ON THE STABILITY OF SOME SPLINE COLLOCATION IMPLICIT
DIFFERENCE SCHEME

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Abstract. Boundary problem for linear partial differential algebraic equations system with multiple
characteristic curves is considered. It is supposed that matrix-functions pencil of the system under
consideration is smoothly equivalent to special canonical form. For this problem, with the help of the
spline collocation method, a difference scheme of arbitrary degree of approximation with respect to
each variable is constructed. Sufficient conditions for its absolute stability is found.

1. Introduction

When modeling some processes of hydrodynamics, gas dynamics, atmospheric physics, plasma
physics etc., it appears systems with identically degenerate matrix-functions in its domain of defi-
nition at all higher partial derivatives [1]-[8]. Such equations are known in the literature as partial
differential algebraic equations, equations not resolved with respect to higher derivatives, degener-
ate systems of partial differential equations or Sobolev’s equations. There exists a number of lines
of research of these equations. One of these lines is based on investigation of canonical structures
of matrix pencils [3], [9], [10]-[13]. At the moment in the literature, it is well studied the issues of
the existence and numerical solution of partial differential algebraic equations with constant matrix
coefficients [3], [4], [8], [14]. It is explained by the fact that canonical structures of constant matrix
pencils are well studied. Insufficiently studied global properties of matrix-functions pencils impede
the investigation of the systems with variable matrix coefficients. In the works [10]-[13] we considered
the systems with simple characteristic curves whose matrix-functions pencils satisfy the criterion known
as “rank-degree” or double “rank-degree” one. Matrices in these pencils depend only on two variables
[15]. For such systems we constructed two-layer and three-layer implicit difference schemes of a first
and second orders of approximation [10]-[13]. This accuracy is not always enough, for example, in the
case when the Lipschitz constant and a domain of definition are quite large. In this case, it requires
difference schemes of higher orders of approximation. In the work [16], we studied matrix-functions
pencils depending on many variables. In a result we obtained sufficient conditions of a smoothly equiva-
ence of these pencils to canonical structure similar to Kronecker form of a regular pencil of constant
matrices. This structure is in fact a generalization of a canonical form of the pencil satisfying to the
“rank-degree” criterion. This structure allows to investigate linear degenerate hyperbolic systems with
arbitrary number of independent variables with multiple characteristic curves.

In this paper we consider linear partial differential algebraic equations system of hyperbolic type
with the pencil smoothly equivalent to its canonical form similar to the Kronecker form [10]. With
the help of spline collocation method, the foundations of which are set out in [17], we construct a high-performance implicit difference scheme of higher order of approximation and then we prove its stability.

The rest of the paper is organized as follows. In the next section, we give a statement of the problem. In section 3, we perform an approximation of unknown function on the uniform grid by a spline of degree \( m_1 \) and \( m_2 \) with respect to each variable, respectively. Then we write down our difference scheme. This approach for constructing of difference scheme was used in [13]. In the work [13], for an approximation of unknown function we used the bi-cubic spline with defect. Accuracy of the calculations in this case was low. It is worth remarking that in the present work the corresponding results significantly improved. In section 4, we give some notations and auxiliary propositions necessary to justify correctness of difference scheme are given. The section 5 is designed to cast the difference scheme to some canonical form. In this form one can easily analyze the spectrum of matrix coefficients. It is shown that for any values of steps, the spectrum is entirely contained in the circle of unit radius. In section 6 we prove an absolute stability property of our difference scheme. Finally, in section 7, for two test examples we show results of numerical experiments.

2. Statement of the problem

Let us consider a boundary problem for the linear partial differential equations system

\[
A(x,t)\partial_t u + B(x,t)\partial_x u + C(x,t)u = f, 
\]

\[
u(x_0,t) = \psi(t), \quad u(x,t_0) = \phi(x),
\]

where \( A(x,t) \), \( B(x,t) \) and \( C(x,t) \) are \( n \times n \) matrices whose elements are supposed to depend on \( x \in \mathbb{R}^1 \) and \( t \in \mathbb{R}^1 \). In the following we suppose that \( (x,t) \in U = [x_0;X] \times [t_0;T] \) and \( U \) is therefore a domain of definition of all functions under consideration. It is supposed that the elements of matrices \( A(x,t), B(x,t), C(x,t) \) and free term \( f(x,t) \) belong to \( C^2(U) \). The vectors \( \psi(t) \) and \( \phi(x) \) are supposed to be some given vector-functions of its arguments whose elements belong to \( C^2([t_0,t]) \) and \( C^2([x_0,X]) \), respectively.

Let us suppose now that

\[
det A(x,t) = 0 \quad and \quad det B(x,t) = 0 \quad \forall \ (x,t) \in U.
\]

The system (2.1) with the condition (2.3) is therefore partial differential algebraic equations one. Its investigation is closely related to analysis of global properties of the pencil \( A(x,t) + \lambda B(x,t) \). In the work [16], we obtained the conditions of smooth equivalence of matrix-functions pencil to its canonical form similar to canonical structure of regular pencil of constant matrix. in this connection, let us remember the definition of smooth equivalency of matrix-functions pencils.

**Definition 2.1.** [16] Two \( n \times n \) matrices pencils \( A(x,t) + \lambda B(x,t) \) and \( \tilde{A}(x,t) + \lambda \tilde{B}(x,t) \) with elements belonging to \( C^s(U) \), where \( \lambda \) is a some parameter are called \( s \)-smoothly equivalent if there exist square matrices \( P(x,t) \) and \( Q(x,t) \) which do not depend on \( \lambda \) and satisfy following conditions:

1. the elements of matrix \( P(x,t) \) and \( Q(x,t) \) belong to \( C^s(U) \);
2. \( \forall (x,t) \in U \) there exist \( P^{-1}(x,t) \) and \( Q^{-1}(x,t) \);
(3) the relation $P(x,t)(A(x,t) + \lambda B(x,t))Q(x,t) = \hat{A}(x,t) + \lambda \hat{B}(x,t) \ \forall (x,t) \in U$ holds.

It is useful following theorem \cite{[16]}.

**Theorem 2.2.** \cite{[16]} Let the following conditions be fulfilled:

1. all roots of the characteristic polynomial $\det(A(x,t) + \lambda B(x,t))$ are real and have a constant multiplicity in a domain of definition $U$;
2. ranks of matrices $A(x,t)$ and $B(x,t)$ are constant at each point of a domain $U$ and less than $n$.

Then the pencil $A(x,t) + \lambda B(x,t)$ is smoothly equivalent to the canonical one

$$\text{diag}\{E_d, M(x,t), E_p\} + \lambda \text{diag}\{J(x,t), E_l, N(x,t)\}, \quad (2.4)$$

where $E_d$ is an identity matrix of an order $d$; $M(x,t)$ and $N(x,t)$ are upper (right) triangular blocks with zero diagonal of orders $l$ and $p$, respectively; $O_l$ is a zero square matrix of order $l$; $J(x,t) = \text{diag}\{J_1(x,t), J_2(x,t), \ldots, J_k(x,t)\}$, where $J_i(x,t)$, for $i = 1, \ldots, k$ are nonsingular matrices of orders $d_i$, respectively; $d = \sum_{i=1}^k d_i$; each block $J_i$ has a unique eigenvalue $-1/\lambda_i(x,t)$ in the domain of definition $U$; $\lambda_i(x,t)$ are eigenvalues of characteristic polynomial $\det(A(x,t) + \lambda B(x,t))$, different from zero in the domain of definition $U$; $p = n - d - l$.

**Remark 2.3.** Let conditions of the theorem \cite{[22]} are valid. Then if one requires

$$\text{rank}(B(x,t)) = \deg(\det(A(x,t) + \lambda B(x,t)))$$

or

$$\text{rank}(A(x,t)) = \deg\left(\hat{\lambda}(\det(A(x,t) + B(x,t)))\right),$$

then the pencil $A(x,t) + \lambda B(x,t)$ is $s$-smoothly equivalent to the pencil \cite{[24]}, in which $N(x,t) \equiv O_l$ or $M(x,t) \equiv O_p$, respectively. In this case one says that the pencil $A(x,t) + \lambda B(x,t)$ satisfy the criterion “rank-degree”. A structure of such a pencil was investigated in \cite{[14]}.

Let us suppose that matrix-functions pencil $A(x,t) + \lambda B(x,t)$ of the system \cite{[24]} satisfy the conditions of theorem \cite{[22]} and remark \cite{[23]} In the next section, we proceed to construct our difference scheme.

### 3. Difference scheme

To construct the difference scheme, we perform the partition of the domain $U$ by the lines $x = x_i$, where $x_i = x_0 + ih$ and $t = t_j$, where $t_j = t_0 + j\tau$ for $i = 0, \ldots, n_1$ and $j = 0, \ldots, n_2$. In a result, we obtain a uniform grid $U_\Delta$ with the steps $h$ and $\tau$ with respect to the space and time variable, respectively. Clearly, $h = (X - x_0)/n_1$ and $\tau = (T - t_0)/n_2$, where $0 < \tau \leq \tau_0$ and $0 < h \leq h_0$. The points with coordinates $(x_i, t_j)$ are referred to as the nodes of the grid $U_\Delta$ while the lines $x = x_i$ and $t = t_j$ are called its layers. In each domain $U_{i,j} = [x_i, x_i + m_1 h] \times [t_j, t_j + m_2 \tau] \subseteq U_\Delta$, where $m_1 \leq n_1$ and $m_2 \leq n_2$, we seek the approximation of the solution $u(x,t)$ of the problem \cite{[24]}, \cite{[22]} in the form of a polynomial $L_{m_1,m_2}^{m_1,m_2}(x,t)$ with indeterminate coefficients of degree $m_1$ and $m_2$ with respect
to variables $x$ and $t$, respectively. We require that the values of this polynomial $L_{i,j}^{m_1,m_2}(x,t)$ in the nodes $(x_i + l_1 h, t_j + l_2 \tau)$ for $l_1 = 0, \ldots, m_1$ and $l_2 = 0, \ldots, m_2$ of a grid domain $U_{i,j}$ coincide with the values of the desired solution $u(x,t)$ at these nodes. In order our approximation will be continuous, we require that the relations

$$L_{i,j}^{m_1,m_2}(x_i + l_1 h, t_j) = L_{i+1,j}^{m_1,m_2}(x_i + l_1 h, t_j), \quad L_{i,j}^{m_1,m_2}(x_0, t_j) = \psi(t_j),$$

$$L_{i,j}^{m_1,m_2}(x_i, t_j + l_2 \tau) = L_{i,j+1}^{m_1,m_2}(x_i, t_j + l_2 \tau), \quad L_{i,0}^{m_1,m_2}(x_i, t_0) = \phi(x_i),$$

hold at horizontal and vertical line $x = x_i$ and $t = t_j$, respectively. Applying differenceless formulas of numerical differentiation for equidistant nodes ([19], p. 161), to derivatives $\partial_x u(x,t)$ and $\partial_x u(x,t)$ at layers $x = x_i$ and $t = t_j$, we obtain

