The Astrophysical Journal

TOPOLOGY OF A LARGE-SCALE STRUCTURE AS A TEST OF MODIFIED GRAVITY

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ABSTRACT

The genus of the isodensity contours is a robust measure of the topology of a large-scale structure, and it is relatively insensitive to nonlinear gravitational evolution, galaxy bias, and redshift-space distortion. We show that the growth of density fluctuations is scale dependent even in the linear regime in some modified gravity theories, which opens a new possibility of testing the theories observationally. We propose to use the genus of the isodensity contours, an intrinsic measure of the topology of the large-scale structure, as a statistic to be used in such tests. In Einstein’s general theory of relativity, density fluctuations grow at the same rate on all scales in the linear regime, and the genus per comoving volume is almost conserved as structures grow homologously, so we expect that the genus–smoothing-scale relation is basically time independent. However, in some modified gravity models where structures grow with different rates on different scales, the genus–smoothing-scale relation should change over time. This can be used to test the gravity models with large-scale structure observations. We study the cases of the $f(R)$ theory, DGP braneworld theory as well as the parameterized post-Friedmann models. We also forecast how the modified gravity models can be constrained with optical/IR or redshifted 21 cm radio surveys in the near future.

Key word: large-scale structure of universe

1. INTRODUCTION

The large-scale structure has long been a major source of information on cosmic evolution. The most frequently used statistics of the large-scale structure, such as the two-point correlation function and the density power spectrum, and especially the baryon acoustic oscillation (BAO) signatures in the power spectrum have been widely used in tests of cosmological models and precise measurements of cosmological parameters (Tegmark et al. 2006; Percival et al. 2007; Gaztanaga et al. 2009). However, these two-point statistics do not exhaust all of the information content about the large-scale structure—at least not in the non-Gaussian case. There are also some limitations on the application of these statistics. As the perturbations grow larger, their evolution becomes nonlinear, resulting in a distorted power spectrum and the emergence of gravity-induced non-Gaussianity. This has a crucial influence in the extraction of the absolute BAO scale: at $k = 0.1 \sim 0.15 \text{ Mpc} h^{-1}$ where the first BAO peak is located, the nonlinear evolution of the power spectrum amounts to 2%–5% differences between $z = 0$ and $z = 0.5$ (Kim et al. 2009). Accurately modeling the nonlinear evolution is highly non-trivial, and practical application in actual model test/parameter determination also requires a fast and easy-to-implement algorithm. Although progress has been made in more sophisticated perturbative expansion (Crocce & Scoccimarro 2006a, 2006b, 2008; Matsubara 2008a, 2008b; Smith et al. 2007) or reconstruction (Padmanabhan et al. 2009; Noh et al. 2009) procedures to correct for the effect of nonlinear evolution, a precise yet practical method to take these effects into account has yet to be developed. Other systematic effects, such as the clustering bias and the redshift distortion, could also affect the outcome of the measurement. Such effects may significantly limit the accuracy of cosmological tests for future redshift surveys, where the statistical error on the BAO scale would be at the 1% level.

An alternative approach to the correlation function and power spectrum is the topology of the isodensity contours, in particular the genus of these contours. This offers another way to characterize the statistical property of the large-scale structure, and is insensitive to the systematic effects discussed above since the intrinsic topology does not change as the structures grow, at least not until the isodensity contours eventually break at shell crossing (Park & Kim 2010; Park et al. 2005). This provides a robust statistic for cosmology. Compared with the power spectrum, the genus changes by only 0.5% between $z = 0.5$ and $z = 0$ (Y.-R. Kim et al., in preparation). This is particularly true when using the volume fraction for the threshold levels to identify the contours. According to the second-order perturbation theory (Matsubara 1994), there is no change in the genus at median density threshold due to the weakly nonlinear evolution, because the isodensity contours enclosing a given fraction of volume do not change as long as the gravitational evolution conserves the rank ordering of the density. Similarly, a monotonic clustering bias does not result in any difference in the isodensity contours, and a continuous coordinate mapping into redshift space does not affect the genus statistics either. In the $\Lambda$CDM model with general relativity, the genus is conserved during the different epochs in the linear regime of evolution. Therefore, by observing the genus curve at the different redshifts and smoothing scales, one can use it as a robust standard ruler for cosmological measurements (Park & Kim 2010).

In the present paper, we consider the phenomenological consequences of alternative gravity theories. A number of such models, e.g., the $f(R)$ theory (Carroll et al. 2004; Capozziello et al. 2003; Nojiri & Odintsov 2003; Song et al. 2007a) and the DGP braneworld model (Dvali et al. 2000), have been proposed to explain the accelerated expansion of the universe. Unlike general relativity (GR), where the growth of the density fluctuation is at the same rate on all scales during
the linear evolution, the modified large-scale gravitational forces induced by an extra scalar field can in general introduce new scale dependence in the growth of structure, and therefore distort the genus curve in a time-dependent way. This provides us with a new tool to distinguish GR from its various alternatives. It probes the combined effects of background expansion and the growth of the structures, and therefore is able to break the degeneracy between the dark energy and modified gravity.

