Abstract. We consider the Springer correspondence in the case of symmetric spaces. In this setting various new phenomena occur which are not present in the classical Springer theory. For example, we obtain representations of (the Tits extension) of the braid group rather than just Weyl group representations. These representations arise from cohomology of families of certain (Hessenberg) varieties. In particular, in this paper we encounter the universal families of hyperelliptic curves. As an application we calculate the cohomology of Fano varieties of $k$-planes in the smooth intersection of two quadrics in an even dimensional projective space.

Contents

1. Introduction 2
2. Notation and preliminaries 5
3. Resolution of the nilpotent cone 10
4. Springer correspondence for symmetric spaces 16
5. Nilpotent orbits of order two 23
6. Matching for nilpotent orbits of order two 29
7. Stalks of the IC sheaves on nilpotent orbits of order two 38
8. Cohomology of Fano varieties of $k$-planes in complete intersections 46
References 51

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1. Introduction

In this paper we consider the Springer correspondence in the case of symmetric spaces. We will concentrate on the split case of type A, i.e., the case of $SL(n, \mathbb{R})$. The case of $SL(n, \mathbb{H})$ was considered by Henderson in [H] and Lusztig in [L5] and the case of $SU(p, q)$ was considered by Lusztig in [L4] where he treats the general case of semi-simple inner automorphisms. In both of these cases Springer theory closely resembles the classical situation. This turns out not to be so in the split case we consider here. For example, we not only encounter representations of Coxeter groups but also representations of the (Tits extension of) braid groups arising from cohomology of interesting families of varieties, the Hessenberg varieties of $[GKM]$. In a loose sense, in the classical Springer correspondence only Artin motives appear but in our case we also encounter motives coming from universal families of hyperelliptic curves.

Our results can, conversely, be used to compute the cohomology of Hessenberg varieties. We introduce these varieties more formally in the companion paper [CVX1]. In this paper simpler versions of these varieties occur and as an application we explain how to compute the cohomology of Fano varieties of $k$-planes in the smooth intersection of two quadrics in an even dimensional projective space. The case of Fano varieties of $k$-planes in the smooth intersection of two quadrics in an odd dimensional projective space will be treated in another companion paper [CVX2]. In that setting the Fano varieties have interesting moduli interpretations and in the simplest cases they amount to the Jacobian [Re, Do] and the moduli space of rank two stable vector bundles with fixed odd determinant on a hyperelliptic curve [DR]. In these cases the cohomology is known, of course, but we also treat the case of general $k$, where the Fano varieties can also have an interpretation of moduli spaces of bundles with extra structure [Ra].

To describe the results in this paper in more detail let us briefly recall how the classical Springer correspondence can be realized and analyzed via the Fourier transform. Let $\mathfrak{g}$ be a semi-simple Lie algebra and let us write $\mathcal{B}$ for the flag manifold and $G$ for the adjoint group. In its original form (generalized later by Lusztig) the Springer correspondence postulates a bijection between (certain) pairs $(\mathcal{O}, \mathcal{E})$ of a $G$-orbit $\mathcal{O}$ on the nilpotent cone $\mathcal{N}$ and an irreducible $G$-equivariant local system $\mathcal{E}$ on $\mathcal{O}$ on one hand and irreducible representations of the Weyl group $W$ on the other. It can be implemented via the Fourier transform on $\mathfrak{g}$ as follows. Let us consider those pairs $(\mathcal{O}, \mathcal{E})$ such that the Fourier transform of the intersection cohomology complex associated to $(\mathcal{O}, \mathcal{E})$ has its support all of $\mathfrak{g}$. This intersection cohomology complex on $\mathfrak{g}$ comes from an irreducible local system on the locus $\mathfrak{g}^{rs}$ of regular semi-simple elements and the local system in turn comes from an irreducible representation of $W$ via the map $\pi_1(\mathfrak{g}^{rs}) \to W$. We obtain all irreducible representations of $W$ in this manner.

Let us recall how one can construct the Springer correspondence via Fourier transform. The Grothendieck simultaneous resolution $\hat{\pi}: \hat{\mathfrak{g}} \to \mathfrak{g}$ restricts to the Springer resolution of...
the nilpotent cone $\pi : \tilde{N} \to N$. These maps fit into the following commutative diagram:

$$
\begin{array}{c}
\tilde{N} & \longrightarrow & \tilde{g} \\
\pi \downarrow & & \downarrow \pi \\
N & \longrightarrow & g
\end{array}
$$

where the map $\pi$ is gotten from $\tilde{\pi}$ simply by base change.

The Fourier transform $\mathcal{F}$ of the constant sheaf $\mathbb{C}_{\tilde{N}}$ is the constant sheaf $\mathbb{C}_{\tilde{g}}$, appropriately shifted; here we take the Fourier transform on the trivial bundle $g \times B \to B$. By functoriality of the Fourier transform we obtain

$$
\mathcal{F}(\pi^* \mathbb{C}_{\tilde{N}}) \cong \tilde{\pi}^* \mathbb{C}_{\tilde{g}} \quad \text{(up to shift)}.
$$

As $\pi$ is semi-small and $\tilde{\pi}$ is small, it is easy to decompose each side into irreducibles. To obtain an explicit matching of both sides still requires some work and is explained, for example, in [L1]. In our case the situation is the same, i.e., the explicit matching is the most challenging part.

Let $G$ be a reductive group and $\theta$ an involution of $G$. We write $K = (G^\theta)^0$ for the connected component of the fixed point set. This gives rise to a symmetric pair $(G, K)$. We also have the corresponding decomposition of the Lie algebra $g = g_0 \oplus g_1$ where $g_0$ is the fixed point set and $g_1$ is the $(-1)$-eigenspace of $\theta$, respectively. We write $N_1 = N \cap g_1$.

We focus on the following question which can be regarded as a symmetric space analogue of Springer theory:

**Question 1.1.** What are the Fourier transforms of $K$-equivariant perverse sheaves on $N_1$? In particular, when are the Fourier transforms supported on all of $g_1$?

We concentrate on the case of full support because of the following

**Conjecture 1.2.** We can obtain the $K$-equivariant perverse sheaves on $N_1$ by induction from those of smaller groups whose Fourier transforms have full support.

One could also consider the more general case of semi-simple automorphisms $\theta$, as is done in [L4]. However, in this paper we concentrate on the case of (outer) involutions and, as was stated at the beginning, our main focus is the split symmetric pair $(SL(N), SO(N))$. We will also assume, mainly for simplicity, that $N$ is odd. In this case the $K$-orbits in $N_1$ are parametrized by partitions of $N$.

Note that if the Fourier transforms have full support then they are IC-sheaves associated to $K$-equivariant local systems on the regular semisimple locus $g_1^{ss}$ of $g_1$. Thus, they are representations of the $K$-equivariant fundamental group of $g_1^{ss}$ which, as explain in §2.6 can

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1It is shown in [Ho] that the matching in [L1] coincides with the one via the Fourier transform up to tensoring by the sign representation.

2For the formulation of some of the results it is perhaps best not to take the connected component, but it makes little difference in this paper.
be identified with $A[2] \times B_N$ where $A[2]$ denotes the order two elements in a maximal (split) torus $A$ and $B_N$ stands for the braid group.

We begin in a manner completely analogous to the classical case by considering a resolution of singularities of $N_1$. In this case we obtain an analogue of diagram (1.1) and in this case $\pi$ is semi-small and $\tilde{\pi}$ is small just as in the classical case. We show in §4 that in this way we obtain a Springer correspondence which is induced from the classical Springer correspondence for $\mathfrak{gl}(n)$ where $2n + 1 = N$. This result can be viewed as evidence for the conjecture above. In particular, in this fashion we obtain a Springer correspondence for IC complexes supported on very special orbits whose Fourier transforms do not have full support. In §4 we also identify a family of IC sheaves whose Fourier transforms do not have full support. In addition, we show that if an IC sheaf on $N_1$ is induced, then its Fourier transform does not have full support.

In the rest of the paper we concentrate in the simplest possible $K$-orbits on $N_1$, namely those which correspond to partitions with only 2’s and 1’s; we call them nilpotent orbits of order 2. Even to handle these cases, as well as in our further work, we are forced to consider several pairs of maps $(\pi, \tilde{\pi})$ and diagrams analogous to (1.1):

\[
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \tilde{\mathcal{X}} \\
\pi \downarrow & & \downarrow \ast \\
N_1 & \longrightarrow & \mathfrak{g}_1
\end{array}
\]

where $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are certain families of Hessenberg varieties, see [CVX1]. The image of $\pi$ is a nilpotent orbit closure $\mathcal{O}$ but neither $\pi$ nor $\tilde{\pi}$ are semi-small in general. Thus, in the analysis of the two sides in the equation (1.2) we have to use the general form of the decomposition theorem which makes things much more complicated than in the classical case. Also, the map $\pi$ is not obtained by base change from $\tilde{\pi}$, but instead $\mathcal{X}$ is a proper subvariety of $\mathcal{O} \times_{\mathfrak{g}_1} \tilde{\mathcal{X}}$ in general.

In §5 and §6 we obtain a complete description of Fourier transforms of IC-sheaves for nilpotent orbits of order 2. Those results are stated at the beginning of §6. We show that for trivial local systems on these nilpotent orbits the Fourier transforms are IC-sheaves arising from representations of the Tits group $A[2] \times S_N$, which do not factor through $S_N$. In the case of non-trivial local systems the Fourier transforms are IC-sheaves arising from representations of the braid group on the cohomology of a universal family of hyperelliptic curves.

In §7 we establish a remarkable isomorphism of the cohomology of the stalks of IC-sheaves attached to nilpotent orbits of order 2 and the cohomology of stalks of IC-sheaves for $\mathfrak{sp}$ in the classical case. The proofs of results in this section are independent of the other sections, and the results are used in §6 and in §8.

In §8 we give an application of our ideas and obtain an explicit formula for the cohomology of the Fano varieties $\text{Fano}_{i-1}^{2n}$ of $(i - 1)$-planes in the intersection of two quadrics in an even...
dimensional projective space $\mathbb{P}^{2n}$. This is part of the following general strategy:

\[(1.4) \quad \text{The computation of the cohomology of the general fiber of } \mathfrak{X} \text{ can be reduced, via the Fourier transform, to the analysis of the boundary family } \mathfrak{X}.\]

For this to work, we need to know the Fourier transforms of the IC-sheaves that occur in the decomposition of $\pi_* (\mathbb{C}\mathfrak{X})$. We apply this principle in our setting and obtain the following theorem. Let us write $g_{k,m}(q) = \prod_{l=m-k+1}^m (1 - q^l) / \prod_{l=1}^k (1 - q^l)$ for the Poincare polynomial of the Grassmannian of $k$-dimensional subspaces in $\mathbb{C}^m$. Then:

**Theorem.** *(Theorem 8.1)* We have

$$H^{2k+1}(\text{Fano}_{2n-1}, \mathbb{C}) = 0,$$

$$H^{2k}(\text{Fano}_{2n-1}, \mathbb{C}) \cong \bigoplus_{j=0}^i L^{\oplus M_i(k,j)},$$

where $M_i(k,j)$ is the coefficient of $q^{k-j(n-i)}$ in $g_{i-j,2n-i-j}(q)$ and the $L_j$ are vector spaces of dimension $\binom{2n+1}{j}$.

The paper is organized as follows. In §2 we set up some notation and discuss some preliminaries. In §3 we discuss the case when $\pi$ is the resolution of the nilpotent cone $N_1$ and prove the semi-smallness of $\pi$. In particular, we show that the fibers of $\pi$, unlike the Springer fibers, are not equi-dimensional in general. In §4 we consider Springer correspondence for special nilpotent orbits (Richardson orbits) which include the regular orbit on $N_1$ and reduce it to the classical Springer correspondence. In §5 we start to analyze the Springer correspondence for nilpotent orbits of order 2 and in §6 we complete this analysis. In §7 we analyze the stalks of IC-sheaves attached to nilpotent orbits of order 2. Finally, in §8 we present our application to the cohomology of Fano varieties of $k$-planes in the intersection of two quadrics in projective space.

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2. Notation and preliminaries

For convenience we work over $\mathbb{C}$ and in the classical topology. However, everything in this paper goes through over an arbitrary algebraically closed field as long as we avoid characteristic 2. At one point in this paper we make use of characteristic cycles but that
part can be replaced by Euler characteristic arguments. We adopt the usual convention of cohomological degrees for perverse sheaves by having them be symmetric around 0. We also use the convention that all functors are derived, so we write, for example, \( \pi_* \) instead of \( R\pi_* \). If \( X \) is smooth we write \( \mathbb{C}_X[-] \) for the constant sheaf placed in degree \(-\dim X\) so that \( \mathbb{C}_X[-] \) is perverse. If \( U \subset X \) is a smooth open dense subset of a variety \( X \) and \( \mathcal{L} \) is a local system on \( U \), we write \( \mathrm{IC}(X, \mathcal{L}) \) for the IC-extension of \( \mathcal{L}[-] \) to \( X \); in particular, it is perverse. For simplicity of notation, when we have a pair \( (\emptyset, \mathcal{E}) \), where \( \emptyset \) is an orbit and \( \mathcal{E} \) is a local system on \( \emptyset \), we also write \( \mathrm{IC}(\emptyset, \mathcal{E}) \) instead of \( \mathrm{IC}(\emptyset, \mathcal{E}) \). For \( F \in D(X) \) and \( x \in X \), we write \( \mathcal{H}^i_x(F) \) for the stalk of the cohomology sheaf \( \mathcal{H}^i F \) at \( x \). This should not be confused with local cohomology.

We denote by \( \mathbf{P}(N) \) the set of partitions of \( N \). For \( \lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbf{P}(N) \), we use the convention that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0 \). We also write \( (\mu_1)^{m_1} \cdots (\mu_k)^{m_k} \) as a partition having distinct parts \( \mu_i \) with multiplicities \( m_i \). The conjugate partition of \( \lambda \) is denoted by \( \lambda' \).

We denote by \( \mathrm{OGr}(k, n) \) the variety of \( k \)-dimensional isotropic subspaces in \( \mathbb{C}^n \) with respect to a non-degenerate bilinear form, and by \( \mathrm{SpGr}(k, 2n) \) the variety of \( k \)-dimensional isotropic subspaces in \( \mathbb{C}^{2n} \) with respect to a non-degenerate symplectic form.

### 2.1. Involutions on reductive groups.

We recall some basic facts about involutions on reductive groups. The main reference for this subsection is the paper of Kostant and Rallis [KR], who work over \( \mathbb{C} \); see [Le] for the finite characteristic case. Although we will primarily work in the type \( A_i \) in this paper, we state many results in this subsection in a more general setting.

Let \( G \) be a reductive algebraic group over \( \mathbb{C} \). Let \( \theta : G \to G \) be an involution. Denote by \( G^\theta \) the subgroup of fixed points of \( \theta \) and by \( K = (G^\theta)^0 \) the identity component of \( G^\theta \). Let \( A \) be a maximal \( \theta \)-split (i.e. \( \theta(a) = a^{-1} \) for all \( a \in A \)) torus. A pair \( (G, K) \) is called split if \( A \) is a maximal torus of \( G \).

Let \( B \) be a \( \theta \)-stable Borel subgroup of \( G \) with unipotent radical \( U \) and \( T \subset B \) a \( \theta \)-stable maximal torus. The intersections \( T_K = T \cap K \), \( B_K = B \cap K \) are maximal torus and Borel subgroup of \( K \), respectively. We denote by \( W \) (resp. \( W_K \)) the Weyl group of \( G \) (resp. \( K \)). We write \( \mathfrak{g} = \text{Lie} \, G \), \( \mathfrak{b} = \text{Lie} \, B \), \( \mathfrak{n} = \text{Lie} \, U \) etc.

The involution \( \theta \) defines a grading \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), where \( \mathfrak{g}_i = \{ x \in \mathfrak{g} \mid d\theta(x) = (-1)^i x \} \). Similarly, we have \( \mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{g}_1 \), \( \mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \), etc. The group \( K \) acts on \( \mathfrak{g}_1 \) by adjoint action.

Let \( \mathfrak{a} = \text{Lie} \, A \) and \( \mathfrak{w}_1 = N_G(A)/Z_G(A) \) the ‘baby Weyl group’. For a split pair, \( \mathfrak{w}_1 = W \).

**Lemma 2.1** (Kostant-Rallis-Levy). The natural inclusion \( \mathfrak{a} \hookrightarrow \mathfrak{g}_1 \) induces an isomorphism \( k[\mathfrak{a}]^{W_1} = k[\mathfrak{g}_1]^K \). We have the relative Chevalley map \( \chi : \mathfrak{g}_1 \to \mathfrak{g}_1 \parallel K \simeq \mathfrak{a}/W_1 := \mathfrak{c} \).

An element \( x \in \mathfrak{g}_1 \) is called regular if \( \dim Kx \geq \dim Ky \) for any \( y \in \mathfrak{g}_1 \). Let \( \mathfrak{g}_1^{\text{reg}} \) (resp. \( \mathfrak{g}_1^{*\text{reg}} \)) denote the set of regular elements (resp. regular semi-simple elements) in \( \mathfrak{g}_1 \). For a
split pair we have $g_1^{reg} = g_1 \cap g^{reg}$. We define $a^{reg} = a \cap g_1^{reg}$. In the split case $a$ is a Cartan subalgebra in $g$ and in this case the elements in $a^{reg}$ are also regular semi-simple in $g$.

2.2. Resolution of nilpotent orbit closures. Let $N$ be the nilpotent cone of $g$ and we define $N_1 = N \cap g_1$. The group $K$ acts on $N_1$ by adjoint action and by $[KR]$ there are only finitely many $K$-orbits in $N_1$. In $[R]$, Reeder constructs resolutions of the nilpotent orbit closures in $N_1$. We recall his construction here.

For $x \in N_1$, we denote by $O_x \subset N_1$ the corresponding $K$-orbit of $x$. Let $\{x, h, y\}$ be a normal $sl_2$-triple containing $x$ (see $[KR]$), where normal means that $x, y \in g_1$ and $h \in g_0$. It defines a grading $g = \oplus_{i \in \mathbb{Z}} g(i)$, where $g(i) = \{v \in g \mid [h, v] = iv\}$. Consider the parabolic subalgebra $p = \oplus_{i \geq 0} g(i)$ of $g$ and the corresponding parabolic subgroup $P$ of $G$. Both $p$ and $P$ are $\theta$-stable. We define $P_K = P \cap K$, which is a parabolic subgroup of $K$, and $p_i = p \cap g_i$. The following map is a resolution of the orbit closure $O_x$

$$\{(z, kP_K) \in g_1 \times K/P_K \mid \text{Ad}(k^{-1})z \in p^x\} \rightarrow \bar{O}_x, \ (z, kP_K) \mapsto z$$

where

$$p^x = \bigoplus_{i \geq 2} g(i) \cap g_1.$$ 

Note that $P$ and $p$ do not depend on the choice of $h$; we call $P$ and $p$ the canonical parabolic subgroup and the canonical parabolic subalgebra associated to $x$, respectively. Moreover, we have $Z_K(x) = Z_{P_K}(x)$.

2.3. The pair $(SL(N), SO(N))$. For this paper we focus on the pair $(G, K) = (SL(N), SO(N)) = (SL(V), SO(V, Q))$ where $\dim V = N = 2n + 1$. We think of $Q$ concretely as a non-degenerate quadratic form on $V$ and we write $\langle \cdot, \cdot \rangle_Q$ for the non-degenerate bilinear form on $V$ associated to $Q$. If we diagonalize $Q$ then the Cartan involution is given by $g \mapsto (g)^{-1}$ and then $g_1$ consists of symmetric matrices.

The nilpotent cone $N_1$ is irreducible in this case. We recall the classification of $K$-orbits in $N_1$ (see $[S]$). The $K$-orbits in $N_1$ are parametrized by $P(N)$, the set of partitions of $N$. More precisely, let $O_\lambda$ denote the $K$-orbit corresponding to $\lambda = (\lambda_1, \ldots, \lambda_s) \in P(N)$, then $x \in O_\lambda$ if and only if the Jordan blocks of $x$ have sizes $\lambda_1, \ldots, \lambda_s$. Moreover, for $x \in O_\lambda$, we have (see $[S]$)

$$\dim Z_K(x) = \sum_{i=1}^{s} (i - 1)\lambda_i. \quad (2.1)$$

The following lemma allows us to choose convenient basis for $V$.

Lemma 2.2. Let $x \in O_\lambda \subset N_1$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \in P(2n + 1)$. There exist $v_i \in V$, $i \in [1, s]$ such that $V = \text{Span}\{x^{a_i}v_i, \ a_i \in [0, \lambda_i - 1], \ i \in [1, s]\}$ and

$$x^{\lambda_i}v_i = 0, \ \langle x^{a_i}v_i, x^{a_j}v_j \rangle_Q = \delta_{i,j}\delta_{a_i+a_j, \lambda_i-1}, \ i, j \in [1, s].$$
Proof. We prove that the lemma holds also for even dimensional $V$ (the fact that some orbits when $\dim V$ is even are parametrized by the same partition does not affect the proof). We prove by induction on $\dim V$. It is clear when $\dim V = 1$. Assume that $\dim V > 1$. One checks easily that there exists $v \in V$ such that $\langle v, x^{\lambda_1-1}v \rangle_Q \neq 0$. We can assume that $\langle v, x^{\lambda_1-1}v \rangle_Q = 1$. We have $x^{\lambda_1}v = 0$ and $v, xv, \ldots, x^{\lambda_1-1}v$ are linearly independent. Let $W = \text{Span}\{v, xv, \ldots, x^{\lambda_1-1}v\}$. Then $V = W \oplus W^\perp$ as $\langle , \rangle_Q|_W$ is non-degenerate. Note that $W^\perp$ is $x$-stable and $x|_{W^\perp} \in \theta_N$, where $\lambda = (\lambda_2, \ldots, \lambda_s)$. The lemma follows from induction. 

\hfill $\square$

2.4. Induced nilpotent orbits. Consider the pair $(G, K) = (\text{SL}(N), \text{SO}(N))$, $N = 2n+1$. Let $L$ be a $\theta$-stable Levi subgroup of a $\theta$-stable parabolic subgroup $P \subset G$. We denote $\mathfrak{l} = \text{Lie } L$, $\mathfrak{p} = \text{Lie } P$ and $\mathfrak{n}_P$ the nilpotent radical of $\mathfrak{p}$. We have the grading $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$, $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ and $\mathfrak{n}_P = (\mathfrak{n}_P)_0 \oplus (\mathfrak{n}_P)_1$ induced by $\theta$. Let $\mathfrak{O} \subset \mathfrak{N}_{l_1} = \mathfrak{N}_{l_1} \cap \mathfrak{l}_1$. There exists a unique $K$-orbit $\mathfrak{O} \in \mathfrak{N}_1$ such that $\mathfrak{O} \cap (\mathfrak{O} + (\mathfrak{n}_P)_1)$ is dense in $\mathfrak{O} + (\mathfrak{n}_P)_1$. Following [LS], we write $\tilde{\mathfrak{O}} = \text{Ind}_{\mathfrak{n}_1 \subset \mathfrak{p}_1} \mathfrak{O}$ and say that $\tilde{\mathfrak{O}}$ is an induced nilpotent orbit in $\mathfrak{N}_1$.

We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{P}(N)$ has gaps, if there exists $i \in [1, s]$ such that $\lambda_i - \lambda_{i+1} \geq 2$ (here $\lambda_{s+1} = 0$). The aim of this subsection is to show the following

Proposition 2.3. An orbit $\mathfrak{O}_\lambda \subset \mathfrak{N}_1$ is induced if and only if $\lambda$ has gaps.

Let us fix a basis $\{e_i, \ i \in [1, 2n+1]\}$ of $V$ such that $\langle e_i, e_j \rangle_Q = \delta_{i+j, 2n+2}$. For a subspace $W \subset V$, we write $W^\perp = \{v \in V \mid \langle v, W \rangle_Q = 0\}$.

Let $d = (d_1, \ldots, d_s)$ be a partition of some $k \in [1, n]$. Let $P_d \subset G$ be the parabolic subgroup that stabilizes the following standard partial flag

$$0 \subset V^1 \subset V^2 \subset \cdots \subset V^l \subset (V^l)^\perp \subset \cdots \subset (V^1)^\perp \subset \mathbb{C}^N,$$

where $V^i = \text{span}\{e_j, j \in [1, \sum_{a=1}^i d_a]\}$. Then $P$ is $\theta$-stable and every $\theta$-stable parabolic subgroup is $K$-conjugate to one of this form.

Let $L_d = L$ be the standard $\theta$-stable Levi subgroup of $P_d$, namely, the subgroup consisting of block diagonal matrices with block sizes $d_1, \ldots, d_s, N-2k, d_t, \ldots, d_1$. Any $\theta$-stable Levi subgroup of $P_d$ is $K$-conjugate to $L_d$ (see for example [BH]). We have

$$L^\theta \cong GL(d_1) \times GL(d_2) \times \cdots \times GL(d_l) \times SO(N-2k),$$

$$I_1 \cong \mathfrak{gl}(d_1) \oplus \cdots \oplus \mathfrak{gl}(d_l) \oplus (\mathfrak{sl}(N-2k)_1).$$

Let $\mathfrak{O}_{L^\theta} \subset I_1$ be a nilpotent $L^\theta$-orbit corresponding to

$$((\mu^1), \ldots, (\mu^l), (\mu^0)),$$

where $\mu^i = (\mu^i_1, \ldots, \mu^i_s) \in \mathbb{P}(d_i)$ represents a $GL(d_i)$-nilpotent orbit in $\mathfrak{gl}(d_i)$ and $\mu^0 = (\mu^0_1, \ldots, \mu^0_s)$ represents a nilpotent $SO(N-2k)$-orbit in $\mathfrak{sl}(N-2k)_1$. 

