NONSEPARABLY CONNECTED COMPLETE METRIC SPACES

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Abstract. A topological space is nonseparably connected if it is connected but all of its connected separable subspaces are singletons. We show that each connected first countable space is the image of a nonseparably connected complete metric space under a continuous monotone hereditarily quotient map.

1. Introduction

In this paper we solve the problem of constructing a nonseparably connected complete metric space, posed in [8, Problem 2]. A topological space is separably connected if any two of its points lie in a connected separable subspace. On the other hand, a topological space is nonseparably connected if it is connected but all of its connected separable subspaces are singletons.

The first example of a nonseparably connected metric space was constructed by Pol in 1975, [9]. Another example was given by Simon in 2001, [13]. In 2008, Morayne and Wójtik obtained a nonseparably connected metric group as a graph of an additive function from the real line, [8]. None of these nonseparably connected spaces are completely metrizable.

Definition 1.1. The separablewise component of a point $x_0$ of a topological space $X$ is the union of all separable connected subsets containing $x_0$. A space is separably connected if it has only one separablewise component. A space is nonseparably connected if it is connected but all of its separablewise components are singletons.

Let $A$ be a separablewise component of a point $x$ in a sequential space $X$. We will show that $A$ is closed. If $a \in \overline{A}$, then there is a sequence of points $(a_n)_{n=1}^{\infty} \subset A$ converging to $a$. There is a sequence of connected separable sets $A_n$ with $x \in A_n$ and $a_n \in A_n$. Notice that the set $\{a\} \cup \bigcup\{A_n : n \in \mathbb{N}\}$ is connected and separable. So it must be contained in $A$. In particular, $a \in A$.

This is a long paper so we decided to precede it with a short overview to acquaint the reader with our main results and to help navigate through the different sections.

For any connected metric space $(X, d)$ we construct a connected complete metric space $\text{Cob}(X, d)$ called the cobweb over $(X, d)$ and define a continuous monotone hereditarily quotient compression map $\pi : \text{Cob}(X, d) \to X$ whose fibers coincide with the separablewise components of $\text{Cob}(X, d)$.

The cobweb space $\text{Cob}(X, d)$ is embedded in $\Gamma(X)$, called the complete oriented graph over $X$, which is simply a graph in which any two distinct vertices $x, u$ are joined by two separate edges denoted $[x, u]$ and $[u, x]$ in such a way that the edges

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carry the Euclidean metric and become arcs and the distance between points on different edges is measured as the shortest distance of traveling along the edges.

Next we iterate the cobweb construction taking the cobweb over some connected metric space \((X, d)\) and then the cobweb over this cobweb and so on. We take the inverse limit of this sequence of cobwebs, \(\text{Cob}^\omega(X, d)\), to kill off all separable connected subsets and end up with a nonseparably connected complete metric space.

Finally, we notice that the cobweb construction works for all distance spaces \((X, d)\), which is a huge generalization of metric spaces, and study such spaces in some detail to formulate our strongest result.

As an application of the cobweb construction we present a non-constant continuous real-valued function defined on a connected complete metric space that has a local minimum or a local maximum at every point.

Since the notion of a quotient map plays a crucial role in our constructions we decided to collect all the necessary definitions and basic theorems into an appendix.

2. The cobweb over a distance space

Our fundamental tool is the cobweb construction which was originally conceived for metric spaces, but turns out to work for a much broader class of spaces.

**Definition 2.1.** For any function \(d : X \times X \to [0, \infty)\) such that \(d(x, x) = 0\) for all \(x \in X\), we call the ordered pair \((X, d)\) a distance space.

Let each set of the form \(B_d(x, r) = \{z \in X : d(x, z) < r\}\) be called a ball of radius \(r > 0\) centered at \(x \in X\).

We equip each distance space \((X, d)\) with the topology generated by the distance function \(d\) so that a set \(E \subset X\) is defined to be open in \((X, d)\) if for every point \(x \in E\) there is an \(r > 0\) such that \(B_d(x, r) \subset E\).

A distance space \((X, d)\) is called well-behaved if \(x \in \text{Int}(B_d(x, r))\) for all \(x \in X\) and \(r > 0\).

It is enough to note that every metric space is a well-behaved distance space to go through our basic constructions. Later we will characterize distance spaces and well-behaved distance spaces when formulating the strongest result in Section 5.

**Definition 2.2.** Let \(\kappa\) be a cardinal number. A hedgehog with \(\kappa\) spikes, each of length \(\varepsilon > 0\), is the space \(H = \{(0, 0)\} \cup (\kappa \times (0, \varepsilon])\) equipped with the metric \(\rho\) given by

\[
\rho((x, t), (u, s)) = \begin{cases} |t - s| & \text{if } x = u, \\ t + s & \text{if } x \neq u. \end{cases}
\]

It is easy to see that a hedgehog is an arcwise connected complete metric space.

**Definition 2.3.** The complete oriented graph \(\Gamma(X)\) over a set \(X\) of vertices is defined to be the union

\[
\Gamma(X) = \bigcup \{[x, y] : x, y \in X, x \neq y\}
\]

of all oriented edges \([x, y]\) with distinct vertices \(x, y \in X\), so that

\([x, y] \cap [y, x] = \{x, y\}\),

and the oriented edge \([x, y]\) is defined as

\([x, y] = \{x, y\} \cup \{(x, y, t) : t \in (0, 1)\}\).
Construction 2.4. We are going to define a complete metric $\rho$ on $\Gamma(X)$ such that

1. each edge $[x, y]$ is isometric to the unit interval $[0, 1]$,
2. each edge without end points $[x, y] \setminus \{x, y\}$ is an open set,
3. if two points $a, b$ lie on disjoint edges then $\rho(a, b) > 1$.

