Gauge Transformations, Twisted Poisson Brackets and Hamiltonization of Nonholonomic Systems

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Abstract

In this paper we study the problem of Hamiltonization of nonholonomic systems from a geometric point of view. We use gauge transformations by 2-forms (in the sense of Ševera and Weinstein [29]) to construct different almost Poisson structures describing the same nonholonomic system. In the presence of symmetries, we observe that these almost Poisson structures, although gauge related, may have fundamentally different properties after reduction, and that brackets that Hamiltonize the problem may be found within this family. We illustrate this framework with the example of rigid bodies with generalized rolling constraints, including the Chaplygin sphere rolling problem. We also see how twisted Poisson brackets appear naturally in nonholonomic mechanics through these examples.

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1 Introduction

It is well known that the equations of motion for a mechanical system with nonholonomic constraints do not arise from a variational principle in the usual sense. As a consequence, they cannot be formulated as a classical Hamiltonian system. Instead, they are written with respect to an almost Poisson bracket that fails to satisfy the Jacobi identity. This formulation has its origins in [30, 25, 21] and others.

On the other hand, after a symmetry reduction, the equations of motion of a number of examples allow a Hamiltonian formulation (sometimes after a time reparametrization) and one talks about Hamiltonization\(^1\) (see [11, 14, 15, 20, 17, 19, 22, 28] and others).

In this paper we employ recent developments of Poisson geometry to study this phenomenon from a geometric perspective. We use gauge transformations by 2-forms as introduced by ˇSevera and Weinstein in [29] to construct different almost Poisson brackets describing the dynamics of the same nonholonomic system. Although our interest is in almost Poisson geometry, we consider more general objects known as almost Dirac structures [12], as they provide the most natural setting for the definition and study of gauge transformations.

We illustrate the need for our methods by working out the Hamiltonization of the motion of rigid bodies that are subject to generalized rolling constraints. These are nonholonomic constraints that relate the angular velocity \(\mathbf{\omega}\) of the body and the linear velocity \(\dot{\mathbf{x}}\) of its center of mass in a linear way (i.e. \(\dot{\mathbf{x}} = A\mathbf{\omega}\) for a 3 \(\times\) 3 matrix \(A\)). This type of constraints contain the celebrated Chaplygin sphere problem as a special case. The latter concerns the motion of an inhomogeneous sphere whose center of mass coincides with its geometric center, that rolls without slipping on the plane. As a consequence of

\(^1\) A different meaning to Hamiltonization is given in [5] where the authors study the unreduced system in connection with the inverse problem of the calculus of variations.
the Hamiltonization, we are able to show complete Liouville integrability of the reduced dynamics of any rigid body subject to generalized rolling constraints.

Incidentally, during our discussion, we discover that twisted Poisson brackets [29] appear in the study of nonholonomic systems. In particular, we show that in the original physical time (before the time reparametrization), the reduced dynamics of the Chaplygin sphere are formulated in terms of a twisted-Poisson bracket. Although these structures do not in general satisfy the Jacobi identity, they possess a fair amount of properties, including foliations, that might imply an interesting interplay with dynamical features. To date, the interest in these brackets has been mainly geometrical.

1.1 Hamiltonization

Perhaps the most interesting example of Hamiltonization concerns the Chaplygin sphere. Even though the formulation and integration of the equations of motion by Chaplygin dates back to 1903 [10], the Hamiltonian structure of the reduced equations (after a time reparametrization) was only discovered in 2001 by Borisov and Mamaev [6].

Recently, Jovanović [22] proved that the multidimensional version of the Chaplygin sphere problem introduced in [16] is also integrable and Hamiltonizable when the vertical angular momentum is zero. This gives a partial solution to a problem that remained open for many years. His approach to prove integrability involves in a crucial way the Hamiltonization of the problem. Another important example where the integration of a nonholonomic system follows from its Hamiltonization is the multidimensional Veselova system treated by Fedorov and Jovanović in [15]. We also mention the recent work of Ohsawa, Fernandez, Bloch and Zenkov [28] in connection with Hamilton-Jacobi theory.

The relationship between Hamiltonization and integrability may have been the original motivation for Chaplygin to consider the problem of Hamiltonization back in 1911 [11]. In this work, Chaplygin proved the famous Chaplygin reducing multiplier Theorem that applies to the so-called G-Chaplygin systems. These are nonholonomic systems with the property that the tangent space to the orbits of a symmetry group \( G \) exactly complements the constraint distribution on the tangent space \( TQ \) of the configuration manifold \( Q \). Stated in modern geometric terms, the Theorem says that if the shape space \( Q/G \) is two-dimensional, and the reduced equations have an invariant measure, then they can be put in Hamiltonian form in the new time \( \tau \) defined by \( d\tau = \frac{1}{\varphi} dt \). The positive function \( \frac{1}{\varphi} : Q/G \to \mathbb{R} \) is known as the reducing multiplier\(^2\). There is a very neat interpretation of the multiplier \( \frac{1}{\varphi} \) in terms of the invariant measure and as a conformal factor for an almost symplectic form that describes the dynamics, see [14, 15, 20, 28]. This interpretation suggests that geometric methods may be useful to understand Hamiltonization in more general scenarios of nonholonomic systems with symmetry.

Recently, Fernandez, Mestdag and Bloch [17], derived a set of coupled first order partial differential equations for the multiplier \( \frac{1}{\varphi} \) for G-Chaplygin systems whose shape space has arbitrary dimension. Even more, the set of equations found by the authors applies to general nonholonomic systems with symmetry that are not necessarily G-Chaplygin. This is done by writing the reduced equations of motion in Hamilton-Poincaré-D’Alembert form as described in [5, 4]. The issue of Hamiltonization is thus reformulated as a problem of existence of a solution for the aforementioned system of partial differential equations.

\(^2\)We denote the reducing multiplier by \( \frac{1}{\varphi} \) instead of \( \varphi \) to be consistent with our exposition which takes the Poisson rather than the symplectic perspective.
differential equations.

Our approach to Hamiltonization contains the same degree of generality but is more intrinsic. Denote by $X_R$ the vector field describing the dynamics on the reduced space $\mathcal{R}$ and by $\mathcal{H}_R$ the reduced Hamiltonian. We formulate the reduced equations of motion in almost Poisson form with respect to a collection of bivector fields $\pi^{\text{red}} B$. Each member in this collection describes the reduced dynamics in (almost) Hamiltonian form (i.e. $(\pi^{\text{red}} B)^\sharp(d\mathcal{H}_R) = -X_R$), and arises as the reduction of a bivector field $\pi^{\text{B}}$ associated to a bracket $\{\cdot,\cdot\}^{\text{B}}_{\text{nh}}$. Such bracket $\{\cdot,\cdot\}^{\text{B}}_{\text{nh}}$ is obtained through what we define as a dynamical gauge transformation by a 2-form $B$ of the noholonomic bracket $\{\cdot,\cdot\}_{\text{nh}}$ defined in [30, 25, 21] (see discussion in subsection 1.2 below).

Having said this, we reformulate the issue of Hamiltonization by requiring that one of the bivector fields $\pi^{\text{red}} B$ in the collection described above is conformally Poisson, i.e.

$$[\varphi \pi^{\text{red}} B, \varphi \pi^{\text{red}} B] = 0,$$

(1.1)

for a positive function $\varphi$, and where $[\cdot,\cdot]$ is the Schouten bracket. The scaling of $\pi^{\text{red}} B$ by $\varphi$ is dynamically interpreted as the time reparametrization $d\tau = \frac{1}{\varphi} \, dt$ (see Section 4.4). Note that, for each $\pi^{\text{red}} B$, equation (1.1) locally defines a set of coupled first order partial differential equations for $\varphi$. This seems to be in agreement with the results of Fernandez, Mestdag and Bloch [17].

If the symmetries are of $G$-Chaplygin type, the bivector fields $\pi^{\text{red}} B$ are everywhere non-degenerate and the equations of motion can be written with respect to the associated almost symplectic form $\Omega^{\text{red}} B$. If a multiplier $\varphi$ satisfying (1.1) exists, then $\Omega^{\text{red}} B$ is conformally closed, $(d(\frac{1}{\varphi} \Omega^{\text{red}} B) = 0)$, and one speaks of Chaplygin Hamiltonization [14].

The term Poissonization was introduced by Fernandez, Mestdag and Bloch in [17] to refer to the case where the bivector field $\pi^{\text{red}} B$ satisfying (1.1) is degenerate. Their motivation to distinguish this case is to study the relationship between Hamiltonization and the existence of invariant measures for the reduced equations. To simplify our exposition and to treat the problem in a unified manner, we will not use their terminology and simply talk about Hamiltonization whenever there exists a solution to (1.1). We give a discussion on the existence of invariant measures for nonholonomic systems admitting a Hamiltonization in this generality in subsection 4.4.

In general terms, the main contributions of this paper to the problem of Hamiltonization are the clear geometric formulation of the problem using recent developments on the field of Poisson geometry (mainly those in Ševera, Weinstein [29]) and the illustration of the usefulness of these techniques in the study of rigid bodies subject to generalized rolling constraints.

1.2 Gauge transformations in nonholonomic mechanics

The Hamiltonization of the Chaplygin sphere can be obtained in two ways:

In the first one, described in Borisov and Mamaev [7], one performs the reduction in two stages. In the first step one reduces the translational symmetries of the problem corresponding to the abelian action of $\mathbb{R}^2$. On the second stage one reduces the internal symmetry corresponding to rotations about the vertical axis described by the action of $S^1$. The second stage is performed in Borisov and Mamaev [7].

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3In fact the function $\varphi : \mathcal{R} \rightarrow \mathbb{R}^+$ is not arbitrary. It is required to be basic with respect to the fibered structure of the reduced space $\mathcal{R}$. Just as for $G$-Chaplygin systems, we think of $\varphi : Q/G \rightarrow \mathbb{R}^+$. 

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using Routh’s reduction method. The Hamiltonian interpretation of the second reduction is delicate
and is studied in Hochgerner, García-Naranjo [20]. Effectively, it is shown that in order to further
reduce the system in a Marsden-Weinstein fashion, the geometric data needs to be modified and the
authors propose a method for this called truncation.

A second manner to achieve the Hamiltonization of the Chaplygin sphere is described in García-
Naranjo [19]. In this approach one first formulates the equations of motion in terms of an affine almost
Poisson bracket that differs from the usual nonholonomic bracket \( \{ \cdot , \cdot \} _{nh} \) considered in [30, 25, 21] and
others. In this framework, the Hamiltonization can be obtained as a single (one step) reduction by the
symmetry group of rigid transformations on the plane, SE(2).

In this paper we further elaborate on the latter approach. Our main tool for this is to incorporate
gauge transformations as introduced in ˇSevera and Weinstein [29].

Recall that the nonholonomic bracket \( \{ \cdot , \cdot \} _{nh} \) considered in [30, 25, 21] is defined on functions on the
constraint phase space \( \mathcal{M} \). The vector field \( X_{nh} \) on \( \mathcal{M} \) that describes the nonholonomic dynamics is
(almost) Hamiltonian with respect to this bracket and with respect to the constrained Hamiltonian \( \mathcal{H}_\mathcal{M} \).
Now, let \( B \) be a 2-form on \( \mathcal{M} \). If \( B \) satisfies a certain technical condition, then a gauge transformation
of the nonholonomic bracket by \( B \) defines a new bracket \( \{ \cdot , \cdot \} _{nh}^B \) on \( \mathcal{M} \) with the same (non-integrable)
characteristic distribution. As a consequence, the (almost) Hamiltonian vector fields associated to
\( \{ \cdot , \cdot \} _{nh}^B \) satisfy the nonholonomic constraints. If in addition, the 2-form \( B \) satisfies \( i_{X_{nh}} B = 0 \), then we
say that \( B \) defines a dynamical gauge transformation and the vector field \( X_{nh} \) is (almost) Hamiltonian
with respect to the gauged bracket \( \{ \cdot , \cdot \} _{nh}^B \) and the constrained Hamiltonian \( \mathcal{H}_\mathcal{M} \). In this way, we
distinguish a family \( \mathfrak{F} \) of almost Poisson structures that describe the dynamics of our nonholonomic
system corresponding to different dynamical gauge transformations. We show (Remark 4.4) that the
affine almost Poisson brackets defined in García-Naranjo [19] are particular members in \( \mathfrak{F} \).

The main motivation to consider the large family \( \mathfrak{F} \) of almost Poisson brackets for our nonholonomic
system is to have a larger choice of structures to describe the reduced dynamics and hope to find one
amongst them that Hamiltonizes the problem. In particular, the Hamiltonization of the Chaplygin
sphere arises as the reduction of a dynamically gauged bracket \( \{ \cdot , \cdot \} _{nh}^B \) that differs from the standard
nonholonomic bracket \( \{ \cdot , \cdot \} _{nh} \).

In fact, the members of the family \( \mathfrak{F} \) behave quite differently after reduction. Our examples show
that while some of them might yield a true Poisson structure after reduction, others yield an almost
Poisson bracket with a non-integrable characteristic distribution.

In order to study the gauge transformations of almost Poisson brackets, we formulate the dynamics
of nonholonomic systems on almost Dirac structures [12]. These are more general geometric objects
that provide the framework in which gauge transformations are more natural. These structures had
been already considered in connection to nonholonomic mechanics by Yoshimura and Marsden [32, 33],
and by Jotz and Ratiu [23]. However, the issue of Hamiltonization and the incorporation of gauge
transformations are not treated in these works.

1.3 Twisted Poisson brackets in nonholonomic mechanics

It is well known that nonholonomic systems are formulated in terms of almost Poisson brackets that
fail to satisfy the Jacobi identity. However, very little research, if any, has been done in understanding
how far away these brackets are from being true Poisson.

A very strong property of a Poisson manifold is that the characteristic distribution is integrable and defines a foliation by even dimensional leaves. This property is also shared by conformally Poisson brackets whose bivector field $\pi$ satisfies $[\varphi \pi, \varphi \pi] = 0$, for a certain positive function $\varphi$ called the conformal factor. These brackets have been considered in the study of Hamiltonization of nonholonomic systems, and, as mentioned before, the conformal factor $\varphi$ defines the time reparametrization $d\tau = \frac{1}{\varphi} dt$.

Another example of almost Poisson brackets that possess a foliation by even dimensional leaves, is given by twisted Poisson brackets that were introduced by Klimeš and Stříbí in [24] and later in Ševera and Weinstein [29] from a more geometric point of view. Twisted Poisson brackets correspond to almost Poisson structures whose associated bivector field $\pi$ satisfies

$$\frac{1}{2} [\pi, \pi] = \pi^\sharp(\phi),$$

for a certain closed 3-form $\phi$. To our knowledge, the present paper is the first one to explore the connection between this type of structures and nonholonomic mechanics.

In this paper we show that an almost Poisson structure with a regular (constant rank), integrable characteristic distribution is twisted (Corollary 3.7). As a consequence we show (Remark 3.8) that the reduced equations of the classical Veselova problem [31] can be formulated in terms of a twisted Poisson bracket in the original physical time (prior to any time reparametrization).

We also show (Theorems 5.7 and 5.8) that the reduced equations of some examples of rigid body motion with generalized rolling constraints, that contain the Chaplygin sphere as a special case, are described by a twisted Poisson bracket in the original physical time. Moreover, we give an explicit formula for the twisting closed 3-form $\phi$.

1.4 Outline and main results of the paper

The paper is organized as follows. In Section 2 we introduce our motivating examples, rigid bodies subject to generalized rolling constraints. As mentioned before, these are nonholonomic constraints that relate the linear velocity of the body to its angular velocity via a $3 \times 3$ matrix $A$. After writing down the reduced equations of motion, we define two different (almost) Poisson structures for the reduced equations according to the rank of $A$, that varies from 0 to 3. For each value of the rank of $A$, we show that only one of the brackets is Poisson (conformally Poisson if rank $A = 1, 2$) while the other one possesses a non-integrable characteristic distribution. The geometric interpretation and construction of these brackets is one of the main goals of the paper and is postponed to Section 5 after the necessary tools are developed in Sections 3 and 4.

In Section 3 we develop the geometric background needed for our purposes. We focus on almost Dirac structures and their gauge transformations, introduced respectively in [12] and [29], and we collect some new results that are important in our study of nonholonomic systems. Proposition 3.1 gives a characterization of regular almost Dirac structures that is used in Corollary 3.4 to describe the structure of almost Poisson brackets having a regular characteristic distribution. Corollary 3.7 shows that an almost Poisson bracket possessing a regular, integrable, characteristic distribution is twisted. In fact this result is proved in the more general setting of almost Dirac structures in Theorem 3.5. We
also mention Theorem 3.11 that asserts that any two regular almost Dirac structures defining the same
distribution are gauge related.

In Section 4 we make the connection between the geometric methods developed in Section 3 and
nonholonomic mechanics. In particular, we construct the nonholonomic bracket of [30, 25, 21] using
Corollary 3.4 and the framework for nonholonomic mechanics described in Bates and Sniatycki [2]. In
Proposition 4.3 we show that the dynamics associated with this bracket coincide with the formulation
of nonholonomic mechanics on almost Dirac structures considered in [32, 33, 23]. Next, we define the
notion of dynamical gauge transformations for a nonholonomic system, and define a family \( \mathfrak{F} \) of almost
Poisson brackets, possessing the same characteristic distribution, and that describe our nonholonomic
system. Finally, we discuss the reduction of these brackets in the presence of symmetries and introduce
our working definition of Hamiltonization.

In Section 5 we resume the study of rigid bodies subject to generalized rolling constraints. In
subsection 5.1 we show that the brackets given in Section 2 to describe the reduced dynamics, arise as
a reduction of different members of the family \( \mathfrak{F} \) (Theorems 5.3 and 5.4). In subsection 5.2 we establish
the Hamiltonization of the reduced equations in detail and we conclude their integrability. Finally, in
subsection 5.3 we focus on the twisted nature of the brackets that Hamiltonize the problem for the
cases Rank \( A \) = 1, 2, prior to the time reparametrization.

2 Motivating Examples: Rigid bodies with Generalized Rolling Constraints

Consider the motion of a rigid body in space that evolves under its own inertia and is subject to the
constraint that enforces the linear velocity of the center of mass, \( \mathbf{x} \), to be linearly related to the angular
velocity of the body \( \omega \), i.e.,

\[
\dot{\mathbf{x}} = r A \omega.
\]  

(2.2)

Both vectors \( \dot{\mathbf{x}} \) and \( \omega \) belong to \( \mathbb{R}^3 \) and are written with respect to an inertial frame. The constant
scalar \( r \) has dimensions of length and is a natural length scale of the system. The dimensionless constant
\( 3 \times 3 \) matrix \( A \) is given and satisfies certain conditions that are made precise in the following Definition.

**Definition 1.** The matrix \( A \) is said to define a generalized rolling constraint if it satisfies one of the
following conditions according to its rank:

(i) \( A = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \), with \( C \in \text{SO}(2) \), if rank \( A \) = 3.

(ii) \( A = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \), with \( C \in \text{SO}(2) \), if rank \( A \) = 2.

(iii) \( A = e_3 e_3^T \), if rank \( A \) = 1, where \( e_3 \) is the third canonical vector in \( \mathbb{R}^3 \), and \( T \) denotes transpose.

(iv) \( A = 0 \) if rank \( A \) = 0.

The above conditions on \( A \) can be relaxed (see Remark 2.1 ahead). However, for simplicity, we will
assume that \( A \) has the form given by one of the items of the above Definition. If \( A \) satisfies any of the
three conditions of the above Definition we say that (2.2) is a generalized rolling constraint.
Our terminology is motivated by a particular example: the Chaplygin sphere. The problem, introduced by Chaplygin in 1903 [10], concerns the motion of a ball whose center of mass coincides with its geometric center that rolls on the plane without slipping. In this case, the matrix \( A \) is given by
\[
A = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
and \( r \) is the radius of the sphere.

The motion of the Chaplygin ball has been the subject of much research to our days. An important property is that (after a time reparametrization) the reduced equations can be given a Hamiltonian structure [6, 19]. The geometry of the Hamiltonization of the problem is intricate. In order to study this phenomenon in a mathematically systematic fashion, we consider more general possibilities for the matrix \( A \).

The crucial property of \( A \) that determines many of the dynamical and geometrical features of the problem is its rank. The Chaplygin sphere corresponds to the case \( \text{rank } A = 2 \). Another familiar case occurs when \( \text{rank } A = 0 \). In this case the constraint (2.2) becomes \( \dot{x} = 0 \) which can be interpreted as a conservation law for the free system that states that the center of mass of the body is at rest in the inertial frame. The motion of the system reduces to that of the classical free rigid body.

We will also consider the cases where the rank of \( A \) equals 3 and 1 which, to our knowledge, have not yet been considered in the literature.

2.1 Generalities

The configuration space for the system is \( Q = \text{SO}(3) \times \mathbb{R}^3 \). Elements in \( Q \) are of the form \( q = (g, x) \in \text{SO}(3) \times \mathbb{R}^3 \). The vector \( x \in \mathbb{R}^3 \) is the position of the center of mass in space and the orthogonal matrix \( g \) specifies the orientation of the ball by relating two orthogonal frames, one attached to the body and one that is fixed in space. We will assume that the body frame has its origin at the center of mass and is aligned with the principal axes of inertia of the body. These frames define the so-called space and body coordinates respectively.

Recall that the Lie algebra \( \mathfrak{so}(3) \) can be identified with \( \mathbb{R}^3 \) equipped with the vector product via the hat map:
\[
\eta = (\eta_1, \eta_2, \eta_3) \mapsto \hat{\eta} = \begin{pmatrix}
0 & -\eta_3 & \eta_2 \\
\eta_3 & 0 & -\eta_1 \\
-\eta_2 & \eta_1 & 0
\end{pmatrix}. \tag{2.3}
\]
Given a motion \( (g(t), x(t)) \in Q \), the angular velocity vector in space coordinates, \( \omega \in \mathbb{R}^3 \), and the angular velocity vector in body coordinates, \( \Omega \in \mathbb{R}^3 \), are respectively given by
\[
\dot{\omega}(t) = \dot{g}(t)g^{-1}(t), \quad \dot{\Omega}(t) = g^{-1}(t)\dot{g}(t),
\]
and satisfy \( \Omega = g^{-1}\omega \). It will be useful to write the constraint (2.2) in terms of the body angular velocity as
\[
\dot{x} = rAg\Omega. \tag{2.4}
\]
The kinetic energy of the rigid body defines the Lagrangian $L : TQ \to \mathbb{R}$ by

$$L(g, \dot{g}, x, \dot{x}) = \frac{1}{2}(\|\Omega\| \cdot \Omega + \frac{m}{2}||\dot{x}||^2), \quad (2.5)$$

where “$\cdot$” denotes the Euclidean scalar product on $\mathbb{R}^3$, $m$ is the mass of the body and the $3 \times 3$ diagonal matrix $\mathbb{I}$ is the inertia tensor with positive entries $I_1, I_2, I_3$.

**Remark 2.1.** It is not hard to see that if the space axes are rotated by an element $h \in SO(3)$, the Lagrangian $L$ is invariant and the constraint (2.2) is rewritten as

$$\dot{x} = rh^{-1}Ah\omega.$$ 

Therefore, the conditions for $A$ given in Definition 1 can be relaxed by allowing conjugation by matrices $h \in SO(3)$.

### 2.2 The equations of motion

Let $p = mx$ be the linear momentum of the body. In accordance with the Lagrange-D’Alembert principle, the constraint forces must annihilate any velocity pair ($\dot{x}, \Omega$) satisfying (2.4). Therefore, the equations of motion are given by

$$\dot{p} = \mu, \quad \mathbb{H} = \mathbb{I} \times \Omega - rg^{-1}A^T\mu, \quad (2.6)$$

where “$\times$” denotes the vector product in $\mathbb{R}^3$ and the multiplier $\mu \in \mathbb{R}^3$ is determined uniquely from the constraint (2.4).

Differentiating (2.4) and using $\dot{g}\Omega = 0$ we find $\mu = mrAg\dot{\Omega}$. Thus, the second equation in (2.6) decouples from the first to give

$$\mathbb{H} = \mathbb{I} \times \Omega - mr^2g^{-1}A^TAg\dot{\Omega}. \quad (2.7)$$

In principle, this equation should be complemented with the reconstruction equation $\dot{g} = g\dot{\Omega}$. It will be shown ahead that it suffices to consider the evolution of the Poisson vector $\gamma := g^{-1}e_3$ that represents the vector $e_3$ written in body coordinates. A direct calculation gives

$$\dot{\gamma} = \gamma \times \Omega.$$ 

The decoupling in (2.6) is due to the presence of symmetries that will be discussed in detail in Section 2.4. Once this equation is solved for $(g, \Omega)$, we obtain $p = mrAg\dot{\Omega}$ that follows from (2.4).

We introduce the kinetic momentum $K \in \mathbb{R}^3$ by

$$K := \mathbb{I}\Omega + mr^2g^{-1}A^TAg\dot{\Omega}. \quad (2.8)$$

This definition of the kinetic momentum allows us to define the (reduced) Hamiltonian

$$\mathcal{H}_R = \frac{1}{2}(K \cdot \Omega), \quad (2.9)$$
which coincides with the kinetic energy on the constraint space \( \mathcal{M} \).

A direct calculation using \( (2.7) \) gives our final set of equations

\[
\dot{K} = K \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega.
\]  

To understand why the above equations define a closed system for \((K, \gamma) \in \mathbb{R}^3 \times \mathbb{R}^3\), and to understand their structure, it is useful to perform a separate study for different values of the rank of the matrix \(A\). This will also show that the Hamiltonian \( H_R \) can be considered as a function of \(K\) and \(\gamma\).

### 2.3 A pair of (almost) Poisson brackets for the equations of motion

For each value of the rank of \(A\) we will give two different brackets that define the equations of motion \( (2.10) \), with respect to the reduced Hamiltonian \( H_R \). In general, these brackets are almost Poisson, i.e. they do not satisfy the Jacobi identity but we will argue that one of them is more convenient than the other. They will be denoted by \{\cdot, \cdot\}_{\text{Rank}j} \text{ and } \{\cdot, \cdot\}'_{\text{Rank}j} \text{ where } j \text{ denotes the rank of the matrix } A. \text{ Both brackets define the equations of motion } (2.10) \text{ in the sense that the directional derivative of any function } f = f(\gamma, K) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \text{ along the flow is given by } \dot{f} = \{f, H_R\}_{\text{Rank}j} = \{f, H_R'\}_{\text{Rank}j}. \text{ The geometric interpretation of these brackets is the subject of the subsequent sections. Concretely, in section } 5 \text{ (Theorems 5.3 and 5.4), we will show that the bracket } \{\cdot, \cdot\}_{\text{Rank}j} \text{ arises as the reduction of the nonholonomic bracket introduced in } [30] \text{, and that } \{\cdot, \cdot\}'_{\text{Rank}j} \text{ arises as the reduction of a gauge transformation of the nonholonomic bracket.}

The following definitions will be useful in our discussion of the utility of the two brackets:

**Definition 2.** Let \(P\) be a manifold equipped with an almost Poisson bracket \{\cdot, \cdot\}.

1. The (almost) **Hamiltonian vector field** \(X_f\) of a function \(f \in C^\infty(P)\) is the vector field on \(P\) defined as the usual derivation \(X_f(g) = \{g, f\}\) for all \(g \in C^\infty(P)\).

2. The **characteristic distribution** of \{\cdot, \cdot\} is the distribution on the manifold \(P\) whose fibers are spanned by the (almost) Hamiltonian vector fields.

3. Due to Leibniz condition of \{\cdot, \cdot\}, there is a bivector field \(\pi \in \Gamma(\bigwedge^2(TP))\) such that for \(f, g \in C^\infty(P)\) we have \(\pi(df, dg) = \{f, g\}\). We say that \(\pi\) is the **bivector field associated to** \{\cdot, \cdot\} and we denote by \(\pi^2 : T^*P \to TP\) the map such that \(\beta(\pi^2(\alpha)) = \pi(\alpha, \beta)\). We will occasionally refer to bivector fields simply as bivectors. Note that the characteristic distribution is the image of \(\pi^2\) and the Hamiltonian vector field \(X_f = -\pi^2(df)\). The 3-vector field \([\pi, \pi]\), where \([\cdot, \cdot]\) is the Schouten bracket, may be different from zero, and it measures the failure of the Jacobi identity through the relation

\[
\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \frac{1}{2}[\pi, \pi](df, dg, dh), \tag{2.11}
\]

for \(f, g, h \in C^\infty(P)\).

4. The bracket is called **conformally Poisson** if there exists a strictly positive function \(\varphi \in C^\infty(P)\) such that the bracket \(\varphi \{\cdot, \cdot\}\) satisfies the Jacobi identity, i.e. \([\varphi \pi, \varphi \pi] = 0\).
The well known symplectic stratification theorem states that the characteristic distribution of a Poisson bracket is integrable and its leaves are symplectic manifolds. Since multiplication of an almost Poisson bracket by a positive function does not change the characteristic distribution, a necessary condition for an almost Poisson bracket to be conformally Poisson is that its characteristic distribution be integrable.

We now come back to the discussion of our example for the different values of the rank of $A$.

**If $A$ has rank 3.** In this case $A^{-1} = A^T$ and $K = (I + mr^2E)\Omega$ where $E$ denotes the $3 \times 3$ identity matrix. It follows from (2.10) that the rotational motion of the body is the same as that of a free rigid body whose total inertia tensor is $I + mr^2E$. It is trivial to write $\Omega = (I + mr^2E)^{-1}K$ and it is clear that equations (2.10) define a closed system in $\mathbb{R}^3 \times \mathbb{R}^3$.

The two brackets for the system for functions $f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ are given by

$$
\{ f, g \}_\text{Rank3} = - (K + mr^2\Omega) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right),
$$

$$
\{ f, g \}'_\text{Rank3} = - K \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right).
$$

(2.12)

The above brackets are quite different. On the one hand, the bracket $\{ \cdot, \cdot \}'_\text{Rank3}$ satisfies the Jacobi identity. It in fact coincides with the Lie-Poisson bracket on the dual Lie algebra $\mathfrak{se}(3)^*$. On the other hand, the bracket $\{ \cdot, \cdot \}_\text{Rank3}$ is not even conformally Poisson as the following Proposition shows.

**Proposition 2.2.** The characteristic distribution of the almost Poisson bracket $\{ \cdot, \cdot \}_\text{Rank3}$ defined in (2.12) is not integrable.

**Proof.** The (almost) Hamiltonian vector field $X_f$ of a function $f \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ corresponding to the bracket $\{ \cdot, \cdot \}_\text{Rank3}$ is given by

$$
X_f = \left( (K + mr^2\Omega) \times \frac{\partial f}{\partial K} + \gamma \times \frac{\partial f}{\partial \gamma} \right) \cdot \frac{\partial}{\partial \gamma} + \left( \gamma \times \frac{\partial f}{\partial K} \right) \cdot \frac{\partial}{\partial \gamma},
$$

and it is annihilated by the non-closed one-form

$$
\chi = \gamma \cdot dK + (K + mr^2\Omega) \cdot d\gamma.
$$

We have

$$
X_{K_1} = (K_3 + mr^2\Omega_3) \frac{\partial}{\partial K_2} - (K_2 + mr^2\Omega_2) \frac{\partial}{\partial K_3} + \gamma_3 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_3},
$$

and thus

$$
\chi([X_{\gamma_1}, X_{K_1}]) = -d\chi(X_{\gamma_1}, X_{K_1}) = -mr^2 \left( \frac{\gamma_3^2}{I_2 + mr^2} + \frac{\gamma_2^2}{I_3 + mr^2} \right) \neq 0.
$$

This shows that the commutator $[X_{\gamma_1}, X_{K_1}]$ does not belong to the characteristic distribution which is therefore not integrable.

Therefore, to obtain a true Hamiltonian formulation of the reduced equations of motion in the case where the rank of $A$ is 3, one needs to work with the bracket $\{ \cdot, \cdot \}'_\text{Rank3}$. 

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If $A$ has rank 2. As mentioned before, this case has the Chaplygin sphere as a particular example. The analysis of the two brackets has been done in [19]. We include it here for completeness and to link it with clarity to other results of the present work.

In view of the form of $A$ given in item (ii) of Definition 1, we can write $A^T A = E - e_3 e_3^T$ and thus, according to (2.8), we get

$$K = (I + mr^2 E) \Omega - mr^2 (\Omega \cdot \gamma) \gamma,$$

which is precisely the expression for the angular momentum about the contact point for the Chaplygin sphere.

The angular velocity $\Omega$ can be written in terms of $K$ and $\gamma$ as

$$\Omega = (I + mr^2 E)^{-1} K + mr^2 \left( \frac{K \cdot (I + mr^2 E)^{-1} \gamma}{||\gamma||^2 - mr^2 \gamma \cdot (I + mr^2 E)^{-1} \gamma} \right) (I + mr^2 E)^{-1} \gamma,$$

so both the equations (2.10) and the Hamiltonian $H_R$ are well defined on $\mathbb{R}^3 \times \mathbb{R}^3$.

In this case, the two brackets for the system for functions $f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ are given by

$$\{f, g\}_{\text{Rank2}} = -(K + mr^2 \Omega - mr^2 (\Omega \cdot \gamma) \gamma) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right),$$

$$\{f, g\}'_{\text{Rank2}} = -(K - mr^2 (\Omega \cdot \gamma) \gamma) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right).$$

None of the above brackets satisfies the Jacobi identity but it is preferable to consider $\{\cdot, \cdot\}'_{\text{Rank2}}$. The reason is that this bracket is conformally Poisson with conformal factor

$$\varphi(\gamma) = \sqrt{||\gamma||^2 - mr^2 (\gamma \cdot (I + mr^2 E)^{-1} \gamma)}.$$

This important observation was first made in [6]. The characteristic distribution of $\{\cdot, \cdot\}'_{\text{Rank2}}$ is thus integrable. The generic leaves are the level sets of the Casimir functions $C_1(K, \gamma) = K \cdot \gamma$ and $C_2(\gamma) = ||\gamma||^2$. Another important feature of this bracket is that it is twisted Poisson (in the sense of [24, 29]) as will be shown in Section 5.3 (Theorem 5.7).

