Worst-Case Welfare of Item Pricing in the Tollbooth Problem

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ABSTRACT

We study the worst-case welfare of item pricing in the tollbooth problem. The problem was first introduced by Guruswami et al. [27], and is a special case of the combinatorial auction in which (i) each of the m items in the auction is an edge of some underlying graph; and (ii) each of the n buyers is single-minded and only interested in buying all edges of a single path. We consider the competitive ratio between the hindsight optimal welfare and the optimal worst-case welfare among all item-pricing mechanisms, when the order of the arriving buyers is adversarial. We assume that buyers own the tie-breaking power, i.e. they can choose whether or not to buy the demand path at 0 utility. We prove a tight competitive ratio of 3/2 when the underlying graph is a single path (also known as the highway problem), whereas item-pricing can achieve the hindsight optimal if the seller is allowed to choose a proper tie-breaking rule to maximize the welfare [6, 11]. Moreover, we prove an O(1) upper bound of competitive ratio when the underlying graph is a tree.

For general graphs, we prove an Ω(m^{1/2}) lower bound of the competitive ratio. We show that an m^{O(1)} competitive ratio is unavoidable even if the graph is a grid, or if the capacity of every edge is augmented by a constant factor c. The results hold even if the seller has tie-breaking power.

CCS CONCEPTS
• Theory of computation → Algorithmic mechanism design.

KEYWORDS
mechanism design, item pricing, tollbooth problem, welfare maximization, tie-breaking rule

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1 INTRODUCTION

In an item-pricing mechanism, each item is given an individual price upfront. Buyers come to the auction sequentially and choose the favorite bundle (among the remaining items) that maximizes their own utilities. For a single-minded buyer, she will purchase her demand set if and only if all items in the set are available, and the total price of the items is at most her value. The performance of an item-pricing mechanism is measured by its worst-case welfare, that is, the minimum sum of buyers’ value achieved by the mechanism when the buyers arrive in any adversarial order. In this setting, the Walrasian equilibrium [1, 15] for gross-substitutes buyers provides a set of prices as well as a welfare-maximizing allocation, such that every buyer receives her favorite bundle [32], which allows us to maximize the welfare using an item-pricing mechanism. However, such an equilibrium (or a set of prices) may not exist when the buyers are single-minded.

In this paper we focus on a special case of the above problem known as the tollbooth problem [27], where the set of commodities and demands are represented by edges and paths in a graph. This setting has found various real-world applications. For instance, a network service provider wants to sell bandwidth along with the links of a network by pricing on every single link, and each customer is only interested in buying a specific path in the network. In this setting, every item in the auction is an edge in some graph G representing the network, and every buyer is single-minded and is only interested in buying all edges of a specific path of G. Both the buyer’s demanding path and value are known to the seller. We additionally impose the restriction that each commodity (edge) may be given to at most one buyer, which means that if an edge was taken by some previous buyer, then buyers who come afterward whose demand set contains this edge may no longer take it.

Similar to the general case, Walrasian equilibrium is not guaranteed to exist for the tollbooth problem. Moreover, Chen and Rudra [10] proved that the problem of determining the existence of a Walrasian equilibrium and the problem of computing such an equilibrium (if it exists) are both NP-hard for the tollbooth problems on general graphs. Therefore, an investigation of the power and limits of item pricing is a natural next step towards a deeper understanding of the tollbooth problem. For an item-pricing mechanism, its competitive ratio is the ratio between the hindsight optimal welfare, i.e. the optimal welfare achieved by any feasible allocation of items to buyers, and the worst-case welfare of the item-pricing mechanism. In the paper, we study the best (smallest) competitive ratio among all item-pricing mechanisms in a given instance of the tollbooth problem.

In addition to the set of prices, a key factor that can significantly affect the welfare of an item-pricing mechanism is the tie-breaking rule. For example, if Walrasian equilibrium exists, item pricing can achieve the optimal welfare, but requires carefully breaking ties among all the favorite bundles of every buyer. However, in real markets, buyers often come to the mechanism themselves and simply purchase an arbitrary favorite bundle that maximizes their own utility. Therefore it is possible that the absence of tie-breaking...
power may influence the welfare achieved by the mechanism. In the paper, we assume that buyers own the tie-breaking power, i.e. they can choose whether or not to buy the demand path at 0 utility.

1.1 Our Results and Techniques

**Competitive Ratio for Path Graphs.** If the seller is allowed to allocate the edges via a proper tie-breaking rule, there indeed exists an item pricing that achieves the offline optimal welfare when the underlying graph \( G \) is a single path \([6, 11]\). Interestingly, we show that this result does not hold when the seller has no tie-breaking power. We present an instance where \( G \) is a single path, that no item pricing can achieve more than \(2/3\)-fraction of the offline optimal welfare (Theorem 3.2). On the other hand, we prove that such a \(3/2\)-approximation is achievable via item pricing (Theorem 3.3).

**Result 1.** The competitive ratio for any tollbooth problem instance on a single path is at most \(3/2\), if buyers own the tie-breaking power. Moreover, the ratio is tight.

The lower bound is achieved by an example with 3 edges and 4 buyers (Table 1). The upper bound result is more involved. The proof is enabled by constructing three sets of edge-disjoint paths from all buyers’ demand paths such that: (i) every edge in the graph is contained in exactly two paths of the three sets; (ii) each set of paths \( Q \) satisfies a special property called uniqueness, which intuitively means that there does not exist another set of paths among the rest paths, whose union is the same as the one of \( Q \). We prove that given any unique set of paths \( Q \), we can design prices to serve all buyers whose demand path is in \( Q \) for any buyers’ arrival order. With this lemma, we can design prices that achieve at least \(2/3\) of the offline optimal, by picking one of the three sets and aiming to serve all buyers whose demand path is in the set.

**Competitive Ratio for Trees.** We also study the case when \( G \) is a tree. When seller owns the tie-breaking power, we show a tight competitive ratio of \(\frac{2}{3}\) (Theorem 3.4). The upper bound is proved by combining Lemma 3.1 with the integrality gap result of multicommodity flow problem on tree \([8, 38]\). On the other hand, we provide an instance on a star to show the competitive ratio is at least \(3/2\).

When the seller has no tie-breaking power, we prove that the competitive ratio of any tree instance is also upper bounded by an absolute constant.

**Result 2.** For any \(\epsilon > 0\), the competitive ratio for any tollbooth problem instance on a tree is at most \(7 + \epsilon\), if buyers own the tie-breaking power.

To prove the result, we start by analyzing the competitive ratio for a special class of graphs called spider, which is obtained by replacing each edge in a star graph with a single path. Then given an offline optimal allocation (which corresponds to a set \( P \) of demand paths) in a tree instance, we partition the paths of \( P \) into two subsets \( P = P_1 \cup P_2 \), such that for each \( t \in \{1, 2\} \), the graph obtained by taking the union of all paths in \( P_t \) is a union of node-disjoint spider graphs. Thus the task of computing the prices on the edges of a tree is reduced to that of a spider, while losing a factor of 2 in the competitive ratio.

**Competitive Ratio for General Graphs.** Next we study the tollbooth problem for general graphs. For general single-minded combinatorial auctions, where the demand of every agent is an arbitrary set rather than a single path, the competitive ratio between item pricing and the offline optimal is proved to be \(O(\sqrt{m})\) \([11, 12]\), which is tight up to a constant \([23]\). For our problem, we first show that the competitive ratio on general graphs can also be polynomial in the number of its edges, in contrast to the constant competitive ratio for path graphs and trees. This polynomially large ratio is unavoidable even if the graph is a grid and if the seller owns the tie-breaking power. On the other hand, we prove an upper bound of \(O(m^{0.4} \log^2 m \log n)\) on the competitive ratio in any tollbooth problem instance (Theorem 4.3). When \( n \), the number of buyers in the auction, is subexponential on \( m \), our competitive ratio is better than the previous ratio \(O(\sqrt{m})\) for general single-minded combinatorial auctions.

