Finite Linear Spaces, Plane Geometries, Hilbert spaces and Finite Phase Space.

M. Revzen and A. Mann
Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel
(Dated: August 4, 2015)

Finite plane geometry is associated with finite dimensional Hilbert space. The association allows mapping of q-number Hilbert space observables to the c-number formalism of quantum mechanics in phase space. The mapped entities reflect geometrically based line-point interrelation. Particularly simple formulas are involved when use is made of mutually unbiased bases (MUB) representations for the Hilbert space entries.

The geometry specifies a point-line interrelation. Thus underpinning d-dimensional Hilbert space operators (resp. states) with geometrical points leads to operators termed "line operators" underpinned by the geometrical lines. These "line operators", $\hat{L}_j$ (j designates the line) form a complete orthogonal basis for Hilbert space operators. The representation of Hilbert space operators in terms of these operators form the phase space representation of the d-dimensional Hilbert space.

The complete $d^2$ dimensional MES i.e. the "line states" are shown to provide a transparent geometrical interpretation to the so called Mean King Problem and its variant.

The "line operators" (resp. "line states") are studied in detail. The paper aims at self sufficiency and to this end all relevant notions are explained herewith.

I. INTRODUCTION

Phase space formulation of continuum (i.e. pertaining to Hilbert space dimensionality $d \rightarrow \infty$) quantum mechanics was initiated with [4, 21] and developed into a coherent autonomous approach to quantization by [33, 42, 43]. The formalism was (and is) clarified and developed by numerous workers (cf. references in, e.g., [29, 31]). It finds wide use in quantum optics [7, 8, 18, 30, 32], quantum cryptography [44] and foundation of quantum mechanics [29, 31, 46]. It may be epitomized, perhaps, by Glauber's coherent state [7, 18, 32] which relates directly to phase space.

The present study is concerns with finite dimensional phase space. It deals with mapping finite dimensional Hilbert space onto (finite dimensional) phase space [10, 12, 16, 19, 25, 28, 39, 53]. These mappings are epitomized, to a large extent, with wave functions of mutually unbiased bases (MUB) (a brief review of MUB is given in Section III below). Physically these states encapsulate complementarity - a fundamental quantum mechanical feature [9, 26, 39]. Mathematically MUB were related to algebraic (Galois) fields [14, 25, 26, 39]. Of special interest in the present work is a deep relation, first noted in [14] and [11], between (finite) affine plane geometry (APG) and MUB. Such a relation is implied by the known (e.g. [1]) relation between algebraic fields and geometry. Thus finite dimensional algebraic fields (Galois fields,GF ) were (and are) used extensively to extend analysis of unitary bases in finite dimensional Hilbert space [19] to finite dimensional phase space quantum mechanics [10, 16, 17, 24, 25, 28, 53]. In the "triangle" GF- MUB - APG the emphasis of the present study is on the MUB - APG side. More specifically we study the interrelation between the dual of APG, viz. DAPG and MUB. (both APG and DAPG are reviewed in Section II.) For example, we show [11, 12, 28, 39, 53, 54] in Section IV, that letting the points of the geometry underpin MUB state projectors - the physical entities (i.e. the Hilbert space operators) that the corresponding geometrical lines underpin, dubbed "line operators" $\hat{L}_j$ (j designates a line) form a complete orthonormal basis for operators in the d-dimensional (Hilbert) space understudy. The expansion coefficients (i.e. the representation) of Hilbert space
operators in terms of these "line operators" are the finite dimensional phase space mappings of the operators. In particular the representation of the density matrix, $\rho$, in the space of these $\hat{L}_j$ is the finite dimensional Wigner function [10, 12, 28, 36]; lines being parametrized by $j = m; m_0$ which corresponds to phase space coordinates $q,p$ in the continuum [47, 48, 52, 53].

The paper is organized as follows. The succeeding section, Section II, contains an explanatory discussion of the basic notions of linear spaces and outline the approach [3], that we adopt, of viewing geometries as constrained linear spaces, $S$. The section, Section II, contains the definitions and postulates of finite geometry [1, 3] and the definition of the corresponding interrelations among the Hilbert space entities underpinned by the geometry. Section III presents the definitions and essential features of mutual unbiased bases (MUB) and collective coordinates in finite dimensional Hilbert spaces [55] that are convenient for the labeling of Hilbert space states underpinned by the geometrical lines where the geometrical points underpin two particles product MUB states. For a d-dimensional Hilbert space the maximal number of MUB is $d+1$ [10, 24, 39]. However realization of $d+1$ MUB is known only for $d = p^m$, $p$ a prime and $m$ a (positive) integer. i.e. precisely for the order for which (finite) algebraic (Galois) fields are known to exist, and precisely the order for which (finite) projective (and both affine and dual affine discussed in Section II) geometries exist. (There is no mathematical proof for the exclusiveness of $d = p^m$ as the dimensionalities within which MUB and the geometries exist. However no counter example is known [1, 39].) Thus it is possible to construct a geometry of order $d$ given that an algebraic field of this order exist. Indeed it was conjectured in [12] that "the existence of $d+1$ MUB for d dimensional Hilbert space, if d differs from a power of prime is intimately linked with the existence of projective planes of this order. The present work is limited to the simplest cases, viz $d=\text{prime} \neq 2$. For these cases MUB and the geometries are realizable with relative ease. The extension to $d=p^m$, i.e. power of prime, including 2, is possible and is briefly discussed in an appendix.

The next section, Section IV, accounts the actual underpinning of MUB projectors and two particle states with finite geometry. It is argued that the most convenient geometry for underpinning MUB entities is the dual affine plane geometry (DAPG) which is discussed in detail.

In Section V mappings of (selected) Hilbert space entities onto finite dimensional phase space is presented. We give here the derivation of finite dimensional Wigner function, Radon transform and the parity operator as c number function in phase space.

d-dimensional Hilbert space may accommodate $d+1$ mutually unbiased bases (MUB) [24, 25, 39]. The existence of the $d+1$ bases is known only for $d = p^m$ ($p$ a prime and $m$ a positive integer). The order, $d$, referred to as dimension in this paper, for which finite affine plane geometry (APG) and it dual (DAPG) are known to exist is likewise $d = p^m$ [1]. The analysis of this paper is confined to $d=p$ (a prime) and extension to power of prime is only briefly discussed.

Some use is made of mathematically known results that are not studied in detail. The three appendices are our struggle at providing a descriptive account of the most important of these. Thus Appendix A illustrates the idea of field extension. In Appendix B we considers a case where few attributes of a linear space allows the deduction of all the essential characteristics of a geometry. Appendix C provides a proof of the equality of the number of pencils in DAPG (the notions involved are elaborated on in Section II) with the number of points on a DAPG line.

II. FINITE GEOMETRIES, LINEAR SPACES

We begin with the definition of linear space $S$: $S$ is made of pair entities, points and lines. There are $\nu$ points, $S_\alpha; \alpha = 1, 2..., \nu$ and there are $B$ sets of points termed lines $L_j; j = 1, 2..., B$. These are interrelated via the following axioms:

$\lambda_1$. Given two distinct points there is exactly one line common to both. Every point is common to at least two lines.

$\lambda_2$. There are at least two distinct points in a line. There are three non co-linear points.

Correspondingly we define dual linear space $\hat{S}$ which is made up of the same entities but their interrelation is gotten by exchanging points $\leftrightarrow$ lines. Thus the axioms for $\hat{S}$ are:
\[ \lambda_1 \]. Given two distinct lines there is exactly one point common to both. Every line is common to at least two points.

\[ \lambda_2 \]. There are at least two distinct lines with a common point. There are three lines with no three-fold common point.

Our interest is with constrained linear spaces: Adding a third axiom, \( A \), to \( \lambda_1, \lambda_2 \) defines a constrained linear space \( \mathcal{A} \):

A: Given a line \( L \) and a point \( S_\alpha \) not on \( L \), there exists exactly one line, \( L' \), containing \( S_\alpha \) such that \( L \cap L' = \emptyset \).

i.e. \( L' \parallel L \).

The three axioms \( \lambda_1, \lambda_2 \) and \( A \) define (finite) affine plane geometry, APG \([1]\), i.e. \( \mathcal{A} \equiv \text{APG} \).

Correspondingly, adding a third axiom \( \tilde{A} \) to those of the dual linear space above, \( \tilde{S} \) defines the linear space \( \tilde{\mathcal{A}} \):

\( \tilde{A} \): Given a point \( S_\alpha \) and a line \( L \) not containing the point there exists exactly one point \( S_\alpha' \) on \( L \) with no line containing them both.

The three axioms \( \tilde{\lambda}_1, \tilde{\lambda}_2 \) and \( \tilde{A} \) define dual affine plane geometry (DAPG), i.e. \( \tilde{\mathcal{A}} \equiv \text{DAPG} \).

In addition to the above two linear spaces, \( \mathcal{A} \)- the APG and \( \tilde{\mathcal{A}} \)- the DAPG , we consider a third linear space, \( \mathcal{P} \)- the (finite) projective geometry, FPP. This is defined by constraining \( \lambda_1, \lambda_2 \) with \( \lambda_3 \).

Thus the linear space \( \mathcal{P} \)- FPP - is defined by \( \lambda_1, \lambda_2, \lambda_3 \).

Some general mathematically proved results for finite geometries will now be listed. \( \nu \) is the number of points and \( B \), the number of lines of the geometry.

It can be shown that for the three linear spaces which are the three finite geometries, \( \mathcal{A} \)- affine, \( \tilde{\mathcal{A}} \)- dual affine and \( \mathcal{P} \)- projective geometry the number of points on a line \( L \), i.e. \( k_L \), is independent of the line, \( j \) and the number of lines sharing a point, \( S_\alpha \), i.e. \( r_p=S_\alpha \), is independent of the point \( \alpha \) they are designated by \( k_L \) and \( r_p \) respectively. The number of points on a line, \( k_L \), referred to as the "order" of the geometry in \([1,3]\) is dubbed the "dimensionality" of the geometry in the present paper since it is what relates to the dimensionality of the underpinned Hilbert space.

We now list for each of the geometries four characterizations two of which are sufficient to define the geometry pertaining to the dimensionality, \( d \), for which the geometry is defined.

\[ \mathcal{A}: \nu = d^2; B = d(d+1); k_L = d; r_p = d+1. \]

\[ \tilde{\mathcal{A}}: \nu = d(d+1); B = d^2; k_L = d+1; r_p = d. \]

\[ \mathcal{P}: \nu = B = d^2 + d + 1; k_L = r_p = d+1. \]

Thus, as an example: A linear space with \( \nu = d^2 \) and \( k_L = d \) is necessarily an affine plane geometry, viz, \( \mathcal{A} \) with \( B=d(d+1) \) and \( r_p=d+1 \). The proof is given in an appendix B.

We designate a set of parallel (i.e. having no common point) lines as "pencil". This is it’s mathematical designation \([1]\). (It is referred to as striation by Wootters \([11]\)). Thus the structure of the linear space, \( \mathcal{A} \) (i.e. APG) is accounted by: The \( d \)-dimensional \( \mathcal{A} \) is made of \( d+1 \) pencils, each made of \( d \) lines. Each line contains \( d \) points and has one common point with every other line of distinct pencil. Note that the numbers fit: \( B = d(d+1), \nu = d^2, k_L = d; r_p = d+1 \). For APG, \( r_S = r_p, r_S \) being the number of pencils. A proof is given in Appendix C.