$$\partial_x u(x_i + l_1 h, t_j) = \partial_x L_{i,j}^{m_1,m_2}(x_i + l_1 h, t_j) + \epsilon_1(h^{m_1}), \quad l_1 = 1, \ldots, m_1,$$

$$\partial_t u(x_i, t_j + l_2 \tau) = \partial_t L_{i,j}^{m_1,m_2}(x_i, t_j + l_2 \tau) + \epsilon_2(\tau^{m_2}), \quad l_2 = 1, \ldots, m_2,$$

where

$$\partial_x L_{i,j}^{m_1,m_2}(x_i + l_1 h, t_j) = \frac{1}{h} \sum_{l_3=0}^{m_1} \bar{\gamma}_{l_1,l_3} u(x_i + l_1 h, t_j),$$

$$\partial_t L_{i,j}^{m_1,m_2}(x_i, t_j + l_2 \tau) = \frac{1}{\tau} \sum_{l_3=0}^{m_2} \gamma_{l_2,l_3} u(x_i, t_j + l_3 \tau),$$

$$\epsilon_1(h^{m_1}) = \frac{h^{m_1} \partial^{m_1+1}(\zeta_1, t_j) / \partial x^{m_1+1}}{(m_1 + 1)!} \left. \frac{d}{ds} \prod_{\nu=0}^{m_1} (s - \nu) \right|_{s=l_1}, \quad x_i < \zeta_1 < x_{i+1},$$

$$\epsilon_2(\tau^{m_2}) = \frac{\tau^{m_2} \partial^{m_2+1}(x_i, \zeta_2) / \partial t^{m_2+1}}{(m_2 + 1)!} \left. \frac{d}{ds} \prod_{\nu=0}^{m_2} (s - \nu) \right|_{s=l_2}, \quad t_j < \zeta_2 < t_{j+1}.$$

The coefficients $\bar{\gamma}_{l_1,l_3}$ and $\gamma_{l_2,l_3}$ are calculated with the help of following formulas ([19], p. 161):

$$\bar{\gamma}_{l_1,l_3} = H(m,s,l_3)|_{m=m_1, s=l_1}, \quad \gamma_{l_2,l_3} = H(m,s,l_3)|_{m=m_2, s=l_2},$$

$$H(m,s,l_3) = (-1)^{m+l_3} \frac{C_{l_3}^m}{m!} \frac{d}{ds} \left( \prod_{\nu=0}^{m} (s - \nu) / (s - l_3) \right),$$

where $C_{l_3}^m$ stand for the binomial coefficients. Writing the system (3.1) at the nodes $(x_i + l_1 h, t_j + l_2 \tau)$ for $l_1 = 1, \ldots, m_1$ and $l_2 = 1, \ldots, m_2$ of the domain $U_{i,j}$ and substituting into it the values of desired function $u(x_i + l_1 h, t_j + l_2 \tau)$ and approximation of its derivatives (3.1) and (3.2) in these points, we obtain following difference scheme:

$$A_{i+1,l_1+l_2} u_{i+1,l_1+l_2} + B_{i+1,l_1+l_2} u_{i+1,l_1+l_2} + C_{i+1,l_1+l_2} u_{i+1,l_1+l_2} = f_{i+1,l_1+l_2},$$

$$u_{0,j} = \psi_j, \quad u_{i,0} = \phi_i, \quad i = 0, \ldots, n_1 - 1, \quad i = 0, \ldots, n_2 - 1,$$

where

$$A_{i+1,l_1+l_2} = A(x_i + l_1 h, t_j + l_2 \tau), \quad B_{i+1,l_1+l_2} = B(x_i + l_1 h, t_j + l_2 \tau),$$

$$C_{i+1,l_1+l_2} = C(x_i + l_1 h, t_j + l_2 \tau), \quad f_{i+1,l_1+l_2} = f(x_i + l_1 h, t_j + l_2 \tau),$$

$$u_{i+1,l_1+l_2} = u(x_i + l_1 h, t_j + l_2 \tau), \quad \phi_i = \phi(x_i), \psi_j = \psi(t_j).$$
The difference scheme (3.4) at each node of the grid $U_\Delta$ is given by a system of linear algebraic equations of order $\tilde{n} = m_1 m_2 n$ with unknown vector

$$\tilde{u}_{i+1,j+1} = (u_{i+1,j+1}, \ldots, u_{i+1,j+m_2}, u_{i+2,j+1}, \ldots, u_{i+2,j+m_2}, \ldots, u_{i+m_1,j+1}, \ldots, u_{i+m_1,j+m_2})^\top.$$ 

In what follows to avoid a confusion in notations, we denote the solution of the system (3.4) in the nodes $(x_{i+1}, t_{j+1})$ of the grid $U_\Delta$ by $v_{i+1,j+1}$, while the values of unknown function $u(x, t)$ in the same nodes we denote as $u(x_{i+1}, t_{j+1}) = u_{i+1,j+1}$. Then the system (3.4) has the following form:

$$A_{i+l_1,j+l_2} \frac{1}{\tau} \sum_{l_3=1}^{m_2} \gamma_{l_2,l_3} v_{i+l_1,j+l_3} + B_{i+l_1,j+l_2} \frac{1}{h} \sum_{l_3=1}^{m_1} \tilde{\gamma}_{l_1,l_3} v_{i+l_1,j+l_3} + C_{i+l_1,j+l_2} v_{i+j+l_2} = f_{i+l_1,j+l_2} - \frac{1}{\tau} A_{i+l_1,j+l_2} \gamma_{l_2,0} v_{i+1,j} - \frac{1}{h} B_{i+l_1,j+l_2} \tilde{\gamma}_{l_1,0} v_{i+1,j},$$

$$v_{0,j} = \psi_j, \quad v_{i,0} = \phi_i, \quad i = 0, \ldots, n_1 - 1, \quad i = 0, \ldots, n_2 - 1, \quad l_1 = 1, \ldots, m_1, \quad l_2 = 1, \ldots, m_2,$$

where

$$\tilde{v}_{i+1,j+1} = (v_{i+1,j+1}, \ldots, v_{i+1,j+m_2}, v_{i+2,j+1}, \ldots, v_{i+2,j+m_2}, \ldots, v_{i+m_1,j+1}, \ldots, v_{i+m_1,j+m_2})^\top.$$ 

Let us remark that the difference scheme (3.5) in fact represents the whole set of implicit difference schemes. Setting different values of the orders $m_1$ and $m_2$ of approximating polynomials $L_{m_1,m_2}^{m_1,m_2}(x, t)$, we get according to (3.1) and (3.2) difference schemes with corresponding approximation orders $O(h^{m_1}) + O(\tau^{m_2})$. In addition, the scheme (3.5) can be both two-layer and multi-layer depending on given information.

In our case we have in mind two-layer difference scheme, that is, we suppose that the values of unknown function are given only at one left layer $x = x_i$ and on one lower layer $t = t_j$ of $U_\Delta$ (at the nodes of these layers the values of grid function were defined in the previous step), but in actual calculations are used $m_1 + 1$ vertical and $m_2 + 1$ horizontal layers, as is shown on the template (Fig. 1). The movement on the grid is carried along its layers and within each layer by steps. Therefore we have constructed the difference scheme (3.5) or more exactly a whole set of difference schemes with different orders of approximation. To proceed, we must to find out the conditions of solvability for the difference scheme (3.5) for any data and for sufficiently small grid steps and finally to prove its

![Figure 1. The grid $U_\Delta$ and $(m_1+1)(m_2+1)$-point template.](Image 204x184 to 421x328)
Lemma 4.1. Let the eigenvalues of the matrices \( \overline{\gamma} \) be fulfilled, then the matrix \( \Omega \) then the blocks \( \Omega \) where it is sufficient to prove this property for its blocks. Since the matrices \( \overline{\gamma} \) and \( \Omega \) section.
In the following, we need in some auxiliary propositions which we formulate and prove in the next section.

4. Preliminaries

Let us spend some lines to fix auxiliary notations used throughout the paper. Denote \( \tau/h = r \) and define diagonal matrices
\[
\bar{\gamma}_0^0 = \text{diag}\{\gamma_{1,0}, \gamma_{2,0}, \ldots, \gamma_{m_1,0}\}, \quad \gamma_0^m = \text{diag}\{\gamma_{1,0}, \gamma_{2,0}, \ldots, \gamma_{m_2,0}\},
\]
\[
\bar{\gamma}_m^0 = (\bar{\gamma}_{i,j})^m_{11} \quad \gamma_m^2 = (\gamma_{i,j})^m_{11} \tag{4.1}
\]
whose elements are given by (3.3). Remark, that \( \bar{\gamma}_m^1 \) and \( \gamma_m^2 \) coincides if \( m_1 = m_2 \). Also we define following block diagonal matrices:
\[
\Omega_{i+1,j+1}^1 = \text{diag}\{\Omega_{i+1,j+1}^1, \Omega_{i+1,j+1}^2, \Omega_{i+1,j+1}^3\}, \\
\text{mathcal}A = \text{diag}\{A^1, A^2, A^3\}, \quad \mathcal{B}_{i+1,j+1}^1 = \text{diag}\{B_{i+1,j+1}^1, B^2, B^3\}, \tag{4.2}
\]
where
\[
\Omega_{i+1,j+1}^1 = E_{m_1} \otimes \gamma_m^2 \otimes E_d + r\bar{\gamma}_m^1 \otimes E_{m_2} \otimes J_{i+1,j+1}, \\
\Omega^2 = r\bar{\gamma}_m^1 \otimes E_{m_2}l, \quad \Omega^3 = E_{m_1} \otimes \gamma_m^2 \otimes E_p, \\
A^1 = E_{m_1} \otimes \gamma_{m_2} \otimes E_d, \quad A^2 = \mathcal{O}_{m_1m_2}, \quad A^3 = E_{m_1} \otimes \gamma_0^0 \otimes E_p, \\
B_{i+1,j+1}^1 = \gamma_0^0 \otimes E_{m_2} \otimes J_{i+1,j+1}, \quad B^2 = \bar{\gamma}_m^0 \otimes E_{m_2} \otimes E_l, \quad B^3 = \mathcal{O}_{m_1m_2p}.
\]

Lemma 4.1. Let \( \xi_s^1 \), \( \xi_s^2 \), and \( \xi_s^3 \) for \( s_1 = 1, \ldots, m_1 \), \( s_2 = 1, \ldots, m_2 \), and \( s_3 = 1, \ldots, k \) are eigenvalues of the matrices \( \bar{\gamma}_m^1 \), \( \gamma_m^2 \) and \( J_{i+1,j+1} \), respectively. If at each node of the grid \( U_\Delta \) the inequalities
\[
r\xi_s^1 \xi_s^3 \neq -\xi_s^2, \quad \forall s_1, s_2 \text{ and } s_3, \tag{4.3}
\]
be fulfilled, then the matrix \( \Omega_{i+1,j+1} \) in (4.2) is nonsingular on the grid \( U_\Delta \).

