CHOW GROUPS AND $L$-DERIVATIVES OF AUTOMORPHIC MOTIVES FOR UNITARY GROUPS, II.

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ABSTRACT. In this article, we improve our main results from [LL] in two direction: First, we allow ramified places in the CM extension $E/F$ at which we consider representations that are spherical with respect to a certain special maximal compact subgroup, by formulating and proving an analogue of the Kudla–Rapoport conjecture for exotic smooth Rapoport–Zink spaces. Second, we lift the restriction on the components at split places of the automorphic representation, by proving a more general vanishing result on certain cohomology of integral models of unitary Shimura varieties with Drinfeld level structures.

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1. Introduction

In [LL], we proved that for certain cuspidal automorphic representations $\pi$ on unitary groups of even ranks, if the central derivative $L'(1/2, \pi)$ is nonvanishing, then the $\pi$-nearly isotypic localization of the Chow group of a certain unitary Shimura variety over its reflex field does not vanish. This proved part of the Beilinson–Bloch conjecture for Chow groups and $L$-functions. Moreover, assuming the modularity of Kudla’s generating functions of special cycles, we further proved the arithmetic inner product formula relating $L'(1/2, \pi)$ and the height of arithmetic theta liftings. In this article, we improve the main results from [LL] in two directions: First, we allow ramified places in the CM extension $E/F$ at which we consider representations that are spherical with respect to a certain special maximal compact subgroup, by formulating and proving an analogue of the Kudla–Rapoport conjecture for exotic smooth Rapoport–Zink spaces. Second, we lift the restriction on the components at split places of the automorphic representation, by proving a more general vanishing result on certain cohomology of integral models of unitary Shimura varieties with Drinfeld level structures. However, for technical reasons, we will still assume $F \neq \mathbb{Q}$ (see Remark 4.31).

Readers may read the introduction of [LL] for more background of those results.

1.1. Main results. Let $E/F$ be a CM extension of number fields with the complex conjugation $c$. Denote by $V_F^{(\infty)}$ and $V_F^{\text{fin}}$ the set of archimedean and non-archimedean places of $F$, respectively; and $V_F^{\text{spl}}, V_F^{\text{int}},$ and $V_F^{\text{ram}}$ the subsets of $V_F^{\text{fin}}$ of those that are split, inert, and ramified in $E$, respectively. For every $v \in V_F^{\text{fin}}$, we denote by $q_v$ the residue cardinality of $F_v$.

**Definition 1.1.** We define the subset $V_F^{\diamond}$ of $V_F^{\text{spl}} \cup V_F^{\text{int}}$ consisting of $v$ satisfying that for every $v' \in V_{E_v}^{(p)} \cap V_{E_v}^{\text{ram}}$, where $p$ is the underlying rational prime of $v$, the subfield of $\overline{F_v}$ generated by $F_v$ and the Galois closure of $E_{v'}$ is unramified over $F_v$.

**Remark 1.2.** The purpose of this technical definition is that for certain places $v$ in $V_F^{\text{spl}} \cup V_F^{\text{int}}$, we need to have a CM type of $E$ such that its reflex field does not contain more ramification over $p$ than $F_v$ does – this is possible for $v \in V_F^{\diamond}$. Note that

- the complement $(V_F^{\text{spl}} \cup V_F^{\text{int}}) \setminus V_F^{\diamond}$ is finite;
- when $E$ is Galois, or contains an imaginary quadratic field, or satisfies $V_F^{\text{ram}} = \emptyset$, we have $V_F^{\diamond} = V_F^{\text{spl}} \cup V_F^{\text{int}}$.

Take an even positive integer $n = 2r$. We equip $W_r := E^n$ with the skew-hermitian form (with respect to the involution $c$) given by the matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Put $G_r := U(W_r)$, the unitary group of $W_r$, which is a quasi-split reductive group over $F$. For every $v \in V_F^{\text{fin}}$, we denote by $K_{r,v} \subseteq G_r(F_v)$ the stabilizer of the lattice $O_{E_v}^n$, which is a special maximal compact subgroup.

**Setup 1.3.** Suppose that $F \neq \mathbb{Q}$, that $V_F^{\text{spl}}$ contains all 2-adic places, and that every prime in $V_F^{\text{ram}}$ is unramified over $\mathbb{Q}$. We consider a cuspidal automorphic representation $\pi$ of $G_r(A_F)$ realized on a space $\mathcal{V}_\pi$ of cusp forms, satisfying:

1. For every $v \in V_F^{(\infty)}$, $\pi_v$ is the holomorphic discrete series representation of Harish-Chandra parameter $\left\{ \frac{-1}{2}, \frac{n-3}{2}, \ldots, \frac{3-n}{2}, \frac{1-n}{2} \right\}$.
2. For every $v \in V_F^{\text{ram}}$, $\pi_v$ is spherical with respect to $K_{r,v}$, that is, $\pi_v^{K_{r,v}} \neq \{0\}$.
3. For every $v \in V_F^{\text{int}}$, $\pi_v$ is either unramified or almost unramified (see Remark 1.4 below) with respect to $K_{r,v}$; moreover, if $\pi_v$ is almost unramified, then $v$ is unramified over $\mathbb{Q}$.
(4) For every \( v \in V^\text{fin}_F \), \( \pi_v \) is tempered.\(^1\)

(5) We have \( \mathcal{R}_\pi \cup \mathcal{S}_\pi \subseteq V^\text{\#}_F \) (Definition 1.1), where

- \( \mathcal{R}_\pi \subseteq V^\text{spl}_F \) denotes the (finite) subset for which \( \pi_v \) is ramified,
- \( \mathcal{S}_\pi \subseteq V^\text{inf}_F \) denotes the (finite) subset for which \( \pi_v \) is almost unramified.

Comparing Setup 1.3 with [LL, Setup 1.3], we have lifted the restriction that \( V^\text{ram}_F = \emptyset \) (by allowing \( \pi_v \) to be a certain type of representations for \( v \in V^\text{ram}_F \)), and also the restriction on \( \pi_v \) for \( v \in V^\text{spl}_F \). Note that (5) is not really a new restriction since when \( V^\text{ram}_F = \emptyset \), it is automatic by Remark 1.2.

**Remark 1.4.** The notion of almost unramified representations of \( G_r(F_v) \) at \( v \in V^\text{fin}_F \) is defined in [Liub, Definition 5.3]. Roughly speaking, an irreducible admissible representation \( \pi_v \) of \( G_r(F_v) \) is almost unramified (with respect to \( K_{r,v} \)) if \( \pi_v^{L,r,v} \) contains a particular character as a module over \( \mathbb{C}[I_{r,v}\backslash K_{r,v}/I_{r,v}] \), where \( I_{r,v} \) is an Iwahori subgroup contained in \( K_{r,v} \), and that the Satake parameter of \( \pi_v \) contains the pair \( \{q_v, q_v^{-1}\} \); it is not unramified. By [Liub, Theorem 1.2], when \( q_v \) is odd, *almost* unramified representations are exactly those representations whose local theta lifting to the non-quasi-split unitary group of the same rank \( 2r \) has nonzero invariants under the stabilizer of an *almost* self-dual lattice.

Suppose that we are in Setup 1.3. Denote by \( L(s, \pi) \) the doubling \( L \)-function. Then we have \( \varepsilon(s) = (-1)^{r[F : \mathbb{Q}] + |S_\pi|} \) for the global (doubling) root number, so that the vanishing order of \( L(s, \pi) \) at the center \( s = \frac{1}{2} \) has the same parity as \( r[F : \mathbb{Q}] + |S_\pi| \). The cuspidal automorphic representation \( \pi \) determines a hermitian space \( V_\pi \) over \( \mathbb{A}_E \) of rank \( n \) via local theta dichotomy (so that the local theta lifting of \( \pi_v \) to \( U(V_\pi)(F_v) \) is nontrivial for every place \( v \) of \( F \)), unique up to isomorphism, which is totally positive definite and satisfies that for every \( v \in V^\text{fin}_F \), the local Hasse invariant \( \varepsilon(V_\pi \otimes_{\mathbb{A}_E} F_v) = 1 \) if and only if \( v \not\in S_\pi \).

Now suppose that \( r[F : \mathbb{Q}] + |S_\pi| \) is odd hence \( \varepsilon(\pi) = -1 \), which is equivalent to that \( V_\pi \) is incoherent. In what follows, we take \( V = V_\pi \) in the context of [LL, Conjecture 1.1], hence \( H = U(V_\pi) \). Let \( \mathcal{R} \) be a finite subset of \( V^\text{fin}_F \). We fix a special maximal subgroup \( L_\mathcal{R} \) of \( H(\mathbb{A}_E^{\infty,R}) \) that is the stabilizer of a lattice \( \Lambda^R \) in \( V \otimes_{\mathbb{A}_E} \mathbb{A}_E^{\infty,R} \) (see Setup 4.2(H6) for more details). For a field \( \mathbb{L} \), we denote by \( \mathcal{T}^R_{\mathbb{L}} \) the (abstract) Hecke algebra \( \mathbb{L}[L^R \backslash H(\mathbb{A}_E^{\infty,R})/L^R] \), which is a commutative \( \mathbb{L} \)-algebra. When \( \mathbb{R} \) contains \( \mathcal{R} \), the cuspidal automorphic representation \( \pi \) gives rise to a character

\[ \chi^R_\pi : \mathcal{T}^R_{\mathbb{L}^{ac}} \rightarrow \mathbb{L}^{ac}, \]

where \( \mathbb{L}^{ac} \) denotes the subfield of \( \mathbb{C} \) of algebraic numbers; and we put \( \mathfrak{m}^R_\pi := \ker \chi^R_\pi \), which is a maximal ideal of \( \mathcal{T}^R_{\mathbb{L}^{ac}} \). This is the following is the first main theorem of this article.

**Theorem 1.5.** Let \( (\pi, V_\pi) \) be as in Setup 1.3 with \( r[F : \mathbb{Q}] + |S_\pi| \) odd, for which we assume [LL, Hypothesis 5.6]. If \( L'(\frac{1}{2}, \pi) \neq 0 \), that is, \( \text{ord}_{s=\frac{1}{2}} L(s, \pi) = 1 \), then as long as \( \mathcal{R} \) satisfies \( \mathcal{R}_\pi \subseteq \mathcal{R} \) and \( |\mathcal{R} \cap V^\text{spl}_F \cap V^\text{\#}_F| \geq 2 \), the nonvanishing

\[ \lim_{L_\mathcal{R}} \left( \text{CH}^r(X_{L_\mathcal{R}})_{\mathbb{L}^{ac}}^\mathfrak{m}^R_\pi \right) \neq \{0\} \]

holds, where the colimit is taken over all open compact subgroups \( L_\mathcal{R} \) of \( H(F_\mathcal{R}) \).

Our remaining results rely on Hypothesis 4.11 on the modularity of Kudla’s generating functions of special cycles, hence are conditional at this moment.

\(^1\)In fact, (4) is implied by (1). See [LL, Remark 1.3(2)].
**Theorem 1.6.** Let \((\pi, \mathcal{V}_\pi)\) be as in Setup 1.3 with \(r[F : \mathbb{Q}] + |\mathcal{S}_\pi|\) odd, for which we assume [LL, Hypothesis 5.6]. Assume Hypothesis 4.11 on the modularity of generating functions of codimension \(r\).

1. For every test vectors
   - \(\varphi_1 = \otimes_v \varphi_{1v} \in \mathcal{V}_\pi\) and \(\varphi_2 = \otimes_v \varphi_{2v} \in \mathcal{V}_\pi\) such that for every \(v \in \mathcal{V}_F^{(\infty)}\), \(\varphi_{1v}\) and \(\varphi_{2v}\) have the lowest weight and satisfy \(\langle \varphi_{1v}^\infty, \varphi_{2v}^\infty \rangle_{\pi_v} = 1\),
   - \(\phi_{1v}^\infty = \otimes_v \phi_{1v}^\infty \in \mathcal{S}(V_r \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty})\) and \(\phi_{2v}^\infty = \otimes_v \phi_{2v}^\infty \in \mathcal{S}(V_r \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty})\),
   the identity
   \[
   \langle \Theta_{\phi_{1v}^\infty}(\varphi_1), \Theta_{\phi_{2v}^\infty}(\varphi_2) \rangle_{X,E}^r = \frac{L'(\frac{1}{2}, \pi)}{b_{2v}(0)} \cdot C_r^{[F : \mathbb{Q}]} \cdot \prod_{v \in \mathcal{V}_F^{(\infty)}} 3_{\pi_v, V_v}(\varphi_{1v}^\infty, \varphi_{2v}^\infty, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^c)
   \]
   holds. Here,
   - \(\Theta_{\phi_{1v}^\infty}(\varphi_1) \in \lim_{L} CH^r(X_L)_C^0\) is the arithmetic theta lifting (Definition 4.12), which is only well-defined under Hypothesis 4.11;
   - \(\langle \Theta_{\phi_{1v}^\infty}(\varphi_1), \Theta_{\phi_{2v}^\infty}(\varphi_2) \rangle_{X,E}^r\) is the normalized height pairing (Definition 4.17), which is constructed based on Beilinson’s notion of height pairing;
   - \(b_{2v}(0)\) is defined in Setup 4.1(F4), which equals \(L(M_r^{(1)}(1))\) where \(M_r\) is the motive associated to \(G_r\) by Gross [Gro97], and is in particular a positive real number;
   - \(C_r = (-1)^{2(r-1)} \pi^2 r^{\Gamma(1) - \Gamma(r)} r^{\Gamma(r+1) - \Gamma(2r)}\), which is the exact value of a certain archimedean doubling zeta integral; and
   - \(3_{\pi_v, V_v}(\varphi_{1v}^\infty, \varphi_{2v}^\infty, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^c)\) is the normalized local doubling zeta integral [LL, Section 3], which equals 1 for all but finitely many \(v\).

2. In the context of [LL, Conjecture 1.1], take \(V = V_\pi\) and \(\tilde{\pi}_\infty\) to be the theta lifting of \(\pi_\infty\) to \(H(\mathbb{A}_F^{\infty})\). If \(L'(\frac{1}{2}, \pi) \neq 0\), that is, \(\text{ord}_{s=\frac{1}{2}} L(s, \pi) = 1\), then
   \[
   \text{Hom}_{H(\mathbb{A}_F^{\infty})}(\tilde{\pi}_\infty, \lim_{L} \text{CH}^r(X_L)_C^0) \neq \{0\}
   \]
   holds.

**Remark 1.7.** We have the following remarks concerning Theorem 1.6.

1. Part (1) verifies the so-called arithmetic inner product formula, a conjecture proposed by one of us [Liu11a, Conjecture 3.11].

2. The arithmetic inner product formula in part (1) is perfectly parallel to the classical Rallis inner product formula. In fact, suppose that \(V\) is totally positive definite but coherent. We have the classical theta lifting \(\theta_{\phi_{1v}^\infty}(\varphi)\) where we use standard Gaussian functions at archimedean places. Then the Rallis inner product formula in this case reads as
   \[
   \langle \theta_{\phi_{1v}^\infty}(\varphi_1), \theta_{\phi_{2v}^\infty}(\varphi_2) \rangle_H = \frac{L(\frac{1}{2}, \pi)}{b_{2v}(0)} \cdot C_r^{[F : \mathbb{Q}]} \cdot \prod_{v \in \mathcal{V}_F^{(\infty)}} 3_{\pi_v, V_v}(\varphi_{1v}^\infty, \varphi_{2v}^\infty, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^c),
   \]
   in which \(\langle , \rangle_H\) denotes the Petersson inner product with respect to the Tamagawa measure on \(H(\mathbb{A}_F)\).

In the case where \(\mathcal{R}_\pi = \emptyset\), we have a very explicit height formula for test vectors that are new everywhere.
Corollary 1.8. Let \((\pi, \mathcal{V}_\pi)\) be as in Setup 1.3 with \(r[F : \mathbb{Q}] + |S_\pi|\) odd, for which we assume \([LL, Hypothesis 5.6]\). Assume Hypothesis 4.11 on the modularity of generating functions of codimension \(r\). In the situation of Theorem 1.6(1), suppose further that

- \(R_\pi = \emptyset\);
- \(\varphi_1 = \varphi_2 = \varphi \in \mathcal{V}_\pi^{|r|\emptyset}\) (see Setup 4.3(G8) for the precise definition of the one-dimensional space \(\mathcal{V}_\pi^{|r|\emptyset}\) of holomorphic new forms) such that for every \(v \in \mathcal{V}_F\), \(\langle \varphi^c_v, \varphi_v \rangle_{\pi_v} = 1\); and
- \(\phi_1^\infty = \phi_2^\infty = \phi^\infty\) such that for every \(v \in \mathcal{V}_F^\fin\), \(\phi_v^\infty = 1_{(\Lambda_v^0)^r}\).

Then the identity

\[
\langle \Theta_{\phi^\infty}(\varphi), \Theta_{\phi^\infty}(\varphi) \rangle_{X,E}^r = (-1)^r \cdot \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot |C_t|^{[F : \mathbb{Q}]} \cdot \prod_{v \in S_\pi} \frac{q_v^{r-1}(q_v + 1)}{(q_v^{2r-1} + 1)(q_v^{2r} - 1)}
\]

holds.

Remark 1.9. Assuming the conjecture on the injectivity of the étale Abel–Jacobi map, one can show that the cycle \(\Theta_{\phi^\infty}(\varphi)\) is a primitive cycle of codimension \(r\). By \([Be\˘ı87, Conjecture 5.5]\), we expect that \((-1)^r \langle \Theta_{\phi^\infty}(\varphi), \Theta_{\phi^\infty}(\varphi) \rangle_{X,E}^r \geq 0\) holds, which, in the situation of Corollary 1.8, is equivalent to \(L'(\frac{1}{2}, \pi) \geq 0\).

Example 1.10. Suppose that \(E/F\) satisfies the conditions in Setup 1.3 and that \(r \geq 2\). Consider an elliptic curve \(A\) over \(F\) without complex multiplication, satisfying that \(\text{Sym}^{2r-1} A\) hence \(\text{Sym}^{2r-1} A_F\) are modular. Let \(\Pi\) be the cuspidal automorphic representation of \(\text{GL}_n(\mathbb{A}_E)\) corresponding to \(\text{Sym}^{2r-1} A_F\), which satisfies \(\Pi^\vee \simeq \Pi \circ c\). Then there exists a cuspidal automorphic representation \(\pi\) of \(G_r(\mathbb{A}_E)\) as in Setup 1.3 with \(\Pi\) its base change if and only if \(A\) has good reduction at every \(v \in \mathcal{V}_F^\fin \setminus \mathcal{V}_F^\pl\). Moreover, if this is the case, then we have \(S_\pi = \emptyset\) hence \(\varepsilon(\pi) = (-1)^{[F : \mathbb{Q}]}\); in particular, above results apply when both \(r\) and \([F : \mathbb{Q}]\) are odd.

1.2. Two new ingredients. The proofs of our main theorems follow the same line in \([LL]\), with two new (main) ingredients, responsible for the two improvements we have mentioned at the beginning.

The first new ingredient is formulating and proving an analogue of the Kudla–Rapoport conjecture in the case where \(E/F\) is ramified and the level structure is the one that gives the exotic smooth model. Here, \(F\) is a \(p\)-adic field with \(p\) odd. Let \(L\) be an \(O_E\)-lattice of a nonsplit (nondegenerate) hermitian space \(V\) over \(E\) of (even) rank \(n\). Then one can associate an intersection number \(\text{Int}(L)\) of special divisors on a formally smooth relative Rapoport–Zink space classifying quasi-isogenies of certain unitary \(O_F\)-divisible groups, and also the derivative of the representation density function \(\partial \text{Den}(L)\) given by \(L\). We show in Theorem 2.9 the formula

\[
\text{Int}(L) = \partial \text{Den}(L).
\]

This is parallel to the Kudla–Rapoport conjecture proved in \([LZ]\), originally stated for the case where \(E/F\) is unramified. The proof follows from the same strategy as in \([LZ]\), namely, we write \(L = L^\flat + \langle x \rangle\) for a sublattice \(L^\flat\) of \(L\) such that \(V_{L^\flat} := L^\flat \otimes_{O_F} F\) is nondegenerate, and regard \(x\) as a variable. Thus, it motivates us to define a function \(\text{Int}_{L^\flat}\) on \(V \setminus V_{L^\flat}\) by the formula \(\text{Int}_{L^\flat}(x) = \text{Int}(L^\flat + \langle x \rangle)\) and similarly for \(\partial \text{Den}_{L^\flat}\). For \(\text{Int}_{L^\flat}\), there is a natural decomposition \(\text{Int}_{L^\flat} = \text{Int}_{L^\flat}^h + \text{Int}_{L^\flat}^v\) according to the horizontal and vertical parts of the special cycle defined by \(L^\flat\). In a parallel manner, we have the decomposition \(\partial \text{Den}_{L^\flat} = \partial \text{Den}_{L^\flat}^h + \partial \text{Den}_{L^\flat}^v\) by simply matching \(\partial \text{Den}_{L^\flat}^h\) with \(\text{Int}_{L^\flat}^h\). Thus, it suffices to
show that $\text{Int}^\nu_{L^\flat} = \partial \text{Den}^\nu_{L^\flat}$. By some sophisticated induction argument on $L^\flat$, it suffices to show the following remarkable property for both $\text{Int}^\nu_{L^\flat}$ and $\partial \text{Den}^\nu_{L^\flat}$: they extend (uniquely) to compactly supported locally constant functions on $V$, whose Fourier transforms are supported in the set $\{ x \in V \mid (x, x)_V \in O_F \}$. However, there are some new difficulties in our case:

- The isomorphism class of an $O_E$-lattice is not determined by its fundamental invariants, and there is a parity constraint for the valuation of an $O_E$-lattice. This will make the induction argument on $L^\flat$ much more complicated than the one in [LZ] (see Subsection 2.7).
- The comparison of our relative Rapoport–Zink space to an (absolute) Rapoport–Zink space is not known. This is needed even to show that our relative Rapoport–Zink space is representable, and also in the $p$-adic uniformization of Shimura varieties. We solve this problem when $F/\mathbb{Q}_p$ is unramified, which is the reason for us to assume that every prime in $\mathfrak{v}_F$ is unramified over $\mathbb{Q}$ in Setup 1.3. See Subsection 2.8.
- Due to the parity constraint, the computation of $\text{Int}^\nu_{L^\flat}$ can only be reduced to the case where $n = 4$ (rather than $n = 3$ in [LZ]). After that, we have to compute certain intersection multiplicity, for which we use a new argument based on the linear invariance of the K-theoretic intersection of special divisors. See Lemma 2.56.

Here comes three more remarks:

- First, we need to extend the result of [CY20] on a counting formula for $\partial \text{Den}(L)$ to hermitian spaces over a ramified extension $E/F$ (Lemma 2.20).
- Second, we have found a simpler argument for the properties of $\partial \text{Den}^\nu_{L^\flat}$ (Proposition 2.23), which does not use any functional equation or induction formula. This argument is applicable to [LZ] to give a new proof of the main result on the analytic side there. Also note that we prove the vanishing property in Proposition 2.23 directly, while in [LZ] it is only deduced after proving $\text{Int}^\nu_{L^\flat} = \partial \text{Den}^\nu_{L^\flat}$.
- Finally, unlike the case in [LZ], the parity of the dimension of the hermitian space plays a crucial role in the exotic smooth case. In particular, we will not study the case where $V$ has odd dimension.

The second new ingredient is a vanishing result on certain cohomology of integral models of unitary Shimura varieties with Drinfeld level structures. For $v \in V^{\text{spl}}_F \setminus V^{\otimes}_F$ with $p$ the underlying rational prime, we have a tower of integral models $\{ X_m \}_{m \geq 0}$ defined by Drinfeld level structures (at $v$), with an action by $\mathcal{H}^{\text{R}^\nu(p)}_{\text{Qac}}$ via Hecke correspondences. We show in Theorem 4.21 that

$$H^{2r}(X_m, \overline{\mathbb{Q}}(r))_m = 0$$

with $\ell \neq p$ and $m := m_v \cap S_{\text{Qac}}^{\text{R}^\nu(p)}$, where $S_{\text{Qac}}^{\text{R}^\nu(p)}$ is the subalgebra of $\mathcal{H}^{\text{R}^\nu(p)}_{\text{Qac}}$ consisting of those supported at split places. We reduce this vanishing property to some other vanishing properties for cohomology of Newton strata of $X_m$, by using a key result of Mantovan [Man08] saying that the closure of every refined Newton stratum is smooth. For the vanishing properties for Newton strata, we generalize an argument of [TY07, Proposition 4.4]. However, since in our case, the representation $\pi_v$ has arbitrary level and our group has nontrivial endoscopy, we need a more sophisticated trace formula, which was provided in [CS17].

1.3. Notation and conventions.
• When we have a function $f$ on a product set $A_1 \times \cdots \times A_m$, we will write $f(a_1, \ldots, a_m)$ instead of $f((a_1, \ldots, a_m))$ for its value at an element $(a_1, \ldots, a_m) \in A_1 \times \cdots \times A_m$.

• For a set $S$, we denote by $\mathbb{1}_S$ the characteristic function of $S$.

• All rings (but not algebras) are commutative and unital; and ring homomorphisms preserve units.

• For a (formal) subscheme $Z$ of a (formal) scheme $X$, we denote by $\mathcal{I}_Z$ the ideal sheaf of $Z$, which is a subsheaf of the structure sheaf $\mathcal{O}_X$ of $X$.

• For a ring $R$, we denote by $\text{Sch}/R$ the category of schemes over $R$, by $\text{Sch}^\text{f}/R$ the subcategory of locally Noetherian schemes over $R$, and when $R$ is discretely valued, by $\text{Sch}^\text{f}/R$ the subcategory of schemes on which uniformizers of $R$ are locally nilpotent.

• If a base ring is not specified in the tensor operation $\otimes$, then it is $\mathbb{Z}$.

• For an abelian group $A$ and a ring $R$, we put $A_R := A \otimes R$.

• For an integer $m \geq 0$, we denote by $0_m$ and $1_m$ the null and identity matrices of rank $m$, respectively. We also denote by $w_m$ the matrix $\begin{pmatrix} -1_m & 1_m \end{pmatrix}$.

• We denote by $c : \mathbb{C} \to \mathbb{C}$ the complex conjugation. For an element $x$ in a complex space with a default underlying real structure, we denote by $x^c$ its complex conjugation.

• For a field $K$, we denote by $\overline{K}$ the abstract algebraic closure of $K$. However, for aesthetic reason, we will write $\overline{\mathbb{Q}}_p$ instead of $\overline{\mathbb{Q}}_p$ and will denote by $\overline{\mathbb{F}}_p$ its residue field. On the other hand, we denote by $\mathbb{Q}^{ac}$ the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$.

• For a number field $K$, we denote by $\psi_K : K\backslash A_K \to \mathbb{C}^\times$ the standard additive character, namely, $\psi_K := \psi_{\mathbb{Q}} \circ \text{Tr}_{K/\mathbb{Q}}$ in which $\psi_{\mathbb{Q}} : \mathbb{Q}\backslash A \to \mathbb{C}^\times$ is the unique character such that $\psi_{\mathbb{Q},\infty}(x) = e^{2\pi i x}$.

• Throughout the entire article, all parabolic inductions are unitarily normalized.

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2. Intersection of special cycles at ramified places

Throughout this section, we fix a ramified quadratic extension $E/F$ of $p$-adic fields with $p$ odd, with $c \in \text{Gal}(E/F)$ the Galois involution. We fix a uniformizer $u \in E$ satisfying $u^c = -u$. Let $k$ be the residue field of $F$ and denote by $q$ the cardinality of $k$. Let $n = 2r$ be an even positive integer.

In Subsection 2.1, we introduce our relative Rapoport–Zink space and state the main theorem (Theorem 2.9) on the relation between intersection numbers and derivatives of representation densities. In Subsection 2.2, we study derivatives of representation densities. In Subsection 2.3, we recall the Bruhat–Tits stratification on the relative Rapoport–Zink space from [Wu] and deduce some consequences. In Subsection 2.4, we prove the linear invariance on the $K$-theoretic intersection of special divisors, following [How19]. In Subsection 2.5, we prove Theorem 2.9 when $r = 1$, which is needed for the proof when $r > 1$. In Subsection 2.6, we study intersection numbers. In Subsection 2.7, we prove Theorem 2.9 for general $r$. In Subsection 2.8, we compare our relative Rapoport–Zink space to certain (absolute) Rapoport–Zink space assuming $F/\mathbb{Q}_p$ is unramified.

Here are two preliminary definitions for this section:

• A hermitian $O_E$-module is a finitely generated free $O_E$-module $L$ together with an $O_E$-bilinear pairing $(\ , \ )_L : L \times L \to E$ such that the induced $E$-valued pairing on $L \otimes_{O_E} F$
is a nondegenerate hermitian pairing (with respect to $c$). When we say that a hermitian $O_F$-module $L$ is contained in a hermitian $O_E$-module or a hermitian $E$-space $M$, we require that the restriction of the pairing $(\ , \ )_M$ to $L$ coincides with $(\ , \ )_L$.

- Let $X$ be an object of an additive category with a notion of dual.
  - We say that a morphism $\sigma_X : X \to X^\vee$ is a symmetric* if $\sigma_X$ is an isomorphism and the composite morphism $X \to X^\vee \xrightarrow{\sigma_X^\vee} X^\vee$ coincides with $\sigma_X$.
  - Given an action $\iota_X : O_E \to \text{End}(X)$, we say that a morphism $\lambda_X : X \to X^\vee$ is $\iota_X$-compatible* if $\lambda_X \circ \iota_X(\alpha) = \iota_X(\alpha^c) \circ \lambda_X$ holds for every $\alpha \in O_E$.

2.1. A Kudla–Rapoport type formula. We fix an embedding $\varphi_0 : E \to \mathbb{C}_p$ and let $\bar{E}$ be the maximal complete unramified extension of $\varphi_0(E)$ in $\mathbb{C}_p$. We regard $E$ as a subfield of $\bar{E}$ via $\varphi_0$ hence identify the residue field of $\bar{E}$ with an algebraic closure $\overline{k}$ of $k$.

**Definition 2.1.** Let $S$ be an object of $\text{Sch}_{/O_{\bar{E}}}$*. We define a category $\text{Exo}_{(n-1,1)}(S)$ whose objects are triples $(X, \iota_X, \lambda_X)$ in which

- $X$ is an $O_F$-divisible group* over $S$ of dimension $n$ and (relative) height $2n$;
- $\iota_X : O_E \to \text{End}(X)$ is an action of $O_E$ on $X$ satisfying:
  - (Kottwitz condition): the characteristic polynomial of $\iota_X(u)$ on the $\mathcal{O}_S$-module $\text{Lie}(X)$ is $(T - u)^n(T + u) \in \mathcal{O}_S[T]$,
  - (Wedge condition): we have
    \[ \bigwedge^2 (\iota_X(u) - u \mid \text{Lie}(X)) = 0, \]
- (Spin condition): for every geometric point $s$ of $S$, the action of $\iota_X(u)$ on $\text{Lie}(X_s)$ is nonzero;
- $\lambda_X : X \to X^\vee$ is a $\iota_X$-compatible polarization such that $\ker(\lambda_X) = X[\iota_X(u)]$.

A morphism (resp. quasi-morphism) from $(X, \iota_X, \lambda_X)$ to $(Y, \iota_Y, \lambda_Y)$ is an $O_E$-linear isomorphism (resp. quasi-isogeny) $\rho : X \to Y$ of height zero such that $\rho^* \lambda_Y = \lambda_X$.

When $S$ belongs to $\text{Sch}^\text{\acute{e}}_{/O_{\bar{E}}}$, we denote by $\text{Exo}^b_{(n-1,1)}(S)$ the subcategory of $\text{Exo}_{(n-1,1)}(S)$ consisting of $(X, \iota_X, \lambda_X)$ in which $X$ is supersingular.

**Remark 2.2.** Giving a $\iota_X$-compatible polarization $\lambda_X$ of $X$ satisfying $\ker(\lambda_X) = X[\iota_X(u)]$ is equivalent to giving a $\iota_X$-compatible symmetrization $\sigma_X$ of $X$. In fact, since $\ker(\lambda_X) = X[\iota_X(u)]$, there is a unique morphism $\sigma_X : X \to X^\vee$ satisfying $\lambda_X = \sigma_X \circ \iota_X(u)$, which is in fact an isomorphism, satisfying
\[ \sigma_X^\vee = \iota_X(u^{-1})^{\vee} \circ \lambda_X^\vee = -\iota_X(u^{-1})^{\vee} \circ \lambda_X = -\lambda_X \circ \iota_X(u^{-1})^c = \lambda_X \circ \iota_X(u^{-1}) = \sigma_X, \]
and is clearly $\iota_X$-compatible. Conversely, given a $\iota_X$-compatible symmetrization $\sigma_X$ of $X$, we may recover $\lambda_X$ as $\sigma_X \circ \iota_X(u)$. In what follows, we call $\sigma_X$ the symmetrization of $\lambda_X$.

To define our relative Rapoport–Zink space, we fix an object $(X, \iota_X, \lambda_X) \in \text{Exo}^b_{(n-1,1)}(\overline{k})$.

**Definition 2.3.** We define a functor $\mathcal{N} := \mathcal{N}_{(X, \iota_X, \lambda_X)}$ on $\text{Sch}^\text{\acute{e}}_{/O_{\bar{E}}}$ such that for every object $S$ of $\text{Sch}^\text{\acute{e}}_{/O_{\bar{E}}}$, $\mathcal{N}(S)$ consists of quadruples $(X, \iota_X, \lambda_X; \rho_X)$ in which

- $(X, \iota_X, \lambda_X)$ is an object of $\text{Exo}^b_{(n-1,1)}(S)$;

\[ \text{An } O_F\text{-divisible group is also called a strict } O_F\text{-module.} \]
• \( \rho_X \) is a quasi-morphism from \((X, \iota_X, \lambda_X) \times S (S \otimes_{O_E} \mathbb{k})\) to \((X, \iota_X, \lambda_X) \otimes_k (S \otimes_{O_E} \mathbb{k})\) in the category \( \text{Exo}^{(n-1,1)}_{O_E}(S \otimes_{O_E} \mathbb{k}) \).

**Hypothesis 2.4.** The functor \( \mathcal{N} \) is (pro-)represented by a separated formal scheme over \( \text{Spf} \, O_E \).

**Remark 2.5.** When \( F \) is unramified over \( \mathbb{Q}_p \), Hypothesis 2.4 is known. In fact, by Corollary 2.66, \( \mathcal{N} \) is isomorphic to an absolute Rapoport–Zink space \( \mathcal{N}^\Phi \) which is known to be a separated formal scheme over \( \text{Spf} \, O_E \) by [RZ96].

In what follows, we will assume Hypothesis 2.4.

**Lemma 2.6.** The functor \( \mathcal{N} \) is a separated formal scheme formally smooth over \( \text{Spf} \, O_E \) of relative dimension \( n - 1 \). Moreover, \( \mathcal{N} \) has two connected components.

**Proof.** The formal smoothness of \( \mathcal{N} \) follow from the smoothness of its local model, which is [RSZ17, Proposition 3.10]; and the dimension also follows. For the last assertion, our moduli functor \( \mathcal{N} \) is the disjoint union of \( \mathcal{N}_{(0,0)} \) and \( \mathcal{N}_{(0,1)} \) from [Wu, Section 3.4], each of which is connected by [Wu, Theorem 5.18(2)].\(^3\)

To study special cycles on \( \mathcal{N} \), we fix a triple \((X_0, \iota_{X_0}, \lambda_{X_0})\) where

• \( X_0 \) is a supersingular \( O_F \)-divisible group over \( \text{Spec} \, O_E \) of dimension 1 and height 2;
• \( \iota_{X_0} : O_E \to \text{End}(X_0) \) is an \( O_E \)-action on \( X_0 \) such that the induced action on \( \text{Lie}(X_0) \) is given by \( \varphi_0 \);
• \( \lambda_{X_0} : X_0 \to X_0^\vee \) is a \( \iota_{X_0} \)-compatible principal polarization.

Note that \( \iota_{X_0} \) induces an isomorphism \( \iota_{X_0} : O_E \xrightarrow{\sim} \text{End}_{O_E}(X_0) \). Put

\[
V := \text{Hom}_{O_E}(X_0 \otimes_{O_E} \mathbb{k}, X) \otimes \mathbb{Q},
\]

which is a vector space over \( E \) of dimension \( n \). We have a pairing

\[
(\ , \ )_V : V \times V \to E
\]

sending \((x, y) \in V^2\) to the composition of quasi-homomorphisms

\[
X_0 \xrightarrow{\iota_X} X \xrightarrow{\lambda_X} X^\vee \xrightarrow{y^\vee} X_0 = \xrightarrow{\iota_{X_0}^{-1}} X_0
\]

as an element in \( \text{End}_{O_E}(X_0) \otimes \mathbb{Q} \) hence in \( E \) via \( \iota_{X_0}^{-1} \). It is known that \((\ , \ )_V\) is a nondegenerate and nonsplit hermitian form on \( V \) [RSZ17, Lemma 3.5].\(^4\)

**Definition 2.7.** For every nonzero element \( x \in V \), we define the *special divisor* \( \mathcal{N}(x) \) of \( \mathcal{N} \) to be the maximal closed formal subscheme over which the quasi-homomorphism

\[
\rho_X^{-1} \circ x : (X_0 \otimes_{O_E} \mathbb{k}) \otimes_k (S \otimes_{O_E} \mathbb{k}) \to X \times_S (S \otimes_{O_E} \mathbb{k})
\]

lifts (uniquely) to a homomorphism \( X_0 \otimes_{O_E} S \to X \).