8
**Lemma 2.4.** Consider the following map
\[ \alpha : K \times^{PK} (\tilde{O}_L + (n_P)_1) \to N_1. \]
Then \( \text{Im} \alpha = \tilde{O}_K \), where \( O_K = O_\lambda, \lambda = (\lambda_1, \ldots, \lambda_s) \), and \( \lambda_i = \mu^0_i + \sum_{j=1}^{l} 2\mu^j_i \). In particular, we have
\[ \text{Ind}^{g_1}_{i_1 \subset p_1} O_L = O_\lambda. \]

**Proof.** Let \( O_L = L \cdot O_{L^\theta} \) be the unique \( L \)-orbit in \( L \) containing \( O_{L^\theta} \). Let \( \tilde{O} = G \cdot O_K \) be the unique \( G \)-orbit containing \( O_K \). Using the results for induced nilpotent orbits in type \( A \) (see for example [CM, Lemma 7.2.5]), one checks that
\[ \tilde{O} \cap (O_L + n_P) \text{ is dense in } O_L + n_P. \]
Thus \( \tilde{O} \cap (O_{L^\theta} + (n_P)_1) \neq \emptyset \). Let \( x \in \tilde{O} \cap (O_{L^\theta} + (n_P)_1) \). Then \( K \cdot x = O_K \). We have
\[ \dim \text{Im} \alpha \leq \dim(K \times_{K^0} (\tilde{O}_{L^\theta} + (n_P)_1)) = \dim O_{L^\theta} + 2 \dim(n_P)_1 - k \]
and it is easy to check that
\[ \dim O_K = \dim O_{L^\theta} + 2 \dim(n_P)_1 - k. \]
Since \( O_K \subset \text{Im} \alpha \) and \( \text{Im} \alpha \) is irreducible, we conclude that \( \text{Im} \alpha = \tilde{O}_K. \)

**Proof of Proposition 2.5.** Assume that \( O_\lambda \) is induced. Then \( \lambda \) is of the form as in the lemma above. It follows that \( \lambda \) has gaps.

Conversely, let \( \lambda \) be a partition with gaps. We can find a partition \( p = (p_1, \ldots, p_s) \) such that \( \sum_{i=1}^{s} p_i := k \geq 1 \) and a partition \( \mu = (\mu_1, \ldots, \mu_s) \) of \( N - 2k \) such that \( \lambda_i = \mu_i + 2p_i, i = 1, \ldots, s \). Let \( d \) be the conjugate partition of \( p \) and let \( P_d \) and \( L_d \) be defined as above. Then \( O_\lambda = \text{Ind}^{g_1}_{i_1 \subset p_1} O_0 \), where \( O_0 \subset I_1 \) is a nilpotent \( L^\theta \)-orbit corresponding to \( ((1^{p_1}), \ldots, (1^{p_s}), (\mu_1, \ldots, \mu_s)). \)

**Remark 2.5.** It follows from the lemma above that the Richardson orbits, i.e. those dense in \( K \cdot (n_P)_1 \) for \( \theta \)-stable parabolics \( P \), are of the form \( (2\mu_1 + 1, \ldots, 2\mu_t + 1, 2\mu_{t+1}, \ldots, 2\mu_s). \)

### 2.5. Fourier transform and induction

As we are working over the complex numbers in this paper, we use [KS] as a general reference. We write \( \mathfrak{F} : D_K(g_1) \to D_K(g_1) \) for the Fourier transform, where we identify \( g_1 \) with \( g_1^* \) using a \( K \)-invariant non-degenerate from \( \langle \cdot, \cdot \rangle_1 \) on \( g_1 \). We recall that the Fourier transform is an equivalence of categories and it preserves perverse sheaves \( \mathfrak{F} : \text{Perv}_K(g_1) \to \text{Perv}_K(g_1) \).

In our setting we have the usual yoga of geometric induction functors which were defined in this setting in [H] and later in greater generality in [L4]. In the notation of the previous subsection the induction functor \( \text{Ind}^{g_1}_{i_1 \subset p_1} : D_{L^\theta}(i_1) \to D_K(g_1) \) can be defined as follows. Consider the diagram
\[
\begin{align*}
I_1 \xrightarrow{p_1} K \times^{PK} p_1 \xrightarrow{p_2} K \times^{PK} p_1 \xrightarrow{p} g_1,
\end{align*}
\]
where $U_K = U \cap K$ and $U$ is the unipotent radical of $P$. Let $A$ be a complex in $D_{L^0}(\mathfrak{l}_1)$. Then $p_1^* A \cong p_2^* A'$ for a well-defined complex $A'$ in $D_K(K \times P_K \mathfrak{p}_1)$. Define
\[
\operatorname{Ind}_{\mathfrak{i}_1 \subset \mathfrak{p}_1}^A A = p_1 A'[\dim U].
\]
It is shown in [H, L4] that the induction functor commutes with Fourier transform, i.e.,
\[
\mathfrak{F}(\operatorname{Ind}_{\mathfrak{i}_1 \subset \mathfrak{p}_1}^A A) \cong \operatorname{Ind}_{\mathfrak{i}_1 \subset \mathfrak{p}_1}^A (\mathfrak{F}(A)) .
\]

2.6. The equivariant fundamental group. We begin by general remarks valid for split groups of all types. Recall that our primary goal is to study $K$-equivariant IC-sheaves and the Fourier transforms under the assumptions that the Fourier transforms are supported on all of $\mathfrak{g}_1$. Those Fourier transforms will then be of the form $\operatorname{IC}(\mathfrak{g}_1, \mathcal{L})$ where $\mathcal{L}$ is a $K$-equivariant local system on the locus $\mathfrak{g}_1^{rs}$ of regular semisimple elements. Hence, they can be regarded as representations of the $K$-equivariant fundamental group $\pi_1^K(\mathfrak{g}_1^{rs}, a)$; here $a \in \mathfrak{a}^{rs}$ is a base point.

To describe this equivariant fundamental group more concretely let us write $c \in \mathfrak{c}^{rs}$ for the image of $a$ under $\mathfrak{a}^{rs} \to \mathfrak{c}^{rs} = \mathfrak{a}^{rs}/W$. Now, $\mathcal{B}_W := \pi_1(\mathfrak{c}^{rs}, c)$ is the braid group attached to the Weyl group $W$. Then the projection $\chi : \mathfrak{g}_1 \to \mathfrak{c}$ (see Lemma 2.1) gives rise to the following commutative diagram with exact rows:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_K(a) & \longrightarrow & \pi_1^K(\mathfrak{g}_1^{rs}, a) & \longrightarrow & \mathcal{B}_W & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Z_K(a) & \longrightarrow & \tilde{W} = N_K(A) & \longrightarrow & W & \longrightarrow & 0 . \\
\end{array}
\]

Note that $\tilde{W}$ is the Tits group which in the split case coincides with $N_K(A)$. The group $\mathcal{B}_W$ acts on $Z_K(a)$ through the quotient $\mathcal{B}_W \to W$. Note that $Z_K(a) \simeq A[2]$, the group of order 2 elements in the split torus $A$. Choosing a section we can split the short exact sequence above and obtain
\[
\pi_1^K(\mathfrak{g}_1^{rs}, a) \simeq A[2] \rtimes \mathcal{B}_W.
\]
However, note that the second row in the diagram is not split in general. In the setting of this paper with $(G, K) = (SL(N), SO(N))$ we have $W = S_N$ and $\mathcal{B}_W = B_N$ is the classical braid group and in the case when $N$ is odd the lower exact sequence also splits and we have
\[
\tilde{W} = N_K(A) \simeq A[2] \rtimes S_N .
\]

3. Resolution of the nilpotent cone

The nilpotent cone $\mathcal{N}_1$ has a nice $K$-equivariant resolution which we discuss in this section in our special case when $(G, K) = (SL(N), SO(N))$ with $N = 2n + 1$. All $\theta$-stable Borel
subgroups of $G$ are $K$-conjugate in this case. Fix a $\theta$-stable Borel subgroup $B$ of $G$. Let $B_K = B \cap K$ and define

$$\tilde{N}_1 = K \times^{B_K} n_1.$$  

We have the following proper map which is a resolution of singularities (see [S, R])

$$\pi: \tilde{N}_1 \to N_1, \ (k, n) \mapsto \text{Ad}_k n.$$  

In this section we show that $\pi$ is semismall, as mentioned in the introduction, and determine the relevant orbits for $\pi$, i.e. those $\Omega_{\lambda} \subset N_1$ such that $2 \dim B_x = \dim N_1 - \dim \Omega_{\lambda}$ for $x \in \Omega_{\lambda}$, where $B_x := \pi^{-1}(x)$.

**Proposition 3.1.** The map $\pi$ is semismall. More precisely, for $x \in \Omega_{\lambda} \subset N_1$, we have $2 \dim \pi^{-1}(x) = 2 \dim B_x \leq \text{codim}_{N_1} \Omega_{\lambda}$, and the equality holds if and only if

$$\lambda = (2p_1 + 1, 2p_2, 2p_3, \ldots, 2p_s),$$

where $p_1 + \cdots + p_s = n$ and $p_1 \geq \cdots \geq p_s$.

**Remark 3.2.** In general $\pi^{-1}(x)$ is not equidimensional. See Example 3.7.

Let us denote by $P_1(2n + 1) \subset P(2n + 1)$ the subset consisting of partitions of the form $(2p_1 + 1, 2p_2, \ldots, 2p_s)$ (where $\sum p_i = n$). The top dimensional irreducible components of $B_x$ for $x \in \Omega_{\lambda}$, where $\lambda \in P_1(2n + 1)$, are described in the following proposition.

**Proposition 3.3.** Let $x \in \Omega_{\lambda}$, where $\lambda = (2p_1 + 1, 2p_2, \ldots, 2p_s) \in P_1(2n + 1)$. Let $I_1, \cdots, I_k$ be the irreducible components of $\pi^{-1}(x)$ of top dimension $(\text{codim} \Omega_{\lambda})/2$. Then $\bigcup_{i \in [1, k]} I_i$ is isomorphic to the Springer fiber of $x' \in \Omega_{\lambda'} \subset \mathfrak{gl}(n)$, where $\lambda' = (p_1, \ldots, p_s)$.

3.1. **Semismallness of $\pi$.** We start with the following lemma. For any $w \in W_K$ we define $b^w = b \cap \text{Ad}_w b$, $n^w = n \cap \text{Ad}_w n$. Both $b^w$ and $n^w$ are $\theta$-stable. Let $b^w = b_0^w \oplus b_1^w$, $n^w = n_0^w \oplus n_1^w$ be the grading induced by $\theta$.

**Lemma 3.4.** We have $\dim b_0^w \geq \dim n_0^w$.

**Proof.** We have $\Delta = \Delta_{im} \cup \Delta_{com}$, where $\Delta_{im}$ (resp. $\Delta_{com}$) is the set of imaginary (resp. complex) roots corresponding to the maximal compact $\theta$-stable torus $t$. We have $\Delta_{im} = \Delta_n \cup \Delta_c$, here $\Delta_n$ (resp. $\Delta_c$) is the subset of noncompact roots (resp. compact roots), i.e., those $\alpha \in \Delta_{im}$ such that the corresponding root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{gl}$ (resp. $\mathfrak{g}_0$). Since $\theta(b) = b$, the intersection $\Delta^+_{com} = \Delta_{com} \cap \Delta^+$ is $\theta$-stable.

Let $w \in W_K$. We define $\Delta^+_w = \{ \alpha \in \Delta^+ : w(\alpha) \in \Delta^+ \}$. We have

$$n_0^w = \bigoplus_{\alpha \in \Delta^+_c, w} g_{\alpha} \oplus \bigoplus_{\alpha \in \Delta^+_nc, w} (g_{\alpha} + g_{\theta(\alpha)})_0, \quad n_1^w = \bigoplus_{\alpha \in \Delta^+_c, w} g_{\alpha} \oplus \bigoplus_{\alpha \in \Delta^+_nc, w} (g_{\alpha} + g_{\theta(\alpha)})_1,$$

here $\Delta^+_c, w = \Delta_c \cap \Delta^+_w$ (resp. $\Delta^+_nc, w = \Delta_{nc} \cap \Delta^+_w$) and $(g_{\alpha} + g_{\theta(\alpha)})_i$ is the $(-1)^i$-eigenspace of $\theta$ on $g_{\alpha} + g_{\theta(\alpha)}$. Thus we have

$$\dim n_1^w - \dim n_0^w = |\Delta^+_nc, w| - |\Delta^+_c, w|.$$  

\footnote{Since $t$ is maximal compact there are no real roots (see [K, Proposition 6.70]).}
There are no compact roots, i.e., $\Delta_c = \emptyset$, moreover, we have $|\Delta_{nc}^+| = |\Delta_{nc}^+| = n$. Thus the above equality implies

$$\dim n_1^w - \dim n_0^w = |\Delta_{nc,w}^+| \leq n = \dim t_0$$

and this gives the desired inequality

$$\dim b_0^w = \dim t_0 + \dim n_0^w \geq \dim n_1^w.$$

\[\square\]

**Remark 3.5.** One can also prove the lemma above using concrete model. Let $g = \mathfrak{sl}_{2n+1}$. Then the involution $\theta$ can be realized as $\theta(x) = -^t x$, where $^t x$ is the transpose with respect to the anti-diagonal. We can assume that $b$ consists of upper-triangular matrice and $t$ consists of diagonal matrices. In this case $n^w \subset n$ is equal to the nilpotent radical of a $\theta$-stable parabolic $p$ containing $b$. From this one can easily see that $\dim n_1^w - \dim n_0^w \leq n$ (in fact, the only properties we need to conclude the inequality are 1) $n^w$ is $\theta$-stable 2) $n^w$ is a sum of root spaces $g_\alpha$, $\alpha \in \Delta^+$).

**Corollary 3.6.** Consider the fiber product $Y = \tilde{N}_1 \times_{N_1} n_1$. We have $\dim Y = \dim n_1$.

**Proof.** The scheme $Y$ can be identified with $Y = \{(x, kB_K) \mid k \in K, \ x \in n_1 \cap \text{Ad}_k n_1\}$. Consider the projection $Y \to K/B_K$, $(x, kB_K) \to kB_K$. For any $w \in W_K$ let $Y_w$ be the pre-image of the $B_K$-orbit $O_w := B_K w B_K/B_K$. We have $Y = \cup Y_w$ and it is enough to show that $\dim Y_w \leq \dim n_1$ for each $w \in W_K$. For this, observe that $Y_w$ is isomorphic to $B_K \times B_K n_1^w$, where $B_K n_1^w := B_K \cap \text{Ad}_w B_K$. Thus by Lemma 3.4.1 we have

$$\dim Y_w = \dim B_K + \dim n_1^w - \dim B_K n_1^w = \dim b_0 + \dim n_1^w - \dim b_0^w \leq \dim b_0.$$ 

Note that $\dim b_0 = \dim n_1$. Hence $\dim Y_w \leq \dim n_1$.  

\[\square\]

**Proof of Semi-smallness of $\pi$.** Let $x \in N_1^0$ and $O_x$ the corresponding $K$-orbit. Define $B_x = \pi^{-1}(x)$. Consider $\tilde{O}_x = \pi^{-1}(O_x) = K \times B_K (n_1 \cap O_x)$. Then we have

$$\dim O_x + \dim B_x = \dim K + \dim n_1 \cap O_x - \dim B_K = \dim n_0 + \dim n_1 \cap O_x.$$ 

On the other hand, we have

$$\dim n_1 \cap O_x + \dim B_x = \dim \tilde{N}_1 \times_{N_1} (n_1 \cap O_x) \leq \dim Y \overset{\text{Cor} \ 3.6}{=} \dim n_1.$$ 

It implies

$$\dim O_x + 2 \dim B_x \leq \dim n_1 + \dim n_0 = \dim N_1.$$

This finishes the proof of semi-smallness of $\pi$.  

\[\square\]
3.2. Completion of the proof of Proposition 3.1. We fix a \( \theta \)-stable Borel subgroup \( B \) of \( G \) that stabilizes the standard flag
\[
0 \subset V_1^0 \subset V_2^0 \subset \cdots \subset V_n^0 \subset V_n^{0\perp} \subset \cdots \subset V_1^{0\perp} \subset V = \mathbb{C}^{2n+1}
\]
where \( V_i^0 = \text{Span}\{e_1, \ldots, e_i\} \). (We have fixed a basis \( (e_i) \) of \( V \) such that \( \langle e_i, e_j \rangle_Q = \delta_{i+j,2n+2} \).)

Let us denote by \( B_K \cong K/B \) the variety of all flags of the form \( 0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_{2n+1} = V \), where \( \dim V_i = i \), \( i \in [0,2n+1] \) and \( V_{n+k} = V_{n+1-k}^\perp, k \in [1,n] \). We write an element in \( B_K \) as \( (V_i) \).

The resolution map \( \pi : \tilde{N}_1 = K \times B_K n_1 \to N_1 \) can be identified with the following map
\[
\pi : \{ (x, (V_i)) \in N_1 \times B_K \mid xV_i \subset V_{i-1}, i \in [1,2n+1] \} \to N_1, \ (x, (V_i)) \mapsto x.
\]

We prove Proposition 3.1 by induction on \( N \). The fiber \( \pi^{-1}(x) \) at \( x \in N_1 \) is \( \{ (V_i) \in B_K \mid xV_i \subset V_{i-1} \} \). Assume that \( N = 3 \). There are three orbits \( \mathcal{O}_3, \mathcal{O}_2 \), \( \mathcal{O}_1 \) in \( N_1 \), of dimensions 3, 2, 0 respectively. It is easy to check that the fiber at \( \mathcal{O}_2 \) is a point and that at 0 is 1 dimensional. Thus proposition holds when \( N = 3 \).

Suppose by induction hypothesis that the proposition holds for all \( N < 2n + 1 \) odd. Assume now that \( N = 2n + 1 \). Consider the following map
\[
p : \pi^{-1}(x) \to \mathbb{P}(\ker x), (V_i) \mapsto V_1.
\]
It is easy to see that we have
\[
p^{-1}(V_1) \cong (\pi')^{-1}(x'),
\]
where \( x' : V_1^\perp/V_1 := V' \to V' \) is the map induced by \( x \) and \( \pi' \) is the analogous map defined for the pair \( (SL(V'), SO(V', Q')) \) (here \( V' \) is equipped with the non degenerate quadratic form \( Q' \) induced by \( Q \) on \( V \)). Let us assume that \( x' \in \mathcal{O}_{\lambda'} \), where \( \lambda' \in \mathbb{P}(N-2) \).

Suppose that \( \lambda = (\mu_1)^{m_1} \cdots (\mu_l)^{m_l} \), where \( \mu_i \) has multiplicities \( m_i \). Let
\[
W_{\mu_i} = \ker x \cap \text{Im}(x^{\mu_{i-1}}) \quad \text{and} \quad K_i = \mathbb{P}(W_{\mu_i}) - \mathbb{P}(W_{\mu_{i+1}}).
\]
Let \( K_i^0 = \{ [v] \in K_i \mid v = x^{\mu_{i-1}}u, \langle v, v \rangle_Q = 0, \langle u, x^{\mu_{i-1}}u \rangle_Q \neq 0 \} \).

Suppose that \( V_1 \in K_i^0 \). Using a basis \( \{ x^u v_i \} \) of \( V \) as in Lemma 2.2, one can check that
\[
(3.1) \quad \lambda' = \text{rearrangement of } (\mu_1)^{m_1} \cdots (\mu_l)^{m_l - 1}(\mu_l - 2)(\mu_{l+1})^{m_{l+1}} \cdots (\mu_l)^{m_l},
\]
i.e. the Young diagram of \( \lambda' \) is obtained from that of \( \lambda \) by removing two boxes from a row with \( \mu_i \) boxes and then rearranging the rows so that it is again a Young diagram.

Suppose that \( V_1 \in K_i - K_i^0 \). Then necessarily \( m_i \geq 2 \). One can check that
\[
(3.2) \quad \lambda' = (\mu_1)^{m_1} \cdots (\mu_l)^{m_l - 2}(\mu_l - 1)^2(\mu_{l+1})^{m_{l+1}} \cdots (\mu_l)^{m_l}.
\]
Let us write $\mathcal{N}_1'$ for the nilpotent cone of the symmetric space given by $(SL(V'), SO(V', Q'))$. By a direct calculation using (2.1), one can verify that

\[
(3.3) \quad \text{codim}_{\mathcal{N}_1'} \mathcal{O}_\lambda = \begin{cases} 
\text{codim}_{\mathcal{N}_1'} \mathcal{O}_\lambda + 2(\sum_{j=1}^i m_j - 1) \text{ in the case (3.1) when } \mu_{i+1} \leq \mu_i - 2 \\
\text{codim}_{\mathcal{N}_1'} \mathcal{O}_\lambda + 2(\sum_{j=1}^i m_j - 1) + m_{i+1} \text{ in the case (3.1) when } \mu_{i+1} = \mu_i - 1 \\
\text{codim}_{\mathcal{N}_1'} \mathcal{O}_\lambda + 2(\sum_{j=1}^i m_j - 1) - 1 \text{ in the case (3.2)}.
\end{cases}
\]

Let $d_i = \dim K_0^1 + \dim(\pi')^{-1}(x')$ (where $x' \in \mathcal{O}_\lambda'$ with $\lambda'$ as in (3.1)). By induction hypothesis and (3.3), we have

\[
d_i \leq \sum_{j=1}^i m_j - 1 + \frac{\text{codim}_{\mathcal{N}_1'} \mathcal{O}_{\lambda'}}{2} \leq \frac{\text{codim}_{\mathcal{N}_1} \mathcal{O}_\lambda}{2}
\]

and both equalities hold if and only if $\mu_{i+1} \leq \mu_i - 2$ and $\lambda' = (\mu_1)^{m_1} \cdots (\mu_i)^{m_i-1}(\mu_i - 2)(\mu_{i+1})^{m_{i+1}} \cdots \in \mathbb{P}_1(N - 2)$, i.e. of the form $(2p'_1 + 1, 2p'_2, \ldots, 2p'_{s'})$.

Let $e_i = \dim(K_i - K_i^0) + \dim(\pi')^{-1}(x')$ (where $x' \in \mathcal{O}_\lambda'$ with $\lambda'$ as in (3.2)). By induction hypothesis and (3.3), we have

\[
e_i \leq \sum_{j=1}^i m_j - 2 + \frac{\text{codim}_{\mathcal{N}_1'} \mathcal{O}_{\lambda'}}{2} < \frac{\text{codim}_{\mathcal{N}_1} \mathcal{O}_\lambda}{2}.
\]

Note that $\dim \pi^{-1}(x) \leq \max\{d_i, e_i\}$. Thus we conclude from the above discussion that $2 \dim \pi^{-1}(x) \leq \text{codim}_{\mathcal{N}_1} \mathcal{O}_\lambda$, and the equality holds if and only if $\lambda \in \mathbb{P}_1(2n + 1)$. This finishes the proof of Proposition 3.1.

**Example 3.7.** This example shows that the fibers $\pi^{-1}(x)$ are not necessarily equidimensional. Consider the orbit $\mathcal{O}_\lambda$, where $\lambda = (3, 2, 2)$. Let $x \in \mathcal{O}_\lambda$ be such that $xe_1 = e_2, xe_2 = 0, xe_3 = e_4, xe_4 = e_5, xe_5 = 0, xe_6 = e_7, xe_7 = 0$, where $e_i, i \in [1, 7]$ is a basis of $V$ with $(e_i, e_j) = \delta_{i+j, 8}$. It is not difficult to check that the $x$-stable maximal isotropic subspaces of $V$ are the following

\[
V^1_3 = \text{Span}\{e_2, e_5, e_7\}, \quad V^2_3 = \text{Span}\{e_2, e_5, ae_1 + be_4 + ce_7\}, \quad \text{where } 2ac + b^2 = 0
\]

\[
V^3_3 = \text{Span}\{e_5, e_7, ae_2 + be_4 + ce_6\}, \quad \text{where } 2ac + b^2 = 0.
\]

One can then verify that $\pi^{-1}(x)$ has five irreducible components $I_i$, $i \in [1, 5]$, one of 3 dimensional and four of 2 dimensional, as follows.

- $I_1$ consists of flags $0 \subset V_1 \subset V_2 \subset V_3 \subset V^1_3 \subset V^2_3 \subset V^1_3 \subset \mathbb{C}^7$, i.e. $I_1$ is isomorphic to the flag manifold of $SL(V^1_3)$
- $I_2$ consists of flags $0 \subset \text{Span}\{xe_2 + ye_5\} \subset \text{Span}\{e_2, e_5\} \subset \text{Span}\{e_2, e_5, ae_1 + be_4 + ce_7\}$ $\subset \cdots$, where $2ac + b^2 = 0$
- $I_3$ consists of flags $0 \subset \text{Span}\{ae_2 + be_5\} \subset \text{Span}\{ae_2 + be_5, x(ae_1 + be_4 + ce_7) + ye_2 + ze_5\}$ $\subset \text{Span}\{e_2, e_5, ae_1 + be_4 + ce_7\} \subset \cdots$, where $2ac + b^2 = 0$
\[ I_4 \text{ consists of flags } 0 \subset \text{Span}\{xe_5 + ye_7\} \subset \text{Span}\{e_5, e_7\} \subset \text{Span}\{e_5, e_7, ae_2 + be_4 + ce_6\} \subset \cdots, \text{ where } 2ac + b^2 = 0 \]

\[ I_5 \text{ consists of flags } 0 \subset \text{Span}\{be_5 + ce_7\} \subset \text{Span}\{be_5 + ce_7, x(ae_2 + be_4 + ce_6) + ye_5 + ze_7\} \subset \text{Span}\{e_5, e_7, ae_2 + be_4 + ce_6\} \subset \cdots, \text{ where } 2ac + b^2 = 0. \]

3.3. Proof of Proposition 3.3. Suppose that \((V_i) \in I_j\) for some \(j \in [1, k]\). We claim that \((3.4)\)

\[ V_n = V_n^x, \]

where \(V_n^x \subset V\) is a maximal isotropic subspace uniquely determined by \(x\) as follows.

