Effective Theory Approach to the Skyrme Model
and Application to Pentaquarks

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The Skyrme model is reconsidered from an effective theory point of view. From the most general chiral Lagrangian up to and including terms of order \( p^4, N_c \) and \( \delta m^2 \) (\( \delta m \equiv m_s - m \)), new interactions, which have not been previously considered, appear upon collective coordinate quantization. We obtain the parameter set that best fits the observed low-lying baryon masses by carrying out second-order perturbative calculations with respect to \( \delta m \). We calculate the masses and the decay widths of the other members of (mainly) anti-decuplet pentaquark states. The formula for the decay widths is reconsidered, and its baryon mass dependence is clarified.

§1. Introduction

Evidence of a new baryonic resonance state, called \( \Theta^+(1540) \), has been claimed recently by Nakano et al.,1) with \( S = +1 \) and a very narrow width of \( \Gamma \leq 15 \text{ MeV} \). Several other experimental groups have confirmed its existence.2)–4) It appears in the recent version of Reviews of Particle Physics5) with a *** rating, though its parity has not been established. Evidence of less certain exotic pentaquark states, \( \phi(1860) \)6) and \( \Theta^c_0(1710) \)7),8) has also been claimed. The discovery of pentaquarks is expected to lead to a deeper understanding of strong interactions at low energies. Actually, it stimulates the formulation of new ideas and reconsideration of old theories and experimental data.

The discovery of pentaquarks was motivated by a paper by Diakonov, Petrov and Polyakov.9) They predicted the masses and widths of the anti-decuplet, of which \( \Theta^+ \) is presumed to be a member, within the framework of the chiral quark-soliton model(\( \chi \)QSM).† (See Ref. 12) for a review of the \( \chi \)QSM.) The chiral quark-soliton model prediction is reexamined in Refs. 13) and 14). (See also Refs. 15)–17) for earlier papers on the anti-decuplet in the Skyrme model.)

In a related work, Jaffe and Wilczek18) proposed a quark model picture of pentaquarks based on diquark correlation. (See Ref. 19) for a review of this approach and other interesting topics.)

Since Witten pointed out that baryons can be considered solitons25) in the large-
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and showed that the soliton (“Skyrmion”) has the correct spin statistics,27) owing to the Wess-Zumino-Witten (WZW) term,28),29) much effort has been made exploring the consequences. However, the results do not appear to be very successful. In order to fit the results to the observed values of the masses, coupling constants, such as the pion decay constant, that appear in the chiral Lagrangian must be very different from their experimentally determined values. For example, in the SU(3) Skyrme model, the pion decay constant typically must be one third of the experimental value in order to reproduce the correct mass splitting.30) It also predicts an anti-decuplet,15)–17) which, until recently, many did not believe to exist. However, because this anti-decouplet, \( \Theta^+ \), has now been discovered, it is time to take a serious look at the Skyrme model again.

One of the most important aspects of \( \Theta^+ \) is its narrowness. Several analyses of older data indicate that the width may be less than 1 MeV.20)–23) Interestingly, the Skyrme model is believed to be capable of explaining the narrowness. Specifically, it is claimed that the width becomes very narrow due to a strong cancellation.24) We are thus lead to the following question: Is this a general result of the Skyrme model, or a “model-dependent” one? Also, what is the most general Skyrme model? However, note that the usual Skyrme model is just a model, not based on a general principle. But, as Witten emphasized, the soliton picture of baryons is a general consequence of the large-\( N_c \) limit of QCD. Assuming that large-\( N_c \) QCD bears a close resemblance to real QCD, we may consider an effective theory (not just a model) of baryons based on the soliton picture, which we called the “Skyrme-Witten large-\( N_c \) effective theory”.

Our key observation is that the Skyrme model conventionally starts with a particular chiral Lagrangian, which consists of the kinetic term, the Skyrme term (which stabilizes the soliton), the WZW term and the leading \( SU(3) \) breaking term. Explicitly we have

\[
S_{\text{Skyrme}} = \frac{F^2}{16} \int d^4x \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{32e^2} \int d^4x \text{Tr} \left( \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \right) + N_c \Gamma[U]
\]

\[
+ \frac{F^2 B}{8} \int d^4x \text{Tr} \left( \mathcal{M}^\dagger U + MU^\dagger \right),
\]

where \( \mathcal{M} \) is the quark mass matrix\(^*)

\[
\mathcal{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m_s \end{pmatrix},
\]

and

\[
\Gamma[U] = \frac{i}{240\pi^2} \int_Q d\Sigma^{ijklm} \text{Tr} \left[ (\partial_i U)^\dagger (\partial_j U) (\partial_k U)^\dagger (\partial_l U) (\partial_m U) U^\dagger \right]
\]

\(^*)\text{We do not consider the isospin breaking in this paper.}
is the WZW term. Chiral perturbation theory\textsuperscript{31,32} (\textit{\textchi}PT) is, however, an effective field theory with infinitely many operators.\textsuperscript{*} We should keep in mind that there are (infinitely) many other terms, and the expansion must be systematic.

In this paper, we explore such an “effective theory” approach, i.e., an approach based on a systematic expansion of the operators based on the symmetry and power-counting. The parameters are determined by fitting the results to the experimental values. Once the parameters are fixed, we can calculate other things as predictions. As an application of our approach, we predict the masses and the decay widths of the other (mainly) anti-decuplet baryons.

Generalizations of the Skyrme model have been investigated by several authors, by including vector mesons\textsuperscript{34–36} or the radial modes.\textsuperscript{37,38} Inclusion of heavier vector mesons may correspond to the inclusion of higher-order terms in the usual pseudoscalar Lagrangian, while the inclusion of the radial modes is beyond the scope of the present paper.

This paper is organized as follows. In §2, we derive the collective coordinate quantized Hamiltonian on the basis of the effective theory approach. The Hamiltonian contains several \textit{SU}(3) flavor symmetry breaking interactions that have not yet been considered in the literature. In §3, we consider the eigenstates of the Hamiltonian, with the eigenvalues being the baryon masses. If the symmetry breaking terms were absent, the eigenstates form the flavor \textit{SU}(3) representations. The symmetry breaking terms mix the representations. We calculate the masses in the perturbation theory up to second order with respect to the symmetry breaking parameter $\delta m \equiv m_s - m$. The masses are represented as functions of the parameters of the theory. In §4, we numerically fit the calculated masses to the experimental values in order to determine the parameters. After determining the parameters, we obtain the masses of mainly anti-decuplet baryons, $N'$, an excited nucleon with $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$, and $\Sigma'$ with $1(\frac{1}{2}^+)$. The decay widths are calculated in §5, after reconsidering the derivation of the formula for the widths in the collective coordinate quantization. Unfortunately, our calculation of the widths is quite ambiguous. We summarize our results and discuss some further issues in §6. Notation, conventions, and derivations of several useful mathematical formulae are presented in Appendix A. Various matrix elements used in the calculations are summarized in Appendix B. Many tables are given there. The reason we present such results is that most of them have not previously appeared in the literature and it requires a great deal of effort to calculate them. The conventional approach to the Skyrme model is reconsidered from a new perspective in Appendix C. The results obtained in Appendix C may be regarded as evidence that our basic strategy is valid. Finally, in Appendix D, we carry out a parallel analysis with the \textit{\textchi}QSM symmetry breaking terms,

\begin{equation}
H_{1}^{DPP} = \alpha D_{88}^{(8)} + \beta Y + \frac{\gamma}{\sqrt{3}} D_{86}^{(8)} F^i, \tag{1.4}
\end{equation}

without considering how these interactions are derived. The parameters $\alpha$, $\beta$, and $\gamma$ are determined in a similar way, and the decay widths are also calculated.

\textsuperscript{*} A long time ago, Kindo and Yukawa\textsuperscript{33} considered the Skyrme model in the framework of the \textit{\textchi}PT context, but their work did not seem to attract much attention at that time.
§2. Collective Hamiltonian

2.1. Chiral Lagrangian up to and including $O(p^4)$ and $O(N_c)$

Effective field theories are not simply models. They represent very general principles, such as analyticity, unitarity, cluster decomposition of quantum field theory and the symmetries of the systems.\(^{39}\) Chiral perturbation theory (\(\chi\)PT), for example, represents the low-energy behavior of QCD (at least) in the meson sector.

Although baryons in the large-\(N_c\) limit behave like solitons, it is not clear in what theory they appear. A natural possibility is \(\chi\)PT, because, as stated above, it is a very general framework in which low-energy QCD is represented. It seems that if baryons can appear as solitons, they should appear in \(\chi\)PT, with infinitely many operators.

At low energies, only a few operators are important in the \(\chi\)PT Lagrangian. We can systematically expand the results with respect to the typical energy/momentum scale, \(p\). This is the usual power counting in \(\chi\)PT, and we assume it is valid even in the soliton (i.e., baryon) sector.

To summarize, it may be the case that a general Skyrme-Witten soliton theory takes the form of a systematic expansion of the soliton sector of \(\chi\)PT, with respect to \(N_c\) and \(p\). Therefore, our starting point is the \(SU(3)\) \(\chi\)PT action (without external gauge fields) up to and including \(O(p^4)\),

$$S_{\chi\text{PT}} = S_0 + S_1 + O(p^6),$$  \hspace{1cm} (2.1)

with

$$S_0 = \frac{F_0^2}{16} \int d^4x \text{Tr} \left( \partial_\mu \partial_\mu U \right) + \frac{F_0^2 B_0}{8} \int d^4x \text{Tr} \left( \mathcal{M}^\dagger U + \mathcal{M} U^\dagger \right) + N_c \Gamma[U],$$ \hspace{1cm} (2.2)

$$S_1 = L_1 \int d^4x \left[ \text{Tr} \left( \partial_\mu U \partial_\mu U^\dagger \right) \right]^2 + L_2 \int d^4x \text{Tr} \left( \partial_\mu U^\dagger \partial_\nu U \right) \text{Tr} \left( \partial_\mu U^\dagger \partial_\nu U \right)$$

$$+ L_3 \int d^4x \text{Tr} \left( \partial_\mu U^\dagger \partial_\nu U \partial_\sigma U^\dagger \partial_\sigma U \right)$$

$$+ L_4 B_0 \int d^4x \text{Tr} \left( \partial_\mu U^\dagger \partial_\sigma U \right) \text{Tr} \left( \mathcal{M}^\dagger U + \mathcal{M} U^\dagger \right)$$

$$+ L_5 B_0 \int d^4x \left[ \partial_\mu U^\dagger \partial_\nu U \left( \mathcal{M}^\dagger U + U^\dagger \mathcal{M} \right) \right]$$

$$+ L_6 B_0^2 \int d^4x \left[ \text{Tr} \left( \mathcal{M}^\dagger U + \mathcal{M} U^\dagger \right) \right]^2$$

$$+ L_7 B_0^2 \int d^4x \left[ \text{Tr} \left( \mathcal{M}^\dagger U - \mathcal{M} U^\dagger \right) \right]^2$$

$$+ L_8 B_0^2 \int d^4x \text{Tr} \left( \mathcal{M}^\dagger U \mathcal{M} U^\dagger + \mathcal{M} U^\dagger \mathcal{M} U \right),$$ \hspace{1cm} (2.3)

where \(L_i (i = 1, \ldots, 8)\) are dimensionless constants. The definition of these parameters is the same as that given in Ref. 32), except for \(F_0\). Our normalization of \(F_0\) is more common in the Skyrme model literature. Their \(N_c\) dependence is known:\(^{32},^{40}\)

$$B_0, 2L_1 - L_2, L_4, L_6, L_7 \cdots O(N_c^0),$$  \hspace{1cm} (2.4)
In the following, we retain only the operators whose coefficients are of order \( N_c \). Experimentally, these constants are not known very accurately. We further assume that the constants \( L_1, L_2 \) and \( L_3 \) are related as

\[
L_1 : L_2 : L_3 = 1 : 2 : -6,
\]

which is consistent with the experimental results, \( L_1 = 0.4 \pm 0.3, 2L_1 - L_2 = -0.6 \pm 0.5, \) and \( L_3 = -3.5 \pm 1.1 \times 10^{-3} \).\(^{41}\) Note that vector meson dominance also implies this ratio.\(^{42}\) This assumption simplifies the analysis greatly, due to the identity

\[
\text{Tr}(ABAB) = -2\text{Tr}(A^2B^2) + \frac{1}{2}\text{Tr}(A^2)\text{Tr}(B^2) + (\text{Tr}(AB))^2, \tag{2.7}
\]

which holds for any \( 3 \times 3 \) traceless matrices \( A \) and \( B \). Using this identity, it can be shown that the \( L_1, L_2 \) and \( L_3 \) terms are given by a single expression,

\[
\frac{1}{32e^2}\text{Tr} \left( \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \right), \tag{2.8}
\]

where we have introduced \( L_2 = 1/(16e^2) \). This term is the Skyrme term.\(^*\) (If we do not assume the above exact ratios among \( L_1, L_2 \) and \( L_3 \), we would have extra terms that lead to terms quartic in time derivatives of the collective coordinates, and this would make the quantization slightly more difficult. Because we consider the case in which the “rotation” is sufficiently slow, such terms can be ignored.)

We thus end up with the action

\[
S[U] = \frac{F_0^2}{16} \int d^4x \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{32e^2} \int d^4x \text{Tr} \left( \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \right) + N_c \Gamma[U]
+ \frac{F_0^2 B_0}{8} \int d^4x \text{Tr} \left( M^\dagger U + M U^\dagger \right)
+ L_5 B_0 \int d^4x \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \left( M^\dagger U + U^\dagger M \right) \right)
+ L_8 B_0^2 \int d^4x \text{Tr} \left( M^\dagger U M^\dagger U + M U^\dagger M U^\dagger \right), \tag{2.9}
\]

which includes all terms up to and including those of \( \mathcal{O}(N_c) \) and \( \mathcal{O}(p^4) \).

It is important to note that, although (2.9) is our starting point, we always need to keep in mind that there are infinitely many higher-order contributions. It is the symmetry and power counting that are actually important. The treatment given in this section (and in Appendix C) may be considered a heuristic derivation, but it is very convenient, because it yields explicit forms of the (leading) orders of \( N_c, m \) and \( \delta m \) for various parameters.

In the usual \( \chi \)PT, one loop quantum effects of mesons can be incorporated consistently to the presently considered order, and physical parameters, such as the

\(^*\) We do not know who first noticed this fact, but probably this is widely known. We learned it from Ref. 12).
pion decay constant $F_\pi$, are defined so as to include one-loop corrections. In our analysis, we ignore all the quantum effects of mesons, but there are still tree-level contributions to physical parameters from the higher-order terms. For the decay constants, we have

\begin{align}
F_\pi &= F_0 (1 + (2m)K_6), \\
F_K &= F_0 (1 + (m + m_s)K_6), \\
F_\eta &= F_0 \left(1 + \frac{2}{3}(m + 2m_s)K_6\right),
\end{align}

where

\begin{equation}
K_6 = \frac{16B_0}{F_0^2}L_5.
\end{equation}

Meson masses are obtained from the quadratic terms in the Lagrangian expanded about $U = 1$,

\begin{align}
M_\pi^2 &= B_0(2m)(1 + (2m)K_3), \\
M_K^2 &= B_0(m + m_s)(1 + (m + m_s)K_3), \\
M_\eta^2 &= \frac{2}{3}B_0(m + 2m_s)\left(1 + \frac{2}{3}(m + 2m_s)K_3\right) + K_5,
\end{align}

where

\begin{align}
K_3 &= \frac{32B_0}{F_0^2}(2L_8 - L_5), \\
K_5 &= (m_s - m)^2 \frac{512B_0^2}{9} \frac{1}{F_0^2}L_8.
\end{align}

2.2. Collective coordinate quantization

In this subsection, we derive the Hamiltonian that describes the baryons by using the collective coordinate quantization. In this treatment, we consider the soliton as a “rigid rotator” and do not consider the breathing degrees of freedom, though such “radial” excitations should be important if we consider other states, such as those with negative parity.

There are two important criticisms of the above-mentioned treatment. The first is an old one, namely, that the flavor $SU(3)$ symmetry breaking is so large that the perturbation theory does not work. The so-called “bound-state” approach has been advocated by Callan and Klebanov\cite{43}, and it was recently reconsidered after the discovery of the $\Theta^+$ resonance\cite{44}. Yabu and Ando\cite{45} showed that the “exact” treatment of the symmetry breaking term gives good results even though the collective coordinate quantization is employed. Subsequently, it was shown that the perturbation theory is capable of reproducing qualitatively equivalent results if one includes mixings with a sufficient number of representations\cite{17,46}. The second

\footnote{We assume that the pentaquark states have positive parity. Otherwise they would not be “rotational” modes of the soliton, and the analysis given in this paper would not make sense.}
criticism was made by Cohen.\textsuperscript{47,48} He claimed that, in the large-$N_c$ limit, the “rotation” is not slow enough for the collective treatment to be justified. However, Diakonov and Petrov\textsuperscript{49} argued that due to the WZW term, the “rotation” is slow enough, even in the large-$N_c$ limit.

The collective coordinate quantization with the flavor $SU(3)$ symmetry is different from that with the isospin $SU(2)$,\textsuperscript{50,51} due to the existence of the WZW term. (See Refs. 27, 30, 52–54.)

The rotational collective coordinates are introduced as a time-dependent $SU(3)$-valued variable $A(t)$ through

$$U(t, x) = A(t)U_c(x)A^\dagger(t), \quad (2.19)$$

where $U_c(x)$ is the classical hedgehog soliton ansatz,

$$U_c(x) = \begin{pmatrix} \exp(i\tau \cdot \hat{x} F(r)) & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.20)$$

with baryon number (topological charge) $B = 1$. The profile function $F(r)$ satisfies the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$. By substituting Eq. (2.19) into Eq. (2.9), we obtain the Lagrangian

$$L = -M_{cl} + \frac{1}{2} \omega^\alpha I_{\alpha\beta}(A)\omega^\beta + \frac{N_c}{2\sqrt{3}} \omega^8 - V(A), \quad (2.21)$$

where we have introduced the “angular velocity” $\omega^\alpha(t) (\alpha = 1, \cdots, 8)$ through

$$A^\dagger(t) \dot{A}(t) = \frac{i}{2} \sum_{\alpha=1}^{8} \lambda_\alpha \omega^\alpha(t), \quad (2.22)$$

with $\lambda_\alpha (\alpha = 1, \cdots, 8)$ being the usual Gell-Mann matrices.

The first term represents the rest mass energy of the classical soliton. In the conventional approach, the $A$-independent part is a functional of the profile function $F(r)$. The classical mass $M_{cl}$ is obtained by minimizing the quantity $-M_{cl}$ by varying $F(r)$ subject to the boundary condition. One might think that the parameters in the $\chi$PT action (2.9) have already been obtained and that $M_{cl}$ could be determined completely in terms of these parameters. This is not true, however, because the $\chi$PT action contains infinitely many terms, and therefore there are infinitely many contributions from higher orders. We do not know all of these higher-order couplings, and therefore, in actual practice, we cannot calculate $M_{cl}$. A more physically oriented procedure is to fit it to its experimentally determined value. This is our basic strategy in the effective theory approach: The operators are determined by the $\chi$PT action, reflecting the fundamental principles and symmetries of QCD, while the coefficients are fitted to the experimental values. Because we do not know the higher-order contributions, the number of parameters is different from that in (2.9). (For comparison with the conventional approach, see Appendix C, which also presents as a “derivation” of the terms discussed below.)
The most important feature of the Lagrangian (2.21) is that the “inertia tensor” \( I_{\alpha\beta}(A) \) depends on \( A \). From the symmetry of the ansatz and the structure of the symmetry breaking, it has the form

\[
I_{\alpha\beta}(A) = I^0_{\alpha\beta} + I'_{\alpha\beta}(A),
\]

with

\[
I^0_{\alpha\beta} = \begin{cases} 
I_1 \delta_{\alpha\beta} & \alpha, \beta \in \mathcal{I}, \\
I_2 \delta_{\alpha\beta} & \alpha, \beta \in \mathcal{J}, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
I'_{\alpha\beta}(A) = \begin{cases} 
\mp \delta_{\alpha\beta} D_{88}^{(8)}(A) & (\alpha, \beta \in \mathcal{I}), \\
\mp d_{\alpha\beta\gamma} D_{8\gamma}^{(8)}(A) & (\alpha \in \mathcal{I}, \beta \in \mathcal{J} \text{ or } \alpha \in \mathcal{J}, \beta \in \mathcal{I}), \\
\mp \bar{z} \delta_{\alpha\beta} D_{88}^{(8)}(A) + \bar{w} d_{\alpha\beta\gamma} D_{8\gamma}^{(8)}(A) & (\alpha, \beta \in \mathcal{J}), \\
0 & (\alpha = 8 \text{ or } \beta = 8),
\end{cases}
\]

where \( \mathcal{I} = \{1, 2, 3\} \) and \( \mathcal{J} = \{4, 5, 6, 7\} \). We denote the \( SU(3) \) representation matrix in the adjoint (octet) representation as

\[
D_{\alpha\beta}^{(8)}(A) = \frac{1}{2} \text{Tr} \left( A^\dagger \lambda_\alpha A \lambda_\beta \right), \quad (\alpha, \beta = 1, \ldots, 8)
\]

and \( d_{\alpha\beta\gamma} \) is the usual symmetric tensor. The parameters \( I_1, I_2, \mp, \mp, \), and \( \mp \) are to be determined. Note that \( I_1 \) and \( I_2 \) are \( O(1) \) while \( \mp, \mp, \), and \( \mp \) are \( O(\delta m) \).