Proof. To prove the nonsingularity of the matrix \( \Omega_{i+1,j+1} \) on \( U_\Delta \) under condition of this lemma, it is sufficient to prove this property for its blocks. Since the matrices \( \bar{\gamma}_m^1 \) and \( \gamma_m^2 \) are nonsingular then the blocks \( \Omega^2 \) \( \Omega^3 \) also share this property. Let us consider now the block \( \Omega_{i+1,j+1}^1 \). Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are constant matrices and \( \mathcal{P}_3(x, t) \) is a matrix-function, such that
\[
\mathcal{P}_1 \bar{\gamma}_m^1 \mathcal{P}_1^{-1} = \bar{\gamma}_m^1, \quad \mathcal{P}_2 \gamma_m^2 \mathcal{P}_2^{-1} = \gamma_m^2, \quad \mathcal{P}_3(x, t) J(x, t) \mathcal{P}_3^{-1}(x, t) = \tilde{J}(x, t),
\]
In the work [24], we proved the identity
\[ y_{\text{Resolving (4.9) in favor of points } t} = \text{the vector-function of differenceless formulas of numerical differentiation (3.2)}, \]
write down the approximation for the derivative of the vector-function of the segment \( \tilde{\gamma} \),
\[ \text{Divide the segment } \tilde{\gamma} = 1_{l} \otimes \ldots \otimes d_{m}, \ldots, 1_{m} \otimes 1_{d}, \ldots, 1_{d} \otimes 1_{m}, \ldots. \]
The following lemma is particularly important because it gives characteristic properties of splines and allows to transform the difference scheme (3.5) to its canonical form in which one easily sees spectra of matrix coefficients.

**Lemma 4.2.** Let \( x_0 \in \mathbb{R}^d \) be an arbitrary vector and \( y_0 = (x_0, x_0, \ldots, x_0)^\top \in \mathbb{R}^{md} \). Let \( \alpha \) be an arbitrary parameter and \( \beta \) be an arbitrary constant \( d \times d \) matrix with corresponding eigenvalues \( \xi_l^\beta \) for \( l = 1, \ldots, d \) satisfying \( \xi_l^\beta \neq -\xi_m^\gamma/\alpha, \forall l, s \), where \( \xi_l^\gamma \) for \( s = 1, \ldots, m \) are eigenvalues of \( m \times m \) matrix \( \gamma_m \equiv (\gamma_{l,m})^m_l \) with elements defined through (3.3). Then
\[ (E_{md} + \alpha(\gamma_m^{-1} \otimes \beta))^{-1} y_0 = -\text{diag}\{\exp(-\alpha\beta), \exp(-2\alpha\beta), \ldots, \exp(-m\alpha\beta)\} y_0 + O(\tau^m). \] (4.4)

**Proof.** Let us consider Cauchy problem for homogeneous system of ordinary differential equations
\[ \dot{x}(t) = -\frac{\alpha}{\tau} \beta x(t), \quad x(t_0) = x_0, \quad t \in I = [0, m\tau]. \] (4.5)
The solution of (4.5), as is known, is
\[ x(t) = \exp\left(-\frac{\alpha}{\tau} \beta t\right) x_0. \] (4.6)
Divide the segment \( I \) into \( (m - 1) \) equal parts by the points \( t_j = j\tau \) for \( j = 0, \ldots, m \). Making use of differenceless formulas of numerical differentiation [3.2], we write down the approximation for the derivative of the vector-function \( x(t) \) at the points \( t_j \) for \( j = 1, \ldots, m \)
\[ \tau \dot{x}(j\tau) = \sum_{l=0}^{m} \gamma_{l,j} x(t_l) + O(\tau^m). \] (4.7)
Rewrite the relations (3.7) in the matrix form
\[ Y = (\gamma^0_m \otimes E_d) y_0 + (\gamma_m \otimes E_d) y + O(\tau^{m+1}), \] (4.8)
where \( Y \) and \( y \) are vectors of the form \( Y = (\dot{x}(h), \dot{x}(2h), \ldots, \dot{x}(mh))^\top \) and \( y = (x(h), x(2h), \ldots, x(mh))^\top \), respectively. With the help of (4.5) we find the values of the derivative of unknown function at the points \( t_j \) of the segment \( I \). Substituting these values into (4.8), we get
\[ -\alpha(E_m \otimes \beta) y = (\gamma^0_m \otimes E_d) y_0 + (\gamma_m \otimes E_d) y + O(\tau^{m+1}). \] (4.9)
Resolving (4.9) in favor of \( y \) gives
\[ y = -\left(\alpha(E_m \otimes \beta) + \gamma_m \otimes E_d\right)^{-1} (\gamma^0_m \otimes E_d) y_0 + O(\tau^{m+1}). \] (4.10)
In the work [24], we proved the identity
\[ (\gamma_m^{-1} \gamma^0_m \otimes E_d) e_{md} = -e_{md} + O(\tau^m). \] (4.11)
Here $e_{md}$ stands for a $md$-dimensional vector, each element of which ia a unit. Remark, that relation (4.11) is in fact a particular case of the following one:

$$\left(\gamma_m^{-1} \gamma_m \otimes E_d\right) y_0 = -y_0 + O(\tau^m). \quad (4.12)$$

Remark, that it is proved in the same way as (4.11), but instead of the interpolated vector-function $g(t) = (1 + t^m, 1 + t^m, \ldots, 1 + t^m)^\top$, we take here $\tilde{g}(t) = x_0 - e_d + g(t)$. Making use of (4.12), we transform (4.10) to get

$$y = - (E_{md} + \alpha(\gamma_m^{-1} \otimes \beta))^{-1} y_0 + O(\tau^m). \quad (4.13)$$

Finally, substituting into (4.13) the values of unknown function (4.6) at the nodal points of $I$, we obtain the relation (4.4). Therefore the lemma is proved.

5. TRANSFORMATION OF DIFFERENCE SCHEME UNDER CONSIDERATION

It is quite difficult to investigate the difference scheme (3.5) without preliminary transformation of it to convenient form. Thus, let us first to cast it to special canonical form. As a result we write down this canonical form in the end of this section.

We suppose that the pencil $A(x, t) + \lambda B(x, t)$ of the system (2.1) satisfies the conditions of the theorem 2.2. Remember that this means that one can find a pair of matrix-functions $P(x, t)$ and $Q(x, t)$ with properties of definition 2.1 which serve for transformation of our pencil to the canonical form (2.4). Prepare following square matrix of order $\tilde{n}$:

$$\tilde{P} = \text{diag} \{P_{t+1,j+1}, \ldots, P_{t+1,j+m_2}, P_{t+2,j+1}, \ldots, P_{t+2,j+m_2}, \ldots, P_{t+m_1,j+1}, \ldots, P_{t+m_1,j+m_2}\}.$$

Multiply the left- and right-hand sides of the system (3.5) on the left by the matrix $\tau \tilde{P}$ and perform a change of variable $v_{i,j} = Q_{i,j} w_{i,j}$, where $w_{i,j}$ is some unknown $n$-dimensional vector-function, calculated in the node $(i, j)$. In a result we obtain the following difference scheme:

$$P_{t+1,j+2} \left\{A_{i+1,j+1+2} \sum_{l_3=1}^{m_2} \gamma_{l_2} Q_{i+1,j+1+2} w_{i+1,j+1+2} + r B_{i+1,j+1+2} \sum_{l_3=1}^{m_1} \tilde{\gamma}_{l_2} Q_{i+1,j+1+2} w_{i+1,j+1+2} \right.$$

$$+ \tau C_{i+1,j+1+2} Q_{i+1,j+1+2} w_{i+1,j+1+2} \left.\right\} = P_{t+1,j+2} \left\{\tau f_{i+1,j+2} - r A_{i+1,j+1+2} \gamma_{l_2} Q_{i+1,j+1+2} w_{i+1,j+1+2} \right.$$

$$- r B_{i+1,j+1+2} \tilde{\gamma}_{l_2} Q_{i+1,j+1+2} w_{i+1,j+1+2} \right\}, \quad (5.1)$$

$$w_{0,j} = Q^{-1}_{0,j} \psi_j, \quad w_{i,0} = Q^{-1}_{i,0} \phi_i,$$

for $i = 0, \ldots, n_1 - 1$, $i = 0, \ldots, n_2 - 1$, $l_1 = 1, \ldots, m_1$ and $l_2 = 1, \ldots, m_2$ with unknown vector

$$\bar{w}_{i+1,j+1} = (w_{i+1,j+1}, \ldots, w_{i+1,j+m_2}, w_{i+2,j+1}, \ldots, w_{i+2,j+m_2}, \ldots, w_{i+m_1,j+1}, \ldots, w_{i+m_1,j+m_2})^\top.$$

With Taylor’s formula we can represent the matrix $Q_{i+1,j+1+2}$ in the following form:

$$Q_{i+1,j+1+2} = Q_{i+1,j+1+2} + \sigma_{i+1,j+1+2}^1 \chi_{l_1} + \sigma_{i+1,j+1+2}^2 \chi_{l_2} \chi_{l_2} \theta \tau,$$

for $\tilde{l}_1 = 1, \ldots, m_1$ and $\tilde{l}_2 = 1, \ldots, m_2$, where

$$\sigma_{i+1,j+1+2}^1 \equiv \partial_x Q(x_{i+1}, t_{j+1+2}) \quad \text{and} \quad \sigma_{i+1,j+1+2}^2 \equiv \partial_x Q(x_{i+1}, t_{j+1+2} + \theta \tau).$$
Here $\theta$ is some parameter, by assumption, obeying the condition $0 < \theta < 1$ and $\hat{t}_k$ are suitable numbers for which one has $\chi_k = |\hat{t}_k - t_k|$, while $0 \leq \chi_k \leq m_k$. Let us denote $\hat{C}_{i,j} = P_{i,j}C_{i,j}Q_{i,j}$ and rewrite system \((5.1)\), taking into account canonical form \((2.4)\) of the pencil $A(x,t) + \lambda B(x,t)$. In a result, we obtain

\[
\hat{\Omega}_{i+1,j+1}w_{i+1,j+1} = q_{i+1,j+1},
\]

\[
\hat{\Omega}_{i+1,j+1} = \hat{\Omega}_{i+1,j+1} + h\sigma_1 + \tau\sigma_2,
\]

where $\hat{\Omega}_{i+1,j+1}$, $\hat{A}$ and $\hat{B}_{i+1,j+1}$ are $\tilde{n} \times \tilde{n}$ matrices of the form

\[
\hat{\Omega}_{i+1,j+1} = E_{m_1} \otimes \gamma_{m_2} \otimes \text{diag}\{E_d, \Omega_1, E_p\} + r\gamma_{m_1} \otimes E_{m_2} \otimes \text{diag}\{J_{i+1,j+1}, E_l, \Omega_p\},
\]

\[
\hat{A} = E_{m_1} \otimes \gamma_{m_2} \otimes \text{diag}\{E_d, \Omega_1, E_p\}, \quad \hat{B}_{i+1,j+1} = \tilde{\gamma}_{m_1} \otimes E_{m_2} \otimes \text{diag}\{J_{i+1,j+1}, E_l, \Omega_p\};
\]

and

\[
\hat{f}_{i+1,j+1} = \tau\hat{P}\hat{f}_{i+1,j+1},
\]

\[
\hat{f}_{i+1,j+1} = (f_{i+1,j+1}, \ldots, f_{i+1,j+m_2}, f_{i+2,j+1}, \ldots, f_{i+2,j+m_2}, \ldots, f_{i+1,m_1+1}, \ldots, f_{i+m_1,j+1}, \ldots, f_{i+m_1,j+m_2})^T;
\]

\[
\hat{w}_{i+1,j} = (e_{m_2} \otimes w_{i+1,j}, e_{m_2} \otimes w_{i+2,j}, \ldots, e_{m_2} \otimes w_{i+m_1,j})^T,
\]

\[
\hat{w}_{i+1,j} = e_{m_1} \otimes (w_{i,j+1}, w_{i,j+2}, \ldots, w_{i,j+m_2})^T.
\]

Here $\sigma_k$ are bounded in the domain of definition matrices obtained as a result of expanding $Q_{i+l_1,j+l_2}$ and $J_{i+1,j+l_2}$ by Taylor’s formula.

