Opinion control in complex networks

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Keywords: network, voter model, synchronization

Abstract
In many political elections, the electorate appears to be a composite of partisan and independent voters. Given that partisans are not likely to convert to a different party, an important goal for a political party could be to mobilize independent voters toward the party with the help of strong leadership, mass media, partisans, and the effects of peer-to-peer influence. Based on the exact solution of classical voter model dynamics in the presence of perfectly partisan voters (i.e., zealots), we propose a computational method that uses pinning control strategy to maximize the share of a party in a social network of independent voters. The party, corresponding to the controller or zealots, optimizes the nodes to be controlled given the information about the connectivity of independent voters and the set of nodes that the opposing party controls. We show that controlling hubs is generally a good strategy, but the optimized strategy is even better. The superiority of the optimized strategy is particularly eminent when the independent voters are connected as directed (rather than undirected) networks.

1. Introduction
We often flip our opinions in response to what others are doing. In various situations ranging from voting to adoption of new habits, it is widely recognized that opinion formation on a large scale is considerably influenced by peer-to-peer interaction among individuals embedded in social networks [1–4]. On the other hand, such contagion effects seem to be irrelevant to some individuals that are resolute on certain matters and do not feel peer pressure. For example, in political elections, the electorate is usually a composition of independent voters and partisan voters who favor specific parties [5, 6]. Because partisans are reluctant to change, the main goal for a party would be to mobilize independent voters toward its candidate. The aim in the present theoretical study is to explore efficient strategies for attracting independent voters when independent voters are connected as a social network and different parties are allowed to influence some independent voters.

In models of collective opinion formation, perfectly partisan voters, or the parties themselves, have been analyzed under the names ‘pinning controllers’ and ‘zealots’. Pinning control of networks, in which a controller unidirectionally affects some selected nodes, is an effective method to guide the state of nodes in a coupled dynamical system to a desired state, such as the synchronized state. The idea has been explored in linear and nonlinear dynamical systems coupled in networks [7–9]. We apply this idea to opinion control in complex networks. Theoretical studies of pinning opinion control in collective opinion dynamics date back at least to studies of zealots in the voter model. The voter model is a paradigmatic model of collective opinion formation, in which individuals would flip their opinions at a rate proportional to the number of neighboring others who select the opposite opinion [1, 3, 10–12]. Mobilia first showed that this dynamic dramatically changed upon the introduction of zealots (i.e., stubborn voters) to the population of ordinary voters [13]. Zealots correspond to perfect partisans, leaders in a community, or mass media and are mathematically the same as pinning controllers (see [14, 15] for the case of error-prone zealots). With the voter model and other opinion formation models, it has also been shown that the existence of zealots in favor of opposing opinions, like competing parties, induces the coexistence of different opinions in the equilibrium [16–21]. This phenomenon contrasts with that of the...
voter model without zealots [1, 3, 10] and with a single type of zealot [13]; in these cases, consensus is always reached in finite networks and low-dimensional infinite lattices.

Controlling opinion dynamics implies that there is a control objective such as the desired final state or cost minimization. Most of the existing literature on voter models in the presence of zealots described phenomena induced by zealots, but it has not treated opinion control in a proactive sense. Among the studies that treated opinion control explicitly, some studies assumed the presence of just a single type of zealot [22–25]. To the best of our knowledge, two studies considered opinion control in networks by competing zealots, similar to the present study. In [26], the authors considered a control strategy in which the highest-degree nodes are controlled. They solved an optimization problem numerically by transforming the problem into that of a random walk in the network and numerically running the random walk. Their objective function was the control cost, under the condition that the fraction of votes in favor of the controller’s opinion must be at least a prescribed threshold. In [21], the authors treated optimal selection of newly controlled nodes under the condition that the two subsets of nodes that the focal zealot and the opposing type of zealot controlled at the moment were known. They proposed a greedy algorithm that exploited the submodularity of the objective function. However, the main focus of the article was the theoretical evaluation of the algorithm, and the proposed algorithm was not numerically tested on networks.

