Variation of local systems and parabolic cohomology

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Abstract

Given a family of local systems on a punctured Riemann sphere, with moving singularities, its first parabolic cohomology is a local system on the base space. We study this situation from different points of view. For instance, we derive universal formulas for the monodromy of the resulting local system. We use a particular example of our construction to prove that the simple groups $\text{PSL}_2(p^2)$ admit regular realizations over the field $\mathbb{Q}(t)$ for primes $p \not\equiv 1, 4, 16 \mod 21$. Finally, we compute the monodromy of the Euler-Picard equation, reproving a classical result of Picard.

Introduction

Local systems on the punctured Riemann sphere arise in various branches of mathematics and have been intensively studied, see e.g. [5], [17]. One way to produce such local systems with interesting properties is the following. Suppose we are given a family $\mathcal{V}_s$ of local systems on the punctured sphere with moving singularities, parameterized by some base space $S$. More precisely, let $D \subset \mathbb{P}^1 \times S$ be a smooth relative divisor and let $\mathcal{V}$ be a local system on $U := \mathbb{P}^1 \times S - D$. Let $\bar{\pi} : \mathbb{P}^1 \times S \to S$ denote the second projection and $j : U \hookrightarrow \mathbb{P}^1 \times S$ the canonical inclusion. Then the first higher direct image sheaf

$$\mathcal{W} := R^1\bar{\pi}_*(j_*\mathcal{V})$$

is a local system on $S$, whose stalk at a point $s \in S$ is the first parabolic cohomology group of the local system $\mathcal{V}_s$, the restriction of $\mathcal{V}$ to the fiber $U_s := \pi^{-1}(s)$. Choose a base point $s_0 \in S$ and set $\mathcal{V}_0 := \mathcal{V}_{s_0}$. We call $\mathcal{V}$ a variation of the local system $\mathcal{V}_0$ over the base $S$ and $\mathcal{W}$ the first parabolic cohomology of the variation $\mathcal{V}$.

A special case of this construction is the middle convolution studied by N. Katz in [17] and by Dettweiler and Reiter in [10]. Starting with some local system $\mathcal{V}_0$ on the punctured Riemann sphere $U_0$, one constructs a variation of local systems over the base $U_0$ by twisting $\mathcal{V}_0$ with a 1-dimensional system with
two singularities, one of which is moving over all points of $U_0$. The parabolic cohomology of this variation gives rise to a local system on $U_0$, called the middle convolution of $V_0$. In [17], Katz proves that all rigid local systems can be constructed from 1-dimensional systems by successive application of middle convolution and ‘scaling’.

A local system on the punctured Riemann sphere corresponds to an $r$-tuple of invertible matrices $g = (g_1, \ldots, g_r) \in \text{GL}_m(\mathbb{C})^r$. Völklein [29] and, independently, Dettweiler and Reiter [9] have defined an operation $g \mapsto \tilde{g}$ on tuples of invertible matrices over any field $K$ corresponding to the middle convolution (in [29], it is called the braid companion functor). The definition of this operation needs only simple linear algebra. Therefore, the tuple $\tilde{g}$ can be easily computed, whereas in the original work of Katz the matrices $\tilde{g}_i$ are computed only up to conjugation in $\text{GL}_m$. This construction has had many applications to the Regular Inverse Galois Problem, see e.g. [29] and [9].

The goal of the present paper is to study the parabolic cohomology of an arbitrary variation of local systems (see the beginning of this introduction), both from an analytic and from an arithmetic point of view.

In the first part of our paper, we treat the analytic aspect, i.e. we use singular cohomology. Given a variation $V$ of local systems over a base $S$, we present an effective method to compute the monodromy representation of $\pi_1(S)$ on the parabolic cohomology of $V$. The result depends on the tuple of matrices corresponding to the fibers of the variation $V$, and on the map from $\pi_1(S)$ to the Hurwitz braid group which describes how the singularities of these fibres move around on $\mathbb{P}^1$. Formally, our method is a straightforward extension of Völklein’s braid companion functor. Apart from the greater generality, the main difference to Völklein’s approach is that we provide a cohomological interpretation of our computation. Such an interpretation has the advantage that it makes it very easy, using comparison theorems, to translate results from one world into another. For instance, one can use topological methods to compute the monodromy of a local system. Then one can apply the results of this computation either to the Galois representations or to the differential equation attached to the local system in question.

We give two applications which illustrate this principle. The first example is concerned with the Regular Inverse Galois Problem and generalizes the method of [9] and [29]. Using this generalization, we prove the following theorem.

**Theorem:** The simple group $\text{PSL}_2(p^2)$ admits a regular realization as Galois group over $\mathbb{Q}(t)$, for all primes $p \neq 1, 4, 16 \mod 21$.

The only other cases where the group $\text{PSL}_2(p^2)$ is known to admit a regular realization over $\mathbb{Q}(t)$ are for $p \neq \pm 1 \mod 5$, by a result of Feit [11], and for $p \neq \pm 1 \mod 24$, by a result of Shiina [25], [26] (see also [19], [7] and [30]).

If $q < n$ then one knows that the group $\text{PSp}_{2n}(q)$ occurs regularly over $\mathbb{Q}(t)$, see [27] and [9]. Similar bounds exist also for other classical groups. On the other hand, experience shows that it is much harder to realize classical groups
of small Lie rank. The realizations of \(\text{PSL}_2(p^2)\) in the above theorem all come from one particular variation of local systems. It is very likely that, by choosing different variations, one can realize many more series of classical groups of small rank. We have chosen one particular example leading to the above theorem, because the case of rank one seems to us the hardest case.

In the last section, we compute the monodromy of the Picard–Euler equation, reproving a classical result of Picard [21],[22]. The Picard–Euler equation is the Fuchsian system of partial differential equations associated to the universal family of Picard curves (tricyclic covers of the Riemann sphere with five branch points). One can identify the local system of solutions to this equation with the parabolic cohomology of a variation of local systems. Therefore, the method developed in the first part of this paper can be used to compute the monodromy of the Picard–Euler equation. We expect that the same approach will prove useful in the study of more general differential equations. There is a vast literature dealing with special classes of Fuchsian systems, many of which arise as the parabolic cohomology of a variation. For instance, this is the case for hypergeometric systems, see [17], [6]. So far, there seemed to be no general method available to compute the monodromy of a local system coming from a variation. Here we present such a method. It is very general, explicit and can easily be implemented on a computer.

The first author would like to thank the mathematical department of the University of Tel Aviv for their hospitality during his stay in spring 2003, especially M. Jarden and D. Haran. Both authors acknowledge the financial support provided through the European Community’s Human Potential Program under contract HPRN-CT-2000-00114, GTEM.

1 Parabolic cohomology

We study the first parabolic cohomology of a local system on the punctured sphere. In particular, we show that it is isomorphic to a certain module \(W_g\), defined in [29].

1.1 Let \(X\) be a connected and locally contractible topological space. Let \(R\) be a commutative ring with unit. A local system of \(R\)-modules on \(X\) is a locally constant sheaf \(\mathcal{V}\) on \(X\) whose stalks are free \(R\)-modules of finite rank. We denote by \(\mathcal{V}_x\) the stalk of \(\mathcal{V}\) at a point \(x \in X\). If \(f : Y \to X\) is a continuous map then \(\mathcal{V}_f\) denotes the group of global sections of the sheaf \(f^*\mathcal{V}\). Note that if \(Y\) is simply connected then the natural morphism

\[
\mathcal{V}_f \longrightarrow \mathcal{V}_{f(y)}
\]

is an isomorphism, for all \(y \in Y\). Therefore, a path \(\alpha : [0, 1] \to X\) gives rise to an isomorphism

\[
\mathcal{V}_{\alpha(0)} \xrightarrow{\sim} \mathcal{V}_{\alpha(1)},
\]
obtained as the composition of the isomorphisms $V_{\alpha(0)} \cong V_\alpha$ and $V_\alpha \cong V_{\alpha(1)}$. The image of $v \in V_{\alpha(0)}$ under the above isomorphism is denoted by $v^\alpha$. It only depends on the homotopy class of $\alpha$.

Let us fix a base point $x_0 \in X$ and set $V := V_{x_0}$. We let elements of $\text{GL}(V)$ act on $V$ from the right. Then the map

$$\rho : \pi_1(X, x_0) \to \text{GL}(V),$$

defined by $v \cdot \rho(\alpha) := v^\alpha$, is a group homomorphism, i.e. a representation of $\pi_1(X, x_0)$. It is a standard fact that the functor $V \mapsto V := V_{x_0}$ is an equivalence of categories between local systems on $X$ and representations of $\pi_1(X, x_0)$.

1.2 Let $X$ be a compact (topological) surface of genus 0 and $D \subset X$ a subset of cardinality $r$. We set $U := X - D$. There exists a homeomorphism $\kappa : X \to \mathbb{P}^1_\mathbb{C}$ between $X$ and the Riemann sphere which maps the set $D$ to the real line $\mathbb{R} \subset \mathbb{P}^1_\mathbb{C}$. Such a homeomorphism is called a marking of $(X, D)$.

Let us, for the moment, identify $X$ with $\mathbb{P}^1_\mathbb{C}$ using the marking $\kappa$. Write $D = \{x_1, \ldots, x_r\}$ with $x_1 < x_2 < \ldots < x_r$ and choose a base point $x_0 \in U$ lying in the upper half plane. There is a standard presentation

$$(1) \quad \pi_1(U, x_0) = \langle \alpha_1, \ldots, \alpha_r \mid \prod_1^r \alpha_i = 1 \rangle$$

of the fundamental group of $U$, depending only on $\kappa$. The generators $\alpha_i$ is generated by a simple closed loop which intersects the real line exactly twice, first the interval $(x_{i-1}, x_i)$, then the interval $(x_i, x_{i+1})$.

Let $\mathcal{V}$ be a local system of $\mathbb{R}$-modules on $U$, corresponding to a representation $\rho : \pi_1(U, x_0) \to \text{GL}(V)$. For $i = 1, \ldots, r$, set $g_i := \rho(\alpha_i) \in \text{GL}(V)$. Then we have

$$\prod_{i=1}^r g_i = 1.$$

Conversely, given a free $\mathbb{R}$-module $V$ of finite rank and a tuple $g = (g_1, \ldots, g_r)$ of elements of $\text{GL}(V)$ satisfying the above relation, we obtain a local system $\mathcal{V}$ which induces the tuple $g$, as above.

1.3 We continue with the notation introduced in the previous subsection. Let $j : U \hookrightarrow X$ denote the inclusion. The parabolic cohomology of $\mathcal{V}$ is defined as the sheaf cohomology of $j_* \mathcal{V}$, and is written as

$$H^n_p(U, \mathcal{V}) := H^n(X, j_* \mathcal{V}).$$

We have natural morphisms $H^n_p(U, \mathcal{V}) \to H^n_c(U, \mathcal{V})$ and $H^n_p(U, \mathcal{V}) \to H^n(U, \mathcal{V})$ ($H_c$ denotes cohomology with compact support).

