A WEIGHTED MODULE VIEW OF INTEGRAL CLOSURES
OF AFFINE DOMAINS OF TYPE I

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(Communicated by Tom Hoeholdt)

Abstract. A type I presentation \( S = R/J \) of an affine (order) domain has a
weight function injective on the monomials in the footprint \( \Delta(J) \). This can
be extended naturally to a presentation, \( \mathcal{R}/J \), of the integral closure \( \text{ic}(S) \).
This presentation over \( P := F[x_n, \ldots, x_1] \) as an affine \( P \)-algebra relative to
a corresponding grevlex-over-weight monomial ordering is shown to have a
minimal, reduced Gröbner basis (for the ideal of relations \( J \)) consisting only
of \( P \)-quadratic relations defining the multiplication of the \( P \)-module generators
and possibly some \( P \)-linear relations if those generators are not independent
over \( P \). There then may be better choices for \( P \) to minimize the number of
\( P \)-module generators needed. The intended coding theory application is to the
description of one-point AG codes, not only from curves (with \( P = F[x_1] \)) but
also from higher-dimensional varieties.

1. Introduction

To properly describe a curve \( X \) to be used to define a one-point AG code, it is
necessary to put it in special position relative to that one special point \( P_\infty \), with
variables corresponding to rational homogeneous functions modulo \( X \), with no poles
except possibly at \( P_\infty \). Then the generator and/or parity-check functions come from
the vector space \( L(mP_\infty) \) of said functions with pole order at most \( m \), contained
in the ring \( L(\infty P_\infty) \) of all such functions. The footprint \( \Delta(J) \) of the affine domain
\( S = R/J \) of type I defining the curve does not usually define all of \( L(\infty P_\infty) \), but
the footprint of its integral closure (in its field of fractions) does. So there is a
compelling reason to study integral closures in the context of AG coding.

If one views the pole orders as corresponding to the (negatives of the) trailing
exponents in the Laurent series expansions in terms of a local parameter \( t_\infty \) at \( P_\infty \),
then the obvious generalization to \( n \)-dimensional surfaces is in terms of the trailing
exponent vector \( \alpha \in N^n_0 \) in an expansion involving \( n \) independent local parameters.

The view in this paper is to start with a polynomial ring \( P := F[x_n, \ldots, x_1] \) with
a monomial ordering (which can be viewed as a weight function \( wt_P \) injective on
the set of monomials), deal with type I integral extensions \( S := P[y]/(f(y)) \) so that
\( wt_P \) can be naturally extended to a weight function \( wt_S \), which naturally induces
a weight function on the integral closure \( \text{ic}(S) \) of \( S \). While similar to the general
theory of order domains in [5], there are two important differences here. The view
there is that one starts with \( S \), may or may not care about \( P \), (since the weight

2000 Mathematics Subject Classification: Primary: 11T71, 13B22, 94B27.
Key words and phrases: AG codes, integral closure, normalization, order domains.

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function on $S$ may not be a monomial ordering on every choice of $P$), and the $\Delta$-set is viewed as an infinite-dimensional vector space over $\mathbb{F}$, not in terms of a finite $P$-module generating set.

The presentation of $ic(S) = \mathbb{R}/J$ desired here is relative to this $P$-module generating set $y_0 := 1, y_1, \ldots, y_{s-1}$, so that $J$ has a minimal, reduced Gröbner basis with elements of the form

$$y_i y_j - \sum_k c_{i,j,k} y_k, \quad c_{i,j,k} \in P$$

defining the algebra multiplication, and possibly some elements of the form

$$a_{j,i} y_i - a_{i,j} y_j \quad \sum_{k \neq i,j} b_{i,j,k} y_k, \quad a_{j,i}, a_{i,j}, b_{i,j,k} \in P$$

if the $P$-module generators are not independent over $P$. (This means that, in general, $s$ is not necessarily the same as the degree of the integral extension.)

This is followed by a possible minimization of such a presentation by making a more enlightened choice for $P$. After all, when $n = 1$, it is known ([10]) that there is an $\mathbb{F}[f_{\rho}]$-module basis, $\rho$ being the smallest positive pole order, with one basis element of smallest positive pole order congruent to $i \mod \rho$ for each $i$. This leaves open the question of what the minimum number of $P$-module generators is when $n > 1$, not only because it may not be easy to see whether an example is minimized, but also because there is no longer any definite relation between the degree of the extension and that number even when the problem is given as a minimized integral extension.

But what is missing from the *generic* presentations of integral closures given by most current implementations is a nice description of the monomials in the footprint. Here the footprint necessarily has monomials all of the form $y_i a_{\alpha}$ for $y_i$ one of the $P$-module generators, and $a_{\alpha}$ a monomial of $P$, whether or not a complete minimization is found.

Section 2 will be primarily concerned with notation and the implications of said notation. Section 3 contains as small an example as known by the author in which the $P$-module generators are not independent. Section 4 gives the theorem summarizing the structure of $S$ and $ic(S)$. Section 5 gives some discussion of the limitations of various implementations. Section 6 contains examples of minimization and a theorem guaranteeing the structure is not compromised by minimization. Examples here and on the website were done using an implementation of the author’s $q$th-power algorithm [9], [11], written in MAGMA [12] and available on the [http://www.dms.auburn.edu/~leonada](http://www.dms.auburn.edu/~leonada) website. (There are also examples on the website of input and output of current implementations, which may be useful in understanding the following commentary in this section, as well as other points made later on.)