On the other hand, similar to Proposition 2.2 we have

**Proposition 2.3 ([19]).** The characteristic distribution of the almost Poisson bracket $\{\cdot, \cdot\}_{\text{Rank2}}$ defined in (2.13) is not integrable.

This can be shown exactly as we did for Proposition 2.2. Therefore if the rank of $A$ is 2, just as in the case of rank 3, a Hamiltonian formulation of the reduced equations can only be obtained if we work with the bracket $\{\cdot, \cdot\}'_{\text{Rank2}}$. However, in this case one needs to multiply the bracket by a conformal factor. This can be interpreted as a time reparametrization, see the discussion in Section 4.4.

If $A$ has rank 1. Taking into account the form of $A$ given in item (iii) of Definition 1 we have $A^T A = e_3 e_3^T$ and thus, in view of (2.8), we get

$$K = I \Omega + mr^2 (\Omega \cdot \gamma) \gamma.$$
The expression for the angular velocity \( \Omega \) in terms of \( K \) and \( \gamma \) is

\[
\Omega = \mathbb{I}^{-1}K - mr^2 \left( \frac{K \cdot \mathbb{I}^{-1}\gamma}{||\gamma||^2 + mr^2(\gamma \cdot \mathbb{I}^{-1}\gamma)} \right) \mathbb{I}^{-1}\gamma,
\]

so again, both the equations (2.10) and the Hamiltonian \( H \) Proposition 2.5. The characteristic distribution of the almost Poisson bracket for functions \( \{\cdot, \cdot\}_\text{Rank1} \) level sets of the Casimir functions \( C \) is not integrable.

This time, the two brackets for the system are given by

\[
\{f, g\}_\text{Rank1} = -(K + mr^2(\Omega \cdot \gamma)) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial \gamma} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial K} \right),
\]

\[
\{f, g\}'_\text{Rank1} = -(K - mr^2\Omega + mr^2(\Omega \cdot \gamma)) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial \gamma} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial K} \right),
\]

for functions \( f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \).

The properties of the brackets above are very similar to those obtained in the case where the rank of \( A \) is 2 except that the roles of \( \{\cdot, \cdot\}_\text{Rank1} \) and \( \{\cdot, \cdot\}'_\text{Rank1} \) are reversed.

This time one can show that

**Proposition 2.4.** The bracket \( \{\cdot, \cdot\}_\text{Rank1} \) defined in (2.15) is conformally Poisson with conformal factor

\[
\varphi(\gamma) = \sqrt{||\gamma||^2 + mr^2(\gamma \cdot \mathbb{I}^{-1}\gamma)}.
\]

**Proof.** We have to prove that the scaled bracket on \( \mathcal{R} \) defined as \( \varphi(\cdot, \cdot)_\text{Rank1} \) satisfies the Jacobi identity, i.e.,

\[
\varphi\{\varphi\{f_1, f_2\}_\text{Rank1}, f_3\}_\text{Rank1} + \varphi\{\varphi\{f_2, f_3\}_\text{Rank1}, f_1\}_\text{Rank1} + \varphi\{\varphi\{f_3, f_1\}_\text{Rank1}, f_2\}_\text{Rank1} = 0
\]

for all \( f_1, f_2, f_3 \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \). In view of the derivation properties of the bracket, is enough to show the identity for the coordinate functions \( K_i, \gamma_i \). In our case, since \( \{\gamma_i, \gamma_j\}_\text{Rank1} = 0 \), it is immediate to check that the identity holds if two of the three functions are \( \gamma_i \)'s. A long but straightforward computation shows that the identity holds for the following three choices of functions \( f_1 = K_1, f_2 = K_2, f_3 = \gamma_1; f_1 = K_1, f_2 = K_2, f_3 = \gamma_3 \) and \( f_1 = K_1, f_2 = K_2, f_3 = K_3 \). Since the definition of the bracket is symmetric with respect to the coordinate functions \( K_i, \gamma_i \), and since the Jacobi identity trivially holds if two of the three functions \( f_1, f_2, f_3 \) are equal, all of the other cases are either trivial or analogous.

Hence, the characteristic distribution of \( \{\cdot, \cdot\}_\text{Rank1} \) is integrable and the generic leaves are again the level sets of the Casimir functions \( C_1(K, \gamma) = K \cdot \gamma \) and \( C_2(\gamma) = ||\gamma||^2 \). It will also be shown in Section 5.3 that \( \{\cdot, \cdot\}_\text{Rank1} \) is twisted Poisson.

On the other hand, analogous to Propositions 2.2 and 2.3 we have

**Proposition 2.5.** The characteristic distribution of the almost Poisson bracket \( \{\cdot, \cdot\}'_\text{Rank1} \) defined in (2.15) is not integrable.

The proof is again similar.

Thus, this time the Hamiltonian structure of the reduced equations can only be obtained with the bracket \( \{\cdot, \cdot\}_\text{Rank1} \), again through the multiplication by a conformal factor that is interpreted as a time reparametrization.
If $A$ has rank 0. In this case $A$ is the zero matrix and the constraints are holonomic and can be seen as a conservation law for the standard free rigid body. We have $K = \Omega$ and clearly the equations (2.10) and the Hamiltonian $H_R$ are well defined on $\mathbb{R}^3 \times \mathbb{R}^3$. The two brackets are given by

\[
\{f, g\}_{\text{rank} 0} = -K \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right),
\]

\[
\{f, g\}'_{\text{rank} 0} = -(K - mr^2 \Omega) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right).
\]

The situation is analogous to that of the case when the rank of $A$ is 3 but, once more, the roles of the brackets are reversed. While $\{\cdot, \cdot\}_{\text{rank} 0}$ coincides with the Lie-Poisson bracket in the dual Lie algebra $\mathfrak{se}(3)^*$ (and hence satisfies the Jacobi identity), we have

**Proposition 2.6.** The characteristic distribution of the almost Poisson bracket $\{\cdot, \cdot\}'_{\text{rank} 0}$ defined in (2.16) is not integrable.

The proof is identical to that of Proposition 2.2.

So in this case, the Hamiltonian structure of the reduced equations (2.10) can only be seen by working with the bracket $\{\cdot, \cdot\}_{\text{rank} 0}$.

### 2.4 Symmetries

The reduced equations (2.10) can be interpreted as the output of a reduction process that we now explain. We begin by noticing that the configuration space $Q = SO(3) \times \mathbb{R}^3$ can be endowed with the Lie group structure of the three dimensional euclidean transformations $SE(3)$. The group multiplication is given by

\[
(g_1, x_1)(g_2, x_2) = (g_1 g_2, g_1 x_2 + x_1).
\]

Let $H$ be the Lie subgroup of $SE(3)$ defined by

\[
H = \{(h, y) \in SE(3) : he_3 = e_3\}.
\]

For matrices $A$ satisfying any of the conditions of Definition 1 it follows that $hA = Ah$ whenever $(h, y) \in H$. We consider the left action of $H$ on $Q$ by left multiplication. The tangent lift of the action to $TQ$ maps

\[
(h, y) : (g, x, \omega, \dot{x}) \mapsto (hg, hx + y, h\omega, h\dot{x}) \quad \text{or} \quad (h, y) : (g, x, \Omega, \dot{x}) \mapsto (hg, hx + y, \Omega, h\dot{x}),
\]

depending on the trivialization of $SO(3)$ that one is working with. Notice that the Lagrangian $L$ given by (2.5) is invariant under the lifted action. Moreover, since $h$ commutes with $A$ for any $(h, y) \in H$, the constraint (2.2) is also invariant.

The momenta $(K, p)$ are geometrically interpreted as coordinates on the fibers of the (trivial) cotangent bundle $T^*Q$. The constraint space $M \subset T^*Q$ is determined by the condition $p = mrAg\Omega$, so the triple $(g, x, K) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$ specifies a unique point in $M$. Reciprocally, any point in $M$ can be represented by a triple $(g, x, K)$.
By invariance of the Lagrangian and the constraints, the lifted action of $H$ to $T^*Q$ leaves $M$ invariant and, therefore, restricts to $\mathcal{M}$. The restricted action is free and proper so the orbit space $\mathcal{R} := \mathcal{M}/H$ is a smooth manifold. The reduced space $\mathcal{R}$ can be identified with $S^2 \times \mathbb{R}^3$; the projection $\rho : \mathcal{M} \to \mathcal{R}$ is given by
\[ \rho(g, x, K) = (\gamma, K), \] (2.17)
where $\gamma = g^{-1}e_3 \in S^2$, and is a surjective submersion. The conditions $he_3 = e_3$ and $hA = Ah$, that are satisfied for $(h, y) \in H$, ensure that the above mapping is well defined (in particular notice that $K$ is invariant). The reduced equations on $\mathcal{R}$ are precisely (2.10) when restricted to the level set $||\gamma|| = 1$.

In this sense, the entries of $\gamma$ should be considered as redundant coordinates for the sphere $S^2$ and the entries of $K$ as coordinates on the fibers of $\mathcal{R}$. Notice that, for any $j = 0, \ldots, 3$, both brackets $\{\cdot, \cdot\}_{\text{Rank}_j}$ and $\{\cdot, \cdot\}'_{\text{Rank}_j}$ restrict to the level set $||\gamma|| = 1$ since $C_2(\gamma) = ||\gamma||^2$ is a Casimir function.

### 2.5 Kinematics and integrability of the constraint distribution

The constraint distribution on $Q$ defined by equation (2.2) has fundamentally different properties according to the rank of the matrix $A$ satisfying the conditions of Definition 1. On one extreme we have the case where $A = 0$ and the distribution is integrable (the 3-dimensional integral leaves are given by $\text{SO}(3) \times \{x\}$ for $x \in \mathbb{R}^3$). As mentioned before, in this case the constraints are holonomic and the problem reduces to the classical free rigid body problem (the center of mass of the body $x$ remains constant in our inertial frame).

The extreme opposite case occurs when rank $A = 3$. In this case the corresponding distribution is completely nonholonomic or bracket-generating, see e.g. [27]. By Chow’s theorem, any two points in the configuration space $Q$ can be joined by a curve $(g(t), x(t))$ satisfying the constraints. Thus, at least at the kinematical level, there are no restrictions on the values of $x$.

The cases where the rank of $A$ is 1 or 2 lie in between the situations described above. If the rank of $A = 2$, the third component $x_3$ of $x$ remains constant during the motion. This is in agreement with our observation that the Chaplygin sphere problem is a particular case of this type of constraints - the sphere rolls on a horizontal plane $x_3 = \text{const}$. In this case, the constraint distribution is non-integrable but is nevertheless tangent to the foliation of $Q$ by 5-dimensional leaves defined by constant values of $x_3$.

Finally, for the case where the rank of $A$ equals 1, the first two components $x_1$, $x_2$, of $x$ remain constant during the motion. The body goes up or down along the $x_3$ axis at a speed that is proportional to its angular velocity about this axis. This time the constraint distribution is non-integrable but tangent to the 4-dimensional leaves given by constant values of $x_1$ and $x_2$.

Without going into technical definitions, we simply state that the degree of non-integrability of the constraint distribution increases with the rank of $A$, passing from an integrable distribution if $A = 0$ to a completely nonholonomic distribution if rank $A = 3$. It is interesting to see how this correlates with the need of a gauge-transformation to Hamiltonize the problem (Remark 5.6).
3 Geometric Setting

This section is concerned with the basic features of (almost) Dirac structures \cite{12}, with focus on the regular case, as well as their gauge transformations \cite{29}. As we will see, these geometric structures provide the setup that gives rise to the different brackets introduced in Section 2. Although we will be mostly interested in the geometry of bivector fields, our discussion is presented at the general level of (almost) Dirac structures, as they provide the framework in which gauge transformations are most natural.

3.1 Dirac and almost Dirac structures

A Dirac structure on a manifold $P$ is a subbundle $L$ of the Whitney sum $TP \oplus T^*P$ such that

(i) $L$ is a maximal isotropic subbundle of $TP \oplus T^*P$ with respect to the pairing $\langle \cdot, \cdot \rangle$ given by

$$\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X), \quad \text{for } (X, \alpha), (Y, \beta) \in TP \oplus T^*P.$$ 

(ii) $\Gamma(L)$ is closed with respect to the Courant bracket defined on $\Gamma(TP \oplus T^*P)$ given by

$$\[(X, \alpha), (Y, \beta)\] = ([X, Y], \ell_X \beta - i_Y \alpha),$$

for $(X, \alpha), (Y, \beta) \in \Gamma(TP \oplus T^*P)$, i.e., $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$.

The underlying manifold $P$ is sometimes referred to as a Dirac manifold.

Let $pr_1 : TP \oplus T^*P \rightarrow TP$ be the projection onto the first factor of $TP \oplus T^*P$. A Dirac structure $L$ on the manifold $P$ carries a Lie algebroid structure with anchor $pr_1|_L : L \rightarrow TP$ and bracket given by the Courant bracket $[\cdot, \cdot]$ restricted to $\Gamma(L)$. It follows that, for a Dirac structure $L$, the distribution $pr_1(L) \subseteq TP$ is integrable, i.e., $P$ can be decomposed into leaves $O$ such that for each $x \in P$, $T_xO = pr_1(L_x)$. If $pr_1(L)$ has constant rank (i.e., $pr_1(L_x) \subseteq T_xP$ has the same dimension for all $x \in P$), then we say that $L$ is a regular Dirac structure, and $pr_1(L)$ defines a regular foliation. Just as a Poisson manifold $P$ is the disjoint union of its symplectic leaves, each leaf of a Dirac manifold $P$ carries a presymplectic form.

Examples Let $\Omega$ be a closed 2-form and $\pi$ be a Poisson bivector field, and consider the maps $\Omega^\circ : TP \rightarrow T^*P$ given by $\Omega^\circ(X) = \Omega(\cdot, X) = -i_X \Omega = \text{for } X \in TP$ and $\pi^\circ : T^*P \rightarrow TP$ as in Definition 2. Then

$$L_\Omega := graph(\Omega^\circ) = \{(X, \alpha) \in TP \oplus T^*P : i_X \Omega = -\alpha\}$$

and

$$L_\pi := graph(\pi^\circ) = \{(X, \alpha) \in TP \oplus T^*P : \pi^\circ(\alpha) = X\}$$

are Dirac structures. Note that $pr_1$ identifies $L_\Omega$ with $TP$ as Lie algebroids. Similarly, $L_\pi$ can be naturally identified with $T^*P$, and the Lie algebroid structure induced on $T^*P$ by $L_\pi$ has anchor $\pi^\circ : T^*P \rightarrow TP$, and bracket

$$[\alpha, \beta]_\pi = L_{\pi^\circ(\alpha)}(\beta) - L_{\pi^\circ(\beta)}(\alpha) - d(\pi(\alpha, \beta)) = L_{\pi^\circ(\alpha)}\beta - i_{\pi^\circ(\beta)}d\alpha.$$

This bracket is uniquely characterized by $[df, dg]_\pi = d\{f, g\}$ and the Leibniz identity.
If a subbundle $L$ of $TP \oplus T^*P$ satisfies (i) above, but not necessarily (ii), then $L$ is called an *almost Dirac structure*. Condition (ii) is called the integrability condition. We say that $L$ is a *regular* almost Dirac structure when the distribution $pr_1(L)$ on $P$ has constant rank. Notice that this distribution might not be integrable in the almost Dirac case.

**Examples** If $\Omega$ is an arbitrary 2-form or $\pi$ an arbitrary bivector field, then their graphs $L_\Omega$ and $L_\pi$ are almost Dirac structures. The failure of the integrability with respect to the Courant bracket of $L_\Omega$ and $L_\pi$ is measured by $d\Omega$ and $\frac{1}{2}[\pi,\pi]$, respectively. For $L_\pi = graph(\pi^t)$, the distribution $\pi^t(T^*P) = pr_1(L_\pi)$ is generally non-integrable. If it has constant rank we call the almost Poisson structure *regular*. Note that the bracket $[\cdot,\cdot]_\pi$ defined as in (3.1) is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Leibniz identity. However, in general, $\pi^t$ does not necessarily preserve the bracket; instead, (see e.g. [8]),

$$
\pi^t([\alpha,\beta]_\pi) = [\pi^t(\alpha),\pi^t(\beta)] - \frac{1}{2}i_{\alpha \wedge \beta}[\pi,\pi], \quad \text{for } \alpha, \beta \in \Omega^1(P). \tag{3.18}
$$

Note that an almost Dirac structure $L$ on $P$ is of the form $L_\pi = graph(\pi^t)$ for a bivector $\pi$ if and only if

$$
TP \cap L = \{0\}, \tag{3.19}
$$

and $L$ is of the form $L_\Omega = graph(\Omega^\flat)$ for a 2-form $\Omega$ if and only if $T^*P \cap L = \{0\}$, see [12]. Another example of an almost Dirac structure that will be very useful for our purposes is given by $L \subset TP \oplus T^*P$ defined as

$$
L := \{(X,\alpha) \in TP \oplus T^*P : X \in F, \ i_X\Omega|_F = -\alpha|_F \}, \tag{3.20}
$$

where $F \subset TP$ is a subbundle, $\Omega$ is a 2-form on $P$ and $\cdot|_F$ denotes the point-wise restriction to $F$. If the subbundle $F$ is an integrable distribution and $\Omega$ is closed, then $L$ is a Dirac structure.

**Proposition 3.1.** The following statements hold:

(i) There is a one-to-one correspondence between regular almost Dirac structures $L \subset TP \oplus T^*P$ and pairs $(F,\Omega_F)$, where $F$ is a regular distribution on $P$ and $\Omega_F \in \Gamma(\wedge^2 F^*)$.

(ii) Let $F \subset TP$ be a regular distribution on $P$. Given a section $\Omega_F \in \Gamma(\wedge^2 F^*)$, there exists a 2-form $\Omega$ on $P$ such that $\Omega|_F = \Omega_F$.

**Proof.** (i) Let $L \subset TP \oplus T^*P$ be a regular almost Dirac structure with distribution $F := pr_1(L) \subset TP$ on $P$ (not necessarily integrable). Consider the section $\Omega_F$ in $\Gamma(\wedge^2 F^*)$ given, at each $x \in P$, by

$$
\Omega_F(x)(X(x),Y(x)) = -\alpha(x)(Y(x)), \quad \text{for } X,Y \in \Gamma(F) \text{ such that } (X(x),\alpha(x)) \in L_x.
$$

It is a straightforward computation to see that $\Omega_F$ is well defined, i.e., it is independent of the choice of $\alpha$. Conversely, given a regular distribution $F$ on $P$ and $\Omega_F \in \Gamma(\wedge^2 F^*)$, we may define the subbundle $L \subset TP \oplus T^*P$ as the pairs $(X,\alpha)$ such that $X \in F$ and $i_X\Omega = -\alpha|_F$.

(ii) Let $W \subset TP$ be a regular smooth distribution such that it is a complement of $F$ on $P$, i.e., $T_xP = F_x \oplus W_x$ for each $x \in P$ (e.g., $W_x$ can be chosen to be the orthogonal complement of $F_x$ with respect to a Riemannian metric). The 2-form $\Omega$ on $P$ can be defined by

$$
\Omega(X,Y) = \Omega_F(X_F,Y_F),
$$

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for $X, Y \in \mathfrak{X}(P)$ such that $X = X_F + X_W$ and $Y = Y_F + Y_W$, where $X_F, Y_F \in \Gamma(F)$ and $X_W, Y_W \in \Gamma(W)$. Differentiability of $\Omega$ follows from its definition and the smoothness of $F$ and $W$.

**Corollary 3.2.** Given a regular almost Dirac structure $L$, there exists a 2-form $\Omega$ on $P$ and a regular distribution $F \subseteq TP$ such that $L$ is written in the form (3.20).

**Remark 3.3.** Note that the 2-form $\Omega \in \Omega^2(P)$ is not uniquely defined and, in general, there is no canonical choice for it.

Given a subbundle $F \subseteq TP$, we say that a section $\Omega_F$ in $\Gamma(\bigwedge^2 F^*)$ is nondegenerate if it is nondegenerate as a bilinear form on $F$ at each point. It follows from (3.20) that $\text{Ker}(\Omega_F) = L \cap TP$, and as a consequence of condition (3.19) we obtain

**Corollary 3.4.** Let $L$ be a regular almost Dirac structure and $(F, \Omega_F)$ the pair associated to it in the sense of Proposition 3.1. Then $\Omega_F$ is nondegenerate if and only if $L$ is the graph of a bivector field $\pi$. Explicitly, the relation between $(F, \Omega_F)$ and $\pi$ is

$$\pi^X(\alpha) = -X \quad \text{if and only if} \quad i_X \Omega_F = \alpha|_F,$$

where $X \in \Gamma(F)$ and $\alpha \in \Omega^1(P)$.

Following notation of Definition 2, if $\{\cdot, \cdot\}$ is the bracket associated to the bivector field $\pi$ in the above Corollary, then $\{f, g\} = \Omega_F(X_f, X_g)$, for all $f, g \in C^\infty(P)$.

### 3.2 Twisted Poisson and twisted Dirac structures

Poisson structures may be viewed as encoding integrability in two levels: first, the characteristic distribution $\pi^2(T^* P) \subseteq TP$ is integrable, i.e., tangent to leaves; second each leaf carries a nondegenerate 2-form that is closed (and this leads to the Jacobi identity). Twisted Poisson structures are special types of almost Poisson structures that retain the integrability of $\pi^2(T^* P)$ but allow the leafwise 2-form to be non closed. These objects turn out to be related to Hamiltonization. We start with the more general notion of twisted Dirac structures.

Consider a closed 3-form $\phi$ on $P$, and define the $\phi$-twisted Courant bracket [29] as follows:

$$[(X, \alpha), (Y, \beta)]_{\phi} = ([X, Y], L_X \beta - i_Y d\alpha + i_X \wedge Y \phi),$$

for $(X, \alpha)$ and $(Y, \beta)$ in $\Gamma(TP \oplus T^* P)$. Now, a subbundle $L$ of $TP \oplus T^* P$ is a $\phi$-twisted Dirac structure [29] if $L$ is maximal isotropic with respect to $\langle \cdot, \cdot \rangle$ and the integrability condition

$$[\Gamma(L), \Gamma(L)]_{\phi} \subseteq \Gamma(L)$$

is satisfied.

As in the ordinary case, a twisted Dirac structure $L$ on $P$ induces a Lie algebroid on $L$ given by the anchor map $pr_1|_L$ and the bracket $[\cdot, \cdot]_{\phi}|_{\Gamma(L)}$. Therefore $pr_1(L)$ is an integrable distribution on $P$. 

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Examples If $\Omega$ is any 2-form on $P$, then $L_\Omega = \text{graph}(\Omega^\flat)$ is $(d\Omega)$-twisted Dirac. One may check that a bivector field $\pi$ on $P$ such that $L_\pi = \text{graph}(\pi^\flat)$ is $\phi$-twisted Dirac verifies (see [29])

$$\frac{1}{2}[\pi, \pi] = \pi^\sharp(\phi).$$

(3.22)

The following result gives more examples:

**Theorem 3.5.** Let $L \subset TP \oplus T^*P$ be a regular almost Dirac structure such that $pr_1(L) \subset TP$ is an integrable distribution on $P$. Then, there exists an exact 3-form $\phi$ with respect to which $L$ is a $\phi$-twisted Dirac structure.

**Proof.** Let $F := pr_1(L) \subset TP$ and $\Omega_F$ be the section in $\Gamma(\bigwedge^2 F^*)$ associated to $L$ given by Proposition 3.1. Since $F$ is integrable, $\Omega_F$ defines a 2-form $\Omega_O$ on each leaf $O$ where $F_x = T_xO$ at each $x \in P$. By Corollary 3.2 there exists a 2-form $\Omega$ on $P$ such that $\iota^*\Omega = \Omega_O$ where $\iota_O: O \hookrightarrow P$ is the inclusion.

We assert that $L$ is a $(d\Omega)$-twisted Dirac structure. In fact, for $(X, \alpha)$ and $(Y, \beta)$ in $\Gamma(L)$,

$$[(X, \alpha), (Y, \beta)](d\Omega) = ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + i_{X\wedge Y} d\Omega) \in \Gamma(L)$$

if and only if

$$i_{[X,Y]}\Omega|_F = -(\mathcal{L}_X \beta - i_Y d\alpha + i_{X\wedge Y} d\Omega)|_F.$$ 

Since $F$ is an integrable distribution we obtain that,

$$-(\mathcal{L}_X \beta - i_Y d\alpha + i_{X\wedge Y} d\Omega)|_F = -\mathcal{L}_X (\beta|_F) + i_Y d(\alpha|_F) - i_{X\wedge Y} d(\Omega|_F) = \mathcal{L}_X i_Y \Omega_F - i_Y dX \Omega_F - i_{X\wedge Y} d\Omega_F = i_{[X,Y]}\Omega_F.$$ 

which completes the proof. \qed

**Remark 3.6.**

(i) Note that if $L$ is a $\phi$-twisted Dirac structure then $L$ is also twisted with respect to any closed 3-form $\phi'$ such that $(\phi - \phi')$ vanishes on the leaves.

(ii) There is no canonical choice for the 3-form $\phi$ given in Theorem 3.5.

Twisted Poisson bivectors.

Bivector fields $\pi$ such that $L_\pi = \text{graph}(\pi^\flat)$ is a $\phi$-twisted Dirac structure are called $\phi$-twisted Poisson bivectors [24, 29], i.e., $\pi$ verifies condition (3.22). We are especially interested in these kind of bivector fields since, as we will see, they appear naturally in the examples of nonholonomic systems introduced in Section 2. If $\{\cdot, \cdot\}$ is the bracket given by the $\phi$-twisted Poisson structure $\pi$, then relation (2.11) becomes

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + \phi(X_f, X_g, X_h) = 0,$$

for $f, g, h \in C^\infty(P)$ and $X_f = \{\cdot, f\}$. So the failure of the Jacobi identity is controlled by the closed 3-form $\phi$. 19
Example If \( \pi : T^*P \to TP \) is an isomorphism, then \( \pi \) is \( \phi \)-twisted and \( \phi = d\Omega \) where \( \Omega \) is the nondegenerate 2-form associated with \( \pi \), i.e., \( \Omega^\circ \circ \pi^\sharp = \text{Id} \) where as usual \( \Omega^\sharp (X) = -i_X \Omega \). Other, less trivial examples, will be presented ahead in Corollary 3.7.

Let \( \pi \) be a \( \phi \)-twisted Poisson structure on the manifold \( P \) and \( [\cdot, \cdot]_\pi \) be the bracket on \( T^*P \) given by (3.1). Note that \( \pi^\sharp \) does not preserve this bracket. However, using (3.18) and (3.22) we obtain

\[
[\pi^\sharp(\alpha), \pi^\sharp(\beta)] = \pi^\sharp \left( [\alpha, \beta]_\pi + i_{\pi^\sharp(\alpha)} \wedge i_{\pi^\sharp(\beta)} \partial \phi \right),
\]

for 1-forms \( \alpha, \beta \) on \( P \). The \( \phi \)-twisted Courant bracket induces a modification of the bracket (3.1) via the identification of \( T^*P \) and \( L_\pi \)

\[
[\alpha, \beta]_{\phi} = \mathcal{L}_{\pi^\sharp(\alpha)} \beta - i_{\pi^\sharp(\beta)} d\alpha + i_{\pi^\sharp(\alpha)} \wedge i_{\pi^\sharp(\beta)} \partial \phi,
\]

such that \((T^*P, [\cdot, \cdot]_{\phi}, \pi^\sharp)\) is a Lie algebroid [29] (see also [8]). The characteristic distribution \( \pi^\sharp(T^*P) \) defines an integrable distribution on \( P \) (that may be singular). Each leaf \( \mathcal{O} \) of the corresponding foliation of \( P \) is endowed with a non-degenerate 2-form \( \Omega_\mathcal{O} \) that is not necessarily closed. If \( \pi \) is \( \phi \)-twisted, then \( d\Omega_{\mathcal{O}} = \iota_{\mathcal{O}}^* \phi \), where \( \iota_{\mathcal{O}} : \mathcal{O} \hookrightarrow P \) is the inclusion.

Important examples of twisted Poisson structures are contained in the following Corollary of Theorem 3.6:

**Corollary 3.7.** Let \( \pi \) be a bivector field on \( P \) with an integrable regular characteristic distribution. Then, there exists an exact 3-form \( \phi \) on \( P \) with respect to which \( \pi \) is \( \phi \)-twisted.

**Remark 3.8.** Mechanical Example. It is shown in [18] that the (semi-direct) product reduction of the Veselova system yields a regular conformally Poisson bracket on the reduced space. It follows that its characteristic distribution is integrable and thus, by Corollary 3.7, it is also twisted-Poisson. This is a first example of a nonholonomic system whose reduced equations are formulated in terms of a twisted-Poisson bracket. Other examples (related to the motion of a rigid body with generalized rolling constraints) are made explicit in Section 5.3.

**Remark 3.9.** An interesting question, that remains to be answered, is to give a characterization of almost Poisson brackets possessing an integrable characteristic distribution that is non-regular.

**Regular conformally Poisson bivectors.**

An interesting class of almost Poisson structures admitting an integrable characteristic distribution is given by conformally Poisson structures. Recall from Section 2 that they are bivector fields \( \pi \) for which exists a strictly positive function \( \varphi \in C^\infty(P) \), such that \( \varphi \pi \) is Poisson. A conformally Poisson manifold \((P, \pi)\) is the disjoint union of conformally symplectic leaves.

Note that this property is stronger than asking for \((P, \pi)\) to be a Jacobi manifold since a conformal factor implies the global existence of a function such that \( \varphi \pi \) is Poisson, while in Jacobi manifolds the factor \( \varphi \) may be only locally defined.

From Theorem 3.5 we observe that any regular bivector admitting a conformal factor is also a twisted Poisson bivector. The following Proposition explains the relation between these two properties.
Proposition 3.10. Let \( \pi \) be a regular conformally Poisson bivector field on \( P \) with conformal factor \( \varphi \). Let \( \Omega \in \Omega^2(P) \) be as in Corollary 3.2. Then any closed 3-form verifying (3.22) for \( \pi \) coincides with \( \frac{1}{\varphi}d\varphi \wedge \Omega \) on the leaves.

Proof. Since \( \pi \) admits a conformal factor \( \varphi \in C^\infty(P) \), then \( [\pi, \pi] = \frac{2}{\varphi}X_\varphi \wedge \pi \). On the other hand, if \( \Omega \) is the 2-form on \( \pi \) associated to \( \pi \) given by Corollary 3.2, then for \( g_1, g_2 \in C^\infty(P) \) we have \( \Omega(X_{g_1}, X_{g_2}) = \pi(dg_1, dg_2) \). Thus, for \( g_1, g_2, g_3 \in C^\infty(P) \) we obtain

\[
\frac{1}{2} [\pi, \pi](dg_1, dg_2, dg_3) = \frac{1}{\varphi}X_\varphi \wedge \pi(dg_1, dg_2, dg_3) = -\frac{1}{\varphi}d\varphi \wedge \Omega(X_{g_1}, X_{g_2}, X_{g_3}).
\]

Then we conclude that any closed 3-form \( \phi \) satisfying (3.22) coincides with \( \frac{1}{\varphi}d\varphi \wedge \Omega \) on the leaves. \( \square \)

Let \( (P, \pi) \) be a regular conformally Poisson manifold. A 2-form \( \Omega \) on \( P \) satisfying the conditions of Corollary 3.2 verifies that \( \varphi \Omega \) is conformally symplectic on each leaf \( \mathcal{O} \). However, \( \Omega \) may not necessarily be conformally closed.

3.3 Gauge transformations.

In this section we will consider a natural action of the abelian group of 2-forms on \( P \) on the almost Dirac structures on \( P \). This action is given by gauge transformations of almost Dirac structures by 2-forms, and it was introduced in [29].

More precisely, consider an almost Dirac structure \( L \) in \( TP \oplus T^*P \). A gauge transformation by the 2-form \( B \) is a map \( \tau_B : L \to TP \oplus T^*P \), given by \( \tau_B((X, \alpha)) = (X, \alpha + i_X B) \) for \( (X, \alpha) \in L \). The subbundle \( \tau_B(L) \) of \( TP \oplus T^*P \) given by

\[
\tau_B(L) = \{(X, \alpha + i_X B) : (X, \alpha) \in L\}
\]

is an almost Dirac structure. If the 2-form \( B \) is closed and \( L \) is Dirac then \( \tau_B(L) \) is again Dirac. Thus, the 3-form \( dB \) is what determines the integrability with respect to the Courant bracket. It is a direct computation to see that if \( L \) is a \( \phi \)-twisted Dirac structure, then the gauge transformation of \( L \) by the 2-form \( B \) is \( (\phi - dB) \)-twisted Dirac (see e.g. [29]).

If \( L_1 \) and \( L_2 \) are almost Dirac structures on \( P \) and there exist a 2-form \( B \) on \( P \) such that \( \tau_B(L_1) = L_2 \), then we say that \( L_1 \) and \( L_2 \) are gauge equivalent or gauge related.