The third term in the Lagrangian (2.21) comes from the WZW term and gives rise to a first-class constraint, which selects possible representations. The last term is a potential term,

\[
V(A) = V^{(1)}(A) + V^{(2)}(A),
\]

\[
V^{(1)}(A) = \frac{\gamma}{2} \left( 1 - D_{88}^{(8)}(A) \right),
\]

\[
V^{(2)}(A) = v \left( 1 - \sum_{\alpha \in \mathcal{I}} \left( D_{8\alpha}^{(8)}(A) \right)^2 - \left( D_{88}^{(8)}(A) \right)^2 \right),
\]

where \( \gamma \) is \( O(\delta m) \) and \( v \) \( O(\delta m^2) \). Note that \( V^{(2)}(A) \) is of higher order in \( \delta m \) than any of the other operators. The reason we include it is that it is leading order in \( N_c \). Equivalently, we assume

\[
\frac{\delta m}{\Lambda} < \frac{1}{N_c^2},
\]

with some relevant mass scale \( \Lambda \).

Collective quantization of the theory is a standard procedure. The only difference is due to the fact that the “inertia tensor” \( I_{\alpha\beta} \) depends on the “coordinates” \( A \). Here,

\footnote{A mechanical analogy is a top with a thick axis and a round pivot, rotating on a smooth floor. For such a system, the moment of inertia depends on the tilt of the axis.}
the operator ordering is relevant, and we adopt the standard one. The kinetic term now involves the inverse of the “inertial tensor,” and we expand it up to and including the terms of order $\delta m$. We then obtain the Hamiltonian

$$H = M_{cl} + H_0 + H_1 + H_2,$$

(2.31)

with

$$H_0 = \frac{1}{2I_1} \sum_{\alpha \in I} (F_{\alpha})^2 + \frac{1}{2I_2} \sum_{\alpha \in J} (F_{\alpha})^2,$$

(2.32)

$$H_1 = x D_{88}^{(8)}(A) \sum_{\alpha \in I} (F_{\alpha})^2 + y \left[ \sum_{\alpha \in I, \beta \in J} + \sum_{\alpha \in J, \beta \in I} \right] \sum_{\gamma=1}^{8} d_{\alpha\beta\gamma} F_{\alpha} D_{8\gamma}^{(8)}(A) F_{\beta}$$

$$+ z \sum_{\alpha \in J} F_{\alpha} D_{88}^{(8)}(A) F_{\alpha} + w \sum_{\alpha, \beta \in J} \sum_{\gamma=1}^{8} d_{\alpha\beta\gamma} F_{\alpha} D_{8\gamma}^{(8)}(A) F_{\beta}$$

$$+ \frac{\gamma}{2} \left( 1 - D_{88}^{(8)}(A) \right),$$

(2.33)

$$H_2 = v \left( 1 - \sum_{\alpha \in I} \left( D_{8\alpha}^{(8)}(A) \right)^2 - \left( D_{88}^{(8)}(A) \right)^2 \right).$$

(2.34)

Here,

$$x = -\frac{\bar{x}}{2I_1^2}, \quad y = -\frac{\bar{y}}{2I_1 I_2}, \quad z = -\frac{\bar{z}}{2I_2^2}, \quad w = -\frac{\bar{w}}{2I_2^2},$$

(2.35)

and $F_{\alpha}$ ($\alpha = 1, \cdots, 8$) are the $SU(3)$ generators,

$$[F_{\alpha}, F_{\beta}] = i \sum_{\gamma=1}^{8} f_{\alpha\beta\gamma} F_{\gamma},$$

(2.36)

where $f_{\alpha\beta\gamma}$ is the totally anti-symmetric structure constant of $SU(3)$. Note that they act on $A$ from the right.

The WZW term leads to the following first-class constraint, which provides an auxiliary condition on the physical states $\Psi(A)$:

$$Y_R \Psi(A) \equiv -\frac{2}{\sqrt{3}} F_8 \Psi(A) = \frac{N_c}{3} \Psi(A).$$

(2.37)

(See Refs. 27, 52–54 for more details.) In the following, we set $N_c = 3$.

A novel feature of the above Hamiltonian is the existence of the interactions quadratic in $F_{\alpha}$. Generalizations of the Skyrme model have been studied by several authors, but interactions as complicated as those appearing above have not previously been considered.

It is also important to note that we do not have any interactions linear in $F_{\alpha}$. The reason for this can be traced back to the fact that the action does not include terms linear in the time derivative, except for the WZW term. The absence of such terms is due to the time reversal invariance of QCD.\footnote{The vacuum angle $\theta$ is assumed to be zero.}
3. Mixing among representations and the masses of baryons

3.1. Symmetric case

In the absence of SU(3) flavor symmetry breaking interactions, the eigenstates of the collective Hamiltonian provide the SU(3) representations. The symmetric part $H_{\text{sym}} = H_{\text{cl}} + H_0$ can be written

$$H_{\text{sym}} = H_{\text{cl}} + \frac{1}{2} \left( \frac{1}{I_1} - \frac{1}{I_2} \right) C_2(SU(2)) + \frac{1}{2I_2} \left( C_2(SU(3)) - \frac{N_c^2}{12} \right),$$

(3.1)

where $C_2(SU(2))$ is the spin SU(2) quadratic Casimir operator,

$$C_2(SU(2)) = \sum_{\alpha \in \mathcal{I}} (F_\alpha)^2,$$

(3.2)

with eigenvalue $J(J+1)$. The operator $C_2(SU(3))$ is the flavor SU(3) quadratic Casimir,

$$C_2(SU(3)) = \sum_{\alpha=1}^{8} (F_\alpha)^2,$$

(3.3)

with eigenvalue

$$C_2(p, q) = \frac{1}{3} \left[ p^2 + q^2 + pq + 3(p + q) \right].$$

(3.4)

Here, $(p, q)$ is the Dynkin index of the representation. Note that we have used the constraint $F_8 = N_c/2\sqrt{3}$ in Eq. (3.1).

The eigenstate is given by the SU(3) representation matrix,

$$\Psi^{(p,q)}(Y, I_3; Y_R, J_3)(A) \equiv \sqrt{\text{dim}(p, q)(-1)^J Y_{R/2}} \left( Y, I, I_3 \right) D^{(p,q)}(A) \left| Y_R, J, -J_3 \right>^*,$$

(3.5)

with $Y_R = 1$, where dim$(p, q)$ is the dimension of the representation $(p, q)$ and is given by

$$\text{dim}(p, q) = (p + 1)(q + 1) \left( 1 + \frac{p + q}{2} \right).$$

(3.6)

For the properties of the wave function (3.5), see Appendix A. Using these properties, we readily calculate the symmetric mass $M_R$ of the representation $\mathcal{R}$, finding

$$M_8 = M_{cl} + \frac{3}{8} \left[ \frac{1}{I_1} + \frac{2}{I_2} \right], \quad M_{10} = M_{cl} + \frac{3}{8} \left[ \frac{5}{I_1} + \frac{2}{I_2} \right],$$

$$M_{10} = M_{cl} + \frac{3}{8} \left[ \frac{1}{I_1} + \frac{6}{I_2} \right], \quad M_{27_d} = M_{cl} + \frac{1}{8} \left[ \frac{3}{I_1} + \frac{26}{I_2} \right],$$

$$M_{27_q} = M_{cl} + \frac{1}{8} \left[ \frac{15}{I_1} + \frac{14}{I_2} \right],$$

(3.7)

and so on. Because 27 contains the spin-$\frac{1}{2}$ part and the spin-$\frac{3}{2}$ part, the former is denoted as $27_d$, and the latter as $27_q$. 

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3.2. Matrix elements of the symmetry breaking operators

The $SU(3)$ flavor symmetry breaking interactions mix the representations, and as a result, an eigenstate of the full Hamiltonian $H$ is a linear combination of the (infinitely many) states of the representations. In this paper, we consider the perturbative expansion up to and including $O(\delta m^2)$.

To calculate the perturbative corrections, we need the matrix elements of the symmetry breaking operators. Though the calculations are group theoretical, they are quite complicated, because of the generators $F_\alpha$. Several useful mathematical tools are summarized in Appendix A.

Note that the symmetry breaking operators preserve spin $SU(2)$, isospin $SU(2)$ and hypercharge $U(1)$ symmetries. For this reason, the matrix elements are classified according to the value of $(I,Y)$, the magnitude of the isospin and the hypercharge, and $J$, the magnitude of the spin. The $SU(3)$ representation and the magnitude of spin are related by the constraint (2.37).

We introduce the following notation:

$O_\gamma \equiv D_{88}^{(8)}(A)$, $O_x \equiv D_{88}^{(8)}(A) \sum_{\alpha \in I} (F_\alpha)^2$, 

(3.8)

$O_y \equiv \left[ \sum_{\alpha \in I, \beta \in J} + \sum_{\alpha \in J, \beta \in I} \right] \sum_{\gamma=1}^8 d_{\alpha\beta\gamma} F_\alpha D_{8\gamma}^{(8)}(A) F_\beta$, 

(3.9)

$O_z \equiv \sum_{\alpha \in J} F_\alpha D_{88}^{(8)}(A) F_\alpha$, $O_w \equiv \sum_{\alpha, \beta \in J} \sum_{\gamma=1}^8 d_{\alpha\beta\gamma} F_\alpha D_{8\gamma}^{(8)}(A) F_\beta$, 

(3.10)

$O_{v1} \equiv \left( D_{88}^{(8)}(A) \right)^2$, $O_{v2} \equiv \sum_{\alpha \in I} \left( D_{8\alpha}^{(8)}(A) \right)^2$. 

(3.11)

We denote these quantities collectively by $O_i$.

3.2.1. $J = \frac{1}{2}$

The $SU(3)$ representations with spin $\frac{1}{2}$ are $8, \overline{10}, 27_d$, and so on. States possessing the same values of $(I,Y)$ can mix.

Because all the symmetry breaking operators behave as $8$, the octet states can mix with $10$ and $27_d$ at first order and with $35_d$ and $64_d$ at second order. The anti-decuplet states can mix with $8, 27_d$ and $35_d$ at first order and with $64_d$ and $81_d$ at second order. To calculate the masses to second order, we do not need the second-order mixings, but they are used for the decay amplitudes.

The matrix elements for the $8$ representation are given in Table I. Note that the matrix elements of the operator $O_z$ are 0.

Similarly, the matrix elements for $\overline{10}$ are given in Table II.

In order to calculate the masses to second order, we also need the matrix elements off-diagonal in the representation. They are given in Tables III, IV, V and VI.

Other matrix elements, which are necessary for the calculation of the mixings, are presented in Appendix B.

Using the matrix elements given in Tables I – VI and Appendix B, we can write
Table I. \( \langle 8 | O_1 | 8 \rangle \).

| \( I, Y \) | \( \gamma \) | \( x \) | \( y \) | \( z \) | \( w \) | \( v_1 \) | \( v_2 \) |
|-----------|-------|--------|-------|------|------|-------|-------|
| \( N_8 \) | \( \frac{1}{2}, +1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( \Sigma_8 \) | \( 1, 0 \) | \( -\frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{3}{2} \) |
| \( \Xi_8 \) | \( \frac{3}{2}, -1 \) | \( -\frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{3}{2} \) |
| \( \Lambda_8 \) | \( 0, 0 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{3}{2} \) |

Table II. \( \langle 10 | O_1 | 10 \rangle \).

| \( I, Y \) | \( \gamma \) | \( x \) | \( y \) | \( z \) | \( w \) | \( v_1 \) | \( v_2 \) |
|-----------|-------|--------|-------|------|------|-------|-------|
| \( \Theta_{10} \) | \( 0, +2 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{1}{2} \) |
| \( N_{10} \) | \( \frac{1}{2}, +1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{1}{2} \) |
| \( \Sigma_{10} \) | \( 1, 0 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{1}{2} \) |
| \( \Xi_{10} \) | \( \frac{1}{2}, -1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( \frac{1}{2} \) |

Table III. \( \langle 10 | O_1 | 8 \rangle \).

| \( I, Y \) | \( \gamma \) | \( x \) | \( y \) | \( z \) | \( w \) | \( v_1 \) | \( v_2 \) |
|-----------|-------|--------|-------|------|------|-------|-------|
| \( \frac{1}{2}, +1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( 1, 0 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |

Table IV. \( \langle 27_d | O_1 | 8 \rangle \).

| \( I, Y \) | \( \gamma \) | \( x \) | \( y \) | \( z \) | \( w \) | \( v_1 \) | \( v_2 \) |
|-----------|-------|--------|-------|------|------|-------|-------|
| \( \frac{1}{2}, +1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( 1, 0 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( \frac{1}{2}, -1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( 0, 0 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |

Table V. \( \langle 27_d | O_1 | 10 \rangle \).

| \( I, Y \) | \( \gamma \) | \( x \) | \( y \) | \( z \) | \( w \) | \( v_1 \) | \( v_2 \) |
|-----------|-------|--------|-------|------|------|-------|-------|
| \( \frac{1}{2}, +1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( 1, 0 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |
| \( \frac{3}{2}, -1 \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( 0 \) | \( -\frac{3}{2} \) |

The perturbation theory results for the masses and the representation mixings. For example, the nucleon [mainly the octet \((I, Y) = (\frac{1}{2}, +1)\) state] mass is calculated as

\[
M_N = M_8 + \frac{\gamma}{2} \left( 1 - \frac{3}{10} \right) + \frac{9}{40} x + \frac{\sqrt{3}}{20} y + \frac{\sqrt{3}}{5} w + \left( 1 - \frac{1}{5} - \frac{1}{5} \right) v
\]

\[
- \frac{\Delta H^2_{10-8}}{M_{10} - M_8} - \frac{2}{3} \frac{\Delta H^2_{27d-8}}{M_{27d} - M_8},
\]

(3.12)
3.2.2. $J = \frac{3}{2}$

Spin $\frac{3}{2}$ states are composed of the representations $10$, $27_q$, and so on. The matrix elements for the $10$ representation are given in Table VII. Note that the matrix elements of $O_z$ are zero.

The decuplet states can mix with $27_q$ and $35$ at first order and with $35_q$, $64_q$ and $81$ at second order. In order to calculate the mass to second order, we need the matrix elements of $10$ with $27_q$ and $35_q$ representations. They are given in Tables VIII and IX.

### §4. Numerical determination of parameters

We can calculate the baryon masses once the Skyrme model parameters are given. In the effective theory approach, however, we have to solve in the opposite direction. Namely, we need to determine the Skyrme model parameters so as to best match the observed baryon masses.
Table IX. \( \langle 35 | O_i | 10 \rangle \).

| \((I,Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) | \(v_1\) | \(v_2\) |
|---|---|---|---|---|---|---|---|
| \((\frac{3}{2},+1)\) | \(\frac{5}{3}\) | \(15\) | \(\frac{5}{3}\) | \(25\sqrt{3}\) | \(15\) | \(5\sqrt{3}\) | \(5\) | \(-\frac{5}{3}\) |
| \((1,0)\) | \(\frac{3}{2}\) | \(4\) | \(\sqrt{2}\) | \(15\sqrt{3}\) | \(3\sqrt{3}\) | \(\frac{15}{2}\) | \(\frac{15}{2}\) | \(-\frac{3}{2}\) |
| \((\frac{1}{2},-1)\) | \(\frac{3}{2}\) | \(4\) | \(\sqrt{2}\) | \(8\sqrt{3}\) | \(4\sqrt{3}\) | \(3\sqrt{3}\) | \(0\) | \(-\frac{3}{2}\) |
| \((0,-2)\) | \(\frac{3}{2}\) | \(4\) | \(\sqrt{2}\) | \(8\sqrt{3}\) | \(4\sqrt{3}\) | \(3\sqrt{3}\) | \(-\frac{15}{2}\) | \(-\frac{3}{2}\) |

Table X. Experimental values of baryon masses and their deviations used in our calculation. In the last row, we also give the baryon masses calculated using the best fit set of parameter values, \((4.2)\).

| \((\text{MeV})\) | \(N\) | \(\Sigma\) | \(\Xi\) | \(\Lambda\) | \(\Delta\) | \(\Sigma^+\) | \(\Xi^\ast\) | \(\Omega\) | \(\Theta\) | \(\phi\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(M_i^{\text{exp}}\) | 939 | 1193 | 1318 | 1116 | 1232 | 1385 | 1533 | 1672 | 1539 | 1862 |
| \(\sigma_i\) | 0.06 | 4.0 | 3.2 | 0.01 | 2.0 | 2.2 | 1.6 | 0.3 | 1.6 | 2.0 |
| \(M_i\) | 940 | 1180 | 1332 | 1116 | 1228 | 1389 | 1537 | 1672 | 1539 | 1862 |

fit to the experimental values of the baryon masses. In order to measure the quality of the fit, we introduce the evaluation function

\[
\chi^2 = \sum_i \frac{(M_i - M_i^{\text{exp}})^2}{\sigma^2_i},
\]

where \(M_i\) represents the calculated mass of baryon \(i\), and \(M_i^{\text{exp}}\) the corresponding experimental value. The accuracy of the \(i\)th experimental value is represented by \(\sigma_i\). Because we ignore the isospin violation effect completely, we use the average among the members of an isospin multiplet for the mass and the range of variation within the isospin multiplet for the values \(\sigma_i\). This is the reason that our estimate of \(\sigma_i\) for isospin singlets gives a small number, while \(\sigma_N\) is quite large, even though the masses of the proton and neutron are very accurately determined. In any case, these numbers should not be taken too seriously.

The sum in Eq. (4.1) is taken over the octet and decuplet baryons, as well as \(\Theta^+(1540)\) and \(\phi(1860)\). Note that we have nine parameters to be determined. We need at least one state in addition to the low-lying octet and the decuplet. In this sense, our effective theory cannot predict the \(\Theta^+\) mass. In our calculations, we use the values of \(M_i^{\text{exp}}\) and \(\sigma_i\) given in Table X.

The problem is a multi-dimensional minimization of the function \(\chi^2\) of nine variables. In general, such a problem is very difficult, but in our case, thanks to the fact that the function is a polynomial in the variable of the perturbation theory, a stable numerical solution can be obtained. Our method is basically that of Powell, but we tried several minimization algorithms, obtaining equivalent values in all cases. The best fit set of parameters corresponds to the bottom of a very shallow (and narrow) “valley” of the function, and \(\chi^2\) is largely insensitive to changes in the parameter values along the “valley”.

The best fit set of parameter values is

\[
M_{cl} = 389 \text{ MeV}, \quad I_1^{-1} = 174 \text{ MeV}, \quad I_2^{-1} = 585 \text{ MeV}, \quad \gamma = 832 \text{ MeV},
\]
Table XI. Mixing coefficients for the (mainly) octet states at first-order in the perturbation theory. The numbers in parentheses are the second-order contributions.

| $R_i$ | $N$ | $\Sigma$ | $\Xi$ | $\Lambda$ |
|-------|-----|----------|-------|---------|
| 8     | 1 (−0.056) | 1 (−0.051) | 1 (−0.014) | 1 (−0.021) |
| 10    | 0.288 (0.036) | 0.288 (0.086) | 0 (0) | 0 (0) |
| 27, d | 0.169 (0.087) | 0.138 (0.087) | 0.169 (0.036) | 0.207 (0.075) |
| 35, d | 0 (0.085) | 0 (0.080) | 0 (0) | 0 (0) |
| 64, d | 0 (0.038) | 0 (0.028) | 0 (0.038) | 0 (0.051) |

Table XII. Mixing coefficients for the (mainly) dec uplet states at first-order in the perturbation theory. The numbers in parentheses are the second-order contributions.

| $R_i$ | $\Delta$ | $\Sigma^*$ | $\Xi^*$ | $\Omega$ |
|-------|----------|----------|--------|-------|
| 10    | 1 (−0.150) | 1 (−0.082) | 1 (−0.033) | 1 (−0.003) |
| 27, q | 0.543 (0.122) | 0.397 (0.031) | 0.243 (−0.017) | 0 (0) |
| 35    | 0.065 (0.076) | 0.082 (0.072) | 0.087 (0.051) | 0.082 (0.024) |
| 35, q | 0 (0.138) | 0 (0.049) | 0 (0) | 0 (0) |
| 64, q | 0 (0.053) | 0 (0.050) | 0 (0.033) | 0 (0) |
| 81    | 0 (0.009) | 0 (0.013) | 0 (0.015) | 0 (0.013) |

$x = 27.8$ MeV, $y = −104$ MeV, $z = −306$ MeV, $w = 111$ MeV, $v = −148$ MeV,

$\chi^2 = 4.4 \times 10^4$.

