Quasimodular Hecke algebras and Hopf actions

Abhishek Banerjee

Dept. of Mathematics, Indian Institute of Science, Bangalore, Karnataka - 560012, India.
Email: abhishekbannerjee1313@gmail.com

Abstract

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In this paper, we extend the theory of modular Hecke algebras due to Connes and Moscovici to define the algebra $Q(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$. Then, $Q(\Gamma)$ carries an action of “the Hopf algebra $H_1$ of codimension 1 foliations” that also acts on the modular Hecke algebra $A(\Gamma)$ of Connes and Moscovici. However, in the case of quasimodular Hecke algebras, we have several additional operators and we can describe them in terms of a new Hopf algebra $H$ acting on $Q(\Gamma)$. Furthermore, for each $\sigma \in SL_2(\mathbb{Z})$, we introduce the collection $Q_\sigma(\Gamma)$ of quasimodular Hecke operators twisted by $\sigma$. Then, $Q_\sigma(\Gamma)$ is a right $Q(\Gamma)$-module and is endowed with a pairing $(\cdot,\cdot) : Q_\sigma(\Gamma) \otimes Q_\sigma(\Gamma) \to Q_\sigma(\Gamma)$. We show that there is a “Hopf action” of a certain Hopf algebra $h_1$ on the pairing on $Q_\sigma(\Gamma)$. Finally, for any $\sigma \in SL_2(\mathbb{Z})$, we consider operators acting between the levels of the graded module $Q_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} Q_{\sigma(m)}(\Gamma)$, where $\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$ for any $m \in \mathbb{Z}$. The pairing on $Q_\sigma(\Gamma)$ can be extended to a graded pairing on $Q_\sigma(\Gamma)$ and we show that there is a Hopf action of a larger Hopf algebra $h_\mathbb{Z} \supseteq h_1$ on the pairing on $Q_\sigma(\Gamma)$.

Keywords: Modular Hecke algebras, Hopf actions

1 Introduction

Let $N \geq 1$ be an integer and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. In [3], [4], Connes and Moscovici have introduced the “modular Hecke algebra” $A(\Gamma)$ that combines the pointwise product on modular forms with the action of Hecke operators. Further, Connes and Moscovici have shown that the modular Hecke algebra $A(\Gamma)$ carries an action of “the Hopf algebra $H_1$ of codimension 1 foliations”. The Hopf algebra $H_1$ is part of a larger family of Hopf algebras $\{H_n| n \geq 1\}$ defined in [2]. The objective of this paper is to introduce and study quasimodular Hecke algebras $Q(\Gamma)$ that similarly combine the pointwise product on quasimodular forms with the action of Hecke operators. We will see that the quasimodular Hecke algebra $Q(\Gamma)$ carries several other operators in addition to an action of $H_1$. Further, we will also study the collection $Q_\sigma(\Gamma)$ of quasimodular Hecke operators twisted by some $\sigma \in SL_2(\mathbb{Z})$. The latter is a generalization of our theory of twisted modular Hecke operators introduced in [1].

We now describe the paper in detail. In Section 2, we briefly recall the notion of modular Hecke algebras of Connes and Moscovici [3], [4]. We let $QM$ be the “quasimodular tower”, i.e., $QM$ is the colimit over all $N$ of the spaces $QM(\Gamma(N))$ of quasimodular forms of level $\Gamma(N)$ (see [2,3]). We define a quasimodular Hecke operator of level $\Gamma$ to be a function of finite support from $\Gamma \setminus GL_2^+(\mathbb{Q})$ to the quasimodular tower $QM$ satisfying a certain covariance condition (see Definition 2.3). We then show that the collection $Q(\Gamma)$ of quasimodular Hecke
operators of level $\Gamma$ carries an algebra structure $({Q}(\Gamma), *)$ by considering a convolution product over cosets of $\Gamma$ in $GL_2(\mathbb{Z})$. Further, the modular Hecke algebra of Connes and Moscovici embeds naturally as a subalgebra of $Q(\Gamma)$. We also show that the quasimodular Hecke operators of level $\Gamma$ act on quasimodular forms of level $\Gamma$, i.e., $QM(\Gamma)$ is a left $Q(\Gamma)$-module. In this section, we will also define a second algebra structure $({Q}(\Gamma), ^*)$ on $Q(\Gamma)$ by considering the convolution product over cosets of $\Gamma$ in $SL_2(\mathbb{Z})$. When we consider $Q(\Gamma)$ as an algebra equipped with this latter product $^*$, it will be denoted by $Q^*(\Gamma) = (Q(\Gamma), ^*)$.

In Section 3, we define two different Hopf algebra actions on $Q(\Gamma)$. Given a quasimodular form $f \in QM(\Gamma)$ of level $\Gamma$, it is well known that we can write $f$ as a sum

$$f = \sum_{i=0}^{s} a_i(f) \cdot G^i$$  \hspace{1cm} (1.1)

where the coefficients $a_i(f)$ are modular forms of level $\Gamma$ and $G^i$ is the classical Eisenstein series of weight 2.

Therefore, we can consider two different sets of operators on the quasimodular tower $QM$: those which act on the powers of $G^i$ appearing in the expression for $f$ and those which act on the modular coefficients $a_i(f)$. The collection of operators acting on the modular coefficients are studied in Section 3.2. These induce on $Q(\Gamma)$ analogues of operators acting on the modular Hecke algebra $A(\Gamma)$. Further, we verify that $A(\Gamma)$ of Connes and Moscovici and we show that $Q(\Gamma)$ carries an action of the same Hopf algebra $A_1$ of codimension 1 foliations that acts on $A(\Gamma)$. On the other hand, by considering operators on $QM$ that act on the powers of $G^i$ appearing in (1.1), we are able to define additional operators $D, \{T_k^l\}_{k \geq 1, l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $Q(\Gamma)$ (see Section 3.1). Further, we show that these operators satisfy the following commutator relations:

$$[D, \phi^{(m)}] = 0 \hspace{1cm} [T_k^l, \phi^{(m)}] = 0 \hspace{1cm} [\phi^{(m)}, \phi^{(m')} ] = 0$$ \hspace{1cm} (1.2)

We then consider the smallest Lie algebra $L$ containing the symbols $D, \{T_k^l\}_{k \geq 1, l \geq 0}, \{\phi^{(m)}\}_{m \geq 1}$ along with the relations (1.2) between the commutators. Then, there is a Lie action of $L$ on $Q(\Gamma)$. However, we want to describe these operators in terms of a Hopf action on the algebra $({Q}(\Gamma), *)$, i.e., a Hopf algebra $H$ acting on $Q(\Gamma)$ such that:

$$h(F^1 \ast F^2) = \sum h_{(1)}(F^1) \ast h_{(2)}(F^2) \hspace{1cm} \forall h \in H, \ F^1, F^2 \in Q(\Gamma)$$ \hspace{1cm} (1.3)

where the coproduct $\Delta : H \rightarrow H \otimes H$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in H$. As an algebra, we let $H$ be the universal enveloping algebra of the Lie algebra $L$. Then, we define the coproduct $\Delta : H \rightarrow H \otimes H$ on $H$ as follows:

$$\Delta(D) = D \otimes 1 + 1 \otimes D - \phi^{(1)} \otimes T^0_1$$
$$\Delta(T_k^l) = T_k^l \otimes 1 + 1 \otimes T_k^l + \frac{24}{k} \phi^{(l)} \otimes T^0_k \hspace{1cm} \forall \ l \geq 0$$
$$\Delta(T_k^l) = T_k^l \otimes 1 + 1 \otimes T_k^l + \frac{24}{k} \phi^{(l)} \otimes T^0_k \hspace{1cm} \forall \ l \geq 0$$
$$\Delta(\phi^{(m)}) = \phi^{(m)} \otimes 1 + 1 \otimes \phi^{(m)} \hspace{1cm} \forall \ m \geq 1$$ \hspace{1cm} (1.4)

The antipode on $H$ is defined by setting:

$$S(D) = -D - \phi^{(1)} T^0_1 \hspace{1cm} (1.5)$$
$$S(T_k^l) = -T_k^l + \frac{24}{k} \phi^{(l)} T^0_k \hspace{1cm} \forall \ l \geq 0$$
$$S(T_k^l) = -T_k^l + \frac{24}{k} \phi^{(l)} T^0_k \hspace{1cm} \forall \ l \geq 0$$
$$S(\phi^{(m)}) = -\phi^{(m)} \hspace{1cm} \forall \ m \geq 1$$

We show in Section 3.1 that, as a consequence of the commutator relations in (1.2), the definitions in (1.4) and (1.5) determine a coproduct and an antipode on all of $H$. Further, we verify that $H$ equipped with the coproduct in (1.4) and the antipode in (1.5) is actually a Hopf algebra. Then, we show that $H$ has a Hopf action.
on $Q(\Gamma)$ in the sense of \[1.3\] and this action captures the operators $D, \{T^l_k\}_{k \geq 1, l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$ on $Q(\Gamma)$. Additionally, we show that corresponding to the product $\ast^r$ on $Q^r(\Gamma)$, we have a smaller Hopf algebra $h$ with a Hopf action on $Q^r(\Gamma)$. Let $I$ be the smallest Lie algebra containing the symbols $D, \{T^l_k\}_{k \geq 1, l \geq 0}$ along with the commutator relations:

\[
[T^l_k, D] = \frac{5}{24}(k-1)T^{l+1}_{k-1} - \frac{1}{2}(k-3)T^l_{k+1} \quad \forall \, k \geq 1, l \geq 0 \tag{1.6}
\]

Let $h$ be the universal enveloping algebra of the Lie algebra $I$ equipped with its natural Hopf algebra structure. Then, we prove that $h$ has a Hopf action on the algebra $Q^r(\Gamma)$ in the sense of \[1.3\].

In Section 4, we develop the theory of twisted quasimodular Hecke operators. For any $\sigma \in SL_2(\mathbb{Z})$, we define in Section 4.1 the collection $Q_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. When $\sigma = 1$, this reduces to the original definition of $Q(\Gamma)$. In general, $Q_\sigma(\Gamma)$ is not an algebra but we show that $Q_\sigma(\Gamma)$ carries a pairing:

\[
(\ldots , \ldots ) : Q_\sigma(\Gamma) \otimes Q_\sigma(\Gamma) \to Q_\sigma(\Gamma) \tag{1.7}
\]

Further, we show that $Q_\sigma(\Gamma)$ may be equipped with the structure of a right $Q(\Gamma)$-module. We can also extend the action of the Hopf algebra $H_1$ of codimension 1 foliations to $Q_\sigma(\Gamma)$. In fact, we show that $H_1$ has a “Hopf action” on the right $Q(\Gamma)$ module $Q_\sigma(\Gamma)$, i.e.,

\[
h(F^1 \ast F^2) = \sum \overline{h}_{(1)}(F^1) \ast \overline{h}_{(2)}(F^2) \quad \forall \, h \in H_1, \, F^1 \in Q_\sigma(\Gamma), \, F^2 \in Q(\Gamma) \tag{1.8}
\]

where the coproduct $\Delta : H_1 \to H_1 \otimes H_1$ is given by $\Delta(h) = \sum \overline{h}_{(1)} \otimes \overline{h}_{(2)}$ for any $h \in H_1$. We recall from \[3\] that $H_1$ is equal as an algebra to the universal enveloping algebra of the Lie algebra $L_1$ with generators $X, Y, \{\delta_n\}_{n \geq 1}$ satisfying the following relations:

\[
[Y, X] = X \quad [X, \delta_n] = \delta_{n+1} \quad [Y, \delta_n] = 2n \delta_n \quad [\delta_k, \delta_l] = 0 \quad \forall \, k, l, n \geq 1 \tag{1.9}
\]

Then, we can consider the smaller Lie algebra $l_1 \subseteq L_1$ with two generators $X, Y$ satisfying $[Y, X] = X$. If we let $h_1$ be the Hopf algebra that is the universal enveloping algebra of $l_1$, we show that the pairing in \[1.7\] on $Q_\sigma(\Gamma)$ carries a “Hopf action” of $h_1$. In other words, we have:

\[
h(F^1, F^2) = \sum \overline{h}_{(1)}(F^1), \overline{h}_{(2)}(F^2) \quad \forall \, h \in h_1, \, F^1, F^2 \in Q_\sigma(\Gamma) \tag{1.10}
\]

where the coproduct $\Delta : h_1 \to h_1 \otimes h_1$ is given by $\Delta(h) = \sum \overline{h}_{(1)} \otimes \overline{h}_{(2)}$ for any $h \in h_1$. In Section 4.2, we consider operators between the modules $Q_\sigma(\Gamma)$ as $\sigma$ varies over $SL_2(\mathbb{Z})$. More precisely, for any $\tau, \sigma \in SL_2(\mathbb{Z})$, we define a morphism:

\[
X_{\tau} : Q_\sigma(\Gamma) \to Q_{\tau \sigma}(\Gamma) \tag{1.11}
\]

In particular, this gives us operators acting between the levels of the graded module

\[
Q_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} Q_{\sigma(m)}(\Gamma) \tag{1.12}
\]

where for any $\sigma \in SL_2(\mathbb{Z})$, we set $\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$. Further, we generalize the pairing on $Q_\sigma(\Gamma)$ in \[1.7\] to a pairing:

\[
(\ldots , \ldots ) : Q_{\tau_1 \sigma}(\Gamma) \otimes Q_{\tau_2 \sigma}(\Gamma) \to Q_{\tau_1 \tau_2 \sigma}(\Gamma) \tag{1.13}
\]

where $\tau_1, \tau_2$ are commuting matrices in $SL_2(\mathbb{Z})$. In particular, \[1.13\] gives us a pairing $Q_{\sigma(m)}(\Gamma) \otimes Q_{\sigma(n)}(\Gamma) \to Q_{\sigma(m+n)}(\Gamma), \, \forall \, m, n \in \mathbb{Z}$ and hence a pairing on the tower $Q_\sigma(\Gamma)$. Finally, we consider the Lie algebra $l_2 \supseteq l_1$ with generators $\{Z, X_n \}_{n \in \mathbb{Z}}$ satisfying the following commutator relations:

\[
[Z, X_n] = (n + 1)X_n \quad [X_n, X_{n'}] = 0 \quad \forall \, n, n' \in \mathbb{Z} \tag{1.14}
\]
Then, if we let $h_Z$ be the Hopf algebra that is the universal enveloping algebra of $l_Z$, we show that $h_Z$ has a Hopf action on the pairing on $Q_\sigma(\Gamma)$. In other words, for any $F_1, F_2 \in Q_\sigma(\Gamma)$, we have

$$h(F_1, F_2) = \sum (h^{(1)}(F_1), h^{(2)}(F_2)) \quad \forall \ h \in h_Z$$

where the coproduct $\Delta : h_Z \rightarrow h_Z \otimes h_Z$ is defined by setting $\Delta(h) := \sum h^{(1)} \otimes h^{(2)}$ for each $h \in h_Z$.

## 2 The Quasimodular Hecke algebra

We begin this section by briefly recalling the notion of quasimodular forms. The notion of quasimodular forms is due to Kaneko and Zagier [5]. The theory has been further developed in Zagier [7]. For an introduction to the basic theory of quasimodular forms, we refer the reader to the exposition of Royer [6].

Throughout, let $\mathbb{H} \subseteq \mathbb{C}$ be the upper half plane. Then, there is a well known action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

$$z \mapsto \frac{az + b}{cz + d} \quad \forall \ z \in \mathbb{H}, \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL_2(\mathbb{Z})$$

(2.1)

For any $N \geq 1$, we denote by $\Gamma(N)$ the following principal congruence subgroup of $SL_2(\mathbb{Z})$:

$$\Gamma(N) := \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL_2(\mathbb{Z}) \bigg| \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) (mod \ N) \right\}$$

(2.2)

In particular, $\Gamma(1) = SL_2(\mathbb{Z})$. We are now ready to define quasimodular forms.

**Definition 2.1.** Let $f : \mathbb{H} \rightarrow \mathbb{C}$ be a holomorphic function and let $N \geq 1, k, s \geq 0$ be integers. Then, the function $f$ is a quasimodular form of level $N$, weight $k$ and depth $s$ if there exist holomorphic functions $f_0, f_1, ..., f_s : \mathbb{H} \rightarrow \mathbb{C}$ with $f_s \neq 0$ such that:

$$(cz + d)^{-k}f\left(\frac{az + b}{cz + d}\right) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz + d}\right)^{j}$$

(2.3)

for any matrix $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma(N)$. The collection of quasimodular forms of level $N$, weight $k$ and depth $s$ will be denoted by $QM^s_k(\Gamma(N))$. By convention, we let the zero function $0 \in QM^0_k(\Gamma(N))$ for every $k \geq 0, N \geq 1$.

