EXISTENCE OF ABSOLUTELY CONTINUOUS SOLUTIONS FOR CONTINUITY EQUATIONS IN HILBERT SPACES

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Abstract. We prove existence of solutions to continuity equations in a separable Hilbert space. We look for solutions which are absolutely continuous with respect to a reference measure $\gamma$ which is Fomin–differentiable with exponentially integrable partial logarithmic derivatives.

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1. Introduction

We are given a separable Hilbert space $H$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), a Borel vector field $F : [0,T] \times H \to H$ and a Borel probability measure $\zeta$ on $H$. We are concerned with the following continuity equation,

$$
\int_0^T \int_H \left[ D_t u(t,x) + \langle D_x u(t,x), F(t,x) \rangle \right] \nu_t(dx) dt = - \int_H u(0,x) \zeta(dx), \quad \forall u \in \mathcal{F}C^1_{b,T},
$$

(1.1)

where the unknown $\nu = (\nu_t)_{t \in [0,T]}$ is a probability kernel such that $\nu_0 = \zeta$. Moreover, $D_x$ represents the gradient operator and $\mathcal{F}C^1_{b,T}$ is defined as follows: let $\mathcal{F}C^k_b$, for $k \in \mathbb{N} \cup \{\infty\}$, denote the set of all functions $f : H \to \mathbb{R}$ of the form

$$
f(x) = \tilde{f}(\langle h_1, x \rangle, \ldots, \langle h_N, x \rangle), \quad x \in H,
$$

where $N \in \mathbb{N}$, $\tilde{f} \in C^k_b(\mathbb{R}^N)$ and $h_1, \ldots, h_N \in Y$, where $Y$ is a dense linear subspace of $H$ to be specified later. Then $\mathcal{F}C^k_{b,T}$ is defined to be the $\mathbb{R}$–linear span of all functions $u : [0,T] \times H \to \mathbb{R}$ of the form

$$
u(t,x) = g(t)f(x), \quad (t,x) \in [0,T] \times H,
$$

where $g \in \mathcal{F}C^k_b([0,T])$ and $f \in \mathcal{F}C^k_b(\mathbb{R}^N)$. Theorem 1.2.1 (existence) states that if $H = L^2(\Gamma, \mu)$ is a Hilbert space, then $\mathcal{F}C^1_{b,T} \subset C_b(\Gamma, \mu)$.

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where \( g \in C^1([0, T]; \mathbb{R}) \) with \( g(T) = 0 \) and \( f \in \mathcal{F}_b^k \). Correspondingly, let \( \mathcal{V} \mathcal{F}_b^k \) be the set of all maps \( G : [0, T] \times H \to H \) of the form

\[
G(t, x) = \sum_{i=1}^{N} u_i(t, x) h_i, \quad (t, x) \in [0, T] \times H,
\]

where \( N \in \mathbb{N} \), \( u_1, \ldots, u_N \in \mathcal{F}_b^k \), and \( h_1, \ldots, h_N \in Y \). Clearly, \( \mathcal{F}_b^\infty \) is dense in \( L^p([0, T] \times H, \nu) \) for all finite Borel measures \( \nu \) on \([0, T] \times H \) and all \( p \in [1, \infty) \). \( \mathcal{F}_b^k \) denotes the set of all \( G \) as in (1.2) with \( u_i \in \mathcal{F}_b^k \) replaced by \( u_i \in \mathcal{F}_b^k \).

It is well known that problem (1.1) in general admits several solutions even when \( H \) is finite dimensional. So, it is natural to look for well posedness of (1.1) within the special class of measures \( (\nu_t)_{t \in [0,T]} \) which are absolutely continuous with respect to a given reference measure \( \gamma \). In this case, denoting by \( \rho(t, \cdot) \) the density of \( \nu_t \) with respect to \( \gamma \),

\[
\nu_t(dx) = \rho(t, x) \gamma(dx), \quad t \in [0, T],
\]

equation (1.1) becomes

\[
\int_0^T \int_H [D_t u(t, x) + \langle D_x u(t, x), F(t, x) \rangle] \rho(t, x) \gamma(dx) dt = -\int_H u(0, x) \rho_0(x) \gamma(dx), \quad \forall \ u \in \mathcal{F}_b^1.
\]

Here \( \rho_0 := \rho(0, \cdot) \) is given and \( \rho(t, \cdot), \ t \in [0, T], \) is the unknown.

In this paper we concentrate on existence of solutions to (1.3). Corresponding uniqueness results under somewhat more stringent conditions are in preparation.

Our basic assumption on \( \gamma \) is the following

**Hypothesis 1.** \( \gamma \) is a nonnegative measure on \((H, \mathcal{B}(H))\) with \( \gamma(H) < \infty \) such that there exists a dense linear subspace \( Y \subset T \) having the following properties:

- For all \( h \in Y \) there exists \( \beta_h : H \to \mathbb{R} \) Borel measurable such that for some \( c_h > 0 \)
  \[
  \int_H e^{c_h |\beta_h|} d\gamma < \infty
  \]
  and
  \[
  \int_H \partial_h u d\gamma = -\int_H u \beta_h d\gamma,
  \]
  where \( \partial_h u \) denotes the partial derivative of \( u \) in the direction \( h \).

Assume from now on that \( \gamma \) satisfies Hypothesis 1.

**Remark 1.1.** It is well known that the operator \( D_x = \text{Fréchet–derivative with domain } \mathcal{F}_b^1 \) is closable in \( L^p(H, \gamma) \) for all \( p \in [1, \infty) \), see e.g. [AlRoe90]. Its closure will again be denoted by \( D_x \) and its domain will be denoted by \( W^{1,p}(H, \gamma) \).

Let \( D_x^* : \text{dom}(D_x^*) \subset L^2(H, \gamma; H) \to L^2(H, \gamma) \) denote the adjoint of \( D_x \).

**Lemma 1.2.** \( \mathcal{V} \mathcal{F}_b^1 \subset \text{dom}(D_x^*) \) and for \( G \in \mathcal{V} \mathcal{F}_b^1 \), \( G = \sum_{i=1}^{N} u_i h_i \) we have

\[
D_x^* G = -\sum_{i=1}^{N} (\partial_h u_i + \beta_h u_i).
\]
Proof. For $v \in \mathcal{F}C^1_b$ we have

$$\int_H \langle D_x v, G \rangle_H d\gamma = \sum_{i=1}^N \int_H \partial_{h_i} v u_i d\gamma$$

$$= \sum_{i=1}^N \int_H \partial_{h_i} (v u_i) d\gamma - \sum_{i=1}^N \int_H v \partial_{h_i} u_i d\gamma$$

$$= - \int_H v \sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i) d\gamma.$$

We stress that if $H$ is infinite dimensional, $\beta_h$ is typically not bounded and not continuous. Here are some examples. For $G$ as in Lemma 1.2, below we sometimes use the notation $\text{div } G := \sum_{i=1}^N \partial_{h_i} u_i$.

Example 1.3. (i) Let $Q$ be a symmetric positive defined operator of trace class on $H$ and $\gamma := N(0, Q)$, i.e. the centered Gaussian measure on $H$ with covariance operator $Q$. Assume that $\ker Q = \{0\}$ and let $Y$ be the linear span of all eigenvectors of $Q$. Then Hypothesis 1 is fulfilled with this $Y$ and for $h \in Y$, $h = c_1 h_1 + \cdots + c_N h_N$ with $Qh_i = \lambda_i^{-1} h_i$, we have

$$\beta_h(x) = - \sum_{i=1}^N c_i \lambda_i \langle h_i, x \rangle_H, \quad x \in H.$$  

This, in particular, covers the case studied in [DaPlRoe14], where only uniqueness of solutions to (1.3) was studied.

