Radial propagation in population dynamics with density-dependent diffusion

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(Dated: May 13, 2013)

We investigate the spatiotemporal pattern formation in a population dynamics in which the system is modeled by a density-dependent reaction-diffusion equation, called the generalized Fisher equation. A new class of radial symmetric solution in two- and three-dimensional space has been presented. These analytical solutions indicate the self-similar evolution of population density that propagates as traveling wave for the large time scale. In addition, we found that the wavefront does not linearly depend on time as in one-dimensional space. However, its behaviors are approximately the same for all dimensions at the large length scale.

PACS numbers: 87.23.Cc, 82.40.Ck, 87.18.Hf, 05.45.-a

The growth and dispersal of species in populations generate attractive spatiotemporal pattern formations. This phenomenon becomes an active research topic for many decades. In theoretical framework, the dynamics of population can be modeled as diffusion with reaction processes. The paradigmatic model is known as the Fisher equation, which has been originated as a model for the spread of an advantageous gene in a population. This equation is attractive because its solution has exemplified the propagating wave in reaction-diffusion system. The similar type of this equation has also appeared in various systems, including combustion theory, flow in porous media, chemical kinetics, nerve pulse propagation and bacterial colony pattern formation.

In the Fisher model, the evolution of population density $u(r,t)$ at spatial position $r$ and time $t$ is governed by the simplest nonlinear reaction-diffusion equation:

$$
\frac{\partial u}{\partial t} = \nabla \cdot \text{diffusion term} + \text{tic growth term} \quad \text{where} \quad u_m > 0.
$$

This equation is attractive because its solution has exemplified the propagating wave in reaction-diffusion system, which has been originated as a model for the spread of an advantageous gene in a population. The reaction term is modeled as logistic law and it describes the growth of population with a limited supply. The movement of individual is usually modeled as random walk where the diffusion coefficient is constant. However, the motion of the biological population is not exactly random but they move with sense. To remedy, the directed motion model has been proposed, which individuals tend to move in the direction of decreasing populations. The diffusion coefficient in this model is dependent on the population density. Later, a general form of the logistic law has been also proposed. With these modifications, the generalized Fisher equation (in dimensionless form) is arisen,

$$
\frac{\partial u}{\partial t} = \nabla^2 u^m + u - u^m, \quad (1)
$$

where $m > 1$. Eq. (1) contains the generalized logistic growth term $u(1-u^{m-1})$ and the density-dependent diffusion term $\nabla \cdot (mu^{m-1}\nabla u)$. The solution of Eq. (1) in one-dimensional space has been known, however its behavior in higher dimensions has not well understood. Typically, the population dynamics takes place in two dimensions, sometimes in three dimensions. It has been mentioned that the effects of curvature can make the front propagation somewhat different from the planar case. Therefore, the solution of Eq. (1) in more than one dimension is of great interest. It could provide better insight into the pattern formation dynamics in populations.

Recently, a new technique has been developed to solve for the analytical solutions of Eq. (1) in one-dimensional space. The solutions show that the population density evolves, from a certain initial condition, as a self-similar pattern that converges to the traveling wave at large time scale. In this paper, we extend that technique to find for the solution of Eq. (1) in two and three dimensions. We describe the procedures as follow.

The linear growth term in Eq. (1) can be eliminated by introducing the transformations

$$
u(r,t) = e^t f(r,t), \quad \tau(t) = \frac{1}{m-1} \left(e^{(m-1)t} - 1\right),
$$

where $f$ and $\tau$ are new variables. These transformations reduce Eq. (1) to

$$
\frac{\partial f}{\partial \tau} = \nabla^2 f^m - f^m. \quad (4)
$$

Eq. (4) is a nonlinear partial differential equation, thus it can not be solved simply by the separation of variables method. Fortunately, Eq. (1) can be reduced further by looking its solution in the following form

$$
f(r, \tau) = \phi^\pm(r) w(r, \tau), \quad (5)
$$

where $\phi(r)$ is purely spatial solution and $w(r, \tau)$ is an inseparable solution. Substituting Eq. (5) into Eq. (1), we have $\phi^\pm \frac{\partial w}{\partial \tau} = \phi \nabla^2 w^m + 2\nabla \phi \cdot \nabla w^m + (\nabla^2 \phi - \phi) w^m$. By choosing

$$
\nabla^2 \phi - \phi = 0, \quad (6)
$$

we obtain

$$
\frac{\partial w}{\partial \tau} = \phi^{1-\pm} \nabla^2 w^m + 2\phi^{-\pm} \nabla \phi \cdot \nabla w^m. \quad (7)
$$

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We now seek for the solution in the radial symmetric form, where the Laplacian and gradient operator, respectively, are given by $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\gamma}{r} \frac{\partial}{\partial r}$ and $\nabla = \mathbf{\hat{r}} \frac{\partial}{\partial r}$, where $r = |\mathbf{r}|$, $\gamma (\equiv 0, 1, \text{and} \ 2)$ is corresponded to dimensions $\gamma + 1$. In this geometry, the solution for Eq. (6) is known as the modified Bessel function \[ I_n(r) = \begin{cases} I_0(r) & \text{for 2D} \\ i_0(r) & \text{for 3D}, \end{cases} \tag{8} \]
where $I_n(r)$ and $i_n(r) = \sqrt{\frac{\pi}{2r}} I_{n+\frac{1}{2}}(r)$ are the modified Bessel function and the modified spherical Bessel function of order $n$, respectively. For large value of $r \gg 1$, the asymptotic form of the Bessel functions are \[ I_n(r) \sim \frac{\gamma}{\sqrt{2\pi r}}, \tag{9} \]
where $\gamma = \nabla_{n+\frac{1}{2}}$.