Note that a gauge transformation does not modify the distribution \( pr_1(L) \). So, for a Dirac structure \( L \), the foliation associated to \( L \) will be the same as the one associated to \( \tau_B(L) \). However, the presymplectic form on each leaf is modified by the pullback of \( B \) to the leaf. If \( L \) is a regular almost Dirac structure determined by the pair \( (F, \Omega_F) \) in the sense of item (i) of Proposition 3.1, a gauge transformation by the 2-form \( B \) corresponds to the operation:

\[ \tau_B : (F, \Omega_F) \to (F, \Omega_F - B|_F). \]

Theorem 3.11. Any two regular almost Dirac structures \( L_1 \) and \( L_2 \) are gauge related if and only if \( pr_1(L_1) = pr_1(L_2) \).
Proof. It remains to prove the “only if” part of the statement. Let us denote \( F := pr_{1}(L_{1}) = pr_{1}(L_{2}) \) and let \( \Omega_{F}^{1} \) and \( \Omega_{F}^{2} \) be the 2-sections associated to \( L_{1} \) and \( L_{2} \) respectively (Proposition 3.1 (i)). Define the section \( B_{F} \in \Gamma(\Lambda^{2}(F^{*})) \) by \( B_{F} := \Omega_{F}^{1} - \Omega_{F}^{2} \) and let \( B \in \Omega^{2}(P) \) such that \( B|_{F} = B_{F} \) (Prop. 3.1 (ii)). We claim that \( \tau_{B} L_{1} = L_{2} \). In fact, if \( (F, \Omega_{F}^{B}) \) is the pair associated to the almost Dirac structure \( \tau_{B} L_{1} \), then by equation (3.23),
\[
\Omega_{F}^{B} = \Omega_{F}^{1} - B_{F} = \Omega_{F}^{2}.
\]
Since the sections associated to \( \tau_{B} L_{1} \) and \( L_{2} \) coincide, by Proposition 3.1 (i) we conclude that \( \tau_{B} L_{1} = L_{2} \) which means that \( L_{1} \) and \( L_{2} \) are gauge related.

\[ \square \]

We are especially interested in gauge transformations of almost Poisson structures. Consider the almost Poisson manifold \((P, \pi)\) and a 2-form \( B \) on \( P \). Then, the gauge transformation of \( L_{\pi} := graph(\pi^{B}) \) is \( \tau_{B}(L_{\pi}) = \{(X, \alpha + i_{X}B) \in TP \oplus T^{*}P : X = \pi^{\sharp}(\alpha)\} \) which does not necessarily correspond to the graph of a new bivector \( \pi^{B} \). A necessary and sufficient condition for this to happen is that
\[
\tau_{B}(L_{\pi}) \cap TP = \{0\},
\]
which is equivalent to the fact that the endomorphism \( \text{Id} + B^{\sharp} \circ \pi^{\sharp} : T^{*}P \to T^{*}P \) is invertible \([29]\). Indeed, if such a bivector field \( \pi^{B} \) exists, then, in view of (3.3), for any 1-form \( \alpha \) on \( P \) we have
\[
\tau_{B} \left( (\pi^{\sharp}(\alpha), \alpha) \right) = \left( (\pi^{\sharp}(\alpha), \alpha + i_{\pi^{\sharp}(\alpha)}B) = \left( (\pi^{B})^{\sharp}(\alpha + i_{\pi^{\sharp}(\alpha)}B), \alpha + i_{\pi^{\sharp}(\alpha)}B \right). \right.
\]
Thus, \( \pi^{B} \) is characterized by the condition
\[
(\pi^{B})^{\sharp}(\alpha + i_{\pi^{\sharp}(\alpha)}B) = \pi^{\sharp}(\alpha), \tag{3.24}
\]
and we can write
\[
(\pi^{B})^{\sharp} = \pi^{\sharp} \circ (\text{Id} - B^{\sharp} \circ \pi^{\sharp})^{-1}. \tag{3.25}
\]
In particular, if \( \pi \) and \( \pi^{B} \) are non-degenerate bivector fields, equation (3.24) is equivalent to
\[
((\pi^{B})^{\sharp})^{-1} = (\pi^{\sharp})^{-1} - B^{\sharp}.
\]
Therefore, any two 2-forms on a manifold are gauge related. This is not necessarily the case with bivector fields. A necessary condition is that their characteristic distributions coincide. In view of Theorem 3.11 this condition is also sufficient if such distributions are regular.

Although gauge related bivectors have the same characteristic distribution, their Schouten brackets \([\pi, \pi] \) and \([\pi^{B}, \pi^{B}] \) may not coincide. The following Proposition makes this precise.

**Proposition 3.12** \([29]\). If a \( \phi \)-twisted Poisson bivector \( \pi \) is gauge related with another bivector \( \pi^{B} \) via the 2-form \( B \), then \( \pi^{B} \) is \((\phi - dB)\)-twisted. That is,
\[
\frac{1}{2}[\pi^{B}, \pi^{B}] = (\pi^{B})^{\sharp}(\phi - dB).
\]

In particular, if \( \pi \) is Poisson, then \( \pi^{B} \) is \((-dB)\)-twisted.
4 Applications to Nonholonomic Systems and Hamiltonization

In what follows, we will analyze the geometry of nonholonomic systems in the framework presented in the previous section. We introduce gauge transformations of the bracket describing the nonholonomic dynamics in order to study the process of Hamiltonization.

4.1 Nonholonomic systems

A nonholonomic system consists of an \( n \)-dimensional configuration manifold \( Q \) with local coordinates \( q \in U \subset \mathbb{R}^n \), a Lagrangian \( L : TQ \rightarrow \mathbb{R} \) of the form \( L(q, \dot{q}) = \frac{1}{2}G(q)(\dot{q}, \dot{q}) - V(q) \), where \( G \) is a kinetic energy metric on \( Q \) and \( V : Q \rightarrow \mathbb{R} \) is a potential, and a regular non-integrable distribution \( D \subset TQ \) that describes the kinematic nonholonomic constraints. In coordinates, the distribution \( D \) is defined by the equation

\[
\epsilon(q) \dot{q} = 0,  \tag{4.26}
\]

where \( \epsilon(q) \) is a \( k \times n \) matrix of constant rank \( k \) where \( k < n \) is the number of constraints. The entries of \( \epsilon(q) \) are the components of the \( \mathbb{R}^k \)-valued constraint 1-form on \( Q \), \( \epsilon := \epsilon(q) \, dq \).

The dynamics of the system are governed by the Lagrange-D’Alembert principle. This principle states that the forces of constraint annihilate any virtual displacement, so they perform no work during the motion. The equations of motion take the form

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \mu^T \epsilon(q).  \tag{4.27}
\]

Here \( \mu : TQ \rightarrow \mathbb{R}^k \) is an \( \mathbb{R}^k \)-valued function whose entries are referred to as Lagrange multipliers. Under our assumptions, it is uniquely determined by the condition that the constraints \( \epsilon(q) \) are satisfied. The equations \( \text{(4.27)} \) together with the constraints \( \text{(4.26)} \) define a vector field \( Y_{\text{nh}} \) on \( D \) whose integral curves describe the motion of the nonholonomic system. A short calculation shows that along the flow of \( Y_{\text{nh}} \), the energy function \( E_L := \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L \), is conserved.

The above equations of motion can be written as a first order system on the cotangent bundle \( T^*Q \) via the standard Legendre transform, \( \text{Leg} : TQ \rightarrow T^*Q \), that defines canonical coordinates \( (q, p) \) on \( T^*Q \) by the rule \( \text{Leg} : (q, \dot{q}) \mapsto (q, p = \partial L/\partial \dot{q}) \). The Legendre transform is a global diffeomorphism by our assumption that \( G \) is a metric.

The Hamiltonian function, \( H : T^*Q \rightarrow \mathbb{R} \), is defined in the usual way \( H := E_L \circ \text{Leg}^{-1} \). The equations of motion \( \text{(4.27)} \) are shown to be equivalent to

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + \mu^T \epsilon(q),  \tag{4.28}
\]

and the constraint equations \( \text{(4.26)} \) become

\[
\epsilon(q) \frac{\partial H}{\partial p} = 0.  \tag{4.29}
\]

The above equation defines the constraint submanifold \( \mathcal{M} = \text{Leg}(D) \subset T^*Q \). Since the Legendre transform is linear on the fibers, \( \mathcal{M} \) is a vector sub-bundle of \( T^*Q \) that for each \( q \in Q \) specifies an \( n - k \) vector subspace of \( T_q^*Q \).
Equations (4.28) together with (4.29) define the vector field \( X_{nh} \) on \( \mathcal{M} \), that describes the motion of our nonholonomic system in the Hamiltonian side and is the push forward of the vector field \( Y_{nh}^D \) by the Legendre transform. The vector field \( X_{nh} \) is defined uniquely in an intrinsic way by the equation

\[
i_{X_{nh}} \iota^* \Omega_Q = \iota^*(dH + \mu^T \tau^* \epsilon),
\]

where \( \Omega_Q \) is the canonical symplectic form on \( T^*Q \), \( \iota : \mathcal{M} \to T^*Q \) is the inclusion and \( \tau : T^*Q \to Q \) is the canonical projection. The constraints (4.29) and their derivatives are intrinsically written as

\[
X_{nh} \in \mathcal{C} := TM \cap F
\]

where \( F \) is the distribution on \( T^*Q \) defined as

\[
F := \{ v \in T(T^*Q) : \langle \tau^* \epsilon, v \rangle = 0 \}
\]

Denote by \( \Omega_M \) the pull-back of \( \Omega_Q \) to \( \mathcal{M} \), i.e. \( \Omega_M := \iota^* \Omega_Q \). The following Proposition is of great importance for our setup of the equations of motion as an almost Hamiltonian system.

**Proposition 4.1** ([34, 2]). The distribution \( \mathcal{C} \) on \( \mathcal{M} \) defined by (4.31) is regular, non-integrable, and the point-wise restriction of \( \Omega_M \) to \( \mathcal{C} \), denoted by \( \Omega_C \), is non-degenerate.

The non-integrability of \( \mathcal{C} \) is a direct consequence of the non-integrability of \( D \). One shows that the rank of \( \mathcal{C} \) is \( 2(n - k) \) and that along \( \mathcal{M} \) we have the symplectic decomposition

\[
T_M(T^*Q) = C \oplus C^{\Omega_Q}
\]

where \( C^{\Omega_Q} \) denotes the symplectic orthogonal complement to \( C \).

Since \( \tau^* \epsilon \) vanishes on \( C \), by restricting (4.30) to \( C \) and denoting \( H_M := \iota^*H \in C^\infty(\mathcal{M}) \), the equations of motion can be written in the appealing format

\[
i_{X_{nh}} \Omega_C = (dH_M)|_C,
\]

where \( (dH_M)_C \) is the point-wise restriction of \( dH_M \) to \( C \). The above equation uniquely defines the vector field \( X_{nh} \) and is central in our treatment; with this in mind, we collect the data of the nonholonomic system in the triple \( (\mathcal{M}, \Omega_C, H_M) \).

Even though (4.33) defines the vector field \( X_{nh} \) uniquely, and resembles a classical Hamiltonian system, notice that since the distribution \( \mathcal{C} \) is non-integrable, then \( \Omega_C \) is a section in \( \bigwedge^2 C^* \to \mathcal{M} \) (not a 2-form).

Let \( (\mathcal{M}, \Omega_C, H_M) \) be a nonholonomic system. For every \( f \in C^\infty(\mathcal{M}) \), let \( X_f \) denote the unique vector field on \( \mathcal{M} \) with values in \( \mathcal{C} \) defined by the equation

\[
i_{X_f} \Omega_C = (df)_C,
\]

where \( (df)_C \) denotes the point-wise restriction of \( df \) to \( C \). The vector field \( X_f \) defined by equation (4.34) is called the \textit{the (almost) Hamiltonian vector field associated to} \( f \).

Since \( \Omega_C \) is nondegenerate, by Corollary 3.4 there is a unique bivector field \( \pi_{nh} \) on \( \mathcal{M} \) associated to the pair \( (\mathcal{C}, \Omega_C) \), that is \( \pi_{nh}^C(\alpha) = -X \) if and only if \( i_X \Omega_C = \alpha|_C \). On exact forms we have, for \( f \in C^\infty(\mathcal{M}) \),

\[
i_{X_f} \Omega_C = (df)_C \text{ if and only if } \pi_{nh}^C(df) = -X_f,
\]

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which is consistent with notation of Definition 2. The bracket \( \{ \cdot, \cdot \}_\text{nh} \) on functions on \( M \) associated to the bivector \( \pi_{\text{nh}} \) describes the dynamics in the sense that

\[
X_{\text{nh}}(f)(m) = X_{\mathcal{H}_M}(f)(m) = \{ f, \mathcal{H}_M \}_\text{nh}(m) \quad \text{for all } f \in C^\infty(M).
\]

It follows from (4.35) that the characteristic distribution of the bracket \( \{ \cdot, \cdot \}_\text{nh} \) is \( \mathcal{C} \). Since \( \mathcal{C} \) is non-integrable then \( \{ \cdot, \cdot \}_\text{nh} \) is an almost Poisson bracket that does not satisfy the Jacobi identity.

**Remark 4.2.** If the constraint distribution \( \mathcal{D} \) were integrable, the same would be true for the distribution \( \mathcal{C} \). Let \( N \subset M \) be a leaf of the corresponding (regular) foliation of \( M \) (i.e. \( \mathcal{C}_x = T_xN \) for all \( x \in \mathcal{N} \)). In view of (4.32) the submanifold \( N \) is symplectic. Therefore, in this case, our construction of \( \{ \cdot, \cdot \}_\text{nh} \) coincides with the usual construction of the Dirac bracket on each leaf \( N \) of the foliation of \( M \) (see [13, 12]). Hence, in this case, the Jacobi identity holds.

Inspired by equation (3.20) and our data, it is natural to define the almost Dirac structure \( L_{\pi_{\text{nh}}} \) on \( M \) by

\[
L_{\pi_{\text{nh}}} := \text{graph}(\pi_{\text{nh}}^\sharp),
\]

as considered (up to a minus sign) in [32, 33], and subsequently in [23].

**Proposition 4.3.** Let \( \pi_{\text{nh}} \) be the bivector field defined in (4.35) and \( \{ \cdot, \cdot \}_\text{nh} \) the corresponding bracket. The following statements hold:

(i) The almost Dirac structures \( L_{\pi_{\text{nh}}} := \text{graph}(\pi_{\text{nh}}^\sharp) \) and \( L_{\text{nh}} \) given in (4.36) coincide.

(ii) The almost Poisson bracket \( \{ \cdot, \cdot \}_\text{nh} \) coincides with the classical almost Poisson bracket for nonholonomic systems defined in [30, 25].

**Proof.** (i) It is immediate since both almost Dirac structure are defined by the same pair \( (\mathcal{C}, \Omega_\mathcal{C}) \).

(ii) The bracket on \( M \) for nonholonomic systems introduced in [30, 25] was shown in [9] to be given by

\[
\{ f, g \} = \Omega_M(\mathcal{P}Y_f, \mathcal{P}Y_g), \quad \text{for } f, g \in C^\infty(M),
\]

where \( \mathcal{P} : T_M(T^*Q) \to \mathcal{C} \) is the projector associated to the symplectic decomposition (4.32), and \( Y_f \) the free Hamiltonian vector field on the symplectic manifold \( (T^*Q, \Omega_Q) \) defined by \( i_{Y_f} \Omega_Q = df \), where \( \bar{f} \) is an arbitrary smooth extension of \( f \) to \( T^*Q \).

It is easy to check that along \( M \) one has \( \mathcal{P}Y_f \mathcal{C} = (df)_\mathcal{C} \), so \( \mathcal{P}Y_f \) coincides with the almost Hamiltonian vector field \( X_f \) defined by equation (4.34). Therefore, for any \( f, g \in C^\infty(M) \),

\[
\{ f, g \} = \Omega_M(\mathcal{P}Y_f, \mathcal{P}Y_g) = \mathcal{C}(X_f, X_g) = -df(\pi_{\text{nh}}^\sharp (dg)) = \{ f, g \}_\text{nh}.
\]

The second item of the above Proposition should not be surprising since the expression (4.37) is the nonholonomic version of the Dirac bracket (see discussion in [21, 9]). Hence, its description naturally falls in the ambit of almost Dirac structures as described above. As a consequence of the above Proposition, we can equivalently describe our nonholonomic system with the triple \( (M, \pi_{\text{nh}}, \mathcal{H}_M) \).
Definition 3. Let $(\mathcal{M}, \pi_{nh}, \mathcal{H}_M)$ be a nonholonomic system.

1. The bivector field $\pi_{nh}$ on $\mathcal{M}$ given by (4.35) is called the nonholonomic bivector field and the bracket $\{\cdot, \cdot\}_{nh}$ is called the nonholonomic bracket.

2. We say that an almost Dirac structure $L$ describes the dynamics of the nonholonomic system if the pair $(-X_{nh}, d\mathcal{H}_M) \in \Gamma(L)$.

4.2 Gauge transformations of the nonholonomic bracket

The main idea of using gauge transformations in our setting is that it opens the possibility to modify the geometric structure on $\mathcal{M}$ that describes the dynamics.

Consider the nonholonomic system $(\mathcal{M}, \pi_{nh}, \mathcal{H}_M)$ and continue to denote $L_{\pi_{nh}} = \text{graph}(\pi_{nh})$. The gauge transformation of $\pi_{nh}$ associated to a 2-form $B$ on $\mathcal{M}$ gives

$$\tau_B(L_{\pi_{nh}}) = \{ (X, \alpha + i_X B) \in T\mathcal{M} \oplus T^*\mathcal{M} : \pi_{nh}(\alpha) = X \}.$$  \hfill (4.38)

First of all we are interested in knowing when the pair $(-X_{nh}, d\mathcal{H}_M)$ is a section of $\tau_B(L_{\pi_{nh}})$. On the other hand, we would also like to know whether the almost Dirac structure $\tau_B(L_{\pi_{nh}})$ corresponds to the graph of a bivector field or not.

If the 2-form $B$ on $\mathcal{M}$ verifies $i_{X_{nh}} B = 0$, then from equation (4.38) we see that the pair $(-X_{nh}, d\mathcal{H}_M)$ belongs to $\Gamma(\tau_B(L_{\pi_{nh}}))$. Moreover, in view of (3.23), the gauge transformation of $\pi_{nh}$ by the 2-form $B$ has the form

$$\tau_B(L_{\pi_{nh}}) = \{ (X, \alpha) \in T\mathcal{M} \oplus T^*\mathcal{M} : X \in \mathcal{C}, i_X(\Omega_M - B)|_\mathcal{C} = -\alpha|_\mathcal{C} \}.$$  \hfill (4.39)

Thus the equations of motion (4.33) are equivalently written as

$$i_{X_{nh}} (\Omega_M - B)|_\mathcal{C} = (d\mathcal{H}_M)|_\mathcal{C},$$

where $B|_\mathcal{C}$ is the point-wise restriction of $B$ to $\mathcal{C}$.

Therefore, as a particular case of Corollary 3.4 we observe that if the section $\Omega_M - B|_\mathcal{C}$ is non-degenerate then the gauge transformation of $\pi_{nh}$ associated to the 2-form $B$ is again a bivector field $\pi_B$. It follows from (3.25) that the non-degeneracy of $\Omega_M - B$ on $\mathcal{C}$ is equivalent to the invertibility of the endomorphism $(\text{Id} - B^p \circ \pi^2_{nh})$ on $T^*\mathcal{C}$.

Definition 4. Let $(\mathcal{P}, \pi)$ be an almost Poisson manifold with a distinguished Hamiltonian function $H \in C^\infty(\mathcal{P})$. Given a 2-form $B$ on $\mathcal{P}$, the gauge transformation of $\pi$ associated to the 2-form $B$ is said to be a dynamical gauge transformation if

(i) $i_{X_H} B = 0$, where $X_H$ is the (almost) Hamiltonian vector field associated to $H$ and

(ii) $\tau_B(\text{graph}(\pi^B))$ corresponds to the graph of a new bivector $\pi^B$, i.e., the endomorphism $(\text{Id} - B^p \circ \pi^2)$ on $T^*\mathcal{P}$ is invertible.
Of course we are interested in dynamical gauge transformations of the nonholonomic bracket where the distinguished Hamiltonian function is $H_M$ and the corresponding (almost) Hamiltonian vector field is $X_{nh}$.

Note that if $\pi$ is regular, by equation (3.23), the gauge transformation defined by $B$ is determined by the restriction $B_F$ of $B$ to $F$ where $F := \pi^2(T^*P)$. Then condition $(i)$ of the above Definition is equivalent to $i_{X_H} B_F = 0$.

**Remark 4.4.** The definition of an affine almost Poisson bracket for a nonholonomic system made in [19] corresponds to a dynamical gauge transformation of the nonholonomic bracket by a 2-form $B = -i^*\Omega_0$ where $\Omega_0$ is a semi-basic form 2-form on $T^*Q$. The proof is analogous to that of item $(ii)$ of Proposition 4.3. In this case, the hypothesis that $\Omega_0$ is semi-basic implies that the condition $(ii)$ of Definition 4 is satisfied (see Proposition 4.6 below).

After this discussion, we observe that it is more appropriate to describe a nonholonomic system by the triple $(M, \mathfrak{F}, H_M)$ where $\mathfrak{F}$ is the family of bivector fields that are related to $\pi_{nh}$ through a dynamical gauge transformation. Notice that $C$ is the characteristic distribution of any bivector field in $\mathfrak{F}$. Thus, the (almost) Hamiltonian vector fields defined by the corresponding brackets satisfy the nonholonomic constraints. It follows that the bivector fields in the family $\mathfrak{F}$ are almost Poisson bivectors in a “strong” sense since the non-integrability of $C$ prevents them from being twisted or conformally Poisson. Our interest in considering this big family of brackets relies on reduction. In the presence of symmetries, the bracket that Hamiltonizes the reduced equations may arise as the reduction of a member of $\mathfrak{F}$ that is not necessarily $\pi_{nh}$.

Since the distribution $C$ is regular we observe

**Corollary 4.5.** [of Theorem 3.11]. All bivectors with characteristic distribution equal to $C$ are gauge related (in particular, gauge related to the nonholonomic bivector $\pi_{nh}$).

We finish this section by discussing some cases for which the second condition in Definition 4 is satisfied. Recall that $M \subset T^*Q$ is a vector bundle over $Q$. We have

**Proposition 4.6.** If $B$ is a semi-basic 2-form on $M$, then the gauge transformation of $\pi_{nh}$ associated to $B$ corresponds again to a bivector field.

**Proof.** The graph of $\pi_{nh}$ is an almost Dirac structure corresponding to the pair $(C, \Omega_C)$ in the sense of Proposition 3.1 (see Proposition 4.3 $(i)$). Thus, in view of (3.23), the gauge transformation of $\pi_{nh}$ is the almost Dirac structure corresponding to the pair $(C, (\Omega_M - B)|_C)$. It is shown in [19] that if $B$ a is a semi-basic 2-form, then the point-wise restriction of $(\Omega_M - B)$ to $C$ is non-degenerate. Thus, by Corollary 3.2, $\tau_B(L_{\pi_{nh}})$ corresponds to the graph of a bivector.

In fact this Proposition is a special case of the following result:

**Proposition 4.7.** Let $P \to Q$ be a vector bundle and $\pi$ a regular almost Poisson bivector on $P$. If for all semi-basic 1-forms $\alpha$ on $P$ the vector field $\pi^2(\alpha)$ is vertical, then the gauge transformation of $\pi$ associated to a semi-basic 2-form $B$ corresponds again to a bivector.
Proof. Consider local bundle coordinates \((q, p) \in U \times V \subset \mathbb{R}^n \times \mathbb{R}^m\), on \(P\) such that \(q\) are local coordinates on the base manifold \(Q\). Since \(\pi^\sharp(dq)\) is vertical and \(B\) is semi-basic we obtain

\[
B^\flat \circ \pi^\sharp dq = b(q, p) dq,
\]

where \(b(q, p)\) denotes the \(m \times n\) matrix with entries \(b_{aj}(q, p) = \langle B^\flat \circ \pi^\sharp dp_a, \partial \partial q_j \rangle\), for \(a = 1, ..., m\), and \(j = 1, ..., n\). Thus, the matrix representation of the endomorphism \((\text{Id} - B^\flat \circ \pi^\sharp)\) on \(T^*P\) is

\[
\begin{pmatrix}
\text{Id}_{n \times n} & -b(q, p)^T \\
0 & \text{Id}_{m \times m}
\end{pmatrix}.
\]

This matrix has full rank and hence \((\text{Id} - B^\flat \circ \pi^\sharp) : T^*P \rightarrow T^*P\) is invertible. \(\square\)

Remark 4.8. It is interesting for future work to drop the condition \((ii)\) in Definition 4 that requires \(\tau_B(\text{graph}(\pi^\sharp_{\text{nh}}))\) to define a bivector field. In this case, the family \(\mathcal{F}\) consists of all the almost Dirac structures that are gauge related to \(L_{\text{nh}}\) and that describe the nonholonomic dynamics. In this sense, the Hamiltonization of the problem is achieved if the reduction of a member of \(\mathcal{F}\) is a Dirac structure (but not necessarily a Poisson structure). This approach requires the consideration of a general reduction scheme for almost Dirac structures. However, we are unaware of any examples of nonholonomic systems that justify the need of such a general framework. \(\diamond\)

4.3 Reduction by a group of symmetries

We now add symmetries to the problem and perform the reduction. Our interest from the point of view of Hamiltonization is to find a bivector field in the family \(\mathcal{F}\) whose reduction is either Poisson or conformally Poisson (see Section 4.4).

Let \(G\) be a Lie group acting freely and properly on \(Q\). We say that \(G\) is a symmetry of the nonholonomic system if the lifted action to \(TQ\) is free and proper, and leaves the constraint distribution \(\mathcal{D} \subset TQ\) and the Lagrangian \(L : TQ \rightarrow \mathbb{R}\) invariant.

Denote by \(\Psi : G \times T^*Q \rightarrow T^*Q\) the cotangent lift of the action to \(T^*Q\). If \(G\) is a symmetry for our nonholonomic system, then \(\Psi\) leaves both the constraint submanifold \(\mathcal{M}\) and the Hamiltonian \(\mathcal{H} : T^*Q \rightarrow \mathbb{R}\) invariant. We continue to denote by \(\Psi\) the restricted action to \(\mathcal{M}\).

One can show that the tangent lift of \(\Psi\) to \(T\mathcal{M}\), preserves the distribution \(\mathcal{C}\) and the section \(\Omega_C\). As a consequence, if \(G\) is a symmetry group for our nonholonomic system, then the action \(\Psi\) preserves the standard nonholonomic bracket \(\{\cdot, \cdot\}_{\text{nh}}\). That is, for \(f, g \in C^\infty(\mathcal{M})\), we have

\[
\{f \circ \Psi, g \circ \Psi\}_{\text{nh}} = \{f, g\}_{\text{nh}} \circ \Psi.
\]

By freeness and properness of the action, the reduced space \(\mathcal{R} := \mathcal{M}/G\) is a smooth manifold and the orbit projection map \(\rho : \mathcal{M} \rightarrow \mathcal{R}\) is a surjective submersion. Notice that \(\mathcal{R}\) inherits a vector bundle structure from \(\mathcal{M}\) over the shape space \(Q/G\). Moreover, \(\mathcal{R}\) is equipped with the reduced nonholonomic bracket \(\{\cdot, \cdot\}_{\text{red}}\) that is characterized by

\[
\{f, g\}_{\text{red}} \circ \rho(m) := \{f \circ \rho, g \circ \rho\}_{\text{nh}}(m) \quad \text{for} \ m \in \mathcal{M} \text{ and } f, g \in C^\infty(\mathcal{R}).
\]
The corresponding bivector field will be denoted $\pi_{\text{red}}$. The reduced nonholonomic bracket describes the reduced dynamics in the sense that the nonholonomic vector field $X_{\text{nh}}$ is $\rho$-related to the (almost) Hamiltonian vector field $X_{\mathcal{H}_R} = \{ \cdot, \mathcal{H}_R \}_{\text{red}}$ associated to the reduced Hamiltonian $\mathcal{H}_R$ defined by the condition $\mathcal{H}_M = \mathcal{H}_R \circ \rho$.

**Remark 4.9.** The reduction of nonholonomic systems performed by Bates and Sniatycki in [2] shows that it is possible to define a 2-form $\omega_{\text{red}}$ on $\mathcal{R}$ which is non-degenerate along a distribution $\overline{C} \subset T\mathcal{R}$. The definition of $\overline{C}$ is given by $\overline{C} := T\rho(U)$ where $U := C \cap (C \cap V)^{\mathcal{R}M}$ and $V$ is the distribution on $\mathcal{M}$ tangent to the orbits of $G$. In fact, the pair $(\overline{C}, \omega_{\text{red}})$ is just the pair associated to the almost Dirac structure given by the bivector field $\pi_{\text{red}}$ in the sense of Corollary 3.4.

The analysis of the reduction of the bivector fields $\pi^B_{\text{nh}}$ in the family $\mathfrak{F}$ that are related to $\pi_{\text{nh}}$ by a dynamical gauge $B$ follows from:

**Proposition 4.10.** Let $(P, \pi)$ be an almost Poisson manifold and $B$ a 2-form on $P$ such that the endomorphism $(\text{Id} - B^* \circ \pi^2) : T^* P \to T^* P$ is invertible. If the Lie group $G$, acting freely and properly on $P$, preserves the almost Poisson structure $\pi$ and leaves $B$ invariant, then $G$ preserves the bivector field $\pi^B$ obtained by the gauge transformation of $\pi$ associated to $B$.

**Proof.** In view of equation (3.25), we see that $(\pi^B)^\sharp$ is a composition of invariant maps and we conclude that $\pi^B$ is invariant as well. \qed

As a direct consequence of this Proposition we have:

**Proposition 4.11.** If $G$ is a symmetry group of the nonholonomic system $(\mathcal{M}, \pi_{\text{nh}}, \mathcal{H}_M)$ and $B$ is a $G$-invariant dynamical gauge on $\mathcal{M}$, then $\{ \cdot, \cdot \}_{\text{red}}^B$ is $G$ invariant. In particular there is a reduced bivector field $\pi_{\text{red}}^B$ on the reduced space $\mathcal{R}$ that determines a well defined bracket $\{ \cdot, \cdot \}_{\text{red}}^B$ on $\mathcal{R}$ satisfying

$$\{ f, g \}_{\text{red}}^B \circ \rho(m) = \{ f \circ \rho, g \circ \rho \}_{\text{nh}}^B(m), \quad \text{for } f, g \in C^\infty(\mathcal{R}).$$

Moreover, the reduced bracket $\{ \cdot, \cdot \}_{\text{red}}^B$ also describes the reduced dynamics in the sense that $X_{\mathcal{H}_R} = \{ \cdot, \mathcal{H}_R \}_{\text{red}}^B$.

There is a good reason why we did not denote the reduced bivector field $\pi_{\text{red}}^B$ by $\pi_{\text{red}}^B$ and that is that in general the reduced bivector fields $\pi_{\text{red}}$ and $\pi_{\text{red}}^B$ need not be gauged related. We shall see this explicitly with the analysis of our mechanical examples (Remark 5.5).

**Remark 4.12.** Since the almost Dirac structure $L_{\text{nh}}$ given in (4.36) is the graph of a bivector (see Proposition 4.3 (i)), then the reduction of $L_{\text{nh}}$ as an almost Dirac structure is simply the reduction of the bivector $\pi_{\text{nh}}$ in the classical way. The same observation is valid for the reduction of the almost Dirac structure $\tau_B(L_{\pi_{\text{nh}}})$ defined in (4.39) since $\tau_B(L_{\pi_{\text{nh}}})$ is also the graph of a bivector ($\Omega_M - B$ is non-degenerate on $\mathcal{C}$). \qed
The \( G \)-Chaplygin case.

If the reduced bivector fields \( \pi^{\text{red}} \) and \( \pi^{\text{red}B} \) happen to be everywhere non-degenerate, then they are gauge-related. This is the scenario that one finds after reduction of external symmetries of \( G \)-Chaplygin systems (see \cite{[14]}). These systems are characterized by the property that the tangent space to the orbits of the symmetry group exactly complements the constraint distribution on the tangent space \( TQ \) of the configuration manifold \( Q \).

In this case there exist unique non-degenerate 2-forms \( \Omega^{\text{red}} \) and \( \Omega^{\text{red}B} \) on \( \mathcal{R} \) satisfying

\[
\pi^{\sharp} \circ \Omega^{\flat}_{\text{red}} = \text{Id}, \quad \pi^{\sharp}_{\text{red}B} \circ \Omega^{\flat}_{\text{red}B} = \text{Id},
\]

where as usual, \( \Omega^{\text{red}}(X) = -i_X \Omega_{\text{red}} \) and \( \Omega^{\text{red}B}(X) = -i_X \Omega_{\text{red}B} \) for all \( X \in \mathfrak{h}(\mathcal{R}) \). In particular, the reduced equations can be written as:

\[
i_X \mathcal{H}_{\mathcal{R}} \Omega^{\text{red}} = i_X \mathcal{H}_{\mathcal{R}} \Omega^{\text{red}B} = d\mathcal{H}_{\mathcal{R}},
\]

which gives different almost symplectic formulations of the reduced equations (compare with the results in \cite{[20]}). The bivector fields \( \pi^{\text{red}} \) and \( \pi^{\text{red}B} \) are gauge related via the 2-form \( B^{\text{red}} := \Omega^{\text{red}} - \Omega^{\text{red}B} \) on \( \mathcal{R} \).

Moreover, the 2-forms \( \Omega^{\text{red}} \) and \( \Omega^{\text{red}B} \) satisfy

\[
(\rho^*\Omega_{\text{red}})^C = \Omega_C \quad \text{and} \quad (\rho^*\Omega_{\text{red}B})^C = (\Omega_M - B)^C,
\]

and thus \( B_{\text{red}} \) verifies that \( (\rho^*B_{\text{red}})^C = B_C \). In other words, we have the following commutative diagram

\[
\begin{array}{ccc}
L_{\pi_{\text{nh}}} & \xrightarrow{\tau_B} & L_{\pi_{\text{nh}}B} \\
\downarrow \rho & & \downarrow \rho \\
L_{\pi_{\text{red}}} & \xrightarrow{\tau_{B_{\text{red}}}} & L_{\pi_{\text{red}}B}.
\end{array}
\]

We stress that one does not have such a commutative diagram in more general situations, see Remark 5.5.

4.4 Hamiltonization

We continue with the notation of the previous sections and suppose that \( G \) is a symmetry group for our nonholonomic system. The solutions of the reduced equations on the reduced space \( \mathcal{R} = \mathcal{M}/G \) are the integral curves of the reduced vector field \( X_{\mathcal{H}_{\mathcal{R}}} \) and preserve the reduced Hamiltonian \( \mathcal{H}_{\mathcal{R}} \).