(ii) Let $H := L^2((0,1), dx)$ and $A := -\Delta$ with zero boundary conditions. Define

$$\gamma(dx) := \frac{1}{Z} e^{-\frac{1}{2} \int_0^1 |x(\xi)|^4 d\xi} N(0, -\frac{1}{2} A^{-1})(dx),$$

where

$$Z := \int_H e^{-\frac{1}{2} \int_0^1 |x(\xi)|^4 d\xi} N(0, -\frac{1}{2} A^{-1})(dx).$$

Then with $Y$ as in (i) for $Q = -\frac{1}{2} A^{-1}$ we find for $h = c_1 h_1 + \cdots + c_N h_N$ as in (i)

$$\beta_h(x) = - \sum_{i=1}^N c_i \left( \lambda_i \langle h_i, x \rangle_H - \int_0^1 h_i(\xi) x(\xi)^3 d\xi \right), \quad \text{for } N(0, -\frac{1}{2} A^{-1})-a.e. \ x \in H$$

and obviously also the exponential integrability condition holds in Hypothesis 1.

(iii) Let $H$ and $A$ be as in (ii) and let $\gamma$ be the invariant measure of the solution to

$$\left\{ \begin{array}{l} dX(t) = [AX(t) + p(X(t))] dt + BdW(t), \\
X(0) = x, \quad x \in H, \end{array} \right.$$

where $p$ is a decreasing polynomial of odd degree equal to $N > 1$, $B \in L(H)$ with a bounded inverse and $W$ is an $H$–valued cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ (see [DaDe17]). Then it was proved in [DaDe17, Proposition 3.5] that Hypothesis 1 holds with
\( Y := D(A) \), where \( A \) is as in (ii) above except that each \( \beta_h \) was only proved to be \( L^p(L^2(0, 1), \gamma) \) for every \( p \geq 1 \). More precisely, it was proved (see [DaDe17, eq. (3.17)]) that for all \( h \in D(A) \)

\[
\left( \int_{L^2(0,1)} |\beta_h|^p \, d\gamma \right)^{\frac{1}{p}} \leq C_p |Ah|, \quad \forall \, p \geq 2,
\]

where \( C_p \) is the constant of the Burkholder–Davis–Gundy inequality for \( p \geq 2 \) which (when proved by Itô’s formula) can easily be seen to be smaller than \( 12p \) if \( p \geq 4 \). For the reader’s convenience we include a proof in Appendix B below. Hence, because for all \( n \in \mathbb{N} \) by Stirling’s formula

\[
\left( \frac{1}{n!} 12^n n^n \right)^{\frac{1}{n}} \leq 12n \left( \frac{1}{\sqrt{2\pi}} n^{-\frac{1}{2}} e^n \right)^{\frac{1}{n}} = 12e \left( \frac{1}{\sqrt{2\pi}} \right)^{\frac{1}{n}} e^{-\frac{1}{2} \ln n} \to 12e \quad \text{as } n \to \infty,
\]

we have for all \( \epsilon \in (0, (12e|Ah|)^{-1}), h \in D(A) \setminus \{0\} \),

\[
\int_{L^2(0,1)} e^{\epsilon |\beta_h|} \, d\gamma \leq \sum_{n=0}^{\infty} \frac{1}{n!} e^n 12^n n^n |Ah|^n < \infty.
\]

So, for any \( c_H \in (0, (12e|Ah|)^{-1}) \), exponential integrability holds for \( |\beta_h| \) and Hypothesis 1 is satisfied.

Concerning \( F \) in (1.1) we assume:

**Hypothesis 2.**

1. \( F : [0, T] \times H \to H \) is Borel measurable and bounded.
2. There exist \( F_j \in \mathcal{F}C_{b,T}^2, j \in \mathbb{N} \), such that each \( F_j \) satisfies (2.8) below and

\[
\begin{align*}
\lim_{j \to \infty} F_j &= F \quad \text{dt} \otimes \gamma\text{-a.e.} \\
\sup_{j \in \mathbb{N}} \|F_j\|_\infty &< \infty, \\
M := \sup_{j \in \mathbb{N}} C_{F_j} &< \infty,
\end{align*}
\]

where \( C_{F_j} \) is defined in Lemma 2.4 below.

**Definition 1.4.** Let \( \rho_0 \in L^1(H, \gamma) \). A solution of the continuity equation (1.3) is a function \( \rho \in L^1(0,T; L^1(H, \gamma)) \) such that \( \rho(0, \cdot) = \rho_0 \) and (1.3) is fulfilled.

If \( \rho_0 \ln \rho_0 \in L^1(H, \gamma) \), in Section 2, we shall prove existence of a solution of (1.3) by introducing the following approximating equation, where \( F \) is replaced by \( (F_j) \) (fulfilling Hypothesis 2) and \( \rho_0 \) by \( \rho_{j,0} \), where \( (\rho_{j,0}) \) is a sequence in \( \mathcal{F}C_{b}^1 \), converging to \( \rho_0 \) in \( L^1(H, \gamma) \):

\[
\begin{align*}
\int_0^T \int_H [D_t u(t,x) + \langle D_x u(t,x), F_j(t,x) \rangle] \, \rho_j(t,x) \, \gamma(dx) \, dt \\
&= -\int_H u(0,x) \rho_{j,0}(x) \, \gamma(dx), \quad \forall \, u \in \mathcal{F}C_{b,T}^1,
\end{align*}
\]

which has a solution \( \rho_j \) since \( F_j \) is regular. Then we shall show that a subsequence of \( (\rho_j) \) converges weakly to a solution of (1.3).

To our knowledge, earliest existence (and uniqueness) results for equation (1.3) concern the case where \( H \) is finite dimensional and the reference measure is the Lebesgue measure, see the seminal papers [DiLi89] and [Am04]. If \( H \) is infinite dimensional and \( \gamma \) is a Gaussian measure problem (1.1) has been studied in [AmFi10], [PaLu10], [DaFlRoe14] and [KoRoe14]. A very general approach in
metric spaces has been presented in [AmTr14]. Our assumptions for getting existence of solutions are, however, weaker than the corresponding ones in these papers.

We finish this section with some notations and preliminaries. \( \mathcal{B}(H) \) denotes the set of all Borel subsets and \( \mathcal{P}(H) \) the set of all Borel probabilities on \( H \). A probability kernel in \([0, T]\) is a mapping

\[
[0, T] \to \mathcal{P}(H), \quad t \mapsto \mu_t,
\]

such that the mapping \([0, T] \to \mathbb{R}, \ t \mapsto \mu_t(I)\) is measurable for any \( I \in \mathcal{B}(H) \). \( L(H) \) is the set of all linear bounded operators in \( H, C_b(H), C_b(H; H) \) the space of all real continuous and bounded mappings \( \varphi: H \to \mathbb{R} \) and \( \varphi: H \to H \) respectively, endowed with the sup norm

\[
\|\varphi\|_{\infty} = \sup_{x \in H} |\varphi(x)|,
\]

whereas \( C_b^k(H), k > 1 \), will denote the space of all real functions which are continuous and bounded together with their derivatives of order less or equal to \( k \). \( B_b(H) \) will represent the space of all real, bounded and Borel mappings on \( H \). Moreover, we shall denote by \( \|\cdot\|_p \) the norm in \( L^p(H, \gamma) \), \( p \in [1, \infty] \). For any \( x, y \in H \) we denote either by \( \langle x, y \rangle \) or by \( x \cdot y \) the scalar product between \( x \) and \( y \). Finally, if \( (e_h) \) is an orthonormal basis in \( H \) we set \( x_h = \langle x, e_h \rangle \) for all \( x \in H \) and \( G_h = \langle G, e_h \rangle \), \( h \in \mathbb{N} \), for all \( G \in L^2(H, \nu; H) \). Finally, we state a lemma, needed in what follows, whose straightforward proof is left to the reader.

