SYMMETRIC PRODUCTS AND MODULI SPACES OF VECTOR BUNDLES OF CURVES

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ABSTRACT. Let $X$ be a smooth projective curve of genus $g \geq 2$ and $M$ be the moduli space of rank 2 stable vector bundles on $X$ whose determinants are isomorphic to a fixed odd degree line bundle $L$. There has been a lot of works studying the moduli and recently the bounded derived category of coherent sheaves on $M$ draws lots of attentions. It was proved that the derived category of $X$ can be embedded into the derived category of $M$ (cf. [11, 27, 28]). In this paper we prove that the derived category of the second symmetric product of $X$ can be embedded into derived category of $M$ when $X$ is non-hyperelliptic and $g \geq 16$.

1. INTRODUCTION

Let $X$ be a smooth projective curve of genus $g \geq 2$ and $M$ be the moduli space of rank 2 stable vector bundles on $X$ whose determinants are isomorphic to a fixed odd degree line bundle $L$. The moduli space $M$ is a smooth projective Fano variety of dimension $3g - 3$, index 2 (cf. [34]).

There has been a lot of works studying the moduli and recently the bounded derived category of $M$ draws lots of attentions. It seems natural to expect that the derived category of $M$ will be closely related to $X$. Let $E$ be a Poincaré bundle on $X \times M$. Then one can define the Fourier-Mukai transform $\Phi_E : D(X) \to D(M)$ with kernel $E$ and a natural question is whether the functor $\Phi_E$ is fully-faithful or not. Indeed, it was proved that the derived category of $X$ can be embedded into the derived category of $M$ via $\Phi_E$ (cf. [11, 27, 28]). See [3, 24] for investigations of similar questions about moduli spaces of higher rank vector bundles on $X$. Then a next task is to understand full semiorthogonal decomposition of $D(M)$. The second named author conjectured that the derived category of $M$ will have the following semiorthogonal decomposition. We were informed that Belmans, Galkin and Mukhopadhyay stated the same conjecture independently. See [2, 23] for more details.

Conjecture 1.1. The derived category of coherent sheaves on $M$ has the following semiorthogonal decomposition

$$D(M) = \langle D(pt), D(pt), D(X), D(X), \cdots, D(X_k), D(X_k), \cdots, D(X_{g-1}) \rangle,$$

i.e., two copies of $D(X_k)$ for $0 \leq k \leq g - 2$ and one copy of $D(X_{g-1})$. Here $X_k$ means the $k$-th symmetric product of $X$.

It turns out that the motive of $M$ has a motivic decomposition which is compatible with the above conjectural semiorthogonal decomposition. See [12, 23] for precise statement and more details. Therefore a natural question is whether derived categories of symmetric products of $X$ can be embedded into the derived category of $M$. In this paper we prove the following result.

Theorem 1.2. If $X$ be a non-hyperelliptic curve with genus $g \geq 16$, then $D(X_2)$ can be embedded into $D(M)$. 
In order to prove the above theorem, we construct a vector bundle $F$ on $X_2 \times M$ which becomes a Fourier-Mukai kernel. On the way of proving the above theorem, we can see that $F_{x,y}$ and $F_{z,w}$ are distinct vector bundles on $M$ when $(x,y)$ and $(z,w)$ are distinct points in $X_2$.

**Proposition 1.3.** There exist a vector bundle $F$ on $X_2 \times M$ such that $X_2$ is a parameter space of a family of vector bundle $F_{x,y}$ on $M$ where $(x,y) \in X_2$.

We also need to compute various cohomology groups in order to prove the embedding.

**Theorem 1.4.** Let $X$ be a non-hyperelliptic curve with genus $g \geq 16$. The we have the following.

1. If one of the four points $x, y, z, w \in X$ is different from all the others, then we have
   \[ H^i(M, E_x \otimes E_y \otimes E_z^* \otimes E_w^*) = 0 \]
   for every $i \in \mathbb{Z}$.

2. For two distinct points $x, z \in X$ we have
   \[ H^i(M, E_x \otimes E_x \otimes E_z^* \otimes E_z^*) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ \mathbb{C}^2 & \text{if } i = 1, \\ \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases} \]

3. For $x \in X$ we have
   \[ H^i(M, E_x \otimes E_x \otimes E_x^* \otimes E_x^*) = \begin{cases} \mathbb{C}^2 & \text{if } i = 0, \\ \mathbb{C}^3 & \text{if } i = 1, \\ \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases} \]

From the above computation and construction of $F$, we can show the Fourier-Mukai transform with kernel $F$ gives us the desired embedding hence obtain Theorem 1.2.

**Remark 1.5.** In the above Theorem, we assume that $X$ to be a non-hyperelliptic curve with genus $g \geq 16$ due to some technical reasons. However we do not need these conditions when we construct the vector bundle $F$ and the cohomology computation above is valid for many $i \in \mathbb{Z}$ for any curve $X$ with $g \geq 3$. We conjecture that the cohomology computation above will be the same and the vector bundle $F$ will induce an embedding $\Phi_F : D(X_2) \to D(M)$ for any smooth projective curve $X$ with $g \geq 3$.

Moreover, we can prove that the embedded copy of $D(X_2)$ form a part of semiorthogonal decomposition of $D(M)$ which is compatible with the copies constructed in [27, 28].

**Theorem 1.6.** Let $X$ be a non-hyperelliptic curve with $g \geq 16$, there exist a semiorthogonal decomposition $D(M) = \langle \mathcal{A}, \mathcal{B} \rangle$ whose component $\mathcal{A}$ has the following semiorthogonal decomposition.

\[ \mathcal{A} = \langle D(pt), D(X), D(X_2) \rangle \]

And each component of the above decomposition does not admit a nontrivial semiorthogonal decomposition.

**Remark 1.7.** When $g \geq 2$, it was proved that $D(X)$ does not admit a nontrivial semiorthogonal decomposition in [37]. When $g \geq 3$, it was proved that $D(X_2)$ does not admit a nontrivial semiorthogonal decomposition in [4].
Conventions. We will work over $\mathbb{C}$. For a variety $Y$, we will use $D(Y)$ to denote the bounded derived category of coherent sheaves on $Y$. We often use the same notation to denote a sheaf on $Y$ (or a morphism from $Y$) and its restriction to an open subset of $Y$ if the meaning is clear from the context. In this paper, $X$ denotes a smooth projective curve of genus $g \geq 2$ and $X_k$ denotes the $k$-th symmetric product of $X$. Let $\theta$ be the ample divisor on $M$ which generates the Picard group of $M$. We will fix a normalized Poincaré bundle $E$ on $X \times M$ such that $\det(E_x) \cong \theta$ for each $x \in X$.

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2. Construction of functors

Let us consider the following diagram.

Then we have the following exact sequence.

$$0 \to \tilde{F} \to p_{13}^* E \otimes p_{23}^* E \to 2\bigwedge E|_{\Delta \times M} \to 0$$

Lemma 2.1. The above vector bundle $\tilde{F}$ on $X \times X \times M$ descents to a vector bundle $F$ on $X_k \times M$.

Proof. First, let us consider the point $(x_1, x_2) \in X \times X \setminus \Delta$. Then the natural $S_2$-action on $X \times X \setminus \Delta$ lifts to $\tilde{F}$. On the diagonal, the above sequence leads the following exact sequence.

$$0 \to 2\bigwedge E|_{\Delta \times M} \otimes N_{\Delta|X \times X \times M}^\nu \to \tilde{F}|_{\Delta \times M} \to \text{Sym}^2 E|_{\Delta \times M} \to 0$$

The permutation group $S_2$ acts on the conormal bundle by multiplying $(-1)$ and hence the stabilizer groups of the $S_2$-action on $\tilde{F}$ are trivial. Therefore $\tilde{F}$ descents to a vector bundle $F$ on $X_k \times M$. □

In general, we can use equivariant derived categories in order to construct functors. We know that there exist a fully faithful embedding $D(X_k) \subset D([X^k/S_k])$ from [33]. Then let us consider the following diagram.
Let us consider the bundle $\tilde{E} = p_{1,k+1}^*E \otimes \cdots \otimes p_{k,k+1}^*E$ on $X^2 \times M$. There is a natural $S_k$-action on $X^2 \times M$ and the bundle $\tilde{E}$ is a $S_k$-equivariant bundle. Therefore we have a natural functor $D([X^k/S_k]) \to D(M)$ which is the descent of the Fourier-Mukai transform $\Phi_F : D(X^k) \to D(M)$ (cf. [18]). By composing these two functors we obtain the functor $D(X_k) \to D(M)$.

**Conjecture 2.2.** When $1 \leq k \leq g - 1$, the functor constructed above induce a fully faithful embedding $D(X_k) \to D(M)$.

In this paper, we will prove that the above conjecture is true when $k = 2$, $X$ is a non-hyperelliptic curve with $g \geq 16$. When $k = 2$, the functor constructed above can be explicitly described using the vector bundle $F$ on $X_2 \times M$.

**Proposition 2.3.** When $k = 2$ the functor is isomorphic to the Fourier-Mukai transform whose kernel is $F$.

**Proof.** The equivariant derived category of $X^2$ has the following semiorthogonal decomposition (cf. [8])

$$D([X^2/S_2]) = \langle D(X) \otimes \zeta, D(X_2) \rangle$$

where the first component is given by direct image functor via diagonal embedding $\Delta : X \to X^2$ tensored by a nontrivial character $\zeta$ of $S_2$ and the second component is given by pullback via the quotient map $X^2 \to X_2$.

