Transmit Designs for the MIMO Broadcast Channel with Statistical CSI

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Abstract—We investigate the multiple-input multiple-output broadcast channel with statistical channel state information available at the transmitter. The so-called linear assignment operation is employed, and necessary conditions are derived for the optimal transmit design under general fading conditions. Based on this, we introduce an iterative algorithm to maximize the linear assignment weighted sum-rate by applying a gradient descent method. To reduce complexity, we derive an upper bound of the linear assignment achievable rate of each receiver, from which a simplified closed-form expression for a near-optimal linear assignment matrix is derived. This reveals an interesting construction analogous to that of dirty-paper coding. In light of this, a low complexity transmission scheme is provided. Numerical examples illustrate the significant performance of the proposed low complexity scheme.

Index Terms—Broadcast channel, multiple-input multiple-output, statistical CSI

I. INTRODUCTION

The multiple-input multiple-output (MIMO) broadcast channel (BC) with Gaussian noise has attracted tremendous research interest in recent years. Dirty paper coding (DPC) has been proved to achieve the capacity region of this channel [1], whereas various linear precoding techniques have also been developed to reduce complexity (see, e.g., [2–7]). Much current work dealing with the MIMO BC model assume that instantaneous channel state information (CSI) is available at both the transmitter and receivers, in order to fully capitalize on the spatial multiplexing advantages of the MIMO transmission. Whilst this assumption may be plausible for fixed or low mobility applications for which the channel realizations change slowly enough to be monitored at the transmitter (e.g., via a feedback link or by exploiting channel reciprocity), for other applications it becomes less feasible. In particular, as mobility increases, the channel fluctuations begin to vary more rapidly, and tracking these gains accurately at the transmitter becomes problematic.

For mobile applications, an alternative approach is to exploit statistical CSI at the transmitter [8–15]; a technique which has drawn much attention in MIMO system design recently [16]. Compared with instantaneous CSI, the statistical parameters typically vary over a much longer time window, and therefore can be monitored more easily at the transmitter. With statistical CSI at the transmitter, an important problem is to understand the information-theoretic limits of the MIMO BC model. In the special case of “more capable” channels, where the power of signals to each receiver can be ordered, the ergodic capacity region was analyzed in [17], developing upon earlier work [19,28]. For the fading single-input single-output (SISO) BC model, an achievable inner bound of the ergodic capacity region was proposed in [20]. This work was extended to the fading multiple-input single-output (MISO) BC model in [21], where the distributions of the fading coefficients were assumed isotropic and the ergodic capacity region was proved to collapse to that of the fading SISO BC model. Zhang et al. [22] examined an outage achievable rate region for the fading single-input multiple-output (SIMO) and MISO BC models. Very recently, simple linear precoding designs for some special MISO BC models were proposed in [23,24].

Despite significant advances as described above, for the fading MIMO BC model, the capacity region remains unknown. To simplify the problem, a so-called linear assignment operation was proposed in [25]. Also, a linear assignment capacity was defined in [25] which, to the best of our knowledge, is the most systematic result revealing the information-theoretic limits of the fading MIMO BC model so far. However, comprehensive and explicit transmit designs based on this linear assignment operation were not given in [25] and are still missing in general.

In this paper, starting with the definition of the linear assignment capacity in [25], we consider the transmit design...
problem aimed at optimizing the linear assignment weighted sum-rate (LAWSR) of the fading MIMO BC model with statistical CSI at the transmitter. Based on an exact expression of the LAWSR, we reveal two key elements that need to be properly designed: 1) The linear assignment matrices; 2) The precoding matrices of the receivers.

We make the following key contributions:

1) We establish necessary conditions for the optimal linear assignment matrices and precoding matrices. Accordingly, the joint design of these can be formulated as a multidimensional optimization problem, which is solved by an alternating optimization method with a gradient descent update.

2) For the linear assignment matrix of each receiver, we provide a heuristic design with reduced computational complexity. This is done by rewriting the expression for the linear assignment achievable rate (LAAR) of each receiver as a difference of two terms, and applying Jensen’s inequality to each. This operation results in similar “bounding errors” for each of the two terms, but we prove it still leads to a strict upper bound of LAAR, which turns out to be fairly tight.

3) The derived upper bound motivates the establishment of a near-optimal construction for the linear assignment matrix to maximize the LAAR. Based on this construction, a simplified closed-form expression for the linear assignment matrix via second-order statistics of the CSI is obtained. This expression resembles the DPC structure, where the interference signal power has no impact on determining the linear assignment matrix. Moreover, for the design based on the derived upper bound and the simplified linear assignment matrix, it is shown that an interference elimination effect analogous to that of DPC transmission with instantaneous CSI exists. In light of this, a low complexity algorithm without numerical averaging is proposed to design the precoding matrices.

4) We reveal that if all the channels of the receivers are independent and identically distributed (i.i.d.) fading, the time division multiple access (TDMA) transmission is optimal. Also, we reveal that if the transmitter has only one antenna, the opportunistic scheduling transmission based on the statistical CSI is optimal.

Numerical results are presented to examine the proposed transmission designs. These indicate that the proposed designs perform close to the no-interference upper bound of the MIMO BC with statistical CSI [25], and achieve significant performance gains compared to the TDMA transmission in various scenarios.

The reminder of the paper is organized as follows. Section II describes the fading MIMO BC model under consideration. In Section III, we establish necessary conditions of the optimal linear assignment matrices and precoding matrices for all the receivers, and propose an iterative algorithm to search for the optimal solution. In Section IV, we derive an upper bound of the LAAR of each receiver, based on which we investigate low complexity transmit designs. Numerical results are provided in Section V and the main results are summarized in Section VI. Main mathematical proofs have been placed in the Appendices.

The following notation is adopted throughout the paper: Vectors are represented as columns and are denoted in lower case bold-face, and matrices are represented in upper case bold-face. The superscripts $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ stand for the matrix transpose, conjugate and conjugate-transpose operations, respectively. We use $\det(\cdot)$ and $\tr(\cdot)$ to denote the matrix determinant and trace operations respectively, and $A^{-1}$ denote the inverse of matrix $A$. $0$ denotes the Hadamard product of two matrices. $A \succeq 0$ means that $A$ is Hermitian positive semi-definite, and $A \succ 0$ means that $A$ is Hermitian positive definite. $\|X\|_F$ denotes the Frobenius norm of matrix $X$. The $M \times M$ identity matrix is denoted by $I_M$, and the all-zero matrix is denoted by $0$. The complex number field is represented by $\mathbb{C}$, and $E[\cdot]$ evaluates the expectation of all the random variables within the bracket.

II. SYSTEM MODEL

Consider a fading MIMO BC scenario, where the transmitter has $N_t$ antennas, while each of the $L$ receivers has $N_r$ antennas. The received vector for the $l$-th receiver can be written as

$$y_l = H_l x_l + z_l$$  \hspace{1cm} (1)

where $x_l \in \mathbb{C}^{N_r \times 1}$ is the transmitted vector designed to meet the power constraint

$$E\left[\tr(x_l x_l^H)\right] \leq P.$$  \hspace{1cm} (2)

$z_l \in \mathbb{C}^{N_r \times 1}$ contains circularly symmetric complex Gaussian noise with zero-mean and covariance $E[z_l z_l^H] = N_0 I_{N_r}$, and $H_l \in \mathbb{C}^{N_r \times N_t}$ is a random channel matrix. The channels are assumed independent across $l$, i.e., for different receivers. Moreover, the $l$-th receiver is assumed to perfectly estimate its own channel matrix, $H_l$, whilst the transmitter only knows statistical CSI\(^2\) for each receiver.

Here, the transmitted signal $x_T$ in (1) is constructed as $x_T = \sum_{i=1}^{L} x_i$, where $x_i$ contains the transmitted signal destined for the $l$-th receiver with the covariance matrix $\Sigma_i = E[x_i x_i^H]$. Similar to [25], we assume that $\Sigma_i$ is nonsingular. Moreover, we follow [25] to assume that the transmitter generates the transmitted data for each user in ascending order. Thus, when designing the transmitted signal for the $l$-th receiver, the transmitter possesses the full non-causal knowledge of the transmitted codewords for the receivers $1, 2, \cdots, l-1$. To this end, we reexpress (1) as

$$y_l = H_l (x_l + s_l) + H_l \sum_{i=l+1}^{L} x_i + z_l$$  \hspace{1cm} (3)

where $s_l = \sum_{i=1}^{l-1} x_i$. In the absence of the instantaneous CSI at the transmitter, the pair $(y_l, H_l)$ constitutes the channel output. The channel transition probability $\Pr[y_l, H_l | x_l, s_l]$ of the fading MIMO BC model in (3) is a function of the

\(^1\)For the single antenna i.i.d. case, the TDMA transmission and the opportunistic scheduling transmission are equivalent.

\(^2\)It is noted that throughout this paper, we make no assumption on the distribution of $H_l$, beyond it having finite energy.
transmitted signal $x_t$, the channel matrix $H_t$, and the state of the non-causally known interference $s_t$, which is an instance of the general class of side-information channels [26, 27]. Hence the capacity for the $l$-th receiver in model (3) is defined as

$$C_l = \sup_{P_t[u_l|s_l], f_l} \left\{ I (u_l; y_l, H_l) - I (u_l; s_l) \right\}$$

(4)

where $u_l$ is an auxiliary random vector with the conditional distribution $P_t[u_l|s_l]$ and $f_l (\cdot)$ is a deterministic function which constructs the transmitted signal as $x_l = f_l(u_l, s_l)$ and satisfies $E \left[ f_l(u_l, s_l) f_l(u_l, s_l)^H \right] = \Sigma_l$. We note that for any particular choice of $P_t[u_l|s_l]$ and $f_l (\cdot)$, expression (4) becomes an achievable transmission rate.

