SUCCESSIVE MINIMA AND LATTICE POINTS

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Abstract. The main purpose of this note is to prove an upper bound on the number of lattice points of a centrally symmetric convex body in terms of the successive minima of the body. This bound improves on former bounds and narrows the gap towards a lattice point analogue of Minkowski’s second theorem on successive minima.

Minkowski’s proof of his second theorem is rather lengthy and it was also criticised as obscure. We present a short proof of Minkowski’s second theorem on successive minima, which, however, is based on the ideas of Minkowski’s proof.

1. Introduction

In 1896 Hermann Minkowski’s fundamental and guiding book “Geometrie der Zahlen” [Min96] was published, which may be considered as the first systematic study on relations between convex geometry, Diophantine approximation, and the theory of quadratic forms (cf. GRÜBER [Gru93]). One of the basic problems in geometry of numbers is to decide whether a given set in the $d$-dimensional Euclidean space $\mathbb{R}^d$ contains a non-trivial lattice point of a $d$-dimensional lattice $\Lambda \subset \mathbb{R}^d$. With respect to the class $\mathcal{K}_d^0$ of all 0-symmetric convex bodies in $\mathbb{R}^d$ with non-empty interior and the volume $\operatorname{vol}(\cdot)$ – $d$-dimensional Lebesgue measure – Minkowski settled this problem:

$$\operatorname{vol}(K) \geq 2^d \cdot \det \Lambda \quad \text{then } K \text{ contains a non-zero lattice point of } \Lambda.$$ (1.1)

Here $\det \Lambda$ denotes the determinant of the lattice $\Lambda$ and the space of all lattices $\Lambda \subset \mathbb{R}^d$ with $\det \Lambda \neq 0$ is denoted by $\mathcal{L}^d$.

Minkowski assessed his result as “ein Satz, der nach meinem Dafürhalten zu den fruchtbarsten in der Zahlenlehre zu rechnen ist” ([Min96], p. 75) and indeed this theorem has many applications (cf. [EGHS9], sec. 3.3). Minkowski proved even a stronger result, for which we have to introduce his “kleinste System von unabhängig gerichteten Strahlendistanzen im Zahlengitter” ([Min96], p. 178).

Definition 1.1. Let $K \in \mathcal{K}_d^0$ and $\Lambda \in \mathcal{L}^d$. For $1 \leq i \leq d$

$$\lambda_i(K, \Lambda) = \min \{ \lambda \in \mathbb{R}_{\geq 0} : \lambda K \text{ contains } i \text{ linearly independent lattice points of } \Lambda \}$$

is called the $i$-th successive minimum of $K$ with respect to $\Lambda$.

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Obviously, we have \( \lambda_1(K, \Lambda) \leq \lambda_2(K, \Lambda) \leq \cdots \leq \lambda_d(K, \Lambda) \) and the first successive minimum \( \lambda_1(K, \Lambda) \) is the smallest dilation factor such that \( \lambda_1(K, \Lambda) K \) contains a non-zero lattice point. With this notation Minkowski’s first theorem on successive minima reads (cf. [Min96], pp. 75)

**Theorem 1.2 (Minkowski).** Let \( K \) in \( \mathcal{K}_d^0 \) and \( \Lambda \in \mathcal{L}^d \). Then

\[
\lambda_1(K, \Lambda)^d \text{vol}(K) \leq 2^d \det \Lambda.
\]

So \( \text{vol}(K) \geq 2^d \det \Lambda \) implies \( \lambda_1(K, \Lambda) \leq 1 \), and we get (1.1). Minkowski’s second theorem on successive minima is a deep improvement of the first one and says (cf. [Min96], pp. 199)

**Theorem 1.3 (Minkowski).** Let \( K \in \mathcal{K}_d^0 \) and \( \Lambda \in \mathcal{L}^d \). Then

\[
\lambda_1(K, \Lambda) \cdot \lambda_2(K, \Lambda) \cdot \ldots \cdot \lambda_d(K, \Lambda) \cdot \text{vol}(K) \leq 2^d \det \Lambda.
\]

