Generalized Huberman-Rudnick scaling law and robustness of q-Gaussian probability distributions

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Abstract – We generalize Huberman-Rudnick universal scaling law for all periodic windows of the logistic map and show the robustness of q-Gaussian probability distributions in the vicinity of chaos threshold. Our scaling relation is universal for the self-similar windows of the map which exhibit period-doubling subharmonic bifurcations. Using this generalized scaling argument, for all periodic windows, as chaos threshold is approached, a developing convergence to q-Gaussian is numerically obtained both in the central regions and tails of the probability distributions of sums of iterates.

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Introduction. – It is well known that many complex systems exhibit transitions from periodic motion to chaos through a period-doubling route like Rayleigh-Benard system in a box [1], forced pendulum [2], logistic map [3], Chirikov map [4] etc. The logistic map, defined as

$$x_{t+1} = 1 - ax_t^2, \quad (1)$$

(where $0 < a \leq 2$ is the control parameter and the phase space $x_t$ is between $[-1,1]$ with $t = 0, 1, 2, \ldots$), is a good example to observe Feigenbaum route generated by pitchfork bifurcations and its universal features. This map has its critical point, denoted by $a_c$, at $a_c = 1.401155189 \ldots$, which can be approached from the left (i.e., from the periodic region) via the period-doubling procedure where $2^n$ periods accumulate at this critical point usually described as chaos threshold. This point can also be approached from the right (i.e., from the chaotic region) via the band merging procedure where an infinite number of bands merge at the critical point [5]. A sketchy view of these approaches to $a_c$ from left and from right is given in fig. 1(a). In the chaotic region ($a > a_c$), there exist many windows of higher periodic cascades of $s 2^n$ periods, where $s$ is an integer and $n$ is the degree of period doublings. It is worth noting that, within each window, reverse bifurcations of $s 2^{n+1}$ bands merging into $s 2^n$ bands can easily be detected although the width of these windows decreases rapidly as long as $s \neq 1$. As an example, the $s = 3$ case (namely, the period-3 case) is illustrated in fig. 1(b).

This self-similar structure of the map is of sensible importance and has led Feigenbaum to develop a scaling theory for the non-chaotic period-doubling region of the bifurcations, which enables one to localize the control parameter value at the bifurcation from the $2^n$ period to $2^{n+1}$ via the scaling relation

$$|a - a_c| \sim \delta^{-n}, \quad (2)$$

where $\delta = 4.699 \ldots$ is the Feigenbaum constant [6]. On the other hand, it is possible to obtain eq. (2) from the famous Huberman-Rudnick scaling law [7], which shows that the envelope of the Lyapunov exponents near $a_c$ exhibits a universal behavior, similar to that of an order parameter close to the critical point of the phase transition. This relation can be written as

$$\lambda = \lambda_0 |a - a_c|^{\nu}, \quad (3)$$

where $a > a_c$, $\nu = \ln 2/\ln \delta$, $\lambda$ is the Lyapunov exponent and $\lambda_0$ is a constant. Equation (2) can easily be obtained from eq. (3). For $a$ values slightly above the chaos threshold, there exist $2^n$ ($n = 1, 2, \ldots, \infty$) chaotic bands, which approach the Feigenbaum attractor as $n \to \infty$ by the band splitting procedure. In that region, if we start
exponent into eq. (3) immediately gives
\[ 2^{-n} = |a - a_c|^{n/2 ln \delta}, \]
from where eq. (2) is easily obtained.

This scaling relation, in fact, is exactly the one used in [8], where the probability distributions of the sums of the iterates of the logistic map, as \( a_c \) is approached from the band merging region, have been shown to be well approached by \( q \)-Gaussians provided that the appropriate number of iterations \( (N^*) \) is obtained from the above-mentioned scaling relation.

