Zero-error classical capacity of qubit channels cannot be superactivated

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It was shown [T.S. Cubitt et al., IEEE Trans. Inform. Theory 57, 8114 (2011)] that there exist quantum channels where a single use cannot transmit classical information perfectly yet two uses can. This phenomenon is called the superactivation of the zero-error classical capacity which does not occur in classical channels. In this paper, it is shown that qubit channels cannot generate the superactivation.

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I. INTRODUCTION

Quantum information theory has provided us with many surprising and interesting results that cannot be explained by classical information theory. Among such results, there is a peculiar one, called superactivation, which can be described as the quantum phenomenon to obtain a useful object with capacity from several objects without capacity by activating their hidden capabilities.

For example, there are quantum states which have nondistillable entanglement, but from which distillable entanglement can be ingeniously extracted by local operations and classical communication [1]. There are two quantum channels which have zero quantum capacity, but whose joint quantum channel has nonzero quantum capacity [2], and there are also two quantum channels which have no zero-error classical capacity, but whose joint quantum channel has a positive zero-error classical capacity [3, 4] or a positive zero-error quantum capacity [5]. They are called the superactivation of bound entanglement, the superactivation of quantum channel capacity, the superactivation of the zero-error classical capacity of a quantum channel, and the superactivation for quantum zero-error capacities, respectively.

Even though it has been known that all bipartite entangled states including bound entangled states are useful for quantum information processing [6], all bound entangled states do not seem to be superactivated. Similarly, although it may be shown that all quantum channels without a certain kind of capacity are useful in a sense, this does not imply that the capacity can be superactivated. Thus, the quantum effect called superactivation might be such a rare phenomenon even in a quantum world that it cannot be readily regarded as a quantum feature, and hence it could be important to decide whether superactivation is feasible in a given situation.

We here take into account the zero-error classical capacity of a quantum channel, which is the amount of classical information that can be perfectly transmitted through the quantum channel. In particular, a quantum channel $E$ from system $A$ to system $B$ with a positive one-shot zero-error classical capacity can be clearly expressed in a mathematical form as follows [3]:

$$\text{tr}[(E^{\dagger})^jE(j)] = 0 \quad (1)$$

for some pure states $|\psi\rangle$ and $|\phi\rangle$ in $\mathcal{H}_A$. Therefore, the superactivation of the one-shot zero-error classical capacity can be mathematically described as follows: The one-shot zero-error classical capacity of quantum channels $E_1$, $E_2$, ..., $E_k$ can be superactivated if and only if

$$\text{tr}[(E_j^{\dagger})^jE_j(j)] \neq 0, \quad (2)$$

for all $1 \leq j \leq k$ and all pure states $|\psi\rangle$ and $|\phi\rangle$ in $\mathcal{H}_A$, but there exist two pure states $|\Psi\rangle$ and $|\Phi\rangle$ in $\mathcal{H}_A^{\otimes k}$ such that

$$\text{tr}[(E_1 \otimes \cdots \otimes E_k)(\Psi)(E_1 \otimes \cdots \otimes E_k)(\Phi)] = 0. \quad (3)$$

Here, Eq. 2 means that no channel $E_j$ has any one-shot zero-error classical capacity, and Eq. 3 means that the joint channel has a positive one-shot zero-error classical capacity.

We note that there are quantum channels whose one-shot zero-error classical capacity is superactivated [3], but, for most quantum channels without the one-shot zero-error classical capacity, their joint channels are unlikely to have a positive one-shot zero-error classical capacity. On this account, in order to figure out when or why the superactivation occurs, it may be important to learn what situation causes the superactivation, or does not cause the superactivation.

In this paper, we present one situation which cannot cause the superactivation of the one-shot zero-error classical capacity. More precisely, we here show that the one-shot zero-error classical capacity of any finitely many qubit channels (with one qudit channel) cannot be superactivated, which can be extended to the case of zero-error classical capacity for qubit channels.

II. CHOI-JAMIOŁKOWSKI ISOMORPHISM AND SUPERACTIVATION

In this section, we introduce a necessary and sufficient condition for a positive one-shot zero-error classical capacity based on the Choi-Jamiołkowski isomorphism between matrices and linear operators [7].

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We remark that there is an isomorphism between \( \mathcal{L}(M_n) \) and \( M_n \otimes M_n \), where \( M_n \) is the space of \( n \times n \) matrices and \( \mathcal{L}(M_n) \) is the space of all linear operators on \( M_n \). By the isomorphism, a quantum channel \( \mathcal{E} \) corresponds to a \( \sigma_{AA'} = (\mathbb{I}_A \otimes \mathcal{E}_{A'})(\omega_{AA'}) \), where \( \omega_{AA'} = \sum_j |j\rangle_A \langle j|_{A'} \) and \( \omega_{AA'} = |\omega\rangle_{AA'} \). This isomorphism is called the Choi-Jamiolkowski (CJ) isomorphism, and the matrix \( \sigma_{AA'} \) is called the CJ matrix of \( \mathcal{E} \).