This inequality is best possible. For instance, with respect to the integral lattice \( \mathbb{Z}^d \), each box with axes parallel to the coordinate axes gives equality. Although Theorem 1.3 has not so many applications as the first theorem on successive minima, it shows a beautiful relation between the volume of \( K \) and the expansion of \( K \) with respect to independent lattice directions of a lattice. The importance of Theorem 1.3 is also reflected in the number of different proofs, see e.g. Bambah, Woods & Zassenhaus [BWZ65], Cassels [Cas59], Danicic [Dan69], Davenport [Dav39], Estermann [Est46], Siegel [Sie89] and Weyl [Wey42].

In [BHW93] it was conjectured that an inequality analogue to Theorem 1.3 holds for the lattice point enumerator \( \#(K \cap \Lambda) \). More precisely,

**Conjecture 1.4.** Let \( K \in \mathcal{K}_d^0 \) and \( \Lambda \in \mathcal{L}^d \). Then

\[
\#(K \cap \Lambda) \leq \prod_{i=1}^d \left\lfloor \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rfloor.
\]

(1.2)

Here \( \lfloor x \rfloor \) denotes the smallest integer not less than \( x \). An analogous statement to the first theorem of Minkowski on successive minima was already shown in [BHW93], namely

\[
\#(K \cap \Lambda) \leq \left\lfloor \frac{2}{\lambda_1(K, \Lambda)} + 1 \right\rfloor^d.
\]

(1.3)

It seems to be worth mentioning that if Conjecture 1.4 were true then we could write by the definition of the Riemann integral

\[
\frac{\text{vol}(K)}{\det \Lambda} = \lim_{r \to 0} \frac{r^d \#(K \cap r\Lambda)}{\det \Lambda} \leq \lim_{r \to 0} \prod_{i=1}^d \left[ \frac{2}{\lambda_i(K, r\Lambda)} + 1 \right] = \prod_{i=1}^d \frac{2}{\lambda_i(K, \Lambda)}.
\]
Thus Conjecture 1.4 implies Minkowski’s second theorem on successive minima (Theorem 1.3). In [BHW93] the validity of the conjecture was proven in the case $d = 2$. Moreover, it was shown that an upper bound of this type exists, if in the above product $2^{\lambda_i(K, \Lambda)}$ is replaced by $2^{2 \lambda_i(K, \Lambda)}$. So, roughly speaking, (1.2) holds up to a factor $d!$. Here we shall improve this bound.

**Theorem 1.5.** Let $d \geq 2$, $K \in K^d_0$ and $\Lambda \in \mathcal{L}^d$. Then

$$\#(K \cap \Lambda) < 2^{d-1} \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rfloor.$$

The proof of this theorem will be given is the next section. Minkowski’s original proof ([Min96], 199-218) of his second theorem on successive minima was sometimes criticised as lengthy and obscure (cf. [Dav77], p.91). One reason might be that in the scope of the proof he also proves many basic facts about the volume of a convex body, like the computation of the volume through successive integrations, etc., which cloud a little bit the simple and nice geometrical ideas of his proof. Based on these ideas we present a short proof of Theorem 1.3 in the last section.

For more information on lattices, successive minima and their role in the geometry of numbers we refer to the books of ERD˝OS, GRUBER AND HAMMER [EGH89], GRUBER AND LEEKERKERKER [GL87] and the survey of GRUBER [Gru93]. For an elementary introduction to the geometry of numbers see [OLD00].

