In this paper, the linear differential expression of order \( n \geq 2 \) with distribution coefficients of various singularity orders is considered. We obtain the associated matrix for the regularization of this expression. Furthermore, we present new statements of inverse spectral problems that consist in the recovery of differential operators from the Weyl matrix on the half-line and on a finite interval. The uniqueness theorems for these two inverse problems are proved by developing the method of spectral mappings. To the best of the author’s knowledge, inverse problems for higher order differential operators with distribution coefficients on the half-line have not been studied before. For the finite interval case, we for the first time consider the issue of recovering coefficients of the boundary conditions.

**KEYWORDS**
distribution coefficients, higher order differential operators, inverse spectral problems, regularization, uniqueness theorem, Weyl matrix

**MSC CLASSIFICATION**
34A55, 34B09, 34B40, 34L05, 46F10

## 1 | INTRODUCTION

This paper deals with the differential expression \( \ell_n(y) \), \( n \geq 2 \), defined as follows:

\[
\ell_{2m+r}(y) := y^{(2m+r)} + \sum_{k=0}^{m-1} (-1)^{k+r} (\sigma_{2k}^{(i_k,j_k)}(x)y^{(k)})^{(k)} + \sum_{k=0}^{m+r-2} (-1)^{m+r+1+k+1} \left[ (\sigma_{2k+1}^{(i_{k+1},j_{k+1})}(x)y^{(k+1)})^{(k+1)} + (\sigma_{2k+1}^{(i_{k+1},j_{k+1})}(x)y^{(k+1)})^{(k)} \right], \quad x \in \mathbb{R}_+,
\]

where \( m \in \mathbb{N}, \ r = 0, 1, n = 2m + r, \mathbb{R}_+ = (0, \infty) \); \((i_k)_{k=0}^{n-2}\) are integers such that \( 0 \leq l_{2k+j} \leq m - k - j \), \( j = 0, 1 \); \((\sigma_v)_{v=0}^{n-2}\) are regular functions, and the derivatives \( \sigma_v^{(k)} \) are understood in the sense of distributions.

Note that, if the functions \((\sigma_v)_{v=0}^{n-2}\) are sufficiently smooth, then the differential expression (1.1) can be reduced to the form

\[
\ell_n(y) = y^{(n)} + \sum_{k=0}^{n-2} P_k(x)y^{(k)}.
\]
However, if the coefficients are distributional, then it is convenient to consider differential expression in form (1.1). In particular, Mirzoev and Shkalikov\textsuperscript{1–3} have developed the regularization approach to the differential operators generated by $\mathcal{L}_n(y)$ in the case of the maximal singularity orders $i_{2k+j} = m - k - j$, $j = 0, 1$.

Linear differential operators $\mathcal{L}_n(y)$ have a variety of physical applications, especially for $n = 2, 3, 4$. The second-order Sturm–Liouville (Schrödinger) operator $-\mathcal{L}_2(y) = -y'' + q(x)y$ models string vibrations in classical mechanics, electron motion in quantum mechanics, and is also widely used in other branches of science. The third-order linear differential operators arise in the inverse problem method for integration of the nonlinear Boussinesq equation (see Deift et al\textsuperscript{4} and McKean\textsuperscript{5}), in mechanical problems of modeling thin membrane flow of viscous liquid and elastic beam vibrations (see Braeutigam and Polyakov\textsuperscript{6} and references therein). Spectral properties of the operator

$$y^{(4)} + (p(x)y)' + q(x)y,$$

which is the special case of $\mathcal{L}_4(y)$, were studied in connection with the Euler–Bernoulli operator $\frac{1}{b(x)}(a(x)u'')''$ describing the beam vibrations (see, e.g., in Badanin and Korotyaev\textsuperscript{7}). In Polyakov\textsuperscript{8}, the operator (1.3) was considered in relation to the analysis of thin liquid polymer films of nanometer thickness.

The Schrödinger operators with distribution potentials are widely used in quantum mechanics for describing the interaction between individual particles.\textsuperscript{9} Some aspects of spectral theory for the fourth-order differential operators with distribution coefficients were recently investigated in Ugurlu and Bairamov\textsuperscript{10} and Zhang et al\textsuperscript{11}. The development of the general theory for higher order differential operators with distribution coefficients could unify the approaches to specific problems in various applications, as well as causes interest from the purely mathematical point of view.

The goal of this paper is twofold. First, we construct the regularization matrix for the differential expression (1.1) with any derivative orders $(i, n_{\text{odd}} - 2)$. Second, we aim to study inverse spectral problems for differential operators generated by $\mathcal{L}_n(y)$. Let us describe the background and the main results for each of these two issues in more detail.

### 1.1 Regularization

To the best of the author’s knowledge, ordinary differential operators with distribution coefficients have been investigated for more than 60 years. The existence and uniqueness issues of initial value problem solution were considered in previous studies\textsuperscript{12–17} and other papers. A short overview of the early results in this direction can be found in Konechnaja and Mirzoev\textsuperscript{18}.

The most common way to treat differential equations with distribution coefficients is the reduction of such equations to first-order systems by introducing quasi-derivatives. Savchuk and Shkalikov\textsuperscript{19} started the systematic study of spectral theory for the second-order Sturm–Liouville equation

$$-y'' + q(x)y = \lambda y, \quad q = \sigma', \quad \sigma \in L_{2,\text{loc}}(0, 1), \quad \lambda \in \mathbb{C},$$

by reducing it to the system

$$\begin{bmatrix} y \\ y^{[1]} \end{bmatrix}' = \begin{bmatrix} \sigma & 1 \\ -\lambda - \sigma^2 & -\sigma \end{bmatrix} \begin{bmatrix} y \\ y^{[1]} \end{bmatrix},$$

where $y^{[1]} := y' - \sigma y$. The comprehensive spectral analysis of the Sturm–Liouville operators with distribution potentials $q$ of classes $W^p_\theta$, $\theta \geq -1$, was performed in Savchuk and Shkalikov\textsuperscript{20} where four different approaches for defining such operators were suggested and asymptotic properties of eigenvalues and eigenfunctions were studied. One of those four approaches, based on quadratic forms, was developed by Neiman-Zade and Shkalikov\textsuperscript{21,22} for higher order differential operators as well as for operators with partial derivatives.

In the monograph,\textsuperscript{23} Weidmann studied the minimal and the maximal operators, deficiency indices, self-adjoint extensions, and some other issues of spectral theory for a class of higher order matrix differential operators which includes operators generated by (1.1) with $i_{2k} = 1$, $i_{2k+1} = 0$, in particular, the Sturm–Liouville operator with potential of $W^{\tilde{p}}_\sigma(a, b)$.

Later on, Mirzoev and Shkalikov\textsuperscript{1} have obtained the regularization matrix for the even-order differential expressions generalizing (1.1) with $n = 2m, i_{2k+j} = m - k - j$, $j = 0, 1$. The analogous construction for the odd-order operators has been provided in Mirzoev and Shkalikov\textsuperscript{2} and Shkalikov.\textsuperscript{3} It is worth noting that, on a finite interval, the Mirzoev–Shkalikov case generalizes all the others with $0 \leq i_{2k+j} \leq m - k - j$, $j = 0, 1$. However, on the half-line, the classes of $\sigma \in L_1(\mathbb{R}_+)$ and $\sigma' \in L_1(\mathbb{R}_+)$ are not nested with one another. Therefore, the expression $\mathcal{L}_n(y)$ is worth being studied for various $(i, n_{\text{odd}} - 2)$. In
particular, Mirzoev and Shkalikov\textsuperscript{24} obtained the regularization matrices for the two-term differential operators $y^{(2n)} + qy$, where $q = \sigma^{(k)}$, $k = 0, 1, \ldots, n$, and $\sigma$ is a regular function. That research was continued in Konechnaja et al.,\textsuperscript{25,26} where the solution asymptotics as $x \to \infty$ were investigated for the two-term differential equation

$$(p(x)y^{(n)})^{(n)} + q(x)y = \lambda y, \ x > 0.$$ \hfill (1.1)$$

The analogous solution asymptotics for some other classes of higher order differential equations with distribution coefficients were obtained in Konechnaja and Mirzoev\textsuperscript{18} and Konechnaja.\textsuperscript{27}

Relying on the ideas of Neiman-Zade and Shkalikov\textsuperscript{21} and Vladimirov,\textsuperscript{28} Vladimirov\textsuperscript{29} has developed an alternative approach to construction of regularization matrices for differential operators represented by bilinear forms. The construction of Vladimirov\textsuperscript{29} can be applied to a wider class of operators than Mirzoev and Shkalikov.\textsuperscript{1,2} In particular, we use it in the present paper to obtain the regularization matrix for (1.1).

In this paper, we assume that the coefficients at $y^{(n)}$ and $y^{(n-1)}$ equal 1 and 0, respectively. This assumption is natural for studying inverse spectral problems. However, in previous studies,\textsuperscript{1,2,29} the coefficients at $y^{(n)}$, $y^{(n-1)}$ can be arbitrary functions of certain classes. Also, note that we consider the cases of even and odd $n$ together, and the odd case appears to be easier for the purposes of this paper.

Following the strategy of Vladimirov,\textsuperscript{29} we obtain the quadratic form and then construct the matrix $F(x) = \{f_{k,j}(x)\}_{k,j=1}^n$ by a certain rule $F = \mathcal{F}_I(\Sigma)$, $I = (i_s)_{s=0}^{n-2}$, $\Sigma = (\sigma_s)_{s=0}^{n-2}$, associated with the differential expression (1.1). The matrix $F(x)$ has to fulfill the following conditions:

$$f_{k,j}(x) \equiv 0, \ k + 1 < j,$$ \hfill (1.4)$$

and $f_{k,j} \in L_1(\mathbb{R}^+)$, $0 \leq j \leq k \leq n$.

By using the matrix $F(x)$, define the quasi-derivatives:

$$y^{[0]} := y, \ y^{[k]} = (y^{[k-1]})' - \sum_{j=1}^k f_{k,j}y^{[j-1]}, \ k = 1, n,$$ \hfill (1.6)$$

and the domain

$$D_F = \{ y : y^{[k]} \in AC_{loc}([\mathbb{R}^+), k = 0, n-1\}. $$ \hfill (1.7)$$

Our goal is to determine the rule $\mathcal{F}_I$ in such a way that $\mathcal{F}_n(y) = y^{[n]}$ for any $y \in D_F$. Then, instead of the equation $\mathcal{F}_n(y) = \lambda y$, we can consider the equivalent system

$$\tilde{y}' = (F(x) + \Lambda)\tilde{y},$$ \hfill (1.8)$$

where $\tilde{y}(x) = \text{col}(y^{[0]}(x), y^{[1]}(x), \ldots, y^{[n-1]}(x)), y \in D_F$, $\lambda$ is the spectral parameter, $\Lambda := \lambda E_{n,1}, E_{n,1}$ is the $(n \times n)$ matrix whose element at $(n, 1)$ equals 1, and all the other elements equal 0. Indeed, the first $(n-1)$ rows of the system (1.8) correspond to the quasi-derivative definition (1.6) and the $n$th row is $y^{[n]} = \lambda y$. For the regular case $i_v = 0$, $v = 0, n-2$, such construction is well known (see Everitt and Marcus\textsuperscript{30}). The regularization in this paper generalizes the both regular and the Mirzoev–Shkalikov cases.

By using the special construction of the matrix $F(x)$ for the differential expression (1.1), we prove some important assertions (Lemmas 2.3 and 2.4) for study of inverse problems.

