Comment on: “Quantum aspects of the Lorentz symmetry violation on an electron in a nonuniform electric field” Eur. Phys. J. Plus (2020) 135:623

Paolo Amore¹ and Francisco M. Fernández²

¹ Facultad de Ciencias, CUICBAS, Universidad de Colima, Bernal Díaz del Castillo 340, Colima, Colima, Mexico e-mail: paolo@ucol.mx
² INIFTA, División Química Teórica, Blvd. 113 y 64 (S/N), Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina e-mail: fernande@quimica.unlp.edu.ar

Abstract. We analyze recent results concerning the hypothesis of a privileged direction in the space-time that is made by considering a background of the Lorentz symmetry violation determined by a fixed spacelike vector field and the analysis of quantum effects of this background on the interaction of a nonrelativistic electron with a nonuniform electric field produced by a uniform electric charge distribution. We show that the conclusions derived by the authors are an artifact of the truncation of the Frobenius series by means of the tree-term recurrence relation for the expansion coefficients. Thus, the existence of allowed angular frequencies stemming from this procedure is meaningless and unphysical.

PACS. 03.65.Ge Solutions of wave equations: bound states – 31.15.xt Variational techniques
In a recent paper Oliveira et al. consider the hypothesis of a privileged direction in the space-time. To this end they take into account a background of the Lorentz symmetry violation determined by a fixed spacelike vector field and analyze quantum effects of this background on the interaction of a nonrelativistic electron with a nonuniform electric field produced by a uniform electric charge distribution. Thus, they derive a Schrödinger equation that is separable in cylindrical coordinates. The resulting radial eigenvalue equation, which exhibits Coulomb plus harmonic interactions, is treated by means of the Frobenius method. Since the expansion coefficients satisfy a three-term recurrence relation the authors can force a truncation of the power series to obtain polynomial solutions and exact eigenvalues. From such results the authors predict the existence of allowed angular frequencies. In what follows we discuss the effect of the truncation of the Frobenius series on the physical conclusions drawn by the authors.

The starting point of present discussion is the eigenvalue equation for the dimensionless radial function

\[ u''(x) + \frac{1}{x} u'(x) - \frac{\nu^2}{x^2} u(x) - \delta x u(x) - x^2 u(x) + W u(x) = 0, \]

\[ W = \frac{2\tau}{m\omega}, \quad \tau = 2mE - k^2, \quad \omega = \sqrt{\frac{2|q|\rho}{m}}, \quad \nu = l + \frac{1}{2}(1 - s), \]

\[ \delta = \frac{gb\sqrt{2m\omega}}{|q|}, \]

where \( l = 0, \pm 1, \pm 2, \ldots \) is the rotational quantum number, \( s = \pm 1, m \) the mass of the particle, \( q = -|q| \) an electric charge, \( b > 0 \) the magnitude of a fixed spaced-like vector field, \( \rho > 0 \) is related to the uniform electric charge distribution and \( E \) the energy. The constant \( -\infty < k < \infty \) comes from the fact that the motion is unbounded along the \( z \) axis; therefore the spectrum is continuous and bounded from below \( E \geq \frac{\tau^2}{2m} \). The authors simply set \( \hbar = 1, c = 1 \) though there are well known procedures for obtaining suitable dimensionless equations in a clearer and more rigorous way [2].

In what follows we focus on the discrete values of \( W \) that one obtains from the bound-state solutions of equation (1) that satisfy

\[ \int_0^\infty |u(x)|^2 x \, dx < \infty. \]

(2)

Notice that we have bound states for all \( -\infty < \delta < \infty \) and that the eigenvalues \( W \) satisfy

\[ \frac{\partial W}{\partial \delta} = \langle x \rangle > 0, \]

(3)

according to the Hellmann-Feynman theorem [3].

The eigenvalue equation (1) is an example of conditionally solvable (or quasi-exactly solvable) problems that have been widely studied by several authors and exhibit a hidden algebraic structure (see [4] and references therein).

