ON A GENERALISED BOOTSTRAP PRINCIPLE

E. Corrigan\textsuperscript{1}, P.E. Dorey\textsuperscript{2}, R. Sasaki\textsuperscript{1,3}

\textsuperscript{1}Department of Mathematical Sciences
University of Durham, Durham DH1 3LE, England

\textsuperscript{2}Theory Division CERN
1211 Geneva 23, Switzerland

\textsuperscript{3}Uji Research Center
Yukawa Institute for Theoretical Physics
Kyoto University, Uji 611, Japan

Abstract

The S-matrices for non-simply-laced affine Toda field theories are considered in the context of a generalised bootstrap principle. The S-matrices, and in particular their poles, depend on a parameter whose range lies between the Coxeter numbers of dual pairs of the corresponding non-simply-laced algebras. It is proposed that only odd order poles in the physical strip with positive coefficients throughout this range should participate in the bootstrap. All other singularities have an explanation in principle in terms of a generalised Coleman-Thun mechanism. Besides the S-matrices introduced by Delius, Grisaru and Zanon, the missing case $f_4^{(1)}, e_6^{(2)}$, is also considered and provides many interesting examples of pole generation.
1. Introduction

Affine Toda field theory [1,2] is a theory of $r$ scalar fields in two-dimensional Minkowski space-time, where $r$ is the rank of a compact semi-simple Lie algebra $g$. The classical field theory is determined by the lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi)$$  \hspace{1cm} (1.1)

where

$$V(\phi) = \frac{m^2}{\beta^2} \sum_0^r n_i e^{\beta \alpha_i \cdot \phi}.$$  \hspace{1cm} (1.2)

In (1.2), $m$ and $\beta$ are real, classically unimportant constants, $\alpha_i$ $i = 1, \ldots, r$ are the simple roots of the Lie algebra $g$, and $\alpha_0 = -\sum_1^r n_i \alpha_i$ is an integer linear combination of the simple roots; it corresponds to the extra spot on an extended Dynkin-Kac diagram for $\hat{g}$. The coefficient $n_0$ is taken to be one. If the term $i = 0$ is omitted from (1.2) in the lagrangian (1.1), then the theory, both classically and after quantisation is conformal; with the term $i = 0$, the conformal symmetry is broken but the theory remains classically integrable, in the sense that there are infinitely many independent conserved charges in involution [3]. In the ‘real coupling’ Toda theory, the fields are supposed to be real. However, there have also been recent studies [4] of the classical soliton solutions to the equations of motion following from (1.1); for these, the fields are complex. The discussion in this article will be restricted to the real-coupling theories.

As quantum field theories, the real-coupling affine Toda field theories fall into two classes. There are those based on the simply-laced root systems corresponding to the diagrams for $a_n^{(1)}, d_n^{(1)}, e_n^{(1)}$, and the others based on the non-simply laced root systems. These fall into dual pairs (dual in this context meaning the replacement $\alpha_i \rightarrow \tilde{\alpha}_i = 2\alpha_i/\alpha_i^2$), namely, $(b_n^{(1)}, a_{2n-1}^{(2)}), (c_n^{(1)}, d_{n+1}^{(2)}), (g_2^{(1)}, d_4^{(3)}), (f_4^{(1)}, e_6^{(2)})$, except for $a_{2n}^{(2)}$ which is self-dual. The simply-laced root systems are also self-dual since $\alpha_i^2 = 2$. Based on a bootstrap principle and a number of checks within perturbation theory, it has proved possible to conjecture [1,3,6,7,8,9] the exact S-matrices for each affine Toda field theory associated with a self-dual root system. These have many interesting properties which have been extensively reviewed elsewhere. In the context of this paper it is intended to concentrate on just a couple of them.
The first important property concerns the bootstrap \[10,11\]. The S-matrix element \( S_{ab}(\theta_a - \theta_b) \) corresponding to the elastic scattering of a pair of particles \( a, b \), is a meromorphic function of the rapidity difference \( \Theta = \theta_a - \theta_b \). For real rapidity, \( S_{ab} \) is a phase given by a product of elementary blocks \( \{x\} \) defined by

\[
\{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)} \quad (x) = \frac{\sinh \left( \frac{\Theta}{2} + \frac{x\pi i}{2\pi} \right)}{\sinh \left( \frac{\Theta}{2} - \frac{x\pi i}{2\pi} \right)}
\]

where \( 0 \leq B \leq 2 \), in the simply-laced cases, there is evidence \[1,5,7,8,12\] for

\[
B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi},
\]

and \( h = \sum_i n_i \) is the Coxeter number for the chosen root system. In the self-dual theories, \( S_{ab} \) has fixed position poles and moving (coupling dependent) zeroes in the physical strip \( (0 \leq \text{Im}\Theta \leq \pi) \), and moving poles and fixed position zeroes outside the physical strip. The odd order poles, with a coefficient equal to \( i \) times a function of \( B \) which is positive throughout the range of \( B, 0 \leq B \leq 2 \) participate in the bootstrap. The poles indicating a bound state ‘fusing’ \( ab \rightarrow c \) occur at precisely the rapidity necessary for energy-momentum conservation given that the particles have mass ratios identical with those derived from the classical lagrangian. Moreover, such fusings occur if and only if there is a corresponding three-point coupling between the three mass eigenstates \( (a, b, c) \) in the classical lagrangian. The magnitude of a three-point coupling is always classically proportional to the area of the triangle whose sides have lengths equal to the masses of the particles participating in the coupling. The precise nature of the odd-order poles in a particular scattering matrix element, and the existence of even order fixed poles on the physical strip are explicable within perturbation theory, in terms of Landau singularities. Detailed checks have been made \[13\] for second and third order poles but not for the others (up to order twelve in the theory associated with \( e_8^{(1)} \)). However, even in the latter cases the origin of the poles is known in principle in the sense that some Feynman diagrams in the perturbation expansion have been identified which contribute to each of them.

Besides the bootstrap, the masses and the eigenvalues of the conserved quantities\[3,14\] are known to be components of the eigenvectors of the adjacency matrix of the simple roots \( \alpha_i, \ i = 1 \ldots r \), a fact discovered classically for the masses\[3,4,15\] and which is preserved in the quantum field theory. The possible couplings have been characterised succinctly in \[16\] where it is noted that the particles of an affine Toda field theory are each associated
with an orbit of a simple root under the action of the Coxeter element \(w\) in the Weyl group of the selected root lattice. The S-matrix itself may be expressed \([16,17]\) in several elegant ways using these Coxeter orbits, but such expressions will not be needed here.

Very few of these facts or formulae work in the quantum theories based on root systems which are not self-dual and it remains an outstanding problem to find their generalisations. The classical data for these types of affine Toda theory may be found in \([7]\), however, it was clear in early renormalisation calculations that the masses of the particles in these theories could not be in the same ratios as the classical masses\([7,8]\). Hence the S-matrices would have a different character from those of the self-dual theories.