1.2 Inverse problems

Inverse problems of spectral analysis consist in the recovery of operators from their spectral information. In terms of application, such problems correspond to determining unknown medium properties from some measured quantities or to constructing systems with desired characteristics.

The classical results of inverse spectral theory have been obtained for the Sturm–Liouville operator $-y'' + q(x)y$ with integrable potential $q(x)$ by Borg,\textsuperscript{31} Marchenko,\textsuperscript{32} Levitan,\textsuperscript{33} and so on. The recovery of the Sturm–Liouville operators
with singular potentials of classes $W^\theta_2$, $\theta \geq -1$, has been studied by Hryniv et al., 34–37 Freiling et al., 38 Savchuk and Shkalikov, 39–41 and Guliyev. 42 We also mention the studies of Mykytyuk and Trush, 43 Eckhardt et al., 44,45 and Bondarenko 46,47 concerning inverse problems for the matrix Sturm–Liouville operator with distribution potentials.

Investigation of inverse problems for the higher order differential operators (1.2) with $n > 2$ causes principal difficulties, since the classical transformation operator method (see Marchenko 32 and Levitan 33) is ineffective for them. Relying on the ideas of Leibenson, 48,49 Yurko has developed the method of spectral mappings. This method has been used to construct the inverse problem theory for the higher order differential operators (1.2) with regular coefficients on a finite interval and on the half-line (see previous studies 50–54). Inverse problems on the line were studied by Beals et al. 55,56 However, for the differential operators (1.1) with distribution coefficients, there is still no general inverse problem theory. The first steps in this direction have been taken in Bondarenko, 57 where the uniqueness theorem has been proved for the inverse problem on a finite interval. In Bondarenko, 58 a general approach to constructive solution of inverse problems for higher order differential operators with distribution coefficients have been suggested and reconstruction formulas have been obtained for differential expression coefficients of some classes.

In this paper, we mostly focus on the inverse problem for the differential expression (1.1) on the half-line. To the best of the author’s knowledge, inverse problems for higher order differential operators with distribution coefficients have not been studied before. Let us provide the inverse problem statement.

Let $U = [u_{k,j}]^n_{k,j=1}$ be a constant $(n \times n)$ matrix of form $U = P_U L_U$, where $P_U$ is a permutation matrix and $L_U$ is a unit lower triangular matrix. This means that the matrix $P_U$ has the elements equal to 1 at the positions $(k, p_k + 1)$, $k = 1, \ldots, n$, where $\{p_k\}_{k=1}^n$ is the permutation of the numbers $\{0, 1, \ldots, n - 1\}$, and all the other elements are zero. The entries of $L_U = [l_{k,j}]_{k,j=1}^n$ satisfy $l_{kj} = \delta_{kj}$ for $1 \leq k \leq j \leq n$, where $\delta_{kj}$ is the Kronecker delta.

Consider the boundary value problem $L_n(\Sigma, U)$ for the equation

$$L_n(y) = \lambda y, \ x \in \mathbb{R}_+,$$

with the boundary conditions

$$U_s(y) := y^{[p_s]}(0) + \sum_{j=1}^{p_s} u_{s,j} y^{[j-1]}(0) = 0, \ s = 1, \ldots, n. \quad (1.10)$$

Denote by $\{C_k(x, \lambda)\}_{k=1}^n$ and $\{\Phi_k(x, \lambda)\}_{k=1}^n$ the solutions of Equation (1.9) satisfying the initial conditions

$$U_s(C_k) = \delta_{k,s}, \ s = 1, \ldots, n, \quad (1.11)$$

and the boundary conditions

$$U_s(\Phi_k) = \delta_{k,s}, \ s = 1, \ldots, n, \quad \Phi_k(x, \lambda) = O(\exp(\rho_0 k x)), \ x \to \infty, \quad (1.12)$$

respectively. Here, $\lambda = \rho^n$ and $\{\omega_k\}_{k=1}^n$ are the roots of the equation $\omega^n = 1$ numbered so that

$$\text{Re}(\rho \omega_1) \leq \text{Re}(\rho \omega_2) \leq \ldots \leq \text{Re}(\rho \omega_n).$$

The functions $\Phi_k(x, \lambda)$ are called the Weyl solutions of (1.9).

Consider the matrix functions $C(x, \lambda) = [\hat{C}_k(x, \lambda)]_{k=1}^n$ and $\Phi(x, \lambda) = [\Phi_k(x, \lambda)]_{k=1}^n$. Since the columns of $C(x, \lambda)$ and $\Phi(x, \lambda)$ form two fundamental systems of solutions (FSS) of (1.8), the relation $\Phi(x, \lambda) = C(x, \lambda) M(\lambda)$ is fulfilled, where $M(\lambda) = [M_{sk}(\lambda)]_{k=1}^n$ is called the Weyl matrix and its entries $M_{sk}(\lambda)$ are called the Weyl functions. The conditions (1.11) and (1.12) imply that $M(\lambda)$ is a unit lower triangular matrix.

The Weyl functions and their generalizations are natural spectral characteristics in the inverse problem theory for various classes of differential operators and pencils (see, e.g., Freiling and Yurko 59). The defined $M(\lambda)$ is analogous to the Weyl matrix used by Yurko 50,51,54 for solving inverse problems for the higher order differential operators with regular coefficients. In this paper, we consider the following problem.

Inverse Problem 1.1. Suppose that $P_U$ is known a priori. Given the Weyl matrix $M(\lambda)$, find $\Sigma = (\sigma_i)_{i=0}^{n-2}$ and $U = [u_{k,j}]_{k,j=1}^n$. 

We study analytic properties of \( M(\lambda) \) (Theorem 4.1) and obtain the asymptotics of the Weyl solutions (Lemma 4.2), by using the Birkhoff solutions of Equation (1.9) with certain asymptotics as \(|\rho| \to \infty\) (Theorems 3.3 and 3.4). By using the method of spectral mappings, we prove the uniqueness theorem for Inverse Problem 1.1. Furthermore, we consider the inverse problem on a finite interval previously studied in Bondarenko.57 We discuss the usage of regularizations with various \((i_n)^{m-2}\) and the recovery of the boundary condition coefficients in a finite interval case. Note that our results are novel even in the regular case \(i_n = 0, n \neq n-2\). Since the method of spectral mappings is constructive, in the future, it can be used for developing an algorithm solving the studied inverse problems and for investigating existence of their solution.

In addition, we consider the examples for the orders \(n = 2\) and \(n = 4\), which often arise in applications. It is shown that the results of this paper generalize the previously known results for the Sturm–Liouville operators with regular and singular potentials. Also, we compare the inverse problems on the half-line and on a finite interval.

The paper is organized as follows. Section 2 is devoted to the regularization issues. In Section 3, theorems on Birkhoff systems are formulated. In Section 4, we consider the inverse problem on the half-line, and in Section 5, on a finite interval. Section 6 contains examples for \(n = 2\) and \(n = 4\).

## 2 | REGULARIZATION

In this section, we obtain the associated matrix \( F(\lambda) \) for regularization of the differential expression (1.1) and study the properties of this matrix needed for the inverse problem theory.

Suppose that

\[
I = (i_n)_{n=0}^{n-2} \in I_n, \quad I_{2m+r} = \begin{cases} (i_n)_{n=0}^{2m+r-2} : & 0 \leq i_{2k} \leq m - k, k = 0, m - 1, \\
 & 0 \leq i_{2k+1} \leq m - k - 1, k = 0, m + r - 2 \end{cases},
\]

\[
\Sigma = (\sigma_n)_{n=0}^{n-2} \in \Sigma_{j,loc}, \quad \Sigma_{j,loc} = \begin{cases} (\sigma_n)_{n=0}^{n-2} : & \sigma_n \in L_1, \sigma_n \in L_2, \nu = 0, n - 2, \\
 & \sigma_n \in L_1, \nu = K(I), \sigma_n \in K(I) \end{cases},
\]

\[
K(I) \subseteq \{0, 1, \ldots, n - 2\}, \quad K(I) = \emptyset \text{ if } n = 2m + 1,
\]

\[
\nu = 2k \in K(I) \iff i_{2k} = m - k, \quad \nu = 2k + 1 \in K(I) \iff i_{2k+1} = m - k - 1 \text{ if } n = 2m.
\]

\(\Sigma\) is defined similarly to \(\Sigma_{j,loc}\) with \(L_{j,loc}\) replaced by \(L_{j,loc} \cap \{0, 1, \ldots, n - 2\}\). The regularity of \(\sigma_n(x)\) on \(\mathbb{R}_+\) is important in the next sections for obtaining the Birkhoff solutions and for investigation of inverse problems.

Denote by \(\mathcal{D}\) the space \(C_0^\infty(\mathbb{R}_+)\) of infinitely differentiable functions with a finite support on \(\mathbb{R}_+\) and by \(\mathcal{D}'\) the space of all the continuous linear functionals on \(\mathcal{D}\). For \(z \in \mathcal{D}\) and \(f \in \mathcal{D}'\), we use the notation \((f, z) = f(z)\). In particular, \((f, z) = \int_0^\infty f(x)z(x) \, dx\) if \(f \in L_{1,loc}(\mathbb{R}_+)\).

**Lemma 2.1.** Suppose that \(y \in W_{1,loc}^m(\mathbb{R}_+)\) if \(K(I) = \emptyset\) and \(y \in W_{2,loc}^m(\mathbb{R}_+)\) otherwise. Then, \(\ell_n(y) \in \mathcal{D}'\) and

\[
(\ell_n(y), z) = (-1)^m(y^{(m+r)}, z^{(m)}) + \sum_{r,j=0}^m (q_{r,j} y^{(r)}, z^{(j)}), \quad z \in \mathcal{D}, \tag{2.1}
\]

where

\[
[q_{r,j}]_{r,j=0}^m = \mathcal{D}(\Sigma) = \sum_{v=0}^{n-2} \sigma_v(x) \chi_{v,i,j}, \quad \chi_{v,i,j} = [\chi_{v,i,s} r_{j}]_{r,j=0}^m, \tag{2.2}
\]

\[
\chi_{2k,i,s+k,i-s+k} = C_i^s, \quad s = 0, i, \quad \chi_{2k+1,i,s+k,i+s+k} = C_{i+1}^s - 2C_i^{s-1}, \quad s = 0, i + 1, \tag{2.3}
\]

and all the other entries \(\chi_{v,i,s} r_{j}\) equal zero. Here and below, \(C_i^s = \binom{i}{s} / S(i-s)\) are the binomial coefficients, \(C_{-1}^i := 0\).
Note that, at the right-hand side of (2.1), all the functions are regular, so (2.1) describes an action of the functional $\ell_n(y) \in \mathfrak{D}'$ on an arbitrary $z \in \mathfrak{D}$. In fact, the relation (2.1) holds for $z$ of a wider class than $\mathfrak{D}$.

**Proof of Lemma 2.1.** Let $i, k \geq 0$. Formal calculations show that

$$\sigma^{(i)} y = \sum_{s=0}^{i} (-1)^s C_s^i (\sigma y^{(i)})(i-s),$$

$$\sigma^{(i)} y^{(k)} = \sum_{s=0}^{i} (-1)^s C_s^i (\sigma y^{(i+k)})(i+s+k),$$

$$((\sigma^{(i)} y^{(k)}), z) = \sum_{s=0}^{i} (-1)^{i+k} C_s^i (\sigma y^{(i+k)} z^{(i-s+k)}), \quad z \in \mathfrak{D}.$$ (2.4)

Clearly, under the conditions of the lemma, $\sigma_{2k} y^{(i+k)} \in L_{1, loc}(\mathbb{R}_+)$ for $s = 0, i_{2k}$, $k = 0, n - 2$, so we conclude that $\sigma_{2k} y^{(i+k)} \in \mathfrak{D}'$.

