Thermal metal in network models of a disordered two-dimensional superconductor

J. T. Chalker$^1$, N. Read$^2$, V. Kagalovsky$^{3,4}$, B. Horovitz$^4$, Y. Avishai$^1$, and A. W. W. Ludwig$^5$

$^1$Theoretical Physics, Oxford University, Oxford OX1 3NP, United Kingdom
$^2$Department of Physics, Yale University, P.O. Box 208120, New Haven, CT 06520-8120
$^3$Negev Academic College of Engineering, Beer-Sheva 84100, Israel
$^4$Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
$^5$Physics Department, University of California, Santa Barbara, CA 93106-4030

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We study the universality class for localization which arises from models of non-interacting quasiparticles in disordered superconductors that have neither time-reversal nor spin-rotation symmetries. Two-dimensional systems in this category, which is known as class D, can display phases with three different types of quasiparticle dynamics: metallic, localized, or with a quantized (thermal) Hall conductance. Correspondingly, they can show a variety of delocalization transitions. We illustrate this behavior by investigating numerically the phase diagrams of network models with the appropriate symmetry, and for the first time show the appearance of the metallic phase.

The properties of quasiparticles in disordered superconductors have been a subject of much recent interest. Within a mean field treatment of pairing, the quasiparticles are noninteracting fermions, governed by a quadratic Hamiltonian which may contain effects of disorder in both the normal part and the superconducting gap function. Such Hamiltonians are representatives of a set of universality classes different from the three classes which are familiar both in normal disordered conductors and in the Wigner-Dyson random matrix ensembles. A list of additional random matrix ensembles, determined by these new symmetry classes, has been established relatively recently. These additional random matrix ensembles describe zero-dimensional problems, and are appropriate to model a small grain of a superconductor in the ergodic limit. In the corresponding higher-dimensional systems from the same symmetry classes, there can be transitions between metallic, localized, or quantized Hall phases for the quasiparticles. The associated changes in quasiparticle dynamics must be probed by energy transport or (in singlet superconductors) spin transport, rather than charge transport, since quasiparticle charge density is not conserved. There are various possibilities for behavior, depending on the particular symmetry class considered. These have been studied theoretically using nonlinear sigma model methods, numerically, and in quasi-one dimensional models. An important question not addressed in such work so far, and which will not be considered here, is whether the self-consistent solution to the gap equation in the presence of disorder affects the universal statistical properties of the ensembles.

In this paper we present extensive numerical results on a symmetry class with particularly rich phase diagram in two dimensions, class D. The symmetry may be realized in superconductors with broken time-reversal invariance, and either broken spin-rotation invariance (as in d-wave superconductors with spin-orbit scattering) or spinless or spin-polarized fermions (as in certain p-wave states). The nonlinear sigma model for class D has been shown, in the two-dimensional case, to flow under the renormalization group to weaker values of the coupling constant. The coupling constant is proportional to the inverse of the thermal conductivity of the superconductor, and this flow implies that there is a phase in which there is a nonzero (indeed, diverging) density of extended fermion eigenstates at zero excitation energy. A superconductor described by this model would be in a thermal metal phase. We will refer to such a phase simply as a metallic phase. In addition, a phase with localized quasiparticles is a natural possibility, and—since time-reversal symmetry is broken—so is one with quantized Hall conductance. Our aim in the following is to investigate the appearance of these phases in simple models.

As our starting point, we take versions of the network model for a single-component fermion, which we specify in detail after first summarizing our findings. Disorder appears in the network model in the form of random scattering phases, and the symmetries of class D restrict scattering phases to the values 0 and π. Remarkably, within this framework, different particular forms of disorder result in quite distinct physical behavior. We discuss three alternative choices. The first of these (CF) was introduced in work by Cho and Fisher with the intention of modeling the two-dimensional random bond Ising model (RBIM), which possesses a fermion representation with the symmetries of class D. In fact, as noted subsequently, a precise mapping of the Ising model leads to a second version of the network model, which we label RBIM. In both these models, scattering phases with the value π appear in correlated pairs. A third model (also discussed in Ref.) arises naturally if one instead takes scattering phases to be independent random variables. Each model has two parameters: a disorder concentration, $p$ ($0 \leq p \leq 1$), and a tunneling amplitude $\alpha$. 

