A PRIORI ESTIMATES FOR THE FRACTIONAL $p$–LAPLACIAN WITH NONLOCAL NEUMANN BOUNDARY CONDITIONS AND APPLICATIONS

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Abstract. We first prove that solutions of fractional $p$–Laplacian problems with nonlocal Neumann boundary conditions are bounded and then we apply such a result to study some resonant problems by means of variational tools and Morse theory.

1. Introduction. In this paper we deal with problems of the form

\[
\begin{aligned}
(-\Delta)^s_p u &= f(x,u) \quad \text{in } \Omega, \\
\mathcal{N}_{s,p} u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

where $p \in (1, \infty)$, $s \in (0, 1)$, $\Omega$ is a bounded domain with Lipschitz boundary and $(-\Delta)^s_p$ is the fractional $p$-Laplacian, defined as (up to a multiplicative constant)

\[
(-\Delta)^s_p u(x) := P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}} dy
\]

for $x \in \Omega$ and $P.V.$ is the Cauchy Principal value.

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Quite recently such an operator has been associated to a nonlocal Neumann boundary condition, namely
\[ N_{s,p}u(x) := \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, dy = 0 \]
for every \( x \in \mathbb{R}^N \backslash \Omega \). Such a condition was introduced in [8] for \( p = 2 \) and generalized in [2] and [14] for \( p > 1 \), and it turns out that it is the natural \( p \)-Neumann boundary condition associated to \((-\Delta)^s_p\). Indeed, this boundary condition allows us to give a formulation of weak solutions of problem (1.1) in a variational way, see [2, 8, 14]. Moreover, in the case \( p = 2 \), it happens that among all functions in the associated fractional Sobolev space (to be defined below), the ones minimizing the Gagliardo seminorm, automatically satisfy the nonlocal Neumann boundary condition above, see [7]. Let us also remark that, far from being an abstract mathematical study, this condition also has an interesting probabilistic motivation, see [19].

As for \( f \), it is a Carathéodory function satisfying standard growth conditions, up to the natural critical growth, see Section 3 below for the precise assumptions.

Problems like (1.1) are not deeply investigated yet. There are many reasons for this fact: first, the nonlocality of the operator, as well as the nonlocality of the boundary conditions, for sure are responsible for this fact. However, a crucial point is that the usual tools which are available in classical elliptic equations, such as a priori estimates on the \( L^\infty \)-norm of the solutions, were lacking so far, except for the case \( p = 2 \), recently studied in [1], where also Hölder continuity of solutions is proved.

This paper aims to start a regularity theory for solutions of problem (1.1), showing that, under natural growth conditions on \( f \), all solutions are bounded, see Theorem 3.1. As far as we know, this is the first regularity result for solutions of (1.1) for general \( p \in (1, \infty) \). In the case \( p = 2 \), as a consequence of related results for the Dirichlet case, we obtain that every weak solution of problem (1.1) is continuous on the whole of \( \mathbb{R}^N \), see Theorem 3.2, a result related to those in [1, 6].

The proof of Theorem 3.1 relies on a fractional version of De Giorgi’s iteration method, which has been usefully employed in the case of fractional Dirichlet boundary conditions (for instance, see [10, 11]), and also for eigenvalues of the Neumann fractional problem (see [14]). Of course, due to the non local nature of the problem, the classical steps cannot be followed verbatim and several novelties are needed.

In the second part of the paper, as an application of the \( L^\infty \) bound, we study problem (1.1) in the asymptotically \( p \)-linear case, that is when
\[ \lim_{|t| \to \infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda \in \mathbb{R}. \]
In particular, we prove that both in the resonant and the non-resonant case there exists a nontrivial solution of problem (1.1). The proofs of these results are both based on critical group theory.

We conclude with an application to \( p \)-superlinear problems associated to linking-type structures, that we face through the notion of cohomological local splitting.

The results proved in this paper are closely related to the ones in [11], where fractional problems were studied under Dirichlet boundary conditions, and are obtained also by exploiting some properties already established for the Neumann case in [14].

The paper is organized as follows. In Section 2 we define the functional space \( X \) where the problem is settled and we give some basic properties. We also recall
some notions of critical group theory and Morse theory that will be useful in the next sections.

In section 3 we first give the $L^\infty$ a priori bound for all solutions of problem (1.1) in Theorem 3.1, while the continuity result for the case $p=2$ is covered by Theorem 3.2.

In Section 4 we deal with the asymptotically $p$–linear problem. First we deal with the non-resonant case, and under suitable conditions on the nonlinearity, we prove the existence of nontrivial solutions in Theorem 4.1. The resonant case is covered by Theorem 4.2, in which we ensure the existence of nontrivial solutions, as well.

Finally, in Section 5 we face $p$–superlinear problems whose associated functional exhibit a cohomological local splitting near 0. Moreover, when the parameter $\lambda$ doesn’t belong to the spectrum of the operator, we show that a ground solution exists.

2. Variational setting. We start defining the functional space in which we seek solutions. For $p \in (1, \infty)$, $s \in (0, 1)$ and $N \geq 1$, we set

$$\|u\| := \left(\int_{Q} |u(x) - u(y)|^p |x - y|^{N+ps} \, dxdy + \|u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

where $Q = \mathbb{R}^{2N} \setminus (C\Omega)^2$, $C\Omega = \mathbb{R}^N \setminus \Omega$, and

$$X := \{u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable such that } \|u\| < \infty\}.$$  

We recall that $X$ is a reflexive Banach space with respect to the norm $\| \cdot \|$, see [14, Proposition 2.2].

Solutions of the studied problem are sought in $X$. Indeed, we say that $u \in X$ is a weak solution of problem (1.1) if

$$\frac{1}{2} \int_{Q} J_p(u(x) - u(y))(v(x) - v(y)) |x - y|^{N+ps} \, dxdy = \int_{\Omega} f(x, u)v \, dx$$

for every $v \in X$, provided that the right hand side is well defined. Here we have used the standard notation

$$J_p(t) = |t|^{p-2}t$$

for every $t \in \mathbb{R}$.

While for the regularity results we need only the standard control on the growth of $f$ at infinity, when dealing with the existence results we need other notions and conditions. More precisely, first we recall that the operator $(-\Delta)^s_p$ admits an unbounded sequence of eigenvalues $\lambda_k$ (see [14]). Since, as in the local case, we don’t know if this sequence covers the entire spectrum when $p \neq 2$, we also need to introduce the spectrum of $(-\Delta)^s_p$, which we denote by $\sigma(s, p)$.

Finally, we recall some notions of critical group theory, see for example [12] and [17]. Let $\Phi : X \rightarrow \mathbb{R}$ be of class $C^1$. We denote by $K(\Phi)$ the set of critical points of $\Phi$,

$$K(\Phi) = \{u \in X : \Phi'(u) = 0\}.$$  

We also use the notation

$$\Phi^a = \{u \in X : \Phi(u) \leq a\}, \quad a \in \mathbb{R}.$$
Assuming that \( x \in X \) is an isolated critical point of \( \Phi \), the \( k \)-th (cohomological) critical groups of \( \Phi \) at \( x \) is defined by

\[
C^k(\Phi, x) = H^k(\Phi^c \cap U, \Phi^c \cap U \setminus \{x\})
\]

for all \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), where \( c = \Phi(x) \), \( U \) is a neighborhood of \( x \) such that \( K(\Phi) \cap \Phi^c \cap U = \{x\} \), and \( H^*(A, B) \) denotes the \( * \)-th Alexander-Spanier cohomology group with coefficients in \( \mathbb{Z}_2 \) of the topological pair \((A, B)\). By the excision property of the cohomology groups, the definition of critical groups is independent of the particular choice of the neighborhood \( U \).