Note that the best fit set of parameter values is quite reasonable, though we do not impose the constraint that the higher-order (in $\delta m$) parameters be small. The parameter $\gamma$ is unexpectedly large (even though it is of leading order in $N_c$), but considerably smaller than the value ($\gamma = 1573$ MeV) for case (3) considered by Yabu and Ando.45) The parameter $z$ also seems too large, and we do not know the reason.

Once we determine the best fit set of parameter values, we can predict the masses of the (mainly) anti-decuplet members,

$$M_N' = 1711 \text{ MeV}, \quad M_{\Sigma'} = 1819 \text{ MeV}.$$  \hspace{1cm} (4.3)

These values should be compared with the $\chi$QSM prediction,14)

$$M_N' = 1646 \text{ MeV}, \quad M_{\Sigma'} = 1754 \text{ MeV}.$$  \hspace{1cm} (4.4)

It is tempting to identify $N'$ with the N(1710) (***) state, while for $\Sigma'$ there are two possibilities, $\Sigma(1770)$ (*) and $\Sigma(1880)$ (**), neither of whose masses is very close to the calculated value. We therefore do not identify them.

The mixing coefficients of the eigenstates have also been obtained. For the (mainly) octet states, the coefficients are given in Table XI. Similarly, for the (mainly) decuplet and (mainly) anti-decuplet states, they are given in Tables XII and XIII. The numbers in the parentheses are the second-order contributions calculated by using the parameters given in (4.2), which are used in §5.2.

It is seen that the mixings are rather large, and for this reason, one may think that the perturbation theory does not work. For the (mainly) octet states, the
Table XIII. Mixing coefficients for the (mainly) anti-decuplet states at first-order perturbation theory. The numbers in parentheses are the second-order contributions.

| \( R_i \) | \( \Theta \) | \( N' \) | \( \Sigma' \) | \( \phi \) |
|---|---|---|---|---|
| 8 | 0 (0) | -0.288 (-0.089) | -0.288 (-0.157) | 0 (0) |
| 10 | 1 (-0.061) | 1 (-0.160) | 1 (-0.235) | 1 (-0.285) |
| 27_d | 0 (0) | 0.314 (0.304) | 0.513 (0.219) | 0.703 (-0.079) |
| 35_d | 0.351 (0.130) | 0.372 (0.193) | 0.352 (0.233) | 0.277 (0.225) |
| 64_d | 0 (0) | 0 (0.136) | 0 (0.203) | 0 (0.215) |
| 81_d | 0 (0.116) | 0 (0.128) | 0 (0.116) | 0 (0.082) |

Second-order contributions are much smaller than the first-order contributions, while for the (mainly) decuplet and the (mainly) anti-decuplet states, the mixings with \( 27_d \) and \( 35_d \) are large, and the second-order and first-order contributions are of similar magnitudes.

§5. Decay widths

In this section, we calculate the decay widths of various channels on the basis of the calculations given in the previous sections. Since our treatment of the baryons is quantum mechanical, a full-fledged field theoretical calculation is unfeasible. What we actually carry out is a perturbative evaluation of the decay operators in the framework of collective coordinate quantum mechanics.

5.1. Formula for the decay width

Because there seems to be confusion\(^{55)-57}\) concerning the factors in the decay widths, we reconsider the derivation of their formula. (See Ref. 14) for discussion of the calculations of the decay widths in the Skyrme model.)

Decay of a baryon into another baryon with a pseudoscalar meson can be described by interactions of the type

\[
L_{\text{decay}} = -ig \partial_\mu \phi^\alpha J_5^{\alpha\mu},
\]

where \( J_5^{\alpha\mu} \) is the baryon axial-vector current and \( \phi^\alpha \) is a pseudoscalar meson field. The coupling \( g \) has the dimension \((\text{mass})^{-1}\) and is usually related to the pion decay constant \( F_\pi \) as \( g \sim F_\pi^{-1} \). In the nonrelativistic limit, the time component can be dropped, and it is useful for us to work in the Hamiltonian formulation, in which case we have

\[
H_{\text{decay}}(t) = \int d^3x H_{\text{decay}} = ig \int d^3x \partial_k \phi^\alpha(x) J_5^{\alpha\mu}(x).
\]

At leading order, the amplitude of the decay \( B \rightarrow B' \phi \) is given by

\[
A = \int dt \langle \phi^\alpha(p) B'(P') | H_{\text{decay}} | B(P) \rangle = ig \int d^4x \langle \phi^\alpha(p) | \partial_k \phi^\beta(x) | 0 \rangle \left< B'(P') \left| J_5^{\alpha\beta}(x) \right| B(P) \right>.
\]

(5.3)
With the relativistic normalization of the state, the first matrix element can be written as

$$\langle \phi^\alpha(p) | \partial_k \phi^\beta(x) | 0 \rangle = ip^k e^{ipx} \delta^{\alpha \beta}. \quad (5.4)$$

In our treatment of baryons, the state $|B(P)\rangle$ has the wave function

$$\sqrt{2E_B(P)} \psi_B(A) e^{-iE_B(P)t + iP \cdot X}, \quad (5.5)$$

where $E_B(P) = \sqrt{P^2 + M_B^2}$, and $X$ is the position of the baryon. This state satisfies the relativistic normalization

$$\langle B'(P') | B(P) \rangle = 2E_B(P)(2\pi)^3 \delta^3(P' - P), \quad (5.6)$$

where the inner product is defined as the integration over $X$ and $A$. The axial-vector current can be obtained from the $\chi$PT action by using Noether’s method. After replacing $U(x)$ with $A(t)U_c(x - X)A^\dagger(t)$, we obtain the collective coordinate quantum mechanical operator by representing the “angular velocity” $\omega^\alpha$ in terms of the generator $F_\alpha$ (“angular momentum”), according to the usual rule. Note that $J_5^{ak}$ depends on $F_\alpha$, but no longer on $t$. Also, note that it depends on $X$ through the combination $x - X$. Now, the second matrix element can be written as

$$\langle B'(P') | J_5^{ak}(x - X) | B(P) \rangle = \sqrt{2E_B'(2E_B - E_B')t} \int dA \psi_{B'}^*(A) \left( \int d^3X e^{-i(P' - P) \cdot X} J_5^{ak}(x - X) \right) \psi_B(A). \quad (5.7)$$

Then, by making the shift $x \rightarrow x + X$ and integrating over $X$ and $t$, we obtain

$$A = (2\pi)^4 \delta^4(P' + P - P') \sqrt{2E_B'} \int dA \psi_{B'}^*(A) \left( g p^k \int d^3x e^{-ip \cdot x} J_5^{ak}(x) \right) \psi_B(A) \equiv (2\pi)^4 \delta^4(P' + P - P) \mathcal{M}. \quad (5.8)$$

In the nonrelativistic approximation, we have $e^{-ip \cdot x} \approx 1$. The integral of the axial-vector current,

$$\frac{1}{\Lambda} \mathcal{O}^{\alpha}_{\text{decay}} = gp^k \int d^3x J_5^{ak}(x), \quad (5.9)$$

depends on $A$ and $F_\alpha$, and it has the correct transformation properties under flavor $SU(3)$ transformations. Here we introduce the mass scale $\Lambda$, which we take as $\Lambda = 1$ GeV. It is known that the leading-order result is

$$\frac{1}{\Lambda} \mathcal{O}^{\alpha}_{\text{decay}} \sim gC D_{\alpha k}^{(8)}(A)p^k, \quad (5.10)$$

$^a \mathcal{M}$ corresponds to the invariant amplitude in relativistic field theory, although, in our formulation, relativistic properties have already been lost. In particular, the spin of the wavefunction (5.5) does not transform properly under boosts. The decoupling of the spin reflects the fact that baryons are now considered to be (almost) static, i.e., $|P| \ll M_B$ and $|P'| \ll M_B'$. 

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Effective Theory Approach to the Skyrme Model
where $C$ is a dimensionless constant. In the conventional approach, this is a functional of the profile function $F(r)$, and it is obtained by explicitly integrating $J_5^{a_k}$ (expressed in terms of $U_c(x)$, $A$, and $F_\alpha$) over $x$. Note that it depends on neither $M_B$ nor $M_{B'}$.

In the effective theory approach, by contrast, the coefficient of the decay operator can be determined by fitting the widths to the experimental values. We follow this approach.

The leading operators are well known:

$$O_{\text{decay}}^a = 3 \left[ G_0 D_{\alpha k}^{(8)}(A) - G_1 \sum_{\beta,\gamma \in \mathcal{J}} d_{k\beta\gamma} D_{\alpha \beta}^{(8)}(A) F_\gamma - \frac{G_2}{\sqrt{3}} D_{\alpha 8}^{(8)}(A) F_k \right] p^k,$$

(5.11)

where the index $k$ takes the values 1, 2 and 3. The constants $G_a$ ($a = 0, 1, 2$) are dimensionless, because we employ the explicit mass scale $\Lambda$.

Once we calculate the amplitude $M$, we are ready to obtain the decay width,

$$\Gamma_{B \to B' \phi} = \frac{|p|^3}{8\pi M_B^2 A^2} |\mathcal{M}|^2.$$

(5.12)

Here, $|p|$ represents the magnitude of the meson momentum in the rest frame of the initial baryon $B$,

$$|p| = \frac{1}{2M_B} \sqrt{[(M_B + M_{B'})^2 - m^2][(M_B - M_{B'})^2 - m^2]},$$

(5.13)

where $m$ is the mass of the meson. The quantity $|\mathcal{M}|^2$ is defined as

$$|\mathcal{M}|^2 = \frac{1}{|p|^2} \sum_{\text{spin, isospin}} |M|^2.$$

(5.14)

The symbol $\sum_{\text{spin, isospin}}$ represents the operations of averaging over the spin and isospin of the initial state baryon as well as and summing over the spin and isospin of the final state baryon. By extracting the normalization factor, $4 M_B E_{B'}(|p|)$ as

$$|\mathcal{M}|^2 = 4M_B E_{B'}(|p|) \left| \tilde{\mathcal{M}} \right|^2,$$

(5.15)

we can rewrite this as

$$\Gamma_{B \to B' \phi} = \frac{|p|^3}{2\pi A^2 M_B} \left| \tilde{\mathcal{M}} \right|^2.$$

(5.16)

This is our formula for the decay width.

The widely used formula,

$$\Gamma_{B \to B' \phi} = \frac{|p|^3}{8\pi M_B M_{B'}} |\mathcal{M}|^2,$$

(5.17)

which corresponds to Eq. (5.16), seems to be based on an interaction of the Yukawa type,

$$\mathcal{L} = g_{\phi B' B} \bar{\psi}_{B'} i\gamma_5 \lambda^\alpha \psi_B \phi^\alpha.$$

(5.18)
The coupling constant $g_{\phi B'B}$ here depends on the initial and final baryons. Actually, the Goldberger-Treiman relation relates $g_{\phi B'B}$ with $g$ in Eq. (5.1). From this point of view, the inverse mass factors in the coefficients in Ref. 9) can be understood. In most of papers appearing to this time, little consideration is given to the normalization (relativistic or nonrelativistic) of the states.

We prefer Eq. (5.1) over Eq. (5.18), because in the latter the derivative coupling is a general sequence of the emission or absorption of a Nambu-Goldstone boson at low energies. Contrastingly, the universality (i.e., the independence of the initial and final baryons) of the coupling $g$ is generally less transparent. It is, however, very naturally understood in the Skyrme model, in which, as we see above, the axial-vector current comes from a single expression for all the baryons.

It is interesting that, although the reasoning employed there seems very different from ours, the decay width formula with the ratio of baryon masses given in Ref. 9),

$$\Gamma = \frac{|p|^3}{2\pi (M_B + M_{B'})^2 M_B} |\mathcal{M}|^2,$$

is similar to ours given in Eq. (5.16), if the factor $(M_B + M_{B'})$ is identified with our common mass scale, $\Lambda$.

5.2. The best fit values of the couplings and the predictions

In this subsection, we first calculate the important factor $|\tilde{\mathcal{M}}|^2$, and then the decay widths. To obtain $|\tilde{\mathcal{M}}|^2$, we need to calculate the matrix element

$$\langle \Psi_{B'} | O_\alpha^{\text{decay}} | \Psi_B \rangle = \int dA \psi_{B'}^{*}(A) O_\alpha^{\text{decay}} \psi_B(A).$$

The baryon wave function $\Psi_B(A)$ is a linear combination of the states in various representations,

$$\Psi_B(A) = \sum_i c_i^{B} \psi_{F_S}^{R_i}(A),$$

where $\psi_{F_S}^{R_i}(A)$ is the eigenstate of $H_0$ with $F = (Y, I_3, I)$ and $S = (Y_R = +1, J_3, J)$ in the representation $R_i$, and the coefficients $c_i^{B}$ are those we obtained in the previous section. Therefore, we first calculate $\langle \psi_{F'_S'}^{R_j} | O_\alpha^{\text{decay}} | \psi_{F_S}^{R_i} \rangle$. Furthermore, the spin and flavor structure is completely determined by the Clebsch-Gordan (CG) coefficients. For example, consider the matrix elements of the $G_0$ decay operator,

$$\int dA \left( \psi_{F'_S'}^{R_j}(A) \right)^* D^{(8)}_{\alpha k}(A)p^k \psi_{F_S}^{R_i}(A) = \frac{\sqrt{\text{dim} R_i}}{\text{dim} R_j} p^*(S') P(S) \sum_r \left( \begin{array}{ccc} 8 & R_i & R_j \\ F & F' & S' \end{array} \right)_r \left( \begin{array}{ccc} 8 & R_i & R_j \\ S & S' & \end{array} \right)_r^{*} p^k. \quad (5.22)$$

Because we have

$$\left( \begin{array}{ccc} 8 & R_i & R_j \\ k & S & S' \end{array} \right)_r p^k = \frac{p^1 - ip^2}{\sqrt{2}} (J', -J'_3 | 1, 1; J, -J_3)$$
Before presenting the factors, however, we introduce the following combinations of factors, and so on. The first two factors are irrelevant when we calculate the average baryon, we obtain the isospin of the initial baryon, and summing over the spin and isospin of the final element can be written as

\[
\int dA \left( \psi^\alpha_{F^j} (A) \right) \cdot \mathcal{O}^{\text{decay}}_{R^i} \psi^\alpha_{F} (A) = \text{(isospin CG)} \times \text{[spin CG with } p^k] \times \tilde{\mathcal{M}}_{R^i R^j} \alpha, (F^i, F) \right),
\]

where \( \tilde{\mathcal{M}}_{R^i R^j} \) contains all the other factors, such as the phases, the isoscalar factors, and so on. The first two factors are irrelevant when we calculate the average and sum of the spins and isospins, and they simply give \( \frac{1}{3} |p|^2 \). Here, only \( \tilde{\mathcal{M}}_{R^i R^j} \) is important. Actually, after squaring the amplitude, averaging over the spin and the isospin of the initial baryon, and summing over the spin and isospin of the final baryon, we obtain \( |\tilde{\mathcal{M}}|^2 \) as

\[
|\tilde{\mathcal{M}}|^2 = \left| \frac{1}{3} \sum_{i,j} \left( \epsilon^B_{ij} \right)^* \tilde{\mathcal{M}}_{R^i R^j} \alpha, (F^i, F) \epsilon^B_{ij} \right|^2.
\]

In the following, we give the factors \( \tilde{\mathcal{M}}_{R^i R^j} \) for various decays in matrix form. Before presenting the factors, however, we introduce the following combinations of couplings:

- \( G_{8,10} = G_0 + \frac{1}{2} G_1 \)
- \( G_{10,35} = G_0 - G_1 \)
- \( G_{27,35} = G_0 + \frac{1}{2} G_1 \)
- \( G_{27,64} = G_0 - \frac{1}{2} G_1 \)
- \( G_{35,35} = G_0 + 11 G_1 \)
- \( G_{64,27} = G_0 - \frac{1}{2} G_1 \)
- \( G_{64,64} = G_1 \)

- \( F_{10,8} = G_0 + \frac{1}{2} G_1 \)
- \( F_{27,10} = G_0 + G_1 \)
- \( F_{27,35} = G_0 - 2 G_1 \)
- \( F_{35,64} = G_0 - \frac{3}{2} G_1 \)
- \( F_{35,35} = G_0 + 11 G_1 \)
- \( F_{35,35} = G_0 - \frac{3}{2} G_1 \)
- \( F_{35,35} = G_0 + \frac{3}{2} G_1 \)
- \( F_{81,64} = G_0 + \frac{3}{2} G_1 \)
These are useful in representing $\tilde{M}_{\alpha(F',F)}^{R_j R_i}$. Here, $G$ represents the decay of the (mainly) decuplet to the (mainly) octet, $F$ the decay of the (mainly) anti-decuplet to the (mainly) decuplet, and $H$ the decay of the (mainly) anti-decuplet to the (mainly) octet. The subscripts indicate the components. The superscript, (1) or (2), distinguishes the outer degeneracy. For example, $G_{8,10}$ represents the coupling that appears in the matrix elements between $10$ and $8$. It is needed for the calculation of the decay of a (mainly) decuplet baryon to a (mainly) octet.

The first-order results are given in Refs. 58) and 14). The second-order results, which correspond to some of ours, are given in Ref. 59). Our results, given below, are very lengthy, but they are a very important part of the present paper, and also useful for $\chi_{QSM}$ calculations.

For the decays of (mainly) decuplet baryons, we have

$\Delta \rightarrow N + \pi :$

\[
\begin{pmatrix}
\frac{1}{\sqrt{5}} G_{8,10} & \frac{\sqrt{7}}{\sqrt{2}} G_{8,27} & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{5}}{\sqrt{6}} G_{10,27} & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{30}} G_{27,10} & \frac{\sqrt{5}}{2} G_{27,27} & \frac{\sqrt{7}}{\sqrt{21}} G_{27,35} & \frac{3}{\sqrt{14}} G_{10,35} & 0 & 0 \\
0 & \frac{1}{2\sqrt{105}} G_{35,27} & 0 & \frac{1}{8} G_{10}^{(1)} (G_{35,35} + 25 G_{35,35}) & \frac{25}{4\sqrt{3}} G_{27,64} & 0 \\
0 & \frac{1}{56\sqrt{3}} G_{64,27} & \frac{3\sqrt{7}}{8\sqrt{3}} G_{64,35} & \frac{3}{8\sqrt{3}} G_{64,35} & \frac{3}{\sqrt{7}} G_{64,64} & \frac{9}{8} G_{64,81}
\end{pmatrix}
\]

(5.29)

$\Sigma^* \rightarrow \Lambda + \pi :$

\[
\begin{pmatrix}
\frac{3}{\sqrt{10}} G_{8,10} & \frac{\sqrt{7}}{\sqrt{2}} G_{8,27} & 0 & 0 & 0 & 0 \\
\frac{\sqrt{5}}{3\sqrt{5}} G_{27,10} & \frac{6\sqrt{2}}{3\sqrt{5}} G_{27,27} & \frac{5\sqrt{2}}{3\sqrt{3}} G_{27,35} & \frac{10}{3\sqrt{7}} G_{27,35} & \frac{25}{42\sqrt{2}} G_{27,64} & 0 \\
0 & \frac{1}{56\sqrt{3}} G_{64,27} & \frac{3\sqrt{7}}{8\sqrt{3}} G_{64,35} & \frac{3}{8\sqrt{3}} G_{64,35} & \frac{25}{7} G_{64,64} & \frac{3\sqrt{7}}{8} G_{64,81}
\end{pmatrix}
\]

(5.30)
\[
\Sigma^* \to \Sigma + \pi : \\
\begin{pmatrix}
\frac{\sqrt{3}}{2} G_{8,10} & 0 & 0 & 0 \\
0 & \frac{\sqrt{7}}{2} G_{10,27} & 0 & \frac{\sqrt{3}}{2\sqrt{14}} G_{35,35} \\
-\frac{\sqrt{15}}{4\sqrt{6}} G_{27,10} & \frac{5\sqrt{3}}{4\sqrt{7}} G_{27,35} & -\frac{5\sqrt{2}}{4\sqrt{7}} G_{27,35} & 0 \\
0 & \frac{\sqrt{3}}{2\sqrt{14}} G_{64,35} & -\frac{\sqrt{3}}{2\sqrt{14}} G_{64,35} & 0 \\
\end{pmatrix}
\] (5.31)

\[
\Xi^* \to \Xi + \pi : \\
\begin{pmatrix}
\frac{3}{4\sqrt{6}} G_{8,10} & 0 & 0 & 0 \\
\frac{57}{28\sqrt{14}} G_{27,10} & \frac{25}{4\sqrt{6}} G_{27,27} - \frac{10}{15} G_{35,35} & 0 & \frac{5\sqrt{2}}{4\sqrt{7}} G_{27,64} \\
0 & \frac{9}{4\sqrt{14}} G_{64,35} & -\frac{2\sqrt{3}}{2\sqrt{14}} G_{64,64} + \frac{11}{7\sqrt{21}} G_{64,64} & 0 \\
\end{pmatrix}
\] (5.32)

With the mixing coefficients, one can easily calculate \(|\tilde{M}|^2\). For example, for the decay \(\Delta \to N + \pi\), it is given to first order by

\[
|\tilde{M}|^2 = \frac{1}{3} \begin{pmatrix}
\sum N_8 & \sum N_{10} & \sum c_{27d} \\
\end{pmatrix}
\begin{pmatrix}
\frac{3}{\sqrt{5}} G_{8,10} & \frac{\sqrt{7}}{2} G_{8,27} & 0 \\
0 & \frac{\sqrt{2}}{6} G_{10,27} & 0 \\
\frac{1}{\sqrt{30}} G_{27,10} & \frac{3}{7} G_{27,27} & \frac{5}{2\sqrt{21}} G_{27,35} \\
\end{pmatrix}
\begin{pmatrix}
\sum c_{10} \\
\sum c_{27q} \\
\sum c_{35} \\
\end{pmatrix}^2
\] (5.33)

with \(c_N^N\) and \(c_A^A\) given in Tables XI and XII. Note that the size of the matrix depends on the quantum numbers \((I,Y)\) of the initial and final baryons and corresponds to our mixing coefficients given in Tables XI–XIII.

