Bogolyubov Invariant via Relative Spectral Invariants on Manifolds

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We introduce and study new spectral invariant of two elliptic partial differential operators of Laplace and Dirac type on compact smooth manifolds without boundary that depends on both the eigenvalues and the eigensections of the operators, which is equal to the regularized number of created particles from the vacuum when the dynamical operator depends on time. We study the asymptotic expansion of this invariant for small adiabatic parameter and compute explicitly the first two coefficients of the asymptotic expansion.
1 Introduction

We introduce and study a new relative spectral invariant of two elliptic operators. We hope that these new invariant could shed new light on the old questions of spectral geometry: “Does the spectral information of two elliptic operators determine the geometry of a manifold?” This invariant appears naturally in the study of particle creation in quantum field theory and quantum gravity [11, 12]. It is equal to the number of created particles from the vacuum when the dynamical operator depends on time. That is why, we will just call it the Bogolyubov invariant. This is a continuation of our work on the systematic study of generalized spectral invariants initiated in [7, 8] where we studied heat determinant and quantum heat traces and, in particular, [9], where we studied relative spectral invariants.

In Sec. 2 we consider an idealized case of an instantenous change of the operators and express the Bogolyubov invariant formally in terms of the heat traces and relative spectral invariants. In Sec. 3 we introduce a rigorous definition of the Bogolyubov invariant as a trace of some non-trivial functions of elliptic operators and describe some of its properties. In Sec 4 we derive some reduction formulas that express the Bogolyubov invariant in terms of the combined heat traces. In Sec. 5 we establish the asymptotics of the combined heat traces. In Sec. 6 we use Mellin transform to compute the asymptotics of the Bogolyubov invariant and explicitly compute the first two coefficients. In Sec. 7 we consider some examples.

1.1 Particle Creation in Quantum Field Theory

To motivate the introduction of this invariant we describe now the standard method for calculation of particles creation via the Bogolyubov transformation. Let $(\mathcal{M}, h)$ be a pseudo-Riemannian $(n+1)$-dimensional spin manifold with a Lorentzian metric $h$. We assume that $(\mathcal{M}, h)$ is globally hyperbolic so that there is a foliation of $\mathcal{M}$ with space slices $M_t$ at a time $t$, moreover, we assume that there is a global time coordinate $t$ varying from $-\infty$ to $+\infty$ and that at all times $M_t$ is a compact $n$-dimensional Riemannian manifold without boundary. We will also assume that there are well defined limits $M_\pm$ as $t \to \pm \infty$. For simplicity, we will just assume that the manifold $\mathcal{M}$ has two cylindrical ends, $(-\infty, \beta) \times M$ and $(\beta, \infty) \times M$ for some positive parameter $\beta$. So, the foliation slices $M_t$ depend on $t$ only on a compact interval $[-\beta, \beta]$. Let $g_t$ be the induced Riemannian metric on $M_t$ and $d\text{vol}_{g_t} = (\det g_{ij})^{1/2} dx$ be the corresponding Riemannian volume element. We label the space-time indices that run over $(0, 1, \ldots, n)$ by Greek letters and the space
indices that run over \((1, \ldots, n)\) by Latin letters. Let \(\mathbf{W}\) be a real vector bundle over \(\mathcal{M}\) and \(\mathbf{V}_t\) be the corresponding time slices (vector bundles over \(\mathcal{M}_t\)) (any complex vector bundle can be made real by just doubling the dimension). Henceforth we will omit the index \(t\) on \(M, g\) and \(\mathbf{V}\) when it does not cause a confusion.

We define the natural \(L^2\) inner product

\[
(\varphi_1, \varphi_2)_M = \int_M d\text{vol}_g \langle \varphi_1, \varphi_2 \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the fiber product in \(\mathbf{V}\), and the space \(L^2(\mathbf{V})\) of square integrable sections of the bundle \(\mathbf{V}\).

In quantum field theory there are two types of particles, bosons and fermions. The bosonic fields are described by second order Laplace type partial differential operators whereas the fermionic fields are described by first order Dirac type partial differential operators.

### 1.2 Bosonic Fields

Bosonic fields are described by sections of tensor bundles (or more generally by twisted tensor bundles); so, let \(\mathbf{W}\) be a tensor bundle. Let \(H_t\) be a one-parameter family of self-adjoint elliptic second-order partial differential operators acting on smooth sections of the bundle \(\mathbf{V}_t\). We assume that there are well defined limits \(H_{\pm}\) as \(t \to \pm \infty\), that is, \(H_{\pm}\) are elliptic self-adjoint second-order partial differential operators acting on sections of \(\mathbf{V}\) over \(M\).

Let \(m\) be a sufficiently large positive parameter so that the operator \(H + m^2\) is positive and the following pseudo-differential operators

\[
\omega_t = (H_t + m^2)^{1/2}
\]

are well defined. Henceforth, we will omit the index \(t\) on all operators.

We define the hyperbolic operator for bosonic fields by

\[
L = \partial_t^2 + H.
\]

Then the dynamics of the bosonic fields is described by the space \(\mathcal{D}\) of solutions of the hyperbolic equation

\[
(L + m^2)\varphi = 0
\]

with the inner product

\[
(\varphi_1, \varphi_2)_\mathcal{D} = (\varphi_1, i\partial_t \varphi_2)_M + (i\partial_t \varphi_1, \varphi_2)_M.
\]
It is easy to show that this inner product is well defined since it does not depend on the time \( t \).

### 1.3 Fermionic Fields

The fermionic fields are described by sections of spin-tensor bundles (or, more generally, by twisted spin-tensor bundles); so, let \( \mathcal{W} \) be a spin-tensor bundle. Let \( \xi \) be a self-adjoint involutive endomorphism of the bundle \( \mathcal{V} \),

\[
\xi^2 = I, \quad \xi^* = \xi,
\]

with \( I \) denoting the identity endomorphism, defining a natural decomposition of the vector bundle \( \mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_- \), so that \( \dim \mathcal{V}_+ = \dim \mathcal{V}_- \) and in the canonical basis

\[
\xi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]

Let \( \eta \) be another self-adjoint involutive endomorphism of the bundle \( \mathcal{V} \) that anti-commutes with \( \xi \),

\[
\eta^2 = I, \quad \eta^* = \eta, \quad \xi\eta = -\eta\xi,
\]

and has the form (in the same basis)

\[
\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\]

This enables one to define an anti-self-adjoint anti-involution

\[
C = \xi\eta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]

that anti-commutes with both \( \xi \) and \( \eta \). (Of course, \( \xi, \eta \) and \( iC \) are nothing but Pauli matrices.)

Let \( A_t \) be a one-parameter family of self-adjoint first-order elliptic partial differential operators acting on sections of the bundles \( \mathcal{V}_\pm(\pm) \) such that its square

\[
H_t = A_t^2
\]

is a self-adjoint second-order elliptic partial differential operator. We assume that there are well defined limits \( A_\pm \) as \( t \to \pm\infty \), that is, \( A_\pm \) are elliptic self-adjoint
Let $K$ be a self-adjoint elliptic first-order partial differential operator acting on smooth sections of the bundle $\mathcal{V}$ defined by

$$K = \xi \otimes A = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}.$$  \hfill (1.13)

Note that the operator $K$ anti-commutes with $\eta, C$ and commutes with $\xi$

$$K\eta = -\eta K, \quad KC = -CK, \quad K\xi = \xi K.$$  \hfill (1.14)

Therefore, the operator

$$K + m\eta = \begin{pmatrix} A & m \\ m & -A \end{pmatrix},$$  \hfill (1.15)

with some mass parameter $m$, is also a self-adjoint elliptic first-order operator.

The square of the operator $K$

$$K^2 = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$$  \hfill (1.16)

is a self-adjoint second-order operator, and, therefore, the operator

$$(K + m\eta)^2 = \begin{pmatrix} H + m^2 & 0 \\ 0 & H + m^2 \end{pmatrix}$$  \hfill (1.17)

is also self-adjoint and positive for any non-zero mass parameter, $m > 0$. We will find it useful to define also the pseudo-differential operator

$$\omega = |K + m\eta| = (H + m^2)^{1/2}.$$  \hfill (1.18)

It is easy to see that the operator $(K + m\eta)$ anti-commutes with the endomorphism $C$,

$$(K + m\eta)C = -C(K + m\eta),$$  \hfill (1.19)

and, therefore, its spectrum (which is obviously real and non-zero) is symmetric with respect to zero, since for every eigensection $\varphi$ with an eigenvalue $\lambda$ there is an eigensection $C\varphi$ with the opposite eigenvalue ($-\lambda$). Note, though, that it does not mean that the spectrum of the operator $A$ is symmetric with respect to the
origin. That is why, it is useful to define the spectral projections on the positive part of the spectrum by

\[ P = \frac{1}{2} (I + F), \]  

(1.20)

where

\[ F = (K + m\eta)\omega^{-1} = \begin{pmatrix} A\omega^{-1} & m\omega^{-1} \\ m\omega^{-1} & -A\omega^{-1} \end{pmatrix}, \]  

(1.21)

is an involution,

\[ F^2 = I. \]  

(1.22)

Obviously, the operator

\[ I - P = \frac{1}{2} (I - F) \]  

(1.23)

is the projection on the negative part of the spectrum.

We define the dynamical hyperbolic operator for the fermionic fields by

\[ D = i\partial_t + K; \]  

(1.24)

then

\[ -(\eta D)^2 = \begin{pmatrix} \partial_t^2 + H & 0 \\ 0 & \partial_t^2 + H \end{pmatrix}. \]  

(1.25)

The dynamical space \( \mathcal{D} \) is now the space of solutions of the hyperbolic equation

\[ (D + m\eta)\varphi = 0 \]  

(1.26)

with the inner product

\[ (\varphi_1, \varphi_2)_\mathcal{D} = (\varphi_1, \varphi_2)_\mathcal{M}. \]  

(1.27)

This inner product is well defined since it does not depend on the time. Notice also that

\[ (\eta D + m)(-\eta D + m) = \begin{pmatrix} \partial_t^2 + H + m^2 & 0 \\ 0 & \partial_t^2 + H + m^2 \end{pmatrix}. \]  

(1.28)

### 1.4 Particle Creation

Now, we introduce two different bases in the dynamical space \( \mathcal{D} \) (the space of solutions of the dynamical equations (1.4) and (1.26)). First, we introduce the so-called positive-frequency in-modes, \( \{\varphi_k\}_{k=1}^{\infty} \), by requiring them to satisfy the asymptotic conditions

\[ \lim_{t \to -\infty} (\partial_t + i\omega_k)\varphi_k = 0 \]  

(1.29)
for some positive constants $\omega_k^-$, and normalize them by
\[(\varphi_i, \varphi_j)_D = \delta_{ij} . \tag{1.30}\]

We also introduce another basis, so called negative-frequency out-modes, $\{\psi_k\}_{k=1}^\infty$, by requiring them to satisfy the asymptotic conditions
\[\lim_{t \to +\infty} (\partial_t - i\omega_k^+)\psi_k = 0 \tag{1.31}\]
for some positive constants $\omega_k^+$, and normalize them by
\[(\psi_i, \psi_j)_D = -\delta_{ij} , \tag{1.32}\]
for bosons and
\[(\psi_i, \psi_j)_D = \delta_{ij} \tag{1.33}\]
for fermions.

