Generalized Integrals and Solvability

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Abstract

Based on previous work we construct an equation (Lagrange equation) and relate it with a system of generalized integrals and differential equations in such a way to provide useful evaluations and connections between them.

1 The inversion problem in the complex analog

The Lagrange inversion formula states: If $f(A)$ is analytic in a disk $D \subset \mathbb{C}$ of center zero and $f(A) \neq 0$ in $D$, then in some neighborhood around 0 the equation

$$\frac{w}{f(w)} = q \quad (1)$$

have solution

$$w = w(q) = \sum_{n=1}^{\infty} c_n q^n. \quad (2)$$

The coefficients $c_n$ are given from

$$c_n = \frac{1}{n!} \left[ \left( \frac{D}{Dh} \right)^{n-1} (f(h))^{n} \right]_{h=0}, \quad n = 1, 2, \ldots. \quad (3)$$

Moreover we can extend the above theorem (formula) to

$$g_0(w(q)) = g_0(0) + \sum_{n=1}^{\infty} \frac{q^n}{n!} \left[ \left( \frac{D}{Dh} \right)^{n-1} (g_0(h)(f(h))^n) \right]_{h=0}, \quad (4)$$

where $g_0(A)$ is also analytic. Setting $g_0(A) = e^A$, we get

$$e^{w(q)} = 1 + \sum_{n=1}^{\infty} \frac{q^n}{n!} \left[ \left( \frac{D}{Dh} \right)^{n-1} (e^h(f(h))^n) \right]_{h=0} \quad (5)$$
Also in view of [3] Theorem 20, it holds the following formula
\[
ed^w(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-\frac{1}{d}} \sum_{d|n} \frac{\mu(n/d)}{n/d} \left[ (\frac{d}{n})^{d-1} f(h) h \right]_{h=0}. \tag{6}
\]
Here \(\mu(n)\) is the Moebius \(\mu\) function.

Set now
\[
a_n = n c_n. \tag{7}
\]
Then in view of [2], in the function
\[
\frac{1}{P(z)} = \sum_{n=1}^{\infty} a_n q^n, \quad q = e(z) := e^{2\pi iz}, \quad Im(z) > 0, \tag{8}
\]
is attached a differential equation
\[
X'(A) + 2^{4/3} A^{-2/3} (1 - A^2)^{-1/3} P(X(A)) = 0. \tag{9}
\]
If \(m^*(z)\) is the elliptic singular modulus defined as (see [4]):
\[
m^*(z) := \left( \frac{\theta_2 \left( e^{i\pi z} \right)}{\theta_3 \left( e^{i\pi z} \right)} \right)^2, \quad Im(z) > 0, \tag{10}
\]
where \(|q| < 1\):
\[
\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \quad \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{11}
\]
are the "null" Jacobi theta functions. Then also in view of [3] we have that the function
\[
Y(z) = X \left( m^*(2z) \right) \tag{12}
\]
satisfies
\[
Y'(z) = 4\pi i \cdot \eta(z)^4 P(Y(z)). \tag{13}
\]
Moreover if
\[
F(z) := \int_{Y(z)}^{z} \frac{dt}{P(t)}, \tag{14}
\]
then
\[
F(Y(z)) = -\sqrt{2} B_0 \left( m^* \left( 2z \right)^2, \frac{1}{6}, \frac{2}{3} \right), \tag{15}
\]
(here \(B_0(z; a, b) := \int_0^z t^{a-1} (1 - t)^{b-1} dt\) is the incomplete Beta function) and
\[
F \left( Y \left( \frac{1}{z} \right) \right) + F(Y(z)) = -\frac{\sqrt{3} \Gamma \left( \frac{1}{3} \right)^3}{\pi \sqrt{2}}. \tag{16}
\]
Also
\[
\exp \left( 2\pi i \int_{Y(i\infty)}^z \frac{dt}{P(t)} \right) = \prod_{n=1}^{\infty} \left( 1 - q^n \right)^{-\frac{1}{2}} \sum_{d|n} \mu(d) n^{1/2} = \\
= \exp \left( 8\pi^2 \int_{i\infty}^{Y(z')} \eta(t)^4 dt \right). \tag{17}
\]

Note that
\[
2\pi i \int_{z_1}^{z_2} \eta(t)^4 dt = \left[ \frac{1}{\sqrt{2}} B_0 \left( m^* (2t)^2 ; \frac{1}{6}; \frac{2}{3} \right) \right]_{t=z_1}^{t=z_2}, \quad \text{Im}(z_1), \text{Im}(z_2) > 0 \tag{18}
\]
and \(\eta(z)\) is the Dedekind’s eta function
\[
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e(z), \quad \text{Im}(z) > 0. \tag{19}
\]

Now according to connection (7) we have
\[
2\pi i \int_{i\infty}^{z} \frac{dt}{P(t)} = w(q). \tag{20}
\]
Hence
\[
P(z) = \frac{1}{qw'(q)}. \tag{21}
\]

From (15) then we get
\[
w \left( e^{2\pi i Y(z)} \right) = -2\pi i \sqrt{2} B_0 \left( m^* (2z)^2 ; \frac{1}{6}; \frac{2}{3} \right) + c. \tag{22}
\]
Hence
\[
w \left( e^{2\pi i X(A)} \right) = -2\pi i \sqrt{2} B_0 \left( A^2 ; \frac{1}{6}; \frac{2}{3} \right) + c. \tag{23}
\]
But it is known that (see [3] Theorem 12)
\[
X(A) = h \left( \sqrt{2} B_0 \left( A^2 ; \frac{1}{6}; \frac{2}{3} \right) \right) \tag{24}
\]
Hence easily
\[
e^{2\pi i h(A)} = w^{-1}\left( -2\pi i A + c \right) \tag{25}
\]
and
\[
e^{2\pi i h(A)} = -\frac{2\pi i A + c}{f(-2\pi i A + c)}. \tag{26}
\]
But relation (24) gives (we use the notation \(q_A = e(A)\)):
\[
w \left( e^{2\pi i A} \right) = -2\pi ih_i(A) + c \Rightarrow
\]
\[ h_i(A) = -\frac{w(q_A)}{2\pi i} + \frac{c}{2\pi i} \]  
(27)

and

\[ h'_i(A) = -w'(q_A)q_A. \]  
(28)

Hence we get also that

\[ h_i(i\infty) = \frac{c}{2\pi i} \Rightarrow h\left(\frac{c}{2\pi i}\right) = i\infty \]  
(28.1)

and

\[ Y(i\infty) = X(0). \]  
(28.2)

From the analysis given in [3] we have (Corollary 1, eq 33) we have

\[ \int_0^{\mu(A)} \frac{dt}{t^{\psi t^{-5} - 11 - t^5}} = -\frac{w\left(e^{2\pi i A}\right)}{2\pi i} + \frac{c}{2\pi i}. \]  
(29)

However if we introduce the function \( F_1(A) \) (as in [1]) such that

\[ F_1'(A) = 5^{-1}F_1(A)\psi F_1(A)^{-5} - 11 - F_1(A)^5, \]  
(30)

then

\[ F_1^{-1}(A) = 6A^{5/6}F_{Ap}\left[\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{7}{6}, \frac{-2A^5}{11 + 5\sqrt{5}}, \frac{-2A^5}{11 - 5\sqrt{5}}\right], \]  
(31)

where

\[ F_{Ap}(a, b_1, b_2; c; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}x^m y^n, |x| < 1, |y| < 1, \]  
(32)

is the first Appell function (not to confused with \( F_1 \) defined in this article) and

\[ \int_0^{F_1(A)} \frac{dt}{t^{\psi t^{-5} - 11 - t^5}} = A. \]  
(33)

For this reason we can write

**Theorem 1.**

The functions \( y(A) \) and \( w(q_A) \) are related with the following identity:

\[ F_1\left(-\frac{w(q_A)}{2\pi i} + \frac{c}{2\pi i}\right) = y(A). \]  
(34)

Moreover we have

\[ \frac{1}{G(y(A))} = h'_i(A) = -w'(q_A)q_A. \]  
(35)
Hence
\[-w'(qA)qAG\left(F_1\left(-\frac{w(qA)}{2\pi i} + \frac{c}{2\pi i}\right)\right) = 1 \Leftrightarrow \] (36)

\[G(y(A)) = -P(A). \] (37)

Continuing our arguments we have
\[G\left(F_1\left(-\int_{i\infty}^{P_i(A)} \frac{dt}{P(t)} + \frac{c}{2\pi i}\right)\right) = -A. \] (38)

This is true because
\[-\frac{w(qA)}{2\pi i} = -\int_{i\infty}^{A} \frac{dt}{P(t)}. \]

Hence (38) gives
\[-\int_{i\infty}^{P_i(A)} \frac{dt}{P(t)} + \frac{c}{2\pi i} = F_1^{(-1)}\left(G^{(-1)}(-A)\right) \Rightarrow \]
\[-\frac{P_i(A)}{A} = \frac{d}{dA}\left(F_1^{(-1)}\left(G^{(-1)}(-A)\right)\right) \Rightarrow \]
\[P^{(-1)'}(A) = -A \frac{d}{dA}\left(F_1^{(-1)}\left(G^{(-1)}(-A)\right)\right). \] (39)

Also
\[P^{(-1)'}(e(A))2\pi i = -e(A)2\pi iF_1^{(-1)'}\left(G^{(-1)}(-e(A))\right)G^{(-1)'}(-e(A)) \Leftrightarrow \]
\[2\pi i \int P^{(-1)'}(e(A))dA = F_1^{(-1)}\left(G^{(-1)}(-e(A))\right). \]

Hence assuming that \(P^{(-1)'}(A) = H'(A)A\) we get
\[2\pi iH(A) = F_1^{(-1)}\left(G^{(-1)}(-A)\right) + c. \]

However we have the next theorems.

**Theorem 2.**

Assuming \(q = e^{2\pi iA}\), \(Im(A) > 0\), we have
\[
\int_{e^{2\pi i}}^{e^{2\pi i+c/(2\pi i)}} G(F_1(t))dt = A. \] (40)

Hence given \(G\) we can find \(w(q)\) and the opposite.

**Proof.**

Integrate (36).
Theorem 3. Knowing $G(A)$ (resp. $w(q_A)$) we can find $w(q_A)$ (resp. $G(A)$) from Theorem 2 and then $P(A)$ from the identity

$$G \left( F_1 \left( -\frac{w(q_A)}{2\pi i} + \frac{c}{2\pi i} \right) \right) = -P(A).$$

(41)

Also there exists the relations

$$w(q_A) = 2\pi i \int_{\infty}^{A} \frac{dt}{P(t)} \quad \text{and} \quad P(A) = \frac{1}{q_A w'(q_A)},$$

(42)

where $q_A = e^{2\pi i A}, \ Im(A) > 0$.

Theorem 4.
Given the function $G(A)$ and assuming function $y(A)$ is solution to the problem

$$5 \int_0^{y(A)} \frac{G(t)}{t^{\sqrt{t} - 11 - t}} \, dt = A,$$

(43)

then $P(A)$ is such that

$$P^{(-1)'}(A) = -A \frac{d}{dA} \left( F_1^{(-1)} \left( G^{(-1)}(-A) \right) \right)$$

(44)

and $y(A)$ is solution of the semialgebraic equation

$$G(y(A)) + P(A) = 0.$$  

(45)

The function $F_1(A)$ is the known function defined in (30),(31),(33). Also $P(A)$ is given from

$$P(A) = -\frac{1}{q_A w'(q_A)}, \ q_A = e(A).$$  

(46)

Theorem 5.
If $P(A)$ is the function (8), then the function

$$w(q) = 2\pi i \int_{\infty}^{z} \frac{dt}{P(t)} \left( q = e(z), \ Im(z) > 0 \right),$$

(47)

is the solution of

$$\frac{w(q)}{f(w(q))} = q,$$

(48)

where $f(A)$ is given from

$$\frac{f'(A)}{f(A)} = \frac{1}{A} + G \left( F_1 \left( -\frac{A}{2\pi i} + \frac{c}{2\pi i} \right) \right).$$

(49)

Also then

$$y(A) = F_1 \left( -\frac{w(q_A)}{2\pi i} + \frac{c}{2\pi i} \right).$$

(50)
Theorem 6.  
Assume that exists function $P_0(A)$ such that
\[ G(F_1(A)) = -\frac{1}{c - 2\pi i A} + P_0(c - 2\pi i A). \]  
(51)

Then
\[ y(A) = F_1\left(-\frac{w\left(e^{2\pi i A}\right)}{2\pi i} + \frac{c}{2\pi i}\right), \]  
(52)

where $w(q)$ is solution of the equation
\[ w(q) \exp\left(C - \int w(q) P_0(t)dt\right) = q. \]  
(53)

Remark. The opposite also can be used: To evaluate a root $w(q)$ of (53), we evaluate $G(A)$ from (51) and then
\[ w(q) = c - 2\pi i F_1^{(-1)}(y(A)). \]  
(53.1)

Theorem 7.  
i) It holds
\[ F(z) = \frac{1}{2\pi i} w(q) + c_1, \ q = e(z), \ Im(z) > 0, \]  
(54)

where
\[ c_1 = \lim_{\sigma \to +\infty} \int_{Y(\sigma)}^{+\infty} \frac{dt}{P(t)} = \int_{Y(\sigma)}^{+\infty} \frac{dt}{P(t)}. \]  
(54.1)

ii) If we define the function $g(A)$ to be
\[ g(A) = A \exp\left(-\int_{c}^{A} P_0(t)dt\right), \]  
(55)

and $P_0$ defined as in Theorem 6. Then the relation
\[ g^{(k)}(2\pi i (c_0 - 2c_1)) = (-1)^k g^{(k)}(0), \ c_0 = -\frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3}{\pi \sqrt{2}}, \]  
(56)

is impossible when $g$ is not constant.

