Approximation on compact sets of functions and all derivatives

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Abstract

In Mergelyan type approximation we uniformly approximate functions on compact sets $K$ by polynomials or rational functions or holomorphic functions on varying open sets containing $K$. In the present paper we consider analogous approximation, where uniform convergence on $K$ is replaced by uniform approximation on $K$ of all order derivatives.

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1 Introduction

Let $K$ be a compact set in the complex plane $\mathbb{C}$, or more generally in $\mathbb{C}^d$. By $O(K)$ we denote the set of all complex functions $f$, such that there exists an open set $V_f$ containing $K$ with the property that $f$ is defined and is holomorphic on $V_f$. One endows $O(K)$ with the supremum norm on $K$. Then the completion of the metric space is denoted by $\overline{O}(K)$ and coincides with the closure of $O(K)$ in $C(K)$. It is well known that $\overline{O}(K)$ is contained in the classical algebra $A(K) = \{f : K \to \mathbb{C} \text{ continuous on } K \text{ and holomorphic in } K^o \}$. If $K^o = \emptyset$, then $A(K) = C(K)$. Mergelyan type theorems are those where under some assumptions on $K$ we can conclude that $A(K) = \overline{O}(K)$. In particular, if $K \subset \mathbb{C}$ and $\mathbb{C} - K$ is connected, then every function $f$ in $A(K)$ can be uniformly on $K$ approximated by polynomials; thus, $A(K) = \overline{O}(K)$ in this case. More generally, if $K \subset \mathbb{C}$ and $\mathbb{C} - K$ has a finite number of components, then every $f$ in $A(K)$ can be uniformly on $K$ approximated by rational functions with prescribed poles, one in each component of $\mathbb{C}^\infty - K$; thus, in this case we also have $A(K) = \overline{O}(K)$. For $K \subset \mathbb{C}$ a characterization of whether it holds that $A(K) = \overline{O}(K)$ or not has been obtained by Vitushkin using continuous analytic capacity. ([4], [5])

In the present paper, for $K$ a compact planar set, we replace uniform on $K$ approximation by uniform
on $K$ approximation of all order derivatives. More precisely, we endow $O(K)$ with the metric

$$d(f, g) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sup_{z \in K} \frac{|f^{(k)}(z) - g^{(k)}(z)|}{1 + \sup_{z \in K} |f^{(k)}(z) - g^{(k)}(z)|},$$

where $F^l$ denoted the $l^{th}$ derivative of $F \in O(K)$, which is well defined on $K$ because $F$ is holomorphic on an open set containing $K$. Now the analogue of $O(K)$ is the completion $B(K)$ of the metric space $(O(K), d)$. Every element of $B(K)$ is a sequence $g = \langle g_l \rangle_{l=0}^{\infty} \in A(K)^{\aleph_0}$. Our effort is to determine explicitly which subset of $A(K)^{\aleph_0}$ is $B(K)$. At the end of the paper we show that, if $K$ is denumerable then $B(K) = A(K)^{\aleph_0}$. If $K = \{ z \in C : |z| \leq 1 \}$, then $B(K) = \{ g = \langle g_l \rangle_{l=0}^{\infty} : \text{there exists a holomorphic function } f : \{ z \in C : |z| < 1 \} \rightarrow C, \text{ such that each derivative } f^l \text{ extends continuously on } \{ z \in C : |z| \leq 1 \} \text{ and } g_l = f^l \}$. Roughly speaking $B(K)$ can be identified with $A^\infty(D)$, $D$ being the unit disk. If $K = [0, 1]$ then $B(K)$ can be identified with $C^\infty[0, 1]$.

In the general case we define a subset $\Gamma(K)$ of $C(K)^{\aleph_0}$ containing $B(K)$ and we give sufficient conditions on $K$ so that $B(K) = \Gamma(K)$. This happens in all previously mentioned cases. We also prove a lemma stating that rational functions with poles outside $K$ are always ”dense” in $B(K)$, where $f = (f^0, f^1, f^2, \ldots)$. Extensions in several variables are possible and will be treated in future papers. When $K \subset C^d, d > 1$, then $A(K)$ will be replaced by a smaller algebra $A_D(K) = \{ f \in C(K) : f \text{ is holomorphic on any analytic disk contained in } K, \text{ even meeting the boundary } K \}$, which satisfies $\overline{O(K)} \subset A_D(K) \subset A(K)$. (2).

