Quantum state merging and negative information

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We consider a quantum state shared between many distant locations, and define a quantum information processing primitive, state merging, that optimally merges the state into one location. As announced in [Horodecki, Oppenheim, Winter, Nature 436, 673 (2005)], the optimal entanglement cost of this task is the conditional entropy if classical communication is free. Since this quantity can be negative, and the state merging rate measures partial quantum information, we find that quantum information can be negative. The classical communication rate also has a minimum rate: a certain quantum mutual information. State merging enabled one to solve a number of open problems: distributed quantum data compression, quantum coding with side information at the decoder and sender, multi-party entanglement of assistance, and the capacity of the quantum multiple access channel. It also provides an operational proof of strong subadditivity. Here, we give precise definitions and prove these results rigorously.

I. INTRODUCTION

The field of quantum information theory is still in its infancy, with many of the key building blocks of the theory not yet in place or not well understood. This is perhaps not surprising, since the important elements of classical information theory have only been in place since the 70’s. The notion of classical information was first introduced by Shannon [1] who defined it operationally, as the minimum number of bits needed to communicate the message produced by a statistical source. This gave meaning to the entropy $H(X)$ of the source producing a random variable $X$. The amount of information that two random variables $X$ and $Y$ have in common was given a meaning through the mutual information $I(X : Y) = H(X) + H(Y) - H(XY)$. Operationally it is the rate of communication possible through a noisy channel taking $X$ to $Y$. The fundamental Shannon theorems treated two basic questions: how many bits does one need to transmit a message from a source? How many bits can one send via a noisy channel?

Another basic brick in classical information theory, which is a generalization of the noiseless coding problem, is the notion of partial information. The question is now, how many bits does the sender (Alice) need to send to transmit a message conditioned on the fact that Bob has $Y$. Entropy of Alice’s source is its full information content. The difference between the two is the information that Bobs needs to complete his prior knowledge about Alice’s source (Figure I). It thus provided an information theoretic basis for the conditional entropy. It should be noted that it is a highly non-trivial operation, since Alice is able to communicate to Bob the full information about her string $X_1 \ldots X_n$, even though she is unaware of what string $Y_1 \ldots Y_n$ Bob has.

The quantities and operational meaning of the entropy, mutual information, and conditional entropy thus form the basic building blocks of classical information theory. We are interested in finding the corresponding basic elements in quantum information theory. The first step was done by Schumacher [3], who showed that the von Neumann entropy plays an analogous role to Shannon entropy: it has the operational interpretation of the number of qubits needed to transmit quantum states emitted by a statistical source.
The rate at which a source can convey messages (Shannon compression)

For an input $X$ which produces $Y$ after being sent down a channel, $I(X : Y)$ is the rate at which information can be sent reliably (channel coding)

The rate at which messages $X$ can be sent to a party who has prior information $Y$ (Slepian-Wolf theorem)

TABLE I: Key concepts in classical information theory

The next step was to find an analogue of the noisy coding theorem. Here it turned out that the analogy was not very strict: the quantum analogue of mutual information cannot be obtained by replacing Shannon entropies with von Neumann ones. It was found that the capacity of the quantum channel is determined by a different quantity – the coherent information [4, 5]. The coherent information, defined for a bipartite state $\rho_{AB}$ is

$$I(A)B = S(B) - S(AB),$$

and the channel capacity is obtained [6–8] by maximising it over input states $\rho_A$. Here, $S(B)$ and $S(AB)$ are the von Neumann entropy of states $\rho_B = \text{Tr}_{A} \rho_{AB}$ and $\rho_{AB}$, and we adopt the notation of dropping the explicit dependence on $\rho$ when such dependence is obvious.

With the coherent information, there was a persistent mystery – for any particular input $\rho_A$, the quantity $S(A) - S(AB)$ could be negative, and it was not known how to interpret such a quantity, as it indicated a sense in which the channel capacity could be negative for such input distributions. Thus it is often the case that for a particular channel, no inputs will give positive distributions, and one should set the coherent information to zero, by inputting the null distribution (any pure state).

Turning next to a quantum analogue of prior and partial information, there had previously not been any such notion – a quantum scenario like that of Slepian-Wolf appeared intractable [9]. Another serious obstacle in the quantum world is that there are no conditional probabilities, hence conditional entropy cannot be defined. Conditional probabilities only exist after one performs a measurement which of course destroys the state. One may try to overcome this difficulty, by naïvely replacing Shannon entropies with von Neumann one in the formula for conditional entropy, so that quantum conditional entropy would be the difference between the total entropy and the entropy of subsystem.

$$S(A|B) = S(AB) - S(B)$$

Such an approach has been strongly advocated [10], however while this $H$ goes to $S$ rule works for defining information, it doesn’t work for channel capacity, as mentioned above. It is thus not clear that it is the correct thing to do. However there is more serious obstacle here: the conditional entropy defined by taking $H$ to $S$ can be negative [10–12]. In [12] this problem was connected with quantum entanglement. Likewise for maximally entangled states, it was connected with the ability to perform teleportation [10]. It had already been noted by Schrödinger, that entangled state may possess a weird feature: if a system is in such a state we may know more about the whole system than about subsystems. In [12], Schrödinger’s intuition was quantified by von Neumann entropies, and it was found that the entropy of subsystem can be greater than the entropy of the total system only when the state is entangled. It was however also found that there are entangled states that do not exhibit this weird property. Thus there was a question: what does it mean, that for some states we have such behaviour, and not for other states?

It doesn’t help that $-S(A|B)$ is nothing but the coherent information, that determines channel capacity [6–8]! How can the duality between channel coding and Slepian-Wolf compression be conserved in any quantum analog?

In our recent paper [13], we approached the problem of quantifying partial and prior information from a purely operational point of view. Inspired by the classical Slepian-Wolf theorem, we consider the scenario in which an unknown quantum state is distributed over two systems. We determined how much quantum communication is needed to transfer the full state to one system. This communication measures the partial information one system needs conditioned on its prior information. We found that the partial information is given by the conditional entropy, just as in the classical case. However, in the classical case, partial information must always be positive, while in the quantum world we find this physical quantity can be negative. If the partial information is positive, its sender needs to communicate this number of quantum bits to the receiver to achieve state transfer; if it is negative, the state can be transferred, and in addition, the sender and receiver gain the corresponding potential for future
FIG. 1: A graphical representation of the building blocks of classical information theory. The total information of the source producing pairs of random variables $X, Y$ is $H(XY)$, while the information contained in just variable $X$ ($Y$) is $H(X)$ ($H(Y)$). The information common to both variables is the mutual information $I(X : Y)$, while the partial informations are $H(X | Y)$ and $H(Y | X)$. In the quantum case, the quantum mutual information $I(A : B)$ can be greater than the total information $S(AB)$, which can be also greater than the local informations $S(A)$ and $S(B)$. To compensate, the partial informations $S(A|B)$ and $S(B|A)$ can be negative.

| concept                     | quantity | operational meaning                                                                 |
|-----------------------------|----------|-------------------------------------------------------------------------------------|
| quantum information         | $S(A)$   | The rate at which a source can convey quantum states (Schumacher compression)       |
| coherent information        | $I(A : B)$ | For an input which produces $\rho_{AB}$ after being sent down a channel, $I(A : B)$ is the rate at which quantum information can be sent reliably down the channel (quantum channel coding). Merging allows us to interpret the negative values of this quantity |
| partial quantum information | $S(A|B)$ | The rate at which quantum states with density matrix $\rho_A$ can be sent to a party who has prior quantum information $\rho_B$ |

TABLE II: Key concepts in quantum information theory with additions due to merging highlighted in bold

quantum communication. This potential communication is in the form of pure entangled states which can be used to teleport quantum states. Thus viewing entanglement as a potential for quantum communication, we see that when the conditional entropy is positive, entanglement needs to be consumed, while when it is negative, entanglement is gained.

One can view it in another way – the entropy $S(B)$ quantifies how much Bob knows (in the sense of possessing the state), while the entropy $S(AB)$ quantifies how much there is to know. Since quantum distributions can have $S(AB) \leq S(B)$, there is a sense in which Bob knows too much. If Alice were to send her full state to him, at a cost of $S(A)$, then he ends up having entropy $S(AB)$ – in the quantum world, after you receive negative information, you know less.

The primitive which (optimally) transfers partial information we call quantum state merging, as Alice’s state is effectively merged with Bob’s state, arriving at his site. With this primitive in hand, one can gain a systematic understanding of quantum network theory, including several important applications such as distributed compression, multiple access channels and assisted entanglement distillation (localizable entanglement), and compression with quantum side information.

The purpose of the current paper is to provide full proofs for the result of [13]. In Section II we formally define the notion of quantum state merging, and state the main result. In Section III we exhibit a general condition to ensure state merging and derive a one-shot protocol based on random measurements. In Section IV we prove the main theorem, show that our protocol has the optimal classical communication rate, and provide a heuristic explanation of why the conditional entropy comes into play.

Once the primitive of state merging has been put on a firm footing, we are able to use it to solve a number of previously intractable problems. A broad outline of these applications was given in [13], and here we provide more details. In Section V we look at the problem of distributed compression, where several parties at different sites
individually compress a source, which is then decoded by a single party. It is found that the parties can compress at the ideal rate of the total entropy, even though they are distributed. In Section VI, we look at noiseless coding with side information, i.e. we consider the problem where one party (Alice) wishes to compress her state to send to a decoder, and a second party (Bob) who holds part of the total state can aid her by sending part of his state. The decoder only wishes to decode the state of Alice, while Bob’s state is only used to help in the decoding. As a corollary, we find that if there is a single encoder Alice who has access to side information, then this can help her in sending information to a decoder, a situation impossible in the classical case.

Next, in Section VII, we treat entanglement of assistance [14] in the case of many helping parties (a concept similar to localizable entanglement [15]). A pure state is shared by many parties, and the goal is to distill the maximum amount of entanglement between two of the parties. The other parties can aid in this distillation through local operations and classical communication. We find that state merging gives the optimal rate of distillation.

We then consider the quantum multiple access channel, in Section VIII. Two parties, Alice and Bob wish to send quantum states to a decoder through a channel which acts on both their states. We find optimal rates using state merging, and derive the full rate region. We are also able to provide an interpretation to the longstanding puzzle of negative coherent information in the formula for the capacity of the quantum channel. Namely, if one party’s rate is negative, than this is the amount of entanglement he or she must invest in order to help the other party achieve the maximum rate.

Before concluding in Section X, we provide a quick and intuitive proof of strong subadditivity using state merging in Section IX.

II. STATE MERGING: CONCEPT, DEFINITIONS AND MAIN RESULT

Consider a source emitting a sequence of unknown bipartite pure states \(|\psi_1\rangle_{AB}, |\psi_2\rangle_{AB}, |\psi_3\rangle_{AB}, \ldots\) from a distribution, with average density matrix \(\rho_{AB}\). As with Schumacher compression, we assume the density matrix of the source is known to the two parties Alice and Bob, but they don’t know the ensemble which realises it. I.e., for any given state they possess, the state is unknown, although the statistics of the source are. We are interested in information theoretic quantities, and in particular, we are interested in quantifying quantum information. We thus allow free classical communication between the two parties, and consider many copies \(n\) of the state \(\rho_{AB}\). We now ask how much quantum communication is needed for Alice to transfer the unknown sequence of states \(|\psi_1\rangle_{AB}, |\psi_2\rangle_{AB}, |\psi_3\rangle_{AB}, \ldots\) to Bob’s site. This we call *quantum state merging*. Notice that because classical communication is free, we can replace quantum communication by entanglement due to teleportation [16] – this will be a more convenient way of accounting for the quantum resources. Faithful state merging means that the fidelity of the sequence of states is kept for any realisation of the density matrix.

