Degree Heterogeneity in a Graph Facilitates Quicker Meeting of Random Walkers*

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SUMMARY  Multiple random walks are the movement of several independent random walkers on a graph, and are applied to various graph algorithms. In order to design an efficient graph algorithm based on multiple random walks, it is essential to study multiple random walks theoretically for deeply understanding their characteristics. The first meeting time is one of the important metrics for the multiple random walks. The first meeting time on a graph is defined by the time it takes for multiple random walkers to meet at the same node in graph. The first meeting time is closely related to the rendezvous problem, which is a fundamental problem in the field of computer science. In previous works, the first meeting time of multiple random walks has been analyzed. However, many of these previous works focus on regular graphs. In this paper, we analyze the first meeting time of multiple random walks in arbitrary graphs, and clarify the effect of graph structures on its expected value. First, we derive the spectral formula of the expected first meeting time on the basis of the spectral graph theory. Then, we examine the principal component of the expected first meeting time using the derived spectral formula. The clarified principal component reveals that (a) the expected first meeting time is almost dominated by $n/(1 + d_{\text{st}}^2/d_{\text{avg}}^2)$ and (b) the expected first meeting time is independent of the starting nodes of random walkers where $n$ is the number of nodes of the graph. $d_{\text{avg}}$ and $d_{\text{st}}$ are the mean and the standard deviation of weighted node degrees, respectively. The characteristics (a) is useful to understand the effect of the graph structure on the first meeting time. According to the revealed effect of graph structures, the variance of coefficient $d_{\text{st}}^2/d_{\text{avg}}^2$ (degree heterogeneity) for weighted degrees facilitates quicker the meeting of random walkers.

key words: First Meeting Time, Random Walk, Spectral Graph Theory

1. Introduction

Multiple random walks are the movement of several independent random walkers on a graph, and are applied to various graph algorithms because of their advantages such as ease of analysis and light weight of processing. The notable applications include (a) the search algorithm for finding a particular node on the graph [1,2], (b) the algorithm for spreading information across graphs by only exchanging information between nodes [3], (c) the rendezvous algorithm for the efficient meeting of multiple random walkers at the same node [4]. In order to design an efficient graph algorithm based on multiple random walks, it is essential to study multiple random walks theoretically for deeply understanding their characteristics.

There are several important metrics (e.g., first hitting time, recurrent time, cover time, re-encountering time, and first meeting time) for multiple random walks. The first hitting time is the time it takes for any random walker to arrive at a given node, and is important to evaluate the performance of search algorithms. The recurrent time is the time required to return any one of random walkers to the starting node, and so it is the particular case of the first hitting time. The cover time is the time it takes for any random walker to reach all nodes, and corresponds to the maximum value of first hitting times. The cover time strongly affects the information dissemination speed in the graph. In addition, the re-encountering time and the first meeting time are the times it takes for multiple random walkers to meet at the same node. The re-encountering time discusses random walkers starting from the same node, and the first meeting time discusses those starting from different nodes. In particular, the first meeting time is closely related to the rendezvous problem, which is a fundamental problem in the field of computer science. The rendezvous problem appears in several engineering problems (e.g., self-stabilizing token management system problem [5,6] and $k$-server problem [7]). In order to design efficient algorithms for the rendezvous problem, the characteristics of the first meeting time should be clarified.

The first meeting time of multiple random walks has been analyzed in [6,12]. However, many of these previous works focus on regular graphs. In [12], George et al provided a pioneering work for multiple random walks on non-regular graphs, and derive a closed-form formula to calculate the expected value of the first meeting time in arbitrary graphs. However, the effect of graph structures on the expected first meeting time has not been clarified. In order to design effective algorithms using multiple random walks for realistic graphs (e.g., social networks and communication networks), it is desirable to understand the effect of graph structures on the expected first meeting time. Since it is difficult to numerically clarify this effect using the closed-form formula derived in [12], the effect should be examined using the analysis of multiple random walks.

In this paper, we analyze the first meeting time of multiple random walks in arbitrary graphs, and clarify the effect of graph structures on its expected value. First, we derive the spectral formula of the expected first meeting time on the basis of the spectral graph theory that is used to analyze the characteristics of graphs. Then, we examine the principal component of the expected first meeting time using the derived spectral formula. The clarified principal component reveals that (a) the expected first meeting time is almost dominated by $n/(1 + d_{\text{st}}^2/d_{\text{avg}}^2)$ and (b) the expected first meeting time is independent of the starting nodes of random walkers. Therefore, the effect of the graph structure on the first meeting time can be easily understood by analyzing the coefficient $d_{\text{st}}^2/d_{\text{avg}}^2$. This result provides a useful tool for designing efficient graph algorithms.
walkers where \( n \) is the number of nodes in the graph. \( d_{\text{avg}} \) and \( d_{\text{std}} \) are the mean and the standard deviation of weighted node degrees, respectively. From the characteristics (a), we understand the effect of the graph structure on the first meeting time. In addition, we confirm the validity of the analysis results through numerical examples.

The contributions in this paper are summarized as follows.

- We extend the analysis of a single random walk using the spectral graph theory to multiple random walks.
- We derive the spectral formula of the expected first meeting time.
- We clarify the principal component of the expected first meeting time.
- We reveal the effect of graph structures on the expected first meeting time.
- We confirm the validity of the derived spectral formula and the clarified principal component for various networks with different scales and different structures.

