Non-Classical Expected Utility Theory∗

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Abstract

In this paper we extend Savage’s theory of decision-making under uncertainty from a classical environment into a non-classical one. We formulate the corresponding axioms and provide representation theorems for qualitative measures and expected utility.

1 Introduction

In this paper we propose an extension of the standard approach to decision-making under uncertainty in Savage’s style from the classical model into the more general model of non-classical measurement theory corresponding the basic mathematical structure of Quantum Mechanics (see [4]).

Formally, this means that we substitute the Boolean algebra model with a more general ortholattice structure (see [4]). In order to provide a first line of motivation for our approach we turn back to Savage’s theory in a very simplified version. In Savage [11], the issue is about the valuation of “acts” with uncertain consequences or results. One of our simplifications is to assume that we are able to provide a cardinal measure of the results - in utils (we below clarify this somehow imprecise formulation). Acts lead to results (measurable in utils), but the results are uncertain. How can one formalize acts with uncertain outcomes?

The classical approach amounts to the following. There exists a set $X$ of states of nature, which may in principle occur. An act corresponds to a

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function \( f : X \rightarrow \mathbb{R} \). If the state \( s \in X \) is realized, our agent receives a utility of \( f(s) \) utils. But before hand it is not possible to say which state \( s \) is going to be realized. To put it differently, the agent has to choose among acts before he learns about the state \( s \). This is the heart of the problem.

Among possible acts there are “constant” acts, i.e., acts with a result that is known before hand, independently of the state of nature \( s \). The constant act is described by a (real) number \( c \in \mathbb{R} \). It is therefore natural to link an arbitrary act \( f \) with its “utility equivalent” \( CE(f) \in \mathbb{R} \). This corresponds to defining the constant act \( c \) (with utility outcome \( CE(f) \)) such that our decision-maker is indifferent between the act \( f \) and the constant act (with utility value) \( CE(f) \). The first postulate of this simplified (we assume also that the set \( X \) is finite) Savage model asserts the existence of the certainty equivalent:

- **S1.** There exists a certainty equivalent \( CE : \mathbb{R}^X \rightarrow \mathbb{R} \) and for the constant act \( 1_X \) we have \( CE(1_X) = 1 \).

It is rather natural to require monotonicity of the mapping \( CE \):

- **S2.** If \( f \leq g \) then \( CE(f) \leq CE(g) \).

The main property we impose on \( CE \) is linearity:

- **S3.** \( CE(f + g) = CE(f) + CE(g) \) for any \( f \) and \( g \in \mathbb{R}^X \).

Presented in such a way, this requirement looks like a very strong condition indeed. Savage himself and his followers preferred to ”hide” it behind the so-called ”sure thing principle” so that the linearity is derived from some other axioms. But as we understand it, this is more of an artifice.

In fact axiom S3 should be understood as a condition of additivity rather linearity. But together with monotonicity axiom S3 implies true linearity, that is \( CE(\alpha f + \beta g) = \alpha CE(f) + \beta CE(g) \) for any \( \alpha, \beta \in \mathbb{R} \). As a linear functional on the vector space \( \mathbb{R}^X \), \( CE \) can be written in a form \( CE(f) = \sum_x f(x)\mu(x) \).

By axiom S2, \( \mu \geq 0 \); by \( CE(1_X) = 1 \) we have \( \sum_x \mu(x) = 1 \). Therefore \( \mu(x) \) can be interpreted as the “probability” for the realization of state \( x \). Sometimes this probability is called subjective or personal, because it only expresses the likelihood that a specific decision-maker assigns to event \( x \). With such an interpretation, \( CE(f) \) becomes the ”expected” utility of the uncertain act \( f \) (or it is better to say - of the act \( f \) with uncertain outcome).

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With such a view we may assign probabilities not only to single state $x$ but also to any subset of states (or to any event) $A \subset X$. $\mu(A)$ can be understood either as the sum $\sum_{x \in A} \mu(x)$, or as $CE(1_A)$, where $1_A$ is the characteristic function of subset $A$. The interpretation in the second approach is clear: the act $1_A$ is a bet on event $A$ such that we receive 1 util if event $A$ is realized and 0 util otherwise (if the opposite or complementary event $\overline{A}$ occurs). The decision-maker can compare such bets on events and thereby compare events with respect to their likelihood. So we arrive at the notion of qualitative probability measure.

Our main idea is to substitute the Boolean lattice of events with a more general ortholattice. The move in that direction was initiated long ago, in fact with the creation of Quantum Mechanics. The Hilbert space entered into the theory immediately, beginning with von Neumann [12] who proposes the lattice of projectors in the Hilbert space as a suitable model instead of the classical (Boolean) logic. Birkhoff and von Neumann in their seminal paper [2] have investigated the necessary properties of such a non-distributive logic (modularity, ortho-modularity?). Recently a few decision-theoretical papers appear (see for example, [5, 9, 6, 8, 7]) in which the standard expected utility theory was transposed into Hilbert space model. Beside the formal arguments, a motivation for this research is that a more general description of the world allows to explain some behavioral anomalies e.g., the Eldsberg paradox (see [6]).

