Neutrino masses and mixing angles in a model with six Higgs triplets and $A_4$ symmetry

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Abstract

We have considered a model [8], where masses and a mixing pattern for neutrinos are governed by six Higgs triplets and $A_4$ symmetry. In this model we have applied a certain diagonalisation procedure through which we have shown that neutrino masses can have both normal or inverted hierarchy. We have also shown that current neutrino oscillation data can be explained in this model.

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1 Introduction

Neutrino masses and mixing angles play a vital role in our understanding about physics beyond the standard model \cite{1}. For a review on neutrino masses and mixing angles, see ref.\cite{2}. One of the unknown facts about neutrino masses is that we do not know how these masses have been ordered. Data from experiments indicate that neutrino masses can be arranged in either normal or inverted hierarchy \cite{2}. The problem related to neutrino mixing angles is explained below. From the fits to various neutrino oscillation data, three mixing angles and the CP violating Dirac phase ($\delta_{CP}$) in the neutrino sector have been found \cite{3}. Out of the three mixing angles, the values of $\theta_{12}$ and $\theta_{23}$ are consistent with $\sin^2 \theta_{12} = 1/3$ and $\sin^2 \theta_{23} = 1/2$, respectively. The third mixing angle is small and it is found that $\sin^2 \theta_{13} \sim 10^{-2}$ \cite{3}. To a good approximation the three neutrino mixing angles are close to the following pattern: $\sin^2 \theta_{12} = 1/3$, $\sin^2 \theta_{23} = 1/2$, $\sin^2 \theta_{13} = 0$. This is known as Tribimaximal (TBM) mixing \cite{4}. From this we can infer that the mixing angles in the neutrino sector are not arbitrary but could emerge from a pattern. Based on this, one would like to know if there is any underlying physics that is responsible for the pattern among the neutrino mixing angles.

To address the above mentioned problem, several theoretical models based on discrete symmetries have been proposed. For a review on these models, see ref.\cite{5}. Out of these, models based on $A_4$ symmetry \cite{6,7} are elegant in explaining the mixing pattern in the neutrino sector. Among these various models of $A_4$ symmetry, here we particularly focus on one model \cite{8}, which is proposed by Ma and Wegman. In this model, six Higgs triplets are introduced along with the standard model (SM) fields \cite{8}. After the Higgs triplets get vacuum expectation values (vevs), neutrinos acquire non-zero masses. By choosing certain $A_4$ symmetric charges for SM fields and Higgs triplets, mixing pattern among neutrinos has been explained in this model. Some details related to these are given in the next section.

Although the above mentioned model is versatile, we explain below that there are few limitations about the results obtained in ref.\cite{8}. In the work of ref.\cite{8}, results are obtained after assuming vevs of some particular two Higgs triplets be equal and opposite. We elaborate on this assumption in the next section where we briefly describe their work. After making this assumption, one conclusion from the results of ref.\cite{8} is that the neutrino masses in this model can only be in normal hierarchy. In the present work, we have analysed the same model as it is proposed in ref.\cite{8}, but we make some assumptions about vevs of Higgs triplets which are different from that in ref.\cite{8}. Following from our
assumptions, we have shown that not only normal but also inverted hierarchy for neutrino masses is possible in this model. Moreover, we have shown that this model is compatible with any currently acceptable values for neutrino mixing angles and $\delta_{\text{CP}}$.

In the model of ref. [8], after the six Higgs triplets get vevs, neutrinos acquire a mixing mass matrix in the flavour basis. This mass matrix should be diagonalised by a unitary matrix and from this we can find the neutrino mixing angles and $\delta_{\text{CP}}$. In this work, in order to diagonalise this mass matrix we develop an approximation scheme, after making some assumptions about the vevs of the Higgs triplets. From our approximation scheme, we obtain the leading order expressions for the three neutrino mixing angles and $\delta_{\text{CP}}$. The approximation scheme that is applied in this work can have similarities with that in other works of refs. [9]. But difference can be seen in the way the mixing angles and $\delta_{\text{CP}}$ are computed in our work as compared to that in other works.

