Stability of Arakelov bundles and tensor products without global sections

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Introduction

G. Faltings has proved that for each semistable vector bundle $E$ over an algebraic curve of genus $g$, there is another vector bundle $F$ such that $E \otimes F$ has slope $g - 1$ and no global sections. (Note that any vector bundle of slope $> g - 1$ has global sections by Riemann–Roch.) See [3] and [4] where this result is interpreted in terms of theta functions and used for a new construction of moduli schemes of vector bundles.

In the present paper, an arithmetic analogue of that theorem is proposed. The algebraic curve is replaced by the set $X$ of all places of a number field $K$; we call $X$ an arithmetic curve. Vector bundles are replaced by so-called Arakelov bundles, cf. section 3. In the special case $K = \mathbb{Q}$, Arakelov bundles without global sections are lattice sphere packings, and the slope $\mu$ measures the packing density.

We will see at the end of section 4 that the maximal slope of Arakelov bundles of rank $n$ without global sections is $d(\log n + O(1))/2 + (\log \mathfrak{d})/2$ where $d$ is the degree and $\mathfrak{d}$ is the discriminant of $K$. Now the main result is:

**Theorem 0.1** Let $E$ be a semistable Arakelov bundle over the arithmetic curve $X$. For each $n \gg 0$ there is an Arakelov bundle $F$ of rank $n$ satisfying

$$\mu(E \otimes F) > \frac{d}{2}(\log n - \log \pi - 1 - \log 2) + \frac{\log \mathfrak{d}}{2}$$

such that $E \otimes F$ has no nonzero global sections.

The proof is inspired by (and generalizes) the Minkowski-Hlawka existence theorem for sphere packings; in particular, it is not constructive. The principal ingredients are integration over a space of Arakelov bundles (with respect to some Tamagawa measure) and an adelic version of Siegel’s mean value formula. Section 2 explains the latter, section 3 contains all we need about Arakelov bundles, and the main results are proved and discussed in section 4.

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1 Notation

Let $K$ be a number field of degree $d$ over $\mathbb{Q}$ and with ring of integers $\mathcal{O}_K$. Let $X = \text{Spec}(\mathcal{O}_K) \cup X_\infty$ be the set of places of $K$; this might be called an ‘arithmetic
curve' in the sense of Arakelov geometry. $X_{\infty}$ consists of $r_1$ real and $r_2$ complex places with $r_1 + 2r_2 = d$. $w(K)$ is the number of roots of unity in $K$.

For every place $v \in X$, we endow the corresponding completion $K_v$ of $K$ with the map $| \cdot |_v : K_v \to \mathbb{R}_{\geq 0}$ defined by $\mu(a \cdot S) = |a|_v \cdot \mu(S)$ for a Haar measure $\mu$ on $K_v$. This is the normalized valuation if $v$ is finite, the usual absolute value if $v$ is real and its square if $v$ is complex. The well known product formula $\prod_{v \in X} |a|_v = 1$ holds for every $0 \neq a \in K$. On the adele ring $\mathbb{A}$, we have the divisor map $\text{div} : \mathbb{A} \to \mathbb{R}^X_{\geq 0}$ that maps each adele $a = (a_v)_{v \in X}$ to the collection $(|a_v|_v)_{v \in X}$ of its norms.

Let $O_v$ be the set of those $a \in K_v$ which satisfy $|a|_v \leq 1$; this is the ring of integers in $K_v$ for finite $v$ and the unit disc for infinite $v$. Let $O_\mathbb{A}$ denote the product $\prod_{v \in X} O_v$; this is the set of all adeles $a$ with $\text{div}(a) \leq 1$. By $D \leq 1$ for an element $D = (D_v)_{v \in X}$ of $\mathbb{R}^X_{\geq 0}$, we always mean $D_v \leq 1$ for all $v$.