The factors for the (mainly) anti-decuplet baryons are obtained similarly. First, for \(\Theta^+\), we have

\[
\Theta^+ \to N + K : \\
\begin{pmatrix}
-\frac{3}{\sqrt{5}} H_{8,10} & 0 \\
-\frac{3}{4} H_{10,27} & 0 \\
\frac{7\sqrt{3}}{4\sqrt{14}} H_{27,10} & 0 \\
\frac{3}{4\sqrt{14}} H_{35,10} & -\frac{10}{17\sqrt{2}} H_{35,35} \\
0 & \frac{9}{\sqrt{70}} H_{64,35} \\
\end{pmatrix}
\] . (5.34)

There are several interesting decay channels for the (mainly) anti-decuplet excited nucleon \(N'\), for the (mainly) anti-decuplet \(\Sigma'\), and for the (mainly) anti-decuplet \(\phi\) decays.
\[
\begin{align*}
\text{N'} \rightarrow \text{N + \pi :} & \quad \left( \begin{array}{c}
-\frac{27}{20} H_{8,8}^{(1)} - \frac{3}{4} H_{8,8}^{(2)} \\
\frac{1}{2\sqrt{5}} H_{27,8} \\
0 \\
0
\end{array} \right), \\
\text{N'} \rightarrow \text{N + \eta :} & \quad \left( \begin{array}{c}
-\frac{9}{20} H_{8,8}^{(1)} + \frac{3}{4} H_{8,8}^{(2)} \\
\frac{3}{2\sqrt{5}} H_{27,8} \\
0 \\
0
\end{array} \right), \\
\text{N'} \rightarrow \Delta + \pi : & \quad \left( \begin{array}{c}
\frac{6}{\sqrt{2}} F_{10,8} \\
\frac{2\sqrt{5}}{3} F_{27,8} \\
0 \\
0
\end{array} \right), \\
\text{N'} \rightarrow \Lambda + K : & \quad \left( \begin{array}{c}
-\frac{9}{20} H_{8,8}^{(1)} - \frac{3}{4} H_{8,8}^{(2)} \\
\frac{1}{4\sqrt{2}} H_{27,8} \\
0 \\
0
\end{array} \right),
\end{align*}
\]
\[ N' \rightarrow \Sigma + K:\]
\[
\begin{pmatrix}
\frac{28}{25} H_{8,8}^{(1)} - \frac{3}{4} H_{8,8}^{(2)} & \frac{1}{2 \sqrt{3}} H_{8,10} & \frac{1}{5 \sqrt{3}} H_{8,27} & 0 & 0 & 0 \\
\frac{3}{\sqrt{3}} H_{10,10} & -\frac{3}{\sqrt{3}} H_{10,10} & \frac{1}{2 \sqrt{3}} H_{10,10} & 0 & 0 & 0 \\
-\frac{1}{2} H_{27,8} & \frac{1}{2 \sqrt{3}} H_{27,10} & \frac{171 \sqrt{3} H_{27,27} - \frac{1}{4 \sqrt{3}} H_{27,27}}{280} & 0 & 0 & 0 \\
0 & 0 & \frac{5 \sqrt{3}}{2 \sqrt{3} H_{35,35}} & \frac{29}{4 \sqrt{3} H_{35,35}} & 0 & 0 \\
0 & 0 & \frac{-5 \sqrt{3}}{28 \sqrt{3} H_{64,27}} & 0 & \frac{9}{28 \sqrt{3} H_{64,64}} & \frac{-\sqrt{3}}{28 \sqrt{3} H_{64,64}} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
(5.39)

\[ \Sigma' \rightarrow N + K:\]
\[
\begin{pmatrix}
-\frac{3}{\sqrt{3}} H_{8,8}^{(1)} + \frac{\sqrt{3}}{2 \sqrt{3}} H_{8,8}^{(2)} & \frac{1}{5 \sqrt{3}} H_{8,10} & \frac{1}{2 \sqrt{3}} H_{8,27} & 0 & 0 & 0 \\
\frac{3}{\sqrt{3}} H_{10,10} & -\frac{3}{\sqrt{3}} H_{10,10} & \frac{1}{2 \sqrt{3}} H_{10,10} & 0 & 0 & 0 \\
-\frac{1}{2} H_{27,8} & \frac{1}{2 \sqrt{3}} H_{27,10} & \frac{171 \sqrt{3} H_{27,27} + \frac{1}{4 \sqrt{3}} H_{27,27}}{280} & 0 & 0 & 0 \\
0 & 0 & \frac{5 \sqrt{3}}{2 \sqrt{3} H_{35,35}} & \frac{29}{4 \sqrt{3} H_{35,35}} & 0 & 0 \\
0 & 0 & \frac{-5 \sqrt{3}}{28 \sqrt{3} H_{64,27}} & 0 & \frac{9}{28 \sqrt{3} H_{64,64}} & \frac{-\sqrt{3}}{28 \sqrt{3} H_{64,64}} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
(5.40)

\[ \Sigma' \rightarrow \Sigma + \pi:\]
\[
\begin{pmatrix}
-\frac{1}{2 \sqrt{3}} H_{8,8}^{(2)} & \frac{1}{5 \sqrt{3}} H_{8,10} & \frac{1}{2 \sqrt{3}} H_{8,27} & 0 & 0 & 0 \\
\frac{3}{\sqrt{3}} H_{10,10} & -\frac{3}{\sqrt{3}} H_{10,10} & \frac{1}{2 \sqrt{3}} H_{10,10} & 0 & 0 & 0 \\
0 & 0 & \frac{171 \sqrt{3} H_{27,27} - \frac{1}{4 \sqrt{3}} H_{27,27}}{280} & 0 & 0 & 0 \\
\frac{-1}{2 \sqrt{3}} H_{35,35} & \frac{1}{2 \sqrt{3}} H_{35,35} & \frac{171 \sqrt{3} H_{35,35} + \frac{1}{4 \sqrt{3}} H_{35,35}}{280} & 0 & 0 & 0 \\
0 & 0 & \frac{5 \sqrt{3}}{2 \sqrt{3} H_{64,35}} & \frac{29}{4 \sqrt{3} H_{64,35}} & 0 & 0 \\
0 & 0 & \frac{-5 \sqrt{3}}{28 \sqrt{3} H_{64,27}} & 0 & \frac{9}{28 \sqrt{3} H_{64,64}} & \frac{-\sqrt{3}}{28 \sqrt{3} H_{64,64}} \\
\end{pmatrix}
\]
(5.41)

\[ \Sigma' \rightarrow \Sigma + \eta:\]
\[
\begin{pmatrix}
-\frac{3}{8 \sqrt{3}} H_{8,8}^{(1)} & \frac{1}{5 \sqrt{3}} H_{8,10} & \frac{1}{2 \sqrt{3}} H_{8,27} & 0 & 0 & 0 \\
\frac{3}{\sqrt{3}} H_{10,10} & -\frac{3}{\sqrt{3}} H_{10,10} & \frac{1}{2 \sqrt{3}} H_{10,10} & 0 & 0 & 0 \\
0 & 0 & \frac{171 \sqrt{3} H_{27,27} - \frac{1}{4 \sqrt{3}} H_{27,27}}{280} & 0 & 0 & 0 \\
\frac{-3}{8 \sqrt{3}} H_{35,35} & \frac{1}{8 \sqrt{3}} H_{35,35} & \frac{171 \sqrt{3} H_{35,35} + \frac{1}{4 \sqrt{3}} H_{35,35}}{280} & 0 & 0 & 0 \\
0 & 0 & \frac{5 \sqrt{3}}{2 \sqrt{3} H_{64,35}} & \frac{29}{4 \sqrt{3} H_{64,35}} & 0 & 0 \\
0 & 0 & \frac{-5 \sqrt{3}}{28 \sqrt{3} H_{64,27}} & 0 & \frac{9}{28 \sqrt{3} H_{64,64}} & \frac{-\sqrt{3}}{28 \sqrt{3} H_{64,64}} \\
\end{pmatrix}
\]
(5.42)
Effective Theory Approach to the Skyrme Model

\[ \Sigma' \rightarrow \Lambda + \pi : \]
\[
\begin{pmatrix}
\frac{9}{10} H_{8,10}^{(1)} & \frac{3}{2^\sqrt{5}} H_{8,10} & \frac{1}{5} H_{8,27} & 0 & 0 & 0 \\
\frac{1}{10} H_{27,8} & \frac{7}{6 \sqrt{7}} H_{27,10} & \frac{27}{10} H_{27,35} & \frac{27}{2^\sqrt{7}} H_{27,35} & \frac{5 \sqrt{7}}{42 \sqrt{2}} H_{27,64} & 0 \\
0 & 0 & \frac{\sqrt{7}}{2 \sqrt{2}} H_{64,27} & \frac{7}{14} H_{64,35} & 0 & 0 \\
\end{pmatrix}
\]  
\[ (5.43) \]

\[ \Sigma' \rightarrow \Xi + K : \]
\[
\begin{pmatrix}
-\frac{9 \sqrt{7}}{10 \sqrt{2}} H_{8,10}^{(1)} - \frac{3 \sqrt{7}}{2^\sqrt{2}} H_{8,10}^{(2)} & \sqrt{\frac{3}{10}} H_{8,10} & -\frac{\sqrt{7}}{2^\sqrt{3}} H_{8,27} & 0 & 0 & 0 \\
-\frac{1}{15} H_{27,8} & \frac{7}{3 \sqrt{5}} H_{27,10} & -\frac{171}{280} H_{27,35} & -\frac{3}{8} H_{27,35} & \frac{5 \sqrt{7}}{6 \sqrt{7}} H_{27,64} & 0 \\
0 & 0 & -\frac{5}{28 \sqrt{2}} H_{64,27} & -\frac{3 \sqrt{7}}{4 \sqrt{7}} H_{64,35} & -\frac{5 \sqrt{7}}{2 \sqrt{10}} H_{64,64} & 0 \\
\end{pmatrix}
\]  
\[ (5.44) \]

\[ \Sigma' \rightarrow \Sigma^* + \pi : \]
\[
\begin{pmatrix}
-\frac{\sqrt{7}}{2^\sqrt{2}} F_{10,8} & 0 & -\frac{\sqrt{7}}{2^\sqrt{3}} F_{10,27} & 0 & 0 & 0 \\
0 & -\frac{\sqrt{7}}{2^\sqrt{2}} F_{27,10} & \frac{\sqrt{3}}{2^\sqrt{3}} F_{27,27} & -\frac{1}{2^\sqrt{7}} F_{27,35} & 0 & 0 \\
0 & 0 & -\frac{5}{2^\sqrt{3}} F_{35,27} & 0 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{7}}{2^\sqrt{3}} F_{35,10} & \frac{1}{2^\sqrt{2}} F_{35,35} & -\frac{\sqrt{7}}{2^\sqrt{3}} F_{35,64} & 0 \\
0 & 0 & 0 & 0 & -\frac{\sqrt{7}}{2^\sqrt{3}} F_{64,64} & 0 \\
\end{pmatrix}
\]  
\[ (5.45) \]

\[ \Sigma' \rightarrow \Delta + K : \]
\[
\begin{pmatrix}
\frac{2 \sqrt{2}}{\sqrt{3}} F_{10,8} & 0 & \frac{1}{3 \sqrt{5}} F_{10,27} & 0 & 0 & 0 \\
-\frac{2 \sqrt{2}}{3} F_{27,8} & \frac{2 \sqrt{2}}{3} F_{27,10} & \frac{9}{14} F_{27,27} & \frac{27}{35} F_{27,35} & \frac{\sqrt{7}}{42 \sqrt{2}} F_{27,64} & 0 \\
0 & 0 & \frac{5 \sqrt{7}}{2 \sqrt{3}} F_{35,27} & \frac{13}{6 \sqrt{7}} F_{35,35} & \frac{1}{2^\sqrt{7}} F_{35,64} & 0 \\
0 & 0 & -\frac{27}{2^\sqrt{7}} F_{35,10} & \frac{13}{2^\sqrt{7}} F_{35,35} & \frac{2 \sqrt{2}}{2^\sqrt{7}} F_{35,64} & 0 \\
0 & 0 & -\frac{25}{14 \sqrt{3}} F_{64,27} & \frac{13}{2^\sqrt{7}} F_{64,35} & \frac{2 \sqrt{2}}{2^\sqrt{7}} F_{64,64} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2^\sqrt{7}} F_{64,64} & 0 \\
\end{pmatrix}
\]  
\[ (5.46) \]
\[ \phi \rightarrow \Sigma + K : \]
\[
\begin{pmatrix}
-\frac{3}{\sqrt{10}} H_{8,10} & \frac{1}{\sqrt{5}} H_{8,27} & 0 & 0 & 0 \\
\frac{3}{4\sqrt{2}} H_{10,10} & -\frac{1}{8} \sqrt{3} H_{10,27} & -\frac{3\sqrt{5}}{8\sqrt{3}} H_{10,35} & 0 & 0 \\
-\frac{7}{8\sqrt{10}} H_{27,10} & \frac{1}{16\sqrt{21}} H_{27,27} + \frac{15\sqrt{3}}{32\sqrt{5}} H_{27,27} & -\frac{9}{4\sqrt{14}} H_{27,64} & 0 & 0 \\
\frac{3}{8\sqrt{14}} H_{35,10} & \frac{25}{28\sqrt{6}} H_{35,27} & \frac{5\sqrt{2}}{28\sqrt{6}} H_{35,64} & \frac{7\sqrt{6}}{20\sqrt{2}} H_{35,81} & 0 \\
0 & \frac{25}{28\sqrt{6}} H_{64,27} & -\frac{9}{4\sqrt{14}} H_{64,64} - \frac{31}{30} H_{64,64} & \frac{7\sqrt{6}}{20\sqrt{2}} H_{64,81} & 0 \\
\end{pmatrix}
\]
(5.47)

\[ \phi \rightarrow \Xi + \pi : \]
\[
\begin{pmatrix}
\frac{1}{\sqrt{10}} H_{8,10} & \frac{5}{\sqrt{14}} H_{8,27} & 0 & 0 & 0 \\
\frac{7}{4\sqrt{15}} H_{27,10} & \frac{1}{28\sqrt{10}} H_{27,27} & \frac{5\sqrt{3}}{4\sqrt{6}} H_{27,35} & \frac{3\sqrt{5}}{4\sqrt{3}} H_{27,64} & 0 \\
0 & \frac{25}{28\sqrt{6}} H_{64,27} & \frac{11\sqrt{3}}{4\sqrt{14}} H_{64,64} & \frac{5\sqrt{2}}{4\sqrt{14}} H_{64,81} & 0 \\
\end{pmatrix}
\]
(5.48)

\[ \phi \rightarrow \Xi^* + \pi : \]
\[
\begin{pmatrix}
\frac{1}{\sqrt{3}} F_{27,10} & \frac{1}{\sqrt{3}} F_{27,27} & 0 & 0 & 0 \\
-\frac{3\sqrt{3}}{7} F_{27,27} & \frac{\sqrt{3}}{4\sqrt{2}} F_{27,35} & -\frac{5\sqrt{3}}{28\sqrt{3}} F_{27,64} & 0 & 0 \\
0 & \frac{3\sqrt{3}}{4\sqrt{2}} F_{27,64} & \frac{3\sqrt{3}}{4\sqrt{2}} F_{64,35} & -\frac{4\sqrt{3}}{24} F_{64,64} & 0 \\
-\frac{5}{28\sqrt{3}} F_{64,27} & \frac{3\sqrt{3}}{4\sqrt{2}} F_{64,35} & -\frac{4\sqrt{3}}{24} F_{64,64} & \frac{3\sqrt{3}}{4} F_{64,81} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
(5.49)
Table XIV. Experimental values of the widths of the (mainly) decuplet baryons and their uncertainties.

| (MeV)   | $\Delta \to N + \pi$ | $\Sigma^* \to \Lambda + \pi$ | $\Sigma^* \to \Sigma + \pi$ | $\Xi^* \to \Xi + \pi$ |
|---------|----------------------|--------------------------------|----------------------------|------------------------|
| $\Gamma^{\exp}$ | 120                  | 32.6                           | 4.45                       | 9.50                   |
| $\sigma$   | 5                    | 0.74                           | 0.74                       | 1.8                    |

Let us gather all of these ingredients. In perturbation theory, it is important to keep clear the order of the expansion. In order to obtain the widths to second order in $\delta m$, we need to drop the higher-order terms consistently.

Because the mixings among the representations are large, however, we often get negative decay widths if we drop higher-order terms in the squares of the amplitudes. Of course such results are physically meaningless. Considering that the perturbative contributions are large, we also give the results in which the squares are not expanded. (In Appendix D, we show that negative decay widths can appear even if the mixings are smaller.)

We perform the second-order calculation as well as the first-order calculation. In the following, case $(a)$ is the first-order result with the square of the amplitude expanded, and case $(a')$ that without it expanded. Cases $(b)$ and $(b')$ are the corresponding second-order results. The second-order results differ drastically from the first-order results. However, before discussing the results, let us explain the procedure.

First, we found the best fit of the couplings, $G_0$ and $G_1$, to the experimental decay widths of the (mainly) decuplet baryons. The procedure is similar to that described in the previous section. We minimize

$$\chi^2 = \sum_i \frac{(\Gamma_i - \Gamma_i^{\exp})^2}{\sigma_i^2},$$

where $\Gamma_i$ stands for the value calculated for channel $i$, $\Gamma_i^{\exp}$ is its experimental value, and $\sigma_i$ represents the experimental uncertainty. The values we use are given in Table XIV. We give the values of the best fit set of couplings $G_0$ and $G_1$ in Table XV. We then determine $G_2$ using the $F/D$ ratio,

$$F/D = \frac{5}{9} \frac{H^{(2)}_{8,8}}{H^{(1)}_{8,8}}.$$ 

The experimental value of $F/D$ is $0.56 \pm 0.02$, though many authors prefer to use $F/D = 0.59$. The values of $G_2$ for various cases are also given in Table XV.

The best fit values of the (mainly) decuplet decay widths are listed in Table XVI, where we also give the phase space factor $K$:

$$\Gamma = K |\tilde{M}|^2, \quad K = \frac{|p|^3}{2\pi A^2 M_B} E_{B'}.$$ 

We use the experimental values for the baryon and meson masses, if they are known, in all decay width calculations. For the (mainly) anti-decuplet $N'$ and $\Sigma'$, we use our predicted values given in Eq. (4.3).
Table XV. Coefficients of the decay operators.

$G_0$ and $G_1$ are obtained by fitting the calculated decay widths of the (mainly) decuplet baryons to the experimental values, while $G_2$ is fixed by the $F/D$ ratio. The results $(a)$ and $(a')$ are those at first order with the squares of the amplitudes expanded and not expanded, respectively. The results $(b)$ and $(b')$ are those at second order. The $\chi^2$ values for these cases are also given.

|       | $(a)$ | $(a')$ | $(b)$ | $(b')$ |
|-------|-------|--------|-------|--------|
| $G_0$ | 4.74  | 5.29   | 5.64  | 5.77   |
| $G_1$ | 13.5  | 8.81   | 15.4  | 9.93   |
| $\chi^2$ | 38.8 | 4.73   | 13.1  | 6.43   |
| $G_2$ | 0.07  | 0.08   | 0.08  | 0.09   |

Table XVI. Decay widths for the (mainly) decuplet baryons. The calculations were carried out with the coupling constants given in Table XV. The kinematical factor $K$ is explained in Eq. (5.52).

|       |       |       |       |       |
|-------|-------|-------|-------|
| $\Delta \to N\pi$ | 1.47  | 92.1  | 114   | 110   |
| $\Sigma^* \to \Lambda\pi$ | 1.18  | 33.9  | 32.8  | 32.8  |
| $\Sigma^* \to \Sigma\pi$ | 0.26  | 5.08  | 5.50  | 5.73  |
| $\Xi^* \to \Xi\pi$ | 0.49  | 13.0  | 11.4  | 13.9  |

Table XVII. Predictions for the decay widths of the (mainly) anti-decuplet baryons.

