COUNTING PARABOLIC PRINCIPAL $G$-BUNDLES WITH NILPOTENT ENDOMORPHISMS OVER $\mathbb{P}^1$

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Abstract. Let $G$ be a split connected reductive group over $\mathbb{F}_q$ and let $\mathbb{P}^1$ be the projective line over $\mathbb{F}_q$. Firstly, we give an explicit formula for the number of $\mathbb{F}_q$-rational points of generalized Steinberg varieties of $G$. Secondly, for each principal $G$-bundle over $\mathbb{P}^1$, we give an explicit formula counting the number of triples consisting of parabolic structures at 0 and $\infty$ and a compatible nilpotent section of the associated adjoint bundle. In the case of $GL_n$ we calculate a generating function of such volumes re-deriving a result of Mellit.

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1. INTRODUCTION

Let $X$ be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$ with $q$ elements. O. Schiffmann in a breakthrough paper [23] computed the number of stable Higgs bundles over $X$ when char($\mathbb{F}_q$) is sufficiently large (see [23, Theorem 1.2]). In a later paper [20] with Mozgovoy, the condition on char($\mathbb{F}_q$) was removed. A major step in their calculation is computing the weighted number of vector bundles over $X$ with nilpotent endomorphisms. A. Mellit in [18] has generalized the result of Mozgovoy and Schiffmann to the parabolic case. In particular, Mellit counts vector bundles over $X$ with nilpotent endomorphisms preserving parabolic structures at marked points. An important part of his calculation is the case of $P^1$ and two marked points, this case allows him to relate the count with modified Macdonald polynomials. It is a natural question to generalize Mellit’s calculations to arbitrary reductive groups. In this paper, we complete this step, namely, we count the number of principal $G$-bundles over $\mathbb{P}^1$ with nilpotent sections of adjoint bundles compatible with parabolic structures at 0 and $\infty$ for any split connected reductive group over $\mathbb{F}_q$ (see Corollary 5.2.1).

In the case of $\mathbb{P}^1$, Mellit uses Hall algebras, which are unavailable for a general reductive group. Instead, we use geometric techniques in our proof. We also derive the result in the case of $GL_n$ using our methods. The counting has two important steps. In the first step, we give an explicit formula for the number of points of generalized Steinberg varieties
in Theorem \([\mathbb{X}.1]\). To this end we introduce a coproduct for any reductive group, which might be of independent interest.

In the second step, we reduce the problem to counting the number of points of generalized Steinberg varieties using Białynicki–Birula decomposition in Theorem \([\mathbb{X}.2]\). We note that the applicability of Białynicki–Birula decomposition is not obvious since the schemes that we work with are neither smooth nor projective.

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## 2. Preliminaries

### 2.1. Split reductive groups and its Lie algebras

A torus over \(\mathbb{F}_q\) is said to be split if is isomorphic to \(\mathbb{G}_m^r \mathbb{F}_q\) for some \(r\). Recall that a connected reductive group over \(\mathbb{F}_q\) is called split if it contains a maximal torus that is split.

We refer to \([19]\) Section 10.6 for the notion of the Lie algebra of an affine algebraic group over a field. For an affine algebraic \(\mathbb{F}_q\)-group \(G\), we will denote its Lie algebra by \(\mathfrak{g}\) or sometimes by \(\text{Lie}(G)\). We say an element \(x \in \mathfrak{g}\) is nilpotent if \(r(x)\) is nilpotent for every Lie algebra homomorphism \(r : \mathfrak{g} \to \mathfrak{g}(V)\), where \(V\) varies over all finite dimensional vector spaces over \(\mathbb{F}_q\).

### 2.2. Parabolic and Levi \(\mathbb{F}_q\)-subgroups

Recall that a smooth closed \(\mathbb{F}_q\)-subgroup \(P \subset G\) is parabolic if the coset space \(G/P\) is \(\mathbb{F}_q\)-proper (see \([\mathbb{X}]\) Section 1.3)). Since \(G/P\) is quasi-projective (see \([\mathbb{X}]\) Theorem 18.1.1)), we see that for a parabolic \(\mathbb{F}_q\)-subgroup \(P\) of \(G\), \(G/P\) is \(\mathbb{F}_q\)-projective. By a Levi \(\mathbb{F}_q\)-subgroup of \(G\) we mean a Levi factor of a parabolic \(\mathbb{F}_q\)-subgroup.

In the rest of the paper, \(G\) will denote a split connected reductive group over \(\mathbb{F}_q\) with a fixed split maximal torus \(T\) and a Borel \(\mathbb{F}_q\)-subgroup \(B\) containing \(T\). Denote by \(W\) the Weyl group of \(G\) relative to \(T\). Next, \(\Pi \subset \Phi^+ \subset \Phi \subset X^*(T)\) will denote the corresponding simple roots, the positive roots and the root system (see \([\mathbb{X}]\) Proposition 11.3.8)). Further, \(X^*(T) := \text{Hom}_{\mathbb{F}_q}(T, \mathbb{G}_m)\) and \(X_*(T) := \text{Hom}_{\mathbb{F}_q}(\mathbb{G}_m, T)\) will denote the lattices of \(\mathbb{F}_q\)-characters of \(T\) and \(\mathbb{F}_q\)-cocharacters of \(T\) respectively.

### 2.3. Parametrization of parabolic \(\mathbb{F}_q\)-subgroups

Let us now recall the description of standard parabolic and Levi \(\mathbb{F}_q\)-subgroups of \(G\). Pick \(J \subset \Pi\) and let \(L_J\) be the subgroup generated by \(T\) as well as \(U_\alpha, U_{-\alpha}, \alpha \in J\) (root subgroups). This is a Levi \(\mathbb{F}_q\)-subgroup (see \([\mathbb{X}]\) Example 12.3.2)) and in fact, it is the scheme-theoretic centralizer of the identity component of \((\bigcap_{\alpha \in J} \ker \alpha)_{\text{red}}\) (recall that \(\ker \alpha\) is an \(\mathbb{F}_q\)-subtorus of \(T\)) (see \([\mathbb{X}]\) Example 12.3.2)). Next, \(P_J := L_JB\) is a parabolic \(\mathbb{F}_q\)-subgroup (see \([\mathbb{X}]\) Example 12.3.2)). The subgroups \(P_J\) are called standard parabolic \(\mathbb{F}_q\)-subgroups and the subgroups \(L_J\) are called standard Levi \(\mathbb{F}_q\)-subgroups. It is known that every parabolic \(\mathbb{F}_q\)-subgroup is \((\mathbb{F}_q)-\text{conjugate to} P_J\) for a unique \(J \subset \Pi\) (see \([\mathbb{X}]\) Corollary 11.4.8) and the discussion afterwards). It follows that in the case of \(G = GL_n\), parabolic \(\mathbb{F}_q\)-subgroups are precisely the stabilizers of flags in \(\mathbb{F}_q^n\) (see \([\mathbb{X}]\) Example 11.4.9)) and that the Levi \(\mathbb{F}_q\)-subgroups are precisely the stabilizers of ordered direct sum decompositions \(\mathbb{F}_q^n = V_1 \oplus \ldots \oplus V_m\), see \([\mathbb{X}]\) Chapter 6)).

**Notation 2.1.** We denote by \(X_{-}(T)\) the semilattice of anti-dominant \(\mathbb{F}_q\)-cocharacters of \(T\), i.e., \(\lambda \in X_{-}(T)\) if and only if \((\alpha, \lambda) \in \mathbb{Z}_{\leq 0}\) for all \(\alpha \in \Phi^+\). We note that \(X_{-}(T) \cong X_{*}(T)/W\).

**Convention 2.2.** We make the following convention about fibre products of schemes over \(\mathbb{F}_q\). For any two schemes \(X\) and \(Y\) over \(\mathbb{F}_q\), we will denote \(X \times_{\mathbb{F}_q} Y\) by \(X \times Y\).
2.4. Principal $G$-bundles over $\mathbb{P}^1$. Let us review the definition of principal $G$-bundles. Let $Y$ be a scheme over $\mathbb{F}_q$. Recall that a $Y$-scheme $\mathcal{P}$ equipped with an action

$$G \times \mathcal{P} \to \mathcal{P}$$

of $G$ such that the morphism $\mathcal{P} \to Y$ is $G$-invariant is called a principal $G$-bundle over $Y$, if $\mathcal{P}$ is faithfully flat and quasi-compact over $Y$ and the action is simply transitive, i.e., the natural morphism $G \times \mathcal{P} \to \mathcal{P} \times_Y \mathcal{P}$ is an isomorphism. Let us recall the construction of associated bundles. Let $H$ be a connected algebraic group over $\mathbb{F}_q$. Let $Y$ be any quasi-projective $\mathbb{F}_q$-scheme and let $\mathcal{E}$ be a principal $H$-bundle over an $\mathbb{F}_q$-scheme $S$. Then we denote by $\mathcal{E} \times^H Y$ the associated bundle with fibre type $Y$, which is the following scheme (see [13, Proposition 3.1]): $\mathcal{E} \times^H Y = (\mathcal{E} \times Y)/H$ for the twisted action of $H$ on $\mathcal{E} \times Y$ given by $h \cdot (e, y) = (e \cdot h, h^{-1} \cdot y)$.

**Definition 1.** Let $H$ and $G$ be connected algebraic groups over $\mathbb{F}_q$ and $\mathcal{E}$ be a principal $H$-bundle over an $\mathbb{F}_q$-scheme $S$. If $\rho: H \to G$ is a homomorphism of groups defined over $\mathbb{F}_q$, then the associated bundle $\mathcal{E} \times^H G$ for the action of $H$ on $G$ by left multiplication through $\rho$, is naturally a principal $G$-bundle over $S$. We denote this principal $G$-bundle over $S$ often by $\rho_*(\mathcal{E})$ and we say this principal $G$-bundle is obtained from $\mathcal{E}$ by extension of structure group.

Consider the $\mathbb{G}_m$-bundle $\mathcal{O}(1)^X$ over $\mathbb{P}^1$, which is $\mathcal{O}(1)$ minus the zero section. Let $\mu \in X_+(T)$, define a principal $G$-bundle over $\mathbb{P}^1$ as:

$$\mathcal{E}_\mu := \mu_* \mathcal{O}(1)^X$$

where we view $\mu$ as a morphism $\mu: \mathbb{G}_m \to G$. Next, every principal $G$-bundle $\mathcal{E}$ over $\mathbb{P}^1$ is isomorphic to exactly one $\mathcal{E}_\mu$, $\mu \in X_-(T)$ (see [21, Theorem 4.2], [15, Theorem 3.8a]) and [16]). Let $\text{ad}(\mathcal{E}_\mu)$ denote the adjoint vector bundle over $\mathbb{P}^1$ associated to $\mathcal{E}_\mu$. Note that $\text{ad}(\mathcal{E}_\mu) = \mathcal{O}(1)^X \times^G \mathfrak{g}$, i.e., it is the quotient of $\mathcal{O}(1)^X \times \mathfrak{g}$ under the action of $\mathbb{G}_m$ given by $g \cdot (e, f) = (e \cdot g, \text{Ad}_g^{-1}(f))$, $e \in \mathcal{O}(1)^X$, $f \in \mathfrak{g}$, $g \in \mathbb{G}_m$. Nilpotent elements of the Lie algebra $H^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu))$ are called nilpotent sections of $\text{ad}(\mathcal{E}_\mu)$.

3. Main Results

In this section we formulate the main results of this paper. In the special case $G = GL_n$, they give a counting result of Mellit [15, Section 5.4].

3.1. Coproduct. Let $H$ be a split connected reductive group over $\mathbb{F}_q$ with a split maximal torus $T_H$ and let $B_H$ be a Borel $\mathbb{F}_q$-subgroup containing $T_H$. Let $\Pi_H \subset X^+(T_H)$ denote the corresponding set of simple roots of $H$. For $J \subset \Pi_H$, let $P_J$ denote the standard parabolic $\mathbb{F}_q$-subgroup of $H$ and let $\mathcal{L}_J$ denote the standard Levi factor of $P_J$ (see Section 2.3). Let $W_H$ denote the Weyl group of $H$ relative to $T_H$ and $J_1, J_2 \subset \Pi_H$. We let $W_i$ denote the subgroup of $W_H$ generated by $s_\alpha$, $\alpha \in J_i$, $i = 1, 2$. We need the following notation:

**Notation 3.1.** It is known that every double coset in $W_1 \backslash W_H / W_2$ has a unique minimal length representative (see [4, Proposition 2.7.3]) and we denote this set of representatives by $D_{J_1, J_2}^H$. When we write $w \in W_1 \backslash W_H / W_2$, by abuse of notation we will mean that $w$ is any representative of the corresponding double coset.

Let $\mathcal{P}(\Pi_H)$ denote the set of subsets of $\Pi_H$. We let $Z[\mathcal{P}(\Pi_H)]$ denote the lattice of functions on $\mathcal{P}(\Pi_H)$ taking values in $\mathbb{Z}$. For any $f \in Z[\mathcal{P}(\Pi_H)]$, define

$$\Delta_H(f): \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H) \to \mathbb{Z},$$

which is given by

$$\Delta_H(f)(J_1, J_2) := \sum_{w \in D_{J_1, J_2}^H} f(J_1 \cap w \cdot J_2).$$

We will call $\Delta_H(f)$ as the coproduct of $f$. We have:

$$\Delta_H: Z[\mathcal{P}(\Pi_H)] \to Z[\mathcal{P}(\Pi_H)] \otimes Z[\mathcal{P}(\Pi_H)] \cong Z[\mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H)].$$
For any $J \subset \Pi_H$, let $Sp_H(J)$ denote the generalized Springer variety of $H$ with respect to $J$, which is defined as the following scheme of pairs:

$$Sp_H(J) := \{(n, P) : P \text{ is } F_q\text{-conjugate to } P_J, n \text{ is nilpotent}, n \in \text{Lie}(P)\}.$$ 

In other words, $P$ is a parabolic subgroup defined over $F_q$. For any two subsets $J_1, J_2 \subset \Pi_H$, let $St_H(J_1, J_2)$ denote the generalized Steinberg variety of $H$ with respect to $J_1$ and $J_2$, which is defined as the following scheme of triples:

$$St_H(J_1, J_2) := \{(n, P, Q) : P \text{ (resp. } Q) \text{ is } F_q\text{-conjugate to } P_{J_1}, \text{ (resp. } P_{J_2}), n \text{ is nilpotent}, n \in \text{Lie}(P) \cap \text{Lie}(Q)\}.$$ 

In other words, $P$ and $Q$ are parabolic subgroups defined over $F_q$. Observe that $Sp_H(J) \cong St_H(\Pi_H, J)$. Define

$$[Sp_H] : \mathcal{P}(\Pi_H) \to \mathbb{Z}, J \mapsto |Sp_H(J)|$$

and define

$$[St_H] : \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H) \to \mathbb{Z}, (J_1, J_2) \mapsto |St_H(J_1, J_2)|.$$ 

Let $\Phi_H$ denote the root system of $H$ with respect to $T_H$ and let $\Phi^+_H$ denote the set of positive roots with respect to $B_H$ and $T_H$. For $J \subset \Pi_H$, let $\Phi_J$ denote the root system of $L_J$ with respect to $T_H$ and let $\Phi^+_J$ denote the set of positive roots with respect to $B_H \cap L_J$ and $T_H$.

**Notation 3.2.** Let $M$ be an algebraic group over $\mathbb{F}_q$ and let $\mathfrak{m}$ be the associated Lie algebra. Recall that the rank of $M$ is the dimension of a maximal torus of $M$ or equivalently the dimension of a Cartan subalgebra of $\mathfrak{m}$. We will denote the rank of $M$ by $\text{rk}(M)$ or $\text{rk}(\mathfrak{m})$.

The following theorem gives an explicit formula for the number of points of generalized Steinberg varieties:

**Theorem 3.1.** With notations as above, we have

(i) $|Sp_H(J)| = q^{l(w)\cdot |\Phi_H^+| + |\Phi^+_J|} \sum_{w \in W_H} q^{l(w)}$, where $l(w)$ represents the minimal length of the elements in $wW_J$ and also $|\Phi_H^+| + |\Phi_J^+| = \text{dim}(P_J) - \text{rk}(P_J)$.

(ii) $\Delta_H([Sp_H]) = [St_H].$

We give the proof of Theorem 3.1 in Section 3.