Remarks.  
i) Condition (56) is equivalent to say that $g$ is analytic in $D_0$ and
\[ g(2\pi i (c_0 - 2c_1) - z) = g(z), \ \forall z \in D_0. \]  
(57)

ii) The set $D_0$ is subset of $C$ containg at least one circle with origin 0 and radius greater than $2\pi |c_0 - 2c_1| > 0$.  

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Proof.
Assuming that (56) holds for every $k = 0, 1, 2, \ldots$. We consider the Taylor series of $g$ around 0 and $2\pi i (c_0 - 2c_1) \neq 0$. We then have

$$e(Y(z)) = w(e(Y(z))) \exp \left( - \int_c^{w(e(Y(z))} P_0(t) \, dt \right)$$

and

$$e(Y(z)) = \sum_{k=0}^{\infty} g^{(k)}(2\pi i (c_0 - 2c_1)) \frac{(w(e(Y(z)) - 2\pi i (c_0 - 2c_1))^k}{k!} =$$

$$= \sum_{k=0}^{\infty} g^{(k)}(2\pi i (c_0 - 2c_1)) \frac{(2\pi i)^k (F(Y(z)) - c_0 + c_1)^k}{k!} =$$

$$= \sum_{k=0}^{\infty} g^{(k)}(2\pi i (c_0 - 2c_1)) \frac{(2\pi i)^k \left( -F \left( Y \left( \frac{-1}{z} \right) \right) + c_1 \right)^k}{k!} =$$

$$= \sum_{k=0}^{\infty} g^{(k)}(0) \left( w \left( e \left( Y \left( \frac{-1}{z} \right) \right) \right) \right)^k = e \left( Y \left( \frac{-1}{z} \right) \right).$$

Hence $2\pi i Y \left( \frac{-1}{z} \right) = 2\pi i Y(z) + 2\pi i k_0$, $k_0 \in \mathbb{Z}$. Hence $Y \left( \frac{-1}{z} \right) = Y(z) + k_0$ and from relation (16) $F(z) + F(k_0 + z) = c_0$.

for all $z \in D_0$. But from periodicity of $\frac{1}{P(z)}$ we have the existence of another constant $c_2$ such that $F(z + 1) - F(z) = c_2$. Hence $F(z + k_0) - F(z) = k_0 c_2$ and $F(z) + k_0 c_2 + F(z) = c_0$. Hence $F$ is constant, which is impossible.

Example.
Assume

$$G(A) = -\frac{1}{c - 2\pi i F_1^{-1}(A)} + \cos \left( c - 2\pi i F_1^{-1}(A) \right).$$

Then $P_0(A) = \cos A$, and the function $y(A)$ such that

$$5 \int_0^{y(A)} \frac{G(t)}{t \sqrt{t^2 - 11 - t^5}} = A$$

is

$$y(A) = F_1 \left( \frac{c}{2\pi i} - \frac{w(e(A))}{2\pi i} \right).$$
where $w(q)$ is solution of

$$w(q)e^{C_1 - \sin(w(q))} = q.$$ 

**Theorem 8.**

Assume the function $h_0$ defined from the relations

$$g^{(k)}(2\pi i(c_0 - 2c_1)) = (-1)^k g^{(k)}(0), \quad c_0 = -\frac{\sqrt{3}\Gamma(\frac{4}{3})}{\pi \sqrt{2}}, \quad (58)$$

where

$$g_2(z) = h_0(g(z)). \quad (59)$$

Then if $c_1$ denotes the constant $c_1 = F(z) - \frac{w(e(z))}{2\pi i}$ we have

$$g(2\pi i(c_0 - 2c_1) - z) = h_0(g(z)), \quad (60)$$

$$w(A) + w(h_0(A)) = c_2 = 2\pi i(c_0 - 2c_1) = \text{constant}, \quad (61)$$

$$e \left( Y \left( \frac{-1}{z} \right) \right) = h_0 \left( e \left( Y(z) \right) \right) \quad (62)$$

and

$$h_0(h_0(z)) = z. \quad (63)$$

**Proof.**

Relation (62) can be shown as in Theorem 7. For to show (61) we have

$$F(z) = \frac{w(e(z))}{2\pi i} + c_1$$

and

$$F(Y(z)) + F \left( Y \left( \frac{-1}{z} \right) \right) = c_0.$$ 

Hence

$$w(e(Y(z))) + 2\pi i c_1 + w \left( e \left( Y \left( \frac{-1}{z} \right) \right) \right) + 2\pi i c_1 = 2\pi i c_0 \Leftrightarrow$$

$$w(e(Y(z))) + w(h_0(e(Y(z)))) = 2\pi i(c_0 - 2c_1).$$

Hence

$$w(A) + w(h_0(A)) = 2\pi i(c_0 - 2c_1).$$

Setting where $h_0 \to h_0^{(-1)}$ in the last equation, we have $w(h_0^{(-1)}(A)) + w(A) = 2\pi i(c_0 - 2c_1)$. Hence

$$w \left( h_0^{(-1)}(A) \right) = w(h_0(A)) \Rightarrow h_0(h_0(A)) = A.$$
QED.

**Theorem 9.**
We define the $B$ function to be such
\[ h_0(A) = e(B(A)), \]  
and the $\lambda$ function
\[ \lambda(A) = B(e(A)). \]  
Then
\[ \int_{\infty}^{A} \frac{dt}{P(t)} + \int_{\infty}^{\lambda(A)} \frac{dt}{P(t)} = c_0 - 2c_1, \]  
\[ \lambda'(A) = -\frac{P(\lambda(A))}{P(A)}, \]  
\[ F(A) + F(\lambda(A)) = \text{constant}, \]  
\[ F(\lambda(\lambda(A))) = F(A), \]  
where $F, \lambda$ are 1-periodic
\[ h_0(e(A)) = e(\lambda(A)). \]  
There exists always integer $k = k(z)$ such that
\[ Y\left(-\frac{1}{z}\right) = \lambda(Y(z)) + k \]  
and
\[ 2\pi iF(B(A)) + w(A) = 2\pi i(c_0 - c_1). \]  

**Proof.**
It holds
\[ w(e(A)) + w(h_0(e(A))) = c_2 \Leftrightarrow \]
\[ 2\pi i \int_{\infty}^{A} \frac{dt}{P(t)} + 2\pi i \int_{\infty}^{B(e(A))} \frac{dt}{P(t)} = c_2 \Leftrightarrow \]
\[ \int_{\infty}^{A} \frac{dt}{P(t)} + \int_{\infty}^{\lambda(A)} \frac{dt}{P(t)} = \frac{c_2}{2\pi i}. \]
Setting $A \rightarrow \lambda(A)$, we get
\[ \int_{\infty}^{\lambda(A)} \frac{dt}{P(t)} = \frac{c_2}{2\pi i} - \int_{\infty}^{\lambda(A)} \frac{dt}{P(t)} \Leftrightarrow \]
\[ \int_{\infty}^{\lambda(A)} \frac{dt}{P(t)} = \int_{\infty}^{A} \frac{dt}{P(t)}. \]
Hence

\[ F(\lambda(\lambda(A))) = F(A). \]

For (68) we have

\[ Y \left( \frac{-1}{A} \right) = F^{(-1)}(c_0 - F(Y(A))) \Leftrightarrow \]
\[ e \left( Y \left( \frac{-1}{A} \right) \right) = e \left( F^{(-1)}(c_0 - F(Y(A))) \right) \Leftrightarrow \]
\[ h_0(e(Y(z))) = e \left( F^{(-1)}(c_0 - F(Y(A))) \right) \Leftrightarrow \]
\[ h_0(e(A)) = e \left( F^{(-1)}(c_0 - F(A)) \right) \Leftrightarrow \]
\[ e(B(e(A))) = e \left( F^{(-1)}(c_0 - F(A)) \right) \Leftrightarrow \]
\[ B(e(A)) = F^{(-1)}(c_0 - F(A)) + k \Rightarrow \]

Hence if \( k = 0 \) we get

\[ F(B(e(A))) + F(A) = c_0 \Rightarrow \]
\[ (F \circ B)(e(B(A))) + (F \circ B)(A) = c_0 \Leftrightarrow \]
\[ (F \circ B)(h_0(A)) + (F \circ B)(A) = c_0 \Rightarrow \]
\[ F(B(e(A))) + F(B(e(B(e(A))))) = c_0 \Leftrightarrow F(\lambda(A)) + F(\lambda(\lambda(A))) = c_0 \Leftrightarrow \]
\[ F(\lambda(A)) + F(A) = c_0 \]

For (72) we have

\[ F(B(A)) = \frac{w(e(B(A)))}{2\pi i} + c_1 = \frac{w(h_0(A))}{2\pi i} + c_1 \Rightarrow \]
\[ F(B(A)) = \frac{2\pi i(c_0 - 2c_1) - w(A)}{2\pi i} + c_1 \Leftrightarrow \]
\[ 2\pi iF(B(A)) + w(A) = 2\pi ic_1 + 2\pi i(c_0 - 2c_1) \]

which give us immediately (72). The proof of other identities are similar and easy.

Now set

\[ c' = c_1 - c_0 + \frac{c}{2\pi i}. \]

Hence easily

\[ c' + F(B(qA)) = \frac{c}{2\pi i} - \frac{w(qA)}{2\pi i} = h_i(A) \Rightarrow \]
\[ c' + F(\lambda(A)) = h_i(A). \]

(73)
Hence \( h_i(A + 1) = h_i(A) \). Also
\[
\int_{Y(i\infty)}^{\lambda(A)} \frac{dt}{P(t)} = \int_{Y(i\infty)}^{i\infty} \frac{dt}{P(t)} + \int_{i\infty}^{\lambda(A)} \frac{dt}{P(t)} = h_i(A) \Rightarrow
\]
\[
c_0 - c_1 - \int_{i\infty}^{A} \frac{dt}{P(t)} = h_i(A) \Rightarrow h_i(A) = c'' - \int_{i\infty}^{A} \frac{dt}{P(t)} \Rightarrow
\]
\[
A = c_0 - c_1 - \int_{i\infty}^{h_i(A)} \frac{dt}{P(t)} \Rightarrow h_i(A) = -P(h_i(A)),
\]
where \( c'' = c_0 - c_1 \). Also from (27):
\[
w(q_A) = -2\pi i h_i(A) + c
\]
and from (61):
\[
w(e(A)) + w(h_0(e(A))) = c_0 - 2c_1 \Rightarrow
\]
\[
w(e(A)) + w(e(\lambda(A))) = c_0 - 2c_1 \Rightarrow
\]
\[
-2\pi i h_i(A) - 2\pi i h_i(\lambda(A)) + 2c = c_0 - 2c_1.
\]
Hence we get the next

**Theorem 10.**

We have
\[
h_i(A) = c' + F(\lambda(A)), \quad h_i(A + 1) = h_i(A), \tag{74}
\]
\[
h_i'(A) = -P(h_i(A)), \quad \lambda(A) = h_i(F(A)) \tag{75}
\]
and
\[
h_i(A) + h_i(\lambda(A)) = \frac{-c_0 + 2c_1 + 2c}{2\pi i}. \tag{76}
\]

**Remark.** \( h_i(A) \) denotes inversion i.e. \( h_i(A) = h^{(i-1)}(A) \), \( f_i'(A) = f^{(i-1)'}(A) \), etc.

**Theorem 11.**

There exists constants \( c, c_1' \) such that
\[
h_i(A) = \frac{c}{2\pi i} - \frac{w(q_A)}{2\pi i} = \frac{c_1'}{2\pi i} - F(A). \tag{77}
\]

