CHARACTERIZING CURVES BY THEIR ODD THETA-CHARACTERISTICS

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1. Introduction

Consider, for every $g \geq 4$, a non-singular, complex, canonical curve of genus $g$, that is, a curve $X$ embedded in $\mathbb{P}^{g-1}$ by its complete canonical series $[\omega_X]$. Each such a curve possesses $2^g$ line bundles $L$ of degree $g-1$ such that $L^\otimes 2 = \omega_X$, the theta-characteristics of $X$, and precisely $N_g := \binom{g}{2}$ of them are odd (i.e. $h^0(X, L)$ is odd). To a non-zero section $\sigma$ of a theta-characteristic one associates a “half canonical” divisor $D = (\sigma)$, whose double $2D$ is cut on $X$ by a hyperplane $H$ in $\mathbb{P}^{g-1}$ (whose scheme-theoretic intersection with $X$ is, of course, everywhere non reduced). For obvious reasons, such a hyperplane $H$ will be called a theta-hyperplane of $X$.

Assume that $X$ is general. Then all odd theta-characteristics $L$ satisfy $h^0(X, L) = 1$, while the even ones have no non-zero sections. Therefore $X$ has exactly $N_g$ theta-hyperplanes; the set of such hyperplanes will be denoted $\theta(X)$, and considered as an element of $\text{Sym}^{N_g}(\mathbb{P}^{g-1})^*$. This said, the main result of this paper is (Theorem 6.1.1)

Main Theorem. Let $X$ and $X'$ be general canonical curves of genus $g \geq 4$. If $\theta(X) = \theta(X')$ then $X = X'$.

The case $g = 3$, not considered here, has recently been settled in [CS] (see below).

The above theorem can be put in a different perspective as follows. Let $J(X) = \text{Pic}^0(X)$ be the jacobian variety of $X$, fix a line bundle $L_0$ of degree $g-1$ and let $\Theta = \Theta_{L_0} = \{ \xi \in J(X) : h^0(\xi \otimes L_0) > 0 \}$ be the corresponding theta divisor. If we choose $L_0$ to be a theta-characteristic we obtain a symmetric theta-divisor. Then the 2-torsion points of $J(X)$ correspond, via multiplication by $L_0$, to the theta-characteristics on $X$. If $X$ is general, $\Theta$ contains exactly $N_g$ 2-torsion points, corresponding to the odd theta-characteristics, and they are nonsingular by Riemann Singularity Theorem. Therefore, by the well known geometric interpretation of the Gauss map for the theta divisor, we obtain that the set of Gauss images of the 2-torsion points of $\Theta$, call it $\gamma(J(X), \Theta_2)$, coincides with $\theta(X) \in \text{Sym}^{N_g}(\mathbb{P}^{g-1})^*$ (after the choice of a basis of $H^1(O_X)$). We deduce that our theorem implies the following

Theorem. If $X$ and $X'$ are general abstract curves of genus $g \geq 4$ such that $\gamma(J(X), \Theta_2)$ and $\gamma(J(X'), \Theta'_2)$ are projectively equivalent, then $X \cong X'$.

Observe that this result gives, in particular, a refinement of the classical theorem of Torelli in the generic case, because $\gamma(J(X), \Theta_2)$ only depends on the first order behaviour at finitely many points of the principally polarized abelian variety $(J(X), \Theta)$. 

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Notice moreover that the definition of $\gamma(J(X), \Theta_2)$ can be extended to any principally polarized abelian variety $(A, \Theta)$, provided that the 2-torsion points of the divisor $\Theta$, which can be assumed to be symmetric, are all nonsingular. This condition is satisfied on a dense open subset $U$ of the moduli space $A_g$ of principally polarized abelian varieties. It is therefore natural to ask whether a result analogous to the above theorem is valid for principally polarized abelian varieties.

This might lead to an “odd counterpart” of the coordinatization of $A_g$ by theta constants (see [Mu1] or [Mu3]). We plan to investigate this problem in a different paper. For a study of similar issues, using the classical theta functions, we refer to the work of R. Salvati Manni in [SM].

Our approach to the Main Theorem is indirect. In fact, we are also interested in limits of line bundles and linear series for families of smooth curves specializing to singular ones. A part of this paper is devoted to certain aspects of such a problem. Our results are applied to construct a degeneration argument that proves our Theorem.

From this point of view, the first issue is to study theta-hyperplanes and theta-characteristics for singular curves, and to relate the abstract and the projective description. This is the topic of Section 2. For the abstract question, we use the so-called moduli space (i.e. stack or scheme) of spin curves, $\overline{S_g}$, constructed by M. Cornalba in [Co]. $\overline{S_g}$ is a geometrically meaningful compactification of the moduli space of theta-characteristics; it is endowed with a natural, finite morphism $\pi : \overline{S_g} \rightarrow \overline{M_g}$ onto the moduli space of stable curves $\overline{M_g}$. The fibers of $\pi$ over singular curves parametrize their “generalized theta-characteristics”.

On the projective side, the configuration $\theta(X)$ of theta-hyperplanes is naturally defined (in [C]) for certain singular curves in $\mathbb{P}^{g-1}$; we thus have a regular morphism $\theta : V \rightarrow \text{Sym}^{N_g}(\mathbb{P}^{g-1})^*$

where $V$ is a suitable subscheme of the Hilbert scheme containing canonical curves in $\mathbb{P}^{g-1}$. The curves parametrized by $V$ include general smooth curves and what we call split curves. A split curve is the union of two rational normal curves meeting transversely at $g + 1$ points.

In this framework, the crucial result is that split curves satisfy our main Theorem; that is, a split curve $X_0$ is uniquely determined, among all curves in $V$, by the configuration $\theta(X_0)$ ([C] Theorem 5, strengthened here by 4.4.5). This is what makes it possible to prove the Main Theorem by degeneration.

Here is the outline of the argument and a brief description of the other tools that we use. Consider a general 1-parameter family $\mathcal{X} \rightarrow T$, ($T$ a smooth curve) of non-singular canonical curves specializing to a split curve $X_0$; denote by $X_t$ the generic fiber. If the Theorem were false, there would exist a second family of canonical curves $\mathcal{X}' \rightarrow T$ with nonsingular generic fiber $X'_{t_0}$, and such that $\theta(X_t) = \theta(X'_{t_0})$. The first difficulty is to control the special fiber of the second family, $X'_{t_0}$, over which there is no a priori information. In particular, $\theta(X'_{t_0})$ might fail to be well defined (if it were defined, it would obviously equals $\theta(X_0)$). To handle the situation, we study the stable reduction of $\mathcal{X}' \rightarrow T$ and transfer our projective problem into an abstract one, using the existence of $\overline{S_g}$ and its properties.

We proceed to analyze the combinatorial side of the issue; the goal is to abstractly characterize split curves, by means of their generalized theta-characteristics; loosely speaking, we are after an abstract counterpart of the projective characterization by theta-hyperplanes mentioned above. This is the topic of Section 3, which is of independent interest. A purely combinatorial invariant for an abstract nodal curve (the “set of exponents”) is defined and it is proved to uniquely determine split curves among all stable curves (3.4.1, 3.4.2). This set of exponents turns out to encode the numerical data of the ramification of the structural morphism $\pi : \overline{S_g} \rightarrow \overline{M_g}$. Whence its relevance for our central problem.

From the above results we obtain that the stable reduction of $\mathcal{X}' \rightarrow T$ has a split curve as special fiber.
Assume now, for simplicity, that $\mathcal{X} \to T$ is the canonical model of its stable reduction. Then $X'_0$ is a split curve and we can apply the result mentioned before (4.4.5) to infer that, since $\theta(X'_0) = \theta(X_0)$, then $X'_0 = X_0$.

The technical problem, behind the above simplification, is dealt with in Section 4, by studying the natural action of $PGL(g)$ on the spaces of configurations of hyperplanes; we here use methods of Geometric Invariant Theory.

To conclude that $X_t = X'_t$, it suffices to show that the morphism $\theta$ is an immersion locally at a split curve. This is done in Section 5, where we investigate the tangent space to the locus of deformations of a split curve $X_0$ which remain tangent to the hyperplanes in $\theta(X_0)$. We apply the theory of elementary transformations of vector bundles.

We conclude this introduction with a few words about the case of genus 3. In [CS] we proved the Theorem using a far more simple version of a similar approach. In that case, there is no need to consider the abstract and the combinatorial side of the problem. The projective analysis suffices, using the techniques of Geometric Invariant Theory. Just recently, D. Lehavi in [L] strengthened our result for $g = 3$, showing that every smooth quartic (i.e., not just a general one) can be recovered by its 28 bitangents. We thank him for sending us his preprint.

The combinatorial Theorem 3.4.1 is a strengthening of our original statement, found by Cinzia Casagrande, to whom we are grateful.

1.1. Notations and Conventions. We work over the field of complex numbers. A semistable curve in the sense of Deligne and Mumford is a connected, reduced, projective curve $Y$ having at most nodes as singularities and such that if $E \subset Y$ is a smooth rational component, then $E$ meets the union of the remaining components of $Y$ in at least 2 points. An $E$ such that $E \cap Y - E = 2$ is called a destabilizing component of $Y$. If $Y$ is semistable and has no destabilizing component, $Y$ is called a stable curve (in the sense of Deligne and Mumford). Stable curves of genus $g$ admit a coarse moduli space, denoted by $\overline{M}_g$; it is a projective, integral scheme.

If $Y$ is semistable and no two of its destabilizing components meet, then $Y$ is called a quasistable curve (a “decent” curve in the terminology of M. Cornalba, in [Co]).

An abstract, stable curve $Y$ of genus $g$ is a split curve if $Y$ is the union of two rational, nonsingular curves meeting transversely at $g + 1$ points. A curve $X \subset \mathbb{P}^{g-1}$ is a projective split curve if and only if $X$ is the union of two rational normal curves meeting transversely at $g + 1$ points.

Clearly, a projective split curve is the canonical image of an abstract split curve.

We shall consider “families” (of curves, almost always) over a one-dimensional pointed base $T$, where $T$ denotes a nonsingular, connected, affine, curve of finite type and $t_0 \in T$ a marked point. A family $\mathcal{U} \to T$ is a flat, proper morphism of schemes; the fiber over the marked point $t_0$ will be denoted by $U_0$ or simply by $U$, and called the “special” fiber. The fiber over $t \neq t_0$ will be denoted by $U_t$ and called the “generic” fiber.

In certain contexts, a family $\mathcal{U} \to T$ as above will be called a one parameter deformation of its central fiber $U_0$.

We shall denote $P_{N_g} := Sym^{N_g}(\mathbb{P}^{g-1})^*$.

For $S$ a topological space, $\gamma_c(S)$ is the number of its connected components.

1.2. Let $L$ and $M$ be two finite sets of integers. We say that $L$ dominates $M$ (in symbols $L \geq M$) iff there exists a surjective map $\alpha : L \to M$ such that for every $l \in L$ we have $\alpha(l) \geq l$. It is very easy to see that, if $L \geq M$ and $M \geq L$, then $L = M$. In particular, the above definition gives a partial ordering on the set of all finite sets of integers.

Let $S$ be a purely dimensional scheme (that is, every irreducible component of $S_{\text{red}}$ has the
same dimension). We associate to $S$ its multiplicity set $L(S)$ as follows

$$L(S) := \{ n : \exists \text{ some irreducible component } Z \text{ of } S \text{ such that } \text{mult}_Z S = n \}$$

## 2. The moduli theoretic framework

### 2.1. The basic projective set up.

Fix $g \geq 3$ and consider the Hilbert scheme $\text{Hilb}^{p(x)}[\mathbb{P}^{g-1}]$ of curves in $\mathbb{P}^{g-1}$ having Hilbert polynomial $p(x) = (2g-2)x - g + 1$. In it we find the locus of connected curves $X$ of degree $2g - 2$ and arithmetic genus $g$, satisfying the following three conditions:

1. $X$ is reduced and has at most nodes as singularities.
2. $X$ is embedded in $\mathbb{P}^{g-1}$ by the complete linear series $|\omega_X|$ (where $\omega_X$ denotes the dualizing line bundle of $X$).
3. No irreducible component of $X$ is contained in a hyperplane.

A curve satisfying 1. and 2. above is, clearly, stable in the sense of Deligne and Mumford; we shall call it a canonical curve throughout the paper.

**Definitions.** Let $X \subset \mathbb{P}^{g-1}$ be a curve satisfying 1, 2 and 3 above. A hyperplane $H \subset \mathbb{P}^{g-1}$ will be called a theta-hyperplane of $X$ if $H \cap X$ is everywhere non reduced and supported at $g - 1$ distinct points. Let $i$ be an integer with $0 \leq i \leq g - 1$. We shall say that the theta-hyperplane $H$ is of type $i$ if $H$ contains exactly $i$ singular points of $X$. Such a curve $X$ will be called theta-generic if it has finitely many theta-hyperplanes.

As noticed in the introduction, a general nonsingular canonical curve has $N_g = 2g^{-1}(2^g - 1)$ distinct theta-hyperplanes, corresponding bijectively to its odd theta characteristics. Therefore it is theta-generic.

We shall say that an abstract stable curve is theta-generic if its dualizing line bundle is very ample and if its canonical model is theta-generic. Notice that theta-generic curves can be of only two topological types: either they are irreducible, or they are the union of two rational components meeting at $g + 1$ distinct points, i.e., they are split curves.

We shall denote by $V \subset \text{Hilb}^{p(x)}[\mathbb{P}^{g-1}]$ the open subset parametrizing theta-generic curves and by $V_0 \subset V$ the open subset parametrizing nonsingular theta-generic curves. As in [C] and [CS] we define a morphism $\theta : V_0 \to Sym^{N_g}(\mathbb{P}^{g-1})^*$ by $\theta(X) = \{ H_1, \ldots, H_{N_g} \}$, where $H_1, \ldots, H_{N_g}$ are the (distinct) theta-hyperplanes of $X$. Then, arguing as in [CS], Lemma 2.3.1, it is easy to see that $\theta$ can be extended to a morphism (called again $\theta$)

$$\theta : V \to Sym^{N_g}(\mathbb{P}^{g-1})^*$$

It will often be convenient to view $\theta(X)$ as a (not necessarily reduced) hypersurface of degree $N_g$ in $\mathbb{P}^{g-1}$, all of whose irreducible components are (possibly multiple) hyperplanes; such a hypersurface will be also denoted $\theta(X)$, by abuse of notation, and called the theta-hypersurface of $X$. From [C] and [CS] we get that $\theta(X)$ is reduced if and only if $X$ is smooth; if $X$ is singular, all hyperplanes of type $i$ appear in $\theta(X)$ with multiplicity $2^i$.