The structure of this paper is as follows. In Sect. 2, we describe the definition of graphs and random walks, and introduce the previous analysis of a single random walk using the spectral graph theory. Then, in Sect. 3, we derive the spectral formula of the expected first meeting time using the spectral graph theory, and clarify the principal component of the expected first meeting time on the basis of the derived spectral formula. In addition, Sect. 4 confirms the validity of the analysis results through numerical examples. Finally, Sect. 5 concludes this paper, and discuss the future work.

2. Preliminary

In this section, we describe the definition of graphs and random walks used in our analysis. In addition, we introduce the existing analysis results of a single random walk based on the spectral graph theory.

The graph is given by \( G = (V, E) \) where \( V \) and \( E \) are the set of nodes and the set of links, respectively. In \( E \), self-loop link \((i, i)\) for \( i \in V \) is not included. The weight for link \((i, j)\) is given by \( w_{ij} \) where \( w_{ij} > 0 \) and \( w_{ij} = w_{ji} \). We denote the set of adjacent nodes of node \( i \in V \) by \( \partial i \). Let \( d_i \) be the weighted degree of node \( i \in V \), and \( d_i \) is defined by

\[
d_i := \sum_{k \in \partial i} w_{ik}. \tag{1}
\]

We describe a random walk starting from node \( a \in V \). In this random walk, the random walker existing at node \( i \in V \) moves to adjacent node \( j \in \partial i \) using transition probability \( p_{i \rightarrow j} \), which is given by

\[
p_{i \rightarrow j} = \frac{w_{ij}}{d_i}. \tag{2}
\]

Let \( x_{a;i}(t) \) be the existing probability that a random walker starting from node \( a \in V \) exists at node \( i \in V \) at time \( t \) where \( \sum_{i \in V} x_{a;i}(t) = 1 \). If Eq. (2) is used, \( x_{a;i}(t + 1) \) is given by

\[
x_{a;i}(t + 1) = \sum_{j \in \partial i} x_{a;j}(t) p_{j \rightarrow i}, \tag{3}
\]

Using column vector \( x_a(t) = (x_{a;i}(t))_{i \in V} \), Eq. (3) for all nodes \( \forall i \in V \) can be written simultaneously by

\[
x_a(t + 1) = AD^{-1} x_a(t), \tag{4}
\]

where \( D \) and \( A \) are degree matrix and adjacency matrix, which are defined by

\[
D := \text{diag}(d_i)_{i \in V}, \tag{5}
\]

\[
A := \begin{cases} w_{ij} & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}, \tag{6}
\]

respectively. \( AD^{-1} \) is the matrix whose \((i, j)\)-th element is transition probability \( p_{j \rightarrow i} \). Equation (4) describes the behavior of the random walk. Since \( AD^{-1} \) is an asymmetric matrix, Eq. (4) is not easy to handle analytically with linear algebra. Therefore, we modify Eq. (4) to

\[
D^{-1/2} x_a(t + 1) = D^{-1/2} AD^{-1/2} D^{-1/2} x_a(t) = W D^{-1/2} x_a(t) = \hat{x}_a(t + 1) = W \hat{x}_a(t), \tag{7}
\]

where \( W = D^{-1/2} AD^{-1/2} \) and \( \hat{x}_a(t) = D^{-1/2} x_a(t) \). Since \( W \) is a symmetric matrix, Eq. (7) is easier to handle than Eq. (4). In general, in the spectral graph theory, the behavior of an analysis target is expressed with a matrix like Eq. (7), and the characteristics of the target is analyzed using the eigenvalues and eigenvectors of the matrix on the basis of linear algebra.

\( W \) can be always diagonalized using orthogonal matrix \( Q \) that satisfies \( ^t Q = Q^{-1} \). Let \( \lambda_k \) be the \( k \)-th largest eigenvalue of \( W \). Note that maximum eigenvalue \( \lambda_1 \) is always 1. In this paper, we assume that \( G \) is connected and not a bipartite graph. In this case, eigenvalues \( \lambda_k \) for \( 2 \leq k \leq n \) satisfy

\[
-1 < \lambda_n \leq \cdots \leq \lambda_2 < 1. \tag{8}
\]

We denote the eigenvector for eigenvalue \( \lambda_k \) by \( q_k \), and then \( Q \) is given by \( Q = \{ q_k \}_{1 \leq k \leq n} \). Since \( Q \) is the orthogonal matrix, \( q_k \) and \( q_l \) for \( 1 \leq k, l \leq n \) satisfy

\[
^t q_k q_l = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}. \tag{9}
\]

In particular, maximum eigenvector \( q_1 \) is given by

\[
q_1 = \frac{1}{s_1} \left( \sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n} \right), \tag{10}
\]

where \( s_1 = \sum_{i \in V} d_i \). Note that \( s_1 \) is related to a statistic of the graph structure of \( G \), and can be written by \( s_1 = n d_{\text{avg}} \) where \( d_{\text{avg}} \) is the average weighted degree.
We derive the spectral formula of expected first meeting time \( \mu_{a,b} \) of two random walkers starting from node \( a \in V \) and \( b \in V \) in graph \( G \) on the basis of the spectral graph theory. We first derive the spectral formula of \( \mu_{a,b} \). Then, we clarify the principal component of \( \mu_{a,b} \) using the derived spectral formula. Finally, we reveal the effect of the graph structure on \( \mu_{a,b} \) on the basis of the clarified principal component.

Our analysis results are important to also understand the first meeting time of \( n_{RW} \) random walkers where \( n_{RW} > 2 \) because it is strongly affected by the first meeting time of two random walkers. In order for the meeting of \( n_{RW} \) random walkers to be efficient, after two of \( n_{RW} \) random walkers meet at the same node, they should move together. In this case, the first meeting time of \( n_{RW} \) random walkers is obtained as the sum of the first meeting times of two random walkers. Therefore, the characteristics of the first meeting time for \( n_{RW} \) random walkers should be strongly associated with that of two random walkers.

