Carleman estimate for stochastic parabolic equations and inverse stochastic parabolic problems

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Abstract
In this paper, we establish a global Carleman estimate for stochastic parabolic equations. Based on this estimate, we study two inverse problems for stochastic parabolic equations. One is concerned with a determination problem of the history of a stochastic heat process through the observation at the final time \( T \) for which we obtain a conditional stability estimate. The other is an inverse source problem with observation on the lateral boundary. We derive the uniqueness of the source.

1. Introduction
In this paper, we study two different inverse problems for stochastic parabolic equations by establishing a global Carleman estimate. We first introduce some notation.

Let \( T > 0, G \subset \mathbb{R}^n (n \in \mathbb{N}) \) be a given bounded domain with a \( C^2 \) boundary \( \Gamma \). Put

\[ Q \triangleq (0, T) \times G, \quad \Sigma \triangleq (0, T) \times \Gamma. \]

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete filtered probability space on which a one-dimensional standard Brownian motion \( \{B(t)\}_{t \geq 0} \) is defined. Let \( H \) be a Banach space. Denote by \( L^2_p(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( X(\cdot) \) such that \( \mathbb{E}(\|X(\cdot)\|_{L^2_p(0, T; H)}^2) < \infty \), with the canonical norm; by \( L^\infty_p(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted bounded processes; and by \( L^2_p(\Omega; C([0, T]; H)) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( X(\cdot) \) satisfying that \( \mathbb{E}(\|X(\cdot)\|_{C([0, T]; H)}^2) < \infty \), with the canonical norm (similarly, one can define \( L^2_p(\Omega; C^k([0, T]; H)) \) for any positive \( k \)).

Throughout this paper, we make the following assumptions on the coefficients:

\[ b^{ij} : \Omega \times Q \to \mathbb{R} \quad (i, j = 1, 2, \ldots, n) : \]
Obviously, a strong solution of equation (1) well-posedness results for equation (1).

There exists a unique weak solution of equation (1.2). Furthermore, it holds that

\[ \sum_{i,j=1}^{n} b^{ij}(\omega, t, x) \xi^i \xi^j \geq \sigma |\xi|^2, \quad (\omega, t, x, \xi) \equiv (\omega, t, x, \xi^1, \ldots, \xi^n) \in \Omega \times Q \times \mathbb{R}^n. \]

(1.1)

Let

\[ a_1 \in L^\infty_{T}(0, T; \mathbb{L}^\infty(G; \mathbb{R}^n)), \quad a_2 \in L^\infty_{T}(0, T; \mathbb{L}^\infty(G)) \quad a_3 \in L^\infty_{T}(0, T; \mathbb{W}^{1,\infty}(G)), \]

\[ f \in L^2_T(0, T; \mathbb{L}^2(G)) \quad \text{and} \quad g \in L^2_T(0, T; \mathbb{H}^1(G)). \]

Consider the following stochastic parabolic equation:

\[
\begin{aligned}
\frac{dy}{dt} - \sum_{i,j=1}^{n} b^{ij}(y, p) \frac{\partial y}{\partial p} &= \left( a_1, \nabla y \right) + a_2 y + f \quad \text{in } Q, \\
y &= 0 \quad \text{on } \Sigma, \\
y(t_0) &= y_0.
\end{aligned}
\]

(1.2)

where \( y_0 \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{L}^2(G)) \) and \( y_x = \frac{\partial y}{\partial x}. \)

We first recall the definition of the weak and strong solution of equation (1.2) and give some well-posedness results.

**Definition 1.1.** We call a stochastic process \( y \in L^2_T(\Omega; \mathbb{C}([0, T]; \mathbb{L}^2(G))) : L^2_T(0, T; \mathbb{H}^1(G)) \) a weak solution of equation (1.2) if for any \( t \in [0, T] \) and any \( p \in \mathbb{H}^1(G) \) it holds that

\[
\int_{G} y(t, x) p(x) \, dx - \int_{G} y_0(x) p(x) \, dx = \int_{0}^{t} \int_{G} \left\{ - \sum_{i,j=1}^{n} b^{ij}(s, x) y_x(s, x) p_x(x) \right. \\
+ \left[ (a_1, \nabla y(s, x)) + a_2(s, x) y(s, x) + f(s, x) \right] p(x) \biggr\} \, dx \, ds \\
+ \int_{0}^{t} \int_{G} [a_3(s)x(y(s, x) + g(s, x)] p(x) \, dx \, d\mathbb{B}, \quad P - \text{a.s.}
\]

(1.3)

**Definition 1.2.** A process \( y \in L^2_T(\Omega; \mathbb{C}([0, T]; \mathbb{H}^2(G)) \cap \mathbb{L}^2_T(L^p, 0, T; \mathbb{H}^1(G))) \) is said to be a strong solution of equation (1.2) if for any \( t \in [0, T] \) it holds that

\[
\begin{aligned}
y(t) &= y_0 + \int_{0}^{t} \left\{ - \sum_{i,j=1}^{n} \left( b^{ij}(s) y_i(s) \right)_j + \left[ (a_1, \nabla y(s)) + a_2(s) y(s) + f(s) \right] \right\} \, ds \\
&\quad + \int_{0}^{t} \int_{G} [a_3(s)x(y(s) + g(s))] \, d\mathbb{B}, \quad P - \text{a.s.}
\end{aligned}
\]

(1.4)

Obviously, a strong solution of equation (1.2) is also its weak solution. We have the following well-posedness results for equation (1.2), whose proof can be found in [11, chapter 6].

**Lemma 1.1.** There exists a unique weak solution of equation (1.2). Furthermore, it holds that

\[
|y|_{L^2_T(\Omega; \mathbb{C}([0, T]; \mathbb{L}^2(G)))} + |y|_{L^2_T(0, T; \mathbb{H}^1(G))} \leq C_{R_1} |y_0|_{L^2_T(\Omega; \mathbb{F}_0, P; \mathbb{L}^2(G))}.
\]

(1.5)
Here and in what follows,
\[ r_1 = |a_1|^2 + |a_2|^2 + |a_3|^2 + 1. \]

**Lemma 1.2.** Let \( y_0 \in L^2(\Omega, \mathcal{F}_0, P; H^2(G) \cap H^1_0(G)), \) \( b_1 \in L^2_\infty(0, T; W^{1, \infty}(G; \mathbb{R}^n)), \) \( b_2 \in L^2_\infty(0, T; \mathbb{R}^n), \) and \( b_3 \in L^2_\infty(0, T; W^{1, \infty}(G)). \) Then, there exists a unique strong solution of equation (1.2).

Next, we present Itô’s formula, which plays a key role in what follows.

**Lemma 1.3 (Itô’s formula).** Let \( X(\cdot) \in L^2_\infty(0, T; H^1_0(G)) \) be a continuous process with values in \( H^{-1}(G). \) Suppose that there exist \( X_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G)), \) \( \Phi(\cdot) \in L^2_\infty(0, T; H^{-1}(G)) \) and \( \Psi(\cdot) \in L^2_\infty(0, T; L^2(G)) \) such that for any \( t \in [0, T] \) it holds that
\[
X(t) = X_0 + \int_0^t \Phi(s) \, ds + \int_0^t \Psi(s) \, dB, \quad \text{P-a.s.}
\]
in \( H^{-1}(G). \) Then, we have that
\[
|X(t)|^2_{L^2(G)} = |X(0)|^2_{L^2(G)} + 2 \int_0^t (X(s), \Phi(s))_{H^1_0(G) \cap H^{-1}(G)} \, ds + 2 \int_0^t (X(s), \Psi(s))_{L^2(G)} \, dB + \int_0^t |\Psi(s)|^2_{L^2(G)} \, ds
\]
for arbitrary \( t \in [0, T]. \)

**Remark 1.1.** Here we only present a special case for Itô’s formula. It is enough for the proof in our paper. The general form can be found in [25, chapter 1].

