SOME APPLICATIONS OF GRIFFITHS THEOREM IN THEORY OF FEYNMAN INTEGRALS

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Abstract. The present paper provides a method for finding partial differential equations satisfied by the Feynman integrals for diagrams of various types, using the Griffiths theorem on the reduction of poles of rational differential forms. As an application, an algorithm for computing partial differential equations satisfied by Feynman integrals for diagrams of a ladder type is described.

Keywords: Feynman integrals, partial differential equations, Griffiths theorem

1. Introduction

Feynman integrals studied in the present paper enter the expressions for the elements of the $S$-matrix in scattering problems of quantum field theory. To fix ideas, consider, for example, an initial state of a quantum system at the moment $t = -\infty$, characterized by some numbers of fermions and photons in specified individual states. The problem consists in describing the state of this system at the moment $t = +\infty$. The answer is given in terms of a wave function $\Phi(t)$, that describes the states of the fields under consideration and satisfies the Schrödinger equation

\begin{equation}
    i \frac{d\Phi(t)}{dt} = V(t)\Phi(t).
\end{equation}

The general solution of this equation allows one to connect the values of the wave function at $t = -\infty$ and $t = +\infty$ by a transformation of the form

\begin{equation}
    \Phi(+\infty) = S\Phi(-\infty).
\end{equation}

The operator $S$ entering this expression is unitary and is called the scattering matrix (or the $S$-matrix). The perturbation theory is the most widely known method for solving the scattering problem. The asymptotic series for the elements of the $S$-matrix provided by the perturbation theory are sums of Feynman integrals corresponding to the Feynman diagrams with fixed number of internal lines.

The Feynman integrals of the quantum field theory are analytic functions with singularities, depending on certain physical parameters. The singularities occur along...
hypothesis, called singular varieties. The behavior of a Feynman integral in the neighborhood of its singular variety is similar to the behavior of hypergeometric functions. Analytical properties of Feynman integrals can be studied by various methods, for example, by means of series expansions near the singular variety (see [10]). These series expansions are derived from systems of partial differential equations for these integrals (see [1, 9]). In the present paper, we give a method for constructing such a system of partial differential equations for general Feynman integrals, based on the Griffiths theorem [5] on the reduction of poles of rational differential forms.

The structure of the paper is as follows. In Sec. 2, Feynman diagrams and parametric representations of Feynman integrals are defined. In Sec. 3, we introduce the necessary algebraic tools: the Griffiths Theorem and the Macaulay Theorem. In Sec. 4, we apply the Griffiths theorem to the problem of finding systems of partial differential equations for Feynman integrals corresponding to different types of diagrams (Theorems 1 and 2).

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2. Parametric representations of Feynman integrals

Each term of the expansion of the $S$-matrix in a series corresponds to some Feynman diagram. The Feynman diagram is a graph with lines topologically representing the propagation of elementary particles and vertices representing interactions of the elementary particles [2].

