Uncertainty and auto-correlation in Measurement

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Abstract

Although a system is described by a well-known set of equations leading to a deterministic behavior, in the real world the value of a measurand obtained by an experiment will mostly scatter. Accordingly, an uncertainty is associated with that value of the measurand due to apparently random fluctuation. This paper deals with the question why this discrepancy exists. Furthermore, it will be shown how the uncertainty of one individual observation is calculated and consequently how the best estimate and its corresponding uncertainty considering auto-correlations is determined.

Introduction

A measurand is determined from other quantities through a functional relationship $f$ by

$$y = f(x_1, x_2, ..., x_n)$$

where $x_1, x_2, ..., x_n$ are input parameters. These quantities are often in turn influenced by other quantities through a functional relationship $g$ by,
\[ x_i = g_i (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \] (0.2)

where the parameters \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \) are fundamental since they determine the characteristics of the measurand via Eq. (0.1,0.2).

Unfortunately, not all fundamental parameters may be known, e.g., convection of air causing dynamic pressure during a weighing measurement. This leads to the generation of additional chaotic forces depending on the velocity pattern of the air flow in the close vicinity of the pan of the balance. If the velocity pattern is not determined as well as the impact of it on the balance, its influence on the measurand is not known. Or, vibrations cause forces acting on the balance due to accelerations. But if accelerations are not measured and additionally, their impact are not known, the influence of them on the measurand is not known either.

Therefore, the fundamental parameters can be distinguished between known and unknown quantities such as,

\[ x_i = g_i (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l, h_1, h_2, \ldots, h_j) \] (0.3)

where \( h_j \) are “hidden” parameters. Thus the measurand is given by,

\[ y = f (g_1 (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l, h_1, h_2, \ldots, h_j), \ldots, g_n (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l, h_1, h_2, \ldots, h_j)) \] (0.4)

Consequently, the measurand becomes a function of fundamental known and hidden parameters,

\[ y = \tilde{f} (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l, h_1, h_2, \ldots, h_j) \] (0.5)
Origin of apparent random fluctuations of the Measurand

Let us first assume that hidden fundamental parameters do not exist and consequently the system is fully described by a well-known set of equations. That means, that all fundamental parameters and their impact on the behavior of the measurand are known. Hence the measurand is given by,

\[ y = \tilde{f} (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l) \quad (0.6) \]

Thus, at time \( t = t_0 \), the measurand is given by,

\[ y(t_0) = \tilde{f} (\varepsilon_1|_{t_0}, \varepsilon_2|_{t_0}, \ldots, \varepsilon_l|_{t_0}) \quad (0.7) \]

at the time \( t = t_0 + \Delta t \), the measurand is given by,

\[ y(t_0 + \Delta t) = \tilde{f} (\varepsilon_1|_{t_0+\Delta t}, \varepsilon_2|_{t_0+\Delta t}, \ldots, \varepsilon_l|_{t_0+\Delta t}) \quad (0.8) \]

If the values of all known fundamental parameter at time \( t = t_0 + \Delta t \), are equal to the values at time \( t = t_0 \), the value of the measurand would be the same,

\[
\begin{align*}
\varepsilon_1|_{t_0+\Delta t} &= \varepsilon_1|_{t_0} \\
\varepsilon_2|_{t_0+\Delta t} &= \varepsilon_2|_{t_0} \\
&\vdots \\
\varepsilon_l|_{t_0+\Delta t} &= \varepsilon_l|_{t_0}
\end{align*}
\]

\[ \Rightarrow y(t_0 + \Delta t) = y(t_0) \quad (0.9) \]

Consequently the system is fully deterministic in that case.