Recently, Delius, Grisaru and Zanon \([18]\) have made a number of fascinating conjectures concerning the nature of the masses and S-matrices for the non-simply-laced theories. One apparent consequence of their work is that each dual pair of root lattices corresponds to a one parameter set of quantum field theories with particle masses ‘floating’ between the classical masses of the partners in each pair (see also \([19]\)). This feature is compatible with a generalised bootstrap principle which will be outlined below. The candidate S-matrices were not obtained for all the pairs. The one which in many ways is the most interesting (for the dual pair \(f_4^{(1)}, e_6^{(2)}\)), was omitted from \([18]\) and will be described in detail in section(3). It displays a variety of properties not shared by the other examples.

Besides the work of ref\([18]\), there is numerical evidence from a simulation of the \(g_2^{(1)}\) theory by Watts and Weston \([20]\) that the masses in these theories based on dual pairs of root lattices do indeed flow with the coupling \(\beta\).

2. Generalised bootstrap

It was noted in the introduction that for the theories based on self-dual lattices the odd order poles always participate in the bootstrap and occur with coefficients whose sign does not change as the coupling constant varies in the range \(0 \leq \beta \leq \infty\). The direct channel poles have a positive coefficient (multiplied by \(i\)), whereas the crossed channel poles have a negative coefficient. The poles with this property will be referred to as positive or negative definite, respectively. Other poles of even order have a real coefficient, do not participate in the bootstrap and, together with higher order odd poles, are explicable within standard perturbation theory in terms of Landau singularities in Feynman diagrams.

In the theories based on non-self-dual root systems, the masses must float and therefore the S-matrices will have moving poles on the physical strip. A criterion is needed to decide
on the basis of the S-matrix alone which moving poles should participate in the bootstrap. Given the experience with the simply-laced theories, it is reasonable to suppose the odd order poles are again the relevant ones. However, as Delius et al. note, their conjectured S-matrices contain poles of odd order on the physical strip which cannot be consistently interpreted as bound states with a mass as indicated by the position of the pole. Some comments on this will be made later. A careful study of the coefficients of these poles reveals that they do not share in all respects the characteristic behaviour noted in the self-dual cases. Specifically, the coefficients of these poles change sign at least once for some value of the floating parameter as it floats between the two partners in the dual pair. Such poles will be referred to as semi-positive. The other odd order poles, which do participate in the bootstrap, have coefficients which do not change sign over the floating interval. As will be seen, this is a persistent feature in all possible cases.

The theory associated with the pair \((g_2^{(1)}, d_4^{(3)})\) will be reviewed first to illustrate the ideas.

3. The case \((g_2^{(1)}, d_4^{(3)})\)

The floating masses for this theory are conjectured\(^{[18]}\) to be

\[
m_1 = \sin \frac{\pi}{H} \quad m_2 = \sin 2\frac{\pi}{H}
\]

(3.1)

up to an overall factor \((2\sqrt{2m})\) which is ignored. The parameter \(H\) floats in the range \(6 \leq H \leq 12\), ie between the Coxeter numbers of the partners in the pair. Each particle couples to itself but there is also expected to be a fusing \(1 \rightarrow 2\) in addition. The rapidity \((\Theta = 2i\pi/H)\) at which the latter fusing occurs also floats, since

\[
m_2^2 = \sin^2 2\pi/H = 2m_1^2 + 2m_1^2 \cosh \Theta = 2\sin^2(\pi/H) (1 + \cos 2\pi/H).
\]

(3.2)

Each particle is self-conjugate. Finally, it may be supposed that for \(H = 6\) or 12, the S-matrix elements are unity.

With these points in mind, the simplest choice\(^{[18]}\) for \(S_{11}\) is:

\[
S_{11}(\Theta) = \frac{(0) (2)}{(H/3 - 2)(4 - H/3)} \frac{(H/3) (2H/3)}{(H - 4)(4)} \frac{(H - 2) (H)}{(2 + 2H/3)(4H/3 - 4)}
\]

(3.3)

where the bracket notation has been adjusted slightly and is now defined by

\[
(x) = \frac{\sinh (\frac{\Theta}{2} + \frac{x\pi i}{2H})}{\sinh (\frac{\Theta}{2} - \frac{x\pi i}{2H})}.
\]

(3.4)
The terms in (3.3) have been grouped together in order to facilitate the inspection of the coefficients of the moving poles. In fact, for $H$ within the stated range, the poles at $\Theta = 2\pi i/3$ and at $2\pi i/H$ are positive definite direct channel poles throughout the range of $H$. The self-coupling bootstrap for the pole at $2\pi/3$,

$$S_{11}(\Theta) = S_{11}(\Theta - i\pi/3)S_{11}(\Theta + i\pi/3),$$

(3.5)

is satisfied.

The fusing $11 \rightarrow 2$ can be used to define the S-matrix elements $S_{12}$ and $S_{22}$ via the bootstrap. In other words,

$$S_{12}(\Theta) = S_{11}(\Theta - i\pi/H)S_{11}(\Theta + i\pi/H),$$

$$S_{22}(\Theta) = S_{12}(\Theta - i\pi/H)S_{12}(\Theta + i\pi/H).$$

(3.6)

This gives:

$$S_{12}(\Theta) = \frac{(1) (2H/3 - 1)}{(H - 5)(5 - H/3)} \frac{(H/3 + 1) (H - 1)}{(4H/3 - 5)(5)},$$

(3.7)

which has a physical pole with a positive residue at $\Theta = i\pi(1 - 1/H)$, corresponding to the fusing $12 \rightarrow 1$, together with its crossed partner, and an extra pair of poles at $\Theta = i\pi(2/3 - 1/H)$, $i\pi(1/3 + 1/H)$. The first of these two extra poles has a negative coefficient, the second positive for $H$ in the range $6 \leq H \leq 9$. On the other hand, for $9 \leq H \leq 12$, the coefficients of these two poles have the opposite sign. These poles are therefore semi-positive. This phenomenon occurs due to a crossing over at $H = 9$ of the two factors $(H - 5)$ and $(5)$ in the denominator of (3.7) with the two factors $(H/3 + 1)$ and $(2H/3 - 1)$, respectively in the numerator.

The second relation of (3.6) gives:

$$S_{22}(\Theta) = \frac{(0) (2H/3 - 2)}{(H - 6)(4 - H/3)} \frac{(2) (2H/3)}{(H - 4)(6 - H/3)} \frac{(H/3) (H - 2)}{(4H/3 - 6)} \frac{(H/3 + 2) (H)}{(4H/3 - 4)(6)},$$

(3.8)

This has a variety of simple poles on the physical strip but only one of them, at $\Theta = 2i\pi/3$, has a positive residue throughout the range of $H$. Another, its crossed partner, has a negative residue throughout the range. The rest have coefficients which change sign at least once (and sometimes twice) over the interval.