**Proposition 1.1**

(i) The group $H^n(U, \mathcal{V})$ is canonically isomorphic to the group cohomology $H^n(\pi_1(U, x_0), V)$. In particular, we have

$$H^0(U, \mathcal{V}) \cong V^{\langle g_1, \ldots, g_r \rangle},$$
and \( H^n(U, V) = 0 \) for \( n > 1 \).

(ii) The map \( H^1_c(U, V) \to H^1_p(U, V) \) is surjective and the map \( H^1_p(U, V) \to H^1(U, V) \) is injective. In other words, \( H^1_p(U, V) \) is the image of the cohomology with compact support in \( H^1(U, V) \).

**Proof:** Part (i) follows from the Hochschild–Serre spectral sequence and the fact that the universal cover of \( U \) is contractible. For (ii), see e.g. [18], Lemma 5.3.

Let \( \delta : \pi_1(U) \to V \) be a cocycle, i.e. we have \( \delta(\alpha \beta) = \delta(\alpha) \cdot \rho(\beta) + \delta(\beta) \). Set \( v_i := \delta(\alpha_i) \). It is clear that the tuple \( (v_i) \) is subject to the relation

\[
(2) \quad v_1 \cdot g_2 \cdots g_r + v_2 \cdot g_3 \cdots g_r + \ldots + v_r = 0.
\]

Conversely, any tuple \( (v_i) \) satisfying (2) gives rise to a unique cocycle \( \delta \). This cocycle is a coboundary if and only if there exists \( v \in V \) such that \( v_i = v \cdot (g_i - 1) \) for all \( i \). By Proposition 1.1 there is a natural inclusion

\[
H^1_p(U, V) \hookrightarrow H^1(\pi_1(U), V).
\]

We say that \( \delta \) is a parabolic cocycle if the class of \( \delta \) in \( H^1(\pi_1(U), V) \) lies in the image of \( H^1_p(U, V) \).

**Lemma 1.2** The cocycle \( \delta \) is parabolic if and only if \( v_i \) lies in the image of \( g_i - 1 \), for all \( i \).

**Proof:** Let \( U_i \subset X \) be pairwise disjoint disks with center \( x_i \), and set \( U_i^* := U_i - \{x_i\} \). We have a long exact sequence

\[
(3) \quad \cdots \to H^n_{x_i}(U_i, (j_i V)|U_i) \to H^n(U_i, (j_i V)|U_i) \to H^n(U_i^*, V|U_i^*) \to \cdots.
\]

Given a class \( c \) in \( H^n(U_i, (j_i V)|U_i) \) we can find a smaller disk \( U_i' \subset U_i \) with center \( x_i \) such that the restriction of \( c \) to \( U_i' \) vanishes (one way to see this is to use Čech cohomology). On the other hand, the cohomology groups \( H^n(U_i, (j_i V)|U_i) \) and \( H^n(U_i^*, V|U_i^*) \) do not change if we shrink the disk \( U_i \). Therefore, by the exactness of (3) we have \( H^n(U_i, (j_i V)|U_i) = 0 \) and hence

\[
(4) \quad H^n_{x_i}(U_i, (j_i V)|U_i) \cong H^{n-1}(U_i^*, V|U_i^*) \cong \begin{cases} \ker(g_i - 1), & n = 1, \\ \text{coker}(g_i - 1), & n = 2, \\ 0, & \text{otherwise}. \end{cases}
\]

For the second isomorphism we have used \( H^{n-1}(U_i^*, V|U_i^*) \cong H^{n-1}(\pi_1(U_i), V) \).

Consider the long exact sequence

\[
(5) \quad \cdots \to H^2_p(X, j V) \to H^2_p(U, V) \to H^2(U, V) \to \cdots.
\]

By (4), the image in \( H^2_p(X, j V) \) of the class of a 1-cocycle \( \delta : \pi_1(U) \to V \) vanishes if and only \( v_i := \delta(\alpha_i) \in \text{Im}(g_i - 1) \). The lemma follows now from the exactness of (5) and from Proposition 1.1 (ii). \( \square \)
The preceding lemma shows that the association \( \delta \mapsto (v_i) \) yields an isomorphism
\[
H_1^p(U, V) \cong W_{\mathbf{g}} := H_{\mathbf{g}} / E_{\mathbf{g}}.
\]
where
\[
H_{\mathbf{g}} := \{ (v_1, \ldots, v_r) \mid v_i \in \text{Im}(g_i - 1), \text{relation (2) holds} \}
\]
and
\[
E_{\mathbf{g}} := \{ (v \cdot (g_1 - 1), \ldots, v \cdot (g_r - 1)) \mid v \in V \}.
\]
The \( R \)-module \( W_{\mathbf{g}} \) has already been defined in [29], where it is called the braid companion of \( V \).

**Remark 1.3** Suppose that \( R = K \) is a field and that the stabilizer \( V_{\pi_1(U)} \) is trivial. Then the Ogg-Shafarevic formula implies the following dimension formula:
\[
dim_K H_1^p(U, V) = (r - 2) \dim_K V - \sum_{i=1}^r \dim_K \text{Ker}(g_i - 1).
\]
This formula can also be verified directly using the isomorphism \( H_1^p(U, V) \cong W_{\mathbf{g}} \).

## 2 Variation of a local system

We study variations of local systems on the punctured sphere, with moving singularities. The main result is the computation of the monodromy of the parabolic cohomology of the variation. This computation is based on a natural generalization of results of Völklein [28] [29].

**2.1** Let \( S \) be a connected complex manifold, and \( r \geq 3 \). An \( r \)-configuration over \( S \) consists of a smooth and proper morphism \( \bar{\pi} : X \rightarrow S \) of complex manifolds together with a smooth relative divisor \( D \subset X \) such that the following holds. For all \( s \in S \) the fiber \( X_s := \bar{\pi}^{-1}(s) \) is a Riemann surface of genus 0, and the divisor \( D \cap X_s \) consists of \( r \) pairwise distinct points \( x_1, \ldots, x_r \).

Let us fix an \( r \)-configuration \( (X, D) \) over \( S \). We set \( U := X - D \) and denote by \( j : U \hookrightarrow X \) the natural inclusion. Also, we write \( \pi : U \rightarrow S \) for the natural projection. Choose a base point \( s_0 \in S \) and set \( X_0 := \bar{\pi}^{-1}(s_0) \) and \( D_0 := X_0 \cap D \). Write \( D_0 = \{x_1, \ldots, x_r\} \) and \( U_0 := X_0 - D_0 = \bar{\pi}^{-1}(s_0) \). Choose a base point \( x_0 \in U_0 \). The projection \( \pi : U \rightarrow S \) is a topological fibration and yields a short exact sequence
\[
1 \rightarrow \pi_1(U_0, x_0) \rightarrow \pi_1(U, x_0) \rightarrow \pi_1(S, s_0) \rightarrow 1.
\]
(6)

From now on, we shall drop the base points from our notation. Let \( \mathcal{V}_0 \) be a local system of \( R \)-modules on \( U_0 \), corresponding to a representation \( \rho_0 : \pi_1(U_0) \rightarrow \text{GL}(V) \), as in §1.2.
Definition 2.1 A variation of $\mathcal{V}_0$ over $S$ is a local system $\mathcal{V}$ of $R$-modules on $U$ whose restriction to $U_0$ is identified with $\mathcal{V}_0$. The parabolic cohomology of a variation $\mathcal{V}$ is the higher direct image sheaf

$$W := R^1\pi_* (j_* \mathcal{V}).$$

A variation $\mathcal{V}$ of $\mathcal{V}_0$ corresponds to a representation $\rho : \pi_1(U) \to GL(V)$ whose restriction to $\pi_1(U_0)$ is equal to $\rho_0$. By definition, the parabolic cohomology $W$ of the variation $\mathcal{V}$ is a sheaf of $R$-modules on $S$. Locally on $S$, the configuration $(X, D)$ is topologically trivial, i.e., there exists a homeomorphism $X \cong X_0 \times S$ which maps $D$ to $D_0 \times S$. It follows immediately that $W$ is a local system with fibre

$$W := H^1_p(U_0, \mathcal{V}_0).$$

In other words, $W$ corresponds to a representation $\eta : \pi_1(S) \to GL(W)$. The following lemma provides a description of $\eta$ in terms of cocycles.

Lemma 2.2 Let $\beta \in \pi_1(S)$ and $\delta : \pi_1(U_0) \to V$ be a parabolic cocycle. We write $[\delta]$ for the class of $\delta$ in $W$. Let $\tilde{\beta} \in \pi_1(U)$ be a lift of $\beta$. Then $[\delta]^{\eta(\tilde{\beta})} = [\delta']$, where $\delta' : \pi_1(U_0) \to V$ is the cocycle

$$\alpha \mapsto \delta(\tilde{\beta} \alpha \tilde{\beta}^{-1}) \cdot \rho(\tilde{\beta}), \quad \alpha \in \pi_1(U_0).$$

Proof: We consider $\beta$ as a continuous map $\beta : I := [0,1] \to S$. Since $I$ is simply connected, there exists a continuous family of homeomorphisms $\tilde{\phi}_t : X_0 \to X_t := \pi^{-1}(t)$, for $t \in I$, such that $\tilde{\phi}_0(D_0) = D_t := X_t \cap D$ and such that $\phi_0$ is the identity. Let $\phi_t$ denote the restriction of $\tilde{\phi}_t$ to $U_0$. Note that $\phi_1 : U_0 \cong U_0$ is a homeomorphism of $U_0$ with itself, whose homotopy class depends only on $\beta \in \pi_1(S)$. We may further assume that $\phi_1(x_0) = x_0$. Then $\tilde{\beta} : t \mapsto \phi_t(x_0)$ is a closed path in $U$ with base point $x_0$. The class of $\tilde{\beta}$ in $\pi_1(U)$ (which we also denote by $\tilde{\beta}$) is a lift of $\beta \in \pi_1(S)$. It is easy to check that for all $\alpha \in \pi_1(U_0)$ we have

$$\phi_1(\alpha) = \tilde{\beta}^{-1} \alpha \tilde{\beta}.$$

Since $\mathcal{V}$ is a local system on $U$, there exists a unique continuous family $\psi_t : \mathcal{V}_0 \cong \phi_1^*(\mathcal{V}|_{U_t})$ of isomorphisms of local systems on $U_0$ such that $\psi_0$ is the identity on $\mathcal{V}_0$. Evaluation of $\psi_t$ at the point $\tilde{\beta}(t) = \phi_t(x_0)$ yields a continuous family of isomorphism $\psi_t(x_0) : V \cong \mathcal{V}_0$. This family corresponds to a trivialization of $\bar{\beta}^* \mathcal{V}$, and we get

$$\rho(\tilde{\beta}) = \psi_1(x_0).$$

The pair $(\phi_t, \psi_t)$ induces a continuous family of isomorphisms

$$\lambda_t : W \cong W_t = H^1_p(U_t, \mathcal{V}|_{U_t}).$$

Using (7) and (8), one finds that $\lambda_t([\delta]) = [\delta']$, where

$$\delta'(\alpha) = \psi_1(\delta(\phi_1^{-1}(\alpha))) = \delta(\tilde{\beta} \alpha \tilde{\beta}^{-1}) \cdot \rho(\tilde{\beta}).$$
By definition of the representation $\eta$, we have $[\delta]^{\eta(\beta)} = \lambda_1([\delta]) = [\delta']$. This completes the proof of the lemma.

\[\square\]

Remark 2.3 With the notation introduced above: let $\mathcal{V}_0$ be local system of $R$-modules, corresponding to a representation $\rho_0 : \pi_1(U_0) \to GL(V)$.