Now let us assume that hidden fundamental parameters exist. The measurand in that case is given by,

\[ y = \tilde{f} (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l, h_1, h_2, \ldots, h_j) \quad (0.10) \]
Again, at time $t = t_0$, the measurand is given by,

$$y(t_0) = \tilde{f}(\varepsilon_1|_{t_0}, \varepsilon_2|_{t_0}, \ldots, \varepsilon_l|_{t_0}, h_1|_{t_0}, h_2|_{t_0}, \ldots, h_j|_{t_0}) \quad (0.11)$$

and at time $t = t_0 + \Delta t$, the measurand is given by,

$$y(t_0) = \tilde{f}(\varepsilon_1|_{t_0+\Delta t_0}, \varepsilon_2|_{t_0+\Delta t_0}, \ldots, \varepsilon_l|_{t_0+\Delta t_0}, h_1|_{t_0+\Delta t_0}, h_2|_{t_0+\Delta t_0}, \ldots, h_j|_{t_0+\Delta t_0}) \quad (0.12)$$

However, since the impact of hidden parameters can not be evaluated leads to the fact, that although in the case that the values of the known parameters at both times are equal, the values of the measurand at both times are not necessarily equal,

$$\begin{align*}
\varepsilon_1|_{t_0+\Delta t} &= \varepsilon_1|_{t_0} \\
\varepsilon_2|_{t_0+\Delta t} &= \varepsilon_2|_{t_0} \\
\vdots \\
\varepsilon_l|_{t_0+\Delta t} &= \varepsilon_l|_{t_0} \\
\end{align*}
\Rightarrow \begin{cases} 
\begin{align*}
y(t_0 + \Delta t) &= y(t_0) \\
\text{if } &h_1|_{t_0+\Delta t} = h_1|_{t_0} \\
\text{if } &h_2|_{t_0+\Delta t} = h_2|_{t_0} \\
&\vdots \\
\text{if } &h_l|_{t_0+\Delta t} = h_l|_{t_0} \\
\end{align*} \\
y(t_0 + \Delta t) &\neq y(t_0) \quad \text{if } h_i|_{t_0+\Delta t} \neq h_i|_{t_0} \\
\end{cases} \quad (0.13)
$$

Strictly speaking, only in the case that the values of all input parameters (known and hidden) are exactly the same, the value of the measurand at both times would be equal. If only the value of one hidden parameter is different at different times, the value of the measurand would be different too. This leads to the fact, that for equal sets of known input parameters the measurand can reach different values. Thus due to lack of information of the system, it apparently behaves not necessarily deterministic but rather reveals a stochastic behavior. This phenomenon is depicted in Fig. (0.1)

Nota bene, the stochastic behavior of the system is a consequence of the existence of hidden parameters.
Figure 0.1: *Measurand as a function of hidden and known parameter.* a) The measurand value is given for instance by $Y = -2T^2 - 4T \cdot P + 1000$. The parameter $T$ denotes the known temperature and $P$ is the unknown pressure. The red balls indicate values of the measurand given by specific values of the known and unknown variable. Here the impact of $P$ on the measurand value is considered to be known. b) Due to the fact, that the hidden parameter is not accessible, only the projection (plane $TY$) of $Y$ is “visible”. Thus, the measurand values apparently scatter.

**Uncertainty of individual value of measurand**

Basically, the true value of a quantity is often not known. For instance, considering hydrostatic weighing for the determination of liquid density. Usually a solid body is immersed into the liquid and the apparent loss of its mass (due to a lift force) is measured by using a balance. The lift depends on the volume of the body which in turn depends on temperature. But, the temperature of the body is not measured directly since one avoids any generation of contact forces acting on the body. Solely, the temperature of the fluid is measured in the vicinity of the body. Thus, one can only estimate the true value of the body temperature.

