BPS Vortices, $Q$-balls, and $Q$-vortices in $\mathcal{N} = 6$ Chern-Simons Matter Theory

Gyungchoon Go$^1$, Chanju Kim$^2$, Yoonbai Kim$^3$, O-Kab Kwon$^3$, Hiroaki Nakajima$^4$

$^1$Center for Nanotubes and Nanostructured Composites, Sungkyunkwan University, Suwon 440-746, Korea
gcgo@skku.edu

$^2$Institute for the Early Universe and Department of Physics, Ewha Womans University, Seoul 120-750, Korea
cjkim@ewha.ac.kr

$^3$Department of Physics, BK21 Physics Research Division, and Institute of Basic Science Sungkyunkwan University, Suwon 440-746, Korea
yoonbai, okab@skku.edu

$^4$Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan, R.O.C.
nakajima@phys.ntu.edu.tw

Abstract

We investigate the vortex-type BPS equations in the ABJM theory without and with mass-deformation. We systematically classify the BPS equations in terms of the number of supersymmetry and the R-symmetries of the undeformed and mass-deformed ABJM theories. For the undeformed case, we analyze the $\mathcal{N} = 2$ BPS equations for $U(2) \times U(2)$ gauge symmetry and obtain a coupled differential equation which can be reduced to either Liouville- or Sinh-Gordon-type vortex equations according to the choice of scalar functions. For the mass-deformed case with $U(N) \times U(N)$ gauge symmetry, we obtain some number of pairs of coupled differential equations from the $\mathcal{N} = 1, 2$ BPS equations, which can be reduced to the vortex equations in Maxwell-Higgs theory or Chern-Simons matter theories as special cases. We discuss the solutions. In $\mathcal{N} = 3$ vortex equations Chern-Simons-type vortex equation is not allowed. We also show that $\mathcal{N} = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ BPS equations are equivalent to those with higher integer supersymmetries.
Contents

1 Introduction

2 ABJM Theory without and with Mass Deformation

3 Vortex-type Objects with $\mathcal{N}=3$ Supersymmetry
   3.1 BPS equations and bound
   3.2 BPS objects in the massless theory
   3.3 BPS objects in the mass-deformed theory

4 Vortex-type Objects with $\mathcal{N}=2$ Supersymmetry
   4.1 BPS equations and bound
   4.2 BPS objects in the massless theory
   4.3 BPS objects in the mass-deformed theory
      4.3.1 $U(2)\times U(2)$ gauge group
      4.3.2 $U(3)\times U(3)$ gauge group
      4.3.3 $U(N)\times U(N)$ gauge group

5 Vortex-type Objects with $\mathcal{N}=1$ Supersymmetry
   5.1 BPS equations and bound
   5.2 BPS objects in the mass-deformed theory

6 Absence of Vortex-type Objects with $\mathcal{N} = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ Supersymmetries
   6.1 $\mathcal{N} = \frac{5}{2}$ supersymmetry
   6.2 $\mathcal{N} = \frac{3}{2}$ supersymmetry
      6.2.1 $\omega_{12} = 0$, $\gamma^i \omega_{13} = \omega_{24}$ case
      6.2.2 $\gamma^1 \omega_{12} = \omega_{34}$, $\gamma^1 \omega_{13} = \omega_{42}$, $\gamma^1 \omega_{14} = \omega_{23}$ case
   6.3 $\mathcal{N} = \frac{1}{2}$ supersymmetry

7 Conclusion

A Vortex-type Objects with Other $\mathcal{N} = 1, 2$ Supersymmetries
   A.1 $\mathcal{N} = 2$ supersymmetry ($\omega_{13} = 0$)
   A.2 $\mathcal{N} = 1$ supersymmetry ($\omega_{12} = \omega_{13} = 0$)

1 Introduction

After the first construction of the $\mathcal{N} = 8$ superconformal Chern-Simons theory (SCS) based on the three algebra by Bagger-Lambert-Gustavsson (BLG) [1, 2], an $\mathcal{N} = 6$ SCS theory with
U(N)×U(N) gauge symmetry was constructed by Aharony-Bergman-Jafferis-Maldacena (ABJM) [3]. The latter theory includes a large class of SCS theories depending on the rank of the gauge group N and the Chern-Simons level k. This ABJM theory was proposed as a low energy effective action of N coincident M2-branes on the C^4/Z_k orbifold fixed point and was conjectured to be dual to type IIA string theory on AdS_4 × CP^3 (N^1/2 ≪ k ≪ N) or M-theory on AdS_4 × S^7/Z_k (k ≪ N^1/2) in the large N limit.

There has been significant progress in understanding the dynamics of M2-branes in M-theory by the BLG and ABJM theories. One direction of this progress might be obtaining solitonic objects which can be identified with M-theory branes, such as M2- and M5-branes. In the BLG theory, the composite of M2- and M5-branes [4, 5], the domain wall solutions [6], and some vortex-type BPS configurations [7] were obtained without and with mass-deformation and also possible BPS equations were classified [8, 9]. Similarly, in the ABJM theory, the composite of M-branes and domain wall solutions [10, 11, 12], the vortex-type solutions [13] [14, 15], and the classification of BPS conditions of intersecting M-branes [16] were studied. For the vortex-type solutions in the ABJM theory, N = 1 Chern-Simons vortex-type BPS equations [13] and Yang-Mills Higgs vortex-type N = 3 BPS ones were obtained [14, 15] and the existence of the corresponding vortices with N = 3 supersymmetry was discussed [17]. Vortex solutions in the nonrelativistic limit of the ABJM theory have been studied in Ref. [18].

As we already know, the mass-deformed BLG and ABJM theories have sextic bosonic potentials with the symmetric and broken vacua. In this reason, these theories can be understood as more complicated Chern-Simons-Higgs theories. So in specific limits of the vortex-type BPS equations in the BLG and ABJM theory, one can obtain the vortex configurations which were widely studied in Abelian and non-Abelian Chern-Simons-Higgs theories [19, 20, 21, 22, 23].

In Ref. [16] the BPS equations preserving various supersymmetries in the undeformed ABJM theory were classified and the corresponding BPS configurations were interpreted as known BPS objects in M-theory. In this paper, we recapitulate the classification of the vortex-type BPS configurations of the undeformed and mass-deformed ABJM theories in terms of the number of remaining supersymmetry and SU(4) and SU(2)×SU(2)×U(1) R-symmetries, respectively. Then we reproduce the resulting BPS equations by reshuffling the energy expression, thereby obtaining the BPS energy bound. An unusual quantity we consider here is the stress tensor which is the spatial component of the energy-momentum tensor. It has been argued [24] from the viewpoint of supersymmetry algebra that the stress vanishes for BPS configurations. We explicitly check it for the ABJM theory. We will see that this can actually be a useful method to obtain consistency conditions for BPS equations in case of lower supersymmetries.

The main concern of the paper is to study possible solutions of the vortex-type BPS equations for various supersymmetries. We have already considered the half-BPS (N = 3) case in [14]. In this paper we extend our analysis to less supersymmetric cases: N = 2, 1 and N = 5/2, 3/2, 1/2. Since the BPS equations get complicated as the number of supersymmetries are smaller, it would not be feasible to find the most general solutions. With suitable ansatizes, however, we show that
for some $\mathcal{N} = 2$ and $\mathcal{N} = 1$ cases the equations reduce to well-known equations such as the vortex equation in the Maxwell-Higgs theories and Chern-Simons-Higgs theories [19, 21, 22]. We compute the energies and the angular momenta for the solutions.

For half-integer supersymmetric cases, we show that the supersymmetries of the solutions to the BPS equations in mass-deformed theory are actually enhanced to integer ones. For example, $\mathcal{N} = 5/2$ BPS equations are shown to be identical to $\mathcal{N} = 3$ BPS equations. Also $\mathcal{N} = 3/2$ cases are enhanced to either $\mathcal{N} = 3$ or $\mathcal{N} = 2$ cases depending on the supersymmetric conditions. Similarly $\mathcal{N} = 1/2$ BPS equations are equivalent to $\mathcal{N} = 1$ BPS equations. It turns out that the stress tensor is a useful quantity to show these enhancement.

The remaining part of this paper is organized as follows. In section 2 we briefly review the ABJM theory without and with mass-deformation to fix the notation. In section 3 we recapitulate the $\mathcal{N} = 3$ vortex-type BPS equations obtained in Refs. [14, 15] and consider more general BPS configurations with finite energy bounds in the mass-deformed case. In sections 4 and 5 we systematically analyze the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ BPS equations, respectively. For the undeformed case in section 4, we obtain a coupled differential equation which can be reduced to either Liouville- or Sinh-Gordon-type vortex equations in special choices of scalar functions. For the mass-deformed case, we obtain some number of pairs of coupled differential equations which have appeared in $U(1)\times U(1)$ Chern-Simons system [25]. We show that these differential equations are reduced to vortex equations in Maxwell-Higgs theory or Chern-Simons matter theories, according to the choices of scalar functions. Difference between $\mathcal{N} = 2$ case and $\mathcal{N} = 1$ case is also discussed. In section 6 we show that the $\mathcal{N} = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ BPS equations are equivalent to those with higher integer supersymmetries. Section 7 is devoted to conclusions and discussions. In appendix A we add the analyses of two more cases of BPS vortex equations with $\mathcal{N} = 1, 2$ symmetries.

2 ABJM Theory without and with Mass Deformation

In this section we briefly review the ABJM theory and its mass deformation to fix the notation. The ABJM theory is an $\mathcal{N} = 6$ superconformal $U(N)\times U(N)$ Chern-Simons gauge theory with level $(k, -k)$, coupled to four complex scalars and four fermions in the bifundamental representation,

$$S_{\text{ABJM}} = \int d^3x \left\{ \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) 
- \text{tr} \left( D_\mu Y_\lambda^A D^\mu Y^\lambda A \right) + \text{tr} \left( \psi^A i \gamma^\mu D_\mu \psi_A \right) - V_{\text{ferm}} - V_0 \right\}, \quad (2.1)$$

where $A = 1, \ldots, 4$ and

$$D_\mu Y^A = \partial_\mu Y^A + i A_\mu Y^A - i Y^A \hat{A}_\mu.$$
We choose real gamma matrices $\gamma^\mu$ with the convention $\gamma^2 = \gamma^0\gamma^1$. An explicit representation would be

$$\gamma^0 = i\sigma^2, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^3. \quad (2.2)$$

The products of spinors are expressed by $\xi\chi \equiv \xi^\alpha\chi_\alpha$ and $\xi\gamma^\mu\chi = \xi^\alpha\gamma_\mu^\alpha\beta\chi_\beta$ with explicit spinor indices for two component spinors $\xi$ and $\chi$.

In the action (2.1), $V_{\text{ferm}}$ is the Yukawa-type quartic-interaction term,

$$V_{\text{ferm}} = \frac{2i\pi k}{k} \text{tr}(Y_A^\dagger Y_A Y_B^\dagger Y_B Y_C^\dagger Y_C + 2Y_A^\dagger Y_A Y_B^\dagger Y_B Y_C^\dagger Y_C - 2Y_A^\dagger Y_B^\dagger Y_A Y_B Y_C^\dagger Y_C - 2Y_A^\dagger Y_B^\dagger Y_A Y_B Y_C^\dagger Y_C),$$

and $V_0$ is the sixth-order scalar potential,

$$V_0 = -\frac{4\pi^2}{3k^2} \text{tr}(Y_A^\dagger Y_A Y_B^\dagger Y_B Y_C^\dagger Y_C + 2Y_A^\dagger Y_A Y_B^\dagger Y_B Y_C^\dagger Y_C - 2Y_A^\dagger Y_B^\dagger Y_A Y_B Y_C^\dagger Y_C),$$

It can be written in a manifestly positive-definite form [26, 27],

$$V_0 = \frac{2}{3} \text{tr} \left| \beta^{BC}_A + \delta^{[B}_A \beta^{C]}_D \right|^2, \quad (2.5)$$

where we have introduced the notation $|\mathcal{O}|^2 \equiv \mathcal{O}^\dagger \mathcal{O}$, and $\beta^{AB}_C$ is defined by

$$\beta^{AB}_C = \frac{4\pi k}{k} Y_A^\dagger Y_C Y_B^\dagger.$$

By adding mass terms to the action, the theory can be deformed [28, 29] in the unique way which preserve the full $\mathcal{N} = 6$ supersymmetry [6],

$$\Delta V_{\text{ferm}} = \mu \psi^A M^B_A \psi_B,$$

$$\Delta V_0 = \text{tr} \left( \frac{4\pi k}{k} Y_A^\dagger Y_A Y_B^\dagger M^C_B Y_C^\dagger - \frac{4\pi k}{k} Y_A^\dagger Y_A Y_B^\dagger M^B_C Y_C^\dagger + \mu^2 Y_A^\dagger Y_A \right),$$

where $\mu$ is the mass deformation parameter and $M^B_A = \text{diag}(1,1,-1,-1)$. Combined with (2.3), the potential $V_m$ in the mass-deformed theory can also be written in a manifestly positive-definite form [14],

$$V_m = V_0 + \Delta V_0 = \frac{2}{3} \text{tr} \left| \beta^{BC}_A + \delta^{[B}_A \beta^{C]}_D + \mu M^B_A \right|^2, \quad (2.8)$$

It is not difficult to see that the theory is invariant under the following $\mathcal{N} = 6$ supersymmetry
transformation [3, 6, 30, 10],

\[
\delta Y_A = i\omega^{AB}\psi_B, \\
\delta \psi_A = \gamma^\mu\omega_{AB}D_\mu Y^B + \frac{2\pi}{k} \left[ -\omega_{AB}(Y^C\psi^B_C - Y^B\psi^A_C) + 2\omega_{BC}Y^BY^A_C \right] + \mu M^B_A\omega_{BC}Y^C \\
= \gamma^\mu\omega_{AB}D_\mu Y^B + \omega_{BC} \left( \beta^{BC} + \delta^B_A\beta^C_D \right) + \mu M_A^B\omega_{BC}Y^C, \\
\delta A_\mu = -\frac{2\pi}{k}(Y^A\psi^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A^BY^B), \\
\delta A_\mu = \frac{2\pi}{k}(\psi^A\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A^BY^B),
\]

where \(\omega_{AB}\) are supersymmetry transformation parameters with

\[
\omega^{AB} = \omega_\omega = \frac{1}{2}\epsilon^{ABCD}\omega_{CD}.
\]

Note that the mass deformation affects only the transformation of the fermion fields by an additional term,

\[
\delta m\psi_A = \mu M_A^B\omega_{BC}Y^C.
\]

Equation (2.7) is not the only form of the mass-deformed theory. One can also get mass-deformed theories in \(\mathcal{N} = 1\) or \(\mathcal{N} = 2\) superfield formalism for which only part of the supersymmetry is manifest. It can, however, be shown [14] that they are all equivalent to (2.7) by a suitable field redefinition.