(i) A necessary condition for the existence of a variation of $V_0$ over $S$ is the following. For every element $\tilde{\beta} \in \pi_1(U_0)$ there exists an element $g \in GL(V)$ such that

$$\rho_0(\tilde{\beta} \alpha \tilde{\beta}^{-1}) = gp_0(\alpha)g^{-1}$$

holds for all $\alpha \in \pi_1(U_0)$.

(ii) Suppose that $S$ is a smooth affine curve, and that (i) holds. Then there exists a variation $V$ of $V_0$ over $S$. This follows easily from the fact that $\pi_1(S)$ is a free group.

(iii) Suppose, moreover, that $R$ is an integral domain and that $V_0$ is irreducible. If $V'$ is another variation of $V_0$ over $S$, then there exists a local system $L$ of rank one on $S$ such that $V' \cong V \otimes R \pi^* L$. Let $W$ (resp. $W'$) denote the parabolic cohomology of $V$ (resp. of $V'$). By the projection formula we have

$$W' \cong R^1 \tilde{\pi}_*(j_* V \otimes \tilde{\pi}^* L) \cong W \otimes L.$$ 

Therefore, the projective representation associated to $W$,

$$\lambda : \pi_1(S) \to \text{PGL}(W),$$

is uniquely determined by $V_0$.

2.2 The Artin braid group and the cocycles $\Phi(g, \beta)$ Let $D_0 \subset \mathbb{C}$ be a set of $r$ distinct complex numbers and set $U_0 := \mathbb{P}^1_{\mathbb{C}} - D_0$. We choose a marking $\kappa$ of $(\mathbb{P}^1_{\mathbb{C}}, D_0)$ which maps $\infty$ into the upper half plane, see §1.2. The choice of $\kappa$ induces a presentation of $\pi_1(U_0, \infty)$, with generators $\alpha_1, \ldots, \alpha_r$ and relation $\prod_i \alpha_i = 1$.

Define

$$\mathcal{O}_r := \{ D \subset \mathbb{C} \mid |D| = r \}.$$ 

The fundamental group $A_r := \pi_1(\mathcal{O}_r, D_0)$ is called the Artin braid group on $r$ strands. The group $A_r$ has $r-1$ standard generators $\beta_1, \ldots, \beta_{r-1}$ with relations

$$\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}, \quad \beta_i \beta_j = \beta_j \beta_i,$$

for $1 \leq i < r$ and $i < j - 1 < r - 1$. The element $\beta_i$ is represented by the path $t \mapsto \{x_1, \ldots, \delta_i^-(t), \delta_i^+(t), \ldots, x_r\}$, where $\delta_i^+$ (resp. $\delta_i^-$) is a path from $x_{i+1}$ to $x_i$ through the inverse image under $\kappa$ of the upper half plane (resp. from $x_i$ to $x_{i+1}$ through the inverse image of the lower half plane).
Define
\[ \mathcal{O}_{r,1} := \{ (D, x) \mid D \in \mathcal{O}_r, x \in \mathbb{P}_C^1 - D \}. \]
The natural projection \( \mathcal{O}_{r,1} \to \mathcal{O}_r \) is a topological fibration with fiber \( U_0 \), and admits a section \( D \mapsto (D, \infty) \). It yields a split exact sequence of fundamental groups
\[ 1 \to \pi_1(U_0, \infty) \to \pi_1(\mathcal{O}_{r,1}, (D_0, \infty)) \to A_r \to 1. \]
We may identify \( A_r \) with its image in \( \pi_1(\mathcal{O}_{r,1}) \) under the splitting induced from the section \( D \mapsto (D, \infty) \). Then \( A_r \) acts, by conjugation, on \( \pi_1(U_0, \infty) \). We have the following well known formulas for this action:
\[ \beta_i^{-1} \alpha_j \beta_i = \begin{cases} \alpha_i \alpha_{i+1} \alpha_i^{-1}, & \text{for } j = i, \\ \alpha_i, & \text{for } j = i + 1, \\ \alpha_j, & \text{otherwise}. \end{cases} \]

Let \( R \) be a commutative ring and \( V \) a free \( R \)-module of finite rank. Define
\[ \mathcal{E}_r := \{ (g = (g_1, \ldots, g_r)) \mid g_i \in \text{GL}(V), \prod_i g_i = 1 \}. \]
An element \( g \in \mathcal{E}_r \) corresponds to a representation \( \rho_0 : \pi_1(U_0) \to \text{GL}(V) \) (set \( \rho_0(\alpha_i) := g_i \)) and hence to a local system \( V_0 \) on \( U_0 \). Given \( \beta \in A_r \), we set
\[ \rho_0^\beta(\alpha) := \rho_0(\beta \alpha \beta^{-1}) \]
and call the local system \( V_0^\beta \) corresponding to the representation \( \rho_0^\beta \) the twist of \( V_0 \) by \( \beta \). We denote by \( g^\beta \) the element of \( \mathcal{E}_r \) corresponding to \( \rho_0^\beta \). This defines an action of \( A_r \) on \( \mathcal{E}_r \), from the right. From (10) we get the following formula for the effect of the standard generators \( \beta_i \) on \( \mathcal{E}_r \):
\[ g^\beta_i = (g_1, \ldots, g_{i+1}, g_i^{-1}, g_{i+1}, \ldots, g_r). \]

Given \( g \in \mathcal{E}_r \), we have defined in §1.3 the \( R \)-module
\[ H_g = \{ (v_1, \ldots, v_r) \mid v_i \in \text{Im}(g_i - 1), \text{ relation (2) holds} \}. \]
An element \( (v_1, \ldots, v_r) \in H_g \) corresponds to a parabolic cocycle \( \delta : \pi_1(U_0) \to V \), determined by \( \delta(\alpha_i) = v_i \), for \( i = 1, \ldots, r \). Here the \( \pi_1(U_0) \)-module structure on \( V \) is induced by \( g \). We say that \( \delta \) is a parabolic cocycle with respect to \( g \). Given \( \beta \in A_r \) and \( \alpha \in \pi_1(U_0) \), set
\[ \delta^\beta(\alpha) := \delta(\beta \alpha \beta^{-1}). \]
One easily checks that \( \delta^\beta : \pi_1(U_0) \to V \) is a parabolic cocycle with respect to \( g^\beta \). Moreover, the association \( \delta \mapsto \delta^\beta \) defines an \( R \)-linear map
\[ \Phi(g, \beta) : H_g \to H_{g^\beta}. \]
In order to maintain compatibility with our convention of ‘acting from the right’, we write \((v_i)_{\Phi(g, \beta)}\) for the image of \((v_i) \in H_g\) under \(\Phi(g, \beta)\). Using (10) and the fact that \(\delta\) is a cocycle with respect to \(g\), we get
\[
(v_1, \ldots, v_r)_{\Phi(g, \beta)} = (v_1, \ldots, v_i, v_{i+1}(1 - g_{i+1}^{-1}g_i) + v_i g_{i+1}, \ldots, v_r)_{(i+1)\text{th entry}}.
\]
Moreover, we have the ‘cocycle rule’
\[
\Phi(g, \beta) \cdot \Phi(g^\beta, \beta') = \Phi(g, \beta \beta').
\]
(The product on the left hand side of (14) is defined as the function from \(H_g\) to \(H_{g^\beta}^{\beta'}\) obtained by first applying \(\Phi(g, \beta)\) and then \(\Phi(g^\beta, \beta')\).)

The submodule
\[
E_g := \{ (v \cdot (g_1 - 1), \ldots, v \cdot (g_r - 1) \mid v \in V \}
\]
of \(H_g\) corresponds to cocycles \(\delta\) which are coboundaries. It is easy to see that \(\Phi(g, \beta)\) maps \(E_g\) into \(E_{g^\beta}\) and therefore induces an isomorphism
\[
\Psi(g, \beta) : W_g := H_g/E_g \sim W_{g^\beta}.
\]
One can compute \(\Psi(g, \beta)\) explicitly for all \(\beta \in A_r\) using (13) and (14), provided that \(\beta\) is given as a word in the standard generators \(\beta_i\). Moreover, this computation can easily be implemented on a computer.

Given \(g \in E_r\) and \(h \in \text{GL}(V)\), we set
\[
g^h = (h^{-1}g_1h, \ldots, h^{-1}g_rh),
\]
and we define an isomorphism
\[
\Psi(g, h) : \begin{cases}
H_{g^h} & \sim H_g \\
(v_1, \ldots, v_r) & \mapsto (v_1 \cdot h, \ldots, v_r \cdot h).
\end{cases}
\]
It is clear that \(\Psi(g, h)\) maps \(E_{g^h}\) to \(E_g\) and therefore induces an isomorphism
\[
\Psi(g, h) : W_{g^h} \sim W_g.
\]

2.3 Explicit computation of the monodromy Let us go back to the situation of §2.1: we are given an \(r\)-configuration \((X, D)\) over a connected complex manifold \(S\). We have also chosen a base point \(s_0 \in S\). As usual, we set \(U := X - D\), and denote by \(U_0\) the fiber of \(U \to S\) over \(s_0\).

**Definition 2.4** An **affine frame** for the configuration \((X, D)\) is an isomorphism of complex manifolds \(\lambda : X \cong \mathbb{P}^1_S\), compatible with the projection to \(S\), such that \(\lambda(D)\) either contains or is disjoint from \(\{\infty\} \times S\).
In this subsection, we shall assume that there exists an affine frame for $(X, D)$, and we use it to identify $X$ with $\mathbb{P}_C^1$. We remark that there exist configurations $(X, D)$ which do not admit an affine frame (e.g. because $X \not\cong \mathbb{P}_C^1$). It seems, however, that such examples have no practical relevance for the problems this paper is about.

By the nature of Definition 2.4, there are two cases to consider. Suppose first that $D$ is disjoint from $\{\infty\} \times S$. Then $D$ gives rise to a map $p : S \to \mathcal{O}_r$ which sends $s \in S$ to the fiber of $D \subset \mathbb{A}_C^1 \to S$ over $s$. Set $D_0 := p(s_0) \subset C$. Choose a marking $\kappa$ of $(\mathbb{P}_C^1, D_0)$. We will use $\kappa$ to identify the fundamental group $\pi_1(\mathcal{O}_r, D_0)$ with the Artin braid group $A_r$, as in the previous subsection. Let $\varphi : \pi_1(S, s_0) \to A_r$ denote the group homomorphism induced by $p$. The exact sequence

$$1 \to \pi_1(U_0, \infty) \to \pi_1(U, (\infty, s_0)) \to \pi_1(S, s_0) \to 1$$

of the fibration $U \to S$ can be identified with the pullback of the sequence (9) along $\varphi$. Using the splitting of (15) coming from the $\infty$-section, we will consider $\pi_1(S)$ as a subgroup of $\pi_1(U)$. By construction, the action of $\pi_1(S)$ on $\pi_1(U_0, \infty)$ factors through the map $\varphi$ and is given by the formulas (10).