Analogously,

$$\sigma^{(i)} y^{(k+1)} + \sigma^{(i)} y^{(k+1)} = \sum_{s=0}^{i+1} (-1)^s (C_s^i - 2C_{i+1}^i) (\sigma y^{(i+k)})(i+1-s+k),$$

$$((\sigma^{(i)} y^{(k+1)}), z) + ((\sigma^{(i)} y^{(k+1)}), z) = \sum_{s=0}^{i+1} (-1)^{i+k+1} (C_s^i - 2C_{i+1}^i) (\sigma y^{(i+k)} z^{(i+1-s+k)}), \quad z \in \mathfrak{D}.$$ (2.5)

where $z \in \mathfrak{D}$. Taking $\sigma = \sigma_{2k+1}, i = i_{2k+1}$, we conclude that the terms with odd indices $(2k + 1)$ in (1.1) belong to $\mathfrak{D}'$ under the conditions of the lemma. Therefore, $\ell_n(y) \in \mathfrak{D}'$. Combining (1.1), (2.4) with $\sigma = \sigma_{2k}, i = i_{2k}$, and (2.5) with $\sigma = \sigma_{2k+1}, i = i_{2k+1}$, we arrive at (2.1).

Now, following the approach of Vladimirov, we are going to construct the matrix $F(x)$ of quasi-derivative coefficients by using the matrix $Q = [q_{Is}]_{i,s=0}^{m}$ of the quadratic form in (2.1). Define the spaces of matrix functions $\mathfrak{Q}_{n, loc}$ and $\mathfrak{G}_{n, loc}$ as follows:

$$\mathfrak{Q}_{n, loc} = \left\{ Q = [q_{Is}]_{i,s=0}^{m} : q_{Is} \in L_{1, loc}(\mathbb{R}_+), q_{Im}, q_{m,s} \in L_{2, loc}(\mathbb{R}_+) \text{ if } n = 2m, \quad l, s = 0, m \right\},$$

$$\mathfrak{G}_{n, loc} = \left\{ F = [f_{k,i}]_{i,j=1}^{n} : f_{k,j} \in L_{1, loc}(\mathbb{R}_+), k = m + 1, 2m + \tau, \quad j = 1, m + \tau, \quad f_{k,m+1}, f_{m,j} \in L_{2, loc}(\mathbb{R}_+), \quad k = m + 1, 2m, \quad j = 1, m, \text{ if } n = 2m \right\},$$

where

$$f_{k,j} = 0, \quad k = 1, m - 1 + \tau, \quad j \leq k \text{ and } j = m + 2, n, \quad k \geq j.$$ (2.6)

The spaces $\mathfrak{Q}_n$ and $\mathfrak{G}_n$ are defined similarly with $L_{1, loc}$ and $L_{2, loc}$ replaced by $L_1$ and $L_1 \cap L_2$, respectively. It follows from the definition of the mapping $\mathcal{D}_l$ in Lemma 2.1 that

$$\mathcal{D}_l : \Sigma_{l, loc} \to \mathfrak{Q}_{n, loc} \text{ and } \mathcal{D}_l : \Sigma_l \to \mathfrak{Q}_n.$$ (2.7)

Define the mapping $\mathcal{J}_n : \mathfrak{Q}_{n, loc} \to \mathfrak{G}_{n, loc}$ acting as follows:

$$F = \mathcal{J}_n(Q), \quad Q = [q_{Is}]_{i,s=0}^{m} \in \mathfrak{Q}_{n, loc}, \quad F = [f_{k,i}]_{i,j=1}^{n} \in \mathfrak{G}_{n, loc},$$

$$n = 2m : \quad \begin{cases} f_{m,j} := (-1)^{m+1} q_{j-1, m}, \quad j = 1, m, \\ f_{k,m+1} := (-1)^{k+1} q_{m, 2m - k}, \quad k = m + 1, 2m, \\ f_{k,j} := (-1)^{k+1} q_{j-1, 2m - k} + (-1)^{m+1} q_{j-1, m} q_{m, 2m - k}, \quad k = m + 1, 2m, \quad j = 1, m. \end{cases}$$ (2.8)

$$n = 2m + 1 : \quad \begin{cases} f_{k,j} := (-1)^{k} q_{j-1, 2m+1 - k}, \quad k = m + 1, 2m + 1, \quad j = 1, m + 1. \end{cases}$$

All the elements $f_{k,j}$ undefined here are uniquely specified by (1.4), (1.5), and (2.6). Obviously,

$$\mathcal{J}_n : \mathfrak{Q}_{n, loc} \to \mathfrak{G}_{n, loc} \text{ and } \mathcal{J}_n : \mathfrak{Q}_n \to \mathfrak{G}_n.$$ (2.9)
The inverse mapping $\mathcal{J}_n : Q_{n,\text{loc}} \rightarrow \mathfrak{S}_{n,\text{loc}}$ is given by the formulas:

$$Q = \mathcal{J}_n^{-1}(F), \quad F = [f_{k,j}]_{k,j=1}^n \in \mathfrak{S}_{n,\text{loc}}, \quad Q = [q_{i,j}]_{i=0}^m \in Q_{n,\text{loc}},$$

$$n = 2m : \begin{cases}
q_{j-1,m} := (-1)^m f_{m,j}, & j = 1, m, \\
q_{m,2m-k} := (-1)^{k+1} f_{k,m+1}, & k = m + 1, 2m, \\
qu_{j-1,2m-k} := (-1)^{k+1} (f_{k,j} - f_{k,m+1}), & k = m + 1, 2m, j = 1, m, \\
qu_{m,m} := 0, \\
qu_{j,2m+1-k} := (-1)^k f_{k,j}, & k = m + 1, 2m + 1, j = 1, m + 1.
\end{cases} \tag{2.10}$$

Define the mapping $\mathcal{F}_i(\Sigma) = \mathcal{J}_n(\mathcal{D}_i(\Sigma))$. For a fixed $I \in I_n$, the relations (2.7) and (2.9) imply

$$\mathcal{F}_I : \Sigma_{I,\text{loc}} \rightarrow \mathfrak{S}_{I,\text{loc}} \quad \text{and} \quad \mathcal{F}_I : I \rightarrow \mathfrak{S}_n. \tag{2.11}$$

In fact, the above formulas defining the mapping $\mathcal{J}_n$ are the special case of the formulas on p. 6 of Vladimirov. In the Mirzoev–Shkalikov case $i_{2k+1} = m - k - j, j = 0, 1$, the matrix function $F(\mathbf{x}) = \mathcal{F}_I(\Sigma)$ coincides with the associated matrices obtained in previous studies.

The following theorem establishes the equivalence of Equation (1.9) and the system (1.8) for $y \in D_F, F(\mathbf{x}) = \mathcal{F}_I(\Sigma)$.

**Theorem 2.2.** Suppose that $\Sigma \in \Sigma_{I,\text{loc}}$ and $F = \mathcal{F}_I(\Sigma)$. Let the quasi-derivatives $y^{[j]}, j = 0, n$, and the domain $D_F$ be defined by (1.6) and (1.7), respectively. Then, for any $y \in D_F$, $\ell_n(y) \in L_{1,\text{loc}}(\mathbb{R}_+)$ and $\ell_n(y) = y^{[n]}$.

**Proof.** For definiteness, consider $n = 2m$. The proof for $n = 2m + 1$ is analogous and even easier. Since $F \in \mathfrak{S}_{n,\text{loc}}$, then the assumption (2.6) holds, which together with (1.6) imply

$$y^{[j]} = y^{(j)}, \quad j = 0, m - 1. \tag{2.12}$$

Therefore, $y \in D_F$ implies $y \in W_{1,\text{loc}}^m(\mathbb{R}_+)$. Moreover,

$$y^{[m]} = y^{[m]} + \sum_{j=1}^m f_{m,j} y^{[j-1]} \in L_{2,\text{loc}}(\mathbb{R}_+), \tag{2.13}$$

so $y \in W_{2,\text{loc}}^m(\mathbb{R}_+)$. Thus, $y$ satisfies the conditions of Lemma 2.1. Hence, $\ell_n(y) \in \mathfrak{S'}$ and (2.1) holds.

Using (1.6) and (2.12), we obtain

$$y^{[k]} = (y^{[k-1]})' - f_{k,m+1} y^{[m]} - \sum_{j=1}^m (f_{k,j} - f_{k,m+1}) y^{[j-1]}, \quad k = m + 1, 2m. \tag{2.14}$$

Substituting (2.10) into (2.14), we derive

$$y^{[k]} = (y^{[k-1]})' + (-1)^k \sum_{j=0}^m q_{j,2m-k} y^{[j]}, \quad k = m + 1, 2m. \tag{2.15}$$

Using the relation

$$(y^{[k]}, z) = -(y^{[k-1]}, z') + (-1)^k \sum_{j=0}^m (q_{j,2m-k} y^{[j]}), \quad z \in \mathfrak{S},$$

recursively for $k = n, n-1, \ldots, m + 1$, we conclude that

$$(y^{[n]}, z) = (-1)^m (y^{[m]}, z^{(m)}) + \sum_{j=0}^m \sum_{s=0}^{m-1} (q_{s,s} y^{[j]}), \quad z^{(s)}.$$
It follows from (2.13) and (2.10) that
\[
y^{[m]} = y^{(m)} + (-1)^m \sum_{j=0}^{m-1} q_{j,m} y^{(j)}.
\] (2.16)  

Note that, in view of the definition in Lemma 2.1, we have \( q_{m,m} = 0 \). Therefore, combining (2.15), (2.16) and comparing the result with (2.1), we get
\[
(\mathcal{E}_n(y), z) = (y^{[n]}, z), \ z \in \mathcal{D}.
\]

Hence, \( \mathcal{E}_n(y) = y^{[n]} \) in \( \mathcal{D}' \).

On the other hand, taking (2.14) for \( k = n \) and (2.13) into account, we conclude that \( y^{[n]} \in L_{1, \text{loc}}(\mathbb{R}^+) \) for \( y \in D_F \).

Thus, \( \mathcal{E}_n(y) \) is also a regular function, which completes the proof. \( \Box \)

For investigation of inverse spectral problems, we need the following technical lemma, which generalizes Bondarenko's Lemma 2.1 and transfers it to the half-line case.

**Lemma 2.3.** Suppose that \( \Sigma, \Sigma \in \Sigma_I \), \( F = \mathcal{F}_I(\Sigma) \), \( F = \mathcal{F}_I(\Sigma) \), and a unit lower triangular matrix function \( P(x) = [p_{k,j}(x)]_{k,j=1}^n \) satisfies the equation
\[
P'(x) + P(x)\tilde{F}(x) = F(x)P(x), \ x \in \mathbb{R}^+.
\] (2.17)  

Then, \( P(x) \) on \( \mathbb{R}^+ \) identically equals the \((n \times n)\) unit matrix \( I_n \) and \( \Sigma = \hat{\Sigma} \), that is, \( \sigma_\nu(x) = \tilde{\sigma}_\nu(x) \) a.e. on \( \mathbb{R}^+, \nu = 0, n - 2 \).