The paper is organised as follows. In the next section we describe the model of ref. [8]. In section 3 we explain the assumptions we make in our work and describe a procedure for diagonalising the mixing mass matrix for the neutrinos. In section 4 we obtain leading order expressions for the neutrino mixing angles and $\delta_{\text{CP}}$. In section 5 we present numerical results of our work. We conclude in the last section.

2 The model

The model we consider is an extension of SM where the additional fields are 2 extra Higgs doublets and 6 Higgs triplets [8]. In this model, $A_4$ symmetry is imposed in addition to the SM gauge symmetry. $A_4$ has the following 4 irreducible representations: $\mathbb{1}, \mathbb{1}', \mathbb{1}'', \mathbb{3}$. Under $A_4$, SU(2) doublets and SU(2) singlets of leptons are assigned as: $L_i = (\nu_i, \ell_i) \sim \mathbb{3}, \ell_1 \sim \mathbb{1}, \ell_2 \sim \mathbb{1}', \ell_3 \sim \mathbb{1}''$. Here, $i = 1, 2, 3$. In the above mentioned model, altogether there are 3 Higgs doublets which are assigned under $\mathbb{3}$ of $A_4$. Assuming that these 3 Higgs doublets acquire the same vev after the electroweak symmetry breaking, we get a mixing mass matrix for charged leptons. This mass matrix can be diagonalized with the following transformations on the charged lepton fields [6].

$$\Psi_L \rightarrow U_L \Psi_L, \quad \Psi_R \rightarrow U_R \Psi_R,$$

$$\Psi_L = (\ell_1^T, \ell_2^T, \ell_3^T)^T, \quad \Psi_R = (\ell_1^c, \ell_2^c, \ell_3^c)^T,$$

$$U_L = U_{\text{CW}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$
Here, $\omega = e^{2\pi i/3}$.

The neutrinos in this model acquire masses when the 6 Higgs triplets get vevs. Denoting these 6 Higgs triplets as $\xi_i, i = 1, \cdots, 6$, under $A_4$ their charges are assigned as follows: $\xi_1 \sim 1, \xi_2 \sim 1', \xi_3 \sim 1''$, $\xi_j \sim 3$. Here, $j = 4, 5, 6$. After these Higgs triplets get vevs, mass terms for neutrinos can be written as follows [8].

$$
\mathcal{L} = \Psi^c \mathcal{M}_\nu \Psi_\nu + \text{h.c.}, \quad \Psi_\nu = (\nu_1, \nu_2, \nu_3)^T, \quad \Psi^c_\nu = C\Psi^T_\nu,
$$

$$
\mathcal{M}_\nu = \begin{pmatrix}
a + b + c & f & e \\
f & a + \omega b + \omega^2 c & d \\
e & d & a + \omega^2 b + \omega c
\end{pmatrix}.
\tag{2}
$$

Here, $C$ is the charge conjugation matrix. In the above equation, $a, b, c, d, e, f$ come from $\langle \xi_1 \rangle, \langle \xi_2 \rangle, \langle \xi_3 \rangle, \langle \xi_4 \rangle, \langle \xi_5 \rangle, \langle \xi_6 \rangle$, respectively [8]. After applying the following transformation on $\Psi_\nu$ as

$$
\Psi_\nu \rightarrow U_{CW}U_{TBM}^\dagger \Psi_\nu, \quad U_{TBM} = \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix},
\tag{3}
$$

the matrix $\mathcal{M}_\nu$ of Eq. (2) would transform to

$$
\mathcal{M}'_\nu = \begin{pmatrix}
a - (b + c)/2 + d & (f + e)/\sqrt{2} & (b - c)/(2\sqrt{2}) \\
(f + e)/\sqrt{2} & a + b + c & i(e - f)/(\sqrt{2}) \\
(b - c)/(2\sqrt{2}) & i(e - f)/(\sqrt{2}) & -a + (b + c)/(2 + d)
\end{pmatrix}.
\tag{4}
$$