Let \(\{x, y, h\}\) be a normal \(\mathfrak{sl}_2\)-triple. Then \(h\) determines a grading on \(V\), i.e. \(V = \bigoplus_{j \in \mathbb{Z}} V^j\), where \(V^j = \{v \in V | hv = jv\}\). Let \(V^{\geq k} = \bigoplus_{j \geq k} V^j\). This defines a (partial) flag \(\cdots \subset V^{\geq k+1} \subset V^{\geq k} \subset \cdots\), which is the one stabilized by the canonical parabolic subgroup associated to \(x\); thus does not depend on the choice of \(h\). Define \(V_n^x = V^{\geq 1}\). One can check that for \(x \in P_1(2n + 1)\), \(\dim V^{\geq 1} = n\).

In concrete terms, let \(x^a_i v_i, a_i \in [0, \lambda_i - 1], i \in [1, s]\), be a basis of \(V\) as in Lemma 2.2 where \(\lambda_1 = 2p_1 + 1, \lambda_i = 2p_i, i \in [2, s]\). We can define \(h\) and \(y\) as follows \(hx^a_i v_i = (2a_i - \lambda_i + 1) x^a_i v_i, yx^a_i v_i = a_i(\lambda_i - a_i) x^a_i v_i\). Then we obtain that

\[ V_n^x = \text{Span}\{x^a_i v_i, 1 \leq i \leq s, \frac{\lambda_i}{2} \leq a_i \leq \lambda_i - 1\}, \]

which does not depend on the choice of \(v_i\)'s.

Proof of Claim 3.4. Suppose that \((V_i) \in I_i\), where \(\dim I_i = \dim \pi^{-1}(x)\). Then \(V_1 = \text{Span}\{x^{\lambda_j - 1} v\}\) for some \(1 \leq j \leq s\) and \(v \in V\) with \(x^{\lambda_j} v = 0\), moreover, \(\lambda_j \geq \lambda_{j+1} + 2\) and \(v, x^{\lambda_j - 1} v \neq 0\).

Let \(W := \text{Span}\{x^p v, 0 \leq p \leq \lambda_j - 1\}\). Then \((,)_W\) is non degenerate and \(W\) is \(x\)-stable. We can find \(v_j \in W\) such that \(x^{\lambda_j - 1} v_j = x^{\lambda_j - 1} v\) and \((v_j, x^p v_j) = \delta_{p, \lambda_j - 1}\). We have \(V = W \oplus W^1\) and \(x|_{W^1}\) has Jordan block sizes \(\lambda_1, \ldots, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_s\). Let \(x^a_i v_i, 1 \leq i \leq s, i \neq j, 0 \leq a_i \leq \lambda_i - 1\) be a basis of \(W^1\) as in Lemma 2.2. Let \(x^j' : V' = V_1^1/V_1 \to V'\) be the map induced by \(x\). We have \(V' \cong ((V_1^1 \cap W)/V_1) \oplus W^1\). Then \(v_j' := x v_j, \ldots, x^{\lambda_j - 2} v_j' = x^{\lambda_j - 1} v_j, x^a_i v_i, 1 \leq i \leq s, i \neq j, 0 \leq a_i \leq \lambda_i - 1\) is a basis of \(V'\) as in Lemma 2.2 (for \(x'\)). By induction hypothesis,

\[ V_n/V_1 = \text{Span}\{x^{a_i} v_i, 1 \leq i \leq s, i \neq j, \frac{\lambda_i}{2} \leq a_i \leq \lambda_i - 1\} \]

\[ \oplus \text{Span}\{x^{a_j} v_j' = x^{a_j + 1} v_j, \frac{\lambda_j}{2} - 2 \leq a_j \leq \lambda_j - 3\} \]

It follows that \(V_n = V_n^x\). \(\square\)

Note that \(x|_{V_n^x}\) belongs to the nilpotent orbit in \(\mathfrak{gl}(V_n^x)\) given by the partition \((p_1, \ldots, p_s)\). Proposition 3.3 follows from Claim 3.4.
3.4. Semismall maps arising from \( \theta \)-stable parabolic subgroups. In this subsection we introduce a family of semismall maps \( \pi_i \) arising from \( \theta \)-stable parabolic subgroups, of similar nature as the resolution map \( \pi \), and we determine the relevant orbits, i.e. those \( \mathcal{O}_\lambda \) such that \( 2 \dim \pi_i^{-1}(x) = \text{codim} \mathcal{O}_\lambda \) (\( x \in \mathcal{O}_\lambda \)). In particular, the map \( \pi_i \) is a resolution of the orbit \( \mathcal{O}_\lambda \), where \( \lambda = (2i + 1)^{2n-2i} \).

For \( 1 \leq i \leq n-1 \), let \( P^i \) be the \( \theta \)-stable parabolic subgroup corresponding to the following (partial) flag
\[
0 \subset V_1^0 \subset V_2^0 \subset \cdots \subset V_i^0 \subset V_i^{0\perp} \subset \cdots \subset V_1^{0\perp} \subset \mathbb{C}^{2n+1},
\]
where \( V_j^0 \)'s are as in \( \S 3.2 \). \( P_K^i = K \cap P^i \) and \( n_{P^i} \) is the nil radical of the Lie algebra of \( P^i \). Consider the following maps
\[
\pi_i : K \times^{P^i_K} (n_{P^i})_1 \to N_1, \ (k, x) \mapsto \text{Ad}_k(x).
\]

**Proposition 3.8.** The maps \( \pi_i \) are semismall. More precisely, for \( x \in \mathcal{O}_\lambda \subset \text{Im} \pi_i \), we have \( 2 \dim \pi_i^{-1}(x) \leq \text{codim}_{\mathcal{O}_\lambda} \mathcal{O}_\lambda \), and the equality holds if and only if
\[
\lambda = (2p_1 + 1, 2p_2 + 1, \ldots, 2p_{2n-2i+1} + 1, 2p_{2n-2i+2} + 1, \ldots, 2p_s)
\]
where \( p_1 + \cdots + p_s = i \) and \( p_1 \geq p_2 \geq \cdots \geq p_s \).

**Proof.** The proof is entirely similar to that of Proposition \( \S 3.1 \). \( \square \)

4. Springer correspondence for symmetric spaces

In this section, we determine \( \mathfrak{g}(\text{IC}(\mathcal{O}_\lambda, \mathbb{C})) \) when \( \lambda \) is of the form \( \lambda = (2\mu_1 + 1, \ldots, 2\mu_i + 1, 2\mu_{i+1} + 1, \ldots, 2\mu_s) \), i.e. when \( \mathcal{O}_\lambda \) is a Richardson orbit (see Remark \( \S 2.5 \)). The story here is very similar to that of the classical Springer correspondence for \( \mathfrak{g}(k) \).

Let us denote by \( S_k^\vee \) the set of irreducible representations of \( S_k \). We write \( \rho_\mu \in S_k^\vee \) as the irreducible representation corresponding to a partition \( \mu \in \mathbf{P}(k) \). We use the convention that the partition \( (k) \) corresponds to the trivial representation and \( (1^k) \) corresponds to the sign representation.

Consider the map
\[
\tilde{\pi} : \mathfrak{g}_1 := K \times^{B_K} \mathfrak{b}_1 \to \mathfrak{g}_1, \ (k, b) \mapsto \text{Ad}_k b.
\]
Note that \( \tilde{\pi} \) is proper. Let \( \mathfrak{g}_1^0 \) denote the image of \( \tilde{\pi} \). Then \( \mathfrak{g}_1^0 = K \cdot \mathfrak{b}_1 \) consists of elements in \( \mathfrak{g}_1 \) with eigenvalues \( a_1, a_1, a_2, a_2, \ldots, a_n, a_n \), and \( -2(a_1 + \cdots + a_n) \).

Let \( Y^r \) be the set of regular elements in \( \mathfrak{g}_1 \) with eigenvalues \( a_1, \ldots, a_n \), each of multiplicity 2, and \( -2\sum_{i=1}^n a_i \), where \( a_i \neq a_j \) for \( i \neq j \). It is easy to check that \( \dim Y^r = \dim \mathfrak{g}_1^0 = \dim \mathfrak{g}_1 - n \). Thus \( Y^r \) is open and dense in \( \mathfrak{g}_1^0 \). We have
\[
\pi_1^K (Y^r) \to B_n \to S_n,
\]
where \( \pi_1^K \) stands for the equivariant fundamental group.
Let us denote by \( \mathcal{L}_\mu \) the irreducible \( K \)-equivariant local system on \( Y^r \) given by the irreducible \( S_n \)-representation \( \rho_\mu \) via \( \pi^K_1(Y^r) \to S_n \), where \( \mu \in \mathbb{P}(n) \).

The main result of this section is the following theorem.

**Theorem 4.1.** Let \( x \in \mathcal{O}_\lambda \), where \( \lambda = (2\mu_1 + 1, 2\mu_2, \ldots, 2\mu_s) \) and \( \sum \mu_i = n \). We have
\[
\mathfrak{f}(\text{IC}(\mathcal{O}_\lambda, \mathbb{C})) \cong \text{IC}(\mathfrak{g}^0_1, \mathcal{L}_{\mu^t}),
\]
where \( \mu^t \) denote the conjugate partition of \( \mu = (\mu_1, \ldots, \mu_s) \).

**Remark 4.2.** The conjugate of a partition appears in the above formula for the same reason that it appears in the classical Springer correspondence if we implement the Springer correspondence via the Fourier transform.

Entirely similarly, let \( P^i, i \in [1, n-1] \) be the \( \theta \)-stable parabolic subgroups defined in §3.4 and let \( p_i = \text{Lie } P^i \). Consider the proper maps
\[
\bar{\pi}_i : K \times B^r_k P^i_1 \to \mathfrak{g}_1, (k, x) \mapsto \text{Ad}_k(x).
\]
Let \( \mathfrak{g}^i_1 \) denote the image of \( \bar{\pi}_i \). Then \( \mathfrak{g}^i_1 \) consists of elements in \( \mathfrak{g}_1 \) with eigenvalues \( a_1, a_1, \ldots, a_i, a_i, a_j, j \in [2i + 1, 2n] \), where \( \sum_{k=1}^{i} 2a_k + \sum_{k=2i+1}^{2n+1} a_j = 0 \). Let \( Y^i_r \) be the set of regular elements in \( \mathfrak{g}_1 \) with eigenvalues \( a_1, a_1, \ldots, a_i, a_i, a_j, j \in [2i + 1, 2n+1] \), where \( a_i \neq a_j \) for \( i \neq j \). One checks that \( Y^i_r = \mathfrak{g}^i_1 \). The equivariant fundamental group \( \pi^K_r(Y^i_r) \) has a quotient \( S_i \). For \( \mu \in \mathbb{P}(i) \), we write \( \mathcal{L}_\mu \) as the irreducible \( K \)-equivariant local system on \( Y^i_r \) given by \( \rho_\mu \in S_i^\vee \) via \( \pi^K_r(Y^i_r) \to S_i \).

Using this set up we have the following:

**Theorem 4.3.** Let \( x \in \mathcal{O}_\lambda \), where \( \lambda = (2\mu_1 + 1, \ldots, 2\mu_{2n-2i+1} + 1, 2\mu_{2n-2i+2}, \ldots, 2\mu_s) \) and \( \sum \mu_j = i \). We have
\[
\mathfrak{f}(\text{IC}(\mathcal{O}_\lambda, \mathbb{C})) \cong \text{IC}(\mathfrak{g}^i_1, \mathcal{L}_{\mu^t}),
\]
where \( \mu^t \) denote the conjugate partition of \( \mu = (\mu_1, \ldots, \mu_s) \).

As the proof of this theorem is completely analogous to that of Theorem 4.1, we only provide the proof in the special case of Theorem 4.1 where the notation is less heavy.

**Remark 4.4.** Theorem 4.1 and Theorem 4.3 are special cases of the general procedure of induction from a \( \theta \)-stable parabolic subgroup (see §2.5).

### 4.1. Proof of Theorem 4.1
We begin with the following lemmas.

**Lemma 4.5.** Consider the fiber product \( X = \bar{\mathfrak{g}}_1 \times_{\mathfrak{g}_1} \mathfrak{b}_1 \). We have \( \dim X = \dim \mathfrak{b}_1 \).

**Proof.** The proof is entirely similar to that of Corollary 3.6. We use the notations there. We have \( X = \{(x, kB_K) \mid k \in K, x \in \mathfrak{b}_1 \cap \text{Ad}_k \mathfrak{b}_1 \} \). Let \( X_w \) be the pre-image of \( \mathcal{O}_w \) under the projection \( X \to K/B_K \). We have \( X_w \simeq B_K \times B^K \mathfrak{b}_1^w \). Using Lemma 3.4 we see that
\[
\dim X_w = \dim \mathfrak{b}_0 + \dim \mathfrak{b}_1^w - \dim \mathfrak{b}_0^w \leq \dim \mathfrak{b}_0 + \dim \mathfrak{t}_1 = \dim \mathfrak{b}_1.
\]

\[\square\]
Lemma 4.6. The map $\tilde{\pi} : \tilde{g}_1 \rightarrow g_1^0$ is a small map.

Proof. Consider the stratification $\{S_i\}$ of $g_1^0$, $S_i := \{x \in g_1^0 \mid \dim B_x = i\}$. We have $\tilde{\pi}^{-1}(S_i) = K \times B_K (b_1 \cap S_i)$ and it implies that

$$\dim S_i + i = \dim K + \dim b_1 \cap S_i - \dim B_K = \dim n_0 + \dim b_1 \cap S_i.$$ 

On the other hand, we have

$$\dim b_1 \cap S_i + i = \dim \tilde{g}_1 \times_{g_1} (b_1 \cap S_i) \leq \dim X \dim b_1.$$ 

It implies

$$(4.1) \quad \dim S_i + 2i \leq \dim b_1 + \dim n_0 = \dim g_1^0.$$ 

To prove the smallness, we need to show that the inequality in $(4.1)$ is strict for $i > 0$. For this, it is enough to show that $Z = \tilde{\pi}^{-1}(S_0 \cap b_1)$ is dense in $X$. Since each $X_w$ is irreducible, it is enough to show that $X_w \cap Z$ is non-empty.

Let $x \in t_1$ satisfy $\dim Z_g(x) \leq \dim Z_g(x')$ for all $x' \in t_1$. We claim that $\dim B_x = 0$. To see this, observe that $\dim B_x = \dim B_L$ where $L = Z_K(x)^0$ and $B_L$ is the flag variety of $L$. On the other hand, we have $\text{Lie} L = Z_{g_0}(x) = t_0$ which is abelian. This implies $\dim B_L = 0$ hence finishes the proof of the claim. The claim implies $x \in S_0 \cap b_1$. Clearly, $x$ is also in the image of $\tilde{\pi}|_w : X_w \rightarrow b_1$. This shows that $X_w \cap Z \neq \emptyset$.

Consider the following diagram

$$(4.2) \quad \tilde{\pi} : \tilde{g}_1 \rightarrow g_1^0$$

Taking the Fourier transform on the trivial vector bundle $g_1 \times B_K \rightarrow B_K$, we obtain $\mathcal{F}(C_{\tilde{N}_1}[-]) \cong C_{\tilde{g}_1}[-]$. As in the proof of the classical Springer correspondence, using functoriality of Fourier transform we observe that

$$(4.3) \quad \mathcal{F}(\pi_* C_{\tilde{N}_1}[-]) \cong \hat{\pi}_* C_{\tilde{g}_1}[-].$$

We prove Theorem 4.1 by decomposing both sides of $(4.3)$ into irreducibles. To this end, let us introduce some auxiliary maps.
Let $P \supset B$ be the $\theta$-stable parabolic subgroup that stabilizes the standard partial flag $0 \subset V_n^0 \subset V_n^{0\perp} \subset V$, $p = \text{Lie}(P)$ and $p_1 = p \cap g_1$. Define

\[
P_K = P \cap K, \quad \tilde{g}_1^P = K \times^{P_K} p_1, \quad \tilde{N}_1^P = K \times^{P_K} (p_1 \cap N_1)
\]

\[
\tilde{\pi}_1 : \tilde{g}_1 \to \tilde{g}_1^P, (k, x) \mapsto (k, x), \quad \tilde{\pi}_2 : \tilde{g}_1^P \to g_1^0, (k, x) \mapsto \text{Ad}_k x
\]

\[
\pi_1 : \tilde{N}_1 \to \tilde{N}_1^P, (k, x) \mapsto (k, x), \quad \varphi_2 : \tilde{N}_1^P \to N_1, (k, x) \mapsto \text{Ad}_k x.
\]

We decompose the diagram (4.2) into two steps as follows

\[
\begin{array}{c}
\tilde{g}_1 \downarrow \pi_1 \\
\downarrow \pi_1 \\
\tilde{N}_1^P \downarrow \varphi_2 \\
\downarrow \pi_1 \\
g_1^0 \end{array}
\]

and break the analysis of equation (4.3) accordingly into these two steps.

Let us write $\mathcal{P}_K = K/P_K$ and consider a more detailed view of the top of the diagram (4.4)

\[
\begin{array}{c}
\tilde{g}_1 \downarrow \pi_1 \\
\downarrow \pi_1 \\
\tilde{N}_1^P \downarrow \varphi_2 \\
\downarrow \pi_1 \\
g_1^0 \end{array}
\]

By functoriality of the Fourier transform we conclude that

\[
\mathcal{F}(\tilde{\pi}_1^* \mathcal{C}_{\tilde{g}_1}[-]) = \mathcal{F}(\pi_1^* \mathcal{C}_{\tilde{g}_1}[-]) = \pi_1^* \mathcal{F}(\mathcal{C}_{\tilde{g}_1}[-]) = \pi_1^* \mathcal{C}_{\tilde{N}_1}[-] = \mathcal{F}(\pi_1^* \mathcal{C}_{\tilde{N}_1}[-]);
\]

note that the Fourier transforms are taken on the trivial bundles $g_1 \times B_K \to B_K$ and $g_1 \times \mathcal{P}_K \to \mathcal{P}_K$, respectively.

Next we study how both sides of formula (4.6) decompose into irreducibles. By $K$-equivariance, the sheaves in the formula (4.6) are smooth over $\mathcal{P}_K$. Thus, to understand
the decomposition it suffices to restrict them to a fiber above a point in $\mathcal{P}_K$. When restricting the diagram (4.5) to our chosen base point in $\mathcal{P}_K$ we obtain the following diagram:

$$\begin{array}{c}
\widetilde{\mathcal{N}}_1 \cap p_1 = P_K \times B_K \mathfrak{n}_1 \xrightarrow{\pi_1} \mathcal{N}_1 \cap p_1 . \\
P_K/B_K \times \mathfrak{g}_1 \xrightarrow{p_1} \mathfrak{g}_1 \\
\widetilde{p}_1 = P_K \times B_K \mathfrak{b}_1 \xrightarrow{\tilde{\pi}_1} p_1
\end{array}$$

After the restriction the formula (4.6) becomes

$$\mathfrak{F}(\tilde{\pi}_1^* \mathcal{C}_{\tilde{p}_1} [-]) = \pi_1^* \mathcal{C}_{\mathcal{N}_1 \cap p_1} [-].$$

We will now further restrict this identity to the Levi $L = P/U_p$. To do so, we make use of the notation

$$\begin{array}{c}
\mathfrak{g}_1 \leftarrow i \mathfrak{p}_1 \xrightarrow{p} \mathfrak{l}_1 \\
\mathfrak{g}_1^* \leftarrow i^* \mathfrak{p}_1^* \xrightarrow{i^* p} \mathfrak{l}_1^*
\end{array}$$

First we observe that the functors $i_* p^*$ and $i^* i^* p_*$ from sheaves on $\mathfrak{l}_1$ to sheaves on $\mathfrak{g}_1$ coincide under our usual identification of $\mathfrak{g}_1$ with $\mathfrak{g}_1^*$ and the compatible identification of $\mathfrak{l}_1$ with $\mathfrak{l}_1^*$. The functorial properties of the Fourier transform then imply that the functor $i_* p^*$ commutes with the Fourier transform. The same statement holds on the level of resolutions, i.e., on the level of vector bundles

$$\begin{array}{c}
P_K/B_K \times \mathfrak{g}_1 \xleftarrow{i} \tilde{\mathfrak{p}}_1 \xrightarrow{p} \tilde{\mathfrak{l}}_1 = L_K \times B_K \cap L_K \left( \mathfrak{b}_1 \cap \mathfrak{l}_1 \right) \\
P_K/B_K = L_K / (B_K \cap L_K)
\end{array}$$

where $L_K = L^\theta$. After restriction to $L$, the map $\pi_1$ becomes $\tilde{\pi}_1 : \mathcal{N}_1 = L_K \times B_K \cap L_K (\mathfrak{n}_1 \cap \mathfrak{l}_1) \to \mathcal{N}_1 \cap \mathfrak{l}_1$, and the map $\tilde{\pi}_1$ becomes $\pi_1 : \tilde{\mathfrak{l}}_1 = L_K \times B_K \cap L_K (\mathfrak{b}_1 \cap \mathfrak{l}_1) \to \mathfrak{l}_1$. Note that $B_K \cap L_K$ is a Borel subgroup of $L_K$. Let us write $B_{L_K} := B_K \cap L_K$. The situation is summarized in the
respectively. Note that \( L_K \cong GL(n) \) and \( \mathfrak{l}_0 \) is the Lie algebra of \( L_K \). Thus we are in the situation of classical Springer correspondence for \( \mathfrak{gl}(n) \). This gives us the following identity

\[
\bigoplus_{\rho \in S_n^\vee} \mathcal{F}(\text{IC}(\mathfrak{l}_1, \mathcal{L}_\rho)) \otimes V_\rho = \mathcal{F}(\rho_{\mathfrak{l}_1}^* \mathbb{C}_{\mathfrak{N}_{\mathfrak{l}_1}} [-]) = \bigoplus_{\rho \in S_n^\vee} \text{IC}(\mathcal{O}_{\mathfrak{l}_1}^\rho, \mathbb{C}) \otimes V_\rho
\]

and the direct sum decompositions on the two sides match. Here \( \mathcal{L}_\rho \) is the irreducible local system on \( \mathfrak{l}_1^r \) (the set of regular semisimple elements in \( \mathfrak{l}_1 \) after identification with \( \mathfrak{gl}(n) \)) given by \( \rho \in S_n^\vee \). Assume that \( \rho = \rho_\lambda, \lambda \in \mathbb{P}(n) \). Then \( \mathcal{O}_{\mathfrak{l}_1}^\rho \) stands for the nilpotent orbit in \( \mathfrak{l}_1 \) corresponding to the partition \( \lambda' \) (again we identify \( \mathfrak{l}_1 \) with \( \mathfrak{gl}(n) \)).

Applying the functor \( i_*p^* \) to (4.8) and using functorial properties of the Fourier transform in the diagram (4.7) we obtain

\[
\bigoplus_{\rho \in S_n^\vee} \mathcal{F}(i_*p^*\text{IC}(\mathfrak{l}_1, \mathcal{L}_\rho)) \otimes V_\rho = \mathcal{F}(i_*p^*\rho_{\mathfrak{l}_1}^* \mathbb{C}_{\mathfrak{N}_{\mathfrak{l}_1}} [-]) = \mathcal{F}(\pi_{\mathfrak{l}_1}^* \mathbb{C}_{\mathfrak{N}_{\mathfrak{l}_1}} [-]) = \bigoplus_{\rho \in S_n^\vee} i_*p^*\text{IC}(\mathcal{O}_{\mathfrak{l}_1}^\rho, \mathbb{C}) \otimes V_\rho
\]

From this we conclude that

\[
\mathcal{F}(i_*p^*\text{IC}(\mathfrak{l}_1, \mathcal{L}_\rho)) = i_*p^*\text{IC}(\mathcal{O}_{\mathfrak{l}_1}^\rho, \mathbb{C}).
\]
Recall that this is an equality after we restrict to a fiber above the base point in $\mathcal{P}_K$. By $K$-equivariance this gives us the following statement on $\mathfrak{g}_1^P$

\[(4.9) \quad \mathfrak{F} (\mathfrak{g}_1^P, \mathcal{L}_\rho) = IC(K \times P_K (\mathcal{O}_{\rho'}^1 + (n_P)_1), \mathbb{C})\]

where $\mathcal{L}_\rho$ is the irreducible $K$-equivariant local system on $K \times P_K (\mathcal{O}_{\rho'}^1 + (n_P)_1)$ given by the irreducible $S_n$-representation $\rho$. Note that the equivariant fundamental group $\pi_1^K (K \times P_K (\mathcal{O}_{\rho'}^1 + (n_P)_1))$ maps subjectively to $S_n$.

