What is Statistics? ; The Answer by Quantum Language

Shiro Ishikawa

Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kouhoku-ku, Yokohama, Japan. E-mail: ishiyawa@math.keio.ac.jp

Abstract

Since the problem: "What is statistics?" is most fundamental in science, in order to solve this problem, there is every reason to believe that we have to start from the proposal of a worldview. Recently we proposed measurement theory (i.e., quantum language, or the linguistic interpretation of quantum mechanics), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. This turn from physics to language does not only extend quantum theory to classical theory but also yield the quantum mechanical world view (i.e., the (quantum) linguistic world view, and thus, a form of quantum thinking, in other words, quantum philosophy ). Thus, we believe that the quantum linguistic formulation of statistics gives an answer to the question: "What is statistics?". In this paper, this will be done through the studies of inference interval, statistical hypothesis testing, Fisher maximum likelihood method, Bayes method and regression analysis in measurement theory.

1 Introduction

1.1 [Animistic worldview] → [Mechanical worldview] → [Linguistic worldview]

Although research of the worldview (or, world-description) has 3000 years or more of history, it was always the central theme in science. The leap (paradigm shift) from the animistic worldview (i.e., life dwells in thing) to the mechanical worldview occurred spontaneously through work of Galileo, bacon, Descartes, Newton and so on (cf. ① in Figure 1). This power was greatest and opened the door from medieval times to modernization. Here, the mechanical worldview means:

(A1) Investigate every science (other than physics) by a model of mechanics.

Figure 1. The development of the worldviews. For the explanations of this figure, see [6, 8].
Although it is simple, it is now accepted to be the best norm of world description, and it has reigned over modern science. In spite of the great success of the (A1), in this paper we shall propose “the linguistic worldview” in Figure 1. This says:

\((A_2)\) Describe every science (other than physics) by quantum language.

(cf. refs. [1–11]).

Note that Figure 1 does not include the (A1) but the (A2). Thus, we have an opinion that the (A1) should not be regarded as the worldview but a hint of scientific thinking. That is, we think that the (A1) plays an auxiliary role in the linguistic worldview (A2).

1.2 Quantum language (=Measurement theory)

It is well known that quantum mechanics has many interpretations (i.e., the Copenhagen interpretation, many worlds interpretation, Born interpretation, probabilistic interpretation, etc.). This fact is never desirable. In 1991, we proposed the mathematical formulation of Heisenberg’s uncertainty principle (cf. [12]). However, we should just have discussed it under a firm interpretation. Recently we proposed the linguistic quantum interpretation (called quantum and classical measurement theory), which was characterized as a kind of metaphysical and linguistic turn of the Copenhagen interpretation. This turn from physics to language does not only extend quantum theory to classical systems but also yield the linguistic worldview (A2) (i.e., the philosophy of quantum mechanics, in other words, quantum philosophy). In fact, we can consider that traditional philosophies have progressed toward quantum philosophy (cf. Figure 1). Thus, we expect that the linguistic interpretation is the only one interpretation of quantum mechanics (see (G) later).

In this paper, we first review the linguistic quantum interpretation (in Section 2), and further, we discuss inference interval (in Section 3), statistical hypothesis test (in Section 4), Fisher maximum likelihood method, Bayes method and regression analysis (in Sections 5 and 6) in measurement theory. The essential parts of Sections 5 and 6 were published in [7].

The purpose of this paper is to answer the question:

\((A_3)\) What is statistics? Or, where is statistics in science?

This will be answered in the framework of Figure 1. (C2) MT \[
\begin{cases}
\text{quantum MT} & \text{when } A = B_c(H) \\
\text{classical MT} & \text{when } A = C_0(\Omega) \\
\end{cases}
\]

2 Measurement Theory (Axioms and Interpretation)

2.1 Mathematical Preparations

In this section, we prepare mathematics, which is used in measurement theory (or in short, MT).

Measurement theory is, by a hint of quantum mechanics (i.e., the “normal → 2” in Figure 1), constructed as the scientific language formulated in a certain \(C^*\)-algebra \(A\) (i.e., a norm closed subalgebra in the operator algebra \(B(H)\) composed of all bounded linear operators on a Hilbert space \(H\)). MT is composed of two theories (i.e., pure measurement theory (or, in short, PMT) and statistical measurement theory (or, in short, SMT). That is, we see:

\((B)\) MT (measurement theory=quantum language)

\[
\begin{align*}
(B_1) &: \text{ [PMT]} \\
&= \text{ [pure measurement] + [causality]} \\
&= \text{ [Axiom}^P 1) \text{ + (Axiom 2)} \\
(B_2) &: \text{ [SMT]} \\
&= \text{ [statistical measurement] + [causality]} \\
&= \text{ [Axiom}^S 1) \text{ + (Axiom 2)}
\end{align*}
\]

where Axiom 2 is common in PMT and SMT. For completeness, note that measurement theory (B) (i.e., (B1) and (B2)) is a kind of language based on “the quantum mechanical worldview”. It may be understandable to consider that

\((C_1)\) PMT and SMT is related to Fisher’s statistics and Bayesian statistics respectively.

When \(A = B_c(H)\), the \(C^*\)-algebra composed of all compact operators on a Hilbert space \(H\), the (B) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when \(A\) is commutative (that is, when \(A\) is characterized by \(C_0(\Omega)\), the \(C^*\)-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space \(\Omega\) (cf. [14]), the (B) is called classical measurement theory. Thus, we have the following classification:
Now we shall explain the measurement theory (B). Let $A(\subseteq B(H))$ be a $C^*$-algebra, and let $A^*$ be the dual Banach space of $A$. That is, $A^* = \{ \rho \mid \rho$ is a continuous linear functional on $A \}$, and the norm $\| \rho \|_A$ is defined by sup${\{ |\rho(F)| : F \in A \text{ such that } \|F\|_A (= \| F \|_{B(H)}) \leq 1 }$. The bi-linear functional $\rho(F)$ is also denoted by $\langle \rho, F \rangle_A$, or in short $(\rho, F)$. Define the mixed state $\rho \in A^*$ such that $\| \rho \|_A = 1$ and $\rho(F) \geq 0$ for all $F \in A$ satisfying $F \geq 0$. And put

$$\mathcal{S}^m(A^*) = \{ \rho \in A^* \mid \rho \text{ is a mixed state} \}.$$ 

A mixed state $\rho (\in \mathcal{S}^m(A^*))$ is called a pure state if it satisfies that $\rho = \theta \rho_1 + (1-\theta) \rho_2$ for some $\rho_1, \rho_2 \in \mathcal{S}^m(A^*)$ and $0 < \theta < 1^*$. The mixed state $\rho \in A^*$ such that $\| \rho \|_A = 1$ and $\rho(F) \geq 0$ for all $F \in A$ satisfying $F \geq 0$. And put

$$\mathcal{S}^p(A^*) = \{ \rho \in \mathcal{S}^m(A^*) \mid \rho \text{ is a pure state} \}.$$ 

which is called a state space. Riesz’s theorem (cf. [15]) says that $C_0(\hat{\Omega})^* = \mathcal{M}(\Omega) = \{ \rho \mid \rho \text{ is a signed measure on } \Omega \}$, $\mathcal{S}^m(C_0(\hat{\Omega})^*) = \mathcal{M}_{f+1}(\Omega) = \{ \rho \mid \rho \text{ is a measure on } \Omega \text{ such that } (\rho(\Omega) = 1) \}$. Also, it is well known (cf. [14]) that $\mathcal{S}^p(B_c(H)^*) = \{ |u|/| \|u\|_H = 1 \}$, and $\mathcal{S}^p(C_0(\hat{\Omega})^*) = \mathcal{M}_{\bar{\omega}}(\Omega) = \{ \bar{\omega}_x \mid \bar{\omega}_x \text{ is a point measure at } \omega \in \Omega \}$, where $\int_{\Omega} f(\omega) \langle \delta_{\omega} \rangle f(\omega) = f(\omega) \langle \forall \delta \in C_0(\Omega) \rangle$.

The latter implies that $\mathcal{S}^p(C_0(\hat{\Omega})^*)$ can be also identified with $\Omega$ (called a spectrum or spectrum space) such as

$$\mathcal{S}^p(C_0(\hat{\Omega})^*) \ni \delta_x \leftrightarrow \omega \in \Omega \quad \text{(spectrum)} \quad \text{(1)}$$

From here, $C_0(\Omega)$ (or, commutative unital $C^*$-algebra that includes $C_0(\Omega)$) is, for simplicity, denoted by $C(\Omega)$. Thus, we put $A^* = C(\Omega)^* = \mathcal{M}(\Omega)$, $\mathcal{S}^m(A^*) = \mathcal{S}^m(C(\Omega)^*) = \mathcal{M}_{f+1}(\Omega)$, and $\mathcal{S}^p(A^*) = \mathcal{S}^p(C(\Omega)^*) = \mathcal{M}_{\bar{\omega}}(\Omega) \approx \Omega$. And, for any mixed state $\nu \in \mathcal{M}_{\bar{\omega}}(\Omega)$ and any observable $O = (X,F,F)$ in $C(\Omega)$, we put:

$$\nu(F(\Xi)) = C(\Omega)^* \langle \nu, F(\Xi) \rangle C(\Omega) = \mathcal{M}(\Omega) \langle \nu, F(\Xi) \rangle C(\Omega) \approx \int_{\Omega} [F(\Xi)](\omega) \nu(d\omega). \quad \text{(2)}$$

Also, put $\nu(D) = \int_D \nu(d\omega)$ (or $D \in B_\Omega$ : Borel $\sigma$-field). In order to avoid the confusion between $\nu(F(\Xi))$ in (2) and $\nu(D)$, we do not use $\nu(F(\Xi))$. Also, for any $\delta_{\omega}$ in $\mathcal{M}_{\bar{\omega}}(\Omega) \approx \Omega$, we put:

$$c(\omega) \langle \delta_{\omega}, F(\Xi) \rangle C(\Omega) = \mathcal{M}(\Omega) \langle \delta_{\omega}, F(\Xi) \rangle C(\Omega) \approx \int_{\Omega} [F(\Xi)](\omega) \delta_{\omega}(d\omega) = [F(\Xi)](\omega).$$

Here, assume that the $C^*$-algebra $A(\subseteq B(H))$ is unital, i.e., it has the identity $I$. This assumption is not unnatural, since, if $I \notin A$, it suffices to reconstruct the $A$ such that it includes $A \cup \{ I \}$.