**Result 3.** There exists a tollbooth problem instance such that the competitive ratio is \(\Omega(m^{1/8})\). Moreover, there exist a constant \(\alpha \in (0, 1)\) and an instance on a grid such that the competitive ratio is \(\Omega(m^\alpha)\). Both results hold even if the seller owns the tie-breaking power. On the other hand, the competitive ratio for any tollbooth problem instance is \(O(m^{0.4} \log^2 m \log n)\). Here \( m \) is the number of edges and \( n \) is the number of buyers.

The hard instance for the \(\Omega(m^{1/8})\) competitive ratio is constructed on a simple series-parallel graph (see Figure 1). In the hard instance, every buyer demands a path connecting the left-most vertex to the right-most vertex, and the value for each demand path is roughly the same. The demand paths are constructed carefully, such that (i) each edge of the graph is contained in approximately the same number of demand paths; and (ii) the demand paths are intersecting in some delicate way. We can then show that, for any price vector \( p \), if we denote by \( P \) the set of affordable (under \( p \)) demand paths, then either the maximum cardinality of an independent subset of \( P \) is small, or there is a path in \( P \) that intersects all other paths in \( P \). Either way, we can conclude that the optimal worst-case welfare achieved by any item-pricing mechanism is small.

![Figure 1: A series-parallel graph with large competitive ratio.](image)

**Resource augmentation.** At last, we study the problem where the seller has more resource to allocate. More specifically, comparing to the offline optimal allocation with supply 1 for each item (denoted as \(OPT\)), the seller has augmented resources and is allowed to sell \( c \) copies of each item to the buyers. In the literature studying offline path allocation problems on graphs (e.g. \([13]\)) and previous work using the techniques of resource augmentation (e.g. \([5, 31, 34, 41, 42]\)), even slightly increased resources usually improves the competitive ratio significantly. However in our problem, we prove that for any constant \( c > 1 \), there exists an instance such that the competitive ratio with augmented resources is \( m^{\Omega(1/c)} \), even we allow different item prices for different copies (Theorem 5.3). In other words, a
competitive ratio that is polynomial in $m$ is unavoidable in the tollbooth problem, even if the capacity of each edge is augmented by a constant. We also prove an almost-maximizing upper bound of $O(m^{1/3})$ in this setting (Theorem 5.1). The upper bound holds for any single-minded welfare maximization problem, where each buyer may demand any set of items instead of edges in a path.

**Result 4.** For any constant integer $c > 0$, consider the tollbooth problem where each edge has $c$ copies to sell. There is an instance, such that for any set of prices $\{p^k(i) | i \in \{m\}, 1 \leq k \leq c\}$ where $p^k(i)$ represents the price for the $k$-th copies of edge $i$, the item-pricing mechanism with above prices achieves worst-case welfare an $O(m^{-1/(2c+6)})$-fraction of OPT. On the other hand, there exists an item pricing that achieves worst-case welfare at least an $\Omega(m^{-1/c})$-fraction of OPT.

### 1.2 Other Related Work

**Profit maximization for the tollbooth problem.** The tollbooth problem has been extensively studied in the literature. One line of work [21, 25–27] aims to efficiently compute prices of items as well as a special subset of buyers called winners while maximizing the total profit, such that it is feasible to allocate the demand sets to all winners, and every winner can afford her bundle. There are two major differences between all works above and our setting. Firstly, the seller owns the tie-breaking power in the above works. Secondly and more importantly, in all works above, it is only required that the set of buyers who get their demand sets can afford their demand sets. But there might be other buyers who could afford their demand sets as well but eventually did not get them (or equivalently, not selected as winners). Since the arriving order is adversarial in our setting, these buyers might come before the winners and take their demand sets. The winners may no longer get their demand sets since some items in their demand sets are already taken. Therefore, the set of prices computed in the works above may not end up achieving the worst-case welfare equal to the total value of all winners. It is not hard to see that our item-pricing mechanisms are stronger than the settings in the works above: If a set of prices has a competitive ratio $\alpha$ in our setting, then such a set of prices is automatically an $\alpha$-approximation in the setting of the works above, but the converse is not true.

Moreover, the tollbooth problem on star graphs is similar to the graph pricing problem (where prices are given to vertices, and each buyer takes an edge) studied by Balcan and Blum [2], while they considered the unlimited supply setting. They obtained a 4-approximation, which was later shown to be tight by Lee [36] unless the Unique Games Conjecture is false. For the multiple and limited supply case, Friggstad et al. [24] obtained an 8-approximation.

**Walrasian equilibrium for single-minded buyers.** A closely related problem of our setting is the problem of finding market-clearing item prices for single-minded buyers. Unlike the resource allocation setting, a Walrasian equilibrium requires every buyer with a positive utility to be allocated. The existence of the Walrasian equilibrium is proven to be NP-hard, while satisfying $\frac{2}{3}$ of the buyers is possible [9, 10, 17, 30]. The hardness of the problem extends to selling paths on graphs, and is efficiently solvable when the underlying graph is a tree [10].

**Profit maximization for single-minded buyers.** For the general profit maximization for single-minded buyers with unlimited supply, Guruswami et al. [27] proved an $O(\log n + \log m)$-approximation. The result was improved to an $O(\log B + \log \ell)$-approximation ratio by Bries et al. [4], and then to an $O(\log B)$-approximation by Cheung and Dhamelincourt [11]. Here $B$ is the maximum number of sets containing an item and $\ell$ is the maximum size of a set. Balcan and Blum [2] gave an $O(\ell^2)$-approximation algorithm. Hartline and Koltun [28] gave an FPTAS with a bounded number of items. On the other hand, the problem was proved to be NP-hard for both the limited-supply [26] and unlimited-supply [4, 27] case, and even hard to approximate [16].

**Pricing for online welfare maximization with tie-breaking power.** The problem of online resource allocation for welfare maximization has been extensively studied in the prophet inequality literature. In the full-information setting where all buyers’ values are known, bundle pricing achieves 2-approximation to optimal offline welfare [14], even when the buyers’ values are arbitrary over sets of items. In a Bayesian setting where the seller knows all buyers’ value distributions, item pricing achieves a 2-approximation in welfare for buyers with fractionally subadditive values [22, 33, 35, 39], and $O(\log \log m)$-approximation for subadditive buyers [19]. For general-valued buyers that demand at most $k$ items, item pricing can achieve a tight $O(k)$-approximation [18]. [6] studied the problem of interval allocation on a path graph, and achieves $(1 - \varepsilon)$-approximation via item pricing when each item has supply $O(\ell^2)$, and each buyer has a fixed value for getting allocated any path she demands. [7] further extends the results to general path allocation on trees and gets a near-optimal competitive ratio via anonymous bundle pricing.

**Pricing for online welfare maximization without tie-breaking power.** When the seller does not have tie-breaking power, [14, 37] show that when there is a unique optimal allocation for online buyers with gross-substitutes valuation functions, static item pricing can achieve the optimal welfare. When the optimal allocation is not unique, [3, 14] show that a dynamic pricing algorithm can obtain the optimal welfare for gross-substitutes buyers, but for not more general buyers. [29] shows that if the buyers have matroid-based valuation functions, when the supply of each item is more than the total demand of all buyers, the minimum Walrasian equilibrium prices achieve near-optimal welfare, [14, 20] shows that for an online matching market, when the seller has no tie-breaking power, static item pricing gives at least 0.51-fracitonal of the optimal offline welfare, and no more than $\frac{2}{3}$.

### 1.3 Organization

In Section 2 we describe the settings of the problems studied in the paper in detail. In Section 3, we present our results on the competitive ratio when the graph is a single path (Section 3.2) or tree (Section 3.3). In Section 4, we prove upper and lower bounds on the competitive ratio for general graphs and lower bounds for grids. In Section 5, we present our results in the setting the capacity of edges in the graph is augmented. Finally we discuss possible future directions in Section 6.
2 OUR MODEL

In this section, we introduce our model in more detail. A seller wants to sell m heterogeneous items to n buyers. Each buyer j is single-minded: She demands a set \( Q_j \subseteq [m] \) with a positive value \( v_j \).1 Her value for a subset \( S \subseteq [m] \) of items is \( v_j \) if \( Q_j \subseteq S \), and 0 otherwise. For every buyer j, the set \( Q_j \) and the value \( v_j \) are known to the seller. The seller aims to maximize the welfare, that is, the sum of all buyers' values which get their demand sets. As a special case of the above problem, in the tollbooth problem, there is an underlying graph \( G \). We denote \( V(G) \) and \( E(G) \) the vertex and edge set of \( G \). Every item in the auction corresponds to an edge in \( E(G) \). Let \( E(G) = \{e_1, \ldots, e_m\} \). For simplicity, we use the index \( i \) to represent the edge \( e_i \) as well. For every agent \( j \), her demand \( Q_j \) is a single path in graph \( G \). For a set of paths \( Q \), denote \( E(Q) = \bigcup_{Q \subseteq Q} E(Q) \). We say that paths in \( Q \) are edge-disjoint (node-disjoint, resp.) if all paths in \( Q \) do not share edges (vertices, resp.).

