Infinite families of congruences modulo 7 for Ramanujan’s general partition function

Nipen Saikia · Jubaraj Chetry

1 Introduction

In a letter to Hardy written from Fitzroy House late in 1918 [6, pp. 192–193], Ramanujan introduced the general partition function $p_r(n)$ which is defined for non-zero integer $r$ and non-negative integer $n$ by

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{(q^r; q)^\infty}, \quad |q| < 1,$$  \hfil (1.1)

Keywords Ramanujan’s general partition function · Partition congruence · $q$-identities

Mathematics Subject Classification 11P83 · 05A15 · 05A17
where here and throughout the paper
\[(a; q)_\infty = \prod_{k=1}^{\infty} (1 - aq^{k-1}).\]

For \( r = 1 \), \( p_1(n) \) is the usual unrestricted partition function which counts the number of unrestricted partitions of \( n \). Ramanujan published three papers [12–14] on the partition function \( p_1(n) \), where congruence properties are established. Ramanujan [6] further claimed that, if \( \lambda \) is a positive integer and \( w \) is a prime of the form \( 6\lambda - 1 \), then
\[p_{-(\lambda+4)}(n w - \frac{(w + 1)}{6}) \equiv 0 \pmod{w}. \quad (1.2)\]

On page 182 of [15], Ramanujan recorded the following congruence satisfied by an infinite set of primes: Let \( \delta \) denote any integer, \( n \) denote a non-negative integer, and \( w \) a prime of the form \( 6\lambda - 1 \). Then
\[p_{\delta w - 4}(n w - \frac{(w + 1)}{6}) \equiv 0 \pmod{w}. \]

After Ramanujan, Newman [9], Ramanathan [10,11], Atkin [2], Andrews [1], Gandhi [7], Kiming and Olsson [8] studied congruence properties of the partition function \( p_r(n) \). Baruah and Ojah [3] also proved some congruences for \( p_{-3}(n) \). Recently, Baruah and Sharma [4] proved some arithmetic identities and congruences for the general partition function \( p_r(n) \) for negative values of \( r \).

In sequel, in this paper we prove many new infinite families of congruences modulo 7 for the general partition function \( p_r(n) \) with negative \( r \) by using \( q \)-identities. In particular, we prove the following congruences:

**Theorem 1.1** For any non-negative integer \( \lambda \) and \( k = 3, 4, 6 \), we have
\[p_{-(7\lambda+1)}(7n + k) \equiv 0 \pmod{7}.\]

**Theorem 1.2** For any non-negative integers \( \lambda \) and \( k = 2, 4, 5, 6 \), we have
\[p_{-(7\lambda+3)}(7n + k) \equiv 0 \pmod{7}.\]

**Theorem 1.3** For any non-negative integer \( \lambda \), we have
\[p_{-(7\lambda+6)}(7n + 5) \equiv 0 \pmod{7}.\]

**Theorem 1.4** For any non-negative integer \( \mu \) and \( k = 1, 2, 3, 4, 5, 6 \), we have
\[p_{-(49\mu+1)}(49n + (7k + 2)) \equiv 0 \pmod{7}.\]

**Theorem 1.5** For any non-negative integer \( \mu \) and \( k = 1, 2, 3, 4, 5, 6 \), we have
\[p_{-(49\mu+2)}(49n + (7k + 4)) \equiv 0 \pmod{7}.\]

2 Preliminaries

From [5, p. 303, Entry 17(v)], we have
\[(q; q)_\infty = (q^{49}; q^{49})_\infty (A(q^7) - q B(q^7) - q^2 + q^5 C(q^7)). \quad (2.1)\]
where
\[ A(q^7) = \frac{f(-q^{14}, -q^{35})}{f(-q^7, -q^{42})}, \quad B(q^7) = \frac{f(-q^{21}, -q^{28})}{f(-q^{14}, -q^{35})}, \quad C(q^7) = \frac{f(-q^7, -q^{42})}{f(-q^{21}, -q^{28})}, \]
and
\[ f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \]

Squaring (2.1), we find that
\[
(q; q)^2_\infty = (q^{49}; q^{49})^2_\infty ((A(q^7)^2 - 2q^7 C(q^7)) - 2q A(q^7) B(q^7) + q^2 (B(q^7)^2 - 2A(q^7)))
+ q^3 (2B(q^7) + q^7 C(q^7)^2) + q^4 + 2q^5 A(q^7) C(q^7) - 2q^6 B(q^7) C(q^7))
\]
\[ = \frac{q^{49}}{q^{49}_\infty} = \frac{q^{49}}{q^{49}_\infty} \quad \text{(2.2)} \]