\(^3\)The article [Wu] only studied the case \( F = \mathbb{Q}_p \). In fact, except for Hypothesis 2.4, all arguments hence results work for general \( F \). This footnote applies to the proof of Proposition 2.29 as well.

\(^4\)Readers may notice that we have an extra factor \( u^{-2} \) in the definition of the hermitian form. This is because we want to ensure that \( \mathcal{N}(x) \) is nonempty if and only if \((x, x)_V \in O_F \).
Definition 2.8. For an $O_E$-lattice $L$ of $V$, the Serre intersection multiplicity
\[ \chi \left( O_{N(x_1)} \otimes \cdots \otimes O_{N(x_n)} \right) \]
does not depend on the choice of a basis $\{x_1, \ldots, x_n\}$ of $L$ by Corollary 2.36, which we define to be $\text{Int}(L)$.

**Theorem 2.9.** Assume Hypothesis 2.4. For every $O_E$-lattice $L$ of $V$, we have
\[ \text{Int}(L) = \partial \text{Den}(L), \]
where $\partial \text{Den}(L)$ is defined in Definition 2.17.

By Remark 2.5, this theorem is unconditional if $F$ is unramified over $Q_p$.

The strategy of proving this theorem described in Subsection 1.2 motivates the following definition, which will be frequently used in the rest of Section 2.

Definition 2.10. We define $b(V)$ to be the set of hermitian $O_E$-modules contained in $V$ of rank $n - 1$. In what follows, for $L^\flat \in b(V)$, we put $V_{L^\flat} := L^\flat \otimes_{O_F} F$ and write $V_{L^\flat}^\perp$ for the orthogonal complement of $V_{L^\flat}$ in $V$.

**Remark 2.11.** Let $S$ be an object of $\text{Sch}_{/O_E}$. We have another category $\text{Exo}_{(n,0)}(S)$ whose objects are triples $(X, \iota_X, \lambda_X)$ in which
- $X$ is an $O_F$-divisible group over $S$ of dimension $n$ and (relative) height $2n$;
- $\iota_X : O_E \to \text{End}(X)$ is an action of $O_E$ on $X$ such that $\iota_X(u) - u$ annihilates $\text{Lie}(X)$;
- $\lambda_X : X \to X^\vee$ is a $\iota_X$-compatible polarization such that $\ker(\lambda_X) = X[\iota_X(u)]$.

Morphisms are defined similarly as in Definition 2.1. The category $\text{Exo}_{(n,0)}(S)$ is a connected groupoid.

For later use, we fix a nontrivial additive character $\psi_F : F \to \mathbb{C}^\times$ of conductor $O_F$. For a locally constant compactly supported function $\phi$ on a hermitian space $V$ over $E$, its Fourier transform $\hat{\phi}$ is defined by
\[ \hat{\phi}(x) = \int_V \phi(y) \psi_F(\text{Tr}_{E/F}(x, y)_V) \, dy \]
where $dy$ is the self-dual Haar measure on $V$.

2.2. **Fourier transform of analytic side.** In this subsection, we study local densities of hermitian lattices. We first introduce some notion about $O_E$-lattices in hermitian spaces.

**Definition 2.12.** Let $V$ be a hermitian space over $E$ of dimension $m$, equipped with the hermitian form $(\cdot, \cdot)_V$.

1. For a subset $X$ of $V$,
   - we denote by $X^\text{int}$ the subset $\{x \in X \mid (x, x)_V \in O_F\}$;
   - we denote by $\langle X \rangle$ the $O_E$-submodule of $V$ generated by $X$; when $X = \{x, \ldots\}$ is explicitly presented, we simply write $\langle x, \ldots \rangle$ instead of $\langle \{x, \ldots\} \rangle$.

2. For an $O_E$-lattice $L$ of $V$, we put
\[ L^\vee := \{x \in V \mid \text{Tr}_{E/F}(x, y)_V \in O_F \text{ for every } y \in L\} \]
\[ = \{x \in V \mid (x, y)_V \in u^{-1}O_E \text{ for every } y \in L\}. \]

We say that $L$ is
- *integral* if $L \subseteq L^\vee$;
• vertex if it is integral such that $L^\vee/L$ is annihilated by $u$; and
• self-dual if $L = L^\vee$.

(3) For an integral $O_E$-lattice $L$ of $V$, we define

• the fundamental invariants of $L$ unique integers $0 \leq a_1 \leq \cdots \leq a_m$ such that $L^\vee/L \cong O_E/(u^{a_1}) \oplus \cdots \oplus O_E/(u^{a_m})$ as $O_E$-modules;
• the type $t(L)$ of $L$ to be the number of nonzero elements in its fundamental invariants; and
• the valuation of $L$ to be $\text{val}(L) := \sum_{i=1}^{m} a_i$; when $L$ is generated by a single element $x$, we simply write $\text{val}(x)$ instead of $\text{val}(\langle x \rangle)$.

The above notation and definitions make sense without specifying $V$, namely, they apply to hermitian $O_E$-modules.

**Remark 2.13.** For an integral hermitian $O_E$-module $L$ of rank $m$ with fundamental invariants $(a_1, \ldots, a_m)$, we have

1. $L$ is vertex if and only if $a_m \leq 1$ and self-dual if and only if $a_m = 0$;
2. $t(L)$ and $\text{val}(L)$ must have the same parity with $m$.

**Remark 2.14.** For a hermitian $O_E$-module $L$, we say that a basis $\{e_1, \ldots, e_m\}$ of $L$ is a normal basis if its moment matrix $T = ((e_i, e_j))_L^{m \times m}$ is conjugate to

$$
\begin{pmatrix}
\beta_1 u^{2b_1} & 0 & \cdots & 0 \\
0 & -u^{2c_1-1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -u^{2c_t-1}
\end{pmatrix}
$$

by a permutation matrix, for some $\beta_1, \ldots, \beta_s \in O_F^\times$ and $b_1, \ldots, b_s, c_1, \ldots, c_t \in \mathbb{Z}$. We have

1. normal basis exists;
2. the invariants $s, t$ and $b_1, \ldots, b_s, c_1, \ldots, c_t$ depend only on $L$;
3. when $L$ is integral, the fundamental invariants of $L$ are the unique nondecreasing rearrangement of $(2b_1 + 1, \ldots, 2b_s + 1, 2c_1, 2c_2, \ldots, 2c_t)$.

**Definition 2.15.** Let $M$ and $L$ be two hermitian $O_E$-modules. We denote by $\text{Herm}_{L,M}$ the scheme of hermitian $O_E$-module homomorphisms from $L$ to $M$, which is a scheme of finite type over $O_F$. We define the local density to be

$$
\text{Den}(M, L) := \lim_{N \to +\infty} \frac{|\text{Herm}_{L,M}(O_F/(u^{2N}))|}{q^{N-d_{L,M}}}
$$

where $d_{L,M}$ is the dimension of $\text{Herm}_{L,M} \otimes_{O_F} F$.

Denote by $H$ the standard hyperbolic hermitian $O_E$-module (of rank 2) given by the matrix

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

For an integer $s \geq 0$, put $H_s := H^\oplus s$. Then $H_s$ is a self-dual hermitian $O_E$-module of rank $2s$. The following lemma is a variant of a result of Cho–Yamauchi [CY20] when $E/F$ is ramified.

**Lemma 2.16.** Let $L$ be a hermitian $O_E$-module of rank $m$. Then we have

$$
\text{Den}(H_s, L) = \sum_{L \subseteq L' \subseteq L^\vee} |L'/L|^{m-2s} \prod_{s-m+t(L') \leq i \leq s}(1 - q^{-2i})
$$

for every integer $s \geq m$, where the sum is taken over integral $O_E$-lattices of $L \otimes_{O_F} F$ containing $L$. 
Proof. Put $V := L \otimes_{O_F} F$. For an integral $O_E$-lattice $L'$ of $V$, we equip the $k$-vector space $L'_k := L' \otimes_{O_E} O_E/(u)$ with a $k$-valued pairing $\langle \cdot, \cdot \rangle_{L'_k}$ by the formula

$$\langle x, y \rangle_{L'_k} := u \cdot (x^s, y^s)_V \mod (u)$$

where $x^s$ and $y^s$ are arbitrary lifts of $x$ and $y$, respectively. Then $L'_k$ becomes a symplectic space over $k$ of dimension $m$ whose radical has dimension $t(L')$. Similarly, we have $H_{s,k}$, which is a nondegenerate symplectic space over $k$ of dimension $2s$. We denote by $\text{Isom}_{L'_k,H_{s,k}}$ the $k$-scheme of isometries from $L'_k$ to $H_{s,k}$.

By the same argument in [CY20, Section 3.3], we have

$$\text{Den}(H_s, L) = q^{-m(4s-m+1)/2} \sum_{L' \subseteq L''} |L'/L|^{m-2s} |\text{Isom}_{L'_k,H_{s,k}}(k)|. $$

Thus, it remains to show that

$$(2.2) \quad |\text{Isom}_{L'_k,H_{s,k}}(k)| = q^{m(4s-m+1)/2} \prod_{s-m+2 \leq i \leq s} (1 - q^{-2i}).$$

We fix a decomposition $L'_k = L_0 \oplus L_1$ in which $L_0$ is nondegenerate and $L_1$ is the radical of $L'_k$. We have a morphism $\pi : \text{Isom}_{L'_k,H_{s,k}} \to \text{Isom}_{L_0,H_{s,k}}$ given by restriction, such that for every element $f \in \text{Isom}_{L_0,H_{s,k}}(k)$, the fiber $\pi^{-1} f$ is isomorphic to $\text{Isom}_{L_1,\text{im}(f) \perp}$. As $\text{im}(f) \perp$ is isomorphic to $H_{s-m+(2s-1),k}$, it suffices to show (2.2) in the two extremal cases: $t(L') = 0$ and $t(L') = m$.

Suppose that $t(L') = 0$, that is, $L'_k$ is nondegenerate. Note that $\text{Sp}(H_{s,k})$ acts on $\text{Isom}_{L'_k,H_{s,k}}$ transitively, with the stabilizer isomorphic to $\text{Sp}(H_{s-m,k})$. Thus, we have

$$|\text{Isom}_{L'_k,H_{s,k}}(k)| = \left| \text{Sp}(H_{s,k})(k) \right| / \left| \text{Sp}(H_{s-m,k})(k) \right| = q^{s^2} \prod_{i=1}^{s} (q^{2i} - 1) / q^{s-m} \prod_{i=1}^{s} (q^{2i} - 1) = q^{m(4s-m+1)/2} \prod_{s-m+2 \leq i \leq s} (1 - q^{-2i}),$$

which confirms (2.2).

Suppose that $t(L') = m$, that is, $L'_k$ is isotropic. Note that $\text{Sp}(H_{s,k})$ acts on $\text{Isom}_{L'_k,H_{s,k}}$ transitively, with the stabilizer $Q$ fits into a short exact sequence

$$1 \to U_m \to Q \to \text{Sp}(H_{s-m,k}) \to 1$$

in which $U_m$ is a unipotent subgroup of $\text{Sp}(H_{s,k})$ of Levi type $\text{GL}_{m,k} \times \text{Sp}(H_{s-m,k})$. Thus, we have

$$|\text{Isom}_{L'_k,H_{s,k}}(k)| = \left| \text{Sp}(H_{s,k})(k) \right| / \left| U_m(k) \right| \left| \text{Sp}(H_{s-m,k})(k) \right| = q^{s^2} \prod_{i=1}^{s} (q^{2i} - 1) / q^{(2s-2m)+\frac{(m+1)}{2}} \cdot q^{s-m} \prod_{i=1}^{s} (q^{2i} - 1) = q^{m(4s-m+1)/2} \prod_{s-m+2 \leq i \leq s} (1 - q^{-2i}),$$

which confirms (2.2).

Thus, (2.2) is proved and the lemma follows. \qed

Now we fix a hermitian space $V$ over $E$ of dimension $n = 2r$ that is nonsplit.
Definition 2.17. For an $O_E$-lattice $L$ of $V$, define the (normalized) local Siegel series of $L$ to be the polynomial $\text{Den}(X, L) \in \mathbb{Z}[X]$, which exists by Lemma 2.20 below, such that for every integer $s \geq 0$,
\[
\text{Den}(q^{-s}, L) = \frac{\text{Den}(H_{r+s}, L)}{\prod_{i=s+1}^{i=s-s+1} (1 - q^{-2i})},
\]
where $\text{Den}$ is defined in Definition 2.15. We then put
\[
\partial \text{Den}(L) := -\frac{d}{dX} \bigg|_{X=1} \text{Den}(X, L).
\]

Remark 2.18. Since $V$ is nonsplit, we have $\text{Den}(1, L) = 0$.

Remark 2.19. Let $L$ be an $O_E$-lattice of $V$. Let $T \in \text{GL}_n(E)$ be a representing matrix of $L$, and consider the $T$-th Whittaker function $W_T(s, 1_{4r}, \mathbb{1}_{H^{2r}})$ of the Schwartz function $\mathbb{1}_{H^{2r}}$ at the identity element $1_{4r}$. By [KR14, Proposition 10.1], we have
\[
W_T(s, 1_{4r}, \mathbb{1}_{H^{2r}}) = \text{Den}(H_{r+s}, L)
\]
for every integer $s \geq 0$. Thus, we obtain
\[
\log q \cdot \partial \text{Den}(L) = \frac{W_T'(0, 1_{4r}, \mathbb{1}_{H^{2r}})}{\prod_{i=1}^{i=s-1} (1 - q^{-2i})}
\]
by Definition 2.17.

Lemma 2.20. For every $O_E$-lattice $L$ of $V$, we have
\[
\text{Den}(X, L) = \sum_{L \subseteq L \subseteq L^\vee} X^{\text{length}_{O_E}(L/L)} \prod_{i=0}^{\frac{t(L)}{2}-1} (1 - q^{2i}X^2),\tag{2.3}
\]
and
\[
\partial \text{Den}(L) = 2 \sum_{L \subseteq L \subseteq L^\vee} \prod_{i=1}^{\frac{t(L)}{2}-1} (1 - q^{2i}),\tag{2.4}
\]
where both sums are taken over integral $O_E$-lattices of $V$ containing $L$.

Proof. The identity (2.3) is a direct consequence of Lemma 2.16 and Definition 2.17. The identity (2.4) is a consequence of (2.3). $\Box$

Definition 2.21. Let $L^\flat$ be an element of $\mathfrak{b}(V)$ (Definition 2.10). For $x \in V \setminus V_{L^\flat}$, we put
\[
\partial \text{Den}_{L^\flat}(x) := \partial \text{Den}(L^\flat + \langle x \rangle),
\]
\[
\partial \text{Den}_L^h(x) := 2 \sum_{L^\flat \subseteq L \subseteq L^\vee} \mathbb{1}_L(x),
\]
\[
\partial \text{Den}_L^v(x) := \partial \text{Den}_{L^\flat}(x) - \partial \text{Den}_L^h(x).
\]
Here in the second formula, $L$ in the summation is an $O_E$-lattice of $V$.

Remark 2.22. We have
\[\text{In [KR14, Proposition 10.1] and its proof, the lattice } L_{r,r} \text{ should be replaced by } H_{r}.\]
\[\text{In (2.4), when } t(L) = 2, \text{ we regard the empty product } \prod_{i=1}^{i=s-1} (1 - q^{2i}) \text{ as 1.}\]
(1) The summation in $\partial \text{Den}^h_{L^b}(x)$ equals twice the number of integral $O_E$-lattices $L$ of $V$ that contains $L^b + \langle x \rangle$ and such that $t(L \cap V_{L^b}) = 1$.

(2) There exists a compact subset $C_L^b$ of $V$ such that $\partial \text{Den}_{L^b}$, $\partial \text{Den}^h_{L^b}$, and $\partial \text{Den}^v_{L^b}$ vanish outside $C_L^b$ and are locally constant functions on $C_L^b \setminus V_{L^b}$.

(3) For an integral $O_E$-lattice $L$ of $V$, if $t(L \cap V_{L^b}) = 1$, then $t(L) = 2$ by Lemma 2.24(1) below and the fact that $V$ is nonsplit.

(4) By (3) and Lemma 2.20, we have

$$\partial \text{Den}^v_{L^b}(x) = 2 \sum_{L^b \subseteq L \subseteq L'} \left( \prod_{i=1}^{t(L)} (1 - q^{2i}) \right) \mathbb{1}_L(x)$$

for $x \in V \setminus V_{L^b}$.

The following is our main result of this subsection.

**Proposition 2.23.** Let $L^b$ be an element of $\mathcal{B}(V)$. Then $\partial \text{Den}^v_{L^b}$ extends (uniquely) to a (compactly supported) locally constant function on $V$, which we still denote by $\partial \text{Den}^v_{L^b}$. Moreover, the support of $\partial \text{Den}^v_{L^b}$ is contained in $V^\text{int}$ (Definition 2.12).

We need some lemma for preparation.

**Lemma 2.24.** Let $L$ be an integral hermitian $O_E$-module of with fundamental invariants $(a_1, \ldots, a_m)$.

1. If $T = (((e_i, e_j)_L)_{i,j=1}^m$ is the moment matrix of an arbitrary basis $\{e_1, \ldots, e_m\}$ of $L$, then for every $1 \leq i \leq m$, $a_1 + \cdots + a_i - i$ equals the minimal $E$-valuation of the determinant of all $i$-by-$i$ minors of $T$.

2. If $L = L' + \langle x \rangle$ for some (integral) hermitian $O_E$-module $L'$ contained in $L$ of rank $m - 1$, then we have

$$t(L) = \begin{cases} t(L') + 1, & \text{if } x' \in uL' \cap L', \\ t(L') - 1, & \text{otherwise,} \end{cases}$$

where $x'$ is the unique element in $L'$ such that $\langle x', y \rangle_L = \langle x, y \rangle_L$ for every $y \in L'$.

**Proof.** Part (1) is simply the well-known method of computing the Smith normal form of $uT$ (over $O_E$) using ideals generated by determinants of minors. For (2), take a normal basis $\{x_1, \ldots, x_{m-1}\}$ of $L$ (Remark 2.14) such that $\langle x_1, \ldots, x_{m-1}, t(L') \rangle$ is self-dual. Applying (1) to the basis $\{x_1, \ldots, x_{m-1}, x\}$ of $L$, we know that $t(L) = t(L') + 1$ if $(x_i, x)_L \in O_E$ for every $m - t(L') \leq i \leq m - 1$; otherwise, we have $t(L) = t(L') - 1$. In particular, (2) follows. \qed

In the rest of this subsection, in order to shorten formulae, we put

$$\mu(t) := \prod_{i=1}^{\frac{t-1}{2}} (1 - q^{2i})$$

for every positive even integer $t$.

**Lemma 2.25.** Take $L^b \in \mathcal{B}(V)$ that is integral. For every compact subset $X$ of $V$ not contained in $V_{L^b}$, we denote by $\delta_X$ the maximal integer such that the image of $X$ under the projection map $V \rightarrow V_{L^b}^\perp$ induced by the orthogonal decomposition $V = V_{L^b} \oplus V_{L^b}^\perp$ is contained in $u^\delta \times (V_{L^b}^\perp)^{\text{int}}$. We denote by $\Sigma$ the set of $O_E$-lattices of $V$ containing $L^b$, and by
the set of triples \((L', \delta, \varepsilon)\) in which \(L'\) is an \(O_E\)-lattice of \(V_{L'}\) containing \(L^b\), \(\delta \in \mathbb{Z}\), and \(\varepsilon: u^\delta(V_{L'}^\perp)^{\text{int}} \to L'^{\alpha} \otimes_{O_F} F/O_F\) is an \(O_E\)-linear map.

(1) The map \(\mathfrak{L} \to \mathfrak{E}\) sending \(L\) to the triple \((L \cap V_{L'}, \delta_L, \varepsilon_L)\) is a bijection, where \(\varepsilon_L\) is the is the extension map \(u^\delta_x(V_{L'}^\perp)^{\text{int}} \to (L \cap V_{L'}) \otimes_{O_F} F/O_F\) induced by the short exact sequence
\[
0 \to L \cap V_{L'} \to L \to u^\delta_x(V_{L'}^\perp)^{\text{int}} \to 0.
\]

Moreover, \(L\) is integral if and only if the following hold:
- \(L \cap V_{L'}\) is integral;
- the image of \(\varepsilon\) is contained in \((L \cap V_{L'})^\vee/(L \cap V_{L'})\);
- \(\varepsilon_L(x) + x \subseteq V^{\text{int}}\) for every \(x \in u^\delta_x(V_{L'}^\perp)^{\text{int}}\).

(2) For \(L \in \mathfrak{L}\) that is integral and corresponds to \((L', \delta, \varepsilon)\) in \(\mathfrak{E}\), we have
\[
t(L) = \begin{cases} 
t(L') + 1, & \text{if the image of } \varepsilon \text{ is contained in } (u(L')^\vee + L')/L', \\
t(L') - 1, & \text{otherwise.} \end{cases}
\]

(3) For every fixed integral \(O_E\)-lattice \(L'^{\alpha}\) of \(V_{L'}\) containing \(L^b\), the sum
\[
\sum_{\substack{L \subseteq L' \\ L \cap V_{L'}=L'}} q^{-\delta_L} |\mu(t(L))|
\]
for every \(z \in V \setminus V^{\text{int}}\). Put
\[
\Omega := \{x \in V^{\text{int}} \mid x' \in (L')^\vee\}, \quad \Omega^\circ := \{x \in V^{\text{int}} \mid x' \in u(L')^\vee + L'\}.
\]

(4) For every fixed integral \(O_E\)-lattice \(L'^{\alpha}\) of \(V_{L'}\) containing \(L^b\) with \(t(L') > 1\), we have
\[
\sum_{\substack{L \subseteq L' \\ L \cap V_{L'}=L'^{\alpha} \delta_L=0}} \mu(t(L)) = 0.
\]

Proof. For (1), the inverse map \(\mathfrak{E} \to \mathfrak{L}\) is the one that sends \((L', \delta, \varepsilon)\) to the \(O_E\)-lattice \(L\) generated by \(L'^\alpha\) and \(\varepsilon_L(x) + x\) for every \(x \in u^\delta_x(V_{L'}^\perp)^{\text{int}}\). The rest of (1) is straightforward. Part (2) is simply Lemma 2.24(2).

Part (4) follows by applying (3) to generators \(z\) of \(O_E\)-modules \(u^{-1}(V_{L'}^\perp)^{\text{int}}\) and \(u^{-2}(V_{L'})^\perp\) and then taking the difference.

Now we prove (3), which is the most difficult one. For every \(x \in V\), we denote by \(x' \in V_{L'}\) the first component of \(x\) with respect to the orthogonal decomposition \(V = V_{L'} \oplus V_{L'}^\perp\). Put
\[
\Omega := \{x \in V^{\text{int}} \mid x' \in (L')^\vee\}, \quad \Omega^\circ := \{x \in V^{\text{int}} \mid x' \in u(L')^\vee + L'\}.
\]

Note that both \(\Omega\) and \(\Omega^\circ\) are open compact subsets of \(V\) stable under the translation by \(L^b\). For an element \(L \in \mathfrak{L}\) corresponding to \((L', \delta, \varepsilon)\) in \(\mathfrak{E}\) from (1), \(L\) is integral if and only

\text{For } (L', \delta, \varepsilon) \in \mathfrak{E}, \text{ we regard } \varepsilon(x) + x \text{ as an } L'^\alpha\text{-coset in } V \text{ as long as we write } \varepsilon(x) + x \subseteq \Omega \text{ for a subset } \Omega \text{ of } V.
Thus, (2.5) follows from Lemma 2.26 below. Part (3) is proved.

\[ q^{-\delta} \mu(t(L)) = \frac{1}{C} \left( \int_{\Omega_z \setminus V_L} |\mu(t(L^b) + 1)| \, dx + \int_{\Omega_z \setminus u(V_L) \cup V_L} |\mu(t(L^b) - 1)| \, dx \right) \]

which is convergent, where

\[ C = \text{vol}(L^b) \cdot \text{vol}((V^b_L)^\text{int} \setminus u(V^b_L)^\text{int}). \]

Now we take an element \( z \in V \setminus V^\text{int} \). We may assume \( z' \in (L^b)^\vee \) since otherwise the summation in (3) is empty. Put

\[ \Omega_z := \{ x \in \Omega \mid (x, z)_V \in u^{-1}O_E \}, \quad \Omega_z^\circ := \{ x \in \Omega^\circ \mid (x, z)_V \in u^{-1}O_E \}, \]

both stable under the translation by \( L^b \) so \( z' \in (L^b)^\vee \). Similarly, we have

\[ \sum_{L \subseteq L^b, \ L \cap V_L = L^b} \sum_{L \cap V_L^\text{int} = L^b} q^{-\delta} \mu(t(L)) = \frac{1}{C} \left( \int_{\Omega_z \setminus V_L} \mu(t(L^b) + 1) \, dx + \int_{\Omega_z \setminus u(V_L) \cup V_L} \mu(t(L^b) - 1) \, dx \right) \]

\[ = \frac{\mu(t(L^b) - 1)}{C} \left( \text{vol}(\Omega_z \setminus \Omega_z^\circ) + (1 - q^{t(L^b) - 1}) \text{vol}(\Omega_z^\circ) \right) \]

\[ = \frac{\mu(t(L^b) - 1)}{C} \left( \text{vol}(\Omega_z) - q^{t(L^b) - 1} \text{vol}(\Omega_z^\circ) \right), \]

where we have used \( t(L^b) > 1 \) in the second equality. Thus, it remains to show that

\[ (2.5) \quad \text{vol}(\Omega_z) = q^{t(L^b) - 1} \text{vol}(\Omega_z^\circ). \]

We fix an orthogonal decomposition \( L^b = L_0 \oplus L_1 \) in which \( L_0 \) is self-dual and \( L_1 \) is of both rank and type \( t(L^b) \). Since both \( \Omega_z \) and \( \Omega_z^\circ \) depend only on the coset \( z + L^b \), we may assume \( z' \in L_1^\vee \) and anisotropic. Let \( V_2 \subseteq V \) be the orthogonal complement of \( L_0 + (z) \). We claim

(\#) There exists an integral \( O_E \)-lattice \( L_2 \) of \( V_2 \) of of type \( t(L^b) \) such that

\[ (u^i L_2^\vee)^\text{int} = \{ x \in V^\text{int}_2 \mid x' \in u^i L_1^\vee \} \]

for \( i = 0, 1 \).

Assuming (\#), by construction, we have

\[ \{ x \in V \mid (x, z)_V \in u^{-1}O_E \} = L_0 \otimes O_F \ F + \langle z \rangle^\vee \oplus V_2. \]

Now we use the condition \( z \notin V^\text{int} \), which implies that \( \langle z \rangle^\vee \subseteq u(z) \cap V^\text{int} \). Combining with (2.6), we obtain

\[ \Omega_z = L_0 \times (z)^\vee \times (L_2^\vee)^\text{int}, \quad \Omega_z^\circ = L_0 \times (z)^\vee \times (uL_2^\vee)^\text{int}. \]

Thus, (2.5) follows from Lemma 2.26 below. Part (3) is proved.

Now we show (\#). There are two cases.
First, we assume $z \neq z'$, that is, $z \notin V_{L'}$. Let $L_2$ be the unique $O_E$-lattice of $V_2$ satisfying (2.7)
\[
L^\vee_2 = \{ x \in V_2 \mid x' \in L_1^\vee \}.
\]
Then (2.6) clearly holds. Thus, it remains to show that $L_2$ is integral of type $t(L'^\vee)$. Put $w := z - z' \in V_{L'}^\perp$ which is nonzero hence anisotropic. Then
\[
\tilde{z} := z' - \frac{(z', z')_V}{(w, w)_V} w
\]
is the unique element in $V_2$ such that $\tilde{z}' = z'$. To compute $L_2$, we write
\[
L^\vee_1 = M + \langle y + \alpha z' \rangle
\]
for some $y \in V_{L'} \cap V_2$ and $\alpha \in E \setminus uO_E$, where $M := L^\vee_1 \cap V_2$. Then
\[
M^\dagger := L_1 \cap V_2 = \{ x \in M^\vee \mid (x, y)_V \in u^{-1}O_E \}.
\]
Since $M^\vee/M^\dagger$ is isomorphic to an $O_E$-submodule of $E/u^{-1}O_E$, we may take an element $y^\dagger \in M^\vee$ that generates $M^\vee/M^\dagger$. Then we have
\[
L_1 = M^\dagger + \langle y^\dagger + \alpha^\dagger z' \rangle
\]
for some $\alpha^\dagger \in E^\times$ such that $(y^\dagger, y)_V + \alpha^\dagger \alpha^c(z', z')_V \in u^{-1}O_E$. Now by (2.7), we have
\[
L^\vee_2 = M + \langle y + \alpha \tilde{z} \rangle.
\]
By the same argument, we have
\[
L_2 = M^\dagger + \langle y^\dagger + \alpha^\dagger \rho \tilde{z} \rangle,
\]
where
\[
\rho := \frac{(z', z')_V}{(\tilde{z}, \tilde{z})_V}.
\]
By Lemma 2.24(2), we have $t(L_2) = t(L_1) = t(L'^\vee)$ as long as $L_2$ is integral. Thus, it suffices to show that $y^\dagger + \alpha^\dagger \rho \tilde{z} \in V^\text{int}$. We compute
\[
(y^\dagger + \alpha^\dagger \rho \tilde{z}, y^\dagger + \alpha^\dagger \rho \tilde{z})_V - (y^\dagger + \alpha^\dagger z', y^\dagger + \alpha^\dagger z')_V = (\alpha^\dagger \rho \tilde{z}, \alpha^\dagger \rho \tilde{z})_V - (\alpha^\dagger z', \alpha^\dagger z')_V
\]
\[
= \text{Nm}_{E/F}(\alpha^\dagger) \left( \frac{(z', z')_V^2}{(\tilde{z}, \tilde{z})_V} - (z', z')_V \right) = \text{Nm}_{E/F}(\alpha^\dagger)(z', z')_V \left( \frac{(z', z')_V}{(z', z')_V + (w, w)_V} - 1 \right)
\]
\[
= \text{Nm}_{E/F}(\alpha^\dagger)(z', z')_V \left( \frac{(w, w)_V}{(z', z')_V + (w, w)_V} - 1 \right) = -\frac{(\alpha^\dagger)^c (z', z')_V^2}{(z, z)_V}.
\]
As $z' \in L_1^\vee$, we have $(\alpha^\dagger z', z')_V \in u^{-1}O_E$. As $z \notin V^\text{int}$, we have $(z, z)_V \notin u^{-1}O_E$. Together, we have $\frac{(\alpha^\dagger z', z')_V}{(z, z)_V} \in O_E$. Thus, $y^\dagger + \alpha^\dagger \rho \tilde{z} \in V^\text{int}$ as $y^\dagger + \alpha^\dagger z' \in V^\text{int}$, hence $L_2$ meets the requirement in (*).

Second, we assume $z = z'$, that is, $z \in V_{L'}$. Take $L_2 = (L_1^\vee \cap V_2)^\vee + u^\delta(V_{L'})^\text{int}$ for some integer $\delta \geq 0$ determined later. We show that $(L_1^\vee \cap V_2)^\vee$ is an integral hermitian $O_E$-module of type $t(L'^\vee) - 1$. As in the previous case, we write
\[
L^\vee_1 = M + \langle y + \alpha z' \rangle
\]
for some $y \in V_{L'} \cap V_2$ and $\alpha \in E \setminus uO_E$, where $M := L^\vee_1 \cap V_2$. Then
\[
L_1 = M^\dagger + \langle y^\dagger + \alpha^\dagger z' \rangle
\]
so that \( M^\vee \) is generated by \( m^\dagger \) and \( y^\dagger \). As \( L_1 \) is of type \( t(L^\vee) \) which is its rank, we have \( L_1 \subseteq uL_1^\vee \), that is,

\[
M^\dagger + \langle y^\dagger + \alpha^\dagger z' \rangle \subseteq uM + u\langle y + \alpha z' \rangle
\]

hence \( M^\dagger \subseteq uM \). As \( z' \in L_1^\vee \), we have \( (\alpha z', z')_V \in u^{-1}O_E \). As \( z' = z \notin V^\text{int} \), we have \( (z', z')_V \notin u^{-1}O_E \) hence \( \alpha^\dagger \in uO_E \). Again as \( z' \in L_1^\vee \), we have \( \alpha^\dagger z' \in uL_1^\vee \) hence \( y^\dagger \in uL_1^\vee \cap V = uM \). Together, we obtain \( M^\vee \subseteq uM \), that is, \( (L_1^\vee \cap V_2)^\vee \) is an integral hermitian \( O_E \)-module of type \( t(L^\vee) - 1 \).

Consequently, \( L_2 \) is an integral \( O_E \)-lattice of \( V_2 \) of type \( t(L^\vee) \). Since \( L_2^\vee = (L_1^\vee \cap V_2) \oplus u^{-1}V_2^\text{int} \), it is clear that for \( \delta \) sufficiently large, \((2.6)\) holds for \( i = 0, 1 \). Thus, \((*)\) is proved.

The lemma is all proved.

\[\Box\]

**Lemma 2.26.** Let \( L \) be an integral hermitian \( O_E \)-module of rank \( 2m + 1 \) for some integer \( m \geq 0 \) with \( t(L) = 2m + 1 \). Then we have

\[
(2.8) \quad \left| \frac{(L^\vee)^\text{int}}{L} \right| = q^{2m} \cdot \left| \frac{(uL^\vee)^\text{int}}{L} \right|.
\]

Note that both \((L^\vee)^\text{int}\) and \((uL^\vee)^\text{int}\) are stable under the translation by \( L \) as \( t(L) = 2m + 1 \).

**Proof.** Put \( V := L \otimes_{O_F} F \). We prove by induction on \( \text{val}(L) \) for integral \( O_E \)-lattices \( L \) of \( V \) with \( t(L) = 2m + 1 \) that \((2.8)\) holds.

The initial case is such that \( \text{val}(L) = 2m + 1 \), that is, \( L^\vee = u^{-1}L \). The pairing \( u^2( , )_V \) induces a nondegenerate quadratic form on \( L^\vee / L \). It is clear that \( (L^\vee)^\text{int} / L \) is exactly the set of isotropic vectors in \( L^\vee / L \) under the previous form. In particular, we have

\[
\left| \frac{(L^\vee)^\text{int}}{L} \right| = q^{2m} \cdot \left| \frac{(uL^\vee)^\text{int}}{L} \right|.
\]

Now we consider \( L \) with \( \text{val}(L) > 2m + 1 \), and suppose that \((2.8)\) holds for such \( L' \) with \( \text{val}(L') < \text{val}(L) \). Choose an orthogonal decomposition \( L = L_0 \oplus L_1 \) in which \( L_0 \) is an integral hermitian \( O_E \)-module with fundamental invariants \((1, \ldots, 1)\) and such that all fundamental invariants of \( L_1 \) are at least 2. In particular, \( L_1 \) has positive rank. It is easy to see that we may choose a hermitian \( O_E \)-module \( L_1' \) contained in \( u^{-1}L_1 \) satisfying \( L_1 \not\subseteq L_1' \) and \( t(L_1') = t(L_1) \). Put \( L' := L_0 \oplus L_1' \). By the induction hypothesis, we have

\[
\left| \frac{(L^\vee)^\text{int}}{L} \right| = q^{2m} \cdot \left| \frac{(uL^\vee)^\text{int}}{L'} \right|.
\]

It remains to show that

\[
(2.9) \quad \left| \frac{(L^\vee)^\text{int} \setminus (L_1^\vee)^\text{int}}{L} \right| = q^{2m} \cdot \left| \frac{(uL^\vee)^\text{int} \setminus (uL_1^\vee)^\text{int}}{L} \right|.
\]

We claim that the map

\[
((L^\vee)^\text{int} \setminus (L_1^\vee)^\text{int}) / L \rightarrow ((uL^\vee)^\text{int} \setminus (uL_1^\vee)^\text{int}) / L
\]

given by the multiplication by \( u \) is \( q^{2m} \)-to-1. Take an element \( x \in (uL^\vee)^\text{int} \setminus (uL_1^\vee)^\text{int} \). Its preimage is bijective to the set of elements \((y_0, y_1) \in L_0/uL_0 \oplus L_1/uL_1 \) such that \( u^{-1}(x + (y_0, y_1)) \in V^\text{int} \), which amounts to the equation

\[
(x, x)_V + \text{Tr}_{E/F}(x, y_0)_V + \text{Tr}_{E/F}(x, y_1)_V + (y_0, y_0)_V \in u^2O_F.
\]

Since \( x \in (uL_0^\vee) \times ((uL_1^\vee)^\text{int} \setminus (u^2L_1^\vee)^\text{int}) \), there exists \( y_1 \in L_1 \) such that \( (x, y_1)_V \in O_E^\times \). In other words, for each \( y_0 \), the above relation defines a nontrivial linear equation on \( L_1/uL_1 \).

