BRANES AND THE DYNAMICS OF QCD

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A brane configuration is described that is relevant to understanding the dynamics of $N = 1$ supersymmetric Yang-Mills theory. Confinement and spontaneous breaking of a discrete chiral symmetry can be understood as consequences of the topology of the brane. Because of the symmetry breaking, there can be domain walls separating different vacua; the QCD string can end on such a domain wall. The model in which these properties can be understood semiclassically does not coincide with supersymmetric Yang-Mills theory but is evidently in the same universality class.

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1. Introduction

Lately, many models of supersymmetric gauge theory in different dimensions have been studied using various approaches based on string theory. These models have varying applications and shed light on many different aspects of strongly interacting gauge theories. The purpose of the present paper is to apply some of these methods to the most classical riddles of four-dimensional gauge dynamics – such as confinement, chiral symmetry breaking, and the relation, if any, between them – in a situation as realistic as possible while still more or less tractable.

To this aim, we want to explore a supersymmetric model that is a relatively close analog of ordinary non-supersymmetric QCD. Such a model is the minimal $N = 1$ model with an $SU(n)$ vector multiplet only and no chiral superfields. This model is believed to exhibit confinement, a mass gap, and spontaneous breaking of a (discrete) chiral symmetry. These are some of the main properties that we would like to understand in ordinary QCD. Many other interesting supersymmetric models contain massless scalars, as a result of which they are not such close relatives of QCD.

Another important property that the minimal $N = 1$ model is believed to share in common with ordinary QCD is a large $n$ limit in which confinement and the mass gap persist, hadron masses approach fixed limits, and the residual interactions among hadrons vanish \footnote{Understanding of this limit, perhaps via a new kind of string theory, is probably well out of reach with present methods, but may offer the best long term hope of a much better understanding of QCD than we now possess. Note that $N = 2$ supersymmetric Yang-Mills theory is not believed to have a large $n$ limit of the conventional sort \footnote{A quite different way to study gauge theory dynamics via string theory involves encoding four-dimensional gauge dynamics in Type IIA and Type IIB geometry near singularities, a process initiated in \cite{20} and developed subsequently in many directions. In \cite{21}, such methods were used to propose a fivebrane interpretation of some four-dimensional field theories, and in \cite{22} this was extended to give exact exact solutions of a very large family of four-dimensional $N = 2$ models.}}. Understanding of this limit, perhaps via a new kind of string theory, is probably well out of reach with present methods, but may offer the best long term hope of a much better understanding of QCD than we now possess. Note that $N = 2$ supersymmetric Yang-Mills theory is not believed to have a large $n$ limit of the conventional sort \footnote{A quite different way to study gauge theory dynamics via string theory involves encoding four-dimensional gauge dynamics in Type IIA and Type IIB geometry near singularities, a process initiated in \cite{20} and developed subsequently in many directions. In \cite{21}, such methods were used to propose a fivebrane interpretation of some four-dimensional field theories, and in \cite{22} this was extended to give exact exact solutions of a very large family of four-dimensional $N = 2$ models.}.

To study the pure $N = 1$ gauge theory, we follow a path that has been followed for a variety of three, four, and five-dimensional models \footnote{A quite different way to study gauge theory dynamics via string theory involves encoding four-dimensional gauge dynamics in Type IIA and Type IIB geometry near singularities, a process initiated in \cite{20} and developed subsequently in many directions. In \cite{21}, such methods were used to propose a fivebrane interpretation of some four-dimensional field theories, and in \cite{22} this was extended to give exact exact solutions of a very large family of four-dimensional $N = 2$ models.}: we construct a configuration of branes in a weakly coupled string theory that realizes the field theory of interest at low energies, and then we solve the model in the strong coupling limit by using some of the string theory dualities. A method of doing this in the case at hand will be described in

\footnote{A quite different way to study gauge theory dynamics via string theory involves encoding four-dimensional gauge dynamics in Type IIA and Type IIB geometry near singularities, a process initiated in \cite{20} and developed subsequently in many directions. In \cite{21}, such methods were used to propose a fivebrane interpretation of some four-dimensional field theories, and in \cite{22} this was extended to give exact exact solutions of a very large family of four-dimensional $N = 2$ models.}
section 2. We analyze the symmetries of the model and show that the $\mathbb{Z}_n$ chiral symmetry is spontaneously broken.

Then in sections 3 and 4, we analyze further properties of the model. In section 3, we show that the model is confining: it has strings or flux tubes, which can terminate on an external quark, and can annihilate in groups of $n$. The ability of $n$ identical strings to annihilate reflects the fact that $n$ quarks – that is $n$ copies of the fundamental representation of $SU(n)$ – can be combined to make an $SU(n)$-invariant “baryon.”

Chiral symmetry breaking means that the theory has $n$ distinct vacua, and therefore there are domain walls connecting them. It has been argued [23] that the domain walls of this model are BPS-saturated. This is also supported by the existence (see p. 635 of [24]) of BPS-saturated domain walls in an effective field theory [25] that applies both to the two-dimensional $\mathbb{CP}^n$ model and to super Yang-Mills domain walls. In section 4, we describe how a BPS-saturated domain wall could be represented in the present formulation via a “supersymmetric three-cycle” in the sense of [26,27]. (But we do not actually prove the existence of such a cycle obeying the appropriate boundary conditions.) The model thus has both one-branes – the QCD strings – and two-branes – the domain walls. We establish by a topological argument a new result which has not been guessed previously: the two-branes are $D$-branes for the QCD string, that is a QCD string can end on a domain wall. This is related to the behavior of the chiral order parameters inside the domain wall. An intuitive explanation of this new effect has been given by S.-J. Rey [28] and will be mentioned in section 4. Finally, we show that – as one might have guessed intuitively – the chiral symmetry breaking of the model is reflected in a non-zero value of the gluino condensate. From a field theory point of view this is of course an old result [29]. To establish it via branes, we introduce and evaluate an expression for the superpotential of an $N = 1$ brane configuration; this expression is likely to have other applications.

In section 5, it comes time to pay the piper. Why has it been possible to understand by semi-classical methods rather subtle, strong coupling properties of QCD (or a good analog of it) that are traditionally out of reach of such methods? Are branes magic? The answer is that what we have realized via the branes is a generalization of the supersymmetric Yang-Mills theory that depends on one extra parameter, essentially the Type IIA string coupling constant. The limit in which the semi-classical methods of this paper are effective is quite different from the limit in which the theory reduces to standard $N = 1$ super Yang-Mills theory. They both are (very likely) in the same universality class, justifying the application
of the results of sections 2-4 to the standard field theory, but quantitative details of the mass spectrum and interactions will be different.

The “new” theory is best understood in terms of compactification of the six-dimensional (0, 2) supersymmetric field theory [30] which can be interpreted in terms of parallel fivebranes [31]. The best approach to the large $n$ limit of QCD might involve tackling first the large $n$ limit of this theory. It is also at least remotely conceivable that the sort of generalization of standard gauge theory considered here could show up directly in accelerator experiments (presumably with a small value of the deformation parameter and not the large value that makes possible semiclassical analysis via branes), maybe even permitting new approaches to the gauge hierarchy problem.

The brane approach to gauge theories is not necessarily limited to supersymmetric theories. To illustrate this, we will in section 6 construct an $M$-theory brane configuration relevant to ordinary bosonic $SU(n)$ gauge theory just as the configuration of section 2 is related to the supersymmetric case. As we explain, the mystery is not how to construct non-supersymmetric brane configurations associated with interesting field theories, but what can be learned by studying them.

While carrying out the present work, I learned of work by K. Hori, H. Ooguri, and Y. Oz in which related results were obtained, including a description of the brane configuration of section 2 and generalizations to include chiral superfields [32]. Another closely related paper is [33].

2. The Brane Configuration

2.1. Preliminaries

We work in Type IIA superstring theory with spacetime coordinates $x^0, x^1, \ldots, x^9$. The brane configuration we will study is similar to ones discussed in papers cited in the introduction. In fact, it is a special case (with some branes omitted) of a configuration studied in [7].

We consider an NS fivebrane located at $x^6 = x^7 = x^8 = x^9 = 0$ (its worldvolume is thus spanned by $x^0, x^1, \ldots, x^5$) and an NS' fivebrane located at $x^6 = S_0, x^4 = x^5 = x^9 = 0$ (its worldvolume is thus spanned by $x^0, \ldots, x^3$ and $x^7, x^8$). Here $S_0$ is an arbitrary length. We set $v = x^4 + ix^5, w = x^7 + ix^8$.

Between the two fivebranes we suspend $n$ Dirichlet fourbranes, whose worldvolumes are defined at the classical level by $v = w = x^9 = 0, 0 \leq x^6 \leq S_0$ (and so are spanned
by \( x^0, \ldots, x^3 \) along with \( x^6 \)). Quantum mechanically, \( v, w, \) and \( x^9 \) are free to fluctuate as fields on the fourbrane, with boundary conditions which, if one ignores fluctuations in the fivebrane position, are \( w = x^9 = 0 \) at \( x^6 = 0 \) and \( v = x^9 = 0 \) at \( x^6 = S_0 \).

Interest will focus on the quantum field theory on the fourbrane. This will be a four-dimensional field theory, since the fourbrane worldvolume (being finite in the \( x^6 \) direction) is 3+1-dimensional macroscopically. Because of the \( n \) parallel fourbranes which classically are coincident, the theory will be an \( SU(n) \) gauge theory. The effective four-dimensional theory has \( N = 1 \) supersymmetry and has no massless bosons other than the gauge fields since all massless scalars on the fourbrane are projected out at \( x^6 = 0 \) (\( w \) and \( x^9 \)) or at \( x^6 = S_0 \) (\( v \) and \( x^9 \)). So this theory is the \( SU(n) \) gauge theory with \( N = 1 \) supersymmetry and no chiral multiplets.

The gauge coupling \( g \) in this theory is related to the Type IIA gauge coupling \( g_{st} \) by

\[
\frac{1}{g^2} = \frac{S_0}{g_{st}}
\]

up to a universal constant multiple. Since \( g \) (which ultimately sets the QCD mass scale) is the only parameter in the pure gauge theory, we see the fundamental fact that the gauge theory depends on only one combination of \( S_0 \) and \( g_{st} \). What happens in the brane theory will be examined in section 5.

The justification for claiming that this brane configuration can realize the pure four-dimensional supersymmetric gauge theory – decoupled from all other complications of the string theory – is as follows. If \( g_{st} \) is small and \( S_0 \) is large, then the theory on the fourbrane is weakly coupled at the string scale. But, being infrared unstable, it flows at very low energies to strongly coupled supersymmetric QCD. At such low energies, gravitation and all the other complications of the string theory can be neglected.

\(^3\) The gauge group is really \( SU(n) \) and not \( U(n) \), as one might have supposed. This was explained at the \( N = 2 \) level in [9] and continues to hold after rotating the branes to get to \( N = 1 \) supersymmetry. (Indeed, the \( M \)-theory brane configuration found below has genus zero – no massless photons – and not genus one – one massless photon.) Apart from arguments given in [9], this can apparently be understood in terms of Higgsing of the center of \( U(n) \) by the antisymmetric tensor fields on the fivebranes.
2.2. Solution Via M-Theory

As in [9], the solution of the model is made, at least at a formal level, by going to M-theory. We take \( g_{str} \) large, whereupon an eleventh dimension appears, a circle \( S^1 \) with angular coordinate \( x^{10} \). We henceforth measure distances in M-theory units. We take \( 0 \leq x^{10} \leq 2\pi \) and put the radius \( R \) of the circle in the eleven-dimensional metric:

\[
ds^2 = \sum_{i,j=0}^{9} \eta_{ij} dx^i dx^j + R^2 (dx^{10})^2.
\] (2.2)

If \( C = 2\pi R \) is the circumference of the circle, and \( S \) is the separation between the fivebranes measured in M-theory units, then the formula (2.1) becomes

\[
\frac{4\pi}{g^2} = \frac{S}{C}.
\] (2.3)

(This assertion is a consequence of the assertion [34] that in compactification of a chiral two-form theory and hence of a fivebrane on a two-torus, the induced \( \tau \) parameter of the four-dimensional \( U(1) \) gauge field derived from the two-form is the \( \tau \) of the two-torus. In the present case, the fourbrane corresponds to a long tube connecting the fivebranes, of circumference and length \( C \) and \( S \), and this gives (2.3).)

