Mathematical simulation of a steady process of anisotropic filtration

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Abstract. This article discusses the methods of approximate solution of mixed variational inequalities with operators of monotone type. The functional, which is included in this variational inequality, is separable, in other words, it is the sum of a number of non-differentiable functionals. These variational inequalities appear, in particular, in the description of steady incompressible filtration processes of highly viscous fluids in anisotropic medium.

1. Introduction
We consider a steady process of underground filtration of an incompressible fluid in an anisotropic porous media [1-5]. The generalized formulation of these problems is formulated in a form of mixed variational inequalities with operators of monotone type and separable, generally, non-differentiable, functional in Hilbert space. We establish the properties of an operator which is included in the variational inequality (inversely strongly monotone, coercivity) and functionals properties (Lipschitz continuity and convexity). This allows the opportunity to apply for prove of the existence theorem the known results of the theory of monotone operators [6]. To solving a variational inequality splitting iterative methods are proposed, which do not require to find the inverse operator to the original operator. Investigation of the convergence of the iterative process based on the method of successive approximations for finding a fixed point of some operator (transition operator). We prove that the transition operator is asymptotically regular [7]. This allow to prove the weakly convergence of successive approximations. At the same time for the filtration problem, each step of the iterative process actually reduces to solving boundary value problems for the Laplace operator or strongly elliptic operator. We note that the proposed methods allow to find the approximate values of not only the solution, but its characteristics (for filtration problems it is the approximate values of the gradient of solution, as well as the approximate value of flow velocities on the sets corresponding the points of multivalence in the filtration law), which is very useful from a practical points of view [8-12].

2. Statement and investigation of nonlinear stationary anisotropic filtration problems
We consider the steady-state filtration process of an incompressible fluid in a porous medium, which occupies a bounded domain \( \Omega \subset \mathbb{R}^m \), \( m \geq 2 \) with Lipschitz continuous boundary \( \Gamma = \Gamma_1 \cup \Gamma_2 \), \( \Gamma_1 \cap \Gamma_2 = \emptyset \), \( \text{mes} \Gamma_1 > 0 \), the pressure is equal to zero on the \( \Gamma_1 \), \( \Gamma_2 \) is impermeable part of the boundary. It is necessary to find a stationary fields of pressure \( u \) and fluid velocity \( v \), that satisfy the continuity equation

\[
\text{div} v(x) = f(x), \quad x \in \Omega,
\]
and the boundary conditions

\[ u(x) = 0, \quad x \in \Gamma_1, \quad (v(x), n) = 0, \quad x \in \Gamma_2, \]

where \( n \) is the external normal to \( \Gamma_2 \), and the assumption that the fluid follows to the filtration multivalued anisotropic law

\[ v_i(u) = \frac{1}{m} \sum_{k=1}^{m} \left[ \sum_{i=1}^{m} \alpha_{ij}^{(i)} g_i(D_i^2 u(x)) \right] \frac{\partial u(x)}{\partial x_k}, \quad x \in \Omega \]

where \( D_i^2 u = (Y_i \nabla u, \nabla u) \), \( \xi \rightarrow g_i(\xi^2) \xi \), \( i = 1, \ldots, m \), are the multivalued functions that determine the filtration law, and \( \tilde{f} \) is the function characterizing the density of the external sources.

We assume that the functions \( \xi \rightarrow g_i(\xi^2) \xi \) can be represented as \( g_i(\xi^2) \xi = g_{i0}(\xi^2) \xi + g_{i1}(\xi^2) \xi \), where \( \xi \rightarrow g_{i0}(\xi^2) \xi \) are the single-valued functions satisfying conditions

\[ g_{i0}(\xi^2) \xi = 0 \quad \text{when} \quad \xi \leq \beta_i \quad (\beta_i \text{ are the limiting gradients}), \]

\[ c_{i1}(\xi - \beta_i) \leq g_{i0}(\xi^2) \xi \leq c_{i2} \xi, \quad \xi \geq \beta_i, \quad c_{i1}, c_{i2} > 0, \]

\[ g_{i0}(\xi^2) \xi - g_{i0}(\xi^2) \xi \geq 0 \quad \text{when} \quad \xi \geq \xi, \]

\[ |g_{i0}(\xi^2) \xi - g_{i0}(\xi^2) \xi| \leq c_{i1} |\xi - \xi|, \quad c_{i1} > 0 \quad \text{when} \quad \xi \geq \xi, \]