Therefore we have the following decomposition (cf. [19][25]).

$$D([X^2/S_2] \times M) = \langle D(X \times M) \otimes \zeta, D(X_2 \times M) \rangle$$

Because the bundle $\tilde{E} = p_{1,3}^*E \otimes p_{2,3}^*E$ on $X^2 \times M$ has a natural linearization, $\tilde{E}$ gives an element in $D([X^2/S_2] \times M)$ and hence it can be projected into $D(X_2 \times M)$ and $D(X \times M) \otimes \zeta$. The projections can be described via the following exact sequence.

$$0 \to \tilde{E} \to p_{1,3}^*E \otimes p_{2,3}^*E \to \bigwedge^2 E|_{\Delta \times M} \to 0$$

Moreover, the Fourier-Mukai functor $\Phi_F : D(X^2) \to D(M)$ descents to a functor $\Phi_F : D([X^2/S_2]) \to D(M)$. (Here we use the same notation to denote the descent.) Therefore we see that the composition of the embedding $D(X_2) \to D([X^2/S_2])$ and the descent of $\Phi_F : D([X^2/S_2]) \to D(M)$ is same as the Fourier-Mukai transform whose kernel is $F$. \hfill $\Box$

**Remark 2.4.** Let $F$ be the vector bundle $X_2 \times M$ constructed above and $\Phi_F$ be the Fourier-Mukai transform whose kernel is $F$. Let $x, y \in X$ and $(x, y) \in X_2$ and $F_{x,y} = \Phi_F(\mathcal{O}((x,y)))$.

1. When $x \neq y$, $F_{x,y} \cong E_x \otimes E_y$.
2. When $x = y$, we have the following short exact sequence.

$$0 \to \bigwedge^2 E_x \to F_{x,x} \to \text{Sym}^2 E_x \to 0$$

In the rest of the paper, we will prove that the Fourier-Mukai transform $\Phi_F$ is fully faithful when $X$ is non-hyperelliptic with $g \geq 16$.

3. **Geometry of projective bundles**

In this section, we prove several results which will be useful later. Let $x$ be a point of $X$ and let us assume that $L \cong \mathcal{O}(x)$ in this section.
3.1. **Projective bundles.** Let \( \pi : \mathbb{P}E_x \to M \) be the projective bundle of \( E_x \). We can compute the (relative) dualizing sheaf of \( \mathbb{P}E_x \) as follows.

**Proposition 3.1.** We have the following isomorphisms.

\[
\omega_{\mathbb{P}E_x/M} \cong \pi^* \mathcal{O}(-2)
\]

\[
\omega_{\mathbb{P}E_x} \cong \pi^* \mathcal{O}^{-1}(-2)
\]

**Proof.** From the relative Euler sequence

\[
0 \to \mathcal{O}_{\mathbb{P}E_x} \to \pi^* E_x \otimes \mathcal{O}(1) \to T_{\mathbb{P}E_x/M} \to 0
\]

we have

\[
\omega_{\mathbb{P}E_x/M} \cong \pi^* \mathcal{O}(-2).
\]

We can compute \( \omega_{\mathbb{P}E_x} \) from the following formula.

\[
\omega_{\mathbb{P}E_x} \cong \omega_{\mathbb{P}E_x/M} \otimes \pi^* \omega_M \cong \pi^* \mathcal{O}^{-1} \otimes \mathcal{O}(-2)
\]

\[\Box\]

3.2. **Hecke transforms.** Let us recall the Hecke transforms and constructions of \([27, 30]\). We have the following short exact sequence on \( X \times \mathbb{P}E_x \).

\[
0 \to H(E) \to (1 \times \pi)^* E \to p_X^* C(x) \otimes p_{\mathbb{P}E_x}^*(\mathcal{O}_{\mathbb{P}E_x}(1)) \to 0
\]

Because \( H(E) \) parametrizes a family of semistable vector bundles of degree 0, we have the following diagram.

\[
\begin{array}{ccc}
X \times \mathbb{P}E_x & \xrightarrow{1 \times \phi} & X \times M_0 \\
\downarrow{1 \times \pi} & & \downarrow{1 \times \pi} \\
X \times M & &
\end{array}
\]

Taking dual of the above sequence we have the following sequence.

\[
0 \to (1 \times \pi)^* E^* \to K(E) \to \mathcal{E}xt^1(p_X^* C(x) \otimes p_{\mathbb{P}E_x}^*(\mathcal{O}_{\mathbb{P}E_x}(1)), \mathcal{O}_{X \times \mathbb{P}E_x}) \to 0
\]

Here \( K(E) \) is the dual of \( H(E) \).

**Remark 3.2.** Ramanan proved that there is no Poincaré bundle on any open subset of \( X \times M_0 \) in \([34]\). However we can define “adjoint bundle” \( \text{ad}G \) on \( X \times M_0 \). See \([1]\) Lemma 2.2, Remark 2.3 for more details.

**Proposition 3.3.** For \( y \) be a point of \( X \) different from \( x \). Over \( (1 \times \phi)^{-1}(X \times M_0^x) \), we have the following isomorphism.

\[
\pi^* \text{ad}E_y \cong \text{ad}K(E)_y \cong \phi^* \text{ad}G_y
\]

**Proof.** For \( y \neq x \), Hecke transform does not change the bundle. Therefore we obtain the first isomorphism. The second isomorphism comes from the construction of \( \text{ad}G \).

\[\Box\]
3.3. **k-ample.** Let us recall works of Sommese which is useful to prove vanishing of certain cohomology groups of tensor products of vector bundles (cf. [3, 20]).

**Definition 3.4.** A line bundle $\mathcal{L}$ on a smooth projective variety $Y$ is $k$-ample if (1) $\mathcal{L}$ is semiample, i.e. $\mathcal{L}^\otimes m$ is base point free for sufficiently large $m$ and (2) the fibers of the morphism $Y \to \mathbb{P}(H^0(Y, \mathcal{L}^\otimes m)^*)$ have dimension $\leq k$.

**Definition 3.5.** A vector bundle $\mathcal{E}$ on a projective variety $Y$ is semiample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is semiample. It is $k$-ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is $k$-ample.

Let us recall Sommese vanishing theorem.

**Theorem 3.6** (Sommese vanishing theorem). Suppose that $E$ is a $k$-ample bundle of rank $r$. Then we have the following equality

$$H^i(Y, \omega_Y \otimes \wedge^j E) = 0$$

for $j > 0$ and $i + j > r + k$.

Let us consider the following morphism (cf. [28]).

$$\begin{array}{ccc}
\mathbb{P}E_x & \xrightarrow{\varphi} & M_0 \\
\pi & \downarrow & \\
M & & 
\end{array}$$

The morphism $\varphi$ is analyzed in [28, 31]. The dimension of the fiber of $\varphi$ is less than or equal to $g - 1$. Therefore we see that $E_x$ is $(g - 1)$-ample. From the following result, we see that $E_x \otimes E_y \otimes E_z \otimes E_w$ is $(g - 1)$-ample for any four points $x, y, z, w$ of $X$.

**Theorem 3.7.** If $\mathcal{E}$ is a $k$-ample vector bundle and $\mathcal{F}$ is a semiample vector bundle on a compact complex manifold $Y$. Then $\mathcal{E} \otimes \mathcal{F}$ is also $k$-ample.

**Proposition 3.8.** Let $x, y, z, w$ be four (not necessarily distinct) points of $X$. Then $E_x \otimes E_y \otimes E_z \otimes E_w^{-1} \otimes \omega_M^{-1}$ is a $(g - 1)$-ample bundle.

**Proof.** From the isomorphism

$$E_x \otimes E_y \otimes E_z \otimes E_w \otimes \omega_M^{-1} \cong E_x \otimes E_y \otimes E_z \otimes E_w$$

we see that $E_x \otimes E_y \otimes E_z \otimes E_w \otimes \omega_M^{-1}$ is a $(g - 1)$-ample bundle. \hfill \Box

**Proposition 3.9.** Let $x, y, z, w$ be four (not necessarily distinct) points of $X$. For $i \geq g + 15$ we have the following vanishing.

$$H^i(M, E_x \otimes E_y \otimes E_z \otimes E_w) = 0$$

**Proof.** From the previous Proposition and Sommese vanishing theorem we have the following vanishing

$$H^i(M, E_x \otimes E_y \otimes E_z \otimes E_w) \cong H^i(M, E_x \otimes E_y \otimes E_z \otimes E_w \otimes \omega_M^{-1} \otimes \omega_M)$$

$$\cong H^i(M, \omega_M \otimes E_x \otimes E_y \otimes E_z \otimes E_w) = 0$$

for $i > g - 1 + 16 - 1 = g + 14$. \hfill \Box
4. Cohomology groups

In this section we will assume that $X$ is a non-hyperelliptic curve with genus $g \geq 16$ and compute cohomology group $H^i(M, E \otimes E_i \otimes E_1^* \otimes E_0^*)$ for all $i$. Let us explain why we need these assumptions. As in [27], we can try to use the morphism $\varphi : \mathbb{P}E_x \rightarrow M_0$ to compute the cohomology groups. Note that the codimension of $\varphi^{-1}(\mathcal{X})$ is $g - 1$. From Sommese vanishing theorem we see that $H^i(M, E \otimes E_i \otimes E_1^* \otimes E_0^*) = 0$ for $i \geq g + 15$, and we see that analyzing the morphism $\varphi$ only over $M_0'$ is not enough to compute $H^i(M, E \otimes E_i \otimes E_1^* \otimes E_0^*)$ for all $i$. Therefore we will use a similar strategy used in [28]. In [31] the second named author and Ramanan constructed modular desingularization $\psi : H_0 \rightarrow M_0$ when $X$ is non-hyperelliptic curve of genus $g \geq 3$. Let $\mathcal{X}$ be the singular locus of $M_0$ and $\mathcal{X}_0$ be the singular locus of $\mathcal{X}$. Let $Z$ be $\psi^{-1}(M_0 \setminus \mathcal{X}_0)$ and $\mathcal{O}$ be the conic bundle over $Z$. Now the codimension of $\varphi^{-1}(\mathcal{X}_0)$ in $\mathbb{P}E_x$ is $3g - 2 - (g - 1) = 2g - 1$. If $g \geq 16$ and we can compute the cohomology groups of the vector bundles over $\mathbb{P}E_0 \setminus \varphi^{-1}(\mathcal{X}_0)$, then we can compute the cohomology groups $H^i$ for all $i$ via cohomology extension and Sommese vanishing theorem. The situation can be summarized as in the following diagram. Let $D$ be the inverse image $\psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$ of $Z$. We will use the same notation for a morphism (e.g. $\varphi$) and its restriction to an open subset of the domain.