3.1 Spectral Formula of Expected First Meeting Time \( \mu_{a,b} \)

We derive the spectral formula of expected first meeting time \( \mu_{a,b} \) with the method to derive that of expected first hitting time \( \mu_{a,i} \) in [13]. In [13], the spectral formula was derived using the generating function of existing probability \( x_{a,i}(t) \). In general, generating function \( F(z) \) of probability \( f(t) \) is defined as

\[
F(z) := \sum_{t=0}^{\infty} f(t) z^t. \tag{16}
\]

Using generating function \( F(z) \), expectation \( E(t) \) with probability \( f(t) \) is given by

\[
E(t) = \sum_{t=1}^{\infty} t f(t) = \frac{dF(z)}{dz} \bigg|_{z=1}. \tag{17}
\]

Importantly, even if we do not know the solution of probability \( f(t) \), we can derive expectation \( E(t) \) using generating function \( F(z) \) on the basis of the above equation. We first obtain the generating function of the first meeting probability. Without the solution of the first meeting probability, we then derive the spectral formula of expected first meeting time \( \mu_{a,b} \) by substituting the obtained generating function into Eq. (17).

Let \( r_{a,b,c}(t) \) be the probability that the two random walkers first meet at node \( c \) at time \( t \). Since the two random walkers can meet at any node, first meeting probability \( r_{a,b,c}(t) \) is given by

\[
r_{a,b,c}(t) = \sum_{c \in V} r_{a,b,c}(t). \tag{18}
\]
In $r_{a,b,c}(t)$, symbol $*$ means any node in $V$. In order to derive the spectral formula of expected first meeting time $\mu_{a,b}$ using Eq. (17), generating function $R_{a,b,c}(z)$ of $r_{a,b,c}(t)$ is needed.

The probabilities that the two random walkers exist at node $i$ at time $t$ are given by $x_{a;i}(t)$ and $x_{b;i}(t)$, respectively. Since the solutions of $x_{a;i}(t)$ and $x_{b;i}(t)$ are obtained as Eq. (11), the generating functions of $x_{a;i}(t)$ and $x_{b;i}(t)$ can be derived using Eq. (16). However, since it is not easy to obtain the solution of $r_{a,b,c}(t)$, we obtain generating function $R_{a,b,c}(z)$ of $r_{a,b,c}(t)$ from those of $x_{a;i}(t)$ and $x_{b;i}(t)$, and derive the spectral formula using Eq. (17).

To obtain generating function $R_{a,b,c}(z)$ of first meeting probability $r_{a,b,c}(t)$, we discuss the relationship between $r_{a,b,c}(t)$, $x_{a;i}(t)$ and $x_{b;i}(t)$. Let $x_{a,b,c}(t)$ be the probability that the two random walkers meet at same node $c \in V$ at time $t$. Meeting probability $x_{a,b,c}(t)$ at any node is given by

$$x_{a,b,c}(t) = \sum_{c \in V} x_{a,b,c}(t).$$

Since each random walker moves independently, $x_{a,b,c}(t)$ is given by

$$x_{a,b,c}(t) = \sum_{c \in V} x_{a,k,c}(t) = \sum_{c \in V} x_{a,c}(t) x_{b,c}(t).$$

Note that $x_{a,b,c}(t)$ includes not only first meeting probability $r_{a,b,c}(t)$, but also the probability of the second and subsequent meetings. Hence, as shown in Fig. 1, we divide the transition of the two random walks from time 0 to time $t$ into two transitions: (a) the transition until they first meet at time $s$, and (b) the rest transition. The probability for the former transition is first meeting probability $r_{a,b,c}(t)$. The probability for the latter transition is the probability that the two random walkers starting from same node $c' \in V$ at time $s$ meet again at same node $c \in V$ at time $t$. Since node $c'$ and node $c$ can be any node, we denote such a probability by $x_{c',c,*}(t-s)$. With these probabilities, $x_{a,b,c}(t)$ is given by

$$x_{a,b,c}(t) = \sum_{s=0}^{t} r_{a,b,c}(s) x_{c',c,*}(t-s).$$

Using probability $x_{c',c',c}(t)$ that the two random walkers starting at node $c' \in V$ at time 0 meet again at node $c \in V$ at time $t$, we give $x_{c',c',*}(t)$ by

$$x_{c',c',*}(t) = \sum_{c' \in V} \frac{d_{c'}}{s_2} \sum_{c \in V} x_{c',c',c}(t),$$

where $s_2 = \sum_{c \in V} d_{c}^2$. The reason why $x_{c',c',*}(t)$ is not given as a simple sum of $x_{c',c',c}(t)$ in Eq. (22) is as follows. According to Eq. (12), existing probability $x_{a;i}$ in the steady state is proportional to weighted degree $d_i$ of node $i$. Therefore, the probability of the first meeting of the two random walkers at node $c'$ should be proportional to $d_{c'}^2$. Therefore, in the sum of Eq. (22), $x_{c',c',c}(t)$ is weighted by $d_{c'}^2/s_2$. In

![Fig. 1: Random walks starting from node $a$ and node $b$ at time 0 to time $t$](image)

Sect. 4, the validity of Eq. (22) will be confirmed by the numerical example.