The graph $G$ consists of elements of two types, the lines $\{l_i, i \in \Omega\}$ and the vertices $\{V_k, k \in \Theta\}$. There is the map $i = i_1 \times i_2 : \Omega \to \Theta \times \Theta$ sending each line $l_j$ ($j \in \Theta$) to the pair of vertices $i_1(j), i_2(j)$, called the initial and final vertices of the line respectively. A line $l$ is incident with a vertex $V$ if it has this vertex as the initial or final vertex. A (minimal) loop of the graph is a sequence of lines $l_1, l_2, ..., l_k$ such that $l_i, l_{i+1}$ are incident with some vertex $V_{i+1}$ for each $i = 1, ..., k$, while $l_1$ and $l_k$ are incident with some vertex $V_1$, and the vertices $V_1, ..., V_k$ are pairwise distinct. A loop can be formed by just one line. A path $\gamma$ in the graph $G$ is a sequence of lines $l_{\gamma(1)}, l_{\gamma(2)}, ..., l_{\gamma(r)}$ such that: 1) neither $l_{\gamma(i)}$ is a loop; 2) $\gamma(i) \neq \gamma(j)$ whenever $1 \leq i \neq j \leq r$; 3) $l_{\gamma(i)}$, $l_{\gamma(i+1)}$ are incident with a vertex $V_i$ ($1 \leq i \leq r-1$); 4) $V_i \neq V_{i+1}$ for $i = 1, ..., r-2$. Denote by $V_0$ the vertex of $l_{\gamma(1)}$, different from $V_1$, and by $V_r$ the vertex of $l_{\gamma(r)}$, different from $V_{r-1}$. If the vertices
$V_0, V_1, ..., V_r$ are pairwise distinct, we will say that the path is simple. In this case the path $\gamma$ starts at $V_0$ and end at $V_r$. Denote by $N$ and $n$ the numbers of elements of the sets $\Omega$ and $\Theta$, i.e. $|\Omega| = N, |\Theta| = n$. If $c$ is the number of connected components of $G$, then the number of independent loops of $G$ is $h = N - n + c$. If several graphs are considered, then the above sets and numbers corresponding to a graph $G$ will be denoted by $\Omega_G, \Theta_G, N(G), n(G)$.

A graph $G$ is called a Feynman diagram [3], if partitions of $\Theta$ and $\Omega$ of the following form are given:

$$
\Theta = \Theta^E \sqcup \Theta^I, \quad \Omega = \Omega^M \sqcup \Omega^0.
$$

The vertices $V_j$ for $j \in \Theta^E$ are called external and the vertices $V_j$ for $j \in \Theta^I$ are called internal; the lines $l_j$ for $j \in \Omega^M$ are called massive and the lines $l_j$ for $j \in \Omega^0$ are called massless.

If $G$ is connected, then we denote by $G^\infty$ the graph which is obtained from $G$ by adding the infinity vertex $V_\infty$ that is connected with all external vertices by lines. Each line of a Feynman diagram is endowed with some orientation, but the Feynman integral defined below is independent of this orientation.

A subgraph $H$ of $G$ is a graph such that $\Omega_H \subset \Omega_G, \Theta_H \subset \Theta_G$ and $i_H = i_G|_{\Theta_H}$. A subgraph $T_r$ of a graph $G$ is called a $r$-tree if it contains no loops and if the number of its connected components is equal to $r$. A spanning $r$-tree of $G$ is a $r$-tree which includes all the vertices of $G$.

To each line $l_j$ ($j \in \Omega$) of $G$, one associates a parameter $\alpha_j$ ($j = 1, ..., N$) (these parameters are called the Feynman parameters). For every subset $\eta \subset \Omega$, we will write $\alpha(\eta) = \prod_{j \in \eta} \alpha_j$. Define the polynomial $U(\alpha) = \sum_{T_1 \subset G} \alpha(\Omega - T_1)$, where the sum is taken over all spanning trees $T_1$ of $G$. Let $\chi \subset \Theta^E$ be an arbitrary subset of the set of external vertices of $G$. Define another polynomial, $W(\chi)(\alpha) = \sum_{T_2 \subset G} \alpha(\Omega - T_2)$, where the sum is taken over all spanning 2-trees $T_2$ of $G$ such that its connected components contain subsets $\chi$ and $\Theta^E - \chi$ respectively.

Let $G$ be a connected graph. Define the space $X_G \subset (\mathbb{C}^D)^{n^E}$ of external momentum vectors satisfying the conservation law

$$
X_G = \left\{ p_k, k \in \Theta^E \mid \sum_{k \in \Theta^E} p_k = 0 \right\}.
$$
Assume that a symmetric bilinear form is given on \((\mathbb{C}^D)^{n_E}\) whose quadratic form is positive definite when restricted to the real subspace \((\mathbb{R}^D)^{n_E}\) of \((\mathbb{C}^D)^{n_E}\). So we can form the invariants