There is an equivalent, yet more elegant way to conceive of this problem. We imagine that the state \(\rho_{AB}\) is part of a larger pure state \(\psi_{ABR} = |\psi\rangle_{AB} \otimes \rho_{ABR}\), with a state vector \(|\psi\rangle_{ABR}\) which also lives on a reference (or environment) system \(R\). Faithful state transfer means that the transferred state has high fidelity with the original state \(|\psi\rangle_{AB} \otimes \rho_{ABR}\). More formally, we define:

**Definition 1 (State merging)** Consider a pure state \(|\Psi\rangle_{ABR}\) shared between two parties \(\tilde{A}, \tilde{B}\) and a reference \(\tilde{R}\). Let Alice and Bob have further registers \(A_0, A_1, B_0, B_1\), respectively. We call a joint operation \(\mathcal{M} : \tilde{A}_0 \otimes \tilde{B}_0 \rightarrow A_1 \otimes B_1 \tilde{B}' \tilde{B}\) state merging of \(\Psi\) with error \(\epsilon\), if it is LOCC and, with \(\rho_{A_1B_1 \tilde{B}' \tilde{B}R} = (\mathcal{M} \otimes \text{id}_R)(\Psi_{ABR} \otimes (\Phi_K)_{A_0B_0})\),

\[
F\left(\rho_{A_1B_1 \tilde{B}' \tilde{B}R}, (\Phi_L)_{A_1B_1} \otimes \Psi_{\tilde{B}' \tilde{B}R}\right) \geq 1 - \epsilon,
\]

with maximally entangled states \(\Phi_K, \Phi_L\) on \(A_0B_0, A_1B_1\) of Schmidt rank \(K, L\), respectively. Here, \(\tilde{B}'\) is a local ancilla of Bob’s of the same size as \(\tilde{A}\). The number \(\log K - \log L\) is called the entanglement cost of the protocol.

In the case of many copies of the same state, \(\Psi = |\psi\rangle^{\otimes n}\), we call \(\frac{1}{n}(\log K - \log L)\) the entanglement rate of the protocol. A real number \(R\) is called an achievable rate if there exist, for \(n \rightarrow \infty\), merging protocols of rate approaching \(R\) and error approaching 0. The smallest achievable rate is the merging cost of \(\psi\).

The main purpose of this paper is to prove in detail the result announced in [13], namely, that the merging cost is equal to the conditional entropy of the state \(\rho_{AB}\) shared by Alice and Bob, \(S(A|B) - S(AB)\).

**Theorem 2 (Quantum State Merging)** For a state \(\rho_{AB}\) shared by Alice and Bob, the entanglement cost of merging is equal to the quantum conditional entropy \(S(A|B) = S(B) - S(AB)\), in the following sense. When the \(S(A|B)\) is positive, then merging is possible if and only if \(R > S(A|B)\) ebits per input copy are provided. When \(S(A|B)\) is negative, then merging is possible by local operations and classical communication, and moreover \(R < -S(A|B)\) maximally entangled states are obtained per input copy.
Our strategy of proof will be the following. We first show that if the quantity is negative, then merging can be done by LOCC (indeed, with only one-way communication from Alice to Bob), and the entanglement rate that can be obtained is equal to minus the conditional entropy. Using this we will show that in the case of positive conditional entropy it is enough to spend $S(A|R)$ ebits of entanglement.

Finally, we will show that the rates given by the conditional entropy are optimal. We will also show that the classical communication cost is equal to the quantum mutual information between Alice and the reference system $R$,

$$I(A : R) = S(A) + S(R) - S(AR)$$  \hfill (5)

and prove its optimality.

### III. ONE-SHOT STATE MERGING

In this section, we first formulate a general sufficient condition on a measurement of Alice that ensures that Bob can complete state merging by local operations; then we show how random measurements succeed with high probability in realising this condition.

#### A. Condition for merging with one-way LOCC

Here we will provide a condition that is sufficient to obtain state merging with only LOCC. We formulate it in the one-shot setting of definition 1. It is based on Alice performing a measurement which takes the original state $\Psi_{\hat{A}\hat{B}\hat{R}}$ to another pure state, with the essential features that: (1) the reference system $\hat{R}$ is unchanged, and (2) Alice’s and the Reference’s states are in product form. Since all purifications are equal up to a local unitaries, this implies that Bob can perform a local unitary which transforms his state into $\rho_{\hat{A}\hat{B}}$.

More formally, we consider a protocol, whose basic constituent is Alice’s incomplete measurement given by Kraus operators $P_j$ mapping $\hat{A}$ to $A_1$ (in our actual solution, it will be a von Neumann measurement followed by a unitary). Given the outcome was $j$, the state $\Psi_{\hat{A}\hat{B}\hat{R}}$ collapses to a state which we will denote by $\Psi_j^{A_1\hat{B}\hat{R}'}$,

$$|\Psi_j^{A_1\hat{B}\hat{R}'}\rangle = \frac{1}{\sqrt{p_j}} (P_j \otimes I_{\hat{B}\hat{R}}) |\Psi\rangle_{\hat{A}\hat{B}\hat{R}},$$  \hfill (6)

where $p_j$ is probability of obtaining outcome $j$,

$$p_j = \langle \Psi | (P_j \otimes I_{\hat{B}\hat{R}}) |\Psi\rangle.$$  \hfill (7)

Suppose for the moment that $|\Psi_j^{A_1\hat{B}\hat{R}'}\rangle$ has the property

$$\rho_j^{A_1\hat{B}} = \tau_{A_1} \otimes \rho_{\hat{R}'}$$  \hfill (8)

where $\rho_j^{A_1\hat{B}}$ is the reduced density matrix of $\Psi_j^{A_1\hat{B}\hat{R}'}$, $\tau_{A_1}$ is the maximally mixed state of dimension $L$ on Alice’s system $A_1$, and $\rho_{\hat{R}'}$ is the reduced density matrix of the original state $\Psi_{\hat{A}\hat{B}\hat{R}}$. Then (see [17]) there exists an isometry $U_j : \hat{B} \rightarrow B_1\tilde{B}'\hat{B}$ on Bob’s side, such that

$$(I_{A_1\hat{B}} \otimes U_j) |\Psi_j^{A_1\hat{B}\hat{R}'}\rangle = |\Phi_L\rangle_{A_1B_1} \otimes |\Psi\rangle_{\tilde{B}'\hat{B}\hat{R}}.$$  \hfill (9)

where $|\Psi\rangle_{\tilde{B}'\hat{B}\hat{R}}$ is the original state $|\Psi\rangle_{\hat{A}\hat{B}\hat{R}}$ with the system $\tilde{B}'$ substituted for $\hat{A}$. This is because $\Psi$ is the purification of $\rho_{\hat{R}}$ and $\Phi_L$ that of $\tau_{A_1}$, so both $\Psi_j^{A_1\hat{B}\hat{R}'}$ and $(\Phi_L)_{A_1B_1} \otimes |\Psi\rangle_{\tilde{B}'\hat{B}\hat{R}}$ are purifications of $\tau_{A_1} \otimes \rho_{\hat{R}'}$. Hence, by Uhlmann’s theorem, they are related by a unitary on Bob’s system.

Since we require fidelity approaching 1 only in the asymptotic limit, we obtain the following merging condition:

**Proposition 3 (Merging condition)** Consider Alice’s measurement with outcomes $j$, which occur with probability $p_j$. Denote the state after the measurement result $j$ was obtained by $|\Psi_j^{A_1\hat{B}\hat{R}'}\rangle$, and its reduced density matrix by $\rho_j^{A_1\hat{R}}$. The following condition implies the existence of a merging protocol with entanglement cost $-\log L$ and error $2\sqrt{\epsilon}$: that the so-called quantum error $Q_e$ satisfies

$$Q_e := \sum_j p_j \| \rho_j^{A_1\hat{R}} - \tau_{A_1} \otimes \rho_{\hat{R}} \|_1 \leq \epsilon,$$  \hfill (10)

where $\tau_{A_1}$ is the maximally mixed state of dimension $L$ on $A_1$. 
Proof. The proof is based on the above considerations concerning the ideal situation. Using the relation eq. (A4) between the trace distance and the fidelity, we get

\[ \sum_j p_j \sqrt{F(\rho_{A_1 R_1}^j, \tau_{A_1} \otimes \rho_{R_1})} \geq 1 - \frac{\epsilon}{2}. \]  

(11)

Then, by Uhlmann’s theorem \[18, 19\] there exist isometries \( U_j \) of Bob such that

\[ F(\rho_{A_1 R_1}^j, \tau_{A_1} \otimes \rho_{R_1}) = F((I_{A_1 R_1} \otimes U_j) |\Psi_J\rangle_{A_1 B_1, R_1} \otimes |\Psi\rangle_{B_1 B_1, \bar{R}_1}), \]  

(12)

hence

\[ \sum_j p_j F((I_{A_1 R_1} \otimes U_j) |\Psi_J\rangle_{A_1 B_1, R_1} \otimes |\Psi\rangle_{B_1 B_1, \bar{R}_1}) \geq \left( \sum_j p_j \sqrt{F((I_{A_1 R_1} \otimes U_j) |\Psi_J\rangle_{A_1 B_1, R_1} \otimes |\Psi\rangle_{B_1 B_1, \bar{R}_1})} \right)^2 \geq \left( 1 - \frac{\epsilon}{2} \right)^2 \geq 1 - \epsilon. \]  

(13)

So, with the output state of the protocol,

\[ \rho_{A_1 B_1, \bar{A} \bar{B} \bar{R}} = \sum_j (I_{A_1 R_1} \otimes U_j) |\Psi_J\rangle_{A_1 B_1, \bar{R}_1} (I_{A_1 R_1} \otimes U_j) \dagger, \]  

(14)

we obtain

\[ F(\rho_{A_1 B_1, \bar{A} \bar{B} \bar{R}}, (\Phi_L)_{A_1 B_1} \otimes \Psi_{\bar{B} \bar{R}}) \geq 1 - \epsilon. \]  

(15)

And using the relation (A4) between fidelity and trace distance once more, we arrive at

\[ \|\rho_{A_1 B_1, \bar{A} \bar{B} \bar{R}} - (\Phi_L)_{A_1 B_1} \otimes \Psi_{\bar{B} \bar{R}}\|_1 \leq 2\sqrt{\epsilon}, \]  

(16)

which concludes the proof. \( \square \)

Note that for any protocol which achieves merging, the condition (10) must necessarily be met at some stage of the protocol. This is because in order for the final state to be close to the original state, \( \rho_{\bar{R}} \) must necessarily be virtually unchanged, and in order for the state to be at Bob’s site, Alice’s state must necessarily be in a product state with the reference system \( \bar{R} \).