The above matrix would be in diagonal form if $e = f = 0$ and $b = c$ and in this case, from the transformations of charged leptons and neutrinos, we can notice that $U_{TBM}$ is the unitary matrix which diagonalises the neutrino mass matrix in a basis where charged lepton masses are already diagonalised. Hence $U_{TBM}$ can be identified as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. We can parametrise the PMNS matrix ($U_{PMNS}$) in terms of neutrino mixing angles, which we have given in section 4. After equating $U_{TBM}$ with $U_{PMNS}$ we can find that the neutrino mixing angles fit the TBM pattern, which we have described in the previous section. But in the above mentioned case, where $e = f = 0$ and $b = c$, the angle $\theta_{13}$ would become zero and this possibility is ruled out by the oscillation data. Hence, in order to get $\theta_{13} \neq 0$, at least some of $e, f$ and $b - c$ should have non-zero values.

Based on the observations made in the previous paragraph, in ref. [8], $\theta_{13}$ has been shown to be non-zero by assuming $e = -f \neq 0$ and $b - c \neq 0$. But by considering this
possibility it has been concluded that the neutrinos can only have normal mass hierarchy. Although we should assume $e$ and $f$ to be non-zero, in general there need not be any constraint between them. In this work we consider non-zero values for $e$, $f$ and $b - c$, but otherwise do not assume any relation between $e$ and $f$.

3 Diagonalisation procedure and neutrino masses

In this section we explain our methodology of diagonalising the matrix $M_\nu$ of Eq. (2). As explained in the previous section that after applying the transformation of Eq. (3) on $M_\nu$ of Eq. (2), we have got the mixing mass matrix among neutrinos which is given by $M'_\nu$. We can notice that $M'_\nu$ is nearly diagonal if we assume $e$, $f$ and $b - c$ are small values. After assuming that these are small, we can expect that $M'_\nu$ can be diagonalised by a unitary matrix which is nearly equal to unit matrix. This unitary matrix can be parametrised, upto first order, as

$$U_\epsilon = \begin{pmatrix} 1 & \epsilon_{12} & \epsilon_{13} \\ -\epsilon_{12}^* & 1 & \epsilon_{23} \\ -\epsilon_{13}^* & -\epsilon_{23}^* & 1 \end{pmatrix}$$

(5)

In the above equation, $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$ are small and complex.

In the above described methodology, in order to diagonalise the matrix $M_\nu$ of Eq. (2), we are applying the following transformation on the neutrino fields

$$\Psi_\nu \rightarrow U_{CW} U_{TBM} U_\epsilon \Psi_\nu$$

(6)

Now, from the transformations of charged leptons and neutrinos, we can notice that the PMNS matrix in this model would be

$$U_{\text{PMNS}} = U_{\text{TBM}} U_\epsilon$$

(7)

As explained before that $U_{\text{PMNS}}$ can be parametrised in terms of neutrino mixing angles. Hence from the above relation we may hope to get $\theta_{13}$ to be non-zero for some particular values of $\epsilon$-parameters. As mentioned before that these $\epsilon$-parameters need to be small, since in our diagonalisation procedure we have assumed that $e$, $f$ and $b - c$ of $M'_\nu$ should be small. Here we quantify how small these parameters need to be. As mentioned previously that the neutrino oscillation data predicts that $\sin^2 \theta_{13} \approx 2 \times 10^{-2}$ which is very small in comparison to unity. So we can take $\sin \theta_{13} \approx 0.15$ to be a small value. Based on this observation, we assume that the real and imaginary parts of $\epsilon$-parameters to be atmost
of the order of \( \sin \theta_{13} \). By making this assumption we show later that we get consistent results in our work.