\[
s(\chi) = \left( \sum_{i \in \chi} p_i \right)^2
\]

from the external momentum vectors \(p_i\) \((i \in \Theta^E)\), where \(\chi\) is a non-empty proper subset of the set \(\Theta^E\). Due to the conservation law these invariants satisfy the relations

\[
s(\chi) = s(\Theta^E - \chi),
\]

\[
s(\chi_1 \cup \chi_2 \cup \chi_3) = s(\chi_1 \cup \chi_2) + s(\chi_1 \cup \chi_3) + s(\chi_2 \cup \chi_3) - s(\chi_1) - s(\chi_2) - s(\chi_3),
\]

where \(\chi_1, \chi_2, \chi_3\) are non-empty proper disjoint subsets of \(\Theta^E\). If \(D \geq n_E - 1\), then the relations (5), (6) are all the relations on the invariants \(s(\chi)\) that are consequences of their definition (5) (about the relations on the invariants for the case \(D < n_E - 1\) see [4]).

Hence, we can fix a collection of subsets \(X = \{\chi_i \mid \chi_i \subset \Theta^E\}, |X| = \frac{1}{2}n^E(n^E - 1)\), such that all invariants \(s(\chi)\) are expressed in terms of \(s_i = s(\chi_i)\), and these invariants are linearly independent (for example, \(X = \{\{i\}, \{j, k\} \mid i, j, k \in \Theta^E - i_0\}\), where \(i_0 \in \Theta^E\) is some fixed external vertex).

So to each graph \(G\) we can associate a set of complex parameters \(s_i\), which are the invariants defined above. Besides, to each internal line \(l_j\) \((j \in \Omega)\) of \(G\), one associates a complex parameter \(z_j\) called the squared mass for that line. Then the parametric representation of the Feynman integral for the given graph \(G\) is defined by the following formula:

\[
F(s, z) = \int_{\Gamma} \frac{U(\alpha)^{N - \frac{D}{2}(h+1)}}{Q(\alpha, s, z)^{N - \frac{D}{2}h}} \omega,
\]

where

\[
\omega = \sum_{i=1}^{N} (-1)^i \alpha_i d\alpha_1 \wedge ... \wedge d\alpha_i \wedge ... \wedge d\alpha_N,
\]

\[
Q(\alpha, s, z) = \frac{1}{2} \sum_{\chi \subset \Theta^E} s(\chi)W_\lambda(\alpha) - U(\alpha) \sum_{i=1}^{N} \alpha_i z_i,
\]

and \(\Gamma\) is a \((N - 1)\)-dimensional piecewise smooth hypersurface lying in the first quadrant of the real subspace \(\mathbb{R}^N\) of \(\mathbb{C}^N\) such that its boundary lies on the coordinate hyperplanes.
\{\alpha_1 = 0\}, ..., \{\alpha_N = 0\}. This hypersurface can be chosen in such a way that it does not intersect the variety \{Q = 0\}.

3. Griffiths theorem on reduction of a pole of a rational differential form

Consider a rational differential \(n\)-form on the projective space \(\mathbb{CP}^n\) with homogeneous coordinates \(\alpha_0, \ldots, \alpha_n\),

(11) \[ \gamma = \frac{R(\alpha)}{Q(\alpha)^k} \omega \quad (k \geq 1), \]

where \(R\) and \(Q\) are homogeneous polynomials of degrees \(p\) and \(q\) in \(\alpha = (\alpha_0, \ldots, \alpha_n)\), such that \(kq = p + n + 1\). The form \(\gamma\) is a rational differential form with pole of order \(k\) on the algebraic variety \(\{Q = 0\}\) in \(\mathbb{CP}^n\). Below we will use the following Griffiths theorem on reduction of a pole of a rational differential form.