B. One-shot merging by random measurement

Here we will prove an abstract, one-shot version of the main theorem, showing that a random orthogonal measurement of rank-\( L \) projectors (and a little remainder) achieves merging.

Proposition 4 (One-shot merging) Let \( \Psi_{\bar{A} \bar{B} \bar{R}} \) be a pure state, with local dimensions \( d_{\bar{A}}, d_{\bar{B}}, d_{\bar{R}} \), and \( \text{Tr} \rho_{\bar{R}}^2 \leq \frac{1}{2} \). Then there exists a POVM consisting of \( N = \left\lfloor \frac{d_{\bar{A}}}{L} \right\rfloor \) projectors of rank \( L \) and one of rank \( L' = d_{\bar{A}} - NL < L \) such that

\[ Q_{\epsilon} \leq 2 \sqrt{L \frac{d_{\bar{R}}}{D}} + 2 \frac{L}{d_{\bar{A}}}, \]  

(17)

and there is a merging protocol with error at most \( 2 \sqrt{2 \sqrt{L \frac{d_{\bar{R}}}{D}} + 2 \frac{L}{d_{\bar{A}}}} \).

In fact, by choosing the measurement at random according to the Haar measure on \( \bar{A} \), the expectation of the left hand side of eq. (17) is upper bounded by the right hand side.
Remark 5 Let us explain here briefly how we will use the Lemma in the proof of Theorem 2 in the case of negative $S(A|B)$. Namely, we will apply this Lemma with the following parameters: $d_R \approx 2^{nS_R} = 2^{nS_B}$, $d_{\tilde{A}} \approx 2^{nS_{\tilde{A}}} = 2^{nS(A)}$, $D \approx 2^{nS(R)} = 2^{nS(B)}$, where $n$ is the number of copies of initial state $\psi_{ABR}$ shared by Alice, Bob and reference system. Moreover $L$ will be related to the rate $r$ of singlets obtained between Alice and Bob in the process of merging: $L \approx 2^{nr}$. Then the expression for quantum error will be

$$Q_c \approx 2^{\frac{1}{2}n(S(AB)-S(B)+r)} + 2^{n(r-S(A))+1}$$

(18)

Thus if only $r < S(AB) - S(B)$, then the quantum error will decay exponentially with $n$.

The crucial technical result in the proof of Proposition 4 will be the following statement about random (Haar distributed) rank-$L$ projectors:

Lemma 6 Let $P : \tilde{A} \rightarrow A_1$ be a random partial isometry of rank $L$, i.e. $P^\dagger P$ is a projection onto a $L$-dimensional subspace of $\tilde{A}$. For example, one might put $P = P_0U$ with some fixed rank $L$-projector $P_0$ onto a subspace $A_1$ of $\tilde{A}$, and a Haar distributed unitary $U$ on $\tilde{A}$. For the subnormalized density matrix

$$\omega_{\tilde{A}R} = (P \otimes I_R)\rho_{\tilde{A}R}(P \otimes I_R)^\dagger,$$

observe that its average over unitaries $U$ is

$$\langle \omega_{\tilde{A}R} \rangle = \frac{L}{d_{\tilde{A}}} Tr_{A_1} \otimes \rho_{\tilde{R}}.$$

And we have:

$$\langle \omega_{\tilde{A}R} - \frac{L}{d_{\tilde{A}}} Tr_{A_1} \otimes \rho_{\tilde{R}} \rangle^2 \leq \frac{L^2}{d_{\tilde{A}}^2} \frac{1}{D},$$

(19)

$$\langle 2 \omega_{\tilde{A}R} \rangle \leq \frac{L^2}{d_{\tilde{A}}^2} \frac{1}{L} Tr \rho_{\tilde{R}}.$$

(20)

Proof. In Appendix A we recall the basic properties of the trace norm $\| \cdot \|_1$ and the Hilbert-Schmidt norm $\| \cdot \|_2$. From there (Lemma 13) we take that $\|X\|_1 \leq \sqrt{d}\|X\|_2$ for an operator on a $d$-dimensional space. This, and the concavity of the square root function, show that eq. (19) implies eq. (20).

To prove eq. (19), we use the fact that it has the form of a variance, so

$$\langle \omega_{\tilde{A}R} - \frac{L}{d_{\tilde{A}}} Tr_{A_1} \otimes \rho_{\tilde{R}} \rangle^2 = \langle \omega_{\tilde{A}R} \rangle^2 - \langle \omega_{\tilde{A}R} \rangle + \frac{L^2}{d_{\tilde{A}}^2} \frac{1}{L} Tr \rho_{\tilde{R}}.$$

(21)

To evaluate the average of $Tr \omega_{\tilde{A}R}^2$, we use the well-known equation

$$Tr \omega_{\tilde{A}R}^2 = Tr ((\omega_{A_1 R} \otimes \omega_{A_1 R})(F_{A_1 A_1} \otimes F_{R R})),$$

(22)

where we have introduced copies of all systems involved, and with the swap (or flip) operator $F$ exchanging the two systems. (Note that $F_{\tilde{A}R,\tilde{A}R} = F_{\tilde{A}A} \otimes F_{R R}$.) With this, and w.l.o.g. assuming that $A_1$ is a subspace of $\tilde{A}$,

$$\langle Tr \omega_{\tilde{A}R}^2 \rangle = \langle Tr ((\omega_{A_1 R} \otimes \omega_{A_1 R})(F_{A_1 A_1} \otimes F_{R R})) \rangle$$

$$= \langle Tr ((UU_{\tilde{A}A} \otimes I_{R R})(\rho_{\tilde{A}R} \otimes \rho_{\tilde{A}R})(UU_{\tilde{A}A} \otimes I_{R R}) (F_{A_1 A_1} \otimes F_{R R})) \rangle$$

$$= \langle Tr ((\rho_{\tilde{A}R} \otimes \rho_{\tilde{A}R})(UU_{\tilde{A}A} \otimes I_{R R})(F_{A_1 A_1} \otimes F_{R R})(UU_{\tilde{A}A} \otimes I_{R R})) \rangle$$

$$= \langle Tr ((\rho_{\tilde{A}R} \otimes \rho_{\tilde{A}R})(UU_{\tilde{A}A} \otimes I_{R R})(F_{A_1 A_1} \otimes F_{R R})(UU_{\tilde{A}A} \otimes I_{R R})) \rangle.$$

(23)
where we have used the shorthand $UU_{\tilde{A}\tilde{A}} := U_\tilde{A} \otimes U_\tilde{A}$. In Appendix B we demonstrate, how using elementary arguments from the representation theory of $U \otimes U$, one can calculate that

$$\langle (UU_{\tilde{A}\tilde{A}})^\dagger F_{A_1 A_1} (UU_{\tilde{A}\tilde{A}}) \rangle = \frac{L}{d_{\tilde{A}}} \frac{d_{\tilde{A}} - L}{d_{\tilde{A}}^2 - 1} I_{\tilde{A}\tilde{A}} + \frac{L}{d_{\tilde{A}}} \frac{L d_{\tilde{A}} - 1}{d_{\tilde{A}}^2 - 1} F_{\tilde{A}\tilde{A}}. \tag{24}$$

Inserting this into eq. (23) gives

$$\langle \text{Tr} \omega_{A_1, R}^2 \rangle = \frac{L}{d_{\tilde{A}}} \frac{d_{\tilde{A}} - L}{d_{\tilde{A}}^2 - 1} \text{Tr} \rho_R^2 + \frac{L}{d_{\tilde{A}}} \frac{L d_{\tilde{A}} - 1}{d_{\tilde{A}}^2 - 1} \text{Tr} \rho_{A_1 R}^2 \leq \frac{L}{d_{\tilde{A}}} \text{Tr} \rho_R^2 + \frac{L^2}{d_{\tilde{A}}^2} \frac{1}{D},$$

and looking at eq. (21) we are done. \hfill \Box

**Proof of Proposition 4.** Fix a random measurement according to the description of the Proposition. One way of doing this is picking $N$ fixed orthogonal subspaces of dimension $L$, and one of dimension $L' = d_{\tilde{A}} - NL < L$. The projectors onto these subspaces followed by a fixed unitary mapping it to $A_1$ we denote by $Q_j$, $j = 0, \ldots, N$. Then put $P_j := Q_j U$ with a Haar distributed random unitary $U$ on $A$.

Then, by Lemma 6, with $\omega_{A_1 R}^j = (P_j \otimes I_R) \rho_{A_1 R} (P_j \otimes I_R)^\dagger$,

$$\left\langle \sum_{j=1}^{N} \left\| \omega_{A_1 R}^j - \frac{L}{d_{\tilde{A}}} \tau_{A_1} \otimes \rho_R \right\|_1 \right\rangle \leq N \frac{L}{d_{\tilde{A}}} \sqrt{\frac{L d_R}{D}} \leq \sqrt{\frac{L d_R}{D}}. \tag{26}$$

This is almost what we want, except that we haven’t taken into account the normalisation: with $p_j = \text{Tr} \omega_{A_1 R}^j$, and $\rho_{A_1 R}^j = \frac{1}{p_j} \omega_{A_1 R}^j$, we need to argue that on average, the $p_j$ are close to $\frac{L}{d_{\tilde{A}}}$. Indeed, eq. (26) implies

$$\left\langle \sum_{j=1}^{N} p_j \left( \frac{L}{d_{\tilde{A}}} - \frac{L}{d_{\tilde{A}}} \right) \right\rangle \leq \sqrt{\frac{L d_R}{D}}. \tag{27}$$

hence we obtain

$$\left\langle \sum_{j=1}^{N} p_j \left( \rho_{A_1 R}^j - \tau_{A_1} \otimes \rho_R \right) \right\|_1 \right\rangle \leq 2 \sqrt{\frac{L d_R}{D}}. \tag{28}$$

Finally, it is clear that $\langle \text{Tr} \rho_{A_1 R}^0 \rangle = \frac{L'}{d_{\tilde{A}}} < \frac{L}{d_{\tilde{A}}}$, and since the trace distance of two states is at most 2, we get the result as advertised, because the quantum error is composed of the probability of hitting $P_0$ and the sum of the error terms of the $P_j$, weighted by their probabilities. Now we can apply Proposition 3. \hfill \Box

So, if $d_R \ll D$ there is a merging LOCC protocol with small error and entanglement cost up to $\log d_R - \log D$ (i.e., the negative of this is the amount of entanglement produced). If $d_R \not\ll D$, consider the state $\Psi_{A_1 B R} \otimes (\Phi_K)_{A_0 B_0}$ with a maximally entangled state of Schmidt rank $K \gg d_R/D$. Now merging is possible (with $L = 1$); the entanglement cost is $\log K$, and it can be made as small as $\log d_R - \log D$.

**IV. PROOF OF THE MAIN THEOREM**

**A. Achievability of merging**

**Proof of Theorem 2.** We will first prove the direct part saying that the rates are achievable. Consider $n$ copies of the state $|\psi\rangle_{A B R}$, and assume first that $S(A|B) < 0$.

We would like to use our one-shot version, Proposition 4, but cannot do so directly, since the dimension $d_R^n$ and the number $\langle \text{Tr} \rho_R^n \rangle^n$ are not information theoretically meaningful.

