On the minimal driver sets for linear dynamical systems on graphs

Johannes G. Maks
Delft Institute of Applied Mathematics
Delft University of Technology
Mekelweg 4, 2628 CD Delft, The Netherlands
Email: j.g.maks@tudelft.nl

Abstract

Let $G = (V, E)$ be a simple, undirected graph on the vertex set $V = \{1, 2, \ldots, n\}$ and let $A$ be the adjacency matrix of $G$. A non-empty subset $S = \{i_1, i_2, \ldots, i_k\}$ of $V$ is called a driver set for $G$ if the system $\dot{x} = Ax + B_fu$, or the pair $(A, B_S)$, is controllable, where $B_S$ is the $(n \times k)$-matrix with columns $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$. Let $D(G)$ denote the minimum cardinality of a driver set for $G$ and let $M(G)$ denote the maximum of all geometric multiplicities of eigenvalues of $A$. It is well-known that $D(G) \geq M(G)$ for all graphs $G$.

Let $\gamma$ denote the Plücker embedding of $Gr(k, n)$ into $P^{\binom{n}{k}-1}$. We prove that in all cases of equality $D(G) = M(G) = k$ a necessary condition for $S$ to be a minimal driver set is the condition that $S$ lies in the support of $\gamma(E_{\lambda})$ for each eigenspace $E_{\lambda}$ of maximal dimension $k$. We classify the minimal driver sets for the path and cycle graphs $P_n$ and $C_n$ for all values of $n$.

Let $\text{Sym}_0(G)$ be the set of symmetric $(n \times n)$-matrices $X$ with zero diagonal and off-diagonal elements $x_{ij}$ unequal to zero if and only if $(i, j) \in E$. The pair $(G, S)$ is called strongly $\text{Sym}_0(G)$-controllable if $(X, B_S)$ is controllable for all $X \in \text{Sym}_0(G)$. We show that this property of $(G, S)$ is respected by the automorphism group $\text{Aut}(G)$.

We determine all orbits of minimal driver sets $S$ for $G = P_n$ and $G = C_n$ for which $(G, S)$ is strongly $\text{Sym}_0(G)$-controllable.
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1 Introduction

1.1 Background and motivation

Let $G = (V, E)$ be a simple, undirected graph on the vertex set $V = \{1, 2, \ldots, n\}$ with adjacency matrix $A$. For each non-empty subset $S = \{i_1, i_2, \ldots, i_k\}$ of $V$ let $B_S$ be the $(n \times k)$-matrix with columns $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$. A non-empty subset $S$ is called a driver set for $G$ if the system $\dot{x} = Ax + B_S u$, or the pair $(A, B_S)$, is controllable.

Let $\text{Sym}(G)$ be the set of symmetric $(n \times n)$-matrices $X$ with free diagonal elements and off-diagonal elements $x_{ij}$ unequal to zero if and only if $(i, j) \in E$. If $(A, B_S)$ is controllable then $(X, B_S)$ is controllable not just for $X = A$ but for almost all $X \in \text{Sym}(G)$, a property which is referred to as structural controllability in the literature. The subject of structural controllability of networks has been studied intensively during the last two decades by many researchers in the systems and control community, in view of applications where the weights of the edges are not fixed due to lack of information or numerical instability.

A stronger version of structural controllability is the property that $(X, B_S)$ is controllable for all $X \in \text{Sym}(G)$. This property is referred to as strong structural controllability in the literature.

Perhaps surprisingly, it turned out that this notion of strong structural controllability of a network is connected to the notion of a zero forcing set of the underlying graph. It has been proved in [5] that $(X, B_S)$ is controllable for all $X \in \text{Sym}(G)$ if and only if $S$ is a zero forcing set of $G$.

A zero forcing set is a special type of driver set but in general not every driver set is a zero forcing set. The discovery of the connection between strong structural controllability and zero forcing sets has understandably caused a surge of research in the latter. We believe there are several good reasons for studying all minimal driver sets for the system $(A, B_S)$, such as:

\footnote{The notion of a zero forcing set (briefly summarized in section 4) had been introduced several years earlier in a different context [1].}
• The minimal size of a driver set could be smaller than the minimal size of a zero forcing set, which could be relevant in applications where using a driver set of minimum cardinality is essential.

• Additional requirements about the relative positions of the vertices in a driver set may exist which might not be satisfied by the zero forcing sets.

• Strong structural controllability with respect to Sym$(G)$ allows for $|V| + |E|$ degrees of freedom in the system matrix. In applications where there is no lack of information or numerical instability regarding the diagonal entries, the $|V|$ degrees of freedom on the diagonal are superfluous. In such cases it is actually more natural to study strong structural controllability with respect to the smaller family Sym$_0(G)$ consisting of all matrices in Sym$(G)$ with zeros on the diagonal, allowing for $|E|$ degrees of freedom only. Driver sets $S$ for which $(G, S)$ is strongly Sym$_0(G)$-controllable are not necessarily a zero forcing set.

It is well-known that $S = \{i_1, i_2, \ldots, i_k\}$ is driver set for $G$ if and only if there is no eigenvector $v$ of $A$ with $v_{i_1} = v_{i_2} = \cdots = v_{i_k} = 0$, i.e., if and only if $\text{Nul}\ B_S^T \cap E_\lambda = \{0\}$ for each eigenspace $E_\lambda$ of $A$. If $|S| = \dim E_\lambda = 1$ the condition $\text{Nul}\ B_S^T \cap E_\lambda = \{0\}$ simply means that $S$ should be contained in the support of any basis vector of $E_\lambda$ (note the slight abuse of notation here, where the singleton $S = \{i\}$ is identified with its element $i$). In the course of our research we realised that a generalization of this interpretation exists if $|S| = \dim E_\lambda = k > 1$, based on the existence of the Plücker embedding $\gamma : \text{Gr}(k, n) \rightarrow \mathbb{P}^{\binom{n}{k}}$. The homogeneous coordinates of the Plücker image $\gamma(E_\lambda)$ of a $k$-dimensional eigenspace $E_\lambda$ are indexed by the $\binom{n}{k}$ subsets $S$ of $V$ of cardinality $k$ and the condition $\text{Nul}\ B_S^T \cap E_\lambda = \{0\}$ is equivalent to the condition that $S$ lies in the support of $\gamma(E_\lambda)$. This observation reveals an interesting aspect of the Plücker images of the eigenspaces of $A$ which can be a useful tool for constructing minimal driver sets for $G$.