Now suppose that $D$ contains the section $\{\infty\} \times S$. We denote by $\pi_1(U_0, \infty)$ the fundamental group of $U_0$ with $\infty$ as ‘tangential base point’. More precisely, consider subsets of $U_0 \subset C$ of the form $\Omega_t = \{ z \in \mathbb{C} \mid |z| > t, z \not\in (-\infty, 0) \}$, for $t \gg 0$. The fundamental group $\pi_1(U_0, \Omega_t)$ is independent of $t$, up to canonical isomorphism, so we may define $\pi_1(U_0, \infty) := \lim_{t \to \infty} \pi_1(U_0, \Omega_t)$. With this convention, the sequence (15) is still well defined and admits a canonical section. In fact, the fibration $U \to S$ admits a section $\xi : S \to U$, unique up to homotopy, such that for all $s \in S$ we have $\xi(s) \in \Omega_t \subset U_s$, for some $t > 0$. As in the first case, we will identify $\pi_1(S)$ with the image of this section.

The Hurwitz braid group $B_r$ is defined as the fundamental group of the set

$$U_r = \{ D \subset \mathbb{P}_C^1 \mid |D| = r \},$$

with base point $D_0$. The natural map $\mathcal{O}_r \to U_r$ identifies $B_r$ with the quotient of $A_r$ by the relation

$$\beta_1 \beta_2 \cdots \beta_{r-1} \cdots \beta_2 \beta_1 = 1.$$

The configuration $(X, D)$ induces a map $p : S \to U_r$ and a homomorphism $\varphi : \pi_1(S) \to B_r$. If $\{\infty\} \times S \subset D$ then the image of $\varphi$ is contained in the subgroup of $B_r$ generated by the first $r-2$ standard braids $\beta_1, \ldots, \beta_{r-2}$, which is isomorphic to $A_{r-1}$. Moreover, just as in the first case, the action of $\pi_1(S, s_0)$ on $\pi_1(U_0, \infty)$ by conjugation factors through the map $\varphi$ and is given by the formulas (10). From now on, we will treat both cases of Definition 2.4 simultaneously.

Let $V_0$ be a local system of free $R$-modules on $U_0 = \mathbb{P}_C^1 \setminus D_0$, corresponding to a representation $\rho_0 : \pi_1(U_0) \to \text{GL}(V)$. A variation of $V_0$ over $S$ corresponds, by definition, to a representation $\rho : \pi_1(U) \to \text{GL}(V)$ whose restriction to $\pi_1(U_0)$ equals $\rho_0$. Obviously, $\rho$ is uniquely determined by its restriction to $\pi_1(S)$, which we denote by $\chi : \pi_1(S) \to \text{GL}(V)$. Then

$$\rho_0(\gamma \alpha \gamma^{-1}) = \chi(\gamma) \rho_0(\alpha) \chi(\gamma)^{-1}$$

(16)
holds for all $\alpha \in \pi_1(U_0)$ and $\gamma \in \pi_1(S)$. With $g \in \mathcal{E}_r$ corresponding to $\rho_0$ (via the choice of the marking $\kappa$), this is equivalent to

\begin{equation}
(17) \quad g^{\varphi(\gamma)} = g^{\chi(\gamma)^{-1}}.
\end{equation}

Let $\mathcal{W}$ be the parabolic cohomology of $\mathcal{V}$ and $\eta : \pi_1(S) \to \text{GL}(W_g)$ the corresponding representation (here we identify the fiber of $\mathcal{W}$ at $s_0$ with the $R$-module $W_g = H_g/E_g$, see the previous subsection).

**Theorem 2.5** For all $\gamma \in \pi_1(S)$ we have

$$
\eta(\gamma) = \tilde{\Phi}(g, \varphi(\gamma)) \cdot \tilde{\Psi}(g, \chi(\gamma)),
$$

where $\tilde{\Phi}(g, \beta) : W_g \xrightarrow{\sim} W_{g^\beta}$ and $\tilde{\Psi}(g, h) : W_{g^h} \to W_g$ are the isomorphisms defined in §2.2.

**Proof:** Straightforward, using Lemma 2.2, the definition of $\tilde{\Phi}(g, \beta)$ and $\tilde{\Psi}(g, h)$, and (17).

\[ \square \]

3 Étale local systems

We transfer the situation considered in the first two sections into the étale world, and we state a comparison theorem. We also prove a theorem which is useful to bound the field of linear moduli of an (étale) local system which is obtained as the parabolic cohomology of a variation.

3.1 Recall In this section, we fix a prime number $l$ and a finite extension $K/\mathbb{Q}_l$. We denote by $R$ one of the following rings: (a) $R := K$, (b) $R := \mathcal{O}_K$, the ring of integers of $K$, or (c) $R := \mathcal{O}_K/\ell^m$, where $\ell$ is the prime ideal of $\mathcal{O}_K$.

Let $k$ be a field of characteristic 0 and $X$ a smooth, geometrically irreducible scheme over $k$. Also, let $x : \text{Spec} \ k \to S$ be a geometric point. We denote by $\pi_1(X) = \pi_1(X, x)$ the algebraic fundamental group of $X$ with base point $x$.

An étale local system $\mathcal{V}$ of $R$-modules on $X$ is, by definition, a locally constant and constructible sheaf of $R$-modules [20], whose stalks are free $R$-modules of finite rank. In case $R = K$, this is also called a lisse $\ell$-adic sheaf [14]. It is a standard fact that $\mathcal{V}$ corresponds to a continuous representation

$$
\rho : \pi_1(X, x) \longrightarrow \text{GL}(V),
$$

where $V := V_x$ is the stalk of $\mathcal{V}$ at $x$.

Now suppose that $k \subset \mathbb{C}$ is a subfield of the complex numbers. The set of $\mathbb{C}$-rational points of $X$ has a canonical structure of a complex manifold, which we denote by $X^{\text{an}}$. Moreover, there is a functor $\mathcal{F} \mapsto \mathcal{F}^{\text{an}}$ from sheaves (of abelian groups) on $X_{\text{et}}$ to sheaves on $X^{\text{an}}$, called analytification (see e.g. [12], §I.11). If $\mathcal{V}$ is an étale local system on $X$ corresponding to a representation $\rho : \pi_1(X, x) \to \text{GL}(V)$, then the analytification $\mathcal{V}^{\text{an}}$ of $\mathcal{V}$ is the local system corresponding to the composition of $\rho$ with the natural homomorphism $\pi_1^{\text{top}}(X^{\text{an}}, x) \to \pi_1(X, x)$. 

12
3.2 Parabolic cohomology of an étale local system  Let \( k \) be a field of characteristic 0 and let \( S \) be a smooth, affine and geometrically connected variety over \( k \). Let \((X, D)\) be an \( r\)-configuration over \( S \). By this we mean that \( \bar{\pi} : X \to S \) is a proper smooth curve of genus 0 and \( D \subset X \) is a smooth relative divisor of relative degree \( r \) (compare with §2.1). We denote by \( j : U := X - D \to X \) the inclusion and by \( \pi : U \to S \) the natural projections. We fix a \( k \)-rational point \( s_0 \) on \( S \) as a base point. We write \( U_0 := \pi^{-1}(s_0) \) for the fiber of \( \pi \) over \( s_0 \) and we choose a geometric point \( x_0 : \text{Spec} \, \bar{k} \to U_0 \) as base point.

**Definition 3.1**  Let \( V_0 \) be an étale local system of \( R \)-modules on \( U_0 \). A variation of \( V_0 \) over \( S \) is an étale local system \( V \) on \( U \) whose restriction to \( U_0 \) is equal to \( V_0 \). The parabolic cohomology of the variation \( V \) is the sheaf of \( R \)-modules on \( S_{\text{ét}} \)

\[
\mathcal{W} := R^1\bar{\pi}_*(j_*V).
\]

See [20].

**Theorem 3.2**  Suppose that \( k \subset \mathbb{C} \).

(i) \( \mathcal{W} \) is an étale local system of \( R \)-modules.

(ii) There is a natural isomorphism of local systems of \( R \)-modules on \( X^{\text{an}} \)

\[
\mathcal{W}^{\text{an}} \cong R^1\bar{\pi}_*(j_*\mathcal{V}^{\text{an}}),
\]

functorial in \( \mathcal{V} \).

**Proof:** Using standard arguments (see e.g. [12], §12), one reduces the claim to the case \( R = O_K/\ell^m \). Let \( \mathcal{F} \) be a constructible sheaf of \( R \)-modules on \( X \). By the comparison theorem between étale and singular cohomology there is a natural isomorphism of sheaves on \( S^{\text{an}} \)

\[
(R^1\bar{\pi}_*\mathcal{F})^{\text{an}} \cong R^1\bar{\pi}_*(\mathcal{F}^{\text{an}}).
\]

(See e.g. [12], Theorem 11.6, for the case where \( k = \mathbb{C} \). The general case follows immediately, using the Proper Base Change Theorem, [12], Theorem 6.1.) It is easy to see (e.g. using [12], Proposition 11.4) that \( (j_*\mathcal{V})^{\text{an}} = j_*\mathcal{V}^{\text{an}} \). Therefore, Part (ii) of the theorem follows from the comparison theorem. By [12], Theorem 8.10, the sheaf \( \mathcal{W} = R^1\bar{\pi}_*(j_*\mathcal{V}) \) is constructible. But we have just proved that \( \mathcal{W}^{\text{an}} \) is locally constant, which shows that \( \mathcal{W} \) is locally constant as well. This finishes the proof of the theorem. \( \square \)

3.3 The field of linear moduli  As in the previous subsection, \( S \) denotes a smooth and geometrically connected \( k \)-variety and \((X, D)\) an \( r \)-configuration over \( S \). We assume that \( k \subset \mathbb{C} \) and denote by \( \bar{k} \) the algebraic closure of \( k \) inside \( \mathbb{C} \).

Unlike in the previous subsection, \( \mathcal{V}_0 \) is now an étale local system of \( R \)-modules on the geometric fibre \( U_{0,\bar{k}} := U_0 \otimes \bar{k} \), and \( \mathcal{V} \) denotes a variation of \( \mathcal{V}_0 \).
over $S_k := S \otimes_k \bar{k}$. Let $W$ be the parabolic cohomology of $V$. By construction, $W$ is an étale local system of $R$-modules on $S_k$. For $\sigma \in \text{Gal}(\bar{k}/k)$, we denote by $\hat{\sigma} : U_{0, k} \to U_{0, \bar{k}}$ the semi-linear automorphism corresponding to $\sigma$ and the $k$-model $U_0$. The twist of $V_0$ by $\sigma$ (with respect to the $k$-model $U_0$) is the étale local system $V_0^\sigma := \hat{\sigma}^* U_0 V_0$.

