Sharp eigenvalue estimates and related rigidity theorems

Yanlin Deng†, Feng Du†, Jing Mao†‡*, Yan Zhao‡

†School of Mathematics and Physics Science,
Jingchu University of Technology, Jingmen, 448000, China
‡Faculty of Mathematics and Statistics,
Key Laboratory of Applied Mathematics of Hubei Province,
Hubei University, Wuhan 430062, China
Emails: dyl690926@163.com (Y. L. Deng),
defengdu123@163.com (F. Du), jiner120@163.com (J. Mao).

Abstract

In this paper, sharp bounds for the first nonzero eigenvalues of different type have been obtained. Moreover, when those bounds are achieved, related rigidities can be characterized. More precisely, first, by applying the Bishop-type volume comparison proven in [10, 13] and the Escobar-type eigenvalue comparisons for the first nonzero Steklov eigenvalue of the Laplacian proven in [25], for manifolds with radial sectional curvature upper bound, under suitable preconditions, we can show that the first nonzero Wentzell eigenvalue of the geodesic ball on these manifolds can be bounded from above by that of the geodesic ball with the same radius in the model space (i.e., spherically symmetric manifolds) determined by the curvature bound. Besides, this upper bound for the first nonzero Wentzell eigenvalue can be achieved if and only if these two geodesic balls are isometric with each other. This conclusion can be seen as an extension of eigenvalue comparisons in [9, 25]. Second, we prove a general Reilly formula for the drifting Laplacian, and then use the formula to give a sharp lower bound for the first nonzero Steklov eigenvalue of the drifting Laplacian on compact smooth metric measure spaces with boundary and convex potential function. Besides, this lower bound can be achieved only for the Euclidean ball of the prescribed radius.

1 Introduction

Throughout this paper, assume that \((M, g)\) is an \(n\)-dimensional \((n \geq 2)\) complete Riemannian manifold with the metric \(g\). Let \(\Omega \subseteq M\) be a compact domain with smooth\(^1\) boundary \(\partial \Omega\). Denote

\(^1\)Coresponding author

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We do know that maybe smoothness assumption is a little bit stronger for the eigenvalue problem (1.1). For instance, when \(\beta = 0\), (1.1) degenerates into the classical Steklov eigenvalue problem (1.3) below, and in this situation (1.3) only has discrete spectrum even if the boundary \(\partial \Omega\) is only Lipschitz continuous (not necessary to be smooth) – see [18, Theorem 6.2] for details. Based on this fact, when considering different eigenvalue problems in this paper, in order to avoid the discussion of regularity assumptions for the boundary, we always assume that the boundary (if exists) is smooth.
by $\Delta$ and $\Delta$ the Laplace-Beltrami operators on $\Omega$ and $\partial \Omega$, respectively. Consider the eigenvalue problem with the Wentzell boundary condition as follows

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \frac{\partial u}{\partial \eta} = \tau u & \text{on } \partial \Omega,
\end{cases} \tag{1.1}$$

where $\eta$ is the outward unit normal vector of the boundary $\partial \Omega$, and $\beta$ is a given real number. The boundary value problem (1.1) is called the Wentzell eigenvalue problem of the Laplacian. It is known that for $\beta \geq 0$, the eigenvalue problem (1.1) only has the discrete spectrum and its elements, called eigenvalues, can be listed non-decreasingly as follows

$$0 = \tau_0(\Omega) < \tau_1(\Omega) \leq \tau_2(\Omega) \leq \tau_3(\Omega) \leq \cdots \uparrow \infty.$$ 

Besides, by the variational principle, it is not hard to know that the first nonzero eigenvalue $\tau_1(\Omega)$ of (1.1) can be characterized as follows

$$\tau_1(\Omega) = \min \left\{ \frac{\int_{\Omega} |\nabla u|^2 dv + \beta \int_{\partial \Omega} |\nabla u|^2 dA}{\int_{\partial \Omega} u^2 dA} \right\}, \quad u \in W^{1,2}(\Omega), \text{Tr}_{\partial \Omega}(u) \in W^{1,2}(\partial \Omega), u \neq 0, \int_{\partial \Omega} u dA = 0, \tag{1.2}$$

where $\nabla$, $\nabla$ are the gradient operators on $\Omega$ and $\partial \Omega$ respectively, $\text{Tr}_{\partial \Omega}$ is the trace operator, $W^{1,2}(\Omega)$ is the completion of the set of smooth functions $C^\infty(\Omega)$ under the Sobolev norm $\|u\|_{1,2} = \left( \int_{\Omega} u^2 dv + \int_{\partial \Omega} |\nabla u|^2 dA \right)^{1/2}$, and $W^{1,2}(\partial \Omega)$ is defined similarly. Here, $dv$, $dA$ are volume elements of the domain $\Omega$ and its boundary $\partial \Omega$, respectively. This usage of notations for volume elements of a domain and its boundary would be used in the sequel also. For the eigenvalue problem (1.1), there are some interesting estimates for eigenvalues $\tau_i$ recently – see, e.g., [4, 5, 6, 24]. Besides, if the Laplace operators $\Delta$, $\Delta$ were replaced by their weighted versions, then (1.1) would become exactly the Wentzell eigenvalue problem of the weighted Laplacian (1.15) considered in [25], where we can give sharp lower and upper bounds for the first nonzero eigenvalue provided suitable constraints imposed for the weighted Ricci curvature, the weighted mean curvature and the second fundamental forms – see [25, Theorems 4.1 and 4.2] for details.

Clearly, when $\beta = 0$, the eigenvalue problem (1.1) degenerates into the following classical Steklov eigenvalue problem

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} = pu & \text{on } \partial \Omega,
\end{cases} \tag{1.3}$$

which only has the discrete spectrum and all the eigenvalues can be listed non-decreasingly as follows

$$0 = p_0(\Omega) < p_1(\Omega) \leq p_2(\Omega) \leq p_3(\Omega) \leq \cdots \uparrow \infty.$$ 

Besides, the first nonzero Steklov eigenvalue can be characterized as follows (see, e.g., [2, p. 144])

$$p_1(\Omega) = \min \left\{ \frac{\int_{\Omega} |\nabla u|^2 dv}{\int_{\partial \Omega} u^2 dA} \right\}, \quad u \in W^{1,2}(\Omega), u \neq 0, \int_{\partial \Omega} u dA = 0. \tag{1.4}$$
By (1.4), one can easily get the Sobolev trace inequality, which makes an important role in the study of existence and regularity of solutions of some boundary value problems, as follows

\[ \int_{\partial \Omega} |u - u_0|^2 dA \leq \frac{1}{p_1(\Omega)} \int_{\Omega} |\nabla u|^2 dv, \]

where \( u_0 \) is the mean value of the function \( u \) when restricted to the boundary.

By (1.2) and (1.4), it is not hard to get the fact:

- **Fact 1.** For \( \beta > 0 \), one has

\[ \tau_1(\Omega) \geq \beta \lambda_1^c(\partial \Omega) + p_1(\Omega), \]

where \( \lambda_1^c(\partial \Omega) \) denotes the first nonzero closed eigenvalue of the Laplacian on the boundary \( \partial \Omega \). Moreover, the equality can be obtained if and only if any eigenfunction \( u \) of \( \tau_1(\Omega) \) is also the eigenfunction corresponding to \( p_1(\Omega) \) and \( u|_{\partial \Omega} \) is the eigenfunction corresponding to \( \lambda_1^c(\partial \Omega) \) on \( \partial \Omega \).