2. Proof of Theorem 1.5

Before giving the proof we list some basic facts on lattices, for which we refer to [GL87]. Every lattice $\Lambda \in \mathcal{L}^d$ can be written as $\Lambda = AZ^d$, where $A$ is a non-singular $(d \times d)$-matrix, i.e., $A \in \text{GL}(d, \mathbb{R})$. In particular we have $\lambda_i(K, \Lambda) = \lambda_i(A^{-1}K, \mathbb{Z}^d)$ and $\#(K \cap \Lambda) = \#(A^{-1}K \cap \mathbb{Z}^d)$. A lattice $\Lambda \in \mathcal{L}^d$ is called a sublattice of $\Lambda \in \mathcal{L}^d$ if $\Lambda \subset \Lambda$. For $a, \overline{a} \in \Lambda$ and a sublattice $\Lambda \subset \Lambda$ we write

$$a \equiv \overline{a} \mod \Lambda \iff (a - \overline{a}) \in \Lambda.$$

In words, $a, \overline{a}$ belong to the same residue class (coset) of $\Lambda$ with respect to $\Lambda$.

We note that there are precisely $\text{det} \Lambda / \text{det} \Lambda$ different residue classes of $\Lambda$ with respect to $\Lambda$. For every set of $d$-linearly independent lattice points $a^1, \ldots, a^d$ of a lattice $\Lambda$ there exists a basis $b^1, \ldots, b^d$ of $\Lambda$ such that $\text{lin}\{a^1, \ldots, a^i\} = \text{lin}\{b^1, \ldots, b^i\}$, where $\text{lin}$ denotes the linear hull. In particular, given $d$ linearly independent lattice vectors $z^i \in \mathbb{Z}^d$, $1 \leq i \leq d$, with $z^i \in \lambda_i(K, \mathbb{Z}^d)K$ then there exists an unimodular matrix $U$, i.e., $U \in \text{GL}(d, \mathbb{R}) \cap \mathbb{Z}^{d \times d}$, such that

$$Uz^i \in \left(\lambda_i(UK, \mathbb{Z}^d)UK\right) \cap \text{lin}\{e^1, \ldots, e^i\}, \quad 1 \leq i \leq d,$$

where $e^i \in \mathbb{R}^d$ denotes the $i$-th unit vector. Furthermore we note that for $d$ linearly independent lattice points $a^1, \ldots, a^d$ of a lattice $\Lambda \in \mathcal{L}^d$ satisfying
\[ a^i \in \lambda_i(K, \Lambda) K, \] the definition of the successive minima implies

\[(2.2) \quad \text{int}(\lambda_i(K, \Lambda) K) \cap \Lambda \subset \text{lin}\{0, a^1, \ldots, a^{i-1}\} \cap \Lambda, \quad 1 \leq i \leq d, \]

where int denotes the interior.

For the proof of Theorem 1.5 we need the following simple lemma.

\textbf{Lemma 2.1.} Let \( K \in C_0^d, \Lambda \in \mathcal{L}^d \) and let \( \Lambda \) be a sublattice of \( \Lambda \). Then

\[ \#(K \cap \Lambda) \leq \frac{\det \Lambda}{\det \Lambda} \#(2K \cap \Lambda). \]

\textit{Proof.} Let \( m = \#(2K \cap \tilde{\Lambda}) \) and suppose there exist at least \( m + 1 \) different lattice points \( a^1, \ldots, a^{m+1} \in K \cap \Lambda \) such that \( a^i \equiv a^1 \mod \Lambda, \quad 1 \leq i \leq m + 1 \). Then we have

\[ a^i - a^1 \in (K - K) \cap \tilde{\Lambda} = 2K \cap \tilde{\Lambda}, \quad 1 \leq i \leq m + 1, \]

which contradicts the assumption \( \#(2K \cap \tilde{\Lambda}) = m \). Thus we have shown that every residue class of \( \Lambda \) with respect to \( \tilde{\Lambda} \) does not contain more than \( m \) points of \( K \cap \Lambda \). Since there are precisely \( \det \Lambda/\det \Lambda \) different residue classes, we get the desired bound. \( \square \)