In contrast to these studies [21, 26], in the present study we establish a procedure that uses linear algebra to maximize the number of votes by independent voters. We assume zealots (equivalently, pinning controllers) possessing different opinions can influence some independent voters, represented by nodes in a social network, to try to coax independent voters into their own parties. On the basis of the exact solution of the mean number of votes in favor of each opinion in the presence of zealots, we establish a method to compute the solution, and we use power iteration to heuristically optimize the mean number of votes. Then, we apply the proposed method to several complex networks. In particular, we examine whether controlling high-degree nodes (i.e., hubs) is efficient, depending on networks.

2. Model

We assume a directed and weighted network with \( N \) nodes, which may have self-loops. Each node is occupied by an independent voter. Therefore, we also refer to nodes as independent voters. The weight of the link from node \( i \) to \( j \) is denoted by \( w_{ij} \geq 0 \) and represents the strength of the influence of independent voter \( i \) on independent voter \( j \) in opinion-formation dynamics. The \( N \) independent voters dynamically switch their opinions between two options, A and B, as we will specify in the following.

We also assume that the network is strongly connected so that there is no root node or root component to send out directed links to the remainder of the network without receiving directed links from outside. Such a root would effectively function as a zealot because the opinion of the root is not affected by the remainder of the network; we will introduce zealots externally in the following. For undirected networks, the strong connectedness is reduced to the usual connectedness.

Apart from the \( N \) independent voters, we assume zealots (i.e., perfect partisans) each of which favors one opinion. By definition, zealots never change their opinions and are given as elements external to the network of independent voters (figure 1). The control gain represents the strength with which a zealot influences an independent voter. We denote by \( p_{A,i} ( \geq 0 ) \) and \( p_{B,i} ( \geq 0 ) \) the control gain on independent voter \( i \) for a zealot in favor of opinions A and B, respectively. The control gain plays the same role as the link weight, \( w_{ij} \). We use different terms and symbols to distinguish interaction between a pair of independent voters (link weight, \( w_{ij} \)) and that between a zealot and an independent voter (control gain, \( p_{A,i} \) and \( p_{B,i} \)). We consider a single zealot in favor of opinion A (called A zealot in the following) and a single zealot in favor of B (B zealot) such that they influence some independent voters, perhaps with different control gains. We do not lose generality by assuming a single zealot of each type because two A zealots influencing independent voter \( i \) with a unity control gain is equivalent to one A zealot influencing \( i \) with a control gain equal to two.

From the zealots’ perspectives, each ‘party’ (i.e., zealot) may be interested in wisely selecting \( p_{A,i} \) and \( p_{B,i} \) (\( 1 \leq i \leq N \)) under given constraints. For example, the sum of the control gain may be upper-bounded because the cost of control is proportional to the total gain. Alternatively, each type of zealot may be able to control a given number of nodes, each with a unity gain, if the party has to make a binary decision regarding whether or not to persuade each independent voter and accessing individual independent voters is a costly process.

In each update event, one of the independent voters in the network, node \( i \), is selected with equal probability, \( 1/N \). Then, \( i \) copies the opinion of one of its upstream neighbors in the network (usual neighbors in the case of undirected network) or that of a zealot if \( i \) is directly influenced by the zealot. The independent voter or the zealot whose opinion \( i \) mimicks is selected with a probability proportional to the link weight or the control gain.

Figure 1 is a schematic example of an undirected and unweighted network. Suppose that node \( i \), which has
degree four and is controlled by the A zealot (with the unity gain assumed, i.e., \( p_{A,i} = 1 \)), is selected for updating. Among the four neighboring independent voters and the A zealot, three possess opinion A (filled circles and square) and two possess opinion B (open circles). Therefore, \( i \) adopts opinion A and B with probability \( \frac{3}{5} \) and \( \frac{2}{5} \), respectively. Generally speaking, a selected node, \( i \), selects the opinion of independent voter \( j \) for copying with probability
\[
\sum_{\ell=1}^{N_i} w_{i\ell} / \left( \sum_{\ell=1}^{N_i} w_{i\ell} + p_{A,i} + p_{B,i} \right),
\]
the A zealot with probability
\[
p_{A,i} / \left( \sum_{\ell=1}^{N_i} w_{i\ell} + p_{A,i} + p_{B,i} \right),
\]
and the B zealot with probability
\[
p_{B,i} / \left( \sum_{\ell=1}^{N_i} w_{i\ell} + p_{A,i} + p_{B,i} \right).
\]
The update rule is the same as the so-called VM (voter model) rule among the three main update rules proposed in [27, 28]. The other update rules are briefly discussed in section 4.