More generally, for any holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ and any matrix $\alpha = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in GL_2^+(\mathbb{Q})$, we define:

$$(f| k \alpha)(z) := (cz + d)^{-k}f\left(\frac{az + b}{cz + d}\right) \quad \forall \ k \geq 0$$

(2.4)

Then, we can say that $f$ is quasimodular of level $N$, weight $k$ and depth $s$ if there exist holomorphic functions $f_0, f_1, ..., f_s : \mathbb{H} \rightarrow \mathbb{C}$ with $f_s \neq 0$ such that:

$$(f| k \gamma)(z) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz + d}\right)^{j} \quad \forall \ \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma(N)$$

(2.5)
When the integer \( k \) is clear from context, we write \( f|_k \alpha \) simply as \( f|\alpha \) for any \( \alpha \in GL_2^+(\mathbb{Q}) \). Also, it is clear that we have a product:

\[
\mathcal{Q}M_k^s(\Gamma(N)) \otimes \mathcal{Q}M_l^t(\Gamma(N)) \to \mathcal{Q}M_{k+l}^{s+t}(\Gamma(N))
\]

on quasi-modular forms. For any \( N \geq 1 \), we now define:

\[
\mathcal{Q}M(\Gamma(N)) := \bigoplus_{s=0}^{\infty} \bigoplus_{k=0}^{\infty} \mathcal{Q}M_k^s(\Gamma(N))
\]

We now consider the direct limit:

\[
\mathcal{Q}M := \lim_{\to} \mathcal{Q}M(\Gamma(N))
\]

which we will refer to as the quasimodular tower. Additionally, for any \( k \geq 0 \) and \( N \geq 1 \), we let \( \mathcal{M}_k(\Gamma(N)) \) denote the collection of usual modular forms of weight \( k \) and level \( N \). Then, we can define the modular tower \( \mathcal{M} \):

\[
\mathcal{M} := \lim_{\to} \mathcal{M}(\Gamma(N)) \quad \mathcal{M}(\Gamma(N)) := \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma(N))
\]

We now recall the modular Hecke algebra of Connes and Moscovici [3].

**Definition 2.2.** (see [3, § 1]) Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup of \( SL_2(\mathbb{Z}) \). A modular Hecke operator of level \( \Gamma \) is a function of finite support

\[
F : \Gamma \backslash GL_2^+(\mathbb{Q}) \to \mathcal{M} \quad \Gamma \alpha \mapsto F_\alpha
\]

such that for any \( \gamma \in \Gamma \), we have:

\[
F_{\alpha \gamma} = F_\alpha|_{\gamma}
\]

The collection of all modular Hecke operators of level \( \Gamma \) will be denoted by \( \mathcal{A}(\Gamma) \).

Our first aim is to define a quasimodular Hecke algebra \( \mathcal{Q}(\Gamma) \) analogous to the modular Hecke algebra \( \mathcal{A}(\Gamma) \) of Connes and Moscovici. For this, we recall the structure theorem for quasimodular forms, proved by Kaneko and Zagier [5].

**Theorem 2.3.** (see [5, § 1, Proposition 1.]) Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup of \( SL_2(\mathbb{Z}) \). For any even number \( K \geq 2 \), let \( G_K \) denote the classical Eisenstein series of weight \( K \):

\[
G_K(z) := -\frac{B_K}{2K} + \sum_{n=1}^{\infty} \left( \sum_{d|n} \frac{d^{K-1}}{d^{K-1}} \right) e^{2\pi inz}
\]

where \( B_K \) is the \( K \)-th Bernoulli number and \( z \in \mathbb{H} \). Then, every quasimodular form in \( \mathcal{Q}M(\Gamma) \) can be written uniquely as a polynomial in \( G_2 \) with coefficients in \( \mathcal{M}(\Gamma) \). More precisely, for any quasimodular form \( f \in \mathcal{Q}M_k(\Gamma) \), there exist functions \( a_0(f), a_1(f), ..., a_s(f) \) such that:

\[
f = \sum_{i=0}^{s} a_i(f) G_2^i
\]

where \( a_i(f) \in \mathcal{M}_{k-2i}(\Gamma) \) is a modular form of weight \( k - 2i \) and level \( \Gamma \) for each \( 0 \leq i \leq s \).
We now consider a quasimodular form \( f \in \mathcal{QM} \). For sake of definiteness, we may assume that \( f \in \mathcal{QM}_s^r(\Gamma(N)) \), i.e. \( f \) is a quasimodular form of level \( N \), weight \( k \) and depth \( s \). We now define an operation on \( \mathcal{QM} \) by setting:

\[
  f|\alpha = \sum_{i=0}^{s} (a_i(f)|_{k-2i}\alpha)G^i_2 \quad \forall \alpha \in GL^+_2(Q) \tag{2.14}
\]

where \( \{a_i(f) \in \mathcal{M}_{k-2i}(\Gamma(N))\}_{0 \leq i \leq s} \) is the collection of modular forms determining \( f = \sum_{i=0}^{s} a_i(f)G^i_2 \) as in Theorem 2.3. We know that for any \( \alpha \in GL^+_2(Q) \), each \( (a_i(f)|_{k-2i}\alpha) \) is an element of the modular tower \( \mathcal{M} \).

This shows that \( f|\alpha = \sum_{i=0}^{s} (a_i(f)|_{k-2i}\alpha)G^i_2 \in \mathcal{QM} \). However, we note that for arbitrary \( \alpha \in GL^+_2(Q) \) and \( a_i(f) \in \mathcal{M}_{k-2i}(\Gamma(N)) \), it is not necessary that \( (a_i(f)|_{k-2i}\alpha) \in \mathcal{M}_{k-2i}(\Gamma(N)) \). In other words, the operation defined in (2.14) on the quasimodular tower \( \mathcal{QM} \) does not descend to an endomorphism on each \( \mathcal{QM}_s^r(\Gamma(N)) \).

From the expression in (2.14), it is also clear that:

\[
  (f \cdot g)|\alpha = (f|\alpha) \cdot (g|\alpha) \quad f|\alpha|\beta = (f|\alpha)|\beta \quad \forall \alpha, \beta \in GL^+_2(Q) \tag{2.15}
\]

We are now ready to define the quasimodular Hecke operators.

**Definition 2.4.** Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup. A quasimodular Hecke operator of level \( \Gamma \) is a function of finite support:

\[
  F : \Gamma \setminus GL^+_2(Q) \to \mathcal{QM} \quad \Gamma \alpha \mapsto F_\alpha \tag{2.16}
\]

such that for any \( \gamma \in \Gamma \), we have:

\[
  F_{\alpha \gamma} = F_\alpha|\gamma \tag{2.17}
\]

The collection of all quasimodular Hecke operators of level \( \Gamma \) will be denoted by \( \mathcal{Q}(\Gamma) \).

We will now introduce the product structure on \( \mathcal{Q}(\Gamma) \). In fact, we will introduce two separate product structures \( (\mathcal{Q}(\Gamma), *) \) and \( (\mathcal{Q}(\Gamma), \star) \) on \( \mathcal{Q}(\Gamma) \).

**Proposition 2.5.** (a) Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup and let \( \mathcal{Q}(\Gamma) \) be the collection of quasimodular Hecke operators of level \( \Gamma \). Then, the product defined by:

\[
  (F \ast G)_\alpha := \sum_{\beta \in \Gamma \setminus GL^+_2(Q)} F_{\beta} \cdot (G_{\alpha \beta}^{-1}|\beta) \quad \forall \alpha \in GL^+_2(Q) \tag{2.18}
\]

for all \( F, G \in \mathcal{Q}(\Gamma) \) makes \( \mathcal{Q}(\Gamma) \) into an associative algebra.

(b) Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup and let \( \mathcal{Q}(\Gamma) \) be the collection of quasimodular Hecke operators of level \( \Gamma \). Then, the product defined by:

\[
  (F \ast^r G)_\alpha := \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} F_{\beta} \cdot (G_{\alpha \beta}^{-1}|\beta) \quad \forall \alpha \in GL^+_2(Q) \tag{2.19}
\]

for all \( F, G \in \mathcal{Q}(\Gamma) \) makes \( \mathcal{Q}(\Gamma) \) into an associative algebra which we denote by \( \mathcal{Q}^r(\Gamma) \).

**Proof.** (a) We need to check that the product in (2.18) is associative. First of all, we note that the expression in (2.18) can be rewritten as:

\[
  (F \ast G)_\alpha = \sum_{\alpha_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot G_{\alpha_2}|\alpha_1 \quad \forall \alpha \in GL^+_2(Q) \tag{2.20}
\]
where the sum in (2.20) is taken over all pairs \((\alpha_1, \alpha_2)\) with \(\alpha_2 \alpha_1 = \alpha\) modulo the following equivalence relation:

\[
(\alpha_1, \alpha_2) \sim (\gamma \alpha_1, \alpha_2 \gamma^{-1}) \quad \forall \gamma \in \Gamma
\]  

(2.21)

Hence, for \(F, G, H \in \mathcal{Q}(\Gamma)\), we can write:

\[
(F \ast (G \ast H))_\alpha = \sum_{\alpha'_{2}\alpha_1 = \alpha} F_{\alpha_1} \cdot (G \ast H)_{\alpha'_2} ||_{\alpha_1} \\
= \sum_{\alpha'_2 \alpha_1 = \alpha} F_{\alpha_1} \cdot (\sum_{\alpha_2 = \alpha'_2} G_{\alpha_2} \cdot H_{\alpha_3} ||_{\alpha_2}) ||_{\alpha_1}
\]

(2.22)

where the sum in (2.22) is taken over all triples \((\alpha_1, \alpha_2, \alpha_3)\) with \(\alpha_3 \alpha_2 \alpha_1 = \alpha\) modulo the following equivalence relation:

\[
(\alpha_1, \alpha_2, \alpha_3) \sim (\gamma \alpha_1, \gamma' \alpha_2 \gamma^{-1}, \alpha_3 \gamma'^{-1}) \quad \forall \gamma, \gamma' \in \Gamma
\]  

(2.23)

On the other hand, we have

\[
((F \ast G) \ast H)_\alpha = \sum_{\alpha'_{3}\alpha_2 = \alpha} (F \ast G)_{\alpha'_3} \cdot H_{\alpha_3} ||_{\alpha'_2} \\
= \sum_{\alpha'_{3}\alpha_2 = \alpha} (\sum_{\alpha_1 = \alpha'_{3}} F_{\alpha_2} \cdot G_{\alpha_1} ||_{\alpha_1}) \cdot H_{\alpha_3} ||_{\alpha_2}
\]

(2.24)

where the sum in (2.24) is taken over all triples \((\alpha_1, \alpha_2, \alpha_3)\) with \(\alpha_3 \alpha_2 \alpha_1 = \alpha\) modulo the equivalence relation in (2.23). From (2.22) and (2.24) the result follows.

\[\]

We know that modular forms are quasimodular forms of depth 0, i.e., for any \(k \geq 0, N \geq 1\), we have \(M_k(\Gamma(N)) = \mathcal{Q}M_k(\Gamma(N))\). It follows that the modular tower \(\mathcal{M}\) defined in (2.19) embeds into the quasimodular tower \(\mathcal{Q}\mathcal{M}\) defined in (2.25). We are now ready to show that the modular Hecke algebra \(\mathcal{A}(\Gamma)\) of Connes and Moscovici embeds into the quasimodular Hecke algebra \(\mathcal{Q}(\Gamma)\) for any congruence subgroup \(\Gamma = \Gamma(N)\).

**Proposition 2.6.** Let \(\Gamma = \Gamma(N)\) be a principal congruence subgroup of \(\text{SL}_2(\mathbb{Z})\). Let \(\mathcal{A}(\Gamma)\) be the modular Hecke algebra of level \(\Gamma\) as defined in Definition 2.4 and let \(\mathcal{Q}(\Gamma)\) be the quasimodular Hecke algebra of level \(\Gamma\) as defined in Definition 2.7. Then, there is a natural embedding of algebras \(\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)\).

**Proof.** For any \(\alpha \in \text{GL}_2^+(\mathbb{Q})\) and any \(f \in \mathcal{Q}M_k(\Gamma)\), we consider the operation \(f \mapsto f||\alpha\) as defined in (2.14):

\[
f||\alpha = \sum_{i=0}^{s} (a_i(f)|_{k-2}\alpha)G_2^i \in \mathcal{Q}\mathcal{M}
\]

(2.25)

In particular, if \(f \in \mathcal{M}_k(\Gamma) = \mathcal{Q}M_k(\Gamma)\) is a modular form, it follows from (2.25) that:

\[
f||\alpha = a_0(f)|_{k}\alpha = f|_{k}\alpha = f|\alpha \in \mathcal{M}
\]

(2.26)

Hence, using the embedding of \(\mathcal{M}\) in \(\mathcal{Q}\mathcal{M}\), it follows from (2.21) in the definition of \(\mathcal{A}(\Gamma)\) and from (2.17) in the definition of \(\mathcal{Q}(\Gamma)\) that we have an embedding \(\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)\) of modules. Further, we recall from [3, § 1] that the product on \(\mathcal{A}(\Gamma)\) is given by:

\[
(F \ast G)_\alpha := \sum_{\beta \in \text{GL}_2^+(\mathbb{Q})} F_{\beta} \cdot (G_{\alpha_{\beta^{-1}}}|_{\beta}) \quad \forall \alpha \in \text{GL}_2^+(\mathbb{Q}), F, G \in \mathcal{A}(\Gamma)
\]

(2.27)

Comparing (2.27) with the product on \(\mathcal{Q}(\Gamma)\) described in (2.18) and using (2.26) it follows that \(\mathcal{A}(\Gamma) \hookrightarrow \mathcal{Q}(\Gamma)\) is an embedding of algebras.

\[\]
We end this section by describing the action of the algebra $\mathcal{Q}(\Gamma)$ on $\mathcal{QM}(\Gamma)$.

**Proposition 2.7.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$. Then, for any element $f \in \mathcal{QM}(\Gamma)$ the action of $\mathcal{Q}(\Gamma)$ defined by:

$$F \ast f := \sum_{\beta \in \Gamma \setminus GL_2^+(\mathbb{Q})} F_\beta \cdot f|_|\beta \quad \forall F \in \mathcal{Q}(\Gamma)$$

(2.28)

makes $\mathcal{QM}(\Gamma)$ into a left module over $\mathcal{Q}(\Gamma)$.

