In this paper we present quantum-like (QL) representation of the Shafir-Tversky statistical effect. We apply so called contextual approach. The Shafir-Tversky effect is considered as a consequence of combination of a number of incompatible contexts which are involved e.g. in Prisoner’s Dilemma or in more general games inducing the disjunction effect. As a consequence, the law of total probability is violated for experimental data obtained by Shafir and Tversky (1992) as well as Tversky and Shafir (1992). Moreover, we can find a numerical measure of contextual incompatibility (so called coefficient of interference) as well as represent contexts which are involved in Prisoner’s Dilemma (PD) by probability amplitudes – normalized vectors (“mental wave functions”). We remark that statistical data from Shafir and Tversky (1992) and Tversky and Shafir (1992) experiments differ crucially from the point of view of mental interference. The second one exhibits the conventional trigonometric (cos-type) interference, but the first one exhibits so called hyperbolic (cosh-type) interference. We discuss QL processing of information by cognitive systems, especially, QL decision making as well as classical and QL rationality and ethics.

Keywords: Tversky-Shafir and Shafir-Tversky experiments, law of total probability, trigonometric and hyperbolic interference.
probability, the sure thing principle, classical and nonclassical probabilistic models, mental contexts, contextual probability, mental interference, trigonometric and hyperbolic interference, interference of mental alternatives i Prisoner’s Dilemma

1 Introduction

The author really wish that this paper would be readable by psychologists, researchers working in cognitive science, sociology, economics. Therefore an extended introduction contains all basic ideas and methods of this paper. The corresponding (simplified) mathematical considerations are placed at the end of this paper, see section 7. On the other hand, one could not totally escape the use of mathematics, since problems under considerations are probabilistic and corresponding experiments, Tversky and Shafir [35] and Shafir and Tversky [36], are statistical experiments.

1.1 On applications of quantum formalism in psychology

Already Bohr pointed out [2] to the possibility to apply the mathematical formalism of quantum mechanics outside of physics, in particular, in psychology. The complementarity principle was considered as the starting point for application of the quantum formalism outside of physics. Originally Bohr borrowed this principle from psychology. Therefore he was sure that in turn the formalism corresponding to this principle could be applied to psychology. We also mention a correspondence between Pauli and Jung [19], [20] in the years 1932-1958.

Studies of psychologist Wright [38] on possibilities to apply the quantum formalism to macroscopic (in particular, cognitive systems) played an important role in understanding of the probabilistic structure of quantum mechanics. The work of D. Aerts and S. Aerts [1] stimulated applications of quantum probability to psychology. It influenced essentially the author of this paper. Quantum modelling in behavioral finances was performed by Choustova [6], [7] and Haven [16]. QL games approach to modelling of financial processes was performed by Piotrowski et al. [31], [32]. Danilov and Lambert-Mogiliansky [9], [10] applied quantum logic type calculus of
noncommutative actions to modelling of decision making, in particular, in economics.

We point out that the complementarity principle is a general philosophical principle. In applications to quantum physics it is quantatively exhibited through interference phenomenon for discrete variables, see Dirac [11]. In purely probabilistic terms interference can be represented as interference of probabilities of alternatives. Detailed analysis of this problem was performed in [23]–[26]. It was shown that interference of probabilities can be represented as violation of the law of total probability (also called the law of alternatives) which is widely used in classical statistics. This effect was confirmed (at least preliminary) experimentally by Conte et al. [23].

Recently a similar viewpoint to the role of the law of total probability was presented by Busemeyer et al. [24]–[26], see also [27], who described the well known disjunction effect (violating Savage STP [28]) by using the quantum formalism, see on this effect: Shafir and Tversky [29], [30] and also Rapoport [31], Hofstader [32], [33] and Groson [34].

1.2 Law of total probability and its violations

We recall this law in the simplest case of dichotomous random variables, $a = \pm$ and $b = \pm$, see e.g. wikipedia – the article “Law of total probability”:

$$P(b = \pm) = P(a = +)P(b = \pm|a = +) + P(a = -)P(b = \pm|a = -) \quad (1)$$

Thus the probability $P(b = \pm)$ can be reconstructed on the basis of conditional probabilities $P(b = \pm|a = \pm)$. This formula plays the fundamental role in modern science. Its consequences are strongly incorporated in modern scientific reasoning. It was a source of many scientific successes, but at the same time its unbounded application induced a number of paradoxes.

In [23]–[26] it was pointed out that the quantum formalism induces a modification of this formula. An additional term appears in the right hand side of (1), so called interference term. Violation of the law of total probability can be considered as an evidence that the classical probabilistic description

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1 Interference for continuous field-type variables is well known in classical physics.

2 “The prior probability to obtain the result e.g. $b = +$ is equal to the prior expected value of the posterior probability of $b = +$ under conditions $a = \pm$."

3 I think that the first paradox of this type was disagreement between classical and quantum physics.
could not be applied (or if it were applied, one could derive paradoxical conclusions). Our aim is to show that QL probabilistic descriptions could be applied. The terminology “quantum-like” and not simply “quantum” is used to emphasize that violations of (1) are not reduced to those which can be described by the conventional quantum model.

Contexts which are nonclassical (in the sense of violation of (1)), but at the same time cannot be described by the conventional quantum formalism may appear outside quantum physics. Nevertheless, the QL approach which was developed in [23]–[26] could be applied even for such contexts (neither classical nor quantum).

1.3 Mental contexts

What are the sources of violation of the law of total probability?

The most natural explanation can be provided in so called contextual probabilistic framework [23]–[26]. The basic notion of this approach is context. In quantum mechanics it is a complex of experimental physical conditions. In the present paper it will be a complex of mental conditions, see also [25]. In particular, we shall consider contexts corresponding to Prisoner’s Dilemma (PD) as well as contexts for Tversky and Shafir [36] gambling experiments. The crucial point is that probabilities in the law of total probability correspond to different contexts. A priori there is no reason to assume that all those (essentially different contexts) could be “peacefully combined.” Therefore in the contextual framework one could not use Boolean algebra for contexts. We recall that Boolean algebra is used in classical probability theory. It is important to remark that in the latter conditioning is considered not with respect to a context, but with respect to an event.