According to the noted idea (cf. [16]) in quantum mechanics, an observable $O = (X,F,F)$ in $\mathcal{A}$ is defined as follows:

$$(D_1) \quad \text{Field } X \text{ is a set, } \mathcal{F} \subseteq \mathcal{P}(X) \text{, the power set of } X \text{ is a field of } X \text{, that is, } \Xi_1, \Xi_2 \in \mathcal{F} \Rightarrow \Xi_1 \cup \Xi_2 \in \mathcal{F} \text{, } \Xi \in \mathcal{F} \Rightarrow \Xi \setminus \Xi \in \mathcal{F}.$$

$$(D_2) \quad \text{Countably additivity } F \text{ is a mapping from } \mathcal{F} \text{ to } \mathcal{A} \text{ satisfying: (a): for every } \Xi, F(\Xi) \text{ is a non-negative element in } \mathcal{A} \text{ such that } 0 \leq F(\Xi) \leq I, \text{ (b): } F(\emptyset) = 0 \text{ and } F(X) = I, \text{ where } 0 \text{ and } I \text{ is the 0-element and the identity in } \mathcal{A}. \text{ (c): for any countable decomposition } \{ \Xi_1, \Xi_2, \ldots \} \text{ of } \Xi \in \mathcal{F} \text{ (i.e., } \Xi_k, \Xi \in \mathcal{F} \text{ such that } \bigcup_{k=1}^{\infty} \Xi_k = \Xi, \Xi \cap \Xi_j = \emptyset (i \neq j) \text{), it holds that}$$

$$\lim_{K \to \infty} \rho(F(\bigcup_{k=1}^{K} \Xi_k)) = \rho(F(\Xi)) \quad \forall \rho \in \mathcal{S}^m(A^*) \quad \text{(i.e., in the sense of weak convergence).}$$

Remark 1. By the Hopf extension theorem (cf. [15]), we have the mathematical probability space $(X,F,F)$ where $F$ is the smallest $\sigma$-field such that $F \subseteq \mathcal{F}$. For the other formulation (i.e., $W^*$-algebraic formulation), see the appendix in [5].

### 2.2 Pure Measurement Theory in (B)

Now we shall explain the pure measurement theory (B). With any system $S$, a $C^*$-algebra $A(\subseteq B(H))$ can be associated in which the pure measurement theory (B) of that system can be formulated. A state of the system $S$ is represented by an element $\rho \in \mathcal{S}^p(A^*)$ and an observable is represented by an observable $O = (X,F,F)$ in $\mathcal{A}$. Also, the measurement of the observable $O$ for the system $S$ with the state $\rho$ is denoted by $M_A(O,S_{\rho})$ (or more precisely, $M_A(O,S_{\rho})$). An observer can obtain a measured value $x (\in X)$ by the measurement $M_A(O,S_{\rho})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born’s probabilistic interpretation of quantum mechanics. And thus, it is a statement without reality.
Axiom 1 [Pure Measurement]. The probability that a measured value $x \in X$ obtained by the measurement $M_{\mathcal{A}}(O \equiv (X, \mathcal{F}, F), S_{[\rho]})$ belongs to a set $\Xi(\mathcal{F})$ is given by $\rho_0(\mathcal{F}(\Xi))$.

Next, we explain Axiom 2 in (B). Let $(T, \leq)$ be a tree, i.e., a partial ordered set such that “$t_1 \leq t_3$ and $t_2 \leq t_3$” implies “$t_1 \leq t_2$ or $t_2 \leq t_1$”. In this paper, we assume that $T$ is finite. Assume that there exists an element $t_0 \in T$, called the root of $T$, such that $t_0 \leq t$ $(\forall t \in T)$ holds. Put $T^2_\leq = \{(t_1, t_2) \in T^2 \mid t_1 \leq t_2\}$. The family $\{\Phi_{t_1, t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1, t_2) \in T^2_\leq}$ is called a causal relation (due to the Heisenberg picture), if it satisfies the following conditions (E1) and (E2).

(E1) With each $t \in T$, a $C^*$-algebra $\mathcal{A}_t$ is associated.

(E2) For every $(t_1, t_2) \in T^2_\leq$, a Markov operator $\Phi_{t_1, t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}$ is defined (i.e., $\Phi_{t_1, t_2} \geq 0$, $\Phi_{t_1, t_2}(I_{\mathcal{A}_{t_2}}) = I_{\mathcal{A}_{t_1}}$). And it satisfies $\Phi_{t_1, t_2}(\Phi_{t_2, t_3}) = \Phi_{t_1, t_3}$ holds for any $(t_1, t_2)$, $(t_2, t_3) \in T^2_\leq$.

The family of dual operators $\{\Phi_{t_1, t_2}^* : \mathcal{S}(\mathcal{A}_{t_1}^*) \to \mathcal{S}(\mathcal{A}_{t_2}^*)\}_{(t_1, t_2) \in T^2_\leq}$ is called a dual causal relation (due to the Schrödinger picture). When $\Phi_{t_1, t_2}^* (\mathcal{S}(\mathcal{A}_{t_2}^*)) \subseteq (\mathcal{S}(\mathcal{A}_{t_1}^*))$ holds for any $(t_1, t_2) \in T^2_\leq$, the causal relation is said to be deterministic.

Now Axiom 2 in the measurement theory (B) is presented as follows:

Axiom 2 [Causality]. The causality is represented by a causal relation $\{\Phi_{t_1, t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1, t_2) \in T^2_\leq}$.

2.3 Linguistic Interpretation

Next, we have to study how to use the above axioms as follows. That is, we present the following interpretation (F) $\equiv (F_1) \equiv (F_3)$, which is characterized as a kind of linguistic turn of so-called Copenhagen interpretation (cf. [5, 6]). That is, we propose:

(F1) Consider the dualism composed of “observer” and “system (=measuring object)”. And therefore, “observer” and “system” must be absolutely separated.

(F2) Only one measurement is permitted. And thus, the state after a measurement is meaningless since it can not be measured any longer. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited.

(F3) Also, the observer does not have the space-time. Thus, the question: “When and where is a measured value obtained?” is out of measurement theory. And thus, Schrödinger’s cat is out of measurement theory, and so on.

And therefore, in spite of Bohr’s realistic view, we propose the following linguistic view:

(G) In the beginning was the language called measurement theory (with the interpretation (F)). And, for example, quantum mechanics can be fortunately described in this language. And moreover, almost all scientists have already mastered this language partially and informally since statistics (at least, its basic part) is characterized as one of aspects of measurement theory (cf. [3, 4, 7]).

Remark 2. As seen in the “→②” of Figure 1, measurement theory was constructed by a hint of quantum mechanics (or, by the linguistic turn of quantum mechanics). However, in the sense of (G), it should be substantially rewritten by the “→②” (description). Of course, we do not deny that another quantum physics exists beyond ② in Figure 1.
2.4 Sequential Causal Observable and Its Realization

For each \( k = 1, 2, \ldots, K \), consider a measurement \( M_A(\mathcal{O}_k \equiv \langle X_k, F_k, F_k \rangle, S_{[\rho]} \) ). However, since the \( \text{F}_2 \) says that only one measurement is permitted, the measurements \( \{ M_A(\mathcal{O}_k, S_{[\rho]} \}_{k=1}^K \) should be reconsidered in what follows. Under the commutativity condition such that

\[
\mathcal{F}(\Xi_i) \mathcal{F}(\Xi_j) = \mathcal{F}(\Xi_j) \mathcal{F}(\Xi_i) \quad (\forall \Xi_i \in \mathcal{F}_i, \forall \Xi_j \in \mathcal{F}_j, i \neq j),
\]

we can define the product observable \( \times_{k=1}^K \mathcal{O}_k = (\times_{k=1}^K X_k, \Xi_{k=1}^K F_k, \times_{k=1}^K F_k) \) in \( \mathcal{A} \) such that

\[
(\times_{k=1}^K \mathcal{F}_k)(\times_{k=1}^K \Xi_k) = \mathcal{F}_1(\Xi_1) \mathcal{F}_2(\Xi_2) \cdots \mathcal{F}_K(\Xi_K)
\]

we know the commutativity condition (3), represented by the simultaneous measurement \( M_A(\times_{k=1}^K \mathcal{O}_k, S_{[\rho]} \).

