Alternative coherent-mode representation of a random source

A S Ostrovsky¹, A M Zemliak, M Á Olvera, and P C Romero
Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, Puebla 72000, México
E-mail: andreyo@fcfm.buap.mx

Abstract. The coherent-mode representation (CMR) of an optical random source is a very powerful tool in contemporary optics. However, the practical value of the CMR is essentially restricted because of the complexity of solving the Fredholm integral equation with the field cross-spectral density as a kernel. Moreover, in practice, the analytical expression for the cross-spectral density of the field, as a rule is unknown, a fact that makes this solution impossible in general. Here we propose a technique for determination of the field CMR that does not involve solving the Fredholm integral equation but is based on usual radiometric measurement. We illustrate the proposed technique with the results of mathematical simulation.

1. Introduction
The coherent-mode representation (CMR) of an optical field broached for the first time by Gamo [1] and later on developed by Wolf [2] is an essential tool in describing the processes and systems in optics. From a physical point of view, the CMR describes an optical field of any state of coherence as a linear superposition of uncorrelated completely coherent modes, a fact that gives new insight into the physics of generation, propagation, and transformation of optical radiation. From a mathematical standpoint, it expresses the cross-spectral density function of an optical field as a sum of terms that are separable in space, a fact that allows significant simplification of the analysis of statistical optical processes and systems.

Some interesting applications of the CMR can be found in Ref. 3. However, the practical value of the CMR is essentially restricted because of the complexity of solving the Fredholm integral equation with the field cross-spectral density as a kernel. Moreover, in practice, the analytical expression for the cross-spectral density of the field, as a rule is unknown, a fact that makes this solution impossible in general. Recently, we proposed a technique for determination of the field CMR that does not involve solving the Fredholm integral equation but is based on usual radiometric measurements [4]. Here we propose somewhat different experimental technique, which seems to be much better justified and more convenient in its practical realization. We illustrate the proposed technique with the results of mathematical simulation.

2. Background
Consider a stochastic, stationary, quasi-monochromatic, scalar, secondary source, occupying a finite closed domain $D$ in the plane $z = 0$ and radiating into the half-space $z > 0$. As well known [5], the second-order statistical properties of such a source can be completely characterized by the cross-spectral density (from now on, we omit the explicit dependence of the considered quantities on the optical frequency $\nu$)

¹ To whom any correspondence should be addressed.

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where $V(x)$ is the fluctuating wave-field at a point specified by a position vector $x$, the asterisk and the angle brackets denote the complex conjugate and the ensemble average, respectively. Equation (1) implies that the function $W(x_1, x_2)$ is Hermitian, i.e.,

$$W(x_2, x_1) = W^*(x_1, x_2).$$

In addition it may be shown that, for any physically realizable source, $W(x_1, x_2)$ is square integrable in $D$, i.e.,

$$\iint_D|W(x_1, x_2)|^2 \, dx_1 \, dx_2 < \infty,$$

and non-negatively defined, i.e.,

$$\iint_D W(x_1, x_2) f^*(x_1) f(x_1) \, dx_1 \, dx_2 \geq 0,$$

where $f(x)$ is any square-integrable function.

In consequence of the properties given by Eqs. (2) – (4) the cross-spectral density can be expressed in the form of Mercer’s expansion as

$$W(x_1, x_2) = \sum_{n=0}^{\infty} \lambda_n \phi_n^*(x_1) \phi_n(x_2),$$

where $\lambda_n$ are the eigenvalues and $\phi_n(x)$ are the eigenfunctions of the Fredholm integral equation of the second kind,

$$\int_D W(x_1, x_2) \phi_n(x_1) \, dx_1 = \lambda_n \phi_n(x_2).$$

The eigenvalues are necessarily real and non-negative, and then eigenfunction may be taken to be orthonormalized, i.e.,

$$\int_D \phi_m^*(x) \phi_n(x) \, dx = \delta_{mn},$$

where $\delta_{mn}$ is the Kronecker symbol. The terms

$$W_n(x_1, x_2) = \lambda_n \phi_n^*(x_1) \phi_n(x_2),$$

describe the natural modes of the source, which are completely coherent and mutually uncorrelated, and therefore the expansion (5) is called the source CMR.