In the dual spaces pencils are sets of points with no common line. Thence the structure of the dual space \( \tilde{\mathcal{A}} \) is: A \( d \)-dimensional dual linear space \( \tilde{\mathcal{A}} \) is made of \( d+1 \) pencils each of \( d \) points not connected by a line. Each point is common to \( d \) lines and has one common line with every point belonging to distinct pencil. Every two lines have one point in common, i.e. there are no parallel lines. For DAPG, \( \tilde{\mathcal{A}} \), we have \( r_S = k_L \).
The remarkable feature characterizing FPP is the dual role played by points and lines, i.e., interchanging (with the suitable linguistic adjustment) the words point and line leaves FPP unaffected. Thus the dual of $\lambda_1, \lambda_2, P_1, P_2$ holds for FPP equally well.

It can be shown [1] that the removal from FPP, of any one line and its points leaves a DAPG. Removal of any point and the lines going through it leaves an APG. Conversely given a APG upon addition of a point connecting members of each pencil (set of parallel lines) and then forming a line with these $d+1$ points gives a FPP. And, correspondingly, connecting in DAPG each set of disconnected points with a line and then having these $d+1$ lines intersect at a point reproduces FPP. Thus a $d$ for which any one of the three geometries exists implies the existence of the other two.

For example, given that APG exists for $d=5$ - implies the existence, for $d=5$, of DAPG and FPP. The characteristics of the APG are, as stipulated above: $d=5 \Rightarrow, \nu = 25, B= 30, r_p = 5$ and $k_L = 6$. For the DAPG (with $d=5$) these are $\nu = 30, B = 25, r_p = 6$ and $k_L = 5$. For FPP, $d=5$, they are $\nu = B = 31, r_p = k_L = 6$. For $\mathcal{P}$, i.e. FPP $r_S = 0$.

No general rule specifying the dimensionalities, $d$, for which the geometries exist is known [1]. For $d=p^n$ (power of prime) the geometries may be constructed. We now outline its construction for $d=p$ (a prime).

APG is the most intuitive geometry this as it is closely related to two dimensional vector space in terms of which it may be coordinatized. We now outline its construction for $d=p$ (a prime).

Define a two dimensional vector space $V : ((x_i,y_j); x_i, y_j \in \mathbb{F}_d \equiv \mathbb{Z}/d\mathbb{Z}, i,j = 0,1,...,d-1)$. i.e. $x_i, y_j$ may be considered as numbers abiding modular algebra. We have, $(x_i,y_j) \in V$ with addition being modular addition component wise and multiplication by $r \in \mathbb{F}_d$ gives $(rx_i, ry_j) \in V$. Now consider a square array of $d^2$ points: $d$ along the "x-axis" and above each $x_i$ a column of $d$ $y_j$. The "tip" of a vector, $(x_i,y_j)$, defines a "point".

Points (i.e. vectors) whose components satisfy an equation of the form

$$y = rx + s \mod d, \ r,s \in \mathbb{F}_d,$$

form the line $L_j, j = (r,s), r,s = 0,1,2,...,d-1$. Thus the equation specifies $d^2$ lines which, with the $d$ lines given by $x = s; s = 0,1,...,d-1$, gives the $d(d+1)$ lines of APG. There are $d+1$ pencils (striations): the $d$ lines given by the $d$ values of $s$ for each pencil: two lines with distinct values of $s$ (holding $r$ fixed or, for the last pencil, holding $x_i = s$) gives lines with no common point. If we consider arbitrary pair of lines belonging to different pencils it is obvious that they share one point: the unique point $(x_i,y_j) \in V$ that satisfies the two equations ($r \neq r', r,r' \neq 0$)

$$y_i = rx_j + s \mod d, \ y_i = r'x_j + s', \mod d$$

$$\Rightarrow \ x_j = \frac{s' - s}{r' - r}, \ y_i = \frac{r'r^{-1}s - s'}{r'r^{-1} - 1}.$$  \((2)\)

(That each line of the vertical pencil, $x = s$, has a unique common point with each line of the other pencils is obvious.) Since each point must be common to $d+1$ lines, one from each pencil, the construction has $r_p = d+1$. Having the array as square implies $k_L = d$. Thus we have constructed APG for $d=p$. The line is parameterized as, $L_j=(r,s)$.

The procedure may be described alternatively [27] as follows. Define a one dimensional vectorial subspace, $W$ via:

$$W := \{(x_i,rx_i)|r, x_i \in \mathbb{F}_d\}.$$  \(W\)

The line, $L_j$ is defined by,

$$L_{j=(r,s)} = (0,s) + W; \ (r,s) \in V \ s \in \mathbb{F}_d,$$

i.e. a line is a vectorial coset  [27].

Forming APG for dimension, $d=p$ (a prime) as was shown above was based on having, in such cases, consistent modular algebra: The elements (e.g. $x,y$) are elements of an (algebraic) field, $\mathbb{F}_d \equiv GF(p)$.

As is well known [1], [2] algebraic fields are also possible for $d = p^m$, m a positive integer, forming thereby an (algebraic) field $GF(p^m)$, with the elements $z = 0,1,...,p^m-1$ which is an extension of $GF(p)$. An illustration of such extension, an extension of $d=3$ to $d = 3^2$, is given in Appendix A.

Our interest is in "realization" of the geometries, i.e. utilizing geometrical points, $S_\alpha$, as underpinning Hilbert space states or operators, i.e.

$$S_\alpha \Rightarrow \hat{S}_\alpha \text{ or } |S(\alpha)\rangle,$$
the latter being Hilbert space entities. Now, given the geometrical interrelation between lines $L_j$ and points $S_\alpha$, 

$$L_j \equiv \bigcup_{\alpha \in j} S_\alpha. \quad (3)$$

- we seek an implied interrelation among the Hilbert space operators (resp. states) underpinned with geometrical points - with the corresponding "line" operators, $\hat{L}_j$, (resp. "line states", $|S(\alpha)\rangle$) that relate to those points via the geometry. (E.g. APG realization as vector field is discussed above.) Addition is defined for Hilbert space operators and states, we thus define "line operators" (resp. "line state"), $\hat{L}_j$ (resp. $|S(\alpha)\rangle$) within APG, by (We use script lettering to emphasize that we deal with APG.)

$$\hat{L}_j \equiv \frac{1}{k_L} \sum_{\alpha \in j} \hat{S}_\alpha, \quad (4)$$

to implement the interrelation among points and lines given by Eq.(3). Thus the present study considers the implied Hilbert space "line operators" (resp. "line states") that follows from the underpinning of Hilbert space operators (resp. states) with geometrical points.

Since, as is shown in Section IV, DAPG is more convenient for our study, we transcribe Eq.(4) to DAPG:

$$\hat{S}_\alpha = \frac{1}{d} \sum_{j \in \alpha} \hat{L}_j. \quad (5)$$

We used $r_p = d$ within DAPG.

This definition leads directly to the universal quantity

$$\frac{1}{k_L} \sum_{\alpha} \hat{S}_\alpha = \frac{1}{r_p} \sum_j \hat{L}_j \quad (6)$$

It is universal in that it involves quantities that are independent of either lines or points. Here $r_p = d$, $k_L = d + 1.$ (The relation holds for APG upon interchanging lines with points.)

Proof: (p designates pencil, there are $d+1$ pencils in a $d$ dimensional DAPG, cf. Appendix C.)

$$\hat{S}_\alpha = \frac{1}{r_p} \sum_{j \in \alpha} \hat{L}_j \Rightarrow \sum_{\alpha \in p} \hat{S}_\alpha = \frac{1}{r_p} \sum_{\alpha \in p} \sum_j \hat{L}_j \Rightarrow \sum_{p} \frac{1}{r_p} \sum_{\alpha \in p} \hat{S}_\alpha = \frac{d+1}{d} \sum \hat{L}_j \Rightarrow \frac{1}{d+1} \sum_{\alpha} \hat{S}_\alpha = \frac{1}{d} \sum_j \hat{L}_j. \quad QED \quad (7)$$

This relation reflects the fundamental linear space identity, $[3]$

$$\sum_j d_{L_j} = \sum_\alpha d_{S_\alpha} \quad (8)$$

where $d_{L_j}, d_{S_\alpha}$ are, respectively the number of points on the line $j$ and the number of lines on the point $\alpha$. The identity is between two different ways $[3]$ of summing.

The definition, Eq.(4) within APG, or respectively, Eq.(5) within its dual, DAPG, constrains the allowed operators (resp. states) that may be underpinned with either geometry. Thus let the operators (resp. states) be separated into (necessarily) mutually exclusive pencils. Only when the sum of the distinct members of the distinct pencils are equal, the operators (resp. states) may be underpinned with a geometry. The proof is given in Appendix E.

The Hilbert space operators and states considered relates to Mutually Unbiased Bases that is reviewed below.
III. MUTUALLY UNBIASED BASES (MUB) AND MUTUALLY UNBIASED COLLECTIVE BASES (MUCB)

In a d-dimensional Hilbert space two complete, orthonormal vectorial bases, $\mathcal{B}_1$, $\mathcal{B}_2$, are said to be MUB if and only if $(\mathcal{B}_1 \neq \mathcal{B}_2)$

$$\forall |u\rangle, |v\rangle \in \mathcal{B}_1, \mathcal{B}_2 \text{ resp.}, \ |\langle u|v\rangle| = 1/\sqrt{d}. \tag{9}$$

Maximal number of MUB allowed in a d-dimensional Hilbert space is $d+1$ \[24, 26\]. Variety of methods for construction of the d+1 bases for $d = p^m$, $m > 1$ in the Appendix. In such cases the modular variables $n, b$ etc., are elements of an (algebraic) field $GF(p^m)$. It is convenient \[12, 16, 25, 34\] to list the d+1 MUB bases in terms of the so called computational basis (CB). The CB states $|n\rangle$, $n = 0, 1,..,d-1$, $|n+d\rangle = |n\rangle$, are eigenfunctions of $\hat{Z}$,

$$\hat{Z}|n\rangle = \omega^n|n\rangle; \ \omega = e^{2\pi i/d}, \tag{10}$$

We now give explicitly the MUB states in conjunction with the algebraically complete operators \[19, 22\] set, $\hat{Z}$, and the shift operator, $\hat{X}|n\rangle = |n+1\rangle$ : In addition to the CB the d other bases, each labeled by $b$, are \[22\]

$$|m; b\rangle = \frac{1}{\sqrt{d}} \sum_{0}^{d-1} \omega^{nb(n-1)-nm}|n\rangle; \ b, m = 0, 1,..,d-1, \tag{11}$$

the $m$ labels states within a basis. Each basis relates to a unitary operator, \[22\], $\hat{X}\hat{Z}^b|m; b\rangle = \omega^m|m; b\rangle$. For later reference we shall refer to the computational basis (CB) by $b = 0$. Thus the d+1 bases, are, $b = 0$ and $b=0,1,...,d-1$. The total number of states is $d(d+1)$ they are grouped in d+1 sets each of d states. When no confusion may arise we abbreviate the states in the CB $|m, 0\rangle$, i.e. the state $m$ in the basis $0$, by $|\bar{m}\rangle$, or simply $|m\rangle$; we abbreviate $|m, 0\rangle$, i.e. the $m$ state in the basis $b=0$ by $|m_0\rangle$.