The maximum achievable rate of the fading MIMO BC model in (4) remains an open problem. However, by generating $x_l = f_l(u_l, s_l) = u_l - F_l s_l$, where $F_l$ represents the $N_t \times N_t$ constant linear assignment matrix, we have an achievable transmission rate

$$R_{L,A,l} = \sup_{P_t[u_l|s_l]} \left\{ R_l \right\}$$

(5)

where

$$R_l = \sup_{P_t[u_l|s_l]} \left\{ \left\{ I (u_l; y_l, H_l) - I (u_l; s_l) \right\} \right\}$$

(6)

Given any fixed linear assignment matrix $F_l$, it was proved in [25, Theorem 1] that the maximum rate in (6) is achieved by choosing $x_l$ to be jointly Gaussian with $s_l$, which in turn determines the conditional distribution $P_t[u_l|s_l]$ as follows

$$P_t[u_l|s_l] = \frac{1}{\sqrt{\det (\Sigma_{u_l|s_l})}} \exp \left( - (u_l - J_l s_l)^H \Sigma_{u_l|s_l}^{-1} (u_l - J_l s_l) \right)$$

(7)

where

$$J_l = (F_l \Sigma_{s_l} + \Sigma_{x_l|s_l}) \Sigma_{s_l}^{-1}$$

(8)

and

$$\Sigma_{u_l|s_l} = (F_l \Sigma_{s_l} + \Sigma_{x_l|s_l}) \Sigma_{s_l}^{-1} (F_l \Sigma_{s_l} + \Sigma_{x_l|s_l})^H$$

(9)

in which $\Sigma_{x_l|s_l}$ and $\Sigma_{s_l}$ denotes the covariance matrix of $x_l$, the covariance matrix of $s_l$, and the cross-covariance matrix of $x_l$ and $s_l$, respectively. Then, the achievable transmission rate in (5) under the linear assignment operation $f_l (u_l, s_l) = u_l - F_l s_l$ and the corresponding conditional distribution in (7) is defined as the linear assignment capacity [25]. Furthermore, it was proved in [25, Theorem 2] that the linear assignment capacity region of model (1) is found by choosing the transmitted vector $x_l$ independent of the interference vector $s_l$, $l = 1, 2, \ldots, L$. In this case, we have $\Sigma_{x_l} = \Sigma_l$, $\Sigma_{s_l} = \Sigma_l$, $\Sigma_{l-1} = \Sigma_l$, and the covariance matrix of the transmitted signal $x_T$ can be expressed as $\Sigma_T = E [x_T x_T^H] = \sum_{l=1}^L \Sigma_l$. The transmitter is subject to an average power constraint $P$, which implies $\text{tr} (\Sigma_T) \leq P$. Also, the conditional distribution $P_t[u_l|s_l]$ in (7) reduces to

$$P_t[u_l|s_l] = \frac{1}{\sqrt{\det (\Sigma_{u_l|s_l})}} \exp \left( - (u_l - F_l s_l)^H \Sigma_{u_l|s_l}^{-1} (u_l - F_l s_l) \right)$$

(10)

Then, by adapting the LAAR expression in [25, Eq. (35)] to complex channels, the LAAR of the $l$-th receiver in (6) can be written as

$$R_l = \log_2 \det (\Sigma_{u_l|s_l}) - E \left[ \log_2 \det (\Sigma_{u_l|y_l|x_l}|H_l = H_l) \right]$$

(11)

where

$$\Sigma_{u_l|y_l|x_l}|H_l = (F_l \Sigma_{s_l} + \Sigma_{x_l|s_l} + \Sigma_{s_l}^H s_l + \Sigma_{x_l|s_l}^H x_l) H_l^H$$

(12)

in which $\Sigma_{x_l|s_l} = H_l \Sigma_{x_l|s_l} H_l^H + N_0 I_{N_x}$. Since $x_l$ and $s_l$ are independent $(\Sigma_{x_l|s_l} = 0)$, (9) and (12) can be simplified as follows

$$\Sigma_{u_l|s_l} = \Sigma_l$$

(13)

$$\Sigma_{u_l|y_l|x_l}|H_l = (F_l \Sigma_{s_l} + \Sigma_{x_l|s_l} + \Sigma_{s_l}^H s_l + \Sigma_{x_l|s_l}^H x_l) H_l^H$$

(14)

Here we present the encoding and decoding process to achieve the linear assignment achievable rate in (11).

Encoding process at the transmitter:

1. First select a transmitted signal $x_1$ for receiver 1.
2. Generate $e^{\mu_1 (u_2, y_2, H_2)}$ independent sequences.
3. Distribute these sequences into the $e^{\mu_2 R_2}$ bin codebook uniformly.
4. Given the non-causally known interference $s_2 = x_1$ and the message $W_2 = k$ for receiver 2, look for a joint typical pair [28] $(u_2, s_2)$ among the sequences in bin $k$.
5. The signal $x_2$ for receiver 2 is constructed using the linear assignment operation $x_2 = u_2 - F_2 s_2$.
6. The signal $x_l$ for receiver $l$ is constructed in a similar manner above. The signal $s_l = x_1 + x_2 + \cdots + x_{l-1}$ is regarded as non-causally known interference. This process continues to the $l$-th receiver.

Decoding process at the receiver $l$:

1. For the received signal $y_l$, look for a joint typical pair $(u_l, y_l, H_l)$ among the sequences in the codebook.
2. Declare an error when more than one joint typical pairs $(u_l, y_l, H_l)$ are found. Also, declare an error when no joint typical pair $(u_l, y_l, H_l)$ is found.
3. Set the estimate $\hat{W}_l$ equal to the index of the bin containing this sequence $u_l$.

With the LAAR in (11), the LAWSR of model (1) is given by

$$R_{\text{sum}} = \sum_{l=1}^L \mu_l R_l$$

(15)

where $R_l$ is evaluated in (11) and $\mu_l$, $l = 1, 2, \ldots, L$ are non-
negative weights satisfying \( \sum_{l=1}^{L} \mu_l = L \). The sequel resorts to develop a transmit design of \( F_l \) and \( \Sigma_l \), \( l = 1, 2, \ldots, L \), under the constraint \( \text{tr} \left( \sum_{l=1}^{L} \Sigma_l \right) \leq P \), by maximizing the LAWSR in (15).

III. TRANSMIT DESIGN FOR MAXIMIZING THE LAWSR

In this section, we investigate the transmit design to maximize the LAWSR in (15). We begin by establishing necessary conditions that the optimal \( F_l \) and \( \Sigma_l \) must satisfy. Then, an algorithm is developed to optimize \( F_l \) and \( \Sigma_l \) iteratively.

A. Necessary Conditions for the Optimal Design

From (15), we know that the objective function \( R_{\text{sum}}^w \) is a non-convex function of the matrices \( F_l \) and \( \Sigma_l \), \( l = 1, 2, \ldots, L \). Thus, we obtain a set of necessary conditions for the optimal linear assignment matrices and precoding matrices below.

**Theorem 1:** The optimal transmit design, which maximizes the LAWSR in (15), satisfies the following conditions:

\[
-\mu_l \log_2 e \mathbb{E}_{H_l} \left[ \Sigma_{u_l|y_l, H_l} (H_l)^{-1} (F_l - T_l(H_l)) \Sigma_{z,l} \right] = 0, \quad l = 1, 2, \ldots, L
\]

\[
(\Sigma_l) = \begin{bmatrix} F_l \end{bmatrix}, \quad l = 1, 2, \ldots, L
\]

\[
(\Sigma_l) = \begin{bmatrix} F_l \end{bmatrix}^{-1} \log_2 e \left[ \sum_{l=1}^{L} \mu_l \mathbb{E}_{H_l} \left[ G_{2,l}(H_l) \right] \right] + \frac{\mu_l \mathbb{E}_{H_l} \left[ G_{3,l}(H_l) \right]}{\sum_{l=1}^{L} \mu_l \mathbb{E}_{H_l} \left[ G_{3,l}(H_l) \right]}, \quad l = 1, 2, \ldots, L
\]

\[
\left( \begin{array}{c}
\begin{array}{c}
\mu_l \mathbb{E}_{H_l} \left[ G_{1,l}(H_l) \right] \\
\mu_l \mathbb{E}_{H_l} \left[ G_{2,l}(H_l) \right]
\end{array}
\end{array} \right)
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\mu_l \mathbb{E}_{H_l} \left[ G_{2,l}(H_l) \right]
\end{array}
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\]

\[
\sum_{l=1}^{L} \text{tr} \left( (\Sigma_l) \right) = P, \quad l = 1, 2, \ldots, L
\]

\[
\theta \left( \sum_{l=1}^{L} \text{tr} \left( \begin{bmatrix} F_l \end{bmatrix} \right) \right) = 0
\]

\[
\sum_{l=1}^{L} \text{tr} \left( \begin{bmatrix} F_l \end{bmatrix} \right) = P, \quad l = 1, 2, \ldots, L
\]

**Proof:** See Appendix A.

Equations (16)–(21) provide elementary conditions that characterize the optimal designs of \( F_l \) and \( \Sigma_l \), \( l = 1, 2, \ldots, L \). In general, finding closed-form expressions for the optimal designs from Theorem 1 is a difficult task, if not intractable. The problems are complex because of their non-convexity and highly involved representation. Indeed, the expectation operation \( \mathbb{E} \cdot \cdot \cdot \) in (16) and (18) requires averaging of all possible realizations of channel matrices \( H_l \). The inverse operation \( (\cdot)^{-1} \) results in an involved structure of \( F_l \) and \( \Sigma_l \). These pose serious challenges. Nevertheless, Theorem 1 provides gradient descent directions of the LAWSR in (15) with respect to \( F_l \) and \( \Sigma_l \), from which a numerical algorithm can be formulated to search for the optimal designs iteratively.

