The Vertex-Face Correspondence and Correlation Functions of the Fusion Eight-Vertex Model
I: The General Formalism

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Abstract

Based on the vertex-face correspondence, we give an algebraic analysis formulation of correlation functions of the $k \times k$ fusion eight-vertex model in terms of the corresponding fusion SOS model. Here $k \in \mathbb{Z}_{>0}$. A general formula for correlation functions is derived as a trace over the space of states of lattice operators such as the corner transfer matrices, the half transfer matrices (vertex operators) and the tail operator. We give a realization of these lattice operators as well as the space of states as objects in the level $k$ representation theory of the elliptic algebra $U_{q,p}(\hat{sl}_2)$. 
1 Introduction

The eight-vertex model was solved by Baxter in the series of seminal papers [1–4]. One of the key insights in these papers was the realization that by a suitable change of basis it was possible to map the model to an ‘Ising-like’ model [4]. This model, which we now refer to as an SOS model, possessed the property of charge conservation through a vertex, and its transfer matrix could be diagonalised using a conventional Bethe ansatz. The height restricted versions of the SOS model, now called RSOS or ABF models, later achieved independent fame, largely due to their connection with conformal field theory models [5, 6].

The method of fusion, leading to higher-spin analogues of the eight-vertex model, was developed in [7] and [8]. Baxter’s vertex-face correspondence between the eight-vertex and SOS models was then extended to these fusion models in [9]. As a result, fusion, or higher-spin, SOS models were defined and studied in great detail [10, 11].

In the early 1990s a new approach to solvable lattice models was developed by Jimbo, Miwa and their collaborators [12, 13]. The approach was applied originally and most fully to the 6-vertex model. The central idea was to exploit to the fullest possible extent an underlying $U_q(\widehat{sl}_2)$ symmetry of the infinite lattice model. The transfer matrix and its associated vector space, the eigenstates, and the local operators of the model were all constructed in terms of this algebra and its associated vertex operators. This enabled this group to express all correlation functions of the six-vertex model in terms of traces of algebraic objects over highest-weight representations of $U_q(\widehat{sl}_2)$. One final ingredient, a free-field realization of the algebra, was then used to compute these traces, thus yielding multiple integral expressions for correlation functions [13].

The success of this method which we shall call the ‘algebraic analysis’ approach led to a great deal of subsequent work in which a whole variety of models were considered from a similar point of view. In particular, a parallel discussion of the eight-vertex model in terms of an elliptic algebra, $A_{q,p}(\widehat{sl}_2)$ [14–16], was presented in [17, 18]. However, a free-field realization of $A_{q,p}(\widehat{sl}_2)$ was and still is lacking. The difficulty is again essentially related to the lack of charge conservation for the eight-vertex model. Thus, while expressions for correlation functions as traces over highest-weight modules exist, it has not proved possible in general to evaluate these traces.

RSOS models were also considered using the algebraic analysis approach [19, 20]. In this case, a free-field realization of the vertex operators appearing in the trace formula was constructed and the trace was computed [20]. What is more, this was originally done, perhaps surprisingly, in the absence of a full understanding of the underlying symmetry algebra.
The correlation functions of the eight-vertex model were finally computed in a beautiful piece of work by Lashkevich and Pugai [21,22]. Their approach was to use and extend Baxter’s vertex-face correspondence in order to map the trace expression for correlation functions of the eight-vertex model into one for SOS models. They then computed the latter using the free-field realization of [20]. An interesting aspect of this work was that correlation functions of the eight-vertex model were found to correspond to correlation functions of the SOS model incorporating a certain dislocation or ‘tail’ operator. Furthermore, this tail operator had a surprisingly simple realization in terms of the SOS free-field realization.

The elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ associated with fusion SOS models was first defined in terms of elliptic Drinfeld currents in [23]. This algebra, or more precisely the closely related algebra $B_{q,\lambda}(\hat{\mathfrak{sl}}_2)$ [16], was later interpreted as a quasi-Hopf twisting of $U_q(\hat{\mathfrak{sl}}_2)$ in [24]. A free-field realization of $U_{q,p}(\hat{\mathfrak{sl}}_2)$ was constructed in [23,24] and the level one case was shown to be equivalent to the ‘phenomenologically’ derived realization of [20].

In this paper, we revisit the approach of Lashkevich and Pugai [21] in light of these recent developments in the understanding of the underlying symmetry algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$. Our twin motivations to carry out this work were: to generalise the approach of [21] to the higher-spin fusion vertex models, and obtain as many explicit results as possible; and to construct the objects appearing in the correlation function traces, such as vertex and tail operators, directly in terms of the algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$.

In Section 2, we construct the higher-spin vertex and SOS weights, and the higher-spin intertwiners that relate them. In Section 3, we review the relevant aspects of the algebraic analysis approach for vertex and SOS models. In Section 4, we generalise the graphical arguments of [21] in order to connect correlation functions of the higher-spin vertex models to those of SOS models with an associated tail operator. In Section 5, we review the construction of $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and give a direct algebraic construction of the vertex operators and the tail operators occurring in the SOS trace expression for vertex model correlation functions. One of our key results is contained in Conjecture 5.9, which gives a remarkably simple algebraic picture of the tail operator as a simple integer powers of one of the half-currents occurring in the definition of $U_{q,p}(\hat{\mathfrak{sl}}_2)$. In a subsequent paper [25], we shall consider the spin-1 generalisation of the eight-vertex model in detail. This corresponds to level $k = 2$ case of the general formalism given in the present paper.
2 The Fusion Vertex and SOS Models

In this section, we define and relate the fusion vertex and SOS models that are of interest to us in this paper. They are statistical-mechanical models whose Boltzmann weights are expressed in terms of elliptic theta functions.

2.1 Notation

First, let us fix our notation for theta functions. Let
\[ p = e^{-\frac{xK'}{K}}, \quad q = e^{-\frac{\pi\lambda}{2K}} \quad \text{and} \quad \zeta = e^{-\frac{\pi\lambda u^2}{2K}}. \]

We introduce \( x, \tau \) and \( r \) by
\[ x = -q, \quad p = e^{-\frac{2\pi i}{\tau}} = x^{2r}, \quad \text{i.e.,} \quad \tau = \frac{2iK'}{K} \quad \text{and} \quad r = \frac{K'}{\lambda}. \]
Throughout this paper \( \Im \tau > 0 \).

We use the theta functions defined in terms of \( \tilde{p} = e^{2\pi i \tau} \) by
\[
\vartheta_1(u|\tau) = 2\tilde{p}^{1/8}(\tilde{p}; \tilde{p})_\infty \sin \pi u \prod_{n=1}^{\infty} (1 - 2\tilde{p}^{n} \cos 2\pi u + \tilde{p}^{2n}),
\]
\[
\vartheta_0(u|\tau) = -ie^{\pi i(u+\tau/4)} \vartheta_1 \left( u + \frac{\tau}{2} \right),
\]
\[
\vartheta_2(u|\tau) = \vartheta_1 \left( u + \frac{1}{2} \right),
\]
\[
\vartheta_3(u|\tau) = e^{\pi i(u+\tau/4)} \vartheta_1 \left( u + \frac{\tau+1}{2} \right).
\]

We define the symbols \([u]^{(s)}, [u] \) and \([u]^*\), by
\[
[u]^{(s)} = x^{\frac{u^2}{2}} \Theta_{2s}(x^{2u}) = C \vartheta_1 \left( \frac{u}{\tau} \right), \quad C = x^{-\frac{\tau}{4}} e^{-\frac{\pi}{8}} \tau^{1/2},
\]
\[
[u] = [u]^{(r)}, \quad [u]^* = [u]^{(r-k)}.
\]

with
\[
\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty, \quad (z;p_1,p_2,\ldots,p_m)_\infty = \prod_{n_1,n_2,\ldots,n_m=0}^{\infty} (1 - z p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}).
\]

Furthermore, we also make use of functions defined in terms of \([u]\) by
\[
[A]_M = [A][A-1] \cdots [A-M+1],
\]
\[
[A,B] = [A][A+1] \cdots [B] \quad (A < B), \quad [A,A-1] = 1,
\]
\[
\begin{bmatrix} A \\ B \end{bmatrix} = [A][A-1] \cdots [A-B+1] \begin{bmatrix} [B][B-1] \cdots [1] \end{bmatrix},
\]
\[
(a,b)_M = (b,a)_M = \begin{bmatrix} M \\ a-b+M \end{bmatrix}^{-1} \begin{bmatrix} a+b-M \atop a+b+M \end{bmatrix} \sqrt{[a][b]}.
\]
In order to distinguish the above notation from that of the \( q \)-integer, we use the following notation for the later:

\[
[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}.
\]

Finally, the relation of our \( \tau, \lambda, r, u \) to \( \tau, \lambda, L, u \) in Date et.al. [11] is as follows. \( \tau_{DJKMO} = -\frac{1}{\tau} \), \( \lambda_{DJKMO} = i\lambda \), \( L_{DJKMO} = -\tau r \) and \( u_{DJKMO} = -\lambda u \).

2.2 The Fusion Vertex Models

2.2.1 The eight-vertex model

The eight-vertex model is a two-dimensional square lattice model. The dynamical variable \( \varepsilon_j \) takes the values + or -. For each vertex, we associate a variable \( \varepsilon_j \) with each edge \( j \). We allow only the eight-configurations for each vertex as depicted in Figure 2.1 (b).

We assign the following Boltzmann weight ( the \( R \) matrix ) \( R(u)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_1' \varepsilon_2'} \) to each configuration:

\[
R(u) = R_0(u) \begin{pmatrix} a(u) & d(u) \\ b(u) & c(u) \\ c(u) & b(u) \\ d(u) & a(u) \end{pmatrix},
\]  

(2.1)
where

\[ R_0(u) = z^{-\frac{1}{2}} \frac{(pq^2 z; q^4, p)_\infty(q^2 z; q^4, p)_\infty(p/z; q^4, p)_\infty(q^4/z; q^4, p)_\infty}{(pq^2/z; q^4, p)_\infty(q^2/z; q^4, p)_\infty(pz; q^4, p)_\infty(q^4z; q^4, p)_\infty}, \quad (2.2) \]

\[ a(u) = \frac{\vartheta_2 \left( \frac{1}{2z} \mid \frac{z}{2} \right) \vartheta_2 \left( \frac{w}{2} \mid \frac{z}{2} \right)}{\vartheta_2 \left( 0 \mid \frac{z}{2} \right) \vartheta_2 \left( \frac{1+w}{2z} \mid \frac{z}{2} \right)}, \quad b(u) = \frac{\vartheta_2 \left( \frac{1}{2z} \mid \frac{z}{2} \right) \vartheta_1 \left( \frac{w}{2z} \mid \frac{z}{2} \right)}{\vartheta_2 \left( 0 \mid \frac{z}{2} \right) \vartheta_1 \left( \frac{1+w}{2z} \mid \frac{z}{2} \right)}, \quad (2.3) \]

\[ c(u) = \frac{\vartheta_1 \left( \frac{1}{2z} \mid \frac{z}{2} \right) \vartheta_2 \left( \frac{w}{2z} \mid \frac{z}{2} \right)}{\vartheta_2 \left( 0 \mid \frac{z}{2} \right) \vartheta_1 \left( \frac{1+w}{2z} \mid \frac{z}{2} \right)}, \quad d(u) = -\frac{\vartheta_1 \left( \frac{1}{2z} \mid \frac{z}{2} \right) \vartheta_1 \left( \frac{w}{2z} \mid \frac{z}{2} \right)}{\vartheta_2 \left( 0 \mid \frac{z}{2} \right) \vartheta_2 \left( \frac{1+w}{2z} \mid \frac{z}{2} \right)}, \quad (2.4) \]

with \( z = \zeta^2 = x^{2u} \). The extra parameter \( u \) is called the spectral parameter. Let \( V \) denote a two-dimensional vector space spanned by \( v_+, v_- \). We regard \( R(u) \) as an operator acting on \( V \otimes V \).

\[ R(u) v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon'_1, \varepsilon'_2} R(u)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} v_{\varepsilon'_1} \otimes v_{\varepsilon'_2}. \]

The \( R \)-matrix satisfies the Yang-Baxter equation, unitarity and crossing symmetry relations given as follows.

\[ R_{12}(u_1 - u_2) R_{13}(u_1 - u_3) R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3) R_{13}(u_1 - u_3) R_{12}(u_1 - u_2), \quad (2.5) \]

\[ R(u) P R(-u) P = I, \quad (2.6) \]

\[ (PR(u)P)^{t_1} = (\sigma^y \otimes 1) R(-u - 1) (\sigma^y \otimes 1)^{-1}, \quad (2.7) \]

where \( P(v_{\varepsilon_1} \otimes v_{\varepsilon_2}) = v_{\varepsilon_2} \otimes v_{\varepsilon_1} \), and \( t_1 \) denotes transposition with respect to the first vector space in the tensor product. Note that \( R_0(u) \) satisfies the following inversion relations.

\[ R_0(u) R_0(-u) = 1, \]

\[ R_0(u) R_0(u + 1) = -\frac{[u + 1]}{[u]}. \]

### 2.2.2 Fusion of the eight-vertex model

Define the operator \( \Pi_{1...k} \) by

\[ \Pi_{1...k} = \frac{1}{k!} (P_{1k} + \cdots + P_{k-1k} + I) \cdots (P_{13} + P_{23} + I)(P_{12} + I). \]

This yields the projection operator onto the space \( V^{(k)} \) of symmetric tensors in \( V \otimes k \). A basis for \( V^{(k)} \) is given by \( \{ v_{\varepsilon}^{(k)} \}_{\varepsilon = -k, -k+2, \ldots, k} \), where \( \varepsilon = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k \) (with \( \varepsilon_i \in \{ +, - \} \)), and

\[ v_{\varepsilon}^{(k)} = \Pi_{1...k} v_{\varepsilon_1} \otimes v_{\varepsilon_2} \otimes \cdots \otimes v_{\varepsilon_k} \]

\[ = \frac{1}{k!} \sum_{\sigma \in S_k} v_{\varepsilon_{\sigma(1)}} \otimes v_{\varepsilon_{\sigma(2)}} \otimes \cdots \otimes v_{\varepsilon_{\sigma(k)}}, \quad (2.8) \]
where \( S_k \) is the symmetric group.

We then define an operator
\[
R_{1\ldots k,j}(u) = \Pi_{1\ldots k} R_{1j}(u + k - 1) \cdots R_{k-1j}(u + 1) R_{kj}(u) \in \text{End}(V(k) \otimes \bar{V}_j).
\]

This satisfies
\[
R_{1\ldots k,j}(u) \Pi_{1\ldots k} = R_{1\ldots k,j}(u).
\]

The \( k \times k \) fusion of the \( R \)-matrix is then given by
\[
R^{(k,k)}(u) = \Pi_{1\ldots k} R_{1\ldots k,k}(u) R_{1\ldots k,k-1}(u - 1) \cdots R_{1\ldots k,1}(u - k + 1).
\] (2.9)

This is an operator in \( \text{End}(V(k) \otimes V(k)) \). It satisfies
\[
R^{(k,k)}(u) = R^{(k,k)}(u) \Pi_{1\ldots k} = R^{(k,k)}(u) \Pi_{1\ldots k}.
\] (2.10)

Using the YBE (2.5) repeatedly and (2.10), we verify that \( R^{(k,k)}(u) \) satisfies the YBE on \( V(k) \otimes V(k) \circledast V(k) \). \( R^{(k,k)}(u) \) also satisfies the unitarity condition which is the simple higher \( k \) version of (2.6).

The situation with crossing symmetry is somewhat more complicated. For general \( k \) we have in general the following relation [13].

\[
(P^{(k)} R^{(k,k)}(u) P^{(k)})^{t_1} = (Q \otimes \text{id}) R^{(k,k)}(-u - 1)(Q^{-1} \otimes \text{id}),
\] (2.11)

where \( P^{(k)} \) is the permutation operator acting on \( V^{(k)} \otimes V^{(k)} \), and \( Q \) is a \( k + 1 \) dimensional matrix whose entries are independent of \( u \). Clearly, for \( k = 1 \), it follows from (2.11) that \( Q = \sigma^y \).

For \( k = 2 \), we find by explicit calculation that [26]

\[
Q = \frac{1}{2} \begin{pmatrix}
1 + y^2 & 0 & 1 - y^2 \\
0 & x^2 & 0 \\
1 - y^2 & 0 & 1 + y^2
\end{pmatrix}
\]

where

\[
x^2 = \frac{1}{2} \frac{\psi_2(0|\tau)}{\psi_0(0|\tau)} \frac{\varphi_3(-\frac{1}{2}|\tau)}{\varphi_3(0|\tau)},
\]

\[
y^2 = -\frac{\psi_2(0|\tau)}{\psi_2(-\frac{1}{2}|\tau)} \frac{\varphi_3(\frac{1}{2}|\tau)}{\varphi_3(0|\tau)}.
\]

The matrix elements of \( R^{(k,k)}(u) \) then define the \( k \times k \) fusion eight-vertex model whose dynamical variables takes values in \( \{-k, -k + 2, \ldots, k\} \).
2.2.3 The ground states

We consider the principal regime specified by

\[ 0 < k < 1, \quad 0 < \lambda < K', \quad -1 < u < 0, \]

where \( k \) denotes the elliptic modulus. Hence \( r > 1 \). In this regime, we find that for the \( k = 1 \) eight-vertex model we have \( c > a, |b|, d \), with \( a > 0, b < 0, d > 0 \). Maximal Boltzmann weight configurations, which we refer to as ground states, will therefore only involve the weight \( c \). More generally, for arbitrary \( k > 0 \), we find that a maximal-weight configuration of edge variables is labelled by \( \ell \in \{0, 1, \cdots, k\} \), and is a periodic repetition of the pattern in Figure 2.2, where \( \varepsilon_\ell = k - 2\ell \).

