Determining lower bounds on a measure of multipartite entanglement from few local observables

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Abstract

We introduce a method to lower bound an entropy-based measure of genuine multipartite entanglement via nonlinear entanglement witnesses. We show that some of these bounds are tight and explicitly work out their connection to a framework of nonlinear witnesses that were published recently. Furthermore, we provide a detailed analysis of these lower bounds in the context of other possible bounds and measures. In exemplary cases, we show that only a few local measurements are necessary to determine these lower bounds.

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I. INTRODUCTION

Quantum entanglement is central to the field of quantum information theory. Due to its numerous applications in upcoming quantum technology much research has been devoted to its understanding (for a recent overview consider Ref. [1]).

Especially in systems comprised of many particles entanglement provides numerous challenges and of course potential applications, such as building quantum computers (see Ref. [2]), performing quantum algorithms (the connection to multipartite entanglement is demonstrated in Ref. [3]) and multi-party cryptography (see e.g. Ref. [4]). Furthermore, the understanding of the behavior of complex systems seems to be closely linked to the understanding of multipartite entanglement manifestations, demonstrated by the connection to phase transitions and ionization in condensed matter systems (e.g. [5]), the properties of ground states in relation to entanglement (as shown e.g. in Ref. [6]), or potentially even biological systems (such as e.g. bird navigation [5]).

In order to judge the relevance of entanglement in such systems it is crucial to not only detect its presence, but also quantify the amount. The structure of entangled states, especially in multipartite systems [6], is very complex and the question whether a given state is entangled is even NP-hard [10]. Thus, in general, it will not be possible to derive a computable measure of entanglement that reveals all entangled states to be entangled and discriminates between different entanglement classes. Furthermore, full information about the state of the system requires a number of measurements that grows exponentially in the size of the system. For the detection of entanglement in multipartite systems most researchers have therefore made it a primary goal to develop entanglement witnesses, which via a limited amount of local measurements can detect the presence of entanglement, even in complex systems (for an overview of multipartite entanglement witnesses consider Ref. [11]).

The expectation value of witness-operators are usually expressed in terms of inequalities, which if violated show the presence of entanglement. Nonlinear witnesses (first introduced in Ref. [12] see also early discussions in e.g. Ref. [13]) provide a generalization that is no longer a linear function of density matrix elements, but a nonlinear one. Thus one cannot reformulate the criteria in terms of an expectation value of a hermitian operator (unless one considers coherent measurements on multiple copies of the state, which out of experimental infeasibility we do not discuss in our manuscript). We will henceforth refer to inequalities that involve nonlinear functions of density matrix elements as nonlinear entanglement witnesses.

Recently some authors pointed out a connection between the possible amount of violation of these nonlinear inequalities and quantification of entanglement in multipartite systems (in Ref. [14] and Ref. [15]). The aims of this paper are twofold. First to systematically show the connection of numerous witnesses to a meaningful measure of genuine multipartite entanglement and second to use this established relation for the development of novel witnesses, which by construction give lower bounds on that measure. To that end we follow and generalize the approach from Ref. [15].

It turns out that only a small number of density matrix elements enters into our lower bounds, making the construction experimentally feasible even in larger systems of high dimensionality.
II. A MEASURE OF MULTIPARTITE ENTANGLEMENT AND ITS LOWER BOUNDS

A. A measure of genuine multipartite entanglement (GME)

The entropy of subsystems has often been used, in order to quantify entanglement contained in multipartite pure states (e.g., see \[1\] [12] [13]). In this paper we will follow the definition first presented in Ref. [10] and define a measure of GME for multipartite pure states as

\[ E_m(|\psi\rangle\langle\psi|) := \min_{\gamma} \sqrt{S_L(\rho_\gamma)} = \min_{\gamma} \sqrt{2 \left(1 - \text{Tr}(\rho_\gamma^2)\right)}, \]

where \( S_L(\rho_\gamma) \) is the linear entropy of the reduced density matrix of subsystem \( \gamma \), i.e., \( \rho_\gamma := \text{Tr}_\gamma(|\psi\rangle\langle\psi|) \). The minimum is taken over all possible reductions \( \gamma \) (where the complement is denoted as \( \bar{\gamma} \)), which corresponds to a bipartite split into \( \gamma|\bar{\gamma} \).

As any proper measure of multipartite entanglement for pure states can be generalized to mixed states via a convex roof, i.e.

\[ E_m(\rho) := \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_m(|\psi_i\rangle\langle\psi_i|). \]

Due to its construction this measure fulfills almost all desirable properties one would expect from measures of GME (see Ref. [15] for details). Because computing all possible pure state decompositions of a density matrix is computationally impossible even if one is given the complete density matrix, we require lower bounds to be calculable for this expression.

Also note that a lower bound on the linear entropy directly leads to a lower bound on the Rényi 2-entropy \( S_R^{(2)}(\rho_\gamma) \) via the relation \( S_R^{(2)}(\rho_\gamma) = -\log_2\left(\frac{2 - S_L(\rho_\gamma)}{2}\right) \), which also provides one of the physical interpretations of this measure. The Rényi 2-entropy in itself is a lower bound to the von Neumann entropy \( S(\rho) \), and the mutual information can be expressed as \( I_{\gamma \bar{\gamma}} := S(\rho_\gamma) + S(\rho_{\bar{\gamma}}) - S(\rho) = 2S(\rho_\gamma) \). Thus by our lower bound we gain a lower bound on the average minimal mutual information across all bipartitions of the pure states in the decomposition, minimized over all decompositions.