Each node is updated once per unit time on average. Owing to the presence of zealots in the opposite opinions, the consensus (also called fixation) of either opinion never occurs [16–21]. We are interested in the fraction of nodes in favor of each opinion in the equilibrium.

3. Results

3.1. Mean-field analysis

We start with a mean-field analysis assuming a well-mixed infinite population corresponding to an undirected and unweighted network of infinite size. For this case, more elaborate theoretical results regarding distributions of the number of independent voters in either opinion are available [16, 18, 19]. Denote by \( x \) \((0 \leq x \leq 1)\) the fraction of independent voters possessing opinion A. The fraction of independent voters possessing opinion B is equal to \( 1 - x \). We assume that there are additional fractions of \( a \) and \( b \) zealots in favor of opinions A and B, respectively, where the fraction refers to that relative to the number of independent voters. For example, \( a = 0.1 \) implies that the number of A zealots is equal to 10% of the number of the independent voters. We also assume that the zealots affect each independent voter with a unity gain. Finally, we assume \( a, b > 0 \). Although the original model has a single zealot in each opinion, in this section we translated it to the fraction of zealots because a single zealot does not have an influence in an infinite population unless the number of independent voters that the zealot influences or the control gain is infinite.

In this situation, the opinion formation dynamic is given by
\[
\frac{dx}{dt} = (1 - x)(x + a) - x(1 - x + b),
\]
where \( t \) is time. The steady state is given by
\[
x^* = -\frac{a}{a + b}.
\]
This steady state is stable with the eigenvalue \(-a - b < 0\); if the fraction of zealots is large, the convergence to the equilibrium given by equation (2) is fast. Equation (2) implies that the fraction of the opinion of independent voters in the equilibrium is equal to that of zealots.

3.2. Average opinion of independent voters in general networks

Denote by \( u_{A,i} \) \((1 \leq i \leq N)\) the probability that independent voter \( i \) takes opinion A. The probability that \( i \) takes opinion B is given by \( u_{B,i} = 1 - u_{A,i} \). The master equation of the opinion dynamics is given by

![Figure 1. Schematic of opinion control and opinion updating. The circles represent independent voters. The squares represent zealots. The filled and open symbols correspond to opinions A and B, respectively. Upon updating, node \( i \) adopts opinions A and B with probability 3/5 and 2/5, respectively.](image)
Consider the undirected and unweighted complete graph with control gains $p_{A,i} = a$ and $p_{B,i} = b$ ($1 \leq i \leq N$). We substitute $L_{ij} = \delta_{ij}(N-1) - (1 - \delta_{ij})$ in equation (4) to obtain $u_{A,i} = a/(a+b)$ ($1 \leq i \leq N$). This result is consistent with that obtained from the mean-field analysis in section 3.1.

Consider a directed and unweighted star with N nodes (figure 2). Node 1 is the unique hub adjacent to all other nodes, each of which is only adjacent to the hub. Assume that the A zealot controls the hub, $p_{B,1} = b\delta_{1,i}$, so that the hub attracts the independent voters. If the A zealot controls the hub, we obtain $u_{A,i} = (ab + a)/(ab + a + b)$ ($1 \leq i \leq N$). If the A zealot controls node 2, which is adjacent only to the hub, we obtain $u_{A,i} = a/(a + b)$ ($1 \leq i \leq N$) and $S_A = a/(a + b)$. If the A zealot controls node $i$ ($3 \leq i \leq N$), we obtain $u_{A,i} = a/(a + b)$ ($1 \leq i \leq N$) and $S_A = a/(a + b)$.
obtain \( u_{A,i} = (ab + a)/(2ab + a + b) \) (i = 1, 4 ≤ i ≤ N), \( u_{A,2} = a/(2ab + a + b) \), and \( S_A = (ab + a)/(2ab + a + b) \).