Going to M-theory makes possible a solution of the theory as follows. In M-theory the fourbranes are just fivebranes wrapped on the \( S^1 \). In fact, all branes in the problem are fivebranes, and the fivebrane world-volume can be described as \( \mathbb{R}^4 \times \Sigma \) where \( \mathbb{R}^4 \) is a copy of four-dimensional Minkowski space (with the coordinates \( x^0, \ldots, x^3 \) that the various branes have in common) and \( \Sigma \) a two-dimensional surface. \( \Sigma \) is in fact a complex Riemann surface in a three-dimensional space with coordinates \( v, w \), and \( t = e^{-s} \), where \( s = R^{-1} x^6 + ix^{10} \).

To determine \( \Sigma \), we may simply proceed as follows. On \( \Sigma \), \( v \) goes to infinity only at one point, which is infinity on the original NS fivebrane; and \( v \) has only a single pole there, since there is only one NS fivebrane. So \( v \) is a holomorphic function on \( \Sigma \) with only one pole at infinity. \( \Sigma \) can therefore be identified as the complex \( v \)-plane (perhaps with some points deleted where other variables have poles). Likewise \( w \) has only a single pole – as \( w \to \infty \) only on the NS’ fivebrane. \( v = 0 \) where \( w \) has a pole (since the NS’ fivebrane is at \( v = 0 \)) and likewise \( w = 0 \) at \( v = \infty \). The most general function with these properties

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4 The \( R^{-1} \) is present in this formula because of the \( R^2 \) multiplying \((dx^{10})^2\) in (2.2).
is \( w = \zeta v^{-1} \) for some complex constant \( \zeta \). \( \Sigma \) should go to infinity only at the positions of one of the two fivebranes, so \( \Sigma \) is precisely the \( v \)-plane with the two points \( v = 0 \) and \( v = \infty \) deleted.

\( t \) should be given by a formula \( t = F_n(v) \), where \( F_n(v) \) is such that the equation \( F_n(v) = t \), regarded as an equation for \( v \) with fixed \( t \), has \( n \) roots – corresponding in the Type IIA description to the fact that for given \( t \), there are \( n \) fourbranes. Any \( F_n \) obeying this condition is of the form \( F_n(v) = P(v)/Q(v) \), where \( P \) and \( Q \) are polynomials of degree \( n \). Since \( t = \infty \) and \( t = 0 \) are at infinity (they correspond to \( x^6 = \pm\infty \) ), \( t \) should have no zeroes or poles as a function of \( v \) except at \( v = 0 \) (which is \( w = \infty \)) or \( v = \infty \). The most general possibility obeying these conditions is essentially \( t = v^n \). (A multiplicative constant, that is \( t = cv^n \), can be absorbed in adding a constant to \( x^6 \); this acts by \( t \to \lambda t \). The alternative solution \( t = v^{-n} \) is equivalent modulo an exchange of the two fivebranes. The solution \( t = v^n \) corresponds to having the NS fivebrane at smaller \( x^6 \) than the NS' fivebrane, so that \( x^6 \to -\infty \) and \( t \to \infty \) on the NS fivebrane while \( x^6 \to \infty \) and \( t \to 0 \) on the NS' fivebrane.)

The curve \( \Sigma \) is thus described by the equations

\[
\begin{align*}
w &= \zeta v^{-1} \\
v^n &= t \\
w^n &= \zeta^n t^{-1}.
\end{align*}
\] (2.4)

Of course, this description is redundant; the second or third equation could be dropped.

**Symmetries**

Now, let us analyze the behavior at infinity. Infinity has two components, \( v \to \infty \) and \( w \to \infty \). One component of infinity can be described asymptotically by \( w \to 0 \) and \( v^n = t \). The other component at infinity can be described asymptotically by \( v \to 0 \) and \( w^n = \zeta^n t^{-1} \). We see that only \( \zeta^n \), and not \( \zeta \), enters in the behavior at infinity.

In studying quantum field theory on branes, the behavior at infinity determines a particular quantum problem. After fixing the behavior at infinity, one looks for the possible supersymmetric or lowest area branes with the given asymptotic behavior. They represent possible quantum ground states in the quantum problem in question.

In the case at hand, for given behavior at infinity, there are \( n \) possible choices for the interior behavior of the brane. Indeed, the number \( \zeta^n \) appears in the behavior at infinity.
The possible interior behaviors or quantum states correspond to the $n$ possible values of $\zeta$ for given $\zeta^n$.

The supersymmetric $SU(n)$ gauge theory that we are trying to solve is believed to have $n$ vacua resulting from a spontaneously broken $\mathbb{Z}_n$ chiral symmetry. It is natural to try to identify the $n$ vacua that we have just found with the $n$ vacua expected from chiral symmetry breaking.

The symmetries of the quantum problem are symmetries of the asymptotic behavior of the branes at infinity. A symmetry is unbroken if it leaves fixed the entire brane, and not just the behavior at infinity. In our case, there is a $U(1)$ symmetry group $U$ that acts by

$$
t \to e^{in\delta} t,
$$

$$
v \to e^{i\delta} v
$$

$$
w \to e^{-i\delta} w.
$$

There is also an additional $\mathbb{Z}_n$ symmetry group $H$ generated by

$$
t \to t
$$

$$
v \to v
$$

$$
w \to \exp(2\pi i/n) w.
$$

Finally there is a $\mathbb{Z}_2$ symmetry $K$ that acts by

$$
v \to w
$$

$$
w \to v
$$

$$
t \to \zeta^n t^{-1}.
$$

Of these symmetries, the $\mathbb{Z}_n$ is spontaneously broken; it is a symmetry of the brane at infinity but is not an exact symmetry of the brane as it does not leave invariant the first equation in (2.4). The other symmetries are unbroken, as they are invariances of (2.4).

To identify these symmetries, first note the following. Let $Y$ be the complex three-fold with coordinates $v, w, t$. Note that topologically $Y = \mathbb{R}^5 \times S^1$. $Y$ can be regarded as a flat Calabi-Yau manifold with metric

$$
 ds^2 = |dv|^2 + |dw|^2 + R^2 |ds|^2 = |dv|^2 + |dw|^2 + R^2 \frac{|dt|^2}{|t|^2}
$$

and holomorphic three-form

$$
 \Omega = R \frac{dv \wedge dw \wedge dt}{t}.
$$


Thus, $\Omega \wedge \overline{\Omega}$ is the volume form of the Riemannian metric on $Y$. As in Calabi-Yau compactification of superstring theory, the superspace measure $d^2\theta$ (the $\theta$’s being the chiral odd coordinates of $N = 1$ superspace) transforms like $\Omega^{-1}$, and a symmetry is an $R$-symmetry if and only if it transforms $\Omega$ non-trivially. By this criterion, we see that the spontaneously broken $\mathbb{Z}_n$ symmetry group found above is a group of $R$-symmetries; this is the expected spontaneously broken $\mathbb{Z}_n$ chiral symmetry group of the model.

What about the other symmetries? $K$, since it reverses the orientation of the Type IIA spacetime (whose coordinates are the original ten variables $x^0, \ldots, x^9$), reverses the orientation of all elementary Type IIA strings. Gauge bosons and gluinos in the adjoint representation of $SU(n)$ are built from elementary strings that connect different fourbranes. Reversing the orientation of the elementary strings exchanges positive and negative roots of $SU(n)$ and hence exchanges the fundamental and antifundamental representations of $SU(n)$. This operation is usually called “charge conjugation.” It is expected to be unbroken in the $SU(n)$ gauge theory, and this agrees with what we have just found.

More mysterious at first sight is the proper interpretation of the $U(1)$. In the Type IIA description, there appear to be three $U(1)$ symmetries. Rotations of the $v$-plane, the $w$-plane, and the $x^{10}$ circle appear to be separate invariances of the classical brane configuration. In actuality, one does not have all three $U(1)$ symmetries separately because the end of a fourbrane on a fivebrane looks like a “vortex.” In fact, on the two fivebranes one has respectively $x^{10} \sim n \arg(v)$ for large $v$ and $x^{10} \sim -n \arg \ln(w)$ for large $w$. These expressions are invariant only under one linear combination of the three $U(1)$’s, namely the one defined in (2.5).

This one $U(1)$ symmetry acts trivially on both gluons and gluinos and hence on the entire $SU(n)$ gauge theory that we are aiming to investigate in the present paper. This is clear in the Type IIA description. States of the Type IIA superstring theory on which this symmetry acts non-trivially are, for instance, zerobranes that carry momentum in the $x^{10}$ direction, and all sorts of things that carry angular momentum in the $v$-plane or $w$-plane. Gluons and gluinos are invariant under this symmetry. In any limit in which the brane theory actually reproduces the supersymmetric Yang-Mills theory quantitatively, all modes carrying the $U(1)$ charge must decouple from the Yang-Mills physics.

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5 The symmetry group of the theory is actually $\mathbb{Z}_{2n}$, but the element $-1$ of $\mathbb{Z}_{2n}$ is a $2\pi$ rotation of spacetime and is unbroken. The group that acts on bosonic variables such as the gluino bilinear $\text{Tr} \lambda\lambda$ is the quotient $H = \mathbb{Z}_n$ of $\mathbb{Z}_{2n}$, and this from a field theory point of view is the spontaneously broken chiral symmetry group.
2.3. Rotation Of $N=2$ Solution

For completeness, we will now show how the brane configuration that we found directly could have been found by “rotating,” in the sense of [35], the brane configuration that describes $SU(n)$ gauge theory with $N=2$ supersymmetry. We recall [9] that the latter is described by the brane configuration

$$t^2 + P_n(v)t + 1 = 0 \quad (2.10)$$

in $v-t$ space, at $w=0$. This describes $n$ fourbranes suspended between two parallel fivebranes. $P_n(v)$ is a polynomial of the form

$$P_n(v) = v^n + u_2v^{n-2} + \ldots + u_n, \quad (2.11)$$

where the $u_i$ are the “order parameters” of the theory.

“Rotating” one of the fivebranes will break $N=2$ supersymmetry to $N=1$. This can only be done at points in moduli space at which the genus $n-1$ curve (2.10) degenerates to a curve of genus zero [39]. A curve $\Sigma$ of genus zero can be “parametrized rationally.” This means the following. One can introduce an auxiliary complex parameter $\lambda$, and identify $\Sigma$ as the complex $\lambda$ plane, perhaps with some points deleted or a point at infinity added. Then $v, w, t$ can be expressed as rational functions of $\lambda$.

As $v$ has poles only at “ends” of fivebranes, of which there are precisely two, $v$ has exactly two simple poles as a function of $\lambda$. We can hence take $v = \lambda + c\lambda^{-1}$, for some constant $c$, with no loss of generality. Also, $t$ can go to zero or infinity only at poles of $v$, that is at $\lambda = 0$ or $\infty$; this implies that $t$ is a constant multiple of a power of $\lambda$. Requiring that (2.10) should be obeyed for $\lambda \to \infty$ and $\lambda \to 0$ (and recalling the form (2.11) of $P_n$), we find that $t = -\lambda^n$ and that $c^n = 1$, and moreover for each choice of $c$ the polynomial $P_n$ is uniquely determined. (There are also solutions with $t$ a multiple of $\lambda^{-n}$ but these differ by $\lambda \to \lambda^{-1}$ and merely give another way to parametrize the same branes.)

