Weak Gravity Conjecture from Conformal Field Theory

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Abstract

In this paper first, we investigate the relationship between weak gravity conjecture and conformal field theory by challenging the hyperscaling violating and Kerr-Newman-AdS black holes. In that case, we take advantage of calculations of the correlation function of conformal field theory on the corresponding black holes and prove the weak gravity conjecture. Here we note that using the conformal field theory correlation function and the mentioned assumption helps us to calculate the critical points of systems. By calculating the critical points and using its poles, we obtain the energy of black holes. We note here, our black hole is disturbed by the imaginary part of the energy, and it leads us to discuss \textit{WGC} in the corresponding point. By using this method, we show that, if we consider $z = 1, d = 1$ and $\theta \to 0^-$, we will have the best case for the \textit{WGC} to be established in the hyperscaling violating black holes. Because such black hole in the mentioned case contains $r_H$ larger and smaller than one. The \textit{WGC} condition for the Kerr-Newman-AdS black hole is related to the rotation and radius parameters. Here also we show that if the charged particle near the black hole is $\frac{1}{a}$ and also have a ratio $\frac{a}{\ell} \ll 1$, the weak gravity conjecture will be satisfied. In addition to proving for \textit{WGC}, we show an exciting close relation between these two ideas, namely weak gravity conjecture and conformal field theory.

Keywords: Weak Gravity Conjecture; Conformal Field theory; Hyperscaling Violating Black Holes; Kerr-Newman-AdS Black Holes.
1 Introduction

The AdS/CFT correspondence has recently been introduced as one of the essential dualities between quantum field theory (QFT) and gravity. Also, with a closer look at the mentioned duality, there is a relation as the quantum physics of correlated many-body systems related to another theory in a higher dimension, namely the classical dynamics of gravity. This duality is a subset of holography known as gauge/ gravity correspondence. As we know for the first time, this correspondence was the relationship between four dimensions conformal field theory to the Anti-de Sitter space (AdS) in five dimensions [1–3].

The geometric holographic explanation of quantum dynamics shows that systems with greater freedom degrees indicate a deep connection between quantum field theory and gravity. The gauge/gravity duality is derived from string theory, which is very common in that field theories are realized on super surfaces embedded in a higher-dimensional space, the space that describes gravity theories. However, investigation of such correspondence to various fields as strong-coupling dynamics (QCD), the physics of black holes and quantum gravity, electroweak theories, relativistic hydrodynamics and applications in condensed matter physics are given by Ref.s. [4–36]. In general, the AdS/CFT correspondence reflects that a five-dimensional theory of gravity in bulk can examine by a four-dimensional gauge theory at the boundary by considering a specific set of rules. Also here we note that the AdS/CFT correspondence is formulated for gravity in asymptotically AdS space-time.

On the other hand, one of the most important issues in physics right now is finding the theory of fields compatible with quantum gravity, known as Landscape. In that case many other field theories are incompatible, they are called Swampland. One of the universal tests to distinguish between these two groups is weak gravity conjecture (WGC). It means that gravity is always the weakest force and shows the extremality state of the black hole [37–82]. Therefore, using CFT and weak gravity conjecture and benefiting from two black holes such as Hyperscaling violating and Kerr-Newman-AdS, we are looking to prove weak gravity conjecture from the calculation of the CFT body. Since the idea of weak gravity conjecture is more consistent at critical points, we use CFT calculations to obtain the critical points associated with each of these black holes and then challenge the weak gravity conjecture at these points. Such studies have been studied for the first time.

According to the above description, we will organize the article as: In section 2, we will review the weak gravity conjecture and its relation to the CFT. In Section 3, we will examine the relationship between the CFT and the WGC in Hyper scaling violating black holes and study the same computational process for finding the relationship between the two ideas concerning Kerr-Newman-AdS black hole in Section 4. The results are described in detail in Section 5.
2 Connecting WGC to CFT

We know that the lack of global symmetries and completeness of the spectrum of charge exists at the heart of the Swampland program. Regardless, they absent phenomenological influence unless one can restrain how approximate a global symmetry can be and whether there exists an upper bound on the mass of some of the charged conditions [83–87]. Differently, they exclusively restrain the complete theory while not the EFT with low energy. In unique, it is phenomenologically pertinent whether whole charged particles can be keen heavy and actual correspond to Black holes’, or there exist several concepts of the spectrum completeness that prevails at low energies. The swampland conjectures discussed some questions such as they aspire to point these assertions how near we can obtain to the status of recovering the global symmetries. Researchers can a priori continually refresh a global symmetry of U(1) by transmitting the gauge coupling to 0, which should not be permitted in quantum gravity.

