Delayed and Predictive Dynamics with Stochastic Time

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Abstract

We present simple classical dynamical models to address the question of introducing a stochastic nature in a time variable. These models include noise in the time variable but not in the “space” variable, which is opposite to the normal description of stochastic dynamics. The notable feature is that these models can induce a “resonance” with varying noise strengths in the time variable. Thus, they provide a different mechanism for “stochastic resonance”, which has been discussed within the normal context of stochastic dynamics. In a broader context, we expect these simple models and associated behaviours may serve as one of the starting points to construct and investigate an extended classical dynamical theory where both the space and time variables are taken as stochastic. We need to explore if such approaches can be further developed for both classical and quantum applications.

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INTRODUCTION

“Time” is a concept that has gained a lot of attention from thinkers from virtually all disciplines [1]. In particular, our ordinary perception of space and time is not the same, and this gap has been appearing in a variety of contemplations of nature. It appears to be the main reason for the theory of relativity, which has conceptually brought space and time closer to receiving equal treatment, continues to fascinate and attract arguments from diverse fields. Also, issues such as “directions” or ”arrows” of time are current interests of research [2]. Another manifestation of this gap is the treatment of noise or fluctuations in dealing with dynamical systems. When we consider dynamical systems, whether classical, quantum, or relativistic, time is commonly viewed as not having stochastic characteristics. In stochastic dynamical theories, we associate noise and fluctuations with only “space” variables, such as the position of a particle, but not to the time variables. In quantum mechanics, the concept of time fluctuation is well accepted through the time-energy uncertainty principle. However, time is not treated as a dynamical quantum observable, and a clearer understanding has been explored [3].

Against these backgrounds, our main theme of this paper is to consider fluctuation of time in classical dynamics through a presentation of simple models. We consider two cases. The first type includes fluctuations for the time point of events to affect the dynamical change. The second type is such that the fluctuation is in the flow of time variable itself. Both types exhibit behaviors which are similar to stochastic resonance.

STOCHASTICITY IN TIME OF EVENTS

The general differential equation form of the class of dynamics we discuss here is given as follows.

\[
\frac{dx(t)}{dt} = f(x(t), \bar{t}).
\]  

(1)

Here, \(x\) is the dynamical variable given as a function of time \(t\), and \(f\) is the “dynamical function” governing the dynamics. The difference from the normal dynamical equation appears in \(\bar{t}\), which contains stochastic characteristics, and \(t \neq \bar{t}\) in general. In other words, the change of \(x(t)\) is governed by a function \(f\), not from its ”current” state \(x(t)\), but its state at \(\bar{t}\), which is given by some stochastic rules. We can define \(\bar{t}\) in a variety of ways, as well as
the function $f$ in normal dynamical equations. In the following, we will present two models and investigate their properties through computer simulations. In order to avoid ambiguity and for simplicity, these models are dynamical map systems incorporating the basic ideas of the general definition given above. The first model we start with is a special case, which is given as follows.

$$x_{n+1} = x_n + r \cos(\bar{t}_n),$$  
$$(2)$$

$$\bar{t}_n = \Delta t(n + \sigma \xi).$$  
$$(3)$$

Here, $x_n$ is the dynamical variable and $\bar{t}_n$ is the time containing noise term $\xi$. The dynamics progress by incrementing $n$ with constant parameters, $r$, $\Delta t$, and $\sigma$. We fix the noise $\xi$ to take a value between $(-1, 1)$ with a uniform probability. This is a special case as the dynamical function $f = r \cos(t)$ depends only on time and not on $x_n$. When $\sigma = 0$, it recovers ordinary dynamics with a sinusoidal path. We have performed computer simulations for this dynamical map with different sets of constants. The examples of the time series and associated power spectrum are shown in Fig. 1. The most notable characteristics are as we change the noise “strength” or “width” given by $\sigma$, a rhythmic behavior appears and disappears. This is shown by the change in the peaks of the power spectrums, an example of which we plotted in Figure 1f. The spacing of the peaks appearing in Figure 1f is one half the period of the dynamical function, indicating $\sigma$ is a source for producing in and out of phase dynamics. At the same time, we note that the role of noise is important, as when we set $\xi$ to be a constant rather than a random variable, the resulting dynamics is a simple sinusoidal time series and a change in the peak height is not observed. The periodical oscillatory dynamics emerging by “tuning of noise” has been studied under the name of “stochastic resonance” [4, 5, 6] and investigated in a variety of fields [7, 8, 9, 10, 11]. We observe here a similar behavior with stochasticity in time rather than in space variable.