To derive a relation for the uncertainty in that case, one calculates the change of the value of the measurand for a small change of the input parameters. This is given by,

$$
\begin{align*}
\frac{dy}{dt} &= \sum_i \left. \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right|_{\varepsilon = \varepsilon_0} \cdot (\varepsilon_i - \varepsilon_{0i}) + \sum_j \left. \frac{\partial \tilde{f}}{\partial h_j} \right|_{h = h_0} \cdot (h_j - h_{0j})
\end{align*}
$$

(0.14)
Now it is assumed that the true value of $\varepsilon_i$ and $h_k$ lies between $[\varepsilon_{i0} - \Delta \varepsilon_i, \varepsilon_{i0} + \Delta \varepsilon_i]$ respectively $[h_{k0} - \Delta h_k, h_{k0} + \Delta h_k]$ so that $\Delta \varepsilon_i$ and $\Delta h_k$ defines the range in which we believe the true value lies with a specified likelihood. Hence it is reasonable to chose $\varepsilon_i = \varepsilon_{i0} + \Delta \varepsilon_i$, and $h_k = h_{k0} + \Delta h_k$ thus Eq. (0.14) becomes,

$$dy = \sum_l \left. \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \Delta \varepsilon_i + \sum_j \left. \frac{\partial \tilde{f}}{\partial h_k} \right|_{h_{10}, \ldots, h_{j0}} \Delta h_k$$

(0.15)

The value $(dy)^2$ is a measure for the measurand uncertainty ($\text{(dy)}^2$ instead of $dy$ since the uncertainty should be positive). Thus,

$$u^2(y) = (dy)^2 = \sum_l \left( \left. \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \Delta \varepsilon_i \right)^2 + 2 \sum_l \sum_{k=i+1}^l \left. \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \left. \frac{\partial \tilde{f}}{\partial \varepsilon_k} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \Delta \varepsilon_i \Delta \varepsilon_k + \sum_j \left( \left. \frac{\partial \tilde{f}}{\partial h_i} \right|_{h_{10}, \ldots, h_{j0}} \Delta h_i \right)^2 + 2 \sum_j \sum_{k=1}^{j-1} \left. \frac{\partial \tilde{f}}{\partial h_i} \right|_{h_{10}, \ldots, h_{j0}} \left. \frac{\partial \tilde{f}}{\partial h_k} \right|_{h_{10}, \ldots, h_{j0}} \Delta h_i \Delta h_k + 2 \sum_l \sum_j \left. \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \left. \frac{\partial \tilde{f}}{\partial h_k} \right|_{h_{10}, \ldots, h_{j0}} \Delta \varepsilon_i \Delta h_k$$

(0.16)

Obviously, since the impact of hidden parameters can not be quantified, the uncertainty of the value of the measurand can not be determined either. However, a reasonable procedure to determine the uncertainty is to consider the variance of the measurand at constant known fundamental parameters. This is clear if we look on equation (0.9). For constant known fundamental parameters the value of the measurand is also constant. Hence, if any fluctuation (scatter) of the measurand is observed at constant known parameters one can readily conclude that this fluctuations must be caused by hidden parameters (Fig. 0.1).

Thus all values of the measurand must be transformed to the same set of known parameters (Fig. 0.3). This can be achieved by calculating a fit function of the measurand values. Thus the transformed measurand values, $\gamma_{ji}$, are given by,
\[
\gamma_i = f : y(\varepsilon_{1i}, \varepsilon_{2i}, ..., \varepsilon_{ni}) \mapsto y(\varepsilon_{1i} = A_1, \varepsilon_{2i} = A_2, ..., \varepsilon_{ni} = A_n)
\] (0.17)

where the parameters \(A_n\) are constants.

Nota bene, this fluctuation of the measurand value does not really exist. They are quasi existing due to the lack of full information of the system.

Figure 0.2: Scatter of the measurand. (a) At constant known parameter \(T\) the measurand exhibits a characteristic behavior dependent on the hidden parameter \(P\). (b) The measurand value apparently shows an stochastic behavior in the accessible projection plane \(TY\). Thus, the measurand values scatters due to lack of information of the system. As it is depicted, it is obvious that the scatter interval \((A,B,C)\) may depent on the known parameter \(T\).