In order to solve the eigenvalue equation (1) the authors propose the ansatz

\[ u(x) = x^\gamma \exp \left(-\frac{\delta x - x^2}{2}\right) f(x), \quad f(x) = \sum_{j=0}^{\infty} a_j x^j, \quad \gamma = |\nu|, \]

(4)
and derive the three-term recurrence relation

\[
\begin{align*}
  a_{j+2} &= \frac{\delta (2j + 2\gamma + 3)}{2(j + 2)[j + 2(\gamma + 1)]}a_{j+1} + \frac{2j - \Theta}{(j + 2)[j + 2(\gamma + 1)]}a_j, \\
  \Theta &= W - 2(\gamma + 1) + \frac{\delta^2}{4}, \quad j = -1, 0, \ldots, a_{-1} = 0, \quad a_0 = 1.
\end{align*}
\]

(5)

If the truncation condition \(a_{n+1} = a_{n+2} = 0\) has physically acceptable solutions then one obtains exact eigenvalues and eigenfunctions. The reason is that \(a_j = 0\) for all \(j > n\) and the factor \(f(x)\) in equation (4) reduces to a polynomial of degree \(n\). This truncation condition is equivalent to \(\Theta = 2n\) and \(a_{n+1} = 0\). The latter equation is a polynomial function of \(\delta\) of degree \(n + 1\) and it can be proved that all the roots \(\delta_{(n,i)}^{(n,i)}\), \(i = 1, 2, \ldots, n + 1\), are real [5, 6]. If \(V(\delta, x) = \delta x + x^2\) denotes the parameter-dependent potential for the model discussed here, then it is clear that the truncation condition produces eigenvalues \(W_{(n,i)}^{(n,i)} = 2(n + \gamma + 1) - \frac{[\delta_{(n,i)}^{(n,i)}]^2}{4}\) for \(n + 1\) different potential-energy functions \(V_{(n,i)}^{(n,i)}(x) = V(\delta_{(n,i)}^{(n,i)}, x)\). It is worth noticing that the truncation condition only yields some particular eigenvalues and eigenfunctions because not all the solutions \(u(x)\) satisfying equation (2) have polynomial factors \(f(x)\). From now on we will refer to them as follows

\[ u_{(n,i)}^{(n,i)}(x) = x^\gamma f_{(n,i)}^{(n,i)}(y) \exp \left( -\frac{\delta}{2} x - \frac{x^2}{2} \right), \quad f_{(n,i)}^{(n,i)}(x) = \sum_{j=0}^{n} a_{j,\gamma}^{(n,i)} x^j. \]

(6)

Let us discuss the first cases as illustrative examples. When \(n = 0\) we have \(\delta_0^{(0)} = 0\) and the eigenfunction \(u_0^{(0)}(x)\) has no nodes. We may consider this case trivial because the problem reduces to the exactly solvable harmonic oscillator. Probably for this reason it was not explicitly taken into account by Oliveira et al [1].

When \(n = 1\) there are two roots

\[ \delta_1^{(1,1)} = \frac{2\sqrt{7}}{\sqrt{2\gamma + 3}}, \quad \delta_1^{(1,2)} = -\frac{2\sqrt{7}}{\sqrt{2\gamma + 3}}, \]

(7)

with the corresponding coefficients

\[ a_{1,\gamma}^{(1,1)} = \frac{\sqrt{7}}{\sqrt{2\gamma + 3}}, \quad a_{1,\gamma}^{(1,2)} = -\frac{\sqrt{7}}{\sqrt{2\gamma + 3}}, \]

(8)

respectively. We appreciate that the eigenfunction \(u_1^{(1,1)}(x)\) is nodeless and \(u_1^{(1,2)}(x)\) has one node.

For \(n = 2\) the results are

\[ \delta_2^{(2,1)} = 4 \sqrt{\frac{4\gamma + 7}{(2\gamma + 3)(2\gamma + 5)}}, \quad a_{1,\gamma}^{(2,1)} = 2 \sqrt{\frac{4\gamma + 7}{(2\gamma + 3)(2\gamma + 5)}}, \]

\[ a_{2,\gamma}^{(2,1)} = \frac{2}{2\gamma + 5}, \]

\[ \delta_2^{(2,2)} = 0, \quad a_{1,\gamma}^{(2,2)} = 0, \quad a_{2,\gamma}^{(2,2)} = -\frac{1}{1 + \gamma}, \]

\[ \delta_2^{(2,3)} = -4 \sqrt{\frac{4\gamma + 7}{(2\gamma + 3)(2\gamma + 5)}}, \quad a_{1,\gamma}^{(2,3)} = -2 \sqrt{\frac{4\gamma + 7}{(2\gamma + 3)(2\gamma + 5)}}, \]

\[ a_{2,\gamma}^{(2,3)} = \frac{2}{2\gamma + 5}. \]

(9)
In this case \( u^{(2,1)}(x) \), \( u^{(2,2)}(x) \) and \( u^{(2,3)}(x) \) have zero, one and two nodes, respectively, in the interval \( 0 < x < \infty \). It is convenient to arrange the roots of \( a_{n+1} = 0 \) as \( \delta^{(n,i)} > \delta^{(n,i+1)} \) so that \( u^{(n,i)}(x) \) has \( i - 1 \) nodes in \( 0 < x < \infty \).