The algebra $A_D(K)$ is more appropriate for approximation in several complex variables than the classical algebra $A(K)$.

2 A function Algebra and an Important Lemma

In this section we introduce an algebra of functions, which will be of interest throughout this paper. Namely, we have the standard definition:

**Definition 2.1** Let $K \subseteq C$ be a planar compact set. We call $O(K)$ the set of all functions $f$ such that there can be found an open set $V_f \supseteq K$ (dependent on $f$), such that $f$ is holomorphic on $V_f$.

Typically this set is endowed with the supremum norm on $K$ and is thus considered as a subset of $C(K)$, the space of continuous functions on $K$. However, in this paper we will provide this space with another topology. We give the next definition:

**Definition 2.2** We endow the space $O(K)$ of Definition 1, with the metric introduced by the denumerable family of seminorms $\sup_{z \in K} |f^i(z)|$ where $f^i$ denotes the derivative of $i$-order of $f$. Namely $O(K)$ is given the metric

$$d(f, g) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sup_{z \in K} \frac{|f^{(k)}(z) - g^{(k)}(z)|}{1 + \sup_{z \in K} |f^{(k)}(z) - g^{(k)}(z)|}$$
From now on we will refer to this particular metric as $d$.

Notice that Definition 2.2 is well posed, since each function in $O(K)$ is defined and holomorphic in a neighborhood of $K$.

**Definition 2.3** We call $B(K)$ the completion of $O(K)$ with respect to the metric introduced in Definition 2.2.

**Remark 2.4** Notice that taking an element $f$ of $O(K)$, for some compact set $K \subset \mathbb{C}$, we can define the sequence $(f, f', f'', \ldots)$ of its derivatives and call this sequence, the sequence corresponding to $f$.

Note now that $d(f_n, f) \to 0$ for some functions $f_n, f$ of $O(K)$ if and only if the corresponding sequence of $f_n$ converges on the corresponding sequence of $f$, where the limit is realized as the term to term limit via the supremum norm on $K$. These sequences, which correspond to elements of $O(K)$ are of course sequences in $A(K)^N_0$. In this way we can refer to elements of the complete space $B(K)$ as sequences in $A(K)^N_0$. Often we identify a function $f$ with the sequence of its derivatives. Thus, we can say that a subset of $O(K)$ is also a subset of $B(K)$.

In this paper one of the questions we are interested is in what length can we determine the set $B(K)$ explicitly. To answer this we will use a lemma, making use of Runge’s Theorem, which we now state.

**Theorem (Runge)** Let $K \subset \mathbb{C}$ be a compact planar set and $L$ be a set, containing at least one point for every connected component of the set $\mathbb{C}_\infty - K$. Then every function $f$ in $O(K)$ can be uniformly on $K$ approximated by rational functions with poles in $L$.

We also give another version of Runge’s Theorem:

**Theorem (Runge)** Let $U \subset \mathbb{C}$ be an open planar set and $L$ be a set, containing at least one point from every connected component of the set $\mathbb{C}_\infty - U$. Then every function $f$ in $H(U)$ can be approximated uniformly on compact subsets of $U$ by rational functions with poles in $L$.

Runge’s Theorem gives us that given a compact set $K$ and a set $L$ as that in the theorem, $R_L$ is dense in $O(K)$ with respect to the supremum norm on $K$, where $R_L$ denotes the set of rational functions with poles only in $L$. However, we have already stated that throughout this paper the metric on $O(K)$ that concerns us is not the supremum one, but the metric $d$. It is therefore reasonable to ask whether $R_L$ is dense in $O(K)$ with respect to $d$. The answer to that is affirmative and for the proof of this we will need the following lemma:

**Lemma 2.5** Let $K$ be a compact planar set and $L$ a set containing one point from every connected component of $\mathbb{C}_\infty - K$. If $V$ is an open set containing $K$, there exists an open set $W$, with
$K \subset W \subset V$, such that every connected component of $\mathbb{C}_\infty - W$ contains a point of $L$.

We now prove our proposition:

**Proposition 2.6** Let $K \subset \mathbb{C}$ be a compact planar set and $L$ be a set, containing at least one point from every connected component of $\mathbb{C}_\infty - K$. Let $R_L$ be the set of rational functions with poles in $L$. $(R_L \subset O(K))$. Then $R_L$ is dense in $B(K)$.

