Criticality in a Hadron Resonance Gas model with the van der Waals interaction

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The van der Waals interaction is implemented in a Hadron Resonance Gas model. It is shown that this model can describe Lattice QCD data of different thermodynamical quantities satisfactorily with the van der Waals parameters $a = 1250 \pm 150$ MeV fm$^3$ and $r = 0.7 \pm 0.05$ fm. Further, a liquid-gas phase transition is observed in this model with the critical point at temperature, $T = 62.1$ MeV and baryon chemical potential, $\mu_B = 708$ MeV.

PACS numbers: 25.75.-q, 25.75.Nq, 12.38.Mh, 24.10.Pa

Keywords: Hadron Resonance Gas model, QCD phase diagram

I. INTRODUCTION

Lattice quantum chromo dynamics (LQCD) provides a first principle approach to study strongly interacting matter at zero chemical potential ($\mu_B$) and finite temperature ($T$). LQCD calculations indicate a smooth cross over transition from hadronic to a quark-gluon plasma (QGP) phase at zero baryon chemical potential and finite temperature. On the other hand, at high baryon chemical potential and low temperature the nuclear matter is expected to have a first-order phase transition.

The nuclear matter shows the critical point at temperature, and baryon chemical potential, for all the baryons in VDWHRG model LQCD data can be described qualitatively in the cross over region. The motivation of the present work is to carry out the reverse prescription, that is to find out van der Waals parameters $a$ and $b$ that gives the best description of LQCD data at zero chemical potential using VDWHRG model and then extend this work to the finite chemical potential and try to locate the existence of a critical point in the QCD phase diagram.

The paper is organized as follows. In the Sec. II we describe the ideal HRG as well as VDWHRG model. In Sec. III we present our results. Finally in the Sec. IV we summarize our findings for this work.

II. MODEL DESCRIPTION

There are varieties of HRG models which exist in the literature. Different versions of this model and some of the recent works using these models may be found in Refs. [17, 33, 43–74]. Some of the HRG models are non-interacting and some of them consider interaction among the particles. Next we will briefly discuss the non-interacting HRG model and the HRG model with van der Waals type interaction.

In the ideal HRG model, the thermal system consists of non-interacting point like hadrons and resonances. The logarithm of the partition function of a hadron resonance gas in the grand canonical ensemble can be written as

$$\ln Z^{id} = \sum_i \ln z_i^{id},$$

where the sum is over all the hadrons and resonances. $id$ refers to ideal i.e., non-interacting HRG model. For particle species
\[ i, \]

\[ \ln Z_i^{vdw} = \pm \frac{V g_i}{2\pi^2} \int_0^\infty p^2 \, dp \ln[1 \pm \exp(-(E_i - \mu_i)/T)], \quad (2) \]

where \( V \) is the volume of the thermal system, \( g \) is the degeneracy, \( E = \sqrt{p^2 + m^2} \) is the single particle energy, \( m \) is the mass of the particle and \( \mu_i = B_i \mu_B + S_i \mu_S + Q_i \mu_Q \) is the chemical potential. In the last expression, \( B_i, S_i, Q_i \) are respectively the baryon number, strangeness and electric charge of the particle, \( \mu_B \) are the corresponding chemical potentials. The upper and lower sign of \( \pm \) corresponds to fermions and bosons, respectively. We have incorporated all the hadrons and resonances listed in the particle data book up to a mass of 3 GeV [75]. The pressure \( p^{vdw} \), the energy density \( \varepsilon^{vdw} \) and the number density \( n^{vdw} \) of the thermal system are given by the following equations,

\[ p^{vdw} = \sum_i \left( \pm \frac{g_i T}{2\pi^2} \int_0^\infty p^2 \, dp \ln[1 \pm \exp(-(E_i - \mu_i)/T)] \right), \quad (3) \]

\[ \varepsilon^{vdw} = \sum_i \frac{g_i T}{2\pi^2} \int_0^\infty p^2 \, dp \frac{\exp[(E_i - \mu_i)/T]}{1 \pm 1}, \quad (4) \]

\[ n^{vdw} = \sum_i \frac{g_i T}{2\pi^2} \int_0^\infty p^2 \, dp \exp[(E_i - \mu_i)/T] \pm 1. \quad (5) \]

Once we know the partition function or the pressure of the system we can calculate other thermodynamic quantities.