In these forms, we can show (for $r \gg 1$) that \[ \frac{\partial \phi}{\partial r} \approx \phi. \tag{11} \]

Applying the transformation $\frac{d\phi}{\partial r} = \frac{\partial \phi}{\partial \chi} \frac{\partial \chi}{\partial r}$ to some term in Eq. (7), we rewrite it as \[ \frac{\partial w}{\partial r} = \phi^{-1} \left( \frac{\partial \phi}{\partial \chi} \right)^2 \frac{\partial^2 \phi}{\partial \chi^2} + 2 \phi^{-1} \frac{\partial \phi}{\partial \chi} \frac{\partial^2 \phi}{\partial \chi^2} + 2 \phi^{-\frac{1}{2}} \frac{\partial^2 \phi}{\partial \chi^2} = \phi^{-2}. \tag{12} \]
which can be evaluated further to \[ \frac{\partial \phi}{\partial r} \approx \phi^{-3}. \tag{11} \]

Using Eq. (11), we calculate $\frac{\partial \phi}{\partial \chi} = \frac{\partial \chi}{\partial \phi}$, which is reduced further to \[ \frac{\partial w}{\partial r} = \frac{\partial}{\partial \xi} \left( \phi^{3+\frac{1}{2}} \frac{\partial w^m}{\partial \xi} \right), \tag{14} \]
by the transformation $\phi^{3+\frac{1}{2}} d\chi = d\xi$, where $\xi$ is new variable. Using Eq. (11), we obtain $\frac{\partial \phi}{\partial \xi} = \phi^{-3}$. This leads previous transformation to $\phi^{3+\frac{1}{2}} d\phi = d\xi$, which can be evaluated to \[ \xi = \frac{m}{m+1} \phi^{-\frac{m}{m+1}}. \tag{13} \]

With Eq. (13) and Eq. (12), we obtain the reduced form of Eq. (7) as \[ \frac{\partial w}{\partial r} = \frac{\partial}{\partial \xi} \left( k \xi \frac{\partial w^m}{\partial \xi} \right), \tag{14} \]
where $k = \left( \frac{m+1}{m} \right)^l$ and $l = \frac{3m+4}{4m+4}$. Eq. (14) is known as the anomalous diffusion equation, whose solution is assumed to be the scaling function $w(\xi, \tau) = \frac{1}{T(T)} F \left( \frac{\xi}{\tau} \right)$.

The evolution of population density obtained from Eq. (10) is demonstrated in Fig. (1). Illuminating from this, the population density evolves in the radial symmetric profile. The density initiates from a profile $u_0(r) = u(r, 0)$ then they grow locally to the saturated value, while invading to unoccupied region. At sufficient large time $t \gg t'$, that $e^{\rho t'} \gg 1$ and $\rho e^{\rho t'} \gg 1$ thus $t' \approx -\ln \rho$, \[ F(x; \tau) \rightarrow \begin{cases} 1 & \text{for } x < 0 \\ \frac{1}{\rho} & \text{for } x > 0 \end{cases}. \tag{15} \]

Substituting Eq. (15) back into main solution Eq. (2), with Eq. (7), Eq. (3), Eq. (9) and Eq. (4), we obtain the initial condition \[ u_0(r) = u(r, 0) = (a')^{-1/p} \left[ 1 + b' \left( r - \frac{r_0}{p} \right) \right]^{1/p}, \tag{16} \] where $a' = (m - 1)$, $b' = b/(m-1)$, $p = 3$ with initial conditions $\rho = 0.2$ and $r_0 = 4$. The solid lines represent solutions in 2D ($\gamma = 1$) and the dashed lines represent solutions in 3D ($\gamma = 2$).
the solution (16) emerges a pattern form of the traveling wave

$$\tilde{u}(r' - t) = \left\{ 1 - \left[ \rho^{-1} e^{c(r' - t - \frac{\gamma}{2} \ln(r/r_0) - t - r_0)} \right]^{\frac{1}{p-1}} \right\}^{\frac{1}{p}} \quad (17)$$

where $r' = r - \frac{\gamma}{2} \ln(r/r_0) - r_0$. The wavefront position $R(t)$ can be tracked by determining $\tilde{u}(R, t) = 0$. Then we have

$$R - \frac{\gamma}{2} \ln \frac{R}{r_0} - r_0 = t - t'.$$

(18)

In Eq. (18), we see that the wavefront position does not linearly depend on time as in 1D case [13]. When we calculate the front speed $c = \frac{dR}{dt}$, we have

$$c = \frac{R}{R - \frac{\gamma}{2}}. \quad (19)$$

Once again, we found that the front speed depends on the front position which differs from in 1D spatial system [6, 7, 12, 13]. However, at sufficient large distance that $R \gg \gamma$, the front speed can be approximated as a constant $c \approx 1$. This is equal to the front speed in 1D case ($\gamma = 0$) [6, 7, 12, 13]. In the terms of transformed density $w(\phi, \tau)$ as a function of transformed space $\phi$ and time $\tau$, the population evolution in the radial form still hold the self-similarity with the scaling law

$$w = \frac{1}{\tau^\beta} F \left( \frac{\phi}{\tau^\beta} \right),$$

where $\beta = \frac{m+1}{m(m-1)}$ as in 1D [13].

In summary, we study the spatiotemporal pattern formation in population dynamics that is governed by the generalized Fisher equation in two- and three dimensional space. We have found a new class of solution in the radial symmetric form. These analytical results show that the evolution of population density is self-similar. At large time scale, the population density propagates as the traveling wave. Thought the density profiles are not identical, the propagating speed in 2D and 3D for large distance behave like in 1D.

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