SETS WITH TOPOLOGY, THE ANALYST’S TST, AND APPLICATIONS

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Abstract. This paper was motivated by three questions. First: in a recent paper, Azzam and Schul asked what sort of sets could play the role of curves in the context of the higher dimensional analyst’s traveling salesman theorem. Second: given a set in the euclidean space which has some ‘lower bound’ on its topology, and some upper bound on its size (in terms of Hausdorff measure), what can we say about its geometric complexity? This was initially raised by Semmes in the mid-nineties. Third: in a paper from 1997, Bishop and Jones proved that if a connected set in the plane is uniformly non-flat (the non-flatness being quantified in terms of the Jones $\beta$ coefficients), then its dimension must be strictly larger than one — how much larger depending on how non-flat the set is; can one prove a similar result for higher dimensional sets?

In this paper we try to give some answers to these questions. We show that if put on $E$ a certain topological non degeneracy condition — introduced by David in a paper from 2004, giving $E$ a robust $d$-dimensionality, then, first, $E$ will satisfy an analyst’s traveling salesman type quantitative estimate, that is, an estimate that looks like

$$\mathcal{H}^d(E) \sim \sum_Q \beta^2(Q) \ell(Q)^d + \text{diam}(E)^d.$$

Second, if we also assume that $E$ is upper Ahlfors regular, then $E$ is uniformly rectifiable. Third, we prove an exact analogue of the theorem of Bishop and Jones, with an explicit dependency of the dimensional lower bound to the non-flatness parameter.

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2010 Mathematics Subject Classification: 28A75, 28A12 28A78.

Key words and phrases. Rectifiability, uniform rectifiability, quasiminimizers, beta numbers, Hausdorff content, Travelling salesman theorem.

M. Villa was supported by The Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (grant EP/L016508/01), the Scottish Funding Council, Heriot-Watt University and the University of Edinburgh.
1. INTRODUCTION

This paper was motivated by three questions, which we introduce below.

1.1. Higher dimensional ‘curves’ for the analyst’s traveling salesman theorem. Let $E$ be a subset of $\mathbb{R}^n$ and suppose that $E$ has Hausdorff dimension larger than one. The first motivating question is the following: what simple geometric or topological condition should we impose on $E$ so that $E$ 'is like a Jordan curve'? Let us explain what is the context of this question (and thus what this question really mean).

We begin by recalling the Travelling Salesman Problem (TSP). Given a finite set of points, it asks to find the shortest path that goes through each one of them. Motivated by problems in harmonic measure, Peter Jones posed the following variant of the TSP: given a general set $E$ in the plane, can we find the length of the shortest curve (up to a constant multiple) passing through it? This makes sense whenever we can actually cover $E$ with a rectifiable curve (that is, a curve of finite length). Hence a restatement of Jones’ question is: give a characterisation of subsets $E$ of rectifiable curves in the plane. This problem came to be known as the Analyst’s TSP. Note that if we know from the start that $E$ is contained in a line $L$, then we are immediately done: the shortest curve $\Gamma$ will be the appropriate line segment, and its length will be exactly equal to the diameter of $E$. However, if $E$ presents some curvature, then the length of the covering curve will necessarily increase; to quantify this increment, one needs to quantify how much $E$ deviates from being a line, that is, we need a measurement of its curvature. Hence it makes sense to
measure how much $E$ deviates from lines at all points and scales: so Jones defined what came to be known as the Jones $\beta$ numbers; for a dyadic square $Q$, set

$$\beta_{E,\infty}(Q) := \inf_{L \text{ a line} \supseteq z \in E \cap 3Q} \frac{\text{dist}(z, L)}{\ell(Q)},$$

(1.1)

where $\ell(Q)$ denotes the side length of $Q$, the infimum is taken over all (affine) lines in $\mathbb{C}$ and $\text{dist}(z, L)$ is the usual Hausdorff distance given by $\inf_{l \in L} |z - l|$. Note that this quantity measure exactly what was mentioned above, i.e. how much $E$ deviates from a line ($Q$ determines the scale and location at which we make the measurement). Jones in [J90] then proved that $E$ is a subset of a curve $\Gamma$ and $\Gamma$ is rectifiable if and only if

$$\beta^2_{E,\infty} := \sum_{Q \text{ dyadic}} \beta^2_{E,\infty}(Q) \ell(Q) < \infty,$$

(1.2)

where the sum is taken over all dyadic cubes. Moreover, this statement is quantitative, that is, let $\Gamma_0$ denotes the shortest curve containing $E$; there exists a universal constant $C_0$ such that for all $E$ we have

$$C_0^{-1} \leq \frac{\text{diameter}(E) + \beta^2_{E,\infty}(E)}{\mathcal{H}^1(\Gamma_0)} \leq C_0.$$

(1.3)

Here $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure, and so $\mathcal{H}^1(\Gamma_0)$ is the length of $\Gamma_0$. See also [J91] for a more thorough introduction to the Analyst’s TSP and some indication of why it is relevant in the theory of harmonic measure. The same characterisation was later generalised to curves in arbitrary euclidean space by K. Okikiolu in [Oki92] and to curves in Hilbert space by R. Schul in [Sch07]. The question on whether a similar theorem could be proved for higher dimensional sets was completely open until very recently. In [AS18] J. Azzam and R. Schul proved a version of Jones’ theorem for sets of dimension larger than one in Euclidean space. It is in this context that the initial question makes perhaps a little more sense. If in the plane we characterise subsets of rectifiable curves, subsets of what type of geometric object do we want to consider now? Or, in other words, what sort of sets in $\mathbb{R}^n$ should play the role that curves played in the plane? One could legitimately think about, for example, topological spheres; see Figure 1.1 for why this would not be a good candidate.

Azzam and Schul decided for a slightly different approach to that of Jones. They chose to focus on obtaining a quantitative result of type (1.3) for a set $E$ lying in $\mathbb{R}^n$ by imposing a certain size condition directly on $E$. This size condition is the following: we say that a set $E \subset \mathbb{R}^n$ is lower content $d$-regular with constant $c_0 < 1$ if

$$\mathcal{H}^d_\infty(B(x, r) \cap E) \geq c_0 r^d$$

(1.4)

for all $(x, r) \in E \times (0, \text{diam}(E))$; here $\mathcal{H}^d_\infty$ is the Hausdorff content, see P. Mattila’s book [Mat95], Chapter 4, for definitions and properties. Note in particular that a curve is lower content 1-regular; hence $E$ is, in this respect, ‘similar’ to a curve (it is at the same time true that there are no assumption on $E$ in Jones’ theorem: some questions remain unanswered). Azzam and Schul also had to think of a new variant of the $\beta$ numbers; indeed, in any situation where one is dealing with sets of dimension larger than one, the $\beta$ coefficients as defined in (1.1) become rather useless: in his PhD thesis, X. Fang constructed a Lipschitz graph $K$ with
Figure 1. Given the 2-dimensional 8-corner Cantor set in $\mathbb{R}^3$, one can construct a 2-dimensional surface with finite measure, so that the closure of this surface will contain the Cantor set and will be homeomorphic to the 2-sphere.

$\beta^2_{2\infty}(K) = \infty$, see [Fa90]. Shortly after, David and Semmes in [DS91] introduced an averaged (or ‘$L^p$’) version of these coefficients, defined as

$$
(1.5) \quad \beta^d_{E,p}(x,r) := \inf_L \left( \frac{1}{r^d} \int_{B(x,r)} \left( \frac{\text{dist}(y,L)}{r} \right)^p d\mathcal{H}^d|_{E}(y) \right)^\frac{1}{p},
$$

where the infimum is taken over all affine $d$-dimensional planes in $\mathbb{R}^n$. David and Semmes were dealing with Ahlfors $d$-regular sets, i.e. sets satisfying

$$
(1.6) \quad C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C r^d \quad \text{for} \quad x \in E \text{ and } 0 \leq r \leq \text{diam}(E).
$$

Clearly with this assumption on $E$, (1.5) makes sense. Note however that the assumption on lower content $d$-regularity does not exclude that $E$ may be of dimension larger than $d$. If that was the case, $\mathcal{H}^d|_{E}$ would not be a locally finite measure, and thus (1.5) would not have any meaning at all. This is the reason why Azzam and Schul introduce the following variant of the $L^p$-type $\beta$ coefficients. They put

$$
(1.7) \quad \beta^{d,p}_{E}(x,r) = \inf_L \left( \frac{1}{r^d} \int_0^1 \mathcal{H}^d_{\infty}(\{y \in B(x,r) \cap E \mid \text{dist}(y,L) > tr\}) t^{p-1} \, dt \right)^\frac{1}{p},
$$

where the infimum is taken over all affine $d$-planes $L$ in $\mathbb{R}^n$. The integral on the right hand side of (1.7) is a Choquet integral. With these Jones coefficients and with the assumption of lower content regularity (i.e. (1.4)) on $E$, Azzam and Schul proved an estimate of the type (1.3), with a difference, however: rather than having only the $d$-dimensional Hausdorff measure of $E$ at the denominator (as in (1.3)),
they had $\mathcal{H}^d(E)$ plus a term, BWGL($E$), which quantify the amount of ‘holes’ present in $E$; that is, they proved an estimate like

\begin{equation}
C^{-1} \leq \frac{\text{diameter}(E)^d + \beta_{d,p}^E(E)^2}{\mathcal{H}^d(E) + \text{BWGL}(E)} \leq C,
\end{equation}

where $\beta_{d,p}^E(E)^2$ is defined as in (1.2), but using the content $\beta$ numbers. We will define precisely BWGL($E$) later; for the moment, let us say that its presence is somewhat natural: in Jones’s theorem, we had at the numerator the length of the minimal curve covering $E$; now, a curve has no holes. However, $E$ may very well be quite broken (even while being lower content regular). Thus, if we imagine our set $E$ being covered by ‘a higher dimensional curve’ $\Gamma$, we would have $\mathcal{H}^d(\Gamma) \sim \mathcal{H}^d(E) + \text{BWGL}(E)$, where BWGL($E$) $\sim \mathcal{H}^d(\Gamma \setminus E)$.

We will state all this more precisely later; but for now, let us go back to our initial question: recall that we asked for conditions on $E$ so that $E$ ‘looks like a curve’. In view of the correspondence between (1.3) and (1.8), we can make our question more precise: what conditions on a set $E$ guarantee an estimate of the type (1.3) without the term BWGL($E$)? The main result of this paper is an answer to this question. We put on $E$ a topological condition (TC) which guarantees precisely this: an estimate of the type (1.3), or, in other words, (1.8) without the BWGL($E$) term. This condition is introduced below, see Subsection 1.4.

1.2. Topological non-degeneracy of sets and geometric complexity. In [S95], Semmes stated the following guiding principle to understand the relation between the topology of some set, and its ‘mass’ distribution.

‘Suitable topological condition on a space in combination with upper bounds on the mass often implies serious restriction on the geometric complexity of the space.’

In the monograph [DS00], David and Semmes made this principle into the following theorem — to avoid introducing extra notation, we state it in a somewhat imprecise manner.

**Theorem 1.1** ([DS00], Theorem 0.10). Let $E$ be a compact subset of $\mathbb{R}^n$ and let $A$ be a union of dyadic cubes in $\mathbb{R}^n$ containing $E$. If $\mathcal{H}^d(E) < +\infty$ and if, given a constant $\theta > 0$, $\mathcal{H}^d(f(E)) > \theta$ for every continuous mapping $f : E \to A$ which is homotopic to the identity through mappings from $E$ to $A$, then the following holds. For any $\tau > 0$, there is a compact set $Z \subset \mathbb{R}^n$ such that

- $Z$ is Ahlfors regular, uniformly rectifiable and contains big pieces of Lipschitz graphs;
- $\mathcal{H}^d(Z) > \theta'$, where $\theta'$ depends on $n$ and $A$, but not on $\tau$.
- $\mathcal{H}^d(Z \setminus E) \leq \tau \mathcal{H}^d(E)$.

Here, we will give a result which is similar to Theorem 1.1; however the initial assumptions are somewhat different: in particular, we will be looking at deformations which are only Lipschitz.

1.3. Uniformly non-flat sets and their Hausdorff dimension. The third question which motivates this paper stems from a result of Bishop and Jones, [BJ97]. Here, they proved that if a connected compact subset of the plane is uniformly non-flat, then its dimension is strictly larger than one. For definitions and the precise statement of this, see Section Moreover, their theorem showed explicitly how the non-flatness of the set affects the lower bound on its dimension.
In [Dav04], David gave a corresponding result for higher dimensional sets, which however is qualitative in nature, i.e. it doesn’t present how the non-flatness of $E$ affect the dimensionality of $E$. Here, as a further application of our main result, we give an exact analogue of Bishop and Jones’s result.

1.4. The topological condition on $E$. Let us now define precisely the topological condition mentioned above. Let $E$ be a subset of $\mathbb{R}^n$.

**Definition 1.2 (Allowed Lipschitz deformations with parameter $\alpha_0$).** Fix a constant $0 < \alpha_0 < 1$. Consider a one parameter family of Lipschitz maps $\{\varphi_t\}$,

0 $\leq$ $t$ $\leq$ 1 and defined on $\mathbb{R}^n$. We say that $\{\varphi_t\}_{0 \leq t \leq 1}$ is an allowed Lipschitz deformation with parameter $\alpha_0$, or an $\alpha_0$-ALD, if it satisfies the following four conditions:

(1.9) $\varphi_t(B(x, r)) \subset \overline{B}(x, r)$ for each $t \in [0, 1]$;

(1.10) for each $y \in \mathbb{R}^n$, $t \mapsto \varphi_t(y)$ is a continuous function on $[0, 1]$;

(1.11) $\varphi_0(y) = y$ and $\varphi_t(y) = y$ for $t \in [0, 1]$ whenever $y \in \mathbb{R}^n \setminus B(x, r)$;

(1.12) $\text{dist}(\varphi_t(y), E) \leq \alpha_0 r$ for $t \in [0, 1]$ and $y \in E \cap B(x, r)$, where $0 < \alpha_0 < 1$.

The topological condition that we impose on $E$ is the following.

**Definition 1.3 (Topological Condition).** Fix four parameters:

(1.13) $r_0$, the scale parameter,

(1.14) $\alpha_0$, the distance parameter,

(1.15) $\delta_0$, the lower regularity parameter,

(1.16) $\eta_0$, the boundary parameter.

We say that a subset $E \subset \mathbb{R}^n$ satisfies the topological condition with parameters $r_0, \alpha_0, \delta_0$ and $\eta_0$, or the $(r_0, \alpha_0, \delta_0, \eta_0)$-(TC), or just (TC), if for all $\alpha_0$-ALD $\{\varphi_t\}$, and for all $x_0 \in E$ and $0 < r < r_0$, we have

(1.17) $H^d(B(x, (1 - \eta_0)r) \cap \varphi_1(E)) \geq \delta_0 r^d$.

We may refer to a set $E$ satisfying the topological condition above as a topologically stable $d$-surface, or, for short, TS $d$-surface.

**Remark 1.4.** Let us remark once more that this condition is not new. As stated it was introduced by G. David in [Dav04], where he proved that a set $E$ endowed with such a condition and so that its $\beta$ numbers are large, then it must have dimension strictly larger than $d$.

1.5. Statement of the main result and some consequences. Let $\mathcal{D}(E) = \mathcal{D}$ denote the family of Christ-David cubes relative to $E$ (see Theorem 2.1 for definitions). Our main result in the following.

**Theorem 1.5.** Let $E \subset \mathbb{R}^n$ and let $0 < \eta_0, \delta_0, \alpha_0 < 1$ and $r_0 > 0$. If $E$ satisfies the $(r_0, \alpha_0, \delta_0, \eta_0)$-(TC), then, for any $R \in \mathcal{D}$ such that

(1.17) $\ell(R) < r_0$,

and any $C_0 \geq 1$,

(1.18) $\diam(R)^d + \sum_{Q \in \mathcal{D}} \beta^{p,d}_E(C_0Q)^2 \ell(Q)^d \leq C H^d(R),$
where the constant $C$ depends on $\alpha_0$, $\delta_0$ and $\eta_0$.

**Remark 1.6.** The assumption (1.17) is a natural one and cannot be avoided. Assuming the topological condition from a certain scale, i.e. from $r_0$, means that at larger scale there could be holes. This would make the term $\text{BWGL}(E)$ come back.

Theorem 1.5 together with the main result from [AS18], see Theorem 2.4 below, gives the following corollary.

**Corollary 1.7.** Let $E \subset \mathbb{R}^n$. If there are parameters $\eta_0, \delta_0, \alpha_0$ and $r_0$ so that $E$ satisfies the topological condition (TC), then with $1 \leq p < p(d)$, where

\[
p(d) := \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ \infty & \text{if } d \leq 2 \end{cases},
\]

and $C_0$ sufficiently large (depending only on $n$), we have

\[
diam(R)^d + \sum_{Q \in \mathcal{D}} \beta_{Q}^{p,d}(C_0Q)^2 \ell(Q)^d \sim \mathcal{H}^d(R),
\]

where $R \in \mathcal{D}$ is so that $\ell(R) < r_0$, and where the constant behind the symbol $\sim$ depends on $C_0, n, p$, the parameters coming from the topological condition, and the parameters behind the constants appearing in Theorem 2.4 (see below).

In some cases, the estimate (1.20) was already known: for $\epsilon$-Reifenberg flat (see Definition 3.3), for example. Another known case was for $n = d - 1$ and $E$ is satisfying Condition B (for definitions, see [AS18] and the references therein). We will see below that both Reifenberg flatness and Condition B imply the topological condition that we introduced above.

Another consequence of Theorem 1.5 and of the techniques coming from [AV19], is the following. See Section 11.

**Theorem 1.8.** Let $E \subset \mathbb{R}^n$ be a lower content $d$-regular set with constant $c_0$ and let $Q_0 \in \mathcal{D}$. Given two parameters $0 < \epsilon, \kappa < 1$, there exists a set $\Sigma = \Sigma(\epsilon, \kappa, Q_0)$ such that

1. $Q_0 \subset \Sigma$.
2. $\Sigma$ is a TC surface with constant $r_0 = \text{diam}(Q_0)/2$, $0 < \eta_0 < 1/100$, and $\alpha_0$ and $\delta_0$ sufficiently small with respect to $\kappa$ and $\epsilon$.
3. We have the estimate

\[
\beta(Q_0)^2 + \text{diam}(Q_0)^d \sim \mathcal{H}^d(\Sigma).
\]

Moreover, if $\mathcal{H}^d(\Sigma) < +\infty$, then $E$ is $d$-rectifiable.

That is, given any lower content regular set, we can cover it with a topologically stable $d$-surface. If this surface has finite measure, the set will be rectifiable. This is basically what happens in the original analyst’s TST by Peter Jones.

Concerning the type of questions raised in Subsection 1.2, we have the following corollary, which pops out immediately from Theorem 1.5 and the theory of uniformly rectifiable sets (see [DS93]).

**Theorem 1.9.** Let $E$ be a topologically stable $d$-surface. If $E$ is upper Ahlfors $d$-regular, then $E$ is uniformly rectifiable.

Finally, concerning the Hausdorff dimension of uniformly non-flat sets (see Section 12 for precise definitions), we have the following theorem.
Theorem 1.10. Let $E \subset \mathbb{R}^n$ be a topologically stable $d$-surface. Let $R \in \mathcal{D}$ be such that, for any $Q \in \mathcal{D}(R)$, we have that
\begin{equation}
\beta_{E}(C_0 Q)^2 > \beta_0 > 0. \tag{1.21}
\end{equation}
Then
\begin{equation}
\dim(R) > d + c\beta_0^2. \tag{1.22}
\end{equation}
See Section 12 for a sketch of proof of this.

1.6. Acknowledgement. I would like to thank Jonas Azzam, my supervisor, for suggesting the problem, for his help and support. I would also like to thank PCMI/IAS and the organisers of the graduate school of 2018 on Harmonic Analysis: it was here that I learned many tools used in this paper, as explained by Guy David (whom I also thank for the clear explanations).

2. Preliminaries

We gather here some notation and some results which will be used later on. We write $a \lesssim b$ if there exists a constant $C$ such that $a \leq Cb$. By $a \sim b$ we mean $a \lesssim b \lesssim a$; by $a \lesssim_{\epsilon} b$, we mean that $a \leq Cb$, with the constant $C$ depending on $\epsilon$.

For sets $A, B \subset \mathbb{R}^n$, we let
\[ \text{dist}(A, B) := \inf_{a \in A, b \in B} |a - b|. \]
For a point $x \in \mathbb{R}^n$ and a subset $A \subset \mathbb{R}^n$,
\[ \text{dist}(x, A) := \text{dist}(\{x\}, A) = \inf_{a \in A} \text{dist}(x, a). \]
We write
\[ B(x, t) := \{y \in \mathbb{R}^n | |x - y| < t\}. \]
For $j \in \mathbb{Z}$, we will denote by $\Delta_j$ the family of dyadic cubes with side length $2^{-j}$.
We also set
\[ \Delta := \bigcup_{j \in \mathbb{Z}} \Delta_j. \]
For a cube $I \in \Delta$, we write
\[ \partial_d I \]
the $d$-dimensional skeleton of $I$. We also set
\begin{equation}
S_{j, d} := \bigcup_{I \in \Delta_j} \partial_d I. \tag{2.2}
\end{equation}
Let us remark that for a set $V$, we write $\partial I$ to mean the standard boundary of $V$; so in particular $\partial I = \partial_{n-1} I$. 


2.1. Christ-David cubes. We recall the following version of “dyadic cubes” for metric spaces, first introduced by David [Dav88] but generalized in [C90] and [HM12].

**Theorem 2.1.** Let $X$ be a doubling metric space. Let $X_k$ be a nested sequence of maximal $\lambda^k$-nets for $X$ where $\lambda < 1/1000$ and let $c_5 = 1/500$. For each $n \in \mathbb{Z}$ there is a collection $D_k$ of “cubes,” which are Borel subsets of $X$ such that the following hold.

1. For every integer $k$, $X = \bigcup_{Q \in D_k} Q$.
2. If $Q, Q' \in D = \bigcup D_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
3. For $Q \in D$, let $k(Q)$ be the unique integer so that $Q \in D_k$ and set $\ell(Q) = 5\lambda^k(Q)$. Then there is $\zeta_Q \in X_k$ so that

\[(2.3) \quad B_X(\zeta_Q, c_5\ell(Q)) \subseteq Q \subseteq B_X(\zeta_Q, \ell(Q))
\]
and $X_k = \{ \zeta_Q \mid Q \in D_k \}$.

