A hybrid post-Newtonian – effective-one-body scheme for spin-precessing compact-binary waveforms

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We introduce TEOBResumSP: an efficient yet accurate hybrid scheme for generating gravitational-waveforms from spin-precessing compact binaries. Our approach Euler-rotates aligned-spin TEOBResumS effective-one-body waveforms from a precessing frame to an inertial frame. We obtain the Euler angles by solving the post-Newtonian precession equations expanded to (next-to)$^4$ leading (second post-Newtonian) order and use them in the waveform mode rotations to extend non-precessing TEOBResumS waveforms to generic spin configurations. The scheme is compared to current state-of-the-art precessing approximants NRSur7dq4 and SEOBNRv4PHM in terms of frequency-domain matches of the $\ell = 2$ gravitational-wave strain for 200 and 1100 binaries, respectively, with the initial gravitational-wave frequencies between 20 and 50 Hz and the precessing spin parameter $\chi_p$ ranging up to one. The matches are better than 0.965 for 85% of the NRSur7dq4 and 75% of the SEOBNRv4PHM sets. The largest disagreements occur for large mass ratios and for large spin components along the orbital plane quantified in terms of a new parameter, $S_{\perp,\text{max}}$, that we introduce. The disagreements stem from Euler-rotating non-precessing waveforms with constant spins, which can be replaced by time-varying $z$-components of spins. Our scheme provides a robust alternative precessing approximant to be employed in the parameter estimation of generic-spin compact binaries with TEOBResumSP waveforms.

I. INTRODUCTION

Gravitational-wave events have become routine in observational astronomy: the Advanced LIGO [1]-Virgo [2] interferometers detected ten binary black hole and one binary neutron star mergers during the first and second observing runs. [3–9]. The third observing run began on 1 April 2019 and delivered by its [premature] end a year later the second binary neutron star merger [10], the first binary black hole with a significant mass asymmetry [11], and additionally more than four dozen triggers with false alarm rates of less than one per year [12]. Most of these triggers are likely due to the inspiral and merger of binary black holes.

In order for triggers to become confirmed events, with estimated properties such as masses and spins, parameter estimation studies must be conducted on a sufficiently “cleaned” version of the relevant segment of the detector data. This requires a large set of “realistic” theoretical gravitational waveform templates which can be cross-correlated with the data. For stellar-mass compact binary systems, there are currently three different approaches to generating the theoretical gravitational waves (GWs) resulting from the inspiral and merger of the systems: post-Newtonian theory [13], numerical relativity, and the effective-one-body theory [14, 15].

Post-Newtonian (PN) theory employs a large-separation (weak-field) expansion to the Einstein field equations. Current PN technology for the evolution of quasi-circular inspirals is at the 3.5PN level with partial higher-order PN information available [16, 17]. As PN information is fully analytical, the resulting waveforms can be evaluated very quickly. Consequently, the LIGO-Virgo Collaboration (LVC) has at its disposal a plethora of PN-based Taylor waveform approximants summarized in Ref. [18]. These approximants are employed to model the inspiral phase of binary neutron stars at low frequencies and extract the chirp mass [10, 19].

Binary black hole (BBH) systems are more massive, thus transit through the LIGO-Virgo detection bandwidth much more quickly than binary neutron stars (BNSs), e.g., GW150914 lasted less than 20 milliseconds [3] whereas GW170817 lasted nearly a minute [4]. As such, we can detect only the last few dozen cycles of their GWs before merger. Such GWs are generated in the very-strong-gravity regime where PN approximation is not reliable. This is the domain of numerical relativity (NR). Since the breakthroughs of 2005 [20–22], it has become routine to evolve strongly gravitating spacetimes of compact binary mergers on large computing clusters. There are now several NR catalogs containing thousands of simulations of compact binary inspirals [23–31]. Of these, the most comprehensive is the 2019 SXS catalog which contains 2018 simulations of precessing systems with the dimensionless Kerr spin parameter up to 0.998 [24].

The effective-one-body (EOB) approach bridges PN theory and NR. It maps the two-body PN motion to a geodesic motion in an effective spacetime via a deformation performed in terms of a quantity called the symmetric mass ratio [14, 15]. In its core, EOB contains an effective Hamiltonian which resums the PN series in a suitable way to better embody the effects of the strong-field regime [32]. The inspiral is driven by a specially factorized/resummed radiation-reaction force [33]. The resulting multipolar gravitational waveforms are also written in a factorized form [34, 35]. The analytical EOB model is further supplemented with input from NR, thus extending the EOB evolution through the merger and, if exists, ringdown stages. These so-
called EOBNR models \([36–38]\) have been incorporated into several waveform-generating models (approximants) \([37, 39–41]\) that are used for parameter estimation studies of LIGO-Virgo GW events. The main advantages of employing EOB-based waveform approximants for parameter estimation are that they (i) contain strong-field information (ii) can be extended to the full parameter space, and (iii) are much faster to evolve than NR simulations. In particular, EOB models are the only available method to accurately model binary neutron star coalescences from low frequencies and up to merger \([19, 42–45]\). Thus, EOB is optimal for providing accurate waveforms for parameter estimation studies of long merger signals, and for extracting information about tides.

Although it has thus far been very difficult to distinguish the effects of precession on the gravitational waves from the few dozen sources hitherto detected, there are at least three GW events for which it has been inferred, at the 99% credible level, that the pre-merger binary components have nonzero spin. These are GW151226 \([5]\), where at least one black hole has dimensionless spin \(> 0.28\) \([46]\), GW170729 \([9, 47]\), where at least one black hole has dimensionless spin \(> 0.27\) \([46]\), and GW190412 where either the primary \([11]\) or the secondary \([48]\) has positive dimensionless spin depending on the priors used.

In binaries containing spinning black holes and/or millisecond pulsars, the spin-orbit and the spin-spin interactions contribute significantly to the phase and modulate distinguishably the amplitude of the emitted GWs. For example, there are more than 20 precession cycles contributing to the phasing of the GWs for a BNS with total mass of \(3 M_\odot\) inspiralling from 30 Hz. \([49]\). Therefore, given that the required relative phase accuracy of the theoretical waveform templates must be better than \(\sim 10^{-3}\) \([50]\), the templates must incorporate the effects of precession. Neglecting precession for high-mass ratio binaries can cause event rate losses of \(\sim 15\%\) and as high as 25% - 60% for the worst cases \([51, 52]\).

There has been a dedicated and an ever-increasing effort to produce accurate gravitational waveforms from precessing compact binary systems. Initial developments were made in post-Newtonian theory \([53, 54]\) after the pioneering work of Mathisson, Papapetrou, and Dixon (MPD) on the motion of spinning test particles in curved spacetimes \([55–57]\). There are now several waveform approximants available for precessing spin analysis (and implemented in the LIGO Algorithm Library (LAL) \([58]\)), which are, chiefly: (i) SpinTaylorT class of approximants which employ 1.5PN analytical expressions of Ref. \([59]\) for the waveform harmonic modes as functions of the spherical angles of the Newtonian orbital angular momentum vector. (ii) (NR)PhenomP class of approximants which initially transformed non-precessing waveforms into precessing waveforms using a single parameter \([60]\) then were later upgraded to transformations consisting of Euler rotations of the modes via three Euler angles \([61–64]\). (iii) SEOBNR class of approximants \([39, 65, 66]\) which evolve the EOB dynamics and precession equations as a coupled system to determine the Euler angles for the rotation of the non-precessing waveform modes. (iv) NRsur class which are surrogate waveform models that interpolate large sets of numerical relativity waveforms in a certain non-inertial co-precessing frame \([67, 68]\). All of these approximants solve the same precession equations, albeit truncated at different PN orders or suitably incorporated into a particular EOB Hamiltonian. The solutions to the precession equations are then translated into the spherical angles of the Newtonian orbital angular momentum. On the other hand, the approximants construct the waveforms differently, but these can be grouped into two main approaches: (1) Using analytical 1.5 PN expressions of Ref. \([59]\). (2) Using the so-called twist method of Ref. \([69]\) which is what concerns us in this article so we provide some details next. Note that a twisting procedure is used also in the NRsur approximants where the surrogate is trained using precessing waveforms that are Euler-rotated to the non-inertial co-orbital frame.

The seeds of the twist method were sown in Ref. \([70]\) \(^1\), where it was shown that in a special non-inertial frame, called the precessing frame, the orbital phase agreed with the PN orbital phase of a non-precessing system. In other words, the modulations in the gravitational wave phase due to precession factored out. Subsequently, Ref. \([71]\) obtained rigorous expressions for the transformation waveform multipoles under rotations, which were then employed by Ref. \([72]\) in order to generate precessing post-Newtonian waveforms to compare with their numerical results.

Motivated perhaps by these early works, Ref. \([69]\) then introduced the quadrupole-aligned (QA) frame defined by the direction toward which the amplitudes of the \((2, \pm 2)\) modes are maximized which turned out to coincide with the instantaneous direction of the total orbital angular momentum vector. Similarly, Ref. \([73]\) constructed a new non-inertial frame by equating the radiation axis with the eigenvector of the rotation group generators which had the largest absolute eigenvalue. Subsequently, Ref. \([74]\) demonstrated that the preferred precessing frames of Refs. \([69, 73]\) are the same if one employs only the \((2, \pm 2)\) modes in the \(m\)-mode sum. Additionally, Ref. \([74]\) rigorously showed the necessity for a third Euler angle \(\gamma\) in order to obtain a unique precessing frame which they dubbed the minimal-rotation frame.

