraju. Faster algorithms for some geometric graph problems in higher dimensions. In Proc. 4th ACM-SIAM Symposium on Discrete Algorithms, pages 291–300, 1993.

[6] P. B. Callahan. Optimal parallel all-nearest-neighbors using the well-separated pair decomposition. In Proc. 34th IEEE Symposium on Foundations of Computer Science, pages 332–340, 1993.

[7] P. B. Callahan and S. R. Kosaraju. Algorithms for dynamic closest-pair and n-body potential fields. In Proc. 6th ACM-SIAM Symposium on Discrete Algorithms, pages 263–272, 1995.

[8] P. B. Callahan and S. R. Kosaraju. A decomposition of multidimensional point sets with applications to k-nearest-neighbors and n-body potential fields. J. ACM, 42:67–90, 1995.

[9] H. T.-H. Chan, A. Gupta, B. M. Maggs, and S. Zhou. On hierarchical routing in doubling metrics. In SODA ’05: Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, pages 762–771, 2005.

[10] T.-H. H. Chan and A. Gupta. Small hop-diameter sparse spanners for doubling metrics. In SODA ’06: Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm, pages 70–78, New York, NY, USA, 2006. ACM.

[11] B. Chandra, G. Das, G. Narasimhan, and J. Soares. New sparseness results on graph spanners. Internat. J. Comput. Geom. Appl., 5:125–144, 1995.

[12] Y. Chu, S. Rao, S. Seshan, and H. Zhang. Enabling conferencing applications on the internet using an overlay multicast architecture. SIGCOMM Comput. Commun. Rev., 31(4):55–67, 2001.

[13] J. Corbo and D. Parkes. The price of selfish behavior in bilateral network formation. In PODC ’05: Proceedings of the twenty-fourth annual ACM symposium on Principles of distributed computing, pages 99–107, New York, NY, USA, 2005. ACM.

[14] G. Das, P. Heffernan, and G. Narasimhan. Optimally sparse spanners in 3-dimensional Euclidean space. In Proc. 9th Annu. ACM Sympos. Comput. Geom., pages 53–62, 1993.
[15] G. Das and G. Narasimhan. A fast algorithm for constructing sparse Euclidean spanners. *Internat. J. Comput. Geom. Appl.*, 7:297–315, 1997.

[16] G. Das, G. Narasimhan, and J. Salowe. A new way to weigh malnourished Euclidean graphs. In *Proc. 6th ACM-SIAM Sympos. Discrete Algorithms*, pages 215–222, 1995.

[17] D. Eppstein. Spanning trees and spanners. In J.-R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pages 425–461. Elsevier Science Publishers B.V. North-Holland, Amsterdam, 2000.

[18] J. Erickson. Dense point sets have sparse Delaunay triangulations. In *Proc. 13th ACM-SIAM Symposium on Discrete Algorithms*, pages 125–134, 2002.

[19] A. Fabrikant, A. Luthra, E. Maneva, C. H. Papadimitriou, and S. Shenker. On a network creation game. In *PODC ’03: Proceedings of the twenty-second annual symposium on Principles of distributed computing*, pages 347–351, 2003.

[20] J. Feigenbaum and S. Shenker. Distributed algorithmic mechanism design: recent results and future directions. In *DIALM ’02: Proceedings of the 6th international workshop on Discrete algorithms and methods for mobile computing and communications*, pages 1–13, New York, NY, USA, 2002. ACM.

[21] M. Feldman and J. Chuang. Overcoming free-riding behavior in peer-to-peer systems. *SIGecom Exch.*, 5(4):41–50, 2005.

[22] M. Feldman, K. Lai, I. Stoica, and J. Chuang. Robust incentive techniques for peer-to-peer networks. In *EC ’04: Proceedings of the 5th ACM conference on Electronic commerce*, pages 102–111, New York, NY, USA, 2004. ACM.

[23] J. Gao, L. Guibas, and A. Nguyen. Deformable spanners and their applications. *Computational Geometry: Theory and Applications*, 35(1-2):2–19, 2006.

[24] L.-A. Gottlieb and L. Roditty. Improved algorithms for fully dynamic geometric spanners and geometric routing. In *SODA ’08: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 591–600, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics.

[25] J. Gudmundsson, C. Levcopoulos, G. Narasimhan, and M. Smid. Approximate distance oracles for geometric graphs. In *Proc. 13th ACM-SIAM Symposium on Discrete Algorithms*, pages 828–837, 2002.

[26] A. Gupta, R. Krauthgamer, and J. R. Lee. Bounded geometries, fractals, and low-distortion embeddings. In *FOCS ’03: Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science*, pages 534–543, 2003.

[27] A. Habib and J. Chuang. Incentive mechanism for peer-to-peer media streaming. In *Proc. of the 12th IEEE International Workshop on Quality of Service (IWQoS’04)*, June 2004.