### 3.2. Stratification of triples.

**Definition 2.** Fix an $\mathbb{F}_q$-rational point $x$ of $\mathbb{P}^1$. For $J \subset \Pi$, a **parabolic structure** on a principal $G$-bundle $\mathcal{E}$ over $\mathbb{P}^1$ at $x$ of type $J$ is a choice of an $\mathbb{F}_q$-rational point $P_x$ of $\mathcal{E}_x/P_J$ where $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at $x$.

Let $\mu \in X_{-}(T)$ and $J_0, J_\infty \subset \Pi$, define $\text{Trip}_\mu(J_0, J_\infty)$ to be the scheme parameterizing triples $(P_0, P_\infty, \Psi)$ such that $\Psi$ is a nilpotent section of $\text{ad}(\mathcal{E}_\mu)$, $P_0$ (resp. $P_\infty$) is a parabolic structure at $0$ (resp. $\infty$) of type $J_0$ (resp. $J_\infty$) and $\Psi_0 \in \text{Lie}(P_0)$, $\Psi_\infty \in \text{Lie}(P_\infty)$. We note that $\text{Trip}_\mu(J_0, J_\infty)$ is a scheme because it is the closed subscheme of $\mathcal{E}_0/P_{J_0} \times E_\infty/P_{J_\infty} \times H^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu))$ given by three closed conditions which are: $\Psi$ is nilpotent, $\Psi_0 \in \text{Lie}(P_0)$, $\Psi_\infty \in \text{Lie}(P_\infty)$.

Now let us explain the meaning of $\text{Lie}(P_x)$, $x = 0, \infty$ in the definition of $\text{Trip}_\mu(J_0, J_\infty)$. For $x = 0, \infty$, we view $(\mathcal{E}_\mu)_x$ as a principal $G$-bundle over the point $x$ and we let $\text{Aut}((\mathcal{E}_\mu)_x)$ denote the $\mathbb{F}_q$-group scheme whose $R$-valued points are the principal $G \times \text{Spec}(R)$-bundle automorphisms of $(\mathcal{E}_\mu)_x \times \text{Spec}(R)$. By Lang’s theorem (see [16]), $(\mathcal{E}_\mu)_x$ is a trivial principal $G$-bundle over the point $x$ and therefore $\text{Aut}((\mathcal{E}_\mu)_x)$ can be non-canonically identified with $G$. Now, $\text{Aut}((\mathcal{E}_\mu)_x)$ acts on $(\mathcal{E}_\mu)_x/P_{J_\infty}$ and the stabilizer of $P_x$ is a parabolic subgroup of $\text{Aut}((\mathcal{E}_\mu)_x)$. We denote by $\text{Lie}(P_x)$ the Lie algebra of this stabilizer. This is a parabolic subalgebra of $\text{Lie}(\text{Aut}((\mathcal{E}_\mu)_x)) = \text{ad}(\mathcal{E}_\mu)_x$.

Since $O(1)^{\times}$ is a principal $\mathbb{G}_m$-bundle over $\mathbb{P}^1$, $\mathbb{G}_m$ acts on $\mathcal{E}_\mu = (O(1)^{\times} \times G)/\mathbb{G}_m$ by acting on the first component. This gives a $\mathbb{G}_m$-action on the parabolic structures and on $\text{ad}(\mathcal{E}_\mu)$, which gives a $\mathbb{G}_m$-action on $H^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu))$. On combining these actions, we get a $\mathbb{G}_m$-action:

$$\mathbb{G}_m \curvearrowright \text{Trip}_\mu(J_0, J_\infty).$$
The proof of Theorem 3.2 will be given in Section 6.

For any $\mu \in X_+(T)$, let $\Pi_{\mu} \subset \Pi$ denote the set of simple roots that are orthogonal to $\mu$ and denote by $L_\mu$ the identity component of the centralizer of $\mu(G_m)$ in $G$. We have $L_\mu$ is a Levi $\mathbb{F}_q$-subgroup of $G$ (see [5 Proposition 2.1.8(2)]) and use the fact that the identity component of the centralizer of a torus is reductive [5 Proposition A.8.12]). We have $\Pi_{\mu}$ is the set of simple roots of $L_\mu$ corresponding to $T$ and $B \cap L_\mu$. In the special case $G = GL_n$, if $\mu$ is of the form $t \mapsto \text{diag}(t^{m_{i_1}}, \ldots, t^{m_{i_s}})$, $m_i \neq m_j$ for $i \neq j$, $m_j \in \mathbb{Z}$ for $1 \leq j \leq s$, then $L_\mu = GL_{i_1} \times \ldots \times GL_{i_s}$.

**Notation 3.3.** For $J \subset \Pi$, denote by $W_J \subset W$ the subgroup generated by $s_\alpha, \alpha \in J$, here $s_\alpha$ denotes the reflection corresponding to $\alpha$. For any $\mu \in X_+(T)$, let $\Pi_{\mu} \subset \Pi$ denote the set of simple roots that are orthogonal to $\mu$ and denote by $L_\mu$ the identity component of the centralizer of $\mu(G_m)$ in $G$. We have $L_\mu$ is a Levi $\mathbb{F}_q$-subgroup of $G$ (see [5 Proposition 2.1.8(2)]) and use the fact that the identity component of the centralizer of a torus is reductive [5 Proposition A.8.12]). We have $\Pi_{\mu}$ is the set of simple roots of $L_\mu$ corresponding to $T$ and $B \cap L_\mu$. In the special case $G = GL_n$, if $\mu$ is of the form $t \mapsto \text{diag}(t^{m_{i_1}}, \ldots, t^{m_{i_s}})$, $m_i \neq m_j$ for $i \neq j$, $m_j \in \mathbb{Z}$ for $1 \leq j \leq s$, then $L_\mu = GL_{i_1} \times \ldots \times GL_{i_s}$.

**Notation 3.4.** Let $X$ be a scheme over a field $K$ and let $H$ be an algebraic group over $K$ acting on $X$. We will denote the fixed point locus of this action by $X^H$.

In this paper, we want to count the number of $\mathbb{F}_q$-points of $\text{Trip}_\mu(J_0, J_\infty)$ for each $\mu \in X_+(T)$, $J_0, J_\infty \subset \Pi$. For this, we would like to apply the Białynicki-Birula decomposition to $\text{Trip}_\mu(J_0, J_\infty)$. Note that it is not immediate in this case because $\text{Trip}_\mu(J_0, J_\infty)$ is neither smooth nor projective in general but nevertheless we have the following theorem, which allows to reduce counting $|\text{Trip}_\mu(J_0, J_\infty)|$ to counting generalized Steinberg varieties. We prove the following Theorem in Section 6.

**Theorem 3.2.** Keep notations as above. Let $G_m$ act on $\text{Trip}_\mu(J_0, J_\infty)$ as in (1). Then there exists a stratification of $\text{Trip}_\mu(J_0, J_\infty)$ by locally closed subsets as:

$$\text{Trip}_\mu(J_0, J_\infty) = \bigsqcup_{w \in W_{\Pi_{\mu}} \setminus W/W_{J_0}} \text{Trip}_\mu(J_0, J_\infty)^+_{w, w'}$$

and a decomposition of $\text{Trip}_\mu(J_0, J_\infty)^{G_m}$ as:

$$\text{Trip}_\mu(J_0, J_\infty)^{G_m} = \bigsqcup_{w \in W_{\Pi_{\mu}} \setminus W/W_{J_0}} \text{Trip}_\mu(J_0, J_\infty)^{G_m}_{w, w'},$$

where $\text{Trip}_\mu(J_0, J_\infty)^{G_m}_{w, w'}$ are the connected components of $\text{Trip}_\mu(J_0, J_\infty)^{G_m}$ with morphisms

$$\text{Trip}_\mu(J_0, J_\infty)^+_{w, w'} \to \text{Trip}_\mu(J_0, J_\infty)^{G_m}_{w, w'},$$

which are given by the limit map as $t \to 0$ and are affine fibrations for $w \in W_{\Pi_{\mu}} \setminus W/W_{J_0}, w' \in W_{\Pi_{\mu}} \setminus W/W_{J_\infty}$ of relative dimensions $\dim(\text{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)$. Moreover, the schemes $\text{Trip}_\mu(J_0, J_\infty)^{G_m}_{w, w'}, w \in D_{\Pi_{\mu}}^{\infty}, w' \in D_{\Pi_{\mu}}^{\infty}$ are isomorphic to the generalized Steinberg varieties $St_{L_\mu}(\Pi_{\mu} \cap w \cdot J_0, \Pi_{\mu} \cap w' \cdot J_\infty)$ defined in Section 3.1.

The proof of Theorem 3.2 will be given in Section 6.

For $\mu \in X_-(T)$, define $\pi_{\mu} : Z[\mathcal{P}(\Pi_{\mu})] \to Z[\mathcal{P}(\Pi)]$ as:

$$\pi_{\mu}(f)(J) := \sum_{w \in D_{\Pi_{\mu}}^{\infty}} f(\Pi_{\mu} \cap w \cdot J), \quad f \in Z[\mathcal{P}(\Pi_{\mu})].$$

and define $|\text{Trip}_\mu| : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to Z$ as:

$$|\text{Trip}_\mu|(J_0, J_\infty) := |\text{Trip}_\mu(J_0, J_\infty)|$$

As an easy corollary of Theorem 3.2, we get:

**Corollary 3.2.1.** Keeping the above notations, we have:

$$|\text{Trip}_\mu| = q^{\dim(\text{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)}(\pi_{\mu} \otimes \pi_{\mu})(|St_{L_\mu}|).$$
Proof. Let \( J_0, J_\infty \subset \Pi \). From Theorem \( 3.2 \) we have

\[
[Trip_\mu](J_0, J_\infty) = \sum_{w \in W_{\mu,J_0} \setminus W/W_{J_0}} |Trip_\mu(J_0, J_\infty)_{w,w'}^+| = \sum_{w \in W_{\mu,J_0} \setminus W/W_{J_0}} q^{\dim(\Aut(\xi_\mu)) - \dim(L_\mu)} |Trip_\mu(J_0, J_\infty)_{w,w'}^G|.
\]

Since the schemes \( Trip_\mu(J_0, J_\infty)^G_{w,w'} \) are isomorphic to the generalized Steinberg varieties \( St_{L_\mu}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty) \) (see Theorem \( 3.2 \)), we have

\[
[Trip_\mu](J_0, J_\infty) = q^{\dim(\Aut(\xi_\mu)) - \dim(L_\mu)} \sum_{w \in D_{\Pi,J_0}^L} \sum_{w' \in D_{\Pi,J_\infty}^L} q^{\Phi_{\Pi_\mu \cap w \cdot J_0 \cap w' \cdot J_\infty}} |St_{L_\mu}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty)|.
\]

Now the corollary follows from the definition of \( \pi_\mu \).

More explicitly, we have the following corollary.

**Corollary 3.2.2.** Keep notations as above. Then we have

\[
[Trip_\mu](J_0, J_\infty) = |L_\mu| q^{\dim(\Aut(\xi_\mu)) - \dim(L_\mu)} \sum_{w \in D_{\Pi,J_0}^L} \sum_{w' \in D_{\Pi,J_\infty}^L} q^{\Phi_{\Pi_\mu \cap w \cdot J_0 \cap w' \cdot J_\infty}} |St_{L_\mu}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty)|,
\]

where \( \Phi_{\Pi_\mu \cap w \cdot J_0 \cap w' \cdot J_\infty} \) is the root system of \( L_{\Pi_\mu \cap w \cdot J_0 \cap w' \cdot J_\infty} \) with respect to \( T \).

**Proof.** By Theorem \( 3.1 \) (ii) and Corollary \( 3.2.1 \) we get

\[
[Trip_\mu](J_0, J_\infty) = q^{\dim(\Aut(\xi_\mu)) - \dim(L_\mu)} \sum_{w \in D_{\Pi,J_0}^L} \sum_{w' \in D_{\Pi,J_\infty}^L} \Delta_{L_\mu}([Sp_{L_\mu}])(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty).
\]

Now the corollary follows from the definition of \( \Delta_{L_\mu} \) and \( [T] \).

It follows from definitions that \( Trip_0(J_0, J_\infty) = St_G(J_0, J_\infty) \). We note the following corollary.

**Corollary 3.2.3.** Keep notations as above and assume that \( \mu \in X_-(T) \) is a central cocharacter. Then \( [Trip_\mu] = [St_G] \).

**Proof.** It follows from \( [2] \) that \( [Trip_\mu] = [Trip_0] \).

**Remark 3.1.** (i) For deducing Corollary \( 3.2.1 \) from Theorem \( 3.2 \) it is crucial that all fibers of the morphism \( Trip_\mu(J_0, J_\infty)^G_{w,w'} \rightarrow Trip_\mu(J_0, J_\infty)^G_{w,w'} \) have the same dimension.
(ii) Since we know \( [Sp_{L_\mu}] \) for \( \mu \in X_-(T), J \subset \Pi_\mu \) explicitly (see Theorem \( 3.2 \), Corollary \( 3.2.1 \)) gives an explicit formula for \( [Trip_\mu] \).
(iii) Notice that \( \pi_\mu \) is an instance of \( \Delta_G \). More precisely, let \( f \in \mathbb{Z}[\mathcal{P}(\Pi_\mu)] \) and let \( \tilde{f} \) be any extension of \( f \) to \( \mathcal{P}(\Pi) \), i.e., \( \tilde{f} \in \mathbb{Z}[\mathcal{P}(\Pi)] \) and \( \tilde{f}|_{\mathcal{P}(\Pi_\mu)} = f \). Then we have \( \pi_\mu(f) = \Delta_G(f)(\Pi_{\mu^*}) \).

3.3. **Comparison between different groups.** We will compare \( [Trip_\mu(J_0, J_\infty)] \) for different groups below. For this, we introduce the following notation.

**Notation 3.5.** Let \( H, T_H, B_H, \Pi_H \) be as in Section \( 3.4 \) and let \( \xi_\nu \) be the principal \( H \)-bundle over \( \mathbb{P}^1 \) induced by \( \nu \). For \( J_0, J_\infty \subset \Pi_H \), as before we let \( Trip_{\nu,H}(J_0, J_\infty) \) denote the scheme parameterizing triples \((P_0, P_\infty, \Psi)\) such that \( \Psi \) is a nilpotent section of \( \text{ad}(\xi_\nu) \), \( P_0 \) (resp. \( P_\infty \)) is a parabolic structure at \( 0 \) (resp. \( \infty \)) of type \( J_0 \) (resp. \( J_\infty \)) and \( \Psi_0 \in \text{Lie}(P_0), \Psi_\infty \in \text{Lie}(P_\infty) \). Again as before, define \( [Trip_{\nu,H}] : \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H) \rightarrow \mathbb{Z} \) by

\[
[Trip_{\nu,H}](J_0, J_\infty) := [Trip_{\nu,H}(J_0, J_\infty)]
\]
Consider the following two situations:

(i) Recall $G, T, B, \Pi$ from Section 2.2. Let $G' := [G, G]$ be the derived group of $G$. Let $j : G' \to G$ be the natural inclusion. Denote the split maximal torus $T \cap G'$ by $T'$ and the Borel $\mathbb{F}_q$-subgroup $B \cap G'$ of $G'$ by $B'$. Let $\mu' \in X_-(T')$, we have $\mu := j \circ \mu' \in X_-(T)$. Since the root systems of $G$ and $G'$ are isomorphic, we will consider $\mathcal{T}(\mu', G')$ and $\mathcal{T}(\mu, G)$ as functions with domain $M \times M$.

(ii) Recall that a morphism $u : G_1 \to G_2$ of connected algebraic groups over $\mathbb{F}_q$ is called a central isogeny if it is a finite flat surjection such that $\ker(u)$ is central in $G_1$ (see [16] Definition 3.3.9). Now let $u : G_1 \to G_2$ be a central isogeny of split connected reductive groups over $\mathbb{F}_q$. Let $T_1$ be a split maximal torus of $G_1$ and let $B_1$ be a Borel $\mathbb{F}_q$-subgroup of $G_1$ containing $T_1$. Then $T_2 := u(T_1)$ is a split maximal torus of $G_2$ and $B_2 := u(B_1)$ is a Borel $\mathbb{F}_q$-subgroup of $G_2$ containing $T_2$ (see [16] Section 3.3). Let $\mu_1 \in X_-(T_1)$, we have $\mu_2 := u \circ \mu_1 \in X_-(T_2)$. Since the root systems of $G_1$ and $G_2$ are isomorphic, we will consider $[\mathcal{T}(\mu_1, G_1)]$ and $[\mathcal{T}(\mu_2, G_2)]$ as functions with domain $M \times M$, where $\Pi$ is the set of simple roots of $G_1$ with respect to $(B_1, T_1)$.

