Ambiguities in Quantizing a Classical System

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Abstract

One classical theory, as determined by an equation of motion or set of classical trajectories, can correspond to many unitarily inequivalent quantum theories upon canonical quantization. This arises from a remarkable ambiguity, not previously investigated, in the construction of the classical (and hence the quantized) Hamiltonian or Lagrangian. This ambiguity is illustrated for systems with one degree of freedom: An arbitrary function of the constants of motion can be introduced into this construction. For example, the nonrelativistic and relativistic free particles follow identical classical trajectories, but the Hamiltonians or Lagrangians, and the canonically quantized versions of these descriptions, are inequivalent. Inequivalent descriptions of
other systems, such as the harmonic oscillator, are also readily obtained.
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I. INTRODUCTION

Although there are quantum systems which cannot be obtained by quantizing a classical system, the route through a classical theory, via canonical quantization or path integrals, is by far the most important way of constructing a quantum theory. One of the most profound problems in physics today is to quantize the classical theory of relativity, an issue to which enormous and diverse efforts have been devoted in the past 50 years. It may therefore be useful to re-examine this quantization procedure with care. Here we discuss one interesting point discovered in such a re-examination, although its implications (if any) specifically for the quantization of gravity are as yet unclear. Rather, it appears to bear on the matter of quantizing classical systems in general.

Given a classical system, we can directly observe the particle trajectories. Thus we can determine the classical equations of motion, but not directly the Hamiltonian or the Lagrangian. From the equations of motion, we (i) construct the Hamiltonian, Lagrangian or action, which (ii) by canonical quantization or a path integral, gives us the quantum description. The quantum mechanics of the system is determined by the action, but the classical paths picked out by the equations of motion concern only the extrema; one certainly does not expect, in general, a functional to be uniquely determined from its extrema. From this point of view, it is remarkable that the quantum mechanics can be determined to a large extent from the classical equations of motion. The issue we address in this paper is: To what extent?

There are well-known ambiguities in both steps (i) and (ii) of this procedure. In step (i), different Hamiltonians or Lagrangians can be constructed from the same equations of motion. A famous example is the Aharonov-Bohm effect [1]. For an electron with charge $-e$ which is kept from entering a long solenoid, the classical equation of motion is the same whether the magnetic field inside the solenoid is turned on or not — but the Hamiltonians (or Lagrangians) are different. In the absence of a magnetic field, outside the solenoid the Hamiltonian is
while with the magnetic field turned on, it is
\[ H = \frac{(p + eA)^2}{2m}. \]  
(1.2)

Although classically this difference in the Hamiltonians is not observable, it leads to observable effects upon quantization.

In step (ii) of the quantization procedure, we have the well-known ordering ambiguity, which appears in various guises. In canonical quantization, it appears as different ways of ordering products of the operators \( \hat{x} \) and \( \hat{p} \); different orderings are possible as long as they lead to Hermitian operators with the same classical limit [2]. In the configuration-space path-integral formulation, it appears as the ambiguity in choosing the measure in the space of paths, while in the phase-space path integral it appears as the ambiguity in skeletonization [3]. Corresponding to these different manifestations are various proposals to fix this ambiguity [4].

In this paper we address an ambiguity in step (i) of the procedure, similar to but not the same as that of the Aharonov-Bohm effect. It is similar in that different Hamiltonians or Lagrangians can be constructed for the same classical equations of motion. However, in the case of the Aharonov-Bohm effect the Hamiltonians (1.1) and (1.2) are related by a canonical transformation
\[ x \mapsto x' = x, \]  
(1.3)

\[ p \mapsto p' = p - eA(x), \]  
(1.4)

with a generating function of the second type [5],
\[ F_2 = p' \cdot x + e \int A(x) \cdot dx. \]  
(1.5)

With (1.3) – (1.5) we can readily obtain [4] the Hamiltonian (1.2) from form (1.1). Since the transformation is canonical, Poisson brackets are preserved [6]. With the Poisson brackets
turned into commutators upon quantization \[2\], it is guaranteed that the two Hamiltonians lead quantum-mechanically to the same local physics, although globally they are different, as manifested in the Aharonov-Bohm effect. In comparison, the ambiguity we examine here is that, in general, for the same classical motions, there exist different Hamiltonians which are \emph{not} canonically equivalent (even locally); these will lead to different quantum behaviors. It is amazing that such an ambiguity, which could have been discovered 70 years ago, has not to our knowledge previously appeared in the literature.