We also recall some notions of critical group theory. For this, we recall that a real functional \( \Phi \) defined in a Banach space \( X \) satisfies the (PS) condition if every sequence \((u_n)_n \subset X\) such that \( \Phi(u_n) \) is bounded and \( \Phi'(u_n) \to 0 \) in \( X' \) as \( n \to \infty \) admits a strongly converging subsequence. Moreover, we say that \( \Phi \) satisfies the (C) condition if, for every sequence \((u_n)_n \subset X\) such that \( \Phi(u_n) \) is bounded and \( (1 + \|u_n\|)\Phi'(u_n) \to 0 \) in \( X' \) as \( n \to \infty \), there exists a strongly converging subsequence of \((u_n)_n\).

Finally, we denote by \( K(\Phi) \) the set of critical points of \( \Phi \).

We recall the following well-known facts (see [3, 4]):

**Proposition 1.** Let \( X \) be a Banach space, \( u \in X \) and for all \( \tau \in [0, 1] \) let \( \Phi_\tau \in C^1(X) \) be a functional such that \( u \in K(\Phi_\tau) \). If there exists a closed neighborhood \( U \subset X \) of \( u \) such that

1. \( \Phi_\tau \) satisfies (PS) in \( U \) for all \( \tau \in [0, 1] \),
2. \( K(\Phi_\tau) \cap U = \{u\} \) for all \( \tau \in [0, 1] \),
3. the mapping \( \tau \mapsto \Phi_\tau \) is continuous from \([0, 1]\) to \( C^1(U) \),

then \( C^k(\Phi_1, u) = C^k(\Phi_0, u) \) for all \( k \in \mathbb{N}_0 \).

On the other hand, it is almost trivial to compute critical groups in extremal points:

**Proposition 2.** Let \( X \) be a Banach space with \( \dim(X) = \infty \), let \( \Phi \in C^1(X) \) be a functional satisfying (C), and let \( u \in K(\Phi) \) be an isolated critical point of \( \Phi \). Then:

1. if \( u \) is a local minimizer of \( \Phi \), then \( C^k(\Phi, u) = \delta_{k,0}\mathbb{Z}_2 \) for all \( k \in \mathbb{N}_0 \),
2. if \( u \) is a local maximizer of \( \Phi \), then \( C^k(\Phi, u) = 0 \) for all \( k \in \mathbb{N}_0 \).

Here, as usual, \( \delta_{ij} \) denotes the Kronecker symbol.

**Definition 2.1.** A functional \( \Phi \) has a cohomological local splitting near 0 in dimension \( k \in \mathbb{N} \) if there exist symmetric cones \( X_\pm \subset X \) with \( X_+ \cap X_- = \{0\} \) and \( \rho > 0 \) such that

1. \( i(X_- \setminus \{0\}) = i(X \setminus X_+) = k \),
2. \( \Phi(u) \leq \Phi(0) \) for all \( u \in \overline{B}_\rho \cap X_- \), and \( \Phi(u) \geq \Phi(0) \) for all \( u \in \overline{B}_\rho \cap X_+ \).

Here \( i \) denotes the \( \mathbb{Z}_2 \) cohomological index introduced in [9].

We shall use the following result (see [5, Proposition 2.1]):

**Proposition 3.** If \( X \) is a Banach space and \( \Phi \in C^1(X) \) has a cohomological local splitting near \( 0 \) in dimension \( k \in \mathbb{N} \), and \( 0 \) is an isolated critical point of \( \Phi \), then \( C^k(\Phi, 0) \neq 0 \).

3. **Regularity.** In this section we prove some \( L^\infty \) a priori estimates and some regularity results for solutions of problems of the type

\[
\begin{cases}
(-\Delta)_p^s u + |u|^{p-2}u = f(x, u) \quad &\text{in } \Omega, \\
\mathcal{N}_{s, p} u = 0 \quad &\text{in } \mathbb{R}^N \setminus \overline{\Omega}.
\end{cases}
\tag{3.1}
\]
We just suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

\((f_1)\) there exists $a > 0$ such that

$$|f(x, t)| \leq a(|t|^q + |t|^{r-1})$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$, with $1 \leq q \leq p \leq r \leq p^*_s$.

As usual, $p^*_s$ is the critical fractional Sobolev exponent, namely

$$p^*_s = \begin{cases} \frac{pN}{N-ps} & \text{if } N < ps, \\ \infty & \text{if } N \geq ps. \end{cases}$$

We tacitly assume that in $(f_1)$ we have $r < \infty$ also when $N \geq ps$.

**Remark 1.** Clearly, if we prove an estimate for a solution of (3.1), then it will also be true for the problem

\[
\begin{cases}
(-\Delta)^s_p u = f(x, u) & \text{in } \Omega, \\
\mathcal{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

since $\tilde{f}(x, u) = f(x, u) - |u|^{p-2}u$ still satisfies condition $(f_1)$.

We follow the lines of the analogous proofs in [11] for the Dirichlet case, but, while adapting the original proofs to our situation, we take the opportunity to give more details.

The first result is the following.

**Theorem 3.1.** If hypothesis $(f_1)$ holds with $1 \leq q \leq p \leq r \leq p^*_s$ satisfying

$$1 + \frac{q}{p} > \frac{r}{p} + \frac{r}{p^*_s},$$

then there exist $K > 0$ and $\alpha > 1$, both depending on $p, q, r, s, a, |\Omega|$, such that every weak solution $u$ of (3.1) belongs to $L^\infty(\Omega)$ and

$$\|u\|_\infty \leq K(1 + \|u\|^p).$$

**Proof.** First, we fix a weak solution $u \in X$ of (3.1) with $u^+ \neq 0$. Take $\rho \geq \max\{1, \|u\|^{-1}_r\}$ and set $v := (\rho\|u\|^{-1}_r)u$. In this way, $v \in X$, and setting $\|v\|_r = \rho^{-1}$, we have that $v$ is a weak solution of the problem

\[
\begin{cases}
(-\Delta)^s_p v + |v|^{p-2}v = (\rho\|u\|^{-1}_r)^{1-p}f(x, \rho\|u\|^{-1}_r)v & \text{in } \Omega, \\
\mathcal{N}_{s,p} v = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

(3.2)

For all $n \in \mathbb{N}$ we set $v_n := (v - 2^{-n})^+$, so that $v_n \in X, v_0 = v^+, 0 \leq v_{n+1} \leq v_n(x)$ for all $n \in \mathbb{N}$ and $v_n(x) \to (v(x) - 1)^+$ a.e. in $\Omega$ as $n \to \infty$. We also have the following inclusion:

$$\{v_{n+1} > 0\} \subseteq \{0 < v < (2^{n+1} - 1)v_n\} \cup \{v_n > 2^{-n-1}\}. \quad (3.3)$$

Now, we set $R_n := \|v_n\|_p^p$ for all $n \in \mathbb{N}$. So, $R_0 = \|u^+\|_p^p \leq \rho^{-r}$ and $(R_n)_n$ is nonincreasing in $[0, 1]$. We claim that $R_n \to 0$ as $n \to \infty$.