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{\varphi} & Z \\
\downarrow \psi & & \downarrow \psi \\
\mathbb{P}E_x \setminus \varphi^{-1}(\mathcal{X}_0) & \xrightarrow{\varphi} & M_0 \setminus \mathcal{X}_0 \\
\pi & & \downarrow \\
M & & \\
\end{array}
\]

**Proposition 4.1.** [31] Remark 5.17] The restriction of $T_\pi$ to $\varphi^{-1}(m_0)$ for $m_0 \in M_0'$ is isomorphic to $\omega_{\mathcal{O}}$.

**Proof.** Drezet and the second named author proved that the canonical divisor of $M_0$ is $\Theta_0^{-4}$ in [9]. Dualizing sheaf of $\mathbb{P}E_x$ is $\pi^*\theta^{-1} \otimes \mathcal{O}(-2)$ so the relative dualizing sheaf $\omega_\theta$ is $\pi^*\theta^{-1} \otimes \mathcal{O}(2)$. Therefore we can check that it is isomorphic to $T_\pi$. 

We can generalize many properties of $\varphi$ to $\bar{\varphi}$ as follows.

**Proposition 4.2.** The restriction of $\bar{\psi}T_\pi$ to a Hecke curve of degenerate type is $\omega_{\mathcal{O}}$.

**Proof.** Let us compare the difference between $K_{E_1} - \bar{\varphi}^{-1}K_Z$ and $\bar{\psi}^{-1}(K_{\mathbb{P}E_x} - \varphi^{-1}K_{M_0})$. We have

$$K_{E_1} - \bar{\varphi}^{-1}K_Z - \bar{\psi}^{-1}(K_{\mathbb{P}E_x} - \varphi^{-1}K_{M_0}) = K_{E_1} - \bar{\psi}^{-1}K_{\mathbb{P}E_x} - \bar{\varphi}^{-1}(K_Z - \psi^{-1}K_{M_0})$$

The relation between $Z$ and $M_0$ was studied in [7] [16] [32]. The map $\psi$ is an isomorphism on $M_0'$. Therefore the difference of the canonical divisors of $Z$ and $M_0 \setminus \mathcal{X}_0$ is a multiple of the divisor $\psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$. Kiem and Li computed discrepancy of the desingularization $H_0 \rightarrow M_0$ in [16] (note that over $M_0 \setminus \mathcal{X}_0$, $H_0$ is isomorphic to the desingularization constructed by Seshadri in [35] and we see that the coefficient of $\psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$ in $K_Z - \psi^{-1}K_{M_0}$ is $g - 2$. From general argument (cf. [10] Proposition IV-21) we see that the morphism $\mathcal{O} \rightarrow \mathbb{P}E_x \setminus \mathcal{X}_0$ is the blow-up of $\psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$. See [31] Theorem 8.14] and [32] Lemma 3.7] for more details. The codimension of $\varphi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$ is $3g - 2 - (2g - 1) = g - 1$. From [17] we see that the discrepancy of $\bar{\psi}$ is $g - 2$. Therefore we obtain the desired conclusion. 

Now we compute cohomology groups of $H^i(M, E \otimes E_i \otimes E_1^* \otimes E_0^*)$. We prove the following Theorem.
Theorem 4.3. (1) If one of the four points \(x, y, z, w \in X\) is different from all the others, then we have

\[ H^i(M, E_x \otimes E_y \otimes E_z^* \otimes E_w^*) = 0 \]

for every \(i \geq 3\).

(2) For two distinct points \(x, z \in X\) we have

\[ H^i(M, E_x \otimes E_x \otimes E_z^* \otimes E_z^*) = \begin{cases} 
\mathbb{C} & \text{if } i = 0, \\
\mathbb{C}^2 & \text{if } i = 1, \\
\mathbb{C} & \text{if } i = 2, \\
0 & \text{if } i \geq 3.
\end{cases} \]

(3) For \(x \in X\) we have

\[ H^i(M, E_x \otimes E_x \otimes E_x^* \otimes E_x^*) = \begin{cases} 
\mathbb{C}^2 & \text{if } i = 0, \\
\mathbb{C}^3 & \text{if } i = 1, \\
\mathbb{C} & \text{if } i = 2, \\
0 & \text{if } i \geq 3.
\end{cases} \]

Sometimes, it is convenient to assume that determinant line bundles of vector bundles which \(M\) is parametrizing are isomorphic to \(\mathcal{O}(x)\). We need the following Lemma.

Lemma 4.4. Let \(M\) (resp. \(M'\)) be the moduli space of stable vector bundle of rank 2 and determinant isomorphic to a fixed line bundle \(L\) (resp. \(L'\)) of degree 1 on \(X\) and \(E\) (resp. \(E'\)) be the normalised Poincaré bundle on \(X \times M\) (resp. \(X \times M'\)). Let \(x, y, z, w\) be four (not necessarily distinct) points of \(X\). Then we have the following isomorphism.

\[ H^i(M, E_x \otimes E_y \otimes E_z^* \otimes E_w^*) \cong H^i(M', E'_x \otimes E'_y \otimes E'_z^* \otimes E'_w^*) \]

for every \(i\).

Proof. Consider the bundle \(E \otimes p_X^* \eta\) where \(\eta\) is a square root of \(L^{-1} \otimes L'\). From the universal property, this bundle defines an isomorphism \(u : M \to M'\) such that \(E \otimes p_X^* \eta \simeq (id \times u)^* E'\) because both \(E\) and \(E'\) are normalised Poincaré bundles. Therefore there is an isomorphism

\[ E_x \otimes E_y \otimes E_z^* \otimes E_w^* \simeq u^*(E'_x \otimes E'_y \otimes E'_z^* \otimes E'_w^*) \]

and from this isomorphism we have

\[ H^i(M, E_x \otimes E_y \otimes E_z^* \otimes E_w^*) \cong H^i(M', E'_x \otimes E'_y \otimes E'_z^* \otimes E'_w^*). \]

\[ \square \]

From now on we assume that \(M\) is the moduli space of rank 2 stable vector bundles on \(X\) whose determinants are isomorphic to \(\mathcal{O}(x)\).
4.1. Cohomology groups $H^i(M, E_i \otimes E_{j} \otimes E_{m}^*)$ and its consequences. Let $x$ be a point in $X$ which is different from $y, z, \psi \in X$. First, we need the following Lemma.

**Lemma 4.5.** We have $\tilde{\phi}_x \tilde{\psi}_x^* \pi^* (E_x \otimes E_{m}^*) = 0$.

**Proof.** Let $m_0$ be a point in $M_{j_0}$. When we restrict $\pi^*(E_x \otimes E_{m}^*)$ to $\varphi^{-1}(m_0)$ is $\mathcal{O}(-1)^{\oplus 8}$ (cf. [27] Proposition 3.4) hence $\phi_x \pi^*(E_x \otimes E_{m}^*) = 0$ over $M_{j_0}$. By [14] Corollary 1.9, $\tilde{\phi}_x \tilde{\psi}_x^* \pi^* (E_x \otimes E_{m}^*)$ is torsion free on $Z$, hence it is zero. □

Next we need to compute $R^1 \tilde{\phi}_x \tilde{\psi}_x^* \pi^* (E_x \otimes E_{m}^*)$. Note that it is supported on $\psi^{-1}(X \setminus X_0)$. Let $\mathcal{P}$ be the Poincaré bundle on $X \times J$ and $\pi_J : X \times J \to J$ be the projection to $J$. In order to describe $R^1 \tilde{\phi}_x \tilde{\psi}_x^* \pi^* (E_x \otimes E_{m}^*)$, let consider the following situation.

![Diagram](image)

We have the following sequence.

$$0 \to \mathcal{P} \otimes \pi_1^* \mathcal{O}(-x) \to \mathcal{P} \otimes \mathcal{O}_{x \times J} \to 0$$

By pushforwarding the above sequence to $J$ we have the following short exact sequence and let $\mathcal{E}_+ \cong R^1 \pi_{J*} (\mathcal{P} \otimes \pi_1^* \mathcal{O}(-x))$ and $\mathcal{E}_- \cong R^1 \pi_{J*} (\mathcal{P} \otimes \pi_2^* \mathcal{O})$.

$$0 \to \mathcal{E}_+ \to \mathcal{E}_- \to 0$$

Similarly, we have the following sequence and let $\mathcal{E}_- \cong R^1 \pi_{J*} (\mathcal{P} \otimes \pi_1^* \mathcal{O}(-x))$ and $\mathcal{E}_- \cong R^1 \pi_{J*} (\mathcal{P} \otimes \pi_2^* \mathcal{O})$.

$$0 \to \mathcal{E}_+ \to \mathcal{E}_- \to 0$$

Let $J' = J \setminus X_0$ and let us consider the above bundles over $J'$. (We will use the same notation to denote the restricted bundles over $J'$. Then the lift of $D^i$ to $J'$ is isomorphic to $\mathbb{P}(\mathcal{E}_+^\vee) \times_J \mathbb{P}(\mathcal{E}_-^\vee)$ and the lift of $\phi^{-1}$ to $J'$ is isomorphic to the union of $\mathbb{P}(\mathcal{E}_+^\vee)$ and $\mathbb{P}(\mathcal{E}_-^\vee)$. Let us consider the following diagram.

![Diagram](image)

Over $X \times \mathbb{P}(\mathcal{E}_+^\vee)$ we have the following sequence.

$$0 \to \mathcal{P} \otimes \mathcal{E}_+^* \mathcal{O}_{\mathbb{P}(\mathcal{E}_+) \times x} (1) \to (1 \times \pi)^* E |_{X \times \mathbb{P}(\mathcal{E}_+^\vee)} \to \mathcal{P}^{-1} \otimes \pi_1^* \mathcal{O}(x) \to 0$$

where we use abuse of notation by using $\mathcal{P}$ to denote the pullback of $\mathcal{P}$ from $X \times J'$ to $X \times \mathbb{P}(\mathcal{E}_+^\vee)$ and also $(1 \times \pi)^* E$ to denote the pullback of the $(1 \times \pi)^* E$ on $X \times \mathbb{P}E_x$ to $X \times \mathbb{P}(\mathcal{E}_+^\vee)$. Note that $\mathbb{P}(\mathcal{E}_+^\vee)$ parametrizes stable vector bundles on $X$ whose determinants are isomorphic to $\mathcal{O}(x)$.