**Remark 1.2.** Obviously, both the weak and strong solutions of equation (1.2) satisfy the assumptions for lemma 1.3. In this paper, for simplicity of notation, we sometimes use the differential form of Itô’s formula, that is, \( d(X^2) = 2X \, dX + (dX)^2. \)

In this paper, we establish a Carleman estimate for equation (1.2). The so-called Carleman estimate is a class of weighted energy estimates which is in connection with (stochastic) differential operators. As far as we know, the first example of such an estimate appeared in Carleman’s pioneering work on the uniqueness of the solution of the first-order elliptic system with two variables (see [8]). The idea was generalized to get the uniqueness of the solutions for general Cauchy problems in [6]. Now it is a useful tool for studying the uniqueness and unique continuation property for partial differential equations (see [15], for example). Such an estimate has been introduced to solve inverse problems in [4], and was comprehensively studied in [18, 22]. Now it is a helpful methodology for solving inverse problems (e.g. [18, 21, 22, 29, 30]). Although the form of Carleman estimates seems to be very complex, the idea behind them is very simple. One can understand it by the following example.

Let
\[
\begin{cases}
\frac{dx}{dt} = a(t)x \\
x(0) = x_0.
\end{cases}
\]
Here \( x_0 \in \mathbb{R} \) and \( a(\cdot) \in L^\infty(0, T). \) We prove that there exists a constant \( C > 0 \) such that for any \( x_0 \in \mathbb{R}, |x(T)| \leq C|x_0| \) by the Carleman estimate. This result is almost trivial and one can prove it without utilizing the Carleman estimate. However, the proof employed here shows all the ideas of the Carleman estimate.

Let \( \dot{x}(t) = e^{\xi t}x(t) \) with \( \xi \geq 0. \) Then, we have
\[
\frac{d\dot{x}^2}{dt} = 2a(t)x^2 - 2\xi \dot{x}^2 = 2[a(t) - \xi] \dot{x}^2.
\]
If we choose $\varsigma \geq |a|_{L^\infty(0, T)}$, then we know that $\frac{\partial^2 y}{\partial x^2} \leq 0$, which implies that $\hat{x}^2(T) \leq \hat{x}^2(0)$. Hence, we get
\[
e^{-2\varsigma T} \hat{x}^2(T) \leq \hat{x}^2(0).\tag{1.9}
\]
From this, we obtain $|x(T)| \leq e^{T |a|_{L^\infty(0, T)}} |x_0|$ immediately. Thus, we prove the desired result and we know $C$ can be chosen to be $e^{T |a|_{L^\infty(0, T)}}$.

Inequality (1.9) is a kind of Carleman estimate. The function $e^{-\varsigma t}$ is called the weight function and $\varsigma$ is a parameter which can be chosen for our purpose. By means of the choice of $\varsigma$, we control the lower order term $a(t)x$ and obtain inequality (1.9). For (stochastic) partial differential equations, both the choice of the weight function and the computation are much more complex. However, they enjoy the same idea.

Now we introduce the Carleman estimate to be established in this paper. To start with, we give some functions. Let $s \in (0, +\infty)$, $t \in (0, +\infty)$ and $\psi \in C^\infty(\mathbb{R})$ with $|\psi| \geq 1$, which is independent of the $x$-variable. Put
\[
\varphi = e^{s\psi} \text{ and } \theta = e^{sp}.\tag{1.10}
\]

We have the following result.

**Theorem 1.1.** Let $\delta \in [0, T)$. Let $\varphi$ and $\theta$ be given in (1.10). There exists a $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$, there exists an $s_0(\lambda_1) > 0$ so that for all $s \geq s_0(\lambda_1)$, it holds that
\[
\lambda \mathbb{E} \int_0^T \int_G \theta^2 |\nabla y|^2 \, dx \, dt + s\lambda^2 \mathbb{E} \int_0^T \int_G \varphi \theta^2 y^2 \, dx \, dt
\leq C \mathbb{E} \left[ \theta^2(T) |\nabla y(T)|^2_{L^2(G)} + \theta^2(\delta) |\nabla y(\delta)|^2_{L^2(G)} + s\lambda \psi(T) \theta^2(T) |y(T)|^2_{L^2(G)} + s\lambda \psi(\delta) \theta^2(\delta) |y(\delta)|^2_{L^2(G)} + \int_0^T \int_G (1 + \varphi) \theta^2 (f^2 + g^2 + |\nabla g|^2) \, dx \, dt \right].\tag{1.11}
\]

Here $y$ is an arbitrary weak solution of equation (1.2).

Here and in what follows, the constant $C$ depends only on $G$, $(b^j)_{1 \leq j \leq n}$, $T$, $\delta$ and $\psi$, which may change from line to line.

Although there are numerous results for the global Carleman estimate for deterministic parabolic equations (see [13, 29], for example), people know very little about the stochastic counterpart. In fact, as far as we know, references [2, 28] are the only two published papers addressing the global Carleman estimate for stochastic parabolic equations. In [2, 28], some Carleman-type inequalities were established for deriving the null controllability of stochastic parabolic equations. Note further that the weight function $\theta$ used in this paper (which plays a key role in what follows) is quite different from that in [2, 28]. It seems that the Carleman estimate in [2, 28] cannot be applied to studying the inverse problems introduced in what follows. Indeed, the weight function $\theta$ in [2, 28] is supposed to vanish at 0 and $T$, and therefore it does not serve the purpose of proving theorems 1.2 and 1.4.

As applications of theorem 1.1, we study two inverse problems for stochastic parabolic equations. There are abundant works addressing the inverse problems for PDEs, and it is impossible to list all related papers owing to the large amount. However, there exist a few works addressing inverse problems for stochastic PDEs (see [3, 9, 16], for example). Although there are some people considering the inverse source problem for parabolic equations with random noise in the measurement (see [20], for example), to the best of our knowledge, there is no paper that considers the inverse problem for stochastic parabolic equations.
Now we introduce the inverse problems studied in this paper. Consider the following stochastic parabolic equation:

$$\begin{aligned}
\frac{dy}{dt} - \sum_{i,j=1}^{n} (b^{ij}y_{x_i})_{x_j} dt &= [(a_1, \nabla y) + a_2y] dt + a_3y dB \\ y &= 0 \\ y(0) &= y_0
\end{aligned}$$

in $Q$, \quad on $\Sigma$, \quad in $\Omega$. \hfill (1.12)

Here, $y_0 \in L^2(\Omega, F_0, \mathbb{P})$.

The first inverse problem is concerned with the following problem.

**Stochastic parabolic equation backward in time.** Let $0 \leq t_0 < T$. Determine $y(\cdot, t_0)$, $\mathbb{P}$-a.s. from $y(\cdot, T)$.

For deterministic parabolic equations, this kind of problem has lots of applications in mathematical physics (e.g. [1]) and is studied extensively (see [30] for a nice survey). Generally speaking, the problem of (stochastic) parabolic equations backward in time is ill-posed. Small errors in measuring the terminal data may cause huge deviations in the final results, that is, there is no stability in this problem. Fortunately, if we assume an $a$ priori bound for $y(0)$ (this assumption is reasonable from a practical viewpoint), then we can regain the stability in some sense. The concept of conditional stability is used to describe this kind of stability. In a general framework, the conditional stability problem can be formulated as follows.

Let $t_0 \in [0, T)\), \alpha_1 \geq 0, \alpha_2 \geq 0$ and $M > 0$. Put

$$U_{M, \alpha_1} \overset{\Delta}{=} \{ f \in L^2(\Omega, \mathbb{P}, \mathcal{F}_0; H^{\alpha_1}(G)) : |f|_{L^2(\Omega, \mathbb{P}, \mathcal{F}_0, H^{\alpha_1}(G))} \leq M \}.$$

If $y_0 \in U_{M, \alpha_1}$, we can choose a function $\beta \in C(0, +\infty)$ satisfying the following properties:

1. $\beta \geq 0$ and $\beta$ is strictly increasing;
2. $\lim_{\eta \to 0} \beta(\eta) = 0$;
3. $|y(t_0)|_{L^2(\Omega, \mathbb{P}, \mathcal{F}_0, P, H^{\alpha_1}(G))} \leq \beta(|y(T)|_{L^2(\Omega, \mathbb{P}, \mathcal{F}_T, P, H^{\alpha_1}(G))})$.