Let us split each block component $w_{i+1,j+1}$ of the vector $\hat{w}_{i+1,j+1}$ into three ones

\[
w_{i+1,j+1} = (w_{i+1,j+1}^1, w_{i+1,j+1}^2, w_{i+1,j+1}^3)^T
\]

of the size $d, l$ and $p$, respectively. Prepare then following $\tilde{n} \times \tilde{n}$ permutation matrix

\[
T = \text{colon}(T_1, T_2, T_3),
\]

\[
T_1 = E_{m_1} \otimes E_{m_2} \otimes (E_d \otimes_{d\times l} \Omega_{d\times p}), \quad T_2 = E_{m_1} \otimes E_{m_2} \otimes (\Omega_{l\times d} E_l \otimes_{l\times p}),
\]

\[
T_3 = E_{m_1} \otimes E_{m_2} \otimes (\Omega_{p\times d} E_p),
\]

where the blocks $T_1$, $T_2$ and $T_3$ are supposed to have the sizes $m_1 m_2 d \times \tilde{n}$, $m_1 m_2 l \times \tilde{n}$ and $m_1 m_2 p \times \tilde{n}$, respectively. It is easy to prove that $T$ is orthogonal matrix. The matrix $T$ perform the permutation of the components of the vector $\hat{w}_{i+1,j+1}$ so that

\[
z_{i+1,j+1} = T\hat{w}_{i+1,j+1} = (\hat{w}_{i+1,j+1}^1, \hat{w}_{i+1,j+1}^2, \hat{w}_{i+1,j+1}^3)^T,
\]

where

\[
\hat{w}_{i+1,j+1} = (w_{i+1,j+1}^k, \ldots, w_{i+1,j+m_2}^k, w_{i+2,j+1}^k, \ldots, w_{i+2,j+m_2}^k, \ldots, w_{i+m_1,j+1}^k, \ldots, w_{i+m_1,j+m_2}^k)^T.
\]

One sees that the vectors $\hat{w}_{i+1,j+1}^1$, $\hat{w}_{i+1,j+1}^2$ and $\hat{w}_{i+1,j+1}^3$ have the sizes $m_1 m_2 d$, $m_1 m_2 l$ and $m_1 m_2 p$, respectively. With the help of $T$ we also reshuffle the elements of $\hat{\Omega}_{i+1,j+1}$, $\hat{A}$ and $\hat{B}_{i+1,j+1}$ in the following way:

\[
T\hat{\Omega}_{i+1,j+1}T^\top = \Omega_{i+1,j+1}, \quad T\hat{A}T^\top = \tilde{A}, \quad T\hat{B}_{i+1,j+1}T^\top = \tilde{B}_{i+1,j+1},
\]

where $\Omega_{i+1,j+1}$, $\tilde{A}$ and $\tilde{B}_{i+1,j+1}$ were defined in \((4.2)\).
Multiply the system (5.2) on the left by the matrix \( T \) and write the unknown vector \( \bar{w}_{i+1,j+1} \) as \( \bar{w}_{i+1,j+1} = T^\top z_{i+1,j+1} \). Then, taking into account (5.3) and (5.4), we obtain from (5.2) the following system:

\[
\tilde{\Omega}_{i+1,j+1} z_{i+1,j+1} = \tau g_{i+1,j+1} - (\bar{A} + \tau \tilde{\sigma}_3) z_{i+1,j} - r(\bar{B}_{i+1,j+1} + \tau \tilde{\sigma}_4) z_{i,j+1},
\]

(5.5)

where

\[
\tilde{\Omega}_{i+1,j+1} = \Omega_{i+1,j+1} + h \tilde{\sigma}_1 + \tau \tilde{\sigma}_2, \quad g_{i+1,j+1} = T f_{i+1,j+1}
\]

and \( \tilde{\sigma}_k \equiv T \sigma_k T^\top \).

Next we suppose that in each node of the grid \( U_\Delta \) inequalities (4.3) from the lemma 4.1 are valid and \( r \) being a ratio of two steps is a constant. In virtue of the lemma 4.1, the matrix \( \Omega_{i+1,j+1} \) is nonsingular on the grid \( U_\Delta \). Thus, the matrix \( \tilde{\Omega}_{i+1,j+1} \) in (5.2) can be presented in the following form:

\[
\tilde{\Omega}_{i+1,j+1} = \Omega_{i+1,j+1} M_{i+1,j+1},
\]

where \( M_{i+1,j+1} = E_n + \Omega_{i+1,j+1}^{-1}(h \tilde{\sigma}_1 + \tau \tilde{\sigma}_2) \). Since, by assumption, the elements of matrix coefficients of the system (2.1) belong to \( C^2(U) \), then in virtue of the theorem 1 in the work [22], the elements of \( J(x,t) \) also belong to \( C^2(U) \). This means that there is such a constant \( K \), for which the condition \( ||\Omega_{i+1,j+1}^{-1}|| \leq K \) is fulfilled.

In virtue of the theorem from [21, p. 195] there exists the inverse matrix \( M_{i+1,j+1}^{-1} \). It can be presented in the form \( M_{i+1,j+1}^{-1} = E_n + M_{i+1,j+1} h \), where

\[
M_{i+1,j+1} = \sum_{\nu=1}^{\infty} (-1)^\nu \left[ \Omega_{i+1,j+1}^{-1}(\sigma_1 + r \sigma_2) \right]^\nu h^{\nu-1}
\]

Therefore we see that the matrix \( \tilde{\Omega}_{i+1,j+1}^{-1} \) exists. It is worth remarking, that therefore we proved the solvability of the difference scheme (3.5) in each node of the grid \( U_\Delta \) for sufficiently small steps \( h \) and \( \tau \).

To proceed a transformation of the difference scheme (3.5), we multiply (5.5) on the left by the matrix \( \tilde{\Omega}_{i+1,j+1}^{-1} \) to get

\[
z_{i+1,j+1} = \tilde{g}_{i+1,j+1} - (F_{i+1,j+1} + \tilde{e}_{i+1,j+1}(h,\tau)) z_{i+1,j} - r(K_{i+1,j+1} + \tilde{e}_{i+1,j+1}(h,\tau)) z_{i,j+1},
\]

(5.6)

where \( \tilde{g}_{i+1,j+1} \) is the \( n \)-dimensional vector given by \( \tilde{g}_{i+1,j+1} = \tau M_{i+1,j+1}^{-1} \Omega_{i+1,j+1}^{-1} g_{i+1,j+1}, \)

\[
F_{i+1,j+1} = \Omega_{i+1,j+1}^{-1} \bar{A}; \quad K_{i+1,j+1} = \Omega_{i+1,j+1}^{-1} \bar{B}_{i+1,j+1},
\]

and

\[
\tilde{e}_{i+1,j+1}(h,\tau) = h M_{i+1,j+1} \tilde{\Omega}_{i+1,j+1}^{-1} \tilde{A} + \tau \tilde{\Omega}_{i+1,j+1}^{-1} \tilde{\sigma}_3 + h r M_{i+1,j+1} \tilde{\Omega}_{i+1,j+1}^{-1} \tilde{\sigma}_4,
\]

\[
\tilde{e}_{i+1,j+1}(h,\tau) = h M_{i+1,j+1} \tilde{\Omega}_{i+1,j+1}^{-1} \tilde{B}_{i+1,j+1} + \tau \tilde{\Omega}_{i+1,j+1}^{-1} \tilde{\sigma}_4 + h r M_{i+1,j+1} \tilde{\Omega}_{i+1,j+1}^{-1} \tilde{\sigma}_4
\]

being \( n \times n \) matrices.

Taking into account (4.2), the system (5.6) is decomposed into three subsystems

\[
\tilde{u}^{k}_{i+1,j+1} = \tilde{g}^{k}_{i+1,j+1} - F^{k}_{i+1,j+1} \tilde{u}^{k}_{i+1,j} - r K^{k}_{i+1,j+1} \tilde{u}^{k}_{i,j+1} - \sum_{l=1}^{3} \tilde{c}^{k,l}_{i+1,j+1}(h,\tau) \tilde{w}^{l}_{i+1,j} - \sum_{l=1}^{3} \tilde{e}^{k,l}_{i+1,j+1}(h,\tau) \tilde{w}^{l}_{i,j+1},
\]

(5.7)
Here
\[ F^k_{i+1,j+1} = \left[ \Omega^k_{i+1,j+1} \right]^{-1} A^k, \quad R^k_{i+1,j+1} = \left[ \Omega^k_{i+1,j+1} \right]^{-1} B^k_{i+1,j+1}, \]
while \( \tilde{e}^k_{i+1,j+1}(h, \tau) \) and \( \tilde{c}^k_{i+1,j+1}(h, \tau) \) are matrix blocks of
\[ \tilde{e}_{i+1,j+1}(h, \tau) = \left( \tilde{c}_{i+1,j+1}(h, \tau) \right)^3_1 \quad \text{and} \quad \tilde{e}_{i+1,j+1}(h, \tau) = \left( \tilde{c}_{i+1,j+1}(h, \tau) \right)^3_1, \]
respectively. The sizes of these blocks correspond to the decomposition of \( z_{i+1,j+1} \). Finally, free terms \( \tilde{g}^k_{i+1,j+1} \) are blocks of the vector
\[ \tilde{g}_{i+1,j+1} = (\tilde{g}_{i+1,j+1}^1, \tilde{g}_{i+1,j+1}^2, \tilde{g}_{i+1,j+1}^3)^T \]
of the sizes \( m_1m_2d, m_1m_2l \) and \( m_1m_2p \), respectively, and the vectors \( \tilde{w}^k_{i+1,j} \) and \( \tilde{w}^k_{i,j+1} \) have the following form
\[ \tilde{w}^k_{i+1,j} = (c_{m_2} \otimes w^k_{i+1,j}, c_{m_2} \otimes w^k_{i+2,j}, \ldots, c_{m_2} \otimes w^k_{i+m_2,j})^T, \]
\[ \tilde{w}^k_{i,j+1} = c_{m_1} \otimes (w^k_{i,j+1}, w^k_{i,j+2}, \ldots, w^k_{i,j+m_2})^T. \]

