Three-boson relativistic bound states with zero-range interaction

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Abstract

For the zero-range interaction providing a given mass $M_2$ of the two-body bound state, the mass $M_3$ of the relativistic three-boson state is calculated. We have found that the three-body system exists only when $M_2$ is greater than a critical value $M_c \approx 1.43 m$ ($m$ is the constituent mass). For $M_2 = M_c$ the mass $M_3$ turns into zero and for $M_2 < M_c$ there is no solution with real value of $M_3$.

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Zero-range two-body interaction provides an important limiting case which qualitatively reflects characteristic properties of nuclear \cite{1} and atomic \cite{2} few-body systems. In the non-relativistic three-body system it generates the Thomas collapse \cite{3}. The latter means that the three-body binding energy tends to $-\infty$, when the interaction radius tends to zero. Several ways to regularize this interaction have been proposed in the literature \cite{4,5}.

When the binding energy or the exchanged particle mass is not negligible in comparison to the constituent masses, the nonrelativistic treatment becomes invalid and must be replaced by a relativistic one. Two-body calculations show that in the scalar case, relativistic effects are repulsive (see e.g. \cite{6}). Relativistic three-body calculations with zero-range interaction have been performed in a minimal relativistic model \cite{7} and in the framework of the Light-Front Dynamics \cite{8}. In these works it was concluded that, due to relativistic repulsion, the three-body binding energy remains finite and the Thomas collapse is consequently avoided.

In the present paper we reconsider, in the Light-Front Dynamics approach, the problem of three equal mass ($m$) bosons interacting via zero-range forces. We will show that instead of the Thomas collapse, its relativistic counterpart takes place. Namely, when the two-body bound state mass $M_2$ decreases, the mass $M_3$ of the three-body system decreases as well and vanishes at some critical value of $M_2 = M_c \approx 1.43 m$. For $M_2 < M_c$ there are no solutions with real value of $M_3$ what means – from physical point of view – that the three-body system collapses.

Our starting point is the explicitly covariant formulation of the Light-Front Dynamics (see for a review \cite{9}). The wave function is defined on the light-front plane given by the equation

\begin{equation}
\end{equation}
\(\omega \cdot x = 0\), where \(\omega\) is a four-vector with \(\omega^2 = 0\), determining the light-front orientation. In the particular case \(\omega = (1, 0, 0, -1)\) we recover the standard approach \([10, 11]\).

The three-body equation is represented graphically in figure 1. It concerns the vertex function \(\Gamma\), related to the wave function \(\psi\) in the standard way:

\[
\psi(k_1, k_2, k_3, p, \omega \tau) = \frac{\Gamma(k_1, k_2, k_3, p, \omega \tau)}{\mathcal{M}^2 - M_3^2}, \quad \mathcal{M}^2 = (k_1 + k_2 + k_3)^2 = (p + \omega \tau)^2.
\]

All four-momenta are on the corresponding mass shells \((k_i^2 = m^2, p^2 = M_3^2, (\omega \tau)^2 = 0)\) and satisfy the conservation law \(k_1 + k_2 + k_3 = p + \omega \tau\) involving \(\omega \tau\). The four-momenta \(\omega \tau\) and \(\omega \tau'\) are drawn in figure 1 by dash lines. The off-energy shell character of the wave function is ensured by the scalar variable \(\tau\). In the standard approach, the minus-components of the momenta are not conserved and the only non-zero component of \(\omega\) is \(\omega_\tau = \omega_0 - \omega_z = 2\). Variable \(2\tau\) is just the non-zero difference of non-conserved components \(2\tau = k_1 - k_2 + k_3 - p_\tau\).