Griffiths theorem \([5]\). Let \(\gamma\) be a rational differential \(n\)-form on the projective space \(\mathbb{CP}^n\) with pole of order \(k \geq 1\) on the variety \(\{Q(\alpha) = 0\}\), given by formula (11). If \(R\) belongs to the ideal in the polynomial ring \(\mathbb{C}[\alpha_0, \ldots, \alpha_n]\), generated by \(Q, \alpha\), \(\ldots, Q, \alpha\), i.e. if \(R\) can be represented in the form \(R = \sum_{\nu=0}^{n} \lambda_{\nu} Q_{\alpha_{\nu}}\), where \(\lambda_{\nu}\) are polynomials in \(\alpha\) and \(Q, \alpha, \ldots, Q, \alpha\) stands for \(\partial Q/\partial \alpha\), then there exists a rational differential \((n - 1)\)-form \(\phi\), such that \(\gamma - d\phi\) has a pole of order \(k - 1\) on the variety \(\{Q(\alpha) = 0\}\).

Proof. Assume \(R\) belongs to the ideal \((Q, \alpha), \ldots, Q, \alpha\), i.e. \(R = \sum_{\nu=0}^{n} \lambda_{\nu} Q_{\alpha_{\nu}}\) for some polynomials \(\lambda_{\nu}\) of degree \((k - 1)q - n\). Consider the \((n - 1)\)-form with pole of order \(k - 1\) on the variety \(\{Q = 0\}\):

(12) \[ \phi = \frac{1}{k - 1} Q^{k-1} \sum_{i<j} (-1)^{i+j} [\alpha_i \lambda_j - \alpha_j \lambda_i] (\ldots \widehat{d\alpha_i} \ldots \widehat{d\alpha_j} \ldots). \]

We have

(13) \[ d\phi = \left[ \frac{1}{Q^k} \left( \sum_{\nu=0}^{n} \lambda_{\nu} Q_{\alpha_{\nu}} \right) - \frac{1}{k - 1} Q^{k-1} \sum_{j=0}^{n} \frac{\partial \lambda_j}{\partial \alpha_j} \right] \omega. \]

So we obtain

(14) \[ \gamma - d\phi = \left[ \frac{1}{k - 1} Q^{k-1} \sum_{j=0}^{n} \frac{\partial \lambda_j}{\partial \alpha_j} \right] \omega. \]

□
It is necessary to note that under condition \( \lambda_{\nu}|_{\alpha_{\nu}=0}, \nu = 0, ..., n \) the exact differential form \( d\phi \) does not contribute to the integral \( \int_{\Gamma} \) by Stokes’ theorem, if the hypersurface \( \Gamma \) satisfies the condition \( \partial \Gamma \subset \bigcup_{\nu=0}^{n} \{ \alpha_{\nu} = 0 \} \). Besides, the following theorem gives the condition under which \( R \) belongs to the ideal.

**Macaulay Theorem** \([6]\). Consider \( n+1 \) homogeneous polynomials \( P_0, P_1, ..., P_n \) on the projective space \( \mathbb{CP}^n \) of degrees \( p_0, p_1, ..., p_n \), such that the equations

\[ P_0(\alpha) = 0, P_1(\alpha) = 0, ..., P_n(\alpha) = 0 \]

have no solution in \( \mathbb{CP}^n \). Then all the homogeneous polynomials of degree greater or equal to \( p_0 + p_1 + ... + p_n - n \) belong to the ideal \( (P_0, P_1, ..., P_n) \).

We do not give a proof of this theorem. The proof consists in an application of the algorithm of constructing expansions of the form \( R = \sum_{i=0}^{n} \lambda_i P_i \) for polynomials \( R \) from the ideal \( (P_0, P_1, ..., P_n) \). Applying the Macaulay theorem to the homogeneous polynomials \( Q_{\alpha_0}, Q_{\alpha_1}, ..., Q_{\alpha_n} \) of degree \( q-1 \), we see that any polynomial \( R \) of degree \( kq - n - 1 \) belongs to the ideal \( (Q_{\alpha_0}, Q_{\alpha_1}, ..., Q_{\alpha_n}) \) provided \( kq - (n+1) \geq (q-2)(n+1) + 1 \).

