A study of random walks on wedges

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Abstract

In this paper we develop the idea of Lyons and gives a simple criterion for the recurrence and the transience. We also show that a wedge has the infinite collision property if and only if it is a recurrent graph.

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1 Introduction

Let us recall briefly the definition of a wedge of $\mathbb{Z}^{d+1}$. Let $f_1, \cdots, f_d$ be a collection of $d$ increasing functions from $\mathbb{Z}^+ \to \mathbb{R}^+ \cup \{+\infty\}$. They induces a wedge, $\text{Wedge}(f_1, \cdots, f_d) = (V, E)$, which has vertex set

$$V = \{(x_1, \cdots, x_d, n) \in \mathbb{Z}^{d+1} : n \geq 0, \ 0 \leq x_i \leq f_i(n) \text{ for each } 1 \leq i \leq d\}$$

and edge set

$$E = \{[u, v] : \|u - v\|_1 = 1, u, v \in V\}.$$

Is a wedge recurrent or transient? (A locally finite connect graph is called transient or recurrent according to the type of simple random walk on it.) Lyons[8] first give

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the result that suppose (A) holds, then \( \text{Wedge}(f_1, \cdots, f_d) \) is recurrent if and only if

\[
\sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{1}{f_i(n) + 1} = \infty. \tag{1.1}
\]

Where

(A): \( f_i(n+1) - f_i(n) \in \{0, 1\} \) for all \( 1 \leq i \leq d \) and all \( n \geq 0 \).

Readers can refer to [1][9] for more background about wedge and the reference therein.

We develop the idea of Lyons in this paper. However, our result does not rely on the condition (A). Define \( d \) increasing integer valued functions \( h_1, \cdots, h_d \). Let \( h_i(0) = 0 \) for each \( 1 \leq i \leq d \). For each \( 1 \leq i \leq d \) and \( n \geq 1 \), if \( h_i(n-1) + 1 > f_i(n) \) then let

\[
h_i(n) = h_i(n-1);
\]

otherwise, if \( h_i(n-1) + 1 \leq f_i(n) \) then let

\[
h_i(n) = h_i(n-1) + 1.
\]

Then we have our first result.

**Theorem 1.1** \( \text{Wedge}(f_1, \cdots, f_d) \) is recurrent if and only if

\[
\sum_{n=0}^{\infty} \prod_{i=1}^{d} \frac{1}{h_i(n) + 1} = \infty. \tag{1.2}
\]

**Example.** Suppose \( d = 2 \), \( f_1(x) = 2^x \) and \( f_2(x) = \log(x + 1) \). Obviously (1.1) does not succeed. On the other hand, \( h_1(n) = n \) and \( h_2(n) = \lfloor \log(n + 1) \rfloor \). Then (1.2) holds and \( \text{Wedge}(f_1, f_2) \) is recurrent.

Now we turn to another question. As usual, we say that a graph has the infinite collision property if two independent simple random walks on the graph will collide infinitely many times, almost surely. Likewise we say that a graph has the finite collision property if two independent simple random walks on the graph collide finitely many times almost surely. It is interesting to known whether or not a graph
has the infinite collision property. Refer to Polya [10], Liggett [7] and Krishnapur & Peres [6] for details. To my interest is the type of a wedge. Other graphs, such as wedge combs, trees or random environment, are studied in [2] [3] [4] [5] [11] etc..

**Theorem 1.2** \( \text{Wedge}(f_1, \cdots, f_d) \) has the infinite collision property if and only if \( \text{Wedge}(f_1, \cdots, f_d) \) is recurrent.

To understand the conditions better, it is worthwhile to compare a wedge with a wedge comb. \( \text{Wedge}(g) \) always has the infinite collision property since any subgraph of \( \mathbb{Z}^2 \) is recurrent. However, \( \text{Comb}(\mathbb{Z}, g) \) may have the finite collision property [2] [6]. Refer to Figure 1 and Figure 2. It implies that our theorem holds owing to the monotone property of the profile \( f_i(\cdot) \) of the wedge.
2 A partition of vertex set \( \mathbb{V} \)

Obviously, the functions \( h_1, \ldots, h_d \) defined in Section 1 satisfy that for each \( 1 \leq i \leq d \) and each \( n \geq 0 \),

\[
0 \leq h_i(n) \leq f_i(n) \quad \text{and} \quad h_i(n + 1) - h_i(n) \in \{0, 1\}.
\]  (2.1)

We shall define a class of subsets \( \Delta_i(n) \) and \( \partial_n \) through these functions. We shall show later that \( \{\partial_n : n \geq 0\} \) is a partition of \( \mathbb{V} \). For each \( 1 \leq i \leq d + 1 \), let

\[
\Delta_i(0) = \{(0, \ldots, 0)\} \in \mathbb{Z}^{d+1}.
\]