**Notation 2.2.** First, for a cube $Q \in D_k$, we put

\[(2.4) \quad \text{Child}(Q) := \{ Q' \in D_{k+1} \mid Q' \subset Q \}.
\]

For $Q \in D$, we denote by $B_Q$ the ball $B(\zeta_Q, \ell(Q))$. For a cube $Q \in D$ and a constant $C \geq 1$, we put

\[(2.5) \quad \beta_{d,p}(Q) := \beta_{d,p}(C B_Q) = \beta_{d,p}(\zeta_Q, C \ell(Q)).
\]

**Definition 2.3.** A collection $F \subseteq D$ is a stopping-time region or tree if the following hold:

1. There is a cube $Q(F) \in F$ that contains every cube in $F$.
2. If $Q \in F$, $R \in F$, and $Q \subseteq R \subseteq Q(F)$, then $R \in F$.
3. $Q \in F$ and there is $Q' \in \text{Child}(Q) \backslash F$, then Child$(Q) \subset F^c$.

2.2. The higher dimensional analyst’s TST. We introduce a little more notation, so to state precisely the results from [AS18] and so to obtain from them a corollary of our results. By Theorem 2.1, for each cube $Q \in D$, there is a ball $B_Q$ centered on and containing $Q$ of comparable size. Given two closed sets $E$ and $F$, and $B$ a set we denote

\[(2.6) \quad d_B(E, F) = \frac{2}{\text{diam} B} \max \left\{ \sup_{y \in E \cap B} \text{dist}(y, F), \sup_{y \in F \cap B} \text{dist}(y, E) \right\}
\]

For $A > 0$, and $\epsilon > 0$, let

\[\text{BWGL}(A, \epsilon) = \{ Q \in D \mid d_{AB_Q}(E, P) \geq \epsilon \text{ for all } d\text{-planes } P \}.
\]

We can now state the result from [AS18]. It is phrased slightly differently from there, but the interested reader can find the justification of this reformulation in the Appendix of [AV19].

**Theorem 2.4.** Let $1 \leq d < n$ and $E \subseteq \mathbb{R}^n$ be a closed set. Suppose that $E$ is $d$-lower content regular with constant $c_0$; let $A > 1$. Then there is $\epsilon > 0$ small enough so that the following holds. Let $1 \leq p < p(d)$ where

\[(2.7) \quad p(d) := \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ \infty & \text{if } d \leq 2 \end{cases}
\]
For \( R \in \mathcal{D} \), let
\[
\text{BWGL}(R) = \text{BWGL}(R, \epsilon, A) = \sum_{Q \subseteq R, \ell(Q) \leq \epsilon} \ell(Q)^d.
\]
and
\[
\beta_{E,C_0,p}(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_{E,p}^d(C_0 B_Q)^2 \ell(Q)^d.
\]
Then for \( R \in \mathcal{D} \),
\[
H^d(R) + \text{BWGL}(R, \epsilon, A) \sim_{C_0, n, c_0, p, A, \epsilon} \beta_{E,p}(R).
\]

2.3. **Constants.** We collect here all the various different constants that will be used throughout the paper.

1. \( n, d \): the dimension of the ambient space and of the set under consideration, respectively.
2. \( c_0 \): the lower content regularity constant.
3. \( C_0 \): it determines how much we are inflating the ball where we are measuring the \( \beta \) number.
4. \( A \): it determines the expansion of the ball where we are measuring the BWGL.
5. \( \epsilon \): tolerance parameter in BWGL, and in the definition of Reifenberg-flatness.
6. \( c_1 \): how far the sphere has to be from the set \( E \) in the definition of Semmes surfaces.
7. \( C_1 \): expansion factor of top cubes in Lemma 4.3.
8. \( \tau \): smoothing parameter in Lemma 4.3.
9. \( k_0 \): generation parameter in Lemma 4.3.
10. \( M \): constant for the stopping time in the construction of Lemma 4.3.
11. \( \lambda \): nets parameter in Theorem 2.1.
12. \( c_5 \): containment parameter in Theorem 2.1.
13. \( r_0, \alpha_0, \eta_0, \delta_0 \): parameters of the topological condition (TC).
14. \( r_1, \alpha_1, \eta_1, \delta_1 \): parameters for the skeletal topological condition (see (5.1)).
15. \( C_2 \): constant of the skeletal topological condition.
16. \( C_3 \): Ahlfors regularity constant of the approximating set \( E_R \) (and of \( E_\rho \)).
17. \( \rho \): scale parameter of the approximating set \( E_\rho \) (see section 8).
18. \( \sigma \): scale parameter for the construction of the domain of the functional \( J \).
19. \( M \): large constant in the functional \( J \) (not the same \( M \) as above!).
20. \( c_2 \): small constant in the definition of \( M \).
21. \( k \): quasiminimality: Hausdorff measure constant.
22. \( \delta \): quasiminimality: locality constant.
23. \( c_3 \): small constant in the definition of \( \delta \).
24. \( C_4 \): inflation constant for the \( \beta \) numbers on \( Z_Q \).

3. **Some remarks on the topological condition**

We would like to motivate a little bit our choices: why would one use the topological condition as in Definition 1.3? A quantitative bound as in (1.20) was already
known for surfaces satisfying the so called Condition B and for Reifenberg flat sets; as mentioned above, both of them imply the topological condition (TC).

As Condition B applies only to subsets of codimension one, let us consider instead a more general property which make sense in any codimension. Subsets satisfying this property are called Semmes surfaces. They were first introduced by G. David in [Dav88].

**Definition 3.1.** Let \( n, d \) be two integers with \( 0 \leq d \leq n - 1 \). A **Semmes surface** is a subset \( E \subset \mathbb{R}^n \) so that the following holds. Let \( c_1 < 1 \) be a constant. For all points \( x_0 \in E \) and radii \( r > 0 \), we can find an affine subspace \( W \) of dimension \( n - d \) and a sphere \( S \) of dimension \( n - d - 1 \) which is contained in \( W \) and so that

\[
\begin{align*}
(3.1) & \quad S \subset B(x_0, r) \\
(3.2) & \quad \text{dist}(S, E) \geq c_1 r \\
(3.3) & \quad S \text{ links } E.
\end{align*}
\]

Let us explain what we mean by \( S \text{ links } E \); we say that \( S \) and \( E \) are linked if it is not possible to find an homotopy \( F(x, t) \) defined and continuous for all \((x, t) \in \mathbb{R}^n \times [0, 1]\) such that

\[
\begin{align*}
(3.4) & \quad F(x, t) = x \text{ for } t = 0 \text{ and for } x \in \mathbb{R}^n \setminus B(x_0, 10r); \\
(3.5) & \quad F(x, 1) \in \mathbb{R}^n \setminus B(x_0, 10r) \text{ for all } x \in E; \\
(3.6) & \quad F(x, t) \in \mathbb{R}^n \setminus S \text{ for all } x \in E \text{ and for all } 0 \leq t \leq 1.
\end{align*}
\]

Note that a set satisfying Condition B is just a \( d \)-dimensional Semmes surface with \( d = n - 1 \). G. David shows the following.

**Lemma 3.2** ([Dav04], Lemma 2.16). A \( d \)-dimensional Semmes surface satisfies the topological condition (TC) with parameters depending on \( c_1 \).

Let us now turn to Reifenberg flat sets.

**Definition 3.3.** Let \( n, d \) as above, and fix a positive constant \( \epsilon > 0 \). A subset \( E \subset \mathbb{R}^n \) is called a \( d \)-dimensional \( \epsilon \)-Reifenberg flat set if for all \((x, r) \in E \times \mathbb{R}_+\), there exists a \( d \)-dimensional affine plane \( P \) so that

\[
d_{x,r}(E, P) < \epsilon,
\]

where \( d_{x,r} = d_{B(x,r)} \) is as in (2.6).

**Lemma 3.4.** Fix \( \epsilon > 0 \) and let \( E \subset \mathbb{R}^n \). If for each \((x, r) \in E \times (0, \text{diam}(E)) \) there exists a \( d \)-dimensional plane \( P \) with

\[
d_{x,r}(E, P) < \epsilon,
\]

then there are constants \( \alpha_0, \delta_0, \eta_0 \) which depend only on \( \epsilon, \) \( n \) so that for any \( \epsilon_0 \)-ALD \( \varphi_t \), we have

\[
\mathcal{H}^d(\varphi_1(E) \cap B(x, (1 - \eta_0)r)) \geq \delta_0 r^d.
\]

The proof of this follows quite quickly from two observation. First, note that if there exists a \( d \)-plane \( P \) so that \( d_{x,r}(E, P) < \epsilon \), then the image of \( E \) under an \( \epsilon \)-ALD will satisfy \( d_{x,r}(\varphi_1(E), P) < 2\epsilon \). Indeed, recall that if \( \varphi_t \) is an \( \epsilon \)-ALD, then (1.12) tell us that

\[
\text{dist}(\varphi_1(y), E) \leq \epsilon r \text{ for all } 0 \leq t \leq 1 \text{ and } y \in B(x, r) \cap E.
\]
Second, it is easy to see that if a pair \((x, r) \in E \times \mathbb{R}_+\) satisfies \(d_{x,r}(E, P) < 2\epsilon\) for some \(d\)-plane \(P\), then \(d_{x, r/2}(E, P) < 4\epsilon\) and so \(\mathcal{H}^d(E \cap B(x, (1 - \eta_0)r)) \geq c(\epsilon)r^d\), where \(c(\epsilon) < 1\) is a positive constant which depends only of \(n, d\) and \(\epsilon\), and \(\eta_0\) is a positive constant that can be chosen to be smaller that 1/100, say. In particular, these parameters do not depend on the set \(E\). For a proof of this, see Lemma 13.2 in [DT12]. Putting together these two facts, we immediately see that the topological condition TC is satisfied.

4. First reductions and the construction of approximating skeleta

4.1. First reductions. Let us get started with the proof of Theorem 1.5. Fix a top cube \(R \in \mathcal{D}\). First, we see that if \(\mathcal{H}^d(R) = \infty\), then there is nothing to prove. Thus, we may assume that

\[ (4.1) \quad \mathcal{H}^d(R) < +\infty \]

We will use the following lemma.

**Lemma 4.1** ([AS18] Lemma 2.13). Let \(1 \leq p < \infty\), \(E \subset \mathbb{R}^n\) a closed subset and \(B\) a ball centered on \(E\) with \(\mathcal{H}^d_p(B \cap E) > 0\). Then

\[ \beta_{E}^{1, d}(B) \lesssim_n \beta_{E}^{p, d}(B). \]

**Remark 4.2.** From Lemma 4.1, we see that it is enough to show \((1.18)\) for \(\beta_{E}^{p, d}\), with \(p \geq 2\).

4.2. Construction of the approximating skeleta \(E_R\). In this subsection we introduce a construction from [AV19]; this is a corona decomposition of a lower regular set \(E\) (in the sense of \((1.4)\)) in terms of Ahlfors regular sets \(E_R\) (as in \((1.6)\)). The following lemma from [AV19] makes this precise.

**Lemma 4.3** ([AV19], Main Lemma). Let \(k_0 > 0\), \(\tau > 0\), \(d > 0\) and \(E\) be a set that is \((c_0, d)\)-lower content regular. Let \(Q_0 \in \mathcal{D}_0\) and \(\mathcal{D}(k_0) = \bigsqcup_{k=0}^{k_0} \{Q \in \mathcal{D}_k | Q \subseteq Q_0\}\). Then we may partition \(\mathcal{D}(k_0)\) into stopping-time regions \(\text{Tree}(R)\) for \(R\) from some collection \(\text{Top}(k_0) \subseteq \mathcal{D}(k_0)\) with the following properties:

1. We have

\[ (4.2) \quad \sum_{R \in \text{Top}(k_0)} t(R)^d \lesssim_{c_0, d} \mathcal{H}^d(Q_0). \]

2. Given \(R \in \text{Top}(k_0)\) and a stopping-time region \(\mathcal{F}(Q) \subseteq \text{Tree}(R)\) with maximal cube \(Q\), let \(\mathcal{F}(Q)\) denote the minimal cubes of \(\mathcal{F}(Q)\) and

\[ (4.3) \quad d_{\mathcal{F}(Q)}(x) = \inf_{P \in \mathcal{F}(Q)} (t(P) + \text{dist}(x, P)) \]

For \(C_1 > 4\) and \(\tau > 0\), there is a collection \(\mathcal{C}_Q\) of disjoint dyadic cubes covering \(C_1B_Q \cap E\) so that if

\[ E_{\mathcal{F}(Q)} = \bigcup_{I \in \mathcal{C}_Q} \partial_d I, \]

where \(\partial_d I\) denotes the \(d\)-dimensional skeleton of \(I\), then the following hold:

(a) \(E_{\mathcal{F}(Q)}\) is Ahlfors regular with constants depending on \(C_1, \tau, d\) and \(c_0\).

(b) We have the containment

\[ (4.4) \quad C_1B_Q \cap E \subseteq \bigcup_{I \in \mathcal{C}_Q} I \subseteq 2C_1B_Q. \]
(c) $E$ is close to $E_{\mathcal{F}(Q)}$ in $C_1 B_Q$ in the sense that
\begin{equation}
\text{dist}(x, E_{\mathcal{F}(Q)}) \lesssim \tau d_{\mathcal{F}(Q)}(x) \quad \text{for all } x \in E \cap C_1 B_Q.
\end{equation}

(d) The cubes in $\mathcal{C}_Q$ satisfy
\begin{equation}
\ell(I) \sim \tau \inf_{x \in I} d_{\mathcal{F}(Q)}(x) \quad \text{for all } I \in \mathcal{C}_Q.
\end{equation}

For the sake of completeness and notation, we give a short sketch of the proof of Lemma 4.3; for details, we refer the reader to [AV19]. The idea is akin to Frostmann’s Lemma and its proof (both can be found in [Mat95], Chapter 8, from page 112).

Assume without loss of generality that $Q_0 \subset [0, 1]^n$; let us fix some notation:
\begin{align*}
\Delta_j(Q_0) &:= \{I \in \Delta_j \mid Q_0 \cap I \neq \emptyset\}; \\
\Delta(Q_0) &:= \bigcup_{j \geq 0} \Delta_j(Q_0).
\end{align*}

We also set
\[ V_j(Q_0) := \bigcup_{I \in \Delta_j(Q_0)} I. \]

We are going to iteratively define a measure on the set the approximating set $V_j$; we then put all those dyadic cubes where this measure is too large in a family called Bad, to then perform a stopping time algorithm on the David-Christ cubes of $Q_0 \subset E$, stopping whenever a David-Christ cubes hits a dyadic cubes in Bad with comparable side length, and restarting after skipping one generation of cubes. Through this stopping time procedure we obtain the decomposition of $D(k_0)$ mentioned in Lemma 4.3.

Let us define the measure mentioned above. For $m \in \mathbb{N}$
\[ \mu^m_m := \mathcal{H}^n \big|_{V_m} 2^{(n-d)m}. \]

Note that then, if $I \in \Delta_m(Q_0)$,
\[ \mu^m_m(I) = \ell(I)^d. \]

We now define a family of cubes $\text{Bad}(m)$ as follows. First, we immediately impose that
\[ \Delta_m(Q_0) \subset \text{Bad}(m). \]

Next, we look at the cubes one level up, that is, at the cubes in $\Delta_{m-1}(Q_0)$. If for one such cube $J$, we have
\[ \mu^m_m(J) > 2\ell(J)^d, \]
the we put $J$ in $\text{Bad}(m)$ and define
\[ \mu^m_{m-1}|J := \ell(J)^d \frac{\mu^m_m|J}{\mu^m_m(J)} < \frac{1}{2} \mu^m_m|J. \]

Otherwise, we set
\[ \mu^m_{m-1}|J := \mu^m_m|J. \]
Note that in this way, it is always true that, for a cube $I \in \Delta_{m-1}(Q_0)$, $\mu_{m-1}^m(I) \leq 2\ell(I)^d$. Continuing inductively in this fashion, we define $\mu_{m-2}^m$, $\mu_{m-3}^m$ and so on; suppose we defined $\mu_k^m$, for $1 \leq k \leq m$. We consider the cubes $I \in \Delta_{k-1}$: if

$$\mu_k^m(I) > 2\ell(I)^d,$$

then we put $I \in \text{Bad}(m)$ and set

$$\mu_{k-1}^m|_I := \ell(I)^d \frac{\mu_k^m|_I}{\mu_k^m(I)} < \frac{1}{2} \mu_k^m.$$

Otherwise, we set

$$\mu_{k-1}^m|_I := \mu_k^m|_I.$$

We stop when we reach $k = 1$ (and so $\mu_0^m$ is defined). One can then show the packing condition

$$(4.7) \sum_{I \in \text{Bad}(m)} \ell(I)^d \leq C(n, d)H^d(Q_0),$$

which is independent of $m \in \mathbb{N}$. For a proof of this, see [AV19], in particular equation (3.5).

Let now $k_0 > 0$ be an arbitrary integer number, $M > 1$ a constant to be fixed later and $C_0 > 1$ the inflation constant for the $\beta$ numbers (see Constant (3)). We now perform the stopping time mentioned above. We start with putting $Q_0 \in \text{Tree}(Q_0)$. If there exists a cube $Q \in \text{Child}(Q_0)$ such that

$$(4.8) MB_Q \cap I \neq \emptyset \quad \text{and} \quad (4.9) \lambda \ell(I) \leq \ell(Q) \leq \ell(I),$$

where $\lambda$ is as in Theorem 2.1, then we stop. Otherwise we put all the cubes $Q \in \text{Child}(Q_0)$ in $\text{Tree}(Q_0)$; next we scan $\text{Child}(Q_0)$ to see if any cube here has a child satisfying (4.8). We proceed recursively in this fashion; the process will eventually terminate because we stopped at all cubes, or because we reached the bottom of $D(k_0)$. Furthermore, we consider all cubes $Q$ of the same generation of $Q_0$, so that

$$2C_0Q_0 \cap Q \neq \emptyset.$$

We denote this family by $\mathcal{N}(Q_0)$. On each of these cubes, we perform the same stopping time, so to construct the relative $\text{Tree}(Q)$. Finally we put

$$(4.10) \text{Forest}(Q_0) := \bigcup_{Q \in \mathcal{N}(Q_0)} \text{Tree}(Q),$$

and also

$$\text{Stop}(Q_0) := \{Q \in D(k_0) | Q \text{ is minimal in } \text{Forest}(Q_0)\}.$$

Next, we put

$$\text{Next}(Q_0) := \bigcup_{Q \in \text{Stop}(Q_0)} \text{Child}(Q).$$
We now repeat the stopping time on each $R \in \text{Next}(Q_0)$. Thus, if we set $\text{Top}_0(k_0) := \{Q_0\}$, then $\text{Top}_1(k_0) := \text{Next}(Q_0)$; proceeding inductively, supposed that $\text{Top}_m(k_0)$ has been defined, for $m \in \mathbb{N}$: we put

$$\text{Top}_{m+1}(k_0) := \bigcup_{R \in \text{Top}_m(k_0)} \text{Next}(R).$$

Finally, we set

$$\text{Top}(k_0) = \bigcup_{k=0}^{\infty} \text{Top}_k,$$

Hence, to each element $R \in \text{Top}$, there correspond a forest $\text{Forest}(R)$ and a family of minimal cubes $\text{Stop}(R)$. Now, for each $R \in \text{Top}$, let us define

$$d_R(x) := \inf_{Q \in \text{Stop}(R)} (\ell(Q) + \text{dist}(x,Q)),$$

and

$$d_R(I) := \inf_{x \in I} d_R(x),$$

whenever $I \in \Delta$.

This is a now standard smoothing procedure which goes back to David and Semmes’ [DS91]. Hence, for a parameter $\tau > 0$, we put

$$C_R := \{\text{maximal } I \in \Delta | I \cap 2C_0R \neq \emptyset \text{ and } \ell(I) < \tau d_R(I)\}.$$

Finally we set

$$\bar{E}_R := \bigcup_{I \in C_R} \partial_d I.$$ 

Thus $\bar{E}_R$ is the union of $d$-dimensional skeleta (see (2.1)) of cubes belonging to $\bar{E}_R$.

**Lemma 4.4** ([AV19], Lemma 3.6). The set $E$ is Ahlfors $d$-regular.

**Remark** 4.5. The constant $M > 1$ is fixed here: it has to be sufficiently large (depending on $\tau$). See the proof of Lemma 3.6 in [AV19].

Let us state a couple of standard technical lemmas which will come handy later on in the paper. We add their proof in the appendix for the sake of completeness.

**Lemma 4.6.** Let $S$ be a cube in $\overline{\text{Stop}}(Q)$ for some $Q \in \text{Next}(R)$, $R \in \text{Top}(k_0)$. Then there exists a dyadic cube $I_S := I \in C_Q$ so that $I_S \subset \frac{1}{2}B_S$ and $\ell(I_S) \sim \tau \ell(S)$.

**Lemma 4.7.** Let $I \in C_Q$ for $Q \in \text{Next}(R)$, $R \in \text{Top}(k_0)$. Then there exists a cube $Q_I \in \overline{\text{Tree}}(Q)$ so that

$$\ell(I) \leq \ell(Q_I) \leq c\tau^{-1} \ell(I);$$

$$\text{dist}(I, Q_I) \leq c\tau^{-1} \ell(I).$$

We want to apply Lemma 4.3, to obtain a corona decomposition of our set $E$, endowed with the topological condition (TC). To do so, however, we first need to show that $E$ is lower content $d$-regular.

**Lemma 4.8.** Let $E \subset \mathbb{R}^n$ be compact subset which satisfies the topological condition (TC) with parameters $r_0, \alpha_0, \delta_0$ and $\eta_0$. Then $E$ satisfies

$$\mathcal{H}_\infty^d(E \cap B(x, r)) \gtrsim c_0 r^d$$

for all $x \in E$ and $r < r_0$; the lower regularity constant $c_0$ will depend on $\delta_0$ and $\eta_0$. 
This fact is essentially present in Chapter 12 of [DS00], although in a somewhat different form. We give a proof for this reason. We will first prove the following Sublemma, which will imply Lemma 4.8.

**Sublemma 4.9.** Let $E$ be a compact subset of $\mathbb{R}^n$ and let $(x,r) \in E \times \mathbb{R}_+$ be a pair so that
\begin{equation}
\mathcal{H}_\infty^d (B(x,r) \cap E) < \mu \nu \delta_0 r^d
\end{equation}
for a parameter $\nu$ (sufficiently small depending on $\eta_0$) and a number $\mu > 0$ which depend only on $\eta_0$ and $\delta_0$ (as in Lemma 4.8). Then there exists a one parameter family of Lipschitz mappings $\{\varphi_1\}$ which satisfies (1.9)-(1.12) and so that $\varphi_1$ maps $B(x,(1-\eta_0)r) \cap E$ into the $(d-1)$-dimensional skeleton of cubes from $\Delta_j$, where $j = j(\rho) \in \mathbb{N}$ is such that $2^{-j} \sim \rho$, and $\rho = (\nu \delta_0)^{1/d_\sigma}$.