It remains to push $(4.9)$ down to $\mathfrak{g}_1$. We first observe that

**Lemma 4.7.** The map $\tilde{\pi}_2$ is a small map, generically one-to-one.

**Proof.** The map $\tilde{\pi}_2$ can be identified as $\{(x, 0 \subset V_n \subset V_n^+ \subset V) \in \mathfrak{g}_1 \times \mathcal{P}_K \mid xV_n \subset V_n \} \to \mathfrak{g}_1^0$, $(x, (V_n)) \mapsto x$. Similar argument as before shows that

\[(4.10) \quad \mathfrak{F} (\tilde{\pi}_2^* \mathbb{C}[-]) \cong \pi_2^* \mathbb{C}[-],\]

where $\pi_2 : \{(x, (V_n)) \in \mathfrak{g}_1 \times \mathcal{P}_K \mid xV_n = 0, xV_n^+ \subset V_n \} \to N_1$, $(x, (V_n)) \mapsto x$. It is easy to check that $\text{Im} \, \tilde{\pi}_2 = \mathcal{O}_{312^{n-1}}$ and that $\pi_2$ is a small map. In fact, the fiber of $\pi_2$ at $\mathcal{O}_{312^{n-1}}$ is a point, the fiber at $\mathcal{O}_{312^{j+2n-2j}}$ (resp. at $\mathcal{O}_{212^{n-1-2j}}$) is isomorphic to $\text{OGr}(n - 1 - j, 2n - 2 - 2j)$ (resp. $\text{OGr}(n - i, 2n + 1 - 2i)$). Hence we have

\[(4.11) \quad \pi_2^* \mathbb{C}[-] \cong IC(\mathcal{O}_{312^{n-1}}, \mathbb{C}).\]

Let $Y^r$ be the set of regular elements defined in the beginning of this section. It is easy to see that for each $y \in \mathfrak{g}_1$, there exists a unique maximal isotropic subspace $V_n \subset V$ such that $V_n$ is $y$-stable (note also that $y|V_n$ is regular semisimple). This shows that $\tilde{\pi}_2$ is generically one-to-one. Using $(4.10)$ and $(4.11)$ we conclude that $\tilde{\pi}_2^* \mathbb{C}[-] \cong IC(Y^r, \mathbb{C})$. Thus $\tilde{\pi}_2$ is small. 

When we push down $(4.9)$ by $\tilde{\pi}_2$ we obtain

\[(4.12) \quad \mathfrak{F} (\tilde{\pi}_2^* \mathfrak{g}_1^P, \mathcal{L}_\rho) = \varphi_2^* IC(K \times P_K (\mathcal{O}_{\rho'}^1 + (n_P)_1), \mathbb{C}).\]

Note that the proof of Lemma 4.7 shows that

\[(4.13) \quad \tilde{\pi}_2^* \mathfrak{g}_1^P, \mathcal{L}_\rho) \cong IC(Y^r, \mathcal{L}_\rho).\]

Suppose that $\rho \in S_n^r$ corresponds to the conjugate partition $\mu'$ of $\mu = (\mu_1, \ldots, \mu_s)$.

**Lemma 4.8.** We have that

\[\varphi_2^* IC(K \times P_K (\mathcal{O}_{\rho'}^1 + (n_P)_1), \mathbb{C}) = IC(\mathcal{O}_\lambda, \mathbb{C})\]

where $\lambda = (2\mu_1 + 1, 2\mu_2, \ldots, 2\mu_s)$.

**Proof.** First note that $\varphi_2^* IC(K \times P_K (\mathcal{O}_{\rho'}^1 + (n_P)_1), \mathbb{C})$ is irreducible by $(4.12)$ and $(4.13)$. It follows from Lemma 2.4 that $\mathcal{O}_\lambda$ is dense in $K.(\mathcal{O}_{\rho'}^1 + (n_P)_1)$. Thus the lemma holds. 

$\Box$
Now Theorem 4.1 follows from (4.12), (4.13) and Lemma 4.8.

4.2. IC sheaves whose Fourier transform have smaller support. In this subsection, we give a sufficient condition for the Fourier transform of IC(O_\lambda, \mathbb{C}) to have smaller support. Namely,

**Theorem 4.9.** Assume that \( \lambda \in \mathcal{P}(N) \) has gaps. Then

\[
\text{supp}(\mathfrak{F}(\text{IC}(O_\lambda, \mathbb{C}))) \subset \mathfrak{g}_1.
\]

**Proof.** Let \( \lambda \) be a partition with gaps and let \( O_0 \subset I_1 \) be the orbit defined as in the proof of Proposition 2.3 so that Ind_{\mathfrak{l}_1}^{\mathfrak{g}_1} \mathcal{O}_0 = \bar{O}_\lambda. \) Using Lemma 2.4, it is easy to see that IC(O_\lambda, \mathbb{C}) is a direct summand of Ind_{\mathfrak{l}_1}^{\mathfrak{g}_1} IC(O_0, \mathbb{C}). Using (2.2) (see §2.5) we see that

\[
\text{Supp}(\mathfrak{F}(IC(O_\lambda, \mathbb{C}))) \subset \text{Supp}(\text{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}_1} \mathfrak{F}(IC(O_0, \mathbb{C})))) \subset \mathfrak{g}_1.
\]

In the last equation we use that Supp(\text{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}_1} A) \subset K. \mathfrak{p}_1 \subset \mathfrak{g}_1.

\[\square\]

**Remark 4.10.** Note that the proof of the theorem above shows that if IC(O, E) is induced, then its Fourier transform has smaller support.

5. Nilpotent orbits of order two

In this section we compute Fourier transforms of IC complexes supported on nilpotent orbits of order two, i.e., nilpotent \( K \)-orbits of the form \( O_{2i+1, 2n+1-i} \), \( i \in [1, n] \). Note that each orbit \( O_{2i+1, 2n+1-i} \) carries a unique non-trivial irreducible \( K \)-equivariant local system. It turns out that the Fourier transforms of such IC complexes are supported on all of \( \mathfrak{g}_1 \); thus we obtain from them irreducible \( K \)-equivariant local systems on \( \mathfrak{g}_1^* \).

The local systems on \( \mathfrak{g}_1^* \) which are obtained from the Fourier transforms of IC(O_{2i+1, 2n+1-i}, \mathbb{C}) turn out to be representations of the Tits group \( \tilde{W} \). They are the irreducible representations which occur in \( \mathbb{C}[A[2]] \) viewed as a representation of \( \tilde{W} \). First, let us write \( L_\chi \) for the representation of \( A[2] \) associated to the character \( \chi \in A[2]^\vee \) and so we have

\[
\mathbb{C}[A[2]] = \bigoplus_{\chi \in A[2]^\vee} L_\chi.
\]

The space \( A[2]^\vee \) breaks into orbits \( \{ \Lambda_i \}_{i=0, \ldots, n} \) under the action of \( W = S_N \), where we have numbered the orbits so that \( |\Lambda_i| = \binom{2n+1}{i} \). Note that here it is crucial that \( N = 2n + 1 \) is odd. Concretely, the \( \Lambda_i \) consists of characters that attain the value \( -1 \in \mathbb{G}_m \) precisely \( i \) times if \( i \) is even and \( 2n + 1 - i \) times if \( i \) is odd. Thus, for each \( i = 0, \ldots, n \) we obtain an irreducible representation of \( \tilde{W} \) as follows

\[
L_i = \bigoplus_{\chi \in \Lambda_i} L_\chi.
\]

\[
L_i = \bigoplus_{\chi \in \Lambda_i} L_\chi.
\]

\[\text{We have chosen this parametrization to get a clean statement for our theorem in } \S 6.\]
We write $L_i$ for the irreducible $K$-equivariant local system on $g_1^{rs}$ corresponding to the representation $L_i$ via the map $\pi_1^K(g_1^{rs}, a) \to \bar{W}$. The first main result of this section is the following:

**Theorem 5.1.** There is a permutation $s$ of the set $\{0, \ldots, n\}$ such that

$$\mathfrak{F}(\text{IC}(\mathcal{O}_{2k-1,2n+2k-1}, \mathbb{C})) \simeq \text{IC}(g_1^{rs}, L_{s(k)})$$

for $k \in \{0, \ldots, n\}$.

Let $\mathcal{E}_i$ denote the unique nontrivial irreducible $K$-equivariant local system on $\mathcal{O}_{2k-1,2n+2k-1}$. We will now define irreducible $K$-equivariant local systems $\mathcal{F}_i$ on $g_1^{rs}$ which, as it turns out, are obtained from the Fourier transforms of IC($\mathcal{O}_{2k-1,2n+2k-1}, \mathcal{E}_i$)'s. These local systems arise as representations of $\pi_1^k(g_1^{rs}, a) = B_N$. To this end we consider the following universal family $C$ of hyperelliptic curves of genus $n$:

To each $a = (a_1, \ldots, a_{2n+1}) \in a^{rs}$ we associate the hyperelliptic curve $C_a$ which ramifies at $\{a_1, \ldots, a_{2n+1}, \infty\}$.

(Here we have chosen a Cartan subspace $a$ of $g_1$ such that it consists of diagonal matrices.) This family gives us a monodromy representation $B_N \to Sp(H^1(C_a, \mathbb{C}))$. Note that, by [A] (see also [KaS]) this monodromy representation has a Zariski dense image, in particular, the monodromy is infinite. From this we get a monodromy representation on the Jacobian of $C_a$ which we break into primitive parts:

$$B_N \to H^i(\text{Jac}(C_a), \mathbb{C})_{\text{prim}} \simeq (\wedge^i H^1(C_a, \mathbb{C}))_{\text{prim}} \quad i \in [1, n].$$

Associated to this representation we obtain a local system $\mathcal{F}_i$ on $g_1^{rs}$. Note that the part $A[2]$ of $\pi_1^k(g_1^{rs})$ acts trivially on $\mathcal{F}_i$.

The second main result of this section is the following:

**Theorem 5.2.** There is a permutation $s \in S_n$ such that

$$\mathfrak{F}(\text{IC}(\mathcal{O}_{2k-1,2n+2k-1}, \mathcal{E}_i)) \simeq \text{IC}(g_1, \mathcal{F}_{s(i)}), \quad i \in [1, n].$$

5.1. **Proof of Theorem 5.1.** By Reeder, we have the resolution of singularities of $\bar{O}_{2n+1}$

$$(5.2) \quad \pi_{2n+1} : K \times^P K \times [n_P, n_P] \to \bar{O}_{2n+1},$$

where $P$ is the stabilizer subgroup in $G$ of the partial flag $0 \subset V_n \subset V_n^{0,1} \subset V$, where $V_n = \text{span}\{e_1, \ldots, e_n\}$ (here $(e_i)$ is a basis of $V$ such that $(e_i, e_j)_Q = \delta_{i+j,2n+2}$, $P_K = K \cap P$, and $n_P$ is the Lie algebra of the unipotent radical of $P$.

For $x \in \bar{O}_{2n+1}$, we have

$$(5.3) \quad \pi_{2n+1}^{-1}(x) = \{V_n \text{ maximal isotropic subspace in } V \mid xV_n^{0} = 0, \ xV \subset V_n\}.$$

**Lemma 5.3.** The map $\pi_{2n+1}$ is semi-small and we have

$$(5.4) \quad \pi_{2n+1*} \mathbb{C}[-] \cong \bigoplus_{k=0}^n \text{IC}(\mathcal{O}_{2k-1,2n+2k-1}, \mathbb{C}).$$
Let \( x \in \mathcal{O}_{2k - 1} \). Suppose that \( V_n \in \pi^{-1}_{2n}(x) \). Then \( \text{Im} \ x \subset V_n \) and \( \dim \text{Im} \ x = k \). Using (5.3), one can identify the fiber \( \pi^{-1}_{2n}(x) \) with

\[
\pi^{-1}_{2n}(x) \cong \{ n - k \text{ dimensional isotropic subspaces of } (\text{Im} \ x)^{\perp} / \text{Im} \ x \}.
\]

Thus \( \pi^{-1}_{2n}(x) \) is irreducible and it is easy to check that \( 2 \dim \pi^{-1}_{2n}(x) = (n - k)(n - k + 1) = \text{codim}_{\mathcal{O}_{2n}} \mathcal{O}_{2k - 1} \). The lemma follows from the decomposition theorem.

Consider the map

\[
\tilde{\pi}_{2n} : K \times^{PK} [n, n]^{\perp} \rightarrow \mathfrak{g},
\]

where \([n, n]^{\perp}\) is the orthogonal complement of \([n, n] \) in \( \mathfrak{g} \) with respect to the non-degenerate form \( \langle \cdot, \cdot \rangle \). The fibers \( \tilde{\pi}^{-1}_{2n}(x) \) at \( x \in \mathfrak{g} \) can be described as follows

\[
\tilde{\pi}^{-1}_{2n}(x) = \{ V_n \text{ maximal isotropic subspace in } V \mid xV_n \subset V^{\perp} \}.
\]

For \( x \in \mathfrak{g}_{1}^{rs} \) there are \( 2^{2n} \) such maximal isotropic subspaces in \( V \) and the centralizer \( Z_{K}(x) \) acts simply transitively on those subspaces (see [Re, BG]).

Consider the \( K \)-equivariant local system \( \mathcal{L} = (\tilde{\pi}_{2n})_{*} \mathbb{C}[\mathfrak{g}]^{rs} \) of rank \( 2^{2n} \) on \( \mathfrak{g}_{1}^{rs} \). Our first goal is to describe this local system. Fix \( a \in \mathfrak{a}^{rs} \). The stalk \( \mathcal{L}_{a} : = \mathcal{L}_{a} \) carries an action of the \( K \)-equivariant fundamental group \( \pi_{1}^{K}(\mathfrak{g}_{1}^{rs}, a) = Z_{K}(a) \rtimes B_{N} \) (see [2.6]). As \( Z_{K}(a) \) acts simply transitively on \( \tilde{\pi}_{2n}^{-1}(x) \), as was remarked above, we can identify \( \mathcal{L} \) with \( \mathbb{C}[Z_{K}(a)] = \mathbb{C}[A[2]] \). Furthermore, the action of \( \pi_{1}^{K}(\mathfrak{g}_{1}^{rs}, a) \) factors through the Tits group \( \bar{W} \) and it coincides with the canonical representation of \( \bar{W} \) on \( \mathbb{C}[A[2]] \).

Let us recall the irreducible \( K \)-equivariant local systems \( \mathcal{L}_{i} \) from the beginning of this section and then

\[
\mathcal{L} = \bigoplus_{i=0}^{n} \mathcal{L}_{i}.
\]

**Lemma 5.4.** We have

\[
(\tilde{\pi}_{2n})_{*} \mathbb{C}[-] = \mathbb{C}[\mathfrak{g}_{1}^{rs}, \mathcal{L}] = \bigoplus_{i=0}^{n} \mathbb{C}[\mathfrak{g}_{1}^{rs}, \mathcal{L}_{i}].
\]

In particular, the map \( \tilde{\pi}_{2n} \) is small.

**Proof.** By the decomposition theorem, \( \bigoplus_{i=0}^{n} \mathbb{C}[\mathfrak{g}_{1}^{rs}, \mathcal{L}_{i}] \) is a direct summand of \( (\tilde{\pi}_{2n})_{*} \mathbb{C}[-] \). On the other hand, since

\[
\mathfrak{g}((\pi_{2n})_{*} \mathbb{C}[-]) \cong (\tilde{\pi}_{2n})_{*} \mathbb{C}[-],
\]

the decomposition in (5.4) implies that \( (\tilde{\pi}_{2n})_{*} \mathbb{C}[-] \) has exactly \( n + 1 \) irreducible summands. The lemma follows from (5.6). \( \square \)

Theorem 5.1 follows from (5.4), (5.7) and (5.8).
5.2. Proof of Theorem 5.2. Let $V = \mathbb{C}^{2n+1}$ and $W = V \oplus \mathbb{C}$. For any $x \in \mathfrak{g}_1$, consider the following two quadrics in $\mathbb{P}(W)$:
\[ \tilde{Q}(v,a) = \langle v, v \rangle_Q = 0 \text{ and } \tilde{Q}_x(v,a) = \langle v, xv \rangle_Q + a^2 = 0. \]
Define
\[ F = \{(x, W_n \subset W) \mid x \in \mathfrak{g}_1, \dim W_n = n, \mathbb{P}(W_n) \subset \tilde{Q} \cap \tilde{Q}_x\}. \]
The map $W \to W$ given by $(v,a) \mapsto (v, -a)$ induces an involution $\sigma$ on $F$ and the set $F^\sigma$ of fixed points is equal to
\[ F^\sigma = \{(x, W_n \subset V) \mid x \in \mathfrak{g}_1, W_n \text{ maximal isotropic in } V, xv_n \subset V_n^\perp\}, \]
i.e. $F^\sigma \cong K \times_{F_2} [\mathfrak{n}_P, \mathfrak{n}_P]_1^\perp$ (see (5.3)).

Note that for $x \in \mathfrak{g}_1^{rs}$, the pencil of quadrics spanned by $\tilde{Q}$ and $\tilde{Q}_x$ is non-degenerate and contains exactly $2n + 2$ singular elements, namely, the quadric $\tilde{Q}$ at infinity and the $2n + 1$ quadrics $\lambda_i\tilde{Q} - \tilde{Q}_x$, where $\lambda_1, ..., \lambda_{2n+1}$ are the roots of $f(t) = \det(t \cdot \text{id} - x)$.

We denote by $\pi : F \to \mathfrak{g}_1$ the natural projection. The fiber $F_x = \pi^{-1}(x)$ of $\pi$ over $x \in \mathfrak{g}_1^{rs}$ is the Fano variety of $(n-1)$-dimensional subspaces contained in the smooth complete intersection $\tilde{Q} \cap \tilde{Q}_x$. According to [Re], $F_x$ is a torsor over $\text{Jac}(C_x)$, where $C_x$ is the smooth projective hyperelliptic curve with affine equation:
\[ y^2 = \prod_{i=1}^{2n+1} (t - \lambda_i). \]
Moreover, the action of the involution $\sigma$ on $F_x$ is compatible with the inversion on $\text{Jac}(C_x)$. In particular, the set $F_x^\sigma$ of fixed points is a $\text{Jac}(C_x)[2]$-torsor, where $\text{Jac}(C_x)[2]$ consists of 2-torsion points of $\text{Jac}(C_x)$.

The discussion above has the following relative version. Namely, let $\pi^C : C \to \mathfrak{g}_1^{rs}$ be the family of curves $C_x$ and let $\text{Jac}(C) \to \mathfrak{g}_1^{rs}$ denote the corresponding relative Jacobian. Let $F|_{\mathfrak{g}_1^{rs}} \to \mathfrak{g}_1^{rs}$ be the family of Fano varieties of $(n-1)$-dimensional subspaces contained in the smooth complete intersection $\tilde{Q} \cap \tilde{Q}_x$. Then $\text{Jac}(C)$ acts naturally on $F|_{\mathfrak{g}_1^{rs}}$, and $F|_{\mathfrak{g}_1^{rs}}$ is a $\text{Jac}(C)$-torsor under this action. Similarly, $\text{Jac}(C)[2]$ acts on $F^\sigma$, and $F^\sigma$ is a $\text{Jac}(C)[2]$-torsor under this action.

We have the following observation.

Lemma 5.5. 1) The action map $\text{Jac}(C) \times \mathfrak{g}_1^{rs} F^\sigma|_{\mathfrak{g}_1^{rs}} \to F|_{\mathfrak{g}_1^{rs}}$ factors through an isomorphism
\[ (\text{Jac}(C) \times \mathfrak{g}_1^{rs} F^\sigma|_{\mathfrak{g}_1^{rs}})/\text{Jac}(C)[2] \simeq F|_{\mathfrak{g}_1^{rs}}. \]
Here $\text{Jac}(C)[2]$ acts on $\text{Jac}(C) \times \mathfrak{g}_1^{rs} F^\sigma|_{\mathfrak{g}_1^{rs}}$ via the diagonal action.

2) For any $a \in \mathfrak{g}_1^{rs}$, there is a canonical isomorphism
\[ H^i(\text{Jac}(C_a), \mathbb{C}) \simeq H^i((\text{Jac}(C_a) \times F^\sigma_a)/\text{Jac}(C_a)[2], \mathbb{C}) \]
compatible with the monodromy actions of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ on both sides.
Proof. Part 1) is clear. For part 2), we observe that, by K"unneth formula, we have
\[ H^i((\text{Jac}(C_a) \times F^a_\sigma)/\text{Jac}(C_a)[2], \mathbb{C}) \simeq (H^i(\text{Jac}(C_a), \mathbb{C}) \otimes H^0(F^a_\sigma, \mathbb{C}))^{\text{Jac}(C_a)[2]} \]
where the right hand side is the $(\text{Jac}(C_a)[2])$-fixed vectors in $H^i(\text{Jac}(C_a), \mathbb{C}) \otimes H^0(F^a_\sigma, \mathbb{C})$. Since the action of $\text{Jac}(C_a)[2]$ on $H^i(\text{Jac}(C_a), \mathbb{C})$ is trivial,
\[ (H^i(\text{Jac}(C_a), \mathbb{C}) \otimes H^0(F^a_\sigma, \mathbb{C}))^{\text{Jac}(C_a)[2]} \simeq H^i(\text{Jac}(C_a), \mathbb{C}), \]
Combining (5.10) and (5.11), we obtain a canonical isomorphism
\[ H^i((\text{Jac}(C_a) \times F^a_\sigma)/\text{Jac}(C_a)[2], \mathbb{C}) \simeq H^i(\text{Jac}(C_a), \mathbb{C}). \]
Since the isomorphisms in (5.10), (5.11) are compatible with the monodromy actions, so is the composition in (5.12). Thus part 2) follows. This finishes the proof of the lemma. \qed

The following is immediate.

**Corollary 5.6.** There is a canonical isomorphism $H^i(F_a, \mathbb{C}) \simeq H^i(\text{Jac}(C_a), \mathbb{C})$ compatible with the monodromy actions of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$.

We use the corollary above to study the monodormy of the family of Fano varieties $F|_{\mathfrak{g}_1^{rs}} \rightarrow \mathfrak{g}_1^{rs}$. To begin with, we observe that over the Kostant section $\kappa : \mathfrak{c}_1^{rs} \hookrightarrow \mathfrak{g}_1^{rs}$, the family $\pi^C : C \rightarrow \mathfrak{g}_1^{rs}$ is the universal family of hyperelliptic curves of genus $n$. As mentioned before, the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ on $H^1(C_a, \mathbb{C})$ is irreducible and the image of $\pi_1^K(\mathfrak{g}_1^{rs}, a) \rightarrow \text{Sp}(H^1(C_a, \mathbb{C}))$ is Zariski dense. This fact together with the corollary above implies that the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ on
\[ H^i(F_a, \mathbb{C})_{\text{prim}} \simeq H^i(\text{Jac}(C_a), \mathbb{C})_{\text{prim}} \simeq (\wedge^i H^1(C_a, \mathbb{C}))_{\text{prim}} \]
is irreducible for $i = 1, \ldots, n$, moreover, the corresponding monodromy group is infinite. Thus the corresponding irreducible $K$-equivariant local systems on $\mathfrak{g}_1^{rs}$ are
\[ (R^i \pi_* \mathbb{C}|_{\mathfrak{g}_1^{rs}})_{\text{prim}} = \mathcal{F}_i \]
where $\mathcal{F}_i$ is the local system defined at the beginning of this section. We have
\[ \dim \mathcal{F}_i = \binom{2n}{i} - \binom{2n}{i-2} \]
and $\mathcal{F}_i \not\cong \mathcal{F}_j$ for $i \neq j$. We show that

**Proposition 5.7.** For $i = 1, \ldots, n$, $\mathfrak{g}(\text{IC}(\mathfrak{g}_1, \mathcal{F}_i))$ is supported on $\bar{\mathfrak{g}}_{2n+1}$.

Recall the local systems $\{\mathcal{L}_i\}_{i=0,\ldots,n}$ from the beginning of this section. Since each $\mathcal{L}_i$ has finite monodromy, we have $\mathcal{F}_i \not\cong \mathcal{L}_j$ for all $i, j$. As there is only one nontrivial irreducible $K$-equivariant local system $\mathcal{E}_i$ on each $\mathfrak{g}_{2n+2i-1}$ for $i \geq 1$, Theorem 5.1 and Proposition 5.7 imply Theorem 5.2.

---

\(^6\)To see this, we observe that the action of $\text{Jac}(C_a)[2]$ on $H^i(\text{Jac}(C_a))$ is the restriction of the action of $\text{Jac}(C_a)$ on $H^i(\text{Jac}(C_a))$. Since $\text{Jac}(C_a)$ is connected the latter action is trivial.
Proof of Proposition 5.7. Since \( \pi : F \to g_1 \) is proper, the decomposition theorem implies that \( \text{IC}(g_1, \mathcal{I}) \) (up to shift) is a summand of \( \pi_* \text{IC}(F, \mathbb{C}) \). Thus it is enough to show that \( \mathcal{I}(\pi_* \text{IC}(F, \mathbb{C})) \) is supported on \( \bar{\Theta}_{2n} \).

Recall the maps \( \pi_{2n}^* : K \times_{P_K} [n_P, n_P]_1 \to \bar{\Theta}_{2n} \) and \( \bar{\pi}_{2n}^* : K \times_{P_K} [n_P, n_P]_1 = \{(x, 0 \subset V_n \subset V_n^\perp \subset V) | xV_n \subset V_n^\perp \} \to g_1 \) defined in (5.2) and (5.3). For simplicity, let us write \( L = [n_P, n_P]_1 \) and \( L^\perp = [n_P, n_P]_1^\perp \).