As explained previously that we are applying the transformation of Eq. (6) on \( \mathcal{M}_\nu \) of Eq. (2). As a result of this, we can notice that, effectively the matrix \( \mathcal{M}_\nu' \) is diagonalised by \( U_\epsilon \). Relation for the diagonalisation of \( \mathcal{M}_\nu' \) can be expressed as

\[
\mathcal{M}_\nu' = U_\epsilon^* \cdot \text{diag}(m_1, m_2, m_3) \cdot U_\epsilon^\dagger
\]  

Here, \( m_1, m_2, m_3 \) are masses of neutrinos. Neutrino masses can be estimated from the global fits to the neutrino oscillation data \[3\]. From these global fits we know that there are two mass-square differences among the neutrino masses, which are given below \[3\].

\[
m^2_{\text{sol}} = m^2_2 - m^2_1 = 7.39 \times 10^{-5} \text{ eV}^2,
\]

\[
m^2_{\text{atm}} = \begin{cases} 
m^2_3 - m^2_2 = +2.525 \times 10^{-3} \text{ eV}^2 & \text{(normal hierarchy)} \\
m^2_3 - m^2_2 = -2.512 \times 10^{-3} \text{ eV}^2 & \text{(inverted hierarchy)} \end{cases}
\]  

In the above we have given the best fit values. Here \( m_{\text{sol}} \) and \( m_{\text{atm}} \) represent solar and atmospheric mass scales respectively. To fit the above mass-square differences we can take neutrino masses as

\[
m_1 \lesssim m_{\text{sol}}, \quad m_2 = \sqrt{m^2_1 + m^2_{\text{sol}}}, \quad m_3 = \sqrt{m^2_1 + m^2_{\text{atm}}} \quad \text{(NH)}
\]

\[
m_3 \lesssim m_{\text{sol}}, \quad m_2 = \sqrt{m^2_3 - m^2_{\text{atm}}}, \quad m_4 = \sqrt{m^2_2 - m^2_{\text{sol}}} \quad \text{(IH)}
\]

Here, NH(IH) indicate normal(inverted) hierarchy. In the case of IH, by taking \( m_3 = m_{\text{sol}} \) we would get \( \sum m_\nu = m_1 + m_2 + m_3 = 0.11 \text{ eV} \). This value is just below the upper bound on the sum of neutrino masses obtained by Planck, which is 0.12 eV \[10\]. On the other hand, in the case of NH, even if we take \( m_1 = m_{\text{sol}} \) we would get \( \sum m_\nu = 0.07 \text{ eV} \), which is reasonably below the above mentioned upper bound.

In the diagonalisation procedure that we have described above, to find the neutrino masses we need to solve the relations in Eq. (8). We can notice here that the matrix \( \mathcal{M}_\nu' \) contain all the model parameters related to neutrino masses. From Eq. (8) it is clear that these model parameters are related to mass eigenvalues of neutrinos and \( \epsilon \)-parameters. In the next section we will show that these \( \epsilon \)-parameters can be determined from the neutrino mixing angles and \( \delta_{\text{CP}} \), whose values are found the oscillation data \[3\]. As for the mass eigenvalues of neutrinos we have described above that they be chosen from mass-square differences which are also found from the oscillation data. Now, after using Eq. (8) we can proceed to find the model parameters of \( \mathcal{M}_\nu' \) in terms of observables from oscillation
data. Before doing that let us mention that the oscillation data predict that there is a hierarchy between the two neutrino mass-square differences. In fact, from the global fits to oscillation data, we can notice that

\[ m^2_{\text{sol}} \sim m^2_{\text{atm}} \sim \sin^2 \theta_{13} \sim 10^{-2} \]  

[3]. As mentioned previously, quantities which are of the order of \( m^2_{\text{sol}} \) or \( m^2_{\text{atm}} \) or \( \sin^2 \theta_{13} \) are very small in comparison to unity and so we neglect them in our analysis. As a result of this, we compute terms which are of up to first order in \( \sin \theta_{13} \sim m_{\text{sol}} \), in the right hand side of Eq. (8). We do this computation in both the cases of NH and IH. In either of these cases, the mass eigenvalues of neutrinos in terms of model parameters are found to be same, which are given below.

\[
m_1 = a + d - \frac{b + c}{2}, \quad m_2 = a + b + c, \quad m_3 = -a + d + \frac{b + c}{2}.
\]