Attempting to comprehend how string theory prohibits this issue and what drives incorrect if one attempts to accomplish this issue can supply facts about the restrictions an EFT must meet to be compatible with quantum gravity. The Weak Gravity Conjecture prohibits this methodology by signaling the existence of unique light-charged forms invalidating the definition of EFT. It correspondingly supplies an upper bound on the mass of these charged states. The WGC includes two pieces: The magnetic and electric version provided a gauge theory that coupled to gravity. So we will have for electrically charged in Planck units as $Q/m \geq Q/M|_{ext} = \mathcal{O}(1)$ [83–87]. Here, Q and M are charge and M mass in the extremal black hole respectively. Also, $Q = qg$, where q and g are the quantized charges of the state and the gauge coupling, respectively. To have a WGC with existence of charge and mass need to apply the charge/mass greater than the one in extremal black hole. The most specific topic coordinates to a theory of Maxwell, which is coupled to gravity in which there exist no massless scalar fields. So, we can assemble solutions of R-N black hole, so provided a p-form gauge field in d dimensions, the WGC indicates the presence of a (p-1)-brane can meet the $p(d - p - 2)T^2/d - 2 \leq Q^2M_p^{d-2}$. [67–76, 84–87]. Here we note that the motivation for the understanding WGC is well seen in the physics of black holes. This conjecture expresses that in an approach with a gauge coupled to gravity, the entire lattice of authorized gauge charges must be settled by physical conditions. This is not required in quantum field theory since a charged particle can be decoupled from this theory by transmitting the mass to $\infty$. The second issue is the breaking global symmetries. There are fascinating relationships between the lack of global symmetries in quantum gravity and spectrum completeness. A typical method of breaking higher form global symmetries is absolutely by including charged conditions. While exclusively if the charged conditions spectrum exists entire, can one break the whole group [39–47,67–76].

In general, in this article, we seek to prove or emerge the weak gravity conjecture from the body of conformal field theory equations. Hence we discuss the mixed Klein Gordon equation against the background of the black hole as a general perturbation, and we focus on a charged
scalar $\Phi$ with charge $q$ and mass $m$ \[88\].

$$\frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \Phi) - 2i q g^{\mu\nu} A_\mu \partial_\nu \Phi - q^2 g^{\mu\nu} A_\mu A_\nu \Phi - m^2 \Phi = 0$$  \hspace{1cm} (1)$$

By inserting $\Phi = e^{-i\omega t} \phi(\vec{x})$, we get its scalar function also $\omega$ is related to the energy part. Then we use $CFT$ and write the two-point correlation function for the scalar operator $J_k$. We can also write the correlation function based on the ratio of its sub-coefficients as follows: \[89–91\]

$$\Upsilon^{(k)}_R(\omega) = < J_k(-\omega) J_k(\omega) > = \frac{B_k(\omega)}{A_k(\omega)}$$  \hspace{1cm} (2)$$

By putting $A_k(\omega) = 0$ one can find the location of the poles in the Green function. In that case will have $\omega = Re(\omega) + Im(\omega)$, it has real part (normal mode) and imaginary part (quasi-normal mode). The imaginary amount of energy indicates that our system is in perturbation. One of the places where can be found is in perturbation black holes. In $CFT$ for $\tau_d = -\frac{2\pi}{Im(\omega)}$ when $T \to 0$, we can define a lower band. We also use \[89\] and \[92\] and write the following relation \[93–95\].

$$\tau_d \geq \frac{1}{T}$$  \hspace{1cm} (3)$$

In the next section, to discuss $WGC$ conditions from CFT side, we will use the above description for the two black holes as Hyperscaling violating black hole and Kerr-Newman-AdS (KN AdS) black hole.