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| $\Theta^+ \to NK$ | 1.91  | 217   | 74.7  | 378   |
| $N' \to N\pi$ | 20.9  | 870   | 246   | 1417  |
| $N' \to N\eta$ | 6.71  | -43.1 | 0.07  | -84.7 |
| $N' \to \Delta\pi$ | 7.36  | 0     | 72.6  | 144   |
| $N' \to \Lambda K$ | 2.04  | 25.1  | 8.37  | 58.5  |
| $N' \to \Sigma K$ | 0.22  | 7.04  | 2.50  | 13.6  |
| $\Sigma' \to NK$ | 15.1  | -1747 | 22.1  | -164  |
| $\Sigma' \to \Sigma\pi$ | 14.6  | 443   | 86.6  | 757   |
| $\Sigma' \to \Sigma\eta$ | 1.62  | -4.11 | 0.61  | -13.4 |
| $\Sigma' \to \Lambda\pi$ | 18.7  | 476   | 114   | 721   |
| $\Sigma' \to \Xi K$ | 0.03  | 0.10  | 0.04  | 0.11  |
| $\Sigma' \to \Sigma^*\pi$ | 5.94  | 0     | 6.71  | 2.25  |
| $\Sigma' \to \Delta K$ | 1.93  | 0     | 41.2  | 25.2  |
| $\phi \to \Sigma K$ | 5.12  | -137  | 14.9  | -181  |
| $\phi \to \Xi\pi$ | 10.8  | 416   | 62.9  | 294   |
| $\phi \to \Xi^*\pi$ | 2.68  | 0     | 8.63  | 47.2  |

Once we determine the best fit values of $G_a$, the decay widths of the (mainly) anti-decuplet baryons can be calculated. The results are given in Table XVII. From Table XVII, one can easily see that the widths change rather randomly in going to second order from first order. It becomes wider in one channel, while narrower in another. The behavior also depends on whether we expand the amplitude or not.

We have also carried out calculations with the calculated baryon masses even for $N$, $\Delta$, etc., and have found that the results change drastically.

Our result for the width of $\Theta^+$ is of order 100 MeV, which clearly contradicts reported experimental results. Still, our calculations offer some insight. First, we see that the widths decrease as the coupling $G_1$ becomes smaller. Second, our results are quite insensitive to the value of $G_2$. We carried out the similar calculations with
$F/D = 0.59$, but the results are quantitatively only slightly different and qualitatively the same, despite the fact that the value of $G_2$ changes considerably.

§6. Summary and discussion

In this paper, we have reconsidered the Skyrme model from an effective theory point of view. In this approach, the Skyrme model parameters, which appear in the collective coordinate quantized Hamiltonian, are determined by fitting the calculated baryon masses to the experimental values. Once the Skyrme model parameters have been fixed, various physical quantities can be calculated. In particular, we made a prediction for the masses of $N'$ and $\Sigma'$.

The idea behind this approach is that $\chi$PT provides a framework that represents QCD at low energies in which the soliton picture of baryons, which is a consequence of the large-$N_c$ limit, emerges. We start with the action up to $O(p^4)$ and keep only the terms that are of leading order in $N_c$. After quantization, we calculate everything as a systematic expansion in powers of $\delta m$. Note, however, that we keep in mind that there are infinitely many terms that contribute to the Skyrme model parameters. Thus the number of parameters in the Hamiltonian appears to be greater than that in the starting action. From the effective theory point of view, it is not the number of independent parameters of the “model” but the symmetry and the power counting that is important. The “derivation” given in Appendix C is, however, convenient to generate relevant operators that respect them.

The basic idea that the higher-order contributions improve the Skyrme model picture seems to be justified in Appendix C by comparing the conventional Skyrme model with that with $L_5$ and $L_8$ terms in the conventional approach.

We have performed the complete second-order calculations for the masses and determined the Skyrme model parameters. We find that, although the octet behaves well, the decuplet and anti-decuplet have large mixings, and the perturbative treatment may be questioned.

We also re-examined the decay calculation in the Skyrme models by deriving the formula for the decay width from the derivative coupling interaction to Nambu-Goldstone bosons. A careful derivation reveals how the decay widths depend on the initial and final baryons.

We calculated the widths of several interesting decays by using the decay width formula. In particular, our calculation predicts a wide decay width for $\Theta^+$, in contradiction to the experimental results. If $\Theta^+$ actually has a very narrow width, as reported, our theory fails to reproduce it. A possible explanation of this failure is that our perturbative treatment is poor for the (mainly) decuplet and the (mainly) anti-decuplet states. Because the decay parameters are determined by using the (mainly) decuplets, this could have a strong influence. Another possible explanation comes from the very subtle nature of the decay width calculations. The results depend strongly on the kinematics, i.e., the masses of the baryons and the factors in the formula. A few percent change in the mass can often cause a hundred percent (or even more) change in the decay width. Thus, the theoretical ambiguity is extremely large. The results given in Appendix D seem to support this explanation.
Is our fitting procedure appropriate? We vary all of the parameters as free parameters and treat them equally. But there must be a natural hierarchy for them: Leading-order parameters must be fitted to the bulk structure, and subleading parameters should account for fine structure. We have attempted to find such a systematic procedure, but to this time, the presented method is the most appropriate.

What should we do to improve the results? It is difficult to go to the next order in perturbation theory, because at the next order, we need to include more operators, and thus more parameters must be be fitted. A diagonalization, rather than a perturbative expansion, may be an option, but somehow it goes beyond the controlled effective theory framework.

Does $\Theta^+$ really exist? Is it narrow? Why so? Is the narrowness a general feature of the Skyrme model? We do not yet have definite answers to these questions, but our results suggest that if it does exist and is indeed narrow, it seems very peculiar, even from the Skyrme model point of view. As shown in Appendix D, this is not just because of the difference in the symmetry breaking interactions.

Praszalowicz\textsuperscript{24} investigated how $\Theta^+$ becomes narrow in the large-$N_c$ limit, and he showed that the narrowness results from the interplay between the cancellation in $G_{10} \equiv H_{8,10}$ (which becomes exact in the nonrelativistic limit) and the phase space volume dependence. The important factor of his argument is, of course, the cancellation in $G_{10}$, but this comes from the $\chi$QSM calculations. In our effective theory treatment, on the other hand, the couplings $G_a$ are parameters to be fitted. The only possible way to understand such “cancellation” in the effective theory context seems through consideration of symmetry. However, we do not know if such symmetry exists, nor its nature.

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Appendix A

--- Mathematical Tools ---

In this appendix, we summarize some basic mathematical formulae for the calculation of the matrix elements.

A.1. Properties of the wave function and basic formulae for matrix elements

Let us first introduce notation. A baryon wave function $\Psi$ has the flavor index $F = (Y, I_3, I)$ and the “spin” index $S = (Y_R, J_3, J)$. The eigenstate wave function

\[ \Psi_{\alpha} = \sum \lambda^\alpha_{\beta} \Psi_{\beta} \]

where $\lambda^\alpha_{\beta}$ are the matrix elements of the transformation matrix.

\[ \langle \Psi_{\alpha} | \mathcal{O} | \Psi_{\beta} \rangle = \sum \lambda^\alpha_{\beta} \langle \Psi_{\alpha} | \mathcal{O} | \Psi_{\beta} \rangle \]

where $\mathcal{O}$ is an operator.
(3.5) of $H_0$ can be written

$$\Psi^R(g)_{FS} = \sqrt{\dim R} P(S) \left( D^R_{F,S}(g) \right)^*, \quad g \in SU(3), \quad (A.1)$$

where $P(S) = (-1)^{J_3 - Y_R/2}$, $\tilde{S} = (Y_R, -J_3, J)$, and $D^R(g)$ is the representation matrix of $g$ for representation $R$. Note that physical states must satisfy the constraint (2.37), $Y_R = 1$, but this is irrelevant to most of the results given in this section. The wave function is normalized as

$$\langle \Psi^R_{FS} | \Psi^R_{FS'} \rangle = \delta_{RR'} \delta_{FF'} \delta_{SS'} . \quad (A.2)$$

The flavor transformation $f \in SU(3)$ acts from the left, i.e., $g \mapsto fg$. The corresponding unitary operator $U_{\text{flavor}}(f)$ acts as

$$U_{\text{flavor}}(f) \Psi^R(g)_{FS} = \Psi^R(f^{-1}g)_{FS} = \Psi^R(g)_{F'S} D^R_{F'}(f). \quad (A.3)$$

Here and hereafter, the summation over repeated indices is understood. On the other hand, the “spin” transformation $s \in SU(3)$ acts from the right, i.e., $g \mapsto gs^{-1}$. The corresponding unitary operator $U_{\text{spin}}(s)$ acts as

$$U_{\text{spin}}(s) \Psi^R(g)_{FS} = \Psi^R(gs)_{FS} = \sqrt{\dim R} P(S) \left( D^R_{F'S}(g) \right)^* \left( D^R_{S'S}(s) \right)^*$$

$$= \sqrt{\dim R} P(S') \left( D^R_{F'S}(g) \right)^* D^R_{S'S}(s), \quad (A.4)$$

where $\overline{R}$ stands for the representation conjugate to $R$, and $S^*$ stands for $(-Y_R, J_3, J)$. We have used the phase convention of Ref. 60). When $s$ is restricted to the “upper-left” $SU(2)$ subgroup, it reduces to the usual spin transformation law.

The infinitesimal transformation (Lie derivative) of the “spin” transformation defines the operator $F_\alpha$ introduced in Eq. (2.36),

$$F_\alpha \Psi^R(g)_{FS} = \sqrt{\dim R} P(S) \left( D^R_{F'S}(g) \right)^* \left( -T_\alpha^R \right)_{\tilde{S},\tilde{S}'} , \quad (A.5)$$

where $T_\alpha^R$ is the $SU(3)$ generator in the representation $R$. In particular, because

$$(-T_8^R)_{\tilde{S},\tilde{S}'} = -\frac{\sqrt{3}}{2} Y_R \delta_{Y_R} Y_R' \delta_{JJ'} \delta_{J3} J_3', \quad (A.6)$$

we have

$$F_8 \Psi^R(g)_{FS} = -\frac{\sqrt{3}}{2} Y_R \Psi^R(g)_{FS}. \quad (A.7)$$

The basic calculational tool for various matrix elements is the orthogonality of irreducible representations, expressed by

$$\int dg \left( D^R_{ij}(g) \right)^* D^R_{kl}(g) = \frac{1}{\dim R} \delta_{R,R'} \delta_{i,k} \delta_{j,l} , \quad (A.8)$$

*) Only the $SU(2)$ subgroup corresponds to the usual spatial rotation.
where \( dg \) is a normalized Haar measure. For a compact group such as \( SU(3) \), it is left- and right-invariant.

Another important tool is the \( SU(3) \) Clebsch-Gordan (CG) coefficients.\(^{60)-62}\)

It enables us to calculate the integral

\[
\int dg \left( D^{R}_{ij}(g) \right)^{*} D^{R_1}_{i_1,j_1}(g) D^{R_2}_{i_2,j_2}(g) = \frac{1}{\dim R} \sum_{r=1}^{m} \left( \begin{array}{ccc} R_1 & R_2 & R \\ i_1 & i_2 & i \end{array} \right)_{r}^{*} \left( \begin{array}{ccc} R_1 & R_2 & R \\ j_1 & j_2 & j \end{array} \right)_{r},
\]

where the subscript \( r \) counts the multiplicity \( m \) of the representation \( R \) in the direct product representation \( R_1 \otimes R_2 \), or, equivalently, the multiplicity of \( \bar{R}_1 \) in \( R \otimes \bar{R}_2 \). Note that in the following, we do not always work with the “physical basis” that diagonalizes the (right-) hypercharge and (iso-)spin, the CG coefficients are not necessarily real.

All of the operators whose matrix elements we need to evaluate involve the octet (adjoint) representation. Thus, our first formula is

\[
\langle \Psi^{R}_{FS} \left| D^{(8)}_{\alpha\beta} F_{\delta} \right| \Psi^{R'}_{F'S'} \rangle = \int dg \left( \Psi^{R}_{FS}(g) \right)^{*} D^{(8)}_{\alpha\beta}(g) \Psi^{R'}_{F'S'}(g)
\]

\[
= \sqrt{\dim R} \sqrt{\dim R'} P^*(S) P(S') \left( \int dg \left( D^{R}_{FS}(g) \right)^{*} D^{(8)}_{\alpha\beta}(g) D^{R'}_{F'S'}(g) \right)^{*}
\]

\[
= \sqrt{\frac{\dim R'}{\dim R}} P^*(S) P(S') \sum_{r=1}^{m} \left( \begin{array}{ccc} 8 & R' & \bar{R} \\ \alpha & F' & \bar{F} \end{array} \right)_{r} \left( \begin{array}{ccc} 8 & R' & \bar{R} \\ \beta & F' & \bar{F} \end{array} \right)_{r}^{*}.
\]

(A-10)

A decay operator contains (at most) one \( F_{\delta} \) with \( D^{(8)} \). Thus our second formula is

\[
\langle \Psi^{R}_{FS} \left| D^{(8)}_{\alpha\beta} F_{\delta} \right| \Psi^{R'}_{F'S'} \rangle = \int dg \left( \Psi^{R}_{FS}(g) \right)^{*} D^{(8)}_{\alpha\beta}(g) \left( F_{\delta} \Psi^{R'}_{F'S'}(g) \right)
\]

\[
= \sqrt{\dim R} \sqrt{\dim R'} P^*(S) P(S') \left( \int dg \left( D^{R}_{FS}(g) \right)^{*} D^{(8)}_{\alpha\beta}(g) D^{R'}_{F'S'}(g) \right)^{*} \left( -T^{R'}_{\delta} \right)_{\bar{S}, \bar{S}'}
\]

\[
= -\sqrt{\frac{\dim R'}{\dim R}} P^*(S) P(S') \sum_{r=1}^{m} \left( \begin{array}{ccc} 8 & R' & \bar{R} \\ \alpha & F' & \bar{F} \end{array} \right)_{r} \left( \begin{array}{ccc} 8 & R' & \bar{R} \\ \beta & F' & \bar{F} \end{array} \right)_{r}^{*} \left( T^{R'}_{\delta} \right)_{\bar{S}, \bar{S}'}.
\]

Most of the symmetry breaking operators in \( H_1 \) contain two \( F_{\alpha} \)’s with \( D^{(8)} \). The third formula is useful for evaluating their matrix elements,

\[
\langle \Psi^{R}_{FS} \left| F_{\delta} D^{(8)}_{\alpha\beta} F_{\eta} \right| \Psi^{R'}_{F'S'} \rangle = \int dg \left( F_{\delta} \Psi^{R}_{FS}(g) \right)^{*} D^{(8)}_{\alpha\beta}(g) \left( F_{\eta} \Psi^{R'}_{F'S'}(g) \right)
\]

\[
= \sqrt{\dim R} \sqrt{\dim R'} P^*(S) P(S') \times \left( -T^{R'}_{\delta} \right)_{\bar{S}, \bar{S}'} \left( \int dg \left( D^{R}_{FS\bar{S}}(g) \right)^{*} D^{(8)}_{\alpha\beta}(g) D^{R'}_{F'S'\bar{S}'}(g) \right)^{*} \left( -T^{R'}_{\eta} \right)_{\bar{S}', \bar{S}'_1}
\]

\[
= \sqrt{\frac{\dim R'}{\dim R}} P^*(S) P(S') \sum_{r=1}^{m} \left( \begin{array}{ccc} 8 & R' & \bar{R} \\ \alpha & F' & \bar{F} \end{array} \right)_{r} \left( T^{R'}_{\delta} \right)_{\bar{S}, \bar{S}'} \left( \begin{array}{ccc} 8 & R' & \bar{R} \\ \beta & S'_{\bar{S}} & S_1 \end{array} \right)_{r}^{*} \left( T^{R'}_{\eta} \right)_{\bar{S}', \bar{S}'_1}.
\]

(A-12)
The sum of the last three factors here can be rewritten as
\[(T^\mathcal{R}_\delta)_{S_1,\bar{S}_1} \left( \begin{array}{ccc} 8 & \mathcal{R}' & \mathcal{R} \\ \beta & S'_1 & S_1 \end{array} \right)_r^* \left( T^{\mathcal{R}'}_\eta \right)_{\bar{S}'_1, S'_1}. \] (A.13)

A.2. Simplification in the diagonal case

A certain simplification occurs in the case \( R = R' \). In this case, the CG coefficients can be considered \( \dim R \times \dim R \) matrices:
\[\left( M^{(R,r)}_\alpha \right)_{ij} \equiv \left( \begin{array}{cc} 8 & \mathcal{R} \\ \alpha & j \end{array} \right)_r \] (A.14).

In the following, we derive several useful formulae by giving the explicit forms of the matrices \( M^{(R,r)}_\alpha \). The analysis can be easily generalized to more general compact groups (at least to \( SU(n) \)).

First, we show that \( M^{(R,r)}_\alpha \) satisfies the commutation relation
\[\left[ T^{\mathcal{R}_\alpha}, M^{(R,r)}_\beta \right] = i\delta_{\alpha\beta} M^{(R,r)}_\gamma. \] (A.15)

Consider the integral
\[\int dg \left( D^{R}_{ij}(g) \right)^* D_{\alpha\beta}(g) D^{R}_{kl}(g) = \frac{1}{\dim R} \sum_{r=1}^m \left( \begin{array}{ccc} 8 & \mathcal{R} & \mathcal{R} \\ \alpha & k & i \end{array} \right)_r^* \left( \begin{array}{ccc} 8 & \mathcal{R} & \mathcal{R} \\ \beta & l & j \end{array} \right)_r. \] (A.16)

Let us change the integration variable as \( g \to gh \). Then, because the measure is right invariant, we have
\[\int dg \left( D^{R}_{ij}(gh) \right)^* D_{\alpha\beta}(gh) D^{R}_{kl}(gh) = \int dg \left( D^{R}_{ij}(g) \right)^* D_{\alpha\beta}(g) D^{R}_{kl}(g). \] (A.17)

For an infinitesimal \( h \), this leads to
\[- \left( T^{\mathcal{R}_\eta}_{\delta\eta} \right)_{j'i'} \left( M^{(R,r)}_\beta \right)_{j'\beta} + \left( T^{(8)}_{\eta} \right)_{\beta'\beta} \left( M^{(R,r)}_\beta \right)_{j'l} + \left( M^{(R,r)}_\beta \right)_{j'l'} \left( T^{\mathcal{R}_\eta}_{\eta} \right)_{\beta'l} = 0, \] (A.18)

where the independence of the CG coefficients has been used. From this, the commutation relation (A.15) follows directly.

Next, we show that any matrix that satisfies Eq. (A.15) is a linear combination of \( T^{\mathcal{R}_\alpha} \) and \( D^{\mathcal{R}}_\alpha \equiv d_{\alpha\beta\gamma} T^{\mathcal{R}}_\beta T^{\mathcal{R}}_\gamma \). (It is important not to confuse \( D^{\mathcal{R}}_\alpha \) with the representation matrix \( D^{\mathcal{R}}_{ij}(g) \).) If the Dynkin index of \( \mathcal{R} \) is \((0,q)\) or \((p,0)\), they are not independent: \( D^{\mathcal{R}}_\alpha \) is proportional to \( T^{\mathcal{R}_\alpha} \). The proof goes as follows. It is easy to show that \( T^{\mathcal{R}_\alpha} \) and \( D^{\mathcal{R}}_\alpha \) satisfy Eq. (A.15). The point is that the multiplicity \( m \) of \( 8 \) in \( \mathcal{R} \otimes \overline{\mathcal{R}} \) is at most 2, as we demonstrate below, and hence \( T^{\mathcal{R}_\alpha} \) and \( D^{\mathcal{R}}_\alpha \) span the complete set.

Let us consider the CG decomposition of \( \mathcal{R} \otimes \overline{\mathcal{R}} \) using the Young tableaux (Littlewood’s method). (See, for example, Ref. 63.) Suppose that the Dynkin index of the representation \( \mathcal{R} \) is \((p,q)\), and, without loss of generality, that \( p \geq q \). Then, there are \((p+2q)\) boxes in the product \( \mathcal{R} \otimes \overline{\mathcal{R}} \). As shown in Fig. 1, there are at most two ways to form an adjoint representation. This explicitly
shows that the multiplicity $m$ of $8$ in $\mathcal{R} \otimes \overline{\mathcal{R}}$ is at most $2$.