**Definition 3.3** We say that $k$ is a field of linear moduli for $V_0$ if the étale local system $V_0^\sigma$ is isomorphic to $V_0$, for all $\sigma \in \text{Gal}(\bar{k}/k)$. We say that $k$ is a field of projective moduli for $V_0$ if for all $\sigma \in \text{Gal}(\bar{k}/k)$ there exists an étale local system $L_\sigma$ of rank one such that $V_0^\sigma \cong V_0 \otimes_R L_\sigma$. Similarly, one defines the notion of ‘field of linear/projective moduli’ for the local systems $V$ and $W$.

**Theorem 3.4**

(i) If $k$ is a field of linear moduli for $V$ then it is also a field of linear moduli for $W$.

(ii) Suppose that $R = O_K$ or $R = K$, and that $V_0$ is irreducible. Then if $k$ is a field of projective moduli for $V_0$, it is also a field of projective moduli for $W$. Moreover, the projective representation $\lambda^\text{geo} : \pi_1(S_k) \to \text{PGL}(W)$ associated to $W$ extends to a projective representation $\lambda : \pi_1(S) \to \text{PGL}(W)$.

**Proof:** Let $\sigma \in \Gamma_{k_0}$ with $V^\sigma \cong V$. Using the Proper Base Change Theorem we get

$$W^\sigma = \hat{\sigma}^* R^1 \bar{\pi}_{k, *} (j_{k, *}(V)) = R^1 \bar{\pi}_{k, *} (j_{k, *}(V^\sigma)) \cong W.$$  

This proves (i). The proof of (ii) is a combination of the preceeding argument and Remark 2.3 (iii). $\square$

**Remark 3.5** Theorem 3.4 can be used to give a new proof of the main result of [30] (which essentially states that the braid companion functor preserves the field of linear moduli). In the rest of the paper, we will not use Theorem 3.4 and we will not need the concept ‘field of linear moduli’. The point is that in our main example in §5, the variation $V$ is already known to be defined over $\mathbb{Q}$, which means that the resulting local system $W$ is defined over $\mathbb{Q}$ as well, by construction. In fact, this seems to be the case for all known applications of these and similar methods to the Regular Inverse Galois Problem. Nevertheless, the authors think that Theorem 3.4 may be useful for future applications.

## 4 Local systems on Hurwitz curves

A finite Galois cover $f : Y \to \mathbb{P}^1$ together with a representation $G \to \text{GL}_n(K)$ of its Galois group corresponds to a local system on $\mathbb{P}^1$ with finite monodromy. Therefore, a representation $G \to \text{GL}_n(K)$ gives rise to a variation of local systems on a certain Hurwitz space $H$. Since Hurwitz spaces are algebraic varieties, the parabolic cohomology of this variation corresponds to a Galois representations of the function field of $H$. In case $H$ is a rational variety, this has potential applications to the Regular Inverse Galois Problem.
4.1 In this section we fix a finite group $G$. For each integer $r \geq 3$, we set

$$\mathcal{E}_r(G) := \{ g = (g_1, \ldots, g_r) \mid G = \langle g_i \rangle, \prod g_i = 1 \}.$$ 

An element $g$ of this set is called a generating system of length $r$ for the group $G$. The group $G$ acts on the set $\mathcal{E}_r(G)$ by simultaneous conjugation. We write $\mathrm{Ni}_r(G)$ for the sets of orbits of this action. Elements of $\mathrm{Ni}_r(G)$ are called Nielsen classes and written as $[g]$, with $g \in \mathcal{E}_r(G)$.

The Artin braid group $A_r$ acts on the set $\mathcal{E}_r(G)$ from the right, in a standard way, see e.g. [13] and §2.2. This action extends to an action of its quotient $B_r$, the Hurwitz braid group. By abuse of notation, we denote the image of the standard generator $\beta_i \in A_r$ in $B_r$ by the same name.

Suppose, for the moment, that $r = 4$. The elements $\beta_1, \beta_2, \beta_3$ generate a normal subgroup $Q \trianglelefteq B_4$, isomorphic to the Klein 4-group. The quotient $B_4 := B_4/Q$ is called the mapping class group. The set of $Q$-orbits of $\mathrm{Ni}_r(G)$ is denoted by $\mathrm{Ni}_r^\text{red}(G)$. Elements of this set are called reduced Nielsen classes, and are written as $[g]^\text{red}$. The action of $B_4$ on $\mathrm{Ni}_4(G)$ descents to an action of the mapping class group $B_4$ on $\mathrm{Ni}_4^\text{red}(G)$.

Let $C = (C_1, \ldots, C_r)$ be an ordered $r$-tuple of conjugacy classes of the group $G$. We say that $g \in \mathcal{E}_r(G)$ has type $C$ if there exist an integer $n$, prime to the order of $G$, and a permutation $\sigma \in S_r$ such that $g_i^n \in C_{\sigma(i)}$ for $i = 1, \ldots, r$. The subset of $\mathcal{E}_r(G)$ of all elements of type $C$ is denoted by $\mathcal{E}(C)$. We also obtain subsets $\mathrm{Ni}(C) \subset \mathrm{Ni}_r(G)$ and $\mathrm{Ni}_r^\text{red}(C) \subset \mathrm{Ni}_r^\text{red}(G)$.

4.2 Let $T$ be a scheme over $\text{Spec}(\mathbb{Q})$ and $X$ a smooth projective curve over $T$ of genus 0. A $G$-cover of $X$ is a finite morphism $f : Y \to X$, together with an isomorphism $G \cong \text{Aut}(Y/X)$, such that the following holds. First, $f$ is tamely ramified along a smooth relative divisor $D \subset X$ with constant degree $r := \text{deg}(D/S)$, and étale over $U := X - D$. Second, for each geometric point $t : \text{Spec} k \to T$, the pullback $f_t : Y_t \to X_t$ is a $G$-Galois cover (in particular, $Y_t$ is connected). We say that two $G$-covers $f_1 : Y_1 \to X_1$ and $f_2 : Y_2 \to X_2$ defined over $T$ are isomorphic if there exist isomorphisms of $T$-schemes $\psi : Y_1 \to Y_2$ and $\phi : X_1 \to X_2$ such that $\phi \circ f_1 = f_2 \circ \psi$.

By the result of [13], [4] and [32], there exists a certain $\mathbb{Q}$-scheme, denoted by

$$H_r^\text{red}(G),$$

which is a coarse moduli space for $G$-Galois covers of curves of genus 0 with $r$ branch points. In particular, to each $G$-Galois cover $f : Y \to X$ over a $\mathbb{Q}$-scheme $T$, we can associate a map $\varphi_f : T \to H_r$ called the classifying map for $f$. The association $f \mapsto \varphi_f$ is functorial in $T$. For $T = \text{Spec} k$, where $k$ is an algebraically closed field of characteristic 0, it induces a bijection between isomorphism classes of $G$-covers of curves of genus 0 with 4 branch points, defined over $k$, and $k$-rational points on $H$. The scheme $H_r^\text{red}(G)$ is called the reduced inner Hurwitz space. It is a smooth and affine variety over $\mathbb{Q}$. 
For the rest of this section, we will assume that $r = 4$. In this case, the variety $H_4^\text{red}(G)$ is a smooth affine curve, equipped with a finite separable cover

$$j : H_4^\text{red}(G) \rightarrow \mathbb{A}_K^1,$$

which is at most tamely ramified at $0, 1728$ and étale over $\mathbb{A}_K^1 - \{0, 1728\}$. The map $j$ is characterized by the following property. Let $t : \text{Spec } k \rightarrow H_4^\text{red}(\mathbb{C})$ be a geometric point, corresponding to a $G$-cover $f : Y \rightarrow X$ with branch locus $D = \{x_1, \ldots, x_4\}$. Then $j(t) \in k$ is the $j$-invariant of the configuration $(X, D)$.

Let $x_1, \ldots, x_4 \in \mathbb{C}$ be four distinct complex numbers. Set $X_0 := \mathbb{P}_C^1$ and $D_0 := \{x_1, \ldots, x_4\} \subset X_0$. Let $z_0 \in \mathbb{R}$ denote the $j$-invariant of the configuration $(\mathbb{P}_C^1, D_0)$. We assume that $z_0 \neq 0, 1728$. After choosing a marking $\kappa$ of $(X_0, D_0)$, we obtain a presentation of $\pi_1(U_0)$ with generators $\alpha_1, \ldots, \alpha_4$ and relation $\prod_i \alpha_i = 1$. This presentation yields a bijection

$$(18) \quad j^{-1}(z_0) \sim Ni_4^\text{red}(G).$$

(Recall that the left hand side of (18) may be identified with the set of isomorphism classes of $G$-covers of $X_0$ with branch locus $D_0$.) The fundamental group of $\mathbb{C} - \{0, 1728\}$ acts on the left hand side of (18), and the mapping class group $\tilde{B}_4$ acts on the right hand side. There is a natural identification of these two groups which makes the bijection (18) equivariant. In particular, we obtain a bijection between the set of connected components of $H_4^\text{red}(G)_C$ and the $\tilde{B}_4$-orbits of $Ni_4^\text{red}(G)$.

### 4.3

Let us now fix the following objects:

- an $\tilde{B}_4$-orbit $O \subset Ni_4^\text{red}(G)$, and
- a faithful and irreducible linear representation $G \hookrightarrow \text{GL}_n(K)$, with coefficients in a number field $K$.

The orbit $O$ corresponds to a connected component $H(O)$ of $H_4^\text{red}(G)_C$. Let $k_O$ be its field of definition, i.e. the smallest subfield $k \subset \mathbb{C}$ such that the natural action of $\text{Aut}(\mathbb{C}/k)$ on the set of connected components of $H_4^\text{red}(G)$ stabilizes $H(O)$. We may and will consider $H(O)$ as a geometrically irreducible variety over $k_O$ (i.e. as a connected component $H_4^\text{red}(G) \otimes k_O$).

Let $f : Y \rightarrow X$ be a versal family over $H(O)$, i.e. a $G$-cover defined over a $k_O$-scheme $S$ whose classifying map $S \rightarrow H_4^\text{red}(G)$ is étale and factors through the natural map $H(O) \rightarrow H_4^\text{red}(G)$. (It is known that such a versal family always exists. Under some favorable conditions, one can take $S = H(O)$, see §5.1.) Let $k$ be the field of definition of $S$, i.e. the algebraic closure of $\mathbb{Q}$ inside the function field of $S$. Since the map $S \rightarrow H_4^\text{red}(G)$ is étale, $S$ is a smooth, affine and geometrically connected $k$-curve. Moreover $k$ is a finite extension of $k_O$.

Choose an integer $N$ such that $G \subset \text{GL}_n(K)$ is contained in $\text{GL}_n(R)$, where $R = \mathcal{O}_K[1/N]$. Let $D \subset X$ be the branch locus of $f : Y \rightarrow X$, and set $U :=
The $G$-cover $f : Y \to X$ gives rise to a surjective group homomorphism $\pi_1(U) \to G$. We denote by
\[ \rho : \pi_1(U) \to \text{GL}_n(R) \]
the composition of this homomorphism with the injection $G \hookrightarrow \text{GL}_n(R)$.