3 The investigation of CFT and WGC with hyperscaling violating black hole

The metric of a Hyperscaling violating black hole with mass $M$ and charge $Q$ is written by, \[89,92,96,99\]

$$ds^2 = r^{-2\theta} \left[ -r^{2x} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\Omega_k^2 \right]$$  \hspace{1cm} (4)$$

where $z$ and $\theta$ are dynamical and hyperscaling violation parameters, respectively. Also, $f(r)$ will be as,

$$f(r) = 1 + \frac{k}{r^2} \left( \frac{(d-1)^2}{z+d-\theta-2} - \frac{M}{r^{x+d-\theta}} + \frac{Q^2}{r^{2(z+d-\theta)}} \right)$$  \hspace{1cm} (5)$$

Here $k = -1, 0, 1$ determine the hyperboloid, planar and spherical topology for the black hole horizon, they have limitation \[96\]. In this article, we consider the $k = 0$, so we will have,

$$ds^2 = r^{-2\theta} \left[ -r^{2x} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2 \right]$$

$$f(r) = 1 - \frac{M}{r^{x+d-\theta}} + \frac{Q^2}{r^{2(z+d-\theta)}}$$  \hspace{1cm} (6)$$
where \( dx^2 = \sum_{i=1}^d dx_i^2 \) and \( x_i \) are spatial coordinates of a \( d \) dimensional space. By placing \( f(r_H) = 0 \), we obtain the event horizon radius, \( r_H \), for charged black hole solution,

\[
r_{H}^{2(d+z-\theta-1)} - M r_{H}^{d+z-\theta-2} + Q^2 = 0
\]  

(7)

Using the formula \( T = \frac{r_{H}}{4\pi}\frac{1}{|\dot{f}(r_H)|} \) and (6), one can obtain the Hawking temperature, which is given by [97].

\[
T = \frac{(z+d-\theta)r_{H}^2}{4\pi}
\left( 1 - \frac{(z+d-\theta-2)Q^2}{z+d-\theta}r_{H}^{2(z-d+\theta+1)} \right)
\]

(8)

Therefore, by placing \( T = 0 \) the extremality bound is obtained by,

\[
r_{H}^{2z+2d-\theta-1} = \frac{(z+d-\theta-2)}{z+d-\theta}Q^2
\]

(9)

In this case, we use relations (7) and (9) and rewrite \( f(r) \) in terms of \( r_H \),

\[
f(r) = 1 - \frac{2(z+d-\theta-1)}{z+d-\theta-2} \left( \frac{r}{r_H} \right)^{z+d-\theta} + \frac{z+d-\theta}{z+d-\theta-2} \left( \frac{r}{r_H} \right)^{2(z+d-\theta-1)}
\]

(10)

Also the potential is given by [96],

\[
A_t = \sqrt{\frac{2(z+d-\theta)(d-\theta)}{z+d-\theta-2}r_{H}^{z+d-\theta-1}} \left( \frac{1}{r_{H}^{z+d-\theta-2}} - \frac{1}{r_{H}^{x+z+d-\theta-2}} \right)
\]

(11)

By changing the corresponding coordinates and examining \( r \) near the event horizon, we can easily write the geometry of \( AdS_2 \times \mathbb{R}^{d-1} \) [97]

\[
r = r_H + \frac{\epsilon r_{H}^2}{(z+d-\theta)(z+d-\theta-1)\zeta}, \quad t = \frac{\tau}{\epsilon r_{H}^2}
\]

(12)

Here we note that equations (12), (10) and (6) are obtained by the limit \( \epsilon \to 0 \) as follows

\[
ds^2 = \frac{2\zeta}{r_{H}^{z+2}} \left[ \frac{-d\tau^2 + d\zeta^2}{(z+d-\theta)(z+d-\theta-1)\zeta^2} + dx^2 \right]
\]

(13)

also, we will have,

\[
A_r = \frac{\sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)(z+d-\theta-1)\zeta} \times r_{H}^{-z+2}
\]

(14)
Now, using (1), (13) and (14) also with respect to a fact that metric allows for the separation of variables, \( \Phi(\tau, \zeta, \vec{x}) = e^{-i\omega \tau} e^{i\vec{k} \cdot \vec{x}} \phi(\zeta) \), one can calculate

\[
\partial^2_{\zeta} \phi(\zeta) + \left( \omega + \frac{q r_H^{-z+2} \sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)(z+d-\theta-1)} \right)^2 \times \phi(\zeta) \]

\[
- \frac{k^2 + m^2 r_H^2 (1-\frac{\theta}{d})}{(z+d-\theta)(z+d-\theta-1) \zeta^2} \times \phi(\zeta) = 0 \tag{15}
\]