Using (4.12), we transform the product \( F^1_{i+1,j+1} \tilde{w}^1_{i+1,j} \) in the first equation of the system (4.7) to get
\[ F^1_{i+1,j+1} \tilde{w}^1_{i+1,j} = (E_{m_2} \otimes \gamma_{m_2} \otimes E_d + r_{\gamma_{m_2}} \otimes E_{m_2} \otimes J_{i+1,j+1})^{-1} (E_{m_2} \otimes \gamma_{m_2} \otimes E_d) \tilde{w}^1_{i+1,j} \]
\[ = (E_{m_2d} + r_{\gamma_{m_2}} \otimes \gamma_{m_2}^{-1} \otimes J_{i+1,j+1})^{-1} \tilde{w}^1_{i+1,j} + O(\tau^{m_2}). \]
Let \( R \) be a \( m_1 \times m_1 \) matrix transforming \( \gamma_{m_1} \) to the normal Jordan form. Remark that the matrix \( \gamma_{m_1} \) is simple for any value of \( m_1 \), that is,
\[ R\gamma_{m_1}R^{-1} = \gamma_{m_1}^* \quad \text{where} \quad \gamma_{m_1}^* = \text{diag} \left\{ \xi_{\gamma_{m_1}}^1, \xi_{\gamma_{m_1}}^2, \ldots, \xi_{\gamma_{m_1}}^{m_1} \right\}. \]
Then
\[ F^1_{i+1,j+1} \tilde{w}^1_{i+1,j} = (R^{-1} \otimes E_{m_2d}) \tilde{R} (R \otimes E_{m_2d}) \tilde{w}^1_{i+1,j} + O(\tau^{m_2}), \]
with \( \tilde{R} = \text{diag}\{\tilde{R}_{11}, \tilde{R}_{22}, \ldots, \tilde{R}_{m_1m_1}\} \), where
\[ \tilde{R}_{ss} = \left( E_{m_2d} + r \xi_{\gamma_{m_1}}^s (\gamma_{m_2}^{-1} \otimes J_{i+1,j+1}) \right)^{-1} \]
for \( s = 1, \ldots, m_1 \). In virtue of the lemma [4.2] we have
\[ \tilde{R}(R \otimes E_{m_2d})\tilde{w}^1_{i+1,j} = D^1_{i+1,j+1}(R \otimes E_{m_2d})\tilde{w}^1_{i+1,j}, \]
where
\[ D^1_{i+1,j+1} = \text{diag} \left\{ \exp \left( -r \xi_{\gamma_{m_1}}^1 J_{i+1,j+1} \right), \exp \left( -2r \xi_{\gamma_{m_1}}^2 J_{i+1,j+1} \right), \ldots, \exp \left( -m_2r \xi_{\gamma_{m_1}}^{m_1} J_{i+1,j+1} \right), \right\}, \]
\[ \exp \left( -r \xi_{\gamma_{m_1}}^2 J_{i+1,j+1} \right), \exp \left( -2r \xi_{\gamma_{m_1}}^2 J_{i+1,j+1} \right), \ldots, \exp \left( -m_2r \xi_{\gamma_{m_1}}^{m_1} J_{i+1,j+1} \right), \ldots, \exp \left( -r \xi_{\gamma_{m_1}}^{m_1} J_{i+1,j+1} \right), \exp \left( -2r \xi_{\gamma_{m_1}}^{m_1} J_{i+1,j+1} \right), \ldots, \exp \left( -m_2r \xi_{\gamma_{m_1}}^{m_1} J_{i+1,j+1} \right) \right\}. \]
Then, taking into account (5.9), the relation (5.8) takes the form
\[ F^1_{i+1,j+1} \tilde{w}^1_{i+1,j} = (R^{-1} \otimes E_{m_2d})D^1_{i+1,j+1}(R \otimes E_{m_2d})\tilde{w}^1_{i+1,j} + O(\tau^{m_2}). \]
Next we consider block permutation matrix $\mathcal{T} = (T_{i,j})$, where $i = 1, \ldots, m_2$ and $j = 1, \ldots, m_1$, where each block $T_{i,j}$ itself consists of blocks $T_{i,j} = (\tilde{T}_{k,s})$, for $k = 1, \ldots, m_1$ and $s = 1, \ldots, m_2$. The blocks $\tilde{T}_{k,s}$ are defined by

$$\tilde{T}_{k,s} = \begin{cases} E_d, & \text{for } k = j, \ s = i, \\ \varnothing_d, & \text{for } k \neq j \text{ or } s \neq i \end{cases}$$

Remark, that the matrix $\mathcal{T}$ is in fact a square orthogonal matrix of the order $m_1 m_2 d$. Also it is worth remarking, that

$$\mathcal{T} (R^{-1} \otimes E_{m_2 d}) \mathcal{D}^{1}_{i+1,j+1} (R \otimes E_{m_2 d}) \mathcal{T}^T = \mathcal{D}^{1}_{i+1,j+1}, \quad (5.11)$$

where

$$\mathcal{D}^{1}_{i+1,j+1} = \text{diag} \left\{ \exp \left( -r \gamma_{m_1} \otimes J_{i+1,j+1} \right), \exp \left( -2r \gamma_{m_1} \otimes J_{i+1,j+1} \right), \ldots, \exp \left( -m_2 r \gamma_{m_1} \otimes J_{i+1,j+1} \right) \right\}.$$ 

Then the relation (5.10) taking into account (5.11) becomes

$$F^{1}_{i+1,j+1} \tilde{w}^{1}_{i+1,j} = \mathcal{T}^T \mathcal{D}^{1}_{i+1,j+1} \mathcal{T} \tilde{w}^{1}_{i+1,j} + O(r^{m_2}). \quad (5.12)$$

Therefore we finished a transformation of $F^{1}_{i+1,j+1} \tilde{w}^{1}_{i+1,j}$ and now let us turn to $r K^{1}_{i+1,j+1} \tilde{w}^{1}_{i+1,j}$ in the first equation of system (5.7). Using (4.12) and taking into account (4.12) we obtain

$$r K^{1}_{i+1,j+1} \tilde{w}^{1}_{i+1,j+1} = r \left( (E_{m_1} \otimes \gamma_{m_2} \otimes E_d + r \gamma_{m_1} \otimes E_{m_2} \otimes J_{i+1,j+1})^{-1} \left( \gamma_{m_1} \otimes E_{m_2} \otimes J_{i+1,j+1} \right) \right) \tilde{w}^{1}_{i+1,j+1}$$

$$= \left( E_{m_1 m_2 d} + \frac{1}{r} \gamma^{-1}_{m_1} \otimes \gamma_{m_2} \otimes J_{i+1,j+1}^{-1} \right) \tilde{w}^{1}_{i+1,j+1} + O(h^{m_1}).$$

Again, let $R_1$ be $m_2 \times m_2$ matrix, transforming $\gamma_{m_2}$ to normal Jordan form $\gamma_{m_2}^*$, that is,

$$R_1 \gamma_{m_2} R_1^{-1} = \gamma_{m_2}^*, \quad \text{where} \quad \gamma_{m_2}^* = \text{diag} \left\{ \xi_{\gamma_{m_2}}, \xi_{\gamma_{m_2}}^2, \ldots, \xi_{\gamma_{m_2}}^{m_2} \right\}.$$ 

Then

$$r K^{1}_{i+1,j+1} \tilde{w}^{1}_{i+1,j+1} = \left( (E_{m_1} \otimes \gamma_{m_2}^{-1} \otimes E_d) \tilde{R}_1 (E_{m_1} \otimes R_1 \otimes E_d) \right) \tilde{w}^{1}_{i+1,j+1} + O(h^{m_1}), \quad (5.13)$$

where

$$\tilde{R}_1 = \left( E_{m_1 m_2 d} + \frac{1}{r} \gamma^{-1}_{m_1} \otimes \gamma_{m_2}^* \otimes J_{i+1,j+1}^{-1} \right)^{-1}.$$ 

Remark, that

$$\mathcal{T} \tilde{R}_1 \mathcal{T}^T = \tilde{R}, \quad (5.14)$$

with $\tilde{R} = \text{diag} \{ \tilde{R}_{11}, \tilde{R}_{22}, \ldots, \tilde{R}_{m_2 m_2} \}$, where

$$\tilde{R}_{ss} = \left( E_{m_1 d} + \frac{1}{r} \xi_{\gamma_{m_2}}^s \left( \gamma^{-1}_{m_1} \otimes J_{i+1,j+1}^{-1} \right) \right)^{-1},$$

for $s = 1, \ldots, m_2$. From (5.13) and (5.14), taking into account lemma 4.12 we get

$$r K^{1}_{i+1,j+1} \tilde{w}^{1}_{i,j+1} = \left( (E_{m_1} \otimes \gamma_{m_2}^{-1} \otimes E_d) \mathcal{T}^T \mathcal{D}^{2}_{i+1,j+1} \mathcal{T} (E_{m_1} \otimes R_1 \otimes E_d) \right) \tilde{w}^{1}_{i,j+1} + O(h^{m_1}), \quad (5.15)$$

where

$$\mathcal{D}^{2}_{i+1,j+1} = \text{diag} \left\{ \exp \left( -\frac{\xi_{\gamma_{m_2}}^1}{r} J_{i+1,j+1}^{-1} \right), \exp \left( -\frac{\xi_{\gamma_{m_2}}^2}{r} J_{i+1,j+1}^{-1} \right), \ldots, \exp \left( -\frac{m_1 \xi_{\gamma_{m_2}}^1}{r} J_{i+1,j+1}^{-1} \right), \right\}.$$

$$\exp \left( -\frac{\xi_{\gamma_{m_2}}^1}{r} J_{i+1,j+1}^{-1} \right), \exp \left( -\frac{\xi_{\gamma_{m_2}}^2}{r} J_{i+1,j+1}^{-1} \right), \ldots, \exp \left( -\frac{m_1 \xi_{\gamma_{m_2}}^1}{r} J_{i+1,j+1}^{-1} \right),$$