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for the CF model in the (\( p, \alpha \)) plane is shown in Fig. 1. It contains a region of metallic phase, and two distinct localized phases, which can be identified with the ordered and disordered phases of the RBIM, or as regions with different quantized Leduc-Righi (thermal Hall) conductivities. As a consequence, three potentially different critical points occur: an insulator-to-insulator quantum-Hall–type transition, an insulator-to-metal transition; and a multicritical point at which all three phases meet. This phase diagram has the form proposed generically for class D in Ref. [7]. In contrast, neither the RBIM nor the O(1) model supports all three phases: arguments that the metallic phase cannot appear in RBIMs with real Ising couplings are given in Ref. [14]; while in the O(1) model we find no localized phase, in striking distinction to all network models studied previously. We show below how these differences can be understood by solving the models in one dimension and by considering them in two dimensions at weak tunneling.

All these models represent coherent propagation of quantum-mechanical flux on a square lattice of directed links which meet at nodes, as illustrated in Fig. 2. Plaquettes of the lattice can be divided into two sets, according to the direction of circulation around them. For general values of \( \alpha \), all plaquettes are coupled, but for \( \alpha = 0 \) the system separates into uncoupled plaquettes with clockwise circulation, while for \( \alpha = \pi/2 \) it consists of uncoupled anticlockwise plaquettes. Disorder is introduced via a phase \( \phi_l \) associated with each link, \( l \). To make clear the constraints imposed in class D, recall that a Bogoliubov-de Gennes Hamiltonian with this symmetry may be written in terms of a purely imaginary Hermitian matrix [1]. The corresponding time evolution operator is purely real, restricting the generalized phase factors to be O(\( N \)) matrices for a model in which \( N \)-component fermions propagate on links, and to the values \( \pm 1 \) for \( N = 1 \), the case we treat. It is useful to consider the gauge-invariant total phase, modulo 2\( \pi \), accumulated on passing around each elementary plaquette. In place of individual link phases, randomness can be characterized by the positions of flux lines which thread a subset of plaquettes, adding \( \pi \) to their phases.

The models we study and the important distinctions between them are as follows. The CF model has transfer matrix tunneling parameters chosen negative with probability \( p \), and positive with probability \( 1 - p \) [12]. In consequence, flux lines appear in pairs at a node with probability \( p \): both members of a pair pass through plaquettes with the same circulation, but different pairs may belong to plaquettes with opposite circulation. The RBIM similarly has a pair of flux lines introduced at each node with probability \( p \), but with the difference that all flux lines thread plaquettes of the same circulation [11, 14]. Finally, the O(1) model has link phase factors chosen negative with probability \( p \) and positive with probability \( 1 - p \), so that the two members of a flux line pair are associated with plaquettes of opposite circulation. Each of the models is invariant under the transformation \( p \rightarrow 1 - p \), and so we consider only \( 0 \leq p \leq 1/2 \). The CF and the O(1) models are both (statistically) self-dual for all \( p \) under a Kramers-Wannier transformation that takes \( \alpha \) to \( \pi/2 - \alpha \), leaving the line \( \alpha = \pi/4 \) invariant. The RBIM is not self-dual, except at \( p = 0 \). Finally, the CF and O(1) models are equivalent, under a gauge transformation, on the line \( p = 1/2 \).

Some of the differences in the behavior of these three models can be illustrated by solving their one-dimensional versions, which consist of a single chain of links and nodes. In one dimension, disorder in the sign of the nearest-neighbor exchange interaction can be removed from the RBIM by gauge transformation, and the
inverse localization length has the disorder-independent value \( \xi_{\text{RBIM}}^{-1} = \text{arctanh}(|\sin(\alpha)|) \), finite for all \( \alpha \neq 0, \pi/2 \). An elementary calculation gives for the CF model

\[
\xi_{\text{CF}}^{-1} = |1 - 2p|\xi_{\text{RBIM}}^{-1},
\]

so that \( \xi_{\text{CF}} \) diverges as \( p \to 1/2 \) but is otherwise finite, while for the \( O(1) \) model \( \xi_{\text{O}(1)}^{-1} = 0 \) for all \( p \neq 0,1 \). Thus, the localization properties of the one-dimensional CF model at \( p \neq 1/2 \) are like those of models belonging to the Wigner-Dyson universality classes, in that states are localized, while the absence of localization in the \( O(1) \) model mirrors that found previously in quasi-one dimensional class D systems [3].

