Dual pairs and infinite dimensional Lie algebras

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Abstract. We construct and study various dual pairs between finite dimensional classical Lie groups and infinite dimensional Lie algebras in some Fock representations. The infinite dimensional Lie algebras here can be either a completed infinite rank affine Lie algebra, the $\mathcal{W}_{1+\infty}$ algebra or its certain Lie subalgebras. We give a formulation in the framework of vertex algebras. We also formulate several conjectures and open problems.

Introduction

The theory of dual pairs of R. Howe [H1, H2] has many important applications to representation theory of reductive groups and $p$-adic groups etc. A dual pair is, roughly speaking, a pair of “maximally” commuting (reductive) Lie groups/algebras acting on a certain minimal module. The theory of dual pairs also offers new insights to various subjects such as invariant theory, spherical harmonics, tensor product problems and so on.

The aim of this paper is to give a somewhat informal discussion of some of our recent results [W] [KWY] (also see [FKRW]) generalizing dual pairs to an infinite dimensional setting and to formulate some conjectures and open problems. The paper is mostly expository except the last section which contains some new points.

A dual pair in our infinite dimensional setting consists of a finite dimensional classical reductive Lie group and an infinite dimensional Lie algebra acting on some Fock representations of free fermions or bosons. In [W], we constructed various dual pairs involving Lie groups $O_n$, $S_{p_{2n}}$, $Spin_n$ and Lie supergroup $Osp_{1,2n}$ on one hand, and Lie subalgebras of a completed infinite rank affine algebra $\hat{gl}_\infty$ of $B, C, D$ types on the other hand. We obtained isotypic decompositions with respect to dual pair actions and found explicit formulas for highest weight vectors in all cases. In the hindsight, a duality of this sort between a finite dimensional general linear group and $\hat{gl}_\infty$ was first given in [F2], although heavy machinery rather than the principles of dual pairs was used there.

It turns out that dual pairs in our infinite dimensional setting exhibit a new phenomenon which is not seen in the finite dimensional cases [FKRW] [KWY]: a natural Lie algebra may replace another natural one acting on a same minimal
module with its dual pair partner fixed. Let us denote by \( \hat{D} \) the universal central extension of the Lie algebra of differential operators on the circle. In the literature \( \hat{D} \) is often denoted by \( \mathcal{W}_{1+\infty} \) as well thanks to its connections to \( \mathcal{W} \)-algebras [Ba, FeF]. There is an intimate relation between modules of \( \hat{gl}_\infty \) and of \( \hat{D} \) via a homomorphism \( \phi \) from \( \hat{D} \) to \( \hat{gl}_\infty \). It is realized [FKR] that \( GLl \) and \( \hat{D} \) form a dual pair acting on the Fock space of \( l \) pairs of free fermions by replacing \( \hat{gl}_\infty \) by \( \hat{D} \).

While \( \hat{gl}_\infty \) is easier to understand because of its similarity with finite dimensional general linear Lie algebras, \( \hat{D} \) is a more natural object of study from the viewpoint of vertex algebras [B, FLM, K].

In view of the homomorphism \( \phi \) from \( \hat{D} \) to \( \hat{gl}_\infty \), it is natural to study Lie subalgebras of \( \hat{D} \) corresponding via \( \phi \) to Lie subalgebras of \( \hat{gl}_\infty \) of \( B, C, D \) types. We found [KWY] there are two (families of) Lie subalgebras, denoted by \( \hat{D}^{\pm} \), of \( \hat{D} \) fixed by two (families of) anti-involutions of \( \hat{D} \) which do the job. In this way we obtain various dual pairs between classical Lie groups and \( \hat{D}^{\pm} \) by substituting the Lie subalgebras of \( \hat{gl}_\infty \) of \( B, C, D \) types appearing in the dual pairs obtained in [W] with appropriate \( \hat{D}^{+} \) or \( \hat{D}^{-} \). The duality results formulated in terms of \( \hat{D} \) and \( \hat{D}^{\pm} \) admit a natural reformulation in the framework of vertex algebras.

The length and complication of [W] and [KWY] are to a large extent caused by the fact that we had to deal with a dozen of different dual pairs. In order to simplify matters and to focus on the main ideas, we restrict ourselves in this paper to the consideration of dual pairs of \( (A,A) \) and \( (D,D) \) types in detail and comment on how other dual pairs can be constructed while omitting almost all proofs. In the last section we work out some low rank cases explicitly and show how these recapture and shed new light on some known results [Ka, DG, DN]. We discuss several open problems and conjectures in the end.

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1. Lie algebras \( \hat{gl}_\infty \) and \( d_\infty \)

1.1. Lie algebra \( \hat{gl}_\infty \). We denote by \( gl \) the Lie algebra of all matrices \((a_{ij})_{i,j}\in\mathbb{Z}\) such that \( a_{ij} = 0 \) for \( |i-j| \) large enough. Denote by \( E_{ij} \) the infinite matrix with 1 at the \((i,j)\)-th place and 0 elsewhere. Let the weight of \( E_{ij} \) be \( j - i \). This defines a \( \mathbb{Z} \)-principal gradation \( gl = \bigoplus_{j\in\mathbb{Z}} gl_j \). Denote by \( \hat{gl}_\infty = gl \bigoplus \mathbb{C}C \) the central extension given by the following 2-cocycle [DJM, KP]:

\[
C(A, B) = \text{Tr} ([J, A]B)
\]

where \( J = \sum_{i\leq 0} E_{ii} \).

The \( \mathbb{Z} \)-gradation of Lie algebra \( gl \) extends to \( \hat{gl}_\infty \) by putting weight \( C \) to be 0. In particular, we have a triangular decomposition

\[
\hat{gl}_\infty = \hat{gl}_+ \bigoplus \hat{gl}_0 \bigoplus \hat{gl}_-
\]

where

\[
\hat{gl}_\pm = \bigoplus_{j\in\mathbb{N}} \hat{gl}_{\pm j}, \quad \hat{gl}_0 = gl_0 \oplus \mathbb{C}C.
\]
Denote by \( \epsilon_i \) the linear function on \( \mathfrak{gl}_0 \), s.t. \( \epsilon_i(E_{jj}) = \delta_{ij} (i, j \in \mathbb{Z}) \). Then the root system of \( \mathfrak{gl} \) is \( \Delta = \{ \epsilon_i - \epsilon_j, i, j \in \mathbb{Z}, i \neq j \} \). The compact anti-involution \( \omega \) is defined by \( \omega(E_{ij}) = E_{ji} \).

Given \( c \in \mathbb{C} \) and \( \Lambda \) in the restricted dual \( \hat{\mathfrak{gl}}_0^* \) of \( \mathfrak{gl}_0 \), we let

\[
\begin{align*}
^a\lambda_i &= \Lambda(E_{ii}), & i \in \mathbb{Z}, \\
^aH_i &= E_{ii} - E_{i+1,i+1} + \delta_{i,0}C, \\
^ah_i &= \Lambda(^aH_i) = \lambda_i - \lambda_{i+1} + \delta_{i,0}c.
\end{align*}
\]

The superscript \( a \) here denotes \( \hat{\mathfrak{gl}}_\infty \) which is of \( \Lambda \) type.

Denote by \( L(\hat{\mathfrak{gl}}_\infty; \Lambda, c) \) (or simply \( L(\hat{\mathfrak{gl}}_\infty; \Lambda) \) when the central charge is obvious from the text) the \( \mathfrak{gl}_\infty \)-module with highest weight \( \Lambda \) and central charge \( c \). It is easy to see that \( L(\hat{\mathfrak{gl}}_\infty; \Lambda, c) \) is quasifinite (namely having finite dimensional graded subspaces according to the principal gradation of \( \hat{\mathfrak{gl}}_\infty \)) if and only if all but finitely many \( h_i, i \in \mathbb{Z} \) are zero. A quasifinite representation of \( \mathfrak{gl}_\infty \) is unitary if an Hermitian form naturally induced by \( \omega \) is positive definite.

