Review Article

Control Sets of Linear Control Systems on Matrix Groups and Applications

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We review recent results on control sets of linear control systems on matrix groups. We also mention some applications of systems with algebraic structures.

1. Introduction

This review is about control system on matrix groups and its applications. Given a matrix group $G$ with matrix algebra $\mathfrak{g}$ we introduce the notion of normalizer $\eta$ of $\mathfrak{g}$, and we mention some classes of control systems inside of the normalizer which have relevant applications in many areas. We concentrate the review in the class of linear control systems on Euclidean spaces and matrix groups. A fundamental notion in this theory is the controllability property of a control system answering the following hard question. Given a specific state of the system, is it possible to reach any arbitrary state through admissible trajectories in positive time? Or better, are there some regions of the space of state where controllability holds? For that, we introduce the notion of a control set in Euclidean spaces and after on matrix groups, which are the main topics of this short review.

Why do we need to consider dynamics or even control systems on matrix groups? Well, many relevant applications are coming from physical problems where the state space is a matrix group. The Noether Theorem [1] states that every differentiable symmetry of the action of a physical system has a corresponding conservation law. And, it is possible to associate symmetry with dynamic through the notion of invariant vector fields on matrix groups.

For instance, the splendid challenge problem called The Brachistochrone introduced by Bernoulli in Acta Eruditorum in 1696:

"Find the shape of the curve down which a ball sliding from rest and accelerated by gravity will slip (without friction) from one point to another".

This problem was solved by I. Newton, J. Bernoulli, and others and open a new era in mathematics [2]. Recently, the authors in [3] show that this problem like many others can be modeled as a control system on some specific matrix group as follows. Let us consider the set

$$G = \left\{ g = (x, y, v, w) : \dot{x} = v, \dot{y} = w, v = -uv, w = -G + uv \right\}, \quad u \in U \quad (1)$$

Here, $G$ is the gravitational constant, $U$ is the set of admissible control functions with values on a compact set $\Omega \subset \mathbb{R}$ containing $0$, and $u \in U$ is a control. It turns out that this model is equivalent to the control system

$$\Sigma_G : \dot{g} = X(g) + uY(g), \quad u \in \Omega. \quad (2)$$

The dynamic defined by the transposed column vectors

$$X = \begin{pmatrix} v & w & 0 & -G \end{pmatrix}^T$$

and

$$Y = \begin{pmatrix} 0 & 0 & -w & v \end{pmatrix}^T$$

generates the dimension 4 diamond Lie algebra

$$\mathfrak{g} = \text{Span}_\mathbb{R} \{X, Y\} = \text{Span} \{X, Y, Z, L\} \quad (4)$$
with the bracket relationships \([X, Y] = Z, [L, X] = X, \) and \([L, Y] = -Y\) and \(Z\) belongs to the center of \(g\). These two vectors are invariant vector fields on the space state which is a matrix group \(G\) with matrix Lie algebra \(g\).

The model of controlling the attitude of a satellite in orbit is given by the matrix group \(G = SO(3) \otimes \mathbb{R}^3\) which is the semidirect product between the rotation group \(SO(3)\) of dimension 3 with the Euclidean space \(\mathbb{R}^3\) [4, 5]. In this case, the dynamic of the system is determined by two invariant vector fields on \(G\), i.e., elements of the matrix Lie algebra \(g = \mathfrak{so}(3) \otimes \mathbb{R}^3\) defined by the semidirect product between \(\mathfrak{so}(3)\) of skew-symmetric matrices of order three and \(\mathbb{R}^3\).

We also mention the problem of optimal controls for a two-compartmental model for cancer chemotherapy with a quadratic objective [6]. Here, the system is determined by two elements of \(\mathfrak{sl}(2, \mathbb{R})\), the algebra of trace zero matrices of order two. Controllability of this kind of control systems means the possibility of transforming any initial state; let us say sick in another one healthy. Mathematical controllability conditions for this kind of systems can be found in [7] and the references therein. It is worth mentioning that in this reference the authors use fundamental notions of the Lie theory as the Cartan-Killing form for semisimple algebras. This is the beauty of mathematics: no matter how abstract a mathematical concept can be, there is always the possibility of using it in a specific application.

For further references on real applications of the Geometric Control Theory on Matrix Groups we mention [8–15]. For more theoretical point of view we recommend [16–20]. For specific theoretical results on linear control system on groups we refer the reader to [7] and the references therein.

2. Control Sets of Linear Control System on Euclidean Spaces

The Euclidean space of dimension \(n\) is given by the Cartesian product of \(n\) copies of the real numbers \(\mathbb{R}\) by

\[
\mathbb{R}^n = \{ x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n \}. \quad (5)
\]

considered with it canonical topology and differential structure.

2.1. A Classical Example. We start this section with a very well-known example from the Pontryagin, Book, [19]. Stop a train in a station in minimum time. Consider the ideal case of the train moving on a straight line without friction and denote by \(x(t)\), the distance from the train to the station at time \(t\), which is considered as the origin of the line. From the Newton law, the force is given by \(f = mu\), the mass by the acceleration. Of course, we can assume \(m = 1\). We get a couple of ordinary differential equations controlled by the possible combinations between acceleration and brake as follows:

\[
\dot{x}(t) = y(t) \quad \text{the velocity}
\]

and

\[
\dot{y}(t) = u(t) \quad \text{the acceleration}. \quad (6)
\]

Here, \(U\) is a family of functions that we call admissible, depending on the possible strategies \(u\) you consider to control the train,

\[
u : [0, T_u] \rightarrow \Omega = [-1.1] \subset \mathbb{R}. \quad (7)
\]

The numbers 1 and −1 represent the maximum and minimum standard normalized speeds. From the mathematical points of view we could consider \(u \in U = L_{loc}^1(\mathbb{R})\), the set of locally integrable measurable functions defined on intervals of the real line \(\mathbb{R}\). But, also the space of continuous functions or even piecewise constant functions are possible and appropriate. These three kinds of controls guarantee the existence and uniqueness of the solution associated with each initial state \((x_0, y_0)\) and for each strategy \(u \in U\). Now, is there a solution to this problem? In the affirmative case, how to prove its existence, or even better, how to compute the optimal solution?

As we can see, this model can be represented in a matrix form. In fact,

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} = \begin{pmatrix}
y(t) \\
u(t)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} + \begin{pmatrix}
u(t) \\
0
\end{pmatrix}. \quad (8)
\]

Each choice of control \(u(\cdot)\) generates an ordinary differential equation. If the solution with control \(u\) and a specific initial condition reaches the origin at some positive time, then \(u\) is a successful control. There may be no such control or many. If there are several, one of them can be preferred depending on the considered criteria, in our case, minimum time. The control problem here is as follows: given \((x_0, y_0) \in \mathbb{R}^2\) find \(u \in U\) such that the integral curve of the system starting in \((x_0, y_0)\) and control \(u\) reaches the origin \((0, 0)\) of the plane at minimum time.

