LANGLANDS FUNCTORIALITY CONJECTURE FOR $\text{SO}^*_{2n}$
IN POSITIVE CHARACTERISTIC

HÉCTOR DEL CASTILLO

ABSTRACT. In this article we are concerned with the Langlands functoriality conjecture. Cogdell, Kim, Piatetski-Shapiro and Shahidi proved functoriality conjecture in the case of a globally generic cuspidal automorphic representation for the split classical groups, unitary groups or even quasi-split special orthogonal groups in characteristic zero. Lomelí extends this result to split classical groups and unitary groups in positive characteristic. Thus, in this article we prove the Langlands functoriality conjecture for the even quasi-split non-split special orthogonal groups in positive characteristic i.e. we lift globally generic cuspidal automorphic representations of quasi-split non-split even special orthogonal groups to generic automorphic representations of suitable general linear groups in positive characteristic. As an application of this result, we prove the compatibility of the local gamma factors and the unramified Ramanujan conjecture.

INTRODUCTION

The Langlands program plays an important role in Number Theory and Representation Theory. A crucial aspect of this program is the functoriality conjecture: let $F$ be a global field with ring of adèles and two connected (quasi-split) reductive groups $G$ and $H$ over $F$. Let

$$\rho: {}^L G \to {}^L H,$$

be a given $L$-homomorphism between the $L$-groups of $G$ and $H$, respectively. Then, according to this conjecture, for every cuspidal automorphic representation $\pi = \bigotimes_x' \pi_x$ of $G(\mathbb{A}_F)$, there exists an automorphic representation $\Pi = \bigotimes_x' \Pi_x$ of $H(\mathbb{A}_F)$ such that, at almost all places $x$ where $\pi_x$ is unramified, $\Pi_x$ is unramified and its Satake parameter corresponds to the image under $\rho$ of the Satake parameter of $\pi_x$. Such representation will be called a lift of $\pi$. Furthermore the lift process should respect arithmetic information coming from $\gamma$-factors, $L$-functions and $\varepsilon$-factors, and lead to a local version of functoriality at the ramified places as well.

When $G$ is a classical group, $^L G$ has a natural representation into $^L H$ for a specific general linear group $H = \text{GL}_N$. That case has been studied by many people using different tools. When $F$ is a number field, two main tools have been used: converse theorem and trace formulas. The former was used by Cogdell, Kim, Piatetski-Shapiro and Shahidi in combination with the Langlands-Shahidi method to prove the conjecture for a globally generic automorphic representation $\pi$ when $G$ is a quasi-split symplectic, unitary or special orthogonal group. For the latter, Arthur and his continuators used trace formulas to get more complete results, not restricted to globally generic cuspidal automorphic representations of quasi-split groups in characteristic zero.

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Lomelí extended the converse theorem method to global function fields, establishing functoriality for globally generic automorphic representations of split classical groups and unitary groups. The present article complements this treatment of the quasi-split classical groups, over a function field $F$, by establishing the functoriality conjecture when $G$ is a quasi-split non-split even special orthogonal group, that we denote by $SO^*_{2n}$, and $\pi$ a globally generic representation.

**Theorem.** Let $F$ be a global function field and $\pi = \bigotimes_x' \pi_x$ a globally generic cuspidal automorphic representation of $SO^*_{2n}(\mathbb{A}_F)$. Then, $\pi$ lifts to an irreducible automorphic representation $\Pi$ of $GL_{2n}(\mathbb{A}_F)$. Furthermore, $\Pi$ can be expressed as an isobaric sum

$$\Pi = \Pi_1 \oplus \cdots \oplus \Pi_d,$$

where each $\Pi_i$ is a unitary self-dual cuspidal automorphic representation of $GL_{N_i}(\mathbb{A}_F)$ for some $N_i$, and where $\Pi_i \not\cong \Pi_j$ for $i \neq j$. Moreover, if we write $\Pi = \bigotimes_x' \Pi_x$, then for every place $x$ of $F$ and every irreducible generic representation $\tau_x$ of $GL_m(F_x)$ we have that

$$\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x),$$

where the $\gamma$-factors on the right are obtained by the Rankin-Selberg method and those on the left by the Langlands-Shahidi method, as extended by Lomelí to positive characteristic.

As in Cogdell, Kim, Piatetski-Shapiro and Shahidi, and Lomelí, the method of proof uses the converse theorem and $L$-functions to construct an automorphic representation of $GL_n(\mathbb{A}_F)$: we provide a proof of a twisted version in positive characteristic of the converse theorem of Cogdell and Piatetski-Shapiro. To apply the converse theorem, one needs analytic properties of the Langlands-Shahidi $L$-functions, and to establish them we adapt Lomelí’s arguments to our new case. We first obtain a lift that has the desired properties at almost all places. Then further properties of partial $L$-functions give that there is a lift which is an isobaric sum of unitary cuspidal automorphic representations. We prove the compatibility between the local $\gamma$-factors of $\pi$ and the lift $\Pi$ at all places. Our lift is close to what is known as the strong lift, which states the compatibility of $L$-functions and $\varepsilon$-factors.

As an application of the functoriality and the validity of the Ramanujan conjecture for general linear groups established by L. Lafforgue, we prove the unramified Ramanujan conjecture for globally generic cuspidal automorphic representations of our classical group in positive characteristic. A strong lift would imply the Ramanujan conjecture at all places.

**Theorem.** Let $\pi = \bigotimes_x' \pi_x$ be a globally generic cuspidal automorphic representation of $SO^*_{2n}(\mathbb{A}_F)$. Then, if $\pi_x$ is unramified, its Satake parameters have absolute value 1.

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1. **Notation**

We let $F$ be a function field in one variable over a finite field or a locally compact field of positive characteristic. Whenever $F$ is a function field in one variable over a finite field, we denote by $|F|$ the set of places of $F$, by $F_x$ the completion at a place $x \in |F|$ and by $q_F$
or simply by \( q \) the cardinality of its field of constants. We also denote the ring of adèles of \( F \) by \( \mathbb{A}_F \). When \( F \) is a locally compact field of positive characteristic, we denote by \( q_F \) the cardinality of its residue field and by \(| \cdot |_F \) or simply by \(| \cdot |\) the absolute value of \( F \).

For an arbitrary algebraic group \( H \) over \( F \), we denote \( H(R) \) its set of \( R \)-points, where \( R \) is a \( F \)-algebra (usually, \( R \) will be \( F \) or \( \mathbb{A}_F \)). In order to reduce the size of some indices, we sometimes denote \( H(F) \) simply by \( H \).

Let \( G \) be a quasi-split reductive group over \( F \) with a maximal split subtorus \( S \). Fix a Borel subgroup \( B \) containing \( S \). We denote the set of (relative) simple roots of \( S \) in \( B \) by \( \Delta \). The Weyl group \( N \) sometimes denote \( F \) is a subset of \( \Delta \) and parabolic subgroups containing \( B \) respectively. Furthermore, we recall that there is an inclusion preserving bijection between \( \theta \) that \( M \) and \( N \) are the Levi subgroup containing \( S \) and \( N \) is the unipotent radical of \( P \) respectively. Furthermore, we recall that there is an inclusion preserving bijection between subsets of \( \Delta \) and parabolic subgroups containing \( B \). We denote the parabolic subgroups associated to \( \theta \subset \Delta \) via that bijection by \( \mathbf{P}_\theta \).

Suppose that \( F \) is a locally compact field of positive characteristic. Let \( P = MN \) be a parabolic subgroup of \( G \) containing \( B \), then we denote by \( i_P^G \) or \( i_{P(F)}^G \), the normalized parabolic induction functor from \( P(F) \) to \( G(F) \). More precisely, if \((\sigma, W)\) is a smooth representation of \( M \), then we denote by \( i_P^G(W) \) the space of all locally constant functions \( f : G \rightarrow W \) such that \( f(mng) = \delta_p^{-1/2}(m)\sigma(m)f(g) \) for \( m \in M, n \in N, g \in G \), where \( \delta_p \) is the modulus character of \( P \). The action of \( G \) on \( i_P^G(W) \) is given by \( g \cdot f(x) = f(xg) \).

2. Quasi-split even special orthogonal group

We start with the general construction of \( \text{SO}(q) \), for an even dimensional non-degenerate quadratic space \( Q = (V, q) \). Two references for this section are [10, Appendix C] and [28].

2.1. Definitions. First we recall some definitions: for an \( F \)-algebra \( R \), an \( R \)-quadratic space is a pair \((V, q)\) consisting of a finite free \( R \)-module \( V \) and a quadratic form \( q : V \rightarrow R \) i.e.

\[
\begin{enumerate}
  \item \( q(rv) = r^2q(v) \), for all \( r \in R \) and \( v \in V \),
  \item the map \( B_q : V \times V \rightarrow R \), defined by \( B_q(x, y) = q(x + y) - q(x) - q(y) \), is \( R \)-bilinear.
\end{enumerate}
\]

The orthogonal group \( \text{O}(q) \) for a general \( F \)-quadratic space \((V, q)\) over \( F \), is a closed subscheme of \( \text{GL}(V) \), which represents the functor

\[
R \mapsto \{ g \in \text{GL}(V \otimes_F R) : q_R(gx) = q_R(x) \quad \text{for all} \quad x \in V \otimes_F R \}.
\]

The special orthogonal group \( \text{SO}(q) \) is defined as the kernel of the Dickson morphism \( D_q \). The Dickson morphism \( D_q \) is a smooth surjection, identifying \( \mathbb{Z}/2\mathbb{Z} \) with \( \text{O}(q)/\text{SO}(q) \). The special orthogonal group \( \text{SO}(q) \) is connected, smooth and reductive of dimension \( n(n-1)/2 \).

Its center is the group of 2-root of unity \( \mu_2 \) as a group scheme.

We consider two families of quadratic spaces: for \( n \geq 1 \), let \( Q_n = (F^{2n}, q_n) \) be the quadratiic space with

\[
q_n(x_1, \ldots, x_{2n}) = x_1x_{2n} + \cdots + x_{n-1}x_n + 1.
\]

And for a separable quadratic extension \( E \) of \( F \) and \( n \geq 1 \) let \( Q_{E,n} = (F^{n-1} \oplus E \oplus F^{n-1}, q_{E,n}) \), where

\[
q_{E,n}(x_1, \ldots, x_{n-1}, x, x_{n+2}, \ldots, x_{2n}) = x_1x_{2n} + \cdots + x_{n-1}x_{n+2} + N_{E/F}(x)
\]

and \( N_{E/F} \) is the norm map.
To simplify the notation, when the base field \( F \) is clear from context, we let

\[ \text{SO}_{2n} := \text{SO}(q_n). \]

Also, when the separable quadratic extension \( E \) of \( F \) is clear from context, we let

\[ \text{SO}^*_n := \text{SO}(q_{E,n}). \]

**Remark 2.1.1.**
- If we allow \( E \) to be \( F \times F \), we get that \( \text{SO}(q_{F \times F,n}) \cong \text{SO}_{2n} \). In this case we have that \( N_{F \times F/F}(x,y) = (xy,xy) \).
- There is a similar construction of special orthogonal groups for odd dimensional non-degenerate quadratic spaces. In particular, we get the (split) odd special orthogonal groups \( \text{SO}_{2n+1} \).

2.2. **Structure.** We now recall some structural properties of the group \( \text{SO}^*_n = \text{SO}_{2n}(q_{E,n}) \). First, since \( E \otimes_F E \cong E \times E \) and Remark 2.1.1, we have that the group \( \text{SO}(q_{E,n}) \) is an \( F \)-form of the split group \( \text{SO}_{2n} \), which splits over \( E \). Thus, the absolute root system is of type \( D_n \).

We note that the orthogonal decomposition \( Q_{E,n} = H_1 \perp \cdots \perp H_{n-1} \perp (E,N_{E/F}) \), for \( n-1 \) hyperbolic planes \( H_i = (F e_i \oplus F e_{n-i+1}, x_i x_{2n-i+1}) \) and an (anisotropic) non-degenerate quadratic space \( (E,N_{E/F}) \), allows us to obtain the following subtori of \( \text{SO}(q_{E,n}) \)

\[ S \cong \prod_{i=1}^{n-1} \text{SO}(H_i) \quad \& \quad T \cong S \times \text{SO}(E,N_{E/F}). \]

Since the dimension of \( S \) is \( n - 1 \) and the dimension of \( T \) is \( n \), we get that \( S \) is a maximal split \( F \)-torus of \( \text{SO}(q_{E,n}) \) and \( T \) is a maximal \( F \)-torus of \( \text{SO}(q_{E,n}) \).

The \( F \)-points of \( \text{SO}_2 \) and \( \text{SO}(E,N_{E/F}) \) can be identified with the multiplicative group \( F^* \) and the group \( E^1 \) of norm one elements of \( E^* \), respectively. More precisely, we can describe the action of \( t = (x_1, \ldots, x_{n-1}, x) \in (F^*)^{n-1} \times E^1 \cong T(F) \) on the \( F \)-vector space \( Q_{E,n} \), by \( t \cdot e_i = x_i e_i \) for \( 1 \leq i \leq n-1 \), \( t \cdot l = x l \) for \( l \in E \) and \( t \cdot e_{2n-i+1} = x_i^{-1} e_{2n+1-i} \) for \( 1 \leq i \leq n-1 \). Moreover,

\[ Z_G(S) = T. \]

We also note that, by restricting \( q_{E,n} \) to \( H_1 \perp \cdots \perp H_{n-1} \perp F \cdot 1 \), we get the split quadratic form of \( 2(n-1) + 1 \) variables

\[ x_1 x_2 + \cdots + x_{n-1} x_{n+2} + x^2. \]

Thus, we get a copy of \( \text{SO}_{2n-1} \cong \text{SO}(W \perp F \cdot 1) \) inside \( \text{SO}(q_{E,n}) \), where \( W = H_1 \perp \cdots \perp H_{n-1} \). And, the (relative) root system \( \Phi(\text{SO}(q_{E,n}), S) = \Phi(\text{SO}(W \perp F \cdot 1), S) \) is of type \( B_{n-1} \).

Let \( B \) be the Borel subgroup of \( \text{SO}(q_{E,n}) \) containing \( S \) and stabilizing the standard flag \((W_{\text{std}}, F_{\text{std}})\), where \( W_{\text{std}} = \text{span}(e_0, \ldots, e_{n-1}) \) and \( F_{\text{std}} = \{ F e_1, F e_2 \oplus F e_1, \ldots, W_{\text{std}} \} \). We have that

\[ B = T \ltimes R_{u,F}(B), \]

where \( R_{u,F}(B) \) is the radical unipotent subgroup of \( B \), and that \( \text{SO}(q_{E,n}) \) is quasi-split.
2.3. **Relative rank one.** We relate \( \text{Res}_{E/F} \text{SL}_2 \) to the simply connected cover of \( \text{SO}(q_{E,2}) \).

Let \( \text{Gal}(E/F) = \{1, \sigma\} \). Then, we note that \( Q_{E,2} \) is isomorphic to the quadratic space \((H, q)\) of Hermitian \(2 \times 2\) matrices, i.e. \(2 \times 2\) matrices \((x_{i,j})\) with entries in \(E\) such that \(x_{i,j} = \sigma(x_{j,i})\) for every \(1 \leq i, j \leq 2\), with quadratic form \(q = -\det\), via

\[
(x_1, x_2, x) \mapsto \begin{pmatrix} -x_1 & x \\ \sigma(x) & x_2 \end{pmatrix}.
\]

Furthermore, we have an action of \( \text{Res}_{E/F} \text{SL}_2 \) on \(H\), via

\[
a \mapsto ga^*,
\]

where \(g^* = \sigma(g)\). We observe that \(q(a) = q(ga^*)\) and \(\det(a \mapsto ga^*) = 1\). Thus the action gives us a morphism

\[
\text{Res}_{E/F} \text{SL}_2 \to \text{SO}'(q) = \ker(\det |_{O(q)}).
\]

As \( \text{Res}_{E/F} \text{SL}_2 \) is connected and \(O(q)/\text{SO}(q) = \mathbb{Z}/2\mathbb{Z}\), this morphism factors through \(\text{SO}(q) \cong \text{SO}(q_{E,2})\). Finally as the kernel of this morphism is \(\mu_2\) and the dimensions of the groups \(\text{SO}(q_{E,2})\) and \(\text{Res}_{E/F} \text{SL}_2\) are the same, we have

\[
1 \to \mu_2 \to \text{Res}_{E/F} \text{SL}_2 \to \text{SO}(q_{E,2}) \to 1.
\]

2.4. **L-group.** We finish this section by describing the \(L\)-groups of \(\text{SO}_{2n}^*, \text{SO}_{2n}\) and \(\text{GL}_m\).