**Remark 1.3.** Here we expect the existence of $\beta$ with the assumptions that $y_0$ belongs to a special set $U_{M, \alpha_1}$, which means that $y_0$ enjoys an $a$ priori bound in some sense. Generally speaking, $\beta$ depends on $M$ and $\alpha_1$. Once we choose $M$ and $\alpha_1$, we add some conditions to the initial data of equation (1.12). Hence, the stability result implied by $\beta$ depends on our choice of the initial data. This is why we call it ‘conditional stability’.

**Remark 1.4.** The first property for $\beta$ means that we only choose $\beta$ in a special class of functions, that is, the strictly increasing functions. The second and the third property guarantee the conditional stability. Without assuming property 2, we can always construct $\beta$ as $\beta(x) = C + x$ with a constant $C$ which is large enough. However, this kind of function does not make any sense for conditional stability.

**Remark 1.5.** Once $\beta$ exists, it is not unique. For example, $\tilde{\beta}(x) = \beta(x) + x$ is another function satisfying the three properties.

In this paper, we obtain the following interpolation inequality for the weak solution of equation (1.12), which implies a conditional stability result for equation (1.12) backward in time.

**Theorem 1.2.** Let $t_0 \in [0, T)$. Then, there exist a constant $\theta \in (0, 1)$ and a constant $C > 0$ such that

$$|y(t_0)|_{L^2(\Omega, \mathbb{P}, \mathcal{F}_0, L^2(G))} \leq C |y|_{L^2(0, T; L^2(G))}^{1-\theta} |y(T)|_{L^2(\Omega, \mathcal{F}_T, P, H^{\alpha_1}(G))}^\theta \hfill (1.13)$$

for any $y$ solving equation (1.12) in the sense of a weak solution.
As a consequence, we obtain the following result.

**Theorem 1.3.** Let \( y_0 \in U_{M,0}, \alpha_2 = 1 \) and \( \beta(x) = CM^{1-\theta}x^\theta \) with a constant \( C \) independent of \( y(0) \). Then, we have

\[
|y(t_0)|_{L^2(\Omega, \mathcal{F}_{t_0}, P; L^2(G))} \leq \beta(|y(T)|_{L^2(\Omega, \mathcal{F}_T, P; H^1(G))}).
\]

The proof of theorem 1.3 follows lemma 1.1 and theorem 1.2 immediately. We omit it here.

In the deterministic setting, a result which is stronger than theorem 1.2 was obtained in [26], where the authors study the following equation:

\[
\begin{cases}
y_t - \Delta y = by & \text{in } Q, \\
y = 0 & \text{on } \Sigma.
\end{cases}
\] (1.14)

Here \( b \) is a suitable function. With the assumption that \( G \) is convex, they get

\[
|y(0)|_{L^2(G)}^2 \leq C \exp \left( \frac{|y(0)|_{L^2(G)}}{|y(0)|_{H^1(G)}} \right) |y(T)|_{L^2(G_0)}^2.
\] (1.15)

Here \( G_0 \) is any open subset of \( G \). Compared with theorem 1.2, only \( |y(T)|_{L^2(G_0)}^2 \) is involved on the right-hand side of the inequality. They prove this result by employing some special frequency functions, which were first constructed for proving the doubling property of the solution of heat equations. However, since the solution of equation (1.12) is non-differentiable with respect to \( t \), it seems that their method cannot be easily adopted to solve our problem.

As another consequence of theorem 1.2, we get a backward uniqueness for equation (1.12).

**Corollary 1.1.** Assume that \( y \) is a weak solution of equation (1.12). If \( y(T) = 0 \) in \( G, P\text{-a.s.} \), then \( y(t) = 0 \) in \( G, P\text{-a.s.} \), for all \( t \in [0, T] \).

The uniqueness problems for the solutions of both deterministic and stochastic partial differential equations have been studied for a long time. There are a great many positive results and some negative results. In the case of time-reversible systems, the backward uniqueness is equivalent to the classical (forward) uniqueness. If one considers time-irreversible systems, such as parabolic equations, the situation is quite different. The backward uniqueness implies the classical (forward) uniqueness; however, generally speaking, the converse conclusion is untrue.

On account of plentiful applications, such as studying the long time behavior of solutions and establishing the approximate controllability from the null controllability, the backward uniqueness for parabolic equations draws lots of attention (see [12, 14, 23, 24, 27] and the references cited therein). It is well understood now. In contrast, as far as we know, [5] is the only paper concerned with backward uniqueness for stochastic parabolic equations in the literature. In [5], the authors obtained the backward uniqueness for semilinear stochastic parabolic equations with deterministic coefficients. They employed some deep tools in stochastic analysis to establish the result. However, it seems that their method depends on the very fact that the coefficients are deterministic and one cannot simply mimic their method to obtain corollary 1.1, since the coefficients are random.

The other inverse problem studied in this paper is about the global uniqueness of an inverse source problem for stochastic parabolic equations. We first give a precise formulation of the problem.
Let \( x = (x_1, x') \in \mathbb{R}^n \) and \( x' = (x_2, \ldots, x_p) \in \mathbb{R}^{n-1} \). Consider a special \( G \) as \( G = (0, l) \times G' \), where \( G' \subset \mathbb{R}^{n-1} \) is a bounded domain with a \( C^2 \) boundary. We consider the following stochastic parabolic equation:

\[
\begin{align*}
\begin{cases}
\frac{\partial y}{\partial t} - \Delta y &= [(b_1, \nabla y) + b_2 y + h(t, x') R(t, x) ] dt + b_3 y dB(t) & \text{in } Q, \\
y(0) &= 0 & \text{on } \Sigma,
\end{cases}
\end{align*}
\]

Here,

\[
b_1 \in L^\infty_\mathcal{F}(0, T; W^{1,\infty}(G; \mathbb{R}^n)), \quad b_2 \in L^\infty_\mathcal{F}(0, T; W^{1,\infty}(G)), \quad b_3 \in L^\infty_\mathcal{F}(0, T; W^{2,\infty}(G))
\]

and

\[
R \in C^2([0, T] \times \overline{G}), \quad h \in L^2_\mathcal{F}(0, T; H^1(G')).
\]

The inverse source problem studied here is as follows. Let \( R \) be given and \( 0 < t_0 < T \). Determine the source function \( h(t, x') \), \( (t, x') \in (0, t_0) \times G' \) by means of the observation of

\[
\frac{\partial y}{\partial n} \bigg|_{(t, x') \in \partial G}.
\]

Here, \( v = (v^1, \ldots, v^n) \in \mathbb{R}^n \) is the outer normal vector of \( \Gamma \).

We have the following uniqueness result for the above problem.

**Theorem 1.4.** Let

\[
|R(t, x)| \neq 0 \text{ for all } (t, x) \in [0, t_0] \times \overline{G}.
\]

If

\[
\frac{\partial y}{\partial v} = 0 \text{ on } [0, t_0] \times \partial G, \text{ P-a.s.},
\]

then

\[
h(t, x') = 0 \text{ for all } (t, x') \in [0, t_0] \times G', \text{ P-a.s.}
\]

**Remark 1.6.** One can follow the proof of theorem 1.4 to show that theorem 1.4 also holds when \( \Delta y \) is substituted by \( \sum_{j=1}^n (h_j y_j) \). Here, we consider equation (1.16) for the sake of presenting the key idea in a simple way.

In practical problems, it is important to specify some proper data so that the parameter to be reconstructed is uniquely identifiable. In our model, the data utilized is the boundary normal derivative of the solution. This type of inverse problem is important in many branches of engineering sciences. For example, an accurate estimation of a pollution source in a river, a determination of the magnitude of groundwater pollution sources, etc.