In the paper we focus on a special class of mechanisms called item pricing mechanisms. In an item-pricing mechanism, the seller first computes a posted price \( p(e_i) \) (or \( p_i \)) for every edge \( e_i \) in the graph.2 The buyers then arrive one-by-one in some order \( \sigma \). When each buyer \( j \) arrives, if any edge in her demand set \( Q_j \) is unavailable (taken by previous buyers), then she gets nothing and pays 0. Otherwise, she compares her value \( v_j \) with the total price \( p(Q_j) = \sum_{e \in Q_j} p(e_i) \):

1. If \( p(Q_j) < v_j \), she takes all edges in \( Q_j \) by paying \( p(Q_j) \); edges in \( Q_j \) then become unavailable;
2. If \( p(Q_j) > v_j \), she takes nothing and pays 0;
3. If \( p(Q_j) = v_j \), then whether she takes all edges in \( Q_j \) at price \( p(Q_j) \) depends on the specification about tie-breaking.

We say that the seller has the tie-breaking power, if the item-pricing mechanism is also associated with a tie-breaking rule. Specifically, whenever \( p(Q_j) = v_j \) happens for some buyer \( j \), the mechanism decides whether the buyer takes the edges or not, according to the tie-breaking rule. Given any price vector \( p = \{p_i\}_{i \in [m]} \) and arrival order \( \sigma \), we denote by \( \text{Wel}_{\sigma}(G, v, p, \sigma) \) the maximum welfare achieved by the mechanism among all tie-breaking rules. On the other hand, the seller does not have the tie-breaking power (or buyer owns the tie-breaking power) if, whenever \( p(S_j) = v_j \) happens for some buyer \( j \), the buyer can decide whether she takes the edges or not. For every price vector \( p \) and arrival order \( \sigma \), we denote by \( \text{Wel}_{\sigma}(Q, v, p, \sigma) \) the worst-case (minimum) welfare achieved by the mechanism, over all tie-breaking decisions made by the buyers. In this paper, by default we assume the seller has no tie-breaking power, and will state explicitly otherwise.

For any graph \( G \), an instance in this problem can be represented as a tuple \( \mathcal{F} = (Q, \varnothing) = ((Q_j)_{j \in [n]}, \{v_j\}_{j \in [n]}) \) that we refer to as a buyer profile. An allocation of the items to the buyers is a vector \( y \in \{0, 1\}^n \), such that for each item \( i \in [m] \), \( \sum_{j \in [n]} y_{ij} \leq 1 \). Namely, for every \( j, y_j = 1 \) if and only if buyer \( j \) takes her demand set \( Q_j \). The welfare of an allocation \( y \) is therefore \( \sum_{i \in [n]} v_i y_i \). We denote by \( \text{OPT}(G, \mathcal{F}) \) the optimal welfare over all allocations, and use \( \text{OPT} \) for short when the instance is clear from the context.

Given any item-pricing mechanism, we define the competitive ratio as the ratio of the following two quantities: (i) the offline optimal welfare, which is the total value of the buyers in the optimal offline allocation; and (ii) the maximum among all choices of prices, of the worst-case welfare when the buyers' arrival order \( \sigma \) is adversarial. Formally, for any instance \( \mathcal{F} = (Q, v) \), if the seller does not have tie-breaking power, we define

\[
\text{gap}_{NT}(G, \mathcal{F}) = \frac{\text{OPT}(G, \mathcal{F})}{\max_{p, \min_{\sigma} \text{Wel}_{\sigma}(Q, v, p, \sigma)}}.
\]

In the paper, we analyze the competitive ratio when \( G \) has different special structures. For ease of notation, for any graph \( G \), denote \( \text{gap}_{NT}(G) \) the largest competitive ratio \( \text{gap}_{NT}(G, \mathcal{F}) \) for any instance \( \mathcal{F} \) with underlying graph \( G \). And given a graph family \( \mathcal{G} \), we denote \( \text{gap}_{NT}(\mathcal{G}) = \max_{G \in \mathcal{G}} \text{gap}_{NT}(G) \). For instance, \( \text{gap}_{NT}(\text{Tree}) \) represents the worst competitive ratio among all trees. For the case when the seller has tie-breaking power, we define \( \text{gap}_{\sigma}(G, \mathcal{F}), \text{gap}_{\sigma}(G) \) and \( \text{gap}_{\sigma}(\mathcal{G}) \) similarly.

3 COMPETITIVE RATIO FOR SPECIAL GRAPHS

In this section, we study the competitive ratio when the underlying graph \( G \) is a single path or tree. Although our main focus is on the scenario where buyers own the tie-breaking power, we will start with the setting where \( G \) is a single path and the seller owns the tie-breaking power to illustrate the basic idea of how to use the linear program to generate the desired prices.

3.1 Warm up: Path Graphs with Tie-Breaking Power

Throughout this subsection, we assume that the seller has tie-breaking power. Given any instance \( \mathcal{F} \), the hindsight optimal welfare is captured by an integer program. The relaxed linear program (LP-Primal) and its dual (LP-Dual) are shown as follows.

(LP-Primal) \[
\begin{align*}
\text{max} & \quad \sum_{j \in [n]} v_j \cdot y_j \\
\text{s.t.} & \quad \sum_{j \in [m]} y_{ij} \leq 1 \quad \forall i \in [m] \\
& \quad y_{ij} \geq 0 \quad \forall j \in [n]
\end{align*}
\]

(LP-Dual) \[
\begin{align*}
\text{min} & \quad \sum_{i \in [m]} p_i \\
\text{s.t.} & \quad \sum_{i \in [m]} p_i \geq v_j \quad \forall j \in [n] \\
& \quad p_i \geq 0 \quad \forall i \in [m]
\end{align*}
\]

We denote \( \text{OPT}_{LP}(Q, v) \) or \( \text{OPT}_{LP} \) if the instance is clear from context) the optimum of (LP-Primal), so \( \text{OPT}_{LP}(Q, v) \geq \text{OPT}(Q, v) \). The following lemma shows that for any feasible integral solution achieved by rounding from the optimal fractional solution, we are able to compute prices to guarantee selling to the exact same set of buyers via (LP-Dual). It follows from the complementary slackness of the LP.
Table 1: Counterexample for path graph

| buyer | 1  | 2  | 3  | 4  |
|-------|----|----|----|----|
| path  | e1 | e2 | e1,2| e2,3|
| value | 1  | 1  | 2  | 2  |

3.2 Path Graphs without Tie-Breaking Power

In this section, we analyze the competitive ratio for path graphs where the seller has no tie-breaking power. We notice that in the item-pricing mechanism with set of prices $p^* = \{p^*_i\} \subset \mathbb{R}$ as suggested in Lemma 3.1, every buyer $j$ with $y^*_j = 0$ has 0 utility of buying the path. When buyers own the tie-breaking power, they can make arbitrary decisions and the worst-case welfare may become lower. In Theorem 3.2, we prove that when the seller has no tie-breaking power, the competitive ratio for path graphs can be strictly larger than 1. The example contains 3 edges $e_1, e_2, e_3$ (from left to right) and 4 buyers. The demand path and value for all buyers are shown in Table 1.

Theorem 3.2. $\text{gap}_{\text{NT}}(\text{Path}) \geq 3/2$.

The main result of this subsection is shown in Theorem 3.3, where we prove that the competitive ratio 3/2 is tight for paths.

Theorem 3.3. $\text{gap}_{\text{NT}}(\text{Path}) \leq 3/2$.