(11)

Whereas, relations for other model parameters are found to be dependent on neutrino mass hierarchy. These relations are given below.

\[
\begin{align*}
\text{NH} : & \quad e + f = 0, \quad \sqrt{3} \frac{2}{\sqrt{2}} (b - c) = m_3 \epsilon_{13}^*, \quad i \sqrt{2} (e - f) = m_3 \epsilon_{23}^*. \\
\text{IH} : & \quad \frac{e + f}{\sqrt{2}} = -m_1 \epsilon_{12} + m_2 \epsilon_{12}^*, \quad \sqrt{3} \frac{2}{\sqrt{2}} (b - c) = -m_1 \epsilon_{13}, \quad i \sqrt{2} (e - f) = -m_2 \epsilon_{23}.
\end{align*}
\]

(12)

Using the above relations we can see that the diagonal elements of the matrix \( \mathcal{M} \nu \), up to first order approximation, would be same as the mass eigenvalues of neutrinos. Whereas, the off-diagonal elements in \( \mathcal{M} \nu \) are related to neutrino masses and \( \epsilon \)-parameters. Previously we have assumed that the real and imaginary parts of \( \epsilon \)-parameters to be around \( \sin \theta_{13} \). As a result of this, the relations in Eq. (12) suggest that the off-diagonal elements of the matrix \( \mathcal{M} \nu \) are suppressed by \( O(\sin \theta_{13}) \) as compared to the neutrino mass eigenvalues. This result is consistent with the assumption we made before that \( e, f \) and \( b - c \) should be small values.

Using the relations of Eqs. (11) & (12), depending on the case of NH or IH, we can determine all the model parameters in terms of neutrino mass eigenvalues and the \( \epsilon \)-parameters. As stated previously that these \( \epsilon \)-parameters can be found from the neutrino mixing angles and \( \delta_{\text{CP}} \), which is the subject of the next section. So we can state that by appropriately choosing the model parameters we can explain either the normal or inverted hierarchy mass spectrum for neutrinos in this model. Here it is worth to mention that in the case of NH, we have \( e = -f \). This is exactly what it is assumed in ref.[8] and as a result of this it has been concluded that neutrinos can only have normal mass hierarchy. So our results are agreeing with that of ref.[8] in the case of NH. But in addition to this,
we have shown that inverted mass hierarchy for neutrinos can also be possible in this model.

4 Neutrino mixing angles

In the previous section we have explained that in order to get $\theta_{13}$ to be non-zero, we have chosen to follow a certain diagonalisation procedure through which we have shown that the PMNS matrix in our model could be given by Eq. (7). The PMNS matrix can be parametrised in terms of neutrino mixing angels and a Dirac CP-violating phase, $\delta_{\text{CP}}$. After using this parametrisation in Eq. (7) we can get relations among neutrino mixing angles, $\delta_{\text{CP}}$ and $\epsilon$-parameters. In this section, we will solve these relations and show that all the three neutrino mixing angles get deviations away from the TBM pattern and hence $\theta_{13} \neq 0$.

To express the PMNS matrix in terms of neutrino mixing angles and $\delta_{\text{CP}}$, we follow the PDG convention, which we have given below [11].

$$U_{\text{PMNS}} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\text{CP}}} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{23}c_{13}
\end{pmatrix}$$

(13)

Here, $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. We use the above form of $U_{\text{PMNS}}$ in Eq. (7) and determine the neutrino mixing angles and $\delta_{\text{CP}}$ in terms of $\epsilon$-parameters. Since these $\epsilon$-parameters are complex we can write them as

$$\epsilon_{ij} = Re(\epsilon_{ij}) + iIm(\epsilon_{ij}), \quad i, j = 1, 2, 3.$$  \hspace{1cm} (14)

Here, $Re(\epsilon_{ij})$ and $Im(\epsilon_{ij})$ are real and imaginary parts of $\epsilon_{ij}$.