B. Linear entropy and its convex roof

The state vector of an \( n \)-partite qudit state can be expanded in terms of the computational basis

\[ |\psi\rangle = \sum_{i_1, i_2, \ldots, i_n=0}^{d-1} c_{i_1, i_2, \ldots, i_n} |i_1, i_2, \ldots, i_n\rangle =: \sum_{\eta \in N^d_2} c_{\eta} |\eta\rangle, \]

where a basis vector is denoted by \( \eta = (i_1, i_2, \ldots, i_n) \in N^d_2 \). This vector notation will facilitate the upcoming derivations. A crucial element of the notation in this paper will be the permutation operator acting upon two vectors, exchanging vector components corresponding to the set of indices. E.g. the permutation operator \( P_{\{1,3\}}(\eta_1, \eta_2) \) will exchange the first and third component of the vector \( \eta_1 \) with the corresponding component of the vector \( \eta_2 \), i.e.

\[ P_{\{1,3\}}(0123, 30121) = (31113, 00221). \]

Using this notation one can write down a very simple expression for the linear entropy of a reduced state \( \rho_\gamma \) (derivation see section 1A in the appendix)

\[ S_L(\rho_\gamma) = \sum_{\eta_1 \neq \eta_2} |c_{\eta_1} c_{\eta_2} - c_{\eta_2} c_{\eta_1}|^2, \]

where \( (\eta_1^*, \eta_2^*) = P_\gamma(\eta_1, \eta_2) \). For pure states we can of course find upper bounds on \( E_m(|\psi\rangle\langle\psi|) \) by lower bounding the linear entropy for all possible bipartitions. For mixed states we can then provide a lower bound for the convex roof \( E_m(\rho) \). We now illustrate our method in one exemplary case and then continue to articulate the main theorem.

Note that the linear entropy of subsystems has been widely used for lower bounding measures of entanglement due to the well known and simple structure of eq. (2).

None of the previous methods, however, work for lower bounding the inherently multipartite measure \( E_m(\rho) \), due to the additional minimization over all bipartitions in each decomposition element of the convex roof.

C. W-states

In order to demonstrate how our framework works let us start by deriving the explicit lower bound detecting the three-qubit \( W \) state \(|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \). For three-qubit states there are three bipartitions \(|12, 23|13, 32\rangle \) and thus we have three linear entropies to look at in order to calculate \( E_m(|\psi\rangle\langle\psi|) \),

\[ \sqrt{S_L(\rho_1)} = \frac{2\sqrt{|001\rangle \langle 100| - |c_{01}c_{10}|^2} + |c_{10}c_{01} - c_{11}c_{00}|^2 + \cdots,} \]

\[ \sqrt{S_L(\rho_2)} = \frac{2\sqrt{|001\rangle \langle 100| - |c_{01}c_{10}|^2} + |c_{10}c_{01} - c_{11}c_{00}|^2 + \cdots,} \]

\[ \sqrt{S_L(\rho_3)} = \frac{2\sqrt{|001\rangle \langle 100| - |c_{01}c_{10}|^2} + |c_{10}c_{01} - c_{11}c_{00}|^2 + \cdots.} \]

Now using \( a^2 + b^2 \geq \frac{1}{2}(a + b) \) (which is a specific case of the inequality \[\Delta^2\] in appendix A) and \(|a-b| \geq |a|-|b| \).
it is obvious that
\[ \sqrt{S_L(\rho)} \geq \frac{2(|c_{001}c_{100}| - |c_{011}c_{100}| + |c_{010}c_{100}| - |c_{110}c_{100}|)}{2} \] (6)
\[ \sqrt{S_L(\rho_2)} \geq \frac{2(|c_{010}c_{100}| - |c_{110}c_{100}| + |c_{010}c_{001}| - |c_{011}c_{001}|)}{2} \] (7)
\[ \sqrt{S_L(\rho_3)} \geq \frac{2(|c_{001}c_{100}| - |c_{101}c_{100}| + |c_{010}c_{001}| - |c_{111}c_{001}|)}{2} \] (8)

Then using \( |ab| - \frac{1}{2}(a^2 + b^2) \leq 0 \) we can add one negative term for each entropy and it will still be a lower bound, i.e. we add \( |c_{010}c_{001}| - \frac{1}{2}(|c_{010}|^2 + |c_{001}|^2) \) in the first lower bound, \( |c_{100}c_{001}| - \frac{1}{2}(|c_{100}|^2 + |c_{001}|^2) \) in the second and \( |c_{010}c_{100}| - \frac{1}{2}(|c_{010}|^2 + |c_{100}|^2) \) in the third. Then we can use that \( \min(P - N_1, P - N_2, P - N_3) \geq P - N_1 - N_2 - N_3 \) and end up with
\[ E_m(\rho) \geq \sqrt{2}(|c_{001}c_{100}| + |c_{011}c_{100}| + |c_{010}c_{100}|) \]
\[ \geq \frac{\sqrt{2}}{2}(|c_{010}|^2 + |c_{100}|^2 + |c_{001}|^2) - \sqrt{2}(|c_{101}c_{000}| + |c_{110}c_{000}| + |c_{111}c_{000}|) \] (9)

Finally we can bound the convex roof using the following two relations
\[ \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i |c_{i1}^i c_{10}^i| \geq |\langle \eta_1 |\rho| \eta_2 \rangle| \] (10)
\[ \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i |c_{i1}^i c_{12}^i| \leq \sqrt{\langle \eta_1 |\rho| \eta_1 \rangle \langle \eta_2 |\rho| \eta_2 \rangle} \] (11)
and end up with a lower bound for mixed states as
\[ E_m(\rho) \geq \sqrt{2}(|\langle 001|\rho|000\rangle| + |\langle 011|\rho|000\rangle| + |\langle 100|\rho|010\rangle|) - \frac{\sqrt{2}}{2}(|010|\rho|001\rangle + |100|\rho|100\rangle + |001|\rho|001\rangle) - \frac{\sqrt{2}}{2}\sqrt{\langle 110|\rho|110\rangle \langle 000|\rho|000\rangle} - \frac{\sqrt{2}}{2}\sqrt{\langle 011|\rho|011\rangle \langle 000|\rho|000\rangle}. \] (12)

Surprisingly this leads directly to the nonlinear entanglement witness inequality presented in Refs. [21, 22] up to a factor of \( \sqrt{2} \). Using only simple algebraic relations we have thus shown how to lower bound the convex roof construction. The first apparent strength of this lower bound is the limited number of density matrix elements that are needed to be computed. E.g. in our exemplary three-qubit case only ten out of possibly sixty-four elements need to be measured. Obviously we can extend the analysis using the same techniques to systems beyond three qubits.