Before proceeding, we should note that this approach is definitely not taken by anyone else. Weight functions, fundamental to the study of order domains [5], [8], are virtually unused or actively ignored in the study of integral closures. The result is usually a generic form of integral closure presentation, with a default monomial ordering, having little to do with the original monomial order of the ring. It is perhaps more surprising, however, that there is never any coherent attempt to give a readable presentation for the integral closure, such as the one suggested here, which is, after all, along the lines of structure constant algebras defined by $s^3$ structure constants (from $P$). (It is even more surprising given that current algorithms are
based on producing sequences of rings each defined in terms of quadratic and linear relations over the previous ring.) But then again, the prevailing viewpoint, at least for \( n > 1 \), is not relative to \( P \), and sometimes not relative to a presentation at all, but merely relative to a set of fractions generating \( ic(S) \) over \( S \). At the very least, one might have hoped for a presentation consisting of:

- a ring \( \mathcal{R} := F[y_{s-1}, \ldots, y_1; x_n, \ldots, x_1] \) as an extension of \( P := F[x_n, \ldots, x_1] \);
- an ideal of relations \( \mathcal{J} \), so that \( ic(S) = \mathcal{R}/\mathcal{J} \);
- an inclusion map \( \phi : S \to ic(S) \) describing what the variables in \( S \) look like in \( ic(S) \);
- an element \( c \) of the conductor, and an inclusion map \( \psi : ic(S) \to \frac{1}{c}ic(S) \) describing what the new variables look like in terms of the field of fractions of \( S \).

even if one doesn’t ask for \( c \in P \), \((y_0 := 1, y_1, \ldots y_{s-1})\) to be \( P \)-module generators, \( \mathcal{J} \) to have the particularly nice form above, or a weight function injective on the monomials of the footprint \( \Delta(\mathcal{J}) \).

### 2. Notation

Let \( F \) be a field (which here and in the intended applications is either a finite field or the algebraic closure of a finite field) and \( P := F[x_n, \ldots, x_1] \) a multivariate polynomial ring over \( F \) in \( n \) independent variables.

Consider the following adaptation of more standard definitions of weight functions in order domains [5], written in terms of the ring \( R \) and ideal \( J \) instead of implicitly in terms of the quotient ring \( S = R/J \).

**Definition 2.1.** Let \( S = R/J \) be an affine domain. A function 

\[
wt_S : R \to \mathbb{N}_0^n \cup \{-\infty\}
\]

(with \(-\infty < \alpha \) for all \( \alpha \in \mathbb{N}_0^n \)) is a weight function on \( S \) iff:

1. \( wt_S(f) = -\infty \) iff \( f \in J \);
2. \( wt_S(f) = 0 \) iff \( f = c + J, c \in F \setminus \{0\} \);
3. \( wt_S(fg) = wt_S(f) + wt_S(g) \) for all \( f, g \notin J \);
4. \( wt_S(\alpha f + \beta g) \leq \max\{wt_S(f), wt_S(g)\} \) for all \( \alpha, \beta \in F \);
5. if \( wt_S(f) = wt_S(g) \geq \lambda \), then \( wt_S(f - \lambda g) < wt_S(g) \) for an unique \( \lambda \in F \).

Let \( A_P \) be a non-singular \( n \times n \) matrix over \( \mathbb{N}_0 \) defining the (global) monomial ordering on \( P \) (with the default here being the grevlex ordering, \( x_n \succ \cdots \succ x_1 \)), and hence a weight function given by \( wt_P(\alpha) := A_P \alpha \), with distinct monomials obviously having distinct weights (since \( A_P \) defines a total order on monomials, or equivalently since it is non-singular).

Let \( f(T) := \sum_{i=0}^{d} f_i T^i \in P[T] \) be a monic, absolutely irreducible polynomial of degree \( d \). Use it to define an integral extension, the affine domain \( S := P[y]/J \) for \( J := \langle f(y) \rangle \). To extend \( wt_P \) to a weight function \( wt_S \) on \( S \), define \( wt_S(\alpha) := d \cdot wt_P(\alpha) \), and \( wt_S(y) := \max\{\frac{wt_P(f_i)}{d} : 0 \leq i < d\} \). If that max is taken on at only one value of \( i \), that value is \( i = 0 \), \( LM(f_0) := \alpha_0 \), and

\[
gcd\{d, gcd\{\alpha_i : 0 \leq i < d\}\} = 1,
\]

then \( S \) is said to be type I. (This terminology probably dates back to [4], but with a more rudimentary definition. The current definition can be found at least as early as [11].)
The standard grevlex order is defined by the $s \times s$ $(0, 1)$ matrix $G^{(s)}$, with $G_{i,j}^{(s)} := 1$ iff $1 \leq i + j \leq s + 1$. Regardless of monomial order, $NF(g, I)$ will always mean the normal form of $g$, meaning the remainder after division by elements of $I$, $LC(g)$, $LM(g)$, and $LT(g) = LC(g)LM(g)$, the leading coefficient, leading monomial, and leading term of $g$ (relative to the given ordering). $SP(f, g)$ will denote the s-polynomial of $f$ and $g$.