![Figure 2.2: The ground state configuration of the fusion vertex model labelled by \( \ell \)](image)

2.3 Fusion SOS models

2.3.1 The eight-vertex SOS model

The eight-vertex SOS model, usually referred to as simply the SOS model, is also a two-dimensional square lattice model [5]. The dynamical variables \( a_j \) are called local heights. They take values in \( \mathbb{Z} \). For each face, we associate a local height \( a_j \) with each vertex \( j \). We allow only the configurations satisfying the so-called admissibility condition \(|a_j - a_k| = 1\) for any two adjacent local heights \( a_j \) and \( a_k \). Then we have only the six possible configurations for each face depicted in Figure 2.3 (b).

\[
W \left( \begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) = \begin{array}{c|c} a & b \\ \hline d & c \end{array} \]

(a)
Figure 2.3: The SOS model: (a) the face weight; (b) the six possible configurations.

We assign the following Boltzmann weight (or face weight) $W\left( \begin{array}{c} a_1 \\ a_4 \end{array} \mid \begin{array}{c} a_2 \\ a_3 \end{array} \mid u \right)$ to each configuration.

$$W\left( \begin{array}{c} n \\ n \end{array} \mid \begin{array}{c} n + 1 \\ n + 2 \end{array} \mid u \right) = R_0(u),$$

$$W\left( \begin{array}{c} n \\ n \end{array} \mid \begin{array}{c} n + 1 \\ n \end{array} \mid u \right) = R_0(u) \frac{[n + u][1]}{n[1 + u]},$$

$$W\left( \begin{array}{c} n \\ n \end{array} \mid \begin{array}{c} n + 1 \\ n \end{array} \mid u \right) = R_0(u) \frac{n + 1}{n[1 + u]}.$$  (2.12)

The face weights satisfy the following face-type Yang-Baxter equation, unitarity, and crossing symmetry relations:

$$\sum_g W\left( \begin{array}{c} a \\ f \\ g \\ e \\ d \end{array} \mid \begin{array}{c} b \\ c \end{array} \mid u - v \right) = \sum_g W\left( \begin{array}{c} a \\ f \\ g \\ e \\ d \end{array} \mid \begin{array}{c} b \\ c \end{array} \mid u - v \right) W\left( \begin{array}{c} a \\ g \end{array} \mid \begin{array}{c} b \\ c \end{array} \mid v \right) W\left( \begin{array}{c} b \\ e \\ d \end{array} \mid \begin{array}{c} c \end{array} \mid u \right),$$  (2.13)

$$\sum_e W\left( \begin{array}{c} a \\ e \\ c \end{array} \mid \begin{array}{c} b \\ d \end{array} \end{array} \mid u \right) W\left( \begin{array}{c} a \\ e \\ c \end{array} \mid \begin{array}{c} b \\ d \end{array} \end{array} \mid -u \right) = 1,$$  (2.14)

$$W\left( \begin{array}{c} a \\ b \\ d \\ c \end{array} \mid u \right) = (-)^{\frac{a+b+c}{2}} |b| |a| W\left( \begin{array}{c} d \\ a \\ c \\ b \end{array} \mid -u - 1 \right).$$  (2.15)

### 2.3.2 Fusion of the SOS model

The $k \times k$ fusion of face weights is obtained as follows: define

$$W^{(k,1)}\left( \begin{array}{c} a \\ d \\ c \end{array} \mid u \right) = \sum W\left( \begin{array}{c} a \\ d \\ c \end{array} \mid \begin{array}{c} a_1 \\ a \end{array} \mid u + k - 1 \right) W\left( \begin{array}{c} a_1 \\ d \\ c \end{array} \mid \begin{array}{c} a_2 \\ d \end{array} \mid u + k - 2 \right) \times \cdots W\left( \begin{array}{c} a_{k-1} \\ d \\ c \end{array} \mid \begin{array}{c} b \\ d \end{array} \mid u \right).$$
Then the RHS is independent of the choice of \( a_1, \ldots, a_{k-1} \) provided \( |a - a_1| = |a_1 - a_2| = \cdots = |a_{k-1} - b| = 1 \) [11]. We then define

\[
W^{(k,k)} \left( \begin{array}{ccc|c} a & b & u \\ d & c & \end{array} \right) = \sum_{a_1, \ldots, a_{k-1}} W^{(k,1)} \left( \begin{array}{ccc|c} a & b & u - k + 1 \\ a_1 & b_1 & \end{array} \right) W^{(k,1)} \left( \begin{array}{ccc|c} a_1 & b_1 & u - k + 2 \\ a_2 & b_2 & \end{array} \right) \times \cdots W^{(k,1)} \left( \begin{array}{ccc|c} a_{k-1} & b_{k-1} & u \end{array} \right). \tag{2.16}
\]

Here the RHS is independent of the choice of \( b_1, \ldots, b_{k-1} \) provided \( |b - b_1| = |b_1 - b_2| = \cdots = |b_{k-1} - c| = 1 \). In \( W^{(k,k)} \), the admissible condition for the dynamical variables is extended to \( a_j - a_k \in \{-k, -k + 2, \ldots, k\} \) for any two adjacent local heights \( a_j, a_k \). The fused face weight \( W^{(k,k)} \) satisfies the face type YBE, unitarity and crossing symmetry relations. The latter two relations are given by

\[
\sum_s W^{(k,k)} \left( \begin{array}{ccc|c} a & s & -u \\ d & c & \end{array} \right) W^{(k,k)} \left( \begin{array}{ccc|c} a & b & u \\ s & c & \end{array} \right) = \delta_{b,d}, \tag{2.17}
\]

\[
W^{(k,k)} \left( \begin{array}{ccc|c} d & c & u \\ a & b & \end{array} \right) = \frac{G^{(k)}_{a,d}}{G^{(k)}_{b,c}} W^{(k,k)} \left( \begin{array}{ccc|c} a & d & -1 - u \\ b & c & \end{array} \right). \tag{2.18}
\]

where \( g_a = s_a \sqrt{|a|} \) \( s_a = \pm 1 \), \( s_0 s_{a+1} = (-)^a \) and \( G^{(k)}_{a,b} = \frac{g_a}{g_b(a,b)_k} \). In addition, we have a symmetry for \( k \in \mathbb{Z}_{>0} \) [11]

\[
W^{(k,k)} \left( \begin{array}{ccc|c} d & a & u \\ c & b & \end{array} \right) = \frac{(a,b)_k(d,a)_k}{(d,c)_k(c,b)_k} W^{(k,k)} \left( \begin{array}{ccc|c} d & c & u \\ a & b & \end{array} \right). \tag{2.19}
\]

### 2.3.3 The ground states

We discuss regime III specified by the region

\[
0 < p < 1, \quad -1 < u < 0.
\]

In this regime, the ground states of the level \( k \) fusion SOS weights are of the form shown in Figure 2.4, where \( m \in \mathbb{Z}, \ell \in \{0, 1, \ldots, k\} \), and we define \( \ell = k - \ell \).

![Figure 2.4: A ground state configuration of the fusion SOS model](image-url)
The ground state indicated in Figure 2.4 with height variable $m + \ell$ on a specified reference site is labelled by the pair $(m, \ell)$.

2.4 The Vertex-Face correspondence

In order to solve the eight-vertex model by the Bethe ansatz, Baxter discovered the celebrated identity referred to as the vertex-face correspondence [3]. This correspondence was later generalised to higher fusion level $k$ in [9].

2.4.1 The simple $k = 1$ case

Let us consider the following vector (Figure 2.5 (a))

$$\psi(u)_{a}^{b} = \psi_{+}(u)_{b}^{a} v_{+} + \psi_{-}(u)_{b}^{a} v_{-},$$

$$\psi_{+}(u)_{b}^{a} = \vartheta_{0} \left( \frac{(a - b)u + a}{2r} \right), \quad \psi_{-}(u)_{b}^{a} = \vartheta_{3} \left( \frac{(a - b)u + a}{2r} \right) \tag{2.20} \tag{2.21}$$

with $|a - b| = 1$. Baxter showed the following identity (Figure 2.6 (a)).

$$\sum_{\varepsilon_{1}', \varepsilon_{2}'} R_{\varepsilon_{1}' \varepsilon_{1} \varepsilon_{2}' \varepsilon_{2}} \psi_{\varepsilon_{1}'}(u)_{b}^{a} \psi_{\varepsilon_{2}'}(v)_{c}^{b} = \sum_{b' \in \mathbb{Z}} \psi_{\varepsilon_{2}}(v)_{b}^{b'} \psi_{\varepsilon_{1}}(u)_{c}^{u'} W \left( \begin{array}{c|c} a & b \\ \hline u & v \end{array} \right). \tag{2.22}$$

We hence call $\psi(u)_{b}^{a}$ the intertwining vector. This identity is a key formula throughout this paper. One should note that Baxter’s original intertwining vector intertwines the eight-vertex model in the disordered regime with the SOS model in regime III [3, 11]. In order to consider the eight-vertex model in the principal regime, we have derived the eight-vertex $R$-matrix (2.2), the SOS face weight (2.12) and the intertwining vector (2.21) from those of Baxter [3] (which are the same as those used in [11]) by using Jacobi’s imaginary transformation.

$$\begin{array}{c}
\psi_{\varepsilon}(u)_{b}^{a} = \quad a & b \\
\varepsilon & u
\end{array} \quad \psi_{\varepsilon}^{*}(u)_{b}^{a} = \quad u & \varepsilon \\
\varepsilon & a
\end{array} \tag{a} \tag{b}$$

Figure 2.5: (a) The intertwining vector ; (b) the dual intertwining vector

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In addition to the intertwining vector, it is necessary to introduce its dual counterpart and a second intertwining vector. The dual intertwining vector $\psi^*(u)_b^a$ (Figure 2.5 (b)) is defined by

$$\psi^*(u)_b^a v_{\varepsilon} = \psi^*_\varepsilon(u)_b^a,$$

$$\psi^*_\varepsilon(u)_b^a = -\varepsilon \frac{a-b}{2|b|} C^2 \psi_{-\varepsilon}(u-1)_b^a,$$  (2.23)

whereas the second intertwining vector $\psi'(u)_b^a$ ($b = a \pm a$) is given by

$$\psi'(u)_b^a = \sum_{\varepsilon = \pm} \psi'_\varepsilon(u)_b^a v_{\varepsilon}, \quad \psi'_\varepsilon(u)_b^a = \frac{|u| a}{|u-1| b} \psi_{\varepsilon}(u-2)_b^a$$

with $|a-b| = 1$ in both cases. Then by direct calculation, one can verify the following inversion relations (Figure 2.7)

$$\sum_{\varepsilon = \pm} \psi^*_\varepsilon(u)_b^a \psi_{\varepsilon}(u)_c^b = \delta_{a,c},$$  (2.24)

$$\sum_{a=b\pm 1} \psi^*_\varepsilon(u)_b^a \psi_{\varepsilon}(u)_a^b = \delta_{\varepsilon',\varepsilon},$$  (2.25)

$$\sum_{\varepsilon = \pm} \psi^*_\varepsilon(u)_b^a \psi_{\varepsilon}(u)_a^c = \delta_{b,c},$$  (2.26)

$$\sum_{b=a\pm 1} \psi^*_\varepsilon(u)_b^a \psi'_{\varepsilon'}(u)_a^b = \delta_{\varepsilon,\varepsilon'}. $$  (2.27)

These inversion properties are the reason that we call $\psi^*(u)_b^a$ the dual intertwining vector.
It then follows from the crossing symmetry properties of \( R \) and \( W \) that the following vertex-face correspondence holds:

\[
\sum_{c_1, \ldots, c_k} R(u - v)^{c_1, \ldots, c_k}_{c_1, \ldots, c_k} \psi^*_{c_1}(u)^{a}\psi^*_{c_2}(v)^{b} = \sum_{a \in \mathbb{Z}} \psi^{c_1}_{c_2}(v)^{b}\psi^*_{c_1}(u)^{a} W(a \begin{array}{c} c \ b' \\ b \ a \end{array} u - v). \tag{2.28}
\]

This relation is represented by Figure 2.6 (b).

### 2.4.2 The general \( k \) case

Let us now discuss the fusion of the vertex-face relationship (2.22). We define fused intertwining vectors by

\[
\psi^{(k)}(u)^a_b = \Pi_{1 \ldots k} \psi(u + k - 1)^{a}_{c_1} \otimes \psi(u + k - 2)^{c_1}_{c_2} \cdots \otimes \psi(u)^{c_{k-1}}_b, \tag{2.29}
\]

The RHS is independent of the choice of \( c_1, \ldots, c_{k-1} \) provided \(|a - c_1| = \ldots = |c_{k-1} - b| = 1\). The components of \( \psi^{(k)}(u)^a_b \) are given by the following formula.

\[
\psi^{(k)}(u)^a_b = \sum_{\varepsilon \in \{-k, -k+2, \ldots, k\}} \psi^{(k)}(u)^a_{\varepsilon} \psi^{(k)}(u)^{\varepsilon}_b,
\]

\[
\psi^{(k)}(u)^a_b = \sum_{\varepsilon_1, \ldots, \varepsilon_{k} \in \{-k, \ldots, k\}} \psi^{(k)}(u)^{a}_{1} \psi^{(k)}(u + k - 2)^{c_1}_{c_2} \cdots \psi^{(k)}(u)^{c_{k-1}}_b.
\]

From (2.29), (2.24), and (2.23), it follows that \( \psi^{(k)}(u)^a_b \) satisfies the \( k \times k \) fusion vertex-face correspondence relations with respect to \( R^{(k,k)} \) and \( W^{(k,k)} \). That is, we have

\[
\sum_{c_1, \ldots, c_k} R^{(k,k)}(u - v)^{c_1, \ldots, c_k}_{c_1, \ldots, c_k} \psi^{(k)}(u)^{a}_{c_1} \psi^{(k)}(v)^{b}_{c_2} = \sum_{a \in \mathbb{Z}} \psi^{(k)}(v)^{a}_{c_1} \psi^{(k)}(u)^{b}_{c_2} W^{(k,k)}(a \begin{array}{c} a \ b' \\ b \ a \end{array} u - v). \tag{2.30}
\]

Similarly, we fuse the second intertwining vector \( \psi'(u)^a_b \) as follows.

\[
\psi'(u)^a_b = \Pi_{1 \ldots k} \psi'(u + k - 1)^{a}_{c_1} \otimes \psi'(u + k - 2)^{c_1}_{c_2} \cdots \otimes \psi'(u)^{c_{k-1}}_b.
\]

Then we find that the components of \( \psi'(u)^a_b \) are given by

\[
\psi'(u)^a_b = \sum_{\varepsilon \in \{-k, -k+2, \ldots, k\}} \psi'(u)^{a}_{\varepsilon} \psi'(u)^{b}_{1},
\]

\[
\psi'(u)^a_b = \left[\frac{a + k - 1}{|a + k - 1|} \right] \psi'(u + k - 2)^{a}_b.
\]

In addition, we define the fusion of the dual intertwining vector in the following way.

\[
\psi^{* (k)}(u)^a_b = \sum_{c_1, \ldots, c_{k-1}} \psi^{* (k)}(u + k - 1)^{c_1}_a \otimes \psi^{* (k)}(u + k - 2)^{c_2}_{c_1} \cdots \otimes \psi^{* (k)}(u)^{c_{k-1}}_b. \tag{2.31}
\]
with the property
\[ \Pi_{1...k} \, \psi^{*\langle k \rangle}(u)_{a}^{b} = \psi^{*\langle k \rangle}(u)_{a}^{b} \, \Pi_{1...k}. \]

Written out in component form, the last relation indicates that the RHS of
\[ \psi^{*\langle k \rangle}(u)_{a}^{b} = \sum_{c_{1}...c_{k-1}} \psi_{\varepsilon_{1}}^{*}(u + k - 1)c_{1}^{1}\psi_{\varepsilon_{2}}^{*}(u + k - 2)c_{2}^{2} \cdots \psi_{\varepsilon_{k}}^{*}(u)_{c_{k-1}}^{b} \]
is independent of the choice of \( \varepsilon_{1}, \ldots, \varepsilon_{k} \) provided \( \varepsilon = \varepsilon_{1} + \cdots + \varepsilon_{k} \).

As above, it follows immediately from (2.28), (2.9) and (2.16), that we have
\[ \sum_{c_{1},c_{2}} R^{(k,k)}(u-v)_{c_{1}}^{c_{2}} \psi^{*\langle k \rangle}(u)_{b}^{a} \psi^{*\langle k \rangle}(v)_{c}^{b} = \sum_{b' \in \mathbb{Z}} \psi^{*\langle k \rangle}(v)_{b'}^{a} \psi_{\varepsilon_{1}}^{*}(u)_{c}^{b'} W^{(k,k)} (u-v) \]  \( \varepsilon = 2 \) case in [26].