In Section II we describe the construction of different Hamiltonians from a given equation of motion, for systems with one degree of freedom. It is shown that the difference cannot be eliminated, even locally, by a canonical transformation. Section III gives simple examples, demonstrating that the different Hamiltonians lead to different quantum descriptions. Section IV gives the Lagrangian treatment, in which the result can in many cases be relatively simple. Section V is a brief conclusion, including remarks on coupled systems with more than one degree of freedom. Where needed, units with \(\hbar = c = 1\) are used.

II. FREEDOM IN CONSTRUCTING THE HAMILTONIAN

Suppose the equations of motion of a system with one degree of freedom are found experimentally to be

\[
\dot{p} + \frac{\partial H}{\partial x} = 0 ,
\]

\[
\dot{x} - \frac{\partial H}{\partial p} = 0 ,
\]

for a certain function \(H = H(p,x)\). No doubt one can identify this function \(H\) as the Hamiltonian, and base canonical quantization on it. However, classically (2.1) and (2.2) are indistinguishable from

\[
g_1(p,x) \left[ \dot{p} + \frac{\partial H}{\partial x} \right] + g_2(p,x) \left[ \dot{x} - \frac{\partial H}{\partial p} \right] = 0 ,
\]
\[ g_3(p, x) \left[ \dot{p} + \frac{\partial H}{\partial x} \right] + g_4(p, x) \left[ \dot{x} - \frac{\partial H}{\partial p} \right] = 0 \quad , \tag{2.4} \]

as long as

\[ \Delta \equiv g_1 g_4 - g_2 g_3 \neq 0 \tag{2.5} \]

obtains. In general, we can form not just linear combinations of (2.1) and (2.2), and the \( g \)'s can depend on \( p, x \) and the time \( t \). For simplicity, we shall restrict to linear combinations in (2.1) and (2.2), and time-independent \( g \)'s. This will suffice to demonstrate the ambiguity in constructing the Hamiltonian.

We examine the possibility that (2.3) and (2.4) are the Hamiltonian equations of motion for another Hamiltonian \( H'(p', x) \), where the coordinate \( x \) (being directly observable) is unchanged, i.e., \( x' = x \), but the momentum \( p' \) could be different from \( p \). This then requires

\[ \dot{p}' + \frac{\partial H'}{\partial x} \bigg|_{p'} = g_1 \left[ \dot{p} + \frac{\partial H}{\partial x} \right] + g_2 \left[ \dot{x} - \frac{\partial H}{\partial p} \right] \quad , \tag{2.6} \]

\[ \dot{x} - \frac{\partial H'}{\partial p'} \bigg|_{x} = g_3 \left[ \dot{p} + \frac{\partial H}{\partial x} \right] + g_4 \left[ \dot{x} - \frac{\partial H}{\partial p} \right] \quad . \tag{2.7} \]

The partial derivatives on the RHS of (2.6) and (2.7) are understood to be at fixed \( p \) or \( x \). By taking

\[ p' = p'(p, x) \quad , \tag{2.8} \]

\[ H' = H'(p'(p, x), x) \quad , \tag{2.9} \]

the LHS of (2.6) and (2.7) can be rewritten as functions of \( x, \dot{x}, p \) and \( \dot{p} \), as can the RHS. Since \( x, \dot{x}, p \) and \( \dot{p} \) are independent, the coefficients of \( \dot{x} \) and \( \dot{p} \) in (2.6) must separately match. By converting the partial derivatives on the LHS from fixed \( p' \) to fixed \( p \) and \( x \), we find

\[ \frac{\partial p'}{\partial x} = g_2(p, x) \quad , \tag{2.10} \]
\[ \frac{\partial p'}{\partial p} = g_1(p, x) \quad . \] (2.11)