In the literature, determining a spacewise dependent source function for parabolic equations has been considered comprehensively (see [7, 10, 18, 19, 30] and the references cited therein). A classical result for the deterministic setting is as follows.

Consider the following parabolic equation:

\[
\begin{align*}
\begin{cases}
y_t - \Delta y &= c_1 \nabla y + c_2 y + Rf & \text{in } Q, \\
y(0) &= 0 & \text{on } \Sigma.
\end{cases}
\end{align*}
\]

Here, \( c_1 \) and \( c_2 \) are suitable functions on \( Q, R \in L^\infty(Q), \) and \( R(t_0, x) \neq 0 \) in \( \overline{G} \) for some \( t_0 \in (0, T), f \in L^2(G) \) is independent of \( t \). The authors in [17] proved the following result.

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Assume that \( y \in H^{2,1}(Q) \) and \( y_t \in H^{2,1}(Q) \); then, there exists a constant \( C > 0 \) such that

\[
|f|_{L^2(G)} \leq C \left( |y(t_0)|_{H^2(G)} + \left| \frac{\partial y_t}{\partial v} \right|_{L^2(0,T;L^2(\Gamma_0))} \right),
\]

where \( \Gamma_0 \) is any open subset of \( \Gamma \).

Compared with theorem 1.4, inequality (1.19) gives an explicit estimate for the source term by \( |y(t_0)|_{H^2(G)} \) and \( |\frac{\partial y_t}{\partial v}|_{L^2(0,T;L^2(\Gamma_0))} \). A key step in the proof of equality (1.19) is to differentiate the solution of (1.18) with respect to \( t \). Unfortunately, the solution of (1.16) does not enjoy differentiability with respect to \( t \) due to the effect of the stochastic noise. However, we can borrow an idea from the proof of inequality (1.19). Although it is impossible for us to assume that the solution of equation (1.16) is differentiable with respect to \( t \), we can show that it is differentiable with respect to \( x \) with some assumptions (the assumptions in this paper are enough). In this case, we can show the uniqueness of \( h \) if \( h \) is independent of some \( x_i (i = 1, \ldots, n) \). Here, we suppose that \( h \) is independent of \( x_i \).

Obviously, both equations (1.12) and (1.16) are special examples of equation (1.2).

**Remark 1.7.** As we have pointed out, the non-differentiability with respect to the variable with noise (say the time variable considered in this paper) of the solution of a stochastic PDE usually leads to substantially new difficulties in the study of inverse problems for stochastic PDEs. Another difficulty in studying the inverse problem of stochastic PDEs is that the usual compactness embedding result does not remain true for the solution spaces related to stochastic PDEs. Due to these new difficulties, some methods of solving inverse problems for deterministic PDEs (see [18, 22], for example) cannot be used to solve the corresponding inverse problems in the stochastic setting.

The remainder of the paper is organized as follows. In section 2, we prove theorem 1.1. Section 3 addresses the proof of theorem 1.2. Finally, in section 4, we give a proof of theorem 1.4.

### 2. Carleman estimate for stochastic parabolic equations

In this section, we prove theorem 1.1.

We first give a weighted identity, which plays an important role in the proof of theorem 1.1.

**Proposition 2.1.** Assume that \( u \) is an \( H^2(\mathbb{R}^n) \)-valued continuous semi-martingale. Put \( v = \theta u \) (recall (1.10) for the definition of \( \theta \)). Then, we have the following equality:

\[
\begin{align*}
-\theta \left[ \sum_{i,j=1}^{n} (b^{ij} u_{x_i}) s_j + s \lambda \phi \psi \theta_t v \right] & \left[ du - \sum_{i,j=1}^{n} (b^{ij} u_{x_i}) s_j \, dt \right] + \frac{1}{4} \lambda \theta v \left[ du - \sum_{i,j=1}^{n} (b^{ij} u_{x_i}) s_j \, dt \right] \\
= & -\frac{1}{2} \sum_{i,j=1}^{n} b^{ij} v_{x_i} \, dv_{x_j} + \frac{1}{2} \sum_{i,j=1}^{n} b^{ij} v_{x_i} v_{x_j} \, dt - \frac{1}{4} \lambda \sum_{i,j=1}^{n} b^{ij} v_{x_i} v_{x_j} \\
& + \frac{1}{2} s \lambda^2 \phi \psi^2 \, dv \, \theta_t v^2 + \frac{1}{2} s \lambda \phi \psi v^2 \, dv - \frac{1}{4} \phi^2 \psi \phi v^2 \, dv + \frac{1}{2} s \lambda \phi \psi (dv)^2 - \frac{1}{8} \lambda (dv)^2 \\
& + \left[ \sum_{i,j=1}^{n} (b^{ij} v_{x_i}) s_j + s \lambda \phi \psi v \right]^2 \, dt.
\end{align*}
\]

(2.1)
Proof. The proof is based on some direct computation by Itô’s stochastic calculus. The first term on the left-hand side of equality (2.1) reads as

\[
-\theta \left[ \sum_{i,j=1}^{n} (b^j v_n)_x \right] \left[ du - \sum_{i,j=1}^{n} (b^i u_n)_x \right] dr \\
= - \left[ \sum_{i,j=1}^{n} (b^j v_n)_x \right] \left[ du - \sum_{i,j=1}^{n} (b^i u_n)_x \right] dr \\
= - \sum_{i,j=1}^{n} (b^j v_n)_x \, dv_s - s \lambda \psi \theta v \, dv_s \\
= - \sum_{i,j=1}^{n} (b^j v_n)_x \, dv_s + \left[ \sum_{i,j=1}^{n} (b^j v_n)_x \right] + s \lambda \psi \theta v \, dv_s \\
= - \sum_{i,j=1}^{n} (b^j v_n)_x \, dv_s + \frac{1}{2} d(\lambda \psi \theta v)^2 + \frac{1}{2}s \lambda^2 \psi^2 \theta v^2 \\
+ \frac{1}{2}s \lambda \psi \theta v^2 \\

The second term on the left-hand side of equality (2.1) satisfies

\[
\frac{1}{4} \lambda v \left[ du - \sum_{i,j=1}^{n} (b^i u_n)_x \right] dr \\
= \frac{1}{4} \lambda v \left[ du - \sum_{i,j=1}^{n} (b^i v_n)_x \right] dr \\
= \frac{1}{4} \lambda v \left[ du - \sum_{i,j=1}^{n} (b^i v_n)_x \right] dr - \frac{1}{4}s \lambda^2 \psi \theta v^2 \\
= \frac{1}{8} \lambda v^2 - \frac{1}{8} \lambda dv_s - \frac{1}{4} \lambda \sum_{i,j=1}^{n} (b^i v_n)_x \\
+ \frac{1}{4} \lambda v \sum_{i,j=1}^{n} (b^i v_n)_x \, dt - \frac{1}{4}s \lambda^2 \psi \theta v^2 \\

This, together with equality (2.2), implies equality (2.1). \square

Now we are in a position to prove theorem 1.1.