### 4. Partial differential equations for some diagrams

Consider an arbitrary Feynman diagram and notice that for any homogeneous polynomial \( P \) in \( N \) variables of degree \( p \), we have the following formula:

\[
P \left( \frac{\partial}{\partial z} \right) F = \int_{\Gamma} P \left( \frac{\partial}{\partial z} \right) U^{N - \frac{D}{2}(h+1)} \frac{Q^{N - \frac{D}{2}h}}{\omega} = \frac{(N - \frac{D}{2}h + p - 1)!}{(N - \frac{D}{2}h - 1)!} \int_{\Gamma} P(\alpha) U^{N - \frac{D}{2}(h+1)+p} \frac{Q^{N - \frac{D}{2}h+p}}{\omega},
\]

where \( h \), as before, denotes the number of independent loops, and \( \frac{\partial}{\partial z} = \left( \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_N} \right) \).

**Theorem 1.** The Feynman integral \( F(s,z) \) corresponding to an arbitrary Feynman diagram is a solution of the system of partial differential equations

\[
\left[ Q_{\alpha_i} \left( \frac{\partial}{\partial z} \right) \frac{\partial}{\partial z_i} - U \left( \frac{\partial}{\partial z} \right) - \left( N - \frac{D}{2}(h+1) + q \right) \frac{\partial}{\partial z_i} U_{\alpha_i} \left( \frac{\partial}{\partial z_i} \right) \right] F = 0, \quad i = 1, ..., N.
\]

**Proof.** Consider the differential operator \( \mathcal{D}_i = Q_{\alpha_i} \left( \frac{\partial}{\partial z} \right) \frac{\partial}{\partial z_i} \). According to (15) we have

\[
\mathcal{D}_i F = (-1)^q \frac{(N - \frac{D}{2}h + q - 1)!}{(N - \frac{D}{2}h - 1)!} \int_{\Gamma} R \frac{Q^{N - \frac{D}{2}h+q}}{\omega},
\]
where \( R = (-1)^q \alpha \alpha U^{N-D(h-1)+q} \). It is clear that \( R \) belongs to the ideal generated by \( Q_{\alpha_1}, ..., Q_{\alpha_N} \). Moreover, each polynomial \( \lambda \) in the expression \( R = \sum_{\nu=0}^{n} \lambda_{\nu} Q_{\alpha_\nu} \) is divisible by \( \alpha_\nu \). So after applying the Griffiths theorem to the right hand side of (17), the contribution of the exact differential form (13) to the integral is zero. Hence, we have the following relation:

\[
D_i F = \frac{(N - D \frac{h}{2} + q - 2)!}{(N - D \frac{h}{2} - 1)!} \int_{\Gamma} \frac{U^{q-1}(U + (N - D \frac{h}{2} + q)\alpha \alpha U)}{Q^{N-D \frac{h}{2} + q-1}} U^{N-D(h+1)} \omega.
\]

On the other hand, according to (15) we have

\[
\left[ U \left( \frac{\partial}{\partial z} \right) + \left( N - D \frac{h}{2} + q \right) U_{\alpha_i} \left( \frac{\partial}{\partial z_i} \right) \right] F = \frac{(N - D \frac{h}{2} + q - 2)!}{(N - D \frac{h}{2} - 1)!} \int_{\Gamma} \frac{U^{q-1}(U + (N - D \frac{h}{2} + q)\alpha \alpha U)}{Q^{N-D \frac{h}{2} + q-1}} U^{N-D(h+1)} \omega.
\]

Consequently, comparing the right hands of both previous identities we obtain the statement.

\[
\square
\]

Recall that, for a given Feynman diagram, we have introduced a collection \( \mathcal{X} \subset 2^{\Theta^E} \) of subsets \( \chi_1, ..., \chi_r \subset \Theta^E \), such that all the invariants \( s(\chi) \) are expressed in terms of the invariants \( s_i = s(\chi_i) \) and the latter ones are linearly independent. Now assume that \( D \geq n^E - 1 \) and consider the Feynman diagrams satisfying the following property:

(P) \( \mathcal{X} \) can be chosen in such a way that \( W_\chi = 0 \) for all \( \chi \in 2^{\Theta^E} \setminus \{ \Theta^E - \chi_1, ..., \Theta^E - \chi_r \} \).