$$\ldots, \exp \left( -\frac{\xi_{\gamma_{m_2}}^1}{r} J_{i+1,j+1}^{-1} \right), \exp \left( -\frac{\xi_{\gamma_{m_2}}^2}{r} J_{i+1,j+1}^{-1} \right), \ldots, \exp \left( -\frac{m_1 \xi_{\gamma_{m_2}}^1}{r} J_{i+1,j+1}^{-1} \right) \right\}. $$
Remark, that
\[(E_{m_1} \otimes R^{-1}_1 \otimes E_d) \mathcal{T} \tilde{D}_{i+1,j+1} \mathcal{T} (E_{m_1} \otimes R_1 \otimes E_d) = \tilde{D}^2_{i+1,j+1}, \quad (5.16)\]
where
\[\tilde{D}^2_{i+1,j+1} = \text{diag}\left\{ \exp\left( -\frac{1}{r} \gamma_{m_2} \otimes J^{-1}_{i+1,j+1} \right) \right\}.\]
Therefore, from (5.15) and (5.16) we get
\[r K^1_{i+1,j+1} \tilde{w}^1_{i,j+1} = \tilde{D}^2_{i+1,j+1} \tilde{w}^1_{i,j+1} + O(h^{m_1}). \quad (5.17)\]
To transform the coefficients of the second and third equation of the system (5.7), it is enough to make use of (4.2) and (4.12). In a result we obtain
\[F^2_{i+1,j+1} \tilde{w}^2_{i,j+1} = 0, \quad r K^2_{i+1,j+1} \tilde{w}^2_{i,j+1} = \tilde{w}^2_{i,j+1} + O(h^{m_1}), \quad (5.18)\]
Therefore, a transformation of all coefficients of the system (5.7) is finished. Let us write down transformed difference scheme using (5.12), (5.17) and (5.18). We have
\[\tilde{w}^1_{i+1,j+1} = \tilde{g}^1_{i+1,j+1} - \mathcal{T}^\top \tilde{D}_{i+1,j+1} \tilde{w}^1_{i,j+1} - 3 \sum_{l=1}^3 \epsilon^1_{i+1,j+1}(h, \tau) \tilde{w}^l_{i+1,j+1} - \sum_{l=1}^3 \epsilon^1_{i+1,j+1}(h, \tau) \tilde{w}^l_{i+1,j+1} + O(h^{m_1}) + O(\tau^{m_2}), \quad (5.19)\]
\[\tilde{w}^2_{i+1,j+1} = \tilde{g}^2_{i+1,j+1} - \tilde{w}^2_{i,j+1} - 3 \sum_{l=1}^3 \epsilon^2_{i+1,j+1}(h, \tau) \tilde{w}^l_{i+1,j+1} - \sum_{l=1}^3 \epsilon^2_{i+1,j+1}(h, \tau) \tilde{w}^l_{i+1,j+1} + O(h^{m_1}) + O(\tau^{m_2}), \quad (5.19)\]
\[\tilde{w}^3_{i+1,j+1} = \tilde{g}^3_{i+1,j+1} - \tilde{w}^3_{i+1,j+1} - 3 \sum_{l=1}^3 \epsilon^3_{i+1,j+1}(h, \tau) \tilde{w}^l_{i+1,j+1} - \sum_{l=1}^3 \epsilon^3_{i+1,j+1}(h, \tau) \tilde{w}^l_{i+1,j+1} + O(\tau^{m_2}). \quad (5.19)\]
The system (5.19) represents a canonical form of the difference scheme (1.2). In this form one immediately sees spectral characteristics of matrix coefficients. This information is necessary to prove a uniform boundedness of the grid solution in the domain \(U\). Therefore the goal of this section is achieved and we are in position to turn to the main part of the paper.

6. THE PROOF OF A STABILITY OF THE DIFFERENCE SCHEME

In this section we prove the following.

**Theorem 6.1.** Let in the difference scheme (5.7) eigenvalues of the matrices \(\tilde{\gamma}_{m_1}, \gamma_{m_2}\) and \(J_{i+1,j+1}\) with given \(m_1, m_2\) satisfy the condition (4.3) in the domain \(U_{\Delta}\) and \(\epsilon_i^s > 0 \quad \forall \ i, j \quad s = 1, \ldots, k\). Let \(r\) being a ratio of two steps is a constant. Then the difference scheme (5.7) is absolutely stable with respect to initial-boundary data and right-hand side. Following estimation for its solution
\[
\|s_{i+1,j+1}\|_{C(U_{\Delta})} \leq M_1\|s_{i+1,j+1}\|_{C(U_{\Delta})} + M_2\|\phi_{i+1}\|_{C(U_{\Delta})} + M_3\|\psi_{j+1}\|_{C(U_{\Delta})} \quad (6.1)
\]
is valid, where \(M_k\) are constants and \(i = 1, \ldots, n_1 - 1, \quad j = 1, \ldots, n_2 - 1\).
Proof. Remark that the existence of a unique solution of the difference scheme (3.5) was in fact proven in previous section together with its transformation. It remains to prove the second part of the theorem.

The difference scheme (3.5) in the previous section was transformed to the form (5.19). To describe the structure of difference scheme (5.19), we rewrite it in matrix form

\[(L + L(h, \tau))V = G,\]

where \(V\) is unknown \(n_1n_2\)-dimensional vector, consisting of three block components \(V = (V^1 \ V^2 \ V^3)\), where

\[V^k = \begin{pmatrix} w_{1,1}^k, w_{1,2}^k, \ldots, w_{1,n_2}^k, \bar{w}_{2,1}^k, w_{2,2}^k, \ldots, \bar{w}_{2,n_2}^k, \ldots, w_{n_1,1}^k, \bar{w}_{n_1,2}^k, \ldots, w_{n_1,n_2}^k \end{pmatrix}^\top.\]

Let us describe in detail system (6.2). In (6.2), \(L\) is a square block diagonal matrix of the order \(n_1n_2\), that is, \(L = \text{diag}\{L^1, L^2, L^3\}\). In turn each block \(L^k\) is block two-diagonal matrix \(L^k = (L^k_{i,j})\) for \(i = 1, \ldots, n_1\) and \(j = 1, \ldots, n_2\). The blocks \(L^k_{i,j}\) at the main diagonal, also have the block two-diagonal form

\[L^k_{i,j} = \begin{pmatrix} E_s & 0_s & 0_s & \ldots & 0_s & 0_s \\ \mathcal{T}_{2,1}^k & E_s & 0_s & \ldots & 0_s & 0_s \\ 0_s & \mathcal{T}_{3,2}^k & E_s & \ldots & 0_s & 0_s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_s & 0_s & 0_s & \ldots & E_s & 0_s \\ 0_s & 0_s & 0_s & \ldots & \mathcal{T}_{n_2,2}^k & E_s \end{pmatrix}.\] (6.3)

Here the parameter \(s\) take values \(d, l\) and \(p\) for \(k = 1, 2\) and \(3\), respectively. The blocks \(\mathcal{T}_{i,j}^k\) of the matrix (6.3) have the form

\[\mathcal{T}_{i,j}^1 = \mathcal{T}_{i,j}^\top \hat{D}_{i,j}^1, \quad \mathcal{T}_{i,j}^2 = 0_l, \quad \mathcal{T}_{i,j}^3 = E_p.\] (6.4)

The blocks \(L^k_{i,j}\), situated under the main diagonal, that is, for \(i = 2, \ldots, n_1\) and \(j = i - 1\), have the following block diagonal form:

\[L^k_{i,j} = \text{diag}\{\mathcal{X}_{1,1}^k, \mathcal{X}_{i,2}^k, \ldots, \mathcal{X}_{i,n_2}^k\}, \quad \mathcal{X}_{1,1}^k = \hat{D}_{i,j}^2, \quad \mathcal{X}_{i,j}^2 = E_l, \quad \mathcal{X}_{i,j}^3 = 0_p.\] (6.5)

All the rest blocks \(L^k_{i,j}\) are zero ones of corresponding sizes.

The vector \(G\) in the system (6.2) is as follows:

\[G = \hat{g} + ((S + S(h, \tau))w_0 + (Q + \Omega(h, \tau))w^0 + O(h^{m_1}) + O(\tau^{m_2}),\]

with \(\hat{g}\) are \(n_1n_2\)-dimensional vector, consisting of blocks \(\hat{g} = (g^1, g^2, g^3)^\top\) of the sizes \(n_1n_2d, n_1n_2l\) and \(n_1n_2p\), respectively,

\[g^k = (\bar{g}_{1,1}^k, g_{1,2}^k, \ldots, \bar{g}_{1,n_2}^k, \bar{g}_{2,1}^k, g_{2,2}^k, \ldots, \bar{g}_{2,n_2}^k, \ldots, \bar{g}_{n_1,1}^k, g_{n_1,2}^k, \ldots, \bar{g}_{n_1,n_2}^k)^\top;\]

\(w_0\) and \(w^0\) are known vectors of the size \(n_1n_2\) the elements of which are defined by initial-boundary data (2.2), namely, \(w_0 = (w_0^1, w_0^2, w_0^3)^\top\), where

\[w_0^k = \begin{pmatrix} \bar{w}_{1,0}^k, \bar{w}_{2,0}^k, \ldots, \bar{w}_{n_1,0}^k \end{pmatrix}^\top \otimes e_{n_2}\]

and \(w_0 = (w_0^{0,1}, w_0^{0,2}, w_0^{0,3})^\top\), where

\[w_0^{0,k} = e_{n_2} \otimes \begin{pmatrix} \bar{w}_{0,1}^k, \bar{w}_{0,2}^k, \ldots, \bar{w}_{0,n_2}^k \end{pmatrix}^\top.\]

The matrices $S$ and $Q$ are known square block diagonal ones of the order $n_1n_2\vec{n}$, that is,

$$S = \text{diag}\{S^1, S^2, S^3\} \quad \text{and} \quad Q = \text{diag}\{Q^1, Q^2, Q^3\}.$$  

Each square block $S^k$ and $Q^k$ has the order $n_1n_2d$, $n_1n_2l$ and $n_1n_2p$, corresponding to the value of $k$. Each block $S^k$ is also block diagonal matrix

$$S^k = \text{diag}\left\{S^k_{1,1}, S^k_{2,2}, \ldots, S^k_{n_1n_1}\right\},$$

where $S^k_{i,i}$ are square blocks of the orders $n_2d$, $n_2l$ and $n_2p$, respectively $k$, of the following form:

$$S^k_{i,i} = \text{diag}\left\{-\Phi^k_{i,i}, \Theta^k_{i,i}, \Theta^k_{i,i}, \ldots, \Theta^k_{i,i}\right\},$$

for $i = 1, \ldots, n_1$, where $s$ takes values $d$, $l$ and $p$, respectively $k$. Each block $Q^k$ is block diagonal matrix

$$Q^k = \text{diag}\left\{Q^k_{1,1}, Q^k_{2,2}, \ldots, Q^k_{n_1n_1}\right\},$$

where

$$Q^k_{i,i} = \text{diag}\left\{-\Psi^k_{i,i}, -\Psi^k_{i,i}, \ldots, -\Psi^k_{i,i}\right\}.$$  

Finally, the matrices $L(h, \tau)$, $S(h, \tau)$ and $Q(h, \tau)$ in (6.2) and (6.6) are quadratic matrices of the order $n_1n_2\vec{n}$ built from the blocks $\vec{e}_{i,j}^{s,l}(h, \tau)$ and $\vec{e}_{i,j}^{s,l}(h, \tau)$.