**Proof.** It is, of course, sufficient to show that $R_L$ is dense in $O(K)$. Let $f \in O(K)$. Thus, there exists an open set $V_f \supset K$ such that $f$ is analytic on $V_f$. By Lemma 2.5 we can find an intermediate open set $W$, with $K \subset W \subset V_f$, such that every connected component of $\mathbb{C}_\infty - W$ contains a point of $L$. By Runge's Theorem (second version) $f$ can be uniformly approximated on compact subsets of $W$ by rational functions $r_n$ with poles in $L$. By Weierstrass’ Theorem $f^l$ is approximated by $r_n^l$ uniformly on $K$ for every $l \in \mathbb{N}$. Hence, we have our result. □

We will also state a form of Mergelyan’s Theorem for future reference.

**Theorem (Mergelyan)** ([3]) Let $K$ be a compact planar set such that $\mathbb{C}_\infty - K$ has finitely many connected components. Let $L$ denote a set containing one point from each connected component of $\mathbb{C}_\infty - K$. Then the rational functions with poles in $L$ are uniformly dense in $A(K) = \{f : K \rightarrow \mathbb{C} \text{ continuous on } K \}$.

### 3 The function algebra $\Gamma(K)$ and its relation with $B(K)$

In this section we will introduce a new function algebra, which has the advantage of being intuitively clearer than $B(K)$. To motivate for the following definition we noted in Remark 2.4 that taking an element $f$ of $O(K)$, for some compact set $K \subset \mathbb{C}$, we can define the sequence $(f, f', f'', \ldots)$ of its derivatives and call this sequence, the sequence corresponding to $f$. These sequences, which correspond to elements of $O(K)$ are sequences in $A(K)^N$. A property of them is that taking $f \in O(K)$ and $\gamma$ a rectifiable curve in $K$ starting at a point $a$ and ending at a point $b$ and $\phi$ a holomorphic function in some neighborhood of $\gamma$, then

$$\int_\gamma \phi(z)f^{l+1}(z)\,dz = f^l(z)\phi(z)|_a^b - \int_\gamma f^l(z)\phi'(z)\,dz$$

(1)

Now if we take $g \in B(K)$, it can be expressed as a sequence $(g_0, g_1, \ldots) \in A(K)^N$, where there exist $f_n \in O(K)$ such that for every $k \in \mathbb{N}$

$$\sup_{z \in K} |f_n^k(z) - g_k(z)| \rightarrow 0$$
By (1):
\[ \int_{\gamma} \phi(z) f_n^{l+1}(z) \, dz = f_n^l(b) \phi(b) - f_n^l(a) \phi(a) - \int_{\gamma} f_n^l(z) \phi'(z) \, dz \]
Taking \( n \to \infty \) we obtain:
\[ \int_{\gamma} \phi(z) g_{l+1}(z) \, dz = g_l(b) \phi(b) - g_l(a) \phi(a) - \int_{\gamma} g_l(z) \phi'(z) \, dz \]
Keeping that in mind we define:

**Definition 3.1** Let \( K \) be a planar compact set. We define \( \Gamma(K) \) as the set of sequences \( (g_k)_{k=0}^\infty \) in \( A(K) \) such that for every rectifiable curve \( \gamma \) in \( K \) starting at some point \( a \) and ending at some point \( b \), and every \( \phi \) holomorphic in some neighborhood of \( \gamma \) we have that
\[ \int_{\gamma} \phi(z) g_{l+1}(z) \, dz = g_l(b) \phi(b) - g_l(a) \phi(a) - \int_{\gamma} g_l(z) \phi'(z) \, dz \]
for every \( k \in \mathbb{N} \).

**Remark 3.2** We provide \( \Gamma(K) \) with the metric \( d \), where
\[
d((f_k)_{k=0}^\infty, (g_k)_{k=0}^\infty) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\sup_{z \in K} |f_k(z) - g_k(z)|}{1 + \sup_{z \in K} |f^{(k)}(z) - g^{(k)}(z)|}
\]
Due to the conversation preceding Definition 3.1, we have that \( B(K) \subset \Gamma(K) \) and the inclusion map is an isometry.

We will now show that in many familiar cases \( B(K) = \Gamma(K) \).