The behavior of the share of opinion A when we set \( b = 1, N \to \infty \), and vary \( a \) is shown in Figure 3. As expected, \( S_A \) increases with \( a \) in all cases. When the A zealot controls the hub (black line), \( S_A \) is larger than when the A zealot controls a nonhub (red and blue lines). When the A zealot controls a nonhub node, \( S_A \) is equal to 0.5 at \( a = 1 \) because the other zealot also controls a nonhub with the same gain. However, when \( a \neq 1 \), \( S_A \) depends on whether the nonhub nodes that the two zealots control are the same (red line) or different (blue line).

3.4. Numerical results

In this section, we numerically evaluate the effect of control protocols on one model network and three real-world networks. We do not assume self-connection in the following.

3.4.1. Numerical procedure

We numerically calculate \( S_A \) and \( S_B \) as follows. Because \( L + \text{diag}(p_{A,1} + p_{B,1}, \ldots, p_{A,N} + p_{B,N}) \) is diagonally dominant, we can obtain its solution by iteration methods such as the Jacobi and Gauss-Seidel iterations [32]. Starting from some initial conditions, which we set to \( u_{A,i} = u_{B,i} = 1/2 \) (1 ≤ i ≤ N), we repeat

\[
\begin{align*}
    u_{A,i} & \leftarrow \frac{p_{A,i} + \sum_{j=1}^{N} w_{ij} u_{A,j}}{\sum_{j=1}^{N} w_{ij} + p_{A,i} + p_{B,i}}, \\
\end{align*}
\]

where \( \leftarrow \) means that we substitute the right-hand side in the left-hand side. We carry out equation (8) simultaneously for all \( i \) in the case of the Jacobi iteration, which we use in the following numerical simulations. Finally, we use equation (6) to calculate \( S_A \) (and \( S_B = 1 - S_A \)).

3.4.2. Barabási-Albert model

We start with a Barabási-Albert (BA) model network that has \( N = 200 \) nodes. The model produces scale-free networks with the power-law exponent of the degree distribution equal to three when \( N \to \infty \) [33]. To generate a network, we assume that there are initially two nodes that are connected by a link and that any arriving node brings in two links. The generated network has 397 undirected and unweighted links.

First, we examine the case in which both zealots control a single node with the same gain. The B zealot is assumed to control the node with either the largest or the smallest degree. If there are multiple nodes with the same degree, we randomly select one. The share of opinion A when the A zealot controls one of the \( N \) nodes is shown in figures 4(a) and (b) for the gains equal to 1 and 10, respectively. In these figures, the circles and squares represent the results when the B zealot controls the node with the smallest degree (circles) or different (squares).

Figure 4 indicates that the share of opinion A is large when the A zealot controls a node with a large degree. The share of A is larger when the B zealot controls the node with the smallest degree (circles) than when the B zealot controls the node with the largest degree (squares). These results indicate that it is more efficient to control hubs than small-degree nodes, which is consistent with the previous results [16, 26]. However, the degree is a dominant determinant, but not the sole determinant of the share of A, which is consistent with a previously made remark [21]. Controlling nodes with the same degree generally leads to different \( S_A \) values. In
addition, in figure 4(b), $S_A$ is slightly larger for the node with degree 22 than it is for the largest-degree node (i.e., degree 34) when the B zealot controls the node with the smallest degree (circles). As a separate observation, for a larger control gain (figure 4(b)), the share of A is more sensitive to the degree than for a smaller control gain (figure 4(a)). This is intuitively because a large control gain implies that the influence of the zealots is relatively important in opinion-formation dynamics when compared to peer-to-peer interaction between independent voters.

