To continue the optical-mechanical analogy

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Abstract. On the basis of the V-function method, an optical-mechanical analogy is carried out; and further it receives a new continuation. In particular, the equality of the phase velocity of the wave and the velocity of the particle is shown. It is noted that according to its physical meaning the wave characterizes the properties of the action that manifests itself in the motion of the particle. The V-function method is based on the concept of a process-state and it consists of a local variational principle, a new formulation of the direct and inverse problem of dynamics. According to the V-function method, the trajectory motion of an object is associated with its wave motion. In this case it is assumed that there is an explicit connection between the trajectory and wave equations of the object. The application of the V-function method for the case of rectilinear motion of an object is shown.

1. Introduction

If one looks at the development of the theory of the nature of light before the beginning of L. de Broglie's work, one can see that in one case the wave properties become visible, and in the other one, the properties of the particle and these properties did not overlap with each other, but only mutually complemented.

A look at the nature of light in physical science is carried out, first of all, through an optical-mechanical analogy. It was the optical-mechanical analogy that allowed L. de Broglie to gain new knowledge about nature. His research [1-4] became the starting point for one of the deepest revolutions in physics. It is precisely the complementary character of the parallel existence of two concepts of light, i.e. particle-wave dualism, led him to the idea of the possibility of the same relationship for particles of matter.

The idea that wave motion is hidden behind the motion of particles has become especially fruitful for physics [5]. L. de Broglie's scientific works served as the basis for Schrödinger's formulation of the wave equation [6], which is now one of the foundations of quantum mechanics [7].

In this work the optical-mechanical analogy is carried out on the basis of the V-function method [8-11]. The V-function method is based on the concept of the process-state [12,13]; it consists of a local variational principle, a new formulation of the direct and inverse problems of dynamics [8-11]. It should be noted the difference from the positivist approach [14,15] of the concept of the process-state.

2. To the continuation of the optical-mechanical analogy.

It follows from the V-function method that the trajectory motion of an object is described by a system of differential equations:

\[ \dot{x} = f(x), \]  

(1)
where \( x(t) = (x_1, x_2, \ldots, x_n)^T \) - phase coordinate vector, \( x \in \mathbb{R}^n \), it is conjugated with wave motion satisfying the equation:

\[
\frac{\partial^2 V}{\partial t^2} - x^T W \dot{x} = 0, \quad W = \left[ \frac{\partial^2 V(x, t)}{\partial x_i \partial x_j} \right]
\]  

(2)

where \( V(x, t) \) - single-valued, finite, continuous, twice continuously differentiable wave function (V-function) \( (x \in \mathbb{R}^n, t \in T) \).

The initial and boundary conditions for equation (2), based on the V-function method, take the form:

\[
V(x, t) \big|_{t=0} = V(x, 0) = 0
\]

(3)

\[
V(x, t) \big|_{t=0} = V(0, t) = 0
\]

(4)

\[
\left. \frac{\partial V(x, t)}{\partial t} \right|_{t=0} = \left. \frac{\partial V(x, 0)}{\partial t} \right| = \text{const}
\]

(5)

\[
\left. \frac{\partial V(x, t)}{\partial x} \right|_{x=0} = \left. \frac{\partial V(0, t)}{\partial x} \right| = k^{-1} \dot{x}(t) = k^{-1} f(x = 0).
\]

(6)

We will consider a one-dimensional case of trajectory-wave motion of an object (particle). The trajectory motion of a particle based on the V-function method is described by the equation:

\[
\dot{x} = k \frac{\partial V}{\partial x}
\]

(7)

The trajectory motion of the particle corresponds to the wave motion that satisfies equation (2), which in this case takes the form:

\[
\frac{\partial^2 V}{\partial t^2} - \dot{x}^2 \frac{\partial^2 V(x, t)}{\partial x^2} = 0,
\]

(8)

Equation (8) taking into account (7) takes the form:

\[
\frac{\partial^2 V(x, t)}{\partial t^2} - \left( k \frac{\partial V}{\partial x} \right)^2 \frac{\partial^2 V(x, t)}{\partial x^2} = 0
\]

(9)

Boundary conditions (3) - (6) in this case take the form:

\[
V(x, t) \big|_{t=0} = V(x, 0) = 0
\]

(10)

\[
\left. \frac{\partial V(x, t)}{\partial t} \right|_{t=0} = \left. \frac{\partial V(x, 0)}{\partial t} \right| = \bar{C}_1
\]

(11)

\[
V(x, t) \big|_{x=0} = V(0, t) = 0
\]

