The Lovelock Black Holes

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Abstract

Lovelock theory is a natural extension of Einstein theory of gravity to higher dimensions, and it is of great interest in theoretical physics as it describes a wide class of models. In particular, it describes string theory inspired ultraviolet corrections to Einstein-Hilbert action, while admits the Einstein general relativity and the so called Chern-Simons theories of gravity as particular cases. Recently, five-dimensional Lovelock theory has been considered in the literature as a working example to illustrate the effects of including higher-curvature terms in the context of AdS/CFT correspondence.

Here, we give an introduction to the black hole solutions of Lovelock theory and analyze their most important properties. These solutions can be regarded as generalizations of the Boulware-Deser solution of Einstein-Gauss-Bonnet gravity, which we discuss in detail here. We briefly discuss some recent progress in understanding these and other solutions, like topological black holes that represent black branes of the theory, and vacuum thin-shell wormhole-like geometries that connect two different asymptotically de-Sitter spaces. We also make some comments on solutions with time-like naked singularities.
1 Introduction

Why higher-curvature corrections?

It is a common belief that General Relativity, despite its fabulous success in describing our Universe at middle and large scale, has to be corrected at short distance. In particular, the apparent tension between Einstein’s theory and quantum field theory supports the idea that General Relativity is merely an effective model that would be replaced in the UV regime by a different theory, and such a new theory would ultimately permit us to make sense of what we call Quantum Gravity. The natural scale at which one expects such short distance corrections to manifestly appear is the Planck scale \( l_P \), determined by the Newton’s coupling constant \( G = l_P^2/16\pi \).

At present, the most successful candidate to represent a quantum theory of gravity is String Theory (or its mother theory, M-theory). In fact, one of the predictions of string theory is the existence of a massless particle of spin 2 whose dynamics at classical level is governed by Einstein equations

\[
R_{\mu\nu} = 0.
\]  

(1)

In addition, string theory also predicts next-to-leading corrections to (1), which would be relevant at distances comparable with the typical length scale of the theory \( l_s = \sqrt{\alpha'} \). These short-distance corrections are typically described by supplementing Einstein-Hilbert action by adding higher-curvature terms \( R^2 \), correcting General Relativity in the UV regime. As a result, the stringy spin 2 interaction turns out to be finite, and this raises the hope to finally have access to a consistent theory of quantum gravity.

To investigate black hole physics in higher-curvature gravity theories, the first question we have to answer is whether such theories actually induce short-distance modifications to the black hole geometry or not. Despite expectations that the inclusion of higher-curvature terms in the gravitational action yields modifications to General Relativity, it is not necessarily the case that such modifications manifestly appear in the static spherically symmetric sector of the space of solutions. In fact, as we will see below, Schwarzschild geometry usually resists modifications. In turn, first it is important to identify the theories of gravity that yield modifications to the spherically symmetric solution.

Schwarzschild metric as a persistent solution

To warm up, let us start by considering a very simple example of higher-curvature term. Consider the action

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda + \alpha R^2 \right)
\]  

(2)

which corresponds to Einstein-Hilbert action in four dimensions augmented with the square of the curvature scalar, where \( \alpha \) is a coupling constant with dimensions of \([\alpha] = \text{length}^2\). This action is a particular case of the so-called \( f(R) \)-gravity theories, which are defined by adding to the Einstein-Hilbert Lagrangian a function of the Ricci scalar \( f(R) \). It is well known that \( f(R) \)-gravity theories are equivalent (after field redefinition that involves a conformal transformation) to General Relativity coupled to a scalar field \( \phi \), provided a suitable self-interaction potential
$V(\phi)$ that depends on the function $f$ (see [4] and references therein). In this sense, these theories are not different from particular models of quintessence. Here, we are interested in less simple models; however, let us consider (2) as the starting point of our discussion.

A remarkable point is that the theory defined by action (2) admits (Anti-) de Sitter-Schwarzschild metric as its static spherically symmetric solution. In particular, when $\Lambda = 0$ the theory still admits the Schwarzschild solution even for $\alpha \neq 0$, and it is due to the property $R_{\mu\nu} = 0$.

The theory defined by action (2) is not the only theory of gravity that admits Schwarzschild metric as a persistent solution. Actually, this is a rather common feature of theories with higher-curvature terms. In the case of quadratic terms in four dimensions this is an indirect consequence of the Gauss-Bonnet theorem\(^1\). A second example we can consider is Einstein gravity coupled to conformally invariant gravity; namely

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda + c C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \right), \quad (3)$$

where $c$ is a coupling constant and $C_{\alpha\beta\mu\nu}$ is the Weyl tensor, whose quadratic contraction reads

$$C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} = \frac{1}{3} R^2 - 2 R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \quad (4)$$

The equations of motion associated to this action read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + c W_{\mu\nu} = 0, \quad (5)$$

where $W_{\mu\nu}$ is the Bach tensor,

$$W_{\mu\nu} = \Box R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} \Box R - \frac{1}{3} \nabla_\mu \nabla_\nu R + 2 R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{1}{2} g_{\mu\nu} R_{\rho\sigma} R^{\rho\sigma} - \frac{2}{3} R R_{\mu\nu} + \frac{1}{6} g_{\mu\nu} R^2. \quad (6)$$

It is easy to show that, when $\Lambda = 0$, Schwarzschild metric solves equations (5) as well. This follows from the fact that Bach tensor (6) vanishes if Ricci tensor vanishes, and thus all solutions to General Relativity are also solutions of (5).

Another example of a modified theory that admits Schwarzschild metric as a solution is the Jackiw-Pi theory [6]. This theory has recently attracted much attention [7]. It is defined by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda + \frac{\theta}{4} * R_{\mu\nu} R^{\mu\nu} \right), \quad (7)$$

\(^1\)The simplest pure gravitational theory that excludes Schwarzschild solution in four dimensions is a cubic contraction of the Weyl tensor [5]. In dimension $D > 4$, and because the Kretschmann invariant $R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}$ is independent from the quadratic scalars $R^2$ and $R_{\mu\nu} R^{\mu\nu}$, quadratic deformation of Einstein gravity may exclude Schwarzschild-Tangherlini solution.
where the function $\theta$ is a Lagrange multiplier that couples to the Pontryagin density $^\star R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, constructed via the dual curvature tensor

$$^\star R_{\alpha\beta}^{\mu\nu} = \frac{1}{2} \varepsilon_{\rho\sigma\mu\nu} R^\alpha_{\rho\sigma},$$

where $\varepsilon_{\rho\sigma\mu\nu}$ is the volume 4-form. The inclusion of the non-dynamical field $\theta$ comes from the fact that the Pontryagin density $^\star R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is a total derivative. Action (7) is often called Chern-Simons modified gravity; however, this has to be distinguished from the Chern-Simons gravitational theories we will discuss in the section 2.

It is not hard to see that the equations of motion derived from action (7) are solved by the Schwarzschild metric. Actually, this is because the Pontryagin density $^\star R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$ of Schwarzschild metric vanishes. In contrast, Kerr metric has non-vanishing Pontryagin form, and thus it is not a solution of Jackiw-Pi theory. In fact, the rotating solution of this theory has not yet been found, and this represents an interesting open problem as the Jackiw-Pi theory is considered as a phenomenologically viable correction to Einstein theory.

Summarizing, there are several models that, while representing short distance corrections to General Relativity, still admit the Schwarzschild metric as an exact solution. In particular, this implies that such models can not be the solution to problems like the issue of the black hole singularity. On the other hand, there are other models which, still being integrable, do yield deviations from General Relativity solutions even in the static spherically symmetric sector. In this paper we will be concerned with one of such models. We will study a very special case of higher-curvature corrections to Einstein gravity in higher dimensions, and we will see that substantial modifications to Schwarzschild solution are found at short distances.

### Higher-curvature terms in higher dimensions

In addition to higher-curvature corrections to Einstein theory, string theory makes other strong predictions about nature. Probably, the most important ones are the existence of supersymmetry and the existence of extra dimensions. In fact, one of the requirements for superstring theory to be consistent is the space-time to have $9 + 1$ dimensions; and we learn from our daily experience that six of these extra dimensions have to be hidden somehow.

This digression convinces us that studying higher-curvature modification of General Relativity in higher dimensions seems to be important to address the problem of quantum gravity, at least within the context of string theory. This is precisely the subject we will study here. More precisely, in this paper we will investigate how the string inspired higher-curvature corrections to Einstein-Hilbert action modify the black hole physics in the UV regime. This turns out to be a very important question since the black holes are known to be a fruitful arena to explore gravitational phenomena beyond the classical level.

The prototypical example we will analyze is $D$-dimensional quadratic Lovelock Lagrangian. But, first, before introducing this theory, let us begin by considering a much more general example. Consider the action

$$S = \int d^Dx \sqrt{-g} \left( R - 2\Lambda + \alpha R^2 + \beta R_{\alpha\beta}R^{\alpha\beta} + \gamma R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \right)$$

(8)
where the constants $\alpha$, $\beta$, and $\gamma$ are the coupling constants for each quadratic term. The field equations obtained by varying the action (8) with respect to the metric read

\[
0 = G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{2} (4\alpha + \beta) g_{\mu\nu} \Box R + (2\alpha + \beta + 2\gamma) \nabla_\mu \nabla_\nu R + 2\gamma R_{\mu\gamma\alpha\beta} R_\nu^{\gamma\alpha\beta} + 2(\beta + 2\gamma) R_{\mu\nu\beta\delta} R^{\alpha\beta} +
\]

\[
-4\gamma R_{\mu\alpha\nu} R^{\alpha} + 2\alpha R R_{\mu\nu} - \frac{1}{2} (\alpha R^2 + \beta R_{\alpha\beta} R^{\alpha\beta} + \gamma R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}) g_{\mu\nu}
\]

Action (8) is the most general quadratic action one can write down in $D$-dimensions. For $D \leq 4$, the Gauss-Bonnet theorem permits to fix $\gamma = 0$ without loss of generality. In $D > 4$, however, three quadratic invariants are needed to describe the most general Lagrangian of this type.

For generic values of the coupling constants $\alpha$, $\beta$ and $\gamma$, the equations of motion (9) are fourth-order differential equations for the metric (i.e. there are terms proportional to $\nabla_\mu \nabla_\nu R$, $\Box R$ and $\Box R_{\mu\nu}$). Nevertheless, a remarkable property of (8) is that there exists one particular choice of the coupling constants $\alpha$, $\beta$ and $\gamma$ that results in the cancellation of all these higher order terms, yielding second order differential equations. It is easy to see that this choice is $\alpha = \gamma = -\beta/4$, which only gives a non-trivial modification to Einstein theory for $D > 4$. Actually, in $D = 5$ and $D = 6$ this choice corresponds to Lovelock theory (see (16) below); namely

\[
S_L = \int d^5x \sqrt{-g} \left(R - 2\Lambda + \alpha \left(R^2 - 4R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}\right)\right)
\]

It is worth emphasizing that this choice of coupling constants that yields second order equations of motion is unique (up to a free parameter $\alpha$). This feature also holds in $D$ dimensions, and is a consequence of a more general result known as the Lovelock theorem [1].

In this paper we will be mainly concerned with the theory defined by action (8) with $\alpha = \gamma = -\beta/4$. Besides the uniqueness of the choice $\alpha = \gamma = -\beta/4$, which already makes this model interesting in its own right, let us say that this is exactly the effective Lagrangian that appears in the low energy action of heterotic string theories in the appropriate frame (and in M-theory compactifications too).

**Higher-curvature terms from string theory effective action**

Now, let us sketch how the five-dimensional Lovelock theory arises in the low energy limit of M-theory (and, consequently, of string theory) when the theory is compactified from 11$D$ (resp. 10$D$) to 5$D$.

M-theory is supposed to be an extension of string theory; a fundamental theory that, in certain regime, would flow to string theory [8].

This Mother-theory, if it exists, is yet to be found; nevertheless, we do know what it has to look like in certain low energy limit: it has to look like eleven-dimensional supergravity augmented with higher-curvature terms. That is, the bosonic sector of the M-theory effective action is given by the graviton $g_{\mu\nu}$ (i.e. the metric) and the 3-form gauge field $A_{\mu\nu\rho}$ (with field
strength \( F_{\mu \nu \rho \sigma} = \frac{1}{4} \partial_{[\mu} A_{\nu \rho \sigma]} \)). Including the pure gravitational fourth order corrections \( \mathcal{O}(R^4) \), this effective action takes the form\[2\] \[9\]

\[
S_M = \frac{1}{(2\pi)^6 l_p^6} \left[ \int d^{11}x \sqrt{g} R - \frac{1}{48} \int d^{11}x \sqrt{g} F_{\mu_1 \mu_2 \mu_3 \mu_4} F^{\mu_1 \mu_2 \mu_3 \mu_4} + \frac{1}{36(4!)} \int d^{11}x \epsilon_{\mu_1 \mu_2 \cdots \mu_7} A^{\mu_1 \mu_2 \cdots \mu_7} F_{\mu_1 \mu_2 \cdots \mu_7} F^{\mu_1 \mu_2 \cdots \mu_7} + \frac{l_p^6}{27} \left( \frac{3}{2} \right)^{\frac{5}{3}} \int d^{11}x \sqrt{g} t_{\mu_1 \mu_2 \cdots \mu_8} t^{\mu_1 \mu_2 \cdots \mu_8} \right. \\
\left. \times \left( \epsilon_{\nu_1 \nu_2 \cdots \nu_7} \epsilon^{\nu_1 \nu_2 \cdots \nu_7} R_{\mu_1 \mu_2 \cdots \mu_7}^{\nu_1 \nu_2 \cdots \nu_7} R_{\mu_1 \mu_2 \cdots \mu_7}^{\nu_1 \nu_2 \cdots \nu_7} R_{\mu_1 \mu_2 \cdots \mu_7}^{\nu_1 \nu_2 \cdots \nu_7} + \cdots \right) \right] + \ldots
\]

where the ellipses stand for the fermionic content and higher-order contributions. These higher order contributions include terms like \( \mathcal{O}(F^4) \) and also couplings of the form \( \mathcal{O}(A R^4) \); we will not consider these terms here. However, let us mention that the existence of the terms \( \mathcal{O}(R^4) \) in the action above are related to the terms \( \mathcal{O}(A R^4) \) through supersymmetry, although indirectly.

The tensor \( t_{\mu_1 \cdots \mu_8} \) in the third line of (11) is defined in terms of the way it acts on antisymmetric tensors of second rank, namely

\[
t_{\mu_1 \mu_2 \cdots \mu_8} B_{\mu_1 \mu_2} B_{\mu_3 \mu_4} B_{\mu_5 \mu_6} B_{\mu_7 \mu_8} = 24 \text{tr}(B^4) - 6 \text{tr}(B^2)^2,
\]

where \( \text{tr}(B^n) \) refers to the trace of \( B^n \).

The term in the fourth line in (11) is actually one of the terms that appear in the Lagrangian of Lovelock theory (see (16) below, where this term is expressed in an alternative way). In contrast, the term in the third line, which is of the same order, does not correspond to a term in the Lovelock theory\[3\].