Lehrer and Shmaya write “We adopt a similar approach and apply it to the quantum framework... While classical probability is defined over subsets (events) of a state space, quantum probability is defined over subspaces of a Hilbert space.” Gyntelberg and Hansen (2004) apply a general event-lattice theory (with axioms that resemble those of von Neumann and Morgenstern) to a similar framework. One could expect that Gyntelberg and Hansen truly would have been working with general ortholattices. But no, they also worked with subspaces of a Hilbert space. Our first aim is to show that there is no need for a Hilbert space, the Savage approach can just as well (and even easier) be developed within the frame of more general ortholattices. Another line of motivation that we share with other decision theoretical papers in this vein is that this model maybe a better representation of the subjectively perceived world. For non-classical features of perception see for instance [1].
2 Ortholattices

A lattice is an ordered set such that any of its subsets (including the empty subset) has a greatest lower bound (∨ or sup) and a lowest higher bound (∧ or inf), which guarantees the existence of a maximal element \(1\) and a minimal element \(0\). An ortholattice is a lattice \(L\) equipped with an operation of orthocomplementation \(\perp: L \to L\). This operation is assumed to be involutive \((a^{\perp\perp} = a)\), to reverse the order \((a \leq b\) if and only if \(b^{\perp} \leq a^{\perp})\) and to satisfy the following property \(a \lor a^{\perp} = 1\) (or, equivalently, \(a \land a^{\perp} = 0\)).

Example 1. Let \(X\) be a set and \(L = 2^X\) the set of all subsets of \(X\). The order is defined by set-theoretical inclusion. For \(A \subset X\), \(A^{\perp} = X - A\), is the set-theoretical complement. It is the classical situation.

Example 2. Take some finite dimensional Hilbert space \(H\) (over the field of real or complex numbers). Let \(L\) be the lattice of vector subspaces of \(H\) and \(\perp\) be the usual orthogonal complementation.

This example is standard in Quantum Mechanics as well as in the above mentioned works. But it was early understood that the lattice \(L(H)\) (sometimes it is called the lattice of projectors) is endowed with a number of special properties. We discuss a significantly more general case in the next example.

Example 3. Let \((X, \perp)\) be an orthospace that is a set \(X\) equipped with an irreflexive and symmetric binary relation (orthogonality) \(\perp\). For \(A \subset X\),

\[ A^{\perp} = \{ x \in X, x \perp a \text{ for all } a \in A \}. \]

The sets of the form \(A^{\perp}\) are called orthoclosed subsets or flats. When equipped with the relation \(\subset\) and the operation \(\perp\), flats form an ortholattice \(F(X, \perp)\) (for details see [4]). Moreover almost any (at least any finite) ortholattice has the form of \(F(X, \perp)\) for a suitable orthospace \((X, \perp)\).

In order to get better acquainted with this subject, let us consider a few concrete examples.

a) Assume that all distinct points of \(X\) are pairwise orthogonal. Then any subset of \(X\) is orthoclosed and the ortho-complementation coincides with the usual set-theoretical complementation. That is we obtain the Boolean model of Example 1.

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1It is more natural to call what we just defined, a complete ortholattice. Usually one only requires the existence of finite bounds. However we shall not interest us much for the general case, assuming finiteness of \(L\).
b) Let $X$ be consist of four points $r, l, r', l'$. The orthogonality relation is represented by the graph below, where we connect points with a plain line when they are NON-ORTHOGONAL (so that orthogonal points as "far from each other" remain unconnected).

\[ \begin{array}{c}
\circ r' \quad \circ l' \quad \circ r \quad \circ l \\
\end{array} \]

The point $r$ is orthoclosed since $r = \{r', l\}^\perp$; similarly the point $l$ is orthoclosed. There are two other (nontrivial) flats: the set $\{r', l\} = r^\perp$ and $\{l', r\} = l^\perp$. The corresponding ortholattice is represented below

\[ \begin{array}{c}
1 \quad r^\perp \quad l^\perp \\
\circ r \quad \circ l \quad 0 \\
\end{array} \]

c) Let us consider the orthospace represented by the following graph

\[ \begin{array}{c}
B \\
L \quad R \\
F \\
\end{array} \]

The corresponding ortholattice is

\[ \begin{array}{c}
1 \quad R = L^\perp \quad F = B^\perp \\
L \quad R \quad F \quad B \\
0 \\
\end{array} \]

d) On the left side below we depicted another orthospace and on the right side the corresponding ortholattice.
We want to defend the thesis that ortholattices is a natural structure for applying all the concepts that are used in the classical theory of decision-making under uncertainty. As in the Boolean model we may speak of the intersection ($\land$) and union ($\lor$), as well as of the complementation (or as the negation, and understand it as ortho-complementation). All the usual relations between these operations are preserved with one exception: the law of distributivity is not satisfied in the general case. But how often is it used? In the proofs of some theorems and propositions, perhaps. But hardly in the formulation of the concepts.

A central point is that it is possible to speak about probabilities which can be considered as a quantified saturation of the ortholattice skeleton.