The remainder of this subsection is dedicated to the proof of Theorem 3.3. According to Theorem 3.1, we start with an integral optimal solution of (LP-Primal). Denote $y^*$ the integral optimal solution of (LP-Primal) that maximizes $\sum_{j \in \mathbb{N}} y^*_j$. Define $\hat{Y} = \{j | y^*_j = 1\}$ and $Q^* = \{Q_j | y^*_j = 1\}$. We prove the following lemma, which is useful to guarantee that the constructed price vector is positive in the proof of Theorem 3.3.

Lemma 3.2. There is an $\varepsilon > 0$ and an optimal solution $p^*(e) \in E(G)$ for (LP-Dual), such that (i) for each edge $e \in E(Q^*)$, $p^*(e) \geq \varepsilon$; and (ii) for each $j \in \mathbb{N}$, either $p^*(Q_j) = v^*_j$, or $p^*(Q_j) > v^*_j + \varepsilon$.

Now consider the parameter $\varepsilon > 0$ and prices $p^*$ as suggested by Lemma 3.2. We define $A = \{j | v^*_j = p^*(Q_j)\}$ as the set of buyers who have 0 utility at prices $p^*$. Let $Q_A = \{Q_j | j \in A\}$ be the set of their demand paths. We need the following definition.

Definition 3.1. A set $Q \subseteq Q_A$ of edge-disjoint paths is unique (in $Q_A$), if there does not exist a set $Q' \subseteq Q_A \setminus Q$ of $|Q'| \geq 2$ edge-disjoint paths, such that $\bigcup_{j \in Q} e \cap Q_j = \bigcup_{j \in Q_j \in Q} Q_j$.

Intuitively, a set of edge-disjoint paths $Q$ is unique if the union of all paths in $Q$ is a single path or there does not exist another set of paths among the rest paths, whose union is the same as the one of $Q$. We prove the following lemma for unique edge-disjoint paths. The lemma shows that given any unique set of edge-disjoint paths, we can design proper prices so that any buyer whose demand path is in $Q$ can afford his path, while other buyers can not afford. It implies that the mechanism can serve all buyers whose demand paths are in the set in any arrival order.

Lemma 3.3. Given any unique set $Q$ of edge-disjoint paths, there exists a set of positive prices $p = \{p(e)\}_{e \in Q}$ that achieves worst-case welfare at least $\sum_{j \in Q} \pi_j$.

Proof of Theorem 3.3: We will prove Theorem 3.3 using Claim 3.3. Denote $Q^* = \{Q_1, Q_2, \ldots, Q_3\}$, where the paths are indexed according to the order in which they appear on $G$. First, for each edge $e \not\in E(Q^*)$, we set its price $p(e) = +\infty$. Therefore, any buyer $j$ whose demand path contains an edge not in $E(Q^*)$ cannot take her demand path. In fact, we may assume without loss of generality that $\bigcup_{j \in Q} Q_j = Q$, since otherwise $\bigcup_{j \in Q} Q_j$ is a union of node-disjoint paths and can be divided into separate sub-instances of path graphs.

The crucial step is to compute three sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3 \subseteq Q_A$ of edge-disjoint paths, such that

(1) every edge of $E(G)$ is contained in exactly two paths of $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$; and
(2) for each $t \in \{1, 2, 3\}$, the set of paths in every connected component in the graph generated from paths in $\hat{Q}_t$ is a unique set of edge-disjoint paths.

By property (1) and the fact that $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3 \subseteq Q_A$, $\sum_{j \in Q} v^*_j + \sum_{j \in Q \in Q_A} v_j = 2 \cdot \sum_{e \in E(G)} p^*(e) = 2 \cdot \text{OPT}(G, F)$. Here the last inequality is because: Since $\bigcup_{j \in Q} Q_j = G$, the offline optimal welfare equals to the optimum of (LP-Primal) and the optimum of (LP-Dual). Assume without loss of generality that $\sum_{j \in Q \in Q_A} v_j \geq (2/3) \cdot \text{OPT}(G, F)$. Then we can set prices $\{p(e)\}$ on edges in $E(\hat{Q}_1)$ according to Lemma 3.3, and $+\infty$ price for all other edges. By Lemma 3.3, the item pricing $p$ achieves worst-case welfare at least $\sum_{j \in Q \in Q_A} v_j \geq (2/3) \cdot \text{OPT}(G, F)$.

We now compute the desired sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$ of edge-disjoint paths, which, from the above discussion, completes the proof of Theorem 3.3.

We start by defining $\hat{Q}$ to be the multi-set that contains, for each path $Q_j \in Q_Y$, two copies $Q'_j, Q''_j$ of $Q_j$. We initially set

$\hat{Q}_1 = \{Q'_0, Q'_3, Q'_6, Q'_{r+3}, 0 \leq r < k/6\} \cup \{Q''_0, Q''_3, 0 \leq r < k/6\}$;

$\hat{Q}_2 = \{Q'_r, Q'_r+2, 0 \leq r < k/6\} \cup \{Q''_r, Q''_r+5, 0 \leq r < k/6\}$;

$\hat{Q}_3 = \{Q'_r, Q'_r+3, 0 \leq r < k/6\} \cup \{Q''_r, Q''_r+5, 1 \leq r < k/6\}$.

See Figure 2(a) for an illustration. Clearly, sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$ partition $Q$, each contains edge-disjoint paths, and every edge appears twice in paths of $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$. However, sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$ may not satisfy Property 2. We will then iteratively modify sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$, such that at the end Property 2 is satisfied.

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Throughout, we also maintain graphs $G_t = \bigcup_{Q \in Q_t} Q$, for each $t \in \{1, 2, 3\}$. As sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$ change, graphs $G_t, G_2, G_3$ evolve. We start by scanning the path $G$ from left to right, and process, for each each connected component of graphs $G_1, G_2, G_3$, as follows.

We first process the connected component in $G_1$ formed by the single path $Q_1'$. Clearly, set $\{Q_1''\}$ is unique, since if there are other paths $\hat{Q}, \hat{Q}' \in Q_A$ such that $\hat{Q}, \hat{Q}'$ are edge-disjoint and $\hat{Q} \cup \hat{Q}' = Q_1$, then the set $\{\hat{Q}, \hat{Q}', \hat{Q}_2, \ldots, \hat{Q}_3\}$ corresponds to another integral optimal solution $\hat{y}''$ of (LP-Primal) with $\sum_{j \in [n]} \hat{y}_{j}'' = k + 1 > k = \sum_{j \in [n]} \hat{y}_{j}'$, a contradiction to the definition of $\hat{y}'$. We do not modify path $Q_1''$ in $\hat{Q}_1$ and continue to the next iteration.

We then process the connected component in $G_2$ formed by the paths $Q'', Q''$. If the set $\{Q_1', Q_2'\}$ is unique, then we do not modify this component and continue to the next iteration. Assume now that the set $\{Q_1', Q_2'\}$ is not unique. From similar arguments, there exist two other paths $Q_1', Q_2'$ in $Q_A$, such that $Q_1', Q_2'$ are edge-disjoint and $Q_1' \cup Q_2' = Q_1 \cup Q_2$. We then replace the paths $Q_1', Q_2'$ in $\hat{Q}_2$ by paths $Q_1', Q_2'$. Let $\hat{v}_1'$ be the vertex shared by paths $Q_1', Q_2'$, so $\hat{v}_1' \neq v_1$. We distinguish between the following cases.

**Case 1.** $\hat{v}_1'$ is to the left of $v_1$ on path $G$. As shown in Figure 2(b), we keep the path $Q_2'$ in $\hat{Q}_2$, and move path $Q_1'$ to $\hat{Q}_3$. Clearly, we create two new connected components: one in $G_3$ formed by a single path $Q_1'$, and the other in $G_2$ formed by a single path $Q_2'$. From similar arguments, the corresponding singleton sets $\{Q_1''\}, \{Q_2''\}$ are unique.

