On the action for Weyssenhoff spin fluid and the Barbero-Immirzi parameter

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Abstract.

It was showed by Perez and Rovelli in 2006 that the Holst action in gravity with torsion with massless and minimally coupling Dirac fermions gives rise to the four-fermion coupling term, whose coefficient is a function of the Barbero-Immirzi (BI) parameter. This parameter is present in the Holst action, which is an object of investigation in a non-perturbative formalism of quantum gravity. The key feature is the torsion because its absence implies no effect from Holst term in dynamical equations. In this paper we consider a spin fluid (also called Weyssenhoff fluid), which is a perfect fluid that has intrinsic spin. We study the Host action in gravity with torsion and spin fluid, and we include, for completeness, minimally coupling massive fermions. We find the equivalent action containing the same four-fermion interaction term previously calculated by Perez and Rovelli without the spin fluid, also the quadratic spin tensor term and finally an interaction term between the axial current and the spin tensor, which can be relevant for new effects and features in Cosmology. In all three cases, the dependence on BI parameter is the same.

PACS: 04.20.-q 98.80.-k 04.50.Kd

Keywords: Dirac fields, Torsion, Barbero-Immirzi parameter, Weyssenhoff Fluid

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1 Introduction

Dirac fields in curved spacetime is a subject which has been studied in a number of works for many years, especially in the last decade, as shown in Refs. [12, 8, 9, 6, 7, 8, 9, 10]. In this paper, we consider classical aspects of Dirac particles in curved spacetime in the presence of the BI parameter [11, 12] and the Weyssenhoff fluid. The BI parameter, $\beta$, carried by the Holst action term [13],

$$\frac{1}{2\beta} \epsilon_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu},$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor, was introduced in the non-perturbative quantum gravity scenario, and is indeed a new dimensionless parameter emerging from a more fundamental theory. In General Relativity (GR) this parameter has no dynamical effect, because $\epsilon_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 0$ by cyclic symmetry of the Riemann tensor ($R^{\alpha\beta\mu\nu} + R^{\alpha\nu\beta\mu} + R^{\alpha\mu\nu\beta} = 0$). The situation changes for the non-Riemannian case of Einstein-Cartan action, where $\tilde{R}^{\alpha\beta\mu\nu} + \tilde{R}^{\alpha\nu\beta\mu} + \tilde{R}^{\alpha\mu\nu\beta} \neq 0$. Here we represent curvatures with tilde as the curvature constructed with torsion. For an introduction and review on the theories with torsion, see for example references [14, 15, 16]. In the theory with Einstein-Hilbert and Holst action, together with minimal coupling fermions in the presence of torsion, the BI parameter does affect the gravitational dynamics, providing an interesting way to investigate its classical and quantum effects, through 4-fermion interaction term [17, 18]. In Refs. [19, 20, 21, 22, 23] another coupling scenarios and features are explored on the issue of the role of BI parameter in physical theories.

The Weyssenhoff fluid [24] is a fluid with intrinsic spin density, which is described by the spin tensor $\Sigma_{\alpha\beta}$. In principle, this fluid can not be associated with a Lagrangian description based on spinor fields. One has to postulate spin extra terms in the energy-momentum tensor. There are several papers considering this fluid, most of all concerned with cosmological applications as a realization of torsion effects in Cosmology (see, for example, [25] and references therein). One can cite, for instance, the work of Gasperini [26] which considered the energy momentum tensor (of the Weyssenhoff fluid) previously improved by Ray and Smalley [27], where spin is a thermodynamical variable. In Ref. [26], the torsion, algebraically related to the quantities describing the sources (what happens in most of the papers in this theme, including this one), provides singularity avoidance and early accelerated expansion, but the expansion factor of the cosmological scale is too small. Obukhov and Korotky [28] formulated a more general variational theory describing the Weyssenhoff fluid and also applied to cosmological models with rotation, shear and expansion.
In this paper we study the effect of the Holst action in a spacetime with torsion together with minimally coupling Dirac fields and also a Weyssenhoff fluid. We find the equivalent action which is composed by the Riemannian Einstein-Hilbert term and the following source terms: a four-fermion interaction expression, $J^\mu J_\mu$, a quadratic spin tensor term, $\Sigma_{\mu\nu} \Sigma^{\mu\nu}$ and a mixing term (which can provide an interesting study of new effects in Cosmology). The coefficients of these three terms are proportional to the same function of the BI parameter. In the next section, we write the starting action and some basic definitions, then we derive the equations with respect to torsion in Section 3. The conclusions is then drawn in Section 4.