Let \( \mathcal{E} \) be the channel, \( \mathcal{E}^* \) be the dual map of \( \mathcal{E} \) with respect to the Hilbert-Schmidt inner product on the left-hand side of Eq. (1), \( \sigma \) be the CJ matrix of \( \mathcal{E}^* \circ \mathcal{E} \), and \( S = \text{supp}(\sigma) \) be the support of \( \sigma \). Then the following proposition can be obtained \([3]\).

**Proposition 1.** \( \mathcal{E} \) has a positive one-shot zero-error classical capacity if and only if there exist two pure states \( |\psi\rangle \) and \( |\phi\rangle \) in \( \mathcal{H}_A \) such that \( |\psi\rangle \otimes |\phi\rangle \in S^\perp \).

By employing Proposition 1, the superactivation of the one-shot zero-error classical capacity of two quantum channels \( \mathcal{E}_1 \) on system \( A_1 \) and \( \mathcal{E}_2 \) on system \( A_2 \) can be redescribed as follows: For each \( j \), let \( \sigma_j \) be the CJ matrix of \( \mathcal{E}_j^* \circ \mathcal{E}_j \), and \( S_{A_j}^\perp = \text{supp}(\sigma_j) \), and let \( S_{A_1A_2}^\perp A_{1}A_{2}' = S_{A_1}^\perp \otimes S_{A_2}^\perp \). Then the one-shot zero-error classical capacity of two quantum channels \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) can be superactivated if and only if, for all pairs of pure states \( |\psi\rangle \) and \( |\phi\rangle \) in each \( \mathcal{H}_{A_1} \), \( |\psi\rangle \otimes |\phi\rangle \notin S_{A_1A_2}^\perp \), but there exist two pure states \( |\Psi\rangle \) and \( |\Phi\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \) such that \( |\Psi\rangle \otimes |\Phi\rangle \in S^\perp \).

### III. MAIN RESULTS

In this section, we show our main results on the superactivation of the one-shot zero-error classical capacity.

#### A. Main Lemma

In this subsection, we introduce the main lemma, from which our main results can be straightforwardly derived.

Our main lemma is as follows:

**Lemma 2.** For each \( j \), let \( \mathcal{E}_j \) be a quantum channel on system \( A_j \), \( \sigma_j \) be the CJ matrix of \( \mathcal{E}_j^* \circ \mathcal{E}_j \), and

\[
S_j^\perp = \text{supp}(\sigma_j),
\]

and let

\[
S_{A_1A_2}^\perp A_{1}A_{2}' = S_{A_1}^\perp \otimes S_{A_2}^\perp.
\]

Assume that \( \dim S_{A_1}^\perp \leq 1 \). Then the one-shot zero-error classical capacity of the channels cannot be superactivated, that is, if \( S_{A_1}^\perp \) does not contain any product states with respect to partition \( A_1 - A_j \) for each \( j = 1, 2 \), then \( S_{A_1A_2}^\perp A_{1}A_{2}' \) does not contain any product states with respect to partition \( A_1A_2 - A_jA_j' \), either.

In order to prove Lemma 2 we first consider the case that \( S_{A_1}^\perp \) is one-dimensional. Assume that \( \dim S_{A_1}^\perp = 1 \).

Then we may let \( \{ |\psi_1\rangle \} \) be a basis for \( S_{A_1}^\perp \), \( \{ |\psi_i\rangle \}_{i=1}^{n^2} \) be a basis for \( S_1 \), \( \{ |\phi_i\rangle \}_{i=1}^{m^2} \) be a basis for \( S_2 \), and \( \{ |\phi_i\rangle \}_{i=k+1}^{m^2} \) be a basis for \( S_2 \), where \( n = \dim \mathcal{H}_{A_1} \), and \( m = \dim \mathcal{H}_{A_2} \). Thus, for any state \( |\Psi\rangle \in (S_1 \otimes S_2)^\perp \), it is clear that

\[
|\Psi\rangle = \sum_{j=k+1}^{m^2} a_j |\psi_j\rangle |\phi_j\rangle + \sum_{i=1}^{n^2} b_{ij} |\psi_i\rangle |\phi_j\rangle
\]