**Case 2.** $\hat{v}_1'$ is to the right of $v_1$ on path $G$. As shown in Figure 2(c), we keep the path $Q_2'$ in $\hat{Q}_2$, move path $Q_1'$ to $\hat{Q}_1$ and additionally move the path $Q_1''$ processed in previous iteration to $\hat{Q}_2$. Clearly, we create two new connected components: one in $G_1$ formed by a single path $Q_1'$, and the other in $G_2$ formed by a single path $Q_2'$. From similar arguments, the corresponding singleton sets $\{Q_1''\}, \{Q_2''\}$ are unique. Note that we have additionally moved $Q_1''$ to $\hat{Q}_2$, but since we did not change the corresponding component, the singleton set $\{Q_1''\}$ is still unique.

![Figure 2: Illustrations of the algorithm for computing path sets $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$.](image)

We continue processing the remaining connected components in the same way until all components are unique. We will show that, every time a connected component is not unique and the corresponding two paths are replaced with two new paths, the connected components in $G_1, G_2, G_3$ that we have processed in previous iterations will stay unique. Therefore, the algorithm will end up producing unique components in $G_1, G_2, G_3$ consisting of a unique set of one or two edge-disjoint paths.

To see why this is true, consider an iteration where we are processing a component consisting of paths $Q''_1, Q''_4$, and there exists edge-disjoint paths $Q''_1, Q''_4$ such that $Q''_1 \cup Q''_4 = Q''_1 \cup Q''_4$, while the endpoint $v_4$ shared by $Q''_1$ and $Q''_4$ is an endpoint of a processed component, as shown in Figure 2(d). Note that this is the only possibility that the new components may influence the previous components. However, we will show that this is impossible. Note that $Q''_1 \subseteq Q_Y$. We denote by $Q''_1$ the path with endpoints $v_1$ and $v_4$, then clearly paths $Q''_1, Q''_4$ are not in $Q_Y$ edge-disjoint and satisfy that $Q_1 = Q''_1 \cup Q''_4$. Consider now the set $(Q_Y \setminus \{Q_1\}) \cup \{Q''_1, Q''_4\}$. It is clear that this set corresponds to another integral optimal solution $\hat{y}''$ of (LP-Primal) with $\sum_{j \in [n]} \hat{y}_{j}'' = k + 1 > k = \sum_{j \in [n]} \hat{y}_{j}'$, a contradiction to the definition of $\hat{y}'$. □

### 3.3 Competitive Ratio for Trees

In this subsection, we study the competitive ratio when graph $G$ is a tree. When the seller owns the tie-breaking power, we prove in Theorem 3.4 a tight competitive ratio of $\frac{3}{2}$. The upper bound is proved by combining Lemma 3.1 with the integrality gap result of the multi-commodity flow problem on tree [8, 38]. On the other hand, we provide an instance on a star (Table 2) to show the competitive ratio is at least $3/2$. We notice that the lower bound also implies that $\text{gap}_T(\text{Star}) \geq \frac{3}{2}$ and $\text{gap}_{NT}(\text{Tree}) \geq \frac{3}{2}$.

![Table 2: Counterexample for stars](image)

**Theorem 3.4.** $\text{gap}_T(\text{Tree}) = \frac{3}{2}$.

When the seller has no tie-breaking power, we show that the competitive ratio for trees can also be upper-bounded by a constant.

**Theorem 3.5.** For any $\varepsilon > 0$, $\text{gap}_{NT}(\text{Tree}) \leq 7 + \epsilon$.

As discussed in the previous subsection for path graphs, the LP-based approach requires the seller to own the tie-breaking power. To prove Theorem 3.5, we use a different approach. We first prove the following structural lemma, which partitions a set of edge-disjoint paths into two sets, such that each set of paths forms a union of vertex-disjoint spider graphs. Here a spider graph $G$ is a tree with one vertex $u$ of degree at least 3 and all others with a degree at most 2. In other words, $E(G)$ can be decomposed into $k$ paths, where any two paths only intersect at $u$. A star is a special spider graph, where all vertices other than $u$ have degree 1.

**Lemma 3.4.** Let $P_1, \ldots, P_n$ be edge-disjoint paths, such that the graph $G = \bigcup_{j \in [n]} P_j$ is a tree, then the set $\mathcal{P} = \{P_1, \ldots, P_n\}$ can be partitioned into two sets $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$, such that both the graph induced by paths in $\mathcal{P}'$ and the graph induced by paths in $\mathcal{P}''$ are the union of vertex-disjoint spiders.
The lemma follows from a constructive proof as follows: Choose any vertex \( v \) and put all paths including \( v \) into \( P' \). Now the remaining graph becomes a forest. For each tree in the forest, choose any vertex and put all paths including this vertex into \( P'' \). Continue this process until all paths are assigned to either \( P' \) or \( P'' \). Both sets can be proved as a union of vertex-disjoint spiders. See the arXiv version for the formal proof.

The following lemma directly follows from Lemma 3.4.

**Lemma 3.5.** \( \text{gap}_{NT}(\text{Tree}) \leq 2 \cdot \text{gap}_{NT}(\text{Spider}). \)

With Lemma 3.5 it’s sufficient to bound the competitive ratio for spider graphs. As a warm-up, we prove here that \( \text{gap}_{NT}(\text{Star}) \leq 2 + \epsilon \) for any \( \epsilon > 0 \). The prices are designed as follows. For any path \( Q_j \) in the offline optimal solution, set the prices of every edge in \( Q_j \) (there are at most 2 edges since \( G \) is a star) to \((\frac{1}{3} - \epsilon) \cdot \alpha_j \) for some small \( \epsilon \). Then at least one edge will always be sold, which contributes welfare at least \((\frac{1}{2} - \epsilon) \cdot \alpha_j \). If neither edge is sold when buyer \( j \) arrives, she can afford her path and will purchase both edges.

For spider graphs, we also aim to design prices for each edge based on the value for the path in the offline optimal. We take any offline optimal solution. For any path \( Q_j \) in the optimal solution that goes through the center of the spider, instead of spreading the value evenly into both legs, we design the prices carefully according to the solution of each leg (a path graph). We choose the set of prices that achieves more welfare from two strategies: either designing prices to sell \( Q_j \) only in the corresponding two legs, or designing prices according to Theorem 3.3 to sell the remaining part of the two legs (which is two separate paths).

**Lemma 3.6.** For any \( \epsilon > 0 \), \( \text{gap}_{NT}(\text{Spider}) \leq 7/2 + \epsilon \). Moreover, \( \text{gap}_{NT}(\text{Star}) \leq 2 + \epsilon \) for any \( \epsilon > 0 \).

**Proof of Theorem 3.5:** It directly follows from Lemmas 3.5 and 3.6. \( \square \)

## 4 COMPETITIVE RATIO FOR GENERAL GRAPHS

In this section we study the competitive ratio when \( G \) is a general graph. We will start by showing a \( poly(m) \) lower bound by constructing an instance in a serial-parallel graph (Section 4.1). Then we use a modification of this instance to prove lower bounds in grids (Sections 4.2). At last, we prove an upper bound for the competitive ratio in general graphs, that depends on the number of buyers (Section 4.3). For results in this section, we assume the strongest dependence on tie-breaking power: the lower bound results hold even when the seller has tie-breaking power, and the upper bound results hold when the seller has no tie-breaking power.

### 4.1 Lower Bound for General Graphs

**Theorem 4.1.** \( \text{gap}_T(\text{General Graphs}) = \Omega(m^{1/8}) \), i.e., there exists a graph \( G \) with \( |E(G)| = m \) and a buyer profile \( F \) on \( G \), such that no set of prices on edges of \( G \) can achieve worst-case welfare \( \Omega(\text{OPT}(G, F)/m^{1/8}) \) even when the seller has tie-breaking power.

The remainder of this subsection is dedicated to the proof of Theorem 4.1. We will construct the graph \( G \) as follows. For convenience, we will construct a family of graphs \( \{H_{ab}\}_{a,b \in \mathbb{Z}} \), in which each graph is featured by two parameters \( a, b \) that are positive integers. We will set the exact parameters in the proof of Theorem 4.1 later.