The other terms in (2.6) give
\[ \frac{\partial H'}{\partial x} - \frac{\partial p'/\partial x}{\partial p'/\partial p} \frac{\partial H'}{\partial p} = g_1 \frac{\partial H}{\partial x} - g_2 \frac{\partial H}{\partial p} \quad . \] (2.12)

Likewise, (2.7) leads to
\[ g_4(p, x) = 1 \quad , \] (2.13)
\[ g_3(p, x) = 0 \quad , \] (2.14)
\[ \frac{1}{\partial p'/\partial p} \frac{\partial H'}{\partial p} = \frac{\partial H}{\partial p} \quad . \] (2.15)

The conditions (2.10) – (2.15) imply the pair of equations
\[ \frac{\partial H'}{\partial x} = g_1(p, x) \frac{\partial H}{\partial x} \quad , \] (2.16)
\[ \frac{\partial H'}{\partial p} = g_1(p, x) \frac{\partial H}{\partial p} \quad . \] (2.17)

Hence the integrability condition for \( H' \) is
\[ \frac{\partial H}{\partial x} \frac{\partial g_1}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial g_1}{\partial x} \equiv [H, g_1]_{x,p} = 0 \quad , \] (2.18)
with \([ \quad ]_{x,p}\) being the Poisson bracket with respect to the independent variables \( x \) and \( p \).

This means that \( g_1(p, x) \) is a constant of motion.

In one dimension, there is only one nontrivial constant of motion, namely \( H \). Thus, we must have
\[ g_1 = F(H) \equiv \frac{d\tilde{F}(H)}{dH} \quad . \] (2.19)

From (2.10) and (2.17), \( \partial H'/\partial x = \partial \tilde{F}/\partial x \), and \( \partial H'/\partial p = \partial \tilde{F}/\partial p \), so that we get immediately
\[ H' = \tilde{F}(H) \quad , \] (2.20)

up to an irrelevant additive constant.

This expresses \( H' \) in terms of \((p, x)\). To express \( H' \) in terms of \((p', x)\), we note that (2.11) determines \( p' \) as

\[ p' = \int_0^p dp \, g_1(p, x) + S(x) \quad , \] (2.21)

up to the arbitrary function \( S(x) \). In view of (2.10), this is the only remaining freedom in \( g_2 \). In many examples, \( H \) is even in \( p \); if we want to maintain the same condition for \( H' \) in terms of \( p' \), then we would have to choose \( S(x) = 0 \).

To be more explicit, the formal solution can be expressed as follows. We write

\[ F(H) = \sum_n a_n(x) p^{2n} \quad . \] (2.22)

(For simplicity we take only even powers, though this restriction is easily lifted.) This yields

\[ p' = \sum_n \frac{1}{2n + 1} a_n(x) p^{2n+1} \quad , \] (2.23)

where, consistent with the assumption that \( H \) is even in \( p \), we have set the integration constant to \( S(x) = 0 \). It is easily checked that (2.23) solves (2.10) as well, because we have guaranteed the integrability condition. The inverse transformation is formally

\[ p = \sum_n b_n(x) p^{2n+1} \quad , \] (2.24)

with

\begin{align*}
  b_0 &= \frac{1}{a_0} \quad , \\
  b_1 &= -\frac{1}{3} \frac{a_1}{a_0^3} \quad , \\
  b_2 &= \frac{1}{3} \frac{a_2}{a_0^5} - \frac{1}{5} \frac{a_2}{a_0^6} \quad ,
\end{align*}

(2.25)

etc.