The size of the parameter space for generic precessing binaries presents another formidable challenge for parameter estimation as the number of intrinsic parameters increases from three (mass ratio and two spin magnitudes) for configurations where the spins are (anti)parallel to the orbital angular momentum, which we refer to as either non-precessing or aligned-spin configurations, to seven for binary black holes, and even more in the case of binary neutron stars to parametrize their tidal interaction.

\(^1\) Though, there seems to be a hint of the twist method in App. B of Ref. \([50]\).
As brute-force coverage of such a large space is computationally expensive, approaches based on reduced-order modelling have recently been developed starting with Ref. [75], where the number of parameters is reduced to two in the QA frame by introducing an effective spin parameter \( \chi_{\text{eff}} \). The precessing waveform is then obtained by twisting the QA waveform with three Euler angles as already described. Ref. [76] took this approach further by packaging the four in-plane (perpendicular to the Newtonian angular momentum) components of the binary’s spin vectors into a single precession parameter, \( \chi_p \), thereby reducing the dimensionality of the parameter space to four. Though most of the reduced-order models have been built for non-precessing systems [77–82], they have eventually led to the emergence of NR “trained” precessing surrogates [67, 68, 83, 84]. Additionally, new approaches are being developed such as “the two-harmonic approximation” [85].

In summary, there now exist several diverse precessing waveform approximants of which the most prominent ones are SpinTaylorT4, NRsur7dq4, PhenomPv3HM (previously PhenomPv2), and SEOBNRv4PHM (previously SEOBNRv3). The latter three approximants have quickly become the preferred waveform models for parameter estimation by the LVC. Crucially, there seems to be a disagreement between the latter two for binary systems with large spins and mass ratios [86], highlighting what one should always keep in mind: waveform approximants are approximate as the name implies so they can disagree.

This paper is the first of a series that develops TEOBResumSP, a generic-spin approximant based on the Euler rotation of aligned-spin waveforms generated by TEOBResum [41]. TEOBResumSP is a state-of-the-art aligned-spin EOBNR model with enhanced spin-orbit, spin-spin, and tidal interactions [45, 87] that is very fast [88] and robustly produces inspiral-merger-ringdown waveforms for five additional modes beside the dominant (2,2) mode [89, 90]. TEOBResumSP is very different in its design from SEOBNRv4PHM, in particular in the spin sector [91], thus provides the only fully independent waveform model from the approximants currently in use for GW analysis (e.g., PhenomPv3 uses fits of SEOBNR waveforms [66]). Our goal in this initial implementation of TEOBResumSP is to introduce minimal modifications to the existing TEOBResum infrastructure. Therefore, we opt for an approach whereby we produce aligned, constant spin waveforms using TEOBResum then generate precessing waveforms by twisting the non-precessing waveforms as is done in, e.g., PhenomPv3HM.

This article is organized as follows. We start by introducing the PN precession equations in Sec. II. In Sec. III, we present some mathematical details for the waveform twist operation. In Sec. IV, we compare TEOBResumSP waveforms with the following waveform approximants: SpinTaylorT4, NRsur7dq4, and SEOBNRv4PHM. We summarize our results in Sec. V. We work in geometrized units setting \( G = c = 1 \) from which one can recover the SI units via \( GM_\odot/c^3 \approx 4.925491 \times 10^{-6} \) sec, where \( M_\odot \) denotes a solar mass. We use bold font to denote Euclidean three-vectors with an overhat representing three-vectors of unit length. Overdots denote derivatives with respect to time.

II. AN OVERVIEW OF PRECESSING COMPACT BINARY SYSTEMS

Let us consider a compact binary system in a quasi-spherical inspiral with the subscripts labelling the primary and 2 labelling the secondary component. Accordingly, the individual masses are denoted by \( m_1 \) and \( m_2 \) with \( m_1 \geq m_2 \). The total mass is defined as \( M = m_1 + m_2 \). Let us also introduce the mass ratio \( q \equiv m_2/m_1 \leq 1 \), the reduced mass \( \mu \equiv Mq/(1+q)^2 \), and the symmetric mass ratio \( \eta \equiv q/(1+q)^2 \). Note that in this article, we often set \( M = 1 \), e.g., Eqs. (1a)-(1c), but sometimes restore solar-mass units \( (M_\odot) \) for \( M \), cf. Eqs. (11), (14). We additionally endow the binary components with spins \( \mathbf{s}_1, \mathbf{s}_2 \), respectively.

A. Spin-orbit Precession Equations

The Newtonian orbital angular momentum for the binary is given by \( \mathbf{L}_N = \mu \mathbf{r} \times \mathbf{v} \), where \( \mathbf{r}, \mathbf{v} \) are the relative separation and velocity vectors of the binary in the usual center-of-mass frame. Note that \( \mathbf{L}_N \) is different from its non-Newtonian counterpart \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \), where \( \mathbf{p} \) is the relative momentum. This distinction, due to \( \mu \mathbf{v} \neq \mathbf{p} \), is a consequence of the fully general relativistic MPD equations for the motion of a spinning test mass in curved spacetime. From PN theory, one obtains \( \mathbf{L} = \mathbf{L}_N + \Delta \mathbf{L}_{\text{PN}} + \ldots \) with correction terms, \( \Delta \mathbf{L}_{\text{PN}} \), known up to 3.5PN (see, e.g., Eq. (4.7) of Ref. [92]). Note that, by definition, the Newtonian \( \mathbf{L}_N \) remains perpendicular to the orbital plane.

Let \( \omega \) be the orbital frequency then via Kepler’s third law: \( v \equiv |\mathbf{r}| = \omega^{-2/3} \). Accordingly, \( \mathbf{L}_N \equiv |\mathbf{L}_N| = \mu r^2 \omega = m_1 m_2 / \omega^{1/3} = \eta / v \), where we have introduced \( v \equiv |\mathbf{v}| = \omega^{1/3} \), i.e., the relative speed between the binary’s components in the usual center-of-mass frame. Clearly, \( v < 1 \) and furthermore, \( v \ll 1 \) for most of the inspiral (recall, \( v = v/c \) in restored units). Note that each power of \( v \) corresponds to a half PN order. In this work, we will use \( v \) to track the orders in the precession equations. Consequently, we reserve expressions such as next-to-leading order (NLO) to verbally track each power of \( v \) beyond a given leading-order (LO) expression. One can start with the general MPD equations of motion and obtain the PN expansions for the time evolution of \( \mathbf{s}_1, \mathbf{s}_2 \). The details of this derivation can be found in, e.g., Secs. II, III of Ref. [93], and Sec. II of Ref. [94]. Up

To our knowledge, a similar parameter was first introduced in Ref. [50].
to NLO, i.e., 0.5PN, the orbital angular momentum and spin precession equations are given by \[49, 95\]

\[
\dot{S}_1^{NLO} = v^5 \eta \left(2 + 3 \frac{\eta}{2g}\right) \left(\mathbf{L}_N \times \mathbf{S}_1\right) + \frac{v^6}{2} \left(\mathbf{S}_2 - 3[\eta \mathbf{S}_1 + \mathbf{S}_2] \cdot \mathbf{\hat{L}}_N\right) \times \mathbf{S}_1,
\]

\[
\dot{S}_2^{NLO} = v^5 \eta \left(2 + 3 \frac{\eta}{2g}\right) \left(\mathbf{L}_N \times \mathbf{S}_2\right) + \frac{v^6}{2} \left(\mathbf{S}_1 - 3[\mathbf{S}_1 + \eta^{-1} \mathbf{S}_2] \cdot \mathbf{\hat{L}}_N\right) \times \mathbf{S}_2,
\]

\[
\dot{L}_N^{NLO} = -\frac{v}{\eta} \left(\dot{S}_1^{NLO} + \dot{S}_2^{NLO}\right).
\]  

Note that, as is usual in the literature, we present the orbit-averaged evolution equations. As such, our solutions to these equations do not capture the mutation of \(\mathbf{L}_N\), but this is of no consequence for parameter estimation purposes at the sensitivity of the advanced GW detectors [96]. For non-averaged versions, cf. App. A of Ref. [93].

The particular form of Eq. (1c) above is the result of total angular momentum conservation: \(\mathbf{\dot{J}} = 0\), where \(\mathbf{J} = \mathbf{L} + \mathbf{S}\) with \(\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2\). The forms of Eqs. (1a-1c) have the added benefit that the evolution of the Newtonian orbital angular momentum can be written as a classical mechanical precession equation:

\[
\dot{\mathbf{L}}_N^{NLO} = \mathbf{\Omega}_{NLO} \times \mathbf{L}_N,
\]

where \(\mathbf{\Omega}_{NLO}\) can be extracted straightforwardly from Eqs. (1a - 1c).

The effect of radiation reaction is implicit in \(v = v(t)\) in Eqs. (1a-1c). For nonspinning systems, \(\dot{v} = \dot{v}(v)\) is fully known as a PN series starting from \(\sim v^9\) and going up to 3.5PN order \(\sim v^{16}\). For systems with spin, spin-orbit terms enter first at 1.5PN and spin-spin terms at 2PN. Here, we employ the Taylor4 resummed form of \(\dot{v}(v)\) [18, 70] as adopted in the SpinTaylor4 approximant. The series coefficients for \(\dot{v}(v)\) can be found, e.g., in App. A of Ref. [97].