Finally, using (2.24) - (2.27), it is easy to verify the following inversion relations.
\[ \sum_{\varepsilon \in \{-k,-k+2,\ldots,k\}} \psi^{*\langle k \rangle}(u)_{b}^{a} \psi^{\langle k \rangle}(u)_{c}^{b} = \delta_{a,c} \]  \( 2.34 \)
\[ \sum_{a+b \in \{-k,-k+2,\ldots,k\}} \psi^{*\langle k \rangle}(u)_{b}^{a} \psi^{\langle k \rangle}(u)_{a}^{b} = \delta_{\varepsilon',\varepsilon} \]  \( 2.35 \)
\[ \sum_{\varepsilon \in \{-k,-k+2,\ldots,k\}} \psi^{\langle k \rangle}(u)_{b}^{a} \psi^{\langle k \rangle}(u)_{a}^{c} = \delta_{b,c} \]  \( 2.36 \)
\[ \sum_{b \in a+\{-k,-k+2,\ldots,k\}} \psi^{\langle k \rangle}(u)_{b}^{a} \psi^{\langle k \rangle}(u)_{a}^{b} = \delta_{\varepsilon',\varepsilon} \]  \( 2.37 \)

2.4.3 The \( L \)-Matrix

In the next section, we will make use of the ‘\( L \)-matrix’, defined in terms of the intertwiner and dual intertwiner by
\[ L^{(k)}_{\varepsilon} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi^{\langle k \rangle}(u)_{c}^{d} \psi^{\langle k \rangle}(u)_{b}^{a}. \]  \( 2.38 \)

The graphical representation is given in Figure 2.8.
It is also useful to define the matrix

$$L^{(k)} \left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| \ u \right) = \sum_{\varepsilon \in \{-k,-k+2,\cdots,k\}} L^{(k)}_{\varepsilon} \left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| \ u \right).$$

(2.39)

If we restrict $-1 < u + \frac{k-1}{2} < 0$, $m \geq 1 + \frac{k}{2}$, and choose $\ell \in \{0,1,\cdots,k\}$, we find that the $L-$matrix with $a,b,c,d$ specified as follows

$$L^{(k)}_{\varepsilon} \left( \begin{array}{cc} m + \ell & m + \tilde{\ell} \\ m + \ell & m + \tilde{\ell} \end{array} \bigg| \ u \right),$$

has a maximum absolute value for the choice $\varepsilon_{\ell} = k - 2\ell$. Thus, maximal weight $L-$matrix configurations are of the form shown in Figure 2.9.

$$ \begin{array}{cc} m + \ell & m + \tilde{\ell} \\ m + \ell & m + \tilde{\ell} \end{array} \bigg| \ u \bigg| \varepsilon_{\ell}$$

Figure 2.9: A maximal weight $L-$matrix configuration

### 3 The Corner Transfer Matrix and Half Transfer Matrices

In this and the next section, we consider correlation functions of the fusion eight-vertex models introduced above. We express and manipulate these correlation functions using two ideas: the expression for correlation functions in terms of the corner transfer matrix (CTM) and half-transfer matrices (HTMs) [13, 27]; and the vertex-face correspondence [21]. We here review the first part, i.e., the algebraic analysis approach to both the fusion eight-vertex and the SOS models.
3.1 Fusion Eight-Vertex Models

Correlation functions of the fusion vertex models correspond to the probabilities of the \( \epsilon \) edge variables taking certain values on some specified set of edges of a lattice. More concretely, consider the following dimension \((2L + N) \times 2L\) lattice on which the edge variables at the indicated sides have the values \( \epsilon_1, \cdots, \epsilon_N \) (where \( N \) in Figure 3.1 as shown is actually 3, and for simplicity we assume that \( L \) is even).

![Figure 3.1: The restriction of edge variables associated with our correlation function](image)

The correlation function we consider specifies the probability of such a configuration. It is the ratio of the weighted sum over such restricted configurations to the weighted sum over all configurations (the latter sum being the partition function). The total weight of any configuration is the product of the local vertex Boltzmann weights.

The algebraic analysis approach of [13,27] gives a way of computing such sums in the infinite \( L \) limit. To be more specific, it allows the computation of this correlation function for the infinite-volume lattice in which sums are taken over edge variable configurations which are fixed to one of the ground state configurations of Figure 2.2 beyond a finite, but arbitrarily large, distance from the centre of the lattice. We denote this correlation function by \( P^{(\ell)}(\epsilon_1, \epsilon_2, \cdots, \epsilon_N) \), where \( \ell \in \{0,1,\cdots,k\} \) labels the chosen ground state configuration.

3.1.1 The space of states

The starting point is to replace the weighted sum by a trace over a vector space \( \mathcal{H}^{(\ell)} \) representing a line running from the centre to the boundary of the infinite lattice. \( \mathcal{H}^{(\ell)} \) is called the space of
states and defined in terms of basis vectors $v^{(k)}_{\varepsilon}(\varepsilon = -k, -k + 2, \ldots, k)$ by

$$H^{(\ell)} = \text{Span}_C \left\{ \cdots \otimes v^{(k)}_{\varepsilon(1)} \otimes v^{(k)}_{\varepsilon(0)} | \varepsilon(i) \in \{-k, -k + 2, \ldots, k\}, \varepsilon(i) = \varepsilon^{(\ell)}(i) \text{ for } i \gg 0 \right\}$$

$$\varepsilon^{(\ell)}(i) = \begin{cases} 2\ell - k & \text{for } i = 0 \mod 2 \\ k - 2\ell & \text{for } i = 1 \mod 2 \end{cases}$$

The correlation function $P^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)$ is then represented in terms of a ratio of traces over $H^{(\ell)}$ of CTMs and HTMs.

### 3.1.2 CTMs and the partition function

These operators are best defined graphically. The North-West corner transfer matrix $A(u) : H^{(\ell)} \rightarrow H^{(\bar{\ell})}$ is represented in Figure 3.2, where $\cdots$ represent the infinite directions.

![Figure 3.2: The North-West vertex model corner transfer matrix $A(u)$](image)

This and other graphical representations should be read as follows: the matrix element $A(u)^{-\varepsilon_2,\varepsilon_1}$ is obtained by computing the weighted sum associated with the lattice in Figure 3.2, with all internal edge variables summed over, and with the West external horizontal edge variables and North vertical edge variables fixed to the values

$$\cdots \varepsilon_2' \varepsilon_1' \quad \text{and} \quad \varepsilon_2 \varepsilon_1$$

respectively. Clearly, one can then define South-West, South-East and North-East corner transfer matrices in an analogous manner. A simple consideration of the boundary conditions establishes that these operators act as

$$A_{SW}(u) : H^{(\ell)} \rightarrow H^{(\bar{\ell})}, \quad A_{SE}(u) : H^{(\bar{\ell})} \rightarrow H^{(\bar{\ell})}, \quad A_{NE}(u) : H^{(\bar{\ell})} \rightarrow H^{(\ell)} \quad \text{where } \bar{\ell} = k - \ell.$$  

It is an easy exercise to show that the crossing symmetry relation (2.11) implies that these new operators are related to $A(u)$ by

$$A_{SW}(u) = Q A(-1 - u), \quad A_{SE}(u) = Q A(u) Q^{-1}, \quad A_{NE}(u) = A(-1 - u) Q^{-1}, \quad (3.1)$$
where \( Q : \oplus \ell \mathcal{H}(\ell) \to \oplus \ell \mathcal{H}(\ell) \) is the operator

\[
Q = \cdots \otimes Q \otimes Q \otimes Q.
\]

Baxter’s key observation about the corner transfer matrix is that in the infinite volume limit we have \( A(u) \sim x^{-2uH(\ell)} \), where \( 2H(\ell) \), the corner Hamiltonian, has discrete and equidistant eigenvalues bounded from below, and \( \sim \) means equal up to a scalar. Thus, from (3.1), we have

\[
A_{NE}(u)A_{SE}(u)A_{SW}(u)A(u) \sim x^{4H(\ell)}.
\]

In terms of CTMs, the partition function \( Z(\ell) \) is expressed by

\[
Z(\ell) = \text{Tr}_{H(\ell)} \left( A_{NE}(u)A_{SE}(u)A_{SW}(u)A(u) \right) \sim \text{Tr}_{H(\ell)} x^{4H(\ell)}.
\]

It is remarkable that this partition function is known to coincide with the principally specialised character of the level \( k \) irreducible highest-weight \( \widehat{sl}_2 \)-module \( V(\lambda_\ell) \) with highest weight \( \lambda_\ell = (k - \ell)\lambda_0 + \ell\Lambda_1 \) \( (\ell = 0, 1, \ldots, k) \) [28]. Here \( \Lambda_i \) \( (i = 0, 1) \) denotes the fundamental weight of \( \widehat{sl}_2 \). Namely we have

\[
Z(\ell) \sim \text{Tr}_{H(\ell)} x^{4H(\ell)} = \chi^{(k)}_{\ell}(\tau),
\]

\[
\chi^{(k)}_{\ell}(\tau) = \frac{x^2}{(x^2; x^2)_{\infty}(x^4; x^4)_{\infty}} [\ell + 1]^{(k+2)}.
\]

Here we set \( e^{2\pi i \tau} = x^4 \). Later we will use another expression of \( \chi^{(k)}_{\ell}(\tau) \) in terms of the string function [29]:

\[
\chi^{(k)}_{\ell}(\tau) = \sum_{n \in \mathbb{Z}} \sum_{M \equiv 0 \mod 2k}^{2k-1} c^{\lambda_{\ell}}_{\lambda_M}(\tau) x^{4k(n+\frac{M}{2k})^2-2k(n+\frac{M}{2k})},
\]

where \( c^{\lambda_{\ell}}_{\lambda_M}(\tau) \) denotes the string function defined by

\[
c^{\lambda_{\ell}}_{\lambda_M}(\tau) = (x^4)^{\frac{\ell(\ell+2)}{4(k+2)} - \frac{M^2}{4k}} \frac{k}{8(k+2)} \sum_{n \geq 0} \text{dim} V(\lambda_\ell)\lambda_M - n\delta \ (x^4)^n
\]

and \( c^{\lambda_{\ell}}_{\lambda_M}(\tau) = 0 \) for \( M \not\equiv \ell \mod 2 \). Here \( V(\lambda_\ell)\lambda \) denotes the weight space of \( V(\lambda_\ell) \); \( \delta \) denotes the null root of \( \widehat{sl}_2 \) satisfying \( (\delta, \delta) = 0 = (\delta, \alpha) \), \( (\delta, d) = 1 \) with a standard symmetric bilinear form \( (\ , \ ) : P \times P \to \frac{1}{2}\mathbb{Z}, P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta \). Note that the string function satisfies the following relations.

\[
c^{\lambda_{\ell}}_{\lambda_M}(\tau) = c^{\lambda_{\ell}}_{\lambda_M}(\tau) = c^{\lambda_{\ell-\delta}}_{\lambda_M}(\tau) = c^{\lambda_{\ell}}_{\lambda_{M+2k}}(\tau).
\]
3.1.3 HTMs

To express the correlation functions in a similar way to (3.3), we need to introduce HTMs, that is, North and South half-transfer matrices, denoted by $\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u) : \mathcal{H}^{(\ell)} \rightarrow \mathcal{H}^{(\bar{\ell})}$ and $\phi_{\varepsilon;\varepsilon}^{(\ell,\bar{\ell})}(u) : \mathcal{H}^{(\ell)} \rightarrow \mathcal{H}^{(\bar{\ell})}$ respectively, and defined graphically by Figure 3.3.

These operators are viewed as acting in an anti-clockwise direction about the finite end, i.e., the end whose edge variable is fixed to the value $\varepsilon$. Again, crossing symmetry implies that we have the relation

$$\phi_{\varepsilon;\varepsilon}^{(\ell,\bar{\ell})}(u) = Q \phi_{\varepsilon}^{*(\ell,\bar{\ell})}(u) Q^{-1}$$

(3.8)

where we define the dual operator $\phi_{\varepsilon}^{*(\ell,\bar{\ell})}(u)$ by

$$\phi_{\varepsilon}^{*(\ell,\bar{\ell})}(u) = \sum_{\varepsilon'} Q_{\varepsilon'} \phi_{\varepsilon}^{(\ell,\bar{\ell})}(u - 1).$$

We will often suppress the $(\ell,\bar{\ell})$ superscripts on these various operators.

The heuristic graphical arguments of [27] then lead to the following relations for half transfer and corner transfer matrices:

$$\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u_2)\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u_1) = \sum_{\varepsilon_1, \varepsilon_2 \in \{-k, \ldots, -k + 2, \ldots, k\}} R^{(k,k)}(u_1 - u_2)\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u_1)\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u_2),$$

(3.9)

$$\sum_{\varepsilon \in \{-k, \ldots, -k + 2, \ldots, k\}} \phi_{\varepsilon}^{*(\ell,\bar{\ell})}(u)\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u) = \text{id},$$

(3.10)

$$A(u)\phi_{\varepsilon}(v) = \phi_{\varepsilon}(v - u)A(u).$$

(3.11)

Furthermore (3.2) and (3.11) yield

$$\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u)x^{4H^{(\ell)}} = x^{4H^{(\ell)}}\phi_{\varepsilon}^{(\ell,\bar{\ell})}(u - 2).$$

(3.12)
3.1.4 Correlation functions

Now let us divide the lattice depicted in Figure 3.1 into the pieces corresponding to CTMs and HTMs. We obtain Figure 3.4.

According to this picture, we can express the correlation function $P^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)$ as follows.

$$P^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) = \frac{1}{Z^{(\ell)}} F^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)$$

(3.13)

where

$$F^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)$$

$$= \text{Tr}_{H^{\sigma^N(\ell)}} \left( \mathcal{A}_{NE}(u) \mathcal{A}_{SE}(u) \phi_{S \varepsilon_N}(u) \cdots \phi_{S \varepsilon_1}(u) \mathcal{A}_{SW}(u) \mathcal{A}(u) \phi_{\varepsilon_1}(u) \cdots \phi_{\varepsilon_N}(u) \right)$$

(3.14)

with $\sigma(\ell) = k - \ell$. One can then use the relations (3.11), (3.8) and (3.11) to write all operators in terms of $\phi_{\varepsilon}(u)$ and $\mathcal{A}(u)$ and to re-order them. We thus obtain the following simplified expression for the correlation function:

$$P^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)$$

$$= \frac{1}{\chi^{(k)}(\ell)} \text{Tr}_{H^{\sigma^N(\ell)}} \left( \mathcal{A}_{H^{(\ell)}}(u) \phi_{\varepsilon_N}(u) \cdots \phi_{\varepsilon_1}(u) \phi_{\varepsilon_{\ell}}(u) \phi_{\varepsilon_{\ell^2}}(u) \cdots \phi_{\varepsilon_{\ell^N}}(u) \right).$$

(3.15)
3.2 Fusion SOS Models

We next recall the analogous technology developed in [27] to write the infinite-volume limit of SOS correlation functions as traces of CTMs and HTMs.

3.2.1 The space of states

The first step is to define the space of states $\mathcal{H}^{(\ell)}_{m,a}$ on which our various SOS operators act. We define $\mathcal{H}^{(\ell)}_{m,a}$ ($\ell \in \{0, 1, \cdots, k\}$, $m \in \mathbb{Z}$, $a \in 2\mathbb{Z} + m + \ell$) by

$$\mathcal{H}^{(\ell)}_{m,a} = \text{Span}_\mathbb{C} \{ \cdots \otimes v_{s(2)} \otimes v_{s(1)} \otimes v_{s(0)} \mid s(i) \in \mathbb{Z}, s(i + 1) - s(i) \in \{-k, -k + 2, \cdots, k\}, s(i) = s^{(\ell)}_m(i) \text{ for } i \gg 0, s(0) = a \}$$

$$s^{(\ell)}_m(i) = \begin{cases} m + \ell & i = 0 \text{ mod } 2 \\ m + \bar{\ell} & i = 1 \text{ mod } 2 \end{cases}$$

$\mathcal{H}^{(\ell)}_{m,a}$ is the vector space associated with the height variables along a line running from the centre of a lattice to the boundary, for which the central height is fixed to $a$, and far from the centre the boundary heights are fixed to the ground state configuration $\cdots, m + \bar{\ell}, m + \ell, m + \ell, m + \ell, m + \ell, \cdots$.

3.2.2 CTMs and the partition function

The infinite volume North-West corner transfer matrix $A_a(u)$ is now defined graphically by

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{north-west-corner-transfer-matrix.png}
\caption{The North-West corner transfer matrix $A_a(u)$}
\end{figure}

where we suppress the appearance of the spectral parameter $u$ associated with each SOS face weight. Note that the centre height is fixed to $a$ and all internal height variables are summed over. $A_a(u)$ preserves the boundary conditions, and can be viewed as an operator that acts in an anti-clockwise direction about the centre of the lattice as $A_a(u) : \mathcal{H}^{(\ell)}_{m,a} \rightarrow \mathcal{H}^{(\ell)}_{m,a}$. One can define the South-West, South-East and North-East corner transfer matrices with fixed central heights in an analogous manner. Using the crossing symmetry relations $\mathcal{C}$, allows us to write each
of these in terms $A_a(u)$. As for vertex models, it is a simple, but illuminating exercise, to show that we have

$$A_{SW:a}(u) = g_a \Gamma A_a(-1 - u), \quad A_{SE:a}(u) = \Gamma A_a(u) \Gamma^{-1}, \quad A_{NE:a}(u) = g_a A_a(-1 - u) \Gamma^{-1}$$

(3.16)

where $\Gamma : \mathcal{H}_{m,a}^{(\ell)} \rightarrow \mathcal{H}_{m,a}^{(\ell)}$ is defined by

$$\cdots \otimes v_{s(2)} \otimes v_{s(1)} \otimes v_{s(0)} \mapsto \cdots (s(2), s(1))_k (s(1), s(0))_k (\cdots \otimes v_{s(2)} \otimes v_{s(1)} \otimes v_{s(0)})$$

and $g_a$ and $(a, b)_k$ are as previously defined.