For $W_{ind}$ a non-singular $n \times n$ matrix over $N_0$, and $W_{dep}$ some $n \times s$ matrix over $N_0$, we introduce grevlex-over-weight order defined by the matrix

\[ M := \begin{pmatrix} G^{(s)} & 0 \\ W_{dep} & W_{ind} \end{pmatrix} \]

and weight-over-grevlex orders defined by the matrix

\[ M := \begin{pmatrix} W_{dep} & W_{ind} \\ G^{(s)} & 0 \end{pmatrix}, \]

with the former emphasizing the $P$-module structure, the latter the weights. In either case, define the weight of the monomial $\omega_p$ as $W_{dep}\omega_p + W_{ind}\omega_p$. The monomial ordering on the extension $S$ of $P$ above, is given by $AS := \left( \frac{1}{wt_S(y)^t} \ 0 \right)$.

The following short lemma is included here only because it explains the use of the gcd condition in the definition of type I above.

**Lemma 2.2 (Folklore).** The (standard) monomials in the footprint $\Delta(J)$ of a type I affine domain have distinct weights.

**Proof.** Suppose, for some $0 \leq j < i < d$, $wt_S(y^i x^j) = wt_S(y^j x^i)$ for these two distinct monomials in the footprint $\Delta(J)$. Then

\[(i - j)wt_S(y) = wt_S(y^i x^j) - wt_S(y^j x^i) = d(wt_P(x^i) - wt_P(x^j)).\]

But because of the gcd condition, this forces either $d|i - j$ or $\alpha = \beta$. \hfill $\square$

**Example 1.** The Klein quartic is most often given in terms of the affine model (with 2 points at infinity) $F_2[y, x]/(y^3 + x^3y + x)$, probably because this has nice symmetry in its homogeneous form. It is already integrally closed. If one were dealing with 2-point codes here, it would be possible to write this as

\[ic(S) = S = F_2[y_2, y_1; x_1]/J, \ J := (y_1^2 \ y_2, y_2y_1 - (y_1x_1^3 + x_1), y_2^2 - (y_2x_1^2 + y_1x_1));\]

While one can try to define a weight function with $wt(y) := 3$, and $wt(x) := 2$, both $y^2 = y_2$ and $x^3 = x_1^3$ are standard monomials with the same weight, 6. This happened because $wt(y^3) = wt(yx^2) > wt(x)$ instead of $wt(y^3) = wt(x) > wt(yx^3)$.

The one-point description: $F_2[y_2; f_3]/(f_3^5 + f_3^3 + f_3^2 + f_3)$, gotten by using $f_5 := yx$ and $f_3 := y$, is not integrally closed since $f_7 := f_2^2 + f_3$ is integral over $F_2[f_3]$. The integral closure $F_2[f_2, f_3, f_5]/(f_5^2 + f_2^2 + f_2f_3, f_2f_5 + f_5 + f_3^2, f_2^2 + f_5 + f_3^2, f_2^2 + f_2 + f_5, f_3^2 + f_2 + f_3 + f_5^2, f_3^2 + f_2 + f_3 + f_5^2, f_3^2 + f_2 + f_3 + f_5^2, f_3^2 + f_2 + f_3 + f_5^2)$, has the obvious weight function (implied by the subscripts used), with standard monomials of the forms $f_3^i, f_5^i f_3$, and $f_7 f_3^i, i \geq 0$, having all possible distinct weights different (those different from 1, 2, 4). And the original 2-point presentation can be recovered from this readily by resubstitution.
Example 2. The presentation $S := F_2[y, x]/(x^{10} + y^4 + y)$, is clearly an integral extension of $P := F_2[x]$, and is already integrally closed. It has a weight function defined by $wt_S(y) := 10$ and $wt_S(x) := 4$, so it is an order domain, but not with semi-group $\Gamma = (4, 10)$. Similarly it is not a type I presentation because $gcd(4, 10) = 2 \neq 1$, meaning that $y_{10}^2$ and $x_{10}^4$ are both standard monomials with the same weight, $20$. But $z := y_{10}^2 + x_{10}^2$ satisfies $z^2 = y_{10}^2$, so weight$(z) = 5$.

This gives a type I presentation $F_2[z_5; x_4]/(z_5^5 + z_5 + x_4^2)$. Depending on the monomial ordering chosen, this can have footprint either $\{x_4^i z_5^j : 0 \leq i < 5, 0 \leq j\}$ or $\{z_5^i x_4^j : 0 \leq i < 4, 0 \leq j\}$, both infinite, but the latter having fewer (4 rather than 5) $P$-module generators. So the presentation suggested here is relative to the $F_2[x_4]$-module basis $y_0 := 1$, $y_1 := z_5$, $y_2 := z_{10} := z_5^2$, and $y_3 := z_{15} := z_5^4$, with the ideal

$$
\langle z_5^2 - z_{10}, z_{10} z_5 - z_{15}, z_{10}^2 - (z_5 + x_4^2),
\rangle
z_{15} z_5 - (z_5 + x_4^2), z_{15} z_{10} - (z_{10} + z_5 x_4^2), z_{15}^2 - (z_{10} + z_{10} x_4^2)\rangle.
$$

(Note that the common module orderings generally referred to as TOP (Term-Over-Position) and POT (Position-Over-Term), at least in the standard text [1] in the section on Gröbner bases for modules, assume no interplay between module positions and terms in those positions, whereas we have a monomial ordering on a ring viewed as a module. There are many places where block orders are implemented and used, but not so for orderings of the type suggested here, which can only be used in various computational algebra packages by defining the whole matrix.)