For a given $X$ in $V$ we will denote by $t_i(X)$ the number of distinct theta-hyperplanes of type $i$. The numbers $t_i(X)$ only depend on the number of nodes and of irreducible components of $X$, and have been computed in [C].
Let

$$X \hookrightarrow T \times \mathbb{P}^{g-1}$$

$$\downarrow$$

$$T$$

be a family of curves whose Hilbert polynomial is \(p(x)\); the natural morphism to the Hilbert scheme is denoted by

$$\psi_X : T \rightarrow \text{Hilb}^{p(x)}[\mathbb{P}^{g-1}].$$

Assume that the generic fiber is in \(V\), that is \(\psi_X(T - \{t_0\}) \subset V\). Since \(T\) is a nonsingular curve and \(P_{N_g} := \text{Sym}^N\mathbb{P}^{g-1} \ast\) is projective the composition \(\theta \circ \psi_X\) extends to the whole of \(T\); it will be denoted by

$$\theta_X : T \rightarrow P_{N_g}.$$ 

If \(\psi_X(T - \{t_0\}) \subset V_0\) we consider the curve

$$J_X \subset T \times (\mathbb{P}^{g-1})^*$$

defined as the closure of the incidence correspondence

$$\{(t, H) : H \subset \theta(X_t), \ t \neq t_0\} \subset T \times (\mathbb{P}^{g-1})^*$$

Away from \(t_0\), \(J_X \rightarrow T\) is an unramified covering of degree \(N_g\).

In the next section, we describe the abstract counterpart of the above set-up.

2.2. Cornalba’s moduli space of spin curves. In [Co] M. Cornalba constructed a geometrically meaningful compactification \(\overline{S}_g\), over \(\overline{M}_g\), of the moduli space of theta-characteristics of smooth curves of genus \(g\). We need to recall some basic features of \(\overline{S}_g\), which is called the moduli space of stable spin curves.

Both the scheme description and the stack description are available, we shall confine ourselves to the schematic definition, which suffices for our purposes.

\(\overline{S}_g\) is a normal, projective scheme which admits a proper morphism \(\pi\) onto \(\overline{M}_g\)

$$\pi : \overline{S}_g \rightarrow \overline{M}_g.$$ 

As expected, the degree of \(\pi\) is \(2^{2g}\), and \(\overline{S}_g\) is a disjoint union of two irreducible components, \(\overline{S}_g^+\) and \(\overline{S}_g^-\), corresponding, respectively, to even and odd theta-characteristics. We are mostly interested in odd theta characteristics in this paper. The degree of the restriction of \(\pi\) to \(\overline{S}_g^-\) is \(N_g\).

The points of \(\overline{S}_g\) are described as line bundles on certain quasistable curves of arithmetic genus \(g\). Some more notation: let \(Y\) be a stable curve and let \(\Sigma\) be a set of nodes of \(Y\), denote by \(Y_{\Sigma}^\nu\) the normalization of \(Y\) at all nodes in \(\Sigma\) and by \(Y_{\Sigma}^\nu\) the quasistable curve obtained by “blowing-up” \(Y\) at all nodes in \(\Sigma\). Thus \(Y_{\Sigma}^\nu\) is the closure of what remains of \(Y_{\Sigma}\) after removing all of its destabilizing components.

A point in \(\overline{S}_g\) is (the isomorphism class of) a spin curve \(\xi\), that is the following set of data. First, a subset \(\Sigma_{\xi} = \Sigma\) of nodes of \(Y\) and the quasistable curve \(Y_{\Sigma}\). Second, a line bundle \(L_{\xi} = L\) on \(Y_{\Sigma}\) and a homomorphism \(\alpha : L^2 \rightarrow \omega_{Y_{\Sigma}}\) such that \(L\) has degree 1 on all destabilizing components, and the restriction of \(\alpha\) to \(Y_{\Sigma}^\nu\) is an isomorphism (see [Co] for details). We shall say that \(\xi\) is supported on \(\Sigma\). As it will be made clear later in the paper (Section 3), not all subsets of nodes of
Y are the support of a spin curve: those that are will be called “admissible” and will be studied extensively later.

The scheme structure on $\overline{S}_g$ is obtained in [Co] after constructing the “universal deformation space” of a spin curve $\xi$. First (in Section 2) he describes its group of automorphisms $\text{Aut}(\xi)$ and shows that there is an exact sequence of finite groups

$$1 \longrightarrow \text{Aut}_0(\xi) \longrightarrow \text{Aut}(\xi) \longrightarrow \text{Aut}(Y)$$

where $\text{Aut}_0(\xi)$ is the so-called group of “inessential” automorphisms, (acting trivially on $Y$). Call $B_\xi$ the base of the universal deformation space of $\xi$ and $B_Y$ the base of the universal deformation space of $Y$; $B_\xi$ and $B_Y$ will be viewed, as in [Co], as analytic spaces, that is, as $3g-3$-dimensional discs. There is a commutative diagram of morphisms

$$\begin{array}{cccc}
B_\xi & \longrightarrow & B_\xi/\text{Aut}_0(\xi) & \longrightarrow & B_\xi/\text{Aut}(\xi) & \hookrightarrow & \overline{S}_g \\
\delta \downarrow & & \pi_\xi \downarrow & & \downarrow \pi & & \\
B_Y & = & B_Y & \overset{\rho_Y}{\longrightarrow} & B_Y/\text{Aut}(Y) & \hookrightarrow & \overline{M}_g
\end{array}$$

with $B_\xi/\text{Aut}_0(\xi) \subset B_Y \times_{\overline{M}_g} \overline{S}_g$; everything is shown to satisfy the necessary compatibility conditions. We recall how the covering $\delta$ is defined ([Co] section 5): choose coordinates $u_1, \ldots, u_{3g-3}$ for $B_Y$ so that the first $\#\Sigma$ correspond to the loci where the $i$-th node of $\Sigma$ is preserved. Choose coordinates $t_1, \ldots, t_{3g-3}$ on $B_\xi$ and define $\delta$ as the base change $u_i = \delta^*(t_i)$ for $i \leq \#\Sigma$ and $u_i = \delta^*(t_i)$ for $i > \#\Sigma$.

Let $Y \longrightarrow T$ be a family of generically stable curves. Since $T$ is nonsingular and $\overline{M}_g$ is projective, its moduli map $T - \{t_0\} \longrightarrow \overline{M}_g$ extends to $T$ and is denoted by $\phi_Y: T \longrightarrow \overline{M}_g$.

The pull-back to $T$ of $\overline{S}_g$ is a curve over $T$ denoted by

$$S_Y := \phi_Y^* \overline{S}_g.$$ 

**Proposition 2.2.1.** Let $Y \longrightarrow T$ be a general one-parameter deformation of the stable curve $Y$.

a ) The curve $S_Y$ is smooth.

b ) Let $\xi$ be a spin curve on $Y$ supported on the set $\Sigma$. Then the index of ramification of the finite covering $S_Y \longrightarrow T$ at the point corresponding to $\xi$ is $2e_Y(\Sigma) - 1$ where $e_Y(\Sigma) = \#\Sigma - \gamma_c(Y^\xi) + 1$.

**Remark.** The same result holds for even spin curves, that is, if we replace $S_Y$ by $\phi_Y^* \overline{S}_g^+$. The proof is the same.

**Proof.** It suffices to show that $S_Y$ is non-singular at the points lying over $t_0$. The moduli morphism $\phi_Y$ factors locally through the map $\phi_Y: \mathcal{O}_{T,t_0} \longrightarrow B_Y$; let $\overline{T} := \text{Im}\phi_Y$. We shall prove, more precisely, that if $\overline{T}$ is transverse in $B_Y$ to the loci where the nodes of $Y$ are preserved, then $S_Y$ is smooth.

It suffices to prove that $\phi_Y^* \circ \rho_Y^* \overline{S}_g$ is smooth. By the above diagram, this is equivalent to show that the curve $\pi_\xi^* \overline{T}$ is smooth. Such a curve is the quotient via a finite group of the curve $\delta^{-1} \overline{T}$; this last curve is smooth, because, by assumption, it is transverse to the branch locus of $\delta$ (look at the explicit description of $\delta$ given above). Hence its quotient by any finite group is smooth; this proves a).
For b) as well, most of the argument is already in Cornalba’s paper (Section 5), together with various explicit examples (from (5.3) to (5.7)).

It suffices to look at how the covering \( \pi_{\xi}^{-1}(\mathcal{T}) \to \mathcal{T} \) ramifies. The natural map \( \delta : B_\xi \to B \) is a finite covering of degree \( 2^{#\Sigma} \) totally ramified over the origin (as explained above); \( \pi_{\xi}^{-1}(\mathcal{T}) \) is the quotient of \( \delta^{-1}(\mathcal{T}) \) via \( \text{Aut}_0(\xi) \). The structure of \( \text{Aut}_0(\xi) \) is described in [Co], Lemma 2.2 in terms of the graph associated to \( \xi \) (having vertices the connected components of \( Y_\Sigma^\nu \) and edges the destabilizing components of \( Y_\Sigma \)); his result can be re-stated by saying that \( \text{Aut}_0(\xi) \) is a vector space over \( \mathbb{Z}/2\mathbb{Z} \) of dimension equal to the number of connected components of \( Y_\Sigma^\nu \) minus 1:

\[
\dim_{\mathbb{Z}/2\mathbb{Z}} \text{Aut}_0(\xi) = \gamma_c(Y_\Sigma^\nu) - 1.
\]

Hence \( \pi_{\xi}^{-1}(\mathcal{T}) \to \mathcal{T} \) ramifies at the point corresponding to \( \xi \) with index \( 2^{#\Sigma-\gamma_c(Y_\Sigma^\nu)}-1 \). \( \square \)

2.3. We now introduce the scheme \( S_Y \) parametrizing odd spin curves having \( Y \) as stable model:

\( S_Y \) is the (scheme-theoretic) special fiber of \( \rho_Y^* \overline{S}_g = B_Y \times_{\overline{\mathcal{M}}_g} \overline{S}_g \to B_Y \).

Thus \( S_Y \) is a zero-dimensional scheme of length \( N_g \). If \( \text{Aut}(Y) \) is trivial, then \( S_Y \) is the fiber of \( \overline{S}_g \) over the point \( Y \in \mathcal{M}_g \). The following statement is an obvious consequence of 2.2.1.

**Corollary 2.3.1.** Let \( Y \) be a stable curve. \( L(S_Y) = \{2^n, \text{ s.t. } \exists \xi \in S_Y : e_Y(\Sigma_\xi) = n\} \).

The number \( e_Y(\Sigma) \), defined in the statement of 2.2.1, will be called the exponent of \( \xi \) and will play a crucial role in the sequel. Notice that, denoting by \( Z = Y_\Sigma^\nu \) and by \( g_Z \) its arithmetic genus, we have that \( e_Y(\Sigma) = g - g_Z \). The following simple observation will be used later.

**Claim.** Assume that \( \xi \in S_Y \) has exponent at least \( g - 3 \); then \( h^0(Z, L_\xi \otimes O_Z) = 1 \).

In fact \( \xi \) is an odd spin curve, by definition; thus \( L_\xi \otimes O_Z \) is an odd (hence effective) square root of \( \omega_Z \), that is, \( h^0(Z, L_\xi \otimes O_Z) \) is odd (see [Co] section 6). Now \( Z \) is a curve of genus at most 3, hence of course \( h^0(Z, L_\xi \otimes O_Z) \leq 2 \).

Denote by \( S_\xi \subset S_Y \) the (unique) connected component of \( S_Y \) supported on \( \xi \). Denote by \( \hat{S}_Y \) the union of all components of \( S_Y \) corresponding to spin curves whose multiplicity is at least \( 2^{g-3} \), that is

\[
\hat{S}_Y := \bigcup_{\xi : e_Y(\Sigma_\xi) \geq g - 3} S_\xi
\]

Of course, \( \hat{S}_Y \) can be empty, for example, it will be empty if \( Y \) is of compact type, in which case \( S_Y \) is reduced (see [Co] and also 3.3.1).

2.4. Relating the abstract and the projective pictures. The set up in this section is the following. \( Y \) is a stable curve and \( Y \to T \) a one-parameter deformation of \( Y \) with smooth and theta-generic general fiber. A generically canonical, birational model of \( Y \to T \) is the following: a projective family of curves: \( \mathbb{P}^{g-1} \times T \supset \mathcal{W} \to T \) and a birational \( T \)-map \( \rho : Y \to \mathcal{W} \) which is an isomorphism away from \( t_0 \) and such that for \( t \neq t_0 \), the induced map \( Y_t \to W_t \) is the canonical morphism.

Unless otherwise specified, no assumption will be made on the central fiber \( W \) of \( \mathcal{W} \), or on the restriction of \( \rho \) on the central fibers (notice that the locus of canonical curves is not closed in \( \text{Hilb}^{p(x)}(\mathbb{P}^{g-1}) \)).

To such a picture we add two, differently defined, families of odd theta-characteristics: to the abstract family \( Y \to T \) we associate \( S_Y \to T \) (defined in 2.2); to the projective family \( \mathcal{W} \to T \)
we associate $J_W \to T$ (defined in 2.1). To make our notation more precise, consider $\theta_W : T \to P_{N_t}$, defined in 2.1. For $t \neq t_0$, $\theta_W(t)$ represents the configuration of the $N_g$ theta-hyperplanes of $W_t$. Since $W_t \in V$, this configuration does not depend on $W$ and we have $\theta_W(t) = \theta(W_t)$. For $t_0$ the configuration $\theta_W(t_0)$ corresponds to a hypersurface of degree $N_g$ in $\mathbb{P}^{g-1}$ for which we shall use the notation

$$\theta_W(W) := \theta_W(t_0)$$

(as usual, the above symbols will be abused to denote both points in $P_{N_t}$ and hypersurfaces in $\mathbb{P}^{g-1}$). If $W \in V$ we have, of course, $\theta_W(W) = \theta(W)$ defined before. Consider now $J_W \subset T \times (\mathbb{P}^{g-1})^*$. If $t \neq t_0$, the fiber of $J_W$ over $t$ only depends on $W_t$ and will be denoted by $J_{W_t}$. The special fiber $(J_W)_{t_0}$ depends, a priori, on $W$. We shall denote

$$J^W_t := (J_W)_{t_0}$$

and, if $W \in V$ we shall simply write $J_W := (J_W)_{t_0}$.