Proof. First, we equip each edge with the euclidean metric by defining an auxiliary function $r$ on a subset of $\Gamma(X) \times \Gamma(X)$. For all $x, y \in X, x \neq y$ and all $t, s \in (0, 1)$, let

$$r(x, x) = 0, \quad r(x, y) = 1,$$
$$r((x, y, t), (x, y, s)) = |t - s|,$$
$$r(x, (x, y, t)) = r((x, y, t), x) = t,$$
$$r(y, (x, y, t)) = r((x, y, t), y) = 1 - t.$$

Next, we extend this auxiliary function to the whole $\Gamma(X) \times \Gamma(X)$. For any points $a, b \in \Gamma(X)$, let $\rho(a, b)$ be the infimum over finite sums

$$\sum_{i=1}^{n} r(a_i, a_{i-1})$$

where $\{a_0, \ldots, a_n\} \subset X$ with $a = a_0, a_n = b$ and consecutive points $a_i, a_{i-1}$ belong to the same edge: $(\forall i)(\exists x, y \in X) \{a_i, a_{i-1}\} \subset [x, y]$.

It is clear that $\rho$ is a metric on $\Gamma(X)$ from the way it is defined.

It is easy to see that each edge $[x, y]$ is isometric to $[0, 1]$.

The set $[x, y] \setminus \{x, y\}$ is open because it is equal to the open ball of radius $1/2$ centered in the middle of $[x, y]$, that is at $(x, y, 1/2)$.

The distance between any two points on disjoint edges is greater than one because to travel from one such point to another along the edges we must move along at least one whole edge.

Each closed ball of radius $1/2$ centered at a vertex $x \in X$ is isometric to a hedgehog with $2(|X| - 1)$ spikes of length $1/2$. In particular, such closed balls are complete metric spaces.

The complete oriented graph $(\Gamma(X), \rho)$ is a complete metric space. Indeed, let $x_n$ be a Cauchy sequence such that $\rho(x_n, x_m) < 1/4$ for all $n, m \in \mathbb{N}$. If $\rho(x_n, u) \geq 1/4$ for all $n \in \mathbb{N}$ and all $u \in X$, then our sequence is contained in one of the edges. Otherwise, there is an index $k$ and a vertex $u \in X$ such that $\rho(x_k, u) < 1/4$. Then $\rho(x_n, u) \leq \rho(x_n, x_k) + \rho(x_k, u) \leq 1/2$ for all $n \in \mathbb{N}$. This means that our sequence is contained in the closed ball of radius $1/2$ centered at $u$, which is a complete metric space. In both cases, our Cauchy sequence converges.

\[\square\]

Definition 2.5. For a distance space $(X, d)$, we construct a metric space $\text{Cob}(X, d)$, the so called cobweb over the distance space $(X, d)$, as a special subspace of $(\Gamma(X), \rho)$ and the so called compression map $\pi : \text{Cob}(X, d) \to X$ in the following way:

Let $d_1 = \min\{d, 1/2\}$. For distinct vertices $a, b \in X$, let $a_b \in [a, b]$ be the unique point satisfying

$$\rho(a_b, b) = d_1(b, a).$$

Note that $a_b \neq b \iff d(b, a) > 0$. Let $[a, a_b]$ be the subarc of $[a, b]$ given by

$$[a, a_b] = \{z \in [a, b] : \rho(a, z) \leq 1 - d_1(b, a)\}.$$

Let

$$Z_a = \bigcup \{[a, a_b] \setminus \{b\} : b \in X \setminus \{a\}\}.$$
Let
\[ \text{Cob}(X, d) = \bigcup_{a \in X} Z_a. \]

Let the so called compression map \( \pi: \text{Cob}(X, d) \to X \) be given by
\[ \pi(Z_a) = \{a\} \text{ for all } a \in X. \]

The next theorem describes the basic properties of the cobweb over a distance space without any reference to the topology of the distance space. In Section 6 devoted to the applications of the cobweb construction, we will refer to these properties only.

**Theorem 2.6.** Let \((X, d)\) be a distance space. Let \( \pi: \text{Cob}(X, d) \to X \) be the compression map. Then for all \( x \in X \) and all \( r \in (0, \frac{1}{2}) \),

1. \( X \subset \text{Cob}(X, d) \),
2. \( \pi(x) = x \),
3. \( \pi^{-1}(x) \) is arcwise connected,
4. \( \pi^{-1}(x) \setminus \{x\} \) is open in \( \text{Cob}(X, d) \), \( (\pi \text{ is locally constant on } \text{Cob}(X, d) \setminus X) \),
5. \( \text{Cob}(X, d, \rho) \) is a complete metric space,
6. \( \pi(B_\rho(x, r)) = B_d(x, r) \),
7. \( |X| \leq \text{dens}(\text{Cob}(X, d)) \leq |\text{Cob}(X, d)| = \epsilon|X| \).

**Proof.** (1)–(4) Evident from the definition.

(5) Notice that \( \text{Cob}(X, d) \) is a closed subset of \((\Gamma(X), \rho)\) because it is obtained by taking away selected open intervals from some of the edges so that in effect it is the intersection of a family of closed subsets. Since \((\Gamma(X), \rho)\) is complete, \( \text{Cob}(X, d, \rho) \) is a complete metric space.

(6) If \( x \in B_\rho(a, r) \), then \( \rho(a, x_a) = d(a, x) < r \). So \( x_a \in B_\rho(a, r) \) and \( x = \pi(x_a) \). Thus \( x \in \pi(B_\rho(a, r)) \).

On the other hand, if \( x \in \pi(B_\rho(a, r)) \), we have a point \( z \in \text{Cob}(X, d) \) such that \( \rho(a, z) < r \) and \( \pi(z) = x \). Consequently, \( z \in [x, x_a] \) for some \( u \in X \setminus \{x\} \). We may assume that \( x \neq a \) because otherwise there is nothing to prove. Now, supposing that \( u \neq a \), we have three distinct vertices \( a, x, u \) and a point \( z \) lying on the edge \([x, u]\), which means that \( \rho(a, z) \geq 1 \). This contradiction shows that \( u = a \) and thus \( z \in [x, x_a] \). Now, the point \( x_a \) lies between \( z \) and \( a \) on the edge \([x, a]\), so that \( d(a, x) = \rho(x_a, a) \leq \rho(a, z) < r \), showing that \( x \in B_\rho(a, r) \).