A string theory \( \mathcal{O}(R^4) \) contribution similar to that of the third and fourth lines of (11) also appears in ten dimensions \[3\]. This can be written as follows\[4\]:

\[
\int d^{10}x \sqrt{g} \left( (R^\mu_{\alpha \beta \gamma})^2 + 2R^\mu_{\nu \rho \sigma} R^\nu_{\mu \alpha \beta} R^\rho_{\nu \alpha \beta} R^\sigma_{\rho \gamma \delta} - 8R^\mu_{\nu \rho \alpha} R^\nu_{\mu \alpha \beta} R^\rho_{\gamma \delta \mu} R^\alpha_{\rho \gamma \delta} \right)
\]

Now, let us analyze what happens when the M-theory effective action we discussed above (including the higher-curvature terms \( \mathcal{O}(R^4) \)) is compactified to five dimensions. Let us assume

\[2\] The eleven-dimensional Newton constant is given by the Planck scale \( G_{(11D)} = 2\pi^4 l_p^9 \).

\[3\] Actually, while second-order terms of heterotic string theory expressed in a particular frame agree with the second-order term of the Lovelock theory, the fourth-order terms of Type IIA and IIB string theories (and M-theory) do not agree with the fourth-order term of the Lovelock theory.

\[4\] Compactifying to four dimensions gives rise to the higher-curvature correction

\[
\int d^4x \sqrt{g} \left( (R^\alpha_{\beta \mu \nu})^2 + (R^\alpha_{\beta \mu \nu})^2 \right).
\]

See \[10\] for a recent discussion on these quartic terms in four dimensions.
we reduce from 11D to 5D by compactifying six of the eleven dimensions in compact Calabi-Yau (CY$_6$) threefold. In that case, the effective action of the five-dimensional theory takes the form \cite{11,12}

\[ S_{\text{eff}} = \int d^5x \sqrt{-g} \left( R + \frac{1}{16} c_{(2)}^I V_I (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}) \right), \] (12)

where we used units such that $G_{(5D)} = 1/16\pi$, and where the coupling $c_{(2)}^I V_I$ is a quantity that depends on the details of the internal CY$_6$ manifold.

In turn, we see that quadratic terms in (12) come from the $O(R^4)$ terms of (11). We observe that action (12) resembles a particular case of (8), namely the case $D = 5$ with $\alpha = \gamma = -\beta/4$, identifying $c_{(2)}^I V_I = 16\alpha$. This is precisely the theory we will study in this paper: the most general quadratic theory of gravity with equations of motion of second order, which, as we have just seen, arises as Calabi-Yau compactifications of M-theory. We already mentioned that a quadratic action similar to (12) also appears in the 1-loop corrected effective action of heterotic string theory. Written in the Einstein frame, the coupling of higher-curvature terms in the heterotic effective action is given by $\alpha \sim \alpha' e^\phi$, where the dilaton field $\phi$ clearly contributes. Black holes solutions in dilatonic Einstein-Gauss-Bonnet theory were studied in Refs. \cite{14,15,16}.

**Higher-curvature terms in AdS/CFT correspondence**

Because an action like (12) also appears in the effective action of the heterotic string, it is also usually referred to as ”string inspired higher-curvature corrections”. In turn, it represents a nice model to explore the effects of next-to-lading contributions of string theory to gravitational physics. In particular, this five-dimensional (Lovelock) model of gravity was recently considered in the context of AdS/CFT holographic correspondence \cite{17}. Actually, one of the applications of the Lovelock theory to AdS/CFT that has attracted attention recently was that of showing that the so-called Kottun-Son-Starinets bound \cite{18,19} may be violated in a theory that contains higher-curvature corrections. The Kottun-Son-Starinets bound (KSS) is the conjecture that states: the ratio between the shear viscosity $\eta$ to the entropy $s$ of all the materials obey the universal relation

\[ \frac{\eta}{s} \geq \frac{1}{4\pi} \] (13)

In Refs. \cite{20,21,22} it was observed that when action (8) with $\alpha = \gamma = -\beta/4$ and $\Lambda = -l^{-2} < 0$ is considered in asymptotically locally AdS$_5$ space, then the conformal field...
theory (CFT) that would be dual to such a theory of gravity would satisfy

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 - \frac{4\alpha}{3l^2}\right)$$

(14)

what then would violate (13) for \(\alpha > 0\). Therefore, the KSS bound would be violated for all the CFTs with a Einstein-Gauss-Bonnet gravity duals with positive \(\alpha\), and this is precisely the sign of \(\alpha\) that comes from string theory.

The consideration of five-dimensional Lovelock theory as a working example to study the effects of including higher-curvature terms in AdS/CFT has been an active line of research in the last years. Just recently, very interesting papers discussing the interplay between causality and higher-curvature terms in the context of AdS/CFT appeared [23, 24]; Ref. [24] considers the case of \(D = 5\) Lovelock theory. Causality in the dual CFT constrains the value of the coupling of the quadratic Gauss-Bonnet term\(^7\). The causality bound comes from demanding that the recidivist gravitons that hit back the boundary after a bulk excursion do not spoil locality in the CFT. The permitted range for the coupling \(\alpha\) turns out to be

\[-\frac{7l^2}{12} < \alpha < \frac{27l^2}{100}\]

(15)

The value \(\alpha = 3l^2/4\) (i.e. \(\lambda = 1/4\)), which is not in this range, corresponds to the Chern-Simons theory of gravity, which we will discuss in section 2. At this value, the ration \(\eta/s\) would vanish (if it were the case that the theory at \(\lambda = 1/4\) has a dual description too\(^8\)).

Other works discussing higher-curvature actions in the context of AdS/CFT appeared recently. See for instance [25], where holographic superconductors in five-dimensional Lovelock gravity are considered, showing that higher-curvature corrections affect the condensation phenomenon. Besides, the \(O(R^2)\) corrections in the non-relativistic version of AdS/CFT [28, 27] were also studied, and the \(D = 5\) Lovelock theory is also used in Ref. [26] as the working example for illustrating the renormalization of the dynamical exponent \(z\). Let us now move to discuss Lovelock theory in detail.

The Lovelock Theory of Gravity

Lovelock theory is the most general metric theory of gravity yielding conserved second order equations of motion in arbitrary number of dimensions \(D\). In turn, it is the natural generalization of Einstein’s general relativity (GR) to higher dimensions [1, 2]. In three and four dimensions Lovelock theory coincides with Einstein theory [29], but in higher dimensions both theories are actually different. In fact, for \(D > 4\) Einstein gravity can be thought of as a particular case of Lovelock gravity since the Einstein-Hilbert term is one of several terms that constitute the Lovelock action. Besides, Lovelock theory also admits other quoted models as particular cases; for instance, this is the case of the so called Chern-Simons gravity theories, which in a sense are actual gauge theories of gravity.

\(^7\)It is usual to define the dimensionless parameter \(\lambda = -\Lambda \alpha/3\). In terms of this parameter, the permitted range reads \(-7/36 < \lambda < 9/100\).

\(^8\)We thank D. Hofman and J. Edelstein for conversations about the case \(\lambda = 1/4\).
On the other hand, Lovelock theory resembles also string inspired models of gravity as its action contains, among others, the quadratic Gauss-Bonnet term, which is the dimensionally extended version of the four-dimensional Euler density. This quadratic term is present in the low energy effective action of heterotic string theory \cite{30,31,32}, and it also appears in six-dimensional Calabi-Yau compactifications of M-theory; see \cite{13} and references therein. In \cite{33} Zwiebach earlier discussed the quadratic Gauss-Bonnet term within the context of string theory, with particular attention on its property of being free of ghost about the Minkowski space. Besides, the theory is known to be free of ghosts about other exact backgrounds \cite{34}. For a nice and concise review on stringy corrections to gravity actions \cite{35,36,37} see the introduction of \cite{38} and references therein. For interesting recent discussions on higher order curvature terms see \cite{13,10,39,40,41} and related works.

The Lovelock theory represents a very interesting scenario to study how the physics of gravity results corrected at short distance due to the presence of higher order curvature terms in the action. In this paper we will be concerned with the black hole solutions of this theory, and we will discuss how short distance corrections to black hole physics substantially change the qualitative features we know from our experience with black holes in GR. So, let us introduce the Lovelock theory.

The Lagrangian of the theory is given as a sum of dimensionally extended Euler densities, and it can be written as follows\footnote{Here we are ignoring the boundary terms. We will consider these terms in section 2.} \footnote{The 2n-dimensional Euler density \( \chi \) is given by \( \chi(M) = \frac{(2n+1)^{2n+2}}{2^{2n+1}\pi^{n+1}} \int_M d^{2n}x \sqrt{-g} \, \mathcal{R}^n \), where, again, we are not considering the boundary terms.} \footnote{See [42] for a related discussion on gravitational dynamics and Lovelock theory.}

\[ \mathcal{L} = \sqrt{-g} \sum_{n=0}^{t} \alpha_n \mathcal{R}^n, \quad \mathcal{R}^n = \frac{1}{2^n} \delta^{\mu_1 \nu_1 \ldots \mu_n \nu_n}_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \prod_{r=1}^{n} R_{\mu_r \nu_r}^{\alpha_r \beta_r}, \] (16)

where the generalized Kronecker \( \delta \)-function is defined as the antisymmetric product

\[ \delta^{\mu_1 \nu_1 \ldots \mu_n \nu_n}_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} = \frac{1}{n!} \delta^{\mu_1}_{\alpha_1} \delta^{\nu_1}_{\beta_1} \cdots \delta^{\mu_n}_{\alpha_n} \delta^{\nu_n}_{\beta_n}. \] (17)

Each term \( \mathcal{R}^n \) in (16) corresponds to the dimensional extension of the Euler density in \( 2n \) dimension\footnote{The \( 2n \)-dimensional Euler density \( \chi \) is given by \( \chi(M) = \frac{(2n+1)^{2n+2}}{2^{2n+1}\pi^{n+1}} \int_M d^{2n}x \sqrt{-g} \, \mathcal{R}^n \), where, again, we are not considering the boundary terms.}, so that these only contribute to the equations of motion for \( n < D/2 \). Consequently, without lack of generality, \( t \) in (16) can be taken to be \( D = 2t \) for even dimensions and \( D = 2t + 1 \) for odd dimensions\footnote{See [42] for a related discussion on gravitational dynamics and Lovelock theory.}

The coupling constants \( \alpha_n \) in (16) have dimensions of [length]\(^{2n-D} \), although it is convenient to normalize the Lagrangian density in units of the Planck scale \( \alpha_1 = (16\pi G)^{-1} = \ell_P^{-2-D} \). Expanding the product in (16) the Lagrangian takes the familiar form

\[ \mathcal{L} = \sqrt{-g} \left( \alpha_0 + \alpha_1 R + \alpha_2 \left( R^2 + R_{\alpha \beta \mu \nu} R^{\alpha \beta \mu \nu} - 4 R_{\mu \nu} R^{\mu \nu} \right) + \alpha_3 O(R^3) \right), \] (18)

where we see that coupling \( \alpha_0 \) corresponds to the cosmological constant \( \Lambda \), while \( \alpha_n \) with \( n \geq 2 \) are coupling constants of additional terms that represent ultraviolet corrections to Einstein
theory, involving higher order contractions of the Riemann tensor $R^{\alpha\beta\mu\nu}$. In particular, the second order term $\mathcal{R}^2 = R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}$ is precisely the Gauss-Bonnet term discussed above. The cubic term still has a moderate form, namely

$$\mathcal{R}^3 = R^3 + 3RR^{\mu\nu\alpha\beta}R_{\alpha\beta\mu\nu} - 12RR^{\mu\nu}R_{\mu\nu} + 24R^{\mu\nu\alpha\beta}R_{\alpha\mu}R_{\beta\nu} + 16R^{\mu\nu}R_{\nu\alpha}R^{\alpha\mu} + 24R^{\mu\nu\alpha\beta}R_{\alpha\beta\mu\rho}R^{\rho}_{\mu} + 8R^{\mu\nu\alpha\beta\rho}R^{\rho}_{\nu\alpha}R_{\mu\beta} + 2R_{\alpha\beta\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}_{\mu\nu}.$$  

(19)

The fourth order term $\mathcal{R}^4$ coincides with the pure gravitational term in the last line of (11).

Even though the way of writing Lovelock action in its tensorial form (18)-(19) may result clear to introduce the theory, it is not the most efficient way for most of the calculations one usually deal with. A more convenient way of working out these expressions is to resort to the so-called first-order formalism, which turns out to be useful both for formal purposes and for practical ones. Nevertheless, it is important to point out that the first-order formalism is not necessarily equivalent to the second-order formalism, so it should not be regarded merely as a different nomenclature. In the first-order formalism, both the vielbein $e^a_{\mu}$ and the spin connection $\omega^{ab}_{\mu}$ are considered as independent degrees of freedom, and the torsion acquires in general propagating degrees of freedom. It is only in the torsion-free sector where both formulations are equivalent; notice that the vanishing torsion condition is always allowed by the equations of motion; see [55], see also [47]. We will make use of the first-order formalism at the end of section 2, as it is almost unavoidable in the discussion of Chern-Simons theory.

However, with the intention to make the exposition as friendly as possible, we will avoid abstruse notation in the rest of the paper. In any case, since we could not afford to give all the definitions necessary to introduce the subject, we will assume the reader is familiarized with basic notions of the theory of gravity and with the standard nomenclature.

Overview

The paper is organized as follows. In section 2, we analyze the spherically symmetric black hole solutions in Lovelock theory [49, 50]. In five-dimensions this is given by the Boulware-Deser solution [34], whose most important properties we review. The special properties of electrically charged black holes [51, 52] are also briefly discussed. In one of the subsections of section 2, we extend the analysis to those black objects whose horizon geometries correspond to more general spaces of constant (but not necessarily positive) curvature [48, 53]. These are the so-called topological black holes, which can be thought of as black brane solutions of the theory. Also, we briefly review the most relevant features of the Lovelock black hole thermodynamics [54], focusing our attention on the qualitative features that have no analogue in GR. Throughout the discussion, the five-dimensional black hole of the Einstein-Gauss-Bonnet theory will serve as prototypical example. In section 3, we discuss the role of boundary terms [55] and the junction conditions these yield [56, 58, 57]. We show how solutions with non-trivial topology can be constructed by a method of geometric surgery. Particular attention is focussed on vacuum wormhole solutions recently found [59, 60]. Finally, we study the spherically symmetric

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12cf. [44], where it was shown that no unambiguous cubic terms arise in string theory effective action; in particular, the Lovelock cubic term is studied. Cubic terms are strongly constrained by supersymmetry.
solutions that develop naked curvature singularities. We study these naked singularities with quantum probes and show that, in spite of the divergence in the curvature, these spaces are well-behaved within a quantum mechanical context.

2 The Lovelock black holes

Spherically symmetric black hole solutions

Let us first consider the theory in five dimensions. Since in \( D < 7 \) the \( R^3 \) term does not contribute to the equations of motion, the five-dimensional Lovelock theory basically corresponds to Einstein gravity coupled to the dimensional extension of the four-dimensional Euler density, i.e. the theory that is usually referred as Einstein-Gauss-Bonnet theory (EGB). The spherically symmetric static solution of EGB theory was obtained by Boulware and Deser in Ref. [34]. The metric takes the simple form

\[
d s^2 = -V^2(r)\, dt^2 + V^{-2}(r)\, dr^2 + r^2 d\Omega_3^2 \tag{20}
\]

where \( d\Omega_3^2 \) is the metric of a unitary 3-sphere, and where the metric function \( V^2(r) \) is given by

\[
V^2(r) = 1 + \frac{r^2}{4\alpha} + \sigma \frac{r^2}{4\alpha} \sqrt{1 + \frac{16\alpha M}{r^4} + \frac{4\alpha\Lambda}{3}}, \tag{21}
\]

with \( \sigma^2 = 1 \). Here we used the standard convention \( \alpha_0/\alpha_1 = -2\Lambda, \alpha_2/\alpha_1 = \alpha \), and, besides, we have set the Newton constant to a specific value for short. From (21) we notice that there exist two different branches of solutions to the spherically symmetric ansatz (20), namely \( \sigma = +1 \) and \( \sigma = -1 \), and this reflects the fact that the equations of motion give a differential equation quadratic in the metric function \( V^2(r) \). As usual, the parameter \( M \) arises here as an integration constant, and it corresponds to the mass of the solution up to the factor we absorbed in \( M \).

It is worth mentioning that (20)-(21) is the most general spherically symmetric solution to EGB theory, provided the fact that the metric is smooth everywhere and that the parameters \( \Lambda \) and \( \alpha \) are generic enough. In turn, a Birkhoff theorem holds for this model [77, 78, 79, 5]. It is important to emphasize that for very particular choices of the set of parameters \( \alpha_n \), degeneracy in the space of solutions can appear, and in those special cases the Birkhoff’s theorem can be circumvented; see [78] for a very interesting discussion. To our knowledge, the most complete analysis of the EGB analogue of Birkhoff’s theorem was performed in [70], where the Nariai-type solutions [80] were also discussed.