Our second model is given as follows, which again shows a resonance-type behavior with a tuning of noise.

$$x_{n+1} = (1 - b)x_n + c\theta[\bar{x}(\bar{t}_n)],$$  
$$(4)$$

$$\bar{t}_n = \phi(n + \sigma \xi).$$  
$$(5)$$

Here, $b$ and $c$ are the parameters, and $\phi(z)$ is a function that gives the closest integer to $z$. 
FIG. 1: Dynamics and power spectrum of sinusoidal function model with stochastic event time. This is an example of the dynamics and associated power spectrum through the simulation of the model given in Eq. (2). The parameters are set as $r = 1, \Delta t = 0.004\pi$, and the noise widths $\sigma$ are set to (a) $\sigma = 0$, (b) $\sigma = 125$, (c) $\sigma = 250$, (d) $\sigma = 375$, and (e) $\sigma = 500$. The simulation is performed up to $L = 10000$ steps and 20 averages are taken for the power spectrum. The unit of frequency $\lambda$ is set as $\frac{1}{L}$, and the power $P(\lambda)$ is in arbitrary units. Graph (f) is the peak height $P^*$ as a function of the noise width $\sigma$. 
FIG. 2: Dynamical functions $f(x)$ with parameters as examples of simulations presented in this paper. (a) Threshold function. The Straight line has slope of $b = 0.08$. (b) Negative feedback function with parameters $\beta = 6$. Straight line has slope of $\alpha = 0.5$.

In this equation, $\xi$ is the same noise term as in the first model and its width is controlled by $\sigma$, and $\theta(x)$ is a “threshold function” (Figure 2(a)) such that,

$$
\theta(x) = \begin{cases} 
-1 & x > 0 \\
0 & x = 0 \\
1 & x < 0 
\end{cases}
$$

In this model, we further need to define $\bar{x}(\bar{t}_n)$, which is given as follows.

$$
\bar{x}(\bar{t}_n) = \begin{cases} 
x_0 & \bar{t}_n \leq 0 \\
x_{\bar{t}_n} & 0 < \bar{t}_n \leq n \\
x_n + (\bar{t}_n - n)(x_n - x_{n-1}) & n < \bar{t}_n 
\end{cases}
$$

The meaning of this definition is that when $\bar{t}_n$ is in the past ($\bar{t}_n \leq n$), the value of $x$ at that past point is used. One the other hand, if $\bar{t}_n$ is in the future ($n < \bar{t}_n$), we estimate $x$ as the value that would be obtained if the same rate of current change extends to the time duration from the present to the future point. Qualitatively, this method of projecting into the future is one of the most commonly used for estimating population and national debt, for instance. The same estimation is used for the recent study in “predictive dynamical systems” [12].

The computer simulation of this model is performed with various parameter sets, examples of which are shown in Fig. 3. Without noise ($\sigma = 0$), the basic dynamics are decaying to the origin, $x = 0$. With the introduction of noise with an increasing width, we begin to observe more rhythmic behaviors. This is seen with the peaks of the power spectrum. The signal to noise ratio, calculated as the ratio of the peak to the background height level of the power spectrum, is the main characterization of the stochastic resonance.
FIG. 3: Dynamics and power spectrum of threshold function model with stochastic event time. This is an example of the dynamics and associated power spectrum through the simulation of the model given in Eq. (3). The parameters are set as $b = 0.08$, $c = 1.0$, and the noise widths $\sigma$ are set to (a) $\sigma = 10$, (b) $\sigma = 30$, (c) $\sigma = 70$, (d) $\sigma = 120$, and (e) $\sigma = 200$. The simulation is performed up to $L = 10000$ steps 20 averages are taken for the power spectrum. The unit of frequency $\lambda$ is set as $\frac{1}{L}$, and the power $P(\lambda)$ is in arbitrary units. Graph (f) is the signal to noise ratio $\frac{S}{N}$ at the peak height as a function of the noise width $\sigma$. 
If we plot this as a function of the noise strength, we can obtain the resonance curve shown in Figure 3f. This indicates that this signal to noise ratio goes through maximum at the optimal noise level with other parameters fixed. Therefore, once again, the time fluctuation is acting in a constructive way.

We would now like to discuss a couple of points with these models. First of all, it should be noted that, in these models, the basic flow and/or “arrow” of time is the same as in ordinary dynamical equations. In other words, the rate of change of the dynamical variable $x_n$ is dictated by stochastically chosen point $\bar{t}_n$ on the time axis, but the ordering of $x_n$ is kept by $n$. From another point of view, these models can be considered a stochastic extension of the combined delayed\cite{13, 14, 15, 16, 17} and predictive\cite{12} dynamical systems. Thus, the stochastic nature of time is only partially implemented here.

