A detailed study of mode II test specimen with elastic or plastic crack tips by using boundary element method

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Abstract. In linear elastic fracture mechanics the stress and strain magnification at the crack tip is characterized by stress intensity factors which have been obtained for a large variety of loading and specimen geometries and of course mode II test specimen included. In this paper not only different kinds of boundary conditions of the contact surfaces of the elastic crack tip will be discussed but the plastic one. The rotation of the center line of the specimen will also be discussed. The results show the powerful application of the BEM model in the specimen.

1. Introduction

There are so many kinds of specimen which deal with mode II stress intensity factor in LEFM. Some of them deal with experiments [1-4]. Some deal with numerical (FEM, BEM) [5-8]. And even some with both [9-10]. However, none of them gives the results about the plastic crack tip. In this paper a mode II test specimen is with BEM developed. The specimen is made from 0.325 in thick 7075-T6 aluminum sheet with a 68,000 MPa Young’s modulus as shown in Figure 1 is now investigation.

![Figure 1. Mode II test specimen (BEM model, dimension in inch).](image-url)
The specimen is treated as two bodies divided along the interface plane on which the crack lies \((x_2 = 0)\) and three different types of crack [(1) non-interfering crack, (2) DBC crack without plastic tip and (3) DBC crack with plastic tip] will be discussed. As described in [11] the dilatant boundary conditions (DBC) are assumed to be idealized uniform sawtooth crack surfaces and effective Coulomb sliding law. In particular to get an accurate experimental value of \(K_{II}\) the rigid body rotation of the crack plane should be as small as possible. Hence, the motion of the crack plane will also be investigated.

2. Accuracy test

For the convenience of comparing the accuracy of present model with the FEM calculation of Gross et al. [12] the NASA specimen, shown in Figure 2 with a hook added and without the crack starter notch, is studied first. Introducing a dimensionless mode II stress intensity factor

\[
Y_II = \frac{K_{II} B \sqrt{W}}{P}
\]

where \(B\) is the thickness of specimen, (0.325 inches in Figure 1) \(W\) is the width of specimen (2.9 inches in Figure 1) \(P\) is the applied load.

![Figure 2. Mode II NASA test spacemen (FEM model, dimension in inch).](image)

The results of this test in Figure 3 show the accuracy of present model is very much like the FEM model and extent the limitations of the value \(a/W\) for good accuracy of [12] in the range between 0.5 and 0.9 has been extended to \(a/W\) between 0 and 1. The discrepancy is consistently on the order of 10% with the BEM always larger as in [11]. In this figure the fine dashed line represents the experimental data from [12] which lie mostly between the BEM and FEM curves. Partial crack opening (loss of contact) does not occur, even after introducing DBC with plastic tip boundary conditions to the crack region of the present model, thus eliminating the need to solve the potential application of the present model to determining the effect of crack face roughness in a realistic experimental specimen. The accuracy among three different approach, i.e., theoretical, numerical (BEM) and experimental of the cracked ring are presented in [13].
3. Boundary conditions

3.1. Non-interfering flat closed crack

As described in [14] the cracked body is divided into two parts $B_\gamma (\gamma = 1, 2)$ along the plane of crack. We call this the interface. Furthermore, the interface of the specimen is subdivided into two regions. $I_1$ is the crack region and $I_2$ is the ligament region as in Figure 1, 3. Let $t_i$ and $u_i$ ($\gamma = 1, 2$ and $i = 1, 2$) denote the $i$th boundary traction and displacement components, respectively, on the boundary $B_\gamma$. The tractions and displacements of all the points in $I_2$ must satisfy the continuity boundary conditions, i.e., $(2t_1)_I = -(1t_1)_I$, $(2t_2)_I = -(1t_2)_I$, $(2u_1)_I = (1u_1)_I$, and $(2u_2)_I = (1u_2)_I$.

![Figure 3. Mode II dimensionless stress intensity factor accuracy of BEM model.](image)

This leaves $(1t_1)_I$, $(1t_2)_I$, $(1u_1)_I$, $(1u_2)_I$ as unknowns. However, due to the closure of the crack the boundary condition for smooth contacting crack surfaces should be as follows (zero shear traction $(1t_1)_I = (2t_1)_I = 0$), and continuity of normal displacement and both stresses $(1u_2)_I = (2u_2)_I$, and $(1t_2)_I = -(2t_2)_I$.