Again, in parallel to the vertex case, it is known that in the infinite volume limit we have

$$A_{NE:a}(u) A_{SE:a}(u) A_{SW:a}(u) A_a(u) \sim [a] x^{4H_{m,a}^{(\ell)}}.$$  

(3.17)

Then the partition function $Z_m^{(\ell)}$ is expressed by CTMs as follows.

$$Z_m^{(\ell)} = \sum_{a \in m + \ell + 2\mathbb{Z}} \text{Tr}_{\mathcal{H}_{m,a}^{(\ell)}} \left(A_{NE:a_N}(u) A_{SE:a'_N}(u) A_{SW:a}(u) A_a(u)\right),$$

$$\sim \sum_{a \in m + \ell + 2\mathbb{Z}} [a] \text{Tr}_{\mathcal{H}_{m,a}^{(\ell)}} x^{4H_{m,a}^{(\ell)}}.$$

It is known that the trace part is given by the string function [28, 30]

$$\text{Tr}_{\mathcal{H}_{m,a}^{(\ell)}} x^{4H_{m,a}^{(\ell)}} = c_{\lambda M}^{\ell}(\bar{\tau}) x^{\frac{(a\tau - a^*_\tau)^2}{k\tau \tau^*}},$$

(3.18)

where $M \equiv a - m \mod 2$. Then by the calculation given in Appendix A we obtain the following expression of the partition function.

**Theorem 3.1.**

$$Z_m^{(\ell)} \sim \sum_{a \in m + \ell + 2\mathbb{Z}} [a] \text{Tr}_{\mathcal{H}_{m,a}^{(\ell)}} x^{4H_{m,a}^{(\ell)}} = [m]^* \chi^{(k)}_{\ell}(\bar{\tau}),$$

(3.19)

were $\chi^{(k)}_{\ell}(\bar{\tau})$ is the principally specialised character given by [34].

The case $k = 1$ was obtained by Lashkevich and Pugai [21]. Note that this represents the vertex-face correspondence between the spaces of states.

It is worth noting the resemblance of (3.19) to the branching formula for the product of characters of irreducible integrable representations of $\hat{\mathfrak{sl}}_2$:

$$\chi^{(k)}_{\ell}(\bar{\tau}) \chi^{(r-k-2)}_{m-1} = \sum_{1 \leq a \leq r-1} b^{(\ell)}_{m,a}(\bar{\tau}) \chi^{(r-2)}_{a-1}(\bar{\tau}),$$

(3.20)
where the principally specialised character $\chi_a^s(\bar{\tau})$ is given in (3.31). The branching function $b_{m,a}^{\ell}(\bar{\tau})$ is known to be the character of the irreducible Virasoro module $Vir_{m,a}$ associated with the coset $(\hat{sl}_2)_k \oplus (\hat{sl}_2)_{r-k-2}/(\hat{sl}_2)_{r-2}$, and with the highest weight $h_{m,a} = \frac{\ell(k-\ell)}{2k(k+2)} + \frac{(mr-ar^*)^2-k^2}{4krr^*}$ and central charge $c_{Vir} = \frac{3k(k+2)}{8k^2} \left(1 - \frac{2(k+2)}{rr^*}\right)$. The main difference between the two formulae are:

1) (3.20) corresponds to the direct sum decomposition of the tensor product representation

$$V(\lambda^{(k)}_\ell) \otimes V(\lambda^{(r-k-2)}_{m-1}) = \bigoplus_{1 \leq a \leq r-1} \Omega^{\ell}_{m,a} \otimes V(\lambda^{(r-2)}_{a-1}).$$

Here $V(\lambda^{(s)}_a)$ denotes the level $s$ irreducible integrable representation with the highest weight $\lambda^{(s)}_a = (s-a)\Lambda_0 + a\Lambda_1$ being dominant integral. $\Omega^{\ell}_{m,a}$ denotes the corresponding irreducible coset Virasoro module $Vir_{m,a}$. For generic $r$, the complete reducibility of the tensor product representation is unknown, and one can not expect a formula like (3.21). However one can define the coset Virasoro algebra even in this case by the Goddard-Kent-Olive construction associated with the tensor product representation. Its irreducible representation will be realised in terms of the representation theory of the elliptic algebra $U_{x,p}(\hat{sl}_2)$ and identified with $\mathcal{H}^{\ell}_{m,a}$ in Section 5.2.1.

2) In terms of lattice models, (3.20) appears in a consideration of the fusion RSOS model with $r > k+2 \in \mathbb{Z}$ and restricted heights $1 \leq a \leq r-1$ and $1 \leq m \leq r - k - 1$ [31]. On the other hand, (3.19) is associated with the fusion SOS model with $r(> k+2)$ generic and with no restriction on the local heights.

### 3.2.3 HTMs

The next object to consider is the North half-transfer matrix $\Phi_{b,a}(u) : \mathcal{H}^{\ell}_{m,a} \rightarrow \mathcal{H}^{\ell}_{m,b}$, which we define graphically by Figure 3.6 a).

![Figures 3.6 a), b), c) and d) : The North, West, South and East half-transfer matrices](image)

Again the centre weights are fixed, and the boundary conditions change as indicated. We view the operator as acting anti-clockwise about the centre of the lattice. There are 3 other half-transfer matrices: $\Phi_{W;b,a}(u)$, $\Phi_{S;b,a}(u)$, and $\Phi_{E;b,a}(u)$, associated with our SOS model that we
might consider, and these are defined graphically by Figures 3.6 b), c) and d). These West, South and East half-transfer matrices are again related to the North half-transfer matrix by crossing symmetry. We find

\[ \Phi_{W;b,a}(u) = (a,b)_k \frac{g_{a}}{g_{b}} \Phi_{b,a}(-1-u), \]

\[ \Phi_{S;b,a}(u) = (b,a)_k \Gamma \Phi_{b,a}(u) \Gamma^{-1}, \]

\[ \Phi_{E;b,a}(u) = \frac{g_{a}}{g_{b}} \Gamma \Phi_{b,a}(-1-u) \Gamma^{-1}. \]

As for vertex models, the graphical arguments of [27], that rely on the Yang-Baxter equation and unitarity, lead to the following relations:

\[ \sum_{d} W^{(k,k)} \begin{pmatrix} c & d \\ b & a \end{pmatrix} u_{1} - u_{2} \Phi_{c,d}(u_{1}) \Phi_{d,a}(u_{2}), \tag{3.22} \]

\[ \sum_{a} \Phi_{b,a}^{*}(u) \Phi_{a,b}(u) = \text{id}, \tag{3.23} \]

\[ A_{b}(u) \Phi_{b,a}(v) = \Phi_{b,a}(v-u) A_{a}(u), \tag{3.24} \]

where we have introduced the dual HTM defined by

\[ \Phi_{b,a}^{*}(u) = \Phi_{W;b,a}(-u) = \frac{1}{G^{(k)}_{b,a}} \Phi_{b,a}(u-1). \]

From [9] and [12], we also have

\[ \Phi_{b,a}(u) x^{4H^{(l)}_{m,a}} = x^{4H^{(l)}_{m,b}} \Phi_{b,a}(u-2). \tag{3.25} \]

### 3.2.4 Correlation functions

The \( N + 1 \) point correlation function \( P^{SOS(l)}_{m}(a,a_{1},..,a_{N}) \) of the fusion SOS model is the probability of the local height variables taking certain values \( a, a_{1},..,a_{N} \) on specified set of vertices \( 0,1,..,N \) of a lattice. Figure 3.7 represents the \( N + 1 \) point function \( P^{SOS(l)}_{m}(a,a_{1},..,a_{N}) \) divided into the parts corresponding to CTMs and HTMs.
According to this picture we have

\[
P_m^{\text{SOS}(\ell)}(a,a_1,\ldots, a_N) = \frac{1}{Z_m^{(\ell)}} F_m^{\text{SOS}(\ell)}(a,a_1,\ldots, a_N),
\]

\[
F_m^{\text{SOS}(\ell)}(a,a_1,\ldots, a_N) = \text{Tr}_{\mathcal{H}_{m,a}^{(\omega N(\ell))}} (\Phi_{a,a_1}(u) \cdots \Phi_{a_{N-1},a_N}(u) A_{N:a_N}(u) A_{SE:a_N}(u)) \times (\Phi_{S:a_N,a_{N-1}}(u) \cdots \Phi_{S:a_1,a}(u) A_{SW:a}(u) A_a(u)).
\]

Using the relations (3.17), (3.24) and (3.25), we obtain the simplified expression.

\[
P_m^{\text{SOS}(\ell)}(a,a_1,\ldots, a_N)
\]

\[
= \frac{[a_N]}{[m]^* \chi_{\ell}^{(k)}(\tau)} \text{Tr}_{\mathcal{H}_{m,a_N}^{(\omega N(\ell))}} (\Phi_{a_N,a_{N-1}}^*(u) \cdots \Phi_{a_1,a}^*(u) \Phi_{a,a_1}(u) \cdots \Phi_{a_{N-1},a_N}(u)).
\]

(3.26)

4 The Vertex-Face Correspondence

While the algebraic analysis approach works well for many models, including fusion six-vertex and fusion SOS models [19, 23, 32–34], it runs into a technical obstacle for the case of the fusion eight-vertex model. The difficulty is the lack of a suitable free-field realization to evaluate the trace occurring in (3.15). In order to overcome this problem, we shall follow Baxter [3] and Lashkevich and Pugai [21] and relate the fusion eight-vertex model to the fusion SOS model. This latter model is more tractable from the point of view of the algebraic analysis approach [20, 23, 24].
4.1 Dressed Vertex Models

The starting point in establishing the connection of the expression (3.15) with SOS models is to dress the boundary of the vertex model defined on the finite \((2L + N) \times 2L\) lattice shown in Figure 3.1 with the intertwining and dual intertwining vectors expressed by 3-vertices in Figure 2.5 \((a)\) and \((b)\). The procedure is shown in Figure 4.1. In this section, we identify the vertex model spectral parameter \(u\) as \(u = u_0 - v\), where \(u_0\) and \(v\) are the spectral parameters associated with vertical and horizontal lines respectively.

![Figure 4.1: The dressed vertex model](image)

As well as the fixed interior edge variables \(\varepsilon_1, \varepsilon_2, \cdots \varepsilon_N\) \((N = 3\) is shown in the Figure\), we fix the boundary edge variables, also marked by bullets, to take the values shown in Figure 4.1, where \(m \geq 1 + \frac{k-1}{2}\) and \(\ell \in \{0, 1, \cdots, k\}\). The total Boltzmann weight associated with a configuration of edge variables is given by the product of \(R\) matrix values around 4-vertices and \(\psi\) and \(\psi^*\) values around 3-vertices. The rational for fixing the boundary condition is that for suitably large lattices it imposes the vertex model boundary condition corresponding to the ground state shown in Figure 2.2. This follows from the observations concerning Figure 2.9.

Let us denote the weighted sum over all edge variably configurations of Figure 4.1, with \(\varepsilon_1, \cdots, \varepsilon_N\) fixed, by \(F^{(\ell)}_{L,m}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)\). The correlation function we are interested in is the
ratios

\[ P_{L;m}^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N) \equiv \frac{1}{Z_{L;m}^{(\ell)}} F_{L;m}^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N) \]

where the partition function \( Z_{L;m}^{(\ell)} \) is the corresponding unrestricted sum, i.e.,

\[ Z_{L;m}^{(\ell)} = \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N} F_{L;m}^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N). \]

Ultimately, we will consider the infinite \( L \) limit of \( P_{L;m}^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N) \). The conjecture is that in this limit the \( m \) dependence associated with the boundary will disappear, and that we can identify the vertex model correlation function as

\[ P^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N) = \lim_{L \to \infty} P_{L;m}^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N). \] (4.1)

4.2 The Relationship with Fusion SOS Correlation Functions

We will next show how to relate the dressed function \( F_{L;m}^{(\ell)}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N) \) corresponding to Figure 4.1 to one associated with SOS models. The argument precedes via a number of diagrammatic equivalences. The first step is to successively use the vertex-face correspondence relations depicted in Figures 2.6 (a) and (b) to turn vertex weights into SOS weights. We use 2.6 (a) starting from the NE corner and 2.6 (b) starting the SW corner of Figure 4.1. A little thought and diagram drawing will convince the reader that you can carrying this procedure through until you end up with a dislocation that extends from the NW to the SE corner of the diagram passing through the \( N \) fixed edges. There are many ways to draw this dislocation - one is shown in Figure 4.2.
The next step is to use the relation of Figure 2.7 (a) to remove the dislocation step by step starting from the NW. We can do this until we reach the leftmost central fixed edge variable $\varepsilon_1$. Hence the sum in Figure 4.2 is equal to that of Figure 4.3 after taking the infinite volume limit and dividing into the parts corresponding to SOS model CTMs and HTMs.

This picture is similar to Figure 3.7 expressing the correlation function of fusion SOS model. But we here have two new ingredients. One is the L-matrix defined in (2.38). The other is the ‘tail operator’ $\tilde{\Lambda}_{b,a}(u_0) : \mathcal{H}^{(l)}_{m,a} \rightarrow \mathcal{H}^{(l)}_{m,b}$ graphically defined by

where the $\varepsilon$’s on the vertical lines are summed over.
4.3 The Tail Operator

The tail operator is characterised by the following commutation relation

\[ \sum_d L^{(k)} \left( \begin{array}{cc} c & d \\ a & b \end{array} \right) v \Phi_{E;c,d}(u) \Lambda_{d,b}(u_0) = \Lambda_{c,a}(u_0) \Phi_{E;a,b}(u), \]

where \( u = u_0 - v \) and the \( L^{(k)} \)-matrix is given in (2.39). This is due to the simple graphical argument given by Figure 4.5. The two steps make successive use of the fundamental intertwining property shown in Figure 2.6 (b).

\[ \ldots \]

\[ \begin{array}{c}
\sum_d \\
\Lambda_{a,a}(u_0) = \tilde{\Lambda}_{a,a}(-u_0 - 1) = \text{id.} \end{array} \]

This follows as a consequence of relation (2.24).

4.4 General Formula for Correlation Functions

Now we return to the problem of computing the vertex model correlation function

\[ P^{(\ell)}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \]

introduced in Section 3.2. In the infinite volume limit, we make the identi-
Here, the sum extends over $F$ where $N$ with tors such as the half transfer matrices, the corner transfer matrix and the space of states.

4.5 The Vertex-Face Correspondence for Lattice Operators

The vertical spectral parameter $u$ be no $u_0$ dependence [21]. A more detailed discussion of this point will be given in [25].
4.5.1 Half transfer matrices

Let us define $\Phi^{(k,k)}(u; u_0)$ and $\Phi^{(k,k)}(u; u_0)$ by

\[\Phi^{(k,k)}(u; u_0) = f(k) (u - u_0) \bigoplus_{a \in \mathcal{A} + 2k} (u - u_0)^{\epsilon_{a'}} (u_0)^{a'} \Phi^{(k,k)}(u_0), \quad (4.6)\]

\[\Phi^{(k,k)}(u; u_0) = \sum_{k'} Q^{k'}(u_0)\Phi^{(k,k)}(u - 1; u_0)
= f(k) (u - u_0)^{k} \bigoplus_{a' \in \mathcal{A} + 2k} (u - u_0)^{a'} \Phi^{(k,k)}(u_0). \quad (4.7)\]

Here the function $f(k)(u)$ is chosen such that

\[f(k)(u)f(k)(u - 1) = C(k)(u), \quad (4.8)\]
\[f(k)(u - 2) = \frac{[u - k - 1]}{[u - 1]} f(k)(u), \quad (4.9)\]

where $C(k)(u)$ is a function appearing in $\Phi(2,3)$. If we consider $u_0$ as a fixed constant and suppress it from the notation from these lattice operators, then we have the following theorem.

**Theorem 4.1.** (i) The lattice operator $\Phi^{(k,k)}(u_0)$ satisfies the commutation relation $\Phi(3,4)$.

(ii) $\Phi^{(k,k)}(u_0)$ and $\Phi^{(k,k)}(u_0)$ satisfy the inversion relation $\Phi(5,6)$.

**Proof.** The commutation relation $\Phi(3,4)$ follows from $\Phi(2,3)$ and $\Phi(3,2)$, whereas the inversion relation $\Phi(5,6)$ follows from $\Phi(2,3)$ and $\Phi(5,3)$.

4.5.2 The corner transfer matrix

Let us define

\[\rho^{(k)} = \bigoplus_{a \in \mathcal{A} + 2k} \Lambda_{a',a}(u_0) [a] x^{H_{m,a}}. \]

**Theorem 4.2.** $\rho^{(k)}$ and $\Phi^{(k,k)}(u; u_0)$ satisfy

\[\rho^{(k)} \Phi^{(k,k)}(u - 2; u_0) = \Phi^{(k,k)}(u; u_0) \rho^{(k)}. \quad (4.10)\]

This should be compared with $\Phi(1,2)$.

**Proof.** Multiply $\Phi(1,2)$ by $f(k)(u - u_0 - 2) \psi^{(k)}(u - u_0) b$ and take the summation in $a$. Then from $\Phi(1,3)$, $\Phi(2,3)$ and $\Phi(4,9)$, we obtain

\[\sum_{a} \Lambda_{c,a}(u_0) [a] f^{(k)}(u - u_0 - 2) \psi^{(k)}(u - u_0 - 2) b \Phi^{(k)}(u_0) = \sum_{a} f^{(k)}(u - u_0) \psi^{(k)}(u - u_0) \Lambda_{c,a}(u_0) \Lambda_{a,b}(u) \Phi^{(k)}(u_0). \]
Now act with $x^{4H_{m,a}^{(\ell)}}$ on right of this expression and use (3.25). Then taking direct sum $\bigoplus_{b,c}$ of the result, we have

$$\sum_a \bigoplus_c \Lambda_{c,a}(u_0)[a]x^{4H_{m,a}^{(\ell)}} \bigoplus_b f^{(k)}(u - u_0 - 2)\psi^{(k)}_\varepsilon(u - u_0 - 2) \Phi_{a,b}(u - 2) = \sum_a \bigoplus_c f^{(k)}(u - u_0)\psi^{(k)}_\varepsilon(u - u_0) \Lambda_{c,a}(u_0) \bigoplus_b \Lambda_{a,b}(u_0)[b]x^{4H_{m,a}^{(l)}}.$$  

We hence obtain (4.10).