As explained above that we use the form for $U_{\text{PMNS}}$ of Eq. (13) in Eq. (7). After equating the 13-elements in the matrix relation of Eq. (7), we get the following relation for $\sin \theta_{13}$.

$$s_{13} = \left( \sqrt{\frac{2}{3}} \epsilon_{13} + \frac{1}{\sqrt{3}} \epsilon_{23} \right) e^{i\delta_{\text{CP}}}.$$ \hspace{1cm} (15)

Since the sine of an angle is real, we need to demand that the imaginary part of the right
hand side of the above relation should be zero. After doing this we get

\[ s_{13} = \left( \sqrt{\frac{2}{3}} Re(\epsilon_{13}) + \frac{1}{\sqrt{3}} Re(\epsilon_{23}) \right) \cos \delta_{\text{CP}} - \left( \sqrt{\frac{2}{3}} Im(\epsilon_{13}) + \frac{1}{\sqrt{3}} Im(\epsilon_{23}) \right) \sin \delta_{\text{CP}}. \]  

(16)

\[ \left( \sqrt{\frac{2}{3}} Re(\epsilon_{13}) + \frac{1}{\sqrt{3}} Re(\epsilon_{23}) \right) \sin \delta_{\text{CP}} + \left( \sqrt{\frac{2}{3}} Im(\epsilon_{13}) + \frac{1}{\sqrt{3}} Im(\epsilon_{23}) \right) \cos \delta_{\text{CP}} = 0. \]  

(17)

From the above two equations we can see that both \( \sin \theta_{13} \) and \( \delta_{\text{CP}} \) can be determined in terms of \( \epsilon_{13} \) and \( \epsilon_{23} \) parameters. Hence, by choosing some particular values for these \( \epsilon \)-parameters we may hope to get consistent values for \( \sin \theta_{13} \) and \( \delta_{\text{CP}} \). We present these numerical results on \( \epsilon \)-parameters in the next section. But before doing that we will apply the above described method to obtain expressions for other sine of the angles, which is explained below.

As stated before that we are neglecting terms of the order of \( s_{13}^2 \) in comparison to unity, hence we have \( c_{13} = \sqrt{1 - s_{13}^2} = 1 + \mathcal{O}(s_{13}^2) \approx 1 \). Now that we have known \( c_{13} \), by equating 12- and 23-elements of the matrix relation of Eq. (7), we can determined \( s_{12} \) and \( s_{23} \) in terms of \( \epsilon \)-parameters. Here again we need to demand that the sine of an angle should be real. After doing this we get the following relations.

\[ s_{12} = \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} Re(\epsilon_{12}), \quad s_{23} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} Re(\epsilon_{13}) + \frac{1}{\sqrt{3}} Im(\epsilon_{23}), \]  

(18)

\[ Im(\epsilon_{12}) = 0, \quad Im(\epsilon_{13}) = \sqrt{2} Im(\epsilon_{23}) \]  

(19)

In the above we have shown that the sine of the three neutrino mixing angles and \( \delta_{\text{CP}} \) can be obtained in terms of \( \epsilon \)-parameters after equating the 12-, 13- and 23-elements of the matrix relation of Eq. (7). In our analysis we have three complex \( \epsilon \)-parameters, whose real and imaginary parts will give us six independent parameters. But from Eq. (19) we can see that \( Im(\epsilon_{13}) \) and \( Im(\epsilon_{23}) \) are not independent parameters and \( Im(\epsilon_{12}) = 0 \). As a result of this the following four parameters can be used to determine the three neutrino mixing angles and \( \delta_{\text{CP}} \): \( Re(\epsilon_{12}), Re(\epsilon_{13}), Re(\epsilon_{23}) \) and \( Im(\epsilon_{13}) \).