By Hölder’s inequality, the fractional Sobolev inequality and (3.3) we have

$$R_{n+1} \leq \|v_{n+1} > 0\|_p^p \leq C|\{v_n > 2^{-n-1}\}|^{1-p/p_s} \|v_{n+1}\|_p^p,$$

for all $n \in \mathbb{N}$. Using Chebichev’s inequality we finally obtain

$$R_{n+1} \leq C 2^{(r-p_s)}(n+1) R_n^{1-p/p_s} \|v_{n+1}\|_p^p. \quad (3.4)$$
Now, we need an estimate on \( \|v_{n+1}\| \). To do that, we introduce the inequality

\[
|\xi^+ - \eta^+|^p \leq |\xi - \eta|^{p-2}(\xi - \eta)(\xi^+ - \eta^+),
\]

(3.5)

for all \( \xi, \eta \in \mathbb{R} \). Testing (3.1) with \( v_{n+1} \) and using (3.5), with \( \xi^+ = v_{n+1}(x) \) and \( \eta^+ = v_{n+1}(y) \), we get

\[
\|v_{n+1}\|^p \leq \langle A(v), v_{n+1} \rangle
\]

\[
= \int_{\Omega} (\|u\|_r)^{1-p} f(x, \rho \|u\|_r) v_{n+1} \, dx
\]

\[
\leq a \int_{\{v_{n+1} > 0\}} ((\|u\|_r)^{q-p} |v|^{q-1} + (\|u\|_r)^{r-p} |v|^{r-1}) v_{n+1} \, dx
\]

\[
\leq a (\|u\|_r)^{r-p} \int_{\{v_{n+1} > 0\}} ((2^{n+1} - 1)^{q-1} v_n^q + (2^{n+1} - 1)^{r-1} v_n^r) \, dx
\]

\[
\leq a 2^{(r-1)(n+1)} (\|u\|_r)^{r-p} \int_{\{v_{n+1} > 0\}} (v_n^q + v_n^r) \, dx.
\]

Now,

\[
\int_{\Omega} v_n^q \leq |\Omega|^{1-\frac{q}{p}} \left( \int_{\Omega} v_n^r \right)^{\frac{q}{r}}
\]

and

\[
\int_{\Omega} v_n^r \leq \left( \int_{\Omega} v_n^r \right)^{1-\frac{q}{p}} \left( \int_{\Omega} (v^+) \right)^{1-\frac{q}{p}} \left( \int_{\Omega} v_n^r \right)^{\frac{q}{r}},
\]

so we finally get

\[
\|v_{n+1}\|^p \leq C 2^{(r-1)(n+1)} (\|u\|_r)^{r-p} R_n^2.
\]

Combining the last estimate with (3.4), we have

\[
R_{n+1} \leq C 2^{(r-\frac{r}{p} + \frac{q}{p} - \frac{r}{p})^+(n+1)} (\|u\|_r)^{\frac{q}{p} - r} R_n^1 + \frac{\beta}{p} - \frac{q}{p},
\]

which can be written as

\[
R_{n+1} \leq C H^n (\|u\|_r)^{\frac{q}{p} - r} R_n^{1+\beta},
\]

(3.6)

with

\[
H := 2^{r-\frac{r}{p} + \frac{q}{p} - \frac{r}{p}} > 0
\]

and

\[
0 < \beta := \frac{q}{p} - \frac{r}{p} < 1.
\]

Setting \( \gamma := r\beta + r - r^2/p > 0 \) and \( \eta := H^{-\frac{\beta}{p}} \in (0, 1) \), we can take

\[
\rho = \max\{1, \|u\|_r^{-1}, \eta^{-\frac{\beta}{p}} \|u\|_r^{\frac{\beta}{p} - r} \}.
\]

Now, we prove by induction that

\[
R_n \leq C \frac{\eta^n}{\rho^n}.
\]

(3.7)

Indeed, \( R_0 = \|v^+\|_r \leq \rho^{-r} \). Now we can assume that (3.7) holds for some \( n \in \mathbb{N} \), and so by (3.6)

\[
R_{n+1} \leq C H^n (\|u\|_r)^{\frac{q}{p} - r} \left( \frac{\eta^n}{\rho^n} \right)^{1+\beta} = C \frac{\eta^n}{\rho^n} \|u\|_r^{\frac{q}{p} - r} \leq C \frac{\eta^{n+1}}{\rho^n}.
\]

Now, we prove by induction that

\[
R_n \leq C \frac{\eta^n}{\rho^n}.
\]

(3.7)

Indeed, \( R_0 = \|v^+\|_r \leq \rho^{-r} \). Now we can assume that (3.7) holds for some \( n \in \mathbb{N} \), and so by (3.6)
Since $\eta \in (0,1)$, from (3.7) $R_n \to 0$ as $n \to \infty$. This implies that $v_n \to 0$ a.e. in $\Omega$, and so $v(x) \leq 1$ a.e. in $\Omega$. A similar argument on $-v$ leads to $v \in L^\infty(\Omega)$ and $\|v\|_\infty \leq 1$. So $u \in L^\infty(\Omega)$ and
\[
\|u\|_\infty \leq \rho \|u\|_r = \max\{\|u\|_r, 1, \eta^{-\frac{r}{p}} \|u\|_r^{1+\frac{r}{p}} \} \leq K(1 + \|u\|_r),
\]
for some $K > 0$ and $\alpha > 1$.

In the next result, we assume $q = p$ in $(f_1)$. Then, if we take $\|u\|_r$ sufficiently small, the $L^\infty$ estimate can be formulated in terms of the $L^r$ norm of the solution.

**Corollary 1.** If hypothesis $(f_1)$ holds with $q = p \leq r < p^*_s$, then there exists a constant $K \in (0,1)$, depending on $s, p, r, N, |\Omega|$, such that, for every weak solution $u \in X$ of (3.1) with $\|u\|_r \leq K$, we have $u \in L^\infty(\Omega)$ and
\[
\|u\|_\infty \leq K^{-1} \|u\|_r.
\]