III. A GENERAL CONSTRUCTION OF LOWER BOUNDS ON THE GME MEASURE \( E_m \)

Now we can generalize the connection of the 3-qubit W state witness and the measure \( E_m \). Just as for three qubits we can always get lower bounds by summing the coefficient pairs \( c_{i1}, c_{i2} \) that belong to a certain target pure state and appear in some or all reduced linear entropies. The construction of such general lower bounds also starts by selecting a subset of coefficient pairs that will be translated into off-diagonal elements \( \rho_{\eta_1, \eta_2} \), where \( (\eta_1, \eta_2) \) is the vector basis pair denoting the row and column of the element in density matrix \( \rho \). We denote the selected vector basis pairs as \( R := \{ (\eta_1, \eta_2) \} \). Then we can repeat the steps analogously to eq.(6-11) and arrive at a general lower bound on the measure as the following theorem:
Theorem 1 (A general lower bound on the GME measure) For a set of row-column pairs \( R = \{(\eta_1, \eta_2)\} \), the genuine multipartite entanglement measure \( E_m \) has the following lower bound:

\[
E_m \geq 2 \frac{1}{|R| - N_R} \left[ \sum_{(\eta_1, \eta_2) \in R} \left( |\rho_{\eta_1, \eta_2}| - \sum_{\gamma \in \Gamma(\eta_1, \eta_2)} \sqrt{|n_1^\gamma n_2^\gamma|^2} \right) - \frac{1}{2} \sum_{\eta \in I(R)} N_{\eta} |\rho_{\eta\eta}| \right].
\]

The right-hand-side of eq. (13) defines a GME witness \( W_R(\rho) \), where \( |\rho_{\eta_1, \eta_2}| := (\eta_1^\gamma \eta_2^\gamma) \notin R \) and \( I(R) := \{\eta : \exists \gamma' \text{ that } (\eta', \eta) \text{ or } (\eta, \eta') \in R \} \) is the set of basis vectors \( \eta \), which appear in the set \( R \).

\( N_R \) is the maximal (or minimal) value of \( |R'| \) over all possible bipartitions \( \gamma | \bar{\gamma} \), where \( R' \) is the set of coefficient pairs \((c_n, c_n') \in R\), which do not contribute to the \( \gamma \)-subsystem entropy.

\( N_{\eta} \) are normalization constants given by the maximal value of \( n_1^\gamma \) over all possible bipartitions \( \gamma | \bar{\gamma} \), where \( n_1^\gamma \) is the number of coefficients \( c_n \) from some pairs in \( R \), which are not counted in the \( \gamma \)-subsystem entropy (and how many are counted depends on whether one chooses \( N_R \) to be maximal or minimal).

Proof. See Appendix 2 for the full proof. \( \blacksquare \)

It is evident that not every choice of coefficient pairs will yield a useful lower bound, because one really needs to select those that are actually contributing to multipartite entanglement. There is however always an obvious choice. The set of coefficient pairs \( R \) must be chosen such that in every subsystem at least one of the elements of \( R \) contribute to the linear entropy of the reduced state. E.g. in the case of GHZ states given in a specific basis \( |GHZ\rangle = (|000\rangle + |111\rangle) / \sqrt{2} \) one would choose the pair \((00\ldots0, 11\ldots1)\), which contributes to all reduced entropies. In the general case however there is still some freedom of choice left to get a valid lower bound. For some sets \( R \) it can happen, that the coefficients do not contribute to every subsystem entropy equally (which we show in an exemplary case in section IV.A). Then one can choose \( N_R \) in different ways, but in all considered cases we found that choosing it maximal or minimal will produce the best bounds (where choosing it maximal usually yields the tightest bounds close to pure states, whereas choosing it minimal improves the noise resistance). Since these coefficients are in general basis dependent, so is also our witness construction. The prefactor \( \sqrt{1/|R| - N_R} \) suggests that the optimal basis for constructing such a lower bound is given by the minimal tensor rank representation of the pure state.

IV. APPLICATIONS AND EXAMPLES

A. Four-qubit singlet state

Let us illustrate how to apply Theorem 1 with an explicit example. In an experimental setting where one expects to produce a four-qubit singlet state (which was e.g. discussed in the context of solving the liar detection problem in Ref. [27]), i.e.

\[
|S_4\rangle = \frac{1}{2\sqrt{3}} (|0011\rangle + 2|1100\rangle - |0110\rangle - |1001\rangle - |1010\rangle - |0101\rangle),
\]

one is confronted with the following expected coefficients: \( c_{0011}, c_{1100}, c_{0111}, c_{1011}, c_{0101}, c_{1001} \). Following the recipe of theorem 1 we now select some coefficient pairs. We could choose e.g. \( R_1 = \{0011, 0101\}, R_2 = \{0111, 1010\}, \) \( R_3 = \{0011, 0110\} \) and \( R_4 = \{0011, 1001\} \), such that \( R = \{R_1, R_2, R_3, R_4\} \). For this selection we use theorem 1 to bound the GME measure. We see that in every subsystem at least two of these pairs appear naturally. Although there are more coefficient pairs we now choose to only take into account two per subsystem entropy and thus choose \( N_{R_1} \) to be the minimal number of coefficient pairs in every subsystem which gives \( N_R = 2 \). Thus we need to add negative terms that compensate for the missing terms just as we did in the three-qubit case, but now we need to do it two times in every subsystem. This results in the following individual prefactors \( N_{R_i} \) for the diagonal elements: \( N_{0011} = 2 \) (as this coefficient appears in two missing pairs in every subsystem), \( N_{0101} = 1 \), \( N_{1001} = 1 \), \( N_{1010} = 1 \) and \( N_{0110} = 1 \) (as those appear maximally once per subsystem entropy). Inserting this in theorem 1 we end up with the lower bound as