We remark that inequality (1.3) is a simple consequence of this lemma. To see this we set \( n_1 = [2/\lambda_1(K, \Lambda) + 1] \) and \( \Lambda = n_1 \Lambda \). Next we observe that \( a \in 2K \cap \Lambda \) implies that \( \frac{1}{n_1}a \in \text{int}(\lambda_1(K, \Lambda) K) \cap \Lambda \) and from (2.2) we conclude \( \#(2K \cap \Lambda) = 1 \). Thus Lemma 2.1 gives

\[ \#(K \cap \Lambda) \leq \frac{\det \Lambda}{\det \Lambda} = (n_1)^d = \left[ \frac{2}{\lambda_1(K, \Lambda)} + 1 \right]^d. \]

Next we come to the proof of Theorem 1.5.

\textit{Proof of Theorem 1.5.} W.l.o.g. let \( \Lambda = \mathbb{Z}^d \) and we may assume that (cf. (2.1) and (2.2))

\[(2.3) \quad \text{int}(\lambda_i(K, \mathbb{Z}^d) K) \cap \mathbb{Z}^d \subset \text{lin}\{0, e^1, \ldots, e^{i-1}\} \cap \mathbb{Z}^d, \quad 1 \leq i \leq d.\]

For abbreviation we set \( q_i = \left[ \frac{2}{\lambda_i(K, \Lambda)} + 1 \right], 1 \leq i \leq d \), and first we determine \( d \) numbers \( n_i \in \mathbb{N} \) such that

\[(2.4) \quad n_d = q_d, \quad q_i \leq n_i < 2q_i, \quad \text{and} \quad n_{i+1} \text{ divides } n_i, \quad 1 \leq i \leq d - 1.\]

Suppose we have already found \( n_1, \ldots, n_{k+1} \) with these properties. In order to determine \( n_k \) we distinguish two cases. If \( n_{k+1} \geq q_k \) we set \( n_k = n_{k+1} \). Since \( q_k \geq q_{k+1} \) we obtain \( q_k \leq n_k = n_{k+1} < 2q_{k+1} \leq 2q_k \). Otherwise, if \( n_{k+1} < q_k \) let \( q_k = m \cdot n_{k+1} + r \) with \( m \in \mathbb{N}, m \geq 1, \) and \( 0 \leq r < n_{k+1} \). In this case we set \( n_k = q_k + n_{k+1} - r \) and obviously, \( n_k \) meets the requirements of (2.4).

Now let \( \Lambda \subset \mathbb{Z}^d \) be the lattice generated by the vectors \( n_1 e^1, n_2 e^2, \ldots, n_d e^d \). Then we have \( \det \Lambda/\det \Lambda = n_1 \cdot n_2 \cdot \ldots \cdot n_d \) and together with the upper bounds
on the the numbers $n_i$, Lemma 2.3 gives

\begin{equation}
\#(K \cap \Lambda) \leq \#(2K \cap \Lambda) \prod_{i=1}^{d} n_i < \#(2K \cap \Lambda) 2^{d-1} \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \Lambda)} + 1 .
\end{equation}

Hence, in order to verify the theorem, it suffices to show $2K \cap \Lambda = \{0\}$. Suppose there exists a $g \in 2K \cap \Lambda \setminus \{0\}$ and let $k$ be the largest index of a non-zero coordinate of $g$, i.e., $g_k \neq 0$ and $g_{k+1} = \cdots = g_d = 0$. Then we may write

$$g = z_1 (n_1 e^1) + z_2 (n_2 e^2) + \cdots + z_k (n_k e^k) \in 2K$$

for some $z_i \in \mathbb{Z}$. Since $n_k$ is a divisor of $n_1, \ldots, n_{k-1}$ and since $2/n_k < \lambda_k(K, \mathbb{Z}^d)$ (cf. (2.3)) we obtain

$$\frac{1}{n_k} g \in \left( \frac{2}{n_k} K \right) \cap \mathbb{Z}^d \subset \text{int}(\lambda_k(K, \mathbb{Z}^d)K) \cap \mathbb{Z}^d.$$ 

However, since $g_k \neq 0$ this relation violates (2.3). Thus we have $2K \cap \Lambda = \{0\}$ and the theorem is proven.

\[ \Box \]

### 3. Proof of Theorem 1.3

Minkowski’s proof of his second theorem on successive minima can be found in his book “Geometrie der Zahlen” ([Min96], 199–219) and for an English translation we refer to [Han64], 570–603.