Fix \( n \geq 1 \), let

\[
\Delta_{d+1}(n) = \{(x_1, \ldots, x_d, n) \in \mathbb{Z}^{d+1} : 0 \leq x_i \leq h_i(n), 1 \leq i \leq d\}.
\]

Then \( \Delta_{d+1}(n) \) is a subset of \( \mathbb{V} \). Fix \( 1 \leq i \leq d \). If \( h_i(n) = h_i(n - 1) + 1 \) then let

\[
\Delta_i(n) = \{(x_1, \ldots, x_d, x_{d+1}) \in \mathbb{V} : x_j \leq h_j(n) \text{ for each } 1 \leq j \leq d, x_i = h_i(n), x_{d+1} \leq n\}.
\]

Otherwise, if \( h_i(n) = h_i(n - 1) \) then let \( \Delta_i(n) = \emptyset \).

For each \( n \geq 0 \) we set

\[
\partial_n = \bigcup_{i=1}^{d+1} \Delta_i(n).
\]

Finally, for each \( x \in \mathbb{R}^{d+1} \) and each \( 1 \leq i \leq d+1 \), we denote by \( x_i \) the \( i \)-th coordinate of \( x \). For each \( x \in \mathbb{V} \) and \( 1 \leq i \leq d \), we set

\[
p_i(x) = \min \{m : h_i(m) \geq x_i\}.
\]

By (2.1)

\[
h_i(p_i(x)) = x_i.
\]

For each \( x \in \mathbb{V} \), set

\[
u(x) = \max \{x_{d+1}, p_1(x), \ldots, p_d(x)\}.
\]

Then we have the following lemma.
Lemma 2.1 For each pair of $m \geq 0$ and $x \in \mathbb{V}$, vertex $x \in \partial_m$ if and only if $u(x) = m$.

Proof. Fix $x = (x_1, \cdots, x_d, n) \in \mathbb{V}$. For conciseness, we write $p_i$ instead of $p_i(x)$. First we shall prove the statement that if $u(x) = m$ then $x \in \partial_m$. Set

$$S = \{i : 1 \leq i \leq d, x_i > h_i(n)\}.$$ 

We consider two cases $S = \emptyset$ and $S \neq \emptyset$.

Case I: $S = \emptyset$. Then for each $1 \leq i \leq d$,

$$x_i \leq h_i(n).$$

As a result,

$$x \in \Delta_{d+1}(n) \subset \partial_n.$$ 

Since $h_i(p_i) = x_i$,

$$h_i(p_i) \leq h_i(n).$$

By the definition of $p_i(\cdot)$,

$$p_i \leq n.$$ 

Therefore, $u(x) = n$ as claimed above.

Case II: $S \neq \emptyset$. Fix $j \in S$ which satisfies that for all $l \in S$,

$$p_l \leq p_j.$$ (2.2)

We shall show that $u(x) = p_j$ and $x \in \partial_{p_j}$. Since $j \in S$,

$$h_j(p_j) = x_j > h_j(n).$$

It implies that

$$n < p_j.$$ (2.3)

Furthermore, for each $l \in \{1, \cdots, d\} \setminus S$

$$h_l(p_l) = x_l \leq h_l(n) \leq h_l(p_j).$$ (2.4)
As a result of that

\[ p_l \leq p_j. \] (2.5)

Owing to (2.2), (2.3) and (2.5),

\[ u(x) = p_j. \]

On the other hand, by the definition of \( p_j(\cdot) \) there has

either \( p_j = 0 \) or \( h_j(p_j - 1) < h_j(p_j) \).

However, there always have

\[ \Delta_j(p_j) = \{(y_1, \cdots, y_d, y_{d+1}) \in \mathbb{V} : y_l \leq h_l(p_j) \text{ for each } 1 \leq l \leq d, y_j = h_j(p_j), y_{d+1} \leq p_j \}. \] (2.6)

By (2.2), for each \( l \in S \)

\[ x_l = h_l(p_l) \leq h_l(p_j). \] (2.7)

By (2.4), (2.6) and (2.7), we have that

\[ x \in \Delta_j(p_j) \subset \partial p_j. \]

Such we have proved the first statement for both cases.