From a standpoint of energy transfer of the emitted radiation the planar source can be characterized by the radiant intensity, that represents the rate of energy per unit solid angle around the direction specified by the 3D unit vector $s$, and is defined as [5]

$$J(s) = \left( \frac{k}{2\pi} \right)^2 \cos^2 \theta \int_{-\infty}^{\infty} W(x_1, x_2) \exp[-i k s_\perp \cdot (x_2 - x_1)] \, dx_1 \, dx_2, $$

where $s_\perp$ is the 2D vector obtained by projecting the unit vector $s$ onto the source plane, $\theta$ is the angle that vector $s$ makes with the $z$-axis, and $k$ is the wave number. The integrals on the right-hand side of Eq. (9) are extended over whole plane $z = 0$ since $W(x_1, x_2)$ has zero values outside the source.
region \( D \). It must be stressed here that the radiant intensity of the original source, given by Eq. (9), is a quantity that can be easily measured in specially performed physical experiment (see, e.g., [5]). Substituting from Eq. (5) into Eq. (9), one finds that

\[
J(s) = \left( \frac{k}{2\pi} \right)^2 \cos^2 \theta \sum_n \alpha_n |\varphi_n(k\mathbf{s})|^2, \tag{10}
\]

where

\[
\varphi_n(k\mathbf{s}) = \int_{-\infty}^{\infty} \varphi_n(x) \exp(-ik\mathbf{s} \cdot x) \, dx. \tag{11}
\]

Formula (10) plays first fiddle in defining the alternative source CMR (see next section).

3. Alternative CMR of a planar source

Let us construct some 4D continuous function that vanishes outside of finite domain \( D \) in the plane \( z = 0 \) and has the form

\[
W_A(x_1, x_2) = \sum_{m=0}^{M-1} \mu_m \psi^*_m(x_1)\psi_m(x_2), \tag{12}
\]

where \( M \) is some integer, \( \mu_m \) are some arbitrary positive values bounded from above, and \( \psi_m(x) \) are some prescribed continuous functions which form an orthonormal set in \( D \), i.e.,

\[
\int_D \psi^*_m(x)\psi_i(x) \, dx = \delta_{mi}. \tag{13}
\]

It may be readily shown that, owing to its construction, \( W_A(x_1, x_2) \) possesses the same properties as does the cross-spectral density \( W(x_1, x_2) \), i.e., it is square integrable, Hermitian and non-negatively defined function. Hence, with the appropriate choice of physical dimensions of quantities \( \mu_m \) and \( \psi_m \), the function \( W_A(x_1, x_2) \) can be accepted as the cross-spectral density of some alternative, not necessary physically realizable, source. Of cause, the radiant intensity of such a source can be written, by analogy with Eqs. (10) and (11), as

\[
J_A(s) = \left( \frac{k}{2\pi} \right)^2 \cos^2 \theta \sum_{m=0}^{M-1} \mu_m |\psi_m(k\mathbf{s})|^2, \tag{14}
\]

where

\[
\psi_m(k\mathbf{s}) = \int_{-\infty}^{\infty} \psi_m(x) \exp(-ik\mathbf{s} \cdot x) \, dx. \tag{15}
\]

Once the radiant intensity of the original source, \( J(s) \), has been measured, one can determine the radiatively equivalent alternative source, minimizing the functional

\[
L(\mu_0, \mu_1, \ldots, \mu_{M-1}) = \int_{(2\pi)^2} [J(s) - J_A(s)]^2 \, d\Omega
\]

\[
= \int_{(2\pi)^2} J^2(s) \, d\Omega + \int_{(2\pi)^2} J^2_A(s) \, d\Omega - 2\int_{(2\pi)^2} J(s)J_A(s) \, d\Omega, \tag{16}
\]