We shall write $\rho^{geo} : \pi_1(U_{\overline{Q}}) \to \text{GL}_n(R)$ (resp. $\rho^{\text{top}} : \pi_1^{\text{top}}(U_{\mathbb{C}}) \to \text{GL}_n(R)$) for the restriction of $\rho$ to the geometric fundamental group of $U$ (resp. to the topological fundamental group of the analytic space associated to $U$). Let $\mathcal{V}^{\text{an}}$ denote the local system of $R$-modules on $U_{\mathbb{C}}$ corresponding to $\rho^{\text{top}}$. Also, let
\[ \mathcal{W}^{\text{an}} := R^1\pi_* (j_* \mathcal{V}^{\text{an}}) \]
be the parabolic cohomology of $\mathcal{V}^{\text{an}}$. Recall that $\mathcal{W}^{\text{an}}$ is a local system of $R$-modules corresponding to a representation $\eta^{\text{top}} : \pi_1^{\text{top}}(S_{\mathbb{C}}) \to \text{GL}(W)$.

On the other hand, for each prime ideal $p$ of $K$ which is prime to $N$ we let $\rho_p : \pi_1(U) \to \text{GL}_n(O_p)$ denote the $p$-adic representation induced by $\rho$. It corresponds to an étale local system $\mathcal{V}_p$ of $O_p$-modules on $U$. Again we can form the (parabolic) higher direct image of $\mathcal{V}_p$,
\[ \mathcal{W}_p := R^1\pi_* (j_* \mathcal{V}_p) , \]
which is an étale local system of $O_p$-modules, thus corresponds to a representation $\eta_p : \pi_1(S) \to \text{GL}(W_p)$.

**Proposition 4.1** There exists a canonical isomorphism $W_p \cong W \otimes_R O_p$ such that the following diagram commutes:
\[
\begin{array}{ccc}
\pi_1^{\text{top}}(S_{\mathbb{C}}) & \xrightarrow{\eta^{\text{top}}} & \text{GL}(W) \\
\downarrow & & \downarrow \\
\pi_1(S) & \xrightarrow{\eta_p} & \text{GL}(W_p) 
\end{array}
\]

Hence the image of $\eta^{geo}_p$ is equal to the topological closure of the image of $\eta^{top}$.

**Proof:** This follows from Theorem 3.2 and the fact that $\pi_1(S_{\overline{Q}})$ is the profinite completion of $\pi_1^{\text{top}}(S_{\mathbb{C}})$.

\[ \square \]

**4.4** We can now use the results of §2.3 to determine the image of $\eta^{\text{top}}$. We will use the notation introduced in the previous subsection, with the following difference. Since we will be working exclusively with complex analytic spaces, we will omit the index $( )_{\mathbb{C}}$. For instance, we write $S$ instead of $S_{\mathbb{C}}$, etc.

Choose a point $s_0 \in S$ with $j(s_0) = z_0$ and let $(X_0, D_0)$ denote the fibre of the configuration $(X, D)$ over $s_0$. Let $\rho_0^{\text{top}} : \pi_1^{\text{top}}(U_0, x_0) \to \text{GL}_n(R)$ denote
the restriction of $\rho^{\top}$ to the subgroup $\pi_1^{\top}(U_0, x_0) \subset \pi_1^{\top}(U, x_0)$, and set $g_i = \rho_0(\alpha_i) \in G$. By construction, the tuple $g := (g_i)$ is a generating system for $G$, and the reduced Nielsen class of $g$ is an element of the $B_4$-orbit $O$. Moreover, $[g]^{\text{red}}$ is stabilized by the image of the the group homomorphism
\[
\bar{\varphi} : \pi_1^{\top}(S, s_0) \rightarrow B_4,
\]
which is induced by the configuration $(X, D)$ over $S$.

For simplicity, we also assume that the configuration $(X, D)$ admits an affine frame (Definition 2.4), which we use to identify $X$ with $\mathbb{P}_S^1$. This assumption will be satisfied in our main example. Note also that (at least in the situation where $S$ is one-dimensional), there always exists an affine frame over a dense open subset of $S$.

The $\infty$-section defines a section of the natural projection $\pi_1^{\top}(U) \rightarrow \pi_1^{\top}(S)$. We identify $\pi_1^{\top}(S)$ with the image of this section. Let $\chi : \pi_1^{\top}(S) \rightarrow G$ denote the restriction of $\rho$ to $\pi_1^{\top}(S)$. Essentially by definition, we have
\[
(19)\quad g^{\varphi(\gamma)} = g^{\chi(\gamma)^{-1}},
\]
for all $\gamma \in \pi_1(S)$, compare with (17). It follows from Theorem 2.5 that
\[
(20)\quad \eta(\gamma) = \bar{\Phi}(g, \varphi(\gamma)) \cdot \bar{\Psi}(g, \chi(\gamma)),
\]
for all $\gamma \in \pi_1^{\top}(S)$. Here $\bar{\Phi}(g, \beta)$ and $\bar{\Psi}(g, h)$ are as in §2.2. Therefore, if we know $\varphi$ and $\chi$ explicitly, we can also compute $\eta^{\top}$.

**Remark 4.2** In practice, it is not always so easy to describe an affine frame $X \cong \mathbb{P}_S^1$ and the induced lift $\varphi$ of $\bar{\varphi}$ explicitly. In many cases, this is possible, using the methods of [8]. However, for applications to the Regular Inverse Galois Problem, it is usually sufficient to determine the image of the projective representation associated to $\eta^{\top}$, and one can proceed as follows.

Let $\varphi : \pi_1^{\top}(S, s_0) \rightarrow A_4$ and $\chi : \pi_1(S, s_0) \rightarrow G$ be any pair of group homomorphisms such that $\varphi$ is a lift of $\bar{\varphi}$ and such that (19) holds. (Using the fact that $\pi_1^{\top}(S, s_0)$ is a free group, it is easy to see that such a pair always exists.) The choice of $(\varphi, \chi)$ determines a representation $\rho' : \pi_1^{\top}(U) \rightarrow \text{GL}_n(R)$ extending $\rho_0$; it corresponds to a variation $\mathcal{V}'$ of $\mathcal{V}_0^\text{an}$. Let $\eta' : \pi_1^{\top}(S, s_0) \rightarrow \text{GL}(W_{\mathfrak{g}})$ be the representation corresponding to the parabolic cohomology of $\mathcal{V}'$. By Remark 2.3 (iii), the projective representations associated to $\eta^{\top}$ and $\eta'$ are equal. See the next section, in particular §5.3.

## 5 An example

We work out one particular example of the construction described in the last section. In this example, the Hurwitz space is a rational curve. As a result, we obtain regular realizations over $\mathbb{Q}(t)$ of certain simple groups $\text{PSL}_2(\mathbb{F}_{\rho^2})$. 






5.1 Let $G := \text{PSL}_2(7) \times \mathbb{Z}/3\mathbb{Z}$. Given a conjugacy class $C$ of elements of the group $\text{PSL}_2(7)$, we denote by $C_i$ the conjugacy class of $(g, i)$ in $G$, where $g \in C$ and $i \in \mathbb{Z}/3\mathbb{Z}$. The conjugacy classes of $\text{PSL}_2(7)$ are denoted in the standard way (see [1]). For instance, $2a$ is the unique class of elements of $\text{PSL}_2(7)$ order 2. Set

$$C := (2a_0, 2a_0, 3a_1, 3a_2).$$

A computer calculation shows that the set $\text{Ni}_{\text{red}}^0(C)$ has 90 elements and that the mapping class group $\bar{B}_4$ acts transitively. Since $C$ is rational (in the sense of [28]), the connected component $S := H_{\text{red}}^0(C)$ of the Hurwitz space $H_{\text{red}}^4(G)$ corresponding to this orbit is defined over $\mathbb{Q}$. So $S$ is a smooth, affine and absolutely irreducible curve over $\mathbb{Q}$. Furthermore, our explicit knowledge of the braid action on $\text{Ni}_{\text{red}}^0(C)$ can be used to show that the complete model $\bar{S}$ of $S$ has genus 0.

**Lemma 5.1**

(i) The curve $S$ is isomorphic to a dense open subset of $\mathbb{P}_\mathbb{Q}^1$.

(ii) There exists a versal $G$-Galois cover $f : Y \to X$ over $S$.

**Proof:** We have to show that $\bar{S} \cong \mathbb{P}_\mathbb{Q}^1$. Since $\bar{S}$ has genus 0, it is well known that it suffices to find a $\mathbb{Q}$-rational effective divisor of odd degree on $\bar{S}$. The description of the covering $j : S_C \to \mathbb{C}$ in terms of the braid action shows that the set of cusps (i.e. the points of $\bar{S} - S$) is such a divisor, of degree 17. This finishes the proof of (i).

Let $s = \text{Spec } \bar{k} \to S$ be a geometric point of $S$ and denote by $f_s : Y_s \to X_s$ the $G$-cover of type $C$ corresponding to $s$. Let $x_1, \ldots, x_4$ denote the branch points of $f_s$, ordered in such a way that $x_i$ corresponds to the conjugacy class $C_i$. By definition, we have an injection $G \hookrightarrow \text{Aut}_k(Y_s)$. We claim that the centralizer of $G$ inside $\text{Aut}_k(Y_s)$ is equal to the center of $G$ (which is cyclic of order 3). Indeed, suppose that $\sigma : Y_s \xrightarrow{\sim} Y_s$ is an automorphism which centralizes the action of $G$. The automorphism $\sigma' : X_s \xrightarrow{\sim} X_s$ induced by $\sigma$ fixes the set $\{x_1, x_2\}$ and the branch points $x_3$ and $x_4$. If $\sigma'$ were nontrivial, it would be of order 2, and there would exist a reduced Nielsen class $[g]_{\text{red}} \in \text{Ni}_{\text{red}}^0(C)$ which is fixed by the element $\beta_1\beta_2\beta_1 \in B_4$. However, one checks that such a Nielsen class does not exist, so $\sigma'$ is the identity. This proves the claim.

The claim implies that for any $G$-cover $f : Y \to X$ over a scheme $T$ whose classifying morphism $\varphi_f : T \to H^1_3(G)$ has its image contained in $S$, the automorphism group of $f$ is canonically isomorphic to the center of $G$. It is shown in [32] that the category of all (families of) $G$-covers of type $C$ is a gerbe over the Hurwitz space $\bar{S} = H(C)$. In our case, the band of this gerbe is simply the constant group scheme $\mathbb{Z}/3\mathbb{Z}$. By general results on non-abelian cohomology, the gerbe is represented by a class $\omega$ in $H^2(S, \mathbb{Z}/3\mathbb{Z})$, and the existence of a global section (i.e., the neutrality of the gerbe) is equivalent to the vanishing of $\omega$. See also [3].