The issue of Hamiltonization in our context concerns answering the question of whether the reduced vector field \( X_{\mathcal{H}_{\mathcal{R}}} \) on \( \mathcal{R} \) is Hamiltonian. Our candidates for the Hamiltonian structure come from the reduction of the (invariant) bivector fields that belong to the family \( \tilde{\mathfrak{h}} \).

It turns out that the above condition is too restrictive. We relax it by asking that the vector field \( X_{\mathcal{H}_{\mathcal{R}}} \) can be rescaled by a basic\(^4\) positive function \( \varphi : \mathcal{R} \to \mathbb{R} \) in such a way that the resulting vector field \( \varphi X_{\mathcal{H}_{\mathcal{R}}} \) is Hamiltonian.

\(^4\)We use the term basic with respect to the fibered structure of \( \mathcal{R} \) inherited from \( \mathcal{M} \). That is \( \varphi = \tilde{\varphi} \circ \tau \) where \( \tau : \mathcal{R} \to Q/G \) is the bundle projection and \( \tilde{\varphi} : Q/G \to \mathbb{R} \).
In view of Proposition 4.11 for any $G$-invariant bivector field $\pi_{\text{nh}}^B$ belonging to $\mathcal{F}$, the rescaled vector field $\varphi X_{\mathcal{H}_R}$ satisfies

$$\left(\varphi \pi_{\text{red}}^{B}\right)(d\mathcal{H}_R) = -\varphi X_{\mathcal{H}_R}.$$  \hfill (4.41)

Hence, we are interested in finding a bivector field $\pi_{\text{red}}^{B}$ satisfying

$$[\varphi \pi_{\text{red}}^{B}, \varphi \pi_{\text{red}}^{B}] = 0,$$  \hfill (4.42)

that is, we want to find $\pi_{\text{red}}^{B}$ conformally Poisson (see Definition 2).

**Definition 5.** If there exist an invariant bivector field $\pi_{\text{nh}}^B$ belonging to $\mathcal{F}$ and a strictly positive, basic function $\varphi : \mathcal{R} \rightarrow \mathbb{R}$ such that (4.41) and (4.42) hold, we say that the nonholonomic system is Hamiltonizable. Moreover, we say that the reduced equations are Hamiltonian in the new time $\tau$ defined by $d\tau = \frac{1}{\varphi}dt$ (see discussion below).

In Section 5 we will extend the discussion of Section 2 and show that all generalized rolling systems are Hamiltonizable. The table (5.48) in Section 5 shows how different scenarios of the Hamiltonization scheme described above are realized according to the rank of the matrix $A$.

**Time reparametrizations**

It is common in the literature to interpret the rescaling of the vector field $X_{\mathcal{H}_R}$ by the basic positive function $\varphi$ as a nonlinear time reparametrization. One argues as follows, let $c(t) \in \mathcal{R}$ be a flow line of $X_{\mathcal{H}_R}$ (i.e. $\frac{dc}{dt}(t) = X_{\mathcal{H}_R}(c(t))$). Introduce the new time $\tau$ by integrating the relation

$$d\tau = \frac{1}{\varphi(c(t))}dt.$$ 

Since $\varphi > 0$, the correspondence between $t$ and $\tau$ is one-to-one and one can express $t$ as a function of $\tau$. The curve $\tilde{c}(\tau) := c(t(\tau))$ is checked to be a flow line of $\varphi X_{\mathcal{H}_R}$ (i.e. $\frac{d\tilde{c}}{d\tau}(\tau) = \varphi(\tilde{c}(\tau))X_{\mathcal{H}_R}(\tilde{c}(\tau))$). This interpretation of the rescaling is quite subtle. The definition of $\tau$ depends on the particular flow line $c(t)$, so different initial conditions induce different reparametrizations. It is therefore not possible to interpret the time rescaling as a “global” operation. This contrasts with the natural procedure of multiplying the vector field $X_{\mathcal{H}_R}$ by the positive function $\varphi$.

**Remark 4.13.** One might wonder why we only care about basic and not arbitrary functions $\varphi : \mathcal{R} \rightarrow \mathbb{R}^+$. A detailed answer to this question would involve a careful study of the structure of the equations of motion that would gear us away from the main subject of this paper. We refer the reader to [15] where one can find a very good discussion on the Hamiltonization of $G$-Chaplygin systems. We simply mention that physically, the fact that $\varphi$ is basic means that it is independent of the momentum variables and only depends on the (reduced) configuration variables. Thus, the time reparametrization changes the speed at which the trajectories are traversed depending on the position of the system but independently of the velocity itself. It is shown in [15] how in order to obtain Darboux coordinates for the reparametrized system, one can keep the same (reduced) configuration variables but should rescale the momenta by $\frac{1}{\varphi}$. Since $\varphi$ is basic, the rescaled momenta continue to depend linearly on the velocities.

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Hamiltonization is strongly related to the existence of invariant measures. Suppose for simplicity that we are dealing with a $G$-Chaplygin system. As mentioned before, in this case the bivector field $\pi_{\text{red}B}$ is everywhere non-degenerate and the reduced equations can be written as:

$$i_{X_{H_R}}\Omega_{\text{red}B} = dH_R,$$

where $\Omega_{\text{red}B}$ is the non-degenerate 2-form on $R$ induced by $\pi_{\text{red}B}$. It follows that the scaled vector field $\varphi X_{H_R}$ satisfies

$$i_{\varphi X_{H_R}}\left(\frac{1}{\varphi}\Omega_{\text{red}B}\right) = dH_R.$$

Hence, Hamiltonization in this setting amounts to finding a positive function $\varphi$ such that the 2-form $\frac{1}{\varphi}\Omega_{\text{red}B}$ is closed (which under our hypothesis, is of course equivalent to (4.42)). Suppose for a moment that this is the case so $(R, \frac{1}{\varphi}\Omega_{\text{red}B})$ is a symplectic manifold. It follows from Liouville’s theorem that the vector field $\varphi X_{H_R}$ preserves the symplectic volume $(\frac{1}{\varphi}\Omega_{\text{red}B})^m$ where $m = \frac{1}{2}\dim R$. Therefore, the volume form $(\frac{1}{\varphi})^{m-1}(\Omega_{\text{red}B})^m$ is preserved by the vector field $X_{H_R}$.

The above argument shows that a Hamiltonizable $G$-Chaplygin system possesses an invariant measure. One might wonder if the reciprocal statement is true, namely, if any $G$-Chaplygin system with an invariant measure is Hamiltonizable. The celebrated Chaplygin’s reducing multiplier theorem [11] demonstrates that the answer is positive if $m = 2$. For $m > 2$, a characterization of the systems for which this is true is an open problem. Interesting examples where this holds for arbitrary values of $m$ have been found by Fedorov and Jovanovic in the study of the multidimensional Veselova problem [15]. See also the discussion in [14] where a candidate for the conformal factor $\frac{1}{\varphi}$ is given under the hypothesis that there exists a preserved measure, and [17] where a set of coupled first order partial differential equations for the multiplier $\frac{1}{\varphi}$ are given.

In the case where the nonholonomic system is Hamiltonizable but the corresponding bivector field $\pi_{\text{red}B}$ is degenerate at some points in $R$, one can repeat the above argument to conclude that the reduced system preserves a measure on every leaf of the symplectic foliation of $R$ corresponding to the Poisson bivector field $\varphi\pi_{\text{red}B}$. However, this does not imply the existence of a smooth invariant measure on $R$ (and this is the motivation for Fernandez, Mestdag, and Bloch [17] to talk about Poissonization). An example of this situation is given by the reduction of the Chaplygin sleigh (see the discussion in [18]) that exhibits asymptotic dynamics that contravene the existence of a global invariant measure. The problem of the existence of a global invariant measure in this case is most naturally attacked by considering the modular class of the Poisson manifold $(R, \varphi\pi_{\text{red}B})$, see [35].

5 Back to the Examples: Hamiltonization and Integrability

In this section, we analyze the generalized rolling systems presented in Section 2 using the geometric framework that was developed in the previous sections. In subsection 5.1 we provide the geometric interpretation for the brackets $\{\cdot, \cdot\}_{\text{Rank1}}$ and $\{\cdot, \cdot\}'_{\text{Rank1}}$ presented in Section 2. In 5.2 we consider the Hamiltonization and integrability of generalized rolling systems in detail and finally, in 5.3 we explicitly show that the brackets $\{\cdot, \cdot\}_{\text{Rank1}}$ and $\{\cdot, \cdot\}'_{\text{Rank1}}$ are twisted Poisson. To our knowledge, this is the first time that the appearance of such structures is made explicit in the field of nonholonomic mechanics.
5.1 The geometry of the rigid bodies with generalized rolling constraints

We begin by computing the nonholonomic bracket for the motion of a rigid body subject to generalized rolling constraints as introduced in Section 2.

Nonholonomic bracket via the non-degenerate 2-section

Consider again, as in Section 2, the motion of a rigid body in space subject to a generalized rolling constraint as in (2.2). That is, the constraint relates the linear and the angular velocities of the body

\[ x = rA \omega = rAg \Omega, \]

where the matrix \( A \) satisfies any of the conditions of Definition 1 and \( \omega, \Omega \) is the angular velocity written in space and body coordinates, respectively.

Recall that the configuration space for the system is \( Q = SO(3) \times \mathbb{R}^3 \). Denote by \( \lambda \) (respectively, \( \rho \)) the left (respectively, right) Maurer-Cartan form on \( SO(3) \). Upon the identification of the Lie algebra \( so(3) \) with \( \mathbb{R}^3 \) by the hat map (2.3) we think of \( \lambda \) and \( \rho \) as \( \mathbb{R}^3 \)-valued 1-forms on \( SO(3) \). For a tangent vector \( v_g \in T_g SO(3) \) we have

\[ \omega = \rho(g)(v_g), \quad \Omega = \lambda(g)(v_g), \]

where \( \omega \) (respectively, \( \Omega \)) denotes the angular velocity vector written in space (respectively, body) coordinates as discussed in Section 2.

The Maurer-Cartan forms \( \lambda \) and \( \rho \) are related by \( \lambda(g) = g^{-1} \rho(g) \) and satisfy the well-known Maurer-Cartan equations

\[ d\rho = [\rho, \rho], \quad d\lambda = -[\lambda, \lambda], \]

where \([\cdot, \cdot]\) is the commutator in the Lie algebra. For the rest of the section we will use three dimensional vector algebra notation in our calculations with differential forms and vector fields. In our convention, the scalar product of differential forms should always be interpreted as a wedge product (and is thus anti-commutative!). The Maurer-Cartan equations take the form

\[ d\rho = \frac{1}{2} \rho \times \rho, \quad d\lambda = -\frac{1}{2} \lambda \times \lambda, \quad (5.43) \]

where “\( \times \)” denotes the standard vector product in \( \mathbb{R}^3 \).

The constraint distribution \( D \), defined by the generalized rolling constraints, can be expressed in the terminology of subsection 4.1 as the annihilator of the \( \mathbb{R}^3 \)-valued 1-form \( \epsilon \) on \( Q \) given by

\[ \epsilon = dx - rA\rho = dx - rAg\lambda. \]

We consider the (global) moving co-frame \( \{\lambda, dx\} \) for \( T^*Q \) that defines fiber coordinates \( (M, p) \) in the following sense. A co-vector \( \alpha_q \in T^*_q Q \) is written uniquely as \( \alpha_q = M \cdot \lambda + p \cdot dx \), for a certain \((M, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \). The Legendre transform \( \text{Leg} : TQ \rightarrow T^*Q \) associated to the kinetic energy Lagrangian (2.5) is defined by the rule:

\[ M = \mathbb{I} \Omega, \quad p = m \dot{x}. \]

Physically, \( p \) is the linear momentum of the body while \( M \) is the angular momentum of the body about the center of mass written in body coordinates.
In order to deal with the constraints, it is more convenient to work with the global moving co-frame \( \{ \lambda, \epsilon \} \) for \( T^*Q \). We denote by \( (K, u) \) the fiber coordinates defined by this co-frame. Putting \( K \cdot \lambda + u \cdot \epsilon = M \cdot \lambda + p \cdot dx \) implies
\[
K = M + rg^T A^T p, \quad u = p.
\]
Along the constraint submanifold \( \mathcal{M} = \text{Leg}(D) \) we have \( p = mrA g \Omega \) so
\[
K = \Omega + mr^2 g^T A^T A g \Omega,
\]
which is the expression for the kinetic momentum obtained in \((2.8)\). Notice that \( M \) is a vector bundle over \( Q \) and that \( K \) is a natural coordinate for the fibers of \( \mathcal{M} \). In what follows we will use the components of \( g, x \), and \( K \), as redundant coordinates on \( \mathcal{M} \).

Denote by \( X_L = (X^1_L, X^2_L, X^3_L) \) the moving frame of \( \text{SO}(3) \) that is dual to \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \). The components of \( X_L \) are the left invariant vector fields on \( \text{SO}(3) \) obtained by left extension of the canonical basis of \( \mathbb{R}^3 \). Along the points of the constraint subbundle \( \mathcal{M} \), the non-integrable distribution \( \mathcal{C} \) defined in \((4.31)\) is given by
\[
\mathcal{C} = \text{span} \left\{ X^L + rg^T A^T \frac{\partial}{\partial x}, \frac{\partial}{\partial K} \right\}.
\]
The canonical 2-form \( \Omega_Q \) on \( T^*Q \) is given by
\[
\Omega_Q = -d(M \cdot \lambda + p \cdot dx) = -d(K \cdot \lambda + p \cdot \epsilon) = \lambda \cdot dK - K \cdot d\lambda - p \cdot d\epsilon + \epsilon \cdot dp,
\]
where \( \cdot \) denotes the usual scalar product in \( \mathbb{R}^3 \).

To compute \( d\epsilon \) we use the identity \( dg = \hat{g} \lambda \) where \( \hat{\cdot} \) denotes the hat map \((2.3)\). Using the Maurer-Cartan equations \((5.43)\) we get
\[
d\epsilon = -rA(dg)\lambda - rAg d\lambda = -rAg(\lambda \times \lambda) - rAg d\lambda = rAg d\lambda.
\]
Therefore,
\[
\Omega_Q = \lambda \cdot dK - K \cdot d\lambda - p \cdot (rAg d\lambda) + \epsilon \cdot dp = \lambda \cdot dK - (K + rg^T A^T p) \cdot d\lambda + \epsilon \cdot dp.
\]

Let \( \iota : \mathcal{M} \hookrightarrow T^*Q \) denote the inclusion. Since \( p = mrAg \Omega \) along \( \mathcal{M} \), we have
\[
\Omega_M := \iota^*(\Omega_Q) = \lambda \cdot dK - (K + mr^2 g^T A^T A g \Omega) \cdot d\lambda + \iota^*(\epsilon \cdot dp).
\]
Since \( \epsilon \) vanishes along the non-integrable distribution \( \mathcal{C} \), we get the following expression for the restriction \( \Omega_C \) of \( \iota^*(\Omega_Q) \) to \( \mathcal{C} \):
\[
\Omega_C = \lambda \cdot dK - (K + mr^2 g^T A^T A g \Omega) \cdot d\lambda.
\]
With this expression for \( \Omega_C \) we are ready to show:
Proposition 5.1. The nonholonomic bracket \{\cdot, \cdot\}_{nh} on \mathcal{M} for the generalized rolling system is given in the redundant coordinates \{g_{ij}, x_k, K_l\}, \ i, j, k, l = 1, 2, 3, for \mathcal{M} by

\[ \{x_i, K_l\}_{nh} = r(Ag)_i, \quad \{g_{ij}, K_l\}_{nh} = -\varepsilon^k_{ij} g_{ik}, \quad \{K_i, K_j\}_{nh} = -\varepsilon^l_{ij} (K + mr^2(g^T A^T Ag))_l, \]

with all other combinations equal to zero. In the above formulas the Einstein convention of sum over repeated indices holds and \varepsilon^k_{ij} denotes the alternating tensor that equals 0 if two indices are equal, it equals 1 if \(i, j, k\) is a cyclic permutation of \(1, 2, 3\), and it is equal to \(-1\) otherwise. The entries of \((g, x) \in Q\) are denoted by \((g_{ij}, x_k)\) and those of \(K\) by \(K_i\).

Proof. We will rely on the identity \((4.35)\) that characterizes the associated bivector field \(\pi_{nh}\) in terms of \(\Omega_C\). Contraction of \(\Omega_C\) by the elements in the basis of \(C\) given in \((5.44)\) gives

\[ i_{(X^l + rg^T A^T \frac{\partial}{\partial x})} \Omega_C = dK - (K + mr^2g^T A^T Ag) \times \lambda, \quad i_{\frac{\partial}{\partial x}} \Omega_C = -\lambda, \]

where we have again made use of the Maurer-Cartan equations \((5.43)\). It follows that

\[ i_{(X^l + rg^T A^T \frac{\partial}{\partial x})} (K + mr^2g^T A^T Ag) \times \frac{\partial}{\partial x} \Omega_C = dK, \quad i_{-rAg} \frac{\partial}{\partial K} \Omega_C = rAg\lambda = dx|_C. \]

Therefore, according to \((4.35)\) we get

\[ \pi_{nh}^\sharp(dK) = -X^l - rg^T A^T \frac{\partial}{\partial x} + (K + mr^2g^T A^T Ag) \times \frac{\partial}{\partial K}, \]

\[ \pi_{nh}^\sharp(\lambda) = \frac{\partial}{\partial K}, \quad \pi_{nh}^\sharp(dx) = rAg \frac{\partial}{\partial K}. \]

In addition, for any canonical vector \(e_i \in \mathbb{R}^3\) we have \(d(g^{-1}e_i) = (g^{-1}e_i) \times \lambda\), so

\[ \pi_{nh}^\sharp(d(g^{-1}e_i)) = (g^{-1}e_i) \times \frac{\partial}{\partial K}. \]

The proof follows using the above formulas and recalling that for any \(f, g \in C^\infty(\mathcal{M})\), we have \(\{f, g\}_{nh} = -df(\pi_{nh}^\sharp(dg))\).

Finally, we state without a formal proof that the nonholonomic vector field \(X_{nh}\) on \(\mathcal{M}\) is given by

\[ X_{nh} = \Omega \cdot X^l + rAg \Omega \cdot \frac{\partial}{\partial x} + (K \times \Omega) \cdot \frac{\partial}{\partial K}. \]

The above expression can be shown by taking into account the equations \((2.10)\), the constraint \((2.2)\), and the definition of \(\Omega\), or, alternatively, by computing the the almost Hamiltonian vector field \(X_{H_M} = -\pi_{nh}^\sharp(dH_M)\) corresponding to the Hamiltonian \(H_M\) that coincides with expression \((2.9)\). The latter approach requires one to write \(\Omega\) in terms of \(K\) and \(g\) as was done in Section 2 for the different values of the rank of \(A\).
The gauge transformation of the nonholonomic bracket

We will now construct a gauge transformation of the nonholonomic bracket in the sense of subsection 4.2. We are interested in describing the same dynamics so we look for a 2-form $B$ that defines a dynamical gauge transformation as introduced in Definition 4. In our case, the distinguished Hamiltonian is $\mathcal{H}_M$ that has $X_{nh}$ as its associated almost Hamiltonian vector field.

Following [19], we consider the bi-invariant volume form $\nu$ on $SO(3)$ oriented and scaled such that $\nu(X^1_L, X^2_L, X^3_L) = 1$. We consider the natural extension of $\nu$ as a 3-form on $Q = SO(3) \times \mathbb{R}^3$. Denote by $\tilde{\nu} \in \Omega^3(T^*Q)$ the 3-form given by $\tilde{\nu} = \nu$ where $\nu : T^*Q \to Q$ is the canonical projection. We can write $\tilde{\nu} = \frac{1}{6} \lambda \cdot (\lambda \times \lambda)$.

Let $B$ be the 2-form on $M$ given by

$$B = mr^2(i_{X_{nh}} \iota^* \tilde{\nu}),$$

where, as before, $\iota : M \to T^*Q$ is the inclusion. Note that $B$ is a semi-basic 2-form on $M$ that vanishes upon contraction with the nonholonomic (almost) Hamiltonian vector field $X_{nh}$. Therefore, by Proposition 4.6, we can perform a dynamical gauge transformation of the nonholonomic bivector field $\pi_{nh}$ by the 2-form $B$ to obtain another bivector field $\pi_{B, nh}$ that also describes the dynamics of our problem.

Using the Maurer-Cartan equations (5.43) and the expression (5.46) for $X_{nh}$, we obtain

$$B = -mr^2 \Omega \cdot d\lambda.$$  \hspace{1cm} (5.47)

To compute the bivector field $\pi_{B, nh}$ associated to the gauge transformation we use equation (3.24). For an arbitrary one-form $\alpha$ on $M$ we have

$$(\pi_{B, nh})^\sharp (\alpha + i_{\pi_{nh}}(\alpha) B) = \pi_{B, nh}^\sharp (\alpha).$$

Setting $\alpha$ equal to $\lambda$ and $d\lambda$ and using (5.45) and (5.47) we obtain

$$(\pi_{nh})_B^\sharp (\lambda) = \frac{\partial}{\partial K}, \hspace{1cm} (\pi_{nh})_B^\sharp (d\lambda) = rAg \frac{\partial}{\partial K}.$$

Similarly, putting $\alpha = dK$ and noticing that

$$i_{x_{nh}}(dK)B = -i_{X_L}B = mr^2 \Omega \times \lambda,$$

we deduce

$$(\pi_{B, nh})^\sharp (dK) = -X^L - rg^T A^T \frac{\partial}{\partial x} + (K + mr^2 (g^T A^T Ag - E) \Omega) \times \frac{\partial}{\partial K},$$

where $E$ denotes the $3 \times 3$ identity matrix.

The above formulas imply

**Proposition 5.2.** The gauged nonholonomic bracket $\{\cdot, \cdot\}_B^{nh}$ on $M$, associated to the bivector field $\pi_{B, nh}$, is given in the redundant coordinates $(g_{ij}, x_k, K_l)$, $i, j, k, l = 1, 2, 3$, for $M$ by

$$\{x_i, K_l\}^{B}_{nh} = r(Ag)_{il}, \hspace{0.5cm} \{g_{ij}, K_l\}^{B}_{nh} = -\varepsilon_{jl}^k g_{ik}, \hspace{0.5cm} \{K_i, K_j\}^{B}_{nh} = -\varepsilon_{ij}^l (K + mr^2 (g^T A^T Ag - E) \Omega)_{il},$$

with all other combinations equal to zero.
Reduction of the symmetries

Recall that the Lie group $H$, introduced in section 2.4, acts on the configuration space $Q$ and that its lift to $TQ$ leaves both the Lagrangian and the constraints invariant. From the discussion in section 4.3 (and the regularity of the action) it follows that the reduced space $\mathcal{R} := \mathcal{M}/H$ is equipped with a reduced bracket $\{\cdot,\cdot\}_{\text{red}}$ determined by condition (4.40) and that describes the reduced dynamics.

We are now ready to give the geometric interpretation of the bracket $\{\cdot,\cdot\}_{\text{rank} j}$ introduced in section 2 for the different values of the rank of $A$.

**Theorem 5.3.** The reduced bracket $\{\cdot,\cdot\}_{\text{red}}$ on $\mathcal{R}$ is precisely the restriction of the bracket $\{\cdot,\cdot\}_{\text{rank} j}$ (defined in section 2) to the Casimir level set $||\gamma|| = 1$, for the different values $j = 0, 1, 2, 3$, of the rank of $A$.

**Proof.** Recall from section 2.4 that the reduced space $\mathcal{R}$ can be identified with $S^2 \times \mathbb{R}^3$ with redundant coordinates $(\gamma, K)$. Therefore, it makes sense to compare the two brackets on the Casimir level set $||\gamma|| = 1$ of the space $(\gamma, K) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Moreover, from the expression of the projection $\rho : \mathcal{M} \to \mathcal{R}$ given by (2.17), and condition (4.40), it follows that the reduced bracket of the (redundant) coordinate functions $(\gamma, K)$ can be computed using the formulas obtained in Proposition 5.1 (notice that $\gamma = (g_{31}, g_{32}, g_{33})$).

The proof is completed by considering the particular form of $A$ for the different values of its rank given in Definition 1, and by writing the bracket $\{f_1, f_2\}_{\text{red}}$ of arbitrary functions $f_1, f_2 \in C^\infty(\mathcal{R})$ in terms of the derivatives $\frac{\partial f_i}{\partial \gamma}$ and $\frac{\partial f_i}{\partial K}$ using Leibniz rule. 

We now turn to the study of the reduction of the gauged nonholonomic bracket $\{\cdot,\cdot\}_{\text{B}}$. First of all notice that the 2-form $B$ that defines the gauge transformation is written in (5.47) in terms of left invariant objects on SO(3). Since the symmetry group $H$ acts by left multiplication on the SO(3) factor of $Q$, it follows that $B$ is invariant under the cotangent lifted action. Therefore, in accordance with Proposition 4.11 the gauged bracket $\{\cdot,\cdot\}_{\text{B}}$ drops to $\mathcal{R}$ where it defines the bracket $\{\cdot,\cdot\}_{\text{red} B}$ that determines the dynamics. As usual, the corresponding bivector field on $\mathcal{R}$ will be denoted by $\pi_{\text{red} B}$.

In analogy with Theorem 5.3 we have

**Theorem 5.4.** The reduced bracket $\{\cdot,\cdot\}_{\text{red} B}$ on $\mathcal{R}$ is precisely the restriction of the bracket $\{\cdot,\cdot\}_{\text{rank} j}$ (defined in section 2) to the Casimir level set $||\gamma|| = 1$, for the different values $j = 0, 1, 2, 3$, of the rank of $A$.

The proof is identical to that of Theorem 5.3 except that one uses the formulas obtained in Proposition 5.2.

According to the discussion in Section 2 the following table summarizes the properties of the reduced bivector fields $\pi_{\text{red}}$ and $\pi_{\text{red} B}$ for a generalized rolling system according to the different values of the rank of $A$:
Remark 5.5. Notice that the reduced bivector fields $\pi_{\text{red}}$ and $\pi_{\text{red}B}$ are not gauge related. Indeed, from the table above one sees that for any value of the rank of $A$, only one of the two bivector fields $\pi_{\text{red}}$ or $\pi_{\text{red}B}$ has an integrable characteristic distribution. It follows from Theorem 3.11 that there cannot exist a gauge transformation between their graphs.

 Remark 5.6. Recall from subsection 2.5 that as the rank of $A$ increases, the constraint distribution is less integrable or “more nonholonomic”. The table (5.48) above seems to suggest that it is appropriate to perform a gauge transformation by the 2-form $B$ when the nonholonomic effects are more important, while the reduction of the standard nonholonomic bracket works better for weaker nonholonomic effects.

5.2 Hamiltonization and integrability of rigid bodies with generalized rolling constraints

According to the notion of Hamiltonization introduced in Section 4.4 (Definition 5), and the table (5.48), it immediately follows that the problem of the motion of a rigid body subject to a generalized rolling constraint is Hamiltonizable for any value of the rank of $A$.

If the rank of $A$ equals 0 (respectively, 3) the reduced equations are Hamiltonian with respect to the bracket $\{\cdot,\cdot\}_{\text{red}}$ (respectively, $\{\cdot,\cdot\}_{\text{red}B}$). Recall that in both cases the reduced dynamics correspond to classical rigid body motion (with modified inertia tensor $I + mr^2E$ if the rank of $A$ equals 3).

If the rank of $A$ equals 1 or 2, the analysis of the Hamiltonization is a bit more delicate but it also follows directly from Definition 5 and the table (5.48). In the case rank $A = 2$, it follows that the reduced equations are Hamiltonian in the new time $\tau_2$ defined by $d\tau_2 = \frac{1}{\varphi_2} dt$ and with respect to the bracket $\varphi_{2\{\cdot,\cdot\}_{\text{red}B}}$ where

$$\varphi_2(\gamma) = \sqrt{1 - mr^2 (\gamma \cdot (I + mr^2E)^{-1}\gamma)}.$$  \hspace{1cm} (5.49)

Note that $\varphi_2$ is a basic function on $R$ corresponding to the restriction of (2.14) to the level set $||\gamma|| = 1$.

Analogously, if the rank of $A$ equals 1, the reduced equations are Hamiltonian in the new time $\tau_1$ defined by $d\tau_1 = \frac{1}{\varphi_1} dt$ and with respect to the bracket $\varphi_{1\{\cdot,\cdot\}_{\text{red}}}$ where

$$\varphi_1(\gamma) = \sqrt{1 + mr^2 (\gamma \cdot I^{-1}\gamma)}.$$  \hspace{1cm} (5.50)
Integrability of the reduced equations

In view of the Hamiltonization of the problem, the integrability of the reduced equations of motion (2.10) can be easily established using the celebrated Arnold-Liouville Theorem for classical Hamiltonian systems, see e.g. [1].

Indeed, for any value of the rank of $A$, the reduced equations are Hamiltonian on $\mathcal{R}$ (after a time reparametrization if rank $A = 1, 2$). Independently of the rank of $A$, the symplectic leaves $O_a$ of the foliation of $\mathcal{R}$ correspond to the level sets $C_1(K, \gamma) = K \cdot \gamma = a$ and can be shown to be diffeomorphic to the tangent bundle $T S^2$ of the sphere (see the discussion in chapter 14 of [26] for the coadjoint orbits on $\mathfrak{se}(3)^*$).

Once the value of $a$ is fixed, the reduced equations (2.10) can be seen as a two degree of freedom classical Hamiltonian system on $O_a$ (again, after a time reparametrization if rank $A = 1, 2$). These equations possess two independent integrals, the Hamiltonian $H_{\mathcal{R}}$, and $F = K \cdot K$, whose joint level sets are compact in $O_a$. It follows from the Arnold-Liouville Theorem that these level sets are invariant two-tori and the dynamics are quasi-periodic on them (notice that the flow on the tori is rectilinear but not uniform if the rank of $A$ is 1 or 2).

The Arnold-Liouville Theorem also tells us that the reduced equations are integrable by quadratures (after the time reparametrization if the rank of $A$ is 1 or 2).

Finally, we state without proof that the reduced equations of motion (2.10) preserve the measure $\mu(\gamma) = \sigma \wedge dK_1 \wedge dK_2 \wedge dK_3$ where $\sigma$ is the area form of the sphere $S^2$, and the basic density $\mu : S^2 \to \mathbb{R}$ is given by

$$
\mu(\gamma) = \begin{cases} 
1 & \text{if rank } A = 0, 3, \\
\frac{1}{\varphi_1(\gamma)} & \text{if rank } A = 1, \\
\frac{1}{\varphi_2(\gamma)} & \text{if rank } A = 2,
\end{cases}
$$

where $\varphi_1, \varphi_2 \in C^\infty(S^2)$ are defined in (5.50) and (5.49) respectively.

5.3 Twisted Poisson structures for rigid bodies with generalized rolling constraints

In Section 3.2, we presented twisted Poisson structures which have been extensively studied in other contexts but not in mechanics. Now, we will show explicitly that twisted Poisson structures appear naturally in the study of nonholonomic systems.

Rigid body with generalized rolling constraints of rank 2

Here we show that the bracket $\{\cdot, \cdot\}_{\text{Rank2}}'$, in addition to being conformally Poisson, is twisted Poisson. Note that this cannot be the case for the other bracket $\{\cdot, \cdot\}_{\text{Rank2}}$ that describes the dynamics since, as shown in Section 2, its characteristic distribution is not integrable.

Recall from the discussion in [2.3] that $\{\cdot, \cdot\}_{\text{Rank2}}'$ should be considered as a bracket on the reduced space $\mathcal{R} = S^2 \times \mathbb{R}^3$ with redundant coordinates $(\gamma, K)$. The characteristic distribution of the bracket is integrable and the leaves $O_a$ of the foliation are the level sets $C_1(\gamma, K) = \gamma \cdot K = a$. By regularity and integrability of the characteristic distribution, it follows from Corollary 3.7 that the bracket is $\phi$-twisted. The value of the 3-form $\phi$ is given in the following,
Theorem 5.7. The bracket \( \{ \cdot, \cdot \}_\text{Rank2}' \) defined in (2.13) (that in particular describes the reduced dynamics of the Chaplygin sphere for the appropriate choice of \( A \)), is a \( \phi \)-twisted Poisson bracket with \( \phi = -d \mathcal{B} \)

\[
\mathcal{B} = mr^2(\Omega \cdot \gamma) \sigma,
\]

and where \( \sigma \) denotes the area form of the sphere \( ||\gamma|| = 1 \).

Proof. The idea of this proof is to show that the bracket is gauge related to a Poisson bracket via the 2-form \( -B \). Thus, by Proposition 3.12, the bracket is \((-d \mathcal{B})\)-twisted Poisson. More precisely, we will show that \( \{ \cdot, \cdot \}_\text{Rank2}' \) is \(-B\)-gauge related with the bracket \( \{ \cdot, \cdot \}_\text{Rank0} \) defined in (2.16) and that coincides with the Lie-Poisson bracket on \( \mathfrak{se}(3)^* \).

According to Theorem 5.4, we denote the bivector field associated to the bracket \( \{ \cdot, \cdot \}_\text{Rank2}' \) by \( \pi_{\text{red}}^B \).

