COMPACTIFIED JACOBIANS

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Preliminary version

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0. INTRODUCTION

0.1. Let C be a reduced projective curve over a field k such that \( C_k \) has only nodes as singularities. The jacobian \( \text{Pic}^0 C \) is a semiabelian variety over \( k \) which parameterizes invertible sheaves on \( C \) of degree 0 on each irreducible component. It need not be proper. The problem of finding a good compactification for it goes back at least to the work of Igusa [Igu56] and the notes of Mumford and Mayer [Mum64, May70]. For an irreducible curve the answer was given already by D'Souza in [D'S79]. Altman, Kleiman and others extended this work to the families of irreducible curves with more general (for example, nonplanar) singularities in a series of works [AK80, AK79, AIK76, KK81, Reg80], and more recently [Sou94, Est95].

0.2. In the case when \( C \) is reducible the situation is more complicated. In a classical paper [OS79] Oda and Seshadri constructed a family of compactified jacobians \( \text{Jac}_\phi \) parameterized by an element \( \phi \) of a certain real vector space. The construction is very general and covers a lot of cases. At the same time it poses a question of giving a more natural definition for \( \text{Jac}_\phi \) and explaining where exactly the multitude of answers comes from. A related paper is [Ses82].

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Remark 0.3. It is important to note that the term “compactified jacobian” is a misnomer. Most of the varieties Jac discussed here do not naturally contain Pic⁰(C). This becomes especially clear when one works over a nonclosed field or with families. Instead, there is always an action of Pic⁰(C) and Jac is stratified into locally closed subschemes so that every stratum is a homogeneous space over Pic⁰(C) and the maximal-dimensional strata are principal homogeneous spaces.

However, we will use the term since it is widely accepted.

0.4. The work of Simpson [Sim94] on the moduli of coherent sheaves on projective schemes implies, as a very special case of a much more general situation, a natural definition of the compactified jacobian Jacd,L(C) which depends on an integer d, the degree, and on an ample invertible sheaf L on C, the polarization. The definition is functorial and therefore also works for families. It turns out that Jacd,L(C) and Oda-Seshadri’s Jacφ(C) coincide and there is a simple formula for φ as a function of d and L. An immediate corollary of this is that they are all reduced and Cohen-Macaulay schemes.

0.5. In the case when the curve C is stable, we can further narrow the choices by using for L the dualizing sheaf ωC. Then for every d ∈ Z the schemes Jacd,ωC / Aut(C) can be put in a family over the moduli space Mₙ, where n is the arithmetical genus of the curve C. This is the result of the work [Pan94] of Pandharipande (he also considers sheaves of rank ≥ 2). A yet another family Pd → Mₙ for d ≥ 10(2g − 2) was earlier constructed by Caporaso in [Cap94] as the compactification of the universal jacobian. The interpretation of the fiber Pd(C) over [C] ∈ Mₙ is in terms of invertible sheaves on certain semistable curves that have C as a stable model. Pandharipande shows that Caporaso’s construction of Pd is equivalent to his.

0.6. Another approach is to look at a one-parameter family of smooth curves Ct degenerating to C and try to find a limit of the family of “Jacobians” Jacd(Ct) = Pic⁰(Ct), perhaps after a finite ramified base change. In the complex analytic situation Namikawa [Nam77] constructed infinitely many toroidal degenerations of principally polarized abelian varieties that depend on polyhedral decompositions. We note a related work [Kaj93] where the compactified jacobians corresponding to polyhedral decompositions appear in the context of log geometry (under the restriction that the irreducible components of the curve C are nonsingular).

Among various polyhedral decompositions Namikawa explicitly distinguished one called the Voronoi decomposition (and the Delaunay decomposition dual to it).

0.7. The degeneration corresponding to the Delaunay and Voronoi decompositions also appears in [AN96] as a result of the “simplified Mumford’s construction”. There it is shown that a family of principally polarized abelian
varieties with theta divisors over spectrum of a complete DVR has the canonical limit (perhaps after making a finite ramified base change first). This limit was called a stable quasiabelian variety (SQAV), and when considered as a pair \((P, B)\) with the theta divisor – a stable quasiabelian pair (SQAP). This poses a question of whether the SQAP which appears as the limit of jacobians is one of the compactified jacobians \(\text{Jac}_\phi\), \(\text{Jac}_{d,L}(C)\) above, and if yes, then which one.

0.8. As explained in [Ale96], an SQAV corresponding to a smooth curve \(C\) coincides with \(\text{Pic}^{g-1}(C)\) and not with \(\text{Pic}^0(C)\). This gives a motivation to look at the case \(d = g - 1\) more closely.

0.9. We show that precisely for one degree, \(d = g - 1\), the scheme \(\text{Jac}_{d,L}(C)\) does not depend on the polarization \(L\), so one can simply write \(\text{Jac}_{g-1}\). For this reason we call it the canonical compactified jacobian. We show that \(\text{Jac}_{g-1}\) possesses a natural ample sheaf with a natural section which we call the theta divisor \(\Theta_C\). We give a very simple explicit combinatorial description of the stratification of \(\text{Jac}_{g-1}\) and the restrictions of \(\Theta_C\) on each stratum. The description goes in terms of the orientations on complete subgraphs of the dual graph \(\Gamma(C)\) and invertible sheaves on the partial normalizations of \(C\) of multidegrees that correspond to these orientations.