Thus, the preimage of \( x \) has cardinality \( q^{2m} \). We obtain \((2.9)\) hence complete the induction process. \[\Box\]
Proof of Proposition 2.23. We fix an element $L^y \in \mathcal{L}(V)$. If $L^y$ is not integral, then $\partial \text{Den}^y_{L^y} \equiv 0$ hence the proposition is trivial. Thus, we now assume $L^y$ integral and will freely adopt notation from Lemma 2.25.

To show that $\partial \text{Den}^y_{L^y}$ extends to a compactly supported locally constant function on $V$, it suffices to show that for every $y \in V_{L^y}/L^y$, there exists an integer $\delta(y) > 0$ such that $\partial \text{Den}^y_{L^y}(y + x)$ is constant for $x \in u^\delta(y)(V_{L^y}^\perp)^{\text{int}} \setminus \{0\}$. If $L^y + \langle y \rangle$ is not integral, then there exists $\delta(y) > 0$ such that $L^y + \langle y + x \rangle$ is not integral for $x \in u^\delta(y)(V_{L^y}^\perp)^{\text{int}} \setminus \{0\}$, which implies $\partial \text{Den}^y_{L^y}(y + x) = 0$.

Now we fix an element $y \in V_{L^y}/L^y$ such that $L^y + \langle y \rangle$ is integral. We claim that we may take $\delta(y) = a_{n+1}$, which is the maximal element in the fundamental invariants of $L^y$. It amounts to showing that for every fixed pair $(f_1, f_2)$ of generators of the $O_F$-module $(V_{L^y}^\perp)^{\text{int}}$, we have

\begin{equation}
(2.10) \quad \partial \text{Den}^y_{L^y}(y + u^\delta f_1) - \partial \text{Den}^y_{L^y}(y + u^\delta^{-1} f_2) = 0
\end{equation}

for $\delta > a_{n-1}$. For every $\delta' \in \mathbb{Z}$, we define two sets

\begin{align*}
\mathfrak{L}^y_1 &= \{ L \in \mathfrak{L} \mid L \subset L^y, \delta_L = \delta', y + u^\delta f_1 \in L \}, \\
\mathfrak{L}^y_2 &= \{ L \in \mathfrak{L} \mid L \subset L^y, \delta_L = \delta', y + u^{\delta^{-1}} f_2 \in L \}.
\end{align*}

By Remark 2.22(4), we have

\begin{align*}
\partial \text{Den}^y_{L^y}(y + u^\delta f_1) &= 2 \sum_{\delta' \leq \delta} \sum_{L \in \mathfrak{L}^y_1} \mu(t(L)) = 2 \sum_{L \in \mathfrak{L}^y_1} \sum_{\delta' \leq \delta} \mu(t(L)), \\
\partial \text{Den}^y_{L^y}(y + u^{\delta^{-1}} f_2) &= 2 \sum_{\delta' \leq \delta^{-1}} \sum_{L \in \mathfrak{L}^y_2} \mu(t(L)) = 2 \sum_{L \in \mathfrak{L}^y_2} \sum_{\delta' \leq \delta^{-1}} \mu(t(L)).
\end{align*}

Now we claim that

\begin{equation}
(2.11) \quad \sum_{\delta' \leq \delta} \sum_{L \in \mathfrak{L}^y_1} \mu(t(L)) - \sum_{\delta' \leq \delta^{-1}} \sum_{L \in \mathfrak{L}^y_2} \mu(t(L)) = 0
\end{equation}

for every $L^y$ in the summation. Since $\delta > a_{n+1}$, it follows that for $\delta' < 0$, we have

\begin{equation}
\mathfrak{L}^y_1 = \mathfrak{L}^y_2 = \{ L \in \mathfrak{L} \mid L \subset L^y, \delta_L = \delta', y \in L \}.
\end{equation}

Thus, the left-hand side of (2.11) equals

\begin{equation}
(2.12) \quad \sum_{\delta' = 0}^{\delta} \sum_{L \in \mathfrak{L}^y_1} \mu(t(L)) - \sum_{\delta' = 0}^{\delta^{-1}} \sum_{L \in \mathfrak{L}^y_2} \mu(t(L)).
\end{equation}

However, we also have $\mathfrak{L}^y_1 = \{ L \in \mathfrak{L} \mid L \subset L^y, \delta_L = \delta', y \in L \}$, which implies

\begin{equation}
\sum_{L \in \mathfrak{L}^y_1} \mu(t(L)) = \mathbbm{1}_{L^y}(y) \sum_{L \subset L^y \delta_L = 0} \mu(t(L)),
\end{equation}

and

\begin{equation}
\sum_{L \in \mathfrak{L}^y_2} \mu(t(L)) = \mathbbm{1}_{L^y}(y) \sum_{L \subset L^y \delta_L = 0} \mu(t(L)).
\end{equation}
which vanishes by Lemma 2.25(4). Thus, we obtain

\begin{align}
(2.13) \quad (2.12) = \sum_{\delta'=1}^{\delta} \sum_{L \in \mathcal{L}_{1}^{\delta'}} \mu(t(L)) - \sum_{\delta'=0}^{\delta-1} \sum_{L \in \mathcal{L}_{2}^{\delta'}} \mu(t(L)).
\end{align}

Finally, the automorphism of \( \mathfrak{c} \) sending \((L^\nu, \delta', \varepsilon)\) to \((L^\nu, \delta' - 1, \varepsilon \circ (u\alpha_{\varepsilon}))\), where \( \alpha \in O_{E}^{\circ} \) is the element satisfying \( f_1 = \alpha f_2 \), induces a bijection from \( \mathcal{L}_1^{\delta'} \) to \( \mathcal{L}_2^{\delta'-1} \) preserving both \( L \cap V_{L^\nu} \) and \( t(L) \). Thus, \((2.13)\) vanishes hence \((2.11)\) and \((2.10)\) hold.

Now we show that the support of \( \partial \operatorname{Den}_{L^\nu}^\nu \) is contained in \( V^\int \). Take an element \( z \in V \setminus V^\int \). Using Remark 2.22(4), we have

\[
\partial \operatorname{Den}_{L^\nu}^\nu(z) = \int_{V} \partial \operatorname{Den}_{L^\nu}^\nu(x) \psi(\operatorname{Tr}_{E/F}(x, z)_V) \, dz
= 2 \sum_{L^\nu \subseteq L \subseteq L^\nu, \ t(L) > 1} \mu(t(L)) \, \operatorname{vol}(L) \mathbb{1}_{L^\nu}(z)
= 2 \sum_{L^\nu \subseteq L' \subseteq (L^\nu)^\nu, \ t(L') > 1} \sum_{L \subseteq L^\nu, \ L \cap V_{L^\nu} = L^\nu, \ z \in L^\nu} \mu(t(L)) \, \operatorname{vol}(L)
= 2 \sum_{L^\nu \subseteq L' \subseteq (L^\nu)^\nu, \ t(L') > 1} \operatorname{vol}(L^\nu) \operatorname{vol}((V_{L^\nu})^\int) \sum_{L \subseteq L^\nu, \ L \cap V_{L^\nu} = L^\nu, \ z \in L^\nu} q^{-\delta_{L}} \mu(t(L)),
\]

which is valid and vanishes by Lemma 2.25(3).

Proposition 2.23 is proved. \( \square \)

2.3. Bruhat–Tits stratification. Let the setup be as in Subsection 2.1. We assume Hypothesis 2.4.

We first generalize Definition 2.7 to a more general context. For every subset \( X \) of \( V \) such that \( \langle X \rangle \) is finitely generated, we put

\[
\mathcal{N}(X) := \bigcap_{x \in X} \mathcal{N}(x),
\]

which is always a finite intersection, and depends only on \( \langle X \rangle \). Clearly, we have \( \mathcal{N}(X') \subseteq \mathcal{N}(X) \) if \( \langle X \rangle \subseteq \langle X' \rangle \). When \( X = \{x, \ldots\} \) is explicitly presented, we simply write \( \mathcal{N}(x, \ldots) \) instead of \( \mathcal{N}(\{x, \ldots\}) \).

Remark 2.27. When \( \langle X \rangle \) is an \( O_{E} \)-lattice of \( V \), the formal subscheme \( \mathcal{N}(X) \) is a proper closed subscheme of \( \mathcal{N} \). This can be proved by the same argument for \([LZ, \text{Lemma 2.10.1}].\)

Definition 2.28. Let \( \Lambda \) be a vertex \( O_{E} \)-lattice of \( L \) (Definition 2.12).

1. We equip the \( k \)-vector space \( L^\nu/\Lambda \) with a \( k \)-valued pairing \( (\ , \ )_{L^\nu/\Lambda} \) by the formula

\[
(x, y)_{L^\nu/\Lambda} := u^{2} \operatorname{Tr}_{E/F}(x^2, y^2)_{V} \quad \text{mod} \ (u^{2})
\]

where \( x^2 \) and \( y^2 \) are arbitrary lifts of \( x \) and \( y \), respectively. Then \( L^\nu/\Lambda \) becomes a nonsplit (nondegenerate) quadratic space over \( k \) of (even positive) dimension \( t(\Lambda) \).
(2) Let $\mathcal{V}_\Lambda$ be the reduced subscheme of $\mathcal{N}(\Lambda)$, and put

$$\mathcal{V}_\Lambda^\circ := \mathcal{V}_\Lambda - \bigcup_{\Lambda \subseteq \Lambda'} \mathcal{V}_{\Lambda'}.$$  

**Proposition 2.29** (Bruhat–Tits stratification, [Wu]). The reduced subscheme $\mathcal{N}_{\text{red}}$ is a disjoint union of $\mathcal{V}_\Lambda^\circ$ for all vertex $O_E$-lattices $\Lambda$ of $V$ in the sense of stratification, such that $\mathcal{V}_\Lambda \cap \mathcal{V}_{\Lambda'}$ coincides with $\mathcal{V}_{\Lambda+\Lambda'}$ (resp. is empty) if $\Lambda + \Lambda'$ is (resp. is not) a vertex $O_E$-lattice. Moreover, for every vertex $O_E$-lattice $\Lambda$,

1. $\mathcal{V}_\Lambda$ is canonically isomorphic to the generalized Deligne–Lusztig variety of $O(\Lambda^\vee/\Lambda)$ over $\overline{k}$ classifying maximal isotropic subspaces $U$ of $(\Lambda^\vee/\Lambda) \otimes_k \overline{k}$ satisfying

$$\dim(U \cap \delta(U)) = \frac{t(\Lambda)}{2} - 1,$$

where $\delta \in \text{Gal}(\overline{k}/k)$ denotes the Frobenius element;

2. the intersection of $\mathcal{V}_\Lambda$ with each connected component of $\mathcal{N}_{\text{red}}$ is connected, nonempty, and smooth projective over $\overline{k}$ of dimension $\frac{t(\Lambda)}{2} - 1$.

**Proof.** This follows from [Wu, Proposition 5.13 & Theorem 5.18]. Note that we use lattices in $V$, which is different from the hermitian space $C$ used in [Wu], to parameterize strata. By the obvious analogue of [KR11, Lemma 3.9], we may naturally identify $V$ with $C$, after which the stratum $S_\Lambda$ in [Wu] coincides with our stratum $\mathcal{V}_\Lambda^\circ$. \qed

**Remark 2.30.** In the above proposition, when $t(\Lambda) = 4$, $\mathcal{V}_\Lambda$ is isomorphic to two copies of $\mathbb{P}^1_{\overline{k}}$, though we do not need this explicit description in the following.

**Corollary 2.31.** For every nonzero element $x \in V$, we have

$$\mathcal{N}(x)_{\text{red}} = \bigcup_{x \in \Lambda} \mathcal{V}_\Lambda^\circ$$

where the union is taken over all vertex $O_E$-lattices of $V$ containing $x$.

**Proof.** Since $\mathcal{N}(x)_{\text{red}}$ is a reduced closed subscheme of $\mathcal{N}_{\text{red}}$, it suffices to check that

$$\mathcal{N}(x)(\overline{k}) = \bigcup_{x \in \Lambda} \mathcal{V}_\Lambda^\circ(\overline{k}).$$

By Definition 2.28(2), we have

$$\mathcal{N}(x)(\overline{k}) \supseteq \bigcup_{x \in \Lambda} \mathcal{V}_\Lambda^\circ(\overline{k}).$$

For the other direction, by Proposition 2.29, we have to show that if $\Lambda$ does not contain $x$, then $\mathcal{N}(x)(\overline{k}) \cap \mathcal{V}_\Lambda^\circ(\overline{k}) = \emptyset$. Suppose that we have $s \in \mathcal{N}(x)(\overline{k}) \cap \mathcal{V}_\Lambda^\circ(\overline{k})$, then $s$ should belong to $\mathcal{V}_{\Lambda'}(\overline{k})$ where $\Lambda'$ is the $O_E$-lattice generated by $\Lambda$ and $x$. In particular, $\Lambda'$ is vertex and strictly contains $\Lambda$. But this contradicts with the definition of $\mathcal{V}_\Lambda^\circ$. The corollary follows. \qed

**Corollary 2.32.** Suppose that $r \geq 2$. For every $x \in V$, the intersection of $\mathcal{N}(x)$ with each connected component of $\mathcal{N}_{\text{red}}$ is strictly a closed subscheme of the latter.

**Proof.** By Corollary 2.31 and Proposition 2.29(2), it suffices to show that the intersection of all vertex $O_E$-lattices of $V$ is $\{0\}$.

Take a nonsplit hermitian subspace $V_2$ of $V$ of dimension 2 and an $O_E$-lattice $L_2$ of $V_2$ of fundamental invariants $(1, 1)$. Then the orthogonal complement $V_2^\perp$ of $V_2$ in $V$ admits a self-dual $O_E$-lattice $L_1$. Choose a normal basis (Remark 2.14) $\{e_1, \ldots, e_{2r-2}\}$ of $L_1$ under which
Thus, we assume Hypothesis 2.4. Linear invariance of intersection numbers.

\[ \Lambda_a := L_2 \oplus \langle u^a_1 e_1, \ldots, u^{a_{2r-2}}_2 e_{2r-2} \rangle \]

is integral with fundamental invariants \((0, \ldots, 0, 1, 1)\), hence vertex. It is clear that the intersection of all such \(\Lambda_a\) is \(L_2\). Since \(r \geq 2\), the intersection of all 2-dimensional nonsplit hermitian subspaces of \(V\) is \(\{0\}\). Thus, the intersection of all vertex \(O_E\)-lattices of \(V\) is \(\{0\}\).

\[ \blacksquare \]

**Lemma 2.33.** Let \(\Lambda\) be a vertex \(O_E\)-lattice of \(V\). For each connected component \(V_{\Lambda}^+\) of \(V_{\Lambda}\) and integer \(d \geq 0\), the group of \(d\)-cycles of \(V_{\Lambda}^+\), up to \(\ell\)-adic homological equivalence for every rational prime \(\ell \neq p\), is generated by \(V_{\Lambda'} \cap V_{\Lambda}^+\) for all vertex \(O_E\)-lattices \(\Lambda'\) containing \(\Lambda\) with \(t(\Lambda') = 2d + 2\).

**Proof.** Let \(k'\) be the quadratic extension of \(k\) in \(\overline{k}\). Note that \(V_{\Lambda}^+\) has a canonical structure over \(k'\), so that \(V_{\Lambda}^{o+} := V_{\Lambda}^+ \cap V_{\Lambda}^+\) (over \(k'\)) is the classical Deligne–Lusztig variety of \(SO(\Lambda^\vee/\Lambda)\) of Coxeter type.

Recall that \(\delta\) is the Frobenius element of \(\text{Gal}(\overline{k}/k)\). Fix a rational prime \(\ell\) different from \(p\). For every finite dimensional \(\overline{\mathbb{Q}}_\ell\)-vector space \(V\) with an action by \(\delta^2\), we denote by \(V^\dagger\) the subspace consisting of elements on which \(\delta^2\) acts by roots of unity. Then for the lemma, it suffices to show that for every \(d \geq 0\), \(H^d(V_{\Lambda}^+, \overline{\mathbb{Q}}_\ell(-d))\) is generated by \((\text{the cycle class of})\) \(V_{\Lambda} \cap V_{\Lambda}^+\) for all vertex \(O_E\)-lattices \(\Lambda'\) containing \(\Lambda\) with \(t(\Lambda') = 2d + 2\). By the same argument for \([LZ, \text{Theorem 5.3.2}]\), it reduces to the following claim:

\((*)\) The action of \(\delta^2\) on \(V := \bigoplus_{j \geq 0} H^{2j}(V_{\Lambda}^{o+}, \overline{\mathbb{Q}}_\ell(j))\) is semisimple, and \(V^\dagger = H^0(V_{\Lambda}^{o+}, \overline{\mathbb{Q}}_\ell)\).

There are three cases.

- When \(t(\Lambda) = 2\), \(V_{\Lambda}^{o+}\) is isomorphic to \(\text{Spec } \overline{k}\) hence \((*)\) is trivial.
- When \(t(\Lambda) = 4\), \(V_{\Lambda}^{o+}\) is an affine curve hence \((*)\) is again trivial.
- When \(t(\Lambda) \geq 6\), by Case 2\(D_n\) (with \(n = \frac{t(\Lambda)}{2} \geq 3\)) in \([\text{Lus76, Section 7.3}]\), the action of \(\delta^2\) on \(\bigoplus_{j \geq 0} H^j(V_{\Lambda}^{o+}, \overline{\mathbb{Q}}_\ell)\) has eigenvalues \(\{1, q^2, q^4, \ldots, q^{(\Lambda)-2}\}\) and that the eigenvalue \(q^{2j}\) appears in \(H^{2j+\frac{t(\Lambda)}{2}-1}(V_{\Lambda}^{o+}, \overline{\mathbb{Q}}_\ell)\). Moreover by \([\text{Lus76, Theorem 6.1}]\), the action of \(\delta^2\) is semisimple. Thus, \((*)\) follows from the Poincaré duality.

The lemma is proved. \[ \blacksquare \]

### 2.4. Linear invariance of intersection numbers.

Let the setup be as in Subsection 2.1. We assume Hypothesis 2.4.

For every nonzero element \(x \in V\), we define a chain complex of locally free \(\mathcal{O}_N\)-modules

\[ C(x) := \left( \cdots \to 0 \to \mathcal{I}_N(x) \to \mathcal{O}_N \to 0 \right) \]

supported in degrees 1 and 0 with the map \(\mathcal{I}_N(x) \to \mathcal{O}_N\) being the natural inclusion. We extend the definition to \(x = 0\) by setting

\[ C(0) := \left( \cdots \to 0 \to \omega \stackrel{0}{\to} \mathcal{O}_N \to 0 \right) \]

supported in degrees 1 and 0, where \(\omega\) is the line bundle from Definition 2.39.

The following is our main result of this subsection.
Proposition 2.34. Let $0 \leq m \leq n$ be an integer. Suppose that $x_1, \ldots, x_m \in V$ and $y_1, \ldots, y_m \in V$ generate the same $O_E$-submodule. Then we have an isomorphism

$$H_i(C(x_1) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} C(x_m)) \simeq H_i(C(y_1) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} C(y_m))$$

of $\mathcal{O}_N$-modules for every $i$.

Proposition 2.34 has the following two immediate corollaries.

Corollary 2.35. Let $0 \leq m \leq n$ be an integer. Suppose that $x_1, \ldots, x_m \in V$ and $y_1, \ldots, y_m \in V$ generate the same $O_E$-submodule. Then we have

$$[C(x_1) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} C(x_m)] = [C(y_1) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} C(y_m)]$$

in $K_0(N)$, where $K_0(N)$ denotes the $K$-group of $N$ [LL, Section B].

Corollary 2.36. Suppose that $x_1, \ldots, x_n \in V$ generate an $O_E$-lattice of $V$. The Serre intersection multiplicity

$$\chi\left(\mathcal{O}_N(x_1) \otimes_{\mathcal{O}_N} \cdots \otimes_{\mathcal{O}_N} \mathcal{O}_N(x_n)\right)$$

$$:= \sum_{i,j \geq 0} (-1)^{i+j} \text{length}_{O_E} H^j \left(\mathcal{N}, H_i \left(\mathcal{O}_N(x_1) \otimes_{\mathcal{O}_N} \cdots \otimes_{\mathcal{O}_N} \mathcal{O}_N(x_n)\right)\right)$$

depends only on the $O_E$-lattice of $V$ generated by $x_1, \ldots, x_n$. Note that by Remark 2.27, the above number is finite.

Now we start to prove Proposition 2.34, following [How19]. Let $(X, \iota_X, \lambda_X)$ be the universal object over $\mathcal{N}$. We have a short exact sequence

$$0 \rightarrow \text{Fil}(X) \rightarrow D(X) \rightarrow \text{Lie}(X) \rightarrow 0$$

of locally free $\mathcal{O}_N$-modules, where $D(X)$ denotes the covariant crystal of $X$ restricted to the Zariski site of $\mathcal{N}$. Then $\iota_X$ induces actions of $O_E$ on all terms so that the short exact sequence is $O_E$-linear.

We define an $\mathcal{O}_N$-submodule $F_X$ of $\text{Lie}(X)$ as the kernel of $\iota_X(u) - u$ on $\text{Lie}(X)$, which is stable under the $O_E$-action.

Lemma 2.37. The $\mathcal{O}_N$-submodule $F_X$ is locally free of rank $n-1$ and is locally a direct summand of $\text{Lie}(X)$.

Proof. Let $s \in \mathcal{N}(\overline{F})$ be a closed point. By the Wedge condition and the Spin condition in Definition 2.1, we know that the map

$$\iota_X(u) - u : \text{Lie}(X) \otimes_{\mathcal{O}_X} \mathcal{O}_{N,s} \rightarrow \text{Lie}(X) \otimes_{\mathcal{O}_X} \mathcal{O}_{N,s}$$

has rank 1 on both generic and special fibers. Thus, $F_X \otimes_{\mathcal{O}_N} \mathcal{O}_{N,s}$ is a direct summand of $\text{Lie}(X) \otimes_{\mathcal{O}_X} \mathcal{O}_{N,s}$ of rank $n-1$. The lemma follows. $\square$

The symmetrization $\sigma_X$ of the polarization $\lambda_X$ (Remark 2.2) induces a perfect symmetric $\mathcal{O}_N$-bilinear pairing

$$(\ , ) : D(X) \times D(X) \rightarrow \mathcal{O}_N$$

satisfying $(\iota_X(\alpha)x, y) = (x, \iota_X(\alpha^c)y)$ for every $\alpha \in O_E$ and $x, y \in D(X)$. As Fil$(X)$ is a maximal isotropic $\mathcal{O}_N$-submodule of $D(X)$ with respect to $(\ , )$, we have an induced perfect $\mathcal{O}_N$-bilinear pairing

$$(\ , ) : \text{Fil}(X) \times \text{Lie}(X) \rightarrow \mathcal{O}_N,$$
under which we denote by $F^\perp_X \subseteq \text{Fil}(X)$ the annihilator of $F_X$. Then the $\mathcal{O}_X$-submodule $F^\perp_X$ is locally free of rank 1 and is locally a direct summand of $\text{Fil}(X)$.

Following [How19, Section 3], we put

$$
\epsilon := u \otimes 1 + 1 \otimes u \in O_E \otimes_{O_F} \mathcal{O}_N;
$$

$$
\epsilon^c := u \otimes 1 - 1 \otimes u \in O_E \otimes_{O_F} \mathcal{O}_N.
$$

**Lemma 2.38.** There are inclusions of $\mathcal{O}_U$-modules $F^\perp_X \subseteq \epsilon D(X) \subseteq D(X)$ which are locally direct summands. The map $\epsilon: D(X) \to \epsilon D(X)$ descends to a surjective map

$$
\text{Lie}(X) \overset{\epsilon}{\rightarrow} \epsilon D(X)/F^\perp_X,
$$

whose kernel $L_X$ is locally a direct summand $\mathcal{O}_U$-submodule of $\text{Lie}(X)$ of rank 1. Moreover, the $O_E$-action stabilizes $L_X$, and acts on $\text{Lie}(X)/L_X$ and $L_X$ via $\varphi_0$ and $\varphi_0^c$, respectively.

**Proof.** This follows from the same proof for [How19, Proposition 3.3].

**Definition 2.39.** We define the line bundle of modular forms $\omega$ to be $L_X^{-1}$, where $L_X$ is the line bundle on $\mathcal{N}$ from Lemma 2.38.

For every closed formal subscheme $Z$ of $\mathcal{N}$, we denote by $\bar{Z}$ the closed formal subscheme defined by the sheaf $\mathcal{I}^2_Z$. Take a nonzero element $x \in V$. By the definition of $\mathcal{N}(x)$, we have a distinguished morphism

$$
X_0|_{\mathcal{N}(x)} \overset{\tilde{x}}{\rightarrow} X|_{\mathcal{N}(x)}
$$

of $O_F$-divisible groups, which induces an $O_E$-linear map

$$
D(X_0)|_{\mathcal{N}(x)} \overset{\tilde{x}}{\rightarrow} D(X)|_{\mathcal{N}(x)}
$$

of vector bundles. By the Grothendieck–Messing theory, the above map admits a canonical extension

$$
D(X_0)|_{\mathcal{N}(x)} \overset{\tilde{x}}{\rightarrow} D(X)|_{\mathcal{N}(x)},
$$

which further restricts to a map

$$
(2.15) \quad \text{Fil}(X_0)|_{\mathcal{N}(x)} \overset{\tilde{x}}{\rightarrow} \text{Lie}(X)|_{\mathcal{N}(x)}.
$$

From now on, we fix a generator $\gamma$ of the rank 1 free $O_E$-module $\text{Fil}(X_0)$.

**Lemma 2.40.** The image $\tilde{x}(\gamma)$ is a section of $L_X$ over $\mathcal{N}(x)$, whose vanishing locus coincides with $\mathcal{N}(x)$, where $\tilde{x}$ is the map (2.15).

**Proof.** This follows from the same proof for [How19, Proposition 4.1].

The following lemma is parallel to [KR11, Proposition 3.5].

**Lemma 2.41.** For every nonzero element $x \in V$, the closed formal subscheme $\mathcal{N}(x)$ of $\mathcal{N}$ is either empty or a relative Cartier divisor.

**Proof.** The case $r = 1$ has been proved in [RSZ17, Proposition 6.6]. Thus, we now assume $r \geq 2$.

We may assume that $\mathcal{N}(x)$ is nonempty. By the same argument in the proof of [How19, Proposition 4.3], $\mathcal{N}(x)$ is locally defined by one equation. It remains to show that such equation is not divisible by $u$. Since $r \geq 2$, this follows from [KR11, Lemma 3.6], Lemma 2.6, and Corollary 2.32.
Proof of Proposition 2.34. The proof of [How19, Theorem 5.1] can be applied in the same way to Proposition 2.34, using Lemma 2.40 and Lemma 2.41.

To end this subsection, we prove some results that will be used later.

Lemma 2.42. The $\Theta_N$-submodule $L_X$ from Lemma 2.38 coincides with the image of the map $\iota_X(u) - u$: $\text{Lie}(X) \to \text{Lie}(X)$.

Proof. Denote by $L_X'$ the image of the map $\iota_X(u) - u$: $\text{Lie}(X) \to \text{Lie}(X)$. As we have $L_X \cong \text{Lie}(X)/F_X$, by Lemma 2.37, for every closed point $s \in N(\bar{k})$, the induced map $L_X' \otimes_{\Theta_N} \bar{k} \to \text{Lie}(X) \otimes_{\Theta_N} \bar{k}$ over the residue field at $s$ is injective. Thus, the quotient $\Theta_N$-module $\text{Lie}(X)/L_X'$ is locally free. It remains to show that $L_X' \subseteq L_X$.

By definition, every section of $L_X'$ can be locally written as the image of $(\iota_X(u) - u)x$ for some section $x$ of $D(X)$. We need to show that

1. $(\iota_X(u) - u)x$ is a section of $\text{Fil}(X)$;
2. $(\iota_X(u) - u)x,y = 0$ for every section $y$ of $F_X$.

For (1), we have $\epsilon(\iota_X(u) - u)x = (\iota_X(u) + u)(\iota_X(u) - u)x = (\iota_X(u^2) - u^2)x$. Since $\iota_X(u^2) - u^2$ acts by zero on $\text{Lie}(X)$, (1) follows. For (2), we have

$$(\iota_X(u) - u)x,y) = ((\iota_X(u) - u)x,(-\iota_X(u) + u)y) = 0$$

as $y$ is a section of $\ker(\iota_X(u) - u)$. Thus, (2) follows.

The lemma is proved.

Lemma 2.43. Let $\Lambda$ be a vertex $O_E$-lattice of $V$ with $t(\Lambda) = 4$. Then $\omega$ has degree $q - 1$ on each connected component of $(\text{the smooth projective curve}) \mathcal{V}_\Lambda$ (Definition 2.28).

Proof. Let $\delta$ be the Frobenius element of $\text{Gal}(\bar{k}/k)$.

Let $s \in N(\bar{k})$ be a closed point represented by the quadruple $(X, \iota_X, \lambda_X; \rho_X)$. Let $M$ be the covariant $O_E$-Dieudonné module of $X$ equipped with the $O_E$-action $\iota_X$, which becomes a free $O_{\bar{k}}$-module. We have $\text{Lie}(X) = M/\text{VM}$. By Definition 2.39 and Lemma 2.42, the fiber $\omega^{-1}|_s$ is canonically identified with $((u \otimes 1)M + \text{VM})/\text{VM}$, which is further canonically isomorphic to $((u \otimes 1)V^{-1}M + M)/M$. By the identification between $\mathcal{V}_\Lambda$ and the generalized Deligne–Lusztig variety of $O(\Lambda^\vee/\Lambda)$ in Proposition 2.29 given in [Wu, Proposition 4.29], we know that $\omega^{-1}|_{\mathcal{V}_\Lambda}$ coincides with $(\delta(U) + U)/U$ where $U$ is the tautological subbundle of $(\Lambda^\vee/\Lambda) \otimes_k \Theta_{\mathcal{V}_\Lambda}$.

To compute the degree of $(\delta(U) + U)/U$, let $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$ be the two connected components of $\mathcal{V}_\Lambda$. Let $\mathcal{L}_\Lambda$ be the scheme over $\bar{k}$ classifying lines in $\Lambda^\vee/\Lambda$ with the tautological bundle $L$. We may identify $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$ as two closed subschemes of $\mathcal{L}_\Lambda$ via the assignment $U \mapsto \delta(U)\cap U$ (see [HP14, Section 3.2] for more details). Then, $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$ are the two irreducible components of the locus where $L$ and $\delta(L)$ generate an isotropic subspace, and the assignment $L \mapsto \delta(L)$ switches $\mathcal{V}_\Lambda^+$ and $\mathcal{V}_\Lambda^-$. Let $\mathcal{I}_\Lambda$ be the locus where $L$ is isotropic and $L = \delta(L)$. Then $\mathcal{I}_\Lambda$ is a disjoint union of $q^2 + 1$ copies of $\text{Spec} \bar{k}$ since there are exactly $q^2 + 1$ isotropic lines in $\Lambda^\vee/\Lambda$, and is contained in $\mathcal{V}_\Lambda^+ \cap \mathcal{V}_\Lambda^-$. Note that the map $\delta(U)/(\delta(U) \cap U) \to (\delta(U) + U)/U$ is an isomorphism, and there is a short exact sequence

$$0 \to \delta(\delta(U) \cap U) \to \delta(U)/(\delta(U) \cap U) \to \Theta_{\mathcal{I}_\Lambda} \to 0$$
Lemma 2.44. Let $L$ be an integral $O_E$-lattice of $V$ with fundamental invariants $(2b_1 + 1, 2b_2 + 1)$. Then

$$\partial \text{Den}(L) = 2 \sum_{j=0}^{b_1} \left( 1 + q + \cdots + q^j + (b_2 - j)q^j \right).$$

Proof. We denote by $\mathfrak{L}$ the set of integral $O_E$-lattices of $V$ containing $L$. We now count $\mathfrak{L}$.

Fix an orthogonal basis $\{e_1, e_2\}$ of $V$ with $(e_1, e_1)_V \in O_F^+$ and $(e_2, e_2)_V \in O_F^+$ such that $L = \langle u^{b_1}e_1 \rangle + \langle u^{b_2}e_2 \rangle$. For every $L \in \mathfrak{L}$, we let $j(L)$ be the unique integer such that $L \cap \langle e_1 \rangle \otimes_{O_F} F = \langle u^{j(L)}e_1 \rangle$ and let $k(L)$ be the unique integer such that image of $L$ under the natural projection map $V \to \langle e_2 \rangle \otimes_{O_F} F$ is $\langle u^{k(L)}e_2 \rangle$. Then by Lemma 2.24(1), $L$ is uniquely determined by $j(L), k(L)$, and the extension map $\varepsilon_L : \langle u^{k(L)}e_2 \rangle \to \langle u^{j(L)}e_1 \rangle \otimes_{O_F} F/O_F$. The condition that $L$ contains $L$ is equivalent to that $j(L) \leq b_1$, $k(L) \leq b_2$, and that $\varepsilon_L$ vanishes on $\langle u^{b_2}e_2 \rangle$. Since $L$ is non-split, the condition that $L$ is integral is equivalent to that $j(L) \geq 0$, $k(L) \geq 0$, and that the image of $\varepsilon_L$ is contained $\langle e_1 \rangle / \langle u^{j(L)}e_1 \rangle$. Thus, the number of $L \in \mathfrak{L}$ with $j(L) = j$ for some fixed $0 \leq j \leq b_1$ equals $1 + q + \cdots + q^j + (b_2 - j)q^j$. Summing over all $0 \leq j \leq b_1$, we obtain

$$|\mathfrak{L}| = \sum_{j=0}^{b_1} \left( 1 + q + \cdots + q^j + (b_2 - j)q^j \right).$$

The lemma then follows from (2.4) as $t(L) = 2$. □

Proposition 2.45. Theorem 2.9 holds when $r = 1$. More explicitly, for an integral $O_E$-lattice $L$ of $V$ with fundamental invariants $(2b_1 + 1, 2b_2 + 1)$, we have

$$\text{Int}(L) = \partial \text{Den}(L) = 2 \sum_{j=0}^{b_1} \left( 1 + q + \cdots + q^j + (b_2 - j)q^j \right).$$
Proof. If $L$ is not integral, then $\text{Int}(L) = \partial \text{Den}(L) = 0$. If $L$ is integral with fundamental invariants $(2b_1 + 1, 2b_2 + 1)$. We may take an orthogonal basis $\{x_1, x_2\}$ of $L$ such that $\text{val}(x_1) = 2b_1 + 1$ and $\text{val}(x_2) = 2b_2 + 1$.

Put $D := \text{End}_{O_F}(X_0) \otimes \mathbb{Q}$, which is a division quaternion algebra over $F$ with the $F$-linear embedding $i_{X_0}: E \to D$. By the Serre construction, we may naturally identify $D$ with $V$, and we have an identity

$$N(x_1) = \sum_{j=0}^{b_1} W_{E_j}$$

of divisors, decomposing the special divisor as a sum of quasi-canonical lifting divisors (see [RSZ17, Section 6 & Proposition 7.1]).

We claim that for every $0 \leq j \leq b_1$, the identity

$$\text{length}_{O_E} W_{E_j} \cap N(x_2) = 2 \left(1 + q + \cdots + q^j + (b_2 - l)q^j\right)$$

holds. In fact, this can be proved in the same way as for [KR11, Proposition 8.4] using Keating’s formula [Vol07, Theorem 2.1]. Notice that in [KR11, Proposition 8.4] we replace $e_s$ by $2q^j$ since $E/F$ is ramified, and that the factor 2 comes from the fact that $Z_t$ has two connected components. By (2.16) and (2.17), we have

$$\text{Int}(L) = \text{length}_{O_E} N(x_1) \cap N(x_2) = \sum_{j=0}^{b_1} 2 \left(1 + q + \cdots + q^j + (b_2 - l)q^j\right).$$

The proposition follows by Lemma 2.44.

Definition 2.46. For $L^b \in b(V)$, we put

$$N(L^b)^\circ := N(L^b) - N(u^{-1}L^b)$$

as an effective divisor by (the $r = 1$ case of) Lemma 2.41.