### 4.5.3 The space of states

Figure 4.3 with $N = 0$ gives the partition function for the dressed vertex model. From (4.3), this coincides with the SOS partition function $Z_m^{(\ell)}$ as mentioned above. Then from Theorem 3.1, we have

$$Z_m^{(\ell)} = [m]^* Z^{(\ell)}$$

The RHS is the partition function of the $k \times k$ fusion eight-vertex model multiplied by $[m]^*$. The multiplicity by the factor $[m]^*$ can be regarded as the effect of dressing the boundary of the vertex model. Understanding this multiplicity and considering [110], we can roughly make the following vertex-face correspondences for the spaces of states and corner transfer matrices.

$$\mathcal{H}_m^{(\ell)} \longleftrightarrow \bigoplus_{a \in \mathbb{Z} + m + 2\mathbb{Z}} \mathcal{H}_{m,a}^{(\ell)}, \quad x^{4H_{m,a}^{(l)}} \longleftrightarrow \rho^{(\ell)}$$

### 5 The Realization of Fusion SOS Models in Terms of The Elliptic Algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$

In this section, we show how the space of the states $\mathcal{H}_{m,a}^{(\ell)}$, the corner Hamiltonian $H_{m,a}^{(\ell)}$, the half transfer matrix $\Phi_{b,a}(u)$ and the tail operator $\Lambda_{b,a}(u)$ appearing in (5.5) can all be constructed in terms of the representation theory of the elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$. One of the key results is the remarkably simple algebraic form of the tail operator given by Conjecture 5.3.

#### 5.1 The Elliptic Algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$

We first present a brief review of the elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$ and its associated vertex operators [23,24]. The elliptic algebra $U_{x,p}(\widehat{\mathfrak{sl}}_2)$ provides the Drinfeld realization of the face-type elliptic quantum group $B_{x,\lambda}(\widehat{\mathfrak{sl}}_2)$ [16] tensored by a Heisenberg algebra.

---

1$U_{x,p}(\widehat{\mathfrak{sl}}_2)$ is conventionally referred to as $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ - the difference is merely a change of notation.
5.1.1 The Definition and a realization of $U_{x,p}(\mathfrak{sl}_2)$

Definition 5.1. [23, 24] (The Elliptic Algebra $U_{x,p}(\mathfrak{sl}_2)$) The elliptic algebra $U_{x,p}(\mathfrak{sl}_2)$ ($p = x^{2r}$, $r \in \mathbb{C}$) is the associative algebra of the currents $E(u), F(u), K(u)$ and the grading operator $\hat{d}$ satisfying the following relations:

\[
\begin{align*}
K(u_1)K(u_2) &= \rho(u_1 - u_2)K(u_2)K(u_1), \\
K(u_1)E(u_2) &= \frac{[u_1 - u_2 + \frac{1-r^2}{r^2}]}{[u_1 - u_2 - \frac{1+r^2}{r^2}]} E(u_2)K(u_1), \\
K(u_1)F(u_2) &= \frac{[u_1 - u_2 - \frac{1+r^2}{r^2}]}{[u_1 - u_2 + \frac{1-r^2}{r^2}]} F(u_2)K(u_1), \\
E(u_1)E(u_2) &= \frac{[u_1 - u_2 + 1]}{[u_1 - u_2 - 1]} E(u_2)E(u_1), \\
F(u_1)F(u_2) &= \frac{[u_1 - u_2 - 1]}{[u_1 - u_2 + 1]} F(u_2)F(u_1),
\end{align*}
\]

\[
[\hat{d}, E(u)] = \left(-z \frac{\partial}{\partial z} + \frac{1}{r^*}\right) E(u), \quad [\hat{d}, F(u)] = \left(-z \frac{\partial}{\partial z} + \frac{1}{r^*}\right) F(u), \\
[E(u_1), F(u_2)] = \frac{1}{x - x^{-1}} \left(\delta(x^{-k}z_1/z_2)H^+ \left(u_2 + \frac{k}{4}\right) - \delta(x^kz_1/z_2)H^- \left(u_2 - \frac{k}{4}\right)\right),
\]

\[
H^\pm(u) = \kappa K \left(u \pm \left(\frac{r}{2} - \frac{k}{4}\right) + \frac{1}{2}\right) K \left(u \pm \left(\frac{r}{2} - \frac{k}{4}\right) - \frac{1}{2}\right).
\]

Here $r^* = r - k$, $z = x^{2u}$, $z_i = x^{2u_i}$ ($i = 1, 2$) and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. The constant $\kappa$ is given by

\[
\kappa = \frac{\xi(x^{-2}; p^*, x)}{\xi(x^{-2}; p, x)} \quad \text{with} \quad \xi(z; p, x) = \frac{(x^2 z; p, x^4)_\infty (px^2 z; p, x^4)_\infty}{(x^4 z; p, x^4)_\infty (pz; p, x^4)_\infty},
\]

and the scalar function $\rho(v)$ is given by

\[
\rho(v) = \frac{\rho^+(v)^*}{\rho^+(v)}, \quad \text{with} \quad \rho^+(v) = z^{1 - \frac{x^2}{2}} \frac{\left(px^2 z; p, x^4\right)_\infty \left(z^{-1}; p, x^4\right)_\infty \left(x^4 z^{-1}; p, x^4\right)_\infty}{\left(pz; p, x^4\right)_\infty \left(px^4 z; p, x^4\right)_\infty \left(x^2 z^{-1}; p, x^4\right)_\infty}.
\]

The * symbol always indicates the replacement $r \to r^*$. For example, $p^* = x^{2r^*}$, $[u]^* = x^2 u^* \Theta_{x^{2r^*}}(x^{2u})$.

The algebra $U_{x,p}(\mathfrak{sl}_2)$ is realized by tensoring $U_x(\mathfrak{sl}_2)$ and a Heisenberg algebra [24]. For this realization, it is convenient to introduce the Drinfeld realization of $U_x(\mathfrak{sl}_2)$.

Definition 5.2. (The Drinfeld Realization of $U_x(\mathfrak{sl}_2)$) The quantum affine algebra $U_x(\mathfrak{sl}_2)$ is the associative algebra generated by $h, a_m, x_n^\pm (m \in \mathbb{Z}_{\neq 0}, n \in \mathbb{Z}), d$ and the central element $k$.
Theorem 5.1. \cite{24} The elliptic algebra

\[ [h,d] = 0, \quad [d,a_n] = na_n, \quad [d,x_n^\pm] = nx_n^\pm, \]
\[ [h,a_n] = 0, \quad [h,x^\pm(z)] = \pm 2x^\pm(z), \]
\[ [a_n,a_m] = \frac{[2n]_x}{n} [kn]_x x^{-k|n|}\delta_{n+m,0}, \]
\[ [a_n,x^+(z)] = \frac{[2n]_x}{n} x^{-k|n|} z^n x^+(z), \]
\[ [a_n,x^-(z)] = -\frac{[2n]_x}{n} z^n x^-(z), \]
\[ (z - x^{\pm 2}w)x^\pm(z)x^\pm(w) = (x^{\pm 2}z - w)x^\pm(w)x^\pm(w), \]
\[ [x^+(z), x^-(w)] = \frac{1}{x - x^{-1}} \left( \delta(x^{-k}z/w)\psi(x^{k/2}w) - \delta(x^kz/w)\varphi(x^{-k/2}w) \right), \]

where \( x^\pm(z), \psi(z) \) and \( \varphi(z) \) denote the Drinfeld currents defined by

\[
\begin{align*}
x^\pm(z) &= \sum_{n \in \mathbb{Z}} x^n z^{-n}, \\
\psi(x^{k/2}z) &= x^h \exp \left( (x - x^{-1}) \sum_{n > 0} a_n z^{-n} \right), \\
\varphi(x^{-k/2}z) &= x^{-h} \exp \left( -(x - x^{-1}) \sum_{n > 0} a_{-n} z^n \right).
\end{align*}
\]

Let us denote by \( \mathbb{C}\{\hat{H}\} \) the Heisenberg algebra generated by the pair \( P,Q \) with \( [Q,P] = 1 \).

Then we have the following realization of \( U_{x,p}(\hat{sl}_2) \).

Theorem 5.1. \cite{24} The elliptic algebra \( U_{x,p}(\hat{sl}_2) \) is realized by tensoring \( U_x(\hat{sl}_2) \) and the Heisenberg algebra \( \mathbb{C}\{\hat{H}\} \). The generators \( E(u), F(u), K(u) \) and \( \hat{d} \) are given by

\[
\begin{align*}
K(u) &= k(z)e^{Qz\left( \frac{1}{4r} - \frac{1}{4r^*} \right) + \frac{1}{2}\frac{1}{r} + \frac{1}{2}\frac{1}{r^*}}, \\
E(u) &= u^+(z,p)x^+(z)e^{2Qz\frac{1}{2r^*}(-P+1)}, \\
F(u) &= x^-(z)u^-(z,p)z^{\frac{1}{2}(P+h-1)}, \\
\hat{d} &= d - \frac{1}{4r^*}(P-1)(P+1) + \frac{1}{4r}(P + h - 1)(P + h + 1),
\end{align*}
\]

where \( k(z) \) and \( u^\pm(z,p) \) are given by

\[
\begin{align*}
k(z) &= \exp \left( \sum_{n > 0} \frac{[n]_x}{[2n]_x [rn]_x} a_{-n} (x^k z)^n \right) \exp \left( -\sum_{n > 0} \frac{[n]_x}{[2n]_x [rn]_x} a_n z^{-n} \right), \\
u^+(z,p) &= \exp \left( \sum_{n > 0} \frac{1}{[rn]_x} a_{-n} (x^r z)^n \right), \quad u^-(z,p) = \exp \left( -\sum_{n > 0} \frac{1}{[rn]_x} a_n (x^{-r} z)^{-n} \right).
\end{align*}
\]

The following commutation relations are important.
Proposition 5.2.

\[ [K(u), P] = K(u), \quad [E(u), P] = 2E(u), \quad [F(u), P] = 0, \quad (5.2) \]
\[ [K(u), P + h] = K(u), \quad [E(u), P + h] = 0, \quad [F(u), P + h] = 2F(u). \quad (5.3) \]

Next we define the ‘half currents’ and the \( L \)-operator.

**Definition 5.3.** \([24]\) (Half Currents) We define the half currents \( E^+(u), F^+(u) \) and \( K^+(u) \) by

\[
K^+(u) = K \left( u + \frac{r+1}{2} \right), \quad E^+(u) = \hat{E}^+(u) \frac{[1]^*}{[P - 1]^*}, \quad F^+(u) = \hat{F}^+(u) \frac{[1]}{[P + h - 1]},
\]

where

\[
\hat{E}^+(u) = a^* \oint_{C^*} \frac{dw}{2\pi i w} E(v) \frac{[u - v + \frac{k}{2} - P + 1]^*}{[u - v + \frac{k}{2}]^*},
\]
\[
\hat{F}^+(u) = a \oint_C \frac{dw}{2\pi i w} F(v) \frac{[u - v + P + h - 1]}{[u - v]}.
\]

The contours \( C^* \) and \( C \) are defined as follows

\[ C^* : \left| p^* x^k z \right| < \left| w \right| < \left| x^k z \right|, \quad C : \left| pz \right| < \left| w \right| < \left| z \right|, \]

and the constant \( a, a^* \) are chosen to satisfy

\[
\frac{a^* a [1]^* \kappa}{x - x^{-1}} = 1.
\]

**Definition 5.4.** \([24]\) (L-operator) We define the \( L \)-operator \( \hat{L}^+(u) \in \text{End}(\mathbb{C}^2) \otimes U_{x,p}(\hat{\mathfrak{sl}}_2) \) by

\[
\hat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u - 1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E^+(u) \end{pmatrix}.
\]

5.1.2 The vertex operators of \( U_{x,p}(\hat{\mathfrak{sl}}_2) \)

Let \( V(\lambda_\ell) \) be the level \( k \) irreducible \( U_x(\mathfrak{sl}_2) \)-module with highest weight \( \lambda_\ell \). Let us denote by \((\pi_{n,z}, V_{n,z}) (n = 0, 1, \ldots, k)\) the \( n + 1 \)-dimensional evaluation representation of \( U_x(\mathfrak{sl}_2) \): \( V_{n,z} = V^{(n)} \otimes \mathbb{C}[z, z^{-1}], \quad V^{(n)} = \bigoplus_{m=0}^{n} \mathbb{C} v_m^{(n)} \). In physical applications, we only have to consider the case \( n = k \). The co-algebra structure of \( B_{x,\lambda}(\mathfrak{sl}_2) \) allows us to define the type I intertwining operator \( \Phi(u, P) : V(\lambda_\ell) \to V(\lambda_{k-\ell}) \otimes V_{k,z} \) of \( B_{x,\lambda}(\mathfrak{sl}_2) \)-modules. Note that \( B_{x,\lambda}(\mathfrak{sl}_2) \cong U_x(\mathfrak{sl}_2) \).

Now let us consider the \( U_{x,p}(\mathfrak{sl}_2) \)-modules. According to the realization of \( U_{x,p}(\mathfrak{sl}_2) \) given by Theorem 5.1 we define the the level \( k \) \( U_{x,p}(\mathfrak{sl}_2) \)-module \( \hat{V}(\lambda_\ell) \) by

\[
\hat{V}(\lambda_\ell) = \bigoplus_{m \in \mathbb{Z}} V(\lambda_\ell) \otimes e^{-mQ}.
\]

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The type I vertex operator $\Phi(u)$ of $U_{x,p}(\mathfrak{sl}_2)$ is simply defined to be the same as $\Phi(u,P)$:

$$\hat{\Phi}(u) = \Phi(u,P) : \hat{V}(\lambda_{\ell}) \to \hat{V}(\lambda_{k-\ell}) \otimes V_{k,z}.$$ 

Throughout this paper we consider only the type I vertex operator.

From the intertwining relation for $\Phi(u,P)$, we obtain the following relation which characterises the vertex operator uniquely up to a normalisation factor:

$$\hat{\Phi}(u_2)\hat{L}^+(u_1) = R^{+,(13)}_{1,k}(u_1 - u_2, P + h)\hat{L}^+(u_1)\hat{\Phi}(u_2). \quad (5.4)$$

Here $R^{+}_{1,k}(u,s)$ is an image of the $L$-operator: $R^{+}_{1,k}(u-v,s) = (\text{id} \otimes \pi_{k,w})L^{+}(u,s)$ with $z = x^{2u}, w = x^{-2v}$. The finite dimensional representations of the elliptic currents as well as the expression for the matrix $R^{+}_{1,k}(u,s)$ can be found in Appendix C of [24]. Inputting the realization of $\hat{L}^+(u)$ given by Definition 5.4 into (5.4), we can solve (5.4) for $\hat{\Phi}(u_k)$. Let us define the components of the vertex operator $\hat{\Phi}(u)$ as follows:

$$\hat{\Phi} \left( u - \frac{1}{2} \right) = \sum_{m=0}^{k} \Phi_{k,m}(u) \otimes v_m.$$ 

Then we obtain the following realization of the vertex operators [24].

**Theorem 5.3.** The highest component $\Phi_{k,k}(u)$ is given by

$$\Phi_{k,k}(u) = \exp \left\{ \sum_{n \neq 0} \frac{[r^n]_x \alpha_n}{[2n]_x [rn]_x} z^{-n} \right\} : e^{\frac{h}{2} \left( -z \right)^{\frac{r}{2}} - \frac{h}{2} (P + h)} :$$

where

$$\alpha_n = \begin{cases} a_n & \text{for } n > 0 \\ \frac{[rn]_x}{[r^n]_x} x^{|n|} a_n & \text{for } n < 0 \end{cases},$$

such that $[\alpha_m, \alpha_n] = \delta_{m+n,0} \frac{[2m]_x [km]_x [rn]_x}{m [r^m]_x}$.

For the remaining components $m = 0, 1, \cdots, k$, we have the formula

$$\Phi_{k,m}(u) = \Phi_{k,k}(u)\hat{F}^+ \left( u + \frac{k}{2} - r \right) \prod_{j=1}^{k-m} \frac{[1][P + h + 2k - m - 1 + 2j]}{[P + h + k - 1 + 2j][P + h + k + 2j]}$$

$$\cdot \int_{C_1} \frac{dw_1}{2\pi i w_1} \cdots \int_{C_{k-m}} \frac{dw_{k-m}}{2\pi i w_{k-m}} \Phi_{k,k}(u)F(v_1) \cdots F(v_{k-m})$$

$$\times \prod_{j=1}^{k-m} \frac{[u - v_j + P + h + \frac{k}{2} - 1 - 2(k - m - j)][1][P + h + 2k - m - 1 + 2j]}{[u - v_j + \frac{k}{2}][P + h + k - 1 + 2j][P + h + k + 2j]}, \quad (5.5)$$

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where $z = x^{2u}, w_j = x^{2v_j}$ and the integral contours $C_j$ ($1 \leq j \leq k - m$) are given by

\[
C_1: |x^k z| < |w_1| < |p^{-1} x^k z|, |x^{-k} z|,
\]

\[
C_j: |x^k z| < |w_j| < |p^{-1} x^k z|, |x^{-k} z|, |x^2 w_{j-1}|, \quad (2 \leq j \leq k - m).
\]

The expression of $\Phi_{k,k}(u)$ is equivalent but slightly different from the one given in [24] in the zero-mode part. Note the following properties.