with due regard for non-negativeness of the values \( \mu_m \). In Eq. (16) \( d\Omega \) is the element of a solid angle around a direction specified by vector \( \mathbf{s} \), and the integration extends over the \( 2\pi \) solid angle subtended by a hemisphere in the half-space, into which the source radiates. We note that the first term in the
second line of Eq. (16) does not depend on optimization parameters $\mu_n$ and, hence, can be omitted from consideration. Then, substituting for $J_A(s)$ from Eq. (14) and making use of the relations $\cos \theta d\Omega = ds_\perp$ and $\cos \theta = (1 - s_\perp^2)^{1/2}$, we may rewrite the problem of minimizing the functional given by Eq. (16) in the following explicit form:

$$
\min \quad L'(\mu_0, \mu_1, \ldots, \mu_{M-1}) = \frac{1}{2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \mu_m \mu_n Q_{mn} + \sum_{m=0}^{M-1} \mu_m P_m,
$$

$$
\text{subject to} \quad \mu_m \geq 0
$$

where

$$
Q_{mn} = \left(\frac{k}{2\pi}\right)^2 \int_{|s| = 1} (1 - s_\perp^2)^{1/2} |\bar{\psi}_m^\dagger (ks_\perp)|^2 |\bar{\psi}_m (ks_\perp)|^2 ds_\perp,
$$

$$
P_m = -\left(\frac{k}{2\pi}\right)^2 \int_{|s| = 1} (1 - s_\perp^2)^{1/2} J(s) |\bar{\psi}_m (ks_\perp)|^2 ds_\perp.
$$

The optimization problem given by Eq. (17) represents a classical quadratic programming problem, which can be solved by well-known methods (see, e.g., Ref. 6).

4. Choice of alternative CMR basis

As follows from the previous section, the alternative coherent-mode functions $\psi_m(x)$ must be chosen to form an orthonormal set in $D$. For practical reasons, we will also require that these functions admit calculation in the closed form of the Fourier transform given by Eq. (15). The simplest choice of such functions may be done using the basis formed by Hermitian polynomials $H_m(u)$ of the integral order $m$ which obey the orthogonality relation [7]

$$
\int_{-\infty}^{\infty} \exp(-x^2) H_m(x) H_n(x) dx = 2^m m! \sqrt{\pi} \delta_{mn}.
$$

Comparing Eqs. (20) and (13), and taking into account that

$$
H_m(x) = H_m(x) H_m(y),
$$

one finds that, in this basis,

$$
\psi_m(x) = \frac{1}{2^m m! \sqrt{2\pi \sigma_s}} \exp\left(-\frac{x^2}{4\sigma_s^2}\right) H_m\left(\frac{x}{\sqrt{2\sigma_s}}\right),
$$

where $\sigma_s$ is the effective size of the original source. Then, substituting from Eq. (22) into Eq. (15) and making use of the well-known relations [7]

$$
\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) H_m(x) \exp(ixu) dx = i^n \sqrt{2\pi} H_m(u) \exp\left(-\frac{u^2}{2}\right)
$$

and

$$
H_m(-u) = (-1)^n H_m(u),
$$

The optimization problem given by Eq. (17) represents a classical quadratic programming problem, which can be solved by well-known methods (see, e.g., Ref. 6).
we find that
\[
\psi_m(kS,\perp) = (-1)^m \frac{2\sqrt{2\pi} \sigma_s}{2^n m!} \exp\left[-2(\sigma^2_s, kS,\perp)^2\right] H_n(\sqrt{2\sigma_s, kS,\perp}).
\] (25)

Concluding this section, we note that Hermitian polynomials which appear in Eq. (25) can be readily calculated using the recurrence formula [7]
\[
H_{m+1}(u) = 2uH_m(u) - 2mH_{m-1}(u),
\] (26)
with due regard for \(H_0 = 1\) and \(H_1 = 2u\).