**STOCHASTICITY IN THE TIME FLOW**

We now turn our attention to the second type of models, in which the noise is in the time flow itself. There are several ways to achieve this effect. The discussion here is limited with a stochastic time flow combined with a delayed dynamics. The general form of the dynamical equation is given as follows.

$$\frac{dx(\bar{t})}{d\bar{t}} = f(x(t), x(t - \tau)), \quad (8)$$

where $\bar{t}$ is again a stochastic variable. It should be noted that we now have a stochasticity in time flow. In particular, we focus on the following corresponding map.

$$x_{n+\xi} = f(x_n, x(n - \tau)). \quad (9)$$

Here, $\xi$ is the stochastic variable which can take either +1 or −1 with certain probabilities. Thus even though the dynamics progress by iteration of $n$, it occasionally “goes back in time” with the occurrence of $\xi = -1$. Let the probability of $\xi = -1$ be given by $p$. Then naturally, with $p = 0$, this map reduces to a normal delayed map. The dynamical function is chosen to be a negative feedback function (Figure 2(b)) and the concrete map model we study is given as follows.

$$x_{n+\xi} = x_n + d\delta(-\alpha x_n - \frac{2}{1 + e^{-\beta x_{n-\tau}}} - 1). \quad (10)$$
When there is no noise in time flow, this map has the origin as the stable fixed point with no delay. The linear stability analysis gives the critical delay \( \tau_c \), at which the stability of the fixed point is lost, as \( 0.59 \sim \tau_c d\delta \). The larger delay gives an oscillatory dynamical path. We have found, through computer simulations, that an interesting behavior arises when delay is smaller than this critical delay. The tuned noise in the time flow gives the system more tendency for oscillatory behavior. In other words, adjusting the value of \( p \) controlling \( \xi \) induces oscillatory behaviors. Some examples are shown in Figure 4. With increasing probability for a time flow to reverse, i.e., with \( p \) increasing, we observe oscillatory behaviors as shown both in the sample dynamical path as well as in the corresponding power spectrum. However, with \( p \) increased enough, the oscillatory behaviors begins to deteriorate. The change of the peak heights is shown in Figure 5. Again, we see a phenomena which resembles stochastic resonance. A resonance with delay and noise, called “delayed stochastic resonance” [21], has been proposed with an additive noise in “space”. Analytical understanding of the mechanism of our model here is yet to be explored. However, this mechanism with stochastic time flow is clearly of different type and new.

**DISCUSSION**

We would like to now discuss a couple of points with respect to these models. First, we can extend these models to include fluctuations in the variable \( x_n \). Proceeding in this way, we have a picture of dynamical systems with fluctuations on both the time and space axes. The analytical frameworks and tools for such descriptions need to be developed along with a search for appropriate applications. For an example of appropriate applications, we may think of modeling a stick balancing on a human fingertip. Recent experiments have found that most of the corrective motion observed has shorter time scales compared to the human reaction time [18, 19, 20]. This may be the result of intricate mixtures of physiological delays, predictions, and physical fluctuations. Models incorporating special fluctuations and delays have been considered, but bringing in the effect from both the past and future through the stochastic time may help to further develop it.

Another direction may be to extend the path integral formalisms along this approach. Whether this type of extension bridges into quantum pictures and/or leads to an alternative understanding of such properties as time-energy uncertainty relations requires further
FIG. 4: Dynamics (left) and power spectrum (middle) of delayed dynamical model with stochastic time flow (right). This is an example of the dynamics and associated power spectrum through the simulation of the model given in Eq. (10) with the probability $p$ of stochastic time flow varied. The parameters are set as $\alpha = 0.5$, $\beta = 6$, $d\delta = 0.01$, $\tau = 25$ and the stochastic time flow parameter $p$ are set to (a) $p = 0$, (b) $p = 0.1$, (c) $p = 0.2$, (d) $p = 0.3$, and (e) $p = 0.45$. The simulation is performed up to $L = 1000$ steps and 20 averages are taken for the power spectrum. The unit of frequency $\lambda$ is set as $\frac{1}{\tau}$, and the power $P(\lambda)$ is in arbitrary units.

Finally, if these models are capturing some aspects of reality, particularly with respect to stochasticity in time flow, this resonance may be used as an experimental indication for probing fluctuations or stochasticity in time. We have previously proposed “delayed stochastic resonance”[21], a resonance by the interplay of noise and delay. It was theoretically
FIG. 5: The signal to noise ratio $\frac{S}{N}$ at the peak height as a function of the probability $p$ of stochastic time flow. The parameter settings are the same as in the Figure 4 with the delay varied as (a) $\tau = 40$, (b) $\tau = 25$ and (c) $\tau = 15$.

extended[22], and recently the effect was experimentally observed through a solid-sate laser system with a feedback loop[23]. It is left for the future to see if an analogous experimental test could be developed with respect to stochastic time.

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