0.10. By considering a degenerating family \(C_t \rightarrow C\) of curves and the corresponding degenerating family \(\text{Pic}^{g-1}(C_t) \rightarrow \text{Jac}_{g-1}(C)\) one can show that the canonical compactified jacobian is an SQAV (we do not include this argument here). Therefore, the functor associating to each stable curve \(C\) its canonical compactified jacobian \((\text{Jac}_{g-1}, \Theta)\) should define a map from the Deligne-Mumford compactification \(\overline{\text{M}}_g\) to the complete moduli space of SQAVs if the latter exists.

0.11. Finally, an SQAV was defined in [AN96, Ale96] in terms of some explicit combinatorial data. We give the corresponding data for a curve \(C\) (about a half of this description can already be found in [Nam76, §18] and [Nam80, 9.D] where it is attributed to Mumford). Further, we explain how this second description is related to the previous one.

0.12. Although the main definitions and results including 1.12 and 1.13 hold over arbitrary base, for most of the paper we will be working over an algebraically closed field for simplicity.

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1. Definition of $\text{Jac}_{d,L}$

In this section I give the definition for compactified jacobians which I feel is the easiest and the most natural, and formulate the main existence theorem, due to Simpson. To take the bull by the horns, here it is:

**Definition 1.1.** For every integer $d$ and a polarization $L$ on $C$, the “compactified jacobian” $\text{Jac}_{d,L}$ is the coarse moduli space of semistable w.r.t. $L$ admissible sheaves on $C$ of degree $d$ up to the gr-equivalence.

Let us now explain the terms used in this definition. The polarization $L$ is an ample invertible sheaf. By admissible (for our purposes) sheaf we mean a coherent $\mathcal{O}_C$-module $F$ such that for every $x \in C$ the stalk $F_{x,C}$ is of depth 1. Equivalently, any nonzero subsheaf of $F$ has support of dimension 1.

The latter condition is what Seshadri [Ses82] calls a depth 1 sheaf and what Simpson [Sim94] calls a purely dimensional sheaf.

1.2. As well known (see f.e. [Ses82]) admissible sheaves have a very simple description:

(i). if $x$ is nonsingular, $F_{x,C} \simeq \mathcal{O}_{x,C}$
(ii). if $x$ is a node, $F_{x,C}$ is either $\mathcal{O}_{x,C}$ or the maximal ideal $m_{x,C}$. In the latter case $F$ is isomorphic to $\pi(x)_* F(x)$ where $\pi(x)$ is a partial normalization of $C$ at $x$ and $F(x) = \pi(x)^* F/torsion$.

Moreover, if we are interested in the depth 1 sheaves that have rank 0 or 1 at every generic point, we need to add the sheaves such that (iii). $F_{x,C} = 0$
(iv). if $x$ is a node lying on two irreducible components $C_1$ and $C_2$ with the inclusions $i_k : C_k \to C$, then $F_{x,C} = i_k_* \mathcal{O}_{C_k}$, $k = 1$ or 2.

**Lemma 1.3.** For each admissible sheaf $F$ on $C$ denote by $\pi' : C' \to C$ the partial normalization of $C$ at the nodes where $F$ is not invertible and by $F' = \pi'^* F/tors$. Then $F = \pi'_* F'$. Therefore, every admissible sheaf $F$ on $C$ can be identified with a unique invertible sheaf $F'$ on a unique partial normalization $C'$ of $C$.

**Proof.** Well known, see f.e. [Ses82].

**Definition 1.4.** For an admissible sheaf $F$ on $C$ the degree is defined as $\deg F = \chi(F) - \chi(\mathcal{O}_C) = \chi(F) + g - 1$,

where $g$ is the arithmetical genus of $C$.

**Remark 1.5.** Note that $\deg F'$ if defined on $C'$ itself is $\deg F$ minus the number of nodes where $F$ is not invertible.
Definition 1.6. Let $\lambda = (\lambda_1 \ldots \lambda_s)$ be the multidegree of $L$, and $r = (r_1 \ldots r_s)$ be the multirank of a depth 1 sheaf $F$. The Seshadri slope is
$$\mu_L(F) = \frac{\chi(F)}{\sum \lambda_i r_i}$$

Definition 1.7. A depth 1 sheaf $F$ on $C$ is said to be stable (resp. semistable) w.r.t. the polarization $L$ if for any nonzero subsheaf $E \subset F$ one has
$$\mu_L(E) < \mu_L(F)$$
(resp $\leq$).

Remark 1.8. This definition, due to Seshadri [Ses82], is a particular case of a much more general one given by Simpson in [Sim94] that applies to a pure-dimensional sheaf on any projective scheme whatsoever.

Lemma 1.9. A depth 1 sheaf is (semi)stable iff the inequality $\mu_L(E) < (resp. \leq) \mu_L(F)$ is satisfied for finitely many subsheaves of the form $F_D = i_*(i^*F/tors)$ for every subcurve $D \subset C$, where $i: D \to C$ is the inclusion morphism.

Proof. Indeed, for any depth 1 subsheaf $E$ with support $D$ one has $E \subset F_D$ and $\mu_L(E) \leq \mu_L(F_D)$.