Two of the semi-positive poles, at rapidity value $\Theta = i\pi(2/3 - 1/H)$ in (3.7) and at $\Theta = i\pi(1 - 2/H)$ in (3.8), are positive in the region near $H = 12$ and reflect the existence
of an extra 221 coupling in the $d_4^{(3)}$ classical theory which is absent in the $g_2^{(1)}$ theory. However, except at the limit $H = 12$ (where the S-matrix elements are unity), the position of these poles do not coincide with the floating masses. Because these poles are semi-positive, they should not participate in the bootstrap. They are artifacts of the bootstrap which can be explained in the same spirit as that invoked by Coleman and Thun to explain the double poles of the Sine-Gordon breather S-matrix elements. As is usual in Toda theory, the mechanism is subtle and will be described in section (8).

The S-matrix elements (3.3), (3.7), and (3.8) look quite strange relative to the elegant formulae in the self-dual cases in terms of the basic block of eq (1.3). However, they may each be written in a similar fashion in terms of a slightly more general block $\{x\}_\nu$ defined as follows. For this purpose, it is convenient to set $H = 6 + 3B$, and to suppose $0 \leq B \leq 2$. However, this parametrisation is not intended to imply $B$ has the form suggested in (1.4). Then, define

$$\{x\}_\nu = \frac{(x - \nu B - 1)(x + \nu B + 1)}{(x + \nu B + B - 1)(x - \nu B - B + 1)}.$$  (3.9)

Clearly the old block (1.3) corresponds to $\{x\}_0$ when $H$ reverts to $h$. In terms of these new building blocks, the S-matrix elements are

$$S_{11}(\Theta) = \{1\}_0 \{H/2\}_{1/2} \{H - 1\}_0,$$

$$S_{12}(\Theta) = \{H/3\}_1 \{2H/3\}_1,$$

$$S_{22}(\Theta) = \{H/3 - 1\}_1 \{H/3 + 1\}_1 \{2H/3 - 1\}_1 \{2H/3 + 1\}_1.$$  (3.10)

Unfortunately, although these expressions are quite compact, the extended blocks do not behave so naturally as (1.3) with respect to the bootstrap.

Finally, it is worth remarking that the eigenvalues of the conserved quantities, compatible with the bootstrap for the conserved charges are

$$q_1^s = \sin s\pi/H \quad q_2^s = \sin 2s\pi/H$$  (3.11)

where $s$, the spin of the conserved charges is equal to 1 or 5 mod 6.
4. The case \((f_4^{(1)}, e_6^{(2)})\)

It is proposed that the floating masses for the theory based on the pair of non-simply laced root systems \(f_4^{(1)}\) and \(e_6^{(2)}\) should be (up to an overall factor \(\sqrt{3m}\)),

\[
m_1 = \sin \pi/H \sin 2\pi/H' \\
m_2 = \sin 3\pi/H \sin \pi/H' \\
m_3 = \sin 2\pi/H \sin 2\pi/H' \\
m_4 = \sin 3\pi/H \sin 2\pi/H'
\]

where

\[
\frac{1}{H} + \frac{1}{H'} = \frac{1}{6}.
\]

It is not hard to check that in the range \(12 \leq H \leq 18\), and up to an overall factor, the masses float between the masses of the particles of the classical \(f_4^{(1)}\) theory and those of the \(e_6^{(2)}\) theory, (as given in \([7]\)). Again, it is sometimes convenient notationally to set \(H = 12 + 3B\). However, as in the previous case, it is not intended in this article to relate \(B\) explicitly to the coupling constant.

In this case, not all of the couplings of either classical theory will float throughout the range in \(H\), and thus be able to participate in the bootstrap. The relevant couplings that do float are:

\[
111, 112, 113, 123, 134, 222, 224, 333, 444,
\]

with the corresponding fusing angles (in units of \(\pi/H\))

\[
U^2_{11} = H/3 + 2, \quad U^1_{12} = 5H/6 - 1, \\
U^3_{11} = 2, \quad U^1_{13} = H - 1, \\
U^4_{22} = H/3 - 2, \quad U^2_{24} = 5H/6 + 1, \\
U^3_{12} = H/2 - 1, \quad U^2_{23} = 2H/3 + 1, \quad U^1_{23} = 5H/6, \\
U^4_{13} = 3, \quad U^3_{14} = H - 2, \quad U^1_{34} = H - 1,
\]

and the self couplings correspond in the same units to the fusing angle \(2H/3\). Of course, these were actually discovered by checking the consistency of the bootstrap in the sense indicated above. Here, the S-matrix elements will be given but the detailed checks of the bootstrap will be omitted. Noting that all S-matrix elements are crossing symmetric, it is convenient to set

\[
[x]_\nu = \{x\}_\nu \{H - x\}_\nu.
\]
In terms of this, the S-matrix elements are given by

\[ S_{11}(\Theta) = [1]_0 \left\{ \frac{H}{3} + 1 \right\}_0 \]
\[ S_{12}(\Theta) = [H/6 + 2]_0 \left\{ \frac{H}{2} + 2 \right\}_0 \]
\[ S_{13}(\Theta) = [2]_0 \left\{ \frac{H}{3} + 2 \right\}_0 \]
\[ S_{14}(\Theta) = [3]_0 \left\{ \frac{H}{3} - 1 \right\}_0 \left\{ \frac{H}{3} + 1 \right\}_0 \left\{ \frac{H}{3} + 3 \right\}_0 \]
\[ S_{22}(\Theta) = [1]_0 \left\{ \frac{H}{3} - 3 \right\}_0 \left\{ \frac{H}{3} + 1 \right\}_0 \left\{ \frac{H}{3} + 3 \right\}_0 \]
\[ S_{23}(\Theta) = [H/6 + 1]_0 \left\{ H/6 + 3 \right\}_0 \left\{ H/2 + 1 \right\}_0 \left\{ H/2 + 3 \right\}_0 \]
\[ S_{24}(\Theta) = [H/6]_0 \left\{ H/6 + 2 \right\}_0 \left\{ H/6 + 4 \right\}_0 \left\{ H/2 \right\}_0 \left\{ H/2 + 2 \right\}_0 \left\{ H/2 + 4 \right\}_0 \]
\[ S_{33}(\Theta) = [1]_0 [3]_0 \left\{ \frac{H}{3} - 1 \right\}_0 \left\{ \frac{H}{3} + 1 \right\}_0 ^2 \left\{ \frac{H}{3} + 3 \right\}_0 \]
\[ S_{34}(\Theta) = [2]_0 [4]_0 \left\{ \frac{H}{3} - 2 \right\}_0 \left\{ \frac{H}{3} + 2 \right\}_0 ^2 \left\{ \frac{H}{3} + 3 \right\}_0 \]
\[ S_{44}(\Theta) = [1]_0 [3]_0 [5]_0 \left\{ \frac{H}{3} - 3 \right\}_0 \left\{ \frac{H}{3} - 1 \right\}_0 ^3 \left\{ \frac{H}{3} + 1 \right\}_0 ^3 \left\{ \frac{H}{3} + 3 \right\}_0 ^3 \left\{ \frac{H}{3} + 5 \right\}_0 . \]