Similarly, we have the following sequence.

$$0 \to \mathcal{P} \otimes \mathcal{E}_-^* \mathcal{O}_{\mathbb{P}(\mathcal{E}_-) \times x} (1) \to (1 \times \pi)^* E |_{X \times \mathbb{P}(\mathcal{E}_-^\vee)} \to \mathcal{P} \otimes \pi_2^* \mathcal{O}(x) \to 0$$

over $X \times \mathbb{P}(\mathcal{E}_-^\vee)$ where we use abuse of notation by using $\mathcal{P}$ to denote the pullback of $\mathcal{P}$ from $X \times J'$ to $X \times \mathbb{P}(\mathcal{E}_-^\vee)$ and also $(1 \times \pi)^* E$ to denote the pullback of the $(1 \times \pi)^* E$ on $X \times \mathbb{P}E_x$ to $X \times \mathbb{P}(\mathcal{E}_-^\vee)$. Note that
\( \mathbb{P}(E^*_x) \) parametrizes stable vector bundles on \( X \) whose determinants are isomorphic to \( \mathcal{O}(x) \).

The map \( \text{Bl}_{\mathbb{P}(\mathcal{O}^* \oplus \mathcal{O}(1))} \mathbb{P}(E^*_x) \to \mathbb{P}(\mathcal{F}^*_x) \) gives a \( \mathbb{P}^1 \)-fibration parametrized by \( \mathbb{P}(\mathcal{F}^*_x) \). Similarly, the map \( \text{Bl}_{\mathbb{P}(\mathcal{O}^* \oplus \mathcal{O}(1))} \mathbb{P}(E^*_x) \to \mathbb{P}(\mathcal{F}^*_x) \) gives a \( \mathbb{P}^1 \)-fibration parametrized by \( \mathbb{P}(\mathcal{F}^*_x) \). The union of the two fibrations is isomorphic to the lift of \( \phi \) to \( D \to D' \). Over a point \( j \in J \), we can describe the geometry as follows.

Let \( V \) be a \( n+1 \)-dimensional vector space and \( l \) be a 1-dimensional subspace in \( V \). There is a natural projection \( \pi_l : PV \to \mathbb{P}(V/l) \) be the point \([l] \in PV \) corresponding to \( l \) and we can make the projection into morphism by taking blow-up \( PV \) at \([l] \in PV \). Let \( E_l \) be the exceptional divisor of the blow-up. We use abuse of notation to denote the morphism \( \pi_l : \text{Bl}_{[l]} PV \to \mathbb{P}(V/l) \). The following description is standard.

**Lemma 4.6.** The morphism \( \pi_l : \text{Bl}_{[l]} PV \to \mathbb{P}(V/l) \) is isomorphic to the natural projection from projective bundle \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(V/l) \). The projection induces an isomorphism between \( E_l \) and \( \mathbb{P}(V/l) \).

**Proof.** We can use standard coordinate system of \( \mathbb{P}V \) and definition of blow-up to obtain the claim. \( \square \)

We can compute push forward of some sheaves using the above description.

**Lemma 4.7.** Let \( H \) be the pullback of the hyperplane of \( \mathbb{P}V \) of the blow-up. Then \( R\pi_* \mathcal{O}(-2H) \) is isomorphic to \( \mathcal{O}(-1)[-1] \).

**Proof.** From the short exact sequence
\[
0 \to \mathcal{O} \to \mathcal{O}(E_l) \to \mathcal{O}_{E_l}(E_l) \to 0
\]
we have the following exact sequence.
\[
0 \to \mathcal{O}(-2H) \to \mathcal{O}(-2H + E_l) \to \mathcal{O}_{E_l}(E_l) \to 0
\]

Because restriction of \( \mathcal{O}(-2H + E_l) \) to each fiber is \( \mathcal{O}(-1) \) we see that \( R\pi_* \mathcal{O}(-2H + E_l) \) is 0. By pushing forward the second exact sequence we have \( R\pi_* \mathcal{O}(-2H) \) is isomorphic to \( \mathcal{O}(-1)[-1] \). \( \square \)

From the above consideration we have the following isomorphisms.

**Proposition 4.8.** Let \( j \oplus j^{-1} \in \mathcal{X} \setminus \mathcal{X}_0 \) and consider the fiber \( \psi^{-1}(j \oplus j^{-1}) \cong \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \). For three points \( y, z, w \in X \) which are distinct from \( x \), the restriction of \( R^1 \phi_*(\overline{\psi}^* \pi^*(E_y \oplus E^*_y \oplus E^*_w)) \) on the fiber is isomorphic to \( \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0) \).

**Proof.** Let us consider the following Mayer-Vietoris sequence where \( l_1 \cup l_2 \) is the fiber of \( \overline{\psi} \) over a point in \( \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \) over \( j \oplus j^{-1} \) and \( S \) is the restriction of \( \overline{\psi}^* \pi^*(E_y \oplus E^*_y \oplus E^*_w) \) to \( l_1 \cup l_2 \).
\[
0 \to H^0(l_1 \cup l_2, S) \to H^0(l_1, S|_{l_1}) \oplus H^0(l_2, S|_{l_2}) \to H^0(l_1 \cap l_2, S|_{l_1 \cap l_2})
\]
\[
\to H^1(l_1 \cup l_2, S) \to H^1(l_1, S|_{l_1}) \oplus H^1(l_2, S|_{l_2}) \to H^1(l_1 \cap l_2, S|_{l_1 \cap l_2}) \to 0
\]
The morphism \( H^0(l_1, S|_{l_1}) \oplus H^0(l_2, S|_{l_2}) \to H^0(l_1 \cap l_2, S|_{l_1 \cap l_2}) \) is as follows.
\[
\mathcal{H}^0((j_y \oplus \mathcal{O}_l(1) \oplus j^{-1}_y) \oplus (j_z \oplus j^{-1}_z \oplus \mathcal{O}_l(-1)) \oplus (j_w \oplus j^{-1}_w \oplus \mathcal{O}_l(-1)))
\]
\[
\oplus \mathcal{H}^0((j_y^{-1} \oplus \mathcal{O}_l(1) \oplus j_y) \oplus (j_z^{-1} \oplus j_z \oplus \mathcal{O}_l(-1)) \oplus (j_w^{-1} \oplus j_w \oplus \mathcal{O}_l(-1)))
\]
\[
\to (j_y \oplus j^{-1}_y) \oplus (j_z \oplus j^{-1}_z) \oplus (j_w \oplus j^{-1}_w)
\]
This map is surjective and the kernel is generated by \( j_y \odot j_z \odot j_w \odot j_y^{-1} \odot j_z^{-1} \odot j_w^{-1} \). Therefore we see that \( H^1(l_1 \cup l_2, S) \cong C^2 \) and isomorphic to \( H^1(l_1, S|_{l_1}) \oplus H^1(l_2, S|_{l_2}) \). It is isomorphic to the following cohomology group.

\[
H^1((j_y \odot \mathcal{O}_{l_1}(1) \odot j_y^{-1}) \odot (j_z \odot j_z^{-1} \odot \mathcal{O}_{l_1}(-1)) \odot (j_w \odot j_w^{-1} \odot \mathcal{O}_{l_1}(-1)))
\]

\[\oplus H^1((j_y^{-1} \odot \mathcal{O}_{l_1}(1) \odot j_y) \odot (j_z^{-1} \odot j_z \odot \mathcal{O}_{l_1}(-1)) \odot (j_w^{-1} \odot j_w \odot \mathcal{O}_{l_1}(-1)))\]

Therefore we have a canonical isomorphism \( H^1(l_1 \cup l_2, S) \cong j_y \odot j_z^{-1} \odot j_y^{-1} \odot j_z \odot j_w \).

Let us discuss about the morphism \( \phi \). See [28] for more details. Let \( z \in \psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0) \) such that \( \psi(z) = j \odot j^{-1} \) and then \( z \) corresponds to a point \((a, b) \in \mathbb{P}H^1(X, j^{02}) \times \mathbb{P}H^1(X, j^{0-2}) \). Then \( \psi^{-1}(z) \) is a pair of lines one in \( \mathbb{P}_j = \mathbb{P}H^1(X, j^{02} \odot \mathcal{O}(−x)) \) and another in \( \mathbb{P}_{j^{-1}} = \mathbb{P}H^1(X, j^{0-2} \odot \mathcal{O}(−x)) \). And all the lines are passing through \( \mathbb{P}_j \cap \mathbb{P}_{j^{-1}} \). Let us fix \( b \) and vary \( a \in \mathbb{P}H^1(X, j^{02}) \). Then the lines parametrized by \( \mathbb{P}H^1(X, j^{02}) \) are proper transforms of lines in \( \mathbb{P}_j \) passing through the point \( \mathbb{P}_j \cap \mathbb{P}_{j^{-1}} \). Applying the previous Lemma to the family of degenerating conics we obtain the desired conclusion.

\[
\square
\]

**Corollary 4.9.** We have the following vanishing

\[
R \psi, R^1 \phi_* (\psi^* \pi^* (E_y \odot E_z^* \odot E_w^*)) = 0
\]

**Proof.** From the above description of \( R^1 \phi_* (\pi^* (E_y \odot E_z^* \odot E_w^*)) \) we see that its derived pushforward to \( \mathcal{X} \setminus \mathcal{X}_0 \) is 0. Hence we obtain the desired result. \( \square \)

**Proposition 4.10.** If one of the four points \( x, y, z, w \in X \) is different from all the others, then we have

\[
H^1(M, E_x \odot E_y \odot E_z^* \odot E_w^*) = 0
\]

for every \( i \).