Let us consider the multiplicity sets; if $t \neq t_0$ we have, obviously, $L(\theta(W_t)) = L(J_{W_t}) = \{1\}$. For the special fibers we have, by construction, $L(\theta(W)) = L(J^W_t)$.

**Lemma 2.4.1.** Let $\mathcal{Y} \to T$ be a general one-parameter deformation of the stable curve $Y$; let $\mathcal{W} \to T$ be a generically canonical, birational image of $\mathcal{Y} \to T$. (a) There exists a natural, surjective, birational $T$-morphism

$$\mu : S_Y \to J_W.$$  

(b) Assume furthermore that $K_Y$ is very ample and that the central fiber $W$ of $\mathcal{W} \to T$ is a canonical image of $Y$, with $W \in V$. Then the morphism $\mu$ above induces an isomorphism on the central fibers. In particular, $L(S_Y) = L(J_W)$.

**Proof.** By assumption, the generic fiber $Y_t$ of $\mathcal{Y}$ is smooth and non-hyperelliptic and the generic fibers of $\mathcal{W} \to T$ are in $V$. There is a natural $T$-birational map

$$\mu : S_Y \dasharrow J_W$$

which is an isomorphism away from $t_0$. It associates to an odd theta-characteristic on $Y_t$ the corresponding theta-hyperplane of its canonical model $W_t$. By part (a) of Proposition 2.2.1 the morphism $\mu$ extends to the whole of $S_Y$. This proves (a).

For (b), notice that by (a) it is enough to show that $\mu$ induces a set theoretic bijection on the central fibers, that is the set theoretic map

$$\mu_0 : (S_Y)_{\text{red}} \to (J_W)_{\text{red}}$$

is a bijection. The proof is a bit long, but very simple and standard.

Throughout the rest of the argument, we shall identify $Y$ with $W$ by the canonical isomorphism given in the statement. Let $\xi \in S_Y$ be a spin curve supported on the set of nodes $\Sigma_\xi = \Sigma$. To $\xi$ there corresponds a line bundle $L_\xi$ on the quasistable curve $Y_\Sigma$ such that, if $Z := Y_\Sigma^\nu \subset Y_\Sigma$, we have $(L_\xi \otimes O_Z)^2 \equiv \omega_Z$ and such that, for every destabilizing component $E$ of $Y_\Sigma$, the restriction of $L_\xi$ to $E$ is $O_E(1)$. Since $W \in V$, there exists a unique effective divisor $D_\xi \in \text{Pic}Z$ such that

$$O_Z(D_\xi) = L_\xi \otimes O_Z.$$  

Moreover, $D_\xi$ is reduced and $\text{supp}D_\xi \subset (Y_\Sigma)_{\text{smooth}}$.

Consider the canonical morphism $\sigma : Y_\Sigma \to W \subset \mathbb{P}^{g-1}$ (which of course contracts all the destabilizing components); locally at every point of $D_\xi$, $\sigma$ is an isomorphism. The set of points
\[ \Sigma \cup \sigma(D_{\xi}) \subset W \] is a set of \( g - 1 \) points in general position (recall that \( W \) is theta-generic), which therefore spans a unique hyperplane \( H_{\xi} \) containing \( \Sigma \). It is clear that \( H_{\xi} \) is tangent to \( W \) at every point of \( \sigma(D_{\xi}) \).

In conclusion, we have explicitely described \( \mu_0(\xi) \); this has the advantage of proving that \( \mu_0 \) is injective. In fact \( \mu_0(\xi) = \mu_0(\xi') \iff H_{\xi} = H_{\xi'} \), this implies that \( \Sigma_{\xi} = \Sigma_{\xi'} \) and that \( D_{\xi} = D_{\xi'} \). This is enough to ensure that \( \xi = \xi' \) (see [Co]).

It remains to show that \( \mu_0 \) is surjective. Let \( H \) be a theta-hyperplane of \( W \), so that \((t_0, H) \in J_W \), and let \( \Sigma = H \cap W_{\text{sing}} \). Then \( H \) is tangent to \( W \) at \( g - 1 - \# \Sigma \) smooth points of \( W \), let \( D \) be the divisor defined by \( H \cap W_{\text{smooth}} = 2D \). Denote by \( \sigma : Y_{\Sigma} \longrightarrow W \) the canonical morphism, let \( Z := Y_{\Sigma}^x \subset Y_{\Sigma} \) and denote by \( \sigma_Z := \sigma|_Z \) the restriction to \( Z \). Let \( \Delta := \sigma_Z^*(\Sigma) \) so that \( \deg \Delta = 2 \# \Sigma \) and

\[ \sigma_Z^* H = 2\sigma_Z^*(D) + \Delta. \]

On the other hand, \( W \) is a canonical image of \( Y_{\Sigma} \) hence

\[ \sigma_Z^* H = \omega_{Y_{\Sigma}} \otimes O_Z = \omega_Z \otimes O_Z(\Delta). \]

Hence \( \omega_Z = 2\sigma_Z^*(D) \).

Finally, by gluing \( \sigma_Z^*(D) \) to \( O_E(1) \) on every destabilizing component \( E \) of \( Y_{\Sigma} \), we obtain a line bundle \( L \) which (regardless of the gluing data, which are proven to be irrelevant in [Co]) corresponds to a spin curve \( \xi \) on \( Y \) such that \( \mu_0(\xi) = H \). Proving that \( \mu_0 \) is surjective. \( \square \)

Let \( H \subset \text{Hilb}^p(x)[P^{g-1}] \) be the projective scheme defined as the closure of \( V \).

Let \( P \) be a projective scheme and consider the space of morphisms from \( T \) to \( P \)

\[ \text{Hom}[P] := \{ \tau : T \longrightarrow P \}. \]

Since \( T \) is a smooth curve, any rational map from \( T \) to \( P \) extends to a (uniquely defined) regular morphism \( \overline{T} \longrightarrow P \), where \( \overline{T} \) is the smooth compactification of \( T \). Thus \( \text{Hom}[P] \) has a natural scheme structure. We can apply this to the projective varieties \( \overline{M}_g \), \( H \) and \( P_{N_g} \). We shall thus consider the scheme of maps from \( T \) to \( \overline{M}_g \): \( \text{Hom}[\overline{M}_g] \); the scheme of maps from \( T \) to \( H \): \( \text{Hom}[H] \), and the scheme \( \text{Hom}[P_{N_g}] \) as above.

There is a rational map \( \phi : H \longrightarrow \overline{M}_g \) (regular at least on \( V \)), which is the moduli map associated to the universal family over \( H \). Recall also that we have a rational map \( \theta : H \longrightarrow P_{N_g} \) which is regular on \( V \). Therefore we have two morphisms of schemes:

\[ \phi_* : \text{Hom}[H] \longrightarrow \text{Hom}[\overline{M}_g] \]

given by composing with \( \phi \) (that is, for \( \psi \in \text{Hom}[H] \), we define \( \phi_*(\psi) := \phi \circ \psi \));

\[ \theta_* : \text{Hom}[H] \longrightarrow \text{Hom}[P_{N_g}] \]

given by composing with \( \theta \) (as above). We have

**Lemma 2.4.2.** Let \( \mathcal{Y} \longrightarrow T \) be a one-parameter deformation of \( Y \), with smooth and theta-generic general fiber. Let \( \mathcal{W} \longrightarrow T \) be a generically canonical, birational image of \( \mathcal{Y} \). Then

\[ L(S_{\mathcal{Y}}) \geq L(J_{\mathcal{W}}). \]

**Proof.** First we consider the case in which \( \mathcal{Y} \longrightarrow T \) is a general one parameter deformation of \( Y \).

By 2.2.1, we have that \( S_{\mathcal{Y}} \) is a smooth curve and, by 2.4.1(a), there is a surjective \( T \)-morphism

\[ S_{\mathcal{Y}} \longrightarrow J_{\mathcal{W}}. \]
By comparing the fibers over $t_0$ we obtain $L(S_Y) \geq L(J_W^M)$ which is what we wanted.

Consider now a one-parameter deformation $Y \longrightarrow T$ of $Y$ and a generically canonical model $W \subset T \times \mathbb{P}^{g-1}$ as stated. As usual, denote by $\psi_W \in \text{Hom}[H]$ and $\phi_W \in \text{Hom}([M_g])$ the corresponding moduli morphisms; of course, $\phi_W = \phi \circ \psi_W$. Pick now a local curve $U^M \subset \text{Hom}([M_g])$ whose special point is $\phi_W$. More precisely, let $R$ be a discrete valuation ring and let

$$U^M := \text{Im}\{\phi_R : \text{Spec}R \longrightarrow \text{Hom}([M_g])\}$$

be such that the image of the generic point, $\phi_R(\eta) = \phi_\eta$, is a general one-parameter deformation of $Y$, and the image of the special point is the given $\phi_R(s) = \phi_W$.

Let $U^H \subset \text{Hom}[H]$ be a lifting of $U^M$ passing through $\psi_W$; that is,

$$U^H := \text{Im}\{\psi_R : \text{Spec}R \longrightarrow \text{Hom}[H]\}$$

such that the image of the generic point, $\psi_R(\eta) = \psi_\eta$ is such that $\phi \circ \psi_\eta = \phi_\eta$ while the image of the special point is the original $\psi_W$. Denote by $\mathcal{W}^\eta \subset T \times \mathbb{P}^{g-1}$ the family of curves corresponding to $\psi_\eta$ (that is, $W^\eta$ is the pull-back, via $\psi_\eta$, of the universal family over $H$ and $\psi_\eta = \psi_{\mathcal{W}^\eta}$).

Lastly, let $U^P \subset \text{Hom}[P_{N_g}]$ be defined as $U^P := \theta_s(U^H)$. Thus

$$U^P = \text{Im}\{\theta_R : \text{Spec}R \longrightarrow \text{Hom}[P_{N_g}]\}$$

such that, denoting, as usual, $\theta_\eta$ and $\theta_s$ the images of the generic and special point (respectively) of $\text{Spec}R$, we have $\theta_\eta = \theta \circ \psi_{\mathcal{W}^\eta} = \theta_{\mathcal{W}^\eta}$ and $\theta_s = \theta_{\mathcal{W}}$. Now we shall construct a universal family of theta-hypersurfaces over $U = \text{Spec}R$ using the morphism defined above

$$\theta_R : U \longrightarrow \text{Hom}[P_{N_g}] \mapsto \text{Hilb}(T \times P_{N_g})$$

(where the immersion on the right is the structural one, used to give a scheme structure to $\text{Hom}[P_{N_g}]$). To do that, we shall proceed in a standard way, using the various universal families over the Hilbert schemes in our set-up.

There is a universal family $\mathcal{F} \longrightarrow \text{Hilb}(T \times P_{N_g})$ which we can pull back to $U$ via $\theta_R$. Denote by $\mathcal{G} := \mathcal{F} \times \text{Hilb}(T \times P_{N_g})_U \longrightarrow U$ this pull-back. All of its fibers are thus isomorphic to $T$. Since $U = \text{Spec}R$, we get that this fibration is in fact trivial, so that $\mathcal{G} \cong T \times U$; let us fix one of such isomorphisms from now on.

Consider also the universal family of (reducible) hypersurfaces over $P_{N_g}$, denoted by $P_{N_g} \times \mathbb{P}^{g-1} \supset \mathcal{I} \longrightarrow P_{N_g}$. We have a diagram

$$
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
U & \longrightarrow & \text{Hilb}(T \times P_{N_g})
\end{array}
$$

so that we can pull back $\mathcal{I} \longrightarrow P_{N_g}$ to $\mathcal{G}$. Denote $\overline{\Theta} := \mathcal{G} \times_{P_{N_g}} \mathcal{I}$ such a pull back. Finally

$$\overline{\Theta} \longrightarrow \mathcal{G} \cong T \times U \longrightarrow U$$

is the family of theta-hypersurfaces that we wanted. Let us now consider the restriction $\overline{\Theta}^0$ of $\overline{\Theta}$ to $\{t_0\} \times U$, i.e. $\overline{\Theta}^0 \longrightarrow \{t_0\} \times U \cong \text{Spec}R$. This is a family of schemes of pure dimension $g - 2$, for which we get $L((\overline{\Theta}^0)_\eta) \geq L((\overline{\Theta}^0)_s)$. By construction, the special fiber is the original
\(\theta^{W}(W) = (\Theta^{0}),\) For the generic fiber, of course, \((\Theta^{0})_{y} = \theta^{W^{y}}(W^{y})\) (where, as usual, \(W^{y}\) denotes the central fiber of \(W^{y}\)). Thus the above inequality translates into \(L(\theta^{W^{y}}(W^{y})) \geq L(\theta^{W}(W))\).

Recall now that \(W^{y}\) is a birational, canonical image of a general one-parameter deformation of \(Y\). Therefore we can apply to \(W^{y}\) the result of the first part of the proof, that is: \(L(S_{Y}) \geq L(J_{W^{y}}^{W^{y}})\); since \(L(J_{W^{y}}^{W^{y}}) = L(\theta^{W^{y}}(W))\), we conclude that \(L(S_{Y}) \geq L(\theta^{W}(W)) = L(J_{W}^{W^{y}}).\) □

3. Combinatorics of stable curves

This section is essentially independent of the rest of the paper; its goal is 3.4.2, which will play a crucial role later on. We have seen that, for a stable curve \(Y\), the multiplicity set \(L(S_{Y})\) only depends on the combinatorial data of \(Y\) (2.2.1 and 2.3.1). Here we shall make this more precise by re-defining it in purely combinatorial terms and thus making its computation rather simple. The key point is that \(L(S_{Y})\) is a fine invariant for certain types of curves: namely, it completely determines the topological and combinatorial structure of \(Y\) (among all stable curves) if \(Y\) is of compact type (this is easy: 3.3.1) and if \(Y\) is split (this is more interesting: 3.4.1, 3.4.2).

3.1. Let \(Y\) be a nodal curve. \(\Gamma_{Y}\) is the dual graph of \(Y\), that is, the vertices of \(\Gamma_{Y}\) correspond to the irreducible components of \(Y\), the edges joining two vertices correspond to the nodes contained in the two corresponding components.