Next we shall show that \( \partial_0, \partial_1, \cdots \) are disjoined. Fix \( n > m \geq 0 \). Since that for any \( x \in \Delta_{d+1}(n) \) and any \( y \in \partial_m \),

\[ x_{d+1} = n > m \geq y_{d+1}. \]

So,

\[ \partial_m \cap \Delta_{d+1}(n) = \emptyset. \] (2.8)

Fix \( 1 \leq i \leq d \) and \( 1 \leq j \leq d \). We will show that \( \Delta_i(m) \cap \Delta_j(n) = \emptyset \). Otherwise, suppose \( \Delta_i(m) \cap \Delta_j(n) \neq \emptyset \). Then

\[ \Delta_i(m) = \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : x_l \leq h_l(m) \text{ for each } 1 \leq l \leq d, x_i = h_i(m), x_{d+1} \leq m \}, \]
\[ \Delta_j(n) = \{(x_1, \cdots, x_d, x_{d+1}) \in \mathbb{V} : x_l \leq h_l(n) \text{ for each } 1 \leq l \leq d, x_j = h_j(n), x_{d+1} \leq n \}. \]
And then
\[ h_j(n) = h_j(n-1) + 1. \]
Furthermore, since \( \Delta_i(m) \cap \Delta_j(n) \neq \emptyset \) there exists \( z \in \Delta_i(m) \cap \Delta_j(n) \). Then
\[ z_j = h_j(n) \text{ and } z_l \leq \min\{h_l(m), h_l(n)\} \text{ for each } 1 \leq l \leq d. \]
Hence,
\[ h_j(n) \leq h_j(m). \tag{2.9} \]
On the other hand, since \( h_j(\cdot) \) is an increasing function and \( n > m \),
\[ h_j(n-1) \geq h_j(m). \]
It deduces that
\[ h_j(n) = h_j(n-1) + 1 \geq h_j(m) + 1 > h_j(m). \]
This contradict (2.9). Therefore,
\[ \Delta_i(m) \cap \Delta_j(n) = \emptyset. \tag{2.10} \]
Similarly, we can prove that
\[ \Delta_i(n) \cap \Delta_{d+1}(m) = \emptyset. \tag{2.11} \]
Taking (2.8), (2.10) and (2.11) together, we get that \( \partial_n \) and \( \partial_m \) are disjoined. We have finished the proof of the lemma. \( \square \)

The next lemma shows that the neighbor of \( \partial_n \) are \( \partial_{n-1} \) and \( \partial_n \) for each \( n \geq 1 \). It implies that \( \partial_n \) is a cutset of the graph \( \text{Wedge}(f_1, \cdots, f_d) \). We write
\[ e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \]
for the \( i \)-th unit vector of \( \mathbb{R}^{d+1} \).

**Lemma 2.2** Let \( x \in \mathbb{V} \) and \( 1 \leq i \leq d+1 \). If \( x + e_i \in \mathbb{V} \) then
\[ u(x+e_i) - u(x) = 0 \text{ or } 1. \]
Proof. Fix $x \in V$. Obviously for each $1 \leq i \leq d + 1$ and $1 \leq l \leq d + 1$ with $i \neq l$, if $x + e_l \in V$ then

$$p_i(x + e_l) = p_i(x).$$

First we consider the easy case $i = d + 1$. Obviously, $x + e_{d+1} \in V$. Hence

$$u(x + e_{d+1}) - u(x) = \max\{x_{d+1} + 1, p_1(x + e_{d+1}), \ldots, p_d(x + e_{d+1})\} - \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\}$$

$$= \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\} - \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\}$$

$$= 0 \text{ or } 1.$$

Next we consider the case $1 \leq i \leq d$. Fix $x \in V$ and $x + e_i \in V$. If $f_i(p_i(x) + 1) \geq x_i + 1$, then

$$f_i(p_i(x) + 1) \geq x_i + 1 = h_i(p_i(x)) + 1.$$

Hence

$$h_i(p_i(x) + 1) = h_i(p_i(x)) + 1 = x_i + 1.$$

Such

$$p_i(x + e_i) = p_i(x) + 1.$$

Similarly we have

$$u(x + e_i) - u(x)$$

$$= \max\{x_{d+1}, p_1(x + e_i), \ldots, p_d(x + e_i)\} - \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\}$$

$$= \max\{x_{d+1}, p_1(x), \ldots, p_{i-1}(x), p_i(x) + 1, p_{i+1}(x), \ldots, p_d(x)\} - \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\}$$

$$= 0 \text{ or } 1.$$

Otherwise, $f_i(p_i(x) + 1) < x_i + 1$. Let

$$\eta_i = \min\{m : f_i(m) \geq x_i + 1\}.$$

Then

$$\eta_i > p_i(x) + 1.$$
Furthermore,

\[ h_i(\eta_i - 1) \geq h_i(p_i(x)) = x_i. \]

On the other hand

\[ h_i(\eta_i - 1) \leq f_i(\eta_i - 1) < x_i + 1. \]

Since \( h_i(\cdot) \) is integer valued,

\[ h_i(\eta_i - 1) = x_i. \]

As a result,

\[ f_i(\eta_i) \geq x_i + 1 = h_i(\eta_i - 1) + 1. \]