5. Example
As an example, we construct the alternative CMR of the 1D Gaussian-Schell model (GSM) source for which the cross-spectral density has the form [5]
\[
W(x_1, x_2) = S(0)\exp\left(-\frac{x_1^2 + x_2^2}{4\sigma_s^2}\right)\exp\left(-\frac{(x_1 - x_2)^2}{2\sigma^2}\right),
\] (27)
where \(S(0)\) is the spectral power at the source center, \(\sigma_s\) is the r.m.s. width of the power spectrum distribution across the source, and \(\sigma^2\) is the r.m.s. width of the spectral degree of coherence. It can be readily shown that the radiant intensity is
\[
J(s, \cos \theta) = S(0) \frac{2k\sigma_s^2\sigma^2}{(4\sigma_s^2 + \sigma^2)^{3/2}} \cos^2 \theta \exp\left[-\frac{\sigma^2}{4\sigma_s^2 + \sigma^2}(\sqrt{2\sigma_s, kS,\perp})^2\right],
\] (28)
It must be noted also that for 1D case Eq. (25) takes the form
\[
\psi_m(kS,\perp) = (-i)^m \left(2 \frac{\sqrt{2\pi} \sigma_s}{2^n m!}\right)^{1/2} \exp\left[-(\sigma_s, kS,\perp)^2\right] H_m\left(\sqrt{2\sigma_s, kS,\perp}\right).
\] (29)

On substituting from Eqs. (28) and (29) into the 1D versions of Eqs. (18) and (19), we obtain the following expressions for coefficients of the quadratic programming problem (17):
\[
Q_m = \frac{4k\sigma_s}{\sqrt{2\pi} 2^n m! 2^n!} \int^{2\sigma_s, \perp}_{0} \left(1 - \frac{u^2}{2(k\sigma_s)^2}\right)^{3/2} \exp(-2u^2)H_m^2(u)H_m^2(u) du,
\] (30)
\[
P_m = -\frac{4S(0)k^2(\sigma_s^2 + \sigma^2)}{\sqrt{\pi} 2^n m! (4\sigma_s^2 + \sigma^2)^{1/2}} \int^{2\sigma_s, \perp}_{0} \left(1 - \frac{u^2}{2(k\sigma_s)^2}\right)^{3/2} \exp\left(-2\frac{\sigma^2}{4\sigma_s^2 + \sigma^2}u^2\right) H_m^2(u) du.
\] (31)

We performed the numerical calculation of coefficients (30) and (31) and solved problem (17) using the standard Matlab program. In our calculations we accepted \(S(0) = 1\), \(\sigma_s = 1\) and \(\sigma^2 = 0.5\), the values which correspond to truly partially coherent source. Also, to simplify calculations, we accepted \(k\sigma_s = 10\) (typically this value can be of order \(10^3\) and larger). We evaluated the quality of alternative CMR by means of the relative error.
\[ \varepsilon = \min_{L} \left( \int_{|s| \leq 1} \left( 1 - s^2 \right)^{-1/2} J^2 (s, \cos \theta) ds \right)^{-1}. \] (32)

The corresponding results for different values of \( M \) are given in Table 1. As can be seen from this table, the relative error (32) decreases rapidly with the increase of \( M \) and, when \( M = 15 \), it becomes less than 1\%, the result which is quite admissible in the majority of practical applications.