For a pair \( a, b \) of integers, graph \( H_{ab} \) is defined as follows. The vertex set is \( V(H_{ab}) = V_1 \cup V_2 \), where \( V_1 = \{u_0, \ldots, u_b\} \) and \( V_2 = \{u_j | 1 \leq j \leq b, 1 \leq j \leq a\} \). The edge set is \( E(H_{ab}) = \bigcup_{1 \leq i \leq 2} E_i \), where \( E_i = \{(u_{i-1}, u_{i+j}), (u_j, u_{i+j}) | 1 \leq j \leq a\} \). See Figure 3 for an example. Equivalently, if we define the multi-graph \( L_{ab} \) to be the graph obtained from a length-\( b \) path by duplicating each edge for \( a \) times, then we can view \( H_{ab} \) as obtained from \( L_{ab} \) by subdividing each edge by a new vertex.

![Figure 3: An illustration of graph \( H_{ab} \).](image)

Let \( a, b \) be such that \( b = a + 3a^3 \) and choose \( G = H_{ab} \). Clearly \( m = |E(G)| = 2ab \), so \( a = \theta(m^{1/4}) \) and \( b = \Theta(m^{3/4}) \). For convenience, we will simply work with graph \( L_{ab} \), since every path in \( L_{ab} \) is also a path in \( H_{ab} \). Note that \( V(L_{ab}) = \{v_0, \ldots, v_b\} \), and we denote by \( e_1, \ldots, e_n \) the edges in \( L_{ab} \), connecting \( v_i \) and \( v_i+1 \). For brevity, we use the index sequence \((j_1, j_2, \ldots, j_b)\) to denote the path consisting of edges \( e_{j_1}, e_{j_2}, \ldots, e_{j_b} \), where each index \( j_t \in [a] \), for each \( t \in [b] \). It is clear that a pair of paths \((j_1, j_2, \ldots, j_b)\) and \((j'_1, j'_2, \ldots, j'_b)\) are edge-disjoint iff for each \( t \in [b] \), \( j_t \neq j'_t \). In the proof we will construct a buyer profile \( F \) on the multi-graph \( L_{ab} \). Clearly it can be converted to a buyer profile on graph \( H_{ab} \), with the same lower bound of competitive ratio. We prove the following lower bound of competitive ratio for \( L_{ab} \). Theorem 4.1 follows directly from Lemma 4.1 where \( a = \Theta(m^{1/4}) \) and \( b = \Theta(m^{3/4}) \). The proof of Lemma 4.1 is deferred to Section A.

**Lemma 4.1.** \( \text{gap}_T(L_{ab}) \geq \sqrt{a} \).

### 4.2 Lower Bound for Grids

We notice that in the graph \( L_{ab} \) and \( H_{ab} \) that we constructed in Theorem 4.1, the maximum degree among all vertices is \( 2a \), which is a polynomial of \( m \). Readers may wonder if the large polynomial competitive ratio is due to the existence of high-degree vertices that are shared by many demand paths. In this section, we show a negative answer to this question. We prove that a \( poly(m) \) lower bound of the competitive ratio is unavoidable even when \( G \) is restricted to be a grid.

**Theorem 4.2.** Let \( G \) be the \( (\sqrt{m} \times \sqrt{m}) \)-grid (so that \( G \) has \( \Theta(m) \) edges). Then \( \text{gap}_T(G) = \Omega(m^{1/20}) \).

The proof is enabled by replacing each high-degree vertex in the graph \( H_{ab} \) with a gadget, so that every vertex in the modified graph \( G \) has degree at most 4. Formally, the graph \( G = R_{ab} \) is constructed as follows. Consider a high-degree vertex \( v \in V(H_{ab}) \). Recall that it has \( 2a \) incident edges \( \{(u_i, u_{i+j}), (u_j, u_{i+j}) | j \in [a]\} \) in graph \( H_{ab} \) (see Figure 4(a)). The gadget for vertex \( v \) is constructed as follows. We first place the vertices \( u_{i,1}, \ldots, u_{a,1}, u_{i,a} \) on a circle in this order, and then for each \( j \in [a] \), we draw a line segment connecting \( u_{i,j} \) with \( u_{i-a,j} \), such that every pair of these segments intersects, and no three segments intersect at the same
point. We then replace each intersection with a new vertex. See Figure 4(b) for an illustration.

\[ \text{(a) The vertex } n_i \text{ and its incident edges in } H_{n,b}. \]

\[ \text{(b) The gadget graph } K_i. \]

Figure 4: An illustration of the gadget construction.

Now the modified graph can be embedded in the grid since each vertex has degree at most 4. The proof appears in Appendix B.

### 4.3 Upper Bound for General Graphs

At last, we prove an upper bound for the competitive ratio in any tollbooth problem (on general graphs). The competitive ratio depends on the number of buyers in the auction. Note that when \( n \) is sub-exponential on \( m \), the competitive ratio proved in Theorem 4.3 is better than the competitive ratio \( \Omega(\sqrt{m}) \) proved in [11, 12].

**Theorem 4.3.** For any given instance \((G, F)\) with \(|E(G)| = m\) and \(|F| = n\), \(\text{gap}_{MP}(G) = O(m^{0.4} \cdot \log^m m \cdot \log n)\).

Here we provide a sketch of how we construct the prices. Take any offline optimal solution \(Q\). As a pre-processing, we first select a subset \(Q'\) of \(Q\) such that each path in the subset has both length and value within 2 times of other paths, by losing a ratio of \(\log^2 m\). Then we increase the prices for each edge in two steps. In the first step, let \(Q''\) be a random subset of \(Q'\) by including each path independently with probability 1/2. We set the price for each edge not contained in any path of \(Q''\) to be \(\infty\). In the second step, we pick a special set of short paths and increase the prices of each edge uniformly for every path in the set. With the above prices, we are able to prove a lower bound of the size of any set of paths sold in the selling process, which contributes enough welfare compared to the \(OPT\).

### 5 RESOURCE AUGMENTATION

In Sections 3 and 4, we studied the competitive ratio of the tollbooth problem, in which each edge can be allocated to at most one buyer. In this section, we consider the case where each item has augmented resources. We prove results in general combinatorial auctions with single-minded buyers, which is a generalization of the tollbooth problem. In a combinatorial auction, let \(U = \{1, \ldots, m\}\) be the item set. Given a buyer profile \(F = \{(Q_j, v_j)\}_{j \in [n]}\), we denote by \(\text{OPT}(U, F)\) the maximum welfare by allocating items in \(U\) to the buyers, such that each item is assigned to at most one buyer. The seller, however, has more resources to allocate during the selling process. For each item \(i \in U\), the seller has \(c\) copies of the item, and each copy is sold to at most one buyer. In an item-pricing mechanism, the seller is allowed to set different prices for different copies. Formally, for each item \(i \in U\), the seller sets \(c\) prices \(p^1(i) \leq \ldots \leq p^c(i)\), such that for each \(1 \leq k \leq c\), the \(k\)-th copy of item \(i\) is sold at price \(p^k(i)\). When a buyer comes, if \(k - 1\) copies of item \(i\) has already been sold, the buyer can purchase item \(i\) with price \(p^k(i)\). Again we define the worst-case welfare of an item pricing as the minimum welfare among all the buyers’ arriving order.

With the augmented resources, the seller can certainly achieve more welfare than in the case with a single unit per item. We show that item pricing can achieve worst-case welfare \(Ω(m^{-1/2}) \cdot \text{OPT}(U, F)\). The result implies that when \(c = Ω(\log m)\), item pricing on \(c\) copies achieves at least a constant factor of the offline optimal welfare when each edge has supply 1.

**Theorem 5.1.** For any buyer profile \(F = \{(Q_j, v_j)\}_{j \in [n]}\) and any integer \(c > 0\), there exists a set \(\{p^k(i)\}_{i \in U, 1 \leq k \leq c}\) of prices on items of \(U\), that achieves worst-case welfare \(Ω(m^{-1/c}) \cdot \text{OPT}(U, F)\), even when the seller has no tie-breaking power.

On the other hand, in Theorem 5.2 we show that a polynomial dependency on \(m\) in the competitive ratio is in fact unavoidable.

**Theorem 5.2.** For any integer \(c > 0\), there exists a ground set \(U\) with \(|U| = m\) and a buyer profile \(F\) such that any item-pricing mechanism with set of prices \(\{p^k(i)\}_{i \in U, 1 \leq k \leq c}\) achieves worst-case welfare \(O(c \cdot m^{-1/(c+1)}) \cdot \text{OPT}(U, F)\), even when the seller has tie-breaking power.