**Proof.** It is easy to check that the right hand side of (2.28) is independent of the choice of coset representatives. Further, since $F \in \mathcal{Q}(\Gamma)$ is a function of finite support, we can choose finitely many coset representatives $\{\beta_1, \beta_2, ..., \beta_n\}$ such that

$$F \ast f = \sum_{j=1}^{n} F_{\beta_j} \cdot f|_|\beta_j$$

(2.29)

It suffices to consider the case $f \in \mathcal{QM}_s^1(\Gamma)$ for some weight $k$ and depth $s$. Then, we can express $f$ as a sum:

$$f = \sum_{i=0}^{s} a_i(f)G_i^2$$

(2.30)

where each $a_i(f) \in \mathcal{M}_{k-2i}(\Gamma)$. Similarly, for any $\beta \in GL_2^+(\mathbb{Q})$, we can express $F_\beta$ as a finite sum:

$$F_\beta = \sum_{r=0}^{t} a_{\beta,r}(F_\beta) \cdot G_r^2$$

(2.31)

with each $a_{\beta,r}(F_\beta) \in \mathcal{M}$. In particular, we let $t = \max\{t_{\beta_1}, t_{\beta_2}, ..., t_{\beta_n}\}$ and we can now write:

$$F_{\beta_j} = \sum_{r=0}^{t} a_{\beta_j,r}(F_{\beta_j}) \cdot G_r^2$$

(2.32)

by adding appropriately many terms with zero coefficients in the expression for each $F_{\beta_j}$. Further, for any $\gamma \in \Gamma$, we know that $F_{\beta_j,\gamma} = F_{\beta_j}||\gamma = \sum_{r=0}^{t} (a_{\beta_j,r}(F_{\beta_j})|\gamma) \cdot G_r^2$. In other words, we have, for each $j$:

$$F_{\beta_j,\gamma} = \sum_{r=0}^{t} a_{\beta_j,\gamma,r}(F_{\beta_j,\gamma}) \cdot G_r^2$$

(2.33)

The sum in (2.29) can now be expressed as:

$$F \ast f := \sum_{j=1}^{n} F_{\beta_j} \cdot f|_|\beta_j = \sum_{i=0}^{s} \sum_{r=0}^{t} \sum_{j=1}^{n} a_{\beta_j,r}(F_{\beta_j}) \cdot (a_i(f)|\beta_j) \cdot G_r^{r+i}$$

(2.34)

For any $i, r$, we now set:

$$A_{ir}(F, f) := \sum_{j=1}^{n} a_{\beta_j,r}(F_{\beta_j}) \cdot (a_i(f)|\beta_j)$$

(2.35)

Again, it is easy to see that the sum $A_{ir}(F, f)$ in (2.35) does not depend on the choice of the coset representatives $\{\beta_1, \beta_2, ..., \beta_n\}$. Then, for any $\gamma \in \Gamma$, we have:

$$A_{ir}(F, f)|\gamma = \sum_{j=1}^{n} (a_{\beta_j,r}(F_{\beta_j})|\gamma) \cdot (a_i(f)|\beta_j) = \sum_{j=1}^{n} a_{\beta_j,\gamma,r}(F_{\beta_j,\gamma}) \cdot (a_i(f)|\beta_j) = A_{ir}(F, f)$$

(2.36)
where the last equality in (2.36) follows from the fact that \{β_1γ, β_2γ, ..., β_nγ\} is another collection of distinct cosets representatives of Γ in \(GL_2^+(\mathbb{Q})\). From (2.36), we note that each \(A_{ir}(F, f)\) ∈ \(\mathcal{M}(\Gamma)\). Then, the sum:

\[
F \ast f = \sum_{i=0}^{s} \sum_{r=0}^{t} A_{ir}(F, f) \cdot G_{2}^{i+r}
\]

(2.37)
is an element of \(\mathcal{QM}(\Gamma)\). Hence, \(\mathcal{QM}(\Gamma)\) is a left module over \(\mathcal{Q}(\Gamma)\).

\[\square\]

3 The Lie algebra and Hopf algebra actions on \(\mathcal{Q}(\Gamma)\)

Let Γ = Γ(N) be a principal congruence subgroup of \(SL_2(\mathbb{Z})\). In this section, we will describe two different sets of operators on the collection \(\mathcal{Q}(\Gamma)\) of quasimodular Hecke operators of level Γ. Given a quasimodular form \(f \in \mathcal{QM}(\Gamma)\) of level Γ, we have mentioned in the last section that \(f\) can be expressed as a finite sum:

\[
f = \sum_{i=0}^{s} a_i(f) \cdot G_2^i
\]

(3.1)

where \(G_2\) is the classical Eisenstein series of weight 2 and each \(a_i(f)\) is a modular form of level Γ. Then in Section 3.1, we consider operators on the quasimodular tower that act on the powers of \(G_2\) appearing in (3.1). These induce operators \(D, \{T_k\}_{k \geq 1, t \geq 0}\) on the collection \(\mathcal{Q}(\Gamma)\) of quasimodular Hecke operators of level Γ. In order to understand the action of these operators on products of elements in \(\mathcal{Q}(\Gamma)\), we also need to define the operators \(\{\phi(m)\}_{m \geq 1}\). Finally, we show that these operators may all be described in terms of a Hopf algebra \(\mathcal{H}\) with a “Hopf action” on \(\mathcal{Q}(\Gamma)\), i.e.,

\[
h(F^1 \ast F^2) = \sum h_{(1)}(F^1) \ast h_{(2)}(F^2) \quad \forall h \in \mathcal{H}, \ F^1, F^2 \in \mathcal{Q}(\Gamma)
\]

(3.2)

where the coproduct \(\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}\) is given by \(\Delta(h) = \sum h_{(1)} \otimes h_{(2)}\) for any \(h \in \mathcal{H}\). In Section 3.2, we consider operators on the quasimodular tower \(\mathcal{QM}\) that act on the modular coefficients \(a_i(f)\) appearing in (3.1). These induce on \(\mathcal{Q}(\Gamma)\) analogues of operators acting on the modular Hecke algebra \(\mathcal{A}(\Gamma)\) of Connes and Moscovici [3]. Then, we show that \(\mathcal{Q}(\Gamma)\) carries a Hopf action of the same Hopf algebra \(\mathcal{H}_1\) of codimension 1 foliations that acts on \(\mathcal{A}(\Gamma)\).

3.1 The operators \(D, \{T_k\}\) and \(\{\phi(m)\}\) on \(\mathcal{Q}(\Gamma)\)

For any even number \(K \geq 2\), let \(G_K\) be the classical Eisenstein series of weight \(K\) as in (2.12). Since \(G_2\) is a quasimodular form, i.e., \(G_2 \in \mathcal{QM}\), its derivative \(G_2'\) ∈ \(\mathcal{QM}\). Further, it is well known that:

\[
G_2' = \frac{5\pi i}{3} G_4 - 4\pi i G_2^2
\]

(3.3)

where \(G_4\) is the Eisenstein series of weight 4 (which is a modular form). For our purposes, it will be convenient to write:

\[
G_2' = \sum_{j=0}^{2} g_j G_2^j
\]

(3.4)
Proof. We start by proving part (a). For the sake of definiteness, we assume that 

\[ g_0 = \frac{5\pi i}{3} G_4 \quad g_1 = 0 \quad g_2 = -4\pi i \] (3.5)

We are now ready to define the operators \( D \) and \( \{ W_k \}_{k \geq 1} \) on \( \mathcal{QM} \). The first operator \( D \) differentiates the powers of \( G_2 \):

\[
D : \mathcal{QM} \rightarrow \mathcal{QM} \\
f = \sum_{i=0}^{s} a_i(f) G_2^i \mapsto -\frac{1}{8\pi i} \left( \sum_{i=0}^{s} i a_i(f) G_2^{i-1} \cdot G_2' \right) = -\frac{1}{8\pi i} \sum_{i=0}^{s} 2 i a_i(f) g_i G_2^{i+j-1} (3.6)
\]

The operators \( \{ W_k \}_{k \geq 1} \) are “weight operators” and \( W_k \) also steps up the power of \( G_2 \) by \( k - 2 \). We set:

\[
W_k : \mathcal{QM} \rightarrow \mathcal{QM} \quad f = \sum_{i=0}^{s} a_i(f) G_2^i \mapsto \sum_{i=0}^{s} i a_i(f) G_2^{i+k-2} (3.7)
\]

From the definitions in (3.6) and (3.7), we can easily check that \( D \) and \( W_k \) are derivations on \( \mathcal{QM} \). Finally, for any \( \alpha \in GL_2^+ (\mathbb{Q}) \) and any integer \( m \geq 1 \), we set

\[
\nu^{(m)}_\alpha = -\frac{5}{24} (G_4^m | \alpha - G_4^m) (3.8)
\]

Lemma 3.1. (a) Let \( f \in \mathcal{QM} \) be an element of the quasimodular tower and \( \alpha \in GL_2^+ (\mathbb{Q}) \). Then, the operator \( D \) satisfies:

\[
D(f)||\alpha = D(f)||\alpha + \nu^{(1)}_\alpha \cdot (W_1(f)||\alpha) (3.9)
\]

where, using (3.8), we know that \( \nu^{(1)}_\alpha \) is given by:

\[
\nu^{(1)}_\alpha := -\frac{1}{8\pi i} (g_0 | \alpha - g_0) = -\frac{5}{24} (G_4 | \alpha - G_4) \quad \forall \ \alpha \in GL_2^+ (\mathbb{Q}) (3.10)
\]

(b) For \( f \in \mathcal{QM} \) and \( \alpha \in GL_2^+ (\mathbb{Q}) \), each operator \( W_k, k \geq 1 \) satisfies:

\[
W_k(f)||\alpha = W_k(f)||\alpha (3.11)
\]

Proof. We start by proving part (a). For the sake of definiteness, we assume that \( f = \sum_{i=0}^{s} a_i(f) G_2^i \) with each \( a_i(f) \in \mathcal{M} \). For \( \alpha \in GL_2^+ (\mathbb{Q}) \), it follows from (3.6) that:

\[
D(f)||\alpha = -\frac{1}{8\pi i} \left( \sum_{i} \sum_{j} i a_i(f) g_j G_2^{i+j-1} \right) ||\alpha = D(f)||\alpha = D \left( \sum_{i} (a_i(f)|\alpha) G_2^i \right) = -\frac{1}{8\pi i} \sum_{i} \sum_{j} i(a_i(f)|\alpha) g_j G_2^{i+j-1} (3.12)
\]

From (3.12) it follows that:

\[
D(f)||\alpha - D(f)||\alpha = -\frac{1}{8\pi i} \sum_{i=0}^{s} \sum_{j=0}^{2} i(a_i(f)|\alpha) (g_j|\alpha - g_j) G_2^{i+j-1} (3.13)
\]
From (3.5), it is clear that \( g_j |\alpha - g_j = 0 \) for \( j = 1 \) and \( j = 2 \). It follows that:

\[
D(f)||\alpha - D(f||\alpha) = -\frac{1}{8\pi i} \sum_{i=0}^{s} i(a_i(f)||\alpha)(g_0|\alpha - g_0)G_2^{-1} = -\frac{1}{8\pi i}(g_0|\alpha - g_0) \cdot \left( \sum_{i=0}^{s} i(a_i(f)||\alpha)G_2^{-1} \right)
\]

This proves the result of (a). The result of part (b) is clear from the definition in (3.7).

We note here that it follows from (3.8) that for any \( \alpha, \beta \in GL_2^+ (\mathbb{Q}) \), we have:

\[
\nu_{\alpha \beta}^{(m)} = \nu_{\alpha}^{(m)}|\beta + \nu_{\beta}^{(m)} \quad \forall \, m \geq 1
\]  

(3.14)

Additionally, since each \( G_4^m \) is a modular form, we know that when \( \alpha \in SL_2(\mathbb{Z}) \):

\[
\nu_{\alpha}^{(m)} = -\frac{5}{24}(G_4^m|\alpha - G_4^m) = 0 \quad \forall \, \alpha \in SL_2(\mathbb{Z}), \, m \geq 1
\]  

(3.15)

Moreover, from the definitions in (3.6) and (3.7) respectively, it is easily verified that \( D \) and \( \{W_k\}_{k \geq 1} \) are derivations on the quasimodular tower \( \mathcal{Q}\mathcal{M} \). We now proceed to define operators on the quasimodular Hecke algebra \( \mathcal{Q}(\Gamma) \) for some principal congruence subgroup \( \Gamma = \Gamma(N) \). Choose \( F \in \mathcal{Q}(\Gamma) \). We set:

\[
D, W_k, \phi^{(m)} : \mathcal{Q}(\Gamma) \rightarrow \mathcal{Q}(\Gamma) \quad k \geq 1, \, m \geq 1
\]

\[
D(F)_{\alpha} := D(F_{\alpha}) \quad W_k(F)_{\alpha} := W_k(F_{\alpha}) \quad \phi^{(m)}(F)_{\alpha} := \nu_{\alpha}^{(m)} \cdot F_{\alpha} \quad \forall \, \alpha \in GL_2^+(\mathbb{Q})
\]  

(3.16)

From Lemma 3.1 and the properties of \( \nu_{\alpha}^{(m)} \) described in (3.14) and (3.15), it may be easily verified that the operators \( D, W_k \) and \( \phi^{(m)} \) in (3.16) are well defined on \( \mathcal{Q}(\Gamma) \). We will now compute the commutators of the operators \( D, \{W_k\}_{k \geq 1} \) and \( \{\phi^{(m)}\}_{m \geq 1} \) on \( \mathcal{Q}(\Gamma) \). In order to describe these commutators, we need one more operator \( E \):

\[
E : \mathcal{Q}\mathcal{M} \rightarrow \mathcal{Q}\mathcal{M} \quad f \mapsto G_4 \cdot f
\]  

(3.17)

Since \( G_4 \) is a modular form of level \( \Gamma(1) = SL_2(\mathbb{Z}) \), i.e., \( G_4|\gamma = G_4 \) for any \( \gamma \in SL_2(\mathbb{Z}) \), it is clear that \( E \) induces a well defined operator on \( \mathcal{Q}(\Gamma) \):

\[
E : \mathcal{Q}(\Gamma) \rightarrow \mathcal{Q}(\Gamma) \quad E(F)_{\alpha} := E(F_{\alpha}) = G_4 \cdot F_{\alpha} \quad \forall \, F \in \mathcal{Q}(\Gamma), \, \alpha \in GL_2^+(\mathbb{Q})
\]  

(3.18)

We will now describe the commutator relations between the operators \( D, E, \{E^iW_k\}_{k \geq 1, \, i \geq 0} \) and \( \{\phi^{(m)}\}_{m \geq 1} \) on \( \mathcal{Q}(\Gamma) \).

**Proposition 3.2.** Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup and let \( \mathcal{Q}(\Gamma) \) be the algebra of quasimodular Hecke operators of level \( \Gamma \). The operators \( D, E, \{E^iW_k\}_{k \geq 1, \, i \geq 0} \) and \( \{\phi^{(m)}\}_{m \geq 1} \) on \( \mathcal{Q}(\Gamma) \) satisfy the following relations:

\[
[E, E^iW_k] = 0 \quad [E, D] = 0 \quad [E, \phi^{(m)}] = 0 \quad [D, \phi^{(m)}] = 0 \quad [W_k, \phi^{(m)}] = 0 \quad [\phi^{(m)}, \phi^{(m')}] = 0
\]  

(3.19)

Proof. For any \( F \in \mathcal{Q}(\Gamma) \) and any \( \alpha \in GL_2^+(\mathbb{Q}) \), by definition, we know that \( D(F)_{\alpha} = D(F_{\alpha}) \), \( W_k(F)_{\alpha} = W_k(F_{\alpha}) \) and \( E(F)_{\alpha} = E(F_{\alpha}) \). Hence, in order to prove that \( [E, W_k] = 0 \) and \( [E, D] = 0 \), it suffices to show that \( [E, W_k](f) = 0 \) and \( [E, D](f) = 0 \) respectively for any element \( f \in \mathcal{Q}\mathcal{M} \). Both of these are easily verified from the definitions of \( D \) and \( W_k \) in (3.6) and (3.7) respectively. Further, since \( [E, W_k] = 0 \), it is clear that \( [E, E^iW_k] = 0 \).
Similarly, in order to prove the expression for \([E^lW_k, D]\), it suffices to prove that:

\[
[E^lW_k, D](f) = \frac{5}{24}(k-1)(E^{l+1}W_{k-1})(f) - \frac{1}{2}(k-3)E^lW_{k+1}(f)
\]

(3.20)

for any \(f \in \mathcal{Q} \mathcal{M}\). Further, it suffices to consider the case where \(f = \sum_{i=0}^{s} a_i(f)G_2^i\) where the \(a_i(f) \in \mathcal{M}\). We now have:

\[
W_kD(f) = -\frac{1}{8\pi i} W_k \left( \sum_{i=0}^{s} \sum_{j=0}^{2} i a_i(f) g_j G_2^{i+j-1} \right) = -\frac{1}{8\pi i} \sum_{i=0}^{s} \sum_{j=0}^{2} i(i + j - 1) a_i(f) g_j G_2^{i+j+k-3}
\]

\[
DW_k(f) = D \left( \sum_{i=0}^{s} i a_i(f) G_2^{i+k-2} \right) = -\frac{1}{8\pi i} \sum_{i=0}^{s} \sum_{j=0}^{2} i(i + k - 2) a_i(f) g_j G_2^{i+j+k-3}
\]

(3.21)

It follows from (3.21) that:

\[
[W_k, D](f) = -\frac{1}{8\pi i} \sum_{i=0}^{s} \sum_{j=0}^{2} i j a_i(f) g_j G_2^{i+j+k-3} + \frac{1}{8\pi i} \sum_{i=0}^{s} \sum_{j=0}^{2} i(k - 1) a_i(f) g_j G_2^{i+j+k-3}
\]

\[
= -\frac{2}{8\pi i} \sum_{i=0}^{s} a_i f G_2^{i+k-1} + (k-1) \frac{1}{8\pi i} \sum_{i=0}^{s} i a_i(f) g_0 G_2^{i+k-3} + (k-1) \frac{2}{8\pi i} \sum_{i=0}^{s} i a_i(f) G_2^{i+k-1}
\]

where the second equality uses the fact that \(g_1 = 0\). Further, since \(g_0 = \frac{5\pi i}{4} G_4\) and \(g_2 = -4\pi i\), it follows from (3.1) that we have:

\[
[W_k, D](f) = \frac{5}{24}(k-1) \sum_{i=0}^{s} G_4 a_i(f) G_2^{i+k-3} - \frac{1}{2}(k-3) \sum_{i=0}^{s} i a_i(f) G_2^{i+k-1}
\]

\[
= \frac{5}{24}(k-1)(E^l W_{k-1})(f) - \frac{1}{2}(k-3)W_{k+1}(f)
\]

(3.22)

Finally, since \(E\) commutes with \(\{W_k\}_{k \geq 1}\) and \(D\), it follows from (3.22) that:

\[
[E^lW_k, D] = \frac{5}{24}(k-1)(E^{l+1}W_{k-1}) - \frac{1}{2}(k-3)E^lW_{k+1} \quad \forall \ k \geq 1, l \geq 0
\]

(3.23)

as operators on \(\mathcal{Q}(\Gamma)\). Finally, it may be easily verified from the definitions that \([E, \phi^{(m)}] = [D, \phi^{(m)}] = [W_k, \phi^{(m)}] = 0\).

\[
\square
\]

The operators \(\{E^lW_k\}_{k \geq 1, l \geq 0}\) appearing in Proposition 3.2 above can be described more succintly as:

\[
T_k^l : \mathcal{Q} \mathcal{M} \rightarrow \mathcal{Q} \mathcal{M} \quad T_k^l := E^lW_k \quad \forall \ k \geq 1, l \geq 0
\]

(3.24)

and

\[
T_k^l : \mathcal{Q}(\Gamma) \rightarrow \mathcal{Q}(\Gamma) \quad T_k^l(F)_{\alpha} := T_k^l(F_\alpha) = E^lW_k(F_\alpha) \quad \forall \ F \in \mathcal{Q}(\Gamma), \alpha \in GL_2^+(\mathbb{Q})
\]

(3.25)

We are now ready to describe the Lie algebra action on \(\mathcal{Q}(\Gamma)\).