| M   | 1     | 5     | 10    | 15    | 20    | 25    | 30    | 35    |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| \( \mu_0 \) | 0.8341 | 0.3249 | 0.2902 | 0.2825 | 0.2800 | 0.2791 | 0.2787 | 0.2786 |
| \( \mu_1 \) | 0.3071 | 0.2608 | 0.2519 | 0.2492 | 0.2482 | 0.2478 | 0.2477 |       |
| \( \mu_2 \) | 0.3026 | 0.2353 | 0.2248 | 0.2218 | 0.2208 | 0.2203 | 0.2202 |       |
| \( \mu_3 \) | 0.3299 | 0.2133 | 0.2009 | 0.1975 | 0.1964 | 0.1958 | 0.1957 |       |
| \( \mu_4 \) | 0.5137 | 0.1950 | 0.1797 | 0.1568 | 0.1568 | 0.1747 | 0.1742 | 0.1740 |
| \( \mu_5 \) | 0.1805 | 0.1611 | 0.1398 | 0.1554 | 0.1549 | 0.1547 |       |       |
| \( \mu_6 \) | 0.1708 | 0.1448 | 0.1248 | 0.1383 | 0.1378 | 0.1376 |       |       |
| \( \mu_7 \) | 0.1687 | 0.1306 | 0.1115 | 0.1225 | 0.1225 | 0.1223 |       |       |
| \( \mu_8 \) | 0.1854 | 0.1185 | 0.0998 | 0.1096 | 0.1090 | 0.1088 |       |       |
| \( \mu_9 \) | 0.2800 | 0.1083 | 0.0895 | 0.0977 | 0.0969 | 0.0967 |       |       |
| \( \mu_{10} \) | 0.1003 | 0.0804 | 0.0870 | 0.0863 | 0.0860 |       |       |       |
| \( \mu_{11} \) | 0.0950 | 0.0725 | 0.0776 | 0.0768 | 0.0765 |       |       |       |
| \( \mu_{12} \) | 0.0941 | 0.0658 | 0.0693 | 0.0683 | 0.0680 |       |       |       |
| \( \mu_{13} \) | 0.1042 | 0.0602 | 0.0619 | 0.0609 | 0.0605 |       |       |       |
| \( \mu_{14} \) | 0.1524 | 0.0558 | 0.0554 | 0.0543 | 0.0538 |       |       |       |
| \( \mu_{15} \) | 0.0529 | 0.0497 | 0.0483 | 0.0479 |       |       |       |       |
| \( \mu_{16} \) | 0.0525 | 0.0446 | 0.0431 | 0.0426 |       |       |       |       |
| \( \mu_{17} \) | 0.0585 | 0.0403 | 0.0385 | 0.0379 |       |       |       |       |
| \( \mu_{18} \) | 0.0828 | 0.0366 | 0.0344 | 0.0338 |       |       |       |       |
| \( \mu_{19} \) |       | 0.0334 | 0.0308 | 0.0301 |       |       |       |       |
| \( \mu_{20} \) |       | 0.0310 | 0.0276 | 0.0268 |       |       |       |       |
| \( \mu_{21} \) |       | 0.0294 | 0.0248 | 0.0239 |       |       |       |       |
| \( \mu_{22} \) |       | 0.0293 | 0.0224 | 0.0214 |       |       |       |       |
| \( \mu_{23} \) |       | 0.0328 | 0.0203 | 0.0191 |       |       |       |       |
| \( \mu_{24} \) |       | 0.0449 | 0.0186 | 0.0171 |       |       |       |       |
| \( \mu_{25} \) |       |       | 0.0172 | 0.0153 |       |       |       |       |
| \( \mu_{26} \) |       |       | 0.0164 | 0.0138 |       |       |       |       |
| \( \mu_{27} \) |       |       | 0.0164 | 0.0124 |       |       |       |       |
| \( \mu_{28} \) |       |       | 0.0184 | 0.0113 |       |       |       |       |
| \( \mu_{29} \) |       |       | 0.0244 | 0.0103 |       |       |       |       |
| \( \mu_{30} \) |       |       |       | 0.0096 |       |       |       |       |
| \( \mu_{31} \) |       |       |       | 0.0091 |       |       |       |       |
| \( \mu_{32} \) |       |       |       | 0.0092 |       |       |       |       |
| \( \mu_{33} \) |       |       |       | 0.0103 |       |       |       |       |
| \( \mu_{34} \) |       |       |       | 0.0132 |       |       |       |       |

\( \varepsilon \% \) | 52.95 | 12.23 | 2.670 | 0.6200 | 0.1340 | 0.0144 | 0.0004 | 0.0004 |
6. Conclusions
We have proposed an experimental technique for calculating the alternative CMR of the field that does not involve solving the Fredholm integral equation but is based on usual radiometric measurements. The technique is well-posed in the sense that the construction of the alternative cross-spectral density assures all the properties of a genuine one, namely, it is square integrable, Hermitian and non-negatively defined function. The alternative CMR calculation is made through the minimization of the pseudo-distance between the measured and constructed radiant intensities associated to the real and constructed cross-spectral densities. Numerical simulation (with the values corresponding to a truly partially coherent source) shows the efficiency of the proposed technique and the obtained results suggests that it can be implemented in the vector case which is of special interest for describing the electromagnetic sources with the arbitrary state of polarization. We hope to report on this subject in subsequent works.

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