In Theorem 5.3, we prove that a polynomial welfare gap also exists in the tollbooth problem. We adapt the series-parallel graph \(H_{a,b}\) used in Theorem 4.1 and show that a polynomial competitive ratio is unavoidable in the tollbooth problem, even each edge has a constant number of copies.

**Theorem 5.3.** In the tollbooth problem, for any constant integer \(c > 0\), there exists a graph \(G\) with \(m\) edges and a buyer profile \(F = \{(Q_j, v_j)\}_{j \in [n]}\), such that any set \(\{p^k(e)\}_{e \in E(G), k \in [c]}\) of prices achieves worst-case welfare \(O(m^{-1/(2c+6)}) \cdot \text{OPT}(G, F)\).

### 6 FUTURE WORK

We study the worst-case welfare of item pricing in the tollbooth problem. There are several future directions following our results. Firstly, in the paper we assume that all buyers’ value are all public. A possible future direction is to study the Bayesian setting where the seller does not have direct access to each buyer’s value, but only know the buyers’ value distributions. Secondly, we focus on the tollbooth problem where each buyer demands a fixed path on a graph. An alternative setting is that each buyer has a starting vertex and a terminal vertex on the graph, and she has a fixed value for getting routed through any path on the graph. Such a setting is equivalent to our problem when the underlying graph is a tree, where a constant competitive ratio is proved in this paper even if the seller does not have tie-breaking power. However, there may exist more than one path between two vertices in a graph with cycles, and thus the buyer is single-minded in this setting. In the paper we have shown that item pricing may not approximate the optimal welfare well in the tollbooth problem. It remains open whether the item pricing performs well in the alternative setting. Thirdly, the power of tie-breaking hasn’t been studied much in the literature of mechanism design. In this paper we show that tie-breaking may cause a difference in the tollbooth problem even when the graph is a single path. It would be interesting to see other scenarios where the tie-breaking power also makes a difference.
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A PROOF OF LEMMA 4.1

We define \( F \) on graph \( L_{a,b} \) as follows. We will first define a set \( F^* \) of buyer profile, and then define, for each subset \( S \subseteq [a] \) with \( |S| \geq \sqrt{a} \), a buyer profile \( F_S \), where the buyers in different sets are distinct. Then we define \( F = F^* \cup (\bigcup S \subseteq [a], |S| \geq \sqrt{a}) F_S \).

Let the set \( F^* \) contain, for each \( r \in [a] \), a buyer \( r \) with \( Q(r) = (r, r, \ldots , r) \) and \( \nu_r = 1 \). Clearly, demand paths of buyers in \( F^* \) are edge-disjoint. In the construction we will ensure that \( \text{OPT}(L_{a,b}, F) = a \), achieved by giving each buyer in \( F^* \) her demand path.

Before we construct the sets \( \{F_S\}_{S \subseteq [a], |S| \geq \sqrt{a}} \), we will first state some desired properties of \( \{F_S\} \), and use it to finish the proof of Lemma 4.1. Let \( \epsilon > 0 \) be an arbitrarily small constant.

1. For each \( S \), set \( F_S \) contains \(|S| \) buyers, and every pair \( Q, Q' \) of demand paths in \( F_S \) share an edge. The value for every buyer in \( F_S \) is \( 1 + \epsilon \).
2. For each demand path \( Q \) in \( F_S \), the index sequence \( (j_1, \ldots , j_m) \) that \( Q \) corresponds to satisfies that (i) \( j_k \in S \) for each \( i \in [b] \); and (ii) the set \( \{j_1, \ldots , j_m\} \) contains all elements of \( S \).
3. The union of all demand paths in \( F_S \) covers the graph \( \bigcup_{r \in S} Q(r) \) exactly twice. In other words, for each \( i \in [b] \) and every \( r \in S \), there are exactly two demand paths \( Q \) in \( F_S \) that satisfy \( \nu_r = r \).
4. For any pair \( S, S' \subseteq [a] \) such that \( |S|, |S'| \geq \sqrt{a} \) and \( S \cap S' \neq \emptyset \), and for any demand path \( Q \) in \( F_S \) and \( Q' \) in \( F_{S'} \), \( Q \) and \( Q' \) share some edge.

Suppose we have successfully constructed the sets \( \{F_S\}_{S \subseteq [a], |S| \geq \sqrt{a}} \) that satisfy all the above properties. We then define \( F = F^* \cup (\bigcup S \subseteq [a], |S| \geq \sqrt{a}) F_S \). From the above properties, it is easy to see that \( \text{OPT}(L_{a,b}, F) = a \), which is achieved by giving each buyer in \( F^* \) her demand path. We will prove that any prices on edges of \( L_{a,b} \) achieve worst-case welfare \( O(\sqrt{a}) \).

Consider now any set of prices on edges of \( L_{a,b} \). We distinguish between the following two cases.

Case 1. At least \( \sqrt{a} \) buyers in \( F^* \) can afford their demand paths. We let \( S \) be the set that contains all indices \( r \in [a] \) such that the buyer \( r \) can afford her demand path \( Q(r) \), so \( |S| \geq \sqrt{a} \). Consider the set \( F_S \) of buyers that we have constructed. We claim that at least some buyers of \( F_S \) can also afford her demand path. To see why this is true, note that by Property 1, \( F_S \) contains \(|S| \) buyers with total value \( (2 + 2\epsilon)|S| \), while the total price of edges in \( \bigcup_{r \in S} Q(r) \) is at most \( |S| \). Therefore, by Property 3, there must exist a buyer in \( F_S \) that can afford her demand path. We then let this buyer come first and get her demand path \( Q \). Then from Property 2, all buyers \( r \in S \) can not get their demand paths since their demand paths share an edge with \( Q \). All buyers \( r \in [a]/S \) can not afford their demand paths. Moreover, for any buyers’ arriving order, let \( Q = (Q_1, \ldots , Q_K) \) \( (Q_1 = Q) \) be the set of demand paths that are allocated eventually. We argue that \( K \leq \sqrt{a} \). For every \( k = 1, \ldots ; K \), \( Q_k \) must come from the profile \( \bigcup_{S \subseteq [a], |S| \geq \sqrt{a}} F_S \). Let \( S_k \subseteq [a] \) be the set that \( Q_k \) appears in \( F_{S_k} \). Then by Property 1 and 4, we have \( S_k \cap S_k' = \emptyset \) for any \( k_1, k_2 \in [K], k_1 \neq k_2 \). Thus we have \( K \leq \sqrt{a} \) since \( |S_k| \geq \sqrt{a} \) for every \( k \). Hence, the achieved welfare is at most \((1 + \epsilon)\sqrt{a}\), for any buyers’ arriving order.

Case 2. Less than \( \sqrt{a} \) buyers in \( F^* \) can afford their demand paths. Similar to Case 1, at most \( \sqrt{a} \) buyers from sets \( \bigcup S \subseteq [a], |S| \geq \sqrt{a} \) can get their demand sets simultaneously. Therefore, the total welfare is at most \((1 + \epsilon)\sqrt{a} + \sqrt{a} = O(\sqrt{a}) \).

It remains to construct the sets \( \{F_S\}_{S \subseteq [a], |S| \geq \sqrt{a}} \) that satisfy all the above properties. We now fix a set \( S \subseteq [a] \) with \( |S| \geq \sqrt{a} \) and construct the set \( F_S \). Denote \( s = |S| \). Since each path can be represented by a length- \( b \)-sequence, we simply need to construct a \((2 \times b)\) matrix \( M_S \), in which each row corresponds to a path in \( L_{a,b} \). We first construct the first \( s \) columns of the matrix. Let \( N_S \) be an \( s \times s \) matrix, such that every row and every column contains each element of \( S \) exactly once (it is easy to see that such a matrix exists). We place two copies of \( N_S \) vertically, and view the obtained \((2s \times s)\) matrix as the first \( s \) columns of matrix \( M_S \). We then construct the remaining \( b - s \geq 3a^2 \) columns of matrix \( M_S \). Let \( S' \) be the multi-set that contains two copies of each element of \( S \). We then let each column be an independent random permutation on elements of \( S' \). This completes the construction of the matrix \( M_S \). We then add a buyer associated with every path above with value \( 1 + \epsilon \). This completes the construction of the set \( F_S \).