A second useful approach illustrating differences between these models is to consider their two-dimensional versions in the limit of weak inter-plaquette tunneling \( (\alpha \ll 1 \text{ or } \pi/2 - \alpha \ll 1) \) and weak disorder \( (p \ll 1) \). We do this in terms of the discrete-time evolution operator \( U \) in a closed system, which is an \( N_f \times N_f \) unitary matrix for a network of \( N \) links [3,16]; a similar approach to a different free fermion representation of the RBIM was used in Ref. [17]. The eigenvalues of \( U \) lie on the unit circle and may be written as \( e^{-i\epsilon} \). The real \( \epsilon (-\pi < \epsilon \leq \pi) \) play the role of excitation energy eigenvalues, and are distributed symmetrically in pairs around \( \epsilon = 0 \) because \( U \) is a real orthogonal matrix. Long-time properties are determined by the part of the spectrum near \( \epsilon = 0 \), on which we now focus. At zero tunneling, it is sufficient to examine an isolated plaquette. In our disorder-free reference system, the evolution operator for a single plaquette satisfies \( U^4 = -1 \), and hence \( \epsilon = \pm \pi/4, \pm 3\pi/4 \). For a single plaquette with a flux line added, \( U^4 = 1 \) and \( \epsilon = 0, \pi, \pm \pi/2 \). Turning on weak tunneling, it is clear that the spectrum near \( \epsilon = 0 \) for a large system will arise by hybridisation of the \( \epsilon = 0 \) states from plaquettes with flux lines. In both the RBIM and CF models, there are two scales for this hybridisation, because flux lines appear in the system in adjacent pairs associated with plaquettes of the same circulation. The first consequence of tunneling is to remove the degeneracy within each pair, yielding approximate eigenvalues \( \epsilon = \pm \epsilon_0 \). At small \( p \), pairs are dilute and tunneling between different pairs is not sufficient to generate extended states at \( \epsilon = 0 \). By contrast, for the \( O(1) \) model in this regime, there is only one scale for hybridisation, since single flux lines appear independently on the set of weakly-coupled plaquettes. As a result, metallic behavior is not excluded even at \( p, \alpha \ll 1 \).

Our results from numerical simulation supplement this qualitative discussion. We study the CF and \( O(1) \) models in cylindrical geometry via the transfer matrix \( T \), obtaining the positive Lyapunov exponents, \( 0 \leq \nu_1 \leq \ldots \leq \nu_M \) in a system of width \( M' = 2M \) links. A crucial technical aspect of these calculations is our discovery that the standard algorithm [18,20] has a serious instability to roundoff errors throughout much of the phase diagram of both models. More specifically, we find that the smallest positive Lyapunov exponent, \( \nu_1 \), may be either identically zero or exceptionally small \( (\nu_1 \ll M^{-1}) \). (The first happens in the \( O(1) \) model for all \( p \) and \( \alpha \), and in the CF model on the self-dual line \( \alpha = \pi/4 \); the second happens in the metallic phase of the CF model.) Under these circumstances, numerical noise from roundoff errors generates a systematic positive error in the value obtain for \( \nu_1 \). From an analytical theory [18] of the instability, we find that the error in \( \nu_1 \) decreases with reduced noise amplitude \( \eta \) only as \( |\log(\eta)|^{-1} \). This instability can be cured by making explicit use in numerical calculations of the structure imposed on \( T \) by current conservation and the symmetry of class D.