2.2. Linear Control System. In a more general setting, we introduce the classical linear control system as follows [21]. A classical linear control system \(\Sigma_{\mathbb{R}^n}\) on the space state \(\mathbb{R}^n\) is determined by the family of ordinary differential equations,

\[
\Sigma_{\mathbb{R}^n} : \dot{x}(t) = Ax(t) + u_1(t) b_1^1 + u_2(t) b_2^2 + \cdots + u_m(t) b_m^m, \quad (9)
\]

\[
A \in \mathbb{R}^{n \times n}, b_j^i \in \mathbb{R}^n, \quad u \in U
\]
guided by \(m\) real functions \((u_1, u_2, \ldots, u_m)\). In other words, the drift \(\dot{x}(t) = Ax(t)\) is controlled by \(m\) engines \(b_j^i\) through the different component of the integrable function \(u = (u_1, u_2, \ldots, u_m) : [0, T_u] \rightarrow \Omega \subset \mathbb{R}^m\), where \(\Omega\) is topologically closed with \(0 \in int(\Omega)\). The system is called unrestricted if \(\Omega = \mathbb{R}^m\) and restricted in the other case.

Classically, this system is written as

\[
\Sigma_{\mathbb{R}^n} : \dot{x}(t) = Ax(t) + Bu, \quad u \in U. \quad (10)
\]
The columns of the cost matrix $B = (b^1 \ldots b^m)$ are called control vectors.

Given the initial condition $x_0 \in \mathbb{R}^n$ and the control $u \in U$, the solution of $\Sigma_{\mathbb{R}^n}$ at any time $t$ denoted by $\phi^n_t(x_0)$ is described through the variation of parameters formula

$$\phi^n_t(x_0) = e^{tA} x_0 + \int_0^t e^{(t-s)A} Bu(s) \, ds.$$  \hfill (11)

Actually, from the Carathéodory Theorem, there exists an unique solution $\phi^n_t(x_0)$ of $\Sigma_{\mathbb{R}^n}$ such that $\phi^n_0(x_0) = x_0$. In fact, from the Fundamental Theorem of Calculus, it follows that

$$\phi^n_t(x_0) = A\phi^n_u(x_0) + Bu(t).$$  \hfill (12)

Here, the exponential map $e^{tA}$ of a matrix $tA$ is defined by $e^{tA} = \Sigma_{k=0}^{\infty} (t/k!)A^k$ where $A^0 = 1d : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. In particular, the set $R = \{ \phi^n_t(x_0) : -\infty < t < +\infty \}$ describes a curve in $\mathbb{R}^n$ starting from $x_0$ at the instant $t = 0$, reaching every element of $R$ determined by the control $u$, in positive and negative time.

The positive orbit $O$ of $\Sigma_{\mathbb{R}^n}$ is defined as the union of the positive orbit $O_t$ up to the time $t \geq 0$, of which elements are the reachable points from the origin at the time $t \geq 0$. Precisely, $O_t = \bigcup_{t \geq 0} O_t$ where $O_t = \{ \phi^n_t(0) : u \in U \}$.

On the other hand, the negative orbit of the system which is given by $O^- = \{ y : \text{there exists } u \in U \text{ and } t > 0 \text{ with } \phi^n_u(y) = 0 \}$ is the set of the states which can reach the origin through a solution of $\Sigma_{\mathbb{R}^n}$ in positive time and the same for $O^-$. Finally, for each $x \in \mathbb{R}^n$, $O(x) = \{ e^{tA}x + \int_0^t e^{(t-s)A} Bu(s) \, ds : u \in U \}$ is the reachable set of $\Sigma_{\mathbb{R}^n}$ from $x$.

For a real matrix $A$ of order $n$ we denote by $\text{Spec}(A)$ the spectrum of $A$, i.e., the set of $A$-eigenvalues and by $\text{Spec}_{\mathbb{R}}(A)$ the Lyapunov spectrum of $A$, which means the sets of the real parts of the eigenvalues in $\text{Spec}(A)$. Let us denote by $\langle A, B \rangle$, the smallest $A$-invariant subspace of $\mathbb{R}^n$ which contains the image $\text{Im}(B)$ of $B$. That is, it is possible to restrict the linear map $A : \langle A, B \rangle \to \langle A, B \rangle$ and $\text{Im}(B) \subset \langle A, B \rangle$, where $B : \mathbb{R}^m \to \mathbb{R}^n$. Denote by $K$

$$K = (B \, AB \, A^2B \ldots A^{n-1}B)$$  \hfill (16)

the Kalman matrix, determined by the products of the powers of $A$ with $B$. We say that $\Sigma_{\mathbb{R}^n}$ satisfy the Kalman rank condition if $\text{rank}(K) = n$, which is the topological dimension of $\mathbb{R}^n$.

2.3. Control Sets and Controllability. The controllability property of a control system is the possibility of connecting any two points of the state space through concatenation of controls in positive time. However, this property is too strong and it is true just for few systems. Thus, it is important to know some regions where controllability holds. This is the main idea to introduce the notion of control set.

Definition 1. $\Sigma_{\mathbb{R}^n}$ is controllable in $\mathbb{R}^n$ if for each $x, y \in \mathbb{R}^n$, $y \in O(x)$.

A more realistic approach is the notion of control set [16].

Definition 2. A subset $\mathcal{C} \subset \mathbb{R}^n$ is called a control set of $\Sigma_{\mathbb{R}^n}$ if for each $x \in \mathcal{C}$

(i) there exists $u \in U$ such that $\phi^n_t(x) \in \mathcal{C}$, for any $t \geq 0$;
(ii) $\mathcal{C} \subset \text{cl} (\mathcal{O}(x))$, where $\text{cl}$ denote the topological closure;
(iii) $\mathcal{C}$ is maximal with respect to conditions (i) and (ii).

This concept is applied on arbitrary restricted control systems on surfaces or even manifolds. For Euclidian linear control systems, there exists a fundamental result due to Colonius-Kliemann [16], as follows.

Theorem 3. Let $\Sigma_{\mathbb{R}^n}$ be a restricted linear control system which satisfies the Kalman rank condition. Hence,

(1) There exists a unique control set $\mathcal{C}$ with nonempty interior with shape

$$\mathcal{C} = \text{cl} (\mathcal{O}) \cap \mathcal{O}^-.$$  \hfill (17)

Furthermore, $\Sigma_{\mathbb{R}^n}$ is controllable in the interior $\text{int}(\mathcal{C})$ of the control set.