Thus, as expected we have found $n$ polynomials $P_n$, corresponding to the $n$ roots of $c^n = 1$, for which the curve $\Sigma$ degenerates to genus zero and hence can be “rotated.” We now set $c = 1$, for brevity, and attempt to make the rotation.

After the rotation, $w$, instead of being zero, should be a non-zero holomorphic function on $\Sigma$. As we only want to rotate one fivebrane, $w$ should get a pole only at one “end” of $\Sigma$ and should vanish at the other end. There is hence no essential loss of generality in setting $w = \zeta\lambda^{-1}$ for some complex constant $\zeta$. Moreover, after redefining $v$ by $v \to v - \zeta^{-1}w$, we reduce to $v = \lambda$, so finally (after a sign change of $t$) the equations defining the curve are the ones found above: $w = \zeta v^{-1}$, $t = v^n$. 

9
2.4. Mass Scale

In the forgoing, we have simply taken $\zeta$ to be an arbitrary complex number. As long as one considers only the universality class of the model, the value of $\zeta$ does not matter. If one considers only holomorphic quantities, $\zeta$ can be scaled out by redefining $w$.

Such a scaling will, however, affect the particle masses and interactions, as it does not leave invariant the metric of space-time. In fact, the parameters of the model from the $M$-theory point of view are $\zeta$ as well as $R$, the radius of the eleventh dimension. We cannot expect to get a quantitative description of supersymmetric Yang-Mills theory if these are taken to be generic (say of order one in $M$-theory units), since that will give no mechanism to decouple all of the extra degrees of freedom and complications of $M$-theory. One certainly can recover super Yang-Mills theory for $R \to 0$ (weakly coupled Type IIA superstring theory) and small $\zeta$. More details about the extent to which this theory behaves like super Yang-Mills theory will gradually become clear.

3. Confinement

3.1. Membranes And Strings

One of the main mysteries of QCD is confinement. Accordingly, we would now like to analyze confinement in the present context.

The signal of confinement is that the theory should contain strings, which we will call QCD strings\(^6\), which could terminate on an external quark (a charge in the fundamental representation of $SU(n)$), but which, in the absence of dynamical quarks, are stable and cannot break. Thus, an external quark and antiquark connected by such a string and separated a distance $r$ would have an energy linear in $r$, a phenomenon referred to as confinement.

Since $n$ copies of the fundamental representation of $SU(n)$ can combine to a singlet, the QCD string is only conserved modulo $n$; $n$ parallel QCD strings can annihilate.

A natural guess, in the present context, is that the QCD string should be identified as an $M$-theory membrane, whose boundary is on the fivebrane as in \cite{31,37}.

\(^6\) From the discussion below and in section 6, it appears likely that the strings we consider are in the same universality class as the conventional non-supersymmetric QCD string, at least if $R^2\zeta^{-1/2}$ is sufficiently small.
The membrane lives in our $M$-theory world $\mathbb{R}^4 \times Y \times \mathbb{R}$, where $\mathbb{R}^4$ (parametrized by $x^0, \ldots, x^3$) is the effective Minkowski space, $Y$ is a complex three-fold, with coordinates $v, w$, and $t = e^{-s}$, that contains the complex curve $\Sigma$, and the last factor $\mathbb{R}$ (parametrized by the last coordinate $x^9$) plays no role in the present paper.

We consider a membrane that is the product of a string or onebrane in $\mathbb{R}^4$ times a onebrane in $Y$. Such a membrane will look like a string to a four-dimensional observer. We could and eventually will consider closed onebranes in $Y$. But more immediately pertinent to the QCD problem are open one-branes that end on $\Sigma$. We consider an open curve $C$ in $Y$, parametrized by a parameter $\sigma$ with $0 \leq \sigma \leq 1$; we orient $C$ in the direction of increasing $\sigma$. We require that the endpoints of $C$ (the points with $\sigma = 0, 1$) are in $\Sigma$. With $\Sigma$ defined by the familiar equations

$$w = \zeta v^{-1}$$
$$v^n = t$$
$$w^n = \zeta^n t^{-1},$$

we consider a curve $C$ determined by setting $t$ to a fixed constant $t_0$ ($t_0$ is necessarily non-zero, since the point $t = 0$ is at $w = \infty$ and so is omitted), and

$$v = t_0^{1/n} \exp(2\pi i \sigma/n)$$
$$w = \zeta v^{-1}.$$  \hspace{1cm} (3.2)

Here $t_0^{1/n}$ is any fixed $n^{th}$ root of the complex number $t_0$.

By picking this particular $C$, we get a string in spacetime which we will call the candidate QCD string. We will show that $n$ such strings can annihilate and then show that they can annihilate only in groups of $n$.\footnote{Twisted open strings that can annihilate in groups of $n$ were described in another context in \cite{38}.}

Consider $n$ strings determined by curves $C_j \subset Y$, $j = 1, \ldots, n$ with the $C_j$ defined by $t = t_0$ and

$$v = t_0^{1/n} \exp(2\pi i \sigma/n) \exp(2\pi ij/n)$$
$$w = \zeta v^{-1}.$$  \hspace{1cm} (3.3)

The $C_j$'s all have boundary on $\Sigma$, so they all determine strings in $\mathbb{R}^4$. The boundaries of the $C_j$ (with boundary points weighted by orientation) add up to zero, so these $n$ strings

\footnote{\cite{38}.}
can detach themselves from Σ, forming a closed loop in the v-plane. This loop can be
contracted to a point, so the n strings determined by the C_j can annihilate.

On the other hand, the C_j are all homotopic to each other (by a homotopy that keeps
the boundary on Σ). A homotopy between them can be made by taking t_0 → t_0e^{2πiα},
with 0 ≤ α ≤ 1. This homotopy deforms C_j to C_j+1. So the process described in the last
paragraph represents the annihilation of n identical candidate QCD strings.

To show that these candidate QCD strings can annihilate only in groups of n, I will
introduce some mathematical machinery that will have further applications below. Given
a topological space Y, one defines the homology groups H_k(Y, Z) to consist of closed
k-dimensional submanifolds (or more generally k-cycles) in Y modulo boundaries of k+1-
manifolds. Homology groups H_k(Σ, Z) of a subspace Σ of Y are defined in the same way,
using cycles in Σ. Finally, one defines the relative homology groups H_k(Y/Σ, Z) to consist
of k-dimensional submanifolds of Y (or more generally chains) which are not necessarily
closed, but whose boundaries are in Σ (modulo an equivalence relation in which a chain or
submanifold is considered trivial if after adding a chain that lies in Σ it is a boundary).
These homology groups are linked by an exact sequence

\[ \cdots \rightarrow H_{k+1}(Y/Σ, Z) \rightarrow H_k(Σ, Z) \rightarrow H_k(Y, Z) \rightarrow H_k(Y/Σ, Z) \rightarrow H_{k-1}(Σ, Z) \rightarrow \cdots \]  
(3.4)

The map H_k(Σ, Z) → H_k(Y, Z) maps a cycle in Σ to the “same” cycle in Y; the map
H_k(Y, Z) → H_k(Y/Σ, Z) maps a cycle in Y (which has no boundary) to the “same” cycle,
regarded as a relative cycle in Y/Σ (i.e., as a submanifold or chain which is allowed to
have a boundary in Σ but may have zero boundary); finally the map from H_k(Y/Σ, Z)
to H_{k−1}(Σ, Z) maps a k-dimensional submanifold S of Y whose boundary is in Σ to the
boundary of S, which is a k−1-dimensional closed manifold in Σ.

In M-theory on R^1 × X, where R^1 is “time” and X is a ten-manifold representing
“space,” membranes are classified topologically by H_2(X, Z). If in space-time there is a
fivebrane with world-volume R^1 × B, so that the membranes can have boundary on B,
then membranes are classified topologically by H_2(X/B, Z). In our case, X = R^4 × Y and
B = R^4 × Σ. Moreover, we want a membrane that is a product of a string in R^4 times a
onebrane in Y. These are classified topologically by H_1(Y/Σ, Z).

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8 If X is unorientable, one wants homology with twisted coefficients, since the three-form field
of eleven-dimensional supergravity (which couples to the membrane world-volume) is odd under
parity. In the present paper, spacetime will always be orientable.
To compute this group, we use the exact sequence (3.4). The relevant part of this exact sequence reads

\[ \ldots \to H_1(\Sigma, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \to H_1(Y/\Sigma, \mathbb{Z}) \to 0. \]  
(3.5)

(We are here using the fact that as \( \Sigma \) is connected, the map from \( H_0(\Sigma, \mathbb{Z}) \) to \( H_0(Y, \mathbb{Z}) \) is an isomorphism, both groups being isomorphic to \( \mathbb{Z} \); so the map from \( H_1(Y/\Sigma, \mathbb{Z}) \) to \( H_0(\Sigma, \mathbb{Z}) \) is zero.) Hence if \( i: H_1(\Sigma, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) is the map in (3.5), then

\[ H_1(Y/\Sigma, \mathbb{Z}) \cong H_1(Y, \mathbb{Z})/i(H_1(\Sigma, \mathbb{Z})). \]  
(3.6)

Now, with \( \Sigma \) being the complex \( v \) plane with the origin deleted, \( H_1(\Sigma, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \), being generated by a simple closed curve that wraps once around the origin in \( v \). With \( Y = \mathbb{R}^5 \times S^1 \), \( H_1(Y, \mathbb{Z}) \) is likewise isomorphic to \( \mathbb{Z} \), being generated by a simple closed curve that wraps once around the \( S^1 \) factor at a fixed point in \( \mathbb{R}^5 \). But the angular variable on the \( S^1 \) is the phase of the complex variable \( t \). So the formula \( t = v^n \) means that the map \( i \) multiplies a loop that wraps once around the origin in the \( v \)-plane to a loop that wraps \( n \) times around the origin in the \( t \)-plane. In other words, \( i \) is multiplication by \( n \) and hence

\[ H_1(Y/\Sigma, \mathbb{Z}) \cong \mathbb{Z}_n. \]  
(3.7)

The argument also shows that the generator of \( H_1(Y/\Sigma, \mathbb{Z}) \) is the image in that group of a generator of \( H_1(Y, \mathbb{Z}) \), that is of a loop \( C' \) that wraps once around the \( S^1 \). Such a \( C' \) can take the form

\[ v = t_0^{1/n}, \]
\[ w = \zeta v^{-1}, \]
\[ t = t_0 \exp(-2\pi i \sigma). \]  
(3.8)

This is actually a closed one-brane, the boundary values at \( \sigma = 0 \) and 1 being equal; it merely corresponds to a closed membrane wrapping once around the eleventh dimension, and is identified in double dimensional reduction [39] with the elementary Type IIA string! To make sense of this, first note that the curve \( C' \) really is homotopic (via a family of open curves with boundary on \( \Sigma \)) to \( C \). Such a homotopy can be made by the one-parameter family of curves \( C_\alpha \), \( 0 \leq \alpha \leq 1 \), with

\[ v = t_0^{1/n} \exp(2\pi i \alpha \sigma/n), \]
\[ w = \zeta v^{-1}, \]
\[ t = t_0 \exp(2\pi i (\alpha - 1) \sigma). \]  
(3.9)
Here $C_0 = C'$ and $C_1 = C$. So it is true that the elementary Type IIA superstring is equivalent topologically to the candidate QCD string. What does this mean?

**Energy Scales**

To proceed farther, we really need a discussion of energy scales in this problem.