Hence

\[ h_i(\eta_i) = h_i(\eta_i - 1) + 1 = x_i + 1. \]

Therefore,

\[ p_i(x + e_i) \leq \eta_i. \] (2.12)

Since \( x + e_i \in \mathbb{V} \),

\[ f_i(x_{d+1}) \geq x_i + 1. \]

and then

\[ \eta_i \leq x_{d+1}. \]

By (2.12),

\[ p_i(x + e_i) \leq x_{d+1}. \]

So that,

\[
\begin{align*}
    u(x + e_i) - u(x) &= \max\{x_{d+1}, p_1(x + e_i), \ldots, p_d(x + e_i)\} - \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\} \\
    &\leq \max\{x_{d+1}, p_1(x), \ldots, p_{i-1}(x), x_{d+1}, p_{i+1}(x), \ldots, p_d(x)\} - \max\{x_{d+1}, p_1(x), \ldots, p_d(x)\} \\
    &\leq 0.
\end{align*}
\]

By the increasing property of \( u(\cdot) \), we get that

\[ u(x + e_i) - u(x) = 0. \]

\[ \square \]
At the end of this section, we shall estimate the cardinality of \( \partial_n \).

**Lemma 2.3** For each \( n \geq 0 \),

\[
\prod_{i=1}^{d} (h_i(n) + 1) \leq |\partial_n| \leq (d + 1) \prod_{i=1}^{d} (h_i(n) + 1).
\]

**Proof.** For each \( n \geq 0 \)

\[
|\partial_n| \geq |\Delta_{d+1}(n)| = \prod_{i=1}^{d} (h_i(n) + 1),
\]

since \( \Delta_{d+1}(n) \subseteq \partial_n \).

Fix \( n \geq 1 \) and \( 1 \leq i \leq d \). Without making confusion, we set

\[
p_i = p_i(n) = \min \{ m : h_i(m) = n \}.
\]

Then

\[
\Delta_i(p_i) = \{(x_1, \ldots, x_d, x_{d+1}) \in V : x_l \leq h_l(p_i) \text{ for each } 1 \leq l \leq d, \ x_i = n, \ x_{d+1} \leq p_i \}.
\]

As we have known that if \( x \in V \) with \( x_i = n \) then \( f_i(x_{d+1}) \geq n \). Let

\[
k = \min \{ u \in \mathbb{Z}^+ : f_i(u) \geq n \}.
\]

Then

\[
\Delta_i(p_i) = \{(x_1, \ldots, x_d, x_{d+1}) \in V : 0 \leq x_l \leq h_l(p_i) \text{ for each } 1 \leq l \leq d, \ x_i = n, \ k \leq x_{d+1} \leq p_i \}
\]

\[
\subseteq \{(x_1, \ldots, x_d, x_{d+1}) \in \mathbb{Z}^{d+1} : 0 \leq x_l \leq h_l(p_i) \text{ for each } 1 \leq l \leq d, \ x_i = n, \ k \leq x_{d+1} \leq p_i \}.
\]

Therefore,

\[
|\Delta_i(p_i)| \leq \frac{p_i - k + 1}{h_i(p_i)} \prod_{i=1}^{d} (h_i(p_i) + 1).
\]

If \( k \leq \eta < p_i \), then

\[
h_i(\eta) + 1 \leq h_i(p_i - 1) + 1 = h_i(p_i) = n \leq f_i(k) \leq f_i(\eta).
\]
And then
\[ h_i(\eta) = h_i(\eta - 1) + 1. \]

Therefore,
\[ h_i(p_i) - h_i(k) = p_i - k. \]

Such
\[ |\Delta_i(p_i)| \leq \frac{h_i(p_i) - h_i(k) + 1}{h_i(p_i) + 1} \prod_{l=1}^{d} (h_l(p_i) + 1) \leq \prod_{l=1}^{d} (h_l(p_i) + 1). \]

So that for any \( m \geq 0 \), if \( m \in \{p_i(n) : n \geq 1\} \), then
\[ |\Delta_i(m)| \leq \prod_{l=1}^{d} (h_l(m) + 1). \tag{2.13} \]

Obviously, (2.13) is true for \( m = 0 \) since \( \Delta_i(0) = \{(0, \cdots , 0)\} \). Notice that \( p_i(0) = 0 \) and the fact that if \( m \in \mathbb{Z}\{p_i(n) : n \geq 0\} \) then \( \Delta_i(m) = \emptyset \). Therefore, (2.13) are true for all \( m \geq 0 \). Finally, for any \( m \geq 0 \)
\[ |\partial_m| \leq \sum_{i=1}^{d+1} |\Delta_i(m)| \leq \sum_{i=1}^{d+1} \prod_{l=1}^{d} (h_l(m) + 1) \leq (d + 1) \prod_{l=1}^{d} (h_l(m) + 1). \]