Proof of theorem 1.1. Applying proposition 2.1 to equation (1.2) with \( u = \gamma \), integrating equality (2.1) on \( [\delta, T] \times G \) for some \( \delta \in [0, T) \) and taking mathematical expectation, we get

\[
-\mathbb{E} \int_{\delta}^{T} \int_{G} \left[ \sum_{i,j=1}^{n} (b^j v_n)_x \right] \left[ du - \sum_{i,j=1}^{n} (b^i u_n)_x \right] dx \\
+ \frac{1}{4} \lambda \mathbb{E} \int_{\delta}^{T} \int_{G} \theta v \left[ dy - \sum_{i,j=1}^{n} (b^i v_n)_x \right] dx \\
= -\mathbb{E} \int_{\delta}^{T} \int_{G} \left( b^j v_n \right) dv_s + \frac{1}{4} \lambda b^j v_n \left( dx ight)_{x_j} \\
\]
\[ + \frac{1}{2} E \int_{\delta}^{T} \int_{G} d \left( \sum_{i,j=1}^{n} b^i v_i v_j - s \lambda \psi_t v^2 + \frac{1}{8} \lambda v^2 \right) \, dx \]
\[ - E \int_{\delta}^{T} \int_{G} \left( \frac{1}{4} \sum_{i,j=1}^{n} b^i v_i v_j \, dr + \frac{1}{2} \sum_{i,j=1}^{n} b^i v_i \, dv \right) \, dx \]
\[ + E \int_{\delta}^{T} \int_{G} \left[ \frac{1}{2} \lambda^2 \psi_t^2 v^2 \, dt + \frac{1}{2} s \lambda \psi_t v^2 \, dt \right. \]
\[ \left. - \frac{1}{4} \lambda \psi_t v^2 \, dt + \frac{1}{2} s \lambda \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] \, dx \]
\[ + E \int_{\delta}^{T} \int_{G} \left[ \sum_{i,j=1}^{n} (b^i v_i)_x + s \lambda \psi_t v \right]^2 \, dx \, dt. \] (2.2)

Now we estimate the terms on the right-hand side of equality (2.2) one by one. For the first one, since \( v |_{\Gamma} = 0 \), we have that \( v |_{\Gamma} = 0 \). Therefore, it holds that
\[ - E \int_{\delta}^{T} \int_{G} \left( \sum_{i,j=1}^{n} b^i v_i \, dv + \frac{1}{4} \lambda b^i v_i \, dv \right) \, dx \]
\[ = - E \int_{\delta}^{T} \int_{G} \left( \sum_{i,1}^{n} b^i \left( v_i \, dv + \frac{1}{4} \lambda v_i \, dv \right) \right) \, v^i \, d \Gamma = 0. \] (2.3)

For the second one, we have
\[ \frac{1}{2} E \int_{\delta}^{T} \int_{G} d \left( \sum_{i,j=1}^{n} b^i v_i v_j - s \lambda \psi_t v^2 + \frac{1}{8} \lambda v^2 \right) \, dx \]
\[ \geq - C E \left( |\nabla v(T)|_{L^2(G)}^2 + |\nabla v(\delta_t)|_{L^2(G)}^2 + s \lambda \psi_{T^*} |v(T)|_{L^2(G)}^2 + s \lambda \psi_{T^*} |v(\delta_t)|_{L^2(G)}^2 \right). \] (2.4)

Since
\[ E \int_{\delta}^{T} \int_{G} \frac{1}{2} \sum_{i,j=1}^{n} b^i v_i \, dv \, dx = \frac{1}{2} E \int_{\delta}^{T} \int_{G} \sum_{i,j=1}^{n} b^i \theta^2 (a_i + g) \, dx \, dt. \]
the third one reads as
\[ E \int_{\delta}^{T} \int_{G} \left( \frac{1}{4} \lambda \sum_{i,j=1}^{n} b^i v_i v_j \, dr + \frac{1}{2} \sum_{i,j=1}^{n} b^i v_i \, dv \right) \, dx \]
\[ \geq E \int_{\delta}^{T} \int_{G} \left[ \frac{1}{4} \lambda |\nabla v|^2 - C |\nabla v|^2 + C |a_1|^2 |\nabla v|^2 + |a_2|^2 |v|^2 + \theta^2 |\nabla g|^2 + \theta^2 |g|^2 \right] \, dx \, dt \]
\[ \geq \frac{1}{4} \lambda E \int_{\delta}^{T} \int_{G} |\nabla v|^2 \, dx \, dt - C (|a_3|^2_{L^2(G)} + 1) E \int_{\delta}^{T} \int_{G} (|\nabla v|^2 + |v|^2) \, dx \, dt \]
\[ - C E \left( T^* |\theta|^2 + g^2 \right) \, dx \, dt. \] (2.5)

For the fourth one, recalling that \( |\psi_t| \geq 1 \) and utilizing that
\[ E \int_{\delta}^{T} \int_{G} \left[ \frac{1}{2} s \lambda \psi_t (dv)^2 - \frac{1}{8} \lambda (dv)^2 \right] \, dx \]
\[ = E \int_{\delta}^{T} \int_{G} \theta^2 \left[ \frac{1}{2} s \lambda \psi_t (a_3 + g)^2 - \frac{1}{8} \lambda (a_3 + g)^2 \right] \, dx \, dt. \]
we see
\[
\mathbb{E} \int_{\delta}^{T} \int_{G} \left[ \frac{1}{2} s \lambda \psi_{i}^{2} v^{2} \, dt + \frac{1}{2} s \lambda \psi_{i}^{2} u^{2} \, dt - \frac{1}{4} s \lambda \psi_{i} v^{2} \, dt + \frac{1}{2} s \lambda \psi_{i} (dv)^{2} - \frac{1}{8} \lambda (dv)^{2} \right] \, dx \\
\geq \frac{1}{4} s \lambda^{2} \mathbb{E} \int_{\delta}^{T} \int_{G} \psi v^{2} \, dx \, dt + sO(\lambda) \mathbb{E} \int_{\delta}^{T} \int_{G} \psi v^{2} \, dx \, dt \\
- C s \lambda \mathbb{E} \int_{\delta}^{T} \int_{G} (1 + \psi) \theta^{2} \, dx \, dr. 
\] (2.6)

Thus, we know that there exists a \( \lambda_{0} > 0 \) such that for all \( \lambda \geq \lambda_{0} \), it holds that
\[
\mathbb{E} \int_{\delta}^{T} \int_{G} \left[ \frac{1}{2} s \lambda \psi_{i}^{2} v^{2} \, dt + \frac{1}{2} s \lambda \psi_{i}^{2} u^{2} \, dt - \frac{1}{4} s \lambda \psi_{i} v^{2} \, dt + \frac{1}{2} s \lambda \psi_{i} (dv)^{2} - \frac{1}{8} \lambda (dv)^{2} \right] \, dx \\
\geq \frac{1}{8} s \lambda^{2} \mathbb{E} \int_{\delta}^{T} \int_{G} \psi v^{2} \, dx \, dt - C s \lambda \mathbb{E} \int_{\delta}^{T} \int_{G} (1 + \psi) \theta^{2} \, dx \, dr. 
\] (2.7)

Now, we estimate the terms on the left-hand side one by one. From equation (1.2) and noting that
\[-E \int_{\delta}^{T} \int_{G} \theta \left[ \sum_{i,j=1}^{n} (\beta^{i} v_{n})_{x_{j}} + s \lambda \psi_{i} v \right] (a_{3} y + g) \, dB \, dx = 0,
\]
we know that
\[-E \int_{\delta}^{T} \int_{G} \theta \left[ \sum_{i,j=1}^{n} (\beta^{i} v_{n})_{x_{j}} + s \lambda \psi_{i} v \right] \left[ (a_{1}, \nabla y) + a_{2} y + f \right] \, dx \, dr \]
\[= - E \int_{\delta}^{T} \int_{G} \theta \left[ \sum_{i,j=1}^{n} (\beta^{i} v_{n})_{x_{j}} + s \lambda \psi_{i} v \right] (a_{1}, \nabla y) + a_{2} y + f \right] \, dx \, dr \]
\[\leq E \int_{\delta}^{T} \int_{G} \left[ \sum_{i,j=1}^{n} (\beta^{i} v_{n})_{x_{j}} + s \lambda \psi_{i} v \right] \left[ (a_{1}, \nabla y) + a_{2} y + f \right] \, dx \, dr \]
\[+ 3 E \int_{\delta}^{T} \int_{G} \theta^{2} (|a_{1}|^{2} |\nabla u|^{2} + a_{2}^{2} u^{2} + f^{2}) \, dx \, dr \]
\[\leq E \int_{\delta}^{T} \int_{G} \left[ \sum_{i,j=1}^{n} (\beta^{i} v_{n})_{x_{j}} + s \lambda \psi_{i} v \right] \left[ (a_{1}, \nabla y) + a_{2} y + f \right] \, dx \, dr \]
\[+ 3 |a_{1}|^{2} \mathbb{E}_{\mathcal{L}^{2}(0,T;L^{\infty}(G);\mathbb{R})} \int_{\delta}^{T} \int_{G} |\nabla v|^{2} \, dx \, dr \]
\[+ 3 |a_{2}|^{2} \mathbb{E}_{\mathcal{L}^{2}(0,T;L^{\infty}(G);\mathbb{R})} \int_{\delta}^{T} \int_{G} v^{2} \, dx \, dr + 3 E \int_{\delta}^{T} \int_{G} \theta^{2} \, dx \, dr, \] (2.8)