\( q \)-Gaussians, defined as
\[ P(y) = \begin{cases} P(0) \left[ 1 - \beta(1-q)y^2 \right]^{-\frac{1}{q-1}}, & \text{for } \beta(1-q)y^2 < 1, \\ 0, & \text{otherwise}, \end{cases} \]

(6)

(where \( q < 3 \) and \( \beta > 0 \) are parameters and the latter controls the width of the distribution) are the distributions that optimize, under appropriate constraints, the nonadditive entropy \( S_q \) (defined to be \( S_q \equiv (1 - \sum_i p_i^q)/(q-1) \), on which nonextensive statistical mechanics is based [9–12]. As \( q \to 1 \), \( q \)-Gaussians recover the Gaussian distribution.

Although in Nature many stochastic processes, consisting of the sum of many independent or nearly independent variables, are known to converge to a Gaussian distribution due to the standard central-limit theorem [13,14], in recent years several complex systems such as low-dimensional dissipative maps in the vicinity of chaos threshold [15–17], high-dimensional dissipative systems [18] and conservative maps [19,20] are shown to exhibit probability distributions that are well approached by \( q \)-Gaussians.

Our main aim in this paper is twofold: firstly, we try to generalize the Huberman-Rudnick scaling law to all periodic windows of the logistic map, secondly, using this generalized version of the Huberman-Rudnick scaling law, we analyse the robustness of the probability distribution of the sums of iterates of the logistic map as chaos threshold is approached.

**Generalization of the Huberman-Rudnick scaling law.** – In order to find a generalized version of the Huberman-Rudnick scaling law, let us start by denoting the accumulation point of a particular periodic cycle \( s \) as \( a_{c}^{(s)} \). For example, \( a_{c}^{(3)} \) and \( a_{c}^{(5)} \) stand for the accumulation points of period-3 and period-5 windows inside the chaotic region. These points can be found as \( a_{c}^{(3)} = 1.779818075 \ldots \) and \( a_{c}^{(5)} = 1.631019835 \ldots \). Then, one needs to check whether the form of the Huberman-Rudnick scaling law is valid for all other periodic cycles in the chaotic region.

More precisely, one needs to check whether the exponent \( \nu \) in the scaling law is equal to \( \ln 2/\ln \delta \) as in the standard case. Therefore, we have first checked this and found that, for all periodic windows, the envelope of the Lyapunov exponents, in the \( \log \lambda \) vs. \( \log(a - a_c) \) plot, is given by a
slope 0.449, which is nothing but $\ln 2/\ln \delta$. Hence we can now write the Huberman-Rudnick scaling law for all periodic cycles as

$$\lambda = \lambda_0[a - a_c^{(s)}]^{2n/\ln \delta}. \quad (7)$$

At this point, it should be recalled that, for a particular period-\(s\) window, a trajectory that starts in one band will be back in the same band after $2^n$ ($n = 1, 2, \ldots, \infty$) iterations. If we use this feature in the definition of the effective Lyapunov exponent, namely $\lambda_0 = \lambda s 2^n$, then one can write the Huberman-Rudnick scaling law for other periodic windows as

$$\lambda = \lambda s 2^n [a - a_c^{(s)}]^{2n/\ln \delta} \quad (8)$$

and

$$2^{-n} = |a - a_c^{(s)}|^{2n/\ln \delta}. \quad (9)$$

This equation enables us to obtain the generalized Huberman-Rudnick scaling law as

$$|a - a_c^{(s)}| = \delta^{-n} - \frac{\ln s}{\ln 2}. \quad (10)$$

This new scaling relation is valid for all periodic windows including the standard case for $s = 1$, which immediately recovers the standard scaling law given in eq. (2).