If \( \alpha > 0 \), the solution corresponding to \( \sigma = -1 \) in (21) may represent a black hole solution whose horizon, in the case \( \Lambda = 0 \), is located at \( r_+ = \sqrt{2(M - \alpha)} \). On the other hand, as long

\[13\] For the discussion on the computation of charges in this theory see the list of references [62, 64, 65, 66, 67, 68, 69, 70, 72]; see also [71, 74, 75, 70, 154].

\[14\] More precisely, in the definition of \( M \) we absorbed a factor \( \frac{8\pi G}{(D-2)(D-2-2)} \) where \( \Omega_n = \frac{(n+1)!^{(n+1)/2}}{\Gamma((n+3)/2)} \) is the surface of the \( n \)-sphere, and where \( G \) is the Newton constant, given by \( G \sim \alpha_1^{-1} \), which has been fixed to a specific values such that \( \alpha_1 = 1 \).
as \( \alpha > 0 \) and \( M > 0 \), the branch \( \sigma = +1 \) has no horizon but presents a naked singularity at \( r = 0 \).

Solutions \( \sigma = -1 \) and \( \sigma = +1 \) have substantially different behaviors, and only one of them tends to the GR solution in the small \( \alpha \) limit. In fact, in the limit \( \alpha \to 0 \) the branch \( \sigma = -1 \) looks like

\[
V_{\sigma=-1}^2(r) \simeq 1 - \frac{2M}{r^2} - \frac{\Lambda}{6}r^2,
\]

where we see it approaches the five-dimensional (Anti)-de Sitter-Schwarzschild-Tangherlini solution [81]. On the other hand, in the \( \alpha \to 0 \) limit the solution corresponding to the branch \( \sigma = +1 \) behaves like

\[
V_{\sigma=+1}^2(r) \simeq 1 + \frac{2M}{r^2} + \frac{\Lambda}{6}r^2 + \frac{1}{2\alpha}r^2,
\]

and we see it acquires a large effective cosmological constant term \( \sim r^2/2\alpha \). In particular, this implies that microscopic (A)dS space-time is a solution of the theory even for \( \Lambda = 0 \). This feature was expressed by Boulware and Deser [34] by saying that EGB theory has its own cosmological constant problem, with \( \Lambda_{\text{eff}} \sim -1/\alpha \). In a sense, the branch \( \sigma = +1 \) is commonly believed to be a false vacuum of the theory, and it is known to present ghost instabilities [34]; see also [82].

The branch \( \sigma = -1 \), on the other hand, is well-behaved, and it represents short distance corrections to GR black holes (22). While at short distances the black hole solutions of both theories are substantially different due to the effects of the Gauss-Bonnet term, in the large distance regime \( r^2 \gg \alpha \) the Lovelock black hole (20) with \( \sigma = -1 \) behaves like a GR black hole whose parameters \( M \) and \( \Lambda \) get corrected by finite-\( \alpha \) subleading contributions \( \mathcal{O}(\alpha \Lambda) \); for instance, the parameter of the mass term gets corrected yielding the effective mass \( M\sqrt{1+4\alpha \Lambda/3} \). In the large \( r \) limit, the next-to-leading \( r \)-dependent contribution to (22) goes like \( \mathcal{O}(\alpha r^{-6}) \).

The damping of this additional term, which in \( D \) dimensions goes like \( \mathcal{O}(\alpha r^{4-2D}) \), is actually strong, and, for distance large enough, it is negligible even in comparison with semiclassical corrections to the metric due to field theory backreaction, which typically go like \( \mathcal{O}(\hbar r^{5-2D}) \) (for instance, see [83]).

All these features are essentially due to the nature of the Gauss-Bonnet term, and also hold in higher dimensions. In fact, it is straightforward to generalize solution (20) to the case of EGB gravity in \( D > 5 \) dimensions, and the metric is seen to adopt a very similar form [34]. Actually, it is given by simply replacing the element of the 3-sphere in (20) by the element of the unitary \((D-2)\)-sphere \( d\Omega^2_{D-2} \), and by replacing the piece \( 16M\alpha/r^4 \) in (21) by \( 16M\alpha/r^{D-1} \).

In spite of the non-polynomial form of (21), the horizon structure of Boulware-Deser solution is quite simple, and in \( D \) dimensions the horizon location is given by the roots of the polynomial

\[
\frac{\Lambda}{6}r^{D-1} - r^{D-3} - 2\alpha r^{D-5} + 2M = 0,
\]

where \( \Lambda \) has been appropriately rescaled by a \( D \)-dimensional constant factor.

From (21) we observe that the five-dimensional case is actually a remarkable example since, among other special features, it allows to have massive solutions with naked singularities. We mentioned above that if \( D = 5 \) and \( \Lambda = 0 \) the black hole horizon is located at \( r^2_+ = 2(M - \alpha) \),
and this implies a lower bound for the spherical solution not to develop a naked singularity, namely  \( M > \alpha \). That is, for  \( 0 < M < \alpha \) we do find naked singularities even for the well-behaved branch \( \sigma = -1 \) with positive \( M \). For the model with a second order term \( \mathcal{R}^2 \) this only occurs in \( D = 5 \). In seven dimensions, for instance, the Boulware-Deser solution with \( \Lambda = 0 \) develops horizons at \( r^2 = \alpha\sqrt{1 + 2M/\alpha^2} - \alpha \) and then the horizon always exists provided \( \alpha > 0, M > 0 \). Naked singularities in \( D = 2n + 1 \) dimensions usually arise when a term of order \( \mathcal{R}^n \) is present in the action. So, for the EGB theory this only occurs for \( D = 5 \).

Another special feature of the (uncharged) five-dimensional case is that the metric (20) turns out to be finite at the origin, namely \( V^2(r=0) = 1 + \sigma \sqrt{M/\alpha} \). Nevertheless, the curvature still diverges at the origin, although not in a dramatic way. We will return to this point in the last section where we will discuss naked singularities.

It could be important to mention that the analysis of the dynamical stability of EGB black holes is also special for \( D = 5 \). The stability analysis under tensor mode perturbations has been explored recently, and it has been shown that the EGB theory exhibits some differences with respect to Einstein theory; at least, it seems to be the case for sufficiently small values of mass in five and six dimensions \[84\] where instabilities arise; see also Refs. \[85, 87, 86, 88\]. In this sense, the cases \( D = 5 \) and \( D = 6 \) are special ones. See Ref. \[89\] for an interesting recent discussion. On the other hand, let us be reminded of the fact that in \( D > 6 \) dimensions the Lovelock action \[16\] presents also additional terms of higher order \( n > 2 \), so that in \( D \geq 7 \) the Boulware-Deser black hole geometry \[20\]-\[21\] only corresponds to a very special example of Lovelock black hole.

Spherically symmetric solutions in higher dimensions containing an arbitrary higher order terms \( \mathcal{R}^n \) in \[16\] can be implicitly found by solving a polynomial equation of degree \( n \) whose solutions give the metric function \( V^2(r) \); this was originally noticed by Wheeler in \[49, 50\]. Moreover, several explicit examples containing arbitrary amount of terms \( \mathcal{R}, \mathcal{R}^2, \ldots, \mathcal{R}^{n-1}, \mathcal{R}^n \) are also known. These correspond to particular choices of the couplings \( \alpha_n \) in \[16\]. One of these explicitly solvable cases corresponds to the Chern-Simons theory, which exists in odd dimensions. We will briefly discuss this special case below. A remarkable fact is that in the case a term \( \mathcal{R}^n \) of the Lovelock expansion \[16\] is considered in the action, then the spherically symmetric solution may still take a very simple expression, and, depending on the coupling constants \( \alpha_n \), it may merely correspond to replacing the square root in \[21\] by a power \( 1/n \); see \[90, 91, 92, 93\] for explicit examples.

Adding electric charge

On the other hand, it is quite remarkable that electrically charged black hole solutions in Lovelock theory also present a very simple form. The solutions charged under both Maxwell and Born-Infeld electrodynamics have been known for long time \[51, 52\], and these solutions were reconsidered recently \[94\]. In general, the metric function of a charged solution takes the form \[21\] but replacing the mass parameter \( M \) by a mass function \( M(r) \) that depends on the radial coordinate \( r \). Function \( M(r) \) depends on the particular electromagnetic Lagrangian one considers. In the case of Maxwell theory, and in five dimensions, this function is given by the energy contribution \( M(r) \sim \int_\epsilon^r dr \ 2^2/r^3 \sim -Q^2/r^2 + M_0 \), where \( Q \) represents the electric
charge of the black hole, and where the UV cut-off in the integral is absorbed in the definition of the additive constant $M_0$. More precisely, for charged black holes in Einstein-Gauss-Bonnet-Maxwell theory we have $M(r) - M_0 = -Q^2/6r^2$, as it was originally noticed by Wiltshire \[51\]. On the other hand, in the case of black holes charged under Born-Infeld theory, the function $M(r)$ is given by

$$M(r) - M_0 = \frac{2}{3} \beta^2 \int_0^r ds \sqrt{s^6 + \beta^{-2}Q^2 - \frac{1}{6} \beta^2 r^4},$$

(25)

where the $\beta^2$ is the Born-Infeld parameter, according to the standard form of the Lagrangian $\mathcal{L}_{BI} = \beta^2 - \beta^2 \sqrt{1 + F^2/\beta^2}$. In the large $\beta$ limit $\mathcal{L}_{BI} \simeq -\frac{1}{2} F^2 + \mathcal{O}(F^4/\beta^2)$, and then the metric approaches the charged solution for the Maxwell-Einstein-Gauss-Bonnet theory,

$$M(r) - M_0 \simeq -\frac{Q^2}{6r^2} + \mathcal{O}(Q^4/r^8 \beta^2).$$

(26)

As expected, the five-dimensional Reissner-Nordström black hole is recovered in the large $r$ regime for the case $\sigma = -1$.

Charged solutions of Lovelock theory coupled to Born-Infeld electrodynamics present curious features that are not present in the case of Einstein-Maxwell theory. Perhaps the most relevant one is the existence of single-horizon charged solutions \[94\]. Besides, Lovelock black holes charged under Maxwell electrodynamics, and for certain values of the coupling constants $\alpha_n$, can develop curvature singularities at fixed values of the radial coordinate \[93\], making necessary to exclude a region of the space. This kind of divergence is usually called branch singularity, and it can also be present in uncharged solutions, as it happens for solutions of EGB gravity with $M < 0$ and $\alpha > 0$, \[95, 96\].

As in the case of Hoffmann’s solution in Born-Infeld-Einstein \[97\] theory, the Lovelock black holes charged under Born-Infeld theory induce a contribution to the mass coming from the finite concentration of electromagnetic energy around the singularity. Of course, this happens because both theories coincides at large distances. For finite values of $\beta$, $M(r)$ has a large distance behavior that induces a mass contribution $\Delta M = (2\beta^2/3) \int_0^\infty dr \sqrt{r^6 + Q^2\beta^{-2}}$. In particular, this implies that, for certain range of $\beta$ and $Q$, naked singularities in five dimensions may arise even for values of the effective mass $M_0 + \Delta M$ grater than $\alpha$. Notice that the cosmological constant term also acquires a $\beta$-dependent contribution $\sim \beta^2$.

In the next subsection we will consider a generalization of the black hole solutions reviewed here. We will discuss extended black objects in EGB theory.

**Topological black holes**

One of the interesting aspects of Lovelock theory is that it admits another class of black objects, whose horizons are not necessarily positive curvature hypersurfaces \[53\]. These solutions are usually called topological black holes, and their metric are obtained by replacing the $(D - 2)$-sphere $d\Omega_{D-2}^2$ in \[20\] by a base manifold $d\Sigma_{D-2}$ of constant (but not necessarily positive) curvature, provided a suitable shifting in the metric function $V^2(r)$. Namely, these solutions
where the metric function is now given by \( K^2(r) = V^2(r) + k - 1 \), with \( k = -1, 0, +1 \), being its sign that of the curvature of the horizon hypersurface, whose line element is \( r^2d\Sigma_{D-2}^2 \). For \( k = +1 \) the Boulware-Deser solution \((20)-(21)\) is recovered. In general, the base manifold \( d\Sigma_{D-2}^2 \) here may be given by a more general constant curvature space: For instance, it can be given by the product of hyperbolic spaces \( d\Sigma_{D-2}^2 = dH_{D-2}^2 \) for the case of negative curvature \( k = -1 \), or merely by a flat space piece \( d\Sigma_{D-2}^2 = dx_i dx^i \). In turn, solutions \((27)\) correspond to black brane type geometries. Such black objects represent fibrations over constant curvature \((D-2)\)-dimensional hypersurfaces, implying that the event horizon, in the cases it exists, is not necessarily a compact simply connected manifold.

Consider for example the five-dimensional EGB theory with negative cosmological constant \( \Lambda < 0 \), and its black brane solution of the form

\[
 ds^2 = -K^2_{(k=0)}(r)dt^2 + K^{-2}_{(k=0)}(r)dr^2 + r^2dx_i dx^i \quad (28)
\]

with

\[
 K^2_{(k=0)}(r) = \frac{r^2}{4\alpha} - \sqrt{\frac{r^4 - 1}{16\alpha^2}} (1 - 4V^2(r)^2/\alpha^2) + \frac{M}{\alpha}, \quad (29)
\]

where \( x^i = x^1, x^2, x^3 \). These objects (brane-like configurations and topological black holes) have attracted some attention recently due to their curious properties, and, more recently, these were considered in applications to AdS/CFT; see for instance \([21, 22]\).

In \([98]\), an exhaustive classification of static topological black hole solutions of five-dimensional Lovelock theory was presented. The authors considered an ansatz such that spacelike sections are given by warped product of the radial coordinate \( r \) and an arbitrary base manifold \( d\Sigma_{D-2}^2 \), and they showed that, for values of the coupling constant \( \alpha_2 \) generic enough, the base manifold must be necessarily of constant curvature, and then the solutions of the theory reduce to the topological extension of the Boulware-Deser metric of the form \((27)\). In addition, they showed that for the special case where the coupling \( \alpha_2 \) is appropriately tuned in terms of the cosmological constant \( \Lambda_0 \), then the base manifold could admit a wider class of geometries, and such enhancement of the freedom in choosing \( d\Sigma_{D-2}^2 \) allows to construct very curious solutions with non-trivial topology. We will return to this point in section 2.

The existence of black holes with generic horizon structure was also analyzed in \([99]\), where selection criteria for the base manifold \( d\Sigma_{D-2}^2 \) were discussed\(^{15}\), and the authors concluded that sensible physical models strongly restrict most of the examples of exotic black holes with non-constant curvature horizons. Moreover, the different horizon structures were also studied in \([48, 96]\) together with its relation to the asymptotic behavior of the corresponding solutions; see

\(^{15}\)These are analogues of the topological black holes previously known in four-dimensions, which, at constant \( t \) hypersurfaces, correspond to fibrations of (closed) base manifolds \( \Sigma_2/\Gamma \) with non-trivial topology.

\(^{16}\)The authors of \([99]\) derived a necessary constraint to be obeyed by the Euclidean manifold that is candidate to represent a horizon geometry of a black hole solution in \( D \)-dimensional Einstein-Gauss-Bonnet theory. They proved that such a \( D - 2 \)-manifold has to obey the equation \( C^l_{ikm}C^{kj}_{lm} \propto \delta^i_1 \), where \( C^l_{ik} \) is the Weyl tensor in \( D - 2 \) dimensions.
also [100, 101, 102, 103, 104]. Recently, the electrically charged topological black hole solutions were also analyzed, both for the case of the second order Lovelock theory in [95, 100] and for the case of the third order Lovelock theory in [103].