**Proposition 3.3.** Let \(\mathcal{L}\) be the smallest Lie algebra containing the symbols \(D, \{T_k^l\}_{k \geq 1, l \geq 0}, \{\phi^{(m)}\}_{m \geq 1}\) along with the following relations between the commutators:

\[
[D, \phi^{(m)}] = 0 \quad [T_k^l, \phi^{(m)}] = 0 \quad [\phi^{(m)}, \phi^{(m')} ] = 0
\]

(3.26)

\[
[T_k^l, D] = \frac{5}{24}(k-1)T_k^{l+1} - \frac{1}{2}(k-3)T_k^l
\]

Then, for any principal congruence subgroup \(\Gamma = \Gamma(N)\), we have a Lie action of \(\mathcal{L}\) on the algebra of quasimodular Hecke operators \(\mathcal{Q}(\Gamma)\) of level \(\Gamma\).
Proof. For any \( k \geq 1 \) and \( l \geq 0 \), \( T_k^l \) has been defined to be the operator \( E^l W_k \) on \( \mathcal{Q}(\Gamma) \). Then \( \mathcal{L} \) acts on \( \mathcal{Q}(\Gamma) \) and it follows from comparing (3.13) and (3.26) that this is a Lie action.

**Lemma 3.4.** Let \( f \in \mathcal{Q}M \) be an element of the quasimodular tower and let \( \alpha \in GL_2^+ (\mathbb{Q}) \). Then, for any \( k \geq 1 \), \( l \geq 0 \), the operator \( T_k^l : \mathcal{Q}M \rightarrow \mathcal{Q}M \) satisfies:

\[
T_k^l(f)||\alpha = T_k^l(f)||\alpha - \frac{24}{5} \nu^l_{\alpha} \cdot (T_k^0(f)||\alpha)
\]  

(3.27)

**Proof.** For the sake of definiteness, we assume that \( f = \sum_{i=0}^s \alpha_i(f) G_2^i \) with each \( \alpha_i(f) \in \mathcal{M} \). We now compute:

\[
T_k^l(f)||\alpha = (E^l W_k)(f)||\alpha
\]

(3.28)

Subtracting, it follows that:

\[
T_k^l(f)||\alpha - T_k^l(f)||\alpha = (G_2^i||\alpha - G_4^i) \cdot (E^l W_k) \sum_{i=0}^s (i G_2^i||\alpha) G_2^{i+k-2} = - \frac{24}{5} \nu^l_{\alpha} \cdot (W_k(f)||\alpha)
\]

(3.29)

Putting \( T_k^0 = E^0 W_k = W_k \), we have the result.

**Proposition 3.5.** Let \( \Gamma = \Gamma(N) \) be a principal congruence subgroup and let \( \mathcal{Q}(\Gamma) \) be the algebra of quasimodular Hecke operators of level \( \Gamma \). Then, for any \( k \geq 1 \), \( l \geq 0 \), the operator \( T_k^l \) satisfies:

\[
T_k^l(F^1 * F^2) = T_k^l(F^1) * F^2 + F^1 * T_k^l(F^2) + \frac{24}{5} (\phi(l)(F^1) * T_k^0(F^2))_\alpha \quad \forall \, F^1, F^2 \in \mathcal{Q}(\Gamma)
\]  

(3.30)

Further, the operators \( \{ T_k^l \}_{k \geq 1, l \geq 0} \) are all derivations on the algebra \( \mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), *) \).

**Proof.** We know that \( T_k^0 = E^0 W_k \) and that \( W_k \) is a derivation on \( \mathcal{Q}M \). We choose quasimodular Hecke operators \( F^1, F^2 \in \mathcal{Q}(\Gamma) \). Then, for any \( \alpha \in GL_2^+(\mathbb{Q}) \), we know that:

\[
T_k^l(F^1 * F^2)_\alpha = E^l W_k \left( \sum_{\beta \in \Gamma \setminus GL_2^+(\mathbb{Q})} F^1_\beta \cdot (F_2^0)_{\alpha^{-1}}||\beta) \right)
\]

\[
= \sum_{\beta \in \Gamma \setminus GL_2^+(\mathbb{Q})} E^l W_k \left( F^1_\beta \cdot (F_2^0)_{\alpha^{-1}}||\beta) \right)
\]

\[
= \sum_{\beta \in \Gamma \setminus GL_2^+(\mathbb{Q})} G_4^i \cdot W_k(F^1_\beta) \cdot (F_2^0)_{\alpha^{-1}}||\beta) + \sum_{\beta \in \Gamma \setminus GL_2^+(\mathbb{Q})} F^1_\beta \cdot G_4^i \cdot W_k(F_2^0)_{\alpha^{-1}}||\beta)
\]

\[
= \sum_{\beta \in \Gamma \setminus GL_2^+(\mathbb{Q})} T_k^l(F^1)\cdot (F_2^0)\cdot W_k(F_2^0)_{\alpha^{-1}}||\beta)
\]

\[
= (T_k^l(F^1)\cdot (F_2^0)\cdot W_k(F_2^0)_{\alpha^{-1}}||\beta)
\]

(3.29)
Proof.
From (3.32), we know that \( \Delta(\epsilon) = 0 \). This proves (3.30). Further, since \( \nu^{(l)}_{\beta} = 0 \) for any \( \beta \in SL_2(\mathbb{Z}) \), when we consider the product \( \ast^r \) defined in (2.19) on the algebra \( Q'(\Gamma) \), the calculation above reduces to

\[
T_k^i(F^1 \ast^r F^2) = T_k^i(F^1) \ast^r F^2 + F^1 \ast^r T_k^i(F^2)
\]

(3.31)

Hence, each \( T_k^i \) is a derivation on \( Q'(\Gamma) \).

We now introduce the Hopf algebra \( H \) that acts on \( Q(\Gamma) \). As an algebra, \( H \) is identical to the universal enveloping algebra \( U(\mathcal{L}) \) of the Lie algebra \( \mathcal{L} \) defined in Proposition 3.3. In order to define the coproduct \( \Delta : H \rightarrow H \otimes H \), we set:

\[
\Delta(D) = D \otimes 1 + 1 \otimes D - \phi^{(1)} \otimes T_1^0
\]

\[
\Delta(T_1^i) = T_1^i \otimes 1 + 1 \otimes T_1^i + \frac{24}{5} \phi^{(l)} \otimes T_1^0 \quad \forall l \geq 0
\]

(3.32)

\[
\Delta(T_1^i) = T_1^i \otimes 1 + 1 \otimes T_1^i + \frac{24}{5} \phi^{(l)} \otimes T_1^0 \quad \forall l \geq 0
\]

\[
\Delta(\phi^{(m)}) = \phi^{(m)} \otimes 1 + 1 \otimes \phi^{(m)} \quad \forall m \geq 1
\]

We mention again that in (3.32), \( \phi^{(0)} \) is understood to be 0. In order to define the antipode \( S : H \rightarrow H \), we set:

\[
S(D) = -D - \phi^{(1)} T_1^0
\]

\[
S(T_1^i) = -T_1^i + \frac{24}{5} \phi^{(l)} T_1^0 \quad \forall l \geq 0
\]

\[
S(T_1^i) = -T_1^i + \frac{24}{5} \phi^{(l)} T_1^0 \quad \forall l \geq 0
\]

\[
S(\phi^{(m)}) = -\phi^{(m)} \quad \forall m \geq 1
\]

(3.33)

Finally, the counit \( \epsilon : H \rightarrow \mathbb{C} \) is defined by setting \( \epsilon(h) \) to be the constant term of \( h \) for any \( h \in H = U(\mathcal{L}) \). We first need to verify that \( H \) is indeed a Hopf algebra.

**Lemma 3.6.** Let \( H \) be the universal enveloping algebra \( U(\mathcal{L}) \) of the Lie algebra \( \mathcal{L} \) with generators \( D, \{T_1^i\}_{k \geq 1, l \geq 0} \) and \( \{\phi^{(m)}\}_{m \geq 1} \) satisfying the commutator relations in (3.30). Then, \( \Delta \) and \( S \) as defined in (3.32) and (3.33) can be extended to define a Hopf algebra structure on \( H \) with coproduct \( \Delta : H \rightarrow H \otimes H \), antipode \( S : H \rightarrow H \) and counit \( \epsilon \).

**Proof.** From (3.32), we know that \( \Delta(T_1^i) = T_1^i \otimes 1 + 1 \otimes T_1^i + \frac{24}{5} \phi^{(l)} \otimes T_1^0 \) for any \( l \geq 0 \). In order to define \( \Delta(T_1^i) \) for any \( k \geq 1 \), we proceed by induction. Suppose that \( \Delta(T_1^i) \) has been defined for all \( l \geq 0, k \leq K \), where \( K \geq 4 \). Then, using (3.30), we can define:

\[
\Delta(T_{K+1}^i) := \frac{2}{K-3} \left( \frac{5}{24} \Delta(T_{K-1}^{i+1}) - \Delta(T_K^i), \Delta(D) \right)
\]

(3.34)

In (3.31), we notice that since \( K \geq 4 \), the denominator \( (K - 3) \) is never 0. Since \( \Delta(T_1^i) \) and \( \Delta(T_4^i) \) are already defined for all \( l \geq 0 \), it only remains to define the coproducts \( \Delta(T_2^i) \) and \( \Delta(T_3^i) \). For this, we again use the formulae in (3.30) and set:

\[
\Delta(T_2^i) := [\Delta(T_1^i), \Delta(D)] \quad \Delta(T_3^i) := 2[\Delta(T_2^i), \Delta(D)] - \frac{5}{12} \Delta(T_1^{i+1})
\]

(3.35)

In order to define the antipode \( S \), we proceed similarly. First, we suppose that the antipode \( S \) can be defined for all \( T_k^l, l \geq 0, k \leq K \) where \( K \geq 4 \). Then, using the fact that the antipode should be an “anti-homomorphism” (i.e., \( S(h_1 h_2) = S(h_2) S(h_1) \) for all \( h_1, h_2 \in H \)) along with the relations (3.30), we set:

\[
S(T_{K+1}^i) := \frac{2}{K-3} \left( \frac{5}{24} S(T_{K-1}^{i+1}) - [S(D), S(T_K^i)] \right)
\]

(3.36)
Further, the operators $S(T^i_2)$ and $S(T^i_3)$ can be computed by setting:

$$S(T^i_2) := [S(D), S(T^i_1)] \quad S(T^i_3) := 2[S(D), S(T^i_2)] - \frac{5}{12} S(T^i_1 + 1) \quad (3.37)$$

It now remains to show that $m \circ (id \otimes S) \circ \Delta = \eta \circ \epsilon = m \circ (S \otimes id) \circ \Delta : \mathcal{H} \rightarrow \mathcal{H}$ where $m : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ and $\eta : \mathbb{C} \rightarrow \mathcal{H}$ are respectively the multiplication and unit maps on $\mathcal{H}$. It may be easily verified that this equality holds for each of $D$, $\{T^i_1 \}_{i \geq 0}$, $\{T^i_3 \}_{i \geq 0}$ and $\{\phi^m \}_{m \geq 1}$ and hence for all of $\mathcal{H}$.

We will now show that $\mathcal{H}$ has a Hopf action on the algebra $Q(\Gamma)$. In other words, we have:

$$h(F^1 * F^2) = \sum h(1) (F^1) * h(2) (F^2) \quad \forall \ h \in \mathcal{H}, \ F^1, F^2 \in Q(\Gamma) \quad (3.38)$$

where we are using Sweedler notation for the coproduct on $\mathcal{H}$, i.e., $\Delta(h) = \sum h(1) \otimes h(2)$ for any $h \in \mathcal{H}$. Additionally, we show that corresponding to the product $*^r$ on $Q^r(\Gamma)$, we have a smaller Hopf algebra $\mathfrak{h}$ with a Hopf action on $Q^r(\Gamma)$. Let $l$ be the smallest Lie algebra containing the symbols $D$, $\{T^i_1 \}_{k \geq 1, i \geq 0}$ along with the commutator relations:

$$[T^i_k, D] = \frac{5}{24} (k - 1) T^i_{k-1} - \frac{1}{2} (k - 3) T^i_{k+1} \quad \forall \ k \geq 1, l \geq 0 \quad (3.39)$$

Then, the Hopf algebra $\mathfrak{h}$ is defined to be the universal enveloping algebra of the Lie algebra $l$. In other words, the coproduct $\Delta : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$ on $\mathfrak{h}$ is defined by:

$$\Delta(D) = D \otimes 1 + 1 \otimes D \quad \forall \ k \geq 1, l \geq 0 \quad (3.40)$$

Further, the antipode $S : \mathfrak{h} \rightarrow \mathfrak{h}$ is defined by setting:

$$S(D) = -D \quad S(T^i_k) = -T^i_k \quad \forall \ k \geq 1, l \geq 0 \quad (3.41)$$

**Proposition 3.7.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and let $Q(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$.

(a) The operator $D : Q(\Gamma) \rightarrow Q(\Gamma)$ on the algebra $(Q(\Gamma), *)$ satisfies:

$$D(F^1 * F^2) = D(F^1) * F^2 + F^1 * D(F^2) - \phi^1(F^1) * T^0_1(F^2) \quad \forall \ F^1, F^2 \in Q(\Gamma) \quad (3.42)$$

When we consider the product $*^r$, the operator $D$ becomes a derivation on the algebra $Q^r(\Gamma) = (Q(\Gamma), *^r)$, i.e.:

$$D(F^1 *^r F^2) = D(F^1) *^r F^2 + F^1 *^r D(F^2) \quad \forall \ F^1, F^2 \in Q^r(\Gamma) \quad (3.43)$$

(b) The operators $\{W_k \}_{k \geq 1}$ and $\{\phi^m \}_{m \geq 1}$ are derivations on $Q(\Gamma)$, i.e.,

$$W_k(F^1 * F^2) = W_k(F^1) * F^2 + F^1 * W_k(F^2) \quad \forall \ F^1, F^2 \in Q(\Gamma), \quad \phi^m(F^1 * F^2) = \phi^m(F^1) * F^2 + F^1 * \phi^m(F^2) \quad (3.44)$$

for any $F^1, F^2 \in Q(\Gamma)$. Additionally, $\{\phi^m \}_{m \geq 1}$ and $\{W_k \}_{k \geq 1}$ are also derivations on the algebra $Q^r(\Gamma) = (Q(\Gamma), *^r)$.  