Define \( ^a\Lambda_j \in \mathfrak{gl}_0^* \) as follows:

\[
(1.2) \quad ^a\Lambda_j(E_{ii}) = \begin{cases} 
1, & \text{for } 0 < i \leq j \\
-1, & \text{for } j < i \leq 0 \\
0, & \text{otherwise}.
\end{cases}
\]

Define \( ^a\Lambda_0 \in \hat{\mathfrak{gl}}_0^* \) by

\[
^a\Lambda_0(C) = 1, \quad ^a\Lambda_0(E_{ii}) = 0 \quad \text{for all } i \in \mathbb{Z}
\]

and extend \( ^a\Lambda_j \) from \( \mathfrak{gl}_0^* \) to \( \hat{\mathfrak{gl}}_0^* \) by letting \( ^a\Lambda_j(C) = 0 \). Then

\[
^a\Lambda_j = ^a\Lambda_j + ^a\Lambda_0, \quad j \in \mathbb{Z}
\]

are the fundamental weights, i.e. \( ^a\Lambda_j(H_i) = \delta_{ij} \).

One can prove that \( L(\hat{\mathfrak{gl}}_\infty; \Lambda, c) \) is unitary if and only if \( \Lambda = ^a\Lambda_{m_1} + \ldots + ^a\Lambda_{m_k} \) for some \( m_1, \ldots, m_k \in \mathbb{Z} \), and \( c = k \in \mathbb{Z}_+ \).

### 1.2. Lie algebra \( d_\infty \)

Let us consider the infinite dimensional vector space \( \mathbb{C}[t, t^{-1}] \) and take a basis \( v_i = t^{-i}, i \in \mathbb{Z} \). The Lie algebra \( \mathfrak{gl} \) acts on this vector space naturally, namely \( E_{ij}v_k = \delta_{jk}v_i \). We denote by \( \overline{\mathfrak{d}}_\infty \) the Lie subalgebra of \( \mathfrak{gl} \) preserving the symmetric bilinear form \( D(v_i, v_j) = \delta_{i,1-j}, i, j \in \mathbb{Z} \). Namely we have

\[
\overline{\mathfrak{d}}_\infty = \{ g \in \mathfrak{gl} \mid D(a(u), v) + D(u, a(v)) = 0 \} = \{ (a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -a_{1-j,1-i} \}.
\]

Denote by \( d_\infty = \overline{\mathfrak{d}}_\infty \oplus \mathbb{C}C \) the central extension given by the 2-cocycle (1.1) restricted to \( \overline{\mathfrak{d}}_\infty \). Then \( d_\infty \) has a natural triangular decomposition induced from \( \hat{\mathfrak{gl}}_\infty : d_\infty = d_\infty^+ \oplus d_\infty^0 \oplus d_\infty^- \).

The set of simple coroots of \( d_\infty \), denoted by \( \Pi^\vee \) can be described as follows:

\[
\Pi^\vee = \{ \alpha_i^\vee = E_{00} + E_{-1,-1} - E_{2,2} - E_{1,1} + C, \quad \alpha_i^\wedge = E_{i,i} + E_{-i,-i} - E_{i+1,i+1} - E_{1-i,1-i}, i \in \mathbb{N} \}.
\]
Given $\Lambda \in d^*_\infty$, we let
\[
  \begin{align*}
  d\lambda_i &= \Lambda(E_{ii} - E_{1-i,1-i}) \quad (i \in \mathbb{N}), \\
  dH_i &= E_{ii} + E_{-i,i} - E_{i+1,i+1} + E_{1-i,1-i} \quad (i \in \mathbb{N}), \\
  dh &= \Lambda(dH_i) = \lambda_i - \lambda_{i+1} \quad (i \in \mathbb{N}), \\
  dH_0 &= E_{0,0} - E_{-1,-1} - E_{2,2} - E_{1,1} + 2C, \\
  c &= \frac{1}{2}(dh_0 + dh_1) + \sum_{i \geq 2} dh_i.
  \end{align*}
\]
The superscript $d$ denotes $d^*_\infty$ which is of $D$ type. Then we denote by $d\Lambda_i$, the $i$-th fundamental weight of $d^*_\infty$, namely $d\Lambda_i(dH_j) = \delta_{ij}$. Denote by $L(d^*_\infty; \Lambda, c)$ or simply $L(d^*_\infty; \Lambda)$ the $d^*_\infty$-module with highest weight $\Lambda$ and central charge $c$. We denote again by $\omega$ the anti-involution on $d^*_\infty$ induced from $\omega$ on $\hat{gl}_\infty$ by restriction. A representation is unitary if an Hermitian form naturally defined with respect to the anti-involution $\omega$ on $d^*_\infty$ is positive definite. $L(d^*_\infty, \Lambda)$ is unitary if and only if $\Lambda = d\Lambda_{m_1} + \ldots + d\Lambda_{m_k}$, for some $m_1, \ldots, m_k \in \mathbb{Z}_+$ and $c = \frac{1}{2}k \in \frac{1}{2}\mathbb{Z}_+$.

2. Lie algebras $\hat{D}$ and $\hat{D}^\pm$

2.1. Lie algebra $\mathcal{D}$. Let $\mathcal{D}_{as}$ be the associative algebra of differential operators on the circle. $\mathcal{D}_{as}$ admits two distinguished bases: one is given by
\[
  J^l_k = -t^{l+k}\partial_l^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z})
\]
where $\partial_l$ denotes $\frac{d}{dt}$, the other is given by
\[
  L^l_k = -t^{k+1}D^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z})
\]
where $D = t\partial_t$. It is easy to see that $J^l_k = -t^k[D]_l$. Here and further we use the notation $[x]_l = x(x-1)\ldots(x-l+1)$.

Let $\mathcal{D}$ denote the Lie algebra obtained from $\mathcal{D}_{as}$ by taking the usual Lie bracket. Denote by $\mathcal{D}$ the central extension of $\mathcal{D}$ by a one-dimensional center with a generator $C$: $\mathcal{D} = \mathcal{D} + \mathbb{C}C$. The Lie algebra $\mathcal{D}$ has the following commutation relations (KR)
\[
  [t^r f(D), t^s g(D)] = t^{r+s} (f(D+s)g(D) - f(D)g(D+r)) + \Psi(t^r f(D), t^s g(D))C
\]
where
\[
  \Psi(t^r f(D), t^s g(D)) = \begin{cases}
    \sum_{-r \leq j \leq -1} f(j)g(j+r), & r = -s \geq 0 \\
    0, & r+s \neq 0.
  \end{cases}
\]

Set the weight of $J^l_k$ to be $k$ and the weight of $C$ $0$. This defines the principal $\mathbb{Z}$-gradations of $\mathcal{D}_{as}, \mathcal{D}$ and $\mathcal{D}$:
\[
  \mathcal{D} = \bigoplus_{j \in \mathbb{Z}} \mathcal{D}_j, \quad \hat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathcal{D}}_j.
\]
Thus we have the triangular decomposition
\[
  \mathcal{D} = \mathcal{D}_+ \bigoplus \mathcal{D}_0 \bigoplus \mathcal{D}_-,
\]
where
\[
  \hat{\mathcal{D}}_{\pm} = \bigoplus_{j \in \pm \mathbb{N}} \hat{\mathcal{D}}_j, \quad \hat{\mathcal{D}}_0 = \mathcal{D}_0 \bigoplus \mathbb{C}C.
\]
By abuse of notation, we use again $J_k^0, L_k^1$ to denote the corresponding elements in $\hat{D}$. The elements $J_m^0, m \in \mathbb{Z}$ span a Heisenberg algebra:

$$[J_m^0, J_n^0] = \delta_{m,-n}mc.$$

The elements $J_m^1, m \in \mathbb{Z}$ span a Virasoro algebra:

$$[J_m^1, J_n^1] = (m-n)J_{m+n}^1 - \delta_{m,-n}\frac{m^3-m}{6}c.$$

### 2.2. Lie algebras $\mathcal{D}^\pm$

Recall that an anti-involution $\sigma$ of $\mathcal{D}_{as}$ is an involutive anti-automorphism of $\mathcal{D}$, i.e. $\sigma^2 = I, \sigma(aX + bY) = a\sigma(X) + b\sigma(Y)$ and $\sigma(XY) = \sigma(Y)\sigma(X)$, where $a, b \in \mathbb{C}, X, Y \in \mathcal{D}$.

**Proposition 2.1.** Any anti-involution $\sigma$ of $\mathcal{D}_{as}$ preserving the principal $Z$-gradation is one of the following:

1. $\sigma_{-b}(t) = -t$, $\sigma_{-b}(D) = -D + b$;
2. $\sigma_{+b}(t) = t$, $\sigma_{+b}(D) = -D + b$, $b \in \mathbb{C}$.