**Lemma 1.5.** Assume, besides Hypothesis [1], that \( F \in \text{dom} (D^*_x) \) and \( \varphi \in C_b^1(H) \). Then \( \varphi F \in \text{dom} (D^*_x) \) and we have

\[
D^*_x(\varphi F) = \varphi D^*_x(F) - \langle D_x \varphi, F \rangle.
\]

2. **The main result**

First we notice that if \( F \in \text{dom} (D^*_x) \) then a regular solution \( \rho \) to (1.3) solves the equation

\[
\begin{cases}
D_t \rho + \langle F, D_x \rho \rangle - D^*_xF \rho = 0, \\
\rho(0, \cdot) = \rho_0,
\end{cases}
\]

and vice versa. In fact, since for all \( u \in \mathcal{FC}^1_{b,T} \)

\[
\int_0^T D_t u(t, x) \rho(t, x) \; dt = - \int_0^T u(t, x) D_t \rho(t, x) \; dt - u(0, x) \rho_0(x), \quad x \in H,
\]

and (thanks to Lemma [1.3])

\[
\int_H \langle D_x u(t, x), F(t, x) \rangle \rho(t, x) \gamma(dx) = \int_H \langle D_x u(t, x), \rho(t, x) F(t, x) \rangle \gamma(dx)
\]

\[
= \int_H u(t, x) D^*_x(\rho F)(t, x) \gamma(dx) = \int_H u(t, x) \rho(t, x) D^*_x F(t, x) \gamma(dx)
\]

\[
- \int_H u(t, x) \langle D_x \rho(t, x), F(t, x) \rangle \gamma(dx).
\]

Clearly (2.2) and (2.3) imply that (1.3) is equivalent to

\[
\begin{cases}
\int_0^T \int_H u(t, x) [-D_t \rho(t, x) + D^*_x F(t, x) \rho(t, x) - \langle D_x \rho(t, x), F(t, x) \rangle] \gamma(dx) \; dt = 0, \\
\rho(0, \cdot) = \rho_0,
\end{cases}
\]

for all \( u \in \mathcal{FC}^1_{b,T} \). By the density of \( \mathcal{FC}^1_{b,T} \) in \( L^2([0, T] \times H, dt \otimes d\gamma) \) we obtain (2.1).
Theorem 2.1. Assume that Hypotheses 1 and 2 hold. Let \( \zeta := \rho_0 \cdot \gamma \) be a probability measure on \((H, \mathcal{B}(H))\) such that
\[
\int_H \rho_0 \ln \rho_0 \, d\gamma < \infty. \tag{2.5}
\]
Then there exists \( \rho : [0, T] \times H \to \mathbb{R}_+, \mathcal{B}([0, T] \times H)\)-measurable such that \( \nu_t(dx) = \rho(t, x) \gamma(dx) \), \( t \in [0, T] \), are probability measures on \((H, \mathcal{B}(H))\) such that (1.1) (equivalently (1.3)) holds. In addition,
\[
\int_0^T \int_H \rho(t, x) \ln \rho(t, x) \gamma(dx) \, dt < \infty. \tag{2.6}
\]

Proof. By disintegration we shall reduce the proof to the case \( H = \mathbb{R}^N \) and by regularization to Corollary A.2 in Appendix A.

Case 1. Suppose \( F \in \mathcal{VF}_{b,T}, \rho_0 \in \mathcal{FC}_{b,1}^1, \rho_0 \geq 0 \) such that condition (2.8) below holds.

In this case we can find an orthonormal basis \( \{e_i : i \in \mathbb{N}\} \) of \( H \) which consists of elements in \( Y \) such that for some \( N \in \mathbb{N} \) (which we fix below)
\[
F(t, x) = \sum_{i=1}^N g_i(t) f_i(x) e_i, \quad (t, x) \in [0, T] \times H, \tag{2.7}
\]
where for \( 1 \leq i \leq N \), \( g_i(t) \in C^1([0, T]; \mathbb{R}) \) with \( g_i(T) = 0 \) and \( f_i \in \mathcal{FC}_b^2 \) such that for \( x \in H \)
\[
f_i(x) = \tilde{f}_i(\langle e_1, x \rangle, \ldots, \langle e_N, x \rangle)
\]
and
\[
\rho_0(x) = \tilde{\rho}_0(\langle e_1, x \rangle, \ldots, \langle e_N, x \rangle)
\]
with \( \tilde{f}_i \in C_b^2(\mathbb{R}^N), \tilde{\rho}_0 \in C_b^1(\mathbb{R}^N) \). Assume that
\[
support \ of \ \tilde{f}_i \ is \ compact, \ \forall \ 1 \leq i \leq N. \tag{2.8}
\]
Define
\[
H_N := \text{lin span} \{e_1, \ldots, e_N\}
\]
and let \( \Pi_N : H \to E \) be the orthogonal projection onto \( E := H_N^\perp \), where \( H_N^\perp \) is the orthogonal complement of \( H_N \), i.e.
\[
H = H_N \oplus E \equiv \mathbb{R}^N \times E, \tag{2.9}
\]
hence, for \( z \in H, z = (x, y) \) with unique \( x \in \mathbb{R}^N, y \in E \).

Letting \( \nu := \gamma \circ \Pi_N^{-1} \) be the image measure on \((E, \mathcal{B}(E))\) of \( \gamma \) under \( \Pi_N^{-1} \). Then we have the following well known disintegration result for \( \gamma \):

Lemma 2.2. There exists \( \Psi : \mathbb{R}^N \times E \to [0, \infty), \mathcal{B}(\mathbb{R}^N \times E)\)-measurable such that
\[
\gamma(dz) = \gamma(dx \, dy) = \Psi^2(x, y)dx \, \nu(dy), \tag{2.10}
\]
where \( dx \) denotes Lebesgue measure on \( \mathbb{R}^N \).

Furthermore, for every \( y \in E \)
\[
\Psi(\cdot, y) \in H^{1,2}(\mathbb{R}^N, dx), \tag{2.11}
\]
i.e. the Sobolev space of order 1 in \( L^2(\mathbb{R}^N, dx) \).

Proof. See [AlRoeZh93, Proposition 4.1].
Now let us continue the proof of Theorem 2.1. We have by Hypothesis I that for all $1 \leq i \leq N$ there exists $c_i \in (0, \infty)$ such that
\[
\infty > \int_H e^{c_i |\beta_{i}|} d\gamma = \int_E \int_{\mathbb{R}^N} e^{c_i |\beta_{i}(x,y)|} \Psi^2(x,y) \, dx \, \nu(dy)
\]
\[
= \int_E \int_{\mathbb{R}^N} \exp \left[ c_i \left| \frac{\partial}{\partial x_i} \Psi^2(x,y) / \Psi^2(x,y) \right| \right] \Psi^2(x,y) \, dx \, \nu(dy),
\]
where we used that for $1 \leq i \leq N$
\[
\beta_{i}(x,y) = \frac{\partial}{\partial x_i} \Psi^2(x,y)/\Psi^2(x,y), \quad (x,y) \in \mathbb{R}^N \times E = H,
\]
which is an immediate consequence of the disintegration (2.10), and the right hand side of (2.12) is defined to be zero on $\{\Psi = 0\}$. Hence we can find $E_0 \subset \mathcal{B}(E)$ such that $\nu(E_0) = 1$ and
\[
\int_{\mathbb{R}^N} \exp \left[ c_i \left| \frac{\partial}{\partial x_i} \Psi^2(x,y) / \Psi^2(x,y) \right| \right] \Psi^2(x,y) \, dx < \infty \quad (2.13)
\]
for $y \in E_0$. Below we fix $y \in E_0$.

Define for $M, l \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^N \times E (= H)$
\[
\Psi_M(x,y) := \left( \Psi^2(x,y) \wedge M \right)^{1/2},
\]
\[
\Psi_{M,l}(x,y) := (\Psi_M^2 (\cdot,y) \ast \delta_l)^{1/2}(x), \quad (2.14)
\]
where $\delta_l(x) = l^N \eta(lx), x \in \mathbb{R}^N, \eta \in \mathcal{S}(\mathbb{R}^N) := \text{set of Schwartz test functions}, \eta > 0, \eta(x) = \eta(-x), x \in \mathbb{R}^N,$ and $\int_{\mathbb{R}^N} \eta \, dx = 1.$ We note that then clearly $\Psi_{M,l}(x,y) > 0$ for all $x \in \mathbb{R}^N$. Then by Corollary A.2 applied with the measure $\gamma_{M,l,y}(dx) := \Psi_{M,l}^2(x,y) \, dx$ replacing $\gamma(dx)$, we know that
\[
\rho_{M,l}(t,(x,y)) := \rho_0(\xi(T,T-t,x)) e^{\int_0^t \! D_{M,l}^* F(T-u,\xi(T-u,T-t,x),y) \, du}, \quad (x,y) \in [0,T] \times \mathbb{R}^N, \quad (2.15)
\]
where (see Lemma 1.2 and (2.7)
\[
D_{M,l}^* F(r,(x,y)) := -\sum_{i=1}^N g_i(r) \left[ \partial_{e_i} f_i(x) + f_i(x) \frac{\partial}{\partial x_i} \Psi_{M,l}(x,y)/\Psi_{M,l}^2(x,y) \right], \quad (2.16)
\]
r $\in [0,T], x \in \mathbb{R}^N$, solves
\[
\left\{ \begin{array}{l}
D_t \rho_{M,l}(t,(x,y)) + \langle F(t,x),D_x \rho_{M,l}(t,(x,y)) \rangle - D_{M,l}^* F(t,(x,y)) \rho_{M,l}(t,(x,y)) = 0, \\
\rho_{M,l}(0,(x,y)) = \rho_0(x).
\end{array} \right. \quad (2.17)
\]
We need a few further lemmas of which the first is the most crucial.