1.2 Specification of results

Let $D(G)$ denote the minimum cardinality of a driver set for the graph $G$, as defined in the introduction, and let $N_D(G)$ denote the number of minimal driver sets for $G$. Furthermore, let $M(G)$ denote the maximum of all geometric multiplicities of $A$ and let $Z(G)$ denote the so-called zero forcing number.
of $G$, the latter being defined as the minimum cardinality of a zero forcing set. For each graph $G$ we have

$$M(G) \leq D(G) \leq Z(G).$$

Our research has produced the following results, all of which are new to the best of our knowledge.

1. For the cases $M(G) = D(G) = k$ we prove that a necessary condition for $S$ to be a minimal driver set is the condition that $S$ lies in the support of $\gamma(E_\lambda)$ for each eigenspace $E_\lambda$ of maximal dimension $k$.

2. If $G = P_n$ then $M(G) = D(G) = Z(G) = 1$. It is known that $\{1\}$ and $\{n\}$ are the only two zero forcing sets of $P_n$. We prove more generally that $\{i\}$ is a driver set for $P_n$ if and only if $\gcd(i, n + 1) = 1$. Hence $N_D(P_n) = \phi(n + 1)$, where $\phi$ denotes Euler’s totient function.

3. If $G = C_n$ then $M(G) = D(G) = Z(G) = 2$. It is known that $\{i, j\}$ is a zero forcing set of $C_n$ if and only $i$ and $j$ are adjacent. We prove more generally that $\{i, j\}$ is a driver set for $C_n$ if and only if $\gcd(2d, n) \in \{1, 2\}$, where $d = d(i, j)$ denotes the distance between the vertices $i$ and $j$.

4. We introduce the notion of strong Sym$_0(G)$-controllability as an alternative/addition to the notion of strong Sym$(G)$-controllability.

5. We show that the properties of strong Sym$_0(G)$-controllability and strong Sym$(G)$-controllability are respected by the automorphism group $Aut(G)$.

6. We classify the orbits of minimal driver sets $S$ for $G = P_n$ and $G = C_n$ for which $(G, S)$ is strongly Sym$_0(G)$-controllable but not strongly Sym$(G)$-controllable.

2 Controllability of linear dynamical systems

In this section we list some standard results about controllability. More details and proofs can be found in any textbook on linear systems theory. The matrices $A$ and $B$ are matrices of sizes $(n \times n)$ and $(n \times k)$, respectively.
Definition 1 The system $\dot{x} = Ax + Bu$, or the pair $(A, B)$, is controllable if any initial state vector $x_0 = x(t_0)$ can be steered by the system to any other state vector $x_1$ in finite time, that is, if for any pair $x_0, x_1 \in \mathbb{R}^n$ with $x_0 = x(t_0)$ there exist a time instant $t_1 > t_0$ and an input function $u = u(t)$ on the interval $[t_0, t_1]$ such that $x_1 = x(t_1)$.

Theorem 2 The system $\dot{x} = Ax + Bu$, or the pair $(A, B)$, is controllable if and only if rank $\left[ \begin{array}{ccc} B & AB & \cdots & A^{n-1}B \end{array} \right] = n$.

Remark 3 If the degree of the minimal polynomial of $A$ is equal to $k$ with $k < n$ then it suffices to use $\left[ B \ AB \cdots A^{k-1}B \right]$ in the theorem above.

Theorem 4 The following three statements are equivalent:

1. $(A, B)$ is controllable.
2. rank $\left[ \begin{array}{cc} A - \lambda I & B \end{array} \right] = n$ for all $\lambda \in \mathbb{C}$.
3. rank $\left[ \begin{array}{cc} A - \lambda I & B \end{array} \right] = n$ for all eigenvalues $\lambda$ of $A$.

Theorem 5 The system $\dot{x} = Ax + Bu$, or the pair $(A, B)$, is not controllable if and only if there exists an eigenvector of $A^T$ which is orthogonal to all columns of $B$.

Statement 3 in Theorem 4 implies that rank $B \geq \text{gm} (\lambda)$ for each eigenvalue $\lambda$ of $A$, hence

$$\text{rank } B \geq \max_{\lambda \in \sigma(A)} \{\text{geometric multiplicity } \lambda\}. \quad (1)$$

Definition 6 Two systems $(A, B)$ and $(A', B')$ are equivalent if there exists an invertible matrix $T$ such that

$$\left\{ \begin{array}{l} A' = TAT^{-1} \\ B' = TB \end{array} \right.$$  

Theorem 7 If $(A, B)$ and $(A', B')$ are equivalent then

$(A', B')$ controllable if and only if $(A, B)$ controllable.
3 Minimal driver sets

Now assume that $A = A(G)$ is an adjacency matrix of a graph, hence a symmetric matrix, and $B = B_S = \begin{bmatrix} e_{i_1} & e_{i_2} & \cdots & e_{i_k} \end{bmatrix}$.

Since rank $B_S = |S|$ inequality (1) yields

$$D(G) \geq M(G).$$

(2)

Application of Theorem 5 to the pair $(A, B_S)$ with $A = A(G)$ yields

**Corollary 8** A non-empty subset $S$ of $V$ is not a driver set for the graph $G$ if and only if there exists an eigenvector $v$ of $A (= A^T)$ with $B_S^T v = 0$.