For precessing binaries, there are three time scales of relevance: radiation-reaction timescale \(T_{RR}\), precession time scale \(T_{pr}\), and orbital time scale \(T_{orb}\). Integrating \(\dot{v} \sim v^9\) yields \(T_{RR} \sim v^{-8}\). From \(v = \omega^{1/3}\), we obtain \(T_{orb} \sim v^{-3}\). Finally, the precession equation (1a) gives \(T_{pr} \sim |\mathbf{S}_1|/|\mathbf{S}_1| \sim v^{-5}\). Since \(v \ll 1\) mostly, we have the following separation of time scales:

\[
T_{orb} \ll T_{pr} \ll T_{RR}.
\]  

Thanks to this separation of scales, we expect our hybrid approach, which combines EOB dynamics with PN precession, to work well as we show in Secs. IVB, IVC.

Recall that Eqs. (1a - 1c) are 0.5-PN (NLO) accurate. Though this is the usual order in the literature, we employ versions of the precession ODEs that have been pushed to the limit of the current analytical PN knowledge, which we denote as N4LO (2 PN) here. As far we can tell these have never appeared in a journal article, but exist in written form in several approximants such as SpinTaylor4. Defining \(\delta m = m_1 - m_2\) in natural units (e.g., \(m_1 = 1/(1 + q)\)), the N4LO spin-orbit precession ODEs read

\[
\dot{S}_1^{N4LO} = \dot{S}_1^{NNLO} + v^9 \left[\frac{27}{32} + \frac{3\eta^2}{16} - \frac{105\eta^2}{32} - \frac{\eta^3}{48} + \delta m \left(-\frac{27}{32} + \frac{39\eta}{8} - \frac{5\eta^2}{32}\right)\right],
\]

\[
\dot{S}_2^{N4LO} = \dot{S}_2^{NNLO} + v^9 \left[\frac{27}{32} + \frac{3\eta^2}{16} - \frac{105\eta^2}{32} - \frac{\eta^3}{48} - \delta m \left(-\frac{27}{32} + \frac{39\eta}{8} - \frac{5\eta^2}{32}\right)\right],
\]

\[
\dot{L}_N^{N4LO} = T_{2PN}^{-1} \left[-\frac{v}{\eta} \left(-\dot{S}_1^{N4LO} - \dot{S}_2^{N4LO}\right) - v^4 c_{S1} S_1^{NNLO} + c_{S2} S_2^{NNLO}\right]
\]

\[
- v^4 \left[c_{S1L} \left(-\frac{\dot{v}}{\eta} (S_1^{NLO} + S_2^{NLO}) \cdot \mathbf{\hat{L}}_N \cdot \mathbf{S}_1\right) + \mathbf{\hat{L}}_N \left[-\frac{\dot{v}}{\eta} (S_1^{NLO} \cdot \mathbf{S}_1 + \mathbf{\hat{L}}_N \cdot \mathbf{S}_1^{NLO})\right]\right] + (1 \leftrightarrow 2),
\]

where

\[
S_1^{NNLO} = S_1^{NLO} + v^7 \left[\frac{9}{16} + \frac{5\eta}{4} + \frac{\eta^2}{24} + \delta m \left(-\frac{9}{16} + \frac{5\eta}{8}\right)\right],
\]

\[
L_{2PN} = 1 + v^4 \left[\frac{3}{2} + \frac{\eta}{6}\right] + v^4 \left(\frac{27}{8} - \frac{19\eta}{8} + \frac{\eta^2}{24}\right),
\]

\[
c_{S1} = -\frac{1}{4} \left(3 + \frac{1}{m_1}\right),
\]

\[
c_{S1L} = -\frac{1}{12} \left(1 + \frac{27}{m_1}\right),
\]

\[
\dot{S}_2^{N2LO}, c_{S2}, c_{S2L}\text{ can be obtained via the (1 ↔ 2) exchange.}
\]

Note that from NNLO on, one no longer has a standard
precession equation for $\hat{L}_N$ of the form of Eq. (2). In fact, as can be seen from Eq. (4c), $\hat{L}_N$ has components both perpendicular and parallel to $\hat{L}_N$. Therefore, as is done in the SpinTaylorT4 approximant, we define

$$\dot{\hat{L}}_{N,\perp}^{\text{N4LO}} = \dot{\hat{L}}_N^{\text{N4LO}} - (\hat{L}_N \cdot \dot{\hat{L}}_N^{\text{N4LO}}) \hat{L}_N, \quad (7)$$

which then satisfies

$$\dot{\hat{L}}_{N,\perp}^{\text{N4LO}} = \Omega_L^{\text{N4LO}} \times \hat{L}_N^{\text{N4LO}}. \quad (8)$$

We use the solutions of Eq. (7)\(^3\), to compute $\hat{L}_N(t)$, but we have also used directly the solutions to Eq. (4c) and found relative differences in the components of $\hat{L}_N$ of $\lesssim 10^{-4}$. We present the derivational details of these N4LO expressions in App. A.

There are indications that the PN precession equations converge with increasing PN order despite missing higher-order information \[^98\]. Indeed, we have found it slightly more beneficial to work with the N4LO precessions rather than the NLO versions. We illustrate this in App. B, where we show that the N4LO-Euler-angle-twisted TEODResumSP agrees better with both NRSur7dq4 and SEOBNRv4PHM than its NLO counterpart. This agreement is demonstrated specifically in terms of waveform strain mismatches which we introduce in Sec. IV. The NLO-N4LO disagreement is more severe for systems with more mass asymmetry, i.e., smaller values of $q$, which we show in terms of Euler angles in Fig. 11 in App. A. As the figure exhibits, there is considerable Euler-angle dephasing between NLO, NNLO, and N4LO solutions for small $q$, but no such dephasing between N3LO and N4LO, which we somewhat expect since their difference is at 2PN. Here, as already stated, we work with N4LO ODEs, but provide more discussion on the various ODE orders in App. A.

We should add that instantaneous corrections to the orbit-averaged expressions start entering at N3LO \[^93\] which we do not take into account here, but the precessing approximant SpinTaylorT4 has been upgraded to include these corrections. Our precession ODEs agree with those in the version of SpinTaylorT4 before the addition of these corrections.

As already mentioned, it is useful to package the six spin degrees of freedom into a space of lower dimensions. This is usually done by considering the projections of $S_1, S_2$ parallel and orthogonal to $L_N(t)$, resulting in two commonly employed scalar quantities. The parallel scalar is

$$\chi_{\text{eff}} = M^{-2} \left( (1 + q)S_1 \cdot \dot{\hat{L}}_N + (1 + q^{-1})S_2 \cdot \dot{\hat{L}}_N \right) \quad (9)$$

which is a conserved quantity of the orbit-averaged precession equations over the precession timescale \[^99\]. The orthogonal parameter is $\chi_p$ of Ref. \[^76\] defined as\(^4\)

$$\chi_p \equiv \frac{m_1^{-2}}{(2 + 3q^2/2)} \max \left\{ \left(2 + \frac{3q}{2}\right) S_{1,\perp}, \left(2 + \frac{3}{2q}\right) S_{2,\perp} \right\},$$

where $S_{1,\perp}, S_{2,\perp}$ denote the components of $S_1(t), S_2(t)$ perpendicular to $\hat{L}_N$, respectively. Both $\chi_{\text{eff}}$ and $\chi_p$ are commonly used in the LVC analysis of GW events \[^9\].

We now introduce our new orthogonal parameter

$$S_{1,\perp,\text{max}} \equiv M^{-2} \max [S_{1,\perp} + S_{2,\perp}] \quad (11)$$

which may be just a useful as $\chi_p$ in encoding the precessing degrees of freedom as we show in Secs. IV B, IV C. $S_{1,\perp,\text{max}}$ is bounded above by $(1 + q^2)/(1 + q)^2$ which yields 0.5 for $q = 1$ and 1 in the test-mass limit.

B. Reference Frames

When considering precessing systems, there are three special frames of reference which have their respective z-axes aligned with $J_0 \equiv J_3(0), L_0 \equiv L_3(0)$, and $L_N(t)$. Here, we refer to these frames as the $J_0$ frame, $L_0$ frame, and the co-precessing frame, respectively. Clearly, the $J_0$ and $L_0$ frames are inertial whereas the co-precessing frame is not.

The $L_0$ frame is the preferred frame for most of the currently available precessing approximants such as SpinTaylorT4, NRSur7dq4, and SEOBNRv4PHM as it is clearly the most straightforward frame for solving the precession ODEs (1a) - (1c) even though the precession-induced amplitude modulations are more pronounced in this frame. This will also be our preferred frame here. Accordingly, we label the azimuthal and the polar angles of $L_0(t)$ with respect to $L_0$ by $\alpha$ and $\beta$ as shown in Fig. 1. We delegate the discussion of the $J_0$ frame to Sec. II C.