This expression is quite useful for checking the bootstrap properties but it conceals a number of cancellations between zeroes and poles. An alternative, in which all the cancelling poles and zeroes have been removed, is given by:

\[ S_{11}(\Theta) = [1]_0 \left\{ \frac{H}{3} + 1 \right\}_0 \]
\[ S_{12}(\Theta) = [H/3]_1/2 \]
\[ S_{13}(\Theta) = [2]_0 \left\{ \frac{H}{3} \right\}_0 \left\{ \frac{H}{2} \right\}_1/2 \]
\[ S_{14}(\Theta) = [H/6 + 1]_1/2 \left\{ \frac{H}{2} + 1 \right\}_1/2 \]
\[ S_{22}(\Theta) = [H/6 - 1]_1/2 \left\{ \frac{H}{2} + 1 \right\}_1/2 \]
\[ S_{23}(\Theta) = [H/3 - 1]_1/2 \left\{ \frac{H}{3} + 1 \right\}_1/2 \]
\[ S_{24}(\Theta) = [H/3 - 2]_1/2 \left\{ \frac{H}{3} \right\}_1/2 \left\{ \frac{H}{3} + 2 \right\}_1/2 \]
\[ S_{33}(\Theta) = [1]_0 [H/6 + 1]_1/2 \left\{ \frac{H}{3} + 1 \right\}_0 \left\{ \frac{H}{2} + 11 \right\}_1/2 \]
\[ S_{34}(\Theta) = [H/6]_1/2 \left\{ H/6 + 2 \right\}_1/2 \left\{ H/2 + 2 \right\}_1/2 \left\{ H/2 \right\}_1/2 \]
\[ S_{44}(\Theta) = [H/6 - 1]_1/2 \left\{ H/6 + 1 \right\}_1/2 \left\{ H/3 \right\}_0 \left\{ H/2 + 1 \right\}_1/2 . \]

There is one curious set of terms, however, occurring in \( S_{44} \) and represented by

\[ [x]'_0 = \{ x \}'_0 \{ H - x \}'_0 \quad \{ x \}'_0 = \frac{(x - 2) (x + 2)}{(x - 2 + 2B)(x + 2 - 2B)}. \]
The latter type of extended block occurs otherwise in only one other place, in the S-matrices for $a_{2n}^{(2)}$.

The expressions given in $a_{2n}$ are very compact and conceal much of the detailed information concerning the analytic structure. However, there are no hidden cancellations that occur when these expressions are expanded into their elementary ratios of hyperbolic sines. If these expressions are examined carefully, there are no positive definite poles other than those at rapidities given by the fusing angles $a_{2n}$). However, some of the positive definite poles are not simple, those corresponding to the self-couplings 333 in $S_{33}$ and 444 in $S_{44}$ are cubic poles, while the rest are first order. The S-matrix elements $S_{11}$, $S_{12}$, $S_{13}$ are straightforward, all poles are either positive or negative definite except for the double poles in $S_{13}$ at $\Theta = i(1/3 + 1/H)\pi$ and $\Theta = i(2/3 - 1/H)\pi$, which have a standard explanation in terms of the Coleman-Thun mechanism (see below). The $S_{14}$ element has a positive definite pole at $\Theta = i(1 - 2/H)\pi$ (and a negative definite pole at the crossed angle), a pair of semi-definite poles at $\Theta = i(1/3 + 2/H)\pi$ and $\Theta = i(2/3 - 2/H)\pi$, and a pair of double poles. $S_{22}$ is similar, with a pair of semi-definite poles and four definite simple poles. $S_{23}$ has a positive definite pole and its crossed partner, four semi-positive poles and a double pole. $S_{24}$ is more intricate—there are six semi-positive poles, two double poles and just one positive pole corresponding to the coupling 242, and its crossed partner. $S_{33}$ has four double poles and a positive definite cubic pole with its crossed partner. $S_{34}$ has six semi-positive poles, two of them cubic, a pair of double poles, and a positive definite simple pole with its crossed partner, corresponding to the coupling 341. Finally, $S_{44}$ is the most intricate despite having a single definite cubic pole with its crossed partner; there is also a pair of semi-positive cubic poles, a pair of semi-positive simple poles and a pair of double poles. Even the positive definite cubic poles have that property as a consequence of a miracle in which a pair of zeroes cross at $H = 15$.

The eigenvalues of the conserved quantities compatible with the bootstrap are given by

$$
q_s^* = \sin s\pi/H \sin 2s\pi/H' \\
q_1^* = \sin 3s\pi/H \sin 2s\pi/H' \\
q_2^* = \sin 2s\pi/H \sin 2s\pi/H' \\
q_3^* = \sin 3s\pi/H \sin 2s\pi/H',
$$

for $s = 1, 5 \mod 6$. The possible spins of the conserved quantities are the exponents common to both $e_7$ and $e_6$.  

10
5. The pair $c_n^{(1)}$, $d_n^{(2)}$

For these cases, it was already proposed by Delius et al. that the floating masses ought, except for an overall factor, to be given by

$$m_a = \sin \frac{a\pi}{H} \quad a = 1, 2, \ldots, n$$

(5.1)

where $H$ lies in the range $2n \leq H \leq 2n + 2$. Also, in this case, with the same proviso as before, $H = 2n + B$. The floating couplings are solely of the type

$$ab \rightarrow a + b$$

\[
\begin{align*}
    U_{ab}^{a+b} &= (a + b)\pi/H \\
    U_{a a+b}^b &= (H - b)\pi/H \\
    U_{b a+b}^a &= (H - a)\pi/H,
\end{align*}
\]

(5.2)

where $a + b \leq n$, or their crossed partners.

The S-matrix elements given in [18] are

$$S_{ab}(\Theta) = \prod_{p=|a-b|+1}^{a+b-1} [p]_0. \quad (5.3)$$

However, as in the previous cases, this notation conceals a number of interesting cancellations when $a + b > n$. For these elements, some of the double poles are cancelled by zeroes to leave semi-positive poles, and some of the simple poles disappear. For example, the pole which might occur in (5.3) at $\Theta = i(a + b)\pi/H$, for $a + b > n$ simply disappears. A series of semi-positive simple poles, occuring at $\Theta = i(2n + 2 - a - b)\pi/H$, $\ldots$ $i(a + b - 2)\pi/H$ and their partners, should not be invited to the bootstrap (even though there are classical couplings of this type in the lagrangian for the $c_n^{(1)}$ affine Toda theory), but can be explained, as will be described in section (8), using a modified Coleman-Thun mechanism. In terms of the modified blocks the expressions for the S-matrix elements with $a + b > n$ are

$$S_{ab}(\Theta) = \prod_{p=|a-b|+1}^{2n-a-b-1} [p]_0 \prod_{p=a+b-n-1}^{n+1-a-b} \{H/2 - p\}_1/2. \quad (5.4)$$

The only positive poles occuring in these S-matrix elements are those corresponding to the couplings (5.2).