3. Example

The following example, which could be defined as a single type I extension directly, will be given as two such instead. It is the smallest example known to the author in which the $P$-module generators are not independent. (Such an example, of course, cannot occur when $n = 1$, but it is conceivable that there is an example of degree less than the $d = 9$ here.)

Start with the polynomial ring $P := F_2[x_2, x_1]$ (with the grevlex order). The monic polynomial $f(T) := T^3 + x_2 x_1 T + (x_2^3 x_1^2 + x_2^2 x_1^2)$ can be used to define a type I integral extension $S_1 := P[y]/(f(y))$ with grevlex-over-weight monomial order defined by $\begin{pmatrix} 1 & 0 & 0 \\ 5 & 3 & 3 \\ 3 & 3 & 0 \end{pmatrix}$.

As a $P$-module, $S_1 := F_2[y, x]/(y^3 + y x_2 x_1 + x_2^3 x_1^2 + x_2^2 x_1^3)$ has standard $P$-module basis $(1, y, y^2)$. $S_1$ happens not to be integrally closed, since $\left(\frac{y^2}{x_2 x_1}\right)^3 + \left(\frac{y^2}{x_2 x_1}\right)^2 + (x_2 x_1 + x_2 x_1^2) = 0$. The integral closure $ic(S_1)$ is, in fact a subset of $\delta^{-1}S_1$ for $\delta := x_2 x_1$, which also has a $P$-module basis $(\delta^{-1} y^3, \delta^{-1} y^3, \delta^{-1} y^3)$. So the obvious hope is that $ic(S_1)$ will have a $P$-module basis of the form $(y_0 := 1, y_1, y_2) = (\delta^{-1} f_{0,0}(y), \delta^{-1} f_{1,0}(y), \delta^{-1} f_{2,0}(y))$ with deg$(f_{i,0}(y)) = i$. $ic(S_1)$ can be written in the form $ic(S_1) = F_2[y_2, y_1; x_2, x_1]/J_1$ for $y_1 := y, y_2 := y^2/(x_2 x_1)$, and $J_1$ the ideal of induced realtions describing the polynomial multiplication ignored when viewing $ic(S_1)$ merely as a $P$-module, in that

$$
J_1 := \langle y_1^2 + y_2 x_2 x_1, y_2 y_1 + y_1 + x_2^2 x_1 + x_2 x_1^2, y_2^2 + y_2 + y_1(x_2 + x_1)\rangle.
$$

Advances in Mathematics of Communications. Volume 3, No. 1 (2009), 1–11
defines a further integral extension

\[ NF(y_1 \cdot y_1) = 0 \]

\[ NF(y_2 \cdot y_1) = (x_2^2 x_1 + x_2 x_1^2) \]

\[ NF(y_2 \cdot y_2) = 0 \]

Now let

\[ h(T) := T^3 + (y_2 + y_1)(x_2 + 1)x_1 T + (y_2(x_2 x_1 + 1) + y_1(x_2 + x_1))x_2^2 x_1, \]

define a further integral extension \( S_2 := ic(S_1)[z]/(h(z)) \).

\[ S_2 = F_2[z, y_2, y_1, x_2, x_1]/ \]

\[ (J_1, z^3 + (y_2 + y_1)(x_2 + 1)x_1 z + (y_2(x_2 x_1 + 1) + y_1(x_2 + x_1))x_2^2 x_1) \]

has a standard \( P \)-module basis \((z^0 y_0, z^0 y_1, z^1 y_2, z^0 y_0, z^1 y_1, z^1 y_2, z^2 y_0, z^2 y_1, z^2 y_2)\). Again \( S_2 \) happens not to be integrally closed. But this time, not only is the conjectured form of the variables used to define \( ic(S_2) \) wrong, there are more variables than the 9 above that describe \( S_2 \). Such a set of dependent variables is:

\[ f_{0,0} := y_0; \]
\[ f_{15,9} := y_1; \]
\[ f_{12,9} := y_2; \]
\[ f_{19,12} := z y_0; \]
\[ f_{34,21} := z y_1; \]
\[ x_1 f_{34,30} := z y_1 x_2 + z y_2 + z y_0; \]
\[ f_{31,21} := z y_2; \]
\[ \delta f_{29,24} := z^2 y_0(x_2^3 x_1^2 + x_2^2 x_1^3 + x_2^3 x_1 + x_2 x_1^3 + x_2 x_1 + 1) \]
\[ + z^2 y_2(x_2 x_1 + 1) + z^2 y_1(x_2 + x_1); \]
\[ \delta f_{26,24} := z^2 y_1(x_2^2 x_1 + x_2 x_1^2 + x_2 x_1 + x_1^2 + x_2 + x_1) \]
\[ + z^2 y_2(x_2 x_1 + x_2 x_1 + x_1 + 1) \]
\[ + z^2 y_0(x_2^3 x_1 + x_2^2 x_1^2 + x_2 x_1 + x_2 x_1 + 1); \]
\[ \delta f_{23,15} := z^2 y_2(x_2 x_1^2 + x_1) + z^2 y_1(x_2 x_1 + x_1^2) + z^2 y_0(x_2^2 x_1 + x_2 x_1^3 + x_2 x_1 + x_1 + 1) \]

for \( \delta := (x_2 + x_1)(x_2 + x_1)x_1(x_2 + x_1 + 1) \). The integral closure is then of the form

\[ ic(S_2) = F_2[f_{34,30}, f_{34,21}, f_{31,21}, f_{29,24}, f_{26,24}, f_{23,15}, f_{19,12}, f_{15,9}, f_{12,9}; f_{9,9}, f_{9,0}]/J_2 \]

with \( J_2 \) the ideal of induced relations most of the form \( f_i f_j - NF(f_i f_j, J_2) \) except for the first, of the form \( SP(f_{34,30}, f_{34,21}) - NF(SP(f_{34,30}, f_{34,21}), J_2) \). That one describes the dependence between \( f_{34,30} \) and \( f_{34,21} \):

\[ f_{34,30} f_{9,0} + f_{34,21} (f_{9,9} + f_{9,0}) + f_{31,21} (1) + f_{19,12} (1) = 0. \]

An example of the others is:

\[ f_{12,9} f_{12,9} + f_{15,9} (f_{9,9} + f_{9,0}) + f_{12,9} (1) = 0 \]

describing a multiplication of \( f_{12,9} \) and \( f_{12,9} \). (A full list of all the 45+1 relations for this example can be found on the website.)
4. Weighted integral theorem

To define the P-linear relation hypothesis, it is necessary to introduce non-standard notation, to write $LM(f) = \sum_{i,j} LM_{ij}(f)$, to split $LM(f)$ over $P$ into a “coefficient” $\sum_{i,j} P(f)$ from $P$ and a “monomial” $LM_{ij}(f)$. (These correspond to a leading coefficient and a leading monomial only if one is working with $P$ as the coefficient ring and allows a weight function to deal with all the variables that occur, whether or not they are variables defined over $P$ or variables in $P$ itself. This is not the view taken currently in any computational algebra packages.)

**Theorem 4.1.** Let $P := \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring in $n$ independent variables over the field $\mathbb{F}$. Let

$Q := P[y_{m-1}, \ldots, y_1]/I = \mathbb{F}[y_{m-1}, \ldots, y_1; x_1, \ldots, x_n]/I$

be an integrally closed extension of $P$ (with $Q = P$ corresponding to the case $m = 1$), with grevlex-over-weight monomial order defined by the matrix $M$.

Let $f(T) \in Q[T]$ be a monic, (absolutely irreducible) polynomial (of some degree $d$) define an integral extension $\mathcal{S} := Q[z]/(f(z))$, with grevlex-over-weight monomial order an extension of that on $Q$ so that $LM(f) = z^d$.

Suppose that $\Delta \in P$ satisfies $\mathcal{S} \subseteq \text{ic}(\mathcal{S}) \subseteq \Delta^{-1} \mathcal{S}$.

Then $\text{ic}(\mathcal{S}) = \mathbb{F}[w_1, \ldots, w_n; \Delta^{-1} \mathcal{S}]$ with $\mathcal{S}$ having Gröbner basis $B$ with elements $w_iw_j - NF(w_i, w_j, S)$ for all $i, j$, and $SP(w_i, w_j) - NF(SP(w_i, w_j, S)$ when $LP(w_i) = LP(w_j)$, with $LP(w_i)$ denoting the leading monomial of $w_i$ if $Q$ is viewed with coefficients from $P$.

**Proof.** Since $\text{ic}(\mathcal{S})$ is a ring, the minimal, reduced Gröbner basis will have to contain elements of the form $w_iw_j = \sum_k c_{i,j,k}w_k = NF(w_i, w_j, S)$ for some structure constants $c_{i,j,k} \in P$. Any other elements in a Gröbner basis for $\text{ic}(\mathcal{S})$ must be P-linear combinations of the $w_i$’s. Since $SP(w_i, w_j) = \sum_k b_{i,j,k}w_k = NF(SP(w_i, w_j, S)$, when $LM_P(w_i) = LM_P(w_j)$, the minimal $P$-linear combinations will be of the form $SP(w_i, w_j) = NF(SP(w_i, w_j, S)$, with $LM_P(w_i) = LM_P(w_j)$.