Corollary 2.47. Take an element $L^b \in b(V)$. For every $x \in V \setminus V_{L^b}$, we have

$$\text{length}_{O_E} N(L^b)^\circ \cap N(x) = 2 \sum_{L \subseteq L^b \setminus L \cap V_{L^b}} \mathbb{1}_L(x).$$

Proof. By Proposition 2.45, we have

$$\text{length}_{O_E} N(L^b) \cap N(x) = \text{Int}(L^b + \langle x \rangle) = \partial \text{Den}(L^b + \langle x \rangle) = 2 \sum_{L \subseteq L^b \setminus L \cap V_{L^b}} \mathbb{1}_L(x),$$

in which the last identity is due to (2.4). Similarly, we have

$$\text{length}_{O_E} N(u^{-1}L^b) \cap N(x) = 2 \sum_{u^{-1}L^b \subseteq L \setminus V_{L^b}} \mathbb{1}_L(x).$$

Taking the difference, we obtain the corollary. □
2.6. Fourier transform of geometric side. Let the setup be as in Subsection 2.1. We assume Hypothesis 2.4. We will freely use notation concerning K-groups of formal schemes from [LL, Section B] and [Zha, Appendix B], based on the work [GS87].

Definition 2.48. Let \( X \) be a formal scheme over \( \text{Spf} O_E \).

1. We denote by \( X^h \) the closed formal subscheme of \( X \) defined by the ideal sheaf \( \mathcal{O}_X[p^\infty] \).
2. We say that an element in \( K_0(X) \) has proper support if it belongs to the subgroup \( K_0^Z(X) \) for a proper closed subscheme \( Z \) of \( X \).

Definition 2.49. Let \( X \) be a subset of \( V \) such that \( \langle X \rangle \) is finitely generated of rank \( m \).

1. We denote by \( K\mathcal{N}(X) \in K_0(\mathcal{N}) \) the element \([C(x_1) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} C(x_m)]\) from Subsection 2.4 for a basis \( \{x_1, \ldots, x_m\} \) of the \( O_E \)-module generated by \( X \), which is independent of the choice of the basis by Corollary 2.35.
2. We denote by \( K\mathcal{N}(X)^h \in K_0(\mathcal{N}) \) the class of \( \mathcal{N}(X)^h \).
3. We put \( K\mathcal{N}(X)^v \coloneqq K\mathcal{N}(X) - K\mathcal{N}(X)^h \in K_0(\mathcal{N}) \).

Lemma 2.50. Let \( L^b \) be an element of \( b(V) \) (Definition 2.10). We have

1. \( \mathcal{N}(L^b)^h \) is either empty or finite flat over \( \text{Spf} O_E \);
2. all of \( K\mathcal{N}(L^b)^h \), \( K\mathcal{N}(L^b)^v \), and \( K\mathcal{N}(L^b)^v \) belong to \( F^{n-1}K_0(\mathcal{N}) \);
3. \( K\mathcal{N}(L^b)^v \) has proper support.

Proof. Part (1) follows from Lemma 2.55 and Lemma 2.54.

Take a basis \( \{x_1, \ldots, x_{n-1}\} \) of the \( O_E \)-module \( L^b \).

For (2), it suffices to show \( K\mathcal{N}(L^b) \in F^{n-1}K_0(\mathcal{N}) \) by (1). By definition, \( K\mathcal{N}(L^b) \) is the cup product of the classes in \( K_0(\mathcal{N}) \) of \( \mathcal{N}(x_1), \ldots, \mathcal{N}(x_{n-1}) \), each being a divisor by Lemma 2.41. Thus, \( K\mathcal{N}(L^b) \) belongs to \( F^{n-1}K_0(\mathcal{N}) \) by (the analogue for formal schemes of) [GS87, Proposition 5.5].

For (3), by the same argument for [LZ, Lemma 5.1.1], we know that there exists a proper closed subscheme \( Z \) of \( \mathcal{N} \) such that \( \mathcal{N}(L^b) \) is contained in \( \mathcal{N}(L^b)^h \cup Z \). By (1), the difference

\[
K\mathcal{N}(x_1)^h \cdots K\mathcal{N}(x_{n-1})^h - K\mathcal{N}(L^b)^h
\]

belongs to \( K_0^Z(\mathcal{N}) \). By definition,

\[
K\mathcal{N}(L^b)^v = K\mathcal{N}(x_1) \cdots K\mathcal{N}(x_{n-1}) - K\mathcal{N}(L^b)^h = (K\mathcal{N}(x_1)^v + K\mathcal{N}(x_1)^h) \cdots (K\mathcal{N}(x_{n-1})^v + K\mathcal{N}(x_{n-1})^h) - K\mathcal{N}(L^b)^h = C + (K\mathcal{N}(x_1)^h \cdots K\mathcal{N}(x_{n-1})^h - K\mathcal{N}(L^b)^h),
\]

where

\[
C := \sum_{i=1}^{n-1} K\mathcal{N}(x_i)^v \cdots K\mathcal{N}(x_{n-1})^v \cdot K\mathcal{N}(x_i) \cdots K\mathcal{N}(x_{n-1})
\]

belongs to \( K_0^Z(\mathcal{N}) \). Thus, \( K\mathcal{N}(L^b)^v \) belongs to \( K_0^Z(\mathcal{N}) \) hence has proper support.

Definition 2.51. Let \( L^b \) be an element of \( b(V) \) (Definition 2.10). For \( x \in V \setminus V_L \), we put

\[
\operatorname{Int}_{L^b}(x) := K\mathcal{N}(L^b)^v \cdot K\mathcal{N}(x),
\]

\[
\operatorname{Int}_{L^b}^h(x) := K\mathcal{N}(L^b)^h \cdot K\mathcal{N}(x),
\]

\[
\operatorname{Int}_{L^b}^v(x) := K\mathcal{N}(L^b)^v \cdot K\mathcal{N}(x).
\]
Here, the intersection numbers are well-defined since \( \mathcal{N}(L') \cap \mathcal{N}(x) \) is a proper closed subscheme of \( \mathcal{N} \) by Remark 2.27. Note that \( \text{Int}_{L'}(x) = \text{Int}(L' + \langle x \rangle) \) (Definition 2.8).

The following is our main result of this subsection.

**Proposition 2.52.** Let \( L' \) be an element of \( \mathcal{b}(V) \) (Definition 2.10).

1. We have \( \text{Int}_{L'}^{h}(x) = \partial \text{Den}_{L'}^{h}(x) \) for \( x \in V \setminus V_{L'} \), where \( \partial \text{Den}_{L'}^{h} \) is from Definition 2.21.
2. The function \( \text{Int}_{L'}^{V} \) extends (uniquely) to a (compactly supported) locally constant function on \( V \), which we still denote by \( \text{Int}_{L'}^{V} \). Moreover, we have

\[
\text{Int}_{L'}^{V} = -\text{Int}_{L'}^{V}.
\]

In particular, the support of \( \text{Int}_{L'}^{V} \) is contained in \( V^{\text{int}} \) (Definition 2.12).

The rest of this subsection is devoted to the proof of this proposition.

**Remark 2.53** (Cancellation law for special cycles). Let \( V' \) be a hermitian subspace of \( V \) that is nonsplit and of positive even dimension \( n' \). Let \( L \) be an integral hermitian \( O_{E} \)-module contained in \( V \) such that \( L \cap V'^{\perp} \) is a self-dual \( O_{E} \)-lattice of \( V'^{\perp} \). We may choose

- an object \( (X', \iota_{X'}, \lambda_{X'}) \in \text{Exo}_{(n', -1, 1)}^{b}(\kappa) \) (Definition 2.1),
- an object \( (Y, \iota_{Y}, \lambda_{Y}) \in \text{Exo}_{(n - n', 0)}(O_{E}) \) (Remark 2.11),
- a quasi-morphism \( \varrho \) from \( (Y, \iota_{Y}, \lambda_{Y}) \otimes_{O_{E}} \kappa \oplus (X', \iota_{X'}, \lambda_{X'}) \) to \( (X, \iota_{X}, \lambda_{X}) \) in the category \( \text{Exo}_{(n-1, 1)}^{b}(S \otimes_{O_{E}} \kappa) \) satisfying
  - \( \varrho \) identifies \( \text{Hom}_{O_{E}}(X_{0} \otimes_{O_{E}} \kappa, X') \otimes \mathbb{Q} \) with \( V' \) as hermitian spaces;
  - \( \varrho \) identifies \( \text{Hom}_{O_{E}}(X_{0} \otimes_{O_{E}} \kappa, Y \otimes_{O_{E}} \kappa) \) with \( L \cap V'^{\perp} \) as hermitian \( O_{E} \)-modules.

Let \( \mathcal{N}' := \mathcal{N}(X', \iota_{X'}, \lambda_{X'}) \) be the relative Rapoport–Zink space for the triple \( (X', \iota_{X'}, \lambda_{X'}) \) (Definition 2.3). We have a morphism \( \mathcal{N}' \to \mathcal{N} \) such that for every object \( S \) of \( \text{Sch}_{O_{E}}^{y} \), \( \mathcal{N}(S) \) it sends an object \( (X', \iota_{X'}, \lambda_{X'}; \rho_{X'}) \in \mathcal{N}'(S) \) to the object

\[
(Y \otimes_{O_{E}} S \oplus X', \iota_{Y} \otimes_{O_{E}} S \oplus \iota_{X'}, \lambda_{Y} \otimes_{O_{E}} S \oplus \lambda_{X'}; \varrho \circ (\text{id}_{Y} \otimes_{O_{E}} S \oplus \rho_{X'})) \in \mathcal{N}(S).
\]

We have

1. The morphism \( \mathcal{N}' \to \mathcal{N} \) above identifies \( \mathcal{N}' \) with the closed formal subscheme \( \mathcal{N}(L \cap V'^{\perp}) \) of \( \mathcal{N} \).
2. Suppose that \( L \cap V' \neq \{0\} \), then \( \mathcal{N}(L) \) coincides with the image of \( \mathcal{N}'(L \cap V') \) under the morphism \( \mathcal{N}' \to \mathcal{N} \) above.
3. For a nonzero element \( x \in V \) written as \( x = y + x' \) with respect to the orthogonal decomposition \( V = V'^{\perp} \oplus V' \), we have

\[
\mathcal{N}' \times_{\mathcal{N}} \mathcal{N}(x) = \begin{cases} 
\emptyset, & \text{if } y \notin L \cap V'^{\perp}, \\
\mathcal{N}', & \text{if } y \in L \cap V'^{\perp} \text{ and } x' = 0, \\
\mathcal{N}(x'), & \text{if } y \in L \cap V'^{\perp} \text{ and } x' \neq 0.
\end{cases}
\]

4. If \( L \) is an \( O_{E} \)-lattice of \( V \), then we have \( \text{Int}(L) = \text{Int}(L \cap V') \).

**Lemma 2.54.** Let \( L'' \in \mathcal{b}(V) \) be an element that is integral and satisfies \( t(L'') = 1 \).

1. The formal subscheme \( \mathcal{N}(L'') \) is finite flat over \( \text{Spf} O_{E} \).

\({}^{8}\text{When } n' = n, \text{ we simply ignore } (Y, \iota_{Y}; \lambda_{Y})\).
(2) If we put \( \mathcal{N}(L')^\circ := \mathcal{N}(L') - \mathcal{N}(L'_\alpha) \) as an element in \( F^{n-1}K_0(\mathcal{N}) \), then for every \( x \in V \setminus V_L \),
\[
\mathcal{N}(L')^\circ.\mathcal{K}(x) = 2 \sum_{\substack{L \subseteq L' \subseteq (L')^\vee \\text{for } L \cap V_L = L''}} \mathbb{1}_L(x).
\]
Here, \( L''_\alpha \) is the unique element in \( b(V) \) satisfying \( L'' \subseteq L''_\alpha \subseteq (L')^\vee \) with \( |L''/L''_\alpha| = q \) (so that \( L''_\alpha \) is either not integral, or is integral with \( t(L''_\alpha) = 1 \)).

Proof. Since \( t(L') = 1 \), we may choose a 2-dimensional (nonsplit) hermitian subspace \( V' \) of \( V \) such that \( L'' \cap V'^\perp \) is a self-dual \( \mathcal{O}_E \)-lattice of \( V'^\perp \). We adopt the construction in Remark 2.53.

For (1), we have \( \mathcal{N}(L') = \mathcal{N}'(L'' \cap V') \), which is finite flat over \( \text{Spf} \mathcal{O}_E \) by (the \( r = 1 \) case) of Lemma 2.41.

For (2), we write \( x = y + x' \) with respect to the orthogonal decomposition \( V = V'^\perp \oplus V' \). Since \( x \notin V_L \), we have \( x' \neq 0 \). By Remark 2.53(2), \( \mathcal{N}(L')^\circ \) coincides with (the class of) \( \mathcal{N}'(L'' \cap V')^\circ \) in \( F^1K_0(\mathcal{N}') \) under the map \( F^1K_0(\mathcal{N}') \to F^{n-1}K_0(\mathcal{N}) \). There are two cases:

If \( y \notin L'' \cap V'^\perp \), then \( \mathcal{N}(L')^\circ.\mathcal{K}(x) = 0 \) by Remark 2.53(3), and there is no integral \( \mathcal{O}_E \)-lattice of \( V \) containing \( L'' + \langle x \rangle \). Thus, (2) follows.

If \( y \in L'' \cap V'^\perp \), then by Remark 2.53(3), we have
\[
\mathcal{N}(L')^\circ.\mathcal{K}(x) = \mathcal{N}'(L'' \cap V')^\circ.\mathcal{K}(x') = \text{length}_{\mathcal{O}_E} \mathcal{N}'(L'' \cap V')^\circ \cap \mathcal{N}'(x').
\]
By Corollary 2.47, we have
\[
\text{length}_{\mathcal{O}_E} \mathcal{N}'(L'' \cap V')^\circ \cap \mathcal{N}'(x') = 2 \sum_{\substack{L' \subseteq L'(\subseteq V') \\text{for } L' \cap (V_L \cap V') = L'' \cap V'}} \sum_{\substack{L \subseteq L' \subseteq (L')^\vee \\text{for } L \cap V_L = L''}} \mathbb{1}_L(x).
\]
Thus, (2) follows. \( \square \)

Lemma 2.55. Let \( L \) be an element of \( b(V) \) (Definition 2.10). We have
\[
\mathcal{N}(L)^h = \bigcup_{\substack{L' \subseteq L' \subseteq (L')^\vee \\text{for } L' \cap V_L = L'' \\text{and } t(L'') = 1}} \mathcal{N}(L')^\circ
\]
as closed formal subschemes of \( \mathcal{N} \), and the identity
\[
\mathcal{K}(L)^h = \sum_{\substack{L' \subseteq L' \subseteq (L')^\vee \\text{for } L' \cap V_L = L'' \\text{and } t(L'') = 1}} \mathcal{N}(L')^\circ
\]
in \( F^{n-1}K_0(\mathcal{N})/F^nK_0(\mathcal{N}) \), where \( \mathcal{N}(L')^\circ \) is introduced in Lemma 2.54(2).

Proof. This lemma can be proved by the same way as for \([\text{LZ}, \text{Theorem } 4.2.1]\), as long as we establish the following claim replacing \([\text{LZ, Lemma } 4.5.1]\) in the case where \( E/F \) is ramified:

- Let \( L \) be a self-dual hermitian \( \mathcal{O}_E \)-module of rank \( n \) and \( L' \) a hermitian \( \mathcal{O}_E \)-module contained in \( L \). If \( L/L' \) is free, then \( L' \) is integral with \( t(L') = 1 \). However, this is just a special case of Lemma 2.24(2). \( \square \)

Lemma 2.56. Let \( \Lambda \) be a vertex \( \mathcal{O}_E \)-lattice of \( V \) with \( t(\Lambda) = 4 \). Take an arbitrary connected component \( V^+_\Lambda \) of the smooth projective curve \( V_\Lambda \) from Proposition 2.29, regarded as an element in \( F^{n-1}K_0(\mathcal{N}) \). For every \( x \in V \setminus \{0\} \), put \( \text{Int}_{V^+_\Lambda}(x) := V^+_\Lambda.\mathcal{K}(x) \). Then \( \text{Int}_{V^+_\Lambda} \)
extends (uniquely) to a compactly supported locally constant function on \( V \), which we still denote by \( \text{Int}_{\Lambda}^{+} \). Moreover, we have

\[
\widetilde{\text{Int}}_{\Lambda}^{+} = -\text{Int}_{\Lambda}^{+}.
\]

**Proof.** Since \( t(\Lambda) = 4 \), we may choose a 4-dimensional (non-split) hermitian subspace \( V' \) of \( V \) such that \( \Lambda \cap V'^{\perp} \) is a self-dual \( O_E \)-lattice of \( V'^{\perp} \). We adopt the construction in Remark 2.53. Write \( x = y + x' \) with respect to the orthogonal decomposition \( V = V'^{\perp} \oplus V' \). Put \( \Lambda' := \Lambda \cap V' \). By Remark 2.53(2) and Definition 2.28(2), \( \nu_{\Lambda} \) coincides with \( \nu_{\Lambda'} \) under the natural morphism \( V' \to \nu_{\Lambda'} \). Denote by \( \nu_{\Lambda'}^{+} \) the connected component of \( \nu_{\Lambda'} \) that corresponds to \( \nu_{\Lambda}^{+} \). By Remark 2.53(3), we have

\[
\nu_{\Lambda}^{+} \cdot \nu(x) = \begin{cases} 0, & \text{if } y \notin \Lambda \cap V'^{\perp} \\ \nu_{\Lambda'}^{+} \cdot \nu(x'), & \text{if } y \in \Lambda \cap V'^{\perp}. \end{cases}
\]

In other words, we have \( \text{Int}_{\nu_{\Lambda}^{+}} = \mathbb{1}_{\Lambda \cap V'^{\perp}} \otimes \text{Int}_{\nu_{\Lambda'}^{+}} \). Therefore, it suffices to consider the case where \( n = 4 \).

We now give an explicit formula for \( \text{Int}_{\nu_{\Lambda}^{+}}(x) \) when \( n = 4 \). Let \( \nu^{+} \) be the connected component of \( \nu \) that contains \( \nu_{\Lambda}^{+} \), and put \( \bar{Z}^{+} := Z \cap \nu^{+} \) for every formal subscheme \( Z \) of \( \nu \). Put \( \Lambda(x) := \Lambda + \langle x \rangle \). There are three cases.

1. Suppose that \( \Lambda(x) \) is not integral. By Corollary 2.31, \( \nu_{\Lambda} \) has empty intersection with \( \nu(x) \). Thus, we have \( \text{Int}_{\nu_{\Lambda}^{+}}(x) = 0 \).

2. Suppose that \( \Lambda(x) \) is integral but \( x \notin \Lambda \). Then \( \Lambda(x) \) has fundamental invariants \((0,0,1,1)\). By Corollary 2.31, \( \nu_{\Lambda}^{+} \cap \nu(x) = \nu_{\Lambda}^{+}(x) \) which is a \( \mathbb{k} \)-point. Thus, we have \( \text{Int}_{\nu_{\Lambda}^{+}}(x) \geq 1 \). Choose a normal basis \( \{x_1, x_2, x_3, x_4\} \) of \( \Lambda \) and write \( x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \) with \( \lambda_i \in E \). Without lost of generality, we may assume \( \lambda_4 \notin O_E \). Since \( ux \in \Lambda \), we have \( \Lambda(x) = \langle x_1, x_2, x_3, x \rangle \). By Corollary 2.31, \( \nu(x_1) \cap \nu(x_2) \cap \nu(x_3) \) contains \( \nu_{\Lambda} \) as a closed subscheme. By Remark 2.53 and Proposition 2.45 applied to \( V' \) spanned by \( x_3 \) and \( x_4 \), \( \nu(\Lambda(x)) \) is a 0-dimensional scheme and \( \text{Int}(\Lambda(x)) = 2 \). It follows that

\[
\text{Int}_{\nu_{\Lambda}^{+}}(x) \leq \text{length}_{O_E} (\nu(x_1) \cap \nu(x_2) \cap \nu(x_3)) \cap \nu(x)^{+} = \text{Int}^{+}(\Lambda(x)) = 1
\]

by Lemma 2.57 below. Thus, we obtain \( \text{Int}^{+}(\Lambda(x)) = 1 \) hence \( \text{Int}_{\nu_{\Lambda}^{+}}(x) = 1 \).

3. Suppose that \( x \in \Lambda \). Then \( \nu_{\Lambda}^{+} \) is a closed subscheme of \( \nu(x) \), which implies

\[
\mathcal{O}_{\nu_{\Lambda}^{+}} \otimes_{\mathcal{O}_{\nu}} \mathcal{O}_{\nu(x)} = \left( \mathcal{O}_{\nu_{\Lambda}^{+}} \otimes_{\mathcal{O}_{\nu(x)}} \mathcal{O}_{\nu(x)} \right) \otimes_{\mathcal{O}_{\nu}} \mathcal{O}_{\nu(x)} = \mathcal{O}_{\nu_{\Lambda}^{+}} \otimes_{\mathcal{O}_{\nu(x)}} \mathcal{O}_{\nu(x)}.
\]

However, by Corollary 2.35, we have \( \mathcal{O}_{\nu(x)} \otimes_{\mathcal{O}_{\nu}} \mathcal{O}_{\nu(x)} = \mathcal{O}_{\nu(x)} \otimes_{\mathcal{O}_{\nu}} C(0) \) in \( K_0(\nu) \), where \( C(0) \) is the complex (2.14). Thus, we obtain

\[
\text{Int}_{\nu_{\Lambda}^{+}}(x) = \chi(C(0)_{\nu_{\Lambda}^{+}}) = \deg(\mathcal{O}_{\nu_{\Lambda}^{+}}) - \deg(\omega|_{\nu_{\Lambda}^{+}}) = -\deg(\omega|_{\nu_{\Lambda}^{+}}) = 1 - q.
\]

by Lemma 2.43.

Since there are exactly \( q^2 + 1 \) vertex \( O_E \)-lattices of \( V \) properly containing \( \Lambda \), combining (1–3), we obtain

\[
\text{Int}_{\nu_{\Lambda}^{+}} = -q(1 + q)\mathbb{1}_{\Lambda} + \sum_{\Lambda \subseteq \Lambda' \subseteq \Lambda''} \mathbb{1}_{\Lambda'}.\]
Proof of Proposition 2.52.

If \( x \in \Lambda \), then \( \widehat{\text{Int}}_{\Lambda}^+ (x) = - \frac{1+q}{q} + \frac{q+1}{q} = q - 1 \).

If \( \Lambda(x) \) is integral but \( x \not\in \Lambda \), then the number of \( \Lambda' \) in the summation of (2.18) such that \( x \in \Lambda' \) is exactly 1 (namely, \( \Lambda(x) \) itself). Thus, we have \( \widehat{\text{Int}}_{\Lambda}^+ (x) = - \frac{1+q}{q} + \frac{q+1}{q} = -1 \).

If \( \Lambda(x) \) is not integral but \( x \in \Lambda' \), then the set of \( \Lambda' \) in the summation of (2.18) satisfying \( x \in \Lambda' \) is bijective to the set of isotropic lines in \( \Lambda'/\Lambda \) perpendicular to \( x \). Now since \( \Lambda(x) \) is not integral, \( x \) is anisotropic in \( \Lambda'/\Lambda \), which implies that the previous set has cardinality \( q + 1 \). Thus, we have \( \widehat{\text{Int}}_{\Lambda}^+ (x) = - \frac{1+q}{q} + \frac{q+1}{q} = 0 \).

If \( x \not\in \Lambda' \), then \( \widehat{\text{Int}}_{\Lambda}^+ (x) = 0 \).

Therefore, we have \( \widehat{\text{Int}}_{\Lambda}^- = - \text{Int}_{\Lambda}^+ \). The lemma is proved. \( \square \)

Lemma 2.57. Denote the two connected components of \( \mathcal{N} \) by \( \mathcal{N}^+ \) and \( \mathcal{N}^- \), and \( \text{Int}^\pm (L) \) the intersection multiplicity in Definition 2.8 on \( \mathcal{N}^\pm \). Then

\[
\text{Int}^+ (L) = \text{Int}^- (L) = \frac{1}{2} \text{Int} (L).
\]

Proof. Choose a normal basis (Remark 2.14) \( \{x_1, \ldots, x_n\} \) of \( L \). Since \( V \) is nonsplit, there exists an anisotropic element in the basis, say \( x_n \). Let \( \theta \) the unique element in \( U(V)(F) \) satisfying \( \theta (x_i) = 1 \) for \( 1 \leq i \leq n - 1 \) and \( \theta (x_n) = -x_n \). Then \( \theta \) induces an automorphism of \( \mathcal{N} \), preserving \( \mathcal{N} (x_i) \) for \( 1 \leq i \leq n \), but switching \( \mathcal{N}^+ \) and \( \mathcal{N}^- \) as \( \det \theta = -1 \). Thus, we have \( \text{Int}^+ (L) = \text{Int}^- (L) = \text{Int} (L) \), the lemma follows. \( \square \)

Proof of Proposition 2.52. We first consider (1). By Lemma 2.55, we have for \( x \in V \setminus V_{L^f} \),

\[
\text{Int}_{L^f}^b (x) = \sum_{L^f \subseteq L^f \subseteq (L^f)^\vee \atop t(L^f) = 1} \mathcal{N} (L^f)^\circ \cdot \kappa (\mathcal{N} (x)),
\]

which, by Lemma 2.54, equals

\[
2 \sum_{L^f \subseteq L^f \subseteq (L^f)^\vee \atop t(L^f) = 1} \sum_{L^f \subseteq L^f \subseteq L^f \atop L \cap V_{L^f} = L^f} \mathbb{1}_L (x) = 2 \sum_{L^f \subseteq L^f \subseteq L^f \atop t(L \cap V_{L^f}) = 1} \mathbb{1}_L (x).
\]

Thus, Proposition 2.52(1) follows from Definition 2.21.

We first consider (2). We may assume \( r \geq 2 \) since otherwise \( \text{Int}_{L^f}^b \equiv 0 \) hence (2) is trivial. We write \( \mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \) for the two connected components. For every vertex \( O_E \)-lattice \( \Lambda \) of \( V \), we put \( \mathcal{V}_\Lambda^\pm := \mathcal{V}_\Lambda \cap \mathcal{N}^\pm \). By Lemma 2.50 and Proposition 2.29, there exists finitely many vertex \( O_E \)-lattices \( \Lambda_1, \ldots, \Lambda_m \) of \( V \) of type \( n \) such that

\[
\kappa (\mathcal{N} (L^f)^\circ) \in \sum_{i=1}^m F^{n-1} K_0^{\Lambda_i} (\mathcal{N}) \subseteq F^{n-1} K_0 (\mathcal{N}).
\]

Since the natural map \( F^{t(\Lambda)} - 2 K_0 (\mathcal{V}_\Lambda) \to F^{n-1} K_0^{\Lambda_i} (\mathcal{N}) \) is an isomorphism for \( 1 \leq i \leq m \), by Lemma 2.33, there exist rational numbers \( c_{\Lambda}^\pm \) for vertex \( O_E \)-lattices \( \Lambda \) of \( V \) with \( t(\Lambda) = 4 \),
of which all but finitely many are zero, such that
\[ \mathcal{K}_\mathcal{N}(L^\delta)^\vee - \left( \sum_\Lambda c_\Lambda^+ \cdot V_\Lambda^+ + c_\Lambda^- \cdot V_\Lambda^- \right) \]
has zero intersection with \( F^1 K_0(\mathcal{N}) \). Thus, Proposition 2.52(2) follows from Lemma 2.56. \( \square \)

2.7. Proof of Theorem 2.9. Let the setup be as in Subsection 2.1. In this subsection, for an element \( L^\delta \in \mathfrak{b}(V) \) (Definition 2.10), we set \( \text{val}(L^\delta) = -1 \) if \( L^\delta \) is not integral.

**Lemma 2.58.** Suppose that \( r \geq 2 \) and take an integral element \( L^\delta \in \mathfrak{b}(V) \) whose fundamental invariants \( (a_1, \ldots, a_{n-2}, a_{n-1}) \) satisfy \( a_{n-2} < a_{n-1} \) (in particular, \( a_{n-1} \) is odd). Then the number of integral \( O_E \)-lattices of \( V \) containing \( L^\delta \) with fundamental invariants \( (a_1, \ldots, a_{n-2}, a_{n-1} - 1, a_{n-1} - 1) \) is either 0 or 2. When the number is 2 and those lattices are denoted by \( L^\delta_1 \) and \( L^\delta_2 \), we have

1. \( L^\delta_1 \cap V_{L^\delta} = L^\delta \);
2. \( a_{n-1} \geq 3 \);
3. there are orthogonal decompositions \( L^\delta = L^\delta_1 + L^\delta_2 \) and \( L^\delta_1 = L^\delta_1 \perp L^\delta_2 \), in which \( L^\delta_1 \), \( L^\delta_2 \), and \( L^\delta_1 \pm L^\delta_2 \) are integral hermitian \( O_E \)-modules with fundamental invariants \( (a_1, \ldots, a_{n-2}), (a_{n-1}), (a_{n-1} - 1, a_{n-1} - 1) \), respectively.

**Proof.** Let \( L \) be an integral \( O_E \)-lattice of \( V \) containing \( L^\delta \) with fundamental invariants \( (a_1, \ldots, a_{n-2}, a_{n-1} - 1, a_{n-1} - 1) \).

We first claim that (1) must hold. We have \( \text{val}(L \cap V_{L^\delta}) \geq a_1 + \cdots + a_{n-2} + a_{n-1} - 1 \) by Lemma 2.24(1). Since \( L \cap V_{L^\delta} \) contains \( L^\delta \) and \( \text{val}(L \cap V_{L^\delta}) \) is odd, we must have \( L \cap V_{L^\delta} = L^\delta \).

Choose a normal basis \( (e_1, \ldots, e_{n-1}) \) of \( L^\delta \) (Remark 2.14), and rearrange them such that for every \( 1 \leq i \leq n - 1 \), exactly one of the following three happens:

1. \( (e_i, e_i) V = \beta_i u^{a_i-1} \) for some \( \beta_i \in O_E^\times \);
2. \( (e_i, e_{i+1}) V = u^{a_{i+1}-1} \);
3. \( (e_i, e_{i-1}) V = -u^{a_i-1} \).

By the claim on (1), we may write \( L = L^\delta + \langle x \rangle \) in which

\[ x = \lambda_1 e_1 + \cdots + \lambda_n e_{n-1} + x_n \]

for some \( \lambda_i \in (E \setminus O_E) \cup \{0\} \) and \( 0 \neq x_n \in V_{L^\delta}^\perp \). Let \( T \) be the moment matrix with respect to the basis \( \{e_1, e_{n-1}, x\} \) of \( L \).

We show by induction that for \( 1 \leq i \leq n - 2 \), \( \lambda_i = 0 \). Suppose we know \( \lambda_1 = \cdots = \lambda_{i-1} = 0 \). For \( \lambda_i \) (with \( 1 \leq i \leq n - 2 \)), there are three cases.

- If \( e_i \) is in the situation (a) above, then applying Lemma 2.24(1) to the \( i \)-by-\( i \) minor of \( T \) consisting of rows \( \{1, \ldots, i\} \) and columns \( \{1, \ldots, i - 1, n\} \), we obtain \( \text{val}_E(\lambda_i \beta_i u^{a_i-1}) \geq a_i - 1 \), which implies \( \lambda_i = 0 \).
- If \( e_i \) is in the situation (b) above, then applying Lemma 2.24(1) to the \( i \)-by-\( i \) minor of \( T \) consisting of rows \( \{1, \ldots, i - 1, i + 1\} \) and columns \( \{1, \ldots, i - 1, n\} \), we obtain \( \text{val}_E(-\lambda_i u^{a_i-1}) \geq a_i - 1 \), which implies \( \lambda_i = 0 \).
- If \( e_i \) is in the situation (c) above, then applying Lemma 2.24(1) to the \( i \)-by-\( i \) minor of \( T \) consisting of rows \( \{1, \ldots, i\} \) and columns \( \{1, \ldots, i - 1, n\} \), we obtain \( \text{val}_E(\lambda_i u^{a_i-1}) \geq a_i - 1 \), which implies \( \lambda_i = 0 \).

Note that \( e_{n-1} \) is in the situation (a). Applying Lemma 2.24(1) to the \( (n-1) \)-by-\( (n-1) \) minor of \( T \) consisting of rows \( \{1, \ldots, n-1\} \) and columns \( \{1, \ldots, n-2, n\} \), we obtain...
val\(_E(\lambda_{n-1} \beta_{n-1} u^{a_{n-1}}) \geq a_{n-1} - 2\), which implies \(\lambda_{n-1} \in u^{-1}O_E\). On the other hand, \(\lambda_{n-1} \neq 0\) since otherwise \(a_{n-1}\) will appear in the fundamental invariants of \(L\), which is a contradiction. Thus, we have \(\lambda_{n-1} \in u^{-1}O_E \setminus O_E\). After rescaling by an element in \(O_E^\perp\), we may assume \(\lambda_{n-1} = u^{-1}\). Applying Lemma 2.24(1) to the \((n-1)\)-by-\((n-1)\) minor of \(T\) consisting of rows \(\{1, \ldots, n-2, n\}\) and columns \(\{1, \ldots, n-2, n\}\), we obtain

\[\text{val}_E((x_n, x_n)_V - u^{-2} \beta_{n-1} u^{a_{n-1}}) \geq a_{n-1} - 2.\]

We note the following facts.

- The set of \(x_n \in V^\perp_L\) satisfying (2.19) is stable under the multiplication by \(1 + uO_E\).
- The set of orbits of such \(x_n\) under the multiplication by \(1 + uO_E\) is bijective to the set of \(L\).
- The number of orbits is either 0 or 2.
- If the number is 2, then \(a_{n-1} \geq 3\), since \(V\) is nonsplit.

Thus, the main part of the lemma is proved, with the properties (1) and (2) included. For (3), we simply take \(L' = \langle e_1, \ldots, e_{n-2} \rangle\) with \(L'_{\pm}\) and \(L'_{\pm}^\perp\) uniquely determined.

The lemma is proved. \(\square\)

In the rest of subsection, we say that \(L^b\) is special if \(L^b\) is like in Lemma 2.58 for which the number is 2. We now define an open compact subset \(S_{L^b}\) of \(V\) for an integral element \(L^b \in \mathfrak{b}(V)\) in the following way:

\[
S_{L^b} := \begin{cases} 
L^b_+ \cup L^b_- & \text{if } L^b \text{ is special,} \\
L^b + (V^\perp_L)^\text{int} & \text{if } L^b \text{ is not special.}
\end{cases}
\]

**Lemma 2.59.** Take an integral element \(L^b \in \mathfrak{b}(V)\). Then for every \(x \in V \setminus (V_{L^b} \cup S_{L^b})\), we may write \(L^b + \langle x \rangle = L^{b'} + \langle x' \rangle\) for some \(L^{b'} \in \mathfrak{b}(V)\) satisfying \(\text{val}(L^{b'}) < \text{val}(L^b)\).

**Proof.** Take an element \(x \in V \setminus (V_{L^b} \cup S_{L^b})\). Put \(L := L^b + \langle x \rangle\). If \(L\) is not integral, then by Remark 2.14, we may write \(L = L^{b'} + \langle x' \rangle\) with \(L^{b'} \in \mathfrak{b}(V)\) that is not integral; hence the lemma follows.

In what follows, we assume \(L\) integral and write its fundamental invariants as \((a'_1, \ldots, a'_n)\). By Remark 2.14, it suffices to show that \(a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1} - 2\).

Choose a normal basis \((e_1, \ldots, e_{n-1})\) of \(L^b\) (Remark 2.14), and rearrange them such that for every \(1 \leq i \leq n-1\), exactly one of the following three happens:

(a) \((e_i, e_i)_V = \beta_i u^{a_i-1}\) for some \(\beta_i \in O_E^\perp;\)
(b) \((e_i, e_{i+1})_V = u^{a_i-1};\)
(c) \((e_i, e_{i-1})_V = -u^{a_i-1}.\)

Write \(x = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + x_n\) for some \(\lambda_i \in (E \setminus O_E) \cup \{0\}\) and \(0 \neq x_n \in V^\perp_L\). Let \(T\) be the moment matrix with respect to the basis \(\{e_1, \ldots, e_{n-1}, x\}\) of \(L\).

If \(\lambda_1 = \cdots = \lambda_{n-1} = 0\), then since \(x \notin S_{L^b}\), we have either \(\langle x \rangle\) is not integral, or \(\text{val}(x) \leq a_{n-1} - 2\) (only possible when \(L^b\) is special) which implies \(a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1} - 2\).

If \(\lambda_i \neq 0\) for some \(1 \leq i \leq n-1\) such that \(e_i\) is in the situation (b) or (c), then applying Lemma 2.24(1) to the \((n-1)\)-by-\((n-1)\) minor of \(T\) deleting the \(i\)-th row and the \(i\)-th column, we obtain \(a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1} - 2.\)
If $\lambda_i \not\in u^{-1}O_F$ for some $1 \leq i \leq n-1$ such that $e_i$ is in the situation (a), then applying Lemma 2.24(1) to the $(n-1)$-by-$(n-1)$ minor of $T$ deleting the $i$-th row and the $n$-th column, we obtain $a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1}-2$.