It follows immediately that $\sigma_{\pm b}((\partial t) = \mp (\partial t - (b + 1)t^{-1})$. Given $s \in \mathbb{C}$, denote by $\Theta_s$ the automorphism of $\mathcal{D}_{as}$ which sends $t$ to $t$ and $D$ to $D + s$. Equivalently $\Theta_s$ is given by sending $a \in \mathcal{D}$ to $t^{-s}at^s$, the conjugate of $a$ by $t^s$. Clearly $\Theta_s$ preserves the principal $Z$-gradation of $\mathcal{D}_{as}$. We have

$$\sigma_{\pm b} \circ \Theta_s = \sigma_{\pm b + s}, \quad \Theta_{-s} \circ \sigma_{\pm b} = \sigma_{\pm b + s}.$$ 

Denote by $\mathcal{D}^{\pm b}$ the Lie subalgebra of $\mathcal{D}$ fixed by $-\sigma_{\pm b}$, namely

$$\mathcal{D}^{\pm b} = \{ a \in \mathcal{D} \mid \sigma_{\pm b}(a) = -a \}.$$ 

Since $\sigma_{\pm b}$ preserves the principal $Z$-gradation of $\mathcal{D}$ there is an induced $Z$-gradation on $\mathcal{D}^{\pm b}$: $\mathcal{D}^{\pm b} = \bigoplus_{j \in \mathbb{Z}} \mathcal{D}_j^{\pm b}$, where $\mathcal{D}_j^{\pm b} = \mathcal{D}^{\pm b} \cap \mathcal{D}_j$. The relation among $\mathcal{D}^{\pm b}$ for different $b \in \mathbb{C}$ can be described as follows.

**Lemma 2.2.** The Lie algebras $\mathcal{D}^{\pm b}$ (resp. $\mathcal{D}^{-2b}$) for different $b \in \mathbb{C}$ are all isomorphic. More precisely, we have $\Theta_s(\mathcal{D}^{\pm b}) = \mathcal{D}^{\pm b - 2s}$.

Thanks to Lemma 2.2 we may fix $b$ in $\mathcal{D}^{\pm b}$. It turns out there is a best choice by putting $b = -1$ (partly because $\sigma_{\pm 1}(\partial t) = \mp \partial t$). From now on we set $\mathcal{D}^+ = \mathcal{D}^{+, -1}$ and $\mathcal{D}^- = \mathcal{D}^{-, -1}$ (note that our $\mathcal{D}^-$ is different from the one in [KWY]). Accordingly we set $\mathcal{D}_j^+ = \mathcal{D}_j^{+, -1}$ and so $\mathcal{D}^+ = \sum_{j \in \mathbb{Z}} \mathcal{D}_j^+$. We shall see that there is a canonical vertex algebra structure on the vacuum module of the central extension of $\mathcal{D}^+$.

Let us denote by $\mathbb{C}[w]^{(0)}$ (resp. $\mathbb{C}[w]^{(1)}$) the set of all even (resp. odd) polynomials in $\mathbb{C}[w]$. We let $\overline{k} = 0$ if $k$ is an odd integer and $\overline{k} = 1$ if $k$ is even.

**Lemma 2.3.** We have

$$\mathcal{D}_j^+ = \left\{ t^j g(D + (j + 1)/2) \mid g(w) \in \mathbb{C}[w]^{(1)} \right\},$$

$$\mathcal{D}_j^- = \left\{ t^j g(D + (j + 1)/2) \mid g(w) \in \mathbb{C}[w]^{(j)} \right\}, \quad j \in \mathbb{Z}.$$

We will concentrate on the Lie algebra $\mathcal{D}^+$. The story for $\mathcal{D}^-$ is quite similar in any event. We set

$$W^n_k = -\frac{1}{2} t^k ([D]_n - [-D - k - 1]_n) \quad (k \in \mathbb{Z}, n \in \mathbb{N}).$$
One can check that $W_k^n \ (k \in \mathbb{Z}, n \in \mathbb{N}_{odd} = \{1, 3, 5, \ldots \})$ form a basis of $D^+$. By abuse of notation we again denote by $\Psi$ the restriction of the 2-cocycle $\Psi$ to $D^+$. Denote by $\hat{D}^+$ the central extension of $D^+$ by $C C$ corresponding to $\Psi$. The Lie algebra $\hat{D}^+$ is a subalgebra of $\hat{D}$ by definition. Note that $W_k^n \ (k \in \mathbb{Z})$ span a Virasoro algebra, namely we have

$$[W_m^1, W_n^1] = (m - n)W_{m+n} + \delta_{m, -n} \frac{m^3 - m}{12} C.$$ 

**Remark** 2.4. We observe from Lemma 2.3 that $\hat{D}^+_j$ coincides with $\hat{D}^-_j$ for $j$ even. The relation between $\hat{D}^+$ and $\hat{D}^-$ is analogous to the relation between untwisted affine algebras and twisted affine algebras (it is more than an analog!).

### 3. Quasifinite modules of $\hat{D}$ and $\hat{D}^+$ and a link to $\hat{g}l_\infty$

#### 3.1. Quasifinite modules of $\hat{D}$ and $\hat{D}^+$

Let $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a $\mathbb{Z}$-graded Lie algebra satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Let $\mathfrak{g}_\pm = \oplus_{j \geq 0} \mathfrak{g}_{\pm j}$. A $\mathfrak{g}$-module $V$ is called $\mathbb{Z}$-graded if $V = \oplus_{j \in \mathbb{Z}} V_j$ and $\mathfrak{g}_j V_j \subset V_{j-1}$. A graded $\mathfrak{g}$-module is called quasifinite if $\dim V_j < \infty$ for all $j$.

Since $\mathfrak{g}$ carries a triangular decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$, we can construct as usual a Verma module $M(\mathfrak{g}; \Lambda)$ with a highest weight vector annihilated by $\mathfrak{g}_+$ and highest weight $\Lambda \in \mathfrak{g}_0^\ast$. Denote by $L(\mathfrak{g}; \Lambda)$ the irreducible quotient of $M(\mathfrak{g}; \Lambda)$ by the maximal proper graded submodule.

We specialize $\mathfrak{g}$ to $\hat{D}$ and $\hat{D}^+$ from now on. Since the Cartan subalgebras of $\hat{D}$ and $\hat{D}^+$ are infinite dimensional, the classifications of quasifinite highest weight $\hat{D}$-modules and $\hat{D}^+$-modules turn out to be quite non-trivial.

For a given $\Lambda \in (\mathcal{D}_0^\ast)^+$, we characterize $\Lambda$ by its labels $\Delta_n = -\Lambda(D^n) \ (n \in \mathbb{Z}_+)$ and the central charge $c = \Lambda(C)$. Similarly for a given $\Lambda \in (\mathcal{D}_0^\ast)^-$, we characterize $\Lambda$ by its labels $\Delta_n^+ = -\Lambda((D+1/2)^n) \ (n \in \mathbb{N}_{odd})$ and the central charge $c = \Lambda(C)$.

We introduce the generating functions

$$\Delta_\Lambda(x) = \sum_{n \geq 0} \frac{x^n}{n!} \Delta_n, \quad \Delta_\Lambda^+(x) = \sum_{n \in \mathbb{N}_{odd}} \frac{x^n}{n!} \Delta_n^+.$$ 

We call $\Delta_\Lambda(x)$, resp. $\Delta_\Lambda^+(x)$, the characteristic generating function of $L(\hat{D}; \Lambda)$, resp. of $L(\hat{D}^+; \Lambda)$. Sometimes we simply drop the subscript $\Lambda$. Clearly we have

$$\Delta(x) = -\Lambda \left( e^{xD} \right), \tag{3.1}$$

$$\Delta^+(x) = -\frac{1}{2} \Lambda \left( e^{xD+1/2} - e^{-x(D+1/2)} \right). \tag{3.2}$$

A quasipolynomial is a finite linear combination of functions of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha \in \mathbb{C}$. The following beautiful characterization of quasifiniteness of an irreducible module $L(\hat{D}; \Lambda)$ is due to Kac and Radul [KR].

**Theorem 3.1.** A $\hat{D}$-module $L(\hat{D}; \Lambda)$ is quasifinite if and only if

$$\Delta(x) = \frac{F(x)}{e^x - 1}$$

where $F(x)$ is a quasipolynomial such that $F(0) = 0$. 


Given an irreducible quasifinite highest weight \( \hat{D} \)-module \( V \) with central charge \( c \in \mathbb{C} \) and with characteristic generating function \( \Delta(x) \), we write \( F(x) + c \) as a finite sum of the form \( \sum_i p_i(x)e^{s_i x} \), where \( p_i(x) \) are non-zero polynomials satisfying \( \sum_i p_i(0) = c \) and \( s_i \) are distinct complex numbers.