**Proof.** Consider $\varepsilon \in (0,1)$ and let $u \in X$ be a weak solution of (3.1) with $u^+ \neq 0$ and $\|u\|_r \leq \varepsilon$. Now we set $v := \varepsilon^{-1}u$, so $v \in X$ and $\|v\|_r \leq 1$. As before, for all $n \in \mathbb{N}$ we can set $v_n = (v - 1 + 2^{-n})^+$ and $R_n = \|v_n\|_r$. Arguing as in the proof of Theorem 3.1 we can obtain the inequality
\[
R_{n+1} \leq CH^n R_n^{1+\beta}, \tag{3.8}
\]
for some $H > 1$ and $0 < \beta < 1$. Setting $\eta = H^{-\frac{1}{\beta}}$, we have $K := \eta^{\frac{1}{p^*_s}} < 1$. Now, if $\|u\|_r < K$, we can take $\varepsilon$ such that $\|u\|_r = \delta = K\varepsilon \in (0,\varepsilon)$. Now, we want to prove that
\[
R_n \leq C \frac{\delta^r}{\varepsilon^r} \eta^n. \tag{3.9}
\]
Indeed, by definition $R_0 \leq \delta^r/\varepsilon^r$. Then, if (3.8) holds for some $n \in \mathbb{N}$, by (3.9)
\[
R_{n+1} \leq CH^n \left( C \frac{\delta^r}{\varepsilon^r} \eta \right)^{1+\beta} = C \frac{\delta^r}{\varepsilon^r} \eta^{\frac{1+\beta}{\beta}} \eta^n = C \frac{\delta^r}{\varepsilon^r} \eta^{n+1}.
\]
From (3.8), $R_n \to 0$ as $n \to \infty$ and so $v(x) \leq 1$ a.e. in $\Omega$. Arguing in a similar way on $-v$, we have $\|v\|_\infty \leq 1$, and so
\[
\|u\|_\infty \leq \varepsilon = K^{-1} \|u\|_r.
\]
Since $\varepsilon \in (0,1)$, for every solution $u$ with $\|u\|_r < K$, we can write $\|u\|_r = K\varepsilon$ and obtain the desired estimate. □

**Remark 2.** We recall that from the Neumann boundary condition we have
\[
\|u\|_{L^\infty(\mathbb{R}^N)} = \|u\|_{L^\infty(\Omega)}.
\]
Indeed, if we take $x \in \mathbb{R}^N \setminus \overline{\Omega}$, then from $\mathcal{N}_{s,p} u = 0$ we have
\[
u(x) \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}u(y)}{|x-y|^{N+ps}} dy = \int_{\Omega} |u(x) - u(y)|^{p-2}u(y) |x-y|^{N+ps} dy.
\]
Now we can assume that $u$ is not constant, otherwise it would be bounded, and obtain
\[
\|u(x)\| = \left| \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}u(y)}{|x-y|^{N+ps}} dy \right| \leq \|u\|_{L^\infty(\Omega)},
\]
which proves our claim (see also [14, Proposition 3.4]).
Now we want to prove that every weak solution of problem
\[
\begin{aligned}
&(-\Delta)^s u = f \quad \text{in } \Omega, \\
&\mathcal{N}_s u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]  
(3.10)
is continuous. The goal is to use \cite[Theorem 1.2]{18}. To do this, we need to prove that every weak solution is continuous outside of \( \Omega \).

**Proposition 4.** Let \( u \in X \) be a weak solution of problem (3.10). Then, \( u \in C(\mathbb{R}^N \setminus \Omega) \).

**Proof.** From \cite[Theorem 2.8]{14} we know that \( N_s u = 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \), that is
\[
u(x) = \int_{\Omega} \frac{u(y)}{|x-y|^{N+s}} \, dy = \int_{\Omega} \frac{1}{|x-y|^{N+s}} \, dy,
\]
for a.e. \( x \in \mathbb{R}^N \setminus \Omega \). Clearly, this can be written as a quotient of convolutions:
\[
u \chi_{\Omega} * \frac{1}{|x|^{N+s}} - \chi_{\Omega} * \frac{1}{|x|^{N+s}},
\]
which is continuous in \( \mathbb{R}^N \setminus \Omega \). So, \( u \) is equal a.e. to a continuous function in \( \mathbb{R}^N \setminus \Omega \), hence it is continuous in this set. \( \square \)

**Theorem 3.2.** Let \( p = 2 \) and let \( u \in X \) be a weak solution of problem (3.10). Then, \( u \in C(\mathbb{R}^N) \).

**Proof.** In light of Proposition 4, it is enough to apply \cite[Theorem 1.2]{18} to obtain the continuity for weak solutions of problem (3.10) on the whole of \( \mathbb{R}^N \). \( \square \)

### 4. Asymptotically \( p \)-linear problem.

In this section we consider problem
\[
\begin{aligned}
&(-\Delta)^s u = h(x,u) \quad \text{in } \Omega, \\
&\mathcal{N}_s u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]  
(4.1)
where \( h(x,t) = \lambda |t|^{p-2} t + g(x,t) \) with \( g(x,t) = o(|t|^{p-1}) \) as \( t \to \infty \). In light of this, it is clear that \( h(x,\cdot) \) is asymptotically \( p \)-linear at infinity, that is
\[
\lim_{|t| \to \infty} \frac{h(x,t)}{|t|^{p-2} t} = \lambda
\]
uniformly a.e. in \( \Omega \), for some \( \lambda \in (0, \infty) \). We recall that in general a problem of this kind is said to be of resonant type if \( \lambda \in \sigma(s,p) \). Otherwise, it is said to be of non-resonant type. However, we shall see that problem (4.1) is of resonant type when \( \lambda + 1 \in \sigma(s,p) \). We assume that \( h : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function, with
\[
H(x,t) = \int_0^t h(x,\tau) \, d\tau \text{ for all } (x,t) \in \Omega \times \mathbb{R},
\]
and satisfying the following hypotheses:
\begin{enumerate}[(h1)]  
\item \( |h(x,t)| \leq a(1 + |t|^{r-1}) \) a.e. in \( \Omega \) and for all \( t \in \mathbb{R} \), with \( a > 0 \) and \( 1 < r < p^*_s \),
\item \( \lim_{|t| \to \infty} \frac{h(x,t)}{|t|^{p-2} t} = \lambda \) uniformly a.e. in \( \Omega \), with \( \lambda > 0 \),
\item there exist \( \delta > 0 \) and \( \mu \in (0,p) \) such that
\[
\begin{aligned}
&h(x,t)t > 0, \text{ for } x \in \Omega, \ 0 < |t| < \delta, \\
&\mu H(x,t) - h(x,t)t \geq 0 \text{ for } x \in \Omega, \ |t| < \delta.
\end{aligned}
\]  
\end{enumerate}
As usual, we want the existence of solutions, so we define the functional

\[ J(u) = \frac{1}{2p} \| u \|^p - \frac{1}{2p} \| u \|^p - \int_{\Omega} H(x, t) \, dx, \]

and we look for critical points of \( J \). Here, we have used the usual symbol \([\cdot]\) for the Gagliardo seminorm

\[ [u] := \left( \iint_{Q} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{1/p} \]

for every \( u \in X \).

**Remark 3.** We can write \( J \) as

\[ J(u) = \frac{1}{2p} \| u \|^p - \frac{1}{2p} \| u \|^p - \int_{\Omega} H(x, t) \, dx, \]

which is the functional associated with the problem

\[
\begin{cases}
(-\Delta)^s_p u + |u|^{p-2}u = \tilde{h}(x, u) & \text{in } \Omega, \\
M_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega},
\end{cases}
\]

with \( \tilde{h}(x, u) = |u|^{p-2}u + h(x, u) \). So, this problem admits the same solutions as problem (4.1). In light of this, we can see that the assumptions of [17, Theorem 5.7] and [17, Theorem 5.9] are satisfied, hence we just need to verify if the hypotheses hold in order to use them. Moreover, thanks to the added term we have

\[ \lim_{|t| \to \infty} \tilde{h}(x, t) = \lambda + 1 \]

uniformly a.e. in \( \Omega \), hence we have resonance when \( \lambda + 1 \in \sigma(s, p) \).