In detail, \( T \) has the polar decomposition

\[
T = \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right) \left( \begin{array}{cc} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{array} \right) \left( \begin{array}{cc} A_4^T & 0 \\ 0 & A_3^T \end{array} \right),
\]

(2)

where \( A_1 \ldots A_4 \) are \( M \times M \) real orthogonal matrices and \( \gamma \) is an \( M \times M \) real diagonal matrix. It follows that \( T^T T \) is diagonalized by the transformation \( B^T T^T T B \), where

\[
B = \left( \begin{array}{cccc} A_3 & A_3 & \ldots & A_3 \\ A_4 & -A_4 & \ldots & A_4 \end{array} \right).
\]

(3)

The standard method for calculating Lyapunov exponents numerically involves acting with the transfer matrices for successive slices of the system on a set of \( M \) orthogonal vectors, and re-imposing orthogonality by means of Gram-Schmidt transformations [18,20]. If all Lyapunov exponents are separated by gaps, this set of vectors converges to the eigenvectors of \( T^T T \) associated with the first \( M \) exponents. Convergence rates are determined by the sizes of gaps between successive exponents. In the present case, convergence rates are seriously reduced if \( \nu_1 \) approaches zero, so that the gap between the smallest positive and largest negative exponents vanishes. Moreover, numerical noise ultimately limits the extent of convergence, and leads to an erroneously large value for \( \nu_1 \).

To overcome this, we impose on the \( M \) vectors concerned not simply orthogonality but instead the fact that their first \( M \) components separately form an orthogonal matrix \( A_3 \), and their last \( M \) components form \( A_4 \), as is evident from Eq. [3]. The results we obtain in this way for the CF model differ significantly from those of Ref. [2].

Evidence in support of the phase diagram of Fig. 4 for the CF model is presented in Fig. 5. On the self-dual line \( (\alpha = \pi/4) \) we believe that \( \nu_1 \) is identically zero (as in the one-dimensional \( O(1) \) model). For example, at \( p = 1/2 \) and \( \alpha = \pi/4 \) we obtain in systems of length \( L = 5 \times 10^3 \) the bounds \( \nu_1 < 1.5 \times 10^{-3} \) at width \( M' = 4 \) and \( \nu_1 < 1.5 \times 10^{-4} \) at \( M' = 256 \). In order to search for a possible multi-critical point on the self-dual line, we therefore examine
different phase, and has for the CF model displayed in Fig. 1. $M$ the metal-insulator phase boundary from the variation of $M$ metal and increases with $\alpha$ point, $p$ as 
$\alpha \rightarrow 0$ or $\pi/2$. Recently, it has been emphasized that the target manifold of the class D nonlinear sigma model is not connected $\star$, and this means that domain wall excitations can occur in the sigma model, which must be described by additional parameters, and have not been taken into account in weak-coupling analyses so far. It is likely that these domain walls in the sigma model language are connected with the richness of phases in this symmetry class. In that context, the O(1) model with $p = 1/2$ is known to be a special case, since it maps to a sigma model without domain walls: this fact suggests that proliferation of domain walls may be necessary for localization $\star \star \star \star$. The work was supported in part by: the EPSRC under Grant GR/J78327 (JTC); the NSF under Grants DMR-98-18259 (NR) and DMR-00-75064 (AWWL); and the DIP German Israeli program (BH and YA).

We calculate for $M' = 16$: $\nu_1 < 10^{-3}$ in the O(1) model, while $\nu_1 = 0.83$ in the CF model.

In summary, we find that two-dimensional models for localization in the symmetry class D can have quite different behavior according to the form of disorder. Several additional points deserve emphasis. The metallic phase of the CF model is self-dual, as is its multicritical point. By contrast, the RBIM is not self-dual but has higher supersymmetry at its multicritical point $\star \star$. There is little reason to suppose that these two multicritical points are in the same universality class. Separately, the apparent absence of an insulating phase in the O(1) model is remarkable, because the bare conductivity becomes small when $\alpha \rightarrow 0$ or $\pi/2$. Recently, it has been emphasized that the target manifold of the class D nonlinear sigma model is not connected $\star$, and this means that domain wall excitations can occur in the sigma model, which must be described by additional parameters, and have not been taken into account in weak-coupling analyses so far. It is likely that these domain walls in the sigma model language are connected with the richness of phases in this symmetry class. In that context, the O(1) model with $p = 1/2$ is known to be a special case, since it maps to a sigma model without domain walls: this fact suggests that proliferation of domain walls may be necessary for localization $\star \star \star \star$.