**Robustness of the probability distributions.**—

Now let us concentrate on the probability distributions of the sums of iterates of the logistic map, which can be written as

$$y := \sum_{i=1}^{N} (x_i - \langle x \rangle), \quad (11)$$

where $x_i$ are the iterates of the logistic map and $x_1$ is the initial value regarded as a random variable. It has analytically been proved that, for strongly chaotic systems, the probability distribution of $y$ becomes Gaussian for $N \to \infty$ [21,22]. Here, the average $\langle \ldots \rangle$ is calculated as time average. On the other hand, as mentioned before in the first section, several complex systems of the low- and high-dimensional dissipative and conservative type exist where the probability distribution does not approach a Gaussian, and therefore violates the standard central-limit theorem due to a possible lack of ergodicity and mixing properties. For such systems, it is necessary to take the average over not only a large number of $N$ iterations but also a large number of $M$ randomly chosen initial values, namely,

$$\langle x \rangle = \frac{1}{M} \sum_{j=1}^{M} \sum_{i=1}^{N} x_{ij}. \quad (12)$$

As chaos threshold is approached, the logistic map has already been studied in this respect [8,15,23,24]. As it has already been argued in [8], in principle, in order to attain chaos threshold point exactly (i.e., approaching this point with infinite precision), one needs to take $n \to \infty$, which, in other words, means that the necessary number of iterations to achieve the limit distribution at the chaos threshold is $N^* \to \infty$ since $N^* = 2^n$. Since this is, no doubt, unattainable in any numerical experiment, one can only approach this critical point using the appropriate values for $(a, N^*)$ pairs coming from the Huberman-Rudnick scaling law. As long as this scaling law is obeyed, developing a $q$-Gaussian shape of the limit distribution, as chaos threshold is approached, has been clearly shown in [8]. On the way of approaching chaos threshold, for any approximation level of finite $(a, N^*)$ pairs, if the number of iterations used is too much larger than $N^*$ and therefore violating the Huberman-Rudnick scaling law (i.e., $N \gg N^*$), it is of course not surprising that the probability distribution starts to approach a Gaussian form from its central part since the system starts to feel that it is not exactly at the chaos threshold. Such numerical examples can be found in [8,23]. On the other hand, if the number of iterations used is too much smaller than $N^*$ and therefore violating again the Huberman-Rudnick scaling law (i.e., $N \ll N^*$), then the summation starts to be inadequate to approach the shape of the limit probability distribution and it exhibits a kind of peaked or multifractal distribution. Such numerical examples have already been given in [15,23,24].

In the remainder of this work, we try to provide further evidence on the robustness of the $q$-Gaussian probability distributions that can be seen as the chaos threshold is approached. In order to accomplish this task, we investigate other periodic windows (numerically chosen examples are period 3 and 5) making use of our generalized Huberman-Rudnick law.

As the separated band structure for periodic cycle 2 goes from $2^1$ to $2^\infty$ with $2^n$ ($n = 1, 2, \ldots, \infty$), for any periodic cycle $s$, the same behavior would be to go from $2^{s0}$ to $2^{s\infty}$ with $2^k$ ($k = 0, 1, \ldots, \infty$). It is evident that there are $k \to (n - 1)$ transformations between $k$ and $n$ mathematically. Generically, for any periodic cycle $s$ in the chaotic region, one needs to perform $s 2^k$ iterations of the map for a given initial value with a control parameter $a$ obtained from the generalized scaling law. After $s 2^k$ iterations, the system will basically fall into the same band of the band splitting structure. This means that the sum of the iterates $\sum_{i=1}^{s 2^k} x_i$ will essentially approach a fixed value $w = s 2^k (x)$ plus a small correction $\Delta w_1$ which describes the small fluctuations of the position of the $s 2^k$-th iterate within the chaotic band. Hence, one can write

$$y_1 = \sum_{i=1}^{s 2^k} (x_i - \langle x \rangle) = \Delta w_1. \quad (13)$$

If we continue to iterate for another $s 2^k$ times, we obtain

$$y_2 = \sum_{i=s 2^k+1}^{2s 2^k} (x_i - \langle x \rangle) = \Delta w_2. \quad (14)$$

The new fluctuation $\Delta w_2$ is not expected to be independent of the old one $\Delta w_1$, since correlations of iterates
Table 1: Parameter values used in this work. The values of $n$ obtained from the generalized scaling law using eq. (17), the corresponding $N^*$ values and the values of $q$ and $\beta$ (estimated from simulations) are listed. The $s = 1$ case, already discussed in [8], has also been included in the table for comparison.