Applying to figure 1 the covariant light-front graph techniques \([9]\), we find:

\[
\Gamma(k_1, k_2, k_3, p, \omega \tau) = \frac{\lambda}{(2\pi)^4} \int \Gamma(k_1', k_2', k_3', p, \omega \tau') \times \delta^{(4)}(k_1' + k_2' - \omega \tau' - k_1 - k_2 + \omega \tau) \frac{d\tau'}{\tau'} \frac{d^3k_1'}{2\varepsilon_{k_1'}} \frac{d^3k_2'}{2\varepsilon_{k_2'}} + (23)1 + (31)2, \quad (1)
\]

where \(\varepsilon_k = \sqrt{m^2 + k^2}\). For zero-range forces, the interaction kernel is replaced by a constant \(\lambda\). In \([11]\) the contribution of interacting pair 12 is explicitly written while the contributions of the remaining pairs are denoted by \((23)1 + (31)2\).

Equation (1) can be rewritten in variables \(\vec{R}_{i\perp}, x_i\), \(i = 1, 2, 3\), where \(\vec{R}_{i\perp}\) is the spatial component of the four-vector \(R_i = k_i - x_i p\) orthogonal to \(\vec{\omega}\) and \(x_i = \frac{\omega k_i}{\omega p}\) \([9]\). For this aim we insert in r.h.-side of (1) the unity integral

\[
1 = \int 2(\omega \cdot k_3') \delta^{(4)}(k_3' - k_3 - \omega \tau_3) d\tau_3 \frac{d^3k_3'}{2\varepsilon_{k_3'}}
\]

and recover the usual three-body space volume which, expressed in the variables \((\vec{R}_{i\perp}, x_i)\), reads

\[
\int \delta^{(4)}(\sum_{i=1}^3 k_i' - p - \omega \tau') \prod_{i=1}^3 \frac{d^3k_i'}{2\varepsilon_{k_i'}} 2(\omega \cdot p) d\tau' = \int \delta^{(2)}(\vec{R}_{i\perp}) \delta(\sum_{i=1}^3 x_i' - 1) \prod_{i=1}^3 \frac{d^2R_{i\perp}'}{2x_1'}.
\]
The Faddeev amplitudes $\Gamma_{ij}$ are introduced in the standard way:

$$\Gamma(1, 2, 3) = \Gamma_{12}(1, 2, 3) + \Gamma_{23}(1, 2, 3) + \Gamma_{31}(1, 2, 3),$$

and equation (1) is equivalent to a system of three coupled equations. With the symmetry relations $\Gamma_{23}(1, 2, 3) = \Gamma_{12}(2, 3, 1)$ and $\Gamma_{31}(1, 2, 3) = \Gamma_{12}(3, 1, 2)$, the system is reduced to a single equation for one of the amplitudes, say $\Gamma_{12}$.

In general, $\Gamma_{12}$ depends on all variables $(\vec{R}_{1\perp}, x_i)$, constrained by the relations $\vec{R}_{1\perp} + \vec{R}_{2\perp} + \vec{R}_{3\perp} = 0$, $x_1 + x_2 + x_3 = 1$, but for contact kernel it depends only on $(\vec{R}_{3\perp}, x_3)$ (4). Equation (3) results into:

$$\Gamma_{12}(\vec{R}_{1\perp}, x) = \frac{\lambda}{(2\pi)^3} \int \left[ \Gamma_{12}(\vec{R}_{1\perp}, x) + 2\Gamma_{12} \left( \vec{R}_{1\perp} - x'\vec{R}_{1\perp}, x'(1 - x) \right) \right] \frac{1}{s'_{12} - M_{12}^2} \frac{d^2R'_1}{2x'(1 - x')} = 1,$$

in which

$$s'_{12} = (k'_1 + k'_2)^2 = \frac{R_{1\perp}^2 + m^2}{x'(1 - x')}$$

is the effective on shell mass squared of the two-body subsystem, whereas $M_{12}^2 = (k'_1 + k'_2 - \omega \tau')^2 = (p - k_3)^2$ corresponds to its off-shell mass. It is expressed through $M_3^2, R_{1\perp}, x$ as

$$M_{12}^2 = (1 - x)M_3^2 - \frac{R_{1\perp}^2 + (1 - x)m^2}{x} = \frac{1}{\lambda - 1 - I(M_{12})} \left( \frac{2}{(2\pi)^3} \int \Gamma_{12} \left( \vec{R}_{1\perp} - x'\vec{R}_{1\perp}, x'(1 - x) \right) \frac{1}{s'_{12} - M_{12}^2} \frac{d^2R'_1}{2x'(1 - x')} \right).$$