We choose the phase of the CB nil, and note that the MUB set is closed under complex conjugation,

$$\langle n|m, b\rangle^* = \langle n|\bar{m}, \bar{b}\rangle, \Rightarrow |\bar{m}, \bar{b}\rangle = |d - m, d - b\rangle, \ b \neq \bar{b}, \tag{12}$$

as can be verified from Eq. (11).

Several studies \[28, 29, 15, 53\] consider the entanglement of two d-dimensional particles Hilbert space via MUB state labeling. We shall now outline briefly the approach adopted by \[53\] that will be used in later sections.

Guided by the continuum case, $d \to \infty$ \[53\], where it is natural to consider collective coordinates (and operators) that refer to relative and center of mass coordinates we consider the definitions for "relative" and "center of mass" for the finite dimensional Hilbert spaces. Thus the Hilbert space is spanned by the single particle computational bases, $|n_1\rangle|n_2\rangle$ (the subscripts denote the particles). These are eigenfunctions of $\hat{Z}_i, i = 1, 2$: $\hat{Z}_i|n_i\rangle = \omega_i^n|n_i\rangle$, $\omega_i = e^{i\pi 2/p}$. Similarly $\hat{X}_i|n_i\rangle = (|n_i+1\rangle + |n_i\rangle)/\sqrt{2}$, $i = 1, 2$. We now define our collective coordinates and collective operators (we remind the reader that the exponents are modular variables, e.g. $1/2 \mod d$ or $(d+1)/2$):

$$\hat{Z}_r \equiv \hat{Z}_1^{1/2}\hat{Z}_2^{-1/2}; \ \hat{Z}_c \equiv \hat{Z}_1^{1/2}\hat{X}_2^{1/2} \leftrightarrow \hat{Z}_1 = \hat{Z}_r \hat{Z}_c; \ \hat{Z}_2 = \hat{Z}_r^{-1}\hat{Z}_c, \tag{13}$$

and, in a similar manner,

$$\hat{X}_r \equiv \hat{X}_1\hat{X}_2^{-1}; \ \hat{X}_c \equiv \hat{X}_1\hat{X}_2 \leftrightarrow \hat{X}_1 = \hat{X}_r^{1/2}\hat{X}_c^{1/2}, \ \hat{X}_2 = \hat{X}_r^{-1/2}\hat{X}_c^{1/2}. \tag{14}$$

We note that $\hat{Z}_s = \hat{X}_s = 1$, and $\hat{X}_s\hat{Z}_s = \omega\hat{Z}_s\hat{X}_s$, $s = r, c$; $\hat{X}_s\hat{Z}_s' = \hat{Z}_s'\hat{X}_s$, $s \neq s'$, $|n_1\rangle|n_2\rangle$, the eigenfunctions of $\hat{Z}_i, i = 1, 2$, spans the $d^2$ dimensional Hilbert space. The sets $\hat{Z}_i, \hat{X}_i; i = 1, 2$ are algebraically complete in this space \[19\], i.e. every (non trivial) operator is a function of these operators. The eigenfunctions of $\hat{Z}_q$ are $|n_c, n_r\rangle$ with $\hat{Z}_c|n_c, n_r\rangle = \omega^{n_c}|n_c, n_r\rangle$, $\hat{Z}_r|n_c, n_r\rangle = \omega^{n_r}|n_c, n_r\rangle$. We note, e.g. \[20\], that $|n_c, n_r\rangle$ is equivalent to $|n_c, n_r\rangle$ when, as is the present case, the two sets, $\hat{Z}_q, \hat{X}_q$; $q = c, r$ are compatible.

Clearly $|n_c, n_r\rangle$: $n, n' = 0, 1,..,d-1$, is a $d^2$ orthonormal basis spanning the two d-dimensional particles Hilbert space. We may consider their respective computational eigen-bases and with them the whole set of MUB bases \[53\],

$$\hat{Z}_s|n\rangle_s = \omega^n|n\rangle_s, \ \hat{X}_s\hat{Z}_s'|m_s, b_s\rangle = \omega^{m_s}|m_s, b_s\rangle; \ \langle n_s|m_s, b_s\rangle = \omega^{b_s} \frac{n_s}{d} (n_s - 1) - m_s, n_s. \ s = r, c. \tag{15}$$
States in the particle coordinates may, clearly, be expressed in terms of the product states of the collective coordinates as both form a complete orthonormal basis that span the two particles d-dimensional Hilbert space,

\[ |n_1\rangle|n_2\rangle = \sum_{n_c,n_r} |n_c,n_r\rangle \langle n_c,n_r|n_1\rangle|n_2\rangle. \tag{16} \]

The matrix element \( \langle n_1,n_2|n_r,n_c\rangle \) is readily evaluated \[53\],

\[ \langle n_1,n_2|n_r,n_c\rangle = \frac{\delta_{n_r,(n_1-n_2)/2}\delta_{n_c,(n_1+n_2)/2}}{2}. \tag{17} \]

We have then,

\[ |n_r,n_c\rangle \leftrightarrow |n_1,n_2\rangle, \text{ for } n_r = \frac{(n_1-n_2)}{2}, n_c = \frac{(n_1+n_2)}{2} \leftrightarrow n_1 = n_r + n_c, n_2 = n_c - n_r. \tag{18} \]

There are, of course, d+1 MUB bases for each of the collective modes. Here too, we adopt the notational simplification \( b_s \rightarrow \bar{0}_s, s = r,c. \)

\[ \text{IV. UNDERPINNING MUTUAL UNBIASED BASES (MUB) WITH DUAL AFFINE PLANE GEOMETRY (DAPG)} \]

For \( d=\text{prime} \) consider an array of \( d(d+1) \) points arranged as \( d+1 \) columns of \( d \) points each. We specify each point by two indices \( (b,m) \). \( b \) specifies the column: \( b=\bar{0} \) designated the left-most column, the column next to it is \( b=0 \), the next \( b=1 \) and so on with the right-most column being \( b=d-1 \).

The rows are specified by \( m \). \( m=0 \) is the upper most row, the row below it is dubbed \( m=1 \) and so on. The bottom row is \( m=d-1 \).

We now consider the points as underpinning MUB projector. Recalling, Eq.(6), that for \( d=\text{prime} \)

\[ |m;b\rangle = \frac{1}{\sqrt{d}} \sum_0^{d-1} \omega^{bn(n-1)-mn}|n\rangle; \quad b = 0,1,...d-1; \quad |m;\bar{0}\rangle = |m\rangle, \tag{10a} \]

the MUB projector is given by

\[ P(m,b) = |m;b\rangle\langle b;m|. \tag{19} \]

Thus, e.g., for \( d=3 \) the array is

\[ \begin{pmatrix} m\backslash b & \bar{0} & 0 & 1 & 2 \\ 0 & P(0,\bar{0}) & P(0,0) & P(0,1) & P(0,2) \\ 1 & P(1,\bar{0}) & P(1,0) & P(1,1) & P(1,2) \\ 2 & P(2,\bar{0}) & P(2,0) & P(2,1) & P(2,2) \end{pmatrix}. \tag{20} \]

We use the \( d(d+1) \) DAPG points to underpin the \( d(d+1) \) MUB projectors. A line contain one point from each column. (Ensuring \( k_L = d+1 \).) The \( d+1 \) DAPG pencils (i.e. set of points not connected by a line) underpin the \( d+1 \) MUB bases. The \( d \) member of each pencil underpin the \( d \) orthogonal projectors of a basis \( b \) (cf. Appendix C).

We now derive the line equations in two steps. First we obtain the equation for part of the line, the part containing one point from each column \( b=0 \) to \( b=d-1 \). We refer to this as an amputated line (AL), and designate it by \( L' \). It has \( d \) points. Later we obtain the \( (d+1) \)st point dubbed \( L'' \) that with \( L \) form to full line, \( L \). We require the CB representation of the MUB projectors.

a. Derivation of the amputated line (AL) , \( L' \), equation.

Using Eq.(11) and Eq.(19) for \( b=0,1,...d-1 \)

\[ \langle n|P(m,b)|n'\rangle = \frac{1}{d} \omega^{i(n-n')/2(n+n'-1)-m}; \quad b = 0,1,...d-1. \tag{21} \]
Thus there is a 1-1 relation between c and $\bar{c}$. The complete proof of the common point is given for $L_0$ and $L_\bar{0}$. The barred AL, $\bar{L}$, is the upper part pertaining to the AL. We may thus parameterize both the AL and the (full) line with $\bar{c}$ (e.g. equation is, then, $m(b) = m_0 + \frac{b}{2}(n + n' - 1)$; $m_0 = m(0)$).

The followings are implied (trivially) by Eq. (21):

$$\langle m; b \mid m; n' \rangle = \langle m; b' \mid m'; n' \rangle; \quad b \neq b'$$

$$\Rightarrow \quad \frac{b}{2}(n + n' - 1) - m = \frac{b'}{2}(n + n' - 1) - m'$$

$$\Rightarrow \quad m(b) = m_0 + \frac{b}{2}(n + n' - 1); \quad m_0 = m(0)$$

$$\langle s; m; b \mid m; s' \rangle = \langle s; m; b' \mid m'; s' \rangle; \quad b \neq b'; \quad s \neq s'.$$

$$\Rightarrow \quad s + s' = n + n'.$$

If $\langle m; b \mid m; n' \rangle = \langle m; b' \mid m'; n' \rangle; \quad m \neq m'$;

$$\Rightarrow \quad m(b) = m_0 + \frac{b}{2}(n + n' - 1); \quad m_0 = m(0)$$

$$\bar{m} \neq m \Rightarrow m_0 \neq \bar{m}_0.$$ (22)

Requiring equality of the $n,n'$ matrix elements of the projectors in distinct columns implies a "line equation", viz the value of $m$ (row) as a function of $b$ (column) for which the equality holds. Via Eq. (22) this is shown to yield

$$m(b) = m_0 + bc - b/2; \quad m_0 = m(0), \quad 2c = n + n'; \quad b = 0, 1, 2...d - 1.$$  

In Eq. (22) it is also shown that the equality of the $n,n'$ matrix elements implies the equality of the $s,s'$ matrix elements provided $s + s' = n + n' = 2c$.

Thus selecting arbitrary $m$ in arbitrary $b, b \neq 0$, and requiring the equality of the $n,n'$ matrix elements of in all the columns, $b (b \neq 0)$ determines AL equation $m(b)$ as stipulated above, dubbed $L_{m_0,c}$. i.e. it is parameterized by $m_0, c$. Selecting a different $\bar{m} \neq m$ and imposing the same requirements gives a different AL, $L_{\bar{m}_0,c}$. Thus distinct amputated lines (AL) parameterized with the same value of $c$, have no common point:

$$\bar{m}(b) = \bar{m}_0 + bc - b/2; \quad \bar{m}_0 \equiv \bar{m}(0), \quad 2c = n + n'; \quad b = 0, 1, 2...d - 1.$$  

The barred AL, $L_{\bar{m}_0,c}$, has no common point with the unbarred one, $L_{m_0,c}$.

Since every pair of full DAPG lines do share one point, the full lines extension of all the amputated lines parameterized by $c$ (e.g. $L_{m_0,c}$ and $L_{\bar{m}_0,c}$) must share the point at $b = 0$. I.e. $n + n' \Rightarrow \bar{m}$. The "natural" relation is $n + n' \equiv 2c = 2\bar{m}$.