**Lemma 2.3.** Let $\epsilon > 0$. Then for all $1 \leq i \leq N, l, M \in \mathbb{N}$
\[
\int_{\mathbb{R}^N} \exp \left[ \epsilon \left| \left( \frac{\partial \Psi_{M,l}^2}{\partial x_i} / \Psi_{M,l}^2 \right)(x,y) \right| \right] \Psi_{M,l}^2(x,y) \, dx
\]
\[
\leq \int_{\mathbb{R}^N} \exp \left[ \epsilon \left| \left( \frac{\partial \Psi_M^2}{\partial x_i} / \Psi_M^2 \right)(x,y) \right| \right] \Psi_M^2(x,y) \, dx
\]
\[
\leq \int_{\mathbb{R}^N} \exp \left[ \epsilon |\beta_{e_i}(x,y)| \right] \Psi^2(x,y) \, dx. \quad (2.19)
\]
Proof. Obviously, the left hand side of (2.19) is dominated by
\[
\int_{\mathbb{R}^N} \exp \left[ \epsilon \int_{\mathbb{R}^N} \left( \frac{\partial \Psi_M^2}{\partial x_i} \right) \psi_M^2(x,y) \delta_i(x - \tilde{x}) \, d\tilde{x} \left( \psi_M^2(x,y) \right)^{-1} \right] \psi_M^2(x,y) \, dx,
\]
where we used that \( \frac{\partial \psi_M^2}{\partial x_i} = 0 \) dx–a.e. on \( \{ \psi_M^2 = 0 \} \).

Applying Jensen’s inequality for fixed \( x \in \mathbb{R}^N \) to the probability measure
\[
\psi_M^2(x,y) \delta_i(x - \tilde{x}) \, d\tilde{x}
\]
and the convex function \( r \to e^{\epsilon r} \), we obtain that the right hand side of (2.20) is dominated by
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \exp \left[ \epsilon \left( \frac{\partial \psi_M^2}{\partial x_i} \right) \psi_M^2(x,y) \right] \psi_M^2(x,y) \, dx. 
\]
By Young’s inequality and since \( \| \delta_i \|_{L^1(\mathbb{R}^N)} = 1 \), the latter is dominated by
\[
\int_{\mathbb{R}^N} \exp \left[ \epsilon 1_{\{ \psi < M \}} \left( \frac{\partial \psi_M^2}{\partial x_i} \right) \right] \psi_M^2(x,y) \, dx.
\]
Hence the first inequality in (2.19) is proved. To show the second we note that
\[
\frac{\partial \psi_M^2}{\partial x_i} = 1_{\{ \psi < M \}} \frac{\partial \psi_M^2}{\partial x_i}, \quad dx–a.s.
\]
Hence the integral in (2.21) is dominated by
\[
\int_{\mathbb{R}^N} \exp \left[ \epsilon 1_{\{ \psi < M \}} \left( \frac{\partial \psi_M^2}{\partial x_i} \right) \right] \psi_M^2(x,y) \, dx,
\]
which in turn by (2.12) is dominated by the last integral in (2.19).

Lemma 2.4. Let
\[
\delta := \inf_{1 \leq i \leq N} \frac{c_i}{N(\| g_i \|_{\infty} \| f_i \|_{\infty} + 1)}
\]
Then
\[
C_F := \sup_{M \in \mathbb{N}} \int_0^T \int_{\mathbb{R}^N} \exp \left[ -\delta \sum_{i=1}^N g_i(t) \partial_c f_i(x) \right] \times \exp \left[ \delta \sum_{i=1}^N \| g_i \|_{\infty} \| f_i \|_{\infty} \left( \frac{\partial \psi_M^2}{\partial x_i} \right) \right] \psi_M^2(x,y) \, dx \, dt < \infty.
\]
Proof. By (2.12), (2.13) and the generalized Hölder inequality this follows immediately from Lemma 2.3.

Lemma 2.5. There exist subsequences \( (l_k)_{k \in \mathbb{N}}, (M_k)_{k \in \mathbb{N}} \) such that the following assertions hold:
(i) For dx–a.e. \( x \in \{ \psi > 0 \} = \{ \psi_M > 0 \} \) we have for all \( M \in \mathbb{N} \)
\[
\lim_{k \to \infty} D_{M,M_k}^* F(\cdot, (x,y)) = -\sum_{i=1}^N g_i \left[ \partial_c f_i(x) + f_i(x) \left( \frac{\partial \psi_M^2}{\partial x_i} \right) (x,y) \right] = D_{M}^* F(\cdot, (x,y)),
\]
and
\[
\lim_{k \to \infty} D_{M_k}^* F(\cdot, (x,y)) = -\sum_{i=1}^N g_i \left[ \partial_c f_i(x) + f_i(x) \beta_{c} (x,y) \right] = D_{M}^* F(\cdot, (x,y)),
\]
in \( L^1([0,T]; dt) \).
(ii) Let \( \rho_M \) and \( \rho \) be defined as \( \rho_{M,L} \) with \( D_{M,L}^*F \) replaced by \( D_M^*F \) and \( D_L^*F \) respectively. Then (selecting a further subsequence if necessary) we have for \( dx \)-a.e. \( x \in \{\Psi > 0\} \) that for all \( M \in \mathbb{N} \)

\[
\lim_{k \to \infty} \rho_{M,k}(t, (x, y)) = \rho_M(t, (x, y)), \quad \forall \ t \in [0, T]
\]

and

\[
\lim_{k \to \infty} \rho_{M,k}(t, (x, y)) = \rho(t, (x, y)), \quad \forall \ t \in [0, T].
\]

**Proof.** (i) Obviously, we can choose the subsequence \((l_k)_{k \in \mathbb{N}}\) such that for \( dx \)-a.e. \( x \in \{\Psi > 0\} \) and all \( M \in \mathbb{N} \)

\[
\lim_{k \to \infty} D_{M,l_k}^*F(t, (x, y)) = D_M^*F(t, (x, y)), \quad \forall \ t \in [0, T].
\]

By Lemma 2.4 the sequence \( D_{M,l_k}^*F(\cdot, (\cdot, y)), \ l \in \mathbb{N} \), is uniformly integrable in \( L^1([0, T] \times K, dt \, dx) \) for every compact \( K \subset \mathbb{R}^N \). Consequently,

\[
\lim_{k \to \infty} D_{M,l_k}^*F(\cdot, (\cdot, y)) = D_M^*F(\cdot, (\cdot, y)) \quad \text{in} \ L^1([0, T] \times K, dt \, dx).
\]

Hence (selecting further sequences if necessary) the first assertion follows, since \( K \) was an arbitrary compact set in \( \mathbb{R}^N \). The second assertion follows analogously, because

\[
\left( \frac{\partial \Psi_M^2}{\partial x_i} / \Psi_M^2 \right) (x, y) = \mathbb{1}_{\{\Psi^2 < M\}} \left( \frac{\partial \Psi^2}{\partial x_i} / \Psi^2 \right) (x, y).
\]

(ii) Clearly, for all \( u, t \in [0, T] \), \( x \mapsto \xi(T - u, T - t, x) \) is a \( C^1 \)-diffeomorphism from \( \mathbb{R}^N \) to \( \mathbb{R}^N \) with strictly positive Jacobian which is bounded in \( u, t \in [0, T] \) and \( x \) in any compact subset \( K \subset \mathbb{R}^N \). So, the assertion follows by (i). \( \square \)

**Lemma 2.6.** Let \( \chi \in C_0^1(\mathbb{R}^N), \ \chi \geq 0, \ \chi = 1 \) on \( \cup_{i=1}^N \text{supp } \bar{f}_i \) (see (2.8)) and let \( l, M \in \mathbb{N} \). Then for all \( t \in [0, T] \)

\[
\int_{\mathbb{R}^N} \rho_{M,l}(t, (x, y)) \left( \ln \rho_{M,l}(t, (x, y)) - 1 \right) \Psi_{M,l}^2(x, y) \chi(x) \, dx
\]

\[
\leq e^{t/\delta} \left[ \int_{\mathbb{R}^N} \rho_0(x) \ln \rho_0(x) - 1 |\Psi_{M,l}^2(x, y) \chi(x)| \, dx \right]
\]

\[
+ C_F + \frac{t}{\delta} |\ln \delta| \int_{\mathbb{R}^N} \rho_0(x) \Psi_{M,l}^2(x, y) \chi(x) \, dx + \int_{\mathbb{R}^N} \Psi_{M,l}^2(x, y) \chi(x) \, dx,
\]

where \( \delta, C_F \) are the constants from Lemma 2.4.