Next, we assume that each zealot controls ten nodes with the same gain, and we examine the case in which the A zealot optimizes the set of nodes to be controlled. The B zealot is assumed to control ten randomly selected nodes. The A zealot is allowed to use this information. First, the A zealot starts by controlling ten randomly selected nodes. Second, we calculate the share of opinion A, $S_A$. Third, we tentatively swap one randomly selected controlled node with one randomly selected uncontrolled node. Fourth, we recalculate $S_A$. If the new $S_A$ value is larger than the old value, then we adopt the swapping. Otherwise, we discard the swapping. We repeat swapping attempts $2 \times 10^4$ times unless otherwise stated. The $S_A$ values after the optimization are shown in table 1 for the control gains equal to 1, 10, and 100. In the table, we also show the results for the degree-based protocol [26], which is defined by the A zealot controlling the ten nodes with the largest degrees. We break ties by randomly selecting the required number of nodes with the threshold degree. For example, suppose that the network is undirected and unweighted, there are nine nodes whose degree is at least 20, and there are three nodes with degree 19. Then, we control one of the three nodes with degree 19 and the nine nodes with the larger degrees. For each gain value, the ten nodes that the B zealot controls are the same for the optimized and degree-based A zealot. Table 1 suggests that $S_A$ after optimization is slightly larger than the value obtained from the degree-based protocol. The degree of the ten controlled nodes by the optimized A zealot, degree-based A zealot, and the B zealot are shown in figures 4(c) and (d) for gains equal to 1 and 10, respectively. The B zealot mostly controls nodes with small degrees owing to the scale-free property of the BA model. The optimized A zealot tends to control hubs. However, some nodes that the optimized A zealot controls are not among the ten largest-degree nodes.

Figure 4. Results for the BA model with $N = 200$ nodes and 397 links. The results when each zealot controls a single node are shown in (a) and (b). The control gain is set to 1 in (a) and 10 in (b) for both zealots. The circles and squares in these panels represent the cases in which the B zealot controls the node with the smallest and largest degrees, respectively. The value on the horizontal axis represents the degree of the node that the A zealot controls. In (c) and (d), the degrees of the controlled nodes when each zealot controls ten nodes are shown in descending order. The rank refers to that in terms of the degree. Given ten randomly selected nodes that the B zealot controls, whose degrees are shown by the triangles, the optimized A zealot controls ten nodes whose degrees are shown by the circles. The degrees of the ten largest-degree nodes, which the A zealot using the degree-based protocol controls, are shown by the squares. The control gain is set to 1 in (c) and 10 in (d) per node for both zealots. In (c) and (d), we have used different sets of ten randomly selected nodes for the B zealot. The degrees shown by the squares in (c) and (d) are the same.  

The $S_A$ values after the optimization are shown in table 1 for the control gains equal to 1, 10, and 100. In the table, we also show the results for the degree-based protocol [26], which is defined by the A zealot controlling the ten nodes with the largest degrees. We break ties by randomly selecting the required number of nodes with the threshold degree. For example, suppose that the network is undirected and unweighted, there are nine nodes whose degree is at least 20, and there are three nodes with degree 19. Then, we control one of the three nodes with degree 19 and the nine nodes with the larger degrees. For each gain value, the ten nodes that the B zealot controls are the same for the optimized and degree-based A zealot. Table 1 suggests that $S_A$ after optimization is slightly larger than the value obtained from the degree-based protocol. The degree of the ten controlled nodes by the optimized A zealot, degree-based A zealot, and the B zealot are shown in figures 4(c) and (d) for gains equal to 1 and 10, respectively. The B zealot mostly controls nodes with small degrees owing to the scale-free property of the BA model. The optimized A zealot tends to control hubs. However, some nodes that the optimized A zealot controls are not among the ten largest-degree nodes.
3.4.3 Coauthorship network

As a second example, we examine a coauthorship network of network scientists [34]. The network is undirected and unweighted and has 1589 nodes and 2742 links. We use the largest connected component of this network, containing $N = 379$ nodes and 914 links. The numerical procedure is the same as that used in the case of the BA model. The numerical results are shown in figure 5 and table 1. The general tendency is the same as that for the results of the BA model. However, the degree of the controlled node affects the results less strongly than in the case of the BA model. In fact, controlling various single nodes, including those with small degrees, yields a larger share of opinion A than controlling the largest-degree hub (figure 5). The difference between the share of A

| network | gain | optimized 1 | 0.576 | 0.574 |
|---------|------|-------------|-------|-------|
| BA      | 10   | 0.713       | 0.703 |
|         | 100  | 0.810       | 0.793 |
|         | 1    | 0.599       | 0.580 |
| netsci  | 10   | 0.744       | 0.686 |
|         | 100  | 0.869       | 0.765 |
|         | 1    | 0.570       | 0.570 |
| email   | 10   | 0.679       | 0.670 |
|         | 100  | 0.860       | 0.840 |
|         | 1    | 0.896       | 0.678 |
| online  | 10   | 0.898       | 0.811 |
|         | 100  | 0.947       | 0.929 |

Figure 5. Results for the coauthorship network in network science with $N = 379$ nodes and 914 links. The results when each zealot controls a single node are shown in (a) and (b). The control gain for both zealots is set to 1 in (a) and 10 in (b). The degrees of the controlled nodes when each zealot controls ten nodes are shown in (c) and (d) when the gain is equal to 1 and 10, respectively. See the caption of figure 4 for legends.