We have the following:

**Corollary 3.2.4. (a) With notations as in (i) above, we have**

$$[\mathcal{T}(\mu', G')] = [\mathcal{T}(\mu, G)].$$

(b) With notations as in (ii) above, we have

$$[\mathcal{T}(\mu_1, G_1)] = [\mathcal{T}(\mu_2, G_2)].$$

As special cases, we may take $G = GL_n$ and $G' = SL_n$ in Corollary 3.2.4 (a) and $G_1 = SL_n$ and $G_2 = PGL_n \cong SL_n/\mu_n$ in Corollary 3.2.3 (b).

4. Generalized Steinberg Varieties

In this section, we give a proof of Theorem 3.1. Recall that for any scheme $X$ over $\mathbb{F}_q$, we denote the number of $\mathbb{F}_q$-rational points of $X$ by $|X|$.

We now prove a simple lemma that will be used several times in the paper:

**Lemma 4.1.** Let $M$ be an algebraic group over $\mathbb{F}_q$ and let $M'$ be a connected $\mathbb{F}_q$-subgroup of $M$. Then $|M/M'| = |M|/|M'|$.

**Proof.** Let $x : \text{Spec}(\mathbb{F}_q) \to M/M'$ be an $\mathbb{F}_q$-rational point of $M/M'$ and let $M \xrightarrow{\pi} M/M'$ be the natural morphism giving $M$ a structure of a principal $M'$-bundle over $M/M'$. Pulling back the principal $M'$-bundle $M \xrightarrow{\pi} M/M'$ along $x$, we get a principal $M'$-bundle $x^*M \to \text{Spec}(\mathbb{F}_q)$. Recall that a theorem of Lang ([16]) asserts that for any connected algebraic group $H$ over a finite field $K$, every principal $H$-bundle over $\text{Spec}(K)$ is trivial, thus we get that $x^*M \to \text{Spec}(\mathbb{F}_q)$ is a trivial principal $M'$-bundle and so, $x^*M \cong M'$. Since $\mathbb{F}_q$-rational points of $M$ map to $\mathbb{F}_q$-rational points of $M/M'$ under $\pi$, the number of $\mathbb{F}_q$-rational points of $M$ mapping to $x$ is equal to $|M'|$ and the lemma follows. $\square$

**Notation 4.1.** Let $M$ be an algebraic group over $\mathbb{F}_q$ and let $\mathfrak{m}$ be the associated Lie algebra. We will denote the nilpotent cone of $\mathfrak{m}$ by $N(\mathfrak{m})$.

**Proposition 4.1.** Let $M$ be an arbitrary connected algebraic group over $\mathbb{F}_q$ and $\mathfrak{m}$ be its Lie algebra. Then $|N(\mathfrak{m})| = q^{\dim(\mathfrak{m}) - \text{rk}(\mathfrak{m})}$.

**Proof.** The case of connected reductive groups over $\mathbb{F}_q$ is proved in [25] (7). We claim that the general case follows from the case of connected reductive groups over $\mathbb{F}_q$. Indeed, let $R_u(M)$ denote the $\mathbb{F}_q$-unipotent radical of $M$. We claim that $M/R_u(M)$ is a connected reductive group over $\mathbb{F}_q$. We have $M_{\overline{\mathbb{F}}_q}/R_u(M_{\overline{\mathbb{F}}_q})$ is a connected reductive group over $\overline{\mathbb{F}}_q$ (to see this, consider the natural projection $\pi : M_{\overline{\mathbb{F}}_q} \to M_{\overline{\mathbb{F}}_q}/R_u(M_{\overline{\mathbb{F}}_q})$, assume on the contrary that
there exists a non-trivial connected, unipotent, normal subgroup $U$ of $M_{\mathbb{F}_q}/R_u(M_{\mathbb{F}_q})$, then $\pi^{-1}(U)^o$ satisfies the same properties and strictly contains $R_u(M_{\mathbb{F}_q})$, which contradicts the fact that $R_u(M_{\mathbb{F}_q})$ is the unipotent radical of $M_{\mathbb{F}_q}$.

Now since $(M/R_u(M))_{\mathbb{F}_q} \cong M_{\mathbb{F}_q}/R_u(M_{\mathbb{F}_q})$ and $R_u(M_{\mathbb{F}_q}) = R_u(M)_{\mathbb{F}_q}$ inside $M_{\mathbb{F}_q}$ (see [5 Proposition 1.1.9(1)]), we get that $R_u(M/R_u(M))_{\mathbb{F}_q} = \{1\}$ (see [3 Proposition 1.1.9(1)]). As a consequence, we have $R_u(M/R_u(M)) = \{1\}$, therefore $M/R_u(M)$ is a connected reductive group over $\mathbb{F}_q$. Now let $u$ denote the Lie algebra of $R_u(M)$, we have $\text{Lie}(M/R_u(M)) = \mathfrak{m}/u$.

We need a simple lemma now:

**Lemma 4.2.** With notations as above, we have

$$|N(\mathfrak{m})| = q^{\dim(\mathfrak{m})}|N(\mathfrak{m}/\mathfrak{u})|.$$  

**Proof.** Consider the natural projection $\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{u}$. We will prove that $N(\mathfrak{m}) = \pi^{-1}(N(\mathfrak{m}/\mathfrak{u}))$ from which the lemma would follow easily. Since $\pi$ maps nilpotent elements of $\mathfrak{m}$ to nilpotent elements of $\mathfrak{m}/\mathfrak{u}$, we get $\pi(N(\mathfrak{m})) \subset N(\mathfrak{m}/\mathfrak{u})$.

Now suppose $x \in \pi^{-1}(N(\mathfrak{m}/\mathfrak{u}))$, using Jordan decomposition write $x = x_s + x_n$, where $x_s$ is a semisimple element, $x_n$ is a nilpotent element and $[x_s, x_n] = 0$. Assume on the contrary that $x_s \neq 0$. Since $\pi$ is a Lie algebra morphism, $\pi(x) = \pi(x_s) + \pi(x_n)$ is the Jordan decomposition of $\pi(x)$. Since $x_s \notin \mathfrak{u}$, we have $\pi(x_s) \neq 0$. Therefore, $\pi(x) \notin N(\mathfrak{m}/\mathfrak{u})$, which is a contradiction. Thus, we have $N(\mathfrak{m}) = \pi^{-1}(N(\mathfrak{m}/\mathfrak{u}))$. Since $\pi$ is clearly surjective, we get $|N(\mathfrak{m})| = |\mathfrak{u}|N(\mathfrak{m}/\mathfrak{u})$. Now the lemma follows from $|\mathfrak{u}| = q^{\dim(\mathfrak{u})}$.  

Now that the statement of Proposition 4.1 is known for reductive groups (see [25 (7)]), we obtain

$$|N(\mathfrak{m}/\mathfrak{u})| = q^{\dim(\mathfrak{m}/\mathfrak{u})} - \text{rk}(\mathfrak{m}/\mathfrak{u}).$$

Since $\text{rk}(\mathfrak{m}) = \text{rk}(\mathfrak{m}/\mathfrak{u})$, we get

$$|N(\mathfrak{m}/\mathfrak{u})| = q^{\dim(\mathfrak{m}/\mathfrak{u})} - \text{rk}(\mathfrak{m}).$$

By applying Lemma 4.2 to (3), we get

$$|N(\mathfrak{m})| = q^{\dim(\mathfrak{u})}q^{\dim(\mathfrak{m}/\mathfrak{u})} - \text{rk}(\mathfrak{m}) = q^{\dim(\mathfrak{m})} - \text{rk}(\mathfrak{m}).$$

This finishes the proof of Proposition 4.1.  

**4.1. Proof of Theorem 3.1 (i).** Let $H$ be a split reductive group over $\mathbb{F}_q$ with a split maximal torus $T_H$ and let $B_H$ be a Borel $\mathbb{F}_q$-subgroup containing $T_H$. Let $\Pi_H \subset X^*(T_H)$ denote the corresponding set of simple roots of $H$. Let $W_H$ denote the Weyl group of $H$ relative to $T_H$. For any $J \subset \Pi_H$, let $P_J$ be the corresponding standard parabolic $\mathbb{F}_q$-subgroup of $H$, $i = 1, 2$. Let $L_J$ and $U_J$ be the Levi factor and the unipotent radical of $P_J$, respectively and let $W_J$ be the corresponding subgroup of $W_H$. The number of points of the generalized Springer variety of $H$ corresponding to $J \subset \Pi_H$ is given by

$$|Sp_H(J)| = \frac{|H|}{|P_J|}|N(\text{Lie}(P_J))| = \frac{|H|}{|P_J|}q^{\dim(P_J) - \text{rk}(P_J)},$$

where the first equality holds because the normalizer of $P_J$ is itself and the fact that if $P$ is a parabolic subgroup of $G$ conjugate over $\mathbb{F}_q$ to $P_J$, then $N(\text{Lie}(P)) \cong N(\text{Lie}(P_J))$. The second equality follows from Proposition 4.1.  

Since $H/P_J$ has a stratification by locally closed subsets as $\bigsqcup_{w \in W_H/W_J} l(w)$ (see [2] Proposition 3.16)), where $l(w)$ represents the minimal length of the elements in $wW_J$, using Lemma 4.1 we get that $|H|/|P_J| = \sum_{w \in W_H/W_J} q^{l(w)}$, which gives

$$|Sp_H(J)| = q^{\Phi_J^+| + \Phi_J^-|} \sum_{w \in W_H/W_J} q^{l(w)}.$$  

This finishes the proof of part (i) of Theorem 3.1.
4.2. Proof of Theorem 3.1(ii). In the proof of part (ii) of Theorem 3.1 we will need another formula for $|Sp_H(J)|$, which we now give. Let $U_J$ denote the unipotent radical of $P_J$. Then, we have $P_J \cong L_J \times U_J$ as schemes over $\mathbb{F}_q$ and $|U_J| = q^{\dim(U_J)}$, (see [24 Corollary 14.2.7]). These two facts together with Lemma 4.1 give

$$|Sp_H(J)| = \frac{|H|}{|L_J|} q^{\dim(L_J) - \text{rk}(L_J)}.$$  

Now Proposition 4.1 gives

$$|Sp_H(J)| = |H| \frac{|N(\text{Lie}(L_J))|}{|L_J|}.$$  

For any $J_i \subseteq \Pi_H$, let $P_i$ be the corresponding standard parabolic $\mathbb{F}_q$-subgroup of $H$, $i = 1, 2$. Let $L_i$ and $U_i$ be the Levi factor and the unipotent radical of $P_i$ respectively, and let $W_i$ be corresponding subgroup of $W_H$, $i = 1, 2$. The number of points of generalized Steinberg variety of $H$ corresponding to $J_1$ and $J_2$ is given by

$$|St_H(J_1, J_2)| = \frac{|H|}{|P_1|} \sum_{h \in H(\mathbb{F}_q)/P_1(\mathbb{F}_q)} |N(\text{Lie}(P_1 \cap h \cdot P_2))| \frac{|H|}{|P_1|} \sum_{h \in P_1(\mathbb{F}_q)/P_2(\mathbb{F}_q)} |N(\text{Lie}(P_1 \cap h \cdot P_2))|$$

where the first equality holds because the normalizer of $P_1$ is itself. Now using $P_i(\mathbb{F}_q)/P_1(\mathbb{F}_q) \cong W_i \backslash W_H/W_2$ (see [11 Theorem 65.21], [19 Theorem 21.91]) and use the well-known fact that $H(\mathbb{F}_q)$ is a finite group with a BN-pair for $B = B_H(\mathbb{F}_q), N = NT_H(\mathbb{F}_q)$, where $NT_H$ is the normalizer of $T_H$ in $H$, Lemma 4.1 and Proposition 4.1 we get

$$|St_H(J_1, J_2)| = |H| \sum_{w \in W_1 \backslash W_H/W_2} q^{\dim(P_1 \cap wP_2w^{-1}) - \text{rk}(P_1 \cap wP_2w^{-1})} |P_1 \cap wP_2w^{-1}|.$$  

Next, we have the following decomposition (the statement is easily reduced to $\mathbb{F}_q$ in which case it is given by [12 Proposition 2.15]):

$$P_1 \cap wP_2w^{-1} = (L_1 \cap wL_2w^{-1})(L_1 \cap wU_2w^{-1})(U_1 \cap wL_2w^{-1})(U_1 \cap wU_2w^{-1})$$

which is a direct product of varieties over $\mathbb{F}_q$. Using this decomposition along with the fact that for any unipotent algebraic group $U$ over $\mathbb{F}_q$, we have $|U| = q^{\dim(U)}$ (see [24 Corollary 14.2.7]), we obtain

$$|St_H(J_1, J_2)| = |H| \sum_{w \in W_1 \backslash W_H/W_2} q^{\dim(L_1 \cap wL_2w^{-1}) - \text{rk}(L_1 \cap wL_2w^{-1})} $$

where we use Proposition 4.1 for the second equality. Recall $D_H^{J_1, J_2}$ from Section 3.1 and let $w \in D_H^{J_1, J_2}$. We also have the following decomposition (the statement is easily reduced to $\mathbb{F}_q$ in which it is given by [4 Theorem 2.8.7]):

$$P_1 \cap wP_2w^{-1} = (L_{J_1 \cap w \cdot J_2})(L_1 \cap wU_2w^{-1})(U_1 \cap wL_2w^{-1})(U_1 \cap wU_2w^{-1})$$

By (7), (8) and the fact that $L_{J_1 \cap w \cdot J_2} \subseteq L_1 \cap wL_2w^{-1}$, we get $L_{J_1 \cap w \cdot J_2} = L_1 \cap wL_2w^{-1}$, which gives

$$|St_H(J_1, J_2)| = |H| \sum_{w \in D_H^{J_1, J_2}} |N(\text{Lie}(L_{J_1 \cap w \cdot J_2}))|$$

Recalling that $\Delta_H$ is given by

$$\Delta_H(f)(J_1, J_2) = \sum_{w \in D_H^{J_1, J_2}} f(J_1 \cap w \cdot J_2),$$

we get from (8) and (9) that

$$\Delta_H([Sp_H]) = [St_H].$$

This finishes the proof of part (ii) of Theorem 3.1.
4.3. More on coproduct. In this section, we would like to prove few properties of $\Delta_H$ that are of independent interest and will be used later in Section 7 in the case of $GL_n$. First we need some definitions.

**Definition 3.** Let $J_1, J_2 \subset \Pi_H$, we say $J_1$ and $J_2$ are associates whenever $\Phi(J_2) = w \cdot \Phi(J_1)$ for some $w \in W_H$. This gives an equivalence relation on $\mathcal{P}(\Pi_H)$, which we denote by $\sim_H$. Let $f \in \mathbb{Z}[\mathcal{P}(\Pi)]$, we say $f$ is associate invariant if $f(J_1) = f(J_2)$ whenever $J_1$ and $J_2$ are associates.

Let $O$ be an equivalence class of $\sim_H$. Let $\delta_O \in \mathbb{Z}[\mathcal{P}(\Pi_H)]$ be the function on $\mathcal{P}(\Pi_H)$ that takes the value 1 on $J$ if $J \in O$ and 0 otherwise. Let us fix a representative $J_O$ in each equivalence class $O$. The following lemma states that $\Delta_H$ preserves associate invariant functions.

**Lemma 4.3.** Keep notations as above. Then

$$\Delta_H(\delta_O) = \sum_{(O_1, O_2) \in (\mathcal{P}(\Pi_H)/\sim) \times (\mathcal{P}(\Pi_H)/\sim)} n^{O_1, O_2} \delta_{O_1} \otimes \delta_{O_2},$$

where

$$n^{O_1, O_2} = |\{w \in W_{J_{O_1}} \setminus W_H/W_{J_{O_2}} : \Phi(J_{O_1}) \cap w \cdot \Phi(J_{O_2}) = w' \cdot \Phi(J_O) \text{ for some } w' \in W_H\}|.$$

In particular, $\Delta_H$ preserves the subspace of associate invariant functions.