The genus is given by means of the Gauss–Bonnet theorem (Hamilton et al. 1986)

\[ g = -\frac{1}{4\pi} \int k dS, \]

where \( g \) is related to the integration of the Gauss curvature \( \kappa = 1/(r_1 r_2) \) over the surface, and \( r_1 \) and \( r_2 \) are the principal radii of curvature at the integration point. In practice, after smoothing the galaxy samples or simulation data into a continuous density field, one first makes a fine tessellation of the whole volume of the space with small polyhedra, e.g., cubes or truncated octahedra; the isodensity surfaces can then be approximated by the polyhedral surfaces. Since the only nonzero contributions to the integrated curvature of the polyhedral surfaces are from the vertices and are equal to the angle deficits, the genus of the isodensity contour at \( \delta_c \) can be obtained by the summation of the angle deficit over all vertices (Gott et al. 1986; Weinberg et al. 1987). In this way, the genus of the isodensity contours is measured independently of the two-point statistics such as the power spectrum or correlation function, though mathematically we know that they should be related by Equation (2) in the Gaussian case.

As the structures grow, the fluctuations would gradually become non-Gaussian, and the genus is no longer solely determined from the nonlinear power spectrum of the structure. However, the amplitude of the genus curve \( A \) does not change too much in the weakly nonlinear regime, because the genus is a topological indicator which is independent of simple growth of clustering without merger. A general correction to \( \mathcal{G}(v) \) for the non-Gaussian field has been derived up to the second-order perturbation (Matsubara 1994, 2003):

\[ \mathcal{G}_{2\nu}(v) = A e^{-v^2/2} \left\{ (1 - v^2) - \sigma \left[ \frac{S}{6} H_3(v) + \frac{3T}{2} H_3(v) + 3U H_1(v) \right] + \cdots \right\}, \]

where \( H_n(v) = (-1)^n \frac{e^{v^2/2} (d/dv)^n e^{-v^2/2}}{n!} \) is the \( n \)-th-order Hermite polynomial, and \( S, T, \) and \( U \) are the third-order moments of \( \delta \) and its gradient \( \nabla \delta \). At the median density threshold \( \langle v \rangle = 0 \), these corrections are zero, so the amplitude of the genus curve does not change. In practice, we calculate \( A \) from a set of points measured between \( v = -1 \) and \( +1 \). Since \( H_{2\nu+1}(v) \) are all odd functions in \( v \), \( A \) would not be affected by non-Gaussianity up to the second order. At smaller scales, further investigation (Y.-R. Kim et al., in preparation) shows that the directly measured genus amplitude is better conserved than the shape of the power spectrum (Equation (2)). The insensitivity of the genus to the nonlinear structure formation is not an artifact of the smoothing procedure, but reflects the more fundamental nature of the topological property of the random field.

As demonstrated by Park & Kim (2010), when the correct redshift–distance relationship \( r(z) \) is adopted, one would be using the same comoving volume \( V(z) = (D^2\bar{H})/H(z) \) and smoothing scale \( R_c \) at different epochs. Here \( D_A(z) \) is the angular diameter distance and \( H(z) \) is the Hubble expansion rate. Therefore, the data at different redshifts actually enclose statistically almost the same amount of structures, and the amplitude of genus curve \( A \) would be almost the same at different redshifts. This can also be seen from Equation (2), which shows that \( A \) actually measures the slope of the power

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spectrum around the smoothing scale $R_G$. Since in the linear regime of GR only the growth rate of the structure, not its shape, evolves, $A$ would be conserved. However, if an incorrect $r(z)$ due to an incorrect cosmology is adopted, both $V(z)$ and $R_G$ would be misestimated. Since the topology of the structure is not scale-free, the genus enclosed in a wrongly sized volume and smoothed with a wrong scale would lead to deviation from the actual one; in this case

$$A_Y(z, R_G; Y) R_{G,Y}^3 = A_X(R_G; X) R_{G,X}^3,$$

where $Y$ represents the adopted cosmology and $X$ the true cosmology. The smoothing scales $R_G$ for different cosmologies are related to each other by

$$(R_{G,X}/R_{G,Y})^3 = (D^2_A/H)_X/(D^2_A/H)_Y.$$ (7)

Utilizing this effect, the topology of the large-scale structure can serve as a standard ruler in cosmology.