\[
= |\psi_1\rangle \left( \sum_{j=k+1}^{m^2} a_j |\phi_j\rangle \right) + \sum_{i=1}^{n^2} b_{ij} |\psi_i\rangle |\phi_j\rangle
\]

\[
= \sum_{j=1}^{k+1} |\tilde{\psi}_j\rangle |\tilde{\phi}_j\rangle,
\]

where \( |\tilde{\psi}_j\rangle = \sum_{i=1}^{n^2} b_{ij} |\psi_i\rangle \) and \( |\tilde{\phi}_j\rangle = |\phi_j\rangle \) for \( 1 \leq j \leq k \), \( |\tilde{\psi}_{k+1}\rangle = |\psi_1\rangle \), and \( |\tilde{\phi}_{k+1}\rangle = \sum_{j=k+1}^{m^2} a_j |\phi_j\rangle \).

We now assume that \( \dim S_{A_1}^\perp = 0 \). Then we may let \( \{ |\psi_1\rangle \}_{i=1}^{n^2} \) be a basis for \( S_1 = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_1}' \), \( \{ |\phi_i\rangle \}_{i=1}^{m^2} \) be a basis for \( S_2 \), and \( \{ |\phi_i\rangle \}_{i=k+1}^{m^2} \) be a basis for \( S_2 \). Thus, for any state \( |\Psi\rangle \in (S_1 \otimes S_2)^\perp \), it is also obvious that

\[
|\Psi\rangle = \sum_{i=1}^{n^2} b_{ij} |\psi_i\rangle |\phi_j\rangle
\]

\[
= \sum_{j=1}^{k} \left( \sum_{i=1}^{n^2} b_{ij} |\psi_i\rangle \right) |\phi_j\rangle
\]

\[
= \sum_{j=1}^{k} |\tilde{\psi}_j\rangle |\tilde{\phi}_j\rangle.
\]

It follows from Eqs. (6) and (7) that the case of \( \dim S_{A_1}^\perp = 0 \) is included in the case of \( \dim S_{A_1}^\perp = 1 \). Hence, it suffices to prove Lemma 2 for the case that \( S_{A_1}^\perp \) is one-dimensional.

For convenience of the proof of Lemma 2 we rephrase the statement of Lemma 2 as its matrix-based version by an isomorphism between states and matrices in the following proposition 3.

**Proposition 3.** There exists an isomorphism \( \mathcal{M} \) between (unnormalized) states in \( \mathbb{C}^{d_A \times d_B} \) and \( d_A \times d_B \) matrices defined as follows: In the standard basis, \( |\psi\rangle_{AB} = \sum \langle i|_A |j|_B \mathcal{M}(|\psi\rangle_{AB}) \equiv (M_{ij}) \). The isomorphism \( \mathcal{M} \) has the following properties:

(i) Product states and entangled states correspond to matrices of rank one and matrices of rank greater than one, respectively.
(ii) For any state $|\Psi\rangle_{1'\cdots2'} = \sum_{i=1}^{k+1} |\psi_i\rangle|\phi_i\rangle$, 
\[ M(|\Psi\rangle_{1'\cdots2'}) = \sum_{i=1}^{k+1} A^i \otimes B^i, \tag{8} \]
where $A^i = M(|\psi_i\rangle), B^i = M(|\phi_i\rangle)$.

By applying Proposition 6 to Eq. (6), Lemma 2 can be rewritten as the following lemma.

Lemma 4. Let $P$ and $Q$ be one-dimensional and $k$-dimensional subspaces of $n \times n$ and $m \times m$ matrices which have no matrices of rank one, respectively. Let $\{A^i\}$ be a basis for $P$, $\{B^i\}_{i=1}^{k+1}$ be a basis for $Q$, $B^1$ be an $m \times m$ matrix in $Q^+$, and $A^1$ be an $n \times n$ matrix for $j \geq 2$. Then $M = \sum_{i=1}^{k+1} A^i \otimes B^i$ cannot be of rank one.

We now prove Lemma 4 which directly implies Lemma 2.

Proof. Suppose that the matrix $M = \sum_{i=1}^{k+1} A^i \otimes B^i$ is of rank one. Then, by the singular value decomposition, without loss of generality, we may assume that the matrix $A^1$ is a diagonal one with at least two positive diagonal entries, since $A^1 \in P$ is not of rank one.