Recall that \(Y_{sing}\) is the set of all nodes of \(Y\). If \(\Sigma \subset Y_{sing}\) the curve \(Y_{\Sigma}^{\nu}\) is the normalization of \(Y\) at the nodes contained in \(\Sigma\). Assume that \(Y\) has arithmetic genus \(g\), \(\delta\) nodes and \(\gamma\) irreducible components; let \(Y = \bigcup_{i=1}^{\gamma} C_{i}\) be the decomposition of \(Y\) into irreducible components, and let \(g_{i}\) be the geometric genus of \(C_{i}\). Recall the genus formula: \(g = \sum g_{i} + \delta - \gamma + \gamma_{c}(Y)\).

Let \(N\) be a node of \(Y\). We say that \(N\) is internal if \(N\) is contained in a unique irreducible component of \(Y\), otherwise we say that \(N\) is external. An (external) node \(N\) is called separating if \(\gamma_{c}(Y) = \gamma_{c}(Y - N) - 1\). If all nodes of \(Y\) are separating, \(Y\) is called “of compact type”.

**Definition.** Exponent of a set of nodes. Let \(\Sigma\) be a set of nodes of \(Y\). The exponent of \(\Sigma\) is the number \(e_{Y}(\Sigma)\) below

\[e_{Y}(\Sigma) = p_{a}(Y) - p_{a}(Y_{\Sigma}^{\nu})\]

As we already know (see 2.3), an equivalent definition is \(e_{Y}(\Sigma) := \#\Sigma - \gamma_{c}(Y_{\Sigma}^{\nu}) + \gamma_{c}(Y)\).

The reason for the name “exponent” comes from Proposition 2.2.1. Keeping the same notation, we list a couple of straightforward consequences of the definition.

**Property A.** If \(\Sigma \subset \Sigma'\) then \(e_{Y}(\Sigma) \leq e_{Y}(\Sigma')\). Moreover, for every \(\Sigma\),

\[0 = e_{Y}(\emptyset) \leq e_{Y}(\Sigma) \leq e_{Y}(Y_{sing}) = g - \sum_{i=1}^{\gamma} g_{i}\]

**Property B.** If \(N\) is a node of \(Y\) not contained in \(\Sigma\), then

\[e_{Y}(\Sigma \cup \{N\}) = \begin{cases} 
\ e_{Y}(\Sigma) + 1 & \text{if } N \text{ is not separating for } Y_{\Sigma}^{\nu}; \\
\ e_{Y}(\Sigma) & \text{if } N \text{ is separating for } Y_{\Sigma}^{\nu}.
\end{cases}\]

We shall need to compute the exponent of a distinguished type of sets of nodes:

**Definition.** Admissible sets of nodes. A set \(\Sigma \subset Y_{sing}\) is said to be admissible if for every subcurve \(W\) of \(Y\), the number of nodes in the intersection \(W \cap W^{c}\) which are not in \(\Sigma\) is even. Equivalently, \(\Sigma\) is admissible if for every \(i = 1, \ldots, \gamma\) the number \(\#[(C_{i} \cap C_{i}^{c}) \cap (Y_{sing} - \Sigma)]\) is even.
The equivalence of the two definitions above is an elementary verification. We denote by \( A_Y \) the set of all admissible sets of nodes of \( Y \):

\[
A_Y := \{ \Sigma \subseteq Y^{\text{sing}} \text{ s.t. } \Sigma \text{ is admissible} \}
\]

The previous definition is implicit in Cornalba’s paper; it is motivated by the fact that \( \Sigma \) is admissible if and only if the dualizing sheaf \( \omega_{Y,\Sigma} \) has even degree on every subcurve. It is important to stress that a set \( \Sigma \) is the support of a spin curve on \( Y \) if and only if \( \Sigma \in A_Y \) (see [Co]).

Some simple examples; the set of all admissible sets of nodes, the \( Y \) contains an even number of external nodes on every irreducible component of \( \Sigma = Y \).

**Proof.**

If \( \Sigma \in Y^{\text{sing}} \) is cyclic sets of nodes. Conversely, a curve of compact type does not admit any cyclic sets of nodes. Then \( \Sigma \leq n \) with 1.

Here are a few immediate properties:

**Property C.** If \( \Sigma \in A_Y \), then \( \Sigma \) contains all the separating nodes of \( Y \).

**Property D.** Let \( \Sigma \subset Y^{\text{sing}} \), denote \( Z = Y^{\text{sing}}_\Sigma \) and let \( \Sigma_1 \subset Z^{\text{sing}} \). Then \( \Sigma_1 \in A_Z \) if and only if \( \Sigma_1 \cup \Sigma \in A_Y \). In particular \( \emptyset \in A_Z \) if and only if \( \Sigma \in A_Y \).

### 3.2. Cyclic sets of nodes.

Given our nodal curve \( Y \), consider its dual graph \( \Gamma_Y \). To any subset, \( \Sigma \), of nodes of \( Y \), we can associate a graph \( \Gamma_\Sigma \), which is the subgraph of \( \Gamma_Y \) generated by all edges representing the nodes contained in \( \Sigma \). We shall say that a non-empty \( \Sigma \) is a cyclic set of nodes if its graph \( \Gamma_\Sigma \) is a closed polygon.

An equivalent, graph-free, definition is the following. Up to reordering the irreducible components of \( Y \), we can write a cyclic set of nodes as

\[
\Sigma = \{ N_{1,2}, N_{2,3}, \ldots, N_{h-1,h}, N_{h,1} \}
\]

with \( 1 \leq h \leq \gamma \), meaning that \( N_{i,j} \in C_i \cap C_j \). For example, if \( N \) is an internal node of \( Y \), then \( \{ N \} \) is a cyclic set. Observe that, if \( Y \) is not of compact type, then \( Y \) always admits some cyclic set of nodes. Conversely, a curve of compact type does not admit any cyclic set of nodes.

**Lemma 3.2.1.** Let \( \Sigma \) be the complement in \( Y^{\text{sing}} \) of a cyclic set of nodes. Then \( \Sigma \) is admissible and \( e_Y(\Sigma) = g - \sum g_i - 1 \). More generally, let \( \Sigma \) be the complement in \( Y^{\text{sing}} \) of a disjoint union of \( n \) cyclic sets of nodes. Then \( \Sigma \) is admissible and \( e_Y(\Sigma) \leq g - \sum g_i - n \).

**Proof.** If \( \Sigma_1, \ldots, \Sigma_n \) are cyclic sets of nodes of \( Y \) such that \( \Sigma_i \cap \Sigma_j = \emptyset \) for all \( i \neq j \), then their union contains an even number of external nodes on every irreducible component of \( Y \). Thus \( \Sigma = Y^{\text{sing}} - \cup_1^n \Sigma_i \) is admissible by definition. If \( n = 1 \) so that \( \Sigma \) is the complement of \( \Sigma_1 \) we easily compute

\[
e_Y(\Sigma) = (\delta - \#\Sigma_1) - (\gamma - \#\Sigma_1 + 1) + \gamma_c(Y) = \delta - \gamma + \gamma_c(Y) - 1 = g - \sum g_i - 1.
\]

If \( n \) is arbitrary, the inequality in the statement is straightforward. \( \square \)

**Definition.** \( Y \) exponent set. We shall call the finite set of integers, that occur as exponents of admissible sets of nodes, the \( Y \) exponent set of \( Y \), and we will denote it by \( E_Y \):

\[
E_Y := \{ e, \text{ such that there exists } \Sigma \in A_Y \text{ with } e_Y(\Sigma) = e \}
\]

**Remark.** It is clear that \( E_Y \) depends only on the combinatorial data of \( Y \). More precisely, if two curves \( Y \) and \( Y' \) have the same genus and the same dual graph, then \( E_Y = E_{Y'} \).
Lemma 3.2.2.

a) If \( Y \) is not of compact type, then \( \{ g - \sum g_i - 1, g - \sum g_i \} \subset E_Y \).
b) If \( \Sigma \) is admissible and \( e_Y(\Sigma) = g - \sum g_i \), then \( \Sigma = Y_{\text{sing}} \).
c) If \( N \in Y \) is an internal node, then either \( g - \sum g_i - 2 \in E_Y \) or \( Y^r \) is of compact type and \( E_Y = \{ 0, 1 \} \).

Proof. Since \( Y_{\text{sing}} \) is admissible and \( e_Y(Y_{\text{sing}}) = g - \sum g_i \), we have that \( g - \sum g_i \in E_Y \). To see that \( g - \sum g_i - 1 \in E_Y \), observe that, since \( Y \) is not of compact type, \( Y \) contains a cyclic set of nodes, whose complement is admissible and has exponent precisely \( g - \sum g_i - 1 \), by 3.2.1. This proves a).

For b) we have to show that if \( \Sigma \) is admissible and its exponent is maximum (that is \( e_Y(\Sigma) = e(Y_{\text{sing}}) = g - \sum g_i \)) then \( \Sigma \) contains every node of \( Y \). By contradiction, let \( \Sigma \neq Y_{\text{sing}} \) and let \( Z = Y_{\Sigma^0}^r \); the curve \( Z \) is singular. By Property D, \( \emptyset \in A_Z \) hence \( Z \) is free from separating nodes, thus \( Z \) is not of compact type and it contains a cyclic set of nodes \( \Sigma_0 \). We have thus a cyclic set of nodes \( \Sigma_0 \subset Y_{\text{sing}} \) such that \( \Sigma \subset \Sigma_0 \). Hence we have, by Property A,

\[
e_Y(\Sigma) \leq e_Y(\Sigma_0) = g - \sum g_i - 1
\]

which is a contradiction. To prove c) let \( Z = Y_N^r \); since \( N \) is internal, we have that

\[
A_Y = \{ \Sigma, \Sigma \cup N, \forall \Sigma \in A_Z \};
\]

By Property B, \( e_Y(\Sigma \cup N) = e_Y(\Sigma) + 1 = e_Z(\Sigma) + 1 \) hence

\[
E_Y = \{ n, n + 1, \forall n \in E_Z \}.
\]

If \( Z \) is of compact type, \( E_Z = \{ 0 \} \) hence \( E_Y = \{ 0, 1 \} \). Otherwise, part (a) applied to \( Z \) says that, \( E_Z \) contains the number \( n = (g - 1) - \sum g_i - 1 \) (the arithmetic genus of \( Z \) is, of course, \( g - 1 \)). Hence \( E_Y \) also contains \( n = g - \sum g_i - 2 \) and we are done. \( \square \)

3.3. Let \( Y \) be stable. There is a relation between \( E_Y \) and \( S_Y \):

\[
L(S_Y) = \begin{cases} 
\{ 2^n, \forall n \in E_Y, n \neq g - \sum g_i \} & \text{if } g_i = 0 \forall i \\
\{ 2^n, \forall n \in E_Y \} & \text{otherwise}
\end{cases}
\]

In fact, \( L(S_Y) \) is described in 2.3.1, and recall that for every \( \xi \in S_Y \), the support \( \Sigma_\xi \) of \( \xi \) is admissible. Now, (see [Co] section 6) if \( \Sigma \) is admissible and \( \Sigma \neq Y_{\text{sing}} \), then there exist odd (and even) spin curves supported on \( \Sigma \) (an equal number of odd and even, in fact). By 3.2.2, \( e_Y(\Sigma) = g - \sum g_i \) iff \( \Sigma = Y_{\text{sing}} \). A spin curve \( \xi \) supported on \( Y_{\text{sing}} \) is given by a line bundle \( L \) on \( Y^r \) which restricts to a theta-characteristic \( L_i \) on every irreducible component \( C_i^r \) of \( Y^r \). For \( \xi \) to be odd it is necessary and sufficient that \( L_i \) be an odd theta-characteristic on \( C_i^r \) for an odd number of components of \( Y^r \). Therefore there exist odd spin curves supported on \( Y_{\text{sing}} \) unless all components of \( Y^r \) are rational; in fact there are obviously no odd theta-characteristics on \( \mathbb{P}^1 \). In other words, if \( g_i = 0 \) for every \( i \), a spin curve of exponent \( g \) is necessarily even.

Corollary 3.3.1. A curve \( Y \) is of compact type if and only if \( E_Y = \{ 0 \} \).

Remark. By 3.3, if \( Y \) is stable, this is equivalent to saying that \( Y \) is of compact type if and only if \( S_Y \) is reduced.

Proof. If \( Y \) is of compact type, the only admissible set is \( Y_{\text{sing}} \) (by Property C), whose exponent is 0. The converse follows from Lemma 3.2.2. \( \square \)
Lemma 3.3.2. Let $Y$ be a connected, nodal curve and let $X$ be its stable model. Then $E_X = E_Y$.

Proof. Let $\sigma : Y \to X$ be the natural map, contracting all smooth rational components of $Y$ meeting their complementary curve in less than 3 points. There is a natural bijection $\beta$ between $A_Y$ and $A_X$

\[
\beta : A_Y \to A_X \\
\Sigma \mapsto \sigma(\Sigma) \cap X_{\text{sing}}
\]

whose inverse is, denoting by $Y_{\text{sep}}$ the subset of all separating nodes of $Y$,

\[
\beta^{-1} : A_X \to A_Y \\
\Sigma' \mapsto [\sigma^{-1}(\Sigma') \cap Y_{\text{sing}}] \cup Y_{\text{sep}}
\]

It is a trivial verification to show that the two above maps are each other inverse and that they preserve the exponents, that is $e_Y(\Sigma) = e_X(\beta(\Sigma))$. \hfill \Box

3.4. We now describe $E_X$ for a split curve $X$. Of course $e(X_{\text{sing}}) = g$. Let $\Sigma$ be a subset of nodes such that $\Sigma \neq X_{\text{sing}}$; then $e_X(\Sigma) = \# \Sigma$ and $\Sigma$ is admissible if and only if $\# \Sigma \not\equiv g \mod 2$. Thus if $g$ is odd, the exponents appearing in $E_X$ are $0, 2, 4, \ldots, g - 1, g$ (in particular, the only odd exponent is $g$). Symmetrically for $g$ even. Summarizing:

Let $X$ be a split curve of genus $g$. Then

\[
E_X = \begin{cases} 
\{0, 2, 4, \ldots, g - 3, g - 1, g\} & \text{if } g \text{ is odd;} \\
\{1, 3, 5, \ldots, g - 3, g - 1, g\} & \text{if } g \text{ is even.}
\end{cases}
\]

We shall prove that the converse is also true, in other words split curves are identified in $\overline{M}_g$ by their exponent set. Something more precise is true.

Theorem 3.4.1. Let $Y$ be a DM stable curve of genus $g$. Assume that $g \in E_Y$ and that $g - 2 \not\in E_Y$. Then either $Y$ is a split curve, or $Y$ is the “polygonal” curve of genus 3.