Since the genus would be affected by modifying the density perturbation through Equation (2), we now consider the evolution of density perturbations in modified gravity. A wide range of modified gravity theories, which satisfy the basic requirement of being a metric theory where energy–momentum is covariantly conserved, can be studied in the so-called parameterized post-Friedmann (PPF) framework (Hu & Sawicki 2007). For these theories, on superhorizon scales structure evolution must be compatible with background evolution; on intermediate scales the theory behaves as a scalar–tensor theory with a modified Poisson equation; and on small scales, to pass stringent local tests, the additional scalar degree of freedom must be suppressed. The evolution of linear perturbations in theories which satisfy these conditions can be characterized by a few parameters. With the Newtonian gauge temporal and spatial curvature scalar perturbation $\Psi$ and $\Phi$, we introduce

$$g \equiv \Phi + \Psi, \quad \Phi_- \equiv \Phi - \Psi/2.$$ (8)

In the absence of anisotropic stress, $\Phi = -\Psi$, and $g = 0$ for the GR case. However, in modified gravity $g$ may not be zero. From causality considerations, on superhorizon scales the evolution of metric perturbation must be determined entirely by the expansion rate and $g$ to first order in $k_{H}$, with

$$\lim_{k_{H} \to 0} \left(\Phi' - \Psi' - \frac{H''}{H'} + \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi\right) = O(k_{H}^2).$$ (9)

where $' = d / d \ln a$, $V_{m} \sim k_{H}$ is the velocity perturbation, and $k_{H} = k / H a$. However, to the next order, we can then introduce $f_{\zeta}$ to characterize the curvature perturbation on superhorizon scales. The subscript $\zeta \equiv \Phi - V_{m}/k_{H}$, which is the curvature perturbation in the matter comoving gauge, indicates the fact that the left-hand side of Equation (9) equals $\zeta$ in the matter comoving gauge. On subhorizon scales, the lensing potential satisfies the modified Poisson equation,

$$k^2\Phi_- = \frac{4\pi G}{1 + f_{G}}a^2\rho_{m}\Delta_{m}(k),$$ (10)

where $\Delta_{m}$ is the fractional density perturbation and $f_{G}$ parameterizes modification to the Newton constant. Equivalently, we can introduce a new scalar field $\varphi$ which characterize the extra source felt by nonrelativistic particles,

$$-k^2\Psi(k) = \frac{4\pi G}{1 + f_{G}}a^2\rho_{m}\Delta_{m}(k) + \frac{1}{2}k^2\varphi(k),$$ (11)

where $\varphi(k)$ satisfies the equation

$$-k^2\varphi(k) = \frac{8\pi G}{1 + f_{G}}a^2g(a, k)\rho_{m}\Delta_{m}(k).$$ (12)

Thus in the subhorizon regime, the scale-dependent deviation from GR is described by $\varphi(k)$. In summary, in the PPF parameterization, the modified gravity model is characterized by $g(a, k)$, $f_{G}(a)$, $f_{\zeta}(a)$, and the superhorizon–subhorizon transition scale $c_{T}$. The evolution of these functions depends on the specific model. The fluctuation power spectrum for such theories can be calculated by using the public Boltzmann code camb with the PPF module$^5$ (Fang et al. 2008a, 2008b; Lewis & Bridle 2002).

3. MODELS

As a first example we consider the $f(R)$ theory,

$$S = \int d^4x \sqrt{-g} \left[ \frac{R + f(R)}{16\pi G} + \mathcal{L}_{m} \right].$$ (13)

where $R$ is the Ricci scalar and $\mathcal{L}_{m}$ is the matter Lagrangian density. For a given expansion history $H(a)$, e.g., the effective dark energy equation of state $w_{e} = -1$, in order to explain the late-time acceleration, the form of $f(R)$ can be determined from a second-order differential equation

$$-f_{R}(HH' + H^2) + \frac{1}{6}f + H^2f_{R,R}R' = \frac{8\pi G}{3}\rho - H^2,$$ (14)

where $\rho$ is the total energy density and $f_{R}$ and $f_{R,R}$ are the first and second derivatives of $f(R)$ with respect to $R$. A given expansion history permits a family of $f(R)$ functions; the additional degree of freedom is usually characterized by $B_{0}$, the present-day value of the function $B(a)$, which is the square of the Compton scale given by

$$B(a) \equiv f_{R,R}/f_{R} R'/H'.$$ (15)

$B_{0}$ is thus a model parameter of the $f(R)$ theory, which is to be constrained by structure growth. Given $H(a)$ and $B(a)$, the metric ratio at superhorizon scale $g_{SH}$ can be obtained by solving the differential equation

$$\Phi'' + \left(1 - \frac{H''}{H^2} + \frac{B'}{1 - B} + \frac{B H'}{H}\right)\Phi' + \left(\frac{H'}{H} - \frac{H''}{H'} + \frac{B'}{1 - B}\right) = 0,$$ (16)

and utilizing the relation $\Psi = (-\Phi - B\Phi')/(1 - B)$. As shown by Hu & Sawicki (2007), the scale-dependent $g(a, k)$ can be well fitted by the interpolation function

$$g(a, k) = g_{SH} + g_{SS}(c_{g}k_{H})^{ps}/(1 + (c_{g}k_{H})^{ps}).$$ (17)