\[
E_m(\rho) \geq 2 \sqrt{2}(|\rho_{R_1}| + |\rho_{R_2}| + |\rho_{R_3}| + |\rho_{R_4}|)
\]

\[
-\sqrt{|01110011\rangle\langle01110011| - |01110111\rangle\langle01110111| - |01110001\rangle\langle01110001| - |01110001\rangle\langle01110001|}
\]

\[
\frac{1}{2}(|\rho_{01010101} + |\rho_{01010101} + |\rho_{01010101} + |\rho_{01010101}|)
\]

\[
-\rho_{01010101} \rangle\langle01010101|)
\]

(15)

We have thus created a nonlinear witness function that lower bounds our measure. From an experimental point
of view this is very favorable as few local measurement settings suffice to ascertain the needed thirteen density matrix elements (especially since the nine diagonal elements can be constructed from a single measurement setting). Of course we could also exploit the connection of our lower bound to the Dicke state witness $W_2$ (which is discussed in section IV.C, which also detects GME in this state (although at the cost of more required measurements). In this case even the resistance to white noise is more favorable with our construction method, as for a state $\rho = \alpha |S_{4}\rangle \langle S_{4}| + \frac{|\alpha|^2}{n^2} I$ this exemplary lower bound detects GME until $p = \frac{27}{25} \approx 0.72$, whereas the old witness construction yields a worse resistance up to $p = \frac{37}{35} \approx 0.77$. This shows the versatility of our general approach. By choosing certain coefficients one can tailor these lower bounds to specific experimental situations. If one is confronted with a low noise system it is always beneficial to choose as few coefficients as possible, such that very few local measurements suffice (even a number that is linear in the size of the system is often sufficient). Every additional measurement can then be included in the lower bound and improves the bound and its noise resistance if necessary.

B. Bipartite witnesses and lower bounds on the measure

Although we have presented our theorem and measures in the general case of $n$-qudits, we can always apply the lower bounds also for $n = 2$, as our theorem holds for any $n$ and $d$. Suppose we are given a bipartite qutrit system and want to lower bound the concurrence with only a few local measurements. If the expected state is e.g. $|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle + |22\rangle)$ we can use the lower bounding procedure outlined above, yielding

$$E_m(\rho) \geq \frac{2}{\sqrt{3}} \left( \Re \left[ \langle 00 | \rho | 11 \rangle \right] - \sqrt{\langle 01 | \rho | 01 \rangle \langle 10 | \rho | 10 \rangle} \right)$$

In order to determine the lower bound we have to measure nine different density matrix elements. Of course any density matrix element can always be obtained via local measurements. How these measurements can be performed in a basis consisting of a tensor product of the generalized Gell-Mann matrices we show explicitly in appendix

It turns out that these nine different density matrix elements can be obtained via ten local measurement settings. Let us study the lower bound in the presence of noise. Suppose we have white noise in the system, i.e. $\rho = \alpha |\psi\rangle \langle \psi| + \frac{|\alpha|^2}{n^2} I$. Calculating the lower bound results in $E_m(\rho) \geq \frac{2(4p-1)}{\sqrt{2n}}$, which is equivalent to the analytical expression of Wootters's concurrence for these systems (as proven in Ref. [23]). In this case we have a necessary and sufficient entanglement criterion and a tight lower bound on the concurrence from ten local measurements for a special class of states. Indeed if one generalizes this example to arbitrary dimension $d$, we find that the bound is always tight for bipartite isotropic states.

C. Dicke States

We will now continue to show how this construction relates to an entanglement witness for Dicke-state, which are multi-dimensional generalizations of the W states (which were first introduced in the context of laser emission in Ref. [22]).

In the original article [15], where this approach was first introduced, the authors connected the violation of a witness suitable for GHZ states (first introduced in Ref. [21] and later presented in a more general framework in Ref. [24]) with a lower bound on the measure $E_m$. We want to follow this approach and establish a general connection between a set of witnesses suitable for all generalized Dicke states introduced in Ref. [22] and generalized in Ref. [24]. To that end let us first introduce a concise notation for those states. Let $\alpha$ be a set containing specific subsystems of a multipartite state. We then define the state $|\alpha\rangle$ as a tensor product of states $|l\rangle$ for all subsystems not contained in $\alpha$ and excited states $|l + 1\rangle$ in the subsystems contained in $\alpha$. E.g. for the four-partite state $|\{1,3\}^3\rangle$ we have $|3232\rangle$. Using this abbreviated notation we can define a generalized set of Dicke states, consisting of $n$ $d$-dimensional subsystems, as

$$|D^d_m\rangle = \frac{1}{\sqrt{\binom{n}{m}(d-1)}} \sum_{\alpha} \sum_{|\alpha| = m} |\alpha\rangle,$$

where the parameter $m$ denotes the number of excitations, with $0 < m < n$.