The loop $C'$ corresponds to a string in spacetime with a tension $T'$ which (in $M$-theory units) is of order

$$T' \sim 2\pi R$$

with $R$ the radius of the eleventh dimension.

What about the tension of a string built using the curve $C$? This is proportional to the length of $C$. In traversing $C$, $t$ is constant, $\nu$ changes by an amount of order $t_0^{1/n}/n$, and $w = \zeta^\nu$ changes by an amount of order $\zeta t_0^{-1/n}/n$. To minimize the length of $C$, we must pick $t_0$ of order $\zeta^{n/2}$, whereupon the length of $C$ is of order $\zeta^{1/2}/n$. The tension $T$ of a four-dimensional string built using $C$ is hence of order

$$T \sim \frac{|\zeta|^{1/2}}{n}.$$  \hspace{1cm} (3.11)

Actually, this formula is only valid in a regime in which the string can be treated semiclassically, which as we discuss in section 5 will not be the case for all values of the parameters. (As we discuss momentarily, the candidate QCD string is not BPS-saturated, so its tension is subject to renormalization.) We will actually not determine the precise conditions for validity of (3.11).

To get a reasonable quantitative likeness of QCD, we need $T \ll T'$, so that a string built from $C$ (which is localized near the fivebrane) is energetically favored over a string built from $C'$ (which is an elementary Type IIA string, free to wander in ten dimensions and having nothing much to do with QCD). Thus, while the parameters can possibly vary more widely without spoiling the “universality class” of the theory, to get a quantitative likeness of QCD, we must require at least that

$$\frac{|\zeta|^{1/2}R}{n} \ll 1.$$  \hspace{1cm} (3.12)

If this inequality holds, then the candidate QCD string is much lighter than the elementary Type IIA string, and there is at least some hope that the QCD sector of the theory is decoupled from the complications of superstring theory and $M$-theory. In this situation, the candidate QCD string is, from this point of view, a sort of bound state of
an elementary Type IIA string with a fivebrane, in which the binding energy is almost one hundred percent (of the energy of a Type IIA string far from the fivebrane). There is the further curious fact that although in the absence of the fivebrane, the number of long parallel elementary Type IIA strings would be conserved, in the presence of the fivebrane they can turn into QCD strings and annihilate in groups of $n$.

3.2. Universality Class Of The QCD String

Now we would like to look more critically at the worldsheet properties of our candidate QCD strings and compare to what is expected for actual supersymmetric QCD strings.

Although it is possible to have BPS-saturated strings in $N = 1$ supersymmetric models in four dimensions, the supersymmetric Yang-Mills theory without chiral superfields has no conserved charges that could appear as central charges for strings. Also, the fact that the QCD string is only conserved modulo $n$ shows that it does not carry an additive conserved quantity and so cannot have a central charge. The supersymmetric QCD string of the minimal $N = 1$ super Yang-Mills theory is therefore not invariant under any of the four supersymmetries. So on this string propagate a full set of Goldstone fermions (two left-moving ones and two right-moving ones). It is not clear whether there are additional massless modes on the supersymmetric QCD string.

What about the candidate QCD string? At first sight, it appears to have one crucial difference from the supersymmetric QCD string. As described in eqn. (2.5), the brane configuration has a $U(1)$ symmetry $U$ that acts trivially on all QCD excitations. In particular, for a string to be interpreted as a product of QCD only (independent of all the other degrees of freedom of $M$-theory), it must be invariant under $U$; and $U$ must act trivially on the massless degrees of freedom that propagate on the string.

Since $U$ acts on $t_0$ by $t_0 \rightarrow e^{i\delta}t_0$, any given classical configuration of our candidate QCD string is not invariant under $U$. It appears that $U$ is spontaneously broken along the string, in which case $U$ would certainly act non-trivially on one of the massless modes that propagate on the string, namely the Goldstone boson.

To rescue the hypothesis that our candidate is really in the universality class of the QCD string, we must show that the dynamics on the string is such that at sufficiently long wavelengths all degrees of freedom that propagate on the string and on which $U$ acts non-trivially get mass. There is no issue of whether $U$ is spontaneously broken or not; by Coleman’s theorem, continuous symmetries such as $U$ are not spontaneously broken in two spacetime dimensions, so the $U$ symmetry, even though spontaneously broken classically,
will be restored at sufficiently long distances. The issue is whether after this symmetry restoration, all degrees of freedom propagating on the string that couple to $U$ get mass (which would agree with a conventional supersymmetric QCD string) or there remain massless excitations on the string on which $U$ acts non-trivially (in which case our candidate string is not in the universality class of the conventional supersymmetric QCD string).

The candidate QCD string appears to differ in yet a second way from actual QCD strings. It appears at first sight that the candidate QCD string has a conserved winding number that measures the number of times that it wraps around the circle factor in $Y = \mathbb{R}^5 \times S^1$. This has no analog in QCD, and must somehow disappear to justify a claim that the candidate QCD string has the universality class of an actual QCD string. We will see that this question is closely linked to the decoupling of $U$.

*Decoupling The $U(1)$ And Generating A Mass Gap*

A model with a spontaneously broken $U(1)$ global symmetry at tree level has a classical Lagrangian that looks like

$$L_2 = \frac{r^2}{4\pi} \int d^2 x |d\psi|^2,$$

(3.13)

where $\psi$ is an angular variable that is rotated by the $U(1)$, and $r$ is a constant; there can be additional terms of higher order or involving massive fields or massless but $U(1)$-invariant fields. Any such theory in $1+1$ dimensions has symmetry restoration quantum mechanically. Under what conditions does it develop a mass gap, as a result of which the $U(1)$ symmetry can decouple from the low energy physics?

If one did not have the $U(1)$ symmetry, the obvious perturbation of (3.13) that produces a mass gap would be a $\cos m\phi$ perturbation for some integer $m$; this is a relevant operator if $r$ is large enough, and its addition to the Lagrangian produces a mass gap. However, all such terms are forbidden by the $U(1)$ symmetry.

There is a slightly less obvious type of perturbation of (3.13) that can induce a mass gap. This is the dual (under the usual $r \rightarrow 1/r$ two-dimensional T-duality) of a $\cos m\phi$ perturbation. Duality exchanges $\cos m\phi$ with “twist fields.” If $r$ is sufficiently small, the twist fields are relevant operators and can induce a mass gap. While invariant under the $U(1)$, the twist fields do not preserve the winding number $\oint d\psi/2\pi$. The free theory (3.13) has no other potentially relevant perturbations.

The conclusion, then, is that in a model that classically has a spontaneously broken $U(1)$, the $U(1)$ can decouple at low energies if and only if $r$ is sufficiently small and the winding number is not conserved.
The importance of violation of the winding number conservation can alternatively be explained as follows. (This explanation will perhaps make it obvious that adding more matter fields to (3.13) does not change the picture.) Let \( |\Omega\rangle \) be the vacuum of a model with a \( U(1) \) symmetry in two dimensions, and let \( f \) be a function of the spatial coordinate \( x \) that is 0 at \( x = -\infty \) and 1 at \( x = +\infty \). Let \( J \) be the conserved current that generates the \( U(1) \) symmetry and \( J_0 \) the charge density. Consider the state

\[
|\Psi\rangle = \exp\left(2\pi i \int_{-\infty}^{\infty} dx \, f(x) J_0(x)\right) |\Omega\rangle.
\]

(3.14)

In the limit that \( f \) is very slowly varying, the energy of this state converges to zero (that is, to the energy of \( |\Omega\rangle \)). The state \( |\Psi\rangle \) carries winding number one, if the winding number is conserved. So either (i) the winding number is not conserved, or (ii) the current acting on the vacuum can create excitations of arbitrarily low energy, and thus the \( U(1) \) symmetry does not decouple from the very low energy two-dimensional physics.

A necessary condition for the candidate QCD strings to be in the universality class of actual supersymmetric QCD strings is therefore that there should be no conserved winding number on the string associated with the \( U(1) \) symmetry \( U \). This is necessary for the decoupling of \( U \), and (as we have already noted) is desirable in itself as this winding number has no analog in QCD.

We must show, then, that a candidate QCD string that wraps around the \( S^1 \) (in \( Y = \mathbb{R}^5 \times S^1 \)) can be unwrapped. To investigate this, we let \( \sigma \) and \( \rho \) be the two coordinates on an \( M \)-theory membrane. We have so far suppressed \( \rho \), describing the membrane at fixed \( \rho \) as a curve

\[
C : \ v = t_0^{1/n} e^{2\pi i \sigma/n}, \ w = \zeta v^{-1}, \ t = t_0.
\]

(3.15)

Now as \( \rho \) varies, the curve will move in \( \mathbb{R}^4 \) (which is why a four-dimensional observer interprets it as a string) and in addition \( t_0 \) can change. By giving \( t_0 \) an appropriate \( \rho \) dependence, we can describe a membrane that represents a string that wraps around the circle:

\[
v = t_0^{1/n} e^{2\pi i (\sigma/n + \rho)}
\]

\[
w = \zeta v^{-1}
\]

\[
t = t_0 e^{2\pi i n \rho}.
\]

(3.16)

This string actually wraps \( n \) times around the circle, because of the exponent in the formula for \( t \).
To show that such a wrapped string can actually disappear, it suffices to show that the two ends of the membrane loop around the same closed curve in $\Sigma$, so that the membrane can detach itself from $\Sigma$ and turn into a closed membrane in spacetime. Having done so, the membrane represents a class in $H_2(Y, \mathbb{Z})$, which (for $Y = \mathbb{R}^5 \times S^1$) vanishes, so the membrane can annihilate.

In fact, at $\sigma = 0$, the membrane traverses the loop $v = t_0^{1/n}e^{2\pi i\rho}, t = t_0e^{2\pi in\rho}$. At the other end of the membrane, $\sigma = 1$, the membrane traverses the loop $v = t_0^{1/n}e^{2\pi i(1/n+\rho)}, t = t_0e^{2\pi in\rho}$. After a transformation $\rho \to \rho + 1/n$, these coincide, showing that the membrane is executing the same loop at both ends and so can indeed detach itself and annihilate.

Since the above discussion was somewhat ad hoc and appears to deal only with the case that the winding number is a multiple of $n$, we will now carry out the analysis using a more powerful mathematical language. Winding states of the membrane are classified topologically by $H_2(Y/\Sigma, \mathbb{Z})$. Since $H_2(Y, \mathbb{Z}) = 0$, the exact sequence (3.4) says that this group is the kernel of the map $i : H_1(\Sigma, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$. We have already computed that this map is multiplication by $n$, and in particular is injective, so $H_2(Y/\Sigma, \mathbb{Z})$ vanishes and the membrane has no conserved winding numbers.

So there is no topological obstruction to the hypothesis that the $U$ symmetry decouples completely from the very long wavelength physics on the membrane. Indeed, it should be expected to do so if the effective value of $r$ is sufficiently small. The effective $r^2$ is actually the string tension $T$ times $R^2$. Thus the quantity that should be sufficiently small is

$$y = TR^2 = \frac{|\zeta|^{1/2}R^2}{n},$$

a combination that we will meet again in section 5.

4. Domain Walls

4.1. Preliminaries

A consequence of having a spontaneously broken discrete symmetry is that there can be domain walls separating spatial domains containing different vacua. In this section, we will consider these domain walls in super Yang-Mills theory.
If one contemplates these domain walls in the context of the 1/n expansion of the $SU(n)$ theory, one immediately runs into a puzzle. It is believed that the large $n$ limit of super (or ordinary) Yang-Mills theory is a weakly coupled theory of neutral massive particles (“glueballs”) with an effective coupling of order $1/n^2$. If we represent the particles of the effective theory by fields $B_i$, then their effective Lagrangian is something like

$$L_{e\text{ff}} = n^2 \int d^4x \mathcal{L}(B, \nabla B, \nabla^2 B, \ldots)$$

with corrections of relative order $1/n^2$. Thus in particular, large $n$ is the regime in which this theory can be treated semiclassically.