(2) A global controllability result can be obtained through $\text{Spec}(A)_{\mathbb{R}^p}$,

$$\Sigma_{\mathbb{R}^n} \text{ is controllable in } \mathbb{R}^n \iff \text{Spec}(A)_{\mathbb{R}^p} = \{0\}$$  \hfill (18)

Example 4. The next example appears in the book [16] and we include it because it is the first one in this theory. Consider the two-dimensional restricted linear control system,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u,$$

$$u \in \Omega = [-1, 1].$$  \hfill (19)

The solutions of $\Sigma_{\mathbb{R}^n}$ are explicitly given by

$$x(t) = e^t x_0 + \int_0^t e^{(t-s)A} Bu(s) \, ds$$

and

$$y(t) = e^{-t} y_0 + \int_0^t e^{-(t-s)A} Bu(s) \, ds.$$  \hfill (20)

It turns out that
(1) the singularity \((-1, 1)\) coming from \(u = 1\) and \((1, -1)\) from \(u = -1\) are both saddle points;

(2) \(\Omega = \mathbb{R} \times (-1, 1)\), and \(\Omega^- = (-1, 1) \times \mathbb{R}\).

These facts are easily viewed by drawing a picture of both saddle points. The only control set of \(\Sigma_{\mathbb{R}^n}\) reads as

\[
\mathcal{C} = (-1, 1) \times [-1, 1].
\]  

**Example 5.** In the train example the system is restricted, satisfying the Kalman rank condition, \(\text{rank}(\frac{1}{4} I) = 2\), and \(\text{Spec}(A)_{\mathbb{R}^n} = \{0\}\). Hence, the associated system is controllable. Therefore, \(\mathcal{C} = \mathbb{R}^n\) is the only control set.

Furthermore, in this case, it is possible to show this fact directly; i.e., from any initial condition there exists an explicitly curve transferring this point to the origin through a positive solution of \(\Sigma_{\mathbb{R}^n}\). We inform that the Pontryagin Maximum Principle (Lenin Prize in Russia) solves the optimal problem showing that the optimal control lives in the boundary of \(\Omega\) and it is of the type bang-bang, i.e., a piecewise constant control with values in the corner of the interval [19]. In this case, \(u = 1\) and \(u = -1\) and the minimum time curve is building with at most one change of the control. So, inside the family of parabolas generated by the solutions of

\[
\frac{\dot{x}}{y} = \frac{1}{y},
\]

\[
y(0) = -1
\]

and

\[
\dot{x} = y(0),
\]

\[
y = 1
\]

you find two specific parabolas \(\phi^-\) with control \(-1\) and \(\phi^+\) with control 1, reaching the origin. Hence, starting from any arbitrary initial condition \((x_0, y_0)\) outside these curves, you choose the unique parabola where starting from \((x_0, y_0)\) and moving in positive time hit one of the curves \(\phi^-\) or \(\phi^+\) and then you change the control to reach the target.

Next, we describe without proof the main facts about the controllability of a unrestricted linear control system on \(\mathbb{R}^n\). These properties strongly depend on the unboundedness of the set \(\Omega = \mathbb{R}^m\). Obviously, the unrestricted case gives more possibilities to characterize controllability.

If the system \(\Sigma_{\mathbb{R}^n}\) is unrestricted, the positive orbit \(\mathcal{C}\) is a vectorial subspace of \(\mathbb{R}^n\). In fact, the algebraic structure of \(\mathcal{C}\) depends on the structure of a vector space of \(U\) under the hypotheses of \(\Omega\). Let \(u_1, u_2 \in U\) and \(\lambda\) be a real number. It turns out that \(u_1 + u_2\) and \(\lambda u\) are also elements of \(U\). Furthermore,

\[
\phi_{\lambda u}(0) = \phi_{u_1}^{u_2}(0) + \phi_{u_2}^{u_1}(0) + \lambda \phi_{u_1}^{u_2}(0).
\]  

Thus, \(\mathcal{C}\) is a vector subspace for any \(t \geq 0\). On the other hand, if \(t_1 < t_2\) it follows that \(\mathcal{C}_{t_1} \subset \mathcal{C}_{t_2}\). For that, you just need to consider first the control \(u = 0\) and rest at the origin \(t_2 - t_1\) units of time. Since the union of a family of crescent subspaces is also a subspace we conclude that \(\mathcal{C}\) is also a subspace. In this setting, the Kalman rank condition is equivalent to controllability.

The next results are fundamental for the theory of linear control systems on Euclidean spaces [21].

**Theorem 6.** Let \(\Sigma_{\mathbb{R}^n}\) be a unrestricted linear control system on \(\mathbb{R}^n\). It holds the following:

(1) For each \(\lambda > 0\), \(\mathcal{C}_{\lambda} = \langle A, B \rangle\).

(2) \(\Sigma_{\mathbb{R}^n}\) is controllable \(\iff\) \(\text{rank}(K) = n \iff \dim \mathcal{C} = n\).

(3) \(\mathcal{C} = \mathbb{R}^n \iff \mathcal{C}(x) = \mathbb{R}^n\) for each \(x \in \mathbb{R}^n\).

**Example 7.** Consider the unrestricted linear control system in the plane \(\mathbb{R}^2\) defined by the dynamic

\[
\Sigma_{\mathbb{R}^2} : \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u,
\]

\[u \in \Omega = \mathbb{R}.
\]

We have \(K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). So, \(\text{rank}(K) = 1\) and according to the previous result the system is not controllable. Furthermore, for each \(t > 0\), \(\mathcal{C}_{\lambda} = \langle A, B \rangle = x\text{-axis}\). Hence, \(\mathcal{C} = \mathcal{O}\) is a control set but with empty interior. In particular, any solution starting from the origin cannot leave the \(x\text{-axis}\).

**Example 8.** Consider a linearization at some point \(x_0 \in \mathbb{R}^4\)

\[
\Sigma_{\mathbb{R}^2} : \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u,
\]

\[u \in U\]

of a nonlinear control system \(\Sigma\) coming from a satellite in Earth orbit, where

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

and \(u \in \Omega = \mathbb{R}^2\) with \(0 \in \text{int}(\Omega)\).

The system satisfies the Kalman rank condition, So, if we assume \(\Omega = \mathbb{R}^2\), the linearized system is controllable. Now, from a practical point of view, \(\Sigma_{\mathbb{R}^2}\) is a restricted control system which also satisfies \(\text{Spec}(A)_{\mathbb{R}^n} = \{0\}\). Hence, the linearized system is controllable. Thus, the original nonlinear system is locally controllable in a neighborhood \(N_0\). If for any reason the satellite comes out of the orbit you can travel through a selected optimal path in \(N_0\) as you wish, approaching the original orbit up to some point; let us say \(x_1\). If it is possible, you can repeat the same idea on \(x_1\) and so on, to reach effectively the orbit.
3. From Euclidean Spaces to Matrix Groups

In this section, we first introduce few ingredients about the notion of a linear control system, from Euclidean spaces to matrix groups; see [22–26].

3.1. About Matrix Groups. Let us denote by $GL(n, \mathbb{R})$ the set of all real invertible matrices of order $n$, i.e.,

$$GL(n, \mathbb{R}) = \{ A : \det(A) \neq 0 \}.$$  \hfill (27)

We denote by $GL^+(n, \mathbb{R})$ the connected component of $GL(n, \mathbb{R})$ which contains the identity map $Id$. In other words, $\det(A) > 0$ for any $A \in GL^+(n, \mathbb{R})$.