(12)
\[ \frac{\partial V(x,t)}{\partial x} \bigg|_{x=0} = \frac{\partial V(0,t)}{\partial x} = \tilde{C}_2 \]  
\hspace{1cm} (13)

We will solve equation (9) using the method of separation of variables \( V(x,t) = \varphi(t)\psi(x) \)

\[ \frac{\dot{\varphi}(t)}{\varphi^3(t)} = \left(\frac{\psi'(x)}{m^2\psi(x)}\right)^2 = c = \pm \omega^2 \]

\[ \dot{\varphi}(t) \pm \omega^2 \varphi^3(t) = 0 \]

\[ \psi''(x)(\psi'(x))^2 \pm \omega^2 m^2 \psi(x) = 0 \]  
\hspace{1cm} (14)

where \( m = \frac{1}{k} \).

Let us consider the case when \( c = -\omega^2 < 0 \), then the stationary equation in (14) takes the form:

\[ \psi''(x)(\psi'(x))^2 + \omega^2 m^2 \psi(x) = 0 \]  
\hspace{1cm} (15)

To do this, we introduce a new variable \( \theta = \theta(\psi) \), so that \( \psi' = \theta'(\psi)\psi \), then \( \psi'' = \theta'(\psi)\theta(\psi) \)

Then equation (15) takes the form:

\[ \theta^3 \frac{d\theta}{d\psi} + \omega^2 m^2 \psi = 0 \]  
\hspace{1cm} (16)

In equation (16) the variables are separated, indeed:

\[ \int \theta^3 d\theta = \int -\omega^2 m^2 \psi d\psi \quad \Rightarrow \quad \theta^4 = -2\omega^2 m^2 \psi^2 + c_1 \quad \cdots \quad \Rightarrow \quad \theta = \sqrt[4]{-2\omega^2 m^2 \psi^2 + c_1} \]

Then we find:

\[ \frac{d\psi}{dx} = \sqrt[4]{-2\omega^2 m^2 \psi^2 + c_1} \quad \Rightarrow \quad \int \frac{d\psi}{\sqrt[4]{c_1 - 2\omega^2 m^2 \psi^2}} = x + c_2 \]  
\hspace{1cm} (17)

The integral in equality (17) is non-integrable if the constant \( C_1 \) is nonzero.

If \( c_1 > \sqrt{2\omega m} \), the value of the integral in equation (17) is sought in the set of real numbers \( \mathbb{R} \), otherwise the solution will be determined in the set of complex numbers \( \mathbb{C} \).

In the computer mathematics system Maple, a numerical solution of equality (17) is found for \( c_1 = 1, 2\omega m = 1, c_2 = 0 \) on the interval (0,1) in the form of a graph (figure 1):

It can be seen from solution (17) that the particle is localized in a small neighborhood of the stationary function of the real variable.

Let us now consider the case of uniform rectilinear motion of an object with constant velocity. Then equations (7) takes the form:

\[ \ddot{x} = \theta \]  
\hspace{1cm} (19)
If we substitute the right-hand side of equation (19) into equation (8), we get the classical wave equation:

$$\frac{\partial^2 V}{\partial t^2} - g^2 \frac{\partial^2 V(x,t)}{\partial x^2} = 0, \tag{20}$$

which is solved with the boundary conditions (10) - (13).

To solve equation (20) with the initial and boundary conditions (10) - (13), we also apply the method of separation of variables: $V(x,t) = \varphi(t)\psi(x)$.

Then the general solution of equation (20) takes the form:

$$V(x,t) = \left( C_1 e^{-i\alpha t} + C_2 e^{i\alpha t} \right) \left( C_3 e^{-\frac{i\alpha x}{\beta}} + C_4 e^{\frac{i\alpha x}{\beta}} \right), \tag{21}$$

Using the initial and boundary conditions (10) - (13), we obtain a solution in the form of a plane wave:

$$V(x,t) = \frac{gC_1}{\omega^2} e^{\pm i\frac{\omega}{\beta} x - \omega \tau} = Ae^{\pm i\frac{\omega}{\beta} x - \omega \tau} \tag{22}$$

The V-function method implies the equality:

$$\frac{\partial V}{\partial t} = const, \tag{23}$$

From equality (23), taking into account relation (22), it follows:

$$\frac{\partial V(x,t)}{\partial t} = \mp i\omega e^{\pm i\frac{\omega}{\beta} x - \omega \tau} = const. \tag{24}$$