We have completed the proof of the lemma. \( \square \)

3 Proof of Theorem 1.1

We shall use the notation of electric network. Every edge of \( \text{Wedge}(f_1, \cdots , f_d) \) is assigned a unit conductance. So that, we get an electric network. For sets \( A, B \subset \mathbb{V} \) with \( A \cap B = \emptyset \), denote by \( \mathcal{R}(A \leftrightarrow B) \) the effective resistance between \( A \) and \( B \) in the electric network. For simplicity, we label \( O \) as the origin of \( \mathbb{Z}^{d+1} \) and set
\[ \mathbb{V}_r = \bigcup_{n=0}^{r} \partial_r \]
for each \( r \geq 1 \). Then we have the following lemma.

**Lemma 3.1** For each \( r \geq 1 \)
\[ \mathcal{R}(O \leftrightarrow \partial_r) \geq \frac{1}{2(d + 1)^2} \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}. \]
Proof. Notice that \( \partial_0 = \{O\} \). By Lemma 2.2 for each \( n \geq 1 \) the neighbor of \( \partial_n \) are \( \partial_{n-1} \) and \( \partial_{n+1} \) in \( \text{Wedge}(f_1, \cdots, f_d) \). So that \( \partial_n \) is a cutset which separates \( O \) from \( \partial_{n+s} \). The rest proof is easy and one can refer to [9]. Fix \( r \). The effective resistance from \( O \) to \( \partial_r \) in \((V, E)\) is equal to that in its subgraph with vertex set \( V_r \).

We short together all the vertices in \( \partial_n \) for each \( 0 \leq n \leq r \). And replace the edges between \( \partial_n \) and \( \partial_{n+1} \) by a single edge of resistance \( \frac{1}{b_n} \), where \( b_n \) is the number of edges connect \( \partial_n \) with \( \partial_{n+1} \). This new network is a series network with the same effective resistance from \( O \) to \( \partial_r \). Thus, Rayleigh’s monotonicity law shows that the effective resistance from \( O \) to \( \partial_r \) in \( V_r \) is at least \( \sum_{n=0}^{r-1} \frac{1}{b_n} \). By Lemma 2.3 and the fact that every vertex of \( \text{Wedge}(h_1, \cdots, h_d) \) has at most \( 2(d+1) \) neighbor,

\[
R(\partial_0 \leftrightarrow \partial_r) \geq \frac{1}{2(d+1)} \sum_{n=0}^{r-1} \frac{1}{|\partial_n|} \geq \frac{1}{2(d+1)^2} \sum_{n=0}^{r-1} \sum_{i=1}^{d} \frac{1}{h_i(n)+1}.
\]

\( \blacksquare \)

On the other hand we can estimate the upper bound of \( R(x \leftrightarrow \partial_r) \).

Lemma 3.2 There exists \( C_d > 0 \) which depends only on \( d \) such that for any \( r \geq 1 \) and any \( x \in \mathbb{V}_{r-1} \),

\[
R(x \leftrightarrow \partial_r) \leq C_d \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{h_i(n)+1}.
\]

Proof. Outline of the proof. We shall construct \( 2d \) functions \( g_{\pm i}(\cdot) \) first. These functions will help us to find a subset \( \mathbb{V}_x \) which satisfies that \( x \in \mathbb{V}_x \subseteq \mathbb{V}_r \). Such \( R_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x) \), the resistance between \( x \) and \( \Delta_{d+1}(r) \cap \mathbb{V}_x \) in the subgraph with vertex set \( \mathbb{V}_x \), is greater than \( R(x \leftrightarrow \partial_r) \). Furthermore, we show the relation between \( \mathbb{V}_x \) and \( \text{Wedge}(h_1, \cdots, h_d) \). As known from Lyons [8], the related resistance in \( \text{Wedge}(h_1, \cdots, h_d) \) can be gotten. So do \( R_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x) \).

Fix \( x = (x_1, \cdots, x_d, s) \in \mathbb{V}_{r-1} \). We shall construct \( 2d \) nonnegative integer valued functions on \( \mathbb{Z}^+ \). Fix \( 1 \leq i \leq d \). First set

\[
g_{\pm i}(0) = x_i.
\]
Suppose that the definition of $g_{±i}(n)$ is known, we define $g_{±i}(n + 1)$ in three cases.

1. If $h_i(n + 1) = h_i(n)$, then we set $g_{±i}(n) = g_{±i}(n + 1)$.
2. If $h_i(n + 1) = h_i(n) + 1$ and if $g_{-i}(n) = 0$, then we set $g_{-i}(n + 1) = 0$ and $g_{i}(n + 1) = g_{i}(n) + 1$.
3. Otherwise, if $h_i(n + 1) = h_i(n) + 1$ and if $g_{-i}(n) > 0$, then we set $g_{-i}(n + 1) = g_{-i}(n) - 1$ and $g_{i}(n + 1) = g_{i}(n)$.