Consider a tree \( (T \equiv \{ t_0, t_1, \ldots, t_n \}, \leq ) \) with the root \( t_0 \). This is also characterized by the map \( \pi : T \setminus \{ t_0 \} \rightarrow T \) such that \( \pi(t) = \max \{ s \in T \mid s < t \} \). Let \( \{ \Phi_{t,t'} : \mathcal{A}_t \rightarrow \mathcal{A}_{\pi(t)} \}_{(t,t') \in T_2^\ast} \) be a causal relation, which is also represented by \( \{ \Phi_{\pi(t)}, t : \mathcal{A}_t \rightarrow \mathcal{A}_{\pi(t)} \}_{t \in T \setminus \{ t_0 \}} \). Let an observable \( \mathcal{O}_t \equiv (X_t, \mathcal{F}_t, F_t) \) in \( \mathcal{A}_t \) be given for each \( t \). Note that \( \Phi_{\pi(t)}, t : \mathcal{A}_t \rightarrow \mathcal{A}_{\pi(t)} \) is an observable in the \( \mathcal{A}_{\pi(t)} \).

For each \( s \in T \), put \( T_s = \{ t \in T \mid t \geq s \} \). And define the observable \( \mathcal{O}_s \equiv (X_t, \mathcal{F}_t, \Xi_{t \in T} \mathcal{F}_t, \mathcal{F}_s) \) in \( \mathcal{A}_s \) as follows:

\[
\mathcal{O}_s = \{ \mathcal{O}_t \quad (\text{if } s \in T \setminus \{ \mathcal{P}(T) \} )
\]

\[
\mathcal{O}_s \times (\times_{t \in \pi^{-1}(t)} \mathcal{F}_t) \mathcal{O}_t \quad (\text{if } s \in \pi(T))
\]

if the commutativity condition holds (i.e., if the product observable \( \mathcal{O}_s \times (\times_{t \in \pi^{-1}(t)} \mathcal{F}_t) \mathcal{O}_t \) exists) for each \( s \in \pi(T) \). Using (4) iteratively, we can finally obtain the observable \( \mathcal{O}_{t_0} \) in \( \mathcal{A}_{t_0} \). The \( \mathcal{O}_{t_0} \) is called the realization (or, realized causal observable) of \( [\mathcal{O}_T] \).

2.5 Statistical Measurement Theory in \( (B_2) \)

We shall introduce the following notation: \( \text{It is usual to consider that we do not know the pure state } \rho_0^m (\in \mathcal{S}^p(\mathcal{A}^\ast)) \) when we take a measurement \( M_A(\mathcal{O}, S_{[\rho]} \).

The Axiom \( \text{Axiom}^1 \) presented below is a kind of mathematical generalization of Axiom \( \text{A}^1 \).

**Axiom \( \text{Axiom}^1 \) [Statistical measurement].** The probability that a measured value \( x \in X \) obtained by the statistical measurement \( M_A(\mathcal{O} \equiv (X, \mathcal{F}, F), S_{[\rho]}(\{ \rho_0^m \})) \) belongs to a set \( \Xi \in \mathcal{F} \) is given by \( \rho_0^m \langle F(\Xi) \rangle \)

Thus, we can propose the statistical measurement theory \( (B_2) \), in which Axiom \( \text{A}^2 \) and Interpretation (G) are common.

Let \( \mathcal{O} \equiv (X \times Y, \mathcal{F} \boxtimes \mathcal{G}, H) \) be an observable in a \( C^\ast \)-algebra \( \mathcal{A} \). Assume that we know that the measured value \( (x, y) \in X \times Y \) obtained by a statistical measurement \( M_A(\mathcal{O}, S_{[\rho]}(\{ \rho_0^m \})) \) belongs to \( \Xi \times Y \in (\mathcal{F} \boxtimes \mathcal{G}) \). Then, there is a reason to infer that the unknown measured value \( y \in (\mathcal{E} \times Y) \) is distributed under the conditional probability \( P_\mathcal{E}(\mathcal{G}) \),

\[
P_\mathcal{E}(\mathcal{G}) = \mathcal{A}^\ast \langle \rho_0^m, H(\Xi \times \Gamma) \rangle \mathcal{A} \quad (\forall \Gamma \in \mathcal{G})
\]

Thus, by a hint of Fisher’s maximum likelihood method, we have the following theorem, which is the most fundamental in this paper.

**Theorem 1** [Fisher’s maximum likelihood method in general \( \mathcal{A} \) (cf. \([7]\)]. Let \( \mathcal{O} \equiv (X \times Y, \mathcal{F} \boxtimes \mathcal{G}, H) \) be an observable in a \( C^\ast \)-algebra \( \mathcal{A} \). Let \( K \subseteq \mathcal{S}^m(\mathcal{A}^\ast) \) be a compact set. Assume that we know that the measured value \( (x, y) \in X \times Y \) obtained by a measurement \( M_A(\mathcal{O}, S_{[\rho]}(K)) \) belongs to \( \Xi \times Y \in (\mathcal{F} \boxtimes \mathcal{G}) \). Then, there is a reason to infer that the unknown measured value \( y \in (\mathcal{E} \times Y) \) is distributed under the conditional probability \( P_\mathcal{E}(\mathcal{G}) \),

\[
P_\mathcal{E}(\mathcal{G}) = \mathcal{A}^\ast \langle \rho_0^m, H(\Xi \times \Gamma) \rangle \mathcal{A} \quad (\forall \Gamma \in \mathcal{G})
\]

Here, \( \rho_0^m (\in K \subseteq \mathcal{S}^m(\mathcal{A}^\ast)) \) is defined by

\[
\mathcal{A}^\ast \langle \rho_0^m, H(\Xi \times Y) \rangle \mathcal{A} = \max_{\rho \in K} \mathcal{A}^\ast \langle \rho^m, H(\Xi \times Y) \rangle \mathcal{A}.
\]
Corollary 1. Let $\mathcal{O}(\equiv (X, \mathcal{F}, F))$ and $(Y, \mathcal{G}, G)$ be observables in a $C^*$-algebra $\mathcal{A}$. Let $\mathcal{O} \equiv (X \times Y, \mathcal{F} \otimes \mathcal{G}, F \otimes G)$ be the tensor observable in a tensor $C^*$-algebra $\mathcal{A} \otimes \mathcal{A}$. Let $K \subseteq \mathcal{S}^p(\mathcal{A}^*)$ be a compact set. And put $K_D^2 = \{\rho \otimes \rho \mid \rho \in K\}$. Assume that we know that the measured value $(x, y) \in \times Y$ obtained by a measurement $M_{\mathcal{A} \otimes \mathcal{A}}(\mathcal{O}, (\Sigma(t)(K_D^2)))$ belongs to $\times Y \in (F \otimes G)$. Then, there is a reason to infer that the unknown measured value $y \in Y$ is distributed under the conditional probability $P_{\Xi}(G(\Gamma))$, where

$$P_{\Xi}(G(\Gamma)) = \mathcal{A}^\ast \langle \rho_0^p, G(\Gamma) \rangle_{\mathcal{A}} \quad (\forall \Gamma \in \mathcal{G}). \quad (7)$$

Here, $\rho_0^p \in K \subseteq \mathcal{S}^p(\mathcal{A}^*)$ is defined by

$$\rho_0^p := \frac{1}{\mathcal{A}^\ast \langle \rho_0^p, F(\Xi) \rangle_{\mathcal{A}}} \max_{\rho \in K} \mathcal{A}^\ast \langle \rho^p, F(\Xi) \rangle_{\mathcal{A}}.$$ 

Proof. The (7) is, from (6), derived as follows:

$$P_{\Xi}(G(\Gamma)) = \frac{A^\ast \otimes \mathcal{A}^\ast \langle \rho_0^p \otimes \rho_0^p, (F \otimes G)(\Xi \times \Gamma) \rangle_{\mathcal{A} \otimes \mathcal{A}}}{A^\ast \otimes \mathcal{A}^\ast \langle \rho_0^p \otimes \rho_0^p, (F \otimes G)(\Xi \times \gamma) \rangle_{\mathcal{A} \otimes \mathcal{A}}} = \mathcal{A}^\ast \langle \rho_0^p, G(\Gamma) \rangle_{\mathcal{A}}.$$ 

Remark 3 [The state before a measurement]. In the sense of Corollary 1, the arbitrariness of $(Y, \mathcal{G}, G)$ says that, when we know that the measured value obtained by $M_{\mathcal{A}}(\mathcal{O} = (X, \mathcal{F}, F), S(\sigma))$ belongs to $\Xi \in \mathcal{F}$, we can infer that the state $[\sigma]$ (before the measurement) is $\rho_0^p$.

3 Inference interval

Let $\mathcal{O}(\equiv (X, \mathcal{F}, F))$ be an observable formulated in a $C^*$-algebra $\mathcal{A}$. Assume that $X$ has a metric $d_X$. And assume that the state space $\mathcal{S}^p(\mathcal{A}^*)$ has the metric $d_\mathcal{S}$, which induces the weak* topology $\sigma(\mathcal{A}^*, \mathcal{A})$. Let $E : X \to \mathcal{S}^p(\mathcal{A}^*)$ be a continuous map, which is called "estimator." Let $\gamma$ be a real number such that $0 < \gamma < 1$, for example, $\gamma = 0.95$. For any $\rho^p \in \mathcal{S}^p(\mathcal{A}^*)$, define the positive number $\eta^\gamma_{\rho^p}$ ($> 0$) such that:

$$\eta^\gamma_{\rho^p} = \inf \{\eta > 0 : \mathcal{A}^\ast \langle \rho^p, F(E^{-1}(B(\rho^p; \eta))) \rangle_{\mathcal{A}} \geq \gamma\}$$

where $B(\rho^p; \eta) = \{\rho^p \in \mathcal{S}^p(\mathcal{A}^*) : d_\mathcal{S}(\rho^p, \rho^p) \leq \eta\}$. For any $x \in X$, put

$$D_2^\gamma = \{\rho^p \in \mathcal{S}^p(\mathcal{A}^*) : d_\mathcal{S}(E(x), \rho^p) \leq \eta^\gamma_{\rho^p}\}. \quad (8)$$

The $D_2^\gamma$ is called the $(\gamma)$-inference interval of the measured value $x$.