From (2.8) the vacuum equation of the mass-deformed theory is

\[
\beta^{BC} + \delta^B_A\beta^C_D + \mu M^{[B}_{[A}\beta^{Y^C]} = 0, \tag{2.12}
\]

which reduces to [29, 14]

\[
\beta^{ab} + \mu Y^b = 0, \\
\beta^{pq} - \mu Y^q = 0, \\
\beta^{ba} = \beta^{qa} = \beta^{pq} = \beta^{ab} = 0. \tag{2.13}
\]

where \(a, b = 1, 2\) and \(p, q = 3, 4\). The general solution of these vacuum equations was found in [29]
and refined in \[31\],

\[
Y^a = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix}
0_{m_1 \times (m_1+1)} & \cdots & 0_{m_1 \times (m_1+1)} \\
\mathcal{M}^{(m_1+1)}_{a} & \cdots & \mathcal{M}^{(m_f)}_{a}
\end{pmatrix},
\]

\[
Y^{a+2} = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix}
\mathcal{M}^{(m_1)}_{a} & \cdots & \mathcal{M}^{(m_f)}_{a} \\
0_{(m_1+1) \times m_{I+1}} & \cdots & 0_{(m_f+1) \times m_f}
\end{pmatrix}, \tag{2.14}
\]

where \(\mathcal{M}^m_a\) is an \(m \times (m + 1)\) matrix,

\[
\mathcal{M}^m_1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix}
\sqrt{m} & 0 & 0 & \cdots & \sqrt{2} & 0 & 1 & 0 \\
0 & \sqrt{m-1} & 0 & \cdots & 0 & \sqrt{2} & 0 & 1
\end{pmatrix}, \quad \mathcal{M}^m_2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix}
0 & 1 & \sqrt{2} & \cdots & \sqrt{m-1} & 0 & \sqrt{m} \\
0 & 0 & \sqrt{2} & \cdots & 0 & \sqrt{m} & 0
\end{pmatrix}. \tag{2.15}
\]

For the U\((N) \times U(N)\) gauge group we have the following constraints

\[
\sum_{m=0}^{\infty} [m\tilde{N}_m + (m + 1)\tilde{N}_m] = N, \quad \sum_{m=0}^{\infty} [(m + 1)\tilde{N}_m + m\tilde{N}_m] = N, \tag{2.16}
\]

where \(\tilde{N}_m\) and \(\tilde{N}_m\) denote the numbers of block of \(\mathcal{M}^{(m)}_a\) and \(\mathcal{M}^{(m)}_a\)-types, and \(\tilde{N}_0\) and \(\tilde{N}_0\) represent the numbers of empty columns and empty rows, respectively.

Since we are interested in the classical vortex-type configurations, we consider the Euler-Lagrange equations of gauge fields \(A^\mu\) and \(\hat{A}^\mu\)

\[
\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} F_{\nu\lambda} = j^\mu, \quad \frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \hat{F}_{\nu\lambda} = -\hat{j}^\mu, \tag{2.17}
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \quad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu],
\]

\[
j^\mu = i \left( (D^\mu Y) Y^A - Y^A (D^\mu Y^A) \right), \quad \hat{j}^\mu = i \left( (D^\mu Y^A) Y^A - Y^A (D^\mu Y^A) \right). \tag{2.18}
\]
The U(1) currents are obtained by taking trace,

\[ j_{U(1)}^\mu = \text{tr} j^\mu, \quad \hat{j}_{U(1)}^\mu = \text{tr} \hat{j}^\mu, \]

and the corresponding charges are

\[ Q = \int d^2 x j^0_{U(1)}, \quad \hat{Q} = \int d^2 x \hat{j}^0_{U(1)}. \]

The Gauss’ laws are the time components of (2.17)

\[ B = \frac{2\pi}{k} j^0 = i \frac{2\pi}{k} \left[ Y^A (D^0 Y_A^\dagger) - (D^0 Y_A) Y_A^\dagger \right], \]
\[ \hat{B} = -\frac{2\pi}{k} \hat{j}^0 = -i \frac{2\pi}{k} \left[ Y_A^\dagger (D^0 Y^A) - (D^0 Y^A) Y_A^\dagger \right], \]

where \( B = \partial_1 A_2 - \partial_2 A_1 + i [A_1, A_2] \) and \( \hat{B} = \partial_1 \hat{A}_2 - \partial_2 \hat{A}_1 + i [\hat{A}_1, \hat{A}_2] \) are magnetic fields. The spatial integral of the left-hand sides of (2.21)–(2.22) gives non-abelian magnetic fluxes

\[ \Phi = \int d^2 x B, \quad \hat{\Phi} = \int d^2 x \hat{B}, \]

and taking trace leads to U(1) magnetic fluxes,

\[ \Phi_{U(1)} = \text{tr} \Phi = \oint |x| \to \infty dx^i \text{tr} \hat{A}^i, \quad \hat{\Phi}_{U(1)} = \text{tr} \hat{\Phi} = \oint |x| \to \infty dx^i \text{tr} \hat{A}^i. \]

As we shall see in the subsequent sections, the BPS equations can also be derived by reshuffling the bosonic sector of energy-momentum tensor

\[ T^{\mu\nu} = \text{tr} (D^\mu Y_A^\dagger D^\nu Y^A + D^\nu Y_A^\dagger D^\mu Y^A) - \eta^{\mu\nu} \left[ \text{tr} (D^\mu Y_A^\dagger D_\mu Y^A + V_0) \right]. \]

For later convenience we introduce energy, linear momentum, and angular momentum, respectively

\[ E = \int d^2 x T^{00} = \int d^2 x \left[ \text{tr} (D^0 Y_A^\dagger D^0 Y^A) + \text{tr} (D^0 Y_A^\dagger D^0 Y^A) + V_m \right], \]
\[ p^i = \int d^2 x T^{i0} = \int d^2 x \text{tr} (D^0 Y_A^\dagger D_i Y^A + D_i Y_A^\dagger D^0 Y^A), \]
\[ J = \int d^2 x \epsilon_{ij} x^i T^{j0} = \int d^2 x \epsilon_{ij} x^i \text{tr} (D^j Y_A^\dagger D^0 Y^A + D^0 Y_A^\dagger D^j Y^A). \]

Pressure component is given by every spatial diagonal component of the energy-momentum tensor and spatial stress is obtained from the off-diagonal component,

\[ P^i \equiv T^{ii} \quad \text{(no sum over } i), \quad T^{ij} = \text{tr} (D^i Y_A^\dagger D_j Y^A + D^j Y_A^\dagger D_i Y^A) \quad (i \neq j). \]
Figure 1: Supersymmetric cases for the vortex-type field configurations

The action (2.1) possesses an SU(4) R-symmetry and charge density for the SU(4) rotations are given by

\[ J_{ab}^0 = i \left[ Y^A (T_{ab})_A^B D_0 Y_B^\dagger - D_0 Y^A (T_{ab})_A^B Y_B^\dagger \right], \tag{2.30} \]

where \( T_{ab} \)’s \((a, b = 1, 2, 3, 4 \text{ and } a \neq b)\) are six generators of SU(4) Lie algebra. The mass deformation (2.11) breaks the SU(4) R-symmetry to SU(2) \( \times \) SU(2) \( \times \) U(1) of which the first SU(2) rotates \( (Y^1, Y^2) \), the second SU(2) does \( (Y^3, Y^4) \), and the U(1) transforms \( (Y^1, Y^2) \) and \( (Y^3, Y^4) \) with opposite phases. For instance, consider an SU(2) rotation transforming \( Y^1 \rightarrow e^{-i\alpha} Y^1 \) and \( Y^2 \rightarrow e^{i\alpha} Y^2 \), and then the corresponding R-charge is given by

\[ R_{12} = \int d^2x \text{ tr } J_{12}^0 = \int d^2x \text{ tr } \left[ i(Y^1 D_0 Y_1^\dagger - D_0 Y^1 Y_1^\dagger) - i(Y^2 D_0 Y_2^\dagger - D_0 Y^2 Y_2^\dagger) \right]. \tag{2.31} \]

In order to obtain the vortex-like BPS configurations, we will impose some supersymmetric conditions to the supersymmetric parameters \( \omega^{AB} \), which reduces the number of supersymme-
tries. Possible supersymmetries and the corresponding supersymmetric conditions are depicted schematically in Fig. 1.

3 Vortex-type Objects with $\mathcal{N} = 3$ Supersymmetry

The vortex-type half-BPS solitons have been discussed in \cite{14}. In this section we briefly summarize the result of our previous work and discuss more general solutions in mass-deformed case.

3.1 BPS equations and bound

Supersymmetric variation of the fermion field $\psi_A$ in (2.9) is,

$$0 = \gamma^0 \delta \psi_A = \left[-\delta_A^B D_0 Y^C + \gamma^0 \left(\beta_A^B + \delta_A^D \beta_D^C + \mu M_A^{[B} Y^C]\right)\right] \omega_{BC} - \gamma^2 (D_1 - \gamma^0 D_2) Y^B \omega_{AB}. \quad (3.1)$$

Now we impose the supersymmetric condition $\gamma^0 \omega_{AB} = i s_{AB} \omega_{AB}$ with $s_{AB} = s_{BA} = \pm 1$ to the equations (3.1), which reduces the number of supersymmetries by half. Then we have the following BPS equations \cite{14}:

$$\begin{align*}
(D_1 - i s_{AB} D_2) Y^B &= 0, \\
\delta_A^B D_0 Y^C - i s_{BC} \left(\beta_A^B + \delta_A^D \beta_D^C + \mu M_A^{[B} Y^C]\right) &= 0, \quad \text{(no sum over $B, C$).} \quad (3.2)
\end{align*}$$

More explicitly, assuming that $Y^1$ is nontrivial, we have

$$\begin{align*}
D_1 Y^1 - i s D_2 Y^1 &= 0, & D_1 Y^B = D_2 Y^B &= 0 \quad (B \neq 1), \\
D_0 Y^1 + i s (\beta_1^2 + \mu Y^1) &= 0, & D_0 Y^2 - i s (\beta_1^2 + \mu Y^2) &= 0, \\
D_0 Y^3 - i s \beta_1^3 &= 0, & D_0 Y^4 - i s \beta_1^4 &= 0, \\
\beta_3^1 &= \beta_4^1 = \beta_2^1 + \mu Y^1, & \beta_4^3 &= \mu Y^3, & \beta_3^4 &= \mu Y^4, \\
\beta_3^2 &= \beta_4^2 = \beta_2^2 = \beta_2^4 = 0, & \beta_A^B &= 0 \quad (A \neq B \neq C \neq A), \quad (3.3)
\end{align*}$$

where $s = \pm 1$.

The BPS equations (3.2) can also be obtained from energy expression (2.26),

$$E = \frac{1}{3} \int d^2 x \text{tr} \left\{ 2 \sum_{A,B,C} \left| \delta_A^B D_0 Y^C - i s_{BC} \left(\beta_A^B + \delta_A^D \beta_D^C + \mu M_A^{[B} Y^C]\right)\right|^2 \\
+ \sum_{A \neq B} |(D_1 - i s_{AB} D_2) Y^A|^2 \\
+ i s \text{tr} \int d^2 x \epsilon_{ij} \partial_i \left(Y_1^\dagger D_j Y^1 - \frac{1}{3} \sum_{A=2}^4 Y_A^\dagger D_j Y^A\right) - \frac{s}{3} \mu \text{tr} \int d^2 x (j^0 + 2 J_{12}) \right\}. \quad (3.4)$$
For any well-behaved BPS configuration satisfying the BPS equations (3.3), the energy is bounded by both the U(1) charge (2.20) and the R-charge (2.31),

\[ E \geq \frac{1}{3} |\mu(Q + 2R_{12})|. \quad (3.5) \]

One can also reshuffle the stress components of the energy momentum tensor (2.25),

\[
T_{ij} = \frac{2}{3} \eta_{ij} \text{Re} \begin{array}{c}
\left[ \delta^B_A \delta^C_D Y^C_0 \right]
+ \delta_{BC} (\beta^B_{AC} + \delta_A^n \beta^C_{CD} + \mu M_A^B Y^C_0) \right]
\times \left[ \delta^B_A \delta^C_D Y^C_0 \right]
+ \frac{1}{2} \text{Re} \begin{array}{c}
\left[ \left( D_i + i \epsilon_k D_k \right) Y^A \right]
\left( D_j - i \epsilon_l D_l \right) Y^A \right]
+ \left( i \leftrightarrow j \right) \right]
\end{array}
\]

which clearly vanish if the BPS equations (3.2) are imposed. Note that from the spatial component of the energy-momentum conservation, the force density \( F^i \) at a given spacetime point \((t, x^i)\) is

\[
F^i = \frac{\partial}{\partial t} T^i_0 = \nabla_j T^{ij}. \quad (3.7)
\]

Thus, vanishing \( T_{ij} \) is a sufficient condition of vanishing force everywhere and any static multi-BPS solitons (or anti-solitons) with vanishing \( T_{ij} \) are noninteracting, at least at the classical level.