We call \( s_i \) the exponents of \( V \) with multiplicities \( p_i(x) \). We single out a class of important quasifinite \( \hat{D} \)-modules called positive primitive modules. \( V \) is called positive primitive if all the multiplicities are positive integers. We observe that \( V \) is uniquely determined by its exponent-multiplicity set \( \mathcal{E} \). We will denote \( V \) by \( L(\hat{D}; \mathcal{E}) \).

The following characterization of quasi-finiteness of an irreducible highest weight \( \hat{D}^+ \)-module is obtained in [KWy].

**Theorem 3.2.** A \( \hat{D}^+ \)-module \( L(\hat{D}^+; \Lambda) \) is quasifinite if and only if

\[
\Delta^+(x) = \frac{F(x)}{2 \sinh(x/2)}
\]

where \( F(x) \) is an even quasipolynomial such that \( F(0) = 0 \).

Given an irreducible quasifinite highest weight \( \hat{D}^+ \)-module \( V \) with central charge \( c \in \mathbb{C} \) and with characteristic generating function \( \Delta^+(x) \), we write \( F(x) + c \) as a finite sum of the form

\[
\sum_i p_i(x) \cosh(e_i^+ x) + \sum_j q_j(x) \sinh(e_j^- x),
\]

where \( p_i(x) \) (resp. \( q_j(x) \)) are non-zero even (resp. odd) polynomials and \( e_i^+ \) (resp. \( e_j^- \)) are distinct complex numbers. Clearly \( \sum_i p_i(0) = c \). The expression (3.3) is unique up to a sign of \( e_i^+ \) or a simultaneous change of signs of \( e_j^- \) and \( q_j(x) \).

We call \( e_i^+ \) (resp. \( e_j^- \)) the even type (resp. odd type) exponents of \( V \) with multiplicities \( p_i(x) \) (resp. \( q_j(x) \)). \( V \) is called positive primitive, if there is no odd type exponents and the multiplicities of its (even type) exponents \( e_i^+ \) are integers \( n_i \) such that \( n_i > 0 \) when \( e_i \neq 0 \) and \( -\frac{1}{2} n_{i_0} \leq n_i \in \frac{1}{2} \mathbb{Z} \) when \( e_i = 0 \). Here \( i_0 \) is the index such that \( e_{i_0} = 1 \). We denote by \( \mathcal{E} \) the set of (even type) nonzero exponents with their multiplicities for a positive primitive module \( V \). The pair \( (\mathcal{E}, c) \) determines uniquely the module \( V \). We will denote this primitive module by \( L(\hat{D}^+; \mathcal{E}, c) \).

**3.2. A homomorphism from \( \hat{D} \) to \( \hat{gl}_{\infty} \).** Let \( \mathcal{O} \) be the algebra of all holomorphic functions on \( \mathbb{C} \) with the topology of uniform convergence on compact sets. Define \( \mathcal{O}^o = \{ f \in \mathcal{O} \mid f(w) = -f(-w) \} \). We define a completion \( \mathcal{D}^o \) of \( \mathcal{D} \) consisting of all differential operators of the form \( \psi f(D) \) where \( f \in \mathcal{O} \) and \( j \in \mathbb{Z} \). This induces a completion \( \mathcal{D}^+; \mathcal{O} \) of \( \mathcal{D}^+ \). Many things defined for \( \mathcal{D} \) and \( \mathcal{D}^+ \), such as 2-cocycle \( \Psi \), commutation relations etc. extend naturally to their completions.

Recall that the Lie algebra \( gl \) acts on the vector space \( \mathbb{C}[t, t^{-1}] \) by \( E_{ij}v_k = \delta_{jk}v_i \), where \( v_k = t^{-k} \). Note that the Lie algebras \( \mathcal{D} \) and \( \mathcal{D}^o \) naturally acts on the same space as differential operators: \( \psi f(D) f^k = f(k) f^{k+j} \). In this way we obtain an algebra embedding \( \phi \) of \( \mathcal{D} \) into \( gl \) given by \( \phi(t^k f(D)) = \sum_{j \in \mathbb{Z}} f(-j) E_{j-k,j} \). We remark that \( \phi \) extending to \( \mathcal{D}^o \) becomes a surjective homomorphism.

When restricted to \( \mathcal{D}^+ \) and \( \mathcal{D}^+; \mathcal{O} \), we have

\[
\phi(t^k f(D + (k + 1)/2)) = \sum_{j \in \mathbb{Z}} f(-j + (k + 1)/2) E_{j-k,j}
\]
for \( f \in \mathcal{O}^\circ \). The image of \( \phi : \mathcal{D}^+, \mathcal{O}^\circ \rightarrow \mathfrak{gl} \) can be easily shown to be \( \mathfrak{d}_\infty \).

It turns out that the 2-cocycle \( \Psi(2.1) \) on \( \mathcal{D} \) and \( \mathcal{D}^+ \) and the 2-cocycle given by (1.1) on \( \mathfrak{gl} \) are compatible via the homomorphism \( \phi : \mathcal{D} \rightarrow \mathfrak{gl} \). Thus we can extend \( \phi \) to a homomorphism \( \tilde{\phi} \) from \( \mathcal{D} \) to \( \mathfrak{gl}_\infty \) by letting

\[
\tilde{\phi}(t^k f(D)) = \sum_{j \in \mathbb{Z}} f(-j) E_{j-k,j}, \quad \tilde{\phi}(C) = C.
\]

When restricted to \( \mathcal{D}^+ \), we obtain a homomorphism \( \tilde{\phi} : \mathcal{D}^+ \rightarrow \mathfrak{d}_\infty \).

Note that the \( \mathbb{Z} \)-gradations of \( \mathcal{D} \) and \( \mathcal{D}^+ \) are compatible with those on \( \mathfrak{gl}_\infty \) and \( \mathfrak{d}_\infty \) respectively via \( \tilde{\phi} \). Thus we may regard \( \mathcal{D} \) (resp. \( \mathcal{D}^+ \)) as a dense subalgebra of \( \mathfrak{gl}_\infty \) (resp. \( \mathfrak{d}_\infty \)) via \( \tilde{\phi} \).

**PROPOSITION 3.3.** The pullback of an irreducible quasifinite module over \( \mathfrak{gl}_\infty \) (resp. \( \mathfrak{d}_\infty \)) via the homomorphism \( \tilde{\phi} \) is an irreducible quasifinite module over \( \mathcal{D} \) and (resp. \( \mathcal{D}^+ \)).

4. (A, A) and (D, D) duality

4.1. Some actions on a Fock space \( \mathcal{F}^\otimes l \). Let us take \( l \) pairs of free fermionic fields

\[
\psi^\pm p(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \psi^\pm p_n z^{-n-\frac{1}{2}}.
\]

The anti-commutation relations among \( \psi^\pm p \) are equivalent to the following operator product expansions

\[
\psi^+ p(z) \psi^- q(w) \sim \frac{\delta_{pq}}{z-w}, \quad \psi^+ p(z) \psi^+ q(w) \sim 0, \quad \psi^- p(z) \psi^- q(w) \sim 0,
\]

where \( p, q = 1, \ldots, l \). We denote by \( \mathcal{F}^\otimes l \) the Fock space generated by a vacuum vector \( |0\rangle \) annihilated by \( \psi^\pm p, n > 0, p = 1, \ldots, l \).

Introduce the following generating functions

\[
E(z, w) \equiv \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{j} = \sum_{p=1}^l : \psi^+ p(z) \psi^- p(w) :
\]

\[
e^{pq}_l(z) \equiv \sum_{n \in \mathbb{Z}} e^{pq}_l(n) z^{-n-1} = : \psi^- p(z) \psi^- q(z) :
\]

\[
e^{pq}_{*+}(z) \equiv \sum_{n \in \mathbb{Z}} e^{pq}_{*+}(n) z^{-n-1} = : \psi^+ p(z) \psi^+ q(z) :
\]

\[
e^{pq}_{+*}(z) \equiv \sum_{n \in \mathbb{Z}} e^{pq}_{+*}(n) z^{-n-1} = : \psi^+ p(z) \psi^- q(z) :
\]

where \( p, q = 1, \ldots, l \), and the normal ordering ::: means that the operators annihilating \( |0\rangle \) are moved to the right and multiplied by \(-1\).