Equivalently,

**Corollary 9** $S$ is a driver set for the graph $G$ if and only if

$$\text{Nul } B_S^T \cap E_{\lambda} = \{0\}$$

for each of the eigenspaces $E_{\lambda}$ of $A$.

Let $W$ be an $m$-dimensional subspace of $\mathbb{R}^n$ and $S$ a subset of $V$ with $|S| = k \geq 1$. Then Nul $B_S^T \cap W = \{0\}$ if and only if $\{B_S^T w_1, B_S^T w_2, \ldots, B_S^T w_m\}$ is linearly independent for each basis $\{w_1, w_2, \ldots, w_m\}$ of $W$. This condition can be rephrased as

$$\text{rank } B_S^T M = m$$

for each $(n \times m)$-matrix $M$ which satisfies $\text{Col } M = W$. Note that $B_S^T M$ is a $(k \times m)$-matrix and that the condition rank $B_S^T M = m$ implies $k \geq m$, i.e.,

$$|S| \geq \dim W.$$

If $|S| = \dim W = k$ then $B_S^T M$ is a $(k \times k)$-matrix in which case the condition rank $B_S^T M = k$ is equivalent to the condition $\det B_S^T M \neq 0$.

The determinant of $B_S^T M$ is the homogeneous coordinate indexed by $S$ of the image of $W$ under the Plücker embedding$^2$

$$\gamma : Gr(k, n) \to \mathbb{P}\left(\binom{n}{k}\right)^{-1}.$$ 

Hence we have the following theorem, which is theoretically interesting and which can be a useful tool for constructing minimal driver sets for graphs $G$ with $D(G) = M(G)$.

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$^2$A brief introduction to this realization of Grassmannians is given in the last section of this paper.
Theorem 10  Let $G$ be a graph on $n$ vertices with

$$D(G) = M(G) = k.$$  

If $S$ is a minimal driver set for $G$ then $S \in \text{support} (\gamma(E_\lambda))$ for each eigenspace $E_\lambda$ of $A$ with $\text{dim } E_\lambda = k$.

Regarding the special case $k = 1$, note that the Plücker coordinates of a one-dimensional subspace are simply the coordinates of a basis vector of that subspace.

Example 11  $G = P_5$. The eigenvalues of $A$ are $-1, 0, 1, -\sqrt{3}, \sqrt{3}$. Basis vectors for the corresponding eigenspaces are the columns of the matrix $M$ given by

$$M = \begin{bmatrix}
-1 & 1 & -1 & 1 & 1 \\
1 & 0 & -1 & -\sqrt{3} & \sqrt{3} \\
0 & -1 & 0 & 2 & 2 \\
-1 & 0 & 1 & -\sqrt{3} & \sqrt{3} \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$  

$D(P_5) = M(P_5) = 1$ and $N_D(P_5) = 2$. The two minimal driver sets are $\{1\}$ and $\{5\}$ because rows 1 and 5 of the matrix $M$ do not contain a zero. The two minimal driver sets lie in a single orbit under the action of the automorphism group $Aut(P_5) = \langle (1, 5), (2, 4) \rangle$. The path graphs $P_n$ for general $n$ will be discussed in section 6.

Example 12  $G = C_6$. The eigenvalues of $A$ are $-2, 2, -1^2, 1^2$ hence $M(G) = 2$. Basis vectors for the corresponding eigenspaces are collected in the following block matrix

$$M = [M_1 | M_2 | M_3 | M_4] = \begin{bmatrix}
-1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 \\
-1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 0 & 1 \\
-1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}.$$  

Theorem 10 tells us that if $S = \{i, j\}$ is a driver set for $G$ then $S$ must lie in the support of $\gamma(E_{-1})$ and in the support of $\gamma(E_1)$. It turns out that these
supports are equal, both consisting of the 12 elements \( \{i, j\} \) with \( d(i, j) \in \{1, 2\} \). Since the basis vectors of the remaining eigenspaces \( E_{-2} \) and \( E_2 \) don’t have two zeros in any of these pairs of positions we can conclude \( D(C_6) = 2 \) with \( N_D(C_6) = 12 \). The sets \( \{i, j\} \) with \( d(i, j) = 3 \) are the sets of cardinality 2 that are not a driver set. For example \( \{1, 4\} \) is not a driver set because \( \det B_{\{1,4\}}^T M_3 = 0 \) or \( \det B_{\{1,4\}}^T M_4 = 0 \) (in this example both are true):

\[
B_{\{1,4\}}^T M_3 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad B_{\{1,4\}}^T M_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]

The minimal driver sets fall into the two orbits \( \{i, j\} | d(i, j) = 1 \} \) and \( \{i, j\} | d(i, j) = 2 \} \) under the group \( \text{Aut}(C_6) = \langle (1, 2, 3, 4, 5, 6), (1, 2)(3, 6)(4, 5) \rangle \). The cycle graphs \( C_n \) for general \( n \) will be discussed in section 6.

Note that in the examples above the property of being a minimal driver set is invariant under the action of the automorphism group \( \text{Aut}(G) \). This is true in general:

**Proposition 13** Let \( \pi \in \text{Aut}(G) \). Then \( S \) is a driver set for \( G \) if and only if \( \pi(S) \) is a driver set for \( G \).