Let \( q : K/P_K \times g_1 \simeq K \times_{P_K} g_1 \to K \times_{P_K} (g_1/L^\perp) \) denote the quotient map. Note that the non-degenerate invariant form on \( g_1 \) induces isomorphisms \( L \simeq (g_1/L^\perp)^*, g_1 \simeq g_1^*, \) and under these isomorphisms, the dual map of \( q \) can be identified with the natural embedding

\[
\tilde{q} : K \times_{P_K} L \to K \times_{P_K} g_1 \simeq K/P_K \times g_1.
\]

Let us decompose \( \pi \) as

\[
\pi : F \xrightarrow{f} K/P_K \times g_1 \xrightarrow{pr_{g_1}} g_1,
\]

where \( f : F \to K \times_{P_K} g_1 \simeq K/P_K \times g_1 \) is given by \( (x, W_n) \to (pr_V(W_n), x) \) and \( pr_V : W = V \oplus \mathbb{C} \to V \) is the projection along \( V \). The map \( f \) factors through the closed sub-scheme \( Z \subset K \times_{P_K} g_1 \) defined by

\[
Z = \{(x, 0 \subset V_n \subset V_n^\perp \subset V) | x \in g_1, \text{rank}(\bar{x} : V_n \to V/V_n^\perp) \leq 1\};
\]

here \( \bar{x} \) is the composition of \( V_n \xrightarrow{x} V \) with the projection \( V \to V/V_n^\perp \). The resulting map (by abuse of notation, still denoted by \( f \))

\[
f : F \to Z
\]

is a branched double cover with branch locus

\[
Z_0 := \{(x, 0 \subset V_n \subset V_n^\perp \subset V) | x \in g_1, \text{rank}(\bar{x} : V_n \to V/V_n^\perp) = 0\}.
\]

Note that \( Z_0 \) coincides with \( K \times_{P_K} L^\perp \).

Let \( Z_0 := Z - K \times_{P_K} L^\perp \). Then \( f^0 := f|_{Z_0} : F \times Z Z_0 \to Z^0 \) is a double cover and we have \( f^0_* \mathcal{C} = \mathcal{C} \oplus \mathcal{L} \), where \( \mathcal{L} = (f^0_* \mathcal{C})^{\sigma = -id} \) is a rank one local system on \( Z^0 \). It follows that

\[
f_* (\text{IC}(F, \mathbb{C})) = \text{IC}(Z, \mathbb{C}) \oplus \text{IC}(Z, \mathcal{L}).
\]

We claim that

\[
\text{both IC}(Z, \mathbb{C}) \text{ and IC}(Z, \mathcal{L}) \text{ descend to } K \times_{P_K} (g_1/L^\perp)
\]

via \( q : K \times_{P_K} g_1 \to K \times_{P_K} (g_1/L^\perp) \).

Here we regard IC\((Z, \mathbb{C})\) and IC\((Z, \mathcal{L})\) as perverse sheaves on \( K/P_K \times g_1 \) via the closed embedding \( u : Z \subset K/P_K \times g_1 \).

\footnote{Using (5.9), one can check that \( pr_V(W_n) \) is isotropic and \( \dim pr_V(W_n) = n \), hence the map \( f \) is well-defined.}

\footnote{In fact, the map \( f : F \to Z \) realizes \( Z \) as the (GIT) quotient \( F/\sigma \).}
Assume that the claim holds. Let $P_1$ (resp. $P_2$) denote the descent of $\text{IC}(Z, \mathbb{C})$ (resp. $\text{IC}(Z, \mathcal{L})$), we get

$$\mathfrak{L}(\pi_*(\text{IC}(F, \mathbb{C}))) \simeq \mathfrak{L}(pr_{g_1}^*(\text{IC}(F, \mathbb{C}))) \simeq \mathfrak{L}(pr_{g_1}^*(\text{IC}(Z, \mathbb{C}) \oplus \text{IC}(Z, \mathcal{L})))$$

$$\simeq \mathfrak{L}(pr_{g_1}^*Z^\bullet(P_1 \oplus P_2)) \simeq pr_{g_1}^*(\mathfrak{L}(P_1 \oplus P_2)) \simeq \pi_{2n+1,*}(\mathfrak{L}(P_1 \oplus P_2)) \text{ (up to shift).}$$

This implies that $\mathfrak{L}(\pi_*(\text{IC}(F, \mathbb{C})))$ is supported on $\bar{\mathcal{O}}_{2n+1}$. Hence Proposition 5.7 holds.

It remains to prove claim (5.15). Let $\bar{Z} \hookrightarrow K \times P_K (g_1/L^\perp)$ be the image of $Z$ under the map

$$Z \xrightarrow{\bar{Z}} K \times P_K g_1 \xrightarrow{\bar{q}} K \times P_K (g_1/L^\perp).$$

Let $\bar{q} : Z \to \bar{Z}$ denote the induced map. We have the following Cartesian diagram:

$$\begin{array}{ccc}
Z & \xrightarrow{\bar{q}} & \bar{Z} \\
\downarrow{u} & & \downarrow{\bar{u}} \\
K \times P_K g_1 & \xrightarrow{\bar{q}} & K \times P_K (g_1/L^\perp)
\end{array}$$

Therefore, it is enough to show that $\text{IC}(Z, \mathbb{C})$ and $\text{IC}(Z, \mathcal{L})$ descend to $\bar{Z}$. To prove this, we observe that $\bar{q}|_{Z_0}$ are smooth maps with contractible fibers. It implies that the local systems $\mathbb{C}$ and $\mathcal{L}$ on $Z^n$ descend to $\bar{Z}$. Since $\bar{q}$ is smooth, it implies that $\text{IC}(Z, \mathbb{C})$ and $\text{IC}(Z, \mathcal{L})$ descend to $\bar{Z}$. We are done.

\[ \square \]

Let $a \in \mathfrak{a}^{rs}$. By (5.14), we have

$$(pr_{g_1})_*\text{IC}(Z, \mathbb{C})|_a \simeq H^*(F_A, \mathbb{C})^{\sigma = id}, (pr_{g_1})_*\text{IC}(Z, \mathcal{L})|_a \simeq H^*(F_A, \mathbb{C})^{\sigma = -id}.$$  

Choosing an isomorphism $\text{Jac}(C_a) \simeq F_A$, we may identify $\sigma$ with the inversion on $\text{Jac}(C_a)$. Therefore

$$H^*(F_A, \mathbb{C})^{\sigma = id} \simeq \oplus_{i=2j} \wedge^i H^1(C_A, \mathbb{C})[-i], \quad H^*(F_A, \mathbb{C})^{\sigma = -id} \simeq \oplus_{i=2j+1} \wedge^i H^1(C_A, \mathbb{C})[-i].$$

This implies that

$$(5.16) \quad \text{IC}(g_1, \oplus_{i=2j} \wedge^i H^1(C_A, \mathbb{C})[-i]) \text{ (resp. IC}(g_1, \oplus_{i=2j+1} \wedge^i H^1(C_A, \mathbb{C})[-i])) \text{ appears in } (pr_{g_1})_*\text{IC}(Z, \mathbb{C}) \text{ (resp. (pr}_{g_1})_*\text{IC}(Z, \mathcal{L})) \text{ as a direct summand (up to shift).}$$

Moreover, these are the only IC complexes with full support appearing in the decomposition.

6. Matching for nilpotent orbits of order two

In this section we describe the Fourier transforms of IC complexes supported on nilpotent orbits of order two more precisely.

For IC sheaves with trivial local systems on such orbits, we have the following:

**Theorem 6.1.** We have $\mathfrak{L}(\text{IC}(\mathcal{O}_{2n-2i+1}, \mathbb{C})) = \text{IC}(g_1, \mathcal{L}_i), i = 0, \ldots, n.$
Similarly, for IC sheaves with nontrivial local systems we have:

**Theorem 6.2.** We have $\mathfrak{F}(\text{IC}(O_{2^2 1 2n+1-2^n}, \mathcal{E}_i)) = \text{IC}(g_1, \mathcal{F}_i)$, $i = 1, \ldots, n$.

We prove Theorem 6.1 below, but give a proof of Theorem 6.2 only for orbits $O_{2^2 1 2n+1-2^n}$ when $i$ is even. We defer the case of odd $i$ to [CVX1] Proposition 5.3.

6.1. **Proof of Theorem 6.2 for $i$ even.** Consider the maps

$$v : \{(x, 0 \subset V_{n-1} \subset V_n \subset V_n^1 \subset V) \mid x \in g_1, x V_n^1 = 0, x V_{n-1}^1 \subset V_n-1 \} := E \to \mathcal{N}_1,$$

$$\bar{v} : \{(x, 0 \subset V_{n-1} \subset V_n \subset V_n^1 \subset V = \mathbb{C}^{2n+1}) \mid x \in g_1, x V_{n-1} \subset V_n^1 \} \to g_1.$$

We have that

$$\dim E = n^2 + 2n - 2, \quad \text{Im} v = \bar{O}_{2^n1}$$

and

$$\mathfrak{F}(v_\ast \mathbb{C}[-]) = \bar{v}_\ast \mathbb{C}[-].$$

In the following we prove the theorem by studying the decompositions of $v_\ast \mathbb{C}[-]$ and $\bar{v}_\ast \mathbb{C}[-]$.

We first study the decomposition of $\bar{v}_\ast \mathbb{C}[-]$. Since in the decomposition of $v_\ast \mathbb{C}[-]$ only IC complexes supported on $O_{2^2 1 2n+1-2^n}$ appear and the Fourier transform of such complexes all have full support (see Theorem 5.1 and Theorem 5.2), we conclude that all IC complexes appearing in the decomposition of $\bar{v}_\ast \mathbb{C}[-]$ are supported on all of $g_1$.

Let $Z = \{(x, 0 \subset V_{n-1} \subset V_n \subset V_n^1 \subset V) \mid x \in g_1, \text{rank}(\bar{x} : V_n \to V/V_n^1) \leq 1 \}$ and $Z_0 = K \times^{\mathbb{P}K} L^1 = \{(x, 0 \subset V_n \subset V_n^1 \subset V) \mid x \in g_1, \text{rank}(\bar{x} : V_n \to V/V_n^1) = 0 \}$ be the varieties introduced in the proof of Proposition 5.7. We have $\dim Z = \dim g_1 + n$ and $\dim Z_0 = \dim g_1$.

We have the following factorization of $\bar{v}$

$$\{(x, 0 \subset V_{n-1} \subset V_n \subset V_n^1 \subset V = \mathbb{C}^{2n+1}) \mid x V_{n-1} \subset V_n^1 \} \xrightarrow{\bar{v}_1} Z \xrightarrow{\bar{v}_2} g_1,$$

where $\bar{v}_1 : (x, V_{n-1} \subset V_n) \mapsto (x, V_n)$ and $\bar{v}_2 : (x, V_n) \mapsto x$. Note that $\bar{v}_2 \mid Z_0 : Z_0 \to g_1$ is equal to the map $\bar{\pi}_{2^n1}$ in (5.3).

The map $\bar{v}_1$ is one-to-one over $Z - Z_1^0$ and is a $\mathbb{P}^{n-1}$-bundle over $Z_0$. It follows that

$$\bar{v}_1 \ast \mathbb{C}[-] = \text{IC}(Z, \mathbb{C}) \oplus \bigoplus_{k=0}^{n-2} \mathbb{C} \mathbb{Z}_0[-][\pm k].$$

Since all IC complexes appearing in the decomposition of $\bar{v}_\ast \mathbb{C}[-]$ are supported on all of $g_1$, Lemma 5.4 and (5.16) imply that

$$(6.1) \quad \bar{v}_\ast \mathbb{C}[-] \cong \bar{v}_2 \ast \bar{v}_1 \ast \mathbb{C}[-] \cong \bigoplus_{k=0}^{n} \text{IC}(g_1, \bigoplus_{s=0}^{k} \mathcal{F}_s) \mid [\pm (n-2k)] \oplus \bigoplus_{a=0}^{n-2} \text{IC}(g_1, \bigoplus_{s=0}^{n} \mathcal{L}_s) \mid [n-2-2a],$$

here $\mathcal{F}_0 = \mathbb{C}$.

---

9 The inverse is given by $(x, V_n) \to (x, V_{n-1} \subset V_n)$ where $V_{n-1} \equiv \text{Ker}(\bar{x} : V_n \to V/V_n^1)$.
Remark 6.3. For \( x \in \mathfrak{g}^r \), the fibers
\[
Z_x := \tilde{v}_2^{-1}(x) = \{(0 \subset V_n \subset V_n^\perp \subset V) \mid \operatorname{rank}(\tilde{x} : V_n \to V/V_n^\perp) \leq 1\}
\]
are the over-generalized Kummer varieties introduced in [Re, p.80]. For example, when \( n = 2 \), the map \( f_x : F_x \to Z_x \) in (5.13) realizes \( Z_x \) as the quotient \( F_x/\sigma \) of the Fano variety \( F_x \) of lines in the complete intersection \( \tilde{Q} \cap \tilde{Q}_x \) of 2 quadrics in \( \mathbb{P}^5 \) by the involution \( \sigma \) (see (5.2)). There are 16 fixed points of \( \sigma \) on \( F_x \) corresponding to 16 singular points of \( Z_x \cong F_x/\sigma \). The Hessenberg variety \( H_x := \tilde{v}^{-1}(x) \) is the blow up of \( Z_x \) at those singular points, which is isomorphic to the Kummer K3 surface coming from the Jac(\( C_x \))-torsor \( F_x \) together with the involution \( \sigma \).

We now study the decomposition of \( \nu_* \mathbb{C}[-] \). Our goal is to prove the following
\[
(6.2) \quad \nu_* \mathbb{C}[n^2 + 2n - 2] \cong \bigoplus_{a=0}^{n-2} \bigoplus_{i=0}^{n} \IC(\mathcal{O}_{2i12n-2i+1}, \mathbb{C})[n - 2 - 2a]
\]
\[
\oplus \bigoplus_{i=1}^{n-2} \bigoplus_{a_i=0}^{n} \IC(\mathcal{O}_{2i12n-4i+2i}, \mathcal{E}_{2i})[n - 2i - 2a_i] \oplus \bigoplus_{a=0}^{n} \IC(\mathcal{O}_{12n+1}, \mathbb{C})[n - 2a].
\]

Taking Fourier transform of (6.1) and using Theorem 5.1, we see that
\[
(6.3) \quad \mathfrak{g}(\bigoplus_{a=0}^{n-2} \bigoplus_{s=0}^{n} \mathcal{L}_{s})[n - 2 - 2a] \cong \bigoplus_{a=0}^{n-2} \bigoplus_{i=0}^{n} \IC(\mathcal{O}_{2i12n-2i+1}, \mathbb{C})[n - 2 - 2a]
\]
is a direct summand of \( \nu_* \mathbb{C}[-] \).

Moreover, these contain all the IC complexes supported on \( \mathcal{O}_{2i12n-2i+1}, i \geq 1 \), with trivial local systems, appearing in the decomposition of \( \nu_* \mathbb{C}[-] \). Now we determine those IC complexes supported on \( \mathcal{O}_{2i12n-2i+1}, i \geq 1 \), with nontrivial local systems \( \mathcal{E}_{i} \), appearing in the decomposition of \( \nu_* \mathbb{C}[-] \).

Let \( x_i \in \mathcal{O}_{2i12n-2i+1}, i \in [1, n] \). We have that \( v^{-1}(x_i) \) is a quadric bundle over \( v_0^{-1}(x_i) \cong \operatorname{OGr}(n-i, 2n-2i+1) \), with fibers quadric of the form \( \sum_{k=1}^{i} a_k^2 = 0 \) in \( \mathbb{P}^{n-1} = \{[a_1 : \ldots : a_n]\} \), and
\[
2 \dim v^{-1}(x_i) = \operatorname{codim}_{E} \mathcal{O}_{x_i} + n - 2 = 2(n - 2) + (n - i)(n - i + 1).
\]
Here \( v_0 = \pi_{2n} \) is the resolution map \( \{(x, 0 \subset V_n \subset V_n^\perp \subset V) \mid xV_n^\perp = 0\} \to \bar{O}_{2n} \) defined in (5.2). We have that (see (5.4))
\[
\nu_{0,*} \mathbb{C}[n^2 + n] \cong \bigoplus_{i=0}^{n} \IC(\mathcal{O}_{2i12n-2i+1}, \mathbb{C})
\]
which implies that
\[
(6.4) \quad H^k(v_0^{-1}(x_j), \mathbb{C}) \cong H^{k-n^2-n}_{x_j} \bigoplus_{i=0}^{n} \IC(\mathcal{O}_{2i12n-2i+1}, \mathbb{C})
\]
We have \( H^{\text{odd}}(v_0^{-1}(x_j), \mathbb{C}) = H^{\text{odd}}(v^{-1}(x_j), \mathbb{C}) = 0 \) and

\[
H^{2k}(v^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-2} H^{2a}(Q_j, \mathbb{C}) \otimes H^{2k-2a}(v_0^{-1}(x_j), \mathbb{C}),
\]

where \( Q_j \) is a quadric of the form \( \sum_{s=1}^{j} a_s^2 = 0 \) in \( \mathbb{P}^{n-1} = \{ [a_1 : \ldots : a_n] \} \). Note that \( H^{2a}(Q_j, \mathbb{C}) = \mathbb{C} \) for \( 0 \leq a \leq n-2 \) if \( j \) is odd, or if \( j \) is even and \( 2a \neq 2n-j-2 \), and for \( j \) even, \( H^{2n-j-2}(Q_j, \mathbb{C}) \cong \mathbb{C} \oplus H_{\text{prim}}^{2n-j-2}(Q_j, \mathbb{C}) \), where \( \dim H_{\text{prim}}^{2n-j-2}(Q_j, \mathbb{C}) = 1 \). Thus

\[
(6.5) \quad H^{2k}(v^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-2} H^{2k-2a}(v_0^{-1}(x_j), \mathbb{C}) \text{ if } j \text{ is odd}
\]

\[
(6.6) \quad H^{2k}(v^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-2} H^{2k-2a}(v_0^{-1}(x_j), \mathbb{C}) \oplus H^{2k-2n+j+2}(v_0^{-1}(x_j), \mathbb{C}) \text{ if } j \text{ is even}.
\]

It follows from (6.4) that

\[
(6.7) \quad \mathcal{H}_{x_j}^{2k-n-2+2n+2} \bigoplus_{a=0}^{n-2} \bigoplus_{i=0}^{n} \text{IC}(\mathcal{O}_{2i;12n-2i+1, \mathbb{C})}[n-2-2a] \cong \bigoplus_{a=0}^{n-2} H^{2k-2a}(v_0^{-1}(x_j), \mathbb{C}).
\]

Thus (6.7) together with (6.5) implies that if \( j \) is odd, then

\[
\mathcal{H}_{x_j}^{2k} \bigoplus_{a=0}^{n-2} \bigoplus_{i=0}^{n} \text{IC}(\mathcal{O}_{2i;12n-2i+1, \mathbb{C})}[n-2-2a] \cong \mathcal{H}_{x_j}^{2k}(v_* \mathbb{C}[-]).
\]

In view of (6.3), we conclude that \( \text{IC}(\mathcal{O}_{2j;12n+1-2j, \mathcal{E}_j}, j \text{ odd}, does not appear in the decomposition of } v_* \mathbb{C}[-].
\]

By the discussion above, we can assume that

\[
(6.8) \quad v_* \mathbb{C}[-] = \bigoplus_{a=0}^{n-2} \bigoplus_{i=0}^{n} \text{IC}(\mathcal{O}_{2i;12n-2i+1, \mathbb{C})}[n-2-2a]
\]

\[
\oplus \bigoplus_{i=1}^{k_i} \bigoplus_{a_i=0}^{[\frac{k_i}{2}]} \text{IC}(\mathcal{O}_{2i+1;12n+1-4i, \mathcal{E}_{2i} \oplus m_{2i}^\delta})[k_i - 2a_i] + \cdots
\]

where the \( m_{a_i}^i \)'s are to be determined and \( \cdots \) is a sum of IC complexes supported at 0. We show that

\[
(6.9) \quad k_i = n - 2i, m_{a_i}^i = m_{n-2i}^i = 1.
\]

Proof of (6.9). Note that in (6.6), \( H^{2k-2n+j+2}(v_0^{-1}(x_j), \mathbb{C}) \neq 0 \) if and only if \( 0 \leq 2k - 2n + j + 2 \leq (n-j)(n-j+1) \), i.e. if and only if

\[
2n - j - 2 \leq 2k \leq (n-j)(n-j+1) + 2n - j - 2,
\]

\[
32
\]
since \( \dim v_0^{-1}(x_j) = (n - j)(n - j - 1)/2 \). Thus for all \( l > (n - 2i)(n - 2i + 1) + 2n - 2i - 2 \), we must have

\[
\mathcal{H}^{l-n^2-2n+2}_{x_{2i}} \bigoplus_{a_i=0}^{k_i} \text{IC}(\mathcal{O}_{2i12n+1-4i}, \mathcal{E}_{2i}^{\oplus m^i_{a_i}})[k_i - 2a_i] = 0,
\]

i.e. for all \( l > n - 2i - \dim \mathcal{O}_{2i12n+1-4i} \), \( \mathcal{H}^{l}_{x_{2i}} \bigoplus_{a_i=0}^{k_i} \text{IC}(\mathcal{O}_{2i12n+1-4i}, \mathcal{E}_{2i}^{\oplus m^i_{a_i}})[k_i - 2a_i] = 0 \). Thus \( k_i \leq n - 2i \). It remains to show that

\[
m^i_0 = m^i_{n-2i} = 1.
\]

We argue using induction on \( 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \). Consider \( i = \left\lfloor \frac{n}{2} \right\rfloor \). If \( n \) is even, then it is easy to see that \( \text{IC}(\mathcal{O}_{2n1}, \mathcal{E}_n) \) is a direct summand. Assume that \( n \) is odd. Take \( j = n - 1 \) in (6.6) we get \( H^{2k-2n+j+2}(v^{-1}_0(x_j), \mathbb{C}) \neq 0 \) if and only if \( 2k = n + 1, n - 1 \) (we have \( v^{-1}_0(x_{n-1}) \) is a nonsingular quadric in \( \mathbb{P}^2 \)) and \( H^{2k-2n+j+2}(v^{-1}_0(x_j), \mathbb{C}) \) gives us a one-dimensional non-trivial local system when \( 2k = n + 1, n - 1 \). In view of (6.3), (6.7) and (6.6) we conclude that

\[
\text{IC}(\mathcal{O}_{2n-113}, \mathcal{E}_{n-1})[-1] \oplus \text{IC}(\mathcal{O}_{2n-113}, \mathcal{E}_{n-1})[1] \text{ is a direct summand of } v_n \mathbb{C}[-].
\]

This proves (6.9) for \( i = \left\lfloor \frac{n}{2} \right\rfloor \).

By induction hypothesis, suppose that (6.9) holds for all \( j < i \). Let \( 2k = (n - 2i)(n - 2i + 1) + 2n - 2i - 2 \). Take the stalk at \( x_{2i} \) of \( \mathcal{H}^{2k-n^2-2n+2} \) in (6.8), we get

\[
\dim \mathcal{H}^{2k-n^2-2n+2}_{x_{2i}} \bigoplus_{a_i=0}^{k_i} \text{IC}(\mathcal{O}_{2i12n+1-4i}, \mathcal{E}_{2i}^{\oplus m^i_{a_i}})[k_i - 2a_i] = 1,
\]

i.e. \( \dim \mathcal{H}^{n-2i-n_{2i-12n+1-4i}} \bigoplus_{a_i=0}^{k_i} \text{IC}(\mathcal{O}_{2i12n+1-4i}, \mathcal{E}_{2i}^{\oplus m^i_{a_i}})[k_i - 2a_i] = 1 \),

here we use (6.6), (6.7) and the fact that

\[
\dim \mathcal{H}^{2k-n^2-2n+2} \bigoplus_{j=i+1}^{n-2j} \bigoplus_{a_j=0}^{k_j} \text{IC}(\mathcal{O}_{2j12n+1-4j}, \mathcal{E}_{2j}^{\oplus m^j_{a_j}})[n - 2j - 2a_j] = 0,
\]

as \( 2k - n^2 - 2n + 2 + 2j(2n + 1 - 2j) + n - 2j - 2a_j \geq 4(j - i)(n + 1 - i - j) > 0 \). Thus we can conclude now that \( m^i_{n-2i} = 1 \) which implies \( m^i_0 = 1 \). This completes the proof of (6.9).

\[\Box\]

Comparing with (6.1), we conclude that

\[
\mathfrak{F}(\text{IC}(\mathcal{O}_{2i12n+1-4i}, \mathcal{E}_{2i})) = \text{IC}(\mathfrak{g}_1, \mathfrak{f}_{2i})
\]

and \( m^i_{a_i} = 1 \) for all \( 0 \leq a_i \leq n - 2i \). This finishes the proof of Theorem 6.2.

Finally taking Fourier transform of (6.8) we obtain the decomposition in (6.2).

**Corollary 6.4.**  
(1) We have \( \mathcal{H}^k_{x_{2i}} \text{IC}(\mathcal{O}_{2i12n-2i+1}, \mathbb{C}) = 0 \) for \( k \) odd and \( j \leq i \).

(2) We have \( \mathcal{H}^k_{x_{2i}} \text{IC}(\mathcal{O}_{2i12n-4i+1}, \mathcal{E}_{2i}) = 0 \) for \( k \) odd and \( j \leq i \).
Proof. Part (1) follows from equation (6.4), the fact that \( n^2 + n \) is even and the fact that \( H^{\text{odd}}(v_{0}^{-1}(x_j), \mathbb{C}) = 0 \). Taking \( \mathcal{H}_x^{k} \) on both sides of the equation (6.2), we see that part (2) follows from the fact that \( n^2 + n \) is even and the fact that \( H^{\text{odd}}(v_{-1}(x_j), \mathbb{C}) = 0 \). \( \square \)

6.2. Proof of Theorem 6.1. Let us start with the matching for the \( \text{IC}((\mathcal{O}_{2^i12n-2i+1}, \mathbb{C}))'s \) with \( i \) odd.