**Theorem 3.3** Let \( K \subset \mathbb{C} \) be a compact set, such that:

1. \( \mathbb{C} \setminus K \) has a finite number of connected components, \( V_0, V_1, \ldots, V_n \) and let \( a_0 = \infty \in V_0 \) and \( a_j \in V_j, j = 0, 1, 2, \ldots, n \) be fixed.
2. There exists \( S \subset K \) with \( \overline{S} = K, a \in S \) and \( M < \infty \), so that for all \( z \in S \) there exists a rectifiable curve \( \gamma_{a,z} \) in \( K \) starting at \( a \) and ending at \( z \) with length \( l(\gamma_{a,z}) \leq M \)
3. There exist \( \delta_i \) in \( K, i = 1, 2, \ldots, n \) closed rectifiable curves such that \( \text{Ind}(\delta_i, a_j) = 0 \) for \( j \neq i \) and \( \text{Ind}(\delta_i, a_i) = 1 \).

Then \( \Gamma(K) = B(K) = \overline{R_L} \) where \( L = \{a_0, a_1, \ldots, a_n\} \) and the closure of \( R_L \) is taken in \( B(K) \).

**Proof.** Let \( g = (g_0, g_1, g_2, \ldots) \in \Gamma(K) \) and \( N \in \mathbb{N} \). It suffices to find \( f_n \in R_L \) such that
\[ \sup_{z \in K} |f_n^l(z) - g_l(z)| \xrightarrow{n \to \infty} 0 \quad \text{for } l = 0, 1, 2 \ldots N. \]
By Mergelyan’s Theorem \( \exists h_m \in R_L \), such that
\[ \sup_{z \in K} |h_m(z) - g_N(z)| \xrightarrow{m \to \infty} 0. \] Now define \( h_{m,r,i} \) for \( 1 \leq r \leq N \) and \( 1 \leq i \leq n \) to be \( h_{m,r,i} = \\text{Res}_{z=a_i} h_m(z)(z-a_i)^{r-1} \). Take \( \tilde{h}_m(z) = h_m(z) - \sum_{r=1}^{\infty} \sum_{i=1}^{n} \frac{h_{m,r,i}}{(z-a_i)^r} \), \( \tilde{h}_m \in R_L \) easily.

By construction \( \tilde{h}_m \) has now an \( N^{\text{th}} \) order primitive, that is there exists \( H_m \) in \( R_L \) such that \( H_m^N(z) = \tilde{h}_m(z) \). We can also take \( H_m^r(a) = g_r(a) \) for \( 0 \leq r \leq N-1 \). We will now show that

\[ \sup_{z \in K} |H_m^{(k)}(z) - g_k(z)| \xrightarrow{m \to \infty} 0 \]

for \( k = 0, 1, 2, ..., N \).

We will first show that \( \frac{h_{m,r,i}}{(z-a_i)^r} \to 0 \) uniformly on \( K \), or equivalently, since \( \text{dist}(a_i, K) > 0 \), that \( h_{m,r,i} \to 0 \). To see that notice that:

\[ 2\pi i \text{Res}_{z=a_i} h_m(z)(z-a_i)^{r-1} = \int_{\delta_i} h_m(z)(z-a_i)^{r-1} \, dz \to \int_{\delta_i} g_N(z)(z-a_i)^{r-1} \, dz = -(r-1) \int_{\delta_i} g_{N-1}(z)(z-a_i)^{r-2} \, dz \]

as \( m \to \infty \), since \( \delta_i \) is closed and \( g \in \Gamma(K) \). By induction:

\[ \int_{\delta_i} g_{N-1}(z)(z-a_i)^{r-2} \, dz = 0 \]

for \( 0 \leq r \leq N-1 \). Therefore, \( h_{m,r,i} \to 0 \) as \( m \to \infty \).

Now \( \sup_{z \in K} |h_m(z) - g_N(z)| \xrightarrow{m \to \infty} 0 \) and \( \sup_{z \in K} |\tilde{h}_m(z) - g_N(z)| \xrightarrow{m \to \infty} 0 \).