Combining the Bishop-type volume comparison (see [10, Theorem 4.2] or [13, Theorem 2.3.2]) with the Escobar-type eigenvalue comparisons for the first nonzero Steklov eigenvalue of the Laplacian (see [25, Theorems 1.5 and 1.6]), we can get a comparison for the first nonzero Wentzell eigenvalue \( \tau_1 \) of the Laplacian on complete manifolds with radial sectional curvature bounded from above. More precisely, we have:

**Theorem 1.1.** Assume that \((M, g)\) is an \(n\)-dimensional complete Riemannian manifold having a radial sectional curvature upper bound \( k(t) \) w.r.t. \( t := d(p, \cdot) \) denotes the distance to the point \( p \in M \), and \( k(t) \) is a continuous function w.r.t. \( t \). Let \( B(p, r) \) be the geodesic ball, with center \( p \) and radius \( r \), on \( M \). For the Wentzell eigenvalue problem (1.1) with \( \beta \geq 0 \), we have

- If \( n = 2, 3 \), then

\[ \tau_1(B(p, r)) \leq \tau_1(B(p^+, r)), \]

where \( r < \min\{\text{inj}(p), l\} \) with \( \text{inj}(p) \) the injectivity radius at \( p \), and \( B(p^+, r) \) is the geodesic ball, with center \( p^+ \) and radius \( r \), of the spherically symmetric manifold \( M^+ = [0, l] \times fS^{n-1} \) with the base point \( p^+ \) and the warping function \( f \) determined by

\[ \begin{aligned}
    f''(t) + k(t)f(t) &= 0 \quad \text{on } (0, l), \\
    f'(0) &= 1, \quad f(0) = 0, \\
    f'|_{(0, l)} &> 0.
\end{aligned} \]

Equality in (1.5) holds if and only if \( B(p, r) \) is isometric to \( B(p^+, r) \).

- If \( n \geq 4 \) and furthermore the first nonzero closed eigenvalues of the Laplacian on the boundary satisfy

\[ \lambda_1^c(\partial B(p, r)) \leq \lambda_1^c(\partial B(p^+, r)), \]

then the same conclusion as in the lower dimensional cases \( n = 2 \) and \( n = 3 \) can also be obtained.
Define the shape operator $S$ of $\partial \Omega$ as $S(X) = \nabla_X \tilde{N}$, and then the second fundamental form of $\partial \Omega$ is defined as $II(X,Y) = \langle S(X), Y \rangle$, where $X,Y \in T \partial \Omega$ with $T \partial \Omega$ the tangent bundle of $\Omega$. The eigenvalues of $S$ are called the principal curvatures of $\partial \Omega$, and the mean curvature $H$ of $\partial \Omega$ is given by $H = \frac{1}{n} \text{tr} S$, where $\text{tr} S$ denotes the trace of $S$. In [20], for a given smooth function $f$ on $\Omega$, Reilly proved the following celebrated formula

$$
\int_{\Omega} \left( (\Delta f)^2 - |\nabla^2 f|^2 - \text{Ric}(\nabla f, \nabla f) \right) dv = \int_{\partial \Omega} \left( (n-1)Hu^2 + 2u\Delta z + II(\nabla z, \nabla z) \right) dA,
$$

where $u = \frac{\partial f}{\partial \eta}|_{\partial \Omega}$, $z = f|_{\partial \Omega}$, $\nabla^2 f$ is the Hessian of $f$, and $\text{Ric}(\cdot, \cdot)$ denotes the Ricci curvature on $\Omega$. Reilly’s formula is a useful tool for eigenvalue estimates. For instance, Reilly [20] used the formula to prove a Lichnerowicz type sharp lower bound for the first eigenvalue of the Laplacian on manifolds with boundary. By applying (1.8), Escobar [7], Wang-Xia [21] successfully gave some estimates for the first nonzero Steklov eigenvalue of the Laplacian, respectively. Qiu and Xia [19] extended Reilly’s formula to the following version:

$$
\int_{\Omega} V \left( (\Delta f + Knf)^2 - |\nabla^2 f + Kfg|^2 \right) dv = \int_{\partial \Omega} V \left( 2u\Delta z + (n-1)Hu^2 + II(\nabla z, \nabla z) + (2n-2)Kuz \right) dA
+ \int_{\partial \Omega} \frac{\partial V}{\partial \eta} \left( |\nabla z|^2 - (n-1)Kz^2 \right) dA + \int_{\Omega} (K\Delta V + nK^2 V) f^2 dv
+ \int_{\Omega} (\nabla^2 V - \Delta V g - (2n-2)KV g + VRic) \langle \nabla f, \nabla f \rangle dv,
$$

where $K \in \mathbb{R}$, $V : \overline{\Omega} \to \mathbb{R}$ is a given a.e. twice differentiable function, and other notations have the same meanings as before. Recently, by applying this generalized Reilly’s formula (1.9), under the nonnegative sectional curvature assumption, Xia and Xiong [23] can obtain a sharp lower bound estimate for the first nonzero Steklov eigenvalue of the Laplacian, which gives a partial answer to the following Escobar’s conjecture:

- (X) Let $(N^n, \bar{g})$ be a compact Riemannian manifold with boundary and dimension $n \geq 3$. Assume that $\text{Ric}(\bar{g}) \geq 0$ and that the second fundamental form $II$ satisfies $II \geq cI$ on $\partial N$, $c > 0$. Then

$$p_1(N^n) \geq c,$$

and the equality holds only for the Euclidean ball of radius $\frac{1}{c}$.

For a given complete $n$-dimensional Riemannian manifold $(M, g)$, the triple $(M, g, e^{-\phi} dv)$ is called a smooth metric measure space (SMMS for short), where $\phi$ is a smooth real-valued function on $M$. We call $dv_{\phi} := e^{-\phi} dv$ the weighted volume density (also called the weighted Riemannian density). On a SMMS $(M, g, e^{-\phi} dv)$, we can define the so-called drifting Laplacian (also called weighted Laplacian) $\mathbb{L}_{\phi}$ as follows

$$\mathbb{L}_{\phi} := \Delta - g(\nabla \phi, \nabla \cdot),$$
where, as before, $\nabla$ and $\Delta$ are the gradient operator and the Laplace operator on $M$, respectively. Ma and Du gave a Reilly-type formula for the weighted Laplacian (see [12, Theorem 1]). In fact, for $f \in C^\infty(\Omega)$, they have proven the following Reilly-type formula

$$
\int_\Omega \left( (\mathbb{L}_\phi f)^2 - |\nabla f|^2 - \text{Ric}^\phi(\nabla f, \nabla f) \right) d\nu_f = \int_{\partial \Omega} \left( (n-1)H^\phi u^2 + 2u\partial_{\phi^\phi} z + 2I(\nabla z, \nabla z) \right) dA_\phi,
$$

(1.10)

where $H^\phi = H + \frac{1}{n-1} \frac{\partial \phi}{\partial \eta}$ denotes the $\phi$-mean curvature (also called the weighted mean curvature, see, e.g., [24] for this notion), $\text{Ric}^\phi := \text{Ric} + \nabla(\nabla \phi)$ denotes the Bakry-Émery Ricci tensor of $M$, $dA_\phi := e^{-\phi} dA$ is the induced Riemannian density of the boundary, $\mathbb{L}_\phi$ is the drift Laplacian on $\partial \Omega$ related to $\mathcal{A}$, and other notations have the same meanings as before. Clearly, if $\phi = \text{const}$ is a constant function, then (1.10) becomes the classical Reilly formula (1.8). By using (1.10), Ma and Du successfully gave estimates for eigenvalues of the drifting Laplacian - see [12, Theorems 2 and 3] for details.

We can prove the following Reilly-type formula for the drifting Laplacian.

**Theorem 1.2.** Let $V : \overline{\Omega} \to \mathbb{R}$ be a a.e. twice differential function, where $\Omega$ is a bounded domain with smooth boundary $\partial \Omega$ on an $n$-dimensional SMMS $(M, g, e^{-\phi} dv)$, $n \geq 2$. Given a smooth function $f$ on $\Omega$, we have

$$
\int_\Omega V \left( (\mathbb{L}_\phi f + Knf)^2 - |\nabla f|^2 + Kf(\nabla f, \nabla f) \right) d\nu_f = \int_{\partial \Omega} V \left( 2u\partial_{\phi^\phi} z + (n-1)H^\phi u^2 + 2I(\nabla z, \nabla z) + (2n-2)Kuz \right) dA_\phi
$$

$$
+ \int_{\partial \Omega} \frac{\partial V}{\partial \eta} \left( |\nabla z|^2 - (n-1)Kz^2 \right) dA_\phi + \int_\Omega (n-1) \left( K\mathbb{L}_\phi V + nK^2 V \right) f^2 d\nu_f
$$

$$
+ \int_\Omega \left( \nabla^2 V - \mathbb{L}_\phi V g - (2n-2)KV g + V\text{Ric}^\phi \right) (\nabla f, \nabla f) d\nu_f,
$$

(1.11)

where same notations have the same meanings as those in (1.8), (1.9) and (1.10).

