Invited Paper

Recent progress in mathematical modelling of complex systems

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Abstract: In this paper, I review recent progress in studies on mathematical modelling of complex systems from a general viewpoint. First, I explain our theoretical platform composed of (1) advanced control theory of complex systems, (2) complex networks theory, and (3) nonlinear data analysis and data-driven modelling, that has been developed for mathematical modelling of complex systems. Second, I introduce recent various applications based on the theoretical platform. Finally, I discuss possible future directions of this research on the mathematical modelling of complex systems.

Key Words: mathematical modelling, nonlinear dynamics, complex systems, attention, dynamical network biomarkers (DNB), optoelectronic neural networks

1. Introduction

This research on mathematical modelling of complex systems is mainly based on the past research projects, namely, the ERATO (Exploratory Research for Advanced Technology) Aihara Complexity Modelling project by JST (Japan Science and Technology Agency) from 2003 to 2008 and the FIRST Innovative Mathematical Modelling project by JSPS (Japan Society for the Promotion of Science) through the FIRST (Funding Program for World-Leading Innovative R&D Science and Technology) program from 2010 to 2014, initiated by CSTP (Council for Science and Technology Policy).

These research projects developed mathematical theory for modelling complex systems and its wide-ranging transdisciplinary applications in science and technology from the viewpoint of mathematical engineering as well as chaos engineering [1]. Here, mathematical engineering is a discipline where mathematical models are built and theoretically analysed in order to understand, optimize, control, and predict real-world systems.

In these projects, we aimed not only at systematizing methodology for modelling complex systems mathematically [2, 3] but also at providing satisfactory solutions for complex problems with high importance and urgency for our society, such as innovative treatment strategies for complex diseases like cancer and HIV [4, 5], optimization and control of power grids, communication networks, and traffic flows [6, 7], and development of novel nonlinear electronic technology such as chaos chips, neurochips, optoelectronic neural networks, and AD converters based on $\beta$ encoders [8, 9].

In this paper, I review recent progress in studies on mathematical modelling of complex systems from a general viewpoint.
2. Theoretical platform for mathematical modelling of complex systems

The theoretical platform for mathematical modelling of complex systems is composed of (1) advanced control theory of complex systems, (2) complex networks theory, and (3) nonlinear data analysis and data-driven modelling as shown in Fig. 1.

The advanced control theory of complex systems was developed by fusion of dynamical systems theory in mathematics and control theory in engineering in order to analyse nonlinear dynamics of complex systems towards control [2, 3]. A typical example is robust bifurcation theory that combines bifurcation theory with robust control for analysis of bifurcation phenomena in mathematical models of real-world systems with inevitable uncertainty [10]. Further, dynamical scale transform in tropical geometry was studied in details by T. Kato [11].

The complex systems are generally composed of many elements interacting through complex network structure. The complex network theory analyses such network structure of complex systems, which is also important for network optimization.

Recently, due to advance of technology for sensors and IoT, a lot of big data can be observed from complex systems. To extract essential information hidden in big data, however, nonlinear data analysis is indispensable. Moreover, when we can store big time series data, a mathematical model can be algorithmically constructed by data-driven modelling.

3. Recent applications of mathematical modelling of complex systems

Many real-world problems related to complex systems have been analysed on the basis of the theoretical platform for mathematical modelling of complex systems.

3.1 Applications of complex systems control theory

Living things and their brains are typical examples of complex systems with rich and high functions some of which even state-of-the-art technologies can not realize. For example, bats use echolocating control for flying and foraging. Fujioka et al. clarified both by mathematical modelling and by field observation that Japanese house bats (see Fig. 2) can fly optimal paths to catch multiple prey items by distributed attention [12]. In this research, the flight dynamics is well modelled by the following equations [12]:

\[
\frac{d\varphi_h(t)}{dt} = \frac{1}{\delta_h} \left\{ \alpha_h \sin[\varphi_{bp1}(t) - \varphi_b(t)] + \beta_h \sin[\varphi_{bp2}(t) - \varphi_b(t)] \right\}, \tag{1}
\]

\[
\frac{d\theta_b(t)}{dt} = \frac{1}{\delta_v} \left\{ \alpha_v \sin[\theta_{bp1}(t) - \theta_b(t)] + \beta_v \sin[\theta_{bp2}(t) - \theta_b(t)] \right\}, \tag{2}
\]
where $\varphi_b(t)$ and $\theta_b(t)$ represent the horizontal and vertical angles of the bat’s flight direction, $\delta_h$ and $\delta_v$ represent positive weighting factors, and $\alpha$ ($\alpha_h$ and $\alpha_v$) is the minimization of the angular difference between the bat’s flight direction and the direction to the first prey $[\varphi_{bp1}(t), \theta_{bp1}(t)]$ (similarly for $\beta$ to the second prey) [12].