Therefore we have that \( \sup_{z \in K} |H_m^N(z) - g_N(z)| \xrightarrow{m \to \infty} 0 \). We proceed by induction:

Suppose for \( r \in 1, 2, ..., N-1 \) that \( \sup_{z \in K} |H_m^{r+1}(z) - g_{r+1}(z)| \to 0 \). Then we have that taking \( b \in S \) and \( \gamma_{a,b} \) in \( K \) such that \( l(\gamma_{a,b}) \leq M \):

\[ \int_{\gamma_{a,b}} H_m^{r+1}(z) \, dz = H_m^r(b) - H_m^r(a) \]

and

\[ \int_{\gamma_{a,b}} g_{r+1}(z) \, dz = g_r(b) - g_r(a) \]

Thus, since \( H_m^r(a) = g_r(a) \) we have:

\[ |H_m^r(b) - g_r(b)| \leq \int_{\gamma_{a,z}} |H_m^{r+1}(z) - g_{r+1}(z)| \, dz \leq M \cdot \sup_{z \in K} |H_m^{r+1}(z) - g_{r+1}(z)| \to 0 \]

, as \( m \to \infty \), for all \( b \in S \) and \( 0 \leq r \leq N-1 \).

Since \( S = K \), we have the proof completed. \( \square \)

We now make a standard definition:

**Definition 3.4** Let \( \Omega \in \mathbb{C} \) be a bounded domain such that \( \overline{\Omega} = \Omega \). A holomorphic function \( f : \Omega \to \mathbb{C} \)
belongs to $A^\infty(\Omega)$ if for every $l = 0, 1, 2...$ the derivative $f^l$ can be continuously extended on $\overline{\Omega}$. The topology in $A^\infty(\Omega)$ is induced by the seminorms $\sup_{z \in \overline{\Omega}} |f^l(z)|$.

As Corollaries of Theorem 3.3, we have the following statements:

**Corollary 3.5** Let $\Omega \subset \mathbb{C}$ be a bounded domain such that $\overline{\Omega}' = \Omega$ and such that $\mathbb{C}_\infty - \overline{\Omega}'$ is connected. We also assume that there exists $M < \infty$, such that for every $a, b \in \Omega$ there exists a rectifiable curve $\gamma_{a,b}$ in $\Omega$ joining $a$ and $b$ with length $l(\gamma_{a,b}) \leq M$. Then, if $P$ denotes the set of polynomials we have that $\overline{P} = A^\infty(\Omega) = B(\overline{\Omega}) = \Gamma(\Omega)$, where the closure of $P$ is taken in the topology of $A^\infty(\Omega)$.

**Proof.** Setting $K = \overline{\Omega}$, $S = \Omega$ in Theorem 3.3, we have that $\overline{P} = B(\overline{\Omega}) = \Gamma(\overline{\Omega})$. Since $P \subset A^\infty(\Omega) \subset \Gamma(\overline{\Omega})$ and the inclusion map $A^\infty(\Omega) \subset \Gamma(\overline{\Omega})$ is an isometry, the statement is proved. $\square$

**Corollary 3.6** Let $\Omega \subset \mathbb{C}$ be a bounded domain such that $\overline{\Omega}' = \Omega$ and such that $\Omega$ is bounded by a finite set of disjoint Jordan curves. Let $L = \{a_0, a_1, a_2, ... a_n\}$ containing exactly one point from each connected component of $\mathbb{C}_\infty - \overline{\Omega}$ with $a_0 = \infty$. We also assume that there exists $M < \infty$, such that for every $a, b \in \Omega$ there exists a rectifiable curve $\gamma_{a,b}$ in $\Omega$ joining $a$ and $b$ with length $l(\gamma_{a,b}) \leq M$. Then $R_L = A^\infty(\Omega) = B(\overline{\Omega}) = \Gamma(\overline{\Omega})$, where the closure of $R_L$ is in the topology of $A^\infty(\Omega)$.

**Proof.** The proof is identical to that of Corollary 3.5. $\square$

**Remark 3.7** If $K$ is a compact rectifiable curve $\gamma : [0, 1] \to \mathbb{C}$, such that $\mathbb{C}_\infty - K$ has a finite set of connected components, then the assumptions of Theorem 3.3 are satisfied. Therefore, if we pick a set $L$, containing a point from each connected component of $\mathbb{C}_\infty - K$, then $\overline{R_L} = B(K) = \Gamma(K)$. Now, suppose in addition the curve $K$ is locally one to one and satisfies the following:

For every $a \in [0, 1]$ there exists a constant $C_a$ dependent on $a$ and a $\delta > 0$, such that, for every $t \in (a - \delta, a + \delta)$ we have that:

$$l(a, t) \leq C_a |\gamma(t) - \gamma(a)|$$

where $l(a, t)$ is the length of $\gamma\vert_{[a,t]}$ if $a < t$ or of $\gamma\vert_{[t,a]}$ if $t \leq a$.