The right side of Eq. (21) is a convolutional sum, and so generating function $X_{a,b,c}(z)$ of $x_{a,b,c}(t)$ is given by

$$X_{a,b,c}(z) = R_{a,b,c}(z) X_{c',c',*}(z).$$

(23)

From the above equation, generating function $R_{a,b,c}(z)$ of first meeting probability $r_{a,b,c}$ is derived as

$$R_{a,b,c}(z) = \frac{X_{a,b,c}(z)}{X_{c',c',*}(z)}.$$  

(24)

By substituting the spectral formulas of $x_{a;i}(t)$ and $x_{b;i}(t)$ given by Eq. (11) into Eq. (16), we obtain the spectral formula of $X_{a,b,c}(z)$ as

$$X_{a,b,c}(z) = \sum_{c \in V} \infty \sum_{t=0}^{\infty} x_{a,c}(t) x_{b,c}(t) z^t$$

$$= \sum_{c \in V} \frac{d_{c}}{\sqrt{d_{a}d_{b}}} \sum_{k=1}^{n} \sum_{k'=1}^{n} \sum_{k''=1}^{n} q_{k}(a) q_{k'}(c) q_{k''}(b) q_{k''}(c) \sum_{t=0}^{\infty} (\lambda_{k} \lambda_{k'} z)^{t}$$

$$= \sum_{c \in V} \frac{d_{c}}{\sqrt{d_{a}d_{b}}} \sum_{k=1}^{n} \sum_{k'=1}^{n} q_{k}(a) q_{k'}(c) q_{k''}(b) q_{k''}(c) \sum_{k''=1}^{n} \frac{1 - \lambda_{k} \lambda_{k'} z}{1 - \lambda_{k} \lambda_{k'} z}.$$  

(25)

Note that $z$ takes a value within the range of $|\lambda_{k} \lambda_{k'}| < 1$ so that the sum of infinite geometric series converges. From Eq. (22), generating function $X_{c',c',*}(z)$ is given by

$$X_{c',c',*}(z) = \sum_{c' \in V} \frac{d_{c'}^2}{s_2} \sum_{c \in V} X_{c',c',c}(z).$$

(26)

By substituting generating function $R_{a,b,c}(z)$ into Eq. (17), expected first meeting time $\mu_{a,b}$ is given by

$$\mu_{a,b} = \sum_{s=0}^{\infty} s r_{a,b,c}(s) = \left. \frac{dR_{a,b,c}(z)}{dz} \right|_{z=1}.$$  

(27)

In the above equation, $dR_{a,b,c}(z)/dz$ is

$$\frac{dR_{a,b,c}(z)}{dz} = \frac{d}{dz} \left( \frac{X_{a,b,c}(z)}{X_{c',c',*}(z)} \right)$$

$$= \frac{dX_{a,b,c}(z)}{dz} X_{c',c',*}(z) - X_{a,b,c}(z) \frac{dX_{c',c',*}(z)}{dz}$$

$$= \frac{X_{a,b,c}(z)}{X_{c',c',*}(z)}.$$  

(28)
where

\[ A(z) = \frac{dX_{a,b,c}(z)}{dz} X_{a,b,c}(z), \quad (29) \]

\[ B(z) = X_{a,b,c}(z) \frac{dX_{a,b,c}(z)}{dz}, \quad (30) \]

\[ C(z) = X_{a,b,c}(z). \quad (31) \]

According to Eqs. (25) and (26), \( A(z) \), \( B(z) \), and \( C(z) \) can be written as polynomials for \((1-z)\) because of \(1-\lambda_1\lambda_1 z = 1 - z\). Hence, we also give \( A(z) \) by

\[ A(z) = \frac{A_3(z)}{(1-z)^3} + \frac{A_2(z)}{(1-z)^2} + \frac{A_1(z)}{1-z} + A_0(z). \quad (32) \]

By substituting Eqs. (25) and (26) into the right side of Eq. (29), \( A_3(z) \) and \( A_2(z) \) are obtained as

\[ A_3(z) = \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \]

\[ \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right)^2, \quad (33) \]

\[ A_2(z) = \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \]

\[ \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right)^2 \]

\[ = \frac{1}{s_1^2} \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \quad (34) \]

Note that we do not provide \( A_1(z) \) and \( A_0(z) \) in this paper because \( A_1(z)/(1-z) \) and \( A_0(z) \) disappear when deriving \( \mu_{a,b} \) using Eqs. (27). Similarly, \( B(z) \) is given by

\[ B(z) = \frac{B_3(z)}{(1-z)^3} + \frac{B_2(z)}{(1-z)^2} + \frac{B_1(z)}{1-z} + B_0(z). \quad (35) \]

By substituting Eqs. (25) and (26) into the right side of Eq. (30), \( B_3(z) \) and \( B_2(z) \) are obtained as

\[ B_3(z) = \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \]

\[ \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right)^2, \quad (36) \]

According to the above equations, we find \( A_3(z) = B_3(z) \). Therefore, \( A(z) - B(z) \) in the numerator of Eq. (28) does not have the term of \((1-z)^3\), and so the term of \((1-z)^2\) becomes the highest order in the polynomials for \((1-z)\) in \( A(z) - B(z) \). Then, \( C(z) \) is given by

\[ C(z) = \frac{C_3(z)}{(1-z)^3} + \frac{C_2(z)}{(1-z)^2} + \frac{C_1(z)}{1-z} + C_0(z). \quad (38) \]

By solving the above equation with the same way, \( C_2(z) \) is obtained as

\[ C_2(z) = \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \]

\[ \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right)^2 \]

\[ = \frac{1}{s_1^2} \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \quad (39) \]