Note that,

(H) for any $\rho^p_0 \in \mathcal{S}^p(\mathcal{A}^*)$, the probability, that the measured value $x \in X$ obtained by the measurement $M_{\mathcal{A}}(\mathcal{O} = (X, \mathcal{F}, F), S(\rho^p_0))$ satisfies the following condition (b), is larger than $\gamma$ (e.g., $\gamma = 0.95$).

$$d(E(x), \rho^p_0) \leq \eta^\gamma_{\rho^p_0} \quad (\forall \rho^p_0 \in \mathcal{S}^p(\mathcal{A}^*)).$$

Assume that we get a measured value $x_0$ by the measurement $M_{\mathcal{A}}(\mathcal{O} = (X, \mathcal{F}, F), S(\rho^p_0))$. Then, we see the following equivalences:

(b) $\iff d_\mathcal{S}(E(x_0), \rho^p_0) \leq \eta^\gamma_{\rho^p_0} \iff D_2^\gamma \ni \rho^p_0.$

Summing the above argument, we have the following theorem.

Proposition 1 [Inference interval]. Let $\mathcal{O} := (X, \mathcal{F}, F)$ be an observable in $\mathcal{A}$. Let $\rho^p_0$ be any fixed state, i.e., $\rho^p_0 \in \mathcal{S}^p(\mathcal{A}^*)$. Consider a measurement $M_{\mathcal{A}}(\mathcal{O} := (X, \mathcal{F}, F), S(\rho^p_0))$. Let $E : X \to \mathcal{S}^p(\mathcal{A}^*)$ be an estimator. Let $\gamma$ be such as $0 < \gamma < 1$ (e.g., $\gamma = 0.95$). For any $x \in X$, define $D_2^\gamma$ as in (8). Then, we see,

(\#) the probability that the measured value $x_0 \in X$ obtained by the measurement $M_{\mathcal{A}}(\mathcal{O} := (X, \mathcal{F}, F), S(\rho^p_0))$ satisfies the condition that

$$D_2^\gamma \ni \rho^p_0,$$

is larger than $\gamma$.

Example 1 [The urn problem]. Put $\Omega = [0, 1]$, i.e., the closed interval in $\mathbb{R}$. We assume that each $\omega \in \Omega \equiv [0, 1]$ represents an urn that contains a lot of red balls and white balls such that:

the number of white balls in the urn $\omega$

the total number of balls in the urn $\omega$

$\approx \omega$ (i.e., $\omega \in [0, 1] \equiv \Omega$).
Define the observable \( O = (X \equiv \{r, w\}, \mathcal{P}(\{r, w\}), F) \) in \( C(\Omega) \) such that where
\[
F(\emptyset)(\omega) = 0, \quad F(\{r\})(\omega) = \omega, \quad F(\{w\})(\omega) = 1 - \omega, \quad F(\{r, w\})(\omega) = 1
\]
\((\forall \omega \in [0, 1] \equiv \Omega)\).

Here, consider the following measurement \( M_\omega \):
\[
M_\omega := \text{ "Pick out one ball from the urn } \omega, \text{ and recognize the color of the ball"}
\]
That is, we consider
\[
M_\omega = M_{C(\Omega)}(O, S_{[\delta_{\omega}]})
\]
Moreover, we define the product observable \( O^N \equiv (X^N, \mathcal{P}(X^N), F^N) \), such that:
\[
[F^N(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_{N-1} \times \Xi_N)](\omega) = [F(\Xi_1)](\omega) \cdot [F(\Xi_2)](\omega) \cdots [F(\Xi_N)](\omega)
\]
\((\forall \omega \in \Omega \equiv [0, 1], \ \forall \Xi_1, \Xi_2, \cdots, \Xi_N \subseteq X \equiv \{r, w\})\).

Note that
\[
\text{"take a measurement } M_\omega \text{ N times"} \\
\Leftrightarrow \text{"take a measurement } M_{C(\Omega)}(O^N, S_{[\delta_{\omega}]})"
\]
Define the estimator \( E : X^N (\equiv \{r, w\}^N) \rightarrow \Omega(\equiv [0, 1]) \)
\[
E(x_1, x_2, \cdots, x_{N-1}, x_N) = \frac{\sharp \{n \in \{1, 2, \cdots, N\} \mid x_n = r\}}{N}
\]
\((\forall x = (x_1, x_2, \cdots, x_{N-1}, x_N) \in X^N \equiv \{r, w\}^N)\).

For example, assume that \( N \) is sufficiently large and \( \gamma = 0.95 \). Then we see, from the property of binomial distribution, that
\[
\eta^0_{\omega} \approx 1.96 \sqrt{\frac{\omega(1-\omega)}{N}}
\]
and
\[
D^0_{x, 95} = \lceil E(x) - \eta_-, E(x) + \eta_+ \rceil
\]
where
\[
\eta_- = \eta^0_{E(x) - \eta_+}, \quad \eta_+ = \eta^0_{E(x) + \eta_+}.
\]
Under the assumption that \( N \) is sufficiently large, we can consider that
\[
\eta_- \approx \eta_+ \approx \eta^0_{E(x)} \approx 1.96 \sqrt{\frac{E(x)(1-E(x))}{N}}.
\]
Then we can conclude that
\[
(1) \text{ for any urn } \omega(\in \Omega \equiv [0, 1]), \text{ the probability, that the measured value } x = (x_1, x_2, \cdots, x_N) \text{ obtained by the measurement } M_{A}(O^N, S_{[\delta_{\omega}]}) \text{ satisfies the following condition (2), is larger than } \gamma \text{ (e.g., } \gamma = 0.95). \\
(2) |\omega - E(x)| \leq 1.96 \sqrt{\frac{E(x)(1-E(x))}{N}} \leq 0.98 \sqrt{N}.
\]

4 Statistical Hypothesis Testing

4.1 Problem (Statistical Hypothesis Testing)

It is usual to consider that we do not know the pure state \( \rho_0 \in \mathcal{S}(\mathcal{A}^*) \) when we take a measurement \( M_A(O, S_{[\rho_0]}) \). That is because we usually take a measurement \( M_A(O, S_{[\rho_0]}) \) in order to know the state \( \rho_0 \). Thus, when we want to emphasize that we do not know the state \( \rho_0 \), \( M_A(O, S_{[\rho_0]}) \) is denoted by \( M_A(O, S_{[\rho]}). \)

In what follows we shall study “statistical hypothesis testing.” Consider a measurement \( M_A(O \equiv (X, \mathcal{F}, F), S_{[\sigma]}) \) formulated in \( \mathcal{A} \).

Here, we assume that \( X, \tau_X \) is a topological space, where \( \tau_X \) is the set of all open sets. And assume that \( \mathcal{F} = \mathcal{B}_X \); the Borel field, i.e., the smallest \( \sigma \)-field that contains all open sets in \( X \). Note that we can assume, without loss of generality, that \( F(\Xi) \neq 0 \)
for any open set \( \Xi(\in \tau_X) \) such that \( \Xi \neq \emptyset \). That is because, if \( F(\Xi) = 0 \), it suffices to redefine \( X \) by \( X \setminus \Xi \).

Assume the following hypothesis called “null hypothesis”:

(J) the unknown state \([*] \) belongs to a set \( N_H \ (\subseteq \mathcal{S}^p(A^*) \).

In order to deny this hypothesis (J), we define the rejection region \( \hat{R}_{N_H}^* \ (\in \mathcal{F}) \) as follows.

(K) For sufficiently small significance level \( \alpha \ (0 < \alpha \ll 1 \), e.g., \( \alpha = 0.05 \) ), define the rejection region \( \hat{R}_{N_H} \ (\in \mathcal{F}) \) such that

\[
\begin{align*}
& (K_1) \ A^*(\rho, F(\hat{R}_{N_H}^*)) _\Lambda \leq \alpha \ (\forall \rho \in N_H \subseteq \mathcal{S}^p(A^*)) \\
& (K_2) \text{ If } \hat{R}_{N_H}^* \ (\in \mathcal{F}) \text{ and } \hat{R}_{N_H}^2 \ (\in \mathcal{F}) \text{ satisfy } (K_1) \text{ and } \hat{R}_{N_H}^* \subseteq \hat{R}_{N_H}^2, \text{ then, choose } \hat{R}_{N_H}^*.
\end{align*}
\]

\[\text{Figure 4. Null Hypothesis } N_H\]

Then, Axiom²¹ 1 says that

(L) if \([*] \in N_H \), the probability that a measured value obtained by \( M_A(O) \equiv (X, \mathcal{F}, F), S_{[*]} \) belongs to \( \hat{R}_{N_H}^* \) is less than \( \alpha \). Therefore, if a measured value belongs to \( \hat{R}_{N_H}^* \), and if \( \alpha \) is sufficiently small, then there is a reason to deny the hypothesis (J).