For that, let us fix a separable closure \(F_s\) of \(F\) and a quadratic separable extension \(E\) of \(F\) contained in \(F_s\). Denote \(\Gamma_F = \text{Gal}(F_s/F)\). Finally, we denote by \(\text{SO}(q)_{F_s}\) and \(\mathbf{T}_{F_s}\) the extension of scalar to \(F_s\) of \(\text{SO}(q)\) and \(\mathbf{T}\), respectively.

Let us consider the split special orthogonal group \(\text{SO}_{2n}\) over \(\mathbb{C}\). We choose a split maximal torus \(\mathbf{T}_n\) and a Borel subgroup \(\mathbf{B}_n\) in the following way

\[
\mathbf{T}_n(\mathbb{C}) = \{t = \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) : t_i \in \mathbb{C}^\times, 1 \leq i \leq n\}
\]

and

\[
\mathbf{B}_n = \mathbf{T}_n \times \{M(u)h(L) : v^tJv = 0 \text{ for all } v \in \mathbb{C}^n\},
\]

the subgroup of upper triangular matrices in \(\text{SO}_{2n}\), where \(J = (\delta_{i,n+1-i})\) with \(\delta\) the Kronecker’s delta is the \(n \times n\) matrix with 1’s along the anti-diagonal, and for upper triangular unipotent \(u \in \text{GL}_n\) and \(L\) an \(n \times n\) matrix,

\[
M(u) = \begin{pmatrix} u & 0_n \\ 0_n & J(u)^{-1}J \end{pmatrix} \quad \& \quad h(L) := \begin{pmatrix} 1_n & L \\ 0_n & 1_n \end{pmatrix}.
\]

We observe that the root datum associated to \((\text{SO}_{2n}, \mathbf{T}_n)\) is isomorphic to the dual root datum \(R^\vee\) of \((\text{SO}(q_{E,n})_{F_s}, \mathbf{T}_{F_s})\). We choose a pinning of \((\text{SO}_{2n}, \mathbf{T}_n, \mathbf{B}_n)\) corresponding to the based root data \((R^\vee, \Delta^\vee)\) in the following manner,

\[
X_{\alpha^\vee} = \begin{pmatrix} E_{i,i+1} & 0_n \\ 0_n & -E_{i,i+1} \end{pmatrix} \in \text{Lie}(\mathbf{B}_n) \quad \text{for } 1 \leq i \leq n-1,
\]

where \(E_{i,j} = (\delta_{i,j})\), and

\[
X_{\alpha_i^\vee} = h \begin{pmatrix} 1 & 0_n \\ 0 & -1 \end{pmatrix} \in \text{Lie}(\mathbf{B}_n).
\]

Finally, using the equivalence of categories given in [10, Theorem 6.1.17], we identify \(\text{SO}_{2n}\) over \(\mathbb{C}\) with the dual group of \(\text{SO}_{2n}\) over \(F\). We thus fix this identification.
(The *-action). Since $\text{SO}_n^* = \text{SO}(q_{E,n})$ is non-split, we have a non-trivial action of $\Gamma_F$ on $(R^\vee, \Delta^\vee)$. Moreover, if we denote $w = \begin{pmatrix} 1_{n-1} & 0 & 1 \\ 1 & 0 & 1_{n-1} \end{pmatrix}$, we have that \((g \mapsto wgw^{-1}) \in \text{Aut}(\text{SO}_n, T_n, \{X_a\}_{a \in \Delta^\vee})\) corresponds, via the equivalence of categories \([10, \text{Theorem 6.1.17}]\), to the non-trivial automorphism in $\text{Aut}(R^\vee, \Delta^\vee)$. Thus, the induced action of $\Gamma_F$ is given by

\[
\varphi: \Gamma_F \to \text{Aut}(\text{SO}_n)
\]

\[
\tau \mapsto \begin{cases} 
(g \mapsto wgw^{-1}) & \text{if } \tau \not\in \text{Gal}(F_s/E) \\
(g \mapsto g) & \text{if } \tau \in \text{Gal}(F_s/E)
\end{cases}
\]

Putting these identifications together, we have that the $L$-group of $\text{SO}_{2n}^*$ over $F$ can be identified with the following semidirect product

$$\text{SO}_{2n}(\mathbb{C}) \rtimes \varphi \Gamma_F.$$ 

In the case of the split groups $\text{GL}_m$ and $\text{SO}_{2n}$ over $F$, their $L$-groups can be identified with the following direct product

$$\text{GL}_m(\mathbb{C}) \times \Gamma_F \quad (\text{resp. } \text{SO}_{2n}(\mathbb{C}) \times \Gamma_F).$$

3. Langlands functoriality and Converse theorem

In this section we state the main result of this article and we start with the first steps of the proof of generic functoriality. The strategy is inspired from [9] and [36].

Suppose that $F$ is a function field in one variable over a finite field. We let for $x \in |F|$, $G_x = G_{F_x}$ the group obtained from $G$ by extending scalars along the inclusion $F \hookrightarrow F_x$. We choose a separable closure $F_{x,s}$ of $F_x$ and an embedding $F_s \hookrightarrow F_{x,s}$ for every $x \in |F|$ that extends $F \hookrightarrow F_x$. We write $\Gamma_F = \text{Gal}(F_s/F)$, $\Gamma_{F_x} = \text{Gal}(F_{x,s}/F_x)$ for every $x \in |F|$ and we let $I_{F_x}$ be the inertia subgroup of $\Gamma_{F_x}$, for every $x \in |F|$. These choices give us an injection $\Gamma_{F_x} \hookrightarrow \Gamma_F$.

Now the restriction along $\Gamma_{F_x} \hookrightarrow \Gamma_F$ induces a (continuous) group homomorphism from $L G_x$ to $L G$, that fits into a commutative diagram

$$
\begin{array}{ccc}
L G_x & \longrightarrow & L G \\
\downarrow & & \downarrow \\
\Gamma_{F_x} & \longrightarrow & \Gamma_F
\end{array}
\]

Now given an $L$-homomorphism $\rho: L G \to L H$, for every place $x$ we can form an $L$-homomorphism $\rho_x: L G_x \to L H_x$. 

3.1. Satake parametrization. For every \( x \in |F| \), we denote by \( W_{F_x} \) the Weil group of \( F_x \). We fix a geometric Frobenius element \( \text{Fr}_x \in W_{F_x} \). We denote by \( \Phi(G_x) \) the set of group morphisms \([4, \text{Section 8}]
\begin{align*}
\phi : W'_{F_x} = W_{F_x} \times \text{SL}_2(\mathbb{C}) & \rightarrow L G_x,
\end{align*}

such that \( \phi(\text{Fr}_x) \) is semi simple, \( \phi|_{I_{F_x}} \) is continuous, \( \phi|_{\text{SL}_2(\mathbb{C})} \) is algebraic and \( \phi \) is relevant, i.e. if the image of \( \phi \) is contained in a Levi subgroup of \( L G_x \) then it is the \( L \)-group of a Levi subgroup of \( G_x \), modulo \( G_x^s \)-conjugacy classes of parameters. Moreover, when \( \phi|_{I_{F_x}} \) and \( \phi|_{\text{SL}_2(\mathbb{C})} \) are trivial, \( \phi \) will be called unramified. We denote by \( \Phi_{\text{unr}}(G_x) \) the set of these classes.

Thus, we also obtain a bijection between unramified representations and unramified parameters \([4, \text{Section 9.5}]\),
\begin{align*}
\Pi_{\text{unr}}(G_x) & \rightarrow \Phi_{\text{unr}}(G_x),
\pi & \mapsto \phi_{\pi},
\end{align*}

3.2. Functoriality conjecture. Let
\begin{align*}
\rho : L G \rightarrow L H
\end{align*}
be an \( L \)-homomorphism between the \( L \)-groups of a reductive group \( G \) and a quasi-split reductive group \( H \) over \( F \), respectively. Let \( \pi = \bigotimes'_{x} \pi_x \) be a cuspidal automorphic representation of \( G(\mathbb{A}_F) \). A (weak) lift or a transfer of \( \pi \) through \( \rho \) is an automorphic representation \( \Pi = \bigotimes'_{x} \Pi_x \) of \( H(\mathbb{A}_F) \), such that the following commutative diagram commutes
\begin{align*}
\begin{array}{ccc}
L G_x & \xrightarrow{\rho_x} & L H_x \\
\phi_{\tau x} & & \phi_{\tau x} \\
W'_{F_x} & \xleftarrow{\phi_{\pi x}} & W_{F_x}
\end{array}
\end{align*}
for all places \( x \) such that \( H_x, G_x, \pi_x \) and \( \Pi_x \) are unramified. When \( \rho \) is clear from the context we will just say that it is a lift or a transfer of \( \pi \). The functoriality conjecture states that such \( \Pi \) always exists, for every cuspidal automorphic representation \( \pi \) and \( L \)-homomorphism \( \rho \).

Our main objective is to prove the existence of such a transfer for \( G = \text{SO}_{2n}^*, H = \text{GL}_{2n}, \pi \) globally generic and \( \rho \) given by
\begin{align*}
\rho_{2n}^* : \text{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_F & \rightarrow \text{GL}_{2n}(\mathbb{C}) \times \Gamma_F \\
(g, \tau) & \mapsto \begin{cases} 
(gw, \tau) & \text{if } \tau \not\in \text{Gal}(F_s/E) \\
(g, \tau) & \text{if } \tau \in \text{Gal}(F_s/E)
\end{cases}.
\end{align*}

(3.2.1)

3.3. Candidate lift. For every place \( x \) we choose a character \( \lambda_x \) of \( T(F_x) \) such that for \( \pi_x \) unramified it is the character obtained by the Satake parametrization and for \( \pi_x \) ramified, \( i_{B_{SO_{2n}(F_x)}}(\lambda_x) \) has an (irreducible) generic subquotient \( \pi'_x = \pi_x' \) with the same central character as \( \pi_x \) (See for example \([9, \text{Section 4.2}]\)). Applying the local Langlands correspondence for tori, from \( \lambda_x \) we get \( \phi_{\lambda_x} : W'_{F_x} \rightarrow T_n(\mathbb{C}) \rtimes \Gamma_{F_x} \).

Let \( i_x : T_n(\mathbb{C}) \rtimes \Gamma_{F_x} \hookrightarrow \text{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{F_x} \) be the inclusion homomorphism. Then, applying the local Langlands correspondence for general linear groups to
\begin{align*}
\rho_{2n,x}^* \circ i_x \circ \phi_{\lambda_x} : W'_{F_x} \rightarrow \text{GL}_{2n}(\mathbb{C}) \times \Gamma_{F_x},
\end{align*}
we find an admissible representation $\Pi_x$ of $\text{GL}_{2n}(F_x)$. We put $\Pi = \bigotimes'_x \Pi_x$, which is an irreducible admissible representation of $\text{GL}_{2n}(\mathbb{A}_F)$. We call $\Pi$ (resp. $\Pi_x$) a candidate lift or candidate transfer of $\pi$ (resp. $\pi_x$). The representation $\Pi$ is not necessarily a transfer because it is not necessarily an automorphic representation of $\text{GL}_{2n}(\mathbb{A}_F)$. But, together with the converse theorem of Section 3.6, we will use it in order to construct in Section 5 the desired lift.

3.4. A description of $\Pi_x$. Now for every place $x$, we will give a description of $\Pi_x$. First, let us note that by definition

$$\text{SO}^*_{2n}(F_x) = \text{SO}(q_{E_x,n})(F_x),$$

where $E_x := E \otimes_F F_x$ is a degree two étale algebra over $F_x$. Thus, it is either a product of two (separable and trivial) fields extensions or a (separable) field extension over $F_x$. Let us concentrate on the case when $E_x$ is a quadratic (separable) extension of $F_x$ (i.e. $x$ is an inert place), for which we have an embedding $E_x \hookrightarrow F_{x,s}$, coming from the one fixed in the beginning of this chapter $F_s \hookrightarrow F_{x,s}$.

Now let us consider torus $T' = \mathbb{G}_m^{2n-2} \times \text{Res}_{E_x/F_x}(\mathbb{G}_m)$, where $\mathbb{G}_m$ is the multiplicative group. We have an isomorphism

$$E_x \otimes_{F_x} F_{x,s} \cong \prod_{\sigma \in \text{Hom}(E_x,F_{x,s})} F_{x,s}.$$

This leads us to an isomorphism

$$X_*(T'_{F_{x,s}}) = X_*(\mathbb{G}_m^{2n-2} \times X_*(\text{Res}_{E_x/F_x}(\mathbb{G}_m))_{F_{x,s}}) \cong \mathbb{Z}^{2n-2} \times \mathbb{Z}^2,$$

where $X_*$ denotes the groups of cocharacters and the non-trivial action of the second factor, $\mathbb{Z}^2$, is given by

$$\Gamma_{F_x} \to \text{Aut}(\mathbb{Z}^2)$$

$$\tau \mapsto \begin{cases} (a_1, a_2) \mapsto (a_2, a_1) & \text{if } \tau \notin \text{Gal}(F_{x,s}/E_x) \\ (a_1, a_2) \mapsto (a_1, a_2) & \text{if } \tau \in \text{Gal}(F_{x,s}/E_x) \end{cases}.$$

If we denote by $D_{2n} \subset \text{GL}_{2n}$ the maximal diagonal torus, we can identify

$$D_{2n}(\mathbb{C}) \times \Gamma_{F_x} \cong {^L T}_x'$$

where the action on the left hand side is given by conjugation by $w$.

Now thanks to this description we can construct the following embeddings

$$\iota_x : T_n(\mathbb{C}) \times \Gamma_{F_x} \hookrightarrow D_{2n}(\mathbb{C}) \times \Gamma_{F_x} \cong {^L T}_x'$$

$$(t, \tau) \mapsto ((t, t^{-1}), \tau)$$

and

$$D_{2n}(\mathbb{C}) \times \Gamma_{F_x} \hookrightarrow \text{GL}_{2n}(\mathbb{C}) \times \Gamma_{F_x}$$

$$(d, \tau) \mapsto \begin{cases} (dw, \tau) & \text{if } \tau \notin \text{Gal}(F_{x,s}/E_x) \\ (d, \tau) & \text{if } \tau \in \text{Gal}(F_{x,s}/E_x) \end{cases}.$$
We can thus factor $\rho_{2n,x}^* \circ i_x$.

\[
\begin{array}{c}
LT_x \\
\downarrow \\
\downarrow
\end{array} \xrightarrow{\rho_{2n,x}^* \circ i_x} \GL_{2n}(\mathbb{C}) \times \Gamma_{F_x}.
\]

The definition of the candidate lift and previous factorization lead us to look at

\[
\begin{array}{ccc}
\Phi(T_x) & \longrightarrow & \Phi(T'_x) \\
\downarrow & & \downarrow \\
\Pi(T_x) & \longrightarrow & \Pi(T'_x)
\end{array}
\]

where the first two vertical arrows are the ones given by the local Langlands correspondence for algebraic tori, the third one is given by the local Langlands correspondence for general linear group and the upper horizontal arrows are the ones obtained from composition with $\rho_{2n,x}^* \circ i_x$ (and its factorization).