The proof of this Sublemma will follow quickly if we use the following Proposition from [DS00].

**Proposition 4.10** ([DS00], Proposition 12.61). Let $A$ be a union of dyadic cubes from $\Delta_j$, where $j$ is some integer. There is a possibly small constant $c > 0$ so that if $\theta \sim c 2^{-j}$, the following is true. Let $F$ be a compact subset of $A$ such that
\begin{equation}
\mathcal{H}_\infty^d (F \cap Q) < \theta \text{ for all } Q \in \Delta_j.
\end{equation}
Then there is a Lipschitz mapping $\phi : F \to A$ so that $\phi(F) \subset S_{j,d-1}$ and $\phi(F \cap Q) \subset Q$ for all $Q \in \Delta_j$. Also, $\phi$ is homotopic to the identity through mappings from $F$ to $A$.

**Proof of Sublemma 4.9.** Let $\rho > 0$ and $j(\rho) \in \mathbb{N}$ be as in the statement of the Sublemma, and let $\mu > 0$, $\nu > 0$ two possibly small parameteres to be fixed soon. Set
\begin{align*}
A^1 &= \bigcup \{I \in \Delta_{j(\rho)} \mid I \cap B(x,(1-\mu)r) \neq \emptyset\}, \\
A^2 &= \bigcup \{I \in \Delta_{j(\rho)} \mid I \cap A^1 \neq \emptyset\}.
\end{align*}
We want $\mu$ and $\nu$ to be so that
\begin{equation}
\eta_0 > 10 \mu > 2(\nu \delta_0)^d.
\end{equation}
This choice then implies that
\begin{equation}
E \cap B(x,(1-\eta_0)r) \subset E \cap B(x,(1-\mu)r) \subset E \cap A^1 \subset E \cap A^2 \subset B(x,r) \cap E,
\end{equation}
Now, by the hypothesis (4.15), we see that for any $I \in \Delta_{j(\rho)}$ which is also contained in $A^2$ we have
\begin{equation}
\mathcal{H}_\infty^d (I \cap (E \cap A^1)) < \mu \nu \delta_0 r^d = \mu \rho^d.
\end{equation}
Adjusting the choice of $\mu$ and $\nu$ if needed, we see that this implies (4.16) to hold for all $I \in \Delta_{j(\rho)}$ which also lie in $A^2$ with $F = E \cap A^1$. Moreover, with this $F$, (4.16) holds trivially for any other $I \in \Delta_{j(\rho)}$. Hence we apply Proposition 4.10 with $j = j(\rho)$ (i.e. so that $2^{-j} \sim \rho$), $A = A^2$ as defined in (4.18) and $F = A^1 \cap E$, as defined in (4.17). We obtain a Lipschitz mapping $\phi$ which sends $E \cap A^1$ into $S_{j(\rho),d-1}$ and all the properties listed in the proposition. Note in particular that
with the choice (4.19) of \( \mu \) and \( \nu \) and the fact that \( \phi(E \cap I) \subset I \) for any \( I \in \Delta_{j(\rho)} \), we have that

\[
B(x, (1 - \eta_0)r) \cap \phi(E) = B(x, (1 - \eta_0)r) \cap (\phi(E \cap A^1) \subset \phi(E \cap A^1).
\]

Now we can extend \( \phi \) to be the identity outside of \( A^2 \). Setting

\[
\varphi_t(y) = t\phi(y) + (1 - t)y \text{ for } t \in [0, 1],
\]
it is easy to check that \( \varphi_t \) satisfies (1.9)-(1.12). \( \square \)

Proof of Lemma 4.8. For the sake of contradiction, suppose that for \( x \in E \), and \( r < r_0 \), the inequality (4.15) holds. Then, using the definition of topological condition (TC) (which can be applied since \( r < r_0 \)), we obtain

\[
\delta_0 r^d < H^d (B(x, (1 - \eta_0)r) \cap \varphi_1(E)) = H^d (B(x, (1 - \eta_0)r) \cap \phi(E)) \leq H^d (\phi(E \cap A^1)) = 0.
\]

Thus we must have that for any such a pair \((x, r) \in E \times \mathbb{R}_+\), (4.15) cannot hold. This implies the lower content \( d \)-regularity of \( E \) (for scales smaller than \( r_0 \)), with constant \( c_0 \) depending only on \( \delta_0 \) and \( \eta_0 \). \( \square \)

Remark 4.11. Because all our statements are local, we will be ignoring the fact that our set is lower regular only for (possibly) small scales. In fact, we could assume without loss of generality that \( r_0 = 1 \).

We now see that whenever \( E \) satisfies the topological condition (TC), then it is also lower content \( d \)-regular. We therefore can apply Lemma 4.3 to \( E \) and obtain a decomposition \( \text{Top} \) and for each \( R \in \text{Top} \), an Ahlfors regular set \( \bar{E}_R \) which approximates \( E \) at a specific scale and location. Recall that the decomposition \( \text{Top} \) depends on the integer \( k_0 \); if we want to underline this, we may write \( \text{Top}(k_0) \).

4.3. Modification of \( \bar{E}_R \). In this subsection, we modify slightly the construction of \( \bar{E}_R \); we need to do so to construct a coherent Federer-Fleming projection in the next section.

Fix \( R \in \text{Top} \); recall the definition of \( \mathcal{C}_R \) in (4.14). Take a cube \( I \in \mathcal{C}_R \). Consider one of its \((n - 1)\)-dimensional faces, and denote it by \( T_I \). Set

\[
\text{Adj}^{n-1}(T_I) := \{ J \in \mathcal{C}_R | \ell(J) \leq \ell(I), J \cap T_I \text{ is an } (n - 1)\text{-face of } J \text{ and } J \cap T_I \subset \text{Int}(T_I) \}
\]

We order the cubes in \( \mathcal{C}_R \) from the largest to the smallest one, and we label them as \( I_0, \ldots, I_N \), for some \( N \in \mathbb{N} \). This is true because the cardinality of \( \mathcal{C}_R \) is finite (depending on \( k_0 \)). Let us start our construction with \( I_0 \in \mathcal{C}_R \) (thus \( I_0 \) is the largest cube in \( \mathcal{C}_R \)). We look at one of its \((n - 1)\)-dimensional faces, let us denote it by \( T_{I_0} \). Now, let \( I \) be a cube of minimal side length contained in \( \text{Adj}^{n-1}(T_{I_0}) \); let \( n(I) \in \mathbb{N} \) be such that \( \ell(I) = 2^{-n(I)} \). We consider the family of cubes in \( \Delta_{n(I)} \) such that they have an \((n - 1)\)-dimensional face contained in \( T_{I_0} \). We call this family \( \mathcal{C}_{n(I)}(T_{I_0}) \). Let us denote by

\[
\mathcal{O}_{n-1}(T_{I_0})
\]
the family of \( (n - 1) \)-dimensional faces of the same side length of \( I \), such that they are both an \( (n - 1) \)-dimensional face of a cube \( J \in \Delta_n(I)(T_{I_0}) \) and also they are contained in \( T_{I_j} \). We may refer to this family as the tiles of \( T_{I_j} \). We repeat the same procedure for \( I_1, \ldots, I_N; \) we don’t do anything if \( \text{Adj}^{n-1}(T_{I_j}) = \emptyset \) for some face \( T_{I_j} \) of \( I_j, 1 \leq j \leq N \). Note that the definition of \( \text{Adj}^{n-1}(T_I) \) imposes the following: if two cubes \( I \) and \( I' \) are so that, say, \( \ell(I) > \ell(I') \) and \( I' \in \text{Adj}^{n-1}(T_I) \), then the tiles constructed on \( T_I \) will be the same one that we have on the face \( T_{I'} \subset T_I \). The construction of tiles on the other \( (n - 1) \)-faces of \( I' \) will not change the ones already present in \( T_{I'} \). This procedure terminates since \( \mathcal{C}_R \) is finite. 

Once we constructed \((n - 1)\)-dimensional tiles on all the \((n - 1)\)-dimensional faces of all cubes in \( \mathcal{C}_R \), we rest. After, we proceed as follows. Denote by 
\[
\mathcal{F}^{n-1}
\]
the family of \((n - 1)\)-dimensional faces belonging to some cube in \( \mathcal{C}_R \). If \( T \in \mathcal{F}^{n-1} \) and \( \mathcal{D}^{n-1}(T) \neq \emptyset \), the put the elements of \( \mathcal{D}^{n-1}(T) \) in \( \mathcal{F}^{n-1} \) and take \( T \) away. If \( \mathcal{D}^{n-1}(T) = \emptyset \), then leave \( T \) in \( \mathcal{F}^{n-1} \). 

Next, we repeat the previous construction: order the elements of \( \mathcal{F}^{n-1} \) in decreasing order with respect to side length and consider \( T_0 \) (the largest face in \( \mathcal{F}^{n-1} \)). For each \((n - 2)\)-dimensional face \( F_{T_0} \) of \( T_0 \) we set 
\[
\text{Adj}^{n-2}(F_{T_0}) := \{ T \in \mathcal{F}^{n-1} | \ell(T) < \ell(T_0), T \cap F_{T_0} \text{ is an } (n - 2) - \text{face of } T \text{ and } T \cap F_{T_0} \subset F_{T_0} \}.
\]
We now look for the minimal element of \( \text{Adj}^{n-2}(F_{T_0}) \), and call it \( T \). Let \( n(T) \in \mathbb{Z} \) so that \( \ell(T) = 2^n(T) \); we now tessellate \( F_{T_0} \) with tiles of side length \( 2^n(T) \); by tessellate here we mean the obvious thing, i.e. we substitute \( F_{T_0} \) with its children of size \( 2^n(T) \). Let us denote the tiles so constructed by 
\[
\mathcal{D}^{n-2}(F_{T_0}).
\]
We repeat the same procedure for \( T_1, \ldots, T_N \in \mathcal{F}^{n-1} \). Again, the construction of \((n - 2)\)-dimensional tiles for smaller \((n - 2)\)-dimensional faces does not affect the previously constructed tiles for larger faces. This procedure terminates since \( \mathcal{F}^{n-1} \) is finite, which follows trivially from \( \mathcal{C}_R \) being finite. Next, we set 
\[
\mathcal{F}^{n-2}
\]
to be the family of \((n - 2)\)-dimensional faces coming from elements of \( \mathcal{F}^{n-1} \), and we immediately modify it as above: if \( \mathcal{D}^{n-2}(F_{T}) \neq \emptyset \), for \( T \in \mathcal{F}^{n-1} \), we substitute \( F_T \) with the corresponding family of tiles.

We continue this construction: we obtain \( \mathcal{F}^{n-3} \) from \( \mathcal{F}^{n-2} \), \( \mathcal{F}^{n-4} \) from \( \mathcal{F}^{n-3} \), and so on, until we construct \( \mathcal{F}^d \). We stop at this point and we set 
\[
E_R := \tilde{E}_R \cup \left( \bigcup_{F \in \mathcal{F}^d} F \right).
\]

Lemma 4.12. The set \( E_R \) is Ahlfors \( d \)-regular.

Proof. Lower regularity follows immediately from the definition and the lower regularity of \( \tilde{E}_R \). On the other hand, note that for any cube \( I \in \mathcal{C}_R \), any smaller neighbouring cube \( I' \in \mathcal{C}_R \) will satisfy \( \ell(I') > \tau \ell(I) \). If we envelope \( I \) in cubes of side length \( \ell(I) \tau \) and we consider the \( d \)-dimensional skeleton of this family of
cubes, we see that the overall additional mass will not exceed a constant times \( \ell(I)^d \), where such a constant depends on \( n, d \) and \( \tau \). Thus upper regularity is also preserved. \( \square \)

**Notation 4.13.** From now on, we fix the notation for the regularity constant of \( E_R \): it will be denoted by \( C_3 \).

5. **A topological condition on approximating skeleta**

We now introduce a condition on \( E_R \) which will imply the existence of a uniformly rectifiable sets lying close to it. This is basically the condition that David calls TND (topological nondegeneracy condition) in [Dav04] with a few changes to adapt it to our trees. Let \( R \in \text{Top} \) and \( E_R \) be the set constructed in Section 4.2, i.e. the set given in (4.25).

**Definition 5.1 (STC).** Let \( C_2 \) be an arbitrary big constant and let \( k_0 \in \mathbb{N} \) be as in the statement of Lemma 4.3. Then we say that the family of subsets \( \{ E_R \}_{R \in \text{Top}(k_0)} \) satisfies the **skeletal topological condition** with parameter \( C_2 \), or \( C_2 \)-(STC), if we can find positive constants \( r_1, \alpha_1 < 1, \eta_1, \delta_1 \), (5.1) such that

(5.2) for all \( x_1 \in E \),

(5.3) for all \( R \in \text{Top}(k_0) \) s.t. \( x_1 \in R \) and \( \ell(R) \leq r_1 \),

(5.4) for all \( Q \in \text{Tree}(R) \) s.t. \( x_1 \in Q \),

for which

(5.5) \[ \mathcal{H}^d(E_R \cap B(x_1, \ell(Q))) \leq C_2 \ell(Q)^d \]

holds, there is a ball \( B(x_2, r_2) \) centered on \( E \) and contained in \( B(x_1, \ell(Q)) \) such that, for each one-parameter family \( \{ \varphi_t \}_{0 \leq t \leq 1} \) of Lipschitz functions on \( \mathbb{R}^n \) that satisfy (1.9), (1.10), (1.11) and

(5.6) \[ \text{dist}(\varphi_t(y), E) \leq \alpha_1 \ell(Q) \text{ for } t \in [0, 1] \text{ and } y \in E_R \cap B(x_2, r_2), \]

we have that

(5.7) \[ \mathcal{H}^d(\varphi_1(E_R \cap B(x_2, r_2))) \geq \delta_1 \ell(Q)^d + \mathcal{H}^d(E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2)), \]

where

(5.8) \[ A_{\eta_1 \ell(Q)}(x_2, r_2) := B(x_2, r_2) \setminus B(x_2, r_2 - \eta_1 \ell(Q)). \]

**Remark 5.2.** Note that

(5.9) \[ r_2 > \eta_1 \ell(Q); \]

if \( r_2 \leq \eta_1 \ell(Q) \), then \( A_{\eta_1 \ell(Q)} = B(x_2, r_2) \). Thus if we apply (5.7) with \( \varphi_t(y) = y \), then we would obtain that \( \mathcal{H}^d(E_R \cap B(x_2, r_2)) > \mathcal{H}^d(E_R \cap B(x_2, r_2)) \), a contradiction.
6. Federer-Fleming projections

In this section we will construct a Federer-Fleming projection of $E$ onto a subset of $E_R$; we will use these projections in the next section to prove that the topological condition (TC) on $E$ implies the condition STC on the approximating skeleta. Our construction will mimic the one in [Dav04], which in turn comes from [DS00]. The difference here is that we are dealing with a skeleton of faces coming from cubes of different sizes.

Let $B(x, r)$ be a ball centered on the set $E$ (the construction below will be applied to the ball $B(x_2, r_2)$ as in the definition of STC, Definition 5.1). Set

\begin{equation}
C_R(x, r) := \{ I \in C_R | I \cap B(x, r) \neq \emptyset \}; \tag{6.1}
\end{equation}

\begin{equation}
F_m(x, r) := \{ T \in F_m | T \cap B(x, r) \neq \emptyset \} \text{ for } d \leq m \leq n - 1; \tag{6.2}
\end{equation}

\begin{equation}
D_R(x, r) := \bigcup_{I \in C_R(x, r)} I. \tag{6.3}
\end{equation}

Furthermore, we set $C_R^2(x, r)$ to be the family of dyadic cubes composed by $C_R(x, r)$ together with a maximal subfamily of cubes $J \in \Delta \setminus C_R$ so that

\begin{equation}
\text{Int}(J) \subset \left( \bigcup_{I \in C_R(x, r)} I \right)^c; \tag{6.4}
\end{equation}

there exists a dyadic cube $I \in C_R(x, r)$ s.t. $I \cap J \neq \emptyset$,

and moreover, if we let \( N(J) \) be the family of cubes in $C_R(x, r)$ which intersect $J$, we ask that

\begin{equation}
\min \{ \ell(I) | I \in N(J) \} \leq \ell(J) \leq \max \{ \ell(I) | I \in N(J) \}. \tag{6.5}
\end{equation}

The family $C_R^2(x, r) \setminus C_R(x, r)$ forms a sheath for $C_R(x, r)$ (imagine the plastic covering of some Minecraft electrical wires). Finally we define

\begin{equation}
D_R^2(x, r) := \bigcup_{I \in C_R^2(x, r)} I. \tag{6.6}
\end{equation}

Recall the definition of $E_R$ as in (4.25). The following lemma is similar to Proposition 3.1 in [DS00], and so is the proof. The only difference is that we are working with a non-uniform grid of cubes.

**Lemma 6.1.** Given $(x, r) \in E \times \mathbb{R}_+$, there exists a Lipschitz map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that

\begin{equation}
\pi(y) = y \text{ whenever } x \in \mathbb{R}^n \setminus D_R^2(x, r); \tag{6.7}
\end{equation}

\begin{equation}
\pi(I) \subset I \text{ if } I \in C_R^2(x, r); \tag{6.8}
\end{equation}

\begin{equation}
\pi(E) \cap I \subset E_R \cap I \text{ for any } I \in C_R(x, r). \tag{6.9}
\end{equation}

We will obtain our Federer-Fleming projection as the composition of a finite number of maps which we will define inductively. We start by defining a map, let us call it $\pi_1$, that will send points in $D_R^2(x, r) \cap E$ into $(n - 1)$-dimensional faces. We define $\pi_1$ on each individual cube $I \in C_R^2(x, r)$ as follows. Pick a point $c_I \in I$ such that $c_I \notin E$. This is possible since $\mathcal{H}^d(E) < \infty$ (recall (4.1)) and thus, in
particular, \( \dim_H(E) < d + 1 \); a standard argument then shows that \( E \) is porous, and thus such a point \( c_I \) must exist. Then for \( y \in E \cap \text{Int}(I) \), we set

\[
\pi_1(y) = \text{the point where the line passing through } y \text{ and } c_I \text{ meets } \partial I;
\]

note that, then, \( \pi_1(y) \) belong to some \( (n - 1) \) dimensional face of \( I \). On the other hand, if \( y \in E \cap \partial I \), we set

\[
\pi_1(y) = y.
\]

We then extend \( \pi_1 \) on the whole of \( I \) such that \( \pi_1(I) \subset I \) and \( \pi_1 \) is Lipschitz on \( I \).

(6.13) This can be done via standard extension results, see for example [H05]). Note that this definition is coherent, in the sense that one can glue together the definition of \( \pi_1 \) on each \( I \in \mathcal{C}_R(x,r) \) into a unique map \( \pi_1 \) defined on the whole of \( D_R^2(x,r) \).

Indeed, if \( I, I' \in \mathcal{C}_R(x,r) \) are so that \( I \cap I' \neq \emptyset \), then the definition of \( \pi_1 \) on \( I' \cap I \) must agree, since \( I \cap I' \) is contained in \( \partial I \) and \( \partial I' \). Furthermore, we extend the definition of \( \pi_1 \) to \( \mathbb{R}^n \setminus D_R^2(x,r) \) by setting

\[
\pi_1(y) = y
\]

there. Thus (6.11)-(6.14) give a coherent definition of \( \pi_1 \) on the whole of \( \mathbb{R}^n \).

Figure 2. The first step in the construction of Federer-Fleming projections.

Now, if \( d = n - 1 \), we stop here and we set \( \varphi := \pi_1 \). Otherwise, we continue as follows. We want to send points on the \( (n - 1) \)-dimensional faces of cubes in \( \mathcal{C}_R(x,r) \) to the boundaries of these faces, which are, in turn, \( (n - 2) \)-dimensional faces. To do this, we proceed, as above, by defining the map we need on each individual face. Recall the definition of \( \mathcal{F}^{n-1} \) in (4.24). Let us start by defining \( \pi_2 \) on each \( \partial T \cup (\pi_1(E) \cap T) \), where \( T \in \mathcal{F}^{n-1} \): we repeat the construction above. Namely,

\[
\pi_2(c_T) \in \text{Int}(T) \setminus \pi_1(E) \text{ and then project radially } \pi_1(E) \cap T \text{ onto } \partial T;
\]

once again, this definition leave unchanged those points which already belong to \( \partial T \). Next,

\[
\pi_2(I) \subset I \text{ for any } I \in \mathcal{C}_R(x,r).
\]
Let us say a few words on how to construct such an extension. Given \( I \in \mathcal{C}_R(x,r) \), let \( T \) be one of its \((n-1)\)-dimensional faces. We know \( \pi_2 \) on \( T \cap \pi_1(E) \) and on \( \partial T \). We can then extend \( \pi_2 \) as a Lipschitz map from \( T \) to \( T \) via standard extension results. We now want to extend \( \pi_2 \) from \( \partial I \), to the whole of \( I \), with the requirement that \( \pi_2(I) \subset I \). To do so, let \( c_I \) be the center of \( I \). We set \( \pi_2(c_I) := x^* \), where \( x^* \) is any point in \( \pi_2(\partial I) \). Then for any point \( y \in \partial I \), and a point \( x = tc_I + (1-t)y \), \( t \in [0,1] \), (so that \( x \) belongs to the line segment from \( c_I \) to \( y \)), we set

\[
\pi_2(x) = t\pi_2(c_I) + (1-t)\pi_2(y).
\]

Note that, because both \( \pi_2(c_I) \) and \( \pi_2(y) \) belong to \( \partial I \), and \( I \) is convex, then \( \pi_2(x) \in I \). Let us check that \( \pi_2 \) so defined is Lipschitz on \( I \). Take any two points \( x_1, x_2 \in I \) and write them as

\[
\begin{align*}
(6.17) & \quad x_1 = tc_I + (1-t)y_1, \; t \in [0,1] \text{ and } y_1 \in \partial I; \\
(6.18) & \quad x_2 = sc_I + (1-s)y_2, \; s \in [0,1] \text{ and } y_2 \in \partial I.
\end{align*}
\]

Assume first that \( t = s \). We can assume that \( t = s < 1 \), for otherwise \( x_1 = x_2 \). In this case, we have that

\[
|\pi_2(x_1) - \pi_2(x_2)| = |(1-t)(\pi_2(y_1) - \pi_2(y_2))| \\
\leq C |(1-t)(y_1 - y_2)| \\
= |x_1 - x_2|.
\]

Here the constant \( C \) is the Lipschitz constant of \( \pi_2 \) as function defined on \( \partial I \).