Using (2.13) one gets

\[
(\pi_{\text{red}}^B\mathcal{B})(dK) = (K - mr^2(\Omega \cdot \gamma)) \times \frac{\partial}{\partial K} + \gamma \times \frac{\partial}{\partial \gamma}, \quad (\pi_{\text{red}}^B\mathcal{B})(d\gamma) = \gamma \times \frac{\partial}{\partial K}.
\]

Next, notice that the 2-form \( \mathcal{B} \) defined by (5.51) is written in the redundant coordinates \( (\gamma, K) \) as

\[
\mathcal{B} = \frac{1}{2} mr^2(\Omega \cdot \gamma) \gamma \times (d\gamma \times d\gamma).
\]

The bivector field \( \pi_{\text{red}}^B \) and the 2-form \( \mathcal{B} \) verify hypothesis of Proposition 4.7 and thus the gauge transformation of \( \pi_{\text{red}}^B \) associated to \( \mathcal{B} \) is again a bivector field that we will denote it by \( \pi_{\text{red}}^B \). Relying on equation (3.24), one computes

\[
i(\pi_{\text{red}}^B)(dK) = mr^2(\Omega \cdot \gamma)(E - \gamma T^T) d\gamma, \quad i(\pi_{\text{red}}^B)(d\gamma) = 0,
\]

that complete the proof.

\[\square\]

The conformal factor and the 3-form \( \phi \). In accordance with Proposition 3.10, since the bracket \( \{ \cdot, \cdot \}_\text{Rank2}' \) is both conformally Poisson and twisted Poisson, there is relationship between the conformal factor \( \varphi_2 \) (given by (5.41)), and the twisting 3-form \( \phi \) (defined in Theorem 5.7).

We leave it to the reader to check that on the leaves \( O_\alpha \) of the foliation of \( \mathcal{R} \) corresponding to the bracket \( \{ \cdot, \cdot \}_\text{Rank2}' \), the 3-form \( \phi \) coincides with \( \psi := \frac{1}{\varphi_2} d\varphi_2 \wedge \Omega \), where 2-form \( \Omega \) is given in the redundant coordinates \( (\gamma, K) \) by

\[
\Omega = \frac{1}{2} (K - mr^2(\Omega \cdot \gamma)) \cdot (d\gamma \times d\gamma) - \gamma \cdot (dK \times d\gamma).
\]

This choice of \( \Omega \) satisfies the conditions of Corollary 3.2 for the graph of the bivector field \( \pi_{\text{red}}^B \) corresponding to \( \{ \cdot, \cdot \}_\text{Rank2}' \) on \( T\mathcal{R} \oplus T^*\mathcal{R} \).
Rigid body with generalized rolling constraints of rank 1

A completely analogous analysis can be performed if the rank of the matrix $A$ equals one. This time it is the bracket $\{·, ·\}_{\text{Rank}1}$ that is both twisted and conformally Poisson. In analogy with Theorem 5.7 we have

**Theorem 5.8.** The bracket $\{·, ·\}_{\text{Rank}1}$ defined in (2.13), is a $\phi$-twisted Poisson bracket with $\phi = dB$ with $B$ given by expression (5.51).

The proof is the same to the Rank 2 case. The bracket $\{·, ·\}_{\text{Rank}1}$ is $B$-gauge related with the bracket $\{·, ·\}_{\text{Rank}0}$ defined in (2.16) and that coincides with the Lie-Poisson bracket on $\mathfrak{se}(3)^*$.

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Gauge Transformations, Twisted Poisson Brackets and Hamiltonization of Nonholonomic Systems

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Abstract

In this paper we study the problem of Hamiltonization of nonholonomic systems from a geometric point of view. We use gauge transformations by 2-forms (in the sense of Ševera and Weinstein [33]) to construct different almost Poisson structures describing the same nonholonomic system. In the presence of symmetries, we observe that these almost Poisson structures, although gauge related, may have fundamentally different properties after reduction, and that brackets that Hamiltonize the problem may be found within this family. We illustrate this framework with the example of rigid bodies with generalized rolling constraints, including the Chaplygin sphere rolling problem. We also see how twisted Poisson brackets appear naturally in nonholonomic mechanics through these examples.

Keywords: nonholonomic systems, almost Poisson brackets, Hamiltonization, gauge transformations of Dirac structures, rigid body dynamics with rolling constraints.

Mathematics Subject Classification (2000) 70F25, 70H06, 53D17.

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1 Introduction

It is well known that the equations of motion for a mechanical system with nonholonomic constraints do not arise from a variational principle in the usual sense (see e.g. [4]). As a consequence, they cannot be formulated as a classical Hamiltonian system. Instead, they are written with respect to an almost Poisson bracket that fails to satisfy the Jacobi identity. This formulation has its origins in [34, 29, 23] and others.

On the other hand, after a symmetry reduction, the equations of motion of a number of examples allow a Hamiltonian formulation (sometimes after a time reparametrization), in which case one talks about Hamiltonization\(^1\) (see [12, 16, 17, 19, 21, 22, 24, 32]).

In this paper we employ recent ideas from Poisson geometry to study the Hamiltonization phenomenon from a geometric perspective. The main tool in our approach is the concept of gauge transformation of bivector fields (in the sense of ˇSevera and Weinstein [33]), an operation that uses differential 2-forms to modify bivector fields keeping their characteristic distribution (see Section 2, Definition 2) unchanged.

Let us consider a nonholonomic system on a constraint phase space \(M\), equipped with the almost Poisson bracket \(\{\cdot,\cdot\}_\text{nh}\) [34, 29, 23], known as the nonholonomic bracket, and Hamiltonian function \(H_M\). The (almost) Hamiltonian vector field \(X_{\text{nh}}\), defined by \(\{\cdot,\cdot\}_\text{nh}\) and \(H_M\), governs the dynamics of the system. In this paper, we consider new brackets \(\{\cdot,\cdot\}_B\) obtained by gauge transformations of \(\{\cdot,\cdot\}_\text{nh}\) with respect to suitably chosen 2-forms \(B\) on \(M\). Since gauge transformations do not modify the characteristic distribution, the (almost) Hamiltonian vector fields associated to \(\{\cdot,\cdot\}_B\) satisfy the nonholonomic constraints. If, in addition, \(B\) verifies \(i_{X_{\text{nh}}} B = 0\), we say that it defines a dynamical

\(^1\)A different meaning to Hamiltonization is given in [5] where the authors study the unreduced system in connection with the inverse problem of the calculus of variations.
A key observation is that, although the brackets in $\mathcal{F}$ are all gauge related, they may have fundamentally different features after reduction. For example, depending on the choice of the 2-form $B$, the characteristic distribution of the reduced almost Poisson bracket $\{\cdot,\cdot\}_{\text{red}}$ may or may not be integrable. The function $\varphi : \mathcal{R} \to \mathbb{R}^+$ is not arbitrary. It is required to be basic with respect to the fibered structure of the reduced space $\mathcal{R}$. 

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\[^2\text{In fact the function } \varphi : \mathcal{R} \to \mathbb{R}^+ \text{ is not arbitrary. It is required to be basic with respect to the fibered structure of the reduced space } \mathcal{R}.\]
integrability of the characteristic distribution is of central importance, as it is a necessary condition for \(\{\cdot, \cdot\}_{\text{red}}^B\) to be conformally Poisson (and, hence, for the system to be Hamiltonizable).

An important class of almost Poisson brackets possessing an integrable characteristic distribution is given by the so-called twisted Poisson structures, introduced in [26] and [33]. For these structures the failure of the Jacobi identity is controlled by a global closed 3-form (see Section 3.2). In this paper we show that any almost Poisson structure whose characteristic distribution is both integrable and regular is a twisted Poisson structure. As a consequence, we show that the reduced equations of certain nonholonomic systems (including the Veselova problem [35] and the Chaplygin sphere) can be formulated in terms of twisted Poisson brackets in the original physical time (prior to any time reparametrization). To our knowledge, this establishes the first connection between twisted Poisson structures and nonholonomic mechanics. This should serve as a motivation to investigate the interplay between the rich geometrical properties of twisted Poisson brackets and the dynamical features of the corresponding (almost) Hamiltonian vector fields.

To illustrate the method depicted in the diagram above, we introduce the problem of the motion of rigid bodies that are subject to generalized rolling constraints. These are nonholonomic constraints that relate the angular velocity \(\omega\) of the body and the linear velocity \(\dot{x}\) of its center of mass in a linear way, i.e., \(\dot{x} = A\omega\) for a \(3 \times 3\) matrix \(A\) satisfying certain properties. This type of constraints generalize the Chaplygin sphere problem. In fact, the constraints vary from completely nonholonomic if the rank of \(A\) equals 3, to (holonomic) classical free rigid body motion if \(A = 0\). For the Chaplygin sphere, the rank of \(A\) equals 2, in which case we recover the Hamiltonization method of [21]. By allowing gauge transformations, we prove that the problem is Hamiltonizable independently of the value of the rank of \(A\). Using the Hamiltonian structure of the reduced system, we also show its complete integrability in the sense of Liouville.

For this explicit class of examples, the behavior of the reduced brackets, according to the rank of \(A\), is illustrated in table (1.1) below. In our notation, \(\{\cdot, \cdot\}_{\text{red}}\) corresponds to the reduction of the classical nonholonomic bracket, while the bracket \(\{\cdot, \cdot\}_{\text{red}}^B\) corresponds to the reduction of a bracket obtained by a dynamical gauge transformation by a specific 2-form \(B\) (defined in Section 5).

| Rank of \(A\) | 0 (free rigid body) | 1 (Chaplygin sphere) | 2 (completely nonholonomic) |
|---------------|---------------------|----------------------|-----------------------------|
| \{\cdot, \cdot\}_{\text{red}} | Poisson | Conformally Poisson and Twisted Poisson | Non-integrable characteristic distribution |
| \{\cdot, \cdot\}_{\text{red}}^B | Non-integrable characteristic distribution | Non-integrable characteristic distribution | Conformally Poisson and Twisted Poisson |

\[\text{(1.1)}\]

**Hamiltonization in context:** The most interesting example of Hamiltonization concerns the Chaplygin sphere. Even though the formulation and integration of the equations of motion by Chaplygin dates back to 1903 [11], the Hamiltonian structure of the reduced equations (after a time reparametriza-
tion) was only discovered in 2001 by Borisov and Mamaev [6]. Their result is all the more remarkable as it is in apparent contradiction to the assertions of [15, 10].

Recently, Jovanović [24] proved that the multidimensional version of the Chaplygin sphere problem introduced in [18] is also integrable and Hamiltonizable when the vertical angular momentum is zero. This gives a partial solution to a problem that remained open for many years. His approach to prove integrability involves in a crucial way the Hamiltonization of the problem. Another important example where the integration of a nonholonomic system follows from its Hamiltonization is the multidimensional Veselova system treated by Fedorov and Jovanović in [17]. We also mention the recent work of Ohsawa, Fernandez, Bloch and Zenkov [32] that explores the connection of Hamiltonization with Hamilton-Jacobi theory.

The relationship between Hamiltonization and integrability may have been the original motivation for Chaplygin to consider the problem of Hamiltonization back in 1911 [12]. In this work, Chaplygin proved the famous Chaplygin reducing multiplier Theorem that applies to the so-called $G$-Chaplygin systems. These are nonholonomic systems with the property that the tangent space to the orbits of a symmetry group $G$ exactly complements the constraint distribution on the tangent space $TQ$ of the configuration manifold $Q$. Stated in modern geometric terms, the theorem says that if the shape space $Q/G$ is two-dimensional, and the reduced equations have an invariant measure, then they can be put in Hamiltonian form in the new time $\tau$ defined by $d\tau = \frac{1}{\varphi} dt$. The positive function $\frac{1}{\varphi} : Q/G \to \mathbb{R}$ is known as the reducing multiplier\(^3\). There is a very neat interpretation of the multiplier $\frac{1}{\varphi}$ in terms of the invariant measure and as a conformal factor for an almost symplectic form that describes the dynamics, see [16, 17, 22, 32]. This interpretation suggests that geometric methods may be useful to understand Hamiltonization in more general scenarios of nonholonomic systems with symmetry. The geometric approach to the study of nonholonomic systems has received a lot of interest in the last couple of decades and has its origins in the seminal paper of Koiller [27].

Recently, Fernandez, Mestdag and Bloch [19], derived a set of coupled first order partial differential equations for the multiplier $\frac{1}{\varphi}$ for $G$-Chaplygin systems whose shape space has arbitrary dimension. Even more, the set of equations found by the authors applies to general nonholonomic systems with symmetry that are not necessarily $G$-Chaplygin. This is done by writing the reduced equations of motion in Hamilton-Poincaré-D’Alembert form as described in [3, 4]. The issue of Hamiltonization is thus reformulated as a problem of existence of a solution for the aforementioned system of partial differential equations.

Our approach to Hamiltonization contains the same degree of generality but is more intrinsic. Indeed, our requirement that one of the members $\{\cdot, \cdot\}_{\text{red}}$ in the collection of brackets describing the reduced dynamics is conformally Poisson can be rewritten as

$$[\varphi \pi_{\text{red}}B, \varphi \pi_{\text{red}}B] = 0,$$

that locally defines a set of coupled, partial differential equations for the multiplier $\varphi$. Here $\pi_{\text{red}}B$ is the bivector field associated to $\{\cdot, \cdot\}_{\text{red}}$ and $[\cdot, \cdot]$ is the Schouten bracket. Our approach is also broader than that of [19] since we have the freedom of selecting the bivector field $\pi_{\text{red}}B$ in (1.2) within the collection of brackets describing the reduced dynamics. We stress that this liberty is crucial to Hamiltonize certain

\(^3\)We denote the reducing multiplier by $\frac{1}{\varphi}$ instead of $\varphi$ to be consistent with our exposition which takes the Poisson rather than the symplectic perspective.
examples. As mentioned before, and as at is shown in table (1.1), the Hamiltonization of the Chaplygin sphere (by a “one-step reduction”) depends on this freedom. The approach taken in [7, 19] involves a two-step reduction and the symplectic interpretation of the second reduction is delicate (see [22]).

If the symmetries are of $G$-Chaplygin type, the bivector fields $\pi_{\text{red}}B$ are everywhere non-degenerate and the equations of motion can be written with respect to the associated almost symplectic form $\Omega_{\text{red}}B$. If a multiplier $\varphi$ satisfying (1.2) exists, then $\Omega_{\text{red}}B$ is conformally closed, $(d(\frac{1}{\varphi}\Omega_{\text{red}}B) = 0)$, and one speaks of Chaplygin Hamiltonization [16]. The term Poissonization was introduced in [19] to refer to the case where the bivector field $\pi_{\text{red}}B$ satisfying (1.2) is degenerate. Their motivation to distinguish this case is to study the relationship between Hamiltonization and the existence of invariant measures for the reduced equations. To simplify our exposition and to treat the problem in a unified manner, we will not use their terminology. We give a discussion on the existence of invariant measures for nonholonomic systems admitting a Hamiltonization in this generality in subsection 4.4.

Outline of the paper: The paper is structured as follows. In Section 2 we introduce our motivating examples: rigid bodies subject to generalized rolling constraints. After writing down the reduced equations of motion, we define two different almost Poisson structures for these equations according to the rank of the matrix $A$ that defines the constraints. For each value of the rank, we show that only one of the brackets is Poisson or conformally Poisson, while the other one possesses a non-integrable characteristic distribution. The geometric interpretation and construction of these brackets is postponed to Section 5 after the necessary tools are developed in Sections 3 and 4.

Section 3 presents the geometric background needed for our purposes, with focus on the study of gauge transformations of bivector fields in the sense of [33]. Although our main interest is in almost Poisson brackets, we consider more general objects known as almost Dirac structures [13], as they provide the most natural setting for the definition and study of gauge transformations (Dirac structures have been also considered in connection with nonholonomic mechanics in [36, 37, 25] although the issue of Hamiltonization and the incorporation of gauge transformations are not treated in these works). Using a general description of regular almost Dirac structures in terms of their characteristic distributions (Proposition 3.1), we show that any almost Poisson bracket possessing a regular, integrable, characteristic distribution is a twisted Poisson bracket. We also prove that any two regular almost Poisson (or, more generally, Dirac) structures defining the same characteristic distribution are gauge related (Theorem 3.11).

In Section 4 we make the connection between the geometric methods developed in Section 3 and nonholonomic mechanics. Using Proposition 3.1 and the framework of nonholonomic mechanics described in Bates and Sniatycki [2] we construct the nonholonomic bracket of [34, 29, 23]. Then we introduce the notion of dynamical gauge transformations for a nonholonomic system and define the family $\mathfrak{F}$ of almost Poisson brackets describing the nonholonomic system (as in the diagram above). After discussing the reduction of these brackets in the presence of symmetries, we introduce our working definition of Hamiltonization.

In Section 5 we resume the study of rigid bodies subject to generalized rolling constraints. We show that the brackets given in Section 2 to describe the reduced dynamics arise as a reduction of different members of the family $\mathfrak{F}$ (Theorems 5.3 and 5.4). We then establish the Hamiltonization of the reduced equations in detail and we conclude their integrability. Finally, we consider the twisted nature of the
brackets that Hamiltonize the problem prior to time reparametrization when the rank of $A$ is 1 or 2; in these cases, we also provide explicit expressions for the associated closed 3-form.

2 Motivating Examples: Rigid bodies with Generalized Rolling Constraints

Consider the motion of a rigid body in space that evolves under its own inertia and is subject to the constraint that enforces the linear velocity of the center of mass, $\mathbf{x}$, to be linearly related to the angular velocity of the body $\mathbf{\omega}$, i.e.,

$$\dot{\mathbf{x}} = rA\mathbf{\omega}.$$  \hspace{1cm} (2.3)

Both vectors $\dot{\mathbf{x}}$ and $\mathbf{\omega}$ belong to $\mathbb{R}^3$ and are written with respect to an inertial frame. The constant scalar $r$ has dimensions of length and is a natural length scale of the system. The dimensionless, constant, $3 \times 3$ matrix $A$ is given and satisfies certain conditions that are made precise in the following definition.

**Definition 1.** The matrix $A$ is said to define a generalized rolling constraint if it satisfies one of the following conditions according to its rank:

(i) $A = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$, with $C \in \text{SO}(2)$, if rank $A = 3$.

(ii) $A = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$, with $C \in \text{SO}(2)$, if rank $A = 2$.

(iii) $A = e_3 e_3^T$, if rank $A = 1$, where $e_3$ is the third canonical vector in $\mathbb{R}^3$, and $T$ denotes transpose.

(iv) $A = 0$ if rank $A = 0$.

The above conditions on $A$ can be relaxed (see Remark 2.1 ahead). However, for simplicity, we will assume that $A$ has the form given by one of the items of the above definition. If $A$ satisfies any of the the conditions of the above definition we say that (2.3) is a generalized rolling constraint.

Our terminology is motivated by a particular example: the Chaplygin sphere. The problem, introduced by Chaplygin in 1903 [11], concerns the motion of a ball whose center of mass coincides with its geometric center that rolls on the plane without slipping. In this case, the matrix $A$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $r$ is the radius of the sphere.

The motion of the Chaplygin ball has been the subject of much research to our days. An important property is that (after a time reparametrization) the reduced equations can be given in a Hamiltonian structure [6, 7, 21]. The geometry of the Hamiltonization of the problem is intricate. In order to study this phenomenon in a mathematically systematic fashion, we consider more general possibilities for the matrix $A$.  

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The crucial property of $A$ that determines many of the dynamical and geometrical features of the problem is its rank. The Chaplygin sphere corresponds to the case $\text{rank } A = 2$. Another familiar case occurs when $\text{rank } A = 0$. In this case the constraint (2.3) becomes $\dot{x} = 0$ which can be interpreted as a conservation law for the free system that states that the center of mass of the body is at rest in the inertial frame. The motion of the system reduces to that of the classical free rigid body.

We will also consider the cases where the rank of $A$ equals 3 and 1 which, to our knowledge, have not yet been considered in the literature.

2.1 Generalities

The configuration space for the system is $Q = \text{SO}(3) \times \mathbb{R}^3$. Elements in $Q$ are of the form $q = (g, x) \in \text{SO}(3) \times \mathbb{R}^3$. The vector $x \in \mathbb{R}^3$ is the position of the center of mass in space and the orthogonal matrix $g$ specifies the orientation of the ball by relating two orthogonal frames, one attached to the body and one that is fixed in space. We will assume that the body frame has its origin at the center of mass and is aligned with the principal axes of inertia of the body. These frames define the so-called space and body coordinates respectively.

Recall that the Lie algebra $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ equipped with the vector product via the hat map:

$$\eta = (\eta_1, \eta_2, \eta_3) \mapsto \hat{\eta} = \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix}. \quad (2.4)$$

Given a motion $(g(t), x(t)) \in Q$, the angular velocity vector in space coordinates, $\omega \in \mathbb{R}^3$, and the angular velocity vector in body coordinates, $\Omega \in \mathbb{R}^3$, are respectively given by

$$\hat{\omega}(t) = \dot{g}(t)g^{-1}(t), \quad \hat{\Omega}(t) = g^{-1}(t)\dot{g}(t),$$

and satisfy $\Omega = g^{-1}\omega$. It will be useful to write the constraint (2.3) in terms of the body angular velocity as

$$\dot{x} = rAg \Omega. \quad (2.5)$$

The kinetic energy of the rigid body defines the Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$ by

$$\mathcal{L}(g, \dot{g}, x, \dot{x}) = \frac{1}{2} (\|\Omega\| \cdot \Omega + \frac{m}{2} \|\dot{x}\|^2), \quad (2.6)$$

where "\cdot" denotes the Euclidean scalar product on $\mathbb{R}^3$, $m$ is the mass of the body and the $3 \times 3$ diagonal matrix $I$ is the inertia tensor with positive entries $I_1, I_2, I_3$.

Remark 2.1. It is not hard to see that if the space axes are rotated by an element $h \in \text{SO}(3)$, the Lagrangian $\mathcal{L}$ is invariant and the constraint (2.3) is rewritten as

$$\dot{x} = rh^{-1}A h \omega.$$
2.2 The equations of motion

Let \( p = m \dot{x} \) be the linear momentum of the body. In accordance with the Lagrange-D’Alembert principle, the constraint forces must annihilate any velocity pair \((\dot{x}, \Omega)\) satisfying (2.5). Therefore, the equations of motion are given by

\[
\dot{p} = \mu, \quad \dot{\Omega} = \Omega \times \Omega - rg^{-1}A^T \mu, \tag{2.7}
\]

where “×” denotes the vector product in \( \mathbb{R}^3 \) and the multiplier \( \mu \in \mathbb{R}^3 \) is determined uniquely from the constraint (2.5).

Differentiating (2.5) and using \( \dot{g} \Omega = 0 \) we find \( \mu = mrAg\dot{\Omega} \). Thus, the second equation in (2.7) decouples from the first to give

\[
\dot{\Omega} = \Omega \times \Omega - mr^2g^{-1}A^T Ag\dot{\Omega}. \tag{2.8}
\]

In principle, this equation should be complemented with the reconstruction equation \( \dot{g} = g\dot{\Omega} \). It will be shown ahead that it suffices to consider the evolution of the Poisson vector \( \gamma := g^{-1}\mathbf{e}_3 \) that represents the vector \( \mathbf{e}_3 \) written in body coordinates. A direct calculation gives

\[
\dot{\gamma} = \gamma \times \Omega.
\]

The decoupling in (2.7) is due to the presence of symmetries that will be discussed in detail in Section 2.4. Once this equation is solved for \((g, \Omega)\), we obtain \( p = mrAg\dot{\Omega} \) that follows from (2.5).

We introduce the kinetic momentum \( K \in \mathbb{R}^3 \) by

\[
K := \Omega + mr^2g^{-1}A^T Ag\dot{\Omega}. \tag{2.9}
\]

This definition of the kinetic momentum allows us to define the (reduced) Hamiltonian

\[
\mathcal{H}_R = \frac{1}{2}(K \cdot \Omega), \tag{2.10}
\]

which coincides with the kinetic energy on the constraint space \( \mathcal{M} \).

A direct calculation using (2.8) gives our final set of equations

\[
\dot{K} = K \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega. \tag{2.11}
\]

To understand why the above equations define a closed system for \((K, \gamma)\) \( \in \mathbb{R}^3 \times \mathbb{R}^3 \), and to understand their structure, it is useful to perform a separate study for different values of the rank of the matrix \( A \). This will also show that the Hamiltonian \( \mathcal{H}_R \) can be considered as a function of \( K \) and \( \gamma \).

2.3 A pair of (almost) Poisson brackets for the equations of motion

For each value of the rank of \( A \) we will give two different brackets that define the equations of motion (2.11), with respect to the reduced Hamiltonian \( \mathcal{H}_R \). In general, these brackets are almost Poisson, i.e. they do not satisfy the Jacobi identity but we will argue that one of them is more convenient than the other. They will be denoted by \( \{\cdot, \cdot\}_{Rank_j} \) and \( \{\cdot, \cdot\}'_{Rank_j} \) where \( j \) denotes the rank of the matrix
A. Both brackets define the equations of motion (2.11) in the sense that the directional derivative of any function $f = f(\gamma, K) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ along the flow is given by $\dot{f} = \{f, \mathcal{H}_R\}_{\text{Rank}_j} = \{f, \mathcal{H}_R\}'_{\text{Rank}_j}$. The geometric interpretation of these brackets is the subject of the subsequent sections. Concretely, in section 5 (Theorems 5.3 and 5.4), we will show that the bracket $\{\cdot, \cdot\}_{\text{Rank}_j}$ arises as the reduction of the nonholonomic bracket introduced in [34], and that $\{\cdot, \cdot\}'_{\text{Rank}_j}$ arises as the reduction of a gauge transformation of the nonholonomic bracket.

The following definitions will be useful in our discussion of the utility of the two brackets:

**Definition 2.** Let $P$ be a manifold equipped with an almost Poisson bracket $\{\cdot, \cdot\}$.

1. The (almost) Hamiltonian vector field $X_f$ of a function $f \in C^\infty(P)$ is the vector field on $P$ defined as the usual derivation $X_f(g) = \{g, f\}$ for all $g \in C^\infty(P)$.

2. The characteristic distribution of $\{\cdot, \cdot\}$ is the distribution on the manifold $P$ whose fibers are spanned by the (almost) Hamiltonian vector fields.

3. Due to Leibniz condition of $\{\cdot, \cdot\}$, there is a bivector field $\pi \in \Gamma(\bigwedge^2(TP))$ such that for $f, g \in C^\infty(P)$ we have $\pi(df, dg) = \{f, g\}$. We say that $\pi$ is the bivector field associated to $\{\cdot, \cdot\}$ and we denote by $\pi^\sharp : T^*P \rightarrow TP$ the map such that $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$. We will occasionally refer to bivector fields simply as bivectors. Note that the characteristic distribution is the image of $\pi^\sharp$ and the Hamiltonian vector field $X_f = -\pi^\sharp(df)$. The 3-vector field $[\pi, \pi]$, where $[\cdot, \cdot]$ is the Schouten bracket, may be different from zero, and it measures the failure of the Jacobi identity through the relation

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \frac{1}{2}[\pi, \pi](df, dg, dh),$$

for $f, g, h \in C^\infty(P)$.

4. The bracket is called conformally Poisson if there exists a strictly positive function $\varphi \in C^\infty(P)$ such that the bracket $\varphi\{\cdot, \cdot\}$ satisfies the Jacobi identity, i.e. $[\varphi\pi, \varphi\pi] = 0$.

The well known symplectic stratification theorem states that the characteristic distribution of a Poisson bracket is integrable and its leaves are symplectic manifolds. Since multiplication of an almost Poisson bracket by a positive function does not change the characteristic distribution, a necessary condition for an almost Poisson bracket to be conformally Poisson is that its characteristic distribution be integrable.

We now come back to the discussion of our example for the different values of the rank of $A$.

**If $A$ has rank 3.** In this case $A^{-1} = A^T$ and $K = (I + mr^2E)\Omega$ where $E$ denotes the $3 \times 3$ identity matrix. It follows form (2.11) that the rotational motion of the body is the same as that of a free rigid body whose total inertia tensor is $I + mr^2E$. It is trivial to write $\Omega = (I + mr^2E)^{-1}K$ and it is clear that equations (2.11) define a closed system in $\mathbb{R}^3 \times \mathbb{R}^3$. 

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The two brackets for the system for functions \( f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) are given by

\[
\{ f, g \}_{\text{Rank3}} = - (K + mr^2 \Omega) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right),
\]

\[
\{ f, g \}^\prime_{\text{Rank3}} = - K \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right) \cdot \frac{\partial}{\partial K} + \left( \gamma \times \frac{\partial f}{\partial K} \right) \cdot \frac{\partial}{\partial \gamma}.
\] (2.13)

The above brackets are quite different. On the one hand, the bracket \( \{ \cdot, \cdot \}_{\text{Rank3}} \) satisfies the Jacobi identity. It in fact coincides with the Lie-Poisson bracket on the dual Lie algebra \( \mathfrak{se}(3)^* \). On the other hand, the bracket \( \{ \cdot, \cdot \}^\prime_{\text{Rank3}} \) is not even conformally Poisson as the following proposition shows.

**Proposition 2.2.** The characteristic distribution of the almost Poisson bracket \( \{ \cdot, \cdot \}^\prime_{\text{Rank3}} \) defined in (2.13) is not integrable.

**Proof.** The (almost) Hamiltonian vector field \( X_f \) of a function \( f \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \) corresponding to the bracket \( \{ \cdot, \cdot \}^\prime_{\text{Rank3}} \) is given by

\[
X_f = \left( (K + mr^2 \Omega) \times \frac{\partial f}{\partial K} + \gamma \times \frac{\partial f}{\partial \gamma} \right) \cdot \frac{\partial}{\partial K} + \left( \gamma \times \frac{\partial f}{\partial K} \right) \cdot \frac{\partial}{\partial \gamma},
\]

and it is annihilated by the non-closed one-form

\[
\chi = \gamma \cdot dK + (K + mr^2 \Omega) \cdot d\gamma.
\]

We have

\[
X_{K_1} = (K_3 + mr^2 \Omega_3) \frac{\partial}{\partial K_2} - (K_2 + mr^2 \Omega_2) \frac{\partial}{\partial K_3} + \gamma_3 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_3},
\]

\[
X_{\gamma_1} = \gamma_3 \frac{\partial}{\partial K_2} - \gamma_2 \frac{\partial}{\partial K_3},
\]

and thus

\[
\chi([X_{\gamma_1}, X_{K_1}]) = - d\chi(X_{\gamma_1}, X_{K_1}) = -mr^2 \left( \frac{\gamma_3^2}{I_2 + mr^2} + \frac{\gamma_2^2}{I_3 + mr^2} \right) \neq 0.
\]

This shows that the commutator \([X_{\gamma_1}, X_{K_1}]\) does not belong to the characteristic distribution which is therefore not integrable. \( \square \)

Therefore, to obtain a true Hamiltonian formulation of the reduced equations of motion in the case where the rank of \( A \) is 3, one needs to work with the bracket \( \{ \cdot, \cdot \}^\prime_{\text{Rank3}} \).

**If \( A \) has rank 2.** As mentioned before, this case has the Chaplygin sphere as a particular example. The analysis of the two brackets has been done in [21]. We include it here for completeness and to link it with clarity to other results of the present work.

In view of the form of \( A \) given in item (ii) of Definition [11] we can write \( A^T A = E - e_3 e_3^T \) and thus, according to (2.9), we get

\[
K = (I + mr^2 E) \Omega - mr^2 (\Omega \cdot \gamma) \gamma,
\]

which is precisely the expression for the angular momentum about the contact point for the Chaplygin sphere.
The angular velocity $\Omega$ can be written in terms of $K$ and $\gamma$ as

$$\Omega = (I + mr^2E)^{-1}K + mr^2\left(\frac{K \cdot (I + mr^2E)^{-1}\gamma}{||\gamma||^2 - mr^2\gamma \cdot (I + mr^2E)^{-1}\gamma}\right)(I + mr^2E)^{-1}\gamma,$$

so both the equations (2.11) and the Hamiltonian $\mathcal{H}_R$ are well defined on $\mathbb{R}^3 \times \mathbb{R}^3$.

In this case, the two brackets for the system for functions $f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ are given by

$$\{f, g\}_{\text{rank}2} = -(K + mr^2\Omega - mr^2(\Omega \cdot \gamma)\gamma) \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K}\right) - \gamma \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma}\right),$$

$$\{f, g\}'_{\text{rank}2} = -(K - mr^2(\Omega \cdot \gamma)\gamma) \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K}\right) - \gamma \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma}\right).$$

None of the above brackets satisfies the Jacobi identity but it is preferable to consider $\{\cdot, \cdot\}'_{\text{rank}2}$. The reason is that this bracket is conformally Poisson with conformal factor

$$\varphi(\gamma) = \sqrt{||\gamma||^2 - mr^2(\gamma \cdot (I + mr^2E)^{-1}\gamma)}.$$

This important observation was first made in [6]. The characteristic distribution of $\{\cdot, \cdot\}'_{\text{rank}2}$ is thus integrable. The generic leaves are the level sets of the Casimir functions $C_1(K, \gamma) = K \cdot \gamma$ and $C_2(\gamma) = ||\gamma||^2$. Another important feature of this bracket is that it is twisted Poisson (in the sense of [20] [33]) as will be shown in Section 5.3 (Theorem 5.7).

On the other hand, similar to Proposition 2.2 we have

**Proposition 2.3** ([21]). The characteristic distribution of the almost Poisson bracket $\{\cdot, \cdot\}_{\text{rank}2}$ defined in (2.14) is not integrable.

This can be shown exactly as we did for Proposition 2.2. Therefore if the rank of $A$ is 2, just as in the case of rank 3, a Hamiltonian formulation of the reduced equations can only be obtained if we work with the bracket $\{\cdot, \cdot\}'_{\text{rank}2}$. However, in this case one needs to multiply the bracket by a conformal factor. This can be interpreted as a time reparametrization, see the discussion in Section 4.4.

**If $A$ has rank 1.** Taking into account the form of $A$ given in item (iii) of Definition 1, we have $A^TA = e_3e_3^T$ and thus, in view of (2.9), we get

$$K = I\Omega + mr^2(\Omega \cdot \gamma)\gamma.$$

The expression for the angular velocity $\Omega$ in terms of $K$ and $\gamma$ is

$$\Omega = I^{-1}K - mr^2\left(\frac{K \cdot I^{-1}\gamma}{||\gamma||^2 + mr^2(\gamma \cdot I^{-1}\gamma)}\right)I^{-1}\gamma,$$

so again, both the equations (2.11) and the Hamiltonian $\mathcal{H}_R$ are well defined on $\mathbb{R}^3 \times \mathbb{R}^3$.