We fix a canonical Haar measure $\lambda_v$ on $K_v$ as follows:

- If $v$ is finite, we normalize by $\lambda_v(O_v) = 1$.
- If $v$ is real, we take for $\lambda_v$ the usual Lebesgue measure on $\mathbb{R}$.
- If $v$ is complex, we let $\lambda_v$ come from the real volume form $dz \wedge d\bar{z}$ on $\mathbb{C}$. In other words, we take twice the usual Lebesgue measure.

This gives us a canonical Haar measure $\lambda := \prod_{v \in X} \lambda_v$ on $\mathbb{A}$. We have $\lambda(\mathbb{A} / K) = \sqrt{n}$ where $n = \text{disc}(K)$ denotes (the absolute value of) the discriminant. More details on this measure can be found in [12], section 2.1.

Let $V_n = \frac{n^{n/2}}{(\sqrt{2\pi})^n}$ be the volume of the unit ball in $\mathbb{R}^n$. For $v \in X_{\infty}$, we denote by $O^n_v$ the $n$-fold cartesian product of $O_v \subseteq K_v$. Similarly, $O^n_\mathbb{A} := \prod_{v \in X} O^n_v$ is the $n$-fold product of $O_\mathbb{A} \subseteq \mathbb{A}$. Its volume $\lambda^n(O^n_\mathbb{A})$ is $V_n^{r_1}(2^n V_{2n})^{r_2}$.

## 2 A mean value formula

The following proposition is a generalization of Siegel’s mean value formula to an adelic setting: With real numbers and integers instead of adeles and elements of $K$, Siegel has already stated it in [10], and an elementary proof is given in [7]. (In the special case $l = 1$, a similar question is studied in [11].)

**Proposition 2.1** Let $1 \leq l < n$, and let $f$ be a nonnegative measurable function on the space $\text{Mat}_{n \times l}(\mathbb{A})$ of $n \times l$ adele matrices. Then

$$
\int_{\text{Mat}_{n \times l}(\mathbb{A})} \sum_{M \in \text{Mat}_{n \times l}(K) \text{rk}(M) = l} f(g \cdot M) \ d\tau(g) = \delta^{-nl/2} \int_{\text{Mat}_{n \times l}(K)} f \ d\lambda^{n \times l}
$$

where $\tau$ is the unique $\text{Sl}_n(\mathbb{A})$-invariant probability measure on $\text{Sl}_n(\mathbb{A}) / \text{Sl}_n(K)$.

**Proof:** The case $l = 1$ is done in section 3.4 of [12], and the general case can be deduced along the same lines from earlier sections of this book. We sketch the main arguments here; more details are given in [8], section 3.2.

Let $G$ be the algebraic group $\text{Sl}_n$ over the ground field $K$, and denote by $\tau_G$ the Tamagawa measure on $G(\mathbb{A})$ or any quotient by a discrete subgroup. The two measures $\tau$ and $\tau_G$ on $\text{Sl}_n(\mathbb{A}) / \text{Sl}_n(K)$ coincide because the Tamagawa number of $G$ is one.

$G$ acts on the affine space $\text{Mat}_{n \times l}$ by left multiplication. Denote the first $l$ columns of the $n \times n$ identity matrix by $E \in \text{Mat}_{n \times l}(K)$, and let $H \subseteq G$ be the stabilizer of $E$. This algebraic group $H$ is a semi-direct product of $\text{Sl}_{n-l}$ and
Mat_{n \times (n-1)}$. Hence section 2.4 of [12] gives us a Tamagawa measure $\tau_H$ on $H(\mathbb{A})$, and the Tamagawa number of $H$ is also one.