According to the above equation, \(-k^2\) is the eigenvalue of the Laplacian in the flat base submanifold. Concerning to the above equations, \( \phi(\zeta) \) is obtained in terms of the Whittaker functions, Which is given by

\[
\phi(\zeta) = c_1 \text{WhittakerM} \left[ \frac{-iq r_H^{z+2} \sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)(z+d-\theta-1)}, \nu_k, 2i\omega \right] + c_2 \text{WhittakerW} \left[ \frac{-iq r_H^{z+2} \sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)(z+d-\theta-1)}, \nu_k, 2i\omega \right] \tag{16}
\]

Also, with respect to the above equation, the \( \nu_k \) is defined by,

\[
\nu_k = \sqrt{\frac{1}{4} + \frac{k^2}{(z+d-\theta)(z+d-\theta-1)} - \frac{q^2 r_H^{2z+4} \sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)^2(z+d-\theta-1)^2} + m^2 r_H^2 (1-\frac{\theta}{d})} \tag{17}
\]

We want to examine the obtained function \( \phi(\zeta) \) near the event horizon. To do this, we use the limit \( \zeta \to 0 \). Given the properties of the Whittaker function and the definition of \( \Delta_k = \frac{1}{2} - \nu_k \), we will have,

\[
\phi(\zeta)_{\zeta \to 0} \approx C \left[ \frac{\Gamma(2\nu_k)(-2i\omega)^{\frac{1}{2} - \nu_k}}{\Gamma\left(\frac{1}{2} + \nu_k - \frac{iq r_H^{z+2} \sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)(z+d-\theta-1)}\right)} \zeta^{\frac{1}{2} - \nu_k} \right. \\
\left. + \frac{\Gamma(-2\nu_k)(-2i\omega)^{\frac{3}{2} + \nu_k}}{\Gamma\left(\frac{1}{2} - \nu_k - \frac{iq r_H^{z+2} \sqrt{2(z+d-\theta)(d-\theta)}}{(z+d-\theta)(z+d-\theta-1)}\right)} \zeta^{\frac{3}{2} + \nu_k} \right] \tag{18}
\]

\[
\equiv B_k(\omega) \zeta^{\Delta_k} + A_k(\omega) \zeta^{1-\Delta_k}
\]
Using equations (2) and (18), the correlation function is obtained by following equation,

\[ X = \Gamma(2\nu_k)\Gamma\left(\frac{1}{2} - \nu_k\right) - \frac{iqr_H^{-z+2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)} \]

\[ Y = \Gamma(-2\nu_k)\Gamma\left(\frac{1}{2} + \nu_k\right) - \frac{iqr_H^{-z+2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)} \]

\[ \gamma_{(k)}(\omega) = (2\omega)^{-2\nu_k}e^{i\pi\nu_k} \times \frac{X}{Y} \]

According to equation (19), do not have \( \omega \) with an imaginary part, as a result, WGC cannot be discussed. Now we consider condition \( r_H \rightarrow r_H + \frac{e^{\tau_H}}{(x+d-\theta)(x+d-\theta-1)\xi_0} \) with respect to (13) and (14), so we can obtain,

\[ ds^2 = \frac{r_H^{2-2\nu}r_H^{-x+2}}{(z + d - \theta)(z + d - \theta - 1)\zeta^2} \left[ (1 - \frac{\zeta^2}{\zeta_0^2})d\tau^2 + (1 - \frac{\zeta^2}{\zeta_0^2})^{-1}d\zeta^2 \right] + r_H^{2-2\nu}dx^2 \]

and

\[ a_\tau = \sqrt{2(z + d - \theta)(d - \theta)}(x + d - \theta - 1)\zeta^2 r_H^{-x+2} \left( 1 - \frac{\zeta}{\zeta_0} \right) \]