We adopt this relation. (Alternatives to this are considered at the end of this section, in subsection d.) The full line equation is, then,

$$m(b) = m_0 + \bar{m}b - b/2; \quad b = 0, 1, ...d - 1.$$  

$$\Rightarrow \quad m_0 = \bar{m}; \quad b = 0.$$ (23)

The upper part pertains to the AL. We may thus parameterize both the AL and the (full) line with $\bar{m}, m_0 : L'(\bar{m}, m_0)$ and $L_{\bar{m}, m_0}$ respectively. It is now obvious that every two (full) lines parameterized by $\bar{m}$ (or equivalently with $c$) have one common point - indeed it is $\bar{m}$ in the $b = 0$ column.

The complete proof of the $c \Leftrightarrow \bar{m}$ relation obtains upon noting that for $c \neq c'$ the two AL do have a common point at $b \neq \bar{0}$. Thus,

$$m_0 + bc = m_0 + b\bar{m} \Rightarrow b = (m_0 - \bar{m}_0)/(c' - c),$$

giving for $m_0 \neq \bar{m}_0$ gives as the common point $m = m_0 + bc - b/2$ at $b = (m_0 - \bar{m}_0)/(c' - c)$. For $m_0 = \bar{m}_0$ the common point is $m = m_0$ at $b = 0$. Thus two amputated lines parameterized with distinct values of $c$ do have a common point. Thus there is a 1-1 relation between $c$ and $\bar{m}$.

We pause here to verify that our underpinning arena, viz the $d(d+1)$ points $S_\alpha; \alpha = (m; b)$ is indeed a DAPG. The points array specified by $d+1$ columns labeled with $b, b = 0, 0, 1,...d - 1, \alpha$ (corresponding to MUB bases) and $d$ rows
labeled with m, m=0,1,...d-1, (corresponding to MUB vectors within a basis). The \( d^2 \) lines are specified via, Eq.(23) (corresponding to \( \tilde{m}, m_0 = 0,1,2,...d-1 \)).

Hence,

1. \( \nu = d(d+1); \; \mathcal{B} = d^2 \).
2. \( k_L = d + 1 \).
3. There are \( d+1 \) sets of points, each containing \( d \) points that have no interconnecting lines, i.e. \( d+1 \) pencils. (These are the \( d \) points in each column.)
4. Every line has one common point with every other line.
5. \( r_p = d \).

Items 1., 2. and 3. are obvious. Item 4. may be seen from Eq.(23): two line equations allow one and only one solution for a common \( m \) value. Item 5. follows from 3. and 4. Thus the lines and points considered form a DAPG.

Thus DAPG forms a natural underpinning array for the projectors of MUB, as is illustrated in Eq.(20) above. Both the geometry and MUB may be constructed for \( d=p \) (prime) that is considered here. The point specified by \( (m,b) \) underpins the projector \( P(m,b) \).

b. Explicit form of the AL operator, \( \hat{L}'_{\tilde{m},m_0} \), and the (full) line operator, \( \hat{L}_{\tilde{m},m_0} \).

We now derive the Hilbert space operator \( \hat{L}(\tilde{m}, m_0), (\tilde{m} \equiv m(0), m_0 \equiv m(0) \) define the line) that is underpinned with the geometrical (DAPG) line formed by the points, \( S_{(\alpha=(m,b))} \). These constitutes the line specified by \( \tilde{m}, m_0 \) via Eq.(23).

Utilizing our formulas derived in Section II we extract, quite generally, within DAPG, for arbitrary \( d (=\text{prime}) \) the expression for \( \hat{L}_j \) in terms of \( S_{(\alpha=(m,b))} \). Starting with the definition, Eq.(4).

\[
\hat{S}_{(\alpha=(m,b))} = \frac{1}{r_p} \sum_{j \in \alpha} \hat{L}_j = \sum_{\alpha \subset j} \hat{S}_{(\alpha=(m,b))} = \frac{1}{d} \sum_{\alpha \subset j} \sum_{j' \subseteq \alpha} \hat{L}_{j'} = \frac{1}{d} \left[(d+1)\hat{L}_j + \sum_{j' \neq j} \hat{L}_{j'}\right] = \hat{L}_j + \frac{1}{d} \sum_{j'} \hat{L}_{j'},
\]

\[
\hat{L}_j = \sum_{\alpha \subset j} \hat{S}_{(\alpha=(m,b))} - \frac{1}{d+1} \sum_{\alpha} \sum_{\alpha \subset j} \hat{S}_{(\alpha=(m,b))} - 1 = \sum_{b=0}^{d-1} P(m(b); b) - 1.
\]

Where we used the universal relation, Eq[4] that gives a summation over all the points, i.e. over all MUB projectors. Summing over the MUB projectors gives \( d+1 \) times unity \( \mathbb{I} \), e.g. consider Eq.(20) for \( d=3 \): summing over the projectors in each of the \( 4 (=d+1) \) columns gives \( \mathbb{I} \).

We evaluate the AL contribution first,

\[
\hat{L}'(\tilde{m}, m_0) = \sum_{b=0}^{d-1} P(m(b); b) - 1.
\]

Since the diagonal elements of the \( d \) projectors are \( 1/d \) subtracting \( I \) leaves

\[
\langle n|\hat{L}'(\tilde{m}, m_0)|n\rangle = 0, \; n = 0,1,...d-1.
\]

Since, via our definition of the line, Eq.(22) the \( n,n' \) matrix elements of all the line elements are equal

\[
\langle n|P(m(b); b)|n'\rangle = \frac{\omega^{-(n-n')m_0}}{d} \; \forall n,n' \text{ such that } n + n' = 2\tilde{m}.
\]

Thence, the non vanishing matrix elements for AL are,

\[
\langle n|\hat{L}'(\tilde{m}, m_0)|n'\rangle = \omega^{-(n-n')m_0}; \; n + n' = 2\tilde{m}.
\]

Since for \( n,n' \) with \( n + n' \neq 2\tilde{m} \) no two terms are equal, i.e.

\[
\langle r|P(m(b); b)|r'\rangle \neq \langle r|P(m(b'); b')|r'\rangle; \; b \neq b' \text{ and } r + r' \neq 2\tilde{m}.
\]
Thus the sum over $b$ from $b=0$ to $b=d-1$ sums over the $d$ roots of unity hence
\[
\sum_{b=0}^{d-1} (r | \mathbb{P}(m(b); b)| r') = 0 \quad r + r' \neq 2 \tilde{m}.
\]

Thus,
\[
\langle n | \hat{L}'(\tilde{m}; m_0) | n' \rangle = \begin{cases} 
\delta_{n+n', 2\tilde{m}\omega^{-(n-n')m_0}}, & n \neq n' \\
0, & n = n'.
\end{cases}
\Rightarrow \langle n | \hat{L}_{m,m_0} | n' \rangle = \delta_{n+n', 2\tilde{m}\omega^{-(n-n')m_0}}.
\] (26)

This is illustrated now for the $d=3$. We first give the projectors in the CB representation, cf. Eq. 110a.

\[
\mathbb{P}(0, \tilde{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \mathbb{P}(0, 0) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} ; \quad \mathbb{P}(0, 1) = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ \omega^2 & \omega & 1 \\ \omega^2 & \omega & 1 \end{pmatrix} ; \quad \mathbb{P}(0, 2) = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega^2 \\ \omega & \omega & 1 \\ \omega & \omega & 1 \end{pmatrix} ;
\]

\[
\mathbb{P}(1, \tilde{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \mathbb{P}(1, 0) = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega & \omega & 1 \end{pmatrix} ; \quad \mathbb{P}(1, 1) = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega^2 \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix} ; \quad \mathbb{P}(1, 2) = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix} ;
\]

\[
\mathbb{P}(2, \tilde{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \quad \mathbb{P}(2, 0) = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix} ; \quad \mathbb{P}(2, 1) = \frac{1}{3} \begin{pmatrix} 1 & \omega & 1 \\ \omega^2 & \omega & \omega \\ \omega^2 & \omega^2 & \omega \end{pmatrix} ; \quad \mathbb{P}(2, 2) = \frac{1}{3} \begin{pmatrix} 1 & \omega & 1 \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix} .
\] (27)

We select $n=0, n'=2$ for $m=1, b=0$, i.e. for $\mathbb{P}(1, 0)$: $(0 | \mathbb{P}(1, 0) | 2) = \frac{\omega}{3}$. In $b=1$ we find that $(0 | \mathbb{P}(0, 1) | 2) = \frac{\omega}{3}$, i.e. in the column $b=1, m=0$ gives the same matrix element. For $b=2$ $(0 | \mathbb{P}(2, 2) | 2) = \frac{\omega^2}{3}$. One notes that these three matrices have equal matrix elements for $n=2, n'=0$ i.e. with the same $n+n'$:
\[
\langle 0 | \mathbb{P}(1, 0) | 2 \rangle = \langle 0 | \mathbb{P}(0, 1) | 2 \rangle = \langle 0 | \mathbb{P}(2, 2) | 2 \rangle = \frac{\omega^2}{3}.
\]

The projector for $b = \tilde{0}$ is $\mathbb{P}(1, \tilde{0})$ since $n + n' = 2 \rightarrow 2m = 2 \rightarrow m = 1$. Thus the line, viz $m(b)$ is:
\[
m(\tilde{0}) = 1, m(0) = 1, m(1) = 0, m(2) = 2.
\]

The points forming this line are marked with $*$,
\[
\begin{pmatrix} m \backslash b \tilde{0} 0 1 2 \\ 0 & - & - & * & - \\ 1 & * & * & - & - \\ 2 & - & - & - & * \end{pmatrix}
\]

Evaluating the line operator,
\[
\hat{L}_{m=1; m_0=1} = \mathbb{P}(1; \tilde{0}) + \mathbb{P}(1; 0) + \mathbb{P}(0; 1) + \mathbb{P}(2; 2) - \mathbb{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega & \omega & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}
\] (28)

The line operators, $\hat{L}_{m, m_0}$, are mutually orthogonal,
\[
tr \hat{L}_{m, m_0} \hat{L}_{m', m'_0} = \sum_{n,n'} \langle n | \hat{L}_{m, m_0} | n' \rangle \langle n' | \hat{L}_{m, m_0} | n \rangle = \sum_{n,n'} \delta_{n+n', 2\tilde{m}\omega^{-(n-n')m_0}} \delta_{n+n', 2\tilde{m}\omega^{-(n'-n)m_0}} = d \delta_{m, m'} \delta_{m_0, m'_0}.
\] (29)
Thus an arbitrary Hilbert space operator, $\hat{A}$, is expressible in terms of the "line operators",

$$\hat{A} = \frac{1}{d} \sum_{j=0}^{d^2-1} (\text{tr}\hat{A}\hat{L}_j)\hat{L}_j, \quad j = (\hat{m}, m_0),$$

(30)
in the sense that, for arbitrary two operators $\hat{A}$, $\hat{B}$,

$$\text{tr}\hat{A}\hat{B} = \frac{1}{d} \sum_{j} \text{tr}\hat{A}\hat{L}_j \text{tr}\hat{B}\hat{L}_j,$$

(31)
i.e. we have a finite dimensional phase space (i.e. via c-number functions of $(\hat{m}, m_0)$) map of Hilbert space.