If such a theory has several vacua, one would at first expect domain walls connecting them to be described as soliton-like classical solutions, in which the fields $B_i$ are independent of the time and two spatial coordinates, but are non-trivial functions of the third spatial coordinate, say $x^3$. The idea would be to find a classical solution in which the $B_i$ approach one vacuum state for $x^3 \to -\infty$ and another vacuum state for $x^3 \to +\infty$.

Such a classical solution would have a tension, or energy per unit area, of order $n^2$. However, there are cogent reasons to believe that the super QCD domain wall has a tension that is of order $n$. This follows, as we will see, from the BPS formula for the tension of the domain wall, which is believed to be BPS-saturated [23].

Note that in $N = 1$ supersymmetry in four dimensions, particles cannot be BPS-saturated, as there is no central charge for particles in the supersymmetry algebra, but either strings or domain walls can be, since central charges can appear after compactification. Super Yang-Mills theory does not have a central charge for the QCD string (it hardly could as these strings can annihilate in groups of $n$), but it does have a central charge for domain walls.

In general, in four-dimensional $N = 1$ theories, the central charge governing domain wall tension $T_D$ in a sector with a domain wall interpolating between two vacua $a$ and $b$ is the superpotential difference $W(b) - W(a)$. One way to prove this is to compactify the directions transverse to the domain wall on a very large two-torus of area $A$. The theory then reduces to an effective two-dimensional theory, in which [12] the central charge in a

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9 Somewhat analogous questions about counting of powers of $n$, together with the analogy between string theory and the 1/n expansion, were part of the motivation for some early pre-D-brane work (see for example [10]) and for early conjectures [4] about string theory corrections of order $e^{-1/g}$.  

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sector of states that interpolate between distinct vacua at spatial infinity is the superpotential difference between the two vacua. In the two-dimensional theory, a BPS-saturated domain wall becomes a particle of mass $AT_D$ (or at least very nearly $AT_D$ in the large $A$ limit). This particle is BPS-saturated (or very nearly so for large $A$) so its mass is the absolute value of the superpotential difference. Since the superpotential difference in two dimensions is $A(W(b) - W(a))$ (as the two-dimensional superpotential is $A$ times the four-dimensional one), the tension of a BPS-saturated domain wall in four dimensions is $|W(b) - W(a)|$.

If we normalize the super QCD action in the customary way as

$$L = \frac{1}{4g^2} \int d^4x \text{Tr} \left( F_{IJ} F^{IJ} + \overline{\lambda} \Gamma \cdot D\lambda \right), \quad (4.2)$$

then the large $n$ limit is made by taking $n$ to infinity with

$$g^2 = \tilde{g}^2 = g^2n \quad (4.3)$$

kept fixed. It can be shown \[13\] that the superpotential in any vacuum of this theory is\[4\]

$$W = n\langle \text{Tr}\lambda\lambda \rangle. \quad (4.4)$$

On the other hand, if chiral symmetry breaking is a feature of the leading large $n$ approximation, then the gluino condensate $\langle \text{Tr}\lambda\lambda \rangle$, like the expectation value of any operator that is defined as the trace of a product of elementary fields, is of order $n$. \[11\] So the value of the superpotential in any given vacuum is of order $n^2$. The vacua differ from each other, however, by a chiral rotation, as a result of which, in the $j^{th}$ vacuum, one has

$$\langle \text{Tr}\lambda\lambda \rangle_j \sim Cn \exp(2\pi ij/n), \quad (4.5)$$

with $C$ independent of $j$ and of order $\Lambda_{QCD}^3$. This formula, when inserted in \(4.4\), shows that while $W$ is of order $n^2$ in any given vacuum, the differences between the values of $W$

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10 This superpotential can be measured as the central charge in a sector with a domain wall, as discussed above, or by coupling to gravity, whereupon the superpotential becomes observable.

11 As M. Shifman has explained, this $n$ dependence and hence the hypothesis that the gluino condensation is part of the leading large $n$ approximation can be confirmed using the fact that the gluino condensate of this theory is actually exactly calculable \[29\]. Note that if gluino condensation were not part of the leading large $n$ approximation, then $\langle \text{Tr}\lambda\lambda \rangle$ would be smaller than $n$ for large $n$, and the problem raised in the text would be more severe.
in neighboring vacua are only of order $n$. Hence the tension in a BPS-saturated domain wall is only of order $n$.

So the QCD domain wall is not a soliton in the effective large $n$ theory, an object whose tension would be of order $n^2$. What is it? If we regard $\lambda = 1/n$ as the string coupling constant of the QCD string, then we are looking for a nonperturbative object with a tension of order $1/\lambda$, rather than a conventional soliton with a tension of order $1/\lambda^2$. In critical string theory, such objects have been interpreted as $D$-branes (for a review see [44]). Might the super Yang-Mills domain wall be a $D$-brane of the QCD string? In critical string theory, a $D$-brane is an object on which elementary strings can end. A $D$-brane in large $n$ QCD would presumably be an object on which the QCD string can end.

In what follows, we aim to clarify some of these issues. We will explain how to describe a domain wall in terms of branes and give the criterion for such a domain wall to be BPS-saturated. Then we will argue that a QCD string really can end on a super Yang-Mills domain wall. Finally, we will tie up the loose ends by showing how to calculate the superpotential of an $M$-theory brane configuration and then showing that the brane configurations found in section 2 do have a non-zero superpotential and therefore, via (4.4), a non-zero gluino condensate.

A puzzle analogous to the one just explained arises in the supersymmetric $\mathbf{CP}^n$ model in two dimensions, where a solitonic domain wall in the large $n$ effective theory would have a mass of order $n$, but the BPS formula indicates that the mass is actually of order 1. In that case, the resolution of the problem [44] is that the domain wall is not a soliton but is one of the underlying elementary particles. This is somewhat similar to the fact that in super Yang-Mills theory the domain wall turns out to be, in a sense, a brane rather than a soliton. Also, one is reminded of the fact that baryons in the $1/n$ expansion have masses of order $n$ and appear as solitons in the meson or open string sector of the large $n$ effective theory [2]. From a contemporary point of view, one might wonder whether a baryon should be interpreted as a Dirichlet zero-brane rather than as an open string soliton. The question, however, may not be completely well-defined; the example of Type I fivebranes (which for large scale size are open-string, Yang-Mills solitons and for small scale size are $D$-branes) shows that solitons of the open-string sector, unlike closed-string solitons, can be continuously connected to $D$-branes.
4.2. The Domain Wall

The actual construction of the domain wall is straightforward in concept; solving the
equations is another matter (beyond the scope of the present paper). The domain wall
is a physical situation that for \( x^3 \to -\infty \) looks like one vacuum of the theory and for
\( x^3 \to +\infty \) looks like another vacuum. Here \( x^3 \) is one of the three spatial coordinates in
\( R^4 \). Meanwhile the physics should be independent of the time \( x^0 \) and of the other two
spatial coordinates \( x^1, x^2 \).

Such a physical situation can be described simply as an \( M \)-theory fivebrane that
interpolates between the fivebranes used to describe the vacuum states at the two ends.
The vacuum states were described by fivebranes of the form \( R^4 \times \Sigma \), where \( \Sigma \) was a Riemann
surface embedded in \( Y = R^5 \times S^1 \). For the domain wall we instead use a fivebrane of the
form \( R^3 \times S \), where \( R^3 \) is parametrized by \( x^0, x^1, x^2 \), and \( S \) is a three-surface in the seven
manifold \( \tilde{Y} = R \times Y \) (the copy of \( R \) here being the \( x^3 \) direction). Near \( x^3 = -\infty \), \( S \) should look like \( R \times \Sigma \), where \( \Sigma \) is the familiar Riemann surface defined by
\( w = \zeta v^{-1}, v^n = t \). Near \( x^3 = +\infty \), \( S \) should look like \( R \times \Sigma' \), where \( \Sigma' \) is the Riemann surface of an “adjacent” vacuum, say defined by
\( w = \exp(2\pi i/n)\zeta v^{-1}, v^n = t \). These are the only “ends” of \( S \).

The condition for the domain wall defined by \( R^3 \times S \) to be BPS-saturated is simply
that \( S \) should be a supersymmetric three-cycle in the sense of [26,27]. In other words,
the Riemannian manifold \( \tilde{Y} \), being flat, is a special case of a manifold of \( G_2 \) holonomy.
There are unbroken supersymmetries in \( M \)-theory with an \( R^3 \times S \) fivebrane (\( S \) being a
three-cycle in \( \tilde{Y} \)) if and only if \( S \) obeys the conditions of unbroken supersymmetry, that
is, the conditions for a supersymmetric three-cycle. In view of [23,24], it seems highly
plausible that such an \( S \) exists with the asymptotic behavior that we have stated.

Now recall from eqn. (2.5) that both \( \Sigma \) and \( \Sigma' \) are invariant under a \( U(1) \) symmetry
generated \( U \) by \( n\partial/\partial t - v\partial/\partial v + w\partial/\partial w \). The question then arises of whether \( S \) is invariant
under \( U \) or only asymptotically so. For our domain wall to be in the universality class
of supersymmetric QCD, \( S \) must be invariant under this symmetry. The reason is that
in supersymmetric QCD, \( U \), since it acts trivially in the field theory, acts trivially on all
degrees of freedom that propagate on the domain wall. But if the symmetry is sponta-
nearously broken on the domain wall, it certainly acts non-trivially in the effective theory
on the domain wall. Note that as the domain wall is 2 + 1-dimensional, we do not have
the option that was available in section 3.2 in the superficially similar case of the QCD

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The string world-volume is 1+1-dimensional. In 1+1 dimensions, a symmetry that is broken classically will be restored quantum mechanically at sufficiently big distances, perhaps with generation of a mass gap. In 2+1 dimensions, this is not necessarily so and will in fact not be so in a region in which a semiclassical treatment by branes is valid.

**D-Brane For The QCD String?**

So we have described the domain wall as a brane at least from the point of view of M-theory – and certainly not as a soliton in an effective glueball theory. But would a low energy QCD observer think of the domain wall as a D-brane? Can QCD strings end on the domain wall? We will now prove that they can.

We recall that the QCD string is represented by a one-brane \( C \) in \( Y \) that interpolates, at \( t = t_0 \) (for some fixed \( t_0 \)) from \( v = t_0^{1/n} \) to \( v = t_0^{1/n} e^{2\pi i/n} \). \( C \) does not lie in \( \Sigma \), but its endpoints do. If (keeping the endpoints of \( C \) in \( \Sigma \) throughout the homotopy) \( C \) could be deformed to a one-brane in \( \Sigma \), then \( C \) would represent the zero element of \( H_1(Y/\Sigma, \mathbb{Z}) \) and the QCD string would be unstable.

Now we want to analyze the stability of \( C \) in the field of a domain wall. For this purpose, by fixing a large negative value of \( x^3 \), we regard \( C \) as a one-brane in \( \tilde{Y} = \mathbb{R} \times Y \). Thus \( C \) is described now by \( x^3 = -c \) (\( c \) a large constant), \( t = t_0 \), with \( v \) still interpolating from \( t_0^{1/n} \) to \( t_0^{1/n} e^{2\pi i/n} \). To show that the QCD string can end in the field of the domain wall, it suffices to show that \( C \) can be deformed, while keeping its ends on \( S \), to a one-brane that lies entirely in \( S \).