We now consider an $m \times m$ submatrix $R_{st}$ of the matrix $M$ defined as

\[ R_{st} = \sum_{i=1}^{k+1} (A^i)_{st} B^i. \tag{9} \]

Since the matrix $M$ is of rank one, the submatrix $R_{st}$ must be the zero matrix or a rank-one matrix. In particular, if $s \neq t$ then the submatrix

\[ R_{st} = \sum_{i=1}^{k+1} (A^i)_{st} B^i \tag{10} \]
is contained in $Q$, which has no matrices of rank one, since $A^1$ is diagonal and $\{B^i\}_{i=2}^{k+1}$ is a basis for $Q$. Hence we obtain that the submatrix $R_{st}$ must be the zero matrix for $s \neq t$, and $A^1$ is diagonal for each $2 \leq i \leq k+1$.

We now take into account the submatrix $R_{st}$ for $s = t$, that is, $R_{ss}$. We first assume that $B^1$ is the zero matrix. Then, similar to Eq. (10), the submatrix

\[ R_{ss} = \sum_{i=2}^{k+1} (A^i)_{ss} B^i \tag{11} \]
is contained in $Q$, and hence $R_{ss}$ cannot be of rank one. Thus, for all $s$, $R_{ss}$ must be the zero matrix, and $(A^i)_{ss}$ is zero for each $2 \leq i \leq k+1$, that is, $A^i$ is the zero matrix for each $2 \leq i \leq k+1$. Since $B^1$ is the zero matrix, this implies that the matrix $M$ is the zero matrix, which is a contradiction.

We now assume that $B^1$ is nonzero. Then it follows that when $(A^1)_{ss} \neq 0$, the submatrix

\[ R_{ss} = \sum_{i=1}^{k+1} (A^i)_{ss} B^i \tag{12} \]
cannot be the zero matrix, and thus it must be of rank one. Since $A^1$ has at least two positive diagonal entries, there exist two distinct submatrices $R_{s\alpha}$ and $R_{s\beta}$ of rank one. The matrix $M$ must have rank greater than one, since $R_{s\alpha}$ and $R_{s\beta}$ do not share any entries of $M$ and both $R_{s\beta}$ and $R_{s\alpha}$ are the zero matrices. This leads to a contradiction.

Therefore, the matrix $M$ cannot be a rank-one matrix. \qed

B. Main Theorem and Main Corollaries

In this subsection, we phrase our main theorem and corollaries.

From Lemma 1 (or Lemma 2), we clearly obtain our main theorem.

Theorem 5. For each $j$, let $E_j$ be a quantum channel on system $A_j$, $\sigma_j$ be the CJ matrix of $E_j^* \circ E_j$,

\[ S_{A_j}^{A_j^\prime} = \text{supp}(\sigma_j), \tag{13} \]

and

\[ S_{A_1A_2A_1^\prime A_2^\prime} = S_{A_1}^{A_1^\prime} \otimes S_{A_2}^{A_2^\prime}. \tag{14} \]

Assume that $\dim S_j^\perp \leq 1$. Then the one-shot zero-error classical capacity of the quantum channels $E_j$ cannot be superactivated.

In order to consider the quantum channels on the two-dimensional quantum system, that is, the qubit channels, we use the following lemma.

Lemma 6. For any subspace $S$ of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ whose states all have at least a Schmidt number of $r$, the maximum dimension of $S$ is $(d_A - r + 1)(d_B - r + 1)$.

It follows from Lemma 6 that if a quantum channel $E_j$ on system $A_j$ is a qubit channel and $S_j$ is the subspace induced by channel $E_j$, then dim $S_j^\perp$ is 0 or 1, since dim $H_{A_j} = 2$. Hence, we can readily obtain the following main corollaries.

Corollary 1. The one-shot zero-error classical capacity of two quantum channels including at least one qubit channel cannot be superactivated.

By the induction on the number of quantum channels, we clearly have the following corollary.

Corollary 2. No finitely many qubit channels (with one qudit channel) can cause the superactivation of the one-shot zero-error classical capacity.

We remark that nonsuperactivation of the one-shot zero-error classical capacity for qubit channels implies non-superactivation of zero-error classical capacity by their definitions. As a consequence, our result for the one-shot zero-error classical capacity of qubit channels can be extended to the case of zero-error classical capacity.
IV. CONCLUSIONS

We have investigated whether the superactivation of the zero-error classical capacity arises in quantum channels on low-dimensional quantum systems, and have presented a necessary condition for the superactivation of the one-shot zero-error classical capacity, which can be reduced to the application of the qubit channels. It has been shown that the one-shot zero-error classical capacity of two quantum channels including at least one qubit channels cannot be superactivated, and no finitely many qubit channels (with one qudit channel) can cause the superactivation of the one-shot zero-error classical capacity.

We note that there have never been any examples of the superactivation for channel capacities in the literature, when the underlying space is a two-dimensional one. Therefore, our results could be applied to the superactivation, and could be generalized to the conclusion that the qubit systems cannot cause the superactivation for any channel capacities.

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