\( \xi \rightarrow g_{i1}(\xi^2) \xi \) are the multi-valued functions, which have the form \( g_{i1}(\xi^2) \xi = \partial \eta h(\xi - \beta_i) \),

\[ h(\xi) = \begin{cases} 
0, & \xi < 0, \\
[0, 1], & \xi = 0, \\
1, & \xi > 0,
\end{cases} \]

It is obvious that the set of \( g_{i1}(\xi^2) \xi \) is the subdifferential of the function \( \mu_i \) in the point \( \xi \),

\[ \mu_i(\xi) = \partial \eta \mu(\xi - \beta_i), \quad \mu(\xi) = \begin{cases} 
0, & \xi < 0, \\
\xi, & \xi \geq 0,
\end{cases} \]

i.e.,

\[ \mu_i(\xi) - \mu_i(\xi) \geq \xi^* (\xi - \xi) \quad \forall \xi^* \in g_{i1}(\xi^2) \xi, \quad \forall \xi, \quad \xi \in R^l. \]

Relatively of the coefficients \( \{\alpha_{kl}^{(i)}\}_{k,j=1}^{m} \) of the matrix \( Y_i \) we assume, that \( \alpha_{kl}^{(i)} = \alpha_{kl}^{(i)}, \)

\[ \sum_{k,j=1}^{m} \alpha_{kl}^{(i)} \xi \xi \geq c_{i1} \sum_{k,j=1}^{m} \xi \xi, \quad c_{i1} > 0, \quad \text{i.e.,} \quad Y_i \text{ is the symmetric, positive definite matrix. Let's denote} \]

\[ (\xi, \zeta)_i = \sum_{k,j=1}^{m} \alpha_{kl}^{(i)} \xi \zeta = (Y_i \xi, \zeta). \]

Let us formulate the variational formulation of the problem (1)–(3). Suppose that \( u \) and \( v \) are the solution of this problem. We denote as \( C^\infty_{\Gamma_1}(\Omega) \) the set of infinitely differentiable functions in \( \Omega \), which are equal the zero on \( \Gamma_1 \). Using (8), we find that the function \( u \) satisfies the variational inequality

\[ \int_{\Omega} \tilde{f}(x)\eta(x) dx \leq \frac{1}{m} \sum_{i=1}^{m} \int_{\Omega} g_{i0}(D_i^2 u(x))(\nabla u(x), \nabla \eta(x)) dx + \]

\[ + \frac{1}{m} \sum_{i=1}^{m} \int_{\Omega} \mu(D_i(u(x) + \eta(x)) - \beta_i) dx - \frac{1}{m} \sum_{i=1}^{m} \int_{\Omega} \mu(D_i(u(x)) - \beta_i) dx \quad \forall \eta \in C^\infty_{\Gamma_1}(\Omega) \]
Let $V = \{v \in W_2^1(\Omega) : \eta(x) = 0, x \in \Gamma_1 \}$ be the Sobolev space with the inner product $(u, \eta)_V = \int_{\Omega} (\nabla u(x), \nabla \eta(x)) \, dx$ and the norm $\|u\|_V = (u, u)^{1/2}_V$. It is easy to verify, that under the conditions (4), (5) the forms $a_i(u, \eta) = \int_{\Omega} g_{i0}(D^2_i u(x))(\nabla u(x), \nabla \eta(x)) \, dx$ are linear and bounded to the second argument, and thus they generate the operators $A_i : V \rightarrow V$ by the formula $a_i(u, \eta) = (A_i \mu, \eta)_V$. Let's set $A = \frac{1}{m} \sum_{i=1}^m A_i$. Further, we consider the functionals $F_i : V \rightarrow \mathbb{R}^m$ by the formulas

$$F_i(u) = \frac{1}{m} \int_{\Omega} \mu_i(D_i(u(x)) \, dx = \frac{1}{m} \int_{\Omega} g_{i0}(\zeta^2) \xi \, d\xi \, dx.$$ (12)