The present model is described into 452 nodal points and 7 regions ($I_1$-$I_7$) of different types of boundary conditions shown in Figure 4. The upper and right pins are assumed fixed. A compressive load is applied through the lower pin in the positive $x_1$ direction as in Figure 5 by assuming a uniform distribution of normal traction over $90^\circ$ of lower pin hole surface, i.e., points 437, 438, 439, 451 and 452 (14). Therefore, the boundary conditions of these points are $(2t_1)_I = p \cos \theta_1$ and $(2t_2)_I = p \sin \theta_1$. The unknowns are $(2u_1)_I = (2u_2)_I$, where $p$ is applied normal stress on the lower hole and $\theta_1$ is the angle between the direction normal of each node and $x_1$ axis. The boundary conditions of the three surfaces ($I_5$) are tractions free and the unknowns are $(1u_1)_I$, $(1u_2)_I$, $(2u_1)_I$, $(2u_2)_I$. The horizontal displacement component of point 217, 218, 220, 221, 441, 352, 353, 354 and 355 ($I_6$) is taken to be zero and their two traction components are related by $\tan \theta_2$, where $\theta_2$ is the angle between the direction normal and $x_1$ axis, in order to have zero shear stress on the hole.
Hence, the boundary conditions for points on $I_6$ in the upper half plane are $(u_1)_{I_6} = 0$, $(t_2)_{I_6} = (t_1)_{I_6} \tan(\theta_2)$ and the unknowns are $(u_2)_{I_6}$, $(t_1)_{I_6}$, and in the lower half plane $(u_1)_{I_6} = 0$, $(t_2)_{I_6} = (t_1)_{I_6} \tan(\theta_2)$ and the unknowns are $(u_2)_{I_6}$, $(t_1)_{I_6}$. Point 219 ($I_7$) is a totally fixed point. The boundary conditions of this point are $(u_1)_{I_7} = (u_2)_{I_7} = 0$ and the unknown are $(t_1)_{I_7}$, $(t_2)_{I_7}$.

![Figure 4. Schematic of BEM mesh of mode II test specimen.](image1)

![Figure 5. Free body diagram of mode II test specimen.](image2)

3.2. DBC-elastic tip
The DBC crack is introduced here in an attempt to predict the induced mode I crack face displacement. The open crack region of the interface mentioned above is replaced with a DBC region and therefore, all boundary conditions are the same with as in section 3.1 with exception of the crack region ($I_7$). As described in [6] the dilatant boundary conditions $(t_2) = -\frac{1}{\Gamma}(t_1) + (\sigma_0 + \frac{1}{\Gamma}t_0)$, $(u_1) = (u_1) - \cot \alpha (u_2) - (u_2)_{I_7}$, and $(t_1) = -(t_1) + (t_2) = -(t_2)_{I_7}$ are used in ($I_7$) leaving as the unknowns in the crack, $(t_1)_{I_7}$, $(u_1)_{I_7}$, $(u_2)_{I_7}$, $(t_2)_{I_7}$ where
\[ \Gamma = \frac{\mu + \tan(\alpha)}{1 - \mu \tan(\alpha)} \]  

(2)

is the effective coefficient of friction, \( \alpha \) is the asperity angle of the sawtooth contact crack surfaces, \( \sigma_0 \), \( \tau_0 \) are the applied normal and shear stress, respectively.

### 3.3. DBC-plastic tip

The DBC with plastic tip region are obtained by adding a plastic region \((I_3)\) between the DBC crack region \((I_1)\) and ligament \((I_2)\). As described in [11] the calculation of the length of plastic region \((I_3)\) is based on a reformulation of the Dugdale strip yielding mode in Dugdale [15] to the mixed mode open crack case, in which the normal and shear stresses in the plastic zone have the same ratio as the applied normal and shear stresses, but are both functions of uni-axial tensile yield stress and the applied stresses in [16]. The boundary conditions of \((I_1)\) \( (t_1)_{t_1} = \tau_0 - \tau_p, (t_2)_{t_2} = \sigma_0 - \sigma_p, \) \((z_1)_{t_1} = -(t_1)_{t_1} = \tau_p - \tau_0, (z_2)_{t_2} = -(t_2)_{t_2} = \sigma_p - \sigma_0. \) Not only are the plastic zone stresses \( \sigma_p \) and \( \tau_p \) unknown a-priori, but so is the length of the plastic zone denoted by \( \Delta a \). To find these unknowns which characterize the crack tip fields under small scale yielding a yield criterion must first be imposed on the total state of stress in the plastic zone. The von Mises yielding criterion written in terms of the tensile yield strength of the material and applied to this state of stress gives \( \sigma_p^2 + 3\tau_p^2 = \sigma_y^2 \) for generalized plane stress and \( \beta\sigma_p^2 + 3\tau_p^2 = \sigma_y^2 \) for plane strain, where \( \beta = \frac{15}{16} - \nu + \nu^2, \) \( \nu \) is the Poisson’s ratio. This provides one relation between \( \sigma_p \) and \( \tau_p \). The other two relationships which determine these stresses and \( \Delta a \) is that both the plastic crack opening displacement (COD) and plastic sliding displacement (CSD) must zero slope at the boundary between the plastic zone and the ligament. Hence, the proper length of the plastic zone will be determined by using the 2D Newton’s method.