Let $K$ denote the function field of $S$. By (i), $K = \mathbb{Q}(t)$ is a rational function field. Since $S$ is affine, we may regard $\omega$ as an element of the Galois cohomology group $H^2(K, \mathbb{Z}/3\mathbb{Z})$. We can give a more concrete description of $\omega$, as follows.
Let \( f_K : Y_K \to X_K \) denote the \( G \)-cover of type \( C \) corresponding to the geometric point \( \text{Spec} \bar{K} \to S \). For \( \sigma \in \text{Gal}(\bar{K}/K) \), let \( f^\sigma_K \) denote the conjugate \( G \)-cover. By definition of the field \( K \), the cover \( f^\sigma_K \) is isomorphic to \( f_K \), i.e. there exists a commutative diagram

\[
\begin{array}{ccc}
Y_K & \xrightarrow{\psi_\sigma} & Y^\sigma_K \\
f_K \downarrow & & \downarrow f^\sigma_K \\
X_K & \xrightarrow{\varphi_\sigma} & X^\sigma_K,
\end{array}
\]

where \( \psi_\sigma \) and \( \varphi_\sigma \) are \( \bar{K} \)-linear isomorphisms and \( \psi_\sigma \) is also \( G \)-equivariant. Note that \( \psi_\sigma \) is not uniquely determined by \( \sigma \): we may compose it with an element of the center of \( G \). However, \( \varphi_\sigma \) is uniquely determined by \( \sigma \) and therefore satisfies the obvious cocycle relation. We conclude that there exists a (unique) model \( X_K \) of \( X \) over \( K \) such that \( \varphi_\sigma \) is determined by the isomorphism \( X_K \cong X_K \otimes \bar{K} \), in the obvious way. In the language of [2], we obtain the following result. The field of moduli of the \( G \)-cover \( f_K \) with respect to the extension \( \bar{K}/K \) and the model \( X_K \) of \( X \) is equal to \( K \). Moreover, the class \( \omega \in H^2(K, \mathbb{Z}/3\mathbb{Z}) \) is the obstruction for \( K \) to be a field of definition.

The curve \( X_K \) is isomorphic to the projective line over \( K \) if and only if it has a \( K \)-rational point. Moreover, there exists a quadratic extension \( L/K \) such that \( X_L := X_K \otimes L \) has an \( L \)-rational point and is isomorphic to \( \mathbb{P}^1_L \). It follows from a theorem of Deligne and Douai [2] that \( L \) is a field of definition of \( f_K \) (here we use that the center of \( G \) is a direct summand of \( G \)). In other words, the restriction of \( \omega \) to \( L \) vanishes. But by [23], Chap. I.2, Prop. 9, the restriction map \( H^2(K, \mathbb{Z}/3\mathbb{Z}) \to H^2(L, \mathbb{Z}/3\mathbb{Z}) \) is an isomorphism. We conclude that \( \omega = 0 \), which finishes the proof of the proposition.

**Remark 5.2** The lemma shows that there exist infinitely many non-isomorphic \( G \)-covers \( f_0 : Y_0 \to X_0 \) defined over \( \mathbb{Q} \). However, we do not know whether we can find any such \( G \)-cover with \( X_0 \cong \mathbb{P}^1_{\mathbb{Q}} \). So we do not know whether the lemma produces any regular realizations of the group \( G \) over \( \mathbb{Q}(t) \).

**5.2** Let \( g \in \mathcal{E}(C) \) be any generating system of type \( C \); for instance, we could take

\[
g := (1,2)(3,4)(5,8)(6,7), \quad (1,6)(2,5)(3,7)(4,8), \quad (1,3,8)(4,5,7)(9,11,10), \quad (1,3,7)(2,8,6)(9,10,11)\]

(here we have chosen a faithful permutation representation \( G \to S_{11} \)). The group \( G \) admits a faithful and absolutely irreducible linear representation of dimension 3, defined over the number field \( K := \mathbb{Q}(\sqrt{-3}, \sqrt{-7}) \). This representation is already defined over \( R := \mathcal{O}_K[1/7] \). From now on, we will consider \( G \) as a subgroup of \( \text{GL}_3(R) \). Note that the matrices \( g_1, g_2 \) are conjugate to the diagonal matrix \( \text{diag}(1,1,-1) \), and that \( g_3 \) (resp. \( g_4 \)) is conjugate to \( \text{diag}(1,1, \omega) \).
Let $\Gamma := \pi_1(S, s_0)$; the first step in the proof of Theorem 5.3 is to determine the image of the projective representation $\lambda_{\text{top}} : \Gamma \to \text{PGL}(W_g)$ associated to $\eta_{\text{top}} : \Gamma \to \text{GL}(W_g)$. Since $S$ is isomorphic to the Riemann sphere minus 17 points, there exist generators $\gamma_1, \ldots, \gamma_{17}$ of $\Gamma$, subject to the relation $\prod_j \gamma_j = 1$. Our strategy is to explicitly compute the image of $\gamma_j$ in $\text{PGL}(W_g)$, for a certain choice of the generators $\gamma_j$.

Let $\varphi : \Gamma \to B_4$ be the group homomorphism induced from the branch locus configuration $(X, D)$ of the versal $G$-cover $f : Y \to X$. By construction, there exists a reduced Nielsen class in $\text{Ni}^{\text{red}}(C)$ which is stabilized by the image of $\varphi$. Since the action of $B_4$ on $\text{Ni}^{\text{red}}(C)$ is transitive, we may normalize things in such a way that the class $[g]^{\text{red}}$ of our originally chosen tuple $g$ is stabilized by $\varphi(\Gamma)$. It is well known (see e.g. [4]) that there exist generators $\delta_0, \delta_\infty, \delta_{1728}$ of $\pi_1^{\text{top}}(C - \{0, 1728\})$, with relation $\delta_0\delta_\infty\delta_{1728} = 1$, which are mapped to $\bar{\beta}_1\bar{\beta}_2, \bar{\beta}_1$ and $\bar{\beta}_1\bar{\beta}_2\bar{\beta}_1$, under the natural map

$$\pi_1^{\text{top}}(C - \{0, 1728\}) \to B_4.$$ 

Let $\Gamma' \subset \pi_1^{\text{top}}(C - \{0, 1728\})$ be the inverse image of the stabilizer of the reduced Nielsen class $[g]^{\text{red}}$. We may identify $\Gamma'$ with the fundamental group of $\Sigma' := j^{-1}(C - \{0, 1728\}) \subset S$. It is a straightforward, although combinatorially involved problem to write down a list of generators of the free group $\Gamma'$, given as words in the generators $\delta$. Moreover, one can choose these generators in such a
way that the usual product-1-relation holds and that each of them represents a simple closed loop around one of the points missing from $S'$. Let $\gamma_1, \ldots, \gamma_{17} \in \Gamma$ be those generators representing a loop around a cusp (i.e. a point $s \in \bar{S}$ with $j(s) = \infty$). Note that $\gamma_j$ is conjugate (inside the group $\pi_1^{\text{top}}(\mathbb{C} - \{0, 1728\})$) to a certain power of $\delta_\infty$. The other generators, representing a loop around one of the points of $S-S'$, are conjugate either to $\delta_0^2$ or to $\delta_{1728}^2$, so their image in $B_4$ is 1. It follows that the map $\Gamma' \to B_4$ factors over the natural, surjective map $\Gamma' \to \Gamma$. Denoting the image of $\gamma_j$ in $\Gamma$ by the same name, we have found explicit generators $\gamma_1, \ldots, \gamma_{17}$ of $\Gamma$, with relation $\prod_j \gamma_j = 1$, and their images under the map $\bar{\varphi} : \Gamma \to B_4$.

It is easy to find, for all $j = 1, \ldots, 17$, an element $\gamma_j' \in A_4$ which lifts $\bar{\varphi}(\gamma_j)$ and an element $h_j \in G$ such that

$$g^{\gamma_j'} = g^{h_j}.$$ 

Moreover, we may do this in such a way that $\prod_j \gamma_j' = 1$ and $\prod_j h_j = 1$. In other words, we can choose homomorphisms $\varphi : \Gamma \to A_4$ and $\chi : \Gamma \to G$ as in Remark 4.2. In fact, the lift $\varphi$ is unique, because the Klein four group $Q$ acts faithfully on $\text{Ni}(\mathbb{C})$. On the other hand, $\chi$ is only determined up to multiplication of $h_j = \chi(\gamma_j)$ by a central element of order 3. This corresponds to the fact that the versal $G$-cover $f : Y \to X$ over $S$ may be twisted by characters of order 3. (It is not clear how to find $\chi$ corresponding to a versal cover $f$ defined over $\mathbb{Q}$). By formula (20) and Remark 4.2 we have

$$\eta^{\text{top}}(\gamma_j) = c_j \cdot \bar{\Phi}(g, \gamma_j') \cdot \bar{\Psi}(g, h_j),$$

for some scalar $c_j \in K^\times$. (In fact, $c_j$ is a third root of unity and we have $\prod_j c_j = 1$.) Set $b_j := \bar{\Phi}(g, \gamma_j') \cdot \bar{\Psi}(g, h_j)$. By construction, $b_j$ is an invertible 2-by-2-matrix with entries in $R$ such that $\prod_j b_j = 1$.

Using a computer program written in GAP, the authors have computed the matrices $b_j$ explicitly. It turns out that 12 of the $b_j$ are transvections and 5 are homologies with eigenvalues 1, $\omega$ or 1, $\omega^2$, where $\omega$ denotes a primitive third root of unity (see [9] for notations). One finds that the trace of the matrix $b_1b_2$ is a generator of the extension $K/\mathbb{Q}$. Moreover, one checks that for every prime $p > 7$ one can find a pair of transvections $b_i, b_j$ whose commutator is not congruent to the identity, modulo any prime ideal $p$ above $p$. This information suffices to show that for a prime $p > 7$ which is not totally split in $K/\mathbb{Q}$, the image of the residual projectivized representation $\lambda_p^{\text{geo}}$ associated to $\eta_p^{\text{geo}}$ is equal to $\text{PSL}_2(p^2)$ (we may identify the residue field of $p$ with $\mathbb{F}_{p^2}$). By a well known argument (see e.g. [24]), it follows that the image of the projective representation $\lambda_p^{\text{geo}}$ associated to $\eta_p^{\text{geo}}$ is equal to $\text{PSL}_2(O_p)$.

The only thing left to prove is that the image of the full projective representation $\lambda_p$ is equal to $\text{PSL}_2(O_p)$ as well. Again, it suffices to show that the image of the residual projectivized representation $\lambda_p$ is equal to $\text{PSL}_2(p^2)$.

One observes that there are exactly five ramification points $s_1, \ldots, s_5 \in \bar{S}$ of the map $j : \bar{S} \to \mathbb{P}^1$ above $\infty$ whose ramification index is equal to 4. One also observes that the matrices $b_{s_\mu}$ corresponding to the points $s_\mu$ are transvection
for \( \mu = 1, \ldots, 4 \), whereas \( b_{j5} \) is a homology. It follows that the set \( \{s_1, \ldots, s_4\} \) is rational, i.e. fixed by the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Furthermore, the transvections \( b_{\mu 1}, \ldots, b_{\mu 4} \) are all conjugate to each other by elements of \( \text{SL}_2(K) \). One concludes that the image of these transvections give rise to conjugate transvections in the image of \( \tilde{\eta}^{\text{geo}} \). The conjugacy class of these transvections is a rational class, in the sense of [28].