 Naturally, we must pick an $x$-axis in the $L_0$ frame, with respect to which we measure $\alpha(t)$. Here, we do this by imposing the condition that $S_1(0)$ is in the $x$-z plane as shown in Fig. 1, which yields $\hat{x} = S_0^{\perp,\perp}/|S_0^{\perp,\perp}|$, where $S_0^{\perp,\perp} \equiv S_1(0) - (S_1(0) \cdot \hat{L}_0) \hat{L}_0$ and $L_0 \equiv L_0/|L_0|$, which is the same choice as in Ref. \[^70\]. We can therefore fully specify $S_1(0)$ via the parameters \{$q, \chi_1, \theta_1$\} where $\theta_1 \equiv \cos^{-1}(S_1(0) \cdot \hat{L}_0/S_1), \quad 0 \leq \chi_1 \leq 1$ and $S_1 \equiv |S_1| = \chi_1(1 + q)^{-2}$ with $m_1 = M/(1 + q)$ setting $M = 1$ and assuming $q \leq 1$. Similarly, $S_2(0)$ is specified by \{$q, \chi_2, \theta_2, \phi_2$\} where $\theta_2 \equiv \cos^{-1}(S_2(0) \cdot \hat{L}_0/S_2), 0 \leq \chi_2 \leq 1$, $S_2 \equiv |S_2| = \chi_2 q^2(1 + q)^{-2}$ and $\phi_2$ is the azimuthal angle with respect to $\hat{x}$ defined above. With our

\(^3\) These expressions match their spinOrd = 7 counterparts as given in the SpinTaylorT4 approximant.

\(^4\) Note that the factor in front of max{\ldots} may differ depending on the convention that assigns either $m_1$ or $m_2$ as the primary binary component.
the starting point of each inspiral, satisfying the usual \( \omega_0 = \pi f_0 \) relation.

C. Effects of Precession

Spin-orbit precession occurs when the spins are not (anti)parallel to the orbital angular momentum. The main effect is a slow precession of \( \mathbf{L}_N \) about an axis that is aligned with \( \mathbf{J}_N(0) \). Moreover, because of radiation reaction, \( \mathbf{L}_N \) shrinks, resulting in a precession cone whose opening angle decreases in time. Thus, the projection of \( \mathbf{L}_N \) orthogonal to \( \mathbf{J}_N(0) \) is the initial \((2,2)\)-mode GW frequency marking the starting point of each inspiral, satisfying the usual \( \omega_0 = \pi f_0 \) relation.

\[
\{ f_0(\text{Hz}), M_{\text{tot}}(M_\odot), q, \chi_1, \chi_2, \theta_1, \theta_2, \phi_2 \}, \quad (13)
\]

where \( f_0 \) is the initial \((2,2)\)-mode GW frequency marking the starting point of each inspiral, satisfying the usual \( \omega_0 = \pi f_0 \) relation.

The precession of \( \mathbf{L}_N \) induces amplitude modulations in the waveform and modifies the phase. The modulations depend strongly on the orientation of the orbit with respect to an observer’s line of sight. This is illustrated in Fig. 3, where the gray curve is the precessing
amplitude modulations than their JL

lead to a significant fraction of precessing signals being

obtained from \(\dot{L}_J\). When going from our inertial

We will consider only one case of it.

Thus far, we have talked about simple precession
dubbed so because both L and S precess around J. However,
when \(\mathbf{L} + \mathbf{S} \approx 0\), a phenomenon known as
transitional precession occurs in which J “tumbles” until radiation reaction decays L enough to take the system away from the \(\mathbf{L} + \mathbf{S} \approx 0\) configuration [49]. Since transitional precession requires careful fine-tuning of the parameters, it is expected to be a rare phenomenon [49, 70] so for our comparisons, we will consider only one case of it.

III. TWISTING NON-PRECESSING WAVEFORMS

Having conceptually introduced the twist operation, we next provide mathematical details. Let us recall that \(\alpha(t)\) and \(\beta(t)\) are the azimuthal and polar angles of \(\mathbf{L}_N(t)\) with respect to \(\mathbf{L}_0\) and the third angle \(\gamma(t)\) is obtained from \(\dot{\gamma} = \dot{\alpha} \cos \beta\). The set \{\(\alpha(t), \beta(t), \gamma(t)\)\} is all we need when transforming between the \(\mathbf{L}_N(t)\) and \(\mathbf{L}_0\) frames. Specifically, when going from our inertial \(\mathbf{L}_0\) frame to the \(\mathbf{L}_N(t)\)-frame, we “forward”-Euler rotate using \(R(\alpha, \beta, \gamma) \equiv R_2(\gamma) R_y(\beta) R_z(\alpha)\) where \(R_j(\zeta)\) represent rotations by the angles \(\zeta\) with respect to the j axis\(^5\). In the following, we omit displaying the explicit time dependence of these angles and various other time-dependent quantities, e.g., \(\mathbf{L}_N(t)\), which we restore when necessary.

Under the forward Euler rotation above, the gravitational-wave modes transform as follows

\[
\hat{h}_{\ell m} = \sum_{m' = -\ell}^{\ell} h_{\ell m'} D^{(\ell)}_{m', m} (\alpha, \beta, \gamma) \tag{15}
\]

where \(D^{(\ell)}_{m', m}\) are Wigner’s D matrices which can be related to spin-weighted spherical harmonics via [100]

\[
Y_{\ell m} (\theta, \phi) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}} D^{(\ell)}_{m, s}(\phi, 0). \tag{16}
\]

Note that different versions of this equation exist in the literature due to conventions of Wigner D matrices. Here, we employ the definition introduced in Ref. [101]

\[
D^{(\ell)}_{m', m}(\alpha, \beta, \gamma) = e^{-i m' \alpha} e^{-i m \gamma} d^{(\ell)}_{m', m}(\beta), \tag{17}
\]

where \(d^{(\ell)}_{m', m}(\beta) \in \mathbb{R}\) are the “little” D matrices given by

\[
d^{(\ell)}_{m', m}(\beta) = \sum_k (-1)^{k - m - m'} \sqrt{\frac{k!}{(j + m - k)! (j - k - m')! (j - m')!}} \sqrt{\frac{(j + m - k)! (j - k - m')! (j - m')!}{(k - m + m')!}} \left( \frac{\beta}{2} \right)^{2k - 2k + m - m'} (\cos \beta)^{2j - 2k + m - m'} \left( \frac{\sin \beta}{2} \right)^{2k - m + m'}. \tag{18}
\]

As explained in Sec. I, the key idea is to “unwrap” or twist aligned-spin waveforms generated in the \(\mathbf{L}_N\) frame using Euler rotations. In order to transform from \(\mathbf{L}_N\) to \(\mathbf{L}_0\) frame we “backward” Euler-rotate via the inverse rotation matrices: \(R^{-1} = R(\gamma, -\beta, -\alpha)\). Therefore, to twist we invert Eq. (15)

\[
\hat{h}^{\mathbb{T}}_{\ell m} = \sum_{m' = -\ell}^{\ell} \hat{h}^{\mathbb{NP}}_{\ell m'} D^{(\ell)}_{m', m} (-\gamma, -\beta, -\alpha), \tag{19}
\]

where we introduced the superscripts T and NP to denote the twisted and the non-precessing waveforms, respectively. Using the standard identity \(D^{(\ell)}_{m', m} = (-1)^{m' - m} D^{(\ell)}_{m, -m}\) which translates to

\[
(-1)^{m' - m} d^{(\ell)}_{m', -m}(\beta) = d^{(\ell)}_{m', m}(\beta) \tag{20}
\]

in Eq. (17) we obtain

\[
\hat{h}^{\mathbb{T}}_{\ell m}(t) = e^{-i m \alpha(t)} \sum_{m' = -\ell}^{\ell} e^{i m' \gamma(t)} d^{(\ell)}_{m', -m}(\beta(t)) \hat{h}^{\mathbb{NP}}_{\ell m'}(t), \tag{20}
\]
where we restored the time dependences.

Note that the literature is replete with slightly different versions of Eq. (20) depending on: (i) Euler rotation conventions, (ii) Wigner D and spherical harmonic conventions, and (iii) the sign of the right-hand-side for the $\dot{\gamma}$ equation. Our definitions and conventions agree with Ref. [39] (modulo the sign of $\dot{\gamma}$) and our practical expression (20) agrees with Ref. [61] which interestingly disagrees with its updated version in Ref. [62], but then agrees with the most recent version of Phenom [63]. We tested the performance of the alternate expression of Ref. [62] against ours in terms of the $\ell = 2$ detector strain mismatches of TEODResumSP with SEOBNRv4PHM and NRSur7dq4. We found that our twist given by Eq. (20) performed better in the sense that it produced smaller mismatches. We delegate the details of this comparison to App. B.

In principle, one can also twist the non-precessing waveforms using the angles of PN-corrected $L(t)$ with respect to $L(0)$. Ref. [96] showed that the resulting differences in the twisted waveforms as compared with precessing NR waveforms are marginal, therefore we use only $L_\text{N}(t)$ with respect to $L_\text{N}(0)$ for TEODResumSP.

We now have all the individual ingredients necessary to generate the precessing TEODResumSP waveforms. The procedure for this operation is as follows:

1. Specify the initial parameters listed in Eq. (13).

2. Generate aligned-spin (non-precessing) $\ell = 2$ waveform modes using TEODResumS via the set of parameters $\{f_0, M_{\text{tot}}, q, \chi_1, \chi_2\}$.