Since \((A(z) - B(z))/C(z)\) is the indeterminate form at \( z = 1 \), we discuss \( \lim_{z \to 1} (A(z) - B(z))/C(z) \) to derive the spectral formula of expected first meeting time \( \mu_{a,b} \) using Eqs. (27) and (28). As the limit of \( z \to 1 \), \( \mu_{a,b} \) is given by

\[ \mu_{a,b} = \lim_{z \to 1} \frac{A(z) - B(z)}{C(z)} = \lim_{z \to 1} \frac{(1-z)^2(A(z) - B(z))}{(1-z)^2 C(z)} \]

\[ = \frac{A_3(1) - B_3(1)}{C_2(1)} \]

\[ = \frac{s_1^2}{(s_2)^2} \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right) \]

\[ = \frac{s_1^2}{(s_2)^2} \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right) \]

\[ \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right)^2 \]

\[ = \frac{s_1^2}{(s_2)^2} \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right) \]

\[ = \frac{s_1^2}{(s_2)^2} \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right) \]

\[ \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right)^2 \]

\[ = \frac{s_1^2}{(s_2)^2} \left( \sum_{c \in V} \sum_{c' \in V} \frac{d_c}{\sqrt{d_a d_b}} q_1(a) q_1^2(c) q_1(b) \right) \]
Since the above equation is expressed by the eigenvalues and the eigenvectors of $W$, it is the spectral formula of $\mu_{a,b}$. 

Equation (40) seems to be complicated. However, if we use expected first meeting time $\mu_{a,b;c}$, which is the time until the two random walkers first meet at node $c \in V$, $\mu_{a,b}$ is written as

$$
\mu_{a,b} = \frac{1}{s_2} \sum_{c \in V} d_c^2 \mu_{a,b;c} - \frac{1}{s_2} \sum_{c \in V} d_c^2 \sum_{c' \in V} d_{c'}^2 \mu_{c',c;c},
$$

where the spectral formula of $\mu_{a,b;c}$ is given by

$$
\mu_{a,b;c} = \mu_{a;c} + \mu_{b;c} + \sum_{1 \leq k,k' \leq n} \frac{q_k(c)q_{k'}(c)}{(1 - \lambda_k \lambda_{k'})d_c} \left( \frac{q_k(a)q_{k'}(b)}{d_a^2} - \frac{q_k(a)q_k(b)}{d_a d_b^2} \right),
$$

or

$$
\mu_{a,b;c} = \mu_{a,b} + s_1 \sum_{c \in V} d_c \mu_{a,b;c},
$$

where $\mu_{a,b}$ is the column vector whose $i$-th element $u_a(i)$ is given by

$$
\mu_{a,b} = \begin{cases} 
\frac{1}{\sqrt{d_a}} & \text{if } i = a \\
0 & \text{otherwise}
\end{cases}
$$

Pseudo-inverse matrix $\tilde{N}^\dagger$ of $\tilde{N}$ in Eq. (44) is also given by

$$
\tilde{N}^\dagger = \sum_{1 \leq k,k' \leq n} \frac{\lambda_k \lambda_{k'} \tilde{q}_{kk'}^t \tilde{q}_{kk'}}{1 - \lambda_k \lambda_{k'}} = \sum_{1 \leq k,k' \leq n} \frac{(1 - \lambda_k \lambda_{k'} + \lambda_k \lambda_{k'}) \tilde{q}_{kk'}^t \tilde{q}_{kk'}}{1 - \lambda_k \lambda_{k'}} = \tilde{I} - \tilde{q}_{11}^t \tilde{q}_{11} + \tilde{M},
$$

where $\tilde{M}$ is

$$
\tilde{M} = \sum_{1 \leq k,k' \leq n} \frac{\lambda_k \lambda_{k'} \tilde{q}_{kk'}^t \tilde{q}_{kk'}}{1 - \lambda_k \lambda_{k'}} = \sum_{1 \leq k,k' \leq n} \frac{(1 - \lambda_k \lambda_{k'} + \lambda_k \lambda_{k'}) \tilde{q}_{kk'}^t \tilde{q}_{kk'}}{1 - \lambda_k \lambda_{k'}} = \tilde{W} - \tilde{q}_{11}^t \tilde{q}_{11} + \sum_{s=0}^{\infty} \left( \tilde{W} - \tilde{q}_{11}^t \tilde{q}_{11} \right)^s.
$$

3.2 Principal Component of Expected First Meeting Time $\mu_{a,b}$

We examine the principal component of $\mu_{a,b}$ with the spectral formula of expected first meeting time $\mu_{a,b}$, and mathematically reveal the effect of the graph structure on expected first meeting time $\mu_{a,b}$ on the basis of the clarified principal component. In order to derive the principal component of $\mu_{a,b}$, we use the method to examine that of first hitting time $\mu_{a;i}$ also used in [14].

First, we introduce

$$
\tilde{N} := \tilde{I} \otimes \tilde{I} - \tilde{W} \otimes \tilde{W} = \tilde{I} - \tilde{W},
$$

where $\tilde{I}$ is the unit matrix of $n \times n$, and $\otimes$ is the Kronecker product. According to the definition of the Kronecker product, $\tilde{I}$ and $\tilde{W}$ are $n^2 \times n^2$ matrices. Let $\hat{N}^\dagger$ be the pseudo-inverse matrix of $\hat{N}$ so that $\hat{N}^\dagger \hat{N} = \hat{N}$, and it is given by

$$
\hat{N}^\dagger = \sum_{1 \leq k,k' \leq n} \frac{\hat{q}_{kk'}^t \hat{q}_{kk'}}{1 - \lambda_k \lambda_{k'}},
$$

where $\hat{q}_{kk'}$ is a column vector with $n^2$ elements, and is given by

$$
\hat{q}_{kk'} := q_k \otimes q_{k'}.
$$

By substituting $\hat{N}^\dagger$ into Eq. (40), expected first meeting time $\mu_{a,b}$ is given by

$$
\mu_{a,b} = \frac{s_2}{s_2} \sum_{c \in V} \sum_{c' \in V} d_c^2 d_{c'}^2 \hat{u}_{cc} \hat{N}^\dagger (\hat{u}_{c'} - \hat{u}_{ab}).
$$