**Proposition 5.4.**

\[
[P, \Phi_{k,k}(u)] = 0, \quad [h, \Phi_{k,k}(u)] = k \Phi_{k,k}(u),
\]

\[
F(v) \Phi_{k,k}(u) = \frac{[u - v - \frac{k}{2}]}{[u - v + \frac{k}{2}]} \Phi_{k,k}(u) F(v),
\]

\[
E(v) \Phi_{k,k}(u) = \Phi_{k,k}(u) E(v).
\]

(5.6)

Let us next consider the commutation relations of the vertex operators of $U_{x,p}(\hat{sl}_2)$. In [16], they are expected to be commutation relations with exchange coefficients being exactly the fused face weights $W^{(k,k)}$ (2.16). In order to derive such relations, we consider the following gauge transformation of the vertex operators $\Phi_{k,m}(u) \rightarrow \hat{\Phi}_\varepsilon(u)$ with $\varepsilon = 2m - k$.

\[
\hat{\Phi}_\varepsilon(u) = \Phi_{k,m}(u) \prod_{j=1}^{k-m} \frac{[P + h + k - 1 + 2j][P + h + k + 2j]}{[1][P + h + 2k - m - 1 + 2j]} \quad (\varepsilon = -k, -k + 2, \ldots, k).
\]

From (5.5), we find

\[
\hat{\Phi}_\varepsilon(u) = \Phi_{k,k}(u) \hat{F}^+ \left( u + \frac{k}{2} - r \right)^{\frac{k-\varepsilon}{k+\varepsilon}}.
\]

(5.7)

By using the realisation obtained in Theorem 5.3 and the commutation relation 5.6, we have checked the following commutation relations for levels $k = 1, 2, 3$.

**Conjecture 5.5. (Commutation relations)** The vertex operators $\hat{\Phi}_\varepsilon(u)$, $(\varepsilon = -k, -k + 2, \ldots, k)$ satisfy the commutation relations

\[
\hat{\Phi}_{\varepsilon_2}(u_2) \hat{\Phi}_{\varepsilon_1}(u_1) = \sum_{\nu_1, \nu_2 \in \{-k, -k + 2, \ldots, k\}; \nu_1 + \nu_2 = \varepsilon_1 + \varepsilon_2} W^{(k,k)} \begin{pmatrix} P + h - \varepsilon_1 - \varepsilon_2 & P + h - \varepsilon_2 \\ P + h - \nu_1 & P + h \end{pmatrix} \left| u_1 - u_2 \right| \hat{\Phi}_{\nu_1}(u_1) \hat{\Phi}_{\nu_2}(u_2). \quad (5.8)
\]

where the coefficients $W^{(k,k)}$ are given by (2.16).
5.2 The Realisation of Fusion SOS Models and the Tail Operator

Now we formulate the fusion SOS model in terms of the representation theory of $U_{x,y}(\hat{\mathfrak{sl}}_2)$ and give a realisation of the space of states $\mathcal{H}_{m,a}^{(\ell)}$, the corner Hamiltonian $H_{m,a}^{(\ell)}$, the half transfer matrix $\Phi_{b,a}(u)$ and the tail operator $\Lambda_{b,a}(u)$.

5.2.1 The space of states and the CTM Hamiltonian

We first show that the level $k$ $U_{x,y}(\hat{\mathfrak{sl}}_2)$-modules have a natural decomposition into the Virasoro highest-weight modules associated with the coset $(\hat{\mathfrak{sl}}_2)_k \oplus (\hat{\mathfrak{sl}}_2)_{r-k-2}/(\hat{\mathfrak{sl}}_2)_{r-k}$. Irreducible Virasoro modules are identified with the spaces of states $\mathcal{H}_{m,a}^{(\ell)}$ of the $k \times k$ fusion SOS model.

In order to see such a decomposition, it is convenient to realize the level $k$ $U_{x}(\hat{\mathfrak{sl}}_2)$-module $V(\lambda_\ell)$ in terms of a $q$-deformed $\mathbb{Z}_k$-parafermion module $\mathcal{H}_{PF}$ and the Fock module $\mathcal{F}^a$ of the Drinfeld bosons $a_n$ [23,35] (see also [36] for the CFT case).

The $q$-deformed $\mathbb{Z}_k$-parafermion algebra is conveniently introduced through the $q$-deformed $\mathbb{Z}$-algebra associated with the level $k$ Drinfeld currents of $U_{x}(\hat{\mathfrak{sl}}_2)$. The algebraic structure of the $q$-deformed $\mathbb{Z}$-algebra is quite parallel to the classical case [37]. The $q$-deformed case was considered in [38]. The deformed $\mathbb{Z}$-algebra is generated by $\mathbb{Z}_{\pm,n}$ ($n \in \mathbb{Z}$) whose generating functions $\mathbb{Z}_{\pm}(z) = \sum_{n \in \mathbb{Z}} \mathbb{Z}_{\pm,n} z^{-n}$ are defined by

$$
\mathbb{Z}_+(z) = \exp \left\{ - \sum_{n > 0} \frac{1}{[kn]_x} a_{-n} z^n \right\} x^+(z) \exp \left\{ \sum_{n > 0} \frac{1}{[kn]_x} a_n z^{-n} \right\},
$$

$$
\mathbb{Z}_-(z) = \exp \left\{ \sum_{n > 0} \frac{x^{kn}}{[kn]_x} a_{-n} z^n \right\} x^-(z) \exp \left\{ - \sum_{n > 0} \frac{x^{kn}}{[kn]_x} a_n z^{-n} \right\}.
$$

The $\mathbb{Z}$-algebra commutes with the Drinfeld bosons $a_n$, $n \neq 0$. Then the level $k$ highest-weight $U_{q}(\hat{\mathfrak{sl}}_2)$-module $V(\lambda_\ell)$ with highest weight $\lambda_\ell$ has the structure

$$
V(\lambda_\ell) = \mathcal{F}^a \otimes \Omega_\ell,
$$

(5.9)

where $\mathcal{F}^a = \mathbb{C}[a_{-n} \; (n > 0)]$. The space $\Omega_\ell$ is called the vacuum space defined by

$$
\Omega_\ell = \{ v \in V(\lambda_\ell) \mid a_n v = 0 \; (n > 0) \}.
$$

The space $\Omega_\ell$ is spanned by the vectors $v_\ell(\varepsilon_1, \ldots, \varepsilon_s; n_1, \ldots, n_s)$ ($s \geq 0, \varepsilon_j \in \{\pm\}, n_s \leq 0, n_{s-1} + n_s \leq 0, \ldots, n_1 + \cdots + n_s \leq 0$) given by

$$
\prod_{1 \leq i < j \leq s} \left[ \left( \frac{x - 2 x^{k + \frac{\varepsilon_i + \varepsilon_j}{2}}; x^{2k} \right)_\infty \right]_{\varepsilon_i \varepsilon_j} \mathcal{Z}_{\varepsilon_1}(z_1) \cdots \mathcal{Z}_{\varepsilon_s}(z_s) \cdot 1 \otimes e^{2a}
$$

$$
= \sum_{n_1, \ldots, n_s \in \mathbb{Z}} v_\ell(\varepsilon_1, \ldots, \varepsilon_s; n_1, \ldots, n_s) z_1^{-n_1} \cdots z_s^{-n_s}.
$$

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Here the action of \( \mathbb{Z}_{\pm,n} \) is defined as follows.

\[
\mathbb{Z}_{\pm,n} \cdot (f \otimes e^{\ell \frac{z}{h}}) = \begin{cases}
\mathbb{Z}_{\pm,n} f \otimes e^{\ell \frac{z}{h}} & n \leq 0 \\
[\mathbb{Z}_{\pm,n}, f] \otimes e^{\ell \frac{z}{h}} & n \geq 1
\end{cases}
\]

for \( f \in \mathbb{C}[\mathbb{Z}_{+,n}, \mathbb{Z}_{-,n}] (n \leq 0) \). The weight of \( \nu_{\ell}(\varepsilon_1, \ldots, \varepsilon_s; n_1, \ldots, n_s) \) is \( \lambda_{\ell} + \sum_{j=1}^{s} \varepsilon_j \alpha \) and its degree is \( -\frac{\ell(\ell+2)}{4(k+2)} + n_1 + \cdots + n_s \).

Now let us consider the \( q \)-deformed \( \mathbb{Z}_k \)-parafermion. Define the basic \( \mathbb{Z}_k \)-parafermion currents \( \Psi(z) \) and \( \Psi^\dagger(z) \) through the following relations.

\[
\mathbb{Z}_+(z) = \Psi(z) \otimes e^{\alpha z \frac{h}{k}} ,
\]

\[
\mathbb{Z}_-(z) = \Psi^\dagger(z) \otimes e^{-\alpha z \frac{h}{k}} ,
\]

\[
[\Psi(z), \alpha] = [\Psi(z), h] = [\Psi^\dagger(z), \alpha] = [\Psi^\dagger(z), h] = 0.
\]

To make this expression well-defined, \( \Psi(z) \) and \( \Psi^\dagger(z) \) should have their mode expansions depending on the weight of vectors on which they act. Namely, on the vector with weight \( \lambda \) such that \( (h, \lambda) = m \), we have

\[
\Psi(z) \equiv \Psi^\dagger(z) = \sum_{n \in \mathbb{Z}} \Psi_{+, \frac{m}{k} - n} z^{-\frac{m}{k} + n - 1} ,
\]

\[
\Psi^\dagger(z) \equiv \Psi^\dagger(z) = \sum_{n \in \mathbb{Z}} \Psi_{-, \frac{m}{k} - n} z^{\frac{m}{k} + n - 1} .
\]

The \( q \)-deformed \( \mathbb{Z}_k \)-parafermion algebra is generated by \( \Psi_{+, \frac{m}{k} - n}, \Psi_{-, \frac{m}{k} - n} (n \in \mathbb{Z}) \). Its relations can be expressed as follows.

\[
\left( \frac{z}{w} \right)^{\frac{2}{k}} \left( \frac{x^{-2k} z^k}{w^2 x^{2k}} \right) \Psi^\dagger(z) \Psi^\dagger(w) = \left( \frac{w}{z} \right)^{\frac{2}{k}} \left( \frac{x^{-2k} z^k}{w^2 x^{2k}} \right) \Psi^\dagger(w) \Psi^\dagger(z) ,
\]

\[
\left( \frac{z}{w} \right)^{-\frac{2}{k}} \left( \frac{x^{-2k} w^k}{z^2 x^{2k}} \right) \Psi^\dagger(z) \Psi^\dagger(w) - \left( \frac{w}{z} \right)^{-\frac{2}{k}} \left( \frac{x^{-2k} w^k}{z^2 x^{2k}} \right) \Psi^\dagger(w) \Psi^\dagger(z) = \frac{1}{x - x^{-1}} \left( \delta(x \frac{w}{z}) - \delta(x^{-1} \frac{w}{z}) \right) .
\]

By construction, the following statement is obvious.

**Theorem 5.6.** The following currents \( x^\pm(z) \) and operator \( d \) with \( h \) give a level \( k \) representation of \( U_q(\mathfrak{sl}_2) \) .

\[
x^+(z) = \Psi(z) : \exp \left\{ -\sum_{n \neq 0} \frac{1}{kn} a_n z^{-n} \right\} : e^{\alpha z \frac{1}{k} h} , \quad (5.10)
\]

\[
x^-(z) = \Psi^\dagger(z) : \exp \left\{ \sum_{n \neq 0} \frac{x|n|}{kn} a_n z^{-n} \right\} : e^{-\alpha z \frac{1}{k} h} , \quad (5.11)
\]

\[
d = d^{PF} + d^a , \quad (5.12)
\]
where
\[
d^a = -\sum_{m>0} \frac{m^2 a_{km}}{[2m)_x](km)_x} a_m a_m - \frac{\hbar^2}{4k}
\]  
(5.13)
and \(d^{PF}\) is an operator such that
\[
d^{PF} \cdot 1 \otimes e^\ell_\alpha = -\frac{\ell(k-\ell)}{2k(k+2)} 1 \otimes e^\ell_\alpha,
\]
\[
[d^{PF}, \Psi(z)] = -z \frac{\partial}{\partial z} \Psi(z), \quad [d^{PF}, \Psi^\dagger(z)] = -z \frac{\partial}{\partial z} \Psi^\dagger(z).
\]

We define the \(Z_{2k}\) charge of \(\Psi_{\pm\epsilon_1-\epsilon_n}\) and \(1 \otimes e^\ell_\alpha\) to be \(\pm 2\) and \(\ell \mod 2k\) respectively. For example, the \(Z_{2k}\) charge of the vector
\[
\Psi_{\epsilon_1+2(\epsilon_2+\cdots+\epsilon_s)} - \Psi_{\epsilon_2+2(\epsilon_3+\cdots+\epsilon_s)} - \cdots - \Psi_{\epsilon_s+n_s} \otimes e^\ell_\alpha
\]
is \(\ell + 2 \sum_{j=1}^s \epsilon_j\). Let us denote by \(H^{PF}_{\ell,M}\) the irreducible parafermion module of the \(Z_{2k}\) charge \(M\) defined by the relation
\[
\Omega_\ell = \bigoplus_{\bar{M} \in \ell+2\mathbb{Z}} H^{PF}_{\ell,\bar{M}} \otimes e^\frac{\bar{M}}{2} = \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{M=0}^{2k-1} H^{PF}_{\ell,M} \otimes e^{-\frac{M+2kn}{2}}. 
\]
(5.15)
Here \(H_{\ell,M}^{PF} = \{0\}\) for \(M \neq \ell \mod 2\) and \(H_{\ell,M}^{PF} = H_{\ell,M+2k}^{PF}\). We also assume the symmetry \(36\)
\[
H_{\ell,M}^{PF} = H_{\ell-k,M+k}^{PF} = H_{\ell,-M}^{PF}.
\]
The basic parafermion currents act on the space \(H_{\ell,M}^{PF}\) as the following linear operators.
\[
\Psi(z) : H_{\ell,M}^{PF} \rightarrow H_{\ell,M+2k}^{PF},
\]
\[
\Psi^\dagger(z) : H_{\ell,M}^{PF} \rightarrow H_{\ell,M-2k}^{PF}.
\]
The character of the \(q\)-\(Z_k\)-parafermion space \(H_{\ell,M}^{PF}\) is known to be \(36\)
\[
(x^4)^{-\frac{c^{PF}}{24}} \text{Tr}_{H_{\ell,M}^{PF}} x^{-4d^{PF}} = \eta(\ell) c^{PF}_{\lambda M}(\ell) \]  
(5.16)
where \(c^{PF} = \frac{2(k-1)}{k+2}\). \(c^{PF}_{\lambda M}(\ell)\) and \(\eta(\ell)\) are the string function and Dedekind’s \(\eta\)-function, given by \(37\) and
\[
\eta(\ell) = (x^4)^{\frac{1}{24}} (x^4; x^4)^\infty.
\]
From \(5.9\) and \(5.15\), the level \(k\) irreducible highest-weight module \(V(\lambda_\ell)\) of \(U_x(\hat{\mathfrak{sl}}_2)\) with highest weight \(\lambda_\ell\) is realized as follows:
\[
V(\lambda_\ell) = \mathcal{F}^a \otimes \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{M=0}^{2k-1} H_{\ell,M}^{PF} \otimes e^{(M+2kn)/2}. 
\]
(5.17)
In particular, the highest-weight vector is given by
\[ 1 \otimes 1 \otimes e^{\ell/2}. \] (5.18)

From (5.17), the normalised character of \( V(\lambda_\ell) \) is evaluated as follows:
\[
\chi^k_{\ell}(x^4, y) = (x^4)^{-\frac{1}{4k}} \text{Tr}_{V(\lambda_\ell)} x^{-4d} y^{\frac{d}{2}} = \sum_{n \in \mathbb{Z}} \sum_{M=0 \mod 2k} c_{\lambda_M}^k \left( \frac{1}{2} \right)^{n \frac{k+2}{k+2}} (n+\frac{M}{2k})^2 x^{(n+\frac{M}{2k})} y^{(n+\frac{M}{2k})}. \] (5.19)

By setting \( y = x^{-2} \), we reproduce the level \( k \) principally specialised character (3.6).