5. Integral closure algorithms

Integral closure algorithms([2],[3]) are based on some version of

$\mathcal{S} \subseteq \text{ic}(\mathcal{S}) \subseteq M = \Delta^{-1} \mathcal{S}$

for some $\Delta$, not necessarily in $P$. Most are based on finding an increasing sequence of rings

$S = R_0 \subset \cdots \subset R_t = R_{t+1} = \text{ic}(\mathcal{S})$

with the elements in $R_{t+1}$ of the form $f_i/d_i$ for $f_i \in R_t$ and $d_1 \cdots d_i | \Delta$. The qth power algorithm [11] on the other hand is based on finding a sequence of $P$-modules

$\Delta^{-1} S = M_0 \supset \cdots \supset M_L = M_{L+1} = \text{ic}(\mathcal{S})$

with $M_{i+1} := \{\Delta^{-1} f \in M_i : \text{NormalForm}(\Delta^{-1} f) \notin M_i\}$. (This works best when all the coefficients involved are in $\mathbb{F}_q$, since it is then a linear algorithm with major cost in computing the normal forms involved.)

But other than choosing $\Delta \in P$ it is not so important which type of algorithm is used. What is more important is choosing to find a large enough set of variables, as above, so that the leading terms are all of degree at most 2 in the dependent variables. And secondarily, the monomial order of the base ring should extend.
to the integral closure. (The algorithm used to produce output for this paper is, however, based on the qth-power algorithm.)

There is an example on the website which is a thinly disguised example of a simple Hermitian curve worked out in MAGMA, SINGULAR, and MACAULAY 2, showing the limitations of each implementation. This example was used to show that there were bugs in the stopping criteria in the latter two implementations.

In the example given, the normal.lib function of SINGULAR [6],[7] produces

\[
T(1) := z, \quad T(2) := y_1, \quad T(3) := y_2, \quad T(4) := x_2, \quad T(5) := x_1,
\]

\[
\delta := (x_2 + x_1)(x_2 + 1)x_2(x_1 + 1)x_1^2,
\]

\[
\delta(T(6) + (x_2 + 1)x_1) := z^2y_2(x_2x_1 + x_2^2) + z^2y_1(x_2x_1^2 + x_1) + z^2(x_2x_1^2 + x_2(x_1^3 + x_1^2) + x_1),
\]

\[
\delta(T(7) + y_2(x_2 + 1) + y_1(x_2 + 1)x_1 + x_2x_1 x_2x_1 + x_2x_1 + x_1^2)
\]

\[
:= z^2y_2(x_2^2x_1^2 + x_2(x_1^3 + x_1^2) + x_1^3) + z^2y_1(x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2)
\]

\[
+ z^2(x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2) + z^2(x_2^3x_1^2 + x_2(x_1^3 + x_1^2) + x_1^2),
\]

\[
\delta(T(8) + y_2(x_2^2 + y_1(x_2 + 1) + z(x_2^2 + x_2x_1 + x_2 + x_1 + 1)
\]

\[
+ y_2(x_2^2 + x_2x_1 + x_2^2 + x_2 + 1) + y_1(x_2^3 + x_2x_1 + x_1) + x_2^2x_1 + x_2^3(x_1^2 + x_1) + x_2(x_1^2 + x_1) + x_1^2)
\]

\[
:= + z^2y_2(x_2^2x_1^2 + x_2(x_1^3 + x_1^2) + x_1^3) + z^2y_1(x_2x_1 + x_2(x_1^2 + x_1) + x_1^2)
\]

\[
+ z^2(x_2^3x_1^2 + x_2(x_1^3 + x_1^2) + x_1^2 + x_2x_1 + x_1^2) + z^2x_2^3(x_1^2 + x_1) + x_2^3(x_1^2 + x_1) + x_2(x_1^3 + x_1^2)
\]

\[
+ z_y(x_2^3(x_1^2 + x_1) + x_2^2(x_1^3 + x_1) + x_2(x_1^3 + x_1^2)) + z(x_2^3(x_1^2 + x_1) + x_2^2(x_1^3 + x_1) + x_2(x_1^3 + x_1^2))
\]

\[
\delta(T(9) + y_2 + zy_1(x_2 + 1) + z(x_2^2 + x_2x_1 + x_2 + x_1 + 1)
\]

\[
+ y_2(x_2^2 + x_2^2 + x_2^2 + x_2 + 1) + y_1(x_2^3 + x_2x_1 + x_2 + x_1 + 1)
\]

\[
+ x_2x_1 + x_2^3(x_1^2 + x_1) + x_2^2(x_1^2 + x_1) + x_2(x_1^3 + x_1^2) + x_1^2)
\]

\[
:= + z^2y_2(x_2^2(x_1^2 + x_1) + x_2(x_1^2 + x_1 + 1) + x_1^3 + x_1^2 + x_1)
\]

\[
+ z^2y_1(x_2x_1^2 + x_1^2 + x_1 + 1)
\]

\[
+ z^2(x_2^3x_1^2 + x_1^2 + x_2x_1 + x_2(x_1^3 + x_1^2) + x_1^2 + x_1 + 1)
\]

\[
+ zy_2(x_2^3(x_1^2 + x_1) + x_2^2(x_1^3 + x_1) + x_2(x_1^3 + x_1^2) + x_2(x_1^3 + x_1^2))
\]

\[
+ zy_1(x_2^2(x_1^2 + x_1) + x_2^2(x_1^3 + x_1) + x_2(x_1^3 + x_1^2))
\]