Roughly speaking violation of the law of total probability is not surprising. It is surprising that we were able to find so many situations (in particular, in classical statistical physics, psychology and economics) in which it can be applied and that we were lucky to proceed so far by using classical probability. The latter can be explained if we consider this law as an approximative law. If the additional term which should appear in the general case in the right-hand side of (1), the “interference term”, see section 5, is relatively small, then one could neglect by it and proceed by applying (1) without problem. In fact, the fundamental contribution of Tversky and Shafir [36], [33] is that they found statistical data which violates essentially the law of total probability.
Our contextual approach does not contradict Bayesian approach which nowadays is extremely popular in cognitive science and psychology. We just say that Bayesian analysis is an approximative theory. It has its domain of application. But (as any mathematical model) it has its boundaries of application. From our viewpoint the disjunction effect demonstrated that we have approached these boundaries.

Thus the formula of total probability which is the basis of Bayesian analysis is, in fact, not the precise equality (??), but it should be written as an approximative formula:

\[ P(b = \pm) \approx P(a = +)P(b = \pm|a = +) + P(a = -)P(b = \pm|a = -) \]  

(2)

### 1.4 Numerical measure of mental interference

In [23]–[26] an interference coefficient \( \lambda \) was introduced. It gives a measure of incompatibility of different contexts. It is important that this coefficient can be found numerically on the basis of experimental statistical data. Moreover, by using this coefficient one can construct a quantum-like (vector space) representation of contexts. Such a representation can be used e.g. in psychology or sociology.

Theoretical investigations of [23]–[26] demonstrated that the situation is even more complicated than one might expect. Besides the ordinary (well known) trigonometric cos-type interference (corresponding to the coefficient of interference bounded by one), there exist incompatible contexts producing so called hyperbolic cosh-interference (corresponding to the coefficient of interference larger than one). The latter type of probabilistic behavior could not be derived from the conventional quantum mechanics. Such a hyperbolic interference has been never observed for physical systems.

A cognitive experiment which demonstrated that cognitive systems (students) can behave in the QL way and produce nonzero coefficients of interference was performed [8]. It is interesting that contexts (corresponding to Gestalt ambiguity figures) used in this cognitive experiment produce the coefficients of interference (providing a numerical measure of incompatibility of these contexts) bounded by one. Thus this experiment on deviations of cognitive statistics from classical statistics demonstrated the presence of usual trigonometric interference – as in classical and quantum wave mechanics. Students behaved (with respect to recognition of Gestalt ambiguity figures) in the same way as photons (with respect e.g. to choices of slits in the two
slit experiment – diffraction of photons on two slits).

Can one hope to find the hyperbolic interference in cognitive experiments?

Intuitively there are no reasons to assume a priory that incompatibility of contexts could not be so large that the $\lambda$ would extend one. On the other hand, only the trigonometric interference has been always produced in experiments which have been done in classical and quantum physics. This as well as the result of [8] may induce opinion that the hyperbolic interference is a kind of a theoretical artifact.

1.5 Shafir-Tversky statistical effect

Recently it was pointed out in Busemeyer et al. [3]–[5] and Franco [12] that disjunction effects in cognitive sciences could be explained on the basis of the quantum model. In the present paper we shall continue their activity. We perform QL-modeling of disjunction effects.

We apply the apparatus of contextual probability [23]–[26] to find numerical characteristic – the coefficient of interference (of mental alternatives) for known experiments which demonstrated the violation of the sure thing principle (Savage [34]).

We shall use statistical data from Shafir and Tversky [35] and Tversky and Shafir [36]. We find coefficients of mental interference for these experiments. This will provide a possibility to represent mental states of players (mental contexts) by wave functions – in the abstract approach by normalized vectors of Hilbert space. We recall that in [23]–[26] an algorithm for such a representation was presented, Quantum-Like Representation Algorithm, QLRA.

We found not only that probabilistic behaviors are nonclassical in both experiments (this was already shown in Busemeyer et al. [3]–[5]), but that they differ essentially. We found that Tversky and Shafir [36] experiment produces the conventional trigonometric interference and consequently players behave under game-contexts in the same way as photons behave under contexts of the two slit experiment. Surprisingly Shafir and Tversky [35] experiment does not (!) produce the conventional trigonometric interference. It produces one interference coefficient which is larger than 1 – hyperbolic interference, and another which is less than 1 – trigonometric interference.

\footnote{“If you prefer to compete knowing that your opponent will compete and you prefer to compete knowing that your opponent will cooperate, then you should prefer to compete even when you do not know yours opponent choice.”}
Thus Shafir and Tversky [35] experiment produces the hyper-trigonometric interference!

This is the first experimental evidence of hyperbolic interference! And it was found not in physics, but in cognitive science.

1.6 Quantum-like thinking

As we pointed out, in [23]–[26] an algorithm, QLRA, for mapping probabilistic data into linear space of probability amplitudes was proposed. It represents contextual probabilities by wave functions (or normalized vectors of Hilbert space). We speculate, see also [27], that cognitive systems might develop (in the process of mental evolution) the ability to apply QLRA and to create QL-representations of mental contexts. Thus, instead of operating with probabilities and analyzing (even unconsciously) probabilities of various alternatives, the brain works directly with mental wave functions (probabilistic amplitudes).

Such a QL-processing of information has the following advantages:

a). This is consistent processing of incomplete information. The crucial point is that it is consistent information cut off. Therefore such a processing does not induce “information chaos”, especially under the assumption that all cognitive systems use the same QL-representation.

b). This is linear (vector space) processing of information. From the purely mathematical viewpoint one can consider this procedure as linearization of probabilistic representation of mental contexts. In particular, the mental wave function evolves linearly. Such an evolution is described by mental Schrödinger’s equation.5

We speculate that the biological evolution induced the QL-representation of information long before discovery of quantum mechanics by Planck, Einstein, Bohr, Heisenberg, Schrödinger, Dirac, von Neumann.