Hence, the individual uncertainty of the measurand value becomes,

\[
u^2(y) = (dy)^2 = \underbrace{\sum_{i} \left( \left. \frac{\partial f}{\partial \varepsilon_i} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \right)^2 \Delta \varepsilon_i}_\text{TYPE B} + \underbrace{2 \sum_{i} \sum_{k=i+1}^{l} \left( \left. \frac{\partial f}{\partial \varepsilon_i} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \left. \frac{\partial f}{\partial \varepsilon_k} \right|_{\varepsilon_{10}, \ldots, \varepsilon_{l0}} \Delta \varepsilon_i \Delta \varepsilon_k \right)}_{\text{TYPE A}} + \text{Var} (\gamma)
\] (0.18)

where the first term on the right hand side of Eq. (0.18) determines the Type B contribution and the second term determines the Type A contribution to the overall uncertainty. Type B uncertainties are calculated by deduction from an given probability density function,
Figure 0.3: Transformation of the measurand values. (a) In order to calculate the variance, the measurand values has to be transformed according to the fundamental relationship \( Y = -2T^2 - 4T \cdot P_C + 1000 \) where \( P_C \) is equal to a given pressure value. (b) However, the parameter \( P \) is hidden. Thus in turn it is necessary to approximate the temperature characteristics of the measurand \( Y \). Thus the measurand values have to be transformed according to a fit function to a specific value of the known parameter \( T \) (usually the mean value) (dashed black line). Otherwise the variance would be overestimated due to an over sized scatter interval (A). The red bold line shows the fit function (in that case a linear fit was chosen). The red balls indicates the transformed measurand values. It is also evident, that a linear transformation (linear fit) is just a approximation. In fact, the black solid lines depicts the functional relationship between the measurand and the temperature at given pressures. It is clear, that this functional relationship would be the best fit function to transform every specific data point. But unfortunately, \( P \) is hidden, and one only observes the situation depicted on the right figure without any information of the true functional relationship between \( T \) and \( Y \).
$p_i(\varepsilon)$ by\textsuperscript{29},

$$
\Delta \varepsilon^2_i = \int_{-\infty}^{+\infty} d\varepsilon (\varepsilon - \langle \varepsilon \rangle)^2 p_i(\varepsilon) = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2
$$

(0.19)

whereas Type A contributions are calculated by induction via the variance which is given by,

$$
Var(\gamma) = \frac{\sum_i^N (\gamma_i - \gamma)^2}{N - 1}
$$

(0.20)

The mean value of the measurand is given by,

$$
\gamma = \frac{\sum_i^N \gamma_i}{N}
$$

(0.21)

It is important to emphasize that,

$$
Var(\gamma) \neq \sum_i^j \left( \frac{\partial \tilde{f}}{\partial h_i} \right)_{h_{1i},...,h_{j0}} \Delta h_i \left( \frac{\partial \tilde{f}}{\partial h_k} \right)_{h_{10},...,h_{j0}} \Delta h_k + 2 \sum_i^j \sum_{k=i+1}^j \left( \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right)_{\varepsilon_{1i},...,\varepsilon_{j0}} \Delta \varepsilon_i \left( \frac{\partial \tilde{f}}{\partial h_k} \right)_{h_{10},...,h_{j0}} \Delta h_k
$$

(0.22)

Depending on the magnitude of $\frac{\partial \tilde{f}}{\partial h_k} \Delta h_k$ and on the stability (variation) of the hidden parameters the variance could be,

$$
Var(\gamma) \geq \sum_i^j \left( \frac{\partial \tilde{f}}{\partial h_i} \right)_{h_{1i},...,h_{j0}} \Delta h_i \left( \frac{\partial \tilde{f}}{\partial h_k} \right)_{h_{10},...,h_{j0}} \Delta h_k + 2 \sum_i^j \sum_{k=i+1}^j \left( \frac{\partial \tilde{f}}{\partial \varepsilon_i} \right)_{\varepsilon_{1i},...,\varepsilon_{j0}} \Delta \varepsilon_i \left( \frac{\partial \tilde{f}}{\partial h_k} \right)_{h_{10},...,h_{j0}} \Delta h_k
$$