Remark 1.10. This leads to a series of simple inequalities some of which will be considered in the next sections. Therefore, knowing the multidegree of $F$ and the set of nodes where $F$ is not locally free, it is easy to say whether $F$ is (semi)stable or not.

According to the general theory, for every depth 1 sheaf $F$ there exists a Harder-Narasimhan filtration
$$0 = F_0 \subset F_1 \subset \cdots \subset F_k = F$$
with strictly decreasing slopes and semistable quotients $F_i/F_{i+1}$.

If $F$ is semistable, then there is a similar (Jordan-Holder) filtration with stable $F_i/F_{i+1}$ which is not unique. However, the graded object $\text{gr}(F) = \oplus F_i/F_{i+1}$ is uniquely defined.

Definition 1.11. Two semistable depth 1 sheaves $F$ and $F'$ are said to be gr-equivalent if $\text{gr}(F) \simeq \text{gr}(F')$.

Now all the ingredients of the definition 1.1 have been introduced.

To put this into a functorial perspective, consider a projective morphism of schemes $\pi: C \times S \to S$ whose every geometric fiber is a reduced curve with nodes only as singularities and a relatively ample sheaf $L$ on $C$. We will say that a coherent sheaf $F$ on $C$ is admissible (resp. stable, resp. semistable) if so is its restriction to every geometric fiber of $\pi$. We say that two sheaves are equivalent (resp. gr-equivalent) if the restrictions on the geometric fibers are isomorphic (resp. gr-equivalent). Now define the moduli functor $\text{Jac}_{\lambda,L}(C/S): \text{Schemes} \to \text{Sets}$ in the following way:
Definition 1.12. For any scheme $S'$, $\text{Jac}_{d,L}^{-}(C/S)(S')$ is the set of semistable admissible sheaves on $C' = C \times_{S} S'$ up to the gr-equivalence. The functor $\text{Jac}_{d,L}^{-}(C/S)$ itself is not necessarily a sheaf for the fpff (faithfully flat of finite presentation) topology and therefore cannot be representable even if all the sheaves are stable. This happens basically for the same reason why the functor $\text{Pic} S'$ is not a sheaf: one needs a rigidification to kill the infinite ($= \mathbb{G}_m$) group of automorphisms.

When the smooth locus of $C/S$ has a section, one can use the rigidified version. Or we can follow the same path as for the relative Picard functor, i.e. define $\text{Jac}_{d,L}(C/S)$ to be the fpff-sheafification of $\text{Jac}_{d,L}^{-}(C/S)$.

Theorem 1.13 (Simpson). The functor $\text{Jac}_{d,L}(C/S)$ is coarsely represented by a projective scheme $\text{Jac}_{d,L}(C/S)$.

Proof. This may be proved by the same methods as in [Sim94] and basically is a very special case of [Sim94, 1.21]. Simpson works over $\mathbb{C}$ but in our particular situation there is no need for this. The hardest question involved, the boundedness, is basically obvious.

Alternatively, in 3.3 we will show that our (semi)stability condition is equivalent to the one that was used by Oda and Seshadri, so in the case $S = \text{Spec } \bar{k}$ the theorem follows by [OS79, 12.14].

2. Basic definitions and notations

The purpose of this section is to fix some notations common to the following sections and to introduce the basic examples of curves on which the later descriptions will be illustrated.

Notation 2.1. (i). To any curve $C$ we can associate the unoriented graph $\Gamma(C)$ by assigning a vertex to each irreducible component $C_i$ and an edge to every node. We do not assume $C$ to be connected, so the graph need not be connected either.

(ii). $\pi : \tilde{C} \rightarrow C$ denotes the normalization of $C$.

(iii). $C_i$ are the irreducible components of $C$, $\tilde{C}_i$ are their normalizations.

(iv). $g_i = p_a(C_i)$, $\tilde{g}_i = p_a(\tilde{C}_i)$.

(v). $h(C) = h(\Gamma(C))$ is the cyclotomic number – the number of independent loops in $\Gamma$, i.e. the rank of $H_1(\Gamma(C))$ when $\Gamma$ is considered as a cell complex.

Six simple examples.
Example 1. $C = C_1 \cup C_2$ intersecting at one point with both $C_i$ smooth. $	ilde{g}_i = g_i$, $h(C) = 0$.

Example 2. The generalization of the previous example is a curve whose dual graph is a forest. Still $	ilde{g}_i = g_i$ and $h(C) = 0$.

Example 3. An irreducible curve with one node. $	ilde{g} = g - 1$, $h(C) = 1$. 
Example 4. The generalization of that is an irreducible curve with \( n \) nodes. 
\[ \tilde{g} = g - n, \ h(C) = n. \]

Example 5. The dollar sign curve. \( \tilde{g}_i = g_i, \ h(C) = 2. \)

Example 6. The generalization of the dollar sign curve is a curve \( C = C_1 \cup C_2 \) with both \( C_i \) smooth and intersecting at \( n \) points. \( \tilde{g}_i = g_i, \ h(C) = n - 1. \)
3. Comparison with Oda-Seshadri’s compactified Jacobians

We would like to compare the compactified Jacobians introduced in section 1 with those appearing in the classical paper [OS79] of Oda and Seshadri.