| $s$ | $a$     | $2n$  | $N^*$  | $q$  | $\beta$ |
|-----|---------|-------|--------|------|---------|
| 1   | 1.401588| 10.05 | $10^2$ | 1.70 | 6.2     |
|     | 1.401248| 12.05 | $10^2$ | 1.70 | 6.2     |
|     | 1.401175| 14.05 | $10^2$ | 1.70 | 6.2     |
|     | 1.40115945| 16.05 | $10^2$ | 1.70 | 6.2     |
| 3   | 1.779819805038384| 14.05 | $3 \times 10^2$ | 1.64 | 6.2     |
|     | 1.779818446177396| 16.05 | $3 \times 10^2$ | 1.64 | 6.2     |
|     | 1.779818155150985| 18.05 | $3 \times 10^2$ | 1.64 | 6.2     |
|     | 1.779818928220039| 20.05 | $3 \times 10^2$ | 1.64 | 6.2     |
| 5   | 1.631020391104644| 14.05 | $5 \times 10^2$ | 1.62 | 6.2     |
|     | 1.63101995463619| 16.05 | $5 \times 10^2$ | 1.62 | 6.2     |
|     | 1.63101986115802| 18.05 | $5 \times 10^2$ | 1.62 | 6.2     |
|     | 1.63101984113785| 20.05 | $5 \times 10^2$ | 1.62 | 6.2     |

Table 2: The $s = 3$ case, already discussed in [8], has also been included in the table for comparison.

| $s$ | $a$     | $2n$  | $N^*$  | $q$  | $\beta$ |
|-----|---------|-------|--------|------|---------|
| 1   | 1.401588| 10.05 | $10^2$ | 1.70 | 6.2     |
|     | 1.401248| 12.05 | $10^2$ | 1.70 | 6.2     |
|     | 1.401175| 14.05 | $10^2$ | 1.70 | 6.2     |
|     | 1.40115945| 16.05 | $10^2$ | 1.70 | 6.2     |
| 3   | 1.779819805038384| 14.05 | $3 \times 10^2$ | 1.64 | 6.2     |
|     | 1.779818446177396| 16.05 | $3 \times 10^2$ | 1.64 | 6.2     |
|     | 1.779818155150985| 18.05 | $3 \times 10^2$ | 1.64 | 6.2     |
|     | 1.779818928220039| 20.05 | $3 \times 10^2$ | 1.64 | 6.2     |
| 5   | 1.631020391104644| 14.05 | $5 \times 10^2$ | 1.62 | 6.2     |
|     | 1.63101995463619| 16.05 | $5 \times 10^2$ | 1.62 | 6.2     |
|     | 1.63101986115802| 18.05 | $5 \times 10^2$ | 1.62 | 6.2     |
|     | 1.63101984113785| 20.05 | $5 \times 10^2$ | 1.62 | 6.2     |

decay very slowly if we are close to the critical point. Continuing this $2^k$ times, we finally obtain

$$y_{2^k} = \sum_{i=s2^k-n2^{k+1}}^{s2^{2k}} (x_i - \langle x \rangle) = \Delta w_{2^k}$$

(15)

if we iterate the map $s2^k$ times in total. The total sum of iterates

$$y = \sum_{i=1}^{s2^{2k}} (x_i - \langle x \rangle) = \sum_{j=1}^{2^k} \Delta w_j$$

(16)

can thus be regarded as a sum of $2^k$ random variables $\Delta w_j$, each being influenced by the structure of the $s2^k$ chaotic bands at distance $a - a_c(s) = \delta^{-n-s/n}$ from the Feigenbaum attractor. At this distance from chaos threshold, in order to see the limit distribution the appropriate number of iterations would be $N^* = s2^{2k}$, which corresponds to $N^* = s2^{2n-2}$ after $k \to (n-1)$ transformations.