Since the explicit form of the nonlinear witness from Ref. [24] will be used in the following considerations we will repeat it in appendix. For all biseparable states this witness $Q_m^d$ is strictly smaller equal zero, i.e.

$$Q(\rho) \leq 0 \Rightarrow \rho \text{ is biseparable}$$

$$Q(\rho) > 0 \Rightarrow \rho \text{ is multipartite entangled}.$$

Furthermore, the witness can also detect the “dimensionality” of GME, by which we mean the maximal number of degrees of freedom $f_\rho(\rho) \leq d$ that occurs in the pure states of an ensemble constituting $\rho$, minimised over all ensembles (this is the natural generalization of the concept of Schmidt number [18] to multipartite systems, further explored e.g. in Ref. [24]). I.e. the dimensionality
is defined as

\[
f_\rho := \inf_{\{p_i, |\psi_i\rangle\}} \max (\min (\text{rank}(\rho_\gamma)))
\]  

Since

\[
Q_m^{(d)} (\rho) \leq f_\rho - 1, \quad \forall \rho,
\]

we can directly infer that

\[
Q_m^{(d)} (\rho) > f - 2 \Rightarrow f_\rho \geq f
\]

In Fig. 1 we show how \(Q_m^{(d)}\) detects the GME dimensional-\(\alpha\)ity. The maximal violation of these inequalities is always achieved for \(m\)-excitation Dicke states, i.e. \(Q_m^{(d)} (|D_m^{(d)}\rangle \langle D_m^{(d)}|) = d - 1\).

If we can find a proper \(R\), as a result of theorem 11 that uses the Dicke state coefficients, we can connect a lower bound of the measure \(E_m\) with the GME witness \(Q_m^{(d)} (\rho)\). Indeed choosing the ordered subset \(R_\sigma\) of the set of coefficients \(\sigma\) used in [13,23], i.e.

\[
R_\sigma = \left\{ (\alpha^a, \beta^b) \in \sigma : a \leq b \right\},
\]

we immediately arrive at a lower bound on \(E_m\) as

\[
E_m (\rho) \geq m \sqrt{\frac{1}{|R_\sigma| - N_{\sigma n}}} Q_m^{(d)} (\rho) \geq m \sqrt{\frac{1}{|R_\sigma|} Q_m^{(d)} (\rho)},
\]

where \(|R_\sigma| = \frac{1}{2} (d - 1)^2 \binom{n}{m} m (n - m)\). In this case \(N_{\sigma n} \leq m (n - m - 1) + \Theta (d - 3) (n - m)\), where \(\Theta\) is a Heaviside step function.

D. PPT-Witness and Our Witness

Using the result on entanglement across bipartitions from the previous section we can explore the relation of our lower bounds to other bipartite entanglement witnesses. In our witness construction, the permutation operator \(P_\gamma\) acting on a pure state is a \(\gamma\)-partial transpose operator, i.e. \(P_\gamma |\psi\rangle \langle \psi| = (|\psi\rangle \langle \psi|)^T_{\gamma}\) (in the sense that our permutation operator now acts upon the index pairs of the coefficients of the pure state). It is thus intuitive to believe that there is a certain connection between our witness and a PPT-witness [26]. Indeed our witnesses are related to a standard PPT-witness construction (where the witnesses separate the convex set of states that are positive under partial transpose (PPT) from its complement). E.g. for diagonal GHZ states we can use the standard PPT-witness construction which goes as follows. For \(|GHZ_{1,n_2}\rangle := \frac{1}{\sqrt{2}} (|n_1\rangle + |n_2\rangle)\) with \(n_1 + n_2 = (d - 1, \cdots, d - 1)\), we can use the eigenvector belonging to the negative eigenvalue of the \(\gamma\)-partial transposed \(|GHZ_{\bar{n}_1,n_2}\rangle \langle GHZ_{\bar{n}_1,n_2}|^T_{\gamma}\) which we denote as \(|\bar{n}_1, \bar{n}_2\rangle = \frac{1}{\sqrt{2}} (|n_1\rangle - |n_2\rangle)\). One can then construct the PPT-witness and write its expectation value as

\[
\Omega_{\text{ppt}}^{(d)} (\rho, |\bar{\sigma}_1, \bar{\sigma}_2\rangle) = \text{Tr} \left( |\bar{\sigma}_1, \bar{\sigma}_2\rangle \langle \bar{\sigma}_1, \bar{\sigma}_2| T_{\gamma}^{\text{ppt}} \rho \right),
\]

For instance in the three-qubit case,

\[
|\lambda_0, 1, 1\rangle \langle \lambda_0, 1, 1| T_{\gamma}^{3/2} =
\]
With the PPT-witness construction in eq. (20) we end up with the following PPT-witness expectation value
\[
\Omega_{\text{ppt}}^{\gamma}(\rho, |\lambda_{\text{GHZ}}\rangle) = \frac{1}{2} \left( \rho_{\gamma_1\gamma_1}^{\gamma_2} + \rho_{\gamma_2\gamma_2}^{\gamma_1} - \text{Re}(\rho_{\gamma_1\gamma_2}) \right).
\]
Under the fixed bipartition $\gamma|\bar{\gamma}$, we construct our witness by choosing $R = (\gamma_1, \gamma_2)$ as
\[
-W_{(\gamma_1, \gamma_2)}^\gamma(\rho) = \sqrt{\rho_{\gamma_1\gamma_1}^{\gamma_2} \rho_{\gamma_2\gamma_2}^{\gamma_1} - \rho_{\gamma_1\gamma_2}}.
\]
It is obvious that $-W_{(\gamma_1, \gamma_2)}^\gamma(\rho) \leq \Omega_{\text{ppt}}^{\gamma}(\rho, |\lambda_{\text{GHZ}}\rangle)$. Hence we say that the witness $W_R(\rho)$ is stronger than the PPT-witness $\Omega_{\text{ppt}}^{\gamma}(\rho, |\lambda_{\text{GHZ}}\rangle)$.

The relation between our witness, the PPT-witness and the PPT-convex set is illustrated in fig. 2. For clarity we just draw two PPT-witnesses in the figure. For the $n$-qudit case there are $2^n$ such eigenvectors $|\lambda_{\gamma_1, \gamma_2}\rangle$, corresponding to negative eigenvalues. Every witness $\Omega_{\text{ppt}}^{\gamma}(\rho, |\lambda_{\gamma_1, \gamma_2}\rangle)$ is tangent to the set of PPT states (i.e. there exists one PPT state for which the witness yields zero). However also our witness $W_R(\rho)$ is zero for all these PPT states, i.e. our new witness detects more states than the traditional PPT-witness.