B. Iterative Algorithm for LAWSR Maximization

From (16) and (18), it can be seen that the optimal matrices \( F_l \) and \( \Sigma_l \) depend on each other, which leads to a multi-dimensional optimization problem. We resort to a prevalent approach in dealing with this type of problem, in terms of iteratively optimizing one variable at a time with others fixed. Within each iteration, we exploit the gradient descent update via the partial derivatives of the LAWSR in (15) with respect to \( F_l \) and \( \Sigma_l \). These partial derivatives \( \nabla_{F_l} R_{\text{sum}}^w \) and \( \nabla_{\Sigma_l} R_{\text{sum}}^w \) are specified by the left-hand terms of (16) and (18) in Theorem 1, respectively. Combining this search direction with the backtracking line search conditions [29], Algorithm 1 maximizes the LAWSR over \( F_l \) and \( \Sigma_l \).

**Algorithm 1:** Maximizing the LAWSR over \( F_l \) and \( \Sigma_l \).

1. Initialize \( F_l^{(1)} \) and \( \Sigma_l^{(1)} \), \( l = 1, 2, \ldots, L \). Set \( n = 1 \), the tolerance for the backtracking line search \( \varepsilon_1 > 0 \), the tolerance for stopping the algorithm \( \varepsilon_2 > 0 \), and the maximum iteration number \( N_{\text{max}} \). Select values for the backtracking line search parameter \( \beta \) with \( \beta \in (0, 1) \).
2. Compute \( R_{\text{sum}}^{w(n)} \) and \( \nabla_{F_l} R_{\text{sum}}^{w(n)} \) based on \( F_l^{(n)} \) and \( \Sigma_l^{(n)} \), \( l = 1, 2, \ldots, L \).
3. Initialize step size \( t = 1 \).
4. If \( t < \varepsilon_1 \), then go to step 8.
5. Compute \( F_l^{(n+1)} = F_l^{(n)} + t \nabla_{F_l} R_{\text{sum}}^{w(n)} \), \( l = 1, 2, \ldots, L \).
6. Evaluate \( R_{\text{sum}}^{w(n+1)} \), based on \( F_l^{(n+1)} \) and \( \Sigma_l^{(n)} \), \( l = 1, 2, \ldots, L \).
7. Set \( t : = \beta t \). If \( R_{\text{sum}}^{w(n+1)} < R_{\text{sum}}^{w(n)} \), go to step 4.
8. Compute \( \nabla_{\Sigma_l} R_{\text{sum}}^{w(n)} \), based on \( F_l^{(n+1)} \) and \( \Sigma_l^{(n)} \), \( l = 1, 2, \ldots, L \).
9. Initialize step size \( u = 1 \).
10. If \( u < \varepsilon_1 \), then go to step 15.
11. Compute \( \Sigma_l^{(n+1)} = \Sigma_l^{(n)} + u \nabla_{\Sigma_l} R_{\text{sum}}^{w(n)} \), \( l = 1, 2, \ldots, L \).
12. If \( \sum_{l=1}^{L} \text{tr} \left( (\Sigma_l) \right) > P \), update \( F_l^{(n+1)} := \begin{bmatrix} F_l^{(n+1)} \end{bmatrix} \), \( l = 1, 2, \ldots, L \).
13. Evaluate \( \nabla_{\Sigma_l} R_{\text{sum}}^{w(n+1)} \), based on \( F_l^{(n+1)} \) and \( \Sigma_l^{(n+1)} \), \( l = 1, 2, \ldots, L \).
14. Set \( u := \beta u \). If \( R_{\text{sum}}^{w(n+1)} < R_{\text{sum}}^{w(n)} \), go to step 10.
15) If $R_{\text{sum}}^{w,(n+1)} - R_{\text{sum}}^{w,(n)} > \varepsilon_2$ and $n < N_{\text{max}}$, set $n := n+1$, go to step 2; otherwise, stop the algorithm.

Since it is generally very difficult to obtain a closed-form expression for $R_l$ in (11), we employ Monte Carlo simulation to estimate the value of $R_{\text{sum}}^{w}$, and the gradients $\nabla_{\tilde{R}_l} R_{\text{sum}}^{w}$ and $\nabla_{\tilde{F}_l} R_{\text{sum}}^{w}$ in Algorithm 1. Such an approach has been also used in [30–33].

IV. LOW COMPLEXITY DESIGN BASED ON SECOND-ORDER STATISTICS OF THE CSI

It should be noted that the main drawback of Algorithm 1 is the exhaustive averaging in each iteration, which might cause long execution time. Therefore, in this section, we propose a low complexity transmit design over the fading MIMO BC in (1) based on second-order statistics of the CSI. Before addressing this, we first derive an upper bound on the achievable rate of the $l$-th receiver of the fading MIMO BC model, given in (11), can be upper bounded by

$$
R_l \leq R_{\text{upp},l} = \log_2 \det (\Sigma_l) - \log_2 \det (C_l)
$$

where $\Sigma_l = I + \sum_{t=1}^{L} \Sigma_t + N_0 I_{N_t}$.

**Proof:** See Appendix B.

Here, Jensen’s inequality, a common tool in MIMO capacity analysis and for deriving power allocation strategies [34–38], is applied to obtain the upper bound in (26). Specifically, we apply Jensen’s inequality to two terms in (116) in Appendix B and then subtract them. Since the “bounding errors” for both terms are similar, subtracting these will have a canceling effect which in turn will yield a fairly tight upper bound. This is confirmed in numerical results where the transmit design in the context of the derived upper bound performs close to the design via Algorithm 1 based on the exact LAAR in (11).

It should be noted that employing Jensen’s inequality to two subtraction terms simultaneously is a useful technique for the transmission design in fading channels, whilst having applicability to problems such as those relating to the MIMO BC, the MIMO interference channels, and massive MIMO systems [39–42]. To the best of our knowledge, Theorem 2 reveals for the first time that this type of subtraction results in a strict theoretical upper bound of the original problem, instead of simply an approximation.

B. A Low Complexity Design

We provide a closed-form design of the linear assignment matrix $\tilde{F}_l$ based on the upper bound in (26) as follows.

**Proposition 1:** For a random fading channel $H_l$ satisfying $E[H_l^H H_l] = R_{g,l}$, a closed-form solution for the linear assignment matrix $\tilde{F}_l$, which maximizes the upper bound in (26), is given by

$$
\tilde{F}_l = \Sigma_l \left( \Sigma_l + \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \right)^{-1}, \quad l = 1, 2, \cdots, L.
$$

The achievable rate of the $l$-th receiver $\tilde{R}_l$, under the linear assignment matrix design in (27), can be upper bounded by

$$
\tilde{R}_l \leq \tilde{R}_{\text{upp},l} = \log_2 \det \left( R_{g,l} \sum_{t=1}^{L} \Sigma_t + N_0 I_{N_t} \right)
$$

$$
- \log_2 \det \left( R_{g,l} \sum_{t=1}^{L} \Sigma_t + N_0 I_{N_t} \right)
$$

**Proof:** See Appendix C.

Interestingly, equation (27) demonstrates that the linear assignment matrix $\tilde{F}_l$ can be designed simply by exploiting the second-order statistics of the CSI, whereas the matrix structure is similar to that of DPC, designed via instantaneous CSI in [25]. Moreover, the structure of $\tilde{R}_{\text{upp},l}$ in (28) is similar to dirty-paper transmission rate [43, Eq. (2.18)] with instantaneous CSI, where the impact of non-causally known interference $\sum_{t=l-1}^{L} \Sigma_t$ in (11) does not exist. Henceforth, various highly efficient algorithms for the MIMO BC model with instantaneous CSI can be utilized to design the matrices $\tilde{F}_l$. To summarize, we present an algorithm with reduced computational complexity to design $\tilde{F}_l$ and $\tilde{\Sigma}_l$ as follows.

**Algorithm 2:** A low complexity transmit design over the fading MIMO BC.

1) Find $\tilde{\Sigma}_l$, $l = 1, 2, \cdots, L$, maximizing $\tilde{R}_{\text{upp},\text{sum}} = \sum_{l=1}^{L} \mu_l \tilde{R}_{\text{upp},l}$ by classical algorithms of the conventional MIMO BC model with instantaneous CSI such as in [43, 44].

2) Design $\tilde{F}_l$, $l = 1, 2, \cdots, L$, as in (27) via the obtained $\tilde{\Sigma}_l$.

C. Transmit Strategies in Two Special Cases

In the following, we discuss the transmit strategies in two special cases based on Proposition 1.

**Proposition 2:** For the special case when $N_i = 1$, a near-optimal power allocation strategy is given by

$$
P_l = \begin{cases} P, & l = \tilde{l} \\ 0, & \text{otherwise} \end{cases}
$$

$$
\tilde{l} = \arg \max_{l} r_l, \quad l = 1, 2, \cdots, L
$$

where $r_l = E[|h_{l,i}^H H_l|]$

**Proof:** See Appendix D.

It is noted that here we evaluate the exact achievable rate $R_l$ by substituting $\tilde{F}_l$ in (27) into (11).
Proposition 2 implies that the multiuser diversity gain is achieved by an opportunistic scheduling scheme. This is similar to results in [45], which applied for the case $N_l = 1$ and with instantaneous CSI at the transmitter.

Proposition 3: If all the receivers experience i.i.d. fading conditions with zero-mean and unit variance, a near-optimal transmit strategy is to perform time-sharing, where all the receivers are served one at a time in a round-robin fashion.

Proof: According to (28), the LAWSR $R_{\text{sum}}^w$ in i.i.d. fading channels is upper bounded by

$$R_{\text{sum}}^w \leq \log_2 \det (N_t \Sigma_T + N_0 I_{N_t})$$

where we have the power constraint $\operatorname{tr}(\Sigma_T) \leq P$. It is known from [46] that the maximal value of the right term of (30) is achieved by $\Sigma_T = P/N_t I_{N_t}$. Then, $R_{\text{sum}}^w$ is bounded by the single-user rate

$$R_{\text{sum}}^w \leq N_l \ln \left( \frac{N_t}{N_l} P + N_0 \right).$$

Note that Proposition 3 corresponds to the conclusion in [25, Appendix I], because in i.i.d. fading channels, no statistical CSI can be exploited by the transmitter.