Results. – Now we are ready to check the shape of the probability distribution of any periodic windows obeying our generalized Huberman-Rudnick scaling law. The chosen examples of possible periodic windows are period-3 and period-5 ones since they are the largest two periodic windows available in the chaotic region. Although conceptually nothing is changed for small-sized windows, numerical analysis is getting more difficult as windows sizes are decreasing. Numerically used values are given in table 1 for our period-3 and -5 analysis. The control parameter $a$ values are chosen so that the precision of the corresponding $n$ values, coming from the generalized Huberman-Rudnick scaling law as

$$n = -\frac{\ln |a - a_c(s)|}{\ln \delta} - \frac{\ln s}{\ln 2},$$

(17)

would be the same (see table 1). This means that we are approaching the critical point with $a$ values located on a straight line with a given slope. Our results are given in fig. 2 for period 3 and in fig. 3 for period 5. In both cases four representative points systematically approaching the chaos threshold are given. It is clear from fig. 2(a) and fig. 3(a) that the probability distributions of both periodic windows approach a $q$-Gaussian. It is also evident that, as the chaos threshold is better approached, the tails of the distribution develops better on the $q$-Gaussian, signaling that the limit distribution obtained at the exact chaos threshold point would be a $q$-Gaussian with infinitely long tails. We also present the same data in fig. 2(b) and fig. 3(b) in a different way so that a straight line would be expected for $q$-Gaussians. Only the closest cases to the
chaos threshold for each periodic window are plotted. It is
seen from these plots that the curves develop on top of a
straight line surrounded by log-periodic modulations.

Conclusions. – Our main results obtained in this
paper can be summarized as follows: i) For the logistic
map having self-similar structure, Huberman-Rudnick uni-
versal scaling law has been generalized, which becomes now
consistent to all periodic windows in the chaotic region of
the map. This new generalized scaling law is of sensible
importance since it enables us to produce self-similar
structure of the map and to explain all band merging
structures in all available periodic windows using only one
generalized formula. ii) The standard Huberman-Rudnick
scaling law has already been used in [8,25] and q-Gaussian
probability distributions have been observed as the
standard period-2 accumulation point is approached.
However, in order to test the robustness of q-Gaussian
distributions, a first straightforward attempt should be to
analyse other critical points (chaos thresholds) of different
periodic windows located in the chaotic region of the logis-
tic map. Since the generalized Huberman-Rudnick scaling
law obtained in the first part of this paper now enables us
to localize appropriate \((n,N^*)\) pairs as the accumulation
point is approached, we managed to check two representa-
tive periodic windows. For each case studied here (and
possibly for all other periodic windows) it is numerically
shown that the q-Gaussian probability distributions with
log-periodic oscillations are again the observed distribu-
tions and developing better as the critical point becomes
closer. These results clearly indicate the robustness of the
q-Gaussian probability distributions seen in the vicinity of
chaos threshold. Although the obtained \(q\) values seem
to exhibit a slow decreasing tendency as the size of the
periodic window becomes smaller, we believe that the
genuine limit distribution of the chaos threshold (for all
\(s\) values) might converge to a q-Gaussian with a unique
\(q\) value, which is expected to be in the interval [1.6, 1.75].
This \(q\) value could be thought as the \(q_{stat}\) value of the
q-triplet proposed by Tsallis [12] \((q_{sen} \text{ and } q_{rel} \text{ being the other
ingredients of the triplet}). In this sense, this value
completes the triplet whose other values have been found
previously in the literature as \(q_{sen} = 0.2445...\) [26,27] and
\(q_{rel} = 2.24...\) [28,29] for the logistic map. Other examples of
q-triplet can also be found in the literature for various
systems like ozone layer [30], solar wind [31], scale-
invariantly correlated binary random variables [32] etc.

Finally it is worth mentioning that the results obtained
here are expected to be valid for all other dissipative maps
sharing the same universality class with the logistic map.
As an open question that can be addressed in a future
work, one can mention the analysis of an appropriate
scaling law for the systems exhibiting quasi-periodic route
to chaos.

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