In the above equation, $\hat{u}_{ab}$ is

$$
\hat{u}_{ab} := u_a \otimes u_b,
$$

where $u_a$ is the column vector whose $i$-th element $u_a(i)$ is given by

$$
\mu_{a,b} = \begin{cases} 
\frac{1}{\sqrt{d_a}} & \text{if } i = a \\
0 & \text{otherwise}
\end{cases}.
$$

In the above deviation process, we used

$$
\tilde{q}_{ij} \tilde{q}_{kl} = \begin{cases} 
1 & \text{if } i = k \text{ and } j = l \\
0 & \text{otherwise}
\end{cases},
$$

$$
(\hat{q}_{kk'}^t \hat{q}_{kk'})^s = \hat{q}_{kk'}^t \hat{q}_{kk'}.
$$

By substituting Eq. (49) into Eq. (46), $\mu_{a,b}$ is expressed by

$$
\mu_{a,b} = \frac{s_2}{s_2} \sum_{c \in V} \sum_{c' \in V} d_c^2 d_{c'}^2 \hat{u}_{cc} \hat{M}(\hat{u}_{c'} - \hat{u}_{ab}).
$$
To obtain the above equation, we used

\[ \dot{\bar{u}}_{cc} = \dot{\bar{u}}_{cc} [\dot{\bar{u}}_{c'} - \bar{u}_{ab}] \]

\[ = \dot{\bar{u}}_{cc} \left( \dot{\bar{u}}_{c'} - \bar{u}_{ab} \right) \]

\[ = \frac{1}{\bar{d}_c} \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \quad \text{if } c = c' \]

\[ = \frac{1}{\bar{d}_c} \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \quad \text{otherwise} \]

\[ = \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) . \quad (54) \]

The first term on the right side of Eq. \(53\) corresponds to the principal component of expected first meeting time \(\mu_{a,b}\).

In order to confirm that \(s^2_1/s_2\) is the principal component of expected first meeting time \(\mu_{a,b}\), we discuss

\[ \left| \frac{\mu_{a,b}}{s^2_1} - 1 \right| = \frac{1}{s_2^2} \sum_{e \in V} \sum_{c = 1}^n \bar{d}_c^2 \bar{d}_c^2 \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \right| . \quad (55) \]

The right side of the above equation expresses the error between \(\mu_{a,b}\) and the principal component \(s^2_1/s_2\).

We examine the upper bound on the right side of Eq. \(55\) using

\[ \frac{1}{1 - \lambda_k \lambda_{k'}} \leq \frac{1}{1 - \lambda_2} , \quad (56) \]

for \(2 \leq k, k' \leq n\). Using the above equation, we obtain

\[ \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ = \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{W} - \dot{\bar{q}}_{11} \dot{\bar{q}}_{11}) (\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ + \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{W} - \dot{\bar{q}}_{11} \dot{\bar{q}}_{11})^2 \sum_{1 \leq k,k' \leq n} \frac{\bar{q}_{kk'}}{1 - \lambda_k \lambda_{k'}} (\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ \leq \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{W} - \dot{\bar{q}}_{11} \dot{\bar{q}}_{11}) \right| \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ + \frac{1}{1 - \lambda_2} \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{W} - \dot{\bar{q}}_{11} \dot{\bar{q}}_{11})^2 (\bar{I} - \dot{\bar{q}}_{11} \dot{\bar{q}}_{11}) (\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ \leq \left| \dot{\bar{u}}_{cc} \bar{W}(\bar{u}_{c'} - \bar{u}_{ab}) \right| + \frac{1}{1 - \lambda_2} \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ \leq \left| \dot{\bar{u}}_{cc} \bar{W}(\bar{u}_{c'} - \bar{u}_{ab}) \right| + \frac{1}{1 - \lambda_2} \left| \dot{\bar{u}}_{cc} \bar{M}(\bar{u}_{c'} - \bar{u}_{ab}) \right| \]

\[ \leq \frac{2w_{max}^2}{d_{min}^2} + \frac{1}{1 - \lambda_2} \frac{w_{max} \sqrt{2} w_{max}}{d_{min}^2} \]

\[ \leq \frac{2w_{max}^2}{d_{min}^2} \left( \frac{1}{1 - \lambda_2} + 1 \right) . \quad (57) \]

In the deviation process, we used

\[ \frac{\nu_i \dot{W} \dot{u}_{kl}}{\nu_i \dot{W} \dot{u}_{kl}} = \frac{\nu_i \dot{W} \dot{u}_{kl}}{\nu_i \dot{W} \dot{u}_{kl}} \]

\[ = \frac{w_{i,k} d_i d_k}{w_{j,t} d_j d_t} \leq \frac{d_{max}^2}{d_{min}^2} . \quad (58) \]

\[ \frac{\nu_i \dot{W} \dot{u}_{ij}}{\nu_i \dot{W} \dot{u}_{ij}} = \frac{w_{i,k}^2}{d_i^2 d_j^2} \leq \frac{w_{max}^2}{d_{min}^2} . \quad (59) \]