**Proof.**

Step 1. For definiteness, consider \( n = 2m \). The case \( n = 2m + 1 \) is analogous and even easier. Using the first \((m - 1)\) rows of (2.17), we get \( p_{k,j} = 0 \) for \( k = \overline{1, m}, j < k \). The \( m \)th row of (2.17) implies
\[
p_{m+1,j} = \tilde{f}_{m,j} - \tilde{f}_{m,j}, \ j = \overline{1, m}.
\] (2.18)  

Similarly, considering the columns of (2.17) for \( j = 2m, 2m - 1, \ldots, m + 1 \), we get \( p_{k,j} = 0 \) for \( k > j, j = \overline{m + 1, 2m} \), and
\[
p_{k,m} = f_{k,m+1} - \tilde{f}_{k,m+1}, \ k = \overline{m + 1, 2m}.
\] (2.19)  

For \( k = \overline{m + 1, 2m}, j = \overline{1, m} \), Equation (2.17) yields
\[
p'_{k,j} + p_{k,j-1} + p_{k,m} \tilde{f}_{m,j} + \tilde{f}_{k,j} = f_{k,j} + f_{k,m+1} p_{m+1,j} + p_{k+1,j}.
\] (2.20)  

Here, we assume that \( p_{k,j} = 0 \) if \( j < 1 \) or \( k > 2m \). Substituting (2.18) and (2.19) into (2.20), we get
\[
p'_{k,j} + p_{k,j-1} + (\tilde{f}_{k,j} - \tilde{f}_{k,m+1} \tilde{f}_{m,j}) = p_{k+1,j} + (f_{k,j} - f_{k,m+1} f_{m,j}), \ k = \overline{m + 1, 2m}, j = \overline{1, m}.
\] (2.21)  

Using (2.10), pass to the new variables \([q_{l,s}]_{l=0}^m := \mathcal{J}_n^1(F), [\tilde{q}_{l,s}]_{l=0}^m := \mathcal{J}_n^1(\tilde{F})\) and \( r_{j-1,2m-k} := (-1)^{k+1} p_{k,j}, k = \overline{m, 2m}, j = \overline{1, m + 1} \). Thus, we get the system
\[
r'_{l,s} + r_{l-1,s} + r_{l,s-1} = q_{l,s} - \tilde{q}_{l,s}, \ l, s = \overline{0, m}, (l, s) \neq (m, m),
\] (2.22)  
\[
r_{m,s} = r_{s,m} = 0, \ s = \overline{0, m - 1}.
\] (2.23)  

Note that it is unimportant whether \( r_{m,m-1} = r_{m-1,m} = 0 \) or \( r_{m,m-1} = r_{m-1,m} = 1 \), since these values do not influence on the other entries.

In the case of odd \( n \), we obtain exactly the same system (2.22)–(2.23) with respect to \( r_{j-1,2m+1-k} := (-1)^k p_{k,j}, k = \overline{m + 1, 2m + 1}, j = \overline{1, m + 1} \).

Step 2. Denote \( \tilde{q}_{l,s} := q_{l,s} - \tilde{q}_{l,s}, \sigma_v = \sigma_\nu - \tilde{\sigma}_\nu \). It remains to prove that the relations (2.22), (2.23) imply \( r_{l,s}(x) = 0 \) for \( l, s = \overline{0, m}, (l, s) \neq (m, m) \), and \( \sigma_v = 0, \nu = 0, n - 2 \). Let us show this by induction. Suppose that we have already proved \( \sigma_{2k} = 0, \tilde{\sigma}_{2k+1} = 0 \) for \( k = \overline{0, K - 1} \) with some fixed \( K \in \{0, \ldots, m - 1\} \). This implies
\begin{align*}
\hat{q}_{k,s} = \hat{q}_{s,k} = 0 \text{ for } s = 0, m, k = 0, K - 1. \text{ Therefore, it follows from } (2.22), (2.23) \text{ that } r_{k,s} = r_{s,k} = 0 \text{ for } s = 0, m, k = 0, K - 1. \text{ Denote }
\begin{align*}
r_{s}^{\pm} := \frac{1}{2}(r_{k,s} \pm r_{s,k}), \quad \hat{q}_{s}^{\pm} := \frac{1}{2}(\hat{q}_{k,s} \pm \hat{q}_{s,k}).
\end{align*}
\end{align*}

From (2.22),(2.23), we get the systems
\begin{align}
(r_{s}^{+})' + r_{s-1}^{+} = \hat{q}_{s}^{+}, \quad s = 0, m, \quad r_{m}^{+} = 0.
\end{align}

By virtue of (2.2), (2.3), we have
\begin{align*}
\hat{q}_{s}^{+} = \begin{cases} \hat{\sigma}_{2k}, & s = i_{2k} + K, \\ 0, & \text{otherwise}, \quad \hat{q}_{s}^{-} = \begin{cases} \hat{\sigma}_{2k+1}, & s = i_{2k+1} + K + 1, \\ 0, & \text{otherwise}. \end{cases} \end{cases}
\end{align*}

Therefore, considering (2.24) with “+” for \(s = m, m - 1, \ldots, i_{2k} + K + 1\), we get \(r_{s}^{+} = 0\). Then, \(r_{s}^{+} = 0\).

If \(i_{2k} = 0\), this immediately yields \(\hat{\sigma}_{2k} = 0\). Otherwise, solving (2.24) for \(s = K, K + 1, \ldots, K + i_{2k} - 1\), we obtain
\begin{align}
r_{k,j}^{+}(x) = \sum_{j=0}^{j} c_{j} \frac{(-1)^{j} x^{j}}{j!}, \quad j = 0, i_{2k} - 1,
\end{align}

where \(\{c_{j}\}_{j=0}^{j} = 0\) are arbitrary constants. Since \(\sigma_{2k}, \hat{\sigma}_{2k} \in L_{1}(\mathbb{R}_{+})\), then \(\hat{\sigma}_{2k}^{+} \in L_{1}(\mathbb{R}_{+})\), so \(c_{j} = 0, j = 0, \ldots, i_{2k} - 1\). Hence, \(r_{s}^{+} = 0\) for \(s = K, K + i_{2k} - 1\) and \(\hat{\sigma}_{2k} = 0\). Analogously, we show that \(\hat{\sigma}_{2k+1} = 0\) by using the system (2.24) with “−”. Note that, in the case \(n = 2m\) and \(K = m - 1\), the expression (1.1) does not contain the coefficient \(\sigma_{2k+1}\), so the last step should be omitted.

Returning to the variables \(p_{k,j}\), we arrive at the assertion of the lemma.

In the case \(\Sigma, \hat{\Sigma} \in \Sigma_{f,(0,1)}\),
\begin{align*}
\Sigma_{f,(0,1)} := \left\{ (\sigma_{\nu})_{\nu=0}^{n-2} : \sigma_{\nu} \in L_{1}(0, 1), \nu = 0, n-2, \sigma_{\nu} \in L_{2}(0, 1), \nu \in K(I) \right\},
\end{align*}

the assertion of Lemma 2.3 is valid under additional initial conditions on \(P(x)\). The following lemma generalizes and improves Bondarenko57, Lemma 2.1 and can be used for studying inverse spectral problems on a finite interval.

**Lemma 2.4.** Suppose that \(\mathcal{N} \subseteq \{0, 1, \ldots, n - 2\}, \Sigma, \hat{\Sigma} \in \Sigma_{f,(0,1)}\), \(\sigma_{\nu}(x) = \hat{\sigma}_{\nu}(x) \text{ a.e. on } (0, 1) \text{ for } \nu \in \mathcal{N}\), \(F = \mathcal{F}(\Sigma), \hat{F} = \mathcal{F}(\hat{\Sigma})\), and a unit lower triangular matrix function \(P(x) = [p_{k,j}(x)]_{k,j=1}^{n}\) satisfies the Equation (2.17) on \((0, 1)\) and the initial conditions \(IC(\nu), \nu = 0, n-2, \mathcal{N}\), where
\begin{align}
IC(2k) : \quad p_{n-k,k+1}(0) + p_{n-k+1,k+1}(0) = 0, \quad s = k, k + i_{2k} - 1, \quad k = 0, m - 1
\end{align}

and
\begin{align}
IC(2k + 1) : \quad p_{n-k,k+1}(0) - p_{n-k+1,k+1}(0) = 0, \quad s = k + 1, k + i_{2k+1}, \quad k = 0, m + r - 2
\end{align}

Then, \(P(x) = I_{n}\) and \(\Sigma = \hat{\Sigma}\).

**Proof.** Lemma 2.4 is proved analogously to Lemma 2.3. Step 1 requires no modifications. For simplicity, assume that \(\mathcal{N} = \emptyset\). At Step 2, we consider the system (2.22)–(2.23) together with the initial conditions
\begin{align}
(r_{k,s} + r_{s,k})(0) = 0, \quad k = 0, m - 1, \quad s = k, k + i_{2k} - 1
\end{align}

and
\begin{align}
(r_{k,s} - r_{s,k})(0) = 0, \quad k = 0, m + r - 2, \quad s = k + 1, k + i_{2k+1}
\end{align}

which are equivalent to (2.25). Further, solving (2.24) with “+” for \(s = K, K + 1, \ldots, K + i_{2k} - 1\), we use the initial conditions \(r_{s}^{+}(0) = 0\), which follow from (2.26). Therefore, we get \(r_{s}^{+} = 0\) for \(s = K, K + i_{2k} - 1\), so \(\hat{\sigma}_{2k} = 0\). The equality \(\hat{\sigma}_{2k+1} = 0\) is proved analogously. Obviously, the proof is valid in the case \(\mathcal{N} \neq \emptyset\) with minor modifications. \(\square\)
3 | BIRKHOFF SOLUTIONS

Suppose that \( I \in I_n, \Sigma \in \Sigma, \) and \( F = \mathcal{F}_I(\Sigma). \) In view of Theorem 2.2, we understand the solution of the Equation (1.9):

\[
\ell_n'(y) = \lambda y, \quad x \in \mathbb{R}_+.
\]

in the following sense.

**Definition 3.1.** A function \( y \) is a solution of Equation (1.9) if \( y \in D_F \) and \( \tilde{y} \) satisfies the system (1.8):

\[
\tilde{y}' = (F(x) + \Lambda)\tilde{y}, \quad x \in \mathbb{R}_+.
\]

In this section, we obtain the Birkhoff solutions with the known behavior as \( \rho \to \infty \) of Equation (1.9) with \( \lambda = \rho^n \).

Consider the partition of the \( \rho \)-plane into the sectors

\[
\Gamma_k = \left\{ \rho : \frac{\pi(k-1)}{n} < \arg \rho < \frac{\pi k}{n} \right\}, \quad k = 1, 2, n.
\]  

(3.1)

Below, we assume that if \( \rho \) lies in a fixed sector \( \Gamma = \Gamma_k \), then the roots \( \{\omega_j\}_{j=1}^n \) of the equation \( \omega^n = 1 \) are numbered so that

\[
\text{Re}(\rho\omega_1) < \text{Re}(\rho\omega_2) < \cdots < \text{Re}(\rho\omega_n), \quad \rho \in \Gamma.
\]

Put \( \Omega := [\omega_j^{-1}]_{j,k=1}^n, \lambda = \rho^n \). Applying the change of variables

\[
\tilde{y}(x) = \text{diag}[1, \rho, \ldots, \rho^{n-1}]\Omega v(x)
\]

(see Bondarenko\textsuperscript{57} and Savchuk and Shkalikov\textsuperscript{60} for details), we reduce the system (1.8) to the form

\[
v' = \rho Bv + A(x)v + D(x,\rho)v, \quad x \in \mathbb{R}_+,
\]

\[
B := \text{diag}\{\omega_1, \omega_2, \ldots, \omega_n\}, \quad D(x,\rho) = \sum_{k=1}^{n-1} \rho^{-k}D_k(x),
\]

(3.3)

where \( A(x) \) and \( D_k(x), k = 1, n-1 \), are \((n \times n)\) matrix functions with entries of the classes \((L_2 \cap L_1)(\mathbb{R}_+)\) and \(L_1(\mathbb{R}_+)\), respectively.