**Proof.** First we rewrite the coproduct $\Delta_H$. Set $\Phi(J) = \Phi(L_J)$, so that $\Phi$ is a bijection from $\mathcal{P}(\Pi_H)$ onto the set of root systems of all Levi subgroups of $H$ containing $T_H$. Then

$$\Delta_H(f)(J_1, J_2) = \sum_{w \in W_{J_2} \setminus W_{J_1} \setminus W_H} f(\Phi^{-1}(\Phi(J_1) \cap w \cdot \Phi(J_2))), \quad f \in \mathbb{Z}[\mathcal{P}(\Pi_H)].$$

For any $J'_1, J'_2 \in \mathcal{P}(\Pi_H)$, $\Delta_H(\delta_O)$ evaluated at $(J'_1, J'_2)$ is equal to

$$\delta_O(\Phi^{-1}(\Phi(J'_1) \cap w \cdot \Phi(J'_2)))) = |\{w \in W_{J'_1} \setminus W_H/W_{J'_2} : \Phi(J'_1) \cap w \cdot \Phi(J'_2) = w' \cdot \Phi(J_O) \text{ for some } w' \in W_H\}|.$$

On the other hand, RHS evaluated at $(J'_1, J'_2)$ is equal to $n^{J'_1, J'_2}$, where $J'_1$ (resp. $J'_2$) is the chosen representative of $[J'_1]$ (resp. $[J'_2]$). There exists $w_1, w_2 \in W_H$ such that $J'_1 = w_1 \cdot J_1$, $J'_2 = w_2 \cdot J_2$ and so, $W_{J'_1} = w_1 W_{J_1} w_1^{-1}$ and $W_{J'_2} = w_2 W_{J_2} w_2^{-1}$. Now the lemma follows from the bijection

$$W_{J_1} \setminus W_H/W_{J_2} \to W_{J'_1} \setminus W_H/W_{J'_2}, \quad W_{J_1} w W_{J_2} \to W_{J'_1} (w_1 w w_2^{-1}) W_{J'_2}.$$

This finishes the proof of Lemma 4.3. □

**Remark 4.1.** The proof of Lemma 4.3 suggests that (10) may be a better definition for $\Delta_H$ as it does not use Proposition 2.7.3. In fact, it may be even better to view $f$ as a function on the set of root systems of the Levi subgroups. Moreover, using this formulation it is easy to see that $\Delta_H$ is co-commutative for associate invariant functions.

As a consequence of Lemma 4.3, we have the following corollary.

**Corollary 4.3.1.** Let $[Sp_H]$ and $[St_H]$ be as in Section 3.7. Then $[Sp_H]$ and $[St_H]$ are associate invariant functions.

**Proof.** Let $J, J' \in \Pi_H$ be such that $J \sim_H J'$. Then we have $L_J \cong L_{J'}$ and as a consequence of (6), it follows that $[Sp_H]$ is associate invariant. Now Lemma 4.3 together with Theorem 3.1(ii) imply that $[St_H]$ is associate invariant in each variable. □
Assume that $H = H_1 \times \ldots \times H_n$ and let $\Pi_k$ be the set of simple roots of $H_k$ with respect to some maximal torus and a Borel subgroup containing it. We can identify $\Pi_H$ with the disjoint union $\bigsqcup_k \Pi_k$. Thus, $\mathcal{P}(\Pi_H) = \prod_k \mathcal{P}(\Pi_k)$ and $\mathbb{Z}[\mathcal{P}(\Pi_H)] = \bigotimes_k \mathbb{Z}[\mathcal{P}(\Pi_k)]$. Under this isomorphism, the following lemma follows from the definitions.

**Lemma 4.4.** Keep notations as above. Then

$$[\text{St}_H] = [\text{St}_{H_1}] \otimes \ldots \otimes [\text{St}_{H_n}].$$

5. Bialynicki–Birula decomposition

In this section we recall Bialynicki–Birula decomposition. We will use these facts in the next section to give a proof of Theorem 3.2.

**Definition 4.** Let $X$ and $Z$ be two schemes. A morphism $\phi : X \to Z$ is called an affine fibration of relative dimension $d$ if for every $z \in Z$, there is a Zariski open neighborhood $U$ of $z$ such that $X_U \cong U \times \mathbb{A}^d$ and this isomorphism identifies $\phi_U : X_U \to Z$ with the projection on the first factor.

We use the following result (see [3, Theorem 3.2]), known as Bialynicki–Birula decomposition which is key to our calculation:

**Fact 5.1.** (Bialynicki–Birula, Hesselink, Iversen). Let $K$ be any field. Let $X$ be a smooth, projective scheme over $K$ equipped with a $\mathbb{G}_m$-action. Then the following holds:

(i) The fixed point locus $X^{\mathbb{G}_m}$ is a closed subscheme of $X$ and is smooth.

(ii) There exists a numbering $X^{\mathbb{G}_m} = \bigsqcup_i Z_i$ of the connected components of $X^{\mathbb{G}_m}$, and a filtration of $X$ by closed subschemes:

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

and affine fibrations $\phi_i : X_i - X_{i-1} \to Z_i$.

(iii) The relative dimension of $\phi_i$ is the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space of $X$ at an arbitrary closed point $z \in Z_i$ and $\dim(Z_i) = \dim(T^\mathbb{G}_m_{Z_i}X)$. We have $X$ is stratified by locally closed subsets $X_i^+ := X_i - X_{i-1}$.

**Definition 5.** Let $K$ be a field. Let $Y$ be a separated scheme over $K$. Let $\phi : \mathbb{A}^1 \setminus \{0\} \to Y$ be a morphism. If $\phi$ extends to a morphism $\hat{\phi} : \mathbb{A}^1 \to Y$, we say that $\lim_{t \to 0} \phi(t)$ exists and set it equal to $\hat{\phi}(0)$. Since $Y$ is separated, the extension $\hat{\phi}$ is unique. Note that if, moreover, $Y$ is proper then an extension of $\hat{\phi}$ always exists.

**Remark 5.1.** (see [3, Section 3]) The Bialynicki–Birula decomposition is explicit in the sense that the locally closed subscheme $X_i^+$ is the set of all points $x \in X$ such that $\lim_{t \to 0} t \cdot x \in Z_i$ where $(t, x) \mapsto t \cdot x$ is the $\mathbb{G}_m$-action. Moreover, the map $\phi_i : X_i^+ \to Z_i$ is then given by $x \mapsto \lim_{t \to 0} t \cdot x$.

Let $K$ be a field. Let $S$ be a smooth separated scheme over $K$ equipped with a $\mathbb{G}_m$-action. By [5, Proposition A.8.10], $S^{\mathbb{G}_m}$ is smooth over $K$. By a smooth equivariant compactification of $S$, we will mean a scheme $\overline{S}$ that is smooth and projective over $K$, $S$ is an open and dense subscheme of $\overline{S}$ and $\overline{S}$ is equipped with a $\mathbb{G}_m$-action that extends the $\mathbb{G}_m$-action on $S$. The following proposition is a consequence of Fact 5.1.

**Proposition 5.1.** Assume that there is an equivariant smooth compactification $\overline{S}$ of $S$. Let $S^{\text{fin}}$ be the subset of $S$ consisting of points $x$ in $S$ for which $\lim_{t \to 0} t \cdot x$ exists in $S$. Then $S^{\text{fin}}$ is a constructible subset of $S$ and there exists a stratification of $S^{\text{fin}}$ by locally closed subsets as:

$$S^{\text{fin}} = \bigsqcup_{\alpha \in I} S_{\alpha}^+$$
and a decomposition of $S_{G_m}^\alpha$ as:

$$S_{G_m}^\alpha = \bigsqcup_{\alpha \in I} S_{G_m}^\alpha,$$

where $S_\alpha$ are the connected components of $S_{G_m}^\alpha$, $\alpha \in I$. Moreover, there are affine fibrations $\lim_\alpha : S_\alpha^+ \to S_{G_m}^\alpha$ given by the limit map as $t \to 0$.

**Proof.** By Fact 5.1 applied to $\mathcal{S}$, we get a stratification of $\mathcal{S}$ by locally closed subsets as:

$$\mathcal{S} = \bigsqcup_{\alpha \in I} S_\alpha^+$$

and a decomposition of $\mathcal{S}_{G_m}$ as:

$$\mathcal{S}_{G_m} = \bigsqcup_{\alpha \in I} S_{G_m}^\alpha$$

where $S_{G_m}^\alpha$ are the connected components of $\mathcal{S}_{G_m}$, $\alpha \in I$. Moreover, we get retractions $\lim_\alpha : S_\alpha^+ \to \mathcal{S}_{G_m}^\alpha$, $\alpha \in I$. Note that these retractions are given by the limit map as $t \to 0$ (see Remark 5.1).

Now by base change of $\mathcal{S}_\alpha^+ \xrightarrow{\lim t} \mathcal{S}_{G_m}^\alpha$ along $\mathcal{S}_\alpha^+ \cap S \to \mathcal{S}_{G_m}^\alpha$, we get a scheme say $S_\alpha^+$ with a retraction to $S_{G_m}^\alpha := \mathcal{S}_\alpha^+ \cap S$, which is an affine fibration and we denote it again by $\lim_\alpha : S_\alpha^+ \to S_{G_m}^\alpha$. Next, we claim that $S_\alpha^+ \subset S$. Indeed, since $\mathcal{S} \setminus S$ is projective and $G_m$-stable, $\lim_\alpha$ preserves $\mathcal{S} \setminus S$ and hence $S_\alpha^+ \subset S$. Thus $S_{G_m}^\alpha = \bigsqcup_{\alpha \in I} S_\alpha^+$ and $S_{G_m}^\alpha$ is constructible since $S_\alpha^+$, $\alpha \in I$ are locally closed subsets of $\mathcal{S}$.

Now we show that $S_{G_m}^\alpha$ are the connected components of $S_{G_m}^\alpha$. Since $\mathcal{S}$ is projective, $\mathcal{S}_{G_m}$ is noetherian. Thus there are finitely many irreducible components of $\mathcal{S}_{G_m}$. Since $\mathcal{S}_{G_m}$ is smooth ([5 Proposition A.8.10]), $\mathcal{S}_{G_m}$ is irreducible and we get that $\mathcal{S}_{G_m}^\alpha \cap S$ is irreducible. This gives us that $\mathcal{S}_{G_m}^\alpha \cap S$, $\alpha \in I$ are the connected components of $S_{G_m}$ since the number of connected components of $\mathcal{S}_{G_m}$ is finite.

This finishes the proof of Proposition 5.1.

In the case of an equivariant vector bundle over a smooth projective scheme equipped with a $G_m$-action, we can say a bit more about the strata in Proposition 5.1. Let $K$ be a field and let $X$ be a smooth projective scheme over $K$ equipped with a $G_m$-action. Let $\pi : E \to X$ be an equivariant vector bundle over $X$. Compactify $E$ by considering the projectivization $\mathcal{P}(E \oplus (X \times \mathbb{A}^1)) =: \overline{E}$. We extend the given $G_m$-action on $E$ to a $G_m$-action on $\overline{E}$ by letting $G_m$ act trivially on $\mathbb{A}^1$ and via the given $G_m$-action on $X$. Since projectivization of a vector bundle over a smooth scheme is smooth, $\overline{E}$ is smooth. Thus $\overline{E}$ is an equivariant smooth compactification of $E$.

Now let us consider the Bialynicki–Birula decomposition of $X$. By Fact 5.1, $X$ has a stratification by locally closed subsets as:

$$X = \bigsqcup_{\alpha \in I} X_\alpha^+$$

and a decomposition of $X_{G_m}$ as:

$$X_{G_m} = \bigsqcup_{\alpha \in I} X_\alpha^G$$

where $X_\alpha^G$ are the connected components of $X_{G_m}$, $\alpha \in I$.

Since $G_m$ acts trivially on $X_{G_m}$, $G_m$ acts on the vector bundle $\pi^{-1}(X_{G_m}) \to X_{G_m}$ fibrewise. Therefore, $\pi^{-1}(X_{G_m})$ decomposes according to the characters of $G_m$,

$$\pi^{-1}(X_{G_m}) = \oplus_{n \in \mathbb{Z}} V_{\alpha,n},$$
where $V_{\alpha,n}$ is the subbundle of $\pi^{-1}(X_{\alpha}^G)$ on which $t \in G_m$ acts via multiplication by $t^n$. We have the following proposition.

**Proposition 5.2.** Keep notations as above. Then $E^\fin$ is a constructible subset of $E$ and there exists a stratification of $E^\fin$ by locally closed subsets as:

$$E^\fin = \bigsqcup_{\alpha \in I} E_\alpha^+$$

and a decomposition of $E^G_m$ as:

$$E^G_m = \bigsqcup_{\alpha \in I} V_{\alpha,0},$$

where $V_{\alpha,0}$ are the connected components of $E^G_m$, $\alpha \in I$ and there are affine fibrations $\lim_{t \to 0} E_\alpha^+ \to V_{\alpha,0}$ given by the limit map as $t \to 0$.

**Proof.** Notice that we have $E^G_m = \bigsqcup_{\alpha \in I} V_{\alpha,0}$. Since $V_{\alpha,0} = (\pi^{-1}(X_{\alpha}^G))^G_m$, $V_{\alpha,0}$ is closed, $\alpha \in I$. Moreover, since $V_{\alpha,0}$, $\alpha \in I$ are connected, we get that $V_{\alpha,0}$, $\alpha \in I$ are the connected components of $E^G_m$. It remains to use Proposition 5.1. \qed

6. Counting triples

This section will be devoted to the proof of Theorem 3.2. Let $G, T, B, W, \Phi$ be as in Section 2.2 and let $\mu, J_0, J_\infty$ be as in the statement of Theorem 3.2. Let $g := \text{Lie}(G)$ be the Lie algebra of $G$. Since $\mu, J_0$ and $J_\infty$ are fixed in the statement of Theorem 3.2, we will denote $\text{Trip}_\mu(J_0, J_\infty)$ by $\text{Trip}$ in the proof of Theorem 3.2.

6.1. **Strategy of the proof.** In this section, we outline the strategy of the proof of Theorem 3.2. Let $G_m$ act on $g$ via $\mu$, so $t \in G_m$ acts trivially on $h$ and via multiplication by $t^{(\alpha, \mu)}$ on the root spaces $g_\alpha$. Let $g^0 := h \oplus \oplus_{(\alpha, \mu) = 0} g_\alpha$, $g^+ := \oplus_{(\alpha, \mu) > 0} g_\alpha$ and $g^- := \oplus_{(\alpha, \mu) < 0} g_\alpha$. Then we get the following $G_m$-stable decomposition of $g$:

$$g = g^0 \oplus g^+ \oplus g^-.$$  

(11)

Note that we have $g^0 = \text{Lie}(L_\mu)$.

For $J \subset \Pi$, define $B_J$ to be the scheme of pairs $(P, v)$ such that $P \in G/P_J, v \in \text{Lie}(P)$, where we identify $G/P_J$ with the scheme of parabolic subgroups of $G$ that are conjugate to $P_J$. Note that $B_J$ is vector bundle over $G/P_J$, in fact, it is a vector subbundle of the trivial vector bundle $G/P_J \times g$ over $G/P_J$. As vector bundles over smooth schemes are smooth, we get that $B_J$ is smooth. Note that $G$ acts in a natural way on $G/P_J \times g$ preserving $B_J$. Pulling back this action along $\mu: G_m \to T \to G$, we get an action

$$\text{G}_m \curvearrowright B_J.$$  

(12)

We need the following object for our proof of Theorem 3.2.

**Definition 6.** Let $Quad$ be the closed subscheme of $B_{J_0} \times B_{J_\infty}$ consisting of quadruples $(P_0, v_0, P_\infty, v_\infty)$ such that $v_0$ and $v_\infty$ are nilpotent, $g^-$-components (11) of $v_0$ and $v_\infty$ are zero and their $g^0$-components (11) are equal.

Note that $Quad$ depends on $\mu$, $J_0$ and $J_\infty$.

**Remark 6.1.** The requirement of $v_0$ and $v_\infty$ being nilpotent in the definition of $Quad$ is equivalent to the requirement of the $g^0$-components of $v_0$ and $v_\infty$ being nilpotent.