Then the total number of created in-particles in the out-vacuum is determined by the so-called Bogolyubov coefficients and is equal (in both bosonic and fermionic cases) to \cite{12, 11, 15, 20}
\[N = \sum_{i,j=1}^\infty |(\varphi_i, \psi_j)_D|^2 . \tag{1.34}\]

Although, it is impossible to calculate this invariant in the general case, it is easy to see that it is always non-negative and vanishes if the operators $H$ and $K$ do not depend on time at all.

## 2 Instantaneous Jumps

We will consider two limiting cases: i) $\beta \to \infty$, which corresponds to a slow (adiabatic) variation of the operators $H_t$ and $K_t$, and ii) $\beta \to 0$, which corresponds to an instantaneous change of the operators. In the limit $\beta \to 0$ one can compute the total number of created particles, at least formally.

### 2.1 Bosonic Fields

We consider the bosonic case first. Let $\{\omega_k^\pm\}_{k=1}^\infty$ be the eigenvalues (counted with multiplicities) and $\{u_k^\pm\}_{k=1}^\infty$ be the corresponding orthonormal sequence of the eigensections of the operators $\omega_k^\pm$; obviously, all eigenvalues are positive, $\omega_k^\pm > 0$. Also,
let $P_k^\pm$ be the orthogonal projections to the eigenvectors $u_k^\pm$. Here every eigenvalue and eigensection is taken with its multiplicity. We assume that the dynamical modes are continuously differentiable in time, that is, they are continuous in time and have continuous first partial time derivative. Then the positive-frequency in-modes and the negative frequency out-modes are

$$\varphi_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}}e^{-i\omega_k t}u_k^- & \text{for } t < 0, \\ \sum_{j=1}^\infty \frac{1}{2\omega_j} \sqrt{2\omega_k} \left( (\omega_j^+ + \omega_k^-)e^{-i\omega_j^+ t} + (\omega_j^- - \omega_k^+)e^{i\omega_j^- t} \right) u_j^+ & \text{for } t > 0, \end{cases} \quad (2.1)$$

$$\psi_k = \begin{cases} \sum_{j=1}^\infty \frac{1}{2\omega_j} \sqrt{2\omega_k} \left( (\omega_j^+ + \omega_k^-)e^{i\omega_j^+ t} + (\omega_j^- - \omega_k^+)e^{-i\omega_j^- t} \right) u_j^- & \text{for } t < 0, \\ \frac{1}{\sqrt{2\omega_k}}e^{i\omega_k t}u_k^+ & \text{for } t > 0. \end{cases} \quad (2.2)$$

By using these modes it is easy to show that the total number of created particles in the bosonic case is given by a formal series

$$N_b = \frac{1}{4} \sum_{j,k=1}^\infty \left( \sqrt{\frac{\omega_j^-}{\omega_k^+}} - \sqrt{\frac{\omega_k^-}{\omega_j^+}} \right)^2 \text{Tr} P_j^- P_k^+; \quad (2.3)$$

here we used the obvious relation

$$\text{Tr} P_j^- P_k^+ = |(u_j^-, u_k^+)|^2. \quad (2.4)$$

This can be written in the form of a formal trace

$$N_b = \frac{1}{4} \text{Tr} \left( \omega_- - \omega_+ \right) \left( \frac{1}{\omega_+} - \frac{1}{\omega_-} \right). \quad (2.5)$$

Note that this invariant is non-negative, as it should be, $N_b \geq 0$.

However, since the operators $\omega_\pm$ are unbounded this combination is not a trace class operator; it diverges at the high-end of the spectrum. Another way to see this is as follows. The operators $\omega$ and its inverse $\omega^{-1}$ can be represented in terms of
the heat kernel of its square $\omega^2 = H + m^2$ by

$$
\omega^{-1} = \frac{1}{\sqrt{\pi}} \int_0^\infty dt \ t^{-1/2} e^{-mt^2} \exp(-tH),
$$

(2.6)

$$
\omega = -\frac{1}{\sqrt{\pi}} \int_0^\infty dt \ t^{-1/2} \partial_t \left[ e^{-mt^2} \exp(-tH) \right].
$$

(2.7)

By using these equations we obtain a formal expression of the number of particles in terms of the heat kernels

$$
N_b = \frac{1}{4\pi} \int_0^\infty dt \int_0^\infty ds \ s^{-1/2} t^{-1/2} \partial_t \left\{ e^{-(t+s)m^2} \Psi(t,s) \right\},
$$

(2.8)

where

$$
\Psi(t,s) = \Tr \left\{ \exp(-tH_+) - \exp(-tH_-) \right\} \left\{ \exp(-sH_+) - \exp(-sH_-) \right\};
$$

(2.9)

we will call this function a relative heat trace. The function $\Psi(t,s)$ is symmetric and can be expressed further in terms of the traces

$$
\Psi(t,s) = \Theta_+(t+s) + \Theta_-(t+s) - X(t,s) - X(s,t),
$$

(2.10)

where

$$
\Theta_\pm(t) = \Tr \exp(-tH_\pm),
$$

(2.11)

is the classical heat trace of a Laplace type operator and

$$
X(t,s) = \Tr \exp(-tH_+) \exp(-sH_-)
$$

(2.12)

is a combined heat trace. For more details, see [9].

It is easy to show that the integral (2.8) converges for large $t$ and $s$ for large positive real $m$; however, it diverges as $t, s \to 0$. Of course, this happens because of the idealized situation with the instantaneous jump of the operator $H$. For a smooth deformation of the operators the number of particles is finite. The Bogolyubov invariant $B_b(\beta)$ we define below is a regularized version of the number of particles that is well defined for any $\beta$ and has the asymptotics (2.5) and (2.8) as $\beta \to 0$. 
2.2 Fermionic Fields

As we mentioned above the spectrum of the operators \( A^\pm \) is not necessarily symmetric with respect to zero. Let the eigenvalues (counted with multiplicities) of the operator \( A^\pm \) be \( \{\mu_k^\pm\}_{k=1}^{\infty} \) (both positive and negative; they can be ordered in their absolute value). Then the eigenvalues of the operator \((K + m\eta)\) are \( \{\omega_k^\pm\}_{k=1}^{\infty} \) with \( \omega_k^\pm = \left((\mu_k^\pm)^2 + m^2\right)^{1/2} \). Let and \( \{u_k^\pm, Cu_k^\pm\}_{k=1}^{\infty} \) be the corresponding orthonormal eigensections of the operators \((K_k^\pm + m\eta)\). Also, let \( P_k^\pm \) and \((-CP_k^\pm C)\) be the orthogonal projections to the eigensections \( u_k^\pm \) and \( Cu_k^\pm \).

We require our modes to be continuous in time. Then the positive-frequency in-modes and negative frequency out-modes are

\[
\psi_k = \begin{cases} 
Cu_k^- e^{-i\omega_k^- t}, & \text{for } t < 0, \\
\sum_{j=1}^{\infty} \left( (u_j^-, Cu_k^-)u_j^+ e^{-i\omega_j^- t} + (Cu_j^-, Cu_k^-)Cu_j^+ e^{i\omega_j^- t} \right), & \text{for } t > 0,
\end{cases}
\]  

\[ \tag{2.13} \]

\[
\varphi_k = \begin{cases} 
\sum_{j=1}^{\infty} \left( (u_j^+, Cu_k^-)u_j^- e^{-i\omega_j^+ t} + (Cu_j^+, Cu_k^-)Cu_j^- e^{i\omega_j^+ t} \right), & \text{for } t < 0, \\
u_k^+ e^{i\omega_k^+ t}, & \text{for } t > 0.
\end{cases}
\]  

\[ \tag{2.14} \]

By using these modes it is easy to show that the total number of created fermionic particles is given by the formal series

\[
N_f = \sum_{j,k=1}^{\infty} \text{Tr} \left( -CP_j^- CP_k^+ \right),
\]  

\[ \tag{2.15} \]

where we used the equation \( \text{Tr} \left( -CP_j^- CP_k^+ \right) = |(Cu_j^-, u_k^+)|^2 \). This can be written in the form of a formal trace

\[
N_f = \text{Tr} (I - P^-) P^+ = \frac{1}{8} \text{Tr} (F_+ - F_-)^2.
\]  

\[ \tag{2.16} \]

By using the same heat kernel trick we represent the operators \( F_\pm \) as

\[
F_\pm = \pi^{-1/2} \int_0^\infty dt \ t^{-1/2} e^{-m^2} (K_\pm + m\eta) \exp(-tH_\pm)
\]  

\[ \tag{2.17} \]
to rewrite this formally in the form
\[ N_f = \frac{1}{8\pi} \int_0^\infty dt \int_0^\infty ds \, s^{-1/2} t^{-1/2} e^{-(t+s)m^2} \]
\[ \times \text{Tr} \left\{ (K_+ + m\eta) \exp(-tH_+) - (K_- + m\eta) \exp(-tH_-) \right\} \]
\[ \times \left\{ (K_+ + m\eta) \exp(sH_+ + m\eta) - (K_- + m\eta) \exp(sH_-) \right\}. \]  
(2.18)

Finally, by using the form (1.15) of the operator \( K \), the finite-dimensional trace can be computed to get
\[ N_f = \frac{1}{4\pi} \int_0^\infty dt \int_0^\infty ds \, s^{-1/2} t^{-1/2} e^{-(t+s)m^2} \left\{ \Phi(t, s) + m^2 \Psi(t, s) \right\}, \]  
(2.19)

where
\[ \Phi(t, s) = \text{Tr} [A_+ \exp(-tH_+) - A_- \exp(-tH_-)] [A_+ \exp(sH_+) - A_- \exp(sH_-)]. \]  
(2.20)

is the relative heat trace for Dirac operators. This function can be written in the form
\[ \Phi(t, s) = -\partial_t \Theta_+(t + s) - \partial_s \Theta_-(t + s) - Y(t, s) - Y(s, t), \]  
(2.21)

where \( \Theta_\pm(t) \) are the classical heat traces given by (2.11) and
\[ Y(t, s) = \text{Tr} A_- A_+ \exp(-tH_+) \exp(-sH_-). \]  
(2.22)

is the combined heat trace for the Dirac operators.