It is possible to get rid of the complex coefficient in (24) if the phase takes on the values:

$$\left( \frac{\omega}{\beta} x - \omega \tau \right) = \frac{\pi}{2} + \pi n, \ (n=0,1,2,3,...) \tag{25}$$

Hence, taking into account (19), it follows that equality (24) takes only discrete values, i.e.
|A|\omega = |A|\omega_0(1/2 + n) = \text{const} \tag{26}

And from equality (7):

\begin{equation}
\frac{\partial \mathcal{V}(x, t)}{\partial x} = \pm i \frac{A \omega}{\mathcal{G}} e^{\pm i(\frac{\omega}{\mathcal{G}} x - \omega t)} = k^{-1} \mathcal{G}, \tag{27}
\end{equation}

we obtain, taking into account (22):

|A|\omega = k^{-1} \mathcal{G}^2. \tag{28}

Function (22) will satisfy equation (9) if equality (26) is satisfied. Let here $k^{-1} = m$ - particle mass. Then the amplitude $|A|$ takes the dimension of the action. If we accept $A = \frac{h}{2\pi} = h$, $h$ - Planck's constant, then from (26) follows the energy quantization rule, which is the same as that of Schrödinger for a linear oscillator.

And from equality (28) we obtain:

\[ \hbar \omega = m \mathcal{G}^2. \tag{29} \]

Using the results obtained, one can make such correspondences between the wave and the particle:

\[ \mathcal{G} = \mathcal{G}, \quad \omega = \frac{m \mathcal{G}^2}{\hbar} = \frac{2E}{\hbar}, \quad \lambda = \frac{\hbar}{m \mathcal{G}}, \quad A = \hbar \]

3. Conclusion

In relations (30), the main thing is the equality of the phase velocity of the wave and the velocity of the particle, while the existing optical-mechanical analogies, the velocity of the particle is equal to the group velocity of the waves. It follows from the second relation in (30) that the energy is transferred by the particle. According to the third relation in (30), the momentum of a particle determines the wavelength $\lambda$, which coincides with the formula obtained by Louis de Broglie. The physical meaning of a wave is the properties of an action that manifests itself in the movement of a particle.

In addition, equation (9) has a solution for $k^{-1} = m \rightarrow 0$. The result is a wave function in the form of a monochromatic plane wave without a particle, which propagates at a given speed and frequency. This can explain the interference pattern when a particle (photon) passes through two slits.

References

[1] L. De Broglie 1923 Ondes et quanta Comptes Rendus 177 507-10
[2] L. De Broglie 1925 Recherches sur la théoré des quanta Annales der Physique 3 22-128
[3] De Broglie L 1967 Waves and quanta diffraction and interference Quantum kinetic theory of gases and Fermat's principle Advances in physical sciences 93(1)
[4] De Broglie L 1986 Heisenberg Uncertainty Relations and Probabilistic Interpretation of Wave Mechanics (Moscow: Mir)
[5] Polak L C 1960 Variational Principles of Mechanics and Their Development and Applications in Physics (Moscow: Fizmatgiz) p 549
[6] Schrödinger E 1926 Quantisierung als Eigenwertproblem I Annalen der Physik 79 361-76
[7] Landau L D and Lifshitz E M 1974 *Quantum Mechanics Nonrelativistic theory* (M., Science) p 752

[8] Valishin N T 2016 An Optical-Mechanical Analogy And The Problems Of The Trajectory-Wave Dynamics *Global Journal of Pure and Applied Mathematics* **12(4)** 2935-51

[9] Valishin N T and Valishin F T 2019 *V*-function method: some solutions of direct and inverse dynamics problems in a new statement *Latvian Journal of Physics and Technical Sciences* **1** 70-81

[10] Valishin N and Moiseev S 2017 A method of V-function: ultimate solution to the direct and inverse problems of dynamics for a hydrogen-like atom *Eastern-European Journal of Enterprise Technologies* **4(5(88))** 23-32

[11] Valishin N T 2019 To Physical Statement of a Controllability Problem *Jour of Adv Research in Dynamical & Control Systems* **11(5)** 1708-13

[12] Valishin FT 2018 *The Problem of the Beginning and the Strategy of Dynamism* (Moscow: "Encyclopedist-Maximum") p 180

[13] Valishin FT 1992 The problem of methodology in the concept of dynamism *Methodological concepts and schools in the USSR* 151-4

[14] Heisenberg W 1962 Über Quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen *Naturwissenschaft* **2** (Stuttgart: Battenderg) 31-45

[15] Born M, Heisenberg W and Jordan P 1926 Zur Quantenmechanik *Zeitschrift für Physik* **35** 557-615