Thus, given any \( H \), we can obtain a host of other Hamiltonians by choosing (i) an arbitrary function \( F(H) \) and (ii) an arbitrary function \( S(x) \). The resulting Hamiltonian
equations are guaranteed to be equivalent to the original ones, (2.1) and (2.2), as long as \( F \) is nonzero [cf. Eqs. (2.5), (2.13) and (2.14)].

Finally consider the Jacobian of the transformation \((x, p) \mapsto (x', p')\), where \( x' = x \):

\[
J = \det \begin{pmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial p} \\
\frac{\partial p'}{\partial x} & \frac{\partial p'}{\partial p}
\end{pmatrix} = [x', p']_{x,p} = g_1(p, x) .
\] (2.26)

A canonical transformation must have \( J = 1 \). The so-called extended canonical transformation \( \otimes \) (with scale transformation included) has \( J = \text{constant} \), but not a function of \( p \) or \( x \). Provided \( g_1 = F(H) \) is not chosen to be a constant over all of phase space, the transformation \((x, p) \mapsto (x', p')\) is not canonical.

### III. EXAMPLES

#### A. Free particle

A Newtonian free particle is described by the Hamiltonian

\[
H = \frac{p^2}{2m} .
\] (3.1)

The choice

\[
F(H) = (1 - 2H/m)^{-3/2} = \left(1 - p^2/m^2\right)^{-3/2} ,
\] (3.2)

leads to

\[
H' = \tilde{F} = \left(m^2 - p^2\right)^{-1/2} ,
\] (3.3)

\[
p' = \frac{p}{(1 - p^2/m^2)^{1/2}} .
\] (3.4)

In terms of \( p' \), we have

\[
H' = \left(p'^2 + m^2\right)^{1/2} ,
\] (3.5)

which is the Hamiltonian of a relativistic free particle! It is physically obvious that by observing classical free particles [provided the particles do not travel at speeds greater than
unity, i.e., provided \( p < m \), as per (3.3)], the two theories cannot be distinguished. Although this relationship between the Newtonian and the relativistic free particle might not have a deep physical meaning, this example clearly demonstrates that drastically different-looking theories can correspond to the same classical motions. It is well known that the quantum theory of a relativistic point particle is very different from that of a Newtonian free particle, with the former sharing some common features with quantum gravity [3].

Although the Hamiltonian (3.3) is nonpolynomial in \( p \), and is therefore a non-local operator upon quantization, there is no reason why canonical quantization cannot be based on it. This has been studied by Newton and Wigner [7] (see also Hartle and Kuchar [3]). The result can be stated simply: a Fourier component with wave number \( k \) evolves in time \( t \) with a phase \( \sqrt{k^2 + m^2} t \). This leads to the propagator function \( G' \) for \( H' \), viz.,

\[
G'(x, t; x_0, t_0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0)}e^{-i(k^2+m^2)^{1/2}(t-t_0)}
\]

\[
= \lim_{\epsilon \to 0^+} \left[ \frac{im(t-t_0-i\epsilon)}{\pi \Delta \lambda}K_1(m \Delta \lambda) \right], \quad (3.6)
\]

with \( \Delta \lambda = [(x-x_0)^2 - (t-t_0-i\epsilon)^2]^{1/2} \), and \( K_1 \) the usual modified Bessel function [8]. This \( G' \) is the probability amplitude for the wavefunction originally localized at \( x_0 \) at time \( t_0 \) to be localized at \( x \) at the later time \( t \), and has the same physical meaning as the familiar Newtonian free-particle propagator \( G \):

\[
G(x, t; x_0, t_0) = \sqrt{\frac{m}{2i\pi(t-t_0)}} \exp \left[ \frac{im(x-x_0)^2}{2(t-t_0)} \right], \quad (3.7)
\]

corresponding to the Hamiltonian (3.3). So although both the Hamiltonians (3.1) and (3.5) give straight lines as classical trajectories, the quantum theories they yield via canonical quantization are very different.