The temperature associated to the metric in equation (20) is \( T = 1/2\pi\xi_0 \). Now, using (1), (20) and (21) also, with respect to this issue that the metric allows for the separation of variables, \( \Phi(\tau, \zeta, x) = e^{-i\omega\tau}e^{ik\cdot x} \phi(\zeta) \), one can calculate,

\[ \partial^2_\zeta \phi(\zeta) + \frac{2\zeta}{\zeta^2 - \zeta_0^2} \partial_\zeta \phi(\zeta) + \left[ \omega + \frac{qr_H^{-z+x+2}\sqrt{2(z + d - \theta)(d - \theta)(1 - \frac{\zeta}{\zeta_0})}}{(z + d - \theta)(z + d - \theta - 1)\zeta} \right]^2 \times \phi(\zeta) \]

\[ - \frac{k^2 + m^2r_H^{-2(1-\frac{\theta}{\sigma})}}{(z + d - \theta)(z + d - \theta - 1)\zeta^2(1 - \frac{\zeta^2}{\zeta_0^2})} \times \phi(\zeta) = 0 \]

By solving the above equation, \( \phi(\zeta, \omega) \) is obtained by,

\[ \phi_k(\zeta, \omega) \sim \left( \frac{1}{\zeta} - \frac{1}{\zeta_0} \right)^{-\frac{1}{2} + \nu_k} \left( \frac{\zeta_0 + \zeta}{\zeta_0 - \zeta} \right)^{-\frac{1}{2} + \nu_k} \times F_1 \left\{ \frac{1}{2} \pm \nu_k + i\omega\zeta_0 - i\frac{qr_H^{-z+2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}, \right\} \]

\[ \pm \nu_k + i\frac{qr_H^{-z+2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)} \pm \nu_k, \frac{2\zeta}{\zeta - \zeta_0} \]

\[ 7 \]
We examine the above solution on the AdS boundary. In that case we use the properties of the hypergeometric functions and the equation (2), we obtain the retarded Green’s function as follows,

\[
A = \Gamma(1 + 2\nu_k) \Gamma\left(1 - \nu_k - \frac{i\omega}{2\pi T} + \frac{iqr_H^{-2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}\right) \\
\times \Gamma\left(\frac{1}{2} - \nu_k + \frac{iqr_H^{-2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}\right)
\]

\[
B = \Gamma(1 - 2\nu_k) \Gamma\left(\frac{1}{2} + \nu_k - \frac{i\omega}{2\pi T} + \frac{iqr_H^{-2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}\right) \\
\times \Gamma\left(\frac{1}{2} + \nu_k + \frac{iqr_H^{-2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}\right)
\]

\[
\Upsilon_R^{(k)}(\omega) = (4\pi T)^{-2\nu_k} \times \frac{A}{B}
\]

where we used the explicit definition of \(T\) instead of \(\zeta_0\). When we get the polarity of Equation (2), \(\omega\) has two parts, real and imaginary as \(\omega = \Re(\omega) + \Im(\omega)\).

\[
\omega = \frac{qr_H^{-2}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)} - i2\pi T \left(\frac{1}{2} + n + \nu_k\right)
\]

Here, we can talk about \(WGC\) because we have a quasi-normal mode, it means that the corresponding black hole is not stable. Generally we know that, \(WGC\) holds where the black hole is unstable. So we note here, we are interested in the response of the system at low frequencies [93–95].

\[
\tau_d \equiv \tau_d^{(0,0)} = \frac{1}{T(\frac{1}{2} + \nu_0)}, \quad \nu_0 = \sqrt{1 - \frac{2q^2r_H^{-2\nu_k}(z + d - \theta)(d - \theta)}{(z + d - \theta)(z + d - \theta - 1)^2} + m^2r_H^{-2(1 - \frac{\nu}{2})}}
\]

Now using equations (3) and (26), we get the following condition,

\[
\left(mr_H^{-1 - \frac{\nu}{2}} - \frac{qr_H^{-2\nu_k}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}\right) \left(mr_H^{-1 - \frac{\nu}{2}} + \frac{qr_H^{-2\nu_k}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}\right) < 0
\]

The \(WGC\) condition is satisfied when we have the following expression,

\[
r_H^{-1 - \frac{\nu}{2}} = \frac{r_H^{-2\nu_k}\sqrt{2(z + d - \theta)(d - \theta)}}{(z + d - \theta)(z + d - \theta - 1)}
\]

According to the relation (9), \(r\) is positive when we have relation as \(z + d - \theta > 2\). The best answer to the above relationship is when we have \(z = 1, d = 1\) and \(\theta \to 0^-\) or \(z \to 1^+, d = 1\).
and $\theta = 0$. So, in those cases we have some figures as (1, 2).