The Birkhoff solutions of differential systems generalizing (3.3) on a finite interval have been constructed in previous studies.\textsuperscript{60–62} Savchuk and Shkalikov\textsuperscript{60} have used those results to obtain the Birkhoff FSS of higher order differential equations with distribution coefficients. For the case of the half-line, their proofs are also valid with necessary modifications. However, the finite and half-line cases have some important differences. In order to study inverse spectral problems for higher order operators or differential systems on a finite interval, it is sufficient to have the Birkhoff solutions analytic for \( \rho \in \Gamma, |\rho| > \rho_a \) with some fixed \( \rho_a > 0 \). For the half-line case, Yurko\textsuperscript{54} has used the family of the Birkhoff FSS analytic for \( \rho \in \Gamma, |\rho| > \rho_a \) and depending on the parameter \( a \geq 0 \), where \( \rho_a \to 0 \) as \( a \to \infty \). Such FSS allowed him to study the properties of the spectral characteristics in the neighborhood of \( \lambda = 0 \) for the higher order differential operators and the first-order differential systems. The construction of such Birkhoff systems for the case of regular coefficients is described, for example, in Yurko.\textsuperscript{62} Developing the methods of Yurko\textsuperscript{62} for the system (3.3), we have proved the following theorem.

**Theorem 3.2.** For every \( a \geq 0 \), there exists a FSS \( \{Y_{k,a}(x,\rho)\}_{k=1}^n \) of (3.3) having the following properties:

1. \( Y_{k,a}(x,\rho) \) are continuous for \( x \in [0, \infty), \rho \in \Gamma, |\rho| \geq \rho_a \).
2. For each \( x \in [0, \infty), Y_{k,a}(x,\rho) \) are analytic in \( \rho \in \Gamma, |\rho| \geq \rho_a \).
3. The asymptotic relations

\[
Y_{k,a}(x,\rho) = \exp(\rho\omega_kx)(e_k + o(1)), \quad |\rho| \to \infty,
\]

(3.4)

hold uniformly with respect to \( x \geq a, \rho \in \Gamma \), where \( e_k \) is the \( k \)th column of the unit matrix \( I_n \).
In Theorems 3.2–3.4, it is supposed that \( \lim_{a \to \infty} \rho_a = 0 \).

Theorem 3.2 and the change of variables

\[
\tilde{y}_{k,a}(x) = \text{diag}\{1, \rho, \ldots, \rho^{n-1}\} \Omega Y_{k,a}(x), \ k = 1, n,
\]

readily imply the following result for Equation (1.9).

**Theorem 3.3.** For every \( \alpha \geq 0 \), there exists FSS \( \{y_{k,a}(x, \rho)\}_{k=1}^n \) of Equation (1.9) such that the quasi-derivatives \( y^{[j]}_{k,a}(x, \rho) \) for \( k = 1, n \), \( j = 0, n-1 \) have the following properties:

1. \( y_{k,a}(x, \rho) \) are continuous for \( x \in [0, \infty) \), \( \rho \in \Gamma \), \( |\rho| \geq \rho_a \).
2. For each \( x \in [0, \infty) \), \( y^{[j]}_{k,a} \) are analytic in \( \rho \in \Gamma \), \( |\rho| \geq \rho_a \);
3. The asymptotic relation

\[
y^{[j]}_{k,a}(x, \rho) = (\rho \omega_k)^j \exp(\rho \omega_k x)(1 + o(1)), \ |\rho| \to \infty, \quad (3.5)
\]

holds uniformly with respect to \( x \geq \alpha \) and \( \rho \in \Gamma \).

Fix \( k \in \{1, \ldots, n-1\} \) and consider the region

\[
G_k = \left\{ \rho \in \mathbb{C} : \arg \rho \in \left( (-1)^{n-k} - 1 \right) \frac{\pi}{2n}, (-1)^{n-k} + 3 \frac{\pi}{2n} \right\}, \quad (3.6)
\]

being the union of two neighboring sectors \( \Gamma \). Note that, while passing the boundary between two neighboring sectors, some neighboring values \( \omega_j \) and \( \omega_{j+1} \) are swapped in (3.2). The pair of sectors \( G_k \) defined by (3.6) is chosen in such a way that \( \omega_k \) and \( \omega_{k+1} \) do not change their relative order; in other words, the sets \( \{\omega_j\}^k_{j=1} \) and \( \{\omega_j\}^n_{j=k+1} \) are preserved in \( G_k \).

Analogously to the system \( B_{am} \), in Section 2.1.2 of Yurko, we construct the following FSS.

**Theorem 3.4.** For every \( \alpha \geq 0 \) and \( k \in \{1, \ldots, n-1\} \), there exist solutions \( \{z_{s,k,a}\}_{s=1}^k \) of Equation (1.9) with the quasi-derivatives \( z^{[j]}_{s,k,a}(x, \rho), s = 1, k, j = 0, n-1 \), having the following properties:

1. \( z^{[j]}_{s,k,a}(x, \rho) \) are continuous for \( x \in [0, \infty) \), \( \rho \in \tilde{G}_k \), \( |\rho| \geq \rho_a \).
2. For each \( x \in [0, \infty) \), \( z^{[j]}_{s,k,a}(x, \rho) \) are analytic in \( \rho \in \tilde{G}_k \), \( |\rho| \geq \rho_a \).
3. The following uniform estimates hold:

\[
z^{[j]}_{s,k,a}(x, \rho) = O(\rho^j \exp(\rho \omega_k x)), \ x \geq \alpha, \ \rho \in \tilde{G}_k, \ |\rho| \to \infty.
\]

4. In each of the two sectors \( \Gamma \subset \tilde{G}_k \), the functions \( \{z_{1,k,a}, \ldots, z_{k,k,a}, y_{k+1,a}, \ldots, y_{n,a}\} \) form a FSS of Equation (1.9), where \( y_{s,a} \) are the solutions from Theorem 3.3.

4 | INVERSE PROBLEM ON THE HALF-LINE

Let \( I \in I_n \) and \( \Sigma \in \Sigma_k \) be fixed. By using the matrix function \( F = \mathcal{F}_I(\Sigma) \), define the quasi-derivatives \( y^{[j]}_s, j = 0, n, \) by (1.6). Consider the boundary value problem \( L(\Sigma, U) \) given by (1.9)–(1.10) and its Weyl matrix \( M(\lambda) = [M_{s,k}(\lambda)]^n_{s,k=1} \). Using the Birkhoff systems constructed in Section 3, we obtain the properties of the Weyl matrix, similar to the ones in the case of regular coefficients (see Yurko, Theorem 2.1.1).

**Theorem 4.1.** For each index pair \( (s,k) \): \( 1 \leq k < s \leq n \), the Weyl function \( M_{s,k}(\lambda) \) is analytic in \( \Pi_{(1-1)^{s-k}} := \mathbb{C}\setminus\{\lambda : (-1)^{n-k} \lambda \geq 0\} \) except for an at most countable bounded set of poles. For \( (-1)^{n-k} \lambda \geq 0 \) except for a bounded set, there exist finite limits \( M^\pm_{s,k} = \lim_{z \to 0, \text{Re } z > 0} M_{s,k}(\lambda \pm iz) \).

The proof of Theorem 4.1 repeats the proof of Theorem 2.1.1 in Yurko, so we sketch it briefly.
Theorem 4.3. For each each fixed \( x \), the Birkhoff solutions \( \Phi_k(x, \lambda) \) can be expanded as

\[
\Phi_k(x, \lambda) = \sum_{j=1}^{k} a_{k,j}(\rho) y_j(x, \rho), \quad k = 1, n,
\]

where

\[
a_{k,j}(\rho) : = (-1)^{k+j} \frac{\det[U_j(y_r)]_{j=\Gamma^{-1}-1, r=\Gamma+1}}{\det[U_j(y_r)]_{j=\Gamma}}.
\]

Since \( M_{k,k}(\lambda) = U_k(\Phi_k) \), we obtain

\[
M_{k,k}(\lambda) = \frac{\det[U_j(y_r)]_{j=\Gamma^{-1}-1, r=\Gamma}}{\det[U_j(y_r)]_{j=\Gamma}}.
\]

The similar arguments can be repeated for the Birkhoff systems \( \{y\}_l \}_{l=1}^n \) and \( \{ y^{(n)} \}_{l=1}^n \) from Theorems 3.3 and 3.4, respectively. The relations of form (4.3) together with the properties of solutions established in Theorems 3.3 and 3.4 yield the assertion of the theorem.

Using (4.1), (4.2) and the asymptotics (3.5), we prove the following lemma.

Lemma 4.2. For each each fixed \( x > 0 \) and \( \varphi \) such that \( \{ \rho : \arg \rho = \varphi \} \subset \Gamma \), the following asymptotic relation holds

\[
\Phi_k^{[l]}(x, \lambda) = \rho^{-\nu} a_{k,k}(\rho \omega_k)^l \exp(\rho \omega_k x)(1 + o(1)), \quad |\rho| \to \infty, \quad k = 1, n, \quad j = 0, n - 1,
\]

where \( a_{k,k}^0 := \frac{d_{k-1,k-1}}{d_{k,k}} \neq 0, \quad d_{k,k} = \det[\omega_k^0]_{k=1}^\Gamma, \quad k = 1, n, \quad d_{0,0} := 1.\)

Along with the problem \( \mathcal{L} = \mathcal{L}_l(\Sigma, U) \), consider another problem \( \tilde{\mathcal{L}} = \mathcal{L}_l(\tilde{\Sigma}, \tilde{U}) \) of the same form but with different coefficients \( \tilde{\Sigma}, \tilde{U} \). We agree that if a symbol \( \gamma \) denotes an object related to \( \mathcal{L} \), then the symbol \( \tilde{\gamma} \) with tilde denotes the analogous object related to \( \tilde{\mathcal{L}} \). In particular, \( U = P_{U} L_{U}, \quad \tilde{U} = P_{U} L_{\tilde{U}} \).

Theorem 4.3. If \( P_{U} = P_{U} \) and \( M(\lambda) \equiv \tilde{M}(\lambda) \), then \( \Sigma = \tilde{\Sigma} \) (i.e., \( \sigma_v(x) = \tilde{\sigma}_v(x) \) a.e. on \( \mathbb{R}^+ \), \( v = 0, n - 2 \)) and \( U = \tilde{U} \).

Proof. The proof is based on the method of spectral mappings (see Yurko\textsuperscript{54} and Bondarenko\textsuperscript{57}). Define the matrix of spectral mappings

\[
P(x, \lambda) := \Phi(x, \lambda)(\Phi(x, \lambda))^{-1}.
\]

Using the relations

\[
\Phi(x, \lambda) = C(x, \lambda) M(\lambda), \quad \tilde{\Phi}(x, \lambda) = \tilde{C}(x, \lambda) \tilde{M}(\lambda),
\]

and \( M(\lambda) \equiv \tilde{M}(\lambda) \), we obtain

\[
P(x, \lambda) = C(x, \lambda)(\tilde{C}(x, \lambda))^{-1}.
\]

Due to the definition of \( C(x, \lambda) \), this matrix function solves the initial value problem

\[
C'(x, \lambda) = (F(x) + \Lambda) C(x, \lambda), \quad x \in \mathbb{R}^+, \quad C(0, \lambda) = U^{-1}.
\]

Hence, \( C(x, \lambda) \) is entire in \( \lambda \) for each fixed \( x \in \mathbb{R}^+ \). Since trace \((F(x)) = 0 \), then \( \det(C(x, \lambda)) \) does not depend on \( x \). In view of the initial condition \( C(0, \lambda) = U^{-1} \), we have \( \det(C(x, \lambda)) \neq 0 \). The same arguments are valid for \( \tilde{C}(x, \lambda) \). Therefore, (4.5) implies that \( P(x, \lambda) \) is entire in \( \lambda \) for each fixed \( x \in \mathbb{R}^+ \) and \( P(0, \lambda) = U^{-1} \).