and that
\frac{1}{4} \lambda \mathbb{E} \int_{\mathbb{S}} \int_{G} \left[ \theta v \left[ d_x - \sum_{i,j=1}^{n} (\theta^2 u_{x_i})_{x_j} dr \right] dx \right] = \frac{1}{4} \lambda \mathbb{E} \int_{\mathbb{S}} \int_{G} \left[ \theta v [(a_1, \nabla y) + a_2 y + f] \right] dr \ dx \\
\leq \frac{1}{64} \lambda \mathbb{E} \int_{\mathbb{S}} \int_{G} v^2 dx \ dr + \mathbb{E} \int_{\mathbb{S}} \int_{G} \theta^2 [(a_1, \nabla y) + a_2 y + f]^2 \ dr \ dx \\
\leq \frac{1}{64} \lambda^2 \mathbb{E} \int_{\mathbb{S}} \int_{G} v^2 dx \ dr + 3 \mathbb{E} \int_{\mathbb{S}} \int_{G} \theta^2 (|a_1|^2 |\nabla u|^2 + a_2^2 + f^2) \ dr \ dx \\
\leq \frac{1}{64} \lambda^2 \mathbb{E} \int_{\mathbb{S}} \int_{G} v^2 dx \ dr + 3 |a_1|^2 \mathbb{E} \int_{\mathbb{S}} \int_{G} |\nabla v|^2 \ dr \ dx \\
+ 3 |a_2|^2 \mathbb{E} \int_{\mathbb{S}} \int_{G} |\nabla^2 f|^2 \ dr \ dx. \tag{2.9}

From (2.2)–(2.9), we find
\begin{align*}
\frac{1}{4} \lambda \mathbb{E} \int_{\mathbb{S}} \int_{G} |\nabla v|^2 dx \ dr & = C(|a_1|^2 \mathbb{E} (0, T, L^\infty(G'; R^r)) + |a_2|^2 \mathbb{E} (0, T, W^1 \infty(G)) + 1) \mathbb{E} \int_{\mathbb{S}} \int_{G} |\nabla v|^2 \ dr \ dx \\
& - C(|a_1|^2 \mathbb{E} (0, T, L^\infty(G)) + |a_2|^2 \mathbb{E} (0, T, W^1 \infty(G)) + 1) \mathbb{E} \int_{\mathbb{S}} \int_{G} |\nabla v|^2 \ dr \ dx \\
& + \left( \frac{1}{8} \lambda^2 - \frac{1}{64} \lambda^2 \right) \mathbb{E} \int_{\mathbb{S}} \int_{G} \varphi v^2 \ dr \ dx \\
& \leq C \mathbb{E} \left[ |\nabla v(T)|^2_{L^2(G)} + |\nabla v(\delta)|^2_{L^2(G)} + s \lambda \varphi(T) |v(T)|^2_{L^2(G)} + s \lambda \varphi(\delta) |v(\delta)|^2_{L^2(G)} + |\nabla v(T)|^2_{L^2(G)} + |\nabla v(\delta)|^2_{L^2(G)} + s \lambda \varphi(T) |v(T)|^2_{L^2(G)} + s \lambda \varphi(\delta) |v(\delta)|^2_{L^2(G)} \right].
\end{align*}

Recalling that
\[ r_1 = |a_1|^2 \mathbb{E} (0, T, L^\infty(G'; R^r)) + |a_2|^2 \mathbb{E} (0, T, L^\infty(G)) + |a_3|^2 \mathbb{E} (0, T, W^1 \infty(G)) + 1, \]
from inequality (2.10), we know that there exists a \( \lambda_1 \geq \max \{C r_1, \lambda_0 \} \) such that for all \( \lambda \geq \lambda_1 \) there exists a \( s_0(\lambda_1) > 0 \) so that for all \( s \geq s_0(\lambda_1) \), it holds that
\begin{align*}
\lambda \mathbb{E} \int_{\mathbb{S}} \int_{G} |\nabla v|^2 dx \ dr & + s \lambda^2 \mathbb{E} \int_{\mathbb{S}} \int_{G} \varphi v^2 dx \ dr \\
& \leq C \mathbb{E} \left[ |\nabla v(T)|^2_{L^2(G)} + |\nabla v(\delta)|^2_{L^2(G)} + s \lambda \varphi(T) |v(T)|^2_{L^2(G)} + s \lambda \varphi(\delta) |v(\delta)|^2_{L^2(G)} \right].
\end{align*}
which implies inequality (1.11) immediately. \( \square \)

3. Proof for theorem 1.2

This section is devoted to the proof of theorem 1.2. We borrow some ideas from [30].

**Proof of theorem 1.2.** Choose \( t_1 \) and \( t_2 \) such that \( 0 < t_1 < t_2 < t_0 \). Set \( \alpha_k = e^{\lambda_k} (k = 0, 1, 2) \). Let \( \rho \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \rho \leq 1 \) and that
\begin{equation}
\rho = \begin{cases}
1, & t \geq t_2, \\
0, & t \leq t_1.
\end{cases}
\end{equation}

Let $z = \rho y$, by means of $y$ solving equation (1.12), we know that $z$ solves
\begin{equation}
\begin{aligned}
\frac{dz}{dt} - \sum_{i,j=1}^{n} (b^{ij} y_{x_i})_{x_j} dt &= [(a_1, \nabla z) + a_2 z + \rho(t) y] dr + a_3 z dB(t) \quad \text{in } Q, \\
z(0) &= 0 \quad \text{on } \Sigma, \\
z(0) &= 0 \quad \text{in } G.
\end{aligned}
\end{equation}

Applying theorem 1.1 with $\psi = t$ and $\delta = 0$ to equation (3.2), for $\lambda \geq \lambda_1$ and $s \geq s_0(\lambda_1)$, we have
\begin{align}
\lambda \mathbb{E} \int_Q \theta^2 |\nabla z|^2 \, dx \, dr &+ s \lambda^2 \mathbb{E} \int_Q \theta^2 \psi |z|^2 \, dx \, dr \\
&\leq C \left[ \theta^2(T) |\nabla z(T)|^2_{L^2(G)} + s \lambda \psi(T) \theta^2 |z(T)|^2_{L^2(G)} + \int_Q \theta^2 |\rho(t) y|^2 \, dx \, dr \right].
\end{align}

From the choice of $\rho$, we see that
\begin{align}
\mathbb{E} \int_Q \theta^2 |\rho(t) y|^2 \, dx \, dr &\leq C \int_{t_0}^{T} \int_Q \theta^2 y^2 \, dx \, dr \leq C \theta^2(t_0) |y|^2_{L^2_x(0,T;L^2(G))}.
\end{align}

This, together with inequality (3.3), implies that
\begin{align}
\lambda \theta^2(t_0) \mathbb{E} \int_{t_0}^{T} \int_G |\nabla y|^2 \, dx \, dr &+ s \lambda^2 \theta^2(t_0) \mathbb{E} \int_{t_0}^{T} \int_G |\psi y|^2 \, dx \, dr \\
&\leq \lambda \mathbb{E} \int_Q \theta^2 |\nabla z|^2 \, dx \, dr + s \lambda^2 \mathbb{E} \int_Q \theta^2 \psi |z|^2 \, dx \, dr \\
&\leq C \theta^2(t_1) |y|^2_{L^2_x(0,T;L^2(G))} + C \mathbb{E} \left( \theta^2(T) |\nabla y(T)|^2_{L^2(G)} + s \lambda \psi(T) \theta^2 |y(T)|^2_{L^2(G)} \right).
\end{align}

Here, we utilize the fact that $\theta(t) \leq \theta(s)$ for $t \leq s$.