Similarly, we can show that for $SU(n)$ there are at most $n - 1$ ways of forming an
adjoint representation from the direct product $\mathcal{R} \otimes \overline{\mathcal{R}}$. This number is just the rank
of the group, rank($G$), and it is also the number of invariant tensors. The matrix
$M^{(\mathcal{R},r)}_\alpha$ can be written as a linear combination of $\text{rank}(G)$ quantities,

$$T^{(\mathcal{R},s)}_\alpha \equiv g_{\alpha \alpha_1 \alpha_2 \cdots \alpha_s} T^{\mathcal{R}}_{\alpha_1} T^{\overline{\mathcal{R}}}_{\alpha_2} \cdots T^{\mathcal{R}}_{\alpha_s}, \quad (s = 1, \cdots, \text{rank}(G)) \quad (A.19)$$

where $g_{\alpha \alpha_1 \alpha_2 \cdots \alpha_s}$ is a real symmetric invariant tensor of $SU(n)$, and therefore $T^{(\mathcal{R},s)}_\alpha$ is Hermitian. For $SU(3)$, there are two symmetric invariant tensors, $\delta_{\alpha \beta}$ and $d_{\alpha \beta \gamma}$.

The independence of $T^{(\mathcal{R},s)}_\alpha$ can be examined by defining the $\text{rank}(G) \times \text{rank}(G)$ matrix $C^{st}$:

$$C^{st} \text{id}_{\mathcal{R}} = T^{(\mathcal{R},s)}_\alpha T^{(\mathcal{R},t)}_\alpha. \quad (A.20)$$

Note that the right-hand side commutes with $T^{\mathcal{R}}_{\beta}$ for all $\beta$, and therefore it is proportional to $\text{id}_{\mathcal{R}}$ by Schur’s lemma. In general, we have $m = \text{rank}(C) \leq \text{rank}(G)$. We stipulate that the first $m$ matrices $T^{(\mathcal{R},s)}_\alpha$ be independent.

We are now able to write the matrix $M^{(\mathcal{R},r)}_\alpha$ as a linear combination of independent matrices $T^{(\mathcal{R},s)}_\alpha$:

$$M^{(\mathcal{R},r)}_\alpha = \sum_{s=1}^{m} V_{rs} T^{(\mathcal{R},s)}_\alpha. \quad (A.21)$$

Note that $V_{rs}$ is a regular $m \times m$ matrix, but it is not orthogonal. The normalization
of $M^{(\mathcal{R},r)}_\alpha$ is determined by the orthogonality condition of the CG coefficients,

$$\left( M^{(\mathcal{R},r)}_\alpha \right)^\dagger M^{(\mathcal{R},s)}_\alpha \bigg|_{ij} = \delta_{ij} \delta_{rs}. \quad (A.22)$$

By substituting (A.21) into the above expression, we obtain

$$V C V^\dagger \bigg|_{rs} = \delta_{rs}, \quad (A.23)$$
and thus,
\[
(V^\dagger V)_{rs} = (C^{-1})_{rs}. \tag{A.24}
\]

We can write (A.16) in terms of \(T^{(R,r)}_\alpha\):
\[
\int dg \left( D^R_{ij}(g) \right)^* \mathcal{D}^{(8)}_{\alpha\beta}(g) D^R_{kl}(g) = \frac{1}{\dim R} \sum_{r=1}^{m} \left( M^{(R,r)}_\alpha \right)^*_{ik} \left( M^{(R,r)}_\beta \right)_{jl} = \frac{1}{\dim R} \sum_{s,t=1}^{\dim R} \left( V^\dagger V \right)_{st} \left( T^{(R,s)}_\alpha \right)_{ki} \left( T^{(R,t)}_\beta \right)_{jl}. \tag{A.25}
\]

By substituting (A.24) into this, we finally obtain the expression
\[
\int dg \left( D^R_{ij}(g) \right)^* \mathcal{D}^{(8)}_{\alpha\beta}(g) D^R_{kl}(g) = \frac{1}{\dim R} \sum_{s,t=1}^{\dim R} \left( C^{-1} \right)_{st} \left( T^{(R,s)}_\alpha \right)_{ki} \left( T^{(R,t)}_\beta \right)_{jl}. \tag{A.26}
\]

For \(SU(n)\), \(D^{(8)}(g)\) should be replaced by \(D^{Ad}(g)\).

A.3. Formulae for the matrix elements diagonal in the representation

We now calculate the matrix elements of various operators, which are diagonal in the representation, by using the formulae derived in the previous subsection.

For \(SU(3)\), there are two symmetric invariant tensors \(\delta_{\alpha\beta}\) and \(d_{\alpha\beta\gamma}\), and we have
\[
T^{(R,1)}_\alpha = T^R_\alpha, \quad T^{(R,2)}_\alpha = d_{\alpha\beta\gamma} T^R_\beta T^R_\gamma = D^R_\alpha. \tag{A.27}
\]

The matrix \(C^{rs}\) can be written as
\[
C = \begin{pmatrix}
C_2 & C_3 \\
C_3 & D_2
\end{pmatrix}, \tag{A.28}
\]
where \(C_2\) and \(C_3\) are the quadratic and cubic Casimir operators\(^{64}\) respectively,
\[
C_2 = \frac{1}{3} \left( p^2 + q^2 + pq + 3(p + q) \right), \tag{A.29}
\]
\[
C_3 = \frac{1}{18} \left( p - q \right) \left( 2p^2 + 2q^2 + 5pq + 9(p + q + 1) \right), \tag{A.30}
\]
for the representation \(R = (p, q)\), while \(D_2 \equiv \sum_\alpha D^R_\alpha D^R_\alpha\) can be written as
\[
D_2 = \left( \frac{1}{3} C_2 + \frac{1}{4} \right) C_2. \tag{A.31}
\]

From the determinant of \(C\),
\[
\det C = \frac{1}{12} pq(p + 2)(q + 2)(p^2 + q^2 + 2pq + 4(p + q) + 3), \tag{A.32}
\]
we see that \(T^R_\alpha\) and \(D^R_\alpha\) are not independent for \(p = 0\) or \(q = 0\). Actually, we have
\[
D^R_\alpha = \frac{C_3}{C_2} T^R_\alpha. \tag{A.33}
\]
In this case, the integral (A.26) becomes simpler:
\[
\int dg \left( D_{ij}^R(g) \right)^* D_{\alpha\beta}^{(8)}(g) D_{kl}^R(g) \frac{1}{\dim R C_2(R)} (T^R_\alpha)_{kl} (T^R_\beta)_{ji'}.
\]
(A.34)

When \(\det C \neq 0\), \(T^R_\alpha\) and \(D^R_\alpha\) are independent, and the matrix \(C\) has the inverse
\[
C^{-1} = \frac{1}{C_2^2 (3 + 4 C_2) - 12 C_3^2} \begin{pmatrix} C_2 (3 + 4 C_2) & -12 C_3 \\ -12 C_3 & 12 C_2 \end{pmatrix}.
\]
(A.35)

### A.3.1. Hamiltonian operators

Here we explicitly use \(Y_R = 1\) and present the results for the various flavor \(SU(3)\) operators appearing in the Hamiltonian. They are expressed in terms of flavor \(T^R_8\) and \(D^R_8\), i.e., in the Gell-Mann-Okubo form.

Let us first explicitly consider the simplest case:
\[
\langle \Psi^R_{F_2 S_2} \left| D_{\alpha\beta}^{(8)} \right| \Psi^R_{F_1 S_1} \rangle = \dim R P^*(S_2) P(S_1) \left( \int dg \left( D_{F_2 S_2}^R(g) \right)^* D_{\alpha\beta}^{(8)}(g) D_{F_1 S_1}^R(g) \right)^*,
\]
(A.36)

Then, using
\[
(T^R_8)_{\tilde{S}_1 \tilde{S}_2} = \frac{\sqrt{3}}{2} \delta_{\tilde{S}_1 \tilde{S}_2},
\]
(A.37)
\[
(D^R_8)_{\tilde{S}_1 \tilde{S}_2} = \frac{\sqrt{3}}{2} \left( J^2 - \frac{1}{3} C_2(R) - \frac{1}{4} \right) \delta_{\tilde{S}_1 \tilde{S}_2},
\]
(A.38)
we have
\[
\langle \Psi^R_{F_2 S_2} \left| D_{88}^{(8)} \right| \Psi^R_{F_1 S_1} \rangle = \frac{\sqrt{3}}{2} \left\{ (T_8)_{F_2 F_1} \left( C_{11}^{-1} + C_{12}^{-1} \left( J^2 - \frac{1}{3} C_2 - \frac{1}{4} \right) \right) \\
+ (D_8)_{F_2 F_1} \left( C_{21}^{-1} + C_{22}^{-1} \left( J^2 - \frac{1}{3} C_2 - \frac{1}{4} \right) \right) \right\} \delta_{\tilde{S}_1 \tilde{S}_2},
\]
(A.39)

where \(J^2\) stands for the spin \(SU(2)\) quadratic Casimir. Here, we have dropped all the \(R\) dependence on the right-hand side for simplicity.

Other matrix elements can be calculated similarly. The results are as follows:
\[
\langle \Psi^R_{F_2 S_2} \left| O_i \right| \Psi^R_{F_1 S_1} \rangle = P^*(S_2) P(S_1) \left\{ (T_8)_{F_2 F_1} \left( C_{11}^{-1} O_i^{(1)} + C_{12}^{-1} O_i^{(2)} \right) \right\} \delta_{\tilde{S}_1 \tilde{S}_2}
+ (D_8)_{F_2 F_1} \left( C_{21}^{-1} O_i^{(1)} + C_{22}^{-1} O_i^{(2)} \right) \delta_{\tilde{S}_1 \tilde{S}_2},
\]
(A.40)

where
\[
O_x^{(1)} = \frac{\sqrt{3}}{2} J^2,
\]
(A.41)
Because T

\[ O_x^{(2)} = \frac{\sqrt{3}}{6} \left( 3J^2 - C_2 - \frac{3}{4} \right) J^2, \]

\[ O_y^{(1)} = \frac{3}{8} + \frac{1}{2} C_2 + \frac{2}{3} C_3 - \frac{3}{2} J^2, \]

\[ O_y^{(2)} = -\frac{3}{16} - \frac{1}{4} C_2 - \frac{1}{3} C_3 + \frac{1}{2} J^2 + C_2 J^2 - (J^2)^2, \]

\[ O_z^{(1)} = \frac{\sqrt{3}}{2} \left( C_2 - J^2 - \frac{9}{4} \right), \]

\[ O_z^{(2)} = \frac{\sqrt{3}}{6} \left( 3J^2 - C_2 - \frac{3}{4} \right) \left( C_2 - J^2 - \frac{9}{4} \right), \]

\[ O_w^{(1)} = -\frac{3}{8} + \frac{1}{3} C_3 - \frac{1}{2} J^2, \]

\[ O_w^{(2)} = -\frac{3}{32} + \frac{1}{8} C_2 + \frac{1}{3} (C_2)^2 + \frac{1}{3} C_3 - \frac{3}{4} J^2 - \frac{5}{6} C_2 J^2 + \frac{1}{2} (J^2)^2. \]

The usefulness of these formulæ rests on the fact that they are easily calculated for an arbitrary representation \( \mathcal{R} \).

A.3.2. Decay operators

The matrix elements of the operator \( G_0 \) can be calculated using Eq. (A.36),

\[ \langle \Psi^{\mathcal{R}}_{F_2 S_2} | D^{(8)}_{\alpha i} | \Psi^{\mathcal{R}}_{F_1 S_1} \rangle = P^*(S_2) P(S_1) \sum_{s,t=1}^{2} (C^{-1})_{st} (T^{\mathcal{R},s}_{\alpha}) F_2 F_1 (T^{\mathcal{R},t}_{i}) \tilde{S}_1 \tilde{S}_2 \]

\[ = P^*(S_2) P(S_1) \left\{ (T_{\alpha}) F_2 F_1 \left( C_{11}^{-1} (T_i) \tilde{S}_1 \tilde{S}_2 + C_{12}^{-1} (D_i) \tilde{S}_1 \tilde{S}_2 \right) \right. \]

\[ + (D_{\alpha}) F_2 F_1 \left( C_{21}^{-1} (T_i) \tilde{S}_1 \tilde{S}_2 + C_{22}^{-1} (D_i) \tilde{S}_1 \tilde{S}_2 \right) \}. \]  

(A.49)  

Because \( T_i \) is the usual spin \( SU(2) \) generator, the matrix elements between states with different spins vanish, and only the \( D_i \) terms contribute. When the two states have the same spin, from the transformation properties, \( D_i \) is proportional to \( T_i \). Then, using

\[ \sum_{i=1}^{3} T_i D_i = \frac{1}{3} C_3 + \frac{1}{2\sqrt{3}} C_2 T_8 + \frac{1}{2\sqrt{3}} J^2 T_8 - \frac{1}{6\sqrt{3}} (T_8)^3 + \frac{1}{2\sqrt{3}} T_8, \]

(A.50)

we have for the same spin states (with \( Y_{\alpha} = 1 \),

\[ (D_i) \tilde{S}_1 \tilde{S}_2 = A (T_i) \tilde{S}_1 \tilde{S}_2, \]

(A.51)

with

\[ A = \frac{1}{J^2} \left( \frac{1}{3} C_3 + \frac{1}{4} C_2 + \frac{1}{4} J^2 + \frac{3}{16} \right). \]

(A.52)

The diagonal (in the representation) matrix elements of the decay operators that are linear in \( F_\alpha \) can be calculated in a similar way. For the operator \( G_2 \), we have

\[ \langle \Psi^{\mathcal{R}}_{F_2 S_2} | D^{(8)}_{\alpha i} | \Psi^{\mathcal{R}}_{F_1 S_1} \rangle \]
\[
= P^*(S_2)P(S_1) \sum_{s,t=1}^{2} (C^{-1})_{st} (T^{R,s}_\alpha)_{F_2F_1} (T^{R,t}_{8})_{S_1S_2} (-T^R_i)_{\bar{S}_1\bar{S}_2}. \quad (A.53)
\]

By using Eqs. (A.37) and (A.38), this can be rewritten as

\[
\langle \Psi^R_{F_2S_2} \left| D^{(8)}_{\alpha\beta} F_i \right| \Psi^R_{F_1S_1} \rangle = -\sqrt{3 \over 2} P^*(S_2)P(S_1) \{ (T^R_\alpha)_{F_2F_1} \left( C^{-1}_{11} + C^{-1}_{12} \left( J^2 - {1 \over 3} C_2 - {1 \over 4} \right) \right) \\
+ (D^R_\alpha)_{F_2F_1} \left( C^{-1}_{21} + C^{-1}_{22} \left( J^2 - {1 \over 3} C_2 - {1 \over 4} \right) \right) \} (T_i)_{\bar{S}_1\bar{S}_2}. \quad (A.54)
\]

Similarly, we can calculate the matrix elements for the operator \( G_1 \):

\[
\langle \Psi^R_{F_2S_2} \left| \sum_{\beta,\gamma \in J} d_{i\beta\gamma} D^{(8)}_{\alpha\beta} F_\gamma \right| \Psi^R_{F_1S_1} \rangle = -P^*(S_2)P(S_1) \sum_{s,t=1}^{2} (C^{-1})_{st} (T^{R,s}_\alpha)_{F_2F_1} \sum_{\beta,\gamma \in J} d_{i\beta\gamma} (T^R_\gamma T^{R,t}_\beta)_{S_1S_2}. \quad (A.55)
\]

Noting that

\[
\sum_{\beta,\gamma \in J} d_{i\beta\gamma} T_\beta T_\gamma = D_i - {2 \over \sqrt{3}} T_i T_8,
\]

\[
\sum_{\beta,\gamma \in J} d_{i\beta\gamma} T_\beta D_\gamma = \left( {1 \over 3} C_2 + {1 \over 4} \right) T_i - {1 \over \sqrt{3}} (D_i T_8 + D_8 T_i),
\]

we have

\[
\langle \Psi^R_{F_2S_2} \left| \sum_{\beta,\gamma \in J} d_{i\beta\gamma} D^{(8)}_{\alpha\beta} F_\gamma \right| \Psi^R_{F_1S_1} \rangle = -P^*(S_2)P(S_1) \{ (T_\alpha)_{F_2F_1} \left( C^{-1}_{11} (D_i - T_i) \\
+ C^{-1}_{12} \left( \left( {1 \over 2} C_2 - {1 \over 2} J^2 + {3 \over 8} \right) T_i - {1 \over 2} D_i \right) \right)_{\bar{S}_1\bar{S}_2} \\
+ (D^R_\alpha)_{F_2F_1} \left( C^{-1}_{21} (D_i - T_i) + C^{-1}_{22} \left( \left( {1 \over 2} C_2 - {1 \over 2} J^2 + {3 \over 8} \right) T_i - {1 \over 2} D_i \right) \right)_{\bar{S}_1\bar{S}_2} \}.
\quad (A.58)
\]

When \( \det C = 0 \), the corresponding formulae become much simpler.

**Appendix B**

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**Tables of Various Matrix Elements**

Those matrix elements of the symmetry breaking operators that are needed to calculated the baryon masses are given in §3.2. In this appendix, we list other matrix
Table XVIII. \(\langle 27_d | O_i | 27_d \rangle\).

| \((I,Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \((\frac{1}{2},+1)\) | 137 | \(\frac{3}{2}\) | \(\frac{1}{2}\) | 397 | \(\frac{3}{2}\) |
| \((1,0)\) | 137 | 1120 | \(\frac{3}{2}\) | 1120 | \(\frac{3}{2}\) |
| \((\frac{1}{2},-1)\) | 137 | 256 | 1120 | \(\frac{3}{2}\) | 1120 | \(\frac{3}{2}\) |
| \((0,0)\) | 137 | 256 | \(\frac{3}{2}\) | 1120 | \(\frac{3}{2}\) | 1120 | \(\frac{3}{2}\) |
| \((\frac{3}{2},-1)\) | 1120 | 256 | 1120 | \(\frac{3}{2}\) | 1120 | \(\frac{3}{2}\) | 1120 | \(\frac{3}{2}\) |

Table XIX. \(\langle 35_d | O_i | 35_d \rangle\).

| \((I,Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \((0,+2)\) | \(\frac{1}{4}\) | \(\frac{3}{4}\) | \(\frac{1}{4}\) | \(\frac{1}{4}\) | \(\frac{1}{4}\) |
| \((\frac{1}{2},+1)\) | \(\frac{3}{16}\) | \(\frac{3}{16}\) | \(\frac{3}{16}\) | \(\frac{3}{16}\) | \(\frac{3}{16}\) |
| \((1,0)\) | \(\frac{1}{8}\) | \(\frac{1}{8}\) | \(\frac{1}{8}\) | \(\frac{1}{8}\) | \(\frac{1}{8}\) |
| \((\frac{3}{2},-1)\) | \(\frac{1}{16}\) | \(\frac{1}{16}\) | \(\frac{1}{16}\) | \(\frac{1}{16}\) | \(\frac{1}{16}\) |

Table XX. \(\langle 35_d | O_i | 27_d \rangle\).

| \((I,Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \((\frac{1}{2},+1)\) | \(\sqrt{15}\) | \(\frac{3}{15}\) | \(\frac{3}{15}\) | \(\frac{3}{15}\) | \(\frac{3}{15}\) |
| \((1,0)\) | \(16\sqrt{2}\) | \(16\sqrt{2}\) | \(16\sqrt{2}\) | \(16\sqrt{2}\) | \(16\sqrt{2}\) |
| \((\frac{3}{2},-1)\) | \(8\sqrt{3}\) | \(8\sqrt{3}\) | \(8\sqrt{3}\) | \(8\sqrt{3}\) | \(8\sqrt{3}\) |

Table XXI. \(\langle 64_d | O_i | 27_d \rangle\).

| \((I,Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \((\frac{1}{2},+1)\) | \(\frac{5}{2}\) | \(\frac{5}{2}\) | \(\frac{5}{2}\) | \(\frac{5}{2}\) | \(\frac{5}{2}\) |
| \((1,0)\) | \(2\sqrt{2}\) | \(2\sqrt{2}\) | \(2\sqrt{2}\) | \(2\sqrt{2}\) | \(2\sqrt{2}\) |
| \((\frac{3}{2},-1)\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) |
| \((0,0)\) | \(14\sqrt{2}\) | \(14\sqrt{2}\) | \(14\sqrt{2}\) | \(14\sqrt{2}\) | \(14\sqrt{2}\) |

Table XXII. \(\langle 64_d | O_i | 35_d \rangle\).

| \((I,Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \((\frac{1}{2},+1)\) | \(\frac{1}{4}\sqrt{5}\) | \(\frac{1}{4}\sqrt{5}\) | \(\frac{1}{4}\sqrt{5}\) | \(\frac{1}{4}\sqrt{5}\) | \(\frac{1}{4}\sqrt{5}\) |
| \((1,0)\) | \(\frac{1}{4}\sqrt{14}\) | \(\frac{1}{4}\sqrt{14}\) | \(\frac{1}{4}\sqrt{14}\) | \(\frac{1}{4}\sqrt{14}\) | \(\frac{1}{4}\sqrt{14}\) |
| \((\frac{3}{2},-1)\) | \(\frac{3}{4}\sqrt{7}\) | \(\frac{3}{4}\sqrt{7}\) | \(\frac{3}{4}\sqrt{7}\) | \(\frac{3}{4}\sqrt{7}\) | \(\frac{3}{4}\sqrt{7}\) |

elements that are necessary to calculate the mixings to second order. These matrix elements are calculated with the method explained in the previous section.