3 Non-classical probability

We show here how the basic concepts of classical probability theory carry over to ortholattices.

The theory of probability starts with the definition of a set $X$ of elementary events. Thereafter it moves over to general events. In our language events (or properties) are elements of an ortholattice $L$. The next key concept is a "collection of mutually exclusive events". In the classical model this is simply a partition of the set $X$, that is a decomposition $X = A_1 \sqcup ... \sqcup A_n$. In the general case the notion of a collection of mutually exclusive events should be replaced by the notion of an Orthogonal Decomposition of the Unit.

Definition. An Orthogonal Decomposition of the Unit (ODU) in an ortholattice $\mathcal{L}$ is a (finite) family of $\alpha = (a(i), i \in I(\alpha))$ of elements of $\mathcal{L}$ satisfying the following condition: for any $i \in I(\alpha)$

$$a(i)\perp = \bigvee_{j \neq i} a(j).$$
The justification for this formulation is provided by that \( a(i) \perp a(j) \) for \( i \neq j \) and \( \lor_i a(i) = 1 \). The proof is obvious.

For instance, the single-element family \( 1 \) is a (trivial) ODU. For any \( a \in \mathcal{L} \), the two-element family \((a, a^\perp)\) is an ODU. We call this kind of family the question about property \( a \).

Intuitively, the family \( \alpha \) is to be understood as a measurement (or a source of information) with a set of possible outcomes \( I(\alpha) \). If such a measurement yields an outcome \( i \in I(\alpha) \), we conclude that our system is endowed with property \( a(i) \) (or that the event \( a(i) \) occurs). Assume that we can ”prepare” our system in some state and repeatedly measure the system (each time prepared in that same state) with our measurement apparatus. The measurement outcomes can differ from one trial to another. Imagine that we performed \( n \) such measurements (for \( n \) relatively large) and that outcome \( i \) was obtained \( n_i \) times. Then we can assign each outcome \( i \) a ”probability” \( p_i = n_i/n \). In fact we have that \( p_i \geq 0 \) and \( \sum p_i = 1 \). This leads us to

Definition. An evaluation on an ortholattice \( \mathcal{L} \) is a mapping \( \nu : \mathcal{L} \to \mathbb{R} \). An evaluation \( \nu \) is called

1) nonnegative if \( \nu(a) \geq 0 \) for any \( a \in \mathcal{L} \);
2) monotone if \( \nu(a) \leq \nu(b) \) when \( a \leq b \);
3) normed if \( \nu(1) = 1 \);
4) additive (or a measure) if \( \nu(a \lor b) = \nu(a) + \nu(b) \) for orthogonal events \( a \) and \( b \). We write \( a \oplus b \) instead of \( a \lor b \) to emphasize that \( a \perp b \).
5) probabilistic (or a probability) if it is nonnegative and \( \sum_i \nu(a(i)) = 1 \) for any ODU \( (a(i), i \in I) \).

We make a few simple remarks on links between these concepts. From 4) or 5) it follows easily that \( \nu(0) = 0 \); clearly then 2) \( \implies \) 1). It is also clear that 5) \( \implies \) 3), and 1), 3) and 4) together imply 5). In the classical (Boolean) case 5) implies 1) - 4), but that is not true in the general case. Indeed, let us consider Example 3b, where (excluding the trivial events \( 1 \) and \( 0 \)) we have four events \( r, l, r^\perp, l^\perp \) and where \( r \leq l^\perp \) and \( l \leq r^\perp \). To give a probability is equivalent to give two numbers \( \nu(r) \) and \( \nu(l) \) both between 0 and 1 but otherwise arbitrary. Such a probability is monotone if \( \nu(r) + \nu(l) \leq 1 \) and is additive if \( \nu(r) + \nu(l) = 1 \).

There exists an important case when everything simplifies and approaches the classical case. It is the case of orthomodular lattices. So are called the lattices that satisfy the property of orthomodularity (if \( a \leq b \) then \( b = a \lor
(b \land a^\perp)). It is clear that any Boolean lattice is orthomodular and so are the
lattices from Examples 2, 3c, and 3d. In contrast, the lattice from Example
3b is not orthomodular. We assert that for orthomodular lattices, property 5)
implies 3) and 4).

**Lemma.** If $L$ is orthomodular ortholattice, then any probability on $L$ is
additive and monotonic.

Proof. Let $\nu$ be a probability on $L$. We first establish additivity. Suppose
$a \perp b$ and pose $c = (a \oplus b)^\perp$. Since $(c, c^\perp)$ is an ODU, $\nu(c) + \nu(c^\perp) = 1$.

We assert that $(a, b, c)$ is an ODU as well. To prove that we need to show
that $a^\perp = b \oplus c$. Since $a, b$ and $c$ are pairwise orthogonal, $b \oplus c \leq a^\perp$. By force
of the property of orthomodularity we have $a^\perp = (b \oplus c) \oplus (a^\perp \land (b \oplus c)^\perp)$. But
$a^\perp \land (b \oplus c)^\perp = (a \lor b \lor c)^\perp = (a \oplus b)^\perp \land c^\perp = c \land c^\perp = 0$. Hence $a^\perp = b \oplus c$. Similarly $b^\perp = a \oplus c$. The equality $c^\perp = a \oplus b$ is satisfied by definition. Thus,
the triplet $(a, b, c)$ is an ODU.