We now argue that the "line operator" $\hat{L}_{j=(\hat{m}, m_0)}$ is displaced parity operator \[49–51\]. Thus

$$\langle n|\hat{L}_{(0,0)}|n'\rangle = \delta_{n+n,0} \Rightarrow \hat{L}_{(0,0)} = \sum_{s=0}^{d-1} |s\rangle\langle -s| \equiv \mathbb{I}.$$ 

(32)

Thus "symmetric line operator", i.e. line operator underpinned with line defined by $c = \hat{m}$, is is a displaced parity operator,

$$\hat{L}_{j=(\hat{m}, m_0)} = \hat{X}^{\hat{m}}\hat{Z}^{-m_0}\hat{L}_{(0,0)}\hat{Z}^{m_0}\hat{X}^{-\hat{m}} \Rightarrow \hat{L}_{(\hat{m}, m_0)}^2 = \mathbb{I}.$$ 

(33)

Proof of this is given in Appendix D. It expresses the geometrical origin of the displaced parity operator \[49–51\] present in c-number functions that emulate Hilbert space operators in phase space formulation of quantum mechanics. The formula agrees with Wootters \[11\] where it is given as a point in what is essentially APG. The essential equivalence of both geometries, APG and DAPG is explained in the succeeding section.

c. Affine Plane Geometry (APG) and Dual Affine Plane Geometry (DAPG).

We now outline the reasoning that allow viewing APG as closely reminiscent of classical phase space. Our starting point is to consider $(\hat{m}, m_0)$ coordinate of a point in (a finite dimensional) phase space: $\hat{m}$, CB eigenvalue, playing the role of q and $m_0$, Fourier transform of the CB, playing the role of p. The $\hat{m}$ axis is along the horizontal (i.e. "x axis") and $m_0$ along the vertical, ("y axis"). Within this intuitive language APG points underpin $L_{m,m_0}$, Hilbert space operator (termed "line operator" within the DAPG). Similarly lines are now designated by $S_{\alpha}(\alpha = (m, b))$ (they were designated points in our DAPG considerations). Correspondingly the expression for the line operator within APG is given by

$$\hat{S}_{\alpha} = \frac{1}{d} \sum_{j \in \alpha} \hat{L}_j; \quad j \equiv (\hat{m}, m_0), \quad \alpha \equiv (m, b).$$ 

(34)

This equation reads: The APG "line operator", $\hat{S}_\alpha$ is the (normalized) sum of the $\hat{L}_j$ that form the line $\alpha$, i.e. that belong to the line equation. We now obtain the actual value of $\alpha$ implied by the line equation.

We consider phase space points with linear relations between $m_0$ and $\hat{m}$ emulating thereby "straight lines" that were the starting point in \[11\] \[27\]. We shall show that these will give for the LHS, viz. $\hat{S}_\alpha$, an MUB projector and relate the line parameters to the $(r, s, s')$ appearing in the equation below to $\alpha$. Thus we consider the following $d(d+1)$ lines,

$$m_0 = r\hat{m} + s, \quad Mod[d] \quad r, s = 0, 1, \ldots d-1, \quad d^2 \text{ lines.}$$

$$\hat{m} = s', \quad Mod[d] \quad s' = 0, 1, \ldots d-1 \quad d \text{ lines.}$$

(35)

We deal first with the set of $d^2$ lines. The "line operator" within APG is given by Eq.(34).

The sum involves matrices each with its distinct skew line of non vanishing matrix elements. Their sum gives the MUB projector, i.e.

$$\hat{S}_{(r,s)} = \mathbb{P}(m = s - b/2; b = -r) \equiv |m = s - b/2; b = -r)(b = -r; m = s - b/2|. $$

(36)
The proof follows from the following reasoning. The \( n, n' \) matrix element of the MUB projector, Eq.() is

\[
\langle n | \mathbb{P}(m; b) | n' \rangle = \frac{1}{d} \omega^{(n-n')(b/2)[n+n'-1]-m}.
\]

For these \( n, n' \) the matrix elements of the APG line operator, labeled with \( r \) and \( s \) are, for \( n+n'=2\tilde{m} \) and \( m_0 = r\tilde{m}+s \).

\[
\langle n | \hat{S}_{(r,s)} | n' \rangle = \frac{1}{d} \omega^{-(n-n')(r\tilde{m}+s)}.
\]

(37)

The two expressions are equal for \( r = -b, \) and \( m = s - b/2 \). QED.

We now consider the set of \( d \) lines given by Eq.(35). In these cases the MUB projector is

\[
\mathbb{P}(m = s'; b = 0) = |s'\rangle\langle s'|.
\]

The proof is as follows.

\[
\langle n | \hat{S}_{s'} | n' \rangle = \frac{1}{d} \sum_{m_0=0}^{d-1} \langle n | \hat{L}_{s',m_0} | n' \rangle = \frac{1}{d} \sum_{m_0=0}^{d-1} \delta_{n+n',2s} \omega^{-(n-n')m_0}.
\]

(38)

The non diagonal terms add up to nil. The diagonal, i.e. \( n = n' = s' \), add up to 1. QED.

We illustrate this for \( d=3 \) and \( r=1, s=0 \): The APG line is made of the following three points \((\tilde{m} = 0; m_0 = 0);(\tilde{m} = 1; m_0 = 1); \tilde{m} = 2; m_0 = 2 \). These underpin the "points operators", given in terms of matrix elements,

\[
(\tilde{m} = 0; m_0 = 0) \Rightarrow \delta_{n+n',0}; (\tilde{m} = 1; m_0 = 1) \Rightarrow \delta_{n+n',2} \omega^{-(n-n')}; (\tilde{m} = 2; m_0 = 2) \Rightarrow \delta_{n+n',1} \omega^{-(n-n')2}.
\]

(39)

Hence, for \( r=1 \) and \( s=0 \),

\[
\hat{S}_{(r=1, s=3)} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ \omega & 0 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ \omega^2 & 0 & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & 1 \\ \omega & 1 & 1 \end{pmatrix} = \mathbb{P}(2, 2).
\]

(40)

i.e. \( b=-r=1=2 \text{ Mod}[3] \), and \( m=s-b/2=-b/2=2 \text{ Mod}[3] \).

For the case of vertical, i.e. \( \tilde{m} = s \), lines we have in the APG the three points

\((\tilde{m} = s, m_0 = 0); (\tilde{m} = s; m_0 = 1); (\tilde{m} = s; m_0 = 2) \).

In terms of the full matrices this is for \( s=0 \):

\[
\hat{S}_{(r=1, s=3)} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{P}(0, 0) = |0\rangle\langle 0|.
\]

(41)

d. Alternative phase space mapping.

We now consider alternative schemes for adjoining \( L' \), a point in \( b = 0 \) column, to \( L \), the AL, to that used in the "symmetric" scheme, viz \( c = \tilde{m} \), considered above. We now consider a general linear relation between \( c \) and \( \tilde{m} \). This is expressed in the following line equations, \( r, s \in \mathbb{F}_d, r \neq 0 \):

\[
L_{(r,s),m_0} : m(b) = \begin{cases} m_0 + bc - b/2, & b \neq 0; \\
\tilde{m}(\tilde{0}) \equiv \tilde{m} = rc + s, & b = \tilde{0}. \end{cases}
\]

(42)

Each of the line equations accounts for \( d^2 \) lines, \( \tilde{m}, m_0 = 0, 1 \ldots d - 1 \) for fixed \( r,s \) \((r \neq 0) \). Thus \((r,s)\) defines a family of \( d^2 \) lines. For \( r=1 \) and \( s=0 \) the line equation reduces to the "symmetric" one. Note that the parametrization of \( L_{(r,s),m_0} : t = (r, s) \) are, aside from their classification \((r,s)\), like those of the "symmetric" lines: \( L_{(r,s),m_0} \), viz \( \tilde{m}; m_0 \).

We now prove that the line operators \( \hat{L}_{(r,s),m_0} : t = (s, r) \) form a set of \( d^2 \) orthogonal operators and hence spans the operator space of the \( d \)-dimensional Hilbert space.
Within the dual space, DAPG, the \( d(d+1) \) "points" are, naturally, \((m,b)\) - \(b\) associated with the (MUB) basis and \(m\), vector within the basis. Note that the lines, \( L^t\), \( t = c, r, s \) may be parametrized for fixed \(c, r\) and \(s\) by \(\tilde{m}, m_0\), the vectors in the \(b = 0\) and \(b = 0\) that are on the line. Thus DAPG provides a mapping scheme for (finite dimensional) Hilbert space operators to (finite dimensional) phase space function. In particular it allows a "natural" derivation of the (finite dimensional) Wigner function \[11, 28, 18\]. To this end we now show that the line operators form a \(d^2\) dimensional orthogonal basis that may be used to express, e.g., the density operator. Using Eq\[24\] we write, \(t = c, r, s\),

\[
\text{tr} \hat{L}^{(t)}_{j = (\tilde{m}, m_0)} \hat{L}^{(t)}_{j' = (\tilde{m}', m_0')} = A + B + C + D + E. \tag{43}
\]

\[
A = \sum_b \text{tr}|m(b), b\rangle \langle b, m(b)|m'(b), b\rangle \langle b, m'(b)| = \begin{cases} 1 & j \neq j' \\ d + 1 & j = j' \end{cases}
\]

\[
B = \sum_{b \neq b'} \text{tr}|m(b), b\rangle \langle b, m(b)|m'(b'), b'\rangle \langle b', m'(b')| = d + 1
\]

\[
C = -\sum_b \text{tr}|m(b), b\rangle \langle b, m(b)| = -(d + 1)
\]

\[
D = -\sum_{b'} \text{tr}|m'(b'), b\rangle \langle b', m'(b')| = -(d + 1)
\]

\[
E = \text{tr} \mathbb{1} = d \tag{44}
\]

The value of \(A\) obtains, because for \(j \neq j'\) within DAPG, the lines do have one common point: cf. \(\tilde{\lambda}_1\), Section I. The value of \(B\) obtains, since we deal with MUB the scalar product squared of any two states of different bases gives \(1/d\). There are \(d(d+1)\) terms in the sum thus we have \(B = d + 1\) as stated.

We have thus,

\[
\text{tr} \hat{L}^{t}_{j = (\tilde{m}, m_0)} \hat{L}^{t}_{j' = (\tilde{m}', m_0')} = d\delta_{\tilde{m}, \tilde{m}'}\delta_{m_0, m_0'} \tag{45}
\]

However

\[
(\hat{L}^{t}_{j = (\tilde{m}, m_0)})^2 = \mathbb{1} \text{ only for } r = 1, s = 0. \tag{46}
\]

It is only with the choice of "symmetric" line equation family, \(r=1, s=0\) in Eq\[42\], that the (displaced) parity operator is introduced in the mapping of Hilbert space formalism onto quantum mechanics of phase space.

V. FINITE DIMENSIONAL PHASE SPACE

The line operators, \(\hat{L}_{j = (\tilde{m}, m_0)}\), \(\tilde{m}, m_0 = 0,1,...d - 1\), was shown, within DAPG, Eq\[15\], to form an orthogonal \(d^2\) dimensional set. We adopt an intuitively appealing view and consider \(\tilde{m}\), which is associated with the eigen value of the accounting operator, \(Z\), Eq\[10\], as designating position, while associating \(m_0\), that relates to its Fourier transform, with the momenta. Thus \((\tilde{m}, m_0)\) is viewed as a point in finite dimensional phase space.

It was argued in the previous section that within APG, coordinated with \(\tilde{m}\) along the positive x direction and \(m_0\) along the (positive) you axis, the DAPG line operator, \(\hat{L}_{j = (\tilde{m}, m_0)}\), is underpinned with APG point \((\tilde{m}, m_0)\).

a. Finite Dimensional Wigner Function.