About the "shape" of function \( G \), we assume first that \( G(F_1(A)) \) is analytic and set
\[
H(z) := G(F_1(z)).
\]
Then
\[
H(z + z_0) = \sum_{k=0}^{\infty} \frac{H^{(k)}(z_0)}{k!} z^k.
\]
Hence we can write

\[-P(A) = - \frac{1}{q_A w'(q_A)} = H \left( - \frac{w(q_A)}{2\pi i} + \frac{c}{2\pi i} \right) = \]

\[= \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{k!} \frac{(-1)^k}{(2\pi i)^k} w(q_A)^k \Rightarrow \]

\[- \frac{1}{Aw'(A)} = \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{k!} \frac{(-1)^k}{(2\pi i)^k} w(A)^k \Rightarrow \]

\[- \frac{w^{(-1)'}(A)}{w^{(-1)}(A)} = \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{k!} \frac{(-1)^k}{(2\pi i)^k} A^k \Rightarrow \]

\[- \log w^{(-1)}(A) = c + \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{k!} \frac{(-1)^k}{(2\pi i)^k} A^{k+1} \Rightarrow \]

\[w^{(-1)}(A) = \exp \left( -c - A \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{(k+1)!} \frac{(-1)^k}{(2\pi i)^k} A^{k+1} \right) \Rightarrow \]

\[f(A) = A \exp \left( c + A \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{(k+1)!} \frac{(-1)^k}{(2\pi i)^k} A^{k+1} \right). \] (78)

Hence given a function \( G \), we can find \( f \) setting \( H(z) = G(F_1(z)) \) and

\[f(A) = A \exp \left( c + A \sum_{k=0}^{\infty} \frac{H^{(k)} \left( \frac{c}{2\pi i} \right)}{(k+1)!} \frac{(-1)^k}{(2\pi i)^k} A^{k+1} \right). \] (78)

However we have assumed that \( f(0) \) is not zero arround 0 "say" in \( D \subset \mathbb{C} \). Hence \( G(F_1(A)) \) must have a pole. We can write

\[G(F_1(A)) = - \frac{1}{c - 2\pi i A} + P_0(c - 2\pi i A), \] (78.1)

for some function \( P_0(A) \). Then

\[G(F_1(A)) = \frac{-1}{c - 2\pi i A} + \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c) (-1)^k (2\pi i)^k}{k!} A^k. \]

That is because

\[G \left( F_1 \left( \frac{c}{2\pi i} - \frac{A}{2\pi i} \right) \right) = - \frac{1}{A} + \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c) (-1)^k (2\pi i)^k}{k!} \left( \frac{c}{2\pi i} - \frac{A}{2\pi i} \right)^k =\]

\[= - \frac{1}{A} + \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c)}{k!} (A - c)^k = \frac{1}{A} + P_0(A), \]
which is true. Also
\[-P(A) = -\frac{1}{w(qA)} + P_0(w(qA)) \iff -\frac{1}{qAw'(qA)} = -\frac{1}{w(qA)} + P_0(w(qA)) \iff
\]
\[-\frac{1}{Aw'(A)} = -\frac{1}{w(A)} + P_0(w(A)) \iff \frac{w^{(-1)'}(A)}{w^{(-1)}(A)} = -\frac{1}{A} + P_0(A) \iff
\]
\[-\log w^{(-1)}(A) = -\log A + \int_{C_1}^A P_0(t)dt + C_0 \iff
\]
\[\frac{1}{w^{(-1)}(A)} = \frac{e^{C_0}}{A} \exp \left( \int_{C_1}^A P_0(t)dt \right) \iff w^{(-1)}(A) = Ae^{-C_0} \exp \left( -\int_{C_1}^A P_0(t)dt \right).\]

Hence
\[f(A) = \exp \left( C_0 + \int_{C_1}^A P_0(t)dt \right) \iff \frac{f'(A)}{f(A)} = P_0(A).\]

Hence if \(f\) is analytic and not zero around 0, then so \(P_0(A)\) is also analytic and we have the next

**Theorem 12.**
Assuming the problem (1),(2),(7), be well defined and in accordance with Ramanujan-Jacobi integral (43), the function \(G(F_1(A))\) must be meromorphic with a single simple pole at \(A_0 = \frac{b_1}{2\pi i}.\) The constant \(c\) is given by \(c = w(e(X(0))).\) Moreover it holds
\[G(F_1(A)) = \frac{-1}{c - 2\pi i A} + \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c) (-1)^k (2\pi i)^k}{k!} A^k, \quad (79)\]
where
\[f(A) = \exp \left( C_0 + \int_{C_1}^A P_0(t)dt \right) \iff \frac{f'(A)}{f(A)} = P_0(A). \quad (80)\]

Hence setting \(P_0(A) = \frac{f'(A)}{f(A)},\) then \(G\) is given by (79) and the opposite.

If \(D_1 = b_1^2 - 4a_1c_1\) and
\[U(x) = U(a_1, b_1; m; x) =
\]
\[= (-1)^m+1 a_1^{m-1} D_1^{-m+1/2} B_0 \left( \frac{-b_1 + \sqrt{D_1} - 2a_1x}{2\sqrt{D_1}}; 1 - m, 1 - m \right). \quad (81)\]

Then from [2]:
\[\exp \left( 2\pi i \int_{\omega_1}^{\omega_2} \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} dt \right) = \frac{\exp \left( 2\pi i \left( h(U(\omega_2)) \right) \right)}{\exp \left( 2\pi i \left( h(U(\omega_1)) \right) \right)} =
\]

14
\[
\frac{w(-1)(-2\pi iU(\omega_2) + c)}{w(-1)(-2\pi iU(\omega_1) + c)} = \frac{(-2\pi iU(\omega_2) + c)\exp\left(-\int_{C_1}^{2\pi iU(\omega_2) + c} P_0(t)dt\right)}{(-2\pi iU(\omega_1) + c)\exp\left(-\int_{C_1}^{2\pi iU(\omega_1) + c} P_0(t)dt\right)}.
\]

where
\[
f_1(A) = -\frac{1}{c - 2\pi iU(A)} + P_0(c - 2\pi iU(A)).
\]

But \(h'(A) = G(F_1(A)) = f_1(U^{(-1)}(A))\). Hence \(P_0(A) = \frac{f'(A)}{f(A)}\) and \(\frac{w(q)}{f(w(q))} = q\).

Therefore
\[
f_1\left(U^{(-1)}\left(\frac{c}{2\pi i} - \frac{A}{2\pi i}\right)\right) = -\frac{1}{A} + P_0(A).
\]

Hence
\[
f_1\left(U^{(-1)}(A)\right) + \frac{1}{c - 2\pi iA} = \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c)(-1)^k(2\pi i)^k}{k!} A^k \Rightarrow
\]
\[
h(A) = \frac{1}{2\pi i} \log(c - 2\pi iA) + A \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c)(-1)^k(2\pi i)^k}{(k + 1)!} A^k + C_1
\]
and
\[
h^{(-1)}(A) = -\frac{w(qA)}{2\pi i} + \frac{c}{2\pi i}.
\]

Hence we have the next theorem

**Theorem 13.**

i) Assume that the function \(f_1\) is known and of the form
\[
f_1\left(U^{(-1)}\left(\frac{c}{2\pi i} - \frac{A}{2\pi i}\right)\right) = -\frac{1}{A} + P_0(A),
\]
where \(P_0(A)\) analytic around 0. Then
\[
\exp\left(2\pi i \int_{\omega_1}^{\omega_2} \frac{f_1(t)}{(a_1t^2 + b_1t + c_1)^m} dt\right) = \frac{(-2\pi iU(\omega_2) + c)\exp\left(-\int_{C_1}^{2\pi iU(\omega_2) + c} P_0(t)dt\right)}{(-2\pi iU(\omega_1) + c)\exp\left(-\int_{C_1}^{2\pi iU(\omega_1) + c} P_0(t)dt\right)}.
\]

ii) \(h(A) = \frac{1}{2\pi i} \log(c - 2\pi iA) + A \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c)(-1)^k(2\pi i)^k}{(k + 1)!} A^k + C_1\)

Therefore
\[
\frac{w(-1)(-2\pi iU(\omega_2) + c)}{w(-1)(-2\pi iU(\omega_1) + c)} = \frac{(-2\pi iU(\omega_2) + c)\exp\left(-\int_{C_1}^{2\pi iU(\omega_2) + c} P_0(t)dt\right)}{(-2\pi iU(\omega_1) + c)\exp\left(-\int_{C_1}^{2\pi iU(\omega_1) + c} P_0(t)dt\right)}.
\]

where
\[
f_1(A) = -\frac{1}{c - 2\pi iU(A)} + P_0(c - 2\pi iU(A)).
\]
and $C_1$ being a constant.

**Theorem 13.1**

Given the functions $f_1(A)$ and $U(A)$ ($U$ being that of (81)), such that $f_1 \left( \frac{U^{-1}(A)}{2\pi i} \right)$ is meromorphic with only simple pole at $A_0 = 0$ and residue $-1$ i.e. with Laurent expansion (82) and $P_0(A) = \frac{f'(A)}{f(A)}$ is analytic. Then with the notation of the remarks below we have

\[ \int_{-\rho_1 - \sqrt{D_1}/a_1}^{-\rho_1} \frac{f_1(t)}{\beta(z_1)} \frac{dt}{(a_1 t^2 + b_1 t + c_1)^m} = \frac{1}{2\pi i} \left[ \log \left( \frac{c - 2\pi i t}{f(c - 2\pi i t)} \right) \right]_{t=\Omega(z_2)}. \]

The function $\Omega(z)$ is

\[ \Omega(z) := (-1)^{m+1} a_1^{m-1} D_1^{(-m+1/2)} \frac{\Gamma(1-m)}{\Gamma(2(1-m))} \frac{1}{1 - z^2} \]

and $\rho_1 = \frac{b_1 - \sqrt{D_1}}{2a_1}$.

**Remarks.**

i) Setting $B_\alpha(z) = \sqrt{B_0(z;\alpha,\alpha)}$, $0 < \alpha < 1$, we have the equation

\[ i \frac{B_{1-m}(1-t)}{B_{1-m}(t)} = z, \quad 0 < m < 1, \quad Im(z) > 0, \]

have solution $t = \beta(z)$. For this solution holds

\[ B_\beta(z;1-m,1-m) = \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \frac{1}{1 - z^2}. \]

Also

\[ U \left( -\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z) \right) = (-1)^{m+1} a_1^{m-1} D_1^{(-m+1/2)} \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \frac{1}{1 - z^2}. \]

ii) Hence the beta functions $B_{1-m}(z)$ form (88). For $m$ rational in $0 < m < 1$, we have numerical evidences that $\beta(z)$ are algebraic numbers when $z = x + i\sqrt{y}$, $x, y$ rationals, with $y > 0$. Hence it is of interest to examine these functions. Also it is of interest to reduce the evaluation of general integrals such the left side of (85) with these simple functions.

iii) Therorem 13.1 tell us that if $f_1$ is a function such that $f_1 \circ U_i$ have simple Laurent expansion, then we can evaluate integral (85) using the analytic part $P_0$ of $f_1 \circ U_i$. The evaluation requires only the knowledge of $f$ and $\frac{f'(A)}{f(A)} = P_0(A)$.