3. Solve the orbit-averaged precession ODEs (4a-4c) using SpinTaylorT4 resummed radiation reaction for $\dot{v}$.

4. Retrieve the spherical angles $\{\alpha(t), \beta(t)\}$ from the components of $L_\text{N}(t)$ in the $L_\text{N}(0)$ frame and subsequently obtain $\gamma(t)$ by solving $\dot{\gamma} = \dot{\alpha} \cos \beta$.

5. Construct the precessing $\ell = 2$ TEODResumSP modes via the twist formula (20).

Let us conclude this section with three remarks: (i) We can generate twisted waveforms in the $J_\text{N}(0)$ frame as well as the $L_\text{N}(0)$ frame, but this is slower because the solutions to the ODEs, which are solved in the $L_\text{N}(0)$ frame, must be Euler-rotated to the $J_\text{N}(0)$ frame at each time step. Since the other approximants with which we compare TEODResumSP use the $L_\text{N}(0)$ frame, we do the same. (ii) It is possible to extend the above scheme by coupling the precession ODEs to TEODResumS dynamics, i.e., by setting $\chi_i = \dot{L}_\text{N}(t) \cdot S_i/m_\text{c}^2$ $(i = 1, 2)$ at each time step of the aligned-spin EOB dynamics, where $\dot{L}_\text{N}(t), S_i(t)$ are obtained from the N4LO precession dynamics. We have tested this approach to some extent as discussed below. (iii) For this initial version of TEODResumSP, we truncate our precessing waveforms before the onset of merging-ringdown. As explained in Refs. [39, 66], attaching the merging-ringdown portions of the twisted waveform to its inspiralling portion introduces extra complications which we leave for the next version of TEODResumSP.

IV. ASSESSING THE TWIST: COMPARISONS WITH SPINTAYLORT4, SEOBNRv4PHM, AND NRSUR7DQ4

To assess the faithfulness of TEODResumSP, we compared the twisted TEODResumS waveforms against precessing waveforms generated by the following
three approximants: \texttt{SpinTaylorT4}, \texttt{NRSur7dq4}, and \texttt{SEOBNRv4PHM}. For the comparisons, we considered approximately 1350 precessing inspirals for which we show \{f_0, M_{\text{tot}}, q\} of Eq. (13) in Fig. 4. We additionally show in the figure \{\chi_{\text{eff},0}, S_{\text{L, max}} (\text{and} \chi_{\text{p}})\} which reduce the remaining five spin degrees of freedom of \(S_1(0), S_2(0)\) (recall we set \(S_{\forall} (0) = 0\)) to just two.

Within the set of roughly 1350 inspirals, we employed \texttt{SpinTaylorT4} to generate precessing waveforms for 50 cases, \texttt{NRSur7dq4} for 200 cases, and \texttt{SEOBNRv4PHM} for \(~\sim\) 1100 cases. Approximately, one tenth of the cases involved BNS inspirals, an additional ten involved black hole neutron star systems, and the rest were BBHs within which we also considered a dozen cases for both GW151226 and GW170729 using the best inferred parameters from Ref. [9]. The latter case clearly stands out as our most massive binary in the right panel of Fig. 4.

As an initial assessment, we compared \((2, \pm 2)\) modes, \(h_{2, \pm 2}(t)\), between \texttt{SpinTaylorT4} and \texttt{TEOBResumSP} by computing the time-domain overlaps between the two approximants using

\[ O_{\ell m} = \frac{\Re \left[ \int_{t_0}^{t_f} \tilde{h}_{\ell m}(t) h^{*\dagger}_{\ell m}(t) dt \right]}{\sqrt{\int_{t_0}^{t_f} |\tilde{h}_{\ell m}(t)|^2 dt \int_{t_0}^{t_f} |h^{*\dagger}_{\ell m}(t)|^2 dt}}, \]

where \(T\) is short for \texttt{TEOBResumSP} and \(i\) for any approximant of our choosing (in this case \texttt{SpinTaylorT4}). We set \(t_0 = 0\). For \(t_f\), we use the inspiral time at which \(f = 100\text{ Hz}\) which we determine from

\[ \Delta t = \int_{t_0}^{t_f} \frac{1}{v(t)} dt = t_f - t_0 = t_f, \]

where we once again use the \texttt{SpinTaylorT4} expression for \(v(t)\). The upper limit of the integral (22) is given by \(v_f = [(100\text{ Hz}) \pi G M_{\text{tot}}/c^3]^{1/3}\) where \(M_{\text{tot}}\) has units of kilograms. Recall that we omit the merger-ringdown parts of the binary evolution in this work and since almost the entire SNR due to precession accumulates in the \(f < O(10^5)\text{ Hz}\) regime [49], 100 Hz is a suitable cutoff for waveform comparisons here. Note that some of the more massive BBH inspirals merge before \(f = 100\text{ Hz}\), for which we use \(t_{\text{peak}},\) i.e., when the \((2,2)\)-mode amplitude peaks.

As our main assessment of \texttt{TEOBResumSP}, we computed frequency-domain matches of the detector strains between \texttt{TEOBResumSP} and \texttt{NRSur7dq4} for 200 inspirals, then between \texttt{TEOBResumSP} and \texttt{SEOBNRv4PHM} for approximately 1100 inspirals using \texttt{pycbc} [102]. The match (or faithfulness) between two approximants is defined as

\[ \mathcal{M} \equiv \max_{t_0, \phi_c} \left| \frac{\langle h^i | h^T \rangle}{\sqrt{\langle h^i | h^i \rangle \langle h^T | h^T \rangle}} \right|. \]

where

\[ \langle h^i | h^T \rangle \equiv 4 \Re \left[ \int_{f_0}^{f_f} \tilde{h}_i^*(f) \tilde{h}_m^*(f) S_n(f) df \right] \]

is the detector-strain \((S_n)\)-weighted inner product between Fourier transforms \(\tilde{h}^i, \tilde{h}^T\) with \(f^r = 100\text{ Hz}\). In principle, \(\tilde{h}^i, \tilde{h}^T\) can be the Fourier transforms of an individual mode, but it is more appropriate to use the Fourier transform of the detector GW strain given by

\[ h(t) = F_+(\theta, \phi, \psi) h_+(t) + F_\times(\theta, \phi, \psi) h_\times(t), \]

where

\[ F_+(\theta, \phi, \psi) = \frac{1 + \cos^2 \theta}{2} \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi, \]

\[ F_\times(\theta, \phi, \psi) = \frac{1 + \cos^2 \theta}{2} \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi, \]

where \(D_L\) is the distance to the source, \(-2Y^\ell_m\) are the \((\text{spin} = -2)\)-weighted spherical harmonics, \(\theta, \phi, \psi\) are the sky-position angles, and \(\psi\) is the polarization angle of the GWs in the detector frame, set to zero for our comparisons. Formally, \(h(t)\) is obtained from a sum over all \(\ell, m\) modes, but here, we suffice with the \(\ell = 2\) mode. We will incorporate the \(\ell > 2\) modes, which recently got upgraded [90], in the next version of \texttt{TEOBResumSP}. Note that \(\mathcal{M}\) is computed by maximizing over initial time and phase shifts, \(t_0, \phi_0^6\).

It has become standard in waveform comparisons to use \(\mathcal{M} = 0.965\) as a benchmark. This cut-off translates to the loss of roughly 10% of events due to waveform systematics [103, 104]. We also employ this threshold and its mismatch counterpart \(1 - \mathcal{M} = 0.035\) which we plot as a horizontal dashed orange line in many of our remaining figures.

\section{Comparisons with SpinTaylorT4 Waveforms}

\texttt{SpinTaylorT4} is the spinning variant of the waveform approximant \texttt{TaylorT4} [18, 70], which constructs the precessing waveforms by solving the precession ODEs at a given specified order ranging from LO \((\bar{S}_{i=1,2} \sim v^0)\) up to N4LO, i.e., \(\bar{S}_{i=1,2} \sim v^5 + \ldots + v^9\) [see Eqs. (4a, 4b)]. The solutions to the precession equations yield the angles \(\alpha, \beta,\) and the third angle \(\gamma\) enters as a correction to the orbital phase via \(\Psi \rightarrow \Psi - \gamma\). These are then used to build the precessing waveforms via the expressions in App. B of Ref. [59] which yield 1.5PN-accurate precessing \((\ell, m)\) modes.

\footnote{One can also maximize over \(t_c, \phi_c\): time and phase shift at coalescence.}
The large blue dots represent $1 - O_{22}$, the small red ones $1 - O_{2-2}$, and the green diamonds $1 - O_{22}$ for binary neutron stars. Note that since we plot $1 - O_{\ell m}$ here, the lower the dots, the better the agreement between the two approximants. Correspondingly, the dashed black, orange, gray, and light gray lines represent $(0.99, 0.95, 0.95, 0.9)$ overlaps, respectively. The plot also shows that the agreement between TEOBResumSP and SpinTaylorT4 worsens for cases with larger $\chi_p$, indicating stronger precession.

Overall, we used 50 cases for the SpinTaylorT4-TEOBResumSP waveform comparisons, but we exercised some care in choosing our cases because of the relatively low PN order of the spin sector of SpinTaylorT4 which may lead to significant disagreements between the nonprecessing modes so as to make the comparison of precessing modes pointless. There is also additional [non-precessing waveform] disagreement because of the superior performance of the TEOBResumS in the merger, ringdown stages [41, 89, 90]. Therefore, we formed the comparison set by picking inspirals for which the non-precessing $(2,2)$-mode overlap between SpinTaylorT4 and TEOBResumSP is $> 0.965$. Strictly speaking, this cutoff is meant for assessing waveform matches; we nonetheless use it here for mode overlaps.