These on- and off-shell masses $s'_{12}$ and $M_{12}^2$ differ from each other, since $k'_1 + k'_2 + k_3 \neq p$. On the energy shell, at $\tau' = 0$, the value $M_{12}^2$ turns into $s'_{12}$, what is never reached for a bound state.

Since the first term $\Gamma_{12}(\vec{R}_{1\perp}, x)$ in the integrand does not depend on the integration variables, we can transform (3) as:

$$\Gamma_{12}(\vec{R}_{1\perp}, x) = \frac{1}{\lambda - 1 - I(M_{12})} \left( \frac{2}{(2\pi)^3} \int \Gamma_{12} \left( \vec{R}_{1\perp} - x'\vec{R}_{1\perp}, x'(1 - x) \right) \frac{1}{s'_{12} - M_{12}^2} \frac{d^2R'_1}{2x'(1 - x')} \right).$$

The integral (4) diverges logarithmically and we implicitly assume that a cutoff $L$ is introduced.

The value of $\lambda$ is found by solving the two-body problem with the same zero-range interaction under the condition that the two-body bound state mass has a fixed value $M_2$. From that we get $\lambda^{-1} = I(M_2)$ with $I$ given by (3). It also diverges when the momentum space cutoff $L$ tends to infinity (or, equivalently, the interaction range tends to zero). However, the difference $\lambda^{-1} - I(M_{12}) = I(M_2) - I(M_{12})$ converges in the limit $L \to \infty$. The factor $F(M_{12}) = 1/[I(M_2) - I(M_{12})]$ gives the two-body off-shell scattering amplitude, depending on the off-shell two-body mass $M_{12}$, without any regularization. For $0 \leq M_{12}^2 < 4m^2$ the calculation gives:

$$F(M_{12}) = \frac{8\pi^2}{\arctan \frac{y_{M_{12}}}{y_{M_2}} - \arctan \frac{y_{M_2}}{y_{M_2}}},$$
where \( y_{M_{12}} = \frac{M_{12}}{\sqrt{4m^2 - M_{12}^2}} \) and similarly for \( y_{M_2} \). If \( M_{12}^2 < 0 \), the amplitude obtains the form:

\[
F(M_{12}) = \frac{8\pi^2}{2y_{M_{12}}} \log \frac{1 + y_{M_{12}}}{1 - y_{M_{12}}} - \arctan y_{M_{12}},
\]

where \( y'_{M_{12}} = \frac{-M_{12}^2}{\sqrt{4m^2 - M_{12}^2}} \).

Finally, the equation for the Faddeev amplitude reads:

\[
\Gamma_{12}(R_\perp, x) = F(M_{12}) \frac{1}{(2\pi)^3} \int_0^1 dx' \int_0^\infty \frac{\Gamma_{12}(R'_\perp, x'(1 - x))}{(\vec{R}'_\perp - x' \vec{R}_\perp)^2 + m^2 - x'(1 - x')M_{12}^2} \, d^2 R'_\perp.
\]

The three-body mass \( M_3 \) enters in this equation through variable \( M_{12}^2 \), defined by (3).