**Proof.** Let \( x \) be a point in \( X \) which is different from \( y, z, w \in X \). We have the following isomorphisms for \( i \leq 2g − 3 \).

\[
H^i(M, E_x \odot E_y \odot E_z^* \odot E_w^*) \cong H^i(PE_x, \mathcal{O}(1) \odot \pi^* (E_y \odot E_z^* \odot E_w^*)) \cong H^i(PE_x, \phi^* \theta_0 \odot \pi^* (E_y \odot E_z^* \odot E_w^*))
\]

\[
\cong H^i(PE_x \setminus \phi^{-1} \mathcal{X}_0, \phi^* \theta_0 \odot \pi^* (E_y \odot E_z^* \odot E_w^*)) \cong H^i(\mathcal{X} \setminus \mathcal{X}_0, \phi^* \theta_0 \odot \psi^* \pi^* (E_y \odot E_z^* \odot E_w^*))
\]

\[
\cong H^i(\mathcal{X}, \varphi^* \pi^* (E_y \odot E_z^* \odot E_w^*)) \cong H^{i-1}(Z, \psi^* \theta_0 \odot R^1 \phi_* \psi^* \pi^* (E_y \odot E_z^* \odot E_w^*))
\]

The last isomorphism comes from the computation \( \tilde{\phi}, \tilde{\psi} \pi^* (E_y \odot E_z^* \odot E_w^*) = 0 \). From the above Corollary and Leray spectral sequence, we obtain the desired vanishing for \( i \leq 2g − 3 \). From Proposition 3.8 and from the inclusion \( H^{2g-2}(M, E_x \odot E_y \odot E_z^* \odot E_w^*) \rightarrow H^{2g-2}(PE_x \setminus \phi^{-1} \mathcal{X}_0, \mathcal{O}(1) \odot \pi^* (E_y \odot E_z^* \odot E_w^*)) \), we have desired result. \( \square \)
4.2. Computation of $H^i(M, E_x \otimes E_x \otimes E_x^* \otimes E_x^*)$ and its consequences. In order to compute cohomology groups $H^i(M, E_x \otimes E_x \otimes E_x^* \otimes E_x^*)$, we need to study coherent sheaf $\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z$ and its dual.

**Lemma 4.11.** The coherent sheaf $\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z$ is a rank 3 vector bundle.

**Proof.** When $m \in M_0^*$, the inverse image of $\tilde{\phi}$ is a smooth conic and the restriction of $\tilde{\psi}^* \pi^* \text{ad} E_z$ over the conic is isomorphic to $O^{\oplus 3}$. Therefore $\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z$ is a rank 3 vector bundle on $\psi^{-1}(M_0^*)$.

Now let us analyze $\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z$ over $\psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$. Let us consider the following Mayer-Vietoris sequence where $l_1 \cup l_2$ is the fiber of $\tilde{\psi}$ over a point in $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$ over $f \oplus j^{-1}$ and $S$ is the restriction of $\tilde{\psi}^* \pi^* \text{ad} E_z$ to $l_1 \cup l_2$.

$$0 \to H^0(l_1 \cup l_2, S) \to H^0(l_1, S|_{l_1}) \oplus H^0(l_2, S|_{l_2}) \to H^0(l_1 \cap l_2, S|_{l_1 \cap l_2})$$

$$\to H^1(l_1 \cup l_2, S) \to H^1(l_1, S|_{l_1}) \oplus H^1(l_2, S|_{l_2}) \to H^1(l_1 \cap l_2, S|_{l_1 \cap l_2}) \to 0$$

Let us analyze the morphism $H^0(l_1, S|_{l_1}) \oplus H^0(l_2, S|_{l_2}) \to H^0(l_1 \cap l_2, S|_{l_1 \cap l_2})$. It is isomorphic to $\mathbb{C}^{\oplus 2} \oplus j^{\oplus 2}_{l_2} \oplus j^{\oplus 2}_E \oplus j^{\oplus 2}_E \to \mathbb{C}^{\oplus 3} \oplus j^{\oplus 2}_E \oplus j^{\oplus 2}_E$ and surjective. Therefore we have $H^0(l_1 \cup l_2, S) \cong \mathbb{C}^{\oplus 3} \oplus j^{\oplus 2}_E \oplus j^{\oplus 2}_E$ and $H^1(l_1 \cup l_2, S) \cong 0$.

Therefore we see that $\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z$ is a rank 3 vector bundle. □

**Proposition 4.12.** Let $\pi_1 : Bl_l \mathbb{P}V \to \mathbb{P}(V/l)$ be the morphism as before. Let $H$ be the pullback of the hyperplane of $\mathbb{P}V$ of the blow-up. Then $R^n \pi_1_* \mathcal{O}(H)$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$.

**Proof.** From the short exact sequence

$$0 \to \mathcal{O}(-E_i) \to \mathcal{O} \to \mathcal{O}_E \to 0$$

we have the following exact sequence.

$$0 \to \mathcal{O}(H - E_i) \to \mathcal{O}(H) \to \mathcal{O}_E \to 0$$

Because $H^0(\mathcal{O}(H - E_i)) \cong \mathbb{C}^n$ and $\pi_1_* \mathcal{O}(H - E_i)$ is a line bundle on $\mathbb{P}(V/l)$ we see that $\pi_1_* \mathcal{O}(H - E_i) \cong \mathcal{O}(1)$. By pushing forward the second exact sequence via $\pi_1$ we have the following sequence

$$0 \to \mathcal{O}(1) \to \pi_1_* \mathcal{O}(H) \to \mathcal{O} \to 0$$

and $R^n \pi_1_* \mathcal{O}(H) = 0$ for $i \geq 1$. Because $\text{Ext}^1(H, \mathcal{O}(1)) = 0$, we have $R^n \pi_1_* \mathcal{O}(H) \cong \mathcal{O} \oplus \mathcal{O}(1)$. □

**Lemma 4.13.** We have following two injective morphism $\tilde{\phi}_* (\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z) \to \tilde{\psi}^* \pi^* \text{ad} E_z$ and $\tilde{\psi}^* \pi^* \text{ad} E_z \to \tilde{\phi}_* (\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z)^*$.  

**Proof.** From adjunction formula we have the following morphism.

$$\tilde{\phi}_* (\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z) \to \tilde{\psi}^* \pi^* \text{ad} E_z$$

It is injective since $\tilde{\phi}_* (\tilde{\phi}_* \tilde{\psi}^* \pi^* \text{ad} E_z)$ is a pure sheaf and it is injective over $M_0^*$. 
By taking dual to the above sequence we have

\[ \tilde{\psi}^* \pi^* \text{ad}E_z \to \tilde{\phi}^*(\tilde{\phi}_s \tilde{\psi}^* \pi^* \text{ad}E_z)^* \]

and see that it is also injective due to the same reason. \[ \square \]

By pushing forward the map

\[ \tilde{\psi}^* \pi^* \text{ad}E_z \to \tilde{\phi}^*(\tilde{\phi}_s \tilde{\psi}^* \pi^* \text{ad}E_z)^* \]

via \( \tilde{\phi} \) we have an injective map

\[ \tilde{\phi}_s \tilde{\psi}^* \pi^* \text{ad}E_z \to (\tilde{\phi}_s \tilde{\psi}^* \pi^* \text{ad}E_z)^* . \]

In order to describe its cokernel let us consider the following situation. Note that the cokernel is supported on \( D \) where \( D \) be the inverse image \( \psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0) \) of \( Z \).

From the above discussions we have the following Lemma.

**Lemma 4.14.** We have the following sequence.

\[ 0 \to \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}}^{-2} \otimes \mathcal{O}(1,0) \oplus \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}} \otimes \mathcal{O}(0,1) \to \tilde{\phi}_s \tilde{\psi}^* \pi^* \text{ad}E_z|_P \to \mathcal{O} \to 0 \]

**Proof.** When we restrict the following sequence on \( X \times \mathbb{P}(\mathcal{E}^*_{\mathcal{X}}) \)

\[ 0 \to \mathcal{R} \otimes \pi^*_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} \mathcal{O}_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})}(1) \to (1 \times \pi)^* E_{X \times \mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} \to \mathcal{R}^{-1} \otimes \pi^*_{\mathbb{X}} \mathcal{O}(x) \to 0 \]

to \( z \times \mathbb{P}(\mathcal{E}^*_{\mathcal{X}}) \) we have the following sequence.

\[ 0 \to \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}} \otimes \pi^*_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} \mathcal{O}_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})}(1) \to \pi^* \mathcal{E}_z|_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} \to \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}}^{-1} \to 0 \]

From the above sequence have the following diagram.

\[ \begin{array}{cccccc}
0 & \to & \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}}^{-2} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})}(1) & \to & \cdots & \to & \mathcal{O} & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \cdots & \to & \pi^* (\mathcal{E}_z \otimes \mathcal{E}^*_z)|_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} & \to & \cdots & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{O} & \to & \cdots & \to & \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})}(-1) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & 0 & \to & 0 & \\
\end{array} \]

Similarly, we have the following sequence

\[ 0 \to \mathcal{R}^{-1} \otimes \pi^*_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} \mathcal{O}_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})}(1) \to \pi^* \mathcal{E}_z|_{\mathbb{P}(\mathcal{E}^*_{\mathcal{X}})} \to \mathcal{P}_{\mathcal{E}^*_{\mathcal{X}}} \to 0 \]

and similar diagram. By pulling the above diagrams to \( \mathcal{C} \) and applying relative Mayer-Vietoris sequence we obtain the desired short exact sequence. \[ \square \]

By taking dual we have the following sequence.
Lemma 4.15. We have the following sequence.
\[ 0 \to \mathcal{O} \to (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})|_D \to \mathcal{R}_z^{-1} \otimes \mathcal{O}(0, -1) \oplus \mathcal{R}_z^{\otimes 2} \otimes \mathcal{O}(-1, 0) \to 0 \]

From the above descriptions, one can check that there is a canonical morphism \( \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z}|_D \to (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^*|_D \). We can complete the sequence as follows.

Lemma 4.16. We have the following exact sequence.
\[ 0 \to \mathcal{R}_z^{-1} \otimes \mathcal{O}(1, 0) \oplus \mathcal{R}_z^{\otimes 2} \otimes \mathcal{O}(0, 1) \to (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^*|_D \to \mathcal{R}_z^{-1} \otimes \mathcal{O}(0, -1) \oplus \mathcal{R}_z^{\otimes 2} \otimes \mathcal{O}(-1, 0) \to 0 \]

Proof. By composing \( \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z}|_D \to \mathcal{O} \) and \( \mathcal{O} \to (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^*|_D \) we have the following diagram.

From the snake lemma, we have the desired exact sequence. \( \square \)

From the above discussion, we can compare the cohomology groups of \( H^i(Z, \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z}) \) and \( H^i(Z, (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^\vee) \).