Proof. (a) We choose quasimodular Hecke operators $F^1, F^2 \in \mathcal{Q}(\Gamma)$. We have mentioned before that $D$ is a derivation on $\mathcal{Q}\mathcal{M}$. Then, for any $\alpha \in GL^*_2(\mathbb{Q})$, we have:

\[
D(F^1 \ast F^2)_\alpha = D \left( \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} F^1_\beta \cdot (F^2_{\alpha \beta^{-1}}|\beta) \right) = \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} D \left( F^1_\beta \cdot (F^2_{\alpha \beta^{-1}}|\beta) \right) = \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} D(F^1_\beta) \cdot (F^2_{\alpha \beta^{-1}}|\beta) + \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} F^1_\beta \cdot D(F^2_{\alpha \beta^{-1}}|\beta)
\]

This proves (3.32). In order to prove (3.33), we note that $\nu^{(1)}_\beta = 0$ for any $\beta \in SL_2(\mathbb{Z})$ (see (3.15)). Hence, when we use the product $\ast^r$ defined in (2.19), the calculation above reduces to

\[
D(F^1 \ast^r F^2) = D(F^1) \ast^r F^2 + F^1 \ast^r D(F^2)
\]

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$.

(b) For any $F^1, F^2 \in \mathcal{Q}(\Gamma)$ and knowing from (3.14) that $\nu^{(m)}_\alpha = \nu^{(m)}_\beta + \nu^{(m)}_{\alpha \beta^{-1}}|\beta$, we have:

\[
\phi^{(m)}(F^1 \ast F^2)_\alpha = \nu^{(m)}_\alpha \cdot \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} F^1_\beta \cdot (F^2_{\alpha \beta^{-1}}|\beta) = \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} (\nu^{(m)}_\beta \cdot F^1_\beta) \cdot (F^2_{\alpha \beta^{-1}}|\beta) + \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} F^1_\beta \cdot (\nu^{(m)}_{\alpha \beta^{-1}} \cdot F^2_{\alpha \beta^{-1}}|\beta)
\]

\[
= \phi^{(m)}(F^1) \ast F^2 + \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} F^1_\beta \cdot (\nu^{(m)}_{\alpha \beta^{-1}} \cdot F^2_{\alpha \beta^{-1}}|\beta)
\]

\[
= \phi^{(m)}(F^1) \ast F^2 + \sum_{\beta \in \Gamma \cap GL^*_2(\mathbb{Q})} F^1_\beta \cdot \phi^{(m)}(F^2)_{\alpha \beta^{-1}}|\beta)
\]

The fact that each $W_k$ is also a derivation on $\mathcal{Q}(\Gamma)$ now follows from a similar calculation using the fact that $W_k$ is a derivation on the quasimodular tower $\mathcal{Q}\mathcal{M}$ and that $W_k(f)|\alpha = W_k(f|\alpha)$ for any $f \in \mathcal{Q}\mathcal{M}$, $\alpha \in GL^*_2(\mathbb{Q})$ (from (3.11)). Finally, a similar calculation may be used to verify that $\{W_k\}_{k \geq 1}$ and $\{\phi^{(m)}\}_{m \geq 1}$ are all derivations on $\mathcal{Q}^r(\Gamma)$.

Proposition 3.8. Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Then, there is a Hopf action of $\mathcal{H}$ on the algebra $\mathcal{Q}(\Gamma)$, i.e.,

\[
h(F^1 \ast F^2) = \sum_{h \in \mathcal{H}} h(F^1) \ast h(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}(\Gamma)
\]

where $\Delta(h) = \sum h(1) \otimes h(2)$ for any $h \in \mathcal{H}$. Additionally, the Hopf algebra $\mathcal{H}$ has a Hopf action on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), \ast^r)$.

Proof. In order to prove (3.41), it suffices to verify the relation for each of $D$, $\{T^1_l\}_{l \geq 0}$, $\{T^2_l\}_{l \geq 0}$ and $\{\phi^{(m)}\}_{m \geq 1}$. From Proposition 3.7 and Proposition 3.5 we know that for $F^1, F^2 \in \mathcal{Q}(\Gamma)$ and any $l \geq 0$:

\[
D(F^1 \ast F^2) = D(F^1) \ast F^2 + F^1 \ast D(F^2) - \phi^{(1)}(F^1) \ast T^0(F^2)
\]

\[
T^1_l(F^1 \ast F^2) = T^1_l(F^1) \ast F^2 + F^1 \ast T^1_l(F^2) + 2 \phi^{(1)}(F^1) \ast T^0(F^2)
\]

\[
T^2_l(F^1 \ast F^2) = T^2_l(F^1) \ast F^2 + F^1 \ast T^2_l(F^2) + 2 \phi^{(1)}(F^1) \ast T^0(F^2)
\]

\[
\phi^{(m)}(F^1 \ast F^2) = \phi^{(m)}(F^1) \ast F^2 + F^1 \ast \phi^{(m)}(F^2)
\]
Comparing with the expressions for the coproduct in \(3.32\), it is clear that \(3.47\) holds for each \(h \in \mathcal{H}\).

Finally, we know from Proposition 3.5 and Proposition 3.7 that \(D\) and \(\{T^k_l\}_{k \geq 1, l \geq 0}\) are all derivations on the algebra \(Q'(\Gamma) = (Q(\Gamma), \ast^r)\). Since \(h\) carries the Hopf algebra structure on the universal enveloping algebra \(U(l)\) of \(l\), the coproduct on \(h\) is determined by \(\Delta(x) = x \otimes 1 + 1 \otimes x\) for each \(x \in l\) and hence, in particular for \(D\), \(\{T^k_l\}_{k \geq 1, l \geq 0}\). This proves the result. \(\square\)

### 3.2 The operators \(X, Y\) and \(\{\delta_n\}\) of Connes and Moscovici

Let \(\Gamma = \Gamma(N)\) be a congruence subgroup. In this subsection, we will show that the algebra \(Q(\Gamma)\) carries an action of the Hopf algebra \(H_1\) of Connes and Moscovici [2]. The Hopf algebra \(H_1\) is part of a larger family \(\{H_n\}_{n \geq 1}\) of Hopf algebras defined in [2] and \(H_1\) is the Hopf algebra corresponding to “codimension 1 foliations”.

As an algebra, \(H_1\) is identical to the universal enveloping algebra \(U(L_1)\) of the Lie algebra \(L_1\) generated by \(X, Y, \{\delta_n\}_{n \geq 1}\) satisfying the commutator relations:

\[
[Y, X] = X \quad [X, \delta_n] = \delta_{n+1} \quad [Y, \delta_n] = n\delta_n \quad [\delta_k, \delta_l] = 0 \quad \forall \ k, l, n \geq 1
\]

Further, the coproduct \(\Delta : H_1 \rightarrow H_1 \otimes H_1\) on \(H_1\) is determined by:

\[
\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \quad \Delta(Y) = Y \otimes 1 + 1 \otimes Y \quad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1
\]

Finally, the antipode \(S : H_1 \rightarrow H_1\) is given by:

\[
S(X) = -X + \delta_1 Y \quad S(Y) = -Y \quad S(\delta_1) = -\delta_1
\]

Following Connes and Moscovici [3], we define the operators \(X\) and \(Y\) on the modular tower: for any congruence subgroup \(\Gamma = \Gamma(N)\), we set:

\[
Y : M_k(\Gamma) \rightarrow M_k(\Gamma) \quad Y(f) := \frac{k}{2} f \quad \forall \ f \in M_k(\Gamma)
\]

Further, the operator \(X : M_k(\Gamma) \rightarrow M_{k+2}(\Gamma)\) is the Ramanujan differential operator on modular forms:

\[
X(f) := \frac{1}{2\pi i} \frac{d}{dz}(f) - \frac{1}{12\pi i} \frac{d}{dz} \log \Delta \cdot Y(f) \quad \forall \ f \in M_k(\Gamma)
\]

where \(\Delta(z)\) is the well known modular form of weight 12 given by:

\[
\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi iz}
\]

We start by extending these operators to the quasimodular tower \(QM\). Let \(f \in QM^*_k(\Gamma)\) be a quasimodular form. Then, we can express \(f = \sum_{i=0}^{s} a_i(f)G_2^i\) where \(a_i(f) \in M_{k-2i}(\Gamma)\). We set:

\[
X(f) = \sum_{i=0}^{s} X(a_i(f)) \cdot G_2^i \quad Y(f) = \sum_{i=0}^{s} Y(a_i(f)) \cdot G_2^i
\]

From (3.55), it is clear that \(X\) and \(Y\) are derivations on \(QM\).
Lemma 3.9. Let $f \in QM$ be an element of the quasimodular tower. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$X(f)||\alpha = X(f||\alpha) + (\mu_{\alpha^{-1}} \cdot Y(f)||\alpha)$$

(3.56)

where, for any $\delta \in GL_2^+(\mathbb{Q})$, we set:

$$\mu_\delta := \frac{1}{12\pi i} \frac{d}{dz} \log \frac{\Delta|\delta}{\Delta}$$

(3.57)

Further, we have $Y(f||\alpha) = Y(f)||\alpha$.

Proof. Following [3, Lemma 5], we know that for any $g \in M$, we have:

$$X(g||\alpha = X(g||\alpha) + (\mu_{\alpha^{-1}} \cdot Y(g)||\alpha) \quad \forall \alpha \in GL_2^+(\mathbb{Q})$$

(3.58)

It suffices to consider the case $f \in QM^+_k(\Gamma)$ for some congruence subgroup $\Gamma$. If we express $f \in QM^+_k(\Gamma)$ as $f = \sum_{i=0}^s a_i(f)G_2^i$ with $a_i(f) \in M_{k-2i}(\Gamma)$, it follows that:

$$X(a_i(f)||\alpha = X(a_i(f)||\alpha) + (\mu_{\alpha^{-1}} \cdot Y(a_i(f)))|\alpha \quad \forall \alpha \in GL_2^+(\mathbb{Q})$$

(3.59)

for each $0 \leq i \leq s$. Combining (3.59) with the definitions of $X$ and $Y$ on the quasimodular tower in (3.55), we can easily prove (3.60). Finally, it is clear from the definition of $Y$ that $Y(f||\alpha) = Y(f)||\alpha$.

From the definition of $\mu_\delta$ in (3.57), one may verify that (see [3 § 3]):

$$\mu_{\delta_1\delta_2} = \mu_{\delta_1}|\delta_2 + \mu_{\delta_2} \quad \forall \delta_1, \delta_2 \in GL_2^+(\mathbb{Q})$$

(3.60)

and that $\mu_\delta = 0$ for any $\delta \in SL_2(\mathbb{Z})$. We now define operators $X, Y$ and $\{\delta_n\}_{n \geq 1}$ on the quasimodular Hecke algebra $Q(\Gamma)$ for some congruence subgroup $\Gamma = \Gamma(N)$. Let $F \in Q(\Gamma)$ be a quasimodular Hecke operator of level $\Gamma$; then we define operators:

$$X, Y, \delta_n : Q(\Gamma) \rightarrow Q(\Gamma),$$

$$X(F) := X(F_\alpha) \quad Y(F) := Y(F_\alpha) \quad \delta_n(F) := X^{n-1}(\mu_{\alpha}) \cdot F_\alpha \quad \forall \alpha \in GL_2^+(\mathbb{Q})$$

(3.61)

We will now show that the Lie algebra $L_1$ with generators $X, Y, \{\delta_n\}_{n \geq 1}$ satisfying the commutator relations in (3.49) acts on the algebra $Q(\Gamma)$. Additionally, in order to give a Lie action on the algebra $Q^r(\Gamma) = (Q(\Gamma), *)$, we define at this juncture the smaller Lie algebra $l_1 \subseteq L_1$ with generators $X$ and $Y$ satisfying the relation

$$[Y, X] = X$$

(3.62)

Further, we consider the Hopf algebra $b_1$ that arises as the universal enveloping algebra $U(l_1)$ of the Lie algebra $l_1$. We will show that $H_1$ (resp. $b_1$) has a Hopf action on the algebra $Q(\Gamma)$ (resp. $Q^r(\Gamma)$). We start by describing the Lie actions.

Proposition 3.10. Let $L_1$ be the Lie algebra with generators $X, Y$ and $\{\delta_n\}_{n \geq 1}$ satisfying the following commutator relations:

$$[Y, X] = X \quad [X, \delta_n] = \delta_{n+1} \quad [Y, \delta_n] = n\delta_n \quad [\delta_k, \delta_l] = 0 \quad \forall k, l, n \geq 1$$

(3.63)

Then, for any given congruence subgroup $\Gamma = \Gamma(N)$ of $SL_2(\mathbb{Z})$, we have a Lie action of $L_1$ on the module $Q(\Gamma)$. 
Similarly, since $X$ and $Y$ on the quasimodular tower $\mathcal{QM}$ (see (3.55)) is naturally extended from their action on $\mathcal{M}$, it follows that $[Y, X] = X$ on the quasimodular tower $\mathcal{QM}$. In particular, given any quasimodular Hecke operator $F \in \mathcal{Q}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we have $[Y, X](F) = X(F)$ for the element $F \in \mathcal{QM}$. By definition, $X(F) = X(F)$ and $Y(F) = Y(F)$ and hence $[Y, X] = X$ holds for the action of $X$ and $Y$ on $\mathcal{Q}(\Gamma)$.

Further, since $X$ is a derivation on $\mathcal{QM}$ and $\delta_n(F) = X^{n-1}(\mu_\alpha) \cdot F$, we have

$$[X, \delta_n(F)](\alpha) = X(X^{n-1}(\mu_\alpha) \cdot F) - X^{n-1}(\mu_\alpha) \cdot X(F) = X(X^{n-1}(\mu_\alpha)) \cdot F = X(\mu_\alpha) \cdot F = \delta_{n+1}(F) \alpha$$

(3.64)

Similarly, since $\mu_\alpha \in \mathcal{M} \subseteq \mathcal{QM}$ is of weight 2 and $Y$ is a derivation on $\mathcal{QM}$, we have:

$$[Y, \delta_n(F)](\alpha) = Y(X^{n-1}(\mu_\alpha) \cdot F) - X^{n-1}(\mu_\alpha) \cdot Y(F) = Y(X^{n-1}(\mu_\alpha)) \cdot F = nX^{n-1}(\mu_\alpha) \cdot F = n\delta_n(F) \alpha$$

(3.65)

Finally, we can verify easily that $[\delta_k, \delta_l] = 0$ for any $k, l \geq 1$.

From Proposition (3.10), it is also clear that the smaller Lie algebra $l_1 \subseteq C_1$ has a Lie action on the module $\mathcal{Q}(\Gamma)$.

**Lemma 3.11.** Let $\Gamma = \Gamma(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let $\mathcal{Q}(\Gamma)$ be the algebra of quasimodular Hecke operators of level $\Gamma$. Then, the operator $X : \mathcal{Q}(\Gamma) \rightarrow \mathcal{Q}(\Gamma)$ on the algebra $(\mathcal{Q}(\Gamma), *)$ satisfies:

$$X(F^1 \ast F^2) = X(F^1) \ast F^2 + F^1 \ast X(F^2) + \delta_1(F^1) \ast Y(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}(\Gamma)$$

(3.66)

When we consider the product $\ast^r$, the operator $X$ becomes a derivation on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), \ast^r)$, i.e.:

$$X(F^1 \ast^r F^2) = X(F^1) \ast^r F^2 + F^1 \ast^r X(F^2) \quad \forall F^1, F^2 \in \mathcal{Q}^r(\Gamma)$$

(3.67)

**Proof.** We choose quasimodular Hecke operators $F^1, F^2 \in \mathcal{Q}(\Gamma)$. Using (3.60), we also note that

$$0 = \mu_1 = \mu_{1-1} \beta + \mu_\beta \quad \forall \beta \in GL_2^+(\mathbb{Q})$$

(3.68)

We have mentioned before that $X$ is a derivation on $\mathcal{QM}$. Then, for any $\alpha \in GL_2^+(\mathbb{Q})$, we have:

$$X(F^1 \ast F^2)(\alpha) = X \left( \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} \frac{F^1_{\beta}}{F_{\beta} - F_{\alpha}} \cdot (F^2_{\alpha} \cdot |\beta|) \right)$$

$$= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} X(F^1_{\beta}) \cdot (F^2_{\alpha \beta^{-1}} \cdot |\beta|)$$

$$= \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} X(F^1_{\beta}) \cdot (F^2_{\alpha \beta^{-1}} \cdot |\beta|) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F^1_{\beta} \cdot X(F^2_{\alpha \beta^{-1}} \cdot |\beta|)$$

$$= (X(F^1) \ast F^2)(\alpha) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F^1_{\beta} \cdot (X(F^2_{\alpha \beta^{-1}} \cdot |\beta|) - \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F^1_{\beta} \cdot ((\mu_{\beta-1} \cdot Y(F^2_{\alpha \beta^{-1}} \cdot |\beta|)$$

$$= (X(F^1) \ast F^2)(\alpha) + \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F^1_{\beta} \cdot (X(F^2_{\alpha \beta^{-1}} \cdot |\beta|) - \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F^1_{\beta} \cdot (\delta_1(F^1) \ast Y(F^2_{\alpha \beta^{-1}} \cdot |\beta|)$$

This proves (3.66). In order to prove (3.67), we note that $\mu_\beta = 0$ for any $\beta \in SL_2(\mathbb{Z})$. Hence, if we use the product $\ast^r$, the calculation above reduces to

$$X(F^1 \ast^r F^2) = X(F^1) \ast^r F^2 + F^1 \ast^r X(F^2)$$

(3.69)

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$.