Every matrix group considered in this review will be a subset of $GL(n, \mathbb{R})$ which is an open subset of the vector space $gl(n, \mathbb{R})$ of all real matrices of order $n$. Since $gl(n, \mathbb{R})$ is isomorphic to the Euclidean space $\mathbb{R}^{n^2}$, it follows that the topology and the differentiable structure of the groups we consider come from this Euclidean space of dimension $n^2$.

The analytical map $L_P : G \rightarrow G$ defined by $L_P(x) = Px$ called the left translations on $G$ is a diffeomorphism, which means that $L_P$ and its inverse $L_P^{-1}$ are differentiable maps.

The Lie algebra of $G$ comes from the notion of invariant vector fields. Denote by $\mathcal{X}^{\infty}(G)$ the set of $C^{\infty}$-vector fields on $G$. By definition, an element $X$ of $\mathcal{X}^{\infty}(G)$ satisfies the following: for any $g \in G$ the value of $X$ in $g$ denoted by $X(g)$ or some times $X_g$ is a vector of the tangent space $T_gG$ of $G$ at $g$, where

$$T_gG = \left\{ v : \text{there exist } y : (-\epsilon, \epsilon) \rightarrow G \text{ with } v(0) = g \text{ and } \dot{y}(0) = v \right\}. \hfill (28)$$

We observe that in our case

$$T_gG \subset T_gGL(n, \mathbb{R}) = g + T_{Id}GL(n, \mathbb{R}) = g + gl(n, \mathbb{R}). \hfill (29)$$

Here, $gl(n, \mathbb{R})$ denotes the tangent space of $GL(n, \mathbb{R})$ at the identity element, which is nothing more than the vector space of all real matrices of order $n$. In fact, for any matrix $g \in gl(n, \mathbb{R})$ the differentiable curve $\gamma(t) = e^{tg} \in GL(n, \mathbb{R})$ satisfies $\gamma(0) = Id$ and $\gamma'(0) = g$.

Definition 9. We say that $X \in \mathcal{X}^{\infty}(G)$ is a left-invariant vector field on $G$ if

$$X \circ L_g = (dL_g)_e(X) \quad \text{for every } g \in G. \hfill (30)$$

Here, $(dL_g)_e$ or $L_g$ denotes the differential of $L_g$ at the identity $e = Id$.

In other words, to define a left-invariant vector field on $G$ we just need to determine a tangent vector at the identity element. In fact, at any point $g \in G$ the value $X_g$ is given by the derivative of left translations. Precisely,

$$(dL_g)_e : T_gG \rightarrow T_{g}G \quad \text{with } X_g = (dL_g)_e(X_e) \hfill (31)$$

Since any left-invariant vector field is determined by its value at the identity, it turns out that the set of all left-invariant vector fields on $G$ denoted by $\mathfrak{g}$ is isomorphic to the tangent space $T_eG$.

The vector space $\mathfrak{g}$ with the application called bracket and defined by

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \text{ defined by } [X,Y] = XY - YX \hfill (32)$$

turns $\mathfrak{g}$ into a matrix algebra, which is called the Lie algebra of $G$.

Of course, the bracket is bilinear and skew-symmetric. This last property means that $[X,Y] = -[Y,X]$. Furthermore, its satisfies the Jacobi identity,

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0,$$  \hfill (33)

for any $X, Y, Z \in \mathfrak{g}$.

A subspace $V \subset \mathfrak{g}$ is a subalgebra if $[V, V] \subset V$ and it is an ideal if $[V, \mathfrak{g}] \subset V$.

Example 10. In the sequel we show the Lie algebra $\mathfrak{g} \equiv T_eG$ of the corresponding matrix group:

(1) $T_{Id}\mathbb{R}^n = \mathbb{R}^n$.  
(2) $T_{Id}GL^+(n, \mathbb{R}) = gl(n, \mathbb{R})$, the set of real matrices of order $n$.  
(3) $T_{Id}S^n = \mathbb{R}^n$, where $S^n$ for $n = 1, 3$ and $7$ is the $n$-dimensional sphere.  
(4) $T_{Id}O(n, \mathbb{R}) = O(n) = \{ A \in GL(n, \mathbb{R}) \mid A + A^t = 0 \}$, the skew-symmetric matrices. Here,

$$O(n, \mathbb{R}) = \left\{ A \in GL(n, \mathbb{R}) \mid AA^t = Id \right\}. \hfill (34)$$

is the matrix group of orthogonal matrices.

(5) $T_{Id}SO(n, \mathbb{R}) = \mathfrak{so}(n)$, where $SO(n, \mathbb{R}) \subset O(n, \mathbb{R})$ and for any $A \in SO(n, \mathbb{R})$, $\det(A) = 1$.  
(6) The trace zero matrices $\mathfrak{sl}(n, \mathbb{R}) = \{ A \in gl(n, \mathbb{R}) \mid tr(A) = 0 \}$ are the matrix algebra of the matrix group $SL(n, \mathbb{R}) = det^{-1}(1)$, the order $n$ matrices with determinant $1$.

(7) The Heisenberg Lie algebra $(\mathbb{R}^3, +, [\cdot, \cdot])$ has the basis $\{Y^1, Y^2, Y^3\}$ such that $[Y^1, Y^2] = Y^3$ is the only nonvanishing bracket. In fact, the Heisenberg group has the matrix representation

$$G = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \hfill (35)$$

$\phi_{g} : (x_1, x_2, x_3) \rightarrow \mathbb{R}^3$. 

\[ \phi_{g} = (x_1, x_2, x_3) \]
The derivative of $y_j : \mathbb{R} \rightarrow \mathbb{R}^3$, $y_j(t) = \phi^{-1}(t e_j)$ at $t = 0$, determines $Y^j$,

$\mathfrak{g} = \text{Span} \left\{ Y^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Y^3 \right\}$.

**Definition 11.** A Lie algebra $\mathfrak{g}$ is as follows:

1. **Abelian** if for any $X, Y \in \mathfrak{g}$ we have $[X, Y] = 0$
2. **Nilpotent** if $\exists k \geq 1$ : the central series stabilizes at $0$
3. **Solvable** if $\exists k \geq 1$ : the derivative series stabilizes at $0$
4. **Semisimple** if the largest solvable subalgebra $\tau(\mathfrak{g})$ of $\mathfrak{g}$ is null

A Lie group is said to be Abelian, nilpotent, solvable and semisimple, if its Lie algebra satisfy the same property.

**Remark 12.** It is well known that the exponential map $\exp = e : \mathfrak{g}(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ is a local diffeomorphism. In fact, $d(\exp)_{0} = Id$ is invertible. Thus, from the Inverse Map Theorem, it follows that there exists a neighborhood $V \subset G$ of the identity such that $\exp : V \subset \mathfrak{g} \rightarrow \exp(V) \subset G$ is a diffeomorphism. Furthermore, for nilpotent and simply connected Lie groups such as the Heisenberg group exp is a global diffeomorphism, which means $V = G$.