**Proof.** By the regularity properties of \( \rho_{M,L} \) stated in Corollary A.2 of Appendix A, all integrals below are well defined. Since \( M, l \in \mathbb{N} \) and \( y \in E_0 \) are fixed, for simplicity of notation we denote the maps \( x \mapsto \rho_{M,l}(t, (x, y)) \) and \( x \mapsto \Psi_{M,l}(x, y) \) by \( \rho(t), \Psi \) respectively.
Then for $t \in [0, T]$,
\[
\int_{\mathbb{R}^N} \rho(t)(\ln \rho(t) - 1) \chi \Psi^2 dx
\]
\[
= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1) \chi \Psi^2 dx + \int_{\mathbb{R}^N} \int_0^t \frac{d}{ds} [\rho(s)(\ln \rho(s) - 1)] ds \chi \Psi^2 dx
\]
\[
= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1) \chi \Psi^2 dx + \int_{\mathbb{R}^N} \int_0^t \ln \rho(s) D_s \rho(s) ds \chi \Psi^2 dx
\]
\[
= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1) \chi \Psi^2 dx - \int_0^t \int_{\mathbb{R}^N} \langle F(s, x), D_x(\rho(s)(\ln \rho(s) - 1)) \rangle \chi \Psi^2 dx ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^N} D^*_M \chi F(s, (\cdot, y)) \rho(s) \ln \rho(s) \chi \Psi^2 dx ds
\]
\[
= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1) \chi \Psi^2 dx + \int_0^t \int_{\mathbb{R}^N} D^*_M F(s, (\cdot, y)) \rho(s) \chi \Psi^2 dx ds
\]
\[
\leq \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1) \chi \Psi^2 dx + \int_0^t \int_{\mathbb{R}^N} \left[ \delta(D^*_M F(s, (\cdot, y))^+ + \frac{1}{\delta} \rho(s)(\ln \frac{1}{\delta} \rho(s) - 1) \right] \chi \Psi^2 dx ds,
\]
where in the third inequality we used (2.18), in the fourth equality we used Fubini’s theorem and the definition of $D^*_M$ as well as the fact that $\chi = 1$ on $\bigcup_{l=1}^N \text{supp } \tilde{f}_l$ and finally, in the last inequality we used that for $a, b \geq 0$
\[
ab a \leq c^a + b(\ln b - 1).
\]
Now the assertion follows by Lemma 2.6 and Gronwall’s lemma, since by (2.18)
\[
\int_{\mathbb{R}^N} \rho_M(t, (x, y)) \Psi^2_M(x, y) \chi(x) dx = \int_{\mathbb{R}^N} \rho_0(x) \Psi^2_M(x, y) \chi(x) dx, \quad \forall \in [0, T]
\]
and $r \ln r - r \geq -1$ for all $r \in [0, \infty)$. □

**Lemma 2.7.** Let $M \in \mathbb{N}$, $\rho_M, t(x, y) := \rho_{M,y}(x)$, $t \in [0, T]$, $x \in \mathbb{R}^N$, and $\Psi_M(x, y) := \Psi_M(x, y)$, $x \in \mathbb{R}^N$. Then $\{\rho_M, t : l \in \mathbb{N}\}$ is uniformly integrable with respect to the measure $\chi(x) dx dt$, where $\chi$ is as in Lemma 2.6.

**Proof.** Let $c \in (1, \infty)$. Then for all $l \in \mathbb{N}$ and $\rho_l := \rho_{M,y}$, $\Psi_l := \Psi_{M,y}$,
\[
\int_0^T \int_{\mathbb{R}^N} 1_{\{|\psi_l| > c\}} \rho_l \Psi_l^2 \chi dx dt \leq \frac{1}{\ln c} \int_0^T \int_{\mathbb{R}^N} 1_{\{|\psi_l| > c\}} (\ln \rho_l + \ln \Psi_l^2) \rho_l \Psi_l^2 \chi dx dt
\]
\[
\leq \frac{1}{\ln c} \int_0^T \int_{\mathbb{R}^N} |\rho_l \rho_l| \psi_l^2 \chi dx dt + \frac{\ln(M+1)}{\ln c} \int_0^T \int_{\mathbb{R}^N} \rho_l \psi_l^2 \chi dx dt.
\]
Since $r \ln r - r \geq -1$, $r \in [0, \infty)$, and because of (2.23), it follows by Lemma 2.6 that both integrals on the right hand side of the last inequality are uniformly bounded in $l$ and the assertion follows. □

Now we proceed with the proof of Case 1 of Theorem 2.1. It follows by (2.18) (analogously to (2.1)–(2.4) above) that for all
\[
u(t, x) := g(t)f(x), \quad t \in [0, T], x \in \mathbb{R}^N,
\]
(2.24)
\( g \in C^1([0,T]; \mathbb{R}) \) with \( g(T) = 0 \) and \( f \in C^1_0(\mathbb{R}^N) \) that
\[
\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x) \rangle] \rho_M(t,(x,y)) \Psi_M^2(x,y) \, dx \, dt
= - \int_{\mathbb{R}^N} u(0,x) \rho_0(x) \Psi_M^2(x,y) \, dx.
\] (2.25)

By Lemma 2.5(ii) and Lemma 2.7 we can pass to the limit in (2.24) along the subsequence \((l_k)_{k \in \mathbb{N}}\) from Lemma 2.5 to conclude that for such \( u \)
\[
\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x) \rangle] \rho_M(t,(x,y)) \Psi_M^2(x,y) \, dx \, dt
= - \int_{\mathbb{R}^N} u(0,x) \rho_0(x) \Psi_M^2(x,y) \, dx.
\] (2.26)

We can also pass to the limit in (2.23) to get
\[
\int_{\mathbb{R}^N} \rho_M(t,(x,y)) \Psi_M^2(x,y) \chi(x) \, dx = \int_{\mathbb{R}^N} \rho_0(x) \Psi_M^2(x,y) \chi(x) \, dx, \quad \forall t \in [0,T].
\] (2.27)

Furthermore, by Lemma 2.5(ii) and Lemma 2.6 we deduce from (2.22) by Fatou’s lemma that for all \( t \in [0,T] \)
\[
\int_{\mathbb{R}^N} \rho_M(t,(x,y)) \left[ \Psi_M^2(x,y) \chi(x) \right] \, dx
\leq e^{t/\delta} \left[ \int_{\mathbb{R}^N} \rho_0(x) \left| \ln \rho_0(x) - 1 \right| \Psi_M^2(x,y) \chi(x) \, dx + C_F + \frac{\delta}{4} |\ln \delta| \int_{\mathbb{R}^N} \rho_0(x) \Psi_M^2(x,y) \chi(x) \, dx \right]
+ \int_{\mathbb{R}^N} \Psi_M^2(x,y) \chi(x) \, dx,
\] (2.28)
where we used that \( \Psi_M^2(x,y) \leq M \) for all \( l \in \mathbb{N} \) and \( \delta, \chi, C_F \) are as in Lemma 2.5.