3.1. The $\text{Jac}_\phi$ in [OS79] are constructed using GIT as the moduli spaces of $\phi$-semistable admissible sheaves. Here $\phi$ is an element of a certain real vector space $\partial C_1(\Gamma, \mathbb{R})$ (without loss of generality one can assume that $\phi \in \partial C_1(\Gamma, \mathbb{Q})$). $\Gamma$ is, as in the previous section, the dual graph of $C$ and $C_0, C_1, H_0, H_1, C^0, C^1, H^0$ and $H^1$ are the associated to it chain and (co)homology groups.

3.2. In [OS79] the main object of interest is the depth 1 sheaves of degree 0. Oda and Seshadri give the combinatorial definition of a $\phi$-stable (resp. semistable) sheaf and introduce the $\phi$-equivalence relation. The main result then is that there exists a reduced scheme $\text{Jac}_\phi$ which coarsely represents the functor of $\phi$-semistable sheaves up to $\phi$-equivalence. This is then applied to compactify $\text{Pic}^0 C$.

However, for any depth 1 sheaf of arbitrary degree $d$ one can relate the $\phi$-(semi)stability and equivalence with $(d, L)$-(semi)stability and equivalence. Then whatever is proved for $\text{Jac}_\phi$ immediately applies to $\text{Jac}_{d,L}$. Here is the precise connection.

**Theorem 3.3.** Let $\lambda = (\lambda_i)$ and $\omega = (\lambda_i)$ be the multidegrees of the polarization $L$ and of the dualizing sheaf $\omega_C$ respectively, and $\lambda = \sum \lambda_i$ and $\omega = \sum \omega_i = 2g - 2$ be the total degrees. Pick arbitrary integers $d_i$ with $\sum d_i = d$ and sufficiently large integers $\bar{n}_i$. Define $\phi = (\phi_i) \in \partial C_1(\mathbb{Q})$ to be a solution of the following system of linear equations

$$(\lambda_i/\lambda)(d - \omega/2) = d_i - \omega_i/2 + \bar{n}_i + \phi_i$$

($\phi$ is only defined up to a shift by a lattice). Then an admissible sheaf of degree $d$ is (semi)stable w.r.t. $L$ iff it is $\phi$-(semi)stable. Two semistable w.r.t. $L$ sheaves are gr-equivalent iff they are $\phi$-equivalent.

**Proof.** This can be extracted from [OS79, §11] directly, particularly from the account on pp.52-53.

**Corollary 3.4.** Every $\text{Jac}_{d,L}$ is isomorphic to one of $\text{Jac}_\phi$ and vice versa.

**Corollary 3.5.** Every $\text{Jac}_{d,L}$ is reduced and Cohen-Macaulay.

**Proof.** Indeed, $\text{Jac}_\phi$ is reduced by [OS79, 11.4]. Moreover, the proof shows (pp.60-62) that $\text{Jac}_\phi$ is a good GIT quotient of a certain scheme $R$ and there exists an open subscheme $Y \subset R \times \mathbb{P}(E^*)$ such that the projection $R \to Y$ is surjective, and $Y$ is formally smooth over a Hilbert scheme $H$ which is open in a quotient by the symmetric group of $C \times \cdots \times C$.

Therefore, $H$ is CM, an so is $Y$, and so is $R$, and so is $\text{Jac}_\phi$. 

4. Description of $\text{Jac}_{g-1}$

**Lemma 4.1.** $\text{Jac}_{g-1,L}(C)$ does not depend on the polarization $L$.

**Proof.** Indeed, by definition 1.4 the degree $d = g - 1$ iff $\chi(F) = 0$. Then for any $E \subset F$ the inequality

$$\mu_L(E) \leq (\text{resp.} <) \mu_L(F)$$

is equivalent to

$$\chi(E) \leq (\text{resp.} <) 0$$

For this reason we will call $\text{Jac}_{g-1}(C)$ the canonical compactified jacobian.

**Definition 4.2.** A subgraph $\Gamma' \subset \Gamma$ is said to be generating if $\text{vertices}(\Gamma) = \text{vertices}(\Gamma')$. Every such subgraph corresponds to a partial normalization of $C$ at the nodes $\Gamma - \Gamma'$. We denote this partial normalization by $\pi(\Gamma') : C(\Gamma') \to C$. Note in particular that $C(\Gamma) = C$ and that $\tilde{C}$ is $C(\Gamma')$ where $\Gamma'$ has all the vertices of $\Gamma$ but no edges at all.

**Definition 4.3.** A subgraph $\Gamma' \subset \Gamma$ is said to be complete if $\text{vertices}(\Gamma') \subset \text{vertices}(\Gamma)$ and $\text{edges}(\Gamma')$ are precisely the edges of $\Gamma$ lying inside $\Gamma'$. These graphs correspond to subcurves $D \subset C$. Often we identify such subcurves $D$ with the corresponding subgraphs.

**Definition 4.4.** A multidegree of a graph $\Gamma$ is a set $\underline{d} = (d_i)$ of integers for every vertex $C_i$ of $\Gamma$. We will always assume that

$$\sum d_i = g - 1$$

A normalization of multidegree $\underline{d}$ is a set of integers $\underline{e} = (e_i)$ defined by

$$e_i = d_i - (\tilde{g}_i - 1).$$

It will be called the normalized multidegree. Note that we can use multidegrees $\underline{d}$ and normalized multidegrees $\underline{e}$ interchangeably. Note that $\sum e_i$ equals the number of edges of $\Gamma$.