19
Finally, we describe the Hopf action of $\mathcal{H}_1$ on the algebra $(\mathcal{Q}(\Gamma), \ast)$ as well as the Hopf action of $\mathfrak{h}_1$ on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), \ast^r)$.

**Proposition 3.12.** Let $\Gamma = \Gamma(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$. Then, the Hopf algebra $\mathcal{H}_1$ has a Hopf action on the quasimodular Hecke algebra $(\mathcal{Q}(\Gamma), \ast)$; in other words, we have:

$$h(F^1 \ast F^2) = \sum h_{(1)}(F^1) \otimes h_{(2)}(F^2) \quad \forall h \in \mathcal{H}_1, F^1, F^2 \in \mathcal{Q}(\Gamma)$$  \hspace{1cm} (3.70)

where the coproduct $\Delta : \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{H}_1$. Similarly, there exists a Hopf action of the Hopf algebra $\mathfrak{h}_1$ on the algebra $\mathcal{Q}^r(\Gamma) = (\mathcal{Q}(\Gamma), \ast^r)$.

**Proof.** In order to prove (3.70), it suffices to check the relation for $X, Y$ and $\delta_1 \in \mathcal{H}_1$. For the element $X \in \mathcal{H}_1$, this is already the result of Lemma 3.11. Now, for any $F^1, F^2 \in \mathcal{Q}(\Gamma)$ and $\alpha \in GL^+_2(\mathbb{Q})$, we have:

$$\delta_1(F^1 \ast F^2)_{\alpha} = \mu_\alpha \cdot \left( \sum_{\beta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_\beta \cdot (F^2_{\alpha,\beta^{-1}} || \beta) \right)$$  \hspace{1cm} (3.71)

$$\quad = \sum_{\beta \in \Gamma \setminus GL^+_2(\mathbb{Q})} (\mu_\beta \cdot F^1_\beta) \cdot (F^2_{\alpha,\beta^{-1}} || \beta) + \sum_{\beta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_\beta \cdot ((\mu_{\alpha,\beta^{-1}} \cdot F^2_{\alpha,\beta^{-1}}) || \beta)$$

Further, using the fact that $Y$ is a derivation on $\mathcal{Q}_\mathcal{M}$ and $Y(f || \alpha) = Y(f) || \alpha$ for any $f \in \mathcal{Q}_\mathcal{M}, \alpha \in GL^+_2(\mathbb{Q})$, we can easily verify the relation (3.70) for the element $Y \in \mathcal{H}_1$. This proves (3.70) for all $h \in \mathcal{H}_1$.

Finally, in order to demonstrate the Hopf action of $\mathfrak{h}_1$ on $\mathcal{Q}^r(\Gamma)$, we need to check that:

$$X(F^1 \ast^r F^2) = X(F^1 \ast^r F^2 + F^1 \ast^r X(F^2)) \quad Y(F^1 \ast^r F^2) = Y(F^1 \ast^r F^2 + F^1 \ast^r Y(F^2)$$  \hspace{1cm} (3.72)

for any $F^1, F^2 \in \mathcal{Q}^r(\Gamma)$. The relation for $X$ has already been proved in (3.69). The relation for $Y$ is again an easy consequence of the fact that $Y$ is a derivation on $\mathcal{Q}_\mathcal{M}$ and $Y(f || \alpha) = Y(f) || \alpha$ for any $f \in \mathcal{Q}_\mathcal{M}, \alpha \in GL^+_2(\mathbb{Q})$.

---

4 Twisted Quasimodular Hecke operators

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. For any $\sigma \in SL_2(\mathbb{Z})$, we have developed the theory of $\sigma$-twisted modular Hecke operators in \textbf{1}. In this section, we introduce and study the collection $\mathcal{Q}_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. When $\sigma = 1$, $\mathcal{Q}_\sigma(\Gamma)$ coincides with the algebra $\mathcal{Q}(\Gamma)$ of quasimodular Hecke operators. In general, we will show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathcal{Q}(\Gamma)$-module and carries a pairing:

$$\langle \cdot, \cdot \rangle : \mathcal{Q}_\sigma(\Gamma) \otimes \mathcal{Q}_\sigma(\Gamma) \longrightarrow \mathcal{Q}_\sigma(\Gamma)$$  \hspace{1cm} (4.1)

We recall from Section 3 the Lie algebra $\mathfrak{l}_1$ with two generators $Y, X$ satisfying $[Y, X] = X$. If we let $\mathfrak{h}_1$ be the Hopf algebra that is the universal enveloping algebra of $\mathfrak{l}_1$, we show in Section 4.1 that the pairing in (4.1) on $\mathcal{Q}_\sigma(\Gamma)$ carries a “Hopf action” of $\mathfrak{h}_1$. In other words, we have:

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall h \in \mathfrak{h}_1, F^1, F^2 \in \mathcal{Q}_\sigma(\Gamma)$$  \hspace{1cm} (4.2)
where the coproduct $\Delta : h_1 \rightarrow h_1 \otimes h_1$ is given by $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in h_1$. In Section 4.2, we consider operators $X_\tau : Q_\sigma(\Gamma) \rightarrow Q_{\tau \sigma}(\Gamma)$ for any $\tau, \sigma \in SL_2(\mathbb{Z})$. In particular, we consider operators acting between the levels of the graded module:

$$Q_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} Q_{\sigma(m)}(\Gamma)$$

where for any $\sigma \in SL_2(\mathbb{Z})$, we set $\sigma(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \sigma$. Further, we generalize the pairing on $Q_\sigma(\Gamma)$ in (4.1) to a pairing:

$$(\quad , \quad) : Q_{\sigma(m)}(\Gamma) \otimes Q_{\sigma(n)}(\Gamma) \rightarrow Q_{\sigma(m+n)}(\Gamma) \quad \forall \ m, n \in \mathbb{Z}$$(4.4)

We show that the pairing in (4.4) is a special case of a more general pairing

$$(\quad , \quad) : Q_{\tau_1 \sigma}(\Gamma) \otimes Q_{\tau_2 \sigma}(\Gamma) \rightarrow Q_{\tau_1 \tau_2 \sigma}(\Gamma)$$

where $\tau_1, \tau_2$ are commuting matrices in $SL_2(\mathbb{Z})$. From (4.4), it is clear that we have a graded pairing on $Q_\sigma(\Gamma)$ that extends the pairing on $Q_\sigma(\Gamma)$. Finally, we consider the Lie algebra $l_\mathbb{Z}$ with generators $\{Z, X_n| n \in \mathbb{Z}\}$ satisfying the commutator relations:

$$[Z, X_n] = (n + 1)X_n \quad [X_n, X_{n'}] = 0 \quad \forall \ n, n' \in \mathbb{Z}$$

Then, for $n = 0$, we have $[Z, X_0] = X_0$ and hence the Lie algebra $l_\mathbb{Z}$ contains the Lie algebra $l_1$ acting on $Q_\sigma(\Gamma)$. In other words, for any $F^1, F^2 \in Q_\sigma(\Gamma)$, we have

$$h(F^1, F^2) = \sum (h_{(1)}(F^1), h_{(2)}(F^2)) \quad \forall \ h \in h_\mathbb{Z}$$

where the coproduct $\Delta : h_\mathbb{Z} \rightarrow h_\mathbb{Z} \otimes h_\mathbb{Z}$ is defined by setting $\Delta(h) := \sum h_{(1)} \otimes h_{(2)}$ for each $h \in h_\mathbb{Z}$.

### 4.1 The pairing on $Q_\sigma(\Gamma)$ and Hopf action

Let $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. We start by defining the collection $Q_\sigma(\Gamma)$ of quasimodular Hecke operators of level $\Gamma$ twisted by $\sigma$. When $\sigma = 1$, this reduces to the definition of $Q(\Gamma)$.

**Definition 4.1.** Choose $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. A $\sigma$-twisted quasimodular Hecke operator $F$ of level $\Gamma$ is a function of finite support:

$$F : \Gamma \backslash GL^+_2(\mathbb{Q}) \rightarrow \text{QM} \quad \Gamma \alpha \mapsto F_\alpha \in \text{QM}$$

such that:

$$F_{\sigma \gamma} = F_\alpha ||_{\sigma \gamma} \sigma^{-1} \quad \forall \ \gamma \in \Gamma$$

We denote by $Q_\sigma(\Gamma)$ the collection of $\sigma$-twisted quasimodular Hecke operators of level $\Gamma$. 
We put \( \beta \in \Gamma \) and hence the sum in (4.11) is well defined, i.e., it does not depend on the choice of coset representatives. We have to show that \((F^1, F^2) \in Q_\sigma(\Gamma)\). For this, we first note that \(F^2_{\delta\sigma^{-1}\beta^{-1}} = F^2_{\alpha\sigma^{-1}\beta^{-1}}\) for any \(\gamma \in \Gamma\) and hence from the expression in (4.11), it follows that \((F^1, F^2)_\alpha = (F^1, F^2)\). On the other hand, for any \(\gamma \in \Gamma\), we can write:

\[
(F^1, F^2) = \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} F^1_{\beta\sigma} \cdot (F^2_{\sigma\delta\gamma^{-1}}) \quad \forall \ F^1, F^2 \in Q_\sigma(\Gamma), \alpha \in GL_2^+ (\mathbb{Q})
\]  

(4.11)

We now consider the Hopf algebra \(H_1\) defined in Section 3.2. By definition, \(H_1\) is the universal enveloping algebra of the Lie algebra \(L_1\) with two generators \(X\) and \(Y\) satisfying \([Y, X] = X\). We will now show that \(L_1\) has a Lie action on \(Q_\sigma(\Gamma)\) and that \(H_1\) has a “Hopf action” with respect to the pairing on \(Q_\sigma(\Gamma)\).

**Proposition 4.3.** Let \(\gamma = \Gamma(\mathbb{N})\) be a principal congruence subgroup of \(SL_2(\mathbb{Z})\) and choose some \(\sigma \in SL_2(\mathbb{Z})\). Then there exists a pairing:

\[
\langle \cdot, \cdot \rangle : Q_\sigma(\Gamma) \otimes Q_\sigma(\Gamma) \rightarrow Q_\sigma(\Gamma)
\]

defined as follows:

\[
(F^1, F^2) = \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} F^1_{\beta\sigma} \cdot (F^2_{\sigma\delta\gamma^{-1}}) \quad \forall \ F^1, F^2 \in Q_\sigma(\Gamma), \alpha \in GL_2^+ (\mathbb{Q})
\]

(4.10)
Proof. (a) We need to verify that for any $F \in \mathcal{Q}_\sigma(\Gamma)$ and any $\alpha \in GL^+_2(\mathbb{Q})$, we have $\langle Y, X|F\rangle_\alpha = X(F)_\alpha$. We know that for any element $g \in \mathcal{Q}M$ and hence in particular for the element $F_\alpha \in \mathcal{Q}M$, we have $Y, X|g) = X(g)$.

The result now follows from the definition of the action of $X$ and $Y$ in (4.15).

(b) The Lie action of $I_1$ on $\mathcal{Q}_\sigma(\Gamma)$ from part (a) induces an action of the universal enveloping algebra $\frak{h}_1$ on $\mathcal{Q}_\sigma(\Gamma)$. In order to prove (4.16), it suffices to prove the result for the generators $X$ and $Y$. We have:

$$(X(F^1, F^2))_\alpha = X((F^1, F^2)_\alpha)$$

$$= X \left( \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Q})} F^1_{\beta \sigma} \cdot (F^2_{\alpha \sigma^{-1} \beta^{-1}} || \sigma \beta) \right)$$

$$= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Q})} X(F^1_{\beta \sigma}) \cdot (F^2_{\alpha \sigma^{-1} \beta^{-1}} || \sigma \beta) + \sum_{\beta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_{\beta \sigma} \cdot X(F^2_{\alpha \sigma^{-1} \beta^{-1}} || \sigma \beta)$$

$$= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Q})} X(F^1_{\beta \sigma}) \cdot (F^2_{\alpha \sigma^{-1} \beta^{-1}} || \sigma \beta) + \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Q})} F^1_{\beta \sigma} \cdot (X(F^2_{\alpha \sigma^{-1} \beta^{-1}}) || \sigma \beta)$$

$$= (X(F^1), F^2)_\alpha + (F^1, X(F^2))_\alpha$$

In (4.17), we have used the fact that $\sigma \beta \in SL_2(\mathbb{Z})$ and hence $X(F^2_{\alpha \sigma^{-1} \beta^{-1}} || \sigma \beta) = X(F^2_{\alpha \sigma^{-1} \beta^{-1}}) || \sigma \beta$. We can similarly verify the relation (4.16) for $Y \in \frak{h}_1$. This proves the result.

Our next aim is to show that $\mathcal{Q}_\sigma(\Gamma)$ is a right $\mathbb{Q}(\Gamma)$-module. Thereafter, we will consider the Hopf algebra $\mathcal{H}_1$ defined in Section 3.2 and show that there is a “Hopf action” of $\mathcal{H}_1$ on the right $\mathbb{Q}(\Gamma)$-module $\mathcal{Q}_\sigma(\Gamma)$.

**Proposition 4.4.** Let $\sigma \in SL_2(\mathbb{Z})$ and let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$. Then, $\mathcal{Q}_\sigma(\Gamma)$ carries a right $\mathbb{Q}(\Gamma)$-module structure defined by:

$$(F^1 \ast F^2)_\alpha := \sum_{\beta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_{\beta \sigma} \cdot (F^2_{\alpha \sigma^{-1} \beta^{-1}} || \beta)$$

(4.18)

for any $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and any $F^2 \in \mathbb{Q}(\Gamma)$.