Next, let us suppose that for \( x_1 \) and \( x_2 \) as in (6.17) and (6.18), we have that \( y_1 = y_2 \), hence they lie on the same line segment from \( c_I \) to \( \partial I \). We first note that (assuming without loss of generality that \( t > s \)),

\[
|x_1 - x_2| = |(t-s)(c_I - y_1)| \geq (t-s)\ell(I).
\]

On the other hand, we have that

\[
|\pi_2(x_1) - \pi_2(x_2)| = |(t-s)(\pi_2(c_I) - \pi_2(y_1))| \leq \sqrt{n} (t-s)\ell(I).
\]

Thus \( |\pi_2(x_1) - \pi_2(x_2)| \leq \sqrt{n}|x_1 - x_2| \). Finally, for any two points \( x_1, x_2 \in I \) as in (6.17) and (6.18), put

\[
x_2' := tc_I + (1-t)y_2.
\]

Note that there exists a constant, depending only on \( n \), so that

\[
(6.19) & \quad |x_2 - x_2'| \leq C |x_1 - x_2|,
\]

But then, by the triangle inequality, we also have that

\[
|x_1 - x_2'| \leq C|x_1 - x_2|.
\]

This give us the following:

\[
|\pi_2(x_1) - \pi_2(x_2)| \leq |\pi_2(x_1) - \pi_2(x_2')| + |\pi_2(x_2') - \pi_2(x_2)| \\
\leq C (|x_1 - x_2'| + |x_2' - x_2|) \\
\leq C'|x_1 - x_2|.
\]

This proves that the extension of \( \pi_2 \) to the whole of \( I \) is indeed Lipschitz, with a Lipschitz constant comparable to that of \( \pi_2 \) as defined on \( \partial I \). Now we let \( \pi_2 \) on \( D(x,r) \) to be piecewise defined on each \( I \) of \( \mathcal{C}_R(x,r) \).
Let us see why this definition is coherent. If $T, T' \in \mathcal{F}^{n-1}$, $T \cap T' \neq \emptyset$ and let us assume without loss of generality that $\ell(T') < \ell(T)$, then either
\[(6.20) \quad T' \subset T,\]
or
\[(6.21) \quad T \cap T' \subset (\partial T) \cup (\partial T').\]
If (6.21) holds, then we immediately see that the definition of $\pi_2$ is coherent, since we defined to be the identity on both $\partial T$ and $\partial T'$. We divert a moment from the main construction to show that the former case does not happen.

**Lemma 6.2.** The case (6.20) does not occur.

**Proof.** Let $T \in \mathcal{F}^{n-1}(x, r)$, and assume first that $T$ is an $(n-1)$-dimensional face (as opposed to a tile) of a cube $I \in \mathcal{C}_R(x, r)$. Suppose that there exists an element $T'$ of $\mathcal{F}^{n-1}(x, r)$ such that $T' \subset T$. If $T'$ is an $(n-1)$-dimensional face of a cube $I' \in \mathcal{C}_R(x, r)$, then, by construction of $\mathcal{F}^{n-1}(x, r)$, we must have that $I' \in \text{Adj}^{n-1}(T)$. But then $F$ cannot possibly belong to $\mathcal{F}^{n-1}$. On the other hand, if $T'$ is a tile, then also in this case $T$ cannot be in $\mathcal{F}^{n-1}$, since it should have been tessellated into tiles of the same size of $T'$.

Suppose now that $T$ is a tile itself. But by construction, we cannot have two tiles of different sizes lying on the same $(n-1)$-dimensional face. Thus $T' \subset T$ has to really be $T' = T$, which contradicts the fact that $\ell(T') < \ell(T)$.

Thus the definition of $\pi_2$ is coherent. Let us now define $\pi_2$ on those $(n-1)$-dimensional faces $T'$ of cubes in $\mathcal{C}^2_R(x, r)$ such that $\text{Int}(T') \notin \text{Int}(D_R(x, r))$ (recall the definition of $D_R(x, r)$, (6.3)). These are the faces which form the external boundary of the sheath $\mathcal{C}^2_R(x, r) \setminus \mathcal{C}_R(x, r)$. For these faces we leave everything unchanged, i.e. we let
\[(6.22) \quad \pi_2(y) = y \quad \text{for any} \quad y \in T,
\]
\[(6.23) \quad \text{where} \quad T \text{ is a } (n-1) \text{-dimensional face} \quad T \text{ with} \quad T \notin D_R(x, r).
\]
Finally, we extend $\pi_2$ to the whole of $D_R^2(x, r) \setminus D_R(x, r)$ by requiring that
\[(6.24) \quad \pi_2(I) \subset I \quad \text{for} \quad I \in \mathcal{C}^2_R(x, r) \setminus \mathcal{C}_R(x, r)
\]
\[(6.25) \quad \pi_2(y) = y \quad \text{whenever} \quad y \in \partial D_R^2(x, r).
\]
(This can be done in the same fashion as for (6.16)). We finally set
\[(6.25) \quad \pi_2(y) = y \quad \text{whenever} \quad y \in \mathbb{R}^n \setminus D_R^2(x, r).
\]
Hence (6.15)-(6.25) give us a Lipschitz map $\pi_2$ defined on the whole of $\mathbb{R}^n$. Now, if $d = n - 2$, then we can set $\varphi = \pi_2 \circ \pi_1$, otherwise we continue projecting. To do so, we define a third map $\pi_3$. We follow the procedure above: first, if $F$ is an $(n-2)$-dimensional element of $\mathcal{F}^{n-2}(x, r)$, then we set $\pi_3$ to be the radial projection from some point $c_F \in \text{Int}(F) \setminus \pi_2 \circ \pi_1(E)$ defined on $\partial F \cup (\pi_1 \circ \pi_2(E) \cap F)$. In particular, $\pi_3(y) = y$ if $y \in \partial F$. Next, we extend $\pi_3$ to the whole of $T$, by requiring that $\pi_3(T) \subset T$; if there is an element $E$ of $\mathcal{F}^{n-2}$ such that $(\pi_2 \circ \pi_1)(E) \cap F = \emptyset$, we set $\pi_3(y) = y$ on such an element. Note that this definition is coherent by construction of $\mathcal{F}^{n-2}(x, r)$, as in the definition of $\pi_2$. Next, we extend the definition of $\pi_3$ to the faces $T$ of dimension $(n-1)$, requiring that for any such a face, we have $\pi_3(T) \subset T$ and $\pi_3(\text{Int}(T)) \subset \text{Int}(T)$; we also require that $\pi_3(y) = y$ on those faces $T$ such that
Finally, we extend $\pi_3$ to the whole cubes $I$, requiring again that $\pi_3(I) \subset I$. At this point, note that for $y \in E$, we have

- either $\pi_3 \circ \pi_2 \circ \pi_1(y) \in \mathbb{R}^n \setminus D_R^2(x, r)$ if $y \in E \setminus D_R^3(x, r)$;
- or $\pi_3 \circ \pi_2 \circ \pi_1(y) \in T$, where $T$ is a $(n - 1)$-dimensional face of a cube in $\mathcal{F}_R^2(x, r)$ s.t. $T \not\subset D_R(x, r)$;
- or $\pi_3 \circ \pi_2 \circ \pi_1(y) \in F$, where $F \in \mathcal{F}^{n-3}$.

**Remark 6.3.** The second possibility only occurs for those $y \in E$ so that

$$y \in \bigcup_{I \in \mathcal{F}^2(x, r) \setminus \mathcal{F}(x, r)} I \subset \mathbb{R}^n \setminus B(x, r).$$

We continue constructing projections in this fashion until reaching the $d$-dimensional skeleton. At each step, we construct $\pi_m$, for $n - d \leq m \leq n$, first on the elements of $\mathcal{F}^n(x, r)$ as a radial projection, and second we extend this definition to faces (or tiles) of increasing dimension, asking (if $F'$ represents on such face or tile) that $\pi_m(F') \subset F'$. We stop once $\pi_{n-d}$ has been defined. If $y \in E$, then, setting

$$\pi := \pi_{n-d} \circ \cdots \circ \pi_1,$$

we see that

- either $\pi(y) \in \mathbb{R}^n \setminus D_R^2(x, r)$, if $y \in E \setminus D_R^3(x, r)$;
- or $\pi(y) \in T$, where $T$ is an $(n - 1)$-dimensional face of a cube in $\mathcal{F}_R^2(x, r)$ such that $T \not\subset D_R(x, r)$;
- or $\pi(y) \in F$, where $F \in \mathcal{F}^d(x, r)$.

Note that the definition of $\pi$ is coherent for the same reasons that $\pi_3$ and $\pi_2$ were coherent. In particular, $\pi$ is Lipschitz (with possible a very large Lipschitz constant, but we do not mind this). Moreover, it follows from the construction that the properties (6.8), (6.9) and (6.10) are satisfied; this concludes the proof of Lemma 6.1

### 7. TC implies STC

In this section, we will prove that the topological condition (TC), imposed on $E$, implies the condition STC on the approximating sets $E_R$ (see in Definition 5.1, see (5.1)-(5.8)). Our proof follows that of David’s, with a few tweaks. See [Dav04].

**Lemma 7.1.** Let $E \subset \mathbb{R}^n$ be such that $0 < \mathcal{H}^d(E) < \infty$. Suppose moreover that $E$ satisfies the $(r_0, \alpha_0, \delta_0, \eta_0)$-(TC), for some given parameters $r_0, \alpha_0, \delta_0, \eta_0$ and let $Q_0 \in \mathcal{D}(E)$ be such that $\ell(Q_0) < r_0$. For some $k_0 \in \mathbb{N}$, apply Lemma 4.3 to $Q_0$ to obtain a corona decomposition $\text{Top}(k_0) = \text{Top}(Q_0, k_0)$ and a family of sets $\{E_R\}_{R \in \text{Top}(k_0)}$ with parameter $\tau$. Then we can find parameters $r_1, \alpha_1, \delta_1$ and $\eta_1$, so that the family $\{E_R\}_{R \in \text{Top}(k_0)}$ satisfies the $C_2$-(STC) for $C_2$ sufficiently large.

We will prove this lemma through a few lemmata below.

Set

$$\tau < \frac{1}{100} \min\{\alpha_0, \eta_0\}.$$  

Now, let $\text{Top} = \text{Top}(k_0)$; recall that for a large constant $C_2$, we want to prove the existence of parameters $r_1, \alpha_1 < 1$ and $\delta_1$ (as in (5.1)) so that for all $x_1 \in E$, $R \in \text{Top}$ and $Q \in \text{Tree}(R)$ with $x_1 \in Q$, as in (5.2)-(5.3), for which (5.5) holds,
we have the lower bound (5.7). Let us immediately choose the parameters in (5.1) (our choice is that of David in [Dav04]). We set

\[(7.2) \quad r_1 = r_0, \text{ where } r_0 \text{ is the one given by (TC)};\]

\[(7.3) \quad \alpha_1 = C \min(\eta_0, \alpha_0);\]

\[(7.4) \quad \eta_1 = C \frac{\delta_0}{C_2};\]

\[(7.5) \quad \delta_1 = C \delta_0.\]

We will fix the various absolute constants $C$ as we go along. They will only depend on $n$. Let $x_1$, $R$ and $Q$ as in (5.2), (5.3) and (5.4). We now want to find a ball $B(x_2, r_2)$ with the required properties. We choose

\[(7.6) \quad x_2 = x_1 \text{ and}\]

\[(7.7) \quad r_2 \text{ s.t. } \frac{\ell(Q)}{3} \leq r_2 \leq \frac{2\ell(Q)}{3}.\]

We would also like the quantity $\mathcal{H}^d (E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2))$ to be small. Indeed, if we had that, for some choice of $r_2$, we held

\[(7.8) \quad \mathcal{H}^d (E_R \cap A_{\eta_1 \ell(Q)}(x_2, r_2)) \lesssim \delta_1 \ell(Q)^d,\]

then in order to verify (5.7) (adjusting the constant in the definition of $\delta_1$), we would only have to check that

\[(7.9) \quad \mathcal{H}^d (\varphi_1(E_R \cap B(x_2, r_2))) \geq \delta_1 \ell(Q)^d.\]

Let us show that such a choice of $r_2$ is indeed possible. Let $s_1, \ldots, s_N$ be a family of radii such that each one of them satisfies (7.7) and \{ $A_{\eta_1 \ell(Q)}(x_2, s_k)$ \}$_{k=1}^N$ is a pairwise disjoint family of (concentric) annuli. By the definition of $A_{\eta_1 \ell(Q)}(x_2, r_2)$, we have that

$$N \geq \frac{1}{\eta_1}.$$ 

Moreover, because these annuli are pairwise disjoint, we see that

$$\sum_{k=1}^N \mathcal{H}^d (E_R \cap A_{\eta_1 \ell(Q)}(x_2, s_k)) \leq \mathcal{H}^d (E_R \cap B(x_2, \ell(Q))) \leq C_2 \ell(Q)^d.$$ 

Then by the pigeonhole principle and (7.4) and (7.5), we must have that for some $1 \leq k \leq N$, (7.8) holds putting $r_2 = s_k$.

**Lemma 7.2.** Let $I \in \mathcal{C}_R(x_2, r_2)$, where $\mathcal{C}_R(x, r)$ is defined in (6.1); also, here $x_2 = x_1 \in Q$, for some $Q \in \text{Tree}(R)$, and $r_2$ is as in (7.7). Then

\[(7.10) \quad \ell(I) \leq C \tau \ell(Q).\]

**Proof.** It suffices to prove the lemma for $Q \in \text{Stop}(R)$. If $I \in \mathcal{C}_R$, then we know that $\ell(I) \lesssim \tau d_R(I)$ (recall that $\tau$ depends on $\alpha_0$ and $\eta_0$ and was fixed in (7.1)). Moreover, $d_R(\cdot)$ is 1-Lipschitz. Then if $y \in I \cap B(x_2, r_2)$, we have that

$$d_R(I) \leq d_R(y) \leq |y - x_2| + d_R(x_2).$$

Because $r_2 \sim \ell(Q)$, $|x_2 - y| \leq \ell(Q)$. On the other hand, we see that $d_R(x_2) = \inf_{P \in \text{Stop}(R)} (\ell(P) + \text{dist}(x_2, P)) \leq \ell(Q)$. \qed
Remark 7.3. Note that the same holds for any $I \in C_R^2(x_2, r_2)$, by definition of $C_R^2(x_2, r_2)$ (as defined in (6.4), (6.5) and (6.6)).

Lemma 7.4. Let $(x_2, r_2)$ to be as chosen in (7.6) and (7.7). For any one parameter family of Lipschitz deformations $\{\varphi_t\}_{0 \leq t \leq 1}$ satisfying (1.9), (1.10), (1.11) and (5.6) (relative to $(x_2, r_2)$), the property (7.9) holds, that is, we have

$$\mathcal{H}^d(\varphi_1(E_R \cap B(x_2, r_2))) \geq \delta_1 \ell(Q)^d.$$  

Proof. We have two ingredients we want to put together to achieve (7.9): on one hand, we know that something similar holds for $E$ (i.e. TC); on the other hand, we know that $E$ is locally well approximated by $E_R$, and we have a continuous (actually Lipschitz) way to move from $E$ to $E_R$ (i.e. the Federer-Fleming projection we constructed in the previous section). The idea is therefore the following: pick the one parameter family $\varphi_1$ for which we want to show (7.9), and pick $\pi$ as in Lemma 6.1. We will construct from these a deformation $f$ which satisfies conditions (1.9)-(1.12); hence from (TC), we will deduce (7.9).

Set

$$(7.11) \quad \pi_t(y) := t \pi + (1-t) \pi(y).$$

Note that

$$|\pi_t(y) - y| \leq t|\pi(y) - y| \leq n^{1/2} \ell(I),$$

if $y \in D_R^2(x_2, r_2)$; otherwise this quantity is equal to zero. By Lemma 7.2 we have that

$$(7.12) \quad \ell(I) \leq t \delta_R(y) \leq \tau (|y - x_2| + d_R(x_2)) \leq \tau \ell(Q).$$

Thus we have

$$(7.13) \quad |\pi(y) - y| \leq \tau \ell(Q) \text{ for all } y \in \mathbb{R}^n.$$

Let us now define $\{f_t\}_{0 \leq t \leq 1}$. We set

$$(7.14) \quad f_t(y) := \begin{cases} \pi_2(y) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \varphi_{2t-1}(\pi(y)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We claim that $\{f_t\}$ satisfies the conditions (1.9)-(1.12) applied to the larger ball $B(x_2, \tilde{r})$ where $\tilde{r} := (1 + \eta_0) r_2$.

We verify these conditions one by one. It is immediate from the definition that each $f_t$ is Lipschitz.

Claim. We have that $f_t(B(x, \tilde{r})) \subset \overline{B}(x, \tilde{r})$, i.e. (1.9) holds for $f_t$.

Note that

$$(7.15) \quad B(x_2, r_2) \cap E \subset D_R(x_2, r_2) \subset D_R^2(x_2, r_2),$$

where $D_R(x_2, r_2)$ and $D_R^2(x_2, r_2)$ were defined in (6.3) and (6.7); indeed (7.16) follows immediately from the definitions. Moreover, using Lemma 7.2, we see that any cube which was added to $C_R^2(x_2, r_2) \setminus C_R(x_2, r_2)$, must have side length at most $C \tau \ell(Q)$ (recall that $Q$ satisfies (5.4)). Thus $D_R^2(x_2, r_2) \subset B(x_2, r_2 + C \tau \ell(Q))$ and also, since

$$\tau \ell(Q) \leq \frac{1}{100} \eta_0 \ell(Q) \leq \eta_0 r_2,$$
(where we used (7.1) and (7.7)), we have that

\[(7.17) \quad \mathcal{D}^2_R(x_2, r_2) \subset B(x_2, r_2 + \tau \ell(Q)) \subset B(x_2, (1 + \eta_0) r) \subset B(x_2, \bar{r}),\]

Let us consider a few cases separately.

- If \( y \in \mathcal{D}^2_R(x_2, r_2) \), then \( \pi_t(y) \in \mathcal{D}^2_R(x_2, r_2) \) for all \( 0 \leq t \leq 1 \) (by Lemma 6.1), and so

\[f_s(y) \in B(x_2, \bar{r}) \text{ for } s \leq \frac{1}{2} \text{ whenever } y \in \mathcal{D}^2_R(x_2, r_2).\]

Now, recall that (1.9) holds for \( \varphi_t \) (relative to \( B(x_2, r_2) \)); hence if, together with \( y \in \mathcal{D}^2_R(x_2, r_2) \), we also have that \( \pi(y) \in B(x_2, r_2) \), we then conclude that \( f_s(y) \in \overline{B}(x_2, r_2) \) for \( s \leq 1 \). Also (1.11) holds for \( \varphi_t \): if \( \pi(y) \not\in B(x_2, r_2) \), then, for \( \frac{1}{2} \leq s \leq 1 \) and recalling (7.16), \( f_s(y) = \pi(y) \in \mathcal{D}^2_R(x_2, r_2) \). We obtain then that

\[(7.18) \quad f_s(y) \in B(x_2, \bar{r}) \text{ whenever } y \in \mathcal{D}^2_R(x_2, r_2).\]

Suppose now that \( y \in B(x_2, \bar{r}) \setminus \mathcal{D}^2_R(x_2, r_2) \); by construction \( \pi(y) = y \) whenever \( y \in \mathbb{R}^n \setminus \mathcal{D}^2_R(x_2, r_2) \), hence \( \pi_t(y) = y \) for \( t \in [0, 1] \), and thus

\[(7.19) \quad f_s(y) = y \text{ for } s \leq \frac{1}{2} \text{ whenever } y \in B(x_2, \bar{r}) \setminus \mathcal{D}^2_R(x_2, r_2).\]

- If \( \pi(y) \in B(x_2, r_2) \setminus \mathcal{D}^2_R(x_2, r_2) \), then \( \varphi_t(\pi(y)) \in \overline{B}(x_2, r_2) \) by (1.9) for \( \varphi_t \); if \( \pi(y) \in B(x_2, \bar{r}) \setminus (B(x_2, r_2) \cup \mathcal{D}^2_R(x_2, r_2)) \), then \( \varphi_t(\pi(y)) = \pi(y) = y \). Thus we obtain that

\[(7.20) \quad f_s(y) \in B(x_2, \bar{r}) \text{ for } \frac{1}{2} \leq s \leq 1 \text{ whenever } y \in B(x_2, \bar{r}) \setminus \mathcal{D}^2_R(x_2, r_2).\]

Now (7.18), (7.19) and (7.20) give us the property (1.9) for \( \{f_s\} \) relative to \( B(x_2, \bar{r}) \).

Claim. (1.10) holds for \( f_t \), that is, the path \( s \mapsto f_s(y) \) is continuous. But this is clear, since \( t \mapsto \pi_t(y) \) is continuous; moreover, \( \pi_1(y) = \pi(y) = \varphi_0(\pi(y)) = f_{\frac{1}{2}}(y) \), and \( t \mapsto \varphi_{2t-1}(\pi(y)) \) is also continuous.

Claim. Condition (1.11) holds.

Indeed, \( f_0(y) = \pi_0(y) = y \); if \( y \in \mathbb{R}^n \setminus B(x_2, \bar{r}) \), we have seen above that \( f_t(y) = y \). Thus (1.11) holds for \( \{f_s\} \) in \( B(x_2, \bar{r}) \).