(7) Notice that each fiber of the compression map \( \pi \) contains a hedgehog with \(|X| - 1\) spikes. Thus \( |\text{Cob}(X, d)| = \epsilon|X| \). The nonempty set \( \pi^{-1}(x) \setminus \{x\} \) is open in \( \text{Cob}(X, d) \) for each \( x \in X \), so \( |X| \leq \text{dens}(\text{Cob}(X, d)) \). \( \square \)

We encourage the reader to check that if \((X, d)\) is a distance space such that \( d(a, b) > 0 \) for all distinct \( a, b \in X \), then every fiber of the compression map is homeomorphic to a hedgehog with \(|X| - 1\) spikes; and the set of points at which the space \( \text{Cob}(X, d) \) is locally connected is equal to \( \text{Cob}(X, d) \setminus X \).

Refer to Section 4 for definitions of some terms used in the following theorem.

**Theorem 2.7.** For any distance space \((X, d)\), the compression map \( \pi: \text{Cob}(X, d) \to X \) is a continuous monotone quotient surjection and \( \text{Cob}(X, d) \) is connected iff \((X, d)\) is connected. Moreover, \( \pi \) is hereditarily quotient iff \((X, d)\) is well-behaved.

**Proof.** Since \( \pi(x) = x \) for every \( x \in X \), \( \pi \) is a surjection.
Since the set $\pi^{-1}(x) \setminus \{x\}$ is open for every $x \in X$, the map $\pi$ is locally constant at each point $z \in \text{Cob}(X, d) \setminus X$. In particular, it is continuous at these points. That $\pi$ is continuous at each $x \in X$ follows directly from the inclusion
\[ \pi(B_\rho(x, r)) \subset B_d(x, r) \]
and the definition of the topology on $X$. So $\pi$ is continuous.

To show that $\pi$ is quotient, take any $x \in A$ such that $\pi^{-1}(A)$ is open in $\text{Cob}(X, d)$. We need to show that $A$ is open in $X$. Since $\pi(x) = x$, we have $x \in \pi^{-1}(A)$. Since $\pi^{-1}(A)$ is open, there is an $r > 0$ such that $B_\rho(x, r) \subset \pi^{-1}(A)$. Then
\[ B_d(x, r) \subset \pi(B_\rho(x, r)) \subset A \]
showing that $A$ is open in $(X, d)$.

The compression map is monotone because each fiber $\pi^{-1}(x)$ is arcwise connected. We have just proved that $\pi$ is quotient. So by Lemma 7.3 if $(X, d)$ is connected, then $\text{Cob}(X, d)$ is connected. On the other hand, if $\text{Cob}(X, d)$ is connected, then $(X, d)$ is connected because the compression map is continuous.

Suppose now that $(X, d)$ is well-behaved. To show that $\pi$ is hereditarily quotient, take any $x \in X$ and any open set $U \subset \text{Cob}(X, d)$ such that $\pi^{-1}(x) \subset U$. Then $x \in \pi^{-1}(x) \subset U$, and since $U$ is open, $x \in B_\rho(x, r) \subset U$ for some $r > 0$. Finally, $x \in \text{Int}(B_d(x, r)) \subset B_d(x, r) \subset \pi(B_\rho(x, r)) \subset \pi(U)$ and thus $x \in \text{Int}(\pi(U))$.

Suppose now that the compression map $\pi : \text{Cob}(X, d) \to X$ is hereditarily quotient. To show that $(X, d)$ is well-behaved, take any $x \in X$ and $r > 0$. Let
\[ U = B_\rho(x, r) \cup (\pi^{-1}(x) \setminus \{x\}) \]
Notice that $U$ is open in $\text{Cob}(X, d)$ and that $\pi^{-1}(x) \subset U$. Since $\pi$ is hereditarily quotient, $x \in \text{Int}(\pi(U))$. But $\pi(U) = B_d(x, r)$, so $x \in \text{Int}(B_d(x, r))$. $\square$

To describe the separablewise components of the cobweb space now and to obtain economical metrics later, we will make use of the fact that the compression map is locally constant except on a metrically discrete subset. Naturally, the cardinality of the image $f(X)$ of a locally constant function $f : X \to Y$ does not exceed the density of the domain, $|f(X)| \leq \text{dens}(X)$, and the cardinality of a metrically discrete space does not exceed its density.

**Theorem 2.8.** Let $(X, d)$ be a distance space. Let $\pi : \text{Cob}(X, d) \to X$ be the compression map. Then $|\pi(A)| \leq \text{dens}(A)$ for any $A \subset \text{Cob}(X, d)$. Consequently, for any metric space $(X, d)$, the fibers of $\pi$ coincide with the separablewise components of $\text{Cob}(X, d)$ which in turn coincide with the arcwise components.

**Proof.** Let $A \subset \text{Cob}(X, d)$. Notice that the restriction $\pi|_{(A \setminus X)}$ is locally constant, so $|\pi(A)| \leq |\pi(A \cap X)| + |\pi(A \setminus X)| \leq |A \cap X| + \text{dens}(A \setminus X)$. On the other hand, $|A \cap X| \leq \text{dens}(A \cap X)$ because $A \cap X$ is metrically discrete, $\rho(a, b) = 1$ for distinct $a, b \in A \cap X$. Thus $|\pi(A)| \leq \text{dens}(A \cap X) + \text{dens}(A \setminus X) = \text{dens}(A)$.

Let $E$ be a connected separable subset of $\text{Cob}(X, d)$. Since $\pi$ is continuous, $\pi(E)$ is connected. Since $|\pi(E)| \leq \text{dens}(E) = \aleph_0$, $\pi(E)$ is countable. Now, if the connected countable set $\pi(E)$ is contained in the metric space $X$, it must be a singleton, which means that $E$ lies in one of the fibers of $\pi$. Recall that the fibers are arcwise connected. $\square$

The following theorem reveals the surprising abundance of connected complete metric spaces that fail to be separably connected.
Theorem 2.9. For every connected metric space \((X,d)\) the cobweb \(\text{Cob}(X,d)\) is a connected complete metric space whose separablewise components form a quotient space homeomorphic to \(X\).