Therefore we have the equality $\nu(a) + \nu(b) + \nu(c) = 1$. Hence $\nu(a \oplus b) = \nu(c^\perp) = 1 - \nu(c) = \nu(a) + \nu(b)$, which yields the additivity of $\nu$.

Monotonicity follows trivially from the formula $b = a \oplus (b \land a^\perp)$, the addi-
tivity and the nonnegativity of the number $\nu(b \land a^\perp)$. QED

Thus, for the case of orthomodular lattices, a probability may also be
declared as a nonnegative normed measure.

## 4 Qualitative Measures

As it was already explained above we model uncertainty by an ortholattice of
properties or events. If we understand the elements of the lattice as events, we
may talk of smaller or larger probability for the realization of these events. Fur-
ther we focus on the ”more (or less) likely than” qualitative relation between
events.

**Definition.** A qualitative measure on an ortholattice $L$ is a binary relation
(of “likelihood”) $\preceq$ on $L$ satisfying the following two axioms:

QM1. $\preceq$ is complete and transitive.

QM2. Let $a \preceq b$ and $a' \preceq b'$. Then $a \oplus a' \preceq b \oplus b'$ (recall that it means
that $a \perp a'$ and $b \perp b'$). The last inequality is strict if at least one of the first
inequalities is strict.\(^2\)

\(^2\)The special case of QM2 when $a' = b'$ is referred to in \([8]\) as De Finetti axiom.
A qualitative measure \( \preceq \) is generated by a (quantitative) measure \( \mu \) when \( a \preceq b \) if and only if \( \mu(a) \leq \mu(b) \). In this section we are interested by the question as to when a qualitative measure can be generated by a quantitative measure (or when there exists a probabilistic sophistication). For simplicity we shall assume that the ortholattice \( \mathcal{L} \) is finite. But even in the classical context the answer is generally negative (Kraft, Pratt, Seidenberg, 1959) Therefore in order to obtain a positive answer we have to impose some additional conditions which strengthen QM2. We shall here consider a condition generalizing the classical “cancellation condition”. We prefer to call it “hyperacyclicity”.

**Definition.** A binary relation on \( \mathcal{L} \) is said to be **hyperacyclic** if the following condition holds:

Assume that we have a finite collection of pairs \((a_i, b_i)\) and that \( a_i \preceq b_i \) for all \( i \) and for some \( i \) the inequality is strict. Then \( \sum \mu(a_i) \neq \sum \mu(b_i) \) for some measure \( \mu \) on \( \mathcal{L} \).

It is obvious that hyperacyclicity implies acyclicity as well as .

Clearly, if the qualitative relation \( \preceq \) is generated by a measure \( \mu \) then it is hyperacyclic. The main result of this section (and the analog of Theorem 1 in \[8\]) asserts that for finite ortholattice the reverse is true.

**Theorem 1.** Let \( \preceq \) be a hyperacyclic qualitative measure on a finite ortholattice \( \mathcal{L} \). Then \( \preceq \) is generated by some measure on \( \mathcal{L} \).

A complete proof of Theorem 1 can be found in the Appendix. Here we confine ourselves with describing the logic of the proof: We first embed the ortholattice \( \mathcal{L} \) into a vector space \( V \) and identify linear functionals on \( V \) with measures on \( \mathcal{L} \). With the qualitative measure \( \preceq \) we construct a subset \( P \subset V \) and show that 0 does not belong to the convex hull of \( P \). The separability theorem then guarantees the existence of a linear functional on \( V \) (that is of a measure on \( \mathcal{L} \)) which is strictly positive on \( P \). It is easy to show that this measure generates the relation \( \preceq \).

Clearly, if the relation \( \preceq \) is monotonic (that is \( a \preceq b \) for \( a \leq b \)), then any measure \( \mu \) generating \( \preceq \) is also monotonic. If, in addition, \( 0 \prec 1 \) then \( \mu(1) > 0 \); dividing the measure \( \mu \) by \( \mu(1) \) we can assume that \( \mu \) is a normalized measure. Thus, the measure \( \mu \) is a monotonic probability.
5 Non-classical utility theory

First of all we need to formulate a suitable generalization of the Savagian concept of act. Roughly speaking an act is a bet on the result of some measurement.

**Definition.** An act is a pair \((\alpha, f)\), where \(\alpha = (a(i), i \in I(\alpha))\) is some ODU (or a measurement), and \(f : I(\alpha) \to \mathbb{R}\) is a function.

We call the measurement \(\alpha\) the basis of our act. Intuitively, if an outcome \(i \in I(\alpha)\) is realized as a result of measurement \(\alpha\), then our agent receives \(f(i)\) utils.