If $\lambda_i \neq 0$ and $\lambda_j \neq 0$ for $1 \leq i < j \leq n-1$ such that both $e_i$ and $e_j$ are in the situation (a), then applying Lemma 2.24(1) to the $(n-1)$-by-$(n-1)$ minor of $T$ deleting the $i$-th row and the $j$-th column, we obtain $a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1}-2$.

The remaining case is that $\lambda_i \in u^{-1}O_F$ for a unique element $1 \leq i \leq n-1$ such that $e_i$ is in the situation (a). Then $L^b + \langle x \rangle$ is the orthogonal sum of $\langle e_1, \ldots, e_i, \ldots, e_{n-1} \rangle$ and $\langle e_i, x \rangle$. In particular, if we write the fundamental invariants of $\langle e_i, x \rangle$ as $(b_1, b_2)$, then the fundamental invariants of $L^b + \langle x \rangle$ is the nondecreasing rearrangement of $(a_1, \ldots, a_i, \ldots, a_{n-1}, b_1, b_2)$. We have two cases:

- If $(x, x)_V \not\in u^{-1}O_F$, then $(b_1, b_2) = (a_i - 1, a_i - 1)$. Thus, we have either $a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1} - 2$, or $i = n-1$, $a_{n-2} < a_{n-1}$, and $L^b + \langle x \rangle$ has fundamental invariants $(a_1, \ldots, a_{n-2}, a_{n-1} - 1, a_{n-1} - 1)$ (hence $L^b$ is special). The latter case is not possible as $x \not\in T^\perp$.
- If $(x, x)_V \not\in u^{-1}O_F$, then $b_1 \leq a_i - 2$. Thus we have $a'_1 + \cdots + a'_{n-1} \leq a_1 + \cdots + a_{n-1} - 2$ or $i = n-1$.

The lemma is proved. \hfill \Box

**Proof of Theorem 2.9.** For every element $L^b \in b(V)$, we define a function

$$\Phi_{L^b} \equiv \partial\text{Den}_{L^b} - \text{Int}_{L^b},$$

which is a compactly supported locally constant function on $V$ by Proposition 2.23 and Proposition 2.52(2). It enjoys the following properties:

1. For $x \in V \setminus V_{L^b}$, we have $\Phi_{L^b}(x) = \partial\text{Den}_{L^b}(x) - \text{Int}_{L^b}(x)$ by Proposition 2.52(1).
2. $\Phi_{L^b}$ is invariant under the translation by $L^b$, which follows from (1) and the similar properties for $\partial\text{Den}_{L^b}$ and $\text{Int}_{L^b}$.
3. The support of $\Phi_{L^b}$ is contained in $V^\text{int}$, by Proposition 2.23 and Proposition 2.52(2).

We prove by induction on val($L^b$) that $\Phi_{L^b} \equiv 0$.

The initial case is that val($L^b$) = $-1$, that is, $L^b$ is not integral. Then we have $\partial\text{Den}_{L^b} = \text{Int}_{L^b} = 0$ hence $\Phi_{L^b} \equiv 0$ by (1).

Now consider $L^b$ that is integral, and assume $\Phi_{L^b} \equiv 0$ for every $L'^b \in b(V)$ satisfying val($L'^b$) < val($L^b$). For every $x \in V \setminus (V_{L^b} \cup S_{L^b})$, by Lemma 2.59, we may write $L^b + \langle x \rangle = L'^b + \langle x' \rangle$ with some $L'^b \in b(V)$ satisfying val($L'^b$) < val($L^b$); and we have

$$\Phi_{L^b}(x) = \partial\text{Den}_{L^b}(x) - \text{Int}_{L^b}(x)$$

$$= \partial\text{Den}(L^b + \langle x \rangle) - \text{Int}(L^b + \langle x \rangle)$$

$$= \partial\text{Den}(L'^b + \langle x' \rangle) - \text{Int}(L'^b + \langle x' \rangle)$$

$$= \Phi_{L'^b}(x') = 0$$

by the induction hypothesis. Thus, the support of $\Phi_{L^b}$ is contained in $S_{L^b}$. There are two cases.

Suppose that $L^b$ is not special. By (2), we may write $\Phi_{L^b} = 1_{L^b} \otimes \phi$ for a locally constant function $\phi$ on $V_{L^b}^\perp$, supported on $(V_{L^b}^\perp)^\text{int}$. Then $\hat{\Phi}_{L^b} = C \cdot 1_{(L^b)^\perp} \otimes \hat{\phi}$ for some $C \in \mathbb{Q}^\times$. Now since $\hat{\phi}$ is invariant under the translation by $u^{-1}(V_{L^b}^\perp)^\text{int}$, we must have $\hat{\phi} = 0$ by (3), that is, $\Phi_{L^b} \equiv 0$. 
Suppose that \( L^b \) is special. We fix the orthogonal decompositions \( L^b = L^b_+ \oplus L^b_- \) and \( L^{b \pm} = L^{b \pm}_- \oplus L^{b \pm}_+ \) from Lemma 2.58. Put \( V_\psi := L^b_+ \otimes_{O_p} F \) and denote by \( V_\psi \) the orthogonal complement of \( V_\psi \) in \( V \). Then both \( L^{b \pm}_+ \) and \( L^{b \pm}_- \) are integral \( O_F \)-lattices of \( V_\psi \) with fundamental invariants \( (a_{n-1} - 1, a_{n-1} - 1) \). Moreover, we have \( S_{L^b_+} = L^{b \pm}_- \times (L_{\pm}^{b \pm}_+ \cup L_{\pm}^{b \pm}_-) \). Thus, by (2), we may write \( \Phi_{L^b} = 1_{L^b_+} \otimes \phi \) for a locally constant function \( \phi \) on \( V_\psi \), supported on \( L^{b \pm}_+ \cup L^{b \pm}_- \). Since \( (a_{n-1} - 1, a_{n-1} - 1) \geq 3 \) by Lemma 2.58, we have \( L^{b \pm}_+ \cup L^{b \pm}_- \subseteq uV^\text{int} \), which implies that the support of \( \phi \) is contained in \( uV^\text{int} \). On the other hand, by (3), the support of \( \phi \) is contained in \( V^\text{int} \). Together, we must have \( \phi = 0 \) by the Uncertainty Principle [LZ, Proposition 8.1.6], that is, \( \Phi_{L^b} \equiv 0 \).

By (1), we have \( \partial \text{Den}_{L^b}(x) = \text{Int}_{L^b}(x) \) for every \( x \in V \setminus V_{\psi} \). In particular, Theorem 2.9 follows as every \( O_F \)-lattice \( L \) of \( V \) is of the form \( L^b + \langle x \rangle \) for some \( L^b \in b(V) \). \( \square \)

### 2.8. Comparison with absolute Rapoport–Zink spaces

Let the setup be as in Subsection 2.1. In this subsection, we compare \( \mathcal{N} \) to certain (absolute) Rapoport–Zink space under the assumption that \( F \) is unramified over \( \mathbb{Q}_p \). Put \( f := [F : \mathbb{Q}_p] \) hence \( q = p^f \). This subsection is redundant if \( f = 1 \).

To begin with, we fix a subset \( \Phi \) of \( \text{Hom}(E, C_p) = \text{Hom}(E, \hat{E}) \) containing \( \varphi_0 \) and satisfying \( \text{Hom}(E, \hat{E}) = \Phi \amalg \Phi^c \). Recall that we have regarded \( E \) as a subfield of \( \hat{E} \) via \( \varphi_0 \). We introduce more notation.

- For every ring \( R \), we denote by \( W(R) \) the \( p \)-typical Witt ring of \( R \), with \( F, V, [\ ] \), and \( l(R) \) its \( (p \text{-typical}) \) Frobenius, the Verschiebung, the Teichmüller lift, and the augmentation ideal, respectively. For an \( F^l \)-linear map \( f : P \to Q \) between \( W(R) \)-modules with \( i \geq 1 \), we denote by \( f^l : W(R) \otimes_{F^lW(R)} P \to Q \) its induced \( W(R) \)-linear map.

- For \( i \in \mathbb{Z}/f\mathbb{Z} \), put \( \psi_i := F^l : O_F \to O_F \), define \( \hat{\psi}_i : O_F \to W(O_F) \) to be the composition of \( \psi_i \) with the Cartier homomorphism \( O_F \to W(O_F) \), and denote by \( \varphi_i \) the unique element in \( \Phi \) above \( \psi_i \).

- For \( i \in \mathbb{Z}/f\mathbb{Z} \), let \( \epsilon_i \) be the unique unit in \( W(O_F) \) satisfying \( \mathcal{V}_{\epsilon_i} = [\psi_i(u^2)] \sim \hat{\psi}_i(u^2) \), which exists by [ACZ16, Lemma 2.24]. Then we fix a unit \( \mu_{u} \) in \( W(O_F) \), where \( \hat{F} \) denotes the complete maximal unramified extension of \( F \) in \( \hat{E} \), such that

\[
\frac{F^l \mu_{u}}{\mu_{u}} = \prod_{i=1}^{f-1} e^{f^l-1-i} \epsilon_i,
\]

which is possible since the right-hand side is a unit in \( W(O_F) \).

- For a \( p \)-divisible group \( X \) over an object \( S \) of \( \text{Sch}_{/O_{\hat{E}}} \) with an action by \( O_F \), we have a decomposition

\[
\text{Lie}(X) = \bigoplus_{i=0}^{f-1} \text{Lie}_{\psi_i}(X)
\]

of \( \mathcal{O}_S \)-modules according to the action of \( O_F \) on \( \text{Lie}(X) \).

**Definition 2.60.** Let \( S \) be an object of \( \text{Sch}_{/O_{\hat{E}}} \). We define a category \( \text{Exo}_{(n-1,1)}(S) \) whose objects are triples \( (X, \iota_X, \lambda_X) \) in which

- \( X \) is a \( p \)-divisible group over \( S \) of dimension \( nf \) and height \( 2nf \);
- \( \iota_X : O_F \to \text{End}(X) \) is an action of \( O_F \) on \( X \) satisfying:
  - (Kottwitz condition): the characteristic polynomial of \( \iota_X(u) \) on the \( \mathcal{O}_S \)-module \( \text{Lie}_{\psi_0}(X) \) is \( (T - u)^{n-1}(T + u) \in \mathcal{O}_S[T] \),

where

\[
\text{End}(X) = \bigoplus_{i=0}^{f-1} \text{End}_{\psi_i}(X)
\]
The above decomposition is
\[ 2 (\iota_X(u) - u \mid \text{Lie}_{\psi}(X)) = 0, \]
(Spin condition): for every geometric point \( s \) of \( S \), the action of \( \iota_X(u) \) on \( \text{Lie}_{\psi}(X_s) \) is nonzero;
(Banal condition): for \( 1 \leq i \leq f - 1 \), \( O_F \) acts on \( \text{Lie}_{\psi}(X) \) via \( \varphi_i \);
\[ \lambda_X : X \to X^\vee \] is a \( \iota_X \)-compatible polarization such that \( \ker(\lambda_X) = X[\iota_X(u)] \).

A morphism (resp. quasi-morphism) from \((X, \iota_X, \lambda_X)\) to \((Y, \iota_Y, \lambda_Y)\) is an \( O_E \)-linear isomorphism (resp. quasi-isogeny) \( \rho : X \to Y \) of height zero such that \( \rho^* \lambda_Y = \lambda_X \).

When \( S \) belongs to \( \text{Sch}^v_{/O_E} \), we denote by \( \text{Exo}^{\Phi, b}_{(n-1,1)}(S) \) the subcategory of \( \text{Exo}^{\Phi}_{(n-1,1)}(S) \) consisting of \((X, \iota_X, \lambda_X)\) in which \( X \) is supersingular.

Note that both \( \text{Exo}^{\Phi, b}_{(n-1,1)} \) and \( \text{Exo}^{\Phi, b}_{(n-1,1)} \) are prestacks (that is, presheaves valued in groupoids) on \( \text{Sch}^v_{/O_E} \). Now we construct a morphism
\[ \text{(2.21)} \quad -^{\text{rel}} : \text{Exo}^{\Phi, b}_{(n-1,1)} \to \text{Exo}^{\Phi, b}_{(n-1,1)} \]
of prestacks on \( \text{Sch}^v_{/O_E} \). We will use the theory of displays [Zin02, Lau08] and \( O_F \)-displays [ACZ16].

Let \( S = \text{Spec} R \) be an affine scheme in \( \text{Sch}^v_{/O_E} \). Take an object \((X, \iota_X, \lambda_X)\) of \( \text{Exo}^{\Phi, b}_{(n-1,1)}(S) \). Write \((P, Q, F, \hat{F})\) for the display of \( X \) (as a formal \( p \)-divisible group). The action of \( O_E \) on \( P \) induces decompositions
\[ P = \bigoplus_{i=0}^{f-1} P_i, \quad Q = \bigoplus_{i=0}^{f-1} Q_i, \quad F = \bigoplus_{i=0}^{f-1} F_i, \quad \hat{F} = \bigoplus_{i=0}^{f-1} \hat{F}_i, \]
where \( P_i \) is the \( W(R) \)-submodule on which \( O_F \) acts via \( \hat{\psi}_i \), and \( Q_i = Q \cap P_i \). It is clear that the above decomposition is \( O_E \)-linear, and \( P_i \) is a projective \( O_E \otimes_{O_E, \hat{\psi}_i} W(R) \)-module of rank \( n \).

**Lemma 2.61.** For \( 1 \leq i \leq f - 1 \), we have
\[ Q_i = (u \otimes 1 - 1 \otimes [\varphi_i(u)])P_i + I(R)P_i, \]
and that the map
\[ F'_i := \hat{F}_i \circ (u \otimes 1 - 1 \otimes [\varphi_i(u)]) : P_i \to P_{i+1} \]
is a Frobenius linear epimorphism hence isomorphism.

**Proof.** The Banal condition in Definition 2.60 implies that for \( 1 \leq i \leq f - 1 \),
\[ (u \otimes 1 - 1 \otimes [\varphi_i(u)])P_i + I(R)P_i \subseteq Q_i. \]
To show the reverse inclusion, it suffices to show that the image of \((u \otimes 1 - 1 \otimes [\varphi_i(u)])P_i \) in \( P_i/I(R)P_i = P_i \otimes_{W(R)} R \) is projective of rank \( n \). But the image is \((u \otimes 1 - 1 \otimes \varphi_i(u))P_i \otimes_{W(R)} R \), which has rank \( n \) since \( P_i \) is projective over \( O_E \otimes_{O_E, \hat{\psi}_i} W(R) \) of rank \( n \).

Now we show that \((F'_i)^2 \) is surjective. It suffices to show that \( \text{coker}(F'_i)^2 \otimes_{W(R)} \kappa \) vanishes for every homomorphism \( W(R) \to \kappa \) with \( \kappa \) a perfect field of characteristic \( p \). Since \( W(R) \to \kappa \) necessarily vanishes on \( I(R) \), it lifts to a homomorphism \( W(R) \to W(\kappa) \). Thus, we may just assume that \( R \) is a perfect field of characteristic \( p \). Since
\[ (u \otimes 1 - 1 \otimes [\varphi_i(u)])(-u \otimes 1 - 1 \otimes [\varphi_i(u)]) = [\psi_i(u^2)] - \psi_i(u^2) = \psi_i \]
in which $\epsilon_i$ is a unit in $W(O_F)$, the image of the map

\[(2.22) \quad (u \otimes 1 - 1 \otimes [\varphi_i(u)]) : P_i \to P_i
\]

contains $(u \otimes 1 - 1 \otimes [\varphi_i(u)])P_i + W(R)\mathcal{V}_i \cdot P_i$. As $R$ is a perfect field of characteristic $p$, we have $W(R)\mathcal{V}_i = I(R)$, hence (2.22) is surjective. Thus, $F'_i$ is a Frobenius linear epimorphism as $F_i$ is.

The lemma is proved. \hfill \Box

Now we put

\[
\begin{align*}
P^\text{rel} & := P_0, \quad Q^\text{rel} := Q_0, \quad F^\text{rel} := F'_{f-1} \circ \cdots \circ F'_1 \circ F_0, \quad \hat{F}^\text{rel} := F'_{f-1} \circ \cdots \circ F'_1 \circ \hat{F}_0.
\end{align*}
\]

Then $(P^\text{rel}, Q^\text{rel}, F^\text{rel}, \hat{F}^\text{rel})$ defines an $f(-\mathbb{Z}_p)$-display in the sense of [ACZ16, Definition 2.1] with an $O_E$-action, for which the Kottwitz condition, the Wedge condition, and the Spin condition are obviously inherited. It remains to construct the polarization $\lambda^\text{rel}_X$. By Remark 2.62 below, we have the collection of perfect symmetric $W(R)$-bilinear pairings \{ $(\ , \ )_i | i \in \mathbb{Z}/f\mathbb{Z}$ \} coming from $\lambda_X$. For $x, y \in P_0$, put $x_i := (F'_{i-1} \circ \cdots \circ F'_1 \circ \hat{F}_0)(x)$ and $y_i := (F'_{i-1} \circ \cdots \circ F'_1 \circ \hat{F}_0)(y)$ for $1 \leq i \leq f$, and we have

\[
(\hat{F}^\text{rel} x, \hat{F}^\text{rel} y)_0 = (F'_{f-1} x_{f-1}, F'_{f-1} y_{f-1})_0
\]

\[
= (\hat{F}'_{f-1}((u \otimes 1 - 1 \otimes [\varphi_{f-1}(u)])x_{f-1}), \hat{F}'_{f-1}((u \otimes 1 - 1 \otimes [\varphi_{f-1}(u)])y_{f-1}))_0
\]

\[
= V^{-1}((u \otimes 1 - 1 \otimes [\varphi_{f-1}(u)])x_{f-1}, (u \otimes 1 - 1 \otimes [\varphi_{f-1}(u)])y_{f-1})_{f-1}
\]

\[
= V^{-1}([\psi_{f-1}(u^2)] - \hat{\psi}_{f-1}(u^2)) \cdot (x_{f-1}, y_{f-1})_{f-1}
\]

\[
= \epsilon_{f-1} \cdot F(x_{f-1}, y_{f-1})_{f-1}
\]

\[
= \cdots = \left( \prod_{i=1}^{f-1} F_{f-1-i} \epsilon_i \right) \cdot F_{f-1}(x_1, y_1)_1
\]

\[
= \left( \prod_{i=1}^{f-1} F_{f-1-i} \epsilon_i \right) \cdot F_{f-1}V^{-1}(x, y)_0.
\]

Put $(\ , \ )^\text{rel} := \mu_u(\ , \ )_0$, which satisfies $(\hat{F}^\text{rel} x, \hat{F}^\text{rel} y)^\text{rel} = F_{f-1}V^{-1}(x, y)^\text{rel}$ by (2.20). Then the $f(-\mathbb{Z}_p)$-display $(P^\text{rel}, Q^\text{rel}, F^\text{rel}, \hat{F}^\text{rel})$ with $O_E$-action together with the pairing $(\ , \ )^\text{rel}$ define an object $(X, \iota_X, \lambda^\text{rel}_X)$ of $\text{Exo}_{[n-1,1]}^b(S)$, as explained in the proof of [Mih20, Proposition 3.4] and Remark 2.62 below. It is clear that the construction is functorial in $S$.

**Remark 2.62.** For an object $(X, \iota_X, \lambda_X)$ of $\text{Exo}_{[n-1,1]}^b(S)$ with $(P, Q, F, \hat{F})$ the display of $X$, we have a similar claim as in Remark 2.2 concerning the polarization $\lambda_X$. In particular, as discussed in [Mih20, Section 11.1], the polarization $\lambda_X$, or rather its symmetrization, is equivalent to a collection of perfect symmetric $W(R)$-bilinear pairings

\[
\{ (\ , \ )_i : P_i \times P_i \to W(R) \mid i \in \mathbb{Z}/f\mathbb{Z} \},
\]

satisfying $(\iota_X(\alpha)x, y)_i = (x, \iota_X(\alpha \overline{c})y)_i$ for every $\alpha \in O_E$ and $(\hat{F}_i x, \hat{F}_i y)_{i+1} = V^{-1}(x, y)_i$ for $i \in \mathbb{Z}/f\mathbb{Z}$. 
Similarly, for an object \((X', \iota_{X'}, \lambda_{X'})\) of \(\Exo_{b,0}^{b}(S)\) with \((P', Q', F', \hat{F}')\) the \((-\mathbb{Z}_p)\)-display of \(X'\), the polarization \(\lambda_{X'}\) is equivalent to a perfect symmetric \(W(R)\)-bilinear pairing

\[
(\ , \ '): P' \times P' \to W(R),
\]
satisfying \((\iota_{X'}(\alpha)x, y)' = (x, \iota_{X'}(\alpha^c)y)'\) for every \(\alpha \in O_E\) and \((\hat{F}'x, \hat{F}'y)' = F'^{-1}V^{-1}(x, y)'\).

**Proposition 2.63.** The morphism \((2.21)\) is an isomorphism.

*Proof.* It suffices to show that for every affine scheme \(S = \text{Spec} R\) in \(\text{Sch}_{/O_E}\), the functor \(-^{rel}(S)\) is fully faithful and essentially surjective.

We first show that \(-^{rel}(S)\) is fully faithful. Take an object \((X, \iota_X, \lambda_X)\) of \(\Exo_{b,0}^{b}(S)\).

It suffices to show that the natural map \(\text{Aut}((X, \iota_X, \lambda_X)) \to \text{Aut}((X, \iota_X, \lambda_X)^{rel})\) is an isomorphism, which follows from a stronger statement that the natural map \(\text{End}_{O_E}(X) \to \text{End}_{O_E}(X^{rel})\) is an isomorphism, where \(X^{rel}\) denotes the first entry of \((X, \iota_X, \lambda_X)^{rel}\) which is an \(O_F\)-divisible group. For the latter, it amounts to showing that the natural map

\[
(2.23) \quad \text{End}_{O_E}((P, Q, F, \hat{F})) \to \text{End}_{O_E}((P^{rel}, Q^{rel}, F^{rel}, \hat{F}^{rel}))
\]
is an isomorphism. For the injectivity, let \(f\) be an element in the source, which decomposes as \(f = \sum_{i=0}^{f-1} f_i\) for endomorphisms \(f_i : P_i \to P_i\) which preserve \(Q_i\) and commute with \(F\) and \(\hat{F}\). Since for every \(i \in \mathbb{Z}/f\mathbb{Z}\), \(\hat{F}_i\) is a Frobenius linear surjective map from \(Q_i\) to \(P_{i+1}\), the map \(f\) is determined by \(f_0\). Thus, \((2.23)\) is injective. For the surjectivity, let \(f^{rel}\) be an element in the target. Put \(f_0 := f^{rel} : P_0 \to P_0\). By Lemma 2.64(2) below, there is a unique endomorphism \(f_1\) of \(P_1\) rendering the following diagram

\[
\begin{array}{ccc}
W(R) \otimes e_{W(R)} Q_0 & \xrightarrow{f_0^3} & P_1 \\
1 \otimes (f_0 |_{Q_0}) & \downarrow & \downarrow f_1 \\
W(R) \otimes e_{W(R)} Q_0 & \xrightarrow{f_0^3} & P_1
\end{array}
\]

commute. For \(2 \leq i \leq f - 1\), we define \(f_i\) to be the unique endomorphism of \(P_i\) satisfying that

\[
f_i \circ (F_i' \circ \cdots \circ F_1')^2 = (F_i' \circ \cdots \circ F_1')^2 \circ (1 \otimes f_i).
\]

Then \(f := \sum_{i=0}^{f-1} f_i\) is an \(O_E\)-linear endomorphism of \(P\) which commutes with \(\hat{F}\) hence \(F\). It remains to check that \(f(Q) \subseteq Q\), which follows from Lemma 2.61.

We then show that \(-^{rel}(S)\) is essentially surjective. Take an object \((X', \iota_{X'}, \lambda_{X'})\) of \(\Exo_{b,0}^{b}(S)\) in which \(X'\) is given by an \((-\mathbb{Z}_p)\)-display \((P', Q', F', \hat{F}')\). For \(0 \leq i \leq f - 1\), put \(P_i := W(R) \otimes_{P_i, W(R)} P'\). Denote by \(u_0 : P_0 \to P_0\) the endomorphism given by the action of \(u \in O_E\) on \(P'\). Put \(Q_i := Q'\) and for \(1 \leq i \leq f - 1\), put

\[
Q_i := ((1 \otimes u_0) \otimes 1 - (1 \otimes 1) \otimes [\varphi_i(u)])|P_i + I(R)P_i.
\]

Fix a normal decomposition \(P' = L' \oplus T'\) for \(Q'\) and let \(\hat{F}' := \hat{F}'|_{L'} + \hat{F}'|_{T'} : P' \to P'\) be the corresponding \(F^{f}\)-linear isomorphism. For \(0 \leq i < f - 1\), let \(\hat{F}_i : P_i \to P_{i+1}\) be the Frobenius linear isomorphism induced by the identity map on \(P'\); and finally let \(\hat{F}_{f-1} : P_{f-1} \to P_0\) be the Frobenius linear isomorphism induced by \(\hat{F}'\). Let \(\bar{F}_0 : Q_0 \to P_0\) be the map defined by the formula \(\bar{F}_0(l + w \cdot t) = \bar{F}_0(l) + w\bar{F}_0(t)\) for \(l \in L', t \in T',\) and \(w \in W(R)\), which is a
Frobenius linear epimorphism. By Lemma 2.64(2) below, there is a unique endomorphism $u_1$ of $P_1$ rendering the following diagram

$$
\begin{array}{ccc}
W(R) \otimes_{F,W(R)} Q_0 & \overset{\hat{F}_0^2}{\longrightarrow} & P_1 \\
1 \otimes (u_0|Q_0) & \downarrow & \\
W(R) \otimes_{F,W(R)} Q_0 & \overset{\hat{F}_0^2}{\longrightarrow} & P_1
\end{array}
$$

commute.\(^9\) For $2 \leq i \leq f - 1$, we define $u_i$ to be the unique endomorphism of $P_i$ satisfying that

$$u_i \circ (\hat{F}_{i-1} \circ \cdots \circ \hat{F}_1)^2 = (\hat{F}_{i-1} \circ \cdots \circ \hat{F}_1)^2 \circ (1 \otimes u_1),$$

and define a map $\hat{F}_i: Q_i \to P_{i+1}$ by the following (compatible) formulae

$$
\begin{align*}
\hat{F}_i((u_i \otimes 1 - 1 \otimes [\varphi_i(u)])x) &= \hat{F}_i(x), \\
\hat{F}_i(\nu_x \cdot w) &= \frac{w}{\epsilon_i} \cdot (u_{i+1} \otimes 1 + 1 \otimes F[\varphi_i(u)]) \hat{F}_i(x),
\end{align*}
$$

for $x \in P_i$ and $w \in W(R)$, which is a Frobenius linear epimorphism. Put

$$
\begin{align*}
P &:= \bigoplus_{i=0}^{f-1} P_i, & Q &:= \bigoplus_{i=0}^{f-1} Q_i, & \hat{F} &:= \sum_{i=0}^{f-1} \hat{F}_i, & u &:= \sum_{i=0}^{f-1} u_i.
\end{align*}
$$

Then it is straightforward to check that $(P, Q, F, \hat{F})$ is a display with an action by $O_E$ for which $u$ acts by $u_i$, where $F$ is determined by $\hat{F}$ in the usual way. Now we construct a collection of perfect symmetric $W(R)$-bilinear pairings $\{(\ , \ )_i \mid i \in \mathbb{Z}/f\mathbb{Z}\}$ as in Remark 2.62. Put $\mu_u(\ , \ )_0 := \mu_u^{-1}(\ , \ )'$, where $(\ , \ )'$ is the pairing induced by $\lambda_X$. Define inductively for $1 \leq i \leq f - 1$ the unique (perfect symmetric $W(R)$-bilinear) pairing $(\ , \ )_i$ satisfying $(\hat{F}_{i-1}x, \hat{F}_{i-1}y)_i = \nu_x^{-1}(x,y)_{i-1}$. It is clear that we also have $(\hat{F}_{f-1}x, \hat{F}_{f-1}y)_0 = \nu_x^{-1}(x,y)_{f-1}$. Then the display $(P, Q, F, \hat{F})$ with the $O_E$-action together with the collection of pairings $\{(\ , \ )_i \mid i \in \mathbb{Z}/f\mathbb{Z}\}$ define an object $(X, \iota_X, \lambda_X) \in \text{Ex}^{\phi,b}_{\delta_{n-1,1}}(S)$, which satisfies $(X, \iota_X, \lambda_X)^{\text{rel}} \simeq (X', \iota_{X'}, \lambda_{X'})$ by construction.

The proposition is proved.\(\square\)

**Lemma 2.64.** Let $R$ be a ring on which $p$ is nilpotent. For a pair $(P, Q)$ in which $P$ is a projective $W(R)$-module of finite rank and $Q$ is a submodule of $P$ containing $l(R)P$ such that $P/Q$ is a projective $R$-module, we define $Q^*$ to be the image of $J(R)P$ under the map $W(R) \otimes_{F,W(R)} l(R)P \to W(R) \otimes_{F,W(R)} Q$ that is the base change of the inclusion map $l(R)P \to Q$, where $J(R)$ denotes the kernel of $(\nu^{-1})^2: W(R) \otimes_{F,W(R)} l(R) \to W(R)$. Then for every Frobenius linear epimorphism $\hat{F}: Q \to P'$ with $P'$ a projective $W(R)$-module of the same rank as $P$, we have

(1) the kernel of $\hat{F}^2$ coincides with $Q^*$;

\(\text{We warn the readers that the endomorphism } u_1 \text{ might be different from } 1 \otimes u_0 \text{ as } u \text{ does not necessarily preserve the normal decomposition. However, the image of } u_1 - 1 \otimes u_0 \text{ is contained in } l(R)P_1.\)
(2) for every endomorphism $f : P \to P$ that preserves $Q$, there exists a unique endomorphism $f' : P' \to P'$ rendering the following diagram

$$\begin{array}{c}
W(R) \otimes_{F, W(R)} Q \xrightarrow{f'} W(R) \\
\downarrow \quad \downarrow \quad \downarrow f' \\
W(R) \otimes_{F, W(R)} Q \xrightarrow{f^\sharp} W(R)
\end{array}$$

commute.

Proof. We first claim that $J(R)$ is contained in the kernel of the map

$$W(R) \otimes_{F, W(R)} l(R) \to W(R) \otimes_{F, W(R)} W(R) = W(R)$$

that is the base change of the inclusion map $l(R) \to W(R)$. Take an element $x = \sum a_i \otimes b_i$ in $W(R) \otimes_{F, W(R)} l(R)$. If $x \in J(R)$, then $\sum a_i b_i = 0$. But the image of $x$ under (2.24) is $\sum a_1 F a_i b_i$, which equals $p \sum a_i b_i$. Thus, $J(R)$ is contained in the kernel of (2.24).

For (1), choose a normal decomposition $P = L \oplus T$ of $W(R)$-modules such that $Q = L \oplus l(R)T$. By (the proof of) [Lau10, Lemma 2.5], there exists a Frobenius linear automorphism $\Psi$ of $P$ such that $\hat{F}(l + at) = \Psi(l) + V^1 a \cdot \Psi(t)$ for $l \in L$, $t \in T$, and $a \in l(R)$. Thus $\ker \hat{F}$ equals the submodule $J(R)T$ of $W(R) \otimes_{F, W(R)} Q$. However, by the claim above, the image of $J(R)\mathcal{L}$ under the map $W(R) \otimes_{F, W(R)} l(R)P \to W(R) \otimes_{F, W(R)} Q$ vanishes. Thus, we have $J(R)T = Q^\ast$.

For (2), the uniqueness follows since $\hat{F}$ is surjective; and the existence follows since the map $1 \otimes (f|_Q)$ preserves $Q^\ast$, which is a consequence of the definition of $Q^\ast$. \qed

To define our (absolute) Rapoport–Zink space, we fix an object $(X, \iota_X, \lambda_X) \in \text{Exo}^b_{n-1,1}(\kbar)$.

**Definition 2.65.** We define a functor $\mathcal{N}^\Phi : \mathcal{N}^\Phi_{(X, \iota_X, \lambda_X)}$ on $\mathcal{S}^\vee_{\mathcal{O}_E}$ such that for every object $S$ of $\mathcal{S}^\vee_{\mathcal{O}_E}$, $\mathcal{N}(S)$ consists of quadruples $(X, \iota_X, \lambda_X; \rho_X)$ in which

- $(X, \iota_X, \lambda_X)$ is an object of $\text{Exo}^b_{n-1,1}(S)$;
- $\rho_X$ is a quasi-morphism from $(X, \iota_X, \lambda_X) \times_S (S \otimes_{\mathcal{O}_E} \kbar) \to (X, \iota_X, \lambda_X) \otimes_{\kbar} (S \otimes_{\mathcal{O}_E} \kbar)$ in the category $\text{Exo}^b_{n-1,1}(S \otimes_{\mathcal{O}_E} \kbar)$.

**Corollary 2.66.** The morphism

$$\mathcal{N}^\Phi = \mathcal{N}^\Phi_{(X, \iota_X, \lambda_X)} \to \mathcal{N}^\text{'rel} := \mathcal{N}^\text{'rel}_{(X, \iota_X, \lambda_X)}$$

sending $(X, \iota_X, \lambda_X; \rho_X)$ to $((X, \iota_X, \lambda_X)^\text{'rel}; \rho_X^\text{'rel})$ is an isomorphism.

Proof. This follows immediately from Proposition 2.63. \qed

Now we study special divisors on $\mathcal{N}^\Phi$ and their relation with those on $\mathcal{N}$. Fix a triple $(X_0, \iota_{X_0}, \lambda_{X_0})$ where

- $X_0$ is a supersingular $p$-divisible group over $\text{Spec} \mathcal{O}_E$ of dimension $f$ and height $2f$;
- $\iota_{X_0} : \mathcal{O}_E \to \text{End}(X_0)$ is an $\mathcal{O}_E$-action on $X_0$ such that for $0 \leq i \leq f - 1$, the summand $\text{Lie}_{\phi_i}(X)$ has rank 1 on which $\mathcal{O}_E$ acts via $\varphi_i$;
- $\lambda_{X_0} : X_0 \to X_0^\vee$ is a $\iota_{X_0}$-compatible principal polarization.
Note that \( \iota_{X_0} \) induces an isomorphism \( \iota_{X_0} : O_E \xrightarrow{\sim} \text{End}_{O_E}(X_0) \). Put

\[
V := \text{Hom}_{O_E}(X_0 \otimes_{O_E} \bar{k}, X) \otimes \mathbb{Q},
\]

which is a vector space over \( E \) of dimension \( n \), equipped with a natural hermitian form similar to (2.1). By a construction similar to (2.21), we obtain a triple \( (X_0, \iota_{X_0}, \lambda_{X_0})_{\text{rel}} \) as in the definition of special divisors on \( \mathcal{N} \) (Definition 2.7), and a canonical map

\[
\text{Hom}_{O_E}(X_0 \otimes_{O_E} \bar{k}, X) \to \text{Hom}_{O_E}(X_{0,\text{rel}} \otimes_{O_E} \bar{k}, X_{\text{rel}}),
\]

which induces a map

\[
-_{\text{rel}} : V \to V_{\text{rel}} := \text{Hom}_{O_E}(X_{0,\text{rel}} \otimes_{O_E} \bar{k}, X_{\text{rel}}) \otimes \mathbb{Q}.
\]

For every nonzero element \( x \in V \), we have similarly a closed formal subscheme \( \mathcal{N}^{\Phi}(x) \) of \( \mathcal{N}^{\Phi} \) defined similarly as in Definition 2.7.

**Corollary 2.67.** The map (2.25) is an isomorphism of hermitian spaces. Moreover, under the isomorphism in Corollary 2.66, we have \( \mathcal{N}^{\Phi}(x) = \mathcal{N}(x_{\text{rel}}) \).

**Proof.** By the definition of \( -_{\text{rel}} \), the map (2.25) is clearly an isometry. Since both \( V \) and \( V_{\text{rel}} \) have dimension \( n \), (2.25) is an isomorphism of hermitian spaces. The second assertion follows from Corollary 2.66 and construction of \( -_{\text{rel}} \), parallel to [Mih20, Remark 4.4]. \( \square \)

**Remark 2.68.** Let \( S \) be an object of \( \text{Sch}/O_E \). We have another category \( \text{Exo}^{\Phi}_{\text{n,0}}(S) \) whose objects are triples \( (X, \iota_{X}, \lambda_{X}) \) in which

- \( X \) is a \( p \)-divisible group over \( S \) of dimension \( nf \) and height \( 2nf \);
- \( \iota_{X} : O_E \to \text{End}(X) \) is an action of \( O_E \) on \( X \) such that for \( 0 \leq i \leq f - 1 \), \( O_E \) acts on \( \text{Lie}_{\varphi_i}(X) \) via \( \varphi_i \);
- \( \lambda_{X} : X \to X^{\vee} \) is a \( \iota_{X} \)-compatible polarization such that \( \ker(\lambda_{X}) = X[\iota_{X}(u)] \).