By substituting Eq. \(57\) into Eq. \(55\), the upper bound of the error between \(\mu_{a,b}\) and principal component \(s^2_1/s_2\) is given by

\[ \frac{\mu_{a,b}}{s^2_1} - 1 \leq \frac{2w_{max}^2}{d_{min}^2} \left( \frac{1}{1 - \lambda_2} + 1 \right) . \quad (60) \]

According to the above equation, the error should be small for graph \(G\) where \(\lambda_2\) and \(w_{max}\) are small while \(d_{min}\) is large. In this case, expected first meeting time \(\mu_{a,b}\) is approximated by

\[ \mu_{a,b} \approx \frac{s^2_1}{s_2} = \frac{n}{1 + \frac{d_{std}^2}{d_{avg}^2}} , \quad (61) \]

where \(d_{avg}\) and \(d_{std}\) are the average and the standard deviation of weighted degrees, respectively.

If approximation formula \(61\) holds, as the characteristics of expected first meeting time \(\mu_{a,b}\), we derive the followings: (a) \(\mu_{a,b}\) is small when the coefficient of variation, \(d_{std}/d_{avg}\), is large, and (b) \(\mu_{a,b}\) does not depend on starting nodes \(a\) and \(b\). The characteristics (a) is useful to understand the effect of the graph structure on \(\mu_{a,b}\).

4. Numerical Example

In this section, we confirm the validity of the spectral formula \(40\) and the principal components of expected first meeting time \(\mu_{a,b}\) revealed in Sect. 3. We also examine the error in the approximation formula \(61\) obtained when \(\mu_{a,b}\) is replaced by its principal component.

4.1 Setting

In this section, we use BA (Barabási-Albert) graphs \(15\) and ER (Erdős-Rényi) graphs \(16\). The spectral formula \(40\) and the approximation formula \(61\) depend on the degree distribution of graph. Since the degree distribution of a BA graph is different from that of an ER graph, these graphs are useful to clarify the effect of the degree distribution on these formulas. Due to space limitation, we provide the results with unweighted graphs where \(w_{ij} = 1\) is given for all links \(\forall(i,j) \in E\). In unweighted graphs, weighted degree \(d_i\) of
node \(i\) corresponds to degree \(m_i\), which is the number of links of node \(i\).

The BA model [15] is a typical model for scale-free random graphs. A BA graph is generated using the following procedure. First, a complete graph with \(n_0\) nodes is created. Note that we assume \(n_0 = m\) for the sake of simplicity. Next, each node is added one by one until the number of nodes in the BA graph is equal to \(n\). When adding \(t\)-th node \((t = m + 1, m + 2, \ldots, n)\), \(m_{\text{new}}\) links are created from node \(t\) to nodes \(j \in \{1, 2, \ldots, t - 1\}\) with connection probability \(p^\text{BA}_j(t)\). Connection probability \(p^\text{BA}_j(t)\) is given by

\[
p^\text{BA}_j(t) = \frac{m_j(t)}{\sum_{l=1}^{t-1} m_l(t)},
\]

where \(m_j(t)\) is the degree of node \(j\) when the addition of \(t - 1\)-th node is finished. BA graphs have the power-law degree distribution (i.e., \(\text{Prob} [m_i = m] \sim m^{-3}\)). If \(G\) is unweighted, average weighted degree \(d_{\text{avg}}\) is equal to average degree \(k_{\text{avg}}\). Hence, average weighted degree \(d_{\text{avg}}\) of a BA graph are approximated by

\[
d_{\text{avg}} = \frac{m (m - 1) + 2 m (n - m)}{n} \approx 2 m,
\]

where we assume \(n \gg m\).

On the other hand, the ER model [16] is a classical random graph model. A ER graph is generated using the following procedure. First, \(n\) nodes are created. Next, links are created between any pair of nodes with probability \(p^\text{ER}\). If the graph is not connected, the link creation is tried again. Average weighted degree \(d_{\text{avg}}\) of a ER graph is given by

\[
d_{\text{avg}} = (n - 1) p^\text{ER},
\]

Degree distribution \(\text{Prob} [m_i = m]\) of a ER graph follows the binomial distribution. According to the difference between the power-law and the binomial distributions, standard deviation \(d_{\text{std}}\) of weighted degrees in a BA graph is larger than that in a ER graph.

In order to focus on the difference in standard deviation \(d_{\text{std}}\) of weighted degrees, we set \(m\) and \(p^\text{ER}\) by

\[
m = \left\lfloor d_{\text{avg}} \right\rfloor, \quad p^\text{ER} = \frac{d_{\text{avg}}}{n - 1},
\]

so that average weighted degree \(d_{\text{avg}}\) of a ER graph and a BA graph are roughly equal. Note that minimum weighted degree \(d_{\text{min}}\) in both graphs also increases as \(d_{\text{avg}}\) increases.

In order to examine the validity and the error of the spectral formula (40) and the approximation formula (61), we measure the average of first meeting times in simulation using the following procedure.

1. Generate a BA graph or a ER graph using the above procedures.

2. Put random walkers on node \(a \in V\) and node \(b \in V\), respectively.

3. Move each random walker with transition probability \(p_{i \rightarrow j}\) given by Eq. (2).

4. Repeat step 3 until the two random walks meet at a same node.

5. Repeat step 2 through step 4 10,000 times to calculate the average of first meeting times.

We use the parameter configuration shown in Tab. 1 as a default parameter configuration.

4.2 Validity for the Spectral Formula of Expected First Meeting Time \(\mu_{a,b}\)

We confirm the validity for the spectral formula of expected first meeting time \(\mu_{a,b}\) given by Eq. (40).