$^5$ http://camb.info/ppf/
where \( g_{QS} = -1/3, c_g = 0.71B^{1/2}, \) and \( n_g = 2, \) with other PPF parameters \( f_\xi = -1/3g, f_G = f_R, \) and \( c_\tau = 1. \)

Given the PPF parameters of the \( f(R) \) theory above, especially \( g(a,k), \) the matter power spectrum at relevant scales can be obtained with the help of Equations (11) and (12). In Figure 1, we plot the genus amplitude of the genus as a function of the smoothing scale \( R_G \) at \( a = 1 \) for several \( f(R) \) models characterized by different \( B_0 \) values. For models with greater \( B_0 \) value, the genus is larger.

We may also see how the amplitude of the genus curve varies as a function of redshift. In Figure 2, we plot the redshift evolution of \( A \) for a fixed smoothing scale \( R_G = 15 \) Mpc \( h^{-1}. \) Compared with the horizontal line, which corresponds to GR in the bottom of the figure, the amplitude of the genus curve exhibits a strong time dependence for the \( f(R) \) theory with a total variation of about 10% for \( B_0 \geq 0.01. \)

From these figures it is obvious that for larger \( B_0, \) the deviation from GR is greater, i.e., the variation of the genus is stronger in the \( f(R) \) theory. By inspecting the perturbative variables in detail, with other PPF parameters we find that this is mainly due to the monotonically increasing deviation of \( \Phi_- \) from the GR prediction with respect to the wavenumber \( k. \) The extra scalar degree of freedom \( \varphi = -f_R \) enhances the gravitational force at smaller scales and therefore increase the slope of the clustering.

Next we consider the self-accelerating Dvali–Gabadadze–Porrati (DGP) braneworld model. In this model, our universe is a \((3+1)\)-dimensional brane embedded in an infinite Minkowski bulk; it can be described by the following action:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{\mathcal{R}(5)}{16\pi G(5)} + \delta(x) \left( \frac{\mathcal{R}_i(4)}{16\pi G_i(4)} + \mathcal{L}_m \right) \right],
\]

where \( \mathcal{R}_i \) and \( G_i, i = 4, 5, \) are the Ricci scalar and Newton constant in the brane and the bulk, respectively. At the background level, Dvali et al. (2000) showed that the consequent Friedmann equation

\[
H^2(a) = \left( \sqrt{\frac{8\pi G(4)}{3}} \sum_i a_i + \frac{1}{4r_c^2} + \frac{1}{2r_c} \right)^2 - \frac{K}{a^2} \quad (19)
\]

leads to a de-Sitter phase at late time characterized by the crossover scale \( r_c = \frac{G(4)}{2G(5)} \).

Above the horizon, the evolution of the perturbation can be solved by an iterative scaling method (Sawicki et al. 2007), and the metric ratio \( g_{SH} \) is well fitted by the function

\[
g_{SH}(a) = \frac{9}{8Hr_c - 1} \left( 1 + \frac{0.51}{Hr_c - 1.08} \right). \quad (20)
\]

In the quasi-static regime,

\[
g_{QS}(a) = -\frac{1}{3} \left[ 1 - 2Hr_c \left( 1 + \frac{1}{3} \frac{H'}{H} \right) \right]^{-1}. \quad (21)
\]

Thus, the scale-dependent \( g(a, k) \) can be described by the same interpolation equation, Equation (17), with \( c_\tau = 0.4, n_g = 3. \) Other PPF parameters are \( f_\xi = 0.4g, f_G = 0, \) and \( c_\tau = 1. \)

From the calculation, we find that the major scale-dependent modification to the growth of the structure is in the superhorizon regime \( (k/aH \ll 1). \) With a Gaussian smoothing scale of 15 Mpc \( h^{-1}, \) only about 0.07% deviation of \( A \) is found compared with GR; this is unlikely to be distinguished by observations in the near future. In this case, other techniques such as the integrated Sachs–Wolfe (ISW) effect or the growth rate are more viable. However, it should also be noticed that (Scoccimarro 2009; Chan & Scoccimarro 2009) with the help of the Vainshtein mechanism, which brings the gravity back to GR at small scales, the nonlinear evolution can also induce significant scale-dependent deviations at relevant scales. We leave this topic to our further study.

Finally we consider more generic models in the context of PPF; this allows the method to be applied to other possible
modified gravity models, and the analysis would also help us to determine for which kind of model the topological method is more sensitive. We note that among the PPF variables, \( \phi_4(a) \) and \( f_2(a) \) connect the metric fluctuations to the matter fields and are only functions of time. Rescaling the amplitude of the power spectrum would not affect the genus, so we do not need to consider this as the topological measurements would be insensitive to here. Similarly, for the wavenumber range, we note that \( g_{3\text{H}} \) describes the modification to the superhorizon metric can also be excluded from our consideration.