The van der Waals equation in the canonical ensemble is given by [76]

\[ p = \left( \frac{N}{V} \right)^2 + \frac{a}{N} \left( \frac{N}{V} \right)^2 (V - Nb) = NT, \quad (6) \]

where \( p \) is the pressure of the system, \( V \) is the volume, \( T \) is the temperature, \( N \) is the number of particles and \( a, b \) are respectively the van der Waals parameters. The parameters \( a \) and \( b \) describe the attractive and repulsive interaction respectively. The Eq. [6] can be written as

\[ p(T, n) = \frac{NT}{V - bN} - a \left( \frac{N}{V} \right)^2 = \frac{nT}{1 - bn} - an^2, \quad (7) \]

where \( n = N/V \) is the number density of particles. The first term in the right hand side of the Eq. [7] corresponds to the excluded volume correction where the system volume is replaced by the available volume \( V_{av} = V - bN \), where \( b = \frac{3\pi}{20} r^3 \) is the proper volume of particles with \( r \) being corresponding hard sphere radius of the particle. The second term in Eq. [7] corresponds to the attractive interaction between particles. The importance of van der Waals equation is that this analytical model can describe first order liquid-gas phase transition of a real gas which ends at the critical point. Such a feature is also an expectation for the QCD phase diagram.

The van der Waals equation of state in the Grand canonical ensemble can be written as [34, 35]

\[ p(T, \mu) = p^{vdw}(T, \mu^*) - an^2, \quad \mu^* = \mu - bp(T, \mu) - abn^2 + 2an, \quad (8) \]

where \( n = n(T, \mu) \) is the particle number density of the van der Waals gas.

\[ n = n(T, \mu) \equiv \left( \frac{\partial p}{\partial \mu} \right)_T = \frac{n^{vdw}(T, \mu^*)}{1 + bn^{vdw}(T, \mu^*)}. \quad (9) \]

The entropy density \( s \) for van der Waals gas can be written as

\[ s(T, \mu) \equiv \left( \frac{\partial p}{\partial T} \right)_\mu = \frac{s^{vdw}(T, \mu^*)}{1 + bn^{vdw}(T, \mu^*)}. \quad (10) \]

Further, the energy density can be calculated as

\[ \varepsilon(T, \mu) = Ts + \mu n - p, \quad (11) \]

and is given by

\[ \varepsilon(T, \mu) = \frac{\varepsilon^{vdw}(T, \mu^*)}{1 + bn^{vdw}(T, \mu^*)} - an^2. \quad (12) \]

For a single component nuclear matter (\( g = 4, m = 938 \text{ MeV} \)) the values of van der Waals parameters were obtained as \( a = 329 \text{ MeV fm}^3 \) and \( b = 3.42 \text{ fm}^3 \) \( (r = 0.59 \text{ fm}) [35] \) from the properties of the ground state of the nuclear matter.

For a hadronic system, we assume that interaction exist between all pair of baryons and all pair of antibaryons. We ignore the interaction for mesons in order to avoid divergence of their number densities when modified chemical potentials become close to the masses of the particles. The baryon-antibaryon interaction is also ignored because the short range repulsive interaction between baryon and antibaryon may be dominated by the annihilation processes [17]. These are the limitations of the current model and leaves the scope for further improvement in future. Hence the pressure of VDWHRG model can be written as [38]

\[ p(T, \mu) = p_M(T, \mu) + p_B(T, \mu) + p_{B^*}(T, \mu), \quad (13) \]

with

\[ p_M(T, \mu) = \sum_{k \in M} p^{vdw}_k(T, \mu_k), \quad (14) \]

\[ p_B(T, \mu) = \sum_{k \in B} p^{vdw}_k(T, \mu_k^*) - an^2, \quad (15) \]