We also emphasize that our hypothesis on QL-processing of mental information has nothing to do with so called quantum reductionist theories, e.g. [14], [15], [29], [30]. By the latter processing of information by cognitive systems have some quantum features, because the brain (as any physical system) is composed of quantum particles. Yes, the brain is composed of

5Thus we guess that the brain was able to linearize the mental world via the QL-representation.
e.g. electrons, protons and photons, but this has nothing to do with QL-representation of mental contexts which is performed on the macro level.\footnote{We remark that neuronal models of QL-representation of mental information have not yet been developed. But we expect that our QL cognitive modeling may stimulate neurophysiological studies.}

1.7 Quantum-like decision making

If our hypothesis on QL-processing of information by cognitive systems is correct, then we should consider the QL-process of decision making. We recall that decision making is the cognitive process leading to the selection of a course of action among variations. Every decision making process produces a final choice.

By our model a cognitive system represents a mental context, say $C$, underlying decision making by a mental wave function, probabilistic (complex or even hyperbolic) amplitude $\psi_C$. This mental wave function evolves linearly in the Hilbert state space: $\psi_C(t)$. “Decision making operation” is represented by an observable, say $b$, taking values corresponding to different choices of action. Its value corresponding to the choice between alternatives is generated (by a classical random generator) with the probability given by the Born’s rule for the mental wave function $\psi_C(T)$, where $T$ is the instance of time corresponding to decision making.

On the basis of such a QL-representation this cognitive system selects a course of action among variations purely automatically (i.e., without applying the rule of reason based on the conventional Boolean logic) on the basis of a random generator reproducing the probability distribution of the QL-observable $b$ for the wave function $\psi_C(T)$. This probability distribution is given by Born’s rule.\footnote{We remark that in \cite{23}--\cite{26} Born’s rule was generalized even to QL models which differ from the conventional quantum model.}

To get the probability that an observable $b$ takes a fixed value, the brain should find the scalar product of the wave function $\psi_C(T)$ and the eigenvector corresponding this value. Finally, the absolute value of the result of this procedure should be squared.

Thus we assume that (at least some) cognitive systems have following QL-abilities:
a). To apply QLRA and to create the QL-representation of mental contexts: a context $C$ is mapped into its wave function $\psi_C$;

b). To generate dynamics of the mental wave function described by Schrödinger equation;

c). To represent “decision making observables” by linear operators;

d). To apply Born’s rule and to create random generators for probability distributions based on this rule.

As was already pointed out in footnote 8, the QL-representation is essentially more general than the conventional quantum representation. For example, some mental contexts might be represented not by complex probability amplitudes, but by hyperbolic (or even mixed hyper-trigonometric) amplitudes.

### 1.8 Quantum-like superposition of choices

We remark that QL decision making also includes the QL-dynamics of the mental state $\psi_C$. Of course, in the same way as in the conventional quantum mechanics by making a concrete choice among alternatives a cognitive system disturbs the QL-evolution which is described (at least approximately) by Schrödinger’s equation.

One could say that “collapse of the mental wave function” occurs at the instant of time $t = T$. In opposite to the conventional Copenhagen interpretation, we do not take collapse too seriously. In our model the $\psi_C$-function is simply a special linear space representation of probabilistic data about the context $C$. In the process of decision making the (self) measurement of a decision maker $b$ is realized in purely classical way. It is assumed existence (in the brain) of a random generator which produces possible values of $b$ with probabilities given by Born’s rule. Let e.g. $b$ take two values. These are two alternative decisions: +1, yes, or -1, no. Then the mental wave function

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8Since decision’s spectrum consists of discrete alternatives, it is enough to operate in finite dimensional linear spaces, i.e., with matrices. In quantum mechanics observables are represented by self-adjoint operators, i.e., by symmetric matrices. However, we speak not only about conventional quantum representation of cognitive entities, but about QL-representation which is based on the contextual approach. As was found in \[23\]–\[26\], contextual probabilistic setups could violate not only the classical probabilistic laws, for example, the law of total probability, but even the conventional quantum laws. For example, it might happen that a mental observable could not be represented by a symmetric matrix.
and the decision maker determine two probabilities, \( p_+ \) and \( p_- \). The values \( b = +1 \) and \( b = -1 \) appear randomly with these probabilities.

Suppose that a cognitive system should make the \( b \)-decision. This system runs the above random generator. It takes the value \( b = +1 \). At this moment the Schrödinger evolution is stopped. It starts again with a new initial mental wave function which is equal to the eigenvector corresponding to the value \( b = +1 \). In accordance with quantum terminology we can say that during the period \( 0 \leq t \leq T \) the brain’s mental state was in the superposition of two states \( b = +1 \) and \( b = -1 \).

In section 1.9 we shall consider more complicated process: a new context can be formed and represented by its own mental wave function. Evolution may start with it and not with the eigenvector corresponding to the previous decision.

In general a mental context \( C \) can be created not specially for making the \( b \)-decision. Decision tasks can come later. Suppose that the brain has a collection of decision makers (self-observables) \( a, b, \ldots \).\(^9\) The mental wave function \( \psi_C(t) \) can be considered (by the conventional quantum terminology) as being in superposition of all possible values for any observable. If the cognitive system should make the \( b \)-decision, then the \( b \)-superposition is reduced to a single value, e.g. \( b = +1 \). Suppose that operators (matrices) representing observables \( a \) and \( b \) do not commute. Then the eigenvector of \( b \) for the value \( b = +1 \) need not be at the same time an eigenvector for \( a \). Hence, after taking the decision \( b = +1 \) the brain’s state is still in superposition of all possible values for \( a \).

Although we use the same terminology as in quantum mechanics, states’ superposition, its interpretation is totally different from the conventional one. Therefore we prefer to speak about QL superposition of mental states and not quantum superposition. The first is described in purely classical terms (even Schrödinger’s dynamics can be easily simulated by classical neural network). Therefore it can be exhibited by macroscopic systems. The original quantum superposition is “real superposition” of e.g. two energy levels. It is not clear how it might be realized for macroscopic systems. The model of the brain operating with quantum superpositions of minds is very old. It was proposed by quantum logician Vladimir Orlov \(^{28} \) (in fact, a few years earlier, \( ^9 \)It may be better to consider “activated decision makers”. The total number of possible decision makers can be essentially larger. However, majority of them are in the “sleeping state.”
but it took time to transfer the manuscript from a concentration camp for
deceased). Similar model was considered by Stuart Hameroff [14], [15] and
Roger Penrose [29], [30]. But they understood well the problem, see e.g.
Roger Penrose [30]: “It is hard to see how one could usefully consider a quantum
superposition consisting of one neuron firing, and simultaneously nonfiring.”