(0.23)
or,

\[
Var(\gamma) \leq \sum_{i}^{j} \left( \left. \frac{\partial f}{\partial h_i} \right|_{h_{10},...,h_{j0}} \Delta h_i \right)^2 + 2 \sum_{i}^{j-1} \sum_{k=i+1}^{j} \left. \frac{\partial f}{\partial h_i} \right|_{h_{10},...,h_{j0}} \left. \frac{\partial f}{\partial h_k} \right|_{h_{10},...,h_{j0}} \Delta h_i \Delta h_k \\
+ 2 \sum_{i}^{l} \sum_{k}^{j} \left. \frac{\partial f}{\partial \varepsilon_i} \right|_{\varepsilon_{10},...\varepsilon_{l0}} \left. \frac{\partial f}{\partial h_k} \right|_{h_{10},...,h_{j0}} \Delta \varepsilon_i \Delta h_k
\]

(0.24)

Generally, in most cases (see Fig. 0.2) due to lack of information (existence of hidden parameters) the total uncertainty for a single observation of the measurand value will be overrated by applying statistical methods (Fig. 0.4). It is just a “tool” to account for uncertainties related to hidden parameters.

Nota bene, for a non-linear relationship between the fundamental parameters and the measurand, \(y\), the Type B uncertainty according to Eq. (0.18) would give a wrong contribution to the overall uncertainty of the measurand. In such a case a suitable procedure is given by the Monte Carlo (MC) method to calculate the Type B uncertainty contribution. According to this method, one calculates the measurand several times where the fundamental parameters are picked from a probability density distribution, \(P_{\varepsilon_n}(\varepsilon_{n_i}, \Delta \varepsilon_{n_i})\) with expectation value, \(\varepsilon_{n_i}\), and variance \(\Delta \varepsilon_{n_i}\). It is given by,

\[
\hat{y}_{ij} = (\hat{\varepsilon}_{1ij}, \hat{\varepsilon}_{2ij}, ..., \hat{\varepsilon}_{n_{ij}})
\]

(0.25)

with,

\[
\hat{\varepsilon}_{n_{ij}} = P_{\varepsilon_n}(\varepsilon_{n_i}, \Delta \varepsilon_{n_i})_j
\]

(0.26)

The Type B uncertainty for an individual measurand value by applying the MC method would then be given by,
\[ u^2_B(y_i) = Var(\hat{y}_i) = \frac{\sum_j^M (\hat{y}_{ij} - \bar{y}_i)^2}{M - 1} \]  \hspace{1cm} (0.27)

with,

\[ \bar{y}_i = \frac{\sum_j^M \hat{y}_{ij}}{M} \]  \hspace{1cm} (0.28)

where \( M \) is the number of trials.

Figure 0.4: \textit{Uncertainty of a single observation.} (a) For a deterministic system (no hidden parameters) the uncertainty is given by Eq. (0.16). (b) Due to lack of information the uncertainty for a single measurand value may be overrated depending on the scatter interval (see Fig. (0.3b).

The mean value of the measurand and its uncertainty

According to the statements in sections (1), (2), and (3) it is inevitably clear that the mean value of the measurand has to be evaluated for a constant set of known fundamental parameters. Thus with Eq. (0.17) it is given by,

\[ \bar{y} = \frac{\sum_i^N y_i(\varepsilon_1 = A_1, \varepsilon_2 = A_2, \ldots, \varepsilon_n = A_n)}{N} = \frac{\sum_i^N \gamma_i}{N} = \bar{\gamma} \]  \hspace{1cm} (0.29)
In principle, the constants $A_n$ are arbitrary but it is reasonable to choose the mean values of the known parameters thus $A_n = \bar{\varepsilon}_n$. Hence, the mean value of the measurand becomes,

$$\bar{y} = \frac{\sum_i^N y_i(\varepsilon_1 = \bar{\varepsilon}_1, \varepsilon_2 = \bar{\varepsilon}_2, ..., \varepsilon_n = \bar{\varepsilon}_n)}{N} \quad (0.30)$$