Again by section 2.4 of [12], we have a Tamagawa measure $\tau_{G/H}$ on $G(\mathbb{A})/H(\mathbb{A})$ as well, and it satisfies $\tau_G = \tau_{G/H} \cdot \tau_H$ in the sense defined there. In particular, this implies

$$\int_{G(\mathbb{A})/H(\mathbb{A})} f(g \cdot E) d\tau_G(g) = \int_{G(\mathbb{A})/H(\mathbb{A})} f(g \cdot E) d\tau_{G/H}(g).$$

It is easy to see that the left hand sides of this equation and of (1) coincide. According to lemma 3.4.1 of [12], the right hand sides coincide, too. □

3 Arakelov vector bundles

Recall that a (euclidean) lattice is a free $\mathbb{Z}$-module $\Lambda$ of finite rank together with a scalar product on $\Lambda \otimes \mathbb{R}$. This is the special case $K = \mathbb{Q}$ of the following notion:

**Definition 3.1** An Arakelov (vector) bundle $E$ over our arithmetic curve $X = \text{Spec}(\mathcal{O}_K) \cup X_\infty$ is a finitely generated projective $\mathcal{O}_K$-module $E_{\mathcal{O}_K}$ endowed with

- a euclidean scalar product $\langle \cdot, \cdot \rangle_{E,v}$ on the real vector space $E_K^v$ for every real place $v \in X_\infty$ and
- an hermitian scalar product $\langle \cdot, \cdot \rangle_{E,v}$ on the complex vector space $E_K^v$ for every complex place $v \in X_\infty$

where $E_A := E_{\mathcal{O}_K} \otimes A$ for every $\mathcal{O}_K$-algebra $A$.

A first example is the trivial Arakelov line bundle $\mathcal{O}$. More generally, the trivial Arakelov vector bundle $\mathcal{O}^n$ consists of the free module $\mathcal{O}_K^n$ together with the standard scalar products at the infinite places.

We say that $E'$ is a subbundle of $E$ and write $E' \subseteq E$ if $E'_{\mathcal{O}_K}$ is a direct summand in $E_{\mathcal{O}_K}$ and the scalar product on $E'_{K^v}$ is the restriction of the one on $E_{K^v}$ for every infinite place $v$. Hence every vector subspace of $E_K$ is the generic fibre of one and only one subbundle of $E$.

From the data belonging to an Arakelov bundle $E$, we can define a map

$$\| \cdot \|_{E,v} : E_{K^v} \rightarrow \mathbb{R}_{\geq 0}$$

for every place $v \in X$:

- If $v$ is finite, let $\|e\|_{E,v}$ be the minimum of the valuations $|a|_v$ of those elements $a \in K_v$ for which $e$ lies in the subset $a \cdot E_{\mathcal{O}_v}$ of $E_{K^v}$. This is the nonarchimedean norm corresponding to $E_{\mathcal{O}_v}$.
- If $v$ is real, we put $\|e\|_{E,v} := \sqrt{\langle e, e \rangle_v}$, so we just take the norm coming from the given scalar product.
- If $v$ is complex, we put $\|e\|_{E,v} := \langle e, e \rangle_v$ which is the square of the norm coming from our hermitian scalar product.

Taken together, they yield a divisor map

$$\text{div}_E : E_A \rightarrow \mathbb{R}^X_{\geq 0}, \quad e = (e_v) \mapsto (\|e_v\|_{E,v}).$$

Although $\mathcal{O}_A$ is not an $\mathcal{O}_K$-algebra, we will use the notation $E_{\mathcal{O}_A}$, namely for the compact set defined by

$$E_{\mathcal{O}_A} := \{ e \in E_A : \text{div}_E(e) \leq 1 \}.$$
Recall that these norms are used in the definition of the Arakelov degree: If \( L \) is an Arakelov line bundle and \( 0 \neq l \in L_K \) a nonzero generic section, then
\[
\deg(L) := -\log \prod_{v \in X} \|l\|_{L,v}
\]
and the degree of an Arakelov vector bundle \( E \) is by definition the degree of the Arakelov line bundle \( \det(E) \). \( \mu(E) := \deg(E)/\text{rk}(E) \) is called the slope of \( E \). One can form the tensor product of two Arakelov bundles in a natural manner, and it has the property \( \mu(E \otimes F) = \mu(E) + \mu(F) \).