From inequality (3.5), we see
\begin{align}
\lambda \mathbb{E} \int_{t_0}^{T} \int_G |\nabla y|^2 \, dx \, dr &+ s \lambda^2 \mathbb{E} \int_{t_0}^{T} \int_G |\psi y|^2 \, dx \, dr \\
&\leq C \theta^2(t_1) \theta^{-2}(t_0) |y|^2_{L^2_x(0,T;L^2(G))} + C \mathbb{E} \left( \theta^2(T) |\nabla y(T)|^2_{L^2(G)} + s \lambda \psi(T) \theta^2 |y(T)|^2_{L^2(G)} \right).
\end{align}

By means of $d(y^2) = 2y \, dy + (dy)^2$, we obtain
\begin{align}
\mathbb{E} \int_G |y(t_0)|^2 \, dx &= \mathbb{E} \int_G |y(T)|^2 \, dx - \mathbb{E} \int_{t_0}^{T} \int_G [2y \, dy + (dy)^2] \, dx \\
&= \mathbb{E} \int_G |y(T)|^2 \, dx - \mathbb{E} \int_{t_0}^{T} \int_G \left\{ 2y \left[ \sum_{i,j=1}^{n} (b^{ij} y_{x_i})_{x_j} + (a_1, \nabla y) + a_2 y \right] + (a_3 y)^2 \right\} \, dx \, dt \\
&\leq \mathbb{E} \int_G |y(T)|^2 \, dx + C \mathbb{E} \int_{t_0}^{T} \int_G |\nabla y|^2 \, dx \, dt + \left[ |a_1|^2_{L^2_x(0,T;L^\infty(G))} + |a_2|^2_{L^2_x(0,T;L^\infty(G))} \right] \mathbb{E} \int_{t_0}^{T} \int_G y^2 \, dx \, dt \\
&\leq \mathbb{E} \int_G |y(T)|^2 \, dx + C \mathbb{E} \int_{t_0}^{T} \int_G |\nabla y|^2 \, dx \, dt + C \mathbb{E} \int_{t_0}^{T} \int_G y^2 \, dx \, dt.
\end{align}
Recalling $\varphi \geq 1$, from inequality (3.7), we know that there exists a $\lambda_2 > 0$ such that for all $\lambda \geq \lambda_2$, it holds that
\[
\mathbb{E} \int_G |y(t_0)|^2 \, dx \leq \mathbb{E} \int_G |y(T)|^2 \, dx + C \left( \lambda \mathbb{E} \int_0^T \int_G |\nabla y|^2 \, dx \, dt + s \lambda^2 \mathbb{E} \int_0^T \int_G \varphi y^2 \, dx \, dt \right).
\]
\[\tag{3.8}\]
Combining inequality (3.6) and inequality (3.8), for any $\lambda \geq \max\{\lambda_1, \lambda_2\}$ and $s \geq s_0(\lambda_1)$, we have
\[
\mathbb{E} \int_G |y(t_0)|^2 \, dx \leq C \theta(t_1) \theta^{-2}(t_0) |y|^2_{L^2_2(0, T; L^2(G))} + C \theta^2(T) \mathbb{E} |y(T)|^2_{H^1(G)}.
\]
\[\tag{3.9}\]
Now we fix $\lambda_3 = \max\{\lambda_1, \lambda_2\}$; from inequality (3.9), we get
\[
\mathbb{E} \int_G |y(t_0)|^2 \, dx \leq C e^{\theta(t_1) \theta^{-2}(t_0) |y|^2_{L^2_2(0, T; L^2(G))} + C \theta^2(T) \mathbb{E} |y(T)|^2_{H^1(G)}}.
\]
\[\tag{3.10}\]
Replacing $C$ by $C e^{\theta(t_1) \theta^{-2}(t_0) |y|^2_{L^2_2(0, T; L^2(G))} + C \theta^2(T) \mathbb{E} |y(T)|^2_{H^1(G)}}$, for any $s > 0$, it holds that
\[
\mathbb{E} \int_G |y(t_0)|^2 \, dx \leq C e^{-2t(t_1) - e^{\theta(t_0)}} |y|^2_{L^2_2(0, T; L^2(G))} + C e^{\theta(T) \mathbb{E} |y(T)|^2_{H^1(G)}}.
\]
\[\tag{3.11}\]
Choosing $s \geq 0$ which minimize the right-hand side of inequality (3.11), we obtain
\[
\mathbb{E} |y(t_0)|^2_{L^2(G)} \leq C |y|^2_{L^2_2(0, T; L^2(G))} \mathbb{E} |y(T)|^2_{H^1(G)}
\]
\[\tag{3.12}\]
with
\[\theta = \frac{2(e^{\lambda_1 t_0} - e^{\lambda_2 t_1})}{C + 2(e^{\lambda_1 t_0} - e^{\lambda_2 t_1})}.
\]

4. Proof of theorem 1.4

This section is devoted to proving theorem 1.4. We borrow some ideas from [30] again.

**Proof of theorem 1.4.** From the assumptions on $b_1, b_2, b_3, R$ and $h$, and by lemma 1.2, we know that equation (1.16) admits a unique strong solution. For arbitrary small $\varepsilon > 0$, we choose $t_1$ and $t_2$ such that
\[0 < t_0 - \varepsilon < t_1 < t_2 < t_0.\]
Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \chi \leq 1$ and that
\[\chi = \begin{cases} 1, & t \leq t_1, \\ 0, & t \geq t_2. \end{cases}\]
\[\tag{4.1}\]
Put $y = Rz$ (recall (1.17) for $R$) in $[0, t_2] \times G$. Since $y$ is a strong solution of equation (1.16), we know that $z$ solves
\[
\begin{cases}
dz - \Delta z \, dt = \left[ (b_1, \nabla z) + \left( \frac{2\nabla R}{R}, \nabla z \right) + \left( b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} \right) \right] \, dt + h \, dt + b_3 \, dB(t) \\
z = \frac{\partial z}{\partial y} = 0 \\
z(0) = 0
\end{cases}
\]
on $[0, t_0] \times G$, \quad \text{on } [0, t_0] \times G, \quad \text{in } G.
\[\tag{4.2}\]
Setting $u = z_\alpha$, noting $z$ is the strong solution of equation (4.2) and $z_\alpha = \frac{u}{w_0} = 0$ on $([0] \times G') \cup ([l] \times G')$, we know that $u$ is the weak solution of the following equation:

$$
\begin{aligned}
\frac{du}{dt} - \Delta u &= \left[ (b_1)_{x_1} \cdot \nabla + (b_1) \cdot \nabla u + \left( \frac{2\nabla R}{R} \right)_{x_1} \cdot \nabla + \left( \frac{2\nabla R}{R} \right) \cdot \nabla u \\
&+ \left( b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_0}{R} + \left( \frac{\nabla R}{R}, b_1 \right) \right) w \right] dt \\
&+ (b_3)_{x_1} z \, dt + b_3 u \, dB(t)
\end{aligned}
$$

(4.3)

$u = 0$ on $[0, t_0] \times \Gamma$

$u(0) = 0$ in $G$.