Instead, we consider the vector $|\Omega\rangle_{A B R}$ and state $|\Psi\rangle_{A B R}$, with

$$|\Omega\rangle_{A B R} := (\Pi_{\tilde{A}} \otimes \Pi_{\tilde{B}} \otimes \Pi_R) |\psi\rangle_{A B R}^\otimes n, \quad |\Psi\rangle_{A B R} := \frac{1}{\langle \Omega | \Omega \rangle} |\Omega\rangle_{A B R}. \tag{29}$$
where $\tilde{A}$, $\tilde{B}$ and $\tilde{R}$ are the typical subspaces of $A^n$, $B^n$ and $R^n$, respectively, and $\Pi_{\tilde{A}}$, etc. are the projection operators onto these typical subspaces. In Appendix C we explain what is necessary to know about typicality, in particular we have:

$$\langle \Omega | \Omega \rangle = \langle \psi | \otimes_n (\Pi_{\tilde{A}} \otimes \Pi_{\tilde{B}} \otimes \Pi_{\tilde{R}}) | \psi \rangle \geq 1 - \epsilon,$$

for any $\epsilon > 0$ and large enough $n$. Indeed, we can choose $\epsilon = 3 \exp(-c\delta^2 n)$ with some constant $c$, where $\delta > 0$ is a typicality parameter; namely from eq. (C5) in Appendix C we have $\text{Tr} \rho_{\tilde{A}} \otimes_n \Pi_{\tilde{A}}$, $\text{Tr} \rho_{\tilde{B}} \otimes_n \Pi_{\tilde{B}}$, $\text{Tr} \rho_{\tilde{R}} \otimes_n \Pi_{\tilde{R}} \geq 1 - \exp(-c\delta^2 n)$. We obtain the bound (30) from observing

$$I_{A^n} \otimes I_{B^n} \otimes I_{R^n} - \Pi_{\tilde{A}} \otimes \Pi_{\tilde{B}} \otimes \Pi_{\tilde{R}} \leq (I_{A^n} - \Pi_{\tilde{A}}) \otimes (I_{B^n} - \Pi_{\tilde{B}}) \otimes (I_{R^n} - \Pi_{\tilde{R}}).$$

Furthermore, with $\Omega = |\Omega\rangle \langle \Omega|$, we have (using eqs. (C10), (C9) and (C6) in Appendix C)

$$\text{rank} \Omega_{\tilde{A}} \geq (1 - \epsilon)2^{n[S(A)-\delta]},$$

$$\text{rank} \Omega_{\tilde{R}} \leq 2^{n[S(R)+\delta]},$$

$$\Omega_{\tilde{B}} \leq \Pi_{\tilde{B}} \rho_{\tilde{B}} \otimes_n \Pi_{\tilde{B}} \leq 2^{-n[S(B)-\delta]} \Pi_{\tilde{B}}.$$  

Hence we get, for the normalized $\tilde{\psi}_{\tilde{A}\tilde{B}\tilde{R}}$:

$$d_{\tilde{A}} \geq (1 - \epsilon)2^{n[S(A)-\delta]}, \quad d_{\tilde{R}} \leq 2^{n[S(R)+\delta]}, \quad D \geq (1 - \epsilon)2^{n[S(B)-\delta]}.$$  

By the gentle measurement Lemma 15 (see Appendix A), we obtain from eq. (30)

$$\|\tilde{\psi}_{\tilde{A}\tilde{B}\tilde{R}} - \tilde{\Omega}_{\tilde{A}\tilde{B}\tilde{R}}\|_1 \leq 2\sqrt{\epsilon}, \quad \text{hence } \|\tilde{\psi}_{\tilde{A}\tilde{B}\tilde{R}} - \tilde{\Omega}_{\tilde{A}\tilde{B}\tilde{R}}\|_1 \leq 4\sqrt{\epsilon}.\$$

Now Alice and Bob follow a merging protocol as if they had $\tilde{\psi}_{\tilde{A}\tilde{B}\tilde{R}}$, and with $L = 2^n[S(B)-S(R)-3\delta]$. If the state were actually $\tilde{\psi}_{\tilde{A}\tilde{B}\tilde{R}}$, the quantum error would be

$$Q_{\epsilon} \leq 2 \sqrt{L \frac{d_{\tilde{A}}}{D}} + 2nL \frac{D}{d_{\tilde{A}}} \leq \frac{2}{1-\epsilon} 2^{-n\delta/2} + 2^{1-2n\delta}.$$  

(Observe that $S(B) - S(R) \leq S(A)$ by subadditivity.) So, by Proposition 3 we would get a merging protocol with error $O(2^{-n\delta/4})$. By eq. (34), running the same protocol on $\tilde{\psi}_{\tilde{A}\tilde{B}\tilde{R}}$, we obtain an error of $O(2^{-n\delta/4}) + O(2^{-cn\delta^2/2})$, which vanishes exponentially as $n \to \infty$. Since $\delta > 0$ was arbitrary, the direct part follows.

It remains to consider the case when $S(A|B)$ is non-negative. Here, Alice and Bob share additionally $n(S(A|B)+\Delta)$ maximally entangled states. Each ebit contributes conditional entropy $-1$, so that the final state has negative conditional entropy $-n\Delta$. Then however merging can be done by LOCC, as we have proven above.

**Remark 7** Note that despite the generality of the definition of merging, our protocol is much more special. The definition allows to start end with certain amounts of ebits, but the amount charged is only the difference, so that it would be conceivable that to achieve the conditional entropy some catalytic use of entanglement is necessary. However, our protocol either needs no initial entanglement and outputs some (if $S(A|B) < 0$) or produces no entanglement but needs some initially (if $S(A|B) \geq 0$).

**B. Merging is optimal**

Let us now turn to the converse part. The essence of the proof is that entanglement cannot increase under local operations and classical communication and transmission of $n$ qubits more than by $n$ [20]. We will consider preservation of Bob’s entanglement with Alice and the Reference. The initial entanglement $E_{in}$ includes the entanglement of the shared state plus any initial resource of pure entanglement $\log K$. Initially, it is $nS(B) + \log K$ as the initial state was just $\tilde{\psi}_{\tilde{A}\tilde{B}}$. The final entanglement $E_{out}$ includes the entanglement of the final state plus the final resource, $\log L$ bits of pure state entanglement, and is

$$E_{out} \approx nS(AB) + n \log L.$$  

Since Alice and Bob used only LOCC operations, we have

$$E_{out} \leq E_{in}$$  

(37)
as entanglement could only decrease, giving $R = \log K - \log L < S(AB) - S(B)$.

In more detail, assume $L \leq 2^{O(n)}$ for technical reasons. The LOCC protocol (which is also LOCC between Bob and Alice+Reference) can be thought of as generating an ensemble $\{\varphi^k_{A_1 B_1 B^n B^n R^n} \}, k \}$ of pure states. Monotonicity of the entropy of entanglement under LOCC [21] means
\[ nS(B) + \log K \geq \sum_k q_k S(\varphi^k_{B_1 B^n B^n}). \] (38)

The condition (4) for successful merging translates into
\[ \sum_k q_k F(\varphi^k_{A_1 B_1 B^n B^n R^n}, (\Phi_L)_{A_1 B_1} \otimes \psi^n_{BR}) \geq 1 - \epsilon, \] (39)

thanks to the linearity of the fidelity when one argument is pure. Using eq. (A4) in Appendix A this yields
\[ \sum_k q_k \| \varphi^k_{A_1 B_1 B^n B^n} - (\Phi_L)_{A_1 B_1} \otimes \psi^n_{BR} \|_1 \leq 2\sqrt{\epsilon}, \] (40)

hence by monotonicity of the trace norm under partial tracing,
\[ \sum_k q_k \| \varphi^k_{B_1 B^n B^n} - \tau_{A_1} \otimes \rho^n_{BR} \|_1 \leq 2\sqrt{\epsilon}. \] (41)

By Fannes’ inequality (stated as Lemma 16 in Appendix A), this finally gives
\[ \sum_k q_k \| S(\varphi^k_{B_1 B^n B^n}) - \log L - nS(AB) \| \leq (\log L + n \log d_A + n \log d_B)\eta(2\sqrt{\epsilon}) \leq O(n)\eta(2\sqrt{\epsilon}), \] (42)

using the concavity of the $\eta$-function. With eq. (38), we thus get
\[ \frac{1}{n}(\log K - \log L) \geq S(A|B) - O(1)\eta(2\sqrt{\epsilon}), \] (43)

which results in the converse when $n \to \infty$ and $\epsilon \to 0$. \qed

C. Classical communication cost of merging

In our protocol for quantum state merging, the amount of classical communication that Alice needs to send Bob is given by the number of possible measurement outcomes: at most $\frac{d_A d_B}{\eta^2} + 1$, which in the i.i.d. case $\psi^n_{ABR}$ means a rate of $S(A) + S(R) - S(AB) = I(A : R)$. Note that this is true regardless of $S(A|B) \geq 0$ or $S(A|B) < 0$.

We now show that this amount of communication is needed, and thus our protocol is communication optimal.

**Theorem 8** For a state $|\psi\rangle_{ABR}$ shared by Alice, Bob and the Reference, the classical communication cost of merging is equal to the quantum mutual information between Alice and the reference system $R$, $I(A : R) = S(A) + S(R) - S(AB)$.

**Proof.** We will first need to take a short digression. Consider a protocol which achieves merging with an entanglement rate $R_q$ and classical communication at rate $R_c$. Now let us imagine that the parties do not have access to a classical channel, so must send all their classical communication via the quantum channel, encoded into qubits. This gives a fully quantum version of merging [22] similar to the “mother protocol” (see [23] for an alternative, direct proof). If $R_q = S(A|B)$ and $R_c = I(A : R)$, we have, in the “sloppy” notation of [24],
\[ \frac{1}{2} I(A : R)[q \rightarrow q] \geq \frac{1}{2} I(A : B)[qq] + (\text{id}_{A \rightarrow B'} : \rho_{AB}), \] (44)

where the equation means that a rate of $\frac{1}{2} I(A : R)$ uses of a noiseless qubit channel $[q \rightarrow q]$, and it produces $\frac{1}{2} I(A : B)$ bits of shared entanglement $[qq]$ in addition to achieving state merging from Alice to Bob. The latter is represented by $(\text{id}_{A \rightarrow B'} : \rho_{AB})$, i.e. a identity channel from Alice to Bob working on the source $\rho_{AB}$.

We briefly sketch how state merging gives the protocol of eq. (44). Our merging protocol is expressed in the resource inequality formalism as
\[ S(A|B)[qq] + I(A : R)[c \rightarrow c] \geq (\text{id}_{A \rightarrow B'} : \rho_{AB}), \] (45)
where \([c \rightarrow c]\) stands for the communication resource of 1 classical bit. Recall that for any state merging protocol, the classical communication must be completely decoupled from the sent state for \(|\psi\rangle_{ABR}^\prime\) to remain pure, and thus it can be recycled as \(R_c\) bits of entanglement; the entanglement can further be used to send quantum states. This is what the authors of [24, 25] call Rule I, where each bit of classical communication (denoted as \([c \rightarrow c]\)) can be made coherent: we denote a coherent classical [26] bit by \([q \rightarrow qq]\). At the left hand side of an inequality like (45), Rule I says that it can be replaced by half a bit of a quantum channel on the left and half a bit of shared entanglement on the right hand side (denoted \(\frac{1}{2}[q \rightarrow q] - \frac{1}{2}[qq]\)). One sees this by sending the classical communication used in teleportation as coherent qubits which are then recycled into entanglement. Thus,

\[
[q \rightarrow qq] = \frac{1}{2}[q \rightarrow q] - \frac{1}{2}[qq].
\]

Applying Rule I of eq. (46) to eq. (45), and rearranging the terms gives the mother protocol in the formulation of eq. (44).