1.9 Parallelism in creation and processing of mental
function

It is clear that the brain cannot operate for a long time starting with some
context $C$. A series of Schrödinger’s evolutions and “state updating” after
decision making can be stopped as a consequence of creation of a new mental
context $C'$ induced by new external and internal signals. This context is
represented by its own mental wave function $\psi_{C'}$ which evolves linearly in
the Hilbert state space. The process of decision making and state updating
is repeated starting with $\psi_{C'}$.

If the brain’s evolution was done properly from the point of view of the
information processing architecture, then it is natural to assume that creation
of a new context and its QL representation can go in parallel to processing,
decision making and state updating based on the previous context $C$.

We consider two domains of the brain, classical and QL. In principle, each
domain can be distributed through the brain (for example, if the neural basis
is given by the frequency domain representation).

In the classical domain a probabilistic image of a mental context $C$ is
created\(^\text{10}\). Then these contextual probabilities are represented by the mental
wave function.

This mental wave function is processed in the QL domain: Schrödinger’s
evolution, measurement, updating, and so on.

The classical domain does not “sleep” meanwhile. It works with a new
context, say $C'$. Its amplitude representation will be transferred to the QL
domain later.

There should be a king of control center coupling consistently functioning
of these two domains. In particular, it should control consistency of time

\(^{10}\)As was pointed out in [23], [27], probabilities may be generated by counting frequencies
of neural firings. However, such a model is just a possible candidate for the neuronal
basis. In any event extended neurophysiological investigations should be performed to
find mechanism of neural creation of the QL representation and dynamics as well as self-
measurements in the process of decision making.
scales for state preparation and decision making. On the one hand, the brain saves a lot of computational resources by working only in the QL domain. Here dynamics is linear – in opposite to essentially nonlinear dynamics in the classical domain of the brain. However, new signals change mental context and it should be updated (in the classical domain).

1.10 Quantum-like rationality

If one defines rational behavior on the basis of the law of total probability, then QL-behaviors would be really irrational, see section 2 on rational behavior, PD and so on. However, the only reason for such an interpretation is common application of the law of total probability in modern statistics. Under the assumption that cognitive systems make decisions via the QL decision making procedure, violation of “Boolean rationality” does not look surprising. One must be essentially more surprised that modern science (including economy and finances) was able to proceed so far on the basis of assumptions based on classical “Boolean rationality.”

Therefore one should consider deviations from “Boolean rationality” not as evidences of irrational behavior, but as evidences that cognitive systems are QL-rational.

We point out to another source of QL-rationality. Besides advantages of QL-processing of incomplete information, see section 1.6, we mention presence of social pressure to proceed in the QL-way. If society consists of QL-thinking cognitive systems, then any individual should use the QL-reasoning to proceed consistently with respect to other members of such a QL-society. An individual who tries to use essentially more detailed description of mental contexts and who tries to build classical-like complete representation of contexts could make decisions which are in fact “more rational” (from the point of view of complete information processing). However, such an individual might be rejected by the QL-society.

1.11 Quantum-like ethics

We remark that “nonconsequential reasoning” was studied a lot in cognitive psychology, e.g. Rapoport [33], Hofstader [17], [18], Tversky and Shafir [36], [35]. However, from the QL point of view such a reasoning is not nonconsequential at all. It is consequential, but consequences are taken into account in the QL-representation. For example, preference of cooperative, ethical
decisions in PD is consequential, but from the viewpoint of QL probability. Hence, human ethics is fact a consequence of the QL-representation of mental contexts. If we were involved in purely classical probabilistic reasoning (based on classical Bayesian analysis), we would not be able to demonstrate such a “nonconsequential behavior” as in PD. We would behave as “cognitive automata” (as creations of AI). The essence of human behavior is the presence of the QL-representation of probabilistic reality. Cooperation may arise simply because the mental wave function produces (via Born’s rule) larger probabilities for cooperative actions.

In the absence of decision making the mental wave function evolves according to Schrödinger’s equation. The generator of evolution is represented by a special QL observable – “mental Hamiltonian”, describing a mental analogue of energy, see [21], [22] for details.

We guess that human beings have mental Hamiltonians such that they produce “ethical wave functions”, \( \psi_C(T) \), starting with a large variety of \( \psi_C \). Creation of such “ethical mental Hamiltonian” is a consequence of influence of social environment already in childhood. We could not exclude that some terms of “ethical mental Hamiltonian” are encoded in genom.

2 Rational behavior, Prisoner’s Dilemma

In game theory, PD is a type of non-zero-sum game in which two players can cooperate with or defect (i.e. betray) the other player. In this game, as in all game theory, the only concern of each individual player (prisoner) is maximizing his/her own payoff, without any concern for the other player’s payoff. In the classic form of this game, cooperating is strictly dominated by defecting, so that the only possible equilibrium for the game is for all players to defect. In simpler terms, no matter what the other player does, one player will always gain a greater payoff by playing defect. Since in any situation playing defect is more beneficial than cooperating, all rational players will play defect.

The classical PD is as follows: Two suspects, A and B, are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal: if one testifies for the prosecution against the other and the other remains silent, the betrayer goes free and the silent accomplice receives the full 10-year sentence. If both stay silent, both prisoners are sentenced to only six months
in jail for a minor charge. If each betrays the other, each receives a two-year sentence. Each prisoner must make the choice of whether to betray the other or to remain silent. However, neither prisoner knows for sure what choice the other prisoner will make. So this dilemma poses the question: How should the prisoners act? The dilemma arises when one assumes that both prisoners only care about minimizing their own jail terms. Each prisoner has two options: to cooperate with his accomplice and stay quiet, or to defect from their implied pact and betray his accomplice in return for a lighter sentence. The outcome of each choice depends on the choice of the accomplice, but each prisoner must choose without knowing what his accomplice has chosen to do. In deciding what to do in strategic situations, it is normally important to predict what others will do. This is not the case here. If you knew the other prisoner would stay silent, your best move is to betray as you then walk free instead of receiving the minor sentence. If you knew the other prisoner would betray, your best move is still to betray, as you receive a lesser sentence than by silence. Betraying is a dominant strategy. The other prisoner reasons similarly, and therefore also chooses to betray. Yet by both defecting they get a lower payoff than they would get by staying silent. So rational, self-interested play results in each prisoner being worse off than if they had stayed silent, see e.g. wikipedia – “Prisoner’s dilemma.”