It is clear that the rejection region \( \hat{R}_{N_H}^* \) is not uniquely determined in general. Thus, we have the following problem:

(M₁) Find the most proper rejection region \( \hat{R}_{N_H}^* \).

This will be answered in the following section.

**Remark 4.** Define the observable \( O_1 = \{(0, 1), P\{\{0, 1\}\}, G\) in \( A \) such that \( \rho(G\{\{1\}\}) = 0 \) (\( \forall \rho \in \mathcal{S}^p(A^*) \setminus N_H \)). Consider the measurement \( M_{A^0 \otimes A}(O_1 \otimes O, S_{[*]}) \), where \( * \otimes * \in \{\rho \otimes \rho \mid \rho \in \mathcal{S}^p(A^*)\} \). Then we see that the measured value obtained by \( M_{A^0 \otimes A}(O_1 \otimes O, S_{[*]}) \) belongs to \( \{0\} \times \hat{R}_{N_H}^* \) is less than \( \alpha \). It is interesting to see the similarity between statistical hypothesis testing and fuzzy contraposition (cf. [1]).

### 4.2 Answer; Likelihood Ratio Test

Let \( O \equiv (X, \mathcal{F}, F) \) be an observable in a \( C^* \)-algebra \( A \). Define the map \( \hat{L} : \tau_X \times \mathcal{S}^p(A^*) \to [0, 1] \) such that

\[
\hat{L}(\Xi, \rho) = \frac{A^*(\rho, F(\Xi))_\Lambda}{\sup_{\rho \in \mathcal{S}^p(A^*)} A^*(\rho, F(\Xi))_\Lambda}.
\]

Further, define the likelihood function \( L : X \times \mathcal{S}^p(A^*) \to [0, 1] \) such that

\[
L(x, \rho) = \lim_{\mathcal{F} \ni \Xi \to x} \hat{L}(\Xi, \rho).
\]

That is, for any positive \( \epsilon \), there exists an open set \( \Xi_\epsilon \) such that it holds that \( |L(x, \rho) - L(\Xi, \rho)| < \epsilon \) for any open set \( \Xi(\in \tau_X) \) satisfying \( x \in \Xi \subseteq \Xi_\epsilon \).

Let \( N_H \) be as in (F). And consider a measurement \( M_A(O \equiv (X, \mathcal{F}, F), S_{[*]})) \). Here define the function \( \Lambda_{N_H} : X \to [0, 1] \) such that:

\[
\Lambda_{N_H}(x) = \sup_{\rho \in N_H} L(x, \rho) \ (\forall x \in X) \text{. (10)}
\]

Also, for any \( \epsilon \ (0 < \epsilon \leq 1) \), define \( D_{N_H}^\epsilon \ (\in \mathcal{F}) \) such that

\[
D_{N_H}^\epsilon = \{x \in X \mid \Lambda_{N_H}(x) \leq \epsilon \}. \text{ (11)}
\]

\[\text{Figure 5. } D_{N_H}^\epsilon\]

Consider a positive number \( \alpha \) (called a significance level) such that \( 0 < \alpha \ll 1 \) (e.g. \( \alpha = 0.05 \) ). Thus we can define \( \epsilon(\alpha) \) such that:

\[
\epsilon(\alpha) = \sup\{\epsilon \mid \sup_{\rho \in N_H} A^*(\rho, F(D_{N_H}^\epsilon))_\Lambda \leq \alpha\} \text{. (12)}
\]

It is clear that the \( D_{N_H}^\epsilon(\alpha) \) satisfies the condition (G).

Then, the rejection region \( \hat{R}_{N_H}^* \) is given by \( D_{N_H}^\epsilon(\alpha) \).

**Remark 5.** Note that Problem (M₁) is not yet answered. However, we want to present the following conjecture:

(M₂) Under the general situation mentioned in Section 4.1, the likelihood ratio test (mentioned in Section 4.2) is the only statistical hypothesis testing. That is, it is best.
The reason that we think so is that we can not come up with another proper idea, since our situation in Section 3.1 is too general.

4.3 Typical Examples in Classical Measurements

Our argument in the previous section may be too abstract and general. However, it is surely usual. In this section, this will be shown as easy examples in classical measurements.

Put \( \Omega = \mathbb{R} \), \( \mathcal{A} = C(\Omega) \). Fix \( \sigma > 0 \). And consider the normal observable \( O_\sigma \equiv (\mathbb{R}, B_{\mathbb{R}}, F_\sigma) \) in \( C(\Omega) \) such that:

\[
[F_\sigma(\Xi)](\omega) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left[-\frac{(x - \omega)^2}{2\sigma^2}\right]dx
\]

\( (\forall \Xi \in B_{\mathbb{R}}, \forall \omega \in \Omega = \mathbb{R}) \). (13)

And further, consider the product observable \( O_\tau \equiv (\mathbb{R}^2, B_{\mathbb{R}^2}, F_\tau^2) \) in \( C(\Omega) \). That is,

\[
[F_\tau^2(\Xi_1 \times \Xi_2)](\omega) = [F_\sigma(\Xi_1)](\omega) \cdot [F_\sigma(\Xi_2)](\omega)
\]

\( (\forall \Xi_k \in B_{\mathbb{R}}(k = 1, 2), \forall \omega \in \Omega = \mathbb{R}) \). (14)

In what follows, we consider the measurement \( M_{C(\Omega)}(O_\sigma^2(\mathbb{R}^2, B_{\mathbb{R}^2}, F_\tau^2), S_{[\sigma]}) \).

[Case(I):] Two sided test, i.e., \( \mathcal{N}_H = \{\omega_0\} \). Assume that \( \mathcal{N}_H = \{\omega_0\} \), \( \omega_0 \in \Omega = \mathbb{R} \). Note the identification (1), i.e., \( \delta_{\omega_0} \approx \omega_0 \). Then, we see that, for any \( (x_1, x_2) \in \mathbb{R}^2 \),

\[
\Lambda_{\mathcal{N}_H}(x_1, x_2) = \sup_{\omega \in (\omega_0)} L((x_1, x_2), \delta_\omega)
\]

\[
= \lim_{\Xi_1 \times \Xi_2 \to (x_1, x_2)} \sup_{\omega \in (\omega_0)} \frac{[F_\sigma^2(\Xi_1 \times \Xi_2)](\omega)}{\exp\left[-\frac{(x_1 - \omega)^2 + (x_2 - \omega)^2}{2\sigma^2}\right]}
\]

\[
= \exp\left[-\frac{(x_1 - \omega_0)^2 + (x_2 - \omega_0)^2}{2\sigma^2}\right]
\]

Thus, putting \( \alpha = 0.05 \), we see that

\[
\hat{R}_{\omega_0}^{0.05} = \mathcal{D}_{\omega_0}^{(0.05)}
\]

\[
= \{(x_1, x_2) \in \mathbb{R}^2 | (x_1 + x_2)/2 \leq \omega_0 - 1.96\sigma/\sqrt{2}\}
\]

\[\cup \{(x_1, x_2) \in \mathbb{R}^2 | (x_1 + x_2)/2 \geq \omega_0 + 1.96\sigma/\sqrt{2}\}
\]

= “Slash part in Figure 6” (18)

\[\text{Figure 6. Rejection region } \hat{R}_{\omega_0}^{0.05}\]

[Case(II):] One sided test, i.e., \( \mathcal{N}_H = \{\omega_0, \infty\} \). Assume that \( \mathcal{N}_H = \{\omega_0, \infty\} \), \( \omega_0 \in \Omega = \mathbb{R} \). Then,

\[
\Lambda_{\mathcal{N}_H}(x_1, x_2) = \sup_{\omega \in (\omega_0, \infty)} L((x_1, x_2), \delta_\omega)
\]

\[
= \sup_{\omega \in (\omega_0, \infty)} \lim_{\Xi_1 \times \Xi_2 \to (x_1, x_2)} \frac{[F_\sigma^2(\Xi_1 \times \Xi_2)](\omega)}{\exp\left[-\frac{(x_1 + x_2)^2 - 2\omega^2}{4\sigma^2}\right]}
\]

\[
= \sup_{\omega \in (\omega_0, \infty)} \exp\left[-\frac{((x_1 + x_2)^2 - 2\omega^2)}{4\sigma^2}\right]
\]

\[
= \left\{ \begin{array}{ll}
\exp\left[-\frac{((x_1 + x_2) - 2\omega_0)^2}{4\sigma^2}\right] & \frac{x_1 + x_2}{2} < \omega_0 \\
1 & \text{otherwise}
\end{array} \right.
\]

(19)

Also, for any \( \epsilon > 0 \), define \( \mathcal{D}_{\omega_0, \infty}^\epsilon \) \( (\in B_{\mathbb{R}^2}) \) such that:

\[
\mathcal{D}_{\omega_0, \infty}^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 | \Lambda_{\omega_0, \infty}(x_1, x_2) \leq \epsilon\}
\]

Thus we can define \( \epsilon(\alpha) \) such that:

\[
\epsilon(\alpha) = \sup\{\epsilon | \sup_{\omega \in (\omega_0, \infty)} [F_\sigma^2(\mathcal{D}_{\omega_0, \infty}^\epsilon)](\omega) \leq \alpha\}.
\]

(21)
Therefore, putting $\alpha = 0.05$, we see that
\[
\hat{R}^{0.05, \infty}_{[\omega_0, \infty)} = D^{(0.05)}_{[\omega_0, \infty)}
\]
\[
= \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 + x_2)/2 \leq \omega_0 - 1.65\sigma/\sqrt{2}\}
\]
\[
= \text{“Slash part in Figure 7”}
\]

**Figure 7.** Rejection region $\hat{R}^{0.05}_{[\omega_0, \infty)}$

[Case(III): $\nu_0 = Q$, i.e., the set of all rational numbers]. It is clear that $\Lambda_Q(x_1, x_2) = 1$, $(\forall (x_1, x_2) \in \mathbb{R}^2$). Thus, the rejection region does not exist.