Recall $B_J^{\fin}$ from Proposition 5.1. Apply Proposition 5.2 on $B_J$ to stratify $B_J^{\fin}$. We obtain the required stratification of $\text{Trip}$ in the following manner: trivialize the fibers of the line bundle $O(1)$ at 0 and $\infty$ to identify $\text{ad}(E_0)_0$ and $\text{ad}(E_\infty)_\infty$ with $g$, now evaluating the nilpotent sections at 0 and $\infty$ gives us a $G_m$-equivariant morphism $\text{Trip} \to B_{J_0} \times B_{J_\infty}$ with $G_m$ acting diagonally on $B_{J_0} \times B_{J_\infty}$. We will see in Lemma 6.1 that this evaluation morphism is a trivial affine
6.2. Reduction to Quad. Now consider evaluations of the nilpotent sections of \( \text{ad}(\mathcal{E}_\mu) \) at 0 and \( \infty \) and then use them to reduce Theorem 3.2 to finding a stratification of Quad. Recall that as \( \mathcal{O}(1)^x \) is a \( \mathbb{G}_m \)-bundle over \( \mathbb{P}^1 \), \( \mathbb{G}_m \) acts on \( \text{ad}(\mathcal{E}_\mu) = \mathcal{O}(1)^x \times \mathbb{G}_m \mathfrak{g} \) (Section 2.4) and this gives an action:

\[
\mathbb{G}_m \curvearrowright \mathcal{H}^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)).
\]

First, we describe sections of the adjoint bundle \( \text{ad}(\mathcal{E}_\mu) \) over \( \mathbb{P}^1 \). Since \( \text{ad}(\mathcal{E}_\mu) = \mathcal{O}(1)^x \times \mathbb{G}_m \mathfrak{g} \), the \( \mathbb{G}_m \)-stable decomposition \((11)\) of \( \mathfrak{g} \) gives a \( \mathbb{G}_m \)-stable decomposition of \( \text{ad}(\mathcal{E}_\mu) \) as \( \text{ad}(\mathcal{E}_\mu) = \text{ad}(\mathcal{E}_\mu)^0 \oplus \text{ad}(\mathcal{E}_\mu)^+ \oplus \text{ad}(\mathcal{E}_\mu)^- \), where \( \text{ad}(\mathcal{E}_\mu)^0 := \mathcal{O}(1)^x \times \mathbb{G}_m \mathfrak{g}^0, \text{ad}(\mathcal{E}_\mu)^{+} := \mathcal{O}(1)^x \times \mathbb{G}_m \mathfrak{g}^+ \) and \( \text{ad}(\mathcal{E}_\mu)^- := \mathcal{O}(1)^x \times \mathbb{G}_m \mathfrak{g}^- \). Since \( \text{ad}(\mathcal{E}_\mu)^- \) is a direct sum of the line bundles \( \mathcal{O}(m), m < 0 \) and \( \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}(m)) = 0 \) for \( m < 0 \), we get

\[
\mathcal{H}^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)) = \mathfrak{g}^0 \oplus \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}(\langle \alpha, \mu \rangle)).
\]

Thus

\[
\mathcal{H}^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)) = \mathfrak{g}^0 \oplus \left( \oplus_{\alpha : \langle \alpha, \mu \rangle > 0} \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}(\langle \alpha, \mu \rangle)) \right).
\]

For \( x = 0, \infty \), \( \text{ad}(\mathcal{E}_\mu)_x \) has a structure of a Lie algebra and for \( \Psi \in \mathcal{H}^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)) \), denote the value of \( \Psi \) at \( x \) by \( \Psi_x \), which is an element of \( \text{ad}(\mathcal{E}_\mu)_x \).

We get the following \( \mathbb{G}_m \)-stable decomposition of \( \text{ad}(\mathcal{E}_\mu)_x \) as:

\[
\text{ad}(\mathcal{E}_\mu)_x = \text{ad}(\mathcal{E}_\mu)^0_x \oplus \text{ad}(\mathcal{E}_\mu)^+_x \oplus \text{ad}(\mathcal{E}_\mu)^-_x, \quad x = 0, \infty.
\]

Remark 6.2. Trivializing the fibers of the \( \mathbb{G}_m \)-bundle \( \mathcal{O}(1)^x \) at 0 and \( \infty \), we identify \( \text{ad}(\mathcal{E}_\mu)_x/P_{J_x} \) with \( \mathbb{G}/P_{J_x} \) and we get a \( \mathbb{G}_m \)-equivariant isomorphism (which is fixed from now on) \( \text{ad}(\mathcal{E}_\mu)_x \cong \mathfrak{g} \), which maps \( \text{ad}(\mathcal{E}_\mu)^0 \) isomorphically onto \( \mathfrak{g}^0, \quad x = 0, \infty \). We note that the isomorphism \( \text{ad}(\mathcal{E}_\mu)_x^0 \cong \mathfrak{g}^0 \) is independent of the trivialization. From now on, we will use the isomorphism \( \text{ad}(\mathcal{E}_\mu)_x \cong \mathfrak{g} \) to identify elements of \( \text{ad}(\mathcal{E}_\mu)_x \) with those of \( \mathfrak{g}, \quad x = 0, \infty \).

The \( \mathbb{G}_m \)-action \((12)\) on \( B_{J_x}, \quad x = 0, \infty \) gives a \( \mathbb{G}_m \)-action on \( B_{J_0} \times B_{J_\infty} \) by \( \mathbb{G}_m \) acting diagonally. Since the decomposition \((11)\) is \( \mathbb{G}_m \)-stable, we get an action

\[
\mathbb{G}_m \curvearrowright \text{Quad}.
\]

By Remark 6.2 we consider the evaluation morphism at 0 and \( \infty \) as taking values in \( B_{J_0} \times B_{J_\infty} \):

\[
ev^{0,\infty} : \mathcal{T}_{\text{rip}} \rightarrow B_{J_0} \times B_{J_\infty}, \quad (P_0, P_{\infty}, \Psi) \mapsto (P_0, \Psi_0, P_{\infty}, \Psi_{\infty}).
\]

Consider the evaluation map at 0 and \( \infty \),

\[
eval : \mathcal{H}^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)) \rightarrow \mathfrak{g} \oplus \mathfrak{g}, \quad \Psi \mapsto (\Psi_0, \Psi_{\infty}).
\]

Notice that for \( v \in \mathfrak{g}^0 \), \( \text{eval}(v) = (v, v) \). Since \( \Psi \in \mathcal{H}^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)) \) is nilpotent if and only if the \( \mathfrak{g}^0 \)-component of \( \Psi \) is nilpotent and the evaluation map \( \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}(m)) \rightarrow \mathbb{A}^1 \times \mathbb{A}^1, \phi \mapsto (\phi_0, \phi_{\infty}) \) is surjective for \( m > 0 \), the image of \( \text{ev}^{0,\infty} \) is equal to Quad.

The next lemma relates \( \mathcal{T}_{\text{rip}} \) and Quad via evaluation at 0 and \( \infty \).

\begin{lemma}
The evaluation morphism

\[
ev^{0,\infty} : \mathcal{T}_{\text{rip}} \rightarrow \text{Quad}
\]

\end{lemma}
is $\mathbb{G}_m$-equivariant and a trivial affine fibration of relative dimension $\sum_{(\alpha, \mu) > 0} \left( (\alpha, \mu) - 1 \right)$. Moreover, $ev^{0, \infty}$ gives the following commutative triangle:

\[
\begin{array}{ccc}
\mathcal{T}rip & \xrightarrow{\sim} & \mathcal{Quad} \times W \\
\downarrow_{ev^{0, \infty}} & & \downarrow_{pr_1} \\
\mathcal{Quad} & & \\
\end{array}
\]

where $W$ is a $\mathbb{G}_m$-representation with $\mathbb{G}_m$ acting by positive weights and all morphisms in the above triangle are $\mathbb{G}_m$-equivariant. In particular, $ev^{0, \infty} : \mathcal{T}rip \to \mathcal{Quad}$ induces an isomorphism

\begin{equation}
\label{eq:16}
ev^{0, \infty} : \mathcal{T}rip^{\mathbb{G}_m} \xrightarrow{\sim} \mathcal{Quad}^{\mathbb{G}_m}.
\end{equation}

Proof. Put

\[g_{0, \infty} := \{(v_0, v_{\infty}) \in g \oplus g : g^0\text{-}components of v_x are equal, g^-\text{-}components of v_x are 0, x = 0, \infty\}.
\]

Since the image of the evaluation map lies inside $g_{0, \infty}$, we will consider the evaluation map with codomain $g_{0, \infty}$,

\[eval : H^0(\mathbb{P}^1, \text{ad}(E_\mu)) \to g_{0, \infty}.
\]

Since the evaluation map $H^0(\mathbb{P}^1, \mathcal{O}(m)) \to \mathbb{A}^1 \times \mathbb{A}^1, \phi \mapsto (\phi_0, \phi_\infty)$ is surjective for $m > 0$ and $eval(v) = (v, v)$ for $v \in g^0$, the map $eval$ is surjective.

Let $W := \ker(eval)$. Notice that $W$ is a representation of $\mathbb{G}_m$ with $\mathbb{G}_m$ acting by positive weights. Since $\mathbb{G}_m$ is reductive, we get a $\mathbb{G}_m$-equivariant isomorphism:

\[H^0(\mathbb{P}^1, \text{ad}(E_\mu)) \cong W \times g_{0, \infty}.
\]

Denote the nilpotent elements of $H^0(\mathbb{P}^1, \text{ad}(E_\mu))$ by $H^0(\mathbb{P}^1, \text{ad}(E_\mu))^{nil}$. Let $g_{0, \infty}^{nil}$ denote the set of elements $(v_0, v_{\infty}) \in g_{0, \infty}$ such that the $g^0$-components of $v_0$ and $v_{\infty}$ are nilpotent. Since $\Psi \in H^0(\mathbb{P}^1, \text{ad}(E_\mu))$ is nilpotent if and only if the $g^0$-component of $\Psi$ is nilpotent, we get a $\mathbb{G}_m$-equivariant isomorphism

\[H^0(\mathbb{P}^1, \text{ad}(E_\mu))^{nil} \cong W \times g_{0, \infty}^{nil}.
\]

Since $ev^{0, \infty} : \mathcal{T}rip \to \mathcal{Quad}$ is the pullback of $H^0(\mathbb{P}^1, \text{ad}(E_\mu))^{nil} \xrightarrow{eval} g_{0, \infty}^{nil}$ along the natural projection $\mathcal{Quad} \to g_{0, \infty}^{nil}$, we get a $\mathbb{G}_m$-equivariant isomorphism

\[\mathcal{T}rip \cong W \times \mathcal{Quad}.
\]

The statement about relative dimension follows from the fact that $\Psi \in H^0(\mathbb{P}^1, \text{ad}(E_\mu))$ is nilpotent if and only if the $g^0$-component of $\Psi$ is nilpotent, $\Psi(v) = (v, v)$ for $v \in g^0$ and by the fact that the evaluation map $H^0(\mathbb{P}^1, \mathcal{O}(m)) \to \mathbb{A}^1 \times \mathbb{A}^1, \phi \mapsto (\phi_0, \phi_\infty)$ has nullity $m - 1$ for $m > 0$. This finishes the proof of Lemma 6.1.

Thus we have reduced finding a stratification of $\mathcal{T}rip$ to finding a stratification of $\mathcal{Quad}$.

6.3. Stratification of $\mathcal{B}^m_\mu$. The following example of Bialynicki-Birula decomposition will be important to us.

Let $\mathbb{G}_m$ act on $G/P_J$ via $\mu$. We have an explicit description of the connected components of the fixed point locus given by the following result (the statement follows by reducing to $\overline{\mathbb{F}}_q$ and by noting that the proof of [14, Lemma 1] works for any algebraically closed field):

**Fact 6.1.** Recall from Section 4.2 that $L_\mu$ is the identity component of the centralizer of $\mu(\mathbb{G}_m)$ in $G$. Then

\[(G/P_J)^{\mathbb{G}_m} = \bigsqcup_{w \in W_{\mathbb{G}_m}\backslash W/W_J} Z_w\]
with \( Z_w \) the orbit of \( wP_Jw^{-1} \) under \( L_\mu \). In particular, the connected components \( Z_i \) of the fixed point locus \((G/P_J)^{G_m}\) appearing in the Bialynicki–Birula decomposition of \( G/P_J \) (Fact 5.7(ii)) are in one to one correspondence with the elements of \( W_{\Pi_m} \setminus W/W_J \).

Note that \( Z_w \cong L_\mu/(L_\mu \cap wP_Jw^{-1}) \) (see \cite{19} Proposition 7.12), which is a partial flag variety of the Levi subgroup \( L_\mu \) of \( G \) defined over \( \mathbb{F}_q \). From Fact 6.1 we get:

\[
(G/P_J)^{G_m} \cong \bigcup_{w \in W_{\Pi_m} \setminus W/W_J} L_\mu/(L_\mu \cap wP_Jw^{-1}).
\]

Let \( \pi : \mathcal{B}_J \to G/P_J \) be the projection. Note that \( \pi \) is \( \mathbb{G}_m \)-equivariant where \( \mathbb{G}_m \) acts on \( \mathcal{B}_J \) as in \( \cite{12} \). Thus \( \mathcal{B}_J \) is an equivariant vector bundle over the smooth projective scheme \( G/P_J \). By Proposition 6.2 we have a stratification of \( \mathcal{B}_J^G \) by locally closed subsets as:

\[
(17) \quad \mathcal{B}_J^G = \bigcup_{w \in W_{\Pi_m} \setminus W/W_J} \mathcal{B}_{J,w}^G,
\]

and a decomposition of \( \mathcal{B}_J^{G_m} \) as

\[
(18) \quad \mathcal{B}_J^{G_m} = \bigcup_{w \in W_{\Pi_m} \setminus W/W_J} V_{w,0},
\]

where \( V_{w,0} \) are the connected components of \( \mathcal{B}_J^{G_m} \), \( w \in W_{\Pi_m} \setminus W/W_J \). Moreover, there are affine fibrations \( \lim_w : \mathcal{B}_{J,w}^+ \to V_{w,0} \) given by the limit map as \( t \to 0 \).

**Remark 6.3.** We can describe \( V_{w,0} \) more explicitly, it is isomorphic to \( \mathcal{B}_{\Pi_m \cap wJ} \), where the underlying group is \( L_\mu \). Indeed, identify \( L_\mu/(L_\mu \cap wP_Jw^{-1}) \) with the scheme of parabolic subgroups of \( L_\mu \) that are conjugate to \( L_\mu \cap wP_Jw^{-1} \).

By Fact 6.1 we obtain

\[
(19) \quad V_{w,0} \cong \{(P',v) : P' \in L_\mu/(L_\mu \cap wP_Jw^{-1}), v \in \text{Lie}(P')\},
\]

where the above isomorphism is given by \( (P,v) \mapsto (P \cap L_\mu, v) \). Note that if, for some \( v' \in g \) we have \( \text{Ad}_t(u) \cdot v' = v' \) for all \( t \in \mathbb{G}_m \), then \( v' \in \text{Lie}(L_\mu) \). Therefore, \( v \in \text{Lie}(P) \cap \text{Lie}(L_\mu) = \text{Lie}(P \cap L_\mu) \). Thus \( (P,v) \mapsto (P \cap L_\mu, v) \) is a well-defined morphism.

The next proposition gives the relative dimension of \( \lim_w \).

**Proposition 6.1.** The relative dimension of the affine fibration \( \lim_w : \mathcal{B}_{J,w}^+ \to V_{w,0} \) is \( (\dim G - \dim L_\mu)/2 \).

**Proof.** To calculate the relative dimension of \( \lim_w : \mathcal{B}_{J,w}^+ \to V_{w,0} \), we will use Fact 5.1(iii) on \( \overline{B}_J \) (this gives us the desired relative dimension because \( \lim_w \) is obtained by base change of the affine fibration that we get by applying Bialynicki–Birula decomposition on \( \overline{B}_J \)).