This time, the two brackets for the system are given by

$$\{f, g\}_{\text{rank}1} = -(K + mr^2(\Omega \cdot \gamma)\gamma) \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K}\right) - \gamma \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma}\right),$$

$$\{f, g\}'_{\text{rank}1} = -(K - mr^2\Omega + mr^2(\Omega \cdot \gamma)\gamma) \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K}\right) - \gamma \cdot \left(\frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma}\right).$$

(2.16)
for functions \( f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \).

The properties of the brackets above are very similar to those obtained in the case where the rank of \( A \) is 2 except that the roles of \( \{\cdot,\cdot\}_{\text{Rank}1} \) and \( \{\cdot,\cdot\}_{\text{Rank}1}' \) are reversed.

This time one can show that

**Proposition 2.4.** The bracket \( \{\cdot,\cdot\}_{\text{Rank}1} \) defined in (2.16) is conformally Poisson with conformal factor

\[ \varphi(\gamma) = \sqrt{||\gamma||^2 + mr^2(\gamma \cdot I^{-1}\gamma)} \]

**Proof.** We have to prove that the scaled bracket on \( \mathcal{R} \) defined as \( \varphi\{\cdot,\cdot\}_{\text{Rank}1} \) satisfies the Jacobi identity, i.e.,

\[ \varphi\{\varphi\{f_1,f_2\}_{\text{Rank}1},f_3\}_{\text{Rank}1} + \varphi\{\varphi\{f_2,f_3\}_{\text{Rank}1},f_1\}_{\text{Rank}1} + \varphi\{\varphi\{f_3,f_1\}_{\text{Rank}1},f_2\}_{\text{Rank}1} = 0 \]

for all \( f_1, f_2, f_3 \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \). In view of the derivation properties of the bracket, is enough to show the identity for the coordinate functions \( K_i, \gamma_j \). In our case, since \( \{\gamma_i,\gamma_j\}_{\text{Rank}1} = 0 \), it is immediate to check that the identity holds if two of the three functions are \( \gamma_j \)'s. A long but straightforward computation shows that the identity holds for the following three choices of functions \( f_1 = K_1, f_2 = K_2, f_3 = \gamma_3 \); \( f_1 = K_1, f_2 = K_2, f_3 = \gamma_3 \) and \( f_1 = K_1, f_2 = K_2, f_3 = K_3 \). Since the definition of the bracket is symmetric with respect to the coordinate functions \( K_i, \gamma_j \), and since the Jacobi identity trivially holds if two of the three functions \( f_1, f_2, f_3 \) are equal, all of the other cases are either trivial or analogous. \( \square \)

Hence, the characteristic distribution of \( \{\cdot,\cdot\}_{\text{Rank}1} \) is integrable and the generic leaves are again the level sets of the Casimir functions \( C_1(K, \gamma) = K \cdot \gamma \) and \( C_2(\gamma) = ||\gamma||^2 \). It will also be shown in Section 5.3 that \( \{\cdot,\cdot\}_{\text{Rank}1} \) is twisted Poisson.

On the other hand, analogous to Propositions 2.2 and 2.3 we have

**Proposition 2.5.** The characteristic distribution of the almost Poisson bracket \( \{\cdot,\cdot\}'_{\text{Rank}1} \) defined in (2.16) is not integrable.

The proof is again similar.

Thus, this time the Hamiltonian structure of the reduced equations can only be obtained with the bracket \( \{\cdot,\cdot\}_{\text{Rank}1} \), again through the multiplication by a conformal factor that is interpreted as a time reparametrization.

**If \( A \) has rank 0.** In this case \( A \) is the zero matrix and the constraints are holonomic and can be seen as a conservation law for the standard free rigid body. We have \( K = \Omega \) and clearly the equations (2.11) and the Hamiltonian \( \mathcal{H}_K \) are well defined on \( \mathbb{R}^3 \times \mathbb{R}^3 \). The two brackets are given by

\[
\{f,g\}_{\text{Rank}0} = -K \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right),
\]

\[
\{f,g\}'_{\text{Rank}0} = -(K - mr^2\Omega) \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial K} \right) - \gamma \cdot \left( \frac{\partial f}{\partial K} \times \frac{\partial g}{\partial \gamma} - \frac{\partial g}{\partial K} \times \frac{\partial f}{\partial \gamma} \right). \tag{2.17}
\]

The situation is analogous to that of the case when the rank of \( A \) is 3 but, once more, the roles of the brackets are reversed. While \( \{\cdot,\cdot\}_{\text{Rank}0} \) coincides with the Lie-Poisson bracket in the dual Lie algebra \( \mathfrak{se}(3)^* \) (and hence satisfies the Jacobi identity), we have
**Proposition 2.6.** The characteristic distribution of the almost Poisson bracket \( \{ \cdot, \cdot \}_\text{Rank}0 \) defined in (2.17) is not integrable.

The proof is identical to that of Proposition 2.2.

So in this case, the Hamiltonian structure of the reduced equations (2.11) can only be seen by working with the bracket \( \{ \cdot, \cdot \}_\text{Rank}0 \).

### 2.4 Symmetries

The reduced equations (2.11) can be interpreted as the output of a reduction process that we now explain. We begin by noticing that the configuration space \( Q = \text{SO}(3) \times \mathbb{R}^3 \) can be endowed with the Lie group structure of the three dimensional euclidean transformations \( \text{SE}(3) \). The group multiplication is given by

\[
(g_1, x_1)(g_2, x_2) = (g_1 g_2, g_1 x_2 + x_1).
\]

Let \( H \) be the Lie subgroup of \( \text{SE}(3) \) defined by

\[
H = \{(h, y) \in \text{SE}(3) : he_3 = e_3\}.
\]

For matrices \( A \) satisfying any of the conditions of Definition 1, it follows that \( hA = Ah \) whenever \( (h, y) \in H \). We consider the left action of \( H \) on \( Q \) by left multiplication. The tangent lift of the action to \( TQ \) maps

\[
(h, y) : (g, x, \omega, \dot{x}) \mapsto (hg, hx + y, h\omega, h\dot{x}) \quad \text{or} \quad (h, y) : (g, x, \Omega, \dot{x}) \mapsto (hg, hx + y, \Omega, h\dot{x}),
\]
depending on the trivialization of \( \text{SO}(3) \) that one is working with. Notice that the Lagrangian \( L \) given by (2.6) is invariant under the lifted action. Moreover, since \( h \) commutes with \( A \) for any \( (h, y) \in H \), the constraint (2.3) is also invariant.

The momenta \( (K, p) \) are geometrically interpreted as coordinates on the fibers of the (trivial) cotangent bundle \( T^*Q \). The constraint space \( \mathcal{M} \subset T^*Q \) is determined by the condition \( p = mrAg\Omega \), so the triple \((g, x, K) \in \text{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \) specifies a unique point in \( \mathcal{M} \). Reciprocally, any point in \( \mathcal{M} \) can be represented by a triple \((g, x, K) \).

By invariance of the Lagrangian and the constraints, the lifted action of \( H \) to \( T^*Q \) leaves \( \mathcal{M} \) invariant and, therefore, restricts to \( \mathcal{M} \). The restricted action is free and proper so the orbit space \( \mathcal{R} := \mathcal{M}/H \) is a smooth manifold. The reduced space \( \mathcal{R} \) can be identified with \( S^2 \times \mathbb{R}^3 \); the projection \( \rho : \mathcal{M} \to \mathcal{R} \) is given by

\[
\rho(g, x, K) = (\gamma, K),
\]

where \( \gamma = g^{-1}e_3 \in S^2 \), and is a surjective submersion. The conditions \( he_3 = e_3 \) and \( hA = Ah \), that are satisfied for \( (h, y) \in H \), ensure that the above mapping is well defined (in particular notice that \( K \) is invariant). The reduced equations on \( \mathcal{R} \) are precisely (2.11) when restricted to the level set \( ||\gamma|| = 1 \). Notice that \( \mathcal{R} \) inherits the (trivial) vector bundle structure \( S^2 \times \mathbb{R}^3 \to S^2 \) from \( \mathcal{M} \).

In this sense, the entries of \( \gamma \) should be considered as redundant coordinates for the sphere \( S^2 \) and the entries of \( K \) as coordinates on the fibers of \( \mathcal{R} \). Notice that, for any \( j = 0, \ldots, 3 \), both brackets \( \{ \cdot, \cdot \}_\text{Rank}j \) and \( \{ \cdot, \cdot \}'_\text{Rank}j \) restrict to the level set \( ||\gamma|| = 1 \) since \( C_2(\gamma) = ||\gamma||^2 \) is a Casimir function.
2.5 Kinematics and integrability of the constraint distribution

The constraint distribution on $Q$ defined by equation (2.3) has fundamentally different properties according to the rank of the matrix $A$ satisfying the conditions of Definition 1. On one extreme we have the case where $A = 0$ and the distribution is integrable (the 3-dimensional integral leaves are given by $\text{SO}(3) \times \{x\}$ for $x \in \mathbb{R}^3$). As mentioned before, in this case the constraints are holonomic and the problem reduces to the classical free rigid body problem (the center of mass of the body $x$ remains constant in our inertial frame).

The extreme opposite case occurs when rank $A = 3$. In this case the corresponding distribution is completely nonholonomic or bracket-generating, see e.g. [31]. By Chow’s theorem, any two points in the configuration space $Q$ can be joined by a curve $(g(t), x(t))$ satisfying the constraints. Thus, at least at the kinematical level, there are no restrictions on the values of $x$.

The cases where the rank of $A$ is 1 or 2 lie in between the situations described above. If the rank of $A = 2$, the third component $x_3$ of $x$ remains constant during the motion. This is in agreement with our observation that the Chaplygin sphere problem is a particular case of this type of constraints - the sphere rolls on a horizontal plane $x_3 = \text{const}$. In this case, the constraint distribution is non-integrable but is nevertheless tangent to the foliation of $Q$ by 5-dimensional leaves defined by constant values of $x_3$.

Finally, for the case where the rank of $A$ equals 1, the first two components $x_1, x_2, \text{of } x$ remain constant during the motion. The body goes up or down along the $x_3$ axis at a speed that is proportional to its angular velocity about this axis. This time the constraint distribution is non-integrable but tangent to the 4-dimensional leaves given by constant values of $x_1$ and $x_2$.

Without going into technical definitions, we simply state that the degree of non-integrability of the constraint distribution increases with the rank of $A$, passing from an integrable distribution if $A = 0$ to a completely nonholonomic distribution if rank $A = 3$. It is interesting to see how this correlates with the need of a gauge-transformation to Hamiltonize the problem (Remark 5.6).

3 Regular almost Dirac structures

This section is concerned with the geometry of (almost) Dirac structures [13], with focus on the regular case, as well as their gauge transformations [33]. Our exposition begins with the basic definitions but it reaches out to present some original results. As we will see, these geometric structures provide the setup that gives rise to the different brackets introduced in Section 2. Although we will be mostly interested in the geometry of bivector fields, our discussion is presented at the general level of (almost) Dirac structures, as they provide the framework in which gauge transformations are most natural. We refer the reader who is unfamiliar with this material to the introductory exposition in section 2 of [9].

3.1 Dirac and almost Dirac structures

A Dirac structure on a manifold $P$ is a subbundle $L$ of the Whitney sum $TP \oplus T^*P$ such that
Examples might not be integrable in the almost Dirac case. Let \( \Omega \) be a closed 2-form and \( \pi \) be a Poisson bivector field, and consider the maps \( \Omega^\flat : TP \to T^*P \) given by \( \Omega^\flat(X) = \Omega(\cdot, X) = -i_X \Omega = \Omega(X, \cdot) \) and\( \pi^\sharp : T^*P \to TP \) as in Definition 2. Then

\[
L_\Omega := graph(\Omega^\flat) = \{(X, \alpha) \in TP \oplus T^*P : i_X \Omega = -\alpha\}
\]

and

\[
L_\pi := graph(\pi^\sharp) = \{(X, \alpha) \in TP \oplus T^*P : \pi^\sharp(\alpha) = X\}
\]

are Dirac structures. Note that \( pr_1 \) identifies \( L_\Omega \) with \( TP \) as Lie algebroids. Similarly, \( L_\pi \) can be naturally identified with \( T^*P \), and the Lie algebroid structure induced on \( T^*P \) by \( L_\pi \) has anchor \( \pi^\sharp : T^*P \to TP \), and bracket

\[
[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha, \beta)) = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - i_{\pi^\sharp(\beta)} d\alpha.
\]

This bracket is uniquely characterized by \([df, dg]_\pi = d\{f, g\}\) and the Leibniz identity.

If a subbundle \( L \) of \( TP \oplus T^*P \) satisfies (i) above, but not necessarily (ii), then \( L \) is called an almost Dirac structure. Condition (ii) is called the integrability condition. We say that \( L \) is a regular almost Dirac structure when the distribution \( pr_1(L) \) on \( P \) has constant rank. Notice that this distribution might not be integrable in the almost Dirac case.

Examples If \( \Omega \) is an arbitrary 2-form or \( \pi \) an arbitrary bivector field, then their graphs \( L_\Omega \) and \( L_\pi \) are almost Dirac structures. The failure of the integrability with respect to the Courant bracket of \( L_\Omega \) and \( L_\pi \) is measured by \( d\Omega \) and \( \frac{1}{2}[\pi, \pi] \), respectively. For \( L_\pi = graph(\pi^\sharp) \), the distribution \( \pi^\sharp(T^*P) = pr_1(L_\pi) \) is generally non-integrable. If it has constant rank we call the almost Poisson
structure regular. Note that the bracket $[\cdot,\cdot]_\pi$ defined as in (3.1) is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Leibniz identity. However, in general, $\pi^2$ does not necessarily preserve the bracket; instead, (see e.g. [8]),

$$
\pi^2(\alpha,\beta) = [\pi^2(\alpha),\pi^2(\beta)] - \frac{1}{2}i_{\alpha\wedge\beta}[\pi,\pi], \quad \text{for } \alpha,\beta \in \Omega^1(P). 
$$

Note that an almost Dirac structure $L$ on $P$ is of the form $L = \text{graph}(\pi^2)$ for a bivector $\pi$ if and only if

$$
TP \cap L = \{0\},
$$

and $L$ is of the form $L_\Omega = \text{graph}(\Omega^2)$ for a 2-form $\Omega$ if and only if $T^*P \cap L = \{0\}$, see [13]. Another example of an almost Dirac structure that will be very useful for our purposes is given by $L \subset TP \oplus T^*P$ defined as

$$
L := \{(X,\alpha) \in TP \oplus T^*P : X \in F, \ i_X\Omega|_F = -\alpha|_F\},
$$

where $F \subset TP$ is a subbundle, $\Omega$ is a 2-form on $P$ and $\cdot|_F$ denotes the point-wise restriction to $F$. If the subbundle $F$ is an integrable distribution and $\Omega$ is closed, then $L$ is a Dirac structure.

**Proposition 3.1.** The following statements hold:

(i) There is a one-to-one correspondence between regular almost Dirac structures $L \subset TP \oplus T^*P$ and pairs $(F,\Omega_F)$, where $F$ is a regular distribution on $P$ and $\Omega_F \in \Gamma(\wedge^2 F^*)$.

(ii) Let $F \subset TP$ be a regular distribution on $P$. Given a section $\Omega_F \in \Gamma(\wedge^2 F^*)$, there exists a 2-form $\Omega$ on $P$ such that $\Omega|_F = \Omega_F$.

**Proof.** (i) Let $L \subset TP \oplus T^*P$ be a regular almost Dirac structure with distribution $F := \text{pr}_1(L) \subset TP$ on $P$ (not necessarily integrable). Consider the section $\Omega_F$ in $\Gamma(\wedge^2 F^*)$ given, at each $x \in P$, by

$$
\Omega_F(x)(X(x),Y(x)) = -\alpha(x)(Y(x)), \quad \text{for } X,Y \in \Gamma(F) \text{ such that } (X(x),\alpha(x)) \in L_x.
$$

It is a straightforward computation to see that $\Omega_F$ is well defined, i.e., it is independent of the choice of $\alpha$. Conversely, given a regular distribution $F$ on $P$ and $\Omega_F \in \Gamma(\wedge^2 F^*)$, we may define the subbundle $L \subset TP \oplus T^*P$ as the pairs $(X,\alpha)$ such that $X \in F$ and $i_X\Omega_F = -\alpha|_F$.

(ii) Let $W \subset TP$ be a regular smooth distribution such that it is a complement of $F$ on $P$, i.e., $T_xP = F_x \oplus W_x$ for each $x \in P$ (e.g., $W_x$ can be chosen to be the orthogonal complement of $F_x$ with respect to a Riemmanian metric). The 2-form $\Omega$ on $P$ can be defined by

$$
\Omega(X,Y) = \Omega_F(X_F,Y_F),
$$

for $X,Y \in \mathfrak{X}(P)$ such that $X = X_F + X_W$ and $Y = Y_F + Y_W$, where $X_F,Y_F \in \Gamma(F)$ and $X_W,Y_W \in \Gamma(W)$. Differentiability of $\Omega$ follows from its definition and the smoothness of $F$ and $W$.

**Corollary 3.2.** Given a regular almost Dirac structure $L$, there exists a 2-form $\Omega$ on $P$ and a regular distribution $F \subset TP$ such that $L$ is written in the form (3.21).

**Remark 3.3.** Note that the 2-form $\Omega \in \Omega^2(P)$ is not uniquely defined and, in general, there is no canonical choice for it.
Given a subbundle $F \subseteq TP$, we say that a section $\Omega_F$ in $\Gamma(\Lambda^2 F^*)$ is nondegenerate if it is nondegenerate as a bilinear form on $F$ at each point. It follows from (3.21) that $\text{Ker}(\Omega_F) = L \cap TP$, and as a consequence of condition (3.20) we obtain

**Corollary 3.4.** Let $L$ be a regular almost Dirac structure and $(F, \Omega_F)$ the pair associated to it in the sense of Proposition 3.1. Then $\Omega_F$ is nondegenerate if and only if $L$ is the graph of a bivector field $\pi$. Explicitly, the relation between $(F, \Omega_F)$ and $\pi$ is

$$\pi^\sharp(\alpha) = -X \quad \text{if and only if} \quad i_X \Omega_F = \alpha|_F,$$

where $X \in \Gamma(F)$ and $\alpha \in \Omega^1(P)$.

Following notation of Definition 2, if $\{\cdot, \cdot\}$ is the bracket associated to the bivector field $\pi$ in the above Corollary, then $\{f, g\} = \Omega_F(X_f, X_g)$, for all $f, g \in C^\infty(P)$.

### 3.2 Twisted Poisson and twisted Dirac structures

Poisson structures may be viewed as encoding integrability in two levels: first, the characteristic distribution $\pi^\sharp(T^*P) \subseteq TP$ is integrable, i.e., tangent to leaves; second each leaf carries a nondegenerate 2-form that is closed (and this leads to the Jacobi identity). Twisted Poisson structures are special types of almost Poisson structures that retain the integrability of $\pi^\sharp(T^*P)$ but allow the leafwise 2-form to be non closed. These objects turn out to be related to nonholonomic systems. We start with the more general notion of twisted Dirac structures.

Consider a closed 3-form $\phi$ on $P$, and define the $\phi$-twisted Courant bracket $[\cdot, \cdot]_\phi$ as follows:

$$[[X, \alpha], (Y, \beta)]_\phi = ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + i_X \gamma(Y, \phi),$$

for $(X, \alpha)$ and $(Y, \beta)$ in $\Gamma(TP \oplus T^*P)$. Now, a subbundle $L$ of $TP \oplus T^*P$ is a $\phi$-twisted Dirac structure $[\cdot, \cdot]_\phi$ if $L$ is maximal isotropic with respect to $\langle \cdot, \cdot \rangle$ and the integrability condition

$$[[\Gamma(L), \Gamma(L)]_\phi \subseteq \Gamma(L)$$

is satisfied.

As in the ordinary case, a twisted Dirac structure $L$ on $P$ induces a Lie algebroid on $L$ given by the anchor map $pr_1|_L$ and the bracket $[\cdot, \cdot]_{\phi|\Gamma(L)}$. Therefore $pr_1(L)$ is an integrable distribution on $P$.

**Examples** If $\Omega$ is any 2-form on $P$, then $L_\Omega = \text{graph}(\Omega^\sharp)$ is $(d\Omega)$-twisted Dirac. One may check that a bivector field $\pi$ on $P$ such that $L_\pi = \text{graph}(\pi^\sharp)$ is $\phi$-twisted Dirac verifies (see [33])

$$\frac{1}{2} [\pi, \pi] = \pi^\sharp(\phi).$$

The following result gives more examples:

**Theorem 3.5.** Let $L \subset TP \oplus T^*P$ be a regular almost Dirac structure such that $pr_1(L) \subset TP$ is an integrable distribution on $P$. Then, there exists an exact 3-form $\phi$ with respect to which $L$ is a $\phi$-twisted Dirac structure.
Proof. Let $F := pr_1(L) \subset TP$ and $\Omega_F$ be the section in $\Gamma(\bigwedge^2 F^*)$ associated to $L$ given by Proposition 3.1. Since $F$ is integrable, $\Omega_F$ defines a 2-form $\Omega_O$ on each leaf $O$ where $F_x = T_xO$ at each $x \in P$. By Corollary 3.2 there exists a 2-form $\Omega$ on $P$ such that $\iota^\ast_O \Omega = \Omega_O$ where $\iota_O : O \hookrightarrow P$ is the inclusion.

We assert that $L$ is a $(d\Omega)$-twisted Dirac structure. In fact, for $(X, \alpha)$ and $(Y, \beta)$ in $\Gamma(L)$, 

$[\{X, \alpha\}, \{Y, \beta\}]_{(d\Omega)} = \{[X,Y], \mathcal{L}_X \beta - i_Y d\alpha + i_{X\wedge Y} d\Omega\} \in \Gamma(L)$

if and only if

$i_{[X,Y]} \Omega|_F = -(\mathcal{L}_X \beta - i_Y d\alpha + i_{X\wedge Y} d\Omega)|_F$.

Since $F$ is an integrable distribution we obtain that,

$-(\mathcal{L}_X \beta - i_Y d\alpha + i_{X\wedge Y} d\Omega)|_F = \mathcal{L}_X i_Y \Omega|_F - i_Y d(X_i \Omega|_F) - i_{X\wedge Y} d\Omega|_F = i_{[X,Y]} \Omega|_F$.

which completes the proof.

\[\square\]

Remark 3.6.

(i) Note that if $L$ is a $\phi$-twisted Dirac structure then $L$ is also twisted with respect to any closed 3-form $\phi'$ such that $(\phi - \phi')$ vanishes on the leaves.

(ii) There is no canonical choice for the 3-form $\phi$ given in Theorem 3.5.

Twisted Poisson bivectors.

Bivector fields $\pi$ such that $L_\pi = \text{graph}(\pi^2)$ is a $\phi$-twisted Dirac structure are called $\phi$-twisted Poisson bivectors \[^{26, 33}\] i.e., $\pi$ verifies condition (3.23). We are especially interested in these kind of bivector fields since, as we will see, they appear naturally in the examples of nonholonomic systems introduced in Section 2. If $\{\cdot, \cdot\}_\pi$ is the bracket given by the $\phi$-twisted Poisson structure $\pi$, then relation (2.12) becomes

$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + \phi(X_f, X_g, X_h) = 0,$

for $f, g, h \in C^\infty(P)$ and $X_f = \{\cdot, f\}$. So the failure of the Jacobi identity is controlled by the closed 3-form $\phi$.

Example If $\pi^2 : T^*P \to TP$ is an isomorphism, then $\pi$ is $\phi$-twisted and $\phi = d\Omega$ where $\Omega$ is the nondegenerate 2-form associated with $\pi$, i.e., $\Omega^\pi \circ \pi^2 = \text{Id}$ where as usual $\Omega^\pi(X) = -i_X \Omega$. Other, less trivial examples, will be presented ahead in Corollary 3.7.

Let $\pi$ be a $\phi$-twisted Poisson structure on the manifold $P$ and $[\cdot, \cdot]_\pi$ be the bracket on $T^*P$ given by (3.1). Note that $\pi^2$ does not preserve this bracket. However, using (3.19) and (3.23) we obtain

$[\pi^2(\alpha), \pi^2(\beta)] = \pi^2 \left([\alpha, \beta]_\pi + i_{\pi^2(\alpha) \wedge \pi^2(\beta)} \phi\right),$
for 1-forms $\alpha, \beta$ on $P$. The $\phi$-twisted Courant bracket induces a modification of the bracket (3.1) via the identification of $T^*P$ and $L_\pi$,

$$[\alpha, \beta]_{\phi} = \mathcal{L}_{\pi^2(\alpha)} \beta - i_{\pi^1(\beta)} d\alpha + i_{\pi^2(\alpha) \wedge \pi^2(\beta)} \phi,$$

such that $(T^*P, [\cdot, \cdot]_{\phi}, \pi^\sharp)$ is a Lie algebroid [33] (see also [8]). The characteristic distribution $\pi^\sharp(T^*P)$ defines an integrable distribution on $P$ (that may be singular). Each leaf $O$ of the corresponding foliation of $P$ is endowed with a non-degenerate 2-form $\Omega_O$ that is not necessarily closed. If $\pi$ is $\phi$-twisted, then $d\Omega_O = \iota_O^* \phi$, where $\iota_O : O \hookrightarrow P$ is the inclusion.

Important examples of twisted Poisson structures are contained in the following Corollary of Theorem 3.5:

**Corollary 3.7.** Let $\pi$ be a bivector field on $P$ with an integrable regular characteristic distribution. Then, there exists an exact 3-form $\phi$ on $P$ with respect to which $\pi$ is $\phi$-twisted.

**Remark 3.8.** Mechanical Example. It is shown in [20] that the (semi-direct) product reduction of the Veselova system yields a regular conformally Poisson bracket on the reduced space. It follows that its characteristic distribution is integrable and thus, by Corollary 3.7, it is also twisted-Poisson. This is a first example of a nonholonomic system whose reduced equations are formulated in terms of a twisted-Poisson bracket. Other examples (related to the motion of a rigid body with generalized rolling constraints) are made explicit in Section 5.3.

**Remark 3.9.** An interesting question, that remains to be answered, is to give a characterization of almost Poisson brackets possessing an integrable characteristic distribution that is non-regular.

### Regular conformally Poisson bivectors.

An interesting class of almost Poisson structures admitting an integrable characteristic distribution is given by conformally Poisson structures. Recall from Section 2 that they are bivector fields $\pi$ for which exists a strictly positive function $\varphi \in C^\infty(P)$, such that $\varphi \pi$ is Poisson. A conformally Poisson manifold $(P, \pi)$ is the disjoint union of conformally symplectic leaves.

Note that this property is stronger than asking for $(P, \pi)$ to be a Jacobi manifold since a conformal factor implies the global existence of a function such that $\varphi \pi$ is Poisson, while in Jacobi manifolds the factor $\varphi$ may be only locally defined.

From Corollary 3.7 we observe that any regular bivector admitting a conformal factor is also a twisted Poisson bivector. The following proposition explains the relation between these two properties.

**Proposition 3.10.** Let $\pi$ be a regular conformally Poisson bivector field on $P$ with conformal factor $\varphi$. Let $\Omega \in \Omega^2(P)$ be as in Corollary 3.2. Then any closed 3-form verifying (3.23) for $\pi$ coincides with $\frac{1}{\varphi} d\varphi \wedge \Omega$ on the leaves.

**Proof.** Since $\pi$ admits a conformal factor $\varphi \in C^\infty(P)$, then $[\pi, \pi] = \frac{2}{\varphi} X_\varphi \wedge \pi$. On the other hand, if $\Omega$ is the 2-form on $P$ associated to $\pi$ given by Corollary 3.2, then for $g_1, g_2 \in C^\infty(P)$ we have $\Omega(X_{g_1}, X_{g_2}) = \pi(dg_1, dg_2)$. Thus, for $g_1, g_2, g_3 \in C^\infty(P)$ we obtain

$$\frac{1}{2} [\pi, \pi](dg_1, dg_2, dg_3) = \frac{1}{\varphi} X_\varphi \wedge \pi(dg_1, dg_2, dg_3) = -\frac{1}{\varphi} d\varphi \wedge \Omega(X_{g_1}, X_{g_2}, X_{g_3}).$$
Then we conclude that any closed 3-form $\phi$ satisfying (3.23) coincides with $(\frac{1}{2}d\varphi \wedge \Omega)$ on the leaves. \[\square\]

Let $(P, \pi)$ be a regular conformally Poisson manifold. A 2-form $\Omega$ on $P$ satisfying the conditions of Corollary 3.2 verifies that $\phi^*\Omega$ is conformally symplectic on each leaf $O$. However, $\Omega$ may not necessarily be conformally closed.

### 3.3 Gauge transformations.

In this section we will consider a natural action of the abelian group of 2-forms on the almost Dirac structures on $P$. This action is given by gauge transformations of almost Dirac structures by 2-forms, and it was introduced in [33].

More precisely, consider an almost Dirac structure $L$ in $TP \oplus T^*P$. A gauge transformation by the 2-form $B$ is a map $\tau_B : L \to TP \oplus T^*P$, given by $\tau_B((X,\alpha)) = (X,\alpha + i_X B)$ for $(X,\alpha) \in L$. The subbundle $\tau_B(L)$ of $TP \oplus T^*P$ given by

$$\tau_B(L) = \{(X,\alpha + i_X B) : (X,\alpha) \in L\}$$

is an almost Dirac structure. If the 2-form $B$ is closed and $L$ is Dirac then $\tau_B(L)$ is again Dirac. Thus, the 3-form $dB$ is what determines the integrability with respect to the Courant bracket. It is a direct computation to see that if $L$ is a $\phi$-twisted Dirac structure, then the gauge transformation of $L$ by the 2-form $B$ is $(\phi - dB)$-twisted Dirac (see e.g. [33]).

If $L_1$ and $L_2$ are almost Dirac structures on $P$ and there exists a 2-form $B$ on $P$ such that $\tau_B(L_1) = \Omega$ then we say that $L_1$ and $L_2$ are gauge equivalent or gauge related.

Note that a gauge transformation does not modify the distribution $pr_1(L)$. So, for a Dirac structure $L$, the foliation associated to $L$ will be the same as the one associated to $\tau_B(L)$. However, the presymplectic form on each leaf is modified by the pullback of $B$ to the leaf. If $L$ is a regular almost Dirac structure determined by the pair $(F,\Omega_F)$ in the sense of item (i) of Proposition 3.1, a gauge transformation by the 2-form $B$ corresponds to the operation:

$$\tau_B : (F,\Omega_F) \to (F,\Omega_F - B|_F). \tag{3.24}$$

**Theorem 3.11.** Any two regular almost Dirac structures $L_1$ and $L_2$ are gauge related if and only if $pr_1(L_1) = pr_1(L_2)$.

**Proof.** It remains to prove the “only if” part of the statement. Let us denote $F := pr_1(L_1) = pr_1(L_2)$ and let $\Omega^1_F$ and $\Omega^2_F$ be the 2-sections associated to $L_1$ and $L_2$ respectively (Proposition 3.1 (i)). Define the section $B_F \in \Gamma(\Lambda^2(F^*))$ by $B_F := \Omega^1_F - \Omega^2_F$ and let $B \in \Omega^2(P)$ such that $B|_P = B_F$ (Prop. 3.1 (ii)). We claim that $\tau_B L_1 = L_2$. In fact, if $(F,\Omega^B_F)$ is the pair associated to the almost Dirac structure $\tau_B L_1$, then by equation (3.21),

$$\Omega^B_F = \Omega^1_F - B_F = \Omega^2_F.$$

Since the sections associated to $\tau_B L_1$ and $L_2$ coincide, by Proposition 3.1 (i) we conclude that $\tau_B L_1 = L_2$ which means that $L_1$ and $L_2$ are gauge related. \[\square\]
We are especially interested in gauge transformations of almost Poisson structures. Consider the almost Poisson manifold \((P, \pi)\) and a 2-form \(B\) on \(P\). Then, the gauge transformation of \(L\pi := \text{graph}(\pi^\sharp)\) is \(\tau_B(L\pi) = \{ (X, \alpha + i_X B) \in TP \oplus T^*P \ : \ X = \pi^\sharp(\alpha) \}\) which does not necessarily correspond to the graph of a new bivector \(\pi^B\). A necessary and sufficient condition for this to happen is that

\[
\tau_B(L\pi) \cap TP = \{0\},
\]

which is equivalent to the fact that the endomorphism \(\text{Id} + B^\flat \circ \pi^\sharp : T^*P \to T^*P\) is invertible \cite{33}. Indeed, if such a bivector field \(\pi^B\) exists, then, in view of (3.3), for any 1-form \(\alpha\) on \(P\) we have

\[
\tau_B\left( (\pi^\sharp(\alpha) , \alpha) \right) = (\pi^\sharp(\alpha) , \alpha + i_{\pi^\sharp(\alpha)} B) = \left( (\pi^B)^\sharp \left( \alpha + i_{\pi^\sharp(\alpha)} B \right) , \alpha + i_{\pi^\sharp(\alpha)} B \right).
\]

Thus, \(\pi^B\) is characterized by the condition

\[
(\pi^B)^\sharp \left( \alpha + i_{\pi^\sharp(\alpha)} B \right) = \pi^\sharp(\alpha), \tag{3.25}
\]

and we can write

\[
(\pi^B)^\sharp = \pi^\sharp \circ (\text{Id} - B^\flat \circ \pi^\sharp)^{-1}. \tag{3.26}
\]

In particular, if \(\pi\) and \(\pi^B\) are non-degenerate bivector fields, equation (3.26) is equivalent to

\[
((\pi^B)^\sharp)^{-1} = (\pi^\sharp)^{-1} - B^\flat.
\]

Therefore, any two 2-forms on a manifold are gauge related. This is not necessarily the case with bivector fields. A necessary condition is that their characteristic distributions coincide. In view of Theorem 3.11 this condition is also sufficient if such distributions are regular.