As we can see in figure 1, you will find that if $\theta$ is closer to zero, $WGC$ will hold for more $r_H$ values. So in that case, it is set for both $r_H$ greater and less than one. In Figure 2, also we see

that when for $z$ that is closer to one, $r_H$ can be established for different values, but $z$ is slightly larger than one, $WGC$ for $r_H < 1$ will be established. When $\theta < d$ we have an solution for a value of $r_H < 1$, see figure 3. As we can see in figure 3, when we consider $\theta$ to be a number less than zero and $z$ and $d$ to be greater than one, then $WGC$ will hold for a value of $r_H < 1$. 

Figure 1: Left plot: $z = 1, d = 1,$ and $\theta = -0.01$ Right plot: $z = 1, d = 1,$ and $\theta = -0.1$

Figure 2: Left plot: $z = 1.01, d = 1,$ and $\theta = 0$ Right plot: $z = 1.1, d = 1,$ and $\theta = 0$
4 CFT, WGC and Kerr-Newman-AdS black hole

The metric of Kerr-Newman-AdS black hole in Boyer-Lindquist-type coordinates is as follows [100],

\[ ds^2 = -\frac{f(r)}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\rho^2}{f(r)} dr^2 + \frac{\rho^2}{f(\theta)} d\theta^2 + \frac{f(\theta)}{\rho^2} \sin^2 \theta \left( adt - \frac{(r^2 + a^2)}{\Xi} d\phi \right)^2 \] (29)

where

\[ f(r) = (r^2 + a^2)(1 + \frac{r^2}{\ell^2}) - 2Mr + Q^2 \]
\[ f(\theta) = 1 - \frac{a^2}{\ell^2} \cos^2 \theta \]
\[ \rho^2 = r^2 + a^2 \cos^2 \theta \]
\[ \Xi = 1 - \frac{a^2}{\ell^2} \] (30)

where \( a \) is the rotational parameter, and \( Q \) is the electric charge, as \( \ell^2 \) is related to the cosmological constant. If it is positive, we have \( dS \), and when it is negative, we have \( AdS \). Its vector potential is considered as follows,

\[ A = -\frac{Qr}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right) \] (31)
The Hawking temperature, entropy and angular velocity of the horizon are obtained by,

\[
T_H = \frac{r_+(1 + \frac{a^2}{\ell^2} + \frac{3r_+^2}{\ell^2} - \frac{a^2 + Q^2}{r_+^2})}{4\pi(r_+^2 + a^2)} \\
S = \frac{\pi(r_+^2 + a^2)}{2} \\
\Omega_H = \frac{\alpha}{r_+^2 + a^2}
\]

(32)

We can rewrite \(f(r)\) near the event horizon in quadratic order as follows [101],

\[
f(r) \simeq k(r - r_+)(r - r_*)
\]

where \(r_+\) is the radius of the external event horizon and \(r_*\) is the radius of another horizon. In the extremality limit, the following conditions also apply [101],

\[
r_+^2 \geq \frac{\ell^2}{6} \left( \sqrt{\left(1 + \frac{a^2}{\ell^2}\right)^2 + 12 \frac{a^2}{\ell^2}} - 1 \right)
\]

(34)

In the limit \(\frac{\alpha}{\ell} \ll 1\) we can consider the minimum value of \(r_+\) equal to \(a\). By putting \(f(r) = 0\) and \(T_H = 0\) we can get the extreme state of the black hole with the condition \(r_+ = a\) as follows,

\[
k = \frac{M_{\text{exe}}}{r_+} + \frac{4r_+^2}{\ell^2}
\]

(35)

We now use (1) to solve Klein Gordon’s equation in the background of the Kerr-Newman-AdS-dS black hole and obtain the correlation function, which is calculated by the following equation [101],

\[
\Upsilon_R = \frac{\Gamma(1 - 2h_Q)}{\Gamma(2h_Q - 1)} \times \frac{\Gamma(h_Q + i\frac{Q_L^L - Q_L^R}{2\pi\ell_L})\Gamma(h_Q + \frac{i\omega_R - Q_R^R}{2\pi\ell_R})}{\Gamma(1 - h_Q + i\frac{Q_L^L - Q_L^R}{2\pi\ell_L})\Gamma(1 - h_Q + \frac{i\omega_R - Q_R^R}{2\pi\ell_R})}
\]