Taking now the subsequence \((M_k)_{k \in \mathbb{N}}\) from Lemma 2.5 instead of \( M \) and using exactly analogous arguments as above, we can pass to the limit in (2.26), (2.27) and (2.28) to obtain that for all \( u \) as in (2.24)
\[
\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x) \rangle] \rho(t,(x,y)) \Psi^2(x,y) \, dx \, dt
= - \int_{\mathbb{R}^N} u(0,x) \rho_0(x) \Psi^2(x,y) \, dx,
\] (2.29)

and for all \( t \in [0,T] \)
\[
\int_{\mathbb{R}^N} \rho(t,(x,y)) \Psi^2(x,y) \chi(x) \, dx = \int_{\mathbb{R}^N} \rho_0(x) \Psi^2(x,y) \chi(x) \, dx
\] (2.30)
and
\[
\int_{\mathbb{R}^N} \rho(t, (x, y))(\ln \rho(t, (x, y)) - 1) \Psi^2(x, y) \chi(x) \, dx \\
\leq e^{t/\delta} \left[ \int_{\mathbb{R}^N} \rho_0(x) |\ln \rho_0(x) - 1| \Psi^2(x, y) \chi(x) \, dx + C_F + \frac{1}{\delta} |\ln \delta| \int_{\mathbb{R}^N} \rho_0(x) \Psi^2(x, y) \chi(x) \, dx \right] (2.31)
\]

for \( \delta, \chi, C_F \) as in Lemma 2.5. Hence in the situation of Case 1 the assertion of Theorem 2.1 now follows easily from the disintegration formula (2.10) by approximating the functions \( u \) in (1.1) in the obvious way and letting \( \chi \uparrow 1 \) in (2.31) to get (2.6). \( \square \)

**Remark 2.8.** (i) We here emphasize that in the situation of Case 1 we have an explicit formula for the solution density in (2.29) given by
\[
\rho(t, (x, y)) = \rho_0(\xi(T, T - t, x)) e^{-\int_0^t D_x^*F(T - u, \rho_0(T - u, T - t, y)) \, du} (2.32)
\]
for \( t \in [0, T] \) and \( dx \)-a.e. \( x \in \mathbb{R}^N \) with \( \xi \) given as in Corollary A.2 of Appendix A.

(ii) Letting \( \chi \uparrow 1 \) in (2.31) and then integrating over \( y \in E_0 \) with respect to \( \nu \), from Lemma 2.2 we obtain that for all \( t \in [0, T] \)
\[
\int_H \rho(t, x)(\ln \rho(t, x) - 1) \gamma(dx) \leq e^{t/\delta} \left[ \int_H \rho_0 |\ln \rho_0 - 1| \, d\gamma + C_F + 1 + \frac{1}{\delta} |\ln \delta| \int_H \rho_0 \, d\gamma \right] (2.33)
\]
and likewise from (2.30) that for all \( t \in [0, T] \)
\[
\int_H \rho(t, x) \gamma(dx) = \int_H \rho_0(x) \gamma(dx) = 1. (2.34)
\]

**Case 2.** Let \( F_j, j \in \mathbb{N} \), be as in Hypothesis 2. Choose nonnegative \( \rho_{0,j} \in \mathcal{F}_{b}^1 \) such that
\[
\lim_{j \to \infty} \rho_{0,j} = \rho_0 \quad \text{in } L^1(H, \gamma) (2.35)
\]
and
\[
\sup_{j \in \mathbb{N}} \int_H \rho_{0,j} \ln \rho_{0,j} \, d\gamma < \infty. (2.36)
\]
For existence of such \( \rho_{0,j}, j \in \mathbb{N} \), see Corollary C.3 in Appendix C below.

Let \( \rho_j \) be the corresponding solutions to (1.1) with \( F_j \) replacing \( F \) and \( \zeta := \rho_0 \cdot \gamma \), which exist by Case 1. Then by (2.34) with \( \rho_j, F_j, \rho_{0,j} \) replacing \( \rho, F \) and \( \rho_0 \) respectively, Hypothesis 2 and (2.34) imply that
\[
\sup_{j \in \mathbb{N}} \sup_{t \in [0, T]} \int_H \rho_j(t, x) \ln \rho_j(t, x) \gamma(dx) < \infty. (2.37)
\]

By Case 1 we have for all \( u \in \mathcal{F}_{b,T}^1 \)
\[
\int_0^T \int_H \left[ \frac{d}{dt} u(t, x) + \langle D_x u(t, x), F_j(t, x) \rangle_H \right] \rho_j(t, x) \gamma(dx) \, dt \\
= - \int_H u(0, x) \rho_{0,j}(x) \gamma(dx). (2.38)
\]
So, by (2.35) we only have to consider the convergence of the left hand side of (2.38), more precisely only the part of it involving $F_j$. But
\[
\left| \int_0^T \int_H \langle (D_x u, F_j)_H \rho_j - (D_x u, F)_H \rho \rangle \, d\gamma \, dt \right|
\leq \|Du\|_\infty \int_0^T \int_H |F_j - F|_H \rho_j \, d\gamma \, dt + \int_0^T \int_H \langle F, Du \rangle (\rho_j - \rho) \, d\gamma \, dt
\]  
(2.39)

Because of the boundedness of $\langle F, Du \rangle$ the second term on the right hand side of (2.39) converges to 0 if $j \to \infty$. Let $\epsilon > 0$. Then, by Young’s inequality, the first term on the right hand side of (2.39) is up to a constant dominated by
\[
\epsilon \int_0^T \int_H \rho_j \ln \rho_j \, d\gamma \, dt + \epsilon \ln \epsilon,
\]
which can be made arbitrarily small uniformly in $j$ because of (2.37). Hence putting all this together we conclude that the right hand side of (2.39) converges to 0 as $j \to \infty$.

It remains to prove (2.6). For this we are going to employ a result due to J. Komlos (see [Kom67]). Namely, since $\rho_j$, $j \in \mathbb{N}$, is bounded in $L^1([0, T] \times H, dt \otimes dx)$ by [Kom67, Theorem 1a] (selecting another subsequence if necessary) we may assume that
\[
\rho^{(N)} := \frac{1}{N} \sum_{j=1}^N \rho_j \to \rho, \quad \gamma\text{-a.s.}
\]
Hence, since $r \mapsto r \ln r$ is convex on $[0, \infty)$, by Fatou’s lemma we obtain
\[
\int_0^T \int_H \rho \ln \rho \, d\gamma \, dt \leq \liminf_{N \to \infty} \int_0^T \int_H \rho^{(N)} \ln \rho^{(N)} \, d\gamma \, dt
\leq \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^N \int_0^T \int_H \rho_j \ln \rho_j \, d\gamma \, dt
\leq \sup_{j \in \mathbb{N}} \int_0^T \int_H \rho_j \ln \rho_j \, d\gamma \, dt,
\]
which is finite by (2.37). Finally from (2.34) and (2.35) it follows that $\nu_t(dx) := \rho(t, x) \gamma(dx)$ is a probability measure for all $t \in [0, T]$. Thus Theorem 2.1 is completely proved.

APPENDIX A. DETERMINISTIC FEYNMAN–KAC FORMULA AND THE SOLUTION OF (2.1) FOR SUFFICIENTLY REGULAR $F$

Consider the equation
\[
\left\{ \begin{array}{l}
\frac{d}{dt} \xi(t) = \tilde{F}(t, \xi(t)), \\
\xi(s) = x, \quad x \in \mathbb{R}^d,
\end{array} \right. \quad (A.1)
\]
with \( \tilde{F} \) regular, namely it belongs to the class \( VF^1_c(H) \). Let \( V : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) be also regular. We want to solve

\[
\begin{cases}
v_s(s, x) + \langle D_xv(s, x), \tilde{F}(s, x) \rangle + V(s, x)v(s, x) = 0, & 0 \leq s < T, \\
v(T, x) = \varphi(x), & x \in H.
\end{cases}
\]

(A.2)

**Proposition A.1.** Assume \( \tilde{F} \in C_b([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \) such that \( \tilde{F}(t, \cdot) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \) for all \( t \in [0, T] \) and let \( V \in C([0, T] \times \mathbb{R}^d) \) such that \( V(t, \cdot) \in C^1(\mathbb{R}^d) \) for all \( t \in [0, T] \) such that \( D_xV : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) is continuous. Let \( \varphi \in C^1(\mathbb{R}^d) \). Then the solution to (A.2) is given by

\[
v(s, x) = \varphi(\xi(T, s, x))e^{\int_s^T V(u, \xi(u, s, x))du}, \quad (s, x) \in [0, T] \times \mathbb{R}^d,
\]

(A.3)

where for \( s \leq t \), \( \xi(t, s, x) \) denotes the solution to (A.1) at time \( t \) when started at time \( s \) at \( x \in \mathbb{R}^d \). In particular, \( v(\cdot, x) \in C^1([0, T]) \) for every \( x \in \mathbb{R}^d \) and \( D_tv \in C([0, T] \times \mathbb{R}^d) \).