Following Hu (2008), we adopt the scale-dependent \( g \) as an interpolation function between the superhorizon regime \( g_{3\text{H}} \) and the quasi-static regime \( g_{3\text{S}} \) (cf. Equation (17)). Both \( c_g(a) \) and the power index \( n_g \) can affect the behavior of the transition between the two regimes.

Equation (17) shows that for each Fourier mode the time-dependent factor \( c_g k_H = c_g k / (a H) \) determines the moment and scale at which the gravity deviates from the GR case. In this simple model, \( \lim_{k \to 0} g(a, k) = 0; \) therefore \( \varphi \) would be modified when \( k \gtrsim k_t = (a H) / c_g \).

First, let us consider the case where \( c_g \) is a constant. The transition scale would first slowly decrease as \( k_t \sim a^{-1/2} \) during the matter-dominated era, then start to increase during the era of accelerated expansion. In order to get significant effect on scales of \( k \sim O(0.1) h \text{Mpc}^{-1} \) or above, \( c_g \) should be at least larger than \( (a H) / k \), which is about \( O(0.01) \).

As an example, in Figure 3 we plot the genus per smoothing volume as a function of \( R_g \) for various \( g_{3\text{S}} \), while keeping \( g_{3\text{H}} = 0, c_g(a) = 1, \) and \( n_g = 2 \). Compared with the GR case (i.e., \( g_{3\text{S}} = 0 \)), the difference is quite apparent. At any time, the metric deviation Equation (17) is a monotonic function of the wavenumber; this induces the same monotonicity in the deviation of \( \Phi_\varphi \) from GR. Specifically, the negative value of \( g_{3\text{S}} \) and therefore the positive extra force \( -\nabla \varphi \) will increase \( \Phi_\varphi \) as well as the slope of the power spectrum at \( k \gtrsim k_t \), and ultimately increase the value of the genus. On the other hand, positive \( g_{3\text{S}} \) behaves in the opposite way. Nevertheless, owing to the slow evolution of \( k_t \), the variation of \( \Phi_\varphi \) between the present and \( z = 5 \) is less than 1% for \( |g_{3\text{S}}| < 0.5 \).

Next, we consider models in which \( c_g \) varies with time. Here the deviation scale \( k_t \) of the theory can vary significantly with time. As a toy model, let us assume that \( c_g(a) \) depends on the square of the scale factor

\[
\Phi_\varphi = c_g(a) \cdot a^2 \tag{22}
\]

and parameterize the model by the value of \( c_g(0) \). In this case, \( k_t \) will always decrease, and for a fixed wavenumber the absolute value of the deviation \( |g| \) will always increase with time. Consequently, as illustrated in Figure 4, \( \Phi_\varphi \) (upper panel) and the power spectrum (lower panel) will deviate from the GR case progressively.

In Figure 5, we can see the evolution of the genus for different model parameters. In the upper panel, \( g_{3\text{S}} \) is varied, taking values from \( -0.5 \) to \( -0.1 \), while \( c_g = 0.1 a^2 \). As expected, for the cases with larger value of \( g_{3\text{S}} \), the genus amplitude rises more quickly. In the lower panel, we show the result of varying the value of \( c_g(0) \) while keeping \( g_{3\text{S}} = 0.5 \) fixed. In this case, increasing \( c_g(0) \) would move \( k_t \) to larger scales, and shift the onset of the deviation to an earlier epoch, so the genus should deviate from GR further and earlier. However, when \( c_g(0) \gtrsim O(1) \), the modification would move to the superhorizon regime; then it would become saturated with a nearly constant genus.

4. PROSPECTS OF OBSERVATION

The genus curve of isodensity contours could be measured with future large-scale structure surveys. In the following, we consider how such surveys could be used to constrain the modified gravity models with the topological measurements discussed above.
indeed give a good analytical estimate of smaller.

The Astrophysical Journal

Figure 5. Redshift evolution of genus amplitude for phenomenological models described by Equations (17) and (22). The upper panel shows models with different $g_{Q5}$ while keeping $c_s(0) = 0.1$. The lower panel shows models with different $c_s(0)$ while keeping $g_{Q5} = 0.5$. The dashed line corresponds to the GR case in each panel.