Moreover, the notion of stability is based on slopes: For \( 1 \leq l \leq \text{rk}(E) \), denote by \( \mu_{\text{max}}^{(l)} \) the supremum (in fact it is the maximum) of the slopes \( \mu(E') \) of subbundles \( E' \subseteq E \) of rank \( l \). \( E \) is said to be stable if \( \mu_{\text{max}}^{(l)} < \mu(E) \) holds for all \( l < \text{rk}(E) \), and semistable if \( \mu_{\text{max}}^{(l)} \leq \mu(E) \) for all \( l \).

To each projective variety over \( K \) endowed with a metrized line bundle, one can associate a zeta function as in [5] or [1]. We recall its definition in the special case of Grassmannians associated to Arakelov bundles:

**Definition 3.2** If \( E \) is an Arakelov bundle over \( X \) and \( l \leq \text{rk}(E) \) is a positive integer, then we define
\[
\zeta_{E}^{(l)}(s) := \sum_{E' \subseteq E, \text{rk}(E')=l} \exp(s \cdot \deg(E')).
\]

The growth of these zeta functions is related to the stability of \( E \). More precisely, we have the following asymptotic bound:

**Lemma 3.3** There is a constant \( C = C(E) \) such that
\[
\zeta_{E}^{(l)}(s) \leq C \cdot \exp(s \cdot l \mu_{\text{max}}^{(l)}(E))
\]
for all sufficiently large real numbers \( s \).

**Proof:** Fix \( E \) and \( l \). Denote by \( N(T) \) the number of subbundles \( E' \subseteq E \) of rank \( l \) and degree at least \(-T\). There are \( C_1, C_2 \in \mathbb{R} \) such that
\[
N(T) \leq \exp(C_1 T + C_2)
\]
holds for all \( T \in \mathbb{R} \). (Embedding the Grassmannian into a projective space, this follows easily from [9]. See [9], lemma 3.4.8 for more details.)

If we order the summands of \( \zeta_{E}^{(l)} \) according to their magnitude, we thus get
\[
\zeta_{E}^{(l)}(s) \leq \sum_{\nu=0}^{\infty} (\text{exp}(-l \mu_{\text{max}}^{(l)}(E) + \nu + 1) \cdot \exp(s \cdot (l \mu_{\text{max}}^{(l)}(E) - \nu))))
\]
\[
\leq \exp(s \cdot l \mu_{\text{max}}^{(l)}(E)) \cdot \sum_{\nu=0}^{\infty} \frac{C_3}{\exp((s-C_1)\nu)}.
\]
But the last sum is a convergent geometric series for all \( s > C_1 \) and decreases as \( s \) grows, so it is bounded for \( s \geq C_1 + 1 \). \( \square \)
4 The main theorem

The global sections of an Arakelov bundle $\mathcal{E}$ over $X = \text{Spec}(O_K) \cup X_\infty$ are by definition the elements of the finite set

$$\Gamma(\mathcal{E}) := \mathcal{E}_K \cap \mathcal{E}_{O_\lambda} \subseteq \mathcal{E}_\lambda.$$ 

Note that in the special case $K = \mathbb{Q}$, an Arakelov bundle without nonzero global sections is nothing but a (lattice) sphere packing: $\Gamma(\mathcal{E}) = 0$ means that the (closed) balls of radius $1/2$ centered at the points of the lattice $\mathcal{E}_\mathbb{Z}$ are disjoint. Here larger degree corresponds to denser packings.