**B. A system with a non-trivial potential**

Consider a one-dimensional system with the classical Hamiltonian

\[
H = x^2p^2 + x^3p + \frac{1}{4}x^4, \quad (3.8)
\]
and take

\[ F(H) = H^{-1/2} = \left(xp + \frac{1}{2}x^2\right)^{-1}. \]  

(3.9)

The conditions (2.10) and (2.11) then give

\[ p' = \frac{1}{x} \ln(xp + \frac{1}{2}x^2) + S(x), \]  

(3.10)

for some function \( S(x) \). We choose, again for simplicity, \( S(x) = 0 \). The new Hamiltonian is

\[ H' = 2H^{1/2} = 2 \left(xp + \frac{1}{2}x^2\right) \]

\[ = 2e^{xp'}. \]  

(3.11)

The easiest way to analyse the properties of this Hamiltonian is to perform an additional canonical transformation

\[ \tilde{p} = xp' \quad \tilde{x} = \ln x. \]  

(3.12)

The generating function is of the second type [5]:

\[ F_2(x, \tilde{p}) = \tilde{p} \ln x. \]  

(3.13)

The Hamiltonian (3.11) becomes

\[ \tilde{H} = 2e^{\tilde{p}}. \]  

(3.14)

The Hamiltonian equations of motion are then trivial, yielding the classical paths

\[ \tilde{x}(t) = 2e^{\tilde{p}}t + \text{const}, \]  

(3.15)

which, via (3.12), gives

\[ x(t) = e^{C_1t + C_2}, \]  

(3.16)

where \( C_1 \) and \( C_2 \) are the two integration constants. It can be checked that trajectory (3.16) indeed solves the Hamiltonian equations of the original Hamiltonian (3.8).
The canonical quantization of the Hamiltonian (3.8) yields a different quantum theory than that of (3.11) [which is the same as that of (3.14)]; this can be seen by comparing the Poisson brackets in the two cases. For any pair of functions

\[ u = u(p', x) = u(p'(p, x), x) \]  
\[ v = v(p', x) = v(p'(p, x), x) \]  

(3.17) (3.18)

it is straightforward to verify

\[ [u, v]_{x,p'} \equiv \frac{\partial u}{\partial x} \frac{\partial v}{\partial p'} - \frac{\partial u}{\partial p'} \frac{\partial v}{\partial x} \]

\[ = e^{xp'} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial x} \right) = e^{xp'} [u, v]_{x,p} , \]

(3.19)

where the factor \( e^{xp'} \) is just the inverse of the Jacobian \( g_1 \) of the transformation. As the Poisson bracket gives the leading term in order of \( \hbar \) of the quantum commutator, the Hamiltonians (3.8) and (3.11) must correspond to different quantum theories. In particular, since the commutators are different, the eigenvalue spectra for observables will be different in the two theories.

C. The harmonic oscillator

Even thoroughly understood systems can be subjected to this type of transformation. For the harmonic oscillator, experimental evidence implies that the Hamiltonian must essentially be

\[ H = \frac{p^2}{2} + \frac{x^2}{2} . \]

(3.20)

We consider three examples of transformations.

Example 1

We take

\[ F = 1 + \frac{\epsilon}{2H} = 1 + \epsilon \left( p^2 + x^2 \right)^{-1} , \]

(3.21)
with $\epsilon \ll 1$. Integrating this in $H$, we find

$$\tilde{F} = H + \frac{\epsilon}{2} \ln H .$$  \hfill (3.22)

Using (2.11), we find

$$p' = p + \frac{\epsilon}{x} \arctan \frac{p}{x} .$$ \hfill (3.23)