For this it suffices to find a one-brane \( \eta \) that lies entirely on \( S \), has the same endpoints as \( C \), and has the property that \( t = t_0 b \) where \( b \) is everywhere real and positive. One can then without changing the endpoints of \( \eta \) make a homotopy from \( \eta \) to a one-brane \( \eta' \) with \( b = 1 \) (that is \( t = t_0 \)). \( \eta' \) and \( C \) are two paths in \( v - w \) space with the same endpoints and with \( t \) fixed at \( t_0 \); since \( v - w \) space is topologically trivial (contractible), \( \eta' \) and \( C \) are homotopic by a homotopy that keeps the boundaries fixed. So \( C \) is equivalent to \( \eta \) by such a homotopy, as desired.

In proving existence of \( \eta \), we may as well take \( t_0 \) real and positive and \( \zeta = 1 \). This does not affect the topological question, and will slightly simplify the formulas.

We set \( \rho = \ln |t| \). We construct \( \eta \) as a path in \( S \) with the properties that (a) \( t \) is always real and positive on \( \eta \); (b) \( \eta \), though not a closed path on \( S \), projects if one forgets all variables except \( \rho \) and \( x^3 \), onto a closed circle at infinity in the \( x^3 - \rho \) plane.
The projection of $\eta$ to the $x^3 - \rho$ plane can be described explicitly as a closed curve that is built by joining together four pieces. (1) Start near $x^3 = -\infty$, $\rho = \infty$. In a vacuum with $vw = 1$, vary $\rho$ from $\infty$ to $-\infty$. (2) Near $\rho = -\infty$, vary $x^3$ from $-\infty$ to $+\infty$, going to a vacuum with $vw = e^{2\pi i/n}$. (3) Near $x^3 = +\infty$, vary $\rho$ from $-\infty$ back to $+\infty$. (4) Finally, near $\rho = +\infty$, vary $x^3$ from $\infty$ back to $-\infty$, going back to the starting point.

In step (1), we begin at very large $v$ and with $w$ near zero; this corresponds to being very near $(v, w) = (a, 0)$ with very large positive $a$. We then, while remaining on $S$, interpolate to $\rho = -\infty$, which means small $v$ and large $w$. This is done at $x^3 = -\infty$, so we are on $\Sigma$ with $v^n = t = e^\rho$, and also with $vw = 1$. After varying $\rho$ in this way, one ends up very near $(v, w) = (0, a)$ with large positive $a$. In step (2), $v$ and $w$ remain fixed. In step (3), one starts at large $w$, small $v$, and by varying $\rho$ one interpolates to small $w$, large $v$, on a curve with $v^n = e^\rho = w^{-n}$, $vw = e^{2\pi i/n}$. This brings us to very near $(v, w) = (ae^{2\pi i/n}, 0)$ with $a$ real and positive. In step (4), $v$ and $w$ remain fixed again. Thus, the one-brane $\eta$, starting very near $(a, 0)$, ends very near $(ae^{2\pi i/n}, 0)$, and hence (for $t_0 = a^n$) has the same endpoints as $C$. Existence of such an $\eta$, lying entirely on $S$ and with $t$ always real and positive, completes the proof that the QCD string can end on the domain wall.

By further consideration of the curve $\eta$, it can be proved that if $S$ is invariant under the $U(1)$ symmetry (as it must be to agree with QCD), then there is a point in $S$ at which $v = w = 0$. In other words, there is a point in $S$ at which the chiral symmetries are all restored.

**Heuristic Interpretation**

S.-J. Rey [28] has suggested an intuitive interpretation [12] of this result in terms of 't Hooft’s concept of oblique confinement [14]. According to this idea, QCD confinement arises from condensation of somewhat elusive “QCD monopoles.” More generally, there are $n$ possible confining phases, the condensed object being a dyon, that is a bound state of a QCD monopole and $k$ quarks, for $0 \leq k \leq n - 1$. [13]

If one adiabatically increases the QCD theta angle by $2\pi$, analogy with the abelian case [47] suggests that a monopole picks up an electric charge and becomes a dyon, and more generally that a bound state of a monopole with $k$ quarks is transformed to a bound

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12 I would like to thank him for giving me permission to summarize the argument here.

13 Notice that the quarks in question, like the QCD monopoles themselves, are somewhat elusive, since the theory of pure super Yang-Mills theory without chiral multiplets does not have dynamical quarks as elementary fields.
state with $k + 1$ quarks. Thus the $k^{th}$ vacuum is adiabatically transformed to the $k + 1^{th}$ in increasing $\theta$ by $2\pi$.

The pure super Yang-Mills theory has an anomalous $U(1)_R$ symmetry, broken by the anomaly down to $\mathbb{Z}_n$. As a result of this, a shift in the $\theta$ angle by $\theta \to \theta + \alpha$ can be absorbed in a chiral rotation that acts on the gluino bilinear by $\text{Tr} \lambda \lambda \to e^{2\pi i \alpha/n} \text{Tr} \lambda \lambda$. In particular, a $2\pi$ shift in $\theta$ multiplies the gluino bilinear by $e^{2\pi i/n}$, giving the discrete chiral symmetry whose spontaneous breaking leads to domain walls.

A domain wall thus separates a state with a given value of $\theta$ from a state in which $\theta$ is shifted by $2\pi$. Therefore, according to the notion of oblique confinement, a domain wall separates a QCD phase in which the condensed object is a monopole bound to $k$ quarks from a phase in which the condensed object is a monopole bound to $k + 1$ quarks.

Now Rey’s intuitive observation is that by combining a composite of an antidyon on one side of the domain wall (an antimonopole bound to $k$ antiquarks) with a dyon on the other side of the domain wall (a monopole bound to $k + 1$ quarks), one can make a free quark in the domain wall. Thus the domain wall should support excitations that behave as though they are in the fundamental representation of $SU(n)$, even though there are no such fields in the original super Yang-Mills Lagrangian. But if the domain wall contains quarks, it should be possible for a QCD string to terminate on a domain wall. This then is the proposed intuitive explanation for the ability of a string to end on a domain wall.

4.3. The Superpotential

Finally, we will show how to compute directly the superpotential of an $M$-theory brane configuration. This will enable us to show from the brane picture that the super Yang-Mills vacua have a non-zero superpotential, and hence in particular a gluino condensate. The method of calculating the superpotential is also likely to have other applications.

Consider in general $M$-theory compactification on $\mathbb{R}^4 \times X \times \mathbb{R}$ where $X$ is a Calabi-Yau threefold. We want to consider a situation in which in spacetime there are fivebranes of the form $\mathbb{R}^4 \times \Sigma$, $\Sigma$ being a two-dimensional real surface in $X$. The field theory on the brane then is a model in $\mathbb{R}^4$ that has global $N = 1$ supersymmetry if $\Sigma$ is a complex curve in $X$, and no unbroken supersymmetry at all otherwise. (One gets global rather than local supersymmetry because eleven-dimensional gravitons and gravitinos are not localized on the brane but propagate in a larger number of non-compact dimensions. If $X$ has a sufficient amount of non-compactness, one can replace $\mathbb{R}^4 \times X \times \mathbb{R}$ by $\mathbb{R}^4 \times X \times S^1$, and still get a model with global supersymmetry.)
For some purposes, it is convenient to think of \( \Sigma \) as an abstract surface with a map \( \Phi : \Sigma \to X \). From this point of view, one can introduce local real coordinates \( \lambda^a, a = 1, 2 \), on \( \Sigma \), and local complex coordinates \( \phi^i, i = 1, \ldots, 3 \) on \( X \), and describe \( \Phi \) via functions \( \phi^i(\lambda^a) \). These functions are chiral superfields from the four-dimensional point of view. In this description, the reparametrizations of the \( \lambda^a \) should be regarded as gauge symmetries.

We wish to compute the superpotential \( W \) as a function of \( \Sigma \). It should have the following properties. (1) \( W \) should be a holomorphic function of \( \Sigma \). (In terms of the last paragraph, this means that \( W \) is a holomorphic function of the chiral superfields \( \phi^i(\lambda^a) \) invariant under reparametrizations of \( \Sigma \).) (2) \( W \) should have a critical point precisely when \( \Sigma \) (or \( \Phi(\Sigma) \)) is a holomorphic curve in \( X \), this being the condition for unbroken supersymmetry. Note that as we are doing global supersymmetry, \( W \) need be defined only up to an overall additive constant.

It is not difficult to describe the functional that obeys these conditions. Let \( \Omega \) be the holomorphic three-form of \( X \). We assume first that \( \Sigma \) is compact, and (unrealistically for the sake of most applications) that the homology class of \( \Sigma \) is zero so that \( \Sigma \) is the boundary of a three-manifold \( B \). The superpotential is then

\[
W(\Sigma) = \int_B \Omega. \tag{4.6}
\]

In the description of \( \Sigma \) by chiral superfields \( \phi^i(\lambda^a) \), this means that the variation of \( W \) in a change in \( \Sigma \) is

\[
\delta W = \int_\Sigma \Omega_{ijk} \delta \phi^i d\phi^j \wedge d\phi^k, \tag{4.7}
\]

where of course \( d = \sum_{a=1,2} d\lambda^a \partial/\partial \lambda^a \). From this formula, it is straightforward to show the two desired properties of \( W \). The fact that (4.7) is proportional to \( \delta \phi^i \) with no \( \delta \phi^i \) means that \( W \) is holomorphic. Moreover, \( W \) is stationary for and only for \( \Sigma \) a holomorphic curve, because this condition is equivalent to \( d\phi^i \wedge d\phi^k = 0 \).

This generalizes straightforwardly to the more typical case that the homology class of \( \Sigma \) is non-zero. One picks a fixed \( \Sigma_0 \) in the homology class of \( \Sigma \), and picks a three-manifold (or more generally a three-cycle) \( B \) with boundary \( \Sigma - \Sigma_0 \) (that is, the boundary of \( B \) consists of the union of \( \Sigma \) and \( \Sigma_0 \), which appear with opposite orientation), and one defines

\[
W(\Sigma) - W(\Sigma_0) = \int_B \Omega. \tag{4.8}
\]

\[14\] Closely related issues were discussed by S. Donaldson in a lecture at the Newton Institute at Cambridge University, November, 1996.
This defines $W(\Sigma)$ up to an additive constant $W(\Sigma_0)$.

There are actually two reasons for the indeterminacy of this additive constant. One is the arbitrary choice of $\Sigma_0$. In addition, if $H_3(X, \mathbb{R})$ is non-zero, then there are different possible choices for the homology class of $B$; changing this class will shift $W(\Sigma)$ by a constant. Note finally that if $H_3(X, \mathbb{Z})$ is non-zero and the space of possible $\Sigma$'s is not simply-connected, then $W(\Sigma)$ may change by an additive constant in going around a loop in the space of $\Sigma$'s. Thus, $W$ is really only defined on the universal cover of the space of $\Sigma$'s. In our actual application, $H_3(X, \mathbb{Z}) = 0$ and the subtleties mentioned in the present paragraph do not arise.

In the above, we have assumed that $\Sigma$ is compact. If not, one should require that $\Sigma$ and $\Sigma_0$ have the same asymptotic behavior at infinity, and that $B$ is “constant” at infinity. A slight generalization of the above is that $\Sigma_0$ (and for that matter $\Sigma$) need not actually be submanifolds of space-time. One can use “cycles,” which for our purposes we can take to mean that one can regard $\Sigma_0$ as an abstract real oriented surface with a differentiable map $\Phi_0 : \Sigma_0 \to X$. The map need not be an embedding. Likewise, we can consider $B$ to be an abstract oriented three-manifold (of boundary $\Sigma - \Sigma_0$) with a map $\Phi_B : B \to X$ (again, not necessarily an embedding). The restriction of $\Phi_B$ to $\Sigma$ should give the surface in $X$ for which we wish to evaluate the superpotential, and the restriction of $\Phi_B$ to $\Sigma_0$ should coincide with $\Phi_0$. In this context the definition of the superpotential is

$$W(\Sigma) - W(\Sigma_0) = \int_B \Phi_B^*(\Omega). \quad (4.9)$$

**Computation Of $W$**

Now we want to actually compute $W$ for the specific brane $\Sigma$ that represents the super Yang-Mills theory vacuum. We recall that $\Sigma$ is defined by the equations $v^n = t$, $w = \zeta v^{-1}$.