\[
+ z(x_2^3(x_1^2 + x_1) + x_2^2(x_1^3 + x_1) + x_2(x_1^3 + x_1^2))
\]

though the last 4 are not given explicitly, and the first 5 are only given explicitly in that the embedding map from S maps z, y_1, y_2, x_2, and x_1 onto T(1), T(2), T(3), T(4), and T(5) respectively. The Gröbner basis elements are an unpredictable mess, since the order is grevlex on (T(1),...,T(9)). Our suggested form would be

\[
ic(S) = \mathbf{F}_2[z_9, z_6, z_7, z_6, z_5, z_4, z_3, z_2, z_1, x_2, x_1] \mathbf{I}
\]

with weight function given by the weight matrix \( \mathbf{M} := \begin{pmatrix} 14 & 34 & 34 & 31 & 29 & 26 & 13 & 19 & 22 & 12 & 15 & 9 & 9 & 9 \end{pmatrix} \) and order gotten by completing this to a grevlex-over-weight order.
6. Minimization

Given that this procedure preserves the recursively generated monomial order, it is sometimes possible to minimize the result if some independent variable \( x_j \) is integral over some different choice \( P^* := F[x_n^*, \ldots, x_1^*] \).

**Example 3.** Consider

\[
Q := P := F_2[x_2, x_1], \quad wt(Q) := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},
\]

\[
f(T) := T^4 + T^2 x_2 x_1 + T^0 x_2^2 x_1^2,
\]

\[
S := Q[z] / \langle f(z) \rangle, \quad wt(S) = \begin{pmatrix} 5 & 4 \\ 3 & 0 \end{pmatrix},
\]

\[
\Delta := x_2^3 x_1^2,
\]

\[
ic(S) = F_2[z_3, z_2, z_1; x_2, x_1] / \mathcal{T}, \quad wt(ic(S)) = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix},
\]

\[
\mathcal{T} = \langle z_1^2 + z_2 x_2 x_1, z_2 z_1 + z_3 x_2 + z_1, z_2^2 + z_2 + x_2, z_3 x_2 x_1, z_3 z_2 + z_1, z_2^2 + z_2 x_1 + x_1 \rangle,
\]

\[
\delta := x_2^2 x_1, \quad z_3 \delta := z^3 + z x_2 x_1, \quad z_2 \delta := z^2 x_2, \quad z_1 \delta := z \delta,
\]

\( x_2 = z_3^2 + z_2 \) is integral over \( F_2[z_2, x_1] \) and \( z_1 = z_3 z_2 \).

\[
ic(S) = F_2[\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_1] / \mathcal{J}, \quad wt(ic(S)) = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix},
\]

\[
\mathcal{J} = \langle \overline{\tau}_1^2 + \overline{\tau}_2 \overline{\tau}_1 + \overline{\tau}_1 \rangle,
\]

\[
\delta := x_2^2 x_1, \quad \overline{\tau}_1 \delta := z^3 + z x_2 x_1, \quad \overline{\tau}_2 \delta := z^2 x_2, \quad \overline{\tau}_1 := x_1 \delta,
\]

\[\psi((\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_1) \delta) = (z^3 + z x_2 x_1, \quad z^2 x_2, \quad x_1 \delta),\]

\[
\phi(z, \quad x_2, \quad x_1) = (\overline{\tau}_1 \overline{\tau}_2, \quad \overline{\tau}_2 + \overline{\tau}_1).
\]

This is an especially important step in some applications where the original ring is described in terms of variables which are not necessarily those of most interest in the final ring (such as the example above having 12 or 5 variables in a module generating set).

There is an example on the website, with elements of weights 25 and 21 in characteristic 2 which exceeded the 4GB storage ceiling in SINGULAR, but for which MAGMA, almost immediately produces a module basis with 21 elements, (too messy to include here) with respective weights

\[ 354, 358, 341, 324, 337, 333, 25j, 14 \geq j \geq 0. \]

It is straightforward, but tedious, to produce the simpler 7 elements of weights \([16,15,13,12,11,10,7]\) from this. This takes at most 358*21 reductions. (If this reduction is not immediately clear, see [13].) It is much harder to produce the corresponding ideal, since, among other things, MAGMA thinks this should be an \( F_2(y_{21}) \)-module computation, not an \( F_2(y_{7}) \)-module one.

**Example 4.** Consider a slight variation on example 9.5 of [5], namely

\[ X_1^4 - X_2^3 + X_3^2 = 0. \]

There are numerous ways to view this as an integral extension problem as considered here. Their weight function is \( wt(X_1) := (3,0) \), \( wt(X_2) := (4,0) \) and \( wt(X_3) := \).
(0, 1); their point being that \((X_1, X_2)\) is a set of independent variables, but their weights aren’t. Here that means that this weight function would be perfectly fine if this is viewed as a type I integral extension of \(P := \mathbb{F}[X_2, X_3]\), integrally closed with \(P\)-module basis \((1, X_1, X_1^2, X_1^3)\), but not if it is viewed as a type I integral extension of \(P := \mathbb{F}[X_2, X_1]\). However, for this extension the default \(\text{wt}(X_3) := (3, 3)\), \(\text{wt}(X_2) := (2, 2)\), \(\text{wt}(X_1) := (1, 0)\). is fine, giving \(P\)-module basis \((1, X_1)\).