Let $\lambda = \chi_{1,x} \otimes \cdots \otimes \chi_{n-1,x} \otimes \chi_{n,x}$ be a character of $T(F_x) = (F_x^\times)^{n-1} \times SO_2^*(F_x)$, where $\chi_{i,x}$ is a character of $F_x^\times$ for $1 \leq i \leq n-1$ and $\chi_{n,x}$ is a character of $SO_2^*(F_x)$. The image of $\lambda$ is

\[
\Lambda_x = \chi_{1,x} \otimes \cdots \otimes \chi_{n-1,x} \otimes \chi_{n,x} \otimes \chi_{-1,x} \otimes \cdots \otimes \chi_{1,x},
\]

where $\mu_{n,x} : E_x^\times \rightarrow \mathbb{C}^\times$ is the character obtained from $\chi_{n,x} : SO_2^*(F_x) \rightarrow \mathbb{C}^\times$ via

\[
\Phi(SO_2^*) \rightarrow \Phi(Res_{E_x/F_x} \mathbb{G}_m).
\]

To specify $\mu_{n,x}$, let $\phi_x$ be the parameter of $\chi_{n,x} = \pi_{\phi_x}$. Using [34, p. 235], we have that

\[
\pi_{(\phi_x,E_x)} : SO_2^*(E_x) \xrightarrow{\text{Norm}} SO_2^*(F_x) \xrightarrow{\pi_{\phi_x}} \mathbb{C}^\times,
\]

where $\pi_{(\phi_x,E_x)}$ is the representation of $SO_2^*(E_x)$ corresponding to the parameter in $\Phi(\mathbb{G}_m,E_x)$ obtained from $\phi_x$ via the restriction $E_x \rightarrow W_{E_x/F_x}$. We can give a description of this construction using the identification between $SO_2^*$ and the norm one elements of $Res_{E_x/F_x} \mathbb{G}_m$ in Section 2.2. Indeed, we have that $SO_2^*(F_x) = E_x^\times$, $SO_2^*(E_x) = E_x^\times$ and the corresponding norm map, also denoted by Norm, is given by

\[
\text{Norm} : E_x^\times \rightarrow E_x^1
\]

\[
x \mapsto x\sigma(x)^{-1},
\]

and thus

\[
\pi_{(\phi_x,E_x)} : E_x^\times \rightarrow \mathbb{C}^\times
\]

\[
x \mapsto \chi_{n,x}(x\sigma(x)^{-1}).
\]

On the other hand, we recall that we have an isomorphism (Shapiro's Lemma) [4, Proposition 8.4]

\[
\Phi(\text{Res}_{E_x/F_x} \mathbb{G}_m) \rightarrow \Phi(\mathbb{G}_m,E_x)
\]

\[
(W_F \rightarrow L_{\Gamma_{E_x} \mathbb{G}_m}(\mathbb{C}) \leftarrow \Gamma_{E_x}) \rightarrow (W_{E_x} \rightarrow \mathbb{G}_m(\mathbb{C}) \times \Gamma_{E_x}),
\]

where $L_{\Gamma_{E_x} \mathbb{G}_m}$ is the functor of induction from $\Gamma_{E_x}$ to $\Gamma_{E_x}$. The image of $t_x \circ \phi_x \in \Phi(\text{Res}_{E_x/F_x} \mathbb{G}_m)$ through this isomorphism is the parameter in $\Phi(\mathbb{G}_m,E_x)$ obtained from $\phi_x$ via the restriction.
\( E_x \rightarrow W_{E_x/F_x} \). Therefore, since \( \mu_{n,x} \) is the character of \( E_x^\times \) corresponding to the parameter \( \iota_x \circ \phi_x \), we have that

\[
(3.4.1) \quad \mu_{n,x} = \pi_{(\phi_x, E_x)}.
\]

Now for the second square, first we look at the image of the parameter corresponding to \( \mu_{n,x} \) via

\[
\Phi(\text{Res}_{E_x/F_x} \mathbb{G}_m) \rightarrow \Phi(\text{GL}_2).
\]

First, using again the identification given above (Shapiro’s Lemma),

\[
\Phi(\text{Res}_{E_x/F_x} \mathbb{G}_m) \rightarrow \Phi(\mathbb{G}_{m,E_x}) \rightarrow \Pi(E^\times),
\]

we have that the image of \( \mu_{n,x} \) corresponds to the Weil-Deligne representation

\[
(\text{Ind}_{E_x/F_x}(\mu_{n,x} \circ \text{ar}_{E_x}), 0),
\]

where \( \text{ar}_{E_x} = r_{E_x}^{-1} \) is the reciprocity map [41, IV. (6.3)] and \( \text{Ind}_{E_x/F_x} \) is the functor of smooth induction from \( \Gamma_{E_x} \) to \( \Gamma_{E_x} \). Now using the local Langlands correspondence for \( \text{GL}_2 \) ([6, Chapter 8]), we get our admissible irreducible representation of \( \text{GL}_2 = \text{GL}_2(F_x) \):

\[
\Pi_{\mu_{n,x}} = \begin{cases} 
\text{Ind}_{E_x/F_x}(\nu_{n,x} \otimes \chi_{n,x} \nu_{n,x}), & \text{if } \mu_{n,x} = \nu_{n,x} \circ \text{N}_{E_x/F_x}, \text{ for some character } \nu_{n,x} \text{ of } F_x^\times, \\
\pi_{\mu_{n,x}}, & \text{otherwise},
\end{cases}
\]

where \( \chi_{n,x} = \det(\text{Ind}_{E_x/F_x} 1_{E_x}) \) and \( \pi_{\mu_{n,x}} \) is constructed in [6, Chapter 8], from a character \( \mu_{n,x} : E_x^\times \rightarrow \mathbb{C}^\times \).

Putting all this together we get an expression for \( \Pi_x \). In particular for \( \lambda_x = (\chi_{1,x}, \ldots, \chi_{n-1,x}, 1) \) unramified we have that \( \Pi_x \) is the constituent of

\[
i_B^G(\chi_{1,x} \otimes \cdots \otimes \chi_{n-1,x} \otimes 1 \otimes \chi_x \otimes \chi_{n-1,x}^{-1} \otimes \cdots \otimes \chi_{1,x}^{-1}),
\]

that has a nonzero vector fixed under the special maximal compact subgroup \( \text{GL}_{2n}(\mathcal{O}_x) \).

Finally, we note that this construction gives us that the central character of \( \Pi_x \) is

\[
\chi_{\mu_{n,x}}|_{F_x^\times} = \chi_x.
\]

Thus, the global character

\[
(3.4.2) \quad \chi = \otimes' \chi_x,
\]

is trivial on \( F_x^\times \).

3.5. Rankin-Selberg factors. To obtain an automorphic representation, we use as starting point the converse theorem. This theorem allows us to translate the existence of an automorphic representation \( \Pi \) of \( \text{GL}_{2n}(\mathbb{A}_F) \) to the existence of an admissible representation \( \text{GL}_{2n}(\mathbb{A}_F) \) with the right properties on its \( L \)-functions. We present the \( L \)-functions and then we will describe the properties needed for this translation: the converse theorem.

Recall that we denote by \( q_F \) or simply by \( q \) the cardinality of its field of constants and by \( q_{F_x} \) the cardinality of the residue field of \( F_x \).

(Local) For every \( x \in \mathcal{O}_x \), let us consider a smooth non-trivial character \( \psi_x : F_x \rightarrow \mathbb{C}^\times \) and a pair of irreducible smooth representations \( \pi_x \) and \( \pi'_x \) of \( \text{GL}_n(F_x) \) and \( \text{GL}_m(F_x) \), respectively. We define as in [22] the (local) Rankin-Selberg \( L \)-functions, \( \varepsilon \)-factors and \( \gamma \)-factors:

\[
L(s, \pi_x \times \pi'_x), \quad \varepsilon(s, \pi_x \times \pi'_x, \psi_x) \quad \& \quad \gamma(s, \pi_x \times \pi'_x, \psi_x).
\]
Remark 3.5.1. We remark that from [19, Corollary 3.8], in the where case where \( \pi_x, \pi'_x \) are generic representations of \( \text{GL}_n(F_x) \) and \( \text{GL}_m(F_x) \) respectively, the polynomials \( L(s, \pi_x \times \pi'_x) \) and \( \varepsilon(s, \pi_x \times \pi'_x, \psi_x) \) coincide with the corresponding factors \( L(s, \pi_x \otimes \pi'_x, \rho) \) and \( \varepsilon(s, \pi_x \otimes \pi'_x, \psi_x, r) \) obtained using the Langlands-Shahidi method (Section 4.1). The maximal parabolic subgroup considered contains the upper triangular matrices and its Levi subgroup is isomorphic to \( \text{GL}_n \times \text{GL}_m \).

(Global). Let \( \psi = \bigotimes_x \psi_x : \mathbb{A}_F/F \to \mathbb{C} \) be a continuous non-trivial character, \( \pi = \bigotimes_x \pi_x \) an admissible representation of \( \text{GL}_n(\mathbb{A}_F) \) and \( \pi' = \bigotimes_x \pi'_x \) an admissible representation of \( \text{GL}_{n'}(\mathbb{A}_F) \). We assume that \( \pi_x \) and \( \pi'_x \) are irreducible. We define the (global) Rankin-Selberg \( L \)-functions, \( \varepsilon \)-factors and \( \gamma \)-factors:

\[
L(s, \pi \times \pi') = \prod_x L(s, \pi_x \times \pi'_x), \quad \text{as a formal power series in } q^{-s},
\]

\[
\varepsilon(s, \pi \times \pi', \psi) = \prod_x \varepsilon(s, \pi_x \times \pi'_x, \psi_x) \text{ monomial function in } q^{-s}.
\]

3.6. Converse Theorem. In this Section, based on [7] and [30], we provide a proof of the twisted version of the converse theorem found in [9, Section 2] for an admissible irreducible representation in positive characteristic. This result is stated in [35, Theorem 8.1], and we now take the opportunity to provide a proof.

Let \( S \) be a finite subset of \( |F| \). For each integer \( m \), let

\[
\mathcal{A}_0(m) = \{ \tau \mid \tau \text{ is a cuspidal automorphic representation of } \text{GL}_m(\mathbb{A}_F) \},
\]

and

\[
\mathcal{A}^S_0(m) = \{ \tau \in \mathcal{A}_0(m) \mid \tau_v \text{ is unramified for all } v \in S \}.
\]

For \( n \geq 2 \), we set

\[
\mathcal{T}(n-1) = \prod_{m=1}^{n-1} \mathcal{A}_0(m) \quad \text{and} \quad \mathcal{T}^S(n-1) = \prod_{m=1}^{n-1} \mathcal{A}^S_0(m).
\]

If \( \eta \) is a continuous character \( F^\times \setminus \mathbb{A}_F^\times \), set

\[
\mathcal{T}(S; \eta) = \mathcal{T}^S(n-1) \cdot \eta = \{ \tau = (\tau' \cdot \eta) : \tau' \in \mathcal{T}^S(n-1) \},
\]

where \( \tau' \cdot \eta \) is the representation given by \( \tau' \otimes (\eta \circ \det) \).

Theorem 3.6.1. Let \( n \) be an integer greater or equal than 2, \( \pi = \bigotimes_x \pi_x \) an irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_F) \) and \( \eta \) a continuous character of \( \mathbb{A}_F^\times \) trivial on \( F^\times \).

We suppose that, for a finite subset \( S \) of places, \( \pi \) satisfies the following properties:

1. The central character \( \chi_\pi = \bigotimes_x \chi_{\pi_x} \) of \( \pi \) is invariant by the discrete subgroup \( F^\times \) of \( \mathbb{A}_F^\times \).

2. For all \( \pi' \in \mathcal{T}(S; \eta) \), the formal series

\[
L(s, \pi \times \pi') \quad \text{and} \quad L(s, \tilde{\pi} \times \tilde{\pi}')
\]

are polynomials and they satisfy the functional equation

\[
L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi', \psi)L(1-s, \tilde{\pi} \times \tilde{\pi}').
\]

Then there exists an irreducible automorphic representation \( \rho \) of \( \text{GL}_n(\mathbb{A}_F) \) such that, for each place \( x \notin S \) such that \( \pi_x \) is unramified, \( \rho_x \) is unramified and \( \pi_x \cong \rho_x \). Moreover, \( \rho \) is cuspidal if \( S = \emptyset \).
In order to prove this, first we introduce some notation. Secondly, we review some notions about Whittaker models. Thirdly, we prove a relation between Rankin-Selberg $L$-functions. Finally, we introduce further notations and we give a proof of converse theorem.

We denote by $U_{n}$ the radical unipotent subgroup of the Borel $F$-subgroup $B_{n}$ of upper triangular matrices of $GL_{n}$. We also denote by $M_{m,n}$ the algebraic group of $(m \times n)$-matrices over $F$. Finally, let us also write

$$K = \prod_{x} K_{x} = \prod_{x} GL_{n}(O_{x}).$$

It is the maximal open compact subgroup of $GL_{n}(\mathbb{A}_{F})$, and $GL_{n}(\mathbb{A}_{F})$ is the restricted product of $GL_{n}(F_{x})$ with respect to the $K_{x} = GL_{n}(O_{x})$.

If $\psi$ is a non-trivial character of either $F_{x}$ or $\mathbb{A}_{F}$, then we use $\psi$ to also denote the character of either $U_{n}(F_{x})$ or $U_{n}(\mathbb{A}_{F})$, that associates to $u = (u_{i,j})$ the complex number

$$\psi(u) = \sum_{i=1}^{n} \psi(u_{i,i+1}).$$

(Whittaker models and induced smooth representations of Whittaker type). First, we recall that the induced (smooth) representations of Whittaker type are representations of the form

$$\psi_{\lambda}(\pi_{x},\psi_{x})_{\psi_{\lambda}}(\rho_{1,x}|\det|_{u_{1,x}} \otimes \cdots \otimes \rho_{m_{x},x}|\det|_{u_{m_{x},x}}),$$

where $Q$ is a parabolic subgroup containing $B_{n}$ associated to a partition $(r_{1,x}, \ldots, r_{m_{x},x})$ of $n$, $\rho_{i,x}$ is an irreducible square-integrable representation of $GL_{r_{i,x}}(F_{x})$ for every $1 \leq i \leq m_{x}$ and the $u_{i,x}$’s are real numbers satisfying $u_{1,x} \leq \cdots \leq u_{m_{x},x}$. Every induced representation of Whittaker type $\pi_{x}$ of $GL_{n}(F_{x})$ admits a $\psi_{x}$-Whittaker model for some non-trivial character $\psi_{x}: F_{x} \to \mathbb{C}^{\times}$, i.e. the space $W(\pi, \psi_{x})$ spanned by functions on $GL_{n}(F_{x})$ of the form

$$g \mapsto \lambda_{\psi_{x}}(\pi_{x}(g)\xi_{x}),$$

where $\xi_{x}$ is a vector of $\pi_{x}$ and $\lambda_{\psi_{x}}: V \to \mathbb{C}$ is a functional such that

$$\lambda_{\psi_{x}}(\pi_{x}(u)v) = \psi_{x}(u)\lambda_{\psi_{x}}(v),$$

for all $u$ in $U_{n}(F_{x})$ and $v$ in $\pi_{x}$. Once the non-zero Whittaker functional is fixed, we denote such function by $W_{\xi_{x}}$. Note that each such function $W_{\xi_{x}}$ is right-invariant under some open subgroup of $GL_{n}(F_{x})$ and the collection of these functions satisfies the following relation:

$$W_{\xi_{x}}(u_{x}g_{x}) = \psi_{x}(u_{x})W_{\xi_{x}}(g_{x}),$$

for every $g_{x} \in GL_{n}(F_{x})$, $u_{x} \in U_{n}(F_{x})$.

Globally, let $\pi = \bigotimes_{x}^{\prime} \pi_{x}$ be an admissible representation of $GL_{n}(\mathbb{A}_{F})$, where $\pi_{x}$ is induced of Whittaker type with fixed Whittaker functional. We can choose $K_{x}$-fixed vectors $\xi_{x}^{0}$, for $x$ outside some finite subset $T$ of $|F|$, such that $W_{\xi_{x}^{0}} \in W(\pi_{x}, \psi_{x})$ is invariant under right multiplication by the compact open subgroup $GL_{n}(O_{x})$ and it is equal to 1 at the identity. Now, for every vector $\xi = (\xi_{x})_{x \in |F|}$ of $\pi$, such that $\xi_{x} = \xi_{x}^{0}$ for almost all $x$, we consider the complex valued function on $GL_{n}(\mathbb{A}_{F})$ given by

$$W_{\xi}: g = (g_{x})_{x} \mapsto \prod_{x} W_{\xi_{x}}(g_{x}).$$

Each such $W_{\xi}$ is right-invariant under by some open compact subgroup of $GL_{n}(\mathbb{A}_{F})$ and satisfies

$$W_{\xi}(ug) = \psi(u)W_{\xi}(g),$$

for every $g \in GL_{n}(\mathbb{A}_{F})$, $u \in U_{n}(\mathbb{A}_{F})$. 


This function will be our main in ingredient constructing a non-zero equivariant homomorphism to the space of automorphic forms.