We assume that the function $\tilde{f}$ generates a continuous linear functional $f$ on $V : \int_{\Omega} \tilde{f}(x) \eta(x) \, dx = (f, \eta)_V$. Then, according to (9), under the decision of the stationary problem of filtration of an incompressible fluid, satisfying the nonlinear multi-valued anisotropic filtration law with limiting gradient, we mean the function $u \in V$, which is the solution of the variational inequality (cf. [13])

$$(Au - f, \eta - u)_V + \frac{1}{m} \sum_{i=1}^m F_i(\eta) - \frac{1}{m} \sum_{i=1}^m F_i(u) \geq 0 \quad \forall \eta \in V. \quad (12)$$

In practice the important factor for solving problems of nonlinear filtration with limiting gradient is to find the boundaries of dead zones, i.e. border regions, where there is no fluid flow.

**Theorem 1.** Let the conditions (4)-(7) hold. Then the variational inequality (12) has a non-empty, convex, closed set of solutions. If $u \in V$ is the solution of (12), then there exists the function $v \in H = [L^2(\Omega)]^m$, that almost everywhere on $\Omega$ satisfied the inclusions (3), and there is the continuity equation $\int_{\Omega} (v(x), \nabla \eta(x)) \, dx = \int_{\Omega} \tilde{f}(x) \eta(x) \, dx$ for any function $\eta \in C^\infty_c(\Omega)$.

Indeed, under the conditions (4)-(7) the operator $A$ is inversely strongly monotone, i.e.,

$$(Au - Ar\eta, \eta - u)_V \geq \sigma \|Au - Ar\eta\|_V^2, \sigma > 0,$$

for all $u, \eta \in V$, in particular, it is the monotone and Lipschitz continuous with the constant $1/\sigma$, and the coercive [13]. The functionals $F_i$, defined by formulas (2), and therefore their sum are convex and Lipschitz continuous, therefore weakly lower semicontinuous. So, the existence of solutions of variational inequalities (12), closed convex set its decisions follows from the general theory of monotone operators [13]. If $u \in V$ is the solution of the (12), then according to the subdifferential definition

$$f - Au \in \partial \left( \sum_{i=1}^m F_i(u) \right). \quad (13)$$

Since the $F_i$ are convex and continuous, then [13] $\partial \left( \sum_{i=1}^m F_i(u) \right) = \sum_{i=1}^m \partial F_i(u)$. By analogy with the [14-16], it is easy to get in the point $u$ the subdifferential of the functional $F_i$ is the set of the linear continuous on $V$ functionals $l_i$:

$$(l_i, \eta)_V = \frac{1}{m} \int_{\Omega} \frac{\chi_{i\alpha}(x)}{D_i(u(x))}(\nabla u, \nabla \eta), \quad \chi_{i\alpha} \in L^\infty(\Omega), \quad \chi_{i\alpha}(x) \in \mathcal{B}_{i\alpha} h(D_i u(x)) \beta_j). \quad (13)$$

So, the relation (13) means that the continuity equation is satisfied.
In this case, the question about the construction of the specified field of the filtration velocity $v$ on the sets of corresponding points of multivalence in the filtration law (when $D_i u(x) = \beta_i$) is open.

3. The iterative method for solving nonlinear stationary anisotropic filtration problems

In order to solve the variational inequalities of the form (12) iterative splitting methods are proposed [17-25]. The main difficulty in this case is the solving of arising problems of minimization in each iteration. In the case of an isotropic filtration law this problem was solved in explicit form. Due to this, it is possible to effectively calculate the subdifferential of the conjugate functional to the minimized one. In the case of anisotropic filtration law, when the minimizing functional is the sum of several functionals, the calculation of the conjugate functional is a difficult problem. We consider the splitting algorithm, which solves indicated above problem.