Where recall that the polygonal curve of genus 3 is the nodal curve made of four copies of $\mathbb{P}^1$ meeting pairwise in one point (having a total of six nodes). Its canonical model is the union of four general lines in $\mathbb{P}^2$, whence the name “polygonal”.

We shall apply the Theorem above to obtain the following crucial result:

Corollary 3.4.2. Let $X$ and $Y$ be stable curves of genus $g$; assume that $X$ is split. (a) If $E_X = E_Y$, then $Y$ is split. (b) If $L(S_X) = L(S_Y)$ then $Y$ is split.

Proof of the Corollary. We can exclude that $Y$ is the polygonal curve of genus 3, in fact, in that case $E_Y = \{2, 3\}$ whereas the exponent set of a split curve $X$ of genus 3 is $\{0, 2, 3\}$. Similarly, $L(S_Y)$ does not contain 1, whereas $L(S_X)$ does.

Assume now that $E_X = E_Y = E$. If $X$ is split, then $E$ satisfies the assumptions of Theorem 3.4.1 (see above), hence $Y$ is split.

This proves (a). Let us show that (b) implies (a). For any stable curve $Z$, the relation between $E_Z$ and $L(S_Z)$ has been explained at the beginning of 3.3. This yields that $E_X$ and $E_Y$ contain precisely the same integers $n$ such that $n \leq g - 1$. It remains to show that $g \in E_Y$. Notice that $g - 2 \not\in E_Y$ and $g - 1 \in E_Y$; therefore, if $g \not\in E_Y$, we get a contradiction to 3.2.2(a) ($Y$ is not of compact type by 3.3.1). \hfill \Box
**Proof of the Theorem.** Recall the basic notation: \( Y = \bigcup_{i=1}^{r} C_i \). The proof is in three steps.

**Step 1:** \( Y \) is a union of smooth rational components. The fact that \( g \in E_Y \) implies that \( e_Y(Y_{\text{sing}}) = g \) (see 3.2.2) and that all irreducible components of \( Y \) have geometric genus 0. If \( Y \) contained an internal node, by 3.2.2, part c), \( E_Y \) would contain \( g-2 \), which is not the case.

**Step 2:** It suffices to assume that \( Y \) is free from separating nodes. Given our stable curve \( Y \), denote by \( Y^* \) a stable curve of genus \( g \) obtained by smoothing out all the separating nodes of \( Y \). This is to say that if \( N \) is a separating node, and \( N = C_1 \cap C_2 \), the curve \( Y^* \) is obtained by replacing \( C_1 \cup C_2 \) (which is a curve of arithmetic genus 0 by the Step 1) with a smooth rational curve \( D \).

Notice that, unless \( Y \) is free from separating nodes, \( Y^* \) contains a separating component: \( D \) (that is \( D^c \) is disconnected).

By Property C there is a natural bijection between the admissible sets of \( Y \) and those of \( Y^* \): to an admissible set \( \Sigma \) of \( Y \), which must contain all the separating nodes of \( Y \) we associate the admissible set \( \Sigma^* = \Sigma - \{N : N \text{ is separating}\} \) on \( Y^* \). By Property B we have \( e_Y(\Sigma) = e(\Sigma^*) \), hence \( E_Y^* = E_{Y^*} \).

If we show that \( Y^* \) is a split curve, then \( Y^* \) has no separating components, hence \( Y = Y^* \) and we are done. From now on we shall assume that \( Y \) is free from separating nodes.

**Step 3.** We can assume that \( Y \) is free from internal and separating nodes. We proceed by induction on \( g \). The case \( g = 3 \) is the first case to be treated. A straightforward case by case analysis (there are only 3 cases) yields the result. By the previous steps, there are only three cases. If \( Y \) is not split, then \( Y \) has more than two components. If \( Y \) has three components, then its dual graph is uniquely determined and so is \( E_Y \) (compare with the Remark in 3.2); \( Y \) has 5 nodes and \( E_Y = \{1,2,3\} \). If \( Y \) has four components, then \( Y \) is the polygonal curve \( E_Y = \{2,3\} \). There are no other cases.

Assume \( g \geq 4 \). Let \( N \) be any node of \( Y \) and let \( Z = Y_N^e \) be the normalization of \( Y \) at \( N \). Since \( N \) is not an internal node of \( Y \), to complete the proof of the Theorem it suffices to prove that \( Z \) is a split curve.

Notice that \( Z \) is connected (\( Y \) is free from separating nodes), nodal, and its arithmetic genus is \( g-1 \). Moreover, \( Z \) is a union of smooth, rational components (because \( Y \) is) and hence \( g-1 \in E_Z \).

Let now \( \Sigma \in A_Z \), then clearly \( \Sigma \cup N \in A_Y \) and \( e_Z(\Sigma) = e_Y(\Sigma \cup N) - 1 \). We conclude that \( E_Z \subset E_Y - 1 \), that is, if \( n \in E_Z \) then \( n+1 \in E_Y \). In particular, \( g-3 \notin E_Z \), because, by hypothesis, \( g-2 \notin E_Y \).

Let now \( X \) be the stable model of \( Z \) (possibly equal to \( Z \)). By Lemma 3.3.2, \( E_X = E_Z \). Therefore \( X \) satisfies the assumptions of the Theorem. By induction we obtain that \( X \) is a split curve of genus \( g-1 \), unless \( X \) is the polygonal curve of genus 3.

The rest of the proof consists in showing that \( Z \) is stable (i.e. \( X = Z \)) and that \( Z \) is not the polygonal curve of genus 3.

Assume first that \( X \) is split.

By contradiction, assume that \( Z \) is not stable; then \( Z \) has one or two destabilizing components (since \( Z \) is the partial normalization of the stable curve \( Y \) at a unique node \( N \)). Hence \( Z \), and likewise \( Y \), has three or four irreducible components. We need to distinguish the components of \( Y \) from those of \( Z \); let us denote by \( C_i^Z \) the irreducible component of \( Z \) which naturally (via the normalization at \( N \)) corresponds to \( C_i \) in \( Y \).

We can write \( X = D_1 \cup D_2 \) with \( \#D_1 \cap D_2 = g \). Let \( C_1 \) and \( C_2 \) be the two components of \( Y \) that correspond to \( D_1 \) and \( D_2 \). Then \( \#C_1 \cap C_2 \geq 2 \); pick two nodes in \( C_1 \cap C_2 \) and denote them by \( N_{1,2}, N_{2,1} \). Obviously the set \( \{N_{1,2}, N_{2,1}\} \) is a cyclic set of nodes of \( Y \).

The fact that \( X \) is split implies that every destabilizing component of \( Z \) meets both \( C_i^Z \)
and $C^Z_i$ in one point, thus (on $Y$) we have that, for all $i \geq 3$ (so that $C^Z_i$ is destabilizing in $Z$), $C_i \cap C_1 \neq \emptyset$ and $C_i \cap C_2 \neq \emptyset$. There are two possible cases, according to the number of destabilizing components of $Z$, which is 1 or 2.

First case: $Z$ has one destabilizing component, $C^Z_2$. In this case, (up to switching $C_1$ and $C_2$) $N \in C_2 \cap C_3$ so let us rename $N = N_2, 3$. Since $C^Z_2 \cap C^Z_3 \neq \emptyset$ there must be another node in $C_2 \cap C_3$, let it denote by $N_3, 2$. The set $\{N_2, 3, N_3, 2\}$ is a cyclic set of nodes of $Y$.

Let now $\Sigma$ be the complement of the two cyclic sets we constructed above, that is

$$\Sigma := Y_{\text{sing}} - (\{N_{1, 2}, N_{2, 1}\} \cup \{N_{2, 3}, N_{3, 2}\}).$$

By Lemma 3.2.1, $\Sigma \in A_Y$ and its exponent is easily seen to be equal to $g - 2$. Then $g - 2 \in E_Y$, a contradiction.

In the second case $Z$ has two destabilizing components, $C^Z_1$ and $C^Z_2$. Now $N \in C_3 \cap C_4$, so we rename it: $N = N_{3, 4}$. Since $C^Z_2$ and $C^Z_4$ both meet $C^Z_3$ and $C^Z_2$, we find (in $Y$) nodes $N_{2, 3} \in C_2 \cap C_3$ and $N_{4, 2} \in C_2 \cap C_4$. The set of nodes $\{N_{2, 3}, N_{3, 4}, N_{4, 2}\}$ is cyclic. Continuing as in the previous case, we let $\Sigma$ be the set

$$\Sigma := Y_{\text{sing}} - (\{N_{1, 2}, N_{2, 1}\} \cup \{N_{2, 3}, N_{3, 4}, N_{4, 2}\}).$$

which is admissible, by 3.2.1, and whose exponent is $g - 2$. A contradiction.

We conclude that $Z$ is stable, which is what we wanted.

To complete the proof we must show that $X$ is not the polygonal curve of genus 3. This is done just as above, showing, by a trivial case by case analysis, that $Y$ contains two disjoint cyclic sets of nodes whose complement has exponent $g - 2$. \qed

4. Split Curves

4.1. The theta-hypersurface of a projective split curve is described in details in [C], where it is proved (Theorem 5) that split curves are uniquely determined by their theta-hypersurface, among all curves in $V$. In this section, we shall give a sharper version of such a result (4.4.5). Furthermore, in section 4.3, we will study the behaviour of certain configurations of theta-hyperplanes of a split curve, under the natural action of $G = PGL(g)$.

The results of this analysis will be applied in the sequel.

It is worth pointing out a nice feature of split curves which is to be used often: the projection of a split curve in $\mathbb{P}^{g-1}$ from every subset of $i \leq g - 3$ of its nodes is a split curve in $\mathbb{P}^{g-1-i}$.

Let $H$ be a theta-hyperplane of type $i$ of a projective split curve $X$, thus $H$ contains exactly $i$ nodes of $X$ ($0 \leq i \leq g - 1$) and it is tangent to $X$ at $g - 1 - i$ smooth points, equally distributed among the two irreducible components of $X$. The set of all hyperplanes of type $i$ is denoted by $\Theta_i(X)$ and its cardinality by $t_i(X) = \#\Theta_i(X)$. It is easy to see that $\Theta_i(X)$ is empty for all $i$ having the same parity of $g$.

We need the following

Claim. For every projective split curve $X$, we have $t_{g-3}(X) = 4\binom{g-1}{g-3}$ and $t_{g-1}(X) = \binom{g+1}{g-1}$.

For the first formula, it suffices to check that for any subset $\Sigma$ of $g - 3$ nodes of $X$, there are 4 theta-hyperplanes of type $g - 3$ containing $\Sigma$, and containing no other node of $X$. If $g = 3$, we are simply saying that two plane conics meeting transversely have exactly 4 distinct tangent lines in common (which is clear by looking at the duals) and that these 4 tangent lines do not go through the common points (also easy, see [CS]). The general case is done by projecting $X$ from $\Sigma$ to a split curve of genus 3 in $\mathbb{P}^2$.\]
The second formula is obvious: the nodes of $X$ are in general linear position, therefore there is a unique theta-hyperplane of type $g - 1$ for every set of $g - 1$ nodes of $X$.

4.2. The dualizing sheaf of a split curve $Y$ might fail to be very ample: if this is the case, the canonical image $W$ of $Y$ is a double rational normal curve in $\mathbb{P}^{g - 1}$. In other words, $Y$ has a degree 2 morphism onto $\mathbb{P}^1$ and it is thus in the closure, in $\overline{M}_g$, of the locus of smooth, hyperelliptic curves (see [HM] 3.159 and 3.160). A split curve of this type will be called a hyperelliptic split curve. Of course in such a case, the canonical model $W$ is not theta-generic and its theta-hypersurface is not consistently defined.

We shall leave the projective point of view, for a moment, and examine the situation abstractly. Let $Y$ be an abstract split curve and let $S_Y$ the scheme parametrizing its odd spin curves. We are interested in the spin curves parametrized by $\hat{S}_Y$, whose exponent is at least $g - 3$ and whose space of global sections has dimension 1 (see 2.3).

Let $\xi \in \hat{S}_Y$ have exponent $g - 3$; then $\xi$ is supported on a set of $g - 3$ nodes of $Y$. Moreover, every set $\Sigma \subset Y_{\text{sing}}$ such that $\# \Sigma = g - 3$ is the support of exactly 4 odd, distinct spin curves (this follows from [Co]). Therefore the number of spin curves of $Y$ having exponent $g - 3$ is $4 \binom{g + 1}{g - 3}$.

There are no spin curves having exponent $g - 2$. The set of $\xi \in \hat{S}_Y$ having exponent $g - 1$ is easily seen to have cardinality equal to the number of subsets of $g - 1$ nodes of $Y$, that is $(\binom{g + 1}{g - 1})$.

Summarizing, the zero-dimensional scheme $\hat{S}_Y$ has $4 \binom{g + 1}{g - 3} + \binom{g + 1}{g - 1}$ irreducible components, its length is equal to $2^{g - 3} \cdot 4 \binom{g + 1}{g - 3} + 2^{g - 1} \binom{g + 1}{g - 1}$ and $L(\hat{S}_Y) = \{2^{g - 3}, 2^{g - 1}\}$.

Let $\sigma : Y \rightarrow W \subset \mathbb{P}^{g - 1}$ be a canonical model of the split curve $Y$, thus $W$ is either a split curve or $\sigma$ is a two-to-one morphism onto a rational normal curve $C$. Let

$$\{N_1, ..., N_{g + 1}\} := \sigma(Y_{\text{sing}})$$

so that the points $\{N_1, ..., N_{g + 1}\}$ are all distinct and, of course, in general linear position, since they lie on a rational normal curve.

Let $\mathcal{Y} \rightarrow T$ be a general one-parameter deformation of $Y$ to smooth, theta-generic curves, and let $\mathcal{Y} \rightarrow W \subset \mathbb{P}^{g - 1} \times T$ be a canonical image inducing $\sigma$ on the central fibers. Then $\theta^W(W)$ is defined as the limit of the theta-hypersurfaces of the generic fibers (see 2.4).

We shall now describe a configuration of hyperplanes $\hat{\Theta}(Y, W) \subset \theta^W(W)$ which corresponds to $\hat{S}_Y$ and is independent of the choice of $\mathcal{Y}$ and $W$.