The eigenvalues of the conserved quantities compatible with the bootstrap are

$$q_a^s = \sin \frac{sa\pi}{H} \quad (5.5)$$

where $s$ is any odd integer.
6. The pair $b_n^{(1)}, a_{2n-1}^{(2)}$

Here the spectrum contains one particle, labelled $n$, whose mass is conveniently chosen not to float and the others, labelled $1, 2, \ldots, n-1$, whose masses float, according to Delius et al, in the following manner

$$m_n = 1 \quad m_a = 2 \sin \frac{a\pi}{H}, \quad a = 1, 2, \ldots, n-1,$$  \hspace{1cm} (6.1)$$

with $2n-1 \leq H \leq 2n$. In this case, $B = 4n-2H$.

The floating couplings are the following:

$$nn \to a, \quad U_{nn}^a = (H - 2a)\pi/H \quad U_{na}^n = (H/2 + a)\pi/H \quad a = 1, 2, \ldots, n-1$$

$$ab \to a + b < n \begin{cases} U_{ab}^{a+b} = (a + b)\pi/H \\ U_{a}^{b}a+b = (H - b)\pi/H \\ U_{b}^{a}a+b = (H - a)\pi/H. \end{cases}$$ \hspace{1cm} (6.2)$$

The S-matrix elements are given by

$$S_{nn}(\Theta) = \prod_{p=1-n}^{n-1} \{H/2 - p\}_{-1/4}$$

$$S_{an}(\Theta) = \prod_{p=1}^{2a-1} \{\frac{H}{2} - a + p\}_0$$ \hspace{1cm} (6.3)$$

$$S_{ab}(\Theta) = \prod_{p=|a-b|+1}^{a+b-1} [p]_0$$

In this case, in $S_{ab}$, there is no hidden cancellation of zeroes and poles. The S-matrices $S_{nn}$ and $S_{an}$ have positive definite simple poles, and their crossed partners. The $S_{an}$ S-matrix elements also have double poles which are explained by the standard Coleman-Thun mechanism. The S-matrix elements $S_{ab}$ have a series of double poles and a pair of semi-positive poles at $i(a+b)\pi/H$ and $i(H - a - b)\pi/H$, when $a+b > n$. These are also explained in section(8). If $a + b < n$, these poles are positive/negative definite, respectively.

7. The case $a_{2n}^{(2)}$

The S-matrix corresponding to the theory based on the roots of $a_{2n}^{(2)}$, will be included here as one of the non-simply-laced cases, for completeness. It has a chequered history.
The minimal S-matrix for this theory (ie just the terms without dependence on $B$) is non-unitary, and was written down by Freund, Klassen and Melzer [22]; it generalises the Bullough-Dodd theory. Despite being a non-simply-laced theory, the classical and quantum mass ratios coincide [8], and the floating does not occur. On the other hand, the affine root system is self-dual, and for that reason the floating would not be expected given the new insight furnished by [18].

The S-matrix is [8]:

$$S_{ab}(\Theta) = \prod_{|a-b|+2 \text{ step 4}} \{p\}_0 \{4n+2-p\}_0' \quad a, b = 2, 4, \ldots, 2n$$ (7.1)

using the block defined in (4.7) with $H = h = 4n + 2$. Actually, these S-matrix elements may be thought of from the point of view of folding $d^{(1)}_{2n+2}$ [6]. The spectrum of the $a^{(2)}_{2n}$ theory coincides with the even-labelled particles in the spectrum of the $d^{(1)}_{2n+2}$ theory. The S-matrices of the latter have a multiple pole structure which is not explicable in terms of the truncated spectrum. However, the $a^{(2)}_{2n}$ S-matrix elements are obtained by simply deleting the inexplicable poles. This procedure leads to the above expression in terms of (4.7), and does not upset the bootstrap. Attempting to do the same in terms of the foldings leading to the other non-simply-laced theories simply does not work.

8. Semi-positive and high order poles

It has been noted in earlier sections that the S-matrices with floating couplings have a variety of pole singularities. There are those of odd order whose coefficient has a single sign throughout the floating range: these participate in the bootstrap, and indeed consistently define the floating couplings between the particles of the theory. Others are double poles whose origin lies in the Coleman-Thun mechanism, originally formulated to explain the double poles appearing in the sine-Gordon S-matrix elements[11]. Besides these, the S-matrices have semi-positive simple (and occasionally cubic) poles which do not participate in the bootstrap and whose origin also relies on the Coleman-Thun mechanism, with an interesting extra subtlety not encountered previously. One might say the prosaic explanation for the double poles within the sine-Gordon theory contains some poetry after all.
The basic mechanism relies on the existence of the diagram in fig(1a), in which the scattering particles $a, b$ each ‘defuse’ into the pairs $r, p$ and $r, q$ respectively, with all particles simultaneously on shell. The circle in the centre of the diagram represents the scattering process in which the pair $p, q$ elastically scatter before fusing with the particles $r$ to build the final state of the elastic $a, b$ scattering process. The dual of the on-shell scattering diagram is the figure (1b). There, the triangles represent the couplings and have sides whose lengths are the masses of the particles $a, b, p, q$ and $r$, as indicated. The relative rapidity of the pair $p, q$ as they scatter is $i$ times one of the inner angles, labelled $\psi$ in the diagram. On the other hand, the relative rapidity of the pair $a, b$ is $i$ times the angle labelled $\phi$. In terms of the coupling angles, $\phi$ and $\psi$ are given by

$$ \phi = \bar{U}^a_{rb} + \bar{U}^p_{ra} \quad \psi = U^b_{rq} + U^a_{rp}, $$

where $\bar{U} = \pi - U$. In practice, such diagrams are discovered by inspecting the coupling triangles and using them to provide a partial ‘tiling’ of the parallelogram with sides $m_a, m_b$ and angles $\phi$ and $\bar{\phi}$. Typically, the existence of a diagram like fig(1a), in which all lines may be simultaneously on shell, explains the double pole. The basic reasoning is simple. On shell, according to the Cutkosky rules, each internal propagating particle in the process contributes a delta function $\theta(p_0)\delta(p^2 - m^2)$ and there are two loop integrals, one for each triangular loop containing the particles $p, q, r$. Overall therefore, the six delta functions and two loop integrals combine to yield a double delta function which translates to a double pole in terms of the relative rapidity between particles $a$ and $b$, ie $\Theta$.