Clearly \((h_3)\) implies that \( f(x, 0) = 0 \) a.e. in \( \Omega \), so \( u = 0 \) is a trivial solution of (4.1). We seek the existence of nontrivial solutions, which are critical points of \( J \). Indeed, if there exists a critical point of \( J \) which is not isolated, then we have infinite solutions. So, we can assume that all critical points of \( J \) are isolated.

We first give a result for the non-resonant case:

**Theorem 4.1.** If Hypotheses \((h_1)-(h_3)\) hold with \( \lambda \in (\lambda_k, \lambda_{k+1}) \setminus \sigma(s, p) \) for some \( k \in \mathbb{N} \), then problem (4.1) admits a nontrivial solution.

**Proof.** If \( \lambda > 0 \), then there exists \( k \in \mathbb{N} \) such that \( \lambda_k < \lambda < \lambda_{k+1} \). From [17, Theorem 5.7] there exist some \( u \in X \) such that \( u \) is a critical point of \( J \) and \( C^k(J, u) \neq 0 \).

We now want to prove that \( C^k(J, 0) = 0 \) for every \( k \in \mathbb{N} \). First we note that (4.2) and (4.3) imply that there exist a constant \( c_0 \) such that

\[ H(x, t) \geq c_0 |t|^\mu, \quad \text{for } x \in \Omega, \quad |t| < \delta. \]  

(4.4)

From \((h_1)\) and (4.4) we have that

\[ H(x, t) \geq c_0 |t|^\mu - c_1 |t|^{r'}, \quad \text{for } x \in \Omega, \quad t \in \mathbb{R} \]  

(4.5)

for some \( c_1 > 0 \) and \( r \in (p, p^*_s) \). So, taking \( u \in X \) and \( \tau > 0 \) we have

\[
J(\tau u) = \frac{\tau^p}{2p} \| u \|^p - \int_{\Omega} H(x, \tau u) \, dx \leq \frac{\tau^p}{2p} \| u \|^p - \int_{\Omega} (c_0 |\tau u|^\mu - c_1 |\tau u|^{r'}) \, dx \\
\leq \frac{\tau^p}{2p} \| u \|^p - c_0 \tau^\mu \| u \|_{L^\mu(\Omega)}^\mu + c_1 \tau^{r'} \| u \|_{L^{r'}(\Omega)}^{r'}. 
\]
Since \( \mu < p < r \), for given \( u \in X \) such that \( u \neq 0 \), there exists \( \tau_0 = \tau_0(u) > 0 \) such that
\[ J(\tau u) < 0, \quad \text{for every } \tau \in (0, \tau_0). \] (4.6)
Let \( u \in X \) be such that \( J(u) = 0 \) and \( u \) is not a constant function. So, there exists \( c_2 > 0 \) such that \( \|u\|^p \geq c_2 \|u\|^p \). Then, from \((h_1)\) and the embedding \( X \hookrightarrow L^p(\Omega) \)
\[
\frac{d}{d\tau} J(\tau u)|_{\tau=1} = (J'(\tau u), u) = \frac{1}{2} \|u\|^p - \int_{\Omega} h(x, u) u \, dx
\]
\[
\geq \frac{c_2}{2} \left( 1 - \frac{\mu}{p} \right) \|u\|^p \geq \frac{c_2}{2} \left( 1 - \frac{\mu}{p} \right) \|u\|^p - c_3 \int_{\Omega} |u|^r \, dx
\]
for some \( r \in (p, p^*_\Omega) \) and \( c_3, c_4 > 0 \). So we can conclude that there exists \( \rho > 0 \) such that
\[
\frac{d}{d\tau} J(\tau u)|_{\tau=1} > 0, \quad \forall u \in X \text{ with } J(u) = 0 \text{ and } \|u\| \leq \rho. \] (4.7)
Now we fix \( \rho > 0 \), and we want to prove that (4.7) implies that
\[ J(\tau u) < 0, \quad \text{for } \tau \in (0, 1), \text{ for } u \in X \text{ with } J(u) < 0 \text{ and } \|u\| \leq \rho. \] (4.8)
Clearly, if \( \|u\| \leq \rho \) and \( J(u) < 0 \), then there exists \( \vartheta \in (0, 1) \) such that \( J(su) < 0 \) for all \( \tau \in (1 - \vartheta, 1) \) from the continuity of \( J \). Suppose that there is a \( \tau_0 \in (0, 1 - \vartheta) \) such that \( J(\tau_0 u) = 0 \) and \( J(\tau u) < 0 \) as \( \tau_0 < \tau < 1 \). Denoting \( u_0 := \tau_0 u \), by (4.7) we have
\[
\frac{d}{d\tau} J(\tau u_0)|_{\tau=1} > 0.
\]
On the other hand \( J(\tau u) - J(\tau_0 u) < 0 \) implies that
\[
\frac{d}{d\tau} J(\tau u)|_{\tau=\tau_0} = \frac{d}{d\tau} J(\tau u_0)|_{\tau=1} \leq 0,
\]
which is a contradiction, so (4.8) holds. We notice that if \( u \) is a constant function, (4.4) implies (4.8).

Now we define a mapping \( T : B_p(0) \to [0, 1] \) as
\[
T(u) = \begin{cases} 
1, & \text{for } u \in B_p(0) \text{ with } J(u) \leq 0 \\
\tau, & \text{for } u \in B_p(0) \text{ with } J(u) > 0, J(\tau u) = 0, \tau < 1.
\end{cases}
\]
By (4.6), (4.7) and (4.8), the mapping \( T \) is well defined. Moreover, if \( J(u) > 0 \), then there exists an unique \( T(u) \in (0, 1) \) such that
\[ J(T(u)u) = 0, \quad J(\tau u) < 0 \quad \forall \tau \in (0, T(u)) \quad \text{and} \quad J(\tau u) > 0 \forall \tau \in (T(u), 1). \] (4.9)
The Implicit Function Theorem, (4.7) and (4.9) imply that \( T \) is continuous in \( u \). Now, we define a mapping \( \eta : [0, 1] \times B_p(0) \to B_p(0) \) as
\[
\eta(\tau u) = (1 - \tau)u + \tau T(u)u, \quad \tau \in [0, 1], \ u \in B_p(0)
\]
From the definition of \( T \) we have that \( \eta \) is a continuous deformation \( (B_p(0), B_p(0) \setminus \{0\}) \to (B_p(0) \cap J^0, B_p(0) \cap J^0 \setminus \{0\}) \). Since \( B_p(0) \setminus \{0\} \) is contractible, by the homotopy invariance of cohomology group, we have
\[ C^k(J, 0) = H^k(B_p(0) \cap J^0, B_p(0) \cap J^0 \setminus \{0\}) = H^k(B_p(0), B_p(0) \setminus \{0\}) = 0 \]
for every \( k \in \mathbb{N} \). So, the critical point that we found above cannot be \( u = 0 \). This concludes the proof.