Although gauge related bivectors have the same characteristic distribution, their Schouten brackets \([\pi, \pi]\) and \([\pi^B, \pi^B]\) may not coincide. The following proposition makes this precise.

**Proposition 3.12** \cite{33}. If a \(\phi\)-twisted Poisson bivector \(\pi\) is gauge related with another bivector \(\pi^B\) via the 2-form \(B\), then \(\pi^B\) is \((\phi - dB)\)-twisted. That is,

\[
\frac{1}{2} [\pi^B, \pi^B] = (\pi^B)^\sharp (\phi - dB).
\]

### 4 Applications to Nonholonomic Systems and Hamiltonization

In what follows, we will analyze the geometry of nonholonomic systems in the framework presented in the previous section. We introduce gauge transformations of the bracket describing the nonholonomic dynamics in order to study the process of Hamiltonization.
4.1 Nonholonomic systems

A nonholonomic system consists of an $n$-dimensional configuration manifold $Q$ with local coordinates $q \in U \subset \mathbb{R}^n$, a Lagrangian $L : TQ \to \mathbb{R}$ of the form $L(q, \dot{q}) = \frac{1}{2}G(q)(\dot{q}, \dot{q}) - V(q)$, where $G$ is a kinetic energy metric on $Q$ and $V : Q \to \mathbb{R}$ is a potential, and a regular non-integrable distribution $\mathcal{D} \subset TQ$ that describes the kinematic nonholonomic constraints. In coordinates, the distribution $\mathcal{D}$ is defined by the equation

$$\epsilon(q) \dot{q} = 0,$$

where $\epsilon(q)$ is a $k \times n$ matrix of constant rank $k$ where $k < n$ is the number of constraints. The entries of $\epsilon(q)$ are the components of the $\mathbb{R}^k$-valued constraint 1-form on $Q$, $\epsilon := \epsilon(q) \, dq$.

The dynamics of the system are governed by the Lagrange-D’Alembert principle. This principle states that the forces of constraint annihilate any virtual displacement, so they perform no work during the motion. The equations of motion take the form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \mu^T \epsilon(q).$$

(4.28)

Here $\mu : TQ \to \mathbb{R}^k$ is an $\mathbb{R}^k$-valued function whose entries are referred to as Lagrange multipliers. Under our assumptions, it is uniquely determined by the condition that the constraints (4.27) are satisfied. The equations (4.28) together with the constraints (4.27) define a vector field $Y_{\text{nh}}^D$ on $\mathcal{D}$ whose integral curves describe the motion of the nonholonomic system. A short calculation shows that along the flow of $Y_{\text{nh}}^D$, the energy function $E_L := \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L$, is conserved.

The above equations of motion can be written as a first order system on the cotangent bundle $T^*Q$ via the standard Legendre transform, $\text{Leg} : TQ \to T^*Q$, that defines canonical coordinates $(q, p)$ on $T^*Q$ by the rule $\text{Leg} : (q, \dot{q}) \mapsto (q, p = \partial L/\partial \dot{q})$. The Legendre transform is a global diffeomorphism by our assumption that $G$ is a metric.

The Hamiltonian function, $H : T^*Q \to \mathbb{R}$, is defined in the usual way $H := E_L \circ \text{Leg}^{-1}$. The equations of motion (4.28) are shown to be equivalent to

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + \mu^T \epsilon(q),$$

(4.29)

and the constraint equations (4.27) become

$$\epsilon(q) \frac{\partial H}{\partial p} = 0.$$

(4.30)

The above equation defines the constraint submanifold $\mathcal{M} = \text{Leg}(\mathcal{D}) \subset T^*Q$. Since the Legendre transform is linear on the fibers, $\mathcal{M}$ is a vector sub-bundle of $T^*Q$ that for each $q \in Q$ specifies an $n - k$ vector subspace of $T_q^*Q$.

Equations (4.29) together with (4.30) define the vector field $X_{\text{nh}}$ on $\mathcal{M}$, that describes the motion of our nonholonomic system in the Hamiltonian side and is the push forward of the vector field $Y_{\text{nh}}^D$ by the Legendre transform. The vector field $X_{\text{nh}}$ is defined uniquely in an intrinsic way by the equation

$$\iota_{X_{\text{nh}}} \iota^*\Omega_Q = \iota^*(dH + \mu^T \tau^* \epsilon),$$

(4.31)
where \(Ω_Q\) is the canonical symplectic form on \(T^*Q\), \(ι : M \hookrightarrow T^*Q\) is the inclusion and \(τ : T^*Q \to Q\) is the canonical projection. The constraints (4.30) and their derivatives are intrinsically written as 

\[
X_{\text{nh}} \in C := TM \cap F,
\]

(4.32)

where \(F\) is the distribution on \(T^*Q\) defined as \(F := \{v \in T(T^*Q) : \langle τ^*e, v \rangle = 0\}\). Denote by \(Ω_M\) the pull-back of \(Ω_Q\) to \(M\), i.e. \(Ω_M := τ^*Ω_Q\). The following proposition is of great importance for our setup of the equations of motion as an almost Hamiltonian system.

**Proposition 4.1** (**[38]** [2]). The distribution \(C\) on \(M\) defined by (4.32) is regular, non-integrable, and the point-wise restriction of \(Ω_M\) to \(C\), denoted by \(Ω_C\), is non-degenerate.

The non-integrability of \(C\) is a direct consequence of the non-integrability of \(D\). One shows that the rank of \(C\) is \(2(n-k)\) and that along \(M\) we have the symplectic decomposition

\[
T_M(T^*Q) = C \oplus C^{Ω_Q},
\]

(4.33)

where \(C^{Ω_Q}\) denotes the symplectic orthogonal complement to \(C\).

Since \(τ^*e\) vanishes on \(C\), by restricting (4.31) to \(C\) and denoting \(H_M := i^*H \in C^∞(M)\), the equations of motion can be written in the appealing format

\[
iX_{\text{nh}} Ω_C = (dH_M)_C,
\]

(4.34)

where \((dH_M)_C\) is the point-wise restriction of \(dH_M\) to \(C\). The above equation uniquely defines the vector field \(X_{\text{nh}}\) and is central in our treatment; with this in mind, we collect the data of the nonholonomic system in the triple \((M, Ω_C, H_M)\).

Even though (4.34) defines the vector field \(X_{\text{nh}}\) uniquely, and resembles a classical Hamiltonian system, notice that since the distribution \(C\) is non-integrable, then \(Ω_C\) is a section in \(\bigwedge^2 C^* \to M\) (not a 2-form).

Let \((M, Ω_C, H_M)\) be a nonholonomic system. For every \(f ∈ C^∞(M)\), let \(X_f\) denote the unique vector field on \(M\) with values in \(C\) defined by the equation

\[
iX_f Ω_C = (df)_C,
\]

(4.35)

where \((df)_C\) denotes the point-wise restriction of \(df\) to \(C\). The vector field \(X_f\) defined by equation (4.35) is called the the (almost) Hamiltonian vector field associated to \(f\).

Since \(Ω_C\) is nondegenerate, by Corollary 3.3 there is a unique bivector field \(π_{\text{nh}}\) on \(M\) associated to the pair \((C, Ω_C)\), that is \(π_{\text{nh}}^2(α) = -X\) if and only if \(i_X Ω_C = α|_C\). On exact forms we have, for \(f ∈ C^∞(M)\),

\[
iX_f Ω_C = (df)_C \quad \text{if and only if} \quad π_{\text{nh}}^*(df) = -X_f,
\]

(4.36)

which is consistent with notation of Definition 2. The bracket \{\cdot, \cdot\}_\text{nh} on functions on \(M\) associated to the bivector \(π_{\text{nh}}\) describes the dynamics in the sense that

\[
X_{\text{nh}}(f)(m) = X_{H_M}(f)(m) = \{f, H_M\}_\text{nh}(m) \quad \text{for all} \ f ∈ C^∞(M).
\]

It follows from (4.36) that the characteristic distribution of the bracket \{\cdot, \cdot\}_\text{nh} is \(C\). Since \(C\) is non-integrable then \{\cdot, \cdot\}_\text{nh} is an almost Poisson bracket that does not satisfy the Jacobi identity.

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Remark 4.2. If the constraint distribution \( \mathcal{D} \) were integrable, the same would be true for the distribution \( \mathcal{C} \). Let \( \mathcal{N} \subset \mathcal{M} \) be a leaf of the corresponding (regular) foliation of \( \mathcal{M} \) (i.e. \( \mathcal{C}_x = T_x \mathcal{N} \) for all \( x \in \mathcal{N} \)). In view of (4.33) the submanifold \( \mathcal{N} \) is symplectic. Therefore, in this case, our construction of \( \{\cdot,\cdot\}_{\text{nh}} \) coincides with the usual construction of the Dirac bracket on each leaf \( \mathcal{N} \) of the foliation of \( \mathcal{M} \) (see [14, 13]). Hence, in this case, the Jacobi identity holds.

Inspired by equation (3.21) and our data, it is natural to define the almost Dirac structure \( L_{\text{nh}} \) on \( \mathcal{M} \) by

\[
L_{\text{nh}} := \{(X,\alpha) \in T\mathcal{M} \oplus T^*\mathcal{M} : X \in \mathcal{C}, \ i_X\Omega_{\mathcal{M}|C} = -\alpha|_C\},
\]

as considered (up to a minus sign) in [36, 37], and subsequently in [25].

Proposition 4.3. Let \( \pi_{\text{nh}} \) be the bivector field defined in (4.36) and \( \{\cdot,\cdot\}_{\text{nh}} \) the corresponding bracket. The following statements hold:

(i) The almost Dirac structures \( L_{\pi_{\text{nh}}} := \text{graph}(\pi^\sharp_{\text{nh}}) \) and \( L_{\text{nh}} \) given in (4.37) coincide.

(ii) The almost Poisson bracket \( \{\cdot,\cdot\}_{\text{nh}} \) coincides with the classical almost Poisson bracket for nonholonomic systems defined in [34, 29].

Proof. (i) It is immediate since both almost Dirac structure are defined by the same pair \( (\mathcal{C},\Omega_{\mathcal{C}}) \).

(ii) The bracket on \( \mathcal{M} \) for nonholonomic systems introduced in [34, 29] was shown in [10] to be given by

\[
\{f,g\} = \Omega_{\mathcal{M}}(\mathcal{P}Y\bar{f},\mathcal{P}Y\bar{g}), \quad \text{for} \ f, g \in C^\infty(\mathcal{M}),
\]

where \( \mathcal{P} : T\mathcal{M}(T^*\mathcal{Q}) \to \mathcal{C} \) is the projector associated to the symplectic decomposition (4.33), and \( Y\bar{f} \) the free Hamiltonian vector field on the symplectic manifold \( (T^*\mathcal{Q},\Omega_{\mathcal{Q}}) \) defined by \( i_{Y\bar{f}}\Omega_{\mathcal{Q}} = d\bar{f} \), where \( \bar{f} \) is an arbitrary smooth extension of \( f \) to \( T^*\mathcal{Q} \).

It is easy to check that along \( \mathcal{M} \) one has \( i_{\mathcal{P}Y\bar{f}}\Omega_{\mathcal{C}} = (df)|_C \), so \( \mathcal{P}Y\bar{f} \) coincides with the almost Hamiltonian vector field \( X_f \) defined by equation (4.35). Therefore, for any \( f, g \in C^\infty(\mathcal{M}) \),

\[
\{f,g\} = \Omega_{\mathcal{M}}(\mathcal{P}Y\bar{f},\mathcal{P}Y\bar{g}) = \Omega_{\mathcal{C}}(X_f,X_g) = -df(\pi^\sharp_{\text{nh}}(dg)) = \{f,g\}_{\text{nh}}.
\]

The second item of the above proposition should not be surprising since the expression (4.38) is the nonholonomic version of the Dirac bracket (see discussion in [23, 10]). Hence, its description naturally falls in the ambit of almost Dirac structures as described above. As a consequence of the above proposition, we can equivalently describe our nonholonomic system with the triple \( (\mathcal{M},\pi_{\text{nh}},\mathcal{H}_M) \).

Definition 3. Let \( (\mathcal{M},\pi_{\text{nh}},\mathcal{H}_M) \) be a nonholonomic system.

1. The bivector field \( \pi_{\text{nh}} \) on \( \mathcal{M} \) given by (4.36) is called the nonholonomic bivector field and the bracket \( \{\cdot,\cdot\}_{\text{nh}} \) is called the nonholonomic bracket.

2. We say that an almost Dirac structure \( L \) describes the dynamics of the nonholonomic system if the pair \( (-X_{\text{nh}},d\mathcal{H}_M) \in \Gamma(L) \).
4.2 Gauge transformations of the nonholonomic bracket

The main idea of using gauge transformations in our setting is that it opens the possibility to modify the geometric structure on $\mathcal{M}$ that describes the dynamics.

Consider the nonholonomic system $(\mathcal{M}, \pi_{nh}, \mathcal{H}_\mathcal{M})$ and continue to denote $L_{\pi_{nh}} = graph(\pi_{nh}^\flat)$. The gauge transformation of $\pi_{nh}$ associated to a 2-form $B$ on $\mathcal{M}$ gives

$$\tau_B(L_{\pi_{nh}}) = \{(X, \alpha + i_X B) \in T\mathcal{M} \oplus T^*\mathcal{M} : \pi_{nh}^\flat(\alpha) = X\}. \quad (4.39)$$

First of all we are interested in knowing when the pair $(-X_{nh}, dH_\mathcal{M})$ is a section of $\tau_B(L_{\pi_{nh}})$. On the other hand, we would also like to know whether the almost Dirac structure $\tau_B(L_{\pi_{nh}})$ corresponds to the graph of a bivector field or not.

If the 2-form $B$ on $\mathcal{M}$ verifies $i_{X_{nh}} B = 0$, then from equation (4.39) we see that the pair $(-X_{nh}, dH_\mathcal{M})$ belongs to $\Gamma(\tau_B(L_{\pi_{nh}}))$. Moreover, in view of (3.24), the gauge transformation of $\pi_{nh}$ by the 2-form $B$ has the form

$$\tau_B(L_{\pi_{nh}}) = \{(X, \alpha) \in T\mathcal{M} \oplus T^*\mathcal{M} : X \in \mathcal{C}, i_X(\Omega_\mathcal{M} - B)|_{\mathcal{C}} = -\alpha|_{\mathcal{C}}\}. \quad (4.40)$$

Thus the equations of motion (4.34) are equivalently written as

$$i_{X_{nh}} (\Omega_\mathcal{C} - B_\mathcal{C}) = (dH_\mathcal{M})_\mathcal{C},$$

where $B_\mathcal{C}$ is the point-wise restriction of $B$ to $\mathcal{C}$.

Therefore, as a particular case of Corollary 3.4 we observe that if the section $\Omega_\mathcal{C} - B_\mathcal{C}$ is non-degenerate then the gauge transformation of $\pi_{nh}$ associated to the 2-form $B$ is again a bivector field $\pi_{nh}^B$. It follows from (3.26) that the non-degeneracy of $\Omega_\mathcal{M} - B$ on $\mathcal{C}$ is equivalent to the invertibility of the endomorphism $(\text{Id} - B^\flat \circ \pi_{nh}^\sharp)$ on $T^*\mathcal{M}$.

Definition 4. Let $(P, \pi)$ be an almost Poisson manifold with a distinguished Hamiltonian function $H \in C^\infty(P)$. Given a 2-form $B$ on $P$, the gauge transformation of $\pi$ associated to the 2-form $B$ is said to be a dynamical gauge transformation if

(i) $i_{X_H} B = 0$, where $X_H$ is the (almost) Hamiltonian vector field associated to $H$ and

(ii) $\tau_B(graph(\pi^\sharp))$ corresponds to the graph of a new bivector $\pi^B$, i.e., the endomorphism $(\text{Id} - B^\flat \circ \pi^\sharp)$ on $T^*P$ is invertible.

Of course we are interested in dynamical gauge transformations of the nonholonomic bracket where the distinguished Hamiltonian function is $\mathcal{H}_\mathcal{M}$ and the corresponding (almost) Hamiltonian vector field is $X_{nh}$.

Note that if $\pi$ is regular, by equation (3.24), the gauge transformation defined by $B$ is determined by the restriction $B_F$ of $B$ to $F$ where $F := \pi^\sharp(T^*P)$. Then condition (i) of the above definition is equivalent to $i_{X_H} B_F = 0$. 

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Remark 4.4. The definition of an affine almost Poisson bracket for a nonholonomic system made in [21] corresponds to a dynamical gauge transformation of the nonholonomic bracket by a 2-form $B = -i^*\Omega_0$ where $\Omega_0$ is a semi-basic form 2-form on $T^*Q$. The proof is analogous to that of item (ii) of Proposition 4.3. In this case, the hypothesis that $\Omega_0$ is semi-basic implies that the condition (ii) of Definition 4 is satisfied (see Proposition 4.6 below).

After this discussion, we observe that it is more appropriate to describe a nonholonomic system by the triple $(\mathcal{M}, \mathfrak{F}, \mathcal{H}_M)$ where $\mathfrak{F}$ is the family of bivector fields that are related to $\pi_{nh}$ through a dynamical gauge transformation. Notice that $\mathcal{C}$ is the characteristic distribution of any bivector field in $\mathfrak{F}$. Thus, the (almost) Hamiltonian vector fields defined by the corresponding brackets satisfy the nonholonomic constraints. It follows that the bivector fields in the family $\mathfrak{F}$ are almost Poisson bivectors in a “strong” sense since the non-integrability of $\mathcal{C}$ prevents them from being twisted or conformally Poisson. Our interest in considering this big family of brackets relies on the outcome after reduction. In the presence of symmetries, the bracket that Hamiltonizes the reduced equations may arise as the reduction of a member of $\mathfrak{F}$ that is not necessarily $\pi_{nh}$.

Since the distribution $\mathcal{C}$ is regular we observe

**Corollary 4.5.** [of Theorem 3.11] All bivectors with characteristic distribution equal to $\mathcal{C}$ are gauge related (in particular, gauge related to the nonholonomic bivector $\pi_{nh}$).

We finish this section by discussing some cases for which the second condition in Definition 4 is satisfied. Recall that $\mathcal{M} \subset T^*Q$ is a vector bundle over $Q$. We have

**Proposition 4.6.** If $B$ is a semi-basic 2-form on $\mathcal{M}$, then the gauge transformation of $\pi_{nh}$ associated to $B$ corresponds again to a bivector field.

**Proof.** The graph of $\pi_{nh}^\sharp$ is an almost Dirac structure corresponding to the pair $(\mathcal{C}, \Omega_C)$ in the sense of Proposition 3.1 (see Proposition 4.3 (i)). Thus, in view of (3.24), the gauge transformation of $\pi_{nh}$ is the almost Dirac structure corresponding to the pair $(\mathcal{C}, (\Omega_M - B)|_\mathcal{C})$. It is shown in [21] that if $B$ is a semi-basic 2-form, then the point-wise restriction of $(\Omega_M - B)$ to $\mathcal{C}$ is non-degenerate. Thus, by Corollary 3.4 $\tau_B(L_{\pi_{nh}})$ corresponds to the graph of a bivector. \qed

In fact this proposition is a special case of the following result:

**Proposition 4.7.** Let $P \to Q$ be a vector bundle and $\pi$ a regular almost Poisson bivector on $P$. If for all semi-basic 1-forms $\alpha$ on $P$ the vector field $\pi^\sharp(\alpha)$ is vertical, then the gauge transformation of $\pi$ associated to a semi-basic 2-form $B$ corresponds again to a bivector.

**Proof.** Consider local bundle coordinates $(q, p) \in U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$, on $P$ such that $q$ are local coordinates on the base manifold $Q$. Since $\pi^\sharp(dq)$ is vertical and $B$ is semi-basic we obtain

\[
B^\flat \circ \pi^\sharp(dq) = 0 \\
B^\flat \circ \pi^\sharp(dp) = b(q, p) dq.
\]
where $b(q,p)$ denotes the $m \times n$ matrix with entries $b_{aj}(q,p) = \langle B^\flat \circ \pi^\sharp(dp_a), \frac{\partial}{\partial q^j} \rangle$, for $a = 1, \ldots, m$, and $j = 1, \ldots, n$. Thus, the matrix representation of the endomorphism $(\text{Id} - B^\flat \circ \pi^\sharp)$ on $T^*P$ is

$$
\begin{pmatrix}
\text{Id}_{n \times n} & -b(q,p)^T \\
0 & \text{Id}_{m \times m}
\end{pmatrix}.
$$

This matrix has full rank and hence $(\text{Id} - B^\flat \circ \pi^\sharp) : T^*P \to T^*P$ is invertible.

**Remark 4.8.** It is interesting for future work to drop the condition (ii) in Definition 4 that requires $\tau_B(\text{graph}(\pi^\sharp_{nh}))$ to define a bivector field. In this case, the family $\mathcal{F}$ consists of all the almost Dirac structures that are gauge related to $L_{nh}$ and that describe the nonholonomic dynamics. In this sense, the Hamiltonization of the problem is achieved if the reduction of a member of $\mathcal{F}$ is a Dirac structure (but not necessarily a Poisson structure). This approach requires the consideration of a general reduction scheme for almost Dirac structures. However, we are unaware of any examples of nonholonomic systems that justify the need of such a general framework.

### 4.3 Reduction by a group of symmetries

We now add symmetries to the problem and perform the reduction. Our interest from the point of view of Hamiltonization is to find a bivector field in the family $\mathcal{F}$ whose reduction is either Poisson or conformally Poisson (see Section 4.4).

Let $G$ be a Lie group acting freely and properly on $Q$. We say that $G$ is a symmetry of the nonholonomic system if the lifted action to $TQ$ is free and proper, and leaves the constraint distribution $\mathcal{D} \subset TQ$ and the Lagrangian $L : TQ \to \mathbb{R}$ invariant.

Denote by $\Psi : G \times T^*Q \to T^*Q$ the cotangent lift of the action to $T^*Q$. If $G$ is a symmetry for our nonholonomic system, then $\Psi$ leaves both the constraint submanifold $\mathcal{M}$ and the Hamiltonian $\mathcal{H} : T^*Q \to \mathbb{R}$ invariant. We continue to denote by $\Psi$ the restricted action to $\mathcal{M}$.

One can show that the tangent lift of $\Psi$ to $TM$, preserves the distribution $\mathcal{C}$ and the section $\Omega_C$. As a consequence, if $G$ is a symmetry group for our nonholonomic system, then the action $\Psi$ preserves the standard nonholonomic bracket $\{\cdot, \cdot\}_{nh}$. That is, for $f, g \in C^\infty(\mathcal{M})$, we have

$$\{f \circ \Psi, g \circ \Psi\}_{nh} = \{f, g\}_{nh} \circ \Psi.$$ 

By freeness and properness of the action, the reduced space $\mathcal{R} := \mathcal{M}/G$ is a smooth manifold and the orbit projection map $\rho : \mathcal{M} \to \mathcal{R}$ is a surjective submersion. Notice that $\mathcal{R}$ inherits a vector bundle structure from $\mathcal{M}$ over the shape space $Q/G$. Moreover, $\mathcal{R}$ is equipped with the reduced nonholonomic bracket $\{\cdot, \cdot\}_{\text{red}}$ that is characterized by

$$\{f, g\}_{\text{red}} \circ \rho(m) := \{f \circ \rho, g \circ \rho\}_{nh}(m) \quad \text{for } m \in \mathcal{M} \text{ and } f, g \in C^\infty(\mathcal{R}).$$

(4.41)

The corresponding bivector field will be denoted $\pi_{\text{red}}$. The reduced nonholonomic bracket describes the reduced dynamics in the sense that the nonholonomic vector field $X_{nh}$ is $\rho$-related to the (almost) Hamiltonian vector field $X_{\mathcal{H}_{\mathcal{R}}} = \{\cdot, \mathcal{H}_{\mathcal{R}}\}_{\text{red}}$ associated to the reduced Hamiltonian $\mathcal{H}_{\mathcal{R}}$ defined by the condition $\mathcal{H}_\mathcal{M} = \mathcal{H}_\mathcal{R} \circ \rho$. 

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Remark 4.9. The reduction of nonholonomic systems performed by Bates and Sniatycki in [2] shows that it is possible to define a 2-form $\omega_{\text{red}}$ on $\mathcal{R}$ which is non-degenerate along a distribution $\mathcal{C} \subset T\mathcal{R}$. The definition of $\mathcal{C}$ is given by $\mathcal{C} := T\rho(U)$ where $U := \mathcal{C} \cap (\mathcal{C} \cap V)^{\mathcal{B}_\mathcal{M}}$ and $V$ is the distribution on $\mathcal{M}$ tangent to the orbits of $G$. In fact, the pair $(\mathcal{C}, \omega_{\text{red}})$ is just the pair associated to the almost Dirac structure given by the bivector field $\pi_{\text{red}}$ in the sense of Corollary 3.4.

The analysis of the reduction of the bivector fields $\pi_{\text{Bnh}}$ in the family $\mathfrak{F}$ that are related to $\pi_{\text{nh}}$ by a dynamical gauge $B$ follows from:

Proposition 4.10. Let $(P, \pi)$ be an almost Poisson manifold and $B$ a 2-form on $P$ such that the endomorphism $(\text{Id} - B^2 \circ \pi^2) : T^*P \rightarrow T^*P$ is invertible. If the Lie group $G$, acting freely and properly on $P$, preserves the almost Poisson structure $\pi$ and leaves $B$ invariant, then $G$ preserves the bivector field $\pi^B$ obtained by the gauge transformation of $\pi$ associated to $B$.

Proof. In view of equation (3.26), we see that $(\pi^B)^\#$ is a composition of invariant maps and we conclude that $\pi^B$ is invariant as well.

As a direct consequence of this proposition we have:

Proposition 4.11. If $G$ is a symmetry group of the nonholonomic system $(\mathcal{M}, \pi_{\text{nh}}, \mathcal{H}_\mathcal{M})$ and $B$ is a $G$-invariant dynamical gauge on $\mathcal{M}$, then $\{\cdot, \cdot\}_{\text{Bnh}}^B$ is $G$-invariant. In particular there is a reduced bivector field $\pi_{\text{red}}^B$ on the reduced space $\mathcal{R}$ that determines a well defined bracket $\{\cdot, \cdot\}_{\text{red}}^B$ on $\mathcal{R}$ satisfying

$$\{f, g\}_{\text{red}}^B \circ \rho(m) = \{f \circ \rho, g \circ \rho\}_{\text{Bnh}}^B(m), \quad \text{for } f, g \in C^\infty(\mathcal{R}).$$

Moreover, the reduced bracket $\{\cdot, \cdot\}_{\text{red}}^B$ also describes the reduced dynamics in the sense that $X_{\mathcal{H}_\mathcal{R}} = \{\cdot, \mathcal{H}_\mathcal{R}\}_{\text{red}}^B$.

There is a good reason why we did not denote the reduced bivector field $\pi_{\text{red}}^B$ by $\pi_{\text{Bnh}}^B$ and that is that in general the reduced bivector fields $\pi_{\text{red}}$ and $\pi_{\text{red}}^B$ need not be gauge related. We shall see this explicitly with the analysis of our mechanical examples (Remark 5.5).

Remark 4.12. Since the almost Dirac structure $L_{\text{nh}}$ given in (4.37) is the graph of a bivector (see Proposition 4.3 (i)), then the reduction of $L_{\text{nh}}$ as an almost Dirac structure is simply the reduction of the bivector $\pi_{\text{nh}}$ in the classical way. The same observation is valid for the reduction of the almost Dirac structure $\tau_B(L_{\pi_{\text{nh}}})$ defined in (4.40) since $\tau_B(L_{\pi_{\text{nh}}})$ is also the graph of a bivector ($(\Omega_\mathcal{M} - B)\pi_{\text{nh}}$ is non-degenerate on $\mathcal{C}$).

The $G$-Chaplygin case.

If the reduced bivector fields $\pi_{\text{red}}$ and $\pi_{\text{red}}^B$ happen to be everywhere non-degenerate, then they are gauge-related. This is the scenario that one finds after reduction of external symmetries of $G$-Chaplygin systems (see [16, 27]). These systems are characterized by the property that the tangent space to the orbits of the symmetry group exactly complements the constraint distribution on the tangent space $TQ$ of the configuration manifold $Q$. 

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In this case there exist unique non-degenerate 2-forms $\Omega_{\text{red}}$ and $\Omega_{\text{red}B}$ on $\mathcal{R}$ satisfying

$$\pi_{\text{red}}^\sharp \circ \Omega_{\text{red}}^\flat = \text{Id}, \quad \pi_{\text{red}B}^\sharp \circ \Omega_{\text{red}B}^\flat = \text{Id},$$

where as usual, $\Omega_{\text{red}}^\flat(X) = -i_X \Omega_{\text{red}}$ and $\Omega_{\text{red}B}^\flat(X) = -i_X \Omega_{\text{red}B}$ for all $X \in \mathfrak{X}(\mathcal{R})$. In particular, the reduced equations can be written as:

$$i_{X_{\mathcal{H}_R}} \Omega_{\text{red}} = i_{X_{\mathcal{H}_R}} \Omega_{\text{red}B} = d\mathcal{H}_R,$$

which gives different almost symplectic formulations of the reduced equations (compare with the results in [22]). The bivector fields $\pi_{\text{red}}$ and $\pi_{\text{red}B}$ are gauge related via the 2-form $B_{\text{red}} := \Omega_{\text{red}} - \Omega_{\text{red}B}$ on $\mathcal{R}$ and we recover the Chaplygin Hamiltonization studied in [16]. Moreover, the 2-forms $\Omega_{\text{red}}$ and $\Omega_{\text{red}B}$ satisfy

$$(\rho^* \Omega_{\text{red}})|_\mathcal{C} = \Omega_{\mathcal{C}} \quad \text{and} \quad (\rho^* \Omega_{\text{red}B})|_\mathcal{C} = (\Omega_{\mathcal{M}} - B)|_\mathcal{C},$$

and thus $B_{\text{red}}$ verifies that $(\rho^* B_{\text{red}})|_\mathcal{C} = B_{\mathcal{C}}$. In other words, we have the following commutative diagram

\[
\begin{array}{ccc}
L_{\pi_{\text{nh}}} & \overset{\tau_{\text{B}}}{\longrightarrow} & L_{\pi_{\text{Bnh}}} \\
\downarrow \rho & & \downarrow \rho \\
L_{\pi_{\text{red}}} & \overset{\tau_{\text{Bred}}}{\longrightarrow} & L_{\pi_{\text{redB}}}.
\end{array}
\]

(4.42)

We stress that one does not have such a commutative diagram in more general situations, see Remark 5.5.

### 4.4 Hamiltonization

We continue with the notation of the previous sections and suppose that $G$ is a symmetry group for our nonholonomic system. The solutions of the reduced equations on the reduced space $\mathcal{R} = \mathcal{M}/G$ are the integral curves of the reduced vector field $X_{\mathcal{H}_R}$ and preserve the reduced Hamiltonian $\mathcal{H}_R$.

The issue of Hamiltonization in our context concerns answering the question of whether the reduced vector field $X_{\mathcal{H}_R}$ on $\mathcal{R}$ is Hamiltonian. Our candidates for the Hamiltonian structure come from the reduction of the (invariant) bivector fields that belong to the family $\mathfrak{F}$.

It turns out that the above condition is too restrictive. We relax it by asking that the vector field $X_{\mathcal{H}_R}$ can be rescaled by a basic\footnote{We use the term basic with respect to the fibered structure of $\mathcal{R}$ inherited from $\mathcal{M}$}. Positive function $\varphi : \mathcal{R} \to \mathbb{R}$ in such a way that the resulting vector field $\varphi X_{\mathcal{H}_R}$ is Hamiltonian.

In view of Proposition 4.11, for any $G$-invariant bivector field $\pi_{\text{nh}}^B$ belonging to $\mathfrak{F}_{\text{nh}}$, the rescaled vector field $\varphi X_{\mathcal{H}_R}$ satisfies

\[(\varphi \pi_{\text{red}B})^\sharp(d\mathcal{H}_R) = -\varphi X_{\mathcal{H}_R}.\]  

Hence, we are interested in finding a bivector field $\pi_{\text{red}B}$ satisfying

\[[\varphi \pi_{\text{red}B}, \varphi \pi_{\text{red}B}] = 0,\]

that is, we want to find $\pi_{\text{red}B}$ conformally Poisson (see Definition 2).
Definition 5. If there exist an invariant bivector field $\pi_{\text{nh}}^B$ belonging to $\mathfrak{h}$ and a strictly positive, basic function $\varphi : \mathcal{R} \to \mathbb{R}$ such that (4.43) and (4.44) hold, we say that the nonholonomic system is Hamiltonizable. Moreover, we say that the reduced equations are Hamiltonian in the new time $\tau$ defined by $d\tau = \frac{1}{\varphi} dt$ (see discussion below).

In Section 5 we will extend the discussion of Section 2 and show that all generalized rolling systems are Hamiltonizable. The table (1.1) in Section 1 shows how different scenarios of the Hamiltonization scheme described above are realized according to the rank of the matrix $A$.

Time reparametrizations

It is common in the literature to interpret the rescaling of the vector field $X_{\mathcal{H}_R}$ by the basic positive function $\varphi$ as a nonlinear time reparametrization. One argues as follows, let $c(t) \in \mathcal{R}$ be a flow line of $X_{\mathcal{H}_R}$ (i.e. $\frac{dc}{dt}(t) = X_{\mathcal{H}_R}(c(t))$). Introduce the new time $\tau$ by integrating the relation

$$d\tau = \frac{1}{\varphi(c(t))} dt.$$

Since $\varphi > 0$, the correspondence between $t$ and $\tau$ is one-to-one and one can express $t$ as a function of $\tau$. The curve $\hat{c}(\tau) := c(t(\tau))$ is checked to be a flow line of $\varphi X_{\mathcal{H}_\pi}$ (i.e. $\frac{d\hat{c}}{d\tau}(\tau) = \varphi(\hat{c}(\tau)) X_{\mathcal{H}_\pi}(\hat{c}(\tau))$). This interpretation of the rescaling is quite subtle. The definition of $\tau$ depends on the particular flow line $c(t)$, so different initial conditions induce different reparametrizations. It is therefore not possible to interpret the time rescaling as a “global” operation. This contrasts with the natural procedure of multiplying the vector field $X_{\mathcal{H}_R}$ by the positive function $\varphi$.