To second order, the (mainly) octet states can mix with \(\bar{16}, 27_d, 35_d\), and \(64_d\). The (mainly) anti-decuplet states can mix with \(\bar{81}_d\) in addition to these. We therefore need \(\langle 27_d | O_i | 27_d \rangle\), \(\langle 35_d | O_i | 35_d \rangle\), \(\langle 35_d | O_i | 27_d \rangle\), \(\langle 64_d | O_i | 27_d \rangle\), \(\langle 64_d | O_i | 35_d \rangle\), and \(\langle 81_d | O_i | 35_d \rangle\).
The (mainly) decuplet states can mix with $27_q$, $35$, $35_q$, $64_q$, and $81$. Thus we need $(27_q | O_{v1} | 27_q)$, $(35 | O_{v} | 35)$, $(35 | O_{v} | 27_q)$, $(35_q | O_{v} | 27_q)$, $(64_q | O_{v} | 27_q)$, $(64_q | O_{v} | 35)$ and $(81 | O_{v} | 35)$.

Because the matrix elements listed above contribute only to the second-order calculations, those of $O_{v1}$ and $O_{v2}$ are not necessary, because these operators themselves are of second order. There are, however, extra matrix elements that we need to calculate, namely, $(35_d | O_{v} | 8)$, $(64_d | O_{v} | 8)$, $(64_d | O_{v} | 10)$, $(81_d (O_{v} | 10)$, $(35_q | O_{v} | 10)$, $(64_q | O_{v} | 10)$, and $(81 | O_{v} | 10)$.
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Table XXVIII. \( (64_d | O_v | 27_v) \).

| \((I, Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \(\frac{1}{2}, +1\) | 5\(\sqrt{3}\) & 15 \(\sqrt{3}\) & \(-2\) & 55\(\sqrt{3}\) & 5 |
| \(1, 0\) | 5\(\sqrt{3}\) & 15 \(\sqrt{3}\) & \(-2\) & 55\(\sqrt{3}\) & 5 |
| \(\frac{1}{2}, -1\) | 5\(\sqrt{3}\) & 15 \(\sqrt{3}\) & \(-2\) & 55\(\sqrt{3}\) & 5 |
| \(0, 0\) | 3\(\sqrt{3}\) & 15 \(\sqrt{3}\) & \(-2\) & 55\(\sqrt{3}\) & 5 |

Table XXIX. \( (64_d | O_v | 35) \).

| \((I, Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \(\frac{3}{2}, +1\) | \(\frac{9}{\sqrt{35}}\) & \(\frac{15}{\sqrt{35}}\) & \(\frac{3\sqrt{15}}{7}\) & \(27\sqrt{3}\) & \(-51\sqrt{3}\) |
| \(1, 0\) | \(\frac{3}{\sqrt{7}}\) & \(\frac{15}{\sqrt{7}}\) & \(\frac{3\sqrt{5}}{7}\) & \(16\sqrt{3}\) & \(-16\sqrt{3}\) |
| \(\frac{1}{2}, -1\) | \(\frac{3}{\sqrt{7}}\) & \(\frac{15}{\sqrt{7}}\) & \(\frac{3\sqrt{5}}{7}\) & \(16\sqrt{3}\) & \(-16\sqrt{3}\) |

Table XXX. \( (81 | O_v | 35) \).

| \((I, Y)\) | \(\gamma\) | \(x\) | \(y\) | \(z\) | \(w\) |
|---|---|---|---|---|---|
| \(\frac{3}{2}, +1\) | \(2\sqrt{35}\) & \(4\sqrt{35}\) & \(\frac{3\sqrt{15}}{7}\) & \(27\sqrt{3}\) & \(-51\sqrt{3}\) |
| \(1, 0\) | \(\frac{3}{\sqrt{7}}\) & \(\frac{15}{\sqrt{7}}\) & \(\frac{3\sqrt{5}}{7}\) & \(16\sqrt{3}\) & \(-16\sqrt{3}\) |
| \(\frac{1}{2}, -1\) | \(\frac{3}{\sqrt{7}}\) & \(\frac{15}{\sqrt{7}}\) & \(\frac{3\sqrt{5}}{7}\) & \(16\sqrt{3}\) & \(-16\sqrt{3}\) |

Table XXXI. \( (35_d | O_v | 8) \) & \( (64_d | O_v | 8) \).

| \((I, Y)\) | \(v_1\) | \(v_2\) | \(v_1\) | \(v_2\) |
|---|---|---|---|---|
| \(\frac{1}{2}, +1\) | \(\frac{1}{\sqrt{10}}\) & \(\frac{1}{10}\) & \(\frac{1}{\sqrt{10}}\) & \(\frac{1}{10}\) |
| \(1, 0\) | \(\frac{1}{2\sqrt{35}}\) & \(\frac{1}{\sqrt{35}}\) & \(\frac{28\sqrt{2}}{\sqrt{5}}\) & \(28\sqrt{2}\) |
| \(\frac{1}{2}, -1\) | \(0\) & \(0\) & \(28\sqrt{2}\) & \(28\sqrt{2}\) |
| \(0, 0\) | \(0\) & \(0\) & \(28\sqrt{10}\) & \(28\sqrt{10}\) |

Table XXXII. \( (64_d | O_v | 10) \) & \( (81_d | O_v | 10) \).

| \((I, Y)\) | \(v_1\) | \(v_2\) | \(v_1\) | \(v_2\) |
|---|---|---|---|---|
| \(0, 2\) | \(0\) & \(0\) & \(\frac{3\sqrt{3}}{\sqrt{5}}\) & \(\frac{3\sqrt{3}}{\sqrt{5}}\) |
| \(\frac{1}{2}, +1\) | \(3\) & \(1\) & \(2\sqrt{3}\) & \(2\sqrt{3}\) |
| \(1, 0\) | \(\frac{1}{2\sqrt{10}}\) & \(\frac{1}{\sqrt{10}}\) & \(\frac{28\sqrt{10}}{\sqrt{5}}\) & \(28\sqrt{10}\) |
| \(\frac{1}{2}, -1\) | \(\frac{1}{2\sqrt{35}}\) & \(\frac{1}{\sqrt{35}}\) & \(\frac{28\sqrt{10}}{\sqrt{5}}\) & \(28\sqrt{10}\) |

Appendix C

Conventional Approach to the Skyrme Model

In this section, we consider the conventional approach to the action (2.9). Specifically, we calculate the profile function \( F(r) \) of the soliton. Then all of the Skyrme model parameters are determined by the \( \chi \)-PT parameters and the integrations in-
volving the profile function.

It is known that, in the conventional Skyrme model, in which $\mathcal{O}_\gamma$ is the only symmetry breaking interaction, the physical values of the $\chi$PT parameters do not reproduce the baryon mass spectrum. In particular, the best fit value of the pion decay constant $F_\pi$ is typically less than one third of the experimentally determined value (e.g., we obtain a value of 46 MeV$^{30}$) in the case that the experimental value is 184.8 MeV), while the kaon mass (if it is treated as a parameter) becomes quite large (around 800 MeV).

From the effective theory point of view, this discrepancy can be understood easily. There are an infinite number of operators in the $\chi$PT action, and they contribute to the Skyrme model parameters. The conventional Skyrme model ignores all such contributions from the higher-order terms. Furthermore, we have no information regarding their coupling constants at all. In this paper, we therefore do not attempt to “calculate” the Skyrme model parameters from the $\chi$PT parameters, and instead fit them directly to the experimental values.

An important question is whether the best fit values of the $\chi$PT parameters “improve” as we systematically take increasingly higher-order contributions into account. In this appendix, we address this question. Starting with the action (2.9), we calculate the profile function and the Skyrme model parameters. By fitting them to the experimental values of the baryon masses, we obtain the best fit values for the $\chi$PT parameters.

### C.1. Profile function $F(r)$

First, it is useful to define the subtracted action $S_{\text{sub}}[U]$ as

$$
S_{\text{sub}}[U] \equiv S[U] - \frac{F_\pi^2 B_0}{8} \int d^4x \text{Tr} \left( \mathcal{M}^\dagger + \mathcal{M} \right) - L_8 B_0^2 \int d^4x \text{Tr} \left( \mathcal{M}^\dagger \mathcal{M}^\dagger + \mathcal{M} \mathcal{M} \right),
$$

so that $S_{\text{sub}}[U = 1] = 0$. The profile function $F(r)$ minimizes $M[F]$, defined by

$$
S_{\text{sub}}[U_c] = -M[F] \int dt,
$$

subject to the boundary conditions

$$
F(0) = \pi, \quad F(\infty) = 0.
$$

The minimum is called $M_{\text{cl}}$:

$$
M_{\text{cl}} = \min_{F(r)} M[F].
$$

\begin{table}
\centering
\caption{\( \langle 35 | \mathcal{O}_\gamma | 10 \rangle, \langle 64 | \mathcal{O}_\gamma | 10 \rangle \) \& \( \langle 81 | \mathcal{O}_\gamma | 10 \rangle \).
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( (I,Y) \) & \( v_1 \) & \( v_2 \) & \( v_1 \) & \( v_2 \) & \( v_1 \) & \( v_2 \) \\
\hline
\( (\frac{3}{2}, 1) \) & \( \frac{\sqrt{5}}{6 \sqrt{2}} \) & \( \frac{\sqrt{5}}{18 \sqrt{2}} \) & \( \frac{3 \sqrt{5}}{56 \sqrt{2}} \) & \( \frac{3 \sqrt{5}}{56 \sqrt{2}} \) & \( \frac{3 \sqrt{5}}{56 \sqrt{2}} \) & \( \frac{3 \sqrt{5}}{56 \sqrt{2}} \) \\
\( (1, 0) \) & \( \frac{\sqrt{3}}{18 \sqrt{2}} \) & \( \frac{\sqrt{3}}{18 \sqrt{2}} \) & \( \frac{3 \sqrt{3}}{84 \sqrt{2}} \) & \( \frac{3 \sqrt{3}}{84 \sqrt{2}} \) & \( \frac{3 \sqrt{3}}{84 \sqrt{2}} \) & \( \frac{3 \sqrt{3}}{84 \sqrt{2}} \) \\
\( (\frac{1}{2}, -1) \) & 0 & 0 & \( \frac{3}{56} \) & \( \frac{3}{56} \) & \( \frac{3}{56} \) & \( \frac{3}{56} \) \\
\( (0, -2) \) & 0 & 0 & 0 & 0 & \( \frac{3 \sqrt{5}}{112} \) & \( \frac{3 \sqrt{5}}{112} \) \\
\hline
\end{tabular}
\end{table}
The profile function satisfies the Euler-Lagrange equation
\[
\frac{d}{dr} \frac{\partial M}{\partial (dF/dr)} - \frac{\partial M}{\partial F} = 0. \quad (C.5)
\]
By substituting the solution \( F(r) \) into Eq. (C.4), we obtain \( M_{cl} \) as
\[
M_{cl} = \frac{4\pi F_0}{e} \int_0^\infty \tilde{r}^2 d\tilde{r} \left[ \frac{1}{8} \left\{ \left( F' \right)^2 + 2\sin^2 F \right\} + \frac{\sin^2 F}{\tilde{r}^2} \left\{ 2\left( F' \right)^2 + \frac{\sin^2 F}{\tilde{r}^2} \right\} + \frac{mB_0}{2F_0^2e^2} (1 - \cos F) + \frac{4mB_0L_5}{F_0^2} \cos F \left\{ \left( F' \right)^2 + \frac{2\sin^2 F}{\tilde{r}^2} \right\} + \frac{8m^2B_0^2L_8}{F_0^4e^2} \sin^2 F \right],
\]
where we have introduced \( \tilde{r} = F_0 er \) and \( F' = dF/d\tilde{r} \). Note that there are higher-order contributions, and they affect the behavior of the profile function, and hence the values of the other parameters. The existence of the higher-order interactions affects the values of the parameters in two ways, through the behavior of the profile function and through the new terms resulting from the higher-order interactions.

For a large value of \( r \), \( F(r) \) and its derivative behave as \( F \sim 0 \), \( F' \sim 0 \), implying \( \sin F \sim F \), \( \cos F \sim 1 \). Thus, we have
\[
\frac{\partial M}{\partial F'} \sim \frac{4\pi F_0}{e} \left( \frac{1}{4} + \frac{8mB_0L_5}{F_0^2} \right) \tilde{r}^2 F',
\]
\[
\frac{\partial M}{\partial F} \sim \frac{4\pi F_0}{e} \left( \frac{mB_0}{2F_0^2e^2} + \frac{16m^2B_0^2L_8}{F_0^4e^2} \right) \tilde{r}^2 F.
\]
By solving the Euler-Lagrange equation, the asymptotic behavior
\[
F(\tilde{r}) \sim e^{-\mu \tilde{r}} \quad (C.9)
\]
is obtained, where \( \mu \) is given by
\[
\mu = \sqrt{2mB_0} \left( \frac{1 + \frac{32mB_0L_8}{F_0^2}}{1 + \frac{32mB_0L_5}{F_0^2}} \right)^{1/2} \approx \sqrt{2mB_0} \left( 1 + \frac{16mB_0}{F_0^2} (L_8 - L_5) \right). \quad (C.10)
\]
Note that \( \mu \) is close to the pion mass but not exactly the same.

C.2. Inertia tensor \( I_{\alpha\beta}(A) \)

After substituting \( U = AU_c A^\dagger \) into \( S_{sub}[U] \), the inertia tensor can be easily read off as the coefficients of the terms quadratic in the “angular velocity” \( \omega_\alpha \):
\[
I_{\alpha\beta}(A) = -\frac{F_0^2}{16} \int d^3x \frac{1}{2} \text{Tr} \left[ \lambda_\alpha U_c^\dagger \lambda_\beta U_c + \lambda_\beta U_c^\dagger \lambda_\alpha U_c - \{\lambda_\alpha, \lambda_\beta\} \right]
\]
\[
+ \frac{1}{128 \varepsilon^2} \int d^3x \text{Tr} \left( [U_c^\dagger \lambda \alpha U_c - \lambda \alpha, U_c^\dagger \partial_i U_c] [U_c^\dagger \lambda \beta U_c - \lambda \beta, U_c^\dagger \partial_i U_c] \right) \\
- \frac{2L_5B_0}{3} \frac{2m + m_s}{3} \int d^3x (1 + 2 \cos F) \frac{1}{2} \text{Tr} \left[ \lambda \alpha U_c^\dagger \lambda \beta U_c + \lambda \beta U_c^\dagger \lambda \alpha U_c - \{ \lambda \alpha, \lambda \beta \} \right] \\
+ \lambda \beta U_c^\dagger \lambda \alpha U_c - \{ \lambda \alpha, \lambda \beta \} \right] \\
+ \frac{L_5B_0}{\sqrt{3}} \frac{2m + m_s}{3} \int d^3x \frac{1}{2} \text{Tr} \left[ (A^\dagger \lambda^8 A) \left[ (\lambda \alpha, U_c) U_c^\dagger [\lambda \beta, U_c] + [\lambda \beta, U_c^\dagger] U_c [\lambda \alpha, U_c^\dagger] \right] \right] \\
- \frac{L_5B_0}{\sqrt{3}} \frac{m_s - m}{2} \int d^3x \frac{1}{2} \text{Tr} \left[ (A^\dagger \lambda^8 A) \left[ (\lambda \alpha, U_c) U_c^\dagger [\lambda \beta, U_c] + [\lambda \beta, U_c^\dagger] U_c [\lambda \alpha, U_c^\dagger] \right] \right] \\
= I_{\alpha\beta}^0 + I_{\alpha\beta}'(A). \tag{C.11}
\]

The A-independent part, \( I_{\alpha\beta}^0 \), can be written as in Eq. (2.24), with

\[
I_1 = \frac{2\pi}{3e^3F_0} \int_0^\infty \tilde{r}^2 \tilde{d} \sin^2 F \left[ 1 + 4 \left\{ \left( F' \right)^2 + \frac{\sin^2 F}{\tilde{r}^2} \right\} + \frac{32\pi B_0}{F_0^3} L_5 \cos F \right], \tag{C.12}
\]

\[
I_2 = \frac{\pi}{2e^3F_0} \int_0^\infty \tilde{r}^2 \tilde{d} (1 - \cos F) \left[ 1 + \left( F' \right)^2 + \frac{2 \sin^2 F}{\tilde{r}^2} \right] + \frac{8\pi B_0}{F_0^2} L_5 (1 + \cos F), \tag{C.13}
\]

where

\[
\overline{m} = \frac{2m + m_s}{3} \tag{C.14}
\]

has been introduced. The A-dependent part, \( I_{\alpha\beta}'(A) \), has the complicated structure given in Eq. (2.25). The parameters are now expressed as

\[
\overline{x} = -\frac{64\pi L_5B_0}{9e^3F_0^3} \delta m \int_0^\infty \tilde{r}^2 \tilde{d} \cos F \sin^2 F, \tag{C.15}
\]

\[
\overline{y} = -\frac{32\pi L_5B_0}{3\sqrt{3}e^3F_0^3} \delta m \int_0^\infty \tilde{r}^2 \tilde{d} \sin^2 F, \tag{C.16}
\]

\[
\overline{z} = \frac{32\pi L_5B_0}{9e^3F_0^3} \delta m \int_0^\infty \tilde{r}^2 \tilde{d} (1 - \cos F)(2 - \cos F), \tag{C.17}
\]

\[
\overline{w} = \frac{16\pi L_5B_0}{3\sqrt{3}e^3F_0^3} \delta m \int_0^\infty \tilde{r}^2 \tilde{d} (1 - \cos F)(2 - \cos F). \tag{C.18}
\]

C.3. Potential

The “angular velocity” independent part (excluding \( M_d \)) of the Lagrangian is the potential part given in Eq. (2.27), \(-V(A)\). The parameter \( \gamma \) in the first-order term (2.28) is given by

\[
\gamma = \gamma_0 + \delta_1 \gamma + \delta_2 \gamma, \tag{C.19}
\]
with
\begin{align}
\gamma_0 &= \delta m \frac{4\pi B_0}{3e^3 F_0} \int_0^\infty \tilde{r}^2 d\tilde{r} (1 - \cos F), \\
\delta_1 \gamma &= \delta m \frac{32\pi B_0 L_5}{3e F_0} \int_0^\infty \tilde{r}^2 d\tilde{r} \cos F \left( (F')^2 + \frac{2\sin^2 F}{\tilde{r}^2} \right), \\
\delta_2 \gamma &= \delta m \frac{64\pi m^2 B_0^2 L_8}{3e^3 F_0^3} \int_0^\infty \tilde{r}^2 d\tilde{r} (1 - \cos 2F).
\end{align}

The parameter $v$ in the second-order term (2.29) is given by
\begin{equation}
v = (\delta m)^2 \frac{16\pi B_0^2 L_8}{9e^3 F_0^3} \int_0^\infty \tilde{r}^2 d\tilde{r} (1 - \cos F) (1 - 2\cos F).
\end{equation}

C.4. Numerical calculations

Starting with the $\chi$PT parameters $F_0$, $B_0$, $e$, $L_5$, and $L_8$ and the quark masses, $m$ and $m_s$, we can first calculate $F(r)$ and then, using this, the Skyrme model parameters, $M_{cl}$, $I_1$, $I_2$, $\gamma$, $x$, $y$, $z$, $w$ and $v$. Once these parameters are determined, the baryon masses can be easily calculated to second order in the perturbation theory. In order to best fit the $\chi$PT parameters, we need to solve in a reverse manner. The procedure is similar to that discussed in §4, but it is a bit more complicated. In order to simplify the calculation, we make the following conditions.

1. The quark masses are fixed. As a reference, we adopt the values
\begin{equation}
m = 6 \text{ MeV}, \quad m_s = 150 \text{ MeV}.
\end{equation}

Actually, these only fix the ratio, because a change in the magnitudes of both by the same factor can be absorbed into $B_0$. (Note that $\mathcal{M}$ appears only in the combination $B_0 \mathcal{M}$.) The ratio is more precisely determined experimentally, and we have\(^{5,66}\)
\begin{equation}
\frac{m_s}{(m_u + m_d)/2} = \frac{2(m_s/m_d)}{1 + (m_u/m_d)} \approx 25.8,
\end{equation}
which is close to the value $150/6 = 25$.