One of the most interesting aspects of these objects with non-trivial horizon geometries is that they enable us to construct a very simple class of Kaluza-Klein black holes with interesting properties from the four-dimensional viewpoint. For instance, such a solution was recently studied by Maeda and Dadhich in Ref. [112]. These Kaluza-Klein black holes are given by a product $M_4 \times H_{D-4}$ between a four-dimensional manifold $M_4$ and a $(D-4)$-dimensional hyperbolic space $H_{D-4}$. It turns out that the four-dimensional piece of the geometry asymptotically approaches the charged black hole in locally AdS$_4$ space. In turn, the Gauss-Bonnet term acts by emulating the Reissner-Nordström term for large $r$, while it changes the geometry at short distances [113, 114, 115]. In addition to these solutions, other exotic Kaluza-Klein Lovelock black hole solutions with arbitrary order terms of the form $R^n$ and for a specific values of the coefficients $\alpha_n$ were studied in [116]. These black holes are different from those studied in [112], and are obtained by considering black $p$-brane geometries of the form $M_{D-p} \times T^p$ in the Lovelock theory with $\alpha_i = \delta_i,n$ and $2n = D - p$. These solutions exist for $D - p$ even, and, in addition, the horizon structure also depends on $n$. Analogous toric compactifications of the form $M_{D-p} \times T^p$ were studied in [117], and warped brane-like configurations were also discussed in both [116] and [117].

It was shown in [116] that, in spite of the difference between Lovelock theory and Einstein theory, the qualitative features of thermodynamic stability of brane-like configurations in both theories are considerable similar, although the higher order terms $R^n$ can be seen to contribute. For example, the thermodynamical analogue of Gregory-Laflamme transition between black hole and black string configurations was discussed in [116]. Extended string-like objects in Lovelock theory and their thermodynamics were also discussed in [118, 119, 154]. We discuss black hole thermodynamics in the next subsection.

**Thermodynamics**

The purpose of this section is to describe the general aspects of black hole thermodynamics in Lovelock theory. In fact, one of the most interesting features of the Lovelock theory regards the thermodynamics of its black hole solutions. This is because it is in the analysis of the black hole thermodynamics where the substantial differences between Lovelock theory and Einstein theory manifest themselves.

Pioneer works where the Lovelock black hole thermodynamics was discussed in detail are references [120, 121]; see also [122, 123, 124]. In [54], Jacobson and Myers derived a close expression for the entropy of these solutions in $D$ dimensions, and they showed that the entropy of these black holes does not satisfy the area law, but contains additional terms that are given by a sum of intrinsic curvature invariants integrated over the horizon.

The thermodynamics of charged solutions was originally studied by Wiltshire in Refs. [51, 52], while the thermodynamics of topological black holes was studied more recently, in Refs. 

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15Recently, references [106, 107, 108, 109, 110, 111] discussed other classes of solutions. We will not comment on these solutions here.
The study of charged topological black holes in presence of cosmological constant was addressed in [126], where the most general solution of this type in EGB theory was obtained. References [127, 128, 129] also analyze topological black holes and their thermodynamics; see also [102, 130].

The aim of this section is to discuss the more relevant thermodynamical features of Lovelock solutions. To do this, we will consider again the five-dimensional case (20)-(21). Actually, besides it represents a simple instructive example, the five-dimensional case is also special in what concerns thermodynamical properties. It is the best example to see that substantial differences between Lovelock gravity and Einstein gravity exist.

It is easy to verify that the Hawking temperature associated to the solution in $D = 5$ with $\Lambda = 0$ is given by

$$ T = \frac{\hbar}{2\pi} \frac{r_+}{4\alpha + r_+^2}. $$

(30)

Then, we see that, as expected, (30) behaves like the Hawking temperature of a GR solution if the black hole is large enough, $r_+ >> \alpha$, going like $T \simeq \hbar/8\pi r_+ - \mathcal{O}(\alpha/r_+^3)$. On the other hand, temperature tends to zero for small values of $r_+$, going like $T \simeq \hbar r_+/8\pi \alpha + \mathcal{O}(r_+^3/\alpha^2)$. This implies that the specific heat changes its sign at length scales of order $r_+ \sim \sqrt{\alpha}$, and a direct consequence of this phenomenon is that five-dimensional Lovelock black holes turn out to be thermodynamically stable, as they yield eternal remnants. This can be easily verified by considering the rate of thermal radiation which goes like $\partial_t M \sim -T^5 r_+^3$, behaving like $dt \sim -dr_+/r_+^7$ at short distances.

Nevertheless, it is worth pointing out that for dimension $D > 5$ the functional form of the temperature is substantially different from the case $D = 5$, as it includes an additional term which is actually proportional to $(D - 5)$. The general formula reads

$$ T = \frac{\hbar}{4\pi} \frac{(D - 3)r_+^2 + 2\alpha(D - 5)}{4\alpha r_+ + r_+^3}. $$

(31)

which implies that, in $D > 5$, the short distance limit is given by $T \simeq (D - 5)\hbar/8\pi r_+$, and the specific heat is then negative. This is the reason why the thermodynamic behavior of higher dimensional Einstein-Gauss-Bonnet black holes turns out to be more similar to that in Einstein theory if $D \neq 5$. In general, eternal black holes arise in $D = 2n + 1$ dimensions if an $n^{th}$-order term $R^n$ is present in the action.

So, let us return to our instructive example of five dimensions. The entropy associated to (30) is given by

$$ S = \frac{A}{4G\hbar} + \mathcal{O}(\alpha r_+) \sim r_+^3 + 12\alpha r_+, $$

(32)

from what we observe that black holes of Lovelock theory do not in general obey the Bekenstein-Hawking area law. Actually, some particular solutions, corresponding to topological black holes with flat horizon geometry $d\Sigma_3^2 = dx_i dx_i$, do obey the area law [131, 130], but it is not the case
for spherically symmetric static solutions. A very interesting discussion on the area law is that of Ref. [104], where a version of the area law for symmetric dynamical black holes defined by a future outer trapping horizon was derived. There, the authors discussed the differences between the branches of solutions with GR limit and those without it, and argue how for the latter one still can define a concept of increasing dynamical entropy.

Notice that the second term in the right hand side of (32) implies that if \( \alpha < 0 \) then the entropy turns out to be negative for sufficiently small black holes. This was discussed in [132], where it was argued there that an additive ambiguity in the definition of the entropy could be a solution for the negative entropy contributions; see also the related discussion in [48]. In any case, the theory for negative values of the coupling constant \( \alpha \) is somehow pathological in several respects. It not only gives negative contributions to the entropy, but also ghost instabilities and strange causal structure arise if \( \alpha < 0 \). We will not consider the negative values of \( \alpha \) here.

Because of the current interest in black hole thermodynamics of higher order theories, we consider convenient to mention that the entropy function formalism, recently proposed by A. Sen [135] within the context of the attractor mechanism, works nicely for the case of Lovelock black holes. In particular, this was recently studied in [136] for the case of EGB black holes, and it was explicitly shown that (32) is recovered by analyzing the near horizon geometry. A rather general analysis was presented in Ref. [137]. Very interesting discussions are those of Refs. [138, 139].

The thermodynamic properties of topological black holes are also very interesting; see for instance [140, 130]. As we already mentioned, it can be shown that those black objects whose horizons are of zero curvature do obey the area law for the entropy density. For instance, consider the black brane geometry (28), which is solution of the theory with negative cosmological constant, \( \Lambda < 0 \). It is straightforward to check that the Hawking temperature of this solution is given by

\[
T = \frac{\hbar}{6\pi |\Lambda| r_+},
\]

and that the area formula for the entropy density does hold in this special case. Remarkably, identical expression for the temperature is obtained in the particular case of the Chern-Simons theories of gravity, which we discuss in the next subsection.

**Chern-Simons black holes**

Now, let us move on, and analyze a very particular case of Lovelock theory which exist in odd dimensions. This is the so-called Chern-Simons gravity (CS), and can be thought of as a higher-dimensional generalization of the Chern-Simons description of three-dimensional Einstein gravity [141]. Basically, these theories are those particular cases of Lovelock Lagrangian

\[\text{(33)}\]

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that admit a formulation in terms of a Chern-Simons action. As we will discuss, these models are given by a very precise choice of the set of coefficients $\alpha_n$.

To discuss CS gravity theories\(^{20}\) it is convenient to resort to the first-order formalism which, in spite of its advantage, it is paradoxically avoided in physics discussions. So, let us first review some basic notions: Consider the vielbein $e^a_\mu$, which defines the metric as $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, where we are using the standard notation such that the greek indices $\mu, \nu, ...$ correspond to the space-time while the latin indices $a, b, ...$ are reserved for the tangent space. Now, consider the 1-form associated to the vielbein, defined by $e^a_\mu = e^a_\mu dx^\mu$, and the corresponding 1-form associated to the spin connection $\omega_{ab}^\mu$, defined by $\omega_{ab}^\mu = \omega_{ab}^\mu dx^\mu$. These quantities enable us to define the so-called curvature 2-form, which is given by

$$R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^b_c = R^{ab}_\mu \mu \nu \ dx^\mu \wedge dx^\nu \equiv \frac{1}{2} R^{ab}_\mu \nu \ (dx^\mu dx^\nu - dx^\nu dx^\mu),$$

and is related to the Riemann tensor by $R^a_{\beta\mu\nu} = \eta_{bc} e^a_c e^b_\beta R^{ab}_\mu \nu$. The torsion-free condition is then given by

$$T^a = de^a + \omega^a_b \wedge e^b = 0.$$

In this language, local Lorentz invariance of the theory is expressed in terms of the covariant derivative

$$\delta \lambda e^a_\mu = d\lambda^a_\mu + \omega^a_{\mu\nu} \nu \wedge \lambda^a_\nu, \quad \delta \lambda \omega^a_{\mu\nu} = -\lambda^a_\nu e^b_\mu,$$

where $\lambda^a_\nu$ represent the parameters of the transformation.

The remarkable fact is that, for particular cases of the action (16), if the coupling constants are chosen appropriately, the theory exhibits an additional local symmetry. For instance, if we consider the case $\Lambda = 0$, such additional symmetry turns out to be given by the invariance of the Lagrangian density under the gauge transformation

$$\delta \lambda e^a_\mu = d\lambda^a_\mu + \omega^a_{\mu\nu} \nu \wedge \lambda^a_\nu, \quad \delta \lambda \omega^a_{\mu\nu} = 0.$$  \(35\)

That is, the CS theory possesses a local symmetry under gauge transformation $\delta \lambda e^a_\mu = \partial^a_\mu \lambda + \omega^a_{\mu\nu} \nu \wedge \lambda^b_\nu$, with $\lambda^a_\nu$ being a parameter. This is actually an off-shell local gauge symmetry of the theory (16) that arises for special choices of the coupling constants $\alpha_n$, as far as the boundary conditions are also chosen in the appropriate way. Besides, it can be easily verified that transformation (35), once considered together with (34), satisfies the Poincaré algebra $ISO(2, 1)$, and this is why these theories are usually referred as Poincaré-Chern-Simons gravitational theories \[144\]; see also \[46\] for an excellent introduction to Chern-Simons gravity.

So, let us specify which are the theories that possess the gauge symmetry like (34)-(35), namely the CS theories. To do this, it is convenient to rewrite the Lovelock Lagrangian. In the first-order formalism, the Lovelock action corresponding to (16) in $D = 2t + 1$ dimensions can be written as

\[\text{(16)}\]

\[\text{(20)}\]

\[\text{(21)}\]
\[ S = \int \varepsilon_{a_1 b_1 a_2 b_2 \ldots a_t b_t} \wedge \left( R^{a_n b_n} + l_n^{-2} e^{a_n} \wedge e^{b_n} \right) \wedge e^c \quad (36) \]

where \( l_n^{-2} \) correspond to \( t \) independent coefficients that are a rearrangement of the coefficients \( \alpha_n \). In (36), the convention is such that the \( t^{th} \) coupling \( \alpha_{n=t} \) has been set to 1 (or, alternatively speaking, it has been absorbed in the definition of the curvature \( R^{ab} \)), so that in this notation we have \( |\Lambda| \sim \prod_{n=1}^t l_n^{-2} \), and \( G^{-1} \sim \sum_{m=1}^t \prod_{n \neq m} l_n^{-2} \).

It is worth noticing that, in order to represent the most general form of (16), the coefficients \( l_n^{-2} \) in (36) should be allowed to take complex values. In fact, Lovelock action (16) with real coefficients \( \alpha_n \) can correspond to (36) with imaginary \( l_n^{-2} \). An example is given by the five-dimensional theory whose action reads

\[ S = \int \varepsilon_{abcd} f \left( R^{ab} + i \beta^2 e^a \wedge e^b \right) \wedge \left( R^{cd} - i \beta^2 e^c \wedge e^d \right) \wedge e^f, \quad (38) \]

which leads to the particular form of (16) where no Einstein-Hilbert contribution is present, but only the cosmological constant and the Gauss-Bonnet term appear, with \( \alpha/\Lambda \sim \beta^{-4} \) for a real \( \beta \).

The CS gravity theories, however, are given by real values of \( l_n^{-2} \). More precisely, CS theory correspond to the special case where the coupling \( l_n^{-2} \) in (36) combine to give only one value for the effective cosmological constant \( \Lambda_{\text{eff}} = \pm l_n^{-2} \). In terms of the Lagrangian density (16) this corresponds to taking the coupling constants \( \alpha_n \) to be \( \alpha_n = (-1)^n \frac{n(n+1)2n-P_m!}{(D-2n)(m-n)!n!} \) for \( n > 0 \), while \( \alpha_0 \) is given by the cosmological constant \( \Lambda = -\alpha_0/2\alpha_1 \). It is important to mention that (36) corresponds to the case of negative cosmological constant, which yields the CS theory with the AdS\(D\) group (i.e. the group \( SO(D-1, 2) \)) as the one that generates the gauge symmetry. The case of positive \( \Lambda \) is simply obtained by changing \( l^2 \rightarrow -l^2 \), while the Poincaré invariant theory is obtained through the Inonu-Wigner contraction of (A)dS group; see [46] for details. An example of Poincaré invariant CS is given by the Lagrangian containing only the quadratic Gauss-Bonnet \( \sqrt{-g} R^2 \) term in five dimensions, without the Einstein-Hilbert term and with \( \Lambda = 0 \).

As it is well known, an example of the CS gravity theory is given by three-dimensional Einstein theory, whose action\(^{22}\)

\[ S = \int d^3 x \ L = \int d^3 x \sqrt{-g} \left( R - 2\Lambda \right), \quad (37) \]

admits to be formulated as a CS theory. To see this, and then extend the construction to higher dimensional cases, let us first point out that (37) can be written as follows,

\[ S = \int_{M_3} \varepsilon_{abc} (R^{ab} \wedge e^c - l^2 e^a \wedge e^b \wedge e^c), \quad (38) \]

with \( \Lambda \sim l^{-2} \).

It turns out that (37)-(38) admits to be formulated as a CS theory [141] for the groups \( SO(2, 2) \), \( SO(3, 1) \) and \( ISO(2, 1) \), depending on whether the cosmological constant \( \Lambda \) is negative, positive or zero, respectively. To make contact with the usual form of the CS action, let\(^{22}\)

\(^{22}\)For simplicity here we have fixed the Newton constant according to \( 16\pi G = 1 \).
us introduce a \((D+1)\)-dimensional 1-form \(A^{ab}\) whose indices run over \(a, b = 0, 1, 2, \ldots, 2t + 1\) (recall \(D = 2t + 1\)), and its strength field \(F^{ab} = dA^{ab} + A^{ac} \wedge A^{cb}\), which are given by

\[
A^{ab} = \begin{pmatrix} \omega^{ab} / l & e^a / l \\ -e^b / l & 0 \end{pmatrix}, \quad F^{ab} = \begin{pmatrix} R^{ab} - l^{-2} e^a \wedge e^b & l^{-1} (de^a + \omega^a \wedge e^c) \\ -l^{-1} (de^b + \omega^b \wedge e^c) & 0 \end{pmatrix}.
\]

That is, \(A^{ab} = \omega^{ab}\) for \(a, b = 0, 1, 2, \ldots, 2t\), while \(A^{aD} = -A^{Da} = e^a / l\) for \(a = 0, 1, 2, \ldots, 2t\). Analogously, \(F^{ab} = R^{ab} - l^{-2} e^a \wedge e^b\) for \(a, b = 0, 1, 2, \ldots, 2t\), while \(F^{aD} = -F^{Da} = T^a / l\) for \(a = 0, 1, 2, \ldots, 2t\).