Proof. By Theorem 2.7, \(\text{Cob}(X,d)\) is a connected complete metric space, whose separablewise components, by Theorem 2.8, coincide with the fibers of the compression map, which in turn form a quotient space homeomorphic to \(X\), as may be easily verified by using the following distance function
\[
D(\pi^{-1}(a), \pi^{-1}(b)) = d(a, b) = d(b, a)
\]
to generate the topology of the quotient space formed by the fibers. (In fact, if \((X,d)\) is any symmetric distance space then the topology of the quotient space formed by the fibers of the compression map is generated by this distance function.) \(\square\)

Notice that in this way we encode the metric space \((X,d)\) as a completely different object \(\text{Cob}(X,d)\) from which we can extract the original space by a topological operation.

Let us make two final remarks to conclude this section.

Recall that a connected, locally connected complete metric space must be arcwise connected, [6, 6.3.11]. The cobweb over a connected metric space is not locally connected, although it is locally connected except on a metrically discrete subset. This illustrates how important it is to assume that the space is locally connected at each point if we want to conclude that it is arcwise connected.

For any distance space \((X,d)\) the separablewise component of a point \(x \in X \subset \text{Cob}(X,d)\) coincides with the fiber \(\pi^{-1}(x)\) if and only if each countable connected subset \(C\) of the distance space \(X\) with \(x \in C\) is equal to the singleton \(\{x\}\). This brings us to the following notion. We say that a topological space \(X\) is functionally Hausdorff if for any two distinct points \(a, b \in X\) there is a continuous function \(f: X \to \mathbb{R}\) with \(f(a) \neq f(b)\). Every metric space \((X,d)\) is functionally Hausdorff, because for any distinct points \(a, b \in X\), the function \(f: X \to [0,1]\) given by \(f(x) = d(x,a)/(d(x,a) + d(x,b))\) is continuous and \(f(a) = 0, f(b) = 1\). Each connected subset \(E\) of a functionally Hausdorff space is either a singleton or contains a set of cardinality \(\mathfrak{c}\). Indeed, let \(a,b \in E\) with \(f(a) < f(b)\). Then the connected set \(f(E)\) must contain the interval \([f(a), f(b)]\). In conclusion, Theorem 2.8 could be stated for any functionally Hausdorff distance space \((X,d)\) and not just for metric spaces. (The lexicographical square with the order topology is a connected compact Hausdorff first countable space. Being normal, it is functionally Hausdorff. Being first countable, it may serve as an example of a connected functionally Hausdorff well-behaved distance space that is not metrizable.)

3. The iterated cobweb functor

Definition 3.1. Given a distance space \((X,d)\), we define by induction a sequence of iterated cobweb spaces over \((X,d)\): let \(\text{Cob}^1(X,d) = \text{Cob}(X,d)\) be equipped with the natural cobweb metric \(\rho_1\) induced from the complete graph over \(X\), and let \(\text{Cob}^{n+1}(X,d) = \text{Cob}(\text{Cob}^n(X,d), \rho_n)\) be equipped with the natural cobweb metric \(\rho_{n+1}\) induced from the complete graph over \(\text{Cob}^n(X,d)\).

Let \(\pi_{n+1}: \text{Cob}^{n+1}(X,d) \to \text{Cob}^n(X,d)\) denote the appropriate compression maps.
Let
\[ \text{Cob}^\omega(X, d) = \{(x_n)_{n=1}^\infty \in \prod_{n \in \mathbb{N}} \text{Cob}^n(X, d) : \pi_n^{n+1}(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N} \} \]
be the inverse limit of the sequence of iterated cobweb spaces over \((X, d)\).

**Theorem 3.2.** For any distance space \((X, d)\), \(\text{Cob}^\omega(X, d)\) is a completely metrizable space that is connected iff \((X, d)\) is connected, and its separablewise components are singletons.

**Proof.** \(\text{Cob}^\omega(X, d)\) is completely metrizable because all the factor spaces \(\text{Cob}^n(X, d)\) are completely metrizable.

Suppose that \((X, d)\) is connected. Then by Theorem 2.7 the spaces \(\text{Cob}^n(X, d)\) are connected and the compression maps \(\pi_n^{n+1} : \text{Cob}^{n+1}(X, d) \to \text{Cob}^n(X, d)\) are continuous monotone hereditarily quotient surjections for all \(n \in \mathbb{N}\). Therefore, by Puzio’s Theorem 7.7, \(\text{Cob}^\omega(X, d)\) is connected.

Let \(A \subset \text{Cob}^\omega(X, d)\) be connected and separable. Fix \(n \in \mathbb{N}\). The projection \(\text{pr}_{n+1}(A)\) is connected and separable so, by Theorem 2.8 it lies in one of the fibers of \(\pi_n^{n+1}\). Thus \(\pi_n^{n+1}(\text{pr}_{n+1}(A))\) is a singleton. But \(\pi_n^{n+1}(\text{pr}_{n+1}(A)) = \pi_n(A)\). So \(\text{pr}_n(A)\) is a singleton for each \(n \in \mathbb{N}\), which means that \(A\) is a singleton.

**Corollary 3.3.** For every connected distance space \((X, d)\) having at least two points, \(\text{Cob}^\omega(X, d)\) is a nonseparably connected completely metrizable space.

Recall that a topological space is **punctiform** if all of its connected compact subsets are singletons. For example, any Bernstein subset of the euclidean plane is a connected punctiform metric space. A separable connected punctiform complete metric space was constructed by Kuratowski and Sierpiński in 1922. A variation of their idea was presented in [14]. Our nonseparably connected complete metric spaces are new examples of connected punctiform complete metric spaces.

Using the notion of an economical metric we show that all separable subsets of our nonseparably connected spaces are in fact zero-dimensional.

**Definition 3.4.** Given a metric space \((X, d)\), we say that the metric \(d\) is **economical** if \(\text{card}(|\{d(a, b) : a, b \in A\}|) = |d(A \times A)| \leq \text{dens}(A) = \min\{|D| : D \subset A \subset \overline{D}\}\) for any infinite subset \(A \subset X\). We say that a topological space is **economically metrizable** if its topology can be generated by an economical metric.