**Proof.** Let \( P \) denote the permutation matrix that corresponds to \( \pi \in \text{Aut}(G) \). Then \( B_{\pi(S)} = PB_S \) and \( A = PAP^T \), hence the systems \( (A, B_S) \) and \( (A, B_{\pi(S)}) \) are equivalent. The result now follows from theorem 7. □

### 4 Strong structural controllability

Let \( S \) be a driver set for a graph \( G = (V, E) \) with \( |V| = n \). Let \( \text{Sym}(G) \) be the set of all symmetric \((n \times n)\)-matrices \( X = [x_{ij}] \) satisfying

\[
x_{ij} \neq 0 \iff (i, j) \in E
\]

for all pairs \((i, j)\) with \( i \neq j \). Hence \( \text{Sym}(G) \) is the largest set of symmetric matrices that have their non-zero off-diagonal entries in precisely the same positions as the adjacency matrix \( A \).

**Definition 14** \((G, S)\) is strongly \( \text{Sym}(G) \)-controllable if \((X, B_S)\) is controllable for all \( X \in \text{Sym}(G) \).
Note that this formulation is a succinct alternative to the more elaborate version ‘\((G, S)\) is strongly structurally controllable with respect to \(\text{Sym}(G)\) if \((X, B_S)\) is controllable for all \(X \in \text{Sym}(G)\)’.

More generally we replace ‘strongly structurally controllable with respect to \(F\)’ by ‘strongly \(F\)-controllable’ (where \(F\) is a set of matrices having the same zero/non-zero pattern in the off-diagonal entries as \(A\)).

It has been proved in [5] that \((G, S)\) is strongly \(\text{Sym}(G)\)-controllable if and only if \(S\) is a zero forcing set of \(G\).

The process of zero forcing, which was introduced in [1] and independently in [2], can be briefly summarized in the following way.

Let \(S\) be a non-empty subset of vertices of \(G\) and suppose all vertices from \(S\) are colored black and all vertices from \(V \setminus S\) are colored white. If there exists a black vertex with exactly one white neighbour \(j\) then change the color of \(j\) to black and extend the set \(S\) to \(S \cup \{j\}\) and repeat this process until no color change is possible anymore.

**Definition 15** The set \(S\) is called a zero forcing set if the coloring process described above results in all vertices being colored black.

**Definition 16** The zero forcing number of \(G\), denoted by \(Z(G)\), is the minimum cardinality of a zero forcing set.

The following examples are well-known.

| \(G\)  | \(Z(G)\) | Zero forcing sets \(S\) with \(|S| = Z(G)\) |
|-------|-------|----------------------------------|
| \(P_n\) | 1     | \(\{1\}\) and \(\{n\}\)          |
| \(C_n\) | 2     | \(\{i, j\}\) with \(d(i, j) = 1\) |
| \(K_n\) | \(n - 1\) | any \((n - 1)\)-set                |

Each zero forcing set is a driver set hence for each graph \(G\) we have

\[
D(G) \leq Z(G). \tag{3}
\]

Note that in each of the examples in the table above all minimal zero forcing sets lie in the same orbit under the action of the automorphism group \(\text{Aut}(G)\). In general the minimal zero forcing sets could lie in different orbits but the property of being a zero forcing set is indeed invariant under the action of \(\text{Aut}(G)\). This follows immediately from the definition of a zero forcing set, which is based on the adjacency structure of \(G\) only. Equivalently we have the following property:
Proposition 17 Let $\pi \in Aut(G)$. Then $(G, S)$ is strongly $Sym(G)$-controllable if and only if $(G, \pi(S))$ is strongly $Sym(G)$-controllable.

Proof. Let $P$ denote the permutation matrix that corresponds to $\pi \in Aut(G)$. Then $B_{\pi(S)} = PB_S$. The systems $(X, B_S)$ and $(PXPT, PB_S)$ are equivalent hence due to Theorem 17 controllability of the one is equivalent to controllability of the other. On the other hand, $Sym(G)$ is invariant under the transformation $X \mapsto PXPT$, which permutes the free parameters on the diagonal and the free parameters on the off-diagonal positions $(i, j) \in E$.

Strong $Sym(G)$-controllability allows for $|V| + |E|$ degrees of freedom in the system matrix. In applications where there is no lack of information or numerical instability regarding the diagonal entries, the $|V|$ degrees of freedom on the diagonal are superfluous. In such cases it would be sufficient and more natural to require strong structural controllability with respect to the smaller family $Sym_0(G)$ consisting of all matrices in $Sym(G)$ with zeros on the diagonal, allowing for $|E|$ degrees of freedom only. Driver sets $S$ for which $(G, S)$ is $Sym_0(G)$-controllable are not necessarily a zero forcing set. Note that Proposition 17 holds for the smaller family $Sym_0(G)$ as well, because the transformation $X \mapsto PXPT$ doesn’t change the zeros on the diagonal.

The chain $Sym_0(G) \subset Sym(G)$ gives rise to the following two types of driver sets $S$:

| Type | $(G, S)$ is strongly $F$-controllable |
|------|--------------------------------------|
| I    | for $F = Sym(G)$                     |
| II   | for $F = Sym_0(G)$, not for $F = Sym(G)$ |

Driver sets of type I are zero forcing sets, driver sets of type II are not zero forcing sets but could still be useful for certain applications. Since each of the two types defined above is $Aut(G)$-invariant we could also speak of orbits of type I, II.

To prove that $(G, S)$ is strongly $F$-controllable we can proceed as follows. Due to Theorem 17 $(X, B_S)$ is controllable for each $X \in F$ if and only if

$$\text{rank } \left[ X - \lambda I \quad B_S \right] = n$$

for all $\lambda \in \mathbb{C}$ and $X \in F$. The rows of $\left[ X - \lambda I \quad B_S \right]$ are linearly independent if and only if the rows of $(X - \lambda I)_{V \setminus S}$ are linearly independent, where $(X - \lambda I)_{V \setminus S}$ is the submatrix of $X - \lambda I$ which is obtained by deleting the
rows that are indexed by the elements of $S$. Hence $(X, B_S)$ is controllable for each $X \in F$ if and only if

$$\text{rank } (X - \lambda I)_{V \setminus S} = n - |S|$$

for all $\lambda \in \mathbb{C}$ and $X \in F$. We shall use this method in the next two sections where we determine the orbits of type II for the path and cycle graphs.