V. NUMERICAL RESULTS

This section illustrates the benefits of the transmit designs and examines the efficacy of the proposed algorithms by several examples. In all these examples, we consider $L = 2$ and normalize the average power of the receivers’ channels as $E[\operatorname{tr}(H_i H_i^H)]=N_t N_l$, $l = 1, 2$. The signal-to-noise ratio (SNR) is given by $\text{SNR} = E[\operatorname{tr}(H_i H_i^H)]/P = \frac{\epsilon_1}{N_0}$. Throughout this section, we set $\epsilon_1 = \epsilon_2 = 0.5$. For Algorithm 1, we set $\mu_1 = \mu_2 = 1$. For Algorithm 1, one may only find a local optimum LAWSR. Meanwhile, the explicit expressions in Proposition 1 may provide us some intuitive instructions on the possible structure of the optimal transmission design. In order to exploit this point and avoid local convergence, we combine the initialization based on the design of Algorithm 2 and the multiple random initializations$^6$ together as the final initializations for Algorithm 1. Then, we choose the obtained designs that achieve the maximal LAWSR as the final solution.

A. Comparison of Computational Complexity

The computational complexity of Algorithm 1 is mainly due to the Monte-Carlo estimation of the expectation in $R_{\text{sum}}^w$, $\nabla_P R_{\text{sum}}^w$, and $\nabla_P R_{\text{sum}}^w$. For Algorithm 2, $F_1$ is designed by a closed-form expression in (27). Therefore, the main computational complexity of Algorithm 2 is the calculation of $R_{\text{upp, sum}}^w$ and $\nabla_P R_{\text{upp, sum}}^w$. Here we compare the computational complexity between evaluating $R_{\text{sum}}^w$, $\nabla_F R_{\text{sum}}^w$, and $\nabla_P R_{\text{sum}}^w$ in Algorithm 1 and evaluating $R_{\text{upp, sum}}^w$ and $\nabla_P R_{\text{upp, sum}}^w$ in Algorithm 2.

The accuracy of Monte Carlo estimation can be improved by increasing the number of channel realizations. However, this will also increase the computational complexity. Here we provide a method to find a reasonable number of channel realizations. We define the following sets: 1) the number of channel realizations set $[k, 2k, 3k, \cdots]$, 2) the estimated $R_{\text{sum}}^w$ set $[R_{\text{sum}}^w(1), R_{\text{sum}}^w(2), R_{\text{sum}}^w(3), \cdots]$, 3) the estimated $\nabla_F R_{\text{sum}}^w$ set $[\nabla_F R_{\text{sum}}^w(1), \nabla_F R_{\text{sum}}^w(2), \nabla_F R_{\text{sum}}^w(3), \cdots]$, 4) the estimated $\nabla_P R_{\text{sum}}^w$ set $[\nabla_P R_{\text{sum}}^w(1), \nabla_P R_{\text{sum}}^w(2), \nabla_P R_{\text{sum}}^w(3), \cdots]$. When $ik$ channel realizations are used, $R_{\text{sum}}^w$, $\nabla_F R_{\text{sum}}^w$, and $\nabla_P R_{\text{sum}}^w$ are the corresponding estimated $R_{\text{sum}}^w$, $\nabla_F R_{\text{sum}}^w$, and $\nabla_P R_{\text{sum}}^w$, respectively.

We define

$$i^* = \arg\min_i \left| \frac{R_{\text{sum}}^w(i+1) - R_{\text{sum}}^w(i)}{R_{\text{sum}}^w(i)} \right| < \alpha$$

for a given threshold $\alpha$. Then, $i^*k$ channel realizations are used to evaluate $R_{\text{sum}}^w$.

Similarly, we define

$$t^* = \arg\min_t \left| \left\| \nabla_F R_{\text{sum}}^w(t+1) \right\|_F - \left\| \nabla_F R_{\text{sum}}^w(t) \right\|_F \right| < \gamma$$

$$p^* = \arg\min_p \left| \left\| \nabla_P R_{\text{sum}}^w(p+1) \right\|_F - \left\| \nabla_P R_{\text{sum}}^w(p) \right\|_F \right| < \gamma$$

for a given threshold $\gamma$. Then, $t^*k$ and $p^*k$ channel realizations are used to evaluate $\nabla_F R_{\text{sum}}^w$ and $\nabla_P R_{\text{sum}}^w$, respectively.

Next, we compare the computational complexity of evaluating $R_{\text{sum}}^w$, $\nabla_F R_{\text{sum}}^w$, and $\nabla_P R_{\text{sum}}^w$ in Algorithm 1 and that of evaluating $R_{\text{upp, sum}}^w$ and $\nabla_P R_{\text{upp, sum}}^w$ in Algorithm 2. We set $k = 1000$, $\alpha = 0.01$, and $\gamma = 0.05$ in our simulation. The simulations are performed with Matlab on an Intel Core i7-3770 3.4GHz processor. The running time for evaluating $R_{\text{sum}}^w$, $\nabla_F R_{\text{sum}}^w$, $\nabla_P R_{\text{sum}}^w$ in Algorithm 1 and evaluating $R_{\text{upp, sum}}^w$ and $\nabla_P R_{\text{upp, sum}}^w$ in Algorithm 2 with different antennas numbers is shown in Table I–Table III. We observe from these tables that the computational effort for evaluating $R_{\text{upp, sum}}^w$ and $\nabla_P R_{\text{upp, sum}}^w$ in Algorithm 2 is several orders of magnitude less than that for evaluating $R_{\text{sum}}^w$, $\nabla_F R_{\text{sum}}^w$, and $\nabla_P R_{\text{sum}}^w$ in Algorithm 1.

**TABLE I: The running time for evaluating $R_{\text{sum}}^w$ in Algorithm 1 and $R_{\text{upp, sum}}^w$ in Algorithm 2**

| Case | $R_{\text{sum}}^w$ | $R_{\text{upp, sum}}^w$ |
|------|-------------------|-------------------------|
| $N_t = N_r = 2$ | 0.270 s | 0.001000006 s |
| $N_t = N_r = 4$ | 0.315 s | 0.00100038 s |
| $N_t = N_r = 6$ | 0.461 s | 0.00100039 s |
| $N_t = N_r = 8$ | 0.528 s | 0.00100045 s |

**TABLE II: The running time for evaluating $\nabla_F R_{\text{sum}}^w$ in Algorithm 1**

| Case | $\nabla_P R_{\text{sum}}^w$ | $F_1$ in (27) |
|------|-----------------|----------------|
| $N_t = N_r = 2$ | 0.225 s | $\times$ |
| $N_t = N_r = 4$ | 0.356 s | $\times$ |
| $N_t = N_r = 6$ | 0.426 s | $\times$ |
| $N_t = N_r = 8$ | 0.529 s | $\times$ |
TABLE III: The running time for evaluating $\nabla P_l R_{\text{sum}}$ in Algorithm 1 and $\nabla P_l \tilde{R}_{\text{sum, upper}}$ in Algorithm 2

| Case | $\nabla P_l R_{\text{sum}}$ | $\nabla P_l \tilde{R}_{\text{sum, upper}}$ |
|------|-----------------|-----------------|
| $N_t = N_r = 2$ | 1.250 s | 0.00199955 s |
| $N_t = N_r = 4$ | 2.050 s | 0.00200003 s |
| $N_t = N_r = 6$ | 2.402 s | 0.00200006 s |
| $N_t = N_r = 8$ | 2.924 s | 0.00200010 s |

B. Performance of the Transmission Design

First, we consider the doubly correlated MIMO channels, which can be modeled as

$$H_l = R_{r,l}^{1/2} H_w R_{t,l}^{1/2}, \quad l = 1, 2$$  \(35\)

where $H_w$ is a complex random matrix with independent random entries following $CN(0, 1)$. The matrices $R_{r,l}$ and $R_{t,l}$ denote the receive and transmit correlation matrices of the $l$-th receiver respectively.

Here we assume the receiver correlation matrices are given by

$$R_{r,1} = \begin{bmatrix} 1 & -0.1 - 0.05j \\ -0.1 + 0.05j & 1 \end{bmatrix},$$  \(36\)

$$R_{r,2} = \begin{bmatrix} 1 & -0.05 + 0.1j \\ -0.05 + 0.1j & 1 \end{bmatrix}$$  \(37\)

and compare two examples with different transmit correlation matrices as follows:

1) Example 1:

$$R_{t,1} = \begin{bmatrix} 1 & 0.85 + 0.13j \\ 0.85 - 0.13j & 1 \end{bmatrix},$$  \(38\)

$$R_{t,2} = \begin{bmatrix} 1 & -0.8 + 0.11j \\ -0.8 + 0.11j & 1 \end{bmatrix}$$  \(39\)

2) Example 2:

$$R_{t,1} = \begin{bmatrix} 1 & 0.95 + 0.12j \\ 0.95 - 0.12j & 1 \end{bmatrix},$$  \(40\)

$$R_{t,2} = \begin{bmatrix} 1 & -0.9 + 0.09j \\ -0.9 + 0.09j & 1 \end{bmatrix}$$  \(41\)

Figure 1 and Figure 2 plot the sum-rate performance of Example 1 and Example 2 achieved by different transmit designs respectively. For Algorithm 2, we compute the exact LAWSR in (15) with the obtained $F_l$ and $\Sigma_l$. Then, we plot the exact LAWSR. For comparison purpose, we plot the sum rate performances achieved by the “TDMA” case. Also, to evaluate our proposed design, we propose a no-interference upper bound of the MIMO BC with statistical CSI as follows [25]:

$$\tilde{R}_{\text{upper}} = \sum_{l=1}^{L} \mu_l E \log_2 \det (I_{N_r} + \sum_{k=l+1}^{L} \Sigma_k H_k H_k^H)^{-1} \left[ N_0 I_{N_r} + \sum_{k=l+1}^{L} \Sigma_k H_k H_k^H \right].$$  \(42\)

The upper bound $\tilde{R}_{\text{upper}}$ is denoted as “Upper Bound” in the figures. It is noted that for the MIMO BC with statistical CSI, the upper bound $\tilde{R}_{\text{upper}}$ coincides with the WSR of the no-interference channel, which is clearly the best we can expect. Nevertheless, whether this bound is achievable is still unknown.