**Proposition 3.5.** If \(X\) is an economically metrizable space, then each subspace \(A \subset X\) of density \(\text{dens}(A) < \varepsilon\) is zero-dimensional. Consequently, each connected economically metrizable space is nonseparably connected.

**Proof.** Take any \(a \in A\) and any \(r > 0\). If \((\forall \varepsilon \in (0, r))(\exists b \in A)(d(a, b) = \varepsilon)\), then \(|d(\{a\} \times A)| \geq |(0, r)| = \varepsilon\). This contradiction shows that there is an \(\varepsilon \in (0, r)\) such that \(d(a, b) \neq \varepsilon\) for all \(b \in A\). So the open ball of radius \(\varepsilon\) centered at \(a\) \(B(a, \varepsilon) \cap A\) is relatively clopen in \(A\) and contained in \(B(a, r)\). This means that \(A\) is zero-dimensional.

**Theorem 3.6.** Let \((X, d)\) be a distance space. Then
\[ \rho_\infty((x_n)_{n=1}^\infty, (u_n)_{n=1}^\infty) = \max\{\frac{\rho_n(x_n, u_n)}{n} : n \in \mathbb{N}\} \]
is a complete economical metric for \( \text{Cob}^\omega(X, d) \). Consequently, for every connected distance space \((X, d)\), \((\text{Cob}^\omega(X, d), \rho_\infty)\) is a connected complete economical metric space and its separable subsets are zero-dimensional.

**Proof.** Notice that in the definition above we have \( \rho_n \leq 2 \) for all \( n \in \mathbb{N} \) because the complete graph metrics are always bounded by two.

Notice that the distance function \( \rho_\infty \) is a product metric on \( \prod_{n \in \mathbb{N}} \text{Cob}^n(X, d) \). Since all the factor metric spaces \((\text{Cob}^n(X, d), \rho_n)\) are complete, so is the product metric space \((\prod_{n \in \mathbb{N}} \text{Cob}^n(X, d), \rho_\infty)\). Now, \( \text{Cob}^\omega(X, d) \), as an inverse limit, is a closed subset and thus a complete metric space.

Let us write \( \text{pr}_n(x_k)_{k=1}^\infty = x_n \) for \( n \in \mathbb{N} \).

Notice that for any \( a, b \in \prod_{n \in \mathbb{N}} \text{Cob}^n(X, d) \) we have

\[
\rho_\infty(a, b) \in \bigcup\{n^{-1}\rho_n(\text{pr}_n(a), \text{pr}_n(b)) : n \in \mathbb{N}\}.
\]

Therefore, if \( A \subset \prod_{n \in \mathbb{N}} \text{Cob}^n(X, d) \), we have

\[
|\rho_\infty(A \times A)| \leq \sum_{n \in \mathbb{N}} |\rho_n(\text{pr}_n(A) \times \text{pr}_n(A))| \leq \sum_{n \in \mathbb{N}} |\text{pr}_n(A)|^2.
\]

To show that \((\text{Cob}^\omega(X, d), \rho_\infty)\) is economical, take any infinite \( A \subset \text{Cob}^\omega(X, d) \).

By Theorem 2.8, \( |\pi_{n+1}^n(E)| \leq \text{dens}(E) \) for all \( n \in \mathbb{N} \) and \( E \subset \text{Cob}^{n+1}(X, d) \).

Therefore, for all \( n \in \mathbb{N} \),

\[
|\text{pr}_n(A)| = |\pi_{n+1}^n(\text{pr}_{n+1}(A))| \leq \text{dens}(\text{pr}_{n+1}(A)) \leq \text{dens}(A).
\]

Thus, since \( \text{dens}(A) \) is infinite,

\[
|\rho_\infty(A \times A)| \leq \sum_{n \in \mathbb{N}} |\text{pr}_n(A)|^2 \leq \sum_{n \in \mathbb{N}} (\text{dens}(A))^2 = \text{dens}(A).
\]

\(\square\)

4. More on Distance Spaces

Obviously, every metric space is a well-behaved distance space. Moreover, the topology of every first countable space can be generated by a well-behaved distance function and conversely every well-behaved distance space is first countable. Furthermore, each distance space (not necessarily well-behaved) is weakly first countable in the sense of Arkhangel’skii [1], and conversely the topology of every weakly first countable space can be generated by a distance function.

**Definition 4.1** (Arkhangel’skii [1]). A topological space \( X \) is weakly first countable if to each point \( x \in X \) we can assign a decreasing sequence \((B_n(x))_{n \in \omega}\) of subsets of \( X \) that contain \( x \) so that a subset \( U \subset X \) is open if and only if for each \( x \in U \) there is \( n \in \omega \) with \( B_n(x) \subset U \).

**Lemma 4.2.** Let \( X \) be an arbitrary set. Let \( E_n(x) \subset X, n \in \mathbb{N}, x \in X, \) be arbitrary sets such that \( x \in E_{n+1}(x) \subset E_n(x) \) for all \( x \in X \) and \( n \in \mathbb{N} \). Let \( d: X \times X \to [0, 1] \) be given by

\[
d(x, y) = \inf\{1/n : y \in E_n(x)\}.
\]

Then

\[
E_{n+1}(x) \subset B_d(x, 1/n) \subset E_n(x)
\]

for all \( x \in X \) and \( n \in \mathbb{N} \).
Proof. Fix \( x \in X \) and \( n \in \mathbb{N} \).

If \( y \in B_d(x, 1/n) \), then \( d(x, y) < 1/n \) and so there is a \( k \in \mathbb{N} \) such that \( 1/k < 1/n \) and \( y \in E_k(x) \). Since \( k > n \), \( y \in E_k(x) \subset E_n(x) \). In effect, \( B_d(x, 1/n) \subset E_n(x) \).