Then, making use of these definitions, \((37)-(38)\) can be alternatively expressed in its Chern-Simons form

\[
S = \int_{M_5} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),
\]

where the trace is over the indices \(a, b\) that run from 0 to 3 (corresponding to \(D = 3\), i.e. \(t = 1\)). Local symmetry under \((34)\) and \((35)\) is then gathered by gauge symmetry of \((39)\).

The next example we could consider is the five-dimensional one, which corresponds to the Lovelock theory \((16)\) for the particular case \(\alpha_0 \alpha_2 = 3 / 2\) (i.e. \(\alpha \Lambda = -3 / 4\)). Then, the action reads

\[
S = \int d^5 x \mathcal{L} = \int d^5 x \sqrt{-g} \left( R + \frac{2}{l^2} - \frac{3l^2}{4} (R + R_{\mu \nu \alpha \beta} R^{\mu \nu \alpha \beta} - 4 R^{\mu \nu} R_{\mu \nu}) \right)
\]

where \(\Lambda = -l^{-2}\) and \(\alpha_2 = 3 / 2 \alpha_0 = -3 / 4 \Lambda = 3 l^2 / 4\). This can be also written as

\[
S = \int_{M_5} \varepsilon_{a b c d e} \left( R^{a b} \wedge R^{c d} + \frac{2}{3 l^2} R^{a b} \wedge e^c \wedge e^d + \frac{1}{5 l^2} e^a \wedge e^b \wedge e^c \wedge e^d \right) \wedge e^f
\]

and, again, it admits to be written in its Chern-Simons form

\[
S = \frac{1}{\kappa^2} \int_{M_5} \text{Tr} \left( A \wedge (dA)^2 + \frac{3}{2} (A) \wedge dA + \frac{3}{5} (A) \wedge 5 \right)
\]

Actually, this structure goes on as \(D\) increases, and it expands a whole family of theories which, still being particular cases of Lovelock theory \((16)\), represent odd-dimensional field theories with local off-shell symmetry under the (A)dS (or Poincaré) group.

Now, once we have introduced the theories, let us analyze their black hole solutions. Going back to solution \((20)\), and considering again the five-dimensional case as an example, we observe that replacing the Chern-Simons condition \((23)\) \(\alpha \Lambda = -3 / 4\) in the metric function \((21)\) leads to a rather different geometry, given by

\[
V^2(r) = \frac{r^2}{4 \alpha} - \mathcal{M} \quad \text{with} \quad \mathcal{M} + 1 = -\sigma \sqrt{\mathcal{M} / \alpha}.
\]

This solution still may represent a black hole, provided \(\mathcal{M} > 0\), with the horizon located at \(r_+ = 2 \sqrt{\mathcal{M} / \alpha}\). However, this is a black hole of a different sort. In particular, it does not

\[23\text{It is helthy to consider the case } \alpha > 0 \text{ and } \Lambda < 0.\]
present a limit where GR is recovered, and this can be understood in terms of the condition 
\( \alpha = -\frac{3}{4} \Lambda \) in the following way: While the cosmological constant \( \Lambda \) introduces an infrared cut-off (the length scale \( 1/\sqrt{|\Lambda|} \)) where the cosmological term dominates over the Einstein-Hilbert term, the Gauss-Bonnet term introduces an ultraviolet cut-off (the length scale \( \sqrt{\alpha} \)) where the quadratic terms dominate. Therefore, the condition \( \alpha = -\frac{3}{4} \Lambda \) basically states that in Chern-Simons theory both length scales are of the same order, and consequently there is no range where the Einstein-Hilbert term is the leading one. This explains why there is no range where (43) approaches Schwarzschild-Tangherlini solution. This asphyxia of the Einstein-Hilbert term is a typical feature of Chern-Simons theories for \( D > 3 \), where a unique free parameter \( l^2 \) appears in the action.

The Hawking temperature associated to black hole solution (43) is given by

\[
T = \frac{h}{8\alpha\pi} r_+ = \frac{h}{6\pi} |\Lambda| r_+,
\]

which in turn agrees with (33), although now it corresponds to a spherically symmetric solution. As it is well known \([146, 145]\) in \( D = 3 \) formula (44) agrees with the area law.

Certainly, solution (43) is reminiscent of the Bañados-Teitelboim-Zanelli three-dimensional black hole (BTZ), which, after all, also corresponds to a CS black hole. In fact, this is not a coincidence, and regarding this, let us make a historical remark: It turns out that, even though one could imagine that CS black holes (43) were discovered as higher-dimensional extensions of the BTZ, the story was precisely the opposite: In 1992, Bañados, Teitelboim and Zanelli discovered the BTZ as a particular case of a family of Lovelock black holes they were studying at that time \([147, 148, 149]\).

The analogy between the BTZ black hole and those solutions for higher-dimensional CS theories was discussed in detail in \([94]\). In particular, it was emphasized there that five-dimensional solution (43) shares several properties with its three-dimensional analogue. For instance, it is the case of their thermodynamics properties, which, after all, are actually encoded in the function \( V^2(r) \). This is also why all CS black holes have infinite lifetime.

Notice that the parameter \( \mathcal{M} \) in Eq. (43) plays the role that the mass \( M \) plays in the BTZ solution. Also, as in the three-dimensional case, the Anti-de Sitter space is obtained for a particular value of this parameter, namely \( \mathcal{M} = -1 \), and a naked singularity is developed for the range \(-1 < \mathcal{M} < 0\).

In \([93]\) the CS black holes and their dimensional extensions were exhaustively studied, together with their topological and charged extensions. There, a very interesting class of black holes was found by considering the particular choice of coefficients that leads to the \((2t + 1)\)-dimensional CS theory, but dimensionally extending the action from \( D = 2t + 1 \) to \( D \geq 2t + 1 \). The metrics of such solutions are given by replacing the constant \( \mathcal{M} \) in (43) by the quantity \( 1 - \mathcal{M} r^{(2t+1-D)/t} \). A further generalization of the solutions of \([93]\) would be given by adding a volume term to the gravitational action, which in turn corresponds to shifting the coupling \( \alpha_0 \to \alpha_0 + \delta \Lambda \) but keeping the rest of \( \alpha_n > 0 \) tuned as they are in the \((2t + 1)\)-dimensional CS theory, given in terms of the length scale \( l^2 \). The solution for this case is given by replacing the constant \( \mathcal{M} \) in (43) by a term \( 1 - (r^{2t} + \lambda r^{2t} + \mathcal{M} r^{2t+1-D})^{1/t} / l^2 \), where \( \lambda + 1 \sim \delta \Lambda / \alpha_0 \). These black holes do have a GR limit since now the cosmological length scale can be pushed away by.
choosing $\delta A$ appropriately.

It is also important to mention that black hole solution (43) is also a solution of the CS theory with torsion \[150, 151, 152\].

The solutions of Chern-Simons theory are very special ones, and this is due to the fact that for that specific choice of the coupling constants $\alpha_n$ the equations of motion of Lovelock theory somehow degenerate. In particular, it is remarkable that the obstruction imposed by Birkhoff-like theorems does not hold for CS theories.

A word on spinning black holes

Before concluding this section, a word on the spinning black hole case: The problem of finding a rotating solution in Einstein-Gauss-Bonnet gravity, which would generalize the Kerr’s spinning black hole of GR, is a hard and still unsolved problem. Recently, it was proven in [153] that the Kerr-Schild ansatz does not work in Lovelock theory (except for very special cases as Einstein theory and Chern-Simons theory), and this manifestly shows how difficult this classical problem is.

Nevertheless, despite the difficulty, some advances in this area were recently achieved: In [153] an exact analytic rotating solution was found for Chern-Simons gravity in five dimensions. This Einstein-Gauss-Bonnet solution, however, does not present a horizon, and thus it does not represent a black hole. Nevertheless, the numerical analysis of [154] supports the idea that rotating solutions actually exist. Besides, approximated analytic solutions at first order in the angular momentum parameter were found in [155]. Other solutions are known which represent rotating flat branes; these are a simple extension of topological black holes with $k = 0$.

Despite these recent advances, the problem of finding an exact analytic rotating black hole solution in Lovelock theory still remains an open problem.

3 Including boundary terms

In this section, we will discuss other constructions which, locally, coincide with the Deser-Boulware spherically symmetry metric.

Wormholes

The next class of solutions we would like to discuss is a class of vacuum solutions of Lovelock theory which represents wormhole geometries that connect two disconnected asymptotic regions of the space-time. Recently, several examples of such solutions were found [156, 157, 158, 159, 160], describing vacuum wormholes with different asymptotic behaviors, and in different number of dimensions. So, the first question we might ask is: why do wormholes exist in Lovelock theory?

The main reason why vacuum wormholes exist in a theory like (16) is actually simple, and it can be heuristically explained as follows: Consider the equations of motion corresponding to
Lagrangian (16), which can be always written as
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = 0
\]  
(45)
where the higher order terms act as an effective stress tensor that here we denoted \( T_{\mu\nu} \). In the case of EGB theory it reads
\[
\frac{1}{\alpha} T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left( R_{\rho\sigma\alpha\beta} R^{\rho\sigma\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) - 2 R R_{\mu\nu} + 4 R_{\mu\rho} R^\rho_{\nu} + R_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} - 2 R_{\mu\alpha\beta\rho} R^{\alpha\beta\rho}_{\nu},
\]
where, as usual, \( \alpha = \alpha_2/\alpha_1, 2\Lambda = -\alpha_0/\alpha_1 \). The key point is that this effective stress tensor \( T_{\mu\nu} \), thought of as a kind of matter contribution, can be shown to violate the energy conditions for \( \alpha \) large enough. Actually, this does not represent an actual problem from the conceptual point of view since this "matter" is actually made of pure gravity. However, a consequence of this violation of the energy conditions is that Eqs. (45) allow the existence of vacuum wormhole solutions at scales of order \( \sqrt{\alpha} \), unlike the case of GR, where solutions of this sort require the consideration of exotic matter.

Furthermore, there is a second reason for such curious solutions to exist in Lovelock theory. As mentioned above, when the coefficients \( \alpha_n \) in (16) correspond to the CS theory, the space of solutions experiments an unusual enhancement, which translates into a large degeneracy of the metric of spaces with enough symmetry. Roughly speaking, for such particular cases, Lovelock theory is somehow degenerated enough to admit metric with very special properties, and wormholes are some of them.

Nevertheless, here we will focus our attention on wormhole solutions that exist in five-dimensional EGB theory without requiring the coefficients \( \Lambda \) and \( \alpha \) to be those that correspond to CS theory. Therefore, the existence of such solutions, regarded as an anomaly, is ultimately attributed to the issue of the energy conditions mentioned above.

**Junction conditions**

The particular configurations we will consider are the so-called thin-shell wormholes, which correspond to connecting two regions of the space through a codimension-one hypersurface that plays the role of the wormhole throat. For such a geometry to be constructed, we have to make use of the junction conditions of the EGB theory [55, 60]. In particular, we will consider the configuration of two Boulware-Deser spaces connected through a hypersurface on which the induced stress-tensor vanishes. Such geometries are not possible in GR, where wormholes require the energy conditions to be violated on the thin-shell. However, in Lovelock theory, and because of the higher order terms, spherically symmetric vacuum wormholes with positive mass can be constructed, as shown by Gravanis and Willison in [59]. Let us review the procedure here.

Let \( \Sigma \) be a four-dimensional timelike orientable hypersurface of codimension one, whose normal vector is denoted by \( n^\mu \). Suppose \( \Sigma \) separates two regions of the space, which we call \( \mathcal{M}_I \) and \( \mathcal{M}_{II} \). Then, junction conditions read
\[
\langle K_{ij} - K h_{ij} \rangle_\Sigma + 2\alpha \left( 3 J_{ij} - J h_{ij} + 2 P_{ijkl} K^{kl} \right)_\Sigma = 8\pi S_{ij}
\]
(46)
where $\langle X \rangle_{\Sigma}$ denotes the jump of the quantity $X$ across the hypersurface $\Sigma$, which means $\langle X \rangle_{\Sigma} = X_{\mid I} \pm X_{\mid I}$, where the sign $\pm$ depends on the relative orientation of the regions. Above, tensor $S_{ij}$ represents the induced stress-tensor on the hypersurface $\Sigma$, in complete analogy with the Israel junction conditions in Einstein theory. In fact, we see that the first two terms in (46) actually correspond to the Israel junction conditions constructed with the extrinsic curvature $K_{\ell}^{\hat{\ell}}$ and its trace $K$. In addition, the junction conditions corresponding to the EGB theory contains contributions cubic in the extrinsic curvature $K$, which is:

$$J_{ij} = \frac{1}{3}(K_{kl}K^{kl}K_{ij} + 2KK_{i\ell}K_{j}^{\ell} - K^2K_{ij} - 2K_{ik}K^{kl}K_{lj}),$$  \hspace{1cm} (47)$$

and also contributions that involve the Riemann curvature tensor of the hypersurface $\mathcal{P}_{ijkl} = R_{ijkl} + R_{jk\hat{i}l} - R_{jl\hat{i}k} + R_{il\hat{j}k} - R_{ik\hat{j}l} + \frac{1}{2}Rh_{ik}h_{jl} - \frac{1}{2}Rh_{il}h_{jk}$. \hspace{1cm} (48)

The notation used here is such that latin indices $i, j, k, l$ refer to coordinates on the four-dimensional hypersurface that separate the two five-dimensional regions of the space. The induced metric is denoted by $h_{ij}$. It is worth mentioning that in Ref. [60] the junction conditions were studied in the most general case, including the case of space-like junctures, which corresponds to a cosmological-type geometries that experiment a change of behavior at a given time characterized by the hypersurface $\Sigma$. It was pointed out by H. Maeda that this kind of space-like junction conditions could be used to construct regular black hole solutions by means of geometric surgery procedure inside the black hole horizon.

Here we will be mainly concerned with static spherically symmetric geometries, and, besides, with spherically symmetric boundary conditions. That is, we will consider the time-like hypersurface $\Sigma$ that separates the two regions of the space to be located at fixed radial coordinate $r = a(\tau)$, and the system of coordinates we will parameterize the three angular directions $\phi_1, \phi_2, \phi_3$ of the junction hypersurface, and the proper time $\tau$ of an observer on $\Sigma$.

Then, we introduce the metrics

$$ds^2_{I,II} = -K_{I,II}^2(r)dt^2 + K_{I,II}^{-2}(r)dr^2 + r^2d\Sigma_3^2,$$

on each region $\mathcal{M}_I$ and $\mathcal{M}_{II}$, and the two regions join at $r = a(\tau)$. Since here we consider vacuum solutions, $K_I^2(r)$ and $K_{II}^2(r)$ are given by (27) (or by (21) in the case $k = 1$). In general, there is no reason for the mass parameters $M_{I,II}$ of the two regions to be equal, and the same happens with the choice of the branches $\sigma_{I,II} = \pm 1$. Moreover, the orientation of $\mathcal{M}_I$ and that of $\mathcal{M}_{II}$ with respect to the normal vector $n^\mu$ are also independent one on each other, and we will take this degree of freedom into account by introducing the variables $\eta_I$ and $\eta_{II}$ which indicate whether in each region the radial coordinate $r_{I,II}$ is parallel ($\eta_{I,II} = 1$) or anti-parallel ($\eta_{I,II} = -1$) to $n^\mu$. Therefore, the anti-parallel geometry corresponds to the orientation $\eta_I\eta_{II} = -1$, while the standard shell-like geometry corresponds to the case $\eta_I\eta_{II} = +1$. The freedom in choosing the parameters $M, \sigma, \eta$ independently in each region allows for a wide class of solutions. The whole catalog was recently studied in [60].