From Figure 1 and Figure 2, we can make several observations.

1) The transmit designs in Algorithm 1 and Algorithm 2 have better sum-rate performance than other design methods throughout the entire SNR region.

2) The curves for maximizing the upper bound via Algorithm 2 and maximizing the exact LAWSR directly via Algorithm 1 are virtually the same, but the complexity is different. The method for maximizing the exact LAWSR requires, at each iteration, numerically averaging certain random matrices involving the inverse of instantaneous realizations of the MIMO channels. Thus, the computing effort of Algorithm 2, which does not require such numerical averaging, is significantly less than that of Algorithm 1.

3) The proposed designs offer appreciable gains in sum-rate performance when compared against the “TDMA” design. Specifically, at the sum-rate 10 b/s/Hz, the SNR gains of the proposed precoding designs over the “TDMA” design are approximately 4.5 dB and 7 dB for Example 1 and Example 2 respectively.

4) The proposed designs perform close to the upper bound $\tilde{R}_{\text{upper}}$ and approach the bound as the transmit correlation increases.

The matrices $F_l$ and $P_l$ obtained by Algorithm 1 and Algorithm 2 at SNR = 0 dB are given by

1) Example 1:

- Algorithm 1

$$F_1 = 0$$  \(43\)

$$F_2 = \begin{bmatrix} 0.3240 + 0.0018j & -0.3206 - 0.0463j \\ -0.3200 + 0.0462j & 0.3232 - 0.0018j \end{bmatrix}$$  \(44\)

$$P_1 = \begin{bmatrix} -0.2712 + 0.3459j & -0.2300 - 0.0272j \\ -0.2215 + 0.3845j & 0.2242 + 0.0590j \end{bmatrix}$$  \(45\)

$$P_2 = \begin{bmatrix} -0.3051 + 0.2603j & -0.1300 - 0.2710j \\ -0.3051 + 0.2603j & 0.1687 + 0.2480j \end{bmatrix}$$  \(46\)

- Algorithm 2

$$F_1 = 0$$  \(47\)

$$F_2 = \begin{bmatrix} 0.3240 + 0.0018j & -0.3206 - 0.0463j \\ -0.3200 + 0.0462j & 0.3232 - 0.0018j \end{bmatrix}$$  \(48\)

$$P_1 = \begin{bmatrix} -0.2657 + 0.3435j & -0.2280 - 0.0289j \\ -0.2244 + 0.3838j & 0.2241 + 0.0570j \end{bmatrix}$$  \(49\)

$$P_2 = \begin{bmatrix} -0.3422 - 0.2143j & -0.1301 - 0.2727j \\ -0.3067 + 0.2613j & 0.1699 + 0.2488j \end{bmatrix}$$  \(50\)

2) Example 2:
\[ \mathbf{F}_1 = 0 \]
\[ \mathbf{F}_2 = \begin{bmatrix} 0.1779 + 0.0023j & -0.1879 + 0.0347j \\ -0.3916 - 0.0373j & 0.3853 + 0.0162j \end{bmatrix} \] (51)
\[ \mathbf{P}_1 = \begin{bmatrix} -0.2712 + 0.3459j & 0.2300 - 0.0272j \\ -0.2215 + 0.3845j & 0.2242 - 0.0590j \end{bmatrix} \] (52)
\[ \mathbf{P}_2 = \begin{bmatrix} 0.3406 - 0.2130j & -0.1300 - 0.2710j \\ -0.3051 + 0.2603j & 0.1687 + 0.2480j \end{bmatrix} \] (53)

- Algorithm 2
\[ \mathbf{F}_1 = 0 \]
\[ \mathbf{F}_2 = \begin{bmatrix} 0.3240 + 0.0018j & -0.3206 - 0.0463j \\ -0.3200 + 0.0462j & 0.3232 - 0.0018j \end{bmatrix} \] (55)
\[ \mathbf{P}_1 = \begin{bmatrix} -0.2657 + 0.3435j & 0.2280 - 0.0289j \\ -0.2244 + 0.3838j & 0.2241 - 0.0570j \end{bmatrix} \] (56)
\[ \mathbf{P}_2 = \begin{bmatrix} 0.3422 - 0.2143j & -0.1301 - 0.2727j \\ -0.3067 + 0.2613j & 0.1699 + 0.2488j \end{bmatrix} \] (57)
\[ \mathbf{P}_2 = \begin{bmatrix} 0.3406 - 0.2130j & -0.1300 - 0.2710j \\ -0.3051 + 0.2603j & 0.1687 + 0.2480j \end{bmatrix} \] (58)

It should be noted that \( \mathbf{F}_l \) and \( \mathbf{P}_l \) obtained by Algorithm 1 and Algorithm 2 maximize the exact LAWSR \( R_{\text{sum}}^{\text{upp}} \) and the upper bound \( R_{\text{sum}}^{\text{upp}, \text{sum}} \), respectively. Although \( R_{\text{sum}}^{\text{upp}, \text{sum}} \) is close to \( R_{\text{sum}}^{\text{upp}} \) due to the canceling effect of the bounding error, they are not the same. As a result, the matrices \( \mathbf{F}_l \) and \( \mathbf{P}_l \) obtained by Algorithm 1 and Algorithm 2 are close, but not always the same.

Figure 2: Sum rates in the fading MIMO BC for different transmit designs and Example 2.

Figure 3 and Figure 4 compare the sum-rate performance given different transmit designs. These results show similar observations as with Example 1 and Example 2. We find that the transmit design based on the upper bound in Proposition 1 performs nearly identically to the design based on the exact result. Moreover, the proposed designs outperform the “TDMA” design throughout the entire SNR region. To achieve a target sum-rate of \( 15 \) b/s/Hz, the SNR gains of the proposed designs over the “TDMA” design are almost \( 5.2 \) dB and \( 7.5 \) dB for Example 3 and Example 4 respectively. Also, the performance of the proposed designs is close to the upper bound \( \mathcal{R}_{\text{upper}} \).

To further validate the proposed designs, we provide Example 3 and Example 4 as follows. We assume the receiver correlation matrices in both examples are given in (59), (60) at the top of the next page. Transmit correlation matrices of the two examples are given in (61), (62) and (63), (64) at the top of the next page.
implemented in practice. However, as indicated above, in each iteration for these SNR levels. The fast convergence behavior of Algorithm 1 implies it can potentially be used for other precoding designs in practical systems. To reduce the complexity, we further propose Algorithm 2 which does not require numerical averaging.

In practical scenarios, the channels often include line-of-sight (LOS) paths. To verify the LOS impact on the accuracy of the upper bound, we consider the Rician fading channel and other transmit designs and Example 4.

\[
\mathbf{R}_r, 1 = \begin{bmatrix}
1 & -0.12 - 0.18j & 0.08 + 0.05j & -0.02 - 0.13j \\
-0.12 + 0.18j & 1 & -0.17 - 0.16j & 0.11 + 0.04j \\
0.08 - 0.05j & -0.17 + 0.16j & 1 & -0.17 - 0.16j \\
-0.02 + 0.13j & 0.11 - 0.04j & -0.17 + 0.16j & 1 \\
\end{bmatrix}
\]

\[
\mathbf{R}_r, 2 = \begin{bmatrix}
1 & -0.11 + 0.15j & 0.07 - 0.04j & -0.01 - 0.10j \\
-0.11 - 0.15j & 1 & 0.10 + 0.10j & 0.05 - 0.02j \\
0.07 - 0.04j & 0.10 - 0.10j & 1 & -0.10 - 0.20j \\
-0.01 + 0.10j & 0.05 + 0.02j & -0.10 + 0.20j & 1 \\
\end{bmatrix}
\]

\[
\mathbf{R}_t, 1 = \begin{bmatrix}
1 & 0.61 + 0.34j & 0.28 & 0.61 - 0.34j \\
0.61 - 0.34j & 1 & 0.61 + 0.34j & 0.28 \\
0.28 & 0.61 - 0.34j & 1 & 0.61 + 0.34j \\
0.61 + 0.34j & 0.28 & 0.61 - 0.34j & 1 \\
\end{bmatrix}
\]

\[
\mathbf{R}_t, 2 = \begin{bmatrix}
1 & -0.24 - 0.71j & -0.48 & -0.24 + 0.71j \\
-0.24 + 0.71j & 1 & -0.24 - 0.71j & -0.48 \\
-0.48 & -0.24 + 0.71j & 1 & -0.24 - 0.71j \\
-0.24 - 0.71j & -0.48 & -0.24 + 0.71j & 1 \\
\end{bmatrix}
\]

\[
\mathbf{R}_t, 1 = \begin{bmatrix}
1 & 0.94 + 0.01j & 0.93 & 0.94 - 0.01j \\
0.94 - 0.01j & 1 & 0.94 + 0.01j & 0.93 \\
0.93 & 0.94 - 0.01j & 1 & 0.94 + 0.01j \\
0.94 + 0.01j & 0.93 & 0.94 - 0.01j & 1 \\
\end{bmatrix}
\]

\[
\mathbf{R}_t, 2 = \begin{bmatrix}
1 & 0.00 - 0.92j & -0.92 & 0.00 + 0.92j \\
0.00 + 0.92j & 1 & 0.00 - 0.92j & -0.92 \\
-0.92 & 0.00 + 0.92j & 1 & 0.00 - 0.92j \\
0.00 - 0.92j & -0.92 & 0.00 + 0.92j & 1 \\
\end{bmatrix}
\]

Fig. 4: Sum rates in the fading MIMO BC for different transmit designs and Example 4.