If \( y \notin B_d(x, 1/n) \), then \( d(x, y) \geq 1/n \), and thus \( (\forall k \in \mathbb{N})(y \in E_k(x) \implies 1/k \geq 1/n) \). So \( y \notin E_{n+1}(x) \). In effect, \( X \setminus B_d(x, 1/n) \subset X \setminus E_{n+1}(x) \). \( \square \)

Theorem 4.3. A topological space is weakly first countable if and only if its topology is generated by a distance function. A topological space is first countable if and only if its topology is generated by a well-behaved distance function.

Proof. Notice that for a distance space \((X, d)\), the family of sets

\[
\{B_d(x, 1/n) : x \in X, n \in \mathbb{N}\}
\]

witnesses that it is weakly first countable. If the distance function \( d \) is well-behaved, then the family of open sets

\[
\{Int(B_d(x, 1/n)) : x \in X, n \in \mathbb{N}\}
\]

witnesses that it is first countable.

Let \( X \) be a weakly first countable space. We have sets \( \{B_n(x) : x \in X, n \in \mathbb{N}\} \) as in Definition 4.1. Let \( d : X \times X \to [0, 1] \) be given by

\[
d(x, y) = \inf\{1/n : y \in B_n(x)\}.
\]

To argue that the topology of \( X \) is generated by the distance function \( d \), take any point \( x \in X \) and any open set \( U \subset X \) with \( x \in U \). There is an \( n \in \mathbb{N} \) such that \( B_n(x) \subset U \). By Lemma 4.2, \( B_d(x, 1/n) \subset B_n(x) \subset U \).

If \( X \) is first countable, then the sets \( B_n(x) \) may be assumed to be open. To argue that \( d \) is well-behaved, fix \( x \in X \) and \( n \in \mathbb{N} \). By Lemma 4.2, \( B_{n+1}(x) \subset B_d(x, 1/n) \).

In particular, \( x \in Int(B_d(x, 1/n)) \). \( \square \)

Recall that a topological space \( X \) is

- **sequential** if for each non-closed subset \( A \subset X \) there is a sequence \( \{a_n\}_{n\in\omega} \subset A \) that converges to a point \( a \in \overline{A} \setminus A \);
- **Fréchet-Urysohn** if for any subset \( A \subset X \) and a point \( a \in \overline{A} \setminus A \) there is a sequence \( \{a_n\}_{n\in\omega} \subset A \) that converges to \( a \).

It is rather easy to see from these definitions that a topological space \( X \) is Fréchet-Urysohn if and only if each subspace of \( X \) is sequential. Distance spaces, being weakly first countable, are sequential, see [12]. It is tempting to think that each subspace of a distance space is itself a distance space, and that in effect all distance spaces are Fréchet-Urysohn, but this reasoning is wrong.

Example 4.4. There is a Hausdorff symmetric (non-well-behaved) distance space \((X, d)\) that is not Fréchet-Urysohn and which contains a subspace \( Z \subset X \) that is not sequential, whose subspace topology is not generated by the restricted distance function \( d|_{(Z \times Z)} \).

Proof. Let \( K = \{1/n : n \in \mathbb{N}\}, A = K \times K, Z = \{(0, 0)\} \cup A, X = Z \cup (K \times \{0\}) \).

Let \( d : X \times X \to [0, 1] \) be the Euclidean metric except

\[
d((0, 0), (1/n, 1/m)) = d((1/n, 1/m), (0, 0)) = 1.
\]

Notice that \((X, d)\) is not well-behaved because \( Int(B(a, 0.9)) \) is empty. Notice that \( a \in \overline{A} \setminus A \), but no sequence of points in \( A \) converges to \( a \). This means that \( X \) is not Fréchet-Urysohn and that \( Z = \{a\} \cup A \) is not sequential. Although \( \{a\} \) is not
a relatively open subset of $Z$, we have $B_d(a,0.9) \cap Z = \{a\}$, which shows that the topology of $Z$ is not generated by $d|(Z \times Z)$. \qed

**Theorem 4.5.** For a Hausdorff distance space, being Fréchet-Urysohn is equivalent to being first countable.

**Proof.** Recall that all first countable topological spaces are Fréchet-Urysohn.

Let $(X,d)$ be a Fréchet-Urysohn Hausdorff distance space. By Theorem 2.7, the compression map $\pi : \text{Cob}(X,d) \to X$ is a quotient continuous surjection. According to [10, 2.4.F(c)], a quotient surjection onto a Fréchet-Urysohn Hausdorff space is hereditarily quotient. So $\pi$ is hereditarily quotient. By Theorem 2.7, $(X,d)$ is well-behaved and thus first countable. \qed

5. **The functors Cob and Cob**

Recalling Definition 3.1, let us introduce the following notation.

**Definition 5.1.** Let $(X,d)$ be a distance space and let $\pi_1^0 : \text{Cob}(X,d) \to X$ denote the compression map. Let $\pi_1^\omega : \text{Cob}^\omega(X,d) \to \text{Cob}^1(X,d)$ denote the limit projection, given by $\pi_1^\omega(x_1,\ldots,x_n,\ldots) = x_1 \in \text{Cob}(X,d)$.

Let $\pi_0^\omega = \pi_0^1 \circ \pi_1^\omega$ be called the limit compression map $\pi_0^\omega : \text{Cob}^\omega(X,d) \to X$.

**Theorem 5.2.** For any distance space $(X,d)$ the limit compression map $\pi_0^\omega : \text{Cob}^\omega(X,d) \to X$ is a continuous monotone and quotient surjection. Moreover, it is hereditarily quotient iff $(X,d)$ is well-behaved.

**Proof.** The limit compression map $\pi_0^\omega = \pi_0^1 \circ \pi_1^\omega$ is continuous as the composition of two continuous functions. It is surjective because $\pi_0^\omega((x)_{n=1}^\infty) = x$ for all $x \in X$.

By Theorem 2.7, the maps $\pi_n^{n+1} : \text{Cob}^{n+1}(X,d) \to \text{Cob}^n(X,d)$ are continuous monotone and quotient surjections. Therefore, by [10, Theorem 9], the limit compression map $\pi_1^\omega : \text{Cob}^1(X,d) \to \text{Cob}^0(X,d)$ is hereditarily quotient. By the Corollary to [10, Theorem 11], it is also monotone.