(Twist). Now, we briefly recall the definition of Rankin-Selberg $L$-functions of representation of Wittaker type and we prove a relation between them.

Let $\tau$ and $\tau'$ be induced representations of Whittaker type of $\text{GL}_n(F_x)$ and of $\text{GL}_m(F_x)$, respectively. We define for any $W \in \mathcal{W}(\tau, \psi_x)$, $W' \in \mathcal{W}(\tau', \psi_x)$, and any compactly supported locally constant function $\Phi : F_x^n \rightarrow \mathbb{C}$, the following local integrals, which define rational functions in $\mathbb{C}(q_{F_x}^{-s})$ [22, Theorem 2.7]. In the case where $m < n$, for $0 \leq j \leq n - m - 1$, we denote

$$
\Psi_j(s; W, W') = \int_{U_m(F_x) \backslash \text{GL}_m(F_x)} \int_{M_{j,m}(F_x)} W'(h) \det(h)^{s - (n - m)/2} dh.
$$

In the case where $m = n$, we put

$$
\Psi(s; W, W', \Phi) = \int_{U_m(F_x) \backslash \text{GL}_m(F_x)} W(g) W'(g) \Phi((0, \ldots, 0, 1) g) \det(g)^s dg.
$$

These integrals form $\mathbb{C}[q_{F_x}^{-s}, q_{F_x}^{+s}]$-fractional ideals $I(\tau, \tau')$ in the case where $n = m$, and $I_j(\tau, \tau')$, in the case where $m < n$, for $0 \leq j \leq n - m - 1$, in $\mathbb{C}(q_{F_x}^{-s})$. The unique generator of these ideals has the form

$$
L(s, \tau \times \tau') = \frac{1}{P(q^{-s})}
$$

with $P(X) \in \mathbb{C}[X]$ a polynomial with $P(0) = 1$. This is the Rankin-Selberg $L$-function of $\tau \times \tau'$.

**Proposition 3.6.2.** Let $k$ a locally compact field of positive characteristic, $\eta : k^\times \rightarrow \mathbb{C}^\times$ a character and $(\tau, V), (\tau', V')$ two induced representations of Whittaker type of $\text{GL}_n(k)$ and $\text{GL}_m(k)$, respectively. Then

$$
L(s, \tau \times (\tau' \cdot \eta)) = L(s, (\tau \cdot \eta) \times \tau').
$$

**Proof.** By definition (Section 3.5), we notice that, after choosing a Whittaker functional $\Lambda : V \rightarrow \mathbb{C}$ of $\tau$, we can compute the function $W_\xi \in \mathcal{W}(\tau \cdot \eta, \psi)$ as follows

$$
W_\xi(g) = \eta(\det(g)) \Lambda(\tau(g) \xi) = \Lambda(\tau \cdot \eta(g) \xi).
$$

Now, let $\Lambda : V \rightarrow \mathbb{C}$ and $\Lambda' : V' \rightarrow \mathbb{C}$ be the respective Whittaker functionals of $\tau$ and $\tau'$, and $W_\xi \in \mathcal{W}(\tau, \psi)$ and $W_{\xi'} \in \mathcal{W}(\tau', \psi)$. Then using the identity 3.6 we get that

$$
\Psi(s; W_\xi, W_{\xi'}) = \Psi(s; \Lambda(\cdot) \xi, \eta(\det(\cdot)) \Lambda'(\tau'(\cdot) \xi')),
$$

if $n = m$, and

$$
\Psi_j(s; W_\xi, W_{\xi'}) = \Psi_j(s; \Lambda(\cdot) \xi, \eta(\det(\cdot)) \Lambda'(\tau'(\cdot) \xi')),
$$

if $m < n$ and $0 \leq j \leq n - m - 1$. As these relations imply the equality of the ideals, we have proved our desired relation. \qed
(Further subgroups of $\text{GL}_n$). Finally, we introduce some notations. We fix a normal and proper curve $X_F$ over $\mathbb{F}_q$ with field of fractions $F$. Denote by $P_n \subset \text{GL}_n$ the subgroup of matrices of the form

$$\begin{pmatrix} * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$ 

For every closed subscheme $N$ of $X_F$ supported on $S$ with the ring of global sections denoted by $\mathcal{O}_N$, we consider the finite index subgroup $K'_S(N)$ of $K_S = \prod_{x \in S} \text{GL}_n(\mathcal{O}_x)$ of matrices with image in $\text{GL}_n(\mathcal{O}_N)$ of the form

$$\begin{pmatrix} * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$ 

We denote $\text{GL}_n(\mathbb{A}_F)'(N)$ the open subgroup of $\text{GL}_n(\mathbb{A}_F)$ given by the inverse image of $K'_S(N)$ under $\text{GL}_n(\mathbb{A}_F) \to \prod_{x \in S} \text{GL}_n(F_x)$.

Now we go back to the proof of the converse theorem.

**Proof of Theorem 3.6.1.** For every $x \in |F|$ such that $\pi_x$ unramified, we fix a vector $v_x \in V^K_x$. For every $x$, let $\Xi_x$ be the representation of $\text{GL}_n(F_x)$ that has $\pi_x$ as its unique Langlands’ quotient. Every $\Xi_x$ is of the form

$$\Xi_x = i_{Q(F_x)}^{\text{GL}_n(F_x)}(\rho_{1,x}) \det |u_{1,x} \otimes \cdots \otimes \rho_{m_x,x}| \det |u_{m_x,x}|,$$

where $Q$ is a parabolic subgroup containing $B_n$ associated to a partition $(r_{1,x}, \ldots, r_{m_x,x})$ of $n$, $\rho_{i,x}$ is an irreducible tempered representation of $\text{GL}_{r_i,x}(F_x)$ for every $1 \leq i \leq m_x$ and the $u_{i,x}$'s are real numbers satisfying $u_{1,x} > \cdots > u_{m_x,x}$. We can reduce the theorem to the case $\eta = 1$. Indeed, by definition of Rankin-Selberg $L$-function and using Proposition 3.6.2 we have

$$L(s, \pi_x \times (\pi'_x \cdot \eta_x)) = L(s, \Xi_x \times (\Xi'_x \cdot \eta_x))$$

$$= L(s, (\Xi_x \cdot \eta_x) \times \Xi'_x) = L(s, (\pi_x \cdot \eta_x) \times \pi'_x),$$

we can apply Theorem 3.6.1, with trivial character, to $\pi \cdot \eta$. Therefore we have that there exists an automorphic representation $\Pi'$ such that $\Pi'_x \cong \pi_x \cdot \eta$ for $x \not\in S$ such that $\pi_x$ is unramified. Then $\Pi := \Pi' \cdot \eta^{-1}$ is automorphic and satisfies that $\Pi_x \cong \pi_x$ for $x \not\in S$ such that $\pi_x$ is unramified. Therefore, from now on, we will assume that $\eta = 1$.

Suppose that $S$ is not empty. For every $x \not\in S$ for which $\pi_x$ is unramified, $\Xi_x$ must have a unique $K_x$-fixed vector $\xi^0_x$ which projects to the fixed $K_x$-fixed $v_x$ vector of $\pi_x$. From these choices, we can consider for every $\xi = (\xi_x)_x$ such that $\xi_x = \xi^0_x$ for almost all $x \not\in S$, the global Whittaker function $W_\xi$ (3.6.1).

Now for every $x \in S$ such that $\pi_x$ is ramified, we can choose $\xi^0_x$ such that $(\xi^0_x)_{x \in S}$ is $K'_S(N)$-invariant for some subscheme $N$ of $X_F$, supported on $S$, and ([7, Section 8 & p. 203])

$$W_{\xi^0}(1) = 1.$$
Thus, $\xi^x$ is invariant under right multiplication by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$, with $h \in \text{GL}_{n-1}(O_x)$, for every $x \in S$.

Finally we consider as in [30, Corollaire B.15], the well defined function on $\text{GL}_n(\mathbb{A}_F)$

$$U_{\xi}(g) = \sum_{\gamma \in U_n(F) \backslash P_n(F)} W_{\xi}(\gamma g).$$

Putting these together we are able to consider, for every $\xi^S = (\xi^S)_{x \notin S}$ completed by $\xi = (\xi^S, (\xi^S_x)_{x \in S})$, the function $U_{\xi^S}$ on $\text{GL}_n(\mathbb{A}_F)$ defined by

$$U_{\xi^S}(g) = U_{\xi}(g'),$$

if $g$ can be written as $g = \gamma g'$ with $\gamma \in \text{GL}_n(F)$ and $g' \in \text{GL}_n(\mathbb{A}_F)'_S(N)$ and, if not, by

$$U_{\xi^S}(g) = 0.$$

The map $\xi^S \mapsto U_{\xi^S}$ defines a non-zero [7, Lemma 6.3] equivariant homomorphism of the smooth admissible representation $\Xi^S = \bigotimes_{x \notin S} \Xi_x \otimes \prod_{x \notin S} \text{GL}_n(F_x)$ to the space of functions on $\text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F)$ that are invariant under right multiplication by open compact subgroups of $\text{GL}_n(\mathbb{A}_F)$ [30, p. 237]. The action of the center $Z_n(\mathbb{A}_F)$ of $\text{GL}_n(\mathbb{A}_F)$ on the span of these functions is according to the central character $\chi_\pi$ of $\pi$.

Since $\Xi^S$ has $\Pi^S = \bigotimes_{x \notin S} \Pi_x$ as its unique irreducible quotient, if we take a vector $\xi^S$ which has a non-zero projection to $\Pi^S$, then $\xi^S$ is a cyclic generator of $\Xi^S$. Thus the representation $V$ of $\text{GL}_n(\mathbb{A}_F)$ generated by the space of $U_{\xi^S}$ is admissible [5, Section 5] and cyclic, generated by some element $f_0$. Let $U$ be a maximal $\text{GL}_n(\mathbb{A}_F)$-invariant subspace of $V$ not containing $f_0$. Then $\Pi' = V/U$ is a non-zero subquotient of the space of automorphic forms; $\Pi'$ is automorphic and at every place $x \notin S$ where $\pi_x$ is unramified, its Satake parameter equal to the one of $\pi_x$ [7, Theorem A].

In the case where $S$ is empty, we just consider $(\xi \mapsto U_{\xi})$. As $U_{\xi}$ is cuspidal [21, Proposition 12.3], we can conclude as before. \hfill $\square$

Next, the tool that will allow us to establish the desired properties of the $L$-functions is provided by the Langlands-Shahidi method.

4. The Langlands-Shahidi Method

The Langlands-Shahidi method in positive characteristic [37], studies $\gamma$-factors, $\varepsilon$-factors and $L$-functions for generic representations associated to irreducible constituents of the adjoint representation of $L^1M$ on $L^n$, where $n$ is the Lie algebra of $L^1N$.

Let $P = MN$ be a maximal parabolic subgroup of $\text{SO}^*_n$ containing $B$. Given the structure given in Section 2.2, we have an isomorphism $M \cong \text{GL}_m \times \text{SO}^*_n$, where $m$ and $n$ are greater than 1. In this case, we obtain the following decomposition of the adjoint representation

$$r = r_1 \oplus r_2,$$

where $r_1 = \rho_m \otimes \rho^*_n$ and $r_2 = \wedge^2 \rho_m \otimes 1_{\text{SO}^*_n}$ [44, p. 565]. Here $\rho_m$ is the standard representation of $L^1\text{GL}_m(\mathbb{C})$, $\rho^*_n$ the representation of $L^1\text{SO}^*_n(\mathbb{C})$ constructed in Section 3.2 eq. (3.2.1) and $1_{\text{SO}^*_n}$ is the trivial representation of $L^1\text{SO}^*_n(\mathbb{C})$. Thus we obtain two instances of Langlands-Shahidi $L$-functions, $\varepsilon$-factors and $\gamma$-factors.
4.1. **Local construction.** Suppose $F$ is a locally compact field of positive characteristic. Recall that the cardinality of the residue field is denoted by $q_F$. Let $E$ be a separable quadratic extension of $F$ contained in a fix separable closure $F_s$, with Galois group $\text{Gal}(E/F) = \{1, \sigma\}.

Let $\psi: F \to \mathbb{C}^\times$ be a smooth non-trivial character, $\pi$ a generic representation of $\text{SO}_n^*(F)$ and $\tau$ a generic representation of $\text{GL}_m(F)$. Then $\tau \otimes \bar{\pi}$ ($\bar{\pi}$ is the contragredient of $\pi$) is a generic representation of $\text{M}(F)$.

- For $r_1$, we denote the corresponding local factors by
  $$\gamma(s, \pi \times \tau, \psi) \quad \varepsilon(s, \pi \times \tau, \psi) \quad L(s, \pi \times \tau).$$

  Observe that if we allow $E = F \times F$, we obtain the local construction for the split groups $\text{SO}_{2n}(F)$, already studied in [35].

- For $r_2$, we denote the corresponding local factors by
  $$\gamma(s, \tau, \lambda^2 \rho_m, \psi) \quad \varepsilon(s, \tau, \lambda^2 \rho_m, \psi) \quad L(s, \tau, \lambda^2 \rho_m).$$

These factors have already been studied in detail in [18].

Among the properties of these factors, we would like to highlight the multiplicativity property [36, Section 5].

**(r_1 case).** We have the following two versions. Let $M_1 = \text{GL}_m \times \text{GL}_n \times \cdots \times \text{GL}_{n_1} \times \text{SO}_{2n_0}^* \subset M$, and suppose that $\pi$ is the generic subquotient of

$$i_{P_1}^{\text{SO}_{2n}(F)}(\pi_b \otimes \cdots \otimes \pi_1 \otimes \pi_0),$$

where $P_1 = M_1 N_1$ is the parabolic subgroup of $\text{SO}_{2n}$, containing $B$, $\pi_i$ is a generic representation of $\text{GL}_{n_i}(F)$ for $1 \leq i \leq b$ and $\pi_0$ is a generic representation of $\text{SO}_{2n_0}^*(F)$. Then the multiplicativity property [36, Section 5] gives us

$$(4.1.1) \quad \gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi_0 \times \tau, \psi) \prod_{i=1}^b \gamma(s, \pi_i \times \tau, \psi) \gamma(s, \pi_i \times \tau, \psi),$$

where $\gamma(s, \pi_i \times \tau, \psi)$ is the Rankin-Selberg $\gamma$-function (Section 3.5). For the other case, let $M_2 = \text{GL}_n \times \cdots \times \text{GL}_{n_1} \times \text{SO}_{2n}^* \subset M$ and suppose that $\tau$ is the generic subquotient of

$$i_Q^{\text{GL}_n(F)}(\tau_b \otimes \cdots \otimes \tau_1),$$

where $Q = M_2 N_2$ is the parabolic subgroup of $\text{GL}_m$ containing the upper triangular matrices, $\tau_i$ is a generic representation of $\text{GL}_{n_i}(F)$ for $1 \leq i \leq b$. Then

$$(4.1.2) \quad \gamma(s, \pi \times \tau, \psi) = \prod_{i=1}^b \gamma(s, \pi \times \tau_i, \psi).$$

Before continuing to the $r_2$ case, we make more explicit the principal series case. Let $(\chi_1, \ldots, \chi_{n-1}, \chi)$ be a character of the maximal subtorus $T(F)$ of $\text{SO}_{2n}^* = \text{SO}(q_{E,n})(F)$, where $\chi_i$ is a character of $F^\times$ for each $1 \leq i \leq n - 1$ and $\chi$ is a character of $E^1$. Then, if $\pi$ is the generic subquotient of

$$i_B^{\text{SO}_{2n}^*}(\chi_1 \otimes \cdots \otimes \chi_{n-1} \otimes \chi)$$

and $\xi$ a character of $F^\times$, the multiplicativity formula gives us

$$\gamma(s, \pi \times \xi, \psi) = \gamma(s, \chi \times \xi, \psi) \prod_{i=1}^{n-1} \gamma(s, \chi_i \xi, \psi) \gamma(s, \chi_i^{-1} \xi, \psi).$$
where the $\gamma(s,\chi,\xi,\psi)$ are Tate factors.

Let us study the rank one case. First write $\psi_E = \psi \circ \text{Tr}_{E/F}$ and let $\lambda(E/F,\psi)$ be the Langlands constant [6, Section 30.4]. Now let us recall that we constructed the simply connected cover of $SO(q_E,2)$ (Section 2.3):

$$\text{Res}_{E/F} \text{SL}_2 \to SO(q_E,2).$$

This morphism restricts to

$$\text{diag}(t,t^{-1}) \mapsto [(x_1,x_2,x) \mapsto (N_{E/F}(t)x_1,N_{E/F}(t)^{-1}x_2,t\sigma^{-1}(t)x)].$$

Thus we have the following [36, Proposition 1.3].