Again, it is easy to show that the integrals (2.18), (2.19) converge for large \( t \) and \( s \) for large positive real \( m \) but diverge as \( t, s \to 0 \). This happens because of the idealized situation with the instantaneous jump of the operators \( K \). For a smooth deformation of the operators the number of fermionic particles is also finite. We will define below an invariant, called Bogolyubov invariant, \( B_f(\beta) \), that is a regularized version of the number of particles that is well defined for any \( \beta \) and has the asymptotics (2.16) and (2.19) as \( \beta \to 0 \).

3 Bogolyubov Invariant

3.1 Laplace Type Operators

Let \( H_\pm \) be two elliptic second-order self-adjoint partial differential operators with positive definite leading symbol of Laplace type and \( \omega_\pm = (H_\pm + m^2)^{1/2} \). Our goal
is to generalize the total number of created particles given by the trace (2.5) so that it converges at high-end of the spectrum. We replace \( \omega \) by \( f(\beta \omega) \) where \( f \) is a smooth function that is linear as \( x \to 0 \) and is constant up to exponentially small terms as \( x \to \infty \) and \( \beta \) is a positive real parameter; so we define

\[
B_b(\beta) = \frac{1}{4} \text{Tr} \left\{ f(\beta \omega_+) - f(\beta \omega_-) \right\} \left\{ \frac{1}{f(\beta \omega_+)} - \frac{1}{f(\beta \omega_-)} \right\}.
\]

(3.1)

In particular, we will use the function

\[
f(x) = \tanh \left( \frac{x}{2} \right).
\]

(3.2)

By using the equations

\[
f(x) = 1 - 2E_f(x),
\]

(3.3)

\[
\frac{1}{f(x)} = 1 + 2E_b(x),
\]

(3.4)

where

\[
E_f(x) = \frac{1}{e^x + 1},
\]

(3.5)

\[
E_b(x) = \frac{1}{e^x - 1},
\]

(3.6)

we can rewrite the invariant (3.1) in the form

\[
B_b(\beta) = \text{Tr} \left\{ E_f(\beta \omega_+) - E_f(\beta \omega_-) \right\} \left\{ E_b(\beta \omega_+) - E_b(\beta \omega_-) \right\},
\]

(3.7)

and, further, by using

\[
E_f(x)E_b(x) = E_b(2x),
\]

(3.8)

in the form

\[
B_b(\beta) = \text{Tr} \left\{ E_b(2\beta \omega_+) + E_b(2\beta \omega_-) - E_b(\beta \omega_+)E_f(\beta \omega_-) - E_b(\beta \omega_-)E_f(\beta \omega_+) \right\}.
\]

(3.9)

Notice that this invariant is closely related to the quantum heat traces

\[
\Theta_{f,b}(\beta, \mu) = \text{Tr} E_{f,b} [\beta(\omega - \mu)]
\]

(3.10)

studied in [8].
Let \((\lambda_k^{\pm})_{k=1}^\infty\) and \((P_k^{\pm})_{k=1}^\infty\) be the eigenvalues and the corresponding projections to the eigenspaces of the operators \(H_{\pm}\); then \(\omega_k^{\pm} = (\lambda_k^\pm + m^2)^{1/2}\) are the eigenvalues of the operators \(\omega_{\pm}\). We order the eigenvalues so that they form an increasing sequence of real numbers. In a special case when the operators \(H_-\) and \(H_+\) have the same projections, \(P_k^- = P_k^+\), but different eigenvalues this invariant takes the form

\[
B_b(\beta) = \sum_{k=1}^\infty \frac{\sinh^2[\beta(\omega_k^+ - \omega_k^-)/2]}{\sinh(\beta \omega_k^+) \sinh(\beta \omega_k^-)}.
\] (3.11)

Invariants like this come up in the study of creation of bosonic particles in quantum field theory and quantum gravity in external classical fields [11]. It is equal to the number of created particles from the vacuum when the dynamical operator is changed in time from \(H_-\) at \(t \to -\infty\) to \(H_+\) at \(t \to +\infty\). In fact, this is equal exactly to the number of created particles in a cosmological model considered in [11]. That is why, we will just call it the Bogolyubov invariant.

It is easy to compute the behavior of this invariant for the asymptotic cases \(\beta \to \infty\) and \(\beta \to 0\). In the adiabatic limit \(\beta \to \infty\) the invariant \(B_b\) takes the form

\[
B_b(\beta) \sim \frac{1}{4} \text{Tr} \left( \frac{1}{\omega_- - \omega_+} \left( \frac{1}{\omega_+} - \frac{1}{\omega_-} \right) \right).
\] (3.14)

This expression is closely related to the relativistic heat trace

\[
\Theta_r(\beta) = \text{Tr} \exp(-\beta \omega)
\] (3.13)

studied in [8]. The limit \(\beta \to 0\) is singular; formally, we obtain

\[
B_b(\beta) \sim \frac{1}{4} \text{Tr} \left( \omega_- - \omega_+ \right) \left( \frac{1}{\omega_+} - \frac{1}{\omega_-} \right)
\] (3.14)

which is similar to (2.5). That is why, the invariant \(B_b(\beta)\) is the generalization of the number of created bosonic particles \(N_b\). However, this trace diverges.

Let \(g_{\pm}^{ij}\) be the metrics determined by the leading symbols of the Laplace type operators \(H^\pm\), i.e.

\[
\sigma_L(H_\pm; x, \xi) = I g_{\pm}^{ij}(\xi) \xi_i \xi_j = |\xi_{\pm}|^2.
\] (3.15)

By using the results of our paper [8] (or by using the calculus of pseudodifferential operators) one can show that as \(\beta \to 0\)

\[
B_b(\beta) = \beta^{-n} V_b + O(\beta^{-n+2}),
\] (3.16)
where $V_b$ is an invariant (depending only on the metrics $g_+$ and $g_-$) defined by

$$V_b = N \int_M dx \int_{\mathbb{R}^n} d\xi \sinh^2 \left( \frac{(|\xi_+| - |\xi_-|)}{2} \right) \sinh(|\xi_+|) \sinh(|\xi_-|)$$

(3.17)

with $N = \text{tr} I = \dim \mathcal{V}$. It is easy to see that this integral converges. We will compute it in an alternative form below.

### 3.2 Dirac Type Operators

Let $K_\pm$ be elliptic first-order self-adjoint partial differential operators of Dirac type described above, $|\omega_\pm| = |K_\pm + m\eta| = (H_\pm + m^2)^{1/2}$, and $F_\pm = (K_\pm + m\eta)\omega_\pm^{-1}$.

Our goal is to generalize the trace (2.16) so that it converges at high-end of the spectrum. We replace the operators $F_\pm$ by the operators $F_\pm g(\beta \omega_\pm)$ where $g$ is a smooth function such that $g(x) \sim 1$ as $x \to 0$ and is exponentially small as $x \to \infty$.

That is, we define the fermionic Bogolyubov invariant by the trace of a trace class operator

$$B_f(\beta) = \frac{1}{8} \text{Tr} \left\{ (K_+ + m\eta)E_0(\beta \omega_+) - (K_- + m\eta)E_0(\beta \omega_-) \right\}.$$  

(3.18)

We choose the function $g$ in the form

$$g(x) = \frac{x}{\sinh x}. $$

(3.19)

Let $E_0$ be a function defined by

$$E_0(x) = \frac{1}{2} (E_b(x) + E_f(x)) = \frac{1}{2 \sinh x}; $$

(3.20)

Then the Bogolyubov invariant has the form

$$B_f(\beta) = \frac{\beta^2}{2} \Tr \left\{ (K_+ + m\eta)E_0(\beta \omega_+) - (K_- + m\eta)E_0(\beta \omega_-) \right\}^2.$$  

(3.21)

By computing the final-dimensional trace it can be reduced to

$$B_f(\beta) = \beta^2 \Tr \left\{ [A_+ E_0(\beta \omega_+) - A_- E_0(\beta \omega_-)]^2 + m^2 [E_0(\beta \omega_+) - E_0(\beta \omega_-)]^2 \right\} $$

$$= \beta^2 \Tr \left\{ \omega_+^2 E_0^2(\beta \omega_+) + \omega_-^2 E_0^2(\beta \omega_-) - 2(A_+ A_- + m^2)E_0(\beta \omega_-)E_0(\beta \omega_+) \right\}.$$  

(3.22)
In the adiabatic limit, as $\beta \to \infty$, the invariant $B_f$ takes the form

$$B_f(\beta) \sim \beta^2 \text{Tr} \left\{ \omega_+^2 e^{-2\beta\omega_+} + \omega_-^2 e^{-2\beta\omega_-} - 2(A_+A_- + m^2)e^{-\beta\omega_+} e^{-\beta\omega_-} \right\}. \quad (3.23)$$

As in the bosonic case the limit $\beta \to 0$ is singular; formally we get a divergent trace

$$B_f(\beta) \sim \frac{1}{8} \text{Tr} \left( F_+ - F_- \right)^2, \quad (3.24)$$

similar to (2.16). Therefore, the invariant $B_f(\beta)$ is the generalization of the number of created fermionic particles $N_f$. Obviously, this trace also diverges.

Let $\gamma^j_\pm$ be the Dirac matrices determined by the leading symbol of the Dirac type operators $A_\pm$, i.e.

$$\sigma(A_\pm; x, \xi) = -\gamma^j_\pm(x)\xi_j. \quad (3.25)$$

Then

$$\gamma^j_\pm(x) = e^j_{\pm,a}(x)\gamma^a, \quad (3.26)$$

where $e^j_{\pm,a}$ are the orthonormal bases for the metrics $g^{ij}_\pm$,

$$g^{ij}_\pm = \delta^{ab} e^j_{\pm,a} e^j_{\pm,b}. \quad (3.27)$$

By using the results of our paper [8] (or by using the calculus of pseudodifferential operators) one can show that the leading asymptotics as $\beta \to 0$ is

$$B_f(\beta) = \beta^{-n} V_f + O(\beta^{-n+2}), \quad (3.28)$$

where $V_f$ is an invariant defined by

$$V_f = \frac{N}{4} \int_M dx \int_{\mathbb{R}^n} d\xi \left\{ \frac{|\xi_+|^2}{\sinh^2(|\xi_+|)} + \frac{|\xi_-|^2}{\sinh^2(|\xi_-|)} - 2 \frac{|\xi_+\xi_-|}{\sinh(|\xi_+|) \sinh(|\xi_-|)} \right\}. \quad (3.29)$$

with $N = \text{tr} I = \dim \mathcal{V}(\pm)$ and

$$|\xi_+\xi_-| = \frac{1}{2} \left( \gamma^j_+ \gamma^j_- + \gamma^j_- \gamma^j_+ \right) \xi_i \xi_j = \delta^{ab} e^j_{+,a} e^j_{-,b} \xi_i \xi_j. \quad (3.30)$$

This integral obviously converges. We will compute it in an alternative form later.
4 Heat Traces

4.1 Reduction Formulas

Let $H$ be a positive self-adjoint elliptic operator, $\omega = \sqrt{H}$, and $f(z)$ be a function which is analytic in the right half-plane decreasing at infinity in the sector $|\arg(z)| < \frac{\pi}{4}$. We are trying to find a reduction formula

$$\Tr f(\omega) = \int_0^\infty dt \ h(t) \Tr \exp(-tH), \quad (4.1)$$

with some function $h$, that reduces the calculation of the trace of $f(\omega)$ to the calculation of the trace of $\exp(-tH)$, that is, to the classical heat trace.