Morphisms are defined similarly as in Definition 2.60. The category \( \text{Exo}^{\Phi}_{\text{n,0}}(S) \) is a connected groupoid. Moreover, one can show that there is a canonical isomorphism \( \text{Exo}^{\Phi}_{\text{n,0}} \to \text{Exo}_{\text{n,0}} \) of prestacks after restriction to \( \text{Sch}^{v}/O_E \) similar to (2.21).

**Remark 2.69.** It is desirable to extend the results in this subsection to a general finite extension \( F/\mathbb{Q}_p \). We hope to address this problem in the future.

### 3. Local theta lifting at ramified places

Throughout this section, we fix a ramified quadratic extension \( E/F \) of \( p \)-adic fields with \( p \) odd, with \( c \in \text{Gal}(E/F) \) the Galois involution. We fix a uniformizer \( u \in E \) satisfying \( u^c = -u \), and denote by \( q \) the cardinality of \( O_E/(u) \). Let \( n = 2r \) be an even positive integer. We fix a nontrivial additive character \( \psi_{F} : F \to \mathbb{C}^{\times} \) of conductor \( O_F \).

The goal of this section is to compute the doubling \( L \)-function, the doubling epsilon factor, the spherical doubling zeta integral, and the local theta lifting for a tempered admissible irreducible representation \( \pi \) of \( G_{r}(F) \) that is spherical with respect to the standard special maximal compact subgroup.
3.1. **Weil representation and spherical module.** We equip \( W_r := E^{2r} \) with the skew-hermitian form given by the matrix \( \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \). We denote by \( \{e_1, \ldots, e_{2r}\} \) the natural basis of \( W_r \). Denote by \( G_r \) the unitary group of \( W_r \), which is a reductive group over \( F \). We write elements of \( W_r \) in the row form, on which \( G_r \) acts from the right. Let \( K_r \subseteq G_r(F) \) be the stabilizer of the lattice \( O_E^{2r} \subseteq W_r \), which is a special maximal compact subgroup. We fix the Haar measure \( dg \) on \( G_r(F) \) that gives \( K_r \) volume 1. Let \( P_r \) be the Borel subgroup of \( G_r \) consisting of elements of the form
\[
\begin{pmatrix} a & b \\ t u & c^{-1} \end{pmatrix},
\]
in which \( a \) is a lower-triangular matrix in \( \text{Res}_{E/F} \text{GL}_r \). Let \( P_r^0 \) be the maximal parabolic subgroup of \( G_r \) containing \( P_r \), with the unipotent radical \( N_r^0 \), such that the standard diagonal Levi factor \( M_r^0 \) of \( P_r^0 \) is isomorphic to \( \text{Res}_{E/F} \text{GL}_r \).

We fix a a split hermitian space \( (V, (,)_V) \) over \( E \) of dimension \( n = 2r \), and a self-dual lattice \( \Lambda_V \) of \( V \), namely, \( \Lambda_V = \Lambda_V^\vee := \{ x \in V \mid \text{Tr}_{E/F}(x, y)_V \in O_F \text{ for every } y \in \Lambda_V \} \). Put \( H_V := U(V) \), and let \( L_V \) be the stabilizer of \( \Lambda_V \) in \( H_V(F) \). We fix the Haar measure \( dh \) on \( H_V(F) \) that gives \( L_V \) volume 1.

**Remark 3.1.** We have

1. There exists an isomorphism \( \kappa: W_r \rightarrow V \) of \( E \)-vector spaces satisfying \( (\kappa(e_i), \kappa(e_j))_V = 0, (\kappa(e_{r+i}), \kappa(e_{r+j}))_V = 0, \) and \( (\kappa(e_i), \kappa(e_{r+j}))_V = u^{-1}\delta_{ij} \) for \( 1 \leq i, j \leq r \), and such that \( L_V \) is generated by \( \{\kappa(e_i) \mid 1 \leq i \leq 2r\} \) as an \( O_E \)-submodule.
2. The double coset \( K_r \backslash G_r(F) / K_r \) has representatives
\[
\begin{pmatrix} u^{a_1} \\ & u^{a_2} \\ & & \ddots \\ & & & u^{a_r} \\ (-u)^{-a_1} \\ & (-u)^{-a_2} \\ & & \ddots \\ & & & (-u)^{-a_r} \end{pmatrix}
\]
where \( 0 \leq a_1 \leq \cdots \leq a_r \) are integers.

We introduce two Hecke algebras:
\[
\mathcal{H}_{W_r} := \mathbb{C}[K_r \backslash G_r(F) / K_r], \quad \mathcal{H}_V := \mathbb{C}[L_V \backslash H_V(F) / L_V].
\]
Then by the remark above, both \( \mathcal{H}_{W_r} \) and \( \mathcal{H}_V \) are commutative complex algebras, and are canonically isomorphic to \( \mathcal{H} := \mathbb{C}[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]^{[\mathbb{Z}^r \times \mathfrak{S}_r]} \).

Let \( (\omega_{W_r, V}, \mathcal{H}_{W_r, V}) \) be the Weil representation of \( G_r(F) \times H_V(F) \) (with respect to the additive character \( \psi_F \) and the trivial splitting character). We recall the action under the Schrödinger model \( \mathcal{V}_{W_r, V} \cong C_c^\infty(V^r) \) as follows:

- For \( a \in \text{GL}_r(E) \) and \( \phi \in C_c^\infty(V^r) \), we have
\[
\omega_{W_r, V} \left( \begin{pmatrix} a & 0 \\ t u & c^{-1} \end{pmatrix} \right) \phi(x) = |\det a|_E^r \cdot \phi(xa);
\]
- For \( b \in \text{Herm}_r(F) \) and \( \phi \in C_c^\infty(V^r) \), we have
\[
\omega_{W_r, V} \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \phi(x) = \psi_F(\text{tr} b T(x)) \cdot \phi(x)
\]
where \( T(x) := ((x_i, x_j)_V)_{1 \leq i,j \leq r} \) is the moment matrix of \( x = (x_1, \ldots, x_r) \);
• for $\phi \in C^\infty_c(V_r)$, we have
  $\omega_{W_r,V} \left( \begin{pmatrix} -1 & 1 \\ 1 & r \end{pmatrix} \right) \phi(x) = \hat{\phi}(x)$;
• for $h \in H_V(F)$ and $\phi \in C^\infty_c(V_r)$, we have
  $\omega_{W_r,V}(h)\phi(x) = \phi(h^{-1}x)$.

Here, we recall the Fourier transform $C^\infty_c(V^r) \to C^\infty_c(V^r)$ sending $\phi$ to $\hat{\phi}$ defined by the formula

$$\hat{\phi}(x) := \int_{V^r} \phi(y)\psi_F \left( \sum_{i=1}^r \Tr_{E/F}(x_i, y_i) \right) dy,$$

where $dy$ is the self-dual Haar measure on $V^r$.

**Definition 3.2.** We define the spherical module $S_{W_r,V}$ to be the subspace of $V_{W_r,V}$ consisting of elements that are fixed by $K_r \times L_V$, as a module over $H_{W_r} \otimes_C H_V$ via the representation $\omega_{W_r,V}$. We denote by Sph($V^r$) the corresponding subspace of $C^\infty_c(V^r)$ under the Schrödinger model.

**Lemma 3.3.** The function $1_{\Lambda^c_V}$ belongs to Sph($V^r$).

**Proof.** It suffices to check that

$$\omega_{W_r,V} \left( \begin{pmatrix} -1 & 1 \\ 1 & r \end{pmatrix} \right) 1_{\Lambda^c_V} = 1_{\Lambda^c_V},$$

which follows from the fact that $\Lambda^c_V = \Lambda_V$. The lemma follows. \qed

**Proposition 3.4.** The annihilator of the $H_{W_r} \otimes_C H_V$-module $S_{W_r,V}$ is $\mathcal{I}_{W_r,V}$, where $\mathcal{I}_{W_r,V}$ denotes the diagonal ideal of $H_{W_r} \otimes_C H_V$.

**Proof.** The same proof of [Liu\textsuperscript{b}, Proposition 4.4] (with $\epsilon = +$ and $d = r$) works in this case as well, using Lemma 3.3. \qed

In what follows, we review the construction of unramified principal series of $G_r(F)$ and $H_V(F)$.

We identify $M_r$, the standard diagonal Levi factor of $P_r$, with $(\Res_{E/F} GL_1)^r$, under which we write an element of $M_r(F)$ as $a = (a_1, \ldots, a_r)$ with $a_i \in E^\times$ its eigenvalue on $e_i$ for $1 \leq i \leq r$. For every tuple $\sigma = (\sigma_1, \ldots, \sigma_r) \in (\mathbb{C}/\mathbb{Z})^r$, we define a character $\chi_r^\sigma$ of $M_r(F)$ hence $P_r(F)$ by the formula

$$\chi_r^\sigma(a) = \prod_{i=1}^r |a_i|_{E}^{e_i + i - 1/2}.$$  

We then have the normalized principal series

$$\mathcal{I}_{W_r}^\sigma := \{ \varphi \in C^\infty(G_r(F)) \mid \varphi(ag) = \chi_r^\sigma(a)\varphi(g) \text{ for } a \in P_r(F) \text{ and } g \in G_r(F) \},$$

which is an admissible representation of $G_r(F)$ via the right translation. We denote by $\pi_r^\sigma$ the unique irreducible constituent of $\mathcal{I}_{W_r}^\sigma$ that has nonzero $K_r$-invariants.

For $V$, we fix a basis $\{v_r, \ldots, v_1, v_{-1}, \ldots, v_{-r} \}$ of the $O_E$-lattice $\Lambda_V$, satisfying $(v_i, v_j)_V = u^{-1}\delta_{i,-j}$ for every $1 \leq i, j \leq r$. We have an increasing filtration

$$\{0\} = Z_{r+1} \subseteq Z_r \subseteq \cdots \subseteq Z_1$$

of isotropic $E$-subspaces of $V$ where $Z_i$ be the $E$-subspaces of $V$ spanned by $\{v_r, \ldots, v_i \}$. Let $Q_V$ be the (minimal) parabolic subgroup of $H_V$ that stabilizes (3.1). Let $M_V$ be the Levi factor of $Q_V$ stabilizing the lines spanned by $v_i$ for every $i$. Then we have the canonical
Since \( \pi \) isomorphism \( M_V = (\text{Res}_{E/F} \text{GL}_1)^r \), under which we write an element of \( M_V(F) \) as \( b = (b_1, \ldots, b_r) \) with \( b_i \in E^\times \) its eigenvalue on \( v_i \) for \( 1 \leq i \leq r \). For every tuple \( \sigma = (\sigma_1, \ldots, \sigma_r) \in (\mathbb{C}/2\pi i \mathbb{Z})^r \), we define a character \( \chi_\sigma \) of \( M_V(F) \) hence \( Q_V(F) \) by the formula

\[
\chi_\sigma(b) = \prod_{i=1}^r |b_i|_{E}^{\sigma_i + 1/2}.
\]

We then have the normalized principal series

\[
\Gamma_\sigma := \{ \varphi \in C^\infty(H_V(F)) \mid \varphi(bh) = \chi_\sigma(b) \varphi(h) \text{ for } b \in Q_V(F) \text{ and } h \in H_V(F) \},
\]

which is an admissible representation of \( H_V(F) \) via the right translation. We denote by \( \pi_\sigma \) the unique irreducible constituent of \( \Gamma_\sigma \) that has nonzero \( L \)-invariants.

### 3.2. Doubling zeta integral and doubling \( L \)-factor

In this section, we compute certain doubling zeta integrals and doubling \( L \)-factors for irreducible admissible representations \( \pi \) of \( G_r(F) \) satisfying \( \pi^{K_r} \neq \{0\} \). We will freely use notation from [Liub, Section 5].

We have the degenerate principal series \( \Gamma_\sigma^D \) := \( \text{Ind}_{C_r^T(F)}^{G_r^T(F)}(\varepsilon^{|r|_E} \circ \Delta) \) of \( G_r^T(F) \). Let \( f_r(s) \) be the unique section of \( \Gamma_\sigma^D \) such that for every \( g \in pK_r \) with \( p \in P_r^r(F) \),

\[
f_r(s)(g) = |\Delta(p)|^{s + r}.
\]

It is a holomorphic standard hence good section.

**Remark 3.5.** By definition, we have \( \Gamma_\sigma^D \) \( \subseteq \Gamma_\sigma^D \), where

\[
\sigma^D := (s + r - \frac{1}{2}, s + r - \frac{3}{2}, \ldots, s - r + \frac{3}{2}, s - r + \frac{1}{2}) \in (\mathbb{C}/2\pi i \mathbb{Z})^{2r}.
\]

Moreover, if we denote by \( \varphi^D_\sigma \) the unique section in \( \Gamma_\sigma^D \) that is fixed by \( K_{2r} \) and such that \( \varphi^D_\sigma(1_{4r}) = 1 \), then \( f_r(s) = \varphi^D_\sigma \).

Let \( \pi \) be an irreducible admissible representation of \( G_r(F) \). For every element \( \xi \in \pi^\vee \boxtimes \pi \), we denote by \( H_\xi \in C^\infty(G_r(F)) \) its associated matrix coefficient. Then for every meromorphic section \( f(s) \) of \( \Gamma_\sigma^D \), we have the (doubling) zeta integral:

\[
Z(\xi, f(s)) := \int_{G_r(F)} H_\xi(g) f(s)(w_r(g, 1_{2r})) \, dg,
\]

which is absolutely convergent for \( \text{Re } s \) large enough and has a meromorphic continuation. We let \( L(s, \pi) \) and \( \varepsilon(s, \pi, \psi_F) \) be the doubling \( L \)-factor and the doubling epsilon factor of \( \pi \), respectively, defined in [Yam14, Theorem 5.2].

Take an element \( \sigma = (\sigma_1, \ldots, \sigma_r) \in (\mathbb{C}/2\pi i \mathbb{Z})^r \). We define an \( L \)-factor

\[
L^\sigma(s) := \frac{1}{\prod_{i=1}^r (1 - q^{\sigma_i - s})(1 - q^{-\sigma_i - s})}.
\]

Since \( \pi_{W_r}^\sigma \) is self-dual, the space \( ((\pi_{W_r}^\sigma)^\vee)^{K_r} \boxtimes (\pi_{W_r}^\sigma)^{K_r} \) is one dimensional. Let \( \xi^\sigma \) be a generator of this one dimensional space; it satisfies \( H_{\xi^\sigma}(1_{2r}) \neq 0 \). We normalize \( \xi^\sigma \) so that \( H_{\xi^\sigma}(1_{2r}) = 1 \), which makes it unique.

**Proposition 3.6.** For \( \sigma \in (\mathbb{C}/2\pi i \mathbb{Z})^r \), we have

\[
Z(\xi^\sigma, f_r(s)) = \frac{L^\sigma(s + \frac{1}{2})}{b_{2r}(s)},
\]
where $b_{2r}(s) := \prod_{i=1}^{r} \frac{1}{1 - q^{-s - 2r - 2i}}$.

Proof. We have an isomorphism $m: \text{Res}_{E/F} \text{GL}_r \to M_0^0$ sending $a$ to $\left( a_{i_{\omega,-1}} \right)$. Let $\tau$ be the unramified constituent of the normalized induction of $\bigotimes_{i=1}^{r} \mathbb{Z}_{E_i}$, as a representation of $\text{GL}_r(E)$. We fix vectors $v_0 \in \tau$ and $v_0' \in \tau^\vee$ fixed by $M_0^0(F) \cap K_r = m(\text{GL}_r(O_E))$ such that $\langle v_0', v_0 \rangle_\tau = 1$.

By a similar argument in [GPSR87, Section 6] or in the proof of [Liub, Proposition 6.2], we have
\begin{equation}
Z(\xi^\sigma, \xi^\sigma_0) = C_{W_r}(s) \int_{\text{GL}_r(E)} \varphi^{\omega_i^0, s}(w_r(m(a), 1_{2r}))|\text{det} a|^{-r/2} \langle \tau^\vee(a)v_0^\vee, v_0 \rangle_\tau \, da,
\end{equation}
where
\[ C_{W_r}(s) = \prod_{i=1}^{r} \frac{\zeta_E(2s + 2i)}{\zeta_E(2s + r + i)} \prod_{i=1}^{r} \frac{\zeta_F(2s + 2i - 1)}{\zeta_F(2s + 2i)} = \prod_{i=1}^{r} \frac{\zeta_E(2s + 2i - 1)}{\zeta_E(2s + r + i)}.
\]

See the proof of [Liub, Proposition 6.2] for unexplained notation. By [GPSR87, Proposition 6.1], we have
\[ \int_{\text{GL}_r(E)} \varphi^{\omega_i^0, s}(w_r(m(a), 1_{2r}))|\text{det} a|^{-r/2} \langle \tau^\vee(a)v_0^\vee, v_0 \rangle_\tau \, da = \frac{L(s + \frac{1}{2}, \tau)L(s + \frac{1}{2}, \tau^\vee)}{\prod_{i=1}^{r} \zeta_E(2s + i)}.
\]
Combining with (3.2), we have
\[ Z(\xi^\sigma, \xi^\sigma_0) = \left( \prod_{i=1}^{r} \frac{\zeta_E(2s + 2i - 1)}{\zeta_E(2s + r + i)} \right) \cdot \left( \frac{L(s + \frac{1}{2}, \tau)L(s + \frac{1}{2}, \tau^\vee)}{\prod_{i=1}^{r} \zeta_E(2s + i)} \right) = \frac{L(s + \frac{1}{2})}{b_{2r}(s)}.
\]

The proposition is proved. 

\[ \square \]

**Proposition 3.7.** For $\sigma \in (\mathbb{C}/2\pi i \mathbb{Z})^r$, we have $L(s, \pi_{W_r}^\sigma) = L^\sigma(s)$ and $\varepsilon(s, \pi_{W_r}^\sigma, \psi_F) = 1$.

Proof. It follows from the same argument for [Yam14, Proposition 7.1], using Proposition 3.6. 

\[ \square \]

**Remark 3.8.** It is clear that the base change $\text{BC}(\pi_{W_r}^\sigma)$ is well-defined, which is an unramified irreducible admissible representation of $\text{GL}_n(E)$, and we have $L(s, \pi_{W_r}^\sigma) = L(s, \text{BC}(\pi_{W_r}^\sigma))$ by Proposition 3.7.

For an irreducible admissible representation $\pi$ of $G_r(F)$, let $\Theta(\pi, V)$ be the $\pi$-isotypic quotient of $\mathcal{V}_{W_r, V}$, which is an admissible representation of $H_V(F)$, and $\theta(\pi, V)$ its maximal semisimple quotient. By [Wal90], $\bar{\theta}(\pi, V)$ is either zero, or an irreducible admissible representation of $H_V(F)$, known as the theta lifting of $\pi$ to $V$ (with respect to the additive character $\psi_F$ and the trivial splitting character).

**Proposition 3.9.** For an irreducible admissible representation $\pi$ of $G_r(F)$ of the form $\pi_{W_r}^\sigma$ for an element $\sigma = (\sigma_1, \ldots, \sigma_r) \in (i\mathbb{R}/2\pi i \mathbb{Z})^r$, we have $\theta(\pi, V) \simeq \pi_{W_r}^\sigma$.

Proof. By the same argument in the proof of [Liub, Theorem 6.2], we have $\Theta(\pi, V)^{LV} \neq \{0\}$. By our assumption on $\sigma, \pi$ is tempered. By (the same argument for) [GI16, Theorem 4.1(v)], $\Theta(\pi, V)$ is a semisimple representation of $H_V(F)$ hence $\Theta(\pi, V) = \theta(\pi, V)$. In particular, we have $\theta(\pi, V)^{LV} \neq \{0\}$. By Proposition 3.4, the diagonal ideal $\mathcal{I}_{W_r, V}$ annihilates $(\pi_{W_r}^\sigma)^K \otimes \theta(\pi, V)^{LV}$, which implies that $\theta(\pi, V) \simeq \pi_{W_r}^\sigma$. 

\[ \square \]
4. Arithmetic inner product formula

In this section, we collect all local ingredients and deduce our main theorems, following the same line as in [LL]. In Subsection 4.1 and 4.2, we recall the doubling method and the arithmetic theta lifting from [LL], respectively. In Subsection 4.3, we prove the vanishing of local indices at split places, by proving the second main ingredient of this article, namely, Theorem 4.21. In Subsection 4.4, we recall the formula for local indices at inert places. In Subsection 4.5, we compute local indices at ramified places, based on the Kudla–Rapoport type formula Theorem 2.9. In Subsection 4.6, we recall the formula for local indices at archimedean places. The deduction of the main results of the article is explained in Subsection 4.7, which is a straightforward modification of [LL, Section 11].

4.1. Recollection on doubling method. For readers’ convenience, we copy three groups of setups from [LL, Section 2] to here. The only difference is item (H5), which reflects the fact that we study certain places in $V_{F}^{\text{ram}}$ in the current article.

Setup 4.1. Let $E/F$ be a CM extension of number fields, so that $c$ is a well-defined element in $\text{Gal}(E/F)$.

(F1) We denote by

- $V_{F}$ and $V_{F}^{\text{fin}}$ the set of all places and non-archimedean places of $F$, respectively;
- $V_{F}^{\text{spl}}, V_{F}^{\text{int}},$ and $V_{F}^{\text{ram}}$ the subsets of $V_{F}^{\text{fin}}$ of those that are split, inert, and ramified in $E$, respectively;
- $V_{F}^{(\diamond)}$ the subset of $V_{F}$ of places above $\diamond$ for every place $\diamond$ of $\mathbb{Q}$; and
- $V_{E}$ the places of $E$ above $V_{F}$.

Moreover,

- for every place $u \in V_{E}$ of $E$, we denote by $u \in V_{F}$ the underlying place of $F$;
- for every $v \in V_{F}^{\text{fin}}$, we denote by $p_{v}$ the maximal ideal of $O_{F_{v}}$, and put $q_{v} := |O_{F_{v}}/p_{v}|$;
- for every $v \in V_{F}$, we put $E_{v} := E \otimes_{F} F_{v}$ and denote by $|\ |_{E_{v}} : E_{v}^{\times} \rightarrow \mathbb{C}^{\times}$ the normalized norm character.

(F2) Let $m \geq 0$ be an integer.

- We denote by $\text{Herm}_{m}$ the subscheme of $\text{Res}_{E/F} \text{Mat}_{m,m}$ of $m$-by-$m$ matrices $b$ satisfying $b^{c} = b$. Put $\text{Herm}_{m}^{\circ} := \text{Herm}_{m} \cap \text{Res}_{E/F} \text{GL}_{m}$.
- For every ordered partition $m = m_{1} + \cdots + m_{s}$ with $m_{i}$ a positive integer, we denote by $\partial_{m_{1}, \ldots, m_{s}} : \text{Herm}_{m} \rightarrow \text{Herm}_{m_{1}} \times \cdots \times \text{Herm}_{m_{s}}$ the homomorphism that extracts the diagonal blocks with corresponding ranks. We simply write $\partial$ for $\partial_{1,1,\ldots,1}$.
- We denote by $\text{Herm}_{m}(F)^{+}$ (resp. $\text{Herm}_{m}^{\circ}(F)^{+}$) the subset of $\text{Herm}_{m}(F)$ of elements that are totally semi-positive definite (resp. totally positive definite). We simply write $F^{+}$ for $\text{Herm}_{m}^{\circ}(F)^{+}$.

(F3) For every $w \in V_{F}^{(\infty)}$, we fix an embedding $\iota_{w} : E \hookrightarrow \mathbb{C}$ above $w$, and

- put $uE := \iota_{w}(E)$ as a subfield of $\mathbb{C}$;
- identify $E_{w}$ with $\mathbb{C}$ via $\iota_{w}$; and
- put $uE_{u} := uE \otimes_{E} E_{u}$ for every $u \in V_{E}$.

(F4) Let $\eta := \eta_{E/F} : \mathbb{A}_{F}^{\times} \rightarrow \mathbb{C}^{\times}$ be the quadratic character associated to $E/F$. For every $v \in V_{F}$ and every positive integer $m$, put

$$b_{m,v}(s) := \prod_{i=1}^{m} L(2s + i, \eta_{v}^{m-i}).$$

Put $b_{m}(s) := \prod_{v \in V_{F}} b_{m,v}(s)$. 
(F5) For every element $T \in \text{Herm}_m(\mathbb{A}_F)$, we have the character $\psi_T : \text{Herm}_m(\mathbb{A}_F) \to \mathbb{C}^\times$ given by the formula $\psi_T(b) := \psi_F(\text{tr} b T)$.

(F6) Let $R$ be a commutative $F$-algebra. A (skew-)hermitian space over $R \otimes_F E$ is a free $R \otimes_F E$-module $V$ of finite rank, equipped with a (skew-)hermitian form $(\ , \ )_V$ with respect to the involution $c$ that is nondegenerate.

**Setup 4.2.** We fix an even positive integer $n = 2r$. Let $(V, (\ , \ )_V)$ be a hermitian space over $\mathbb{A}_F$ of rank $n$ that is totally positive definite.

(H1) For every commutative $\mathbb{A}_F$-algebra $R$ and every integer $m \geq 0$, we denote by $T(x) := ((x_i, x_j)_V)_{i,j} \in \text{Herm}_m(R)$ the moment matrix of an element $x = (x_1, \ldots, x_m) \in V^m \otimes_{\mathbb{A}_F} R$.

(H2) For every $v \in \mathcal{V}_F$, we put $V_v := V \otimes_{\mathbb{A}_F} F_v$ which is a hermitian space over $E_v$, and define the local Hasse invariant of $V_v$ to be $c(V_v) := \eta_v((-1)^r \text{det } V_v) \in \{\pm 1\}$. In what follows, we will abbreviate $c(V_v)$ as $\epsilon_v$.

(H3) Let $v$ be a place of $F$ and $m \geq 0$ an integer.

- For $T \in \text{Herm}_m(F_v)$, we put $(V_v^m)_T := \{x \in V_v^m \mid T(x) = T\}$, and
  $$(V_v^m)_\text{reg} := \bigcup_{T \in \text{Herm}_m(F_v)} (V_v^m)_T.$$

- We denote by $\mathcal{S}(V_v^m)$ the space of (complex valued) Bruhat–Schwartz functions on $V_v^m$. When $v \in \mathcal{V}_F^{\infty}$, we have the Gaussian function $\phi_v^0 \in \mathcal{S}(V_v^m)$ given by the formula $\phi_v^0(x) = e^{-2\pi r \text{tr} T(x)}$.

- We have a Fourier transform map $\hat{\cdot} : \mathcal{S}(V_v^m) \to \mathcal{S}(V_v^m)$ sending $\phi$ to $\hat{\phi}$ defined by the formula
  $$\hat{\phi}(x) := \int_{V_v^m} \phi(y) \psi_{E,v} \left( \sum_{i=1}^{m} (x_i, y_i)_V \right) \, dy,$$
  where $dy$ is the self-dual Haar measure on $V_v^m$ with respect to $\psi_{E,v}$.

- In what follows, we will always use this self-dual Haar measure on $V_v^m$.

(H4) Let $m \geq 0$ be an integer. For $T \in \text{Herm}_m(F)$, we put
  $$\text{Diff}(T, V) := \{v \in \mathcal{V}_F \mid (V_v^m)_T = \emptyset\},$$
  which is a finite subset of $\mathcal{V}_F \setminus \mathcal{V}_F^{\text{ram}}$.

(H5) Take a nonempty finite subset $\mathcal{R} \subseteq \mathcal{V}_F^{\text{fin}}$ that contains
  $$\{v \in \mathcal{V}_F^{\text{ram}} \mid \text{either } \epsilon_v = -1, \text{ or } 2 \mid v, \text{ or } v \text{ is ramified over } \mathbb{Q}\}.$$

Let $\mathcal{S}$ be the subset of $\mathcal{V}_F^{\text{fin}} \setminus \mathcal{R}$ consisting of $v$ such that $\epsilon_v = -1$, which is contained in $\mathcal{V}_F^{\text{int}}$.

(H6) We fix a $\prod_{v \in \mathcal{V}_F^{\text{fin}} \setminus \mathcal{R}} O_{E_v}$-lattice $\Lambda^\mathbb{R}$ in $V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty,\mathbb{R}}$ such that for every $v \in \mathcal{V}_F^{\text{fin}} \setminus \mathcal{R}$, $\Lambda_v^\mathbb{R}$ is a subgroup of $(\Lambda^\mathbb{R})^\vee$ of index $q_v^1 - \epsilon_v$, where
  $$((\Lambda^\mathbb{R})^\vee)_v := \{x \in V_v \mid \psi_{E,v}((x, y)_V) = 1 \text{ for every } y \in \Lambda^\mathbb{R}_v\}$$
  is the $\psi_{E,v}$-dual lattice of $\Lambda^\mathbb{R}_v$.

(H7) Put $H := U(V)$, which is a reductive group over $\mathbb{A}_F$.

(H8) Denote by $L^\mathbb{R} \subseteq H(\mathbb{A}_F^{\infty,\mathbb{R}})$ the stabilizer of $\Lambda^\mathbb{R}$, which is a special maximal subgroup. We have the (abstract) Hecke algebra away from $\mathcal{R}$
  $$\mathbb{T}^\mathbb{R} := \mathbb{Z}[L^\mathbb{R} \setminus H(\mathbb{A}_F^{\infty,\mathbb{R}})/L^\mathbb{R}],$$
which is a ring with the unit $1_{L^g}$, and denote by $S^g$ the subring

$$
\lim_{T \leq V_F^{\text{spl}} |R| |T| \to \infty} \mathbb{Z}[(L^g)|_T \setminus H(F_T)/(L^g)|_T] \otimes 1_{(L^g)^t}
$$

of $T^g$.

(H9) Suppose that $V$ is incoherent, namely, $\prod_{v \in V_F} \epsilon_v = -1$. For every $w \in V_F \setminus V_F^{\text{spl}}$, we denote by $^wV$ the $w$-nearby space of $V$, which is a hermitian space over $E$. More precisely,

- for $w \in V_F^{(\infty)}$, $^wV$ is the hermitian space over $E$, unique up to isomorphism, that has signature $(n-1,1)$ at $w$ and satisfies $^wV \otimes_E \mathbb{A}_F^w \simeq V \otimes_{A_F} \mathbb{A}_F^w$;

- for $w \in V_F^{\text{fin}} \setminus V_F^{\text{spl}}$, $^wV$ is the hermitian space over $E$, unique up to isomorphism, that satisfies $^wV \otimes_E \mathbb{A}_F^w \simeq V \otimes_{A_F} \mathbb{A}_F^w$.

We put $^wH := U(^wV)$, which is a reductive group over $F$.

Setup 4.3. Let $m \geq 0$ be an integer. We equip $W_m = E^{2m}$ and $\tilde{W}_m = E^{2m}$ the skew-hermitian forms given by the matrices $w_m$ and $-w_m$, respectively.

(G1) Let $G_m$ be the unitary group of both $W_m$ and $\tilde{W}_m$. We write elements of $W_m$ and $\tilde{W}_m$ in the row form, on which $G_m$ acts from the right.

(G2) We denote by $\{e_1, \ldots, e_{2m}\}$ and $\{\tilde{e}_1, \ldots, \tilde{e}_{2m}\}$ the natural bases of $W_m$ and $\tilde{W}_m$, respectively.

(G3) Let $P_m \subseteq G_m$ be the parabolic subgroup stabilizing the subspace generated by $\{e_{r+1}, \ldots, e_{2m}\}$, and $N_m \subseteq P_m$ its unipotent radical.

(G4) We have

- a homomorphism $m: \text{Res}_{E/F} GL_m \to P_m$ sending $a$ to

$$
m(a) := \begin{pmatrix} a & \varepsilon \cdot -1 \\
\end{pmatrix},
$$

which identifies $\text{Res}_{E/F} GL_m$ as a Levi factor of $P_m$;

- a homomorphism $n: \text{Herm}_m \to N_m$ sending $b$ to

$$
n(b) := \begin{pmatrix} 1_m & b \\
1_m \\
\end{pmatrix},
$$

which is an isomorphism.

(G5) We define a maximal compact subgroup $K_m = \prod_{v \in V_F} K_{m,v}$ of $G_m(A_F)$ in the following way:

- for $v \in V_F^{\text{fin}}$, $K_{m,v}$ is the stabilizer of the lattice $O_{E_v}^{2m}$;

- for $v \in V_F^{(\infty)}$, $K_{m,v}$ is the subgroup of the form

$$
[k_1, k_2] := \frac{1}{2} \begin{pmatrix} k_1 + k_2 & -ik_1 + ik_2 \\
-ik_1 - ik_2 & k_1 + k_2 \\
\end{pmatrix},
$$

in which $k_i \in GL_m(\mathbb{C})$ satisfying $k_i k_i^v = 1_m$ for $i = 1, 2$. Here, we have identified $G_m(F_v)$ as a subgroup of $GL_{2m}(\mathbb{C})$ via the embedding $\iota_v$ in Setup 4.1(F3).

(G6) For every $v \in V_F^{(\infty)}$, we have a character $\kappa_{m,v}: K_{m,v} \to \mathbb{C}^\times$ that sends $[k_1, k_2]$ to $\det k_1 / \det k_2$.

(G7) For every $v \in V_F$, we define a Haar measure $dg_v$ on $G_m(F_v)$ as follows:

\[\text{In fact, both } K_{m,v} \text{ and } \kappa_{m,v} \text{ do not depend on the choice of the embedding } \iota_v \text{ for } v \in V_F^{(\infty)}.\]
Remark 4.5. By [LL, Proposition 3.6(2)] and Proposition 3.9, we know that when $\mathfrak{R} \subseteq V_F^{\mathfrak{spl}}$, $V$ coincides with the hermitian form over $\mathbb{A}_F$ of rank $n$ determined by $\pi$ via local theta dichotomy.

**Definition 4.6.** We define the $L$-function for $\pi$ as the Euler product $L(s, \pi) := \prod_v L(s, \pi_v)$ over all places of $F$, in which

(1) for $v \in V_F^{\text{fin}}$, $L(s, \pi_v)$ is the doubling $L$-function defined in [Yam14, Theorem 5.2];

(2) for $v \in V_F^{(\infty)}$, $L(s, \pi_v)$ is the $L$-function of the standard base change $\text{BC}(\pi_v)$ of $\pi_v$.

By Setup 4.4(1), $\text{BC}(\pi_v)$ is the principal series representation of $\text{GL}_n(\mathbb{C})$ that is the normalized induction of $\text{arg}^{n-1} \otimes \text{arg}^{n-3} \otimes \cdots \otimes \text{arg}^{3-n} \otimes \text{arg}^{1-n}$ where $\text{arg}: \mathbb{C}^\times \to \mathbb{C}^\times$ is the argument character.

**Remark 4.7.** Let $v$ be a place of $F$.

(1) For $v \in V_F^{(\infty)}$, doubling $L$-function is only well-defined up to an entire function without zeros. However, one can show that $L(s, \pi_v)$ satisfies the requirement for the doubling $L$-function in [Yam14, Theorem 5.2].
(2) For \( v \in \mathcal{V}_F^{\mathrm{pl}} \), the standard base change \( BC(\pi_v) \) is well-defined and we have \( L(s, \pi_v) = L(s, BC(\pi_v)) \) by [Yam14, Theorem 7.2].
(3) For \( v \in \mathcal{V}_F^{\mathrm{nt}} \setminus \mathbb{R} \), the standard base change \( BC(\pi_v) \) is well-defined and we have \( L(s, \pi_v) = L(s, BC(\pi_v)) \) by [Liub, Remark 1.4].
(4) For \( v \in \mathcal{V}_F^{\mathrm{am}} \setminus \mathbb{R} \), the standard base change \( BC(\pi_v) \) is well-defined and we have \( L(s, \pi_v) = L(s, BC(\pi_v)) \) by Remark 3.8.

In particular, when \( \mathbb{R} \subseteq \mathcal{V}_F^{\mathrm{pl}} \), we have \( L(s, \pi) = \prod_v L(s, BC(\pi_v)) \).

Recall that we have the normalized doubling integral
\[
3^2_{\pi_v, V_v} : \pi_v^\vee \otimes \pi_v \otimes \mathcal{S}(V_v^{2r}) \to \mathbb{C}
\]
from [LL, Section 3].

**Proposition 4.8.** Let \((\pi, \mathcal{V}_\pi)\) be as in Setup 4.4.

1. For every \( v \in \mathcal{V}_F^{\mathrm{fin}} \), we have
   \[
   \dim_\mathbb{C} \text{Hom}_G(\pi_v^{\square}(0), \pi_v \otimes \pi_v^\vee) = 1.
   \]
2. For every \( v \in (\mathcal{V}_F^{\mathrm{fin}} \setminus \mathbb{R}) \cup \mathcal{V}_F^{\mathrm{pl}} \), \( V_v \) is the unique hermitian space over \( E_v \) of rank \( 2r \), up to isomorphism, such that \( 3^2_{\pi_v, V_v} \neq 0 \).
3. For every \( v \in \mathcal{V}_F^{\mathrm{fin}} \), \( \text{Hom}_G(\pi_v^{\square}(0), \pi_v \otimes \pi_v^\vee) \) is irreducible as a representation of \( H(F_v) \), and is nonzero if \( v \in (\mathcal{V}_F^{\mathrm{fin}} \setminus \mathbb{R}) \cup \mathcal{V}_F^{\mathrm{pl}} \).