**Proposition 4.1.1.** Let $(\chi,\xi)$ be a smooth character of $T(F)$, and $\mu$ the character of $E^\times$ defined by $[t \mapsto (\chi \circ N_{E/F})(t) \cdot \xi(t\sigma^{-1}(t))]$. Then

$$\gamma(s,\chi \times \xi,\psi) = \lambda(E/F,\psi)\gamma(s,\mu,\psi_E).$$

($r_2$ case). The multiplicative property in the case of $r_2$ has the following form. Let $M_1 = GL_{m_1} \times \cdots \times GL_{m_1} \times SO_{2n}^\ast \subset M$ and suppose that $\tau$ is the generic subquotient of

$$i_{GL_m(F)}(\tau_0 \otimes \cdots \otimes \tau_1),$$

where $\tau_i$ is a generic representation of $GL_{m_i}(F)$ for $1 \leq i \leq b$. Then

$$\gamma(s,\tau,\wedge^2 \rho_m,\psi) = \prod_{i=1}^{b} \gamma(s,\tau_i,\wedge^2 \rho_{m_i},\psi) \prod_{i<j} \gamma(s,\tau_i \times \tau_j,\psi).$$

Another important property are the following two stability results for the $\gamma$-functions. First, for any smooth character $\eta : F^\times \to \mathbb{C}^\times$ and any smooth representation $\tau$ of $GL_m(F)$, we write

$$\tau \cdot \eta = \tau \otimes (\eta \circ \det).$$

**Lemma 4.1.2.** [46, Main Lemma 1] Let $\pi$ be a generic representation of $SO_{2n}^\ast(F)$ and $\tau$ a generic representation of $GL_m(F)$. Then there exists a character $\chi$ of $F^\times$ so that $\gamma(s,\pi \times (\tau \cdot \chi),\psi)$ is a monomial in $q_F^{-s}$, for $1 \leq i \leq m$. Moreover $\chi$ can be replaced by any character of $F^\times$ whose conductor is larger than that of $\chi$.

The other important stability result is the following.

**Theorem 4.1.3.** [12, Corollary 6.5] Let $\pi_1$ and $\pi_2$ be two irreducible generic representations of $SO_{2n}^\ast(F)$ having the same central character, and let $\tau$ be an irreducible generic representation of $GL_m(F)$. Then for a sufficiently highly ramified character $\chi$ of $F^\times$, we have

$$\gamma(s,\pi_1 \times (\tau \cdot \chi),\psi) = \gamma(s,\pi_2 \times (\tau \cdot \chi),\psi).$$

### 4.2. Global construction.

Suppose now that $F$ is a function field in one variable over a finite field. Let $\psi = \bigotimes_{x}^\prime \psi_x : A/F \to \mathbb{C}^\times$ be a continuous non-trivial character, $\pi = \bigotimes_{x}^\prime \pi_x$ a globally generic cuspidal automorphic representation of $SO_{2n}^\ast(A_F)$ and $\tau = \bigotimes_{x}^\prime \tau_x$ a cuspidal automorphic representation of $GL_m(A_F)$. Then $\tau \otimes \pi$ is a globally generic cuspidal automorphic representation of $M(A_F)$. If $S$ is a finite subset of $|F|$, such that $\pi_x$, $\tau_x$ and $\psi_x$ are unramified, we denote the partial $L$-function as follows.

($r_1$ case). For $r_1$, we denote the partial $L$-functions by

$$L^S(s,\pi \times \tau) = \prod_{x \notin S} L(s,\pi_x \times \tau_x).$$
and the completed $L$-functions and $\varepsilon$-factors by
\[
L(s, \pi \times \tau) = \prod_{x \in |F|} L(s, \pi_x \times \tau_x) \quad \text{and} \quad \varepsilon(s, \pi \times \tau, \psi) = \prod_{x \in |F|} \varepsilon(s, \pi_x \times \tau_x, \psi_x).
\]

They satisfy the functional equation [36, Section 5.5]
(4.2.1)
\[
L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau)L(1 - s, \bar{\pi} \times \bar{\tau}).
\]
(r$_2$ case) Similarly, for $r_2$, we denote the partial $L$-functions by
\[
L^S(s, \tau, \wedge^2 \rho_m) = \prod_{x \in S} L(s, \tau_x, \wedge^2 \rho_m),
\]
and the completed $L$-functions and $\varepsilon$-factors by
\[
L(s, \tau, \wedge^2 \rho_m) = \prod_{x} L(s, \tau_x, \wedge^2 \rho_m) \quad \text{and} \quad \varepsilon(s, \tau, \wedge^2 \rho_m, \psi_x) = \prod_{x} \varepsilon(s, \tau_x, \wedge^2 \rho_m, \psi_x).
\]

They also satisfy the functional equation [36, Section 5.5]
\[
L(s, \tau, \wedge^2 \rho_m) = \varepsilon(s, \tau, \wedge^2 \rho_m)L(1 - s, \bar{\tau}, \wedge^2 \rho_m).
\]

We now go back to the generic functoriality.

5. Generic functoriality for $\text{SO}^*_{2n}$

Let $\pi = \bigotimes'_x \pi_x$ be a globally generic cuspidal automorphic representation of $\text{SO}^*_{2n}(\mathbb{A}_F)$. We apply Langlands-Shahidi method to $\pi$ and the candidate lift construction made in Section 3.4. In this section, we will focus in the case $x$ is an inert place. The split case version of the result in this section are obtained as in [35]. Now, we start with case when $\pi_x$ is unramified.

5.1. Local lift.

**Proposition 5.1.1.** Let $\pi_x$ be an unramified generic irreducible representation of $\text{SO}^*_{2n}(F_x)$ and $\Pi_x$ a candidate lift as in Section 3.3. Then for a generic irreducible representation $\tau_x$ of $\text{GL}_m(F_x)$ we have the following
(5.1.1)
\[
L(s, \pi_x \times \tau_x) = L(s, \Pi_x \times \tau_x),
\]
\[
\varepsilon(s, \pi_x \times \tau_x, \psi_x) = \varepsilon(s, \Pi_x \times \tau_x, \psi_x).
\]

**Proof.** We start with the setup of the definition of local factors [11, Appendix A]. Let
\[
\iota_{P}^{\text{SO}^*_{2n}}(\pi_{1,x} | \det |^{r_1}, \ldots, \pi_{b,x} | \det |^{r_b}, \pi_{0,x})
\]
be the induced representation such that $\pi_x$ is its Langlands quotient, and where $0 < r_1 < \cdots < r_b$, $P$ is a parabolic subgroup of $\text{SO}^*_{2n}$ containing $B$, $\pi_{i,x}$ is an irreducible tempered representation of $\text{GL}_{r_i}(F_x)$ for $1 \leq i \leq b$, $\pi_{0,x}$ is an irreducible tempered representation of $\text{SO}^*_{2n_0}(F_x)$. Let
\[
\iota_{Q}^{\text{GL}_m}(\tau_{1,x} | \det |^{t_1}, \ldots, \tau_{d,x} | \det |^{t_d})
\]
be the induced representation such that $\tau_x$ is its Langlands quotient, and where $0 < t_1 < \cdots < t_d$, $Q$ is a parabolic subgroup containing the Borel subgroup of $\text{GL}_m$ consisting of upper
triangular matrices, the $\tau_i$’s are generic unitary tempered representation of $\text{GL}_{m_i}(F_x)$. By definition we have (5.1.2)

$$L(s, \pi_x \times \tau_x) = \prod_{j=1}^{d} L(s + t_j, \pi_{0,x} \times \tau_{j,x}) \prod_{i=1}^{b} \prod_{j=1}^{d} L(s + t_j + r_i, \pi_{i,x} \times \tau_{j,x}) L(s - r_i + t_j, \tilde{\pi}_{i,x} \times \tau_{j,x}).$$

With the setup ready, we first study every factor individually. Since $\pi_x$ is unramified, we have that $\pi_{i,x}$ is unramified. Let $\Pi_{0,x}$ be a lift as in Section 3.3 of the unramified representation $\pi_{0,x}$. From multiplicativity we observe that, for $1 \leq j \leq d$,

$$\gamma(s, \pi_{0,x} \times \tau_{j,x}, \psi_x) = \gamma(s, \Pi_{0,x} \times \tau_{j,x}, \psi_x).$$

Since the Satake parameters of $\pi_{0,x}$ have absolute value equal to 1, $\Pi_{0,x}$ is tempered [11]. Thus, for every $0 \leq i \leq b$ and $1 \leq j \leq d$,

$$(5.1.3) \quad \varepsilon(s, \pi_{0,x} \times \tau_{j,x}, \psi_x) \frac{L(1 - s, \tilde{\pi}_{0,x} \times \tilde{\tau}_{j,x})}{L(s, \pi_{0,x} \times \tau_{j,x})} = \varepsilon(s, \Pi_{0,x} \times \tau_{j,x}, \psi_x) \frac{L(1 - s, \tilde{\Pi}_{0,x} \times \tilde{\tau}_{j,x})}{L(s, \Pi_{0,x} \times \tau_{j,x})}.$$  

From the tempered $L$-function conjecture [11], we have that $L(s, \pi_{0,x} \times \tau_{j,x})$ and $L(s, \Pi_{0,x} \times \tau_{j,x})$ are holomorphic on $\text{Re}(s) > 0$. Furthermore, the regions where these $L$-functions have poles do not intersect. Therefore, there are no cancellations involving the numerator and denominator and thus

$$L(s, \pi_{0,x} \times \tau_{j,x}) = L(s, \Pi_{0,x} \times \tau_{j,x}),$$

for every $1 \leq j \leq d$. Using this on the right hand side of (5.1.2), we obtain that

$$L(s, \pi_x \times \tau_x) = L(s, \Pi_{0,x} \times \tau_x) \prod_{i=1}^{b} L(s + r_i, \pi_{i,x} \times \tau_x) L(s - r_i, \tilde{\pi}_{i,x} \times \tau_x).$$

We note that the unramified component of

$$\psi_{Q', n}^{\text{GL}_{2n}}(\pi_{1,x} \cdot \text{det} |^{r_1} \circ \cdots \circ \pi_{b,x} | \cdot \text{det} |^{r_b} \circ \Pi_{0,x} \circ \tilde{\pi}_{b,x} | \cdot \text{det} |^{-r_b} \circ \cdots \circ \tilde{\pi}_{1,x} | \cdot \text{det} |^{r_1}),$$

where $Q'$ is the parabolic subgroup of $\text{GL}_{2n}$ containing $B_{2n}$ associated to the partition $(n_1, \ldots, n_b, 2n_0, n_0, \ldots, n_1)$ of $2n$, is $\pi_x$. Then the right hand side of the last expression is equal to $L(s, \Pi_x \times \tau_x)$. Therefore, we obtain the desired relation between the $L$-functions.

Similarly, using the analogous relation between $L$-function of the contragredient representations and using (5.1.3), we can obtain the desired relations for the $\varepsilon$-factors.

Now, we analyze the case when $\pi_x$ is ramified. In this case, we twist our representation in order to reduce to a $\gamma$-factor relation.

**Proposition 5.1.2.** Let $\pi_x$ be an irreducible generic representation of $\text{SO}^*_{2n}(F_x)$ and $\Pi_x$ a candidate lift as in Section 3.3. Then for any sufficiently ramified enough character $\eta_x$ of $F_x^*$, we have that (5.1.4)

$$L(s, \pi_x \times (\tau_x \cdot \eta_x)) = L(s, \Pi_x \times (\tau_x \cdot \eta_x)),
\varepsilon(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \varepsilon(s, \Pi_x \times (\tau_x \cdot \eta_x), \psi_x),$$

for every unramified irreducible representation $\tau_x$ of $\text{GL}_{m}(F_x)$.
Proof. As before, we start with the setup of the definition of local factors. Let 

$$i_P^{SO_{2n}^*}(|\pi_1| \det |\tau_1| \cdots \det |\pi_b| \det |\tau_0|)$$

be the induced representation such that $\pi_x$ is its Langlands quotient, and where $0 < r_1 < \cdots < r_b$, $P$ is a parabolic subgroup of $SO_{2n}^*$ containing $B$, $\pi_{i,x}$ is an irreducible tempered representation of $GL_{n_i}(F_x)$ for $1 \leq i \leq b$, $\tau_0$ is an irreducible tempered representation of $SO_{2m_0}^*(F_x)$. Let

$$i_Q^{GL_n}(|\tau_1| \cdots \det |\tau_d| \det |\delta_1|)$$

be the induced representation such that $\tau_x$ is its Langlands quotient, and where $0 < t_1 < \cdots < t_d$, $Q$ is a parabolic subgroup containing the Borel subgroup of $GL_m$ consisting of upper triangular matrices, the $\tau_{i,x}$'s are generic unitary tempered representation of $GL_{m_i}(F_x)$. Finally, let

$$i_Q^{GL_{2n}}(|\eta_1| \cdots \det |\eta_t| \cdots |\eta_s|)$$

be the induced representation such that $\eta_x$ is its Langlands quotient, and where $0 < s_1 < \cdots < s_d$, $Q'$ is a parabolic subgroup containing the Borel subgroup of $GL_{2n}$ consisting of upper triangular matrices, the $\eta_{i,x}$'s are generic unitary tempered representation of $GL_{2n}(F_x)$.

Now, making $\eta_x$ sufficiently ramified to obtain (Lemma 4.1.2), we obtain that

$$L(s, \pi_{i,x} \times (\tau_{j,x} \cdot \eta_x)) \equiv 1 \equiv L(s, \Pi_{i,x} \times (\tau_{j,x} \cdot \eta_x)),$$

for every $0 \leq i \leq b$, $1 \leq j \leq d$ and $1 \leq k \leq l$. By definition of $\varepsilon$-factors, this implies that

$$\varepsilon(\pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \gamma(\pi_x \times (\tau_x \cdot \eta_x), \psi_x),$$

$$\varepsilon(\Pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \gamma(\Pi_x \times (\tau_x \cdot \eta_x), \psi_x).$$

Thus, we are left to prove the corresponding identity for the $\gamma$-factors.

Now, as $\pi'_x$ is generic we can use the stability of the gamma factors (Theorem 4.1.3). By this result, if we make $\eta_x$ ramified enough, the following identity also holds

$$\gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \gamma(s, \pi'_x \times (\tau_x \cdot \eta_x), \psi_x).$$

On the other hand, using the explicit description of the $\gamma$-factors in the principal series case in Section 4.1, we have

$$\gamma(s, \pi'_x \times \eta_x, \psi_x) = \gamma(s, \chi_{n,x} \times \eta_x, \psi_x) \prod_{i=1}^{n-1} \gamma(s, \chi_{i,x}^{-1} \eta_x, \psi_x)$$

$$= \gamma(s, \Pi_{\mu_{n,x}} \times \eta_x, \psi_x) \prod_{i=1}^{n-1} \gamma(s, \chi_{i,x} \eta_x, \psi_x) \gamma(s, \chi_{i,x}^{-1} \eta_x, \psi_x)$$

$$= \gamma(s, \Pi_x \times \eta_x, \psi_x)$$

Now, since $\tau_x$ is unramified, it is a subquotient of an induced representation of the form

$$i_B^{GL_m(F_x)}(|\cdot| \det |\nu|)$$

where $b_i \in \mathbb{C}$. Using the multiplicativity of the $\gamma$-factors, we have

$$\gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \prod_{i=1}^{m} \gamma(s - b_i, \pi_x \times \eta_x)$$
and
\[
\gamma(s, \Pi_x \times \tau_x \cdot \eta_x) = \prod_{i=1}^{m} \gamma(s, \Pi_x \times (|b_i| \cdot \eta_x)) = \prod_{i=1}^{m} \gamma(s - b_i, \Pi_x \times \eta_x).
\]
Comparing these two, we obtain the desired identity. \hfill \square

5.2. Global lift. Using the equalities (5.1.1) twisted by any sufficiently ramified enough character and (5.1.4), we have:

**Corollary 5.2.1.** Let \( \pi = \bigotimes_x \pi_x \) be a globally generic cuspidal automorphic representation of \( \text{SO}^*_2(\mathbb{A}_F) \), unramified outside of a non-empty \( S \subset |F| \) and let \( \Pi \) a candidate lift of \( \pi \) as in Section 3.3. Then, for a character \( \eta = \bigotimes_x \eta_x \), sufficiently ramified in \( x \in S \), (so as to satisfy (5.1.4)), we have

\[
L(s, \pi \times \tau) = L(s, \Pi \times \tau),
\]
\[
\varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau),
\]
for every \( \tau \in \mathcal{T}(S; \eta) \) (as in Section 3.6).