We say that these functions $g_{±i}(n)$ has the properties (a), (b) and (c). Where

(a) : $g_{i}(n + 1) - g_{i}(n) \in \{0, 1\}$ and $g_{-i}(n + 1) - g_{-i}(n) \in \{0, -1\}$ for each $n \geq 0$;
(b) : $g_{i}(n) - g_{-i}(n) = h_i(n)$ for each $n \geq 0$;
(c) : $0 \leq g_{-i}(n) \leq g_{i}(n) \leq \min\{f_{i}(n + s), h_i(r)\}$ for each $0 \leq n \leq r - s$.

Obviously, (a) are true for all $n \geq 0$. Next we shall prove (b) by induction to $n$. It is true for $n = 0$ since $h_i(0) = 0$. Suppose (b) is true for $n = m$ and we shall check $n = m + 1$. In any case of (1), (2) and (3), there has

$$h_i(m + 1) - h_i(m) = [g_{i}(m + 1) - g_{i}(m)] - [g_{-i}(m + 1) - g_{-i}(m)].$$

By the assumption that (b) is true for $n = m$, we can get that (b) is still true for $n = m + 1$. Such (b) is true for any $n \geq 0$. Again we prove (c) by induction. Owing to $x \in \mathbb{V}_{r-1}$ and $x_{d+1} = s$,

$$0 \leq x_i \leq h_i(x_{d+1}) = h_i(s) \leq \min\{h_i(r), f_i(s)\}.$$  

So (c) is true for $n = 0$. Suppose (c) is true for $n = m < r - s$ and we shall check $n = m + 1$.

If (1) is true for $n = m + 1$, then by the assumption that (c) is true for $n = m$ and the monotone property of $f_i(\cdot)$, we have (c) for $n = m + 1$.

If (2) is true for $n = m + 1$, then what we need to care is only $g_i(n + 1)$. However, by the result (b) we have proved

$$g_i(n + 1) = h_i(n + 1) + g_{-i}(n + 1) = h_i(n + 1) \leq f_i(n + 1) \leq f_i(s + n + 1).$$

Furthermore, since $n < r - s$,

$$h_i(n + 1) \leq h_i(r).$$
Therefore (c) is true for \( n = m + 1 \).

If (3) is true for \( n = m + 1 \), then what we need to care is only \( g_{-i}(n + 1) \). But by the condition that \( g_i(n) > 0 \), we have

\[
g_{-i}(n + 1) = g_{-i}(n) - 1 \geq 0.
\]

Hence (c) is true, too. Therefore, in any case (c) is true for \( n = m + 1 \) with \( n < r - s \).

As a result, we can define vertex set \( \mathbb{V}_x \) and edge set \( \mathbb{E}_x \). Let

\[
\mathbb{V}_x = \{(u_1, \cdots, u_d, n+s) \in \mathbb{Z}^{d+1} : 0 \leq n \leq r-s, \ g_{-i}(n) \leq u_i \leq g_i(n) \text{ for each } 1 \leq i \leq d\}.
\]

Let

\[
\mathbb{E}_x = \{[u, v] \in \mathbb{E} : u, v \in \mathbb{V}_x\}.
\]

The definition does not make confusion of \( \mathbb{V}_x \) and \( \mathbb{V}_n \) since \( x \) is a vector. By (c),

\[
x \in \mathbb{V}_x \subseteq \mathbb{V}_r.
\]

Hence graph \((\mathbb{V}_x, \mathbb{E}_x)\) is a subgraph of \(\text{Wedge}(f_1, \cdots, f_d)\). Notice that

\[
\partial_r \cap \mathbb{V}_x \supseteq \Delta_{d+1}(r) \cap \mathbb{V}_x.
\]

(Actually \( \partial_r \cap \mathbb{V}_x = \Delta_{d+1}(r) \cap \mathbb{V}_x \), but we omit the proof here since it is irrelevant to our main result.) By the Rayleigh’s monotonicity law, the effective resistance between \( x \) and \( \Delta_{d+1}(r) \cap \mathbb{V}_x \) in the subgraph is greater than that in the old graph. That is,

\[
\mathcal{R}(x \leftrightarrow \partial_r) \leq \mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x). \tag{3.1}
\]

So that we need only to estimate the upper bound of \( \mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x) \).