Suppose that there exists an element \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and a lift \( \alpha \in \pi_1(S) \) of \( \sigma \) such that \( \bar{\eta}_p(\alpha) \in \text{GL}_2(p^2) \) does not lie in \( \text{SL}_2(p^2) \). Using the branch cycle argument (as in the proof of [30], Corollary 4.6), one would conclude that \( \sigma \) does not fix the set \( \{s_1, \ldots, s_4\} \). But this would be a contradiction to the assertion made above. It follows that the image of \( \bar{\lambda}_p \) is equal to \( \text{PSL}_2(p^2) \). The proof of Theorem 5.3 is now complete.

**Remark 5.5** It is also possible to prove Theorem 5.3 by using a generalisation of methods developed by Völklein, see e.g. [30]. Here is a brief outline. One constructs a certain projective system of finite étale covers of \( S \) (these covers are themselves Hurwitz spaces). Our representation \( \eta^{\text{geo}} \) essentially corresponds to the inverse limit of the Galois closures of these covers. One then has to show, using the theory of [13], that this projective system has \( \mathbb{Q} \) as a field of moduli. One deduces that \( \eta^{\text{geo}} \) has \( \mathbb{Q} \) as a field of linear moduli (in our approach this is automatic, as \( \eta^{\text{geo}} \) is the restriction of a representation \( \eta \) of the full arithmetic fundamental group). The rest of the proof goes as above.

### 6 The monodromy of the Picard–Euler system

**6.1** In this section, we use the notation of §4. However, all varieties are defined over the complex numbers. Let \( G \) be a cyclic group of order 3, with generator \( \sigma \). We fix a nontrivial character \( \chi : G \rightarrow \mathbb{C}^\times \). Then \( \omega := \chi(\sigma) \) is a primitive third root of unity. We consider the generating system

\[
g = (\sigma, \sigma, \sigma, \sigma, \sigma^2) \in \mathcal{E}_5(G).
\]

The orbit \( O \) of \( g \) under the action of the braid group consists simply of the five permutations of \( g \).

Let

\[
S := \{(s, t) \in \mathbb{C}^2 \mid s, t \neq 0, 1, s \neq t \},
\]

and let \( X := \mathbb{P}^1_S \) denote the relative projective line over \( S \). The equation

\[
y^3 = x(x - 1)(x - s)(x - t)
\]

(21)

defines a finite Galois cover \( f : Y \rightarrow X \) of smooth projective curves over \( S \), namely ramified along the divisor \( D := \{0, 1, s, t, \infty\} \subset X \). We identify the Galois group of \( f \) with \( G \) in such a way that \( \sigma^* y = \omega \cdot y \). The classifying map \( \varphi_f : S \rightarrow H^\text{red}_5(G) \) is a finite étale cover of the connected component \( H(O) \subset H^\text{red}_5(G) \) corresponding to the braid orbit \( O \) of \( g \). Thus, in the terminology of
§4, $f: Y \to X$ is a versal family of $G$-covers of type $g$. A $G$-cover of $\mathbb{P}^1$ of type $g$ is called a Picard curve, see e.g. [16].

Let $K := \mathbb{Q}(\omega)$ denote the field of third roots of unity and $\mathcal{O}_K = \mathbb{Z}[\omega]$ its ring of integers. The family of $G$-covers $f: Y \to X$ together with the character $\chi$ of $G$ give rise to a local system of $\mathcal{O}_K$-modules on $U := X - D$, see §4.3. Set $s_0 := (2,3) \in S$ and let $\mathcal{V}_0$ denote the restriction of $\mathcal{V}$ to the fibre $U_0 = \mathbb{A}^1_C - \{0,1,2,3\}$ of $U \to S$ over $s_0$. We consider $\mathcal{V}$ as a variation of $\mathcal{V}_0$ over $S$.

Let $\mathcal{W}$ denote the parabolic cohomology of this variation; it is a local system of rank three, see Remark 1.3. Let $\chi': G \hookrightarrow \mathbb{C}^\times$ denote the conjugate character to $\chi$ and $\mathcal{W}'$ the parabolic cohomology of the variation of local systems $\mathcal{V}'$ corresponding to the $G$-cover $f$ and the character $\chi'$. We write $\mathcal{W}_C$ for the local system of $\mathbb{C}$-vectorspaces $\mathcal{W} \otimes \mathbb{C}$.

Proposition 6.1 We have a canonical isomorphism of local systems

$$R^1\pi_{Y,*}C \cong \mathcal{W}_C \oplus \mathcal{W}'_C.$$ 

This isomorphism identifies the fibres of $\mathcal{W}_C$ with the $\chi$-eigenspace of the singular cohomology of the Picard curves of the family $f$.

Proof: The group $G$ has a natural left action on the sheaf $f_!C$. It is easy to see that we have a canonical isomorphism of sheaves on $X$

$$f_!C \cong C \oplus j_*\mathcal{V}_C \oplus j_*\mathcal{V}',$$

which identifies $j_*\mathcal{V}_C$, fibre by fibre, with the $\chi$-eigenspace of $f_!C$. Now the Leray spectral sequence for the composition $\pi_Y = \pi_X \circ f$ gives isomorphisms of sheaves on $S$

$$R^1\pi_{Y,*}C \cong R^1\pi_{X,*}(f_*C) \cong \mathcal{W}_C \oplus \mathcal{W}'_C.$$

Note that $R^1\pi_{X,*}C = 0$ because the genus of $X$ is zero. Since the formation of $R^1\pi_{Y,*}$ commutes with the $G$-action, the proposition follows. 

6.2 The comparison theorem between singular and deRham cohomology identifies $R^1_{\text{sing}}\pi_*C$ with the local system of horizontal sections of the relative deRham cohomology module $R^1_{\text{dR}}\pi_*\mathcal{O}_Y$, with respect to the Gauss-Manin connection. The $\chi$-eigenspace of $R^1_{\text{dR}}\pi_*\mathcal{O}_Y$ gives rise to a Fuchsian system known as the Picard–Euler system. In more classical terms, the Picard–Euler system is a set of three explicit partial differential equations in $s$ and $t$ of which the period integrals

$$I(s,t;a,b) := \int_a^b \frac{dx}{\sqrt{x(x-1)(x-s)(x-t)}}$$

(with $a,b \in \{0,1,s,t,\infty\}$) are a solution. See [21], [15], [16]. It follows from Proposition 6.1 that the monodromy of the Picard–Euler system can be identified with the representation $\eta: \pi_1(S) \to \text{GL}_3(\mathcal{O}_K)$ corresponding to the local system $\mathcal{W}$.
Theorem 6.2 (Picard) For suitable generators $\gamma_1, \ldots, \gamma_5$ of the fundamental group $\pi_1(S)$, the matrices $\eta(\gamma_1), \ldots, \eta(\gamma_5)$ are equal to
\[
\begin{pmatrix}
\omega^2 & 0 & 1 - \omega \\
\omega - \omega^2 & 1 & \omega^2 - 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
\omega^2 & 0 & 1 - \omega \\
1 - \omega^2 & 1 & \omega^2 - 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & \omega^2 - 1 \\
0 & \omega^2 - 1 & -2\omega
\end{pmatrix},
\begin{pmatrix}
\omega^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
\omega^2 & -\omega^2 & 0 \\
0 & 1 & 0 \\
1 - \omega & \omega^2 - 1 & 1
\end{pmatrix}.
\]

Proof: Recall that $A_r$ (resp. $B_r$) denotes the Artin (resp. Hurwitz) braid group on $r$ strands. We identify $A_{r-1}$ with the subgroup of $B_r$ generated by the standard braids $\beta_1, \ldots, \beta_{r-2}$. By the results of §2 and §4, the representation $\eta : \pi_1(S) \to \text{GL}(W_3) \cong \text{GL}_3(O_K)$ factors through the map $\varphi : \pi_1(S) \to A_4 \subset B_5$ induced by the branch divisor $D \subset \mathbb{P}^1_S$ (we write the points of $D$ in the order $(0, 1, s, t, \infty)$). Using standard methods (see e.g. [29] or [9]), one can show that the image of $\varphi$ is indeed generated by the five braids
\[\beta_3^2, \beta_3^2 \beta_2 \beta_3^{-1}, \beta_3 \beta_2 \beta_1 \beta_2^{-1} \beta_3^{-1}, \beta_2^3, \beta_2 \beta_3 \beta_2^{-1} \beta_3^{-1},\]
see Figure 1. It is clear that these five braids can be realized as the image under the map $\varphi$ of generators $\gamma_1, \ldots, \gamma_5 \in \pi_1(S)$.

Let $\rho : \pi_1(U) \to G \subset K^\times$ denote the representation corresponding to the $G$-cover $f : Y \to X$, and $\rho_0 : \pi_1(U_0) \to G$ its restriction to the fibre above $s_0$. Considering the $\infty$-section as a ‘tangential base point’ for the fibration $U \to S$, we obtain a section $\pi_1(S) \to \pi_1(U)$. We use this section to identify $\pi_1(S)$ with a subgroup of $\pi_1(U)$. Let $\alpha_1, \ldots, \alpha_5$ be the standard generators of $\pi_1(U_0)$. Using (21) one checks that $\rho_0$ corresponds to the tuple $g = (\sigma, \sigma, \sigma, \sigma, \sigma^2)$, i.e. that $\rho_0(\alpha_i) = g_i$. Also, since the leading coefficient of the right hand side of (21) is one, the restriction of $\rho$ to $\pi_1(S)$ is trivial. Hence, by Theorem 2.5, we have
\[
\eta(\gamma_i) = \Phi(g, \varphi(\gamma_i)).
\]
A straightforward computation, using (13) and the cocycle rule (14), gives the value of $\eta(\gamma_i)$ (in form of a three-by-three matrix depending on the choice of a basis of $W_3$). For this computation, it is convenient to take the classes of $(1, 0, 0, -\omega^2)$, $(0, 1, 0, -\omega)$ and $(0, 0, 1, 0, -1)$ as a basis. In order to obtain the matrices stated in the theorem, one has to use a different basis, i.e. conjugate with the matrix
\[
B = \begin{pmatrix}
0 & -\omega - 1 & -\omega \\
\omega + 1 & \omega + 1 & \omega + 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

\[\square\]

25
Figure 1: The braids $\gamma_1, \ldots, \gamma_5$

Remark 6.3  
(i) Theorem 6.2 is due to Picard, see [21], p. 125, and [22], p. 181. He obtains exactly the matrices given above, but he does not list all of the corresponding braids. Holzapfel in [15] gives a list of five braids which generated $\pi_1(S)$, see [15], p. 125. But contrary to what is claimed, these braids do not correspond to the five matrices found by Picard.

(ii) Essentially the same computation as in the proof of Theorem 6.2, but for arbitrary cyclic covers of $\mathbb{P}^1$ and with $p$-adic coefficients, can be found in [31].

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