Let \( z = (wP_Jw^{-1},0) \in V_{w,0}(\mathbb{F}_q) \). Since \( z \) is a \( \mathbb{G}_m \)-fixed point (see Fact 6.1), we get an action

\[
\mathbb{G}_m \curvearrowright T_{z,\mathbb{F}_q}(\overline{B}_J) = T_{z,\mathbb{F}_q}(B_J),
\]

where \( T_{z,\mathbb{F}_q}(\overline{B}_J) = T_{z,\mathbb{F}_q}(B_J) \) because \( B_J \) is an open subscheme of \( \overline{B}_J \). Let \( T_{z,\mathbb{F}_q}(B_J) \) (resp. \( T_{z,\mathbb{F}_q}^-(B_J) \)) denote the positive (resp. negative) eigenspace of the \( \mathbb{G}_m \)-action on the tangent space of \( B_J \) at \( z \) and let \( T_{z,\mathbb{F}_q}^\alpha(B_J) \) denote the fixed eigenspace of the \( \mathbb{G}_m \)-action of the tangent space of \( B_J \) at \( z \). Since \( z \in V_{w,0}(\mathbb{F}_q) \), the relative dimension of the affine fibration \( \lim_w : \mathcal{B}_{J,w}^+ \to V_{w,0} \) is equal to \( \dim T_{z,\mathbb{F}_q}^\alpha(B_J) \) by Fact 5.1(iii), so it suffices to calculate \( \dim T_{z,\mathbb{F}_q}^\alpha(B_J) \). Note that \( T_{z,\mathbb{F}_q}(B_J) \) is \( \mathbb{G}_m \)-equivariantly isomorphic to \( g/\text{Lie}(wP_Jw^{-1}) \oplus \text{Lie}(wP_Jw^{-1}) \). Since \( L_\mu \) is the centralizer of \( \mu(\mathbb{G}_m) \), we see that
The schemes $\text{Ad}_{\mu(t)}$ acts via multiplication by $t^{(w: \alpha, \mu)}$ on $\text{Ad}_w(b)$, $t \in \mathbb{G}_m$, $\alpha \in \Phi$ and acts trivially on $\text{Ad}_w(b)$. Thus, $T_2(B_J)$ is $\mathbb{G}_m$-equivariantly isomorphic to $g$, which gives

$$\dim T_2(B_J) = (\dim G - \dim L_{\mu})/2.$$

\[\square\]

### 6.4. Stratification of Quad

We will now work towards obtaining a stratification of $\text{Quad}$ by using the stratification of $\mathbb{G}_m$. Once we have such a stratification, Theorem 3.2 will be an easy consequence of it as explained in Section 6.1. First, let us show that the $\text{Quad}$ is contained in $\mathbb{B}_J^{\text{fin}} \times \mathbb{B}_{J_{\infty}}^{\text{fin}}$.

**Lemma 6.2.** Keep notations as above. We have $\text{Quad} \subset \mathbb{B}_J^{\text{fin}} \times \mathbb{B}_{J_{\infty}}^{\text{fin}}$.

**Proof.** Note that it is enough to show that $\text{Quad}$ is contained in the constructible subset $\mathbb{B}_J^{\text{fin}} \times \mathbb{B}_{J_{\infty}}^{\text{fin}}$ at the level of closed points. Let $K$ be a finite extension of $\mathbb{F}_q$. Let $(P_0, n_0, P_{\infty}, n_{\infty}) \in \text{Quad}(K)$, then $(P_0, n_0) \in B_J(K)$ and $(P_{\infty}, n_{\infty}) \in B_{J_{\infty}}(K)$. The lemma will follow if we show $\lim_{t \to 0} t \cdot (P_x, n_x)$ exists in $B_{J_{\infty}}(K)$, $x = 0, \infty$.

Since $G/P_J$ is a projective scheme, we get that $\lim_{t \to 0} t \cdot P_x$ exists, $x = 0, \infty$. By definition of $\text{Quad}$, $g^0$-component of $n_x$ is 0 and therefore $\lim_{t \to 0} t \cdot n_x$ exists and is equal to the $g^0$-component of $n_x$, $x = 0, \infty$. Thus, $\lim_{t \to 0} t \cdot (P_0, n_0)$ exists in $B_J$ and $\lim_{t \to 0} t \cdot (P_{\infty}, n_{\infty})$ exists in $B_{J_{\infty}}$. \[\square\]

Recall $W_{\Pi_w}, W_{J_0}, W_{J_{\infty}}, L_{\mu}$ from Section 3.2. For $w \in W_{\Pi_w} \setminus W/W_{J_0}, w' \in W_{\Pi_w} \setminus W/W_{J_{\infty}}$, recall $V_{w,0}, V_{w',0}, \lim_w, \lim_{w'}$ from Section 6.3 and put

$$\text{Quad}^{\text{fin}}_{w,w'} := (V_{w,0} \times V_{w',0}) \cap \text{Quad}.$$

Let $\text{Quad}^{\text{fin}}_{w,w'}$ be the pullback of $\lim_w \times \lim_{w'} : \mathbb{B}_J^{\text{fin}} \times \mathbb{B}_{J_{\infty}}^{\text{fin}} \to V_{w,0} \times V_{w',0}$ along $\text{Quad}^{\text{fin}}_{w,w'} : \text{Quad} \to V_{w,0} \times V_{w',0}$, that is, we have the following cartesian square:

\[
\begin{array}{ccc}
\text{Quad}^{\text{fin}}_{w,w'} & \longrightarrow & \mathbb{B}_J^{\text{fin}} \times \mathbb{B}_{J_{\infty}}^{\text{fin}} \\
\downarrow & & \downarrow \\
V_{w,0} \times V_{w',0} & \longrightarrow & \lim_w \times \lim_{w'}
\end{array}
\]

Let us denote the left vertical arrow in the above diagram again by $\lim_w \times \lim_{w'}$. Next, we would like to show that the schemes $\text{Quad}^{\text{fin}}_{w,w'}$ give a stratification of $\text{Quad}$, which is the content of the next lemma.

**Lemma 6.3.** Keep notations as above. We have $\text{Quad}^{\text{fin}}_{w,w'} \subset \text{Quad}$.

**Proof.** Note that it is enough to show that $\text{Quad}^{\text{fin}}_{w,w'}$ is contained in $\text{Quad}$ at the level of closed points. Let $K$ be a finite extension of $\mathbb{F}_q$. Let $(P_0, v_0, P_{\infty}, v_{\infty}) \in \text{Quad}^{\text{fin}}_{w,w'}(K)$, then we have $\lim_{t \to 0} t \cdot (P_0, v_0, P_{\infty}, v_{\infty}) \in \text{Quad}^{\text{fin}}_{w,w'}(K)$.

In particular, $\lim_{t \to 0} t \cdot v$ exists in $g$ and is equal to the $g^0$-component of $v$, $x = 0, \infty$. Since for any $v \in g$, $\lim_{t \to 0} t \cdot v$ exists in $g$ if and only if the $g^-$-components $v_x$ are zero, $x = 0, \infty$. Moreover, since $\text{Quad}^{\text{fin}}_{w,w'}$ is the closed subscheme of $\text{Quad}$ consisting of quadruples $(P_0, n, P_{\infty}, n)$ such that $P_0 \in Z_w$ (resp. $P_{\infty} \in Z_{w'}$), $n$ is a nilpotent element of $g$ such that $n \in \text{Lie}(P_0)$ and $n \in \text{Lie}(P_{\infty})$, we get that the $g^0$-components of $v_x$ are equal and nilpotent, $x = 0, \infty$. The lemma now follows from Remark 6.1. \[\square\]

The following lemma identifies the schemes $\text{Quad}^{\text{fin}}_{w,w'}$, with the generalized Steinberg varieties.

**Lemma 6.4.** Keep notations as above. Then the schemes $\text{Quad}^{\text{fin}}_{w,w'}$ are isomorphic to the generalized Steinberg varieties $\text{St}_{L_2}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_{\infty})$, $w \in D_{\Pi_w,J_0}^G, w' \in D_{\Pi_w,J_{\infty}}^G$. 
Proof. Notice that \( Quad_{w,w}'^m \) is the closed subscheme of \( Quad \) consisting of quadruples \((P_0, n, P_\infty, n)\) such that \( P_0 \in Z_w \) (resp. \( P_\infty \in Z_w \)), \( n \) is a nilpotent element of \( g \) such that \( n \in \text{Lie}(P_0) \) and \( n \in \text{Lie}(P_\infty) \) (note that the \( g^+ \) and \( g^- \) components of \( n \) are 0 since \( G_m \) acts trivially on \( Quad_{w,w}'^m \)). Thus we have

\[
(20) \quad Quad_{w,w}'^m \cong \text{St}_{\nu}((\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty),
\]

where the above isomorphism is given by \((P_0, n, P_\infty, n) \mapsto (P_0 \cap L_\mu, P_\infty \cap L_\mu)\).

Next, we show that the generalized Steinberg varieties (see Section 3.1) are connected.

**Lemma 6.5.** Recall \( \Pi_H, J_1, J_2 \) and \( \text{St}_H(J_1, J_2) \) from Section 3.1. Then \( \text{St}_H(J_1, J_2) \) is connected.

**Proof.** We show that \( \text{St}_H(J_1, J_2) \) is geometrically connected. Let \( K = \overline{\mathbb{F}}_q \). Since closed points of \( \text{St}_H(J_1, J_2)_K \) are dense in \( \text{St}_H(J_1, J_2)_K \) and the connected components are closed, it suffices to show that all the closed points of \( \text{St}_H(J_1, J_2)_K \) are contained in the same connected component. Let \((n, P, Q) \in \text{St}_H(J_1, J_2)(K)\). Consider the morphism

\[
\phi : A^1_K \to \text{St}_H(J_1, J_2)_K, \quad t \mapsto (t \cdot n, P, Q).
\]

Since \( A^1_K \) is connected, the image of \( \phi \) is connected. Therefore, \((n, P, Q)\) and \((0, P, Q)\) are contained in the same connected component of \( \text{St}_H(J_1, J_2)_K \). Since \( H/P_{J_1} \times H/P_{J_2} \) is geometrically connected, each closed point of \( \text{St}_H(J_1, J_2)_K \) is contained in the connected component containing \((0)\) \times_K \((H/P_{J_1})_K \times (H/P_{J_2})_K\). This finishes the proof of the lemma. \( \square \)

Thus we get a stratification of \( Quad \) by locally closed subsets as:

\[
(21) \quad Quad = \bigsqcup_{w \in W_{\Pi_\mu} \setminus W/W_{J_0}} \quad Quad_{w,w}'^+\cap_{w' \in W_{\Pi_\mu} \setminus W/W_{J_\infty}}
\]

and a decomposition of the fixed point locus \( Quad^g_m \) as:

\[
Quad^g_m = \bigsqcup_{w \in W_{\Pi_\mu} \setminus W/W_{J_0}} \quad Quad_{w,w}'^m\cap_{w' \in W_{\Pi_\mu} \setminus W/W_{J_\infty}}
\]

where \( Quad_{w,w}'^m \) are the connected components (see Lemma 6.3 and Lemma 6.4) of \( Quad^g_m \). Moreover, we have retractions

\[
\text{lim}_w \times \text{lim}_{w'} : Quad_{w,w}'^+ \to Quad_{w,w}'^m,
\]

which are affine fibrations.

Finally we calculate the relative dimension of the affine fibration \( \text{lim}_w \times \text{lim}_{w'} : Quad_{w,w}'^+ \to Quad_{w,w}'^m \).

**Corollary 6.5.1.** The relative dimension of the affine fibration

\[
\text{lim}_w \times \text{lim}_{w'} : Quad_{w,w}'^+ \to Quad_{w,w}'^m
\]

is equal to \( \dim G - \dim L_\mu \).

**Proof.** Since the affine fibration \( \text{lim}_w \times \text{lim}_{w'} : Quad_{w,w}'^+ \to Quad_{w,w}'^m \) is obtained by base change of the affine fibration \( \text{lim}_w \times \text{lim}_{w'} : B_{J_0,w}' \times B_{J_\infty,w}' \to V_w,0 \times V_{w',0} \), the corollary follows from Proposition 6.1. \( \square \)
6.5. Completing the proof of Theorem 3.2 Recall $ev^{0,\infty}$ defined in Lemma 6.1. For each $w \in W_{I_0} \setminus W/W_{I_0}$, $w' \in W_{I_0} \setminus W/W_{J_{-\infty}}$, put

$$Trip^+_{w,w'} := (ev^{0,\infty})^{-1}(Quad^+_{w,w'}).$$

Since $Quad^+_{w,w'}$, $w \in W_{I_0} \setminus W/W_{I_0}$, $w' \in W_{I_0} \setminus W/W_{J_{-\infty}}$ form a stratification of $Quad$, we get a stratification of $Trip$ by locally closed subsets as:

$$Trip = \bigsqcup_{w \in W_{I_0} \setminus W/W_{I_0}, w' \in W_{I_0} \setminus W/W_{J_{-\infty}}} Trip^+_{w,w'}.$$

Now let $(ev^{0,\infty})_{w,w'} := ev^{0,\infty}|_{Trip(J_0,J_{-\infty})+}$. Consider the morphism

$$(lim_w \times lim_{w'}) \circ (ev^{0,\infty})_{w,w'} : Trip^+_{w,w'} \rightarrow Quad^+_{w,w'}.$$

Now we will need the following result (see [21, Proposition 5.2]), which describes $Aut(\mathcal{E}_\mu)$ as a scheme:

**Fact 6.2.** Let $\mathcal{E}_\mu$ be as above. Then $Aut(\mathcal{E}_\mu)$ is isomorphic as a scheme to

$$L_\mu \times \prod_{\alpha \in \Phi: (\alpha,\mu) > 0} H^0(\mathbb{P}^1, \mathcal{O}((\alpha,\mu))).$$

Lemma 6.1 and Corollary 6.5.1 have the following consequence.

**Lemma 6.6.** The morphism $(lim_w \times lim_{w'}) \circ (ev^{0,\infty})_{w,w'}$ is an affine fibration of relative dimension $dim(Aut(\mathcal{E}_\mu)) - dim(L_\mu)$.

**Proof.** Since $lim_w \times lim_{w'}$ is a trivial affine fibration (see Lemma 6.1) and $(ev^{0,\infty})_{w,w'}$ is an affine fibration (see Corollary 6.5.1), their composition $(lim_w \times lim_{w'}) \circ (ev^{0,\infty})_{w,w'}$ is an affine fibration.

Now let us calculate the required relative dimension. By Fact 6.2 we have

$$dim(Aut(\mathcal{E}_\mu)) - dim(L_\mu) = dim \left( \prod_{\alpha \in \Phi: (\alpha,\mu) > 0} H^0(\mathbb{P}^1, \mathcal{O}((\alpha,\mu))) \right) = \sum_{(\alpha,\mu) > 0} \left( \alpha + 1 \right) \mu.$$

As $(ev^{0,\infty})_{w,w'}$ is of relative dimension $\sum_{(\alpha,\mu) > 0} \left( \alpha + 1 \right)$ (see Lemma 6.1) and $lim_w \times lim_{w'}$ is of relative dimension $dim G - dim L_\mu$ (see Corollary 6.5.1), we see that $(lim_w \times lim_{w'}) \circ (ev^{0,\infty})_{w,w'}$ is of relative dimension

$$dim G - dim L_\mu + \sum_{(\alpha,\mu) > 0} \left( \alpha + 1 \right) \mu.$$

Now the Lemma follows by noting that $dim G - dim L_\mu = 2|\alpha \in \Phi : (\alpha,\mu) > 0|$.

We define $Trip^+_{\mathcal{E}_\mu}$ to be the subscheme of $Trip^+_{\mathcal{E}_\mu}$ corresponding to $Quad^+_{\mathcal{E}_\mu}$ in (19). Thus, Lemma 6.6 gives the required affine fibration in Theorem 3.2

$$Trip^+_{\mathcal{E}_\mu} \rightarrow Trip^+_{w,w'}.$$

This finishes the proof of Theorem 3.2

**Remark 6.4.** Define a nilpotent parabolic pair of type $(G, \mathbb{P}^1, \{0,\infty\})$ to be a collection $(\mathcal{E}, P_0, P_\infty, \Psi)$, where $\mathcal{E}$ is a principal $G$-bundle over $\mathbb{P}^1$, $P_x$ is a parabolic structure on $\mathcal{E}$ at $x$, $\Psi$ is a nilpotent section of $ad(\mathcal{E})$ such that $\Psi_x \in \text{Lie}(P_x)$, $x = 0, \infty$. We will denote the groupoid of nilpotent parabolic pairs by $Pair_{nilp}(G, \mathbb{P}^1, \{0,\infty\})$. Then $Pair_{nilp}(G, \mathbb{P}^1, \{0,\infty\})$ decomposes into subgroupoids according to the type of parabolic structures at $0$ and $\infty$. We denote these subgroupoids by $Pair_{nilp}(G, \mathbb{P}^1, \{0,\infty\})$, $J_0, J_{-\infty} \subset \Pi$. For $\mu \in X_-(T)$, let $Pair_{nilp}(G, \mathbb{P}^1, \{0,\infty\})$ denote the subgroupoid of $Pair_{nilp}(G, \mathbb{P}^1, \{0,\infty\})$ such that the underlying principal $G$-bundle over $\mathbb{P}^1$ is isomorphic.
to $E_\mu$. Explicitly knowing $|\text{Trip}_\mu(J_0, J_\infty)|$ allows us to calculate the volume of the groupoid $\mathcal{P}air_{L_0, L_\infty}^G, \mathbb{P}^1, \{0, \infty\}$.