**Proof.** We only present the main steps. We shall check that \( v \) defined by (A.3) is a solution to (A.2).

For any decomposition \( \{ s = s_0 < s_1 < \cdots < s_n = T \} \) of \([s, T]\) we write

\[
v(s, x) - \varphi(x) = -\sum_{k=1}^{n} [v(s_k, x) - v(s_{k-1}, x)],
\]

which is equivalent to,

\[
v(s, x) - \varphi(x) = -\sum_{k=1}^{n} [v(s_k, x) - v(s_k, \xi(s_k, s_{k-1}, x))] - \sum_{k=1}^{n} [v(s_k, \xi(s_k, s_{k-1}, x)) - v(s_{k-1}, x)] =: J_1 - J_2.
\]

(A.4)

Concerning \( J_1 \) we write thanks to Taylor’s formula

\[
J_1 \sim \sum_{k=1}^{n} \langle D_xv(s_k, x), \xi(s_k, s_{k-1}, x) - x \rangle \sim \sum_{k=1}^{n} \langle D_xv(s_k, x), \tilde{F}(s_k, x) \rangle (s_k - s_{k-1})
\]

\[
\to \int_s^T \langle D_xv(r, x), \tilde{F}(r, x) \rangle dr.
\]

(A.5)
Proof. Apply Proposition A.1 with $D$ as in the assertion above, $v(s, x) = \varphi(x) + \int_s^T (D_x v(r, x), \bar{F}(r, x)) dr + \int_s^T v(r, x) V(r, x) dr$

and the claim is proved. □

As a trivial consequence we obtain

Corollary A.2. Suppose $H = \mathbb{R}^d$ and $\gamma$ satisfies Hypothesis \[1\]. Let $F \in C_b([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that $F(t, \cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ and $D^*_x F(t, \cdot) \in C^1(\mathbb{R}^d)$ for all $t \in [0, T]$, and $D^*_x F \in C([0, T] \times \mathbb{R}^d)$, $D_x D^*_x F \in C([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. Then for every $\rho_0 \in C^1(\mathbb{R}^d)$, $\rho_0 \geq 0$,

$$\rho(t, x) := \rho_0(\xi(T, T - t, x)) e^{\int_0^t D^*_x F(T - u, \xi(T - u, T - t, x)) du}$$

is a solution of \[2.1\], where $\xi(\cdot, s, x)$ is the solution to \[A.1\] started at time $s$ at $x \in \mathbb{R}^d$, with $\bar{F}(t, x) := -F(T - t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$. Furthermore, $\rho(\cdot, x) \in C^1([0, T])$ for every $x \in \mathbb{R}^d$ and $D_t \rho \in C([0, T] \times \mathbb{R}^d)$.

Proof. Apply Proposition A.1 with $\bar{F}$ as in the assertion above,

$$V(t, x) = D^*_x F(T - t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

and $\varphi := \rho_0$. □

\[1\]In the second line below we use that $\xi(T, s_k, \xi(s_k, s_{k-1}, x)) = \xi(T, s_{k-1}, x)$
APPENDIX B. A REMARK ON THE BURKHOLDER–DAVIS–GUNDY INEQUALITY

Our aim in this section is to prove the following proposition.

**Proposition B.1.** Let $p \geq 4$. Then for every $t \geq 0$,

$$
\mathbb{E} \sup_{s \leq [0, t]} \left| \int_0^t f(s) dW(s) \right|^p \leq c_p \left[ \mathbb{E} \left( \int_0^t \|f(s)\|^2_{L_2^2} ds \right)^{p/2} \right],
$$

where $c_p := 12^p p^p$.

**Proof.** Set

$$
Z(t) = \int_0^t f(s) dW(s), \quad t \geq 0,
$$

and apply Itô’s formula to $f(Z(\cdot))$ where $f(x) = |x|^p$, $x \in H$. Since

$$
f_{xx}(x) = p(p - 2)|x|^{p-4} x \otimes x + p|x|^{p-2} I, \quad x \in H,
$$

we have

$$
\|f_{xx}(x)\| \leq p(p - 1)|x|^{p-2},
$$

therefore

$$
|\text{Tr} \, f_{xx}(Z(t))| \leq p(p - 1)|Z(t)|^{p-2} \|f(t)\|^2_{L_2^2}.
$$

By taking expectation in the identity

$$
|Z(t)|^p = p \int_0^t |Z(s)|^{p-2} \langle Z(s), dZ(s) \rangle + \frac{1}{2} \int_0^t \text{Tr} \, f_{xx}(Z(s)) \|f(s)\|^2_{L_2^2} ds,
$$

we obtain by the Burkholder–Davis–Gundy inequality for $p = 1$

$$
\mathbb{E} \sup_{s \in [0, t]} |Z(s)|^p \leq \frac{p(p - 1)}{2} \mathbb{E} \left( \int_0^t |Z(s)|^{p-2} \|f(s)\|^2_{L_2^2} ds \right)
+ 3p \mathbb{E} \left[ \left( \int_0^t \|f(s)\|^2_{L_2^2} |Z(s)|^{2p-2} ds \right)^{1/2} \right]
\leq \frac{p(p - 1)}{2} \mathbb{E} \left( \sup_{s \in [0, t]} |Z(s)|^{p-2} \int_0^t \|f(s)\|^2_{L_2^2} ds \right)
+ 3p \mathbb{E} \left[ \sup_{s \in [0, t]} |Z(s)|^{p-1} \left( \int_0^t \|f(s)\|^2_{L_2^2} ds \right)^{1/2} \right]
\leq \frac{p(p - 1)}{2} \left[ \mathbb{E} \left( \sup_{s \in [0, t]} |Z(s)|^p \right)^{\frac{p-2}{p}} \mathbb{E} \left( \int_0^t \|f(s)\|^2_{L_2^2} ds \right)^{\frac{2}{p}} \right]^{\frac{1}{p}}
+ 3p \mathbb{E} \left[ \sup_{s \in [0, t]} |Z(s)|^{\frac{p-1}{p}} \mathbb{E} \left( \int_0^t \|f(s)\|^2_{L_2^2} ds \right)^{\frac{2}{p}} \right]^{\frac{1}{p}}
:= J_1 + J_2.
$$

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For $J_1$ we use Young’s inequality with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ and find
\[
J_1 \leq \frac{1}{4} E \left[ \sup_{s \in [0,t]} |Z(s)|^p \right] + 2^{p-1} p \ E \left( \int_0^t \|\Phi(s)\|^2_{L^2} ds \right)^{\frac{p}{2}}.
\]
For $J_2$ we use Young’s inequality with exponents $\frac{p}{p-1}$ and $p$ and find
\[
J_2 \leq \frac{1}{4} E \left[ \sup_{s \in [0,t]} |Z(s)|^p \right] + \frac{1}{2} 12^p p \ E \left( \int_0^t \|\Phi(s)\|^2_{L^2} ds \right)^{\frac{p}{2}}.
\]
Now (B.1) with $c_p := 12^p p^p$ follows. □

Appendix C. Density of $F_{C^1_b}$ in Orlicz spaces

Let $N : \mathbb{R} \to [0, \infty)$ be continuous and a Young function, i.e. convex, even and $N(0) = 0$.