The variance of the actual genus measurement is usually estimated with the help of simulation. For the purpose of making forecasts, an analytical calculation is more desirable. In principle, the uncertainty in the topological measurement is different from the uncertainty in the power spectrum of the large-scale structure; this is especially true for the general (non-Gaussian) case. However, as discussed previously, the Gaussian assumption should be a reasonable approximation for the purpose of making forecasts, as far as the large scales are concerned. In the Appendix, we estimate the minimal amount of variance $\sigma_A$ (Equation (A2)) by propagating from the uncertainty in the power spectrum $\sigma_P(k)$. We also compared our analytical estimation with the measurement from Gott et al. (2009), who utilized two volume-limited subsamples of luminous red galaxies to measure the genus statistics: a dense shallow sample at $0.2 < z < 0.36$ with smoothing length $R_d = 21 \, h^{-1}$ Mpc$^{-1}$, and a sparse deep sample at $0.2 < z < 0.44$ with $R_d = 34 \, h^{-1}$ Mpc$^{-1}$. Their result is consistent with the Gaussian distribution with the amplitude $A = 167.4 \pm 7.0$ and $A = 79.6 \pm 6.0$, respectively. By assuming a reasonable bias factor $b = 2$, our formula (Equation (A2)) gives $\sigma_A/A \sim 4.5\%$ for the shallow sample and $\sigma_A/A \sim 7.4\%$ for the deep sample, very close to the measured values (4.1\% and 7.5\%, respectively). This shows that Equation (A2) could indeed give a good analytical estimate of $\sigma_A$.

Throughout this section, we assume the Gaussian smoothing scale $R_d = 15 \, h^{-1}$ Mpc unless explicitly emphasized otherwise. At high redshifts the nonlinear scale decreases, and one can use smaller $R_d$ which makes the uncertainty of the genus much smaller. Taking $O = \ln A$ as the observable, the Fisher matrix is

$$F_{ij} = \sum_{\text{redshift bins}} \frac{\partial O}{\partial p_i} \frac{\partial O}{\partial p_j} + F_{\text{CMB},ij},$$  

where the $p_i$ include $\Omega_m h^2, \Omega_b h^2, h_0, n_s, \ln A, r, w_0$, as well as other modified gravity parameters, e.g., $B_0$ for $f(R)$ theory. The Planck prior is effectively added by the contribution $F_{\text{CMB}}$, which helps to break the degeneracies between the various parameters. The summation is over different redshift bins of size $dz = 0.1$. For the cosmological parameters ($\Omega_m h^2, h_0, w_0$), both the perturbations and background evolution contribute to the constraints. Therefore, derivatives in Equation (23) should be carefully calculated by taking into account the scaling relation

$$A_{\text{obs}}(R_d) = \lambda A_{\text{true}}(\lambda^{-1/3} R_d)$$  

$$\lambda(z) = \frac{D_{\text{ref}}^2}{D_A^2 H_{\text{ref}}^2}.$$  

Here $D_{\text{ref}}$ and $H_{\text{ref}}$ are the angular diameter distance and the Hubble expansion rate evaluated in the reference cosmology, which is used when reconstructing the position from the redshift, and for simplicity we assumed it to be the same as the fiducial cosmology.

We consider the measurement of large-scale structure topology in extended redshift ranges with a number of optical/near IR galaxy surveys at $z < 2$, such as the LAMOST$^6$ (see also Wang et al. 2009), BOSS,$^7$ WFMOS (Bassett et al. 2005), JDEM,$^8$ and BIGBOSS (Schlegel et al. 2009) surveys, as well as a few 21 cm intensity mapping experiments at $0 < z < 5$, e.g., the cylindrical radio telescope (CRT; Chang et al. 2007; Seo et al. 2010) or the Tianlai telescope (Chen 2011) and the MWA.$^9$ The survey parameters we adopt are listed in Table 1. Needless to say, these parameters are only preliminary estimates based on current planning; the parameters for the actual projects are subject to change.

In Figure 6, we illustrate the likelihood distribution over $B_0$ of the $f(R)$ theory at fiducial values $B_0 = 10^{-2}$ and $10^{-4}$ calculated for various galaxies surveys. As one of the projects under planning, BIGBOSS can provide the most stringent constraints: the 1σ error on $B_0$ is 0.0039 at $B_0 = 10^{-2}$ and $5.38 \times 10^{-5}$ at $B_0 = 10^{-4}$. Even for surveys which are ongoing or will start in the near future (e.g., BOSS and LAMOST), one could also gain considerably rich information about the redshift evolution of the genus amplitude. The 1σ errors at the same fiducial values are 0.011 and $1.1 \times 10^{-4}$ for LAMOST and 0.010 and 1.04 $\times 10^{-4}$ for BOSS, respectively. Notably, the constraining conclusion drawn from a particular survey depends on the true cosmology, not only quantitatively but also qualitatively. For example, the WFMOS survey, which is deep but narrow and is more powerful than the BOSS and LAMOST surveys at $B_0 = 10^{-2}$, becomes insignificant at $B_0 = 10^{-4}$. This is because when $B_0$ is small, the amplitude of the genus at high redshift would quickly approach to the value of GR (cf. Figure 2) and become nearly indistinguishable for a narrow survey.