This is the principle of rational behavior which is basic for rational choice theory which is the dominant theoretical paradigm in microeconomics. It is also central to modern political science and is used by scholars in other disciplines such as sociology. However, Shafir and Tversky [35] found that players frequently behave irrationally.

3 Contextual analysis of Prisoner’s Dilemma

Each contextual model is based on a collection of contexts and a collection of observables. Such observables can be measured\footnote{By measurements we understand even self-measurements which are performed by e.g. the brain.} for each of contexts under consideration, see [26] for the general formalism. The following mental contexts are involved in PD:

Context $C$ representing the situation such that a player has no idea about planned action of another player.
Context $C^A_\pm$ representing the situation such that the $B$-player supposes that $A$ will cooperate and context $C^A_- - A$ will compete. We can also consider similar contexts $C^B_{\pm}$.

We define dichotomous observables $a$ and $b$ corresponding to actions of players $A$ and $B$: $a = +$ if $A$ chooses to cooperate and $a = -$ if $A$ chooses to compete, $b$ is defined in the same way.

A priori the law of total probability might be violated for PD, since the $B$-player is not able to combine contexts. If those contexts were represented by subsets of a so called space of “elementary events” as it is done in classical probability theory (based on Kolmogorov (1933) measure-theoretic axiomatics), the $B$-player would be able to consider the conjunction of the contexts $C$ and e.g. $C^A_+$ and to operate in the context $C \land C^A_+$ (which would be represented by the set $C \cap C^A_+$). But the very situation of PD is such that one could not expect that contexts $C$ and $C^A_\pm$ might be peacefully combined. If the $B$-player obtains information about the planned action of the $A$-player (or even if he just decides that $A$ will play in the definite way, e.g. the context $C^A_+$ will be realized), then the context $C$ is simply destroyed. It could not be combined with $C^A_+$. We can introduce the following contextual probabilities:

$$P(b = \pm | C)$$ – probabilities for actions of $B$ under the complex of mental conditions $C$.

$$P_{\pm, +} \equiv P(b = \pm | C^A_+)$$ and $$P_{\pm, -} \equiv P(b = \pm | C^A_-)$$ – probabilities for actions of $B$ under the complexes of mental conditions $C^A_+$ and $C^A_-$, respectively.

$$P(a = \pm | C)$$ – priori probabilities which $B$ assigns for actions of $A$ under the complex of mental conditions $C$.

As we pointed out, there are no priori reasons for the equality (1) to hold. And experimental results of Shafir and Tversky [35] demonstrated that this equality could be really violated, see Busemeyer et al. [3].

By Shafir and Tversky [35] for PD experiment we have:

$$P(b = -| C) = 0.63$$ and hence $$P(b = +| C) = 0.37;$$

$$P_{-, -} = 0.97, P_{+, -} = 0.03; \quad P_{-, +} = 0.84, P_{+, +} = 0.16.$$  

As always in probability theory it is convenient to introduce the matrix of transition probabilities

$$P = \begin{pmatrix} 0.16 & 0.84 \\ 0.03 & 0.97 \end{pmatrix}. $$
We point out that this matrix is stochastic. It is a square matrix each of whose rows consists of nonnegative real numbers, with each row summing to 1. This is the common property of all matrices of transition probabilities.

We now recall the definition of a doubly stochastic matrix: in a doubly stochastic matrix all entries are nonnegative and all rows and all columns sum to 1. It is clear that the matrix obtained by Shafir and Tversky is not doubly stochastic.

In the simplified framework the prisoner $B$ considers (typically unconsciously) priory probabilities $p = P(a = +|C)$ and $1 - p = P(a = -|C)$ which $B$ assigns for actions of $A$ under the complex of mental conditions $C$. These probabilities are parameters of the model. In the simplest case $B$ assigns some fixed value $p$ to $A$-cooperation. The mental wave function depends on $p$.

However, in reality the situation is essentially more complicated. The $B$ is not able to determine precisely $p$. He considers a spectrum of possible $p$ which might be assigned to $A$-cooperation. Therefore, instead of a pure QL-state (mental wave function), the $B$-brain creates a statistical mixture of mental wave functions corresponding to some range of parameters $p$ which could be assigned to $A$-cooperation. In this statistical mixture different wave functions are mixed with some weights. Instead of the wave function, $B$ creates a von Neumann density matrix which describes $B$’s state of mind. We emphasize that the latter operation of statistical mixing is purely classical. The crucial step is creation of the QL-representation for fixed value of the parameter $p$.

4 Contextual analysis for Tversky and Shafir gambling experiment

Tversky and Shafir [36] proposed to test disjunction effect for the following gambling experiment. In this experiment, you are presented with two possible plays of a gamble that is equally likely to win 200 USD or lose 100USD. You are instructed that the first play has completed, and now you are faced with the possibility of another play.

Here a gambling device, e.g., roulette, plays the role of $A$; $B$ is a real player, his actions are $b = +$, to play the second game, $b = -$, not. Here the context $C$ correspond to the situation such that the result of the first game is unknown for $B$; the contexts $C_{\pm}^A$ correspond to the situations such that
the results $a = \pm$ of the first play in the gamble are known. From Tversky and Shafir \cite{36} we have:

$$P(b = +|C) = 0.36$$

and hence

$$P(b = -|C) = 0.64;$$

$$P_{+, -} = 0.59, \quad P_{-, -} = 0.41; \quad P_{+, +} = 0.69, \quad P_{-, +} = 0.31.$$ 

We get the following matrix of transition probabilities:

$$P = \begin{pmatrix}
0.69 & 0.31 \\
0.59 & 0.41
\end{pmatrix}.$$ 

This matrix of transition probabilities is neither (cf. Shafir-Tversky \cite{35}) doubly stochastic.