We now verify that also (1.12) holds, i.e. we need to show that \( \text{dist}(f_s(y), E) \leq \alpha_0 \bar{r} \) for \( s \in [0, 1] \) and \( y \in E \cap B(x_2, \bar{r}) \). First, consider \( 0 \leq s \leq \frac{1}{2} \); let \( y \in E \cap B(x_2, \bar{r}) \); then we have

\[\text{dist}(f_s(y), E) = \text{dist}(\pi_{2s}(y), E) \leq |\pi_{2s}(y) - y| \]

\[\leq n \tau \ell(Q) \quad \text{(7.1)} \]

\[\leq \frac{1}{3} \alpha_0 \ell(Q) \quad \text{(7.7), (7.15)} \leq \alpha_0 \bar{r}. \]

Now suppose that \( s > \frac{1}{2} \). If \( y \in B(x_2, \bar{r}) \cap E \), then

\[\quad \text{(7.21) either } y \in \mathcal{D}^2_R(x_2, r_2) \]

\[\quad \text{(7.22) or } y \not\in \mathcal{D}^2_R(x_2, r_2) \cup B(x_2, r_2). \]

If \( y \) is so that (7.22) holds, then \( \pi(y) = y \), and moreover, from (1.11) for \( \varphi_t \) relative to \( B(x_2, r_2) \), we see that \( \varphi_{2s-1}(\pi(y)) = \varphi_{2s-1}(y) = y \). Hence (1.12) holds in this
On the other hand, suppose that (7.21) holds. Then
\begin{equation}
\text{(7.23)} \quad \text{either } \pi(y) \in E_R,
\end{equation}
\begin{equation}
\text{(7.24)} \quad \text{or } \pi(y) \in T, \text{ where } T \text{ is an } (n-1)\text{-face with } T \not\subset D_R(x_2, r_2).
\end{equation}

If (7.23) holds, then dist($\varphi_{2s-1}(\pi(y), E) \leq \alpha_1 \ell(Q)$ by (5.6) applied to $\varphi_1$; this immediately implies dist($f_s(y), E \leq \alpha_0 \bar{r}$ for $s > \frac{1}{2}$ by the choice of $\alpha_1$ in (7.3). On the other hand, if (7.24) holds, we must have $\pi(y) \notin B(x_2, r_2)$ (by construction of $\pi$), and therefore $\varphi_{2s-1}(\pi(y)) = \pi(y)$. Now $\pi(y)$ belongs to a cube in $\mathcal{C}^2_{\bar{r}}(x_2, r_2)$ with side length at most $\tau \ell(R)$ and touching $E$, hence we retrieve dist($f_s(y), E \leq \alpha \ell(R) \leq \alpha_0 \bar{r}$). Together with the previous estimates, we obtain that $\{f_s\}$ satisfies (1.12) for all $s \in [0, 1]$. That is to say, we have that by the TC
\begin{equation}
\text{(7.25)} \quad \mathcal{H}^d(B(x_2, (1 - \eta_0)\bar{r}) \cap f_1(E)) \geq \delta_0 \bar{r}^d.
\end{equation}

Since $B(x_2, r_2) \supset B(x_2, (1 - \eta_0)\bar{r})$, (7.25) implies that
\begin{equation}
\text{(7.26)} \quad \mathcal{H}^d(B(x_2, r_2) \cap f_1(E)) \geq \delta_0(1 + \eta_0)^d r_2^d.
\end{equation}

Recall from above that if $y \in E$ and it is such that $y /\notin D_R(x_2, r_2)$, then $f_1(y) /\notin B(x_2, r_2)$. Thus,
\[B(x_2, r_2) \cap f_1(E) \subset f_1(D_R(x_2, r_2) \cap E).\]

Note also that $\pi(E \cap D_R(x_2, r_2)) \subset E_R$. Thus we obtain, using (7.26), (7.8) and (7.5),
\[\mathcal{H}^d(\varphi_1(E_R) \cap B(x_2, r_2)) \geq 2 \delta_1 \ell(Q)^d.\]

This concludes the proof of Lemma 7.4. \hfill \Box

Lemma 7.4 and the considerations given above (7.9) prove Lemma 7.1.

8. A further approximating set

We now construct a dyadic approximation of $E_R$. We will then show that this approximation satisfies the STC; in the next section, we will show that this implies that this dyadic approximation has large intersection with a uniformly rectifiable set.

Let $\rho$ be a small parameter (which we will fix later, and can be assumed to be of the form $2^{-k}, k \in \mathbb{N}$). We write
\[\Delta_\rho := \Delta_{j(\rho)},\]
where $j(\rho)$ is an integer so that $2^{-j(\rho)} = \rho$.

We set
\begin{equation}
\text{(8.1)} \quad \mathcal{C}_{R, \rho} := \{I \in \Delta_\rho | I \cap E_R \neq \emptyset\};
\end{equation}
\begin{equation}
\text{(8.2)} \quad E_\rho = E_{R, \rho} := \bigcup_{I \in \mathcal{C}_{R, \rho}} \partial_d I.
\end{equation}

**Lemma 8.1.** Let $I_*$ be the smallest cube in $\mathcal{C}_R$ (which exists since $\mathcal{C}_R$ is finite). Then for all $\rho < \ell(I_*)$, $E_\rho$ is Ahlfors regular, with the same regularity constant as $E_R$. 

Proof. Let \( T \) be a \( d \)-dimensional face of some cube \( J \in \mathcal{C}_R \). Denote by \( \mathcal{F}_R^d \) the collection of \( d \)-dimensional faces from cubes in \( \mathcal{C}_R,\rho \). Then we can cover \( T \) with a disjoint subcollection of \( \mathcal{F}_R^d \). If we denote such a collection by \( \mathcal{F}_R^d(T) \), then it is obvious that

\[
\mathcal{H}_d(T) = \sum_{F \in \mathcal{F}_R^d} \mathcal{H}_d(F).
\]

To each such a face \( F \in \mathcal{F}_R^d \), there corresponds a bounded number of cubes so that \( F \subset I \in \mathcal{C}_R,\rho \). This bounded number depends only on \( n \) and \( d \). Moreover, each of these cubes has a bounded number of other \( d \)-dimensional faces, and, again, this number depends only on \( n \) and \( d \). Thus, if we denote by \( \Delta_\rho(T) \) the family of cubes in \( \Delta_\rho \) which also meet \( T \), we see that

\[
\sum_{I \in \Delta_\rho(T)} \mathcal{H}_d(I) \leq C(n,d) \sum_{F \in \mathcal{F}_R^d} \mathcal{H}_d(F) = C(n,d) \mathcal{H}_d(T).
\]

Then, we see that

\[
\mathcal{H}_d(E_\rho \cap B(x,r)) \leq \sum_{I \in \mathcal{C}_R,\rho} \mathcal{H}_d(I) \leq \sum_{J \in \mathcal{C}_R(x,r)} \sum_{T \text{ face of } J} \sum_{I \in \mathcal{F}_R^d(T)} \mathcal{H}_d(I) \leq C(n,d) \mathcal{H}_d(J) \leq C_3 r^d.
\]

Lower regularity is even easier since we are adding mass. \( \square \)

Lemma 8.2. The set \( E_\rho \) satisfies the condition STC relative to the scale of \( R \) with the same constants as \( E_R \).

Proof. Let \( C_2 \) be a constant sufficiently large (in particular, we need it to be larger than \( C_3 \), the regularity constant of \( E_R \) and \( E_\rho \)). We know from Section 7, that \( E_R \) satisfies the STC with a choice of constants as in (7.2) - (7.5). Let \( (x_2, r_2) \) as
in (7.6) and (7.7). We now add to the constraint on \( \rho \) given in the statement of Lemma 8.1, the following one: we ask that

\[
\rho < \frac{1}{1000\sqrt{n}} \min \{ \eta_1, \alpha_1 \} \ell(R).
\]

Note that because \( E_\rho \) is Ahlfors regular independently of \( \rho \), this does not cause any trouble.

Note that \( E_\rho \cap B(x_2, r_2) \supset E_R \cap B(x_2, r_2) \), simply because \( E_R \subset E_\rho \). Thus, also \( \varphi_1(E_\rho \cap B(x_2, r_2)) \supset \varphi_1(E_R \cap B(x_2, r_2)) \), and therefore

\[
\mathcal{H}^d(\varphi_1(E_\rho \cap B(x_2, r_2))) \geq \mathcal{H}^d(\varphi_1(E_R \cap B(x_2, r_2))) \geq \delta_1 \ell(R)^d + \mathcal{H}^d(E_R \cap A_{\eta_1, \ell(R)}(x_2, r_2)).
\]

But now note that if we choose a parameter \( c > 0 \) sufficiently small, and we put \( \tilde{\eta}_1 = c\eta_1 \), then we see that

\[
\mathcal{H}^d(E_R \cap A_{\tilde{\eta}_1, \ell(R)}(x_2, r_2)) \geq \mathcal{H}^d(E_\rho \cap A_{\tilde{\eta}_1, \ell(R)}(x_2, r_2)).
\]

Note that \( c \) only depends on \( n \) and \( d \). Hence we obtain that

\[
\mathcal{H}^d(\varphi_1(E_\rho \cap B(x_2, r_2))) \geq \delta_1 \ell(R)^d + \mathcal{H}^d(E_\rho \cap A_{\tilde{\eta}_1, \ell(R)}(x_2, r_2)),
\]

and the lemma is proven. \( \square \)

9. **STC implies that \( E_\rho \) has large intersections with some uniformly rectifiable set**

In this section, we show that the topological condition STC imposed on \( E_R \) (and thus on \( E_\rho \)) tells us that \( E_\rho \) has a large intersection (large with respect to the scale of \( R \)) with a uniformly \( d \)-rectifiable set. The idea is to define a functional whose minimizer \( F \) has large intersection with \( E_\rho \). In turn \( F \), by virtue of being a minimiser of such a functional, will turn out to be a quasiminimiser (in the sense of [DS00]), and thus uniformly rectifiable.

**Remark 9.1.** Once again, we follow David in [Dav04] and we adapt his proof to our current situation.

9.1. **Definition of a functional \( J \).** Let \( C_2 \) be a large constant and \( k_0 \) a sufficiently large integer. Then STC gives us constants \( r_1, \alpha_1 < 1, \eta \) and \( \delta_1 \) (as in (5.1)) such that for every choice of \( x_1 \in E \) and \( R \in \text{Top}(k_0) \) as in (5.2) and (5.3) (here we are applying STC with the cube \( Q \) to be equal to \( R \) itself), so that \( \mathcal{H}^d(E_R \cap B(x_1, \ell(R)) \leq C_2 \ell(R)^d \) holds, we can find a ball \( B(x_2, r_2) \subset B(x_1, \ell(R)) \) for which, given an appropriate one-parameter family of Lipschitz deformations \( \{ \varphi_k \} \), we have the lower bound (5.7). From the previous section, we see that this holds (at this specific scale \( \ell(R) \)) for both \( E_\rho \) and \( E_R \).

To implement the idea described above, we first need a functional: to define a functional, we need to clarify upon which type of sets it will act, and where these sets may be. In other words, we need to determine an appropriate class of subsets as domain of definition of such a functional. To simplify the notation, we put

\[
r := r_2; \\
x := x_2.
\]
For later use, let us set
\[
B_j := B\left(x, r - \eta_1 \ell(R) + \frac{j\eta_1 \ell(R)}{10}\right) \text{ for } 0 \leq j \leq 10.
\]

Then we put
\[
\Delta_\sigma(B_6) := \{I \in \Delta_j(\sigma) \mid I \cap \partial B_6 = \emptyset\}; \\
C_1^\sigma(B_6) := \{I \in \Delta_\sigma(B_6) \mid I \cap E_R = \emptyset\}; \\
C_2^\sigma(B_6) := \{I \in \Delta_\sigma(B_6) \mid \exists J \in C_1^\sigma(B_6) \text{ with } I \cap J = \emptyset\}.
\]

Finally we put
\[
V^1 := \bigcup_{I \in \mathcal{C}_2^\sigma(B_6)} I \quad \text{ and } \\
V^2 := \bigcup_{I \in \mathcal{C}_1^\sigma(B_6)} I.
\]

Note that for any cube $I \in \mathcal{C}_R(B_6)$, there exists a cube $J \in \mathcal{C}_\sigma(B_6)$ so that $J \supset I$.

**Lemma 9.2.** With the notation above, we have that
\[
B_6 \cap E \subset V^2 \subset B_7 \quad \text{and} \\
dist(y, E) \leq \alpha_1 \ell(R) \text{ for all } y \in V^1.
\]

**Proof.** The first inclusion in (9.10) is immediate. To see the second one, note that for any point $y \in V^2$, we have that $dist(y, V^1) \leq \sqrt{n} \rho$. In turn, any point in $V^1 \setminus B_6$ can be at most $\sqrt{n} \rho$ away from $\partial B_6$. Hence, by the choice of $\sigma$ in (9.4), we see that if $y \in V^2 \setminus B_6$, $dist(y, \partial B_6) < \frac{1}{\eta_1} \eta_1 \ell(R)$, and so (9.10) will be satisfied. Similarly for (9.11): first, if $y \in V^1$, then $dist(y, E_R) \leq \rho \leq \frac{1}{1000} \ell(R)$. On the other hand, if $y' \in E_R$, then we have that $dist(y', E) \leq \tau \ell(R)$ (this follows from Lemma 7.2). But also $\tau < \min \{\eta_0, \alpha_0\}$ (by (7.1)) and thus $\tau \leq \alpha_1$ (using (7.3)). Therefore, letting $y' \in E_R$ be such that $dist(y, E_R) \sim |y - y'|$, we obtain
\[
\text{dist}(y, E) \leq |y - y'| + \text{dist}(y', E) \leq \rho + \tau \ell(R) \leq \alpha_1 \ell(R).
\]

We are now ready to fix the class of subsets upon which the said functional will be allowed to act. We set
\[
\mathcal{F}_0
\]
and to be the class of subsets $F$ of $\mathbb{R}^n$ such that
\[
F \text{ is closed (in the topology of } \mathbb{R}^n). \\
F \subset V^2 \\
F = F^* \cup L.
\]
Here $L$ denotes any subset of Hausdorff dimension smaller or equal than $d - 1$; by $F^*$ we mean a finite union of $d$-dimensional faces of cubes coming from $\Delta_\rho$. We will call $F^*$ the coral part of $F$. In other words the class $\mathcal{F}_0$ is composed by subsets that are built out of a finite number of $d$-faces coming from cubes in $\Delta_\rho$.

Let us consider a subclass of $\mathcal{F}_0$: we set
\begin{equation}
\mathcal{F} := \{ F \in \mathcal{F}_0 \mid F = \varphi_1(E_\rho \cap V^2) \},
\end{equation}
where $\{ \varphi_t \}_{0 \leq t \leq 1}$ is a family of Lipschitz mappings on $\mathbb{R}^n$ such that
\begin{itemize}
\item[(9.17)] $\varphi_t(V^2) \subset V^2$ for all $t \in [0, 1]$;
\item[(9.18)] $t \mapsto \varphi_t(y)$ is continuous for all $y \in \mathbb{R}^n$;
\item[(9.19)] $\varphi_t(y) = y$ for $t = 0$ and for $y \in \mathbb{R}^n \setminus V^2$;
\item[(9.20)] $\text{dist}(\varphi_t(y), E_R) < \alpha_1 \ell(R)$ for $y \in E_\rho \cap V^2$ and all $t \in [0, 1]$;
\item[(9.21)] $\varphi_1(y) \in V^1$ for $y \in E_\rho \cap V^2$.
\end{itemize}

**Lemma 9.3.** We have that $E_\rho \cap V^2 \in \mathcal{F}$. In particular, the class $\mathcal{F}$ is nonempty.

**Proof.** We just take the trivial deformation $\varphi_t(y) = y$ for all $y$ and $t$, so that (9.17), (9.18) and (9.19) hold immediately. Moreover, by construction we have that all points in $E_\rho$ are contained in a cube from $\mathcal{C}_{R,\rho}$. The side length of these cubes is (much) less than $C_3 \delta_1$ and they must touch $E$. Hence $\text{dist}(y, E_R) \leq \tau \ell(R)$ and so (9.20) is satisfied. As for condition (9.21) is concerned, we see that if $y \in E_\rho \cap V^2$, then by definition of $E_\rho$ and $\mathcal{C}_{R,\rho}$ in (8.2) and (8.1), we see that $y$ must lies in a cube which belongs to $\mathcal{C}_2(B_6)$ (from the definition of $\sigma$ in (9.4)), and thus it must be in $V^1$. \hfill \Box

Let now define the aforementioned functional. For some $c_2 < 1$ to be chosen later, we put
\begin{equation}
M = \frac{C_3}{c_2 \delta_1},
\end{equation}
where recall that $C_3$ is the Ahlfors regularity constant of $E_R$ (as fixed in Notation 4.13). Then we set
\begin{equation}
J(F) := \mathcal{H}^d(F \cap E_\rho) + M \mathcal{H}^d(F \setminus E_\rho) \text{ for } F \in \mathcal{F}.
\end{equation}
Note that $J(F) = J(F^*)$ (with notation as in (9.15)), and there is only a finite number of sets like $F^*$. Thus there exists a set $\bar{F} \in \mathcal{F}$ such that
\begin{equation}
J(\bar{F}) = \min_{F \in \mathcal{F}} J(F).
\end{equation}

Note that, for a set $F \in \mathcal{F}$ trying to keep $J(F)$ small, it will be very expensive to have a large portion which does not intersect $E_\rho$, as $M$ can be quite large. This is the reason why we expect the minimiser $\bar{F}$ to have a large intersection with $E_\rho$. This also implies that a minimiser of $J$ also will lie close to $E_R$.

**Lemma 9.4.** Let $\bar{F}$ be a minimiser of $J$ (as in (9.23)) in $\mathcal{F}$. Then
\begin{equation}
\mathcal{H}^d(E_\rho \cap \bar{F}) \geq C \delta_1 \ell(R)^d.
\end{equation}

Once again, the proof below follows that of David in [Dav04]. We include it for completeness.
Proof. Because $\bar{F} \in \mathcal{F}$, then $\bar{F} = \varphi_1(E_\rho \cap V^2)$, where $\{\varphi_t\}$ satisfies (9.17)-(9.21). We want to check that this specific one parameter family $\varphi_t$ satisfies also the conditions for the deformations used for STC (see Definition 5.1). Note that $\{\varphi_t\}$ satisfies (1.9), (1.10) and (1.11), since from (9.10), we have that $V^2 \subset B(x, r)$. We want to check that (5.6) holds, that is, that

$$\text{dist}(\varphi_t(y), E_R) \leq \alpha_1 \ell(R) \text{ for } t \in [0, 1] \text{ and } y \in E_\rho \cap B(x, r).$$

So, let $y \in E_\rho \cap B(x, r)$. If $y \notin V^2$, then $\varphi_t(y) = y$ by (9.19) and so (5.6) is satisfied (recall (8.3)). If $y \in V^2 \cap E_\rho$, then by (9.21), $\varphi_t(y)$ must lie in $V^1$, and hence be at most $\sigma$ far away from $E_R$ (by construction); but $\sigma < \alpha_1 \ell(R)$, and so (9.25) holds.

Thus $\{\varphi_t\}$ is a one parameter family of Lipschitz deformations which satisfies the requirements stated in the definition of STC (see Definition 5.1). Recall that $E_\rho$ to satisfies the STC (at scale $\ell(R)$); we therefore have the lower bound (8.4), i.e.

$$\mathcal{H}^d(\varphi_1(E_\rho \cap B(x, r))) \geq \delta_1 \ell(R)^d + \mathcal{H}^d(E_\rho \cap A_{\tilde{\eta}_1 \ell(R)}(x, r)).$$

Now, the family $\{\varphi_t\}$ which we are considering, not only satisfies (1.11), but also (19.19), and so, in particular,

$$\varphi_1 (E_\rho \cap B(x, r)) = \varphi_1 \left( (V^2 \cap E_\rho) \cup (E_\rho \cap (B(x, r) \setminus V^2)) \right)$$

$$= \varphi_1 (E_\rho \cap V^2) \cup \left( E_\rho \cap (B(x, r) \setminus V^2) \right),$$

recalling that by definition (see (9.10)) $V^2 \subset B(x, r)$. Also, note that

$$E_\rho \cap (B(x, r) \setminus V^2) \subset A_{\tilde{\eta}_1 \ell(R)}(x, r) \cap E_\rho;$$

indeed, using $B_6 \cap E_\rho \subset V^2$, we see that $E_\rho \cap (B(x, r) \setminus V^2) \subset E_\rho \cap (B(x, r) \setminus B_6)$, and (recalling (9.3)), $B(x, r) \setminus B_6 \subset A_{\tilde{\eta}_1 \ell(R)}(x, r)$. Thus we have

$$\mathcal{H}^d(\varphi_1(E_\rho \cap V^2)) + \mathcal{H}^d(A_{\tilde{\eta}_1 \ell(R)}(x, r) \cap E_\rho) \geq \delta_1 \ell(R)^d + \mathcal{H}^d(E_\rho \cap A_{\tilde{\eta}_1 \ell(R)}(x, r))$$

and so

$$\mathcal{H}^d(\varphi_1(E_\rho \cap V^2)) \geq \delta_1 \ell(R)^d.$$

In particular, from the definition of $\mathcal{F}$, this inequality holds for any $F \in \mathcal{F}$. Recall now that we decided that $\bar{F}$ was a minimiser of $J$ (as defined in (9.23)). Thus we have that

$$J(\bar{F}) \leq J(\varphi_1(E_\rho \cap V^2)) \leq \mathcal{H}^d(\varphi_1(E_\rho \cap V^2)).$$

Moreover, by definition of $J$,

$$\mathcal{H}^d\left(\bar{F} \setminus E_\rho\right) \leq \frac{J(\bar{F})}{M} \leq \frac{C_3}{M} \ell(R)^d \leq c_2 \delta_1 \ell(R)^d.$$

But then we have that

$$\mathcal{H}^d\left(\bar{F} \cap E_\rho\right) = \mathcal{H}^d(\bar{F}) - \mathcal{H}^d(\bar{F} \setminus E_\rho) \geq \frac{\delta_1 \ell(R)^d}{2}.$$

This proves the Lemma. \qed
9.2. Almgren quasiminimality of $\tilde{F}$. Roughly speaking, a set $S$ in $\mathbb{R}^n$ is a quasiminimiser of the $d$-dimensional Hausdorff measure $\mathcal{H}^d$ if, whenever we deform $S$ in a suitable way, the $d$-measure of such deformations does not shrink too much. Quasiminimality is a form of stability: the set maintain its Hausdorff dimension under a suitable class of perturbations. Heuristically, this is the reason why we need to transfer the topological condition from $E$ to an Ahlfors regular set: in this case, modulo technicalities, quasiminimality roughly coincides with our topological condition.