Then condition (1) implies that for a sequence $g_t$ in $B(K)$ we have:

$$\int_a^t g_{t+1}(z) \, d\gamma = g_t(\gamma(t)) - g_t(\gamma(a))$$

Thus taking derivatives with respect to the position we have:

$$\frac{d g_t}{dz}\bigg|_{z=\gamma(a)} = \lim_{t \to a} \frac{g_t(\gamma(t)) - g_t(\gamma(a))}{\gamma(t) - \gamma(a)} = \lim_{t \to a} \frac{\int_a^t g_{t+1}(z) \, d\gamma}{\gamma(t) - \gamma(a)} = \lim_{t \to a} \frac{\int_a^t g_{t+1}(z) \, d\gamma}{\gamma(t) - \gamma(a)} + g_{t+1}(\gamma(a)) = g_{t+1}(\gamma(a))$$
since
\[
\limsup_{t \to a} \left| \frac{f_{l}(z) - g_{l}(\gamma(a))}{\gamma(t) - \gamma(a)} \right| \leq \limsup_{t \to a} \sup_{|s-a| \leq |t-a|} \left| \frac{g_{l}(\gamma(s)) - g_{l}(\gamma(a))}{\gamma(t) - \gamma(a)} \right| \leq C_{a} \cdot \limsup_{t \to a} \sup_{|s-a| \leq |t-a|} \left| g_{l}(\gamma(s)) - g_{l}(\gamma(a)) \right| \to 0
\]
due to the continuity of \(g_{l}\).

We thus get that the the differentiation of \(g_{l}\) with respect to the position yields \(g_{l+1}\). It follows that \(\mathcal{K} = B(K) = \Gamma(K)\) coincides with the set \(C^{\infty}(K)\) of all functions \(f : K \to \mathbb{C}\), which have derivatives of any order on \(K\) with respect to the position.

**Remark 3.8** Using Remark 3.7, we see that if \(K = [a, b]\), then \(B(K) = C^{\infty}[a, b]\) and if \(K = \{e^{i\theta} : \theta \in \mathbb{R}\} = T\), then \(B(T) = C^{\infty}(T)\).

**Remark 3.9** Using the previous results we can make deductions about \(B(K)\) on more complex situations. For example take \(K = K_{1} \cup K_{2} \cup K_{3}\), where \(K_{1}\) and \(K_{3}\) are disjoint closed disks and \(K_{2} = [A, B]\), where \(A \in \partial K_{1}\), \(B \in \partial K_{3}\) and \(K_{2}\) is disjoint from \(K_{1}\) and \(K_{3}\), except for the points \(A\) and \(B\). Then \(f : K \to \mathbb{C}\) belongs to \(B(K)\) if and only if \(f_{1} = f_{1}|_{K_{1}} \in B(K_{1}), f_{2} = f_{2}|_{K_{2}} \in B(K_{2}), f_{3} = f_{3}|_{K_{3}} \in B(K_{3})\) and for every \(l = 0, 1, 2, 3, ... f_{l}^{1}(A) = f_{l}^{1}(A)\) and \(f_{l}^{3}(B) = f_{l}^{3}(B)\). We know, however, that \(B(K_{1}) = A^{\infty}(K_{1}), B(K_{3}) = A^{\infty}(K_{3})\) and \(B(K_{2}) = C^{\infty}(K_{2})\). We thus have an exact description of \(B(K)\).

We now give a result, concerning the extendability of functions \(g_{k}, k = 0, 1, 2, 3, ..\) where \((g_{k}) \in B(K)\), \(K\) being a compact set with no isolated points.

**Theorem 3.10** Let \(K \subset \mathbb{C}\) be compact without isolated points. Then there exists \((g_{k})_{k=0}^{\infty} \in B(K)\), such that for every \(k = 0, 1, 2, ..\) and every open disk \(D\), such that \(D \cap K \neq \emptyset, D \cap K^{c} \neq \emptyset\), there is no holomorphic function \(F : D \to \mathbb{C}\) such that \(F|_{D \cap K} = g_{k}|_{D \cap K}\). The set of such sequences \((g_{k})_{k=0}^{\infty}\) is \(G_{\delta}\)-dense in \(B(K)\).