The study of prostate cancer treatment by intermittent androgen suppression based on mathematical modelling [2] has been further advanced by combining the mathematical modelling with machine learning [13, 14], using the following mathematical model:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} w_{1,1}^1 & 0 & 0 \\ w_{2,1}^1 & w_{2,2}^1 & 0 \\ w_{3,1}^1 & w_{3,2}^1 & w_{3,3}^1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

while the hormonal therapy is on, and

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} w_{0,1}^1 & 0 & 0 \\ 0 & w_{0,2}^1 & 0 \\ 0 & 0 & w_{0,3}^1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

while the hormonal therapy is off, where $x_1$ represents the number of androgen dependent cancer cells, and $x_2$ and $x_3$ represent the numbers of androgen independent cancer cells.

It is expected that this kind of intermittent treatment based on mathematical modelling can be fundamentally applied to any disease to which effective therapy exists but if we keep the therapy for a long time, the disease will acquire resistance to the therapy. Moreover, it provides optimal scheduling of the treatment as well.

### 3.2 Applications of complex network theory

Complex network structure is important to understand emergence of functions in complex systems as well as design innovative technological systems. For example, hardware systems of optoelectronic neural networks are implemented by degenerate optical parametric oscillators where arbitrary network structure among the oscillators is realized by a measurement and feedback scheme [15, 16]. The nonlinear dynamics of each oscillator is described by the following equations:

$$dc_i = \left[ (-1 + p - c_i^2 - s_i^2)c_i + \sum_j \xi_{ij}\tilde{c}_j \right] dt + \frac{1}{A_s} \sqrt{c_i^2 + s_i^2 + \frac{1}{2}} dW_1,$$

$$ds_i = \left[ -1 + p - c_i^2 - s_i^2 \right] s_i dt + \frac{1}{A_s} \sqrt{c_i^2 + s_i^2 + \frac{1}{2}} dW_2,$$

where $c_i$ and $s_i$ are the in-phase and quadrature phase amplitudes of the $i$th oscillator, respectively, $p$ is the pump rate, $\xi_{ij}$ is the connection weight from the $j$th oscillator to the $i$th one, $\tilde{c}_i$ is the measured in-phase amplitude of the $i$th oscillator, $A_s$ is the saturation amplitude, and $dW_1$ and $dW_2$ are independent Gaussian noise processes.

The complex network theory is also useful to understand emergence of dysfunctions such as diseases. Dynamical network biomarkers (DNB) are formulated to find a pre-disease state (Fig. 3(b)) just before
imminent bifurcation or transition from a healthy state (Fig. 3(a)) to a disease state (Fig. 3(c)) by detecting early warning signals of a pre-disease state around the bifurcation point [17–19].

Recently, the DNB analysis is extended to construct an individual-specific network from a single sample [20] and to quantify critical states of complex diseases by single-sample DNB [21].

3.3 Applications of nonlinear data analysis and data-driven modelling
The nonlinear data analysis and data-driven modelling are used to elucidate nonlinear mechanisms of complex systems such as hierarchical circadian rhythms in Arabidopsis [22], earthquakes [23], geological data on glacial-interglacial cycles [24], and fluctuations of power grid frequency [25]. The methodology is also applied to analysis of music data [26, 27] and single-cell chromosome structure [28] where the recurrence plot is effectively used for elucidating music structure and three-dimensional reconstruction of chromosome structure, respectively.

4. Concluding Remarks
The theoretical platform for mathematical modelling of complex systems and its various applications can be further developed towards the future.

For example, the distributed attention of bats [12] should be important for improvement of artificial intelligence such as robotics and autodriving cars. The treatment based on mathematical modelling can optimize schedules of the treatment in a personalized way, and the DNB will make the precision and preemptive medicine possible.

Mathematical modelling is becoming very much important methodology for various research fields. Even within the University of Tokyo, mathematical modelling is actively studied for bridging brain-scientific research of development and novel AI & brain pathology at International Research Center for Neurointelligence (IRCN) established by WPI (World Premier International Research Center Initiative), for exploring universality of biological principles at Universal Biology Institute (UBI), for understanding human behavior at Center for Integrative Science of Human Behavior (CiSHuB), for analysing infectious dynamics at Anti-infective Research Consortium (ARC-UT), for solving complex social problems through Social Cooperation Programs with Kozo Keikaku Engineering Inc. on Mathematical Engineering for Complex Social Systems in Future, for developing brain-morphic AI through Social Cooperation Programs with NEC on Brain-Morphic AI to Resolve Social Issues, and for further progressing innovative mathematical modelling at Collaborative Research Center for Innovative Mathematical Modelling (IMM) at Institute of Industrial Science.

I expect that mathematical modelling will keep playing essential roles to cope with important complex problems in complex real-world systems of our society.

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