It is not difficult to see that the function $h(t)$ is given by the inverse Laplace transform of the function $f(\sqrt{z})$

$$h(t) = \frac{1}{\pi i} \int_C dz \ z e^{zt} f(z); \quad (4.2)$$

here $C$ is a $<$-shaped contour in the right half-plane going from $e^{-i\pi/4}\infty$ to the point $\varepsilon > 0$ on the real axis and then to $e^{i\pi/4}\infty$ so that all poles of the integrand lie to the left of the contour $C$. We consider the class of functions $f$ such that this integral converges. Then one can show that the function $f$ can be represented by the integral

$$f(x) = \int_0^\infty dt \ h(t) e^{-tx^2}. \quad (4.3)$$

By applying this method to the exponential function we get, in particular,

$$e^{-x} = (4\pi)^{-1/2} \int_0^\infty dt \ t^{-3/2} \exp\left(-\frac{1}{4t}\right) e^{-tx^2}, \quad (4.4)$$

which is valid for any $x \geq 0$.

The functions introduced above can be represented as series which converge
for $x > 0$,

\[
E_f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} e^{-kx}, \quad (4.5)
\]

\[
E_b(x) = \sum_{k=1}^{\infty} e^{-kx}, \quad (4.6)
\]

\[
E_0(x) = \sum_{k=0}^{\infty} e^{-(2k+1)x}. \quad (4.7)
\]

By using the integral (4.4) we find the integral representation of these functions

\[
E_{b,f,0}(x) = \int_{0}^{\infty} dt \, h_{b,f,0}(t) \exp(-tx^2), \quad (4.8)
\]

where

\[
h_f(t) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=1}^{\infty} (-1)^{k+1} k \exp\left(-\frac{k^2}{4t}\right), \quad (4.9)
\]

\[
h_b(t) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=1}^{\infty} k \exp\left(-\frac{k^2}{4t}\right), \quad (4.10)
\]

\[
h_0(t) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=0}^{\infty} (2k + 1) \exp\left(-\frac{(2k + 1)^2}{4t}\right). \quad (4.11)
\]

By computing the inverse Laplace transform these functions can also be written as

\[
h_f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} dp \, p \tan\left(\frac{p}{2}\right) \exp(-tp^2), \quad (4.12)
\]

\[
h_b(t) = \frac{1}{2\pi} \int_{\mathbb{R}} dp \, p \cot\left(\frac{p}{2}\right) \exp(-tp^2), \quad (4.13)
\]

\[
h_0(t) = \frac{1}{2\pi} \int_{\mathbb{R}} dp \, \frac{p}{\sin p} \exp(-tp^2), \quad (4.14)
\]

where the integrals are taken in the principal value sense; the imaginary part cancels out. Obviously, we have

\[
h_0(t) = \frac{1}{2} \left[ h_b(t) + h_f(t) \right]. \quad (4.15)
\]
All the functions \( h_{b,f,0}(t) \) are exponentially small as \( t \to 0 \)

\[
h_{b,f,0}(t) \sim (4\pi)^{-1/2} t^{-3/2} \exp\left(-\frac{1}{4t}\right), \quad (4.16)
\]

As \( t \to \infty \) these functions have the asymptotic expansions

\[
h_f(t) \sim (4\pi)^{-1/2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k} - 1)B_{2k}}{2^{2k}k!} t^{-k-1/2}, \quad (4.17)
\]

\[
h_b(t) \sim \frac{1}{\sqrt{\pi}} t^{-1/2} - (4\pi)^{-1/2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{2k}}{2^{2k-1}k!} t^{-k-1/2}, \quad (4.18)
\]

where \( B_k \) are Bernoulli numbers. The leading terms of the asymptotics are

\[
h_f(t) \sim \frac{1}{8 \sqrt{\pi}} t^{-3/2}, \quad (4.19)
\]

\[
h_b(t) \sim \frac{1}{\sqrt{\pi}} t^{-1/2}, \quad (4.20)
\]

\[
h_0(t) \sim \frac{1}{2 \sqrt{\pi}} t^{-1/2}. \quad (4.21)
\]

### 4.2 Heat Trace Representation

Now, by using these integral representations of the functions \( E_{b,f,0} \) we obtain the heat trace representation for the Bogolyubov invariant. For the bosonic case we get from (3.7)

\[
B_0(\beta) = \int_0^\infty dt \int_0^\infty ds \ h_f(t) h_b(s) \exp\left(-m^2\beta^2(s + t)\right) \Psi\left(\beta^2 t, \beta^2 s\right), \quad (4.22)
\]

where \( \Psi(s, t) \) is the function defined in (2.9). This is the regularized version of the eq. (2.8). For the fermionic case we get from (3.22)

\[
B_f(\beta) = \int_0^\infty dt \int_0^\infty ds \ h_0(t) h_0(s) \exp\left(-m^2\beta^2(s + t)\right) \times \beta^2 \left\{ \Phi\left(\beta^2 t, \beta^2 s\right) + m^2 \Psi\left(\beta^2 t, \beta^2 s\right) \right\}, \quad (4.23)
\]
where $\Phi(t, s)$ is the function defined in (2.20). This is the regularized version of the eq. (2.19). It is easy to see that the integrals for the Bogolyubov invariant converge both as $t, s \to 0$ and (for sufficiently large $m^2$) also as $t, s \to \infty$.

We can also define the corresponding Bogolyubov zeta function

$$Z_{b,f}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} B_{b,f}(\beta).$$  \hfill (4.24)

As will be shown below, as $\beta \to 0$ the Bogolyubov invariants behave as $\beta^{-n}$, with $n = \dim M$, and, therefore, the zeta functions are analytic for $\text{Re } s > n$. Thus, by using the inverse Mellin transform we obtain the Bogolyubov invariant in terms of the zeta functions

$$B_{b,f}(\beta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \beta^{-s} \Gamma(s)Z_{b,f}(s),$$  \hfill (4.25)

where $c > n$.

Thus, we reduced the calculation of the Bogolyubov invariant to the calculation of the classical heat trace and the combined heat traces studied in [9]

$$\Theta_{\pm}(t) = \text{Tr } \exp(-tH_{\pm}),$$  \hfill (4.26)

$$X(t, s) = \text{Tr } \exp(-tH_+) \exp(-sH_-),$$  \hfill (4.27)

$$Y(t, s) = \text{Tr } A_+ \exp(-tH_+) A_- \exp(-sH_-).$$  \hfill (4.28)

Here $\Theta(t)$ is the standard classical heat trace; it has been studied a lot in the literature (see, e.g. [14, 11, 3, 5]). The other traces are new.

For the fermionic case we will find it useful to introduce more general traces (recall that $H_\pm = A_\pm^2$)

$$\Xi_\pm(t, \alpha) = \text{Tr } \exp(-tH_\pm + i\alpha A_\pm),$$  \hfill (4.29)

$$W(t, s; \alpha, \beta) = \text{Tr } \exp(-tH_+ + i\alpha A_+) \exp(-sH_- + i\beta A_-).$$  \hfill (4.30)

Then, obviously,

$$\Theta_\pm(t) = \Xi_\pm(t, 0),$$  \hfill (4.31)

$$X(t, s) = W(t, s; 0, 0),$$  \hfill (4.32)

$$Y(t, s) = -\frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta} W(t, s; \alpha, \beta) \bigg|_{\alpha=\beta=0}. \hfill (4.33)$$

Therefore, all traces can be obtained from the traces (4.29) and (4.30).
Notice that the trace $\Xi(t, \alpha)$ satisfies the one-dimensional heat equation on $\mathbb{R}$,
\begin{equation}
\partial_t \Xi_{\pm}(t, \alpha) = \partial_{\alpha}^2 \Xi_{\pm}(t, \alpha), \tag{4.34}
\end{equation}
and, therefore, it can be written in the form
\begin{equation}
\Xi_{\pm}(t, \alpha) = (4\pi t)^{1/2} \int_{\mathbb{R}} d\alpha' \exp \left\{ -\frac{(\alpha - \alpha')^2}{4t} \right\} T_{\pm}(\alpha'), \tag{4.35}
\end{equation}
where
\begin{equation}
T_{\pm}(\alpha) = \text{Tr} \exp (i\alpha A_{\pm}), \tag{4.36}
\end{equation}
which should be understood in the distributional sense. Similarly, the invariant $W(t, s; \alpha, \beta)$ satisfies the heat equations
\begin{align}
\partial_t W(t, s; \alpha, \beta) &= \partial_{\alpha}^2 W(t, s; \alpha, \beta), \tag{4.37} \\
\partial_s W(t, s; \alpha, \beta) &= \partial_{\beta}^2 W(t, s; \alpha, \beta), \tag{4.38}
\end{align}
and, therefore, it can be written in the form
\begin{equation}
W(t, s; \alpha, \beta) = (4\pi)^{-1}(ts)^{-1/2} \int_{\mathbb{R}^2} d\alpha' d\beta' \exp \left\{ -\frac{(\alpha - \alpha')^2}{4t} - \frac{(\beta - \beta')^2}{4s} \right\} S_{\pm}(\alpha', \beta'), \tag{4.39}
\end{equation}
where
\begin{equation}
S_{\pm}(\alpha, \beta) = \text{Tr} \exp (i\alpha A_{+}) \exp (i\beta A_{-}). \tag{4.40}
\end{equation}