**Proof.** Define \(E_{k} = \{(g_{n})_{n=0}^{\infty} \in B(K)\}, such that for every open disk \(D\) such that \(D \cap K \neq \emptyset, D \cap K^{c} \neq \emptyset\) there is no holomorphic function \(F : D \to \mathbb{C}\), such that \(F|_{D \cap K} = g_{k}|_{D \cap K}\). We want to show that \(\bigcap_{k=0}^{\infty} E_{k}\) is \(G_{\delta}\)-dense. Since \(B(K)\) is a complete metric space, by Baire’s Theorem we need to show \(E_{k}\) is \(G_{\delta}\)-dense for each \(k = \{0, 1, 2, ..\}\).

Let \(k \in N\). Notice that for \((g_{n})_{n=0}^{\infty} \in (E_{k})^{c}\), there exists \(D\), open disk such that \(g_{k}\) is holomorphically extendable on \(D\) and \(D \cap K \neq \emptyset, D \cap K^{c} \neq \emptyset\). If we pick a smaller disk \(D' \subset D\), such that \(D' \cap K \neq \emptyset, D' \cap K^{c} \neq \emptyset\), then the extension on \(D'\) is also bounded. Thus \((E_{k})^{c} = \bigcup_{n \in N} \{(g_{n})_{n=0}^{\infty} \in B(K)\}, such that there exists an open disk \(D\), with \(D \cup K \neq \emptyset, D \cap K^{c} \neq \emptyset\) and a holomorphic function \(F\) on \(D\), such that \(F|_{D \cap K} = g_{k}|_{D \cap K}\) and \(|F(z)| \leq n\) for \(z \in D\) for some natural number \(n\). It is known that disks with rational centers and rational ratios form a subbase of the topology of \(\mathbb{C}\).

Assume that \(r_{1}, r_{2}, ...\) is an enumeration of the rational numbers and define \(S = \{(k_{1}, k_{2}, k_{3})\}, such that
$r_{k_3} > 0$ and $D \cap K \neq \emptyset$,$D \cap K^c \neq \emptyset$ where $D = D(r_{k_1} + ir_{k_2}, r_{k_3})$ is the ball with center $r_{k_1} + ir_{k_2}$ and ratio $r_{k_3} > 0$. Then:

$$(E_k)^c = \bigcup_{n \in \mathbb{N}, (k_1, k_2, k_3) \in S} L_{n,k_1,k_2,k_3}$$

where $L_{n,k_1,k_2,k_3} = \{(g_s)_{s=0}^{\infty} \in B(K), \text{such that there exists a holomorphic function } F \text{ on } D = D(r_{k_1} + ir_{k_2}, r_{k_3}), \text{such that } F_{\mid D \cap K} = g_k_{\mid D \cap K} \text{ and } |F(z)| \leq n \text{ for } z \in D\}$. By Baire’s Theorem we need to show that $L_{n,k_1,k_2,k_3}$ is closed and nowhere dense.

1. It is closed, since if we have $g^1, g^2, g^3, \ldots \in L_{n,k_1,k_2,k_3}$ with $g^n \to g$ then $g_k^n \to g_k$ uniformly on $K$, as $n \to \infty$, and thus $F_n \to g_k$ uniformly on $K$, as $n \to \infty$, where $F_n$ are the extensions of $g_k^n$ on $D$. Since $|F_n(z)| \leq n$ on $D$, by Montel’s Theorem we get a function $F \in H(D)$ such that $F_n \to F$ uniformly on compact subsets of $D$ for some sequence $s_t$. Thus $F_{\mid D \cap K} = g_k_{\mid D \cap K}$ and $|F(z)| \leq n$ for $z \in D$. Therefore $g \in L_{n,k_1,k_2,k_3}$.

2. Finally, $L_{n,k_1,k_2,k_3}$ is nowhere dense, since taking $\varepsilon > 0$, we find $z_0 \in D \cap K^c$ and define $h(z) = \frac{M}{z - z_0}$, where $M$ is small enough such that $d(h,0) < \varepsilon$. Then, if $g \in L_{n,k_1,k_2,k_3}$, then $d(g + h, g) < \varepsilon$ but $g + h \notin L_{n,k_1,k_2,k_3}$, because otherwise $h = (h + g) - g \in L_{2n,k_1,k_2,k_3}$, which is not the case. \(\square\)