We know that a lift \( \Pi \) of \( \pi \) is irreducible and admissible and that its central character is trivial on \( F^\times \) (3.4.2), but we do not necessarily have that it is automorphic. For that we use the converse theorem (Theorem 3.6.1). Thus, we need to make sure that \( L(s, \pi \times \tau) \) is a polynomial for \( \tau \in \mathcal{T}(S; \eta) \), for some character \( \eta \) of \( \mathbb{A}_F^\times \) trivial on \( F^\times \). In order to obtain that we are going to study the local Normalized Intertwining Operator.

5.3. Local Normalized Intertwining Operator. First, let us review the definition of the Local Normalized Intertwining Operator. Thus, assume that \( F \) is a locally compact field of positive characteristic.

Let \( P = MN \) be a maximal parabolic subgroup of \( \text{SO}^*_{2m+2n} \) containing \( B \), such that \( M \cong \text{GL}_m \times \text{SO}^*_2 \), associated to \( \Delta \setminus \{\alpha\} \) and \( \tilde{w}_0 \in G(F) \) a representative of \( w_0 = w_{l,G}w_{l,M} \in W^G \), where \( w_{l,G} \) and \( w_{l,M} \) are the longest element of the Weyl group of \( G \) and of \( M \), respectively. For \( \sigma = \tau \otimes \tilde{\pi} \) a generic representation of \( M(F) \), where \( \tau \) is a representation of \( \text{GL}_m(F) \) and \( \pi \) is a representation of \( \text{SO}^*_2(F) \), we define

\[
r(s, \sigma) = \frac{L(s, \pi \times \tau)L(2s, \tau, \Lambda^2 \rho_m)}{L(1+s, \pi \times \tau)L(1+2s, \tau, \Lambda^2 \rho_m)\varepsilon(s, \pi \times \tau, \psi)\varepsilon(2s, \Lambda^2 \rho_m, \psi)},
\]
and the normalized intertwining operator \( N(s, \sigma, \tilde{w}_0) \) is defined to be such that

\[
A(s, \sigma, \tilde{w}_0) = r(s, \sigma)N(s, \sigma, \tilde{w}_0): \iota^G_P(s, \sigma) \rightarrow \iota^G_{P'}(\tilde{w}_0(s), \tilde{w}_0(\sigma))
\]
as a rational operator in \( s \), where \( A(s, \sigma, \tilde{w}_0) \) is the operator is given by

\[
A(s, \sigma, \tilde{w}_0)f(g) = \int \mu_{|G \mu_{\tilde{w}_0^{-1}G\mu'}\mu'|} f(\tilde{w}_0^{-1}u'g) du'.
\]

The holomorphicity of the local normalized intertwining operator relies on two local properties: the first one is the so-called standard module conjecture. In our case, it states that every generic smooth irreducible representation of \( \text{SO}^*_{2n}(F) \) is the full induced representation

\[
|\iota^{	ext{SO}^*_{2n}(F)}_P(\pi_0) \det |^s \otimes \cdots \otimes \pi_1 \det |^{1} \otimes \pi_0,
\]
where \(0 < r_1 \leq \cdots \leq r_b\), with \(r_b < 1\), \(\pi_i\) is an irreducible tempered representation of \(\text{GL}_{n_i}(F)\) for \(1 \leq i \leq b\) and \(\pi_0\) is an irreducible tempered representation of \(\text{SO}^*_{2n_0}(F)\). In order words, the standard modules of generic irreducible smooth representations are irreducible. This conjecture have been proven for a general quasi-split groups in characteristic zero [15] and in positive characteristic [11].

The second ingredient is that the real numbers \(r_i\)'s appearing in (5.3.1) are less than 1. To obtain that, we will use a local-global result (Proposition 5.4.2) and then we will follow the Kim’s arguments in [24, Section 3] to obtain the bound.

Using these two properties and following [24, Proposition 3.4], we have the following.

**Proposition 5.3.1.** Let \(\pi\) be generic representation of \(\text{SO}^*_{2n}(F)\) such that it is the full induced representation

\[
\tau_p^{\text{SO}_{2n}(F)}(\pi_0 | r^b \otimes \cdots \otimes \pi_1 | r^1 \otimes \pi_0),
\]

where \(0 < r_1 \leq \cdots \leq r_b\), with \(r_b < 1\), \(\pi_i\) is an irreducible tempered representation of \(\text{GL}_{n_i}(F)\) for \(1 \leq i \leq b\) and \(\pi_0\) is an irreducible tempered representation of \(\text{SO}^*_{2n_0}(F)\).

Then \(N(s, \tau \otimes \tilde{\pi}, \tilde{w}_0)\) is holomorphic and non-zero on \(\text{Re}(s) \geq 1/2\) for every generic unitary representation \(\tau\) of \(\text{GL}_{m}(F)\).

**Proof.** We can write \(\tau\) as the full induced representation [50, Section 7]

\[
\tau_Q^{\text{GL}_m}(\xi_1 | t^1 \otimes \cdots \otimes \xi_d | t^d \otimes \xi_{d+1} \otimes \xi_d | t^{-1} \otimes \cdots \otimes \xi_1 | t^{-t_1}),
\]

where \(Q\) is a parabolic subgroup containing the Borel subgroup of \(\text{GL}_m\) consisting of upper triangular matrices, the \(\xi_i\)'s are tempered representations of \(\text{GL}_{n_i}(F)\) and \(0 < t_1 \leq \cdots \leq t_d < 1/2\).

Combining the description for \(\pi\) and \(\tau\) as induced representations, we obtain that \(\tau \otimes \tilde{\pi}\) is full induced from quasi-tempered datum. This allows us to use multiplicativity of the normalized intertwining operators (See [26, Proposition 4.6]), in order to reduce to the following rank one cases \(\text{GL}_k \times \text{GL}_l \subset \text{GL}_{k+l}, \text{SO}^*_{2l} \times \text{GL}_k \subset \text{SO}^*_{2(l+k)}\) and \(\text{GL}_{l-1} \subset \text{SO}^*_{2l}\) \((l \geq 3)\):

1. For the case \(\text{GL}_k \times \text{GL}_l \subset \text{GL}_{k+l}\), we obtain from \(\text{Re}(s \pm r_i \pm t_j) > -1\), for \(\text{Re}(s) \geq 1/2\) the condition, thanks to [39, Proposition I.10].
2. For the case \(\text{SO}^*_{2l} \times \text{GL}_k \subset \text{SO}^*_{2(l+k)}\), we note that \(\text{Re}(s \pm t_d) \geq 0\) for \(\text{Re}(s) \geq 1/2\). As \(\pi_0\) is tempered, we get our condition.
3. Finally for the case \(\text{GL}_{l-1} \subset \text{SO}^*_{2l}\) \((l \geq 3)\), we conclude as in [23, Lemma 3.3, Proposition 3.4] to conclude.

5.4. **Generic functoriality for \(\text{SO}^*_{2n}\).** Now let us go back to the global situation and we are going to check the last points to obtain the generic functoriality.

*Polynomial condition.* To prove the polynomial condition, we need the following holomorphicity result of the \(L\)-functions. The result is inspired from [27, Section 2] and [37, Section 4].

**Proposition 5.4.1.** Suppose that \(\pi\) and \(\tau\) are unramified outside of \(T\) and that \(S'\) is a subset of \(T\) with the property that for \(x \in S'\), the local Normalized Intertwining Operator \(N(s, \tau_x \otimes \tilde{\tau}_x, \tilde{w}_0)\) is holomorphic and non-zero on \(\text{Re}(s) \geq 1/2\) and \(\tilde{w}_0(\tau \otimes \tilde{\pi}) \neq \tau \otimes \tilde{\pi}\). Then
the L-function

\[ L^{T \backslash S'}(s, \pi \times \tau) = \prod_{x \in T \backslash S'} L(s, \pi \times \tau) \]

is holomorphic on Re(s) ≥ 1/2 and non-zero on Re(s) ≥ 1.

**Proof.** In this proof we use the global intertwining operator and its relation with Eisenstein series. For the definition of the global intertwining operator we refer to Section 6.1, but for their properties we rely on [37].

Now, putting the definition of the normalized intertwining operator in the right hand side of the formula [37, Eq. (3.2)], we get

\[ M(s, \sigma, \tilde{w}_0)f = \bigotimes_{x \in T \backslash S'} A(s, \sigma_x, \tilde{w}_0) f_x \cdot \]

\[ \bigotimes_{x \in S'} r(s, \sigma_x)^{-1} N(s, \sigma_x, \tilde{w}_0) f_x \cdot \]

\[ \frac{L^T(s, \pi \times \tau)L^T(2s, \tau, \wedge^2 \rho_m)}{L^T(1 + s, \pi \times \tau)L^T(1 + 2s, \tau, \wedge^2 \rho_m)} \bigotimes_{x \in T} \tilde{f}_x. \]

Since \( \tilde{w}_0(\tau \otimes \tilde{\pi}) \not\subset \tau \otimes \tilde{\pi} \), we have that \( M(s, \sigma, \tilde{w}_0) \) is holomorphic on Re(s) ≥ 0 [37, Lemma 3.3]. Using this holomorphicity result, and that \( A(s, \sigma_x, \tilde{w}_0) \) and ε-factors are non-vanishing [51, p. 283, Eq. (10)], we have that

\[ \frac{L^T(s, \pi \times \tau)L^T(2s, \tau, \wedge^2 \rho_m)}{L^T(1 + s, \pi \times \tau)L^T(1 + 2s, \tau, \wedge^2 \rho_m)} \]

is holomorphic on Re(s) ≥ 0. Furthermore using that \( N(s, \sigma_x, \tilde{w}_0) \) is holomorphic and non-zero on Re(s) ≥ 1/2, we have that

\[ \frac{L^{T \backslash S'}(s, \pi \times \tau)L^{T \backslash S'}(2s, \tau, \wedge^2 \rho_m)}{L^{T \backslash S'}(1 + s, \pi \times \tau)L^{T \backslash S'}(1 + 2s, \tau, \wedge^2 \rho_m)} \]

is holomorphic on Re(s) ≥ 1/2. From the fact that L-functions are holomorphic on some Re(s) > N [4, Section 13.2], we get that

\[ (5.4.1) \quad L^{T \backslash S'}(s, \pi \times \tau)L^{T \backslash S'}(2s, \tau, \wedge^2 \rho_m) \]

is holomorphic on Re(s) ≥ 1/2.

On the other hand, since \( \tilde{w}_0(\tau \otimes \tilde{\pi}) \not\subset \tau \otimes \tilde{\pi} \), \( E(s, \Phi, g, P) \) is holomorphic on Re(s) ≥ 0 [37, Lemma 3.3]. Using this holomorphicity result and that the local L-functions are non-vanishing by definition and the relation [37, Eq. (1.3)] we also get

\[ \prod_{x \in |S'|} L(s, \pi_x \times \tau) L(1 + 2s, \tau, \wedge^2 \rho_{x,m}) L^T(1 + s, \pi \times \tau) L^T(1 + 2s, \tau, \wedge^2 \rho_m) \]

\[ = L^{T \backslash S'}(1 + s, \pi \times \tau)L^{T \backslash S'}(1 + 2s, \tau, \wedge^2 \rho_m) \]

is non-zero on Re(s) ≥ 0.

Now we proceed as in [37, Section 6.1], to get that \( L^T(s, \tau, \wedge^2 \rho_m) \) is holomorphic on Re(s) ≥ 1/2 and non-zero on Re(s) ≥ 1. Indeed we consider the global intertwining operator in a maximal Siegel case. Then as before, staring from [37, Lemma 3.3] and using [37, Eq. (1.3) and (3.2)], we obtain that

\[ L^T(s, \tau, \wedge^2 \rho_m) \]
is holomorphic on $\text{Re}(s) \geq 1/2$ and non-zero on $\text{Re}(s) \geq 1$. Furthermore, since every $\tau_x$ is tempered and using [37, Corollary 5.5] (tempered $L$-function conjecture), we have
\[
L^{\tau \Omega'(s, \tau, \wedge^2 \rho_m)}
\]
is holomorphic on $\text{Re}(s) \geq 1/2$. Thus, $L^{\tau \Omega'(s, \pi \times \tau)}$ is holomorphic on $\text{Re}(s) \geq 1/2$ by (5.4.1). Similarly, but using (5.4.2), we get it is non-zero on $\text{Re}(s) \geq 0$. \hfill \Box

As we mention in the previous section, we already have the standard module conjecture at our disposal. Then, in order to apply Proposition 5.3.1 and Proposition 5.4.1, we need the following local-global results. The result is inspired from [24, Theorem 3.2].

**Proposition 5.4.2.** Let $\tau$ be a (globally generic) cuspidal automorphic representation of $\text{GL}_m(A_F)$ and $\pi$ a globally generic cuspidal automorphic representation of $\text{SO}_{2n}^*(A_F)$ such that $\tau \otimes \bar{\pi} = \sigma \not\cong \bar{w}_0 \sigma$. Then $L(s, \pi_{x_0} \times \tau_{x_0})$ is holomorphic on $\text{Re}(s) \geq 1$, for every $x \in |F|$.

**Proof.** As before we input the definition of the normalized operator in the right hand side of the formula [37, (3.2)] to get
\[
M(s, \sigma, \bar{w}_0)f = \bigotimes_{x \in S \setminus \{x_0\}} A(s, \sigma_x, \bar{w}_0) f_{x_0}
\]
\[
r(s, \sigma_{x_0})^{-1} N(s, \sigma_{x_0}, \bar{w}_0) f_{x_0} \cdot
\]
\[
\frac{L^S(s, \pi \times \tau) L^S(2s, \tau, \wedge^2 \rho_m)}{L^S(1 + s, \pi \times \tau) L^S(1 + 2s, \tau, \wedge^2 \rho_m)} \prod_{x \notin S} f^x_{x_0}
\]
Now let $N_0 \geq 1$ be big enough so that $L(1 + s, \pi_{x_0} \times \tau_{x_0})$ has no poles on $\text{Re}(s) \geq N_0$, i.e. $L(s, \pi_{x_0} \times \tau_{x_0})$ is holomorphic on $\text{Re}(s) \geq N_0 + 1$. This gives us that, if $\text{Re}(s) \geq N_0$, then
\[
L(s, \pi_{x_0} \times \tau_{x_0})
\]
is non-zero. Secondly, since $\tau$ is cuspidal automorphic, then thanks to [30, Théorème VI.10] $\tau_{x_0}$ is tempered. Then, using [37, Corollary 5.5] we have that $L(s, \tau_{x_0}, \wedge^2 \rho_m)$ is holomorphic on $\text{Re}(s) \geq 1$. Therefore
\[
r(s, \pi_{x_0} \otimes \bar{\pi}_{x_0}) = \frac{L(s, \pi_{x_0} \times \tau_{x_0}) L(2s, \tau_{x_0}, \wedge^2 \rho_m)}{L(1 + s, \pi_{x_0} \times \tau_{x_0}) L(1 + 2s, \tau_{x_0}, \wedge^2 \rho_{m,x_0}) \varepsilon(s, \pi_{x_0} \times \tau_{x_0}, \varepsilon_{x_0}) \varepsilon(2s, \tau_{x_0}, \wedge^2 \rho_{m,x_0}, \psi_{x_0})}
\]
is non-zero on $\text{Re}(s) \geq N_0$. Thirdly, using Corollary 5.4.1 for $S' = \emptyset$, we get that
\[
r(s, \pi_{x_0} \otimes \bar{\pi}_{x_0}) = \frac{L^S(s, \pi \times \tau) L^S(2s, \tau, \wedge^2 \rho_m)}{L^S(1 + s, \pi \times \tau) L^S(1 + 2s, \tau, \wedge^2 \rho_m)}
\]
is non-zero on $\text{Re}(s) \geq N_0$.