In such a way the set of acts with basis \(\alpha\) can be identified with the set (vector space, indeed) \(F(\alpha) = \mathbb{R}^{I(\alpha)}\). The set of all acts \(F\) is the disjoint union of \(F(\alpha)\) taken over all ODU \(\alpha\).

We are concerned with the comparison of acts with respect to their attractiveness for our decision-maker. We start with an implicit formula for such a comparison. Assume that the agent knows (more precisely, he thinks he knows) the state of the system, that is he has in his mind a (subjective) probability measure \(\mu\) on the ortholattice \(\mathcal{L}\). Then, for any act \(f\) on the basis \(\alpha = (a(i), i \in I(\alpha))\), he can compute the following number (expected value of the act \(f\))

\[
CE_\mu(f) = \sum_i \mu(a(i))f(i).
\]

Using those numbers our agent can compare different acts.

We now shall (following Savage) go the other way around. We begin with a relation \(\preceq\) representing preferences over the set of all acts \(F\), thereafter we formulate axioms, impose conditions and arrive at the conclusion that the preferences are explained by some probability measure \(\mu\) on \(\mathcal{L}\).

More precisely, instead of a preference relation \(\preceq\) on the set \(F\) of acts, we at once assume the existence of a certainty equivalent \(CE(f)\) for every act \(f \in F\). (Of course that does simplify the task a little. But this step is unrelated to the issue of classicality or non-classicality of the "world"; it is only the assertion of the existence of a utility on the set of acts. It would have been possible to obtain the existence of \(CE\) from yet other axioms. We chose a more direct and shorter way).

Given that we shall only impose three requirements on \(CE\). The first two relate to acts defined on a fixed basis. Such acts are identified with elements of the vector space \(F(\alpha) = \mathbb{R}^\alpha\).
Monotonicity axiom. The restriction of $CE$ on each $F(\alpha)$ is a monotone functional.

Linearity axiom. For any measurement $\alpha$ the restriction of $CE$ on $F(\alpha)$ is a linear functional.

The third axiom links acts between different but in some sense comparable basis. For this we need to be able to compare at least roughly two different measurements. Consider two ODU $\alpha = (a(i), i \in I(\alpha))$ and $\beta = (b(j), j \in I(\beta))$. We say the measurement $\alpha$ is finer than $\beta$ if there exists a mapping $\varphi : I(\alpha) \to I(\beta)$ such that $a(i) \leq b(\varphi(i))$ for any $i \in I(\alpha)$. Simply stated it means that as we know a result $i$ of the first measurement, we know the result of the second measurement without performing it, it is $j = \varphi(i)$. We note also that the transformation mapping $\varphi$ is uniquely defined. In fact assume that $\varphi(i)$ simultaneously belongs to $b(j)$ and $b(k)$. Then $a(i)$ belongs to $b(j) \land b(k)$. But since $b(j)$ and $b(k)$ are orthogonal $b(j) \land b(k) = 0$, so $a(i) = 0$. But this type of event do only formally enter the decomposition of the unit and they can be neglected.

In any case any such mapping $\varphi : I(\alpha) \to I(\beta)$ defines a mapping $\varphi^* : F(\beta) \to F(\alpha)$.

For a function $g$ on $I(\beta)$ the function $\varphi^*(g)$ in a point $i$ has the value $g(\varphi(i))$.

Intuitively, the payoffs from both functions (acts) $g$ and $f = \varphi^*(g)$ are identical in all situations. Therefore our agent should consider them as equivalent and assign them the same certainty equivalent. This is the idea of the following axiom.

Agreement axiom. Suppose that a measurement $\alpha$ is finer than $\beta$ and $\varphi : I(\alpha) \to I(\beta)$ is the corresponding mapping. Then $CE(g) = CE(\varphi^*(g))$ for each $g \in F(\beta)$.

Take for instance $f$ to be the constant function in $I(\alpha)$ with value 1. The agreement axiom says that the agent is indifferent between two acts. The first is to receive one util without performing any measurement. The second is to perform the measurement $\alpha$ and (independently of the outcome) to receive a unit of utils.

The last requirement which cannot really be called an axiom says that the utility of the trivial act with payoff 1 is equal to 1. That is $CE(1) = 1$.

Theorem 2. Suppose that a certainty equivalent $CE$ satisfies the monotonicity, linearity and agreement axioms. Then there exists a probabilistic
valuation \( \mu \) on \( \mathcal{L} \) such that \( CE(f) = \sum_i \mu(a(i))f(i) \) for any act \( f \) on the basis of measurement \( \alpha = (a(i), \ i \in I(\alpha)) \). Moreover this valuation \( \mu \) is uniquely defined.

Proof. For \( a \in \mathcal{L} \) we denote \( 1_a \) the bet on the property \( a \). It gives 1 util if we receive the answer YES on the question \((a, a^\perp)\) and 0 for NO. Let \( \mu(a) = CE(1_a) \). Since \( 1_a \geq 0 \) we have \( \mu(a) \geq 0 \) for any \( a \in \mathcal{L} \).

