Targeted cutting of random recursive trees.

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Outline

1. Random cutting (Fortuitous failures)
2. Targeted cutting (Malicious attack)
3. Results
4. Coupling
5. Work in progress
Increasing trees

- Edges directed towards root
- Labels $[n] = \{1, 2, \ldots, n\}$
- Labels increase away from root
- $\deg_{T_n}(v), \Delta(T_n)$
Building a random recursive tree

A random recursive tree on $n$ vertices is a tree chosen uniformly at random on the class of increasing trees on $[n]$.

- $T_1$ is a tree with a single vertex
- For $i > 1$ build $T_i$ from $T_{i-1}$ by adding
  - vertex $i$
  - arrow $i \rightarrow j$

$$P(i \rightarrow j | T_{i-1}) = \frac{1}{i-1}$$
Building a RRT

Figure: Obtaining $T_8$ from $T_7$
Characteristics

\[ D := \deg_{T_n}(1) = \sum_{i=2}^{n} B_i, \text{ where } B_i \text{ is a Bernoulli } \left( \frac{1}{i-1} \right) \]

\[ \mathbb{E} D = H_{n-1} \sim \ln(n) \]

Goh and Schmutz [2002]. Let \( \Delta(T_n) \) be the maximum degree in \( T_n \). Then

\[ \Delta(T_n) \sim \log_2(n) \approx 1.4 \ln(n) \]
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Na and Rapoport [1970]

\[ Z_i(n) := \#\{1 \leq j \leq n \mid \text{deg}_{T_n}(j) = i\}, \]

\[ \frac{\mathbb{E}Z_i(n)}{n} \xrightarrow{n \to \infty} \frac{1}{2^{i+1}}, \]

\[ \mathbb{E}Z_i(n) \xrightarrow{n \to \infty} \begin{cases} \infty & \text{for } i << \log_2(n) \\ \frac{1}{2} & \text{for } i - \log_2(n) = \Theta(1) \\ 0 & \text{for } i >> \log_2(n) \end{cases} \]
Introduction

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Addario-Berry and Eslava [2017]

\[ Z_{\geq i}(n) = \#\{1 \leq j \leq n \mid \deg_{T_n}(j) \geq i\}. \]

\[ \mathbb{E}Z_{\geq k+\lfloor \log_2(n) \rfloor}(n) \xrightarrow{n \to \infty} \frac{1}{2^k} \]

for \( k \in \mathbb{N} \) and \( k = k(n) \) in a suitable range.
Random cutting

Introduced by Meir and Moon [1974]

- Choose an edge uniformly at random and delete it
- Discard the subtree not containing the root
- Repeat this procedure until the root is isolated

![Tree Diagram]

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Let $J_n = \#$ cuts needed to destroy a random recursive tree with $n$ vertices.
Key property: After each cut the remaining tree is a RRT. Used to obtain

$$\mathbb{E} J_n \sim \frac{n}{\ln n}$$

Meir and Moon [1974]

$$\mathbb{E} J_n \sim \frac{n}{\ln(n)}$$

Panholzer [2004]

$$\frac{\ln (n)}{n} J_n \overset{P}{\to} 1$$

Drmota et al. [2009]

$$\frac{(\ln (n))^2}{n} J_n - \ln (n) - \ln (\ln (n))$$

converges weakly to a random variable $Y$ with characteristic function

$$\varphi_Y(\lambda) = \exp \left\{ i \lambda \ln |\lambda| - \frac{\pi |\lambda|}{2} \right\}$$
**Our proposal: Targeted cutting**

- List the vertices from highest to lowest degree
- Sequentially remove the vertex in the listed order, discarding the subtrees not containing the root
- End when the root is removed
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We lose properties that were known in the random cutting procedure, including
- The distribution of the remaining tree
- The number of vertices deleted after each cut

Let $K_n = \#$ targeted cuts needed to delete a random recursive tree with $n$ vertices. Recall
- $Z_{\geq i}(n) = \# \{ 1 \leq j \leq n \mid \deg_{T_n}(j) \geq i \}$
- $D = \deg_{T_n}(1)$

Note

$$K_n \leq Z_{\geq D}(n).$$
Let $\gamma := 1 - \ln(2) \approx 0.307$

**Theorem (Moments)**

*For any positive integer $k$,*

$$\mathbb{E} \left( \ln \left( Z_{\geq D} \right) \right)^k = \left( \ln \left( n^\gamma \right) \right)^k (1 + O(1)).$$