2 Holst action with Dirac fields and Weyssenhoff fluid

Our starting point is the Holst action in the presence of torsion, as well as minimally coupling fermions and the Weyssenhoff fluid:

$$S = S_H + S_\Sigma + S_\psi + S_{PF},$$  \hspace{1cm} (1)

where $S_H$ is the action for Einstein-Cartan theory with also the Holst term, $S_\Sigma$ is the action describing the interaction between torsion and the intrinsic spin of the fluid, $\Sigma_{\mu\nu}$, the action $S_\psi$ is the minimally coupled Dirac action and $S_{PF}$ is the action for the perfect fluid counterpart of the Weyssenhoff fluid such that the corresponding energy-momentum tensor is the standard one, $T_{\mu\nu} = (p + \rho) U_\mu U_\nu - pg_{\mu\nu}$.

The first contribution to the total action is the one for gravitational sector, $S_H$:

$$S_H = \frac{1}{\kappa} \int d^4x \sqrt{-g} \left\{ - \bar{R} + \frac{1}{2\beta} \epsilon^{\mu\nu\alpha\beta} \bar{R}_{\mu\nu\alpha\beta} \right\} .$$  \hspace{1cm} (2)

Here $\kappa = 16\pi G$, $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and the tilde above curvatures means that they are constructed using the full connection $\bar{\Gamma}^{\rho}_{\alpha\beta}$ which includes the torsion tensor

$$T^{\rho}_{\alpha\beta} = \bar{\Gamma}^{\rho}_{\alpha\beta} - \bar{\Gamma}^{\rho}_{\beta\alpha}.$$  

Using the above definition, the metricity condition gives us the expression of the full connection in terms of the Riemannian connection, $\Gamma^{\rho}_{\alpha\beta}$, as

$$\bar{\Gamma}^{\rho}_{\alpha\beta} = \Gamma^{\rho}_{\alpha\beta} + K^{\rho}_{\alpha\beta},$$  

where the last term is called contortion tensor, given by

$$K^{\rho}_{\alpha\beta} = \frac{1}{2} (T^{\rho}_{\alpha\beta} - T^{\rho}_{\alpha\beta} - T^{\rho}_{\beta\alpha}).$$
The second term in the right hand side of equation (1) is the action for the interaction between torsion and spin of the fluid [27]:

$$S_\Sigma = \alpha \int d^4x \sqrt{-g} \Sigma^\mu U_\lambda K_{\mu \nu \lambda},$$

where $U^\mu$ is the 4-velocity in comoving frame, $U^\mu = (1, 0, 0, 0)$ and $\Sigma^\mu \nu$ is the antisymmetric spin tensor of the fluid which satisfies the Frenkel condition $\Sigma^\mu \nu U_\nu = 0$ [26] [28]. The constant $\alpha$ is any number.

The Dirac action $S_\psi$ is given by

$$S_\psi = \frac{i}{2} \int d^4x \sqrt{-g} \left\{ \bar{\psi} \gamma^\mu \tilde{\nabla}_\mu \psi - \tilde{\nabla}_\mu \bar{\psi} \gamma^\mu \psi + 2im\bar{\psi}\psi \right\}, (4)$$

where the operator $\tilde{\nabla}_\mu$ on spinors is defined using the spinor connection

$$\tilde{\omega}^{ab}_\mu = \frac{1}{2} \left( e^b_\alpha \partial_\mu e^a_\alpha - e^a_\alpha \partial_\mu e^b_\alpha \right) - \frac{1}{2} \tilde{\Gamma}^\rho_\alpha_\mu \left( e^b_\alpha e^a_\rho - e^a_\alpha e^b_\rho \right).$$