iv) The problem also related with Ramanujan-Jacobi integrals (see relation (29) and [2]) and holds $h'(A) = G(F_1(A)) = f_1(U_i(A))$. This last equation and Theorem 2 give rise to Lagrange inversion formula, since it holds

\[ h \left( \frac{c}{2\pi i} - \frac{w(qA)}{2\pi i} \right) = A. \]
Proof.
Assume the Lagrange equation
\[ \frac{w(A)}{f(w(A))} = A. \]
We find \( P_0(A) \) from
\[ P_0(A) = \frac{f'(A)}{f(A)}. \]
Then holds the following integral
\[ \int_{U_1(h_1(z_1))}^{U_1(h_1(z_2))} \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} dt = z_2 - z_1 \]
(91)
where
\[ f_1(A) = -\frac{1}{c - 2\pi i U(A)} + P_0 (c - 2\pi i U(A)). \]
and
\[ w(e(A)) = -2\pi i h_i(A) + c. \]
(92)
Hence
\[ \int_{z_1}^{z_2} \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} dt = \frac{1}{2\pi i} \log \left( \frac{c - 2\pi i U(z_2)}{c - 2\pi i U(z_1)} \right) + \]
\[ + \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c)(-2\pi i)^k}{(k+1)!} \left( U(z_2)^{k+1} - U(z_1)^{k+1} \right). \]
(92.1)
Hence if we set in \( A_2, A_1 \) the values
\[ A_2 = -\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z_2), \quad A_1 = -\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z_1), \]
we get
\[ \int_{-\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z_2)}^{-\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z_1)} \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} dt = \frac{1}{2\pi i} \log \left( \frac{c - 2\pi i U(A_2)}{c - 2\pi i U(A_1)} \right) + \]
\[ + \sum_{k=0}^{\infty} \frac{P_0^{(k)}(c)(-2\pi i)^k}{(k+1)!} \left( -1 \right)^{(m+1)k+1} a_1^{m-1} k^k \frac{\Gamma(1-m)2^{(k+1)}}{\Gamma(2(1-m))} \left[ \frac{1}{(1-t^2)^k} \right]_{t=z_1}^{t=z_2}. \]
Hence if \( P_0^{(k)}(c) = P_1^{(k+1)}(c) \), then
\[ \int_{-\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z_2)}^{-\rho_1 - \frac{\sqrt{D_1}}{a_1} \beta(z_1)} \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} dt = \frac{1}{2\pi i} \log \left( \frac{c - 2\pi i U(A_2)}{c - 2\pi i U(A_1)} \right) - \]
\[ - \frac{1}{2\pi i} \left[ P_1 \left( c - 2\pi i (-1)^{m+1} a_1^{m-1} D_1^1 (-m+1/2) \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \frac{1}{1-t^2} \right) \right]_{t=z_1}^{t=z_2} \]
\[
\int_{-\rho - \sqrt{\pi} \mathcal{H}(z_2)}^{\rho + \sqrt{\pi} \mathcal{H}(z_1)} \frac{f_1(t)}{(-c_1 t^2 + b_1 t + c_1)^m} \, dt = \frac{1}{2\pi i} \log \left( \frac{c - 2\pi i U(A_2)}{c - 2\pi i U(A_1)} \right) - \frac{1}{2\pi i} \int_{-2\pi i(-1)^{m+1}a_1^{m-1}D_1^{(-m+1/2)} \left( \frac{1}{\Gamma(1-m)} \right)} \left( \frac{1}{1 - z_2^2} \right) \rho_0(t + c) \, dt \\
- \frac{1}{2\pi i} \int_{-2\pi i(-1)^{m+1}a_1^{m-1}D_1^{(-m+1/2)} \left( \frac{1}{\Gamma(1-m)} \right)} \left( \frac{1}{1 - z_1^2} \right) \rho_0(t + c) \, dt = \frac{1}{2\pi i} \log \left( \frac{c - 2\pi i U(A_2)}{c - 2\pi i U(A_1)} \right) - \\
\frac{1}{2\pi i} \log \left( \frac{f \left( c - 2\pi i(-1)^{m+1}a_1^{m-1}D_1^{(-m+1/2)} \left( \frac{1}{\Gamma(1-m)} \right) \right)}{f \left( c - 2\pi i(-1)^{m+1}a_1^{m-1}D_1^{(-m+1/2)} \left( \frac{1}{\Gamma(1-m)} \right) \right)} \right) .
\]

**Example.**

If \( \rho_0(A) = 1 \), then we have \( f(A) = C_0 e^A \) and \( w_i(A) = C_0^{-1} A e^{-A} \), with \( w(A) = -W(-C_0 A) \) and \( W(A) \) is the Lambert’s function, \( C_0 = e^{-C} \).

\[
f_1(U_i(A)) = -\frac{1}{c - 2\pi i A} + 1 \iff f_1(A) = -\frac{1}{c - 2\pi i U(A)} + 1 .
\]

Also then

\[
\exp \left( 2\pi i \int_0^\omega \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} \, dt \right) = e(U(\omega)) \left( 1 - 2\pi i e^{-1} U(\omega) \right) .
\]

On the other hand we have \( G(F_1(A)) = \frac{1}{c - 2\pi i F_1^{-1}(A)} + 1 \). Hence

\[
G(A) = -\frac{1}{c - 2\pi i F_1^{-1}(A)} + 1 .
\]

The function \( h_0(A) \) is such that \( w(A) + w(h_0(A)) = c_{11} \). Hence

\[
h_0(A) := e^{C - c_{11}} W(-A e^{-C}) \left( c_{11} + W(-A e^{-C}) \right) , \quad c_{11} = 2\pi i(c_0 - c_1)
\]

and indeed holds \( h_0(h_0(A)) = A \), \( W(x) \) is the Lambert’s \( W \) function. The function \( y(A) \) is

\[
y(A) = F_1 \left( \frac{W(-q e^{-C})}{2\pi i} + \frac{c}{2\pi i} \right) , \quad q = e(A) .
\]
\[ F(A) = \frac{c_1 - c}{2\pi i} - \frac{W(qe^{-C})}{2\pi i}, \quad q = e(A) \]

\[ P(A) = -\frac{1 + W(-qe^{-C})}{W(-qe^{-C})}, \quad q = e(A). \]

Also from (84)
\[ h(A) = \frac{1}{2\pi i} \log \left( A - \frac{c}{2\pi i} \right) + A + C_1 \]

and from (24),(12)
\[ Y(A) = \frac{1}{2\pi i} \log \left( \sqrt[3]{2} B_0 \left( m^*(2\pi^2)^2; \frac{1}{6}, \frac{1}{3} \right) - \frac{c}{2\pi i} \right) + \sqrt[3]{2} B_0 \left( m^*(2\pi^2)^2; \frac{1}{6}, \frac{1}{3} \right) + C_1. \]

Note. Solving equation (9) with "Mathematica" program (I have "Mathematica 11") requires some extra care when using the constants. Also mathematica does not recognizes \( W(xe^x) = x \) and it is better to use \( 6A^{1/3} F_1 \left( \frac{1}{6}, \frac{1}{3}; \frac{1}{2}; A^2 \right) \) in place of \( B_0 \left( A^2; \frac{1}{6}, \frac{1}{3} \right) \). A better example is to take \( P_0(t) = \frac{1}{1+t} \), which is equivalent to \( f(A) = C(A + 1) \).

2 The real analog

Going from the complex to the real analog (see [3]) here we have an equation
\[ \frac{w(q)}{f(w(q))} = q, \quad (93) \]

with \( f(A) \) analytic and \( f(0) \neq 0 \) around 0. The equation (93) have solution
\[ w(q) = \sum_{n=1}^{\infty} c_n q^n, \quad q = e^{-\pi \sqrt{A}}, \quad A > 0. \quad (94) \]

Then if
\[ a_n = c_n n, \quad (95) \]

we will study all the functions in which
\[ \frac{1}{P(A)} = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{-\pi \sqrt{A}}, \quad A > 0, \quad (96) \]

in the sense that \( P(A) \) defines a function \( X(A) \) such that \( X(A) \) is solution of the equation (103) below and is connected with inversion problem in [3] and [2]. Hence due to the connection (95) the class of all functions \( P(A) \) is very wide. A first result is
\[ \int \frac{1}{qP(A)} dq = w(q) + c \quad (97) \]
and
\[ P(A) = \frac{1}{qw'(q)}, \quad q = e^{-\pi\sqrt{A}}, \quad A > 0. \quad (98) \]

Also
\[ -2P(A)h'_i(A) = 1 \quad (99) \]

and
\[
\begin{align*}
    h_i(A) &= c + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{a_n}{n^2} q^n + \frac{\sqrt{A}}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} q^n \\
    &= c + \pi^{-2} \int q^{-1}w(q)dq - \pi^{-2}w(q)\log q = \\
    &= c + \pi^{-2}w(q)\log q - \pi^{-2} \int w'(q)\log(q)dq - \pi^{-2}w(q)\log q \\
    &\Rightarrow h_i(A) = c - \pi^{-2} \int w'(q)\log(q)dq \quad (100)
\end{align*}
\]

and
\[ h'_i(A) = \frac{1}{2}e^{-\pi\sqrt{A}}w'\left(e^{-\pi\sqrt{A}}\right). \quad (101) \]

The function \( X(A) \) is given from
\[
X(A) = h\left(\frac{1}{\sqrt{4}}B_0\left(A^2; \frac{1}{6}, \frac{2}{3}\right)\right) \quad (102)
\]

and satisfies the equation
\[
X'(A) + \frac{2^{4/3}}{A^{2/3}(1 - A^2)^{1/3}} P(X(A)) = 0, \quad (103)
\]

which is equivalent to
\[ h'(A) + 2P(h(A)) = 0. \quad (104) \]

The function
\[ Y(r) = X(kr), \quad (105) \]

satisfies
\[
Y'(r) - \frac{\pi}{\sqrt{r}} q(i\sqrt{r}/2)^4 P(Y(r)) = 0. \quad (106)
\]

Also if
\[ F(A) = \int_{X(0)}^{A} \frac{dt}{P(t)}, \quad (107) \]

then
\[ F(Y(r)) = -\frac{2}{\sqrt{4}}B_0\left(k_r^2; \frac{1}{6}, \frac{2}{3}\right). \quad (108) \]

and
\[ F(Y(4r)) + F\left(Y\left(\frac{4}{r}\right)\right) = c_0 = -\frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{\pi\sqrt{2}} \quad (109) \]
Moreover if $G(A)$ is the function related with the Ramanujan-Jacobi inversion problem, then setting

$$h_1(t) := \left(\frac{1}{h'_i(\cdot)}\right)^{-1}(t), \quad (110)$$

we have

$$G_i(x) = F_1\left(\int_c^x \frac{h_1'(t)}{t} \, dt \right) \quad (111)$$

and

$$G(-1)(-2x) = F_1\left(\frac{-1}{2} \int_c^{P_i(x)} \frac{dt}{P(t)} \right). \quad (112)$$

**Theorem 14.**

$$w'(q) q = -2h'_i(A), \quad q = e^{-\pi \sqrt{A}}, \quad A > 0. \quad (113)$$

Or equivalent

$$h_i(A) = c - \frac{1}{2} \int w'(q) q dA = c' - \pi^{-2} \int w'(q) \log(q) dq = c' - \pi^{-2} C(q) \quad (114)$$

and $F_1(A)$ such that

Again from [2] Corollary 1 eq. 33, we have

$$h_i(A) = 5 \int_0^{\nu(A)} \frac{dt}{t^\frac{3}{2} - 11 - t^2}. \quad (114.1)$$

If $q = e^{-\pi \sqrt{A}}$, then

$$5 \int_0^{\nu(A)} \frac{dt}{t^\frac{3}{2} - 11 - t^2} = c' - \pi^{-2} \int w'(q) \log(q) dq = c - \frac{1}{2} \int w'(q) dA \Rightarrow$$

$$F_1\left(c - \frac{1}{2} \int w'(q) q dA\right) = y(A), \quad (115)$$

where

$$5 \int_0^{\nu_1(A)} \frac{dt}{t^\frac{3}{2} - 11 - t^2} = A. \quad (115.1)$$

**Note.** In view of [2] we have for $A$ real and positive

$$F_1(A) = R\left(e^{-\pi \sqrt{m_0(A)}}\right), \quad (115.1)$$

where $m_0(A)$ is the inverse function of

$$2^{-2/3} B_0 \left(k_A^2; \frac{1}{6}, \frac{2}{5}\right)$$
and \( R(q) \) is the Rogers-Ramanujan continued fraction. The function \( k_r \) being the elliptic singular modulus i.e:
\[
k_r = \left( \frac{\theta_2(q)}{\theta_3(q)} \right)^2, \quad q = e^{-\pi \sqrt{r}}, \quad r > 0.
\]
But \( G(y(A)) = 1/h'_y(A) \). Hence
\[
G \left( F_1 \left( c - \frac{1}{2} \int w'(q)qdA \right) \right) = -2P(A). \tag{116}
\]
Assume (in the same way as we did above) that
\[
G \left( F_1 (A) \right) = H(A),
\]
where \( H(A) \) is analytic. Also set
\[
C(q) = \int w'(q) \log(q) dq = \int q w'(t) \log(t) dt \tag{117}
\]
and \( f(A) = e^{-P'_1(A)} \) analytic and not zero at the origin. Then
\[
C'(A) = w'(A) \log A \Rightarrow C'(w_i(A))w'_i(A) = w'(w_i(A)) \log(w_i(A))w'_i(A) \Rightarrow
\]
\[
C(w_i(A)) = \int \log(w_i(A)) dA + c_1 = \int \log \left( \frac{A}{f(A)} \right) dA + c_1 = \int \log(A) dA + P_1(A) + c_1. \tag{118}
\]
We have
\[
H(A) = \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} A^k.
\]
Setting \( A \to c - \frac{1}{2} \int w'(q)qdA \), we have
\[
G \left( F_1 \left( c - \frac{1}{2} \int w'(q)qdA \right) \right) = \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c - \frac{1}{2} \int w'(q)qdA \right)^k \Rightarrow
\]
\[
-2P(A) = \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - \pi^{-2} \int w'(q)qdq \right)^k \Rightarrow
\]
\[
\frac{-2}{w'(q)q} = \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - \pi^{-2}C(q) \right)^k \Rightarrow
\]
\[
-\frac{2}{w'(A)A} = \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - \pi^{-2}C(A) \right)^k \Rightarrow
\]
\[-2 \frac{w'_i(A)}{w_i(A)} = \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} (c' - \pi^{-2} C(w_i(A)))^k \Rightarrow \]

\[\frac{w'_i(A)}{w_i(A)} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - c_1 - \pi^{-2} \int \log(A) dA - \pi^{-2} P_1(A) \right)^k. \]