Arbitrary Hilbert space operator, \(\hat{A}\), may be expanded in terms of the \(d^2\) orthogonal line operators,

\[
\hat{A} = \frac{1}{d} \sum_j (\text{tr} \hat{A} \hat{L}_j) \hat{L}_j. \tag{47}
\]

\((\text{tr} \hat{A} \hat{L}_j)\) may be viewed within APG, wherein \(j = (\tilde{m}, m_0)\) is a point, as a (finite dimensional) phase space representation of the (finite dimensional) Hilbert space operator, \(\hat{A}\):

\[
\hat{A} \Rightarrow A(\tilde{m}, m_0) = (\text{tr} \hat{A} \hat{L}_j).
\]
Considering, in particular, the case \( \hat{A} = \hat{\rho} \), the density operator,

\[
\hat{\rho} \Rightarrow \rho(\hat{m}, m_0) = \text{tr}\hat{\rho}\hat{L}_{j=\hat{m},m_0} \equiv dW(\hat{m}, m_0).
\]  

(48)

\( W(\hat{m}, m_0) \) is the finite dimensional Wigner function, where the analogy is \( \hat{m} \sim q; m_0 \sim p \). This function is normalized

\[
\sum_{\hat{m}, m_0} W(\hat{m}, m_0) = \text{tr}\rho[\frac{1}{d} \sum_j \hat{L}_{j=\hat{m},m_0}] = \text{tr}\rho = 1.
\]  

(49)

It is real,

\[
W(\hat{m}, m_0)^* = \frac{1}{d} (\sum_{n,n'} (n|\rho|n')\delta_n^* + 2\hat{m}\omega^{-(n'-n)m_0})^* = \frac{1}{d} \sum_{n',n} \langle n|\rho|n'\rangle\delta_n^* - 2\hat{m}\omega^{-(n'-n)m_0} = W(\hat{m}, m_0).
\]  

(50)

It plays the role of a distribution,

\[
\text{tr}\rho\hat{A} = \frac{1}{d} \sum_{j,j'} (\text{tr}\rho\hat{L}_j)(\text{tr}\hat{A}\hat{L}_{j'}) = \sum_{\hat{m}, m_0} W(\hat{m}, m_0)A(\hat{m}, m_0).
\]  

(51)

However the (finite dimensional) Wigner function is not positive definite. Thus consider two orthogonal density matrices, \( \rho_1, \rho_2 \),

\[
0 = \text{Tr}\rho_1\rho_2 = \frac{1}{d} \sum_{j=m_0} (\text{tr}\rho_1\hat{L}_j)(\text{tr}\rho_2\hat{L}_{j'}) = d \sum_{\hat{m}, m_0} W_1(\hat{m}, m_0)W_2(\hat{m}, m_0) = 0,
\]  

(52)

implying that finite dimensional Wigner function is not positive definite and hence is "quasi distribution", in close analogy with the Wigner function within the continuous phase space, [44].

b. Finite Dimensional Radon Transform.

We now review briefly some elements of phase space representation of Hilbert space operators in the continuum, \( d \Rightarrow \infty \). This will guide us in our finite dimensional Radon transform formulation [45].

Consider, within the continuum the operator,

\[
\hat{X}_\theta = \hat{x}C + \hat{\rho}S; \quad C = \cos \theta, \quad S = \sin \theta, \quad \hat{x}, \hat{\rho} \text{ position and momentum operators resp.}
\]  

(53)

Denote its (\( \delta \) function) orthonormalized eigenfunctions, \( |x; \theta\rangle \),

\[
\langle x|x'; \theta\rangle = x'|x'; \theta\rangle, \quad \langle \theta; x''|x'; \theta\rangle = \delta(x'' - x'),
\]

\[
\langle x|x'; \theta\rangle = \frac{1}{\sqrt{2\pi|S|}} e^{\frac{i\theta x x'}{2}}.
\]  

(54)

The phase of the x representative wave function, \( \langle x|x'; \theta\rangle \), was chosen [11] to assure (for \( -\infty \leq x, x' \leq \infty, 0 \leq \theta \leq \pi \))

\[
\lim_{\theta \rightarrow 0} \langle x|x'; \theta\rangle = \delta(x - x'); \quad \lim_{\theta \rightarrow \pi/2} \langle x|x'; \theta\rangle = \frac{e^{-ixx'}}{\sqrt{2\pi}}.
\]  

(55)

The bases, \( |x; \theta\rangle, |x'; \theta\rangle \), for \( \theta \neq \theta' \) form an MUB sets [39]:

\[
|\langle \theta'; x'|x''; \theta''\rangle| = \frac{1}{2\pi|S(\theta' - \theta'')|} \text{ independent of } x', x''.
\]  

(56)

The Moyal [42] mapping of the continuous Hilbert space operators, \( \hat{A} \), onto the continuous phase space, is given by

\[
\hat{A} \Rightarrow A(q,p) = \int dq \text{e}^{-ipy/2} (q - y/2|\hat{A}|q + y/2).
\]  

(57)

For \( \hat{A} = \hat{\rho} \), the density operator, the RHS is \( 2\pi W(q,p) \), where \( W(q,p) \) is the Wigner function and the \( 2\pi \) is introduced to have it normalized to unity.

Reverting to our notation, the Moyal transform of \( \hat{A} \) reads,

\[
\hat{A} \Rightarrow A(q,p) = \text{tr}\hat{A}\hat{L}_{q,p}; \quad \Rightarrow \langle x|\hat{L}_{q,p}|x'\rangle = \delta\left(\frac{x + x'}{2} - q\right)\text{e}^{-ip(x-x')}.
\]  

(58)
The Moyal transform of \( \hat{A} = \hat{P}(x; \theta) = |x; \theta\rangle \langle \theta; x| \) is,

\[
\hat{P}(x; \theta) \rightarrow P(q, p) = \int e^{-iyp}(q - y/2|x; \theta\rangle \langle \theta; x|q + y/2) =
\]

\[
tr\hat{P}(x; \theta) \hat{L}_{(q,p)} = \delta(x - qC - pS),
\]

where we used Eq. (54).

The Moyal transform abides by the overlap formula,

\[
tr\hat{A}\hat{B} = \frac{1}{2\pi} \int dqdp tr(\hat{A}\hat{L}_{(q,p)})tr(\hat{A}\hat{L}_{(q,p)}) = \int dqdpA(q, p)B(q, p),
\]

suggesting that the Radon transform of Wigner function which is

\[
R[W](x; \theta) \equiv \hat{\rho}(x, \theta) = \int dqdpW(q, p)\delta(x - qC - pS),
\]

allows interpreting Eq. (61) by: The Radon transform is the phase space map of an MUB projector of the density operator. Carrying this to the finite dimensional case, the projector of an MUB state mapped onto (finite dimensional) phase space corresponding to the continuum expression, Eq. (54), is

\[
\hat{S}_{\alpha=(m;b)} \equiv \hat{P}(m; b) \rightarrow P(m, m_0) = tr\hat{S}_{\alpha=(m;b)} \hat{L}_{j=(m,m_0)} \equiv \Lambda_{\alpha,j} = \begin{cases} 1, & \alpha \in j \\ 0, & \alpha \notin j. \end{cases}
\]

Thence the finite dimensional Radon transform (of Wigner function) is (cf. Eq. (61))

\[
R[W](m, b) = tr\hat{\rho}\hat{S}_{\alpha} = (\hat{\rho}\hat{L}_{m,m_0})^{\alpha,j} = \sum_{m,m_0} W(m, m_0)\Lambda_{\alpha,j}.
\]

where \( \Lambda \) plays the role of the \( \delta \) function in the continuum.

Generalizing the Radon transform of an arbitrary finite dimensional phase space function, \( Q(m, m_0) \),

\[
\hat{Q} \Rightarrow Q(m, m_0) = \frac{1}{d} tr\hat{Q}\hat{L}_{m,m_0}.
\]

\[
R[W](m, b) = tr\hat{Q}\hat{S}_{\alpha=(m,b)} \equiv \sum_{m,m_0} Q(m, m_0)\Lambda_{(m,m_0),(m,b)}. \tag{64}
\]

c. Phase Space Mappings of Finite Dimensional Wave Functions.

We now consider DAPG underpinning of a two d-dimensional particles wave function, \( |m; b\rangle_1|\tilde{m}; \tilde{b}\rangle_2 \) (the notation are defined in section II) [5]: The \((m,b)\) coordinate scheme for the DAPG now underpins the two particles wave function rather than the operator \( P(m, b) \) considered above.

Underpinning two d-dimensional particles product MUB states with geometrical points, we get, as the definition of the corresponding DAPG "line state",

\[
S_{\alpha=(m,b)} \Rightarrow |S_{\alpha=(m,b)}\rangle = |m; b\rangle_1|\tilde{m}; \tilde{b}\rangle_2
\]

\[
= |m; b\rangle_1|\tilde{m}; \tilde{b}\rangle_2 = \frac{1}{d} \sum_{j \in \alpha=(m,b)} |L_j\rangle. \tag{65}
\]

Inverting, i.e. expressing the "line states" in terms of the "point states", cf Eq. (54), gives

\[
|L_j\rangle = \sum_{\alpha \in j} |S_\alpha\rangle \rightarrow |\mathcal{R}\rangle = \frac{1}{d+1} \sum_{1}^{d(d+1)} |S_\alpha\rangle = \frac{1}{d} \sum_{1}^{d^2} |L_j\rangle. \tag{66}
\]
The universal (wave function) $|\mathcal{R}\rangle$ is

$$|\mathcal{R}\rangle = \frac{1}{d+1} \sum_{(m,b)} d(d+1) |m;b\rangle_1 |\tilde{m};\tilde{b}\rangle_2 = \sum_m |m;b\rangle_1 |\tilde{m};\tilde{b}\rangle_2. \quad (67)$$

The RHS is gotten by noting that

$$\sum_m |m;b\rangle_1 |\tilde{m};\tilde{b}\rangle_2 = \sum_{m'} |m';b'\rangle_1 |\tilde{m'};\tilde{b'}\rangle_2. \quad (68)$$

i.e. the summation over pencil states is independent of the pencil proving Eq. (67).