**Proof.** We take $\gamma \in \Gamma$. Then, since $F^1 \in \mathcal{Q}_\sigma(\Gamma)$ and $F^2 \in \mathbb{Q}(\Gamma)$, we have:

$$F^1_{\gamma \beta \sigma} = F^1_{\beta \sigma} 
\quad \frac{F^2_{\alpha \sigma^{-1} \beta^{-1}, \gamma \beta}}{F^2_{\alpha \sigma^{-1} \beta^{-1}, \gamma \beta}} = \frac{F^2_{\alpha \sigma^{-1} \beta^{-1}, \gamma \beta}}{F^2_{\alpha \sigma^{-1} \beta^{-1}, \gamma \beta}} = \frac{F^2_{\alpha \sigma^{-1} \beta^{-1}}}{F^2_{\alpha \sigma^{-1} \beta^{-1}}}$$

(4.19)

It follows that the sum in (4.18) is well defined, i.e., it does not depend on the choice of coset representatives for $\Gamma$ in $GL^+_2(\mathbb{Q})$. Further, it is clear that $(F^1 \ast F^2)_\gamma = (F^1 \ast F^2)_\alpha$. In order to show that $F^1 \ast F^2 \in \mathcal{Q}_\sigma(\Gamma)$, it remains to show that $(F^1 \ast F^2)_\alpha \gamma = (F^1 \ast F^2)_\alpha || \sigma \gamma \sigma^{-1}$. By definition, we know that:

$$(F^1 \ast F^2)_\alpha \gamma = \sum_{\beta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_{\beta \sigma} \cdot (F^2_{\alpha \gamma \sigma^{-1} \beta^{-1}} || \beta)$$

(4.20)

We now set $\delta = \beta \gamma^{-1} \sigma^{-1}$. This allows us to rewrite (4.20) as follows:

$$(F^1 \ast F^2)_\alpha \gamma = \sum_{\delta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_{\delta \sigma} \cdot (F^2_{\alpha \sigma^{-1} \delta^{-1}} || \delta \sigma \gamma \sigma^{-1})$$

$$= \sum_{\delta \in \Gamma \setminus GL^+_2(\mathbb{Q})} ((F^1_{\delta \sigma} || \delta \sigma \gamma \sigma^{-1}) \cdot ((F^2_{\alpha \sigma^{-1} \delta^{-1}} || \delta \sigma \gamma \sigma^{-1})$$

$$= \left( \sum_{\delta \in \Gamma \setminus GL^+_2(\mathbb{Q})} F^1_{\delta \sigma} \cdot (F^2_{\alpha \sigma^{-1} \delta^{-1}} || \delta) \right) || \sigma \gamma \sigma^{-1}$$

(4.21)

$$= (F^1 \ast F^2)_\alpha || \sigma \gamma \sigma^{-1}$$

23
Hence, \((F^1 \ast F^2) \in Q_\sigma(\Gamma)\). In order to show that \(Q_\sigma(\Gamma)\) is a right \(Q(\Gamma)\)-module, we need to check that \(F^1 \ast (F^2 \ast F^3) = (F^1 \ast F^2) \ast F^3\) for any \(F^1 \in Q_\sigma(\Gamma)\) and any \(F^2, F^3 \in Q(\Gamma)\). For this, we note that:

\[
(F^1 \ast F^2)_\alpha = \sum_{\alpha_2 \alpha_1 = \alpha} F_{\alpha_1}^1 \cdot (F_{\alpha_2}^2 |_{\alpha_1} \sigma^{-1}) \quad \forall \alpha \in GL^+_2(\mathbb{Q}) \tag{4.22}
\]

where the sum in (4.22) is taken over all pairs \((\alpha_1, \alpha_2)\) such that \(\alpha_2 \alpha_1 = \alpha\) modulo the following equivalence relation:

\[
(\alpha_1, \alpha_2) \sim (\gamma \alpha_1, \alpha_2 \gamma^{-1}) \quad \forall \gamma \in \Gamma \tag{4.23}
\]

It follows that for any \(\alpha \in GL^+_2(\mathbb{Q})\), we have:

\[
((F^1 \ast F^2) \ast F^3)_\alpha = \sum_{\alpha_2 \alpha_1 = \alpha} F_{\alpha_1}^1 \cdot ((F^2 \ast F^3)_{\alpha_2} |_{\alpha_1} \sigma^{-1}) \tag{4.24}
\]

where the sum in (4.24) is taken over all triples \((\alpha_1, \alpha_2, \alpha_3)\) such that \(\alpha_3 \alpha_2 \alpha_1 = \alpha\) modulo the following equivalence relation:

\[
(\alpha_1, \alpha_2, \alpha_3) \sim (\gamma \alpha_1, \gamma' \alpha_2 \gamma^{-1}, \alpha_3 \gamma'^{-1}) \quad \forall \gamma, \gamma' \in \Gamma \tag{4.25}
\]

On the other hand, we have:

\[
(F^1 \ast (F^2 \ast F^3))_\alpha = \sum_{\alpha_3 \alpha_2 \alpha_1 = \alpha} F_{\alpha_1}^1 \cdot ((F^1 \ast F^2)_{\alpha_2} |_{\alpha_1} \sigma^{-1}) = \sum_{\alpha_3 \alpha_2 \alpha_1 = \alpha} F_{\alpha_1}^1 \cdot (F^2_{\alpha_2} |_{\alpha_1} \sigma^{-1}) \cdot (F^3_{\alpha_3} |_{\alpha_2 \alpha_1} \sigma^{-1}) \tag{4.26}
\]

Again, we see that the sum in (4.26) is taken over all triples \((\alpha_1, \alpha_2, \alpha_3)\) such that \(\alpha_3 \alpha_2 \alpha_1 = \alpha\) modulo the equivalence relation in (4.25). From (4.24) and (4.26), it follows that \((F^1 \ast (F^2 \ast F^3))_\alpha = ((F^1 \ast F^2) \ast F^3)_\alpha\). This proves the result.

We are now ready to describe the action of the Hopf algebra \(H_1\) on \(Q_\sigma(\Gamma)\). From Section 3.2, we know that \(H_1\) is generated by \(X, Y, \{\delta_n\}_{n \geq 1}\) which satisfy the relations (3.29), (3.30), (3.31).

**Proposition 4.5.** Let \(\Gamma = \Gamma(N)\) be a principal congruence subgroup of \(SL_2(\mathbb{Z})\) and choose some \(\sigma \in SL_2(\mathbb{Z})\).

(a) The collection of \(\sigma\)-twisted quasimodular Hecke operators of level \(\Gamma\) can be made into an \(H_1\)-module as follows: for any \(F \in Q_\sigma(\Gamma)\) and \(\alpha \in GL^+_2(\mathbb{Q})\):

\[
X(F)_\alpha := X(F_\alpha) \quad Y(F)_\alpha := Y(F_\alpha) \quad \delta_n(F)_\alpha := X^{n-1}(\mu_{\alpha \sigma^{-1}} \cdot F_\alpha) \quad \forall n \geq 1 \tag{4.27}
\]

(b) The Hopf algebra \(H_1\) has a “Hopf action” on the right \(Q(\Gamma)\)-module \(Q_\sigma(\Gamma)\); in other words, for any \(F^1 \in Q_\sigma(\Gamma)\) and any \(F^2 \in Q(\Gamma)\), we have:

\[
h(F^1 \ast F^2) = \sum h(1)(F^1) \ast h(2)(F^2) \quad \forall h \in H_1 \tag{4.28}
\]

where the coproduct \(\Delta : H_1 \longrightarrow H_1 \otimes H_1\) is given by \(\Delta(h) = \sum h(1) \otimes h(2)\) for each \(h \in H_1\).

**Proof:** (a) For any \(F \in Q_\sigma(\Gamma)\), we have already checked in the proof of Proposition 4.3 that \(X(F), Y(F) \in Q_\sigma(\Gamma)\). Further, from (3.60), we know that for any \(\alpha \in GL^+_2(\mathbb{Q})\) and \(\gamma \in \Gamma\), we have:

\[
\mu_{\gamma \sigma^{-1}} = \mu_{\gamma} |_{\alpha \sigma^{-1}} + \mu_{\alpha \sigma^{-1}} = \mu_{\alpha \sigma^{-1}} \\
\mu_{\alpha \gamma^{-1}} = \mu_{\alpha \sigma^{-1}} |_{\sigma \gamma \sigma^{-1}} + \mu_{\sigma \gamma \sigma^{-1}} = \mu_{\alpha \sigma^{-1}} |_{\sigma \gamma \sigma^{-1}} \tag{4.29}
\]
Hence, for any \( F \in \mathcal{Q}_\alpha(\Gamma) \), we have:

\[
\delta_n(F)_{\alpha \gamma} = X^{n-1}(\mu_{\alpha \gamma}) \cdot F_{\alpha \gamma} = X^{n-1}(\mu_{\alpha \gamma}) \cdot F_{\alpha \gamma} = \delta_n(F)_{\alpha \gamma},
\]

\[
\delta_n(F)_{\alpha} = X^{n-1}(\mu_{\alpha}) \cdot F_{\alpha} = X^{n-1}(\mu_{\alpha}) \cdot F_{\alpha} = \delta_n(F)_{\alpha}.
\]

Hence, \( \delta_n(F) \in \mathcal{Q}_\alpha(\Gamma) \). In order to show that there is an action of the Lie algebra \( \mathcal{L}_1 \) (and hence of its universal enveloping algebra \( \mathcal{H}_1 \)) on \( \mathcal{Q}_\alpha(\Gamma) \), it remains to check the commutator relations (3.49) between the operators \( X, Y \) and \( \delta_n \) acting on \( \mathcal{Q}_\alpha(\Gamma) \). We have already checked that \( [Y, X] = X \) in the proof of Proposition 3.3. Since \( X \) is a derivation on \( \mathcal{Q}_\alpha \) and \( \delta_n(F)_{\alpha} = X^{n-1}(\mu_{\alpha}) \cdot F_{\alpha} \), we have:

\[
[X, \delta_n](F)_{\alpha} = X(X^{n-1}(\mu_{\alpha}) \cdot F_{\alpha}) - X^{n-1}(\mu_{\alpha}) \cdot X(F_{\alpha}) \]

\[
= X(X^{n-1}(\mu_{\alpha}) \cdot F_{\alpha}) - X^{n-1}(\mu_{\alpha}) \cdot F_{\alpha} = \delta_n(F)_{\alpha}.
\]

Similarly, since \( \mu_{\alpha \gamma} \in \mathcal{M} \subseteq \mathcal{Q}_\alpha \), we have:

\[
[Y, \delta_n](F)_{\alpha} = Y(Y^{n-1}(\mu_{\alpha \gamma}) \cdot F_{\alpha}) - Y^{n-1}(\mu_{\alpha \gamma}) \cdot Y(F_{\alpha}) \]

\[
= Y(Y^{n-1}(\mu_{\alpha \gamma}) \cdot F_{\alpha}) - Y^{n-1}(\mu_{\alpha \gamma}) \cdot F_{\alpha} = n\delta_n(F)_{\alpha}.
\]

Finally, we can verify easily that \( [\delta_k, \delta_l] = 0 \) for any \( k, l \geq 1 \).

(b) In order to prove (3.38), it is enough to check this equality for the generators \( X, Y \) and \( \delta_1 \in \mathcal{H}_1 \). For \( F^1 \in \mathcal{Q}_\alpha(\Gamma), F^2 \in \mathcal{Q}(\Gamma) \) and \( \alpha \in GL_2^+(\mathbb{Q}) \), we have:

\[
(X(F^1 \ast F^2))_\alpha = X((F^1 \ast F^2)_\alpha) = \sum_{\beta \in \Gamma \cap GL_2^+(\mathbb{Q})} X(F^1_{\beta \alpha} \cdot (F^2_{\alpha \beta})_{\beta - 1} | \beta) + \sum_{\beta \in \Gamma \cap GL_2^+(\mathbb{Q})} F^1_{\beta \alpha} \cdot X(F^2_{\alpha \beta})_{\beta - 1} | \beta - \sum_{\beta \in \Gamma \cap GL_2^+(\mathbb{Q})} F^1_{\beta \alpha} \cdot \mu_{\beta - 1} \cdot \delta(F^2_{\alpha \beta})_{\beta - 1} | \beta
\]

\[
\sum_{\beta \in \Gamma \cap GL_2^+(\mathbb{Q})} F^1_{\beta \alpha} \cdot X(F^2_{\alpha \beta})_{\beta - 1} | \beta - \sum_{\beta \in \Gamma \cap GL_2^+(\mathbb{Q})} F^1_{\beta \alpha} \cdot \mu_{\beta - 1} \cdot \delta(F^2_{\alpha \beta})_{\beta - 1} | \beta
\]

\[
= (X(F^1 \ast F^2)_\alpha + (F^1 \ast X(F^2))_\alpha + \delta_1(F^1 \ast Y(F^2))_\alpha).\]

In (3.38) above, we have used the fact that \( \theta = 0 = \mu_{\beta - 1} = \mu_{\beta - 1} + \mu_{\beta} \). For \( \alpha, \beta \in GL_2^+(\mathbb{Q}) \), it follows from (3.60) that

\[
\mu_{\alpha \beta} - 1 = \mu_{\alpha \beta} - 1 + \beta + \mu_{\beta} \quad (3.43)
\]

Since \( F^2 \in \mathcal{Q}(\Gamma) \) we know from (3.01) that \( \delta_1(F^2)_{\alpha \beta} = \mu_{\alpha \beta} \cdot F^2_{\alpha \beta} \). Combining with (3.44), we have:

\[
\delta_1((F^1 \ast F^2)_\alpha) = \mu_{\alpha \beta} - 1 \cdot (F^1 \ast F^2)_\alpha \sum_{\beta \in \Gamma \cap GL_2^+(\mathbb{Q})} (F^2_{\alpha \beta})_{\beta - 1} | \beta
\]

\[
= (\delta_1(F^1 \ast F^2)_\alpha + (F^1 \ast \delta(F^2))_\alpha).\]

Finally, from the definition of \( Y \), it is easy to show that \( (Y(F^1 \ast F^2))_\alpha = (Y(F^1) \ast F^2)_\alpha + (F^1 \ast Y(F^2))_\alpha.\)
4.2 The operators $X_\tau : \mathcal{Q}_\sigma(\Gamma) \to \mathcal{Q}_{\tau \sigma}(\Gamma)$ and Hopf action

Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup and choose some $\sigma \in SL_2(\mathbb{Z})$. In Section 4.1, we have only considered operators $X$, $Y$ and $\{\delta_n\}_{n \geq 1}$ that are endomorphisms of $\mathcal{Q}_\sigma(\Gamma)$. In this section, we will define an operator

$$X_\tau : \mathcal{Q}_\sigma(\Gamma) \to \mathcal{Q}_{\tau \sigma}(\Gamma)$$

for $\tau \in SL_2(\mathbb{Z})$. In particular, we consider the commuting family $\left\{ \rho_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}_{n \in \mathbb{Z}}$. Then, we have operators:

$$X_{\rho_n} : \mathcal{Q}_{\sigma(m)}(\Gamma) \to \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall \ m, n \in \mathbb{Z}$$

acting “between the levels” of the graded module $\mathcal{Q}_\sigma(\Gamma) := \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$. We already know that $\mathcal{Q}_\sigma(\Gamma)$ carries an action of the Hopf algebra $\mathfrak{h}_1$. Further, $\mathfrak{h}_1$ has a Hopf action on the pairing on $\mathcal{Q}_\sigma(\Gamma)$ in the sense of Proposition 4.3. We will now show that $\mathfrak{h}_1$ can be naturally embedded into a larger Hopf algebra $\mathfrak{h}_2$ acting on $\mathcal{Q}_\sigma(\Gamma)$ that incorporates the operators $X_{\rho_n}$ in (4.37). Finally, we will show that the pairing on $\mathcal{Q}_\sigma(\Gamma)$ can be extended to a pairing:

$$(\cdot \cdot, \cdot) : \mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \to \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall \ m, n \in \mathbb{Z}$$

This gives us a pairing on $\mathcal{Q}_\sigma(\Gamma)$ and we prove that this pairing carries a Hopf action of $\mathfrak{h}_2$. We start by defining the operators $X_\tau$ mentioned in (4.36).

Proposition 4.6. (a) Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose $\sigma \in SL_2(\mathbb{Z})$.

(a) For each $\tau \in SL_2(\mathbb{Z})$, we have a morphism:

$$X_\tau : \mathcal{Q}_\sigma(\Gamma) \to \mathcal{Q}_{\tau \sigma}(\Gamma) \quad X_\tau(F)_\alpha := X(F_\alpha)||_{\tau^{-1}} \quad \forall \ F \in \mathcal{Q}_\sigma(\Gamma), \ \alpha \in GL_2^+(\mathbb{Q})$$

(b) Let $\tau_1, \tau_2 \in SL_2(\mathbb{Z})$ be two matrices such that $\tau_1 \tau_2 = \tau_2 \tau_1$. Then, the commutator $[X_{\tau_1}, X_{\tau_2}] = 0$.

Proof. (a) We choose any $F \in \mathcal{Q}_\sigma(\Gamma)$. From (4.39), it is clear that $X_\tau(F)_{\gamma \alpha} = X_\tau(F)_\alpha$ for any $\gamma \in \Gamma$ and $\alpha \in GL_2^+(\mathbb{Q})$. Further, we note that:

$$X_\tau(F)_{\alpha \gamma} = X(F_{\alpha \gamma})||_{\tau^{-1}} = X(F_\alpha)||_{\tau^{-1}} = X(F_\alpha)||_{\tau^{-1}} ||_{\tau^{-1}} ||_{\tau^{-1}} = X(F_\alpha)||_{((\tau \sigma \gamma) \tau^{-1})}$$

It follows from (4.40) that $X_\tau(F) \in \mathcal{Q}_{\tau \sigma}(\Gamma)$ for any $F \in \mathcal{Q}_\sigma(\Gamma)$.