Unlike galaxy surveys, the 21 cm intensity mapping experiments have relatively poor angular resolutions; for the CRT

$$R_{\text{res}}(z) = r(z) \frac{\lambda}{L_{\text{CRT}}}$$  

References:

$^6$ http://www.lamost.org/

$^7$ http://cosmology.lbl.gov/BOSS/, http://www.sdss3.org/

$^8$ http://www.jdem.gsfc.nasa.gov/

$^9$ http://www.MWAtlescope.org/
Table 1

| Galaxy Survey | Redshift Range | Sky Area |
|---------------|----------------|----------|
| BOSS          | 0.2 < z < 0.7  | 100000 deg$^2$ |
| LAMOST        | 0.0 < z < 0.7  | 8000 deg$^2$   |
| WFMOS         | 0.5 < z < 1.3  | 20000 deg$^2$  |
| JDEM          | 0.5 < z < 2.0  | 100000 deg$^2$ |
| BIGBOSS       | 0.2 < z < 2.0  | 240000 deg$^2$ |

| 21 cm Intensity Mapping | Redshift Range | Other Parameter | Integration Time (hr) |
|-------------------------|----------------|-----------------|----------------------|
| MWA                     | 3.5 < z < 5    | $N_a = 500$, $A_s = 16\pi^2$ deg$^2$ | 1000     |
| MWA5000                 | 3.5 < z < 5    | $N_a = 5000$, $A_s = 16\pi^2$ deg$^2$ | 4000     |
| CRT                     | 0 < z < 2.5    | $L = 100$ m, $W = 15$ m x 7 | 100000   |

Figure 6. Likelihood distribution on $B_0$ of $f(R)$ theory for various optical/IR surveys. Two fiducial value ($B_0 = 0.01$ and 0.0001) are shown in the figure.

Figure 7. Same plot as Figure 6 for 21 cm surveys.

Numerous efforts have been made to constrain the $f(R)$ model, with various observations such as the cosmic microwave background (CMB) anisotropies, supernovae, BAO distance, weak gravitational lensing, galaxies flow, and clusters abundance. The currently strongest constraints combining all of the data give $B_0 < 1.1 \times 10^{-3}$ at 95% C.L. (Lombriser et al. 2010). The main constraining power comes from the low-redshift cluster abundance data, while the ISW effects from galaxies–CMB cross-correlation also provide a moderate constraint of $B_0 < 0.42$ at 95% C.L. (Lombriser et al. 2010). On the other hand, the galaxies power-spectrum data (Tegmark et al. 2006) together with the CMB (Spergel et al. 2006) and supernovae (Astier et al. 2006) only place an upper bound on $B_0$ of order unity (Song et al. 2007b), Although theoretical calculation shows significantly enhanced growth of the large-scale structure in the $f(R)$ models (Song et al. 2007a), uncertainties in the nuisance parameters such as the galaxies bias $b$ and nonlinearity parameter $Q_{nl}$ substantially weakened its constraining power. Assuming similar survey parameters as those in Song et al. (2007b), we find that the 1σ uncertainty of $B_0$ from genus measurement is around 0.2 at $B_0 = 10^{-4}$. This demonstrates how the topological measurement, which is insensitive to...
Equations (17) and (22). Similar to the case of the structure constraint on the nonlinearity and clustering bias, could improve the large-scale structure as a measure of the scale-dependent expansion history, thus reducing the effects of these potential errors, and may therefore be a more reliable way to extract information on the large-scale structure.

In Figure 8, we also plot the two-dimensional constraint on $g_{QS}$ and $c_g(0)$ for the phenomenological model considered in Equations (17) and (22). Similar to the case of the $f(R)$ model, BIGBOSS provides a very stringent constraint: the 1σ errors are $0.036$ and $0.014$, respectively. For the CRT, $g_{QS} = 0.042$ and $c_g(0) = 0.028$ (assuming the redshift-varying $R_C$). $g_{QS} = 0.18$ and $c_g(0) = 0.19$ for LAMOST; and $g_{QS} = 0.17$ and $c_g(0) = 0.18$ for BOSS.

5. CONCLUSION

Topological indicators such as the genus of the isodensity contours provide an independent way to characterize the large-scale structure, complementary to the more often used two-point statistics such as the correlation function and power spectrum. A significant advantage of the topological measurement is that it is less susceptible to the nonlinear evolution and bias of the large-scale structure, thus reducing the effects of these potential systematic errors, and may therefore be a more reliable way to extract information on the large-scale structure.

In this paper, we studied the topology of the large-scale structure as a measure of the scale-dependent expansion history of the universe in models of modified gravity theories. In the modified gravity theory models, the structure growth can be scale dependent even in the linear regime; the amplitude of the genus curve varies significantly with redshift, and hence it can be used as a new tool for distinguishing the models from the case of standard gravity (i.e., GR). We illustrated this for the $f(R)$ theory and the DGP braneworld model as well as a phenomenological model parameterized with the PPF variables introduced by Hu & Sawicki (2007). We find that the genus curves for these models are modified and evolve with redshift due to the scale-dependent growth effect; hence the genus curve can be used as an observable to distinguish the modified gravity models from the general relativity theory. Finally, using the Fisher matrix formalism, we also forecasted the sensitivity of this test with some current or future optical/IR and 21 cm redshift surveys, showing that the method is a competitive way to test modified gravity models.