Fig. 5: Convergence of Algorithm 1 for Example 2 at different SNR levels.
model
\[ H_l = \sqrt{\frac{K}{K+1}} \mathbf{H}_t + \sqrt{\frac{1}{K+1}} R_{r,l}^{1/2} \mathbf{R}_t^{1/2}, \quad l = 1, 2 \] (65)
where matrix \( \mathbf{H}_t \) is a deterministic matrix, satisfying \( \text{tr}(\mathbf{H}_t^{H}) = N_r N_t \), and \( K \geq 0 \) is the Rician \( K \)-factor. We assume
\[ \mathbf{H}_1 = \begin{bmatrix} 0.5898 & 1.1795 \\ 0.2949 & 1.4744 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 0.3849 & 1.1547 \\ 0.3849 & 1.5396 \end{bmatrix}. \] (66)
\( R_{r,l} \) and \( R_{t,l} \) are chosen as in Example 1. Figure 7 shows the sum rate performance of Algorithm\(^8\) 2 in the context of different \( K \)-factors and SNRs. The sum rates achieved by the iterative water-filling algorithm [43] with instantaneous channel matrices, where \( H_l = \mathbf{H}_t \), \( l = 1, 2 \), are also plotted as benchmarks. We observe from Figure 7 that the sum rate performance of Algorithm 2 improves and approaches the sum rates obtained with instantaneous channel matrices as \( K \) increases. This is expected because if the channels of all receivers converge to a constant, (28) indicates that the design based on Theorem 2 tends to be optimal and Algorithm 2 becomes equivalent to the DPC design in [43].

VI. CONCLUSION

This paper has considered the transmit design over the fading MIMO BC with statistical CSI. To address this problem, a linear assignment operation was implemented. We first determined a set of necessary conditions for the optimal transmit design, from which an iterative gradient descent algorithm was developed to maximize the LAWSR but with a high computational complexity. Thus, we employed Jensen’s inequality into two substraction terms of the LAAR for each receiver to average the random component and proved that this results in a strict upper bound of the LAAR. In light of this, a concise closed-form expression of the linear assignment matrix was derived for each receiver in terms of second-order statistics of the CSI. The derived expression captures some well known construction properties of the DPC design in the BC model with instantaneous CSI at the transmitter and has achieved a similar interference mitigation effect as DPC. This immediately permits the application of classical maximum weighted sum-rate algorithms in the MIMO BC model to design the precoding matrices. Then, we formulated a low-complexity transmission scheme via second-order statistics of the CSI based on the obtained precoding matrices and the closed-form linear assignment matrices. Moreover, transmit strategies in two special fading channel models were discussed. Finally, we provided concrete simulation results to illustrate the substantial gains achieved by the proposed designs.

APPENDIX A

PROOF OF THE THEOREM 1

We first rewrite \( \Sigma_l = \mathbf{P}_l \mathbf{P}_l^H \). Let \( \theta \) be the Lagrange multiplier associated with the inequality constraint \( \sum_{l=1}^{L} \text{tr}(\mathbf{P}_l \mathbf{P}_l^H) \leq P \). Then, we have the cost function for the optimal design as
\[ L(\mathbf{F}, \mathbf{P}, \lambda) = -\mathcal{P}_\text{sum} + \theta \left( \sum_{l=1}^{L} \text{tr}(\mathbf{P}_l \mathbf{P}_l^H) - P \right). \] (67)
By employing similar approaches as those in [50,51], we define the complex gradient operator as \( \nabla_{\mathbf{W}} f = \frac{\partial f}{\partial \mathbf{W}} \). The \((i,j)^{th}\) element of the matrix \( \mathbf{W} \) with the complex gradient operator is defined as \( \{ \nabla_{\mathbf{W}} f \}_{i,j} = \nabla_{(\mathbf{W})_{i,j}} f = \frac{\partial f}{\partial (\mathbf{W})_{i,j}} \). To this end, the KKT conditions satisfied by the optimal \( \mathbf{P}_l, \mathbf{P}_1, \ldots, L, \) and \( \theta \) can be expressed as [29, Eq. (5.49)]
\[ \nabla_{\mathbf{F}} L(\mathbf{F}, \mathbf{P}, \theta) = 0, \quad l = 1, 2, \ldots, L \] (68)
\[ \nabla_{\mathbf{P}} L(\mathbf{F}, \mathbf{P}, \theta) = 0, \quad t = 1, 2, \ldots, L \] (69)
\[ \theta \left( \sum_{l=1}^{L} \text{tr}(\mathbf{P}_l \mathbf{P}_l^H) - P \right) = 0 \] (70)
\[
\sum_{l=1}^{L} \text{tr} \left( P_l P_l^H \right) - P \leq 0
\] 
\[
\theta \geq 0.
\]

(71) \hspace{1cm} (72)

Next, we consider the calculation of \( \nabla F, L (F, P, \theta) \). We first express the \((m, n)\)th element of the matrix \( C_l \) as
\[
\{ C_l \}_{m, n} = e_m^H \left( F \Sigma_S F^H + \Sigma_l \right) e_n
\]
\[
= \text{tr} \left( (F \Sigma_S F^H + \Sigma_l) e_m e_n^H \right)
\]
\[
= \text{tr} \left( F \Sigma_S F^H e_m e_n^H + \Sigma_l e_m e_n^H \right)
\]
\[
= \{ \Sigma_l \} e_m e_n^H + \text{tr} \left( F \Sigma_S \right) e_m e_n^H
\]
(73)

where \( e_m \) is a unit-vector with one at the \( m \)th element and zeros elsewhere. According to the complex matrix differentiation results [52, Table 4.3], we have
\[
\frac{\partial \{ C_l \}_{m, n}}{\partial \{ F \}_{i, j}} = e_i^H e_m e_n^H F \Sigma_S \Sigma_{l, i} \] 
\[
\frac{\partial \{ C_l \}_{m, n}}{\partial \{ F \}_{i, j}} = e_i^H e_m e_n^H F \Sigma_S \Sigma_{l, i} \]
\[
= e_i^H e_m e_n^H F \Sigma_S \Sigma_{l, i} \]
\[
= e_i^H e_m e_n^H F \Sigma_S \Sigma_{l, i} \]
\[
= e_i^H e_m e_n^H F \Sigma_S \Sigma_{l, i} \]
(74) \hspace{1cm} (75) \hspace{1cm} (76)

Following similar steps as those in (74)–(76), we can obtain
\[
\frac{\partial \{ T_l \} (H_l)}{\partial \{ F \}_{i, j}} = T_l \{ H_l \} \Sigma_S \Sigma_{l, i} e_j e_i^H.
\]
(77)

Based on the expression for the LAAR in (11), we know that \( F_l \) has no relation to \( R_k \) for \( l \neq k \). As a result, recalling the expression for \( L (F, P, \theta) \) in (67) and the definition \( R_{\text{sum}} = \sum_{l=1}^{L} \mu_l R_l \), we have \( \nabla F, L (F, P, \theta) = \mu \nabla F, R_l \). By exploiting the expression for \( R_l \) in (11) and the matrix derivative rule [53, Eq. (38)], it yields (78) and (79) at the top of the next page. Then, we have
\[
\nabla F, L (F, P, \theta) = \mu \nabla F, R_l = -\mu_2 \log_2 e E_{H_l} \left[ \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} (F_l - T_l (H_l)) \Sigma_S \Sigma_l \right) \right]
\]
(80)

Next, we evaluate \( \nabla F, L (F, P, \theta) \). By employing the matrix derivative rule in [53, Eq. (33)] and following similar steps as those in (74)–(80), we have

1) \( t < l \)
\[
\{ \nabla P, R_l \}_{i, j} = -\log_2 e E \left[ \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} F_l P_l e_j e_i^H \right) \right]
\]
\[
- \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} P_l e_j e_i^H T_l^H (H_l) \right)
\]
\[
+ \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l^H (H_l) H_l P_l e_j e_i^H T_l (H_l) \right)
\]
\[
- \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) H_l P_l e_j e_i^H T_l^H (H_l) \right)
\]
\[
\{ \nabla P, R_l \}_{i, j} = -\log_2 e E \left[ F_l^H \Sigma_{u_i, y_l, H_l} (H_l)^{-1} F_l P_l \right]
\]
\[
- T_l^H (H_l) \Sigma_{u_i, y_l, H_l} (H_l)^{-1} F_l P_l + T_l^H (H_l) \Sigma_{u_i, y_l, H_l} (H_l)^{-1} \right]
\]
\[
\times \left( T_l (H_l) P_l - F_l^H \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) P_l \right)
\]
(81) \hspace{1cm} (82)

2) \( t = l \)
\[
\{ \nabla P, R_l \}_{i, j} = \log_2 e \text{tr} \left( \left( P_l^H \right)^{-1} e_j e_i^H \right)
\]
\[
- \log_2 e E \left[ \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} P_l e_j e_i^H \right) \right]
\]
\[
- \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} P_l e_j e_i^H T_l^H (H_l) \right)
\]
\[
+ \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) P_l e_j e_i^H T_l^H (H_l) \right)
\]
\[
- \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) P_l e_j e_i^H T_l^H (H_l) \right)
\]
\[
\{ \nabla P, R_l \}_{i, j} = \log_2 e \left( P_l^H \right)^{-1} - \log_2 e E \left[ \Sigma_{u_i, y_l, H_l} (H_l)^{-1} P_l \right]
\]
\[
- T_l^H (H_l) \Sigma_{u_i, y_l, H_l} (H_l)^{-1} P_l + T_l^H (H_l) \Sigma_{u_i, y_l, H_l} (H_l)^{-1} \right]
\]
\[
T_l (H_l) P_l - \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) P_l \right)
\]
(83) \hspace{1cm} (84)

3) \( t > l \)
\[
\{ \nabla P, R_l \}_{i, j} = -\log_2 e E \left[ \text{tr} \left( \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) P_l \right) \right]
\]
\[
\times e_j e_i^H T_l^H (H_l) \right)
\]
\[
\{ \nabla P, R_l \}_{i, j} = -\log_2 e E \left[ T_l^H (H_l) \Sigma_{u_i, y_l, H_l} (H_l)^{-1} T_l (H_l) \right]
\]
(85) \hspace{1cm} (86)

Based on (15) and \( \nabla P, L (F, P, \theta) \) can be expressed as
\[
\nabla P, L (F, P, \theta) = - \sum_{l=1}^{L} \mu_l \nabla P, R_l + \mu \nabla P, R_l
\]
\[
= + \sum_{l=1}^{L} \mu_l \nabla P, R_l + \theta P_l
\]
(87)

Finally, the theorem can be proved by combining (68)–(72), (80), (82), (84), (86), and (87), along with some simplifications.