Proposition 6.5. If \( i \) is odd, then \( \tilde{\mathfrak{f}}(\text{IC}(\mathcal{O}_{2^i12n-2i+1}, \mathbb{C})) = \text{IC}(\mathfrak{g}_1, \mathcal{L}_i) \).

Proof. Assume that \( 2m \leq n + 1 \). Let us write \( \mathcal{O}_m = \mathcal{O}_{3^m-12^m-n-3m+2} \). Consider the following resolution map \( \tau_m : \tilde{\mathcal{O}}_m \to \mathcal{O}_m \), where

\[
\tilde{\mathcal{O}}_m = \left\{ (x, 0 \subset V_{m-1} \subset V_m \subset V_{m-1}^\perp \subset V = \mathbb{C}^{2n+1}) \mid x \in \mathfrak{g}_1, xV_m = 0, xV_m \subset V_{m-1} \right\}.
\]

We show that

(6.10) \( \text{IC}(\mathcal{O}_{2^i2m-12n-4m+3}, \mathbb{C}) \) is a direct summand of \( \tau_m \mathcal{C}[-] \).

Recall that \( \mathcal{O}_\lambda \subset \tilde{\mathcal{O}}_\mu \) if and only if \( \mu \geq \lambda \) if and only if \( \lambda^t \geq \mu^t \). Thus we have

\[
\tilde{\mathcal{O}}_{3^i2^i2^i12n+1-3i-2j} \supset \mathcal{O}_{3^i2^i2^i12n+1-3i-2j'} \quad \text{if and only if} \quad i \geq i', 2i + j \geq 2i' + j'.
\]

In particular, \( \mathcal{O}_{2^i2m-12n-4m+3} \subset \tilde{\mathcal{O}}_{3^i2^i2^i12n+1-3i+2j} \subset \mathcal{O}_m \) if and only if \( i \leq m - 1 \) and \( 2i + j = 2m - 1 \).

We show in [CVX1] Lemma 2.4, independently of this paper, that for \( i \in [0, m - 1] \) and \( x_i \in \mathcal{O}_{3^i2^i2m-12i12n-4m+3+i} \), \( \tau_m^{-1}(x_i) \cong \text{OGr}(m - 1 - i, 2m - 1 - 2i) \). It is then easy to check that

(6.11) \[
2 \dim \tau_m^{-1}(x_i) = \text{codim}_m \mathcal{O}_{x_i} = (m - i - 1)(m - i).
\]

Thus (6.10) follows from the discussion above, (6.11) and the decomposition theorem.

Consider the map

\[
\tilde{\tau}_m : \{ (x, 0 \subset V_{m-1} \subset V_m \subset V_{m-1}^\perp \subset \mathbb{C}^{2n+1}) \mid x \in \mathfrak{g}_1, xV_m \subset V_m \} \to \mathfrak{g}_1.
\]

We have that

(6.12) \[
\tilde{\mathfrak{f}}(\tau_m \mathcal{C}[-]) \cong \tilde{\tau}_m \mathcal{C}[-].
\]

By Theorem 5.1, \( \tilde{\mathfrak{f}}(\text{IC}(\mathcal{O}_{2^i2m-112n-4m+3}, \mathbb{C})) = \text{IC}(\mathfrak{g}_1, \mathcal{L}_i) \) for some \( i \in [1, n] \). We show in [CVX1] Remark 4.10, again independently of this paper, that among the \( \text{IC}(\mathfrak{g}_1, \mathcal{L}_i)’s \, (i \geq 1) \), only \( \text{IC}(\mathfrak{g}_1, \mathcal{L}_{2j-1}) \), \( 1 \leq j \leq m \), appear in the decomposition of \( \tilde{\tau}_m \mathcal{C}[-] \). Thus by induction on \( m \), (6.10) and (6.12) imply that

\[
\tilde{\mathfrak{f}}(\text{IC}(\mathcal{O}_{2^i2m-112n-4m+3}, \mathbb{C})) = \text{IC}(\mathfrak{g}_1, \mathcal{L}_{2m-1}).
\]

\( \square \)
So Theorem 6.1 is now reduced to the statement that for even $i$,
\[ F(\text{IC}(O_{2i+2n-2i+1}, \mathbb{C})) = \text{IC}(g_1, L_i). \]
For this we observe that $\dim L \neq \dim L_j$ for $i \neq j$, hence it suffices to show that the generic rank of $F(\text{IC}(O_{2i+2n-2i+1}, \mathbb{C}))$ is equal to $\dim L_i$.

We make use of the theory of characteristic cycles to compute the generic rank of $F(\text{IC}(O_{2i+2n-2i+1}, \mathbb{C}))$. Recall the following facts about characteristic cycles (see [KS]):

1. The Fourier transform preserves the characteristic cycle of conic sheaves.
2. Let $F$ be an irreducible perverse sheave on $g_1$ and let $r$ be the generic rank of $F$. Then the characteristic cycle of $F$, denoted by $\text{CC}(F)$, satisfies
\[ \text{CC}(F) = r \cdot (T^*_0 g_1) + \cdots \]

We will prove the following equality of characteristic cycles.

**Proposition 6.6.** Assume that $i$ is even. We have
\[ \text{CC}(\text{IC}(O_{2i+2n-2i+1}, \mathbb{C})) = \text{CC}(\text{IC}(O_{2i-12n-2i+1}, E_i)) + \text{CC}(\text{IC}(O_{2i-12n-2i+3}, \mathbb{C})). \]

In Theorem 6.2 we have shown that for even $i$, the generic rank of $F(\text{IC}(O_{2i-12n-2i+1}, E_i))$ is $\dim F_i$. This together with the equality above imply Theorem 6.1. Indeed, using Propositions 6.5 and 6.6 we see that
\[ \text{CC}(\text{IC}(O_{2i+2n-2i+1}, \mathbb{C})) = \text{CC}(\text{IC}(g_1, F_i)) + \text{CC}(\text{IC}(g_1, L_{i-1})) = r \cdot (T^*_0 g_1) + \cdots \]
where
\[ r = \dim F_i + \dim L_{i-1} = \binom{2n}{i} - \binom{2n}{i-2} + \binom{2n+1}{i-1} = \binom{2n+1}{i} = \dim L_i. \]
Hence the generic rank of $F(\text{IC}(O_{2i+2n-2i+1}, \mathbb{C}))$ is equal to $\dim L_i$, and the theorem follows.

It remains to prove Proposition 6.6. This is done in the next subsection.

6.3. **Proof of Proposition 6.6.** Recall that for any variety $X$ and $\mathcal{F} \in D(X)$ we can consider the corresponding local Euler characteristic function
\[ \chi(\mathcal{F}) : X \to \mathbb{Z} \]
defined by
\[ \chi(\mathcal{F})_x = \sum (-1)^i \dim(\mathcal{H}^i_x \mathcal{F}). \]
We have the following fact

**Fact 6.7 ([KS], Theorem 9.7.11).** Let $\mathcal{F}_1, \mathcal{F}_2 \in D(X)$. Then $\text{CC}(\mathcal{F}_1) = \text{CC}(\mathcal{F}_2)$ if and only if $\chi(\mathcal{F}_1) = \chi(\mathcal{F}_2)$.

By the fact above we are reduced to show the following:
Proposition 6.8. For even $i$, we have
\[ \chi(\text{IC}(O_{2^i12^{n-2}i}, \mathbb{C})) = \chi(\text{IC}(O_{2^i12^{n-2}i}, E_i)) + \chi(\text{IC}(O_{2^i12^{n-2}i+2}, \mathbb{C})). \]

In [7] Theorem 7.1 and Theorem 7.2 we show that
\begin{align*}
(6.13) & \quad \chi(\text{IC}(O_{2^i12^{n-2}i+1}, \mathbb{C})) = \chi(\text{IC}(O_{2^i12^{n-2}i}, \mathbb{C})) \text{ for all } i \\
(6.14) & \quad \chi(\text{IC}(O_{2^i12^{n-2}i}, E_i)) = \chi(\text{IC}(O_{2^i12^{n-2}i}, E'_i)) \text{ for even } i,
\end{align*}
where $O'_{2^i12^{n-2}i}$ is the nilpotent $Sp(2n)$-orbit in $sp(2n)$ corresponding to the partition $2^i12^{n-2i}$ and $E'_i$ is the unique non-trivial irreducible $Sp(2n)$-equivariant local system on $O'_{2^i12^{n-2}i}$.

As such, we prove the proposition above making use of the classical Springer correspondence for symplectic group $Sp(2n)$. As discussed earlier, this will complete the proof of Theorem 6.1.

Proof of Proposition 6.8 The proof is reduced to proving the following: for even $i$, $j \leq i$, and $x'_j \in O'_{2^i12^{n-2}i}$ we have
\[ (6.15) \quad \chi(\text{IC}(O'_{2^i12^{n-2}i}, \mathbb{C}))(x'_j) = \chi(\text{IC}(O'_{2^i12^{n-2}i}, E'_i))(x'_j) + \chi(\text{IC}(O'_{2^i12^{n-2}i+2}, \mathbb{C}))(x'_j). \]

We will show that
\begin{align*}
(6.16) & \quad \chi(\text{IC}(O'_{2^i12^{n-2}i}, \mathbb{C}))(x'_j) = \chi(\text{IC}(O'_{2^i12^{n-2}i}, \mathbb{C}))(0) \quad \text{if } j \text{ is even}, \\
(6.17) & \quad \chi(\text{IC}(O'_{2^i12^{n-2}i}, \mathbb{C})) = \binom{n-j}{i-j} \quad \text{if } j \text{ is even}, \\
(6.18) & \quad \chi(\text{IC}(O'_{2^i12^{n-2}i}, E'_i))(x'_j) = \binom{n-j}{i-j} - \binom{n-j}{i-j-2} \quad \text{if } j \text{ is odd}, \\
(6.19) & \quad \chi(\text{IC}(O'_{2^i12^{n-2}i}, E'_i))(x'_j) = 0 \quad \text{if } j \text{ is odd}.
\end{align*}

The equality (6.15) follows. \qed

Proof of (6.16). The equality (6.16) follows from (6.13) and Proposition 7.11 (see [7.5]), which states that $\mathcal{H}^k_x IC(O_{2^i12^{n-2}i+1}, \mathbb{C}) = \mathcal{H}^k_{x'} IC(O_{2^i12^{n-2}i+2}, \mathbb{C})$ ($s_j = j(2n+1-j)$). Indeed, since $s_j$ is even, we have
\[ \chi(\text{IC}(O'_{2^i12^{n-2}i}, \mathbb{C}))(x'_j) = \sum (-1)^k \dim \mathcal{H}^k_{x'_j} IC(O'_{2^i12^{n-2}i}, \mathbb{C}) = 0. \]

Proof of (6.17) and (6.18). We work in the classical Springer correspondence setting for $sp(2n)$. Let $\tilde{N} = \mathcal{N}_{sp(2n)}$ and $G = Sp(2n)$ in this proof. Recall that we have the Springer resolution $\varphi : \tilde{N} \to N$ and
\[ (6.20) \quad \varphi_* \mathcal{C}[-] \cong \bigoplus_{(\mathcal{O}, \mathcal{L})} \text{IC}(\mathcal{O}, \mathcal{L}) \otimes V_{\mathcal{O}, \mathcal{L}}. \]
Here (𝒪, ℳ) runs through the pairs consisting of a nilpotent G-orbit 𝒪 ⊂ ℕ and an irreducible
G-equivariant local system on 𝒪, that appear in the Springer correspondence. Moreover 𝒪, ℳ
denotes the irreducible Weyl group representation corresponding to the pair (𝒪, ℳ) under Springer correspondence. It follows that
\begin{equation}
(6.21)
\chi(\text{IC}(𝒪, ℳ))_{x} = [V_{𝒪, ℳ} : H^{∗}(\mathcal{B}_{x}, \mathbb{C})]
\end{equation}
for \(x \in \mathcal{O}\). Here \(\mathcal{B}_{x} = \varphi^{-1}(x)\) is the Springer fiber and \([V_{𝒪, ℳ} : H^{∗}(\mathcal{B}_{x}, \mathbb{C})]\) denotes the multiplicity of \(V_{𝒪, ℳ}\) in the Weyl group representation \(H^{∗}(\mathcal{B}_{x}, \mathbb{C})\).

Let us denote the Weyl group of type \(C_{n}\) by \(W_{n}\). It is well known that the irreducible representations of \(W_{n}\) are parametrized by pairs of partitions \((\alpha)(\beta)\) such that \(|\alpha| + |\beta| = n\). Lusztig in [L1] has computed the (generalized) Springer correspondence explicitly. In our case, we have
\begin{equation}
(6.22)
V_{𝒪_{2i12n−2i−44}, ℳ} = (1^{i})(1^{n−i}), \quad V_{𝒪_{2i12i−112n−4i+2}, ℳ} = (1^{n−i+1})(1^{i+1}),
\end{equation}
\begin{equation}
(6.23)
V_{𝒪_{2i12n−44}, ℳ} = (0)(2^{i}1^{n−2i}),
\end{equation}
and \(\text{IC}(𝒪_{2i12n−4i+2}, ℳ)\) does not appear in the decomposition of \(\varphi_{∗}\mathbb{C}[-]\). It follows from (6.21) and (6.22) that
\begin{equation}
\chi(\text{IC}(𝒪_{2i12n−2i}, \mathbb{C}))_{0} = [V_{𝒪_{2i12n−2i}, ℳ} : H^{∗}(\mathcal{B}, \mathbb{C})] = \dim V_{𝒪_{2i12n−2i}, ℳ} = \left(\frac{n}{i}\right).\end{equation}
This proves (6.17).

We prove (6.18). Recall that \(H^{\text{odd}}(\mathcal{B}_{x}, \mathbb{C}) = 0\) (see [CLP]) and if \(x \in \text{Lie} L\) is regular nilpotent, we have (see [L3])
\begin{equation}
(6.24)
\sum H^{2k}(\mathcal{B}_{x}, \mathbb{C}) = \text{Ind}_{W_{L}}^{W_{n}} \mathbb{C},
\end{equation}
where \(L\) is a Levi subgroup of \(G\) and \(W_{L}\) is the Weyl group of \(L\).

Consider our elements \(x'_{j} \in \mathcal{O}_{2i12n−2j}\), where \(j \leq 2i\). Assume that \(j = 2j_{0}\). We can find a Levi subgroup \(L \subset \text{Sp}(2n)\) such that \(L \cong GL(2) \times \cdots \times GL(2)\) \((j_{0}\text{ copies})\) \(\subset L_{0} = GL(n)\), and \(x'_{j} \in \text{Lie} L\) is regular, where \(L_{0}\) is a Levi subgroup of a maximal parabolic subgroup. We have \(W_{L} \cong S_{2} \times \cdots \times S_{2} \) \((j_{0}\text{ copies})\) \(\subset S_{n}\). Thus
\begin{equation}
\chi(\text{IC}(𝒪_{2i12n−44}, ℳ)_{x'_{j}}) = [V_{𝒪_{2i12n−44}, ℳ} : H^{∗}(\mathcal{B}_{x'_{j}}, \mathbb{C})] = \left(\frac{(0)(2^{i}1^{n−2i}), \text{Ind}_{W_{L}}^{W_{n}} \mathbb{C}}{[(0)(2^{i}1^{n−2i}), \text{Ind}_{W_{L}}^{W_{n}} \mathbb{C}]}ight) \left(\frac{\text{Res}_{W_{n}}^{W_{L}}(0)(2^{i}1^{n−2i}), \text{Ind}_{W_{L}}^{W_{n}} \mathbb{C}}{\text{Res}_{W_{n}}^{W_{L}}(0)(2^{i}1^{n−2i}), \text{Ind}_{W_{L}}^{W_{n}} \mathbb{C}}\right)
\end{equation}
Here in the last two equalities, \(B_{2i01n−2j0}^{GL(n)}\) denotes the Springer fiber of an element in the nilpotent orbit corresponding to the partition \(2i01n−2j0\) in \(\mathfrak{g}(n)\), and \(\chi(\text{IC}(𝒪_{2i1n−2i}, ℳ))_{2i01n−2j0}^{GL(n)}\) is defined analogously in \(GL(n)\). Now it follows from the classical result for \(GL(n)\) that
\begin{equation}
\chi(\text{IC}(𝒪_{2i1n−2i}, ℳ))_{2i01n−2j0}^{GL(n)} = K_{2i1n−2i, 2j01n−2j0} = K_{2i−j01n−2i, 1n−2j0} = \left(\frac{n−2j_{0}}{i−j_{0}}\right) - \left(\frac{n−2j_{0}}{i−j_{0}−1}\right),
\end{equation}
where \(K_{λ, μ}\) denotes the Kostka number. This proves (6.18).
Theorem 7.1. Assume that the proofs of these theorems are independent of sp
orbits of order two in
supported on nilpotent orbits of order two and stalks of the IC sheaves supported on nilpotent

Let
It follows that

Proof of (6.19). Let
, where
is odd. In view of (6.14), it suffices to show that
. We prove this using the decomposition in (6.2) (see (6.1)

\[ v_{j} \mathbb{C}[n^2 + 2n - 2] \cong \bigoplus_{a=0}^{n-2} \bigoplus_{i=0}^{n} \text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1}}, \mathbb{C})[n-2-2a] \]

\[ \oplus \bigoplus_{i=1}^{[\frac{n}{2}]} \bigoplus_{a_i=0}^{n-2i} \text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1-4i}}, \mathcal{E}_{2i})[n-2i-2a_i] \oplus \bigoplus_{a=0}^{n} \text{IC}(\mathcal{O}_{1^{2n+1}}, \mathbb{C})[n-2a] \]

Note that \( \mathcal{H}_{x_j} \text{oddIC}(\mathcal{O}_{2^{i}1^{n-2i+1}}, \mathbb{C}) = \mathcal{H}_{x_j} \text{oddIC}(\mathcal{O}_{2^{i}1^{n-2i+1-4i}}, \mathcal{E}_{2i}) = 0 \) (see Corollary 6.4). Thus

\( \chi(\text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1-4i}}, \mathcal{E}_{2i}), \mathbb{C})_{x_j} \geq 0 \). It follows from the decomposition above that

\[ \chi(H^*(v^{-1}(x_j), \mathbb{C}) = (n-1) \sum_{i=j}^{n} \chi(\text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1}}, \mathbb{C})_{x_j}) + \sum_{j \leq 2i \leq n} \chi(\text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1-4i}}, \mathcal{E}_{2i}))_{x_j}. \]

We have \( \chi(H^*(v^{-1}(x_j), \mathbb{C}) = (n-1)2^{n-j} \) (here \( v^{-1}(x_j) \) is a quadric bundle over \( \text{OGer}(n-j, 2n-2j+1) \) with fibers having euler characteristic \( n-1 \), see \( \text{(6.1)} \). Thus equations \( \text{(6.13)} \)

and \( \text{(6.17)} \) imply that

\[ \sum_{i=j}^{n} \chi(\text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1}}, \mathbb{C})_{x_j}) = \sum_{i=j}^{n} \binom{n-j}{i} = 2^{n-j}. \]

It follows that \( \chi(\text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1-4i}}, \mathcal{E}_{2i}), \mathbb{C})_{x_j} = 0. \)

The proof of Proposition \( \text{(6.6)} \) is complete. This completes the proof of Theorem \( \text{(6.1)} \).

7. Stalks of the IC sheaves on nilpotent orbits of order two

In this section we establish a remarkable isomorphism between stalks of the IC sheaves supported on nilpotent orbits of order two and stalks of the IC sheaves supported on nilpotent orbits of order two in \( \text{sp}(2n) \). More precisely, let \( N_{\text{sp}(2n)} \) denote the nilpotent cone of \( \text{sp}(2n) \). Let \( \mathcal{O}_{2^{i}1^{n-2i}} \subset N_{\text{sp}(2n)} \) denote the nilpotent orbit corresponding to the partition \( 2^{i}1^{n-2i} \). We have \( \dim \mathcal{O}_{2^{i}1^{n-2i}} = \dim \mathcal{O}_{2^{i}1^{n-2i+1}} = i(2n+1-i) \), and for each \( i \geq 1 \), there exits a unique nontrivial irreducible \( Sp(2n) \)-equivariant local system \( \mathcal{E}_{i} \) on \( \mathcal{O}_{2^{i}1^{n-2i}} \).

We have the following theorems (which was used in the proof of Theorem \( \text{(6.1)} \) in \( \text{[6]} \). We remark that the proofs of these theorems are independent of \( \text{[6]} \).

**Theorem 7.1.** For \( x \in \mathcal{O}_{2^{i}1^{n-2i+1}}, x' \in \mathcal{O}_{2^{i}1^{n-2i}}, \) we have

\[ \mathcal{H}_{x}^{k} \text{IC}(\mathcal{O}_{2^{i}1^{n-2i+1}}, \mathbb{C}) \cong \mathcal{H}_{x'}^{k} \text{IC}(\mathcal{O}_{2^{i}1^{n-2i}}, \mathbb{C}). \]

**Theorem 7.2.** Assume that \( i \) is even. For \( x \in \mathcal{O}_{2^{i}1^{n-2i+1}}, x' \in \mathcal{O}_{2^{i}1^{n-2i}}, \) we have

\[ \mathcal{H}_{x}^{k} \text{IC}(\mathcal{O}_{2^{i}1^{n-2i-2}}, \mathcal{E}_{i}) \cong \mathcal{H}_{x'}^{k} \text{IC}(\mathcal{O}_{2^{i}1^{n-2i-2}}, \mathcal{E}_{i}'). \]
Remark 7.3. Note that, in general, the singularities for $\tilde{O}_{2|12n-2i+1}$ and $\tilde{O}'_{2|12n-2i}$ are non-isomorphic. For example, the Euler obstruction $\text{Eu}(0, \mathcal{O}'_{\text{min}})$ for the minimal orbit $\mathcal{O}'_{\text{min}} := \tilde{O}'_{2|12n-2}$ is zero (see [EM]). On the other hand, using
\[
\text{CC}(\text{IC}(\mathcal{O}_{\text{min}}, \mathbb{C})) = \text{CC}(\mathfrak{g}((\text{IC}(\mathcal{O}_{\text{min}}, \mathbb{C}))) = \text{CC}(\text{IC}(\mathfrak{g}, \mathcal{L})))
\]
and $\dim \mathcal{L}_1 = 2n + 1$ (see Theorem 6.1), one can check that $\text{Eu}(0, \mathcal{O}_{\text{min}}) = 2n$, where $\mathcal{O}_{\text{min}} := \tilde{O}_{2|12n-1}$.

7.1. Resolutions. For the proof of Theorem 7.1 and Theorem 7.2 we need several preliminary steps. We begin with the construction of resolutions of $\tilde{O}_{2|12n+1-2i}$ and $\tilde{O}'_{2|12n-2i}$.

For the orbit $\tilde{O}_{2|12n+1-2i} \subset N_1$, $1 \leq i \leq n$, consider Reeder’s resolution map of $\tilde{O}_{2|12n+1-2i}$,
\[
\sigma_i : \{ (x, 0 \subset V_i \subset V_1) \mid x \in \mathfrak{g}_1, xV \subset V_i \} \rightarrow \tilde{O}_{2|12n+1-2i}.
\]
For the orbit $\tilde{O}'_{2|12n-2i} \subset N_{\mathfrak{sp}(2n)}$, we have the following resolution map for $\tilde{O}'_{2|12n-2i}$ (HE)
\[
\tau_i : \{ (x, 0 \subset U_i \subset U_1 \subset U = \mathbb{C}^{2n}) \mid x \in \mathfrak{sp}(2n), xU \subset U_i \} \rightarrow \tilde{O}'_{2|12n-1}.
\]
Here $U$ is a 2n-dimensional vector space equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ such that $\mathfrak{sp}(2n) = \mathfrak{sp}(U, \langle \cdot, \cdot \rangle)$, $\dim U_i = i$ and $U_i^\perp = \{ u \in U \mid \langle u, U_i \rangle = 0 \}$.

We show that for $x \in \tilde{O}_{2|12n-2j+1}$ and $x' \in \tilde{O}'_{2|12n-2j}$, $j \leq i$,
\[
\text{(7.1)} \quad \text{We have } \dim \sigma_i^{-1}(x) = \dim \tau_i^{-1}(x'). \text{ Moreover, } H^*(\sigma_i^{-1}(x), \mathbb{C}) \cong H^*(\tau_i^{-1}(x'), \mathbb{C}).
\]
\[
\text{(7.2)} \quad \text{The action of } A(x) := Z_K(x)/Z_K(x^0) \text{ (resp. } A'(x') := Z_{\mathfrak{sp}(2n)}(x')/Z_{\mathfrak{sp}(2n)}(x'^0)) \text{ on } H^*(\sigma_i^{-1}(x), \mathbb{C}) \text{ (resp. } H^*(\tau_i^{-1}(x'), \mathbb{C})) \text{ is trivial.}
\]
Let $s_{ij} = 2\dim(\sigma_i^{-1}(x_j)) - \text{codim } \tilde{O}_{2|12n-2j+1} = 2(i - j)(n - i)$, where $x_j \in \tilde{O}_{2|12n-2j+1}$. It follows from (7.1) and (7.2) that

Lemma 7.4. We have
\[
\text{(7.3)} \quad (\sigma_i)_* \mathbb{C}[-] = \bigoplus_{j=0}^{s_{ij}} \bigoplus_{k=0}^{i} \text{IC}(\tilde{O}_{2|12n-2j+1}, \mathbb{C}^{t_{ik}})[\pm k]
\]
\[
\text{(7.4)} \quad (\tau_i)_* \mathbb{C}[-] = \bigoplus_{j=0}^{s_{ij}} \bigoplus_{k=0}^{i} \text{IC}(\tilde{O}'_{2|12n-2j}, \mathbb{C}^{l_{ik}})[\pm k],
\]
here $t_{ik} = (t_{ik})' = \delta_{k,0}$.