Fourthly, recall that $M(s, \sigma, \bar{w}_0)$ is holomorphic on $\text{Re}(s) \geq 0$, since $\bar{w}_0 \sigma \not\cong \sigma$ [37, Lemma 3.3]. Thus, using that $A(s, \sigma_{x_0}, \bar{w}_0)$ is non-zero and the equality at the beginning of the proof, we have that $N(s, \sigma_{x_0}, \bar{w}_0)$ is holomorphic on $\text{Re}(s) \geq N_0$.

Lastly, since the holomorphicity of $N(s, \sigma_{x_0}, \bar{w}_0)$ implies its non-zeroness [52, Theorem 3], we have that $N(s, \sigma_{x_0}, \bar{w}_0)$ is also non-zero on $\text{Re}(s) \geq N_0$. Hence
\[
\frac{L(s, \pi_{x_0} \times \tau_{x_0})}{L(1 + s, \pi_{x_0} \times \tau_{x_0})}
\]
is holomorphic on $\text{Re}(s) \geq N_0$, and thus $L(s, \pi_{x_0} \times \tau_{x_0})$ has no poles on $\text{Re}(s) \geq N_0$. Arguing inductively, we conclude that $L(s, \pi_{x_0} \times \tau_{x_0})$ is holomorphic on $\text{Re}(s) \geq 1$.

Using the previous local-global result, we are able to prove the second property needed for the local Normalized Intertwining Operator. Again, it is inspired from [24, Lemma 3.3].

**Corollary 5.4.3.** Let $\pi = \bigotimes'_x \pi_x$ be a globally generic cuspidal automorphic representation of $\text{SO}^*_{2n}(\mathbb{A}_F)$. Then $r_{b,x} < 1$ for every $x \in |F|$, where $r_{b,x}$ is as in equation (5.3.1).

**Proof.** Fix a place $x$. Let

$$i_p^{SO^*_{2n}(\pi_{1,x})} \det |r_{1,x} \otimes \cdots \otimes \pi_{b,x}| \det |r_{b,x} \otimes \pi_{0,x}|$$

be the induced representation such that $\pi_x$ is its Langlands quotient, and where $0 < r_{1,x} \leq \cdots \leq r_{b,x}$. $P$ is a parabolic subgroup of $\text{SO}^*_{2n}$ containing $B$, $\pi_{i,x}$ is an irreducible discrete series representation of $\text{GL}_{n_i}(F_x)$ for $1 \leq i \leq b$, and $\pi_{0,x}$ is an irreducible tempered representation of $\text{SO}^*_{2n_0}(F_x)$.

By definition we have that $L(s - r_{b,x}, \pi_{b,x} \times \pi_{b,x})$ is a factor of $L(s, \pi_x \times \pi_{b,x})$. Since

$L(s - r_{b,x}, \pi_{b,x} \times \pi_{b,x})$ is a Rankin-Selberg $L$-functions (Remark 3.5.1), it has a pole for $s = r_{b,x}$. Now, there is a cuspidal automorphic representation $\tau_b = \bigotimes'_x \nu_{b,x}$ of $\text{GL}_{n_0}(\mathbb{A}_F)$, such that $\nu_{b,x} \cong \pi_{b,x}$. Using Proposition 5.4.2, we have that $L(s, \pi_x \times \pi_{b,x})$ is holomorphic on $\text{Re}(s) > 1$, and thus we must have that $r_{b,x} < 1$.

Now, we are finally ready to prove the polynomial condition. First, we can find as in [37, Proposition 4.1] a sufficiently ramified character $\eta_{x_0}$, with $x_0 \in S$ (nonempty by definition), such that $\tilde{\nu}(\pi \otimes \tau \cdot \eta) \not\cong \pi \otimes (\tau \cdot \eta)$. This allows us to obtain the condition needed for Proposition 5.4.2. Furthermore, combining Corollary 5.4.3 and Proposition 5.3.1, we obtain that the local normalized intertwining operator

$$N(s, \pi_{x_0} \otimes \pi_{x_0}, \tilde{\nu}_0)$$

is holomorphic and non-zero on $\text{Re}(s) \geq 1/2$, for every inert place $x \in |F|$. Now, using an analogous result of Proposition 5.3.1 in the split case $\text{SO}_{2n}$ in [11] and applying Proposition 5.4.1 to the set $T = S'$, we obtain that

$$L(s, \pi \times \tau) = \prod_{x \in |F|} L(s, \pi_x \times \tau_x)$$

is holomorphic on $\text{Re}(s) \geq 1/2$. Finally using the Langlands-Shahidi functional equation (4.2.1), we get that $L(s, \pi \times \tau)$ is entire. In addition, using the rationality property of $L$-functions [37, Theorem 1.2] we see that $L(s, \Pi \times \tau)$ is a polynomial.

(Trivial on $F^\times$). Choosing characters $\nu_x$ for $x \in S$ sufficiently ramified as in the polynomial condition and in the ramified case of (5.1.4), we can find a character $\eta$ of $\mathbb{A}_F^\times$ trivial on $F^\times$ [1, X, Theorem 5] and which satisfies $\eta_x = \nu_x$ for $x \in S$.

As we have checked all the hypothesis of the converse Theorem 3.6.1 in the previous section, we find an irreducible automorphic representation $\Pi'$ of $\text{GL}_{2n}(\mathbb{A}_F)$ such that $\Pi'_x = \Pi_x$ for $x \not\in S$.

**Theorem 5.4.4.** Let $\pi = \bigotimes'_x \pi_x$ be a globally generic cuspidal automorphic representation of $\text{SO}^*_{2n}(\mathbb{A}_F)$, unramified outside of a finite set of places $S$. Then there exists an automorphic representation $\Pi = \bigotimes'_x \Pi_x$ of $\text{GL}_{2n}(\mathbb{A}_F)$ such that the representation $\Pi_x$ is unramified for every $x \not\in S$ and that its Satake parameter $\phi_{\Pi_x}$ satisfy that $\phi_{\Pi_x} = \rho^\ast_{2n,x} \circ \phi_{\pi_x}$ and where its central character is given by (3.4.2).
We will study further properties of these lifts, in the next section.

6. Automorphic L-functions and Image of the Functoriality

Inspired by [37] and [49], we prove that the cuspidal factors of the isobaric sum are distinct, unitary and self-dual, in positive characteristic. We check that this lift respects the arithmetic information coming from \( \gamma \)-factors. We finish by proving, as an application of the functoriality, the unramified Ramanujan conjecture for globally generic cuspidal automorphic representations of \( \text{SO}_{2n}^*(A_F) \).

To obtain the result about the isobaric sum, we need the following fact about the unramified unitary dual of the split special orthogonal group \( \text{SO}_{2n} \) [11].

**Proposition 6.0.1.** Let \( \tau \) be a tempered smooth irreducible unramified representation of \( \text{GL}_m(F) \) and \( \pi \) a unitary smooth irreducible unramified representation of \( \text{SO}_{2n}(F) \). Let \( s \) be a complex parameter with \( \text{Re}(s) > 1 \). Then the unramified component of \( i_{P(F)}^{\text{SO}_{2n}(s+1)(F)}(|\det|^s \tau \otimes \pi) \) is not unitary.

**6.1. Global L-functions.** With the unramified unitary dual result at hand, we are ready to continue the study of the image of functoriality.

Let \( P = MN \) be a maximal parabolic subgroup of \( G \) containing \( B \) associated to \( \theta = \Delta - \{ \alpha \} \subset \Delta \) and let \( w_0 = w_l G w_0 M \in W^G \), where \( w_l G \) and \( w_l M \) are the longest element of the Weyl group of \( G \) and of \( M \), respectively. We fix a maximal compact open subgroup \( K = \prod_x K_x \) of \( G = G(A_F) \), as in [38, Section I.1.4].

Let \( \sigma = \bigotimes_x \sigma_x \) be a unitary cuspidal automorphic representation of \( M(A_F) \), where the restricted product is taken with respect to functions \( \{ \varphi_x^0 \} \). We write

\[
i_{P}^{G}(s, \sigma) = \bigotimes_x i_{P(\varphi_x^0)}^{G(F_x)}(s, \sigma_x) = \bigotimes_x i_{P(\varphi_x^0)}^{G(F_x)}(\sigma_x \otimes q^{(s \sigma_x, H_{F_x}(\cdot))}),
\]

where the restricted product is taken with respect to the functions \( f_{x,s}^0 \in i_{P}^{G}(s, \sigma_x) \) such that \( f_{x,s}^0(k_x) = \varphi_x^0 \) for all \( k_x \in K_x \). For \( \bar{w}_0 \) a representative of \( w_0 \), we define the global intertwining operator for \( \text{Re}(s) \) big enough, as in [37, Section 1.2], by

\[
M(s, \sigma, \bar{w}_0): i_{P}^{G}(s, \sigma) \to i_{P'}^{G}(\bar{w}_0(s), \bar{w}_0(\sigma))
\]

\[
f \mapsto \left( g \mapsto \int_{N'(A_F)} f(\bar{w}_0^{-1} n g) dn \right),
\]

where \( N' \) is the radical of \( P' = P_{w_0(\theta)} \).

**Proposition 6.1.1.** Suppose that \( G = \text{SO}_{2n(m+n)}^* \), let \( P = MN \) be a parabolic subgroup containing \( B \) with Levi subgroup \( M \) isomorphic to \( \text{GL}_m \times \text{SO}_{2n}^* \) and \( \bar{w}_0 \in G(F) \) a representative of \( w_0 \in W^G \). Let \( \sigma = \tau \otimes \pi \) be a unitary globally generic cuspidal automorphic representation of \( M(A_F) \). Then \( M(s, \sigma, \bar{w}_0) \) is holomorphic on \( \text{Re}(s) > 1 \).

**Proof.** Let \( S \) be a finite subset of \( |F| \), such that \( \sigma_x \) is unramified for \( x \notin S \).

Thanks to the work of L. Lafforgue [30, Théorème VI.10], we know that each local component of the globally generic cuspidal automorphic representation \( \tau = \bigotimes_x \tau_x \) of \( \text{GL}_m(A_F) \) is tempered. Then, for each \( x \notin S \) we have that \( \tau_x \) is of the form

\[
i_{B_m}^{G_m}(\chi_{1,x} \otimes \cdots \otimes \chi_{m,x}),
\]

where \( \chi_{j,x} \) is unitary unramified character of \( F_x^* \).
Now, if $s_0$ is a pole of $M(s, \sigma, \tilde{w}_0)$, then some subquotient of $i_{P}^{G}(s_0, \sigma)$ would be in the discrete residual spectrum [24, Section 1], thus unitary. Then for such $s_0$, we would have that for almost every $x \in |F|$, the unramified component of $i_{P(F_2)}^{G(F_2)}(s_0, \sigma_x)$ is unitary.

We argue by contradiction, and thus we assume that $\text{Re}(s_0) > 1$. First, we can enlarge $S$, so that $i_{P(F_2)}^{G(F_2)}(s_0, \sigma_x)$ has a unitary unramified component. As the set $\{x \in |F| \setminus S : x \text{ is split in } E\}$ has density $1/2$, by Chebotarev’s theorem, we can always find $x_0 \notin S$ split in $E$. But, thanks to Proposition 6.0.1, we get that the unramified component of $i_{P(F_2)}^{G(F_2)}(s_0, \sigma_x)$ is not unitary, thus a contradiction. 

Theorem 6.1.2. Suppose that $G = \text{SO}^*_2(m+n)$, and $P = MN$ parabolic subgroup with Levi subgroup $M$ isomorphic to $\text{GL}_m \times \text{SO}^*_n$. Let $\sigma = \tau \otimes \tilde{\pi}$ be a globally generic cuspidal automorphic representation of $M(\mathbb{A}_F)$. Then $L^S(s, \pi \times \tau)$ is holomorphic and non-vanishing on $\text{Re}(s) > 1$ and has at most a simple pole at $s = 1$.

Proof. Let $S$ be a finite subset of $|F|$, such that $\sigma_x$ is unramified for $x \notin S$, as in the proof of Proposition 6.1.1. From [37, Eq. (3.2)] and Proposition 6.1.1, we have that

$$L^S(s, \pi \times \tau)L^S(2s, \tau, \wedge^2 \rho_m)$$

is holomorphic on $\text{Re}(s) > 1$. As $L^S(s, \tau, \wedge^2 \rho_m)$ is holomorphic and non-zero on $\text{Re}(s) > 1$ [37, Corollary 6.4], we can conclude that

$$L^S(s, \pi \times \tau)$$

is holomorphic on $\text{Re}(s) > 1$.

On the other hand, as [38, Proposition II.1.7]

$$E_P(s, f, g, P) = \int_{U(F) \setminus U(\mathbb{A}_F)} E(s, f, ug, P)du = f(g) + M(s, \sigma, \tilde{w}_0)f(g)$$

and Proposition 6.1.1, we have that the poles of the constant terms $E_P(s, f, g, P)$ are contained in $\text{Re}(s) \leq 1$. Since $U(F) \setminus U(\mathbb{A}_F)$ is compact, the formula [37, Eq. (1.3)] and that $L^S(s, \tau, \wedge^2 \rho_m)$ is holomorphic and non-zero on $\text{Re}(s) > 1$, we conclude that $L^S(1+s, \pi \times \tau)^{-1}$ is holomorphic and non-vanishing on $\text{Re}(s) > 1$. Thus $L^S(s, \pi \times \tau)$ is holomorphic on $\text{Re}(s) > 1$.

Finally, we have that the poles of the global intertwining operator are all simple [38, Proposition IV.1.11, (c)]. Then, again by [37, Eq. (3.2)] and the non-zeroness of the local intertwining operators $A(s, \tau_x \otimes \tilde{\pi}_x, \tilde{w}_0)$, we obtain

$$L^S(s, \pi \times \tau)L^S(2s, \tau, \wedge^2 \rho_m)$$

has at most a simple pole at $s = 1$. From [37, Corollary 6.4], $L^S(2, \tau, \wedge^2 \rho_m)$ and $L^S(3, \tilde{\tau}, \wedge^2 \rho_m)$ are non-zero. Thus, $L^S(s, \pi \times \tau)$ has at most a simple pole at $s = 1$. 

6.2. Image and isobaric sums. We recall that from [33, Proposition 2], every automorphic representation $\Pi$ of $\text{GL}_{2n}(\mathbb{A}_F)$ arises as a subquotient of a representation induced from cuspidal automorphic representations,

$$(6.2.1) i_{P(\mathbb{A}_F)}^{\text{GL}_{2n}(\mathbb{A}_F)}(\Pi_1 \otimes \cdots \otimes \Pi_d),$$

Thus, for almost every $\Pi \subset \text{GL}_{2n}(\mathbb{A}_F)$, we can conclude that $i_{\mathbb{A}_F}^{\text{GL}_{2n}(\mathbb{A}_F)}(\Pi)$ is unitary.
where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{GL}_{2n}$ containing the Borel subgroup of $\mathbf{GL}_{2n}$ consisting of upper triangular matrices, and with every $\Pi_i$ a cuspidal automorphic representation of $\mathbf{GL}_{n_i}(\mathbb{A}_F)$. We also recall that Langlands isobaric sum gives us a subquotient of (6.2.1), that we denote [32, Section 2]

$$\Pi_1 \boxplus \cdots \boxplus \Pi_d.$$ 

We are going to study the representation $\Pi_j$ for every $1 \leq j \leq d$ obtained from the representations in the image of functoriality.

**Theorem 6.2.1.** Let $\pi$ be a globally generic cuspidal automorphic representation of $\mathbf{SO}^*_n(\mathbb{A}_F)$. Then, $\pi$ transfers to a globally generic irreducible automorphic representation $\Pi$ of $\mathbf{GL}_{2n}(\mathbb{A}_F)$. Its central character is given by (3.4.2) and $\Pi$ can be expressed as an isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each $\Pi_i$ is a unitary self-dual cuspidal automorphic representation of $\mathbf{GL}_{n_i}(\mathbb{A}_F)$, and $\Pi_i \not\sim \Pi_j$ for $i \neq j$.

**Proof.** The existence of the transfer of $\pi$ and its central character is Theorem 5.4.4. We now show the properties of $\Pi_i$. Let $S$ be a finite set of $|F|$ such that $\pi$ is unramified outside of $S$.