## 5 Path graphs

Since $Z(P_n) = 1$ and $D(P_n) \leq Z(P_n)$ it follows that $D(P_n) = 1$ as well. In the following theorem $\phi$ denotes the Euler totient function.

**Theorem 18**\{i\} is a driver set for the graph $P_n$ if and only if

$$\gcd(i, n + 1) = 1,$$

hence $N_D(P_n) = \phi(n + 1)$.

**Proof.** The eigenvalues of $A = A(P_n)$ are given by $\lambda_k = 2 \cos \left(\frac{k\pi}{n+1}\right)$ with $k = 1, 2, \ldots, n$ and all eigenvalues have multiplicity equal to 1. The vector $\left[ \sin \left(\frac{k\pi}{n+1}\right), \sin \left(\frac{2k\pi}{n+1}\right), \ldots, \sin \left(\frac{nk\pi}{n+1}\right) \right]^T$ is an eigenvector of $A$ belonging to the eigenvalue $\lambda_k$. Due to Corollary \[ the singleton \{i\} is not a driver set if and only if there exists an eigenvector of $A$ whose $i$-th entry is equal to 0 hence if and only if $\sin \left(\frac{ik\pi}{n+1}\right) = 0$ for at least one $k \in \{1, 2, \ldots, n\}$. The latter is true if and only if $ik \equiv 0 \mod n + 1$ for at least one $k \in \{1, 2, \ldots, n\}$, which is equivalent to $\gcd(i, n + 1) \neq 1$. ■

The orbits of minimal driver sets under the group $Aut(P_n) \cong S_2$ are simply the pairs \{\{i\}, \{n + 1 - i\}\} with $\gcd(i, n + 1) = 1$ hence the number of orbits is equal to $\frac{1}{2}\phi(n + 1)$.

Driver sets of type I have to be zero forcing sets \[. It is obvious that \{1\} and \{n\} are the only zero forcing sets for $P_n$ and that this is true for all $n \geq 2$. It is easy to see that the orbit \{\{1\}, \{n\}\} is of type I without resorting to the notion of zero forcing sets. We only need to show this for one representative of the orbit. For each $X = [x_{ij}] \in \text{Sym}(P_n)$ the matrix $(X - \lambda I)_{\{2, \ldots, n\}}$ is an echelon matrix with $n - 1$ pivots $x_{12}, x_{23}, \ldots, x_{n-1,n}$ hence rank $(X - \lambda I)_{\{2, \ldots, n\}} = n - 1$ for all $X \in \text{Sym}(P_n)$ and $\lambda \in \mathbb{C}$. Before examining the other orbits of minimal driver sets we present some useful lemmas.
Lemma 19. For each $X = [x_{ij}] \in \text{Sym}_0(P_n)$ we have

$$\det X = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} x_{12}^2 x_{34}^2 \cdots x_{n-1,n}^2 & \text{if } n \text{ is even} \end{cases}$$

**Proof.** Let $d_n = \det X$ with $X = [x_{ij}] \in \text{Sym}_0(P_n)$. Then $d_1 = 0$ and $d_2 = -x_{12}^2$ and expansion along the last column and then along the last row yields the recurrence relation

$$d_n = -x_{n-1,n}^2 d_{n-2}$$

for all $n \geq 3$. ■

For each $X \in \text{Sym}_0(P_n)$ with $n \geq 3$ and $i \in \{2, \ldots, n-1\}$ the matrix $X_{V\backslash\{i\}}$ has the block structure

$$\begin{bmatrix} Y & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & Z \end{bmatrix}$$

with $Y \in \text{Sym}_0(P_{i-1})$ and $Z \in \text{Sym}_0(P_{n-i})$.

Lemma 20. Let $X \in \text{Sym}_0(P_n)$ with $n \geq 3$ and $i \in \{2, \ldots, n-1\}$ and $Y$ and $Z$ as in (4). Then $\text{rank}(X - \lambda I)_{V\backslash\{i\}} < n - 1$ if and only if $Y$ and $Z$ have a common eigenvalue $\lambda$.

**Proof.** The linear system $(X - \lambda I)^T_{V\backslash\{i\}} v = 0$ breaks down into

1. $(Y - \lambda I)v_{(1,2,\ldots,i-1)} = 0$
2. $x_{i-1,i}v_{i-1} + x_{i,i+1}v_i = 0$
3. $(Z - \lambda I)v_{(i+1,i+2,\ldots,n-1)} = 0$

Equation (2) implies that either $v_{i-1} = v_i = 0$ or $v_{i-1}v_i \neq 0$. If $v_{i-1} = v_i = 0$ then it follows from (1) and (3) that $v = 0$. Suppose $\text{rank}(X - \lambda I)_{V\backslash\{i\}} < \ldots$\footnote{A similar result has been proved in \cite{17} with respect to the system $(L_n, B_{(i)})$, where $L_n$ is the Laplacian matrix of $P_n$.}
\(n - 1\), i.e., suppose the system above does have a non-trivial solution \(v\). Then \(v_{i-1} \neq 0\) and \(v_i \neq 0\) hence \(v_{\{1,2,\ldots,i-1\}} \neq 0\) and \(v_{\{i+1,i+2,\ldots,n-1\}} \neq 0\), so (1) and (3) show that \(\lambda\) is an eigenvalue of \(Y\) and \(Z\). Conversely, suppose \(Y\) and \(Z\) have a common eigenvalue \(\lambda\), i.e., suppose (1) and (3) have non-trivial solutions. These solutions can be scaled in such a way that \(v_{i-1}\) and \(v_i\) satisfy equation (2), hence a non-trivial solution of the linear system \((X - \lambda I)_{V\backslash\{i\}} v = 0\) exists. \(\blacksquare\)

Now let us examine the orbit \(\{\{2\}, \{n - 1\}\}\). Due to Theorem \(\text{18}\) \(\{2\}\) is a driver set if and only if \(\gcd(2,n+1) = 1\), i.e., if and only if \(n\) is even. It is a zero forcing set for \(n = 2\), so we consider \(n \geq 4\).