In other words, the corresponding polynomial (10) can be written in the form

\[
Q = \sum_{i=1}^{r} s_i W_i - U \sum_{i=1}^{N} \alpha \alpha z_i, \quad \text{where} \quad W_i = W_{\chi_i}.
\]

For example, the \( h \)-loop ladder diagram (see the figure) and the one-loop \( N \)-point diagram satisfy this property.
As above, notice that for any homogeneous polynomial $P$ in $r + N$ variables of degree $p$ we have the following equation:

\[
P(\frac{\partial}{\partial s}, \frac{\partial}{\partial z}) F = (-1)^p \cdot \frac{(N - \frac{\partial}{\partial h} + p - 1)\int P(W_{i,-\alpha U}) Q^N - \frac{\partial}{\partial h + p} U^{N - \frac{\partial}{\partial h} (h+1)} \omega,}{(N - \frac{\partial}{\partial h} - 1)!}
\]

where $W = (W_1, ..., W_r), \frac{\partial}{\partial s} = (\frac{\partial}{\partial s_1}, ..., \frac{\partial}{\partial s_r}).$

**Theorem 2.** Assume that the Feynman diagram satisfies the property (P). Then the corresponding Feynman integral is a solution of the system of partial differential equations

\[
\left[Q_{\alpha_j} \left(\frac{\partial}{\partial z}\right) \frac{\partial}{\partial s_i} + \left(N - \frac{D}{2} (h + 1) + q - 1\right) U_{\alpha_j} \left(\frac{\partial}{\partial z}\right) \frac{\partial}{\partial s_i} - W_{i,\alpha_j} \left(\frac{\partial}{\partial z}\right)\right] F = 0,
\]

for all $i, j$ such that the subset $\chi_i$ can be connected with the subset $\Theta^E - \chi_i$ by an internal line $l_j$.

**Proof.** Consider the differential operator $D_{ij} = Q_{\alpha_j} \left(\frac{\partial}{\partial z}\right) \frac{\partial}{\partial s_i}$. According to (16) we have

\[
D_{ij} F = (-1)^q \cdot \frac{(N - \frac{\partial}{\partial h} + q - 1)}{(N - \frac{\partial}{\partial h} - 1)!} \int \frac{R}{Q^N - \frac{\partial}{\partial h + q} \omega},
\]

where $R = (-1)^{q-1} W_{i} U^N - \frac{\partial}{\partial h (h+1)} + q - 1 Q_{\alpha_j}$. It is clear that $R$ belongs to the ideal generated by $Q_{\alpha_1}, ..., Q_{\alpha_N}$. Moreover, due to the property (P), each polynomial $W_i$ is divisible by $\alpha_j$. So the polynomials $\lambda_{\nu}$ in the expression $R = \sum_{\nu=0}^n \lambda_{\nu} Q_{\alpha_{\nu}}$ are divisible by $\alpha_{\nu}$. Thus after applying the Griffiths’ theorem to the right hand side of (17) the exact differential form (13) gives no contribution to the integral. Hence, we have the following equality:

\[
D_{ij} F = \frac{(N - \frac{\partial}{\partial h} + q - 2)}{(N - \frac{\partial}{\partial h} - 1)!} \int \frac{(U^{q-1} W_{i,\alpha_j} + (N - \frac{D}{2} (h + 1) + q - 1) U^{q-2} U_{\alpha_j} W_i)}{Q^N - \frac{\partial}{\partial h + q - 1} U^{N - \frac{\partial}{\partial h} (h+1)} \omega}.
\]