(b) Since $\tau_1$ and $\tau_2$ commute, both $X_{\tau_1}X_{\tau_2}$ and $X_{\tau_2}X_{\tau_1}$ are operators from $\mathcal{Q}_\sigma(\Gamma)$ to $\mathcal{Q}_{\tau_1 \tau_2 \sigma}(\Gamma) = \mathcal{Q}_{\tau_2 \tau_1 \sigma}(\Gamma)$. For any $F \in \mathcal{Q}_\sigma(\Gamma)$, we have $(\forall \ \alpha \in GL_2^+(\mathbb{Q})):\]

$$(X_{\tau_1}X_{\tau_2}(F))_\alpha = X(X_{\tau_2}(F)_\alpha)||_{\tau_1^{-1}} = X^2(F_\alpha)||_{\tau_2^{-1} \tau_1^{-1}} = X^2(F_\alpha)||_{\tau_2^{-1} \tau_1^{-1}} = (X_{\tau_2}X_{\tau_1}(F))_\alpha$$

This proves the result. □

As mentioned before, we now consider the commuting family $\left\{ \rho_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}_{n \in \mathbb{Z}}$ of matrices in $SL_2(\mathbb{Z})$ and set $\sigma(n) := \rho_n \cdot \sigma$ for any $\sigma \in SL_2(\mathbb{Z})$. We want to define a pairing on the graded module $\mathcal{Q}_\sigma(\Gamma) := \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$ that extends the pairing on $\mathcal{Q}_\sigma(\Gamma)$. In fact, we will prove a more general result.
Proposition 4.7. Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and choose $\sigma \in SL_2(\mathbb{Z})$. Let $\tau_1, \tau_2 \in SL_2(\mathbb{Z})$ be two matrices such that $\tau_1 \tau_2 = \tau_2 \tau_1$. Then, there exists a pairing
\[
(\cdot, \cdot) : \mathcal{Q}_{\tau_1}(\Gamma) \otimes \mathcal{Q}_{\tau_2}(\Gamma) \to \mathcal{Q}_{\tau_1 \tau_2}(\Gamma)
\] (4.42)
defined as follows: for any $F^1 \in \mathcal{Q}_{\tau_1}(\Gamma)$ and any $F^2 \in \mathcal{Q}_{\tau_2}(\Gamma)$, we set:
\[
(F^1, F^2)_\alpha := \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} \left( F^1_\beta ||\tau_2^{-1} \right) \cdot \left( F^2_{\alpha \sigma^{-1} \beta^{-1}} ||\tau_2 \sigma \tau_1^{-1} \tau_2^{-1} \right) \quad \forall \alpha \in GL_2^+(\mathbb{Q})
\] (4.43)
In particular, when $\tau_1 = \tau_2 = 1$, the pairing in (4.43) reduces to the pairing on $\mathcal{Q}_\sigma(\Gamma)$ defined in (4.11).

Proof. We choose some $\delta \in \mathcal{Q}_{\tau_1}(\Gamma)$. We now set
\[
\Delta := \mathcal{Q}_{\tau_1}(\Gamma) \cap \mathcal{Q}_{\tau_2}(\Gamma)
\]
In particular, it follows from the pairing in (4.42) that for any $m \in \mathbb{Z}$, we have:
\[
\Delta = \{ (F^1, F^2) \in \mathcal{Q}_{\tau_1}(\Gamma) \mid (F^1, F^2)_m = 0 \}
\]
We now set $\rho = \sigma \gamma^{-1} \sigma^{-1}$. Since $F^1 \in \mathcal{Q}_{\tau_1}(\Gamma)$, we know that $F^1_\sigma \rho = F^1_\sigma ||\tau_1 \sigma \gamma^{-1} \tau_1^{-1}$. Then, we can rewrite the expression in (4.42) as follows:
\[
(F^1, F^2)_\alpha = \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} \left( F^1_\sigma ||\tau_2^{-1} \right) \cdot \left( F^2_{\alpha \sigma^{-1} \beta^{-1}} ||\tau_2 \sigma \gamma^{-1} \tau_1^{-1} \tau_2^{-1} \right)
\]
From (4.45) it follows that $(F^1, F^2) \in \mathcal{Q}_{\tau_1 \tau_2}(\Gamma)$.

In particular, it follows from the pairing in (4.42) that for any $m, n \in \mathbb{Z}$, we have a pairing
\[
(\cdot, \cdot) : \mathcal{Q}_{\sigma(m)}(\Gamma) \otimes \mathcal{Q}_{\sigma(n)}(\Gamma) \to \mathcal{Q}_{\sigma(m+n)}(\Gamma)
\] (4.46)
It is clear that (4.46) induces a pairing on $\mathcal{Q}_\sigma(\Gamma) = \bigoplus_{m \in \mathbb{Z}} \mathcal{Q}_{\sigma(m)}(\Gamma)$ for each $\sigma \in SL_2(\mathbb{Z})$. We will now define operators $\{X_n\}_{n \in \mathbb{Z}}$ and $Z$ on $\mathcal{Q}_\sigma(\Gamma)$. For each $n \in \mathbb{Z}$, the operator $X_n : \mathcal{Q}_\sigma(\Gamma) \to \mathcal{Q}_\sigma(\Gamma)$ is induced by the collection of operators:
\[
X_n^m := X_{n \rho_n} : \mathcal{Q}_{\sigma(m)}(\Gamma) \to \mathcal{Q}_{\sigma(m+n)}(\Gamma) \quad \forall m \in \mathbb{Z}
\] (4.47)
where, as mentioned before, $\rho_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then, $X_n : \mathcal{Q}_\sigma(\Gamma) \to \mathcal{Q}_\sigma(\Gamma)$ is an operator of homogeneous degree $n$ on the graded module $\mathcal{Q}_\sigma(\Gamma)$. We also consider:
\[
Z : \mathcal{Q}_{\sigma(m)}(\Gamma) \to \mathcal{Q}_{\sigma(m)}(\Gamma) \quad Z(F)_\alpha := mF_\alpha + Y(F_\alpha) \quad \forall F \in \mathcal{Q}_{\sigma(m)}(\Gamma), \alpha \in GL_2^+(\mathbb{Q})
\] (4.48)
This induces an operator $Z : Q_\sigma(\Gamma) \rightarrow Q_\sigma(\Gamma)$ of homogeneous degree 0 on the graded module $Q_\sigma(\Gamma)$. We will now show that $Q_\sigma(\Gamma)$ is acted upon by a certain Lie algebra $I_2$ such that the Lie action incorporates the operators $\{X_n\}_{n \in \mathbb{Z}}$ and $Z$ mentioned above. We define $I_2$ to be the Lie algebra with generators $\{Z, X_n | n \in \mathbb{Z}\}$ satisfying the following commutator relations:

$$[Z, X_n] = (n + 1)X_n \quad [X_n, X_{n'}] = 0 \quad \forall \ n, n' \in \mathbb{Z}$$

(4.49)

In particular, we note that $[Z, X_0] = X_0$. It follows that the Lie algebra $I_2$ contains the Lie algebra $I_1$ defined in (3.02). We now describe the action of $I_2$ on $Q_\sigma(\Gamma)$.

**Proposition 4.8.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Then, the Lie algebra $I_2$ has a Lie action on $Q_\sigma(\Gamma)$.

**Proof.** We need to check that $[Z, X_n] = (n+1)X_n$ and $[X_n, X_{n'}] = 0$, $\forall \ n, n' \in \mathbb{Z}$ for the operators $\{Z, X_n | n \in \mathbb{Z}\}$ on $Q_\sigma(\Gamma)$. From part (b) of Proposition 4.6 we know that $[X_n, X_{n'}] = 0$. From (4.47) and (4.48), it is clear that in order to show that $[Z, X_n] = (n+1)X_n$, we need to check that $[Z, X_n^m] = (n+1)X_n^m : Q_{\sigma(m)}(\Gamma) \rightarrow Q_{\sigma(m+n)}(\Gamma)$ for any given $m \in \mathbb{Z}$. For any $F \in Q_{\sigma(m)}(\Gamma)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, we now check that:

$$(ZX_n^m(F))_\alpha = (n + m)X_n^m(F)_\alpha + Y(X_n^m(F)_\alpha) = (n + m)X(F_\alpha)||\rho^{-1} + YX(F_\alpha)||\rho^{-1}$$

$$X_n^mZ(F)_\alpha = X(Z(F)_\alpha)||\rho^{-1} = mX(F_\alpha)||\rho^{-1} + XY(F_\alpha)||\rho^{-1}$$

(4.50)

Combining (4.50) with the fact that $[Y, X] = X$, it follows that $[Z, X_n^m] = (n+1)X_n^m$ for each $m \in \mathbb{Z}$. Hence, the result follows.

We now consider the Hopf algebra $h_2$ that is the universal enveloping algebra of the Lie algebra $I_2$. Accordingly, the coproduct $\Delta$ on $h_2$ is given by:

$$\Delta(X_n) = X_n \otimes 1 + 1 \otimes X_n \quad \Delta(Z) = Z \otimes 1 + 1 \otimes Z \quad \forall \ n \in \mathbb{Z}$$

(4.51)

It is clear that $h_2$ contains the Hopf algebra $h_1$, the universal enveloping algebra of $I_1$. From Proposition 4.3, we know that $h_1$ has a Hopf action on the pairing on $Q_\sigma(\Gamma)$. We want to show that $h_2$ has a Hopf action on the pairing on $Q_\sigma(\Gamma)$. For this, we prove the following Lemma.

**Lemma 4.9.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Let $\tau_1$, $\tau_2$, $\tau_3 \in SL_2(\mathbb{Z})$ be three matrices such that $\tau_i \tau_j = \tau_j \tau_i$, $\forall \ i, j \in \{1, 2, 3\}$. Then, for any $F^1 \in Q_{\tau_1}\sigma(\Gamma)$, $F^2 \in Q_{\tau_2}\sigma(\Gamma)$, we have:

$$X_{\tau_3}(F^1, F^2) = (X_{\tau_3}(F^1), F^2) + (F^1, X_{\tau_3}(F^2))$$

(4.52)

**Proof.** Consider any $\alpha \in GL_2^+(\mathbb{Q})$. Then, from the definition of $X_{\tau_3}$, it follows that:

$$X_{\tau_3}(F^1, F^2)_\alpha = \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} X((F^1_{\beta\sigma}||\tau^{-1}_2) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}}||\tau_2\sigma\beta\tau^{-1}_1\tau^{-1}_2))||\tau^{-1}_3$$

$$= \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (X(F^1_{\beta\sigma}||\tau^{-1}_2\tau^{-1}_3) \cdot (F^2_{\alpha\sigma^{-1}\beta^{-1}}||\tau_2\sigma\beta\tau^{-1}_1\tau^{-1}_2\tau^{-1}_3)$$

$$\quad + \sum_{\beta \in \Gamma \setminus SL_2(\mathbb{Z})} (F^1_{\beta\sigma}||\tau^{-1}_2\tau^{-1}_3) \cdot (X(F^2_{\alpha\sigma^{-1}\beta^{-1}}||\tau_2\sigma\beta\tau^{-1}_1\tau^{-1}_2\tau^{-1}_3)$$

(4.53)
Since $F^1 \in Q_{\tau_1}(\Gamma)$, it follows that $X_{\tau_1}(F^1) \in Q_{\tau_1\tau_2}(\Gamma)$. Similarly, we see that $X_{\tau_3}(F^2) \in Q_{\tau_2\tau_3}(\Gamma)$. It follows that:

$$
(X_{\tau_1}(F^1), F^2)_\alpha = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (X_{\tau_1}(F^1)_\beta \beta_2) (F^2_{\alpha - \beta^{-1}}) (\tau_2 \tau_1^{-1} \tau_2^{-1} \tau_3^{-1}) = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (X(F^1_\beta)_{\tau_2^{-1} \tau_3^{-1}}) (F^2_{\alpha - \beta^{-1}}) (\tau_2 \tau_1^{-1} \tau_2^{-1} \tau_3^{-1})
$$

$$
(F^1, X_{\tau_3}(F^2))_\alpha = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F^1_\beta \tau_2^{-1} \tau_3^{-1}) (X_{\tau_3}(F^2)_{\alpha - \beta}) (\tau_2 \tau_3^{-1} \tau_2^{-1} \tau_3^{-1}) = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F^1_\beta \tau_2^{-1} \tau_3^{-1}) (X(F^2_{\alpha - \beta^{-1}})_{\tau_3^{-1}} (\tau_2 \tau_3^{-1} \tau_2^{-1} \tau_3^{-1}) (\tau_2 \tau_3^{-1} \tau_2^{-1} \tau_3^{-1})
$$

(4.54)

Comparing (4.53) and (4.54), the result of (4.52) follows.

As a special case of Lemma 4.9, it follows that for any $F^1 \in Q_{\sigma(m)}(\Gamma)$ and $F^2 \in Q_{\sigma(m')}(\Gamma)$, we have:

$$
X_{\rho_n}(F^1, F^2) = X_n(F^1, F^2) = (X_n(F^1), F^2) + (F^1, X_n(F^2)) \quad \forall \ n \in \mathbb{Z}
$$

(4.55)

We conclude by showing that $\mathfrak{h}_Z$ has a Hopf action on the pairing on $Q_{\sigma}(\Gamma)$.

**Proposition 4.10.** Let $\Gamma = \Gamma(N)$ be a principal congruence subgroup of $SL_2(\mathbb{Z})$ and let $\sigma \in SL_2(\mathbb{Z})$. Then, the Hopf algebra $\mathfrak{h}_Z$ has a Hopf action on the pairing on $Q_{\sigma}(\Gamma)$. In other words, for $F^1, F^2 \in Q_{\sigma}(\Gamma)$, we have

$$
h(F^1, F^2) = \sum (h_{(1)})(F^1), h_{(2)}(F^2))
$$

(4.56)

where the coproduct $\Delta : \mathfrak{h}_Z \rightarrow \mathfrak{h}_Z \otimes \mathfrak{h}_Z$ is defined by setting $\Delta(h) := \sum h_{(1)} \otimes h_{(2)}$ for each $h \in \mathfrak{h}_Z$.

**Proof.** It suffices to prove the result in the case where $F^1 \in Q_{\sigma(m)}(\Gamma)$, $F^2 \in Q_{\sigma(m')}(\Gamma)$ for some $m, m' \in \mathbb{Z}$. Further, it suffices to prove the relation (4.56) for the generators $\{Z, X_n | n \in \mathbb{Z}\}$ of the Hopf algebra $\mathfrak{h}_Z$. For the generators $X_n$, $n \in \mathbb{Z}$, this is already the result of (4.55) which follows from Lemma 4.9. Since $\Delta(Z) = Z \otimes 1 + 1 \otimes Z$, it remains to show that

$$
Z(F^1, F^2) = (Z(F^1), F^2) + (F^1, Z(F^2)) \quad \forall \ F^1 \in Q_{\sigma(m)}(\Gamma), F^2 \in Q_{\sigma(m')}(\Gamma)
$$

(4.57)

By the definition of the pairing on $Q_{\sigma}(\Gamma)$, we know that $(F^1, F^2) \in Q_{\sigma(m+m')}(\Gamma)$. Then, for any $\alpha \in GL_2(\mathbb{Q})$, we have:

$$
Z(F^1, F^2)_\alpha = (m + m')^2(F^1)_\alpha + Y(F^1, F^2)_\alpha + \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} ((F^1_\beta || \rho_m^{-1})) (F^2_{\alpha - \beta^{-1}} || \rho_m \sigma \rho_m^{-1} \rho_m^{-1})
$$

$$
= \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} \left( (mF^1_\beta + Y(F^1_\beta) || \rho_m^{-1}) (F^2_{\alpha - \beta^{-1}} || \rho_m \sigma \rho_m^{-1} \rho_m^{-1})
$$

$$
+ \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} \left( F^1_\beta || \rho_m^{-1}) (mF^2_{\alpha - \beta^{-1}} + Y(F^2_{\alpha - \beta^{-1}}) || \rho_m \sigma \rho_m^{-1} \rho_m^{-1})
$$

$$
+ \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} \left( Z(F^1)_\beta || \rho_m^{-1}) (F^2_{\alpha - \beta^{-1}} || \rho_m \sigma \rho_m^{-1} \rho_m^{-1})
$$

$$
+ \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} \left( F^1_\beta || \rho_m^{-1}) (Z(F^2)_{\alpha - \beta^{-1}} || \rho_m \sigma \rho_m^{-1} \rho_m^{-1})
$$

(4.58)

$$
= (Z(F^1), F^2)_\alpha + (F^1, Z(F^2))_\alpha
$$


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