Again, from [5, p. 39, Entry 24(ii)] we note that
\[
(q; q)^3_\infty = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}. \quad \text{(2.3)}
\]

From (2.3), it follow that
\[
(q; q)^3_\infty = J_0(q^7) - 3q J_1(q^7) + 5q^3 J_3(q^7) - 7q^6 J_6(q^7)
\equiv J_0(q^7) - 3q J_1(q^7) + 5q^3 J_3(q^7) \quad \text{(mod 7)} \quad \text{(2.4)}
\]
and
\[
(q; q)^6_\infty = \frac{J_0(q^7)^2}{J_0(q^7)} + q J_0(q^7) J_1(q^7) + 2q^2 J_1(q^7)^2 + 3q^3 J_0(q^7) J_3(q^7)
+ 5q^4 J_1(q^7) J_3(q^7) + 4q^6 J_3(q^7)^2 \quad \text{(mod 7)},
\]
\[ \quad \text{(2.5)} \]
where \(J_0, J_1, J_3, \) and \(J_6\) are series with integral powers of \(q^7\).

We will also need the following congruence which follows from binomial theorem:
\[
(q; q)^7_\infty \equiv (q^7; q^7)_\infty \quad \text{(mod 7)}. \quad \text{(2.6)}
\]

### 3 Proofs of Theorems 1.1–1.5

In this section, all congruences are to the modulus 7.

**Proof of Theorem 1.1** Setting \(r = -(7\lambda + 1)\) in (1.1), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+1)}(n) q^n = (q; q)_{\infty}^{7\lambda+1} = (q; q)_{\infty}^{7\lambda}(q; q)_{\infty}. \quad \text{(3.1)}
\]

Using (2.6) in (3.1), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+1)}(n) q^n \equiv (q^7; q^7)^{\lambda}_\infty (q; q)_\infty. \quad \text{(3.2)}
\]
Employing (2.1) in (3.2), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+1)}(n)q^n \equiv (q^7; q^7)_{\infty} q^{49\mu+1} (q^4; q^7)_{\infty} (A(q^7) - qB(q^7) - q^2 + q^5C(q^7)).
\]
(3.3)

Extracting the terms involving \(q^{7n+k}\) for \(k = 3, 4\) in both sides of (3.3), we arrive at desired result.

**Proof of Theorem 1.2** Setting \(r = -(7\lambda + 3)\) in (1.1), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+3)}(n)q^n = (q; q)_{\infty}^{7\lambda+3} = (q; q)_{\infty}^{7\lambda}(q; q)^3. \tag{3.4}
\]

Employing (2.4) and (2.6) in (3.4), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+3)}(n)q^n \equiv (q^7; q^7)_{\infty} (J_0(q^7) - 3qJ_1(q^7) + 5q^3J_3(q^7)). \tag{3.5}
\]

Extracting the terms involving \(q^{7n+k}\) for \(k = 2, 4, 5, 6\) in both sides of (3.5), we arrive at the desired result.

**Proof of Theorem 1.3** Setting \(r = -(7\lambda + 6)\) in (1.1), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+6)}(n)q^n = (q; q)_{\infty}^{7\lambda+6} = (q; q)_{\infty}^{7\lambda}(q; q)^6. \tag{3.6}
\]

Employing (2.5) and (2.6) in (3.6), we obtain
\[
\sum_{n=0}^{\infty} p_{-(7\lambda+6)}(n)q^n \equiv (q^7; q^7)_{\infty} (J_0(q^7) + qJ_0(q^7)J_1(q^7) + 2q^2J_1(q^7)^2 \\
+ 3q^3J_0(q^7)J_2(q^7) + 5q^4J_1(q^7)J_3(q^7) + 4q^6J_3(q^7)^2). \tag{3.7}
\]

Extracting the terms involving \(q^{7n+5}\) in both sides of (3.7), we arrive at the desired result.

**Proof of Theorem 1.4** Setting \(r = -(49\mu + 1)\) in (1.1), we obtain
\[
\sum_{n=0}^{\infty} p_{-(49\mu+1)}(n)q^n = (q; q)_{\infty}^{49\mu+1} = (q; q)_{\infty}^{49\mu}(q; q)_{\infty}. \tag{3.8}
\]

Using (2.6) in (3.8), we obtain
\[
\sum_{n=0}^{\infty} p_{-(49\mu+1)}(n)q^n \equiv (q^{49}; q^{49})_{\infty}^\mu (q; q)_{\infty}. \tag{3.9}
\]

Employing (2.1) in (3.9), we obtain
\[
\sum_{n=0}^{\infty} p_{-(49\mu+1)}(7n + 2)q^{7n+2} \equiv (q^{49}; q^{49})_{\infty}^{\mu+1} (A(q^7) - qB(q^7) - q^2 + q^5C(q^7)). \tag{3.10}
\]
Extracting the terms involving $q^{5n+2}$ in both sides of (3.10), dividing by $q^2$, and replacing $q^7$ by $q$, we obtain

$$\sum_{n=0}^{\infty} p_{-(49\mu+1)}(7n+2)q^n \equiv (-1)(q^7; q^7)^{\mu+1}. \quad (3.11)$$

Extracting the terms involving $q^{7n+k}$ for $k = 1, 2, 3, 4, 5, 6$ in both sides of (3.11), we complete the proof.