The above argument represents the generic situation. The $p, q$ scattering matrix element appearing in the middle of the diagram plays a rôle only in the detailed computations for the coefficient of the double pole. However, and this is the subtle part, the $p, q$ S-matrix element may have a zero at $\Theta_{pq} = i\psi$. If that happens then the double pole will be reduced to a simple pole. Moreover, the coefficient of the pole as a function of the floating parameter $(H$ or $B)$ will be composed of two pieces. First of all, there are the four couplings occuring in pairs corresponding to $arp$ and $brq$. These are computed from the coefficients of positive poles in the S-matrix elements $S_{rp}$ and $S_{rq}$, corresponding to the fusings $rp \rightarrow a$ and $rq \rightarrow b$, and, by definition, these factors do not change sign over the floating interval. The other important factor enters as the coefficient of the zero in $S_{pq}$, and this can change sign over the floating interval. Indeed, this is precisely what happens and provides a mechanism for many of the semi-positive poles. If $S_{pq}$ has a double zero, then the potential double pole will be removed completely.
However, this is not the whole story. Occasionally, a semi-positive pole has a more complicated explanation lying beyond the basic Coleman-Thun mechanism. It may happen that there is more than one rescattering in the middle of an on-shell diagram. For example, consider fig(2a) and its dual diagram fig(2b). Here, each particle \( a, b \) defuses into the pairs \( r, p \) and \( r', q \), respectively, and the pair \( p, q \) then fuses to make \( r'' \), all particles being on-shell. The middle circle then represents a three-to-three scattering with the three relative rapidities \( \Theta_{r,r'}, \Theta_{r,r''}, \Theta_{r',r''} \) fixed by energy-momentum conservation. Since the S-matrix theory is factorisable, the three-particle scattering S-matrix is regarded as the product of the three two-particle S-matrices \( S_{rr'}(\Theta_{r,r'}) \cdot S_{r'r''}(\Theta_{r',r''}) \cdot S_{rr''}(\Theta_{r,r''}) \), see fig(2c), evaluated for the appropriate rapidity differences. Specifically,

\[
\begin{align*}
\Theta_{ab} &= i(U_{ap}^r + U_{bq}^{r'} + U_{pq}^{r''}) \\
\Theta_{rr'} &= i(U_{pq}^{r''} + U_{pr}^a + U_{qr'}^b) \\
\Theta_{r'r''} &= i(U_{qpr}^b + U_{pr'}^a)
\end{align*}
\]

(8.2)

In the absence of any zeroes, the existence of such an on-shell diagram would imply a cubic pole. To see this, it is enough to remark the number of internal lines (13) less loop integrations (5) in fig(2c). However, one or more of the three factors in the three-to-three S-matrix element may have a zero, in which case the order of the pole is reduced. That some semi-positive poles are explained in this way is a matter of inspection. Several examples are included below. Generically, diagrams such as fig(2a) will provide cubic poles whose coefficient has a sign determined by the behaviour of the three inner S-matrix elements over the range of \( H \). Occasionally, the opposite is true, one or more of the rescattering S-matrices may also have a pole which indicates that one or more of the three inner parts of fig(2b) (the parallelograms marked with the dashed lines), may also be fully or partially tiled. In those cases, there may be an inner fusing (as for example in figs(3e,3f)) or, the Coleman-Thun mechanism may be repeated (as for example in fig(3g)). Note, there are actually two ways of tiling the central part of fig(2b) using the parallelograms. However, they correspond to identical products of S-matrix elements.

As a first example, consider the \( S_{12} \) matrix element in the \((g_2^{(1)}, d_4^{(3)})\) theory, eq(3.7). This has a semi-positive pole at \((1 + H/3)i\pi/H\). On the other hand, there is a diagram of type fig(1a) in which \( p, q, r \) are each particle 1, and the angle \( \psi = (2 + 2H/3)\pi/H \). At the latter angle, the matrix element \( S_{11} \), eqn(3.3), clearly has a simple zero, which on
inspection is seen to change sign at $H = 9$. The $S_{22}$ matrix element exhibits both types of behaviour. Consider the semi-positive pole at $2i\pi/H$. There is a diagram of type fig(1a) and again the internal particles are all particle 1; this time, $\psi = 4\pi/H$ and an inspection of $S_{11}$ reveals a zero at this angle. Again it has a coefficient which changes sign over the floating range. On the other hand, there is also a semi-positive pole at $\psi = (2H/3 - 2)\pi/H$, and this time there is a process corresponding to the diagram fig(2a). All the internal lines represent particle 1, and the inner angles are given by

$$\Theta_{rr''} = \Theta_{r'r''} = i(2H/3 + 2)\pi/H \quad \Theta_{rr'} = i(2H/3 - 4)\pi/H,$$

and thus it is clear the three-to-three scattering matrix has a double zero, since two of its factors vanish. This time the change of sign cannot come from the coefficients of these zeroes individually since only the square of the coefficient enters. However, the third factor $S_{rr'}(i(2H/3 - 4)\pi/H) = S_{11}(i(2H/3 - 4)\pi/H)$ does change sign over the interval (twice in fact). Apart from these singularities (and their crossed partners), the poles in the $(d_2^{(1)}, d_4^{(3)})$ S-matrix elements are positive or negative definite.