We shall show the relation between \((\mathbb{V}_x, \mathbb{E}_x)\) and \(\text{Wedge}(h_1, \cdots, h_d)\). Let

\[
\mathbb{H} = \{(x_1, \cdots, x_d, n) \in \mathbb{Z}^{d+1} : 0 \leq x_i \leq h_i(n) \text{ for each } 1 \leq i \leq d, 0 \leq n \leq r-s\}.
\]

Obviously, \( \mathbb{H} \) is a subset of vertices of \(\text{Wedge}(h_1, \cdots, h_d)\). By the construction of \( g_{\pm i}(\cdot) \), one can easily check that there has for each \( n \geq 1 \).
either \( g_{-i}(n) = g_{-i}(n-1) \) or \( g_{i}(n) = g_{i}(n-1) \).

So we can define
\[
L_i(n) = \min\{g_{si}(n) : g_{si}(n) = g_{si}(n-1), s \in \{-1, 1\}\}.
\]

Let \( \Gamma(x) = O \). For each \((u_1, \cdots, u_d, n + s) \in \mathbb{V}_x\) with \(n \geq 1\), let
\[
\Gamma(u_1, \cdots, u_d, n + s) = (|u_1 - L_1(n)|, \cdots, |u_d - L_d(n)|, n).
\]

By (b), \( \Gamma \) is a bijection function from \( \mathbb{V}_x \) to \( \mathbb{H} \). Obviously, \([u,v] \in \mathbb{E}_x\) if and only if \([\Gamma(u), \Gamma(v)]\) is an edge of Wedge\((h_1, \cdots, h_d)\) for each pair of \(u\) and \(v\) with \(u_{d+1} = v_{d+1}\). Moreover, for any \(u \in \mathbb{V}_x\) we have that \(u - e_{d+1} \in \mathbb{V}_x\) if and only if \(\Gamma(u) - e_{d+1} \in \mathbb{H}\).

Since \(h_i(\cdot)\) increases at most one at each step, we can use the result of Lyons[8]. That is, there exists a unit flow \(w\) from \(O\) to \(\Delta_{d+1}(r - s)\) in the subgraph of Wedge\((h_1, \cdots, h_d)\) with vertex set \(\mathbb{H}\), such that for each \(u \in \mathbb{H}\) with \(u_{d+1} = n < r - s\),
\[
w(u, u + e_{d+1}) = \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}, \quad (3.2)
\]
and the energy of \(w\) has upper bound
\[
\mathcal{E}(w) \leq C_d \sum_{n=0}^{r-s-1} \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}, \quad (3.3)
\]
where \(C_d < \infty\) and depends only on \(d\). Let \(w_x\) be a function on \(\mathbb{E}_x\) and satisfies that for each \([u,v] \in \mathbb{E}_x\) with \(u_{d+1} = v_{d+1}\),
\[
w_x(u,v) = w(\Gamma(u), \Gamma(v)).
\]

and for each \(u \in \mathbb{V}_{r-1}\) with \(u_{d+1} = n\), let
\[
w_x(u, u + e_{d+1}) = \prod_{i=1}^{d} \frac{1}{h_i(n) + 1}.
\]

Directly calculate
\[
\sum_{v: [u,v] \in \mathbb{E}_x} w_x(u,v)
\]
= \text{w}_x(u, u + e_{d+1}) + \text{w}_x(u, u - e_{d+1})1_{\{u - e_{d+1} \in \mathbb{V}_x\}} + \sum_{v : [u, v] \in E, u_{d+1} = v_{d+1}} \text{w}_x(u, v)

= \prod_{i=1}^{d} \frac{1}{h_i(n) + 1} - \prod_{i=1}^{d} \frac{1}{h_i(n - 1) + 1} 1_{\{u - e_{d+1} \in \mathbb{V}_x\}} + \sum_{v : [u, v] \in E, u_{d+1} = v_{d+1}} \text{w}((u), (v))

= \text{w}((u), (u) + e_{d+1}) + \text{w}((u), (u) - e_{d+1})1_{\{(u) - e_{d+1} \in \mathbb{H}\}} + \sum_{z \in \mathbb{H} : \|u - z\|_1 = 1, u_{d+1} = z_{d+1}} \text{w}((u), (z))

= \sum_{z \in \mathbb{H} : \|u - z\|_1 = 1} \text{w}((u), (z)).

Together with the fact that \text{w} is a unit flow, we get that \text{w}_x is a unit flow from \text{x} to \Delta_{d+1}(r) \cap \mathbb{V}_x in graph (\mathbb{V}_x, E_x). Obviously

\mathcal{E}(	ext{w}_x) = \mathcal{E}(\text{w}). \tag{3.4}

Together (3.1), (3.3) and (3.4), we have

\mathcal{R}(x \leftrightarrow \partial_r) \leq \mathcal{R}_{\mathbb{V}_x}(x \leftrightarrow \Delta_{d+1}(r) \cap \mathbb{V}_x) \leq \mathcal{E}(\text{w}_x) = \mathcal{E}(\text{w}) \leq C \frac{r - 1}{\prod_{n=0}^{d} \frac{1}{h_i(n) + 1}}.