For a subcurve $D \subset C$, i.e. a complete subgraph $\Gamma' \subset \Gamma$, we set

$$d_D = \sum_{C_i \subset D} d_i, \quad e_D = \sum_{C_i \subset D} e_i$$

**Definition-Proposition 4.5.** A normalized multidegree $\underline{e}$ is called semistable (resp. stable) if any of the following equivalent conditions hold:

(i). 

$$|e_D - \#\text{edges}(D) - \frac{1}{2}D(C-D)| \leq \frac{1}{2}D(C-D)$$

for every subcurve $D \subset C$. Here $D(C-D)$ is the number of points in $D \cap (C \setminus \overline{D})$ (resp. <).
(ii). 
\[ e_D \leq \#\text{edges}(D) + D(C - D) \] 
(resp. <).

(iii). there exists an orientation of the graph \( \Gamma \) such that \( e_i \) equals the number of arrows pointing at \( C_i \) (resp. in addition there is no proper subcurve \( D \subset C \) such that all arrows between \( D \) and \( C - D \) go in one direction).

In this case the multidegree \( (d) \) is also called (semi)stable.

**Proof.** The implication (i) \( \Rightarrow \) (ii) is clear and the inverse is obtained by looking at \( D' = C - D \). (iii) obviously implies (ii).

To prove the implication (ii) \( \Rightarrow \) (iii) first assume that the normalized multidegree \( \epsilon \) of the graph of \( C \) is strictly semistable, i.e. there exists a subcurve \( D \subset C \) for which the equality holds. Then consider separately the following multidegrees on \( D \) and \( C - D \). On \( C - D \) simply take the restriction of \( \epsilon \). On \( D \), however, for every vertex \( C_i \) take \( \epsilon'_i = e_i \) minus the number of edges between \( C_i \) and \( C - D \). Then it is easy to show that the two multidegrees thus obtained are semistable. Therefore, the orientations on \( D \) and \( C - D \) exist by the induction on the number of vertices. To complete the orientation of \( C \), orient all the edges between \( D \) and \( C - D \) to point at \( D \).

In general, starting with a semistable multidegree as in (ii) we can fix an arbitrary vertex \( C_{i_0} \) and change the degrees of \( C_{i_0} \) and the neighboring vertices = curves \( C_j \) by 1 to make the multidegree strictly semistable, thus reducing to the previous case. Hence, we get an orientation for the modified multidegree. The orientation for the original multidegree is then obtained by reversing the orientations of edges \((i_0, j)\).

The third condition of the above definition is the easiest to check. We will call an orientation satisfying (iii) semistable (resp. stable). Note that different orientations may well produce the same multidegree.

4.6. This is how the above combinatorial definitions relate to the (semi) stability of admissible sheaves on \( C \). By lemma 1.3 every admissible sheaf \( F \) on \( C \) can be identified with a unique invertible sheaf \( F' \) on a unique partial normalization \( C' = C(\Gamma') \) of \( C \). Denote by \( (d') \) (resp. \( (\epsilon') \)) the corresponding (resp. normalized) multidegrees on \( C' \). Then

**Lemma 4.7.** If \( \deg F = g - 1 \), then for \( (d') \) one has \( \sum d'_i = g' - 1 \).

**Proof.** Obvious.

**Lemma 4.8.**

(i). \( F \) is semistable iff \( (\epsilon') \) is semistable.

(ii). \( F \) is stable iff \( (\epsilon') \) is stable and the graphs \( \Gamma \) and \( \Gamma' \) have the same number of connected components.

**Proof.** Follows easily from 4.5 and 1.9.
We can now describe the points of $\text{Jac}_{g-1}(C)$ as follows.

**Theorem 4.9.** (i). $\text{Jac}_{g-1}(C)$ has a natural stratification into homogeneous spaces over $\text{Pic}^0(C)$. Each stratum corresponds in a 1-to-1 way to a stable multidegree $d'$ (resp. stable normalized multidegree $\tilde{d}'$) on a generating subgraph $\Gamma' \subset \Gamma$. The $k$-points of this stratum can be identified with $k$-points of $\text{Pic}_{d'}(C(\Gamma'))$, i.e. with invertible sheaves on $C(\Gamma')$ of multidegree $d'$. The codimension of this stratum equals $h(\Gamma) - h(\Gamma')$.

(ii). There is a natural Cartier divisor $\Theta$ on $\text{Jac}_{g-1}(C)$. Under the above identification, the restriction of $\Theta$ on each stratum corresponds to the sheaves $L$ with $h^0(L) > 0$.

To illustrate this theorem, let us see what happens in our basic examples.