**Remark 1.3.** Clearly, if $\phi = \text{const}$., our Reilly-type formula (1.11) degenerates into (1.9); if $V \equiv 1$ and $K = 0$, (1.11) becomes (1.10); if $V \equiv 1$, $K = 0$ and $\phi = \text{const}$., our formula (1.11) degenerates into the classical Reilly’s formula (1.8).

Consider the Steklov-type eigenvalue problem of the drifting Laplacian on an $n$-dimensional ($n \geq 2$) SMMS $(M, g, e^{-\phi} dv)$ as follows

$$
\begin{align*}
\mathbb{L}_\phi u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} &= \sigma u \quad \text{on } \partial \Omega,
\end{align*}
$$

(1.12)

where notations have the same meanings as before. One can get that (1.12) has the discrete spectrum and all the eigenvalues can be listed non-decreasingly as follows

$$
0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \sigma_3(\Omega) \leq \cdots \uparrow \infty.
$$

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2Readers might find that the second term $\frac{1}{n-1} \frac{\partial \phi}{\partial \eta}$ of the $\phi$-mean curvature has different forms in literatures (for instance, $-\frac{1}{n-1} \frac{\partial \phi}{\partial \eta} \frac{\partial \phi}{\partial \eta}$, etc), but actually they have no essential difference.
Besides, the first nonzero Steklov eigenvalue $\sigma_1(\Omega)$ of $\mathbb{L}_\phi$ can be characterized as follows

$$\sigma_1(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 dv_\phi}{\int_{\partial\Omega} u^2 dA_\phi} \mid u \in W^{1,2}(\Omega), u \neq 0, \int_{\partial\Omega} u dA_\phi = 0 \right\}, \quad (1.13)$$

where $\tilde{W}^{1,2}(\Omega)$ is the completion of the set of smooth functions $C^\infty(\Omega)$ under the weighted Sobolev norm $\|u\|_{1,2} = \left( \int_\Omega u^2 dv_\phi + \int_{\partial\Omega} |\nabla u|^2 dA_\phi \right)^{1/2}$. Please see Appendix A for the explanation of the spectral structure of the eigenvalue problem (1.12). Applying the Reilly-type formula (1.11), we can give a sharp lower bound for the first nonzero Steklov eigenvalue $\sigma_1(\cdot)$ of the drifting Laplacian. More precisely, we have:

**Theorem 1.4.** Assume that the potential function $\phi$ of the n-dimensional $(n \geq 2)$ SMMS $(M, g, e^{-\phi} dv)$ is convex. For the eigenvalue problem (1.12), if the sectional curvature of $\Omega \subset M$ is nonnegative (i.e., $\text{Sec}(\Omega) \geq 0$) and the principal curvatures of $\partial\Omega$ are bounded below by a constant $c > 0$, then we have

$$\sigma_1(\Omega) \geq c, \quad (1.14)$$

with equality holding if and only if $\Omega$ is isometric to an n-dimensional Euclidean ball of radius $\frac{1}{c}$ and $\nabla^2 \phi = 0$.

**Remark 1.5.** (1) Clearly, if $\phi = \text{const.}$, then the estimate (1.14) degenerates into $p_1(\Omega) \geq c$ and the rigidity also holds. This assertion is exactly the statement of [23, Theorem 1]. That is to say, our Theorem 1.4 covers [23, Theorem 1] as a special case.

(2) In fact, if $\text{Ric}^\phi \geq 0$, the principal curvatures of $\partial\Omega$ are bounded below by a constant $c > 0$ and $H^\phi > c$, for the eigenvalue problem (1.12), one can obtain by applying (1.10)

$$0 \geq \int_\Omega \left( (\mathbb{L}_\phi f)^2 - |\nabla^2 f|^2 - \text{Ric}^\phi(\nabla f, \nabla f) \right) dv_\phi$$

$$= \int_{\partial\Omega} \left( (n-1)H^\phi u^2 + 2u\mathbb{L}_\phi z + 2H(\nabla z, \nabla z) \right) dA_\phi$$

$$> \int_{\partial\Omega} \left[ -2g(\nabla u, \nabla z) + c \cdot g(\nabla z, \nabla z) \right] dA_\phi$$

$$= \int_{\partial\Omega} \left[ -2\sigma_1(\Omega) \cdot g(\nabla z, \nabla z) + c \cdot g(\nabla z, \nabla z) \right] dA_\phi,$$

which implies

$$\sigma_1(\Omega) > \frac{c}{2}. \quad (1.15)$$

So, clearly, if $\phi = \text{const.}$, the estimate (1.15) degenerates into $p_1(\Omega) > c/2$, which is exactly Escobar’s estimate given in [7, Theorem 8] for the case $n \geq 3$. Besides, if $\phi = \text{const.}$, then $\text{Ric}^\phi = \text{Ric}$ and the assumption $H > c$ implies $H^\phi > c$ directly. This is because, in this situation, $H^\phi = H = \frac{\text{Hess}^\phi}{n-1} > c$. In this sense, our estimate (1.15) here covers Escobar’s conclusion [7, Theorem 8] as a special case and of course gives a partial answer to Escobar’s conjecture.

(3) It is easy to find that our estimate (1.15) here is covered by the lower bound estimate for the first nonzero eigenvalue of the Wentzell eigenvalue problem of the drifting Laplacian given in [25].
Theorem 4.2]. In fact, one only needs to choose \( \beta = 0 \) in the estimate (4.4) of [25, Theorem 4.2], and then our estimate (1.15) follows directly. The reason why we do not list our previous result [25, Theorem 4.2] directly is that we would like to show the application of the Reilly-type formula of the drifting Laplacian (1.10) intuitively.

The paper is organized as follows. In Section 2 for complete manifolds with radial sectional curvature bounded from above, the Escobar-type eigenvalue comparison theorem for the first nonzero Wentzell eigenvalue of the Laplacian on geodesic balls (of these manifolds) can be set up. Of course, the equality case in this eigenvalue comparison has been characterized. Besides, a related open problem will be proposed at the end of this section. In Section 3, we firstly give the proof to the Reilly-type formula given in Theorem 1.2, and then by applying this Reilly-type formula, under suitable constraints, a sharp lower bound for the first nonzero Steklov eigenvalue of the drifting Laplacian can be obtained. Moreover, when this sharp bound is achieved, a rigidity result can be obtained. That is to say, by the usage of Theorem 1.2 we devote to give the proof of Theorem 1.4 in the second part of Section 3.

2 The Escobar-type eigenvalue comparison for the first nonzero Wentzell eigenvalue of the Laplacian

We give the proof of Theorem 1.1 as follows:

Proof of Theorem 1.1. Let \( \psi(t) \) be the function satisfying the differential equation

\[
\begin{cases}
\frac{1}{f^{n-1}(t)} \frac{d}{dt} \left( f^{n-1}(t) \frac{d}{dt} \psi(t) \right) - \frac{(n-1)}{f^{n-1}(t)} \psi(t) = 0 & \text{in } (0, 1), \\
\psi'(r) = p_1(B(p^+, r)) \psi(r), \ \psi(0) = 0,
\end{cases}
\]

where naturally \( p_1(B(p^+, r)) \) is the first non-zero Steklov eigenvalue of the Laplacian on \( B(p^+, r) \).