In this experiment (in contrast to Shafir-Tversky \cite{35}) probabilities $P(a = \pm|C)$ are not subject of a priori consideration. They are fixed from the very beginning as $1/2$.

5 Coefficient of interference (incompatibility)

Violation of the law of total probability implies that the left-hand and right-hand sides of (1) do not coincide. Therefore it is natural to consider the difference between them as a measure of incompatibility between contexts $C$ and $C_A^\pm$. We denote it by the symbol $\delta^{\pm}$. It is the measure of impossibility to combine these contexts in a single space of elementary events. In PD $C$ can be called uncertainty context – $B$ has no information about planned actions of $A$. This context is incompatible with the contexts $C_A^\pm$ corresponding to definite actions of $A$. We propose to measure this incompatibility numerically by using $\delta$. This number can be found if one have all probabilities involved in the law of total probability.

The next important question is the choice of normalization of $\delta$. Here we proceed in the following way, see \cite{23}. We are lucky that quantum mechanics has been already discovered. Its formalism implies \cite{11} that for quantum systems (e.g. photons) this coefficient of incompatibility has the form $2\cos\theta$ (where the angle $\theta$ is called phase) multiplied by the normalization factor which is equal to square root of the product $\Pi$ of all probabilities in the right-hand side of (1). Thus

$$\delta = 2\cos\theta \sqrt{\Pi}.$$
We proposed to use the same normalization in the general case of any collection of contextual probabilities. Thus we introduce the normalized coefficient of incompatibility of mental contexts:

$$\lambda = \frac{\delta}{2 \sqrt{\Pi}}.$$ 

As was mentioned, in the conventional quantum mechanics it is always bounded by one. Hence, it can be written as $\lambda = \cos \theta$, where $\theta = \arccos \lambda$.

However, as was found in [26], it could as well be larger than one. In such a case it can be written as $\lambda = \pm \cosh \theta$, where $\theta = \arccosh |\lambda|$.

Since in the conventional quantum mechanics the term $\delta = 2 \cos \theta \sqrt{\Pi}$ describes interference, we can call $\delta$ the interference term even in the general contextual framework. The same terminology we use for the normalized coefficient $\lambda$: the coefficient of interference. It can be considered as a measure of “interference of mental contexts.”

6 Coefficients of interference for disjunction experiments

Since in the Tversky and Shafir [36] gambling experiment the $A$-probabilities are fixed, it is easier for investigation. Simple arithmetic calculations give $\delta_+ = -0.28$, and hence $\lambda_+ = -0.44$. Thus the probabilistic phase $\theta_+ = 2.03$. We recall [23] that $\delta_+ + \delta_- = 0$ (in the general case). Thus $\delta_- = 0.28$, and hence $\lambda_- = 0.79$. Thus the probabilistic phase $\theta_- = 0.66$.

In the case of Shafir and Tversky [35] PD-experiment the $B$-player assigns probabilities of the $A$-actions, $p$ and $1 - p$ (in the simplest case). Thus coefficients of interference depend on $p$. We start with $\delta_- = -(0.21 + 0.13p)$ and $\lambda_- = -(0.12 + 0.07p)/\sqrt{p(1-p)}$. For example, if $B$ would assume that $A$ will act randomly with probabilities $p = 1 - p = 1/2$, then the interference between contexts is given by $\lambda_- = -0.31$ and hence the phase $\theta = 1.89$.

We now find $\delta_+ = (0.21 + 0.13p)$ and $\lambda_+ = -(1.52 + 0.94p)/\sqrt{p(1-p)}$. For example, if $B$ would assume that $A$ will act randomly with probabilities $p = 1 - p = 1/2$, then the interference between contexts is given by $\lambda_+ = 3.98$. Thus interference is very high. It exceeds the possible range of

\[\text{We remark that in general we could expect neither classical nor conventional quantum probabilistic behaviors.}\]
the conventional trigonometric interference. This is the case of hyperbolic interference! Here the hyperbolic phase $\theta_+ = \text{arccosh}(3.98) = 2.06$.

7 Quantum-like representation algorithm - QLRA

This algorithm will produce a probability amplitude from contextual probabilities. We shall consider separately two cases:

7.1 Trigonometric mental interference

The coefficients of interference are bounded by one.

In this case we can represent $\lambda_\pm$ in the form $\lambda_\pm = 2 \cos \theta_\pm \sqrt{\Pi}$. Hence we obtain the following modification of the law of total probability:

$$P(b = \pm) = P(a = +)P_{\pm,+} + P(a = -)P_{\pm,-} + 2 \cos \theta_\pm \sqrt{\Pi}, \quad (3)$$

where $\Pi_\pm = P(a = +|C)P(a = -|C)P_{\pm,+}P_{\pm,-}$. In a special case – for a doubly stochastic matrix of transition probabilities – this law can be derived in the conventional quantum formalism.

We now recall elementary formula from algebra of complex numbers:

$$k = k_1 + k_2 + 2\sqrt{k_1k_2} \cos \theta = |\sqrt{k_1} + e^{i\theta} \sqrt{k_2}|^2,$$

for real numbers $k_1, k_2 > 0, \theta \in [0, 2\pi]$. Thus

$$k = |\psi|^2,$$

where $\psi = \sqrt{k_1} + e^{i\theta} \sqrt{k_2}$.

Let us compare this formula and the interference law of total probability \(3\). We set $k = P(b = \pm), k_1 = P(a = +)P_{\pm,+}, k_2 = P(a = -)P_{\pm,-}$. We introduce the complex probability amplitudes:

$$\psi(\pm) = \sqrt{P(a = +)P_{\pm,+} + e^{i\theta\pm} \sqrt{P(a = -)P_{\pm,-}}}.$$

We call its mental wave function (it is defined on the set \{+, −\} and takes complex values) representing the context $C$ via observables $a$ and $b$.