We start by writing explicitly

$$\hat{\Theta}(Y, W) := \{2^{g - 3} \Theta_{g - 3}(Y, W), 2^{g - 1} \Theta_{g - 1}(Y, W)\}$$

where $\Theta_{g - 3}(Y, W)$ will correspond to spin curves having exponent $g - 3$ and $\Theta_{g - 1}(Y, W)$ to spin curves having exponent $g - 1$.

We have two cases: either $\sigma$ is an isomorphism, or it is not. In the first case, $W$ is a projective split curve, we shall simplify the notation, writing $\hat{\Theta}(Y, W) = \hat{\Theta}(W)$, and we have

$$\hat{\Theta}(Y, W) = \hat{\Theta}(W) := \{2^{g - 3} \Theta_{g - 3}(W), 2^{g - 1} \Theta_{g - 1}(W)\}$$

defined at the beginning of the section. The hyperplanes in $\Theta_{g - 3}(W)$ correspond naturally to spin curves of exponent $g - 3$ and similarly, theta-hyperplanes in $\Theta_{g - 1}(W)$ correspond to spin curves of exponent $g - 1$.

If $K_Y$ is not very ample and the image of $\sigma$ is a rational normal curve $C$, we define

$$\Theta_{g - 3}(Y, W) := \langle N_{i_1}, ..., N_{i_{g - 3}}, T_{N_{i_{g - 2}}}C \rangle, \forall i_1, ..., i_{g - 2} \text{ s.t. } i_j \neq i_h$$

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where by $T_NC$ we denote the tangent line to $C$ at $N$. The other piece of $\Theta(Y,W)$ is defined by

$$\Theta_{g-1}(Y,W) = \{ < N_{i_1} + ... + N_{i_{g-1}} > \text{ s.t. } 1 \leq i_1 < ... < i_{g-1} \leq g + 1 \}$$

**Lemma 4.2.1.** Let $Y$ be a split curve and let $Y \rightarrow W$ be a canonical model. The configuration $\Theta(Y,W)$ defined above has the following naturality property. For every one-parameter deformation $\mathcal{Y} \rightarrow T$ of $Y$ to theta-generic curves, and for any choice of a canonical model $\mathcal{Y} \rightarrow W \subset \mathbb{P}^{g-1} \times T$ restricting to the above $Y \rightarrow W$, we have that

$$\Theta(Y,W) \subset \theta^W(W).$$

Moreover, there is a natural bijection between the irreducible components of $\Theta(Y,W)$ and the irreducible components of $\hat{S}_Y$.

**Proof.** Consider first a general one-parameter deformation $\mathcal{Y}$ of $Y$. Then, by 2.4.1(a), we have a natural, birational $T$-morphism $\mu : S_\mathcal{Y} \rightarrow J_W$; denote by $\hat{\mu} : \tilde{S}_\mathcal{Y} \rightarrow J_W$ its restriction to $\tilde{S}_\mathcal{Y}$. We are stating that its image $\hat{\mu}(\tilde{S}_\mathcal{Y})$ is independent of the choice of the family, and it corresponds to $\Theta(Y,W)$.

If $W$ is a split curve, this follows from the discussion at the beginning of the section.

Otherwise $Y$ is hyperelliptic, thus, as we mentioned above, $Y$ can be expressed as the specialization of a family of smooth hyperelliptic curves $Y_\eta$. We shall use this to describe $\tilde{S}_\mathcal{Y}$.

The Weierstrass points $W_1, ..., W_{2g+2}$ of the generic, smooth fiber $Y_\eta$, specialize in pairs to the $g+1$ nodes of $Y$. Let us fix the notation so that, for $i = 1, ..., g + 1$, the pair $(W_i, W_{i+g+1})$ specializes to the node that maps to $N_i$. Recall now that for smooth, hyperelliptic curves, a complete description of the set of theta-characteristics is given in [Mu2], Proposition 6.1. The hyperplanes of $\Theta_{g-1}(Y,W)$ correspond to the specialization of odd theta-characteristics of type $O_{Y_\eta}(W_{i_1} + W_{i_2} + ... + W_{i_{g-3}} + W_{i_{g-2}} + W_{i_{g-2}+g+1})$, with $1 \leq i_1 < ... < i_{g-2} \leq g + 1$.

By 2.3, there is no ambiguity for the choice of the hyperplane corresponding to a given $\xi \in \tilde{S}_\mathcal{Y}$.

In a completely similar fashion we deal with spin curves of exponent $g - 1$. This is in some sense easier, in fact such spin curves are supported on sets of $g - 1$ nodes (all of such sets will occur). Moreover, for a given set $\Sigma$ of $g - 1$ nodes of $Y$ there exists a unique, odd spin curve supported on $\Sigma$ (by [Co]). As before, viewing $Y$ as specialization of smooth hyperelliptic curves, the hyperplanes of $\Theta_{g-1}(Y,W)$ correspond to the specializations of theta-characteristics of type $O_{Y_\eta}(W_{i_1} + W_{i_2} + ... + W_{i_{g-2}} + W_{i_{g-1}})$ with $1 \leq i_1 < ... < i_{g-1} \leq g + 1$.

So our statement holds for a general $\mathcal{Y}$. Now an arbitrary one-parameter deformation of $Y$ (satisfying the assumptions) can be obtained as a specialization of general ones, for which the result holds. A standard specialization argument (as in 2.4.2) gives the result in general. \[ \square \]

### 4.3.** GIT Stability Criterion.** $\Omega$ is stable if and only if for every $h = 0, ..., g - 2$, we have $\mu_h(\Omega) < \max_h(m)$, where $\max_h(m) := m^{\frac{g-1-h}{g}}$.

We are going to apply it to the next result.
Lemma 4.3.1. Let \( C \subset \mathbb{P}^{g-1} \) be a rational normal curve and let \( N_1, \ldots, N_{g+1} \) be distinct points on \( C \). Let \( \Omega \in \text{Sym}^m(\mathbb{P}^{g-1})^* \) be a configuration of hyperplanes of type a), b) or c) below. Then \( \Omega \) is GIT-stable with respect to the natural action of \( G \) on \( \text{Sym}^m(\mathbb{P}^{g-1})^* \).

\[ a) \ m = 4 \binom{g+1}{g-3} \quad \text{and} \quad \Omega = \mathcal{Q}_{g-3}(X) \text{ where } X \subset \mathbb{P}^{g-1} \text{ is a projective split curve.} \]
\[ b) \ m = 4 \binom{g+1}{g-3} \quad \text{and} \quad \Omega = \{ <N_{i_1}, \ldots, N_{i_{g-3}}, T_{N_{i_{g-2}}}, X> \mid \forall i_1, \ldots, i_{g-2} \text{ s.t. } i_j \neq i_h \} \]
\[ c) \ m = \binom{g+1}{g-1} \quad \text{and} \quad \Omega = \{ <N_{i_1}, \ldots, N_{i_{g-1}}, X> \mid 1 \leq i_1 < \ldots < i_{g-1} \leq g + 1 \} \]

Remark. If \( \Omega \) is as in b), we have, of course, that \( \Omega = \mathcal{Q}_{g-3}(Y, W) \). Similarly, if \( \Omega \) is as in c), then \( \Omega = \mathcal{Q}_{g-1}(X) = \mathcal{Q}_{g-1}(Y, W) \).

Proof. To prove the result for cases a) and b), notice that for every \( h \) we have
\[ \mu_h(\mathcal{Q}_{g-3}(X)) \leq \mu_h(\mathcal{Q}_{g-3}(Y, W)) ; \]
therefore, if \( \mathcal{Q}_{g-3}(Y, W) \) satisfies the criterion, \( \mathcal{Q}_{g-3}(X) \) also does. It is thus enough to deal with case b). Let us compute \( \mu_h(\mathcal{Q}_{g-3}(Y, W)) \). The \( h \)-dimensional linear subspaces contained in \( \mathcal{Q}_{g-3}(Y, W) \), and having the highest multiplicity are, clearly, those spanned by \( h + 1 \) among the generating points \( N_i \). Of course, the definition of \( \mathcal{Q}_{g-3}(Y, W) \) being symmetric with respect to the \( N_i \), varying the \( (h + 1) \)-t-uple does not change the multiplicity. Therefore we can pick a specific set, say \( \{N_1, \ldots, N_{h+1}\} \), of \( h + 1 \) nodes and have
\[ \mu_h(\mathcal{Q}_{g-3}(Y, W)) = \# \left\{ H \subset \mathcal{Q}_{g-3}(Y, W) : \{N_1, \ldots, N_{h+1}\} \subset H \right\} . \]
Now there are two types of hyperplanes \( H \) contributing to \( \mu_h(\mathcal{Q}_{g-3}(Y, W)) \): either \( H \) does not contain the tangent line \( T_{N_i} C \) for any \( i = 1, \ldots, h + 1 \), or it does. The hyperplanes \( H \) of the first type are
\[ \binom{g + 1 - (h + 1)}{g - 3 - (h + 1)} 4 = \binom{g - h}{4} 4 \]
where the binomial coefficient is the number of \( (g - 3) \)-uples containing the fixed set \( \{N_1, \ldots, N_{h+1}\} \), and the coefficient 4 is there because for every chosen \( g - 3 \) as above, we have a choice of 4 points where the hyperplane is tangent to \( C \).

The hyperplanes of the second type contain the tangent line to \( C \) at one of the \( N_i \), for \( i \leq h + 1 \). The total number of them is
\[ \binom{g + 1 - (h + 1)}{g - 3 - h} (h + 1) = \binom{g - h}{3} (h + 1) \]
where the binomial coefficient is the number of \( g - 3 \)-uples containing a fixed subset of \( h \) elements in the set \( \{N_1, \ldots, N_{h+1}\} \); the factor \( (h + 1) \) represents the choice of the \( N_i \) such that \( H \) contains the tangent line to \( C \) at \( N_i \).

In conclusion
\[ \mu_h(\mathcal{Q}_{g-3}(Y, W)) = \binom{g - h}{4} 4 + \binom{g - h}{3} (h + 1) = \frac{(g - h)(g - h - 1)(g - h - 2)(g - 2)}{3!} \]
On the other hand, the strict upper bound allowed for stability by the criterion of GIT is
\[ \text{Max}_h(m) = m \frac{g - 1 - h}{g} = 4 \binom{g + 1}{g - 3} \frac{g - 1 - h}{g} = \frac{(g + 1)(g - 1)(g - h - 1)(g - 2)}{3!} . \]
Since $0 \leq h$, it is clear that for every $h$

$$\mu_h(\Theta_{g-3}(Y, W)) < \text{Max}_h(m)$$

and therefore $\Theta_{g-3}(Y, W)$ and $\Theta_{g-3}(X)$ are GIT-stable.

Now we treat case c), which is simpler. We have for every $h$

$$\mu_h(\Omega) = \binom{g + 1 - h - 1}{g - 1 - h + 1} = \binom{g - h}{2} = \frac{(g - h)(g - h - 1)}{2}.$$

On the other hand, the (strict) upper bound given by the criterion above is, for every $h$

$$\text{Max}_h(m) = m \frac{g - 1 - h}{g} = \binom{g + 1}{2} \frac{g - 1 - h}{g} = \frac{(g + 1)(g - 1 - h)}{2}$$

it is thus evident that, for every $h \geq 0$, $\mu_h(\Omega) < \text{Max}_h(m)$, and hence $\Omega$ is GIT-stable. \hfill \Box

**Remark.** The interested reader can generalize the above computation to see that such stability results are special cases of a more general phenomenon, about the stability of configurations of hyperplanes spanned by sets of points and tangent lines of a rational normal curve.

**Corollary 4.3.2.** Let $X$ be a projective split curve; let $Y$ be an abstract split curve and $Y \to W$ a canonical model. Consider the configurations $\hat{\Theta}(X)$ and $\hat{\Theta}(Y, W)$ with respect to the natural action of $G$. (a) They are GIT-stable. (b) If they are in the same $G$-orbit, then $W$ is a projective split curve.

**Proof.** The proof of (a) is a straightforward application of the criterion already used for the proof of the previous result. Let $m_1 = 4 \binom{g+1}{g-3}$, $m_2 = \binom{g+1}{g-1}$ and $m = 2^{g-3}m_1 + 2^{g-1}m_2$. By definition,

$$\text{Max}_h(m) = 2^{g-3}\text{Max}_h(m_1) + 2^{g-1}\text{Max}_h(m_2)$$

where $0 \leq h \leq g - 2$. Let now $\Omega \in \text{Sym}^m(\mathbb{P}^{g-1})^*$ be one of the two configurations in the statement. Then we can write $\Omega = \{2^{g-3}\Omega_1, 2^{g-1}\Omega_2\}$ where $\Omega_1$ is either $\Theta_{g-3}(X)$ or $\Theta_{g-3}(Y, W)$, and hence a configuration of $m_1$ hyperplanes of type a) or b) in the previous Lemma 4.3.1 a); similarly, $\Omega_2$ is a configuration of $m_2$ hyperplanes of type c) in 4.3.1. We have

$$\mu_h(\Omega) = 2^{g-3}\mu_h(\Omega_1) + 2^{g-1}\mu_h(\Omega_2) < 2^{g-3}\text{Max}_h(m_1) + 2^{g-1}\text{Max}_h(m + 2) = \text{Max}_h(m)$$

(where the inequality comes from 4.3.2), hence we are done.

For (b) we must prove that the $G$-orbits of $\hat{\Theta}(X)$ and $\hat{\Theta}(Y, W)$ are different, if $W$ is not a split curve. This follows from the fact that the two, regarded as hypersurfaces, have different singularities. More precisely, by looking at the points $N_i$, one sees that

$$\mu_0(\Theta_{g-3}(Y, W)) > \mu_0(\Theta_{g-3}(X))$$

(recall in fact that if $H \in \Theta_{g-3}(X)$, then $H$ is tangent to $X$ at two smooth points). Since, as already noticed, $\mu_h(\Theta_{g-1}(Y, W)) = \mu_h(\Theta_{g-1}(X))$ for every $h \geq 0$, we conclude that

$$\mu_0(\hat{\Theta}(Y, W)) > \mu_0(\hat{\Theta}(X))$$

and hence the two configurations cannot possibly be projectively equivalent. \hfill \Box

**4.4.** We conclude with a strengthening of Theorem 5 of [C] which will be used later. The improvement consists essentially in the fact that to recover the curves it is enough to consider theta-hyperplanes of type $g - 3$ and $g - 1$, rather than all of them.
Proposition 4.4.5. Let $X$ and $X'$ be two split curves in $\mathbb{P}^{g-1}$. If $\hat{\Theta}(X) = \hat{\Theta}(X')$, then $X = X'$.