V. CONCLUSIONS

In conclusion we have presented a method to derive lower bounds on a measure of genuine multipartite entanglement. We show that in experimentally plausible scenarios (i.e. one knows which state one aims to produce) we can derive such lower bounds simply based on coefficients of the corresponding pure states. We also connected the lower bound construction to a framework of nonlinear entanglement witnesses developed in Refs. [20–24]. These witnesses are experimentally feasible in terms of required local measurement settings. We provide further evidence in the bipartite case, where we also show that for certain families of mixed states our lower bounds are tight.

Some open questions remain, such as whether this general construction method will work for all kinds of states and how it can be generalized beyond just multi- and bipartite entanglement, but anything in between. We want to point out that recently also other authors have used a similar approach to bound this measure in the multipartite case [25] and for multipartite $W$ states [26].

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Appendix: Proofs

1. The Formulas Used in The Main Article

a. Reduced linear entropy of pure states

Let \( |\psi\rangle = \sum_{\eta_1, \eta_2 \in \mathbb{N}^n} c_{\eta_1} |\eta_1\rangle \) be an \( n \)-qudit pure state. The linear entropy of \( |\psi\rangle \) can be written as

\[
S_L (\rho_\gamma) = \sum_{\eta_1, \eta_2 \in N^m} |c_{\eta_1} c_{\eta_2} - c_{\eta_1}^* c_{\eta_2}^*|^2 ,
\]

(A.1)

where \( (\eta_1, \eta_2) = P_\gamma (\eta_1, \eta_2) \).

Proof. The linear entropy regarding a specific partition \( \gamma |\gamma \rangle \) is defined as \( S_L (\rho_\gamma) = 2 (1 - tr (\rho_\gamma^2)) \), where \( \rho_\gamma \) is the \( \gamma \)-reduced matrix of \( \rho \). The trace of \( \rho_\gamma \) is \( tr (\rho^2) = \sum_{\alpha_1, \alpha_2 \in H_\gamma} (\rho_\gamma)_{\alpha_1 \alpha_2} (\rho_\gamma)_{\alpha_2 \alpha_1} \), where \( H_\gamma \) is the subspace of the reduction \( \gamma \). We separate the summation into diagonal and off-diagonal parts. For the diagonal part we use the normalization condition to evaluate its value.

\[
tr (\rho^2) = \sum_{\alpha_1 = \alpha_2} (\rho_\gamma)_{\alpha_1 \alpha_1} + \sum_{\alpha_1 \neq \alpha_2} (\rho_\gamma)_{\alpha_1 \alpha_2} (\rho_\gamma)_{\alpha_2 \alpha_1} + \sum_{\alpha_1 \neq \alpha_2 \in H_\gamma} (\rho_\gamma)_{\alpha_1 \alpha_2} (\rho_\gamma)_{\alpha_2 \alpha_1} = 1 - \sum_{\alpha_1 \neq \alpha_2 \in H_\gamma} \sum_{\beta_1, \beta_2 \in H_\gamma} |c_{\alpha_1 \beta_1} c_{\alpha_2 \beta_2} - c_{\alpha_1 \beta_2} c_{\alpha_2 \beta_1}|^2 + \sum_{\alpha_1 \neq \alpha_2 \in H_\gamma} \sum_{\beta_1, \beta_2 \in H_\gamma} |c_{\alpha_1 \beta_1} c_{\alpha_2 \beta_2} - c_{\alpha_1 \beta_2} c_{\alpha_2 \beta_1}|^2 . \]

By exchanging the indices \( \alpha_1 \) and \( \alpha_2 \) one has

\[
tr (\rho^2) = 1 - \frac{1}{2} \sum_{\alpha_1, \alpha_2 \in H_\gamma} \sum_{\beta_1, \beta_2 \in H_\gamma} |c_{\alpha_1 \beta_1} c_{\alpha_2 \beta_2} - c_{\alpha_1 \beta_2} c_{\alpha_2 \beta_1}|^2 , \]

(A.2)

where \( \eta = \alpha \otimes \beta \) and \( (\eta_1^\gamma, \eta_2^\gamma) = P_\gamma (\eta_1, \eta_2) \). The linear entropy is then calculated to

\[
S_L (\rho_\gamma) = \sum_{\eta_1 \neq \eta_2 \in N^m} |c_{\eta_1} c_{\eta_2} - c_{\eta_1}^* c_{\eta_2}^*|^2 , \]

(A.4)

b. An Important Inequality

The following is an inequality, which is crucial for derivation of the prefactor \( \sqrt{\frac{1}{|R| \cdot N_R}} \) in the theorem [1]

\[
|I| \sum_{i \in I} |a_i|^2 \geq \sum_{i \in I} |a_i|^2 . \]

(A.5)

Proof. We prove this inequality by constructing two vectors as follows (using \( |I| = n \))

\[
\bar{x} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{pmatrix} \]

(A.6)

The right hand side of (A.5) can be written as the scalar product of \( \bar{x} \) and \( \bar{y} \).

\[
\sum_{i \in I} |a_i|^2 = \sum_{i, j \in I} a_i a_j^* = |\bar{x} \cdot \bar{y}| . \]

(A.7)

According to the Cauchy-Schwarz inequality, one can derive

\[
|I| \sum_{i \in I} a_i^* = |\bar{x} \cdot \bar{y}| \leq \sum_{i \in I} |a_i|^2 . \]

(A.8)