**Theorem 21** For all even \(n \geq 4\) the minimal driver sets \(\{2\}\) and \(\{n - 1\}\) for the graph \(P_n\) are of type II.

**Proof.** We only need to show this for one representative of the orbit. Due to Lemma \(\text{20}\) \((P_n, \{2\})\) is not strongly \(\text{Sym}_0(P_n)\)-controllable if and only if there exists an \(X \in \text{Sym}_0(P_n)\) such that \(Y \in \text{Sym}_0(P_1)\) and \(Z \in \text{Sym}_0(P_{n-2})\) (as defined in [4]) have a common eigenvalue. In this case \(Y = [0]\) so \((P_n, \{2\})\) is not strongly \(\text{Sym}_0(P_n)\)-controllable if and only if \(Z\) is singular. It follows from Lemma \(\text{19}\) that \(\det Z = 0\) if and only if \(n - 2\) is odd. \(\blacksquare\)

Finally we show that the remaining orbits are not of type II.

**Theorem 22** Let \(\{i\}\) be a minimal driver set for \(P_n\) with \(3 \leq i \leq n - 2\). \((P_n, \{i\})\) is not strongly \(\text{Sym}_0(P_n)\)-controllable.

**Proof.** Due to Lemma \(\text{20}\) \((P_n, \{i\})\) is not strongly \(\text{Sym}_0(P_n)\)-controllable if and only if there exists an \(X \in \text{Sym}_0(P_n)\) such that \(Y\) and \(Z\) have a common eigenvalue. For each \(i \in \{3, \ldots, n - 2\}\) such a pair \(Y, Z\) is easily constructed in the following way. Choose any \(Y \in \text{Sym}_0(P_{i-1})\) and \(Z \in \text{Sym}_0(P_{n-i})\) and a pair \(\lambda_0, \mu_0\) of non-zero eigenvalues of \(Y\) and \(Z\) respectively. Then \(\mu_0 Y \in \text{Sym}_0(P_{i-1})\) and \(\lambda_0 Z \in \text{Sym}_0(P_{n-i})\) share the eigenvalue \(\lambda_0 \mu_0\). \(\blacksquare\)

### 6 Cycle graphs

Let \(\omega = \exp(i \frac{2\pi}{n})\). The eigenvalues of the adjacency matrix \(A = A(C_n)\) are given by \(\lambda_k = \omega^k + \omega^{n-k} = 2 \cos \left(\frac{2k\pi}{n}\right)\) with \(k = 0, 2, \ldots, n-1\). The algebraic multiplicities (equal to the geometric ones because \(A\) is symmetric hence diagonalizable) are all equal to 2 with the exceptions of \(\lambda_0 = 2\) for all \(n\) and
\( \lambda_{\frac{n}{2}} = -2 \) for all even \( n \). Hence \( M(C_n) = 2 \) which implies \( D(C_n) \geq 2 \). On the other hand \( Z(C_n) = 2 \) hence \( D(C_n) = 2 \) as well. The following Theorem specifies which pairs of vertices do in fact form a minimal driver set.

**Theorem 23** \( \{i, j\} \) is a driver set for the graph \( C_n \) if and only if

\[
\gcd(2d, n) \in \{1, 2\},
\]

where \( d = d(i, j) \) denotes the distance between the vertices \( i \) and \( j \).

**Proof.** The 1-dimensional eigenspaces of \( A \) is/are given by

\[
\text{Span} \left\{ \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \right\} \text{ for all } n \text{ and }
\text{Span} \left\{ \begin{bmatrix} 1 & -1 & \cdots & 1 & -1 \end{bmatrix}^T \right\} \text{ for all even } n,
\]

hence all entries of the eigenvectors from these eigenspaces are unequal to 0. This implies that \( S = \{i, j\} \) is not a driver set if and only if \( p_{ij} = 0 \) for at least one Plücker coordinate \( p_{ij} \) of at least one 2-dimensional eigenspace of \( A \). Let \( \lambda_k \) be an eigenvalue of \( A \) of multiplicity 2 of \( A \), i.e., let

\[ \lambda_k = \omega^k + \omega^{n-k}, \]

with \( \omega = \exp\left(i\frac{2\pi}{n}\right) \) for \( k \in \{1, \ldots, n\} \) except \( k = \frac{n}{2} \) if \( n \) is even. The corresponding 2-dimensional eigenspace \( E_{\lambda_k} \) is given by

\[
E_{\lambda_k} = \text{Span} \left\{ v_k, \bar{v}_k \right\},
\]

where \( v_k = \begin{bmatrix} 1 & \omega^k & \cdots & \omega^{(n-1)k} \end{bmatrix}^T \). The Plücker coordinates of \( E_\lambda \) are given by

\[
p_{ij} = \det \begin{bmatrix} (v_k)_i & (\bar{v}_k)_i \\ (v_k)_j & (\bar{v}_k)_j \end{bmatrix},
\]

hence

\[
p_{ij} = \det \begin{bmatrix} \omega^{(i-1)k} & \omega^{n-(i-1)k} \\ \omega^{(j-1)k} & \omega^{n-(j-1)k} \end{bmatrix} = \omega^{n+k(i-j)} - \omega^{n+k(j-i)}.
\]