Now if \( \xi \) is positive constant and \( x \) positive variable with \( 0 < x < \xi \), then

\[
\log w_i(x) - \log w_i(\xi) = -\frac{1}{2} \int_{\xi}^{x} \left\{ \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - c_1 - \pi^{-2} \int \log(A) dA - \pi^{-2} P_1(A) \right)^k \right\} dA \Rightarrow \\

w_i(x) = w_i(\xi) \times \exp \left( -\frac{1}{2} \int_{\xi}^{x} \left\{ \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - c_1 - \pi^{-2} \int \log(A) dA - \pi^{-2} P_1(A) \right)^k \right\} dA \right). \\

Finally

\[
f(x) = \frac{\pi f(\xi)}{\xi} \times \exp \left( \frac{1}{2} \int_{\xi}^{x} \left\{ \sum_{k=0}^{\infty} \frac{H^{(k)}(0)}{k!} \left( c' - c_1 - \pi^{-2} \int \log(A) dA - \pi^{-2} P_1(A) \right)^k \right\} dA \right). \\

Since the integral \( \int \log(A) dA \) is continuous and bounded in \([0, \xi]\) and \( P_1(A) \) analytic, we have \( f(0) = 0 \), which is impossible. Hence \( G(F_1(A)) \) is not analytic. However if we assume that

\[
G(F_1(A)) + \frac{2}{L(A)} = P_0^*(A),
\]

where \( L \) is a function such that

\[
L \left( c - \frac{1}{2} \int w'(q) q dA \right) = w(q)
\]

and \( P_0^*(A) \) analytic. Then

\[
G \left( F_1 \left( c - \frac{1}{2} \int w'(q) q dA \right) \right) + \frac{2}{w(q)} = P_0^* \left( c - \frac{1}{2} \int w'(q) q dA \right) \Rightarrow \\

-2P(A) + \frac{2}{w(q)} = P_0^* \left( c' - \pi^{-2} \int w'(q) \log(q) dq \right) \Rightarrow \\

-\frac{2}{w'(q)q} + \frac{2}{w(q)} = P_0^* \left( c' - \pi^{-2} C(q) \right) \Rightarrow \\

-2 \frac{w'_i(A)}{w_i(A)} + \frac{1}{A} = P^* \left( c' - \pi^{-2} C(w_i(A)) \right) \Rightarrow \\

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\[-2 \log(w_i(t)) + 2 \log t \bigg|_{t=\xi}^{t=\xi} = \int_{\xi}^{A} P_0^* \left( c' - \pi^{-2} C(w_i(t)) \right) dt \Rightarrow \]
\[-2 \log \left( \frac{w_i(A)}{A} \right) + 2 \log \left( \frac{w_i(\xi)}{\xi} \right) = \]
\[\int_{\xi}^{A} P_0^* \left( c' - c_1 - \pi^{-2} \int \log(t) dt - \pi^{-2} P_1(t) \right) dt \Rightarrow \]
\[\log(f(A)) - \log(f(\xi)) = \]
\[\frac{1}{2} \int_{\xi}^{A} P_0^* \left( c' - c_1 - \pi^{-2} \int \log(t) dt - \pi^{-2} P_1(t) \right) dt \Rightarrow \]
\[f(A) = f(\xi) \exp \left( \frac{1}{2} \int_{\xi}^{A} P_0^* \left( c' - c_1 - \pi^{-2} \int \log(t) dt - \pi^{-2} P_1(t) \right) dt \right). \]

Hence we get the next

**Theorem 15.**

If we assume the problem

\[\frac{w(q)}{f(w(q))} = q, \quad q = e^{-\pi \sqrt{r}}, \quad r > 0, \quad (119)\]

where \( f \) is analytic arround 0 and \( f(0) \neq 0 \) and assume the solution is

\[w(q) = \sum_{n=1}^{\infty} c_n q^n. \quad (120)\]

Setting \( a_n = nc_n \), we define

\[\frac{1}{P(A)} = \sum_{n=1}^{\infty} a_n q^n. \quad (121)\]

Hence, we define a connection of the Lagrange problem with the Hauptmodul and the Ramanujan-Jacobi problem as described in the above notes of present article (see also [2],[3]). The connection of Ramanujan-Jacobi problem and the Hauptmodul general problem is

\[G_i(-2x) = F_1 \left( \frac{-1}{2} \int_{c}^{P_i(x)} \frac{dt}{P(t)} \right). \quad (122)\]

Then in order the above problem to be well defined, function \( G(F_1(A)) \) must be of the form

\[G \left( F_1(A) \right) = -\frac{2}{L(A)} + P_0^* (A), \quad (123)\]
where \( P^*_0(A) \) is analytic in an interval containing 0 and \( L(A) \) must satisfy
\[
L \left( c - \frac{1}{2} \int w'(q)qdA \right) = w(q). \tag{124}
\]
Also if \( \xi \) is suitable positive constant and \( x > 0 \), then setting \( f(A) = e^{-P^*_i(A)} \), we have
\[
f(A) = f(\xi) \exp \left( \frac{1}{2} \int_A^\xi P^*_0 \left( c' - c_1 - \pi^{-2} \int \log(t) dt - \pi^{-2} P_1(t) \right) dt \right). \tag{125}
\]

The function \( L(A) \) can be written as
\[
L_i(w(q)) = c - \frac{1}{2} \int w'(q)qdA \Rightarrow L_i(w(q))w'(q)q \frac{-\pi}{2\sqrt{A}} = -\frac{1}{2}w'(q)q \Leftrightarrow
\]
\[
L_i'(w(q)) = \frac{\pi}{\sqrt{A}} = 1 \Leftrightarrow -\pi^2 L_i'(w(q)) = -\pi \sqrt{A} = \log q \Leftrightarrow
\]
\[
L_i'(q) = -\frac{1}{\pi^2} \log(w_i(q)) \Leftrightarrow L_i'(A) = -\frac{1}{\pi^2} \log \left( \frac{A}{f(A)} \right).\]
Hence we get the next

**Theorem 16.**
\[
L_i(A) = -\pi^{-2} \int \log \left( \frac{A}{f(A)} \right) dA + c. \tag{126}
\]

**Theorem 17.**
If we set
\[
S(A) := \frac{\pi^{-2}}{2} P^*_0 \left( A \right), \tag{127}
\]
then
\[
-\frac{L''(A)}{L'(A)^3} + \frac{\pi^{-2}}{L(A)} = S(A). \tag{128}
\]

**Proof.**
We have
\[
-\frac{2}{w'(q)q} + \frac{2}{w(q)} = P^*_0 \left( c' - \pi^{-2} C(q) \right) \Leftrightarrow
\]
\[
-\frac{2}{w'(q)q} + \frac{2}{w(q)} = P^*_0 \left( L_i(w(q)) \right) \Leftrightarrow
\]
\[
-2 \log q + 2 \log(w(q)) = \int_{c_1}^{w(q)} P^*_0 \left( L_i(t) \right) dt \Leftrightarrow
\]
\[-2 \log w_i(A) + 2 \log A = \int P^*_0 (L_i(A)) \, dA \Leftrightarrow \]
\[2 \log (f(A)) = \int P^*_0 (L_i(A)) \, dA \Leftrightarrow \]
\[2 \log (f(A)) = \int P^*_0 \left( \pi^{-2} \int \log \left( \frac{f(A)}{A} \right) \right) \, dA. \]
Set 
\[S(A) := \frac{\pi^{-2}}{2} P^*_0 (A). \]
Then
\[L'_i(A) = \int S(L_i(A)) \, dA - \pi^{-2} \log A \Leftrightarrow \]
\[L''_i(A) = S(L_i(A)) - \frac{\pi^{-2}}{A}. \quad (129) \]
Hence if 
\[u(x) := L_i(x), \]
then
\[\frac{d}{dA} (u''(A)) = \frac{d}{dA} (S(u(A))) \Rightarrow \]
\[u''(A) = S(u(A)) + A[u'''(A) - S'(u(A))u'(A)] = 0. \quad (130) \]
A solution of (130) is
\[u_i(A) = \int_{c_1}^A \frac{dt}{\sqrt{c_2 + 2 \int_{c_3}^t S(t) \, dt}}. \]
However if we set \( A \rightarrow L_i(A) \) in (129), then
\[- \frac{L''(A)}{L'(A)^2} + \frac{\pi^{-2}}{L(A)} = S(A). \]
Remarks.
Hence if exists \( S_1(A) \) such that
\[- \frac{u''(A)}{u'_i(A)^3} + \frac{1}{u_i(A)} = S_1 (u_i(A)) = S(A), \]
wec get solving the first equality
\[u_i(A) = \int_{c_0}^A \frac{dt}{c_1 - \log t + 5 S_1(t) \, dt}. \]
Hence
\[S_1 \left( \int_{c_0}^A \frac{dt}{c_1 - \log t + 5 S_1(t) \, dt} \right) = S(A). \]
Now set the functions \( S, S_0 \) such that
\[u''_i(A) = S(u_1(A)) - \frac{1}{A}. \quad (131) \]
and
\[ u''_2(A) = S_0(u_2(A)). \] (132)

Hence
\[ u_2^{(-1)}(x) = \int_{c_1}^{x} \frac{dt}{\sqrt{2} \int_{c_2}^{t} S_0(t_1)dt_1}. \]

But equation (131) can be written as
\[- \frac{u_1^{(-1)r}(A)}{u_1^{(-1)}(A)^3} + \frac{1}{u_1^{(-1)}(A)} = S(A) = S_1(u_1^{(-1)}(A))\]
and have solution
\[ u_1^{(-1)}(A) = \int_{c_0}^{A} \frac{dt}{c_1 - \log t + \int S_1(t)dt}. \]

However \( u_1^{(-1)}(A) = u_2^{(-1)}(A) \). Hence
\[ \int_{c_1}^{A} \frac{dt}{\sqrt{2} \int_{c_2}^{t} S_0(t_1)dt_1} = \int_{c_0}^{A} \frac{dt}{c_1 - \log t + \int S_1(t)dt} \]
\[ S_1 \left( \int_{c_0}^{A} \frac{dt}{c_1 - \log t + \int S_1(t)dt} \right) = S(A) = S_1 \left( \int_{c_1}^{A} \frac{dt}{\sqrt{2} \int_{c_2}^{t} S_0(t_1)dt_1} \right). \]

The solution of (128) is not an easy problem and it might be unsolved.