The variational inequality (12), which characterizes the filtration problem can be written as follows:

$$ (Au - f, \eta - u)_V + \frac{1}{m} \sum_{i=1}^{m} G_i(B_i\eta) - \frac{1}{m} \sum_{i=1}^{m} G_i(B_i u) \geq 0 \quad \forall \eta \in V, $$

if the functionals $F_i$ have the form $G_i = G_i \circ B_i$, where the functionals $G_i$ defined on $H$, according to the formula $G_i(z) = \frac{1}{2m} \int_{\Omega} |\xi|^2 g_{ii}(\xi) d\xi dx$, they are convex and Lipschitz continuous, $B_i : V \to H$ – linear continuous operators, $B_i = Y^{1/2} \nabla$.

To solve the variational inequality (14) we consider the following iterative process. Let’s define $\tau > 0$, $r > 0$, and set $u^{(0)} \in V$, $y_{i}^{(0)}$, $\lambda_{i}^{(0)} \in H$, $i = 1, 2, \ldots, m$, are the arbitrary elements.

For $k = 1, 2, \ldots$ we know $y_{i}^{(k)}$, $\lambda_{i}^{(k)}$, let’s define

$$ u^{(k+1)} = u^{(k)} - \tau (Au^{(k)} - f + r \sum_{i=1}^{m} B_i B_i^* u^{(k)} + \sum_{i=1}^{m} B_i^* (\lambda_{i}^{(k)} - r y_{i}^{(k)})). $$

Then we find $y_{i}^{(k+1)}$ and solve the problem

$$ r (y_{i}^{(k+1)} - y_{i}^{(k)} - \lambda_{i}^{(k+1)})_H + G_i(z_{i}) - G_i(z_{i} - \lambda_{i}^{(k+1)}) \geq 0 \quad \forall z_{i} \in H, \quad i = 1, 2, \ldots, m. $$

Finally, we find

$$ \lambda_{i}^{(k+1)} = \lambda_{i}^{(k)} + r (B_i u^{(k)} - y_{i}^{(k+1)}), \quad i = 1, 2, \ldots, m. $$

In the equation (15) the operator $B_i^* : H \to V$ is the conjugate operator to the operator $B_i : V \to H$.

In order to study the convergence of the described iterative process we write out the explicit form of the transition operator. Let us first recall the definition and properties of the proximal mapping (see., e.g., [13]). Suppose that $V$ is a Hilbert space, $\xi \in V$ is the any element, $\Phi : V \to R^1$ is the convex, propex, lower semicontinuous functional. Then the variational inequality

$$ (u - \xi, \eta - u)_V + \Phi(\eta) - \Phi(u) \geq 0 \quad \forall \eta \in V $$

has a unique solution $u$. Thus, the operator $\text{Prox}_{\Phi} : V \to V$, $u = \text{Prox}_{\Phi}(\xi)$ is defined. From the definition of the proximal mapping using the variational inequality (18), it is implied that the operator is confirmly non-extracted [13, 26], i.e.,
\[
\|\text{Prox}_{\phi}(u) - \text{Prox}_{\phi}(\eta)\|_V^2 \leq (\text{Prox}_{\phi}(u) - \text{Prox}_{\phi}(\eta), u - \eta)_V \quad \forall u, \eta \in V.
\]  

We denote by \( H^m \) as the direct product of the \( H \) spaces and introduce the operator \( T : Q = V \times H^m \times H^m \to Q \), mapping a \( q = (q_0, q_1, \ldots, q_m) = (u, y_1, \ldots, y_m, \lambda_1, \ldots, \lambda_m) \) to the element

\[
T_0 q = q_0 - \tau (Aq_0 - f + r \sum_{i=1}^{m} B_i B_i^* q_0 + \sum_{i=1}^{m} B_i^* (q_{m+i} - r y_i)),
\]

\[
T_i q = \text{Prox}_{G_i(r)}(B_i T_0 q + \frac{1}{r} q_{m+i}),
\]

\[
T_m q = q_{m+i} + r(B_i T_0 q - T_i q),
\]

where the operators \( \text{Prox}_{G_i(r)} \) are proximal (the functionals \( G_i / r \) are the convex, proper, lower semicontinuous functionals, because \( r > 0 \)).