It is well known that the operators \( e^{pq}(n), e^{pq}_{*+}(n), e^{pq}_{+*}(n) \), \( p, q = 1, \ldots, l \), \( n \in \mathbb{Z} \) defined as above form a representation of the affine algebra \( \mathfrak{so}_2 \) of level 1. In particular \( e^{pq}_l(n) \), \( p, q = 1, \ldots, l \), \( n \in \mathbb{Z} \) form a representation of the affine algebra \( \mathfrak{gl} \) of level 1. The operators \( e^{pq}(0), e^{pq}_{*+}(0), e^{pq}_{+*}(0) \) \( p, q = 1, \ldots, l \) form the horizontal subalgebra \( \mathfrak{so}_2 \) in \( \mathfrak{so}_2 \) while \( e^{pq}_{-+}(0) \) \( p, q = 1, \ldots, l \) form the horizontal subalgebra \( \mathfrak{gl} \) in \( \mathfrak{gl} \). We identify the Borel subalgebra \( \mathfrak{b}(\mathfrak{so}_2) \) with
the one generated by \(e_{pq}^+(p \neq q), e_{pq}^-(p \leq q), p, q = 1, \ldots, l\). We take the Borel subalgebra of \(\mathfrak{gl}_l\) to be \(\mathfrak{b}(\mathfrak{gl}_l) = \mathfrak{gl}_l \cap \mathfrak{b}(\mathfrak{so}_{2l})\).

It follows from [1] that
\[
\sum_{i,j,k \in \mathbb{Z}} (E_{ij} - E_{1-j,1-i})z^{i-1}w^{-j} = \sum_{k=1}^l \langle \psi^+, k(z) \psi^+, k(w) : - \psi^+, k(w) \psi^+, k(z) : \rangle.
\]

One can show [2] that \(E_{ij}(i, j \in \mathbb{Z})\) defined above span \(\hat{\mathfrak{gl}}_\infty\) and therefore \(E_{ij} - E_{1-j,1-i}(i, j \in \mathbb{Z})\) span \(d_\infty\) with central charge \(l\).

4.2. Duality involving \(\hat{\mathfrak{gl}}_\infty\) and its subalgebras. We observe that the horizontal subalgebra \(\mathfrak{gl}_l\) acting on \(F^\otimes l\) lifts to \(GL_l\). It is not difficult to show that the action of \(\hat{\mathfrak{gl}}_\infty\) commutes with the action of \(GL_l\) on \(F^\otimes l\). We can show that \(\hat{\mathfrak{gl}}_\infty\) and \(GL_l\) form a dual pair acting on \(F^\otimes l\) by a similar argument as in the finite dimensional case [H1].

It is well known that the set \(\Sigma(A)\) of the \(n\)-tuples \((m_1, \ldots, m_l)\), \(m_1 \geq m_2 \geq \ldots \geq m_l, m_i \in \mathbb{Z}\) parametrizes the finite dimensional irreducible \(GL_l\)-modules. We define a map \(\Lambda^\alpha\) from \(\Sigma(A)\) to \(\hat{\mathfrak{gl}}_0\) as follows: given \(\lambda = (m_1, \ldots, m_l)\), we define
\[
\Lambda^\alpha(\lambda) = \sum_{i=1}^l a_{m_i}.
\]

The following duality of \((A, A)\) type is due to I. Frenkel [3] (also see [4]).

**Theorem 4.1** (Frenkel). Under the joint action of \(GL_l\) and \(\hat{\mathfrak{gl}}_\infty\), we have the isotypic decomposition
\[
F^\otimes l = \bigoplus_{\lambda \in \Sigma(A)} V(GL_l; \lambda) \otimes L\left(\hat{\mathfrak{gl}}_\infty; \Lambda^\alpha(\lambda), l\right)
\]
where \(L\left(\hat{\mathfrak{gl}}_\infty; \Lambda^\alpha(\lambda), l\right)\) is the irreducible highest weight \(\hat{\mathfrak{gl}}_\infty\)-module of highest weight \(\Lambda^\alpha(\lambda)\) and central charge \(l\).

In [3] heavy machinery rather than principles of dual pairs was used in establishing the theorem. An alternative simple proof goes as follows, see [5] for detail. It follows directly from the principles of dual pairs that the decomposition is multiplicity-free. We find explicit highest weight vectors and read off the highest weights with respect to the action of \(GL_l\) and \(\hat{\mathfrak{gl}}_\infty\) respectively. The theorem is now established by the simple observation that all highest weights for finite dimensional irreducible \(GL_l\)-modules already show up.

**Remark 4.2.** All unitary irreducible highest weight \(\hat{\mathfrak{gl}}_\infty\)-modules with central charge \(l\) appear in the above decomposition. They are in one-to-one correspondence with all irreducible finite dimensional \(GL_l\)-modules.

We note that the horizontal subalgebra \(\mathfrak{so}_{2l}\) acting on \(F^\otimes l\) lifts to \(SO_{2l}\) and extends to an action of \(O_{2l}\) naturally. One can show (see [5]) that the actions of \(O_{2l}\) and \(d_\infty\) on \(F^\otimes l\) commute and moreover \(d_\infty\) and \(O_{2l}\) form a dual pair.

\(O(2l)\) is a semi-direct product of \(SO(2l)\) by \(\mathbb{Z}/2\mathbb{Z}\). Below we will use a highest weight to denote the corresponding module for short. If \(\lambda\) is a representation
of $SO(2l)$ of highest weight $(m_1, m_2, \ldots, m_l)$ ($m_l \neq 0$), then the induced representation of $\lambda$ to $O(2l)$ is irreducible and its restriction to $SO(2l)$ is a sum of $(m_1, m_2, \ldots, m_l)$ and $(m_1, m_2, \ldots, -m_l)$. We denote this irreducible representation of $O(2l)$ by $(m_1, m_2, \ldots, m_l)$, where $m_l$ is chosen to be greater than 0. If $m_l = 0$, the representation $\lambda = (m_1, m_2, \ldots, m_{l-1}, 0)$ extends to two different representations of $O(2l)$, i.e. $\lambda$ itself and $\lambda \otimes \text{det}$, where $\text{det}$ is the 1-dimensional non-trivial representation of $O(2l)$.

Thus the irreducible modules of $O_{2l}$ are parametrized by the set

$$
\Sigma(D) = \{(m_1, m_2, \ldots, m_l) \mid m_1 \geq m_2 \geq \ldots \geq m_l > 0, m_i \in \mathbb{Z}; \ (m_1, m_2, \ldots, m_{l-1}, 0) \otimes \text{det},
\ (m_1, m_2, \ldots, m_l \geq 0, m_i \in \mathbb{Z}) \}.
$$

We denote by $V(O_{2l}; \lambda)$ the irreducible $O_{2l}$-module corresponding to $\lambda \in \Sigma(D)$ from now on.

We define a map $\Lambda^{\partial} : \Sigma(D) \rightarrow d_{\infty}^*$ by sending $\lambda = (m_1, \ldots, m_l)$ ($m_l > 0$) to

$$
\Lambda^{\partial}(\lambda) = (l - i) \hat{d}_0 \Lambda_0 + (l - i) \hat{d}_1 + \sum_{k=1}^{i} \hat{d}_{m_k},
$$

sending $(m_1, \ldots, m_j, 0, \ldots, 0)$ ($j < l$) to

$$
\Lambda^{\partial}(\lambda) = (2l - i - j) \hat{d}_0 + (j - i) \hat{d}_1 + \sum_{k=1}^{i} \hat{d}_{m_k},
$$

and sending $(m_1, \ldots, m_j, 0, \ldots, 0) \otimes \text{det}$ ($j < l$) to

$$
\Lambda^{\partial}(\lambda) = (j - i) \hat{d}_0 + (2l - i - j) \hat{d}_1 + \sum_{k=1}^{i} \hat{d}_{m_k},
$$

if $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 1 > m_{l+1} = \ldots = m_l = 0$.

We have the following duality of $(D, D)$ type.

**Theorem 4.3.** Under the joint action of $O_{2l}$ and $d_{\infty}$, we have the isotypic decomposition

$$
\mathcal{F} \otimes l = \bigoplus_{\lambda \in \Sigma(D)} V(O_{2l}; \lambda) \otimes L(d_{\infty}; \Lambda^{\partial}(\lambda), l)
$$

where $L(d_{\infty}; \Lambda^{\partial}(\lambda), l)$ is the irreducible $d_{\infty}$-module of highest weight $\Lambda^{\partial}(\lambda)$ and central charge $l$.