More concretely, we have

$$|\mathcal{P}air_{L_0, L_\infty}^G, \mathbb{P}^1, \{0, \infty\}| = \frac{|\text{Trip}_\mu(J_0, J_\infty)|}{|\text{Aut}(E_\mu)|},$$

where $\mathcal{X}$ denotes the volume of any groupoid $\mathcal{X}$.

### 6.6. Proof of Corollary 3.2.4

In this section, we will give the proof of Corollary 3.2.4. First we need the following notation:

**Notation 6.1.** For any algebraic group $H$ over $\mathbb{F}_q$ and a cocharacter $\mu$ of a maximal torus, we will denote the scheme-theoretic centralizer of $\mu(\mathbb{G}_m)$ in $H$ by $Z_H(\mu)$.

Now we will prove Corollary 3.2.4. By Lemma 6.7, root systems of $L_\mu$ and $L_{\mu'}$ are isomorphic.

**Proof.** We have $L_\mu = Z_G(\mu)$ and

$$(22) \quad L_{\mu'} = Z_{G'}(\mu') = (Z_G(\mu) \cap G')^o = (L_\mu \cap G')^o$$

Clearly by (22), we have $[L_\mu, L_{\mu'}] \subset [L_\mu, L_{\mu'}]$. Now we show the other inclusion. Since $G' = [G, G]$, we have $[L_\mu, L_{\mu'}] \subset G'$. Thus we have $[[L_\mu, L_{\mu'}], [L_\mu, L_{\mu'}]] \subset [G', G']$. Since derived group of any connected reductive group over $\mathbb{F}_q$ is perfect (see [5, Proposition 1.2.6]), $[[L_\mu, L_{\mu'}], [L_\mu, L_{\mu'}]] = [L_\mu, L_{\mu'}]$ and hence $[L_\mu, L_{\mu'}] \subset [G', G']$. Combining it with the fact that $[L_\mu, L_{\mu'}]$ is connected, we get that

$$[L_\mu, L_{\mu'}] \subset (L_\mu \cap G')^o$$

Now (22) gives us that $[L_\mu, L_{\mu'}] \subset L_{\mu'}$ and hence we have the other inclusion $[L_\mu, L_{\mu'}] \subset L_{\mu'}$. This finishes the proof of Lemma 6.7.

We return to the proof of Corollary 3.2.4. By Lemma 6.7, root systems of $L_\mu$ and $L_{\mu'}$ are isomorphic, which gives us that $[\text{Sp}_{L_\mu}] = [\text{Sp}_{L_{\mu'}}]$ as the number of points of the generalized Springer variety depend only on the root system of the underlying algebraic group (see Theorem 3.1(i)), hence $\Delta_{L_\mu}(\text{Sp}_{L_\mu}) = \Delta_{L_{\mu'}}(\text{Sp}_{L_{\mu'}})$ (see the definition of coproduct in 3.1). Now Corollary 3.2.4 (a) follows from the equality $\dim(\text{Aut}(E_\mu)) - \dim(L_\mu) = \dim(\text{Aut}(E_{\mu'})) - \dim(L_{\mu'})$ (see Fact 6.2).

Now we give a proof of Corollary 3.2.4 (b).

Keep notations as in Section 3.3. We have $u_{L_\mu} : L_\mu \rightarrow L_{\mu_2}$ is a flat surjective morphism (see [6, Corollary 2.1.9]). Moreover, $u_{|L_{\mu_2}} : L_\mu \rightarrow L_{\mu_2}$ is finite as the restriction of a finite morphism to closed subschemes is again a finite morphism. Clearly, $\ker(u_{|L_{\mu_2}})$ is central in $L_{\mu_2}$ as $\ker(u)$ is central in $G_1$. Hence, $u_{|L_{\mu_2}} : L_\mu \rightarrow L_{\mu_2}$ is a central isogeny. So we get that the root systems of $L_{\mu_1}$ and $L_{\mu_2}$ are isomorphic (see [6, Proposition 3.4.1]), which gives us that $[\text{Sp}_{L_{\mu_1}}] = [\text{Sp}_{L_{\mu_2}}]$ as the number of points of the generalized Springer variety depend only on the root system of the underlying algebraic group (see Theorem 3.1(i)), hence $\Delta_{L_{\mu_1}}(\text{Sp}_{L_{\mu_1}}) = \Delta_{L_{\mu_2}}(\text{Sp}_{L_{\mu_2}})$ (see the definition of coproduct in 3.1). Now Corollary 3.2.4 (b) follows from the equality $\dim(\text{Aut}(E_{\mu_1})) - \dim(L_{\mu_1}) = \dim(\text{Aut}(E_{\mu_2})) - \dim(L_{\mu_2})$ (see Fact 6.2).

This finishes the proof of Corollary 3.2.4 (b).
7. Special case of vector bundles over $\mathbb{P}^1$

In this section, we derive the Mellit’s result [18, Section 5.4] from our method. Let us recall the notions of lambda rings, plethystic substitutions and plethystic exponentials from [18] Section 2.1, Section 2.2.

Fix a base ring $R$. We will denote the ring of symmetric functions in the sequence of variables $(x_1, x_2, \ldots)$ with coefficients in $R$ by $\text{Sym}_R[X]$, where $X = (x_1, x_2, \ldots)$. We will denote the ring of symmetric functions with coefficients in $R$ that are symmetric in both sequences of variables $X$ and $Y$ by $\text{Sym}_R[X, Y]$. We will denote the degree $n$ component of $\text{Sym}_R[X]$ by $\text{Sym}_R^n[X]$ and we will denote the bidegree $(n, n)$ component of $\text{Sym}_R[X, Y]$ by $\text{Sym}_R^n[X, Y]$.

**Definition 7.** Let $\Lambda$ be a ring such that $\mathbb{Q} \subset \Lambda$. A lambda ring structure on $\Lambda$ is a collection of homomorphisms $p_n : \Lambda \to \Lambda$, $n \in \mathbb{Z}_{>0}$ satisfying:

1. $p_1(x) = x$, $x \in \Lambda$ and
2. $p_m(p_n(x)) = p_{mn}(x)$, $m, n \in \mathbb{Z}_{>0}$, $x \in \Lambda$.

By a lambda ring, we will mean a ring together with a lambda ring structure.

In the above definition, we require $\Lambda$ to contain $\mathbb{Q}$ because of the fact that $\{p_n : n \in \mathbb{Z}\}$ forms a basis of the ring of symmetric functions when the base ring contains $\mathbb{Q}$.

When our base ring $R$ is itself a lambda ring, then we define a lambda ring structure on $\text{Sym}_R[X]$ as follows: note that our ring is freely generated as an $R$-algebra by the Newton polynomials $p_m(X)$. Thus there is a unique homomorphism $p_n : \text{Sym}_R[X] \to \text{Sym}_R[X]$ whose restriction to $R$ is given by the lambda ring structure on $R$ and $p_n(p_m(X)) = p_{nm}(X)$. We define the lambda ring structure on $\text{Sym}_R[X, Y]$ similarly.

The lambda ring structure that we consider on $\mathbb{Q}[[q^{-1}]]$ is defined as:

$p_n : \mathbb{Q}[[q^{-1}]] \to \mathbb{Q}[[q^{-1}]]$, $n \in \mathbb{N}$

$r \mapsto r$, $q^{-1} \mapsto q^{-n}$, $r \in \mathbb{Q}$.

The lambda ring structure that we consider on $\mathbb{Q}[[q^{-1}]][[t]]$ is defined as:

$p_n : \mathbb{Q}[[q^{-1}]][[t]] \to \mathbb{Q}[[q^{-1}]][[t]]$, $n \in \mathbb{N}$

$r \mapsto r$, $q^{-1} \mapsto q^{-n}$, $t \mapsto t^n$, $r \in \mathbb{Q}$.

Note that $\mathbb{Q}(q)[[t]]$ is a sub lambda ring of $\mathbb{Q}[[q^{-1}]][[t]]$.

**Definition 8.** Let $\Lambda$ be a lambda ring containing $\mathbb{Q}$. Let $F \in \text{Sym}_\mathbb{Q}[X]$ and $x \in \Lambda$. We define the plethystic action of $F$ on $x$ as follows: write $F$ as a polynomial in power sum symmetric functions, say $F = f(p_1, p_2, \ldots)$ for some $f \in \mathbb{Q}[p_1, p_2, \ldots]$, we set

$F[x] = f(p_1(x), p_2(x), \ldots)$.

The plethystic action satisfies the following properties:

$(FG)[x] = F[x]G[x]$, $(F + G)[x] = F[x] + G[x]$, $r[x] = r$, $F, G \in \text{Sym}_\mathbb{Q}[X], r \in \mathbb{Q}, x \in \Lambda$.

In other words, for each $x \in \Lambda$, $f \mapsto f[x]$ is a homomorphism of $\mathbb{Q}$-algebras from $\text{Sym}_\mathbb{Q}[X]$ to $\Lambda$.

**Definition 9.** Let $R$ be a base ring such that $\mathbb{Q} \subset R$. Define $\text{Exp}[x]$ for any $x$ in any topological lambda ring $\Lambda$ containing $R$ as follows:

$\text{Exp}[x] = \exp \left( \sum_{n=1}^{\infty} \frac{p_n[x]}{n} \right)$

provided that the right hand side converges.
Let \( \Xi_n := \{ e_1 - e_2, \ldots, e_{n-1} - e_n \} \) denote the set of simple roots of \( GL_n \) relative to the diagonal torus \( T_n \) and the Borel subgroup \( B_n \) consisting of upper-triangular matrices. Consider the standard full flag \( E_\bullet = \{ E_j \} \) in \( \mathbb{F}_q^n \). Let \( J \subset \Xi_n \). Recall from Section \ref{Section2.3} that \( P_J \) denotes the standard parabolic subgroup of \( GL_n \) corresponding to the subset \( J \). Then \( P_J \) is the stabilizer in \( GL_n \) of the flag obtained by removing from \( E_\bullet \) the terms \( E_j \) for \( e_j - e_{j+1} \in J \). From now on, we identify \( \mathcal{P}(\Xi_n) \) with the set of standard parabolic subgroups of \( GL_n \) via \( J \mapsto P_J \). Let \( \Pi_n \) denote the set of partitions of \( \{1, \ldots, n\} \). For any partition \( \nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_l) \in \Pi_n \), set

\[
J(\nu) := \{ e_i - e_{i+1} : i \neq \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \ldots + \nu_l = n, 1 \leq i \leq n-1 \}.
\]

This gives a inclusion from \( \Pi_n \) to \( \mathcal{P}(\Xi_n) \), \( \nu \mapsto P_{J(\nu)} \) where the image consists of partial flag varieties of \( GL_n \) whose types are given by partitions. This gives a bijection between the set of partitions of \( n \) and \( GL_n(\mathbb{F}_q) \)-conjugacy classes of Levi \( \mathbb{F}_q \)-subgroups of \( GL_n \). Define \( \mu(\nu) : \mathbb{G}_m \to T_n \) as:

\[
t \mapsto \text{diag}(t, t, \ldots, t, t, \ldots)
\]

Recall \( L_\mu \) from Section \ref{Section4.2}. We set \( L_\nu := L_{\mu(\nu)} \). Notice that we have \( L_\nu \cong GL_{\nu_1} \times \ldots \times GL_{\nu_l} \) (see Section \ref{Section4.2}).

For \( \lambda \in \Pi_n \), we will denote the monomial symmetric function corresponding to \( \lambda \) by \( m_\lambda \) and the homogeneous symmetric function corresponding to \( \lambda \) by \( h_\lambda \). Before proceeding, we make the following convention.

**Convention 7.1.** We identify symmetric functions of degree \( n \) with the associate invariant functions on \( \mathcal{P}(\Xi_n) \) by identifying \( m_\lambda \) with \( h_{\tilde{\delta}_\lambda(\lambda)} \), \( \lambda \in \Pi_n \).

Recall from Section \ref{Section4.2} that \( \text{Tri}_{\mu} \) is the function on \( \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \) that counts the number of \( \mathbb{F}_q \)-points of \( \text{Tri}_{\mu}(J_0, J_\infty, (J_0, J_\infty) \in \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \). Since \( \pi_\mu = \Delta_G(\Pi_{\mu, \cdot}) \) (see Remark \ref{Remark5.1}(iii)), by Corollary \ref{Corollary3.2.1} Lemma \ref{Lemma4.3} and Corollary \ref{Corollary4.3.1} we consider \( \text{Tri}_{\mu} \) as a symmetric function (see Convention \ref{Convention7.1}). Thus

\[
\text{Tri}_{\mu} = \sum_{(\nu^0, \nu^\infty) \in \Pi_n \times \Pi_n} |\text{Tri}_{\mu}(J(\nu^0), J(\nu^\infty))|m_{\nu^0}(X)m_{\nu^\infty}(Y).
\]

Let \( \mu \in X_-(T_n) \), define \( C_\mu[X, Y; t] \) as:

\[
C_\mu[X, Y; t] = \frac{[\text{Tri}_{\mu}]}{[\text{Aut}(\xi)]}
\]

and consider

\[
\Omega_{\mu, \infty}^0(\mathbb{P}^1)[X, Y; t] = \sum_{\mu \in X_-(T_n)} t^{-\deg(\xi)} C_\mu[X, Y; t].
\]

Explicitly,

\[
\Omega_{\mu, \infty}^0(\mathbb{P}^1)[X, Y; t] = \sum_{\mu \in X_-(T_n)} t^{-\deg(\xi)} \sum_{(\nu^0, \nu^\infty) \in \Pi_n \times \Pi_n} \frac{|\text{Tri}_{\mu}(J(\nu^0), J(\nu^\infty))|}{[\text{Aut}(\xi)]} m_{\nu^0}(X)m_{\nu^\infty}(Y).
\]

Notice that \( \Omega_{\mu, \infty}^0(\mathbb{P}^1)[X, Y; t] \) defined above is the same as the one considered in \cite{Mellit} Section 5.4. Now using our techniques, we would like to re-derive the following result of Mellit \cite{Mellit} Section 5.4.

**Proposition 7.1.** The following holds as formal series with coefficients in \( \text{Sym}_{\mathbb{F}_q[[t]]} \):

\[
\sum_{n=0}^{\infty} \Omega_{\mu, \infty}^0(\mathbb{P}^1)[X, Y; t] = \text{Exp} \left[ \frac{XY}{(q-1)(1-t)} \right], \quad \text{where } XY = \sum_{i,j} x_i y_j.
\]

Recall from Section \ref{Section3.3} that \( \text{Sp}_{GL_n} \) is the function on \( \mathcal{P}(\Xi_n) \) that counts the number of \( \mathbb{F}_q \)-points of \( \text{Sp}_{GL_n}(J) \), \( J \in \mathcal{P}(\Xi_n) \). Using Corollary \ref{Corollary4.3.1} and Remark \ref{Remark4.3.1} we consider \( \text{Sp}_{GL_n} \) as a symmetric function.