In the above expression, the objects $e^a_\alpha$ are the vierbeins such that $\gamma_\alpha = e^a_\alpha \gamma_a$, $e^a_\alpha e^{b}_\alpha = \eta^{ab}$ and $e^a_\alpha e_{a\beta} = g_{a\beta}$. Separating the Riemannian part from the torsion part, we can express the Dirac action after some algebraic manipulation as follows [15]:

$$S_\psi = \frac{i}{2} \int d^4x \sqrt{-g} \left\{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi + 2im\bar{\psi}\psi \right\} + \int d^4x \sqrt{-g} S_\mu J^\mu, (5)$$

where $J^\mu$ is the axial current $\bar{\psi} \gamma^5 \gamma^\mu \psi$ and $S_\mu$ is the pseudo-trace of torsion which can be seen in decomposition of torsion in its irreducible parts (along with trace $T_\alpha := T^\mu_{\alpha \mu}$ and tensorial part $q^\mu_{\alpha \beta}$),

$$T^\mu_{\alpha \beta} = \frac{1}{3} \left( \delta^\mu_\beta T_\alpha - \delta^\mu_\alpha T_\beta \right) - \frac{1}{6} \epsilon^\mu_{\alpha \beta \sigma} S^\sigma + q^\mu_{\alpha \beta}. (6)$$

The pseudo-trace and tensor $q^\mu_{\alpha \beta}$ satisfy the identities

$$\epsilon^{\mu \alpha \beta \sigma} T^\sigma_{\mu \alpha \beta} = S^\sigma, \quad q^\mu_{\alpha \beta} = -q^\mu_{\beta \alpha} \quad \text{and} \quad \epsilon^{\mu \alpha \beta \sigma} q_{\mu \alpha \beta} = 0.$$

We have so far the total action with all terms defined such that in order to write separately the torsion independent part and the torsion dependent part, we should take into account

$$\bar{\mathcal{R}}^{\tau \rho \lambda} = R^{\tau \rho \lambda} + K^{\tau \phi \rho} K^{\phi \lambda} - K^{\tau \phi \lambda} K^{\phi \rho} + \text{total derivatives}. (7)$$

Now we are able to vary the total action in relation to $K^\mu_{\alpha \beta}$ to obtain the equation for torsion. Then, we shall substitute torsion by its sources back into the action.
3 Equation for torsion and equivalent action

Let us write some useful computations in varying the action. After straightforward calculation, one achieves

$$\frac{\delta}{\delta K_{\mu}^{\alpha \beta}} \int d^4x \sqrt{-g} \left( -\tilde{R} \right) = \delta^{\beta}_\mu T^\alpha - g^{\alpha \beta} T_\mu - K_\mu^{\beta \alpha} + K^{\alpha \beta}_\mu,$$

(8)

and

$$\frac{\delta}{\delta K_{\mu}^{\alpha \beta}} \int d^4x \sqrt{-g} \tilde{R}^{\sigma \rho \lambda} \epsilon_{\tau \sigma \rho \lambda} = 2 \epsilon_{\mu \sigma \beta \lambda} K^{\alpha \sigma \lambda} + 2 \epsilon_{\alpha \sigma \rho \beta} K_{\mu \sigma \rho},$$

(9)

Using these results, we obtain the equation for torsion, $\delta S / \delta K^{\alpha \beta}_\mu = 0$, in the form

$$\frac{1}{\kappa} \left\{ \delta^{\beta}_\mu T^\alpha - g^{\alpha \beta} T_\mu + T_\mu^{\beta \alpha} + \frac{1}{\beta} \left( 2 \epsilon_{\mu \sigma \beta \lambda} K^{\alpha \sigma \lambda} + 2 \epsilon_{\alpha \sigma \rho \beta} K_{\mu \sigma \rho} \right) \right\}$$

$$= -\alpha \Sigma_{\mu}^{\alpha} U^\beta - 2 \eta \epsilon_{\mu \alpha \beta \sigma} J_\sigma .$$