**Remark 4.4.** All unitary irreducible highest weight $d_{\infty}$-modules with central charge $l$ appear in the above decomposition. They are in one-to-one correspondence with all irreducible finite dimensional $O_{2l}$-modules.

**Remark 4.5.** We indicate below that there are many variations of Theorems 4.1 and 4.3 which give rise to different dual pairs involving with various Lie groups. See \cite{W} for more detail.

1. If we consider the Fock space of $l$ pairs of fermions together with a neutral fermion in a similar way, we can construct a dual pair between $O_{2l+1}$ and $d_{\infty}$ (with central charge $l + \frac{1}{2}$).
2. If we change the indices for the Fourier components of the fermions \( \psi^{\pm p}(z) \) \( (p = 1, \ldots, l) \) from half-odd-integers to integers, we can construct a dual pair between the pin group \( \text{Pin}_{2l} \) and a Lie subalgebra of \( \hat{gl}_{\infty} \) of type \( B \) acting on the Fock space modified accordingly. If in addition we add a neutral fermion as in the previous case, the spin group \( \text{Spin}(2l + 1) \) will show up instead.

3. If we consider the Fock space of free bosons instead of free fermions, we obtain dual pairs involving the symplectic group \( \text{Sp}_{2n} \) and Lie supergroup \( Osp_{1,2n} \).

4. It turns out that the finite dimensional Lie groups arising in the dual pairs considered so far always arise from lifting of a horizontal subalgebra of an appropriate untwisted affine algebra acting on a Fock space \( FF \). If we consider the horizontal subalgebras of certain twisted affine algebras acting on Fock spaces, we obtain other new dual pairs between finite dimensional Lie groups and infinite dimensional Lie algebras as well.

5. Lie algebras \( \hat{D}, \hat{D}^+ \) and vertex algebras

5.1. Duality involving \( \hat{D} \) and \( \hat{D}^+ \). We obtain an action of \( \hat{D} \) on \( F \otimes l \) by composing the action of \( \hat{gl}_{\infty} \) and the homomorphism \( \hat{\phi} \) given by the formula (3.4).

It follows that the actions of \( GL_l \) and \( \hat{D} \) on \( F \otimes l \) commute and that \( GL_l \) and \( \hat{D} \) form a dual pair on \( F \otimes l \).

Introduce the generating function

\[
J^n(z) = \sum_{k \in \mathbb{Z}} J^n_k z^{-k-n-1}.
\]

**Lemma 5.1.** On \( F \otimes l \) we have

\[
J^n(z) = \sum_{k=1}^{l} \partial^n_k \psi^{-k}(z) \psi^{+k}(z) :.
\]

For \( \lambda = (m_1, \ldots, m_l) \in \Sigma(A) \) where \( m_1 \geq \ldots \geq m_l, m_i \in \mathbb{Z} \), we let \( \mathcal{E}(\lambda) \) be the set of exponents \( m_k \) \( (k = 1, \ldots, l) \) with multiplicity 1. Note that the \( \hat{D} \)-module \( L(\hat{D}; \mathcal{E}(\lambda)) \) is positive primitive.

**Remark 5.2.** We implicitly assumed that \( m_1, \ldots, m_l \) are distinct in defining \( \mathcal{E}(\lambda) \) above. In general if \( m', \ldots, m'_\alpha \) are distinct numbers among \( m_1, \ldots, m_l \) with multiplicities \( a_1, \ldots, a_{\alpha} \), then \( \mathcal{E}(\lambda) \) should be the set of exponents \( m', \ldots, m'_\alpha \) with multiplicities \( a_1, \ldots, a_{\alpha} \). With this understood, we have preferred the oversimplified and slightly wrong way of defining \( \mathcal{E}(\lambda) \) here and below.

**Theorem 5.3 (FKRW).** Under the joint action of \( GL_l \) and \( \hat{D} \), we have the isotypic decomposition

\[
F \otimes l = \bigoplus_{\lambda \in \Sigma(A)} V(GL_l; \lambda) \otimes L \left( \hat{D}; \mathcal{E}(\lambda) \right).
\]

**Corollary 5.4.** The space of invariants of \( GL_l \) in the Fock space \( F \otimes l \) is naturally isomorphic to the irreducible module \( L(\hat{gl}_{\infty}; l^a \Lambda_0) \), or equivalently to the irreducible module \( L \left( \hat{D}; \mathcal{E}(0) \right) \).
We obtain the action of \( \hat{D}^+ \) on \( \mathcal{F}^{\otimes l} \) by composing the action of \( d_\infty \) and the homomorphism \( \hat{\phi} \) given by the formula (3.4). Similarly \( O_{2l} \) and \( d_\infty \) form a dual pair. Introduce the generating function

\[
W^n(z) = \sum_{k \in \mathbb{Z}} W_k^n z^{-k-n-1}.
\]

**Lemma 5.5.** On \( \mathcal{F}^{\otimes l} \) we have

\[
W^n(z) = \frac{1}{2} \sum_{k=1}^l (z \partial_z^n \psi^{-k}(z) \psi^{+k}(z) + z \partial_z^n \psi^{+k}(z) \psi^{-k}(z)).
\]

For \( \lambda = (m_1, \ldots, m_l) \in \Sigma(D) \) where \( m_1 \geq \ldots \geq m_i > m_{i+1} = \ldots = m_l = 1 \), we let \( \mathcal{E}(\lambda) \) be the set of exponents \( m_k \) \( (k = 1, \ldots, i) \) with multiplicity 1 and the exponent \( j \) with multiplicity \( l - i \); for \( \lambda = (m_1, \ldots, m_l) \in \Sigma(D) \) where

\[
m_1 \geq \ldots m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0 \quad (j < l),
\]

we let \( \mathcal{E}(\lambda) \) be the set of exponents \( m_k \) \( (k = 1, \ldots, i) \) of multiplicity \( 1 \), exponent \( 1 \) of multiplicity \( j - i \) for \( (m_1, \ldots, m_l) \otimes \det \in \Sigma(D) \) where

\[
m_1 \geq \ldots m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0 \quad (j < l),
\]

we let \( \mathcal{E}(\lambda) \) be the set of exponents \( m_k \) \( (k = 1, \ldots, i) \) of multiplicity \( 1 \), exponent \( 1 \) of multiplicity \( 2l - i - j \). We will write \((0, \ldots, 0)\) and \((0, \ldots, 0) \otimes \det\) for short as \(0\) and \(\det\) respectively. Note that the \( \hat{D}^+ \)-module \( L(\hat{D}^+; \mathcal{E}(\lambda), l) \) is positive primitive.

**Theorem 5.6** ([KWY]). Under the joint action of \( O_{2l} \) and \( \hat{D}^+ \), we have the isotypic decomposition

\[
\mathcal{F}^{\otimes l} = \bigoplus_{\lambda \in \Sigma(D)} V(O_{2l}; \lambda) \otimes L(\hat{D}^+; \mathcal{E}(\lambda), l).
\]

**Corollary 5.7.** The space of invariants of \( O_{2l} \) in the Fock space \( \mathcal{F}^{\otimes l} \) is naturally isomorphic to the irreducible module \( L(d_\infty; 2l \dA_0) \) of central charge \( l \), or equivalently to the irreducible module \( L(\hat{D}^+; \mathcal{E}(0), l) \).

The Dynkin diagram of \( d_\infty \) admits an automorphism of order 2 denoted by \( \sigma \). \( \sigma \) induces naturally an automorphism of order 2 of \( d_\infty \), which is denoted again by \( \sigma \). \( \sigma \) acts on the set of highest weights for \( d_\infty \) by mapping \( \lambda = d_{h_0} \dA_0 + d_{h_1} \dA_1 + \sum_{i \geq 2}^{d_{h_i}} \dA_i \) to \( \sigma(\lambda) = d_{h_0} \dA_0 + d_{h_1} \dA_1 + \sum_{i \geq 2}^{d_{h_i}} \dA_i \). In this way one obtain an irreducible module of the semi-product \( \sigma \times d_\infty \) on \( L(d_\infty; \lambda) \otimes L(d_\infty; \sigma(\lambda)) \) if \( \sigma(\lambda) \neq \lambda \) and on \( L(d_\infty; \lambda) \) if \( \sigma(\lambda) = \lambda \). Note that \( (SO_{2l}, \sigma \times d_\infty) \) form a dual pair on \( \mathcal{F}^{\otimes l} \).