**Example 1.** Graph $\Gamma$ doesn’t have any stable multidegrees: take $D$ to be one of the vertices. The only possibility then is $\Gamma'$ which is a disjoint union of two vertices and the multidegree $e = (0, 0)$, i.e. $d = (\tilde{g}_1 - 1, \tilde{g}_2 - 1) = (g_1 - 1, g_2 - 1)$. The graph $\Gamma'$ corresponds to the normalization $\tilde{X} = X_1 \sqcup X_2$ and

$$\text{Jac}_{g-1}(C) = \text{Pic}^{g_1-1}(C_1) \oplus \text{Pic}^{g_2-1}(C_2)$$

**Example 2.** Once again, a forest doesn’t have any stable orientations unless all the brunches, i.e. edges, are cut. So, there is only one normalized multidegree $e' = (0, \ldots, 0)$ for a generating subgraph corresponding to the normalization $\tilde{C}$ and

$$\text{Jac}_{g-1}(C) = \oplus_i \text{Pic}^{g_i-1}(C_i)$$

**Example 3.** The stable orientations are

![Diagram](attachment:image.png)

The first corresponds to $\text{Pic}^{\tilde{g}-1+1}(C) = \text{Pic}^{\tilde{g}}(C)$ and the second – to $\text{Pic}^{\tilde{g}-1}(\tilde{X})$. 
Example 4. There are $2^n$ subgraphs $\Gamma'$: each edge is either included in $\Gamma'$ or it’s not. Each graph with $k$ edges obviously defines the multidegree $(d') = (k)$. Therefore there are $\binom{n}{n-k} = \binom{n}{k}$ strata of codimension $n-k$ each corresponding to $\text{Pic}^{d'-1}(C')$.

Example 5 (Dollar sign curve). The possible generating subgraphs are:

It is very easy to list all stable orientations and the corresponding stable multidegrees. Here are some of them:

Here is the complete list:

(i). For the graph $\Gamma$ itself there are two multidegrees $(2, 1)$ and $(1, 2)$ corresponding to invertible sheaves of multidegree $(\tilde{g}_1 + 1, \tilde{g}_2) = (\tilde{g} - 1, \tilde{g}_2 - 1) + (2, 1)$ and $(\tilde{g}_1, \tilde{g}_2 + 1) = (\tilde{g} - 1, \tilde{g}_2 - 1) + (1, 2)$ on $C$.

(ii). In the second column, for each graph we have a unique stable multidegree $(1, 1)$. Hence, there are 3 strata of codimension 1 corresponding to invertible sheaves of multidegree $(\tilde{g}_1, \tilde{g}_2) = (\tilde{g} - 1, \tilde{g}_2 - 1) + (1, 1)$.

(iii). In the third column there are no stable orientations – the graphs are trees.

(iv). From the last column we get the normalized multidegree $(0, 0)$ which corresponds to the invertible sheaves on the normalization $\tilde{C}$ of $C$ of multidegree $(\tilde{g}_1 - 1, \tilde{g}_2 - 1) = (\tilde{g} - 1, \tilde{g}_2 - 1) + (0, 0)$. 
Example 6. This is an exercise no harder than the previous five. Any subgraph $\Gamma'$ has at least one stable orientation with one exception: when $\Gamma'$ contains only one edge. For each such subgraph with $k$ edges the number of possible stable multidegrees is $k - 1$. Therefore, there are $\binom{n}{k}(k - 1)$ strata of codimension $n - k$ for $k > 0$ and one stratum for $k = 0$.

Proof of 4.9. The proof of (i) follows immediately from 1.3 and 1.9. In each gr-equivalence class of strictly semistable sheaves we can choose the one with the minimal graph $\Gamma'$ and it will be stable in our definition.

Next, we have to show the existence of a natural line bundle with a natural section on $\text{Jac}_{g-1}(C)$.

To define the divisor $\Theta$ in a way similar to how it was done in [Sou94, Est95] for irreducible $C$. Consider any family $\pi : C \times S \to S$ and an admissible sheaf $F$ of degree $g - 1$ on $C \times S$. Then there is a natural line bundle $L(F)$ on $S$ defined as

$$L(F) = (\det R\pi_*F)^{-1}$$

see [KM76] for its definition.

If we replace $F$ by $F \otimes \pi^*E$, $L(F)$ will be replaced by $L(F) \otimes E^{-\chi(F)}$. When $d = g - 1$, $\chi = 0$ which means that $L(F)$ will not change, so it is universally defined. Moreover, two gr-equivalent families of semistable sheaves produce the same $L(F)$. The latter follows from the fact that if

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence, then $\det R\pi_*F = (\det R\pi_*F') \otimes (\det R\pi_*F'')$, so only the stable factors are important.

$\text{Jac}_{d,1}$ and $\text{Jac}_{d}$ are constructed using GIT as a quotient of the Grothendieck’s Quot-schemes. By the universality $L(F)$ descends to $\text{Jac}_{g-1}$.

Now fix a point $c \in C$ and consider a universal family $F$ of invertible sheaves of degree $g' - 1$ and multidegree $d'$ over $\text{Pic}^{g'-1}(C')$, where $C' = C(\Gamma')$ is any of the partial normalizations of $C$. When does the formula

$$\Theta = \{s \in \text{Pic}^{g'-1}(C') \mid h^0(F_s) > 0\}$$

define a divisor? The answer is given by a theorem of Beauville [Bea77, 2.1]: it is exactly when the multidegree $d'$ is semistable using the part (iii) of Definition-Proposition 4.5. It is also easy to show directly that if two sheaves of degree $g' - 1$ are semistable and gr-equivalent, then $h^0(F_1) \neq 0$ iff $h^0(F_2) \neq 0$.