Proof. The proof is similar to the proof of Theorem 5 in [C]. The argument there is divided into two parts; the first part shows that, if the two curves have the same theta-hyperplanes of type $g - 1$, then they have the same singularities. We can here use that part, since, by definition, $\Theta(X) = \hat{\Theta}(X')$ implies that $X$ and $X'$ have the same theta-hyperplanes of type $g - 1$ (which are, of course, those hyperplanes having multiplicity $2^{g-1}$).

Denote $X_{\text{sing}} = X'_{\text{sing}} = \{N_1, \ldots, N_{g+1}\}$, $X = C_1 \cup C_2$ and $X' = C'_1 \cup C'_2$, with $C_i$ and $C'_i$ rational normal curves.

Now let, for $j = 1, 2$
\[ \Lambda_j = \langle N_j, N_3, N_4, \ldots, N_{g-3}, N_{g-2} \rangle \]
thus $\Lambda_1$ and $\Lambda_2$ are two linear subspaces of dimension $g - 4$ intersecting in $\langle N_3, \ldots, N_{g-2} \rangle$.

Let, for $j = 1, 2$, $\pi_j : \mathbb{P}^{g-1} - \{\lambda\} \to \mathbb{P}^2$ be the projection from $\Lambda_j$ onto $\mathbb{P}^2$ and let $X_j = \pi_j(X)$ and $X'_j = \pi_j(X')$. $X_j$ and $X'_j$ are two plane quartics with the same singularities, moreover $X_j$ is split, because $X$ is split, notice in fact that $X_j$ (respectively, $X'_j$) is the normalization of $X$ (respectively, of $X'$) at the nodes $N_j, N_3, \ldots, N_{g-2}$. Since $X_{\text{sing}} = X'_{\text{sing}}$, we have that $X_j$ and $X'_j$ have the same singularities. There is a natural bijection between the theta-lines of type 0 of $X_j$ (respectively, of $X'_j$) and the theta-hyperplanes of type $g - 3$ of $X$ (respectively, of $X'$) that contain $N_j, N_3, \ldots, N_{g-2}$. By assumption, $X$ and $X'$ have the same theta-hyperplanes of type $g - 3$, therefore $X_j$ and $X'_j$ have the same theta-lines of type 0. We conclude (see [CS]) that $X_j = X'_j$ for $j = 1, 2$.

We have thus proved that projecting $X$ and $X'$ to $\mathbb{P}^2$ from the same $g - 4$-dimensional linear subspace spanned by $g - 3$ of their (common) nodes, one obtains the same split curve in $\mathbb{P}^2$. It is clear that, up to re-naming the nodes, we can chose $\Lambda_1$ and $\Lambda_2$ so that for $j = 1, 2$ we have that $\pi_j(C_1) = \pi_j(C'_1)$ and, of course, $\pi_j(C_2) = \pi_j(C'_2)$.

Let $S_j$ be the cone over $C_1$ with vertex $\Lambda_j$, that is
\[ S_j := \bigcup_{P \in C_1} \langle \Lambda_j, P \rangle, \]
then the fact that $\pi_j(C_1) = \pi_j(C'_1)$ implies that $C'_1 \subset S_j$ for both $j = 1, 2$. We shall now show that this implies that $C_1 = C'_1$. It suffices to show that that
\[ S_1 \cap S_2 \subset C_1 \cup \langle \Lambda_1, \Lambda_2 \rangle. \]

By contradiction; suppose that there is a point $Q$ such that $Q \in S_1 \cap S_2$ but $Q \notin C_1 \cup \langle \Lambda_1, \Lambda_2 \rangle$. Then the linear space $\langle Q, \Lambda_j \rangle$ must intersect $C_1$ in a point $P_j$; we thus find two new points $P_1$ and $P_2$ lying in the intersection of $C_1$ with the hyperplane $H := \langle Q, \Lambda_1, \Lambda_2 \rangle$. Then
\[ \{N_1, N_2, \ldots, N_{g-2}, P_1, P_2\} \subset C_1 \cap H \]
that is, $\deg C_1 \cap H \geq g$, which is not possible. Therefore $C_1 = C'_1$.

Repeating the argument for $C_2$ and $C'_2$ we conclude that $X = X'$. \(\Box\)

5. The local picture near split curves

The purpose of this section is to analyze the differential of $\theta : V \to \text{Sym}^N (\mathbb{P}^{g-1})^*$ at a point parametrizing a split curve, and to prove (5.2.2) that it is injective.

5.1. Vector bundles on $\mathbb{P}^1$: elementary transformations. We recall a few well known facts concerning vector bundles on rational curves. Denote by $C = \mathbb{P}^1$ and by $\lambda$ the line bundle of degree
Let $E$ be a vector bundle on $C$. An \textit{elementary transformation} of $E$ is a vector bundle $E'$ such that there is an exact sequence

\begin{equation}
0 \to E' \to E \to C_r \to 0
\end{equation}

where $C_r$ is a torsion sheaf supported on a point $x \in C$ with fiber $C$. We have

\[ \text{rk}(E') = \text{rk}(E); \quad \text{deg}(E') = \text{deg}(E) - 1 \]

Let us denote by $\mathbb{P}(E)$ the projective bundle over $C$ associated to $E$. The exact sequence (1) corresponds to a point of $\mathbb{P}(E)$, contained in the fiber $\mathbb{P}(E)_x$, called the \textit{center} of the elementary transformation $E'$. 

It is well known that the projective bundle $\mathbb{P}(E')$ is obtained from $\mathbb{P}(E)$ by blowing up the center and then blowing down the proper transform $\tilde{F}$ of the fiber containing it (see [Ma]); $\mathbb{P}(E)$ is obtained from $\mathbb{P}(E')$ by the analogous “inverse” process with center the image of $\tilde{F}$. For example, if $E$ has rank 2, we write $\mathbb{P}(E) \cong \mathbb{F}_n$ and $\mathbb{P}(E') \cong \mathbb{F}_n'$, with the usual notation $\mathbb{F}_n := \mathbb{P}(\lambda^0 \oplus \lambda^n)$. Of course, in our situation, $|n - n'| = 1$. More precisely, we have that $n = n' - 1$ if and only if the center $p \in \mathbb{P}(E)$ belongs to the $(-n)$-curve of $\mathbb{P}(E)$, if and only if the center $q \in \mathbb{P}(E')$ of the inverse process does not belong to the $(-n - 1)$-curve of $\mathbb{P}(E')$.

An exact sequence of vector bundles

\[ 0 \to F \to E \to L \to 0 \]

with $\text{rk}L = 1$, defines a section $\sigma$ of $\mathbb{P}(E)$; $\sigma$ contains the center of $E'$ if and only if $F \subset E'$. In this case we have an exact sequence:

\[ 0 \to F \to E' \to L\lambda^{-1} \to 0 \]

In particular, if $E = F \oplus L$ then $E' = F \oplus L\lambda^{-1}$.

Given an exact sequence

\[ 0 \to E_n \to E \to F \to 0 \]

where $F = C_{x_1} \oplus \cdots \oplus C_{x_n}$, is a torsion sheaf supported on $n$ points of $C$, we say that $E_n$ is obtained from $E$ by a sequence of $n$ elementary transformations, centered at $n$ points of $\mathbb{P}(E)$.

Conversely, a set $\Gamma$ of $n$ points of $\mathbb{P}(E)$, no two on the same fiber, defines a vector bundle $E_\Gamma$ endowed with an exact sequence

\[ 0 \to E_\Gamma \to E \to F \to 0 \]

where $F$ is as above. Of course, $E_\Gamma$ is obtained applying the elementary transformations corresponding to the points of $\Gamma$.

We need to consider the following special case. Let $r = \text{rk}E$ and let $\Gamma'$ be a set of $r$ points, all contained in a fiber of $\mathbb{P}(E)$, and in general position in such a fiber. Then one can consistently define $E_{\Gamma'} = E \otimes \lambda^{-1}$, so that $\mathbb{P}(E) \cong \mathbb{P}(E_{\Gamma'})$. Similarly, if $n = rq$ and $\Gamma_0$ is a union of disjoint $r$-uples of points, as the $\Gamma'$ above, then $E_{\Gamma_0} = E \otimes \lambda^{-q}$.

We shall need the following very simple Lemma.

\textbf{Lemma 5.1.1.} Let $\mathcal{E} \to T$ be a family of vector bundles over $\mathbb{P}^1$. Let $E_t = \oplus \lambda^{n_t}$ be the generic fiber and let $E_0 = \oplus \lambda^{n_0}$ be the special fiber. Then

\begin{enumerate}
  \item \[ \max_i\{n_i\} \leq \max_i\{n_i^0\} \quad \text{and} \quad \min_i\{n_i\} \geq \min_i\{n_i^0\}. \]
  \item If $E_0$ is balanced (i.e. $n_i = n_j$ for all $i, j$) then $E_t$ is balanced and $E_t \cong E_0$.
\end{enumerate}

\textit{Proof.} Let $M_t := \max_i\{n_i\}$ and $M_0 := \max_i\{n_i^0\}$. By contradiction; if $M_t > M_0$, we can consider the family $\mathcal{E}' \to T$ of vector bundles, obtained by tensoring the given family $\mathcal{E}$ by $\lambda^{-M_t}$. Then $E'_0$
has no global sections, being a sum of line bundles of negative degree. On the other hand $E'_t$ has one summand equal to $\lambda^0$, thus it has non-zero sections. This is a contradiction. The statement about the minimum is obtained in the same way, working on the sequence of dual vector bundles. Part b) follows immediately from a). □

5.2. Normal bundles of split curves. Let now $g \geq 3$ and let $C \subset \mathbb{P}^{g-1}$ be a rational normal curve. Let us consider the normal bundle of $C$:

$$N_C := N_{C/\mathbb{P}^{g-1}} \cong \bigoplus^{g-2} \lambda^{g+1}$$

We have $\text{deg} N_C = (g-2)(g+1), h^0(N_C) = (g-2)(g+2)$ and $\mathbb{P}(N_C) \cong C \times \mathbb{P}^{g-3}$. This isomorphism can be interpreted in terms of the identifications:

$$\mathbb{P}(N_C) \cong \mathbb{P}(N_C(-1)) = \{(x, H) : x \in C, T_xC \subset H\} \subset C \times (\mathbb{P}^{g-1})^*$$

($T_xC$ is the tangent line to $C$ at $x$) coming from the surjections:

$$\bigoplus^g \mathcal{O}_C \rightarrow T_{C/\mathbb{P}^{g-1}}(-1) \rightarrow N_C(-1)$$

More precisely, we have: $\mathbb{P}(N_C) \cong C \times |\lambda^{g-3}|$. Thus a fiber $\mathbb{P}(N_C)_x, x \in C$, can be identified with the complete linear system $|\lambda^{g-3}|$ cut out by all hyperplanes containing $T_xC$.

Let $X = C_1 \cup C_2 \subset \mathbb{P}^{g-1}$ be a split curve of genus $g$ and let $C_1 \cap C_2 = \{N_1, \ldots, N_{g+1}\} = X_{\text{sing}}$. Let $\Omega = \Theta_{g-3}(X)$ be the set of all theta-hyperplanes of $X$ passing through exactly $g - 3$ nodes of $X$ (described in the previous section). We write $\Omega = \Omega_1 \cup \ldots \cup \Omega_{\binom{g+1}{g-3}}$, where $\Omega_i$ is the set of four theta-hyperplanes containing a fixed subset $\Sigma_i$ of $g - 3$ nodes. We denote by $H_{i,j}, j = 1, \ldots, 4$, the four hyperplanes of $\Omega_i$.

Consider the everywhere nonreduced zero-dimensional scheme

$$Z := \left[ \bigcup_{H \in \Omega} H \right] \cap X_{\text{reg}} \subset X$$

and let, for $k = 1, 2$, $Z_k = Z \cap C_k$. Then $Z_k \subset C_k$ is supported at the $4\binom{g+1}{g-3}$ points below

$$\left\{ p^{k}_{i,1}, \ldots, p^{k}_{i,4}, \ i = 1, \ldots, 4\binom{g+1}{g-3} \right\}$$

For a general split curve, such points are distinct and $p^{k}_{i,j} \neq N_h$ for all choices of indeces. We have natural surjections $N_X \rightarrow T^1_Z$ and $N_{C_k} \rightarrow T^1_{Z_k}$, where $T^1$ denotes the first cotangent sheaf of $-1$, and we define

$$N'_X := \ker \{ N_X \rightarrow T^1_Z \}$$

$$N''_{C_k} = \ker \{ N_{C_k} \rightarrow T^1_{Z_k} \}$$
We have an exact and commutative diagram for $k = 1, 2$:

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \mathcal{N}_{C_k}' & \mathcal{N}_{C_k} & T^1_{Z_k} & 0 \\
\downarrow & \downarrow & \parallel \\
0 & (\mathcal{N}_{X})_{|C} & (\mathcal{N}_{X})_{|C_k} & T^1_{Z_k} & 0 \\
\downarrow & \downarrow \\
\mathcal{T} &= \mathcal{T} \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]  

(3)

where $\mathcal{T} = \mathbb{C}_{N_1} \oplus \cdots \oplus \mathbb{C}_{N_{g+1}}$ is a torsion sheaf.

The proof of Theorem 5.2.2 is based on the following key result

**Proposition 5.2.1.** With the above notation, $H^0(X, \mathcal{N}_{X}') = 0$ for all $g \geq 4$.

**Proof.** We have an exact sequence:

\[
0 \rightarrow \mathcal{N}_X' \rightarrow (\mathcal{N}_X')_{|C_1} \oplus (\mathcal{N}_X')_{|C_2} \rightarrow Q \rightarrow 0
\]  

(2)

where $Q = \mathbb{C}_{N_1}^{g-2} \oplus \cdots \oplus \mathbb{C}_{N_{g+1}}^{g-2}$, it therefore suffices to prove that, for $k = 1, 2$,

\[
H^0((\mathcal{N}_{X}')_{|C_k}) = 0.
\]

To prove this, we will consider the above diagram (3). From its first row, we see that $\mathcal{N}_{C_k}'$ is obtained from $\mathcal{N}_{C_k}$ applying a sequence of $4 \binom{g+1}{g-3}$ elementary transformations, centered at the points of the set

\[
\Gamma := \{(p_{i,j}, \Sigma_i), \ i = 1, \ldots, \binom{g+1}{g-3}, \ j = 1, \ldots, 4\}
\]

Let us fix now $k = 1$ and drop the index $k$ for simplicity, denoting $C_1 = C$. Let $E := \mathcal{N}_C = \oplus_{g} \mathbb{C}_{g+1}$. Let $n = 4 \binom{g+1}{g-3}$.