Now assume that \( m \) is rational and set
\[ B_{\alpha}(x) := \sqrt{B_0(x, \alpha, \overline{\alpha})} = \sqrt{\int_{0}^{x} (t - t^2)^{\alpha-1} dt}. \]

It is known that
\[ \int_{z_1}^{z_2} \frac{dt}{(at^2 + bt + c)^m} = U(a, b, c; m; z_2) - U(a, b, c; m; z_1). \] (133)

Also if
\[ A_1 = -\rho_1 - \frac{\sqrt{D}}{a} \beta_{r_1}, \quad A_2 = -\rho_1 - \frac{\sqrt{D}}{a} \beta_{r_2}, \] (134)

where \( \rho_1 = \frac{b - \sqrt{D}}{2a} \) and \( \beta_{r_1}, \beta_{r_2} \) are solutions of
\[ \frac{B_{1-m}(1 - \beta_{r_{1,2}})}{B_{1-m}(\beta_{r_{1,2}})} = \sqrt{r_{1,2}}, \] (135)
(if \( r \) is positive rational, then \( \beta_r \) is algebraic), we have

\[
\int_{A_1}^{A_2} \frac{dt}{(at^2 + bt + c)^m} = (-1)^{m+1}a^{m-1}D^{-m+1/2}B_{1-m}(\beta_r)^2 - (-1)^{m+1}a^{m-1}D^{-m+1/2}B_{1-m}(\beta_r)^2.
\]

But one can easily see that

\[
B_\alpha^2(z) + B_\alpha(1-z)^2 = \int_0^1 (t(1-t))^\alpha dt = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}.
\]

Setting \( z = \beta_r \) in the above formula we have

\[
\frac{B_\alpha(1 - \beta_r)}{B_\alpha(\beta_r)} = \sqrt{r}
\]

and

\[
B_\alpha(\beta_r) = \sqrt{\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)(r+1)}}
\]

and also

\[
B_\alpha(\beta_{n^2r}) = \sqrt{\frac{r+1}{n^2r+1}B_\alpha(\beta_r)}.
\]

Hence we get the next

**Theorem 18.**

If \( r_1, r_2 \) are rational and \( A_1, A_2 \) are that of (134) with \( \beta_{r_1}, \beta_{r_2} \) the algebraic solutions of (135), then

\[
\int_{A_1}^{A_2} \frac{dt}{(at^2 + bt + c)^m} = (-1)^{m+1}a^{m-1}D^{-m+1/2} \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \left( \frac{1}{r_2 + 1} - \frac{1}{r_1 + 1} \right).
\]

Assuming \( r_2 = r \) and \( r_1 = +\infty \) we have

\[
\int_{-\infty}^{-\frac{\sqrt{4\rho_2^2 + 1}}{2}} \frac{dt}{(at^2 + bt + c)^m} = (-1)^{m+1}a^{m-1}D^{-m+1/2} \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \frac{1}{r + 1}
\]

But it holds (see [2]) \( h'(A) = G(F_1(A)) = f_1(U_1(A)) \). Hence from Theorem 16 we have

\[
h'(A) = -\frac{2}{L(A)} + P_0^*(A) \Rightarrow h(U(A)) = -2 \int_c^{U(A)} \frac{dt}{L(t)} + \int_c^{U(A)} P_0^*(t)dt,
\]

where \( U(A) \) is that of (81). Hence assuming \( P_0^*(A) \) is given analytic function, if

\[
f_1(t) = -\frac{2}{L(U(t))} + P_0^*(U(t)),
\]

(140)
we have

$$\int_{A_1}^{A_2} f_1(t) \left( \frac{1}{a_1 t^2 + b_1 t + c_1} \right)^m dt = -2 \int_{U(A_1)}^{U(A_2)} \frac{dt}{L(t)} + \int_{U(A_1)}^{U(A_2)} P_0^*(t) dt, \quad (141)$$

where $A_1, A_2$ may be arbitrary. The function $L(A)$ is determined from Theorem 16 equation (126) and $P_0^*(A)$ from Theorem 17. Hence for $f_1(A)$ we can evaluate the integral (141). Hence as a special case

$$\int_{-\rho_1 - \sqrt{D_1 a_1}}^{\beta_2} f_1(t) \left( \frac{1}{a_1 t^2 + b_1 t + c_1} \right)^m dt = -2 \int_{U(A_2)}^{U(A_1)} \frac{dt}{L(t)} + \int_{U(A_1)}^{U(A_2)} P_0^*(t) dt, \quad (142)$$

where

$$U(A_{1,2}) = (-1)^{m+1} a_1^{m-1} D_1^{-m+1/2} \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \frac{1}{r_{1,2} + 1}. \quad (143)$$

Continuing from Theorem 17 we can write

$$\int_{A_1}^{A_2} \frac{f_1(t)}{(a_1 t^2 + b_1 t + c_1)^m} dt =$$

$$= -2 \int_{U(A_1)}^{U(A_2)} \frac{dt}{L(t)} + \int_{U(A_1)}^{U(A_2)} 2\pi^2 \left( \frac{L''(t)}{L(t)^3} + \frac{\pi^{-2}}{L(t)} \right) dt =$$

$$= -2\pi^2 \int_{U(A_1)}^{U(A_2)} \frac{L''(t)}{L(t)^3} dt = \pi^2 \left( \frac{1}{L'\left(U(A_2)\right)^2} - \frac{1}{L'\left(U(A_1)\right)^2} \right).$$

Also

$$f_1(A) = \pi^2 \left( \frac{d}{dt} \frac{1}{L(t)^2} \right)_{t=U(A)}.$$  

The above will help us to prove the next:

**Theorem 19.**

Assume that $f_1$ is any smooth function of the form

$$f_1(A) = \pi^2 \left( \frac{d}{dt} \frac{1}{L(t)^2} \right)_{t=U(A)}. \quad (144)$$

Knowing $L(A)$, we can assume that $R_{1,2}$ are solutions of the equation

$$c - \frac{1}{2} \int_{c_0}^{R_{1,2}} \frac{dt}{P(t)} = (-1)^{m+1} a_1^{m-1} D_1^{-m+1/2} \frac{\Gamma(1-m)^2}{\Gamma(2(1-m))} \frac{1}{r_{1,2} + 1}. \quad (145)$$
However $\beta_{r_1,2}$ are solutions of (135) and we finally have

$$
\int_{-\rho_1-\sqrt{\frac{r_1}{a_1}}}^{\rho_1-\sqrt{\frac{r_1}{a_1}}} f_1(t) \frac{\rho_1 - \sqrt{D_1 a_1}}{(a_1 t^2 + b_1 t + c_1)^m} dt = R_2 - R_1.
$$

(146)

**Proof.**

Given any $f_1(A)$ and $A_1, A_2$, we have

$$
\int_{U_i(A_1)}^{U_i(A_2)} f_1(t) \frac{\rho_1 - \sqrt{D_1 a_1}}{(a_1 t^2 + b_1 t + c_1)^m} dt = \pi^2 \left( \frac{1}{L'(A_2)^2} - \frac{1}{L'(A_1)^2} \right),
$$

(147)

where $f_1$ and $L$ are related as

$$
f_1(A) = \pi^2 \left( \frac{d}{dt} \frac{1}{L'(t)^2} \right) |_{t=U(A)}.
$$

(148)

But from equation (114) we have

$$
\int_{U_i(A_1)}^{U_i(A_2)} f_1(t) \frac{\rho_1 - \sqrt{D_1 a_1}}{(a_1 t^2 + b_1 t + c_1)^m} dt = A_2 - A_1.
$$

(149)

Assume that $R_{1,2}$ are solutions of

$$
c - \frac{1}{2} \int_{c_0}^{R_{1,2}} w'(q)qdA = (-1)^{m+1} a_1^{m-1} D_1^{-m+1/2} \frac{\Gamma(1-m) \Gamma(2(1-m))}{r_{1,2} + 1} \frac{1}{\Gamma(1-m)}
$$

(150)

and $\beta_{r_1,2}$ are solutions of (135). Then

$$
U_i \left( c - \frac{1}{2} \int_{c_0}^{R_{1,2}} w'(q)qdA \right) =
$$

$$
= U_i \left( (-1)^{m+1} a_1^{m-1} D_1^{-m+1/2} \frac{\Gamma(1-m) \Gamma(2(1-m))}{r_{1,2} + 1} \right) =
$$

$$
= -\rho_1 - \sqrt{D_1} a_1 \beta_{r_1,2}.
$$

Hence we get the proof of the theorem.

**Remarks.** We have

$$
G(y(A)) = 1/h_i(A).
$$

Hence

$$
5 \int_{0}^{y(A)} \frac{dt}{\sqrt{t^{-5} - 11 - t^5}} = h_i(A) \Leftrightarrow F_1(h_i(A)) = y(A) \Leftrightarrow
$$

$$
F_1 \left( c - \frac{1}{2} \int w'(q)qdA \right) = y(A) \Rightarrow
$$

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\[ G \left( F_1 \left( c - \frac{1}{2} \int w'(q) q dA \right) \right) = 1/h_i'(A) = -\frac{2}{w'(q)q} \Rightarrow \]
\[ G(y(A)) + 2P(A) = 0. \]

Also
\[ G(F_1(A)) = -\frac{2}{L(A)} + P_0^*(A) \]
\[ f_1(U_i(A)) = h'(A) = \exp dL' A^2 \Leftrightarrow \]
\[ h(A) = \frac{\pi^2 d}{L(A)^2} + l_1, \quad \text{(151)} \]
where \( l_1 \) is constant. But differentiating (124) we have
\[ L' \left( c - \frac{1}{2} \int w'(q) q dA \right) = \frac{\pi}{ \sqrt{A}}. \quad \text{(152)} \]

Setting in (151) \( A \to c - \frac{1}{2} \int w'(q) q dA \) and using (152),(126), we get
\[ h \left( L_i(w(q)) \right) = A + l_1. \quad \text{(153)} \]

But \( \pi L'_i(w(q)) = \sqrt{A} \). Hence
\[ L'_i(w(q))w'(q)q - \frac{\pi}{2\sqrt{A}} = w'(q)q - \frac{\pi}{2\sqrt{A}} = -\frac{w'(q)q}{2}. \]

Hence from Theorem 14 we get
\[ L_i(w(q)) = h_i(A) + l_2, \quad \text{(154)} \]
where \( l_2 \) is constant. Hence
\[ h(h_i(A) + l_2) = A + l_1 \Rightarrow h_i(A + l_1) = h_i(A) + l_2. \quad \text{(154.1)} \]

Hence the functions \( h(A), L_i \left( w \left( e^{-\pi \sqrt{A}} \right) \right) \) are one to one and hence strictly increasing or decreasing. Also their derivatives are periodic. Another interesting thing is that (using (154)): \[ L(A) = w \left( e^{-\pi \sqrt{h_i(A) - l_i}} \right). \quad \text{(155)} \]

Also from (151) we have
\[ L(A) = \pm \pi \int_{c^*}^{A} \frac{dt}{h(t) - l_i}. \quad \text{(156)} \]

However if we know \( h(A) \) we know by simple inversions the functions \( y(A), G(A), P(A), \int \frac{dA}{h(A)} \) (from relations (99),(112),(114.1),(116)). Hence if we know the expansion
\[ G \left( F_1(A) \right) = h'(A) = -\frac{2}{L(A)} + P_0^*(A), \quad \text{(157)} \]
then we can find \( f(A) \) from Theorem 16 and then solve (93) with respect to \( w \). Hence we have the next

**Theorem 20.**
Assume given a function \( h(A) \) we can write it in the form

\[
h'(A) = -\frac{2}{L(A)} + P_0^*(A),
\]

where \( L(A) \) is solution of the equation

\[
-2\pi^2 \frac{L''(A)}{L'(A)^3} + \frac{2}{L(A)} = P_0^*(A)
\]

and \( P_0^*(A) \) analyitc. Then \( f(A) \) is given from (126) and \( w(A) \) from \( w(A) \) holds

\[
\pm \pi \int_0^A dt \sqrt{h(t) - l_1} = w \left( e^{-\pi \sqrt{h(A) - l_1}} \right) = L(A).
\]

But

\[
\frac{1}{L'(A)} = -\frac{1}{\pi^2} \log \left( \frac{L(A)}{f(L(A))} \right) \Rightarrow
\]

\[
-\frac{\pi^2}{L'(A)} = -\pi \sqrt{h(A) - l_1} = \log \left( \frac{L(A)}{f(L(A))} \right) \Rightarrow
\]

\[
e^{-\pi \sqrt{h(A) - l_1}} = \frac{L(A)}{f(L(A))}.
\] (161)

Set now \( Q(A) \) such \( L(A) = Q \left( e^{-\pi \sqrt{h(A) - l_1}} \right) \) and \( Q(Q(A)) = \phi(A) \). Then (here function \( \phi \) must not confused with \( \phi \) of [2] and \( \phi \) of Section 3 below):

\[
A = \frac{Q(A)}{f(Q(A))} \Rightarrow Q(A) = \frac{\phi(A)}{f(\phi(A))} \Rightarrow L(A) = \frac{\phi \left( e^{-\pi \sqrt{h(A) - l_1}} \right)}{f \left( \phi \left( e^{-\pi \sqrt{h(A) - l_1}} \right) \right)}.
\] (162)

If we assume that

\[
Q(A) = \frac{\phi(A)}{f(\phi(A))} = \phi(\lambda(A)),
\] (163)

we must have equivalently

\[
Q(A) = \frac{\phi(A)}{f(\phi(A))} \Leftrightarrow Q(Q(A)) = \phi(A) \Leftrightarrow \frac{\phi \left( \frac{\phi(A)}{f(\phi(A))} \right)}{f \left( \phi \left( \frac{\phi(A)}{f(\phi(A))} \right) \right)} = \phi(A) \Leftrightarrow
\]

\[
\frac{\phi(\phi(\lambda(A)))}{f(\phi(\phi(\lambda(A))))} = \phi(A).
\] (a)
But from (162) we have
\[
\frac{\phi(\phi(\lambda(A)))}{f(\phi(\phi(\lambda(A))))} = \phi(\lambda(\phi(\lambda(A)))).
\] (b)