**Theorem (Convergence in probability)**

$$\frac{\ln(Z_{\geq D})}{\ln(n)} \xrightarrow{p} \gamma.$$

**Theorem (Growth order)**

*For any $\varepsilon > 0$,*

$$K_n = O_p(n^{\gamma + \varepsilon}).$$
Conditioned on the event

\[(1 - \varepsilon) \ln(n) \leq D \leq (1 + \varepsilon) \ln(n),\]

which we denote \(D \in (1 \pm \varepsilon) \ln(n)\), we have that

\[Z \geq \lceil (1 + \varepsilon) \ln(n) + 1 \rceil \leq Z \geq D \leq Z \geq \lfloor (1 - \varepsilon) \ln(n) \rfloor.\]

Moments of the upper bound can be controlled [Addario-Berry and Eslava, 2017]. Lower bound can be approximated with a Poisson random variable [Eslava, 2020].
The idea is to build simultaneously $T_n$, a RRT, and $T_n^{(\varepsilon)}$, a RRT conditioned on $D^{(\varepsilon)} \in (1 \pm \varepsilon) \ln(n)$.

- $B = (B_2, \ldots, B_n)$ with independent entries where $B_i$ is Bernoulli $\left( \frac{1}{i-1} \right)$.
- $B^{(\varepsilon)} = (B_2^{(\varepsilon)}, \ldots, B_n^{(\varepsilon)})$ with the law of $B$ conditioned on $\sum_{i=2}^n B_i^{(\varepsilon)} \in (1 \pm \varepsilon) \ln(n)$.
- $Y_i$ uniform in $\{2, 3, \ldots, i - 1\}$ for $i \geq 3$. 

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Coupling

\[ Y_3 = 2, \ Y_4 = 2, \ Y_5 = 4, \ Y_6 = 3, \ Y_7 = 4 \]

\[ B = (1, 0, 0, 0, 0, 0) \quad B^{(\varepsilon)} = (1, 0, 1, 0, 0, 0) \]
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\[ B^{(e)} = (1, 0, 1, 0, 0, 0) \]
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Under this coupling

Let \( Z^{(\varepsilon)}_{\geq i}(n) = \#\{1 \leq j \leq n \mid \text{deg}_{T^{(\varepsilon)}_n}(j) \geq i\} \), \( W_d = \frac{1 + Z^{(\varepsilon)}_{\geq d}(n)}{1 + Z_{\geq d}(n)} \)

**Proposition (Similitud en razón)**

Let \( \varepsilon \in (0, 1/3) \), \( \delta \in (0, 1) \). There exists \( \beta_3 \in (0, \varepsilon^2/12) \), \( C \) constant and \( n_0 = n_0(\varepsilon, \delta) \) such that \( 0 < d < (1 + \varepsilon) \ln(n) \) under the coupling

\[
\mathbb{P}(W_d \in (1 \pm \delta)) \geq 1 - Cn^{-\beta_3}.
\]

for each \( n > n_0 \)

**Proposition**

Let \( \varepsilon \in (0, 1/3) \), \( k \in \mathbb{N} \). For \( 0 < d < (1 + \varepsilon) \ln(n) \) under the coupling

\[
\left| \mathbb{E}(\ln(W_d))^k \right| \leq 2C(\ln(n + 1))^k n^{-\beta_3}.
\]
Choose an edge uniformly at random and delete the vertex closest to the root.
Keep the subtree having the root.
The process is finished when the root is deleted.

\[
\frac{(\ln n)^2}{n} \mathbb{E}(Z_n) \to 1.
\]
Open problems

- $\delta$–attack: Generalized cutting process. Let $\delta \in (0, 1)$ and $T_n$ a RRT.
  - List the vertices by their degree.
  - Remove a vertex uniformly from $\{v_1, \ldots, v_{\lceil \delta n \rceil}\}$.
  - End the process when the root is removed.

Case $\delta \geq 1 - \frac{1}{n}$ is the random attack, case $\delta \leq \frac{1}{n}$ is targeted cutting. Interesting case: $\delta = n^{-\alpha}$ for some $\alpha > 0$.

- Weighted random trees.
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Recapitulation

- **Number of random cuts needed to destroy a RRT of size $n$:**
  \[ \mathbb{E} J_n \sim \frac{n}{\ln(n)}. \]
  \[ \frac{\ln(n)}{n} J_n \xrightarrow{P} 1. \]
  \[ \frac{(\ln(n))^2}{n} J_n - \ln(n) - \ln(\ln(n)) \xrightarrow{d} Y. \]

- **Number of targeted cuts needed to destroy a RRT $K_n$:**
  For any $\varepsilon > 0$,
  \[ K_n = O_p \left( n^{1 - \ln(2) + \varepsilon} \right). \]

- **$K_n \leq Z_{\geq D}(n)$:**
  \[ \frac{\ln(Z_{\geq D}(n))}{\ln(n)} \xrightarrow{P} 1 - \ln(2). \]
  \[ \mathbb{E} \left( \ln(Z_{\geq D}(n)) \right)^k \sim \left( \ln(n^{1 - \ln(2)}) \right)^k. \]