We can specialize our curve $X = C \cup C_2$ to a hyperelliptic split curve (see 4.2), whose canonical model $X_0$ is supported on $C$. We can do that by keeping the component $C$ fixed (i.e. $C_2$ specializes to $C$) and by maintaining the nodes of every fiber at the points $N_1, \ldots, N_{g+1}$. Call $X_t$ the generic fiber and $X_0$ the special fiber. Let $\Gamma_t$ be the $n$-uple of points in $\mathbb{P}(E)$ defined as the centers of the $n$ elementary transformation of the first row of diagram (3), which we can now re-write for $X_t$

\[
0 \rightarrow E_{\Gamma_t} \rightarrow E \rightarrow T^1_t \rightarrow 0
\]

As $X_t$ specializes to $X_0$, the configuration of theta-hyperplanes $\Theta_{g-3}(X)$ specializes to a configuration that has been defined and studied in the previous section. Namely, call $Y_0$ the abstract, hyperelliptic split curve whose canonical model is $X_0$. Then the limit configuration is $\Theta_{g-3}(Y_0, X_0)$
defined in 4.2, where \( W = X_0 \). Therefore the set \( \Gamma_t \) specializes to a set \( \Gamma_0 \subset \mathbb{P}(E) \) which has implicitly been described in 4.2:

\[
\Gamma_0 = \{(N_j, \Sigma_i) : \# \Sigma_i = g - 3, \ N_j \not\in \Sigma_i\}
\]

In particular, \( \Gamma_0 \) is entirely contained in the \( g+1 \) fibers over \( N_1, \ldots, N_{g+1} \); each such a fiber contains exactly \( \binom{g}{g-3} \) points of \( \Gamma_0 \), and such points are in general position in every fiber (because \( C \) is a rational normal curve).

At this point, we need to distinguish three cases; first: \( g \not\equiv 0 \pmod{3} \) and \( g \neq 4 \); second: \( g = 3x + 2 \) and third: \( g = 4 \).

• Suppose that \( g \not\equiv 2 \pmod{3} \).

Then we can factor \( n = (g - 2)(g + 1)\frac{g(g-1)}{2} \), so that \( \Gamma_0 \) is a union of disjoint \( g - 2 \)-uples contained in a fiber of \( \mathbb{P}(E) \to C \). We are thus in the situation described in 5.1; we obtain that

\[
E_{\Gamma_0} = E \otimes \lambda^{-(g+1)\frac{g(g-1)}{2}} = \bigoplus \lambda^a
\]

where \( a = g + 1 - (g + 1)\frac{g(g-1)}{2} \). Thus \( E_t \) specializes to the “balanced” vector bundle \( E_0 = \oplus \lambda^a \).

By 5.1.1, \( E_t \) is also balanced and isomorphic to \( E_0 \).

We have therefore proved that \( \mathcal{N}'_C = \oplus \lambda^{-g+2} \).

A straightforward computation shows that, if \( g \geq 5 \), then \( a < -g - 1 \).

The exact sequence in the first column of (3) expresses \( \mathcal{N}'_C \) as obtained from \( (\mathcal{N}'_X)_C \) by a sequence of \( g + 1 \) elementary transformations. In particular,

\[
\deg(\mathcal{N}'_X)_C = \deg \mathcal{N}'_C + g + 1.
\]

If \( g \geq 5 \), we immediately deduce that each line bundle summand in the splitting of \( (\mathcal{N}'_X)_C \) has negative degree, therefore \( H^0((\mathcal{N}'_X)_C) = 0 \) and we are done.

• Let \( g = 3x + 2 \). Here the problem is that for every \( h = 1, \ldots, g+1 \) the number \( f \) of points of \( \Gamma_0 \) contained in the fiber over \( N_h \) is not a multiple of \( g - 2 \). More precisely we have

\[
f := \begin{pmatrix} g \\ g - 3 \end{pmatrix} = x + (g - 2)\frac{3x(x + 1)}{2}
\]

Pick on each of these \( g + 1 \) fibers a subset of \( f - x \) points of \( \Gamma_0 \), and call \( \Gamma'_0 \) the union of these \( g + 1 \) subsets. Thus \( \Gamma'_0 \subset \Gamma_0 \),

\[
\# \Gamma'_0 = (g + 1)(g - 2)\frac{3x(x + 1)}{2}
\]

and, as before,

\[
E_{\Gamma'_0} = E \otimes \lambda^{-(g+1)\frac{3x(x+1)}{2}} = \bigoplus \lambda^{a'}
\]

where \( a' = g + 1 - (g + 1)\frac{3x(x+1)}{2} \). As in the previous case, we conclude that \( \mathcal{N}'_C \subset \oplus \lambda^{-2} \).

One easily sees that, if \( x \geq 1 \), then \( a' < -g - 1 \). So we conclude by looking at the first column of diagram (3), exactly as we did before.

• If \( g = 4 \) the argument of the first case yields \( \mathcal{N}'_C = \lambda^{-5} \oplus \lambda^{-5} \), which is not enough; we use a slightly different strategy. We can write \( (\mathcal{N}'_X)_C = \lambda^u \oplus \lambda^v \) with \( u \leq v \) and \( u + v = -5 \). To show that \( (\mathcal{N}'_X)_C \) has no sections it suffices to show that \( v - u < 5 \).
Consider the second column of diagram (3)

\[ 0 \rightarrow \lambda^5 \oplus \lambda^5 \rightarrow (N_X)_C = \lambda^c \oplus \lambda^d \rightarrow T \rightarrow 0 \]

with \(c + d = 15\) and \(c \leq d\). For \(i = 1, \ldots, 5\) let \(H_i\) be the plane spanned by the tangent lines at \(N_i\) to \(C_1 = C\) and \(C_2\):

\[ H_i := \langle T_{N_i}C_1, T_{N_i}C_2 \rangle. \]

It is a local computation to check that \(T_{N_i}\) is defined by the plane \(H_i\).

Therefore the 5 centers of the elementary transformation above are the points \((q_i, N_i)\) where \(q_i \in C \cap H_i - \{N_i\}\) is the residual point of intersection of \(H_i\) with \(C_i\). It is easy to see that \(p_i \neq p_j\). We deduce that such centers are not all contained in the same section. In particular, \(\mathbb{P}(N_X)_C \neq \mathbb{P}_5\), hence \(d - c < 5\).

Now we look at the second row of diagram (3). By the same argument as in the previous part of the proof, using 5.1.1a), we can conclude that \(v - u \leq d - c < 5\); hence we are done.

We can now prove the

**Theorem 5.2.2.** Let \(X \subset \mathbb{P}^{g-1}\) be a general, projective split curve. Then \(\theta\) is an immersion at (the point parametrizing) \(X\).

**Proof.** Let \(X = C_1 \cup C_2 \subset \mathbb{P}^{g-1}\) be a general split curve. To prove that \(\theta\) is an immersion at \(X\) amounts to showing that \(T_X\theta^{-1}(\theta(X)) = (0)\). Note that \(\theta^{-1}(\theta(X))\) is a closed subfamily of \(V\) consisting of canonical curves which have \(\theta(X)\) as theta-hypersurface. Consider the subsheaf \(N'_X \subset N_X\) studied above. Its sections define a subspace of \(H^0(X,N_X) = T_XV\) consisting of first order deformations of \(X\) which remain tangent to the hyperplanes of \(\Omega\) at the points of \(Z\).

Therefore

\[ T_X\theta^{-1}(\theta(X)) \subset H^0(X,N'_X) \]

But \(H^0(X,N'_X) = 0\) by 5.2.1, thus our statement is proved.

---

**6. Characterizing canonical curves by their theta-hyperplanes**

For the reader’s convenience, we restate our main Theorem, before proving it.

**Theorem 6.1.1.** Let \(X\) and \(X'\) be general canonical curves in \(\mathbb{P}^{g-1}\) having genus \(g \geq 4\). If \(\theta(X) = \theta(X')\) then \(X = X'\).

**Proof.** By contradiction. If the statement is false we can find two families of canonical curves as follows. The first family

\[
\begin{array}{ccc}
\mathcal{X} & \hookrightarrow & T \times \mathbb{P}^{g-1} \\
\downarrow & & \\
T & &
\end{array}
\]

is a general one-parameter deformation of a general split curve \(X\), with \(X_t\) smooth for \(t \neq t_0\). For every \(t \in T\), we assume that \(\psi_X(t) \in V\).

The second family

\[
\begin{array}{ccc}
\mathcal{X}' & \hookrightarrow & T \times \mathbb{P}^{g-1} \\
\downarrow & & \\
T & &
\end{array}
\]

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is such that the generic fiber $X'_t$ is a smooth, theta-generic canonical curve, having the same theta-hypersurface as $X_t$: $\theta(X_t) = \theta(X'_t)$ for every $t \neq t_0$.

We cannot, for the moment, say much about the special fiber $X'$ of the second family, we shall use the existence of stable reduction to analyze it. Modulo replacing $T$ by a finite covering, we can assume that $\mathcal{X}' \to T$ admits stable reduction over $T$; let it denote by $\mathcal{Y} \to T$. We have that, for $t \neq t_0$, $X'_t$ is a canonical (isomorphic) model of $Y_t$. The central fiber $Y$ of $\mathcal{Y} \to T$ is a stable curve, and we have a birational map $\mathcal{Y} \to \mathcal{X}'$ which is an isomorphism away from the central fibers.

* We claim that $Y$ is a split curve.

To prove that, consider first $J_\mathcal{X} \to T$ defined at the end of 2.1, and the family of odd spin curves $S_\mathcal{X} \to T$ defined in 2.2 (abusing notation denoting by the same symbol $\mathcal{X}$ the abstract family and its canonical model). We can compare the two: by our genericity assumption, $S_\mathcal{X}$ is a smooth curve (by 2.2.1) dominating $J_\mathcal{X}$ (by 2.4.1(a)). Notice also that, by construction, $J_\mathcal{X} = J_{\mathcal{X}'}$; in conclusion, we have a birational morphism of $T$-schemes

$$S_\mathcal{X} \to J_\mathcal{X} = J_{\mathcal{X}'}$$

and by 2.4.1(b),

$$L(S_\mathcal{X}) = L(J_\mathcal{X}) \tag{1}$$

Consider now the pull-back to $T$ of the space of odd spin curves of the family $\mathcal{Y} \to T$, as usual denoted by $S_\mathcal{Y}$. Away from their special fibers, $S_\mathcal{Y}$ and $J_{\mathcal{X}'}$ are isomorphic over $T$; by what we have observed above, we have a birational $T$-morphism (recall that $S_\mathcal{X}$ is a smooth curve)

$$\alpha : S_\mathcal{X} \to S_\mathcal{Y}.$$ 

Looking at the central fibers we get that

$$L(S_\mathcal{X}) \geq L(S_\mathcal{Y}),$$

equivalently, by (1)

$$L(J_\mathcal{X}) \geq L(S_\mathcal{Y}) \tag{2}$$

Now we apply 2.4.2, with $W = \mathcal{X}'$. We get that

$$L(S_\mathcal{Y}) \geq L(J_{\mathcal{X}'})$$

but $J_{\mathcal{X}'} = J_\mathcal{X}$ and hence, of course, $J_\mathcal{X} = J_{\mathcal{X}'}$, so that we get

$$L(S_\mathcal{Y}) \geq L(J_\mathcal{X}) \tag{3}$$

Combining (2) and (3) (see 1.2) and using (1), we conclude that

$$L(S_\mathcal{X}) = L(S_\mathcal{Y}).$$

Finally, $X$ being a split curve, we use 3.4.2(b) to conclude that $Y$ is a split curve. The claim is thus proved.

Let now $Y \to W$ be a canonical map. Let

$$\mathcal{Y} \to \mathcal{W} \hookrightarrow T \times \mathbb{P}^{g-1}$$

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be a canonical image of $Y \to T$, whose restriction to the central fibers is the above $Y \to W$. If $t \neq t_0$, $W_t$ is an isomorphic model of $Y_t$ and it is thus projectively equivalent to $X'_t$. That is, there exists a morphism $\gamma : T - \{t_0\} \to G$ such that

$$X'_t = W_t^{\gamma(t)}.$$ 

Hence

$$\theta(X'_t) = \theta(X_t) = \theta(W_t)^{\gamma(t)}.$$ 

By 4.2.1, we can consider the distinguished subconfiguration of theta-hyperplanes $\hat{\Theta}(X) \subset \theta(X)$; by definition, $\hat{\Theta}(X)$ is the subconfiguration (with multiplicities) of all components of $\theta(X)$ having multiplicity at least $2g-3$. Therefore it is the specialization of a well defined subconfiguration of hyperplanes $\hat{\Theta}_t \subset \theta(X_t)$. Similarly, consider the distinguished configuration $\hat{\Theta}(Y, W) \subset \theta^W(W)$. By construction, we have that $\hat{\Theta}(Y, W)$ is the specialization of the family of configurations $(\hat{\Theta}_t)^{\gamma(t)}$. Recall that, by 4.3.2, $\hat{\Theta}(X)$ and $\hat{\Theta}(Y, W)$ are GIT-stable points in $Sym^m(\mathbb{P}^{g-1})^*$. We have just seen that they are specializations of two $G$-conjugate families. Therefore (since they are GIT-stable) they are themselves $G$-conjugate, that is, there exists an element $g_0 \in G$ such that

$$\hat{\Theta}(X) = (\hat{\Theta}(Y, W))^{g_0}.$$ 

Now this implies that $W$ is a split curve, by 4.3.2, (b). Hence $\hat{\Theta}(Y, W) = \hat{\Theta}(W)$ and

$$\hat{\Theta}(X) = \hat{\Theta}(W)^{g_0} = \hat{\Theta}(W^{g_0}).$$ 

Summarizing, $X$ and $W^{g_0}$ are split curves such that $\hat{\Theta}(X) = \hat{\Theta}(W^{g_0})$. We are thus in the position of applying 4.4.5, to conclude that

$$W^{g_0} = X.$$ 

Now $W^{g_0}$ is the central fiber of the family over $T$ whose generic fiber is $W_t^{\gamma(t)}$. By Theorem 5.2.2, we get that $W_t^{\gamma(t)} = X_t$, hence we are done, since by construction $W_t^{\gamma(t)} = X'_t$. 

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