It has already been noted that the situation in the $(f_4^{(1)}, e_6^{(2)})$ S-matrix is much more involved. The element $S_{13}$ has a standard double pole at $i(2H/3 - 1)\pi/H$. $S_{14}$ has a standard double pole at $2i\pi/3$, and a semi-positive pole at $i(H/3 + 2)\pi/H$ which is explained by the above mechanism (with $p, q = 1, 3$ in fig (1a)) since $S_{13}(i(2/3 + 3/H)\pi) = 0$. The semi-positive pole in $S_{22}$ at $i(H/3 + 2)\pi/H$ is explained similarly (with $p, q = 1, 1$ in fig(1a)), since $S_{11}(i(2/3 + 4/H)\pi) = 0$. $S_{23}$ has a standard double pole at $i\pi/2$ and a pair of semi-positive poles at $i(H/6 + 2)\pi/H$ and $i(H/2 - 2)\pi/H$ explained (with $p, q = 1, 1$and $p, q = 1, 3$, respectively in fig(1a)), since $S_{11}(i(1/3 + 4/H)\pi) = 0$ and $S_{13}(i(2/3 + 3/H)\pi) = 0$. $S_{24}$ has standard double poles at $i(H/2 + 1)\pi/H$, and semi-positive poles at $i(H/6 + 1)\pi/H$, $i(H/6 + 3)\pi/H$ and $i(H/2 + 3)\pi/H$ which are explained (with $p, q = 1, 1$, $p, q = 1, 3$ and $p, q = 3, 3$, respectively in fig(1a)), since $S_{11}(i(2/3 + 4/H)\pi) = 0$, $S_{13}(i(1/3 + 5/H)\pi) = 0$ and $S_{33}(i(2/3 + 4/H)\pi) = 0$. $S_{33}$ has a pair of standard double poles at $2i\pi/H$ and $i(H/3 + 2)\pi/H$, and a standard (ie positive coefficient) cubic pole at $2i\pi/3$. The latter cubic pole is interesting for another reason. Alone among all the floating couplings in these theories, the 333 coupling in $(f_4^{(1)}, e_6^{(2)})$ has a subtiling, displayed in fig(3a). Because of this, there are several processes contributing to the cubic pole, among them the pair of vertex corrections in fig(3b). This sub-tiling of the 333 mass triangle also complicates the issue in a discussion of the poles in the $S_{34}$ and $S_{44}$ matrix elements. Specifically, the
The next, consider the pair \( (c_n^{(1)}, a_{n+1}^{(2)}) \). All the additional singularities of \( S_{ab} \) in \( (5.3) \) or \( (5.4) \), the double poles and semi-positive simple poles, can be explained by one type of diagram, fig(1a) with \( r = k, p = a - k, q = b - k, \phi = (a + b - 2k)\pi/H \) and \( \psi = (a + b)\pi/H \) for various possible \( k \)'s. When \( a + b \leq n \), they give the standard double poles for the entire range of \( k \), namely from \( k = 1, 2, \ldots, \min (a, b) - 1 \). While, for \( a + b > n \), the rescattering S-matrix in the middle of the Coleman-Thun mechanism, \( S_{a-k, b-k} \) has a zero at \( iv = i(a + b)\pi/H \) for \( k = 1, 2, \ldots, a + b - n - 1 \). In this range of \( k \), the processes corresponding to the diagrams provide a series of semi-positive simple poles located at

\[ S_{34} \text{ element has a standard double pole at } i(1/3 - 1/H)\pi, \text{ and its crossed partner; a semi-positive pole at } i3\pi/H \text{ explained (with } p, q = 1, 3 \text{ in fig(1a)), since } S_{13}(i5\pi/H) = 0; \text{ a semi-positive pole at } i(1/3 + 3/H)\pi, \text{ explained by the mechanism corresponding to fig(2b), with } r, r', r'' = 3, 1, 1, \text{ since } S_{13}(i(2/3 + 3/H)\pi) = 0 = S_{13}(i(2/3 - 5/H)\pi); \text{ and a semi-positive cubic pole at } i(H/3 + 1)\pi/H \text{ for which there are several contributions exemplified by figs(3c,3d)). The diagram fig(3c) yields a cubic pole in a straightforward manner; the contribution corresponding to fig(3d) relies on the retiling of the 333 coupling, and a zero at } i(2/3 + 3/H)\pi \text{ in the } S_{13} \text{ matrix element to reduce the expected fourth order pole down to cubic. The coefficient of the zero changes sign over the floating interval. Finally, } S_{44} \text{ has a standard double pole at } i(H/3 - 2)\pi/H \text{ explained (with } p, q = 2, 2 \text{ in fig(1a)); a semi-positive pole at } i2\pi/H \text{ explained (with } p, q = 1, 1 \text{ in fig(1a)), since } S_{11}(6i\pi/H) = 0; \text{ and cubic poles at } 2i\pi/3 \text{ and } i(2H/3 - 2)\pi/H, \text{ the latter being semi-positive and the former corresponding to the coupling 444. Fig(3e) again makes use of retiling the 333 coupling triangle to represent a process similar to that of fig(2a) but including a couple of vertex corrections. The expected fifth order pole at } i(2H/3 - 2)\pi/H \text{ is reduced to a cubic pole by a pair of zeroes (at } i(2/3 + 3/H)\pi \text{ in the two } S_{13} \text{ S-matrix elements contributing to the three-to-three scattering in the middle of the diagram. On the other hand, fig(3f) allows a cubic pole at } 2i\pi/3, \text{ since the expected fourth order pole is reduced to cubic by a zero (at } i(2/3 + 4/H)\pi \text{ in the } S_{11} \text{ rescattering process inside the diagram. It is also possible for a potential singularity to be removed completely by the zeroes in the rescattering S-matrices. Two cases have been encountered. The first is in } S_{44} \text{ at } \Theta = i4\pi/H. \text{ The standard Coleman-Thun mechanism (fig(1a) with } r = 1, p = q = 3 \text{) fails to produce any singularity here because } S_{33} \text{ has a double zero at } \psi = i6\pi/H. \text{ The second, and more interesting, case is again in } S_{44}, \text{ at } \Theta = i(H/3 + 4)\pi/H, \text{ which has fig(2a) with } r = r' = 3, r'' = p = q = 1. \text{ The potential cubic pole is removed completely by the three zeroes } S_{13}(i(2H/3 + 3)\pi/H) = 0 \text{ (twice) and } S_{33}(i(H/3 + 6)\pi/H) = 0. \]
\[i(2n - a - b + 2)\pi/H, \ldots, i(a + b - 2)\pi/H.\] For \(k = a + b - n, \ldots, \min(a, b) - 1,\) the element \(S_{a-k, b-k}\) is non-vanishing at \(i\psi = i(a + b)\pi/H\) and a series of standard double poles at \(i(a - b + 2)\pi/H, \ldots, i(2n - a - b)\pi/H\) is generated.

The singularity structure of the S-matrices for the pair \((b_n^{(1)}, a_{2n-1}^{(2)})\) is the simplest. \(S_{an}\) has a series of double poles at \(i(H/2 - a + 2k)\pi/H, k = 1, 2, \ldots, a - 1.\) They are explained by two types of diagrams, the standard Coleman-Thun crossed box together with an uncrossed box. \(S_{ab}\) has a series of double poles at \(i(|a - b| + 2)\pi/H, \ldots, i(a + b - 2)\pi/H,\) which can be explained by the standard Coleman-Thun mechanism as in the pair \((c_n^{(1)}, d_{n+1}^{(2)})\). The only semi-positive simple poles are at \(i(a + b)\pi/H\) with its crossed partner for \(a + b > n.\) It corresponds to \(p = q = r = n\) and \(\psi = (2 - 2(a + b)/H)\pi.\) The S-matrix, \(S_{nn},\) in the middle of fig(1a) has a simple zero at \(i\psi = i(2 - 2(a + b)/H)\pi,\) which reduces the singularity of \(S_{ab}\) at \(i\phi = i(a + b)\pi/H\) to a simple pole. This is the pole whose ‘shifted position’ is discussed in detail from a different point of view in [13].

A detailed checking of the coefficients of these poles will not be attempted here. Nevertheless, for each singularity, there is at least one identified process that can produce it, and sometimes several. There is no situation in which a semi-positive simple or cubic pole fails to have an explanation in these terms, at least in principle.