**Proof.** This is same as [LL, Proposition 3.6] except that in (2) we have to take care of the case where \( v \in \mathcal{V}_F^{\mathrm{am}} \), which is a consequence of Proposition 3.9. \( \square \)

**Proposition 4.9.** Let \((\pi, \mathcal{V}_\pi)\) be as in Setup 4.4 such that \( L\left(\frac{1}{2}, \pi\right) = 0 \). Take

- \( \varphi_1 = \otimes_v \varphi_{1v} \in \mathcal{V}_\pi^{[r]\mathbb{R}} \) and \( \varphi_2 = \otimes_v \varphi_{2v} \in \mathcal{V}_\pi^{[r]\mathbb{R}} \) such that \( \langle \varphi_{1v}^c, \varphi_{2v} \rangle_{\pi_v} = 1 \) for \( v \in \mathcal{V}_F \setminus \mathbb{R} \),\(^{11}\) and
- \( \Phi = \otimes_v \Phi_v \in \mathcal{S}(V_v^{2r}) \) such that \( \Phi_v \) is the Gaussian function (Setup 4.2(H3)) for \( v \in \mathcal{V}_F^{(\infty)} \), and \( \Phi_v = \mathbb{1}_{(\Lambda^R_v)^{2r}} \) for \( v \in \mathcal{V}_F^{\mathrm{fin}} \setminus \mathbb{R} \).

Then we have
\[
\int_{G_r(F) \setminus G_r(\mathbb{A}_F)} \int_{G_r(F) \setminus G_r(\mathbb{A}_F)} \varphi_2(g_2) \varphi_1^c(g_1) E'(0, (g_1, g_2), \Phi) \, dg_1 \, dg_2
= \frac{L\left(\frac{1}{2}, \pi\right)}{b_{2r}(0)} \cdot C_r^{[r]_\mathbb{R}} \prod_{v \in \mathcal{V}_F^{\mathrm{fin}}} 3^2_{\pi_v, V_v}(\varphi_{1v}^c, \varphi_{2v}, \Phi_v)
= \frac{L\left(\frac{1}{2}, \pi\right)}{b_{2r}(0)} \cdot C_r^{[r]_\mathbb{R}} \prod_{v \in \mathcal{V}_F^{\mathrm{fin}}} \frac{(-1)^r q_v^{-1}(q_v + 1)}{(q_v^{2r-1} + 1)(q_v^{2r} - 1)} \cdot \prod_{v \in \mathbb{R}} 3^2_{\pi_v, V_v}(\varphi_{1v}^c, \varphi_{2v}, \Phi_v),
\]
where
\[
C_r := (-1)^r 2^{(r-1)} \pi^{r^2} \frac{\Gamma(1) \cdots \Gamma(r)}{\Gamma(r+1) \cdots \Gamma(2r)},
\]
and the measure on \( G_r(\mathbb{A}_F) \) is the one defined in Setup 4.3(G7).

\(^{11}\)Strictly speaking, what we fixed is a decomposition \( \varphi_1 = \otimes_v (\varphi_{1v}^c) \) and have abused notation by writing \( \varphi_{1v}^c \) instead of \( (\varphi_1)_v \).
Proof. The proof is same as [LL, Proposition 3.7], with the additional input
\[ 3^k_{\pi, v, c} (\varphi_{1v}, \varphi_{2v}, \Phi_v) = 1 \]
for \( v \in \mathcal{V}_{F}^\text{ram} \setminus \mathcal{R} \) by Proposition 3.6.

Suppose that \( V \) is incoherent. By [Liu11b, Section 2B], we have

1. Take an element \( w \in \mathcal{V}_F \setminus \mathcal{V}_{F}^\text{pl} \), and \( \Phi = \otimes_v \Phi_v \in \mathcal{S}(wV^{2r} \otimes_F \mathbb{A}_F) \), where we recall from Setup 4.2(H9) that \( wV \) is the \( w \)-nearby hermitian space, such that \( \text{supp}(\Phi_v) \subseteq (wV_v^{2r})_{\text{reg}} \) (Setup 4.2(H3)) for \( v \) in a nonempty subset \( \mathcal{R}' \subseteq \mathcal{R} \). Then for every \( g \in P_r \cap (F')G_r(\mathbb{A}_F^\infty) \), we have
   \[
   E(0, g, \Phi) = \sum_{T^\square \in \text{Herm}_2, (F)} \prod_{v \in \mathcal{V}_F} W_{T^\square}(0, g_v, \Phi_v).
   \]

2. Take \( \Phi = \otimes_v \Phi_v \in \mathcal{S}(V^{2r}) \) such that \( \text{supp}(\Phi_v) \subseteq (V_v^{2r})_{\text{reg}} \) for \( v \) in a subset \( \mathcal{R}' \subseteq \mathcal{R} \) of cardinality at least 2. Then for every \( g \in P_r \cap (F')G_r(\mathbb{A}_F^\infty) \), we have
   \[
   E'(0, g, \Phi) = \sum_{w \in \mathcal{V}_F \setminus \mathcal{V}_{F}^\text{pl}} \mathcal{E}(g, \Phi)_w,
   \]
where
   \[
   \mathcal{E}(g, \Phi)_w := \sum_{T^\square \in \text{Herm}_2, (F)} W_{T^\square}(0, g_w, \Phi_w) \prod_{v \in \mathcal{V}_F \setminus \{w\}} W_{T^\square}(0, g_v, \Phi_v).
   \]

Here, \( \text{Diff}(T^\square, V) \) is defined in Setup 4.2(H4).

Definition 4.10. Suppose that \( V \) is incoherent. Take an element \( w \in \mathcal{V}_F \setminus \mathcal{V}_{F}^\text{pl} \), and a pair \((T_1, T_2)\) of elements in \( \text{Herm}_r(F) \).

1. For \( \Phi = \otimes_v \Phi_v \in \mathcal{S}(wV^{2r} \otimes_F \mathbb{A}_F) \), we put
   \[
   E_{T_1, T_2}(g, \Phi) := \sum_{T^\square \in \text{Herm}_2, (F) \atop \partial_r, T^\square = (T_1, T_2)} \prod_{v \in \mathcal{V}_F} W_{T^\square}(0, g_v, \Phi_v).
   \]

2. For \( \Phi = \otimes_v \Phi_v \in \mathcal{S}(V^{2r}) \), we put
   \[
   \mathcal{E}_{T_1, T_2}(g, \Phi)_w := \sum_{T^\square \in \text{Herm}_2, (F) \atop \partial_r, T^\square = (T_1, T_2)} W_{T^\square}(0, g_w, \Phi_w) \prod_{v \in \mathcal{V}_F \setminus \{w\}} W_{T^\square}(0, g_v, \Phi_v).
   \]

Here, \( \partial_r : \text{Herm}_2r \to \text{Herm}_r \times \text{Herm}_r \) is defined in Setup 4.1(F2).

4.2. Recollection on arithmetic theta lifting. From this moment, we will assume \( F \neq \mathbb{Q} \).

Take an element \( w \in \mathcal{V}_{F}^{(\infty)} \), and fix an isomorphism
\[
\text{w} - : V \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \iso wV \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty
\]
of hermitian spaces over \( \mathbb{A}_F^\infty \), which induces an isomorphism \( \text{w} - : H(\mathbb{A}_F^\infty) \iso wH(\mathbb{A}_F^\infty) \). For every open compact subgroup \( L \subseteq H(\mathbb{A}_F^\infty) \), we have the Shimura variety \( wX_L \) associated to \( \text{Res}_{F/\mathbb{Q}} wH \) of the level \( wL \), which is a smooth projective scheme over \( wE \) of dimension \( n - 1 \). In what follows, for a place \( u \in \mathcal{V}_E \), we put \( wX_{L, u} := wX_L \otimes_{wE} wE_u \) as a scheme over \( wE_u \).
For every $\phi^\infty \in \mathcal{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$ and $T \in \text{Herm}_m(F)$, we put
\[
\langle uZ_T(\phi^\infty)_L := \sum_{x \in L \setminus \text{Herm}_m(F)^\infty} \phi^\infty(x) uZ(x)_L,
\]
where $uZ(x)_L$ is Kudla’s special cycle recalled in [LL, Construction 4.2]. As the above summation is finite, $uZ_T(\phi^\infty)_L$ is a well-defined element in $\text{CH}^m(\omega X L)_C$. For every $g \in G_m(\mathbb{A}_F)$, Kudla’s generating function is defined to be
\[
uZ(\phi^\infty)(g)_L := \sum_{T \in \text{Herm}_m(F)^+} \omega_{m,\infty}(g_0) \phi^\infty_0(T) \cdot uZ_T(\omega_{m,\infty}(g^\infty)_L
\]
as a formal sum valued in $\text{CH}^m(\omega X L)_C$, where
\[
\omega_{m,\infty}(g_0) \phi^\infty_0(T) := \prod_{v \in V^\infty} \omega_{m,v}(g_v) \phi^\infty_v(T).
\]

Here, we note that for $v \in V^\infty_F$, the function $\omega_{m,v}(g_v) \phi^\infty_v$ factors through the moment map $V^m_v \to \text{Herm}_m(F_v)$ (see Setup 4.2(H1)).

**Hypothesis 4.11** (Modularity of generating functions of codimension $m$). For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, every $\phi^\infty \in \mathcal{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, and every complex linear map $l$: $\text{CH}^m(\omega X L)_C \to \mathbb{C}$, the assignment
\[
g \mapsto l(uZ(\phi^\infty)(g)_L)
\]
is absolutely convergent, and gives an element in $\mathcal{A}^{[r]}(G_m(F) \setminus G_m(\mathbb{A}_F))$. In other words, the function $uZ(\phi^\infty)(-)_L$ defines an element in $\text{Hom}_C(\text{CH}^m(\omega X L)_C, \mathcal{A}^{[r]}(G_m(F) \setminus G_m(\mathbb{A}_F)))$.

**Definition 4.12.** Let $(\pi, V^\pi)$ be as in Setup 4.4. Assume Hypothesis 4.11 on the modularity of generating functions of codimension $r$. For every $\varphi \in V^\pi[r]$, every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, and every $\phi^\infty \in \mathcal{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, we put
\[
uTheta(\varphi)_L := \int_{G_r(F) \setminus G_r(\mathbb{A}_F)} \varphi^c(g) uZ(\phi^\infty)(g)_L \, dg,
\]
which is an element in $\text{CH}^r(\omega X L)_C$ by [LL, Proposition 4.7]. It is clear that the image of $uTheta(\varphi)_L$ in $\text{CH}^r(\omega X L)_C := \lim_{L} \text{CH}^r(\omega X L)_C$

depends only on $\varphi$ and $\phi^\infty$, which we denote by $uTheta(\varphi)$. Finally, we define the arithmetic theta lifting of $(\pi, V^\pi)$ to $uV$ to be the complex subspace $\Theta(\pi, uV)$ of $\text{CH}^r(\omega X)_C$ spanned by $uTheta(\varphi)$ for all $\varphi \in V^\pi[r]$ and $\phi^\infty \in \mathcal{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$.

We recall Beilinson’s height pairing for our particular use from [LL, Section 5]. We have a map
\[
\langle , \rangle_{uX L, u\omega E}: \text{CH}^r(\omega X L)^{(\ell)}_C \times \text{CH}^r(\omega X L)^{(\ell)}_C \to \mathbb{C} \otimes Q \ell
\]
that is complex linear in the first variable, and conjugate symmetric. Here, $\ell$ is a rational prime so that $uX L, u$ has smooth projective reduction for every $v \in V^\infty_E$. For a pair $(c_1, c_2)$ of elements in $Z^r(\omega X L)_C^{(\ell)} \times Z^r(\omega X L)_C^{(\ell)}$ with disjoint supports, we have
\[
\langle c_1, c_2 \rangle_{uX L, u\omega E} = \sum_{u \in V^\infty_E} 2(c_1, c_2)_{uX L, u\omega E} + \sum_{u \in V^\infty_E} \log q_u \cdot \langle c_1, c_2 \rangle_{uX L, u\omega E}.
\]
in which

- \( q_u \) is the residue cardinality of \( E_u \) for \( u \in \mathbb{V}_E^{\text{fin}} \);
- \( \langle c_1, c_2 \rangle_{w, X_{L,u}, w, E_u} \in \mathbb{C} \otimes \mathbb{Q}_l \) is the non-archimedean local index recalled in [LL, Section B] for \( u \in \mathbb{V}_E^{\text{fin}} \), which equals zero for all but finitely many \( u \);
- \( \langle c_1, c_2 \rangle_{w, X_{L,u}, w, E_u} \in \mathbb{C} \) is the archimedean local index for \( u \in \mathbb{V}_E^{(\infty)} \), recalled in [LL, Section 10].

**Definition 4.13.** We say that a rational prime \( \ell \) is \( \mathbb{R} \)-good if \( \ell \) is unramified in \( E \) and satisfies \( \mathbb{V}_F^{(\ell)} \subseteq \mathbb{V}_F^{\text{fin}} \setminus (\mathbb{R} \cup \mathbb{S}) \).

**Definition 4.14.** For every open compact subgroup \( L_R \) of \( H(F) \) and every subfield \( \mathbb{L} \) of \( \mathbb{C} \), we define

1. \((\mathbb{S}_L^\mathbb{R})^0_{L_\mathbb{R}}\) to be the subalgebra of \( \mathbb{S}_L^\mathbb{R} \) (Setup 4.2(H8)) of elements that annihilate
   \[
   \bigoplus_{i \neq 2r - 1} \mathbb{H}^0_{\text{dR}}(w X_{L,u}, w, E) \otimes \mathbb{L},
   \]

2. for every rational prime \( \ell \), \((\mathbb{S}_L^\mathbb{R})^{(\ell)}_{L_\mathbb{R}}\) to be the subalgebra of \( \mathbb{S}_L^\mathbb{R} \) of elements that annihilate
   \[
   \bigoplus_{u \in \mathbb{V}_E^{\text{fin}} \setminus \mathbb{V}_E^{(\ell)}} \mathbb{H}^{2r}(w X_{L,u}, w, E) \otimes \mathbb{L}.
   \]

Here, \( L^R \) is defined in Setup 4.2(H8).

**Definition 4.15.** Consider a nonempty subset \( R' \subseteq \mathbb{R} \), an \( \mathbb{R} \)-good rational prime \( \ell \), and an open compact subgroup \( L \) of \( H(\mathbb{A}_F^{\infty}) \) of the form \( L_R L^R \) where \( L^R \) is defined in Setup 4.2(H8).

An \((\mathbb{R}, R', \ell, L)\)-admissible sextuple is a sextuple \((\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)\) in which

- for \( i = 1, 2 \), \( \phi_i^\infty = \otimes_v \phi_i^\infty_v \in \mathcal{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L \) in which \( \phi_i^\infty_v = 1_{(\mathbb{A}_F^\infty)^r} \) for \( v \in \mathbb{V}_F^{\text{fin}} \setminus \mathbb{R} \), satisfying that \( \text{supp}(\phi_1^\infty_v \otimes (\phi_2^\infty_v)^c) \subseteq (V^2_L)^{\text{reg}} \) for \( v \in \mathbb{R}^\ell \);
- for \( i = 1, 2 \), \( s_i \) is a product of two elements in \((\mathbb{S}^{\text{fin}})^{\ell}_{\mathbb{Q}^\ell})_{L_\mathbb{R}} \);
- for \( i = 1, 2 \), \( g_i \) is an element in \( G_v(\mathbb{A}_e^\mathbb{R}) \).

For an \((\mathbb{R}, R', \ell, L)\)-admissible sextuple \((\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)\) and every pair \((T_1, T_2)\) of elements in \( \text{Herm}^+(F)^+ \), we define

1. the global index \( w \mathcal{I}_{T_1, T_2}(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2) \) to be
   \[
   \langle \omega_{r, \infty}(g_1^1 \phi_1^0(T_1) \cdot s_1^w T_1, (\omega_{r, \infty}(g_1^2 \phi_1^0(T_2) \cdot s_2^w T_2, (\omega_{r, \infty}(g_2^1 \phi_2^0) L)_{w X_{L,u}}, w E)
   \]
   as an element in \( \mathbb{C} \otimes \mathbb{Q}_l \), where we note that for \( i = 1, 2 \), \( s_i^w T_1, (\omega_{r, \infty}(g_i^1 \phi_i^0) L)_{w X_{L,u}}, w E \) belongs to \( CH^{(w X_{L,u})^{\ell}}_{\mathcal{C}} \) by Definition 4.14(2);
2. for every \( u \in \mathbb{V}_E^{\text{fin}} \), the local index \( w \mathcal{I}_{T_1, T_2}(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2) \) to be
   \[
   \langle \omega_{r, \infty}(g_1^1 \phi_1^0(T_1) \cdot s_1^w T_1, (\omega_{r, \infty}(g_1^2 \phi_1^0(T_2) \cdot s_2^w T_2, (\omega_{r, \infty}(g_2^1 \phi_2^0) L)_{w X_{L,u}}, w E_u)
   \]
   as an element in \( \mathbb{C} \otimes \mathbb{Q}_l \);
3. for every \( u \in \mathbb{V}_E^{(\infty)} \), the local index \( w \mathcal{I}_{T_1, T_2}(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2) \) to be
   \[
   \langle \omega_{r, \infty}(g_1^1 \phi_1^0(T_1) \cdot s_1^w T_1, (\omega_{r, \infty}(g_1^2 \phi_1^0(T_2) \cdot s_2^w T_2, (\omega_{r, \infty}(g_2^1 \phi_2^0) L)_{w X_{L,u}}, w E_u)
   \]
   as an element in \( \mathbb{C} \).

Let \((\pi, \mathcal{V}_\pi)\) be as in Setup 4.4, and assume Hypothesis 4.11 on the modularity of generating functions of codimension \( r \).
Remark 4.16. In the situation of Definition 4.12 (and suppose that $F \neq \mathbb{Q}$), suppose that $L$ has the form $L_0 L^r$ where $L^r$ is defined in Setup 4.2(H8). We have, from [LL, Proposition 5.10], that for every elements $\varphi \in V_{\pi}^r$ and $\phi^\infty \in \mathcal{S}(V^r \otimes_{A_F} A_F^\infty)^L$,

1. $s^u \Theta_{\phi^\infty}(\varphi)_L = u^u \Theta_{\phi^\infty}(\varphi)_L$ for every $s \in S_{\infty}$ such that $\chi^r_\pi(s) = 1$;
2. $u^u \Theta_{\phi^\infty}(\varphi)_L \in CH^r(u^u X^L_0)_C$;
3. under [LL, Hypothesis 5.6], $u^u \Theta_{\phi^\infty}(\varphi)_L \in CH^r(u^u X^L_0)_C^{(f)}$ for every $\mathcal{R}$-good rational prime $\ell$.

We recall the normalized height pairing between the cycles $u^u \Theta_{\phi^\infty}(\varphi)$ in Definition 4.12, under [LL, Hypothesis 5.6].

Definition 4.17. Under [LL, Hypothesis 5.6], for every elements $\varphi_1, \varphi_2 \in V_{\pi}^r$ and $\phi^\infty_1, \phi^\infty_2 \in \mathcal{S}(V^r \otimes_{A_F} A_F^\infty)^L$, we define the normalized height pairing

$$\langle u^u \Theta_{\phi^\infty_1}(\varphi_1), u^u \Theta_{\phi^\infty_2}(\varphi_2) \rangle^\natural_{u^u X^L_0} \in \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

to be the unique element such that for every $L = L_0 L^r$ as in Remark 4.16 (with $\mathcal{R}$ possibly enlarged) satisfying $\varphi_1, \varphi_2 \in V_{\pi}^r$, $\phi^\infty_1, \phi^\infty_2 \in \mathcal{S}(V^r \otimes_{A_F} A_F^\infty)^L$, and that $\ell$ is $\mathcal{R}$-good, we have

$$\langle u^u \Theta_{\phi^\infty_1}(\varphi_1), u^u \Theta_{\phi^\infty_2}(\varphi_2) \rangle^\natural_{u^u X^L_0} = \text{vol}^\natural(L) \cdot \langle u^u \Theta_{\phi^\infty_1}(\varphi_1)_L, u^u \Theta_{\phi^\infty_2}(\varphi_2)_L \rangle^\natural_{u^u X^L_0}$$

where $\text{vol}^\natural(L)$ is introduced in [LL, Definition 3.8] and $(\Theta_{\phi^\infty_1}(\varphi_1)_L, u^u \Theta_{\phi^\infty_2}(\varphi_2)_L)^\natural_{u^u X^L_0}$ is well-defined by Remark 4.16(3). Note that by the projection formula, the right-hand side of the above formula is independent of $L$.

Finally, we review the auxiliary Shimura variety that will only be used in the computation of local indices $u^u I_{T_1, T_2}(\phi^\infty_1, \phi^\infty_2, s_1, s_2, g_1, g_2)_L, u$.

Construction 4.18. We define a torus $T_0$ over $\mathbb{Q}$ such that for every commutative $\mathbb{Q}$-algebra $\mathcal{R}$, we have $T_0(\mathcal{R}) = \{ a \in E \otimes_{\mathbb{Q}} \mathcal{R} \mid \text{Nm}_{E/F} a \in \mathcal{R}^\times \}$.

We choose a CM type $\Phi$ of $E$ containing $\iota_w$ and denote by $u^u E'$ the subfield of $\mathbb{C}$ generated by $u^u E$ and reflex field of $\Phi$. For a (sufficiently small) open compact subgroup $L_0$ of $T_0(\mathbb{A}^\infty)$, we have the PEL type moduli scheme $Y$ of CM abelian varieties with CM type $\Phi$ and level $L_0$, which is a smooth projective scheme over $u^u E'$ of dimension 0. In what follows, when we invoke this construction, the data $\Phi$ and $L_0$ will be fixed, hence will not be carried into the notation $u^u E'$ and $Y$. For every open compact subgroup $L \subset H(\mathbb{A}^\infty)$, we put $u^u X'_L := u^u X_L \otimes_{u^u E'} Y$.

as a scheme over $u^u E'$.

The following setup is parallel to [LL, Setup 6.6].

Setup 4.19. In Subsections 4.3, 4.4, and 4.5, we will consider a place $u \in V_{E}^{\text{fin}}$. Let $p$ be the underlying rational prime of $u$. We will fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ under which $\iota_w$ induces the place $u$. In particular, we may identify $\Phi$ as a subset of $\text{Hom}(E, \overline{\mathbb{Q}}_p)$.

We further require that $\Phi$ in Construction 4.18 is admissible in the following sense: if $\Phi_v \subset \Phi$ denotes the subset inducing the place $v$ for every $v \in V_{E}^{(p)}$, then it satisfies

1. when $v \in V_{E}^{(p)} \cap V_{E}^{\text{spl}}$, $\Phi_v$ induces the same place of $E$ above $v$;
2. when $v \in V_{E}^{(p)} \cap V_{E}^{\text{int}}$, $\Phi_v$ is the pullback of a CM type of the maximal subfield of $E_v$ unramified over $\mathbb{Q}_p$. 


void unless \( u \in \mathcal{V}_F^p \) (Definition 1.1): when \( v \in \mathcal{V}_F^{(p)} \cap \mathcal{V}_{F,\text{ram}}^p \), the subfield of \( \overline{\mathbb{Q}}_p \) generated by \( u \mathcal{E}_u \) and the reflex field of \( \Phi_v \) is unramified over \( u \mathcal{E}_u \).

To release the burden of notation, we denote by \( \mathcal{K} \) the subfield of \( \overline{\mathbb{Q}}_p \) generated by \( u \mathcal{E}_u \) and the reflex field of \( \Phi \), by \( k \) its residue field, and by \( \overline{K} \) the completion of the maximal unramified extension of \( K \) in \( \overline{\mathbb{Q}}_p \) with the residue field \( \overline{\mathbb{F}}_p \). It is clear that admissible CM type always exists; and that when \( u \in \mathcal{V}_F^p \), \( K \) is unramified over \( u \mathcal{E}_u \).

We also choose a (sufficiently small) open compact subgroup \( L_0 \) of \( T_0(\mathbb{A}^\infty) \) such that \( L_{0,p} \) is maximal compact. We denote by \( \mathcal{Y} \) the integral model of \( Y \) over \( O_K \) such that for every \( S \in \text{Sch}_{/O_K}^f \), \( \mathcal{Y}(S) \) is the set of equivalence classes of quadruples \( (A_0, \iota_{A_0}, \lambda_{A_0}, \eta_{A_0}^p) \) where

- \( (A_0, \iota_{A_0}, \lambda_{A_0}) \) is a unitary \( O_E \)-abelian scheme over \( S \) of signature type \( \Phi \) (see [LTXZZ, Definition 3.4.2 & Definition 3.4.3])\(^{12}\) such that \( \lambda_{A_0} \) is \( p \)-principal;
- \( \eta_{A_0}^p \) is an \( L_0^p \)-level structure (see [LTXZZ, Definition 4.1.2] for more details).

By [How12, Proposition 3.1.2], \( \mathcal{Y} \) is finite and étale over \( O_K \).

### 4.3. Local indices at split places.

In this subsection, we compute local indices at almost all places in \( \mathcal{V}_E^{p\ell} \). Our goal is to prove the following proposition.

**Proposition 4.20.** Let \( \mathcal{R}, \mathcal{R}', \ell, \) and \( L \) be as in Definition 4.15 such that the cardinality of \( \mathcal{R}' \) is at least 2. Let \( (\pi, \mathcal{V}_\pi) \) be as in Setup 4.4. For every \( u \in \mathcal{V}_E^{p\ell} \) satisfying \( u \notin \mathcal{R} \setminus \mathcal{V}_F^p \) and \( \mathcal{V}_F^{(p)} \cap \mathcal{R} \subseteq \mathcal{V}_F^{p\ell} \) where \( p \) is the underlying rational prime of \( u \), there exist elements \( s_1^u, s_2^u \in \mathbb{S}_{\mathcal{V}_E}^{R,\mathcal{R}} \setminus \mathfrak{m}_\pi^R \) such that

\[
-w_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, s_1^us_1, s_2^us_2, g_1, g_2)_{L,u}^\ell = 0
\]

for every \( (\mathcal{R}, \mathcal{R}', \ell, L) \)-admissible sextuple \( (\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2) \) and every pair \( (T_1, T_2) \) in \( \text{Herm}_r^p(F)^\ast \). Moreover, we may take \( s_1^u = s_2^u = 1 \) if \( u \notin \mathcal{R} \).

**Proof.** This is simply [LL, Proposition 7.1] but without the assumption that \( \pi_\mathcal{V} \) is a (tempered) principal series. The proof is same, after we slightly generalize the construction of the integral model \( \mathcal{X}_m \) to take care of places in \( \mathcal{V}_F^{(p)} \cap \mathcal{V}_{F,\text{ram}}^{p\ell} \), and use Theorem 4.21 below which generalizes [LL, Lemma 7.3]. \( \square \)

From now to the end of this section, we assume \( \mathcal{V}_F^{(p)} \cap \mathcal{R} \subseteq \mathcal{V}_F^{p\ell} \). We also assume \( u \in \mathcal{V}_F^p \) when we need \( m \geq 1 \) below. We invoke Construction 4.18 together with Setup 4.19. The isomorphism \( C \cong \mathcal{Q}_p \) in Setup 4.19 identifies \( \text{Hom}(E, \mathcal{C}) \) with \( \text{Hom}(E, \mathcal{C}_p) \). For every \( v \in \mathcal{V}_F^{(p)} \), let \( \Phi_v \) be the subset of \( \Phi \), regarded as a subset of \( \text{Hom}(E, \mathcal{C}_p) \), of elements that induce the place \( v \) of \( F \).

For every integer \( m \geq 0 \), we define a moduli functor \( \mathcal{X}_m \) over \( O_K \) as follows: For every \( S \in \text{Sch}_{/O_K}^f \), \( \mathcal{X}_m(S) \) is the set of equivalence classes of tuples

\[
(A_0, \iota_{A_0}, \lambda_{A_0}, \eta_{A_0}^p; A, \iota_A, \lambda_A, \eta_A^p; \{\eta_{A,u}\}_{v \in \mathcal{V}_F^{(p)} \cap \mathcal{V}_F^{p\ell} \setminus \{u\}; \eta_{A,u,m})
\]

where

- \( (A_0, \iota_{A_0}, \lambda_{A_0}, \eta_{A_0}^p) \) is an element in \( \mathcal{Y}(S) \);
- \( (A, \iota_A, \lambda_A) \) is a unitary \( O_E \)-abelian scheme of signature type \( n\Phi - \iota_w + \iota_w' \) over \( S \), such that

\(^{12}\)Here, our notation on objects is slightly different from [LTXZZ] or [LL] as we, in particular, retrieve the \( O_E \)-action \( \iota_{A_0} \).
for every \( v \in V_F^{(p)} \setminus V_F^{\text{ram}} \), \( \lambda_A[v^\infty] \) is an isogeny whose kernel has order \( q_v^{1-\epsilon_v} \);
- \( \text{Lie}(A[u^\infty]) \) is of rank 1 on which the action of \( O_E \) is given by the embedding \( \iota_v \);
- for every \( v \in V_F^{(p)} \cap V_F^{\text{ram}} \), the triple \((A_0[v^\infty], \iota_{A_0}[v^\infty], \lambda_{A_0}[v^\infty]) \otimes_{O_K} O_K \) is an object of Exo
  \( \Phi_{(n,0)}(S \otimes_{O_K} O_K) \) (Remark 2.68, with \( E = E_v, F = F_v, \) and \( \ell = K \)).

- \( \eta_A \) is an \( L^p \)-level structure;
- for every \( v \in V_F^{(p)} \cap V_F^{\text{pl}} \setminus \{ \emptyset \}, \eta_{A,v} \) is an \( L_v \)-level structure;
- \( \eta_{A,u,m} \) is a Drinfeld level-\( m \) structure.

See \([LL, \text{Section 7}]\) for more details for the last three items. By \([RSZ20, \text{Theorem 4.5}]\), for every \( m \geq 0 \), \( X_m \) is a regular scheme, flat (smooth, if \( m = 0 \)) and projective over \( O_K \), and admits a canonical isomorphism

\[
X_m \otimes_{O_K} K \cong w \chi_{L_{w,m},L_u} \otimes_{w E_u} K
\]

of schemes over \( K \). Note that for every integer \( m \geq 0 \), \( \mathcal{S}_{\text{RL,R},p} \) naturally gives a ring of \( \text{étale} \) correspondences of \( X_m \).

The following theorem confirms the conjecture proposed in \([LL, \text{Remark 7.4}]\), and the rest of this subsection will be devoted to its proof.

**Theorem 4.21.** Let the situation be as in Proposition 4.20 and assume \( u \in V_F^\infty \). For every integer \( m \geq 0 \),

\[
(H^{2r}(X_m, \mathbb{Q}_L(r)) \otimes_{\mathbb{Q}} \mathbb{Q}^\text{ac})_m = 0
\]

holds, where \( m := m^R_F \cap S_{\text{RL,R},p} \).

We temporarily allow \( n \) to be an arbitrary positive integer, not necessarily even. Put \( Y_m := X_m \otimes_{O_K} k \). For every point \( x \in Y_m(\mathbb{F}_p) \), we know that \( A_x[u^\infty] \) is a one-dimensional \( O_{F_x} \)-divisible group of (relative) height \( n \), and we let \( 0 \leq h(x) \leq n-1 \) be the height of its \( \text{étale} \) part. For \( 0 \leq h \leq n-1 \), let \( Y_m^{[h]} \) be locus where \( h(x) \leq h \), which is Zariski closed hence will be endowed with the reduced induced scheme structure, and put \( Y_m^{(h)} := Y_m^{[h]} - Y_m^{[h-1]} \) \((Y_m^{[-1]} = 0)\). It is known that \( Y_m^{(h)} \) is smooth over \( k \) of pure dimension \( h \).

Now we suppose that \( m \geq 1 \). Let \( \mathcal{S}_m \) be the set of free \( O_{F_x}/\mathfrak{p}_x^m \)-submodules of \((\mathfrak{p}_x^m/O_{F_x})^n \) of rank \( n - h \), and put \( \mathcal{S}_m := \bigcup_{h=0}^{n-1} \mathcal{S}_m^h \). For every \( M \in \mathcal{S}_m \), we denote by \( Y_m^{(M)} \subseteq Y_m^{(h)} \) the (open and closed) locus where the kernel of the Drinfeld level-\( m \) structure is \( M \). Then we have

\[
Y_m^{(h)} = \bigcap_{M \in \mathcal{S}_m^h} Y_m^{(M)}
\]

for every \( 0 \leq h \leq n-1 \). Let \( Y_m^{(M)} \) be the scheme-theoretic closure of \( Y_m^{(M)} \) inside \( Y_m \). Then we have

\[
Y_m^{(M)} = \bigcup_{M' \in \mathcal{S}_m} Y_m^{(M')}
\]

as a disjoint union of strata. Note that Hecke operators away from \( u \) (of level \( L_u \)) preserve \( Y_m^{(M)} \) hence \( Y_m^{(M)} \) for every \( M \in \mathcal{S}_m \).

We need some general notation. For a sequence \((g_1, \ldots, g_t)\) of nonnegative integers with \( g = g_1 + \cdots + g_t \), we denote by \( P_{g_1, \ldots, g_t} \) the standard upper triangular parabolic subgroup of

\[\text{II.57}\]
GL_g of block sizes g_1, \ldots, g_t, and M_{g_1, \ldots, g_t} its standard diagonal Levi subgroup. Moreover, we denote by C_{m}^{g_1, \ldots, g_t} the cardinality of
\[ GL_g(O_{F_u}/p_u^m)/P_{g_1, \ldots, g_t}(O_{F_u}/p_u^m), \]
which depends only on the partition g = g_1 + \cdots + g_t. We also put
\[ L_{u,m}^g := \ker \left( GL_g(O_{F_u}) \to GL_g(O_{F_u}/p_u^m) \right). \]
For an irreducible admissible representation \( \pi \) of \( GL_g(F_u) \) and a positive integer \( s \), we have the representation \( Sp_s(\pi) \) of \( GL_{sg}(F_u) \) defined in [HT01, Section 1.3].

**Lemma 4.22.** For \( (g_1, \ldots, g_t) \) with \( g = g_1 + \cdots + g_t \) as above and another integer \( g' \geq g \), we have
\[ C_{m}^{g'-g,g} C_{m}^{g_1, \ldots, g_t} = C_{m}^{g'-g+g_1, g_2, \ldots, g_t}. \]

**Proof.** It follows from the isomorphism
\[ P_{g'-g,g}(O_{F_u}/p_u^m)/P_{g'-g+g_1, g_2, \ldots, g_t}(O_{F_u}/p_u^m) \cong GL_g(O_{F_u}/p_u^m)/P_{g_1, \ldots, g_t}(O_{F_u}/p_u^m). \]

**Lemma 4.23.** Suppose that \( m \geq 1 \). Take a sequence \( (g_1, \ldots, g_t) \) of nonnegative integers with \( g = g_1 + \cdots + g_t \). Let \( \pi_1 \boxtimes \cdots \boxtimes \pi_t \) be an admissible representation of \( M_{g_1, \ldots, g_t}(F_u) \). Then we have
\[ \dim \left( \text{Ind}_{P_{g_1, \ldots, g_t}(F_u)}^{GL_g(F_u)} \pi_1 \boxtimes \cdots \boxtimes \pi_t \right) L_{u,m}^g = C_{m}^{g_1, \ldots, g_t} t \prod_{i=1}^t \dim \pi_i^{L_{u,m}^g}. \]

**Proof.** Pick a set \( X \) of representatives of the double coset
\[ P_{g_1, \ldots, g_t}(F_u) \backslash GL_g(F_u)/L_{u,m}^g \]
contained in \( GL_g(O_{F_u}) \), which is possible by the Iwasawa decomposition. Then an element
\[ f \in \left( \text{Ind}_{P_{g_1, \ldots, g_t}(F_u)}^{GL_g(F_u)} \pi_1 \boxtimes \cdots \boxtimes \pi_t \right) L_{u,m}^g \]
is determined by \( f|_X \). Since \( GL_g(O_{F_u}) \) normalizes \( L_{u,m}^g \), a function \( f' \) on \( X \) is of the form \( f' = f|_X \) if and only if \( f' \) takes values in \( \bigotimes_{i=1}^t \pi_i^{L_{u,m}^g} \). As \( |X| = C_{m}^{g_1, \ldots, g_t} \), the lemma follows.