and

\[ p_{B^*}(T, \mu) = \sum_{k \in \bar{B}} p^{vdw}_k(T, \mu_k^*) - an^2, \quad (16) \]

where \( M, B, \bar{B} \) stand for mesons, baryons and antibaryons respectively. The modified chemical potential for baryons and antibaryons are given by

\[ \mu_k^{B(\bar{B})} = \mu_k - bp_{B(\bar{B})} - abn^2_{B(\bar{B})} + 2an_{B(\bar{B})}. \quad (17) \]
where \( n_B \) and \( n_k \) are particle number densities of baryons and antibaryons respectively. Once we know the pressure of the system, we can calculate different thermodynamic quantities. The derivative of \( p(\beta_B) \) with respect to the baryon chemical potential will give us the corresponding number densities:

\[
\frac{d n_B^{\beta_B}}{d \beta_B} = \sum_{\beta_B \in \beta_B} \frac{n_B^{\beta_B}(T, \mu_B^{\beta_B})}{1 + b \sum_{\beta_B \in \beta_B} n_B^{\beta_B}(T, \mu_B^{\beta_B})} 
\]

From pressure, we can calculate entropy density, energy density using the Eqs. 19, 21. Further one can calculate specific heat at constant volume as

\[
C_V = \left( \frac{\partial \epsilon}{\partial T} \right)_V
\]

and the susceptibilities of conserved charges as

\[
\lambda_{B_{SQ}} = \frac{\partial \epsilon^{(y+y^2)}(p/T^4)}{\partial (\mu_B/T)^2 \partial (\mu_S/T)^2 \partial (\mu_Q/T)^2}
\]

In the VDWHRG model if we put \( a = 0 \) and \( b = 0 \) we will get the results of the ideal HRG model. While with \( a \) is in VDWHRG model it corresponds to Excluded Volume HRG (EVHRG) model \([17, 52]\), where only repulsive interaction is included. Both ideal HRG model and EVHRG model do not show any kind of phase transition. Still these models are quite successful in describing LQCD data of the bulk properties of hadronic matter in thermal and chemical equilibrium \([3–5, 15–18, 60, 61]\). This model is also successful in describing the ratios of hadron yields, at chemical freeze-out, created in central heavy ion collisions from SIS up to LHC energies \([43–44, 54–57, 62–64]\). The heavy ion collisions at RHIC and LHC have established quark-hadron phase transition.

III. RESULTS

In order to extract the van der Waals parameters \( a \) and \( b \) in VDWHRG model that best describe the LQCD data at \( \mu_B = 0 \), we use \( \chi^2 \) minimization technique where \( \chi^2 \) is defined as

\[
\chi^2 = \frac{1}{N_a} \sum_{i,j} \frac{(R_{i,j}^{LQCD}(T_j) - R_{i,j}^{model}(T_j))^2}{\lambda_{i,j}^{LQCD}(T_j)^2}
\]