By replacing \( x'(1 - x) \rightarrow x' \), equation (6) can be transformed to

\[
\Gamma_{12}(R_\perp, x) = F(M_{12}) \frac{1}{(2\pi)^3} \int_0^{1 - x} dx' \frac{dx'}{x'(1 - x - x')} \int_0^\infty \frac{d^2 R'_\perp}{\mathcal{M}^2 - M_3^2} \Gamma_{12}(R'_\perp, x'),
\]

where

\[
\mathcal{M}^2 = \frac{\vec{R}'_\perp^2 + m^2}{x'} + \frac{\vec{R}_\perp^2 + m^2 - x'(1 - x')M_{12}^2}{1 - x - x'}.
\]

This equation is the same than equation (11) from [8] except for the integration limits of \((\vec{R}'_\perp, x')\) variables. In [8] the integration limits follow from the condition \( M_{12}^2 > 0 \). They read

\[
\int_{\frac{m^2}{M_3^2}}^{1 - x} [\ldots] \, dx' \int_0^{k_{max}^-} [\ldots] \, d^2 R'_\perp.
\]

with \( k_{max}^- = \sqrt{(1 - x')(M_3^2 x' - m^2)} \) and introduce a lower bound on the three-body mass \( M_3 > \sqrt{2m} \). The same condition, though in a different relativistic approach, was used in [7]. The integration limits in (8) restrict the arguments of \( \Gamma_{12} \) to the domain

\[
\frac{m^2}{M_3^2} \leq x \leq 1 - \frac{m^2}{M_3^2}, \quad 0 \leq R_\perp \leq k_{max}^-
\]

and can be considered as a method of regularization. In this case, one no longer deals with the zero-range forces.

Being interested in studying the zero-range interaction, we do not cut off the variation domain of variables \( R_\perp, x \)

\[
0 \leq x \leq 1, \quad 0 \leq R_\perp < \infty.
\]

The integration limits for these variables reflect the conservation law of the four-momenta in the three-body system and they are automatically fulfilled, as far as the \( \delta^{(4)} \)-function in (1) is taken into account. The off-shell variable \( M_{12}^2 \) may take negative values, when \( R_\perp \) and \( x \) vary in their proper limits. Thus, if \( M_3^2 > m^2 \) one has \( - \infty \leq M_{12}^2 \leq (M_3 - m)^2 \) but if \( M_3^2 < m^2 \), \( M_{12}^2 \) is always negative \( - \infty \leq M_{12}^2 \leq 0 \). We would like to notice that \( M_{12}^2 \) is not to be confused with the on-shell effective mass squared \( s'_{12} = (k_1' + k_2')^2 \) which is indeed always positive and
Figure 2: (a) Three-body bound state mass $M_3$ versus the two-body one $M_2$ (solid line). Results obtained with integration limits (8) are in dash line. Dots values are taken from [12]. (b) Zoom of the zero two-body binding region ($M_2 \to 2m, B_2 \to 0$) displaying solid line only.

Even $s^1_{12} \geq 4m^2$. As we will see, this point turns out to be crucial for the appearance of the relativistic collapse.

The results of solving equation (6) are presented in what follows. Calculations were carried out with constituent mass $m = 1$ and correspond to the ground state. We represent in fig. 2a the three-body bound state mass $M_3$ as a function of the two-body one $M_2$ (solid line) together with the dissociation limit $M_3 = M_2 + m$. The zero two-body binding limit $B_2 = 2m - M_2 \to 0$ is magnified in fig. 2b. In this limit the three-boson system has a binding energy $B_3 \approx 0.012$.

When $M_2$ decreases, the three-body mass $M_3$ decreases very quickly and vanishes at the two-body mass value $M_2 = M_c \approx 1.43$. Whereas the meaning of collapse as used in the Thomas paper implies unbounded nonrelativistic binding energies and cannot be used here, the zero bound state mass $M_3 = 0$ constitutes its relativistic counterpart. Indeed, for two-body masses below the critical value $M_c$, the three-body system no longer exists.

The results corresponding to integration limits (8) are included in fig. 2a (dash line) for comparison. Values given in [8] were not fully converged [13]. They have been corrected in [12] and are indicated by dots. In both cases the repulsive relativistic effects produce a natural cutoff in equation (6), leading to a finite spectrum and – in the Thomas sense – an absence of collapse, like it was already found in [7]. However, except in the zero binding limit, solid and dash curves strongly differ from each other.