As explained in the proof of [25, Theorem 1.6] (see page 403 of [25]), we know that \( \psi(t) \) does not change sign on \((0, r)\). Without loss of generality, one can assume \( \psi(t) > 0 \) on \((0, r)\), and then \( \psi'(t) > 0 \) on \((0, r)\) since

\[
\psi'(t) = \frac{n-1}{f^{n-1}(t)} \int_0^t \psi(s)f^{n-3}(s)ds,
\]

where \( f \) is the solution to (1.6). Construct the test function \( \varphi(t, \xi) = a_+(t)e_1(\xi) \), where \( e_1(\xi) \) is the eigenfunction of the first nonzero closed eigenvalue \( \lambda_1^c(\partial B(p, r)) \) of the Laplacian on the boundary \( \partial B(p, r) \), and

\[
a_+(t) := \min\{a(t), 0\}, \quad a(t) := \psi(t) \left( \frac{f^{n-1}(t)}{h(t)} \right)^{1/2} + \int_t^0 \psi(s) \left( \frac{f^{n-1}(s)}{n^{-1} h(s)} \right)^{1/2} ds,
\]

with \( h(t) := \max\left\{ d^*(t), \frac{f^2(t)}{n-1} d^2(t) \right\} \) and

\[
d^*(t) = \int_{\mathbb{S}^{n-1}} |\nabla e_1|^2_{\mathbb{S}^{n-1}}(\xi) f^{n-3}(t, \xi) d\sigma,
\]

\[
d^2(t) = \int_{\mathbb{S}^{n-1}} e_1^2(\xi) \cdot \det A(t, \xi) d\sigma = \int_{\mathbb{S}^{n-1}} e_1^2(\xi) \cdot \sqrt{|g|}(t, \xi) d\sigma = \int_{\mathbb{S}^{n-1}} e_1^2(\xi) f^{n-1}(t, \xi) d\sigma.
\]
Here $d\sigma$ denotes the $(n-1)$-dimensional volume element on $\mathbb{S}^{n-1}$, $\mathbb{A}(t, \xi)$ is the path of linear transformations (see [25, Subsection 1.1] for the definition), and $J^{n-1} = \sqrt{|g|} = \det \mathbb{A}(t, \xi)$ represents the square root of the determinant of the metric matrix. It is easy to check that $h(t)$ is Lipschitz continuous and hence differentiable almost everywhere, and moreover $\varphi(t, \xi) \in W^{1,2}(B(p, r))$. By the characterization (1.2), together with (3.6)-(3.8) in [25] (see pp. 405-406 of [25]), we can obtain

$$
\tau_1(B(p, r)) \leq \frac{\int_{B(p, r)} |\nabla \varphi|^2 dv + \beta \int_{\partial B(p, r)} |\nabla \varphi|^2 dA}{\int_{\partial B(p, r)} \varphi^2 dA} 
\leq \frac{\int_0^r (\psi'(t))^2 (f - 1)(t) dt + (n - 1) \int_0^r \psi^2 f^{n-3}(t) dt}{\psi^2(r) f^{n-1}(r)} + \beta \cdot \frac{\int_{\partial B(p, r)} |a_+(r) \nabla e_1(\xi)|^2 dA}{\int_{\partial B(p, r)} (a_+(r) e_1(\xi))^2 dA} 
= p_1(\mathcal{B}(p^+, r)) + \beta \lambda_1^\alpha(\partial B(p, r)).
$$

(2.16)

As shown in the proof of [25, Theorem 1.5] (see pp. 406-407 of [25]), for $n = 2, 3$, one has $\lambda_1^\alpha(\partial B(p, r)) \leq \lambda_1^\alpha(\partial \mathcal{B}(p^+, r))$ directly. Putting this fact and the assumption (1.7) (for $n \geq 4$) into (2.16), together with Fact1, yields

$$
p_1(B(p, r)) + \beta \lambda_1^\alpha(\partial B(p, r)) \leq \tau_1(B(p, r)) \leq p_1(\mathcal{B}(p^+, r)) + \beta \lambda_1^\alpha(\partial B(p, r)) 
\leq p_1(\mathcal{B}(p^+, r)) + \beta \lambda_1^\alpha(\partial \mathcal{B}(p^+, r)) 
= \tau_1(\mathcal{B}(p^+, r)).
$$

(2.17)

When $\tau_1(B(p, r)) = \tau_1(\mathcal{B}(p^+, r))$, then from (2.16) and (2.17) we infer that

$$
p_1(B(p, r)) = p_1(\mathcal{B}(p^+, r)) \quad \text{and} \quad \lambda_1^\alpha(\partial B(p, r)) = \lambda_1^\alpha(\partial \mathcal{B}(p^+, r))
$$

holds, which, by [25, Theorems 1.5 and 1.6], implies that $B(p, r)$ is isometric to $\mathcal{B}(p^+, r)$. In fact, from the proof of [25, Theorem 1.6] (see page 406 of [25]), we know that $p_1(B(p, r)) = p_1(\mathcal{B}(p^+, r))$ implies $J(t, \xi) = f(t)$ on $(0, r)$. Then the rigidity follows by directly applying the Bishop-type volume comparison theorem for manifolds having a radial sectional curvature upper bound (see [10, Theorem 4.2] or [13, Theorem 2.3.2]).

**Remark 2.1.** (1) As we know, roughly speaking, the main contribution of the Escobar-type eigenvalue comparison theorems ([25, Theorems 1.5 and 1.6]) for the first nonzero Steklov eigenvalue of the Laplacian is that J. Mao and his collaborators therein have successfully weakened the curvature assumption for the classical eigenvalue comparison theorem (of the first nonzero Steklov eigenvalue of the Laplacian) in J. F. Escobar’s influential work [3] — “the sectional curvature is bounded from above by some constant” has been weakened to “the radial sectional curvature has an upper bound (w.r.t. the chosen point) which is given by a continuous function of the Riemannian distance parameter”, and correspondingly, this change leads to the situation that the model spaces in Escobar’s setting, which are space forms, should be replaced by more general model spaces (i.e., spherically symmetric manifolds) in Mao’s setting.

The corresponding author here, Prof. J. Mao, has used spherically symmetric manifolds as the model spaces to get some interesting (volume, eigenvalue, heat kernel) comparison conclusions —
see [10, 13, 14, 15, 16, 17, 25] for details.

(2) As mentioned before, clearly, if $\beta = 0$, then the eigenvalue problem (1.1) degenerates into the classical Steklov eigenvalue problem (1.3) of the Laplacian. Correspondingly, our Theorem 1.1 becomes exactly the eigenvalue comparisons [25, Theorems 1.5 and 1.6] directly. However, people are really caring about the case $\beta > 0$. The proof of our Theorem 1.1 here deeply depends on some arguments in the proof of [25, Theorems 1.5 and 1.6], but if one checks the above proof of Theorem 1.1 carefully, then one would find that the eigenvalue comparison here cannot be obtained by applying the Escobar-type eigenvalue comparisons (for the first nonzero Steklov eigenvalue of the Laplacian) [25, Theorems 1.5 and 1.6] directly. In fact, for complete $n$-manifolds whose radial sectional curvature has an upper bound $k(t)$ w.r.t. $p$, by [25, Theorems 1.5 and 1.6], one has $p_1(B(p, r)) \leq p_1(\mathcal{B}(p^+, r))$ with $r < \min\{\text{inj}(p), l\}$, but together with (2.17), one cannot get $\tau_1(B(p, r)) \leq \tau_1(\mathcal{B}(p^+, r))$ since $p_1(B(p, r)) + \beta \lambda^c_1(\partial B(p, r)) \leq \tau_1(B(p, r))$. On contrary, for complete $n$-manifolds whose radial sectional curvature has an upper bound $k(t)$ w.r.t. $p$, by Theorem 1.1, one has $\tau_1(B(p, r)) \leq \tau_1(\mathcal{B}(p^+, r))$ with $r < \min\{\text{inj}(p), l\}$, which together with (2.17) implies

$$p_1(B(p, r)) + \beta \lambda^c_1(\partial B(p, r)) \leq \tau_1(B(p, r)) \leq \tau_1(\mathcal{B}(p^+, r)) = p_1(\mathcal{B}(p^+, r)) + \beta \lambda^c_1(\partial \mathcal{B}(p^+, r)).$$

Therefore, one finally gets

$$p_1(B(p, r)) \leq p_1(\mathcal{B}(p^+, r))$$

since $\lambda^c_1(\partial B(p, r)) \leq \lambda^c_1(\partial \mathcal{B}(p^+, r))$ is valid in this setting. So, our Theorem 1.1 is a very interesting improvement of [25, Theorems 1.5 and 1.6], and of course covers it as a special case.

(3) The curvature assumption here is reasonable, since for a given complete Riemannian manifold and a chosen point on the boundary, one can always find a sharp upper bound (which is given by a continuous function of the distance parameter) for the radial sectional curvature – see (2.10) in [10] for the accurate expression.