### 3.2 BPS objects in the massless theory

For the original ABJM theory without mass deformation \((\mu = 0)\), it has been shown that the BPS equation (3.3) is equivalent to [14]

\[
(D_1 - i s D_2) Y^1 = 0,
Y^A = v^A I, \quad (A = 2, 3, 4),
B = \hat{B} = -\frac{s}{2} \left( \frac{2\pi v}{k} \right)^2 [Y^1, Y^1]^\dagger, \quad (3.8)
\]

where \( v^A (A = 2, 3, 4) \) are constants and \( v^2 = \sum_{A=2}^{4} |v^A|^2 \). Note that all the constraints in (3.3) are completely solved. The Eq. (3.8) is nothing but the Hitchin equation [32] which is a half-BPS equation of super Yang-Mills theory with the identification \( g_{YM} = \frac{2\pi v}{k} \). This identification has appeared in the context of the compactification of ABJM theory (from M2 to D2) [33, 34, 35]. See also Refs. [36, 37, 38].

Under a suitable ansatz [14], Eq. (3.8) is reduced to (affine-) Toda-type equation,

\[
\partial \bar{\partial} \ln |y_a|^2 = 4v \left( \frac{2\pi}{k} \right)^2 \sum_{b=1}^{N-1} K_{ab} \left( |y_b|^2 - \frac{|G(z)|^2}{|c_b|^2 \prod_{c=1}^{N-1} |y_c|^2} \right),
\]

\[
y_M = \frac{G(z)}{\prod_{a=1}^{N-1} y_a}.
\]

11
where $G(z)$ is an arbitrary holomorphic function. For $SU(2)$, this becomes to Liouville-type equation (with $G = 0$) or Sinh-Gordon-type equation (with $G =$const.).

### 3.3 BPS objects in the mass-deformed theory

In the mass-deformed theory ($\mu \neq 0$), the constraint equations in (3.3) have not been solved completely in general except $N = 2, 3$. Here we briefly summarize $U(2) \times U(2)$ case discussed in [14] and generalize the result to $U(N) \times U(N)$ case.

Solving the constraints in (3.3), it turns out that scalar fields can be written in the form

$$
Y^1 = \sqrt{\frac{k \mu}{2\pi}} \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}, \quad Y^2 = \sqrt{\frac{k \mu}{2\pi}} \begin{pmatrix} a & 0 \\ 0 & \sqrt{a^2 + 1} \end{pmatrix}, \\
Y^3 = Y^4 = 0,
$$

while the magnetic fields take the diagonal form

$$
B = \bar{B} = -2s\mu^2 \begin{pmatrix} a^2(1 + |f|^2) & 0 \\ 0 & (a^2 + 1)(1 - |f|^2) \end{pmatrix},
$$

where $a$ is a nonnegative constant. Combining these two using the equations in the first line of (3.3) results in

$$
\partial \bar{\partial} \ln |f|^2 + i(\partial \bar{\partial} - \bar{\partial} \partial)\Omega = \mu^2 [(2a^2 + 1)|f|^2 - 1],
$$

where $\Omega$ is the phase of the scalar field, $f = |f|e^{i\Omega}$. This is the well-known vortex equation appearing in Maxwell-Higgs theory. Note that the phase of the scalar field $\Omega$ in two spatial dimensions can be decomposed into a smooth part $\Omega_{\text{reg}}$ and a singular part,

$$
\Omega_{\text{sing}} = -\frac{i}{2} \ln \prod_{p=1}^{n} \frac{z - z_p}{\bar{z} - \bar{z}_p}.
$$

This gives the 2-dimensional Green’s function

$$
i(\partial \bar{\partial} - \bar{\partial} \partial)\Omega = i(\partial \bar{\partial} - \bar{\partial} \partial) (\Omega_{\text{reg}} + \Omega_{\text{sing}}) = -\frac{1}{4} \nabla^2 \ln \prod_{p=1}^{n} |z - z_p|^2 = -\pi \sum_{p=1}^{n} \delta(\vec{x} - \vec{x}_p),
$$

where $n$ is interpreted as vorticity of multi-vortex configurations.

The energy of the solution is a sum of two terms,

$$
E = \frac{n k \mu}{2a^2 + 1} + \left| \frac{k \mu}{2\pi} B_0 \text{tr} \int d^2 x \right|,
$$

where $B_0 = -4s\mu^2 a^2(a^2 + 1)$ is the asymptotic value of the magnetic field in (3.11). Therefore solutions with nonzero $a$ may be interpreted as vortices in a constant magnetic field.
Considering the vacuum configurations (2.14), we can generalize the ansatz (3.10) (with $a = 0$) to that of $U(N) \times U(N)$ case,

$$Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0_{m_1 \times (m_1+1)} & & & & & f_1 M^{(m_1+1)}_{1} \\ & \ddots & & & & \vdots \\ & & \ddots & & & f_K M^{(m_1+K)}_{1} \end{pmatrix},$$

where $f_k$'s ($k = 1, \cdots, K$) are arbitrary complex functions. We fix the remaining complex scalar fields $Y^{2,3,4}$ as the vacuum configurations given in (2.14). This ansatz satisfies all constraints in (3.3). Combining the Gauss constraints (2.21), (2.22) and the first order differential equations in the second and the third lines of (3.3), one can reduce the gauged Cauchy-Riemann equation of $Y^1$ in (3.3) to the second order differential equations,

$$\partial \bar{\partial} \ln n_k \frac{|f_k|^2}{\prod_{p=1}^{n_k} |z - z^{(k)}_p|^2} = \mu^2 (|f_k|^2 - 1), \quad (k = 1, \cdots, K),$$

where $n_k$ and $z^{(k)}_p$ denote the vorticity and the position of the zeroes of $f_k$, respectively. Its energy is given by

$$E = k\mu \sum_{k=1}^{K} \frac{m_k(m_k - 1)n_k}{2}.$$

It is worth noting that the angular momentum (2.28) of the solution vanishes contrary to the usual spinning BPS vortices in Chern-Simons Higgs theory \[19, 21]. This is because fields do not carry both charge and vorticity, i.e., either $D_0 Y^A$ or $D_1 Y^A$ vanishes in this case. In the next section, however, we will see that solutions with less supersymmetries have nonzero angular momenta.

4 Vortex-type Objects with $\mathcal{N}=2$ Supersymmetry

In this section we consider vortex-type $\mathcal{N} = 2$ BPS solitons without and with mass deformation. As we see in Fig.1, we can obtain $\mathcal{N} = 2$ configurations by imposing one of the following conditions

(i) $\omega_{12} = 0,$

(ii) $\omega_{13} = 0,$

(iii) $\gamma^1 \omega_{12} = \omega_{34}, \gamma^1 \omega_{14} = \omega_{23},$

(iv) $\gamma^1 \omega_{13} = \omega_{24}, \gamma^1 \omega_{14} = \omega_{23}$
in addition to the condition $\gamma^0 \omega_{AB} = is_{AB} \omega_{AB}$ ($s_{AB} = \pm 1$). Here we only consider the case (i). We will treat the case (ii) in Appendix A.1. In the undeformed ABJM theory, the cases (i) and (ii) are equivalent due to the SU(4) R-symmetry. However, in the mass-deformed case, they are inequivalent in general. As discussed in section 6, the BPS solutions of cases (iii) and (iv) are equivalent to those of $\mathcal{N} = 3$ BPS equations.

The brane interpretations of the cases (i) and (ii), which are equivalent in the massless case, were given in section 5.3 of the Ref. [16]. These cases are interpreted as the configuration of intersecting M2-branes spanning two complex coordinates. If we assume that the intersecting M2-branes span only one complex, then the corresponding configuration becomes that of $\mathcal{N} = 3$ BPS equations discussed in the previous section.

4.1 BPS equations and bound

When $\omega_{12} = 0$, the Killing spinor equation (3.1) leads to the following BPS equations:

\[
(D_1 - isD_2)Y^1 = 0, \quad (D_1 + isD_2)Y^2 = 0, \\
D_1 Y^p = D_2 Y^p = 0, \quad (p = 3, 4), \\
D_0 Y^1 + is(\beta_{21}^{12} + \mu Y^1) = 0, \quad D_0 Y^2 - is(\beta_{12}^{12} + \mu Y^2) = 0, \\
D_0 Y^p + is(\beta_{2p}^{12} - \beta_{1p}^{12}) = 0, \\
\beta_{2a}^{3a} = \beta_{a}^{4a} (a = 1, 2), \quad \beta_{43}^{43} - \mu Y^3 = \beta_{34}^{34} - \mu Y^4 = 0, \\
\beta_{21}^{24} = \beta_{24}^{21} = \beta_{13}^{14} = \beta_{14}^{13} = \beta_{3}^{23} = \beta_{4}^{24} = 0. \quad (4.1)
\]

Compared with the $\mathcal{N} = 3$ BPS equations (3.3), the main difference is that we have nontrivial equations for $Y^2$: a gauged Cauchy-Riemann equation of $Y^2$ field and some constraint equations involving $Y^2$. We expect that the $Y^2$ field is allowed to have some nontrivial configurations instead of vacuum configurations in the $\mathcal{N} = 3$ BPS case (3.3). The obtained BPS objects would be in general different from the Maxwell-Higgs type vortices of the half BPS case. In special case (constant $Y^2$), they would reduce to the $\mathcal{N} = 3$ BPS objects discussed in the previous section.

As we did in section 3, we can obtain the energy bound by reshuffling terms in the energy expression,

\[
E = \text{tr} \int d^2x \left[ \sum_{a,p,q} |\delta^g_p | D_0 Y^a | - is_{qa} \left( \beta_{p}^{qa} + \delta^g_p \beta_A^q + \mu M_p | Y^a | \right) |^2 + (1, 2 \leftrightarrow 3, 4) \\
+ |(D_1 - isD_2)Y^1|^2 + |(D_1 + isD_2)Y^2|^2 \\
+ \frac{1}{2} \sum_{p=3,4} \left( |(D_1 - isD_2)Y^p|^2 + |(D_1 + isD_2)Y^p|^2 \right) \right] \\
+ is \text{tr} \int d^2x \partial_i (Y_1^\dagger D_j Y^1 - Y_2^\dagger D_j Y^2) - s\mu \text{tr} \int d^2x j_{12}^0, \quad (4.2)
\]
where $a = 1, 2$ and $p, q = 3, 4$. By requiring the square terms to vanish, we reproduce the BPS equations (4.1). The first term in the last line is a boundary term which vanishes for any well-behaved field configuration. Note that in the mass-deformed theory with $\mu \neq 0$, unlike the half BPS case in (3.5), the energy is not bounded by the $U(1)$ charge $Q$ but the global $SU(2)$ R-charge $R_{12}$ (2.31),

$$E \geq |\mu \text{tr} R_{12}|.$$  

(4.3)

The stress components of energy-momentum tensor (2.25) are also written as

$$T_{ij} = \eta_{ij} \text{Re} \text{tr} \left\{ \left[ \delta^q_p D_0 Y^a + i s_{qa} (\beta^q_p \beta^a_A + \mu M_p [q Y^a]) \right]^\dagger \right. \times \left. \left[ \delta^q_p D_0 Y^a - i s_{qa} (\beta^q_p \beta^a_A + \mu M_p [q Y^a]) \right] + (1, 2 \leftrightarrow 3, 4) \right\}$$

$$+ \frac{1}{2} \text{Re} \text{tr} \left\{ (D_i + i s \epsilon_{ik} D_k) Y^A \right] \text{tr} (D_j - i s \epsilon_{jl} D_l) Y^A + (i \leftrightarrow j) \right\},$$  

(4.4)

which vanishes everywhere on imposing $\mathcal{N} = 2$ BPS equations (4.1). As discussed in (3.7) this pointwise absence of force guarantees that the obtained BPS objects are classically noninteracting.

### 4.2 BPS objects in the massless theory

In the massless case ($\mu = 0$), the energy (4.2) is bounded by the total derivative term, as already discussed in the massless half-BPS case,

$$E = \left| i s \text{tr} \int d^2 x \epsilon_{ij} \partial_i (Y_1^\dagger D_j Y^1 - Y_2^\dagger D_j Y^2) \right|,$$

(4.5)

which vanishes for any well-behaved field configuration. In this case, we expect that there is no regular soliton solution with finite energy.

With a $U(N) \times U(N)$ gauge transformation, we may assume without loss of generality that $Y^3$ is diagonal,

$$Y^3 = \begin{pmatrix} v_1^3 I_{n_1} & & \\ v_2^3 I_{n_2} & & \\ & \ddots & \vdots \\ & & v_k^3 I_{n_k} \end{pmatrix}, \quad (0 \leq v_1^3 < v_2^3 < \cdots < v_k^3).$$

(4.6)

From the constraints $\beta^{34} = \beta^{43} = 0$ (with $\mu = 0$) in (4.1), $Y^4$ has to be also diagonal. Applying the other constraints $\beta^{34} = \beta^{23} = 0$ in (4.1), we notice that $Y^1$ and $Y^2$ are block diagonal and, in each block diagonal subspace, $Y^4$ should be proportional to the identity. Then, for each subspace
where \( Y^A = v^A I \) \((A = 3, 4)\), nontrivial BPS equations in (4.1) become

\[
(D_1 - isD_2)Y^1 = 0, \quad (D_1 + isD_2)Y^2 = 0,
\]

\[
B = -2s \left(\frac{2\pi}{k}\right)^2 \left\{ [Y^1 Y^\dagger_2, Y^2 Y^\dagger_1] + v^2 \left( [Y^1, Y^\dagger_1] - [Y^2, Y^\dagger_2] \right) \right\},
\]

\[
\hat{B} = -2s \left(\frac{2\pi}{k}\right)^2 \left\{ [Y^\dagger_2 Y^1, Y^\dagger_1 Y^2] + v^2 \left( [Y^1, Y^\dagger_1] - [Y^2, Y^\dagger_2] \right) \right\},
\]

(4.7)

where \( v^2 = \sum_{A=3,4} |v^A|^2 \). When one of \( Y^1 \) and \( Y^2 \) is assumed to be proportional to identity in each block diagonal subspace, (4.7) reduce to (3.8) in the \( \mathcal{N} = 3 \) BPS case.