Hence from (a), (b), we must have
\[
\phi(\lambda(\phi(\lambda(A)))) = \phi(A) \iff \lambda(\phi(\lambda(A))) = A.
\]
Hence
\[
L(A) = \lambda^{-1}(e^{-\pi\sqrt{h(A)-l_1}}).
\]
Hence from (155) we must have
\[
\lambda^{-1}(A) = \frac{\phi(A)}{f(\phi(A))} = w_1(\phi(A)) \iff
\]
\[
w(A) = \phi(\lambda(A)) = \lambda^{-1}(A) = \frac{\phi(A)}{f(\phi(A))} \iff
\]
\[
w(w(A)) = \phi(A), \lambda(A) = \frac{A}{f(A)}. \tag{164}
\]
Also
\[
\lambda(\lambda(A)) = \phi^{-1}(A) \iff \frac{A}{f(A)f\left(\frac{A}{f(A)}\right)} = \phi^{-1}(A)
\]

**Theorem 21.**

We have
\[
y(A) = F_1\left(-\frac{1}{2}\int_c^A \frac{dt}{P(t)}\right), \tag{165}
\]
\[
G(y(A)) + 2P(A) = 0 \tag{166}
\]
and
\[
\int_{c_1}^{-1/2}\int_c^A \frac{dt}{P(t)} G(F_1(t)) dt = A. \tag{167}
\]

**Proof.**

We have \(-2P(A)h_1'(A) = 1\) and \(h_1'(A) = 1/G(y(A))\). Hence
\[
G(y(A)) = -2P(A).
\]
Also \(F_1(h_1(A)) = y(A)\). Hence
\[
y(A) = F_1\left(-\frac{1}{2}\int_c^A \frac{dt}{P(t)}\right).
\]

33
3 Solving polynomial equations

An interesting case of functions are the Lambert functions defined as

\[
\phi(x) = q^2 \sum_{n=1}^{\infty} \frac{A_n q^n}{1 - q^n}, \quad q = e^{-\pi \sqrt{x}}
\]  

(168)

where \( A_n = \sum_{d|n} a_d \mu(n/d) \) and \( a_n \) is arithmetic \( T \)-periodic function. Then we can write

\[
\phi(x) = q^2 \sum_{n=1}^{\infty} \frac{A_n q^n}{1 - q^n} = q^2 \sum_{n=1}^{\infty} \left( \frac{1}{d|n} \sum A_d \right) q^n = q^2 (1 - q^T)^{-1} \sum_{n=1}^{T} \left( \frac{1}{d|n} \sum A_d \right) q^n =
\]

\[
= q^2 (1 - q^T)^{-1} \sum_{n=1}^{T} a_n q^n.
\]  

(169)

For example if \( a_n = \sqrt{2} \cos(\pi n/4) \), then \( T = 8 \) and

\[
\sqrt{2} \sum_{n=1}^{\infty} \sum_{d|n} \cos(\pi d/4) \mu(n/d) \frac{q^n}{1 - q^n} = \frac{q - q^3 - \sqrt{2} q^4 - q^5 + q^7 + \sqrt{2} q^8}{1 - q^8}
\]

Hence the series

\[
\phi(x) = q^2 \sum_{n=1}^{\infty} \frac{A_n q^n}{1 - q^n}, \quad q = e^{-\pi \sqrt{x}}, \quad x > 0
\]  

(170)

is a rational function of \( q = e^{-\pi \sqrt{x}} \). Also holds

\[
\phi(x) = q^2 \sum_{n=1}^{\infty} q^n a_n,
\]  

(171)

where \( a_n = \sum_{d|n} A_d \). Now if \( B_n \) is defined such that \( B_n \) is the arithmetic inverse function of \( X_{n>2}(n) a_n^{-2}, \) \( X_{n>2}(n) \) is 0 if \( n = 1, 2 \) and 1 otherwise), then we have

\[
\sum_{n=1}^{\infty} B_n \phi\left(n^2 x\right) = q.
\]  

(172)

Hence since \( q = e^{-\pi \sqrt{x}}, \quad x > 0 \), we have for some function \( \theta(x) \)

\[
\sum_{n=1}^{\infty} B_n \phi\left(n^2 \theta(x)\right) = e^{-\pi \sqrt{\theta(x)}}.
\]  

(173)

If \( M_n^*(x) := \phi\left(n^2 \theta(x)\right) \), then we have the following expansion for \( e^{-\pi \sqrt{\theta(x)}} \):

\[
\sum_{n=1}^{\infty} B_n M_n^*(x) = e^{-\pi \sqrt{\theta(x)}}.
\]  

(174)
Hence if we define \( \psi(x) \) such that
\[ \phi(x) = \psi\left(e^{-\pi \sqrt{x}} \right) = e^{-2\pi \sqrt{x}} \left(1 - e^{-T \pi \sqrt{x}} \right)^{-1} \sum_{n=1}^{T} a_n e^{-\pi n \sqrt{x}}, \]
(175)
then setting \( x \to \theta(x) \), with \( \theta(x) \) such that \( \psi\left(e^{-\pi \sqrt{x}} \right) = \theta^{(-1)}(x) \), we get
\[ \phi(\theta(x)) = e^{-2\pi \sqrt{\theta(x)}} \left(1 - e^{-\pi T \sqrt{\theta(x)}} \right)^{-1} \sum_{n=1}^{T} a_n e^{-\pi n \sqrt{\theta(x)}} = x. \]
(176)
Hence the equation
\[ x_0 = X_0^2 \left(1 - X_0^{-1}\right)^{-1} \sum_{n=1}^{T} a_n X_0^n, \]
(177)
have solution
\[ X_0 = e^{-\pi \sqrt{\theta(x_0)}}. \]
(178)
An analysis of how we find functions like \( \theta(x) \) is given in [2]. For example let \( G(x) \) be such that
\[ \phi^{(-1)}(k_x) = m_G^{(-1)}(x) = \pi \int_{+\infty}^{+\infty} \eta(it/2)^4 G(R(e^{-\pi t})) dt, \]
(179)
where
\[ \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi iz}, \quad \text{Im}(z) > 0 \]
(180)
and
\[ R(x) = \frac{x^{1/5}}{1 + x^{1/1} + x^{1/2} + x^{1/3} + \ldots}, \quad |x| < 1, \]
(181)
are the Dedekind eta function and the Rogers-Ramanujan continued fraction respectively. Consequently the function \( y \) can be found as
\[ y(x) = R\left(e^{-\pi \sqrt{k} \phi(x)} \right). \]
(182)
Hence with the notation of [2] relation (59), (here Lambert’s function \( \phi(x) \) is not to be confused with the notation of \( \phi(x) \) of [2] relation (14)):
\[ \phi(x) = s(x) \]
(183)
and thus
\[ G(x) = \frac{G_0(x)}{\sigma(F_i(x))}, \]
(184)
where
\[ \sigma(x) = \frac{1}{s_i'(x)} = \frac{1}{\phi^{(-1)'}(x)}. \]
(185)
Hence
\[ G(x) = G_0(x) \phi^{(-1)}(F_i(x)). \] (186)

Continuing, from (22) above and \( q = e^{-\pi \sqrt{x}} \) we have that (we change \( \theta_{(a,p)}(q) \rightarrow \theta_G(q) \))
\[ \theta_G(q) = m_G^{(-1)}(x) = \phi^{(-1)}(k_x). \] (187)

Hence
\[ \phi \left( \theta_G \left( e^{-\pi \sqrt{k_i(x)}} \right) \right) = x. \] (188)

Also must hold
\[ \theta_G \left( e^{-\pi \sqrt{k_i(\phi(x))}} \right) = m_G^{(-1)}(k_i(\phi(x))) = x \]
and
\[ e^{-\pi \sqrt{\theta(x)}} = \psi^{(-1)}(x), \] (189)
or equivalently
\[ \phi(\theta(x)) = x \iff \phi(x) = \theta^{(-1)}(x). \] (190)

Now we have in generality
\[ F_i(h_i'(x)) = y(x). \] (191)

Hence
\[ G(F_i(h_i'(x))) = G(y(x)) \]
and if we define \( X(x) \) such that
\[ X(A) = s_i(A) = \phi^{(-1)}(x), \] (192)
then
\[ G \left( F_1 \left( -\frac{1}{2} \int_c^x \frac{dt}{P(t)} \right) \right) = -2P(x) \] (193)
and
\[ G(y(x)) = -2P(x) \text{ and } -2h'_i(x)P(x) = 1. \] (194)

Hence from [3] Theorem 12 we get the next

**Theorem 22.**
\[ P(x) = -\frac{\left( \phi(x) \sqrt{1 - \phi(x)^2} \right)^{2/3}}{2\sqrt{2}\phi'(x)} \] (195)

and
\[ h_i(x) = c - \frac{1}{2} \int_{c_0}^x \frac{dt}{P(t)} = \frac{1}{\sqrt{3}} B_0 \left( \phi(x)^2; \frac{1}{6}, \frac{2}{3} \right), \] (196)
where \( B_0(a,b) = \int_0^1 t^{a-1}(1 - t)^{b-1} dt \) is the incomplete Beta function.
Someone can also easily see that
\[
\int \frac{dx}{P(x)} = -\frac{2}{\sqrt{4}} B_0 \left( \phi(x)^2 \cdot \frac{1}{6} \cdot \frac{2}{3} \right) + c, \quad (197)
\]
\[
P(X(x)) = -\frac{(x\sqrt{1-x^2})^{2/3}}{2\sqrt{2}} \phi^{(-1)'}(x) \quad (198)
\]
and
\[
Y(x) = \phi^{(-1)}(k_x). \quad (199)
\]
There exist \( \alpha > 0 \) such that for all \( x \in [0, \alpha) \), we have \( \phi(0) = 0 \),
\[
F(x) = \frac{1}{\sqrt{4}} B \left( \phi(x)^2 ; \frac{1}{6} \cdot \frac{2}{3} \right) = h_i(x) \quad (200)
\]
and hence
\[
F \left( \phi^{(-1)} (x^2) \right) + F \left( \phi^{(-1)} \left[ \left( \frac{1-x}{1+x} \right)^2 \right] \right) = \frac{\sqrt{3} \Gamma \left( \frac{1}{3} \right)^3}{2 \pi \sqrt{2}} \Leftrightarrow \quad (201)
\]
\[
F \left( \phi^{(-1)} (x) \right) + F \left( \phi^{(-1)} \left[ \left( \frac{1-x}{1+x} \right)^2 \right] \right) = \frac{\sqrt{3} \Gamma \left( \frac{1}{3} \right)^3}{2 \pi \sqrt{2}} \Leftrightarrow \quad (202)
\]
\[
F(x) + F \left( \phi^{(-1)} \left[ \left( \frac{1-x}{1+x} \right)^2 \right] \right) = \frac{\sqrt{3} \Gamma \left( \frac{1}{3} \right)^3}{2 \pi \sqrt{2}}. \quad (203)
\]
Also \(-\frac{1}{2} F'(x) = -\frac{1}{2P(x)} = h'_i(x)\). Now we define \( a_n^*, b_n^* \) such that
\[
\sqrt{x} = \sum_{n=1}^{\infty} a_n^* e^{-\pi n \sqrt{x}} = -\frac{2\sqrt{2} \phi'(x) \sqrt{x}}{(\phi(x) \sqrt{1-\phi(x)^2})^{2/3}} \quad (204)
\]
and
\[
\sum_{n=1}^{\infty} a_n^* e^{-\pi n \sqrt{x}} = \sqrt{x} \quad (205)
\]
Then we set \( c_n^{(1)} := \frac{a_n^*}{\pi} \) and \( c_n^{(2)} := \frac{b_n^*}{\pi} \) and assume that
\[
w_1(q) = \sum_{n=1}^{\infty} c_n^{(1)} q^n, \quad w_2(q) = \sum_{n=1}^{\infty} c_n^{(2)} q^n, \quad (206)
\]
are the solutions of the equations
\[
\frac{w_1(q)}{f_1(w_1(q))} = q, \quad \frac{w_2(q)}{f_2(w_2(q))} = q, \quad \text{resp.}, \quad (207)
\]
where \( f_1(x), f_2(x) \) are functions such that \( f_1(0), f_2(0) \neq 0 \) and analytic around the origin, we get integrating (202):

\[
\int \left( \sum_{n=1}^{\infty} a_n e^{-\pi \sqrt{x}} \right) dx = -\frac{2}{\pi} \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2; \frac{1}{6} : \frac{2}{3} \right) + c \leftrightarrow \\
-\frac{2}{\pi} w_1(q) = -\frac{2}{\pi} \sum_{n=1}^{\infty} c_n^{(1)} q^n = -\frac{2}{\pi} \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2; \frac{1}{6} : \frac{2}{3} \right) \leftrightarrow \\
w_1(q) = \sum_{n=1}^{\infty} c_n^{(1)} q^n = \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2; \frac{1}{6} : \frac{2}{3} \right).
\]