$\Theta$ provides a section of $(\det R\pi_*F)^{-1}$. \qed

Remark 4.10. From the above proof we have a yet another characterization of semistable admissible sheaves in degree $g - 1$: they have the multidegrees for which the usual definition of the theta-divisor actually gives a divisor.
5. An SQAV Corresponding to a Curve

5.1. An SQAV was defined in AN96 explicitly starting from the following combinatorial data:

(i). a lattice $X \cong \mathbb{Z}^{g'}$ (and a lattice $Y$ isomorphic to it via $\phi : Y \cong X$).

(ii). a symmetric positive definite bilinear form $B : X \times X \to \mathbb{Z}$.

(iii). an abelian variety $A_0$ of dimension $g''$, $g' + g'' = g$, with a principal polarization given by an ample sheaf $\mathcal{M}_0$.

(iv). a homomorphism $c_0 : X \to A_0(k)$ (and a dual homomorphism $c_0^t : Y \to A_0(k)$) defining a semiabelian variety $G_0$ (and a dual semiabelian variety $G^t$).

(v). a trivialization of the biextension $\tau_0 : 1_{X \times X} = 1_{Y \times X} \to (c^t \times c)^* \mathcal{P}_{A_0}^{-1}$, where $\mathcal{P}_{A_0}$ is the Poincare bundle.

When the abelian part $A_0$ is trivial, $\tau_0$ becomes simply a bilinear symmetric function $b_0 : X \times X \to k^*$.

5.2. We now would like to explain how to associate this data to a curve $C$. Part of this description can already be found in Nam76, §18 and Nam80, 9.D where it is attributed to Mumford.

(i). $X = H_1(\Gamma(C), \mathbb{Z})$.

(ii). By choosing arbitrarily an orientation on $\Gamma$, we get a natural embedding of $X$ in a free abelian group $C_1(\Gamma(C), \mathbb{Z}) = \oplus \mathbb{Z}e_j$, each $e_j$ corresponds to an edge of $\Gamma$. The form $B$ is the restriction to $H_1$ of the standard Euclidean form on $C_1$.

(iii). an abelian variety $A_0$ is $\text{Pic}^0(\tilde{C})$. Instead of line bundle on $A_0$ we consider $B_0 = \text{Pic}^{g-1}(\tilde{C})$ and the natural line bundle $M_0$ on it defines by the theta divisor. A choice of an isomorphism $A_0 \to B_0$ doesn’t matter.

(iv). every element of $H_1(\Gamma)$ defines a cycle of multidegree $(0, \ldots, 0)$ on $\tilde{C}$, i.e. an element of $A_0^\vee$. This gives the homomorphism $c$.

(v). Finally, the map $\tau$ is the most interesting part. The quick answer is that $\tau$ is given by a “generalized crossratio”.

5.3. Let $f, g$ be two meromorphic functions on a smooth projective curve $X$ with disjoint divisors. Then, defining

$$(f, g) = f(\text{div } g) = \prod_{x \in C} f(x)^{v_x(g)},$$

one has $(f, g) = (g, f)$ according to A. Weil. For $f = (z - a)/(z - b)$, $g = (z - c)/(z - d)$ this is nothing but the usual crossratio.

In SGA4, XVII Deligne showed that to arbitrary two invertible sheaves $L, M$ on $X$ and their meromorphic sections $f, g$ with disjoint divisors one can associate an element $(f, g)$ of a certain one-dimensional vector space $(L, M)$. These one-dimensional vector spaces are bilinear and symmetric in $L, M$ (in the case of degree 0 they form a symmetric biextension of $\text{Pic}^0 \times \text{Pic}^0$) and
\[(f, g) = (g, f) \text{ if } \deg M \cdot \deg L \text{ is even and } = -(g, f) \text{ otherwise.} \]

We will call this pairing Deligne symbol. A very nice summary of its properties can be found in [BM87].

\[5.4. \text{ Now for every two distinct elements } e_k, e_l \text{ of the standard basis in } C_1(\Gamma) \text{ we have two divisors of the total degree } 0 \text{ on } \tilde{C} = \bigcup \tilde{C}_i. \text{ This defines a one-dimensional vector space } V_{k,l} \text{ and an element } (e_k, e_l) \text{ in it. If } k = l, \text{ we still have a vector space } V_{k,k} \text{ but } (e_k, e_k) \text{ is undefined. We define it arbitrarily.} \]

In particular, restricting this to \(X \times X \subset C_1 \times C_1\), we obtain a pairing on \(X \times X\) with the values in a certain collection of one-dimensional vector spaces. Because every element of \(H_1\) has degree 0 on each irreducible component \(\tilde{C}_i\), this pairing is symmetric. It can be checked that these one-dimensional vector spaces are the fibers of \((c^t \times c)^* P_{A_0}^{-1}\), where \(P_{A_0}\) is the Poincare bundle (= Weil biextension) on \(A_0 \times A_0\). This defines the trivialization \(\tau_0\).