**Corollary 5.8.** The space of invariants of \( SO_{2l} \) in \( \mathcal{F}^{\otimes l} \) is isomorphic to the \( d_\infty \)-module \( L(d_\infty; 2l \dA_0) \otimes L(d_\infty; 2l \dA_1) \) or equivalently the \( \hat{D}^+ \)-module \( L(\hat{D}^+; \mathcal{E}(0), l) \otimes L(\hat{D}^+; \mathcal{E}(\det), l) \).

**Remark 5.9.** The dual pairs as indicated in Remark 4.5 can be reformulated in terms of \( \hat{D}^+ \) or \( \hat{D}^- \) accordingly as in Theorem 5.6.
5.2. Vertex algebras associated to $\hat{D}$ and $\hat{D}^+$. The irreducible quasifinite $\hat{D}$-module $L(\hat{D}; \mathcal{E}(0))$ (resp. $\hat{D}^+$-module $L(\hat{D}^+; \mathcal{E}(0), c)$) is usually referred to as the irreducible vacuum $\hat{D}$-module (resp. $\hat{D}^+$-module), denoted by $V_c$ (resp. $V_c^+$). Denote by $|0\rangle$ the highest weight vector.

The fermionic Fock space $\mathcal{F} \otimes I$ is one of the simplest examples of vertex algebras $\mathcal{B}$, $\mathcal{FLM}$, $\mathcal{K}$. One may view $GL_l$ and $O_{2l}$ as automorphism groups of the vertex algebra $\mathcal{F} \otimes I$. The space $V_i$, being isomorphic to the space of invariants of $GL_l$ of the vertex algebra $\mathcal{F} \otimes I$ by Corollary 5.4, inherits a vertex algebra structure. Similarly it follows from Corollary 5.7 that $V_i^+$ is a vertex subalgebra of $\mathcal{F} \otimes I$.

We want to show that $V_c$ and $V_c^+$ are vertex algebras for general $c$. One can easily check that

$$\lim_{z \to 0} J^i(z)|0\rangle = J^i_{-l-1}|0\rangle, \quad [J^i_0, J^j(z)] = \partial J^j(z),$$
$$\lim_{z \to 0} W^i(z)|0\rangle = W^i_{-l-1}|0\rangle, \quad [W^i_0, W^j(z)] = \partial W^j(z).$$

For a typical element in $V_c$ (resp. $V_c^+$) of the form $J_{k_1,0}^{n_1} \cdots J_{k_l,0}^{n_l} |0\rangle$ (resp. $W_{k_1,0}^{n_1} \cdots W_{k_l,0}^{n_l} |0\rangle$) where $k_i \in \mathbb{Z}_+$, we associated a field (state-field correspondence): $\psi^{(k_1)}_z W^{n_1}(z) \cdots \psi^{(k_l)}_z W^{n_l}(z)$.

One can prove by using Lemmas 5.1 and 5.5 that

$$(z - w)^{m+n+2} [J^m(z), J^n(w)] = 0, \quad (z - w)^{m+n+2} [W^m(z), W^n(w)] = 0,$$

in $\mathcal{F} \otimes I$ for all $l \in \mathbb{Z}_+$. It follows that these equations hold for any $c$.

Combining all the above, we have proved the following theorem [FKRW] [KWy]. The simplicity here follows from the irreducibility of $V_c$ (resp. $V_c^+$) as a module over $\hat{D}$ (resp. $\hat{D}^+$).

**Theorem 5.10.** $V_c$ and $V_c^+$ are simple vertex algebras.

**Corollary 5.11.** $L(\hat{D}; \mathcal{E}(\lambda))$, $\lambda \in \Sigma(A)$ are irreducible modules of the vertex algebra $V_l$; $L(\hat{D}^+; \mathcal{E}(\lambda), l)$, $\lambda \in \Sigma(D)$ are irreducible modules of $V_l^+$.

**Remark 5.12.** We obtain a dual pair between a Lie group and a vertex algebra $(GL_l, V_l)$ (resp. $(O_{2l}, V_l^+)$) by substituting $\hat{D}$ (resp. $\hat{D}^+$) with $V_l$ (resp. $V_l^+$).

6. Some examples and open problems

6.1. Examples. Let us first consider the case $l = 1$. Set $\mathcal{F} = \mathcal{F} \otimes I$ and $\psi^\pm(z) = \psi^\pm_1(z)$. We claim that the $(GL_1, \hat{D})$-duality in Theorem 5.3 is essentially the celebrated boson-fermion correspondence. To see that, we notice an infinitesimal generator of $GL_1$, $J^0 = \sum_{n \in \mathbb{Z}} : \psi^-_n \psi^+_n :$, acts on a typical element $\psi^-_{-m_1} \cdots \psi^-_{-m_j} \psi^+_{-n_1} \cdots \psi^+_{-n_l} |0\rangle$ as multiplication by $j - i$.

Recall that the Fourier components of $J^0(z)$ generate a Heisenberg algebra $\mathcal{H}$. One easily shows that $J^0(z)(n \geq 1)$ can be written as a normally ordered polynomial in terms of $\partial^i J^0(z)$, $i \in \mathbb{Z}^+$. For example, $J^1(z) = \frac{1}{2} (\partial J^0(z) + : J^0(z) J^0(z) :)$.

Thus an irreducible representation of $\hat{D}$ inside $\mathcal{F}$ remains irreducible when restricted to the Heisenberg algebra $\mathcal{H}$. Theorem 5.3 for $l = 1$ says that

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}^n$$
where $\mathcal{H}^n$, the subspace of $\mathcal{F}$ on which $J_0^0$ acts as multiplication by $n$, is an irreducible module over $\mathcal{H}$. In particular the vertex algebra $V_1$ is isomorphic to the Heisenberg vertex algebra $\mathcal{H}^0$ generated by $J^0(z)$.

Now consider the $(O_2, \hat{D}^+)$-duality on $\mathcal{F}$ in Theorem 5.6. $O_2$ is the semidirect product of $SO_2$ by $\mathbb{Z}/2\mathbb{Z}$ generated by the order 2 automorphism $\tau$ which exchange $\psi^+$ and $\psi^-$. An irreducible $O_2$-module except the trivial module and the determinant module (which are of 1-dimension) is two-dimensional. When restricted to $SO_2$, it decomposes into a sum of two $SO_2$-modules, with weight say $m$ and $-m$, $m \in \mathbb{N}$.

Theorem 4.4 (or rather together with Theorem 3.2, [W]) has the following consequences: $\mathcal{H}^m$ ($m \neq 0$) is irreducible as a module over $\hat{D}^+$ and $\mathcal{H}^m$ is isomorphic to $\mathcal{H}^{-m}$ as $\hat{D}^+$-modules; $\mathcal{H}^0$ is a direct sum of the $(\pm 1)$-eigenspaces $\mathcal{H}_\pm^0$ of $\tau$; $\mathcal{H}_+^0$ is isomorphic to $L(\hat{D}^+; E(0), 1)$ (which is in turn isomorphic to $V_1^+$) while $\mathcal{H}_-^0$ is isomorphic to $L(\hat{D}^+; E(\det), 1)$.

The $q$-character formulas for irreducible $\hat{D}^+$-modules (counting the graded dimensions) appearing in $\mathcal{F} \hat{\otimes} 1$ were obtained in [KWWY]. In particular one deduces that

$$ch_q V_1^+ = \frac{1}{2\prod_{j \geq 1} (1-q^j)} \left( \prod_{j \geq 1} (1+q^j) + \prod_{j \geq 1} (1-q^j) \right) = \frac{\sum_{j \geq 0} (-q)^j^2}{\prod_{j \geq 1} (1-q^j)}.$$

The second equality here is obtained by using the Gauss identity

$$\prod_{j \geq 1} (1-q^j^2) = \sum_{j \in \mathbb{Z}} (-q)^j^2.$$

Denote by $L(c, h)$ the irreducible module of Virasoro algebra with highest weight $h$ and central charge $c$. It is well known (cf. e.g. [KRa]) that the character of $L(1, 4m^2)$ is

$$ch_q L(1, 4m^2) = \frac{q^{4m^2} - q^{(2m+1)^2}}{\prod_{j \geq 1} (1-q^j)}.$$ 

It follows from the $q$-character formula of $V_1^+$ that $V_1^+ = \bigoplus_{m \geq 0} L(1, 4m^2)$ as a module over the Virasoro algebra generated by $W^1(z)$. Similarly we can show that

$$L(\hat{D}^+; E(\det), 1) = \bigoplus_{m \geq 0} L(1, (2m + 1)^2)$$

as a module over this Virasoro algebra.