We now recall from [DS00] the precise definitions to make this notion precise. Let $U$ be an open set in $\mathbb{R}^n$ and fix two constants

\begin{equation}
1 \leq k < \infty \text{ and } 0 < \delta \leq +\infty.
\end{equation}

Let $S \subset U$ be so that

\begin{equation}
S \neq \emptyset \text{ and } \overline{S} \setminus S \subset \mathbb{R}^n \setminus U.
\end{equation}

Assume also that

\begin{equation}
\mathcal{H}^d(S \cap B) < +\infty \text{ for all balls } B \subset U.
\end{equation}

Now, let us make precise what we mean by ‘deformations’ or ‘perturbations’. Given a set $S$, deformations of $S$ will be sets of the form $\phi(S)$, where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and satisfies the following properties.

\begin{align*}
\text{diam}(W \cup \phi(W)) &\leq \delta \text{ where } W := \{ x \in \mathbb{R}^n \mid \phi(x) \neq x \}; \\
\text{dist}(W \cup \phi(W), \mathbb{R}^n \setminus U) &> 0; \\
\phi &\text{ is Lipschitz-homotopic to the identity.}
\end{align*}

The last requirement means that there exists a continuous map

\[ h : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n \]

such that $h(x, 0) = x$ and $h(x, 1) = \phi(x)$ for all $x \in \mathbb{R}^n$, such that $h(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz for all $t \in [0, 1]$, and such that

\[ \text{diam}(\hat{W}) < \delta \text{ and } \text{dist}(\hat{W}, \mathbb{R}^n \setminus U) > 0, \]

where

\[ \hat{W} := \bigcup_{t \in [0, 1]} W_t \cup \phi_t(W_t), \quad \phi_t(x) = h(x, t) \text{ and } W_t := \{ x \in \mathbb{R}^n \mid \phi_t(x) \neq x \}. \]

**Definition 9.5.** Let $0 < d < n$; let $U \subset \mathbb{R}^n$ be an open set and fix two constant $k, \delta$ as in (9.32). We say that $S \subset U$ is a $(U, k, \delta)$-quasiminimizer for $\mathcal{H}^d$ if $S$ satisfies (9.33), (9.34) and

\begin{equation}
\mathcal{H}^d(S \cap W) \leq k \mathcal{H}^d(\phi(S \cap W))
\end{equation}

for all Lipschitz mappings $\phi$ which satisfy (9.36), (9.37) and (9.38).

**Lemma 9.6.** The set

\begin{equation}
S := \tilde{F} \cap B_2
\end{equation}

is a $(B_2, k, \delta)$-quasiminimizer for $\mathcal{H}^d$, where

\begin{equation}
k = C 4^{nd} M,
\end{equation}

where $M$ is the Lebesgue measure of $B_2$. \

(here $C$ is a geometric constant), and
\begin{equation}
\delta = c_3 \min \{\alpha_1, \eta_1\} \ell(R).
\end{equation}
Here $0 < c_3 < 1$ is a parameter that we will fix later.

We will need the following lemma from [Dav04]. We tailor it to our current notation.

**Lemma 9.7 ([Dav04], Lemma 5.8)**. Let $V^1$ be a finite union of dyadic cubes belonging to $\Delta_\sigma$, where $\sigma$ should be thought of as in (9.4). There exists a $4^n$-Lipschitz function $h$, defined on $V_1 := \{ y \in \mathbb{R}^n \mid \text{dist}(y, V^1) \leq \frac{\sigma}{4}\}$ (9.43) and such that
\begin{equation}
V_1^1 := \{ y \in \mathbb{R}^n \mid \text{dist}(y, V^1) \leq \frac{\sigma}{4}\}
\end{equation}
and such that
\begin{align}
& h(V_1^1) \subset V^1 \\
& h(y) = y \text{ for } y \in V^1, \\
& |h(y) - y| \leq n^{1/2} \sigma.
\end{align}

Recall that we want to show that $\tilde{F}$ is a quasiminimal set for $\mathcal{H}^d$. Here is the idea to do so. We want to look at $\mathcal{H}^d(\phi(\tilde{F} \cap W))$; what we know about $\tilde{F}$ which makes us hope that it may well be a quasiminimal set is that $\tilde{F}$ is a minimiser of the functional $J$ as defined in (9.23). We want to use this information. In other words, we would like to say that $\phi(\tilde{F})$ is a competitor of $\tilde{F}$ belonging to the class $\mathcal{F}$. Unfortunately, this is not true, in the sense that $\phi(\tilde{F})$ may lie outside $V^1$, and this is not permitted (see (9.21)). What we can do however, is first to retract $\phi(\tilde{F})$ (which we will call $F_1$) back into $V^1$ (using the map $h$ from Lemma 9.7); let us set $F_2 := h(F_1)$. Next, we want to project $F_2$ onto some $d$-dimensional skeleton so that it belongs to $\mathcal{F}_0$ (as defined in (9.12)). This projection will happen in two steps, with two corresponding maps; we will denote the images so obtained by $F_3$ and then $F_4$; this latter one will be the needed competitor. The last step will be to show that these distortions of $\phi(\tilde{F})$ don’t increase the size of $\phi(\tilde{F})$ too much. In this way, first by the minimising property of $\tilde{F}$ we will obtain a bound like $\mathcal{H}^d(\tilde{F}) \leq M \mathcal{H}^d(F_4)$ and then, by this last step, a bound similar to $\mathcal{H}^d(F_4) \leq C \mathcal{H}^d(\phi(\tilde{F}))$ and thus establishing quasiminimality.

**9.2.1. Constuction of $F_1$ and $F_2$.** Let us get started: we want to deform $\tilde{F}$ with Lipshitz maps $\phi$ as in (9.35). Pick one such Lipshitz deformation $\phi$. We are interested in those points $y \in W \cap \tilde{F}$, i.e. those points which are actually being moved by $\phi$. But by (9.36), we must have that $|\phi(y) - y| \leq \delta$. We put
\begin{equation}
\delta = c_3 \min \{\alpha_1, \eta_1\} \ell(R),
\end{equation}
where $c_3$ is a small parameter to be chosen soon. The rationale to choose $c_3$ is that we want $\phi(\tilde{F})$ to lie in $V_1^1$, so that we may apply Lemma 9.7 and send $\phi(\tilde{F})$ back into $V^1$. Recall that $V_1^1$ is the set of points lying at most $\sigma/4$ far away from $V^1$; recall also that $\tilde{F} = \phi_1(E_\rho \cap V^2) \subset V^1$ by (9.16) (the way $\mathcal{F}$ was defined) and the
property of \{\varphi_1\}, (9.21). Hence for an appropriate choice of \(c_3\), say \(c_3 = \frac{1}{301\sqrt{n}}\) (see (9.4)) we have that if \(y \in W \cap \tilde{F}\), then

\[(9.48) \quad \phi(y) \in V_1.\]

We set

\[(9.49) \quad F_1 := \phi(\tilde{F}) \quad \text{and} \quad (9.50) \quad F_2 := h(F_1) = h(\phi(\tilde{F})).\]

In particular, \(F_2 \subset V^1\).

9.2.2. Construction of \(F_3\). We want to project \(F_2\) back into a \(d\)-dimensional skeleton, since this is a requirement to belong to \(F_0\) (and so eventually to \(\mathcal{F}\)). By definition of \(\mathcal{F}_0\), we will be projecting onto the \(d\)-skeleton of cubes coming from \(\Delta_\rho\).

We will use the following Lemma, which is taken from [DS00].

**Lemma 9.8** (Lemma 11.14, [DS00]). Let \(j \in \mathbb{Z}\) and let \(A\) be a compact subset of \(\mathbb{R}^n\) such that \(\mathcal{H}^d(A) < \infty\). Denote by \(N(A)\) the union of all the cubes \(I \in \Delta_j\) that touch a cube in \(\Delta_j\) which intersects \(A\). Then there is a Lipschitz mapping \(f : \mathbb{R}^n \to \mathbb{R}^n\) with the following properties.

\[(9.51) \quad f(x) = x \quad \text{for} \quad x \in \mathbb{R}^n \setminus N(A);\]
\[(9.52) \quad f(x) = x \quad \text{for all} \quad x \in S_{j,d};\]
\[(9.53) \quad f(A) \subset S_{j,d};\]
\[(9.54) \quad f(I) \subset I \quad \text{for all} \quad I \in \Delta_j;\]
\[(9.55) \quad \mathcal{H}^d(f((A \cap I) \setminus S_{j,d})) \leq C \mathcal{H}^d((A \cap I) \setminus S_{j,d}) \quad \text{for all} \quad I \in \Delta_j.\]

Recall the definition of \(S_{j,d}\) in (2.2).

We now apply Lemma 9.8 with \(j = -j(\rho)\) and \(A = F_2 = h(\phi(\tilde{F}))\), and thus we set

\[(9.56) \quad F_3 = f(F_2) = (f \circ h \circ \phi)(\tilde{F}).\]

**Remark 9.9.** Let us note a couple of facts. First, we see that if \(y \in \tilde{F} \setminus W\), then \(\phi(y) = y\) (by definition of \(W\), as in (9.36)); but, still with the same \(y\), also \(h(\phi(y)) = h(y) = y\), since \(y \in V^1\) already, and \(h\) does not move such points (as in (9.45)), and further, \(f(h(\phi(y))) = f(y) = y\) by (9.52), since \(y \in \tilde{F}\), and therefore it belongs to the \(d\)-face of some cubes from \(\Delta_\rho\).

**Lemma 9.10.** With the notation as above, we have

\[(9.57) \quad \dim \left(F_3 \setminus S_{-j(\rho),d}\right) \leq d - 1.\]

**Proof.** By Remark 9.9, we already know that \(f(h(\phi(\tilde{F} \setminus W))) = \tilde{F} \setminus W \subset S_{-j(\rho),d}\).

On the other hand, we must have that \(f(h(\phi(\tilde{F} \cap W))) \subset S_{-j(\rho),d}\) by (9.53). Thus the Lemma follows. \(\square\)
9.2.3. **Construction of** $F_4$ **and** $F_4 \in \mathcal{F}$. Note that $F_3$ is not necessarily a union of full $d$-dimensional faces: the projection $f$ is into and not necessarily onto.

**Lemma 9.11.** There exists a Lipschitz map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ so that $\pi(F_3)$ is precisely the union of those $d$-dimensional faces which were contained in $F_3$.

**Proof.** See [Dav04], pages 211-212. The idea of the proof is to consider those faces $T$ which intersect $F_3$ on a set of positive $d$-dimensional measure but that are not contained in $F_3$. On these faces, exactly because they are not contained in $F_3$, we can define a Lipschitz map $\pi$ which sends whatever lies of $F_3$ in one such face to its $(d-1)$-dimensional boundary. The result, $F_4$, will be a set of full $d$-faces plus a set of dimension smaller or equal to $d-1$. $\square$

We now set

$$F_4 := \pi(F_3).$$

**Lemma 9.12.** With notation as above,

$$F_4 \in \mathcal{F}.$$ 

**Proof.** Once again, see [Dav04], pages 212 to 215. $\square$

Hence $F_4$ is a valid competitor in $\mathcal{F}$ of $\tilde{F}$. But $\tilde{F}$ is a minimiser of the functional $J$ in this class, and therefore we have the inequality

$$J(\tilde{F}) \leq J(F_4).$$

We will use this inequality in the following subsection to finally prove that $\tilde{F}$ is also a quasiminimiser of $\mathcal{H}^d$.

9.2.4. **$\tilde{F}$ is a quasiminimiser.** First, note that $F_4 \subset F_3$, except perhaps from a set of dimension smaller than, or equal to, $d-1$. Thus, using also (9.60), we have that

$$J(\tilde{F}) \leq J(F_3).$$

Recall the definition of $W$ in (9.36) and that of $J$ in (9.23). Writing $\tilde{F} = \left(\tilde{F} \cap W\right) \cup \left(\tilde{F} \setminus W\right)$, and using the additivity of $\mathcal{H}^d$, we have

$$J(\tilde{F}) = J(\tilde{F} \cap W) + J(\tilde{F} \setminus W).$$

Let us set

$$\Phi(y) = (f \circ h \circ \phi)(y).$$

With this notation we see that $F_3 = \Phi(\tilde{F})$ (this is just (9.56)). Moreover, recall from Remark 9.9, that $\Phi(\tilde{F}) = \Phi(\tilde{F} \cap W) \cup \Phi(\tilde{F} \setminus W)$, and in turn, that $\Phi(\tilde{F} \setminus W) = \tilde{F} \setminus W$, and thus $J(\Phi(\tilde{F} \setminus W)) = J(\tilde{F} \setminus W)$, which is immediate from the definition of $J$ as in (9.23). In particular we get that

$$J(F_3) = J(\Phi(\tilde{F}))$$

$$\leq J\left(\Phi(\tilde{F} \cap W)\right) + J\left(\Phi(\tilde{F} \setminus W)\right)$$

$$= J\left(\Phi(\tilde{F} \cap W)\right) + J\left(\tilde{F} \setminus W\right).$$
We thus
\[ J(\tilde{F}) = J(\tilde{F} \cap W) + J(\tilde{F} \setminus W) \leq J(\Phi(\tilde{F} \cap W)) + J(\tilde{F} \setminus W), \]
which, subtracting \( J(\tilde{F} \setminus W) \) from both sides, gives,
\[ J(\tilde{F} \cap W) \leq J(\Phi(\tilde{F} \cap W)). \]
But note that, by the definition of the functional \( J \) in (9.23),
\[ H^d(\tilde{F} \cap W) \leq J(\tilde{F} \cap W) \leq J(\Phi(\tilde{F} \cap W)) \]
\[ = H^d(\Phi(\tilde{F} \cap W) \cap E_{\rho}) + M H^d(\Phi(\tilde{F} \cap W) \setminus E_{\rho}) \leq M H^d(\Phi(\tilde{F} \cap W)). \]
That is,
\[ H^d(\tilde{F} \cap W) \leq M H^d(\Phi(\tilde{F} \cap W)). \]
(9.62)
Note that (9.62) resembles the comparison estimate (9.39): we need to swap \( \Phi \) with \( \phi \). To do so, we need to show that up constants, the maps \( f \) and \( h \) did not increase the mass of \( \phi(\tilde{F} \cap W) \). Let us worry about \( f \) first. We write
\[ A_1 := h(\phi(\tilde{F} \cap W)) \cap S_{-j(\rho),d}; \]
(9.63)
\[ A_2 := h(\phi(\tilde{F} \cap W)) \setminus S_{-j(\rho),d}. \]
(9.64)
Now, because \( f(y) = y \) whenever \( y \in S_{-j(\rho),d} \), we immediately have that
\[ H^d(f(A_1)) = H^d(A_1). \]
(9.65)
Let us look at \( A_2 \). Because of (9.54) and the fact that dyadic cubes have bounded overlap, we can write
\[ H^d(f(A_2)) \leq \sum_{I \in \Delta_{-j(\rho)}} H^d(f(A_2 \cap I)) \leq C \sum_{I \in \Delta_{-j(\rho)}} H^d(f(\overline{A_2} \cap I \setminus S_{-j(\rho),d})). \]
To estimate this last sum, we use (9.55):
\[ C \sum_{I \in \Delta_{-j(\rho)}} H^d(f(\overline{A_2} \cap I \setminus S_{-j(\rho),d})) \]
\[ \leq C' \sum_{I \in \Delta_{-j(\rho)}} H^d(\overline{A_2} \cap I \setminus S_{-j(\rho),d}) \]
\[ \leq C' H^d(h(\phi(\tilde{F} \cap W) \setminus S_{-j(\rho),d})). \]
Putting together these estimates, we see that
\[ H^d(\Phi(\tilde{F} \cap W)) \leq H^d(A_1) + C' H^d(h(\phi(\tilde{F} \cap W)) \setminus S_{-j(\rho),d}). \]
(9.66)
**Lemma 9.13.** With the notation as above, we have
\[ \dim\left(h(\phi(\tilde{F} \cap W)) \setminus (h(\phi(\tilde{F} \cap W)) \cup S_{-j(\rho),d}) \right) \leq d - 1. \]
(9.67)
**Proof.** See the proof of equation 5.60 in [Dav04].
Using (9.67), we then can write
\[ H^d(A_1) + C'H^d(h(\phi(\tilde{F} \cap W)) \setminus S_{-j(\rho),d}) \]
\[ \leq H^d(A_1) + C'H^d(h(\phi(\tilde{F} \cap W))) - j(\rho) \]
\[ = H^d(A_1) + C'H^d(A_2) \]
\[ \leq C'H^d(h(\phi(\tilde{F} \cap W))). \]

(9.68)

Hence, (9.66) and (9.68) tell us that
\[ H^d(\Phi(\tilde{F} \cap W)) \leq C'H^d(h(\phi(\tilde{F} \cap W))). \]

(9.69)

Now note that because \( \pi \) is Lipschitz with constant \( 4^n \) as for Lemma 9.7, we immediately see that
\[ H^d(h \circ \phi(\tilde{F} \cap W)) \leq 4^d H^d(\phi(\tilde{F} \cap W)). \]

(9.70)

The two estimates (9.69) and (9.70) together show that \( \tilde{F} \cap B_2 \) is a \((B_2, k, \delta)\)-quasiminimal set (with \( B_2 \) as defined in (9.3), \( k = 4^n C'M \) and \( \delta = c_3 \min \{ \alpha, \eta \} \ell(R) \)).

9.3. A uniformly rectifiable set covering the minimising set \( \tilde{F} \). In this short subsection, we will use the main result of [DS00], to show that \( \tilde{F} \) can be locally covered by a uniformly rectifiable set.

Lemma 9.14. With notation as above (in particular recall the definition of \( B_1 \) in (9.3)), we have that
\[ \tilde{F} \cap B_1 \subset Z, \]
where \( Z \) is a uniformly \( d \)-rectifiable set.

Let us recall the main result in [DS00].

Theorem 9.15 ([DS00], Theorem 2.11). Let \( U \) be an open set in \( \mathbb{R}^n \), and suppose that \( S \) is a \((U, k, \delta)\)-quasiminimizer for \( H^d \). Let \( S^* \) be the support in \( U \) of the restriction of \( H^d \) to \( S \). Then for each \( x \in S^* \) and radius \( R_0 \) which satisfy
\[ 0 < R_0 < \delta \text{ and } B(x, 3R_0) \subset U, \]
there is a compact, Ahlfors \( d \)-regular set \( Z \) such that
\[ S^* \cap B(x, R_0) \subset Z \subset S^* \cap B(x, 2R_0) \]
and
\[ Z \text{ is uniformly rectifiable and contains big pieces of Lipschitz graphs.} \]
The constants for the Ahlfors regularity and for the uniform rectifiability of \( Z \) depend only on \( n \) and \( k \).

Proof of Lemma 9.14. Recall that \( \tilde{F} \) is a minimiser of the functional \( J \) over \( \mathcal{F} \), see definitions (9.23) and (9.16). By Lemma 9.6, we know that \( S = \tilde{F} \cap B_2 \) is a \((B_2, k, \delta)\)-quasiminimizer for \( H^d \). Also recall that
\[ B_2 = B \left( x, r - \frac{8}{10} \ell(R) \right) \]
(see (9.3)).
that $k = C^{4d}M$, and $\delta = c_3 \min\{\alpha_1, \eta_1\} \ell(R)$. Then, making $c_3$ smaller if needed, we see that for all points $y \in \tilde{F} \cap B_1$, we have

$$B(y, 3\delta) \cap \tilde{F}^* \subset B_2,$$

where recall that $\tilde{F}^*$ is the coral part of $\tilde{F}$ (see (9.15)). By Theorem 9.15, we see that there exists a uniformly rectifiable set $Z_y$ so that

$$\tilde{F}^* \cap B(y, \delta/2) \subset Z_y \subset \tilde{F}^* \cap B(y, \delta),$$

since we can chose $R_0 \geq \delta/2$. Now, clearly

$$\bigcup_{y \in B \cap \tilde{F}} B(y, \delta/2) \supset \tilde{F} \cap B_1.$$

Moreover, we can find a finite subfamily of balls $\{B(y, \delta/2)\}_{j=0}^N$ such that

$$\tilde{F} \cap B_1 \subset \bigcup_{i=0}^N B(y_i, \delta/2)$$

and

$$N \leq C = C(n, \eta_1).$$

To see this, recall that $r(B_1) = r - 9/10\eta_1(\ell(R))$. This, by the choice of $r = r_2$ in (7.7), and choosing the constant $C$ in (7.4) appropriately, gives that $r(B_1) \leq \ell(R)/2$. Hence, since $\delta = c_3 \min\{\alpha_1, \eta_1\}$, we need at most $C$ balls in (9.76), where $C$ depends only on $n, \alpha_1$ and $\eta_1$. Now, for this each one of these balls, we take the corresponding uniformly rectifiable set $Z_{y_i}$ as in (9.75), and we set

$$Z_x := \bigcup_{i=0}^N Z_{y_i}.$$

Then $Z_x$ is a uniformly rectifiable set (with uniform constant now depending on $N$, and thus $\alpha_1$ and $\eta_1$) so that

$$\tilde{F}^* \cap B_1 \subset \bigcup_{i=0}^N B(y_i, \delta/2) \cap \tilde{F}^* \subset \bigcup_{i=0}^N Z_{y_i} = Z_x.$$

This proves the lemma. \hfill $\square$

**Remark 9.16.** A short summary of what has been done so far will be useful to the reader in the subsequent section.

We started off with a surface $E$ satisfying the topological condition (TC) with some prescribed parameters $r_0, \alpha_0, \eta_0$ and $\delta_0$. We took the corona construction from Lemma 4.3, and we showed that TC on $E$ implies a skeletal topological condition on all the approximating $E_R$ in the coronisation (Section 7, Lemma 7.1). Next, we constructed a further approximating Ahlfors regular set $E_{R, \rho}$ (see (8.2)), to then show that for any point $x \in R$ (see the choice of $x_2$ in (7.6)), there is a $(B_2, k, \delta)$-quasiminiser set $\tilde{F} = \tilde{F}(R, x)$ such that, first,

$$\mathcal{H}^d(\tilde{F} \setminus E_{\rho}) \leq c_2 \delta \ell(R)^d,$$

and
— this is equation (9.30)); and second, that there exists a uniformly $d$-rectifiable set $Z_x$ so that

$$B_1 \cap \tilde{F}^* \subset Z_x; \quad (9.79)$$

— this is Lemma 9.14.

**Lemma 9.17.** Let $R \in \text{Top}(k_0)$. Then there exists a uniformly $d$-rectifiable set $Z_R$ and a set $\tilde{F}_R$ which is a the union of a uniformly finite family of quasiminimal sets so that,

$$\mathcal{H}^d(\tilde{F}_R \cap E_{R,\rho}) \geq \delta_0 \ell(R)^d, \quad (9.80)$$

and,

$$\mathcal{H}^d(\tilde{F}_R \setminus E_{R,\rho}) \leq c_2 \delta_1 \ell(R)^d; \quad (9.81)$$

and,

$$\tilde{F}_R \subset Z_R. \quad (9.82)$$

**Proof.** Now, recall from (9.3), that

$$B_1 = B(x, r_2 - \frac{9}{10} \eta_1 \ell(R)), \quad (9.83)$$

and also recall from (5.9), that $r_2 > \eta_1 \ell(R)$. In particular we have that

$$B(x, \eta_1 \ell(R)/10) \subset B_1.$$ 

Note that to cover $R$, we need at most $N' \sim n \eta_1^{-n}$ balls centered on $R$ and with radius $\eta_1 \ell(R)/10$. Pick one such collection that is also minimal, and thus of bounded overlap. Let it be

$$B^j := B(x_j, \eta_1 \ell(R)/10), \quad (9.84)$$

For each $1 \leq j \leq N'$, there correspond a quasiminimal set $\tilde{F} = \tilde{F}(R, x_j)$ (and its coral part $\tilde{F}^*$) satisfying (9.78), and a corresponding uniformly $d$-rectifiable set $Z_{x_j}$ satisfying (9.79). We now set

$$\tilde{F}_R := \bigcup_{j=1}^{N'} \tilde{F}(R, x_j), \quad (9.85)$$

and

$$Z_R := \bigcup_{j=1}^{N'} Z_{x_j}. \quad (9.86)$$

It is then easy to see that (9.80), (9.81) and (9.82) hold.