\footnote{See [161] for a recent review. See also [162] where boundary terms in odd-dimensions are discussed.}
Figure 1: Einstein-Rosen bridge geometry as a vacuum solution.

The metric on $\Sigma$ induced from region $\mathcal{M}_I$ is the same as the one induced from region $\mathcal{M}_{II}$, and is given by

$$ds^2 = -d\tau^2 + a_I^2 d\Sigma_3^2,$$

(50)

where, according to (27), $d\Sigma_3^2$ will be chosen to be the line element of a 3-manifold with (intrinsic) curvature $k = +1, -1, 0$, i.e. it is a unit sphere, a hyperboloid or flat space respectively. The hypersurface $\Sigma$ is the world-volume of the juncture where regions $\mathcal{M}_I$ and $\mathcal{M}_{II}$ join.

To see whether such a wormhole-like (or shell-like) configuration is possible in vacuum, we have to solve junction conditions (46) with $S_i = 0$. To do this we first need to compute the components of the intrinsic curvature. These are given by

$$K_{\phi_i} = \frac{1}{a} \left( (V^2(a) + (\partial_\tau a)^2)^{1/2} \right), \quad K_\tau = (\partial_\tau^2 a + \frac{1}{2} \partial_\tau V^2(a))(V^2(a) + (\partial_\tau a)^2)^{-1/2}$$

with $i = 1, 2, 3$. This also yields

$$3J_\tau^\phi - J = \frac{2}{a^3} (V^2(a) + (\partial_\tau a)^2)^{3/2}, \quad 3J_\phi^\phi - J = \frac{2}{a^2} (V^2(a) + (\partial_\tau a)^2)^{1/2} (\partial_\tau^2 a + \frac{1}{2} \partial_\tau V^2(a)).$$

On the other hand, the components of Riemann tensor $R^{ij}_{\tau \phi}$ and those of $P^{ij}_{\tau \phi}$ are

$$R^{\phi_i \phi_j}_{\tau \phi_i} = P^{\phi_i \phi_j}_{\phi_i} = \frac{1}{a^2} \partial_\tau a, \quad R^{\phi_i \phi_j}_{\phi_i \phi_j} = P^{\phi_i \tau}_{\phi_i \tau} = \frac{1}{a^2} (1 + (\partial_\tau a)^2).$$

Putting all this together, we can evaluate the junction conditions (46) in vacuum. The two independent equations read

$$(\eta_I V_I(a_0) - \eta_{II} V_{II}(a_0))(a_0^2 + \frac{4\alpha}{3} (3k - V_I^2(a_0) - V_{II}^2(a_0) - \eta_I \eta_{II} V_I(a_0) V_{II}(a_0))) = 0,$$

$$J_\phi^\phi - J = 0.$$
\[(\eta_I V_I^{-1}(a_0) - \eta_{III} V_{III}^{-1}(a_0))(k - \frac{\Lambda a_0^2}{3} - \eta_I \eta_{III} V_I(a_0) V_{III}(a_0)) = 0.\]

For the wormhole orientation, \(\eta_I \eta_{III} = -1\), and for the symmetric case \(V_I^2(a) = V_{III}^2(a)\), these equations take the simple form

\[V^2(a_0) = \frac{3}{4\alpha} a_0^2 + 3k, \quad V^2(a_0) = \frac{\Lambda a_0^2}{3} - k.\] (51)

From these equations we see that the radius of the throat of the wormhole is given by

\[a_0^2 = \frac{12\alpha k}{\alpha\Lambda - 9/4},\] (52)

and from this we can also calculate the mass of the wormhole easily. Eq. (52) implies that, in the case of the spherically symmetric wormhole \((k = 1)\), we need \(\Lambda\alpha > 9/4\) for the wormhole to exist. Then, provided \(\Lambda\alpha\) is of order one, the radius of the wormhole throat is or order \(a_0 \sim \sqrt{\alpha}\). Besides, we should ask the throat to be located outside the horizon, namely \(a_0 > r_+\).

It is remarkable that all these conditions can be satisfied [60] for positive values of \(\alpha, k, M\), and \(\Lambda\). However, it is worth mentioning that spherically symmetric wormhole solutions only exist if at least one of the two regions \(\mathcal{M}_{I,III}\) corresponds to the branch \(\sigma = +1\) in (21).

More remarkable is the fact that the analysis of the dynamic case \(a = a(\tau)\) follows straightforward. When \(\partial_\tau a \neq 0\), equations \(S_\tau = 0\) and \(S_{a_0}^i = 0\) are not linearly independent, and it is sufficient to solve the first of them. Considering \(k = 1\), we get

\[(\partial_\tau a)^2 + W(a) = 0 \quad \text{with} \quad W(a) = \frac{1}{4\alpha} a^2 - \frac{\sigma}{2} \sqrt{\frac{a^4}{16\alpha^2}(1 + 4\Lambda\alpha/3) + \frac{MG}{\alpha}} + 1,\] (53)

which has the form of a one-dimensional dynamic equation of motion constrained by the vanishing energy condition. Notice that (51) is recovered by demanding \(W(a) = 0\). Notice also that the effective potential \(W(a)\) has negative derivative, and for large values of \(a\) it goes like \(W(a) \approx a^2(2 - \sqrt{(1 + 4\Lambda\alpha/3)})/8\alpha < 0\). The effective potential \(W(a)\) can be positive and of positive derivative for non-symmetric wormhole configurations.

Summarizing, we have just seen that spherically symmetric (microscopic) thin-shell wormholes in vacuum are admitted as solutions of the five-dimensional Lovelock theory. These solutions are allowed by additional terms arising in the junction conditions of the EGB theory.

It is worth mentioning that the one we discussed here is not the only class of wormhole-like solutions that exists in Lovelock theory. For instance, in [157, 160] a static wormhole solution for gravity in vacuum was found for CS gravity in arbitrary (odd) dimensions \(D = 2t + 1 \geq 5\). This wormhole connects two asymptotic regions whose respective boundaries are locally given by \(R \times S^1 \times H_{d-3}\).

Besides, \(D\)-dimensional static wormhole solutions of the EGB theory were also studied in [158], and explicit wormhole solutions respecting the energy conditions in the whole spacetime were found for the case \(\alpha > 0\). The asymptotic behavior of these solutions is given by \(R \times H_{d-2}\).
Naked singularities

As we have seen in the previous sections, there are many features of Lovelock solutions that are not present in GR. Eternal black holes and wormholes are remarkable examples. Another example is the existence of positive mass solutions with naked singularities. In fact, naked singularities appear in all the catalog of solutions, for both spherically symmetric and extended objects, for both solutions with a suitable GR limit and solutions without it. But, what kind of naked singularities are these? For instance, we could ask whether these are stable under gravitational perturbations; or whether these turn out to be "bad" singularities when probed with wave functions.

Regarding the question about the stability, this issue was studied recently within the framework of the Kodama-Ishibashi formalism, and some evidence of instabilities was found. On the other hand, here we will address the second question, the one about how these naked singularities look like when analyzed with quantum probes. To do this we will employ the method developed by Horowitz and Marolf in Ref. [168], based on the pioneer work of Wald [169]. The basic idea is the following: Unlike what happens in the classical regime, where a singular space is defined by the concept of geodesic incompleteness, in the quantum mechanical regime the singular character of the space-time is defined in terms of the ambiguity in the definition of the Hamiltonian evolution of wave functions on it. More specifically, the singular nature of a given space is determined in terms of the ambiguity when trying to find a self-adjoint extension of the Hamiltonian operator to the whole space. When such self-adjoint extension exists and is unique, then it is said that the space is quantum mechanically regular, in spite of the singularities it might present at classical level. Notice that this is not matter of deforming the space or somehow resolving it, but it is rather a reconsideration of what is the relevant physical dynamics on it. In fact, a space can be classically singular but still regular when it is analyzed with quantum probes.

Here, we will apply the concept of quantum probes to the singular solutions of Lovelock theory discussed above. But, first, let us review the method developed in [168, 170]. Consider the quantum dynamics of a scalar field \( \varphi \) on the spherically symmetric space (20), which is governed by the Klein-Gordon equation

\[
(\nabla_\mu \nabla^\mu - m^2 - 2\xi R) \varphi = 0.
\]

This equation can be written as follows

\[
\partial_t^2 \varphi + \mathcal{H}^2 \varphi = 0, \quad \text{with} \quad \mathcal{H}^2 = -V(r) \nabla^i \left( V(r) \nabla_i \varphi \right) + V(r)^2 m^2 \varphi + 2V(r)^2 \xi R \varphi
\]

where \( \nabla^i \) is the covariant derivative on the spacelike hypersurfaces defined by constant \( t \) foliations, and where the metric function \( V^2(r) \) is given by (21). The piece \( V(r)^2 \nabla^i \left( V(r) \nabla_i \varphi \right) \) in (55) involves the Laplacian operator on the unitary 3-sphere, whose eigenvalues are known to be given by \(-l(l+2)\) with positive integers \( l = 0, 1, 2, 3, ... \). Now, equation (55) can be written in its Schrödinger-like form, schematically,

\[
i\partial_t \varphi = \mathcal{H} \varphi.
\]

\[25\] For a discussion on the formation of naked singularities, see [163, 164].
and then the problem to deal with is to decide whether the Hamiltonian operator $H$ admits a unique self-adjoint extension in spite of the fact the space is singular at the origin $r = 0$. As mentioned, in the quantum mechanical context the existence of singularity is associated to the non-existence of a unique self-adjoint extension of the Hamiltonian operator rather than to a geodesical completeness. Then, the problem of determining whether the space is regular is translated into the problem of verifying whether $H^2$ admits a unique self-adjoint extension $H^2_E$. If such extended operator exists, then the Hamiltonian evolution of the wave function in this space would be given by

$$\varphi(t) = \exp(-it H_E) \varphi(0),$$

and it would be well-defined.

It turns out that a sufficient condition for $H^2_E$ to exist and be unique is that at least one of the solutions of the differential equation

$$\partial_r^2 \phi(r) + \partial_r \log \left(r^3 V(r)^2 \partial_r \phi(r) - V_r^{-2} \left(r^{-2} l(l + 2) - m^2 + \xi R \pm iV_r^{-2} \right) \phi(r)\right) = 0 \quad (56)$$

fails to be of finite norm near the origin for any value of $l$ and for any of the two possible signs $\pm$ in $(56)$; see [168] for details. In other words, for the space to be considered regular quantum mechanically it is necessary to see that at least one solution $\phi$ to $(56)$ is non-normalizable around the origin. This criterion strongly depends on which norm $||\phi||$ is considered.

The well-posedness of an initial value problem requires not only the existence and uniticity of conditions, but also continuous dependence of solutions on initial data. Then, the norm $||\phi||$ to be considered should select a function space that fulfills these requirements. A sensitive norm in this sense is the Sobolev norm [171].

To see how the method works in the case we are interested in, let us consider again the five-dimensional Boulware-Deser space (20)-(21). The branch $\sigma = +1$ of this space presents a naked singularity for all positive values of $M$, while the branch $\sigma = -1$ only presents naked singularities within the range $0 < M < \alpha$. Then, let us solve the wave equation for these spaces. To analyze the solutions of $(56)$ near the singular point $r = 0$ it is convenient to write this equation as

$$\partial_r^2 \phi + r^{-1} p(r) \partial_r \phi + r^{-2} q(r) \phi = 0,$$

with $p(r)$ and $q(r)$ being two functions analytic at the origin. This is a Fuchsian equation and so it admits solutions with the form $\phi(r) = r^\eta f(r)$ for certain analytic function $f(r)$ and a complex number $\eta$ that is known to solve the indicial equation

$$\eta^2 + (p(r=0) - 1)\eta + q(r=0) = 0.$$ 

Then, replacing $(21)$ in $(56)$ we find

$$p(r=0) = 3, \quad q(r=0) = -l(l + 2)/(1 + \sigma \sqrt{M/\alpha}),$$

and two independent solutions to $(56)$ are then given by the two values of $\eta$ that solve $(\eta + 1)^2 = 1 + l(l + 2)/(1 + \sigma \sqrt{M/\alpha})$. Therefore, we find that one of the solutions to $(56)$ always diverges at least as rapidly as $|\phi|^2 \simeq r^{-2}$, and so it fails to be integrable with respect to the Sobolev norm.

Summarizing, there exists a unique self-adjoint extension $H^2_E$, from what we conclude that five-dimensional Boulware-Deser metric turns out to be regular when tested by quantum probes. It is remarkable that the positive (but small) mass solutions of five-dimensional black holes are in a sense regular quantum mechanically, despite the naked curvature singularity they exhibit at the origin.

Before concluding, we wish to make a remark about the consistency of studying naked singularities in this way. Actually, one could wonder whether probing naked singularities in
a theory with a finite higher curvature expansion makes sense or not. For instance, in string
inspired models, as soon as one approaches the singularity, neglecting higher order corrections
seems to be impossible since higher and higher order terms start to dominate as we go close
enough to the singularity. However, let us argue here that, even though this is true, this
is not necessarily an obstruction for testing singularities with quantum probes up to certain
order in the higher curvature expansion. Let us be reminded of what we do when we solve
the Schrödinger equation for the Coulombian potential (e.g. In fact, the analogy with the
hydrogen atom in quantum mechanics is quite good since such problem also corresponds to
solving a wave function equation in presence of a central potential whose classical counterpart
breaks down at the origin). In quantum mechanics, even though the Coulombian potential
diverges at the origin, we know that the quantum problem still makes sense, and we do solve
the wave equation without complaining about the fact that other corrections to the potential
(e.g. effective screening due to quantum effects, or short distance corrections to the Coulombian
potential) could in principle appear at very short distances. Heuristically speaking, what one
really has to do to make sure the whole procedure makes sense is comparing the typical size
of the wave packet with the length scale where the terms that were neglected would dominate.
For example, above we were dealing with the EGB action, and the terms $R^3$ were certainly
neglected, and so the analysis carried out could still make sense as long as the Compton length
of the wave packet is small enough in comparison with the length scale imposed by the coupling
constant $\alpha_n$ with $n > 2$, and provided the fact higher curvature terms act as a perturbation.

For some particular models where the couplings $\alpha_n$ are given in terms of the same funda-
mental scale (like the models inspired in string theory where the scale is given by $l_s^2 \sim \alpha'$) the
story could be a little more subtle, and so the argument above would not be valid in general.
Nevertheless, it is likely the case that higher order terms would contribute by smoothing out
the singularity even more, although not necessarily resolving it in a classical sense.

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References

[1] D. Lovelock, *The Einstein tensor and its generalizations*, J. Math. Phys. 12 (1971) 498.
[2] D. Lovelock, *The four-dimensionality of space and the einstein tensor*, J. Math. Phys. **13** (1972) 874.

[3] D. Gross and E. Witten, *Superstrings modifications of Einstein’s equations*, Nucl. Phys. **B277** (1986) 1.

[4] H. Schmidt, *Fourth order gravity: Equations, history, and applications to cosmology*, Int. J. Geom. Meth. Mod. Phys. **4** (2007) 209, [arXiv:gr-qc/0602017](http://arxiv.org/abs/gr-qc/0602017).

[5] S. Deser and B. Tekin, *Shortcuts to high symmetry solutions in gravitational theories*, Class. Quant. Grav. **20** (2003) 4877, [arXiv:gr-qc/0306114](http://arxiv.org/abs/gr-qc/0306114).

[6] R. Jackiw and S.-Y. Pi, *Chern-Simons modification of General Relativity*, Phys. Rev. D68 (2003) 104012, [arXiv:gr-qc/0308071](http://arxiv.org/abs/gr-qc/0308071).

[7] S. Alexander and N. Yunes, *Chern-Simons Modified General Relativity*, to appear in Phys. Repts., [arXiv:0907.2562](http://arxiv.org/abs/0907.2562).

[8] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. B443 (1995) 85, [arXiv:hep-th/9503124](http://arxiv.org/abs/hep-th/9503124).