In order to deal with the resonant case, we need to assume additional conditions to have compactness of critical sequences. For all \((x, t) \in \Omega \times \mathbb{R}\) we define

\[
\mathcal{H}(x, t) := pH(x, t) - h(x, t)t
\]

We have the following result:

**Theorem 4.2.** If Hypotheses \((h_1)-(h_3)\) hold with \( \lambda + 1 \in \sigma(s, p) \), and there exist \( k \in \mathbb{N}, h_0 \in L^1(\Omega) \) such that one of the following holds:

(i) \( \lambda_k < \lambda + 1 \leq \lambda_{k+1}, \mathcal{H}(x, t) \leq -h_0(x) \) a.e. in \( \Omega \) and for all \( t \in \mathbb{R} \), and

\[
\lim_{|t| \to \infty} \mathcal{H}(x, t) = -\infty
\]

uniformly a.e. in \( \Omega \).

(ii) \( \lambda_k \leq \lambda + 1 < \lambda_{k+1}, \mathcal{H}(x, t) \geq h_0(x) \) a.e. in \( \Omega \) and for all \( t \in \mathbb{R} \), and

\[
\lim_{|t| \to \infty} \mathcal{H}(x, t) = \infty
\]

uniformly a.e. in \( \Omega \).

Then problem \((4.1)\) admits a nontrivial solution.

**Proof.** Since \( \lambda + 1 \in \sigma(s, p) \), there exists some \( k \in \mathbb{N} \) such that \( \lambda + 1 \in [\lambda_k, \lambda_{k+1}] \), which is a non degenerate interval. We set

\[
\Psi(u) = J(u) - \frac{1}{p} \langle J'(u), u \rangle = -\frac{1}{p} \int_{\Omega} \mathcal{H}(x, u) \, dx
\]

for all \( u \in X \). Assume we are in the case (i). In order to apply [17, Theorem 5.9] we need to verify if the \((H_+)\) condition holds, that is, \( \Psi \) is bounded from below and every sequence \((u_n)_n \subset X\) such that \( \|u_n\| \to \infty \) and \( v_n = u_n/\|u_n\| \) converges weakly to some \( v \neq 0 \) as \( n \to \infty \) admits a subsequence such that

\[
\lim_{n \to \infty} \Psi(\tau u_n) = +\infty, \quad \forall \tau \geq 1.
\]

Respectively, to have condition \((H_-)\) we ask that \( \Psi \) is bounded from above and \( \Psi(\tau u_n) \to -\infty \) for every \( \tau \geq 1 \) (see [17, p. 82]). Clearly, for all \( u \in X \) we have

\[
\Psi(u) \geq \frac{1}{p} \int_{\Omega} \mathcal{H}(x, u) \, dx,
\]

so \( \Psi \) is bounded from below in \( X \). Now let \((u_n)_n \subset X\) be a sequence such that \( \|u_n\| \to \infty \) and \( v_n = u_n/\|u_n\| \to v \neq 0 \) as \( n \to \infty \). In particular, we have that \( v_n(x) \to v(x) \) a.e. in \( \Omega \) as \( n \to \infty \). From the Fatou Lemma we have, for all \( \tau \geq 1 \),

\[
\liminf_{n \to \infty} \Psi(\tau u_n) \geq -\frac{1}{p} \int_{\Omega} \liminf_{n \to \infty} \mathcal{H}(x, \|u_n\| \tau v_n) \, dx = \infty,
\]

so we can conclude that the \((H_+)\) condition holds. Applying [17, Theorem 5.9] we have that \( J \) satisfies the \((C)\) condition and there exists a critical point \( u \) such that \( C^k(J, u) \neq 0 \). As in the proof of Theorem 4.1 we can see that \( C^k(J, 0) = 0 \) for every \( k \in \mathbb{N} \), so that \( u \neq 0 \) is a nontrivial solution of \((4.1)\).

For the case (ii) the argument is similar, with the difference that we need to verify condition \((H_-)\) instead of \((H_+)\). \(\square\)
5. $p$-superlinear problem. In this section we study the problem
\[
\begin{cases}
(-\Delta)^s u = \lambda |u|^{p-2}u + g(x, u) & \text{in } \Omega, \\
\mathcal{M}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega},
\end{cases}
\] (5.1)
where $g$ is a Carathéodory function satisfying the following hypotheses:

$(g_1)$ there exist $a > 0$ and $r \in (p, p^*_s)$ such that
\[|g(x, t)| \leq a(1 + |t|^{r-1})\]
a.e. in $\Omega$ and for all $t \in \mathbb{R}$.

$(g_2)$ there exist $\mu > p$ and $R > 0$ such that
\[0 < \mu G(x, t) \leq g(x, t)t\]
a.e. in $\Omega$ and for all $|t| \geq R$, where $G(x, t) = \int_0^t g(x, \tau) d\tau$, and
\[G(x, t) \geq C_0|t|^p - C_1\] (5.3)
a.e. in $\Omega$ and for all $t \in \mathbb{R}$, for some $C_0, C_1 > 0$.

$(g_3)$
\[\lim_{t \to 0} \frac{g(x, t)}{|t|^{p-1}} = 0\]
uniformly a.e. in $\Omega$.

**Remark 4.** Let us remark that, since the Ambrosetti-Rabinowitz condition in $(g_2)$ is weaker than the original one, inequality (5.3) must be assumed a priori, see [13].

The main result is the following one, which corresponds to [11, Theorem 4.1]:

**Theorem 5.1.** If hypotheses $(g_1)$-$(g_3)$ and one of the following hold:

$(i)$ $2\lambda \neq \lambda_k$ for all $k \in \mathbb{N}$,
$(ii)$ $2\lambda = \lambda_k$, $k \in \mathbb{N}$, and $G(x, t) \geq 0$ a.e. in $\Omega$ and for all $|t| \leq \delta$ for some $\delta > 0$,
$(iii)$ $2\lambda = \lambda_k$, $k \in \mathbb{N}$, and $G(x, t) \leq 0$ a.e. in $\Omega$ and for all $|t| \leq \delta$ for some $\delta > 0$,

then problem (5.1) admits a nontrivial solution.

We define the functional associated to (5.1) as
\[E(u) := \frac{1}{2p} |u|^p - \frac{\lambda}{p} \int \Omega |u|^{p-2}u \, dx - \int \Omega G(x, u) \, dx,\]
so that critical points of $E$ are solution of (5.1).

**Lemma 5.2.** The functional $E$ is of class $C^1(X)$ and satisfies (PS). Moreover, for every $\eta < 0$ the set $E^{-}\eta$ is contractible.