From the roots of \( a_2 = 0 \) the authors derive the frequency

\[
\omega_{1,l} = \frac{4q^2}{m (bg)^2 \left( 2|\nu| + 3 \right)},
\]

and state that “Hence, Eq. (26) (present equation (10)) corresponds to the allowed values of the parameter \( \omega \) that yield a polynomial of first degree to the function \( f(x) \). These are the allowed values of \( \omega \) with respect to the radial mode \( n = 1 \). For this reason, we have labelled \( \omega = \omega_{1,l} \). For other radial modes \( (n = 2, 3, 4, \ldots) \), other expressions can be obtained. Therefore, from Eq. (26), we observe that there is a discrete set of values of the parameter \( \omega \) that permit us to construct a polynomial of first degree to \( f(x) \).” They also show an analytical expression for \( \omega_{2,l} \) and for the energies \( E_{1,l,k} \) and \( E_{2,l,k} \) and state that “The allowed energies (28) show us that the effects of the violation of the Lorentz symmetry modify the spectrum of energy for an electron that interacts with the nonuniform electric field. In this case, the effects of Lorentz symmetry violation determine the possible values of the angular frequency (26) associated with the radial mode \( n = 1 \), i.e. the values of \( \omega \) that allow us to obtain a polynomial of degree \( n = 1 \) to the function \( f(x) \).” In what follows we will show that these statements are nonsensical and have no physical significance because they are a product of the wrong belief that the only bound states are those given by the truncation condition.

In the general case one would obtain (in present, more general, notation) something like

\[
\omega^{(n,i)} = \frac{\left[ \delta^{(n,i)} \right]^2 q^2}{2m (bg)^2}, \quad i = 1, 2, \ldots, n + 1,
\]

and (also in present notation)

\[
E^{(n,i)} = \frac{\omega^{(n,i)} (n + \gamma + 1)}{2} - \frac{m (bg)^2}{8q^2} \left[ \omega^{(n,i)} \right]^2 + \frac{k^2}{2m}.
\]

If we omit the solutions \( \delta = 0 \) that appear for \( n \) even then we have \( (n - 1)/2 \) different energies for \( n \) odd and \( n/2 \) for \( n \) even. Such a multiplicity of solutions was not taken into account by Oliveira et al [11]. However, this fact is of no importance because the dependence of \( \omega \) on the node number \( n \) and on the quantum numbers is an artifact of the truncation condition that has no physical significance. All these results are meaningless because \( E^{(n,i)} \) are eigenvalues of the set of models \( V^{(n,i)}(x) \), \( i = 1, 2, \ldots, n + 1 \) while \( E^{(n',i')} \) are eigenvalues of completely different models \( V^{(n',i')}(x) \), \( i' = 1, 2, \ldots, n' + 1 \). In other words, what the authors exhibit as the spectrum of a given quantum-mechanical system are, in fact, some particular eigenvalues of several different models. In what follows we show how to obtain the true energies of the quantum-mechanical model reduced to the eigenvalue equation [11].
Given the model parameters $m$, $q$, $g$, $\omega$ and $\rho$ we obtain a value of $\delta$ and solve equation (11) for those functions $u(x)$ that satisfy the bound-state condition (9). In this way we obtain the allowed eigenvalues $W_{j,\gamma}$, $j = 0, 1, \ldots$, $W_{j,\gamma} < W_{j+1,\gamma}$ and the true allowed model energies

$$\mathcal{E}_{j,\gamma,k} = \frac{W_{j,\gamma}}{4} \omega + \frac{k^2}{2m}. \quad (13)$$