Now let us consider the \( U_{x,p}(\hat{sl}_2) \)-modules \( \hat{V}(\lambda_\ell) = \bigoplus_{m \in \mathbb{Z}} V(\lambda_\ell) \otimes e^{-mQ} \). From (5.17), we have
\[
\hat{V}(\lambda_\ell) = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{n \in \mathbb{Z}} \bigoplus_{M=0 \mod 2k} F_{M;m,\ell,n} \] (5.20)
with
\[
F_{M;m,\ell,n} = F^{a} \otimes \mathcal{H}_{\ell,M}^{PF} \otimes e^{(M+2kn)^2/2} \otimes e^{-mQ}. \] (5.21)

Let \( r \) be generic and note that \( P|_{F_{M;m,\ell,n}} = m, \quad P + h|_{F_{M;m,\ell,n}} = M + m + 2kn. \) (5.22)

From (5.21) and (5.16), the character of the space \( F_{M;m,\ell,n} \) is evaluated as follows.
\[
(x^4)^{-\frac{c_{\ell,2}}{24}} \text{Tr}_{F_{M;m,\ell,n}} x^{-4d} = c_{\lambda_M}^\ell \left( \frac{1}{2} \right)^{r^2} x^{(mr-(M+m+2kn))r^2/2},
\]
where \( c_{Vir} = \frac{3k}{k+2} \left( 1 - \frac{2(k+2)}{rr^2} \right) \). This coincides with the one point function (3.18) of the fusion SOS model for \( a = M + m + 2kn \). We hence make the following identification:

the SOS space of states : \( \mathcal{H}_{m,\alpha}^{(\ell)} \leftrightarrow F_{M;m,\ell,n} \quad a = M + m + 2kn, \quad M \equiv \ell \mod 2 \),
the corner Hamiltonian : \( H_{m,\alpha}^{(\ell)} \leftrightarrow -\hat{d} - \frac{1}{24} c_{Vir} \).

Furthermore let us set
\[
F_{m,\ell}(n) \equiv \bigoplus_{M=0 \mod 2k} F_{M;m,\ell,n}. \]

When \( r \) is generic, the character of \( F_{m,\ell}(n) \) coincides with the one of the irreducible Virasoro module \( Vir_{m,\alpha} \) \( (a \equiv \ell + m \mod 2) \) associated with the coset \((\hat{sl}_2)_k \oplus (\hat{sl}_2)_{r-k-2}/(\hat{sl}_2)_r \). In addition, in Appendix B, we consider the case when \( r \) is an integer \( > k + 2 \). In this case, \( F_{m,\ell}(n) \) is reducible. We observe that the BRST resolution of the complex formed by \( F_{m,\ell}(n) \) yields the irreducible coset Virasoro minimal module \( Vir_{m,\alpha}, \quad (a \equiv m + \ell \mod 2) \). These considerations leads us to the following conjecture:
Conjecture 5.7. The space \( \mathcal{F}_{m,\ell}(n) \) is isomorphic to the irreducible coset Virasoro module \( \text{Vir}_{m,a} \) with the central charge \( c_{\text{Vir}} = \frac{2k}{k+2} \left( 1 - \frac{2(k+2)}{4\pi r^2} \right) \) and the highest weight \( h_{m,a} = \frac{\ell(k-\ell)}{2(k+2)} + \frac{(mr-ar^*)^2-k^2}{4kr^*} \).

5.2.2 The vertex operators

The vertex operator \( \hat{\Phi}_\varepsilon(u) \) (\( \varepsilon = -k, -k+2, \ldots, k \)) of the elliptic algebra \( \mathcal{U}_{x,p}(\hat{sl}_2) \) in (5.7) acts on the space \( \mathcal{F}_{M;m,\ell,n} \) as

\[
\hat{\Phi}_\varepsilon(u) : \mathcal{F}_{M;m,\ell,n} \rightarrow \mathcal{F}_{M+\varepsilon, m, k-\ell,n}
\]

and satisfies the commutation relation (5.8). The relation (5.8) is similar to that of the lattice vertex operators (3.22) but not precisely the same. Noting the symmetry (2.19), it turns out that the following gauge transformation \( \hat{\Phi}(u) \rightarrow \Phi_\varepsilon(u) \) resolves this discrepancy.

\[
\Phi_\varepsilon(u) = \hat{\Phi}_\varepsilon(u) \frac{1}{(P+h, P+h+\varepsilon)_k} = \Phi_{k,k}(u) \hat{F}^+ \left( u + \frac{k}{2} - r \right) \frac{k-\varepsilon}{(P+h, P+h+\varepsilon)_k}. \tag{5.23}
\]

In fact \( \Phi_\varepsilon(u) \) satisfies the commutation relation

\[
\Phi_{\varepsilon_2}(u_2)\Phi_{\varepsilon_1}(u_1) = \sum_{\nu_1+\nu_2=\varepsilon_1+\varepsilon_2} W^{(k,k)} \left( \begin{array}{ll} P+h & P+h-\nu_1 \\ P+h-\varepsilon_2 & P+h-\varepsilon_1-\varepsilon_2 \end{array} \right) \Phi_{\nu_1}(u_1)\Phi_{\nu_2}(u_2). \]

This is exactly the same commutation relation as (3.22) if we make the identification

\[
\Phi_{a+\varepsilon,a}(u) = \Phi_\varepsilon(u) \tag{5.24}
\]

on \( \mathcal{F}_{M;m,\ell,n} = \mathcal{H}^{(\ell)}_{m,a} \) (\( a = M + m + 2kn, \ M \equiv \ell \mod 2 \)). This is the realization of the half transfer matrix in terms of the vertex operator of \( \mathcal{U}_{x,p}(\hat{sl}_2) \).

As discussed in [21], there is a second realization of the vertex operators. This is due to the symmetries of the space of states \( \mathcal{H}^{(\ell)}_{m,a} = \mathcal{H}^{(\ell)}_{-m,-a} \) and the Boltzmann weights

\[
W^{(k,k)} \left( \begin{array}{ll} a & b \\ c & d \end{array} \right) u = W^{(k,k)} \left( \begin{array}{ll} -a & -b \\ -c & -d \end{array} \right) u. \]

In fact from (3.22), we have

\[
\Phi_{c,a}(u_2)\Phi_{a,b}(u_1) = \sum_d W^{(k,k)} \left( \begin{array}{ll} -c & -d \\ -a & -b \end{array} \right) u_1-u_2 \Phi_{c,d}(u_1)\Phi_{d,b}(u_2). \tag{5.25}
\]
On the other hand, we have an operator

\[ \Phi_{-a',-a}(u) : \mathcal{H}_{m-a}^{(\ell)} \to \mathcal{H}_{-m-a'}^{(k-\ell)}, \]  

which can be shown, using the same argument that leads to (5.22), to satisfy the commutation relation

\[ \Phi_{-c, -a}(u_2)\Phi_{-a, -b}(u_1) = \sum_d W^{(k,k)}_{(d)} \begin{pmatrix} -c & -d \\ -a & -b \end{pmatrix} \Phi_{-c, -d}(u_1)\Phi_{-d, -b}(u_2). \]  

(5.27)

Comparing this with (5.25), we can simply make the following identification.

\[ \Phi_{a,b}(u) = \Phi_{-a, -b}(u) \]  

(5.28)

on \( \mathcal{H}_{-m-b}^{(\ell)} \). From (5.24), we have

\[ \Phi_\varepsilon(u) = \Phi_\varepsilon(u) \bigg|_{\mathcal{H}_{-m-a}^{(\ell)}} = \Phi_{k,k}(u) \hat{F}^+ \left( u + \frac{k}{2} - r \right) \frac{1}{(P + h, P + h - \varepsilon)_k} \bigg|_{\mathcal{H}_{-m-a}^{(\ell)}}. \]  

(5.29)

5.2.3 The tail operator

We next consider the realisation of the tail operator introduced in Section 3.5. The tail operator \( \Lambda_{a', a}(u) : \mathcal{H}_{m,a}^{(\ell)} \to \mathcal{H}_{m,a'}^{(\ell)} \) is characterised by the commutation relation (4.2). In addition, it follows from formula (15.21) that we only have to consider the case \( a' - a \in 2\mathbb{Z} \). In a similar way to the case of vertex operators, we seek to realise the tail operator in the following form

\[ \Lambda_{a+\varepsilon, a}(u) = \Lambda_\varepsilon(u) \ (\varepsilon \in 2\mathbb{Z}) \]  

(5.30)

on the space \( \mathcal{F}_{M,m,\ell,n} = \mathcal{H}_{m,a}^{(\ell)} \ (a = M + m + 2kn, \ M \equiv \ell \ \text{mod} \ 2) \). Note that from (5.22), the tail operator should satisfy

\[ [P, \Lambda_\varepsilon(u)] = 0, \ [P + h, \Lambda_\varepsilon(u)] = \varepsilon. \]

Substituting (5.24) and (5.30) into (4.2), we obtain the following commutation relation.

\[ \Lambda_{\varepsilon_1}(u_1)\Phi_\varepsilon_2(u_2) = \sum_{\nu_2 \in (-k, -k+2, \ldots)} L^{(k)} \begin{pmatrix} P + h & P + h - \nu_2 \\ P + h - \varepsilon_1 & P + h - \varepsilon_1 - \varepsilon_2 \end{pmatrix} \left( u_2 - u_1 \right) \Phi_{\nu_2}(u_2)\Lambda_{\nu_1}(u_1). \]

(5.31)

Here \( L^{(k)} \) are the \( k \)-fusion \( L \)-matrices, explicit formulae for which are given in Appendix C.

Let us first consider the case \( \varepsilon < 0 \) in \( \Lambda_\varepsilon(u) \). Setting \( \varepsilon_1 = -2k, \varepsilon_2 = k, \nu_1 = -2s, \nu_2 = -k + 2s \ (s \in \mathbb{N}) \) in (5.31), we have

\[ \Lambda_{-2k}(u_1)\Phi_k(u_2) = \sum_{s=0}^k L^{(k)} \begin{pmatrix} P + h & P + h + k - 2s \\ P + h + 2k & P + h + k \end{pmatrix} \left( u_2 - u_1 \right) \Phi_{-k+2s}(u_2)\Lambda_{-2s}(u_1). \]

(5.32)
where the coefficient \( L^{(k)} \) is given by \([3.34]\) with \( m = P + h, n = P + h + 2k \). This \( L^{(k)} \) has simple poles at \( u = 0, -1, \cdots, -k + 1 \). We take the residue at \( u_1 = u_2 + k - 1 \). Assuming that \( \Lambda_{-2k}(u_1)\Phi_k(u_2) \) on the left hand side of \([5.32]\) does not have a pole at \( u_1 = u_2 + k - 1 \), we have the necessary condition

\[
0 = \sum_{s=0}^{k} [P + h + k]_s [P + h + 2k - s - 1]_{k-s} [k + s - 1]_s [k]_{k-s} \times \Phi_{-k+2s}(u_2) \Lambda_{-2s}(u_2 + k - 1). \tag{5.33}
\]

Substituting in formula \([5.23]\), we obtain the recursion relation for the tail operator

\[
0 = \sum_{s=0}^{k} [P + h + 2k]_s [P + h + 3k - s - 1]_{k-s} [k + s - 1]_s [k]_{k-s} \times F^+ \left( u + \frac{k}{2} - r \right)^{k-s} \Lambda_{-2s}(u_2 + k - 1) \frac{1}{(P + h - 2s, P + h - k)_k}.
\]

To solve this for \( \Lambda_\varepsilon(u + k - 1) \), we make the following ansatz.

\[
\Lambda_{-2s}(u + k - 1) = \hat{F}^+ \left( u + \frac{k}{2} - r \right)^s \lambda_s(P + h), \tag{5.34}
\]

where \( \lambda_s(P + h) \) is a function to be determined. Then the necessary condition \([5.34]\) reduces to the relation

\[
\sum_{s=0}^{k} [P + h]_s [P + h + k - s - 1]_{k-s} [k + s - 1]_s [k]_{k-s} \lambda_s(P + h) \frac{1}{(P + h - 2s, P + h - k)_k} = 0.
\]

This equation is satisfied if we choose \( \lambda_s(P + h) = g_{P+h}^{-1} \cdot g_{P+h-2s} \) (where \( g_a \) is defined below \([2.19]\)). We hence obtain the identification

\[
\Lambda_{-2s}(u + k - 1) = g_{P+h} \hat{F}^+ \left( u + \frac{k}{2} - r \right)^s \frac{g_{P+h}^{-1}}{F^+ (P+h)}.
\]

To check that this is also a sufficient condition, we substitute \([5.34]\) back into the full commutation relation \([5.32]\). We find that \([5.32]\) then reduces to the same theta function identity that occurs in the commutation relation \([5.8]\) for vertex operators, which we have checked for the cases \( k = 1, 2, 3 \). See Appendix [\ref{dist} for details.

Let us next study \( \Lambda_\varepsilon(u) \) for the case \( \varepsilon > 0 \). For this purpose, we use the second realization of the vertex operators \([5.29]\). In addition we have the symmetry

\[
\psi^{(k)}(u)_{-a}^{b} = \psi^{(k)}(u)^a_{b}, \quad \psi^*(k)(u)_{-a}^{b} = \psi^*(k)(u)^a_{b}.
\]

Hence

\[
L^{(k)} \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} u = L^{(k)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} u.
\]
Therefore from (4.2), we have

\[ \Lambda_{c,a}(u_1)\Phi_{a,b}(u_2) = \sum_d L^{(k)} \left( \begin{array}{cc} -c & -d \\ -a & -b \end{array} \right) \Phi_{c,d}(u_2)\Lambda_{d,b}(u_1). \tag{5.35} \]

On the other hand, consider the operator

\[ \Lambda_{-a',-a}(u) : \mathcal{H}^{(f)}_{-m,-a} \to \mathcal{H}^{(f)}_{-m,-a'} . \]

From the same argument that leads to (4.2), we have the commutation relation

\[ \Lambda_{-c,-a}(u_1)\Phi_{-a,-b}(u_2) = \sum_d L^{(k)} \left( \begin{array}{cc} -c & -d \\ -a & -b \end{array} \right) \Phi_{c,d}(u_2)\Lambda_{d,b}(u_1). \tag{5.36} \]

Comparing (5.35) and (5.36) and using the second realisation of the vertex operator (5.29), we identify

\[ \Lambda_{-a,-b}(u) = \Lambda_{a,b}(u) \]

under the identification of \( \mathcal{H}^{(f)}_{-m,-a} \) with \( \mathcal{H}^{(f)}_{m,a} \). Therefore, we obtain the following realisation

\[ \Lambda_{a+2s,a}(u+k-1) = \Lambda_{-a-2s,-a}(u+k-1) \]

\[ = g_{P+h}\hat{F}^+(u + k - r) \left. \left( \begin{array}{c} s \end{array} \right) \right|_{\mathcal{H}^{(f)}_{-m,-a}}^{g_{P+h}} . \]

We are thus lead to the following simple conjecture.

**Conjecture 5.8. (The Realisation of the Tail Operator)** The tail operator \( \Lambda_{a \pm 2s,a}(u) \) (\( s \in \mathbb{N} \)) is realized by the following power of the half-current:

\[ \Lambda_{a \pm 2s,a}(u) = g_{P+h}\hat{F}^+(u - \frac{k}{2} - r + 1) \left. \left( \begin{array}{c} s \end{array} \right) \right|_{\mathcal{H}^{(f)}_{m,\mp a}}^{g_{P+h}} \]

\[ = \int_{J_1} \frac{dw_1}{2\pi i w_1} \cdots \int_{J_s} \frac{dw_s}{2\pi i w_s} F(v_1) \cdots F(v_s) \]

\[ \times (-1)^s \sqrt{\frac{[P+h-2s]}{[P+h]} \prod_{j=1}^{s} \frac{[u-v_j+P+h-k/2-2s+2j]}{[u-v_j-k/2+1]}} \left|_{\mathcal{H}^{(f)}_{m,\mp a}} \right. . \]

The integrations contours \( J_j \) (\( 1 \leq j \leq s \)) are given by

\[ J_j : |x^{-k}z^{-1}|, |x^{-2}w_{j+1}| < |w_j| < |p^{-1}x^{-k}z^{-1}|. \]
6 Summary

In this paper, we have first generalised the approach of [21] in order to obtain the trace formula for N-point correlation functions of the level $k$ fusion analogue of the eight-vertex model. The objects that appear in this trace are the space of states $H^{(\ell)}_{m,a}$, the corner Hamiltonian $H^{(\ell)}_{m,a}$, the half transfer matrices $\Phi_{b,a}(u)$ and $\Phi^*_{b,a}(u)$, and the tail operator $\Lambda_{b,a}(u)$. We have constructed each of these objects in terms of the algebra $U_{x,p}(\hat{\mathfrak{sl}}_2)$ in Section 5. A multiple integral formula for (4.5) then follows rather simply.

In a following paper [25], we shall examine the $k=2$ case in detail. We shall make use of the rather simpler 1-boson/1-fermion free field realisation that exists in this case in order to produce and analyse explicit expressions for certain correlation functions.