**Proof of Theorem 1.3** (following Minkowski). Again w.l.o.g. we may assume that $\Lambda = \mathbb{Z}^d$. For convenience we write $\lambda_i = \lambda_i(K, \mathbb{Z}^d)$ and set $K_i = \frac{\lambda_i}{2} K$. Furthermore, we assume that $z^1, \ldots, z^d$ are $d$ linearly independent lattice points with $z^i \in \lambda_i K \cap \mathbb{Z}^d$ and $\text{lin}\{z^1, \ldots, z^d\} = \text{lin}\{e^1, \ldots, e^d\}$, $1 \leq i \leq d$, (cf. (2.1)). For short, we denote the linear space $\text{lin}\{e^1, \ldots, e^d\}$ by $L_i$.

For an integer $q \in \mathbb{N}$ let $M^d_q = \{z \in \mathbb{Z}^d : |z| \leq q, 1 \leq i \leq d\}$ and for $1 \leq j \leq d - 1$ let $M^d_q = M^d_q \cap L_j$. Since $K$ is a bounded set there exists a constant $\gamma$, only depending on $K$, such that

\begin{equation}
\text{vol}(M^d_q + K_d) \leq (2q + \gamma)^d.
\end{equation}

By the definition of $\lambda_1$ we have $(z + \text{int}(K_1)) \cap (\overline{z} + \text{int}(K_1)) = \emptyset$ for two different lattice point $z, \overline{z} \in \mathbb{Z}^d$, because otherwise we would get the contradiction $z - \overline{z} \in (\text{int}(K_1) - \text{int}(K_1)) \cap \mathbb{Z}^d = \text{int}(K_1 - K_1) \cap \mathbb{Z}^d = \text{int}(\lambda_1 K) \cap \mathbb{Z}^d = \{0\}$. Thus we have

\begin{equation}
\text{vol}(M^d_q + K_1) = (2q + 1)^d \text{vol}(K_1) = (2q + 1)^d \left( \frac{\lambda_1}{2} \right)^d \text{vol}(K).
\end{equation}

In the following we shall show that for $1 \leq i \leq d - 1$

\begin{equation}
\text{vol}(M^d_q + K_{i+1}) \geq \left( \frac{\lambda_{i+1}}{\lambda_i} \right)^{d-i} \text{vol}(M^d_q + K_i).
\end{equation}
To this end we may assume $\lambda_{i+1} > \lambda_i$ and let $z, \bar{z} \in \mathbb{Z}^d$, which differ in the last $d - i$ coordinates, i.e., $(z_{i+1}, \ldots, z_d) \neq (\bar{z}_{i+1}, \ldots, \bar{z}_d)$. Then

\begin{equation}
|z + \text{int}(K_{i+1})| \cap |\bar{z} + \text{int}(K_{i+1})| = \emptyset.
\end{equation}

Otherwise the $i + 1$ linearly independent lattice points $z - \bar{z}, z^1, \ldots, z^i$ belong to the interior of $\lambda_{i+1}K$ which contradicts the minimality of $\lambda_{i+1}$. Hence we obtain from (3.4)

\[\text{vol} \left( M_q^d + K_{i+1} \right) = (2q + 1)^{d-i} \text{vol} \left( M_q^i + K_{i+1} \right),\]
\[\text{vol} \left( M_q^d + K_i \right) = (2q + 1)^{d-i} \text{vol} \left( M_q^i + K_i \right).
\]

and in order to verify (3.3) it suffices to show

\begin{equation}
\text{vol} \left( M_q^i + K_{i+1} \right) \geq \left( \frac{\lambda_{i+1}}{\lambda_i} \right)^{d-i} \text{vol}(M_q^i + K_i).
\end{equation}