The finite result of each different boundary condition is a simultaneous linear algebraic equations for the unknowns nodal displacements and stresses. After the system is solved the crack tip stress intensity factors are calculated by fitting the calculated COD or CSD to the standard square root form at an elastic crack tip.

### 4. Results

The CSD of the specimen, Figure 6 (a) with a uniform distribution loading \( p = 600 \) MPa over 45° and 5.24 mm non-interfering crack in 7075-T6 aluminum are shown in Figure 6(a). The \( \sqrt{R} \) behavior of the CSD is clearly shown. Due to closure there is no COD. The distributions of shear and normal stresses along the interface are shown in Figure 6(b). There is no shear stress in the crack region for smooth contact surface, and compression everywhere along the crack because it is closed. Hence there is a stress singularity at the crack tip for shear stress only. In this case \( K_{II} \) is 38.64 MPa\( \sqrt{m} \) and the dimensionless mode II stress intensity factor \( Y_{II} \) is 1.645. Note that the normal stress is effectively zero in the ligament, but increasingly compressive in the crack as the crack mouth is approached. This will affect the interfering crack problems.
Figure 6. (a) CSD (b) stress distributions of mode II test specimen; in aluminum with 5.24 mm non-interfering crack, 600 MPa applied loading, $\mu = 0.2$, $G = 27000$ MPa.

Figure 7 shows the displaced position of the interface of the specimen. The majority of the displacement shown is normal to the interface (crack plane) and the motion parallel to the interface cannot be seen on the scale shown in the figure. It can be seen, and is expected from Figure 5, that the ligament portion of the interface is rotated about 0.01° counterclockwise from horizontal. This kind of rotation should be reduced as much as possible in order to obtain a correct $K_{II}$ or $Y_{II}$. The maximum motion of the interface due to the rotation is about 52 $\mu$m.

Figures 8 shows the COD, CSD and traction distributions in the interface for a homogeneous aluminum specimen with the same boundary conditions as in Figure 6, except for the DBC crack. The asperity angle is 20° and the coefficient of friction $\mu$ is 0.2 which gives $\Gamma = 0.6083$ according to equation 2. Due to the combined shielding effect the CSD decreases as compared with the CSD in Figure 8(a). Due to the roughness induced mode I crack face displacements are also observed. A stress singularity is observed for both normal and shear stresses in Figure 8(b) because it is the location of the elastic crack tip. The ratio of the CSD to the COD is equal to $\cot \alpha$ and the absolute value of the ratio of shear stress to the normal stress is $\Gamma$, as imposed by the boundary conditions.

Figure 7. Displaced position of interface of mode II test specimen; in aluminum with 5.24 mm non-interfering crack and 600 MPa applied loading, $\mu = 0.2$, $G = 27000$ MPa.
Figure 8. (a) COD and CSD (b) stress distributions of mode II test specimen in aluminum with DBC, 5.24 mm non-interfering crack and 600 MPa applied loading, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa.

The rotation of the ligament portion of the interface has been reduced to $0.08^\circ$ counterclockwise from horizontal as shown in Figure 9 after DBC boundary conditions were introduced. The maximum shift of the interface due to the rotation is now reduced to 34 $\mu$m. This reduction is probably due to the fact that the roughness provides additional resistance to rotation besides that provided by the loading pins.

Figure 9. Displaced position of interface of mode II test specimen in aluminum with DBC, 5.24 mm non-interfering crack and 600 MPa applied loading, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa.