2. Proof of Theorem [1] and Approach of Construction of a GME Witness

Firstly one can estimate the lower bound on \( S_L (\rho_\gamma) \) by summing its elements over a selected Region \( R \), and dropping the other non-negative summands (i.e. lower bounding them with 0),

\[
S_L (\rho_\gamma) \geq 4 \sum_{(\eta_1, \eta_2) \in R} |c_{\eta_1}^* c_{\eta_2} - c_{\eta_1} c_{\eta_2}^*|^2 . \]

(A.9)

Here we add a prefactor \( 4 \) in eq. (A.9), since the symmetric factor of all \( (\eta_1, \eta_2) \) equals 4. That means for every \( (\eta_1, \eta_2) \) there are three other \( (\eta_1, \eta_2) \) having the same value of \( |c_{\eta_1} c_{\eta_2} - c_{\eta_1}^* c_{\eta_2}^*| \) as \( (\eta_1, \eta_2) \). Here we choose a non-degenerate vector basis set \( R \), and therefore need a prefactor 4 in the lower bound. The set \( R^T \) is the subset of
According to eq. (II A) together with eq. (A.11), the lower terms in the density matrix elements. Therefore we estimate the sum-
the minimal value of $\gamma_m \sqrt{\eta_1, \eta_2}$. 

$$S_L(\rho^I_i) \geq \frac{4}{|R| - |R'|} \left( \sum_{n_1, n_2 \in R} |c_{\eta_1}^i c_{\eta_2}^i - P_i c_{\eta_1}^i c_{\eta_2}^i| \right)^2,$$

(A.10)

$$\Downarrow$$

$$\sqrt{S_L(\rho^I_i)} \geq 2 \sqrt{\frac{1}{|R| - |R'|}} \sum_{n_1, n_2 \in R} \left| c_{\eta_1}^i c_{\eta_2}^i - c_{\eta_1}^i c_{\eta_2}^i \right|.$$  

(A.11)

According to eq. (II A) together with eq. (A.11), the lower bound reads

$$E_m \geq 2 \sum_{(p_i, \psi_i)} p_i \left[ \frac{1}{|R| - |R'|} \sum_{n_1, n_2 \in R} \left( |c_{\eta_1}^i c_{\eta_2}^i| - |c_{\eta_1}^i c_{\eta_2}^i| \right) \right].$$

(A.12)

where $\gamma_i$ is the partition in which the linear entropy $S_L(\{|\psi_i\rangle\rangle_\gamma\})$ of $\{|\psi_i\rangle\rangle_\gamma\}$ has its minimum. By defining the normalization factor $N_R := \min_{\gamma} |R^\gamma|$, which is the minimal value of $|R^\gamma|$ over all possible bipartitions $\{|\gamma\rangle\}$, we can extract the prefactor from the convex roof summation.

$$E_m \geq 2 \sum_{(p_i, \psi_i)} p_i \left[ \min_{\gamma} \left( \frac{1}{|R| - |R'|} \sum_{n_1, n_2 \in R} \left( |c_{\eta_1}^i c_{\eta_2}^i| - |c_{\eta_1}^i c_{\eta_2}^i| \right) \right) \right].$$

(A.13)

The most difficult part of detecting entanglement of mixed states is a result of the mixing of the decomposition coefficients $c_{\eta_1}^i c_{\eta_2}^i$. In the lab we have only the information about the mixed density matrix element $\rho_{n_1 n_2}$ but not $c_{\eta_1}^i c_{\eta_2}^i$, therefore we must exchange the two summations in eq. (A.13), and mix the coefficients $c_{\eta_1}^i c_{\eta_2}^i$ into density matrix elements. Therefore we estimate the summands with a bound, which is independent of the specific partition $\gamma_i$, by adding a summation of non-positive terms $\sum_{R^\gamma} \left[ |c_{\eta_1}^{i,1} c_{\eta_2}^{i,1}| - \frac{1}{2} \left( |c_{\eta_1}^{i,2}|^2 + |c_{\eta_2}^{i,2}|^2 \right) \right]$ into the summands.

$$\sum_{(n_1, n_2) \in R \in R'} \left[ |c_{\eta_1}^{i,1} c_{\eta_2}^{i,1}| - \frac{1}{2} \left( |c_{\eta_1}^{i,2}|^2 + |c_{\eta_2}^{i,2}|^2 \right) \right]$$

where $I(R) := \{ \eta \in \mathbb{N}_d^2 : \exists (\eta', \eta') \in (\eta, \eta') \cap R \}$ is the set of indices contained in the set $R$, $\Gamma(n_1, n_2) = \{ \gamma | P(n_1, n_2) \notin R \}$ and $n_0^{\eta}$ is the number of vector pairs in $R^\eta$ containing index $\eta$. In order to eliminate the dependence of the partition $\gamma_i$, we define the maximal value of $n_0^\eta$ over all possible partitions $\{|\gamma\rangle\}$ as $N_0 := \max_\eta n_0^\eta$. Then one can estimate the GME measure with eq. (A.13) and (A.14) as

$$E_m(\rho) \geq 2 \left[ \frac{1}{|R| - N_R} \sum_{n_1, n_2 \in R} \inf_{(p_i, \psi_i)} \sum_{\gamma \in \Gamma(n_1, n_2)} p_i \left( |c_{\eta_1}^i c_{\eta_2}^i| - \sum_{\gamma \in \Gamma(n_1, n_2)} |c_{\eta_1}^i c_{\eta_2}^i| \right) \right].$$

(A.15)

Now one can safely exchange the summation in eq. (A.15) and lower bound it with the triangle inequality (i.e. $\sum_{n_1, n_2} p_i |c_{\eta_1}^i c_{\eta_2}^i| \geq |\rho_{n_1 n_2}|$) and the Cauchy-Schwarz inequality (i.e. $\sum_{n_1, n_2} p_i |c_{\eta_1}^i c_{\eta_2}^i|^2 \leq \sqrt{\sum_{n_1, n_2} p_i |c_{\eta_1}^i c_{\eta_2}^i|^2}$). Finally we arrive at the result

$$E_m(\rho) \geq 2 \left[ \frac{1}{|R| - N_R} \sum_{n_1, n_2 \in R} \sum_{\gamma \in \Gamma(n_1, n_2)} p_i \rho_{n_1 n_2} \sum_{\gamma \in \Gamma(n_1, n_2)} \sqrt{p_{\eta_1} p_{\eta_2} |p_{\eta_1} p_{\eta_2}|} \right].$$

(A.16)

where $\rho_{n_1 n_2} := \langle \eta_1 | \rho | \eta_2 \rangle$.