We believe that the O(1) model has only a metallic phase, and has $\nu_1$ identically zero for all $p \neq 0$. Our calculations cover the range $0.1 \leq p < 0.5$ and $0.1 \leq \sin^2(\alpha) \leq 0.5$. If the model were to support a localized phase, it should appear at small $p, \alpha$. As an illustration of the absence of localization, at $p = 0.1$, $\sin^2(\alpha) = 0.1$, we calculate for $M' = 16$: $\nu_1 < 10^{-3}$ in the O(1) model, while $\nu_1 = 0.83$ in the CF model.

In summary, we find that two-dimensional models for localization in the symmetry class D can have quite different behavior according to the form of disorder. Several additional points deserve emphasis. The metallic phase of the CF model is self-dual, as is its multicritical point. By contrast, the RBIM is not self-dual but has higher supersymmetry at its multicritical point $\star \star$. There is little reason to suppose that these two multicritical points are in the same universality class. Separately, the apparent absence of an insulating phase in the O(1) model is remarkable, because the bare conductivity becomes small when $\alpha \rightarrow 0$ or $\pi/2$. Recently, it has been emphasized that the target manifold of the class D nonlinear sigma model is not connected $\star$, and this means that domain wall excitations can occur in the sigma model, which must be described by additional parameters, and have not been taken into account in weak-coupling analyses so far. It is likely that these domain walls in the sigma model language are connected with the richness of phases in this symmetry class. In that context, the O(1) model with $p = 1/2$ is known to be a special case, since it maps to a sigma model without domain walls: this fact suggests that proliferation of domain walls may be necessary for localization $\star \star \star \star$.

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[1] A. Altland and M.R. Zirnbauer, Phys. Rev. B 55, 1142 (1997); M.R. Zirnbauer, J. Math. Phys. 37, 4986 (1996).
[2] T. Senthil et al., Phys. Rev. Lett. 81, 4704 (1998).
[3] V. Kagalovsky et al., Phys. Rev. Lett. 82, 3516 (1999).
[4] T. Senthil, J.B. Marston, and M.P.A. Fisher, Phys. Rev. B 60, 4245 (1999); I.A. Gruzberg, A.W.W. Ludwig, and N. Read, Phys. Rev. Lett. 82, 4254 (1999).
[5] P.W. Brouwer et al., Phys. Rev. Lett. 85, 1064 (2000).
[6] R. Burschchuh et al., Phys. Rev. B 59, 4382 (1999).
[7] T. Senthil and M.P.A. Fisher, Phys. Rev. B 61, 9690 (2000).
[8] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[9] M. Boquet, D. Serban, and M.R. Zirnbauer, Nucl. Phys. B 578, 628 (2000).
[10] I.A. Gruzberg, N. Read, and A.W.W. Ludwig, cond-mat/0007254
[11] J.T. Chalker and P.D. Coddington, J. Phys. C 21, 2665 (1988).
[12] S. Cho and M.P.A. Fisher, Phys. Rev. B 55, 1025 (1997).
[13] S. Cho, PhD thesis, UC Santa Barbara, 1997, unpublished.
[14] N. Read and A.W.W. Ludwig, cond-mat/0007254
[15] R. Klesse and M. Metzler, Europhys. Lett. 32, 229 (1995).
[16] C.-M. Ho and J.T. Chalker, Phys. Rev. B 54, 8708 (1996).
[17] J.A. Blackman and J. Poulter, Phys. Rev. B 44, 4374 (1991).
[18] J.T. Chalker et al., unpublished.
[19] A. MacKinnon and B. Kramer, Phys. Rev. Lett. 47, 1546 (1981); Z. Phys. B53, 1 (1983).
[20] J.L. Pichard and G. Sarma, J. Phys C 17, 4111 (1981).
[21] A complication arises from the fact that the CF model is not (statistically) invariant under 90° rotations, except at $p = 0$ and $p = 1/2$. Because of this ($p$ and $\alpha$ dependent) anisotropy, the $\nu_n$ have been calculated throughout as the geometric mean of the Lyapunov exponents for cylinders with perpendicular orientations relative to the lattice.