We set \( q^{(k)} = (u^{(k)}, y_1^{(k)}, \ldots, y_m^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_m^{(k)}) \), then the iteration process (15) –(17) can be written as:

\[
q^{(k+1)} = Tq^{(k)}, \quad k = 1, 2, \ldots,
\]

\( q^{(0)} \in Q \) is an arbitrary element, i.e., \( T \) is the transition operator of an iterative process.

The relation between the solution of the problem (14) and the set of fixed points of the operator \( T \) is defined by the following theorem.

**Theorem 2.** The point \( q = (u, y_1, \ldots, y_m, \lambda_1, \ldots, \lambda_m) \) is a fixed point of the operator \( T \) if and only if the following conditions are satisfied:

\[
y_i = B_i u, \quad \lambda_i = \partial G_i(B_i u), \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} B_i^* \lambda_i = Au - f.
\]

In this case, the first component of any fixed point of the \( T \) operator is a solution of problem (12).

Really, if \( q = (u, y_1, \ldots, y_m, \lambda_1, \ldots, \lambda_m) \) is the fixed point of the \( T \) operator, then by virtue of (20), (21) we have the following:

\[
u = u - \tau (A u - f + r \sum_{i=1}^{m} B_i^* B_i u + \sum_{i=1}^{m} B_i^* (\lambda_i - r y_i)), \quad y_i = \text{Prox}_{G_i(r)}(B_i u + \frac{1}{r} \lambda_i), \quad \lambda_i = \lambda_i + r(B_i u - y_i), \quad i = 1, 2, \ldots, m.
\]

Thus, using (18) and the subdifferential definition we get the desired result.

Let us introduce on \( Q \) the bilinear form by the formula:

\[
(p, q)_Q = \frac{1 - m \tau r}{\tau} (p_0, q_0)_V + \frac{r}{\tau} \sum_{i=1}^{m} (p_i, q_i)_H + \frac{1}{\tau} \sum_{i=1}^{m} (p_{m+i}, q_{m+i})_H.
\]

If the condition \( m \tau r < 1 \) then the form (23) satisfies the axioms of a scalar product. We have

**Theorem 3.** Suppose that \( A \) is the invertibly strongly monotone operator with the constant \( \sigma > 0, \quad 0 < \tau < 2 \sigma / (2m r + 1) \), and the problem (12) has at least one solution. Then operator \( T \) is asymptotically regular, the iterative sequence \( \{q^{(k)}\}_{k=0}^{\infty} \), constructed by, converges weakly in \( Q \) for \( k \to \infty \). Its limit is a fixed point of the operator \( T \), \( \|y^{(k)} - B_i u^{(k)}\|_H \to 0 \) for \( k \to \infty \).

Indeed, by virtue of the definition of operator \( T \), and using (19), we can establish that \( T \) is the contractive operator, moreover, for any \( p, q \in Q \) the inequality
\[ \| Tq - Tp \|_Q^2 + \delta (Aq_0 - Ap_0, q_0 - p_0) \nu + \frac{1}{1 - m \tau r} \| (1 - r \tau) (q_0 - T_0 q - (p_0 - T_0 p)) - \tau (Aq_0 - Ap_0) \|_H^2 + r \sum_{i=1}^m \| q_i - B_i T_0 q - (p_i - B_i T_0 p) \|_H^2 \leq \| q - p \|_Q^2, \quad \delta = 2 - \tau f (\sigma (1 - m \tau r)). \]

hold. This inequality implies the asymptotic regularity of the operator \( T \), since the iterative process (22) weakly converge, and also in \( H \) the sequences \( \{ y_i^{(k)} - B_i u^{(k)} \}_{k=0}^\infty \) strongly converge to zero \( i = 1, 2, \ldots, m \).