9. Discussion

The scattering matrices for the dual pairs are reminiscent of the sine-Gordon breather S-matrices. There, because the spectrum of the theory is not perturbative, it was never expected to find a perturbative understanding of the multiple pole singularities. Here, the mass-spectrum floats with a parameter, \(H\) or \(B,\) interpolating the mass sets of the two partners in the dual pair, and is also expected to be non-perturbative. An infinite order of perturbation theory would be needed to see the floating phenomenon over its whole range. The self-dual affine Toda theories are very special, the masses in the full theory occur in the same ratios as in the classical lagrangian, there is no floating, and therefore the existence of the multiple poles in the S-matrix can be inferred from the perturbation theory relatively easily. In those cases, it is the coefficients of the poles that can be computed to finite order only in the coupling constant, \(\beta.\)

Despite these differences, there appears to be a general statement that can be made concerning the bootstrap. Namely, the existence of a genuine bound-state fusing is signalled by the existence of a pole of odd order, whose coefficient is positive throughout.
the range of a parameter, which interpolates two theories with unit S-matrix. This statement applies equally well to both types of theory. Poles of odd order whose coefficients change sign within this interval are not supposed to take part in the bootstrap, which is consistent without including them. Moreover, all such singularities have an explanation in terms of a generalised Coleman-Thun mechanism in which the floating zeroes of the S-matrix elements play a crucial rôle. The fact that zeroes play a part emphasises the non-perturbative nature of the mechanism since a zero in an S-matrix element is not easy to see in perturbation theory. It was perhaps fortunate that for the simply-laced theories these special mechanisms are not apparently necessary since all odd order poles in those cases have a coefficient of a definite sign throughout the range of the coupling constant.

Acknowledgements

All of us are grateful for the opportunity to spend some time at the Isaac Newton Institute for Mathematical Sciences, where this study was begun, and to Gérard Watts for discussions. One of us (PED) is supported by a European Community Fellowship, another (RS) thanks the Japan Society for Promotion of Science for a one year Visiting Fellowship to the University of Durham, and also the Department of Mathematical Sciences for its hospitality.
References

[1] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, Quantum S-matrix of the 1+1 dimensional Toda chain, Phys. Lett. B87 (1979) 389.

[2] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, Two dimensional generalised Toda lattice, Comm. Math. Phys. 79 (1981) 473;
D. I. Olive and N. Turok, The symmetries of Dynkin diagrams and the reduction of Toda field equations, Nucl. Phys. B215 (1983) 470;
D. I. Olive and N. Turok, Local conserved densities and zero-curvature conditions for Toda lattice field theories, Nucl. Phys. B257 (1985) 277;
D. I. Olive and N. Turok, The Toda lattice field theory hierarchies and zero-curvature conditions in Kac-Moody algebras, Nucl. Phys. B265 (1986) 469.

[3] G. Wilson, The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras, Ergod. Th. and Dynam. Sys. 1 (1981) 361;
V.G. Drinfel’d and V.V. Sokolov, J. Sov. Math. 30 (1984) 1975.

[4] T.J. Hollowood, Solitons in affine Toda field theories, Nucl. Phys. B384 (1992) 523;
N.J. MacKay, W.A. McGhee, Affine Toda solitons and automorphisms of Dynkin diagrams, Durham/Kyoto preprint, July 1992;
D. Olive, N. Turok, J.W.R. Underwood, Solitons and the energy momentum tensor for affine Toda theory, Swansea/Imperial College preprint October 1992.

[5] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Extended Toda field theory and exact S-matrices, Phys. Lett. B227 (1989) 411.

[6] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Aspects of perturbed conformal field theory, affine Toda field theory and exact S-matrices, Proceedings of the NATO Conference on Differential Geometric Methods in Theoretical Physics, Lake Tahoe, USA 2-8 July 1989 (Plenum 1990).

[7] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Affine Toda field theory and exact S-matrices, Nucl. Phys. B338 (1990) 689.

[8] P. Christe and G. Mussardo, Integrable systems away from criticality: the Toda field theory and S-matrix of the tri-critical Ising model, Nucl. Phys. B330 (1990) 465;
P. Christe and G. Mussardo, Elastic S-matrices in 1 + 1 dimensions and Toda field theories, Int. J. Mod. Phys. A5 (1990) 4581.

[9] C. Destri and H.J. de Vega, The exact S-matrix of the affine $E_8$ Toda field theory, Phys. Lett. B233 (1989) 336.

[10] M. Karowski, On the bound state problem in 1 + 1 dimensional field theories, Nucl. Phys. B153 (1979) 244.

[11] A.B. Zamolodchikov and Al. B. Zamolodchikov, Factorised S-matrices in 2 dimensions as the exact solutions of certain relativistic quantum field theory models, Ann. Phys. 120 (1979) 253.
[12] H. W. Braden and R. Sasaki, The S-matrix coupling dependence for ADE affine Toda field theory, Phys. Lett. B255 (1991) 343;
H.W. Braden and R. Sasaki, Affine Toda perturbation theory, Nucl. Phys. B379 (1992) 377.
[13] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, Multiple poles and other features of affine Toda field theory, Nucl. Phys. B356 (1991) 469.
[14] T. R. Klassen and E. Melzer, Purely elastic scattering theories and their ultra-violet limits, Nucl. Phys. B338 (1990) 485.
[15] M. D. Freeman, On the mass spectrum of affine Toda field theory, Phys. Lett. B261 (1991) 57;
A. Fring, H.C. Liao and D. Olive, The mass spectrum and coupling in affine Toda field theory, Phys. Lett. 266B (1991) 82.
[16] P. E. Dorey, Root systems and purely elastic S-matrices, Nucl. Phys. B358 (1991) 654.
[17] P. E. Dorey, Root systems and purely elastic S-matrices II, Nucl. Phys. B374 (1992) 741.
[18] G. W. Delius, M.T. Grisaru, D. Zanon, Exact S matrices for non-simply laced affine Toda theories, Nucl. Phys. B382 (1992) 365.
[19] H.G. Kausch and G.M.T. Watts, Duality in quantum Toda theory and W-algebras, Nucl. Phys. B386 (1992) 166;
H.S. Cho, I.G. Koh and J.D. Kim, Duality in the $d_4^{(3)}$ affine Toda theory, KAIST preprint 1992.
[20] G.M.T. Watts, R. A. Weston, $g_2^{(1)}$ affine Toda field theory: A Numerical test of exact matrix results, Phys. Lett. B289 (1992) 61.
[21] S. Coleman and H. Thun, Comm. Math. Phys. 61 (1978) 31.
[22] P. G. O. Freund, T. Klassen and E. Melzer, S-matrices for perturbations of certain conformal field theories, Phys. Lett. B229 (1989) 243.