Lemma 4.17. We have the following isomorphism
\[ H^i(Z, \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z}) \cong H^i(Z, (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^\vee) \]
for all \( i \).

Proof. Because the cokernel of the map \( 0 \to \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z} \to (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^\vee \) is supported on \( D \) so we can complete the short exact sequence as follows.
\[ 0 \to \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z} \to (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^\vee \to \mathcal{R}_z^{-1} \otimes \mathcal{O}(0, -1) \oplus \mathcal{R}_z^{\otimes 2} \otimes \mathcal{O}(-1, 0) \]

Since the pushforward of \( \mathcal{R}_z^{-1} \otimes \mathcal{O}(0, -1) \oplus \mathcal{R}_z^{\otimes 2} \otimes \mathcal{O}(-1, 0) \) to \( \mathcal{X} \setminus \mathcal{X}_0 \) is zero so we obtain desired isomorphism. \( \square \)

Then we have the following computation.

Lemma 4.18. We have the following isomorphism
\[ H^i(\mathcal{E}, \bar{\psi}^* (T_\pi \otimes \pi^* \text{ad}_{E_z})) = 0 \]
for all \( i \).

Proof. We have the following isomorphisms
\[ H^i(\mathcal{E}, \bar{\psi}^* (T_\pi \otimes \pi^* \text{ad}_{E_z})) \cong H^i(\mathcal{E}, \omega_\phi \otimes \bar{\psi}^* \pi^* \text{ad}_{E_z}) \cong H^{i+1}(Z, (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^\vee) \]

From the previous isomorphism, we have
\[ H^{i+1}(Z, (\bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z})^\vee) \cong H^{i+1}(Z, \bar{\phi}, \bar{\psi}^* \pi^* \text{ad}_{E_z}) \cong H^{i+1}(\mathcal{E}, \bar{\psi}^* \pi^* \text{ad}_{E_z}) \cong H^{i+1}(M, \text{ad}_{E_z}) \]
and hence we obtain the desired result.

Therefore we obtain the desired cohomology computation.

**Proposition 4.19.** For two distinct points $x, z \in X$ we have

$$H^i(M, E_x \otimes E_z \otimes E_x^* \otimes E_z^*) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ \mathbb{C}^2 & \text{if } i = 1, \\ \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

**Proof.** First, we have the following isomorphism.

$$H^i(M, E_x \otimes E_z \otimes E_z^*) \cong H^i(M, E_x \otimes E_z \otimes E_z^*) \cong H^i(M, E_x \otimes E_z^*) \oplus H^i(M, \text{ad}E_x \otimes E_z \otimes E_z^*)$$

From $\pi_* T_\pi = \text{ad}E_x$, we see that

$$H^i(M, \text{ad}E_x \otimes E_z \otimes E_z^*) \cong H^i(\mathbb{P}E_x, T_\pi \otimes \pi^* E_z \otimes \pi^* E_z^*) \cong H^i(\mathbb{P}E_x, T_\pi \otimes \pi^* E_z \otimes \pi^* E_z^*)$$

for $i \leq 2g - 3$.

When $i = 0, 1, 2$ we can compute the cohomology groups from previous Lemmas. When $3 \leq i \leq 2g - 2$ we obtain desired vanishing from the inclusion $H^i(M, E_x \otimes E_z^*) \rightarrow H^i(M, \mathbb{P}E_x \setminus \varphi^{-1}(\mathcal{X}_0), T_\pi \otimes \pi^* E_z \otimes \pi^* E_z^*)$. □

From the above computation we have the following consequences.

**Proposition 4.20.** Let $x, z$ be two distinct points in $X$. Then $F_{x,z}$ is a simple vector bundle and $\text{Ext}^i_M(F_{x,z}, F_{x,z}) = 0$ for all $i \geq 3$.

**Proof.** When $x, z$ be two distinct points in $X$ then $F_{x,z} \cong E_x \otimes E_z$. From the above computation we see that $F_{x,z}$ is a simple vector bundle and $\text{Ext}^i_M(F_{x,z}, F_{x,z}) = 0$ for all $i \geq 3$ when $g$ is sufficiently large. □

**Proposition 4.21.** For two distinct points $(x,x), (z,z) \in X_2$ we have $\text{Ext}^i_M(F_{x,x}, F_{z,z}) = 0$ for all $i$.

**Proof.** We have the following exact sequences.

$$0 \rightarrow \bigwedge^2 E_x \rightarrow F_{x,x} \rightarrow \text{Sym}^2 E_x \rightarrow 0$$

$$0 \rightarrow \text{Sym}^2 E_z^* \rightarrow F_{z,z}^* \rightarrow \bigwedge^2 E_z^* \rightarrow 0$$
By tensoring the two exact sequence we have the following commutative diagram.

\[
\begin{array}{cccccc}
0 & \to & \wedge^2 E_x \otimes \text{Sym}^2 E^*_x & \to & F_{x,x} \otimes \text{Sym}^2 E^*_x & \to & \text{Sym}^2 E_x \otimes \text{Sym}^2 E^*_x & \to & 0 \\
0 & \to & \wedge^2 E_x \otimes F^*_x & \to & F_{x,x} \otimes F^*_x & \to & \text{Sym}^2 E_x \otimes F^*_x & \to & 0 \\
0 & \to & \wedge^2 E_x \otimes \wedge^2 E^*_x & \to & F_{x,x} \otimes \wedge^2 E^*_x & \to & \text{Sym}^2 E_x \otimes \wedge^2 E^*_x & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
\]

We have \(H^i(M, \text{Sym}^2 E_x \otimes \wedge^2 E^*_x) \cong H^i(P E_x, \pi^* \theta^{-1} \otimes O(2)) \cong H^i(P E_x, T \pi) \cong H^i(M, \text{ad} E_x)\) and can compute \(H^i(M, \wedge^2 E_x \otimes \text{Sym}^2 E^*_x)\) similarly. Moreover, from

\[
H^i(M, E_x \otimes E_x \otimes E^*_x \otimes E^*_x) = \begin{cases} 
\mathbb{C} & \text{if } i = 0, \\
\mathbb{C}^2 & \text{if } i = 0, \\
\mathbb{C} & \text{if } i = 0, \\
0 & \text{if } i \geq 3.
\end{cases}
\]

we see that

\[
H^i(M, \text{Sym}^2 E_x \otimes \text{Sym}^2 E^*_x) = \begin{cases} 
\mathbb{C} & \text{if } i = 2, \\
0 & \text{otherwise}.
\end{cases}
\]

Then by diagram chasing, we have \(H^i(M, F_{x,x} \otimes F^*_x) = 0\) for every \(i\). \(\square\)

4.3. **Computation of \(H^i(M, E_x \otimes E_x \otimes E^*_x \otimes E^*_x)\) and its consequences.** Now we compute the cohomology group \(H^i(M, E_x \otimes E_x \otimes E^*_x \otimes E^*_x)\). Let us consider the following diagram.

\[
\begin{array}{ccc}
P E_x & \xrightarrow{\phi} & M_0 \\
\pi & \downarrow & \downarrow \\
M & \to & M
\end{array}
\]

**Lemma 4.22.** We have the following short exact sequence.

\[
0 \to \pi^*(E_x \otimes \theta) \otimes O(-1) \to \pi^* \text{Sym}^2 E_x \to O(2) \to 0
\]

**Proof.** From adjunction formula \(\text{Hom}(\text{Sym}^2 E_x, \text{Sym}^2 E_x) \cong \text{Hom}(\pi^* \text{Sym}^2 E_x, O(2))\) we have a natural map \(\pi^* \text{Sym}^2 E_x \to O(2)\). This morphism is a surjection and we have an exact sequence of the following form when \(A\) is an object in \(D(M)\).

\[
0 \to \pi^* A \otimes O(-1) \to \pi^* \text{Sym}^2 E_x \to O(2) \to 0
\]

By twisting \(O(-1)\) we have

\[
0 \to \pi^* A \otimes O(-2) \to \pi^* \text{Sym}^2 E_x \otimes O(-1) \to O(1) \to 0
\]
From our computation of $\omega_{E_{x}/M}$ we have

$$R^1\pi_*\mathcal{O}(-2) \cong \theta^{-1}.$$  

Therefore we have $A \cong E_x \otimes \theta$ by applying $R\pi_*$ to the above sequence. \qedhere

By taking dual we have the following.

**Lemma 4.23.** We have the following short exact sequence.

$$0 \to \mathcal{O}(-2) \to \pi^*\text{Sym}^2E_x^* \to \pi^*(E_x^* \otimes \theta^{-1}) \otimes \mathcal{O}(1) \to 0$$

From the above short exact sequences we can compute the following cohomology groups.

**Lemma 4.24.** For $x \in X$ we have

$$H^i(M, \text{Sym}^2E_x \otimes \theta^{-1}) = \begin{cases} 
0 & \text{if } i = 0, \\
\mathbb{C} & \text{if } i = 1, \\
0 & \text{if } i = 2, \\
0 & \text{if } i \geq 3.
\end{cases}$$

**Proof.** From the following isomorphisms

$$H^i(M, \text{Sym}^2E_x \otimes \theta^{-1}) \cong H^i(M, \text{ad}E_x)$$

and computations in [30] we can compute the cohomology groups. \qedhere

**Lemma 4.25.** For $x \in X$ we have

$$H^i(M, \theta \otimes \text{Sym}^2E_x^*) = \begin{cases} 
0 & \text{if } i = 0, \\
\mathbb{C} & \text{if } i = 1, \\
0 & \text{if } i = 2, \\
0 & \text{if } i \geq 3.
\end{cases}$$

**Proof.** From the following isomorphisms

$$H^i(M, \theta \otimes \text{Sym}^2E_x^*) \cong H^i(M, \text{ad}E_x)$$

and computations in [30] we can compute the cohomology groups. \qedhere

Recall that $D$ is the inverse image $\psi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$ of $Z$.

**Proposition 4.26.** Let $\bar{\varphi} : \mathcal{E} \to Z$ be the conic bundle and $W = R^1\bar{\varphi}_*\omega^\otimes_\varphi^{\otimes 2}$. Then $c_1(W) \cong D$.