Now \( \omega^{n+k(i-j)} - \omega^{n+k(j-i)} = 0 \) if and only if \( k(i-j) \equiv k(j-i) \mod n \), i.e., if and only if \( 2k(j-i) \equiv 0 \mod n \). The trivial solutions \( k = 0 \) and \( k = \frac{n}{2} \) for even \( n \) do not correspond to a 2-dimensional eigenspace hence \( \{i, j\} \) is not a driver set if and only if \( \gcd(2(j-i), n) \notin \{1, 2\} \). This condition can be replaced by \( \gcd(2d, n) \notin \{1, 2\} \) because \( d(i, j) = \min\{j-i, n+i-j\} \) and \( 2k(j-i) \equiv 0 \mod n \) if and only if \( 2k(n+i-j) \equiv 0 \mod n \).
The orbits of minimal driver sets under the group $\text{Aut}(C_n) \cong D_n$ are the sets $\Omega_d$ defined by

$$\Omega_d = \{\{i, j\} \in \binom{V}{2} \mid d(i, j) = d\}$$

for fixed values of $d \in \{1, \ldots, \lfloor n/2 \rfloor\}$ satisfying $\gcd(2d, n) \in \{1, 2\}$. Since the size of each orbit is equal to $n$ the number of orbits is equal to $\frac{1}{n} \mathcal{N}_D(C_n)$. Note that the value $d = 1$ satisfies $\gcd(2d, n) = 1$ for all $n$ and that the corresponding orbit $\Omega_1$ is the (unique) orbit of minimal driver sets that are zero forcing sets. The following table lists the values of $\mathcal{N}_D(C_n)$ for $n \leq 12$.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|----|----|----|
| $\mathcal{N}_D(C_n)$ | 3 | 4 | 10 | 12 | 21 | 16 | 27 | 40 | 55 | 24 |

It is easy to see that the orbit $\Omega_1$ is of type I without resorting to the notion of zero forcing sets. We only need to show this for one representative of the orbit. For each $X = [x_{ij}] \in \text{Sym}(C_n)$ the matrix $(X - \lambda I)_{\{3, \ldots, n\}}$ is row equivalent to an echelon form with pivots $x_{1n}$, $x_{23}$, $x_{34}$, $\ldots$, $x_{n-2,n-1}$ hence $\text{rank}(X - \lambda I)_{\{3, \ldots, n\}} = n - 2$ for all $X \in \text{Sym}(C_n)$ and $\lambda \in \mathbb{C}$. Before examining the other orbits we discuss the analogue of Lemma 20 for the cycle graphs. For each $X \in \text{Sym}_0(C_n)$ with $n \geq 6$ and $j \in \{3, \ldots, \lfloor n/2 \rfloor\}$ the matrix $X_{\{1\} \setminus \{j\}}$ has the block structure

\[
\begin{bmatrix}
  x_{12} & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & x_{i-1,j} \\
  x_{1n} & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
  Y \\
  0 \\
  \vdots \\
  x_{i-1,j} \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  0 \\
  \vdots \\
  x_{1i+1} \\
  \vdots \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  Z \\
\end{bmatrix}
\]

with $Y \in \text{Sym}_0(P_{j-2})$ and $Z \in \text{Sym}_0(P_{n-j})$. The following Lemma can be proved in the same way as Lemma 20.

**Lemma 24** \(^4\) Let $X \in \text{Sym}_0(C_n)$ with $n \geq 6$, $j \in \{3, \ldots, \lfloor n/2 \rfloor\}$ and $Y$ and $Z$ as in (5). Then $\text{rank}(X - \lambda I)_{\{1\} \setminus \{j\}} = n - 2$ if and only if $Y$ and $Z$ have a common eigenvalue $\lambda$.

\(^4\) A similar result has been proved in \cite{7} with respect to the system $(L_n, B_{\{i,j\}})$, where $L_n$ is the Laplacian matrix of $C_n$.  

15
Proof. The linear system \((X − \lambda I)^T v_{\{1, j\}} = 0\) breaks down into

\begin{align*}
(1) & \quad x_{12}v_1 + x_{1n}v_{n-2} = 0 \\
(2) & \quad (Y − \lambda I)v_{\{2, \ldots, j-1\}} = 0 \\
(3) & \quad x_{j-1,j}v_{j-1} + x_{j,j+1}v_j = 0 \\
(4) & \quad (Z − \lambda I)v_{\{j+1, \ldots, n-2\}} = 0
\end{align*}

The existence of a non-trivial solution \(v\) forces \(v_1, v_{j-1}, v_j\) and \(v_{n-2}\) to be non-zero and \(\lambda\) to be an eigenvalue of \(Y\) and \(Z\). Conversely, the existence of non-trivial solutions of (2) and (3) gives rise to a non-trivial solution \(v\). □

Now let us examine the orbit \(\Omega_2\). Due to the theorem above, \(\{i, j\}\) with \(d(i, j) = 2\) is a driver set if and only if \(\gcd(4, n) \in \{1, 2\}\).

Theorem 25 \(\Omega_2\) is a type II orbit of minimal driver sets for \(C_n\) if and only if \(n\) is odd (\(>3\)).

Proof. We only need to show this for one representative of the orbit. We consider \(S = \{i, j\} = \{1, 3\}\). Due to Lemma 24 \((C_n, \{1, 3\})\) is not strongly \((C_n)\)-controllable if and only if there exists an \(X \in \text{Sym}_0(C_n)\) such that \(Y \in \text{Sym}_0(P_1)\) and \(Z \in \text{Sym}_0(P_{n-3})\) (as defined in (5)) have a common eigenvalue. In this case \(Y = [0]\) so \((C_n, \{1, 3\})\) is not strongly \((C_n)\)-controllable if and only if \(Z\) is singular. It follows from Lemma 19 that \(\det Z \neq 0\) for all odd \(n\). Obviously the case \(n = 3\) is not included because \(\{1, 3\} \in \Omega_1\) for the graph \(C_3\). □

Finally we show that the remaining orbits are not of type II.