**Theorem 4.1** Let $\mathcal{E}$ be an Arakelov bundle over the arithmetic curve $X$. If an integer $n > \text{rk}(\mathcal{E})$ and an Arakelov line bundle $\mathcal{L}$ satisfy

$$1 > \frac{n^l}{\lambda(n^l)} \cdot \lambda(n) \cdot \int \exp(l \deg(\mathcal{L})).$$

then there is an Arakelov bundle $\mathcal{F}$ of rank $n$ and determinant $\mathcal{L}$ such that

$$\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0.$$

**Proof:** Note that any global section of $\mathcal{E} \otimes \mathcal{F}$ is already a global section of $\mathcal{E}' \otimes \mathcal{F}$ for a unique minimal subbundle $\mathcal{E}' \subseteq \mathcal{E}$, namely the subbundle whose generic fibre is the image of the induced map $(\mathcal{F}_K)_{\text{dual}} \rightarrow \mathcal{E}_K$. We are going to average the number of these sections (up to $K^*$) for a fixed subbundle $\mathcal{E}'$ of rank $l$.

Fix one particular Arakelov bundle $\mathcal{F}$ of rank $n$ and determinant $\mathcal{L}$. Choose linear isomorphisms $\phi_{\mathcal{E}'} : K^l \rightarrow \mathcal{E}'_K$ and $\phi_{\mathcal{F}} : K^n \rightarrow \mathcal{F}_K$ and let

$$\phi : \text{Mat}_{n \times l}(K) \rightarrow (\mathcal{E}' \otimes \mathcal{F})_K$$

be their tensor product. Our notation will not distinguish these maps from their canonical extensions to completions or adeles.

For each $g \in \text{Sl}_n(K)$, we denote by $g\mathcal{F}$ the Arakelov bundle corresponding to the $K$-lattice $\phi_{\mathcal{F}}(gK^n) \subseteq \mathcal{F}_K$. More precisely, $g\mathcal{F}$ is the unique Arakelov bundle satisfying $(g\mathcal{F})_\mathbb{A} = \mathcal{F}_\mathbb{A}$, $(g\mathcal{F})_{O_\lambda} = \mathcal{E}_{O_\lambda}$ and $(g\mathcal{F})_K = \phi_{\mathcal{F}}(gK^n)$. This gives the usual identification between $\text{Sl}_n(\mathbb{A})/\text{Sl}_n(K)$ and the space of Arakelov bundles of rank $n$ and fixed determinant together with local trivialisations.

Observe that the generic fibre of $\mathcal{E}' \otimes g\mathcal{F}$ is $\phi(g\text{Mat}_{n \times l}(K))$. A generic section is not in $\mathcal{E}'' \otimes g\mathcal{F}$ for any $\mathcal{E}'' \subseteq \mathcal{E}'$ if and only if the corresponding matrix has rank $l$. So according to the mean value formula of section 2, the average number of global sections

$$\int_{\text{Sl}_n(\mathbb{A})/\text{Sl}_n(K)} \text{card} \left( \frac{K^* \Gamma(\mathcal{E}' \otimes g\mathcal{F})}{K^*} \bigg/ \bigcup_{\mathcal{E}'' \subseteq \mathcal{E}'} \frac{K^* \Gamma(\mathcal{E}'' \otimes g\mathcal{F})}{K^*} \right) d\tau(g)$$

is equal to the integral

$$\varphi^{-nl/2} \int_{\text{Mat}_{n \times l}(\mathbb{A})} (f_K \circ \text{div}_{\mathcal{E}' \otimes \mathcal{F}} \circ \phi) d\lambda_{n \times l}.$$

Here the function $f_K : \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0$ is defined by

$$f_K(D) := \begin{cases} 1/\text{card}\{a \in K^* : \text{div}(a) \cdot D \leq 1\} & \text{if } D \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
with the convention \(1/\infty = 0\).