Thus the geometrical considerations gives for the "line" state, $(\alpha = (m,b); j = (\tilde{m}, m_0))$, the expression

$$|L_{j=(\tilde{m},m_0)}\rangle = \sum_b |m(b);b\rangle_1 |\tilde{m}(b);\tilde{b}\rangle_2 - \sum_m |m;b\rangle_1 |\tilde{m};\tilde{b}\rangle_2$$

$$= \sum_{n,n'} |n\rangle_1 |n'\rangle_2 [\sum_b (n|m(b);b\rangle \langle n'|\tilde{m}(b);\tilde{b}\rangle - \sum_m (n|m;b\rangle \langle n'|\tilde{m};\tilde{b}\rangle)$$

$$= \sum_{n,n'} |n\rangle_1 |n'\rangle_2 [\langle n| \sum_b (m(b);b\rangle (b;m(b)| - I) |n'\rangle]$$

$$= \sum_{n,n'} |n\rangle_1 |n'\rangle_2 \delta_{n+n'-2\tilde{m}} \omega^{-(n-n')m_0}$$

$$= \omega^{2\tilde{m}m_0} \sum_n |n\rangle_1 (2\tilde{m} - n) \omega^{-2m_0n}. \quad (69)$$

Where we used the results of Section III. The last expression for the line state is evidently a (not normalized) MES. Written in terms of the collective coordinates, $\text{Eq.}(18)$, and normalizing we write

$$\frac{1}{\sqrt{d}} |L_{j=(\tilde{m},m_0)}\rangle = \frac{\omega^{2\tilde{m}m_0} |\tilde{m};\tilde{b}\rangle_1}{\sqrt{d}} \sum_n |n - \tilde{m};\tilde{b}\rangle_1 \omega^{-2m_0n} = |\tilde{m};\tilde{b}\rangle_1 |2m_0;0\rangle_1 \Rightarrow$$

$$\frac{1}{\sqrt{d}} |L_{0,0}\rangle = |0;\tilde{0}\rangle_1 |0;0\rangle_1 \text{ origin of } "\text{phase space}". \Rightarrow$$

$$\hat{X}^\tilde{m}_c \hat{Z}^{-2m_0}_r \frac{1}{\sqrt{d}} |L_{j=(\tilde{m},m_0)}\rangle. \quad (70)$$

The MUB sets $b = \tilde{0}$ (the computational basis (CB)) and $b = 0$, its Fourier transform basis are, of course, complete orthonormal bases. Thence the "line" states $|L_{j=(\tilde{m},m_0)}\rangle$ form a $d^2$ orthogonal MES basis that spans the $d^2$ dimensional two particle states:

$$\langle L_{m',m_0'}|L_{m,m_0}\rangle = \delta_{m',m} \delta_{m_0',m_0}, \quad m', \tilde{m}, m_0', m_0 = 0, 1, 2...d - 1. \quad (71)$$

This orthogonality may be proved via the approach of Eq. (43).

A conjugate MES basis is,

$$|\tilde{L}_{m_0;\tilde{m}}\rangle = |2m_0;0\rangle_1 |\tilde{m};\tilde{b}\rangle_1 \Rightarrow$$

$$\langle \tilde{L}_{m_0;\tilde{m}}|L_{m',m_0'}\rangle = \langle 0;2m'_{0'}|\tilde{0};\tilde{m}'\rangle_1 \langle \tilde{m};\tilde{b}\rangle_1 |2m_0;0\rangle_1 = \frac{1}{d} \omega^{-2m_0m_0'} \omega^{2m_0'\tilde{m}}. \quad (72)$$

Note that the $d^2$ dimensional orthonormal MES basis, $|2m_0;0\rangle_1 |\tilde{m};\tilde{b}\rangle_1$; $m_0, \tilde{m} = 0, 1, 2,...d - 1$ forms a MUB to the $d^2$ dimensional MES basis, $|2m_0;0\rangle_1 |\tilde{m};\tilde{b}\rangle_1$:

$$|\langle \tilde{L}_{m_0;\tilde{m}}|L_{m',m_0'}\rangle| = \frac{1}{d}. \quad (73)$$

d. Mean King Problem (MKP) and Tracking the Mean King (TMK).
A DAPG underpinning of an Hilbert space allows a direct solution to the MKP as well as its variant, TMK. The MKP is a quantum mechanical retrodiction problem \([5, 15, 57]\). It was posed originally in \([38]\) for spin 1/2 particles, extended in \([45]\) to prime dimensionality, \(d=p\), and to powers of primes in \([39–41]\). Further generalization are discussed in \([5]\). We consider in the following the \(d=p\) cases.

The MKP (i) and TMK (ii) involves a (two particles) state prepared by Alice. One of the particles is availed to the King who measures its state in an MUB, \(b\), of his choice. Subsequent to his measurement, Alice performs a control measurement of the two particle state. Now i. Within the MKP, Alice is challenged to infer the outcome of the King’s measurement, say, \(m\) when, after she completes her control measurement, she is told the basis, \(b\), used by the King in his measurement. ii. Within the TMK she is challenged to deduce, via her control measurement, the basis used by the King.

We consider sequentially the MKP and TMK.

Let Alice prepare the universal "line state", Eq.(67),

\[
|\mathcal{R}\rangle = \sum_{m} |m; b\rangle_1 |\tilde{m}; \tilde{b}\rangle_2
\]

The King measures, \(\hat{K}\),

\[
\hat{K} = \sum_{m'} |m' : b\rangle K_{m'} \langle b; m'|_1,
\]

for some \(b\) of his choice with an outcome, say, \(m\).

i. To handle the MKP, Alice measures \(\hat{\Gamma}\) for her control measurement.

\[
\hat{\Gamma} = \sum_{\tilde{m}, m_0} |\tilde{m}\rangle_c |2m_0; 0\rangle_r \Gamma_{\tilde{m}, m_0} \langle \tilde{m}|_c \langle 0; 2m_0|_r,
\]

with outcome, say \(m_0', \tilde{m}'\). Thence,

\[
\langle \tilde{m}'|_c (2m_0'; 0)|_r| m; b\rangle_1 |\tilde{m}; \tilde{b}\rangle_2 \neq 0.
\]

The matrix element gives the probability amplitude for the "point state" \(|m; b\rangle_1 |\tilde{m}; \tilde{b}\rangle_2\) to be on the "line state" \(|\tilde{m}'|_c (2m_0'; 0)|_r\). This is non nil only if the geometrical point \((m,b)\) is on the geometrical line viz

\[
m = m_0' + b\tilde{m}' - b/2; \ b \neq 0.
\]

For \(b = 0\), evaluation of the matrix element gives, \(m = \tilde{m}'\). Since Alice knows \(m_0', \tilde{m}',\) she may infer the value of \(m\) upon being informed of the value of \(b\).

ii. To deal with TMK, Alice measures \(\hat{\tilde{\Gamma}}\) for her control measurement. With \(\hat{\tilde{\Gamma}}\) conjugate to \(\hat{\Gamma}\), thus,

\[
\hat{\tilde{\Gamma}} = \sum_{\tilde{m}, m_0} |\tilde{m}\rangle_r |2m_0; 0\rangle_c \tilde{\Gamma}_{\tilde{m}, m_0} \langle \tilde{m}|_r \langle 0; 2m_0|_c,
\]

with outcome, say, \(m_0', \tilde{m}'\). We have then,

\[
\langle \tilde{m}'|_r (2m_0'; 0)|_c| m; b\rangle_1 |\tilde{m}; \tilde{b}\rangle_2 \neq 0.
\]

The non vanishing of this matrix element implies,

\[
b = \begin{cases} 
- \frac{m'}{m}; \ & \tilde{m}' \neq 0. \\
0; & m_0' \neq 0, \ \tilde{m}' = 0. \\
\text{undetermined} \ & \tilde{m}' = m_0' = 0.
\end{cases}
\]

Since Alice knows \(m_0', \tilde{m}'\), she can deduce \(b\), the basis used by the King, except for the case, whose probability is \(1/d^2\), when \(m_0' = \tilde{m}' = 0\), that leaves \(b\) undetermined.
Note:

1. In the special case, whose probability is $1/d^2$, that her control measurement yields back the initially prepared line: i.e. $m'_0 = \tilde{m}' = 0$, Alice does not gain any information.

2. Alice’s deduction is independent of the outcome of the king’s measurement. His measurement, in effect, non-selective.

VI. SUMMARY AND CONCLUDING REMARKS

Finite affine plane geometry (APG) and its dual, DAPG, were studied within the general theory of linear spaces. DAPG was found most convenient for our study and we consider lines and points within the theory. (The basic notions of linear spaces, APG and DAPG are given in Section II. We refer to the order of a geometry as its dimension to simplify the notation.) It was shown that for a d-dimensional Hilbert space the existence of $d+1$ mutually unbiased bases (MUB) (MUB are introduced in Section III) implies the existence of d-dimensional DAPG. (Which, in turn, implies the existence of APG as well as projective geometry.) DAPG is used to define lines and points operators that act in the overriding Hilbert space. The interrelation among these operators is determined by the underpinned DAPG.

Inversion of the expression that gives the "point operator" in terms of "line operators", requires the existence of a geometrically based universal operator, i.e. one that is independent of either points or lines. Expressing operators (and states) via mutually unbiased bases (MUB) lead to simple universal operators. Thus having the geometrical points underpin MUB projectors provides for unity as the universal operator.

The "line operators" are shown to form a $d^2$ dimensional orthogonal basis for a d-dimensional Hilbert space operators. These line operators provide a convenient means for the introduction of finite dimensional phase space. Thus representations of Hilbert space operators in terms of these line operators may thus be interpreted as mappings of q-number Hilbert space entities onto c-number phase space functions. An example of such function is, as is well known, the Wigner function viewed here as a mapping of $\rho$, the density matrix, onto (finite dimensional) phase space wherein $\rho$ is expressed in terms of the "line operators".

The definition of the geometries as constrained linear space underscores their non unique lines constituency. (These notions are introduced in Section II.) The existence of several distinct line formation for a d-dimensional DAPG implies that there exist several distinct sets of Hilbert space "line operators" for a d-dimensional Hilbert space. This, in turn, leads to distinct mappings of the Hilbert space onto (finite dimensional) phase space. For example, it is shown that only one such possible map relates to the celebrated Wigner function connection with the parity operator.

Underpinning two particle MUB product state with DAPG points gives as "line states" maximally entangled states (MES). The $d^2$ orthogonal "line states" of this case form a complete MES orthonormal basis for the two particle Hilbert space. The universal state implied here allows a concise solution to the so called Mean King Problem.

The correspondence of finite dimensional line operators within dual affine plane geometry, DAPG, with finite dimensional point operators affine plane geometry, APG, (and vice versa) is established. Thus confirming the basic equivalence of using either for mapping of Hilbert space q functions to phase space c functions.

The $d(d+1)$ points of a d-dimensional DAPG were used also to underpin $d(d+1)$ two particles MUB product states. In this case the so called "line states" $|L_i\rangle$ underpinned with the corresponding DAPG lines are maximally entangled states (MES) of the two particles. These in turn were shown to be product states when accounted for in terms of collective coordinates. The universal function implied in this case allowed a concise solution to the so called Mean King problem and its extension.

The present work underscores the fundamental role played by Mutually Unbiased Bases (MUB) in relating physical and geometrical entities and provides a geometrical approach for a phase space formulation of quantum mechanics.
Appendices

The following appendices contains sample proofs aimed at illustrating some of the mathematics as well as provide examples that hopefully clarifies some perhaps involved mathematical ideas.

A. Extension of GF(3) to GF(3^2).

We now illustrate how the extended field $GF(p) \Rightarrow GF(p^m)$, in a positive integer allows the construction of APG. Our example is $d = 3^2$.