**Proof.** Let $(u_n)_n$ be a sequence in $X$ such that $(E(u_n))_n$ is bounded and $E'(u_n) \to 0$ in $X'$. By (5.3) we have
\[
\left(\frac{\mu}{p} - 1\right) \frac{\|u_n\|^p}{4} = \frac{\mu + p}{2} E(u_n) - \langle E'(u_n), u_n \rangle + \left(\frac{2\lambda + 1}{2p}\right) \|u_n\|^p
\]
\[
+ \int \Omega \left(\frac{\mu + p}{2} G(x, u_n) - g(x, u_n)u_n\right) \, dx
\]
\[
\leq C + \|E'(u_n)\|_{X'} \|u_n\| + \left(\frac{2\lambda + 1}{2p}\right) \|u_n\|^p - \frac{\mu - p}{2} \|u_n\|^\mu,
\]
so $(u_n)_n$ is bounded in $X$. By [15, Proposition 3.2], $E$ satisfies the (PS) condition.
Now, fix \( u \in X \setminus \{0\} \). By (5.3) we have
\[
E(\tau u) \leq \frac{\tau^p |u|^p}{p} - \frac{\lambda \tau^p \|u\|^p}{p} - C(\tau^\mu \|u\|^\mu - 1)
\]
for all \( \tau > 0 \). So
\[
\lim_{\tau \to \infty} E(\tau u) = -\infty
\]
and \( E \) is unbounded below. Moreover, by (5.2)
\[
\langle E'(u), u \rangle = pE(u) - \int_\Omega (pG(x, u) - g(x, u)u) \, dx \leq pE(u).
\]
Taking \( \eta < 0 \) we have
\[
\langle E'(u), u \rangle < 0 \text{ for every } u \in E^n. \quad (5.5)
\]
Let \( \partial B_1 := \{ u \in X : \|u\| = 1 \} \). In light of (5.4) and (5.5), for every \( u \in \partial B_1 \) we can find a maximal \( \tau(u) > 0 \) such that \( E(\tau(u)u) = \eta \). The Implicit Function Theorem implies that \( \tau \in C(\partial B_1) \). We can extend \( \tau \) to all of \( X \setminus \{0\} \) by setting
\[
\tau^*(u) := \frac{1}{\|u\|} \tau \left( \frac{1}{\|u\|} \right) \text{ for all } u \in X \setminus \{0\}.
\]
So \( \tau^* \in C(X \setminus \{0\}) \) and \( E(\tau^*(u)u) = \eta \) for all \( u \in X \setminus \{0\} \). In addition, \( E(u) = \eta \) implies that \( \tau^*(u) = 1 \). Now we set
\[
\hat{\tau} := \begin{cases} 
1, & \text{if } E(u) \leq \eta, \\
\tau^*(u), & \text{if } E(u) > \eta.
\end{cases}
\]
Then \( \hat{\tau} \in C(X \setminus \{0\}) \).

Now we show that \( E^n \) is a strong deformation retract of \( X \setminus \{0\} \). We consider the homotopy \( h : [0, 1] \times (X \setminus \{0\}) \to X \setminus \{0\} \) as
\[
h(t, u) = (1 - t)u + t\hat{\tau}(u)u
\]
for all \( (t, u) \in [0, 1] \times (X \setminus \{0\}) \). Clearly, we have
\[
h(0, u) = u \text{ and } h(1, u) = \hat{\tau}(u)u \in E^n
\]
for all \( u \in X \setminus \{0\} \). Moreover, for all \( (t, u) \in [0, 1] \times E^n \)
\[
h(t, u) = u,
\]
and so \( E^n \) is a strong deformation retract of \( X \setminus \{0\} \).

By the radial retraction \( r_0(u) = \frac{u}{\|u\|} \) we know that \( \partial B_1 \) is a retract of \( X \setminus \{0\} \). So we can use the deformation
\[
\tilde{h}(t, u) = (1 - t)u + t r_0(u),
\]
for all \( (t, u) \in [0, 1] \times (X \setminus \{0\}) \), to see that \( X \setminus \{0\} \) is deformable onto \( \partial B_1 \). Then \( \partial B_1 \) is a deformation retract of \( X \setminus \{0\} \). So \( E^n \) and \( \partial B_1 \) are homotopy equivalent. Since \( X \) is infinite dimensional, \( \partial B_1 \) is contractible, hence \( E^n \) is contractible as well.

Now we want to compute the critical groups of \( E \) at 0. So, for all \( \tau \in [0, 1] \), we define the functional
\[
E_\tau(u) := \frac{1}{2p} |u|^p - \frac{\lambda}{p} \|u\|^p - \int_\Omega G(x, (1 - \tau)u + \tau \theta(u)) \, dx,
\]
Lemma 5.4. If one of the following holds true for some $\lambda > 0$, and $\theta$ is such that
\[
\theta(t) = \begin{cases} 
-\delta & \text{if } t \leq -\delta, \\
t & \text{if } |t| \leq \frac{\delta}{2}, \\
\delta & \text{if } t \geq \delta.
\end{cases}
\] (5.6)

With this definition, it clearly follows that $E_\tau \in C^1(X)$ and $E = E_0$. We have the following result:

**Lemma 5.3.** The point 0 is an isolated critical point of $E_\tau$ uniformly with respect to $\tau \in [0, 1]$. Moreover
\[
C^k(E, 0) = C^k(E_1, 0)
\]
for all $k \in \mathbb{N}_0$.

*Proof.* Since 0 is an isolated critical point of $E$, for $\varepsilon > 0$ small enough we have $K(E) \cap \overline{B}_\varepsilon(0) = \{0\}$. We want to prove the same for $E_\tau$, that is for $\varepsilon > 0$ small enough
\[
K(E_\tau) \cap \overline{B}_\varepsilon(0) = \{0\}
\] for every $\tau \in [0, 1], \quad (5.7)
and we argue by contradiction. So we assume that there exist two sequences, $(\tau_n)_n$ in $[0, 1]$ and $(u_n)_n$ in $X \setminus \{0\}$, such that $E_{\tau_n}'(u_n) = 0$ for all $n \in \mathbb{N}$, and $u_n \to 0$ in $X$. For all $n \in \mathbb{N}$ and $(x, t) \in \Omega \times \mathbb{R}$ we set
\[
g_n(x, t) := (1 - \tau_n + \tau_n \theta(t)) + g(x, (1 - \tau_n)u + \tau_n \theta(u)),
\]
with $\theta \in C(\mathbb{R}, [-\delta, \delta])$ defined as in (5.6). By definition $g_n : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, Hypotheses $(g_1)$ and $(g_3)$ imply that, for all $n \in \mathbb{N}$ and some $a' > 0$,
\[
|\lambda|^p |t|^p - 2 + g_n(x, t)| \leq a'(|t|^{p-1} + |t|^{r-1}) \quad (5.8)
\]
a.e. in $\Omega$ and for all $t \in \mathbb{R}$. Since $u_n$ is a critical point of $E_{\tau_n}'$ for all $n \in \mathbb{N}$, it is also a weak solution of the auxiliary problem
\[
\begin{cases}
(\Delta)_p^* u = \lambda |u|^{p-2} u + g_n(x, u) & \text{in } \Omega, \\
\mathcal{M}_{s,p} u = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\] (5.9)

In light of (5.8) there exists a constant $K > 0$, independent of $n$, such that for every weak solution $u \in X$ of (5.9) with $||u||_r < K$, then $u \in L^\infty(\Omega)$ and $||u||_{\infty} \leq K^{-1} ||u||_r$ (see Corollary 1). From the continuous embedding $X \hookrightarrow L^r(\Omega)$ we know that $u_n \to 0$ in $L^r(\Omega)$, so $u_n \to 0$ in $L^\infty(\Omega)$ as well. Hence, if $n \in \mathbb{N}$ is big enough, we can take $u_n \in \overline{B}_\varepsilon(0)$ and $||u_n||_\infty \leq \delta/2$. Then, from the definition of $E_\tau$ we can see that
\[
E'(u_n) = E_{\tau_n}'(u_n) = 0,
\]
that is $u_n \in K(E) \cap \overline{B}_\varepsilon(0) \setminus \{0\}$, which is a contradiction, and so (5.7) holds true.