The crucial point is that Born’s rule takes place:

$$P(b = \pm) = |\psi(\pm)|^2.$$

We speculate that the brain can apply such an algorithm to probabilistic data about contexts and construct the complex probability amplitude, the mental wave function. Then it operates only with such amplitudes and not with original probabilities.
7.2 Hyperbolic mental interference

The coefficients of interference are larger than one.

Here mathematics is more complicated. One should use so called hyperbolic numbers, instead of complex numbers. We would not like to go in mathematical details. We just mention that one should change everywhere the imaginary unit $i$ (such that $i^2 = -1$) to hyper-imaginary unit $j : j^2 = +1$ and usual trigonometric functions $\cos \theta$ and $\sin \theta$ to their hyperbolic analogues $\cosh \theta$ and $\sinh \theta$, see e.g. [26] for details. Here the probabilistic image of incompatible mental contexts is given by the hyperbolic probabilistic amplitude:

$$\psi(\pm) = \sqrt{P(a = +)P_{\pm,+} \pm e^{j\theta}\sqrt{P(a = -)P_{\pm,-}}}.$$ 

Finally, we remark that some cognitive systems may exhibit (for some mental contexts) hyper-trigonometric interference: one coefficient, e.g., $\lambda_+$ is bounded by one and another is larger than one.

7.3 Quantum-like representation for Tversky-Shafir gambling experiment

This experiment has simpler QL-representation. Both coefficients of interference are bounded by one. Thus we can represent incompatible contexts by the complex probability amplitude:

$$\psi(+) \approx 0.59 + e^{2.03i} 0.54; \quad \psi(-) \approx 0.39 + e^{0.79i} 0.45.$$ 

7.4 Quantum-like representation for Shafir-Tversky PD experiment

Here the $B$-player creates QL-representation by assigning the probabilities $p$ and $1 - p$ to possible actions of $A$. The wave function depends on $p$. For example, suppose that $B$ assigned to the $A$-actions equal probabilities. Then the $B$-brain would represent the PD game by the following hyper-trigonometric amplitude:

$$\psi(+) \approx 0.28 + e^{2.06j} 0.12; \quad \psi(-) \approx 0.65 + e^{1.89i} 0.7$$
8 Non-doubly stochasticity of matrices of transition probabilities in cognitive science

We have seen that matrices of transition probabilities which are based on experimental data of Tversky-Shafir Game and Shafir-Tversky PD experiments are not doubly stochastic. The same is valid for the matrix obtained in the Bari-experiment [8]. On the other hand, matrices of transition probabilities that should be generated by the conventional quantum mechanics in the two dimensional Hilbert space are always doubly stochastic, see [37].

We can present two possible explanations of this “non-doubly stochasticity paradox”:

a). Statistics of these experiments are neither classical nor quantum (i.e., neither the Kolmogorov measure-theoretic model nor the conventional quantum model with self-adjoint operators could describe this statistics).

b). Observables corresponding to real and possible actions are not complete. From the viewpoint of quantum mechanics this means that they should be represented not in the two dimensional (mental qubit) Hilbert space, but in Hilbert space of a higher dimension.  

Personally I would choose the a)-explanation (and not simply because it was my own). It seems that actions of A and B in the PD do not have a finer QL-representation which would be natural with respect to the QL-machinery of decision making.

Of course, there are many brain-variables which are involved in the PD decision making. However, the essence of creation of a QL representation is selection of the most essential variables. Other variables should not be included in the chosen (for a concrete problem) QL representation.

Nevertheless, we could not ignore completely the incompleteness conjecture of Busemeyer and Lambert-Mogiliansky. We would immediately meet a really terrible problem: “How can we find the real dimension of the quantum (or QL) state space?” So, if this dimension is not determined by values of complementary observables a and b, then we should be able to find an answer to the question: “Which are those additional mental observables which could complete the model?” One should find complete families of observables.

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13This latter possibility was pointed to me by Jerome Busemeyer and Ariane Lambert-Mogiliansky during the recent workshop “Can quantum formalism be applied in psychology and economy?” (Int. Center for Math. Modeling in Physics and Cognitive Sciences, University of Växjö, Sweden; 17-18 September, 2007).
\(u_1^a, \ldots, u_m^a\) and \(u_1^b, \ldots, u_m^b\) compatible with \(a\) and \(b\), respectively.

We remark that in the case of the hyperbolic interference we would not be able to solve the “non-doubly stochasticity paradox” even by going to higher dimensions.

My conjecture (similar ideas also were presented by Luigi Accardi and Dierk Aerts, at least in conversations with me and our Email exchange) is that the laws of classical probability theory can be violated in cognitive sciences, psychology, social sciences and economy. However, nonclassical statistical data is not covered completely by the conventional quantum model.

My personal explanation is based on the evidence [23] that violation of the formula of total probability does not mean that we should obtain precisely the formula of total probability with the interference term which is derived in the conventional quantum formalism.

Nevertheless, the conventional quantum formalism can be used as the simplest nonclassical model for mental and social modelling.

**Conclusion.** By using violation of the law of total probability as the starting point we created the QL-representation of mental contexts. As was pointed out by Busemeyer et al., violation of the law of total probability can be used to explain disjunction effect. Therefore the QL-representation can be applied for description of this effect. The essence of our approach is the possibility to introduce a numerical measure of disjunction, so called interference coefficient. In particular, we found interference coefficients for statistical data from Shafir–Tversky and Tversky–Shafir experiments coupled to Prisoner’s Dilemma. We also represent contexts of these experiments by QL probability amplitudes, “mental wave functions.” We found that, besides the conventional trigonometric interference (Tversky–Shafir [36]) in cognitive science can be exhibited so called hyperbolic interference - Shafir and Tversky [35]. Thus the probabilistic structure of cognitive science is not simply nonclassical (cf. [3], [3]), but it is even essentially richer than the probabilistic structure of quantum mechanics.

**References**

[1] Aerts, D. and Aerts, S. (1995). Applications of quantum statistics in psychological studies of decision-processes. *Foundations of Science* 1, 1-12.
[2] Bohr, N. (1987). *The philosophical writings of Niels Bohr*, 3 vols. Woodbridge, Conn.: Ox Bow Press.