(4) Since some arguments in the proof of [25, Theorems 1.5 and 1.6] make an important role in the proof of our Theorem 1.1 here, similar to the statement in [25, Theorems 1.5 and 1.6] and as explained in [25, (3) of Remark 1.7], the restraint on the injectivity radius is necessary.

(5) Clearly, by the Sturm-Picone separation theorem, if $k(t) \leq 0$, then the initial value problem (1.6) has the positive solution on $(0, \infty)$. More precisely, in this situation, $l = \infty$ and $f(t) \geq t$ on $(0, \infty)$. Except the non-positivity assumption of $k(t)$, it is interesting to find other assumptions such that (1.6) has a positive solution on $(0, \infty)$. This problem has close relation with the oscillation of solutions to the ODE $f'''(t) + k(t)f(t) = 0$. There exist some nice results working on this problem – see, e.g., Bianchini-Luciano-Marco [2], Hille [17] and Mao [13, Subsection 2.6] for nice sufficient conditions on $k(t)$ such that (1.6) has a positive solution on $(0, \infty)$.

Naturally, we can propose:

**Open problem.** For $n \geq 4$, is the Escobar-type Wentzell eigenvalue inequality (1.5) also true without the precondition (1.7)?

### 3 The Reilly-type formula and its application

In this section, we first give the proof of the Reilly-type formula (1.11), and then show an application of this formula – the sharp lower bound estimate (1.14) with the related rigidity.
Some ideas of the following proof come from [12, 19].

Proof of the Theorem [7.2]. Let $f_i, f_{ij}, \cdots$ and $f_n$ be covariant derivatives and the normal derivative of a function $f$ w.r.t. the metric $g$. Here, an orthonormal tangent frame field $\{e_1, e_2, \cdots, e_n\}$ has been used in all the calculations carried out at some point in $\Omega$. Then we have $\nabla^2 f = \sum_{i,j=1}^n f_1 f_{ij}$. Noticing $\mathrm{Ric}^\phi = \mathrm{Ric} + \nabla^2 \phi$, we infer from the integration by parts and the Ricci identity that

$$\int_{\Omega} V|\nabla^2 f|^2 e^\phi dv = \int_{\partial \Omega} V \sum_{i,j=1}^n f_{ij} f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} V \sum_{i=1}^n f_{i\eta} f_{i} e^{-\phi} dA - \int_{\partial \Omega} V \sum_{i,j=1}^n V_j f_{ij} f_{ij} e^{-\phi} dv + \int_{\Omega} V \sum_{i,j=1}^n f_{ij} f_{ij} e^{-\phi} dv$$

$$+ \int_{\Omega} V \sum_{i,j=1}^n f_{ij} f_{i\phi} e^{-\phi} dv$$

$$= \int_{\partial \Omega} V \sum_{i=1}^n f_{i\eta} f_{i} e^{-\phi} dA - \int_{\partial \Omega} V \sum_{i,j=1}^n V_j f_{ij} f_{ij} e^{-\phi} dv - \int_{\Omega} V \sum_{i,j=1}^n f_{ij} f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} V \sum_{i=1}^n f_{i\eta} f_{i} e^{-\phi} dA - \int_{\partial \Omega} V \sum_{i,j=1}^n V_j f_{ij} f_{ij} e^{-\phi} dv + \int_{\Omega} V \sum_{i,j=1}^n f_{ij} f_{ij} e^{-\phi} dv$$

$$- \int_{\Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i e^{-\phi} dv - \int_{\Omega} V \sum_{i=1}^n \left( \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\Omega} V \sum_{i,j=1}^n f_{ij} f_{i\phi} e^{-\phi} dv$$

$$= \int_{\partial \Omega} V \sum_{i=1}^n f_{i\eta} f_{i} e^{-\phi} dA - \int_{\partial \Omega} V \sum_{i,j=1}^n V_j f_{ij} f_{ij} e^{-\phi} dv - \int_{\partial \Omega} V \sum_{i,j=1}^n V_j f_{ij} f_{ij} e^{-\phi} dv$$

$$- \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i e^{-\phi} dv - \int_{\partial \Omega} V \sum_{i,j=1}^n \left( f_{ij} f_{ij} \right) e^{-\phi} dv + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$- \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i e^{-\phi} dv - \int_{\Omega} V \sum_{i,j=1}^n \left( \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\Omega} V \sum_{i,j=1}^n f_{ij} f_{i\phi} e^{-\phi} dv$$

$$= \int_{\partial \Omega} \left( \sum_{i=1}^n f_{i\eta} f_i - \frac{1}{2} V\eta |\nabla f|^2 - V_{\phi} f f_{\eta} \right) e^{-\phi} dA + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} \left( \sum_{i=1}^n f_{i\eta} f_i - \frac{1}{2} V\eta |\nabla f|^2 - V_{\phi} f f_{\eta} \right) e^{-\phi} dA + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} \left( \sum_{i=1}^n f_{i\eta} f_i - \frac{1}{2} V\eta |\nabla f|^2 - V_{\phi} f f_{\eta} \right) e^{-\phi} dA + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} \left( \sum_{i=1}^n f_{i\eta} f_i - \frac{1}{2} V\eta |\nabla f|^2 - V_{\phi} f f_{\eta} \right) e^{-\phi} dA + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} \left( \sum_{i=1}^n f_{i\eta} f_i - \frac{1}{2} V\eta |\nabla f|^2 - V_{\phi} f f_{\eta} \right) e^{-\phi} dA + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$

$$= \int_{\partial \Omega} \left( \sum_{i=1}^n f_{i\eta} f_i - \frac{1}{2} V\eta |\nabla f|^2 - V_{\phi} f f_{\eta} \right) e^{-\phi} dA + \int_{\partial \Omega} V \sum_{i,j=1}^n \left( L_{\phi} f \right) f_i f_{ij} e^{-\phi} dv$$

$$+ \int_{\partial \Omega} V \sum_{i,j=1}^n \left( \left( L_{\phi} f + \sum_{j=1}^n f_{j\phi j} \right) f_i f_{ij} e^{-\phi} dv$$
where the first usage of the integration by parts was in the second equality of (3.1), and the Ricci identity

\[ f_{iji} - f_{jii} = \sum_{k=1}^{n} f_k R_{jkij}, \]

with \( R_{jkij} \) the components of the curvature sensor on \( \Omega \subset M \), has been used firstly in the third equality of (3.1). Summing the above Ricci identity w.r.t. the index \( i \) from 1 to \( n \) yields

\[ \sum_{j=1}^{n} (f_{iji} - f_{jii}) = \sum_{k,j=1}^{n} f_k R_{jkij}, \]

which directly implies

\[ \sum_{j=1}^{n} f_{iji} - (\Delta f)_i = \sum_{j=1}^{n} f_j R_{ji}, \]

with \( R_{ji} \) the components of the Ricci tensor Ric on \( \Omega \subset M \). The relation (i.e., \( \mathbb{L}_\phi = \Delta - \langle \nabla \phi, \cdot \rangle \)) between the operators \( \Delta \) and \( \mathbb{L}_\phi \) has been used several times in the rest part of (3.1). We also infer from the integration by parts that

\[ \int_{\Omega} V f \mathbb{L}_\phi f e^{-\phi} dv = \int_{\partial \Omega} V f f_\eta e^{-\phi} dA - \int_{\Omega} \left( V |\nabla f|^2 + \sum_{i=1}^{n} f V_i f_i \right) e^{-\phi} dv. \quad (3.2) \]