On the other hand, using (4.4) and the asymptotics of Lemma 4.2 for the entries of \( \Phi(x, \lambda) \) and \( \tilde{\Phi}(x, \lambda) \), we obtain the following asymptotic relation for the entries of \( P(x, \lambda) = [P_{k,j}]_{k,j=1}^n \):

\[
P_{k,j}(x, \lambda) = \rho^{k-j}(\delta_{k,j} + o(1)), \quad k, j = 1, n,
\]
for each fixed $x > 0$ as $|\lambda| \to \infty$ along any fixed ray arg $\lambda = \beta \not\in \{0, \pi\}$. Applying Pragmen–Lindelöf’s theorem (see Buterin et al.\textsuperscript{63}) and Liouville’s theorem, we conclude that $P(x, \lambda)$ equals a constant unit lower triangular matrix $P(x)$ for each fixed $x > 0$ and

$$ P(0) = U^{-1} \hat{U}. \quad (4.6) $$

Using (4.4) and the relations

$$ \Phi'(x, \lambda) = (F(x) + \Lambda) \Phi(x, \lambda), \quad \Phi'(\lambda, \lambda) = (\hat{F}(x) + \Lambda) \Phi(x, \lambda), $$

we derive

$$ P'(x) + P(x) \hat{F}(x) = F(x) P(x). \quad (4.7) $$

Hence, $P(x)$ satisfies the assumptions of Lemma 2.3, which yields $\Sigma = \tilde{\Sigma}$ and $P(x) \equiv I_n$. Using (4.6), we conclude that $U = \hat{U}$. \hfill \Box

\textbf{Remark 1.} In fact, instead of the boundary value problem (1.9), (1.10), we study the first-order system (1.8) with the boundary condition $U\hat{f}(0) = 0$. Nevertheless, in the proof of Theorem 4.3, the special structure of the matrix $F = F_1(\Sigma)$ constructed by the coefficients $\Sigma$ of the differential expression (1.1) is important. An arbitrary matrix function $F \in \mathcal{F}_n$ cannot be uniquely recovered from the corresponding Weyl matrix, because the assertion of Lemma 2.3 does not hold for an arbitrary $F \in \mathcal{F}_n$. This is shown by Example 1.

\textbf{Example 1.} Suppose that $n = 2, F, \hat{F} \in \mathcal{F}_2$,

$$ F = \begin{bmatrix} a & 1 \\ b & -a \end{bmatrix}, \quad \hat{F} = \begin{bmatrix} \hat{a} & 1 \\ \hat{b} & -\hat{a} \end{bmatrix}, $$

and a unit lower triangular matrix function $P(x)$ satisfies (2.17). It is easy to see that the system (2.17) in this case is equivalent to

$$ p_{2,1} = \hat{a} - a, \quad p'_{2,1} + \hat{a}^2 - a^2 + \hat{b} - b = 0. $$

The latter relations do not imply $p_{2,1} = 0, a = \hat{a}, b = \hat{b}$. For instance, one can take $a = b = 0$, an arbitrary function $\tilde{a} \in (L_1 \cap AC_{loc})(\mathbb{R}_+)$ such that $\hat{a} \in L_1(\mathbb{R}_+), p_{21} := \tilde{a}, \hat{b} := -(p'_{21} + p^2_{21})$. Thus, the condition (2.17) is fulfilled but the assertion of Lemma 2.3 does not hold for this case. The matrix function $\hat{F}(x)$ of form (4.8) defines the quasi-derivatives

$$ y^{[1]} = y' - ay, \quad y^{[2]} = (y^{[1]})' + ay^{[1]} - by. $$

Hence, the equation $y^{[2]} = \lambda y$ turns into the Sturm–Liouville equation $y'' - q(x)y = \lambda y$ with the potential $q = a' + a^2 + b$. Even if we reconstruct the potential $q$ by using some spectral data, we cannot uniquely determine the functions $a$ and $b$.

\section{Inverse Problem on a Finite Interval}

The inverse spectral problem for the differential expression (1.1) on the finite interval $(0, 1)$ has been considered in Bondarenko\textsuperscript{57} for the Mirzoev–Shkalikov case: $i_{2k+j} = m - k - j, j = 0, 1$. In contrast to the half-line, for a finite interval $W_{k_1}^{k_2}(0, 1) \subset W_{k_2}(0, 1)$ if $k_1 < k_2$. Therefore, the case of arbitrary $I \in I_n$ can be reduced to the Mirzoev–Shkalikov case, and the results of Bondarenko\textsuperscript{57} can be applied. However, in this section, we show that the regularization of Section 2 for any $I \in I_n$ can be used for investigating inverse problems. In addition, we discuss the recovery of the boundary conditions, improving the results of Bondarenko.\textsuperscript{57}

Suppose that $I \in I_n, \Sigma \in \Sigma_{I,(0,1)}$. Denote by $\mathcal{L} = \mathcal{L}_I(\Sigma, U, V)$ the differential equation

$$ \mathcal{L}_n(y) = \lambda y, \quad x \in (0, 1), $$

(5.1)
given together with the linear forms (1.10) and

\[ V_s(y) := y^{P_s-1} + \sum_{j=1}^{P_s} v_{s,j} y^{l_j-1}, \ s = 1, n, \]

where \( V = [v_{s,j}]_{s,j=1}^n \) is a constant \((n \times n)\) matrix of form \( V = P_V L_V \). \( P_V \) is the permutation matrix with the unit elements at the positions \((k, p_k + 1), k = \overline{1, n}\), and \( L_V \) is a unit lower triangular matrix.

Denote by \( \{C_k(x, \lambda)\}_{k=1}^n \) and \( \{\Phi_k(x, \lambda)\}_{k=1}^n \) the solutions of Equation (5.1) satisfying the conditions (1.11) and

\[ U_s(\Phi_k) = \delta_{s,k}, \ s = 1, k, \ V_l(\Phi_k) = 0, \ l = k + 1, n, \]

respectively. Define the matrix functions \( C(x, \lambda) = [\tilde{C}_k(x, \lambda)]_{k=1}^n \) and \( \Phi(x, \lambda) = [\tilde{\Phi}_k(x, \lambda)]_{k=1}^n \). Then, \( C(x, \lambda) = \Phi(x, \lambda) M(\lambda) \), where \( M(\lambda) \) is the Weyl matrix. It is shown in Bondarenko\(^{57}\) that \( M(\lambda) \) is a unit lower triangular matrix function meromorphic in \( \lambda \).

It has been proved in Bondarenko\(^{57}\) that the Weyl matrix \( M(\lambda) \) uniquely specifies the coefficients \( \Sigma \) in the Mirzoev–Shkalikov case if the matrices \( U \) and \( V \) are known a priori. Here, we focus on the recovery of the boundary conditions in more details for various \( I \in I_n \).

Using the entries of the matrix \( L_U = [l_{k,j}]_{k,j=1}^n \), define the vectors

\[ L_{2k} := (l_{n-k,k+1} + l_{n-k,k+1}, s = k, k + 1, k + 2, \ldots, k + \ell, k = 0, m - 1, \]
\[ L_{2k+1} := (l_{n-k,k+1} + l_{n-k,k+1}, s = k + 1, k + 2, \ldots, k + \ell, k = 0, m - 2, \ldots, m - 1}. \]

**Inverse Problem 5.1.** Suppose that \( P_U \) and \( L_\nu \), \( \nu = 0, n - 2 \), are known a priori. Given the Weyl matrix \( M(\lambda) \), find \( \Sigma = (\sigma_\nu)_{\nu=0}^{n-2}, U = [u_{k,j}]_{k,j=1}^n \), and \( V = [v_{k,j}]_{k,j=1}^n \).

Note that, in the regular case \( i_\nu = 0, \nu = 0, n - 2 \), no elements of \( L_U \) are required to be known in Inverse Problem 5.1. In the case \( i_{2k} = i_{2k+1} = 1 \), \( j_k \), the values \( L_{2k} \) and \( L_{2k+1} \) can be replaced by \( l_{n-k,k+1} \) and \( l_{n-k,k+1}, s = k, k + 1, \ldots, k + \ell, k = 0, m - 1, \ldots, m - 1 \). In particular, in the Mirzoev–Shkalikov case, the values \( (l_{k,j})_{k=0, m-1, j=1}^m \) are required to be known.

Along with the problem \( L = L(\Sigma, U, V) \), consider another problem \( \tilde{L} = L(\tilde{\Sigma}, \tilde{U}, \tilde{V}) \) of the same form but with different coefficients \( \tilde{\Sigma}, \tilde{U}, \tilde{V} \). We agree that, if a symbol \( \gamma \) denotes an object related to \( L \), then the symbol \( \tilde{\gamma} \) with tilde denotes the analogous object related to \( \tilde{L} \).

In view of Remark 4.1 in Bondarenko\(^{57}\) the right-hand boundary condition coefficients cannot be uniquely recovered from the Weyl matrix. However, some equivalence classes can be considered, so we need the following definition.

**Definition 5.2.** Let \( I \in I_n, \Sigma \in \Sigma(I_{0,1}), \) and \( U = P_U L_U \) be fixed. Then, the matrices \( V = P_V L_V \) and \( \tilde{V} = P_{\tilde{V}} L_{\tilde{V}} \) are called **equivalent** if the corresponding problems \( L = L(\Sigma, U, V) \) and \( \tilde{L} = L(\tilde{\Sigma}, \tilde{U}, \tilde{V}) \) have equal Weyl solutions: \( \Phi_k(x, \lambda) \equiv \tilde{\Phi}_k(x, \lambda), k = 1, n \).

The following uniqueness theorem generalizes Theorem 6.2 from Bondarenko\(^{57}\).

**Theorem 5.3.** If \( P_U = P_C, L_\nu = \tilde{L}_\nu, \nu = 0, n - 2, \) and \( M(\lambda) \equiv \tilde{M}(\lambda) \), then \( \Sigma = \tilde{\Sigma} \) (i.e., \( \sigma_\nu(x) = \tilde{\sigma}_\nu(x) \) a.e. on \((0, 1), \nu = 0, n - 2 \)), \( U = \tilde{U}, \) and \( V \) is equivalent to \( \tilde{V} \) in sense of Definition 5.2.

The proof of Theorem 5.3 is analogous to the proof of Theorem 6.2 in Bondarenko\(^{57}\) and relies on Lemma 2.4, so we omit it.

**Remark 2.** Suppose that \( \mathcal{N} \subseteq \{0, 1, \ldots, n - 2\} \) and the functions \( (\sigma_\nu)_{\nu \in \mathcal{N}} \) are known a priori. Then, it is sufficient to know \( L_\nu \) for \( \nu = 0, n - 2 \setminus \mathcal{N} \) together with \( P_U \) for the unique recovery of the problem \( L(\Sigma, U, V) \) from \( M(\lambda) \).

**Remark 3.** In contrast to Inverse Problem 5.1, Inverse Problem 1.1 on the half-line does not require the boundary condition coefficients \( l_{k,j} \) to be known, roughly speaking, because of the implicit condition at infinity: \( \sigma_\nu \in L_1(\mathbb{R}_+), \nu = 0, n - 2 \).
6 | EXAMPLES

6.1 | Case n = 2.