Figures 2 and 3 show the first meeting times obtained from the simulation and the analysis (i.e., the spectral formula (40)), for different settings of random walker’s starting node \(b\) in BA graphs and ER graphs, respectively. According to Figs. 2 and 3 the analysis results are almost same as the simulation results regardless of \(n\) and \(b\).

We then evaluate the error in the spectral formula (40).

In this evaluation, we use relative error \(\epsilon_{a,b}\) of expected first meeting time \(\mu_{a,b}\). Relative error \(\epsilon_{a,b}\) is defined by

\[
\epsilon_{a,b} := \left| \frac{\mu_{a,b} - \mu_{a,b}^\text{sim}}{\mu_{a,b}^\text{sim}} \right|,
\]

where \(\mu_{a,b}^\text{sim}\) is the average of first meeting times obtained from the simulation. We examine the average and the maximum of relative errors \(\epsilon_{a,b}\) when changing starting node \(b\) while starting node \(a\) is fixed.
Figure 3: Expected first meeting time $\mu_{a,b}$ for different settings of starting node b of the random walker in ER graphs.

Figure 4: Average weighted degree $d_{\text{avg}}$ vs. the average and the maximum of relative error $\epsilon_{a,b}$ for expected first meeting $\mu_{a,b}$.

4.3 Validity for the Principal Component of Expected First Meeting Time $\mu_{a,b}$

We clarify the validity for the principal component of expected first meeting time $\mu_{a,b}$ derived in Sect. 3. Specifically, we examine relative error $\epsilon_{a,b}'$ of the approximation formula (61) obtained when expected first meeting time $\mu_{a,b}$ is given by the principal component (i.e., $s_1^2/s_2$). Relative error $\epsilon_{a,b}'$ is defined by

$$
\epsilon_{a,b}' := \frac{\left| s_1^2/s_2 - \mu_{a,b}^{\text{sim}} \right|}{\mu_{a,b}^{\text{sim}}},
$$

(68)

Figures 6 and 7 show the average of relative errors $\epsilon_{a,b}'$ of the approximation formula (61) for BA graphs and ER graphs with different numbers of nodes, n, respectively. The average of relative errors $\epsilon_{a,b}$ was calculated from 10,000 simulations where starting node a and starting node b are randomly selected. In Fig. 7, we do not plot the result for $n = 10,000$ and $d_{\text{avg}} = 6$ because a connected ER graph cannot be generated. According to the results, if average weighted degree $d_{\text{avg}}$ is large enough, relative error $\epsilon_{a,b}'$ is small, and the derived principal component is valid. This can also be explained by Eq. (60). The right-hand side of Eq. (60) represents the upper bound of the error in the approximation formula (61). If average weighted degree $d_{\text{avg}}$ is large, minimum weighted degree $d_{\text{min}}$ is also large. As the minimum weighted degree $d_{\text{min}}$ increases, the upper bound becomes small, and relative error $\epsilon_{a,b}$ of the approximation formula (61) should decreases. Moreover, according to Figs. 6 and 7, the average of relative errors $\epsilon_{a,b}'$ is constant or becomes smaller as n increases, and so the approximation formula (61) is also effective for large-scale graphs.

From the above results, the derived principal component is valid if the average weighted degree $d_{\text{avg}}$ is sufficiently large (i.e., $d_{\text{avg}} \geq 4$). According to the site (17) collecting the statistical information (e.g., average degree) of various graphs existing in reality, the average degree of a typical graph is greater than 4. Hence, our analysis results is expected to be useful for many graphs in reality.

Finally, we confirm the effect of the graph structure on expected first meeting time $\mu_{a,b}$ revealed in Sect. 3. According to the approximation formula (61), $\mu_{a,b}$ increases as $s_1^2/s_2$ and the average of the first
meeting times obtained in the simulation.

Figures 8 and 9 show the average of first meeting times obtained from the simulation with different settings of $s_1^2/s_2^2$ in BA graphs and ER graphs, respectively. To calculate the average of first meeting times, we execute 10,000 simulations where starting node $a$ and starting node $b$ are randomly selected. In these figures, we plot a straight line of $y = x$ to easily confirm the effect. According to the results, the average of first meeting times is approximately along $y = x$ line, excepting the result for BA graphs with average weighted degree $d_{avg} = 2$. Therefore, the effect is also confirmed from the numerical example if average weighted degree $d_{avg}$ is sufficiently large.

5. Conclusion and Future Work

In this paper, we analyzed the first meeting time of multiple
random walks in arbitrary graphs, and clarified the effect of graph structures on its expected value. First, we derived the spectral formula of the expected first meeting time for two random walkers using the spectral graph theory. Then, we examined the principal component of the expected first meeting time using the derived spectral formula. The clarified principal component reveals that (a) the expected first meeting time is almostly dominated by \( n/(1 + d_{\text{avg}}^2/d_{\text{avg}}^2) \) and (b) the expected first meeting time is independent of the starting nodes of random walkers where \( n \) is the number of nodes. \( d_{\text{avg}} \) and \( d_{\text{std}} \) are the mean and the standard deviation of the weighted degree, respectively. The characteristics (a) is useful to understand the effect of the graph structure on the first meeting time. In addition, we confirmed the validity of the analysis results through numerical examples. According to the revealed effect of graph structures, the variance of coefficient for weighted degrees, \( d_{\text{std}}/d_{\text{avg}} \) (degree heterogeneity), facilitates quicker the meeting of random walkers.

As future work, we are planning to examine the validity of the analysis results with graphs existing in reality, and apply them to develop efficient graph algorithms.

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