To prove the stability property of difference scheme (5.5), there is a need to estimate the norm of unknown vector $V$ in the system (6.3). For this aim we must calculate $L^{-1}$. Since $L$ is a block diagonal matrix, and each its diagonal block is a block two-diagonal matrix, then we can easily write down the matrix $L^{-1}$ in its explicit form. Each its diagonal block component is of the form

$$(L^k)^{-1} = \Lambda^k \Phi^k, \quad \text{for} \quad k = 1, 2, 3,$$  

where $\Lambda^k$ are matrices of the order $n_1n_2d$, $n_1n_2l$ and $n_1n_2p$, respectively value of $k$, which have the form

$$\Lambda^k = \begin{pmatrix} E_s & 0_s & 0_s & \ldots & 0_s \\ \Lambda^k_2 & E_s & 0_s & \ldots & 0_s \\ \Lambda^k_3 & \Lambda^k_4 & E_s & \ldots & 0_s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{i=s_1}^{n_1} A^k_1 & \prod_{i=s_1}^{n_1} A^k_2 & \prod_{i=s_1}^{n_1} A^k_3 & \prod_{i=s_1}^{n_1} A^k_4 & E_s \end{pmatrix},$$  

where $s$ take values $n_2d$, $n_2l$ and $n_2p$, respectively value of $k$. Every block $\Lambda^k_i$, for $i = 2, \ldots, n_1$ is defined as $\Lambda^k_i = (\Phi^k_i)^{-1}Q^k_{i,i}$, where

$$(\Phi^k)^{-1} = \begin{pmatrix} E_s & 0_s & 0_s & \ldots & 0_s \\ -\Phi^k_{2,1} & E_s & 0_s & \ldots & 0_s \\ \Phi^k_{3,2} & -\Phi^k_{4,3} & E_s & \ldots & 0_s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n_2+1}\prod_{i=s_1}^{n_2} \Phi^k_{i,s_1} & (-1)^{n_2+2}\prod_{i=s_1}^{n_2} \Phi^k_{i,s_1} & (-1)^{n_2+3}\prod_{i=s_1}^{n_2} \Phi^k_{i,s_1} & (-1)^{n_2+4}\prod_{i=s_1}^{n_2} \Phi^k_{i,s_1} & E_s \end{pmatrix},$$  

where $s$ take values $d$, $l$ and $p$, respectively value of $k$. The matrices $\Phi^k$ in (6.7) have the form

$$\Phi^k = \text{diag}\{(\Phi^k_1)^{-1}, (\Phi^k_2)^{-1}, \ldots, (\Phi^k_{n_1})^{-1}\}. $
Estimate the norms of the matrices \((L^k)^{-1}\). For \(k = 2\) and \(k = 3\), it follows from (6.4), (6.5) and (6.7)-(6.9) that \(\Lambda^k = \mathbf{0}\) and \((L^k)^{-1} = E_{n_1n_2s}\), where \(s = d, l\) for \(k = 2, 3\), respectively. Thus,
\[
\| (L^k)^{-1} \|_{C(\mathcal{U}_\Delta)} = 1 \quad \text{for} \quad k = 2, 3.
\]

Let us estimate the norm of the first block component \((L^1)^{-1}\). It follows from (6.7) and (6.8) that
\[
\| (L^1)^{-1} \|_{C(\mathcal{U}_\Delta)} \leq \| \Lambda^1 \|_{C(\mathcal{U}_\Delta)} \| \Phi^1 \|_{C(\mathcal{U}_\Delta)},
\]
\[
\| \Lambda^1 \|_{C(\mathcal{U}_\Delta)} \leq 1 + \| \Lambda^1 \|_{C(\mathcal{U}_\Delta)} + \| \Lambda^1 \|_{C(\mathcal{U}_\Delta)}^2 + \cdots + \| \Lambda^1 \|_{C(\mathcal{U}_\Delta)}^{n_1-1}.
\]

In virtue of (6.9) we have
\[
\| \Lambda^1 \|_{C(\mathcal{U}_\Delta)} \leq (\| \Phi^1 \|_{C(\mathcal{U}_\Delta)})^{-1} \| \mathcal{K}^1_{i,j} \|_{C(\mathcal{U}_\Delta)}.
\]

In turn for the first multiplier in the right-hand side of the inequality (6.12) we have
\[
\| (\Phi^1)^{-1} \|_{C(\mathcal{U}_\Delta)} \leq 1 + \| \mathcal{F}^1_{i,n_2} \|_{C(\mathcal{U}_\Delta)} + \| \mathcal{F}^1_{i,n_2} \|_{C(\mathcal{U}_\Delta)}^2 + \cdots + \| \mathcal{F}^1_{i,n_2} \|_{C(\mathcal{U}_\Delta)}^{n_2-1}.
\]

From (6.11) and (6.4) we find the radius \(\mu\) of the spectrum of the matrix \(\mathcal{F}^1_{i,j}\) to get
\[
\mu = \max \left\{ \left| \exp \left( -k_1 r \xi_{m_1}^{k_2} \xi_{m_2}^{k_3} \right) \right| \right\}
\]
for \(i = 1, \ldots, n_1, j = 1, \ldots, n_2, k_1 = 1, \ldots, m_2, k_2 = 1, \ldots, m_1 \) and \(k_3 = 1, \ldots, d\). We have a pair of inequalities: \(\text{Re}(\xi_{m_1}^{k_2}) > 0 \forall k_2\), which can be verified with the help of Routh-Hurwitz criterion and \(\xi_{m_1}^{k_2} > 0 \forall k_3\), which is valid by condition of our theorem. Then, from (6.14) we get \(\mu < 1\). Applying theorem on spectral decomposition for a power function of matrix, namely, \((\mathcal{F}^1_{i,j})^{n_2}\) given in [21, p. 155] or theorem from [25], we get
\[
\| (\mathcal{F}^1_{i,j})^{n_2} \|_{C(\mathcal{U}_\Delta)} \to 0, \quad \text{at} \quad n_2 \to \infty.
\]

From (6.15) it follows, that there exists such a value of the power \(\tilde{m}\), for which
\[
\| (\mathcal{F}^1_{i,j})^{\tilde{m}} \|_{C(\mathcal{U}_\Delta)} < 1,
\]
where \(m\) is a suitable constant. Let us consider the product \(\prod_{j=2}^{\tilde{m}+1} \mathcal{F}^1_{i,j}\). With Taylor’s formula we obtain
\[
\prod_{j=2}^{\tilde{m}+1} \mathcal{F}^1_{i,j} = (\mathcal{F}^1_{i,\tilde{m}})^{\tilde{m}} + O(\tau).
\]

Denote \(\chi = \left| \prod_{j=2}^{\tilde{m}+1} \mathcal{F}^1_{i,j} \right|_{C(\mathcal{U}_\Delta)}\). From (6.16) and (6.17) it follows that \(\chi < 1\). Hence, from (6.13) we get
\[
\| (\Phi^1)^{-1} \|_{C(\mathcal{U}_\Delta)} \leq \eta/(1 - \chi),
\]
where
\[
\eta = 1 + \| \mathcal{F}^1_{i,j} \|_{C(\mathcal{U}_\Delta)} + \| \mathcal{F}^1_{i,j} \|_{C(\mathcal{U}_\Delta)}^2 + \cdots + \| \mathcal{F}^1_{i,j} \|_{C(\mathcal{U}_\Delta)}^{\tilde{m}-1}.
\]

Let us estimate a norm of the matrix \(\mathcal{K}^1_{i,j}\). We learn from [20] that for a constant \(n \times n\) matrix \(\hat{A}\) spectrum of which lies strictly in the left halfplane it is valid following inequality
\[
\| \exp(t\hat{A}) \|_2 \leq \sqrt{e} \exp(-\kappa t), \quad t \geq 0,
\]
where \(\kappa = 1/(2\|X\|_2), \quad c = \|X^{-1}\|_2\|X\|_2\) and \(X\) is Hermitian \(n \times n\) matrix, being a solution of Lyapunov equation \(XA + A^*X = -E_n\) and \(\| \cdot \|_2\) is a norm of a matrix, agreed with Hermitian norm. The constants \(c\) and \(\kappa\) are clarified in [27].
From the above mentioned assumptions with respect to eigenvalues of the matrix $J(x,t)$ and from (5.10), (5.15) and (5.19) it follows, that for the matrix $K_{i,j}^1$ it is valid the following estimation:

$$\|K_{i,j}^1\|_{C(U_\Delta)} \leq \sqrt{\tilde{c}} \exp(-\tilde{\kappa}/r),$$

(6.20)

where $\tilde{c}$ and $\tilde{\kappa}$ are constants. From (6.20) it follows that, reducing $r$, one can achieve a sufficient smallness of the elements of $K_{i,j}^1$ including that the inequality

$$\frac{\eta}{1-\chi} \|K_{i,j}^1\|_{C(U_\Delta)} < 1.$$  

(6.21)

will be valid. So, from (6.12), (6.18) and (6.21) it follows that there are such values of the steps $\tau$ and $h$ for which the inequality $\|\Lambda^1\|_{C(U_\Delta)} < 1$ holds. Taking into account (6.11), we get

$$\|\Lambda^1\|_{C(U_\Delta)} < \tilde{c}, \quad \text{where} \quad \tilde{c} = 1/(1 - \|\Lambda^1\|_{C(U_\Delta)}).$$

(6.22)

Therefore, from (6.11) and (6.18) it follows that

$$\|(L^1)^{-1}\|_{C(U_\Delta)} < \tilde{c}, \quad \text{where} \quad \tilde{c} = \hat{c}\eta/(1 - \chi).$$

(6.23)

These inequalities (6.10) and (6.23) mean uniform boundedness of the matrix $L^{-1}$ in (6.2). From (5.9) and (6.4) it follows that to estimate a norm of the matrix $\Gamma_{i,j}^1$, we can use the inequality (6.19). In a result, we get

$$\|\Gamma_{i,j}^1\|_{C(U_\Delta)} \leq \sqrt{\epsilon} \exp(-\tilde{\kappa}r),$$

(6.24)

where $\epsilon$ and $\tilde{\kappa}$ are some constants. Then the matrices $S$ and $Q$ in (6.6), in virtue of (6.20) and (6.24), are also uniformly boundness at the grid $U_\Delta$. That is, for sufficiently small values of $h$ and $\tau$ there are such constants $\rho_1$ and $\rho_2$, for which

$$\|S + \tilde{s}(h, \tau)\|_{C(U_\Delta)} \leq \rho_1, \quad \|Q + \tilde{q}(h, \tau)\|_{C(U_\Delta)} \leq \rho_2.$$  

(6.25)

From (6.2) and (6.6) we obtain

$$\mathcal{V} = (E_{n_1n_2\tilde{n}} + L^{-1}L(h, \tau))^{-1}L^{-1} \left(\tilde{g} + (S + \tilde{s}(h, \tau))\bar{w}_0 + (Q + \tilde{q}(h, \tau))\tilde{w}^0 + O(h^{m_1}) + O(\tau^{m_2})\right).$$

(6.26)

In virtue of a boundness of the matrix $L^{-1}$, for sufficiently small $\tau$ and $h$, we have

$$\|(E_{n_1n_2\tilde{n}} + L^{-1}L(h, \tau))^{-1}\|_{C(U_\Delta)} \leq \tilde{\eta},$$

where $\tilde{\eta} = 1/(1 - \tilde{\eta})$ and $\tilde{\eta} = \|L^{-1}L(h, \tau)\|_{C(U_\Delta)}$. Denote

$$\mathcal{M}_1 = \tau\rho\tilde{\eta}K\|\tilde{P}\|_{C(U_\Delta)}/(1 - \|\tilde{M}_{i+1,j+1}\|_{C(U_\Delta)}),$$

$$\mathcal{M}_2 = \rho\tilde{\eta}, \quad \mathcal{M}_3 = \rho\tilde{\eta}$$

where $\rho = \max\{\tilde{c}, 1\}$ and $\rho_1$, $\rho_2$ as in (6.25). Letting $h \to 0$ and $\tau \to 0$, from (6.26) we obtain desired estimate (6.11). Therefore the theorem is proven.