Nota bene, if no hidden parameters would exist, each measurand $y_i$ for a constant set of known fundamental parameters would be given according to Eq. (0.6) by,

$$y_i = \tilde{f}(\varepsilon_1 = A_1, \varepsilon_2 = A_2, ..., \varepsilon_n = A_n) \quad (0.31)$$

With Eq. (0.30) this would lead to the fact, that,

$$\bar{y} = \tilde{f}(\varepsilon_1 = \bar{\varepsilon}_1, \varepsilon_2 = \bar{\varepsilon}_2, ..., \varepsilon_n = \bar{\varepsilon}_n) \quad (0.32)$$

However, in that specific case, the measurand is totally deterministic and the concept of a mean value and variance loses their meaning. Furthermore, bear in mind, that the constants $A, B, C, ...$ are totally arbitrary.

The uncertainty of the mean is given with Eq. (0.18) choosing $\varepsilon_{i_0} = \bar{\varepsilon}_1, \ldots, \varepsilon_{l_0} = \bar{\varepsilon}_l$ by,

$$u^2(\bar{y}) = (d\bar{y})^2 = \begin{cases} \text{TYPE B} \\ \sum_i^l \left( \frac{\partial \tilde{f}}{\partial \varepsilon_i} \bigg|_{\varepsilon_1 = \bar{\varepsilon}_1, \ldots, \varepsilon_l = \bar{\varepsilon}_l} \right) \Delta \varepsilon_i \right)^2 + 2 \sum_i^l \sum_{k=i+1}^{l-1} \frac{\partial \tilde{f}}{\partial \varepsilon_i} \bigg|_{\varepsilon_1 = \bar{\varepsilon}_1, \ldots, \varepsilon_l = \bar{\varepsilon}_l} \frac{\partial \tilde{f}}{\partial \varepsilon_k} \bigg|_{\varepsilon_1 = \bar{\varepsilon}_1, \ldots, \varepsilon_l = \bar{\varepsilon}_l} \Delta \varepsilon_i \Delta \varepsilon_k \\ + \frac{1}{N^2} \left[ \sum_{i=1}^N \text{Var}(\gamma_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N \text{COV}(\gamma_i, \gamma_j) \right] \end{cases} \quad (0.33)$$

where $\text{COV}(\gamma_i, \gamma_j)$ accounts the correlation between the transformed measurand value $\gamma_i$ and $\gamma_j$.

The correlation term can be written as,
\[ COV(\gamma_i, \gamma_j) = r_{\gamma_i, \gamma_j} \sqrt{Var(\gamma_i)} \sqrt{Var(\gamma_j)} \]  

(0.34)

where \( r_{\gamma_i, \gamma_j} \) is the correlation coefficient. Since the values \( \gamma_i \) and \( \gamma_j \) belongs to the same measurand, \( r_{\gamma_i, \gamma_j} \) is called auto correlation coefficient. Thus, the Type A contribution to the overall uncertainty of the mean of the measurand becomes,

\[ u^2_A(y) = \frac{1}{N} \left[ Var(\gamma) + \frac{2}{N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} r_{\gamma_i, \gamma_j} \sqrt{Var(\gamma_i)} \sqrt{Var(\gamma_j)} \right] \]  

(0.35)

Case 1: \( Var(\gamma_i) = Var(\gamma_j) = Var(\gamma), \ r_{\gamma_i, \gamma_j} = r \geq 0 \)

If we assume that all variances are equal as well as the auto correlation coefficient between two transformed measurand value \( \gamma_i \) and \( \gamma_j \), Eq. (0.35) becomes

\[ u^2_A(y) = Var(\gamma) \left( r + \frac{1 - r}{N} \right) \]  

(0.36)

It is evident, that for correlated system, the contribution of Type A uncertainties to the overall uncertainty of the mean becomes in the limit of \( N \to \infty \),

\[ \lim_{n \to \infty} u^2_A(y) = r Var(\gamma) \]  