[3] Busemeyer, J. B., Wang, Z. and Townsend, J. T. (2006). Quantum dynamics of human decision making, *J. Math. Psychology*, **50**, 220-241.

[4] Busemeyer, J. B. and Wang, Z. (2007). Quantum information processing explanation for interactions between inferences and decisions. In P. D. Bruza, W. Lawless, K. van Rijsbergen, D. A. Sofge (Eds). *Quantum interaction, AAAI Spring Symposium*, Technical Report SS-07-08 (pp. 91-97). Menlo Park, CA: AAAI Press.

[5] Busemeyer, J. R., Matthew, M., Wang, Z. (2006). A Quantum Information Processing Theory Explanation of Disjunction Effects. *Proceedings of the Cognitive Science Society*.

[6] Choustova, O. A. (2004). Bohmian mechanics for financial processes. *J. Modern Optics*, **51**, 1111.

[7] Choustova, O.A. (2006). Quantum bohmian model for financial market. *Physica A* **374**, 304–314.

[8] Conte, E., Todarello, O., Federici, A., Vitiello, F., Lopane, M., Khrennikov, A. Yu and Zbilut, J. P. (2006). Some remarks on an experiment suggesting quantum-like behavior of cognitive entities and formulation of an abstract quantum mechanical formalism to describe cognitive entity and its dynamics. *Chaos, Solitons and Fractals*, **31**, 1076-1088.

[9] Danilov, V. I. and Lambert-Mogiliansky, A. (2006). Non-classical expected utility theory *Preprint Paris-Jourdan Sc. Economiques*.

[10] Danilov, V. I. and Lambert-Mogiliansky, A. (2006). Non-classical measurement theory: a framework for behavioral sciences [arXiv:physics/0604051].

[11] Dirac, P. A. M. (1930). *The Principles of Quantum Mechanics*, Oxford: Oxford Univ. Press.

[12] Franco, R. (2007). Quantum mechanics, Bayes’ theorem and the conjunction fallacy. [http://www.arxiv.org/abs/quant-ph/0703222](http://www.arxiv.org/abs/quant-ph/0703222).
[13] Groson, R. (1999). The disjunction effect and reasoning-based choice in games. *Organizational Behavior and Human Decision Processes*, **80**, 118-133.

[14] Hameroff, S. (1994). Quantum coherence in microtubules. A neural basis for emergent consciousness? *J. of Consciousness Studies*, **1**, 91-118.

[15] Hameroff, S. (1994). Quantum computing in brain microtubules? The Penrose-Hameroff Orch Or model of consciousness. *Phil. Tr. Royal Sc., London A*, 1-28,

[16] Haven, E. (2006). Bohmian mechanics in a macroscopic quantum system. In A. Yu. Khrennikov (Ed), *Foundations of Probability and Physics-3*. American Institute of Physics, Melville, New York, **810**, 330-340.

[17] Hofstader, D. R. (1983). Dilemmas for superrational thinkers, leading up to a luring lottery. *Scientific American*, June.

[18] Hofstader, D. R. (1985). *Metamagical themas: Questing for the essence of mind and pattern*. New York: Basic Books.

[19] Jung, C.G. and Pauli, W. (1992). In C. A. Meier (Ed), *Ein Briefwechsel*. (pp. 679-702). Berlin: Spinger.

[20] Jung, C.G. and Pauli, W. (1952). Natureklarung und Psyche. Zurich: Rascher Verlag.

[21] Khrennikov, A. Yu. (1999). Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social and anomalous phenomena. *Foundations of Physics*, **29**, 1065-1098.

[22] Khrennikov, A. Yu. (2002). *Classical and quantum mental models and Freud’s theory of unconscious mind*. Växjö: Växjö Univ. Press.

[23] Khrennikov, A. Yu. (2004). *Information Dynamics in Cognitive, Psychological and Anomalous Phenomena*. Dordreht: Kluwer Academic.

[24] Khrennikov, A. Yu. (2005). Interference in the classical probabilistic framework. *Fuzzy Sets and Systems*, **155**, 4-17.

[25] Khrennikov, A. Yu. (2006). Quantum-like brain: Interference of minds. *BioSystems*, **84**, 225-241.
[26] Khrennikov, A. Yu. (2005). The principle of supplementarity: A contextual probabilistic viewpoint to complementarity, the interference of probabilities, and the incompatibility of variables in quantum mechanics. *Foundations of Physics, 35*(10), 1655 – 1693.

[27] Khrennikov, A. Yu. (2005). Hierarchic p-adic encoding of cognitive information based on neuronal trees. *Developments in Experimental and Theoretical Biology, 1*, 19-39.

[28] Orlov, Y.F. (1982). The wave logic of consciousness: A hypothesis *Int. J. Theor. Phys., 21*, 37–53.

[29] Penrose, R. (1989). *The emperor’s new mind*. New-York: Oxford Univ. Press.

[30] Penrose, R. (1994). *Shadows of the mind*. Oxford: Oxford Univ. Press.

[31] Piotrowski, E. W. and Sladkowski, J. (2001). Quantum-like approach to financial risk: Quantum anthropic principle *Acta Physica Polonica B, 32*, 3873-3879.

[32] Piotrowski, E. W., Sladkowski, J., Syska, J. (2003). Interference of quantum market strategies, *Physica A, 318*, 516-528.

[33] Rapoport, A. (1988). Experiments with n- person social traps 1: Prisoner’s Dilemma, weak Prisoner’s Dilemma, Volunteer’s Dilemma, and Largest Number. *Journal fo Conflict Resolution, 32*, 457-472.

[34] Savage, L.J. (1954). *The foundations of statistics*. New York: Wiley and Sons.

[35] Shafir, E. and Tversky, A. (1992). Thinking through uncertainty: non-consequential reasoning and choice. *Cognitive Psychology, 24*, 449-474.

[36] Tversky, A. and Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psychological Science, 3*, 305-309.

[37] von Neumann, J. (1955). *Mathematical foundations of quantum mechanics*. Princeton, N.J.: Princeton Univ. Press.,

[38] Wright, R. (1991). Generalized urn models. *Foundations of physics, 20*, 881-907.