The differential expression (1.1) for \( n = 2 \) takes the form

\[
\ell_2(y) = y'' + (-1)^i_0 \sigma_0^{(i_0)} y,
\]

where \( i_0 \in \{0, 1\}, I = (i_0), \Sigma = (\sigma_0) \). First, consider the inverse problem on the half-line.

1. In the case \( i_0 = 0 \), Equation (1.9) takes the form

\[
y'' + \sigma_0 y = \lambda y, \quad \sigma_0 \in L_1(\mathbb{R}^+). \tag{6.1}
\]

Using (2.2), (2.3), and (2.8), we obtain the matrix functions \( Q = \mathcal{D}_I(\Sigma), F = \mathcal{F}_I(\Sigma): \)

\[
Q = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ -\sigma_0 & 0 \end{bmatrix}.
\]

Hence, \( y^{[1]} = y', \ y^{[2]} = y'' + \sigma_0 y \). For definiteness, suppose that

\[
P_U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L_U = \begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix}, \quad h \in \mathbb{C}. \tag{6.2}
\]

Then,

\[
U_1(y) = y'(0) + hy(0), \quad U_2(y) = y(0).
\]

The Weyl matrix has the form

\[
M(\lambda) = \begin{bmatrix} 1 & 0 \\ M_{2,1}(\lambda) & 1 \end{bmatrix}, \quad M_{2,1}(\lambda) = U_2(\Phi_1), \tag{6.3}
\]

where \( \Phi_1(x, \lambda) \) is the Weyl solution of Equation (6.1) satisfying the boundary conditions

\[
U_1(\Phi_1) = 1, \quad \Phi_1(x, \lambda) = O(\exp(-\rho x)), \quad x \to \infty, \quad \Re \rho \geq 0, \quad \rho \neq 0. \tag{6.4}
\]

Inverse Problem 1.1 takes the following form.

**Inverse Problem 6.1.** Given the Weyl function \( M_{2,1}(\lambda) \), find \( \sigma_0 \) and \( h \).

This is the standard inverse problem for the Sturm–Liouville operator on the half-line by the Weyl function, which has been considered, for example, in Freiling and Yurko.\(^{59, \text{Section 2.2 Theorem 4.3}} \) for this case is equivalent to Theorem 2.2.1 in Freiling and Yurko.\(^{59} \)

2. In the case \( i_0 = 1 \), Equation (1.9) takes the form

\[
y'' - \sigma_0 y = \lambda y, \quad \sigma_0 \in (L_1 \cap L_2)(\mathbb{R}^+), \tag{6.5}
\]

where the derivative is understood in the sense of distributions.

Using (2.2), (2.3), and (2.8), we obtain the matrix functions \( Q = \mathcal{D}_I(\Sigma), F = \mathcal{F}_I(\Sigma): \)

\[
Q = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \sigma_0 & 1 \\ -\sigma_0 & -\sigma_0 \end{bmatrix}.
\]

Thus, \( F(x) \) coincides with the well-known regularization matrix for the Sturm–Liouville operator with singular potential (see, e.g., Savchuk and Shkalikov\(^{19,20} \)). The quasi-derivatives have the form

\[
y^{[1]} = y' - \sigma_0 y, \quad y^{[2]} = (y^{[1]})' + \sigma_0 y^{[1]} + \sigma_0^2 y.
\]
Define $P_U$ and $L_U$ by (6.2). Then,

$$U_1(y) = y^{(1)}(0) + hy(0), \quad U_2(y) = y(0).$$

The Weyl matrix has the form (6.3), where $\Phi_1(x, \lambda)$ is the solution of Equation (6.5) (in the sense of Definition 3.1) satisfying the boundary conditions (6.4). Inverse Problem 1.1 for this case takes the form of Inverse Problem 6.1.

Suppose that the problems $\mathcal{L}$ with $i_0 = 0$, $\sigma_0 \in L_1(\mathbb{R}_+)$ and $\tilde{\mathcal{L}}$ with $\tilde{i}_0 = 1$, $\tilde{\sigma}_0 \in (L_1 \cap L_2)(\mathbb{R}_+)$ are equivalent to each other. Let us show that the corresponding inverse problems are also equivalent to each other. Comparing (6.1) and (6.5), we conclude that $\sigma_0 = -\tilde{\sigma}_0$, so

$$\tilde{\sigma}_0 \in AC[0, \infty), \quad \tilde{\sigma}_0(x) = \tilde{\sigma}_0(0) - \int_0^x \sigma_0(t) dt. \quad (6.6)$$

Note that the linear forms $U_i$ and $\tilde{U}_i$ for the problems $\mathcal{L}$ and $\tilde{\mathcal{L}}$, respectively, differ

$$U_1(y) = y'(0) + hy(0), \quad U_1(y) = y^{(1)}(0) + \tilde{h}y(0) = y'(0) - \tilde{\sigma}_0(0)y(0) + \tilde{h}y(0).$$

These forms coincide with each other and provide the same Weyl function $M_{2,1}(\lambda) = \tilde{M}_{2,1}(\lambda)$ if and only if

$$h = -\tilde{\sigma}_0(0) + \tilde{h}. \quad (6.7)$$

The relations (6.6) and (6.7) together imply

$$\tilde{\sigma}_0(x) = \tilde{h} - h - \int_0^x \sigma_0(t) dt. \quad (6.8)$$

Since $\sigma_0, \tilde{\sigma}_0 \in L_1(\mathbb{R}_+)$, then

$$\tilde{h} - h - \int_0^\infty \sigma_0(t) dt = 0. \quad (6.9)$$

The relations (6.8) and (6.9) give the one-to-one correspondence between the data $(\sigma_0, h) \leftrightarrow (\tilde{\sigma}_0, \tilde{h})$. Thus, the reconstruction of either $(\sigma_0, h)$ or $(\tilde{\sigma}_0, \tilde{h})$ by using $M_{2,1}(\lambda)$ is equivalent.

The situation is different for a finite interval. For definiteness, consider

$$P_V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_V = \begin{bmatrix} 1 & 0 \\ H & 1 \end{bmatrix}, \quad H \in \mathbb{C}.$$

Then, $V_2(y) = y^{(1)}(1) + Hy(1)$. The Weyl matrix has the form (6.3), where $\Phi_1(x, \lambda)$ is the Weyl solution of the equation $\ell_2(y) = \lambda y, \ x \in (0, 1)$, satisfying the boundary conditions $U_1(\Phi_1) = 1$, $V_2(\Phi_1) = 0$, where $U_1(y)$ is defined similarly to the half-line case. In the regular case $i_0 = 0$, Inverse Problem 5.1 takes the following form.

**Inverse Problem 6.2.** Given $M_{2,1}(\lambda)$, find $\sigma_0$, $h$, and $H$.

Inverse Problem 6.2 is the classical problem of the recovery of the Sturm–Liouville operator from the Weyl function, which is equivalent to Borg’s problem by two spectra and to Marchenko’s problem by the spectral function (see, e.g., previous studies\textsuperscript{31,32}.

In the singular case $i_1 = 0$, Inverse Problem 5.1 can be reformulated as follows.

**Inverse Problem 6.3.** Given $h$ and $M_{2,1}(\lambda)$, find $\sigma_0$ and $H$.

Inverse Problem 6.3 in various equivalent formulations was studied in previous studies\textsuperscript{34,35,42} and other papers. The uniqueness Theorem 4.3 for Inverse Problems 6.2 and 6.3 corresponds to the previously known results.

Suppose that the problem $\mathcal{L}$ with $i_0 = 0$, $\sigma_0 \in L_1(0, 1)$ is equivalent to the problem $\tilde{\mathcal{L}}$ with $i_0 = 1$, $\tilde{\sigma}_0 \in W_1^1[0, 1]$, that is, the relations (6.8) and

$$H = -\tilde{\sigma}_0(1) + \tilde{H} \quad (6.10)$$

hold. If $\tilde{h}$ is fixed, then (6.9) and (6.10) give the one-to-one correspondence between the data $(\sigma_0, h, H) \leftrightarrow (\tilde{\sigma}_0, \tilde{H})$. Consequently, Inverse Problem 6.2 for $\mathcal{L}$ and Inverse Problem 6.3 for $\tilde{\mathcal{L}}$ are equivalent to each other in this case.
6.2 | Case $n = 4$

In Konechnaya, the regularization matrices have been provided for the differential expression

$$(py'')' - (q^{(a)})y' + r^{(b)}y,$$  \hspace{1cm} (6.11)

where $p, q, r$ are regular functions, $\alpha \in \{0, 1\}, \beta \in \{0, 1, 2\}$. In the case $p \equiv 1$, (6.11) is equivalent to the differential expression (1.1) for $n = 4$, $I = (i_0, i_1, i_2, i_3)$, $\Sigma = (\sigma_0, \sigma_1, \sigma_2)$ with $\sigma_1 = 0$:

$$\ell_4(y) = y^{(4)} + (-1)^{i_1+1}(\sigma_2^{(i_1)}(x)y' + (-1)^{i_0}\sigma_0^{(i_0)}y.$$  

Here, $i_0 \in \{0, 1, 2\}, i_2 \in \{0, 1\}$. Let us consider all the six possible cases. For convenience, denote $Q_{i_0,i_2} := \mathcal{O}_I(\Sigma)$, $F_{i_0,i_2} := \mathcal{F}_I(\Sigma)$.

Using (2.2), (2.3), and (2.8), we obtain

$$Q_{0,0} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{1,0} = \begin{bmatrix} 0 & \sigma_0 & 0 \\ \sigma_0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{2,0} = \begin{bmatrix} 0 & 0 & \sigma_0 \\ 0 & \sigma_2 + 2\sigma_0 & 0 \\ \sigma_0 & 0 & 0 \end{bmatrix},$$

$$Q_{0,1} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}, \quad Q_{1,1} = \begin{bmatrix} 0 & \sigma_0 & 0 \\ \sigma_0 & 0 & \sigma_0 \\ 0 & \sigma_0 & 0 \end{bmatrix}, \quad Q_{2,1} = \begin{bmatrix} 0 & 0 & \sigma_0 \\ 0 & 2\sigma_0 & \sigma_0 \\ \sigma_0 & \sigma_0 & \sigma_0 \end{bmatrix}.$$

The matrices $F_{i_0,i_2}$ coincide with the ones provided in Konechnaya. In particular, the matrix $F_{0,0}$ corresponds to the well-known regular case (see Everitt and Marcus, Appendix A), and $F_{2,1}$ was obtained in Vladimirov.

For clarity, denote $\mathcal{L}_{i_0,i_2} := \mathcal{L}_I$. Similarly to the case $n = 2$, it can be shown that, if the problem $\mathcal{L} = \mathcal{L}_{i_0,i_2}(\Sigma, U)$ on the half-line is equivalent to $\tilde{\mathcal{L}} = \mathcal{L}_{i_0,i_2}(\tilde{\Sigma}, U)$ with $\tilde{\Sigma} = (\tilde{\sigma}_0, \tilde{\sigma}_2) \neq (i_0, i_2)$, $P_U = P_{U_I}$, then the corresponding inverse problems are equivalent to each other. We obtain the equivalence relations between the problem coefficients $(\Sigma, U) \leftrightarrow (\tilde{\Sigma}, U)$ analogous to (6.8), (6.9) for several cases. The other cases can be investigated similarly.