We consider some simple solutions of (4.7) with \( s = 1 \) for definiteness. For \( U(2) \times U(2) \) case, we take an ansatz,

\[
Y^1 = \begin{pmatrix} 0 & d \\ e & 0 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.
\]

(4.8)

Plugging (4.8) into (4.7) we obtain

\[
ab = C, \quad de = D,
\]

\[
\partial \bar{\partial} \ln |b|^2 - i(\partial \bar{\partial} - \bar{\partial} \partial)\Omega_b = \left(\frac{2\pi}{k}\right)^2 (|b|^2 - |a|^2)(|d|^2 + |e|^2),
\]

(4.9)

\[
\partial \bar{\partial} \ln |d|^2 - i(\partial \bar{\partial} - \bar{\partial} \partial)\Omega_d = \left(\frac{2\pi}{k}\right)^2 (|a|^2 + |b|^2 + 2v^2)(|d|^2 - |e|^2),
\]

(4.10)

where \( \Omega \)'s are phases of scalar fields, \( b = |b|e^{-i\Omega_b}, d = |d|e^{i\Omega_d} \) and \( C, D \) are arbitrary constants. For \( Y^2 = I \), \( a = b \), (4.9) and (4.10) are reduced to the Liouville-type equation (with \( D = 0 \)) or Sinh-Gordon-type equation (with \( D = \text{const} \)) which we obtained in subsection 3.2 as \( \mathcal{N} = 3 \) BPS configurations. On the other hand, with \( a = e = v = 0 \), the equations are further simplified to

\[
\partial \bar{\partial} \ln |b|^2 - i(\partial \bar{\partial} - \bar{\partial} \partial)\Omega_b = \left(\frac{2\pi}{k}\right)^2 |b|^2|d|^2,
\]

\[
\partial \bar{\partial} \ln |d|^2 - i(\partial \bar{\partial} - \bar{\partial} \partial)\Omega_d = \left(\frac{2\pi}{k}\right)^2 |b|^2|d|^2,
\]

(4.11)

which again become a Liouville equation with \( b = d \).

### 4.3 BPS objects in the mass-deformed theory

#### 4.3.1 U(2) × U(2) gauge group

Let us first consider the simplest U(2) × U(2) case. By the same reasoning developed in [14] to obtain reduced equations in \( \mathcal{N} = 3 \) BPS case (3.12), it is readily shown that the constraint
equations in (4.1) lead us to put \( Y^3 = Y^4 = 0 \) for nontrivial solutions. Then we are left with

\[
(D_1 + iD_2)Y^1 = 0, \quad (D_1 - iD_2)Y^2 = 0, \quad D_0 Y^1 - i(\beta_1^2 + \mu Y^1) = 0, \quad D_0 Y^2 + i(\beta_1^2 + \mu Y^2) = 0,
\]

(4.12)
as well as the Gauss' laws (2.21) and (2.22). Comparing with the half-BPS case, we have nontrivial equations for \( Y^2 \) in addition to \( Y^1 \) and they are coupled to each other.

We proceed by adopting a simple ansatz from the broken vacuum (2.14). More specifically, in \( U(2) \times U(2) \) case, we consider

\[
Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}, \quad Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad Y^3 = Y^4 = 0,
\]

(4.13)
which is actually the same ansatz used in the previous section to obtain (4.11). Moreover, comparing with the ansatz (3.10) employed in half-BPS case, we see that (4.13) has essentially the same form as (3.10) (with \( a = 0 \)). Here, thanks to the less supersymmetries, \( Y^2 \) is no longer a constant. We will see below that this freedom allows us to have richer solutions with nonvanishing angular momenta.

With the above ansatz, the Gauss' laws (2.21) and (2.22) take simple diagonal forms,

\[
B = 2\mu^2 \begin{pmatrix} -|d|^2(1 - |b|^2) & 0 \\ 0 & |b|^2(1 - |d|^2) \end{pmatrix}, \quad \hat{B} = 2\mu^2 \begin{pmatrix} 0 & 0 \\ 0 & |b|^2 - |d|^2 \end{pmatrix},
\]

(4.14)

from which we can write the corresponding gauge fields as

\[
A = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

(4.15)
where

\[
B = \frac{2}{i}(\partial \bar{A} - \bar{\partial} A), \quad \hat{B} = \frac{2}{i}(\partial \bar{\hat{A}} - \bar{\partial} \hat{A}).
\]

(4.16)
Substituting (4.15) and (4.13) into the first line of BPS equations (4.12) we have

\[
\bar{u} - \bar{v} = i\bar{\partial} nd, \quad v - \hat{v} = i\partial \hat{n}b.
\]

(4.17)
Inserting (4.17) into the magnetic field (4.16) and comparing it with (4.14), we obtain two equations for scalar fields

\[
\partial \bar{\partial} \ln \frac{|b|^2}{\prod_{n=1}^{m_b} |z - z_p|^2} = \mu^2 |d|^2 (|b|^2 - 1),
\]

(4.18)
\[
\partial \bar{\partial} \ln \frac{|d|^2}{\prod_{q=1}^{m_d} |z - z_q|^2} = \mu^2 |b|^2 (|d|^2 - 1).
\]

(4.19)
The energy bounded by the R-charge (4.3) is rewritten as

$$E = \frac{k\mu^3}{\pi} \int d^2x \left[ |b|^2(1 - |d|^2) + |d|^2(1 - |b|^2) \right].$$  \hfill (4.20)

The reduced equations (4.18) and (4.19) have been considered in self-dual $U(1) \times U(1)$ Chern-Simons system with two scalar fields [25], which may be considered as the abelian part of the theory under consideration. Let us analyze the equations in the present setting. From the expression of the energy (4.20), it follows that there are two classes of boundary conditions at spatial infinity,

$$|b(\infty)| = |d(\infty)| = 1, \quad |b(\infty)| = |d(\infty)| = 0,$$ \hfill (4.21)

which correspond to topological and nontopological solutions, respectively. At every vortex point $z = z_p, z'_q$, singlevaluedness of the fields requires their amplitudes to vanish, $|b(z_p)| = |d(z'_q)| = 0$. (If $n_b = 0$ or $n_d = 0$, the corresponding amplitudes need not vanish.) Depending on the vorticity, nontopological solutions can further be classified as nontopological $Q$-balls ($n_b = n_d = 0$), $Q$-vortices ($n_b \neq 0$ and $n_d \neq 0$), and their hybrids ($n_b \neq 0$ and $n_d = 0$).

These gauged vortices carry diagonal components of magnetic fluxes of which the contributions come from the spatial infinity,

$$\Phi = 4 \oint_{|x^i| \to \infty} dx^i \begin{pmatrix} \partial_i \ln \prod_{p=1}^{n_b} |x - x_p| & 0 \\ 0 & -\partial_i \ln \prod_{q=1}^{n_d} |x - x_q| \end{pmatrix},$$ \hfill (4.22)

$$\hat{\Phi} = 4 \oint_{|x^i| \to \infty} dx^i \begin{pmatrix} 0 & 0 \\ \partial_i \ln \prod_{q=1}^{n_d} |x - x_q| & |d| \prod_{p=1}^{n_b} |x - x_p| \end{pmatrix}.$$ \hfill (4.23)

We parameterize the asymptotic behaviors of the fields as

$$|b| \sim |x^i|^{-\alpha_b}, \quad |d| \sim |x^i|^{-\alpha_d},$$ \hfill (4.24)

where $\alpha_b$ and $\alpha_d$ are positive constants for nontopological solutions, while $\alpha_b = \alpha_d = 0$ for topological ones. Then the magnetic fluxes (4.22) – (4.23) become

$$\Phi = 2\pi \begin{pmatrix} -n_b - \alpha_b & 0 \\ 0 & n_d + \alpha_d \end{pmatrix}, \quad \hat{\Phi} = 2\pi \begin{pmatrix} 0 & 0 \\ -n_b - \alpha_b + n_d + \alpha_d & 0 \end{pmatrix}.$$ \hfill (4.25)

For the topological vortices satisfying the first boundary condition in (4.21), the fluxes are quantized by the integer-valued vorticities $n_b$ and $n_d$, as $\alpha_b$ and $\alpha_d$ are zero.

In Chern-Simons gauge theories, the Gauss’ laws (2.21) – (2.22) with the help of the conserved currents (2.18) imply that

$$Q = \frac{k}{2\pi} \Phi, \quad \hat{Q} = \frac{k}{2\pi} \hat{\Phi},$$ \hfill (4.26)
and hence the flux carrying objects are also charged. Since the energy of $N = 2$ BPS solitons is bounded by the trace of R-charge (4.2)–(4.3), they carry R-charge (2.31) as well,

$$R_{12} = \frac{k\mu^2}{\pi} \int d^2x \begin{pmatrix} \frac{1 - |b|^2}{|d|^2} & 0 \\ 0 & \frac{1 - |d|^2}{|b|^2} \end{pmatrix}.$$

(4.27)

As we mentioned above, a notable difference from the $N = 3$ BPS case is that the solution carries nonzero angular momentum in the present case. In this regard, note in particular that both $D_0Y^a$ and $D_iY^a$ ($a = 1, 2$) are not zero from the BPS equations (4.12). Therefore the angular momentum (2.28) does not vanish,

$$J = -\frac{k}{2\pi} \int d^2x \epsilon_{ij} x_i \left[ (v_j - \partial_j \Omega_d)B_{11} + (u_j + \partial_j \Omega_b)B_{22} \right].$$

(4.28)

The explicit value of $J$ can be computed for rotationally symmetric solutions as seen below.

For rotationally symmetric configurations we take the ansatz

$$\Omega_b = n_b \theta, \quad \Omega_d = n_d \theta,$$

(4.29)

as well as

$$u_i = -\epsilon_{ij} x^j t^2 u_t(r), \quad v_i = -\epsilon_{ij} x^j t^2 v_r(r).$$

(4.30)

Then after a straightforward calculation (see also [25]) we find

$$J = k(\alpha_b \alpha_d - n_b n_d).$$

(4.31)

For topological vortices, the $U(2) \times U(2)$ gauge symmetry is spontaneously broken to $U(1) \times U(1)$. Since the fundamental group of the vacuum manifold is computed as $\pi_1(U(2) \times U(2)/U(1) \times U(1)) = \pi_1(U(1) \times S^2 \times U(1) \times S^2) = \mathbb{Z} \times \mathbb{Z}$, the stability of the composite of two static vortices is topologically guaranteed. $Q$-balls and $Q$-vortices are generated in the symmetric phase of which the vacuum has trivial topology and their stability should be examined energetically [39]. The mass of the transverse scalar fields $Y^A$ and that of the fermions $\psi_A$ are all $\mu$ from (2.8) and (2.7). Since the minimum energy to produce $Q$-balls and $Q$-vortices of R-charge $R_{12}$ is given by (4.3), the rest energy to produce the scalar or fermion particles of the R-charge $R_{12}$ is exactly the same as the minimum energy of $Q$-balls or $Q$-vortices of the same amount of R-charge. Therefore, these $Q$-balls or $Q$-vortices are marginally stable [21] and this marginal stability is a character of Chern-Simons Higgs theory in the BPS limit with single mass scale.

In addition to rotationally symmetric solutions, we can obtain other class of solutions of (4.18) and (4.19) for a few simple cases [25]. When the $|b|$ field takes the Higgs vacuum value $|b| = 1$,
(4.18) becomes trivial and (4.19) reduces to the scalar BPS equation for the Nielsen-Olesen type vortices

\[ \partial \bar{\partial} \ln \left( \frac{|d|^2}{\prod_{q=1} |z - z_q|^2} \right) = \mu^2 (|d|^2 - 1), \]

which has already been discussed in the \( \mathcal{N} = 3 \) case; see (3.12). When \( d \) and \( b \) are parallel, (4.18)-(4.19) become single scalar BPS equation [13] equivalent to that for the vortices in Abelian Chern-Simons-Higgs theory [19].

\[ \partial \bar{\partial} \ln \left( \frac{|d|^2}{\prod_{q=1} |z - z_q|^2} \right) = \mu^2 |d|^2 (|d|^2 - 1). \]

This equation also supports the BPS multi-vortex-type solutions including topological vortices [19, 40] in the broken phase of \( \lim_{r \to \infty} |d|^2 \to 1 \), and \( Q \)-balls and \( Q \)-vortices [41] in the symmetric phase of \( \lim_{r \to \infty} |d|^2 \to 0 \) [21, 41]. In this case the magnetic fluxes (4.25) are traceless so that solutions do not carry U(1) magnetic fluxes, \( \Phi_{U(1)} = \hat{\Phi}_{U(1)} = 0 \) in (2.24), and U(1) charge, \( Q_{U(1)} = \hat{Q}_{U(1)} = 0 \) in (2.20). It carries, however, a fractional angular momentum. For rotationally symmetric configurations, it is given by \( J = k(\alpha_d^2 - n_d^2) \) from (4.31). When the topological vortices are separated from each other, the scalar amplitude is expanded near a vortex point \( z_q \) as

\[ \ln |d|^2 \approx \ln |z - z_q|^2 + a_q(z - z_q) + \mathcal{O}((z - z_q)^2), \]

and then total angular momentum is given by a sum of spin part and orbital part as

\[ J \approx -k|n_d| - k \sum_{q=1}^{n_q} |z_q a_q|. \]

For the sufficiently large separation, \( a_q \to 0 \) and there remains only spin part linearly proportional to the vorticity [42].