Notice also that for equal operators $H_{-} = H_{+}$ and $A_{-} = A_{+}$ the combined heat traces can be expressed in terms of the classical one
\begin{align}
X(t, s) &= \Theta(t + s), \tag{4.41} \\
Y(t, s) &= -\partial_t \Theta(t + s), \tag{4.42}
\end{align}
also,
\begin{equation}
W(t, s; \alpha, \beta) = \Xi(t + s, \alpha + \beta). \tag{4.43}
\end{equation}
More complicated similar relations can be obtained by considering distinct but commuting operators. For example, if the operators differ by just a constant, $M^2$,
\begin{equation}
H_{+} = H_{-} + M^2, \tag{4.44}
\end{equation}
then
\begin{equation}
X(t, s) = e^{-tM^2} \Theta_{-}(t + s). \tag{4.45}
\end{equation}
Also, if the operators $A_+$ and $A_-$ differ by a constant
\[ A_+ = A_- + M, \] (4.46)
then
\[ W(t, s; \alpha, \beta) = e^{-itM^2 + i\alpha M} \Xi_-(t + s, \alpha + \beta + 2itM). \] (4.47)

### 4.3 Spectral Representation of Heat Traces

Let $\{\lambda_k^\pm\}$ be the eigenvalues and $\{\varphi_k^\pm\}$ be the orthonormal sequence of eigensections of the operator $H_\pm$. Then the heat traces (4.26)-(4.28) take the following form
\[ \Theta_\pm(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k^\pm}, \] (4.48)
\[ X(t, s) = \sum_{k, j=1}^{\infty} e^{-t(\lambda_k^+ - s\lambda_j^-)} |(\varphi_j^-, \varphi_k^+)|^2. \] (4.49)

Also, let $\mu_k^\pm$ be the eigenvalues of the operator $A_\pm$ (recall that $H_\pm = A_\pm^2$). Then
\[ Y(t, s) = \sum_{k, j=1}^{\infty} e^{-t(\mu_k^+)^2 - s(\mu_j^-)^2} |\mu_k^+\mu_j^-| |(\varphi_j^-, \varphi_k^+)|^2, \] (4.50)
and the generalized traces (4.29), (4.30), have the form
\[ \Xi_\pm(t, \alpha) = \sum_{k=1}^{\infty} e^{-t(\mu_k^\pm)^2 + i\alpha \mu_k^\pm}, \] (4.51)
\[ W(t, s; \alpha, \beta) = \sum_{k, j=1}^{\infty} e^{-t(\mu_k^+)^2 + i\alpha \mu_k^+ - s(\mu_j^-)^2 + i\beta \mu_j^-} |(\varphi_j^-, \varphi_k^+)|^2. \] (4.52)

### 4.4 Integral Representation of Heat Traces

Laplace type operators $H_\pm$ naturally define Riemannian metrics $g_\pm$ and connections $\nabla^\pm$ on the vector bundle. We use these metrics and connections to define the geodesic distance, the parallel transport, the covariant derivatives etc. Since while working with two different operators we do not have a single metric, then, following [4, 9], we prefer to work with the vector bundle of densities of weight $1/2$ and with the Lebesgue measure $dx$ instead of the Riemannian volume element.
\[ d \text{vol}_g = dx \ g^{1/2}, \text{ with } g = \det(g_{ij}). \] Then the heat kernels \( U_\pm(t; x, x') \) of the heat semigroup \( \exp(-tH_\pm) \) are also densities of weight \( 1/2 \) at each point \( x \) and \( x' \), and the diagonals \( U_\pm(t; x, x) \) are densities of weight 1.

The heat kernel of the operator \( H_\pm \) has the following spectral representation

\[ U_\pm(t; x, x') = \sum_{k=1}^{\infty} \exp\left(-t\lambda_k^\pm \right) \varphi_k^\pm(x)\varphi_k^{\ast\pm}(x'). \] (4.53)

Then the heat traces (4.26)-(4.28) take the following form

\[ \Theta_\pm(t) = \int_M dx \ \text{tr} \ U_\pm(t; x, x), \] (4.54)

\[ X(t, s) = \int_{M \times M} dx \ dx' \ \text{tr} \ U_+(t; x, x')U_-(s; x', x). \] (4.55)

Here and everywhere below \( \text{tr} \) denotes the fiber trace.

The integral kernel of the heat semigroup \( \exp(-tH_\pm + i\alpha A_\pm) \) has the form

\[ V_\pm(t, \alpha; x, x') = \sum_{k=1}^{\infty} \exp \left[-t(\mu_k^\pm)^2 + i\alpha \mu_k \right] \varphi_k^\pm(x)\varphi_k^{\ast\pm}(x'). \] (4.56)

Therefore, (recall that \( \lambda_k^\pm = (\mu_k^\pm)^2 \))

\[ U_\pm(t; x, x') = V_\pm(t, 0; x, x') = \sum_{k=1}^{\infty} \exp \left[-t(\mu_k^\pm)^2 \right] \varphi_k^\pm(x)\varphi_k^{\ast\pm}(x'). \] (4.57)

Then

\[ A_\pm U_\pm(t; x, x') = \sum_{k=1}^{\infty} \exp \left(-t(\mu_k^\pm)^2 \right) \mu_k^\pm \varphi_k^\pm(x)\varphi_k^{\ast\pm}(x'). \] (4.58)

and

\[ Y(t, s) = \int_{M \times M} dx \ dx' \ \text{tr} \ A_+ U_+(t; x, x') A_- U_-(s; x', x), \] (4.59)

where the differential operators act on the first spatial argument of the heat kernel.

The generalized traces (4.29), (4.30), have the form

\[ \Xi_\pm(t, \alpha) = \int_M dx \ \text{tr} \ V_\pm(t, \alpha; x, x), \] (4.60)

\[ W(t, s; \alpha, \beta) = \int_{M \times M} dx \ dx' \ \text{tr} \ V_+(t, \alpha; x, x') V_-(s, \beta; x', x). \] (4.61)
5 Asymptotics of Heat Traces

5.1 Heat Kernel Asymptotics of Laplace Type Operators

First of all, it is easy to see that the asymptotics of the heat trace as $\beta \to \infty$ are determined by the bottom eigenvalues

$$\Theta_{\pm}(\beta^2 t) \sim \exp \left(-\beta^2 t \lambda_{\pm}^1 \right).$$

(5.1)

We will be primarily interested in the asymptotics as $\beta \to 0$.

For Laplace type operators $H_{\pm}$ there is an asymptotic expansion of the heat kernel $U_{\pm}(t; x, x')$ in the neighborhood of the diagonal as $t \to 0$ (see e.g. [1, 3, 5, 6])

$$U_{\pm}(t; x, x') \sim (4\pi)^{-n/2} \exp \left(-\frac{\sigma_{\pm}}{2t} \right) \sum_{k=0}^{\infty} t^{k-n/2} \tilde{a}_{\pm}^k,$$

(5.2)

where $\sigma_{\pm} = \sigma_{\pm}(x, x')$ is the Ruse-Synge function (also called the world function) of the metric $g_{\pm}$ and

$$\tilde{a}_{\pm}^k = \frac{(-1)^k}{k!} D_{\pm}^{1/2} \mathcal{P}_{\pm} a_{\pm}^k,$$

(5.3)

where $D_{\pm} = D_{\pm}(x, x')$ is the Van Vleck-Morette determinant, $\mathcal{P}_{\pm} = \mathcal{P}_{\pm}(x, x')$ is the operator of parallel transport of sections along the geodesic in the connection $\nabla_{\pm}$ and the metric $g_{\pm}$ from the point $x'$ to the point $x$ and $a_{\pm}^k = a_{\pm}^k(x, x')$ are the usual heat kernel coefficients, in particular, [3]

$$[a_0] = I.$$

(5.4)

Therefore, there is the asymptotic expansion of the classical heat trace (4.48) as $\beta \to 0$,

$$\Theta_{\pm}(\beta^2 t) \sim (4\pi)^{-n/2} \sum_{m=0}^{\infty} \beta^{2m-n} t^m A_{\pm}^m,$$

(5.5)

where

$$A_{\pm}^m = \frac{(-1)^m}{m!} \int_M d x \ g_{\pm}^{1/2} \text{tr} [a_{\pm}^m].$$

(5.6)

are the well known global heat trace coefficients for the operators $H_{\pm}$ (notice the different normalization factor compared to our earlier work [1, 3, 5, 6]). This is the classical heat trace asymptotics of Laplace type operators.
5.2 Combined Heat Trace Asymptotics

It is easy to see again that the asymptotics of the combined heat traces as $\beta \to \infty$ are determined by the bottom eigenvalues

$$X(\beta^2 t, \beta^2 s) \sim \exp \left[ -\beta^2 (t\lambda_1^+ + s\lambda_1^-) \right] |(\varphi_1^+, \varphi_1^-)|^2,$$

(5.7)

$$Y(\beta^2 t, \beta^2 s) \sim \exp \left[ -\beta^2 (t(\mu_1^+)^2 + s(\mu_1^-)^2) \right] \mu_1^+\mu_1^- |(\varphi_1^+, \varphi_1^-)|^2.$$

(5.8)

We will be interested mainly in the asymptotics as $\beta \to 0$. In [9] we proved the following theorem.

**Theorem 1** There are asymptotic expansions as $\beta \to 0$

$$X(\beta^2 t, \beta^2 s) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} \beta^{2k-n} B_k(t, s),$$

(5.9)

$$Y(\beta^2 t, \beta^2 s) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} \beta^{2k-2-n} C_k(t, s),$$

(5.10)

where $B_k(t, s)$ are homogeneous functions of $t$ and $s$ of degree $(k-n/2)$ and $C_k(t, s)$ are homogeneous functions of $t$ and $s$ of degree $(k - 1 - n/2)$. They are integrals of some scalar densities built polynomially from the symbols of the operators $H_\pm$ and $A_\pm$.

The coefficients $B_0$, $B_1$, $C_0$, and $C_1$ are computed explicitly in [9].

This enabled us also to compute the asymptotic expansion of the relative spectral invariants as $\beta \to 0$; they have the form [9]

$$\Psi(\beta^2 t, \beta^2 s) \sim (4\pi)^{-n/2} \sum_{m=0}^{\infty} \beta^{2k-n} \Psi_k(t, s),$$

(5.11)

$$\Phi(\beta^2 t, \beta^2 s) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} \beta^{2k-2-n} \Phi_k(t, s),$$

(5.12)

where

$$\Psi_k(t, s) = (t + s)^{k-n/2} (A_1^+ + A_1^-) - B_k(t, s) - B_k(s, t),$$

(5.13)

$$\Phi_k(t, s) = -\left( k - \frac{n}{2} \right) (t + s)^{k-1-n/2} (A_1^+ + A_1^-) - C_k(t, s) - C_k(s, t).$$

(5.14)
Notice that since for equal operators $H = H_+$ the combined trace $X(t, s)$ can be expressed in terms of the classical heat trace (4.41), then, by comparing (5.9) and (5.5) we see that in this case

$$B_k(t, s) = (t + s)^{k-n/2}A_k.$$  

(5.15)

This gives non-trivial relations between the heat kernel coefficients and their derivatives and provides a useful check of the results.

