A NOTE ON SESHADRI CONSTANTS ON GENERAL $K3$ SURFACES

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Abstract. We prove a lower bound on the Seshadri constant $\varepsilon(L)$ on a $K3$ surface $S$ with $\text{Pic} S \simeq \mathbb{Z}[L]$. In particular, we obtain that $\varepsilon(L) = \alpha$ if $L^2 = \alpha^2$ for an integer $\alpha$.

1. Introduction and results

Let $X$ be a smooth, projective variety and $L$ be an ample line bundle on $X$. Then the real number

$$\varepsilon(L, x) := \inf_{C \ni x} \frac{L.C}{\text{mult}_x C},$$

introduced by Demailly [De], is the Seshadri constant of $L$ at $x \in X$ (where the infimum is taken over all irreducible curves on $X$ passing through $x$). The (global) Seshadri constant of $L$ is defined as

$$\varepsilon(L) := \inf_{x \in X} \varepsilon(L, x).$$

We refer to [La, pp. 270–303] for more background, properties and results on these constants.

The subtle point about Seshadri constants is that their exact values are known only in a few cases and even on surfaces it is difficult to control them.

It is known that the global Seshadri constant on a surface satisfies $\varepsilon(L) \leq \sqrt{L^2}$, cf. e.g. [St, Rem. 1], and that $\varepsilon(L)$ is rational if $\varepsilon(L) < \sqrt{L^2}$, cf. [Sz, Lemma 3.1] or [Og, Cor. 2]. (It is not known whether Seshadri constants are always rational, but no examples are known where they are irrational.)

In the case of $K3$ surfaces, Seshadri constants have only been computed for the hyperplane bundle of quartic surfaces [Ba2] and in the particular case of non-globally generated ample line bundles [BDS, Prop. 3.1].

In this note we prove the following result:

**Theorem** Let $S$ be a smooth, projective $K3$ surface with $\text{Pic} S \simeq \mathbb{Z}[L]$. Then either

$$\varepsilon(L) \geq \lfloor \sqrt{L^2} \rfloor,$$

or

$$(L^2, \varepsilon(L)) \in \left\{ \left( \alpha^2 + \alpha - 2, \alpha - \frac{2}{\alpha + 1} \right), \left( \alpha^2 + 1, \frac{1}{2} - \frac{1}{2\alpha + 1} \right) \right\}$$

for some $\alpha \in \mathbb{N}$. (Note that in fact $\alpha = \lfloor \sqrt{L^2} \rfloor$.)

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Remark In the two exceptional cases (1) of the theorem, the proof below shows that there has to exist a point \( x \in S \) and an irreducible rational curve \( C \in |L| \) (resp. \( C \in |2L| \)) such that \( C \) has an ordinary singular point of multiplicity \( \alpha + 1 \) (resp. \( 2\alpha + 1 \)) at \( x \) and is smooth outside \( x \), and \( \varepsilon(L) = L.C/\text{mult}_xC \).

By a well-known result of Chen [Ch1], rational curves in the primitive class of a general K3 surface in the moduli space are nodal. Hence the first exceptional case in (1) cannot occur on a general K3 surface in the moduli space (as \( \alpha \geq 2 \)). If \( \alpha = 2 \), so that \( L^2 = 4 \), this special case is case (b) in [Ba2, Theorem].

As one also expects that rational curves in any multiple of the primitive class on a general K3 surface are always nodal (cf. [Ch2, Conj. 1.2]), we expect that also the second exceptional case in (1) cannot occur on a general K3 surface.

Since \( \varepsilon(L) \leq \sqrt{L^2} \), an immediate corollary of the theorem is the following:

Corollary Let \( S \) be a smooth, projective K3 surface such that \( \text{Pic} S \cong \mathbb{Z}[L] \) with \( L^2 = \alpha^2 \) for an integer \( \alpha \geq 4 \).

Then \( \varepsilon(L) = \alpha \).

2. Proof of the theorem

The reader will recognize the similarity of the proof of the theorem with the proofs of [Ba1] Thm. 4.1 and [St] Prop. 1.

Set \( \alpha := \sqrt{L^2} \) and assume that \( \varepsilon(L) < \alpha \). Then it is well-known (see e.g. [Og, Cor. 2]) that there is an irreducible curve \( C \subset S \) and a point \( x \in C \) such that

\[
C.L < \alpha \text{ mult}_xC.
\]

Set \( m := \text{mult}_xC \). Since a point of multiplicity \( m \) causes the geometric genus of an irreducible curve to drop at least by \( \binom{m}{2} \) with respect to the arithmetic genus, we must have

\[
p_a(C) = \frac{1}{2}C^2 + 1 \geq \binom{m}{2} = \frac{1}{2}m(m-1),
\]

so that

\[
m(m-1) - 2 \leq C^2.
\]

We have that \( C \in |nL| \) for some \( n \in \mathbb{N} \). From (2) we obtain \( nL^2 < m\alpha \), so that, by assumption, \( n\alpha^2 < m\alpha \), whence \( n\alpha < m \). As \( n\alpha \in \mathbb{Z} \) we must have

\[
(5) \quad n\alpha \leq m - 1.
\]

Combining (2), (4) and (5), we obtain

\[
m(m-1) - 2 \leq C^2 = nC.L < n\alpha m \leq m(m-1),
\]

giving the only possibilities \( C^2 = n^2L^2 = m(m-1) - 2 \) and \( n\alpha = m - 1 \). It follows from (3) that \( C \) is a rational curve with a single singular point \( x \) that is an ordinary singularity of multiplicity \( m \geq 2 \).
As
\[ C.L = nL^2 = \frac{m(m-1)-2}{n} = m\alpha - \frac{2}{n} \]
and \( m\alpha \in \mathbb{Z} \), we must have \( \frac{2}{n} \in \mathbb{Z} \), so that \( n = 1 \) or \( 2 \).

If \( n = 1 \), then \( m = \alpha + 1 \), so that \( L^2 = C^2 = m(m-1) - 2 = \alpha(\alpha+1) - 2 \) and
\( \varepsilon(L) = C.L/m = \alpha - \frac{2}{\alpha+1} \) from (6).

If \( n = 2 \), then \( m = 2\alpha + 1 \), so that \( L^2 = \frac{1}{4}C^2 = \frac{1}{4}((2\alpha+1)2\alpha - 2) \) and
\( \varepsilon(L) = \alpha - \frac{1}{2\alpha+1} \) from (6).

This concludes the proof of the theorem.

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