We now show that the mother is an optimal protocol to achieve state merging in the case when one doesn’t have access to a classical channel (see also [23]). We use the fact that a necessary condition for any state merging protocol is that Alice must completely decouple herself from the state \(|\psi\rangle_{ABR}\). This is because the state needs to be shared by \(R\) and \(B\) by definition of state merging.

Whatever Alice does, including measurements and processing, we may consider coherently, as an operation which takes \(\rho_A\) and some ancillas, and produces a part which gets sent down the quantum channel, and a part \(\rho_A'\) she retains. This results in a state \(|\psi'\rangle_{B'B'R}\) which has high fidelity with \(|\psi\rangle_{ABR}^\prime\), plus some entanglement between Alice and Bob. Now, using standard quantum cryptographic reasoning originating in [27], if \(|\psi'\rangle_{B'B'R}\) is (almost) pure, then the system \(A'\) must be virtually in a product state with \(B'B'R\). In particular, the mutual information between the state \(\rho'_A\) and the reference system \(R\) must be close to zero. Each qubit sent can reduce Alice’s mutual information with the reference system by at most 2, thus at a minimum, Alice must send \(\frac{1}{2}I(A : R)\) qubits down the quantum channel. This gives the optimality of Alice’s use of the quantum channel in protocol (44).

That at most \(\frac{1}{2}I(A : B)\) bits of entanglement are obtainable from the shared state, when sending \(\frac{1}{2}I(A : R)\) qubits, can be easily seen as follows. Observe that the \(\frac{1}{2}I(A : R)[q \rightarrow q]\) on the left hand side of eq. (44) can be replaced by \(\frac{1}{2}I(A : R)[qq] + I(A : R)[c \rightarrow c]\) due to teleportation. If the entanglement rate on the right were larger than \(\frac{1}{2}I(A : B)\), we could perform state merging with entanglement rate strictly smaller than \(I(A : R) - \frac{1}{2}I(A : B) = S(A|B)\), contradicting the converse of Theorem 2.

Now, to prove optimality of the classical communication in eq. (45), consider a hypothetical state merging protocol

\[
R_q[qq] + R_c[c \rightarrow c] \geq \langle \text{id}_{A \rightarrow B'} : \rho_{AB} \rangle
\]

which we may transform using Rule I [24, 25] into

\[
\left( R_q - \frac{1}{2}R_c \right)[qq] + \frac{1}{2}R_c[q \rightarrow q] \geq \langle \text{id}_{A \rightarrow B'} : \rho_{AB} \rangle.
\]

Comparing this with the mother protocol (44), we have that \(R_c \geq I(A : R)\) by virtue of the optimality of (44); \(R_q \geq S(A|B)\) comes out again, as it should.

Thus in addition to giving an operational interpretation for the quantum condition entropy, merging gives an operational interpretation for the quantum mutual information. Secondly, the measurement of Alice makes her state completely product with \(R\), thus reinforcing the interpretation of quantum mutual information as the minimum entropy production of any local decorrelating process [28, 29]. This same quantity is also equal to the amount of irreversibility of a cyclic process: Bob initially has a state, then gives Alice her share (communicating \(S(A)\) qubits), which is finally merged back to him (communicating \(S(A|B)\) qubits). The total quantum communication of this cycle is \(I(A : R)\) quantum bits.

Having concluded our proofs regarding state merging, we now turn to its applications.

V. DISTRIBUTED COMPRESSION

In usual Schumacher compression, a single party Alice, receives a state from a source, and must compress the states so that they can be faithfully decoded by another party. For a source emitting states with density matrix \(\rho_A\), this can be done at a rate given by the entropy \(S(A)\) of the source [3]. One can imagine the situation where the states are distributed over many parties, and have to be compressed individually. Each party then sends their compressed share
to a decoder who must be able to decode the full state. In the classical case, this problem was solved by Slepian and Wolf [2] who found that the total rate for distributed compression could equal the compression rate when the parties are not distributed. In the quantum case, previous results [9, 30] were interpreted as indications that one cannot compress at the same rate in the distributed vs. non-distributed case. However, using state merging, we will show that formally the same achievable rate region as in the Slepian-Wolf theorem is obtained.

In detail, we assume that the source emits states with average density matrix $\rho_{A_1A_2...A_m}$, and distributes it over $m$ parties. The parties wish to compress their shares as much as possible so that the full state can be reconstructed by a single decoder. We allow classical side information for free (we will only need classical communication from each encoder to the joint decoder), and only ask about the rate $R_i$ of entanglement between the $i$th encoder and the decoder. A tuple $(R_1,...,R_m)$ is achievable if there exists an $(m+1)$-party LOCC procedure taking in the source $\rho_{A_1...A_m}$, purified to a state $\psi_{RA_1...A_m}^{\otimes n}$, and $n(R + \epsilon)$ ebits between $A_i$ and the decoder $B$, such that the final state is $\rho_{R_{i}B_{i}'}^{n}$ with

$$F(\rho, \psi^{\otimes n}) \geq 1 - \epsilon,$$

and $\epsilon \to 0$ as $n \to \infty$. As always, the reference is passive, and plays no role in the protocol. Note that the rates $R_i$ can be negative here, just as in state merging, meaning that $n(-R_i + \epsilon)$ ebits are returned by the protocol.

Let us first describe the quantum solution for two parties and depict the rate region in Figure 2. If one party compresses at a rate $S(B)$, then the other party can over-compress at a rate $S(A|B)$, by merging her state with the state which will end up with the decoder. The only difference between this scenario and the state merging one, is that Bob first compresses his state, and sends it to the decoder, who then decompresses it; Alice then merges her state with Bob's state which is now at the decoder. This gives us one possible way for the two parties to jointly compress the states. Time-sharing gives the full rate region, since the bounds evidently cannot be improved.

Analogously, for $m$ parties $A_i$, and all subsets $T \subseteq \{A_1, A_2, \ldots, A_m\}$ holding a combined state with entropy $S(T)$, the rate sums $R_T = \sum_{A_i \in T} R_{A_i}$ clearly have to obey

$$R_T \geq S(T|\overline{T}) \quad \text{for all sets } T,$$

with $\overline{T} = \{A_1, A_2, \ldots, A_m\} \setminus T$ the complement of set $T$. This just follows from the converse to Theorem 2: even if

![FIG. 2: The rate region for distributed compression by two parties with individual rates $R_A$ and $R_B$. The total rate $R_{AB}$ is bounded by $S(AB)$. The top left diagram shows the rate region of a source with positive conditional entropies; the top right and bottom left diagrams show the purely quantum case of sources where $S(A|B) < 0$ or $S(B|A) < 0$. It is even possible that both $S(B|A)$ and $S(A|B)$ are negative, as shown in the bottom right diagram, but observe that the rate-sum $S(AB)$ has to be positive.](image-url)
the decoder somehow has all the shares $\mathcal{T}$, a total rate of at least $S(\mathcal{T}|\mathcal{T})$ is necessary to convey the remaining shares $\mathcal{T}$.

That this bound can be achieved simply follows from the fact that with $\mathcal{T}$ at the decoder, each party can in turn merge their state with what will be at the decoder. So, for example, with four parties, an obtainable rate point is obtained when party $A_1$ sends her state at rate $S(A_1)$ just by regular Schumacher compression, party $A_2$ merges her state with the first parties state at the decoder with rate $S(A_2|A_1)$, party $A_3$ merges at a rate $S(A_3|A_1A_2)$, and party $A_4$ at rate $S(A_4|A_1A_2A_3)$, with rate total being the Schumacher rate $S(A_1A_2A_3A_4)$, etc. These rate tuples are however just the corners of the region defined by eq. (50); hence time sharing between various combinations of ordering the encoders gives the full rate region.

VI. QUANTUM SOURCE CODING WITH SIDE INFORMATION AT THE DECODER

Related to distributed compression is the case where only Alice’s state needs to arrive at the decoder, while Bob can send part of his state to the decoder (subject to a rate constraint) in order to help Alice lower her rate. The classical case of this problem was introduced by Wyner [31]. For the quantum case, we demand that the full state $\psi_{ABR}$ be preserved in the protocol, but do not place any restriction on what part of Bob’s state may be at the decoder and what part can remain with him, while Alice’s has to go to the decoder.

To arrive at a formal definition, we would like to speak of two rates $R_A$ and $R_B$ here, of entanglement between Alice and the decoder $C$ and of Bob and the decoder $C$. Starting with $n$ copies of the source, $\Psi_{A^nB^nR^n} = \psi_{ABR}^\otimes n$, we may consider LOCC protocols between $A$, $B$ and $C$, that take in this state and maximally entangled states of Schmidt rank $K_A(K_B)$ between $A$ and $C$ ($B$ and $C$). It is supposed to produce a high-fidelity approximation of $\Psi_{C^nC'\otimes B^nR^n}$ tensored with maximally entangled states of Schmidt rank $L_A(L_B)$ between $A$ and $C$ ($B$ and $C$), where $\Psi_{C^nC'\otimes B^nR^n}$ is obtained from $\psi_{ABR}^\otimes n$ by substituting $C^n$ for $A^n$ and with an isometry (e.g. a unitary operation taking one system to two systems) $B^n \rightarrow C^nB'$. If in the limit of arbitrary block length the fidelity tends to 1 and $\frac{1}{n}(\log K_A - \log L_A) \rightarrow R_A$, $\frac{1}{n}(\log K_B - \log L_B) \rightarrow R_B$, we call the rate pair $(R_A, R_B)$ achievable, and the side information problem is to characterise the achievable pairs as concisely as possible.

Using state merging we can see that for any isometry $T : B \rightarrow U \otimes V$, the rates

$$R_A = S(A|U) \quad \text{and} \quad R_B = E_P(AU : R) - S(A|U)$$

are achievable, where $\psi_{AUVR} = (id_A \otimes T \otimes id_R)\psi_{ABR}$, and

$$E_P(AU : R) = \min_{\Lambda} S((id_A \otimes \Lambda)\rho_{AUV})$$

is the so-called entanglement of purification [32] of the state $\rho_{AU,R}$ with respect to the split $AU-R$. The minimum is taken over all channels $\Lambda$ acting on $V$. The entanglement of purification is in some sense a measure of total correlations, as it can be interpreted as the amount of entanglement needed to create a state, if the only allowed operations is tracing out.

The achievability of rates can be seen as follows: the channel $\Lambda$ can be represented, with the help of an environment $B'$, as another isometry $V \rightarrow W'B'$, so that $\psi_{UV} = \psi_{AUW'B'}$. Now, with many copies, let Bob send the system $U$ to the decoder, at rate $S(U)$, and Alice merge her state to the decoder, at rate $R_A = S(A|U)$. Finally, with the decoder now having $AU$, let Bob merge $W$ to him, which has rate $S(W|AU)$, so that the total of Bob’s rate is $R_B = S(U) + S(W|AU) = S(AUV) - S(A|U)$. The minimisation over $W$ leads to the formula for the entanglement of purification.