Figure 10 shows the linear relationships of the mode I and II shear stress intensity factors with respect to the applied shear stress intensity factor $K_{IIapp}$ which is calculated from non-interfering crack stress intensity factor solution. Each curve in Figure 10 is characterized by $\alpha = 20^\circ$, $\mu = 0.2$ which gives $\Gamma = 0.6083$ and $a/W = 0.45$ as described above. Due to the geometry factor the relationship between the resistance shear stress intensity factor (applied-actual) and the induced mode I stress intensity factor in the Griffith crack problem ($K_{IIr} = K_{IIind}$) is no longer valid. Each curve in Figure 11 is for various values of asperity angle $\alpha$ with $p = 600$ MPa, $\mu = 0.2$ and 5.72 mm crack.
Figure 12 shows the COD, CSD and traction distribution in the interface for a homogeneous aluminum mode II test specimen with DBC and plastic crack tip. The asperity angle is 20° and $\mu = 0.2$ as before. The results in Figure 12(a) show the ratio of the plastic zone length to the crack length is 0.636 with applied uniform distribution loading $p = 600\,\text{MPa}$. The zero slopes of both COD and CSD are

![Graph](image)

**Figure 10.** Linear relationship of $K_I$ and $K_{II}$ with respect to $K_{II\text{app}}$ in aluminum with DBC, elastic tip, 5.72 mm crack and 20° asperity angle, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa.

![Graphs](image)

**Figure 11.** (a) Mode II shielding (b) mode I enhancement of mode II test specimen in aluminum with DBC, elastic tip, 5.72 mm crack and different values of asperity angle, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa, $\sigma_y = 500$ MPa.
Figure 12. (a) COD and CSD (b) stress distributions of mode II test specimen in aluminum with DBC, plastic tip, 5.72 mm crack, 600 MPa applied loading and 20° asperity angle, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa, $\sigma_y = 500$ MPa.

very clear. Because no partial contact occurred the ratio of the CSD to the COD is $\cot \alpha$ everywhere in the crack region. Also, it is not difficult to understand from Figure 13 that the rotation of the ligament portion of the interface is seen to be 0.069° and the maximum shift of the interface is 30 μm after the plastic tip boundary conditions have been added.

As described in [6] we interpret the plastic tip results in terms of “plastic” stress intensity factors. The geometry factor has been calculated using BEM non-interfering crack model is 1.558, i.e.,

$$K_{lp} = 1.158\sigma_{\text{eff}} \sqrt{\pi a}$$

$$K_{lp} = 1.158\tau_{\text{eff}} \sqrt{\pi a}$$

The relationships of $K_{lp}$, $K_p$ with $K_{l\text{app}}$ are shown in Figure 14 for various asperity angle $\alpha$. They are calculated with $\mu = 0.2$, $G = 27000$ MPa for the convenience of being compared to the elastic case. The results show $K_f$ decreases monotonically as $\alpha$ increase. But, only when $\alpha$ is smaller than about 45° does $K_f$ increases with the increasing $\alpha$. Otherwise, $K_f$ decreases all the way to zero as $\alpha$ increase to 78.69°. In Figure 14(b) the elastic $K_f$ is larger than plastic for a while as $\alpha$ increase, but becomes smaller after passing the critical $\alpha$ around 40°. The elastic $K_f$ however, is always larger than the plastic one. Note that $K_{lp}$ vs. $K_p$ curves are concave down.
Figure 13. Displaced position of interface of mode II test specimen in aluminum with DBC, 5.72 mm crack and 600 MPa applied loading and 20°asperity angle, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa $\sigma_y = 500$ MPa.

Figure 14. (a) The relationships of $K_{lp}$ (b) $K_{Ilp}$ with $K_{illapp}$ with DBC, 5.72 mm crack and 600 MPa applied loading and various asperity angle $\alpha$, $\nu = 0.33$, $\mu = 0.2$, $G = 27000$ MPa $\sigma_y = 500$ MPa. (e for elastic crack tip, p for plastic crack tip)

5. Conclusions
The numerical results were presented and show the potential application of the BEM model to the effects of mixed mode loading in mode II test specimen and in particular interface and crack problems with various boundary non-linearity due to rough contact and plasticity. It shown not only the difference of COD, CSD, stress distributions and stress intension factors between the elastic and plastic crack tips but also the rotation of the centre lines of the specimen. Compare the results of elastic crack tip boundary condition the plastic boundary releases the rigidity of the specimen.
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