[9] A. Tseytlin, *$R^4$ terms in 11 dimensions and conformal anomaly of (2,0) theory*, Nucl. Phys. **B584** (2000) 233, [arXiv:hep-th/0005072](http://arxiv.org/abs/hep-th/0005072).

[10] A. Gruzinov and M. Kleban, *Causality Constrains Higher Curvature Corrections to Gravity*, [arXiv:hep-th/0612015](http://arxiv.org/abs/hep-th/0612015).

[11] I. Antoniadis, S. Ferrara, R. Minasian and K. Narain, *$R^4$ Couplings in M and Type II Theories*, Nucl. Phys. **B507** (1997) 571, [arXiv:hep-th/9707013v2](http://arxiv.org/abs/hep-th/9707013v2).

[12] S. Ferrara, R. Khuri and R. Minasian, *M-Theory on a Calabi-Yau Manifold*, Phys. Lett. **B375** (1996) 81, [arXiv:hep-th/9602102](http://arxiv.org/abs/hep-th/9602102).

[13] M. Guica, L. Huang, W. Li and A. Strominger, *$R^2$ Corrections for 5D Black Holes and Rings*, JHEP **0610** (2006) 036, [arXiv:hep-th/0505188](http://arxiv.org/abs/hep-th/0505188).

[14] P. Kanti, N. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, *Dilatonic Black Holes in Higher Curvature String Gravity*, Phys. Rev. **D54** (1996) 5049, [arXiv:hep-th/9511071](http://arxiv.org/abs/hep-th/9511071).

[15] T. Torii, H. Yajima and K.I. Maeda, *Dilatonic Black Holes with Gauss-Bonnet Term*, Phys. Rev. **D55** (1997) 739, [arXiv:gr-qc/9606034](http://arxiv.org/abs/gr-qc/9606034).

[16] S. Alexeyev and M. Pomazanov, *Black hole solutions with dilatonic hair in higher curvature gravity*, Phys. Rev. **D55** (1997) 2110, [arXiv:hep-th/9605106](http://arxiv.org/abs/hep-th/9605106).

[17] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231-252, [arXiv:hep-th/9711200](http://arxiv.org/abs/hep-th/9711200).
[18] P. Kovtun, D. T. Son and A. O. Starinets, Holography and hydrodynamics: diffusion on the stretched horizons, JHEP 0310 (2003) 064, [arXiv:hep-th/0309213].

[19] P. Kovtun, D. T. Son and A. O. Starinets, Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics, Phys. Rev. Lett. 94 (2005) 111601, [arXiv:hep-th/0405231].

[20] Y. Kats and P. Petrov, Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory, JHEP 0901 (2009) 044, [arXiv:0712.0743].

[21] M. Brigante, H. Liu, R. Myers, S. Shenker and S. Yaida, The Viscosity Bound and Causality Violation, [arXiv:0802.3318].

[22] M. Brigante, H. Liu, R. Myers, S. Shenker and S. Yaida, Viscosity Bound Violation in Higher Derivative Gravity, [arXiv:0712.0805].

[23] D. Hofman, Higher Derivative Gravity, Causality and Positivity of Energy in a UV complete QFT, [arXiv:0907.1625].

[24] A. Buchel and R. Myers, Causality of Holographic Hydrodynamics, [arXiv:0906.2922].

[25] R. Gregory, S. Kanno and J. Soda, Holographic Superconductors with Higher Curvature Corrections, [arXiv:0907.3203].

[26] A. Adams, A. Maloney, A. Sinha and S. E. Vazquez, 1/N Effects in Non-Relativistic Gauge-Gravity Duality, JHEP 0903 (2009) 097, [arXiv:0812.0166].

[27] D. T. Son, Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry, Phys. Rev. D78 (2008) 046003, [arXiv:arXiv:0804.3972].

[28] K. Balasubramanian and J. McGreevy, Gravity duals for non-relativistic CFTs, Phys. Rev. Lett. 101 (2008) 061601, [arXiv:0804.4053].

[29] C. Lanczos, A Remarkable property of the Riemann-Christoffel tensor in four dimensions, Ann. Math. 39 (1938) 842.

[30] C. Callan, I. Klebanov and M. Perry, String Theory Effective Actions, Nucl. Phys. B278 (1986) 78.

[31] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Vacuum Configurations for Superstrings, Nucl. Phys. B258 (1985) 46.

[32] D. Gross and J. Sloan, The Quartic Effective Action for the Heterotic String, Nucl. Phys. B291 (1987) 41.

[33] B. Zwiebach, Curvature Squared Terms and String Theories, Phys. Lett. B156 (1985) 315.
[34] D. Boulware and S. Deser, *String Generated Gravity Models*, Phys. Rev. Lett. **55** (1985) 2656.

[35] C. Callan, R. Myers and M. Perry, *Black Holes in String Theory*, Nucl. Phys. **B311** (1989) 673.

[36] R. Myers, *Black Holes in Higher Curvature Gravity*, [arXiv:gr-qc/9811042](https://arxiv.org/abs/gr-qc/9811042).

[37] R. Myers, *Superstring Gravity and Black Holes*, Nucl. Phys. **B289** (1987) 701.

[38] F. Moura and R. Schiappa, *Higher-Derivative Corrected Black Holes: Perturbative Stability and Absorption Cross-Section in Heterotic String Theory*, Class. Quant. Grav. **24** (2007) 361, [arXiv:hep-th/0605001](https://arxiv.org/abs/hep-th/0605001).

[39] I. Bena and P. Kraus, *R* \(^2\) Corrections to Black Ring Entropy, [arXiv:hep-th/0506015](https://arxiv.org/abs/hep-th/0506015).

[40] Y. Kats, L. Motl and M. Padi, *Higher-order corrections to mass-charge relation of extremal black holes*, JHEP **0712** (2007) 068, [arXiv:hep-th/0606100](https://arxiv.org/abs/hep-th/0606100).

[41] A. Sen, *Entropy Function for Heterotic Black Holes*, JHEP **0603** (2006) 008, [arXiv:hep-th/0508042](https://arxiv.org/abs/hep-th/0508042).

[42] N. Dadhich, *On the derivation of the gravitational dynamics*, [arXiv:0802.3034](https://arxiv.org/abs/0802.3034).

[43] F. Müller-Hoissen, *Spontaneous Compactification With Quadratic And Cubic Curvature Terms*, Phys. Lett. **B163** (1985) 106.

[44] R. Metsaev and A. Tseytlin, *Curvature cubed terms in string theory effective actions*, Phys. Lett. **B185** (1987) 52.

[45] R. Troncoso and J. Zanelli, *Higher Dimensional Gravity, Propagating Torsion and AdS Gauge Invariance*, Class. Quant. Grav. **17** (2000) 4451.

[46] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities*, 2nd edition, [arXiv:hep-th/0502193](https://arxiv.org/abs/hep-th/0502193).

[47] Deser, *First-order Formalism and Odd-derivative Actions*, Class. Quant. Grav. **23** (2006) 5773, [arXiv:gr-qc/0606006](https://arxiv.org/abs/gr-qc/0606006).

[48] R. Aros, R. Troncoso and J. Zanelli, *Black holes with topologically nontrivial AdS asymptotics*, Phys. Rev. **D63** (2001) 084015, [arXiv:hep-th/0011097](https://arxiv.org/abs/hep-th/0011097).

[49] J. Wheeler, *Symmetric Solutions to the Gauss-Bonnet Extended Einstein Equations*, Nucl. Phys. **B268** (1986) 737.

[50] J. Wheeler, *Symmetric Solutions To The Maximally Gauss-Bonnet Extended Einstein Equations*, Nucl. Phys. **B273** (1986) 732.
[51] D. Wiltshire, *Black Holes in String-Generated Gravity Models*, Phys. Rev. **D38** (1988) 2445.

[52] D. Wiltshire, *Spherically Symmetric Solutions Of Einstein-Maxwell Theory With A Gauss-Bonnet Term*, Phys. Lett. **B169** (1986) 36.

[53] Rong-Gen Cai, *Gauss-Bonnet Black Holes in AdS Spaces*, Phys. Rev. **D65** (2002) 084014, arXiv:hep-th/0109133.

[54] T. Jacobson, R. Myers, *Entropy of Lovelock Black Holes*, Phys. Rev. Lett. **70** (1993) 3684, arXiv:hep-th/9305016.

[55] R. Myers, *Higher Derivative Gravity, Surface Terms And String Theory*, Phys. Rev. **D36** (1987) 392.

[56] S. Davis, *Generalised Israel Junction Conditions for a Gauss-Bonnet Brane World*, Phys. Rev. **D67** (2003) 024030, arXiv:hep-th/0208205.

[57] E. Gravanis and S. Willison, *Israel conditions for the Gauss-Bonnet theory and the Friedmann equation on the brane universe*, Phys. Lett. **B562** (2003) 118, arXiv:hep-th/0209076.

[58] E. Gravanis and S. Willison, *Intersecting hypersurfaces in AdS and Lovelock gravity*, J. Math. Phys. **47** (2006) 2503, arXiv:hep-th/0412273.

[59] E. Gravanis and S. Willison, *‘Mass without mass’ from thin shells in Gauss-Bonnet gravity*, Phys. Rev. **D75** (2007) 084025, arXiv:gr-qc/0701152.

[60] C. Garraffo, G. Giribet, E. Gravanis and S. Willison, *Gravitational solitons and $C^\infty$ vacuum metrics in five-dimensional Lovelock gravity*, J. Math. Phys. **49** (2008) 042503, arXiv:0711.2992.

[61] Deser and A. Ryzhov, *Curvature invariants of static spherically symmetric geometries*, Class. Quant. Grav. **22** (2005) 3315, arXiv:gr-qc/0505039.

[62] G. Allemandi, M. Francaviglia and M. Raiteri, *Charges and Energy in Chern-Simons Theories and Lovelock Gravity*, Class. Quant. Grav. **20** (2003) 5103, arXiv:gr-qc/0308019.

[63] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Vacuum Energy in Odd-Dimensional AdS Gravity*, arXiv:hep-th/0412046.

[64] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Transgression forms and extensions of Chern-Simons gauge theories*, JHEP **0602** (2006) 067, arXiv:hep-th/0601081.

[65] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved Charges for Even Dimensional Asymptotically AdS Gravity Theories*, Phys. Rev. **D62** (2000) 044002, arXiv:hep-th/9912045.
[66] R. Aros, M. Contreras, Rodrigo Olea, R. Troncoso and J. Zanelli, *Conserved charges for gravity with locally AdS asymptotics*, Phys. Rev. Lett. 84 (2000) 1647, [arXiv:gr-qc/9909015].

[67] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite action principle for Chern-Simons AdS gravity*, JHEP 0406 (2004) 036, [arXiv:hep-th/0405267].

[68] M. Dehghani, N. Bostani and A. Sheykhi, *Counterterm Method in Lovelock Theory and Horizonless Solutions in Dimensionally Continued Gravity*, Phys. Rev. D73 (2006) 104013, [arXiv:hep-th/0603058].

[69] G. Kofinas and R. Olea, *Universal regularization prescription for Lovelock AdS gravity*, JHEP 0711 (2007) 069, [arXiv:0708.0782].

[70] H. Maeda and M. Nozawa, *Generalized Misner-Sharp quasi-local mass in Einstein-Gauss-Bonnet gravity*, Phys. Rev. D77 (2008) 064031, [arXiv:0709.1199].

[71] O. Miskovic and R. Olea, *Counterterms in Dimensionally Continued AdS Gravity*, [arXiv:0706.4460].

[72] S. Deser and B. Tekin, *Gravitational Energy in Quadratic Curvature Gravities*, Phys. Rev. Lett. 89 (2002) 101101, [arXiv:hep-th/0205318].

[73] A. Padilla, *Surface terms and the Gauss-Bonnet Hamiltonian*, Class. Quant. Grav. 20 (2003) 3129, [arXiv:gr-qc/0303082].

[74] N. Okuyama and J. Koga, *Asymptotically anti de Sitter spacetimes and conserved quantities in higher curvature gravitational theories*, Phys. Rev. D71 (2005) 084009, [arXiv:hep-th/0501044].

[75] S. Deser and B. Tekin, *Energy in generic higher curvature gravity theories*, Phys. Rev. D67 (2003) 084009, [hep-th/0212292].

[76] N. Deruelle, J. Katz and S. Ogushi, *Conserved charges in Einstein Gauss-Bonnet theory*, Class. Quant. Grav. 21 (2004) 1971, [arXiv:gr-qc/0310098].

[77] S. Deser and J. Franklin, *Birkhoff for Lovelock Redux*, Class. Quant. Grav. 22 (2005) L103, [arXiv:gr-qc/0506014].

[78] R. Zegers, *Birkhoff’s theorem in Lovelock gravity*, J. Math. Phys. 46 (2005) 072502, [arXiv:gr-qc/0505016].

[79] C. Charmousis and J. Dufaux, *General Gauss-Bonnet brane cosmology*, Class. Quant. Grav. 19 (2002) 4671, [arXiv:hep-th/0202107].

[80] V. Cardoso, O. Dias and J. Lemos, *Nariai, Bertotti-Robinson and anti-Nariai solutions in higher dimensions*, Phys. Rev. D70 (2004) 024002, [arXiv:hep-th/0401192].
[81] F. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, Nuovo Cimento 27 (1963) 636.

[82] D. Boulware and S. Deser, *Effective Gravity Theories with Dilatons*, Phys. Lett. B175 (1986) 409.

[83] A. Garbarz, G. Giribet and F.D. Mazzitelli, *Conformal invariance and apparent universality of semiclassical gravity*, Phys. Rev. D78 (2008) 084014, [arXiv:0807.4885](https://arxiv.org/abs/0807.4885).

[84] G. Dotti and R. Gleiser, *Gravitational instability of Einstein-Gauss-Bonnet black holes under tensor mode perturbations*, Class. Quant. Grav. 22 (2005) L1, [arXiv:gr-qc/0409005](https://arxiv.org/abs/gr-qc/0409005).

[85] G. Dotti and R. Gleiser, *Linear stability of Einstein-Gauss-Bonnet static spacetimes. Part I: tensor perturbations*, Phys. Rev. D72 (2005) 044018, [arXiv:gr-qc/0503117](https://arxiv.org/abs/gr-qc/0503117).

[86] M. Beroiz, G. Dotti and R. Gleiser, *Gravitational instability of static spherically symmetric Einstein-Gauss-Bonnet black holes in five and six dimensions*, [arXiv:hep-th/0703074](https://arxiv.org/abs/hep-th/0703074).

[87] G. Dotti and R. Gleiser, *Linear Stability of Einstein-Gauss-Bonnet Spacetimes, Part II: Vector and Scalar Perturbations*, Phys. Rev. D72 (2005) 124002, [arXiv:gr-qc/0510069](https://arxiv.org/abs/gr-qc/0510069).

[88] R. Konoplya and A. Zhidenko, *(In)stability of D-dimensional black holes in Gauss-Bonnet theory*, [arXiv:0802.0267](https://arxiv.org/abs/0802.0267).

[89] T. Hirayama, *Negative modes of Schwarzschild black hole in Einstein-Gauss-Bonnet Theory*, [arXiv:0804.3694](https://arxiv.org/abs/0804.3694).

[90] M. Dehghani and M. Shamirzaie, *Thermodynamics of Asymptotically Flat Charged Black Holes in Third Order Lovelock Gravity*, Phys. Rev. D72 (2005) 124015, [arXiv:hep-th/0506227](https://arxiv.org/abs/hep-th/0506227).

[91] M. Dehghani and N. Bostani, *Spacetimes with Longitudinal and Angular Magnetic Fields in Third Order Lovelock Gravity*, [arXiv:hep-th/0612103](https://arxiv.org/abs/hep-th/0612103).

[92] R. Cai and N. Ohta, *Black Holes in Pure Lovelock Gravities*, Phys. Rev. D74 (2006) 064001, [arXiv:hep-th/0604088](https://arxiv.org/abs/hep-th/0604088).

[93] J. Crisostomo, R. Troncoso and J. Zanelli, *Black Hole Scan*, Phys. Rev. D62 (2000) 084013, [arXiv:hep-th/0003271](https://arxiv.org/abs/hep-th/0003271).