One can show that $W^n(z)$ for all $n$ can be written as normally ordered polynomials in terms of $\partial W^1(z)$ and $\partial^i W^3(z)$, $i \in \mathbb{Z}_+$. One checks that $W^1(z)$ and $W^3(z)$ generate the $W(2, 4)$ algebra with central charge 1 ([BS]). This shows that $V_1^+$ is isomorphic to the $W(2, 4)$ algebra. The latter is simple as we know that $V_0^+$ is simple for any $c$.

Remark 6.1. The fixed-point vertex subalgebra $\mathcal{H}_+^0$ of $\mathcal{H}^0$ by $\tau$ was studied in detail in [DG, DN], where $\mathcal{H}_+^0$ is denoted by $M(1)^+$. The results derived above mainly as consequences of the $(O_2, \hat{D}^+)$-duality have been obtained in [DG, DN] in a very different way. Irreducible modules over the vertex algebra $\mathcal{H}_+^0$ has also been classified [DN]. We note that this classification can be achieved by the approach of [W2] as well. In view of the isomorphism between $V_1^+$ and $\mathcal{H}_+^0$, the classification of
the irreducible modules over \( V^+ \) is done. Relations (conjecturally being minimal) among the two generators in \( V^+ \) could be obtained by our approach.

Now let us consider the Fock space, denoted by \( \mathcal{F}^{\otimes \frac{1}{2}} \), of a neutral fermion \( \varphi(z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} \varphi_n z^{-n-\frac{1}{2}} \) satisfying \( [\varphi_m, \varphi_n]^+ = \delta_{m,-n} \). One has a duality between \( O_\lambda \) which is \( \mathbb{Z}/2\mathbb{Z} \) and \( \infty \) (or \( D^+ \)) with central charge \( \frac{1}{2} \) (see Remark 4.3 or [W], [KRY] for more detail). Thus the decomposition of \( \mathcal{F}^{\otimes \frac{1}{2}} \) into even and odd parts gives rise to

\[
\mathcal{F}^{\otimes \frac{1}{2}} = L(d_{\infty}; \tilde{\Lambda}_0) \bigoplus L(d_{\infty}; \tilde{\Lambda}_1),
\]

or equivalently,

\[
\mathcal{F}^{\otimes \frac{1}{2}} = L(\hat{D}^+; \mathcal{E}(0), 1/2) \bigoplus L(\hat{D}^+; \mathcal{E}(\text{det}), 1/2).
\]

In this case, we have \( W^i(z) = \partial^i \varphi(z) \varphi(z) \). The Fourier components of \( W^1(z) \) generate the Virasoro algebra with central charge \( \frac{1}{2} \). One can easily show by induction that \( W^n(z) \) for all \( n \) can be written as a normally ordered polynomial in terms of \( W^1(z) \). It follows that \( L(\hat{D}^+; \mathcal{E}(0), \frac{1}{2}) \) (which is the even part of \( \mathcal{F}^{\otimes \frac{1}{2}} \)) is irreducible over the Virasoro algebra generated by \( W^1(z) \) \( (c = \frac{1}{2}) \) with highest weight \( 0 \), and \( L(\hat{D}^+; \mathcal{E}(\text{det}), \frac{1}{2}) \) (which is the odd part of \( \mathcal{F}^{\otimes \frac{1}{2}} \)) is irreducible over this Virasoro algebra with highest weight \( \frac{1}{2} \). This identification of the even (resp. odd) part of \( \mathcal{F}^{\otimes \frac{1}{2}} \) with the irreducible module over the Virasoro algebra was proved (cf. e.g. [KRa]) by using deep results on the structures of modules over the Virasoro algebra. In particular we have proved that the vertex algebra \( V^\frac{1}{2} \) is isomorphic to the Virasoro vertex algebra with central charge \( \frac{1}{2} \). This is indeed the only \( c \) such that \( V^c \) is rational.

**Remark 6.2.** We can obtain an explicit decomposition of a tensor product of two minimal modules of Virasoro algebra with central charge \( \frac{1}{2} \) into irreducible modules of Virasoro algebra with central charge 1 from the duality results discussed above by using the see-saw pair technique in the theory of dual pairs.

### 6.2. Some open problems

We discuss several open problems and conjectures to conclude this paper.

**Problem 1:** determine a minimal set of generators and relations for the vertex algebras \( V_c \) and \( V^+_c \); classify all the irreducible modules of \( V_c \) and \( V^+_c \).

Recall that one can associate a \( \mathcal{W} \)-algebra \( \mathcal{W}_g \) to an arbitrary complex simple Lie algebra \( \mathfrak{g} \) [BS, Fer]. In general such a \( \mathcal{W} \)-algebra can be defined in terms of intersections of screening operators, cf. Fer. With some mild modification, one can define a \( \mathcal{W} \)-algebra \( \mathcal{W}(\mathfrak{gl}_l) \) associated to \( \mathfrak{gl}_l \) [FKRW].

When \( c \notin \mathbb{Z} \), the structure of \( V_c \) is well understood. In [FKRW], it is shown that the vertex algebra \( V_l \) associated to \( \hat{\mathfrak{h}} \) for \( l \in \mathbb{N} \) is isomorphic to \( \mathcal{W}(\mathfrak{gl}_l) \) with central charge \( l \). More precisely, the first \( l \) generating fields \( J^i(z) \), \( i = 0, 1, \ldots, l-1 \) freely generate \( V_l \) while the other fields \( J^i(z) \), \( i \geq l \) can be written as some normally ordered polynomials in terms of \( \partial^n J^i(z) \), \( n \geq 0, i = 0, 1, \ldots, l-1 \). The irreducible modules of \( V_l \) were classified.

**Remark 6.3.** The vertex algebra \( V_c \) can be viewed as some universal \( \mathcal{W} \)-algebra which is sort of “limit” of the \( \mathcal{W}(\mathfrak{gl}_l) \) algebra which are generated by fields of conformal dimension \( 1, 2, \ldots, l \). One naturally asks a similar question for other
series of $W$ algebras associated to classical simple Lie algebras such as $WD_l$. Since the $WD_l$ algebra has generating fields of conformal dimension $2, 4, \ldots, 2l - 2$ and $l$ (which equals 1 plus the exponents of $D_l$), one cannot take the naive “limit” due to the existence of the field with conformal dimension $l$. However $V^+_c$ can be viewed as a universal $W$-algebra which is sort of “limit” of $WD_l$ in the following sense \( [KWy] \): a direct sum of $V_i$ with the distinguished irreducible module $L(\widehat{D}^+; E(\det), l)$ over $V_i$ is isomorphic to $WD_l$.

The structure of $V_c$ for a negative integral $c$ is much more complicated. In $[W1]$, we showed that $V_{-1}$ is isomorphic to the $W(\mathfrak{gl}_3)$ algebra. The irreducible modules of $V_{-1}$ were classified in $[W2, W1]$. A relation among the generating fields of $V_{-1}$ was implicit there and conjecturally it determines all other relations. For a general negative integral $c = -l$ it is conjectured that a minimal set of generating fields in $V_{-1}$ consists of the first $l^2 + 2l - 2l$. The conjecture was verified in the case $c = -1$ $[W2]$.

For $c$ generic which means $c \notin \frac{1}{2}\mathbb{Z}$ in this case, the structure of $V^+_c$ is understood $[KWy]$. For $c \in \frac{1}{2}\mathbb{Z}_+$ we conjecture that a minimal set of generating fields of $V^+_c$ consists of the first $2c$ fields $W^{(i)}(z)$, $i = 1, 3, \ldots, 4c - 1$. The conjecture in the case $c = \frac{1}{2}$ was verified in the preceding subsection.

**Problem 2:** clarify precise relations between $\widehat{D}^+$ and $\widehat{D}^-$.

As remarked in Remark 3.4, $\widehat{D}^-$ should be thought as some kind of twisted algebra of $\widehat{D}^+$. We conjecture that irreducible modules appearing in the duality decomposition involving $\widehat{D}^-$ with central charge $c \in \frac{1}{2}\mathbb{Z}_+$ (see $[KWy]$) can be viewed as irreducible twisted modules over the vertex algebra $V^+_c$. A natural problem then will be to classify all the irreducible twisted modules over $V^+_c$.

**Problem 3:** formulate results in $[KWy]$ in terms of lattice vertex algebras by means of the boson-fermion correspondence.

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