But in either case the weight of the first variable is not really dependent on the weight of the third. So maybe using weights won’t be the best way to see that the latter gives a smaller presentation in terms of the \(\mathbb{F}[y, x]\)-module basis \((1, z)\) than the former, with \(\mathbb{F}[y, z]\)-module basis \((1, x, x^2, x^3)\).

But there are times when it is possible to use the weights to determine a minimal presentation, including the important case \(n = 1\).

**Theorem 6.1.** Suppose that both presentations

\[S = \mathbb{F}[x_n, \ldots, x_1]/(f_1(y))\text{ and }S = \mathbb{F}[y, x_n-1, \ldots, x_1]/(f_2(x_n))\]

are type I integral extensions with the same weight function \(\text{wt}_x\). Then \(\text{wt}_S(y)\) and \(\text{wt}_S(x_n)\) are scalar multiples of each other.

**Proof.**

\[d_1 \text{wt}_S(y) = a_n \text{wt}_S(x_n) + \sum_{i=1}^{n-1} a_i x_i, \quad d_2 \text{wt}_S(x_n) = b_n \text{wt}_S(y) + \sum_{i=1}^{n-1} b_i \text{wt}_S(x_i)\]

for some non-negative integers \(a_i\) and \(b_i\), and some positive integers \(d_1\) and \(d_2\). But then

\[(d_1d_2 - a_n b_n) \text{wt}_S(x_n) = \sum_{i=1}^{n-1} (b_n a_i + d_1 b_i) \text{wt}_S(x_i)\]

This is clearly a dependence among independent weights, so \(b_n a_i + d_1 b_i = 0\) for \(1 \leq i < n\). But this forces \(b_i = 0\) for \(1 \leq i < n\), and \(d_2 \text{wt}_S(x_n) = b_n \text{wt}_S(y)\).

This gives the known (certainly implicitly understood as long ago as the early 1990’s) result for \(n = 1\):

**Corollary 6.2 (Folklore).** For curves (the case \(n = 1\)), there is an \(\mathbb{F}[x_1]\)-module basis \((y_0 := 1, y_1, \ldots, y_{n-1})\) with \(x_1 := f_{\rho}\) of smallest positive pole order \(\rho\) at \(P_\infty\), and \(y_i := f_{\rho_i}\) with \(\rho_i \equiv i(\text{mod } \rho)\) (also smallest). In particular \(\mathcal{L}(\infty P_\infty)\) has vector-space generators \(f_{\rho}, f_{\rho}^j\) for \(0 \leq i < \rho\), and \(0 \leq j \) over \(\mathbb{F}\).

We have seen by the example above that, for higher dimensional varieties \((n > 1)\), it may be the case that a weight function for \(S\) viewed as a type I integral extension relative to one choice of \(P\), cannot possibly be used for \(S\) viewed similarly relative to some other choice of \(P\). If \(f(T) = T^d + \cdots + f_0\) with \(f_0 = x^{\alpha_0}\) and \(\alpha_i \neq 0\) for \(1 \leq i \leq n\), then it would seem that the above theorem would not apply, since \(x_n\) would not be integral over \(\mathbb{F}[y, x_n-1, \ldots, x_1]\), but consider the following example:

**Example 5.** \(S := \mathbb{F}_2[y; x_2, x_1]/(y^6 + yx_2^2x_1^2 + x_2^2x_1^2)\), with \(\text{wt}_S(x_1) := (6, 0)\), \(\text{wt}_S(x_2) := (12, 12)\), and \(\text{wt}_S(y) := (13, 10)\). Then there is an \(\mathbb{F}_2[x_2, x_1]\)-module basis \((1, y, z, zy, z^2, w)\) for \(z := y^2/(x_2x_1)\) and \(w := y^5/(x_2^2x_1^2)\); but since \(\text{wt}_x(z) = (8, 8)\), and \(3(8, 8) = 2(12, 12)\) there is an \(\mathbb{F}_2[z, x_1]\)-module basis \((1, x_2, y, w)\) with \(s = 4\) instead of \(s = 6\).
Example 6. $S := \mathbb{F}_2[y, z]/(y^7 + y^6z + x^6 + x^5z)$ has an integral closure with $\mathbb{F}_2[x, z]$-module basis of size 7, whereas it has $\mathbb{F}_2[y, z]$-module basis of size 6, with no clue from the weights: $(0, 0)$, $(6, 5)$, $(12, 10)$, $(11, 8)$, $(10, 6)$, $(9, 4)$, $(8, 2)$; $(7, 7)$, $(7, 0)$, in the former and $(0, 0)$, $(7, 6)$, $(8, 6)$, $(9, 6)$, $(10, 6)$, $(11, 6)$; $(6, 6)$, $(6, 0)$, in the latter. However using weights $(6, 0), (7, 0)$ and $(0, 7)$ in place of $(6, 5), (7, 7)$, $(7, 0)$, would make it simple to apply the theorem.

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Received June 2008; revised January 2009.

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