**Proof.** From Grothendieck-Riemann-Roch, we see that (cf. [13 page 155]. [26 page 302])

$$\text{ch}(\bar{\varphi}_*\omega^\otimes_\varphi^{\otimes 2}) \cong \bar{\varphi}_*(\text{ch}(\omega^\otimes_\varphi^{\otimes 2}) \cdot \text{td}(T_{\bar{\varphi}}))$$

$$\cong \bar{\varphi}_*((1 + c_1(\omega^\otimes_\varphi^{\otimes 2}) + \frac{1}{2}\omega^\otimes_\varphi^{\otimes 2}) \cdot (1 - \frac{1}{2}\omega^\otimes_\varphi^{\otimes 2}) + \frac{1}{12}(c_1(\omega^\otimes_\varphi^{\otimes 2})^2 + D) + \cdots))$$

$$\cong \bar{\varphi}_*(\frac{3}{2}c_1(\omega^\otimes_\varphi) + (c_1(\omega^\otimes_\varphi)^2 + \frac{1}{12}(c_1(\omega^\otimes_\varphi)^2 + D) + \cdots))$$

where we use abuse of notation so that $D$ denotes the locus where the fiber of $\bar{\varphi}$ has nodal singularity. Because $R^i\bar{\varphi}_*\omega^\otimes_\varphi^{\otimes 2} = 0$ for $i \neq 1$ and $R^1\bar{\varphi}_*\omega^\otimes_\varphi^{\otimes 2} = W$ is a rank 3 bundle we have

$$c_1(W) \cong -\bar{\varphi}_*(c_1(\omega^\otimes_\varphi)^2 + \frac{1}{12}(c_1(\omega^\otimes_\varphi)^2 + D)).$$
From $R^1\bar{\varphi}_*\omega_\varphi \cong \mathscr{O}$ and

\[ \text{ch}(\bar{\varphi}_*\omega_\varphi) \cong \bar{\varphi}_*(\text{ch}(\omega_\varphi) \cdot \text{td}(T_\varphi)) \]

\[ \cong \bar{\varphi}_*((1 + c_1(\omega_\varphi) + \frac{1}{2} c_1(\omega_\varphi)^2 + \cdots) \cdot (1 - \frac{1}{2} c_1(\omega_\varphi) + \frac{1}{12} c_1(\omega_\varphi)^2 + D) + \cdots) \]

we see that $\bar{\varphi}_*(\frac{1}{12}(c_1(\omega_\varphi)^2 + D)) = 0$. Therefore we see that $c_1(W) \cong D$. \hfill \Box

**Proposition 4.27.** The line subbundle of $S^2W^*$ defining conic bundle $\mathcal{C} \subset \mathbb{P}(W)$ is

$\mathscr{O}(-D) \subset S^2W^*$.

**Proof.** Let $q : S^2W \rightarrow \mathcal{L}^*$ be the quadratic form defining the conic bundle $\mathcal{C}$. The discriminant $\det q$ of $q$ is a section of the line bundle $(\wedge^3 W^*)^\otimes 2 \otimes \mathcal{L}^* \otimes 3$ whose 1st Chern class is $D$. From the above Proposition, we see that $\mathcal{L}^* \cong \mathscr{O}(D)$. \hfill \Box

From the quadratic form $q : S^2W \rightarrow \mathcal{L}^*$ we have the map $\bar{\varphi} : W \rightarrow W^* \otimes \mathcal{L}^*$. Then we have the following short exact sequence.

\[ 0 \rightarrow W \rightarrow W^* \otimes \mathcal{L}^* \rightarrow \text{coker } \bar{\varphi} \rightarrow 0 \]

Let $\bar{D} = \bar{\psi}^{-1}(D)$. We have the following exact sequence

\[ 0 \rightarrow \mathscr{O} \rightarrow \mathscr{O}(\bar{D}) \rightarrow \mathscr{O}_D(\bar{D}) \rightarrow 0 \]

which is pullback of

\[ 0 \rightarrow \mathscr{O} \rightarrow \mathscr{O}(D) \rightarrow \mathscr{O}_D(D) \rightarrow 0 \]

since $\bar{\psi}$ is a flat morphism. Then we have the following.

**Proposition 4.28.** We have the following isomorphisms.

(1)

\[ H^i(Z, W^* \otimes \mathcal{L}^*) = \begin{cases} \mathbb{C} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases} \]

(2)

\[ H^i(Z, \text{coker } \bar{\varphi}) = 0, \ i \in \mathbb{Z}. \]

(3)

\[ H^i(Z, W) = \begin{cases} \mathbb{C} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases} \]

**Proof.** From relative duality we see that $W^* \cong \bar{\varphi}_*\omega_\varphi^{-1}$. From the following sequences of isomorphisms, $H^i(Z, W^*) \cong H^i(\mathcal{C}, \omega_\varphi^{-1}) \cong H^i(\mathcal{P}E_\varphi, \omega_\varphi^{-1}) \cong H^i(\mathcal{P}E_\varphi, \omega_\varphi^{-1}) \cong H^{i-1}(M, \mathscr{O})$ we have

\[ H^i(\mathcal{C}, \omega_\varphi^{-1}) = \begin{cases} \mathbb{C} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases} \]

Let us consider the following short exact sequence.

\[ 0 \rightarrow \omega_\varphi^{-1} \rightarrow \omega_\varphi^{-1}(\bar{D}) \rightarrow \omega_\varphi^{-1} \otimes \mathscr{O}_D(\bar{D}) \rightarrow 0 \]
Note that $\mathcal{O}_D(\tilde{D})$ is the pull-back of $\mathcal{O}_D(D)$ so it is isomorphic to $\mathcal{O}(-1, -1)$ on each $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ (cf. [1][2]). Therefore we have $\bar{\psi}_*(\mathcal{O}_\phi^{-1} \otimes \mathcal{O}_D(\tilde{D})) \cong \mathcal{O}_\phi^{-1} \otimes \bar{\psi}_*(\mathcal{O}_D(D)) = 0$. Therefore we have

$$H^i(Z, W^* \otimes \mathcal{L}^*) \cong H^i(\mathcal{E}', \mathcal{O}_\phi^{-1}(\tilde{D})) \cong H^i(\mathcal{E}', \mathcal{O}_\phi^{-1})$$

and hence obtain the desired isomorphisms.

For (2), we know that $\text{coker } \tilde{q} \otimes \mathcal{L}$ is 2-torsion line bundle on $D$. Then $\text{coker } \tilde{q}$ is isomorphic to 2-torsion line bundle on $D$ tensored by the normal bundle $\mathcal{O}_D(D)$ and hence its restriction to $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ is isomorphic to $\mathcal{O}(-1, -1)$. Therefore when we see that the pushforward $\text{coker } \tilde{q}$ to $\mathcal{X} \setminus \mathcal{X}_0$ is 0 and hence obtain the desired result.

From the cohomology long exact sequence and (1), (2) we obtain the desired isomorphisms for (3). □

From the above discussions we have the following isomorphisms.

**Lemma 4.29.** For $x \in X$ we have

$$H^i(M, \text{Sym}^2 E_x \otimes \text{Sym}^2 E_x^*) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ \mathbb{C} & \text{if } i = 1, \\ \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

**Proof.** By tensoring $\mathcal{O}(2)$ to the following sequence

$$0 \to \mathcal{O}(-2) \to \pi^* \text{Sym}^2 E_x^* \to \pi^*(E_x^* \otimes \theta^{-1}) \otimes \mathcal{O}(1) \to 0$$

we have

$$0 \to \mathcal{O} \to \pi^* \text{Sym}^2 E_x^* \otimes \mathcal{O}(2) \to \pi^*(E_x^* \otimes \theta^{-1}) \otimes \mathcal{O}(3) \to 0$$

Note that $\pi^*(E_x^* \otimes \theta^{-1}) \otimes \mathcal{O}(3) \cong \pi^* E_x^* \otimes \mathcal{O}(1) \otimes T_\pi$ and consider the following short exact sequence.

$$0 \to T_\pi \to \pi^* E_x^* \otimes \mathcal{O}(1) \otimes T_\pi \to T_\pi^{\mathcal{O}_2} \to 0$$

Therefore it remains to compute $H^i(\mathbb{P}E_x, T_\mathcal{X}^{\mathcal{O}_2})$. We have the following isomorphisms

$$H^i(\mathbb{P}E_x, T_\mathcal{X}^{\mathcal{O}_2}) \cong H^i(\mathcal{E}', \bar{\psi}^* T_\mathcal{X}^{\mathcal{O}_2}) \cong H^i(\mathcal{E}', \mathcal{O}_\phi^{-2}) \cong H^{i-1}(Z, W)$$

and hence obtain the desired result. □

**Proposition 4.30.** For $x \in X$ we have

$$H^i(M, E_x \otimes E_x \otimes E_x^* \otimes E_x^*) = \begin{cases} \mathbb{C}^2 & \text{if } i = 0, \\ \mathbb{C}^3 & \text{if } i = 1, \\ \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

**Proof.** First, we have the following isomorphism.

$$H^i(M, E_x \otimes E_x \otimes E_x^* \otimes E_x^*) \cong H^i(M, \mathcal{O}) \otimes H^i(M, \theta \otimes \text{Sym}^2 E_x^*) \otimes H^i(M, \text{Sym}^2 E_x \otimes \theta^{-1}) \otimes H^i(M, \text{Sym}^2 E_x \otimes \text{Sym}^2 E_x^*)$$

Then we obtain desired isomorphism via the above discussions. □

From the above computation of cohomology groups, we obtain the following conclusions.