\square

Proof of Theorem 1.1. As it is well known, a connect graph with local finite degree is recurrent if and only if the resistance from any one vertex to the infinity in the graph is infinite (Refer to [9], Proposition 9.1). Together with Lemmas 3.1 and 3.2 we have the desired result. \square

4 Proof of Theorem 1.2

Lemma 4.1 Let G be a graph of bounded degrees with a distinguished vertex o and suppose that there exists a sequence of sets (B_r)_r growing with r and satisfying

\text{g}_{B_r}(o, o) \rightarrow \infty \text{ as } r \rightarrow \infty \text{ and } \text{g}_{B_r}(x, x) \leq C \text{g}_{B_r}(o, o), \ \forall x \in G,

for a uniform constant C > 0. Here, \text{g}_{B}(\cdot, \cdot) is the green function of the simple random walk on G killed when it exits B. Then the graph G has the infinite collision property.
Proof. Refer to [2].

Proof of Theorem 1.2. First suppose $\text{Wedge}(f_1, \ldots, f_d)$ is not a recurrent graph. Then $\text{Wedge}(f_1, \ldots, f_d)$ is a transient graph. It implies that $g_V(O, O)$, the expected number of returning to $O$, is finite. One can easily get that the expected number of collisions between two independent simple random walks starting from $O$ is less than $2(d+1)g_V(O, O)$. So that, almost surely the number of collisions is finite. Hence, $\text{Wedge}(f_1, \ldots, f_d)$ has the finite collision property.

On the other hand, suppose $\text{Wedge}(f_1, \ldots, f_d)$ is recurrent. By Theorem 1.1 we have (1.2). Furthermore, by Lemma 3.1

$$\lim_{r \to \infty} R(O \leftrightarrow \partial_r) \geq \lim_{r \to \infty} \frac{1}{2(d+1)^2} \sum_{n=0}^{r-1} \prod_{i=1}^{d} \frac{1}{f_i(n) + 1} = \infty.$$ 

As it is known to all (refer to [2]) that for each $r \geq 1$

$$R(O \leftrightarrow \partial_{r+1}) = g_{V_r}(O, O).$$

So $\lim_{r \to \infty} g_{V_r}(O, O) = \infty$. By Lemmas 3.1 and 3.2 for all $r \geq 1$ and $x \in \text{Wedge}(f_1, \ldots, f_d)$

$$g_{V_r}(x, x) \leq 2(d+1)^2 C_d g_{V_r}(O, O).$$

By Lemma 4.1 $\text{Wedge}(f_1, \ldots, f_d)$ has the infinite collision property. 

References

[1] Angel, O., Benjamini, I., Berger, N., Peres, Y., Transience of percolation clusters on wedges, Electric Journal of Probability, Vol. 11, No. 25, 655-669, (2006).

[2] Barlow, M.T., Peres, Y., Sousi, P., Collisions of Random Walks, preprint, available at http://arxiv.org/PS_cache/arxiv/pdf/1003/1003.3255v1.pdf

[3] Chen, X., Chen, D., Two random walks on the open cluster of $\mathbb{Z}^2$ meet infinitely often. Science China Mathematics, 53, 1971-1978 (2010).
[4] Chen, X., Chen, D., *Some sufficient conditions for infinite collisions of simple random walks on a wedge comb*, electronic Journal of Probability, No. 49, 1341-1355 (2011)

[5] Chen, D., Wei, B. and Zhang, F., *A note on the finite collision property of random walks*. Statistics and Probability Letters, 78, 1742-1747, (2008).

[6] Krishnapur, M. and Peres, Y., *Recurrent graphs where two independent random walks collide finitely often*. Elect. Comm. in Probab. 9, 72-81, (2004).

[7] Liggett T M. A characterization of the invariant measures for an infinite particle system with interaction II. Trans Amer Math Soc, 1974, 198: 201C213

[8] Lyons, T. *A simple criterion for transience of a reversible Markov chain*, Annal of Probability, 1983, Vol 11, No.2, 393-402.

[9] Peres, Y., *Probability on trees: an introductory climb*, Lectures on probability theory and statistics (Saint-Flour, 1997), 193C280, Lecture Notes in Math., 1717, Springer, Berlin.

[10] Polya, G., *George Polya: Collected Papers*, Volume IV, 582-585, The MIT Press, Cambridge, Massachusetts.

[11] Shan, Z., Chen, D., *Voter model in a random envoirment in Z^d*, Frontiers of Mathematics in China, 2012, Volume 7, No. 5, 895-905.