In order to compute \(\varphi\), we start with the local transformation formula

\[
\lambda_{v}^{n,l} \left( \{ M \in \text{Mat}_{n \times l}(K_v) : c_1 \leq \| \phi(M) \|_{\mathcal{E} \otimes \mathcal{F},v} \leq c_2 \} \right) = \\
\lambda_{v}^{n,l} \left( \{ M \in K^0_v : c_1 \leq \| M \| \leq c_2 \} \right) \cdot \det(\phi)^{-1}_{\det(\mathcal{E} \otimes \mathcal{F},v)}
\]

for all \(c_1, c_2 \in \mathbb{R}_{\geq 0}\). Regarding this as a relation between measures on \(\mathbb{R}_{\geq 0}\) and taking the product over all places \(v \in X\), we get the equation

\[
(\text{div}_{\mathcal{E} \otimes \mathcal{F}} \circ \phi) \lambda^{n,l} = \exp(\text{deg}(\mathcal{E} \otimes \mathcal{F})) \cdot (\text{div}_{\mathcal{O}^{n,l}}) \lambda^{n,l}
\]

of measures on \(\mathbb{R}_{\geq 0}^X\). Hence the integrals of \(f_K\) with respect to these measures also coincide:

\[
\int_{\text{Mat}_{n \times l}((\mathcal{A})} (f_K \circ \text{div}_{\mathcal{E} \otimes \mathcal{F}} \circ \phi) d\lambda^{n,l} = \exp(n \text{deg}(\mathcal{E}')) + l \text{deg}(\mathcal{F})) \cdot \lambda^{n,l} \left( \frac{\mathcal{K}_{\mathcal{O}^{n,l}}}{\mathcal{K}_*} \right).
\]

We substitute this for the integral in \(\varphi\). A summation over all nonzero sub-bundles \(\mathcal{E}' \subseteq \mathcal{E}\) yields

\[
\int \text{card} \left( \frac{\mathcal{K}^* \Gamma(\mathcal{E} \otimes g \mathcal{F}) \setminus 0}{K_*} \right) d\tau(g) = \sum_{l=1}^{\text{rk}(\mathcal{E})} \zeta_{\mathcal{E}}^{(l)}(n) \exp(l \text{deg}(\mathcal{F})) \lambda^{n,l} \left( \frac{\mathcal{K}_{\mathcal{O}^{n,l}}}{\mathcal{K}_*} \right).
\]

But the right hand side was assumed to be less than one, so there there has to be a \(g \in \text{Sl}_n(\mathcal{A})\) with \(\Gamma(\mathcal{E} \otimes g \mathcal{F}) = 0\).

In order to apply this theorem, one needs to compute \(\lambda^n(\mathcal{K}^* \mathcal{O}_K^{\mathcal{N}}/\mathcal{K}_*)\) for \(N \geq 2\). We start with the special case \(K = \mathbb{Q}\). Here each adele \(a \in \mathcal{O}_K^\times\) outside a set of measure zero has a rational multiple in \(\mathcal{O}_K^\times\) with norm 1 at all finite places, and this multiple is unique up to sign. Hence we conclude

\[
\lambda^n \left( \frac{\mathbb{Q}^* \mathcal{O}_K^\times}{\mathbb{Q}^*} \right) = \frac{V_N}{2} \cdot \prod_{p \text{ prime}} \lambda_p^n (\mathbb{Z}_p^N \setminus p\mathbb{Z}_p^N) = \frac{V_N}{2\zeta(N)}.
\]

In particular, the special case \(K = \mathbb{Q}\) and \(\mathcal{E} = \mathcal{O}\) of the theorem above is precisely the Minkowski-Hlawka existence theorem for sphere packings [8], §15.