**Proposition 4.31.** $F_{x, \alpha}$ is a simple vector bundle.
Proof. We slightly modified the arguments used in [29, Lemma 4.1] and [6, Lemma 4.2]. Let us recall $F_{x,x}$ can be obtained from the following nontrivial extension.

$$0 \to \bigwedge^2 E_x \to F_{x,x} \to \text{Sym}^2 E_x \to 0$$

Suppose that $F_{x,x}$ is not simple. Then there is a nontrivial homomorphism $\alpha : F_{x,x} \to F_{x,x}$ whose rank is less than 4. If $\alpha(\theta) = 0$, then the above short exact sequence splits. Therefore we see that $\alpha(\theta) \neq 0$. And since the composition map $\theta \to F_{x,x} \to F_{x,x} \to \text{Sym}^2 E_x$ is zero, the image of $\theta$ under $\alpha$ is $\theta$. Similarly, we see that $\alpha$ induces a homomorphism $\alpha : \text{Sym}^2 E_x \to \text{Sym}^2 E_x$. The induced homomorphism $\alpha : \text{Sym}^2 E_x \to \text{Sym}^2 E_x$ is not zero since the extension is nontrivial. However the only nontrivial homomorphism from $\text{Sym}^2 E_x$ to $\text{Sym}^2 E_x$ is an isomorphism. Therefore we see that $\alpha$ has rank 4 which gives a contradiction. Therefore we see that $F_{x,x}$ is simple. \qed

Proposition 4.32. For a point $(x,x) \in X_2$ we have $\text{Ext}^1_M(F_{x,x},F_{x,x}) = 0$ for all $i \geq 3$.

Proof. From the short exact sequence

$$0 \to \bigwedge^2 E_x \to F_{x,x} \to \text{Sym}^2 E_x \to 0$$

we obtain the following commutative diagram.

$$
\begin{array}{cccccc}
0 & \to & \bigwedge^2 E_x \otimes \text{Sym}^2 E_x & \to & F_{x,x} \otimes \text{Sym}^2 E_x & \to & \text{Sym}^2 E_x \otimes \text{Sym}^2 E_x & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \bigwedge^2 E_x \otimes F_{x,x}^* & \to & F_{x,x} \otimes F_{x,x}^* & \to & \text{Sym}^2 E_x \otimes F_{x,x}^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \bigwedge^2 E_x \otimes \bigwedge^2 E_x^* & \to & F_{x,x} \otimes \bigwedge^2 E_x^* & \to & \text{Sym}^2 E_x \otimes \bigwedge^2 E_x^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
$$

By the above computation of cohomology groups of $H^i(M, \mathcal{O})$, $H^i(M, \theta \otimes \text{Sym}^2 E_x^*)$, $H^i(M, \theta^{-1} \otimes \text{Sym}^2 E_x^*)$, $H^i(M, \text{Sym}^2 E_x \otimes \text{Sym}^2 E_x^*)$ we have the conclusion. \qed

5. PROOF OF THE EMBEDDING OF $D(X_2)$ INTO $D(M)$

In this section we assume that $X$ is a non-hyperelliptic curve with genus $g \geq 16$. In order to prove that $D(X_2)$ can be embedded into $D(M)$ via $\Phi_F$, we check a criterion of Bondal and Orlov (cf. [5]). We need to compute various cohomology groups.

Proposition 5.1. For two distinct points $(x,y), (z,w) \in X_2$ we have $\text{Ext}^i_M(F_{x,y},F_{z,w}) = 0$ for all $i \in \mathbb{Z}$ when $g \geq 16$.

Proof. Suppose that $(x,y) \neq (z,w)$. We have following cases.

1. $(x,y) \notin \Delta$ and $(z,w) \notin \Delta$
(2) \((x, y) \not\in \Delta\) and \((z, z) \in \Delta\)
(3) \((x, x) \in \Delta\) and \((z, w) \not\in \Delta\)
(4) \((x, x) \in \Delta\) and \((z, z) \in \Delta\)

For the case (1), (2), (3), we have at least one point which is different from the other three points. Therefore the claim follows from Proposition 4.10. For the case (4), the claim follows from Proposition 4.21. □

**Theorem 5.2.** Let \(\Phi_F : D(X_2) \to D(M)\) be the Fourier-Mukai transformation. Then \(\Phi_F\) is a fully faithful embedding.

**Proof.** From the criterion of Bondal and Orlov it is sufficient to prove the followings.

1. \(\text{Hom}_M(F_{x,y}, F_{x,y}) = \mathbb{C}\).
2. \(\text{Ext}_i^M(F_{x,y}, F_{x,y}) = 0\) for \(i \geq 3\).
3. \(\text{Ext}_i^M(F_{x,y}, F_{z,w}) = 0\) for all \(i \in \mathbb{Z}\) if \((x, y) \neq (z, w)\).

The isomorphism (1) \(\text{Hom}_M(F_{x,y}, F_{x,y}) = \mathbb{C}\) need to be checked for two cases; \(x = y\) case and \(x \neq y\) case. When \(x = y\), then the isomorphism follows from Proposition 4.31. When \(x \neq y\), it follows from Proposition 4.20.

The vanishing (2) \(\text{Ext}_i^M(F_{x,y}, F_{x,y}) = 0\) for \(i \geq 3\) also need to be checked for two cases; when \(x = y\) and \(x \neq y\). When \(x = y\), then it follows from Proposition 4.32. When \(x \neq y\), it follows from Proposition 4.20.

Finally, it remains to show that (3) \(\text{Ext}_i^M(F_{x,y}, F_{z,w}) = 0\) for all \(i \in \mathbb{Z}\) if \((x, y) \neq (z, w)\) hold. It follows from the previous Proposition.

Therefore we obtain the desired result. □

### 6. Semiorthogonal Decomposition

In this section we prove \(D(M)\) has a semiorthogonal decomposition whose component is equivalent to \((D(pt), D(X), D(X_2))\) when \(X\) is a non-hyperelliptic curve with genus \(g \geq 16\).

**Proposition 6.1.** Let \(g \geq 8\). For any three distinct points \(x, y, z \in X\), we have the following identity.

\[
H^i(M, E_x^* \otimes E_y^*) = 0
\]

\[
H^i(M, E_x \otimes E_x^* \otimes E_y^*) = 0
\]

\[
H^i(M, E_x \otimes E_y^* \otimes E_z^*) = 0
\]

for every \(i \in \mathbb{Z}\).

**Proof.** If \(i \geq g + 3\), then we have

\[
H^i(M, E_x^* \otimes E_y^*) \cong H^i(M, \omega_M \otimes E_x \otimes E_y) = 0
\]

from Sommese vanishing theorem. From Serre duality, we have the following isomorphism.

\[
H^i(M, E_x^* \otimes E_y^*) \cong H^{3g-3-i}(M, E_x^* \otimes E_y^*)^* = 0
\]
when \( i \leq 2g - 6 \). Therefore we have
\[
H^i(M, E_i^* \otimes E_j^*) = 0
\]
for all \( i \in \mathbb{Z} \) when \( g \geq 8 \).

Using Sommese vanishing theorem we have
\[
H^i(M, E_x \otimes E_x^* \otimes E_y^*) \cong H^i(M, \omega_M \otimes E_x \otimes E_y) = 0
\]
for \( i \geq g + 7 \).

For small \( i \), let us choose a point \( w \in X \) which is different from \( x, y \). The proof of Proposition 4.10 implies that
\[
H^i(M, E_x \otimes E_x^* \otimes E_y^*) \sim H^i(M, \mathbb{P}E_x, \pi^*(E_x \otimes E_x^* \otimes E_y^*)) = 0
\]
for \( i \leq 2g - 2 \). Therefore we have
\[
H^i(M, E_x \otimes E_x^*) = 0
\]
for all \( i \) when \( g \geq 8 \).

Similarly, we have \( H^i(M, E_x \otimes E_y^* \otimes E_y^*) = 0 \) for every \( i \in \mathbb{Z} \). \( \square \)

**Proposition 6.2.** Let \( g \geq 8 \). For any distinct pairs \( x, y \in X \), we have the following identity.

\[
H^i(M, F_{x,x}^*) = 0
\]
\[
H^i(M, F_{x,x}^* \otimes E_x) = 0
\]
\[
H^i(M, F_{x,x}^* \otimes E_y) = 0
\]

for every \( i \in \mathbb{Z} \).

**Proof.** As in the previous Proposition, we have
\[
H^i(M, E_x^* \otimes E_x^*) = 0
\]
for all \( i \) when \( g \geq 8 \). Therefore we have \( H^i(M, \text{Sym}^2 E_x^*) = H^i(M, \bigwedge^2 E_x^*) = 0 \) for all \( i \).

From the exact sequence
\[
0 \to \bigwedge^2 E_x \to F_{x,x} \to \text{Sym}^2 E_x \to 0
\]
we have the following exact sequence.
\[
0 \to \text{Sym}^2 E_x^* \to F_{x,x}^* \to \bigwedge^2 E_x^* \to 0
\]
Therefore we have \( H^i(M, F_{x,x}^*) = 0 \) for all \( i \).

Using Sommese vanishing theorem we have
\[
H^i(M, E_x \otimes E_x^* \otimes E_y^*) \cong H^i(M, \omega_M \otimes E_x \otimes E_y) = 0
\]
\[
H^i(M, E_y \otimes E_x^* \otimes E_y^*) \cong H^i(M, \omega_M \otimes E_y \otimes E_x) = 0
\]
for \( i \geq g + 7 \).
For small $i$, let us choose a point $w \in X$ which is different from $x, y, z$. The proof of Proposition 4.10 implies that
\[ H^i(M, E_x \otimes E_y^* \otimes E_z^*) \cong H^i(P E_w, \pi^* (E_x \otimes E_y^* \otimes E_z^*)) = 0 \]
for $i \leq 2g - 2$. Therefore we have
\[ H^i(M, E_x \otimes E_y^* \otimes E_z^*) = 0 \]
and
\[ H^i(M, E_y \otimes E_x^* \otimes E_z^*) = 0 \]
for all $i$ when $g \geq 8$. From these vanishing we have $H^i(M, F^* \otimes E_x \otimes E_y) = H^i(M, F^* \otimes E_x \otimes E_y) = 0$ for all $i \in \mathbb{Z}$. □

From the above discussions, we obtain the following result.

**Theorem 6.3.** The derived category of $M$ has a semiorthogonal decomposition $D(M) = \langle \mathcal{A}, \mathcal{B} \rangle$ where $\mathcal{A}$ is equivalent to $(D(pt), D(X), D(X_2))$ when $X$ is a non-hyperelliptic curve with genus $g \geq 16$.

**Proof.** From [27, 28], we see that $D(M)$ has a semiorthogonal decomposition having the following component.
\[ \langle O, \Phi F(D(X)) \rangle \]
Then the above propositions show that the above component lies on $\langle \Phi F(D(X_2)) \rangle^\perp$. Therefore we obtain the desired conclusion. □

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