A $C^\infty$ homomorphism between two matrix groups $G$ and $H$ is called a matrix group homomorphism. A bijective matrix group homomorphism of $G$ with itself is called a matrix group automorphism. If $G$ is connected, the set $Aut(G)$ of $G$-automorphisms is a matrix group with Lie algebra $\mathfrak{aut}(G)$, [25].

**Remark 13.** An intrinsically relationship between a matrix group homomorphism $\varphi : G \rightarrow H$ and its derivative $(d\varphi)_{e} : T_{e_G} \rightarrow T_{e_H}$, is given by $\varphi(\exp X) = \exp(d\varphi)(X))$, which comes from the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{(d\varphi)_{e}} & \mathfrak{h} \\
\exp_{\mathfrak{g}} \downarrow & \leftrightarrow & \downarrow \exp_{\mathfrak{h}} \\
G & \xrightarrow{\varphi} & H
\end{array}
$$

Since $(d \det)_{Id} = tr$ is the trace map, it follows that

$$
e^{trA} = \det(\exp A), \quad A \in \mathfrak{gl}(d, \mathbb{R}).$$

In particular, if $trA = 0$ its exponential $\exp A$ has determinant $1$.

### 3.2. The Normalizer and Linear Vector Fields.

As we saw, a linear control system $\Sigma_{R^n}$ on $\mathbb{R}^n$ is written as

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + \sum_{j=1}^{m} u_j b_j, \quad u \in U. \quad (41)$$

Essentially, $\Sigma_{R^n}$ depends on two classes of dynamics,

1. the linear differential equation $\dot{x}(t) = Ax(t)$, to be controlled;
2. the control vectors $b_j$, which are invariant vector fields on $\mathbb{R}^n$.

If we would like to extend $\Sigma_{R^n}$ from $\mathbb{R}^n$ to a matrix group $\mathbb{G}$ we first need to understand its dynamic. The solution of the linear differential equation determined by the matrix $A$ with the initial condition $x$ reads as

$$\dot{x}(t) = e^{tA}x. \quad (42)$$

We observe that it associated flow $e^{tA} \in GL(n, \mathbb{R}) = \det^{-1}(0)$, i.e.,

$$GL(n, \mathbb{R}) = \{ P : P \text{ is an invertible matrix of order } n \}. \quad (43)$$

So, for any $t \in \mathbb{R}$, $e^{tA} \in Aut(\mathbb{R}^n)$ belongs to the automorphisms group of $\mathbb{R}^n$.

On the other hand, for any $j = 1, \ldots, m$, the vector field $Y^j$ on $\mathbb{R}^n$ defined by $Y^j(x) = b_j, x \in \mathbb{R}^n$, is invariant by translation, depending just by its value $Y^j(0) = b_j$ at the origin. In fact, the solution of the associated differential equation $\dot{x}(t) = b_j$ with initial condition $x_0$ is given by

$$\dot{x}(t) = x_0 + tb_j, \quad x \in \mathbb{R}^n. \quad (44)$$

A simple computation shows that $[Ax, b_j] = -Ab_j$. Hence, the linear application $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ transforms the invariant vector field $Y^j$ in the invariant vector field $-AY^j$. The previous discussion allows us to reach the concept of normalizer, which plays a role in Geometric Control Systems Theory.

Let $\mathbb{G}$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ as the set of left-invariant vector fields on $\mathbb{G}$. Let us denote by $X^{\infty}(\mathbb{G})$ the Lie algebra of all smooth vector fields on $\mathbb{G}$. The normalizer of $\mathfrak{g}$ in $X^{\infty}(\mathbb{G})$ is the set

$$\eta = \text{norm}_{X^{\infty}(\mathbb{G})}(\mathfrak{g}) = \{ X \in X^{\infty}(\mathbb{G}) : [X, Y] \in \mathfrak{g}, \text{ for every } Y \in \mathfrak{g} \}. \quad (45)$$

Of course, for any constant control $u \in \Omega$, the vector field $A + \sum_{j=1}^{m} u_j b_j$ is an element of the normalizer $\eta$. In fact, since $\mathbb{R}^n$ is an Abelian Lie algebra, then $[b_j, b] = 0$ for any invariant vector filed $b \in \mathbb{R}^n$. So, we obtain $[A + \sum_{j=1}^{m} u_j b_j, b] = -Ab \in \mathbb{R}^n$, which of course is also invariant.

In the sequel, $\mathfrak{aut}(\mathbb{G})$ denotes the Lie algebra of the matrix group $Aut(\mathbb{G})$ and $\mathfrak{g}$ the Lie algebra of all derivations of $\mathfrak{g}$. By definition, an element $\mathfrak{D} \in \mathfrak{g}$ is a linear transformation $\mathfrak{D} : \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies the Leibnitz rule with respect to the Lie brackets, i.e.,

$$\mathfrak{D}[P, Q] = [\mathfrak{D}P, Q] + [P, \mathfrak{D}Q], \quad \text{for any } P, Q \in \mathfrak{g}. \quad (46)$$
Definition 14. A linear vector field $\mathcal{X}$ on a matrix group $G$ is determined by the following requirement: its flows $\{\mathcal{X}_t : t \in \mathbb{R}\}$ is an infinitesimal automorphism of $G$, which means that $\mathcal{X}_t \in \text{Aut}(G)$, $\forall t \in \mathbb{R}$.

It turns out that (see [27])

$$\mathcal{X} \text{ is linear } \iff \mathcal{X} \in \eta \text{ and } \mathcal{X}_e = 0. \quad (47)$$

The relationship between derivations and linear vector fields is given as follows.

Remark 15. For any $t \in \mathbb{R}$, $(d\mathcal{X}_t)_e = e^{t\mathcal{D}}$. In particular, from the commutative diagram

$$\begin{array}{ccc}
g & \xrightarrow{(d\mathcal{X}_t)_e} & g \\
\exp \downarrow & & \downarrow \exp \\
G & \xrightarrow{\mathcal{X}_t} & G
\end{array}$$

then, $\mathcal{X}_t(\exp Y) = \exp (d\mathcal{X}_t)_e Y = \exp (e^{t\mathcal{D}}Y)$, $t \in \mathbb{R}$, $Y \in g$. \quad (48)

Example 16. Consider the solvable Lie group $G$ of dimension two

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, x > 0 \right\} \cong \mathbb{R}^+ \times \mathbb{R} \quad (50)$$

with Lie algebra $g = \text{Span}[Y_1, Y_2]$ and $[Y_1, Y_2] = Y_2$. Since $\mathbb{R}^+ \times \mathbb{R}$ is simply connected, the Lie algebra of derivations of $g$ coincides with the normalizer,

$$\eta = \partial g = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} : \alpha \in \mathbb{R}, \beta \right\}. \quad (51)$$

Take $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \partial g$. The induced linear vector field $\mathcal{X}$ on $G$ given by

$$\mathcal{X}_t(x_1, x_2) = (x_1, x_2 + tx_2), \quad t \in \mathbb{R}. \quad (52)$$