Since $\pi_1^\omega$ is hereditarily quotient and $\pi_1^\omega, \pi_0^1$ are both monotone, by Lemma 7.5 $\pi_0^\omega = \pi_0^1 \circ \pi_1^\omega$ is monotone.

By Theorem 2.7, $\pi_0^1$ is quotient; moreover, it is hereditarily quotient if $(X,d)$ is well-behaved. Now, $\pi_0^\omega$ is the composition of quotient maps so it is quotient. Moreover, if $(X,d)$ is well-behaved, $\pi_0^\omega$ is the composition of hereditarily quotient maps, so it is hereditarily quotient, by Lemma 7.6.

Assume now that $\pi_0^\omega$ is hereditarily quotient. To show that $(X,d)$ is well-behaved, take any $x \in X$ and any $r \in (0,1/2)$. Note that $x \in \text{Cob}^n(X,d)$ for all $n \in \mathbb{N}$. Let

$$U_1 = \text{Cob}^\omega(X,d) \cap \left( B_{p_1}(x,r) \times \prod_{n=2}^\infty \text{Cob}^n(X,d) \right),$$

$$U_2 = \{ z \in \text{Cob}^\omega(X,d) : \pi_0^\omega(z) = x \} \setminus \{(x)_{n=1}^\infty \}.$$ 

Notice that both of these sets are open and that $(\pi_0^\omega)^{-1}(x) \subset U = U_1 \cup U_2$. Since $\pi_0^\omega$ is hereditarily quotient, $x \in \text{Int}(\pi_0^\omega(U))$. For any $(z_n)_{n=1}^\infty \in U_1$, we have $z_1 \in B_{p_1}(x,r)$ and consequently $\pi_0^\omega(z) = \pi_0^1(\pi_1^\omega(z)) = \pi_0^1(z_1) \in B_d(x,r)$. In effect, $\pi_0^\omega(U_1) \subset B_d(x,r)$. This means that $x \in \text{Int}(\pi_0^\omega(U)) \subset \text{Int}(B_d(x,r)). \qed$

We are now ready for our strongest result.
Theorem 5.3. Each (connected) first countable space is the image of a (connected) complete economical metric space under a continuous monotone hereditarily quotient map. Each (connected) weakly first countable space is the image of a (connected) complete economical metric space under a continuous monotone quotient map.

Proof. Let $X$ be a first countable space. By Theorem 4.3, the topology of $X$ is generated by a well-behaved distance function $d: X \times X \to [0, 1]$. By Theorem 5.2, $\pi_d^\circ: \text{Cob}^\circ(X, d) \to X$ is a continuous monotone hereditarily quotient surjection. By Theorem 3.2, $(\text{Cob}^\circ(X, d), \rho_\infty)$ is a complete economical metric space. By Theorem 3.2, $\text{Cob}^\circ(X, d)$ is connected if $X$ is connected. We argue analogously for weakly first countable spaces. \qed

6. Applications of the cobweb functor

In this section we shall apply the cobweb construction to obtain a non-constant continuous locally extremal function defined on a connected complete metric space. We define a function $f: X \to \mathbb{R}$ to be locally extremal if each point $x \in X$ is a point of local maximum or local minimum of $f$. In [11] Sierpiński proved that each continuous locally extremal function $f: \mathbb{R} \to \mathbb{R}$ is constant. This result was generalized in [2] to locally extremal functions $f: X \to \mathbb{R}$ defined on connected metrizable spaces $X$ of density $\text{dens}(X) < \aleph_0$. In this situation it is natural to ask if the result of Sierpiński holds true for connected metrizable spaces of arbitrary density. A counterexample to this problem was constructed by Le Donne and Fedeli in [5], and independently by the authors in [3]. Here we present another example based on the cobweb construction.

Theorem 6.1. There is a connected complete metric space $Z$ with cardinality and density both equal to $\aleph_0$ and a monotone continuous hereditarily quotient surjection $f: Z \to (0, 1)$ that has a local extremum at every point.

Proof. Let $X = (0, 1) \times \{0, 1\}$. Let $d: X \times X \to [0, 1]$ be a symmetric distance function on $X$ such that for all $t \in (0, 1)$ and $\varepsilon \in (0, t) \cap (0, 1-t)$,

(i) $d((t, 0), (t, 1)) = 0$
(ii) $d((t, 0), (t - \varepsilon, 1)) = \varepsilon$
(iii) $d((t, 1), (t + \varepsilon, 0)) = \varepsilon$
(iv) $d(a, b) = 1$ otherwise.

Let $Z = \text{Cob}(X, d)$ with the natural cobweb metric $\rho$ induced from the complete graph over $X$. Then, by Theorem 2.3, $Z$ is a complete metric space such that $|Z| = \text{dens}(Z) = \aleph_0$.

Let $p: X \to (0, 1)$ be given by $p(t, y) = t$. Let $\pi: \text{Cob}(X, d) \to X$ denote the compression map. Let $f = p \circ \pi$. Then $f: Z \to (0, 1)$ is a surjection that is locally constant at each $z \in Z \setminus X$.

Thanks to (i) the fibers of $f$ are connected, so $f$ is monotone. Notice that for all $t \in (0, 1)$ and $\varepsilon \in (0, t) \cap (0, 1-t)$,

$p(\pi(B_\rho((t, 0), \varepsilon))) = p(B_d((t, 0), \varepsilon)) = (t - \varepsilon, t)$, by (ii) and (iv),
$p(\pi(B_\rho((t, 1), \varepsilon))) = p(B_d((t, 1), \varepsilon)) = [t, t + \varepsilon)$, by (iii) and (iv).