where \( R_{i,j}^{model}(T_j) \) is the \( i \)th observable with \( R_{i,j}^{LQCD}(T_j) \) and \( \lambda_{i,j}^{LQCD}(T_j) \) are its values and errors respectively at \( j \)th temperature calculated in LQCD, \( N_a \) is the number of LQCD data points. Here, we assume that van der Waals parameters \( a \) and \( b \) are independent of temperature and chemical potential. Errors on the parameters are obtained by knowing their values at \( \chi^2_{min} + 1 \). In this paper we use the latest continuum limit LQCD data \([2, 16]\) of \( p/T^4, \epsilon/T^4, s/T^3, C_V/T^3 \) and \( \chi_2^2 \) at \( \mu = 0 \) within the temperature range 130 – 180 MeV to calculate \( \chi^2 \) using Eq. 21. We assume that HRG model is valid up to \( T = 180 \) MeV because the transition at \( \mu_B = 0 \) is a crossover. Hence thermodynamic observables do not exhibit sharp changes. LQCD results of quantities like \( p/T^4, \epsilon/T^4 \) and \( s/T^3 \) have the smooth crossover at temperature range up to 180 MeV \([5]\). Depending on the choice of order parameter the QCD crossover temperature \( T_c \) has a range of from 155 MeV to 175 MeV. For example LQCD calculation with chiral condensate gives \( T_c = 155 \) MeV \([77]\). However if one chooses strange quark number susceptibility the \( T_c \sim 170 \) MeV \([78]\). Typical error including systematics and due to the choice of order parameter on is \( T_c \sim 20 \) MeV. Lowest temperature is taken as \( T = 130 \) MeV since the LQCD data of susceptibilities are not available below \( T = 130 \) MeV in Ref. [16]. The best fit in terms of \( \chi^2 \) is achieved for parameter values of \( a = 1250 \pm 150 \) MeV fm\(^3\) and \( r = 0.7 \pm 0.05 \) fm. Relatively smaller parameter values, \( a = 329 \) MeV fm\(^3\) and \( r = 0.59 \) fm, were obtained by \([35]\). With these parameters only a qualitative description of LQCD data at \( \mu = 0 \) is possible \([68]\) which we have already stated. In some previous works hardcore radius has been estimated in the EVHRG model. In Ref. \([23]\) hardcore radii of pion and other hadrons were obtained as 0.62 fm and 0.8 fm respectively by fitting the experimental data of hadronic ratios at AGS and SPS energies. While the value of the hardcore radius was estimated as 0.3 fm in the Ref \([56]\) using the experimental data of hadronic ratios at SPS energies. Also in Refs. \([17, 18]\) it was shown that the LQCD data of different thermodynamic quantities can be described in EVHRG model with the radius parameter between 0.2 – 0.3 fm. Our present estimate of radius parameter is comparable to that of Ref. \([22]\). However, it should be noted that in all those works \([17, 18, 22, 54]\) only repulsive interaction was considered for all mesons and baryons and there was no attractive interaction. To check the sensitivity of value of the parameters on the temperature range we have refitted the LQCD data up to \( T = 165 \) MeV (typical chemical freeze-out temperature from RHIC top energy) and found the new \( a \) value to be 1210 MeV fm\(^3\) which is within the uncertainty of the \( a \) value obtained by fitting up to \( T = 180 \) MeV, i.e., 1250 MeV fm\(^3\) ± 150 MeV fm\(^3\). There is no change in the value of \( r \) parameter.

Figure 1 shows variation of \( p/T^4, \epsilon/T^4, \epsilon - 3p/T^4, \epsilon^2 \) and \( C_V/T^3, \chi_2^2 \) with temperature at \( \mu = 0 \). Blue lines show the results of VDWHRG model using the parameters \( a = 1250 \) MeV fm\(^3\) and \( r = 0.7 \) fm. The bands are due to the errors on the parameters \( a \) and \( r \). Results of ideal HRG model along with the LQCD data of the Wuppertal-Budapest (WB) Collaboration \([2]\) and the Hot QCD Collaboration \([3]\) are also shown in this figure. Our estimations of all these observables in the VDWHRG model are in good agreement with LQCD calculations in the temperature range studied. Compared to ideal HRG model, improvement of the results in VDWHRG model is observed which indicates the interacting nature of baryons especially at high temperature region. Among all these observables, behavior of \( \epsilon^2 \) is most interesting in VDWHRG model. The \( \epsilon^2 \) is a quantity that is sensitive to the phase transition effect. While in ideal HRG model \( \epsilon^2 \) decreases with increasing temperature, in VDWHRG model it shows a minimum near \( T = 150 \) MeV which is consistent with the LQCD data. The minimum of the \( \epsilon^2 \) is known as the softest point where the expansion of the system slows down.
FIG. 1: (Color online) The variation of different thermodynamical quantities with the temperature at $\mu = 0$. Blue lines show the results of VDWHRG model using the parameters $a = 1250$ MeV fm$^3$ and $r = 0.7$ fm. Blue bands are due to the errors on the van der Waals parameters in the VDWHRG model. The continuum extrapolated LQCD data are taken from Refs. [2] (WB) and [3] (HotQCD).