5 **Fisher-Bayes Method in classical MT**

5.1 **Bayes Method in Classical $C(\Omega)$**

Let $O_1 \equiv (X, F, F)$ be an observable in a commutative $C^*$-algebra $C(\Omega)$. And let $O_2 \equiv (Y, G, G)$ be any observable in $C(\Omega)$. Consider the product observable $O_1 \times O_2 \equiv (X \times Y, F \otimes G, F \times G)$ in $C(\Omega)$. The existence will be shown in Section 7 in [7].

Assume that we know that the measured value $(x, y)$ obtained by a simultaneous measurement $M_{C(\Omega)}(O_1 \times O_2, S_\nu(\{\nu_0\}))$ belongs to $\Xi \times Y (\in F \otimes G)$. Then, (5), we can infer that

(N) the probability $P_\Xi(G(\Gamma))$ that $y$ belongs to $\Gamma (\in G)$ is given by

\[
P_\Xi(G(\Gamma)) = \int_{\Gamma} |F(\Xi)|G(\Gamma)|/|F(\Xi)||\nu_0(d\omega) = \nu_0(d\omega) \quad (\forall \Gamma \in G).
\]

Thus, we can assert that:

**Theorem 2** [Bayes method, cf. [3, 4, 7]]. When we know that a measured value obtained by a measurement $M_{C(\Omega)}(O_1 \equiv (X, F, F), S_{\nu_1}(\{\nu_0\}))$ belongs to $\Xi$, there is a reason to infer that the state after the measurement is equal to $\nu_0^\prime (\in M_{+1}^n(\Omega))$ such that

\[
\nu_0^\prime(D) = \int_{\Omega} |F(\Xi)||\nu_0(d\omega) = \nu_0(d\omega) \quad (\forall D \in B_\Omega)
\]

**Proof.** Note that we can regard that $P_\Xi \in M_{+1}^n(\Omega) (\subseteq C(\Omega)^*)$. That is, there exists $\nu_0^\prime (\in C(\Omega)^*)$ such that

\[
P_\Xi(G(\Gamma)) = \int_{\Omega} |G(\Gamma)|/|\nu_0^\prime(d\omega) \quad (\forall \Gamma \in G) \quad (23)
\]

Then, Axiom 3.1 says that the probability that a measured value $y (\in Y)$ obtained by the measurement $M_{C(\Omega)}(O_2 \equiv (Y, G, G), S_{\eta_2}(\{\nu_0\}))$ belongs to a set $\Gamma (\in G)$ is given by $\int_{\Omega} |G(\Gamma)|/|\nu_0^\prime(d\omega)$, which is equal to $P_\Xi(G(\Gamma))$ in (23). Since $O_2 \equiv (Y, G, G)$ is arbitrary, we obtain Theorem 2.

**Remark 6.** The above (N) is, of course, fundamental. However, in the sense mentioned in the above proof, we admit Theorem 2 as the equivalent statement of the (N). That is, in spite of Interpretation (F2), we admit the wavefunction collapse such as

\[
\begin{align*}
(O_1) & \quad \text{(pretest state)} & \quad \text{Bayes} & \quad \text{Theorem 1} & \quad \nu_0 & \quad \text{Theorem 2} & \quad \nu_0^\prime \\
& \quad \text{($\in M_{+1}^n(\Omega)$)} & \quad \text{($\in M_{+1}^n(\Omega)$)} & \quad \text{($\in M_{+1}^n(\Omega)$)} & \quad \text{($\in M_{+1}^n(\Omega)$)}
\end{align*}
\]

Theorem 2 was, for the first time, proposed in [3, 4] without the conscious understanding of Interpretation (F2). Also, note that,

\[
(O_2) \quad \text{in Theorem 2, if } \nu_0 = \delta_{\omega_0} (\in M_{+1}^n(\Omega)), \text{then it clearly holds that } \nu_0^\prime = \delta_{\omega_0}.
\]

Also, for our opinion concerning the wavefunction collapse in quantum mechanics, see [5].

5.2 **Fisher-Bayes Method in Classical $C(\Omega)$**

Combining Theorem 1 (Fisher’s method) and Theorem 2 (Bayes’ method), we get the following corollary.

**Corollary 2** [Fisher-Bayes method (i.e., Regression analysis in a narrow sense)]. When we know that a measured value obtained by a measurement $M_{C(\Omega)}(O_1 \equiv (X, F, F), S_{\eta}(K))$ belongs to $\Xi$, there is a reason to infer that the state after the measurement is equal to $\nu_0^\prime (\in M_{+1}^n(\Omega))$ such that

\[
\nu_0^\prime(D) = \frac{\int_{\Omega} |F(\Xi)|/|\nu_0(d\omega) = \nu_0(d\omega) \quad (\forall D \in B_\Omega)}{\int_{\Omega} |F(\Xi)|/|\nu_0(d\omega) = \nu_0(d\omega)}
\]

where the $\nu_0(\in K)$ is defined by

\[
\int_{\Omega} |F(\Xi)|/|\nu_0(d\omega) = \max_{\nu \in K} \int_{\Omega} |F(\Xi)|/|\nu(d\omega).
\]

**Remark 7.** As mentioned in the above, note that Corollary 2 is composed of the following two procedures:

\[
\begin{align*}
(O_3) & \quad \text{Corollary 2} & \quad \text{Bayes} & \quad \nu_0 & \quad \text{Corollary 1} & \quad \nu_0^\prime \\
& \quad \text{($\in M_{+1}^n(\Omega)$)} & \quad \text{($\in M_{+1}^n(\Omega)$)} & \quad \text{($\in M_{+1}^n(\Omega)$)} & \quad \text{($\in M_{+1}^n(\Omega)$)}
\end{align*}
\]
5.3 A Simple example of Fisher-Bayes Method (Regression Analysis in a Narrow Sense)

In this section, we examine Corollary 2 in a simple example. Readers will find that Corollary 2 can be regarded as regression analysis in a narrow sense.

We have a rectangular water tank filled with water. Assume that the height of water at time \( t \) is given by the following function \( h(t) \):

\[
h(t) = \alpha_0 + \beta_0 t,
\]

where \( \alpha_0 \) and \( \beta_0 \) are unknown fixed parameters such that \( \alpha_0 \) is the height of water filling the tank at the beginning and \( \beta_0 \) is the increasing height of water per unit time. The measured height \( h_m(t) \) of water at time \( t \) is assumed to be represented by

\[
h_m(t) = \alpha_0 + \beta_0 t + e(t),
\]

where \( e(t) \) represents a noise (or more precisely, a measurement error) with some suitable conditions. And assume that we obtained the measured data of the heights of water at \( t = 0, 1, 2 \) as follows:

\[
h_m(0) = 0.5, \quad h_m(1) = 1.6, \quad h_m(2) = 3.3.
\]

Then, we get the deterministic causal operators \( \text{hus}, \{ \Phi_{\pi(t), t} : C(\Omega_t) \to C(\Omega_{\pi(t)}) \}_{t \in \{1, 2\}} \) such that

\[
(\Phi_{0,1}f_1)(\omega_0) = f_1(\phi_{0,1}(\omega_0)) \quad (\forall f_1 \in C(\Omega_1), \forall \omega_0 \in \Omega_0)
\]

\[
(\Phi_{1,2}f_2)(\omega_1) = f_2(\phi_{1,2}(\omega_1)) \quad (\forall f_2 \in C(\Omega_2), \forall \omega_1 \in \Omega_1).
\]

Thus, we have the causal relation as follows.

\[
C(\Omega_0) \xleftarrow{\phi_{0,1}} C(\Omega_1) \xrightarrow{\phi_{1,2}} C(\Omega_2).
\]

Put \( \phi_{0,2}(\omega_0) = \phi_{1,2}(\phi_{0,1}(\omega_0)) \), \( \Phi_{0,2} = \phi_{0,1} \cdot \phi_{1,2} \).

Let \( \mathbb{R} \) be the set of real numbers. Fix \( \sigma > 0 \).