If $H'$ is now expressed in terms of $p'$ and $x$, we get

$$H' = \frac{p'^2}{2} + \frac{x^2}{2} + \epsilon \left[ \frac{1}{2} \ln \frac{p'^2 + x^2}{2} - \frac{p'}{x} \arctan \left( \frac{p'}{x} \right) \right] + O(\epsilon^2) .$$ \hfill (3.24)

**Example 2**

As another example, we choose

$$F = 1 + \epsilon(2H) = 1 + \epsilon(p^2 + x^2) .$$ \hfill (3.25)

Following the same steps, we find

$$H' = \frac{p'^2}{2} + \frac{x^2}{2} + \epsilon \left[ \frac{1}{4}(p'^2 + x^2)^2 - \frac{1}{6} p'^4 - x^2 p'^2 \right] ,$$ \hfill (3.26)

$$p' = p + \epsilon \left( \frac{5}{3} p^3 + x^2 p \right) .$$ \hfill (3.27)

**Example 3**

In this case we take $F = H$ and $\tilde{F} = \frac{1}{2} H^2$. The relation between $p'$ and $p$ is given by (2.11), leading to

$$p' = \frac{1}{6} p^3 + \frac{1}{2} x^2 p ,$$ \hfill (3.28)

where we have set the integration constant to zero under the assumption that parity is maintained (i.e., when $p \mapsto -p$, then $p' \mapsto -p'$). The inverse relation cannot be expressed in closed form, but is formally given by the power series

$$p = \frac{2}{x^2} p' - \frac{8}{9 x^3} p'^3 + \cdots .$$ \hfill (3.29)

When (3.29) is substituted into $H' = \tilde{F} = (p^2 + x^2)^2/8$, we obtain $H' = H'(p', x)$, which is guaranteed to give the same classical motion. However, this is a complicated expression. We shall see later that the complication arises entirely from $p'$, and the same situation has a simple description in the Lagrangian formulation because $p'$ will not appear.
IV. LAGRANGIAN FORMULATION

The preceding examples show that at least part of the complication arises from the transformation from $p$ to $p'$, and the inverse transformation, e.g., (3.29). This would suggest that the formalism may be simpler in a Lagrangian description, in so far as neither $p$ nor $p'$ need appear.

A. Formalism

The Lagrangian equation of motion is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 .$$

(4.1)

with $L = L(x, \dot{x})$. In greater detail, and adopting the notation $(x, \dot{x}, \ddot{x}) \mapsto (x, y, z)$, and $L_x = \partial L/\partial x$ etc., we have

$$z L_{yy} + y L_{xy} - L_x = 0 .$$

(4.2)

For this to be a non-degenerate second-order differential equation, we shall assume $L_{yy} \neq 0$. (In the Newtonian case, $L_{yy} = m$.)

This could just as well be written with an extra multiplicative factor of $F = F(x, y)$. If this modified equation is to be the Euler-Lagrange equation derived from another Lagrangian $L'$, then we must have

$$z L'_{yy} + y L'_{xy} - L'_x = F \left( z L_{yy} + y L_{xy} - L_x \right) .$$

(4.3)

We have used the notation $F$ in anticipation that this function will turn out to be the same as $g_1$ in the Hamiltonian formulation.

In the above equation, $z$ appears only where shown explicitly, and since (4.3) must hold as an identity, the terms with and without $z$ have to be separately equal, leading to two conditions

$$L'_{yy} = FL_{yy} .$$

(4.4)
\( yL'_{xy} - L'_x = F \left( yL_{xy} - L_x \right) \) \hspace{1cm} (4.5)

In these expressions, we write the quantities on the LHS as

\[
\begin{align*}
L'_{yy} &= \partial_y L'_y = \partial_y p' , \\
L'_{xy} &= \partial_x L'_y = \partial_x p' .
\end{align*}
\] (4.6)

Their compatibility requires the integrability condition that \( \partial_x \partial_y p' = \partial_y \partial_x p' \). Some arithmetic then leads to

\[
yF_x L_{yy} - yF_y L_{xy} + F_y L_x = 0 \hspace{1cm} (4.7)
\]