(0.37)

In practice usually one encounters the fact, that the auto correlation of the data is not considered in the total uncertainty of the mean. Thus it is just often calculated by,

\[ u^2_A(y) = \frac{Var(\gamma)}{N} \]  

(0.38)

Hence the uncertainty of the mean vanishes in the limit of \( N \to \infty \). Equation (0.38) should be utilized with care because it can result in a strong underestimation of the uncertainty.

\[ \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} = \frac{1}{2} N (N - 1) \]
Case 2: $\text{Var } (\gamma_i) \neq \text{Var } (\gamma_j), \ r_{\gamma_i, \gamma_j} \neq r \geq 0$

Usually the correlation coefficient between two random variables $x, y$ is given by\[^4\],

$$r_{x,y} = \frac{\sum_i^N (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i^N (x_i - \bar{x})^2 \cdot \sum_i^N (y_i - \bar{y})^2}} \quad (0.39)$$

| $n$ | $x$ | $y$ | $n$ | $x$ |
|-----|-----|-----|-----|-----|
| 1   | ▲   | ▲   | 1   | ▲   |
| 2   | ▲   | ▲   | 2   | ▲   |
| 3   | ▲   | ▲   | 3   | ▲   |
| 4   | ▲   | ▲   | 4   | ▲   |
| 5   | ▲   | ▲   | 5   | ▲   |
| 6   | ▲   | ▲   | 6   | ▲   |
| 7   | ▲   | ▲   | 7   | ▲   |
| 8   | ▲   | ▲   | 8   | ▲   |
| 9   | ▲   | ▲   | 9   | ▲   |
| 10  | ▲   | ▲   | 10  | ▲   |

Figure 0.5: From correlation to auto correlation. Two separated random variables are merged to one data set.

Merging both variables (Fig. (0.5)) the correlation coefficient can be calculated as,

$$r_{x,y} = r_{x,x+5} = \frac{\sum_i^{N/2} (x_i - \bar{x}_{\leq 5})(x_{i+5} - \bar{x}_{>5})}{\sqrt{\sum_i^{N/2} (x_i - \bar{x}_{\leq 5})^2 \cdot \sum_i^{N/2} (y_i - \bar{x}_{>5})^2}} \quad (0.40)$$

with the mean values given by,

$$\bar{x}_{\leq 5} = \frac{1}{N/2} \sum_{i=1}^{N/2} x_i \quad \bar{x}_{>5} = \frac{1}{N/2} \sum_{i=6}^{N/2} x_i \quad (0.41)$$

Hence in general, the auto correlation coefficient is given by,

$$r_{\gamma_i, \gamma_j} = r_{\gamma_k, \gamma_{k+m}} = \frac{\sum_{i=0}^{N-(k+m)} (\gamma_{k+i} - \bar{\gamma}_k)(\gamma_{k+m+i} - \bar{\gamma}_{k+m})}{\sqrt{\sum_{i=0}^{N-(k+m)} (\gamma_{k+i} - \bar{\gamma}_k)^2 \cdot \sum_{i=0}^{N-(k+m)} (\gamma_{k+m+i} - \bar{\gamma}_{k+m})^2}} \quad (0.42)$$

with,
\[
Var(\gamma_k) = \sum_{i=0}^{N-(k+m)} (\gamma_{k+i} - \overline{\gamma_k})^2
\]

\[
Var(\gamma_{k+m}) = \sum_{i=0}^{N-(k+m)} (\gamma_{k+m+i} - \overline{\gamma_{k+m}})
\]

(0.43)

Figure 0.6: Auto correlation. (a) Auto correlation coefficient calculation with a small step size \(m\). (b) Large step size leads to no overlap zone between the data.

For example, Fig. shows the auto correlation coefficient of a data set with \(N = 150\).

Figure 0.7: Auto correlation coefficient for a specific data set. (a) Auto correlation coefficient. (b) Contour plot of the auto correlation coefficient.
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