4 B(K) in Countable Sets

In the last paragraph, we saw various examples of sets of the form $B(K)$, where $K$ is a planar compact set. Notice that in these examples there appears to be a strong correlation between $g_k$ and $g_{k+1}$, where $(g_k)_{k=0}^{\infty}$ is a sequence in $B(K)$. In this paragraph, we show that this is not always the case. To show our result we will prove a more general theorem, which also holds for non-compact countable sets:

**Theorem 4.1** Let $X$ denote a countable planar set and $f_0, f_1, f_2, \ldots$ be a sequence of continuous functions on $X$. It is possible to find functions $h_0, h_1, h_2, \ldots$ such that for $i \in \mathbb{N}, h_i$ is defined in an open neighborhood of $X$, $h_i$ is locally a polynomial and for $j \in \mathbb{N}$

$$\limsup_{i \to +\infty, z \in X} |h^j_i(z) - f_j(z)| \to 0$$

**Proof.** Let $X = \{x_0, x_1, x_2, x_3, \ldots \}$ and let $k \in \mathbb{N}$. For $i = 0, 1, 2, \ldots, k$ we can find an open disk $D_i$ of center $x_i$ and ratio $\delta_i$ such that:

1. The disks $D_i$ are disjoint.

2. There is no point of $X$ lying on the circumference of $D_i$.

3. $\text{diam}(f_j(D_i \cap X)) < \frac{1}{2k}$ for $j = 0, 1, 2, 3, \ldots, k$.  

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4. If \( p_{i,k} \) denotes the unique polynomial of degree \( k \) with derivative of order \( s \) at \( x_i : f_s(x_i) \) for \( s \leq k \), then \( \text{diam}(p_{i,k}^s(D_i)) < \frac{1}{2k} \) for all \( i \leq k \) and \( s \leq k \).

Define now the function \( h_k \) as follows:

\[
h_k = p_{i,k} \text{ on } D_i \text{ for every } i \leq k.
\]

In case \( X \subset \bigcup_{i=0}^{k} D_i \), \( h_k \) is well defined on an open neighborhood of \( X \). If not, pick the least possible \( t \) such that \( x_t \not\in \bigcup_{i=0}^{k} D_i \). Then define a disk \( D_{k+1} \) of center \( x_t \) and ratio \( \delta_{k+1} \), disjoint from the already defined disks \( D_i, i \leq k \) such that:

1. There is no point of \( X \) lying on the circumference of \( D_{k+1} \)
2. \( \text{diam}(f_j(D_{k+1} \cap X)) < \frac{1}{2k} \) for \( j = 0, 1, 2, 3, ..., k \).
3. \( \text{diam}(p_{i,k}^s(D_{k+1})) < \frac{1}{2k} \) for all \( s \leq k \).

Define \( h_k = p_{t,k} \) on \( D_{k+1} \). In case \( X \subset \bigcup_{i=0}^{k+1} D_i \), \( h_k \) is well defined on an open neighborhood of \( X \). If not, once more pick the least possible \( t \) such that \( x_t \not\in \bigcup_{i=0}^{k+1} D_i \) and continue this process inductively.

We now prove that \( h_k^s \to f_s \) uniformly on \( X \) for fixed \( s \in \mathbb{N} \). Indeed for \( k > s \):

\[
\sup_{z \in X} |h_k^s(z) - f_s(z)| = \sup_{i \in N} |h_k^s(x_i) - f_s(x_i)|
\]

1. For \( i \leq k \) we have that \( h_k^s(x_i) = f_s(x_i) \)
2. For \( i > k \) \( x_i \) belongs to some disk \( D \) of center \( x_p \) having the properties listed above. Therefore:

\[
|h_k^s(x_i) - f_s(x_i)| \leq |h_k^s(x_i) - h_k^s(x_p)| + |h_k^s(x_p) - f_s(x_p)| + |f_s(x_p) - f_s(x_i)|
\]

\[
\leq \text{diam}(h_k^s(D)) + \text{diam}(f_s(D \cap X)) \leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}
\]

We thus have that:

**Corollary 4.2** If \( K \) is a countable compact planar set, then \( \overline{P} = B(K) = [C(K)]^{\aleph_0} \), where \( P \) denotes the set of polynomials and its closure is taken in the topology of \( B(K) \).

**Proof.** The fact that \( B(K) = [C(K)]^{\aleph_0} \) is immediate from Theorem 4.1. The part that \( \overline{P} = B(K) \) follows from Proposition 2.6 , since \( \mathbb{C}_\infty - K \) is connected. \( \square \)

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