Also, since for equal operators $A = A_+$ the combined trace $Y(t, s)$ can be expressed via (4.42) in terms of the classical heat trace then, by comparing (5.10) and (5.5) we see that in this case

$$C_k(t, s) = -\left(k - \frac{n}{2}\right)(t + s)^{k-1-n/2}A_k.$$  

(5.16)

This gives non-trivial relations between the heat kernel coefficients and their derivatives and provides a useful check of the results.

6 Asymptotics of Bogolyubov Invariant

6.1 Mellin Transforms

We use the Mellin transform to study the asymptotic expansion of the integrals following [1, 5, 8]. Let $f$ be a smooth function on $\mathbb{R}_+$. Suppose that:

1. it decreases at infinity faster than any power of $t$, that is,

$$\lim_{t \to \infty} t^{\gamma} \partial^N_t f(t) = 0$$  

(6.1)

for any positive constant $\gamma > 0$ and any non-negative integer $N \geq 0$, and

2. there is a constant $\mu$ such that

$$\lim_{t \to 0} t^{\gamma} \partial^N_t [t^\mu f(t)] = 0$$  

(6.2)

for any positive constant $\gamma > 0$ and any non-negative integer $N$.

We consider a slightly modified version of the Mellin transform of the function $f$ introduced in [1]

$$\hat{f}_q = \frac{1}{\Gamma(-q)} \int_0^\infty dt \, t^{-q-1+\mu} f(t).$$  

(6.3)
The integral (6.3) converges for \( \text{Re } q < 0 \). By integrating by parts \( N \) times and using the asymptotic conditions (6.2) we also get

\[
\hat{f}_q = \frac{1}{\Gamma(-q + N)} \int_0^\infty dt \ t^{-q-1+N} (-\partial_t)^N \left[ t^\mu f(t) \right].
\]  

(6.4)

This integral converges for \( \text{Re } q < N \) and, therefore, defines an entire function. Now, by inverting the Mellin transform we obtain a useful integral representation

\[
f(t) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} dq \ t^{\mu + \text{Re } q} \hat{f}_q,
\]  

(6.5)

where \( c < 0 \). By moving the contour of integration to the right we obtain the asymptotic expansion of the function \( f \) as \( t \to 0 \),

\[
f(t) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{k-\mu} \hat{f}_k.
\]  

(6.6)

Now, let \( h \) be another smooth function on \( \mathbb{R}_+ \). Suppose that:

1. it decreases as \( t \to 0 \) faster than any power of \( t \), that is,

\[
\lim_{t \to 0} t^{-\gamma} \partial_t^N h(t) = 0
\]  

(6.7)

with any positive \( \gamma > 0 \) and any non-negative integer \( N \geq 0 \), and

2. there is a positive constant \( \nu > 0 \) such that

\[
\lim_{t \to \infty} t^{-\gamma} \partial_t^N \left[ t^{\nu} h(t) \right] = 0
\]  

(6.8)

with any positive \( \gamma > 0 \) and any non-negative integer \( N \geq 0 \).

We define a modified Mellin transform of the function \( h \) by

\[
\hat{h}_q = \frac{1}{\Gamma(-q)} \int_0^\infty dt \ t^{-1+\gamma} h(t).
\]  

(6.9)

This integral converges and defines an analytic function of \( q \) for \( \text{Re } q < 0 \). By integration by parts we get for any \( N > 0 \),

\[
\hat{h}_q = \frac{1}{\Gamma(-q + N)} \int_0^\infty dt \ t^{-1+\gamma+N} \partial_t^N \left[ t^{\nu} h(t) \right];
\]  

(6.10)
this integral converges for $\text{Re } q < N$ and defines the analytic continuation to an entire function. By inverting the Mellin transform we get an integral representation of the function $h$

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, t^{-q-N} \Gamma(-q)\hat{h}_q$$

(6.11)

where $c < 0$. By moving the contour of integration to the right we get the asymptotic expansion of the function $h$ as $t \to \infty$

$$h(t) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{-k-N} \hat{h}_k.$$  

(6.12)

Lemma 1  Let $f$ and $h$ be the functions described above and $m = |\mu + \nu - 1|$. Then the integral

$$I(\varepsilon) = \int_{0}^{\infty} dt \, h(t)f(\varepsilon t),$$

(6.13)

has the following asymptotic expansion as $\varepsilon \to 0$:

1. If $\mu + \nu$ is not an integer then

$$I(\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^{k-N} c_k^{(1)} + \sum_{k=0}^{\infty} \varepsilon^{k+N-1} c_k^{(2)},$$

(6.14)

where

$$c_k^{(1)} = \frac{(-1)^k}{k!} \Gamma(-k + \mu + \nu - 1) \hat{h}_{k-\mu-\nu+1} \hat{f}_k,$$  

(6.15)

$$c_k^{(2)} = \frac{(-1)^k}{k!} \Gamma(-k - \mu - \nu + 1) \hat{h}_{k+\mu+\nu-1} \hat{f}_k.$$  

(6.16)

2. If $\mu + \nu = 1 + m \geq 1$ is a positive integer with $m \geq 0$, then

$$I(\varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^{k-N} c_k^{(3)} + \sum_{k=0}^{\infty} \varepsilon^{k+N-1} c_k^{(4)} + \log \varepsilon \sum_{k=0}^{\infty} \varepsilon^{k+N-1} c_k^{(5)},$$

(6.17)
where

\[ c^{(3)}_k = \frac{(-1)^k}{k!} (m-k-1)! H_{k-m} F_k, \]
\[ c^{(4)}_k = \frac{(-1)^m}{(k+m)!k!} \left[ [\psi(k+1) + \psi(k+1+m)] \hat{h}_k \hat{f}_{k+m} 
- \hat{h}^{\prime}_k \hat{f}_{k+m} - \hat{h}_k \hat{f}^{\prime}_{k+m} \right], \]
\[ c^{(5)}_k = -\frac{(-1)^m}{(k+m)!k!} \hat{h}_k \hat{f}_{k+m}, \]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the logarithmic derivative of the gamma-function, \( \hat{f}_k = \partial_q \hat{f}_q \big|_{q=k} \) and \( \hat{h}_k = \partial_q \hat{h}_q \big|_{q=k} \). If \( m = 0 \) then the first sum is absent in (6.17).

3. If \( \mu + \nu = 1 - m \geq 0 \) is a non-positive integer with \( m \geq 1 \), then

\[
I(\varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^{k-m-\mu} c^{(6)}_k + \sum_{k=0}^{\infty} \varepsilon^{k-\mu} c^{(7)}_k + \log \varepsilon \sum_{k=0}^{\infty} \varepsilon^{k-\mu} c^{(8)}_k,
\]

where

\[
c^{(6)}_k = \frac{(-1)^k}{k!} (m-k-1)! \hat{h}_k \hat{f}_{k-m},
\]
\[
c^{(7)}_k = \frac{(-1)^m}{(k+m)!k!} \left[ [\psi(k+1) + \psi(k+1+m)] \hat{h}_{k+m} \hat{f}_k 
- \hat{h}^{\prime}_{k+m} \hat{f}_k - \hat{h}_k \hat{f}^{\prime}_k \right],
\]
\[
c^{(8)}_k = -\frac{(-1)^m}{(k+m)!k!} \hat{h}_{k+m} \hat{f}_k.
\]

Proof. By using the Mellin representations of the functions \( f \) and \( h \) we get

\[
I(\varepsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \frac{\Gamma(-q)\Gamma(-q+\mu+\nu-1)}{\Gamma(-q+\mu+\nu+1)\hat{f}_q \varepsilon^{q-\mu}}
\]

with \( c \) being a sufficiently large negative constant such that the arguments of both gamma functions have positive real parts, that is, \( c < \min\{0, \mu + \nu - 1\} \); then all singularities of the integrand lie to the right of the contour of integration. Integrals of this type are a particular case of the so-called Mellin-Barnes integrals. They are
Let
\[ \varphi(q) = \Gamma(-q) \Gamma(-q + \mu + \nu - 1) \hat{h}_{q-\mu-\nu+1} \hat{f}_q \]  
(6.26)

There are three essentially different cases.

Case I. In the case when the number \( \mu + \nu \) is not a integer the function \( \varphi \) is meromorphic with simple poles at the points \( q = k, k = 0, 1, 2, \ldots \) and at the points \( q = k + \mu + \nu - 1, k = 0, 1, 2, \ldots \). By moving the contour to the right and using the well known analytic structure of the gamma function
\[ \Gamma(-k - z) = \frac{(-1)^k}{k!} \left\{ -\frac{1}{z} + \psi(k + 1) + O(z) \right\}, \]  
(6.27)

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \), to evaluate the residues we obtain (6.14).

Case II: Suppose that the number \( \mu + \nu \geq 1 \) is a positive integer, that is, \( \nu = -\mu + 1 + m \) with some non-negative integer \( m = \mu + \nu - 1 \geq 0 \). In this case the function \( \varphi \) is meromorphic with simple poles at the points \( q = 0, 1, 2, \ldots, m - 1 \) (of course, if \( m = 0 \) then there are no simple poles) and double poles at the points \( q = k + m, k = 0, 1, 2, \ldots \). By evaluating the residues we obtain (6.17).

Case III: Suppose that the number \( \mu + \nu \leq 0 \) is a non-positive integer, that is, \( \nu = -\mu + 1 - m \) with some positive integer \( m = -\mu - \nu + 1 \geq 1 \). Then the function \( \varphi \) is meromorphic with simple poles at the points \( q = -m, -m + 1, \ldots, -1 \) and double poles at the points \( q = k, k = 0, 1, 2, \ldots \). By evaluating the residues we obtain (6.21).