Example 17. Let $G$ be the simply connected Heisenberg matrix group of dimension three with Lie algebra $g = \text{Span}[Y_1, Y_2, Y_3]$ and $[Y_1, Y_2] = Y_3$. A simple computation shows that the Lie algebra $\partial g$ of the $g$-derivations is six dimensional and given by

$$\eta = \partial g = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a + d \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}. \quad (53)$$

According to [27], the face of the linear vector field associated with a derivation $\mathcal{D} \in \eta$ is given by

$$\mathcal{X}(x, y, z) = (ax + dy) \frac{\partial}{\partial x} + (bx + cy) \frac{\partial}{\partial y} + \left(\frac{b}{2}x^2 + \frac{d}{2}y^2 + cx + fy + (a + e)z\right) \frac{\partial}{\partial z}. \quad (54)$$

A special class of linear vector field is given by inner automorphisms of the group. Let $g$ be a non-Abelian Lie algebra and $Y \in g$. Define the vector field $\mathcal{X}$ as follows:

$$\mathcal{X}_t(x) = \exp(tY) x \exp(-tY), \quad x \in G \text{ and } t \in \mathbb{R}. \quad (55)$$

As can be seen, $\mathcal{X}_t$ is an automorphism with inverse $(\mathcal{X}_t)^{-1} = \mathcal{X}_{-t}$. The evaluation of the vector field $\mathcal{X}$ at any point $x_0 \in G$ is by definition

$$\mathcal{X}(x) = \left(\frac{d}{dt}\right)_{t=0} \mathcal{X}_t(x) = \left(\frac{d}{dt}\right)_{t=0} \exp(tY) x \exp(-tY). \quad (56)$$

Example 18. Let us consider $G = E(2)$ the group of Euclidean motions of the plane and $g = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & b \\ 0 & -b & a \end{pmatrix} : y \in \mathbb{R} \right\} \subset G$. In this case,

$$g = \text{Span} \left\{ Y^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Y^3 \right\} \quad (57)$$

For the generator $Y^1 \in g$, the associated linear vector field defined by inner automorphisms reads as

$$\mathcal{X}(x_0) = \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 0 & 0 & 0 \\ x + t - at & a & b \\ y + bt & -b & -a \end{pmatrix} \quad (58)$$

For the generator $Y^3 \in g$, we get

$$\mathcal{X}(x_0) = \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 0 & 0 & 0 \\ x \cos t + y \sin t & a & b \\ -x \sin t + y \cos t & -b & a \end{pmatrix} \quad (59)$$
4. Control Sets of Linear Control Systems on Matrix Groups

In [28] the author studies a particular class of linear systems on matrix groups. After that, Ayala and Tirao give a formal definition of a linear control system on Lie groups as follows; see [7] and the references therein.

4.1. The Definition and the Solution

**Definition 19.** A linear control system on a connected matrix group $G$ is determined by the family of differential equations

$$
Σ_G : x(t) = X(x(t)) + \sum_{j=1}^{m} u_j Y^j(x(t)),
$$

$$
x(t) \in G, \quad u \in U.
$$

Here, $X$ denotes the drift which is a linear vector field on $G$. The control vectors $Y^j$, $j = 1, 2, \ldots, m$, belong to $g$ and we shall think of $g$ as the set of left-invariant vector fields. The input functions $u = (u_1, u_2, \ldots, u_m)$ belong to, the class admissible controls. More precisely, the elements of $U$ are locally integrable functions $u : [0, \infty) \to R^m$, with $0 \in int(Ω)$.

Linear control systems are important for at least two reasons. First, as we showed they are a natural generalization of the classical linear control system on the Euclidean space $G = R^n$. Besides that, Jouan [29] proved that $Σ_G$ is relevant from theoretical and practical point of view; see also. Actually, he shows that any general control system,

$$
Σ_M : \dot{x} = X(x(t)) + \sum_{j=1}^{m} u_j X^j(x(t)),
$$

on a differential manifold is equivalent to a an invariant control system on $G$ or equivalent to a linear control systems on a homogeneous space of $G$.

For the sake of completeness we show a formula which allow to compute the $Σ_G$ solutions when you know the flow of the drift $X$.

**Theorem 20.** Let us consider a constant admissible control $u \in R^m$. Therefore, the vector field $X + \sum_{j=1}^{m} u_j X^j \in η$ has the solution given by

$$
ψ^n_t (g) = X_t(x) \exp \left( \sum_{j=1}^{m} (-1)^{j+1} t^j d_j (X^n, D) \right)
$$

where $X^n = \sum_{j=1}^{m} u_j X^j \in g$ and for each $n \geq 1, d_n : g \otimes g \to g$ is a homogeneous polynomial map of degree $n$. The first terms of $d_n$ are obtained by a recursive formula as follows:

$$
d_1 (Y^n, D) = Y^n
$$

$$
d_2 (Y^n, D) = \frac{1}{2} D (Y^n)
$$

$$
d_3 (Y^n, D) = \frac{1}{12} [Y^n, D (Y^n)] + \frac{1}{6} D^3 (Y^n)
$$

$$
d_4 (Y^n, D) = \frac{1}{24} [Y^n, D^2 (Y^n)] + \frac{1}{24} D^4 (Y^n), \quad \text{etc.}
$$

**Example 21.** Let us consider the Lie algebra $g = R X_1 + R X_2 + R X_3$ with the following generators:

$$
Y^1 = \frac{∂}{∂x_1},
$$

$$
Y^2 = x_2 \frac{∂}{∂x_1} + \frac{∂}{∂x_2},
$$

$$
Y^3 = \frac{∂}{∂x_3}
$$

where the only nonnull bracket is $[Y^3, Y^2] = Y^1$. The group $G = R^3$; its elements are of the form $g = (x_1, x_2, x_3)$, with the group operation "⊙" defined by

$$
(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_1 + y_1 + x_2 y_2 - x_3 y_3, x_2 + y_2, x_3 + y_3)
$$

$$
(x_1, x_2, x_3)^{-1} = (−x_1 + x_2 x_3, −x_2, −x_3)
$$

On the other hand, the exponential and Logarithm maps are given by

$$
\exp (a_1 X_1 + a_2 X_2 + a_3 X_3) = \left( a_1 + \frac{1}{2} a_2 a_3 + a_2 a_3, a_3 \right)
$$

$$
\log (x_1, x_2, x_3) = \left( x_1 - \frac{1}{2} x_2 x_3, x_1 + x_2 X_2 + x_3 X_3 \right)
$$

Let us consider the system $Σ_G$ given by

$$
\dot{x} = X(x) + u Y^2(x), \quad u \in R,
$$

where the linear vector field $X$ is determined by its flows as follows:

$$
X_t (x_1, x_2, x_3) = \left( x_1 + x_2 t + \frac{1}{2} x_2^2 t, x_2, x_2 t + x_3 \right).
$$

Hence, the system $Σ$ reads as

$$
Σ : \begin{cases} 
\dot{x}_1 = x_2 + \frac{1}{2} x_2^2 + ux_3 \\
\dot{x}_2 = u \\
\dot{x}_3 = x_2 
\end{cases}
$$