Let us remark, the theorem 6.1 is also valid for initial-boundary problems of the form (2.1) and (2.2) with nondegenerate in a domain of definition matrices $A(x,t)$ and $B(x,t)$. To check this, it is enough to put $p = 0$ and $l = 0$. 
7. Numerical experiments

For numerical solving of boundary problems of the form (2.1)-(2.3) with mentioned above conditions, we have created a program in which the user can set input data, that is, matrix coefficients $A(x,t)$, $B(x,t)$, $C(x,t)$ and vector-functions $f(x,t)$, $\phi(x)$ and $\psi(t)$; the domain of definition $U$; the values of the steps $h$ and $\tau$ and the orders of a spline $m_1$ and $m_2$ with respect to each independent variable.

In this section we present the numerical results of solving of some problems of the form (2.1)-(2.3). Remark, that these examples were made only to demonstrate the stability property of the difference scheme (3.5).

Example 7.1. Let us consider the system (2.1), in which

$$A(x,t) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp(xt) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 + xt & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B(x,t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp(\sin(\vartheta)) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ xt & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C(x,t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2xt & 0 \\ 0 & 0 & 0 & \vartheta & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f(x,t) = \begin{pmatrix} \exp(\vartheta) \\ \exp(xt)(x \exp(xt) + t \sin(\vartheta) + 2xt) \\ \vartheta \\ x(1 + t(\vartheta + 1)) \\ x \\ 1 \end{pmatrix}.$$ (7.1)

with $\vartheta = x + t$. We know that the exact solution of our system with the data (7.1) is

$$u(x,t) = (x, \exp(xt), \exp(\vartheta), xt, 1, \exp(xt) + \vartheta)^\top.$$

With the help of nondegenerate in compact domain $\bar{U} \subset \{(x,t) \in \mathbb{R}^2, xt \neq -1\}$ matrices

$$P(x,t) = \begin{pmatrix} 0 & 1/\exp(xt) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(1 + xt) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$ (7.2)

the system (2.1) with the data (7.1) transforms to the canonical form (2.4), in which $J_1(x,t) = \exp(\sin(\vartheta))/\exp(xt)$, $J_2(x,t) = xt/(1 + xt)$ and $M = N = O_2$. To check this, it is enough to multiply the system (2.1) on the left by the matrix $P(x,t)$ and to make a change of variable: $v(x,t) = Qu(x,t)$.

Remark, that in arbitrary compact domain $\bar{U}$ the first condition of the theorem 2.2 is not fulfilled. To check this, it is enough to write down the characteristic equation of the system (2.1) and to see that its roots have common values in $\bar{U}$. Nevertheless, the system (2.1) with the data (7.1) can be cast to the form (2.4) and therefore the conditions of the theorem 6.1 are fulfilled. Thus, in this case the difference scheme (3.5) is stable in the domain $\bar{U}$. We show the results of numerical solving in the table 1. As stability estimate we take in this case the value of absolute error of the solution $\Delta u = \|u_{i,j} - v_{i,j}\|_{C(U_{\Delta})}$. 
the numerical solutions in the domain of calculations. What we learn from the test 17 is that using third-degree spline the accuracy of degree of the spline impact on the dimension of the difference scheme (3.5) and therefore on a speed enough to increase a degree of the spline with respect to one variable. This is important, because a Example 7.2.

From the table 1 we see, that to achieve more accuracy of numerical solving, in some cases, it is enough to increase a degree of the spline with respect to one variable. This is explained by the impact of a large Lipchitz constant.

Table 1.

| N | h   | τ   | t₀ | T₀ | x₀ | X   | m₁ | m₂ | Δu     |
|---|-----|-----|----|----|----|-----|----|----|--------|
| 1 | 10⁻¹| 10⁻¹| 0  | 0  | 1  | 2   | 2  | 2  | 2.07 × 10⁻² |
| 2 | 10⁻¹| 10⁻¹| 0  | 1  | 0  | 3   | 2  | 2  | 2.07 × 10⁻² |
| 4 | 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 2  | 3  | 1.96 × 10⁻³ |
| 5 | 10⁻¹| 10⁻¹| 0  | 0  | 1  | 3   | 3  | 3  | 1.96 × 10⁻³ |
| 6 | 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 4  | 3  | 1.96 × 10⁻³ |
| 7 | 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 3  | 4  | 1.91 × 10⁻⁴ |
| 8 | 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 4  | 4  | 1.91 × 10⁻⁴ |
| 9 | 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 5  | 5  | 1.91 × 10⁻⁵ |
| 10| 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 6  | 6  | 1.91 × 10⁻⁶ |
| 11| 10⁻¹| 10⁻¹| 0  | 1  | 0  | 1   | 7  | 7  | 1.93 × 10⁻⁷ |
| 12| 10⁻²| 10⁻¹| 0  | 1  | 0  | 1   | 2  | 2  | 2.07 × 10⁻² |
| 13| 10⁻¹| 10⁻²| 0  | 1  | 0  | 1   | 2  | 2  | 1.28 × 10⁻³ |
| 14| 10⁻²| 10⁻²| 0  | 1  | 0  | 1   | 2  | 2  | 1.96 × 10⁻⁴ |
| 15| 5 × 10⁻³| 5 × 10⁻³| 0  | 1  | 0  | 1   | 2  | 2  | 4.95 × 10⁻⁵ |
| 16| 10⁻¹| 10⁻¹| 0  | 1  | 0  | 2   | 2  | 2  | 7.14 × 10⁻² |
| 17| 10⁻¹| 10⁻¹| 0  | 1  | 0  | 2   | 3  | 3  | 1.31 × 10⁻² |
| 18| 10⁻¹| 10⁻¹| 0  | 1  | 0  | 2   | 4  | 4  | 2.52 × 10⁻³ |
| 19| 10⁻¹| 10⁻¹| 0  | 1  | 0  | 2   | 5  | 5  | 4.99 × 10⁻⁴ |

From the table 1 we see, that to achieve more accuracy of numerical solving, in some cases, it is enough to increase a degree of the spline with respect to one variable. This is explained by the impact of a large Lipchitz constant.

Example 7.2. Let us consider the system (2.1) with multiple characteristic curves with the following data

\[ A(x, t) = \text{diag}\{E_5, 0, 1\}, \quad B(x, t) = \text{diag}\{J_1(x, t), J_2(x, t), 1, 0\}, \]

\[ J_1(x, t) = \begin{pmatrix} \exp(\varphi) & 0 & 0 \\ 0 & \exp(\varphi) & 1 \\ 0 & 0 & \exp(\varphi) \end{pmatrix}, \quad J_2(x, t) = \begin{pmatrix} 1 + t \exp(x) & 1 \\ 0 & 1 + t \exp(x) \end{pmatrix}, \]

\[ C(x, t) = \begin{pmatrix} x^2 + t & 0 & 1 + xt & -\exp(\varphi) & 0 & 0 \\ 0 & x^2 & xt & 0 & 0 & 1 \varphi \\ 1 & 0 & 0 & 0 & 1 & xt \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ x \exp(\varphi) & 1 & x^2t & 0 & \varphi & 0 \\ 0 & 0 & \exp(\varphi) & 0 & 0 & 1 \\ \end{pmatrix}, \quad (7.3) \]
Here $\vartheta$ is defined in example 1. The exact solution of (2.1) with the data (7.3) is

$$u(x, t) = (\exp(\vartheta), \vartheta, 2xt, x - t, 1, x \exp(t), x^2t)\top.$$

This system is given in canonical form. It has multiple nontrivial characteristic curves

$$\lambda_{1,2,3} = -\exp(-\vartheta), \quad \lambda_{4,5} = -1/(1 + t \exp(x))$$

and one zero characteristics $\lambda_6 = 0$. In an arbitrary compact domain $\tilde{U} \subset U = \{(x, t), \ t \exp(x) > 0\}$, the conditions of theorems 2.2 and 6.1 and remark 2.3 are valid. Therefore, the difference scheme (3.5) is stable in any domain $\tilde{U}$. The results of numerical solving are given in the table 2.

**Table 2.**

| $N$ | $h$  | $\tau$ | $t_0$ | $T$ | $x_0$ | $X$ | $m_1$ | $m_2$ | $\Delta u$       |
|-----|------|--------|-------|-----|-------|-----|-------|-------|------------------|
| 1   | $10^{-4}$ | 10^{-1} | 0     | 1   | 0     | 1   | 2     | 2     | $3.54 \times 10^{-2}$ |
| 2   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 3     | 2     | $4.96 \times 10^{-3}$ |
| 3   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 1     | 3     | $3.04 \times 10^{-2}$ |
| 4   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 2     | 3     | $3.34 \times 10^{-3}$ |
| 5   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 2     | 3     | $4.78 \times 10^{-4}$ |
| 6   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 3     | 4     | $2.91 \times 10^{-3}$ |
| 7   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 4     | 4     | $3.23 \times 10^{-4}$ |
| 8   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 5     | 3     | $6.19 \times 10^{-4}$ |
| 9   | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 5     | 4     | $4.76 \times 10^{-5}$ |
| 10  | $10^{-1}$  | 10^{-1} | 0     | 1   | 0     | 1   | 5     | 5     | $3.21 \times 10^{-5}$ |
| 11  | $10^{-2}$  | 10^{-1} | 0     | 1   | 0     | 1   | 2     | 2     | $3.23 \times 10^{-4}$ |
| 12  | $10^{-1}$  | 10^{-1} | 0     | 2   | 0     | 2   | 2     | 2     | $7.97$           |
| 13  | $10^{-1}$  | 10^{-1} | 0     | 2   | 0     | 2   | 3     | 3     | $7.64 \times 10^{-1}$ |
| 14  | $10^{-1}$  | 10^{-1} | 0     | 2   | 0     | 2   | 4     | 4     | $7.46 \times 10^{-2}$ |

Remark, that in the tests 12, 13 and 14 on the error has influenced a large Lipchitz constant.

8. Conclusion

Numerous actual calculations show that spline collocation difference scheme given in this paper is a quite effective and gives sufficient accuracy. But, as was mentioned above, the difference scheme (3.5) in fact being a linear system of algebraic equations has the order $m_1m_2n$. Therefore increase of the degree of a used spline yields an increase of an order of this system. Since our system by assumption has general form then to solve it one needs to use, generally speaking, universal methods and therefore procedure of numerical solving might be quite laborious. In this case there is a need of a parallelization of numerical calculations. Following remark is in order. The point is that many actual problems of mathematical physics and mechanics are described by systems of partial differential
algebraic equations, involving more than two independent variables. Examples are given by Sobolev’s system, linearized Navier-Stokes system etc. [1]. Therefore the method given in this work could be more marketable if one generalizes it to the case of many independent variables and this is the problem to be addressed.

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