Above is the proof of theorem IV in the case of $N_R := \min_\eta |R^\eta|$. For the choice of $N_R := \max_\eta |R^\eta|$, one just needs to calculate $\max_\eta |R^\eta|$ at the first step, i.e. eq. (A.10), then pick up $|R| - \max_\eta |R^\eta|$ elements from $R(R')$ as summation region in the second line and then repeat the whole proof above. At the end we will attain the same expression for the lower bound on $E_m$, as eq. (A.16), but with different $N_0$ from the ones before $N_0 := \min_\eta |R^\eta|$. $N_0$ in this maximum choice is greater or equal to the one derived in the minimal-case. In the four-qubit singlet example in sec. IV A, the value of $N_0$ is exactly the same for both choices. Therefore we choose the maximum, i.e. $N_R = 2$, to get a tighter lower bound on $E_m$.

3. Explicit decomposition of the bipartite witness into local observables

The measurements needed to ascertain the relevant density matrix elements in the bipartite scenario can be performed in a basis consisting of a tensor product of the generalized Gell-Mann matrices. We continue to provide for each of the density matrix elements above their respective coefficients. The density matrix elements are either off diagonal elements or diagonal elements. The
off-diagonal elements can be obtained by expectation values of the symmetric and antisymmetric generalized Gell-Mann matrices:

\[
\Lambda^1_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda^2_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^3_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda^4_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\] (A.17)

They can be written as follows:

\[\Re \left[ \langle 00 \vert \rho \vert 11 \rangle \right] = \frac{1}{2} \langle \Lambda^1_4 \otimes \Lambda^1_4 - \Lambda^1_4 \otimes \Lambda^1_4 \rangle, \quad \Re \left[ \langle 00 \vert \rho \vert 22 \rangle \right] = \frac{1}{2} \langle \Lambda^1_4 \otimes \Lambda^1_4 - \Lambda^1_4 \otimes \Lambda^1_4 \rangle, \quad \Re \left[ \langle 11 \vert \rho \vert 22 \rangle \right] = \frac{1}{2} \langle \Lambda^1_4 \otimes \Lambda^1_4 - \Lambda^1_4 \otimes \Lambda^1_4 \rangle.\] (A.19)

We now consider the terms obtained via the diagonal generalized Gell-Mann matrices:

\[
\Lambda^0_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Lambda^1_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Lambda^2_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

We will expand the soughtafter terms into coefficients, utilizing the following basis:

\[
b = \begin{pmatrix} \Lambda^0_0 \otimes \Lambda^0_0 \\ \Lambda^0_0 \otimes \Lambda^1_1 \\ \Lambda^0_0 \otimes \Lambda^2_2 \\ \Lambda^0_0 \otimes \Lambda^3_3 \\ \Lambda^0_0 \otimes \Lambda^4_4 \\ \Lambda^1_1 \otimes \Lambda^1_1 \\ \Lambda^1_1 \otimes \Lambda^2_2 \\ \Lambda^1_1 \otimes \Lambda^3_3 \\ \Lambda^1_1 \otimes \Lambda^4_4 \\ \Lambda^2_2 \otimes \Lambda^2_2 \\ \Lambda^2_2 \otimes \Lambda^3_3 \\ \Lambda^2_2 \otimes \Lambda^4_4 \\ \Lambda^3_3 \otimes \Lambda^3_3 \\ \Lambda^3_3 \otimes \Lambda^4_4 \\ \Lambda^4_4 \otimes \Lambda^4_4 \end{pmatrix}, \quad \texttt{For further reference the coefficients are given as:}
\]

\[
\langle 01 \vert \rho \vert 01 \rangle = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \langle 10 \vert \rho \vert 10 \rangle = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \langle 02 \vert \rho \vert 02 \rangle = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \langle 12 \vert \rho \vert 12 \rangle = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \langle 21 \vert \rho \vert 21 \rangle = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We now recall the explicit form of the nonlinear witness from Ref. [24]. Using the notation for Dicke states introduced in section IV.C, we arrive at the following lower bound

\[
Q^{(d)}_m = \frac{1}{m} \sum_{l'=0}^{d-2} \sum_{\sigma} \left( \left| \langle \alpha^l \vert \rho \vert \beta^{l'} \rangle \right|^2 - \sum_{\delta \in \Delta} \sqrt{\langle \alpha^l \vert \rho \vert \beta^{l'} \rangle \langle \beta^{l'} \vert \rho \otimes \rho^P \vert \alpha^l \rangle} \right) - N_D \sum_{l=0}^{d-2} \sum_{\sigma} \langle \alpha^l \vert \rho \vert \alpha^l \rangle, \quad (A.23)
\]

with

\[
m \in \{1, \ldots, \lfloor n/2 \rfloor \}, \quad N_D = (d - 1) m (n - m - 1),
\]

\[
\sigma := \{ (\alpha, \beta) : |\alpha \cap \beta | = m - 1 \},
\]

\[
\Delta := \left\{ \begin{array}{ll} \alpha & , l' = l \\ \delta \in \frac{\alpha}{|\alpha|} & , l' < l \\ \delta \in \frac{\beta}{|\beta|} & , l' > l \end{array} \right\}.
\]

The properties of this witness are discussed in the main text.