It follows from theorem 3 that the sequence \( \{ u^{(k)} \}_{k=0}^\infty \), \( \{ y_i^{(k)} \}_{k=0}^\infty \), constructed by (15)-(17), converge weakly to \( u \) and to \( B_i u_i \), \( i = 1, 2, \ldots, m \), respectively, where \( u \) is the solution of variational inequality (12).

Let us consider the features of the application of the iterative method (15) - (17) for the solutions of the anisotropic filtration problems.

In order to determine \( u^{(k+1)} \) it is necessary firstly to solve the boundary value problem

\[ \begin{cases}
R_s = f - Au^{(k)} + \sum_{i=1}^m B_i^* (\lambda_i^{(k)} - r y_i^{(k)}) + mR u^{(k)}, & x \in \Omega, \\
s(x) = 0, & x \in \Gamma_1, \quad (\nabla s(x), n) = 0, & x \in \Gamma_2,
\end{cases} \tag{24} \]

where \( R = -\text{div} (\sum_{i=1}^m Y_i \nabla) \), and then put \( u^{(k+1)} = u^{(k)} + \tau s \). We note that in our case \( B_i^* = -\text{div} Y_i^{1/2} \).

In the numerical implementation of the iterative method the main difficulty is to solve the minimization problems (16). We write them in the form:

\[ (r B u^{(k+1)} + \lambda_i^{(k)} - r y_i^{(k+1)} )_H \leq G_i (z) + \frac{r}{2} \| z \|_H^2 - G_i (y_i^{(k+1)}) + \frac{r}{2} \| y_i^{(k+1)} \|_H^2 \forall z \in H, \]

or,

\[ (r B u^{(k+1)} + \lambda_i^{(k)} - r y_i^{(k+1)} )_H \leq G_{i,r} (z) - G_{i,r} (y_i^{(k+1)}) \quad \forall z \in H, \tag{25} \]

where \( G_{i,r} (z) = G_i (z) + \frac{r}{2} \| z \|_H^2 \). Using the definition of the subdifferential \( \partial G_{i,r} \), we write (25) in the form of inclusion \( r B_i u^{(k+1)} + \lambda_i^{(k)} \in G_{i,r} (y_i^{(k+1)}) \), which is equivalent [13] to the inclusion \( y_i^{(k+1)} \in G_{i,r}^* (r B_i u^{(k+1)} + \lambda_i^{(k)}) \), where \( G_{i,r}^* \) is a conjugate functional [13] to \( G_{i,r} \). The proposed scheme for solving the minimization problem (16) allows significantly alleviate the numerical implementation of the splitting method, when we can effectively calculate the subdifferential \( \partial G_{i,r}^* \). In this case the problem of finding the element \( y_i^{(k+1)} \) is reduced to calculations by explicit formulas. We use this way for solving the considered anisotropic filtration problems. We have

\[ G_{i,r}^* (z) = \int_\Omega \frac{1}{2} g_{i,r}^* (\xi^2) \xi d\xi dx, \]

where
The functional $G^*_G$ is convex and differentiable with respect to the Gateaux, therefore [13] the subdifferential $\partial G^*_G$ consists of a single element, that coincides with its gradient, which is determined by the shape $(G^*_G y, z) = \int g^*_G(|z|^2)(z, y)dx$. Thus, the problem (16) consists of computing $y_{i}^{(k+1)}$ using the formula $y_{i}^{(k+1)} = g^*_G(|z|^2)z$, where $z = rB_i^{(k+1)} + \lambda_i^{(k)}$. Therefore, each step of the considered iterative method is actually reduced to solving the boundary problem (24) with the linear strongly elliptic operator. At the same time, it follows from Theorem 2 that the approximate value of filtration velocity $v^{(k)}$ can be calculated by the formula

$$v_{j}^{(k)} = \frac{1}{m} \sum_{i=1}^{m} \sum_{i=1}^{m} \alpha_{ij}^{(i)} g_{ij}^{0}(\|y_{i}^{(k)}\|^2 + \frac{\lambda_i^{(k)}}{|y_{i}^{(k)}|}) y_{j}^{(k)}.$$  

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