Combining (3.1) with (3.2) yields

\[ \int_{\Omega} V f \mathbb{L}_\phi f e^{-\phi} dv = \int_{\partial \Omega} V f f_\eta e^{-\phi} dA - \int_{\Omega} \left( V |\nabla f|^2 + \sum_{i=1}^{n} f V_i f_i \right) e^{-\phi} dv \]

\begin{align*}
&= \int_{\Omega} V \left( \left( \mathbb{L}_\phi f + Knf \right)^2 - |\nabla^2 f + Kf g|^2 \right) dv \\
&= \int_{\Omega} V \left( \left( \mathbb{L}_\phi f \right)^2 - |\nabla^2 f|^2 \right) e^{-\phi} dv + (2n - 2)K \int_{\Omega} V f \mathbb{L}_\phi f e^{-\phi} dv \\
&\quad + n(n-1)K^2 \int_{\Omega} V f^2 e^{-\phi} dv - 2K \int_{\Omega} V f \sum_{i=1}^{n} f_i f_i e^{-\phi} dv \\
&= \int_{\partial \Omega} \left( -V \sum_{i=1}^{n} f_i f_i + \frac{1}{2} V_\eta |\nabla f|^2 + V \mathbb{L}_\phi f f_\eta + (2n - 2)K V f f_\eta \right) e^{-\phi} dA \\
&\quad + \int_{\Omega} \left( -\frac{1}{2} \mathbb{L}_\phi V |\nabla f|^2 - \mathbb{L}_\phi f \sum_{i=1}^{n} V_i f_i + V \text{Ric}^\phi (\nabla f, \nabla f) \right) e^{-\phi} dv \\
&\quad - (2n - 2)K \int_{\Omega} \left( V |\nabla f|^2 + \sum_{i=1}^{n} f V_i f_i \right) e^{-\phi} dv + n(n-1)K^2 \int_{\Omega} V f^2 e^{-\phi} dv \\
&\quad - 2K \int_{\Omega} V f \sum_{i=1}^{n} f_i f_i e^{-\phi} dv \quad (3.3)
\end{align*}
Using the integration by parts again, we have

\[
\int_{\Omega} -\mathbb{L}_\phi f \sum_{i=1}^{n} V_i f_i e^{-\phi} dv \\
= \int_{\partial \Omega} -f_\eta \sum_{i=1}^{n} V_i f_i e^{-\phi} dA + \int_{\Omega} \left( \sum_{i,j=1}^{n} V_{ij} f_i f_j + \sum_{i=1}^{n} V_i \left( \frac{1}{2} |\nabla f|^2 \right)_i \right) e^{-\phi} dv \\
= \int_{\partial \Omega} \left( -f_\eta \sum_{i=1}^{n} V_i f_i + \frac{1}{2} |\nabla f|^2 V_\eta \right) e^{-\phi} dA + \\
\int_{\Omega} \left( \sum_{i,j=1}^{n} V_{ij} f_i f_j - \frac{1}{2} \mathbb{L}_\phi V |\nabla f|^2 \right) e^{-\phi} dv,
\]

(3.4)

and

\[
\int_{\Omega} \sum_{i=1}^{n} V_i f_i e^{-\phi} dv \\
= \int_{\Omega} \sum_{i=1}^{n} V_i \left( \frac{1}{2} f^2 \right)_i e^{-\phi} dv \\
= \int_{\partial \Omega} \frac{1}{2} f^2 V_\eta e^{-\phi} dA - \int_{\Omega} \frac{1}{2} f^2 \mathbb{L}_\phi V e^{-\phi} dv.
\]

(3.5)

Taking (3.4) and (3.5) into (3.3), we have

\[
\int_{\Omega} V \left( (\mathbb{L}_\phi f + Kn f)^2 - |\nabla^2 f + K f g|^2 \right) dv \phi \\
= \int_{\partial \Omega} \left( -V \sum_{i=1}^{n} f_\eta f_i + V_\eta |\nabla f|^2 + V \mathbb{L}_\phi f f_\eta + (2n - 2) K V f f_\eta - f_\eta \sum_{i=1}^{n} V_i f_i \right) e^{-\phi} dA \\
- (n - 1) K f^2 V_\eta e^{-\phi} dA \\
+ \int_{\Omega} \left( -\frac{1}{2} \mathbb{L}_\phi V |\nabla f|^2 - \sum_{i,j=1}^{n} V_{ij} f_i f_j - \frac{1}{2} \mathbb{L}_\phi V |\nabla f|^2 + V \text{Ric}^\phi (\nabla f, \nabla f) \right) e^{-\phi} dv \\
- (2n - 2) K \int_{\Omega} \left( V |\nabla f|^2 - \frac{1}{2} f^2 \mathbb{L}_\phi V \right) e^{-\phi} dv + n(n - 1) K^2 \int_{\Omega} V f^2 e^{-\phi} dv \\
- 2K \int_{\Omega} V f \sum_{i=1}^{n} f_i \phi e^{-\phi} dv.
\]

(3.6)

Choosing an orthonormal frame \{\vec{e}_i\}_{i=1}^{n} such that \vec{e}_n = \vec{\eta} on \partial \Omega. Note that \(z = f|_{\partial \Omega}, u = f_\eta|_{\partial \Omega}\)
and $H^\phi = H + \frac{1}{n-1} \phi \eta$, we infer from the Gauss-Weingarten formula that

$$
\int_{\partial \Omega} V \left( L_\phi f f \eta - \sum_{i=1}^{n} f_{i \eta} f_i \right) e^{-\phi} dA
$$

$$
= \int_{\partial \Omega} V \left( \sum_{i=1}^{n} (f_{ii} + f_i \phi_i) f \eta - \sum_{i=1}^{n} f_{i \eta} f_i \right) e^{-\phi} dA
$$

$$
= \int_{\partial \Omega} V \left( \sum_{i=1}^{n-1} (f_{ii} + f_i \phi_i) f \eta + f_\eta^2 \phi \eta - \sum_{i=1}^{n-1} f_{i \eta} f_i \right) e^{-\phi} dA
$$

and

$$
\int_{\partial \Omega} \left| \nabla f \right|^2 V \eta - \sum_{i=1}^{n} f_{i \eta} V_i f_i \right) e^{-\phi} dA
$$

$$
= \int_{\partial \Omega} \left| \nabla z \right|^2 V \eta - u \left( \nabla V, \nabla z \right) \right) e^{-\phi} dA
$$

Here, in fact, the classical Gauss-Weingarten formula has been used in (3.7), that is, for smooth tangent vector fields $X, Y$, one has

$$\nabla_X Y = \nabla_{X|_{\partial \Omega}} Y|_{\partial \Omega} + II(X|_{\partial \Omega}, Y|_{\partial \Omega}),$$

where, as before, $II(\cdot, \cdot)$ stands for the second fundamental form of the boundary $\partial \Omega$, and $\nabla, \nabla$ are the gradient operators on $\Omega$ and $\partial \Omega$, respectively. Combining (3.7) and (3.8), we have

$$
\int_{\partial \Omega} \left( -V \sum_{i=1}^{n} f_{i \eta} f_i + V \eta \left| \nabla f \right|^2 + V L_\phi f f \eta + (2n - 2)Kf f \eta \right.
$$

$$\left. - f \eta \sum_{i=1}^{n} V_i f_i - (n - 1)Kf^2 V \eta \right) dA_\phi
$$

$$
= \int_{\partial \Omega} V \left( 2u \nabla \phi + (n - 1)H^\phi u^2 + II(\nabla z, \nabla z) + (2n - 2)Kuz \right) dA_\phi
$$

$$
+ \int_{\partial \Omega} V \eta \left( \left| \nabla z \right|^2 - (n - 1)Kz^2 \right) dA_\phi.
$$

Substituting (3.9) into (3.7), we have

$$
\int_{\Omega} V \left( \left( L_\phi f + Knf \right)^2 - \left| \nabla f \right|^2 + Kf g \left( \nabla f, \nabla \phi \right) \right) dv_\phi
$$

$$
= \int_{\partial \Omega} V \left( 2u \nabla \phi + (n - 1)H^\phi u^2 + II(\nabla z, \nabla z) + (2n - 2)Kuz \right) dA_\phi
$$

$$
+ \int_{\partial \Omega} V \eta \left( \left| \nabla z \right|^2 - (n - 1)Kz^2 \right) dA_\phi + \int_{\Omega} (n - 1) \left( K \nabla \phi V + nK^2 V \right) f^2 dv_\phi
$$

$$
+ \int_{\Omega} \left( \nabla^2 V - \nabla \phi V + (2n - 2)KV g + VRic \phi \right) \left( \nabla f, \nabla f \right) dv_\phi.$$
Lemma 3.2. Let $F \in \Gamma(T\Omega)$ be a Lipschitz vector field. Let $u \in H^2(\Omega)$ with $\mathbb{L}_\phi u = 0$ in $\Omega$. Then

$$\int_{\partial\Omega} \left( \frac{\partial u}{\partial \eta} \cdot g(F, \nabla u) - \frac{1}{2} |\nabla u|^2 g(F, \bar{\eta}) \right) dA_\phi = \int_{\Omega} \left( g(\nabla \nabla_u F, \nabla u) - \frac{1}{2} |\nabla u|^2 \text{div}_\phi F \right) dv_\phi, \quad (3.10)$$

where $\text{div}_\phi := \text{div} - g(\cdot, \nabla \phi)$ denotes the weighted divergence operator on $\Omega$, and other notations have the same meanings as before.