As a result the system spends a longer time in this temperature range which may be a crucial indicator of the quark-hadron transition of the system observed in heavy ion collisions [3, 79].

In Fig. 2 temperature dependence of second order fluctuations of different conserved charges at zero chemical potential have been shown. One can see that the qualitative behaviors of all these fluctuations in VDWHRG model are similar to the LQCD data at high temperature which are different from ideal HRG model where all the quantities increase rapidly with increasing temperature. Not only that, the $\chi^2_B$ and $\chi^2_S$ obtained from the VDWHRG model match quantitatively with the LQCD data. However, for $\chi^2_Q$, which is dominated by the non-interacting mesons, hence VDWHRG model overestimates the LQCD data.

Figure 3 shows correlations among conserved charges. Magnitudes of the $\chi^{11}_{BS}$ and $\chi^{11}_{QS}$ increase with increasing temperature and at a very high temperature they are expected to reach at $1/3$, the value at Stefan-Boltzmann limit. $\chi^{11}_{BS}$ and $\chi^{11}_{QS}$ calculated in VDWHRG model are close to the LQCD data in
Since we have used van der Waals interactions for simplicity, errors on the critical point is due to the uncertainties on the van der Waals parameters in the VDWHRG model. The LQCD data are taken from Refs. [5, 16].

The parameters \( a = 1250 \) MeV fm\(^3\) and \( r = 0.7 \) fm. Blue bands are due to the errors on the van der Waals parameters in the VDWHRG model. Figure 4 shows variation of pressure with number density at a fixed temperature in VDWHRG model. The black dot indicates the critical point.

**FIG. 3:** (Color online) The variation of correlations between conserved charges with the temperature at zero chemical potential. Blue lines show the results of VDWHRG model using the parameters \( a = 1250 \) MeV fm\(^3\) and \( r = 0.7 \) fm. Blue bands are due to the errors on the van der Waals parameters in the VDWHRG model. The LQCD data are taken from Refs. [5, 16].

**FIG. 4:** (Color online) The variation of pressure with the number density of the hadronic medium at different temperature. The black dot indicates the critical point.

![Graph](image)

The HRG model does not have QGP phase but with attractive and repulsive interaction for baryons VDWHRG model explains LQCD data which have QGP phase. So if interactions are the sole driving force behind the physics of phase transitions one expects a similar phase transition effect in VDWHRG model. Figure 4 shows variation of pressure with number density at a fixed temperature in VDWHRG model. The parameters \( a \) and \( r \) are fixed from the best fit values of the VDWHRG model to LQCD data at \( \mu_B = 0 \). For simplicity, we assume nature of the interaction is similar to both non-zero and zero \( \mu_B \) regions of the phase diagram. We observe the value of critical temperature to be \( T = 62.1 \) MeV. Below this temperature the number density changes discontinuously which resembles a hadron-liquid first order phase transition. The picture will be more clear in Fig. 5 where we show variations of \( (\partial p/\partial n)_T \) with respect to \( \mu_B \) and \( n \) respectively. One can see that at \( T = 62.1 \) MeV and \( \mu_B = 708 \) MeV, \( (\partial p/\partial n) \) becomes zero and above \( T = 62.1 \) MeV, \( (\partial p/\partial n) \) is always greater than zero. Since we have used van der Waals interaction it is expected that the phase transition which we observed is a liquid-gas phase transition and the critical point \( (T = 62.1^{\pm25.4}_{-19.1} \) MeV, \( \mu_B = 708^{\pm90}_{-146} \) MeV) so obtained is that of a liquid-gas transition. Errors on the critical point is due to the uncertainties on the parameters \( a \) and \( r \). A similar result of critical point with \( T = 89 \) MeV and \( \mu_B = 724 \) MeV is also obtained by using the holographic gauge/gravity correspondence to map baryon number fluctuations in QCD to the charge fluctuations of holographic black holes [84].