Let $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}^d$ be the linear maps given by

\[f_1(x) = \left( \frac{\lambda_{i+1}}{\lambda_i} x_1, \ldots, \frac{\lambda_{i+1}}{\lambda_i} x_i, x_{i+1}, \ldots, x_d \right),\]
\[f_2(x) = \left( x_1, \ldots, x_i, \frac{\lambda_{i+1}}{\lambda_i} x_{i+1}, \ldots, \frac{\lambda_{i+1}}{\lambda_i} x_d \right).
\]

Since $M_q^i + K_{i+1} = f_2(M_q^i + f_1(K_i))$ we get

\[\text{vol} \left( M_q^i + K_{i+1} \right) = \left( \frac{\lambda_{i+1}}{\lambda_i} \right)^{d-i} \text{vol}(M_q^i + f_1(K_i))
\]

and for the proof of (3.3) we have to show

\begin{equation}
\text{vol} \left( M_q^i + f_1(K_i) \right) \geq \text{vol} \left( M_q^i + K_i \right).
\end{equation}

To this end let $L_i^\perp$ be the $(d - i)$-dimensional orthogonal complement of $L_i$. Then it is easy to see that for every $x \in L_i^\perp$ there exists a $t(x) \in L_i$ with $K_i \cap (x + L_i) \subset (f_1(K_i) \cap (x + L_i)) + t(x)$ and so

\[\left( M_q^i + K_i \right) \cap (x + L_i) \subset \left[ \left( M_q^i + f_1(K_i) \right) \cap (x + L_i) \right] + t(x).
\]

Thus we get

\[
\text{vol}(M_q^i + K_i) = \int_{x \in L_i^\perp} \text{vol} \left( \left( M_q^i + K_i \right) \cap (x + L_i) \right) \, dx \\
\leq \int_{x \in L_i^\perp} \text{vol} \left( \left( M_q^i + f_1(K_i) \right) \cap (x + L_i) \right) \, dx \\
= \text{vol}(M_q^i + f_1(K_i)),
\]
where \( \text{vol}_i(\cdot) \) denotes the \( i \)-dimensional volume. This shows (3.6) and so we have verified (3.3). Finally, it follows from (3.1), (3.2) and (3.3)

\[
(2q + \gamma)^d \geq \text{vol} \left( M_q^d + K_d \right) \geq \left( \frac{\lambda_d}{\lambda_{d-1}} \right) \text{vol} \left( M_q^d + K_{d-1} \right) \\
\geq \left( \frac{\lambda_d}{\lambda_{d-1}} \right)^2 \left( \frac{\lambda_{d-1}}{\lambda_{d-2}} \right)^2 \text{vol} \left( M_q^d + K_{d-2} \right) \geq \cdots \\
\geq \left( \frac{\lambda_d}{\lambda_{d-1}} \right)^d \left( \frac{\lambda_{d-1}}{\lambda_{d-2}} \right)^{d-1} \cdots \left( \frac{\lambda_2}{\lambda_1} \right)^{d-1} \text{vol} \left( M_q^d + K_1 \right) \\
= \lambda_d \cdots \lambda_1 \frac{\text{vol}(K)}{2^d} \left( 2q + 1 \right)^q
\]

and so

\[
\lambda_1 \cdots \lambda_d \text{vol}(K) \leq 2^d \left( \frac{2q + \gamma}{2q + 1} \right)^d.
\]

Since this holds for all \( q \in \mathbb{N} \) the theorem is proven. 

\[ \square \]

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