The $Σ$-solutions are given by

$$
x(t) = X(x) \cdot \exp \xi(t)
$$
The derivation $\mathcal{D}$ associated with $\mathcal{X}$ has the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $\mathcal{D}$ is nilpotent with the nilpotency degree 2, $d_n$ is zero for $n \geq 4$. The nonnull terms of the series are listed below:

\[
d_1 = uY^2, \\
d_2 = \frac{1}{2}u(Y^1 + Y^3), \\
d_3 = -\frac{1}{12}u^2Y^1, \\
\text{and } d_4 = d_5 = \cdots = 0.
\]

In such a case, the series is in fact a finite sum

\[
\zeta(t) = td_1 - t^2d_2 + t^3d_3
\]

and

\[
\exp \zeta(t) = \exp \left(-\frac{t^3}{12}u^2 - \frac{t^2}{2}u\right)Y^1 + utY^2 - \frac{t^2}{2}uY^3.
\]

Therefore, it is possible to decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$, where

\[
\mathfrak{g}^+ = \bigoplus_{\alpha: \text{Re}(\alpha) > 0} \mathfrak{a}_\alpha, \\
\mathfrak{g}^0 = \bigoplus_{\alpha: \text{Re}(\alpha) = 0} \mathfrak{a}_\alpha, \\
\text{and } \mathfrak{g}^- = \bigoplus_{\alpha: \text{Re}(\alpha) < 0} \mathfrak{a}_\alpha.
\]

It turns out that $\mathfrak{g}^+$, $\mathfrak{g}^0$, $\mathfrak{g}^-$ are $\mathcal{D}$-invariant Lie algebras and $\mathfrak{g}^+$, $\mathfrak{g}^-$ are nilpotent.

This decomposition allows understanding the topological-dynamical behavior of $\mathcal{X}$. For general properties of vector fields we refer the readers to [30].

4.3. Control Sets and Controllability. The corresponding connected matrix groups of the Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^0$, $\mathfrak{g}^{+0} = \mathfrak{g}^+ \oplus \mathfrak{g}^0$, and $\mathfrak{g}^{-0} = \mathfrak{g}^- \oplus \mathfrak{g}^0$ are denoted by $G^+$, $G^0$, $G^{+0}$, and $G^{-0}$, respectively. These groups play a fundamental role in the understanding of the dynamics of the system as showed in [7] and the references therein. All these groups are closed and invariant by the flow $\{X_t: t \in \mathbb{R}\}$. Moreover, if $G$ is a solvable group, then $G$ is decomposable, which means that $G = G^+G^0G^- = G^+G^0G^+$. Next, we establish the main properties about control sets of a linear control system $\Sigma_\mathcal{G}$ on matrix group $G$. As before, $\mathcal{O}_\mathcal{G}$, $\mathcal{O}^+_\mathcal{G}$ and $\mathcal{O}^-\mathcal{G}$ have the same meaning but now with respect to the system $\Sigma_\mathcal{G}$.

4.3.1. The Existence of Control Sets. Our control system is said to be locally accessible at $g \in G$ if for all $t > 0$ the sets $\mathcal{O}_\mathcal{G}(g)$ and $\mathcal{O}^-\mathcal{G}(g)$ have nonempty interior and locally accessible if it is locally accessible at any state $g \in G$. In this review we always assume that $\mathcal{O}$ is an open set. In particular, the system $\Sigma$ satisfies the Lie algebra rank condition (Larc), which means that

\[
\text{Span}_{\mathbb{R}}\{X, Y^j: j = 1, \ldots, m\} = \mathfrak{g}.
\]

Therefore, $\Sigma_G$ is locally accessible and under this condition, there exists a control set $\mathcal{O}$ with nonempty interior for $\Sigma_G$, So, using the exponential rule, the $\Sigma$-solution is explicitly given by

\[
x(t) = \left(x_1 + \left(x_2 + \frac{1}{2}x_2^2 + ux_3\right)t + \left(ux_2 - \frac{u}{2}\right)t^2\right)
\]

\[
- \frac{t^3}{3}u^2x_2 + utx_2 + x_3 - \frac{t^2}{2}u.
\]
4.3.2. Topological Properties of Control Sets. Let us denote that \( \mathcal{O}_G^- = \mathcal{O} \cap G^- \) and \( \mathcal{O}_G^+ = \mathcal{O} \cap G^+ \). If \( G \) is decomposable, it is possible to prove that
\[
\mathcal{O} = \mathcal{O}_G^{+,0}
\]
and
\[
\mathcal{O}^- = \mathcal{O}_G^{-,0}.
\]
(81)
The sets \( \mathcal{O}_G^- \), \( \mathcal{O}_G^+ \), and \( G^0 \) are contained in \( \mathcal{O} \cap G^- \). Hence, if the control set \( \mathcal{O} \) is bounded we get that \( \text{cl}(\mathcal{O}_G^-) \), \( \text{cl}(\mathcal{O}_G^+) \), and \( G^0 \) are also bounded. Reciprocally, the boundedness of these three sets could imply in some cases the boundedness of \( \mathcal{O} \). In [7] and the references therein the following two results are proved.

**Theorem 22.** Assume that \( G \) is semisimple or nilpotent. If \( \text{cl}(\mathcal{O}_G^-) \), \( \text{cl}(\mathcal{O}_G^+) \), and \( G^0 \) are compact subsets of \( G \) then \( \mathcal{O} \) is bounded.

Recall that a linear transformation \( L \) is said to be hyperbolic if \( L \) has just eigenvalues with nonzero real parts, i.e., \( 0 \notin \text{Spec}_\mathbb{C}(L) \).

**Theorem 23.** Let \( G \) be a nilpotent simply connected Lie group. Then,
\[
\mathcal{O} \text{ is bounded } \iff \text{cl}(\mathcal{O}_G^-), \text{cl}(\mathcal{O}_G^+), \text{G}^0 \text{ are compacts and } \mathcal{D} \text{ is hyperbolic.}
\]
(82)

We start by establishing an appropriate notion to understand the topology about the control set \( \mathcal{O} \) containing the identity element of \( G \).

**Definition 24.** A connected matrix group \( G \) has the finite semisimple center property if any semisimple Lie subgroups of \( G \) has finite center.

For example, any solvable group and \( \text{SL}(2, \mathbb{R}) \) have the semisimple center property. The next results appears in [7]

**Theorem 25.** Assume that \( G \) has the finite semisimple center property and the reachable set \( \mathcal{O} \) of \( \Sigma_G \) is open. For the existent control set \( \mathcal{O} \) it holds that

1. \( \mathcal{O} \) is closed if and only if \( \mathcal{O}^- = G \);
2. \( \mathcal{O} \) is open if and only if \( \mathcal{O} = G \);
3. furthermore, if \( G \) is nilpotent, we have
   (i) \( \mathcal{O} \) is closed if and only if \( \mathcal{D} \) has only eigenvalues with nonpositive real part;
   (ii) \( \mathcal{O} \) is open if and only if \( \mathcal{D} \) has only eigenvalues with nonnegative real part;
   (iii) \( \mathcal{O} = G \) if and only if \( \mathcal{D} \) has only eigenvalues with zero real part.