For a general number field \(K\), we note that the roots of unity preserve \(\mathcal{O}_K^N\). Then we apply Stirling's formula to the factorials occurring via the unit ball volumes and get

\[
\lambda^n \left( \frac{\mathcal{K}^* \mathcal{O}_K^{\mathcal{N}}}{\mathcal{K}_*} \right) \leq \lambda^n(\mathcal{O}_K^N) \leq \left( \frac{2\pi e}{N} \right)^{dN/2} \left( \frac{1}{\pi N} \right)^{(r_1+r_2)/2} \frac{1}{2^{r_2/2}w(K)}.
\]

Using such a bound and the asymptotic statement [8] about \(\zeta^{(l)}_{\mathcal{E}}\), one can deduce the following corollary of theorem [4.4.1]:

**Corollary 4.2** Let the Arakelov bundle \(\mathcal{E}\) over \(X\) be given. If \(n\) is a sufficiently large integer and \(\mu\) is a real number satisfying

\[
\mu_{\text{max}}^{(l)}(\mathcal{E}) + \mu \leq \frac{d}{2} (\log n + \log l - \log \pi - 1 - \log 2) + \frac{\log \vartheta}{2}
\]

for all \(1 \leq l \leq \text{rk}(\mathcal{E})\), then there is an Arakelov bundle \(\mathcal{F}\) of rank \(n\) and slope larger than \(\mu\) such that \(\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0\).
If $\mathcal{E}$ is semistable, this gives the theorem stated in the introduction. Here is some evidence that these bounds are not too far from being optimal:

**Proposition 4.3** Assume given $\epsilon > 0$ and a nonzero Arakelov bundle $\mathcal{E}$. Let $n > n(\epsilon)$ be a sufficiently large integer, and let $\mu$ be a real number such that

$$
\mu_{\text{max}}^{(l)}(\mathcal{E}) + \mu \geq \frac{d}{2}(\log n + \log l - \log \pi - 1 + \log 2 + \epsilon) + \frac{\log d}{2}
$$

holds for at least one integer $1 \leq l \leq \text{rk}(\mathcal{E})$. Then there is no Arakelov bundle $\mathcal{F}$ of rank $n$ and slope $\mu$ with $\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0$.

**Proof:** Fix such an $l$ and a subbundle $\mathcal{E}' \subseteq \mathcal{E}$ of rank $l$ and slope $\mu_{\text{max}}^{(l)}(\mathcal{E})$. For each $\mathcal{F}$ of rank $n$ and slope $\mu$, we consider the Arakelov bundle $\mathcal{F}' := \mathcal{E}' \otimes \mathcal{F}$ of rank $nl$.

By Stirling’s formula, the hypotheses on $n$ and $\mu$ imply

$$
\exp \text{deg}(\mathcal{F}') \cdot \lambda^{nl}(\mathcal{O}^{nl}_A) > 2^{nl} \cdot d^{nl/2}.
$$

Now choose a $K$-linear isomorphism $\phi : K^{nl} \cong \mathcal{F}'_K$ and extend it to adeles. Applying the global transformation formula (3), we get

$$
\lambda^{nl}(\phi^{-1}(\mathcal{F}'_\mathcal{O}^l)) > 2^{nl} \cdot \lambda^{nl}(K^{nl}/K^{nl}).
$$

According to Minkowski’s theorem on lattice points in convex sets (in an adelic version like [11], theorem 3), $\phi^{-1}(\mathcal{F}'_\mathcal{O}^l) \cap K^{nl} \neq \{0\}$ follows. This means that $\mathcal{F}'$ — and hence $\mathcal{E} \otimes \mathcal{F}$ — must have a nonzero global section.

Observe that the lower bound and the upper bound differ only by the constant $d \log 2$. So up to this constant, the maximal slope of such tensor products without global sections is determined by the stability of $\mathcal{E}$, more precisely by the $\mu_{\text{max}}^{(l)}(\mathcal{E})$.

Taking $\mathcal{E} = \mathcal{O}$, we get lower and upper bounds for the maximal slope of Arakelov bundles without global sections, as mentioned in the introduction. In the special case $\mathcal{E} = \mathcal{O}$ and $K = \mathbb{Q}$ of lattice sphere packings, [2] states that no essential improvement of corollary is known whereas several people have improved the other bound by constants.

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