In the matrix relation of Eq. (7) we have equated 12-, 13- and 23-elements and found relations for the three neutrino mixing angles and \( \delta_{\text{CP}} \) in terms of \( \epsilon \)-parameters. By now we have used all the available \( \epsilon \)-parameters in determining the neutrino mixing angles and \( \delta_{\text{CP}} \). These relations for neutrino mixing angles and \( \delta_{\text{CP}} \) can be used in other elements of the matrix relation of Eq. (7) and then we may expect to get some constraints among the
\(\epsilon\)-parameters. Below we will demonstrate that no constraints among these \(\epsilon\)-parameters will happen. Let us equate the 11-elements of the matrix relation of Eq. (7) and this would lead to

\[
\epsilon_{12}\epsilon_{13} = \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}}\epsilon_{12}^*.
\]

(20)

We can check that the above relation is satisfied self consistently upto first order in \(\epsilon\)-parameters, after using Eqs. (16), (18) & (19). Similarly we have checked that the relations we would get by equating other elements of the matrix relation of Eq. (7) are satisfied self consistenly upto first order in \(\epsilon\)-parameters after using Eqs. (16) - (19). As a result of this, we do not get any additional constraints on the \(\epsilon\)-parameters.

5 Results

In the previous section we have explained that the three neutrino mixing angles and \(\delta_{\text{CP}}\) can be determined by \(Re(\epsilon_{12}), Re(\epsilon_{13}), Re(\epsilon_{23}) \) and \(Im(\epsilon_{13})\). In this section we will show that for some particular values of these \(\epsilon\)-parameters, the three neutrino mixing angles and \(\delta_{\text{CP}}\) can be fitted to the observed values as found from the oscillation data [3]. For this purpose in table 1 we mention the 3\(\sigma\) ranges obtained in the cases of NH and IH for the neutrino mixing angles and \(\delta_{\text{CP}}\).

|          | NH       | IH       |
|----------|----------|----------|
| \(\sin^2\theta_{12}\) | 0.275→0.350 | 0.275→0.350 |
| \(\sin^2\theta_{23}\) | 0.418→0.627 | 0.423→0.629 |
| \(\sin^2\theta_{13}\) | 0.02045→0.02439 | 0.02068→0.02463 |
| \(\delta_{\text{CP}}/^{\circ}\) | 125→392 | 196→360 |

Table 1: 3\(\sigma\) ranges in the cases of both NH and IH for the square of the sine of the three neutrino mixing angles and the CP-violating Dirac phase [3].

From the relations of Eq. (16) - Eq. (19), we can obtain all \(\epsilon\)-parameters in terms of neutrino mixing angles and \(\delta_{\text{CP}}\). Using the 3\(\sigma\) range for \(\sin^2\theta_{12}\), we found the allowed range for \(Re(\epsilon_{12})\) as: -6.19\times10^{-2} to 1.77\times10^{-2}. We can see that the magnitude of these allowed values are below \(s_{13} \approx 0.15\). From the 3\(\sigma\) ranges of \(\sin^2\theta_{13}, \sin^2\theta_{23}\) and \(\delta_{\text{CP}}\) we can get allowed regions for \(Re(\epsilon_{13}), Re(\epsilon_{23})\) and \(Im(\epsilon_{13})\). These allowed regions are plotted in figure 1 in the case of NH. From this figure we can see that the values for \(|Re(\epsilon_{13})|\) and \(|Re(\epsilon_{23})|\) can be atmost of 0.2, which is just at the order of \(s_{13} \approx 0.15\). In fact, \(|Re(\epsilon_{13})|\) and \(|Re(\epsilon_{23})|\) get maximum values when \(\delta_{\text{CP}}\) is around 180\(^{\circ}\) or 360\(^{\circ}\). Otherwise, these
Figure 1: Allowed regions for $Re(\epsilon_{13})$, $Re(\epsilon_{23})$ and $Im(\epsilon_{13})$ are shown in the case of NH. $\delta_{CP}$ is expressed in degrees. In all the above plots, $3\sigma$ ranges for $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ have been used.

parameters can take values even less than 0.2. As for the $|Im(\epsilon_{13})|$, we can notice from figure 1 that this parameter can take a maximum of 0.13 when $\delta_{CP}$ is around 270°.