Table 1. Share of opinion A when the A zealot has optimized the controlled nodes and when it has used the degree-based protocol. Each zealot controls ten nodes. BA: Barabási-Albert model. netsci: coauthorship network of network scientists. email: email communication network. online: online social network. We have used different sets of ten randomly selected nodes for the B zealot for different gain values.
attained by the optimization and that attained by the degree-based protocol is also larger for the coauthorship network than for the BA model (table 1).

3.4.4. Email communication network
As a third example, we examine the largest connected component of an email exchange network between members of the University Rovira i Virgili, Tarragona, Spain [35]. This network is undirected and unweighted and has $N = 1133$ nodes and 5451 links. The numerical results for this network are shown in figure 6 and table 1. The correlation between the degree of the controlled node and the share of opinion A is very large for this network (figure 6). As a result, the share of A obtained after the optimization and that based on the degree-based protocol are only slightly different (table 1).

3.4.5. Directed online social network
All networks examined so far have been undirected networks. In the voter model without zealots, the impact of node $i$ as measured by the fixation probability (i.e., the probability that the opinion starting solely from node $i$ is eventually adopted by all nodes in the network) is equal to the degree of node $i$ [27, 28]. This may be the reason why the degree-based protocol is close to optimal for some undirected networks (i.e., the BA model and the email communication network). In contrast, the fixation probability for a node can substantially deviate from the out-degree (i.e., the number of other nodes that the focal node directly influences) in directed networks [36]. Therefore, we postulate that the optimal nodes to be controlled may considerably deviate from high out-degree nodes in directed networks.

To examine this possibility, we use an online social network among students at the University of California, Irvine [37]. The network has $N = 1899$ nodes and 20296 directed and weighted links. We focus on the largest strongly connected component of this network, containing 1294 nodes and 19026 weighted links. The numerical results are shown in figure 7 and table 1. We increased the number of iterations in the optimization procedure to $5 \times 10^4$ because the $S_A$ value was still increasing to some extent at the $2 \times 10^4$ th iterate. The results indicate that the out-degree is not a good predictor of the impact of the node when it is controlled.
4. Discussion

We studied the problem of maximizing votes in opinion-formation dynamics in complex networks when two opposing parties influence subsets of nodes. We proposed a heuristic algorithm based on exact counting of the mean vote and applied it to artificial and real complex networks. We showed that the degree of the controlled node was a main determinant of the efficiency of control in undirected networks, which is consistent with previous results [26]. Controlling hubs is generally a good strategy. However, optimized selection of the controlled nodes realized a larger share of the desired opinion than the degree-based protocol did. The difference between the performance of the two methods was particularly large in directed networks.

We used the VM rule for opinion updating. In non-regular networks (i.e., networks in which some nodes have different degrees), different update rules, namely the invasion process (IP) and link dynamics (LD), substantially change the outcome of opinion-formation dynamics in the absence of zealots [27, 28, 36, 38]. Extending the present framework to these cases is straightforward because the use of a different update rule corresponds to assigning a rescaled weight to each link [36]. In undirected networks without zealots, the fixation probability of the node is proportional to the degree under the VM rule, inversely proportional to the degree under the IP rule, and independent of the degree under the LD rule [27, 28]. In the present study, we showed that in the presence of zealots, the effect of the control is strongly correlated with the degree of the controlled node in undirected networks. Given the two results, it may be better to control small-degree nodes under the IP rule, and the degree of controlled nodes may be irrelevant under the LD rule.

There are several possible extensions of the present work. First, the proposed algorithm was not fast, in particular when the control gain was small. This was why our examples were relatively small networks (i.e., up to \( N \approx 1300 \) nodes). The bottleneck seems to be the Jacobi iteration. In addition, we employed a greedy method for optimization, which may also be inefficient. Perturbation methods with which one can assess the change in the eigenvector upon an external change (i.e., a slightly altered set of controlled nodes in our case) [39–43] and genetic algorithms [24, 25] may be useful for designing better optimization algorithms.