Theorem 26 Let \(\Omega_d\) be an orbit of minimal driver sets for \(C_n\) with \(d \geq 3\). \(\Omega_d\) is not strongly \((C_n)\)-controllable.

Proof. We only need to show this for one representative of the orbit. We consider \(\{1, j\}\) with \(j \in \{4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}\). Due to Lemma 24 \((C_n, \{1, j\})\) is not strongly \((C_n)\)-controllable if and only if there exists an \(X \in \text{Sym}_0(C_n)\) such that \(Y\) and \(Z\) have a common eigenvalue. Choose any \(Y \in \text{Sym}_0(P_{j-2})\) and \(Z \in \text{Sym}_0(P_{n-j})\) and a pair \(\lambda_0, \mu_0\) of non-zero eigenvalues of \(Y\) and \(Z\) respectively. Then \(\mu_0Y \in \text{Sym}_0(P_{j-2})\) and \(\lambda_0Z \in \text{Sym}_0(P_{n-j})\) share the eigenvalue \(\lambda_0\mu_0\). □
7 Exterior algebra and Plücker embedding

In this section we give a brief summary of the algebraic framework that underlies the definition of the so-called Plücker embedding, by which the elements of $\text{Gr}(k, n)$ are mapped to points in the projective space $\mathbb{P}^{\binom{n}{k}-1}$. More details and proofs can be found in any textbook on multilinear algebra.

Let $X$ denote an $n$-dimensional vector space over a field $F$ and $T(X)$ its tensor algebra. In this paper we only need the case $F = \mathbb{R}$ but the structures and theorems reviewed in this section apply to any field $F$.

**Definition 27** For each $k \in \{1, 2, \ldots, n\}$ the Grassmannian $\text{Gr}(k, n)$ is the set of all $k$-dimensional subspaces of $X$.

The exterior algebra (or Grassmann algebra) $\Lambda X$ associated to $X$ is defined as follows:

**Definition 28** The exterior algebra (or Grassmann algebra) $\Lambda X$ associated to $X$ is defined by

$$\Lambda E = T(E)/I(E),$$

where $I(E)$ is the two-sided ideal generated by all elements $x \otimes x$ with $x \in X$.

The product of two elements $a$ and $b$ (in that order) in the algebra $\Lambda X$ is denoted by $a \wedge b$. If $\{e_1, e_2, \ldots, e_n\}$ is a basis of $X$ then $\Lambda X$ can be seen as the associative algebra with identity element 1 which is generated by $e_1, e_2, \ldots, e_n$ subject to the relations

$$e_i \wedge e_j + e_j \wedge e_i = 0.$$

The algebra $\Lambda X$ is a graded algebra with decomposition

$$\Lambda X = \Lambda^0 X \oplus \Lambda^1 X \oplus \cdots \oplus \Lambda^n X,$$

where the scalar part $\Lambda^0 X$ and the vector part $\Lambda^1 X$ are copies of $F$ and $X$, respectively. More generally, the $k$-vector part $\Lambda^k X$ is spanned by the $\binom{n}{k}$ linearly independent elements

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}, \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n,$$

hence $\dim \Lambda E = \sum_{k=0}^{n} \binom{n}{k} = 2^n$.

**Definition 29** A $k$-vector $u \in \Lambda^k X$ with $1 \leq k \leq n$ is called decomposable if there exist $k$ vectors $u_1, u_2, \ldots, u_k \in X$ such that $u = u_1 \wedge u_2 \wedge \cdots \wedge u_k$.
Proposition 30 \( u_1 \land u_2 \land \cdots \land u_k = 0 \), for \( k \) vectors \( u_1, u_2, \ldots, u_k \in X \), if and only if \( \{u_1, u_2, \ldots, u_k\} \) is linearly dependent.

Proposition 31 Let \( U \in Gr(k, n) \). If \( \{u_1, u_2, \ldots, u_k\} \) and \( \{w_1, w_2, \ldots, w_k\} \) are two bases of \( U \) then

\[
u_1 \land u_2 \land \cdots \land u_k = c(w_1 \land w_2 \land \cdots \land w_k)
\]

for some non-zero \( c \in \mathbb{F} \).

Due to the two propositions above the Grassmannian \( Gr(k, n) \) can be represented as a set of points in \( \mathbb{P}(\Lambda^k X) \), the projective space associated to the subspace \( \Lambda^k X \); this is the so-called Plücker embedding of \( Gr(k, n) \).

Definition 32 The mapping \( \gamma : Gr(k, n) \to \mathbb{P}(\Lambda^k X) \cong \mathbb{P}^{\binom{n}{k} - 1} \) is defined by

\[
\gamma(\text{Span} \{u_1, u_2, \ldots, u_k\}) = [u_1 \land u_2 \land \cdots \land u_k],
\]

where \([u_1 \land u_2 \land \cdots \land u_k]\) denotes the set of all nonzero scalar multiples of \( u_1 \land u_2 \land \cdots \land u_k \).

Given a basis \( \{e_1, e_2, \ldots, e_n\} \) of \( X \), the coordinates of a decomposable \( k \)-vector \( u_1 \land u_2 \land \cdots \land u_k \) turn out to be equal to determinants of \((k \times k)\)-matrices made up from coefficients of the vectors \( u_1, \ldots, u_k \) with respect to that basis. More precisely, if \( u_k = \sum_{i=1}^{n} a_{ik} e_i \) then

\[
\begin{vmatrix}
a_{i_11} & \cdots & a_{i_1k} \\
\vdots & & \vdots \\
a_{i_k1} & \cdots & a_{i_kk}
\end{vmatrix}

\begin{vmatrix}
e_{i_1} \land e_{i_2} \land \cdots \land e_{i_k}
\end{vmatrix}.
\]

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