Proof. Since $\mathbb{L}_\phi u = 0$ in $\Omega$, we have

$$0 = \int_{\Omega} \mathbb{L}_\phi u \cdot g(F, \nabla u) dv_\phi = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \cdot g(F, \nabla u) dA_\phi - \int_{\Omega} g(\nabla \nabla_u F, \nabla u) dv_\phi - \int_{\Omega} |\nabla u|^2 \text{div}_\phi F dv_\phi. \quad (3.11)$$

By a direct calculation in an orthonormal local frame chosen for the tangent bundle $T\Omega$, one has

$$\nabla^2 u(F, \nabla u) = u_{ij}F_{,i}F_{,j} = (u_{ij}F_{,i}u_{j})_{,i} - u_{ij}F_{,i}u_{j} - u_{ij}F_{,j}u_{i} = \text{div}(|\nabla u|^2 F) - |\nabla u|^2 \text{div} F - \nabla^2 u(F, \nabla u).$$

Then, we infer from the integration by parts that

$$\int_{\Omega} \nabla^2 u(F, \nabla u) dv_\phi = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \cdot g(F, \bar{\eta}) dA_\phi + \frac{1}{2} \int_{\Omega} |\nabla u|^2 g(F, \nabla \phi) dv_\phi - \frac{1}{2} \int_{\Omega} |\nabla u|^2 \text{div}_\phi F dv_\phi = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \cdot g(F, \bar{\eta}) dA_\phi - \frac{1}{2} \int_{\Omega} |\nabla u|^2 \text{div}_\phi F dv_\phi. \quad (3.12)$$

Then (3.10) follows by substituting (3.12) into (3.11) directly. 

In order to prove Theorem 1.4, we need to choose a special function $V = \rho - \frac{c}{2}\rho^2$ in the Reilly-type formula, where $\rho = \text{dist}(\cdot, \partial\Omega)$ denotes the distance function to the boundary $\partial\Omega$, and the constant $c$ is the positive lower bound of the principal curvatures of $\partial\Omega$. Besides, we also need the following fact:

Lemma 3.2. (Proposition 10) Fix a neighborhood $U$ of Cut($\partial\Omega$) in $\Omega$, with Cut($\partial\Omega$) the cut-locus of points at the boundary $\partial\Omega$. Then for any $\varepsilon > 0$, there exists a smooth nonnegative function $V_\varepsilon$ on $\Omega$ such that $V_\varepsilon = V$ on $\Omega \setminus U$ and

$$\nabla^2 (-V_\varepsilon) \geq (c - \varepsilon)g.$$ 

Proof of Theorem 1.4. Since $V_\varepsilon|_{\partial\Omega} = V|_{\partial\Omega} = 0$ and $\nabla_\eta V_\varepsilon|_{\partial\Omega} = \nabla_\eta V|_{\partial\Omega} = -1$, then taking $V_\varepsilon$ into the Reilly-type formula (1.11) for $K = 0$ we have

$$-\int_{\Omega} V_\varepsilon |\nabla^2 f|^2 dv_\phi = -\int_{\partial\Omega} |\nabla z|^2 dA_\phi + \int_{\Omega} \left( \nabla^2 V_\varepsilon - \mathbb{L}_\phi V_\varepsilon g + V_\varepsilon \text{Ric}_\phi \right) (\nabla f, \nabla f) dv_\phi. \quad (3.13)$$
Taking $F = \nabla V_\varepsilon$ into the generalized Pohozaev identity (3.10), we have
\begin{equation}
\int_{\partial \Omega} |\nabla z|^2 dA_\phi = \int_{\partial \Omega} \left( \frac{\partial f}{\partial \eta} \right)^2 dA_\phi + \int_{\Omega} (2 \nabla^2 V_\varepsilon - \mathbb{I}_\phi V_\varepsilon g) (\nabla f, \nabla f) dv_\phi. \tag{3.14}
\end{equation}

Combining (3.13) and (3.14) results in
\begin{equation}
\int_{\partial \Omega} \left( \frac{\partial f}{\partial \eta} \right)^2 dA_\phi = \int_{\Omega} \left( - \nabla^2 V_\varepsilon + V_\varepsilon |\nabla^2 f|^2 + V_\varepsilon \text{Ric}^\phi \right) (\nabla f, \nabla f) dv_\phi. \tag{3.15}
\end{equation}

Putting the assumptions Sec$(\Omega) \geq 0$, $II > cI$ and $\nabla^2 \phi \geq 0$ into (3.15), and using Lemma 3.2, we can obtain
\begin{equation}
\int_{\partial \Omega} \left( \frac{\partial f}{\partial \eta} \right)^2 dA_\phi \geq (c - \varepsilon) \int_{\Omega} |\nabla f|^2 dv_\phi. \tag{3.16}
\end{equation}

Choosing furthermore $f$ to be an eigenfunction corresponding to $\sigma_1(\Omega)$, and then together with (1.13) and (3.16), it follows that
\begin{equation}
(\sigma_1(\Omega))^2 \int_{\partial \Omega} f^2 dA_\phi = \int_{\partial \Omega} \left( \frac{\partial f}{\partial \eta} \right)^2 dA_\phi \geq c \int_{\Omega} |\nabla f|^2 dv_\phi = c \sigma_1(\Omega) \int_{\partial \Omega} f^2 dA_\phi,
\end{equation}
which implies that $\sigma_1(\Omega) \geq c$. When $\sigma_1(\Omega) = c$, then by [23, Propositions 15 and 16], (3.15) and (3.16), we know that $\nabla^2 \phi = 0$ and $\Omega$ is isometric to an $n$-dimensional Euclidean ball of radius $\frac{1}{c}$. This finishes the proof of Theorem 1.4. 

\section*{Appendix A}

In this part, we would like to give a detailed explanation for the spectrum of (1.12), and this explanation is inspired by an argument shown in [1], pp. 2202-2204].

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and let $\mathcal{V}$ be another Hilbert space, which is embedded as a dense subspace in $\mathcal{H}$, so that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. Assume that $a$ is a closed, symmetric, real-valued, coercive quadratic form, i.e., for all $u \in \mathcal{V}$, the inequality
\begin{equation}
a(u) + w\|u\|^2_{\mathcal{H}} \geq \alpha\|u\|^2_{\mathcal{V}}
\end{equation}
holds for some $w \in \mathbb{R}$ and $\alpha > 0$. Let $\mathcal{D} : \mathcal{V} \rightarrow \mathcal{V}^*$ be the Dirichlet-to-Neumann map. Associated to $a$ is a bounded operator $A_1 : \mathcal{V} \rightarrow \mathcal{V}^*$. Also associated to $a$ is an unbounded self-adjoint operator $A_2$ on $\mathcal{H}$ with domain $\mathcal{D}(A_2) \subset \mathcal{V} \subset \mathcal{H}$. Therefore, one has:

- $x \in \mathcal{D}(A_1)$ and $A_1(x) = y \in \mathcal{V}^*$ if and only if $a(x,v) = \langle y,v \rangle$ for all $v \in \mathcal{V}$. 

It is not hard to know the operator $A_2$ is the part of $A_1$ in $\mathcal{D}(A_2)$, and so one can write either operator as $A$ and drop the subscript. Based on these preparations, one can obtain:

- The form $a$ is accretive (i.e., $a(u) \geq 0$ for all $u \in \mathcal{V}$) if and only if $A$ is nonnegative (i.e., $\langle Au, u \rangle \geq 0$ for all $u \in \mathcal{D}(A)$);
- $A$ has compact resolvent, and hence discrete spectrum, if and only if the inclusion $\mathcal{D}(A) \hookrightarrow \mathcal{H}$ is compact.