On the other hand, according to (16) we have

\[
\left[W_{i,\alpha_j} \left(\frac{\partial}{\partial z}\right) - \left(N - \frac{D}{2} (h + 1) + q - 1\right) U_{\alpha_j} \left(\frac{\partial}{\partial z}\right) \frac{\partial}{\partial s_i}\right] F =
\]

\[
= \left(N - \frac{\partial}{\partial h} + q - 2\right)! \int \frac{(U^{q-1} W_{i,\alpha_j} + (N - \frac{D}{2} (h + 1) + q - 1) U^{q-2} U_{\alpha_j} W_i)}{Q^N - \frac{\partial}{\partial h + q - 1} U^{N - \frac{\partial}{\partial h} (h+1)} \omega}.
\]

Consequently, comparing the right hand sides of both previous identities, we obtain the statement.

Now we will present a general method for deriving partial differential equations of order $p$ on the example of a $h$-loop ladder diagram. This method can be realized by using a computer algebra system.
Let \( \mathcal{X} = \{ \chi_1 = \{1\}, \ldots, \chi_4 = \{4\}, \chi_5 = \{1, 2\}, \chi_6 = \{2, 3\} \} \). Then the Feynman integral of the \( h \)-loop ladder diagram has the following form:

\[
F(s, z) = \int_\Gamma \frac{U^{h-1}}{Q^{h+1}} \omega,
\]

where \( Q = \sum_{i=1}^{6} s_i W_{i} - U \left( \sum_{i=1}^{3h+1} \alpha_i z_i \right), \) \( W_{i} = W_{\chi_{i}}, \) \( s_i = s(\chi_i). \) Consider the differential operators of order \( p \) and \( p - 1 \) respectively with undetermined coefficients depending on the parameters \( s_i, z_j, \)

\[
\mathcal{D} = \sum_{|I|+|J|=p} a_{I,J} \partial_s^I \partial_z^J, \quad \tilde{\mathcal{D}} = \sum_{|I|+|J|=p-1} b_{I,J} \partial_s^I \partial_z^J,
\]

where we use the standard multi-index notation. According to (16) we have

\[
\mathcal{D} F = (-1)^p \frac{(h + p)!}{h!} \int_\Gamma \frac{R}{Q^{h+1+p}} \omega, \quad \tilde{\mathcal{D}} F = (-1)^{p-1} \frac{(h + p - 1)!}{h!} \int_\Gamma \frac{\tilde{R}}{Q^{h+p}} \omega,
\]

where the polynomials \( R \) and \( \tilde{R} \) have the following form:

\[
R = \sum_{|I|+|J|=p} a_{I,J} W^I \alpha^J (-1)^{|J|} U^{|I|}, \quad \tilde{R} = \sum_{|I|+|J|=p-1} b_{I,J} W^I \alpha^J (-1)^{|J|} U^{|I|}.
\]

Next, consider the homogeneous polynomials \( \lambda_{\nu} = \alpha_{\nu} \sum_{|K|=\deg R - q + 1} \lambda_{\nu,K} \alpha^K \) of degree \( \deg R - q + 1 \) with undetermined coefficients and impose the following conditions on the polynomials \( R, \tilde{R}, \lambda_{\nu}: \)

\[
R = \sum_{\nu=1}^{3h+1} \lambda_{\nu} Q_{\alpha_{\nu}}, \quad \sum_{\nu=1}^{3h+1} \frac{\partial \lambda_{\nu}}{\partial \alpha_{\nu}} = \tilde{R}.
\]

After collecting coefficients of appropriate monomials we obtain a system of homogeneous linear equations with unknowns \( a_{I,J}, b_{I,J}, \lambda^I_{\nu}. \) Due to Griffiths theorem each solution of this system corresponds to a relation of the form \( (\mathcal{D} + \tilde{\mathcal{D}}) F = 0, \) which is the wanted partial differential equation satisfied by the Feynman integral \( F \) of a \( h \)-loop ladder diagram.

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