**Corollary 9.18.** For $R \in \text{Top}(k_0)$, let $Q \in \text{Tree}(R)$. Then there exists a uniformly $d$-rectifiable set $Z_Q$ and a set $\tilde{F}_Q$ which is the union of a uniformly finite family of quasiminimal sets so that $\mathcal{H}^d(\tilde{F}_Q \cap E_{R,\rho}) \geq \delta_0 \ell(Q)^d$, $\mathcal{H}^d(\tilde{F}_Q \setminus E_{R,\rho}) \leq c_2 \delta_1 \ell(Q)^d$ and $\tilde{F}_Q \subset Z_Q$. 

$\square$
This follows immediately from the arguments that lead to the previous lemma. Indeed, the only property of \( R \) that was used to show the existence of the quasi-minimiser \( \tilde{F}_R \) was that \( R \) satisfied the STC. However, any \( Q \in \text{Tree}(R) \) satisfies the same property, hence the Corollary.

**Lemma 9.19.** For each pair \((x, r) \in E_{R,\rho} \times (0, \ell(R))\), there exists a uniformly \( d \)-rectifiable set \( Z_{x,r} \) such that

\[
H^d(E_{R,\rho} \cap Z_{x,r} \cap B(x, r)) \geq \delta_0 r^d.
\]

The Ahlfors regularity, uniform rectifiability and BPLG constants for the sets \( Z_{x,r} \) are uniform in \((x, r)\).

**Proof.** If \( x \in E_{\rho} \), then by the construction of \( E_{\rho} \) (as in (8.1) and (8.2)), there exists a dyadic cube \( I \in C_{R} \) such that \( \text{dist}(x, \partial_d I) < \ell(I) \). Recall also that for each \( I \in C_{R} \), there exists a surface cube \( Q_I \in \text{Tree}(R) \) such that \( \text{dist}(Q_I, I) \leq c\tau^{-1} \ell(I) \) and \( \ell(I) \leq \ell(Q_I) \leq c\tau^{-1} \ell(I) \). This cube is given by Lemma 4.7.

Now, take two constants \( C, C' \) to be fixed below, depending on the constants \( c, c' \).

1. Suppose first that

\[
C\tau^{-1} \ell(I) \leq r \leq C' \ell(Q_I).
\]

Choosing \( C \) appropriately, we can insure that \( B(x, r) \supset B_{Q_I} \).

But from Lemma 9.17, we know that

\[
H^d(E_{\rho} \cap Z_{Q_I}) = H^d(E_{\rho} \cap Z_{Q_I} \cap B(x, r)) \geq \delta_1 \ell(Q)^d.
\]

Since \( r \sim \ell(Q_I) \), where the constants behind \( \sim \) depend on \( C, C', c, c' \), then we conclude that there is an absolute constant \( C'' \) so that

\[
H^d(E_{\rho} \cap Z_{Q_I} \cap B(x, r)) \geq C'' \delta_0 r^d.
\]

This gives the Lemma for this case.

2. Suppose now that

\[
0 < r < C\tau^{-1} \ell(I).
\]

Let \( T \) be a \( d \)-face of \( \partial_d I \), and let \( T(r) \) be tile of \( T \) containing \( x \) and with

\[
\ell(T(r)) \sim \min \left\{ \frac{1}{10} r, \ell(I) \right\}.
\]

Then clearly,

\[
T(r) \subset E_{\rho} \cap B(x, r) \text{ for all } r > 0,
\]

and \( T(r) \) is a uniform rectifiable set with constants independent of \( r \). Now, note that if \( C\tau^{-1} > r > \ell(I) \), then

\[
H^d(E_{\rho} \cap T(r) \cap B(x, r)) \geq H^d(T(r)) = c\ell(I)^d \sim \tau, c, c', r^d.
\]

On the other hand, if \( 0 < r < \ell(I) \), we have

\[
H^d(E_{\rho} \cap B(x, r) \cap T(r)) \geq H^d(T(r)) = cr^d.
\]

In any case, we found a uniformly rectifiable set which intersects \( E_{\rho} \) with measure bounded below uniformly. This gives the Lemma in this case.
(3) Now, if $C \ell(Q_1) < r < \ell(R)$, we can repeat the arguments of point (1) for some parent of $Q_1$ appropriately chosen. By construction, this parent will be belong to $\text{Tree}(R)$, and thus the same estimates apply.

□

From the theory of uniformly rectifiable sets (see [DS93]), we deduce the following.

**Corollary 9.20.** With notation as above, $E_\rho = E_{R,\rho}$ is a uniformly $d$-rectifiable set with Ahlfors regularity, uniform rectifiability and BPLG constants dependent only on those of the intersecting UR sets $Z_{x,r}$.

For future use, let us pin down an easy fact about the distance between $R \subset E$ and $E_\rho = E_{R,\rho}$.

**Lemma 9.21.** For each $x \in S$, with $S \in \text{Stop}(R)$, we have

$$\text{dist}(x, E_\rho) \leq C\ell(S).$$

**Proof.** Using again Lemma 4.6, we see that if $S \in \text{Stop}(R)$, then there exists a cube $I_S \in \mathscr{C}_R$ such that $\zeta_S \subset I_S$ and $\ell(S) \sim \tau^{-1}\ell(I_S)$. Thus in particla, if $x \in S$, then $\text{dist}(x, E_R) \lesssim \ell(S) \sim \tau^{-1}\ell(I_S)$. Further, by construction of $E_\rho = E_{R,\rho}$, we have that $E_R \subset E_\rho$. This proves the Lemma. □

10. Estimates on the $\beta$ coefficients and the end of the proof

In this section we give the final estimates on the Jones’ $\beta$ coefficient which will prove Theorem 1.5.

Recall the notation $D(k_0)$ from Lemma 4.3. Theorem 1.5 will easily follow from the Lemma below.

**Lemma 10.1.** Let $p = p(d)$ be as in (2.7) and fix an arbitrary (but sufficiently large) integer $k_0 > 0$. There exists a constant $C_0 \geq 1$ (independent of $k_0$) so that, with the above notation,

$$\sum_{Q \subset Q_0, Q \in D(k_0)} \beta_{E}^{p,d}(C_0Q)^2 \ell(Q)^d \leq C \mathcal{H}^d(E \cap B_{Q_0}),$$

where $C$ depends on $n, d$, but not on $k_0$.

We will use the following lemma.

**Lemma 10.2** ([AS1], Lemma 2.21). Let $1 \leq p < \infty$ and $E_1, E_2 \subset \mathbb{R}^n$. Let $x \in E_1$ and fix $r > 0$. Take some $y \in E_2$ so that $B(x, t) \subset B(y, 2t)$. Assume that $E_1, E_2$ are both lower content $d$-regular. Then

$$\beta_{E_1}^{p,d}(x, t) \lesssim \beta_{E_2}^{p,d}(y, 2t) + \left(\frac{1}{t^d} \int_{E_1 \cap B(x, 2t)} \left(\frac{\text{dist}(y, E_2)}{t}\right)^p \frac{d\mathcal{H}^d(y)}{t}\right)^{\frac{1}{p}}.$$

Let us now get started. Recall that the definition of $\text{Forest}(R)$ in (4.10).

**Sublemma 10.3.** Let $p = p(d)$ be as in (2.7) and fix an arbitrary integer $k_0 > 0$. There exists a constant $C_0 \geq 1$ (independent of $k_0$) so that, with the above notation,

$$\sum_{Q \in \text{Tree}(R), Q \subset D(k_0)} \beta_{E}^{p,d}(C_0Q)^2 \ell(Q)^d \lesssim \ell(R)^d,$$
where the constant behind the symbol $\lesssim$ depends on $n, d, C_0$, and the Ahlfors regularity constant of $E_R$, but not on $k_0$.

**Sublemma 10.4.** Let $R \in \text{Top}(k_0)$. Then

\begin{equation}
\sum_{P \in \text{Tree}(R)} \beta_{E}^{p,d}(P)^2 \ell(P)^d \lesssim \ell(R)^d.
\end{equation}

**Proof.** We want to apply Lemma 10.2 with $E_1 = E$ and $E_2 = E_{R,\rho}$. For $Q \in D$, recall that $\zeta_Q$ denotes the center of $Q$. By the definition of $\text{Forest}(R)$, we see that if $Q \in \text{Forest}(R)$, then there must exists a dyadic cube $I \in \mathcal{C}_R$ which meets $Q$. The $d$-skeleton $\partial_d I$ of $I$ is part of $E_{R,\rho}$. We see that $|I| \lesssim \tau \ell(Q)$. Hence there exists a point $x'_Q \in E_{R,\rho}$ such that $|x_Q - x'_Q| \leq 4\tau \ell(Q)$, and we obtain that

$$B_Q := B(x_Q, \ell(Q)) \subset B(x'_Q, 2\ell(Q)) =: B'_Q.$$ 

This implies that for each cube $Q \in \text{Forest}(R)$ the hypotheses of Lemma 10.2 are satisfied (with $E_1 = E$ and $E_2 = E_{R,\rho}$); we may then write

$$\sum_{P \in \text{Tree}(R)} \beta_{E}^{p,d}(C_0 B_P)^2 \ell(P)^d \lesssim \sum_{P \in \text{Tree}(R)} \beta_{E_{R,\rho}}^{p,d}(2C_0 B'_P)^2 \ell(P)^d$$

$$+ \sum_{P \in \text{Tree}(R)} \left( \frac{1}{\ell(P)^d} \int_{2C_0 B_P \cap E} \left( \frac{\text{dist}(y, E_{R,\rho})}{\ell(P)} \right)^p dH^d_{\infty}(y) \right) \frac{\ell^2}{\ell(P)^d} \ell(P)^d$$

$$:= I_1 + I_2.$$ 

First, let us look at $I_1$. We apply Theorem 2.1 to $E_{R,\rho}$; let us denote the cubes so obtained by $D_{E_{R,\rho}}$. Note that for each $P \in \text{Forest}(R)$ with $P \in D(k_0)$, $x'_P$ belongs to some cube $P' \in D_{E_{R,\rho}}$ so that $\ell(P') \sim \ell(P)$; hence there exists a constant $C_4 \geq 1$ so that

$$2C_0 B'_P \subset C_4 B_{P'}.$$ 

This in turn implies that $\beta_{E_{R,\rho}}^{p,d}(2C_0 B'_P)^2 \ell(P)^d \lesssim_{p,n,d,C_0,C_4} \beta_{E_{R,\rho}}^{p,d}(C_4 B_{P'})^2 \ell(P')^d$. Hence,

\begin{equation}
\sum_{P \in \text{Tree}(R)} \beta_{E}^{p,d}(C_0 B_P)^2 \ell(P)^d \lesssim_{p,n,d,C_0,C_4} \sum_{P \in \text{Tree}(R)} \beta_{E_{R,\rho}}^{p,d}(C_4 B_{P'})^2 \ell(P')^d.
\end{equation}

Since $E_{R,\rho}$ is uniformly rectifiable (by Corollary 9.20), we immediately have that $I_1 \lesssim \ell(R)^d$. Let us now estimate $I_2$. Now let $y \in 2C_0 R$; by Lemma 9.21, there exists a cube $S \in \text{Stop}(R)$ such that

\begin{equation}
\text{dist}(y, E_{R,\rho}) \lesssim \ell(S);
\end{equation}
We can estimate the integral in $I_2$ with (10.5) as follows.

\[
\int_{2C_0B_R \cap E} \left( \frac{\text{dist}(y, E_{R,P})}{\ell(P)} \right)^p \, dH_\infty^d(y) \leq \sum_{\mathcal{P} \in \mathcal{N}(\mathcal{P})} \sum_{S \subset \mathcal{P}'} \sum_{\mathcal{S} \subset \mathcal{P}'} \frac{\ell(S)^d}{\ell(P)^d}.
\]

and so

\[
I_2 \lesssim \sum_{\mathcal{P} \in \text{Tree}(\mathcal{R})} \sum_{\mathcal{P}' \in \mathcal{N}(\mathcal{P})} \sum_{S \subset \mathcal{P}'} \frac{\ell(S)^{d+2}}{\ell(P)^{d+2}}.
\]

We now swap the sums (which are all finite), to obtain that

\[
I_2 \lesssim \sum_{S \in \text{Stop}(\mathcal{R})} \ell(S)^{\frac{d}{p}+2} \sum_{\mathcal{P} \in \text{Tree}(\mathcal{R})} \sum_{\mathcal{P}' \in \mathcal{N}(\mathcal{P}) \cap \text{Stop}(\mathcal{R})} \frac{1}{\ell(P)^{d+2}} \quad \text{if } S \supset \mathcal{P}' \gtrsim_{d,n} \sum_{S \in \text{Stop}(\mathcal{R})} \ell(S)^{\frac{d}{p}+2} \sum_{\mathcal{P} \in \text{Tree}(\mathcal{R})} \sum_{\mathcal{P}' \in \mathcal{N}(\mathcal{P}))} \frac{1}{\ell(P)^{d+2}} \quad \text{if } \mathcal{P}' \supset Q.
\]  

We see that the number of cubes $\mathcal{P} \in \text{Tree}(\mathcal{R})$ of a given generation so that there exists a sibling $\mathcal{P}' \in \mathcal{N}(\mathcal{P})$ for which $\mathcal{P}' \supset S$ is bounded above by a universal constant depending on $n$ and $C_0$. Thus we can sum geometrically the interior series:

\[
\sum_{\mathcal{P} \in \text{Tree}(\mathcal{R})} \sum_{\mathcal{P}' \in \mathcal{N}(\mathcal{P})} \frac{1}{\ell(P)^{d+2}} \approx_n \sum_{S \in \text{Stop}(\mathcal{R})} \ell(S)^{d+2} \sum_{\mathcal{P} \in \text{Tree}(\mathcal{R})} \sum_{\mathcal{P}' \in \mathcal{N}(\mathcal{P})} \frac{1}{\ell(P)^{d+2}}.
\]

Therefore we obtain

\[
(10.6) \lesssim \sum_{S \in \text{Stop}(\mathcal{R})} \ell(S)^{d+2} \sum_{\mathcal{P} \in \text{Tree}(\mathcal{R})} \sum_{\mathcal{P}' \in \mathcal{N}(\mathcal{P})} \frac{1}{\ell(P)^{d+2}} = \sum_{S \in \text{Stop}(\mathcal{R})} \ell(S)^d.
\]

This latter sum is bounded above by $C\ell(\mathcal{R})^d$. This concludes the proof of the lemma.

\begin{proof}[Proof of Lemma 10.1] Let $Q_0$ as above and let $R_0$ be the minimal cube in Top($Q_0$) so that $Q_0 \subset R_0$. Then we write

\[
\sum_{Q \subset Q_0 \atop Q \in \mathcal{D}(k_0)} \beta^p(\mathcal{P}Q)^2 \ell(Q)^d \leq \sum_{R \in \text{Top}} \sum_{Q \in \text{Tree}(\mathcal{R})} \sum_{Q \subset Q_0 \atop Q \in \mathcal{D}(k_0)} \beta^p(\mathcal{P}Q)^2 \ell(Q)^d
\]

\[
+ \sum_{Q \in \text{Tree}(\mathcal{R}_0) \atop Q \subset Q_0 \atop Q \in \mathcal{D}(k_0)} \beta^p(\mathcal{P}Q)^2 \ell(Q)^d =: \Sigma_1 + \Sigma_2.
\]

\end{proof}
First, let us look at $\Sigma_2$: by the same argument of Lemma 10.3, we see that this sum is bound above by $\ell(Q_0)^d$. On the other hand, by Lemma 10.3, we see that
\begin{equation}
\Sigma_1 \lesssim \sum_{R \in \text{Top}, R \subset Q_0} \ell(R)^d.
\end{equation}

Note that each $R \in \text{Top}$ is the child of some stopped cube $R'$. Recall we stopped at a surface cube $R' \in D$ whenever it happened that $R' \cap I$ and $\ell(I) \sim \ell(Q)$ for some $I \in \text{Bad}$. We can therefore associate to each $R \in \text{Top}$ a bad dyadic cube $I$, and thus, by (4.7), we have that
\begin{equation}
\sum_{R \in \text{Top}, R \subset Q_0} \ell(R)^d \lesssim \sum_{I \in \text{Bad}, I \subset B_{Q_0}} \ell(I)^d \lesssim \mathcal{H}^d(E \cap B_{Q_0}).
\end{equation}

The estimate in (4.7) is independent of $k_0$, so is the one we obtained here. All in all, we see that, by lower $d$-regularity,
\begin{equation}
\Sigma_1 + \Sigma_2 \lesssim \ell(Q_0)^d + \mathcal{H}^d(E \cap B_{Q_0}) \lesssim \mathcal{H}^d(E \cap B_{Q_0}).
\end{equation}

This concludes the proof of Lemma 10.1.

**Remark 10.5.** Note that the estimate in (10.1) is independent of $k_0$, and therefore holds for all $k_0 \in \mathbb{N}$. Hence we can sum over all cubes $Q \subset Q_0$ and we have the estimate
\begin{equation}
\sum_{Q \subset Q_0} \beta_E^d(C_0 Q)^2 \ell(Q)^d \lesssim C(n, d, C_0) \mathcal{H}^d(E \cap B_{Q_0}).
\end{equation}

It follows from Theorem I in [AS18] that
\begin{equation}
\mathcal{H}^d(E \cap B_{Q_0}) \lesssim 1 + \sum_{Q \in D, Q \subset Q_0} \beta_E^{dp}(C_0 Q)^2 \text{diam}(Q)^d
\end{equation}

This completes the proof of Corollary 1.7.

### 11. Covering by Topologically Stable Surfaces

In this section, we show that for any set $E$ which is lower content regular, there exists a TC surface $\Sigma$ such that $\Sigma \supset E$. Such a surface will be constructed by ‘filling holes’ with dyadic skeleta of the size of the holes.

Let us briefly sketch this construction. Fix let $Q_0 \in D$ be a chosen top cube (if $E$ is bounded we can let $E = Q_0$). For two constants $\epsilon > 0$ and $A > 1$, put
\begin{align*}
\mathcal{B}_{\text{BWGL}}^{Q_0} &:= \{Q \in D(Q_0) \mid d_{ABQ}(Q, L) > \epsilon \text{ for all } d - \text{planes}\}; \\
\mathcal{B}_{\text{BWGL}}^{Q_0} &:= D(Q_0) \setminus \mathcal{B}_{\text{BWGL}}^{Q_0}.
\end{align*}

Pick now a further parameter $\kappa > 0$ to be fixed below. If $Q \in \mathcal{B}_{\text{BWGL}}^{Q_0}$, we find an integer $k(Q) \in \mathbb{N}$, so that $\frac{1}{\kappa} \ell(Q) \leq 2^{-k(Q)} \leq 2\kappa \ell(Q)$. We then set
\begin{equation}
S(Q) := \{I \in \Delta_{k(Q)} \mid I \cap Q \neq \emptyset\},
\end{equation}
and
\begin{equation}
E_Q := \bigcup_{I \in S(Q)} \partial_4 I.
\end{equation}

It is immediate that $E_Q$ is Ahlfors regular, with constants depending on $n$ and $\kappa$. 
We now define the covering surface: we put
\[ \Sigma = \Sigma(\epsilon, \kappa, Q_0) := Q_0 \cup \bigcup_{Q \in \mathcal{BWGL}(Q_0)} E_Q. \]

Remark 11.1. Let us quickly remark that the surface constructed in (11.1) satisfies the following. For some fixed \( \epsilon > 0 \) and \( A > 1 \), we have that for each \( x \in \Sigma \) and for each \( 0 < r < \text{diam}(\Sigma)/A \), it holds that either there exists a plane \( P_{x,r} \) with \( d_{x,Ar}(\Sigma, P) < \epsilon \) or there exists a family \( \mathcal{F} = \mathcal{F}(x,r) \) of \( d \)-dimensional planes so that \( d_{x,Ar}(\bigcup_{P \in \mathcal{F}} P, \Sigma) < \epsilon \). This follows immediately by the construction of \( \Sigma \).

Lemma 11.2. Let \( \Sigma = \Sigma(\epsilon, \kappa, Q_0) \) be as defined in (11.1). Then we have that
\[ \mathcal{H}^d(\Sigma) \leq C(n,d) \mathcal{BWGL}(Q_0). \]

Recall the definition of \( \mathcal{BWGL}(Q_0) \) in (2.8).

Proof. We have that
\[ \mathcal{H}^d(\Sigma) \leq \mathcal{H}^d(Q_0) + \mathcal{H}^d\left( \bigcup_{Q \in \mathcal{BWGL}(Q_0)} E_Q \right) \]
\[ \leq \mathcal{H}^d(Q_0) + \sum_{Q \in \mathcal{BWGL}(Q_0)} \mathcal{H}^d(E_Q) \]
\[ \leq \mathcal{H}^d(Q_0) + C(n,d) \sum_{Q \in \mathcal{BWGL}(Q_0)} \ell(Q)^d. \]
By definition
\[ \sum_{Q \in \mathcal{BWGL}(Q_0)} \ell(Q)^d = \mathcal{BWGL}(Q_0). \]
The lemma then follows.

Lemma 11.3. Let \( \epsilon, \kappa \) be positive (small) parameters. Then \( \Sigma = \Sigma(\epsilon, \kappa) \) satisfies the topological condition (TC) with constants \( r_0 = \text{diam}(\Sigma)/2 \sim \text{diam}(E), 0 < \eta_0 < 1/100 \) and \( \alpha_0 \) and \( \delta_0 \) sufficiently small depending only on \( \epsilon, \kappa \).