As a result of having to groom our assessment set, we treat the comparisons with SpinTaylorT4 as a “warm-up” for the more detailed comparisons with NRSur7dq4 and SEOBNRv4PHM that we make in Secs. IV B, IV C. Consequently, in this section, we suffice with computing precessing waveform overlaps for only the $(2,\pm2)$ modes instead of detector strain matches.

Within our set of 50 precessing inspirals, we found that approximately half yielded precessing $(2,\pm2)$-mode overlaps, $O_{2\pm2} > 0.965$ and a significant majority yielded overlaps greater than 0.9 as shown in Fig. 5 in terms of $1 - O_{2\pm2}$. The figure also shows a rough correlation between $1 - O_{2\pm2}$ and $\chi_p$. 30 cases correspond to BNSs represented by the green dots in Fig. 5. Once again, about half of these yielded $O_{2\pm2} > 0.965$ and roughly 90% gave $O_{2\pm2} > 0.9$. The correlation between $\chi_p$ and $1 - O_{2\pm2}$ also persisted for this subset.

Fig. 5 also confirms our expectation that $O_{2-2}$ (small red dots) is approximately equal to $O_{22}$ (large blue dots). This is because the two mode overlaps have a relative difference of roughly $A_{\ell m}^{NP}/A_{\ell m}^{NP} \ll 1$, where $A_{\ell m}^{NP}$ denotes the amplitude of a non-precessing $(\ell, m)$ mode. This small difference can be occasionally spotted in Fig. 5 as a small offset between the red and the blue dots.

B. Comparisons with NRSur7dq4 Waveforms

NRSur7dq4 is the most recent NR-surrogate based on ~1500 precessing SXS simulations with a parameter range of $1/4 \leq q \leq 1$ and $\chi_1, \chi_2 \leq 0.8$ [68]. It has been shown to be better than 99% faithful to NR simulations for > 95% of the cases in its extrapolation space [68].

For our assessment here, we employed the gwsurrogate package [105, 106] to generate all the precessing $\ell = 2$ modes for a set consisting of 200 cases. In order to maximize the number of orbital cycles, hence the number of precession cycles, we set $f_0 \in [35, 40]$ Hz and $M_{tot} \in [35, 40]M_\odot$. Making these values any smaller tended to hit the low frequency bound of NRSur7dq4, and setting them higher would miss the one, or at best two, precession cycles that we expect. We further set $\chi_1 = \chi_2 = 0.75$ since NRSur7dq4’s current upper limit is 0.8. As the blue triangles in Fig. 4 show, our set covers the $\{S_{l,\max}, \chi_p\}$ space reasonably well.
For the match computation, we randomly selected 200 pairs of sky angles \( \{\theta_*, \phi_*\} \) with which we constructed the resulting \( \ell = 2 \) GW detector strains from both the precessing NRSur7dq4 modes and the twisted TEOBResumSP modes. We then employed Eq. (23) to compute the matches for these 200 cases. We found that

(i) \( \sim 69\% \) the cases yielded TEOBResumSP-NRSur7dq4 matches of 0.99 or better.

(ii) \( \sim 85\% \) of the cases yielded \( \mathcal{M} \geq 0.965 \).

(iii) Roughly \( 97\% \) yielded \( \mathcal{M} \geq 0.9 \).

As we may expect, the disagreement between TEOBResumSP and NRSur7dq4 seems to be greater for cases with stronger precession. We illustrate this correlation in Fig. 6, where we plot the mismatch, \( 1 - \mathcal{M} \), against both \( \chi_p \) and our new quantity \( S_{\text{max}} \). One may first notice that \( 1 - \mathcal{M} \) seems to better correlate with \( S_{\text{max}} \) instead of \( \chi_p \), which is further supported by Fig. 8. Second, \( \chi_p \) values are often degenerate whereas \( S_{\text{max}} \) produces a better spread across its domain. Therefore, \( S_{\text{max}} \) may be at least as suitable as \( \chi_p \) in inheriting the precessing degrees of freedom in compact binary inspirals.

To further gauge the possible dependence of the match on other binary parameters, we show corner plots of the TEOBResumSP-NRSur7dq4 match in Fig. 7 with “corner” parameters given by the set \( \{ \chi_{\text{eff},0}, S_{\text{max}}, 1/q \} \). As can be deduced from the middle and right panels of Fig. 7, the TEOBResumSP-NRSur7dq4 matches also worsen for the \( q = 0.25 \) subset beside large \( S_{\text{max}} \) cases, i.e., strong precession. Given that \( 0.25 < q \leq 1/3 \) cases exhibit no such degradation, it is likely that the increased mismatch \( q = 0.25 \) mismatches are due to the fact the \( q = 0.25 \) is at the edge of NRSur7dq4’s current interpolation range. However, it is also possible that with more data in the \( 0.25 < q \leq 1/3 \) regime, we may start seeing a relation between the mismatches and \( q \), but this is not one of our goals for this article. Nonetheless, we comment on related future work below.

Edge cases aside, the main reason for TEOBResumSP’s mismatch is the fact that we twist \( S_1(t) = S_{1,0}(0) = \chi_1 n_1^2, S_2(t) = S_{2,0}(0) = \chi_2 n_2^2 \), i.e., \( \chi_1, \chi_2 = \) constant, aligned-spin waveforms to construct the precessing TEOBResumSP modes whereas NRSur7dq4 interpolates time-dependent data for \( S_1(t), S_2(t) \) coming from \( \sim 1500 \) NR simulations. For cases with stronger precession, we expect \( S_{1,0}(0), S_{2,0}(0) \) to disagree more with \( S_1(t), S_2(t) \), thus making the non-precessing constant-spin TEOBResumS waveforms [used in the twist] more prone to disagreeing with NRSur7dq4’s co-precessing waveforms which, recall, are Euler-rotated to give the final detector-frame (\( \ell, m \)) modes of NRSur7dq4 [67, 68].

The other source of disagreement is the non-precessing (2,0) mode. By default, TEOBResumS sets \( h_{20}^p = 0 \), whereas NRSur7dq4 does not. This disagreement can cause considerable mismatches if the binary parameters are such that \( h_{20}^p \)’s contribution is amplified inside the twist and the strain formulae relative to that of \( h_{22}^p \) and \( h_{21}^p \)’s contributions. We found that this occurred only a handful of times within our sample of 200 cases.

It is not in the scope of this article to conduct a more in-depth study of the mismatch’s dependence on the intrinsic parameters. However, a future such study would be beneficial to the waveform community, and even more so if it is conducted ensuing two obvious improvements: (i) extending the NR surrogates’ ranges to \( q < 1/4 \) and \( \chi_1, \chi_2 > 0.8 \) and (ii) twisting aligned-spin TEOBResumS waveforms with varying spins by setting \( \chi_1 = L_1(t) \cdot S_1(t)/m_1^2, \chi_2 = L_2(t) \cdot S_2(t)/m_2^2 \) instead of constant initial values. Since the cost of NR simulations significantly increases for smaller mass ratios and higher spins, it seems unlikely that surrogates with expanded ranges for \{\( q, \chi_1, \chi_2 \} \) will be available in the near future. On the other hand, item (ii) can be accomplished by coupling the precession ODEs to the
The mismatch seems to correlate better with our computed at 100 Hz for the circa 1100 inspirals described in Sec. IV C. The mismatch may at least be as suitable as \( \chi_p \) in capturing the effects of precession on waveform mismatches. The horizontal dashed black, orange, and gray lines represent approximants precessing. Most recent comparisons have yielded \( M \geq 0.99, 0.965, \) and 83% of the cases, respectively. As in the comparisons with \( \text{SEOBNRv3} \), the mismatch seems to correlate with our parameter \( S_{\perp, \text{max}} \) better than \( \chi_p \), which we show in Fig. 8, where we plot \( 1 - M \) vs. \( S_{\perp, \text{max}} \) and \( \chi_p \). On the other hand, the matches appear to be randomly distributed in the sky as we show in Fig. 9. This is expected because we have compared here only the \( \ell = 2 \) detector strain [Eq. (25)]. We obtained \( M \geq 0.99, 0.965, \) and 9% for 60%, 75%, and 83% of the cases, respectively. As such, the matches appear to change once higher modes are included due to the differences in the respective EOBs’ aligned-spin dynamics and the different procedures used to obtain the precessing waveforms.