(11)

Contracting $\beta$ with $\mu$, one gets

$$T^\alpha = \frac{1}{4 \beta} S^\alpha ,$$

(12)

which is also useful to achieve the following result after multiplying (11) by $\epsilon_{\beta \mu \alpha \tau}$:

$$S_\tau = -\alpha \kappa \theta \epsilon_{\beta \mu \alpha \tau} \Sigma^{\mu \alpha} U^\beta + 12 \eta \kappa \theta J_\tau ,$$

(13)

where

$$\theta = \frac{\beta^2}{\beta^2 + 1} .$$

We can use the above equations in (11) and after some algebraic manipulations we get finally

$$T^{\mu \alpha \beta} = \frac{\alpha \kappa \theta}{4 \beta} \left( \delta^{\mu}_\beta \epsilon_{\alpha \rho \lambda \tau} \Sigma^{\rho \lambda} U^\tau - \delta^{\mu}_\alpha \epsilon_{\beta \rho \lambda \tau} \Sigma^{\rho \lambda} U^\tau - 2 \epsilon_{\alpha \beta \rho \lambda} \Sigma^{\rho \lambda} U^{\mu} \right) +$$

$$+ \frac{\eta \kappa \theta}{\beta} \left( \delta^{\mu}_\beta J^\alpha - \delta^{\mu}_\alpha J^\beta \right) + \alpha \kappa \theta \Sigma_{\beta \alpha} U^\mu + 2 \eta \kappa \theta \epsilon^{\mu \alpha}_{\beta \alpha} J_\sigma .$$

(14)

Now we are in a position to substitute all torsion terms in the action (1) by the sources $\Sigma^{\rho \lambda}$ and $J_\nu$ through the equation (14). This task is quite tedious, so the best way to do this is to rewrite the action in terms of Riemannian structures and the
irreducible components of torsion, $T_\mu$, $S_\mu$ and $q^\mu_{\alpha\beta}$, and then not to substitute equation (14) directly, but the corresponding equations for components of torsion, which can be derived by taking into account equations (6), (12), (13) and (14). For both $T_\mu$ and $S_\mu$, we already have the convenient expressions (equations (12), (13)). For the remaining component, $q^\mu_{\alpha\beta}$, we obtain

\[
q^\mu_{\alpha\beta} = \frac{\alpha\kappa\theta}{6\beta} \left( \delta^\mu_\beta \epsilon_{\alpha\rho\lambda\tau} \Sigma^{\rho\lambda} U^\tau - \delta^\mu_\alpha \epsilon_{\beta\rho\lambda\tau} \Sigma^{\rho\lambda} U^\tau - 3\epsilon_{\alpha\beta\rho\lambda} \Sigma^{\rho\lambda} U^\mu \right) - \frac{\alpha\kappa\theta}{3} \left( \Sigma^\mu_\beta U_\alpha - \Sigma^\mu_\alpha U_\beta + 2\Sigma_{\alpha\beta} U^\mu \right). \tag{15}
\]

Actually it would be much more simple to write the action, in the very beginning, in terms of the irreducible components, and then take the variation of the action in relation to $\delta T_\mu$, $\delta S_\mu$ and $\delta q^\mu_{\alpha\beta}$ separately. Nevertheless, this procedure is not equivalent to what we have done because the components $T_\mu$, $S_\mu$ and $q^\mu_{\alpha\beta}$ are not independent each other. Thus after straightforward calculations, with the help of the useful equations

\[
- \tilde{R} = -R + \frac{2}{3} T_\mu T^\mu - \frac{1}{24} S_\mu S^\mu - \frac{1}{2} q_{\mu\rho\sigma\tau} q^{\mu\rho\sigma\tau}, \tag{16}
\]

and

\[
\frac{1}{2\beta} \tilde{R}^{\mu\rho\sigma\tau} \tilde{R}_{\mu\rho\sigma\tau} = -\frac{1}{3\beta} T_\mu S^\mu + \frac{1}{4\beta} \epsilon^{\mu\rho\sigma\tau} q_{\rho\sigma\tau} q^\rho_{\alpha\beta}, \tag{17}
\]