Let now \( \alpha = (a(i), i \in I(\alpha)) \) be an arbitrary ODU, and \( f : I(\alpha) \to \mathbb{R} \) be an act on the basis \( \alpha \). We denote with the symbol \( 1_i \) the act on the basis of \( \alpha \) which yields 1 on \( i \) and 0 on \( F(\alpha) - \{i\} \). By the agreement axiom we have that \( CE(1_i) = \mu(a(i)) \). Since \( f = \sum_i f(i)1_i \) we conclude that

\[
CE(f) = \sum_i \mu(a(i))f(i)
\]

In particular, if \( f = 1 \) we obtain that \( 1 = CE(1) = \sum_i \mu(a(i)) \). Therefore \( \mu \) is a probabilistic valuation. QED

We do not assert that the valuation \( \mu \) is monotone. In the next section we substitute the agreement axiom with a stronger "dominance" axiom and we obtain the monotonicity of \( \mu \).

6 The Dominance axiom

Let \( \alpha = (a(i), i \in I(\alpha)) \) be a measurement (or an ODU). And let \( b \in \mathcal{L} \) be an event (or a property). We say that an outcome \( i \in I(\alpha) \) is impossible under condition \( b \) (or in presence of the property \( b \)), if \( a(i) \perp b \). All other outcomes are in principle possible, and we denote the set of possible outcomes as \( I(\alpha|b) \). Clearly

\[
b \leq \bigvee_{i \in I(\alpha|b)} a(i) = a(I(\alpha|b)),
\]

and \( I(\alpha|b) \) is the smallest subset of \( I(\alpha) \) with that property. In fact if \( b \leq a(J) \) then \( a(J) \perp \leq b \). But \( a(J) \perp = a(I(\alpha) - J) \), therefore for any \( i \), not belonging to \( J \), we have \( a(i) \leq b \perp \), that is \( a(i) \perp b \).

Consider for instance a situation when we have two measurements \( \alpha = (a(i), i \in I(\alpha)) \) and \( \beta = (b(j), j \in I(\beta)) \). Suppose that the measurement \( \alpha \) is finer than \( \beta \) and \( \varphi : I(\alpha) \to I(\beta) \) is the corresponding mapping. Since

\[
b(j) = a(\varphi^{-1}(j)),
\]
it is easily seen that \(I(\alpha|b(j)) = \varphi^{-1}(j)\) and \(I(\beta|a(i)) = \{\varphi(i)\}\).

We go back to acts. Let \(f: I(\alpha) \to \mathbb{R}\) and \(g: I(\beta) \to \mathbb{R}\) be acts on the \(\alpha\) and \(\beta\) basis respectively. We say the \(g\) dominates \(f\) (and write \(f \leq g\)) if for any \(i \in I(\alpha)\) and any \(j \in I(\beta|a(i))\) (that is \(j\) is possible at the event \(a(i)\)) the inequality \(f(i) \leq g(j)\) is true. Intuitively, this means that the act \(g\) always gives no less than the act \(f\). With such an interpretation it is natural to assume that our rational decision-maker must assign to \(g\) no less utility than to \(f\). We formulate this as

**Axiom of dominance.** If \(f \leq g\) then \(CE(f) \leq CE(g)\).

It is clear that the dominance implies monotonicity. We assert that the dominance axiom also implies the axiom of agreement. In fact let \(\beta\) be a measurement coarser than \(\alpha\) and \(f = \varphi^*(g)\) for some act \(g\) on the \(\beta\) basis. From the description above it is clear that \(f \leq g\) and \(g \leq f\) such that \(CE(f) = CE(g)\).

**Theorem 3.** Assume that the axiom of linearity and dominance are satisfied. Then \(CE\) is an expected utility for some monotonic probability measure \(\mu\) on \(\mathcal{L}\).

**Proof.** The first statement follows from earlier remarks and theorems. Therefore we should prove the monotonicity of the measure \(\mu\). Let \(a \leq b\). Consider two measurement-questions \(\alpha = (a, a^{\perp})\) and \(\beta = (b, b^{\perp})\). Let \(f = 1_a\), that is a bet on event (property) \(a\) : the agent receives one util if measurement \(\alpha\) reveals (actualizes) property \(a\), and receives nothing in the opposite case. We define \(1_b\) similarly on the \(\beta\) basis. Clearly \(1_a \leq 1_b\). In fact if the first measurement reveals (actualizes) property \(a\), then \(b\) is true for sure since \(a \leq b\). Therefore \(1_b\) gives the agent one util when \(a\) occurs, and \(\geq 0\) util when \(a^{\perp}\) occurs, which is not worth less than \(1_a\). By force of the axiom of dominance \(CE(\alpha) \leq CE(\beta)\). The first term is equal to \(\mu(a)\) and the second to \(\mu(b)\). QED

**Appendix**

Here we prove Theorem 1.