For each \( t = 0, 1, 2 \), define the normal observable \( O_t = (\mathbb{R}, B_\mathbb{R}, G_\sigma^2) \) in \( C(\Omega_t) \) such that

\[
(G_\sigma^2(\Xi))(\omega_t) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{\Xi} \exp \left( -\frac{(x-\alpha)^2}{2\sigma^2} \right) dx
\]

\[
\forall \Xi \in B_\mathbb{R}, \forall \omega_t = (\alpha, \beta) \in \Omega_t = [0, 2t + 2] \times [0, 2].
\]

Thus, we get the sequential deterministic causal observable \( [O_T] = \{ (O_t)_{t=0,1,2}, \Phi_{\pi(t), t} : C(\Omega_t) \to C(\Omega_{\pi(t)}) \}_{t=1,2} \). Then, the realized causal observable \( \hat{O}_0 = (\mathbb{R}^3, B_{\mathbb{R}^3}, \hat{F}_0) \) in \( C(\Omega_0) \) is, by (4) and (28), obtained as follows:

\[
[\hat{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2)](\omega_0)
\]

\[
= \left( [G_\sigma^2(\Xi_0)](\omega_0) \cdot [G_\sigma^2(\Xi_1)](\omega_1) \cdot [G_\sigma^2(\Xi_2)](\omega_2) \right)
\]

\[
(\forall \Xi_0, \Xi_1, \Xi_2 \in B_\mathbb{R}, \forall \omega_0 = (\alpha, \beta) \in \Omega_0).
\]

Putting \( K = \mathcal{M}_p^0(\Omega_0) \), we have the measurement \( \mathcal{M}_{C(\Omega_0)}(\hat{O}_0, S_{tr}(\mathcal{M}_p^{1,1}(\Omega_0))) \). Recall the (26), that is, the measured value \((x_0, x_1, x_2)\) obtained by the measurement \( \mathcal{M}_{C(\Omega_0)}(\hat{O}_0, S_{tr}(\mathcal{M}_p^{1,1}(\Omega_0))) \) is equal to

\[
(0.5, 1.6, 3.3) \in \mathbb{R}^3.
\]

Define the closed interval \( \Xi_t \) (\( t = 0, 2, 3 \)) such that

\[
\Xi_0 = [0.5 - \frac{1}{2N}, 0.5 + \frac{1}{2N}],
\]

\[
\Xi_1 = [1.6 - \frac{1}{2N}, 1.6 + \frac{1}{2N}],
\]

\[
\Xi_2 = [3.3 - \frac{1}{2N}, 3.3 + \frac{1}{2N}],
\]

for sufficiently large \( N \). Here, Fisher’s method (Theorem 1) says that it suffices to solve the problem.
(Q1) Find \((\alpha_0, \beta_0)\) such as
\[
\max_{(\alpha, \beta) \in \Omega_0} \left[ \tilde{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2)(\alpha, \beta) \right] (32)
\]
Putting
\[
U(x_0, x_1, x_2, \alpha, \beta) = \sum_{k=0}^{2} (x_k - (\alpha + k\beta))^2
\]
we have the following problem that is equivalent to (O₁):
(Q₂) Find \((\alpha_0, \beta_0)\) such as
\[
\begin{align*}
\min_{(\alpha, \beta) \in \Omega_0} & \exp \left( -\frac{U(x_0, x_1, x_2, \alpha, \beta)}{2\sigma^2} \right) \\
\Leftrightarrow \max_{(\alpha, \beta) \in \Omega_0} & U(x_0, x_1, x_2, \alpha, \beta).
\end{align*}
\]
Calculating
\[
\frac{\partial}{\partial \alpha} U(0.5, 1.6, 3.3, \alpha, \beta) = 0,
\frac{\partial}{\partial \beta} U(0.5, 1.6, 3.3, \alpha, \beta) = 0,
\]
we get
\[
(\alpha, \beta) = (0.4, 1.4)
\]
Thus, we see, by the statement (O₂) that
\[
\begin{array}{c}
\text{(R)} \quad \mathcal{M}_{\Omega_0}^{\text{Bayes}} \xrightarrow{\text{Fisher}} \delta_{\text{(0.4,1.4)}} \\
\quad \xrightarrow{\text{Theorem 2}} \delta_{\text{(0.4,1.4)}}
\end{array}
\]
This (i.e., \((\alpha_0, \beta_0) = (0.4, 1.4)\)) is the answer to the problem (P).

**Problem 1.** Since the above example is quite easy, the validity of Bayes’ theorem in (R) may not be clear. If it is so, instead of the problem (O₃), we should present the following simple problem.

(S) Infer the water level at time 1.

Some may calculate and conclude as follows:
\[
h(1) = \alpha_0 + \beta_0 \times 1 = 0.4 + 1.4 = 1.8
\]
However, this calculation is based on the Schrödinger picture, and thus, the justification of this calculation (35) is not assured. That is because measurement theory (particularly, Interpretation (F₂)) says that the Heisenberg picture should be adopted. Therefore, in order to answer the problem (S), we must prepare Corollary 3 (i.e., regression analysis in a wide sense) in the following section.

**Remark 8.** It should be noted that the following two are equivalent:

(T₁) \([= (P)]; \text{Inference}: \) when measured data \((26)\) is obtained, infer the unknown parameter \((\alpha_0, \beta_0)\).

(T₂) \([\text{Control}]: \) Settle the parameter \((\alpha_0, \beta_0)\) such that measured data \((26)\) will be obtained.

That is, we see that
\[\text{“inference” = “control”}\.\]

Hence, from the measurement theoretical point of view, we consider that
\[\text{“Statistics” = “Dynamical system theory”}\.
\]

Though these are superficially different in applications.

### 6 Causal Fisher-Bayes method in classical MT

#### 6.1 Causal Bayes Method in Classical C(Ω)

Let \( t_0 \) be the root of a tree \( T \). Let \( \{\Omega^\omega_0\} = \{|\Omega^\omega_0\| = \{X_t \times Y_t, F_t \equiv G_t, F_t \times G_t\}\}_t \) and \( \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1})\}_{(t_1, t_2) \in T^2_0} \) be sequential causal observable with the realization \( \tilde{O}_{t_0}^\omega \equiv (\times_{\omega\in T}(X_t \times Y_t), \times_{\omega\in T}F_t \equiv G_t, \tilde{H}_{t_0}\) in \( C(\Omega_{t_0}) \). Thus we have the statistical measurement \( \mathcal{M}_{C(\Omega_0)}(\tilde{O}_{t_0}^\omega, S_{[\nu]}\{|\nu_0\})\), where \( \nu_0 \in \mathcal{M}_{\Omega_0}^{\text{Fisher}}(\tilde{O}_{t_0}^\omega, S_{[\nu]}\{|\nu_0\})\). Assume that we know that the measured value \((x, y) = ((x_t)_{t \in T}, (x_t)_{t \in T}) \in (\times_{t \in T}X_t) \times (\times_{t \in T}Y_t)\) obtained by the measurement \( \mathcal{M}_{C(\Omega_0)}(\tilde{O}_{t_0}^\omega, S_{[\nu]}\{|\nu_0\})\) belongs to \((\times_{t \in T}X_t) \times (\times_{t \in T}Y_t) \in (\times_{\omega\in T}F_t \equiv G_t, \tilde{H}_{t_0}\) in \( C(\Omega_{t_0}) \). Then, by (5), we can infer that

\[
P_{\times_{\omega\in T}X_t}(\{G_t(\Gamma_t)\}_{t \in T}) \text{ that } y \in \times_{t \in T} \Gamma_t \in \{\text{Bars} \in T^2_0 F_t\}
\]

\[
= \int_{\tilde{O}_{t_0}} [\tilde{H}_{t_0}((\times_{\omega\in T}X_t) \times (\times_{t \in T}Y_t))] (\omega) \nu_0(\omega) d\omega \\
= \int_{\Omega_{t_0}} [\tilde{H}_{t_0}((\times_{\omega\in T}X_t) \times (\times_{t \in T}Y_t))] (\omega) \nu_0(\omega) d\omega
\]

\((\forall \Gamma_t \in \text{Bars}, t \in T)\).
Note that we can regard that $P_{\times_{t\in T} \Xi_t} \in M^m_{+1}(X_{t\in T} \Omega_t) \subseteq C(X_{t\in T} \Omega_t)$. That is, there uniquely exists $\nu^2_\psi \in M^m_{+1}(X_{t\in T} \Omega_t)$ such that

$$P_{\times_{t\in T} \Xi_t} = \int_{\times_{t\in T} \Omega_t} [\bigotimes_{t\in T} G_t(\Gamma_t)](\omega) \cdot \nu^2_\psi(\omega) (d\omega)$$

(36)

for any observable $(Y_t, G_t, G_t)$ in $C(\Omega_t)$ ($t \in T$). Here, we used the following notation:

$$[\bigotimes_{t\in T} G_t(\Gamma_t)](\omega) = \times_{t\in T} [G_t(\Gamma_t)](\omega_t)$$

(\forall \omega = (\omega_t)_{t\in T} \in \times_{t\in T} \Omega_t).

Define the observable $\tilde{O}_{\psi_0} \equiv (X_{t\in T} X_t, \bigotimes_{t\in T} F_t, \tilde{F}_{\psi_0})$ such that

$$\tilde{F}_{\psi_0}(\times_{t\in T} \Xi_t) = \tilde{H}_{\psi_0}((\times_{t\in T} \Xi_t) \times (\times_{t\in T} Y_t)).$$

Then, we can define the Bayes operator $[B_{\tilde{O}_{\psi_0}}(X_{t\in T} \Xi_t)] : M^m_{+1}(\Omega_{t\in T}) \rightarrow M^m_{+1}(X_{t\in T} \Omega_t)$ by (36).

Thus, as the generalization of Theorem 2, we have:

**Theorem 3** [Causal Bayes’ theorem in classical measurements, cf. [7]]. Let $t_0$ be the root of a tree $T$. Let $[\Omega_T] = \{[\bigcup_{t\in T} \{X_{t\in T}, F_t, F_t\}]_{t\in T}, \{\Phi_{t_1, t_2} : C(\Omega_{t_2}) \rightarrow C(\Omega_{t_1})\}_{(t_1, t_2)\in T^2} \}$ be a sequential causal observable with the realization $\tilde{O}_{\psi_0} \equiv (X_{t\in T} X_t, \bigotimes_{t\in T} F_t, \tilde{F}_{\psi_0})$. Thus the Bayes operator $[B_{\tilde{O}_{\psi_0}}(X_{t\in T} \Xi_t)] : M^m_{+1}(\Omega_{t\in T}) \rightarrow M^m_{+1}(X_{t\in T} \Omega_t)$ is given by (36).