Proof of Theorem 1.5 Setting $r = -(49\mu + 2)$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} p_{-(49\mu+2)}(n)q^n = (q; q)_{49\mu+2} = (q; q)_{49\mu}(q; q)^2. \quad (3.12)$$

Using (2.6) in (3.12), we obtain

$$\sum_{n=0}^{\infty} p_{-(49\mu+2)}(n)q^n \equiv (q^{49}; q^{49})_{\infty}^\mu(q; q)^2. \quad (3.13)$$

Employing (2.2) in (3.13), we obtain

$$\sum_{n=0}^{\infty} p_{-(49\mu+2)}(n)q^n \equiv (q^{49}; q^{49})_{\infty}^\mu((A(q^7)^2 - 2q^7C(q^7)) - 2qA(q^7)B(q^7)$$

$$+ q^2(B(q^7)^2 - 2A(q^7)) + q^3(2B(q^7) + q^7C(q^7)^2)$$

$$+ q^4 + 2q^5A(q^7)C(q^7) - 2q^6B(q^7)C(q^7)). \quad (3.14)$$

Extracting the terms involving $q^{7n+4}$ in both sides of (3.14), dividing by $q^4$, and replacing $q^7$ by $q$, we deduce that

$$\sum_{n=0}^{\infty} p_{-(49\mu+2)}(7n+4)q^n \equiv (q^7; q^7)^{\mu+2}. \quad (3.15)$$

Extracting the terms involving $q^{7n+k}$ for $k = 1, 2, 3, 4, 5, 6$ in both sides of (3.15), we arrive at the desired result.

Acknowledgements The authors are extremely grateful to the anonymous referee for his/her valuable suggestions and comments which have greatly improved the original version of the manuscript. The corresponding author (N. Saikia) thanks Council of Scientific and Industrial Research of India for partially supporting the research work under the Research Scheme No. 25(0241)/15/EMR-II (F. No. 25(5498)/15).

References

1. Andrews, G.E.: A survey of multipartitions: congruences and identities. Dev. Math. 17, 1–19 (2008)
2. Atkin, A.O.L.: Ramanujan congruences for $p_k(n)$. Can. J. Math. 20, 67–78 (1968)
3. Baruah, N.D., Ojah, K.K.: Some congruences deducible from Ramanujan’s cubic continued fraction. Int. J. Number Theory 7, 1331–1343 (2011)
4. Baruah, N.D., Sarmah, B.K.: Identities and congruences for the general partition and Ramanujan $\tau$ functions. Indian J. Pure Appl. Math. 44(5), 643–671 (2013)
5. Berndt, B.C.: Ramanujan’s Notebooks, Part III. Springer, New York (1991)
6. Berndt, B.C., Rankin, R.A.: Ramanujan: Letters and Commentary. American Mathematical Society, Providence (1995)
7. Gandhi, J.M.: Congruences for $p_r(n)$ and Ramanujan’s $\tau$ function. Am. Math. Mon. 70, 265–274 (1963)
8. Kiming, I., Olsson, J.B.: Congruences like Ramanujan’s for powers of the partition function. Arch. Math. (Basel) 59(4), 348–360 (1992)
9. Newman, M.: Congruence for the coefficients of modular forms and some new congruences for the partition function. Can. J. Math. 9, 549–552 (1957)
10. Ramanathan, K.G.: Identities and congruences of the Ramanujan type. Can. J. Math. 2, 168–178 (1950)
11. Ramanathan, K.G.: Ramanujan and the congruence properties of partitions. Proc. Indian Acad. Sci. (Math. Sci.) 89, 133–157 (1980)
12. Ramanujan, S.: Some properties of \( p(n) \), the number of partition of \( n \). Proc. Camb. Philos. Soc. 19, 207–210 (1919)
13. Ramanujan, S.: Congruence properties of partitions. Proc. Lond. Math. Soc. 18, xix (1920)
14. Ramanujan, S.: Congruence properties of partitions. Math. Z. 9, 147–153 (1921)
15. Ramanujan, S.: The Lost Notebook and Other Unpublished Papers. Narosa, New Delhi (1988)