Here, the isometry $T$ acts on many copies of $B$, and up to this “regularisation limit”, the rate pairs (51) are optimal for one-way protocols. To see why this is so, consider that at the end of the protocol, Bob will have sent part of his state to the decoder. This part, $U$, is obtained by some local isometry of Bob’s: $B^n \rightarrow UV$. Likewise, Alice will have sent all her $A^n$ to the decoder. The total amount of entanglement used, $n(R_A + R_B)$, cannot be less than the total entropy of what ends up at the receiver, which has entropy $S(A^nU)$, and this is lower bounded by $E_P(A^nU : R)$. By the converse of Theorem 2, Alice’s entanglement cost, $nR_A$, cannot be less than $S(A^n|U)$. Thus we have proved that the set of achievable pairs is given by

$$\bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} (S(A^n|U), E_P(A^nU : R^n) - S(A^n|U)) \text{ s.t. } T : B^n \rightarrow UV \text{ isometry} \right\}.$$

(Note that since the formula doesn’t mention $V$, we may actually look at channels $B^n \rightarrow U$.)

Because $T$ acts on many copies of $B$, it is unclear whether a single-letter formula for the achievable rate region can be obtained, potentially by finding a better – lower – expression for Bob’s rate. Indeed, in the classical case, this is
what happens [31]. For classical random variables X and Y with Alice and Bob, respectively, the single-letterized rate for Bob is given by imagining a channel Y → W. Bob needs to send only I(W : X) bits of W rather than H(W). While the quantum protocol above is clearly optimal, it may be that the entanglement of purification is non-additive, and thus S(U) may be much lower than nS(U_i) where \( \rho_{U_i} \) is the state obtained by acting a channel on single copies of \( \rho_B \).

Source coding with side information at the encoder

In the classical case, if a party aims to send her variable to the decoder, having herself access to some side information is of no additional value. If Alice wants to send classical variable X to Bob, she cannot lower her rate by sending or even knowing additional information. In the quantum world, this is not the case, as can be seen from the side information problem in the case of one party. We consider Alice, who has state \( \rho_{A_1} \) and is required to send it to Bob. This she can do using state merging at rate \( S(A_1 | B) \). However, if she also has access to state \( \rho_{A_2} \) which may be entangled or correlated to \( \rho_{A_1} \), then she may be able to do better. This better rate is obtained by sending part of \( \rho_{A_2} \) as well – so in some cases, less is more!

Applying an isometry \( T : A_2 \rightarrow A'_2 A''_2 \), and actually merging \( A_1 A'_2 \), she can achieve a rate \( S(A_1 A'_2 | B) \). Hence one would naturally minimize over channels \( T \):

\[
R \geq \min_T S(A_1 A'_2 | B). \tag{54}
\]

As argued in the side information problem, the right hand side is equal to \( E_P (A_1 B : R) - S(B) \). Essentially, due to the non-monotonicity of the von Neumann entropy, it can be beneficial to lower the entropy of what you are sending, by merging additional quantum states which are entangled with what you needed to send.

VII. MULTIPARTITE ENTANGLEMENT OF ASSISTANCE

In this section we consider the multipartite entanglement of assistance [14]. Sometimes it is called localizable entanglement [15], although we operate in the regime of many copies and collective measurements. Consider a pure \( m \)-partite state \( \psi_{A_1, A_2, \ldots, A_m} \). The entanglement of assistance is defined for two fixed nodes \( A_i \) and \( A_j \), as the maximal pure entanglement that can be obtained between those nodes by LOCC operations performed by all the parties. Here is a more precise definition:

**Definition 9** For an \( m \)-partite pure state, consider a measurement performed by LOCC that leads to pure states between chosen nodes \( A_i \) and \( A_j \) for any outcome \( k \) of the measurement. Let the probability of the outcome \( k \) be \( p_k \), and the entropy of the node \( i \) (equal to entropy of the node \( j \)) be denoted by \( S_k(A_i) \). The entanglement of assistance between the nodes \( A_i \) and \( A_j \) is defined as

\[
E_A(\psi, A_i : A_j) = \sup_k \sum p_k S_k(A_i) \tag{55}
\]

where supremum is taken over the above measurements. Asymptotic entanglement of assistance is given by regularization of the above quantity

\[
E^\infty_A (\psi, A_i : A_j) = \lim_{n \to \infty} \frac{1}{n} E_A(\psi^{\otimes n}, A_i : A_j). \tag{56}
\]

Asymptotic entanglement of assistance was determined for pure states of up to four parties in [33]. Namely it was proven that for \( m \leq 4 \) the maximal amount of entanglement that can be distilled between Alice and Bob, with the help of the other \( m - 2 \) parties \( C_1, \ldots, C_{m-2} \), is given by the minimum entanglement across any bipartite cut of the system which separates Alice from Bob:

\[
E^\infty_A (\psi, A : B) = \min_T \{ S(\rho_A T), S(B \overline{T}) \} =: E_{\text{min-cut}}(\psi, A : B), \tag{57}
\]

where the minimum is taken over all possible partitions of the other parties into a group \( T \) and its complement \( \overline{T} = \{ C_1, \ldots, C_{m-2} \} \setminus T \).

In [13] we generalized this result to an arbitrary number of parties, by use of the primitive of state merging. The result is clearly optimal – one cannot increase entanglement by LOCC. The entropy of any splitting \( T \) which divides \( A \)
from $B$ is a measure of the entanglement of the total pure state between $AT$ and $B\overline{T}$ and it cannot increase during the protocol – in fact not by any protocol allowing arbitrary joint operations of the two groups $AT$ and $B\overline{T}$ and classical communication. Thus all entropies under such splitting serve as an upper bound for the amount of entanglement which can be distilled between $A$ and $B$.

The protocol for achieving this optimal rate is as follows: each party in turn merges their state with the remaining parties on its side of the minimal cut, preserving the minimum cut entanglement. The merging protocol we consider will be slightly different from the merging protocol considered previously in two respects. As before, the party who wishes to merge his state with other parties performs a random measurement on their typical subspace. However, since the receiver will consist of many parties who are separated from one another the final decoding step (i.e. the unitary which the receiver performs conditional on the measurement outcome of the sender) will not be performed until the very end. The second difference is that the senders will perform complete measurements, and will not attempt to distill additional entanglement between themselves and the receiving parties. This will not effect the merging condition, but it does mean that the maximally entangled states which would be created between the merging parties and the receiver will be destroyed. This greatly simplifies the analysis, despite some entanglement being lost. We only consider entanglement of assistance – i.e. a protocol which attempts to distill entanglement between $A$ and $B$. More complicated protocols can be constructed which also result in entanglement between other parties.

Before moving to the protocol, we will need to prove an aspect of state merging already implicit in Theorem 2, which will serve as a cornerstone of (among other things) proving a formula for asymptotic entanglement of assistance:

**Proposition 10 (Random measurement gives covering)** Let $\psi_{\bar{A}B\overline{R}}$ be a tripartite pure state with $S(R) < S(B)$, of which we consider $n$ copies, and consider the state $\tilde{\psi}_{\bar{A}B\overline{R}}$ of the proof of Theorem 2 (Section IV) belonging to the typical subspaces $\bar{A}B\overline{R}$. Denote by $\rho_{\bar{A}}$ the state of system $\bar{A}$. Let $\{|\psi\rangle\}$ be a basis on $\bar{A}$ chosen at random according to the Haar measure, and $\rho_{\bar{A}}^j$ be the state obtained on system $\bar{A}$ upon obtaining outcome $j$; let $p_j$ be the probability of this event. Then for any $\epsilon > 0$ and all large enough $n$, we have

$$\left\langle \sum_j p_j \|\rho_{\bar{A}}^j - \rho_{\bar{A}}\|_1 \right\rangle \leq \epsilon,$$

where the average is taken over the choice of basis.

**Proof.** This is just the special case of $L = 1$ in Proposition 4. \hfill \Box

With this tool in hand we can analyze the protocol outline above. Clearly, if $m = 2$, there is only one cut, and its entropy is $S(A)$, the entropy of entanglement, and we are done. So, from now on $m \geq 3$.

Assume for the moment that all $S(\overline{AT})$ are distinct (we’ll come back to this point at the end), and consider helper $C_{m-2}$. For each set $T$, clearly $S(\overline{AT}) = S(B\overline{T})$, by the purity of the overall state. Hence, for the min-cut we can restrict to looking at the entropies $S(\overline{AT})$ and $S(B\overline{T})$, with $C_{m-2} \notin T$. For each such set $T \subset \{1,\ldots,m-3\}$, consider the relative complement $T' := \{1,\ldots,m-3\} \setminus T$. This defines a tripartite system composed of $C_{m-2}$, $\overline{AT}$ and $B\overline{T}$. Let $C_{m-2}$ perform a random measurement on his typical subspace $\overline{AT}$, as in Proposition 10. We get (if only $n$ is large enough), with arbitrarily high probability, states $\tilde{\psi}_{\bar{A}BC_1\ldots C_3}$ which by eq. (58) satisfy:

$$S(A^n T^n)_{\overline{A}} = S(B^n T'^n)_{\overline{A}} = n \left( \min \{S(\overline{AT}) , S(B\overline{T'}) \} \pm \delta \right),$$

with arbitrarily small $\delta$. In other words, for each such $\tilde{\psi}_{\overline{A}}$,

$$E_{\text{min-cut}}(\tilde{\psi}, A^n : B^n) = n \left(E_{\text{min-cut}}(\psi, A : B) \pm \delta\right),$$

and that means that the min-cut entanglement is almost preserved (up to an arbitrarily small variation in the rate), and hence that the reduced state entropies can be assumed to be all distinct (by choosing $\delta$ small enough). Now we recursively apply the same to $C_{m-3},\ldots,C_1$. \hfill \Box

Finally, for the assumption that all reduced state entropies are pairwise distinct: this can be enforced if the parties first “borrow” an arbitrarily small rate of entanglement to distribute singlets between chosen pairs. Then our distinctness assumption becomes true. In the limit, only a sublinear amount of entanglement is needed to do this, but on the other hand [34] shows that the asymptotic entanglement landscape of multiple parties does not change if one allows this sublinear amount – this is due to them being able to always, perhaps inefficiently, extract some entanglement across any given cut unless across that cut they happen to be in a product state.
**Remark 11** Note that a crucial part of the argument of why the minimum cut entropy doesn’t change is the use of random codes. This is because $C_1$’s procedure is universal – it does not depend on the cut. He makes a measurement which only depends on the typical subspace of his state. The measurement thus serves to merge his state with whichever grouping of subsystems has the larger entropy compared with the remaining systems. Not all quantum codes have this feature – for example Devetak codes [8] depend both on the state of the sender, and that of the receiver. The same applies to [33], which is why there even the argument for $m = 4$ has to be quite subtle.

It may seem odd that after performing a random measurement, one’s state goes to any set of parties which has more entropy than the remaining parties. Since there are many possible groupings of the parties, for some groupings a certain party would help receive the state, but for other groupings, that party’s state would be left unchanged by the random measurement. Of course, there is no contradiction, as in the end, at the decoding step, one has to decide on the grouping, and with fidelity approaching 1 only for many copies of the state.