We can notice from table 1 that the $3\sigma$ ranges for the neutrino mixing angles do not change much between NH and IH cases. The only significant difference is that $\delta_{CP}$ has a narrow allowed region in the case of IH as compared that of NH. Because of this, we can expect that the numerical limits quoted for $Re(\epsilon_{12})$, $Re(\epsilon_{13})$, $Re(\epsilon_{23})$ and $Im(\epsilon_{13})$ in
the case of NH would almost be the same even in the case of IH. This we have seen after
computing the above mentioned parameters in the case of IH. In fact, we have found that
the allowed regions shown in figure 1 do not change significantly in the case of IH, except
for the fact that in IH the axis of $\delta_{CP}$ varies from $196^\circ$ to $360^\circ$.

From the numerical results described above we can see that all the $\epsilon$-parameters, in the
case of NH and IH, are less than or of the order of $s_{13}$. This justifies the assumption we
have made for diagonalising the neutrino mass matrix in section 3. This justification also
vindicate one of our results that both NH and IH cases are possible in the model of ref.[8].
Here we comment on the fact that the calculations done in this work are upto first order
in $s_{13}$. By including second order terms we expect the relations mentioned in Eqs. (11)
- (12) & (16) - (19) get corrections with terms which are of $O(s_{13}^2)$. Since these second
order terms contribute very small values in the numerical analysis, we do not expect any
changes in the qualitative conclusions made in this work.

6 Conclusions

In this work we have analysed a model which is proposed in ref.[8]. In this model neutrinos
acquire masses and mixing pattern mainly due to the presence of six Higgs triplets and
$A_4$ symmetry. In order to explain the mixing pattern among neutrinos, we have followed
a certain approximation procedure for diagonalising the neutrino mass matrix of this
model. We then have show that both NH and IH cases are possible for neutrino masses
in this model. Following our approximation procedure, we have computed leading order
expressions for neutrino mixing angles and $\delta_{CP}$. Using these expressions we have shown
that the current oscillation data can be explained in this model.

References

[1] C. Quigg, hep-ph/0404228,
J. Ellis, Nucl. Phys. A 827 (2009) 187C.

[2] M. C. Gonzalez-Garcia and M. Maltoni, Phys. Rept. 460 (2008) 1.

[3] I. Esteban, M.C. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni and T.
Schwetz, JHEP 1901 (2019) 106.
[4] P.F. Harrison, D.H. Perkins and W.G. Scott, Phys. Lett. B\textbf{530} (2002) 167;  
P.F. Harrison and W.G. Scott, Phys. Lett. B\textbf{535} (2002) 163;  
Z.-z. Xing, Phys. Lett. B\textbf{533} (2002) 85.

[5] G. Altarelli, \texttt{hep-ph/0611117}  
S.F. King and C. Luhn, Rept. Prog. Phys. \textbf{76} (2013) 056201.

[6] E. Ma and G. Rajasekaran, Phys. Rev. D\textbf{64} (2001) 113012.

[7] E. Ma, Phys. Rev. D\textbf{70} (2004) 031901;  
G. Altarelli and F. Feruglio, Nucl. Phys. B\textbf{720} (2005) 64.

[8] E. Ma and D. Wegman, Phys. Rev. Lett. \textbf{107} (2011) 061803.

[9] H. Ishimori, S. Khalil and E. Ma, Phys. Rev. D\textbf{86} (2012) 013008;  
H. Ishimori and E. Ma, Phys. Rev. D\textbf{86} (2012) 045030;  
E. Ma, A. Natale and A. Rashed, Int. J. Mod. Phys. A\textbf{27} (2012) 1250134;  
S. Bhattacharya, E. Ma, A. Natale and D. Wegman, Phys. Rev. D\textbf{87} (2013) 013006.

[10] N. Aghanim \textit{et al.} (Planck Collaboration), \texttt{arXiv:1807.06209} [astro-ph.CO].

[11] M. Tanabashi \textit{et al.} (Particle Data Group), Phys. Rev. D\textbf{98}, 030001 (2018).