As a first step in proving Proposition 7.1, we prove the following proposition:
Proposition 7.2. The following holds in $\text{Sym}_{\mathbb{Q}(q)}[X]$:

$$h_n \left[ \frac{X}{q-1} \right] = \frac{1}{|GL_n|} \left[ S_{pGL_n} \right].$$

Proof. By (5), the desired equality can be rewritten as:

(23) $$h_n \left[ \frac{X}{q-1} \right] = \sum_{\nu \in \Pi_n} \frac{q^{\dim(L_{\nu})}}{q^n|L_{\nu}|} m_{\nu}(X).$$

We have the following identity (see [17, Chapter 4, Section 2]) in $\text{Sym}_{\mathbb{Q}(q)}[X,Y]$:

(24) $$h_n(XY) = \sum_{\nu \in \Pi_n} m_{\nu}(X)h_{\nu}(Y).$$

Then the specialization $x_i \mapsto x_i, y_j \mapsto q^{-(j-1)}, i,j \in \mathbb{N}$ gives a homomorphism of lambda rings $\text{Sym}_{\mathbb{Q}(q)}[X,Y] \to \text{Sym}_{\mathbb{Q}[[q^{-1}]]}[X]$. Thus this specialization commutes with the plethystic action and we have

$$h_n \left[ X \left( 1 + \frac{1}{q} + \cdots + \frac{1}{q^r} \right) \right] = \sum_{\nu \in \Pi_n} m_{\nu}(X)h_{\nu} \left[ \frac{1}{q} + \cdots + \frac{1}{q^r} \right] \quad \text{in} \quad \text{Sym}_{\mathbb{Q}[[q^{-1}]]}[X]$$

$$h_n \left[ \frac{qX}{q-1} \right] = \sum_{\nu \in \Pi_n} m_{\nu}(X)h_{\nu} \left[ \frac{q}{q-1} \right] \quad \text{in} \quad \text{Sym}_{\mathbb{Q}[[q^{-1}]]}[X].$$

Since the terms of the above identity lie in $\text{Sym}_{\mathbb{Q}(q)}[X]$, the equality holds in $\text{Sym}_{\mathbb{Q}(q)}[X]$. Since $h_{\nu}[A] = q^{r_{\nu}}h_{\nu}[A]$ for any $A \in \text{Sym}_{\mathbb{Q}(q)}[X]$, we get

(25) $$h_n \left[ \frac{X}{q-1} \right] = \sum_{\nu \in \Pi_n} m_{\nu}(X)h_{\nu} \left[ \frac{1}{q-1} \right].$$

We now need a lemma:

Lemma 7.1. The following holds in $\mathbb{Q}(q)$:

$$h_n \left[ \frac{1}{1-q} \right] = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Proof. Let $H(w) := \sum_{r \geq 0} h_r(X)w^r \in (\text{Sym}_{\mathbb{Q}(q)}[X])[[w]]$ be the generating function for the homogeneous symmetric functions and let $P(w) := \sum_{r \geq 1} p_r(X)w^{r-1} \in (\text{Sym}_{\mathbb{Q}(q)}[X])[[w]]$ be the generating function for the power sum symmetric functions. Then we have the following well-known identity in $(\text{Sym}_{\mathbb{Q}(q)}[X])[[w]]$:

$$H(w) = \exp \left( \int P(w)dw \right).$$

Now,

(26) $$P \left[ \frac{1}{1-q} \right] = \sum_{r \geq 1} p_r \left[ \frac{1}{1-q} \right] w^{r-1} = \sum_{r \geq 1} w^{r-1} \left( \sum_{m \geq 0} (q^r)^m \right) = \sum_{m \geq 0} \frac{q^m}{1-wq^m}.$$ 

By (26) we have

$$H \left[ \frac{1}{1-q} \right] = \exp \left( \int P \left[ \frac{1}{1-q} \right] \right) = \exp \left( \int \sum_{m \geq 0} \frac{q^m}{1-wq^m}dw \right) = \prod_{m \geq 0} \frac{1}{1-wq^m}.$$ 

Now the lemma follows from [17] Chapter 1, Section 2, Example 4. □
The specialization $q \mapsto 1/q, x_i \mapsto x_i, i \in \mathbb{N}$ gives an automorphism of lambda rings $\text{Sym}_{\mathbb{Q}(q)}[X] \to \text{Sym}_{\mathbb{Q}(q)}[X]$. Thus, this specialization commutes with the plethystic action on $\text{Sym}_{\mathbb{Q}(q)}[X]$ and we have

$$h_n \left[ \frac{1}{1 - \frac{1}{q}} \right] = \frac{qq^2 \cdots q^n}{(1 - \frac{1}{q})(1 - \frac{1}{q^2}) \cdots (1 - \frac{1}{q^n})} = \frac{qq^2 \cdots q^n}{(q - 1)(q^2 - 1) \cdots (q^n - 1)}.$$

Since $h_n[qf] = q^n h_n[f]$ for any $f \in \text{Sym}_{\mathbb{Q}(q)}[X]$, we get

$$h_n \left[ \frac{1}{q - 1} \right] = \frac{qq^2 \cdots q^n}{q^n (q - 1)(q^2 - 1) \cdots (q^n - 1)}.$$

Now (28) gives

$$h_n \left[ \frac{X}{q - 1} \right] = \sum_{\nu \in \Pi_n} m_\nu(X) \prod_{i=1}^k h_{\nu_i} \left[ \frac{1}{q - 1} \right] = \sum_{\nu \in \Pi_n} m_\nu(X) \frac{1}{q^n} \prod_{i=1}^k \frac{qq^2 \cdots q^{\nu_i}}{(q - 1)(q^2 - 1) \cdots (q^{\nu_i} - 1)}.$$

The coefficient of $m_\nu(X)$ in equation (27) is equal to

$$\frac{1}{q^n} \prod_{i=1}^k \frac{qq^2 \cdots q^{\nu_i}}{(q - 1)(q^2 - 1) \cdots (q^{\nu_i} - 1)} = \frac{1}{q^n} \prod_{i=1}^k \frac{\nu_i^2}{\nu_i q^{\nu_i - 1}}.$$

Since $L_\nu \cong GL_{\nu_1} \times \cdots \times GL_{\nu_k}$, Proposition 7.2 follows.

Next, consider the standard coproduct on symmetric functions:

$$\Delta^n : \text{Sym}_n^2[X] \to \text{Sym}_n^2[X] \otimes \text{Sym}_n^2[Y] = \text{Sym}_n^2[X, Y], \quad f(X) \mapsto f(XY).$$

Let $\Delta'_{GL_n}$ denote $\Delta_{GL_n}$ restricted to associate invariant functions. We would like to show that $\Delta^n$ agrees with $\Delta'_{GL_n}$ by identifying symmetric functions of degree $n$ with the associate invariant functions on $P(\Xi_n)$ (see Convention 7.1). First we need a notation.

**Notation 7.1.** For any sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that $\sum_{j=1}^m \alpha_j = n$, we will denote the subgroup $S_{\alpha_1} \times \cdots \times S_{\alpha_m}$ of $S_n$ by $S_\alpha$.

We have the following proposition.

**Proposition 7.3.** Keep notations as above. Then we have $\Delta'_{GL_n} = \Delta^n$.

**Proof.** Since $m_\nu, \nu \in \Pi_n$ form a basis of $\text{Sym}_n^2[X]$, it is enough to check that $\Delta'_{GL_n}$ agrees with $\Delta^n$ on this basis. We re-write the conclusion of Lemma 4.4 for $GL_n$. Let $\nu$ be a partition of $n$. It gives an equivalence relation $\sim_\nu$ on \{1, \ldots, n\}, where $i \sim_\nu j$ if and only if there exists a $t$ such that $\nu_1 + \cdots + \nu_t \leq i, j < \nu_1 + \cdots + \nu_{t+1}$. Note that $S_\nu$ acts on \{1, \ldots, n\} and thus on the equivalence relations. For an equivalence relation $\sim$ on \{1, \ldots, n\}, we will write $\text{Part}(\sim) \subset \Pi_n$ for the corresponding partition of $n$, that is, the ordered sequence of sizes of equivalence classes. By Lemma 4.4 we have

$$\Delta'_{GL_n}(m_\nu) = \sum_{\lambda, \mu \in \Pi_n} n^{\lambda, \mu}_\nu m_\lambda \otimes m_\mu,$$

where

$$n^{\lambda, \mu}_\nu = \left| \{ w \in S_\lambda \setminus S_n / S_\mu : \text{Part}(\sim_\lambda \cap w(\sim_\mu)) = \nu \} \right|.$$

If we identify roots for $GL_n$ with pairs of integers, then $\Phi(J_\nu)$ is equal to $\sim_\nu$ without the diagonal. Thus we have

$$n^{\lambda, \mu}_\nu = \sum_{w \in S_n : \text{Part}(\sim_\lambda \cap w(\sim_\mu)) = \nu} \frac{|w^{-1}S_\lambda w \cap S_\mu|}{|S_\mu||S_\lambda|}.$$
Now consider the coproduct $\Delta^n$. We have

$$\Delta^n(m_\nu) = \sum_{[(i_1, j_1), \ldots, (i_n, j_n)]} (X_{i_1}Y_{j_1}) \cdots (X_{i_n}Y_{j_n})$$

where the sum is over all multisets $[(i_1, j_1)]$, where the multiplicities of elements are given by $\nu$.

The group $S_n$ is acting naturally on length $n$ sequences. Let $(\mu)$ be the standard sequence $1, \ldots, 1, 2, \ldots, 2, \ldots, n$ times $\mu_2$ times.

In $\Delta^n(m_\nu)$, $X^\lambda Y^\mu$ occurs as:

$$\sum_{j_1, \ldots, j_n} \frac{1}{|\text{orbit of } S_\lambda \text{ on } j_1, \ldots, j_n|} (X_{Y_{j_1}} \cdots X_{Y_{j_{\lambda_1}}} Y_{X_{j_{\lambda_1}+1}} \cdots Y_{X_{j_{\lambda_2}}}) \cdots,$$

where the summation is over all sequences $j_1, \ldots, j_n$ such that $|j_1, \ldots, j_n| = |(\mu)|$ and $\text{Part}(\sim_j \cap \sim_j) = \nu$, where $\sim_j$ denotes the equivalence relation $t \sim_j s$ iff $j_1 = j_s$. Let $\tilde{\pi}_n^\lambda, \mu$ denote the coefficient of $m_\lambda \otimes m_\mu$ in $\Delta^n(m_\nu)$. The condition $|j_1, \ldots, j_n| = |(\mu)|$ is equivalent to the existence of $w \in S_n$ such that $w \cdot (\mu) = (j_1, \ldots, j_n)$, in which case $w \cdot \sim_j = \sim_j$. Since there are exactly $|S_\mu|$ such $w$, we get

$$\tilde{\pi}_n^\lambda, \mu = \sum_{w \in S_n : \text{Part}(\sim_j \cap \sim_j(w(\sim_j))) = \nu} \frac{|w^{-1}S_\lambda S_\mu|}{|S_\mu||S_\lambda|},$$

which agrees with (28). This finishes the proof of Proposition 7.3.

Recall the vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ in [18, Section 5.4], which is defined as:

$$\mathcal{E} = \mathcal{O}(-d_1)^{\mu_1} \oplus \cdots \oplus \mathcal{O}(-d_m)^{\mu_m}, \quad 0 \leq d_1 < \ldots < d_m, \quad \mu_i > 0, \quad 1 \leq i \leq m, \quad \sum_{i=1}^{m} \mu_i = n.$$

Let $\mu : \mathbb{G}_m \to T_n$ be the cocharacter of the form

$$t \mapsto \text{diag}(t^{-d_1}, \ldots, t^{-d_1}, t^{-d_1}, \ldots, t^{-d_1}), \quad 0 \leq d_1 < \ldots < d_m, \quad \mu_i > 0, \quad 1 \leq i \leq m, \quad \sum_{i=1}^{m} \mu_i = n.$$

Then we have $\mu \in X_-(T_n)$ and we get that $\mathcal{E} = \mathcal{E}_\mu$ (see 2.3).

Let us write $\mu = (\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$, where $\tilde{\mu}_k : \mathbb{G}_m \to T_{\mu_k}, 1 \leq k \leq m$ is the cocharacter

$$t \mapsto \text{diag}(t^{-d_k}, \ldots, t^{-d_k}), \quad d_k \geq 0.$$

We have $\tilde{\mu}_k \in X_-(T_{\mu_k})$. The following is a key factorization result, which is a corollary of Theorem 3.2.

**Corollary 7.1.1.** For the vector bundle $\mathcal{E}$ over $\mathbb{P}^1$, we have

$$C_\mu[X,Y;q] = \prod_{k=1}^{m} C_{\tilde{\mu}_k}[X,Y;q].$$

**Proof.** Let $f_k$ be an associate invariant function on $\mathcal{P}(\Pi_k)$, where $\Pi_k$ is the set of simple roots of $GL_{\mu_k}$. According to our Convention 7.1, $f_k$ is viewed as an element of $\text{Sym}_{\mu_k}^\infty[X]$. However, we can also view $f_k$ as a symmetric function $f'_k$ in the variables $x_{\mu_1 + \ldots + \mu_{k-1} + 1}, \ldots, x_{\mu_1 + \ldots + \mu_k}$. Recall that in Section 3.2 we defined the map $\pi_\mu : \mathbb{Z}[\mathcal{P}(\Pi)] \to \mathbb{Z}[\mathcal{P}(\Pi)]$. In the case of $GL_n$, this map relates products for two different interpretations of symmetric functions.

...
Lemma 7.2. Keep notations as above. Then
\[ f_1 \ldots f_k = \pi_\mu(f_1' \ldots f_k'). \]

Proof. Note that in the case of $GL_n$ the map $\pi_\mu$ is the symmetrization map.

We return to the proof of Corollary 7.1.1. Recall from Section 3.1 that $[St_{GL_\mu}]$ is the function on $P^* \times P^*$ that counts the number of $P$-points of $St_{GL_\mu}$ $(J_1, J_2), (J_1, J_2) \in P^* \times P^*$. Using Corollary 7.3.1 we consider $[St_{GL_\mu}]$ as a symmetric function (see Convention 7.1). We can also view $[St_{\mu}]$ as a symmetric function $[St_{\mu}]'$ in the variables $x_{\mu_1+\ldots+\mu_k-1+1}, \ldots, x_{\mu_1+\ldots+\mu_k}$. Now by Corollary 3.2.1 Fact 6.2 and Lemma 4.4, we have
\[ C_{\mu}[X, Y; q] = (\pi_\mu \otimes \pi_\mu)([St_{\mu}]) / \prod_k |GL_{\mu_k}| = (\pi_\mu \otimes \pi_\mu)(\prod_k [St_{\mu_k}'] / \prod_k |GL_{\mu_k}|. \]

Using Lemma 7.2 in each variable, we get
\[ (\pi_\mu \otimes \pi_\mu)(\prod_k [St_{\mu_k}']) / \prod_k |GL_{\mu_k}| = \prod_k [St_{\mu_k}] / |GL_{\mu_k}| = \prod_k C_{\tilde{\mu_k}}[X, Y; q], \]

where we also used Corollary 3.2.3. This finishes the proof of Corollary 7.1.1.

We have the following corollary.

Corollary 7.2.1. Keep notations as above. Then
\[ C_{\mu}[X, Y; q] = \prod_{k=1}^m h_{\mu_k} \left[ \frac{XY}{q-1} \right]. \]

Proof. We have
\[ C_{\mu}[X, Y; q] = \prod_{k=1}^m C_{\tilde{\mu_k}}[X, Y; q] = \prod_{k=1}^m \Delta_{GL_{\mu_k}}([St_{GL_{\mu_k}}]) / |GL_{\mu_k}|, \]

where the first equality follows from Corollary 7.1.1, the second equality follows from Corollary 3.2.3 and the last equality follows from Theorem 3.1 (ii). Now the lemma follows from Proposition 7.2 and Proposition 7.3.

Now we are ready to prove Proposition 7.1. We have
\[ \sum_{n=0}^\infty \Omega_{n, (0, \infty)}([P]^1)[X, Y; q, t] = \sum_{n=0}^\infty \sum_{\mu \in X_n} t^{-deg(\xi)} C_{\mu}[X, Y; q] = \sum_{n=0}^\infty \sum_{\mu = (\mu_1, \ldots, \mu_n)} t^{\sum_{k=1}^n d_k \mu_k} h_{\mu} \left[ \frac{XY}{q-1} \right], \]

where the second equality follows from Corollary 7.2.1. The above is equal to
\[ \prod_{d=0}^\infty \sum_{k=0}^\infty t^{dk} h_k \left[ \frac{XY}{q-1} \right] = \prod_{d=0}^\infty \text{Exp} \left[ t^d \frac{XY}{q-1} \right] = \text{Exp} \left[ \frac{XY}{q-1} \sum_{d=0}^\infty t^d \right] = \text{Exp} \left[ \frac{XY}{(q-1)(1-t)} \right]. \]

This finishes the proof of Proposition 7.1. \qed
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