**APPENDIX B**

**PROOF OF THE THEOREM 2**

Before we present the proof, we find the following two lemmas useful.

**Lemma 1:** Assume \( X > 0 \) is a \( N \times N \) random matrix with \( E[X] = M_x \). Then, the following result holds
\[
E \left[ X^{-1} \right] \geq M_x^{-1}.
\]
(88)

**Proof:** Let \( \mathcal{H}^N (J) \) denote the space of all Hermitian \( N \times N \) matrices whose eigenvalues all fall within \( J \). Then, we define a matrix function \( f : J \mapsto J \), given by
\[
f (T) = T^{-1}.
\]
(89)

According to [54, Lemma 2.5], we know that \( f \) is a strictly matrix convex function. Let \( T = T_1 + a (T_2 - T_1) = (1 - a) T_1 + a T_2 \), \( 0 \leq a \leq 1 \). Then, recalling the definition of matrix convex function in [54, Definition 2.2], it yields
\[
(1 - a) f (T_1) + a f (T_2) \geq f (T).
\]
(90)

Rewriting (90), we have
\[
f (T_2) - f (T) \geq \frac{1 - a}{a} \left( f (T) - f (T_1) \right).
\]
(91)

Considering the Taylor expansion of the matrix function in [54, Definition 3.1], \( f (T) - f (T_1) \) can be expressed as
\[
f (T) - f (T_1)
\]
\[
= f (T_1) \left[ T - T_1 \right] + o \left( \left\| T - T_1 \right\|_F \right) I_N
\]
\[
= a f (T_1) \left[ T_2 - T_1 \right] + o \left( \left\| T_2 - T_1 \right\|_F \right) I_N
\]
(92)

where \( f (T_1) \) is a Hermitian symmetric multi-linear mapping on the space \( \mathcal{H}^N \). Plugging (92) into (91) and taking the
\[ \{\nabla_{F_i} R_i\}_{i,j} = -\log_2 e E_{H_i} \left[ \text{tr} \left( \Sigma_{u_i|y_i, H_i} (H_i)^{-1} \left( \frac{\partial C_i}{\partial F_i^+_{i,j}} - \frac{\partial (T_1 (H_i) A_i^H)}{\partial F_i^+_{i,j}} \right) \right) \right] \]
\[ = -\log_2 e E_{H_i} \left[ \text{tr} \left( \Sigma_{u_i|y_i, H_i} (H_i)^{-1} \left[ F_i \Sigma_{S_i} e_j e_i^H - T_1 (H_i) \Sigma_{S_i} e_j e_i^H \right] \right) \right]. \]

limit as \( a \to 0 \), the right term of (91) becomes
\[ \lim_{a \to 0} \frac{1}{a} \left( f(T) - f(T_1) \right) \]
\[ = f^{(1)}(T_1) \|T_2 - T_1\|_F I_N \]
\[ = f^{(1)}(T_1) \|T_2 - T_1\|_F I_N. \] (93)

Synchronously, when \( a \to 0 \), the left term of (91) can be written as
\[ \lim_{a \to 0} f(T_2) - f(T) = f(T_2) - f(T_1). \] (94)

Eqs. (93) and (94) imply the following result for arbitrary \( N \times N \) Hermitian matrices \( T_1 \) and \( T_2 \)
\[ f(T_2) - f(T_1) \geq f^{(1)}(T_1) \|T_2 - T_1\|. \] (95)

By setting \( T_1 = M_z \), \( T_2 = X \), and evaluating the expectations of both sides of (95), it yields
\[ E[f(X)] - f(M_z) \geq f^{(1)}(M_z)[E[X] - M_z] = 0 \] (96)
which completes the proof of the lemma.

**Lemma 2:** Assume \( Y \succeq 0 \) is a \( N \times N \) random matrix with \( E[Y] = M_y \), \( A \) and \( B \) are \( N \times N \) constant matrices satisfying \( A \succ 0 \), \( B \succ 0 \), and \( A - B \succeq 0 \). Then, the following inequality holds
\[ E[\log_2 \text{det} \{YA + I_N\}] - E[\log_2 \text{det} \{YB + I_N\}] \leq \log_2 \text{det} (M_yA + I_N) - \log_2 \text{det} (M_yB + I_N). \] (97)

**Proof:** We define the function \( g(T) \) as follows
\[ g(T) = \log_2 \text{det} (M_yT + I_N) - E[\log_2 \text{det} (YT + I_N)]. \] (98)

Also, we construct a composite function \( h(\omega) \) as
\[ h(\omega) := g \left[ (1 - \omega) B + \omega A \right] = g(U), \quad 0 \leq \omega \leq 1. \] (99)

Since \( A \succ 0 \) and \( B \succ 0 \), it can be readily identified that \( U \succ 0 \). Moreover, from (99), we have
\[ h(0) = g(B), \quad h(1) = g(A). \] (100)

Results in (100) suggest to consider
\[ g(A) - g(B) = h(1) - h(0) = \int_0^1 \frac{dh(\omega)}{d\omega} d\omega. \] (101)

Application of the chain rule of the derivative leads to
\[ \frac{dh(\omega)}{d\omega} = \sum_{r=1}^N \sum_{s=1}^N \left\{ \frac{\partial g(U)}{\partial U} \right\}_{r,s} \times \left\{ \frac{d}{d\omega} (\omega A + (1 - \omega) B) \right\}_{r,s} \]
\[ = \sum_{r=1}^N \sum_{s=1}^N \left\{ \frac{\partial g(U)}{\partial U} \right\}_{r,s} \times \{A - B\}_{r,s}. \] (102)

Following similar approaches as those in (73)–(80) and keeping in mind that \( U \succ 0 \), we can obtain
\[ \frac{\partial g(U)}{\partial U} = (M_y U + I_N)^{-1} M_y - E \left[ (YU + I_N)^{-1} Y \right] \]
\[ = (M_y U + I_N)^{-1} [M_y U + I_N - I_N] U^{-1} \]
\[ - E \left[ (YU + I_N)^{-1} [YU + I_N - I_N] U^{-1} \right] \]
\[ = U^{-1} - (M_y U + I_N)^{-1} U^{-1} \]
\[ - U^{-1} + E \left[ (YU + I_N)^{-1} U^{-1} \right] \]
\[ = E \left[ (UYU + U)^{-1} \right] - (UYU + U)^{-1}. \] (107)

Applying Lemma 1 to (106), yields
\[ \frac{\partial g(U)}{\partial U} \geq 0. \] (108)

Recalling \( A - B \succeq 0 \), the result in (108), and the Schur Product Theorem in [55, Theorem 7.5.3], we have
\[ \frac{\partial g(U)}{\partial U} \circ (A - B) \geq 0 \] (109)
which implies that
\[ \sum_{r=1}^N \sum_{s=1}^N \left\{ \frac{\partial g(U)}{\partial U} \right\}_{r,s} \times \{A - B\}_{r,s} \geq 0. \] (110)

Then, combining (101), (103), and (110), we have
\[ g(A) - g(B) \geq 0 \] (111)
which completes the proof of the lemma.

Now we begin to prove Lemma 2. As mentioned in Section II, when discussing the linear alignment capacity, similar to the assumption in [25], we assume \( \Sigma_i > 0 \). Henceforth, we know that \( C_i \) is invertible. To this end, \( \log_2 \text{det} (\Sigma_{u_i|y_i, H_i} (H_i)) \) in (11) can be reexpressed as \( \log_2 \text{det} (\Sigma_{u_i|y_i, H_i} (H_i)) = \log_2 \text{det} (C_i) \)
\[ + \log_2 \text{det} \left( \text{I}_{N_i} - C_i^{-1} A_i H_i^H \left[ H_i B_i H_i^H + \Sigma_{Z,l} \right]^{-1} H_i A_i^H \right) \] (112)
\[ \overset{(a)}{=} \log_2 \text{det} (C_i) \]
\[ + \log_2 \text{det} \left( \text{I}_{N_i} - H_i C_i^{-1} C_i A_i H_i^H \left[ H_i B_i H_i^H + \Sigma_{Z,l} \right]^{-1} \right) \]
\[ = \log_2 \text{det} (C_i) + \log_2 \text{det} (H_i (B_i - A_i H_i^H + \Sigma_{Z,l})) \]
\[ = \log_2 \text{det} (C_i) \]
\[ + \log_2 \text{det} \left( \text{H}_i \left(D_i + \sum_{t=i+1}^L \Sigma_t \right) H_i^H + N_0 I_{N_i} \right) \] (113)
\[ \overset{(a)}{=} \log_2 \text{det} (C_i) \]
\[ + \log_2 \text{det} \left( \text{H}_i (B_i - A_i H_i^H + \Sigma_{Z,l}) H_i^H + N_0 I_{N_i} \right) \]
\[ = \log_2 \text{det} (C_i) \]
\[ + \log_2 \text{det} \left( \text{H}_i \left(D_i + \sum_{t=i+1}^L \Sigma_t \right) H_i^H + N_0 I_{N_i} \right) \] (114)
\[ \overset{(a)}{=} \log_2 \text{det} (C_i) \]
\[ + \log_2 \text{det} \left( \text{H}_i \left(D_i + \sum_{t=i+1}^L \Sigma_t \right) H_i^H + N_0 I_{N_i} \right) \] (115)
\[ -\log_2 \det \left( H_l \left( B_l + \sum_{t=1}^{L} \Sigma_t \right) H_l^H + N_0 I_N \right) \] 
\[ = \log_2 \det (C_l) \quad \text{(b)} \] 
\[ + \log_2 \det \left( H_l^H H_l \left( D_l + \sum_{t=1}^{L} \Sigma_t \right) + N_0 I_N \right) \] 
\[ + (N_c - N_t) \log_2 N_0 \] 
\[ - \log_2 \det \left( H_l^H H_l \left( B_l + \sum_{t=1}^{L} \Sigma_t \right) + N_0 I_N \right) \] 
\[ - (N_c - N_t) \log_2 N_0 \]  
\[ \text{(115)} \]

where equalities (a) and (b) are obtained according to the determinant identity \( \det(X + AB) = \det(X) \det(I + BX^{-1}A) \).