Proof. Let \( \Sigma \) be defined as in (11.1). We want to find parameter \( r_0, \alpha_0, \delta_0 \) and \( \eta_0 \) so that for all \( \alpha_0\text{-ALD} \{ \varphi_1 \} \) and for all \( x_0 \in \Sigma \) and \( 0 < r < r_0 \), we have that
\[ \mathcal{H}^d(B(x,(1 - \eta_0)r) \cap \varphi_1(\Sigma)) \geq \delta_0 r^d. \]
We immediately choose
\[ r_0 = \text{diam}(\Sigma)/2. \]

Now, let \( (x, r) \in \Sigma \times (0, r_0) \). Moreover, let \( \delta_0, \alpha_0 \) and \( \eta_0 \) be positive parameters to be fixed below. We consider two separate cases.

Case 1. Suppose first that \( x \in E \). Let \( \kappa_1 > 0 \) be a parameter to be fixed below. Then there is a cube \( Q_{x,r} \in \mathcal{D} \) such that
\[ 10^{-3}\lambda \kappa_1 r \leq \ell(Q_{x,r}) \leq \kappa_1 r, \text{ and } B_{Q_{x,r}} \subset B(x,r). \]
Here recall that \( \lambda \) is the parameter from the Christ-David cubes construction (Theorem 2.1); the constant \( 10^{-3} \) could be substituted with any other constant sufficiently small. We consider three subcases, depending on what type of cube \( Q_{x,r} \) is.
(1) If

\[ Q_{x,r} \in \mathcal{Q}^{BWGL}(Q_0) \]

then, by construction, \( Q_{x,r} \) must satisfy the following. There exists a plane \( L_{x,r} \) such that

\[ d_{ABQ_{x,r}}(L_{x,r}, E) < \epsilon. \]

Now, by Remark 11.1, we can choose a subset of \( \Sigma \), denoted by \( \Sigma' \), so that for each \( x' \in \Sigma \), we can find a \( d \)-plane so that so that \( d_{x',A'}(P, \Sigma') < \epsilon \). We also make sure that the point \( x \in \Sigma \) under consideration, also satisfies \( x \in \Sigma' \).

Next, we apply a result of David, Theorem 1.10 from [Dav14], to obtain a smooth surface \( \tilde{\Sigma} \), such that \( \text{dist}(x, \Sigma') < C\epsilon r \) for all \( x \in \tilde{\Sigma} \) and \( \text{dist}(x, \Sigma) < C\epsilon r \) for all \( x \in \Sigma' \).

Now, being a smooth surface, \( \tilde{\Sigma} \) satisfies local property of being linked to a \((n-d-1)\)-dimensional sphere, as in the definition of Semmes surfaces in Definition 3.1. This property is stable, thus holds for \( \Sigma' \) at \( x \). One can then use the proof of David of Lemma 3.2, and the fact that \( \Sigma' \subset \Sigma \), to obtain the lower bound

\[ \mathcal{H}^d(\varphi_1(E) \cap B(x, (1 - \eta_0)r)) \geq \delta_0 r, \]

for any \( \alpha_0\text{-ALD} \varphi_t \), with parameters depending on \( \epsilon > 0 \).

(2) Suppose now that \( Q_{x,r} \in \mathcal{Q}^{BWGL}(Q_0) \).

Then, by construction, there is a dyadic cube \( I \in S(Q(x,r)) \cap \Delta_{\ell}(Q) \) such that \( x \in I \) and \( \frac{1}{2}\kappa\ell(Q) \leq \ell(I) \leq 2\kappa\ell(Q) \). In particular,

\[ \ell(I) \geq 10^{-4}\kappa_1\lambda r. \]

Moreover, if we choose \( \kappa_1 \) appropriately (depending also on \( 0 < \eta_0 < 1/100 \)), we can make sure that

\[ E_Q \subset B(x, (1 - \eta_0)r). \]

Hence, if we choose

\[ \alpha_0 < 10^{-5}\lambda\kappa\kappa_1, \]

we get that

\[ \varphi_1(\partial_d I) \subset N_{10^{-5}\lambda\kappa r}(\partial I) \subset B(x, (1 - \eta_0)r). \]

Choosing

\[ 0 < \eta_0 < 1/100 \]

(the precise upper bound for this parameter is not relevant here), we obtain that

\[ \mathcal{H}^d(B(x, (1 - \eta_0)r) \cap \varphi_1(\partial_d I)) = \mathcal{H}^d(\varphi_1(\partial_d I)) \geq C(\kappa_1\lambda\kappa)^d r^d \]

If we now set

\[ \delta_0 = C(\kappa_1\lambda\kappa)^d, \]

and

\[ \alpha_0 < 10^{-5}\kappa_1\lambda\kappa, \]
we obtain that, whenever \( \varphi_t \) is an \( \alpha_0 \)-ALD,
\[
\mathcal{H}^d(\varphi_1(\Sigma) \cap B(x,(1-\eta_0)r)) \geq \mathcal{H}^d(\varphi_1(\partial_d I Q)) \geq \delta_0 r^d.
\]
Hence we see that for \((x,r) \in E \times (0,\text{diam}(\Sigma)/2)\), the topological condition (TC) is satisfied with constants \( \alpha_0 = \min\{c, 10^{-6} r \lambda \kappa\} \), \( \delta_0 = \min\{c', C(\tau \lambda \kappa)^d\} \) and \( 0 < \eta_0 < 1/100 \).

**Case 2.** If \( x \in \Sigma \setminus E \), then by construction,
\[
x \in E_Q \text{ for some } Q \in B^{BWGL}(Q_0).
\]
In particular, \( x \in \partial_d I \), for some \( I \in S(Q) \). We will consider two subcases.

1. Suppose that \( 0 < r < C_c^{-1} \ell(Q) \), for some constant \( C \). Let \( T_I \) be the \( d \)-dimensional face of \( I \) containing \( x \), and let \( T(r) \subset T \) be a tile of \( T \) of side length \( 10^{-1} \epsilon r \) which also contains \( x \). If we choose \( \alpha_0 \) sufficiently small, say \( \alpha_0 < 10^{-5} \epsilon \), and \( 0 < \eta_0 < 1/100 \), then we that for any \( \alpha_0 \)-ALD \( \varphi_t \),
\[
\varphi_1(T(r)) \subset N_{\alpha_0 r}(T(r)) \subset B(x,(1-\eta_0)r),
\]
and thus
\[
\mathcal{H}^d(\varphi_1(\Sigma) \cap B(x,(1-\eta_0)r)) \geq \mathcal{H}^d(\varphi_1(\partial_d I) \cap B(x,(1-\eta_0))) \\
\geq \mathcal{H}^d(\varphi_1(T(r)) \cap B(x,(1-\eta_0)r)) \\
= \mathcal{H}^d(\varphi_1(T(r))) \\
\geq (10^{-2} \epsilon)^d r^d.
\]

2. Suppose now that \( C_c^{-1} \ell(Q) \leq r \leq \text{diam}(\Sigma) \). There exists a cube \( \hat{Q} \in \mathcal{D} \) so that \( \hat{Q} \supset Q \) and so that \( r \sim \ell(\hat{Q}) \) (here we can choose the constant behind \( \sim \) to depend on \( \kappa_1 \) and \( \lambda \) as in (11.2)). Now, if \( \hat{Q} \in B^{BWGL} \), we can argue as Case 1, (1). If \( \hat{Q} \in B^{BWGL}(Q_0) \), we can argue as in Case 1, (2).

We now choose the constants \( \alpha_0, \delta_0 \) and \( \eta_0 \) to be sufficiently small so that the upper bounds given in the various cases above hold, and the lemma is proven. \( \square \)

**Theorem 11.4.** Let \( E \subset \mathbb{R}^d \) be a lower content \( d \)-regular set with constant \( c_0 \) and let \( Q_0 \in \mathcal{D} \). Given two parameters \( 0 < \epsilon, \tau < 1 \), there exists a set \( \Sigma = \Sigma(\tau, \epsilon, Q_0) \) such that

1. \( Q_0 \subset \Sigma \).
2. \( \Sigma \) is a TC surface with constant \( r_0 = \text{diam}(Q_0), 0 < \eta_0 < 1/100 \), and \( \alpha_0 \) and \( \delta_0 \) sufficiently small with respect to \( \tau \) and \( \epsilon \).
3. We have the estimate
\[
\beta(Q_0)^2 + \text{diam}(Q_0)^d \sim_{c_0, n, d, \epsilon} \mathcal{H}^d(\Sigma).
\]

**Proof.** We take \( \Sigma \) to be the surface constructed in (11.1). It holds by definition that \( Q_0 \subset \Sigma \). by Lemma 11.3, we see that also (2) is satisfied. We just need to check (3), but this is an easy consequence of Theorem 1.5 and the Traveling Salesman theorem of Azzam and Schul. Indeed, from Theorem 1.5, we see that
\[
\beta(\Sigma)^2 \leq C(\epsilon, \tau, n, d) \mathcal{H}^d(\Sigma).
\]
On the other hand, using Lemma 11.2 and the TST of Azzam and Schul, we have
\[
\mathcal{H}^d(\Sigma) \leq_{c_0, n, d} \mathcal{H}^d(Q_0) + BWGL_E(Q_0) \leq_{c, c_0, n, d} \beta E(Q_0)^2 + \text{diam}(Q_0)^d.
\]
By containment, we have $\beta(Q_0)^2 + \text{diam}(Q_0)^d \lesssim \beta(\Sigma)^2 + \text{diam}(\Sigma)^d$. This concludes the proof of Theorem 11.4. □

12. AN APPLICATION TO UNIFORMLY NON-FLAT SETS

In [Dav04], David proved that if $E$ is a topologically stable $d$-surface and it is uniformly non-flat, then it must have dimension strictly larger than $d$. As mentioned in the introduction, David’s result was in the spirit of a previous result by Bishop and Jones about uniformly wiggly, or uniformly non-flat, sets.

Definition 12.1. A set $E \subset \mathbb{R}^n$ is called uniformly wiggly or uniformly non-flat (with parameter $\beta_0$) if for all cubes $Q \in \mathcal{D}_E$, we have that $\beta_\infty(Q) > \beta_0 > 0$.

Remark 12.2. Clearly, this definition can be recast in terms of different types of $\beta$ numbers, such as the content beta numbers which we have been using so far in this paper.

Let us now recall the result of Bishop and Jones.

Theorem 12.3 ([BJ97], Theorem 1.1). Let $E \subset \mathbb{R}^2$ be a compact, connected subset which is uniformly wiggly with parameter $\beta_0$. Then $\dim(E) > 1 + C\beta_0^2$, where $C$ is an absolute constant.

Let us go back to David’s result. His is, in a sense, a generalisation of Bishop and Jones’s Theorem. However, it is of qualitative nature, and the dependence of the lower bound on the parameter $\beta_0$ is not explicit. In this section we give a generalisation of Bishop and Jones Theorem where such a dependence is made explicit. This result is a fairly immediate application of Corollary 1.7 and of the scheme of proof from [BJ97].

Theorem 12.4. Let $E \subset \mathbb{R}^n$ be a topologically stable $d$-surface. Let $R \in \mathcal{D}$ be such that, for any $Q \in \mathcal{D}_E$, we have that

\begin{equation}
\beta_m(R) = \sum_{Q \in \mathcal{D}_m(R)} \beta^p_{E}(Q)\ell(Q)^d.
\end{equation}

Then

\begin{equation}
\dim(R) > d + c\beta_0^2.
\end{equation}

The scheme of the proof is the same as that of Bishop and Jones. We also used a clear summary of such proof to be found in Garnett and Marshall’s book, [GM05], page 429. For this reason, we only sketch the proof.

Proof. Given a TS $d$-surface, a cube $R \in \mathcal{D}(E)$ and an integer $m \geq 0$, we put

\begin{equation}
\Delta_{k,c}(R) := \{ I \in \Delta \mid I \cap R \neq \emptyset \text{ and } \ell(I) = c2^{-k} \},
\end{equation}

where $c < 1$ is a constant which is a power of $2$ and will be fixed later (it will depend on the parameter $\lambda > 0$ coming from Theorem 2.1). We then put

$$E_{R,k} := \bigcup_{I \in \Delta_{k,c}(R)} \partial_d I.$$
Claim 1. There exists a constant $C_5$ so that, if
\begin{equation}
R \in D_{N_0}(E)
\end{equation}
with $N_0 \leq k$, then
\begin{equation}
C_5 \left( \ell(R)^d + \sum_{m=N_0}^{k} \beta_m(R) \right) \leq H^d(E_{R,k}).
\end{equation}

To see this, note first that because $E$ satisfies the topological condition (TC) with parameters $r_0, \alpha_0, \eta_0, \delta_0$, then $E_{R,k}$ must also be a TS $d$-surface with comparable parameters (up to constants). Hence, we can apply Corollary 1.7 to see that
\begin{equation}
H^d(E_{R,k}) \sim \beta_{E_{R,k},C_0,p}(R),
\end{equation}
where the constants behind $\sim$ are as in the statement of Corollary 1.7.

We can now check (12.5): we have that
\begin{equation}
H^d(E_{R,k}) \sim \frac{1}{\ell(R)^d} \int_{2C_0B_P} \left( \frac{\text{dist}(y,E_{R,k})}{\ell(P)} \right)^p dH^d(y).
\end{equation}

By construction, we immediately see that $\text{diam}(E_{R,k}) \sim \ell(R)^d$. On the other hand, consider a cube $Q \in D_E$, such that $\ell(Q) > c2^{-k}$, for $c < 1$ as in (12.3). If we choose $c$ sufficiently small, we can apply Lemma 10.2 with $E_1 = E$ and $E_2 = E_{R,k}$, to obtain
\begin{equation}
\beta_{E}^d(C_0 P) \lesssim \beta_{E_{R,k}}(2C_0 P) + \sum_{P \in D_{ER,k}} \beta_{E_{R,k}}(C_0 P) \ell(P)^d
\end{equation}

Thus we see that
\begin{equation}
\sum_{P \in D_E, \ell(P) > c2^{-k}} \beta_{E}^d(C_0 P)
\lesssim \sum_{P' \in D_{ER,k}, \ell(P') > c2^{-k}} \beta_{E_{R,k}}(2C_0 P') + \sum_{P \in D_{ER,k}} \left( \frac{1}{\ell(P)^d} \int_{2C_0B_P} \left( \frac{\text{dist}(y,E_{R,k})}{\ell(P)} \right)^p dH^d(y) \right)^\frac{1}{p}.
\end{equation}

With a calculation similar to that in Sublemma 10.4, we obtain that the second sum above is $\lesssim \ell(R)^d$. This then gives
\begin{equation}
H^d(E_{R,k}) \sim \ell(R)^d + \sum_{P \in D_{ER,k}} \beta_{E_{R,k}}(C_0 P) \ell(P)^d
\end{equation}

\begin{equation}
\gtrsim \ell(R)^d + \sum_{P' \in D_{ER,k}, \ell(P') > c2^{-k}} \beta_{E_{R,k}} + \sum_{P \in D_E, \ell(P) > c2^{-k}} \left( \frac{1}{\ell(P)^d} \int_{2C_0B_P} \left( \frac{\text{dist}(y,E_{R,k})}{\ell(P)} \right)^p dH^d(y) \right)^\frac{1}{p}
\end{equation}

\begin{equation}
\gtrsim C_5 \left( \ell(R)^d + \sum_{P' \in D_{ER,k}, \ell(P') > c2^{-k}} \beta_{E_{R,k}} \right).
\end{equation}

This proves (12.5).
Claim 2. Let $N$ an integer so that $N > N_0$ (recall that $N_0$ is roughly the scale of $R$, see (12.4)). Consider a dyadic cube $I_N \in \Delta_N(\mathbb{R}^n)$ for which $\ell(I_N) < \ell(R)/10$ and such that $\frac{1}{3}I_N \cap E \neq \emptyset$. For $k > N$, we have

$$\sum_{m=N}^{k} \beta_m(R \cap I_N)^2 \geq (k - N)\beta_0^2 2^{-dN}.$$  

By $\beta_m(R \cap I_N)$ here we mean that we sum over those cubes $Q \in \mathcal{D}_m(R)$ such that $Q \cap I_N \neq \emptyset$. To see this, note first that by lower regularity of $E$, there are at least $2^{d(m-N)}$ (up to a a constant depending on the lower regularity parameter) dyadic cubes $J$ of generation $m$ (with $m > N$) such that $J \subset I_N$ and $J \cap E \neq \emptyset$. Hence since $E$ is uniformly non-flat, we see that if $N \leq m \leq k$,

$$\beta_m(R \cap I_N)^2 = \sum_{Q \in \mathcal{D}_m(R)} \beta_E(C_0Q)^2 \ell(Q)^d \geq \beta_0^2 \sum_{Q \in \mathcal{D}_m(R)} \ell(Q)^d \sim \beta_0^2 \sum_{J \in \Delta_m(R)} \ell(J)^d \geq c \beta_0^2 2^{d(m-N)} 2^{-dN} \sim c \beta_0^2 2^{-dN}.$$  

Hence, we have that

$$\sum_{m=N}^{k} \beta_m(R \cap I_N)^2 \geq c (k - N)\beta_0^2 2^{-dN},$$  

and so, using (12.5),

$$\mathcal{H}^d(E_{R,k} \cap I_N) \geq c (k - N)\beta_0^2 2^{-dN}.$$  

Let now $\{z_j\}$, $j$ in some index set $A$, be a maximal $2^{-k}$-separated net of $E_{R,k} \cap I_N$ such that

$$\bigcup_{j \in A} B(z_j, 2^{-k+2}) \supset E_{R,k} \cap I_N.$$  

Then there exists a constant $C_7$ (depending only on $n$) so that

$$\mathcal{H}^d(E_{R,k} \cap I_N) \leq C_7 c^d 2^{-dk} \text{Card}(A).$$  

Thus we obtain

$$C_7 c^d 2^{-dk} \text{Card}(A) \geq C_5, c (k - N)\beta_0^2 2^{-dN},$$  

and therefore

$$\text{Card}(A) \geq c_5, c (k - N)\beta_0^2 2^{d(k-N)}.$$  

Since $k$ was an arbitrary integer with $k \geq N$, we can choose it so that

$$\kappa := k - N \sim \frac{1}{\beta_0^2}.$$  

Hence we see from (12.7) that

$$\text{Card}(A) \geq 2^{d+c\beta_0^2} \kappa,$$  

(12.8)
where $c' = c'(C_7, C_5, c)$.

We now apply this construction recursively for each $N > N_0$, as follows. For $N_0$, we put

$$S_0 := \{ I \in \Delta_{N_0 + \kappa}(R) \mid \exists j \in A \text{ s.t. } z_j \in I \}$$

Then for each $I \in S_0$, we find a maximal net $\{ z_j \}_{j \in A}$ as in (12.6); the cardinality of this net will be again as in (12.8). We put the relative cubes in the subfamily

$$S(I) := \{ J \in \Delta_{N_0 + 2\kappa} \mid \exists j \in A \text{ s.t. } z_j \in J \} .$$

We then put

$$S_1 := \bigcup_{I \in S_0} S(I) .$$

Having defined $S_{j-1}$, we set

$$S_j := \bigcup_{I \in S_{j-1}} S(I),$$

where $S(I) = \{ j \in \Delta_{N_0 + j\kappa} \mid \exists j \in A \text{ s.t. } z_j \in J \}$. Let us record that for each $j \in \mathbb{N}$, we have

1. Each $J \in S_j$ is a subset of some $I \in S_{j-1}$.
2. Each $I \in S_{j-1}$ contains at least $2^{(d+c' \beta_0^2)\kappa}$ cubes $I \in S_{j}$ (as in (12.7)).
3. For each $j \in \mathbb{N}$, if $I \in S_j$, we have $I \cap R \neq \emptyset$.

Claim 3. If $R$ satisfies (1)-(3), then $\dim(R) > d + c' \beta_0^2$.

To prove this claim, we define the $\mu$ on the elements $I$ of $S_j$, for $j \geq 0$, by

$$\mu(I) = \text{Card}(A)^{-j} \leq 2^{-j(d+c' \beta_0^2)} .$$

One can then check that $\text{spt}(\mu) = E$ and that $\mu(R) = 1$. Then, by Frostman’s Lemma (Theorem 8.8 in [Mat95]), we have that

$$\mathcal{H}^{d+c' \beta_0^2}(R) > 0 .$$

This completes the proof of Theorem 12.4. □

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Recall the statement of Lemma 4.6.

**Lemma.** Let $S$ be a cube in $\widetilde{\text{Tree}}(Q)$ for some $Q \in \text{Next}(R)$, $R \in \text{Top}(k_0)$. Then there exists a dyadic cube $I_S := I \in \mathcal{C}_Q$ so that $I_S \subset \frac{1}{2} B_S$ and $\ell(I_S) \sim \tau \ell(S)$.

**Proof of Lemma 4.6.** Let $\zeta_S$ be the center of $S$. Then there exists a dyadic cube $I \in \mathcal{C}_Q$ such that $\zeta_S \in I$; thus for $I$ we have $\text{dist}(I, S) = 0$, and therefore $d_Q(I) \leq \text{dist}(I, S) + \ell(S) = \ell(S)$. In other words, when computing $d_Q(I)$, it suffices to minimise over all cubes $T$ such that

$$\text{dist}(I, T) + \ell(T) \leq \ell(Q).$$

But note that since $S$ is a minimal cube in $\widetilde{\text{Tree}}(Q)$, we must have that $T \subset S'$. Recall also that, by Theorem 2.1, $E \cap B(\zeta_S, c_0 \ell(S)) \subset S$. If we let $\tau$ be small enough, we can insure that $I \subset B(\zeta_S, \frac{c_0 \ell(S)}{2})$; hence we see that

$$\text{dist}(I, T) \gtrsim \ell(S),$$

and therefore $\tau^{-1} \ell(I) \sim d_Q(I) \gtrsim \ell(S) \geq \ell(I)$.

**Lemma.** Let $I \in \mathcal{C}_Q$ for $Q \in \text{Next}(R)$, $R \in \text{Top}(k_0)$. Then there exists a cube $Q_I \in \widetilde{\text{Tree}}(Q)$ so that

$$\ell(I) \leq \ell(Q_I) \leq c \tau^{-1} \ell(I);$$

$$\text{dist}(I, Q_I) \leq c \tau^{-1} \ell(I).$$

**Proof of Lemma 4.7.** Recall that $d_Q(I) \sim \ell(I)$. Now, by definition of $d_Q(I)$, there exists a cube $Q' \in \widetilde{\text{Tree}}(Q)$ such that $\text{dist}(I, Q') + \ell(Q') \leq 1.5 d_Q(I) \sim \tau^{-1} \ell(I)$. This
immediately implies (13.2) and the second inequality in (13.1). As for the first one, if it doesn’t hold, it suffices to take some ancestor of $Q'$ in $\overline{\text{Tree}}(Q)$. We then let this ancestor to be $Q_I$. $\square$