In addition, by evaluating the index of the differential operator associated with the appropriate fluctuation equation, one can find that the number of free parameters of the general solutions of (4.18) and (4.19) [25] is given by \( 2(n_b + n_d + \lceil \alpha_b \rceil + \lceil \alpha_d \rceil) \), where \( \lceil \alpha_b \rceil \) and \( \lceil \alpha_d \rceil \) respectively denote the largest integer less than \( \alpha_b \) and \( \alpha_d \).

### 4.3.2 U(3) × U(3) gauge group

For higher-rank gauge groups, we proceed by setting \( Y^3 \) and \( Y^4 \) to vacuum values. This is a natural choice considering that \( D_i Y^p = 0 \) for all \( i = 1, 2 \) and \( p = 3, 4 \) in the BPS equation (4.1).
which suggest no nontrivial dynamics for $Y^{3,4}$. Then from (2.14), we have the following two possible field configurations,

(1) $Y^3 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y^4 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$, $Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$,

(ii) $Y^3 = Y^4 = 0$, $Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \sqrt{2}b \end{pmatrix}$, $Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & \sqrt{2}d & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$.

(4.36)

For the case (i), the magnetic field profiles are calculated as

$B = 2\mu^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -|b|^2(1 - |d|^2) & 0 \\ 0 & 0 & |d|^2(1 - |b|^2) \end{pmatrix}$, $\hat{B} = 2\mu^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & |d|^2 - |b|^2 \\ 0 & 0 & 0 \end{pmatrix}$,

(4.37)

and then the resulting second order scalar BPS equations are the same as those in $\mathcal{N} = 2$ $U(2) \times U(2)$ case, (4.18)–(4.19). The energy for these solitons, read from the R-charge, is also given by (4.20).

For the configuration (ii), we again end up with (4.18)–(4.19). The energy and the angular momentum, however, have different values,

$E = \frac{k\mu^2}{\pi} \text{tr} \int d^2x \begin{pmatrix} 2|d|^2(1 - |b|^2) & 0 & 0 \\ 0 & |b|^2 + |d|^2 - 2|b|^2|d|^2 & 0 \\ 0 & 0 & 2|b|^2(1 - |d|^2) \end{pmatrix}$

$= 2k\mu \text{tr} \begin{pmatrix} n_b + \alpha_b & 0 & 0 \\ 0 & \frac{1}{2}(n_b + \alpha_b + n_d + \alpha_d) & 0 \\ 0 & 0 & n_d + \alpha_d \end{pmatrix}$

$= 3k\mu(n_b + \alpha_b + n_d + \alpha_d),$

$J = 3k(\alpha_b\alpha_d - n_b n_d),$

(4.38)

where the angular momentum is calculated for rotational symmetric configurations.
4.3.3 $\text{U}(N) \times \text{U}(N)$ gauge group

For the $\text{U}(N) \times \text{U}(N)$ gauge group, based on the vacuum configurations (2.14), we consider the following field ansatz

$$
Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix}
0_{m_1 \times (m_1+1)} \\
\vdots \\
0_{m_I \times (m_I+1)} \\
\vdots \\
0_{m_{I+1} \times (m_{I+1}+1)} \\
\vdots \\
0_{m_K \times (m_K+1)} \\
\vdots \\
\vdots \\
\vdots \\
a_1 \mathcal{M}^{(m_{I+1})}_1 \\
\vdots \\
b_K \mathcal{M}^{(m_{I+K})}_1 \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix},
$$

$$
Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix}
0_{m_1 \times (m_1+1)} \\
\vdots \\
0_{m_I \times (m_I+1)} \\
\vdots \\
0_{m_{I+1} \times (m_{I+1}+1)} \\
\vdots \\
0_{m_K \times (m_K+1)} \\
\vdots \\
\vdots \\
\vdots \\
a_1 \mathcal{M}^{(m_{I+1})}_2 \\
\vdots \\
b_K \mathcal{M}^{(m_{I+K})}_2 \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix},
$$

where $a_k, b_k \ (k = 1, \ldots, K)$ are arbitrary complex functions, and $Y^3$ and $Y^4$ are set to vacuum values. The resulting scalar BPS equations are $K$ pairs of coupled differential equations

$$
\partial \bar{\partial} \ln \frac{|a_k|^2}{\prod_{p=1} |z - z_p|^2} = \mu^2 |b_k|^2 (|a_k|^2 - 1), \quad \partial \bar{\partial} \ln \frac{|b_k|^2}{\prod_{q=1} |z - z_q'|^2} = \mu^2 |a_k|^2 (|b_k|^2 - 1). \tag{4.40}
$$

Summing over all contributions from each block in (4.39), we obtain the energy and angular momentum

$$
E = k\mu \sum_{k=1}^K \frac{m_k(m_k - 1)}{2} (n_{a,k} + \alpha_{a,k} + n_{b,k} + \alpha_{b,k}),
$$

$$
J = k \sum_{k=1}^K \frac{m_k(m_k - 1)}{2} (\alpha_{a,k} \alpha_{b,k} - n_{a,k} n_{b,k}), \tag{4.41}
$$

where the angular momentum is calculated for the rotational symmetric configuration as before.

When we choose $Y^2$ to the vacuum configuration by fixing $b_k = 1$, the second order differential equations (4.40) and the expressions of energy and angular momentum are reduced to those of $\mathcal{N} = 3$ case. In this case the supersymmetry of the corresponding BPS configuration is enhanced to $\mathcal{N} = 3$. 

22
5 Vortex-type Objects with $\mathcal{N}=1$ Supersymmetry

As we see in Fig.1, there are four ways to obtain $\mathcal{N}=1$ BPS equations:

(i) $\omega_{13} = \omega_{14} = 0$,
(ii) $\omega_{12} = \omega_{13} = 0$,
(iii) $\omega_{13} = 0$, $\gamma^1 \omega_{12} = \omega_{34}$, $\gamma^1 \omega_{14} = \omega_{23}$,
(iv) $\omega_{12} = 0$, $\gamma^1 \omega_{13} = \omega_{24}$, $\gamma^1 \omega_{14} = \omega_{23}$

in addition to the condition $\gamma^0 \omega_{AB} = i s_{AB} \omega_{AB}$ ($s_{AB} = \pm 1$). In this section we only consider the case (i). The cases (i) and (ii) are equivalent in the massless limit due to the SU(4) R-symmetry of the undeformed theory. The corresponding configurations are interpreted as the intersecting M2-branes spanning all the transverse coordinates. We will discuss the case (ii) in the mass deformed theory in Appendix A.2. The BPS equations for the cases (iii) and (iv) are equivalent to those of the $\mathcal{N}=2$ supersymmetries with conditions $\omega_{13} = 0$ and $\omega_{12} = 0$, respectively. We postpone the discussions on these phenomena to section 6.

5.1 BPS equations and bound

When $\omega_{13} = \omega_{14} = 0$, the BPS equations are given by

\[
(D_1 - is D_2) Y^a = 0, \quad (a = 1, 2), \\
(D_1 + is D_2) Y^p = 0, \quad (p = 3, 4), \\
D_0 Y^1 + is (\beta_C^1 - 2\beta_2^1 - \mu Y^1) = 0, \\
D_0 Y^2 + is (\beta_C^2 - 2\beta_1^2 - \mu Y^2) = 0, \\
D_0 Y^3 + is (\beta_C^3 - 2\beta_4^3 + \mu Y^3) = 0, \\
D_0 Y^4 - is (\beta_C^4 - 2\beta_3^4 + \mu Y^4) = 0,
\]

\[
\beta_1^{34} = \beta_2^{34} = \beta_3^{34} = \beta_4^{34} = 0, \quad (s = \pm 1).
\]

(5.1)

Note that in this case, all four scalar fields enter the equations in a nontrivial way, i.e., satisfy the gauged Cauchy-Riemann equations in contrast with $\mathcal{N}=3$ and $\mathcal{N}=2$ cases where only one or two scalars have nontrivial profiles.

In accordance with (5.1), the energy expression for bosonic sector (2.26) can be reshuffled as

\[
E = \int d^2 x \text{tr} \left[ \sum_{a=1,2} |D_0 Y^a - is (\beta_A^a - 2\beta_b^a - \mu Y^a)|^2 + \sum_{p=3,4} |D_0 Y^p - is (\beta_A^p - 2\beta_q^p + \mu Y^p)|^2 \\
+ \sum_{a=1,2} |(D_1 - is D_2) Y^a|^2 + \sum_{p=3,4} |(D_1 + is D_2) Y^p|^2 \\
+ 4 (|\beta_3^{12}|^2 + |\beta_4^{12}|^2 + |\beta_3^{34}|^2 + |\beta_2^{34}|^2) \right] \\
+ is \text{tr} \int d^2 x \left[ c_{ij} \partial_i (Y^a_j D_j Y^a - Y^p_j D_j Y^p) \right] + s \mu \text{tr} \int d^2 x j^0,
\]

(5.2)

where $a,b = 1,2$ and $p,q = 3,4$. As we discussed previously, in the massless limit $\mu \to 0$, the energy is bounded by the total derivative term in the fourth line of (5.2). In this section, however,
we only consider the cases with mass deformation. For any well behaved $\mathcal{N} = 1$ BPS configurations with mass deformation, the total derivative term does not contribute to the energy and then the energy is bounded by the $U(1)$ charge (2.20),

$$E \geq |\mu Q_{U(1)}|.$$  \hspace{1cm} (5.3)

By the Gauss’ law (2.21), one can say that the energy is bounded by the magnetic flux (2.24),

$$E \geq \left|\frac{k\mu}{2\pi} \Phi_{U(1)}\right|.$$  \hspace{1cm} (5.4)

The bound in (5.4) is useful when we discuss topological vortices carrying quantized magnetic flux, and the bound in (5.3) is useful when we discuss nontopological $Q$-balls and $Q$-vortices stabilized by conserved charge.

It is a straightforward matter to rewrite the spatial stress components of energy-momentum tensor (2.29) as

$$T_{ij} = \eta_{ij} \text{Re tr} \left\{ \left( (D_0 Y^a + is(\beta^A A_a - 2\beta^b a_b - \mu Y^a)) (D_0 Y^a - is(\beta^A A_a - 2\beta^b a_b - \mu Y^a)) \right)^\dagger + (1, 2, \mu \leftrightarrow 3, 4, -\mu) \right\}$$

$$- 4 \left( |\beta^1_2|^2 + |\beta^2_1|^2 + |\beta^3_4|^2 + |\beta^3_2|^2 \right) \right\} + \frac{1}{2} \text{Re tr} \left\{ \left( [(D_i - is\epsilon_{ik} D_k) Y_A] \right)^\dagger (D_j + is\epsilon_{jl} D_l) Y^A + (i \leftrightarrow j) \right\}.$$  \hspace{1cm} (5.5)

Therefore, for any BPS soliton or anti-soliton configurations satisfying the $\mathcal{N} = 1$ BPS equations (5.1), the spatial stress components of energy-momentum tensor (5.5) vanish everywhere, as it should.

### 5.2 BPS objects in the mass-deformed theory

The $\mathcal{N} = 1$ BPS equations (5.1) include four algebraic constraints of which the number is much less than that of the $\mathcal{N} = 2$ or $\mathcal{N} = 3$ BPS equations. Then even for the case of $U(2) \times U(2)$ gauge group, it would not be feasible to solve the BPS equations in the most general way. As we did for the $\mathcal{N} = 2$ case, here we will be content with simple cases. Using ansätze based on the vacuum solutions (2.14), we investigate possible BPS solutions from the cases of $U(2) \times U(2)$ and $U(3) \times U(3)$ gauge groups and then extend the results to the case of $U(N) \times U(N)$ gauge group.

For the case of $U(2) \times U(2)$ gauge group, the vacuum configuration (2.14) suggests

$$Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, \quad Y^3 = Y^4 = 0.$$  \hspace{1cm} (5.6)

up to a substitution $Y^{1,2} \leftrightarrow Y^\dagger_{3,4}$. Then the BPS equations (5.1) are simplified to the following form:

$$(D_1 - isD_2) Y^a = 0, \quad D_0 Y^a - is(\beta^b a_b + \mu Y^a) = 0, \quad a, b = 1, 2.$$  \hspace{1cm} (5.7)
The resultant BPS equations are equivalent to those of \( \mathcal{N} = 2 \) case in section 4. Therefore, there is no genuine \( \mathcal{N} = 1 \) U(2) \( \times \) U(2) BPS solution within the ansatz (5.6) based on the vacuum solution. In fact, this is natural considering that all four scalar fields have to be nontrivial for \( \mathcal{N} = 1 \) solutions, as pointed out below (5.1), namely all four \( Y^A \)'s satisfy the gauged Cauchy-Riemann equations.