Consider the measure space $(H, \mathcal{B}(H), \gamma)$, where $H$ is as before a separable real Hilbert space with Borel $\sigma$–algebra $\mathcal{B}(H)$ and $\gamma$ a nonnegative finite measure on $(H, \mathcal{B}(H))$. We recall that the Orlicz space $L_N$ corresponding to $N$ is defined as

\[
L_N := L_N(H, \gamma) := \{ f : H \to \mathbb{R} : f \text{ is } \mathcal{B}(H)\text{-measurable and } \int_H N(a f) \, d\gamma < \infty \text{ for some } a > 0 \}
\]
or equivalently

\[
L_N := \{ f : H \to \mathbb{R} : f \text{ is } \mathcal{B}(H)\text{-measurable and } \|f\|_{L_N} < \infty \},
\]
where

\[
\|f\|_{L_N} := \inf \left\{ \lambda > 0 : \int_H N(f/\lambda) \, d\gamma \leq 1 \right\}.
\]

$(L_N, \|\cdot\|_{L_N})$ is a Banach space (see e.g. [RaRe02]).

**Proposition C.1.** $F_{C^1_b}$ is dense in $(L_N, \|\cdot\|_{L_N})$, where $F_{C^1_b}$ is defined as in Section 1. Furthermore, if $f \in L_N$, $f \geq 0$, then there exist nonnegative $f_n \in F_{C^1_b}$, $n \in \mathbb{N}$, such that

\[
\lim_{n \to \infty} \|f - f_n\|_{L_N} = 0.
\]

**Proof.** We need the following lemma whose proof is straightforward, see e.g. [Le, Lemma 1.16]

**Lemma C.2.** Let $f_n \in L_N$, $n \in \mathbb{N}$. Then the following assertions are equivalent:

(i) \[ \lim_{n \to \infty} \|f_n\|_{L_N} = 0 \]

(ii) For all $a \in (0, \infty)$

\[ \limsup_{n \to \infty} \int_H N(a f_n) \, d\gamma \leq 1 \]

(iii) For all $a \in (0, \infty)$

\[ \lim_{n \to \infty} \int_H N(a f_n) \, d\gamma = 0. \]

**Proof of Proposition C.1**

We shall use a monotone class argument. Define

\[ M := \left\{ f : H \to \mathbb{R} : f \text{ bounded, } \mathcal{B}(H)\text{-measurable such that } \lim_{n \to \infty} \|f - f_n\|_{L_N} = 0, \text{ for some } f_n \in F_{C^1_b}, n \in \mathbb{N} \right\}. \]
Obviously, $\mathcal{M}$ is a linear space, $\mathcal{FC}_b^1 \subset \mathcal{M}$ and $\mathcal{FC}_b^1$ is closed under multiplication and contains the constant function $1$. Furthermore, if $0 \leq u_n \in \mathcal{M}$, $n \in \mathbb{N}$, such that $u_n \uparrow u$ as $n \to \infty$ for some bounded $u : H \to [0, \infty)$, then for each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{FC}_b^1$ such that

$$
\|u_n - f_n\|_{L_N} \leq \frac{1}{n}.
$$

(C.1)

But since $N$ is continuous on $\mathbb{R}$, hence locally bounded, we have that for every $a \in (0, \infty)$, $N(a(u - u_n))$, $n \in \mathbb{N}$, are uniformly bounded. Consequently, by Lebesgue’s dominated convergence theorem and Lemma C.2 we conclude that

$$
\lim_{n \to \infty} \|u - u_n\|_{L_N} = 0.
$$

(C.2)

(C.1) and (C.2) imply that $u \in \mathcal{M}$, and therefore $\mathcal{M}$ is a monotone vector space and thus by the monotone class theorem $\mathcal{M}$ is equal to the set of all bounded $\sigma(\mathcal{FC}_b^1)$–measurable functions on $H$. But $\sigma(\mathcal{FC}_b^1) = \mathcal{B}(H)$, since the weak and norm–Borel $\sigma$–algebra on a separable Banach space coincide. Hence $\mathcal{M}$ is equal to all bounded $\mathcal{B}(H)$–measurable functions on $H$. Since by Lemma C.2 and the same arguments as above every $f$ in $L_N$ can be approximated in the norm $\| \cdot \|_{L_N}$ by bounded $\mathcal{B}(H)$–measurable functions, the first assertion of the proposition is proved.

Now let $f \in L_N$, $f \geq 0$. By the argument above we may assume that $f$ is bounded. Then by what we have just proved we can find $f_n \in \mathcal{FC}_b^1$ such that

$$
\lim_{n \to \infty} \|f - f_n\|_{L_N} = 0.
$$

Since $|f - f_n^+| = |f^+ - f_n^+| \leq |f - f_n|$ for all $n \in \mathbb{N}$ and $N$ is even and increasing (because $N$ is convex and $N(0) = 0$), Lemma C.2 immediately implies that

$$
\lim_{n \to \infty} \|f - f_n^+\|_{L_N} = 0.
$$

Fix $n \in \mathbb{N}$ and for $\varepsilon > 0$ take an increasing function $\chi_\varepsilon \in C^1(\mathbb{R})$, $\chi_\varepsilon(s) = s$, $\forall s \in [0, \infty)$ and $\chi_\varepsilon(s) = -\varepsilon$ if $s \in (-\infty, -2\varepsilon)$. Then for each $n \in \mathbb{N}$

$$
\lim_{m \to \infty} \left\| f_n^+ - \left( \chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \right) \right\|_\infty = 0.
$$

So, again by Lemma C.2 and Lebesgue’s dominated convergence theorem it follows that

$$
\lim_{m \to \infty} \left\| f_n^+ - \left( \chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \right) \right\|_{L_N} = 0.
$$

But obviously, $\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \in \mathcal{FC}_b^1$, $m \in \mathbb{N}$, and each such function is nonnegative. Hence the second part of the assertion follows.

**Corollary C.3.** Let $\rho \geq 0$, $\mathcal{B}(H)$–measurable such that

$$
\int_H \rho \log \rho \, d\gamma < \infty.
$$

Then there exist nonnegative $\rho_n \in \mathcal{FC}_b^1$, $n \in \mathbb{N}$, such that

$$
\lim_{n \to \infty} \rho_n = \rho \quad \text{in } L^1(H, \gamma)
$$

and

$$
\sup_{n \in \mathbb{N}} \int_H \rho_n \log \rho_n \, d\gamma < \infty.
$$
Proof. Let \( N(s) := (|s|+1) \ln(|s|+1) - |s|, s \in \mathbb{R} \). Then it is easy to check that \( N \) is a continuous Young function. Hence by Proposition C.1 we can find \( \rho_n \in \mathcal{F}_b^1, \rho_n \geq 0, n \in \mathbb{N}, \) such that
\[
\lim_{n \to \infty} \|\rho - \rho_n\|_{L_N} = 0. \tag{C.3}
\]
Since \( L_N \subset L^1(H, \gamma) \) continuously (see [Le, Proposition 1.15]), the first assertion follows. Furthermore, we have for all \( s \in (0, \infty) \)
\[
s \ln s - s \leq s \ln(s+1) \leq (s+1) \ln(s+1) - s = N(s)
\]
and hence for \( n \in \mathbb{N} \) by the convexity of \( N \) and every \( a \in (0, \infty) \)
\[
\int_H \rho_n \ln \rho_n \, d\gamma = \frac{1}{a} \int_H a \rho_n \ln(a \rho_n) \, d\gamma - \ln a \int_H \rho_n \, d\gamma
\]
\[
\leq \frac{1}{a} \int_H N(a \rho_n) \, d\gamma + |1 - \ln a| \int_H \rho_n \, d\gamma
\]
\[
\leq \frac{1}{2a} \int_H N(2a(\rho_n - \rho)) \, d\gamma + \frac{1}{2a} \int_H N(2a \rho) \, d\gamma + |1 - \ln a| \int_H \rho_n \, d\gamma.
\]
Hence by the first part of the assertion, (C.3) and Lemma C.2, it follows that
\[
\limsup_{n \to \infty} \int_H \rho_n \ln \rho_n \, d\gamma \leq \frac{1}{2a} \int_H N(2a \rho) \, d\gamma + |1 - \ln a| \int_H \rho \, d\gamma.
\]
But since \( \rho \in L_N \) we can find \( a > 0 \) such that the right hand side is finite. Hence the second part of the assertion also follows.

\( \square \)
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