### 6.2 Laplace Type Operators

Now we can compute the asymptotics of Bogolyubov invariant as \( \beta \to 0 \). By introducing the integration variables
\[ \rho = t + s, \quad u = \frac{t}{t + s}, \]  
(6.28)

we can write the Bogolyubov invariant (4.22) in the form
\[ B_b(\beta) = \int_0^1 du \int_0^\infty d\rho h(\rho, u) \psi(\beta^2 \rho, u) \]  
(6.29)
where
\[ h(\rho, u) = \rho h_f(\rho u) h_b(\rho(1-u)), \quad (6.30) \]
\[ \psi(\rho, u) = \exp(-m^2\rho) \Psi(\rho u, \rho(1-u)). \quad (6.31) \]

Now, we can apply Lemma 1 to compute the asymptotics as \( \beta \to 0 \). By using the
asymptotics of the function \( \Psi(t, s), (5.11) \), and the functions \( h_f, h_b, (4.19), (4.20) \),
it is easy to see that the functions \( \psi \) and \( h \) above satisfy all the conditions of the
lemma with \( \mu = n/2 \) and \( \nu = 1 \). We define the Mellin transforms of the functions
\( h \) and \( \psi \) by
\[ \hat{\psi}_q(u) = \frac{1}{\Gamma(-q)} \int_0^{\infty} d\rho \rho^{-q-1+n/2} \psi(\rho, u), \quad (6.32) \]
\[ \hat{h}_q(u) = \frac{1}{\Gamma(-q)} \int_0^{\infty} d\rho \rho^q h(\rho, u). \quad (6.33) \]

It is worth pointing out that the values of the Mellin transform at non-negative in-
teger points, \( k \geq 0 \), are determined by the coefficients of the asymptotic expansion
(5.11) of the relative spectral invariant
\[ \hat{\psi}_k(u) = (4\pi)^{-n/2} \sum_{j=0}^{k} (-1)^{j+k} \frac{k!}{j!} m^{2j} \Psi_k(u, 1-u). \quad (6.34) \]
The values at non-integer points, as well as the values of the derivatives at integer
points, \( \hat{\psi}_k' \), are determined by the global behavior of the relative spectral invariant
and are not locally computable.

The coefficients \( \Psi_0(t, s) \) and \( \Psi_1(t, s) \) are computed explicitly in [9]. In partic-
ular,
\[ \Psi_0(t, s) = (t+s)^{-n/2} (A_0^+ + A_0^-) - B_0(t, s) - B_0(s, t), \quad (6.35) \]
where
\[ A_0^\pm = N \int_M dx \ g_{\pm}^{1/2}, \quad (6.36) \]
with \( N = \text{tr} I = \dim \mathcal{V} \) and \( g_{\pm} = (\det g_{ij}^{-1})^{-1} \), is the standard first heat kernel
coefficient and
\[ B_0(t, s) = N \int_M dx \ g^{1/2}(t, s); \quad (6.37) \]
here $g(t, s) = \det g_{ij}(t, s)$, and $g_{ij}(t, s)$ is the inverse of the matrix

$$g^{ij}(t, s) = t g_{ij} - s g_{ij}.$$  \hspace{1cm} (6.38)

We have to distinguish the cases of even and odd dimension.

Case I. Odd dimension, $n = 2m + 1$. Then $\mu + \nu = m + 3/2$ is not an integer and the asymptotics is given by (6.14).

$$B_0(\beta) \sim \sum_{k=0}^{\infty} \beta^{2k-n} c_k^{(1)} + \sum_{k=0}^{\infty} \beta^{2k} c_k^{(2)},$$ \hspace{1cm} (6.39)

where

$$c_k^{(1)} = \frac{(-1)^k}{k!} \Gamma(-k + n/2) \int_0^1 du \, \hat{h}_{k-n/2}(u) \hat{\psi}_k(u),$$ \hspace{1cm} (6.40)

$$c_k^{(2)} = \frac{(-1)^k}{k!} \Gamma(-k - n/2) \int_0^1 du \, \hat{h}_{k}(u) \hat{\psi}_{k+n/2}(u).$$ \hspace{1cm} (6.41)

Notice that the coefficients $c_k^{(1)}$ of the all odd powers of $\beta$ are locally computable invariants whereas the coefficients $c_k^{(2)}$ of the even non-negative powers of $\beta$ are non-locally computable global invariants.

Case II. Even dimension, $n = 2m$. Then $\mu + \nu = m + 1$ is an integer and the asymptotics is given by (6.17).

$$B_0(\beta) \sim \sum_{k=0}^{m-1} \beta^{2k-n} c_k^{(3)} + \sum_{k=0}^{\infty} \beta^{2k} c_k^{(4)} + \log^2 \beta \sum_{k=0}^{\infty} \beta^{2k} c_k^{(5)},$$ \hspace{1cm} (6.42)

where

$$c_k^{(3)} = \frac{(-1)^k}{k!} \Gamma(-k + n/2) \int_0^1 du \, \hat{h}_{k-n/2}(u) \hat{\psi}_k(u),$$ \hspace{1cm} (6.43)

$$c_k^{(4)} = \frac{(-1)^m}{(k + m)!k!} \int_0^1 du \, \left\{ [\psi(k + 1) + \psi(k + 1 + m)] \hat{h}_k(u) \hat{\psi}_{k+m}(u) - \hat{h}_k(u) \hat{\psi}_{k+m}(u) - \hat{h}_k(u) \hat{\psi}_{k+m}(u) \right\},$$ \hspace{1cm} (6.44)

$$c_k^{(5)} = -\frac{(-1)^m}{(k + m)!k!} \int_0^1 du \, \hat{h}_k(u) \hat{\psi}_{k+m}(u).$$ \hspace{1cm} (6.45)

Notice that the coefficients $c_k^{(3)}$ and $c_k^{(5)}$ of the singular part and the logarithmic part are locally computable invariants whereas the coefficients $c_k^{(4)}$ of the regular part.
are not. Also, the coefficients \( c^{(3)} \), when written for general \( n \), have the same form
as the coefficients \( c^{(1)}_k \). Therefore, the singular part of the asymptotics containing
the negative powers of \( \beta \) has the same form in both cases, regardless where the
dimension \( n \) is even or odd.

The leading asymptotics have the form

\[
B_\alpha(\beta) = \beta^{-n} c^{(1)}_0 + \beta^{-n+2} c^{(1)}_1 + O(\beta^{-n+4}) + O(\log \beta).
\]

(6.46)

By using the Mellin transform (6.54) of the function \( h \) and changing the integra-
tion variables \( (\rho, u) \mapsto (t, s) \) we can rewrite the coefficients \( c^{(1)}_k \) in the form

\[
c^{(1)}_k = \frac{(-1)^k}{k!} \int_0^\infty dt \int_0^\infty ds \ (t+s)^{k-n/2} h_f(t) h_b(s) \hat{\psi}_k \left( \frac{t}{t+s} \right).
\]

(6.47)

Now, by using (6.34) and the homogeneity property of the coefficients \( \Psi_k(t, s) \)
(they are homogeneous functions of \( t \) and \( s \) of degree \( k - n/2 \)) we obtain

\[
c^{(1)}_0 = (4\pi)^{-n/2} \int_0^\infty dt \int_0^\infty ds \ h_f(t) h_b(s) \Psi_0(t, s),
\]

(6.48)

\[
c^{(1)}_1 = (4\pi)^{-n/2} \int_0^\infty dt \int_0^\infty ds \ h_f(t) h_b(s) \left\{ \Psi_1(t, s) - m^2(t+s) \Psi_0(t, s) \right\}.
\]

(6.49)

One can show that the coefficient \( c^{(1)}_0 \) is nothing but the coefficient \( V_b \) computed in (3.17).

6.3 Dirac Type Operators

Following the same strategy we compute the asymptotics of the Bogolyubov in-
variant for the Dirac type operators. We have

\[
B_f(\beta) = \beta^2 \int_0^1 du \int_0^\infty d\rho \ \chi(\rho, u) \varphi(\beta^2 \rho, u),
\]

(6.50)

where

\[
\chi(\rho, u) = \rho h_0(\rho u) h_0(\rho(1-u)),
\]

(6.51)

\[
\varphi(\rho, u) = \exp(-m^2 \rho) \left\{ \Phi(\rho u, \rho(1-u)) + m^2 \Psi(\rho u, \rho(1-u)) \right\}.
\]

(6.52)
Now, we can apply Lemma 1 to compute the asymptotics as $\beta \to 0$. By using the asymptotics of the functions $\Phi(t, s)$, (5.12), and $\Psi(t, s)$, (5.11), and the functions $h_f, h_b$, (4.19), (4.20), it is easy to see that the functions $\varphi$ and $\chi$ above satisfy all the conditions of the lemma with $\mu = n/2 + 1$ and $\nu = 0$. We define the Mellin transforms of the functions $\chi$ and $h$ by

$$\hat{\phi}_q(u) = \frac{1}{\Gamma(-q)} \int_0^\infty d\rho \rho^{-q+n/2} \varphi(\rho, u),$$

$$\hat{\chi}_q(u) = \frac{1}{\Gamma(-q)} \int_0^\infty d\rho \rho^{q-1} \chi(\rho, u).$$

(6.53)

(6.54)

It is worth pointing out that the values of the Mellin transform at non-negative integer points, $k \geq 0$, are determined by the coefficients of the asymptotic expansion (5.11) of the relative spectral invariant

$$\hat{\varphi}_0(u) = (4\pi)^{-n/2} \Phi_0(u, 1 - u)$$

(6.55)

and for $k \geq 1$

$$\hat{\varphi}_k(u) = (4\pi)^{-n/2} \sum_{j=0}^k \frac{(-1)^j k!}{j!} m^{2j} \left( \Phi_{k-j}(u, 1 - u) - j \Psi_{k-j}(u, 1 - u) \right).$$

(6.56)

The values at non-integer points, as well as the values of the derivatives at integer points, $\hat{\chi}_k'$, are determined by the global behavior of the relative spectral invariant and are not locally computable.

The coefficients $\Phi_0(t, s)$ and $\Phi_1(t, s)$ are computed explicitly in [9]. In particular,

$$\Phi_0(t, s) = \frac{n}{2} (t + s)^{-n/2 - 1} \left( A_0^+ + A_0^- \right) - C_0(t, s) - C_0(s, t),$$

(6.57)

where $A_0^\pm$ are the standard first heat kernel coefficients (6.36) and

$$C_0(t, s) = \int_M dx g^{1/2}(t, s) \frac{1}{2} g_{ij}(t, s) \text{tr} \left( \gamma^i \gamma^j \right),$$

(6.58)

where $\gamma_i^j$ are Dirac matrices determined by the leading symbols of the operators $A_\pm$, (3.25).

We have again two cases.