We prove that the randomized construction satisfies all the desired properties, with high probability. For Property 1, clearly \( F_S \) contains \(|S| \) buyers, and the value associated with each demand path is \( 1 + \epsilon \). For every pair of distinct rows in \( M_S \), the probability that their entries in \( j \)-th column for any \( j > s \) are identical is \( 1/2s \), so the probability that the corresponding two paths are edge-disjoint is at most \((1 - 1/2s)^{3a^2} \). From the union bound over all pairs of rows, the probability that there exists a pair of edge-disjoint demand paths in \( F_S \) is at most \((1 - 1/2s)^{3a^2} \leq a^2 \cdot e^{-a^2} \). Property 2 is clearly satisfied by the first \( s \) column of matrix \( M_S \). Property 3 is clearly satisfied by the construction of matrix \( M_S \). Therefore, from the union bound on all subsets \( S \subseteq [a] \) with \( |S| \geq \sqrt{a} \), the probability that Properties 1, 2, and 3 are not satisfied by all sets \( \{F_S\}_{S \subseteq [a], |S| \geq \sqrt{a}} \) is \( a^2 \cdot e^{-a^2} - 2a^2 < e^{-a^2} \).

For Property 4, consider any pair \( S, S' \) of such sets and any row in the matrix \( M_S \) and any row in the matrix \( M_{S'} \). Since \( S \cap S' \neq \emptyset \), the probability that they have the same element in any fixed column is at least \((1/|S|) \cdot (1/|S'|) \geq 1/a^2 \), so the probability that the corresponding two paths are edge-disjoint is at most \((1 - a^{-2})^{3a^2} \leq e^{-3a^2} \). From the union bound, the probability that Property 4 is not satisfied is at most \( e^{-3a^2} \cdot (2s^2 \cdot 2a^2) < e^{-a^2} \). Altogether, our randomized construction satisfies all the desired properties with high probability. Thus there must exist a deterministic construction of \( F_S \) that satisfies all the properties. This completes the proof of Lemma 4.1.

B PROOF OF THEOREM 4.2

Denote by \( K_i \) the resulting graph after constructing the gadget, as shown in Figure 4(b). We use the following lemma.

**LEMMA B.1.** Let \( \sigma \) be any permutation on \([a]\), then graph \( K_i \) contains a set \( P_i = \{P_{i,j} \in [a]\} \) of edge-disjoint paths, such that path \( P_{i,j} \) connects vertex \( u_{i,j} \) to vertex \( x_{i,\sigma(j)} \).

Before proving Lemma B.1, we first give the proof of Theorem 4.2 using Lemma B.1.
Proof of Theorem 4.2: First, the graph $R_{a,b}$ is obtained by taking the union of all graphs $(K_i)_{0 \leq i \leq b}$, while identifying, for each $i \in [b]$ and $j \in [a]$, the vertex $u_{ij}$ in $K_{i-1}$ with the vertex $u_{ij}$ in $K_i$. Clearly, the maximum vertex degree in graph $R_{a,b}$ is 4.

We now show that we can easily convert the buyer-profile $\mathcal{F}$ on graph $H_{a,b}$ into a buyer-profile $\tilde{\mathcal{F}}$ on graph $R_{a,b}$ while preserving all desired properties. Consider first the buyers $1, \ldots, a$ in $\mathcal{F}^*$. Let $\sigma_1$ be the identity permutation on $[a]$, for each $i \in [b]$. From Lemma B.1, there exist sets $\{P^*_i\}_{i \in [b]}$ of edge-disjoint paths, where $P^*_i = \{P^*_{i,j} \mid j \in [a]\}$ for each $i \in [b]$. We then let $\tilde{\mathcal{F}}^*$ contains, for each $j \in [a]$, a buyer $\tilde{B}_j$ demanding the path $\tilde{Q}_j$ with value 1, where $\tilde{Q}_j$ is the sequential concatenation of paths $P^*_1, \ldots, P^*_b$. Clearly, paths $\{\tilde{Q}_j\}_{j \in [a]}$ are edge-disjoint. Consider now a set $S \subseteq [a]$ with $|S| \geq \sqrt{a}$. Recall that in $\mathcal{F}_S$, we have 2|S| buyers, whose demand paths cover the paths $\{Q_{(j)} \mid j \in S\}$ exactly twice. Therefore, for each $i \in [b]$, the way that these paths connect vertices of $U_{i-1} = \{u_{i-1,j} \mid j \in [a]\}$ to vertices of $U_i = \{u_{i,j} \mid j \in [a]\}$ form two perfect matchings between vertices of $U_{i-1}$ and vertices of $U_i$. From Lemma B.1, there are two sets $\mathcal{P}_2, \mathcal{P}'_2$ of edge-disjoint paths connecting vertices of $U_{b-1}$ and vertices of $U_b$. We then define, for each demand path $Q = (j_0, j_1, \ldots, j_b)^3$ in $\mathcal{F}_S$, its corresponding path $\tilde{Q}$ to be the sequential concatenation of, the corresponding path in $\mathcal{P}_2 \cup \mathcal{P}'_2$ that connects $u_{b,j_i}$ to $u_{b,j_{i+1}}$, the corresponding path in $\mathcal{P}_2 \cup \mathcal{P}'_2$ that connects $u_{i,j_{i+1}}$ to $u_{i+1,j_i}$, all the way to the corresponding path in $\mathcal{P}_b \cup \mathcal{P}'_b$ that connects $u_{b-1,j_1}$ to $u_{b,j_0}$. It is easy to verify that all desired properties are still satisfied. Lastly, to ensure that the graph $R_{a,b}$ can be embedded into the $(\sqrt{m} \times \sqrt{m})$-grid, we need $\sqrt{m} = a^2 b$, where $b = a + 3a^3$. Thus $a = \Theta(m^{1/10})$. Theorem 4.2 now follows from Lemma 4.1. □

It remains to prove Lemma B.1. We prove by induction on $a$. The base case where $a = 1$ is trivial. Assume that the lemma is true for all integers $a \leq r - 1$. Consider the case where $a = r$. For brevity of notations, we rename $K_i$ by $K, \mathcal{P}_i$ by $\mathcal{P}$, $u_{i,j}$ by $u_j$ and $u_{i+1,j}$ by $u'_j$. Recall that the graph $K$ is the union of $r$ paths $W_1, \ldots, W_r$, where path $W_j$ connects vertex $u_j$ to vertex $u'_{r+1-j}$, such that every pair of these paths intersect at a distinct vertex. Recall that we are also given a permutation $\sigma$ on $[r]$, and we are required to find a set $\mathcal{P} = \{P_j \mid j \in [r]\}$ of edge-disjoint paths in $K$, such that the path $P_j$ connects $u_j$ to $u'_{\sigma(j)}$.

We first define the graph $K' = K \setminus W_r$, and we define an one-to-one mapping $f : [r-1] \rightarrow \{2, \ldots, r\}$ as follows. For each $j \in [r-1]$ such that $\sigma(j) \in \{2, \ldots, r\}$, we set $f(j) = \sigma(j)$; for $j \in [r-1]$ such that $\sigma(j) = 1$, we set $f(j) = \sigma(r)$. Note that $K'$ is a graph consisting of $(r-1)$ pairwise intersecting paths, and $f$ is an one-to-one mapping from the left set of vertices of $K'$ to the right set of vertices of $K'$. From the induction hypothesis, there is a set $\mathcal{P}' = \{P'_j \mid j \in [r-1]\}$ of edge-disjoint paths, such that path $P'_j$ connects $u_j$ to $u'_{f(j)}$. If $\sigma(r) = 1$, then we simply let $\mathcal{P} = \mathcal{P}' \cup \{W_r\}$, and it is easy to check that the set $\mathcal{P}$ of paths satisfy the desired properties. Assume now

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3We can arbitrarily assign additionally the path $Q$ with some index $j_b \in S$, such that each index of $S$ is assigned to exactly two demand paths in $\mathcal{F}_S$. 