1. **Construction of the vector space \(V\).** Denote \(\mathbb{R} \otimes \mathcal{L}\) the vector space generated by \(\mathcal{L}\). It consists of (finite) formal expressions of the form \(\sum_i r_i a_i\), where \(r_i \in \mathbb{R}\) and \(a_i \in \mathcal{L}\). Denote \(K\) the vector subspace in \(\mathbb{R} \otimes \mathcal{L}\) generated by expressions \(a \oplus b - a - b\) (recall that \(a \oplus b\) means that \(a \oplus b = a \lor b\) and
a \perp b$.) Finally, $V = V(\mathcal{L})$ is the quotient space $\mathbb{R} \otimes \mathcal{L}$ by the subspace $K$, $V = (\mathbb{R} \otimes \mathcal{L})/K$.

The ortholattice $\mathcal{L}$ naturally maps into $V$; the image $1 \cdot a$ of an element $a \in \mathcal{L}$ we denote simply as $a$. Any linear functional $l$ on $V$ restricted to $\mathcal{L}$ gives a valuation on $\mathcal{L}$. Since $l(a \oplus b - a - b) = l(a \oplus b) - l(a) - l(b) = 0$, the valuation $l$ is additive, that is a measure on the ortholattice $\mathcal{L}$. Conversely, let $l$ be a measure on $\mathcal{L}$. We extend it by linearity to $\mathbb{R} \otimes \mathcal{L}$ assuming $l(\sum r_i a_i) = \sum r_i l(a_i)$. By force of additivity, $l$ yields 0 for elements of the form $a \oplus b - a - b$, that is $l$ vanishes on the subspace $K$. Therefore $l$ factors through $V$ and is obtained from a linear functional defined on $V$. We just proved

**Proposition 1.** The vector space of measures on $\mathcal{L}$ is identified with the space $V^*$ of linear functionals on $V$.

Remark. The canonical mapping $\mathcal{L} \rightarrow V(\mathcal{L})$ can be considered as the universal measure on the ortholattice $\mathcal{L}$. It is injective if and only if the ortholattice $\mathcal{L}$ is orthomodular.

2. **Construction of the set of “strictly positive” $P$.** Let $\preceq$ be a binary relation on $\mathcal{L}$; as usual, $\prec$ denote the strict part of $\preceq$. By definition, $P = P(\preceq)$ consists of (finite) expressions of the form $\sum_i (a_i - b_i)$, where $b_i \preceq a_i$ for all $i$ and $b_i \prec a_i$ for some $i$. ($P$ is empty if the relation $\prec$ is empty, that is if all elements in $\mathcal{L}$ are equivalent relatively to $\preceq$.) We note also that $P$ is stable with respect to the addition.

3. Suppose now that a relation $\preceq$ is hyperacyclic. Note that the hyperacyclicity of $\preceq$ means precisely that 0 does not belongs to $P$.

**Proposition 2.** If the relation $\preceq$ is hyperacyclic then 0 does not belong to the convex hull of $P$.

**Proof.** Assume that 0 is a convex combination of elements of $P$, $0 = \sum_i r_i p_i$, where $p_i \in P$, $r_i \geq 0$, and $\sum_i r_i = 1$. By Caratheodory’s theorem we can assume that the $p_i$ are affinely independent (and therefore the coefficients $r_i$ are uniquely defined). We assert that in this case the coefficients are rational numbers.

It would be simplest to say that the set $P$ is defined over the field of rational numbers. But it is not so easy to provide a precise meaning to it. For that purpose we choose and fix some subset $L \subset \mathcal{L}$, such that its image in $V$ is a basis of that vector space. We also choose a subset $M$ of expressions of the form $a \oplus b - a - b$, which constitute a basis of the subspace $K$. The union
of $L$ and $M$ is a basis of the vector space $R \otimes L$. On the other side, $L$ is a basis of $\mathbb{R} \otimes L$ as well. Since elements of $L \cup M$ are rational combinations of elements of the $L$-basis elements of $L$, in turn, can be rationally expressed in terms of $L \cup M$. In particular, the images of elements of $L$ in $V$ are rational combinations of elements of the $L$-basis. All the more, the elements $p_i \in P$ can be rationally expressed in terms of $L \cup M$. In particular, the images of elements of $L$ in $V$ are rational combinations of elements of the $L$-basis. All the more, the elements $p_i \in P$ can be rationally expressed in terms of $L \cup M$. It follows (see, for example, Proposition 6 in [3], Chap. 2, § 6) that 0 can be expressed rationally through $p_i$. Since the coefficients $r_i$ are defined uniquely, they are rational numbers.

Now the proof can be easily completed. We have an equality $0 = \sum_i r_i p_i$, where $p_i \in P$ and $r_i$ are rational numbers (not all equal to zero). Multiplying with a suitable integer we may consider $r_i$ themselves as integers. Since $P$ is stable with respect to addition, we obtain that $0 \in P$, in contradiction with hyperacyclicity of the relation $\preceq$.

4. Together with Separation theorem of convex sets (see [10]) the results above imply existence of a (non-trivial) linear functional $\mu$ on $V$, non-negative on $P$. But we need strict positivity on $P$. To obtain it we show that (in the case of a finite ortholattice $L$) the convex hull of $P$ is a polyhedron.