Second, we assumed during optimization that the opposing zealot (i.e., B) did not change the set of controlled nodes. Only the focal zealot (i.e., A) was allowed to strategically behave to update the set of controlled nodes. This is an important limitation of the present study. In more realistic situations, the optimal selection of

![Figure 7. Results for the online social network with \( N = 1294 \) nodes and 19026 links. The results when each zealot controls a single node are shown in (a) and (b). The out-degrees of the controlled nodes when each zealot controls ten nodes are shown in (c) and (d). In (a) and (c), the gain is set to 1. In (b) and (d), the gain is set to 10. See the caption of figure 4 for legends.](image)
controlled nodes should depend on what the opponent does. Allowing both zealots to strategically behave may be an interesting question. An obtained equilibrium may be formulated as a game-theoretic equilibrium.

Third, we assumed that the complete knowledge of the network of independent voters was available to the controllers. In practical situations such as real voting and social mobilization, this assumption would be violated. Methods applicable when only partial knowledge of the network is available are desired.

Fourth, our analysis was concerned with the mean vote count in the steady state. In fact, the opinion of each agent fluctuates between the two opinions, even in the equilibrium. Previous studies quantified such fluctuations for mean-field populations [16, 18, 19] and mathematically proved the presence of fluctuations under some conditions for networks [20, 21]. Quantifying the fluctuations for general networks may be possible. In addition, actual elections may occur before the equilibrium is reached. Therefore, analysis of transient dynamics seems to be of practical importance. Related to this issue, the convergence rate of opinion dynamics depending on the location of zealots was examined in recent literature [44].

Fifth, we considered the case of two opinions for simplicity. Extension to the case of more than two opinions is straightforward. A theory of node importance in directed networks when there are a general number of root nodes [29] may prove useful to this end.

Sixth, we did not explore the effects of self-loops, although our formulation allowed them. An increase in the weight of a self-loop makes the corresponding independent voter relatively deaf to others’ opinions when updating its own opinion. With the large weight of a self-loop, the independent voter plays a role close to that of a zealot.

Seventh, controlling a single node with a large gain (e.g., 10) and controlling many nodes with a small gain (e.g., 10 nodes with a unity gain) may make a difference. We did not explore this point. Note that even if the total gain exerted by the controller is the same, controlling many nodes with small gains may be more costly than controlling one node with a large gain [9].

Last, the effects of zealots have also been examined for other collective dynamics such as a local majority-vote model [45], the naming game [46–48], Axelrod’s model for cultural dissemination [49], a model with agents in a neutral position [50], and the prisoner’s dilemma game [51–53]. Maximization of the vote by pinning control may also be relevant in these contexts.

Acknowledgments

NM acknowledges the support provided through JST, CREST, and JST, ERATO, Kawarabayashi Large Graph Project.

Appendix: Proof of $u_{A,i} + u_{B,i} = 1$

By summing equation (4) and its equivalent for opinion B, we obtain

$$L + \text{diag}(p_{A,1} + p_{B,1}, \ldots, p_{A,N} + p_{B,N}) \begin{bmatrix} u_{A,1} + u_{B,1} \\ u_{A,2} + u_{B,2} \\ \vdots \\ u_{A,N} + u_{B,N} \end{bmatrix} = \begin{bmatrix} p_{A,1} + p_{B,1} \\ \vdots \\ p_{A,N} + p_{B,N} \end{bmatrix}.$$

(A.1)

Because $L + \text{diag}(p_{A,1} + p_{B,1}, \ldots, p_{A,N} + p_{B,N})$ is diagonally dominant, equation (A.1) has a unique solution for unknowns $u_{A,1} + u_{B,1}, \ldots, u_{A,N} + u_{B,N}$. In fact, $u_{A,i} + u_{B,i} = 1$ ($1 \leq i \leq N$) solves equation (A.1) because $L$ is a Laplacian matrix such that it has a zero right eigenvector $(1, \ldots, 1)^T$, where $T$ denotes the transposition.

Therefore, $u_{A,i} + u_{B,i} = 1$ ($1 \leq i \leq N$) holds true.

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