Since \( \Sigma_t > 0 \), it yields \( B_l > 0 \), \( \sum_{t=1}^{L} \Sigma_t > 0 \), and \( B_l - D_l = A_l^H C_l A_l \geq 0 \). Next, we need to prove that \( D_l > 0 \).

Let \( u \) and \( v \) be two \( N \times 1 \) constant non-zero vectors. We begin by considering
\[ \begin{bmatrix} u^H & v^H \end{bmatrix} \begin{bmatrix} \Sigma_{S,t} & \Sigma_{S,t} F_l^H \\ F_l \Sigma_{S,t} & F_l \Sigma_{S,t} F_l^H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \]
\[ = u^H \Sigma_{S,t} u + u^H F_l \Sigma_{S,t} v + v^H F_l \Sigma_{S,t} u + v^H F_l \Sigma_{S,t} F_l^H v \]
\[ = u^H (\Sigma_{S,t} u + \Sigma_{S,t} F_l^H v) + v^H F_l (\Sigma_{S,t} u + \Sigma_{S,t} F_l^H v) \] 
\[ \begin{align*} \quad \text{(b)} \quad & 
\end{align*} \]
\[ \begin{align*} \quad \text{(117)} \quad & 
\end{align*} \]
\[ \begin{align*} \quad \text{(a)} \quad & 
\end{align*} \]
\[ \begin{align*} \quad \text{(118)} \quad & 
\end{align*} \]

where (a) is obtained based on the fact \( \Sigma_{S,t} = \sum_{t=1}^{L} \Sigma_t > 0 \).

Equation (132) presents a unique closed-form structure of \( \tilde{F}_l \) satisfying the necessary condition in (129). Therefore, the derived \( \tilde{F}_l \) is a global maximizer of the upper bound \( R_{\text{upp},l} \).

Next, by substituting the expression of \( \tilde{F}_l \) in (132) into \( N_l \) in (125), it yields
\[ N_l = \left( I_{N_t} - \Sigma_t W_l - \tilde{F}_l \Sigma_{S,t} W_l \right) \Sigma_l \] 
\[ = \left( I_{N_t} - \Sigma_t \right) \left( I_{N_t} + \left( \Sigma_t + \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \right) \right) \left( \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \right) \] 
\[ \Sigma_l \] 
\[ = \left( I_{N_t} - \Sigma_t \right) \left( \left( \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \right) \left( \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \right) \right) \Sigma_l \] 
\[ \quad \text{(134)} \]
\[ \quad \text{(135)} \]
\[ \quad \text{(136)} \]

Thus, the achievable rate of the \( l \)-th receiver can be upper bounded by
\[ R_l \leq R_{\text{upp},l} = \log_2 \det \begin{vmatrix} \Sigma_l + \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \end{vmatrix} \]
\[ - \log_2 \det \begin{vmatrix} \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \end{vmatrix} \]
\[ = \log_2 \det \begin{vmatrix} R_{g,l} \sum_{t=1}^{L} \Sigma_t + N_0 I_{N_t} \end{vmatrix} \]
\[ = \log_2 \det \begin{vmatrix} R_{g,l} \sum_{t=1}^{L} \Sigma_t + N_0 I_{N_t} \end{vmatrix} \] 
\[ \text{(137)} \]
\[ \text{(138)} \]

APPENDIX C

PROOF OF THE PROPOSITION 1

First, we rewrite the upper bound expression in (26) as given in (122)–(124) at the top of the next page, where
\[ N_l = F_l (\Sigma_{S,t} - \Sigma_{S,t} W_l \Sigma_{S,t}) F_l^H - F_l \Sigma_{S,t} W_l \Sigma_{S,t} \]
\[ - \Sigma_l W_l \Sigma_{S,t} F_l^H + \Sigma_l - \Sigma_l W_l \Sigma_{S,t} \]
\[ \text{(125)} \]
\[ W_l = \left( R_{g,l} \left( B_l + \sum_{t=1}^{L} \Sigma_t \right) + N_0 I_{N_t} \right) \]
\[ = \left( B_l + \sum_{t=1}^{L} \Sigma_t \right) + N_0 R_{g,l}^{-1} \]
\[ \text{(126)} \]
\[ \text{(127)} \]

Now, we find \( F_l \) maximizing \( R_{\text{upp},l} \). Applying similar steps as those in (74)–(80), we can obtain
\[ \{ \nabla F_l R_{\text{upp},l}, \} = -\text{tr} \left( \sum_{l=1}^{L} \left[ \Sigma_l (\Sigma_{S,t} - \Sigma_{S,t} W_l \Sigma_{S,t}) \Sigma_l - \Sigma_l W_l \Sigma_{S,t} \right] \right) \]
\[ \text{(128)} \]

Then, we have a necessary condition for the optimal \( F_l \)
\[ \nabla F_l R_{\text{upp},l} = -\sum_{l=1}^{L} \left[ \Sigma_l (\Sigma_{S,t} - \Sigma_{S,t} W_l \Sigma_{S,t}) - \Sigma_l W_l \Sigma_{S,t} \right] = 0 \]
\[ \text{(129)} \]

From (129), a closed-form expression for the matrix \( \tilde{F}_l \) can be derived as
\[ \tilde{F}_l = \Sigma_l W_l (I_{N_t} - \Sigma_{S,t} W_l) \left( I_{N_t} + \left( \Sigma_l + \sum_{t=1}^{L} \Sigma_t + N_0 R_{g,l}^{-1} \right) \right)^{-1} \]
\[ \Sigma_l \] 
\[ \text{(130)} \]
\[ \text{(131)} \]
\[ \text{(132)} \]

APPENDIX D

PROOF OF THE PROPOSITION 2

We note that when \( N_t = 1 \), the power allocation design to maximize \( R_{\text{upp},\text{sum}} \) lies either on the stationary points or the boundary points of the feasible solution set. For the stationary points, we consider the KKT conditions. To this end, let \( \nu \) be
\[ R_{\text{upp}, t} = \log_2 \det (\Sigma_i) - \log_2 \det (C_i) \]

\[ = \log_2 \det (\Sigma_i) - \log_2 \det \left( I_{N_t} - \left( R_{g,t} + \sum_{l=l+1}^{L} \Sigma_l \right) + N_0 I_{N_t} \right)^{-1} \left( R_{g,t} A_i^H C_i^{-1} A_i \right) \]

(122)

\[ = \log_2 \det (\Sigma_i) - \log_2 \det \left( C_i - A_i \left( R_{g,t} + \sum_{l=l+1}^{L} \Sigma_l \right) + N_0 I_{N_t} \right)^{-1} \left( R_{g,t} A_i^H \right) \]

(123)

\[ = \log_2 \det (\Sigma_i) - \log_2 \det (N_i) . \]

(124)

the Lagrange multiplier for the constraint \( \sum_{l=1}^{L} P_l \leq P \). Then, necessary KKT conditions satisfied by the optimal solution can be written as

\[ \frac{\partial R_{\text{upp, sum}}}{\partial P_t} - \nu = 0, \quad t = 1, 2, \cdots, L \]

(139)

where \( \frac{\partial R_{\text{upp, sum}}}{\partial P_t} = \sum_{l=1}^{L} \frac{\partial R_{\text{upp, sum}}}{\partial P_l} \) denotes the partial derivative of \( R_{\text{upp, sum}} \) with respect to \( P_t, t = 1, 2, \cdots, L \). The derivative of \( \frac{\partial R_{\text{upp, sum}}}{\partial P_t} \) can be expressed as

\[ \frac{\partial R_{\text{upp, sum}}}{\partial P_t} = \begin{cases} \frac{r_l P_t + a_t}{r_l P_t + a_t} - \frac{a_l}{a_l}, & l < t \\ \frac{r_l P_t + a_t}{r_l P_t + a_t}, & l = t \\ 0, & l > t \end{cases} \]

(140)

where \( a_l = r_l \sum_{l=1}^{L} P_l + N_0 \). Substituting (140) into (139) yields

\[ \frac{r_t}{r_t P_t + a_t} + \sum_{l=1}^{t-1} \left( \frac{r_l}{r_l P_t + a_t} - \frac{r_l}{a_l} \right) = \nu. \]

(141)

For arbitrary \( t = n \) and \( t = n + 1 \) in (141), the following results hold

\[ \nu = \frac{r_t}{r_t P_t + a_t} + \sum_{l=1}^{t-1} \left( \frac{r_l}{r_l P_t + a_t} - \frac{r_l}{a_l} \right) \]

(142)

and

\[ \nu = \frac{r_{t+1}}{r_{t+1} P_{t+1} + a_{t+1} + \sum_{l=1}^{t} \left( \frac{r_l}{r_l P_t + a_t} - \frac{r_l}{a_l} \right). \]

(143)

Subtracting (142) from (143), we have

\[ \frac{r_{t+1}}{r_{t+1} P_{t+1} + a_{t+1}} = \frac{r_t}{a_t} \]

(144)

Plugging the expressions of \( a_t \) and \( a_{t+1} \) into (144) and simplifying yields

\[ r_t = r_{t+1}, \quad t = 1, 2, \cdots, L - 1. \]

(145)

The KKT conditions hold only when the equations in (145) are satisfied. When these equations are satisfied, the power allocation in (29) will obviously maximize \( R_{\text{upp, sum}} \). And when these equations are not satisfied, we note that the KKT conditions do not hold under any power allocation solutions. Hence the maximum \( R_{\text{upp, sum}} \) is achieved by boundary points (to allocate the total power to a certain receiver). Then, we know that selecting the receiver with the most robust channel condition achieves the maximum \( R_{\text{upp, sum}} \), which completes the proof.

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