Let us introduce some notations. $A$ denotes the set of expression $a - b$, where $a \succ b$. $B$ denotes the set of rays of the form $\mathbb{R}_+(a - b)$, where $a \succeq b$. Finally, $Q$ is the convex hull of $A \cup B$ in $V$. By definition, $Q$ consists of elements of the form

$$q = \alpha_1(a_1 - b_1) + ... + \alpha_n(a_n - b_n) + \beta_1(c_1 - d_1) + ... + \beta_m(c_m - d_m), \quad (*)$$

where $a_i, b_i, c_j, d_j \in L$ (more precisely, belong to their image in $V$), $a_i \succ b_i$ for any $i$, $c_j \succeq d_j$ for any $j$, $\alpha_i, \beta_i$ are nonnegative, and $\sum_i \alpha_i = 1$.

**Proposition 3.** The convex hull of $P$ coincides with $Q$.

**Proof.** It is clear from the definitions that any element of $P$ belongs to $Q$. By the convexity of $Q$, the convex hull of $P$ is also contained in $Q$.

It remains to show the converse, that any element $q$ of $Q$ belongs to the convex hull of $P$. For that (appealing to the convexity of $\text{co}(P)$) we can assume that $q$ has the form in $(*)$ with $n$ and $m$ equal to 1, that is

$$q = (a - b) + \beta(c - d),$$

where $a \succ b$, $c \succeq d$ and $\beta \geq 0$. If $\beta$ is an integer, it is clear that $q \in P$. In general case $\beta$ is a convex combination of two nonnegative integers $\beta_1$ and $\beta_2$,
then \( q \) is the corresponding convex combination of two points \((a-b) + \beta_1(c-d)\) and \((a-b) + \beta_2(c-d)\) both belonging to \( P \).

**Corollary.** Assume that an ortholattice \( \mathcal{L} \) is finite. Then the convex hull of \( P \) is a polyhedron.

In fact, in this case the sets \( A \) and \( B \) are finite. Therefore (see \[10\], theorem 19.1) \( Q \) is a polyhedra.

Thus, if 0 does not belong to the convex hull of \( P \) (see Proposition 2) then there exists a linear functional \( \mu \) on \( V \) which is strictly positive on \( P \). As we shall see, this immediately provides us with a proof of Theorem 1.

5. **Proof of Theorem 1.** The assertion in the theorem is trivially true if all elements of \( \mathcal{L} \) are equivalent to each other. Therefore we can assume that there exists at least one pair \((a, b)\) such that \( a \succ b \). Let \( \mu \) be a linear functional on \( V \) (we may consider \( \mu \) as a measure on the ortholattice \( \mathcal{L} \)) strictly positive on \( P \). We assert that this measure generates the relation \( \preceq \).

Let us suppose \( c \succeq d \). Since for any integer positive number \( n \) the element \((a-b) + n(c-d)\) belongs to \( P \), we have \( \mu(a) - \mu(b) > n (\mu(d) - \mu(c)) \) for any \( n \). This implies \( \mu(d) \leq \mu(c) \). Conversely, let us suppose \( \mu(c) \geq \mu(d) \) for some \( c, d \in \mathcal{L} \). We have to show that \( c \succeq d \). If this is not the case then, by completeness of the relation \( \succeq \), we have \( d \succ c \). But then \( d - c \) belongs to \( P \) and \( \mu(d - c) = \mu(d) - \mu(c) > 0 \), which contradicts to our first assumption. This completes the proof of Theorem 1.

**References**

[1] Atmanspacher H., Filk T., and Romer H. (2004) "Quantum Zeno features of bistable perception" *Biological Cybernetics* 90, 33-40.

[2] Birkhoff G. and von Neumann J. (1936) The logic of quantum mechanics, *Ann. Math.* 37, 823-843.

[3] Bourbaki N. (1962) *Algebra*, Hermann, Paris.

[4] Danilov V.I. and A. Lambert-Mogiliansky (2005) Non-classical Measurement Theory. xxx.lanl.gov/physics/0604051

[5] Deutsch D. (1999) Quantum Theory of Probability and Desisions. *Proc. R. Soc. Lond. A* 455, 3129-3137. See also xxx.lanl.gov/quant-ph/9906015.
[6] Gyntelberg J. and F. Hansen (2004) Expected utility theory with “small worlds”. http://www.econ.ku.dk/wpa/pink/2004/0420.pdf

[7] La Mura P. (2005) Decision Theory in the Presence of Risk and Uncertainty. mimeo Leipzig Graduate School of Business.

[8] Lehrer E. and Shmaya E. (2005) A Subjective Approach to Quantum probability. mimeo

[9] Pitowsky I. (2003) Betting on the outcomes of measurements. Studies in History and Philosophy of Modern Physics 34, 395-414. See also xxx.lanl.gov/quant-ph/0208121

[10] Rockafeller R.T. (1970) Convex Analysis, Princeton University Press, Princeton.

[11] Savage L. (1954) The Foundations of Statistics. John Wiley, New York.

[12] von Neumann J. (1932) Mathematische Grunlagen der Quantummechanik. Springer-Verlag, Berlin