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APPENDIX

STATISTICAL UNCERTAINTY OF THE GENUS MEASUREMENT

Since the genus amplitude $A$ can be expressed as a function of the power spectrum (Equation (2)), we may approximately estimate the statistical uncertainty $\sigma_A$ by propagating the error from the uncertainty of the power spectrum $\sigma_P(k)$.

Rewriting the observable

$$O = \ln(A) = (3/2)[\ln(M_3) - \ln(M_2)]$$

where $M_n$ is defined as

$$M_n = \int d^3k k^{n-2}P_X(\vec{k})W(kR_C)$$

and $P_X(\vec{k})$ denotes the power spectrum under consideration, then the variance $\sigma_O^2 \equiv \text{Var}(O)$ is simply

$$\sigma_O^2 = \left( \frac{3}{2} \right)^2 \left[ \frac{\text{Var}(M_2)}{M_2^2} + \frac{\text{Var}(M_4)}{M_4^2} - 2\text{Cov}(M_2, M_4) \right].$$

Here $\text{Var}(M_m) \equiv \text{Cov}(M_m, M_m)$, and $\text{Cov}(M_m, M_n)$ is propagated from $\sigma_P^2$.

\[
\text{Cov}(M_m, M_n) = \langle M_m, M_n \rangle - \langle M_m \rangle \langle M_n \rangle = \frac{2(2m)^2}{V_s} \int dk d\mu k^{m+n-2}W^2(kR_C)\sigma_P^2(\vec{k})
\]

where $\mu$ is the cosine of the angle between $\vec{k}$ and the line of sight, and $V_s$ is the survey volume. We have assumed the covariance matrix of power spectrum is diagonal,

$$\text{Cov}(P(\vec{k}_1), P(\vec{k}_2)) = \delta_D(\vec{k}_1 - \vec{k}_2) \frac{2(2\pi)^2}{V_s} \sigma_P^2(\vec{k}).$$

We need to estimate the statistical uncertainty in the power-spectrum measurements for the different experiments. For galaxy surveys, the statistical uncertainty of $P_g(k)$ per Fourier mode includes both the cosmic variance and the shot noise due to the finite number of galaxies:

$$\sigma_P(\vec{k}) = \left[ P_g + \frac{1}{n} \right]^{1/2}.$$
For the 21 cm intensity mapping experiment, e.g., the CRT, in addition to the cosmic variance and the shot noise due to the finite number of galaxies, there is also the noise due to the foreground and receiver (Seo et al. 2010), so \( \sigma_{P_{21\text{ cm}}} = P_{21\text{ cm}}(\mathbf{k}) \), with

\[
P_{21\text{ cm}}(\mathbf{k}) = p_s^2 \left( P_{HI}(\mathbf{k}) + \frac{1}{n} \right) + \left( \frac{k_B g \bar{T}_{\text{sky}} + \bar{T}_a}{\sqrt{\Delta f} \Delta f} \right)^2 V_R
\]

with \( p_s = k_B g \bar{T}_{\text{sig}} \Delta f \), \( g = 0.8 \) is the gain, \( t_{\text{int}} \) is the integration time, and \( V_R \) is the volume of a pixel; \( \bar{T}_{\text{sig}} \) is the average brightness temperature which is estimated as

\[
\bar{T}_{\text{sig}} = 188 \frac{\chi_{HI}(z) \Omega_H \theta b(1 + z)^2}{H(z)/H_0} \text{ mK}, \quad (A6)
\]

with a conservative assumption for the neutral hydrogen fraction \( s_{HI}(z) \Omega_H = 0.00037 \). \( \bar{T}_{\text{sky}} \) and \( \bar{T}_a \) are average sky and antenna noise temperatures, which are assumed to equal 10 K and 50 K, respectively.

For high-redshift 21 cm experiments such as the MWA, we assume that the system temperature of the telescope is dominated by the sky:

\[
\bar{T}_{\text{sys}} \sim 250(1 + z)/71^{2.6} \text{ K},
\]

and the observation time is

\[
t_k = (\lambda_c t_{\text{int}}/\lambda^2) n(\lambda_k)
\]

at \( \lambda_k \), where \( n(\lambda_k) \) is the number of baselines which observe the transverse component of the wavevector. This can be calculated from the array configuration; here we model the antennas distribution of MWA as \( \rho(r) \sim r^{-2} \), with radius \( r_m = 750 \text{ m} \) and a flat core of radius \( r_c = 20 \text{ m} \). For the hypothetical follow-up of MWA, denoted as MWA5000, we have assumed \( r_m = 2 \text{ km} \) and \( r_c = 80 \text{ m} \).

The results of our estimates are given in Section 4.

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