4.3.3. The Uniqueness of Control Sets. In the classical linear systems on Euclidean spaces, the Kalman rank condition implies the existence and uniqueness of one control set with nonempty interior. In [7], the authors prove the same result for any decomposable group.

**Theorem 26.** The set \( \mathcal{O} = \text{cl}(\mathcal{O}) \cap \mathcal{O}^- \) is the only control set of the linear control system \( \Sigma_G \) whose interior intersects \( G^+ G^- \) and \( G^- G^+ \).

From that, we obtain the following consequence [7]

**Corollary 27.** Assume that \( G \) is decomposable. Then, \( \mathcal{O} \) is the only control set with nonempty interior.

Just observe that, for any linear control system which satisfies the \( \text{Larc} \) on a solvable matrix group \( G \), we obtain that \( \mathcal{O} = \text{cl}(\mathcal{O}) \cap \mathcal{O}^- \) is the only control set with nonempty interior.

For general matrix groups we do not know if the control set around the identity is the only one. So, to understand how many control sets with nonempty interior could have \( \Sigma_G \) on an arbitrary connected matrix group \( G \), we should study the situation when \( G \) is semisimple. The Levi decomposition theorem together with the information on the solvable case we have in our hand will solve the question.

We finish this section with a conjecture: the uniqueness of control sets of a linear control system \( \Sigma_G \) on a connected matrix group \( G \) is no longer true. More precisely,

**Claim.** Assume that the derivation \( \mathcal{D} \) associated with the drift vector field \( \mathcal{X} \) of \( \Sigma_G \) is semisimple on \( g \). Then, the control sets of \( \Sigma_G \) are exact translations of \( \mathcal{O} \), which gives, in particular, an upper bound for the number of control sets.

For the two-dimensional solvable group \( \mathbb{R}^+ \times \mathbb{R} \), an important information comes from [31], where the authors proved that a linear control system whose associated derivation admits a nonzero eigenvalue cannot be controllable. Therefore, each control set is a proper subset of \( G \).

Next we give a short abstract about a number of examples that will be published elsewhere in detail.

**Example 28.** Let us consider a linear control system on the solvable matrix group \( G = \mathbb{R}^+ \times \mathbb{R} \), in three cases:

1. First, we consider the dynamic
   \[
   \begin{align*}
   \dot{x} &= ux \\
   \dot{y} &= -y + ux,
   \end{align*}
   \]
   where \( u \in \Omega = [-1, 1] \).
The solution with initial condition \((x, y)\) and control \(u\) is given by concatenations of the curves
\[
(e^u x, m_u (e^u - e^{-t}) x + e^{-t} y),
\]
when \(u \neq -1\), \(m_u = \frac{u}{u + 1}, t \in \mathbb{R}\) \hspace{1cm} (84)
and \((e^{-t} x, e^{-t} (y - tx))\), when \(u = -1\), \(t \in \mathbb{R}\).

Denote by \(l_u\) the line in \(G\) determined by
\[
l_u = \{(x, y) : y = m_u x\}
\]
\hspace{1cm} (85)
It turns out that the only control set containing the identity element \((1, 0)\) in \(G\) is given by
\[
\mathcal{C} = \cup \{l_u : -1 < u \leq 1\}.
\]
\hspace{1cm} (86)
Just observe that \(\mathcal{C}\) is unbounded.

(2) Let us consider now the dynamic
\[
\dot{x} = 0
\]
and \(\dot{y} = x - 1 + ux, \hspace{1cm} (87)
\]
where \(u \in \Omega = [-1.1]\).

The solution with initial condition \((x, y)\) and control \(u\) is given by concatenations of
\[
(x, (x - 1 + ux) t + y), \hspace{1cm} t \in \mathbb{R}.
\]
\hspace{1cm} (88)
There exists a quantity \(r > 0\) small enough such that for any \(x \in (1 - r, 1 + r)\) there are controls \(u_1\) and \(u_2\) in \(\Omega\) such that
\[
x - 1 + u_1 \quad x < 0
\]
and \(x - 1 + u_2 \quad x > 0. \hspace{1cm} (89)
\]
It turns out that the system admits an infinite number of unbounded control sets with empty interior as follows:
\[
\mathcal{C}_x = \{x\} \times \mathbb{R} \quad \text{for any } x \in (1 - r, 1 + r). \hspace{1cm} (90)
\]
(3) The last one is given by the following equations:
\[
\dot{x} = ux
\]
and \(\dot{y} = x - 1 \hspace{1cm} (91)
\]
where \(u \in \Omega = [-1.1]\).

The solution of the system with initial condition \((x, y)\) and control \(u \in \Omega\) is given by concatenations of
\[
\left(e^u x, \frac{e^u - 1}{u} x - t + y\right) \quad \text{if } u \neq 0 \hspace{1cm} (92)
\]
and \((x, (x - 1) t + y)\) if \(u = 0; \ t \in \mathbb{R}\).

It turns out that the system is controllable. In particular,
\[
\mathcal{C} = G = \left\{ \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix} : x, y \in \mathbb{R}, x > 0 \right\} \cong \mathbb{R}^+ \times \mathbb{R}. \hspace{1cm} (93)
\]
Thus, for any \(g = (x, y) \in G\) there exists a control \(u \in \Omega\) transferring the identity \((1, 0)\) to \(g\) in positive time.

Remark 29. The challenge now is to compute the control sets for linear control systems on any 3-dimensional connected matrix group. A starting point is to consider the controllability property of linear control systems on any 3-dimensional solvable matrix group. Here we mention an example.

Example 30. Consider the solvable Lie algebra \(g = \mathbb{R} \times \mathbb{R}^2\) where \(\theta = (\frac{1}{2}, \frac{1}{2})\) and with matrix group \(G\). It is possible to prove that a linear control system on \(G\) is controllable if and only if it satisfies \(Larc\) and \(g = g^0\).

Therefore, if the Lyapunov spectrum of any \(g\)-derivation \(\mathcal{D}\) associated with the drift vector field \(X\) contains a zero real number, the system cannot be controllable. So, it is necessary to compute the control sets of \(\Sigma_G\). On the other hand, we have the following.

Example 31. Let \(\Sigma_G\) be a linear control system on a solvable 3-dimensional matrix group \(G\). Assume that \(\mathcal{D}\) is open, then
\[
G^+0 \subset \mathcal{C}
\]
and \(G^-0 \subset \mathcal{C}^\circ. \hspace{1cm} (94)
\]
In particular, if \(\mathcal{D}\) has only eigenvalues with zero real part, i.e., \(G^0 = G\), the system is controllable. In fact, \(G^+0 = G^-0 = G^0 \implies \mathcal{C} = G\). This result is also true for any dimension; see [7] and the references therein.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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