1. Consider equivalent problems $\mathcal{L} = \mathcal{L}_{0,0}(\Sigma, U), \tilde{\mathcal{L}} = \mathcal{L}_{0,1}(\tilde{\Sigma}, U)$, where $\sigma_0 = \tilde{\sigma}_0 \in L_1(\mathbb{R}_+), \sigma_2 = -\tilde{\sigma}_2', \sigma_2 \in L_1(\mathbb{R}_+), \tilde{\sigma}_2 \in (L_1 \cap L_2)(\mathbb{R}_+)$. The quasi-derivatives for the problems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are defined via (1.6) by using the entries of the matrix functions $F_{0,0}$ and $F_{0,1}$, respectively. Substituting these quasi-derivatives into the equivalence relations for the boundary condition forms: $U_s(y) = \tilde{U}_s(y), s = 1, 4$, we derive

$$l_{2,1} = \tilde{l}_{2,1}, \quad l_{3,1} = \tilde{l}_{3,1}, \quad l_{3,2} = \tilde{\sigma}_2(0) + \tilde{l}_{3,2}, \quad l_{4,1} = \tilde{l}_{4,1}, \quad l_{4,2} = \tilde{l}_{4,2} + \tilde{l}_{4,3} = \tilde{l}_{4,3}.$$

Consequently, the equivalence $(\Sigma, U) \leftrightarrow (\tilde{\Sigma}, U)$ is given by the relations

$$l_{3,2} = \tilde{l}_{3,2} + \int_0^\infty \tilde{\sigma}_2(t) dt, \quad \tilde{\sigma}_2(x) = \int_x^\infty \sigma_2(t) dt,$$

$$(\sigma_0, l_{2,1}, l_{3,1}, l_{4,1}, l_{4,2}, l_{4,3}) = (\tilde{\sigma}_0, \tilde{l}_{2,1}, \tilde{l}_{3,1}, \tilde{l}_{4,1}, \tilde{l}_{4,3}), \quad l_{4,2} = \tilde{l}_{4,2} + \tilde{l}_{4,3} \tilde{\sigma}_2(0) + \tilde{l}_{4,2}. \hspace{1cm} (6.12)$$
2. Consider equivalent problems $\mathcal{L} = \mathcal{L}_{0,1}(\Sigma, U)$, $\tilde{\mathcal{L}} = \mathcal{L}_{1,1}(\tilde{\Sigma}, \tilde{U})$, where $\sigma_0 = -\tilde{\sigma}^0$, $\sigma_0 \in L_1(\mathbb{R}^+)$, $\tilde{\sigma}_0 \in L_1(\mathbb{R}^+)$, $\sigma_2 = \tilde{\sigma}_2 \in (L_1 \cap L_2)(\mathbb{R}^+)$. The one-to-one correspondence $(\Sigma, U) \leftrightarrow (\tilde{\Sigma}, \tilde{U})$ is given by the relations

$$l_{3,1} = l_{3,1} + \int_0^\infty \sigma_0(t)\,dt, \quad l_0(\sigma_0) = \int_x^\infty \sigma_0(t)\,dt,$$

$$\{\sigma_2, l_{2,1}, l_{3,1}, l_{4,2}, l_{4,3}\} = (\tilde{\sigma}_2, l_{2,1}, l_{3,1}, l_{4,2}, l_{4,3}).$$

(6.13)

3. Consider equivalent problems $\mathcal{L} = \mathcal{L}_{0,0}(\Sigma, U)$, $\tilde{\mathcal{L}} = \mathcal{L}_{2,0}(\tilde{\Sigma}, \tilde{U})$, where $\sigma_0 = -\tilde{\sigma}^0$, $\sigma_0 \in L_1(\mathbb{R}^+)$, $\sigma_0$ is continuous at zero, $\tilde{\sigma}_0 \in (L_1 \cap L_2)(\mathbb{R}^+)$, $\sigma_2 = \tilde{\sigma}_2 \in L_1(\mathbb{R}^+)$. The one-to-one correspondence $(\Sigma, U) \leftrightarrow (\tilde{\Sigma}, \tilde{U})$ is given by the relations

$$l_{3,1} = l_{3,1} + \int_0^\infty \sigma_0(t)\,dt, \quad l_0(\sigma_0) = \int_x^\infty \sigma_0(t)\,dt,$$

$$\{\sigma_2, l_{2,1}, l_{3,2}, l_{4,3}\} = (\tilde{\sigma}_2, l_{2,1}, l_{3,2}, l_{4,3}),$$

$$l_{4,2} = l_{4,2} - 2\tilde{\sigma}_0(0), \quad l_{4,1} - \sigma_0(0) = l_{4,3}\tilde{\sigma}_0(0) + \tilde{l}_{4,1}.$$  

(6.14)

Proceed to the finite interval case. Since $\sigma_1 = 0$ is known, put $\mathcal{N} : = \{1\}$. Taking Theorem 4.3 and Remark 2 into account, we conclude that the numbers

$$l_{4,s+1} + l_{4,s+1}, \quad s = 0, l_0 - 1,$$

and $l_{3,2}$ if $l_2 = 1$ have to be given together with $P_U$ and $M(\lambda)$ for the unique reconstruction of $\mathcal{L}_I(\Sigma, U, V)$. Alternatively, one can give either $\{l_{4,1}\}_{s=4-4-s+1}$ or $\{l_{4,1}\}_{s=1}$ instead of (6.15). For definiteness, suppose that we have $\{l_{4,1}\}_{s=4-4-s+1}$.

Let us shortly denote by $\text{IP}_{l_0,l_2}$ the inverse problem for the corresponding $l_0$ and $l_2$. Suppose that $P_U$ and $M(\lambda)$ are given. For the recovery of $\Sigma$, $U$, and $V$ the following boundary condition coefficients $l_{k,i}$ are required:

$$\text{IP}_{l_0} : \text{none}, \quad \text{IP}_{l_1} : l_{4,1}, \quad \text{IP}_{l_2} : l_{3,1}, l_{4,1} \quad \text{IP}_{l_3} : l_{3,2}, l_{4,1} \quad \text{IP}_{l_4} : l_{3,1}, l_{3,2}, l_{4,1}.$$  

It can be shown that if the problem $\mathcal{L}_{l_0,l_2}(\Sigma, U, V)$ is equivalent to $\tilde{\mathcal{L}}_{\tilde{l}_0,\tilde{l}_2}$, with $(\tilde{l}_0, \tilde{l}_2) \neq (l_0, l_2)$, $P_U = P_{\tilde{U}}$, then the corresponding inverse problems $\text{IP}_{l_0,l_2}$ and $\text{IP}_{\tilde{l}_0,\tilde{l}_2}$ are equivalent to each other.

For simplicity, assume that the matrix $P_U$ is defined by the permutation $(p_{1,1}, p_{2,1}, p_{3,1}, p_{4,1}) = (3, 2, 1, 0)$. Thus, the Weyl solutions (5.2) is defined by the following linear forms:

$$V_2(y) = y[2](1) + v_{2,2}y[1](1) + v_{2,1}y(1), \quad V_3(y) = y[1](1) + v_{3,1}y(1), \quad V_4(y) = y(1).$$

Note that the linear form $V_1(y)$ does not participate in (5.2). Moreover, the Weyl solutions $\Phi_k(x, \lambda)$ do not depend on the coefficients $(v_{2,1}, v_{2,2}, v_{3,1})$. Therefore, all matrices $V$ with the fixed $P_U$ are equivalent in the sense of Definition 5.2. Hence, $L_V$ cannot be uniquely recovered from the Weyl matrix $M(\lambda)$ even if $\Sigma$ and $U$ are known. In order to prove the inverse problem equivalence in this case, we only need to obtain the equivalence relations $(\Sigma, U) \leftrightarrow (\tilde{\Sigma}, \tilde{U})$. Below, we consider the Cases 1–3 similar to the ones studied for the half-line.

1. Consider equivalent problems $\mathcal{L} = \mathcal{L}_{0,0}(\Sigma, U)$, $\tilde{\mathcal{L}} = \mathcal{L}_{0,1}(\tilde{\Sigma}, \tilde{U}, \tilde{V})$, where $\sigma_0 = \tilde{\sigma}_0 \in L_1(0, 1)$, $\sigma_2 = -\tilde{\sigma}_0^0 \in L_1(0, 1)$. If $\tilde{l}_{3,2}$ is fixed, then the one-to-one correspondence

$$(\sigma_0, \sigma_2, l_{2,1}, l_{3,1}, l_{3,2}, l_{4,1}, l_{4,2}, l_{4,3}) \leftrightarrow (\tilde{\sigma}_0, \tilde{\sigma}_2, \tilde{l}_{2,1}, \tilde{l}_{3,1}, \tilde{l}_{4,1}, \tilde{l}_{4,2}, \tilde{l}_{4,3})$$

is given by (6.12) and

$$\tilde{\sigma}_2(x) = l_{3,2} - \tilde{l}_{3,2} - \int_0^x \sigma_2(t)\,dt.$$  

Hence, $\text{IP}_{l_0,0}$ is equivalent to $\text{IP}_{l_0,1}$.

2. Consider equivalent problems $\mathcal{L} = \mathcal{L}_{0,1}(\Sigma, U)$, $\tilde{\mathcal{L}} = \mathcal{L}_{1,1}(\tilde{\Sigma}, \tilde{U}, \tilde{V})$, where $\sigma_0 = -\tilde{\sigma}_0^0 \in L_1(0, 1)$, $\sigma_2 = \tilde{\sigma}_2 \in L_2(0, 1)$. If $\tilde{l}_{4,1}$ is fixed, then the one-to-one correspondence

$$(\sigma_0, \sigma_2, l_{2,1}, l_{3,1}, l_{3,2}, l_{4,1}, l_{4,2}, l_{4,3}) \leftrightarrow (\tilde{\sigma}_0, \tilde{\sigma}_2, \tilde{l}_{2,1}, \tilde{l}_{3,1}, \tilde{l}_{3,2}, \tilde{l}_{4,2}, \tilde{l}_{4,3})$$

is given by (6.13) and

$$\tilde{\sigma}_2(x) = l_{3,2} - \tilde{l}_{3,2} - \int_0^x \sigma_2(t)\,dt.$$  

(6.15)
is given by (6.13) and
\[ \tilde{\sigma}_0(x) = \tilde{l}_{4,1} - l_{4,1} - \int_0^x \sigma_0(t) \, dt. \]

Hence, IP_{0,1} is equivalent to IP_{1,1}.

3. Consider equivalent problems \( \mathcal{L} = \mathcal{L}_{1,0}(\Sigma, U, V) \), \( \tilde{\mathcal{L}} = \mathcal{L}_{2,0}(\tilde{\Sigma}, \tilde{U}, \tilde{V}) \), where \( \sigma_0 = -\sigma'_0 \in L_1(0, 1) \), \( \sigma_0 \) is continuous at zero, \( \sigma_2 = \tilde{\sigma}_2 \in L_1(\mathbb{R}_+) \). If \( I_{3,1} \) is fixed, then the one-to-one correspondence

\[ (\sigma_0, \sigma_2, l_{2,1}, l_{3,1}, l_{3,2}, l_{4,1}, l_{4,2}, l_{4,3}) \leftrightarrow (\tilde{\sigma}_0, \tilde{\sigma}_2, \tilde{l}_{2,1}, \tilde{l}_{3,2}, \tilde{l}_{4,1}, \tilde{l}_{4,2}, \tilde{l}_{4,3}) \]

is given by (6.14) and
\[ \tilde{\sigma}_0(x) = l_{3,1} - \tilde{l}_{3,1} - \int_0^x \sigma_0(t) \, dt. \]

Hence, IP_{1,0} is equivalent to IP_{2,0}.

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**CONFLICT OF INTEREST**

The author declares that this paper has no conflict of interest.

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