Since we were able to make comparisons with \( q \) as low as one tenth, we were able to discern that the TEOBResumSP-SEOBNRv4PHM matches deteriorate for small values of \( q \) as can be seen in Fig. 10, where we display corner plots of the match against the set \( \{ S_{\perp, \text{max}}, T_{\text{insp}}, q, \chi_{\text{eff}, 0}\} \). One can see an increasing density of \( M \leq 0.85 \) points in the small \( q \) regions. This disagreement is expected as it is known that the mismatch between the non-precessing TEOBResumSP and \( \text{SEOBNRv3} \) increases as \( q \) decreases [41]. Fig. 10 also reiterates the mismatch-\( S_{\perp, \text{max}} \) correlation as can be seen from the left panels. The \( q-S_{\perp, \text{max}} \) corner plot situated in the left panel of the middle row is especially illustrative. As far as we can tell, there do not seem to be discernible correlations between the match and the remaining two parameters, \( T_{\text{insp}, \chi_{\text{eff}, 0}} \). However, we have seen significant TEOBResumSP-SEOBNRv4PHM mismatches for long inspirals involving BNSs, specifically for \( f_0 \lesssim 30 \) Hz, whereas the BNS runs with \( f_0 \geq 35 \) Hz had all yielded \( M \geq 0.97 \). We believe the reason for this is the fact that TEOBResumSP twists \( \chi_1, \chi_2 \) = constant waveforms. We expect this issue to be resolved with the next version of TEOBResumSP when \( \chi_1, \chi_2 \) will be obtained from the time-dependent solutions \( S_1(t), S_2(t) \). For this reason, in this article, we use BNSs with \( f_0 \geq 35 \) Hz for the TEOBResumSP-SEOBNRv4PHM comparisons.

V. CONCLUSIONS

In this article, we introduced TEOBResumSP: the precessing upgrade to TEOBResumS. TEOBResumSP delivers precessing \( \ell = 2, m \in [-2, 2] \) modes by Euler-rotating non-precessing (aligned spin) TEOBResumS modes from...
the instantaneous, non-inertial $L_N(t)$ frame to the inertial $L_N(0)$ frame. This frame rotation, given by Eq. (20), is performed with Wigner’s D matrices.

We also introduced a new parameter, $S_{\perp, \text{max}}$, in Eq. (11) for representing the strength of precession. Although it is akin to $\chi_p$ [Eq. (10)], $S_{\perp, \text{max}}$ seems to better encode the dependence of the strain mismatch on the strength of precession and is less degenerate.

We assessed the faithfulness of TEOBResumSP by computing first TEOBResumSP-SpinTaylorT4 $(2, \pm 2)$-mode overlaps for 50 inspirals, then $\ell=2$ detector strain matches of TEOBResumSP with NRSur7dq4 for 200 inspirals and with SEOBNRv4PHM for nearly 1100 inspirals. Since TEOBResumSP currently does not compute the merger-ringdown portions of precessing modes, we computed mode overlaps and strain matches at $f = 100\,\text{Hz}$, or at the peak of the $(2,2)$-mode amplitude in the case of heavy binary black hole inspirals which coalesce at $f < 100\,\text{Hz}$.

TEOBResumSP’s performance can be summarized as follows:

(i) TEOBResumSP $(2, \pm 2)$ mode overlaps with SpinTaylorT4 were better than 0.965 for $\geq 50\%$ of the cases. Overall, $\geq 90\%$ of the cases yielded $\geq 0.9$ overlaps. As expected, the overlaps worsened for more strongly precessing systems which we show in Fig. 5 in terms of $\chi_p$. Since, the precessing modes of SpinTaylorT4 are only 1.5PN accurate, this disagreement with stronger precession is not surprising.

(ii) TEOBResumSP $\ell=2$ detector strains matched NRSur7dq4 to better than 0.99 for 69% and better than 0.965 for 85% of the 200 cases with $\chi_p$ ranging up to 1. Overall, roughly 97% of the cases yielded $M \geq 0.9$ with only one case giving $M < 0.70$.

(iii) The TEOBResumSP-NRSur7dq4 mismatches show a clearer correlation with $S_{\perp, \text{max}}$ than $\chi_p$ as exhibited in Fig. 6.

(iv) The worst TEOBResumSP-NRSur7dq4 matches happen when $q = 0.25$ as shown in Fig. 7. This may be due to the fact that $q = 0.25$ is the lower bound of NRSur7dq4’s current range of mass ratios. As Fig. 7 hints, with the exception of the $q = 0.25$ cases, the match does not seem to depend on $\chi_{\text{eff},0}$ or $q$.

(v) Of the circa 1100 TEOBResumSP-SEOBNRv4PHM $\ell=2$ detector strain matches we computed, $\sim 60\%$ gave $M > 0.99$, $\sim 75\%$ gave $M > 0.965$, and $\sim 83\%$ yielded $M > 0.9$.

(vi) The TEOBResumSP-SEOBNRv4PHM mismatch also seems to correlate better with $S_{\perp, \text{max}}$ than $\chi_p$ as shown in Fig. 8.

(vii) The TEOBResumSP-SEOBNRv4PHM match is independent of the source sky location as Fig. 9 demonstrates. This may change when higher modes ($\ell > 2$) are included in the strain computation (25).
FIG. 10. Corner plots of the ℓ = 2 detector strain matches between TEOBResumSP and SEOBNRv4PHM for approximately 1100 precessing compact binary inspirals. The corner parameters are once again χ_{eff,0}, S_{\perp,\text{max}}, q, and additionally, the inspiral time in seconds, T_{\text{insp}} (note the log scale). The cooler colors (purple, blue) represent cases with M ≥ 0.965 while the warmer colors (orange, red) represent M ≤ 0.85. The rough correlation between the mismatch and S_{\perp,\text{max}} shown in Fig. 8 can be discerned in the left panels here. Additionally, the degradation of the match with decreasing q is evident in the middle row and the right-most panel.

(viii) The TEOBResumSP-SEOBNRv4PHM match worsens with decreasing q as shown in Fig. 10. This is expected since the non-precessing TEOBResumS, SEOBNRv3 modes show increasing disagreement for decreasing values of q.

We have already mentioned a few of the improvements that we plan to add to the next version of TEOBResumSP. We now provide the full list: First task will be coupling the precession equations to the TEOBResumS dynamics. This will enable us to generate aligned-spin waveforms with time-varying χ_1, χ_2, and boost the performance of TEOBResumSP for long inspirals such as BNSs from 20 Hz. Second improvement will be adding merger-ringdown. This involves Euler-rotating the inspiralling modes to the J_{\text{peak}} frame to attach the ringdown portion of the modes^7, where J_{\text{peak}} is extracted from the precession ODEs at the peak of the orbital frequency. The stitched inspiral-merger-ringdown GW modes are then rotated to the desired inertial frame. Finally, as a third improvement, we will replace the SpinTaylorT4 radiation reaction for ̇v in the precession ODEs with one which we would obtain from the aligned-spin TEOBResumS dynamics. It is not clear that this would be an improvement, but may prove to be so for BBH mergers.

As it stands, the current version of TEOBResumSP agrees well, i.e., yields matches > 0.965, with the two main precessing approximants in the market, NRSur7dq4 and SEOBNRv4PHM, for 85%, 75% of the cases, respectively. The significantly disagreeing cases either have very strong precession, as indicated by our newly introduced parameter S_{\perp,\text{max}}, or small mass ratio. Another nice feature of TEOBResumSP is that it is fast thanks to the post-adiabatic method implemented in TEOBResumS which “rushes” the inspiral [88]. TEOBResumSP will be added to the TEOBResumS git repository https://bitbucket.org/eob_ihes/teobresums/wiki/Home.

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7 However, the very recent Ref. [66] attached the merger-ringdown portions in the co-precessing frame. Currently, it is not clear to us which frame is more advantageous for this operation.
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Appendix A: Derivation of the post-Newtonian spin precession equations up to N4LO

This section builds upon the work of Ref. [107]. Recall that the \( \mathbf{J} \) equation is obtained by imposing total angular momentum conservation, \( \mathbf{J} = 0 \) which leads to

\[
\dot{\mathbf{J}} = -S_1 - \dot{S}_2. \tag{A1}
\]

\( \mathbf{L} \) is provided up to 3.5PN in, e.g., Eq. (4.7) of Ref. [92] which we rewrite in the following compact form

\[
\mathbf{L} = \frac{\eta v}{\eta v} \left( \mathbf{L}_N \left[ 1 + v^2 \left( \frac{3}{2} + \frac{\eta}{6} \right) \right] + v^3 \left( \frac{27}{8} - \frac{19\eta}{8} + \frac{\eta v^2}{24} \right) + \mathcal{O}(v^6) \right)
\]

\[
+ v^3 \Delta \mathbf{L}_{1.5}^S + v^5 \Delta \mathbf{L}_{2.5}^S + v^7 \Delta \mathbf{L}_{3.5}^S + \mathcal{O}(v^8) \right), \tag{A2}
\]

where we defined the terms \( \Delta \mathbf{L}_{n}^S \) with \( n = 1, 5, 2.5, 3.5 \) with their explicit \( v \) scalings factored out. From Ref. [92], we can extract

\[
\Delta \mathbf{L}_{1.5}^S = \ell \left( -\frac{35}{6} S_\ell - \frac{5}{2} \delta m \Sigma_\ell \right) \tag{A3}
\]

\[
- \lambda \left( \delta m \Sigma_\lambda + \lambda \left( \frac{1}{2} S_n + \frac{1}{2} \delta m \Sigma_n \right) \right),
\]

where \( \ell = \mathbf{L}_N \), \( n = r/|r| \) is the relative separation unit vector, and \( \lambda = \ell \times n \). Moreover, \( S_{\ell}, S_\lambda, S_n, S_{\ell,\lambda, n} \equiv (\ell, \lambda, n) \cdot S \), \( \Sigma_{\ell,\lambda, n} \equiv (\ell, \lambda, n) \cdot \Sigma \), where \( S = S_1 + S_2 \), \( \Sigma = S_2/m_2 - S_1/m_1 \). Defining \( S_{1,\ell} \equiv (\ell, S_1) \) and similarly for \( S_{1,\lambda}, S_{1,n} \), as well as the 1 \( \rightarrow \) 2 counterparts, Eq. (A3) becomes

\[
\Delta \mathbf{L}_{1.5}^S = -\frac{5}{6m_1} (3M + m_1) S_{1,\ell} + \frac{M - m_1}{2m_1} S_{1,n} - \frac{M + m_1}{m_1} S_{1,\lambda} + (1 \rightarrow 2), \tag{A4}
\]

where we restored \( M = m_1 + m_2 \) for clarity in this section. We can now orbit-average this expression using \( \langle \delta \mathbf{v} \rangle = \frac{1}{2} (\delta \mathbf{v}) + (\delta \mathbf{v}) \) which yields \( \langle S_{1,\lambda} \rangle = \frac{1}{2} (S_1 - S_{1,\ell}) \). Substituting these orbit-average terms into Eq. (A4) we arrive at

\[
\Delta \mathbf{L}_{1.5}^S = -\frac{M + 3m_1}{4m_1} S_1 - \frac{(27M + m_1)}{12m_1} \mathbf{L}_N (\mathbf{L}_N \cdot S_1) + (1 \rightarrow 2). \tag{A5}
\]