Clearly, if $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact, then one certainly has $\mathcal{D}(A) \hookrightarrow \mathcal{H}$ is compact and of course $A$ has the discrete spectrum.

In what follows, we prefer to use the above argument related to the Dirichlet-to-Neumann map $\mathcal{D}$ to show the eigenvalue problem (1.12) has discrete spectrum. In fact, one would see that:

- The eigenvalues of the eigenvalue problem of the drifting Laplacian (1.12) can be interpreted as the eigenvalues of the Dirichlet-to-Neumann operator $\mathcal{D}: \tilde{H}^{1/2}(\partial \Omega) \to \tilde{H}^{-1/2}(\partial \Omega)$ which maps a function $f \in \tilde{H}^{1/2}(\partial \Omega)$ to $\mathcal{D}f = \partial_\nu (H_wf) = \frac{\partial (H_wf)}{\partial n}$, where $H_wf$ is the weighted harmonic extension of $f$ to the interior of $\Omega$ (i.e., $\|\phi(H_wf) = 0$ in $\Omega$).

For the eigenvalue problem (1.12), since $\partial \Omega$ is smooth, $\partial \Omega$ is locally a smooth graph such that $\Omega$ lies locally on one side of $\partial \Omega$, and w.r.t. the weighted volume densities $dv_\phi$ and $dA_\phi$, one can define the Hilbert space $\tilde{L}^2(\Omega)$, the Sobolev space $\tilde{H}^1(\Omega)$ and $\tilde{L}^2(\partial \Omega)$ naturally. More precisely, $\tilde{L}^2(\Omega)$ is the set of functions defined over $\Omega$ and satisfying $\int_\Omega f^2 dv_\phi = \int_\Omega f^2 e^{-\phi} dv < \infty$, $\tilde{L}^2(\partial \Omega)$ is the set of functions defined over $\partial \Omega$ and satisfying $\int_{\partial \Omega} f^2 dA_\phi = \int_{\partial \Omega} f^2 e^{-\phi} dA < \infty$, and as mentioned before $\tilde{H}^1(\Omega) := \tilde{W}^{1,2}(\Omega)$ is the completion of the set of smooth functions $C^\infty(\Omega)$ under the weighted Sobolev norm $\| \cdot \|_{1,2}$. Clearly, one can naturally define an inner product $\langle \cdot, \cdot \rangle$ in $\tilde{L}^2(\Omega)$ as follows

$$\langle f, h \rangle = \int_\Omega fhdv_\phi,$$

which leads to the fact that $\tilde{L}^2(\Omega)$ becomes a Hilbert space. Similarly, $\tilde{H}^1(\Omega)$, $\tilde{L}^2(\partial \Omega)$ would be Hilbert spaces by suitably defining inner products. Let $C^\infty_0(\Omega)$ be the set of smooth functions (defined on $\Omega$) with compact support, and $\tilde{H}^1_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in $\tilde{H}^1(\Omega)$ under the weighted norm. The boundary restriction map $u \mapsto u|_{\partial \Omega} := \mathcal{R}_mu$ is well-defined for any $u \in H^1(\Omega) \cap C^0(\Omega)$, and this map can be extended to a bounded operator $\mathcal{R}_m: H^1(\Omega) \to \tilde{L}^2(\partial \Omega)$ with nullspace $\tilde{H}^1_0(\Omega)$. We have the following facts:

- Assume that $u \in \tilde{H}^1(\Omega)$. One says that $\mathcal{L}_0u \in \tilde{L}^2(\Omega)$ if there exists $f \in \tilde{L}^2(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla vdv_\phi = \int_\Omega fvdv_\phi \quad \text{for all } v \in \tilde{H}^1_0(\Omega).$$

\[3\]In fact, by checking the argument in Appendix A carefully, one would find that if the boundary $\partial \Omega$ is only Lipschitz, the eigenvalue problem (1.12) also has discrete spectrum.

\[4\]Here we have used a convention – for the Sobolev space $\tilde{W}^{k,p}(\cdot)$ w.r.t. the weighted volume density, if $p = 2$, we usually write $\tilde{H}^k(\cdot) = \tilde{W}^{k,2}(\cdot)$, $k = 0, 1, 2, \cdots$. 

• Assume that \( u \in \tilde{H}^1(\Omega) \) and \( \mathbb{J}_\phi u \in \tilde{L}^2(\Omega) \). One says that \( \partial_n u \in \tilde{L}^2(\partial\Omega) \) if there exists \( h \in \tilde{L}^2(\partial\Omega) \) such that
\[
\int_{\Omega} \left( \nabla u \cdot \nabla v - \mathbb{J}_\phi u \cdot \nabla v \right) dv = \int_{\partial\Omega} h \tau dA_{\phi} \quad \text{for all} \quad v \in \tilde{H}^1(\Omega),
\]
and then one writes \( \partial_n u = h \).

Here and later, for simplification, we often omit the trace signs under the integral, e.g., simply write \( \int_{\partial\Omega} h v dA_{\phi} = \int_{\partial\Omega} h v |_{\partial\Omega} dA_{\phi} \). Based on these facts, it is not hard to get the following Green’s formula:
\[
\int_{\Omega} \left( \nabla u \cdot \nabla v - \mathbb{J}_\phi u \cdot \nabla v \right) dv = \int_{\partial\Omega} (\partial_n u) \cdot \nabla v dA_{\phi}
\]
holds for all \( v \in \tilde{H}^1(\Omega) \) whenever \( u \in \tilde{H}^1(\Omega) \), \( \mathbb{J}_\phi u \in \tilde{L}^2(\Omega) \) and \( \partial_n u \in \tilde{L}^2(\partial\Omega) \).

For any \( \sigma \in \mathbb{R} \), consider the quadratic form
\[
b_{\sigma}(u,v) = \int_{\Omega} \nabla u \cdot \nabla v d\phi - \sigma \int_{\partial\Omega} u \cdot \tau dA_{\phi}, \quad \text{for} \quad u, v \in \tilde{H}^1(\Omega).
\]

By Green’s formula, one has
\[
b_{\sigma}(u,u) = \int_{\Omega} \nabla u \cdot \nabla u d\phi - \sigma \int_{\partial\Omega} u \cdot \tau dA_{\phi} = \int_{\Omega} (\mathbb{J}_\phi u) u d\phi + \int_{\partial\Omega} (\partial_n u) u dA_{\phi} - \sigma \int_{\partial\Omega} u^2 dA_{\phi}
\]
for \( u \in \tilde{H}^1(\Omega) \). Hence, the form \( b_{\sigma}(u,v) \) associated with the boundary value problem (1.12) should be coercive since \( b_{\sigma}(u,u) \geq 0 \) for \( u \in \tilde{H}^1(\Omega) \). So, this quadratic form \( b_{\sigma}(\cdot,\cdot) \) determines an operator \( (\mathbb{J}_\phi)_{\sigma} \), and letting \( v \in \tilde{H}^1_0(\Omega) \) shows that \( (\mathbb{J}_\phi)_{\sigma} \) should be the drifting Laplacian \( \mathbb{J}_\phi \) in the interior of \( \Omega \). Then we can infer that \( u \in \mathcal{D}( (\mathbb{J}_\phi)_{\sigma} ) \) implies \( \partial_n u = \sigma u \), at least in the weak sense. Therefore, one has
\[
\mathcal{D}( (\mathbb{J}_\phi)_{\sigma} ) = \left\{ u \in \tilde{L}^2(\Omega) \mid \mathbb{J}_\phi u \in \tilde{L}^2(\Omega), \partial_n u \text{ exists and } \partial_n u = \sigma \cdot u |_{\partial\Omega} \right\}.
\]

Since \( \tilde{H}^1(\Omega) \) is compactly included in \( \tilde{L}^2(\Omega) \), using the facts of Dirichlet-to-Neumann map shown at the first part of this section, one knows that the operator \( \mathbb{J}_\phi \) related to the eigenvalue problem (1.12) has discrete spectrum. Once we have this conclusion, the characterization (1.13) follows by using the standard argument of the variational method (see, e.g., [3, Section 5, Chapter I] for Rayleigh’s theorem and Max-min theorem from which one can get the characterizations for eigenvalues of the Laplacian of prescribed type).

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