Consider a polynomial of degree 2 ($m=2$) with coefficients in GF(3) ($p=3$) that has no roots in GF(3). e.g. $(-1=2 \bmod 3)$

\[ q(u) = u^2 - u - 1. \]  \hspace{1cm} (81)

(One can readily check that no $u \in GF(3)$, viz. $u=0,1,2$ is a root.) The field is now extended to include the root of $q(u)$, viz $u \in q(u) = 0$. Thus the elements of the field are the elements of GF(3) and their sum and product with $u$ - the root of the irreducible polynomial, $q(u)$:

$GF(3^2) = \{0, 1, 2, u, u+1, u+2, 2u, 2u+1, 2u+2\}$.  \hspace{1cm} (82)

For example:

\[ (2u + 1) + (2u + 2) = u; \quad (2u + 1)(2u + 2) = u; \quad \frac{1}{2u + 1} = 2u. \]

We now consider a square, 9x9, array whose horizontal direction (x axis) is labelled with

\[ x = 0, 1, 2, u, u+1, u+2, 2u, 2u+1, 2u+2, \]

with similar labelling for the vertical direction (y axis). A "point" is the "coordinate" (x,y) and the lines $L_{(\alpha,\alpha')}$, ($(\alpha,\alpha')$ two distinct points) are given, for 81 ($3^2 \cdot 3^2$) of them by a linear relation,

\[ y = ax + b; \quad x, y, a, b \in GF(3^2) \]  \hspace{1cm} (83)

and the other 9 ($3^2$) by

\[ x = b; \quad x, b \in GF(3^2). \]  \hspace{1cm} (84)

Thus we have $d^2 = 81$ points and $d(d + 1) = 90$ lines with $d + 1 = 10$ pencils (i.e. sets of parallel lines) thence we have constructed a FAPG. We are thus assured that for this dimensionality both DAPG and PPG exist.

B. Proof that in a linear space knowing the number of points is $d^2$ and that the number of points per line is $d$, the same for all lines suffice to show that the linear space is dual affine plane geometry, DAPG.

Given a linear space S wherein the following items holds i. $\nu = d^2$, and ii. $k_{L_j} = k_L = d$. We then prove that S=A i.e. DAPG.

We first show i. and ii. implies $b=d(d+1)$ and $r_{S_a} = r_S = d + 1$;

Consider an arbitrary point, $S_\beta$. We now count the points residing on the $r_{S_a}$ lines that share the point $S_\beta$ excluding the point itself. This number is, given ii above, is $(d-1)r_{S_\beta}$. Since $S_\beta$ is connected uniquely, given $\lambda 1$, to every other point, all points are counted and each point is counted once. Thus we have using i.,

\[(d-1)r_{S_\beta} = \nu - 1 = d^2 - 1 \Rightarrow r_{S_a} = r_S = d + 1.\]  \hspace{1cm} (85)

Now counting incidences in two ways $\mathfrak{B}$ (p. 14), we have quite generally, Eq.(83),

\[ \sum_{\alpha} r_{S_{\alpha}} = \sum_j k_{L_j} \Rightarrow r_{S\nu} = k_L b \Rightarrow (d+1)d^2 = db. \]  \hspace{1cm} (86)

i.e. $b=(d+1)d$. QED Now $r_S = d + 1$ and $k_L = d$ implies A: $\lambda 1$ implies that d lines on $S_\alpha$ not on $L_j$ connects it with $L_j$ and exactly one that does not.
C. Properties of Pencils within DAPG.

Generalities: \( r_s \) = number of pencils, \( r_{cs} \) = number of points in DAPG pencil.

i. Pencils are mutually exclusive.
Let \( \Pi \) define pencil relation: \( \alpha \Pi \alpha' \leftrightarrow \alpha \) and \( \alpha' \) are not connected by a line.
We shall show that \( \alpha \Pi \alpha', \alpha \Pi \alpha'' \Rightarrow \alpha' \Pi \alpha'' \).
Proof. Suppose \( \alpha' \) and \( \alpha'' \) are joined by a line, \( L(\alpha', \alpha'') \). This line has two points not connected to \( \alpha \). Contradiction (note axiom \( \tilde{A} \)).
Thus a pencil is an equivalence class \([1]\).

ii. Pencils are equally populated (\( r_{cs} \)).
Consider a point \( \alpha \). A set of \( d \) lines share it. Consider a line \( L_j \) not in this set. It has one point, \( \alpha' \), not connected to \( \alpha \) (cf. Axiom \( \tilde{A} \)). There are \( d \) lines sharing through this point. Since there are all together \( d^2 \) lines there are \( d-1 \) such sets each with \( d \) lines and each with a point \( \alpha'' \) not connected to \( \alpha \). With \( \alpha \) there are thus \( d \) point in a pencil, \( r_{cs} = d \).

iii. There are \( r_s = k \).
There are \( d+1 \) points on a line, each belonging to a distinct pencil. Each pencil is an exclusive set containing \( d \) points. There are all together \( d(d+1) \) points. Thus \( d(d+1) = d(r_s) \Rightarrow r_s = d + 1 = k \).

iv. The universal function and pencil’s constituents.
We now prove that The definition, Eq.(5):
\[
\hat{S}_\alpha = \frac{1}{d} \sum_{j \in \alpha} \hat{L}_j \Rightarrow \sum_{\alpha \in p} \hat{S}_\alpha = \sum_{\alpha' \in p'} \hat{S}_{\alpha'}.
\]
i.e. the RHS sum is independent of the pencil, \( p \).
Proof:
\[
\hat{S}_\alpha = \frac{1}{d} \sum_{j \in \alpha} \hat{L}_j \Rightarrow \\
\sum_{\alpha \in p} \hat{S}_\alpha = \frac{1}{d} \sum_{j} \hat{L}_j, \Rightarrow \\
\sum_{\alpha \in p} \hat{S}_\alpha = \sum_{\alpha' \in p'} \hat{S}_{\alpha'}. QED.
\]
(87)

D. Proof of Eq.(33).
Consider \( |m(b); b\rangle \) for \( m(b) = m_0 + b\tilde{m} - b/1, \; \text{for} \; b \neq \tilde{0}; \; m(\tilde{0}) = \tilde{m} \).
For \( \tilde{m}, m_0 = 0 \) the line made of the projectors \( | - b/2; b\rangle \langle b; -b/2| \).
We have,
\[
| - b/2; b\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |n\rangle \omega^{\frac{n^2}{2}}.
\]
Thus,
\[
\hat{X}^{\tilde{m}} \hat{Z}^{-m_0} | - b/2; b\rangle = \frac{1}{\sqrt{d}} \sum |n + \tilde{m}\rangle \omega^{\frac{n^2}{2} - m_0 n} =
\]
\[
= \frac{\omega^{\tilde{m}(\tilde{m} + m_0)}}{\sqrt{d}} \sum |n\rangle \omega^{\frac{1}{2}(m_0(n-1) - (m_0 + b\tilde{m} - b/2)n)} = \omega^{\tilde{m}(\tilde{m} + m_0)} |m_0 + b\tilde{m} - b/2; b\rangle.
\]
(88)
Hence
\[
\hat{X}^{\tilde{m}} \hat{Z}^{-m_0} \hat{L}_{0,0} \hat{Z}^{m_0} \hat{X}^{-\tilde{m}} = \hat{L}_{\tilde{m}, m_0}.
\]
(89)
QED.

[1] M. K. Bennett, *Affine and Projective Geometry*, John Wiley & Sons, Inc. New York (1995).
[2] M. R. Schroeder, *Number Theory in Science and Communication*, Springer-Verlag, Berlin (1984).
[3] L. M. Batten and A. Beutelspacher, *The Theory of Finite Linear Spaces*, Cambridge University press, 1993.
[4] E. P. Wigner, Phys. Rev. 40, 749 (1932).
[5] M. Reimpell and R. Werner, Phys. Rev. A., 75, 062334 (2007).
[6] M. Revzen, Quant. Studies: Math. Found. 2, 77 (2014).
[7] R. J. Glauber, Phys. Rev. 130, 2529 (1963).
[8] M. Reimpell and R. Werner, Phys. Rev. A., 75, 062334 (2007).
[9] M. Revzen, Quant. Studies: Math. Found. 2, 77 (2014).
[10] E. P. Wigner, Phys. Rev. 40, 749 (1932).
[11] M. R. Schroeder, *Number Theory in Science and Communication*, Springer-Verlag, Berlin (1984).
[12] L. M. Batten and A. Beutelspacher, *The Theory of Finite Linear Spaces*, Cambridge University press, 1993.
[13] E. P. Wigner, Phys. Rev. 40, 749 (1932).
[14] M. Reimpell and R. Werner, Phys. Rev. A., 75, 062334 (2007).
[15] M. Revzen, Quant. Studies: Math. Found. 2, 77 (2014).
[16] R. J. Glauber, Phys. Rev. 130, 2529 (1963).
[17] K. S. Gibbons, M. J. Hoffman and W. K. Wootters, Phys. Rev. A. 70, 062101 (2004).
[18] J.R. Klauder and B.-S. Skagerstam, editors *Coherent states* World Scientific Publishing Co., Singapore (1985).
[19] J. Schwinger, Proc. Nat. Acad. Sci. USA 46, 560 (1960).
[20] J. Schwinger, *Quantum Mechanics: Symbolism of Atomic Measurements* World Scientific Publishing Co., Singapore (1985).
[21] M. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* Cambridge U. Press (2000).
[22] A. Kalev, M. Revzen and F. C. Khanna, Phys. Rev. A 80, 022112 (2009).
[23] A. B. Klimov, J. L. Romero, G. Bjork and L. L. Sanchez-Soto, J. Phys. A: Math. Theor. Phys. 40, 3987 (2007); Ann. Phys. (NY), 324, 53 (2009).
[24] A. B. Klimov, L. L. Sanchez-Soto and H. de Guise, J. Phys. A, 39, 2747 (2005).
[25] F. E. Schroeck, *Quantum Mechanics in Phase Space*, World Scientific, New Jersey (2005).
[26] J. R. Klauder and E. C. G. Sudarshan, *Fundamental Quantum Optics*, W. A. Benjamin, New-York (Reprinted by Dover, Mineola, (2006)).
[27] L. Cohen, J. Math. Phys. 7, 781 (1966).
[28] A. B. Klimov, C. Munos and J. L. Romero, J. Phys. A 39, 14471 (2006).
[29] W. Schleich, *Quantum Optics in Phase Space*, Wiley, New-York (2001).
[30] T. Durt, B.-G. Englert, I. Bengtsson and K. Zyczkowski, J. Quant. Inf. 8, 535, (2010).
[31] J. Loustau and M. Dillon, Jour. Math. Phys. 50, 1742 (2006).
[32] P. K. Aravind, Naturforsch., 58a, 85 (2003)
[33] J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).
[34] H. Groenewold, Physica (Amsterdam), 12, 405 (1946).
[35] U. Leonhardt, *Measuring the Quantum State of Light*, Cambridge U. Press, 1997.
[36] B.-G. Englert and Y. Aharonov, Phys. Lett. A. 284, 1 (2001).
[37] I. Bengtsson, *Measuring the Quantum State of Light*, Cambridge U. Press, 1997.
[38] D. Elinas and A. J. Bracken, Phys. Rev. A. 78, 052106 (2008).
[39] P. A. Mello and M. Revzen, Phys. Rev. A 89, 012106 (2014).
[40] K. Zyczkowski, J. Phys. A, 39, 2747 (2005).
[41] A. Grossman, Math. Phys. 48, 191 (1976)
[42] A. Royer, Phys. Rev. A 15, 449 (1977); 43, 44 (1991); 45, 793 (1992).
[43] R. F. Bishop and A. Vourdas, Phys. Rev. A 50, 4488 (1994).
[44] A. Mann, P. A. Mello and M. Revzen, unpublished.
[45] W. K. Wootters, Phys. Rev. A 81, 012113 (2010).
[54] M. Revzen, quant-ph/1111.6446v4 (2011).
[55] M. Revzen, EPL, 98, 1001 (2012).
[56] M. Revzen, J. Phys. A: Math. Theor., 46, 075303 (2013).
[57] A. Kalev, A. Mann and M. Revzen, Eur. Phys. Lett., 104, 50008 (2013).