**Conjecture 12** It is awkward that in the recursive procedure described above for $m$ parties we have to first consider a measurement on a long block of states, and then for the second measurement blocks of these blocks, etc.

It seems likely that the simplest random measurement strategy will indeed also work: all $m−2$ helpers $C_1,\ldots,C_{m−2}$ measure in a random basis of their respective typical subspaces and broadcast the result to Alice and Bob. They should then end up, with high probability, with a state of the min-cut entanglement.

VIII. CAPACITY REGION FOR THE MULTIPLE ACCESS CHANNEL

We consider a channel with two senders Alice and Bob, and one receiver Charlie; this is the multiple access channel. For the classical multiple access channel, any rates satisfying the following inequalities are achievable for encoding independent messages from Alice and from Bob at their respective terminals to Charlie who decodes them jointly:

\[
\begin{align*}
R_A &\leq I(A : C|B) \\
R_B &\leq I(B : C|A) \\
R_A + R_B &\leq I(AB : C).
\end{align*}
\]  

(61)

The quantum multiple access channel – where Alice and Bob want to send quantum information was considered in [35], and we refer to that paper for the definitions of codes and rate region. In [13], we found that one could use state merging to find a larger achievable region, including negative rates. Namely, that for the quantum multiple access channel, there is the following region of achievable rates:

\[
\begin{align*}
R_A &\leq I(A|C|B) := I(A)BC \\
R_B &\leq I(B|C|A) := I(B)AC \\
R_A + R_B &\leq I(AB|C).
\end{align*}
\]  

(62)

The state on which the quantities are evaluated is constructed as follows. Consider two pure states $\psi_{AA'}$ and $\psi_{BB'}$. Let $\rho_{ABC}$ be the state, resulting from the halves $A'$ and $B'$ being sent down the channel:

\[
\rho_{ABC} = (I_{AB} \otimes A_{A'} B_{B'} \rightarrow C)(|\psi\rangle\langle\psi|_{AA'} \otimes |\psi\rangle\langle\psi|_{BB'}).
\]  

(63)

In the classical theory, only positive rates make sense. In the quantum case, the rates can be meaningful, even if one of them is negative. For example, when $R_A$ is negative, and $R_B$ is positive, this means that when Alice invests $R_A$ qubits, then Bob can send $R_B$ qubits, as we shall see.

**A. Remarks on coherent information**

In [4] the coherent information was introduced and defined in terms of an input state $\rho_A$ and a channel producing output $\rho_B$ as

\[
I(A|B) = S(B) - S(AB),
\]  

(64)

that is, as the conditional entropy with a minus sign; this was puzzling because it can be negative. Since it gives the channel capacity of a quantum channel (by maximizing it over input distributions $\rho_A$), it was unclear how to interpret
negative uses of a channel. We will see that the negative part will acquire operational meaning, in full accordance with the positive part. We also have defined the conditional coherent information as

\[ I(A)B|C = S(B|C) - S(AB|C). \] (65)

We have the useful identity [consistent with eq. (62)]

\[ I(A_1)B|A_2 = I(A_1)B|A_2. \] (66)

That is, conditioning the coherent information is very simple: just erase the bar. Then we have a chain rule of the same form as the one for mutual information,

\[ I(A_1A_2)B = I(A_2)B + I(A_1)B|A_2. \] (67)

What seems surprising is that conditioning can only increase coherent information! However, this can be explained as follows. Namely, in classical information theory we have to have situations where conditioning decreases information, due to lack of monogamy. Indeed, we can have situation where

\[ I(X_1 : Y) + I(X_2 : Y) > I(X_1X_2 : Y). \] (68)

(E.g., the three variables could be fully correlated.) Therefore, to save the chain rule, conditioning must decrease mutual information. However in the quantum case we always have

\[ I(A_1)B + I(A_2)B \leq I(A_1A_2)B, \] (69)

due to strong subadditivity. Now conditioning very often increases coherent information, because we have equality in the chain rule identity (67).

### B. Direct coding theorem: achievability of rates

To check that the rates satisfying the above conditions are achievable, it is enough to consider one corner, for example

\[ R_A = I(A)BC, \quad R_B = I(B)C, \] (70)

which is an upper corner of the rate region, see Figure 3.

When both \( I(A)BC \) and \( I(B)C \) are negative, they are trivially achievable: Alice and Bob do nothing. So in this case negativity of rates does not appear meaningful, as zero is achievable too, and one always optimises rates over input states. When \( I(A)BC \) is negative and \( I(B)C \) is positive, again, those rates can be achieved by Alice doing nothing, and Bob – by standard quantum coding theorem. So again the negative rate is not interesting. There are therefore two situations, which we have to consider:

\[ I(A)BC \geq 0 \quad \text{and} \quad I(B)C \geq 0, \quad \text{or} \]

\[ I(A)BC \geq 0 \quad \text{and} \quad I(B)C < 0. \] (71) (72)

It is enough to consider the first one in detail, as the second one is its simple consequence. Let us first describe how to achieve those rates, when Bob and Alice can communicate quantum messages to C if classical side-communication is permitted. Alice and Bob prepare \( n \) copies of states \( \psi_{AA'} \) and \( \psi_{BB'} \), respectively, and send halves of them down the channel (inputs \( A^n \) and \( B^n \)). Then Bob performs the merging protocol, i.e. he makes the measurement on his typical subspace in blocks of size \( 2^{nR_B} \). As previously we label blocks (codes) by \( j \). On average, he obtains a state close to a \( 2^{nR_B} \) dimensional maximally entangled state shared with Charlie (who holds the system C), and Bob’s part of the state \( \psi_{ABCR} \) is merged with Charlie (\( \psi_{ABCR} \) is purification of \( \rho_{ABC} \)). Then, Alice shares with Charlie state \( \rho_{ABC} \) where both part B and C is now with Charlie. Random measurement of Alice in blocks \( 2^{nR_A} \), will create a state close to the maximally entangled state of this dimension between Alice and Charlie, after Alice communicates her results to Charlie. In this way she also merges her part to Charlie, however it is not important in the present context.

Let us now show, how Alice and Bob can share with Charlie maximally entangled state of suitable dimensions without classical communication. Namely, both Alice and Bob can perform their measurements before sending halves of their states \( \psi_{AA'} \) and \( \psi_{BB'} \) down the channel. They can then send the states \( \psi_{AA'}, \psi_{BB}, \) that they have obtained
FIG. 3: The rate region for the multiple-access channel for two parties with individual rates $R_A$ and $R_B$. The total rate $R_{AB}$ is bounded by $I(AB|C)$. The top left diagram shows the rate region when both rates are positive; the top right and bottom left diagrams show the case where $I(B|C) < 0$ or $I(A|C) < 0$. I.e., here, Bob (Alice) can invest entanglement so that the other party can send at a rate $I(A|BC) \geq R_A \geq I(A|C)$ ($I(B|AC) \geq R_B \geq I(B|C)$). In the bottom right diagram, both parties may have the option of achieving the higher rate by having the other party invest entanglement.

(here $j_A$ and $j_B$ denote the outcomes of measurement). This still requires communication, as they have to tell Charlie, what outcomes they obtained.

However, instead of measuring, they can prepare already $\psi_{AA}^{j_A, j_B}$ with fixed $j_A$ and $j_B$ known to Charlie. This will have the same effect as before, once they choose such labels, that guarantee that merging conditions are satisfied. Note that the states that Alice and Bob are now sending are close to maximally entangled states (this is guaranteed by the merging condition). The maximally entangled states to which they are close, defines the subspaces, which go through the channel, and allow correction of errors. The subspaces are codes that when used by Alice and Bob, allow them to obtain the above rates. Since our criterion was fidelity with the maximally entangled state, we have obtained here coding theorem with small average error.

In our case it was relatively easy to go from one way to zero because the states that Alice and Bob obtain in our one-way protocol are close to maximally entangled states. For more complicated situations see [36].

Finally, consider the case, where $I(B|C)$ is negative, eq. (72). The reasoning is very similar: in the scenario with classical side-communication, Bob sends $-I(B|C)+\epsilon$ halves of maximally entangled states through a noiseless channel, (keeping the other half), and performs merging, so that after that Alice can achieve her rate as above. However, again Alice and Bob instead of performing measurements, can send the state that would emerge under some outcome of the measurement. The difference is that Bob will send the state not only down the noisy channel, but also down the supplementary noiseless channel, and will share $\epsilon$ rate of maximally entangled states (thus his overall rate is negative).

This is the more interesting rate point: for Alice to achieve the rate $I(A|BC)$, she requires Charlie to have $C$ and $B$. Bob assists in providing this information (which can be understood as additional error correcting information from inside the channel) but that comes at a price, which is exactly $-I(B|C)$. We thus have an interpretation of negative channel capacities.

$\square$

C. Converse coding theorem

Here we briefly argue that (up to regularization) the rate region described by our conditions is optimal. The reasoning is quite standard (see e.g. [5, 37]), therefore we will provide only a sketch of the proof. Suppose that some
rates $R_A$ and $R_B$ are achievable. Consider first the case where they are both positive. This means that Alice and Bob can send halves of singlets down the channels in such a way that after decoding by Charlie, they share with Charlie those singlets with fidelity tending asymptotically to one. Alice shares a singlet of dimension $2^{nR_A}$ with Charlie, and Bob one of dimension $2^{nR_B}$. Would they have exact singlets, the coherent informations would be equal to $I(A|BC) = I(A|C) = nR_A$, $I(B|C) = nR_B$ and $I(AB|C) = n(R_A + R_B)$. Because they share inexact singlets, we apply asymptotic continuity of coherent information [37] (which plays here the role of Fano’s inequality), thanks to which the coherent informations of the real state, per use of channel, approach the ideal values in the asymptotic limit. This means that there exist such states, such that, if Alice and Bob will send halves of them down the channel, then after Charlie’s decoding, the coherent informations approach the values from the coding theorem.

There are still two issues. First, the states may be mixed: Alice and Bob prepared singlets, however the encoding procedure may turn them into mixed states. However, coherent information is convex, so that Alice and Bob will not do worse by sending some pure states. Second, we considered the joint state after Charlie’s decoding, while in the coding theorem, we have state merging just from sending by Alice and Bob. However, due to the data processing inequality [5] (saying that operating on $V$ one cannot increase $I(U|V)$), the coherent information of the state before Charlie’s decoding can be only greater.

Let us now consider the case when one of the rates (suppose $R_B$) is negative. This means that Bob uses the noiseless qubit channel an additional $R_B$ times (per use of the noisy channel), and Alice achieves her rate. It suffices to show that, if rate pair $(R_A, R_B)$ where $R_B$ is negative, is achievable, then Alice and Bob can create the joint state of $ABC$ system, such that $I(A|BC) = R_A$ and $I(A|C) = R_B$ per use of channel. To this end, consider a new channel which consists of the old one supplemented by $-R_B + \epsilon$ uses of the noiseless channel from Bob to Charlie. For the new channel, the rates $(R_A, \epsilon)$ are achievable. They are positive, so that, as explained above, there exist states of Alice and Bob, that sent down the channel produce a joint state having $I(A|C) = R_A$ and $I(B|C) = \epsilon$. Suppose now that Bob will not send part the system that was intended to go through the noiseless channel, but keeps it. In this situation they only use the original channel. We will now see that they achieve the needed coherent informations in this way. Of course $I(A|BC) = R_A$, as this quantity does not depend on whether a given system is with Bob or with Charlie. Let us now estimate the quantity $I(B|C)$. By sending $-R_B + \epsilon$ qubits, Bob could increase it up to $\epsilon$. However, by sending one qubit, one can increase coherent information no more than by one. Thus, coherent information $I(B|C)$ cannot be smaller than $R_B$. This ends the proof of the converse theorem.