Set $w = \chi$. Then, we know that $w$ is a weak solution of the following equation:

$$
\begin{aligned}
\frac{dw}{dt} - \Delta w &= \left[ (b_1)_{x_1} \chi \nabla z + (b_1) \chi \nabla w + \left( \frac{2\nabla R}{R} \right)_{x_1} \chi \nabla z + \left( \frac{2\nabla R}{R} \right) \chi \nabla w \\
&+ \left( b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_0}{R} + \left( \frac{\nabla R}{R}, b_1 \right) \right) \chi w \\
&+ \left( b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_0}{R} + \left( \frac{\nabla R}{R}, b_1 \right) \right) \chi w \right] dt \\
&+ (b_3)_{x_1} \chi \frac{dz}{dt} + b_3 w \frac{dB(t)}{dt} - \chi' u \, dt
\end{aligned}
$$

(4.4)

$w = 0$ on $[0, t_0] \times \Gamma$,

$w = 0$ in $G$.

By means of $u = z_\alpha$ and $z(t, 0, x') = y(t, 0, x') = 0$ for $(t, x') \in (0, t_0) \times G'$, we see that

$$
\chi = \chi \left( \int_{0}^{t_0} u(t, \eta, x') \, d\eta \right) = \chi \int_{0}^{t_0} w(t, \eta, x') \, d\eta.
$$

(5.5)

This, together with equation (4.4), implies that $w$ is the weak solution of the following equation:

$$
\begin{aligned}
\frac{dw}{dt} - \Delta w &= \left[ (b_1) \chi \nabla w + \left( \frac{2\nabla R}{R} \right) \chi \nabla w + \left( (b_1)_{x_1}, \chi \int_{0}^{t_0} w(t, \eta, x') \, d\eta \right) \\
&+ \left( b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_0}{R} + \left( \frac{\nabla R}{R}, b_1 \right) \right) \chi w \\
&+ \left( b_2 + \frac{\Delta R}{R} - \frac{2(\nabla R, \nabla R)}{R^2} - \frac{R_0}{R} + \left( \frac{\nabla R}{R}, b_1 \right) \right) \chi w \right] dt \\
&+ (b_3)_{x_1} \chi \int_{0}^{t_0} u(t, \eta, x') \, d\eta \, dB(t) + b_3 w \frac{dB(t)}{dt} - \chi' u \, dt
\end{aligned}
$$

(4.6)

$w = 0$ on $[0, t_0] \times \Gamma$,

$w = 0$ in $G$.

Applying theorem 1.1 to equation (4.6) with $\psi(t) = -t$, noting that $w(0) = 0$ and that $w(t, 0) = \chi(t_0) u(t, 0) = 0$, we get

$$
\begin{aligned}
\mathbb{E} \int_{0}^{t_0} \int_{G} \theta^2 (\chi \nabla w)^2 + s \partial^2 w^2) \, dx \, dt \leq C \mathbb{E} \int_{0}^{t_0} \int_{G} \theta^2 \chi u^2 \, dx \, dt \\
+ C \mathbb{E} \int_{0}^{t_0} \int_{G} \theta^2 \left( \int_{0}^{t} \chi(w(t, \eta, x') \, d\eta \right)^2 + \int_{0}^{t} |\nabla w(t, \eta, x') \, d\eta|^2 \right) \, dx \, dt.
\end{aligned}
$$

(4.7)
Since
\[ \left| \int_0^{\xi_1} w(t, \eta, x') \, d\eta \right|^2 \leq \int_0^l |w(t, \eta, x')|^2 \, d\eta, \]
we know
\[ \int_0^\rho \int_G \theta^2 \left| \int_0^{\xi_1} w(t, \eta, x') \, d\eta \right|^2 \, dx \, dt \leq \int_0^l \int_G \int_0^{\xi_1} \theta^2 |w(t, \eta, x')|^2 \, d\eta \, dx' \, dt \]
\[ \leq l^2 \int_0^\rho \int_G \theta^2 |w(t, \eta, x')|^2 \, d\eta \, dx' \, dt. \]  
\tag{4.8}

By virtue of
\[ \nabla \int_0^{\xi_1} w(t, \eta, x') \, d\eta = \int_0^{\xi_1} \nabla w(t, \eta, x') \, d\eta + w(t, 0, x') = \int_0^{\xi_1} \nabla w(t, \eta, x') \, d\eta, \]
we get that
\[ \int_0^\rho \int_G \theta^2 \left| \nabla \int_0^{\xi_1} w(t, \eta, x') \, d\eta \right|^2 \, dx \, dt = \int_0^\rho \int_G \theta^2 \left| \int_0^{\xi_1} \nabla w(t, \eta, x') \, d\eta \right|^2 \, dx \, dt \]
\[ \leq l \int_0^l \int_0^\rho \int_G \theta^2 |\nabla w(t, \eta, x')|^2 \, d\eta \, dx' \, dt \]
\[ \leq l^2 \int_0^\rho \int_G \theta^2 |\nabla w(t, \eta, x')|^2 \, d\eta \, dx' \, dt. \]  
\tag{4.9}

From inequality (4.7)–(4.9), we obtain
\[ \mathbb{E} \int_0^\rho \int_G \theta^2 (\lambda |\nabla w|^2 + s\lambda^2 w^2) \, dx \, dt \]
\[ \leq C \mathbb{E} \int_0^\rho \int_G \theta^2 |\nabla w|^2 \, dx \, dt + C \rho^2 \mathbb{E} \int_0^\rho \int_G \theta^2 (|\nabla w|^2 + |w|^2) \, dx \, dt. \]  
\tag{4.10}

Thus, we know that there is a \( \lambda_4 = \max\{C_{r1}, \lambda_1\} \) such that for all \( \lambda \geq \lambda_4 \), there exists an \( s_1(\lambda_4) > 0 \) so that for all \( s \geq s_1(\lambda_4) \), it holds that
\[ \mathbb{E} \int_0^\rho \int_G \theta^2 (\lambda |\nabla w|^2 + s\lambda^2 w^2) \, dx \, dt \leq C \mathbb{E} \int_0^\rho \int_G \theta^2 |\nabla w|^2 \, dx \, dt. \]  
\tag{4.11}

Fix \( \lambda = \lambda_4 \); by the property of \( \chi \) (see (4.1)), we find
\[ \mathbb{E} \int_0^\rho \int_G \theta^2 |\nabla w|^2 \, dx \, dt \leq e^{2\log_2 \lambda_4} \mathbb{E} \int_0^\rho \int_G |w|^2 \, dx \, dt \leq e^{2\log_2 \lambda_4} |w|_{L^2_{\rho}([0,T];L^2(G))}. \]  
\tag{4.12}

This, together with inequality (4.11), implies that for all \( s \geq s_1 \), it holds that
\[ e^{2\log_2 \lambda_4 - (\rho - t)} \int_0^{\rho - \varepsilon} \int_G (|\nabla w|^2 + sw^2) \, dx \, dt \leq \mathbb{E} \int_0^{\rho - \varepsilon} \int_G \theta^2 (|\nabla w|^2 + sw^2) \, dx \, dt \]
\[ \leq \mathbb{E} \int_0^\rho \int_G \theta^2 (|\nabla w|^2 + sw^2) \, dx \, dt \]
\[ \leq C e^{2\log_2 \lambda_4} |w|_{L^2_{\rho}([0,T];L^2(G))}. \]  
\tag{4.13}

From inequality (4.13), we have
\[ |w|_{L^2_{\rho}([0,T];H^1_0(G))} \leq C e^{2\log_2 \lambda_4 - (\rho - t)} |w|_{L^2_{\rho}([0,T];L^2(G))}. \]  
\tag{4.14}

Recalling that \( t_0 - \varepsilon < t_1 \), we know \( e^{-\lambda_4 t_0} - e^{-\lambda_4 (t_0 - \varepsilon)} < 0 \). Letting \( s \to +\infty \), we obtain
\[ w = 0 \text{ in } (0, t_0 - \varepsilon) \times G, \quad P\text{-a.s.} \]
This, together with equality (4.5), implies that
\[ z = 0 \text{ in } (0, t_0 - \varepsilon) \times G, \ P-a.s., \]
which means
\[ h = 0 \text{ in } (0, t_0 - \varepsilon) \times G', \ P-a.s. \]
Since \( \varepsilon > 0 \) is arbitrary, the proof of theorem 1.4 is completed. \( \square \)

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