one finally can write the full action eliminating all torsion terms as

\[
S = S_{EH} + S_{PF} + S_{\text{Dirac}} + \kappa\theta \int d^4x \sqrt{-g} \left\{ \frac{\alpha^2}{4} \Sigma^\alpha_{\alpha\beta} \Sigma^{\alpha\beta} + \alpha\eta \epsilon_{\mu\nu\sigma\tau} J^\mu \Sigma^{\nu\sigma\tau} U^\beta + 6\eta^2 J_\mu J^\mu \right\}, \tag{18}
\]

where $S_{PF}$ is the perfect fluid action and

\[
S_{EH} = -\frac{1}{\kappa} \int d^4x \sqrt{-g} R, \quad S_{\text{Dirac}} = \frac{i}{2} \int d^4x \sqrt{-g} \left\{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi + 2im\bar{\psi} \psi \right\}.
\]

In the above result, the choice $\eta = 1/8$ (minimal coupling) implies the coefficient $3\pi G\theta/2$ for the 4-fermion term $J_\mu J^\mu$. In Ref. [17] this coefficient is the same (see also [18]).

1In this alternative and simpler way to derive the equations for torsion, the equation for $q^\mu_{\alpha\beta}$, analogous to (15), would be such that $\epsilon^{\mu\alpha\beta}\varepsilon_{\mu\alpha\beta} \neq 0$, what violates the defining property of $q^\mu_{\alpha\beta}$.

2Except by a non-essential sign coming from different notations.
4 Conclusions

We have considered the Holst action with fermions and a Weyssenhoff fluid in the spacetime with torsion. Torsion is responsible for the existence of non-trivial effects of the term $\epsilon^{\mu\nu\alpha\beta} \tilde{R}_{\mu\nu\alpha\beta}$ through its coefficient $1/2\beta$. By varying the action in respect to torsion (or contortion), one can write down the dynamical equation for torsion, which actually is not dynamical because it just relates algebraically torsion with the sources (fermionic current and intrinsic spin), such that we found the equivalent action in the form of Einstein-Hilbert gravity in Riemannian spacetime with fermions and a perfect fluid, together to the additional term

$$\kappa \theta \int d^4x \sqrt{-g} \left\{ \frac{\alpha^2}{4} \Sigma_{\alpha\beta} \Sigma^{\alpha\beta} + \alpha \eta \epsilon_{\mu\nu\alpha\beta} J^{\mu} \Sigma^{\nu\alpha} U^{\beta} + 6 \eta^2 J^{\mu} J^{\mu} \right\}.$$ 

The fluids are macroscopic entities, such the intrinsic spin of the fluid is conveniently thought as a quantity subject to statistical procedures. In the literature, the relevant quantity is defined as

$$2\sigma^2 = \langle \Sigma_{\mu\nu} \Sigma^{\mu\nu} \rangle, \quad (19)$$

in such a way that $\langle \Sigma_{\mu\nu} \rangle$ should vanish because of its random character. Notice that the current $J_{\mu}$ is different because it can be treated as a background external preferred direction, which is not random. In what sense a coupling between an external quantity, non-random, and a random quantity, as $\epsilon_{\mu\nu\alpha\beta} J^{\mu} \Sigma^{\nu\alpha} U^{\beta}$, can keep $\Sigma^{\nu\alpha}$ random? The answering of this and similar questions we postpone for another work. Let us note that the $J_{\mu}$ in the form $(0, J_x, J_y, J_z)$ can make $\epsilon_{\mu\nu\alpha\beta} J^{\mu} \Sigma^{\nu\alpha} U^{\beta}$ non-trivial and thus inducing some anisotropy in the cosmological context, opening the possibility of establishing observational limits to $\alpha \eta \theta$.

Acknowledgments

G.B.P. is grateful to CNPq and FAPEMIG for partial support. C.A.S. is grateful to CAPES for PhD program. We acknowledge the comments and suggestions from the audience of our talk in the seminar session of our research group.

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