We prove (7.1) and (7.2). For (7.1), we show that
\[
\text{(7.5)} \quad \sigma_i^{-1}(x) \cong \text{OGr}(i - j, 2n - 2j + 1) \quad \text{and} \quad \tau_i^{-1}(x') \cong \text{SpGr}(i - j, 2n - 2j).
\]
For (7.2), we show that the action of $A(x)$ (resp. $A'(x')$) on $\sigma_i^{-1}(x)$ (resp. $\tau_i^{-1}(x')$) is trivial, thus inducing a trivial action on $H^*(\sigma_i^{-1}(x), \mathbb{C})$ (resp. $H^*(\tau_i^{-1}(x'), \mathbb{C})$).

Let us consider the case of $x \in \tilde{O}_{2|12n+1-2j}$ first. Take a basis $\{ v_l, xv_l, l \in [1, j], w_s, s \in [1, 2n + 1 - 2j] \}$ of $V$ as in Lemma 2.2. In terms of this basis, the fiber $\sigma_i^{-1}(x)$ can be
described as follows. It consists of the flags $0 \subset V_i \subset V_i^\perp \subset \mathbb{C}^{2n+1}$, where $V_i = \text{Span}\{xv_k, \ k \in [1,j]\} \oplus W_{i-j}$ and $W_{i-j} \subset \text{Span}\{w_s, \ s \in [1, 2n+1-2j]\}$ is such that $W_{i-j} \subset W_{i-j}^\perp$. Thus (7.5) holds in this case.

Define $g \in Z_K(x)$ as follows. When $j$ is odd, let $gv_k = -v_k$, $gw_s = w_s$; when $j$ is even, let $gv_1 = v_2$, $gv_2 = v_1$, $gv_k = v_k$, $k \neq 1, 2$, $gw_s = w_s$. (Note that the actions of $g$ on other basis vectors are determined by the property that $g \in Z_K(x)$.) It is easy to see that $g \notin Z_K(x)^0$. Thus we can identify $A(x)$ with $\{g, 1\}$. Now it is easy to see that $A(x)$ fixes each flag in $\sigma^{-1}_i(x)$, thus acting trivially.

The proof for the case of $x' \in O'_{2j12n-2j}$ is entirely similar. There exist vectors $v'_1, \ldots, v'_j, w'_1, \ldots, w'_{n-j}, u'_1, \ldots, u'_{n-j} \in U = \mathbb{C}^{2n}$ such that $x'^2v'_i = 0$, $l \in [1, j]$, $x'u'_s = x'u'_s = 0$, $s \in [1, n-j]$, $U = \text{Span}\{v'_i, x_v' v_i, \ l \in [1, j], w'_s, u'_s, s \in [1, n-j]\}$, and

$$\langle v'_k, v'_l \rangle = \langle v'_k, w'_s \rangle = \langle v'_k, u'_l \rangle = \langle w'_s, u'_l \rangle = 0, \langle v'_k, x_v' v_i \rangle = \delta_{k,l}, \langle w'_s, u'_l \rangle = \delta_{s,l}.$$

(Note that $\langle x'v, w \rangle = -\langle v, x'w \rangle$ as $x' \in \mathfrak{sp}(2n)$.) In terms of this basis, the fiber $\tau^{-1}_i(x')$ can be described as follows. It consists of the flags $0 \subset U_i \subset U_i^\perp \subset \mathbb{C}^{2n}$, where $U_i = \text{Span}\{x'_v v_i, 1 \leq k \leq j\} \oplus W_{i-j}$ and $W_{i-j} \subset \text{Span}\{w'_s, u'_s, 1 \leq s \leq n-j\}$ is such that $W_{i-j} \subset W_{i-j}^\perp$.

Define $g' \in Z_{\mathfrak{sp}(2n)}(x')$ as follows. When $j$ is odd, let $g'v'_k = -v'_k$, $gw'_s = w'_s$, $gu'_s = u'_s$; when $j$ is even, let $g'v'_1 = v'_2$, $g'v'_2 = v'_1$, $g'v'_k = v'_k$, $k' \neq 1, 2$, $g'w'_s = w'_s$, $g'u'_s = u'_s$. As before we can identify $A'(x')$ with $\{g', 1\}$ and it is easy to see that $A'(x')$ fixes each flag in $\tau^{-1}_i(x')$, thus acting trivially.

### 7.2. The maps $\tilde{\sigma}_i$ and $\tilde{\tau}_i$

We preserve the notations from (7.1) For the proof of Theorem 7.2 we need the following auxiliary maps.

For $O'_{2j12n-2j+1} \subset N_1$, consider the map

$$\tilde{\sigma}_i : \{(x, 0 \subset V_{i-1} \subset V_i \subset V_i^\perp \subset V_{i-1}^\perp \subset V) \mid x \in g_1, xV \subset V_i, xV_{i-1} \subset V_{i-1}\} \to \tilde{O}_{2j12n+1-2j}.$$

For $O'_{2j12n-2j} \subset N_{\mathfrak{sp}(2n)}$, consider the following map

$$\tilde{\tau}_i : \{(x', 0 \subset U_{i-1} \subset U_i \subset U_i^\perp \subset U_{i-1}^\perp \subset U) \mid x' \in \mathfrak{sp}_{2n}, x'U \subset U_i, x'U_{i-1} \subset U_{i-1}\} \to \tilde{O}'_{2j12n-2j}.$$

We show that for $x \in O'_{2j12n-2j+1}$ and $x' \in O'_{2j12n-2j}$, $j \leq i$,

(7.6) We have $\dim \tilde{\sigma}_i^{-1}(x) = \dim \tilde{\tau}_i^{-1}(x')$ and $H^*(\tilde{\sigma}_i^{-1}(x), \mathbb{C}) \simeq H^*(\tilde{\tau}_i^{-1}(x'), \mathbb{C})$, the latter isomorphism is compatible with the actions of $A(x)$ and $A'(x')$.

(7.7) For odd $i$, $A(x)$ (resp. $A'(x')$) acts trivially on $H^*(\tilde{\sigma}_i^{-1}(x), \mathbb{C})$ (resp. $H^*(\tilde{\tau}_i^{-1}(x'), \mathbb{C})$).

It follows that
Lemma 7.5. We have
\[
(\bar{\sigma}_i)_* \mathbb{C}[-] = \bigoplus_{k \geq 0} \text{IC}(\mathcal{O}_{\mathbb{P}^{2n-2j+1}}(m_{jk})[\pm k]) \bigoplus_{j \text{ even}, k \geq 0} \text{IC}(\mathcal{O}_{\mathbb{P}^{2n-2j+1}}^{\text{even}}(j_{jk})[\pm k])
\]
(\bar{\tau}_i)_* \mathbb{C}[-] = \bigoplus_{k \geq 0} \text{IC}(\mathcal{O}_{\mathbb{P}^{2n-2j}}^{\text{even}}(m_{jk})[\pm k]) \bigoplus_{j \text{ even}, k \geq 0} \text{IC}(\mathcal{O}_{\mathbb{P}^{2n-2j}}^{\text{even}}(j_{jk})[\pm k]).
\]

We prove (7.6) and (7.7) in the reminder of this subsection.

The fiber $\bar{\sigma}_i^{-1}(x)$ is a quadric bundle over $\sigma_i^{-1}(x) \cong \text{OGr}(i-j, 2n-2j+1)$ with fibers a quadric $\sum_{s=1}^j b_s^2 = 0$ in $\mathbb{P}^{i-1} = \{[b_1 : b_2 : \cdots : b_j]\}$. More precisely, we have an obvious map $\pi : \bar{\sigma}_i^{-1}(x) \to \sigma_i^{-1}(x)$,
\[
(0 \subset V_{i-1} \subset V_i \subset V_i^\perp \subset V_{i-1}^\perp \subset V) := (V_{i-1} \subset V_i) \mapsto (0 \subset V_i \subset V_i^\perp \subset V) := (V_i)
\]
by forgetting $V_{i-1}$. Now we describe the fibers of $\pi$. Recall that if $(V_i) \in \sigma_i^{-1}(x)$, then there exists $W_{i-j} \subset \text{Span}\{w_k, k \in [1, 2n+1-2j]\}$ with $W_{i-j} \subset W_{i-j}'$ such that $V_j = \text{Span}\{x_v, k \in [1,j]\} \oplus W_{i-j}$. Let $[b_1 : b_2 : \cdots : b_j]$ be the homogenous coordinates of $\mathbb{P}(V_i)$ given by the basis $\{x_v, k \in [1, j], \bar{w}_l, l \in [1, i-j]\}$ of $V_i$, where $\{\bar{w}_l\}$ is a basis of $W_{i-j}$. It is easy to check that the fibers of $\bar{\pi}$ are isomorphic to the quadric $Q : \sum_{s=1}^j b_s^2 = 0$ in $\mathbb{P}(V_i) \cong \mathbb{P}^{i-1}$. It follows that
\[
H^*(\bar{\sigma}_i^{-1}(x), \mathbb{C}) \cong H^*(Q, \mathbb{C}) \otimes H^*(\sigma_i^{-1}(x), \mathbb{C}).
\]

We describe the action of $A(x)$ on $H^*(\bar{\sigma}_i^{-1}(x), \mathbb{C})$. As we have shown that $A(x)$ acts trivially on $H^*(\sigma_i^{-1}(x), \mathbb{C})$, it suffices to describe the action of $A(x)$ on $H^*(Q, \mathbb{C})$.

We claim that if $j$ is odd, then $A(x)$ acts on $H^*(Q, \mathbb{C})$ trivially, thus acting trivially on $H^*(\bar{\sigma}_i^{-1}(x), \mathbb{C})$, and if $j$ is even, then $A(x)$ acts on $H^{2k}(Q, \mathbb{C})$ trivially if $2k \neq 2i-j-2$ and $H^{2i-j-2}(Q, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{E}$. This follows from the following lemma.

Lemma 7.6. Let $Q$ be the quadric given by the equation $\sum_{i=1}^n a_i^2 = 0$ in $\mathbb{P}^{n-1}$ with coordinates $[a_1, \ldots, a_n]$ and consider the automorphism $\gamma$ of $\mathbb{P}^{n-1}$ given by $[a_1, a_2, a_3, \ldots, a_n] = [a_2, a_1, a_3, \ldots, a_n]$. If $k$ is odd, then $\gamma$ acts trivially on $H^*(Q, \mathbb{C})$. If $k$ is even, $\gamma$ acts trivially on $H^2(Q)$ for $j \neq 2n-k-2$ and the action on the two dimensional space $H^{2n-k-2}(Q, \mathbb{C})$ has eigenvalues $1$ and $-1$.

Proof. The quadric $Q$ is the join of the nonsingular quadric $\bar{Q}$ in $\mathbb{P}^{k-1} \subset \mathbb{P}^{n-1}$ given by $\sum_{i=1}^k a_i^2 = 0$ and the linear subspace $L$ of dimension $n-k-1$ given by $a_1 = \cdots = a_k = 0$. Now $Q - L$ is an affine space bundle over $\bar{Q}$ of fiber dimension $n-k$. Thus, $H^i(Q - L, \mathbb{C}) = H^{i-2n+2k}(\bar{Q}, \mathbb{C})$. As $\bar{Q}$ and $L$ only have (compactly supported) cohomology in even degrees we conclude that
\[
H^i(Q, \mathbb{C}) = H^{i-2n+2k}(\bar{Q}, \mathbb{C}) \oplus H^i(L, \mathbb{C})
\]
The automorphism $\gamma$ of course acts trivially on the cohomology of $\mathbb{P}^{n-1}$ and hence it acts trivially on the cohomology of $L$ (it even acts trivially on $L$ itself). Thus we are reduced to consider the action of $\gamma$ on the cohomology of $\bar{Q}$. The action on the non-primitive cohomology
is trivial and so the only possibly nontrivial action is on $H_{prim}^{k-2}(\tilde{Q}, \mathbb{C})$. If $k$ is odd, this group is zero and so we are reduced to the case of $k$ even.

Assume that $k$ is even and $k = 2k_0$. The variety of $(k_0 - 1)$-planes contained in $\tilde{Q}$ has two disjoint irreducible components, which can be identified with the two disjoint irreducible components of the variety of maximal isotropic spaces in $\mathbb{C}^{2k_0}$ (equipped with the standard bilinear form). It is clear that $\gamma$, regarded as an element in $O_{2k_0} - SO_{2k_0}$, interchanges these two irreducible components. Now Reid in [Re, Theorem 1.12] has shown that $H^{k-2}(\tilde{Q}, \mathbb{C}) = \text{span}(a, b)$, where $a$ and $b$ are the classes of $(k_0 - 1)$-planes from the two families respectively. Thus our lemma follows.

The fiber $\tilde{\tau}_i^{-1}(x')$ is a quadric bundle over $\tau_i^{-1}(x') \cong \text{SpGr}(i - j, 2n - 2j)$ with fibers a quadric $\sum_{s=1}^j b_s^2 = 0$ in $\mathbb{P}^{i-1}$. Entirely similarly, one checks that (1) and (2) hold and that the action of $A'(x')$ on $\tilde{\tau}_i^{-1}(x')$ is the same as that of $A(x)$ on $\tilde{\tau}_i^{-1}(x)$.

7.3. Proof of Theorem 7.1. Let $\sigma_i$, $\tau_i$ be the resolution of $\mathcal{O}'_{2^i(2n-2i+1)}$, $\mathcal{O}'_{2^i(2n-2i+2)}$, defined in (7.3). We begin with the following lemma.

**Lemma 7.7.** Let $i \in [1, n]$. Assume that $t_{jk}' = (t_{jk})'$ for all $j, k$, and all $i' \leq i$ in (7.3) and (7.4). Then we have

$$
\mathcal{H}^i_x \text{IC}(\mathcal{O}'_{2^i(2n-2i+2)}, \mathbb{C}) \simeq \mathcal{H}^i_x \text{IC}(\mathcal{O}'_{2^i(2n-2i+1)}, \mathbb{C})
$$

for $x \in \mathcal{O}_{2^i(2n-2i+1)}$, $x' \in \mathcal{O}'_{2^i(2n-2i+2)}$, and $j \leq i$.

**Proof.** We prove the lemma by induction on $i$. The case when $i = 1$ is clear. In fact, we only need to check the conclusion of the lemma for $j = 0$. We have

$$(\sigma_i)_* \mathbb{C}[-] = \text{IC}(\mathcal{O}_{2^i2n-2i+1}, \mathbb{C}) \oplus \bigoplus_{k \geq 0} \text{IC}(\mathcal{O}_{2^i2n-2i+1}, \mathbb{C}^{t_{0k}})[\pm k].$$

$$(\tau_i)_* \mathbb{C}[-] = \text{IC}(\mathcal{O}'_{2^i2n-2i+2}, \mathbb{C}) \oplus \bigoplus_{k \geq 0} \text{IC}(\mathcal{O}'_{2^i2n-2i+2}, \mathbb{C}^{t_{0k}}')[\pm k].$$

It is clear that $\mathcal{H}^i_0 \text{IC}(\mathcal{O}_{2^i2n-2i+1}, \mathbb{C}) \cong \mathcal{H}^i_0 \text{IC}(\mathcal{O}'_{2^i2n-2i+2}, \mathbb{C})$ as they are determined by the cohomology of $\sigma_i^{-1}(0) \cong \tau_i^{-1}(0)$ and the numbers $(t_{0k}) = (t_{0k})'$ in the same way.

By induction hypothesis, we can assume that for $s < i$

$$(7.10) \quad \mathcal{H}^i_x \text{IC}(\mathcal{O}_{2^s(2n-2s+1)}, \mathbb{C}) \cong \mathcal{H}^i_x \text{IC}(\mathcal{O}'_{2^s(2n-2s+2)}, \mathbb{C}).$$

Recall that

$$
(\sigma_i)_* \mathbb{C}[-] = \text{IC}(\mathcal{O}_{2^i2n-2i+1}, \mathbb{C}) \oplus \bigoplus_{s<i} \text{IC}(\mathcal{O}_{2^s2n-2s+1}, \mathbb{C}^{t_{ss}})[\pm k].
$$

This implies that the stalks of $\text{IC}(\mathcal{O}_{2^i2n-2i+1}, \mathbb{C})$ are uniquely determined by i) the stalks of $\text{IC}(\mathcal{O}_{2^i2n-2i+1}, \mathbb{C})$ for $s < i$, ii) the cohomology groups of the fibers of the map $\sigma_i$, and iii) the numbers $t_{ss}$, $s < i$. Similarly, the stalks of $\text{IC}(\mathcal{O}'_{2^i2n-2i+2}, \mathbb{C})$ are uniquely determined, in the same way, by i) the stalks of $\text{IC}(\mathcal{O}'_{2^i2n-2i+2}, \mathbb{C})$ for $s < i$, ii) the cohomology groups of the
fibers of the map \( \tau_i \), and iii) the numbers \((t^i_{jk})', s < i\). Now the desired claim follows form (7.10), (7.11), and the assumption that \( t^i_{jk} = (t^i_{jk})' \). \( \square \)

By the lemma above, to prove Theorem 7.1 it suffices to show the following

**Lemma 7.8.** We have \( t^i_{jk} = (t^i_{jk})' \).

**Proof.** We argue by induction on \( t \). The case when \( i = 1 \) is easy to check, i.e. we have that \( t^1_{0k} = (t^1_{0k})' = 1 \) when \( k \in [0, s_{1j}] \) and \( k \) is even, and \( t^1_{0k} = (t^1_{0k})' = 0 \) otherwise.

So by induction hypothesis we can assume that for all \( s < i \), \( t^s_{jk} = (t^s_{jk})' \). By lemma 7.7 we have for \( s < i \),

\[
(7.11) \quad \mathcal{H}_x^k \text{IC}(\mathcal{O}_{2^j12n-2+1}, \mathbb{C}) \simeq \mathcal{H}_x^k \text{IC}(\mathcal{O}_{2^j'12n-2+1}, \mathbb{C})
\]

for \( x \in \mathcal{O}_{2^j12n-2+1}, x' \in \mathcal{O}_{2^j'12n-2} \) and \( j \leq s \).

We show that \( t^j_{jk} = (t^j_{jk})' \) by induction on \( j \). The case \( j = i \) is clear. So by induction, we can assume that \( t^j_{jk} = (t^j_{jk})' \) holds for \( j < j' \leq i \). Then, for \( x_j \in \mathcal{O}_{2^j12n-2+1} \), we have

\[
(7.12) \quad (\sigma_i)_x \mathbb{C}[-] | x_j \simeq \text{IC}(\mathcal{O}_{2^j12n-2+1}, \mathbb{C}) \oplus \bigoplus_{j < j' < i} \text{IC}(\mathcal{O}_{2^j'12n-2+1}, \mathbb{C}^{t^j_{jk}})[\pm k] \big| x_j .
\]

Since the stalk IC(\( \mathcal{O}_{2^j12n-2+1}, \mathbb{C} \)|\( x_j \) is concentrated in degree \( < -\dim \mathcal{O}_{2^j12n-2+1} \) and IC(\( \mathcal{O}_{2^j12n-2+1}, \mathbb{C}^{t^j_{jk}} \)|\( x_j \) is concentrated in degree \( \geq -\dim \mathcal{O}_{2^j12n-2+1} \), the decomposition in (7.12) implies the multiplicity numbers \( t^j_{jk} \) are uniquely determined by i) the cohomology of \( \sigma_i^{-1}(x_j) \) and ii) the stalks IC(\( \mathcal{O}_{2^j'12n-2+1}, \mathbb{C}^{t^j_{jk}} \)|\( x_j \) for \( j < j' < i \). The numbers \( t^j_{jk} \)'s are determined, in the same manner, by the corresponding data. Now since \( t^i_{jk} = (t^i_{jk})' \) for \( j' > j \) (by induction) the desired equality \( t^i_{jk} = (t^i_{jk})' \) follows from (7.11). \( \square \)

This completes the proof of Theorem 7.1

7.4. **Proof of Theorem 7.2.** Let \( \tilde{\sigma}_i, \tilde{\tau}_i \) be the maps introduced in 7.2. We only make use of the maps when \( i \) is even. We begin with the following lemma

**Lemma 7.9.** Let \( i \in [1, n] \) be even. Assume that \( m^i_{jk} = (m^i_{jk})' \), \( a^i_{jk} = (a^i_{jk})' \) for all \( i' < i \) even and all \( j, k \) in (7.8) and (7.9). Then we have

\[
\mathcal{H}_x^i \text{IC}(\mathcal{O}_{2^j12n-2+1}, \mathcal{E}_i) \simeq \mathcal{H}_x^i \text{IC}(\mathcal{O}_{2^j12n-2}, \mathcal{E}_i')
\]

for \( x \in \mathcal{O}_{2^j12n-2+1}, x' \in \mathcal{O}_{2^j12n-2}, \) and \( j \leq i \).

**Proof.** The proof is entirely similar to that of Lemma 7.7. We argue by induction on even \( i \) and we use Theorem 7.1. Note that if \( i \) is even, then in the decomposition of \( (\tilde{\tau}_i)_x \mathbb{C}[-], (\tilde{\sigma}_i)_x \mathbb{C}[-] \), we have \( a^i_{ik} = (a^i_{ik})' = \delta_{0,k} \). Hence

\[
(\tilde{\sigma}_i)_x \mathbb{C}[-] = \text{IC}(\mathcal{O}_{2^j12n-2u+1}, \mathcal{E}_i) \oplus \bigoplus_{j < i, j \text{ even}} \text{IC}(\mathcal{O}_{2^j12n-2+1}, \mathcal{E}_j^{a^i_{ik}})[\pm k]
\]

We have \( (\tilde{\tau}_i)_x \mathbb{C}[-] = \text{IC}(\mathcal{O}_{2^j12n-2}, \mathcal{E}_i) \oplus \bigoplus_{j < i, j \text{ even}} \text{IC}(\mathcal{O}_{2^j12n-2+1}, \mathcal{E}_j)[\pm k] \).
Now the same type of induction argument as in the proof of Lemma 7.7, (7.6), and the assumption $m^i_{jk} = (m^i_{jk})'$, $a^i_{jk} = (a^i_{jk})'$, imply the desired isomorphism. \hfill \Box

By the lemma above, to prove Theorem 7.2 it suffices to show the following

**Lemma 7.10.** We have $m^i_{jk} = (m^i_{jk})'$ and $a^i_{jk} = (a^i_{jk})'$.

**Proof.** We argue by induction on $i$, entirely similarly as in the proof of Lemma 7.8. Consider the complex

\[
(\bar{\sigma}_i)_* \mathbb{C}[-] \mid_{x_j} = IC(\mathcal{O}_{\mathcal{L}_{12n-2j+1}}, \mathcal{E}_i) \mid_{x_j} \bigoplus \bigoplus_{j \leq j' < i} IC(\mathcal{O}_{\mathcal{L}_{12n-2j'+1}}, \mathcal{E}^{a^i_{jk}'}_{j'}) \mid_{x_j} \bigoplus \bigoplus_{j \leq j' < i} IC(\mathcal{O}_{\mathcal{L}_{12n-2j'+1}}, \mathcal{C}^{m^i_{jk}'}_{j'}) \mid_{x_j}
\]

Observe that since $IC(\mathcal{O}_{\mathcal{L}_{12n-2j+1}}, \mathcal{E}_i) \mid_{x_j}$ is concentrated in degree $< -\dim \mathcal{O}_{\mathcal{L}_{12n-2j+1}}$ and both $IC(\mathcal{O}_{\mathcal{L}_{12n-2j+1}}, \mathbb{C}^{m^i_{jk}'}_{j'}) \mid_{x_j}$, $IC(\mathcal{O}_{\mathcal{L}_{12n-2j+1}}, \mathcal{E}^{a^i_{jk}'}_{j'}) \mid_{x_j}$ are concentrated in degree $\geq -\dim \mathcal{O}_{\mathcal{L}_{12n-2j+1}}$, the above decomposition implies that the multiplicities $m^i_{jk}, a^i_{jk}$ are uniquely determined by the cohomology of $\bar{\sigma}_i^{-1}(x_j)$ (as an representation of the $A(x_j)$) and the stalks $IC(\mathcal{O}_{\mathcal{L}_{12n-2j'+1}}, \mathbb{C}^{m^i_{jk}'}_{j'}) \mid_{x_j}$ and $IC(\mathcal{O}_{\mathcal{L}_{12n-2j'+1}}, \mathcal{E}^{a^i_{jk}'}_{j'}) \mid_{x_j}$ for $j < j' < i$. Again, the numbers $(m^i_{jk})'$, $(a^i_{jk})'$ are determined, in the same manner, by the corresponding data for $\bar{\tau}_i$.

Now by induction, we can assume $m^s_{jk} = (m^s_{jk})'$, $a^s_{jk} = (a^s_{jk})'$ for $s < i$. By Lemma 7.3 we can further assume the stalks for the intersection cohomology complexes $IC(\mathcal{O}_{\mathcal{L}_{12n-2j'+1}}, \mathcal{E}_{j'})$ and $IC(\mathcal{O}_{\mathcal{L}_{12n-2j'+1}}, \mathcal{E}_{j'})$ are the same for $j' < i$ and $j'$ even. Since the action of $A(x_j)$ on the fibers $\bar{\sigma}^{-1}(x_j)$ and $\bar{\tau}^{-1}(x_j)$ are isomorphic by (7.6), the identity $m^i_{jk} = (m^i_{jk})'$, $a^i_{jk} = (a^i_{jk})'$ follows from the observation above.

This completes the proof of Theorem 7.2.

### 7.5. Reduction of stalks

In this subsection we prove the following proposition which was used in the proof of (6.16). Again, the proof is independent of results from 6.1.