The lowest rank gauge group, for which all scalar fields are nonvanishing within the ansatz based on the vacuum configurations, is U(3) \( \times \) U(3). In this case we expect that there exist some \( \mathcal{N} = 1 \) BPS solutions with nontrivial configurations for all \( Y^A \)'s in this gauge group. An interesting configuration can be obtained from the following ansatz based on one of the vacuum solutions of the U(3) \( \times \) U(3) case,

\[
Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
Y^3 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y^4 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & e & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(5.8)

where \( a, b, d, e \) are arbitrary complex functions. By the procedure similar to those in the previous sections we obtain two pairs of coupled second order equations,

\[
\partial \bar{\partial} \ln \frac{|a|^2}{n_a} \prod_{p=1}^{n_b} |z - z_p(1)|^2 = \mu^2 |b|^2 (|a|^2 - 1), \quad \partial \bar{\partial} \ln \frac{|b|^2}{n_b} \prod_{q=1}^{n_a} |z - z_q(2)|^2 = \mu^2 |a|^2 (|b|^2 - 1),
\]

(5.9)

\[
\partial \bar{\partial} \ln \frac{|d|^2}{n_d} \prod_{r=1}^{n_e} |z - z_r(3)|^2 = \mu^2 |e|^2 (|d|^2 - 1), \quad \partial \bar{\partial} \ln \frac{|e|^2}{n_e} \prod_{s=1}^{n_d} |z - z_s(4)|^2 = \mu^2 |d|^2 (|e|^2 - 1).
\]

(5.10)

Note that each pair is identical to (4.18)–(4.19) in \( \mathcal{N} = 2 \) case and hence the same solutions. In the present case, one pair of equations (5.9) lies on the \( (Y^1,Y^2) \)-plane while the other pair of equations (5.10) lies on the \( (Y^3,Y^4) \)-plane. The energy and angular momentum of this configuration are given by

\[
E = k\mu (n_a + n_b + n_d + n_e + \alpha_a + \alpha_b + \alpha_d + \alpha_e),
\]

\[
J = k(\alpha_a \alpha_b - n_a n_b + \alpha_d \alpha_e - n_d n_e),
\]

(5.11)

where \( \alpha \) and \( n \) are defined in (4.24) and (4.25), respectively, and we calculated the angular momentum \( J \) for the rotationally symmetric case as before.
We can generalize the ansatz \((5.8)\) of the \(U(3) \times U(3)\) theory to \(U(N) \times U(N)\) case as follows:

\[
Y^1 = \sqrt{\frac{k \mu}{2\pi}} \left( \begin{array}{cccc}
0_{m_1 \times (m_1+1)} & & & \\
& \cdots & & \\
& & 0_{m_I \times (m_I+1)} & \\
& & & a_1 M_1^{(m_I+1)}
\end{array} \right),
\]

\[
Y^2 = \sqrt{\frac{k \mu}{2\pi}} \left( \begin{array}{cccc}
0_{m_1 \times (m_1+1)} & & & \\
& \cdots & & \\
& & 0_{m_I \times (m_I+1)} & \\
& & & b_1 M_1^{(m_I+1)}
\end{array} \right),
\]

\[
Y^3 = \sqrt{\frac{k \mu}{2\pi}} \left( \begin{array}{cccc}
d_1 M_1^{(m_1)} & & & \\
& \cdots & & \\
& & d_I M_I^{(m_I)} & \\
& & & 0_{(m_I+1) \times m_I+1}
\end{array} \right),
\]

\[
Y^4 = \sqrt{\frac{k \mu}{2\pi}} \left( \begin{array}{cccc}
e_1 M_2^{(m_1)} & & & \\
& \cdots & & \\
& & e_I M_I^{(m_I)} & \\
& & & 0_{(m_I+1) \times m_I+1}
\end{array} \right),
\]

\[
(5.12)
\]

where \(a_k, b_k (k = 1, \ldots, K)\) and \(d_i, e_i (i = 1, \ldots, I)\) are arbitrary complex functions. Then we have \((K + I)\)-pairs of coupled differential equations

\[
\partial \bar{\partial} \ln \frac{\prod_{n_{a,k}} |a_k|^2}{|z - z_{p_k}^{(1)}|^2} = \mu^2 |b_k|^2 (|a_k|^2 - 1), \quad \partial \bar{\partial} \ln \frac{\prod_{n_{b,k}} |b_k|^2}{|z - z_{q_k}^{(2)}|^2} = \mu^2 |a_k|^2 (|b_k|^2 - 1),
\]

\[
\partial \bar{\partial} \ln \frac{\prod_{n_{d,i}} |d_i|^2}{|z - z_{r_i}^{(3)}|^2} = \mu^2 |e_i|^2 (|d_i|^2 - 1), \quad \partial \bar{\partial} \ln \frac{\prod_{n_{e,i}} |e_i|^2}{|z - z_{s_i}^{(4)}|^2} = \mu^2 |d_i|^2 (|e_i|^2 - 1).
\]

\[
(5.13)
\]
Here, the \( K \)-pairs of equations in the first line of (5.13) live on the \((Y_1, Y_2)\)-plane and the \( I \)-pairs of equations in the second line of (5.13) live on the \((Y_3, Y_4)\)-plane. The energy and angular momentum of this configuration are

\[
E = k \mu \sum_k \frac{m_k(m_k + 1)}{2} (n_{a,k} + n_{b,k} + \alpha_{a,k} + \alpha_{b,k}) + k \mu \sum_i \frac{m_i(m_i + 1)}{2} (n_{d,i} + n_{e,i} + \alpha_{d,i} + \alpha_{e,i}),
\]

\[
J = k \sum_k \frac{m_k(m_k + 1)}{2} (\alpha_{a,k} - \alpha_{b,k}) + k \sum_i \frac{m_i(m_i + 1)}{2} (\alpha_{d,i} - \alpha_{e,i}),
\]

(5.14)

where for the calculation of angular momentum we again considered rotationally symmetric configurations only.

When we fix \( d_k = e_k = 1 \), the ansatz (5.12) and the corresponding physical quantities are equivalent to case of \( \mathcal{N} = 2 \) in section 4 and hence the supersymmetry of the BPS solutions in this case is enhanced to \( \mathcal{N} = 2 \).

6  Absence of Vortex-type Objects with \( \mathcal{N} = \frac{5}{2}, \frac{3}{2}, \frac{1}{2} \) Supersymmetries

In this section, we discuss the BPS nature of half integer supersymmetries, which are obtained by imposing the supersymmetric conditions \( \gamma^1 \omega_{AB} = \pm \omega_{AB}^* \), in addition to the conditions of integer supersymmetries. These additional conditions deform the gauged Cauchy-Riemann equations in the BPS equations of integer supersymmetries. From the algebraic relations of BPS equations and the vanishing \( T_{ij} \) condition discussed in section 3.1, we will argue that the spectrum of BPS solitons of half-integer supersymmetries are equivalent to those of integer supersymmetries.

6.1 \( \mathcal{N} = \frac{5}{2} \) supersymmetry

As we see in Fig.1, there are two ways to obtain \( \mathcal{N} = \frac{5}{2} \) BPS equations. In addition to \( \gamma^0 \omega_{AB} = is_{AB}\omega_{AB} \) which is necessary to get vortex-type equations, we can impose one of the following conditions

(i) \( \gamma^1 \omega_{12} = \omega_{34} \),

(ii) \( \gamma^1 \omega_{13} = \omega_{24} \).

One of these would kill one real degree of \( \omega_{12} \) or \( \omega_{13} \), leaving five real independent supersymmetric parameters. Here, we will consider the case (i). The same argument can be applied to the case (ii) as well.

In the presence of the additional condition (i), the BPS equations are the same as those in (3.3) except the Cauchy-Riemann equation for \( Y^2 \), which is modified as

\[
(D_1 - isD_2)Y^2 = 0 \quad \rightarrow \quad (D_1 - isD_2)Y^2 = 2\beta_1^{34},
\]

(6.1)
and a constraint $\beta_{34}^1 = 0$, which is no longer zero.

It should be noted however that the resulting BPS equations do not necessarily have nontrivial solutions with the expected number of supersymmetries. In other words, all the solutions may have enhanced supersymmetries and hence the BPS equations are actually equivalent to those of higher symmetries. This is indeed the case with $\mathcal{N} = \frac{5}{2}$ equations which are actually equivalent to $\mathcal{N} = 3$ equations. To see this, we multiply $(\beta_{34}^1)^\dagger$ to the deformed BPS equation (6.1) and take trace,

$$
\text{tr}|\beta_{34}^1|^2 = \frac{1}{2}\text{tr}[(\beta_{34}^1)^\dagger(D_1 - i s D_2)Y^2] = \frac{1}{2}(\partial_1 - i s \partial_2)\text{tr}[(\beta_{34}^1)^\dagger Y^2]
$$

$$
+ \frac{1}{2}\text{tr}[\beta_{12}^1(D_1 - i s D_2)Y_3^\dagger + (\beta_{34}^1)^\dagger(D_1 - i s D_2)Y^1 - \beta_{12}^1(D_1 - i s D_2)Y_4^\dagger].
$$

(6.2)

Note that each term in the second trace vanishes due to the BPS equations for $\mathcal{N} = 3$ and there remains only the first term. But $\text{tr}[(\beta_{34}^1)^\dagger Y^2] = -\text{tr}[(\beta_{34}^1)^\dagger Y^1]$, which again is zero thanks to the constraint $\beta_{34}^1 = 0$. Therefore we reobtain the the missing constraint,

$$
\beta_{34}^1 = 0,
$$

(6.3)

which results in $\mathcal{N} = 3$ equations.

At this point, it is worth examining the BPS nature of $\mathcal{N} = \frac{5}{2}$ from the point of view of the stress components of energy-momentum tensor, $T_{i\ell}$. In section 3.1, we discussed the relation between force and stress components of energy-momentum tensor. From (3.7), we read vanishing $T_{i\ell}$ as a sufficient condition for noninteracting BPS solitons. The terms in the spatial stress components of energy-momentum tensor [22,25] can be reshuffled as

$$
T_{i\ell} = \frac{1}{3}\text{Re}\text{ tr}\left\{\eta_{ij}\left[(\delta_A^{[B}D_0Y^{C]} + i s BC(\beta_A^{BC} + \delta_A^{[B}C_D^{]} + \mu M_A^{[B}Y^{C]})\right]^\dagger
\times (\delta_A^{[B}D_0Y^{C]} - i s BC(\beta_A^{BC} + \delta_A^{[B}C_D^{]} + \mu M_A^{[B}Y^{C]})]\right.
$$

$$
+ ((D_1 + i s \epsilon_{i k}D_k)Y^A)^\dagger((D_j - i s \epsilon_{j l}D_l)Y^A + ((D_1 - i s \epsilon_{i k}D_k)Y^A)^\dagger((D_1 + i s \epsilon_{i j}D_l)Y^A
$$

$$
- (i s)^{i+j}[((D_1 - i s D_2)Y^a + 2\epsilon^{a b}\delta_{34}^1)^\dagger((D_1 + i s D_2)Y^a
$$

$$
+ ((D_1 + i s D_2)Y^p + 2\epsilon^{p q}\beta_{12}^1)((D_1 - i s D_2)Y^p)^\dagger
$$

$$
+ 2\epsilon^{a b}\delta_{34}^1(D_1 + i s D_2)Y^b + 2\epsilon^{p q}\beta_{12}^1((D_1 - i s D_2)Y^q)^\dagger]\right)
$$

$$
- 4\eta_{ij}|\beta_{34}^1|^2\right\}.
$$

(6.4)

where $a, b = 1, 2$ and $p, q = 3, 4$. On imposing the original form of $\mathcal{N} = \frac{5}{2}$ BPS equations, all terms except the last term vanish and we are left with

$$
T_{i\ell} = -\frac{4}{3}\eta_{ij}\text{tr}|\beta_{34}^1|^2.
$$

(6.5)
As seen above, however, the consistency of the equations requires (6.3) and it precisely corresponds to the condition that the stress tensor should vanish to have noninteracting BPS solitons. We will see that this kind of structure reappears in other cases with half-integer supersymmetry. In fact, it turns out that the manipulation of the stress tensor is quite a useful tool to obtain consistency conditions of BPS equations.

6.2 $\mathcal{N}=\frac{3}{2}$ supersymmetry

There are four ways to obtain $\mathcal{N}=\frac{3}{2}$ BPS equations:

(i) $\omega_{12}=0$, $\gamma^1 \omega_{13} = \omega_{24}$,
(ii) $\omega_{13}=0$, $\gamma^1 \omega_{12} = \omega_{34}$,
(iii) $\omega_{13}=0$, $\gamma^1 \omega_{14} = \omega_{23}$,
(iv) $\gamma^1 \omega_{12} = \omega_{34}$, $\gamma^1 \omega_{13} = \omega_{42}$, $\gamma^1 \omega_{14} = \omega_{23}$

in addition to the condition $\gamma^0 \omega_{AB} = i s_{AB} \omega_{AB}$ ($s_{AB} = \pm 1$). We will treat the case (i) and (iv). The cases (ii) and (iii) are similar to the case (i).