[94] M. Aiello, R. Ferraro and G. Giribet, *Exact Solutions of Lovelock-Born-Infeld Black Holes*, Phys. Rev. D70 (2004) 104014, [arXiv:gr-qc/0408078](https://arxiv.org/abs/gr-qc/0408078).

[95] T. Torii and H. Maeda, *Spacetime structure of static solutions in Gauss-Bonnet gravity: charged case*, Phys. Rev. D72 (2005) 064007, [arXiv:hep-th/0504141](https://arxiv.org/abs/hep-th/0504141).
[96] T. Torii and H. Maeda, *Spacetime structure of static solutions in Gauss-Bonnet gravity: neutral case*, Phys. Rev. D71 (2005) 124002, [arXiv:hep-th/0504127].

[97] B. Hoffmann, *Gravitational and Electromagnetic Mass in the Born-Infeld Electrodynamics*, Phys. Rev. 47 (1935) 877.

[98] G. Dotti, J. Oliva and R. Troncoso, *Exact solutions for the Einstein-Gauss-Bonnet theory in five dimensions: Black holes, wormholes and spacetime horns*, [arXiv:0706.1830].

[99] G. Dotti and R. Gleiser, *Obstructions on the horizon geometry from string theory corrections to Einstein gravity*, Phys. Lett. B627 (2005) 174, [arXiv:hep-th/0508118].

[100] M. Dehghani, *Charged Rotating Black Branes in anti-de Sitter Einstein-Gauss-Bonnet Gravity*, Phys. Rev. D67 (2003) 064017, [arXiv:hep-th/0211191].

[101] M. Dehghani, *Asymptotically (anti)-de Sitter solutions in Gauss-Bonnet gravity with cosmological constant*, Phys. Rev. D70 (2004) 064019, [arXiv:hep-th/0405206].

[102] M. Dehghani and R. Mann, *Thermodynamics of Rotating Charged Black Branes in Third Order Lovelock Gravity and the Counterterm Method*, Phys. Rev. D73 (2006) 104003, [arXiv:hep-th/0602243].

[103] M. Dehghani, N. Alinejadia and S. Hendi, *Topological Black Holes in Lovelock-Born-Infeld Gravity*, [arXiv:0802.2637].

[104] M. Nozawa and H. Maeda, *Dynamical black holes with symmetry in Einstein-Gauss-Bonnet gravity*, Class. Quant. Grav. 25 (2008) 055009, [arXiv:0710.2709].

[105] S. Sarkar, S. Shankaranarayan and L. Sriramkumar, *Sub-leading contributions to the black hole entropy in the brick wall approach*, [arXiv:0710.2013].

[106] S. Hendi and M. Dehghani, *Taub-NUT Black Holes in Third order Lovelock Gravity*, [arXiv:0802.1813].

[107] M. Dehghani and R. Mann, *NUT-Charged Black Holes in Gauss-Bonnet Gravity*, Phys. Rev. D72 (2005) 124006, [arXiv:hep-th/0510083].

[108] M. Dehghani and S. Hendi, *Taub-NUT/Bolt Black Holes in Gauss-Bonnet-Maxwell Gravity*, Phys. Rev. D73 (2006) 084021, [arXiv:hep-th/0602069].

[109] S. Habib Mazharimousavi and M. Halilsoy, *5D black hole solution in Einstein-Yang-Mills-Gauss-Bonnet theory*, Phys. Rev. D76 (2007) 087501, [arXiv:0801.1562].

[110] S. Habib Mazharimousavi and M. Halilsoy, *Black Holes in Einstein-Maxwell-Yang-Mills Theory and their Gauss-Bonnet Extensions*, [arXiv:0801.2110].

[111] A. Aliev, H. Cebeci and T. Dereli, *Exact solutions in five-dimensional axi-dilaton gravity with Euler-Poincare term*, Class. Quant. Grav. 24 (2007) 3425, [arXiv:gr-qc/0703011].
[112] H. Maeda and N. Dadhich, Kaluza-Klein black hole with negatively curved extra dimensions in string generated gravity models, Phys. Rev. D74 (2006) 021501, [arXiv:hep-th/0605031].

[113] A. Molina and N. Dadhich, On Kaluza-Klein spacetime in Einstein-Gauss-Bonnet gravity, [arXiv:0804.1194].

[114] N. Dadhich and H. Maeda, Origin of matter out of pure curvature, [arXiv:0705.2490].

[115] H. Maeda and N. Dadhich, Matter without matter: novel Kaluza-Klein spacetime in Einstein-Gauss-Bonnet gravity, Phys. Rev. D75 (2007) 044007, [arXiv:hep-th/0611188].

[116] G. Giribet, J. Oliva and R. Troncoso, Simple compactifications and Black p-branes in Gauss-Bonnet and Lovelock Theories, JHEP 0605 (2006) 007, [arXiv:hep-th/0603177].

[117] D. Kastor and R. Mann, On black strings & branes in Lovelock gravity, JHEP 0604 (2006) 048, [arXiv:hep-th/0603168].

[118] T. Kobayashi and T. Tanaka, Five-dimensional black strings in Einstein-Gauss-Bonnet gravity, Phys. Rev. D71 (2005) 084005, [arXiv:gr-qc/0412139].

[119] C. Sahabandu, P. Suranyi, C. Vaz and L. Wijewardhana, Thermodynamics of static black objects in D dimensional Einstein-Gauss-Bonnet gravity with D-4 compact dimensions, Phys. Rev. D73 (2006) 044009, [arXiv:gr-qc/0509102].

[120] R. Myers and J. Simon, Black Hole Thermodynamics in Lovelock Gravity, Phys. Rev. D38 (1988) 2434.

[121] B. Whitt, Spherically Symmetric Solutions Of General Second Order Gravity, Phys. Rev. D38 (1988) 3000.

[122] T. Jacobson, G. Kang and R. Myers, Entropy increase for black holes in higher curvature gravity, Proceedings of the 7th Marcel Grossmann Meeting on General Relativity (1994) 937.

[123] T. Jacobson, G. Kang and R. Myers, Increase of black hole entropy in higher curvature gravity, Phys. Rev. D52 (1985) 3518.

[124] T. Jacobson, G. Kang and R. Myers, Black hole entropy in higher curvature gravity, [arXiv:gr-qc/9502009].

[125] R. Cai and K. Soh, Topological black holes in the dimensionally continued gravity, Phys. Rev. D59 (1999) 044013, [arXiv:gr-qc/9808067].

[126] M. Cvetic, S. Nojiri and S. Odintsov, Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity, Nucl. Phys. B628 (2002) 295, [arXiv:hep-th/0112045].
[127] Y. Cho, I. Neupane, *Anti-de Sitter black holes, thermal phase transition and holography in higher curvature gravity*, Phys. Rev. D66 (2002) 024044, [arXiv:hep-th/0202140](http://arxiv.org/abs/hep-th/0202140).

[128] I. Neupane, *Black hole entropy in string generated gravity models*, Phys. Rev. D67 (2003) 061501, [arXiv:hep-th/0212092](http://arxiv.org/abs/hep-th/0212092).

[129] I. Neupane, *Thermodynamic and gravitational instability on hyperbolic spaces*, Phys. Rev. D69 (2004) 084011, [arXiv:hep-th/0302132](http://arxiv.org/abs/hep-th/0302132).

[130] R. Cai, *A Note on Thermodynamics of Black Holes in Lovelock Gravity*, Phys. Lett. B582 (2004) 237, [arXiv:hep-th/0311240](http://arxiv.org/abs/hep-th/0311240).

[131] M. Dehghani, G. H. Bordbar and M. Shamirzaie, *Thermodynamics of Rotating Solutions in Gauss-Bonnet-Maxwell Gravity and the Counterterm Method*, Phys. Rev. D74 (2006) 064023, [arXiv:hep-th/0607067](http://arxiv.org/abs/hep-th/0607067).

[132] T. Clunan, S. Ross, D. Smith, *On Gauss-Bonnet black hole entropy*, Class. Quant. Grav. 21 (2004) 3447, [arXiv:gr-qc/0402044](http://arxiv.org/abs/gr-qc/0402044).

[133] S. Nojiri, S. Odintsov and S. Ogushi, *Friedmann-Robertson-Walker brane cosmological equations from the five-dimensional bulk (A)dS black hole*, Int. J. Mod. Phys. A17 (2002) 4809, [arXiv:hep-th/0205187](http://arxiv.org/abs/hep-th/0205187).

[134] J. Lidsey, S. Nojiri and S. Odintsov, *Braneworld Cosmology in (Anti)–de Sitter Einstein–Gauss–Bonnet–Maxwell Gravity*, JHEP 0206 (2002) 026, [arXiv:hep-th/0202198](http://arxiv.org/abs/hep-th/0202198).

[135] A. Sen, *Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity*, JHEP 0509 (2005) 038, [arXiv:hep-th/0506177](http://arxiv.org/abs/hep-th/0506177).

[136] J. Morales and H. Samtleben, *Entropy function and attractors for AdS black holes*, JHEP 0610 (2006) 074, [arXiv:hep-th/0608044](http://arxiv.org/abs/hep-th/0608044).

[137] B. Chandrasekhar, S. Parvizi, A. Tavanfar and H. Yavartanoo, *Non-Supersymmetric Attractors in R’2 Gravities*, JHEP 0608 (2006) 004, [arXiv:hep-th/0602022](http://arxiv.org/abs/hep-th/0602022).

[138] D. Astefanesei, H. Nastase, H. Yavartanoo, S. Yun, *Moduli flow and non-supersymmetric AdS attractors*, JHEP 0804 (2008) 074, [arXiv:0711.0036](http://arxiv.org/abs/0711.0036).

[139] D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen, S. P. Trivedi, *Rotating attractors*, JHEP 0610 (2006) 058, [arXiv:hep-th/0606244](http://arxiv.org/abs/hep-th/0606244).

[140] R. Cai, C. Chen, K. Maeda, N. Ohta and D. Pang, *Entropy Function and Universality of Entropy-Area Relation for Small Black Holes*, [arXiv:0712.4212](http://arxiv.org/abs/0712.4212).

[141] E. Witten, *(2+1)-Dimensional Gravity As An Exactly Soluble System*, Nucl. Phys. B311 (1988) 46.
[142] S. Deser, R. Jackiw and S. Templeton, *Topological massive gauge theories*, Anals Phys. **140** (1982) 372, [Erratum-ibid. 185, 406.1988 APNYA, 281, 409 (1988) APNYA, 281-449.2000].

[143] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive gauge theories*, Phys. Rev. Lett. **48** (1982)

[144] M. Bañados, R. Troncoso and J. Zanelli, *Higher dimensional Chern-Simons supergravity*, Phys. Rev. **D54** (1996) 2605, [arXiv:gr-qc/9601003].

[145] M. Bañados, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. **D48** (1993) 1506. [gr-qc/9302012].

[146] M. Bañados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, Phys. Rev. Lett. **69** (1992) 1849, [hep-th/9204099].

[147] M. Bañados, C. Teitelboim and J. Zanelli, *Lovelock-Born-Infeld Theory of Gravity*, in J.J. Giambiagi Festschrift, Ed. by H. Falomir et al. (1990).

[148] M. Bañados, C. Teitelboim and J. Zanelli, *Dimensionally continued black holes*, Phys. Rev. **D49** (1994) 975, [arXiv:gr-qc/9307033].

[149] M. Bañados, C. Teitelboim and J. Zanelli, *Black hole entropy and the dimensional continuation of the Gauss-Bonnet theorem*, Phys. Rev. Lett. **72** (1994) 957, [arXiv:gr-qc/9309026].

[150] F. Canfora, A. Giacomini and R. Troncoso, *Black holes, parallelizable horizons and half-BPS states for the Einstein-Gauss-Bonnet theory in five dimensions*, to appear in Phys. Rev. **D**, [arXiv:0707.1056].

[151] F. Canfora, A. Giacomini and S. Willison, *Some exact solutions with torsion in 5-D Einstein-Gauss-Bonnet gravity*, Phys. Rev. **D76** (2007) 044021, [arXiv:0706.2891].

[152] F. Canfora, *Some solutions with torsion in Chern-Simons gravity and observable effects*, [arXiv:0706.3538].

[153] A. Anabalon, N. Deruelle, Y. Morisawa, J. Oliva, M. Sasaki, D. Tempo and R. Troncoso, *Kerr-Schild ansatz in Einstein-Gauss-Bonnet gravity: An exact vacuum solution in five dimensions*, to appear in Class. Quant. Grav. (2008), [arXiv:0812.3194] [hep-th].

[154] Y. Brihaye and E. Radu, *Five-dimensional rotating black holes in Einstein-Gauss-Bonnet theory*, Phys. Lett. **B661** (2008) 167, [arXiv:0801.1021].

[155] Hyeong-Chan Kim and Rong-Gen Cai, *Slowly Rotating Charged Gauss-Bonnet Black holes in AdS Spaces*, Phys. Rev. **D77** (2008) 024045, [arXiv:0711.0885].

[156] B. Bhawal and S. Kar, *Lorentzian wormholes in Einstein-Gauss-Bonnet theory*, Phys. Rev. **D46** (1992) 2464.
[157] G. Dotti, J. Oliva and R. Troncoso, Static wormhole solution for higher-dimensional gravity in vacuum, Phys. Rev. D75 (2007) 024002, [arXiv:hep-th/0607062].

[158] H. Maeda and M. Nozawa, Static and symmetric wormholes respecting energy conditions in Einstein-Gauss-Bonnet gravity, [arXiv:0803.1704].

[159] M. Richarte and C. Simeone, Thin-shell wormholes supported by ordinary matter in Einstein-Gauss-Bonnet gravity, Phys. Rev. D76 (2007) 087502, [arXiv:0710.2041].

[160] D. Correa, J. Oliva and R. Troncoso, Stability of asymptotically AdS wormholes in vacuum against scalar field perturbations, [arXiv:0805.1513].

[161] C. Charmousis, Higher order gravity theories and their black hole solutions, arXiv:0805.0568.

[162] O. Miskovic and R. Olea, On boundary conditions in three-dimensional AdS gravity, Phys. Lett. B640 (2006) 101, [arXiv:hep-th/0603092].

[163] H. Maeda, Effects of Gauss-Bonnet term on the final fate of gravitational collapse, Class. Quant. Grav. 23 (2006) 2155, [arXiv:gr-qc/0504028].

[164] M. Nozawa and H. Maeda, Effects of Lovelock terms on the final fate of gravitational collapse: analysis in dimensionally continued gravity, Class. Quant. Grav. 23 (2006) 1779, [arXiv:gr-qc/0510070].

[165] G. Gibbons, S. Hartnoll and A. Ishibashi, On the Stability of Naked Singularities, Prog. Theor. Phys. 113 (2005) 963-978, [arXiv:hep-th/0409307].

[166] G. Dotti and R. Gleiser, Instability of the negative mass Schwarzschild naked singularity, Class. Quant. Grav. 23 (2006) 5063-5078, [arXiv:gr-qc/0604021].

[167] G. Dotti and M. Gleiser, privatized communication, to be presented at Seventh Alexander Friedmann Seminar on Gravitation and Cosmology, Joao Pesoa, Brasil, 2008.

[168] G. Horowitz and D. Marolf, Quantum probes of space-time singularities, Phys. Rev. D52 (1995) 5670, [arXiv:gr-qc/9504028].

[169] R. Wald, Dynamics In Nonglobally Hyperbolic, Static Space-Times, J. Math. Phys. 21 (1980) 2802.

[170] A. Ishibashi and R. Wald, Dynamics in Non-Globally-Hyperbolic Static Spacetimes II: General Analysis of Prescriptions for Dynamics, Class. Quant. Grav. 20 (2003) 3815-3826, [arXiv:gr-qc/0305012].

[171] A. Ishibashi and A. Hosoya, Who’s Afraid of Naked Singularities?, Phys. Rev. D 60 (1999) 104028, [arXiv:gr-qc/9907009].