(36)

where

\[
\omega_L = \frac{r_+^2 + r_*^2 + 2a^2}{2a\Xi}\omega \\
Q_L = Q_R = e \\
\mu_L = \frac{Q(r_+^2 + r_*^2 + 2a^2)}{2a\Xi(r_+ + r_*)} \\
T_L = \frac{k(r_+^2 + r_*^2 + 2a^2)}{4\pi a\Xi(r_+ + r_*)}
\]

\[
\omega_R = \frac{r_+^2 + r_*^2 + 2a^2}{2a\Xi}\omega - M \\
\mu_R = \frac{Q(r_+ + r_*)}{2a\Xi} \\
T_R = \frac{k(r_+ - r_*)}{4\pi a\Xi}
\]

(37)
also we have,

\[ h_Q = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \left( \frac{e^2 Q^2}{k^2} - \frac{K_Q}{k} \right)} \]  \tag{38} 

where \( K_Q \) is the separation constant [101]. The pole of equation (36) is given as follows,

\[ 1 - h_Q + i \frac{\omega_L - Q_L \mu_L}{2 \pi T_L} = -n \quad 1 - h_Q + i \frac{\omega_R - Q_R \mu_R}{2 \pi T_R} = -n \]  \tag{39} 

Since, above two relations are similar to each other, we consider one of them and place it in relation (3) with condition of \( K_Q = -3k \) and obtain following equation,

\[ \frac{eQ}{k} > 1 \]  \tag{40} 

Using equations (34), (35) and (40) also with condition \( \frac{a}{\ell} \ll 1 \), we achieve the following equation,

\[ \frac{eaQ}{M_{exe} + \frac{4a^3}{\ell^2}} > 1 \]  \tag{41} 

The above statement satisfies the WGC condition for \( \frac{a}{\ell} \ll 1 \) and \( e = \frac{1}{a} \).

## 5 Conclusion

As we know, weak gravity conjecture idea derived from string theory, it has been introduced as part of the swampland program. Weak gravity conjecture has been challenged in many cosmological structures, such as inflation, dark energy, and black holes physics [37–82]. On the other hand, conformal field theory in recent years, the researchers have done a lot of work in this field and the famous correspondence (AdS/CFT). Even new theories have been introduced that many researchers are still researching in this field [?, 4–29]. But the connection between the two theories, namely the weak gravity conjecture and the structure of the conformal field theory, can lead to exciting results. However, in recent years various works have been done in this area. Therefore, the authors of this article also seek to examine the relationship between the conformal field theory and the idea of weak gravity conjecture. Therefore, in this paper, the primary purpose was to investigate the relationship between weak gravity conjecture and conformal field theory by challenging the Hyperscaling violating and Kerr-Newman-AdS black holes. In that case, we used the correlation function of conformal field theory. Hence, we obtained proof for the prediction of weak gravity conjecture on the CFT side. It means that the emergence of weak gravity conjecture through correlation function calculations by considering black holes such as Hyperscaling violating and Kerr-Newman-AdS black holes. Since the weak
gravity conjecture at critical points has compatibility results, it also appears in extremality bound of black holes. So with this assumption and conformal field theory correlation function, we calculated the critical points of each mentioned black hole. By calculating the critical points, we can challenge the weak gravity conjecture. Hence, weak gravity conjecture will directly appear for both black holes. In addition to proving WGC, we have also shown the exciting close relation between these two ideas, namely weak gravity conjecture and conformal field theory. In that case, we used the correlation function in CFT and its poles, we obtained the energy spectrum of the black hole and found that the energy consists of an imaginary part, in which case our black hole was disturbed and we can discuss WGC. We found that when $z = 1, d = 1$ and $\theta \to 0^-$, it is the best form to have WGC in the Hyperscaling violating black holes because it contains $r_H$ larger and smaller than one. In other cases, WGC is only valid for $r_H$ less than one. The WGC condition for the Kerr-Newman-AdS black hole is related to the rotation and radius of the AdS of the black hole if the charged particle near the black hole was $\frac{1}{\ell}$ and have a ratio such as $\frac{a}{\ell} \ll 1$. One can challenge the calculations shown in this article for other black holes using different conditions and discuss the results. Since a relationship between the two ideas has somehow emerged, we can do more calculations to look at the results and even reach a correspondence between the two theories, which we will examine as future work.

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