Note that

\[
\phi(0) = (+\infty) \left( \sum_{k=1}^{T} a_k \right), \quad \phi(+\infty) = 0.
\]

Hence \( \phi^{(-1)}(0) = +\infty \Rightarrow X(0) = +\infty \) and

\[
F(x) = \frac{1}{\sqrt{4}} B \left( \phi(x)^2; \frac{1}{6} : \frac{2}{3} \right).
\]

Hence when \( q = e^{-\pi \sqrt{x}}, x > 0 \), we have

\[
w_1(q) = \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2; \frac{1}{6} : \frac{2}{3} \right) = \pi F(x) = \pi h_4(x).
\]

Also

\[
\int \left( \sum_{n=1}^{\infty} b_n e^{-\pi \sqrt{x}} \right) dx = -2 \sqrt{2} \int_{c}^{x} \frac{\phi'(t)}{\sqrt{7} \left( \phi(t) \sqrt{1 - \phi(t)^2} \right)^{2/3}} dt \leftrightarrow \\
-\frac{2}{\pi} w_2(q) = -\frac{2}{\pi} \sum_{n=1}^{\infty} c_n^{(2)} q^n = -2 \sqrt{2} \int_{c}^{x} \frac{\phi'(t)}{\sqrt{7} \left( \phi(t) \sqrt{1 - \phi(t)^2} \right)^{2/3}} dt \leftrightarrow \\
w_2(q) = \pi \sqrt{2} \int_{+\infty}^{x} \frac{\phi'(t)}{\sqrt{7} \left( \phi(t) \sqrt{1 - \phi(t)^2} \right)^{2/3}} dt.
\]

However from (97) it holds

\[
\int \frac{1}{q P(A)} dq = w(q) + c \leftrightarrow w(q) = \int \frac{1}{q P(x)} q^{-\pi/2} dx + c = \\
= -\frac{\pi}{2} \int \frac{1}{P(x) \sqrt{x}} dx = -\frac{\pi}{2} (-2 \sqrt{2}) \int \frac{\phi'(t)}{\sqrt{7} \left( \phi(t) \sqrt{1 - \phi(t)^2} \right)^{2/3}} dt \leftrightarrow 
\]

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\[ w(q) = w_2(q) + c. \]

Hence we get the next

**Theorem 23.**

If \( q = e^{-\sqrt{x}} \), \( x > 0 \), then

\[
w_1(q) = \sum_{n=1}^{\infty} a_n e^{-\pi n \sqrt{x}} = \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2; \frac{1}{6}, \frac{2}{3} \right) = \pi F(x) = \pi h_i(x) \tag{208}
\]

and

\[
w(q) = w_2(q) = \sum_{n=1}^{\infty} b_n e^{-\pi n \sqrt{x}} = \pi \sqrt{2} \int_{+\infty}^{x} \frac{\phi'(t)}{\sqrt{T(\phi(t) \sqrt{1 - \phi(t)^2})^{2/3}}} dt, \tag{209}
\]

where

\[
\sum_{n=1}^{\infty} a_n e^{-\pi n \sqrt{x}} = \sqrt{x}. \tag{210}
\]

**Lemma 1.**

If \( \frac{w_1(q)}{f_1(w_1(q))} = q \), with \( f_1, w_1 \) as above, then

\[
\frac{\pi}{\sqrt{4}} B_0 \left( \phi \left( \pi^{-2} \log^2 \left( x/f_1(x) \right) \right)^2; \frac{1}{6}, \frac{2}{3} \right) = x. \tag{211}
\]

**Proof.**

\[
w_1 \left( e^{-\sqrt{x}} \right) = \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2; \frac{1}{6}, \frac{2}{3} \right) \Leftrightarrow \]

\[
w_1 \left( \frac{1}{x} \right) = \frac{\pi}{\sqrt{4}} B_0 \left( \phi \left( \pi^{-2} \log^2 x \right)^2; \frac{1}{6}, \frac{2}{3} \right). \]

Hence \( w_1(1/x) = w_1(x) \). Also

\[
x = \frac{\pi}{\sqrt{4}} B_0 \left( \phi \left( \pi^{-2} \log^2 \left( x/f_1(x) \right) \right)^2; \frac{1}{6}, \frac{2}{3} \right). \]

In the same way as above

**Lemma 2.**

If \( \frac{w(q)}{f(w(q))} = q \), with \( f(x) \) analytic at the origin and \( f(0) \neq 0 \), then

\[
\pi \sqrt{2} \int_{+\infty}^{\pi^{-2} \log^2 (x/f(x))} \frac{\phi'(t)}{\sqrt{T(\phi(t) \sqrt{1 - \phi(t)^2})^{2/3}}} dt = x. \tag{212}
\]
Also if we define the function $m(x)$ such that

$$\pi \int_{\sqrt{m(x)}}^{+\infty} \eta(it/2)^4 \, dt = x, \quad (213)$$

then

$$\frac{1}{\sqrt{4}} B_0 \left( k(m(x))^2 ; \frac{1}{6}, \frac{2}{3} \right) = x \quad (214)$$

Hence we have the next

**Lemma 3.**

If $f_1(x)$ is analytic around 0 and $f_1(0) \neq 0$, then $\phi_i(0) = +\infty$

$$f_1(x) = x \exp \left[ \pi \sqrt{\phi_i \left( k \left( m \left( \frac{x}{\pi} \right) \right) \right)} \right]. \quad (215)$$

**Proof.**

From

$$w_1 \left( e^{-\pi \sqrt{\phi_i}} \right) = \frac{\pi}{\sqrt{4}} B_0 \left( \phi(x)^2 ; \frac{1}{6}, \frac{2}{3} \right) \Leftrightarrow w_1 \left( e^{-\pi \sqrt{\phi_i(x)}} \right) = \frac{\pi}{\sqrt{4}} B_0 \left( x^2 ; \frac{1}{6}, \frac{2}{3} \right) \Leftrightarrow$$

$$w_1 \left( e^{-\pi \sqrt{\phi_i(k(x))}} \right) = \frac{\pi}{\sqrt{4}} B_0 \left( k(x)^2 ; \frac{1}{6}, \frac{2}{3} \right) \Leftrightarrow w_1 \left( e^{-\pi \sqrt{\phi_i(k(m(x/\pi)))}} \right) = x \Leftrightarrow f_1(x) = x e^{-\pi \sqrt{\phi_i(k(m(x/\pi)))}},$$

we get the result.

Now from Theorem 16 we have

$$w \left( e^{-\pi^2 L_1(x)} \right) = x. \quad (216)$$

Also from (154) we have

$$L \left( \frac{1}{\sqrt{4}} B_0 \left( \phi(x)^2 ; \frac{1}{6}, \frac{2}{3} \right) + l_2 \right) = \pi \sqrt{2} \int_{+\infty}^{x} \frac{\phi'(t)}{\sqrt{t} \left( \phi(t) \sqrt{1 - \phi(t)^2} \right)^{2/3}} \, dt \quad (217)$$

and (from (209),(216))

$$L^{(-1)r(-1)}(x) = \pi \sqrt{2} \int_{+\infty}^{\pi^2 x^2} \frac{\phi'(t)}{\sqrt{t} \left( \phi(t) \sqrt{1 - \phi(t)^2} \right)^{2/3}} \, dt. \quad (218)$$
Hence differentiating the relation (218) we get
\[
2\pi \sqrt{2} \frac{\phi'(x^2 \pi^2)}{(\phi(x^2 \pi^2) \sqrt{1 - \phi(x^2 \pi^2)^2})^{2/3}} = L(\phi^{-1}'(x)) = \pi^2 P(x^2 \pi^2).
\] (219)

From (217) we get
\[
L \left( \frac{1}{\sqrt{4}} B_0 \left( \phi \left( \frac{\pi^2 x^2}{2}; \frac{1}{6}, \frac{2}{3} \right) \right) \right) = L(\phi^{-1}'(x)).
\] (220)

Hence
\[
\frac{1}{L' \left( \frac{1}{\sqrt{4}} B_0 \left( \phi(x^2); \frac{1}{6}, \frac{2}{3} \right) \right)} = \frac{\sqrt{x}}{\pi}.
\] (221)

From this we have the next

**Theorem 24.**
\[
L' \left( \frac{1}{\sqrt{4}} B_0 \left( \phi(x^2); \frac{1}{6}, \frac{2}{3} \right) \right) = \frac{\pi}{\sqrt{x}}
\] (222)

and
\[
\pi \sqrt{2} \int_\pi^+ \frac{\phi'(t)}{\sqrt{t} \sqrt{1 - \phi(t)^2}}^{2/3} dt = L \left( \frac{1}{\sqrt{4}} B_0 \left( \phi(x^2); \frac{1}{6}, \frac{2}{3} \right) + l_2 \right),
\] (223)

which is equivalent to
\[
L \left( \pi^{-1} w_1(x) + l_2 \right) = w(x).
\] (224)

Continuing we have from (222), (using the function \(m(x)\)):
\[
L' \left( \frac{1}{\sqrt{4}} B \left( k(x)^2; \frac{1}{6}, \frac{2}{3} \right) + l_2 \right) = \frac{\pi}{\sqrt{\phi^{-1}(k(x))}} \Rightarrow
\]
\[
L' \left( \pi \int_\pi^+ \frac{\eta(it/2)^3}{} dt + l_2 \right) = \frac{\pi}{\sqrt{\phi^{-1}(k(x))}} \Rightarrow
\]
\[
L' \left( x + l_2 \right) = \frac{\pi}{\sqrt{\phi^{-1}(k(m(x)))}}.
\]

Hence from (154.1) and (160) we have:

**Theorem 25.**
\[
L(x) = \pi \int_\pi^x \frac{1}{\sqrt{\phi^{-1}(k(m(t))) - t_1}} dt = w \left( e^{-\pi \sqrt{\phi^{-1}(k(m(x))) - l_1}} \right) + c_1
\] (225)
and
\[ h(x) = \phi^{-1}(k(m(x))). \] (226)

\[ x = \pi \int_c^{L(x) - l_2} \frac{1}{\sqrt{\phi^{-1}(k(m(t)))}} \, dt \implies 1 = \pi \frac{L^{(-1)'(x)}}{\sqrt{\phi^{-1}(k(m(L(x))))}} \Rightarrow \]
\[ L^{(-1)'}(x) = \pi^{-1} \sqrt{\phi^{-1}(k(m(L(x))))}. \]

4 Evaluation of functions as products and Lagrange’s equation

Using equations (1)-(6) we can easily show that if \( q = e^{-\pi \sqrt{x}}, x > 0 \) and \( g(q) \) is analytic around 0 with \( g(0) = 1 \):

\[ g(q) = e^{w(q)} = 1 + \sum_{n=1}^{\infty} \frac{q^n}{n!} \left( \frac{D}{Dh} \right)^{n-1} (e^h f(h)^n)_{h=0} = \]
\[ = \prod_{n=1}^{\infty} (1 - q^n)^{-1/n} \sum_{d|n} \frac{\mu(n/d)}{\varphi(d)} \left( \left[ \frac{D}{Dh} \right]^{d-1} (f(h)^n) \right)_{h=0}, \] (227)

where \( f(x) \) is defined as follows: It holds
\[ w(x) = \log (g(x)). \] (228)

If \( g_1(x) \) is the inverse of \( \log(g(x)) \), then
\[ f(x) = \frac{x}{g_1(x)}, \] (229)

where \( g_1(x) \) is analytic around 0 and have simple root at \( x = 0 \) i.e. \( g_1(0) = 0 \) and \( g_1'(0) \neq 0 \).

Example.
Asume that
\[ g(q) = \sqrt{1 + q + q^2}, \]
then set \( w(x) = \log (\sqrt{1 + x + x^2}) \). Solving \( w(x) = y \), we get
\[ x = w^{-1}(y) = \frac{1}{2} \left(-1 + \sqrt{-3 + 4e^{2y}}\right). \]

Hence
\[ f(y) = \frac{2y}{-1 + \sqrt{-3 + 4e^{2y}}}. \]

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Set
\[ A(n) = \frac{1}{n} \sum_{d|n} \frac{\mu(n/d)}{\Gamma(d)} \left[ \left( \frac{D}{Dh} \right)^{d-1} (f(h)^d) \right]_{h=0}. \] (230)

Then
\[ g(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-A(n)}. \] (231)

Here (in our example) we get \( A(1) = \frac{1}{2}, \) \( A(3) = -\frac{1}{2} \) and \( A(n) = 0 \) for \( n \neq 1, 3. \) Hence
\[ \sqrt{1 + q + q^2} = (1 - q)^{-1/2}(1 - q^3)^{1/2}. \] (232)
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