**Proposition 7.11.** Let $j \leq i$ and set

\[
s_j = \dim \mathcal{O}_{\mathcal{L}_{12n-2n+1}} - \dim \mathcal{O}_{\mathcal{L}_{12n-2j+1}} = j(2n+1-j).
\]

Then for $x \in \mathcal{O}_{\mathcal{L}_{12n-2j+1}}$, we have

\[
\mathcal{H}^s_x IC(\mathcal{O}_{\mathcal{L}_{12n-2i+1}}, \mathbb{C}) = \mathcal{H}^{s+k+s_j}_x IC(\mathcal{O}_{\mathcal{L}_{12n-2i+1}}, \mathbb{C})
\]

The proof of Proposition 7.11 is quite similar to that of Theorem 7.1. We start with the following lemma. Recall the multiplicity numbers $(t^i_{\bar{l},k})$ in (7.3). In the remainder of this
subsection let us write the maps as $\sigma^n_i$ and the numbers as $(t^i_{l,k})_n$ to indicate that the ambient symmetric pair is $(SL(2n+1), SO(2n+1))$.

**Lemma 7.12.** Let $i \in [1,n]$. Assume that $(t^i_{l,k})_n = (t^{i-j}_{l-j,k})_{n-j}$ for all $j \leq l < i'$ and all $i' \leq i$. Then

$$H^k_x IC(\mathcal{O}_{2i'2n-2i'+1}, \mathbb{C}) = H^0 IC(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C})$$

for $x \in \mathcal{O}_{2i'2n-2i'+1}$.

**Proof.** The conclusion of the lemma obviously holds in the case when $i = 1$. By induction on $i$, we can assume that

$$(7.13) \quad H^k_x IC(\mathcal{O}_{2i'2n-2i'+1}, \mathbb{C}) = H^0 IC(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C})$$

for $s < i$. Consider the following decompositions

$$(7.14) \quad (\sigma^n_i)_* \mathcal{C}[-]_x = IC(\mathcal{O}_{2i'2n-2i'+1}, \mathbb{C})|_x \oplus \bigoplus_{j \leq l < i} IC(\mathcal{O}_{2i'2n-2i'+1}, \mathbb{C}^{(t^i_{l,k})}_n)[\pm k]|_x.$$  

$$(7.15) \quad (\sigma^{n-j}_i)_* \mathcal{C}[-]_0 = IC(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C})|_0 \oplus \bigoplus_{j \leq l < i} IC(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C}^{(t^i_{l,k})}_{n-j})[\pm k]|_0.$$  

Since $(t^i_{l,k})_n = (t^{i-j}_{l-j,k})_{n-j}$ by assumption, by taking $H^k_x$ (resp. $H^0$) on both sides of (7.14) (resp. (7.15)) and using (7.13), we see that

$$H^k_x IC(\mathcal{O}_{2i'2n-2i'+1}, \mathbb{C}) \simeq H^0 IC(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C})$$

if and only if

$$(7.16) \quad H^{k+d^n_i}((\sigma^n_i)^{-1}(x), \mathbb{C}) \simeq H^{k+d^{n-j}_i+s_j}(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C}^{(t^i_{l,k})}_{n-j})^{-1}(0), \mathbb{C}).$$

Here $d^n_i := \dim \mathcal{O}_{2i'2n-2i'+1}$. But since $(\sigma^n_i)^{-1}(x) \simeq (\sigma^{n-j}_i)^{-1}(0) \simeq OGr(i - j, 2n - 2j + 1)$, (7.16) is true. We are done.

\[\square\]

The proof of Proposition 7.11 is reduced to the proof of the following lemma.

**Lemma 7.13.** We have $(t^i_{l,k})_n = (t^{i-j}_{l-j,k})_{n-j}$ for $j \leq l < i$.

**Proof.** We argue by induction on $i$. For $i = 1$ it is clear. By induction hypothesis we can assume that $(t^i_{l,k})_n = (t^{i-j}_{l-j,k})_{n-j}$ for $s < i$ and by Lemma 7.12 we can further assume that

$$H^k_x IC(\mathcal{O}_{2i'2n-2i'+1}, \mathbb{C}) = H^0 IC(\mathcal{O}_{2i-j2n-2i+1}, \mathbb{C})$$

for $s < i$.

We show that $(t^i_{l,k})_n = (t^{i-j}_{l-j,k})_{n-j}$ by induction on $l$. Clearly for $l = i$ we have $(t^i_{l,k})_n = (t^{i-j}_{l-j,k})_{n-j} = \delta_{0,k}$. So by induction, we can assume that $(t^{i}_{l',k})_n = (t^{i-j}_{l'-j,k})_{n-j}$ for $l < l' \leq i$. ...
Let \( x \in \Omega_{2|12n-2l+1}, \quad x' \in \Omega_{2|j12n-2l+1}. \) Since \( \text{IC}(\Omega_{2|12n-2l+1}, \mathbb{C})|_x \) is concentrated in degree \( < -d^m_l \), taking \( \mathcal{H} \mathcal{K}_{l+k} \) on both side of (7.15), we get
\[
(t^i_{lk})_n = \dim H^{k-d^m_l + d^n_i}((\sigma^n_i)^{-1}(x), \mathbb{C}) - \dim \left( \bigoplus_{l<d^m_l < i} \mathcal{H}^{k-d^m_l}_{l<k} \bigoplus_{k'} \text{IC}(\Omega_{2|12n-2l+1}, \mathbb{C})(t^i_{l+k'})[\pm k'] \right)
\]
\[
= \dim H^{k-d^m_l - d^n_i}(\sigma^n_i)^{-1}(x', \mathbb{C})
\]
\[
- \dim \left( \bigoplus_{l<d^m_l < i} \mathcal{H}^{k-d^m_l}_{l<k} \bigoplus_{k'} \text{IC}(\Omega_{2|12n-2l+1}, \mathbb{C})(t^i_{l+k'})[\pm k'] \right) = (t^i_{l-j,k})_n-j.
\]
This finishes the proof of the lemma. \( \square \)

8. Cohomology of Fano varieties of \( k \)-planes in complete intersections

In this section we compute, as an application of our results, the cohomology of Fano varieties of \( k \)-planes in the smooth complete intersection of two quadrics in \( \mathbb{P}^{2n} \); we denote these Fano varieties by \( \text{Fano}^2_{k,n} \). Note that \( \text{Fano}^2_0 \) is the smooth complete intersection of two quadrics in \( \mathbb{P}^{2n} \).

Let \( \text{Gr}(k, n) \) denote the Grassmannian variety of \( k \)-dimensional subspaces in \( \mathbb{C}^n \). Let
\[
g_{k,n}(q) := \sum \dim H^{2l}(\text{Gr}(k, n), \mathbb{C}) q^l = \frac{\prod_{l=n-k+1}^{n} (1 - q^l)}{\prod_{l=1}^{n} (1 - q^l)}
\]
be the Poincare polynomial of \( \text{Gr}(k, n) \).

Recall the monodromy representations \( L_i \) which were defined in [5]. The cohomology of the Fano variety \( \text{Fano}^2_{1,n-1} \) is described as follows.

**Theorem 8.1.** We have
\[
H^{2k+1}(\text{Fano}^2_{1,n-1}, \mathbb{C}) = 0,
\]
\[
H^{2k}(\text{Fano}^2_{1,n-1}, \mathbb{C}) \cong \bigoplus_{j=0}^{i} L_i \otimes M_i(k,j),
\]
where \( M_i(k,j) \) is the coefficient of \( q^{k-j(n-i)} \) in \( g_{i-j,2n-i-j}(q) \).

8.1. **Fano varieties and resolutions for \( \bar{\Omega}_{2|12n-2l+1} \).** We start with the following simple observation, which is a direct consequence of Theorem 6.1.

Let \( \pi : \Sigma \to \mathfrak{g}^{rs}_1 \) be a family of smooth projective varieties over \( \mathfrak{g}^{rs}_1 \) and let \( R^k \pi_* \mathbb{C} \) be the corresponding local system on \( \mathfrak{g}^{rs}_1 \). Suppose that the Fourier transform of \( \text{IC}(\mathfrak{g}_1, R^k \pi_* \mathbb{C}) \) is supported on \( \bar{\Omega}_{2n-1} \) and is given by
\[
\mathfrak{F}(\text{IC}(\mathfrak{g}_1, R^k \pi_* \mathbb{C})) = \bigoplus_{j=0}^{n} \text{IC}(\bar{\Omega}_{2|12n-2l+1}, \mathbb{C}^{mk_j}).
\]
Then the cohomology of the fiber $\Sigma_x := \pi^{-1}(x)$ over $x \in \mathfrak{g}^{rs}$ satisfies

$$H^k(\Sigma_x, \mathbb{C}) \cong \bigoplus_{j=0}^n L_j^{\oplus m_k}. $$

Moreover, the isomorphism above is compatible with the monodromy actions.

Let us apply this observation to the following situation. Consider the maps

$$\tilde{\sigma}_i : \{(x, 0 \subset V_i \subset V_i^\perp \subset \mathbb{C}^{2n+1}) \mid x \in \mathfrak{g}_1, xV_i \subset V_i^\perp\} \to \mathfrak{g}_1. $$

Note that for $x \in \mathfrak{g}_1^{rs}$, we have $\tilde{\sigma}_i^{-1}(x) \cong \text{Fano}_{2n_{i-1}}$, the Fano variety of $(i - 1)$-planes in the smooth complete intersection of two quadrics $Q(v) = 0$ and $\langle xv, v \rangle_Q = 0$ in $\mathbb{P}^{2n}$.

Let us consider $\pi_i = \tilde{\sigma}_i \mid \tilde{\sigma}_i^{-1}(\mathfrak{g}_1^{rs})$, which is a smooth family of Fano varieties, and consider the corresponding local system $R^k\pi_\ast \mathbb{C}$. Recall that we have Reeder’s resolutions $\sigma_i$ for $\bar{O}_{2j12n-2j+1}$ (see §7.1) and

$$\sigma_i \ast \mathbb{C}[i(2n + 1 - i)] = \bigoplus_{j=0}^i \bigoplus_{k=0}^{2(i-j)(n-i)} \text{IC}(\bar{O}_{2j12n-2j+1}, \mathbb{C}^{t_j})[\pm k]. $$

Since the Fourier transform of $\text{IC}(\bar{O}_{2j12n-2j+1}, \mathbb{C})$ is supported on all of $\mathfrak{g}_1$ for all $j$ (Theorem 6.1), the equation

$$\mathcal{F}(\tilde{\sigma}_i \ast \mathbb{C}[-]) \cong \sigma_i \ast \mathbb{C}[-]$$

implies that

$$\mathcal{F}(\tilde{\sigma}_i \ast \mathbb{C}[-]) \cong \bigoplus_{k=0}^{4i(n-i)} \mathcal{F}(\text{IC}(\mathfrak{g}_1, R^k\pi_\ast \mathbb{C})[-k + 2i(n - i)])$$

$$\cong \bigoplus_{j=0}^i \bigoplus_{k=0}^{2(i-j)(n-i)} \text{IC}(\bar{O}_{2j12n-2j+1}, \mathbb{C}^{t_j})[\pm k]. $$

Here $2i(n - i) = \dim \tilde{\sigma}_i^{-1}(x) = \dim \text{Fano}_{2n_{i-1}}$, for $x \in \mathfrak{g}_1^{rs}$.

Hence we see that $\mathcal{F}(\text{IC}(\mathfrak{g}_1, R^k\pi_\ast \mathbb{C}))$ is supported on $\bar{O}_{2j12n-2j+1}$, and has the form

$$\mathcal{F}(\text{IC}(\mathfrak{g}_1, R^k\pi_\ast \mathbb{C})) = \bigoplus_{j=0}^i \text{IC}(\bar{O}_{2j12n-2j+1}, \mathbb{C}^{t_j}[2i(n-i)-k]). $$

So by the observation above and Theorem 6.1, we deduce that the cohomology of the Fano varieties $\text{Fano}_{2n_{i-1}}$ is given by

$$H^k(\text{Fano}_{2n_{i-1}}, \mathbb{C}) \cong \bigoplus_{j=0}^i L_j^{\oplus t_j[2i(n-i)-k]}.$$
8.2. The numbers $t_{jk}^i$. In this subsection let us again use the notation $(t_{jk}^i)_n$ for the numbers $(t_{jk}^i)$ in (8.3) to indicate that the ambient symmetric pair is $(SL(2n+1), SO(2n+1))$.

**Lemma 8.2.** We have $(t_{jk}^i)_n = 0$ for odd $k$.

*Proof.* In the decomposition (8.3), we take the stalk $\mathcal{H}^l_{x_j}$ on both sides for odd $l$, where $x_j \in O_{2j12n+1-2j}$. Since $i(2n-i+1)$ is even and $H^{odd}(\sigma_i^{-1}(x_j), \mathbb{C}) = 0$, we have that $H^l_{x_j} \sigma_i \mathbb{C}[i(2n+1-i)] = 0$ for all odd $l$. Suppose that there exists $k$ odd such that $t_{jk}^i \neq 0$, then there exists an odd $k$ such that $\mathcal{H}^l_{x_j} \text{IC}(O_{2j12n-2j+1}, \mathcal{C}^{i,j})[\pm k] \neq 0$ (note that $\mathcal{H}^j_{x_j}(2n-j+1)\text{IC}(O_{2j12n-2j+1}, \mathbb{C}) \neq 0$). This is a contradiction. Thus the lemma is proved. $\square$

Recall from Lemma 7.13 that we have $(t_{i,k}^j)_n = (t_{i-j,k}^j)_n - j$ for $j \leq l < i$. This implies, in particular, that $(t_{i,k}^j)_n = (t_{i-1,j,k}^j)_n - j$ for $j \geq 1$. Since $(t_{i,k}^j)_n = \delta_{0,k}$, the determination of $(t_{i,k}^j)_n$ are reduced to that of $(t_{0,k}^j)_n$. In the following we describe how to determine the latter numbers inductively. The dimensions of the stalks $\mathcal{H}^j_{x_j} \text{IC}(O_{2j12n-2j+1}, \mathbb{C})$ can be determined simultaneously (in 8.3 we determine these dimensions directly, see 8.8).

Recall that $\mathcal{H}_{0}^{odd} \text{IC}(O_{2j12n-2j+1}, \mathbb{C}) = 0$ (see Corollary 6.1). Note also that $\mathcal{H}_{0}^{odd} \text{IC}(O_{2j12n-2j+1}, \mathbb{C}) \neq 0$ implies that $-\dim O_{2j12n-2j+1} \leq 2k \leq -1$. Let us write

$$m_j = (\dim O_{2j12n-2j+1})/2 = j(2n-j+1)/2,$$

$$f_j(q) = \sum_{k=-m_j}^{-1} (\dim \mathcal{H}_{0}^{2k} \text{IC}(O_{2j12n-2j+1}, \mathbb{C})) q^k,$$

and

$$og_{j,2n+1}(q) = \sum_{k=0}^{d_j} \dim H^{2k}(OGr(j, 2n+1), \mathbb{C}) q^k,$$

where $d_j = \dim OGr(j, 2n+1) = j(4n-3j+1)/2$. The polynomials $og_{j,2n+1}(q)$ are known, i.e.

$$og_{i,2n+1}(q) = \frac{(1 - q^{2(n-i+1)})(1 - q^{2(n-i+2)}) \ldots (1 - q^{2n})}{(1 - q)(1 - q^2) \ldots (1 - q^t)}.$$

Note that $\sigma_i^{-1}(0) \cong OGr(i, 2n+1)$. In view of Lemma 8.2, the decomposition (8.3) implies that

$$og_{i,2n+1}(q) q^{-i(2n-i+1)/2} = f_i(q) + \sum_{j=1}^{i-1} f_j(q) \sum_{k=0}^{(i-j)(n-i)} (t_{j,2k}^i)_n q^{\pm k} + \sum_{k=0}^{i(n-i)} (t_{i,0,2k}^i)_n q^{\pm k}. \quad (8.5)$$

Note that $f_0(q) = 1$. It is easy to check using (8.5) that

$$(t_{0,2k}^i)_n = 1 \text{ for } 0 \leq k \leq n - 1, \quad (t_{0,2k}^i)_n = 0 \text{ otherwise}, \quad \text{and } f_1(q) = q^{-n}.$$ 

This completes the determination of the numbers $(t_{jk}^i)_n$ and $f_i(q)$’s for $n = 1$, and for all $n$ and $i = 1$. By induction on $n$, we can assume that the numbers $(t_{jk}^i)_n'$ have been determined for all $n' < n$. This implies that $(t_{jk}^i)_n$ for $j \geq 1$ have been determined.
now the numbers \((t_{0,2k}^i)_n\) by induction on \(i\). We can assume that all \(f_i'(q),\ i'<i,\) has been determined. Note that \(f_i(q)\) is concentrated in negative degrees. Thus we can determine the numbers \((t_{0,2k}^i)_n\) from (8.5) and then determine \(f_i(q)\).

We have shown that

\[
(8.6) \quad \text{the equations (8.5) determine } f_i(q)\text{'s and } \sum_{k=0}^{i(n-i)} (t_{0,2k}^i)_n q^{\pm k}\text{'s uniquely.}
\]

8.3. The functions \(f_i(q)\). In fact, the functions \(f_i(q)\) can be determined directly making use of our identification of \(H^k IC(\mathcal{O}_{2i'12n−2i}, \mathbb{C})\) with \(H^k IC(\mathcal{O}'_{2i'12n−2i}, \mathbb{C})\) (see Theorem 7.11), where \(\mathcal{O}'_{2i'12n−2i} \subset \mathcal{N}_{\text{sp}}(2n)\), and the classical Springer correspondence.

**Lemma 8.3.** We have that

\[
(8.7) \quad f_i(q) = q^{-i(2n-i+1)/2} g_{[i/2],n}(q^2),
\]

where \([i/2]\) is the integer part of \(i/2\). Namely

\[
(8.8) \quad \dim H^k IC(\mathcal{O}_{2i'12n−2i+1}, \mathbb{C}) = \dim H^{k/2}(\text{Gr}(\lfloor i/2 \rfloor), n), \mathbb{C}) \text{ if } k \equiv 0 \mod 4,
\]

\[
\dim H^k IC(\mathcal{O}_{2i'12n−2i+1}, \mathbb{C}) = 0 \text{ otherwise.}
\]

**Proof.** We have

\[
\dim H^k IC(\mathcal{O}_{2i'12n−2i+1}, \mathbb{C}) = \dim H^k IC(\mathcal{O}'_{2i'12n−2i}, \mathbb{C}) = [V(\mathcal{O}'_{2i'12n−2i}, \mathbb{C}) : H^{k+2n^2}(\mathcal{B}, \mathbb{C})],
\]

where \(V(\mathcal{O}'_{2i'12n−2i}, \mathbb{C})\) is the representation of type \(C\) weyl group attached to the pair \((\mathcal{O}'_{2i'12n−2i}, \mathbb{C})\) under Springer correspondence (see (6.3)). The numbers \([V(\mathcal{O}'_{2i'12n−2i}, \mathbb{C}) : H^{k+2n^2}(\mathcal{B}, \mathbb{C})]\) are the so-called fake degrees and they have been computed explicitly by Lusztig in [L2]. In fact, let us write

\[
P_i(q) = \sum [V(\mathcal{O}'_{2i'12n−2i}, \mathbb{C}) : H^{2i}(\mathcal{B}, \mathbb{C})] q^k.
\]

Using (6.22) and [L2], we see that

\[
P_i(q) = q^{n^2-ni+i(i-1)/2} g_{[i/2],n}(q^2).
\]

Now \(\dim H^{k-2n-i+1} IC(\mathcal{O}_{2i'12n−2i+1}, \mathbb{C})\) is the coefficient of \(q^{k-2n-i+1+2n^2}/2\) in \(P_i(q)\), which is the coefficient of \(q^{k/2}\) in \(g_{[i/2],n}(q^2)\). The equation (8.8) follows. Thus the lemma is proved.

8.4. **Proof of Theorem 8.1** The equation (8.1) in the theorem follows from (8.4) and Lemma 8.2. The equation (8.2) in the theorem follows from (8.4) and the following statement about the numbers \(t_{j,2k}^i\).

**Proposition 8.4.** We have

\[
\sum_{k=0}^{(i-j)(n-i)} (t_{j,2k}^i)_n q^{\pm k} = q^{-(i-j)(n-i)} g_{i-j,2n-i-j}(q).
\]
In view of (8.6), the proposition above follows from (8.5), (8.7), and the following equation

\[(8.9)\]
\[
og_{i,2n+1}(q) = \sum_{j=0}^{i} q^{(i-j)(i-j+1)/2} g_{[j/2],n}(q^2) g_{i-j,2n-i-j}(q),
\]

where \(g_{0,n}(q) = 1\).

**Proof of (8.9).** The proof given here was kindly supplied to us by Dennis Stanton.

Define \((A; q)_l := \prod_{k=0}^{l-1}(1 - Aq^k)\) and \(\left[n\atop j\right]_q := \frac{(q; q)_n}{(q; q)_j (q; q)_{n-j}}\). Let us write

\[S = \sum_{j=0}^{[i/2]} q^{(i-j)(i-j+1)/2} g_{[j/2],n}(q^2) g_{i-j,2n-i-j}(q).\]

We have

\[
S = \sum_{j=0}^{[i/2]} q^{(i-2j-1)(i-2j)/2} \left[\begin{array}{c}
\left[2n - i - 2j - 1\atop i - 2j - 1\right]_q \\
\left[i-2j\atop i - 2j\right]_q
\end{array}\right] + q^{i-2j} \left[\begin{array}{c}
2n - i - 2j \\
2n - i - 2j
\end{array}\right].
\]

\[
= q^{(i)}(q; q)_{2n-i-1} \sum_{j=0}^{[i/2]} (-1)^j \left(\frac{q^{-2n}; q^2) j (q^{-1}; q) j (q^{-1}; q) j q^{j+2i-2n+4} (q^2; q^2) j (q^{-2n+i+1}; q^2) j (q^{-2n+i+2}; q^2) j (1 - q^{2n-4j}).
\]

Consider the terminating very well-poised basic hypergeometric series which, by definition, is given by

\[
6\phi_5\left(a, q^2 a^{1/2}, -q^2 a^{1/2}, b, c, q^{-2m} a^{1/2}, -q^{2m} a^{1/2}; q^2, \frac{aq^{2m+2}}{bc}\right)
\]

\[
= \sum_{j=0}^{\infty} \frac{(a; q^2) j (q^2 a^{1/2}; q^2) j (q^{-a^{1/2}}; q^2) j (q^2 a^{1/2}; q^2) j (c; q^2) j (q^{-2m}; q^2) j (q^{2m+2}; q^2) j (aq^{2m+2}; q^2) j}{(q^2; q^2) j (a^{1/2}; q^2) j (a^{1/2}; q^2) j (a^{1/2}; q^2) j (aq^2/b; q^2) j (aq^2/c; q^2) j (aq^{2m+2}; q^2) j (aq^{2m+2}; q^2) j (aq^{2m+2}; q^2) j}.
\]

Note that if \(m > 0\), then \((q^{-2m}; q^2) j = 0\) for \(j > m\) and thus the sum above is finite. Take \(a = q^{-2n}; m = i/2\) and \(b = q^{1-i}\) for even \(i\); and \(m = (i - 1)/2, b = q^{-i}\) for odd \(i\).

One checks that

\[
S = (1- q^{2n}) q^{(i)}(q; q)_{2n-i-1} \lim_{c \to \infty} 6\phi_5\left(a, q^2 a^{1/2}, -q^2 a^{1/2}, b, c, q^{-2m} a^{1/2}, -q^{2m} a^{1/2}; q^2, \frac{aq^{2m+2}}{bc}\right).
\]

Now use the summation formula (see [GR, Appendix (II.21)])

\[
6\phi_5\left(a, q^2 a^{1/2}, -q^2 a^{1/2}, b, c, q^{-2m} a^{1/2}, -q^{2m} a^{1/2}; q^2, \frac{aq^{2m+2}}{bc}\right) = \frac{(aq^2; q^2)_m (aq^2/b; q^2)_m}{(aq^2/b; q^2)_m (aq^2/c; q^2)_m}.
\]
we get
\[
\lim_{c \to \infty} 6\phi_5 \left( \begin{array}{c}
a, q^2 a^{1/2}, -q^2 a^{1/2}, b, c, q^{-2m} \\
a^{1/2}, -a^{1/2}, aq^2/b, aq^2/c, aq^{2m+2} \\
\end{array} \right) = \left( \frac{aq^2; q^2_m}{aq^2/b; q^2_m} \right).
\]
Thus
\[
S = (1 - q^{2n}) \frac{q^{(i)}(q; q)_{2n-1}}{(q; q)_{2n-2i}} \frac{(aq^2; q^2_m)}{(aq^2/b; q^2_m)} = o_{i,2n+1}(q).
\]

\[\Box\]

**Example 8.5** (Cohomology of \(Fano_2^n\)). The cohomology of \(Fano_2^n\), the Fano variety of lines in the smooth complete intersection of two quadrics in \(\mathbb{P}^n\), can be described as follows:

\[
H^{2k+1}(Fano_2^n, \mathbb{C}) = 0
\]

\[
H^{2(4n-8-k)}(Fano_2^n, \mathbb{C}) = H^{2k}(Fano_1^n, \mathbb{C}) = \begin{cases} 
\mathbb{C}^{(i/2)} & \text{if } 0 \leq k \leq n - 3 \\
\mathbb{C}^{(i/2)} \oplus L_1 & \text{if } n - 2 \leq k \leq 2n - 5 \\
\mathbb{C}^{(i/2)} \oplus L_1 \oplus L_2 & \text{if } k = 2n - 4.
\end{cases}
\]

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52