**Example 2** [The simple case such that $T = \{0, 1, 2\}$]. Consider a particular case such that $T = \{0, 1, 2\}$ is a series ordered set, i.e., $\pi(t) = t - 1$ ($\forall t \in T \setminus \{0\}$). And consider a causal relation $\{C(\Omega_t) \Phi_{t_1, t_2} C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}}$, that is,

$$C(\Omega_0) \Phi_{0,1} C(\Omega_1) \Phi_{1,2} C(\Omega_2).$$

Further consider sequential causal observable $[\Omega_T] = \{[\bigcup_{t\in T} \{X_{t\in T}, \Phi_{t, \pi(t)} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})\}_{t \in T \setminus \{0\}} \}$.

Let $\tilde{O}_0 \equiv (X_{t\in T} X_t, \bigotimes_{t\in T} F_t, \tilde{F}_0)$ be its realization. Note, by the formula (4), that,

$$\tilde{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2) = \Phi_{0,1}(F_0(\Xi_0))(\Phi_{1,2}(F_1(\Xi_1))(\Phi_{1,2}(F_2(\Xi_2))))$$

$$(\forall \Xi_t \in F_t(\forall t \in T)).$$

Putting $K = \{\nu_0\}$, we have the measurement

$$M_{C(\Omega_0)}(\tilde{O}_0) = (X_{t\in T} X_t, \bigotimes_{t\in T} F_t, \tilde{F}_0, S_{[\{\nu_0\}]}, (\nu_0)).$$

(37)

Let $\nu^2_\psi(\in M^m_{+1}(\Omega_0 \times \Omega_1 \times \Omega_2))$ be the posttest state in (V), that is, $\nu^2_\psi = [B_{\tilde{O}_0}(X_{t\in T} \Xi_t)](\nu_0)$. Define $\nu^2_{\psi(1)}(\in M^m_{+1}(\Omega_1))$ such that

$$\nu^2_{\psi(1)}(D_1) = \nu^2_\psi(\Omega_0 \times \Omega_1 \times \Omega_2) \quad (\forall D_1 \in B_{\Omega_1}).$$

Then, we see that

$$\nu^2_{\psi(1)} = \frac{(F_1(\Xi_1)(\Phi_{1,2}(F_2(\Xi_2)))(\Phi_{0,1}(F_0(\Xi_0))\nu_0))}{\nu_0(F_0(\Xi_0)\Phi_{0,1}(F_1(\Xi_1))\Phi_{1,2}(F_2(\Xi_2)))}.$$  

That is because we see that, for any observable $(Y_1, G_1, G_1)$ in $C(\Omega_1),$

$$\nu^2_{\psi(1)} = \frac{\nu_0(F_0(\Xi_0)\Phi_{0,1}(F_1(\Xi_1))\Phi_{1,2}(F_2(\Xi_2)))}{\nu_0(F_0(\Xi_0)\Phi_{1,2}(F_2(\Xi_2)))}.$$  

(38)

**Example 3** [Continued from the above example]. For each $t = 1, 2$, assume that $\Phi_{\pi(t), t} : C(\Omega_t) \rightarrow C(\Omega_{\pi(t)})$ is deterministic, that is, there exists a continuous function $\phi_{\pi(t), t} : \Xi_{\pi(t)} \rightarrow \Omega_t$ satisfying (28). And, putting $K = \{\delta_{\omega_0}\}$, consider the measurement
Thus, we see that
\[ \nu_{1}^{a} = \delta_{\phi_{0,1}(\omega_{0})} \]  

Further easily see that \( \nu_{2}^{a} = [B_{\delta_{0}}(X_{t \in T} \Xi)](\omega_{0}) = \delta_{(\omega_{0},\phi_{0,1}(\omega_{0}),\phi_{0,2}(\omega_{0}))} \in \mathcal{M}_{+1}^{\mathbb{R}}(\Omega_{0} \times \Omega_{1} \times \Omega_{2}) \).

### 6.2 Causal Fisher-Bayes Method in Classical C(Ω)

Now we can present Corollary 4 (i.e., regression analysis in a wide sense) as follows.

\[
(W_{1}) \quad [\text{Corollary } 4] = [\text{Theorem } 1] + [\text{Theorem } 3] \\
(W_{2}) \quad [\text{Theorem } 1] + \top \text{\large (\text{Bayes' method})} \top \\
(W_{3}) \quad \top \text{\large (\text{Fisher's method})} \top
\]

**Corollary 4** [Causal Fisher-Bayes method (i.e., regression analysis in a wide sense), cf. [7].]

Let \( t_{0} \) be the root of a tree \( T \). Let \( \mathcal{O}_{T} = \{ \mathcal{O}_{t} = (X_{t}, F_{t}, F_{t}) \mid t \in T \}, \{ \Phi_{t_{1}, t_{2}} : C(\Omega_{t_{2}}) \to C(\Omega_{t_{1}}) \} \}_{t_{1}, t_{2} \in T} \] be a sequential causal observable with the realization \( \hat{O}_{t_{0}} = (X_{t \in T}, \Xi_{t \in T}, F_{t}, F_{t}) \). Assume the statistical measurement \( M_{C(\Omega_{t_{0}})}(\hat{O}_{t_{0}}, S_{[t]}(K)) \). And assume that we know that a measured value obtained by the measurement \( M_{C(\Omega_{t_{0}})}(\hat{O}_{t_{0}}, S_{[t]}(K)) \) belongs to \( X_{t \in T} \Xi_{t} \). Then, there is a reason to infer that the mixed state \( \nu_{2}^{a}(\mathbb{R}) \in \mathcal{M}_{+1}^{\mathbb{R}}(X_{t \in T} \Omega_{t}) \) after the measurement \( M_{C(\Omega_{t_{0}})}(\hat{O}_{t_{0}}, S_{[t]}(K)) \) is given by \( [B_{\delta_{0}}(X_{t \in T} \Xi)](\nu_{0}) \).

Here, the \( \nu_{0}(\mathbb{R}) \) is defined by

\[
\int_{\Omega} [F_{t_{0}}(X \Xi)](\omega)\nu_{0}(d\omega) = \max_{\nu \in \mathbb{R}} \int_{\Omega} [F_{t_{0}}(X \Xi)](\omega)\nu(d\omega). \tag{40}
\]

**Remark 10.** Note that Fisher maximum likelihood method and Bayes’ theorem are hidden in Corollary 4. That is, Corollary 4 includes the following procedure:

\[
\mathcal{M}_{C(\Omega_{t_{0}})}(\hat{O}_{t_{0}}) = (X_{t \in T} X, \Xi_{t \in T} F, \tilde{F}_{t_{0}}), S_{[t]}(\delta_{\omega_{0}})). \tag{W_{2}}
\]

Then, we see, by (38), that for any \( g_{1} \in \mathcal{C}(\Omega_{1}) \),

\[
\langle \nu_{1}^{a}, g_{1} \rangle = \langle \delta_{\omega_{0}}, F_{0}(\Xi)(\phi_{0,1} g_{1} \Phi_{1,2}(F_{2}(\Xi))) \rangle
\]

\[
= \frac{\langle \delta_{\omega_{0}}, F_{0}(\Xi)(\phi_{0,1} g_{1} \Phi_{1,2}(F_{2}(\Xi))) \rangle}{\langle F_{0}(\Xi)(\phi_{0,1}) \Phi_{1,2}(F_{2}(\Xi)) \rangle} \langle F_{0}(\Xi)(\phi_{0,1}) \Phi_{1,2}(F_{2}(\Xi)) \rangle \phi(0, \omega_{0})
\]

\[
= g_{1}(\phi_{0,1}(\omega_{0})) = \langle \delta_{\phi_{0,1}(\omega_{0})}, g_{1} \rangle.
\]

Thus, we see that

\[
\nu_{1}^{a} = \delta_{\phi_{0,1}(\omega_{0})}, \tag{39}
\]

which is the generalization of the (K).

**Answer 1** [Answer to Problem 1 (S)]. Now we can answer Problem 1 (S) as follows. The (33) says that \( \nu_{0} = \delta_{(\alpha_{0}, \beta_{0})} = \delta_{(0.4, 1.4)} \). Thus, using (39), we see that \( \nu_{1}^{a} = \delta_{(\alpha_{0}+\beta_{0})} = \delta_{(1.8)} \). Also, note that (33) and (39) are consequences of Corollary 4. Hence, the calculation (34) is justified by Corollary 4.

### 7 Conclusions

It is a matter of course that our problem (A₃) (i.e., What is statistics?) is most fundamental in science. Thus, there is every reason to consider that, in order to solve this problem, we have to start from “worldview”. Hence, in this paper we started from the linguistic worldview.

Most scientists may be skeptical about traditional and modern philosophy unless they know the linguistic worldview (A₂) in Figure 1. Thus, we believe that Figure 1 (particularly, 3 and 5) implies and realizes the end of grand narratives (i.e., the 3000 years’ final answer to the worldview problem). If it be so, we can assert that our proposal in this paper is decisive and final. That is, we assert that

\( (X) \) to do science (other than physics) is to describe every phenomenon by quantum language,

which may be also regarded as the answer to the question: “What is science?”

We hope that our proposal (i.e., the quantum linguistic formulation of statistics) will be examined from various points of view.

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