Notice that there are eigenvalues and eigenfunctions for any value of $\omega$ so that this frequency is by no means discrete. In order to illustrate this point we apply the Rayleigh-Ritz variational method with the non-orthogonal basis set

$$\{ \phi_j(x) = x^{\gamma+j} \exp \left( -\frac{x^2}{2} \right), \ j = 0, 1, \ldots \}. \quad \text{As a first example, we arbitrarily choose } \delta = \delta^{(1,1)}_0 = \frac{2\sqrt{5}}{5};$$

the first eigenvalues are: $W_{0,0} = W^{(1)}_0 = \frac{10}{2}$, $W_{1,0} = 8.417789723$, $W_{2,0} = 13.16139523$, $W_{3,0} = 17.76476931$, $W_{4,0} = 22.2869551$. Notice that the truncation condition only yields the ground state and misses all the other eigenvalues. The second example is $\delta = \delta^{(1,2)}_0 = -\frac{2\sqrt{5}}{5}$ and in this case we have $W_{0,0} = 0.3822700980$, $W_{1,0} = W^{(1)}_0 = \frac{10}{2}$, $W_{2,0} = 6.589162760$, $W_{3,0} = 9.984807649$, $W_{4,0} = 13.46286362$. Clearly, in this case the truncation condition yields the first excited state and misses all the other eigenvalues. In this two examples we have chosen values of $\delta$ that are given by the truncation condition. If we choose $\delta = 1$, which is not a root of $a_{n+1} = 0$ for any value of $n$, we have $W_{0,0} = 2.840687067$, $W_{1,0} = 7.506478794$, $W_{2,0} = 11.96275335$, $W_{3,0} = 16.33275291$, $W_{4,0} = 20.65232861$. None of these eigenvalues can be derived from the truncation condition. We appreciate that these results satisfy the Hellmann-Feynman theorem (9) because $W_{c,0} \left( \delta^{(1,2)}_0 \right) < W_{c,0} \left( \delta^{(1,2)}_0 \right) < W_{c,0} \left( \delta^{(1,2)}_0 \right)$, where $\delta^{(1,2)}_0 < 1 < \delta^{(1,1)}_0$. We also realize that $\omega$ is not quantized and that there are bound states for any given value of this parameter.

Figure 1 shows several eigenvalues $W^{(n,i)}_0$ and $W^{(n,i)}_1$ given by the truncation condition (blue points) as well as the eigenvalues $W_{j,0}$ and $W_{j,1}$ obtained by means of the variational method (red lines). The continuous lines show that there are eigenvalues for any value of $\delta$ and, consequently, for any value of $\omega$. The occurrence of discrete values of the angular frequency is an artifact of the truncation procedure that only yields some particular solutions to the eigenvalue equation (11) (red points). A question arises as to the meaning of the roots $W^{(n,i)}_{j,\gamma}$ stemming from the truncation condition. Taking into account the nodes of the solutions and the Hellmann-Feynman theorem we conclude that a pair $\left( \delta^{(n,i)}_{\gamma}, W^{(n,i)}_{\gamma} \right)$ is a point on the curve $W_{i-1,\gamma}(\delta)$ in complete agreement to what is shown in figure 1.

**Summarizing:** all the physical conclusions derived by Oliveira et al (1) regarding the effects of Lorentz symmetry violation that determine the possible values of the angular frequency are obviously wrong and a mere artifact of the truncation method used to obtain some particular solutions to a conditionally solvable quantum-mechanical model (see, for example, (5),(6) and references therein).
Fig. 1. Eigenvalues $W_{j,n,l}^{(\gamma)}$ from the truncation condition (blue points) and variational results $W_{j,\gamma}$ (red lines) for $\gamma = 0$ (left panel) and $\gamma = 1$ (right panel).

Acknowledgements

The research of P.A. was supported by Sistema Nacional de Investigadores (México).

References

1. A. S. Oliveira, K. Bakke, and H. Belich, Eur. Phys. J. Plus 135, 623 (2020).
2. F. M. Fernández, Dimensionless equations in non-relativistic quantum mechanics, arXiv:2005.05377 [quant-ph].
3. R. P. Feynman, Phys. Rev. 56, 340 (1939).
4. A. V. Turbiner, One-dimensional quasi-exactly solvable Schrödinger equations, arXiv:1603.02992v2.
5. M. S. Child, S-H. Dong, and X-G. Wang, J. Phys. A 33, 5653 (2000).
6. P. Amore and F. M. Fernández, On some conditionally solvable quantum-mechanical problems, arXiv:2007.03448 [quant-ph].