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A The Proof of Formula (3.19)

It is convenient to use the parametrisation $a = \ell + m + 2(kn + t) \ (n \in \mathbb{Z}, \ 0 \leq t \leq k - 1 \mod 2)$. From (3.18), we have

$$
\sum_{a \in m + \ell + 2\mathbb{Z}} [a] \operatorname{Tr}_{\mathcal{H}^{(t)}_{m,a}} x^{\mathcal{H}^{(t)}_{m,a}} = \sum_{n \in \mathbb{Z}} \sum_{t=0 \mod k}^{k-1} \left( \ell + m + 2(kn + t) \right) c_{\lambda_M}^x (\bar{\tau}) x^{(m-(\ell+m+2(kn+t))r+)^2} \frac{1}{krt^2} 
$$

$$
= \sum_{n \in \mathbb{Z}} \sum_{M=0 \mod 2k}^{2k-1} [M + m + 2kn] c_{\lambda_M}^x (\bar{\tau}) x^{(m-(M+m+2kn))r+)^2} \frac{1}{krt^2} 
$$

with $M \equiv a - m \mod 2$. Note that $c_{\lambda_M}^x (\bar{\tau}) = 0$ for $M \not\equiv \ell \mod 2$. Using the formula

$$
[u] = x^u - u \sum_{s \in \mathbb{Z}} (-1)^s x^{rs^2} x^{s(2u-r)} 
$$

this can be rewritten as follows:

$$
x^{m^2 - m} \sum_{s \in \mathbb{Z}} (-1)^s x^{rs^2} x^{s(s+1)} x^{-2ms} I(s),
$$

where we define

$$
I(s) = \sum_{n \in \mathbb{Z}} \sum_{M=0 \mod 2k}^{2k-1} c_{\lambda_M}^x (\bar{\tau}) x^{4k(n+M/2)^2} x^{-2k(n+M/2)}. 
$$

We show that $I(s) (s \in \mathbb{Z})$ is independent of $s$. In fact, for the case $s = 2u + 1 \ (u \in \mathbb{Z})$, we can eliminate $u$ by shifting $n \rightarrow n + u$. Then we obtain

$$
I(2u + 1) = \sum_{n \in \mathbb{Z}} \sum_{M=0 \mod 2k}^{2k-1} c_{\lambda_M}^x (\bar{\tau}) x^{4k(n+M/2)^2} x^{-2k(n+M/2)} = \chi_{\lambda}^{(2u)} (\bar{\tau}). 
$$

Let us next set $s = 2u$. We have

$$
I(2u) = \sum_{n \in \mathbb{Z}} \sum_{M=0 \mod 2k}^{2k-1} c_{\lambda_M}^x (\bar{\tau}) x^{4k(n+M/2)^2} x^{2k(n+M/2)}. 
$$

By changing $n \rightarrow -n$, $M \rightarrow -M$ and using the symmetry $c_{\lambda_M}^x (\bar{\tau}) = c_{\lambda_{-M}}^x (\bar{\tau})$, we find that $I(2u)$ coincides with $I(2u + 1)$. Therefore the LHS of (3.19) is

$$
\operatorname{LHS} = x^{m^2 - m} \sum_{s \in \mathbb{Z}} (-1)^s x^{rs^2} x^{s(s+1)} x^{-2ms} \chi_{\lambda}^{(k)} (\bar{\tau}) 
$$

$$
= [m]^s \chi_{\lambda}^{(k)} (\bar{\tau}). 
$$

This coincides with the RHS.
B The BRST Resolution of $\mathcal{F}_{m,\ell}(n)$

Let $r > k + 2 \in \mathbb{Z}$ and fix $m, \ell \in \mathbb{Z}$ with $0 \leq \ell \leq k$. Note that regarding the half currents $E^+ (u)$ and $F^+ (u)$ as the screening currents [23], we can define the $q$-Virasoro algebra associated with the coset $(\hat{sl}_2)_k \oplus (\hat{sl}_2)_{r-k-2}/(\hat{sl}_2)_{r-2}$ as their commutant [39]. As such a $q$-Virasoro module, $\mathcal{F}_{m,\ell}(n)$ is reducible. Consider the BRST operator given by

$$Q^+_s = \hat{E}(u)^s : \mathcal{F}_{m,\ell}(n) \to \mathcal{F}_{m-2s,\ell}(n).$$

**Proposition B.1.** [23] The BRST operator $Q^+_s$ is independent of $u$ and is nilpotent in the following sense.

$$Q^+_s Q^+_{r-s} = Q^+_{r-s} Q^+_s = 0.$$

Setting $Q_{2j} = Q^+_m$, $Q_{2j+1} = Q^+_{r-m}$ ($j \in \mathbb{Z}$), we then have the following complex $C_{m,\ell}$

$$\cdots \xrightarrow{Q_{-2}} \mathcal{F}_{m+2r^*,\ell}(n) \xrightarrow{Q_{-1}} \mathcal{F}_{m,\ell}(n) \xrightarrow{Q_0} \mathcal{F}_{m,\ell}(n) \xrightarrow{Q_1} \cdots .$$

We conjecture the following statement about the cohomology group $H^j(C_{m,\ell})$ of this complex [23]:

$$H^j(C_{m,\ell}) = 0 \quad (j \neq 0).$$

Then by the Euler-Poincaré principle, we can evaluate the character of the 0-th cohomology group as follows:

$$\text{Tr}_{H^0(C_{m,\ell})} x^{-4(\hat{d} + c\text{Vir}_{24})} = \sum_{j \in \mathbb{Z}} \left( \text{Tr}_{\mathcal{F}_{m-2r^*,\ell}(n)} x^{-4(\hat{d} + c\text{Vir}_{24})} - \text{Tr}_{\mathcal{F}_{m-2r^*,\ell}(n)} x^{-4(\hat{d} + c\text{Vir}_{24})} \right),$$

(B.1)

where from (5.21), we have

$$\text{Tr}_{\mathcal{F}_{m-2r^*,\ell}(n)} x^{-4(\hat{d} + c\text{Vir}_{24})} = \sum_{M=0 \mod 2k}^{2k-1} C_{\lambda M}^\lambda (\tau) x^{(\pm mr - (M + m - 2r^* j + 2kn))r^* - 2rr^* j^2}. $$

(B.2) coincides with the branching function $b_{m,a}^{(\ell)}(\tau)$, i.e., the character of the irreducible coset Virasoro minimal module $\text{Vir}_{m,a}$ ($a = m + \ell \mod 2$). This is also known to equal to the one point function of the $k$ fusion RSOS model with height restriction $1 \leq a \leq r - 1$ and $1 \leq m \leq r - k - 1$ [10, 31]. Hence in this case we can make the identification

$$H^0(C_{m,\ell}) \leftrightarrow \mathcal{H}_{m,a}^{RSOS(\ell)}.$$
C Fusion of the $L$-matrix

In this appendix, we give explicit formulae for the fused $L$-matrix. The $L$-matrix $L^{(1)}$ is defined by

$$L^{(1)}(a\ b\ c\ d\ |\ u) = \sum_{\varepsilon} \psi^{s(1)}_{\varepsilon}(u)\psi^{d(1)}_{\varepsilon}(u)_{b}^{a}.$$ 

From (2.21) and (2.23), we obtain the formula

$$L^{(1)}(m\ m\pm 1\ n\ n\pm 1\ |\ u) = \frac{\lfloor u + \frac{n-m}{2} \rfloor}{\lfloor |u|/n \rfloor},$$

$$L^{(1)}(m\ m\mp 1\ n\ n\pm 1\ |\ u) = \frac{\lfloor u + \frac{n+m}{2} \rfloor}{\lfloor |u|/n \rfloor}.$$ 

The $k$-fused $L$-matrix $L^{(k)}$ is given by

$$L^{(k)}(m_{0}\ m_{k}\ n_{0}\ n_{k}\ |\ u) = \sum_{\varepsilon} \psi^{s(k)}_{\varepsilon}(u)^{m_{k}}_{m_{0}}\psi^{d(k)}_{\varepsilon}(u)^{n_{k}}_{n_{0}}.$$ 

According to the fusion formulae for $\psi^{s(k)}_{\varepsilon}(u)$ and $\psi^{d(k)}_{\varepsilon}(u)$ (2.24) and (2.21), the $L^{(k)}$ satisfies the following fusion formula.

$$L^{(k)}(m_{0}\ m_{k}\ n_{0}\ n_{k}\ |\ u) = \sum_{n_{1}, n_{2}, \cdots, n_{k-1}} L^{(1)}(m_{0}\ m_{1}\ n_{0}\ n_{1}\ |\ u + k - 1) L^{(1)}(m_{1}\ m_{2}\ n_{1}\ n_{2}\ |\ u) \cdots L^{(1)}(m_{k-1}\ m_{k}\ n_{k-1}\ n_{k}\ |\ u),$$

where the right hand side is independent of the dynamical variables $m_{1}, m_{2}, \cdots, m_{k-1}$. By induction making use of (C.1), we obtain the following compact expressions.

$$L^{(k)}(m\ m - k + 2i\ n\ n - k + 2j\ |\ u) = \sum_{l = \text{Max}(0, i-j)}^{\text{Min}(i, k-j)} L^{(k-i)}(m\ m - k + i\ n\ n - k + 2j - i + 2l\ |\ u + i) \times L^{(i)}(m - k + i\ m - k + 2i\ n - k + 2j - i + 2l\ n - k + 2j\ |\ u)$$

$$= \sum_{l = \text{Max}(0, i+j-k)}^{\text{Min}(i, j)} L^{(i)}(m\ m + i\ n\ n - i + 2l\ u + k - i) \times L^{(k-i)}(m + i\ m - k + 2i\ n - i + 2l\ n - k + 2j\ u).$$
Here $0 \leq i, j \leq k$ and we have
\[
L^{(k)} \left( \begin{array}{c|c} m & m + k \\ n & n + k - 2j \\ \hline \end{array} \right) \begin{pmatrix} u \end{pmatrix}
= \begin{bmatrix}
\frac{1}{2}(n + m) + k - 1 - j \\
k - j \\
\end{bmatrix} \begin{bmatrix}
\frac{1}{2}(n - m) \\
j \\
\end{bmatrix} \begin{bmatrix}
-u + \frac{1}{2}(n + m) \\
j \\
\end{bmatrix} \begin{bmatrix}
-u + \frac{1}{2}(m - n) \\
k - j \\
\end{bmatrix}
\begin{bmatrix}
n + k - 1 - 2j \\
k - j \\
\end{bmatrix} \begin{bmatrix}
n + k - j \\
j \\
\end{bmatrix} \begin{bmatrix}
-u \\
k \\
\end{bmatrix},
\]
(C.2)

\[
L^{(k)} \left( \begin{array}{c|c} m & m - k \\ n & n + k - 2j \\ \hline \end{array} \right) \begin{pmatrix} u \end{pmatrix}
= \begin{bmatrix}
\frac{1}{2}(n - m) + k - 1 - j \\
k - j \\
\end{bmatrix} \begin{bmatrix}
\frac{1}{2}(n + m) \\
j \\
\end{bmatrix} \begin{bmatrix}
-u + \frac{1}{2}(n - m) \\
j \\
\end{bmatrix} \begin{bmatrix}
-u - \frac{1}{2}(m + n) \\
k - j \\
\end{bmatrix}
\begin{bmatrix}
n + k - 1 - 2j \\
k - j \\
\end{bmatrix} \begin{bmatrix}
n + k - j \\
j \\
\end{bmatrix} \begin{bmatrix}
-u \\
k \\
\end{bmatrix},
\]
(C.3)

In particular, we use the following formulae in Section 5.2.

\[
L^{(k)} \left( \begin{array}{c|c} m & m + k - 2j \\ n & n + k \\ \hline \end{array} \right) \begin{pmatrix} u \end{pmatrix} = \frac{[n + m + k - 1 + j]_k [n - m - 1 + j]_k}{[n + k - 1]_k}
\times \frac{[u - m - n - j]_k [u - m + n + j - k]_k}{[-u]_k},
\]
(C.4)

\[
L^{(k)} \left( \begin{array}{c|c} m & m + k - 2j \\ n & n - k \\ \hline \end{array} \right) \begin{pmatrix} u \end{pmatrix} = \frac{[n + m + 2j]_k [m - n]_k [u + m + n - j]_k [u + m - n + j - k]_k}{[n]_k [-u]_k},
\]
(C.5)

Recently one of the authors obtained an explicit expression of $L^{(k)}$ in terms of the very-well-poised elliptic hyper geometric series [40].

### D Commutation Relations of the Tail and Vertex Operators

In this appendix, we check the commutation relations between the tail operator $\Lambda_\varepsilon(u)$ and the vertex operator $\Phi_\varepsilon(u)$. Let us consider an integral of the form
\[
\oint \frac{dw_1}{2\pi i w_1} \oint \frac{dw_2}{2\pi i w_2} F(v_1) F(v_2) f(v_1, v_2),
\]
where the integration contours for \( w_1 \) and \( w_2 \) are the same. The commutation relation \( F(v_1)F(v_2) = \frac{[v_1-v_2-1]}{[v_1-v_2+1]} F(v_2)F(v_2) \), implies that this integral is equal to

\[
\oint \frac{dw_1}{2\pi iw_1} \oint \frac{dw_2}{2\pi iw_2} F(v_1)F(v_2)f(v_2, v_1) \frac{[v_2-v_1-1]}{[v_2-v_1+1]}.
\]

Observing this, we define the notion of ‘weak equality’. The functions \( f(v_1, v_2) \) and \( g(v_1, v_2) \) are equal in weak sense if

\[
f(v_1, v_2) + \frac{v_2-v_1-1}{v_2-v_1+1} f(v_2, v_1) = g(v_1, v_2) + \frac{v_2-v_1-1}{v_2-v_1+1} g(v_2, v_1).
\]

We write \( f(v_1, v_2) \sim g(v_1, v_2) \) to denote weak equality.

Let us consider the commutation relation

\[
\Lambda_{-2k}(u_2)\Phi_k(u_1) = \sum_{s=0}^k L^{(k)} \begin{pmatrix} P+h & P+h+k-2s \\ P+h+2k & P+h+k \end{pmatrix} u_1-u_2 \Phi_{-k+2s}(u_1)\Lambda_{-2s}(u_2). \tag{D.1}
\]

This reduces to the following weak equality

\[
I_\Lambda(v_1, v_2, \cdots, v_k) \sim 0,
\]

where

\[
I_\Lambda(v_1, v_2, \cdots, v_k) = (-1)^k \sqrt{\frac{[n]}{[n+2k]}} \frac{1}{(n+k, n+2k)_k} \prod_{j=1}^k \frac{[-u_2-v_j+n-k/2+2j-1]}{[-u_2-v_j-k/2]} \\
- \sum_{s=0}^k L^{(k)} \begin{pmatrix} n & n+k-2s \\ n+2k & n+k \end{pmatrix} u_1+u_2+1 \cdot (-1)^s \sqrt{\frac{[n+k-2s]}{[n+k]}} \frac{1}{(n, n+k-2s)_k} \\
\times \prod_{j=1}^{k-s} \frac{[u_1-v_j+n-k/2+2j-1]}{[u_1-v_j+k/2]} \prod_{j=k-s+1}^k \frac{[-u_2-v_j+n-3k/2+2j-1]}{[-u_2-v_j-k/2]}.
\]

Let us consider

\[
\tilde{\Phi}_{-k}(u_2)\tilde{\Phi}_k(u_1) = \sum_{s=0}^k W^{(k,k)} \begin{pmatrix} P+h & P+h+k-2s \\ P+h+k & P+h \end{pmatrix} u_1-u_2 \tilde{\Phi}_{-k+2s}(u_1)\tilde{\Phi}_{k-2s}(u_2). \tag{D.2}
\]

This in turn reduces to the weak equality

\[
I_{\tilde{\Phi}}(v_1, v_2, \cdots, v_k) \sim 0,
\]

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where
\[ I_\Phi(v_1, v_2, \cdots, v_k) = \prod_{j=1}^{k} \frac{[u_1 - v_j - \frac{k}{2}]}{[u_1 - v_j + \frac{k}{2}]} \prod_{j=1}^{k} \frac{[u_2 - v_j + n - \frac{k}{2} + 2j - 1]}{[u_2 - v_j + \frac{k}{2}]} \]
\[ - \sum_{s=0}^{k} \prod_{j=1}^{k-s} \frac{[u_1 - v_j + n - \frac{k}{2} + 2j - 1]}{[u_1 - v_j + \frac{k}{2}]} \prod_{j=s+1}^{k} \frac{[u_2 - v_j + n - \frac{3k}{2} + 2j - 1]}{[u_2 - v_j + \frac{k}{2}]} \]
\[ \times \prod_{j=1}^{k-s} \frac{[u_2 - v_j - \frac{k}{2}]}{[u_2 - v_j + \frac{k}{2}]} \]

We have checked \( I_\Phi(v_1, v_2, \cdots, v_k) \sim 0 \) for the case \( k = 1, 2, 3 \).

**Proposition D.1.** The \( L^{(k)} \)-matrix and the Boltzmann Weight \( W_{k,k} \) are related by
\[ W_{k,k} \left( \frac{n}{n+k-2s} \frac{n+k}{n} \right) = \sqrt{\frac{[n-k][n+2k]}{[n][n+k]}} \left( \begin{array}{c} n+k-2s \\ n+k \end{array} \right) \left( \begin{array}{c} n+k-2s \\ n+2k \end{array} \right) \]

By using this proposition, we have
\[ I_\Phi(v_1, v_2, \cdots, v_k | u_1, u_2) = (-1)^k \sqrt{\frac{[n+2k]}{[n]} (n+k, n+2k)} I_\Lambda(v_1, v_2, \cdots, v_k | u_1, u_2) \left( \prod_{j=1}^{k} \frac{[u_2 - v_j - \frac{k}{2}]}{[u_2 - v_j + \frac{k}{2}]} \right) \]

Therefore [D.1] and [D.2] reduce to the same identity, which we have checked for \( k = 1, 2, 3 \).

This supports our conjecture [5.31] for the explicit form of the tail operator.

Let us check a commutation relation of the tail and vertex operators which does \textit{not} reduce to one for just vertex operators in the above way. Let us consider the commutation relation for \( s \geq c \),
\[ \Lambda_{-2s} (u_2) \Phi_k (u_1) = \sum_{t=0}^{k} L^{(k)} \left( \begin{array}{c} P + h \\ P + h + k + 2t \end{array} \right) \Phi_{-k+2t} (u_1) \Lambda_{2k-2s-2t} (u_2) \]

Taking the residue at \( u_1 = -u_2 - k \), a necessary condition becomes the following theta identity
\[ \sum_{t=0}^{k} [n + k - s] [s] k - t [2k + n - 1 - s - t] k - t [s + t - 1] k \]
\[ \times (-1)^{s+t+k} \sqrt{\frac{[n-2s-2t+2k]}{[n]} \frac{1}{(n+k-2s, n+2k-2s-2t)}} = 0. \]
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