**Lemma 4.24.** Suppose that \( m \geq 1 \). For every positive integer \( g \) and every unramified character \( \phi \) of \( F_u^\times \), we have
\[ \sum_{h=0}^{g} (-1)^h C_{m}^{g-h,h} \dim \text{Sp}_h(\phi) L_{u,m}^g = 0. \]

**Proof.** We claim the identity
\[ \sum_{h=0}^{g} (-1)^h \left[ \text{Ind}_{P_{h,g-h}(F_u)}^{GL_g(F_u)} \text{Sp}_h(\phi) \boxtimes \left( \phi|_{u^{2^{h}} \circ \det_{g-h}} \right) \right] = 0 \]
in Groth(\( GL_g(F_u) \)). Assuming it, we have
\[ \sum_{h=0}^{g} (-1)^h \dim \left( \text{Ind}_{P_{h,g-h}(F_u)}^{GL_g(F_u)} \text{Sp}_h(\phi) \boxtimes \left( \phi|_{u^{2^{h}} \circ \det_{g-h}} \right) \right) L_{u,m}^g = 0. \]

By Lemma 4.23, the lemma follows.
For the claim, put
\[ I(\phi) := \text{Ind}_{\mathbb{P}_1,\ldots,\mathbb{P}_n}^{\text{GL}_n(F_\mathbb{Q})} \phi \boxtimes \phi \big|_{\mathbb{U} \boxtimes \cdots \boxtimes \phi} \big|_{\mathbb{U}}^{g-1}. \]

By the transitivity of (normalized) parabolic induction, every irreducible constituent of
\[ I(\phi)^{h,g-h} := \text{Ind}_{\mathbb{P}_{h,g-h}}^{\text{GL}_n(F_\mathbb{Q})} \text{Sp}_h(\phi) \boxtimes \phi \big|_{\mathbb{U}^{g-h-1}} \circ \det_{g-h} \]
is a constituent of \( I(\phi) \). By [Zel80], there is a bijection between the set of irreducible subquotients of \( I(\phi) \) and the set of sequences of signs of length \( g - 1 \). For such a sequence \( \sigma \), we denote by \( I(\phi)_\sigma \) the corresponding irreducible subquotient. For \( 0 \leq h \leq g - 1 \), we denote by \( \sigma(i) \) the sequence starting from \( h \) negative signs followed by \( g - 1 - h \) positive signs. In particular,
\[ I(\phi)_{\sigma(g-1)} = \text{Sp}_g(\phi) = I(\phi)^{g,0}, \quad I(\phi)_{\sigma(0)} = \phi \big|_{\mathbb{U}^{g-1}} \circ \det_{g} = I(\phi)^{0,g}. \]

By [HT01, Lemma I.3.2], we have
\[ [I(\phi)^{h,g-h}] = [I(\phi)_{\sigma(h)}] + [I(\phi)_{\sigma(h-1)}] \]
in \( \text{Groth}(\text{GL}_g(F_\mathbb{U})) \) for \( 0 < h < g \). Thus, (4.2) follows. \( \square \)

**Proposition 4.25.** Fix an isomorphism \( \overline{\mathcal{O}}_\ell \simeq \mathbb{C} \). Suppose that \( m \geq 1 \). For every \( 0 \leq h \leq n - 1 \) and \( M \in \mathcal{S}_m^h \), we have
\[ H^j(S^M_m \otimes_k \mathbb{F}_p, \overline{\mathcal{O}}_\ell)_m = 0 \]
for every \( j \neq h \).

This is an extension of [TY07, Proposition 4.4]. However, we allow arbitrary principal level at \( u \) and our case involves endoscopy.

**Proof.** In what follows, \( h \) will always denote an integer satisfying \( 0 \leq h \leq n - 1 \). Denote by \( D_{n-h} \) the division algebra over \( F_\mathbb{Q} \) of Hasse invariant \( \frac{1}{n-h} \), with the maximal order \( O_{D_{n-h}} \).

For a \( \mathbb{F} \)-scheme \( Y \) of finite type over \( k \), and a (finite) character \( \chi: T_0(\mathbb{Q}) \backslash T_0(\mathbb{A}^\infty)/L_0 \to \overline{\mathcal{O}}_\ell^\times \), we put
\[ [H^*_\chi(Y, \overline{\mathcal{O}}_\ell)] := \sum_{j \in \mathbb{Z}} (-1)^j H^*_j(Y \otimes_k \mathbb{F}_p, \overline{\mathcal{O}}_\ell)[\chi] \]
as an element in \( \text{Groth}(\text{Gal}(\mathbb{F}_p/k)) \) for \( ? = \{ +, c \} \).

Let \( I^h_m \) be the Igusa variety (of the first kind) introduced in [HT01, Section IV.1] so that \( I^h_m \) is isomorphic to \( Y^M_m \) for every \( M \in \mathcal{S}_m^h \) as schemes over \( k \) (but not as schemes over \( Y^0_{h}(\mathbb{A}^\infty) \)). Combining with (4.1), we obtain the identity
\[ (4.3) \quad [H^*_\chi(Y^M_m, \overline{\mathcal{O}}_\ell)] = \sum_{h' = 0}^{h} \sum_{M' \in \mathcal{S}_m^{h'}} \sum_{M \subseteq M'} (-1)^{h-h'}[H^*_\chi(Y^{M'}_m, \overline{\mathcal{O}}_\ell)] \]
\[ = \sum_{h' = 0}^{h} (-1)^{h-h'} \cdot \left| \left\{ M' \in \mathcal{S}_m^{h'} | M' \subseteq M \right\} \right| \cdot [H^*_\chi(I^h_m, \overline{\mathcal{O}}_\ell)] \]
\[ = \sum_{h' = 0}^{h} (-1)^{h-h'} C_m^{h-h', h'} \cdot [H^*_\chi(I^h_m, \overline{\mathcal{O}}_\ell)] \]
in \( \text{Groth}(\text{Gal}(\mathbb{F}_p/k)) \).
Now to compute $[H_c(I^h_m, \mathbb{Q}_l)]$, we use [CS17, Lemma 5.5.1] in which the corresponding $J_b(\mathbb{Q}_p)$ is $D_{n-h'} \times \text{GL}_{h'}(F_\mathfrak{p})$, and we take $\phi = \phi\mathfrak{u} \phi\mathfrak{w}$ where $\phi\mathfrak{w}$ is the characteristic function of $L^\mathfrak{w}$ and $\phi\mathfrak{u}$ is the characteristic function of $O^\times_{D_{n-h'}} \times L^h_{\mathfrak{u}, m}$. Then we have the identity

\begin{equation}
[H_{c, \chi}(I^h_m, \mathbb{Q}_l)] = \sum_n \sum_{\Pi^u} c(n, \Pi^u) \cdot \text{Red}^h_n(\pi^u_n) O^\times_{D_{n-h'}} \times L^h_{\mathfrak{u}, m}
\end{equation}

in $\text{Groth}(D^\times_{n-h'} / O^\times_{D_{n-h'}})$, where

- $n$ runs through ordered pairs $(n_1, n_2)$ of nonnegative integers such that $n_1 + n_2 = n$, which gives an elliptic endoscopic group $G_n$ of $U(n^\mathfrak{v})$;
- $\Pi^u$ runs through a finite set of certain isobaric irreducible cohomological (with respect to the trivial algebraic representation) automorphic representations of $\mathcal{G}_n(\mathbb{A}_F)$, with $\pi^u_n$ the descent of $\Pi^u_n$ to $G_n(F_\mathfrak{u}) \simeq M_{n_1, n_2}(F_\mathfrak{u})$;
- $c(n, \Pi^u)$ is a constant depending only on $n$ and $\Pi^u$ but not on $h'$;
- $\text{Red}^h_n : \text{Groth}(M_{n_1, n_2}(F_\mathfrak{u})) \to \text{Groth}(D^\times_{n-h'} \times \text{GL}_{h'}(F_\mathfrak{u}))$ is the zero map if $h' < n_2$, and otherwise is the composition of
  - $\text{Groth}(M_{n_1, n_2}(F_\mathfrak{u})) \to \text{Groth}(M_{n-h', h'-n_2, n_2}(F_\mathfrak{u}))$, which is the normalized Jacquet functor,
  - $\text{Groth}(M_{n-h', h'-n_2, n_2}(F_\mathfrak{u})) \to \text{Groth}(M_{n-h', h'}(F_\mathfrak{u}))$, which is the normalized parabolic induction, and
  - $\text{Groth}(M_{n-h', h'}(F_\mathfrak{u})) \to \text{Groth}(D^\times_{n-h'} \times \text{GL}_{h'}(F_\mathfrak{u}))$, which is the Langlands–Jacquet map (on the first factor).

The image of $[H_{c, \chi}(I^h_m, \mathbb{Q}_l)]$ in $\text{Groth}(\text{Gal}(\overline{\mathbb{F}}_p/k))$ is given by the map

\[\text{Groth}(D^\times_{n-h'} / O^\times_{D_{n-h'}}) \to \text{Groth}(\text{Gal}(\overline{\mathbb{F}}_p/k))\]

sending an (unramified) character $\phi \circ \text{Nm}_{D^\times_{n-h'}}$ to $\text{rec}(\phi^{-1}) \cdot \hat{\chi}$, where $\hat{\chi}$ is a finite character of $\text{Gal}(\overline{\mathbb{F}}_p/k)$ determined by $\chi$. In what follows, we will regard

\[\text{Red}^h_n(\pi^u_n) O^\times_{D_{n-h'}} \times L^h_{m, \mathfrak{u}}\]

as an element of $\text{Groth}(\text{Gal}(\overline{\mathbb{F}}_p/k))$ via the above map.

Now let us compute for each $n = (n_1, n_2)$,

\begin{equation}
\sum_{h'=0}^h (-1)^{h-h'} C^h_{m, h', h'} \cdot \text{Red}^h_n(\pi^u_n) O^\times_{D_{n-h'}} \times L^h_{m, \mathfrak{u}}
\end{equation}

in $\text{Groth}(\text{Gal}(\overline{\mathbb{F}}_p/k))$, when $\pi^u_n$ is tempered. Write $\pi^u_n = \pi^1 \boxtimes \pi^2$ where $\pi^\alpha$ is an tempered irreducible admissible representation of $\text{GL}_{n_\alpha}(F_\mathfrak{u})$. In particular, $\pi^1$ is a full induction of the form

\[\text{Ind}_{\text{Sp}_{s_1 g_1, \ldots, s_t g_t}(F_\mathfrak{u})} \text{Sp}_{s_1} (\pi^1_s) \boxtimes \cdots \boxtimes \text{Sp}_{s_t} (\pi^1_t),\]

where $s_1, \ldots, s_t$ and $g_1, \ldots, g_t$ are positive integers satisfying $s_1 g_1 + \cdots + s_t g_t = n_1$; and for $1 \leq i \leq t$, $\pi^1_i$ is an irreducible cuspidal representation of $\text{GL}_{s_i}(F_\mathfrak{u})$ such that $\text{Sp}_{s_1} (\pi^1_1)$ is unitary. Let $I$ be the subset of $\{1, \ldots, t\}$ such that $\pi^1_i$ is an unramified character (hence
Proof of Theorem 4.21. We may assume \( m \geq 1 \) since the morphism \( X_m \to X_0 \) is finite and flat. In what follows, \( h \) is always an integer satisfying \( 0 \leq h \leq n - 1 = 2r - 1 \). For a subset \( \Sigma \subset \mathcal{S}_m \), we put
\[
Y \left( \Sigma \right) := \bigcup_{M \in \Sigma} Y \left( M \right), \quad Y \left[ \Sigma \right] := \bigcup_{M \in \Sigma} Y \left( M \right)
\]
and
\[
Y \left( \Sigma \right) := \bigcup_{M \in \Sigma} Y \left( M \right), \quad Y \left[ \Sigma \right] := \bigcup_{M \in \Sigma} Y \left( M \right)
\]
in which the first union is disjoint. If $h \geq 1$, we also denote by $\Sigma^+$ the subset of $\mathfrak{S}^{h-1}_m$ consisting of $M'$ that contains an element in $\Sigma$.

Fix an arbitrary isomorphism $\mathfrak{Q}_\ell \cong \mathbb{C}$. We show by induction on $h$ that for every $\Sigma \subset \mathfrak{S}^h_m$,

$$H^2(Y^{(\Sigma)}_m \otimes_k \mathbb{F}_p, \mathfrak{Q}_\ell)_m = H^2(Y^{[\Sigma]}_m \otimes_k \mathbb{F}_p, \mathfrak{Q}_\ell)_m = 0$$

if $j > h$. To ease notation, we simply write $H^*_q(-)$ for $H^*_q(- \otimes_k \mathbb{F}_p, \mathfrak{Q}_\ell)_m$ for $? \in \{ \cdot, c \}$.

The case for $h = 0$ is trivial. Suppose that we know (4.8) for $h - 1$ for some $h \geq 1$. For every $M \in \mathfrak{S}^h_m$, we have the exact sequence

$$\cdots \rightarrow H^{j-1}(Y^{(\{M\}^h)}) \rightarrow H^j(Y^{[M]}_m) \rightarrow H^j(Y^{[M]}_m) \rightarrow \cdots$$

By Proposition 4.25 and the induction hypothesis, we have $H^j(Y^{[M]}_m) = 0$ for $j > h$. Now take a subset $\Sigma$ of $\mathfrak{S}^h_m$. Then we have $H^j(Y^{(\Sigma)}_m) = \bigoplus_{M \in \Sigma} H^j(Y^{[M]}_m) = 0$ for $j > h$. By the exact sequence

$$\cdots \rightarrow H^j(Y^{(\Sigma)}_m) \rightarrow H^j(Y^{[\Sigma]}_m) \rightarrow H^j(Y^{[\Sigma]}_m) \rightarrow \cdots$$

and the induction hypothesis, we have $H^j(Y^{[\Sigma]}_m) = 0$ for $j > h$. Thus, (4.8) holds for $h$.

By (4.8) and the Poincaré duality, we have

$$H^j(Y^{(h)}_m \otimes_k \mathbb{F}_p, \mathfrak{Q}_\ell)_m = 0$$

for $j < h$. Now we apply [LL, Corollary B.13(2)] with $Y_j = Y^{2r-1-j}_m$ for $0 \leq j \leq 2r-1$ and $n_j = j + 1$, which is allowed since $2r - 2n_j < 2r-1 - j$. Note that the vanishing of $H^{2r}(X_m \otimes_{Q_K} K, \mathfrak{Q}_\ell(r))_m$ follows from the same argument for [LL, Lemma 7.3]. We obtain $H^{2r}(X_m \otimes_{Q_K} K, \mathfrak{Q}_\ell(r))_m = 0$, hence the theorem follows. \qed

4.4. Local indices at inert places. In this subsection, we compute local indices at places in $\mathcal{V}^p_{E^\text{int}}$ not above $\mathfrak{R}$.

**Proposition 4.26.** Let $\mathfrak{R}, \mathfrak{R}', \ell$, and $L$ be as in Definition 4.15. Take an element $u \in \mathcal{V}^p_{E^\text{int}}$ such that its underlying rational prime $p$ is odd and satisfies $\mathcal{V}^p_{E,F} \cap \mathfrak{R} \subseteq \mathcal{V}^p_{E,F}$.  

(1) Suppose that $u \not\in \mathcal{S}$. Then we have

$$\log q_u \cdot \text{vol}^\Sigma(L).\text{ }u I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)^{\varphi}_{L,u} = \mathcal{E}_{T_1,T_2}((g_1, g_2), \Phi^0_\infty \otimes (s_1 \phi_1^\infty \otimes (s_2 \phi_2^\infty)^c))_u$$

for every $(\mathfrak{R}, \mathfrak{R}', \ell, L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)$ and every pair $(T_1, T_2)$ in $\text{Herm}^\Sigma(F)^+$.  

(2) Suppose that $u \in \mathcal{S} \cap \mathcal{V}^p_{E,F}$ and is unramified over $\mathbb{Q}$. Fix an isomorphism

$$\mathfrak{A}^{\varphi}_{E,F} \cong \mathfrak{A}^\varphi_{E,F}$$

of hermitian spaces over $\mathfrak{A}^\varphi_{E,F}$ and a $\psi_{E,F}$-self-dual lattice $\Lambda^\varphi_u$ of $\mathfrak{A}^\varphi_{E,F}$. Then there exist elements $s_1^u, s_2^u \in \mathbb{R}_{\text{ram}} \setminus \mathfrak{m}_{E,F}$ such that

$$\log q_u \cdot \text{vol}^\Sigma(L).\text{ }u I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, s^u_1 s_1, s^u_2 s_2, g_1, g_2)^{\varphi}_{L,u} = \mathcal{E}_{T_1,T_2}((g_1, g_2), \Phi^0_\infty \otimes (s_1^u s_1 \phi_1^\infty \otimes (s_2^u s_2 \phi_2^\infty)^c) \otimes 1(\Lambda^\varphi_u)^r)$$

for every $(\mathfrak{R}, \mathfrak{R}', \ell, L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)$ and every pair $(T_1, T_2)$ in $\text{Herm}^\Sigma(F)^+$.  

In both cases, the right-hand side is defined in Definition 4.10 with the Gaussian function \( \Phi_\infty \in \mathcal{S}(V^{2r} \otimes_{K_F} F_\infty) \) (Setup 4.2(H3)), and \( \text{vol}^\ddagger(L) \) is defined in [LL, Definition 3.8].

**Proof.** Part (1) is proved in the same way as [LL, Proposition 8.1]. Part (2) is proved in the same way as [LL, Proposition 9.1]. Note that we need to extend the definition of the integral model due to the presence of places in \( V_F^{(p)} \cap V_F^{\text{ram}} \), as we do in the previous subsection. The requirement that \( u \in V_F^\circ \) in (2) is to ensure that \( K \) is unramified over \( wE_u \) (see Setup 4.19). \( \square \)

### 4.5. Local indices at ramified places.

In this subsection, we compute local indices at places in \( V_F^{\text{ram}} \) not above \( \mathbb{R} \).

**Proposition 4.27.** Let \( R, R', \ell, \) and \( L \) be as in Definition 4.15. Take an element \( u \in V_F^{\text{ram}} \) such that its underlying rational prime \( p \) satisfies \( V_F^{(p)} \cap R \subseteq V_F^{\text{spl}} \). Then we have

\[
\log q_u \cdot \text{vol}^\ddagger(L) \cdot u(I_{T_1, T_2}(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2))_{L,u} = \mathcal{E}_{T_1, T_2}((g_1, g_2), \Phi_\infty \otimes (s_1 \phi_1^\infty \otimes (s_2 \phi_2^\infty))c)_{u}
\]

for every \((R, R', \ell, L)\)-admissible sextuple \((\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)\) and every pair \((T_1, T_2)\) in \( \text{Herm}_2^\circ(F)^+ \), where the right-hand side is defined in Definition 4.10 with the Gaussian function \( \Phi_\infty \in \mathcal{S}(V^{2r} \otimes_{K_F} F_\infty) \) (Setup 4.2(H3)), and \( \text{vol}^\ddagger(L) \) is defined in [LL, Definition 3.8].

**Proof.** The proof of the proposition follows the same line as in [LL, Proposition 8.1], as long as we accomplish the following three tasks. We invoke Construction 4.18 together with Setup 4.19.

1. Construct a good integral model \( \mathcal{X}_L \) for \( \mathcal{X}_L \) over \( O_K \) for open compact subgroups \( \bar{L} \subseteq L \) satisfying \( \bar{L}_v = L_v \) for \( v \in V_F^{(p)} \setminus V_F^{\text{spl}} \), which is provided after the proof.
2. Establish the nonarchimedean uniformization of \( \mathcal{X}_L \) along the supersingular locus using the relative Rapoport–Zink space \( \mathcal{N} \) from Definition 2.3, analogous to [LL, (8.1)], and compare special divisors. This is done in Proposition 4.29 below.
3. Show that for \( x = (x_1, \ldots, x_{2r}) \in V^{2r} \) with \( T(x) \in \text{Herm}_2^\circ(F_u) \), we have

\[
\chi\left(\sigma_{\mathcal{N}(x_1)}^{\bar{L}} \otimes \sigma_{\mathcal{N}}^{\cdot \cdot \cdot \otimes \sigma_{\mathcal{N}}^{\cdot \cdot \cdot \otimes \sigma_{\mathcal{N}}^{(x_{2r})}}\right) = \frac{b_{2r,u}(0)}{\log q_u} W^\ddagger (0, 1_{4r}, \mathbb{1}_{(A^\infty_2)^{2r}})
\]

if \( T(x) = T_{\ddagger} \). In fact, this follows from Theorem 2.9, Remark 2.19, and the identity

\[
b_{2r,u}(0) = \prod_{i=1}^r (1 - q_u^{-2i}).
\]

The proposition is proved. \( \square \)

Let the situation be as in Proposition 4.27. The isomorphism \( \mathbb{C} \cong \overline{\mathbb{Q}}_p \) in Setup 4.19 identifies \( \text{Hom}(E, \mathbb{C}) \) with \( \text{Hom}(E, \mathbb{C}_p) \). For every \( v \in V_F^{(p)} \), let \( \Phi_v \) be the subset of \( \Phi \), regarded as a subset of \( \text{Hom}(E, \mathbb{C}_p) \), of elements that induce the place \( v \) of \( F \).

To ease notation, put

\[
U := \{v \in V_F^{(p)} \setminus V_F^{\text{spl}} \mid v \neq u\}.
\]

In particular, \( U \cap R = \emptyset \).

There is a projective system \( \{\mathcal{X}_v\} \), for open compact subgroups \( \bar{L} \subseteq L \) satisfying \( \bar{L}_v = L_v \) for \( v \in V_F^{(p)} \setminus V_F^{\text{spl}} \), of smooth projective schemes over \( O_K \) (see [RSZ20, Theorem 4.7, AT type (2)]) with

\[
\mathcal{X}_L \otimes_{O_K} K = \mathcal{X}_L \otimes_{wE} K = \left(\mathcal{X}_L \otimes_{wE} Y\right) \otimes_{wE} K,
\]
and finite étale transition morphisms, such that for every $S \in \text{Sch}'_{/O_K}$, $\mathcal{X}_L(S)$ is the set of equivalence classes of tuples

$$(A_0, \tau_{A_0}, \lambda_{A_0}, \eta_{A_0}^p, \tau_A, \lambda_A, \eta_A^p, \{\eta_{A,v}\}_{v \in \mathcal{V}_F^{(p)}})$$

where

- $(A_0, \tau_{A_0}, \lambda_{A_0}, \eta_{A_0}^p)$ is an element in $\mathcal{V}(S)$;
- $(A, \tau_A, \lambda_A)$ is a unitary $O_E$-abelian scheme of signature type $n\Phi - \tau_w + \epsilon_w$ over $S$, such that
  - for every $v \in \mathcal{V}_F^{(p)} \setminus \mathcal{V}_F^{\text{ram}}$, $\lambda_A[v^\infty]$ is an isogeny whose kernel has order $q_v^{1-\epsilon_v}$;
  - for every $v \in \mathcal{U} \cap \mathcal{V}_F^{\text{ram}}$, the triple $(A_0[v^\infty], \tau_{A_0}[v^\infty], \lambda_{A_0}[v^\infty]) \otimes_{O_K} O_F$ is an object of $\text{Exo}^{\Phi_v}_{(n,0)}(S \otimes_{O_K} O_K)$ (Remark 2.68, with $E = E_v, F = F_v$, and $\tilde{E} = \tilde{K}$);
- for $v = w$, $(A_0[v^\infty], \tau_{A_0}[v^\infty], \lambda_{A_0}[v^\infty]) \otimes_{O_K} O_K$ is an object of $\text{Exo}^{\Phi_w}_{(n-1,1)}(S \otimes_{O_K} O_K)$ (Definition 2.60, with $E = E_v, F = F_v$, and $\tilde{E} = \tilde{K}$);
- $\eta_A^p$ is an $\tilde{L}$-level structure;
- for every $v \in \mathcal{V}_F^{(p)} \cap \mathcal{V}_F^{\text{pl}}, \eta_{A,v}$ is an $\tilde{L}_v$-level structure.

In particular, $\mathcal{S}^R$ is naturally a ring of étale correspondences of $\mathcal{X}_L$.

Let $\phi^\infty \in \mathcal{V}(V \otimes_{A,E} A_L^{\infty})^L$ be a $p$-basic element [LL, Definition 5.5]. For every element $t \in F^+$, we have a cycle $\bar{Z}_t(\phi^\infty)_L \in Z^1(\mathcal{X}_L)$ extending the restriction of $wZ_t(\phi^\infty)$ to $wX_L$, defined similarly as in [LZ, Section 13.3].

Now we study the nonarchimedean uniformization of $\mathcal{X}_L$ along the supersingular locus. Fix a point $P_0 := (A_0, \tau_{A_0}, \lambda_{A_0}, \eta_{A_0}^p) \in \mathcal{V}(O_K)$. Put

$$\mathcal{X} := \varprojlim_L \mathcal{X}_L$$

and denote by $\mathcal{X}_0$ the fiber of $P_0$ along the natural projection $\mathcal{X} \to \mathcal{V}$. Let $\mathcal{X}_0^\infty$ be the completion along the (closed) locus where $A[u^\infty]$ is supersingular, as a formal scheme over $\text{Spf} O_K$.

Also fix a point $P \in \mathcal{X}_0^\infty(\overline{\mathbb{F}}_p)$ represented by $(P_0 \otimes_{O_K} \overline{\mathbb{F}}_p; A, \tau_A, \lambda_A, \eta_A^p, \{\eta_{A,v}\}_{v \in \mathcal{V}_F^{(p)}})$.

Put $V := \text{Hom}_{O_E}(A_0 \otimes_{O_E} \overline{\mathbb{F}}_p, A) \otimes \mathbb{Q}$. Fixing an element $\varpi \in O_F$ that has valuation 0 (resp. 1) at places in $\mathcal{U} \cap \mathcal{V}_F^{\text{int}}$ (resp., $\mathcal{U} \cap \mathcal{V}_F^{\text{ram}}$), we have a pairing

$$(\ , \ )_V : V \times V \to E$$

sending $(x, y) \in V^2$ to the composition of quasi-homomorphisms

$$A_0 \xrightarrow{x} X \xrightarrow{\lambda_A} A^\vee \xrightarrow{y^\vee} A_0^{\varpi^{-1}\lambda_{A_0}^{-1}} \xrightarrow{\tau_{A_0}} A_0$$

as an element in $\text{End}_{O_E}(A_0) \otimes \mathbb{Q}$ hence in $E$ via $\tau_{A_0}^{-1}$. We have the following properties concerning $V$:

- $V, (\ , \ )_V$ is a totally positive definite hermitian space over $E$ of rank $n$;
- for every $v \in \mathcal{V}_F^{\text{int}} \setminus (\mathcal{V}_F^{(p)} \setminus \mathcal{V}_F^{\text{pl}})$, we have a canonical isometry $V \otimes_F F_v \simeq V \otimes_F F_v$ of hermitian spaces;
- for every $v \in \mathcal{U}$, the $O_{E_v}$-lattice $\Lambda_v := \text{Hom}_{O_E}(A_0 \otimes_{O_E} \overline{\mathbb{F}}_p, A) \otimes_{O_F} O_{F_v}$ is
  - self-dual if $v \in \mathcal{U} \cap \mathcal{V}_F^{\text{int}}$ and $\epsilon_v = 1$,
  - almost self-dual if $v \in \mathcal{U} \cap \mathcal{V}_F^{\text{int}}$ and $\epsilon_v = -1$,
  - self-dual if $v \in \mathcal{U} \cap \mathcal{V}_F^{\text{ram}}$. 

• $V \otimes_F F_{\underline{u}}$ is nonsplit, and we have a canonical isomorphism

$$V \otimes_F F_{\underline{u}} \cong \text{Hom}_{O_Eu}(A_0[u^\infty] \otimes_{O_K} \overline{F}_p, A[u^\infty]) \otimes \mathbb{Q}$$

of hermitian spaces over $E_u$.

We have a Rapoport–Zink space $\mathcal{N}$ (Definition 2.3, with $E = E_0$, $F = F_{\underline{u}}$, $\tilde{E} = \tilde{K}$, and $\varphi_0$ the natural embedding) with respect to the object

$$\left(\underline{X}, \iota_{\underline{X}}, \lambda_{\underline{X}} \right) := (A[u^\infty], \iota_A[u^\infty], \lambda_A[u^\infty])^{rel} \in \text{Exo}_{(n-1,1)}(\overline{F}_p),$$

where $^{\text{rel}}$ is the morphism (2.21). We now construct a morphism

$$(4.9) \quad \Upsilon^{\text{rel}}: \mathcal{X}^\wedge_0 \to U(V)(F) \backslash \left( \mathcal{N} \times U(V)(\overline{A}_{F,F^\infty}) / \prod_{v \in \mathcal{U}} L_v \right)$$

of formal schemes over $\text{Spf} \, O_{\tilde{K}}$, where $L_v$ is the stabilizer of $\Lambda_v$ in $U(V)(F_v)$, as follows.

We have the Rapoport–Zink space $\mathcal{N}^{\Phi_u} = \mathcal{N}^{\Phi_u}_{(A[u^\infty], \iota_A[u^\infty], \lambda_A[u^\infty])}$ from Definition 2.65. We first define a morphism

$$\Upsilon: \mathcal{X}^\wedge_0 \to U(V)(F) \backslash \left( \mathcal{N}^{\Phi_u} \times U(V)(\overline{A}_{F,F^\infty}) / \prod_{v \in \mathcal{U}} L_v \right),$$

and then define $\Upsilon^{\text{rel}}$ as the composition of $\Upsilon$ with the morphism in Corollary 2.66. To construct $\Upsilon$, we take a point

$$P = (P_0 \otimes_{O_K} S; A, \iota_A, \lambda_A, \eta_A, \{\eta_{A,v}\}_{v \in \mathcal{V}(\rho) \cap \text{spl}}) \in \mathcal{X}^\wedge_0(S)$$

for a connected scheme $S$ in $\text{Sch}_{/O_{\tilde{K}}} \cap \text{Sch}_{/O_{\tilde{K}}}$ with a geometric point $s$. In particular, $A[p^\infty]$ is supersingular. By [RZ96, Proposition 6.29], we can choose an $O_E$-linear quasi-isogeny

$$\rho: A \times_S (S \otimes_{O_{\tilde{K}}} \overline{F}_p) \to A \otimes_{\overline{F}_p} (S \otimes_{O_{\tilde{K}}} \overline{F}_p)$$

of height zero such that $\rho^* \lambda_A \otimes_{\overline{F}_p} (S \otimes_{O_{\tilde{K}}} \overline{F}_p) = \lambda_A \times_S (S \otimes_{O_{\tilde{K}}} \overline{F}_p)$. We have

• $(A[u^\infty], \iota_A[u^\infty], \lambda_A[u^\infty]; \rho[u^\infty])$ is an element in $\mathcal{N}^{\Phi_u}(S)$;

• the composite map

$$V \otimes_{Q} A^\infty \cong V \otimes_{Q} A^\infty \otimes_{Q} \text{Hom}_{E \otimes Q A^\infty}(H_1(A_{0,s}, A^\infty), H_1(A_s, A^\infty))$$

is an isometry, which gives rise to an element $h^p \in U(V)(\overline{A}_{F,F^\infty})$;

• the same process as above will produce an element $h^\text{spl}_{v} \in \prod_{v \in \mathcal{V}(\rho) \cap \text{spl}} U(V)(F_v)$;

• for every $v \in U$, the image of the map

$$\rho_{s,s} : \text{Hom}_{O_{E_v}}(A_{0,s}[u^\infty], A_s[u^\infty]) \to \text{Hom}_{O_{E_v}}(A_{0,s}[u^\infty], A_s[u^\infty]) \otimes Q = V \otimes F_v$$

is an $O_{E_v}$-lattice in the same $U(V)(F_v)$-orbit of $\Lambda_v$, which gives rise to an element $h_v \in U(V)(F_v)/L_v$.

Together, we obtain an element

$$\left( (A[u^\infty], \iota_A[u^\infty], \lambda_A[u^\infty]; \rho[u^\infty]), (h^p, h^\text{spl}_v, \{h_v\}_{v \in \mathcal{U}}) \right) \in \mathcal{N}^{\Phi_u}(S) \times U(V)(\overline{A}_{F,F^\infty}) / \prod_{v \in \mathcal{U}} L_v,$$

and we define $\Upsilon(P)$ to be its image in the quotient, which is independent of the choice of $\rho$. 
Remark 4.28. Both $V$ and $\Upsilon_{rel}$ depend on the choice of $P$, while the isometry class of $V$ does not.

Proposition 4.29. The morphism $\Upsilon_{rel}$ (4.9) is an isomorphism. Moreover, for every $p$-basic element $\phi^\infty \in \mathcal{S}(V \otimes_{A_F} A_F^\infty)^L$ and every $t \in F^+$, we have

$$(4.10) \quad \Upsilon_{rel}\left( Z_t(\phi^\infty)L|_{x^c_0} \right) = \sum_{x \in U(V^c)(F)/V} \sum_{h \in U(V^c)(F)\backslash U(V)(A_F^\infty)^{\text{un}}/\prod_v L_v} \phi(h^{-1}x) \cdot (\mathcal{N}(x_{rel}), h),$$

where

- $V^x$ denotes the orthogonal complement of $x$ in $V$;
- $\phi$ is a Schwartz function on $V \otimes_F A_F^{\infty,\un}$ such that $\phi_v = \phi^\infty_v$ for $v \in \mathcal{V}_F^\infty \setminus (\mathcal{V}_F^p \setminus \mathcal{V}_F^{spl})$ and $\phi_v = 1_{A_v}$ for $v \in U$;
- $x_{rel}$ is defined in (2.25); and
- $(\mathcal{N}(x_{rel}), h)$ denotes the corresponding double coset in (4.9).

Proof. By a similar argument for [RZ96, Theorem 6.30], the morphism $\Upsilon$ is an isomorphism. Thus, $\Upsilon_{rel}$ is an isomorphism as well by Corollary 2.66.

For (4.10), by a similar argument for [Liu, Theorem 5.22], the identity holds with $\mathcal{N}(x_{rel})$ replaced by $\mathcal{N}^{\Phi_v}(x)$. Then it follows by Corollary 2.67.

The proposition is proved. $\square$

4.6. Local indices at archimedean places. In this subsection, we compute local indices at places in $\mathcal{V}_E^\infty$.

Proposition 4.30. Let $\mathbb{R}$, $\mathbb{R}'$, $\ell$, and $L$ be as in Definition 4.15. Let $(\pi, \mathcal{V}_c)$ be as in Setup 4.4. Take an element $u \in \mathcal{V}_E^\infty$. Consider an $(\mathbb{R}, \mathbb{R}', \ell, L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, s_1, s_2, g_1, g_2)$ and an element $\varphi_1 \in \mathcal{V}_{E,R}'$. Let $K_1 \subseteq G_r(A_F^\infty)$ be an open compact subgroup that fixes both $\phi_1^\infty$ and $\varphi_1$, and $\mathfrak{H}_1 \subseteq G_r(F^\infty)$ a Siegel fundamental domain for the congruence subgroup $G_r(F) \cap g_1^{-1}K_1(g_1)^{-1}1$. Then for every $T_2 \in \text{Herm}_c^0(F)^+$, we have

$$\text{vol}^c(L) \cdot \int_{\mathfrak{H}_1} \varphi^c(\tau_1 g_1) \sum_{T_1 \in \text{Herm}_c^0(F)^+} w_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, s_1, s_2, \tau_1 g_1, g_2)_{L,u} d\tau_1$$

$$= \frac{1}{2} \int_{\mathfrak{H}_1} \varphi^c(\tau_1 g_1) \sum_{T_1 \in \text{Herm}_c^0(F)^+} \mathcal{E}_{T_1,T_2}((\tau_1 g_1, g_2), \Phi_\infty^0 \otimes (s_1 \Phi^\infty_\infty \otimes (s_2 \Phi^\infty_\infty)^c))_{u} d\tau_1,$$

in which both sides are absolutely convergent. Here, the term $\mathcal{E}_{T_1,T_2}$ is defined in Definition 4.10 with the Gaussian function $\Phi^0_\infty \in \mathcal{S}(V^2r \otimes_{A_F} F^\infty)$ (Setup 4.2(H3)), and $\text{vol}^c(L)$ is defined in [LL, Definition 3.8].

Proof. This is simply [LL, Proposition 10.1]. $\square$

4.7. Proof of main results. The proofs of Theorem 1.5, Theorem 1.6, and Corollary 1.8 follow from the same lines as for [LL, Theorem 1.5], [LL, Theorem 1.7], and [LL, Corollary 1.9], respectively, written in [LL, Section 11]. However, we need to take $\mathbb{R}$ to be a finite subset of $\mathcal{V}_F^{spl} \cap \mathcal{V}_F^\infty$ containing $\mathbb{R}_c$ and of cardinality at least 2, and modify the reference according to the table below.
Remark 4.31. Finally, we explain the main difficulty on lifting the restriction $F \neq \mathbb{Q}$ (when $r \geq 2$). Suppose that $F = \mathbb{Q}$ and $r \geq 2$. Then the Shimura variety $^wX_L$ from Subsection 4.2 is never proper over the base field. Nevertheless, it is well-known that $^wX_L$ admits a canonical toroidal compactification which is smooth. However, to run our argument, we need suitable compactification of their integral models at every place finite place $u$ of $E$ as well. As far as we can see, the main obstacle is the compactification of integral models using Drinfeld level structures when $u$ splits over $F$, together with a vanishing result like Theorem 4.21.

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