[28] A. Habib and J. Chuang. Service differentiated peer selection: An incentive mechanism for peer-to-peer media streaming. *IEEE Transactions on Multimedia*, 8(3):610–621, June 2006.

[29] S. Har-Peled and M. Mendel. Fast construction of nets in low dimensional metrics, and their applications. In *SCG ’05: Proceedings of the twenty-first annual symposium on Computational geometry*, pages 150–158, New York, NY, USA, 2005. ACM.

[30] K. Hildrum, J. Kubiatowicz, S. Ma, and S. Rao. A note on the nearest neighbor in growth-restricted metrics. In *SODA ’04: Proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 560–561, Philadelphia, PA, USA, 2004. Society for Industrial and Applied Mathematics.

[31] K. Hildrum, J. D. Kubiatowicz, S. Rao, and B. Y. Zhao. Distributed object location in a dynamic network. In *SPAA ’02: Proceedings of the fourteenth annual ACM symposium on Parallel algorithms and architectures*, pages 41–52, New York, NY, USA, 2002. ACM.

[32] T. Jansen and M. Theile. Stability in the self-organized evolution of networks. In *GECCO ’07: Proceedings of the 9th annual conference on Genetic and evolutionary computation*, pages 931–938, New York, NY, USA, 2007. ACM.

[33] R. Krauthgamer and J. R. Lee. Navigating nets: simple algorithms for proximity search. In *Proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 798–807, 2004.
[34] M. Kwon and S. Fahmy. Topology-aware overlay networks for group communication. In *NOSSDAV '02: Proceedings of the 12th international workshop on Network and operating systems support for digital audio and video*, pages 127–136, New York, NY, USA, 2002. ACM.

[35] C. Levcopoulos, G. Narasimhan, and M. H. M. Smid. Improved algorithms for constructing fault-tolerant spanners. *Algorithmica*, 32(1):144–156, 2002.

[36] K. Lua, J. Crowcroft, M. Pias, R. Sharma, and S. Lim. A survey and comparison of peer-to-peer overlay network schemes. *Communications Surveys & Tutorials, IEEE*, pages 72–93, 2005.

[37] T. Moscibroda, S. Schmid, and R. Wattenhofer. On the topologies formed by selfish peers. In *PODC '06: Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing*, pages 133–142, New York, NY, USA, 2006. ACM.

[38] G. Narasimhan and M. Smid. Approximating the stretch factor of Euclidean graphs. *SIAM J. Comput.*, 30:978–989, 2000.

[39] G. Narasimhan and M. Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007.

[40] E. Ng and H. Zhang. Predicting Internet network distance with coordinates-based approaches. In *Proc. IEEE INFOCOM*, pages 170–179, 2002.

[41] D. Peleg. *Distributed Computing: A Locality Sensitive Approach*. Monographs on Discrete Mathematics and Applications. SIAM, 2000.

[42] C. G. Plaxton, R. Rajaraman, and A. W. Richa. Accessing nearby copies of replicated objects in a distributed environment. In *Proc. ACM Symposium on Parallel Algorithms and Architectures*, pages 311–320, 1997.

[43] S. Ratnasamy, M. Handley, R. Karp, and S. Shenker. Topologically-aware overlay construction and server selection. In *Proceedings of the 21th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM'05)*, volume 3, pages 1190–1199, 2002.

[44] L. Roditty. Fully dynamic geometric spanners. In *SCG '07: Proceedings of the twenty-third annual symposium on Computational geometry*, pages 373–380, New York, NY, USA, 2007. ACM.

[45] S. Schosser, K. Böhm, R. Schmidt, and B. Vogt. Incentives engineering for structured p2p systems - a feasibility demonstration using economic experiments. In *EC '06: Proceedings of the 7th ACM conference on Electronic commerce*, pages 280–289, New York, NY, USA, 2006. ACM.

[46] S. Schosser, K. Böhm, and B. Vogt. Indirect partner interaction in peer-to-peer networks: stimulating cooperation by means of structure. In *EC '07: Proceedings of the 8th ACM conference on Electronic commerce*, pages 124–133, New York, NY, USA, 2007. ACM.

[47] A. S. Tanenbaum. *Computer networks (3rd ed.).* Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1996.

[48] W. Wang, C. Jin, and S. Jamin. Network overlay construction under limited end-to-end reachability. In *Proceedings of the 24th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM'05)*, volume 3, pages 2124–2134, March 2005.

[49] X. Zhang, Z. Li, and Y. Wang. A distributed topology-aware overlays construction algorithm. In *MG '08: Proceedings of the 15th ACM Mardi Gras conference*, pages 1–6, New York, NY, USA, 2008. ACM.

[50] B. Y. Zhao, J. D. Kubiatowicz, and A. D. Joseph. Tapestry: An infrastructure for fault-tolerant wide-area location and. Technical report, Berkeley, CA, USA, 2001.