By computing $W$, we really mean determining its $\zeta$-dependence and in particular (since the different vacua are permuted by $\zeta \to \zeta e^{2\pi i/n}$) comparing the values of $W$ for the different vacua. However, according to (4.8) or (4.9), to define $W$ we must pick a base-point $\Sigma_0$ in the space of possible $\Sigma$'s. Ideally, we would like to pick a chirally-symmetric $\Sigma_0$ and define $W(\Sigma_0) = 0$, so as to get a chiral-invariant definition of $W$. Since it is at best awkward to find a chiral-invariant $\Sigma_0$, we will proceed by a slight variant of this. (Note that $H_3(Y, \mathbb{Z}) = 0$, so there is no indeterminacy in $W$ coming from periods of $\Omega$.)

In doing the computation, it will be useful to use the formulation of equation (4.9) in which $\Sigma_0$ and $B$ are not embedded in $Y = \mathbb{R}^5 \times S^1$, but are abstract manifolds with maps.
to $Y$. We introduce a complex variable $r$ and take $\Sigma_0$ to be the complex $r$-plane with the origin deleted. We write $r = \exp(\rho + i\theta)$, with $\rho$ and $\theta$ real, and pick an arbitrary smooth function $f$ of a real variable such that $f(\rho) = 1$ for $\rho > 2$ and $f(\rho) = 0$ for $\rho < 1$. Then we define the map $\Phi_0 : \Sigma_0 \to Y$ by\[ t = r^n \]
\[ v = f(\rho)r \]
\[ w = \zeta f(-\rho)r^{-1}. \]

Note that $\Sigma_0$ is asymptotic at infinity to $\Sigma$. $\Sigma_0$ is close enough to being invariant under the $\mathbb{Z}_n$ chiral symmetry to that a chirally-invariant definition of $W$ will be obtained by setting $W(\Sigma_0) = 0$. This will be shown by a slightly technical argument which will occupy the rest of this paragraph.\[ \] Let $\Sigma'$ be the chirally rotated version of $\Sigma_0$ with $\zeta$ replaced by $\zeta e^{2\pi i/n}$. I will show that $W(\Sigma_0) = W(\Sigma')$, which shows that to achieve a chirally-symmetric definition of $W$, one must fix the additive constant in the definition of $W$ so that $W(\Sigma_0) = 0$. To show that $W(\Sigma_0) = W(\Sigma')$, note that $\Sigma_0$ and $\Sigma'$ are described by two different but related maps of the $r$-plane to $Y$. Consider a (non-holomorphic) reparametrization of the $r$-plane defined by $\rho \to \rho, \theta \to \theta + b(\rho)$, where $b$ is any continuous function with $b(\rho) = 0$ for $\rho \geq 1$, and $b(\rho) = -2\pi/n$ for $\rho \leq -1$. This reparametrization maps $\Sigma_0$ into $\Sigma'$ except in the region $-1 \leq \rho \leq 1$, and in that region $v$ and $w$ are both mapped to zero. The region $-1 \leq \rho \leq 1$ of the $r$-plane is an annulus; by gluing together two copies of this annulus along the boundary circles at $\rho = \pm 1$ we make a two-torus $T$. The difference $\Sigma_0 - \Sigma'$ can be represented by a two-cycle consisting of a map $\phi$ of $T$ to $Y$ whose image is entirely at $v = w = 0$. Thus, $\phi$ is really a map of $T$ to the punctured $t$-plane (with $t = 0$ omitted); we call the punctured $t$-plane $V$. Since $H_2(V, \mathbb{Z}) = 0$, we can find a three-manifold $B$, of boundary $T$, and a map $\Phi : B \to V$ whose restriction to $T$ coincides with $\phi$. As $\Phi(B)$ is at $v = w = 0$, we have $\int_B \Phi^*\Omega = 0$, so $0 = W(T) = W(\Sigma_0) - W(\Sigma')$, as was to be proved.

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15 This map fails to be an embedding in the region $-1 \leq \rho \leq 1$ where $f(\rho) = f(-\rho) = 0$. It is hard to avoid this for a chiral-invariant $\Sigma_0$. That is why we have introduced the slightly more abstract formalism with cycles rather than submanifolds.

16 An error in an earlier version of this argument was pointed out by M. Schmaltz.
We take $\Sigma$ to be again the $r$-plane, mapped to $Y$ by similar formulas, but with $f$ replaced by 1:

$$
t = r^n
$$

$$
v = r
$$

$$
w = \zeta r^{-1}.
$$

To compute $W(\Sigma)$, we need to pick a three-manifold with boundary $\Sigma - \Sigma_0$, and an appropriate map $\Phi_B : B \to Y$. We take $B$ to be the product of the punctured $r$-plane (with the origin deleted) and the unit interval $I$, parametrized by a real variable $\sigma$, $0 \leq \sigma \leq 1$. We identify $\Sigma$ with the boundary at $\sigma = 1$ and $\Sigma_0$ with the boundary at $\sigma = 0$. We introduce a smooth bounded function $g(\rho, \sigma)$ with $g(\rho, 1) = 1$, $g(\rho, 0) = f(\rho)$, and $g(\rho, \sigma) = 1$ for $\rho > 2$. The map $\Phi_B : B \to Y$ can then be defined by

$$
t = r^n
$$

$$
v = g(\rho, \sigma)r
$$

$$
w = \zeta g(-\rho, \sigma)r^{-1}.
$$

The definition

$$
W(\Sigma) = \int_B \Phi_B^*(\Omega)
$$

of the superpotential now becomes (with $\Omega = R \, dv \wedge dw \wedge dt/t$ and $dt/t = n \, dr/r$)

$$
W(\Sigma) = Rn \int_B \frac{dr}{r} \wedge dv \wedge dw.
$$

This is more explicitly

$$
W(\Sigma) = iRn\zeta \int_0^1 d\sigma \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\rho \left( \frac{\partial g_+}{\partial \sigma} \frac{\partial g_-}{\partial \rho} - \frac{\partial g_+}{\partial \rho} \frac{\partial g_-}{\partial \sigma} \right),
$$

with $g_{\pm}(\rho, \sigma) = g(\pm \rho, \sigma)$. The integral over $\theta$ gives a factor of $2\pi$. The integral over the remaining variables is the integral over $\mathbb{R} \times I$ (the two factors being parametrized by $\sigma$ and $\rho$) of the exact two-form $d(g_+ dg_- - g_- dg_+)$. Integrating by parts and picking up surface terms at $\rho = \pm \infty$, this integral equals 2. So finally

$$
W(\Sigma) = 4\pi iRn\zeta.
$$

\[\text{As we have not been careful about an overall multiplicative constant in the definition of}\ \Omega, \ \text{the following formula (and others deduced from it later) are uncertain by such a universal constant.}\ \]

\[\text{29}\]
In particular, the theory has a gluino condensate, \( \langle \text{Tr} \lambda \lambda \rangle = 4\pi i R \zeta \). We also can now get some insight about the large \( n \) limit. Since the QCD string had a tension of order \( |\zeta|^{1/2}/n \), a smooth large \( n \) limit of the QCD string tension required \( \zeta \sim n^2 \). To make \( W \) of order \( n^2 \), we hence need \( R \sim 1/n \).

Ordinarily, Kaluza-Klein excitations carrying momentum in the \( x^{10} \) direction have energies of order \( 2\pi/R \), so the result \( R \sim 1/n \) would appear at first sight to show that these modes decouple in the large \( n \) limit. We must, however, be careful in drawing such a conclusion for modes that are localized on the fivebrane \( \mathbb{R}^4 \times \Sigma \). Because \( \Sigma \) wraps \( n \) times around the \( x^{10} \) circle, if the brane can be treated semiclassically one would expect such modes to see an effective radius of order \( nR \), that is, of order 1. Thus, it appears that this sort of Kaluza-Klein excitation on the fivebrane survives for \( n \to \infty \). Of course, we must decouple such modes (along with others) if we are to get super Yang-Mills quantitatively, and not just a theory that shares many qualitative properties with super Yang-Mills theory. What is involved in decoupling such modes will be the subject of the next section.

The mechanism whereby the Kaluza-Klein modes on the fivebrane survive for \( n \to \infty \) – a brane wrapped \( n \) times around a circle with radius of order \( 1/n \) sees an effective radius of order 1 – is highly reminiscent of recent results in black hole theory \([48]\) and matrix string theory \([49-51]\).

Since the domain wall tension is the magnitude of the difference between the values of \( W \) in different vacua, \((4.16)\) leads to a precise formula for the domain wall tension \( T_D \):

\[
T_D = 4\pi n R \left| \zeta (1 - e^{2\pi i/n}) \right|.
\]

(4.17)

5. Interpretation

It is surprising to be able to understand semiclassically properties like those we have explored in this paper – confinement, chiral symmetry breaking, and the fact that the domain wall behaves as a \( D \)-brane. In this section, we will demystify this success a bit, by showing that it depends on the fact that the model we have analyzed, while evidently in the same universality class as the super Yang-Mills four-dimensional field theory, is actually a different theory.

Two of our main results are the estimates for the tension \( T \) of the QCD string and the tension \( T_D \) of the domain wall. In order of magnitude, in units in which the eleven-dimensional Planck scale is one, one has

\[
T \sim \frac{|\zeta|^{1/2}}{n}.
\]

(5.1)
One also has a precise BPS formula \((4.17)\) for the domain wall tension, which in order of magnitude is

\[ T_D = R|\zeta|. \tag{5.2} \]

Since \((5.2)\) is a BPS formula, it holds for all values of the parameters \(R, \zeta\). The same cannot be said for \((5.1)\), which is valid only to the extent that the QCD string can be treated semiclassically. We recall that \((5.1)\) has a simple origin. Our fivebrane wrapped \(n\) times around the \(S^1\) factor in \(Y = R^5 \times S^1\); the distance between neighboring branches was of order \(|\zeta|^{1/2}/n\), and to the extent that fluctuations in the string position are not important, this is the string tension. Such fluctuations will be suppressed if \(|\zeta|^{1/2}/n\) and \(R\) are large (in eleven-dimensional Planck units). But in that case the string tension is much greater than the eleven-dimensional gravitational scale, and one may wonder whether the complexities of \(M\)-theory are really well-separated from the four-dimensional super Yang-Mills physics. After all, in the Yang-Mills field theory, quantum fluctuations in the string are believed to be important – so important that it has been hard to see the existence of the string at all, in the continuum quantum Yang-Mills theory.

To address this question further, let us ask whether it is possible for the brane theory to agree with super Yang-Mills theory for values of the parameters for which \((5.1)\) holds. In super Yang-Mills theory, there is a mass scale \(\Lambda\), and one has \(T \sim \Lambda^2, T_D \sim n\Lambda^3\). So \((5.1)\) and \((5.2)\) enable us to evaluate the \(M\)-theory parameters in terms of \(\Lambda\). We get

\[ \zeta \sim n^2\Lambda^4 \]
\[ R \sim \frac{1}{n\Lambda}. \tag{5.3} \]

Can this theory be super Yang-Mills theory in any region in which \((5.3)\) holds? Kaluza-Klein excitations, with momentum around the \(S^1\) factor in \(Y\), must decouple in any limit in which this happens. It was argued at the end of the last section that such modes have masses of order \(1/Rn\), that is of order \(\Lambda\). Thus they do not decouple; their masses are comparable to the QCD scale.