We would like to remark that for $M_2 \leq M_c$, equation (6) possesses square integrable solutions with negative values of $M_3^2$. They have no physical meaning but $M_3^2$ remains finite in all the
two-body mass range $M_2 \in [0, 2]$. The results of $M_3^2$ are given in figure 3. When $M_2 \to 0$, $M_3^2$ tends to $\approx -11.6$.

In figure 4 is shown the Faddeev amplitude $\Gamma_{12}$ corresponding to a solution with relatively large binding energies $M_2 = 1.5$ and $M_3 = 1$. Figure 4a shows its $R_\perp$-dependence at fixed values of $x$ and figure 4b – its $x$-dependence at fixed values of $R_\perp \in [0, 20]$. One can remark in figure 4b that, in addition to the zeroes at $x = 0$ and $x = 1$, $\Gamma_{12}(R_\perp, x)$ has two zeroes at $x \approx 0.15$ and $x \approx 0.7$. When the mass values $(M_2, M_3)$ are changed, the positions of zeroes are only slightly shifted.

![Figure 3: Three-body bound state mass squared $M_3^2$ versus $M_2$.](image)

![Figure 4: Faddeev amplitude $\Gamma_{12}$ for $M_2 = 1.5$, $M_3 = 1$ (a) as a function of $R_\perp$ at fixed values of $x$ and (b) as a function of $x$ at fixed $R_\perp$ values.](image)
In summary, we have considered the relativistic problem of three equal-mass bosons, interacting via zero-range forces constrained to provide finite two-body mass $M_2$. The Light-Front Dynamics equation has been derived and solved numerically.

We have found that the three-body bound state exists for two-body mass values in the range $M_c \approx 1.43 \, m \leq M_2 \leq 2 \, m$. At the zero two-body binding limit, the three-body binding energy is $B_3 \approx 0.012 \, m$. The Thomas collapse is avoided in the sense that three-body mass $M_3$ is finite, in agreement with [4, 5]. However, another kind of catastrophe happens. Removing infinite binding energies, the relativistic dynamics generates zero three-body mass $M_3$ at a critical value $M_2 = M_c$. For stronger interaction, i.e. when $0 \leq M_2 < M_c$, there are no physical solutions with real value of $M_3$. In this domain, $M_3^2$ becomes negative and the three-boson system cannot be described by zero range forces, as it happens in nonrelativistic dynamics. This fact can be interpreted as a relativistic collapse.

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**References**

[1] G.E. Brown, A.D. Jackson, The nucleon-nucleon interaction, North-Holland, Amsterdam, 1976.

[2] Y.N. Demkov, V.N. Ostrovskii, Zero-range potentials and their applications in atomic physics, Plenum Press, New-York 1988.

[3] L.H. Thomas, Phys. Rev. 47 (1935) 903.

[4] S. K. Adhikari, T. Frederico, I.D. Goldman, Phys. Rev. Lett. 74 (1995) 487; T. Frederico, L. Tomio, A. Delfino, A.E.A Amorin, Phys. Rev. A60 (1999) R9.

[5] D.V. Fedorov, A.S. Jensen, Phys. Rev. A63 (2001) 063608; Nucl. Phys. A697 (2002) 783.

[6] M. Mangin-Brinet, J. Carbonell, Phys. Lett. B474, (2000) 237

[7] J.V. Lindesay and H.P. Noyes, Preprint SLAC-PUB-2932, 1986.

[8] T. Frederico, Phys. Lett. B282 (1992) 409.

[9] J. Carbonell, B. Desplanques, V.A. Karmanov, J.-F. Mathiot, Phys. Reports, 300 (1998) 215.

[10] S.J. Brodsky, H.-C. Pauli, S.S. Pinsky, Phys. Reports, 301 (1998) 299.

[11] B.L.G. Bakker, L.A. Kondratyuk, M.V. Terentyev, Nucl. Phys, B158 (1979) 497.

[12] W.R.B. de Araujo, J.P.B.C. de Melo, T. Frederico, Phys. Rev. C52 (1995) 2733.

[13] T. Frederico, private communication.