6.2.1 $\omega_{12}=0$, $\gamma^1 \omega_{13} = \omega_{24}$ case

In this case the BPS equations are the same as those of the $\mathcal{N}=2$ case (4.1) except that the gauged Cauchy-Riemann equations in the second line of (4.1) are changed to

\begin{align}
(D_1 - i s D_2) Y^3 = 0 & \quad \rightarrow \quad (D_1 - i s D_2) Y^3 = -2 \beta^{24}_1,
(D_1 - i s D_2) Y^4 = 0 & \quad \rightarrow \quad (D_1 - i s D_2) Y^4 = -2 \beta^{13}_2,
\end{align}

and two algebraic constraints $\beta^{24}_1 = 0$ and $\beta^{13}_2 = 0$ disappear. As we did in $\mathcal{N}=\frac{5}{2}$ case, we rewrite a positive semi-definite quantity using the deformed equations (6.6) and (6.7),

\begin{align}
\text{tr}[(|\beta^{24}_1|^2 + |\beta^{13}_2|^2)] &= -\frac{1}{2} \text{tr}[(\beta^{24}_1)^\dagger (D_1 - i s D_2) Y^3 + \beta^{13}_2 (D_1 + i s D_2) Y^4]\n&= -\frac{1}{2}(\partial_1 - i s \partial_2) \text{tr}[(\beta^{24}_1)^\dagger Y^3]
&\quad - \frac{1}{2} \text{tr}[(\beta^{24}_3)^\dagger (D_1 - i s D_2) Y^1 + \beta^{13}_4 ((D_1 + i s D_2) Y^2)^\dagger].
\end{align}

The second trace of (6.8) vanish due to the gauged Cauchy-Riemann equations in the first line of (4.1). Moreover, since $\text{tr}[(\beta^{24}_3)^\dagger Y^3] = -\text{tr}[(\beta^{24}_3)^\dagger Y^1]$ and $\beta^{24}_3 = 0$ still holds, the right hand side vanishes identically. Thus two missing constraints are regained,

$$\beta^{24}_1 = \beta^{13}_2 = 0.$$  \hfill (6.9)

Substituting (6.9) into (6.6)–(6.7) the set of BPS equations for $\mathcal{N} = \frac{3}{2}$ supersymmetry coincides with that for $\mathcal{N} = 2$ supersymmetry. Therefore there is no solution with genuine $\mathcal{N} = \frac{3}{2}$ supersymmetry.
As before, this can be seen by considering the stress tensor. Inserting the $\mathcal{N} = \frac{3}{2}$ BPS equations into the expression of $T_{ij}$, we obtain

$$T_{ij} = -2\eta_{ij}\text{tr}(|\beta_{1}^{24}|^2 + |\beta_{2}^{13}|^2),$$

(6.10)

which vanishes.

6.2.2 $\gamma^{1}\omega_{12} = \omega_{34}, \, \gamma^{1}\omega_{13} = \omega_{42}, \, \gamma^{1}\omega_{14} = \omega_{23}$ case

The supersymmetric condition for this case has an SO(3) symmetry with respect to the indices 2, 3 and 4 and it is reflected on the BPS equations:

$$(D_1 - isD_2)Y^1 = 2\beta_{24}^{34} = 2\beta_{3}^{24},$$

$$(D_1 - isD_2)Y^2 = -2\beta_{1}^{34}, \quad (D_1 + isD_2)Y^2 = -2\beta_{14}^{3} = 2\beta_{3}^{14},$$

$$(D_1 - isD_2)Y^3 = -2\beta_{1}^{42}, \quad (D_1 + isD_2)Y^3 = -2\beta_{12}^{4} = 2\beta_{3}^{12},$$

$$(D_1 - isD_2)Y^4 = -2\beta_{1}^{23}, \quad (D_1 + isD_2)Y^4 = -2\beta_{13}^{2} = 2\beta_{3}^{13},$$

$$D_0Y^1 + is(\beta_{21}^{1} + \mu Y^1) = 0, \quad D_0Y^2 - is(\beta_{12}^{1} + \mu Y^2) = 0,$$

$$D_0Y^3 - is\beta_{1}^{13} = 0, \quad D_0Y^4 - is\beta_{1}^{14} = 0,$$

$$\beta_{3}^{31} = \beta_{4}^{41} = \beta_{2}^{21} + \mu Y^1, \quad \beta_{4}^{43} = \mu Y^3, \quad \beta_{3}^{34} = \mu Y^4,$$

$$\beta_{3}^{32} = \beta_{4}^{42} = \beta_{2}^{23} = \beta_{2}^{24} = 0.$$

(6.11)

When we set the $\beta^{BC}_{A} = 0 (A \neq B \neq C \neq A)$, the BPS equations (6.11) are the same as those of $\mathcal{N} = 3$ case in (3.3).

One may wonder whether there is any supersymmetry enhancement due to consistency between the equations as in the previous subsections. The answer turns out to be positive but the argument is rather subtle in this case. We first calculate the stress tensor using (6.11),

$$T_{ij} = -\frac{2}{3}\eta_{ij}\text{tr} \sum_{A\neq B\neq C\neq A} |\beta_{C}^{AB}|^2.$$

(6.12)

Note that, as in the previous cases, the summation consists of positive semi-definite terms which are actually the missing constraints not existing in (6.11). Then one may suspect that the missing constraints should be obtained from (6.11) by similar manipulations to (6.8). After inserting (6.11) into (6.12), we find

$$T_{ij} = -\frac{1}{3}\eta_{ij}\text{tr}\epsilon_{ijk} (\partial_1 - is\partial_2)[(\beta_{i}^{1j})^1Y^1],$$

(6.13)

where $i, j, k = 2, 3, 4$. From this expression, we cannot directly conclude $T_{ij} = 0$ because none of the terms can be put to zero by itself. Note, however, that the terms inside the total derivative
vanish for vacuum configurations, see (2.13). Integrating over the whole space, it then should go to zero for finite energy configurations, i.e.,
\[ \int d^2 x T_{ij} = 0. \] (6.14)
From (6.12), it is now obvious that each positive semi-definite term should vanish: \( \beta_{BC}^A = 0 \) \((A \neq B \neq C \neq A)\). This concludes the argument that the case (iv) with \( \mathcal{N} = 3/2 \) supersymmetry is actually enhanced to \( \mathcal{N} = 3 \) case.

### 6.3 \( \mathcal{N} = \frac{1}{2} \) supersymmetry

There are two ways to obtain \( \mathcal{N} = \frac{1}{2} \) BPS equations:

(i) \( \omega_{12} = 0, \omega_{13} = 0, \gamma^1 \omega_{14} = \omega_{23}, \)

(ii) \( \omega_{13} = 0, \omega_{14} = 0, \gamma^1 \omega_{12} = \omega_{34}, \)

under the condition \( \gamma^0 \omega_{AB} = i s_{AB} \omega_{AB} \) \((s_{AB} = \pm 1)\). Here we will consider only the case (i). The case (ii) is similar to case (i).

The BPS equations of the case (i) are almost the same as those of \( \mathcal{N} = 1 \) case; the only change is the deformation of gauged Cauchy-Riemann equations in the first line of (5.1)
\[
(D_1 - i s D_2) Y^a = 0 \quad \rightarrow \quad (D_1 - i s D_2) Y^a = 2 \epsilon^{ab} \beta^3_{b}, \quad (a, b = 1, 2), \quad \text{(6.1)}
\]
\[
(D_1 + i s D_2) Y^p = 0 \quad \rightarrow \quad (D_1 + i s D_2) Y^p = 2 \epsilon^{pq} \beta^1_{q}, \quad (p, q = 3, 4) \quad \text{(6.2)}
\]
with nonvanishing \( \beta^3_{b} \) and \( \beta^1_{q} \). Like the case discussed in section 6.2.2, we cannot fix \( \beta^3_{b} \) and \( \beta^1_{q} \) to zero by algebraic manipulations of the BPS equations. In order to figure out the BPS nature of \( \mathcal{N} = \frac{1}{2} \) object we calculate \( T_{ij} \). Applying the BPS equations into the \( T_{ij} \), we have
\[
T_{ij} = -4 \eta_{ij} \text{tr} \left( |\beta^3_3|^2 + |\beta^3_4|^2 + |\beta^1_1|^2 + |\beta^1_2|^2 \right)
\]
\[
= 2 \eta_{ij} \text{tr}(\partial_1 - i \partial_2)[(\beta^1_1)^\dagger Y^2], \quad \text{(6.3)}
\]
which is again a summation of positive semi-definite terms and, at the same time, a total derivative of a term vanishing for vacuum configurations. Therefore we recover the missing constraints, \( \beta^3_3 = 0 \) and \( \beta^1_2 = 0 \). Then the supersymmetry is actually enhanced to \( \mathcal{N} = 1 \).

### 7 Conclusion

We investigated the vortex-type BPS equations with various supersymmetries in the ABJM theory without or with mass-deformation. For a given number of supersymmetry, we classified distinguishable BPS conditions, and then obtained the BPS equations and the energy bound. As a nontrivial consistency check of the BPS equations, we investigated the stress components of energy-momentum tensor and showed that it vanishes. Then, we set a special type of ansatz
which solves constraints of the BPS equations, based on the discrete vacua of the mass-deformed ABJM theory. Using these ansätze for $U(N) \times U(N)$ gauge group, we obtained several types of BPS vortex equations with finite energy.

For the undeformed ABJM theory we obtained $\mathcal{N} = 2$ BPS equations. After solving all the constraint equations of the BPS equations for $U(2) \times U(2)$ gauge group, we showed that the resulting equations are reduced to the Liouville-type or Sinh-Gordon-type vortex equations in special limits.

For the mass-deformed theory with $U(N) \times U(N)$ gauge group, we obtained special types of $\mathcal{N} = 3, 2, 1$ BPS configurations. In constructing these configurations, we used ansätze based on the vacuum solutions to solve the complicated constraint equations and to obtain finite energy configurations. Our BPS vortex equations are summarized as follows:

| $\mathcal{N}$ | gamma matrix projection | vortex-type equation |
|----------|------------------------|---------------------|
| 3        | $\gamma^0_{AB} = \pm i\omega_{AB}$ | K-MH on $Y^1$, vacua along $(Y^2, Y^3, Y^4)$ |
| 2        | $\gamma^0_{AB} = \pm i\omega_{AB}$, $\omega_{12} = 0$ | $K$-pairs of CDE on $(Y^1, Y^2)$, vacua along $(Y^3, Y^4)$ |
| 2        | $\gamma^0_{AB} = \pm i\omega_{AB}$, $\omega_{13} = 0$ | K-MH on $Y^1$, $I$-MH on $Y^3$, vacua along $(Y^2, Y^4)$ |
| 1        | $\gamma^0_{AB} = \pm i\omega_{AB}$, $\omega_{13} = \omega_{14} = 0$ | $K$-pairs of CDE on $(Y^1, Y^2)$, $I$-pairs of CDE on $(Y^3, Y^4)$ |
| 1        | $\gamma^0_{AB} = \pm i\omega_{AB}$, $\omega_{12} = \omega_{13} = 0$ | $K$-pairs of CDE on $(Y^1, Y^2)$, $I$-pairs of CDE on $(Y^3, Y^4)$ |

where MH denotes the vortex equations in Maxwell-Higgs theory and CDE the coupled second-order differential equations discussed in section 4, and $K$ and $I$ indicate the numbers of nonvanishing blocks of vacuum solutions on $(Y^1, Y^2)$- and $(Y^3, Y^4)$-planes, respectively. The two $\mathcal{N} = 1$ cases are actually equivalent. See Appendix A.2 for the details.

In section 6, we also analyzed the cases of half-integer supersymmetries. With the help of the stress tensor $T_{ij}$, we showed that the supersymmetries are actually enhanced to integer ones. In other words, the BPS equations with $\mathcal{N} = \frac{5}{2}$, $\mathcal{N} = \frac{3}{2}$, and $\mathcal{N} = \frac{1}{2}$ supersymmetries are respectively equivalent to those of $\mathcal{N} = 3$, $\mathcal{N} = 2$ or 3 (depending on the supersymmetry conditions), and $\mathcal{N} = 1$ supersymmetries.

The BPS configurations of the $\mathcal{N} = 3, 2, 1$ BPS vortex equations in the undeformed ABJM theory were interpreted as intersecting M2-branes spanning one, two, and four complex coordinates in transverse directions, respectively [16]. However, the brane interpretation of the BPS vortex equations in the mass-deformed ABJM theory in M-theory ($k \ll N^\frac{4}{3}$) is unclear up to now, though the configuration of the $\mathcal{N} = 3$ vortex equations in the Maxwell-Higgs theory obtained in section 3 can be identified with D0-branes in type IIA string theory ($N^\frac{4}{3} \ll k \ll N$) [15]. In this paper, we obtained some pairs of coupled differential equation, which can be reduced to the vortex equations in Maxwell-Higgs theory or Chern-Simons matter theories, in special limits of the $\mathcal{N} = 2, 1$ vortex-type BPS equations. These pairs of coupled differential equation reflect the complicated vacuum structure [29, 31] of the mass-deformed ABJM theory. It would be interesting if we can identify the corresponding configurations for the coupled differential equation in dual gravity limit [43, 44, 45].
Acknowledgements

This work has been supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2011-0011660) (Y.K.), by the World Class University grant R32-2008-000-10130-0 and the NRF grant funded by the Korea government (MEST) (No. 2012-045385) (C.K.), by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2011-0009972) (O.K.), and by National Science Council and National Center for Theoretical Sciences, Taiwan, R.O.C. (H.N.).

A Vortex-type Objects with Other $\mathcal{N}=1, 2$ Supersymmetries

We listed the various supersymmetric cases in Fig.1 by imposing different supersymmetric conditions to the supersymmetric parameters $\omega^{AB}$. We did not treat all of those cases in the body of this paper. In this appendix, we briefly discuss their possible soliton objects.