Remark 4.13. One might wonder why we only care about basic and not arbitrary functions $\varphi : \mathcal{R} \to \mathbb{R}^+$. A detailed answer to this question would involve a careful study of the structure of the equations of motion that would gear us away from the main subject of this paper. We refer the reader to [17] where one can find a very good discussion on the Hamiltonization of $G$-Chaplygin systems. We simply mention that physically, the fact that $\varphi$ is basic means that it is independent of the momentum variables and only depends on the (reduced) configuration variables. Thus, the time reparametrization changes the speed at which the trajectories are traversed depending on the position of the system but independently of the velocity itself. It is shown in [17] how in order to obtain Darboux coordinates for the reparametrized system, one can keep the same (reduced) configuration variables but should rescale the momenta by $\frac{1}{\varphi}$. Since $\varphi$ is basic, the rescaled momenta continue to depend linearly on the velocities.

Measure preservation

Hamiltonization is strongly related to the existence of invariant measures. Suppose for simplicity that we are dealing with a $G$-Chaplygin system. As mentioned before, in this case the bivector field $\pi_{\text{red}B}$ is everywhere non-degenerate and the reduced equations can be written as:

$$i_{X_{\mathcal{H}_R}} \Omega_{\text{red}} = d\mathcal{H}_R,$$
where $\Omega_{\text{red}B}$ is the non-degenerate 2-form on $\mathcal{R}$ induced by $\pi_{\text{red}B}$. It follows that the scaled vector field $\varphi X_{\mathcal{H}R}$ satisfies
\[ i_{\varphi X_{\mathcal{H}R}} \left( \frac{1}{\varphi} \Omega_{\text{red}B} \right) = d\mathcal{H}_R. \]

Hence, Hamiltonization in this setting amounts to finding a positive function $\varphi$ such that the 2-form $\frac{1}{\varphi} \Omega_{\text{red}B}$ is closed (which under our hypothesis, is of course equivalent to (4.44)). Suppose for a moment that this is the case so $(\mathcal{R}, \frac{1}{\varphi} \Omega_{\text{red}B})$ is a symplectic manifold. It follows from Liouville’s theorem that the vector field $\varphi X_{\mathcal{H}R}$ preserves the symplectic volume $\left( \frac{1}{\varphi} \Omega_{\text{red}B} \right)^m$ where $m = \frac{1}{2} \dim \mathcal{R}$. Therefore, the volume form $\frac{1}{\varphi} (\Omega_{\text{red}B})^m$ is preserved by the vector field $X_{\mathcal{H}R}$.

The above argument shows that a Hamiltonizable $G$-Chapligyn system possesses an invariant measure. One might wonder if the reciprocal statement is true, namely, if any $G$-Chaplygin system with an invariant measure is Hamiltonizable. The celebrated Chaplygin’s reducing multiplier Theorem [12] demonstrates that the answer is positive if $m = 2$. For $m > 2$, a characterization of the systems for which this is true is an open problem. Interesting examples where this holds for arbitrary values of $m$ have been found by Fedorov and Jovanovic in the study of the multidimensional Veselova problem [17]. See also the discussion in [16] where a candidate for the conformal factor $\frac{1}{\varphi}$ is given under the hypothesis that there exists a preserved measure, and [19] where a set of coupled first order partial differential equations for the multiplier $\frac{1}{\varphi}$ are given.

In the case where the nonholonomic system is Hamiltonizable but the corresponding bivector field $\pi_{\text{red}B}$ is degenerate at some points in $\mathcal{R}$, one can repeat the above argument to conclude that the reduced system preserves a measure on every leaf of the symplectic foliation of $\mathcal{R}$ corresponding to the Poisson bivector field $\varphi \pi_{\text{red}B}$. However, this does not imply the existence of a smooth invariant measure on $\mathcal{R}$ (and this is the motivation for Fernandez, Mestdag, and Bloch [19] to talk about Poissonization). An example of this situation is given by the reduction of the Chaplygin sleigh (see the discussion in [20]) that exhibits asymptotic dynamics that contravene the existence of a global invariant measure. The problem of the existence of a global invariant measure in this case is most naturally attacked by considering the modular class of the Poisson manifold $(\mathcal{R}, \varphi \pi_{\text{red}B})$, see [39].

5 Back to the Examples: Hamiltonization and Integrability

In this section, we analyze the generalized rolling systems presented in Section 2 using the geometric framework that was developed in the previous sections. In subsection 5.1 we provide the geometric interpretation for the brackets $\{\cdot, \cdot\}_{\text{Rank1}}$ and $\{\cdot, \cdot\}'_{\text{Rank1}}$ presented in Section 2. In 5.2 we consider the Hamiltonization and integrability of generalized rolling systems in detail and finally, in 5.3 we explicitly show that the brackets $\{\cdot, \cdot\}_{\text{Rank2}}$ and $\{\cdot, \cdot\}'_{\text{Rank2}}$ are twisted Poisson. To our knowledge, this is the first time that the appearance of such structures is made explicit in the field of nonholonomic mechanics.

5.1 The geometry of the rigid bodies with generalized rolling constraints

We begin by computing the nonholonomic bracket for the motion of a rigid body subject to generalized rolling constraints as introduced in Section 2.
Nonholonomic bracket via the non-degenerate 2-section

Consider again, as in Section 2, the motion of a rigid body in space subject to a generalized rolling constraint as in (2.3). That is, the constraint relates the linear and the angular velocities of the body $\mathbf{x} = r A \mathbf{\omega} = r A g \mathbf{\Omega}$, where the matrix $A$ satisfies any of the conditions of Definition 1 and $\mathbf{\omega}$, $\mathbf{\Omega}$ is the angular velocity written in space and body coordinates, respectively.

Recall that the configuration space for the system is $Q = \text{SO}(3) \times \mathbb{R}^3$. Denote by $\lambda$ (respectively, $\rho$) the left (respectively, right) Maurer-Cartan form on $\text{SO}(3)$. Upon the identification of the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$ by the hat map (2.4) we think of $\lambda$ and $\rho$ as $\mathbb{R}^3$-valued 1-forms on $\text{SO}(3)$. For a tangent vector $v_g \in T_g \text{SO}(3)$ we have

$$\mathbf{\omega} = \rho(g)(v_g), \quad \mathbf{\Omega} = \lambda(g)(v_g),$$

where $\mathbf{\omega}$ (respectively, $\mathbf{\Omega}$) denotes the angular velocity vector written in space (respectively, body) coordinates as discussed in Section 2.

The Maurer-Cartan forms $\lambda$ and $\rho$ are related by $\lambda(g) = g^{-1} \rho(g)$ and satisfy the well-known Maurer-Cartan equations

$$d\rho = [\rho, \rho], \quad d\lambda = -[\lambda, \lambda],$$

where $[\cdot, \cdot]$ is the commutator in the Lie algebra. For the rest of the section we will use three dimensional vector algebra notation in our calculations with differential forms and vector fields. In our convention, the scalar product of differential forms should always be interpreted as a wedge product (and is thus anti-commutative!). The Maurer-Cartan equations take the form

$$d\rho = \frac{1}{2} \rho \times \rho, \quad d\lambda = -\frac{1}{2} \lambda \times \lambda, \quad (5.45)$$

where “$\times$” denotes the standard vector product in $\mathbb{R}^3$.

The constraint distribution $\mathcal{D}$, defined by the generalized rolling constraints, can be expressed in the terminology of subsection 4.1 as the annihilator of the $\mathbb{R}^3$-valued 1-form $\epsilon$ on $Q$ given by

$$\epsilon = d\mathbf{x} - r A \rho = d\mathbf{x} - r A g \lambda.$$

We consider the (global) moving co-frame $\{\lambda, d\mathbf{x}\}$ for $T^*Q$ that defines fiber coordinates $(\mathbf{M}, \mathbf{p})$ in the following sense. A co-vector $\alpha_q \in T^*_q Q$ is written uniquely as $\alpha_q = \mathbf{M} \cdot \lambda + \mathbf{p} \cdot d\mathbf{x}$, for a certain $(\mathbf{M}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$. The Legendre transform $\text{Leg} : TQ \to T^*Q$ associated to the kinetic energy Lagrangian (2.6) is defined by the rule:

$$\mathbf{M} = \mathbb{I} \mathbf{\Omega}, \quad \mathbf{p} = m \dot{\mathbf{x}}.$$

Physically, $\mathbf{p}$ is the linear momentum of the body while $\mathbf{M}$ is the angular momentum of the body about the center of mass written in body coordinates.

In order to deal with the constraints, it is more convenient to work with the global moving co-frame $\{\lambda, \epsilon\}$ for $T^*Q$. We denote by $(\mathbf{K}, \mathbf{u})$ the fiber coordinates defined by this co-frame. Putting $\mathbf{K} \cdot \lambda + \mathbf{u} \cdot \epsilon = \mathbf{M} \cdot \lambda + \mathbf{p} \cdot d\mathbf{x}$ implies

$$\mathbf{K} = \mathbf{M} + r g^T A^T \mathbf{p}, \quad \mathbf{u} = \mathbf{p}.$$
Along the constraint submanifold $\mathcal{M} = \text{Leg}(\mathcal{D})$ we have $p = mrAg\Omega$ so
\[ K = \Omega + mr^2gTAg\Omega, \]
which is the expression for the kinetic momentum obtained in (2.9). Notice that $\mathcal{M}$ is a vector bundle over $Q$ and that $K$ is a natural coordinate for the fibers of $\mathcal{M}$. In what follows we will use the components of $g, x$, and $K$, as redundant coordinates on $\mathcal{M}$.

Denote by $X^1 = (X_1^L, X_2^L, X_3^L)$ the moving frame of $\text{SO}(3)$ that is dual to $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. The components of $X^i$ are the left invariant vector fields on $\text{SO}(3)$ obtained by left extension of the canonical basis of $\mathbb{R}^3$. Along the points of the constraint subbundle $\mathcal{M}$, the non-integrable distribution $\mathcal{C}$ defined in (4.32) is given by
\[ \mathcal{C} = \text{span} \left\{ X^i + rgTA^T \frac{\partial}{\partial x^j} , \frac{\partial}{\partial K} \right\}. \quad (5.46) \]

The canonical 2-form $\Omega_Q$ on $T^*Q$ is given by
\[ \Omega_Q = -d(M \cdot \lambda + p \cdot dx) = -d(K \cdot \lambda + p \cdot \epsilon) = \lambda \cdot dK - K \cdot d\lambda - p \cdot d\epsilon + \epsilon \cdot dp, \]
where "\cdot" denotes the usual scalar product in $\mathbb{R}^3$.

To compute $d\epsilon$ we use the identity $dg = g\hat{\lambda}$ where $^\}$ denotes the hat map (2.4). Using the Maurer-Cartan equations (5.45) we get
\[ d\epsilon = -rA(dg)\lambda - rAg d\lambda = -rAg(\lambda \times \lambda) - rAg d\lambda = rAg d\lambda. \]
Therefore,
\[ \Omega_Q = \lambda \cdot dK - K \cdot d\lambda - p \cdot (rAg d\lambda) + \epsilon \cdot dp = \lambda \cdot dK - (K + rgTA^T p) \cdot d\lambda + \epsilon \cdot dp. \]

Let $\iota : \mathcal{M} \rightarrow T^*Q$ denote the inclusion. Since $p = mrAg\Omega$ along $\mathcal{M}$, we have
\[ \Omega_{\mathcal{M}} := \iota^*(\Omega_Q) = \lambda \cdot dK - (K + mr^2gTA^T Ag\Omega) \cdot d\lambda + \iota^*(\epsilon \cdot dp). \]
Since $\epsilon$ vanishes along the non-integrable distribution $\mathcal{C}$, we get the following expression for the restriction $\Omega_{\mathcal{C}}$ of $\iota^*(\Omega_Q)$ to $\mathcal{C}$:
\[ \Omega_{\mathcal{C}} = \lambda \cdot dK - (K + mr^2gTA^T Ag\Omega) \cdot d\lambda. \]

With this expression for $\Omega_{\mathcal{C}}$ we are ready to show:

**Proposition 5.1.** The nonholonomic bracket $\{\cdot,\cdot\}_{\text{nh}}$ on $\mathcal{M}$ for the generalized rolling system is given in the redundant coordinates $\{g_{ij}, x_k, K_l\}$, $i, j, k, l = 1, 2, 3$, for $\mathcal{M}$ by
\[ \{x_i, K_l\}_{\text{nh}} = r(Ag)_{il}, \quad \{g_{ij}, K_l\}_{\text{nh}} = -\varepsilon^k_{ij} g_{lk}, \quad \{K_i, K_j\}_{\text{nh}} = -\varepsilon^l_{ij}(K + mr^2gTA^T Ag\Omega)_{kl}, \]
with all other combinations equal to zero. In the above formulas the Einstein convention of sum over repeated indices holds and $\varepsilon^k_{ij}$ denotes the alternating tensor, that equals 0 if two indices are equal, it equals 1 if $(i,j,k)$ is a cyclic permutation of $(1,2,3)$, and it is equal to $-1$ otherwise. The entries of $(g, x) \in Q$ are denoted by $(g_{ij}, x_k)$ and those of $K$ by $K_l$.
Proof. We will rely on the identity (4.36) that characterizes the associated bivector field \( \pi_{nh} \) in terms of \( \Omega_C \). Contraction of \( \Omega_C \) by the elements in the basis of \( C \) given in (5.46) gives

\[
\mathbf{i}_{(X^L + r g^T A^T \frac{\partial}{\partial \pi})} \Omega_C = dK - (K + mr^2 g^T A^T Ag) \times \lambda, \quad \mathbf{i}_{\frac{\partial}{\partial \pi}} \Omega_C = -\lambda,
\]

where we have again made use of the Maurer-Cartan equations (5.45). It follows that

\[
\mathbf{i}_{(X^L + r g^T A^T \frac{\partial}{\partial \pi})} \Omega_C = i_{\frac{\partial}{\partial K}} \Omega_C = -\lambda,
\]

\[
\mathbf{i}_{(-rAg \frac{\partial}{\partial K})} \Omega_C = dK,
\]

\[
\mathbf{i}_{\frac{\partial}{\partial K}} \Omega_C = dK - (K + mr^2 g^T A^T Ag) \times \partial \frac{\partial}{\partial K}.
\]

Therefore, according to (4.36) we get

\[
\pi_{nh}^\sharp(dK) = -(X^L - r g^T A^T \frac{\partial}{\partial \pi}) \times \partial \frac{\partial}{\partial K},
\]

\[
\pi_{nh}^\sharp(\lambda) = \frac{\partial}{\partial K},
\]

\[
\pi_{nh}^\sharp(dx) = rAg \frac{\partial}{\partial K}.
\]

In addition, for any canonical vector \( e_i \in \mathbb{R}^3 \) we have \( d(g^{-1} e_i) = (g^{-1} e_i) \times \lambda \), so

\[
\pi_{nh}^\sharp(d(g^{-1} e_i)) = (g^{-1} e_i) \times \frac{\partial}{\partial K}.
\]

The proof follows using the above formulas and recalling that for any \( f, g \in C^\infty(\mathcal{M}) \), we have \( \{f, g\}_{nh} = -df(\pi_{nh}^\sharp(dg)) \).

\( \square \)

Finally, we state without a formal proof that the nonholonomic vector field \( X_{nh} \) on \( \mathcal{M} \) is given by

\[
X_{nh} = \Omega \cdot X^L + rAg \Omega \cdot \frac{\partial}{\partial x} + (K \times \Omega) \cdot \frac{\partial}{\partial K}.
\]

The above expression can be shown by taking into account the equations (2.11), the constraint (2.3), and the definition of \( \Omega \), or, alternatively, by computing the the almost Hamiltonian vector field \( X_{H_M} = -\pi_{nh}^\sharp(dH_M) \) corresponding to the Hamiltonian \( H_M \) that coincides with expression (2.10). The latter approach requires one to write \( \Omega \) in terms of \( K \) and \( g \) as was done in Section 2 for the different values of the rank of \( A \).

The gauge transformation of the nonholonomic bracket

We will now construct a gauge transformation of the nonholonomic bracket in the sense of subsection 4.2. We are interested in describing the same dynamics so we look for a 2-form \( B \) that defines a dynamical gauge transformation as introduced in Definition 4. In our case, the distinguished Hamiltonian is \( H_M \) that has \( X_{nh} \) as its associated almost Hamiltonian vector field.

Following [21], we consider the bi-invariant volume form \( \nu \) on \( SO(3) \) oriented and scaled such that \( \nu(X^L_1, X^L_2, X^L_3) = 1 \). We consider the natural extension of \( \nu \) as a 3-form on \( Q = SO(3) \times \mathbb{R}^3 \). Denote by \( \bar{\nu} \in \Omega^3(T^*Q) \) the 3-form given by \( \bar{\nu} = \nu^* \nu \) where \( \nu : T^*Q \to Q \) is the canonical projection. We can write \( \bar{\nu} = \frac{1}{6} \lambda \cdot (\lambda \times \lambda) \).
Let $B$ be the 2-form on $\mathcal{M}$ given by

$$B = mr^2 (i_{X_{nh}} \iota^*_\nu),$$

where, as before, $\iota : \mathcal{M} \hookrightarrow T^*Q$ is the inclusion. Note that $B$ is a semi-basic 2-form on $\mathcal{M}$ that vanishes upon contraction with the nonholonomic (almost) Hamiltonian vector field $X_{nh}$. Therefore, by Proposition 4.6 we can perform a dynamical gauge transformation of the nonholonomic bivector field $\pi_{nh}$ by the 2-form $B$ to obtain another bivector field $\pi^B_{nh}$ that also describes the dynamics of our problem.

Using the Maurer-Cartan equations (5.45) and the expression (5.48) for $X_{nh}$, we obtain

$$B = -mr^2 \Omega \cdot d\lambda.$$  \hspace{1cm} (5.49)

To compute the bivector field $\pi^B_{nh}$ associated to the gauge transformation we use equation (3.25). For an arbitrary one-form $\alpha$ on $\mathcal{M}$ we have

$$(\pi^B_{nh})^\sharp (\alpha + \iota^*_{\pi_{nh}}(\alpha) B) = \pi^\sharp_{nh}(\alpha).$$

Setting $\alpha$ equal to $\lambda$ and $dx$ and using (5.47) and (5.49) we obtain

$$(\pi^B_{nh})^\sharp (\lambda) = \frac{\partial}{\partial K}, \quad (\pi^B_{nh})^\sharp (dx) = rAg \frac{\partial}{\partial K}. $$

Similarly, putting $\alpha = dK$ and noticing that

$$\iota^*_{\pi_{nh}}(dK) B = -i_{X_L}B = mr^2 \Omega \times \lambda,$$

we deduce

$$(\pi^B_{nh})^\sharp (dK) = -X_L - rg^T A^T \frac{\partial}{\partial x} + (K + mr^2 (g^T A^T Ag - E) \Omega) \times \frac{\partial}{\partial K},$$

where $E$ denotes the $3 \times 3$ identity matrix.

The above formulas imply

**Proposition 5.2.** The gauged nonholonomic bracket $\{\cdot, \cdot\}^B_{nh}$ on $\mathcal{M}$, associated to the bivector field $\pi^B_{nh}$, is given in the redundant coordinates $(g_{ij}, x_k, K_l)$, $i, j, k, l = 1, 2, 3$, for $\mathcal{M}$ by

$$\{x_i, K_l\}_{nh}^B = r(Ag)_{il}, \quad \{g_{ij}, K_l\}_{nh}^B = -\varepsilon_{ij}^l g_{ik}, \quad \{K_i, K_j\}_{nh}^B = -\varepsilon_{ij}^l (K + mr^2 (g^T A^T Ag - E) \Omega)_{il},$$

with all other combinations equal to zero.

**Reduction of the symmetries**

Recall that the Lie group $H$, introduced in section 2.4, acts on the configuration space $Q$ and that its lift to $TQ$ leaves both the Lagrangian and the constraints invariant. From the discussion in section 4.3 (and the regularity of the action) it follows that the reduced space $\mathcal{R} := \mathcal{M} / H$ is equipped with a reduced bracket $\{\cdot, \cdot\}_\text{red}$ determined by condition (4.41) and that describes the reduced dynamics.

We are now ready to give the geometric interpretation of the bracket $\{\cdot, \cdot\}_{\text{Rank}^j}$ introduced in section 2 for the different values of the rank of $A$. 

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Theorem 5.3. The reduced bracket \( \{ \cdot, \cdot \}_\text{red} \) on \( \mathcal{R} \) is precisely the restriction of the bracket \( \{ \cdot, \cdot \}_\text{Rank } j \) (defined in section 2) to the Casimir level set \( ||\gamma|| = 1 \), for the different values \( j = 0, 1, 2, 3 \), of the rank of \( A \).

Proof. Recall from section 2.4 that the reduced space \( \mathcal{R} \) can be identified with \( S^2 \times \mathbb{R}^3 \) with redundant coordinates \((\gamma, K)\). Therefore, it makes sense to compare the two brackets on the Casimir level set \( ||\gamma|| = 1 \) of the space \((\gamma, K) \in \mathbb{R}^3 \times \mathbb{R}^3\).

Moreover, from the expression of the projection \( \rho : \mathcal{M} \to \mathcal{R} \) given by (2.18), and condition (4.41), it follows that the reduced bracket of the (redundant) coordinate functions \((\gamma, K)\) can be computed using the formulas obtained in Proposition 5.1 (notice that \( \gamma = (g_{31}, g_{32}, g_{33}) \)).

The proof is completed by considering the particular form of \( A \) for the different values of its rank given in Definition 1, and by writing the bracket \( \{f_1, f_2\}_\text{red} \) of arbitrary functions \( f_1, f_2 \in C^\infty(\mathcal{R}) \) in terms of the derivatives \( \frac{\partial f_i}{\partial \gamma} \) and \( \frac{\partial f_i}{\partial K} \) using Leibniz rule.

We now turn to the study of the reduction of the gauged nonholonomic bracket \( \{\cdot,\cdot\}_B \). First of all notice that the 2-form \( B \) that defines the gauge transformation is written in (5.49) in terms of left invariant objects on \( \text{SO}(3) \). Since the symmetry group \( H \) acts by left multiplication on the \( \text{SO}(3) \) factor of \( Q \), it follows that \( B \) is invariant under the cotangent lifted action. Therefore, in accordance with Proposition 4.11 the gauged bracket \( \{\cdot,\cdot\}_B \) drops to \( \mathcal{R} \) where it defines the bracket \( \{\cdot,\cdot\}_\text{red}B \) that determines the dynamics. As usual, the corresponding bivector field on \( \mathcal{R} \) will be denoted by \( \pi_\text{red}B \).

In analogy with Theorem 5.3 we have

Theorem 5.4. The reduced bracket \( \{\cdot,\cdot\}_\text{red}B \) on \( \mathcal{R} \) is precisely the restriction of the bracket \( \{\cdot,\cdot\}_\text{Rank } j \) (defined in section 2) to the Casimir level set \( ||\gamma|| = 1 \), for the different values \( j = 0, 1, 2, 3 \), of the rank of \( A \).

The proof is identical to that of Theorem 5.3 except that one uses the formulas obtained in Proposition 5.2.

According to the above theorems and the discussion in Section 2, the properties of the reduced brackets \( \{\cdot,\cdot\}_\text{red} \) and \( \{\cdot,\cdot\}_\text{red}B \) are summarized in the table (1.1) presented in the introduction (Section 1) according to the different values of the rank of \( A \).

Remark 5.5. Notice that the reduced bivector fields \( \pi_\text{red} \) and \( \pi_\text{red}B \) are not gauge related. Indeed, from table (1.1) one sees that for any value of the rank of \( A \), only one of the two bivector fields \( \pi_\text{red} \) or \( \pi_\text{red}B \) has an integrable characteristic distribution. It follows from Theorem 3.11 that there cannot exist a gauge transformation between their graphs. This situation is opposite to the one described by \( G \)-Chaplygin systems in diagram (4.42).

Remark 5.6. Recall from subsection 2.5 that as the rank of \( A \) increases, the constraint distribution is less integrable or “more nonholonomic”. Table (1.1) seems to suggest that it is appropriate to perform a gauge transformation by the 2-form \( B \) when the nonholonomic effects are more important, while the reduction of the standard nonholonomic bracket works better for weaker nonholonomic effects.
5.2 Hamiltonization and integrability of rigid bodies with generalized rolling constraints

According to the notion of Hamiltonization introduced in Section 4.4 (Definition 5), and the table (1.1), it immediately follows that the problem of the motion of a rigid body subject to a generalized rolling constraint is Hamiltonizable for any value of the rank of $A$.

If the rank of $A$ equals 0 (respectively, 3) the reduced equations are Hamiltonian with respect to the bracket $\{\cdot, \cdot\}_{\text{red}}$ (respectively, $\{\cdot, \cdot\}_{\text{red}B}$). Recall that in both cases the reduced dynamics correspond to classical rigid body motion (with modified inertia tensor $\mathbb{I} + mr^2 E$ if the rank of $A$ equals 3).

If the rank of $A$ equals 1 or 2, the analysis of the Hamiltonization is a bit more delicate but it also follows directly from Definition 5 and the table (1.1). In the case rank $A = 2$, it follows that the reduced equations are Hamiltonian in the new time $\tau_2$ defined by $d\tau_2 = \frac{1}{\varphi_2} dt$ and with respect to the bracket $\varphi_2 \{\cdot, \cdot\}_{\text{red}B}$ where

$$\varphi_2(\gamma) = \sqrt{1 - mr^2 (\gamma \cdot (\mathbb{I} + mr^2 E)^{-1} \gamma)}.$$  \hspace{1cm} (5.50)

Note that $\varphi_2$ is a basic function on $\mathbb{R}$ corresponding to the restriction of (2.15) to the level set $|\gamma| = 1$.

Analogously, if the rank of $A$ equals 1, the reduced equations are Hamiltonian in the new time $\tau_1$ defined by $d\tau_1 = \frac{1}{\varphi_1} dt$ and with respect to the bracket $\varphi_1 \{\cdot, \cdot\}_{\text{red}}$ where

$$\varphi_1(\gamma) = \sqrt{1 + mr^2 (\gamma \cdot \mathbb{I}^{-1} \gamma)}.$$  \hspace{1cm} (5.51)

**Integrability of the reduced equations**

In view of the Hamiltonization of the problem, the integrability of the reduced equations of motion (2.11) can be easily established using the celebrated Arnold-Liouville Theorem for classical Hamiltonian systems, see e.g. [1].

Indeed, for any value of the rank of $A$, the reduced equations are Hamiltonian on $\mathcal{R}$ (after a time reparametrization if rank $A = 1, 2$). Independently of the rank of $A$, the symplectic leaves $\mathcal{O}_a$ of the foliation of $\mathcal{R}$ correspond to the level sets $C_1(K, \gamma) = K \cdot \gamma = a$ and can be shown to be diffeomorphic to the tangent bundle $T S^2$ of the sphere (see the discussion in chapter 14 of [30] for the coadjoint orbits on $\mathfrak{se}(3)^*$).

Once the value of $a$ is fixed, the reduced equations (2.11) can be seen as a two degree of freedom classical Hamiltonian system on $\mathcal{O}_a$ (again, after a time reparametrization if rank $A = 1, 2$). These equations possess two independent integrals, the Hamiltonian $\mathcal{H}_R$, and $F = K \cdot K$, whose joint level sets are compact in $\mathcal{O}_a$. It follows from the Arnold-Liouville Theorem that these level sets are invariant two-tori and the dynamics are quasi-periodic on them (notice that the flow on the tori is rectilinear but not uniform if the rank of $A$ is 1 or 2).

The Arnold-Liouville Theorem also tells us that the reduced equations are integrable by quadratures (after the time reparametrization if the rank of $A$ is 1 or 2).

Finally, we state without proof that the reduced equations of motion (2.11) preserve the measure $\mu(\gamma) \sigma \wedge dK_1 \wedge dK_2 \wedge dK_3$ where $\sigma$ is the area form of the sphere $S^2$, and the basic density $\mu : S^2 \rightarrow \mathbb{R}$

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is given by

$$\mu(\gamma) = \begin{cases} 
1 & \text{if rank } A = 0, 3, \\
\frac{1}{\varphi_1(\gamma)} & \text{if rank } A = 1, \\
\frac{1}{\varphi_2(\gamma)} & \text{if rank } A = 2,
\end{cases}$$

where $\varphi_1, \varphi_2 \in C^\infty(S^2)$ are defined in \[5.51\] and \[5.50\] respectively.

### 5.3 Twisted Poisson structures for rigid bodies with generalized rolling constraints

In Section 3.2, we presented twisted Poisson structures which have been extensively studied in other contexts but not in mechanics. Now, we will show explicitly that twisted Poisson structures appear naturally in the study of nonholonomic systems.

#### Rigid body with generalized rolling constraints of rank 2

Here we show that the bracket $\{\cdot, \cdot\}'_{\text{Rank2}}$, in addition to being conformally Poisson, is twisted Poisson. Note that this cannot be the case for the other bracket $\{\cdot, \cdot\}'_{\text{Rank2}}$ that describes the dynamics since, as shown in Section 2, its characteristic distribution is not integrable.

Recall from the discussion in 2.4 that $\{\cdot, \cdot\}'_{\text{Rank2}}$ should be considered as a bracket on the reduced space $\mathcal{R} = S^2 \times \mathbb{R}^3$ with redundant coordinates $(\gamma, K)$. The characteristic distribution of the bracket is integrable and the leaves $\mathcal{O}_a$ of the foliation are the level sets $C_1(\gamma, K) = \gamma \cdot K = a$. By regularity and integrability of the characteristic distribution, it follows from Corollary 3.7 that the bracket is $\phi$-twisted. The value of the 3-form $\phi$ is given in the following,

**Theorem 5.7.** The bracket $\{\cdot, \cdot\}'_{\text{Rank2}}$ defined in (2.14) (that in particular describes the reduced dynamics of the Chaplygin sphere for the appropriate choice of $A$), is a $\phi$-twisted Poisson bracket with $\phi = -dB$ where

$$B = mr^2(\Omega \cdot \gamma) \sigma,$$

and where $\sigma$ denotes the area form of the sphere $||\gamma|| = 1$.

**Proof.** The idea of this proof is to show that the bracket is gauge related to a Poisson bracket via the 2-form $-B$. Thus, by Proposition 3.12 the bracket is $(-dB)$-twisted Poisson. More precisely, we will show that $\{\cdot, \cdot\}'_{\text{Rank2}}$ is $(-B)$-gauge related with the bracket $\{\cdot, \cdot\}'_{\text{Rank0}}$ defined in (2.17) and that coincides with the Lie-Poisson bracket on $\mathfrak{se}(3)^\ast$.

According to Theorem 5.4, we denote the bivector field associated to the bracket $\{\cdot, \cdot\}'_{\text{Rank2}}$ by $\pi_{\text{redB}}$. The bivector field $\pi_{\text{redB}}$ and the 2-form $B$ verify hypothesis of Proposition 4.7 and thus the gauge transformation of $\pi_{\text{redB}}$ associated to $B$ is again a bivector field that we will denote it by $\pi_{\text{redB}}^\ast B$. Relying on equation (3.25) and writing in the redundant coordinates $(\gamma, K)$ as

$$B = \frac{1}{2}mr^2(\Omega \cdot \gamma) \gamma \cdot (d\gamma \times d\gamma),$$

one gets that

$$(\pi_{\text{redB}}^\ast B)^2(dK) = K \times \frac{\partial}{\partial K} + \gamma \times \frac{\partial}{\partial \gamma}, \quad (\pi_{\text{redB}}^\ast B)^4(d\gamma) = \gamma \times \frac{\partial}{\partial K}.$$
and the proof is complete.

\[\square\]

**The conformal factor and the 3-form \(\phi\).** In accordance with Proposition 3.10, since the bracket \(\{\cdot, \cdot\}_\text{Rank2}'\) is both conformally Poisson and twisted Poisson, there is relationship between the conformal factor \(\varphi_2\) (given by (5.50)), and the twisting 3-form \(\phi\) (defined in Theorem 5.7).

We leave it to the reader to check that on the leaves \(O_a\) of the foliation of \(\mathcal{R}\) corresponding to the bracket \(\{\cdot, \cdot\}_\text{Rank2}'\), the 3-form \(\phi\) coincides with \(\psi := \frac{1}{\varphi_2} d\varphi_2 \wedge \Omega\), where 2-form \(\Omega\) is given in the redundant coordinates \((\gamma, K)\) by

\[
\Omega = \frac{1}{2} \left( K - m r^2 (\Omega \cdot \gamma) \right) \cdot (d\gamma \times d\gamma) - \gamma \cdot (dK \times d\gamma).
\]

This choice of \(\Omega\) satisfies the conditions of Corollary 3.2 for the graph of the bivector field \(\pi_{\text{red}B}\) corresponding to \(\{\cdot, \cdot\}_\text{Rank2}'\) on \(T\mathcal{R} \oplus T^*\mathcal{R}\).

### Rigid body with generalized rolling constraints of rank 1

A completely analogous analysis can be performed if the rank of the matrix \(A\) equals one. This time it is the bracket \(\{\cdot, \cdot\}_\text{Rank1}\) that is both twisted and conformally Poisson. In analogy with Theorem 5.7 we have

**Theorem 5.8.** The bracket \(\{\cdot, \cdot\}_\text{Rank1}\) defined in (2.16), is a \(\phi\)-twisted Poisson bracket with \(\phi = dB\) with \(B\) given by expression (5.52).

The proof is the same to the Rank 2 case. The bracket \(\{\cdot, \cdot\}_\text{Rank1}\) is \(B\)-gauge related with the bracket \(\{\cdot, \cdot\}_\text{Rank0}\) defined in (2.17) and that coincides with the Lie-Poisson bracket on \(\mathfrak{se}(3)^*\).

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