This means, in particular, that $f$ has a local maximum at each point $(t, 0) \in Z$ and a local minimum at each point $(t, 1) \in Z$. So $f$ has a local extremum at every point. Moreover, it follows that $f$ is continuous.
Take any \( t \in (0, 1) \) and any open set \( U \subset Z \) such that \( f^{-1}(t) \subset U \). Then \((t, 0), (t, 1) \in U \). Since \( U \) is open, there is an \( \varepsilon \in (0, \frac{1}{2}) \) such that \( A = B_{\rho}(t, 0, \varepsilon) \cup B_{\rho}(t, 1, \varepsilon) \subset U \). But then \((t - \varepsilon, t + \varepsilon) \cap (0, 1) = f(A) \subset f(U) \), so \( t \in \text{Int}(f(U)) \).

By Definition 7.1, \( f \) is hereditarily quotient. By Lemma 7.3, \( Z \) is connected. \( \square \)

Recall that open surjections and closed surjections are always quotient maps. However, a continuous monotone hereditarily quotient surjection may be arbitrarily irregular as shown in the following theorem, which makes use of the cobweb construction.

**Theorem 6.2** (Krzysztof Omiljanowski). Let \((X, d)\) be a (connected) metric space and let \( E \subset X \) be an arbitrary subset. There is a (connected) complete metric space \((Z, \rho)\) and a monotone continuous hereditarily quotient surjection \( f: Z \to X \) such that for every \( x \in X \) there are points \( a, b \in Z \) such that \( f(a) = f(b) = x \) and for all \( \varepsilon \in (0, \frac{1}{2}) \),

\[
\begin{align*}
&f\left(B_{\rho}(a, \varepsilon)\right) = \{x\} \cup \left(B_{\rho}(x, \varepsilon) \cap E\right) \\
&f\left(B_{\rho}(b, \varepsilon)\right) = \{x\} \cup \left(B_{\rho}(x, \varepsilon) \setminus E\right).
\end{align*}
\]

**Proof.** Let \( Y = X \times \{0, 1\} \). Let \( r: Y \times Y \to [0, \infty) \) be a nonsymmetric distance function on \( Y \) defined by

\[
r((x, a), (y, b)) = \begin{cases}
0 & \text{if } x = y, \\
d(x, y) & \text{if } a = 0 \land b = 0 \land y \in E, \\
d(x, y) & \text{if } a = 1 \land b = 0 \land y \notin E, \\
1 & \text{otherwise.}
\end{cases}
\]

Let \( Z = \text{Cob}(Y, r) \) with the natural cobweb metric induced from the complete graph over \( Y \). Thus \( Z \) is a complete metric space.

Let \( p: Y \to X \) be given by \( p(x, a) = x \). Let \( \pi: \text{Cob}(Y, r) \to Y \) denote the compression map. Let \( f = p \circ \pi \). Then \( f: Z \to X \) is a surjection that is locally constant at each \( Z \setminus Y \), and thus continuous at these points.

Since \( r((0, 0), (1, 1)) = 0 \), the fibers of \( f \) are connected, so \( f \) is monotone.

Notice that for all \( x \in X \) and \( \varepsilon \in (0, \frac{1}{2}) \),

\[
\begin{align*}
&\pi(B_{\rho}(x, 0, \varepsilon)) = B_{\rho}(x, 0, \varepsilon) = \{(x, 0)\} \cup \{(y, 0): y \in E \land d(x, y) < \varepsilon\}, \\
&f(B_{\rho}(x, 0, \varepsilon)) = \{x\} \cup \left(B_{\rho}(x, \varepsilon) \cap E\right), \\
&\pi(B_{\rho}(x, 1, \varepsilon)) = B_{\rho}(x, 1, \varepsilon) = \{(x, 1)\} \cup \{(y, 0): y \notin E \land d(x, y) < \varepsilon\}, \\
&f(B_{\rho}(x, 1, \varepsilon)) = \{x\} \cup \left(B_{\rho}(x, \varepsilon) \setminus E\right).
\end{align*}
\]

This means, in particular, that \( f \) is continuous at each \( z \in Y \subset Z \).

Moreover, \( f \) is hereditarily quotient by the same argument as in the proof of Theorem 6.1. By Lemma 7.3, \( Z \) is connected if \( X \) is connected. \( \square \)

7. Appendix: Quotient Maps

Quotient maps play a fundamental role in our crucial results. In this section we collect a number of definitions and known facts.

**Definition 7.1.** Let \( X \) and \( Y \) be topological spaces. Then a function \( f: X \to Y \) is

1. monotone if \( f^{-1}(y) \) is connected for each \( y \in Y \);
2. quotient if \( A \) is open in \( Y \) whenever \( f^{-1}(A) \) is open in \( X \);
Lemma 7.4. If \( f : X \to Y \) is monotone and hereditarily quotient then \( f^{-1}(E) \) is connected whenever \( E \subset Y \) is connected.

Proof. The restriction \( f|f^{-1}(E) : f^{-1}(E) \to E \) is quotient and evidently monotone. By Lemma 7.3, \( f^{-1}(E) \) is connected if \( E \) is connected.

Lemma 7.5. If \( f : X \to Y \) and \( g : Y \to Z \) are monotone and \( f \) is hereditarily quotient, then \( g \circ f \) is monotone.

Proof. Since \( g^{-1}(z) \) is connected and \( f \) is monotone and hereditarily quotient, by Lemma 7.4 \((g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z))\) is connected.

Lemma 7.6. If \( f : X \to Y \) and \( g : Y \to Z \) are hereditarily quotient then their composition \( g \circ f : X \to Z \) is hereditarily quotient.
Problem 8.1. Is there a nonseparably connected complete metric space in ZF?
In [8], Morayne and Wójcik constructed a nonseparably connected metric group, which is an example of a homogeneous nonseparably connected metric space.

**Problem 8.2.** Is there a nonseparably connected complete metric space that is homogeneous?

**Problem 8.3.** Is there a locally connected nonseparably connected metric space?

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We rely on [10] in our Theorem 3.2 to argue that the inverse limit is connected, and in Theorem 5.2 to argue that the limit projection is hereditarily quotient. In Section 4 on distance spaces we quote some classical results from Engelking’s book [6] and refer to [12] to argue that distance spaces are sequential. Besides that, our proofs do not require any other sources.

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