A.1 $\mathcal{N}=2$ supersymmetry ($\omega_{13} = 0$)

- **BPS equations and BPS bound**: In section 4, we obtained the BPS solutions for $\omega_{12} = 0$ case. Here we will treat the $\omega_{13} = 0$ case (or equivalently $\omega_{14} = 0$ case which is connected by field redefinition, $Y^3 \leftrightarrow Y^4$). Then the BPS equations of this case are given by

\[
\begin{align*}
(D_1 - isD_2)Y^1 &= 0, \\
D_1Y^p &= D_2Y^p = 0, \quad (p = 2, 4), \\
D_0Y^1 + is\beta_{31}^{13} &= 0, \\
D_0Y^3 - is\beta_{41}^{13} &= 0, \\
\beta_4^{24} &= \beta_2^{24} = 0, \\
\beta_1^{23} &= \beta_2^{34} = \beta_2^{14} = \beta_3^{12} = \beta_3^{14} = \beta_4^{12} = \beta_4^{23} = 0.
\end{align*}
\]

By using the BPS equations (A.1) we can show that the energy is bounded by R-charge,

\[
E \geq \mu R_{24}, \quad (A.2)
\]

where

\[
R_{24} = \int d^2x \, tr J_{24}^0 = \int d^2x \, tr \left[ i(Y^2D_0Y_2^\dagger - D_0Y^2Y_2^\dagger) + i(Y^4D_0Y_4^\dagger - D_0Y^4Y_4^\dagger) \right]. \quad (A.3)
\]

- **BPS objects**: As we did in previous sections, we consider special types of solutions, based the vacuum solutions of the mass-deformed ABJM theory. From the shapes of the gauged Cauchy-Riemanns equations in (A.1) one can naturally consider an ansatz, which is constructed from
corresponding second order differential equations for scalar field $s$ as

\[ \partial \bar{\partial} \ln \frac{|a_k|^2}{\prod_{p_k=1} |z - z_{p_k}|^2} = \mu^2(|a_k|^2 - 1), \quad \partial \bar{\partial} \ln \frac{|d_i|^2}{\prod_{r_i=1} |z - z_{r_i}|^2} = \mu^2(|d_i|^2 - 1). \quad (A.4) \]

The energy of this configuration is given by

\[ E = k \mu \sum_k \frac{m_k(m_k + 1)n_{a,k}}{2} + k \mu \sum_i \frac{m_i(m_i + 1)n_{d,i}}{2}. \quad (A.5) \]

The scalar equations in (A.4) represent two sets of Maxwell-Higgs type vortex equations living on $(Y^1, Y^2)$- and $(Y^3, Y^4)$-planes. Each set of the equations appears in the $\mathcal{N} = 3$ BPS configurations. Since there exist only Maxwell-Higgs type of vortex equations, the angular momentum of this configuration vanishes.

### A.2 $\mathcal{N} = 1$ supersymmetry ($\omega_{12} = \omega_{13} = 0$)

- **BPS equations and BPS bound**: If we choose $\omega_{12} = \omega_{13} = 0$, the BPS equations are

\[
\begin{align*}
(D_1 - isD_2)Y^a &= 0, \quad (a = 1, 4), \\
(D_1 + isD_2)Y^p &= 0, \quad (p = 2, 3), \\
D_0Y^1 + is(\beta_C^{14} - 2\beta_4^{11} + \mu Y^1) &= 0, \\
D_0Y^2 - is(\beta_C^{24} - 2\beta_3^{32} + \mu Y^2) &= 0, \\
D_0Y^3 - is(\beta_C^{34} - 2\beta_2^{31} - \mu Y^3) &= 0, \\
D_0Y^4 + is(\beta_C^{34} - 2\beta_1^{13} - \mu Y^4) &= 0,
\end{align*}
\]

\[ \beta_1^{23} = \beta_2^{14} = \beta_3^{14} = \beta_4^{23} = 0. \quad (A.6) \]

Then we can write the energy bound as

\[ E \geq \mu(R_{12} + R_{34}), \quad (A.7) \]

where $R_{12} + R_{34}$ is

\[
R_{12} + R_{34} = \int d^2 x \text{tr} (J_{12}^0 + J_{34}^0) = \int d^2 x \text{tr} \left[ i(Y^1D_0Y_1^\dagger - D_0Y^1Y_1^\dagger) - i(Y^2D_0Y_2^\dagger - D_0Y^2Y_2^\dagger) + i(Y^3D_0Y_3^\dagger - D_0Y^3Y_3^\dagger) - i(Y^4D_0Y_4^\dagger - D_0Y^4Y_4^\dagger) \right].
\]

\[ (A.8) \]

- **BPS objects**: Note that if we change the relative signatures between $D_1Y^{2,4}$ and $D_2Y^{2,4}$ in the Cauchy-Riemann type equations in the first line of (A.6), they are the same as those of $\mathcal{N}=1$ case with $\omega_{13} = \omega_{14} = 0$ (see (5.11)). Therefore the resulting second order differential equations of this case are equivalent to the $\mathcal{N}=1$ case with $\omega_{13} = \omega_{14} = 0$ up to substitutions $b_k \leftrightarrow b_k^*$ and $e_i \leftrightarrow e_i^*$ in the ansatz (5.12). As a result, in this case we have the same BPS soliton solutions as those of $\mathcal{N} = 1$ case in section 5.
References

[1] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. D 75, 045020 (2007) [arXiv:hep-th/0611108]; “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. D 77, 065008 (2008) [arXiv:0711.0955 [hep-th]]; “Comments On Multiple M2-branes,” JHEP 0802, 105 (2008) [arXiv:0712.3738 [hep-th]].

[2] A. Gustavsson, “Algebraic structures on parallel M2-branes,” Nucl. Phys. B 811, 66 (2009) [arXiv:0709.1260 [hep-th]].

[3] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP 0810, 091 (2008) [arXiv:0806.1218 [hep-th]].

[4] C. Krishnan and C. Maccaferri, “Membranes on Calibrations,” JHEP 0807, 005 (2008) [arXiv:0805.3125 [hep-th]].

[5] G. Bonelli, A. Tanzini and M. Zabzine, “Topological branes, p-algebras and generalized Nahm equations,” Phys. Lett. B 672 (2009) 390 [arXiv:0807.5113 [hep-th]].

[6] K. Hosomichi, K. M. Lee and S. Lee, “Mass-Deformed Bagger-Lambert Theory and its BPS Objects,” Phys. Rev. D 78, 066015 (2008) [arXiv:0804.2519 [hep-th]].

[7] J. Kim and B. H. Lee, “Abelian Vortex in Bagger-Lambert-Gustavsson Theory,” JHEP 0901, 001 (2009) [arXiv:0810.3091 [hep-th]].

[8] I. Jeon, J. Kim, N. Kim, S. W. Kim and J. H. Park, “Classification of the BPS states in Bagger-Lambert Theory,” JHEP 0807, 056 (2008) [arXiv:0805.3236 [hep-th]].

[9] I. Jeon, J. Kim, B. -H. Lee, J. -H. Park and N. Kim, “M-brane bound states and the supersymmetry of BPS solutions in the Bagger-Lambert theory,” Int. J. Mod. Phys. A 24, 5779 (2009) [arXiv:0809.0856 [hep-th]].

[10] S. Terashima, “On M5-branes in N=6 Membrane Action,” JHEP 0808, 080 (2008) [arXiv:0807.0197 [hep-th]].

[11] K. Hanaki and H. Lin, “M2-M5 Systems in N=6 Chern-Simons Theory,” JHEP 0809, 067 (2008) [arXiv:0807.2074 [hep-th]].

[12] A. Mohammed, J. Murugan and H. Nastase, “Looking for a Matrix model of ABJM,” Phys. Rev. D 82, 086004 (2010) [arXiv:1003.2599 [hep-th]].

[13] M. Arai, C. Montonen and S. Sasaki, “Vortices, Q-balls and Domain Walls on Dielectric M2-branes,” JHEP 0903, 119 (2009) [arXiv:0812.4437 [hep-th]].
[14] C. Kim, Y. Kim, O. K. Kwon and H. Nakajima, “Vortex-type Half-BPS Solitons in ABJM Theory,” Phys. Rev. D 80, 045013 (2009) [arXiv:0905.1759 [hep-th]].

[15] R. Auzzi and S. P. Kumar, “Non-Abelian Vortices at Weak and Strong Coupling in Mass Deformed ABJM Theory,” JHEP 0910, 071 (2009) [arXiv:0906.2366 [hep-th]].

[16] T. Fujimori, K. Iwasaki, Y. Kobayashi and S. Sasaki, “Classification of BPS Objects in N = 6 Chern-Simons Matter Theory,” JHEP 1010, 002 (2010) [arXiv:1007.1588 [hep-th]].

[17] X. Han and Y. Yang, “Existence Theorems for Vortices in the Aharony–Bergman–Jaferis–Maldacena Model,” arXiv:1209.3078 [math-ph].

[18] S. Kawai and S. Sasaki, “BPS Vortices in Non-relativistic M2-brane Chern-Simons-matter Theory,” Phys. Rev. D 80, 025007 (2009) [arXiv:0903.3223 [hep-th]].

[19] J. Hong, Y. Kim and P. Y. Pac, “On The Multivortex Solutions Of The Abelian Chern-Simons-Higgs Theory,” Phys. Rev. Lett. 64, 2230 (1990); R. Jackiw and E. J. Weinberg, “Selfdual Chern-Simons Vortices,” Phys. Rev. Lett. 64, 2234 (1990).

[20] C. Lee, K. M. Lee and E. J. Weinberg, “Supersymmetry And Selfdual Chern-Simons Systems,” Phys. Lett. B 243, 105 (1990).

[21] R. Jackiw, K. M. Lee and E. J. Weinberg, “Selfdual Chern-Simons solitons,” Phys. Rev. D 42, 3488 (1990).

[22] C. Kim, “Selfdual vortices in the generalized Abelian Higgs model with independent Chern-Simons interaction,” Phys. Rev. D 47, 673 (1993) [arXiv:hep-th/9209110].

[23] L. F. Cugliandolo, G. Lozano, M. V. Manias and F. A. Schaposnik, “Bogomolny equations for nonAbelian Chern-Simons Higgs theories,” Mod. Phys. Lett. A 6, 479 (1991); K. M. Lee, “Selfdual nonabelian Chern-Simons solitons,” Phys. Rev. Lett. 66, 553 (1991); “Relativistic nonAbelian selfdual Chern-Simons systems,” Phys. Lett. B 255, 381 (1991).

[24] E. F. Moreno and F. A. Schaposnik, “BPS Equations and the Stress Tensor,” Phys. Lett. B 673, 72 (2009) [arXiv:0811.2359 [hep-th]].

[25] C. -j. Kim, C. -k. Lee, P. -w. Ko, B. -H. Lee and H. -s. Min, “Schrodinger fields on the plane with U(1)**N Chern-Simons interactions and generalized selfdual solitons,” Phys. Rev. D 48, 1821 (1993) [hep-th/9303131].

[26] M. A. Bandres, A. E. Lipstein and J. H. Schwarz, “Studies of the ABJM Theory in a Formulation with Manifest SU(4) R-Symmetry,” JHEP 0809, 027 (2008) [arXiv:0807.0880 [hep-th]].
[27] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park and P. Yi, “A Nonperturbative Test of M2-Brane Theory,” JHEP **0811**, 058 (2008) [arXiv:0809.1771 [hep-th]].

[28] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, “N=5,6 Superconformal CS Theories and M2-branes on Orbifolds,” JHEP **0809**, 002 (2008) [arXiv:0806.4977 [hep-th]].

[29] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, “A Massive Study of M2-brane Proposals,” JHEP **0809**, 113 (2008) [arXiv:0807.1074 [hep-th]].

[30] D. Gaiotto, S. Giombi and X. Yin, “Spin Chains in N=6 Superconformal Chern-Simons-Matter Theory,” JHEP **0904**, 066 (2009) [arXiv:0806.4589 [hep-th]].

[31] H. C. Kim and S. Kim, “Supersymmetric vacua of mass-deformed M2-brane theory,” Nucl. Phys. B **839**, 96 (2010) [arXiv:1001.3153 [hep-th]].

[32] N. J. Hitchin, “The Selfduality equations on a Riemann surface,” Proc. Lond. Math. Soc. **55**, 59 (1987).

[33] S. Mukhi, C. Papageorgakis, “M2 to D2,” JHEP **0805**, 085 (2008) [arXiv:0803.3218 [hep-th]].

[34] Y. Pang, T. Wang, “From N M2’s to N D2’s,” Phys. Rev. **D78**, 125007 (2008) [arXiv:0807.1444 [hep-th]].

[35] Y. Kim, O-K. Kwon, D. D. Tolla, “Mass-Deformed Super Yang-Mills Theories from M2-Branes with Flux,” JHEP **1109**, 077 (2011) [arXiv:1106.3866 [hep-th]].

[36] G. Go, O-K. Kwon and D. D. Tolla, “$\mathcal{N} = 3$ Supersymmetric Effective Action of D2-branes in Massive IIA String Theory,” Phys. Rev. D **85**, 026006 (2012) [arXiv:1110.3902 [hep-th]].

[37] I. Jeon, N. Lambert and P. Richmond, “Periodic Arrays of M2-Branes,” [arXiv:1206.6699 [hep-th]].

[38] Y. Kim, O-K. Kwon and D. D. Tolla, “Partially Supersymmetric ABJM Theory with Flux,” [arXiv:1209.5817 [hep-th]].

[39] R. Friedberg, T. D. Lee and A. Sirlin, “A Class Of Scalar-Field Soliton Solutions In Three Space Dimensions,” Phys. Rev. D **13**, 2739 (1976);

[40] R. Wang, “The existence of Chern-Simons vortices,” Commun. Math. Phys. **137**, 587 (1991).

[41] J. Spruck and Y. Yang, “The Existence of nontopological solitons in the selfdual Chern-Simons theory,” Commun. Math. Phys. **149** (1992) 361,

D. Chae and O. Y. Imanuvilov, “The existence of nontopological multivortex solutions in the relativistic selfdual Chern-Simons theory,” Commun. Math. Phys. **215**, 119 (2000).
[42] Y. Kim and K. M. Lee, “Vortex dynamics in selfdual Chern-Simons Higgs systems,” Phys. Rev. D 49, 2041 (1994) [arXiv:hep-th/9211035].

S. R. Coleman, “Q Balls,” Nucl. Phys. B 262, 263 (1985) [Erratum-ibid. B 269, 744 (1986)].

[43] I. Bena and N. P. Warner, “A Harmonic family of dielectric flow solutions with maximal supersymmetry,” JHEP 0412, 021 (2004) [hep-th/0406145].

[44] H. Lin, O. Lunin and J. M. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP 0410, 025 (2004) [hep-th/0409174].

[45] S. Cheon, H. C. Kim and S. Kim, “Holography of mass-deformed M2-branes,” arXiv:1101.1101 [hep-th].