In the case \(A_0 = 0\), i.e. when all \(C_i\) are rational, \(\tau_0 = b_0\) is a product of the usual cross ratios.

\[5.5. \text{ Our definition seemingly depends on a choice of } (e_k, e_k). \text{ However, by [AN96] an SQAV depends only on the residue class of } \tau_0 \text{ modulo the following equivalence relation.} \]

\(\tau_0'(x, y) = \tau_0(x, y) \cdot cB_1(x, y)\)

for any \(c \in k\) and any symmetric positive bilinear form \(B_1\) defining the same Delaunay decomposition as \(B\) (for the definitions of the Delaunay decompositions, see [AN96] or [OS79]).

The independence of the choice of \((e_k, e_k)\) now follows because on \(C_1(\mathbb{R})\) the standard Euclidean form and the form \(\sum \lambda_i z_i^2\) for any \(\lambda_i > 0\) define the same Delaunay decomposition. The Delaunay cells are the standard cubes and their faces. Because, as one can easily show ([OS79, §18]) \(C_1(\Gamma(C), \mathbb{Z}) \cap H_1(\Gamma(C), \mathbb{R}) = H_1(\Gamma(C), \mathbb{Z})\), the Delaunay decomposition of \(X \otimes \mathbb{R} = H_1(\mathbb{R})\) is the intersection of this standard Delaunay decomposition with \(H_1(\mathbb{R})\).

Therefore, every cell has the following simple description. For each \(1 \leq i \leq \dim C_1\) we choose an integer \(n_i\) and two numbers \(a_i, b_i\) with either \(a_i = b_i = n_i\) or \(a_i = n_i\) and \(b_i = n_i + 1\). Then we obtain the cell \(\sigma\) in \(C_1(\mathbb{R})\) defined by the inequalities

\[a_i \leq z_i \leq b_i\]

and the cell in \(\sigma \cap H_1(\mathbb{R})\) in \(H_1(\mathbb{R})\) (it may be empty).

\[\text{Example 4. In this case } H_1 = C_1 \text{ and we have the standard Euclidean space } \mathbb{R}^g \supset \mathbb{Z}^g. \text{ The Delaunay cells are standard cubes and their faces. Modulo the translation by } \mathbb{Z}^g \text{ there are exactly } \binom{n}{k} \text{ such cells of codimension } k. \text{ These numbers are the same as in section 4.}\]
Further assume that the curve $C$ is rational for simplicity. Then the symmetric bilinear form $b_0$ is defined by $n(n-1)/2$ crossratios $(e_i, e_j), i < j$.

**Example 5.** In this case $H_1(\Gamma(C), \mathbb{Z}) \subset C_1(\Gamma(C), \mathbb{Z})$ is the hyperplane \{ \(x_1 + x_2 + x_3 = 0\) \}. The Delaunay decomposition is the decomposition of $\mathbb{R}^2$ into unilateral triangles. Modulo the translations there are two cells of dimension 2, 3 cells of dimension 1 and 1 cell of dimension 0. These numbers are the same as in section 4.

This SQAV does not depend on the form $\tau_0$ as all the $3 = \text{dim} S^2(H_1)$ choices are killed by the $3 = \text{dim} C_1$ choices for $(e_k, e_k)$.

**Example 6.** In this case $H_1(\Gamma(C), \mathbb{Z}) \subset C_1(\Gamma(C), \mathbb{Z})$ is the hyperplane \{ \(x_1 + \ldots + x_n = 0\) \}. The lattice is the standard lattice $A_n$. It can be checked that the number of $k$-dimensional cells is given by the same formula as in the section 4.

5.6. [AN96] gives a stratification of an SQAV into locally closed subschemes which are homogeneous spaces over a semiabelian variety. A stratum of dimension $n$ corresponds to a Delaunay cell of dimension $n - \text{dim} A_0$.

On the other hand, in section 4 we have given a similar description for $\text{Jac}_{g-1}$ and the semiabelian variety is $\text{Pic}^0 C$. In all the above examples the numbers of strata of each dimension in both descriptions are the same. We now would like to relate the two descriptions explicitly.

5.7. Consider an arbitrary orientation of the generating subgraph $\Gamma' \subset \Gamma$. By 4.3 it corresponds to a semistable multidegree $d'$ of the graph $\Gamma(C)$. Now, depending on whether the edge $e_i$ is oriented the “right” way (the same that we used defining $H_1$), the “wrong” way, or not present at all, choose $a_i = 0, b_i = 1, a_i = -1, b_i = 0$ or $a_i = b_i = 0$. This gives a Delaunay cell $\sigma$ of $C_1(\mathbb{R})$ and the Delaunay cell $\sigma \cap H_1(\mathbb{R})$ of $H_1(\mathbb{R})$ as in 5.4. Moreover, the orientation is stable iff

$$\dim \sigma = \dim \sigma \cap H_1(\mathbb{R})$$

**Proof.** The above is Oda and Seshadri’s description of the stratification of $\text{Jac}_\phi$ for the case $\phi = \partial e(J)/2$, in which case the Namikawa-Delaunay decomposition of [OS79] coincides with the Delaunay decomposition we have described above. Therefore, everything follows from [OS79] and the following lemma.

**Lemma 5.8.** $\text{Jac}_{g-1}$ corresponds to the choice $\phi = \partial e(J)/2$ in the notations of [OS79].

**Proof.** Follows directly from 3.3.
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