In the Fig. 6 we have plotted a collection of \( (T, \mu_B) \) points to make a comparison of (i) liquid-gas CP from our present analysis (ii) CP from LQCD (iii) chemical freeze-out parameters from heavy ion collision experiments. Blue circular point in Fig. 6 shows the critical point \( (T = 62.1^{\pm25.4}_{-19.1} \) MeV, \( \mu_B = 708^{\pm90}_{-146} \) MeV) of the liquid-gas transition as estimated within the current model calculations in the QCD phase diagram. Critical points calculated in lattice [81, 82] and the chemical freeze-out parameters obtained by different groups [83, 83] at various energies are also shown in this plot.

**IV. SUMMARY**

To summarize, we have used LQCD data of \( p/T^4, \varepsilon/T^4, s/T^3, C_v/T^3 \) and \( \chi_B^2 \) at \( \mu = 0 \) to extract the van der Waals parameters in the VDWHRG model. We assume that baryons are interacting whereas mesons are non-interacting. We get \( a = 1250 \pm 150 \) MeV fm\(^3\) and \( r = 0.7 \pm 0.05 \) fm in our present work which best describes the LQCD data at \( \mu = 0 \) within the temperature range 130 – 180 MeV. The values of the VDWHRG model...
parameters are obtained using a chi-square minimization procedure. With these parameters which explains the QCD matter simulated by lattice, we observe a phase transition in VDWHRG model at large potential with a critical point in the \((T, \mu_B)\) phase diagram at \(T = 62.1\) MeV and \(\mu_B = 708\) MeV. Our result of critical point is comparable with that of Ref. [80] where the critical point is obtained by using the holographic gauge/gravity correspondence to map baryon number fluctuations in QCD to the charge fluctuations of holographic black holes. Several improvements in the future can be carried out to our present idea and work. One of them includes incorporating the mesonic interaction of the system. Another is to incorporate other missing resonances in the hadronic spectrum [84].

Acknowledgements

We are thankful to Sourendu Gupta, Victor Roy and Nu Xu for carefully reading the manuscript and for their valuable comments. We thank Paolo Alba for suggesting useful references related to the work. BM acknowledges financial sup-

Fig. 5: (Color online) Variations of \((\partial p/\partial n)_T\) with respect to \(\mu_B\) and \(n\) respectively.

Fig. 6: (Color online) The critical point (CP) of the liquid-gas transition of the present work in the QCD phase diagram. Critical point calculated in lattice are taken from Ref. [81] (Fodor et al.) and Ref. [82] (Datta et al.). Chemical freeze-out (CFO) parameters shown in this figure are taken from Ref. [63] (Andronic et al.) and Ref. [83] (Cleymans et al.).

[1] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, Nature 443, 675 (2006).
[2] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg and K. K. Szabo, Phys. Lett. B 730, 99 (2014).
[3] A. Bazavov et al. [HotQCD Collaboration], Phys. Rev. D 90, 094503 (2014).
[4] S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti and K. Szabo, JHEP 1201, 138 (2012).
[5] A. Bazavov et al. [HotQCD Collaboration], Phys. Rev. D 86, 034509 (2012).
[6] S. Gupta, X. Luo, B. Mohanty, H. G. Ritter and N. Xu, Science 332, 1525 (2011).
[7] M. Asakawa and K. Yazaki, Nucl. Phys. A 504, 668 (1989).
[8] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. Lett. 81, 4816 (1998).
[9] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. D 60, 114028 (1999).
[10] B. I. Abelev et al. [STAR Collaboration], Phys. Rev. C 81, 024911 (2010).
[11] L. Adamczyk et al. [STAR Collaboration], Phys. Rev. Lett. 112, 032302 (2014).
[12] G. Agakishiev et al. [HADES Collaboration], Eur. Phys. J. A 52, 178 (2016).
[13] T. Ablyazimov et al. [CBM Collaboration], Eur. Phys. J. A 53,
[84] A. Bazavov et al., Phys. Rev. Lett. 113, 072001 (2014).