On the other hand, we can consider the quantity

\[
L_{yy} \dot{F}
\]

\[
= L_{yy} \left( F_x \dot{x} + F_y \dot{y} \right) 
\]

\[
= L_{yy} \left( F_{xy} + F_y z \right) ,
\] \hspace{1cm} (4.8)

and use the equation of motion (4.2) to eliminate \( zL_{yy} \). This shows that \( L_{yy} \dot{F} \) is exactly the LHS of (4.7), implying \( \dot{F} = 0 \), i.e., \( F \) is a constant of motion.

In fact, it is easy to identify what \( F \) is. Note that in (4.3), we can write the bracket as

\[
yL_{xy} - L_x = \partial_x \left( yL_y - L \right) = \partial_x H ,
\] \hspace{1cm} (4.9)

given \( L_y = p \). The same holds for the analogous expression involving \( L' \). Thus (4.3) is equivalent to \( \partial_x H' = F \partial_x H \), confirming that \( F \) is indeed the same as \( g_1 \) introduced previously [cf. (2.16)]. However, in the rest of this Section, we shall not rely on any results from the Hamiltonian treatment.

Thus, once we choose any constant of motion \( F \), we can integrate to find \( L' \), up to a function \( S(x) \).
B. Free particle

We return to the free particle, with $L = \frac{1}{2} my^2$, $L_y = my$, $L_{yy} = m$. We choose $F = (1 - y^2)^{-3/2}$, which is a constant of motion, and solve for

$$L_{yy}' = FL_{yy} = m(1 - y^2)^{-3/2},$$

(4.10)
giving, up to integration constants which we set to zero,

$$L' = -m(1 - y^2)^{1/2},$$

(4.11)
which is the Lagrangian for the relativistic free particle.

C. Harmonic oscillator

For the harmonic oscillator, we have $L = \frac{1}{2} y^2 - \frac{1}{2} x^2$, $L_y = y$, and $L_{yy} = 1$. We choose $F = \frac{1}{2} y^2 + \frac{1}{2} x^2$, which is a constant. Thus, $L'$ is to be found by integrating

$$L_{yy}' = FL_{yy} = \frac{1}{2} y^2 + \frac{1}{2} x^2,$$

(4.12)
leading to

$$L' = \frac{1}{2} y^4 + \frac{1}{4} y^2 x^2 + f(x).$$

(4.13)
There should be two integration constants, but we have set the term linear in $y$ to zero using the assumption of parity. The function $f(x)$ is easily determined by considering the terms without any $y$ in (4.12):

$$- f'(x) = \frac{1}{2} x^2 (\pm x).$$

(4.14)
Thus we have

$$L' = \frac{1}{24} \dot{x}^4 + \frac{1}{4} \dot{x}^2 x^2 - \frac{1}{8} x^4.$$

(4.15)
It can be verified directly, without any of the general formalism given above, that this
Lagrangian gives the same classical motion, with the Euler-Lagrange equation being the
usual one multiplied by $F$.

In fact, this is the same as the last example presented under the Hamiltonian treatment
of the simple harmonic oscillator, but now expressed in closed form. Comparison shows that
the Lagrangian formulation is much simpler in this case, since all the complications lie in
$p' = L'_y$.

D. Path integral

In all these cases the quantum mechanics derived from $L$ and $L'$ are different. This is
most easily seen by verifying that, in the path-integral formulation, the phases associated
with paths are different in nontrivial ways. Consider two paths $x_a(t)$ and $x_b(t)$ with the
same end-points at $t = t_1$ and $t = t_2$ [i.e., $x_a(t_1) = x_b(t_1) = x_1$, $x_a(t_2) = x_b(t_2) = x_2$], and
the action integrals $S_a = \int_{t_1}^{t_2} dt L[x_a]$ etc. The differences $\Delta S = S_a - S_b$ and $\Delta S' = S'_a - S'_b$
are clearly different in general. Therefore the quantum mechanics as defined by the path
integral will be different in the two cases.