Case I. Odd dimension, $n = 2m + 1$. Then $\mu + \nu = m + 3/2$ is not an integer and
the asymptotics is given by (6.14).

\[ B_f(\beta) \sim \sum_{k=0}^{\infty} \beta^{2k-n} d_{k}^{(1)} + \sum_{k=0}^{\infty} \beta^{2k} d_{k}^{(2)}, \]  

(6.59)

where

\[ d_{k}^{(1)} = \frac{(-1)^k}{k!} \Gamma(-k + n/2) \int_0^1 du \, \hat{\chi}_{k-n/2}(u) \hat{\phi}_k(u), \]  

(6.60)

\[ d_{k}^{(2)} = \frac{(-1)^k}{k!} \Gamma(-k - n/2) \int_0^1 du \, \hat{\chi}_k(u) \hat{\phi}_{k+n/2}(u). \]  

(6.61)

Notice that the coefficients \( d_{k}^{(1)} \) of the all odd powers of \( \beta \) are locally computable invariants whereas the coefficients \( d_{k}^{(2)} \) of the even non-negative powers of \( \beta \) are non-locally computable global invariants.

Case II. Even dimension, \( n = 2m \). Then \( \mu + \nu = m + 1 \) is an integer and the asymptotics is given by (6.17).

\[ B_f(\beta) \sim \sum_{k=0}^{m-1} \beta^{2k-n} d_{k}^{(3)} + \sum_{k=0}^{\infty} \beta^{2k} d_{k}^{(4)} + \log \beta^2 \sum_{k=0}^{\infty} \beta^{2k} d_{k}^{(5)}, \]  

(6.62)

where

\[ d_{k}^{(3)} = \frac{(-1)^k}{k!} \Gamma(-k + n/2) \int_0^1 du \, \hat{\chi}_{k-n/2}(u) \hat{\phi}_k(u), \]  

(6.63)

\[ d_{k}^{(4)} = \frac{(-1)^m}{(k + m)!k!} \int_0^1 du \, \{[\psi(k + 1) + \psi(k + 1 + m)] \hat{\chi}_{k}(u) \hat{\phi}_{k+m}(u) \\ -\hat{\chi}'_{k}(u) \hat{\phi}_{k+m}(u) - \hat{\chi}_{k}(u) \hat{\phi}'_{k+m}(u)\}, \]  

(6.64)

\[ d_{k}^{(5)} = \frac{(-1)^m}{(k + m)!k!} \int_0^1 du \, \hat{\chi}_{k}(u) \hat{\phi}_{k+m}(u). \]  

(6.65)

Notice that the coefficients \( d_{k}^{(3)} \) and \( d_{k}^{(5)} \) of the singular part and the logarithmic part are locally computable invariants whereas the coefficients \( d_{k}^{(4)} \) of the regular part are not. Also, the coefficients \( d_{k}^{(3)} \), when written for general \( n \), have the same form as the coefficients \( d_{k}^{(1)} \). Therefore, the singular part of the asymptotics containing the negative powers of \( \beta \) has the same form in both cases, regardless whether the dimension \( n \) is even or odd.
The leading asymptotics have the form
\[ B_f(\beta) = \beta^{-n} d_0^{(1)} + \beta^{-n+2} d_1^{(1)} + O(\beta^{-n+4}) + O(\log \beta). \] (6.66)

By using the Mellin transform (6.54) of the function \( \chi \) and changing the integration variables \((\rho, u) \mapsto (t, s)\) we can rewrite the coefficients \( d_k^{(1)} \) in the form
\[ d_k^{(1)} = \frac{(-1)^k}{k!} \int_0^\infty dt \int_0^\infty ds \ (t + s)^{k-1-n/2} h_0(t) h_0(s) \hat{\Phi}_k \left( \frac{t}{t + s} \right). \] (6.67)

Now, by using (6.56) and the homogeneity property of the coefficients \( \Phi_k(t, s) \) (they are homogeneous functions of \( t \) and \( s \) of degree \((k - 1 - n/2))\) and \( \Psi_k(t, s) \) (they are homogeneous functions of \( t \) and \( s \) of degree \((k - n/2))\) we obtain
\[ d_0^{(1)} = (4\pi)^{-n/2} \int_0^\infty dt \int_0^\infty ds \ h_0(t) h_0(s) \Phi_0(t, s), \] (6.68)
\[ d_1^{(1)} = (4\pi)^{-n/2} \int_0^\infty dt \int_0^\infty ds \ h_0(t) h_0(s) \left[ \Phi_1(t, s) + m^2 \ [-(t + s)\Phi_0(t, s) + \Psi_0(t, s)] \right]. \] (6.69)

One can show that the coefficient \( d_0^{(1)} \) is nothing but the coefficient \( V_f \) computed in (3.29).

7 Solvable Cases

7.1 Equal Operators

First of all, we notice that since for equal operators \( H_- = H_+ \) the combined trace \( X(t, s) \) can be expressed in terms of the classical heat trace
\[ X(t, s) = \Theta(t + s), \] (7.1)
then, by comparing (5.9) and (5.5) we see that in this case
\[ B_k(t, s) = (t + s)^{k-n/2} A_k. \] (7.2)
Similarly, since for equal operators \( A_- = A_+ \) the combined trace \( Y(t, s) \) can be expressed in terms of the classical heat trace
\[
Y(t, s) = -\partial_t \Theta(t + s),
\]
then, by comparing (5.10) and (5.5) we see that in this case
\[
C_k(t, s) = -\left(k - \frac{n}{2}\right)(t + s)^{k-1-n/2}A_k.
\]
It is easy to see then that for equal operators \( L_- = L_+ \) and \( D_- = D_+ \) the relative spectral invariants vanish, \( \Psi(t, s) = \Phi(t, s) = 0 \) and therefore, the Bogolyubov invariant vanishes
\[
B_0(\beta) = B_1(\beta) = 0.
\]

### 7.2 Constant Potential Term

If the Laplace type operators differ by just a constant,
\[
H_+ = H_- + M^2,
\]
then the metrics and the connections are the same and
\[
\Theta_+(t) = e^{-tM^2} \Theta_-(t),
X(t, s) = e^{-tM^2} \Theta_-(t + s),
\]
and, therefore,
\[
\Psi(t, s) = \left(e^{-tM^2} - 1\right)\left(e^{-sM^2} - 1\right) \Theta_-(t + s).
\]
In this case
\[
B_0(t, s) = (t + s)^{-n/2}A_0^-,
B_1(t, s) = (t + s)^{-n/2}A_1^- - t(t + s)^{-n/2}M^2 A_0^-.
\]

For the Dirac case suppose that there is an endomorphism \( M \) such that it anti-commutes with the operator \( A_- \),
\[
A_- M = -MA_-,
\]
and \( M^2 \) is a scalar. Then it is easy to see that
\[
\text{Tr} MA_- \exp(-sA_-^2) = 0.
\]
Now, suppose that
\[ A_+ = A_- + M, \]
so that (recall that \( H_+ = A_+^2 \))
\[ H_+ = H_- + M^2; \]
(7.15)
Then it is easy to show that
\[ Y(t, s) = -e^{-tM^2} \partial_t \Theta_-(t + s), \]
(7.16)
and, hence,
\[ \Phi(t, s) = -\left(e^{-tM^2} - 1\right)\left(e^{-sM^2} - 1\right) \partial_t \Theta_-(t + s) + M^2 e^{-(t+s)M^2} \Theta_-(t + s). \]
(7.17)
Therefore,
\[ C_0(t, s) = \frac{n}{2}(t + s)^{-1-n/2} A_0, \]
(7.18)
\[ C_1(t, s) = \left(\frac{n}{2} - 1\right)(t + s)^{-n/2} A_1 - \frac{n}{2} t(t + s)^{-1-n/2} M^2 A_0. \]
(7.19)
A more general case is the case of commuting operators; then the combined heat traces still simplify significantly, they can be expressed in terms of the classical one
\[ X(t, s) = \text{Tr} \ exp(-tH_+ - sH_-), \]
(7.20)
\[ Y(t, s) = \text{Tr} A_- A_+ \ exp(-tH_+ - sH_-). \]
(7.21)
Therefore, the asymptotics of the combined traces can be obtained from the classical ones. Notice that the leading symbol of the operators \( H(t, s) = tH_+ + sH_- \) is determined exactly by the metric \( g^{ij}(t, s) \). Therefore, in this case the combined traces are given by the classical trace for the operator \( H(t, s) \).

7.3 Nilpotent Lie Algebra

Now, suppose that there are two sets of operators \( \nabla_i^+, \nabla_j^- \) forming the Lie algebra
\[ [\nabla_i^+, \nabla_j^-] = \mathcal{R}_{ij}^+, \]
(7.22)
\[ [\nabla_i^-, \nabla_j^-] = \mathcal{R}_{ij}^-, \]
(7.23)
\[ [\nabla_i^+, \nabla_j^+] = \mathcal{R}_{ij}, \]
(7.24)
where
\[
\mathcal{R}_{ij} = \frac{1}{2} \left( \mathcal{R}^+_{ij} + \mathcal{R}^-_{ij} \right),
\] all other commutators being zero. We define two operators
\[
H^\pm = -g^{ij}_\pm (\nabla_i^\pm + B_i^\pm)(\nabla_j^\pm + B_j^\pm) + Q_\pm.
\]
where \( g^{ij}_\pm \) are constant positive matrices, \( B_i^\pm \) are constant vectors and \( Q_\pm \) are some constants. Then one can prove the following theorem for the heat semigroup \([2, 6]\).

**Theorem 2** *The heat semigroup \( \exp(-tH^\pm) \) can be presented in form of an average over the Lie group with the Gaussian measure*

\[
\exp(-tH^\pm) = (4\pi)^{-n/2}\Omega(t) \exp(-tQ_\pm) \int_{\mathbb{R}^n} d\xi \exp \left\{ -\frac{1}{4} \langle \xi, D(t)\xi \rangle + \langle B^\pm, \xi \rangle \right\} \exp \langle \xi, \nabla^\pm \rangle.
\]

where \( D = (D_{ij}) \) is the matrix defined by
\[
D(t) = \mathcal{R} \coth(tg^{-1}\mathcal{R})
\]
and
\[
\Omega(t) = \det \left( \frac{\sinh(tg^{-1}\mathcal{R})}{\mathcal{R}} \right)^{-1/2}.
\]
By using this representation one can compute the heat semigroup convolution
\[
U(t, s) = \exp(-tH_+) \exp(-sH_-)
\]
exactly. We are going to carry this out in a separate work.

**8 Conclusion**

The goal of this paper was to introduce and to study new spectral invariants of two elliptic operators on manifolds that we call the Bogolyubov invariants, the bosonic one \( B_b(\beta) \) and the fermionic one, \( B_f(\beta) \), which depend on an adiabatic parameter \( \beta \). We established the general asymptotic expansion of these invariants as \( \beta \to 0 \) in terms of the so-called relative spectral invariants studied in our paper \([9]\) and computed the first two coefficients of the asymptotic expansions.
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