Thus, our theory is not super Yang-Mills theory, though it is evidently a close cousin of it. It is not super Yang-Mills theory because it depends on two parameters, \(\zeta\) and \(R\), while super Yang-Mills theory has only one parameter, \(\Lambda\). The formulas \(T \sim |\zeta|^{1/2}/n\), \(T_D \sim R|\zeta|\) make it clear that the theory really depends independently on \(R\) and \(\zeta\). What has apparently happened is that the brane picture has given us a surprising generalization
of the conventional super Yang-Mills theory, in which one extra parameter is introduced in a way that preserves gauge-invariance and relativity and simplifies the dynamics.

An analogous statement applies to many recent papers in which brane configurations are used to study various four-dimensional (or three-dimensional) field theories. Branes achieve their simplification by giving us not the familiar four-dimensional field theories but simpler and for some purposes equivalent cousins.

**Compactification From Six Dimensions**

If the theory that we have been studying in this paper is not really super Yang-Mills theory, then what is it? To answer this question, it is helpful to note that intuitively, if we are to try to get super Yang-Mills theory from the brane configuration, Λ should be extremely small compared to the eleven-dimensional Planck scale.

Thus, in (5.3), R must be very large, and ζ very small. The fact that R is large means that the fivebrane is large in all six world-volume dimensions. The fact that ζ is small means that the different branches of the fivebrane are very close together. The fivebrane, as it wraps n around the S^1 factor in Y, thus looks locally like a system of n almost coincident parallel fivebranes.

This strongly suggests that the system should be understood as an unusual compactification of the six-dimensional (0, 2) superconformal field theory [30], which can be described in terms of parallel fivebranes [31]. In this field theory, the separation ε between neighboring fivebranes is interpreted as a scalar field of dimension two. There is therefore a limit in which ε goes to zero, with all six-dimensional lengths being scaled as 1/ε^{1/2}.

This is precisely the scaling that we see in (5.3). In other words, according to (5.3), the separation ε = |ζ|^{1/2}/n between branes is of order Λ^2. Since Λ is the QCD mass scale, this is the expected statement that ε behaves as a field of dimension two. Meanwhile the formula R ~ 1/nΛ shows that R has dimension −1, as expected in conformal field theory.

The whole setup can now be described more or less precisely as a sort of exotic compactification of the six-dimensional (0, 2) superconformal field theory. We start with the six-manifold V = R^4 × R × S^1, where R^4 is to be interpreted as four-dimensional Minkowski space, and R and S^1 are parametrized by x^6 and x^{10}. As usual, we take R to be the radius of the S^1, and introduce s = R^{-1}x^6 + ix^{10} and t = e^{-s}.

We want to consider the six-dimensional A_{n-1} superconformal field theory (in other words, the theory that can be realized with n parallel fivebranes) on the six-manifold V. In this theory, v = x^3 + ix^4 and w = x^7 + ix^8 can be interpreted (along with x^9, which
we set to zero asymptotically) as order parameters or scalar fields of dimension two. To be more precise, as there are \( n \) fivebranes, these order parameters are \( n \)-valued. Moreover, as the \((0,2)\) superconformal field theory is not well understood, I only claim that these \( n \)-valued order parameters are well-defined when they are large, in units set by some other length scale (such as \( R \)) characteristic of a given physical problem.

I claim that the problem studied in the present paper is equivalent to studying the \( A_{n-1} \) superconformal theory on \( V \) in a situation in which the asymptotic behavior of \( v \) for \( x^6 \to \infty \) is

\[
v^n = t, \tag{5.4}
\]

with \( w \) small, and the asymptotic behavior for \( x^6 \to -\infty \) is

\[
w^n = \zeta t^{-1}, \tag{5.5}
\]

with \( v \) small. To the extent that the \( A_{n-1} \) superconformal theory can be understood semiclassically via branes, these boundary conditions lead back immediately to the configuration studied in the present paper.

Conformal invariance of the \((0,2)\) field theory means that the physics with this sort of compactification is invariant under rescaling of \( \zeta \) or equivalently of \( \epsilon = |\zeta|^{1/2}/n \) provided that one keeps

\[
y = \epsilon R^2 \tag{5.6}
\]

fixed. \( y \) is the extra dimensionless parameter not present in ordinary super Yang-Mills theory. If \( y \) is too large, the candidate QCD string does not have the correct universality class, as we saw in section 3.2.

In what limit might one recover the conventional four-dimensional super Yang-Mills theory? This apparently must be the limit of \( y \to 0 \), which if we go back to the full \( M \)-theory (rather than the \((0,2)\) superconformal field theory which is a limit of \( M \)-theory) can be interpreted to mean that \( R \) is small and \( M \)-theory degenerates to the weakly coupled Type IIA superstring. That, after all, is where our discussion started. Whatever else may happen, conventional Yang-Mills theory can arise as a limit of weakly coupled Type IIA superstring theory.

Another Parameter

Another parameter could be introduced by replacing the \( M \)-theory manifold \( \mathbb{R}^4 \times Y \times \mathbb{R} \), in which we have worked in this paper, by \( \mathbb{R}^4 \times Y \times S^1 \), and judiciously varying the
radius of the new $S^1$. In this fashion, one gets a deformation of super Yang-Mills theory depending on two extra parameters (essentially the radius of the circle factor in $Y$ and of the new circle) not seen in conventional four-dimensional super Yang-Mills theory. By analogy with what has just been said, this theory can be understood in terms of compactification of the six-dimensional non-critical string theory [52] which generalizes the six-dimensional $(0,2)$ field theory.

6. The Non-Supersymmetric Case

To conclude, we would like to argue that the brane method may be relevant to theories that are not supersymmetric, for instance to ordinary four-dimensional Yang-Mills theory without supersymmetry. The problem, as we will see, is not to find the brane configurations but to determine what aspects of the Yang-Mills physics can be understood from them. Also, in the nonsupersymmetric case there is an extra potential pitfall. In going from the field theory to the brane system, there may in the absence of supersymmetry be a first order phase transition, which might make it impossible in principle to learn anything about Yang-Mills theory by studying the branes. We must hope that there is no such first-order transition.

We start in weakly coupled Type IIA superstring theory with $n$ fourbranes suspended between two parallel fivebranes, a system of $N = 2$ supersymmetry. Just as one can “rotate” the fivebranes to break $N = 2$ supersymmetry down to $N = 1$ [35], a further generic “rotation” breaks the supersymmetry completely. This gives the gluinos a bare mass, but one still has the $SU(n)$ gauge symmetry of $n$ parallel fourbranes. So in a certain limit, one obtains the conventional, bosonic $SU(n)$ Yang-Mills theory.

To study this theory via branes, we go to a strongly-coupled limit, which is $M$-theory on $\mathbb{R}^{10} \times S^1$, with a fivebrane that – as in the $N = 1$ case – wraps $n$ times around the $S^1$, but has a supersymmetry-breaking asymptotic behavior at infinity. As before, we write $\mathbb{R}^{10} \times S^1 = \mathbb{R}^4 \times Y \times \mathbb{R}$, where $\mathbb{R}^4$ is the effective Minkowski space and $Y = \mathbb{R}^5 \times S^1$. The desired fivebrane will be of the form $\mathbb{R}^4 \times \Sigma$, where $\Sigma$ is a two-dimensional surface in $W$. $\Sigma$ will go to infinity in certain directions, corresponding to the fact that in the Type IIA description, there are two non-compact fivebranes. $\Sigma$ cannot be found by requiring holomorphy, since (supersymmetry being completely broken) $\Sigma$ is not a holomorphic curve in any complex structure on $W$. $\Sigma$ must be found by minimizing its area subject to a given asymptotic behavior at infinity.
Fortunately, the techniques for doing so are familiar from the theory of the classical bosonic string. $Y$ has a flat Riemannian metric

$$ds^2 = \sum_{i,j=4,\ldots,8,10} \eta_{ij} dx^i dx^j = \sum_{i=4,\ldots,8} (dx^i)^2 + R^2(dx^{10})^2.$$  \hspace{1cm} (6.1)

The embedding of $\Sigma$ in $Y$ induces a metric and therefore a complex structure on $\Sigma$. We can assume that $\Sigma$ has genus zero, since this is the case even before the rotation from $N=1$ to $N=0$, and the genus of a smooth surface does not change under a generic rotation. Hence, $\Sigma$ as a complex manifold is isomorphic to $\mathbb{P}^1$, perhaps with some points deleted. The number of deleted points is precisely two, as we want to study a situation with precisely two infinite fivebranes. Hence we can identify $\Sigma$ with the complex $\lambda$-plane, with the origin deleted.

The embedding of the $\lambda$-plane in $Y$ is not holomorphic in any complex structure on $Y$. However, it is a minimal area embedding. As in the classical theory of the bosonic string, this means that the coordinates $x^4,\ldots,x^8$ and $x^{10}$ are harmonic functions on the $\lambda$-plane which obey the Virasoro constraints

$$\eta_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0.$$  \hspace{1cm} (6.2)

The Virasoro constraints are automatic if the embedding of $\Sigma$ in $Y$ is holomorphic, but as it is we will have to check the Virasoro constraints by hand.

In the $N=1$ case, the embedding of $\Sigma$ in $Y$ is $v = \lambda$, $w = \zeta \lambda^{-1}$, $t = \lambda^n$. With $v = x^4 + ix^5$, $w = x^7 + ix^8$, $t = \exp(-(R^{-1}x^6 + ix^{10}))$, these equations can be written

$$x^4 = \text{Re} \lambda$$

$$x^5 = \text{Im} \lambda$$

$$x^7 = \text{Re} (\zeta \lambda^{-1})$$

$$x^8 = \text{Im} (\zeta \lambda^{-1})$$

$$x^6 = -(Rn) \text{Re} \ln \lambda$$

$$x^{10} = -n \text{Im} \ln \lambda.$$  \hspace{1cm} (6.3)

To rotate this to a nonsupersymmetric configuration that still obeys the Virasoro constraints, it suffices to do the following. Combine $x^4, x^5, x^7,$ and $x^8$ into a real four-vector $\vec{A}$. Generalize the first four equations in (6.3) to

$$\vec{A} = \text{Re} (\vec{p} \lambda + \vec{q} \lambda^{-1})$$  \hspace{1cm} (6.4)
with complex four-vectors $\vec{p}, \vec{q}$. Meanwhile, generalize the last two equations in (6.3) to
\begin{align*}
x^6 &= -(Rnc) \text{ Re } \ln \lambda \\
x^{10} &= -n \text{ Im } \ln \lambda
\end{align*}
with a real constant $c$. The Virasoro constraints turn out to be
\[ \vec{p}^2 = \vec{q}^2 = 0, \quad -\vec{p} \cdot \vec{q} + \frac{R^2 n^2}{2} (1 - c^2) = 0. \] (6.6)
The condition for unbroken supersymmetry is that in addition $\vec{p} \cdot \vec{q} = 0, \ c = \pm 1$. Simply by picking complex null vectors $\vec{p}, \vec{q}$ with $\vec{p} \cdot \vec{q} \neq 0$, and adjusting $c$ appropriately, we get nonsupersymmetric solutions of (6.6).

Clearly, a number of parameters appear in this general description. An amusing special case is that in which $\vec{q}$ is the complex conjugate of $\vec{p}$ and $c = 0$. In this case, $\Sigma$ is embedded as a minimal area surface in a space of only three real dimensions (namely the $x^{10}$ direction and the directions spanned by the real and imaginary parts of $\vec{p}$).

Rotating the brane and deforming away from $c = 1$ does not change the topology of the situation, so the QCD string still exists in this nonsupersymmetric theory. Obviously, our considerations about chiral symmetry breaking and domain walls do not carry over in this situation.

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