(Unitarity). We write $\Pi_i = |\det|^{\nu_i} \Pi'_i$, where $\Pi'_i$ is unitary for every $1 \leq i \leq d$ and $\nu_d \geq \cdots \geq \nu_1$. Given that the central character of $\Pi$ is unitary, we have that $\nu_1 \leq 0$. By (5.1.1) and the multiplicativity property of Rankin-Selberg $L$-functions we have

$$L^S(s, \pi \times \Pi_1) = L^S(s, \Pi \times \Pi'_1) = \prod_j L^S(s, \Pi_j \times \Pi'_j)$$

$$= \prod_j L^S(s + n_j, \Pi'_j \times \Pi'_1).$$

Since the left hand side has at most a pole at $s = 1$ and it is holomorphic and non-vanishing for $\text{Re}(s) > 1$ by Theorem 6.1.2, we must have that $\nu_1 = 0$. Recursively we can check that $\nu_i = 0$ for all $i$. Thus $\Pi_i$ is unitary for all $i$.

As a consequence we have that $\Pi$ is equal to the isobaric sum of the $\Pi_i$’s, as each $\Pi_i$ is unitary and thus $\Pi$ is the full induced representation. Moreover, as the $\Pi_i$’s are globally generic then $\Pi$ is also globally generic.

(Distinct). As before we consider

$$L^S(s, \pi \times \Pi_i) = L^S(s, \Pi \times \Pi_i) = \prod_j L^S(s, \Pi_j \times \Pi_i)$$

$$= \prod_j L^S(s + n_i, \Pi_i \times \Pi_i).$$

Arguing as above, we must have $\Pi_i \not\sim \Pi_j$ for $i \neq j$, because otherwise the right hand side would not have a simple pole [20, II; (3.6)].

(Self-dual). First observe that linear map $\tilde{w}_0$ of $Q_{F,n+m}$ (See Section 2.1), given by $\tilde{w}_0 e_i = e_{2(n+m)-(n-i)}$ for $1 \leq i \leq n$, $\tilde{w}_0 e_i = e_i$ for $n + 1 \leq i \leq 2m + n$, trivial on $l$ and $\tilde{w}_0 e_i = e_{i-n-2m}$ for $n + 2m + 1 \leq i \leq 2n + 2m$ is in $\mathbf{SO}^*_{2n+2m}(F)$ and is a representative of $w_0 = w_{l,G^U_l,M} \in W_{G}$, where $M \cong \mathbf{GL}_m \times \mathbf{SO}^*_{2n}$. The action of $\tilde{w}_0$ on $(g_1, g_2) \in M(\mathbb{A}_F)$ is $(g_1^{-1}, g_2)$. Furthermore, $\tilde{w}_0(\sigma) = \Pi_i \otimes \tilde{\pi}$. Assume that $\Pi_i$ not selfdual. In that we would have $\sigma \not\sim \tilde{w}_0(\sigma)$. In that case, Corollary 5.4.1 implies that the left hand side
\[
L^S(s, \pi \times \tilde{\Pi}_i) = L^S(s, \Pi \times \tilde{\Pi}_i) = \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_i) = \prod_j L^S(s, \Pi_j \times \tilde{\Pi}_i)
\]

is holomorphic on \(\text{Re}(s) > 1/2\). But the right hand side has a pole coming from \(L(s, \Pi_i \times \tilde{\Pi}_i)\) [20, II; (3.6)]. A contradiction, thus the \(\Pi_i\)'s are self-dual. \(\square\)

**Remark 6.2.2.** We can compare this construction with the following construction that uses a combination of recent results of V. Lafforgue and the already used result of L. Lafforgue [30, 31]. More precisely, given a cuspidal automorphic representation \(\pi\) of \(\text{SO}^*_n(\mathbb{A}_F)\), we get a Langlands parameter \(\phi : \Gamma_F \to \text{G}(\mathbb{Q})\) of \(\text{SO}^*_n\). After composing with \(\rho_{2n}^*\) (the \(\ell\)-adic version), we get a (semisimple) \(2n\)-dimensional Galois representation \(V\) [31]. We let \(V_l\) for \(1 \leq l \leq e\) be its irreducible subrepresentations. Every irreducible representation \(V_l\) corresponds to an irreducible cuspidal automorphic representation \(\Pi_{L,i}\) of \(\text{GL}_{M_i}(\mathbb{A}_F)\) [30, Théorème VI.9]. We consider the parabolically induced representation

\[
i_{\text{GL}^{2n}(\mathbb{A}_F)}(\Pi_{L,1} \otimes \cdots \otimes \Pi_{L,e}).
\]

Every irreducible subquotient of this representation is a lift.

If \(\pi\) is globally generic, we will compare the construction above with the construction \(\Pi = \Pi_1 \boxtimes \cdots \boxtimes \Pi_d\), from Theorem 6.2.1. Since the representations

\[
i_{\text{GL}^{2n}(\mathbb{A}_F)}(\Pi_{L,1} \otimes \cdots \otimes \Pi_{L,e}) \quad \& \quad i_{\text{P}(\mathbb{A}_F)}(\Pi_1 \otimes \cdots \otimes \Pi_d)
\]

have the same unramified component, we have that \(d = e\) and that there exists a bijection \(\sigma\) of \(\{1, \ldots, e\}\) such that \(\Pi_i \cong \Pi_{\sigma(i),L}\) [20, Theorem 4.4]. Therefore, as \(\Pi_i\) is also unitary for every \(1 \leq i \leq d\) and thus the induced representation above is irreducible, we have that both constructions coincide. Finally, as the Langlands parameter is orthogonal, we obtain that the representation \(\Pi\) is orthogonal.

We finish this section with a result about the compatibility between the \(\gamma\)-factors of the representation \(\pi\) and its transfer \(\Pi\).

**Theorem 6.2.3.** Let \(\Pi_x\) be the local component of a transfer as in Theorem 6.2.1 and \(\tau_x\) be an irreducible generic representation of \(\text{GL}_m(F_x)\). Then

\[
\gamma(s, \pi_x \times \tau_x, \psi_x) = \gamma(s, \Pi_x \times \tau_x, \psi_x).
\]

**Proof.** We first note that this is true when \(\pi_x\) is unramified, i.e when \(x \not\in S\) (5.1.1).

Let us fix \(x_0 \in |F|\) and suppose first that \(\tau_{x_0}\) is cuspidal. Then there is a cuspidal automorphic representation \(\tau = \bigotimes' \tau_x\) of \(\text{GL}_m(\mathbb{A}_F)\) that is \(\tau_{x_0}\) at \(x_0\) and such that \(\tau_x\) is unramified for \(x \not\in S\) [37, Lemma 3.1]. Furthermore, thanks to the Grunwald-Wang theorem [1, X, Theorem 5] and Section 5.1 (central character of \(\Pi_x\) is \(\zeta_x\)), we can choose a character \(\eta = \otimes \eta_x\) such that \(\eta_x\) is sufficiently ramified for \(x \in S\) and \(x \neq x_0\) so that

\[
\gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) = \gamma(s, \Pi_x \times (\tau_x \cdot \eta_x), \psi_x)
\]

and \(\eta_{x_0} = 1\).
On the other hand, the Langlands-Shahidi functional equation [36, Theorem 5.1 (vi)] gives us that
\[ L^S(s, \pi \times (\tau \cdot \eta)) = \prod_{x \in S \setminus \{x_0\}} \gamma(s, \pi_x \times (\tau_x \cdot \eta_x), \psi_x) L^S(1 - s, \tilde{\pi} \times (\tilde{\tau} \cdot \tilde{\eta})). \]

Similarly for the Rankin-Selberg \( L \)-functions
\[ L^S(s, \Pi \times (\tau \cdot \eta)) = \prod_{x \in S \setminus \{x_0\}} \gamma(s, \Pi_x \times (\tau_x \cdot \eta_x), \psi_x) L^S(1 - s, \tilde{\Pi} \times (\tilde{\tau} \cdot \tilde{\eta})). \]

Thus, after simplifying we get
\[ \gamma(s, \pi_{x_0} \times \tau_{x_0}, \psi_{x_0}) = \gamma(s, \Pi_{x_0} \times \tau_{x_0}, \psi_{x_0}), \]

obtaining thus the relation for the cuspidal representation \( \tau_x \).

The representation \( \tau_x \) can be expressed as subquotient of
\[ i^{GL_m(F_x)}(P(F_x)) (\rho_1 \otimes \cdots \otimes \rho_d), \]
where \( P \) is a parabolic subgroup containing the Borel subgroup of \( GL_m \) consisting of upper triangular matrices and the \( \rho_i \)'s are generic cuspidal representations of \( GL_m(F_x) \). Using the multiplicativity property of \( \gamma \)-factors (4.1.2) we get
\[ \gamma(s, \pi_x \times \tau_x, \psi_x) = \prod_{i=1}^{d} \gamma(s, \pi_x \times \rho_i, \psi_x). \]

Similarly,
\[ \gamma(s, \Pi_x \times \tau_x, \psi_x) = \prod_{i=1}^{d} \gamma(s, \Pi_x \times \rho_i, \psi_x). \]

As we know the desired relation when the representation is cuspidal, we obtain
\[ \gamma(s, \pi_x \times \tau_x, \psi_x) = \prod_{i=1}^{d} \gamma(s, \pi_x \times \rho_i, \psi_x) \]
\[ = \prod_{k=1}^{d} \gamma(s, \Pi_x \times \rho_k, \psi_x) \]
\[ = \gamma(s, \Pi_x \times \tau_x, \psi_x). \]

\[ \square \]

**Remark 6.2.4.**  
- First, we remark that \( \Pi \) only depends on \( \pi \), thanks to the multiplicity one result for isobaric sums [20]. We also note that, combining Theorem 6.2.3 and [17, Théorème 1.1], we can prove that \( \Pi_x \) only depends on \( \pi_x \). Finally, we also mention that, conjecturally, the image is characterised by the condition in the Theorem 6.2.1 and the fact that \( L^T(s, \Pi, \text{Sym}^2) \) has a pole at \( s = 1 \) for any sufficiently large finite set of places \( T \) containing all archimedean places. This is established in the work of Cogdell, Piatetski-Shapiro and Shahidi over number fields [9].
• We expect a relation between $L$-functions and $\varepsilon$-factors, that is similar to the one between $\gamma$-factors in Theorem 6.2.3. In positive characteristic, we have these relations established for the split classical groups [35].

6.3. Unramified Ramanujan conjecture.

Theorem 6.3.1. Let $\pi = \otimes_x \pi_x$ be a globally generic cuspidal representation of $SO_{2n}^*(\mathbb{A}_F)$. If $\pi_x$ is unramified, then its Satake parameter has absolute value 1.

Proof. Let us fix $x \in |F|$ an inert place. The split case is obtained as in [35, Theorem 9.14]. Using Theorem 6.2.1, we can find a transfer $\Pi$ of $\pi$ that is the isobaric sum

$$\Pi = \Pi_1 \oplus \cdots \oplus \Pi_e,$$

where each $\Pi_i$ is a unitary self-dual cuspidal automorphic representation of $GL_{N_i}(\mathbb{A}_F)$, and $\Pi_i \not\cong \Pi_j$ for $i \neq j$. By [30, Théorème VI.10], each $\Pi_{i,x}$ is tempered.

If $\pi_x$ is unramified, let

$$\text{diag}(\alpha_1, \cdots, \alpha_{n-1}, 1) \rtimes \text{Fr}_x$$

be its semisimple conjugacy class. Then, by definition, the semisimple conjugacy class of $\Pi_x$ is given by

$$\text{diag}(\alpha_1, \cdots, \alpha_{n-1}, 1, 1, \alpha_{n-1}^{-1}, \cdots, \alpha_1^{-1}).$$

Every $\alpha_j$ or $\alpha_j^{-1}$ is the Satake parameter of one the representations $\Pi_{i,x}$. As every $\Pi_{i,x}$ is tempered, we must have

$$|\alpha_i| = 1.$$

Remark 6.3.2. If one proves the relation between $L$-functions, mentioned in Remark 6.2.4, then we expect to prove that $\pi_x$ is tempered for every $x \in |F|$. 

References

[1] Emil Artin and John Tate. Class Field Theory. AMS Chelsea Publishing, V. 366, Primary 11 (2008)
[2] Joseph Bernstein and Andrei Zelevinsky. Induced representations of reductive p-adic groups I. Annales Scientifiques de l’École Normale Supérieure, Série 4, 10 (4), pp. 441–472 (1977)
[3] Armand Borel. Admissible representations of a semi-simple group over a local field with vectors fixed under an iwasori subgroup. Invent Math 35, pp. 233–259 (1976)
[4] Armand Borel. Automorphic L-functions. Proc. Symp. Pure Math. 33 (Part 2), pp. 27-61 (1979)
[5] Armand Borel and Hervé Jacquet. Automorphic forms and Automorphic representations. Proc. Symp. Pure Math. 33 (Part 1), pp. 189-202 (1979)
[6] Colin J. Bushnell and Guy Henniart. The Local Langlands Conjecture for GL(2). Springer-Verlag Berlin Heidelberg (2006)
[7] James W. Cogdell and Ilya I. Piatetski-Shapiro. Converse Theorems for $GL_n$. Publications Mathématiques de l’IHÉS, Volume 79, pp. 157-214 (1994)
[8] James W. Cogdell, Ilya I. Piatetski-Shapiro and Freydoon Shahidi. Functoriality for the classical groups. Publications Mathématiques de l’IHÉS, Volume 99, pp. 163-233 (2004)
[9] James W. Cogdell, Ilya I. Piatetski-Shapiro and Freydoon Shahidi. Functoriality for the Quasisplit Classical Groups. On Certain L-Functions. Clay Mathematics Proceedings Vol. 13, pp. 117-140 (2011)
[10] Brian Conrad. Reductive group schemes. Autour des Schémas en Groupes, École d’été "Schémas en Groupes", a celebration of SGA3. Volume I, pp. 93-440. Panoramas et Syntheses Volume: 42-43 (2014)
[11] Héctor del Castillo, Guy Henniart and Luis Lomeli. On three conjectures. Work in progress (2021)
[12] Wee Teck Gan and Luis Lomeli. Globalization of supercuspidal representations over function fields and applications. Journal of the European Mathematical Society, 20, pp 2813-2858 (2018)
...
[40] Goran Muic. A Proof of Casselman-Shahidi’s Conjecture for Quasi-split Classical Groups. Canadian Mathematical Bulletin 44 pp. 298-312 (2001)
[41] Jürgen Neukirch. Algebraic Number Theory. Springer-Verlag Berlin Heidelberg (1999)
[42] Ilya Piatetski-Shapiro. Mutilplicity one Theorems. Proc. Symp. Pure Math. 33 (Part 1), pp. 209-212 (1979)
[43] Freydoon Shahidi. On Certain L-Functions. American Journal of Mathematics Vol. 103, No. 2, pp. 297-355 (1981)
[44] Freydoon Shahidi. On the Ramanujan Conjecture and Finiteness of Poles for Certain L-Functions. The Annals of Mathematics, Second Series, Vol. 127, Issue 3, pp. 547-585 (1988)
[45] Freydoon Shahidi. A Proof of Langlands’ Conjecture on Plancherel Measures; Complementary Series of $p$-adic groups. Annals of Mathematics Second Series, Vol. 132, No. 2, pp. 273-330 (1990)
[46] Freydoon Shahidi. Twists of a General Class of L-Functions by Highly Ramified Characters. Canadian Mathematical Bulletin 43(3) pp. 380-384 (2000)
[47] Joseph A. Shalika. The multiplicity one theorem for $GL(n)$. Ann. Math. V. 100, pp. 171-193 (1974)
[48] Allan J. Silberger. Introduction to Harmonic Analysis on Reductive $p$-adic Groups. (MN-23): based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971-73. Princeton University Press (1979)
[49] David Soudry, On Langlands functoriality from classical groups to $GL_n$. Formes Automorphes (I), Astérisque 298, pp. 335-390 (2005)
[50] Marko Tadić. Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). Annales scientifiques de l’École Normale Supérieure, Serie 4, Volume 19 no. 3, p. 335-382 (1986)
[51] Jean-Loup Waldspurger. La Formule de Plancherel pour les Groupes $p$-adiques. D’après Harish-Chandra. Journal of the Inst. of Math. Jussieu 2(2) pp. 235-333 (2003)
[52] Yuanli Zhang. The holomorphy and nonvanishing of normalized local intertwining operators. Vol. 180, No. 2, pp. 385-398 (1997)

Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile

Email address: hector.math@gmail.com