Similarly, with some more determination, one can obtain

\[
\Delta \mathbf{L}_{2.5}^S = \left( \frac{7M - 31m_1}{16m_1} + \frac{22M + 9m_1}{48m_1} \right) S_1
\]

\[
+ \left[ -\frac{49M + 39m_1}{16m_1} + \frac{59M}{24m_1} + \frac{13}{144} \right] \mathbf{L}_N (\mathbf{L}_N \cdot S_1) + (1 \rightarrow 2). \tag{A6}
\]

Eq. (A3) inside Eq. (A2) together with Eqs. (4a, 4b) give us all the pieces that we need to go to N4LO [Eq. (A6) enters at N5LO so we drop it.] For clarity, let us once again consider NNLO first. At this order, Eq. (A2) becomes

\[
\mathbf{L} = \mathbf{L}_N \frac{\eta v}{\eta v} L_{1PN} + \eta v^2 (c_{1S} S_1 + c_{2S} S_2)
\]

\[
+ \eta v^2 \mathbf{L}_N (c_{1SL} \mathbf{L}_N \cdot S_1 + c_{1SL} \mathbf{L}_N \cdot S_2), \tag{A7}
\]

where \( L_{1PN} \equiv 1 + v^2 \left( \frac{3}{2} + \frac{5}{6} \eta \right) \) and the constants \( c_{1S}, c_{1SL}, c_{2S} \), etc., are given in Eqs. (6a, 6b). Differentiating Eq. (A7) with respect to time, we obtain

\[
\dot{\mathbf{L}}_N = \frac{\eta}{\eta v} \mathbf{L}_{1PN} \left[ -S_{1NN} - S_{2NN}
\right]
\]

\[
- \eta v^2 \left( c_{1S} S_{1LO} + c_{2S} S_{2LO} \right), \tag{A8}
\]

where, e.g., \( S_{1NN} \) implies that only terms that scale as \( v \leq 7 \) should be retained. Several simplifications occurred in reaching Eq. (A8). First, the second \( S_1, S_2 \) terms contribute only at the LO. This is because of the factor of \( v^2 \) in front, which means that at our requested order, i.e., NNLO, the terms multiplying \( v^2 \) can be at most \( v^3 \) which is LO for \( S_1, S_2 \) as can be seen from Eqs. (1a, 1b). Second, all the \( c_{1S}, c_{2S} \) terms have dropped from Eq. (A8) because (i) \( v^3 \mathbf{L}_N \propto v^5 \), i.e., is N3LO and (ii) at NNLO only \( v^2 \mathbf{L}_N \cdot S_{1LO} \) scales as \( v^7 \), but is actually zero because \( \mathbf{L}_N \perp S_{1LO} \) is as clear from Eqs. (1a, 1b).
Pushing now to N4LO, Eq. (A1) becomes
\[ L = \dot{L}_\parallel = \dot{L}_N v_{2PN} + \eta v^2 \Delta L_{1.5PN}^S = -S_1^{\text{N4LO}} - S_2^{\text{N4LO}}, \]  

(A9)

where \( L_{2PN} \) is given in Eq. (5b). Note that we omit the radiation reaction terms starting at NNLO via \( \eta v^2 \Delta L_{1.5PN}^S \) since these terms are all parallel to \( L \). The effects of radiation reaction are incorporated via \( v = v(t) \) in the precession ODEs after the standard change of variables \( d/dt \rightarrow v(v)d/dv \) in Eqs. (1a)-(5a).

Explicitly writing out Eq. (A9) at N4LO then rearranging gives us Eq. (4c), where we used the property that \( S_i \perp S_s \) up to NLO. In terms of powers of \( v \), each term in Eq. (4c) goes up to \( v^3 \), i.e., N4LO as defined.

We can now obtain \( L_N \), therefore, the angles \( \alpha \) and \( \beta \) at any order of our choosing varying from NLO to N4LO, which we show in Fig. 11 as functions of the (2,2)-mode GW frequency for three different precessing compact binary inspirals.

![Fig. 11](image)

**FIG. 11.** The spherical angles of \( L_N(t) \) in the \( L_0 \) frame described in Sec. II B for three separate cases: binary neutron star (left panels), black hole neutron star (middle panels), and binary black hole (right panels) inspirals. \( \alpha \) is the azimuthal angle and \( \beta \) is the polar angle (see Fig. 1). In each panel, we show the angle obtained from solving the precession ODEs truncated at four different orders: NLO, NNLO, N3LO, and N4LO. From left to right, the binaries respectively have \( M = 3, 20, 35M_\odot \) and \( q \approx 0.85, 0.11, 0.35 \). \( f \) represents the (2,2)-mode gravitational wave frequency.

A thorough survey of the effects of the truncation order of the precession ODEs, the instantaneous terms (entering at N3LO), and the neglected terms would be beneficial to the entire gravitational-wave community. Ref. [98] has already done some work in this regard, but a systematic, large-scale analysis quantified in terms of consequences to parameter estimation remains to be undertaken at this point.

**Appendix B: Results of using NLO angles and a different twist formula**

In this section, we briefly show results from two additional tests we conducted: (1) Using Euler angles in the twist formula (20) that are obtained from the precession ODEs truncated at NLO as given in Eqs. (1a)-(5c). (2) Using N4LO Euler angles in an alternate twist formula. Specifically, we have chosen to test the expression provided by Eq. (A2) of Ref. [62]

\[ h_{lm}^T(t) = e^{-i\alpha} \sum_{m'=-l}^l e^{-im'\gamma} d_{m',m}^{\ell} (-\beta) h_{lm'}^{\text{NP}}. \]  

(B1)

This version differs from our twist formula (20) in the signs of the \( \alpha \) and \( \gamma \) exponents. For convenience, we redisplay our expression

\[ h_{lm}^T(t) = e^{-i\alpha} \sum_{m'=-l}^l e^{im'\gamma} d_{m',m}^{\ell} (-\beta) h_{lm'}^{\text{NP}}. \]  

(B2)
For both tests, we used a subset of precessing compact binary inspirals that is a combination of 50 cases from our NRSur7dq4 set and 60 cases from our SEOBNRv4PHM set. Using Eq. (B1) at N4LO and Eq. (B2) at NLO we generated two new sets of twisted $\ell = 2$ TEOBResumS modes with which we then computed the $\ell = 2$ detector strain matches as before. We show how these two alternate twists perform against ours, dubbed Eq. (B2)[N4LO], in Fig. 12, where it is evident that our twist produces consistently the smallest mismatches (red circles). The alternate twist formula of Eq. (B1) is clearly the worst choice producing $M > 0.965$ for only about two thirds of the set (black inverted triangles). The reason why Eq. (B1)[N4LO] still somehow manages to mostly yield $M > 0.965$ is due to both the fact that $\gamma$ remains close to $\alpha$ because $\beta$, starting from zero, is small for most binaries, and that the twisted $(\pm 2)$ modes differ by a small amount as explained in Sec. IV A. Therefore, in binaries for which $\beta(t) \ll 1$ and the precessing $(\pm 2)$ modes dominate the mode-sum in the strain formula (25), Eqs. (B1) and (B2) are nearly equal under the $m \rightarrow -m$ exchange, thus produce twisted waveform strains that are very close to each other.

Returning to Fig. 12, we see that the NLO version of our twist performs somewhat well in the sense that roughly three quarters of the cases yielded $M > 0.965$ (blue triangles). The details of the differences in the plotted NLO, N4LO mismatches lay with the differences in the Euler angles used in the respective twists. We have already shown in Fig. 11 how these Euler angles vary as the ODE truncation order goes from NLO to N4LO. For most cases, the difference in the angles becomes significant only in the last few orbital cycles, corresponding to the small differences between the NLO and N4LO mismatches of Fig. 12. But for cases with small $q$, the differences in the Euler angles becomes more significant as can be seen in the middle panels of Fig. 11. It is possible that the speed-up gained in using NLO-truncated precession ODEs, instead of N4LO, is significant enough to justify their use in parameter estimation. As we have not yet carried out detailed speed tests of our code, we can not verify or refute this hypothesis, but will do so with the next version of TEOBResumSP.

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