2. The values of $L_5$ and $L_8$ are fixed. When we vary these parameters too, we find that the numerical calculation becomes very unstable. In reality, the entire formulation given in this paper assumes these parameters to be small. In searching for the “valley” numerically, this assumption is often ignored, and we believe that it is the reason for the instability. Instead, we fix these parameters to be the central values determined experimentally,\(^{41}\)
\begin{equation}
L_5 = 1.4 \times 10^{-3}, \quad L_8 = 0.9 \times 10^{-3}.
\end{equation}

There is another important point. As Yabu and Ando\(^{45}\) discussed, there is a kind of “zero-point energy” contribution common to all of the calculated baryon masses. This contribution can be calculated as the symmetry breaking effects on the
Table XXXIV. Baryon masses for the best fit values in the conventional approach. The results listed in the first and second rows are those with and without the higher-order contributions, respectively. The “zero-point energy” contribution has been subtracted.

| Baryon | $M_N$ (MeV) | $M_{\Sigma}$ (MeV) | $M_{\Xi}$ (MeV) | $M_{\Lambda}$ (MeV) | $M_{\Delta}$ (MeV) | $M_{\Sigma^*}$ (MeV) | $M_{\Omega}$ (MeV) | $M_{\Theta}$ (MeV) | $M_\phi$ (MeV) |
|--------|-------------|------------------|----------------|------------------|------------------|---------------------|------------------|------------------|------------------|
| Mass, (MeV) | 915 | 1287 | 1411 | 1116 | 1358 | 1518 | 1666 | 1563 | 1965 |
| Mass, $w/o$, (MeV) | 898 | 1269 | 1405 | 1116 | 1118 | 1321 | 1501 | 1639 | 1948 |

fictitious (unphysical) singlet baryon mass,

$$M_{\text{vac}} = \frac{\gamma}{2} \left(1 \left|1 - D^{(8)}_8(A)\right| 1\right) + v \left(1 \left|1 - \sum_{\alpha \in I} \left(D^{(8)}_{8\alpha}(A)\right)^2 - \left(D^{(8)}_8(A)\right)^2\right| 1\right)$$

$$- \frac{1}{M_8 - M_1} \left|\left<- 8 \left|\left(-\frac{\gamma}{2} D^{(8)}_8(A)\right)\right| 1\right>^2\right|$$

$$= \frac{\gamma}{2} + \frac{v}{2} - \frac{\gamma^2 I_2}{48},$$

(C.27)

where we have introduced

$$M_1 = M_{\text{cl}} + \frac{3}{8} \left[\frac{1}{I_1} - \frac{2}{I_2}\right].$$

(C.28)

We subtract this quantity from all of the calculated masses. Note that, in the effective theory approach, this contribution is renormalized into the parameter $M_{\text{cl}}$, and, therefore, it needs not be considered separately, as it has been implicitly taken into account.

Our numerical results are

$$F_0 = 82.7 \text{ MeV}, \quad B_0 = 2697, \quad e = 4.51,$$

(C.29)

which lead to the values

$$F_\pi = 91.4 \text{ MeV}, \quad M_\pi = 185.3 \text{ MeV}, \quad M_K = 867.2 \text{ MeV}$$

(C.30)

for the physical parameters, and the baryon masses for these values are given in Table XXXIV. These values should be compared with those calculated with $L_5 = L_8 = 0$, that is, those without contributions from higher-order terms,

$$F_0 = F_\pi = 58.2 \text{ MeV}, \quad B_0 = 7825, \quad e = 4.06,$$

(C.31)

which lead to

$$M_\pi = 306.4 \text{ MeV}, \quad M_K = 1104.9 \text{ MeV}.$$

(C.32)

The baryon masses are also given in Table XXXIV.

It is interesting that the values of the physical parameters shift in the correct direction. Even though these values are still far from the experimental values, we think that this is an explicit demonstration that our basic strategy is correct.
Table XXXV. The matrix elements of $O_{\text{DPP}}$, $\langle R_i | O_{\text{DPP}} | R_j \rangle$, for the spin doublet states, which are abbreviated as $(R_i, R_j)$.

| $(I, Y)$ | $(8, 8)$ | $(8, 10)$ | $(8, 27d)$ | $(10, 10)$ | $(10, 27d)$ | $(10, 35d)$ |
|----------|---------|---------|---------|---------|---------|---------|
| $(0, +2)$ | 0 | 0 | $-\frac{\sqrt{3}}{2}$ | 0 | $\frac{\sqrt{3}}{2}$ |
| $(1, +1)$ | $-\frac{\sqrt{3}}{20}$ | $\frac{\sqrt{3}}{20}$ | $-\frac{1}{10\sqrt{2}}$ | $-\frac{7}{16\sqrt{10}}$ | $-\frac{16\sqrt{14}}{16\sqrt{14}}$ |
| $(1, 0)$ | $-\frac{3\sqrt{3}}{40}$ | $\frac{3\sqrt{3}}{40}$ | $-\frac{1}{10\sqrt{2}}$ | 0 | $-\frac{7}{8\sqrt{15}}$ | $\frac{8\sqrt{7}}{8\sqrt{7}}$ |
| $(1, -1)$ | $\frac{\sqrt{3}}{20}$ | 0 | $-\frac{1}{10\sqrt{2}}$ | 0 | 0 | 0 |
| $(\frac{3}{2}, -1)$ | 0 | 0 | $\frac{3\sqrt{3}}{20}$ | $-\frac{7}{10\sqrt{2}}$ | $\sqrt{14}$ |
| $(0, 0)$ | $\frac{3\sqrt{3}}{20}$ | 0 | $-\frac{\sqrt{3}}{20}$ | 0 | 0 | 0 |

---

**Symmetry Breaking Interactions in the Chiral Quark-Soliton Model**

In this appendix, we perform a similar “best fit” analysis with the symmetry breaking terms (1.4) which appear in the $\chi$QSM. In the derivation of these terms, they are related to the $\pi$-N $\sigma$ term, soliton moments of the inertia, and so on, but we ignore this fact and just treat the couplings as free parameters. The reason is that this makes comparison with our approach transparent and reveals how the $\chi$QSM predictions depend on the detailed form of the parameters. Note that the values of the parameters given in Refs.9)14) are not obtained through the best fit procedure but in a rather arbitrary way.

**D.1. Best fit to the baryon masses**

To obtain the masses at second order in the perturbation theory, we need the matrix elements of $H_{\text{DPP}}$. The matrix elements of $D_{88}^{(8)}$ are given in §3.2 and in Appendix B, and those for $Y$ are trivial. Only the matrix elements of $O_{\text{DPP}} \equiv D_{8i}^{(8)} F^i$ need to be calculated. We first present the matrix elements for the spin $J = \frac{1}{2}$ states in Tables XXXV and XXXVI. Those for the spin $J = \frac{3}{2}$ states are presented in Tables XXXVII and XXXVIII. Most of them have never been given in the literature. (Incidentally, the matrix elements diagonal in the representation are given by formulae similar to (A.40) with

\[
O_{\text{DPP}}^{(1)} = -J^2, \quad (D.1)
\]

\[
O_{\text{DPP}}^{(2)} = -\left(\frac{1}{3}C_3 + \frac{1}{4}C_2 + \frac{1}{4}J^2 + \frac{3}{16}\right) \quad (D.2)
\]

See §A.3 for an explanation of the notation.)

By using these matrix elements, we can calculate the baryon masses, and by best fitting the calculated values to the observed ones, we can determine the parameters $\alpha$, $\beta$, and $\gamma$ as well as $M_d$, $I_1$, and $I_2$. The procedure is the same as that employed in §4, and therefore we do not explain it again. The best fit set of parameters is

\[
M_d = 837 \text{ MeV}, \quad I_1^{-1} = 163 \text{ MeV}, \quad I_2^{-1} = 394 \text{ MeV},
\]

\[
\alpha = -554 \text{ MeV}, \quad \beta = -40.9 \text{ MeV}, \quad \gamma = 42.0 \text{ MeV}, \quad (D.3)
\]
Table XXXVI. The same as Table XXXV.

\[
\begin{array}{cccccccc}
(I,Y) & (27_d, 27_d) & (27_d, 35_d) & (27_d, 64_d) & (35_d, 35_d) & (35_d, 64_d) & (35_d, 81_d) \\
(0, +2) & 0 & 0 & 0 & -\frac{\sqrt{3}}{8} & 0 & -\frac{3\sqrt{3}}{2} \\
(\frac{1}{2}, +1) & 7\sqrt{3} & \frac{5\sqrt{7}}{112} & -\frac{5}{8} & -\frac{3\sqrt{3}}{32} & \frac{\sqrt{3}}{8\sqrt{7}} & -\frac{3}{8} \\
(1, 0) & 1\sqrt{3} & \frac{5\sqrt{7}}{16} & -\frac{5\sqrt{7}}{16\sqrt{7}} & -\frac{3\sqrt{3}}{16} & \frac{\sqrt{3}}{8\sqrt{7}} & -\frac{3\sqrt{3}}{16} \\
(\frac{1}{2}, -1) & 7\sqrt{3} & 0 & -\frac{5}{8} & 0 & 0 & 0 \\
(\frac{3}{2}, -1) & \frac{5\sqrt{7}}{224} & \frac{3\sqrt{7}}{128} & -\frac{5\sqrt{7}}{32\sqrt{7}} & -\frac{3\sqrt{3}}{32} & \frac{\sqrt{3}}{8\sqrt{7}} & -\frac{3\sqrt{3}}{32} \\
(0, 0) & 1\sqrt{3} & 0 & -\frac{\sqrt{3}}{8\sqrt{7}} & 0 & 0 & 0 \\
\end{array}
\]

Table XXXVII. The same as Table XXXVI.

\[
\begin{array}{cccccccc}
(I,Y) & (10, 10) & (10, 27_q) & (10, 35) & (27_q, 27_q) & (27_q, 35) \\
(\frac{3}{2}, +1) & -\frac{5\sqrt{3}}{10} & \frac{5\sqrt{7}}{16\sqrt{7}} & -\frac{5\sqrt{3}}{16\sqrt{7}} & -\frac{\sqrt{3}}{10} & -\frac{3\sqrt{3}}{4} \\
(1, 0) & 0 & \frac{5}{8\sqrt{7}} & -\frac{\sqrt{3}}{8\sqrt{7}} & \frac{3\sqrt{3}}{16\sqrt{7}} & \frac{\sqrt{3}}{8\sqrt{7}} & -\frac{3\sqrt{3}}{16\sqrt{7}} \\
(\frac{1}{2}, -1) & \frac{5\sqrt{3}}{16} & \frac{5\sqrt{7}}{16} & -\frac{5\sqrt{3}}{16} & \frac{3\sqrt{3}}{16} & \frac{\sqrt{3}}{8\sqrt{7}} & -\frac{3\sqrt{3}}{16} \\
(0, -2) & \frac{5\sqrt{3}}{8} & 0 & -\frac{\sqrt{3}}{8\sqrt{7}} & 0 & 0 & 0 \\
\end{array}
\]

Table XXXVIII. The same as Table XXXVII.

\[
\begin{array}{cccccccc}
(I,Y) & (27_q, 35) & (27_q, 64_q) & (35, 35) & (35, 64_q) & (35, 81) \\
(\frac{3}{2}, +1) & \frac{\sqrt{3}}{2\sqrt{7}} & -\frac{\sqrt{3}}{112} & \frac{\sqrt{3}}{32} & 1\sqrt{3} & -\frac{3\sqrt{3}}{2} & -\frac{3\sqrt{3}}{2} \\
(1, 0) & \frac{\sqrt{3}}{4\sqrt{21}} & -\frac{\sqrt{3}}{112\sqrt{7}} & \frac{\sqrt{3}}{32\sqrt{7}} & 1\sqrt{3} & -\frac{3\sqrt{3}}{2} & -\frac{3\sqrt{3}}{2} \\
(\frac{1}{2}, -1) & 0 & -\frac{\sqrt{3}}{56\sqrt{7}} & \frac{5\sqrt{3}}{32\sqrt{7}} & 1\sqrt{3} & -\frac{3\sqrt{3}}{2} & -\frac{3\sqrt{3}}{2} \\
(0, -2) & 0 & 0 & \frac{\sqrt{3}}{2\sqrt{7}} & 0 & \frac{3\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} \\
\end{array}
\]

Table XXXIX. Baryon masses calculated using the best fit set of parameters (D.3) with the χQSM Hamiltonian.

\[
\begin{array}{cccccccccc}
(M_{NN'}) & N & \Sigma & \Xi & \Lambda & \Delta & \Sigma^+ & \Xi^+ & \Omega & \Theta & \phi \\
942 & 1206 & 1335 & 1116 & 1227 & 1383 & 1531 & 1672 & 1538 & 1868 \\
\end{array}
\]

which leads to the masses given in Table XXXIX, with $\chi^2 = 6.4 \times 10^1$.

Considering that number of parameters is small, this fit is very good. It is also noteworthy that the parameters have expected magnitudes. The best fit values yield the following prediction for the masses of the other members of pentaquarks:

\[
M_{NN'} = 1668 \text{ MeV}, \quad M_{\Sigma N'} = 1777 \text{ MeV}.
\]

These results are not very different from those in (4.4).

D.2. Decays

Next we turn to the calculation of the decay widths. Again, the procedure is the same as that in §5. Actually, the necessary matrix elements are the same. The only difference comes from the mixings. The mixing coefficients that correspond to our results in Tables XI, XII and XIII are given in Tables XL, XLI and XLII.

One can easily see that the mixings are much smaller for the χQSM breaking
Table XL. Mixing coefficients for the (mainly) octet states with the $\chi$QSM symmetry breaking terms. These quantities correspond to those given in Table XI.

| $\mathcal{R}_i$ | $N$ | $\Sigma$ | $\Xi$ | $\Lambda$ |
|-----------------|-----|----------|-------|--------|
| 8               | 1 (−0.030) | 1 (−0.027) | 1 (−0.010) | 1 (−0.015) |
| $\overline{10}$ | 0.202 (−0.023) | 0.202 (0.030) | 0 (0) | 0 (0) |
| $27_d$          | 0.140 (0.003) | 0.114 (0.022) | 0.140 (0.020) | 0.171 (0.008) |
| $35_d$          | 0 (0.022) | 0 (0.021) | 0 (0) | 0 (0) |
| $64_d$          | 0 (0.010) | 0 (0.008) | 0 (0.010) | 0 (0.014) |

Table XLI. Mixing coefficients for the (mainly) decuplet states with the $\chi$QSM symmetry breaking terms. These quantities correspond to those given in Table XII.

| $\mathcal{R}_i$ | $\Delta$ | $\Sigma^*$ | $\Xi^*$ | $\Omega$ |
|-----------------|-----------|------------|--------|--------|
| 10              | 1 (−0.105) | 1 (−0.060) | 1 (−0.026) | 1 (−0.005) |
| $27_q$          | 0.451 (0.005) | 0.329 (0.007) | 0.202 (0.006) | 0 (0) |
| 35              | 0.081 (0.022) | 0.103 (0.023) | 0.109 (0.019) | 0.103 (0.013) |
| $35_q$          | 0 (0.058) | 0 (0.027) | 0 (0) | 0 (0) |
| $64_q$          | 0 (0.026) | 0 (0.025) | 0 (0.017) | 0 (0) |
| 81              | 0 (0.004) | 0 (0.006) | 0 (0.007) | 0 (0.006) |

Table XLII. Mixing coefficients for the (mainly) anti-decuplet states with the $\chi$QSM symmetry breaking terms. These quantities correspond to those given in Table XIII.

| $\mathcal{R}_i$ | $\Theta$ | $N'$ | $\Sigma'$ | $\phi$ |
|-----------------|----------|-----|----------|-------|
| 8               | 0 (0) | −0.202 (−0.009) | −0.202 (−0.049) | 0 (0) |
| $\overline{10}$ | 1 (−0.009) | 1 (−0.036) | 1 (−0.044) | 1 (−0.033) |
| $27_d$          | 0 (0) | 0.105 (−0.037) | 0.171 (−0.029) | 0.234 (0.005) |
| $35_d$          | 0.131 (0) | 0.139 (0.008) | 0.131 (0.016) | 0.104 (0.019) |
| $64_d$          | 0 (0) | 0 (0.012) | 0 (0.018) | 0 (0.019) |
| $81_d$          | 0 (0.009) | 0 (0.010) | 0 (0.009) | 0 (0.006) |

Table XLIII. Coefficients of the decay operators for the $\chi$QSM symmetry breaking terms. The notation here is the same as in Table XV.

| | (a) | (a') | (b) | (b') |
|-----------------|-----|------|-----|------|
| $G_0$           | 5.33 | 4.11 | 3.38 | 3.95 |
| $G_1$           | 9.98 | 13.7 | 21.4 | 17.5 |
| $\chi^2$       | 43.1 | 14.6 | 23.9 | 36.0 |
| $G_2$           | 0.08 | 0.06 | 0.04 | 0.05 |

terms. From the quality of the fit to the masses and the smallness of the mixings, it is reasonable to conclude that the perturbative treatment of the symmetry breaking terms is justified for this model.

With these mixing coefficients, the decay widths are readily calculated using our width formula (5.16). First, we present the best fitted values of $G_0$, $G_1$, and $G_2$ in Table XLIII. These values lead to the decay widths for the (mainly) decuplet baryons given in Table XLIV.

Finally, we obtain the various decay widths for the (mainly) anti-decuplet baryons, which are listed in Table XLV.

Despite the good perturbative behavior in the baryon mass fitting, the decay widths vary considerably. In most cases, the second-order results are very different.
Table XLIV. Decay widths for the (mainly) decuplet baryons with the coupling constants given in Table XLIII.

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| (MeV) | $K$   | $\Gamma_{(a)}$ | $\Gamma_{(a')}$ | $\Gamma_{(b)}$ | $\Gamma_{(b')}$ |
| $\Delta \rightarrow N\pi$ | 1.47 | 90.2 | 106 | 105 | 95.8 |
| $\Sigma^* \rightarrow \Lambda\pi$ | 1.18 | 34.1 | 33.1 | 32.9 | 33.5 |
| $\Sigma^* \rightarrow \Sigma\pi$ | 0.26 | 4.69 | 5.85 | 6.63 | 6.19 |
| $\Xi^* \rightarrow \Xi\pi$ | 0.49 | 12.9 | 12.7 | 13.8 | 13.7 |

Table XLV. Predictions for the decay widths of the (mainly) anti-decuplet baryons with the $\chi$QSM symmetry breaking terms.

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| (MeV) | $K$   | $\Gamma_{(a)}$ | $\Gamma_{(a')}$ | $\Gamma_{(b)}$ | $\Gamma_{(b')}$ |
| $\Theta^+ \rightarrow NK$ | 1.91 | 63.0 | 250 | 727 | 429 |
| $N' \rightarrow N\pi$ | 18.2 | 210 | 669 | 1793 | 1030 |
| $N' \rightarrow N\eta$ | 4.82 | -0.30 | 26.2 | 104 | 56.9 |
| $N \rightarrow \Delta\pi$ | 5.69 | 0 | 132 | 242 | 259 |
| $N \rightarrow \Lambda K$ | 0.86 | 2.19 | 14.7 | 48.1 | 26.7 |
| $N' \rightarrow \Sigma K$ | — | — | — | — | — |
| $\Sigma' \rightarrow NK$ | 12.5 | -21.5 | 8.47 | 102 | 36.8 |
| $\Sigma' \rightarrow \Sigma\pi$ | 12.2 | 101 | 346 | 1175 | 676 |
| $\Sigma' \rightarrow \Sigma\eta$ | 0.50 | 0.40 | 4.63 | 14.1 | 7.79 |
| $\Sigma' \rightarrow \Lambda\pi$ | 16.0 | 93.0 | 328 | 1030 | 619 |
| $\Sigma' \rightarrow \Xi K$ | — | — | — | — | — |
| $\Sigma' \rightarrow \Sigma^*\pi$ | 4.40 | 0 | 4.01 | 15.0 | 5.37 |
| $\Sigma' \rightarrow \Delta K$ | 0.74 | 0 | 13.8 | 20.9 | 17.1 |
| $\phi \rightarrow \Sigma K$ | 5.11 | -17.7 | 25.1 | 5.83 | 35.8 |
| $\phi \rightarrow \Xi\pi$ | 10.8 | 72.7 | 328 | 1095 | 614 |
| $\phi \rightarrow \Xi^*\pi$ | 2.68 | 0 | 0.91 | 5.37 | 2.32 |

from the first-order results. Even though the perturbation theory seems to be effective, some negative values appear when we expand the amplitudes. In particular, the width of $\Theta^+$ is predicted to be much larger than the experimental values.

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