**IX. STRONG SUBADDITIVITY**

Using state merging, we can get a very quick and operationally intuitive proof of strong subadditivity [38], which can be written as

$$S(A|BC) \leq S(A|B).$$

(73)

Strong subadditivity is simply the observation that if Bob has access to an additional register $C$, then Alice surely doesn’t need to send more partial information for him to get the full state $\rho_{AB}$. After all, Bob could always ignore the ancilla on $C$, but if he uses it, Alice may need to send him less. Mathematically, we can use this argument because in the proof that $S(A|B)$ is the optimal merging rate we have used only typical subspaces and elementary probability for the direct part, and ordinary subadditivity in the converse part.

**X. CONCLUSION**

It is very interesting to compare the proof of the classical Slepian-Wolf theorem, with the proof of its quantum version - state merging. The Slepian-Wolf protocol is as follows: the typical sequences of Alice are divided into blocks of size $\approx 2^{nI(A:B)}$. Note that this is the size of a good code. Now, when a particular sequence occurs, Alice lets Bob know in which code is the sequence, and this is enough for him to determine her sequence. Thus the Slepian-Wolf theorem follows solely from the fact that a random code is a good code, which was shown by Shannon.

Interestingly, our protocol is based on the same property, especially for states for which coherent information is positive. (This could be regarded as a situation analogous to the classical case, as the classical mutual information is always positive.) Namely, to prove quantum state merging it is enough to know that a random quantum code is a good quantum code. And in the quantum state merging protocol Alice performs an analogous task: she measures in which quantum code her state is, and tells Bob the result.

What is now extremely surprising, is that those similarities turn out to be quite superficial. Namely, in the Slepian-Wolf protocol, the amount of bits needed to tell Bob the information “which code” is just the cost of transmission of
Alice’s data to Bob. In the quantum case, the information “which code”, since represented by classical bits, is not counted at all, as we count only the quantum information. Thus in this case (positive coherent information) merging does not cost at all, unlike in the classical case. What is more remarkable still is that despite this difference, the cost of sending partial quantum information is the conditional entropy, and thus formally similar to the classical case. This despite the fact that the classical case does not emerge as a limit from the quantum case. In other words, if one takes quantum state merging, and applies it to classical states (i.e. states which are fully decohered, and contain only classical correlations), then the goal is rather different, as one is attempting to retain entanglement between this classical state and the reference system, and one is further allowing free classical communication.

We have two ways of interpreting the classical mutual information: (i) either as the quantity responsible for capacity or (ii) as the quantity that reports the part of information that is common both to Alice and Bob. Indeed, the latter meaning is implied by the fact that the cost of communication needed to transfer full information to Bob is $H(X)$ (full information content of Alice’s state) reduced by the amount of mutual information. Thus the latter represents that part of Alice’s information, that Bob also knows, and it need not be transferred to him.

It turns out that in the quantum case those two notions are no longer represented by the same quantity (see however [39]). Namely, the communication cost is equal to Alice’s information reduced by quantum mutual information. Thus quantum mutual information serves as common information. The capacity is on the other hand represented by the coherent information. The first quantity is sometimes greater than the whole of Alice’s information, and precisely in those instances, the second quantity has the chance to be positive.

It is indeed the beauty of the quantum information world, that both the quantities, into which the classical quantity has split, do their job in an analogous way as it was in the classical case. Indeed, the analogue of common information counts by how much the transmission cost is reduced – exactly as in the classical case, while the analogue of capacity is responsible for protocol, with the same basic elements as in classical case. The additional brick in the quantum protocol is teleportation, which is perhaps the thread that binds the two notions together.

However, as we have noted, the analogy in the protocol is quite superficial. Even though Alice perform the operations that can be called by use of the same name (checking “which code”, and telling it to Bob) the meaning of those operations is completely different. It is extremely mysterious, how the quantum and classical cases can have so much in common, and at the same time can be so different.

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APPENDIX A: MISCELLANEOUS FACTS ABOUT NORMS AND FIDELITY

The following lemma relates the trace norm to the Hilbert-Schmidt norm. Recall that these norms are defined, for an operator $X$, as

$$\|X\|_1 := \text{Tr} \sqrt{X^\dagger X},$$  \hspace{1cm} (trace norm)

$$\|X\|_2 := \sqrt{\text{Tr} X^\dagger X}. \hspace{1cm} \text{(Hilbert-Schmidt norm)}$$

**Lemma 13** For any operator $X$,

$$\|X\|_1^2 \leq d\|X\|_2^2,$$  \hspace{1cm} (A1)

where $d$ is the dimension of the support of operator $X$ (the subspace on which $X$ has nonzero eigenvalues).

**Proof.** It is implied by convexity of function $x^2$, where one takes probabilities $1/d$. \hfill \Box

The fidelity of two states is given by

$$F(\rho, \sigma) = \left(\text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}\right)^2.$$  \hspace{1cm} (A2)

Notice that if one of the states is pure, say $\sigma = |\phi\rangle \langle \phi|$, then

$$F(\rho, |\phi\rangle \langle \phi|) = \langle \phi | \rho | \phi \rangle = \text{Tr} (\rho |\phi\rangle \langle \phi|).$$  \hspace{1cm} (A3)
Lemma 14 The fidelity is related to trace norm as follows [40]:

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}.$$  \hfill (A4)

Lemma 15 (Gentle measurement) Let \( \rho \) be a (subnormalized) state, i.e. \( \rho \geq 0 \) and \( \text{Tr} \rho \leq 1 \), and let \( 0 \leq X \leq I \).

Then, if \( \text{Tr} \rho X \geq 1 - \epsilon \),

$$\|\sqrt{X} \rho \sqrt{X} - \rho\|_1 \leq 2\sqrt{\epsilon}.$$  \hfill (A5)

Proof. See [41], Lemma 9; the better constant above is from [42]. \qed

Lemma 16 (Fannes [43]) For states \( \rho \) and \( \sigma \) on a d-dimensional space, such that \( \|\rho - \sigma\|_1 \leq \epsilon \),

$$|S(\rho) - S(\sigma)| \leq \eta(\epsilon) \log d, \quad \text{with} \quad \eta(x) := \begin{cases} x - x \log x, & \text{if } x \leq \frac{1}{e}, \\ x + \frac{\log e}{x}, & \text{if } x \geq \frac{1}{e}. \end{cases} \hfill (A6)$$

**APPENDIX B: THE TWIRLING AVERAGE OF EQ. (24)**

We use the fact that an operator

$$T(X) := \langle (U U_A)^\dagger X (U U_A)^\dagger \rangle$$  \hfill (B1)

is \( U \otimes U \)-invariant. However, the representation of \( U \otimes U \) decomposes into the two reducible components, the symmetric and the antisymmetric subspace. By Schur’s lemma, the only invariant operators are then linear combinations of the projections onto these subspaces:

$$\Pi^\text{sym}_{A\bar{A}} = \frac{1}{2} (I_{A\bar{A}} + F_{A\bar{A}}), \quad \Pi^\text{anti}_{A\bar{A}} = \frac{1}{2} (I_{A\bar{A}} - F_{A\bar{A}}).$$  \hfill (B2)

Hence, the twirling map \( T \) can be written

$$T(X) = \frac{1}{\text{Tr} \Pi^\text{sym}_{A\bar{A}}} \text{Tr} (X \Pi^\text{sym}_{A\bar{A}}) + \frac{1}{\text{Tr} \Pi^\text{anti}_{A\bar{A}}} \text{Tr} (X \Pi^\text{anti}_{A\bar{A}}).$$  \hfill (B3)

This is enough to evaluate our average:

$$\langle (U U_A)^\dagger F_{A_i A_j} (U U_{\bar{A}_i}) \rangle = \frac{2}{d_A(d_A + 1)} \Pi^\text{sym}_{A\bar{A}} \text{Tr} (F_{A_i A_j} \Pi^\text{sym}_{A\bar{A}}) + \frac{2}{d_A(d_A - 1)} \Pi^\text{anti}_{A\bar{A}} \text{Tr} (F_{A_i A_j} \Pi^\text{anti}_{A\bar{A}})$$

$$= \frac{2}{d_A(d_A + 1)} \frac{L + L^2}{2} + \frac{2}{d_A(d_A - 1)} \frac{L - L^2}{2}$$

$$= \frac{L(L + 1)}{d_A(d_A + 1)} I_{A\bar{A}} + \frac{L(L - 1)}{d_A(d_A - 1)} F_{A\bar{A}}.$$  \hfill (B4)

**APPENDIX C: TYPICALITY**

We shall need the concept and a few properties of typical subspaces [3]. Consider \( n \) copies of a density matrix \( \rho \), \( \rho^{\otimes n} \). Writing \( \rho \) in its eigenbasis, \( \rho = \sum_i p_i |i\rangle \langle i| \), we note first of all that \( S(\rho) = H(p_i) \). Now,

$$\rho^{\otimes n} = \sum_{i^n} p_{i^n} |i^n\rangle \langle i^n|,$$  \hfill (C1)

with

$$i^n = i_1 \cdots i_n,$$

$$p_{i^n} = p_{i_1} \cdots p_{i_n},$$

$$|i^n\rangle = |i_1\rangle \cdots |i_n\rangle.$$  \hfill (C2)
For $\delta > 0$, the set of typical sequences is defined as (see [44])

$$T^\delta_n := \{i^n : | - \log p_{i^n} - nS(\rho)| \leq n\delta\}, \quad (C3)$$

and the typical projector [3] is

$$\Pi^\delta_n := \sum_{i^n \in T^\delta_n} |i^n\rangle \langle i^n|, \quad (C4)$$

The typical projector inherits its properties from the set of typical sequences. We quote the following from [3], and from [30] for the exponential bounds (see also [44]): abbreviating $\Pi = \Pi^\delta_n$,

$$\text{Tr} (\rho \otimes n \Pi) \geq 1 - \exp(-c\delta^2 n) \text{ with a constant } c, \quad (C5)$$

$$\Pi \rho \otimes n \Pi \leq \rho \otimes n, \quad (C6)$$

$$\Pi \rho \otimes n \Pi \leq 2^{n[S(\rho) - \delta]} \Pi, \quad (C7)$$

$$\Pi \rho \otimes n \Pi \geq 2^{n[S(\rho) + \delta]} \Pi, \quad (C8)$$

$$\text{rank } \Pi = \text{Tr } \Pi \leq 2^{n[S(\rho) + \delta]}, \quad (C9)$$

$$\text{rank } \Pi = \text{Tr } \Pi \geq (1 - e^{-c\delta^2 n}) 2^{n[S(\rho) - \delta]} \quad (C10)$$

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