V. DISCUSSION

We have shown that just as in the case of Aharonov-Bohm effect, different Hamiltonians
can be constructed for the same classical equation of motion. However, unlike the case of
the Aharonov-Bohm effect, these Hamiltonians are not related by canonical transformations.
Although they must be regarded as the same classical theory — since it is the trajectories
or equations of motion, not the Hamiltonian, which are observed classically — these variant
Hamiltonians give different quantum theories upon canonical quantization.

We may place the ambiguity discussed here into context by classifying the known quan-
tization ambiguities into three levels. First, there are small differences in the sense that
$H$ and $H'$ agree classically but differ by $O(h)$ quantum-mechanically; the operator-ordering
ambiguity is of this type. Second, there are cases where $H - H'$ is not small (in the above sense), but can be made small by performing a canonical transformation on one of them. The canonical transformation may be global (in which case the situation is the same as the first type), or only local, as in the Aharonov-Bohm effect. The third level are differences between $H$ and $H'$ that cannot be made small by a canonical transformation. The ambiguities discussed in this paper are of this type, and in a sense are the most “serious.”

This possibility of inequivalent Hamiltonians raises several interesting questions, to which we do not at present have complete answers. One might be tempted to impose the requirement that the Hamiltonian must be polynomial in the momentum. However, this seems to be rather ad hoc as a fundamental principle. Moreover, in the case of a free particle, nature favors the more complicated relativistic Hamiltonian over the simpler form (3.1). In the case of the harmonic oscillator, Hamiltonian (3.24), which is quadratic in $p$, seems to be the preferred one. It is not clear whether either of the alternatives (3.8) or (3.11) is more appropriate than the other. Is there any guiding principle in choosing which one to use, in general? Are all the other possibilities truly forbidden?

The discussion in this paper, and all the examples, refer to systems with only one degree of freedom. This may already be of some interest, since some models of mini-superspace contain only one degree of freedom [9].

However, the interesting question is, of course, whether these ambiguities extend to systems with more degrees of freedom. The most general argument is that, for any number of degrees of freedom, one has no right to expect a functional (the action) to be determined by its extrema (the classically allowed trajectories), and therefore the existence of alternate Hamiltonians and Lagrangians should be the rule rather than the exception. The challenge will be to discover interesting examples, and if possible to characterize and classify all the possible ambiguities.
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REFERENCES

[1] Y. Aharonov and B. Bohm, Phys. Rev. 15, 485 (1959); also see, e.g., J. J. Sakurai, *Modern Quantum Mechanics*, Rev. Ed., (Addison-Wesley, N.Y. 1994), p. 136.

[2] See, for example, P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th edition (Oxford University Press, New York, 1958), p. 87.

[3] See, for example, J. B. Hartle and K. Kuchar, Phys. Rev. D34, 2323 (1986).

[4] J. B. Hartle and K. Kuchar op. cit., J. J. Halliwell, Phys. Rev. D D38, 1988 and references cited therein.

[5] See, for example, H. Goldstein, *Classical Mechanics*, 2nd edition (Addison-Wesley, London 1980), p. 378.

[6] See, H. Goldstein, op. cit., p. 397.

[7] T. Newton and E. Wigner, Rev. Mod. Phys. B21, 400 (1949).

[8] I. H. Redmount and W.-M. Suen, Int. J. Mod. Phys. A 8, 1629 (1993).

[9] Some examples of mini-superspace models can be found in S. W. Hawking, Nucl. Phys. B329, 257 (1984); G. T. Horowitz, Phys. Rev. D 31, 1169 (1985), J.J. Halliwell, Nucl. Phys. B266, 228 (1986), J. Louko, Class. Quantum Gravit. 4, 581 (1987), W.-M. Suen and K. Young, Phys. Rev. D 39, 2201 (1989).