For all $\tau \in [0, 1]$, the functional $E_\tau \in C^1(X)$ satisfies hypoteses analogous to $(g_1)$-(g3) so, similarly as in the proof of Lemma 5.2, $E_\tau$ satisfies (PS) in $\overline{B}_\varepsilon(0)$. Moreover, the mapping $\tau \mapsto E_\tau$ is continuous in $[0, 1]$. Now we can apply Proposition 1 to obtain $C^k(E, 0) = C^k(E_1, 0)$ for all $k \in \mathbb{N}_0$, as desired.

Now we want to prove that, for every $\lambda > 0$, $E$ has a non-trivial critical group at zero (recall that $(\lambda_k)_k$ is a sequence of eigenvalues of $(-\Delta)_p^*$ and $\lambda_1 = 0$).

**Lemma 5.4.** If one of the following holds true for some $k \in \mathbb{N}$.

(i) $\lambda_k < 2\lambda < \lambda_{k+1}$,
Now we prove the second condition for the local splitting, that is, for $\rho > 0$.

Proof. We want to prove that $E$ has a cohomological local splitting near $0$ in dimension $k \in \mathbb{N}$. As in [15], we set
\[
C_k^- := \{u \in X : [u]^p \leq \lambda_k [u]_p^p\}, \quad C_k^+ := \{u \in X : [u]^p \geq \lambda_{k+1} [u]_p^p\},
\]
which are symmetric closed cones with $C_k^- \cap C_k^+ = \{0\}$. From this definition, we have
\[
||u||^p \leq (\lambda_k + 1) [u]_p^p \quad \text{for every} \quad u \in C_k^-,
\]
and
\[
||u||^p \geq (\lambda_{k+1} + 1) [u]_p^p \quad \text{for every} \quad u \in C_k^+.
\]
Moreover, defining the manifold $M$ as
\[
M := \{u \in X : [u]_p^p = 1\},
\]
by [15, Theorem 2.6] we have
\[
i(C_k^- \setminus \{0\}) = i(X \setminus C_k^+) = k,
\]
so the first condition for the local splitting is satisfied.

We remark that by $(g_3)$, for every $\varepsilon > 0$ there exists $\rho > 0$ such that, a.e. in $\Omega$, and for every $|t| < \rho$,
\[
|g(x,t)| \leq \varepsilon |t|^{p-1}.
\]
Integrating the inequalities in (5.12) and in $(g_1)$ we get, for all $u \in X$,
\[
\int_{\Omega} G(x,u) \, dx \leq \int_{\Omega \setminus \{|u| \leq \rho\}} \frac{\varepsilon |u|^p}{p} \, dx + \int_{\Omega \setminus \{|u| > \rho\}} a \left( |u| + \frac{|u|^p}{r} \right) \, dx
\]
\[
\leq \frac{\varepsilon [u]_p^p}{p} + C [u]_r^r.
\]
Now, the continuous embeddings of $X$ in $L^p(\Omega)$ and in $L^r(\Omega)$, together with the arbitrariness of $\varepsilon > 0$, imply that
\[
\int_{\Omega} G(x,u) \, dx = o([u]_p^p) \quad \text{as} \quad [u]_p^p \to 0.
\]
(5.13)

Now we prove the second condition for the local splitting, that is, for $\rho$ small enough,
\[
E(u) \leq 0 \quad \text{for all} \quad u \in \mathcal{B}_\rho(0) \cap C_k^-,
\]
\[
E(u) \geq 0 \quad \text{for all} \quad u \in \mathcal{B}_\rho(0) \cap C_k^+,
\]
(5.14)

and we have to consider three different cases.

Assume (i). Then, for every $u \in C_k^- \setminus \{0\}$, by (5.13) and (5.10) we have
\[
E(u) \leq \left( \frac{\lambda_k - 2\lambda}{\lambda_k + 1} \right) \frac{[u]_p^p}{2p} + o([u]_p^p),
\]
where the latter is negative if $[u]_p$ is small enough. On the other hand, for every $u \in C_k^+ \setminus \{0\}$, by (5.13) and (5.11)
\[
E(u) \geq \left( \frac{\lambda_{k+1} - 2\lambda}{\lambda_{k+1} + 1} \right) \frac{[u]_p^p}{2p} + o([u]_p^p),
\]
and now the latter is positive if $[u]_p$ is small enough. So (5.14) holds for (i).
Now assume (ii). From Lemma 5.3 we know that \( C^k(E,0) = C^k(E_1,0) \), so we consider \( E_1 \). Since \( 2\lambda = \lambda_k \), for every \( u \in C_k^− \setminus \{0\} \), from (5.6) and (5.10) we have

\[
E_1(u) \leq - \int_{\Omega} G(x, \theta(u)) \, dx \leq 0.
\]

If \( u \in C_k^+ \setminus \{0\} \), from (5.11) we have

\[
E_1(u) \geq \left( \frac{\lambda_k - 2\lambda}{\lambda_k + 1} \right) \frac{\|u\|^p}{2p} + o(\|u\|^p),
\]

and the latter is positive for small values of \( \|u\| \).

From (5.6) we have \( \theta(t) = t \) if \( t \) is small enough, so taking \( u \in \overline{B}_\rho(0) \) we get

\[
E_1(u) \geq \left( \frac{\lambda_k - 2\lambda}{\lambda_k + 1} \right) \frac{\|u\|^p}{2p} + o(\|u\|^p),
\]

for every \( u \in \overline{B}_\rho(0) \cap C_k^- \), and

\[
E_1(u) \geq - \int_{\Omega} G(x, \theta(u)) \, dx \geq 0
\]

for every \( u \in \overline{B}_\rho(0) \cap C_k^+ \).

So (5.14) holds true in every case. Recalling Lemma 5.3 for the cases (ii) and (iii), we can apply Proposition 3 to obtain \( C^k(E,0) \neq 0 \), which concludes the proof.

Now we are ready to give the proof of Theorem 5.1:

**Proof of Theorem 5.1.** We argue by contradiction, and we assume

\[
K(E) = \{0\}.
\]  
(5.15)

Let \( \eta < 0 \), so by Lemma 5.2 \( E^0 \) is contractible. From (5.15) we know that there is no critical value for \( E \) in \( [\eta,0) \), and from Lemma 5.2 we know that \( E \) satisfies (PS) in \( X \). So, by the Second Deformation Lemma, the set \( E^0 \) is a deformation retract of \( E^0 \setminus \{0\} \). In a similar way, since there is no critical value in \( (0, +\infty) \), the set \( E^0 \) is a deformation retract of \( X \). By the properties of critical groups and since \( E^0 \) is contractible, for all \( k \in \mathbb{N}_0 \) we have

\[
C^k(E,0) = H^k(E^0, E^0 \setminus \{0\}) = H^k(X, E^0) = 0.
\]

On the other hand, in all cases (i)-(iii), if we fix \( k \in \mathbb{N}_0 \), one of the assumptions of Lemma 5.4 has to hold, which implies \( C^k(E,0) \neq 0 \), a contradiction. So (5.15) is false and there exists some \( u \in K(E) \setminus \{0\} \), which is a nontrivial solution of (5.1).

Finally we show that when \( \lambda \notin \sigma(s,p) \) in Theorem 5.1, problem (5.1) admits a ground state solution. Let \( K = \{ u \in X \setminus \{0\} : E'(u) = 0 \} \) be the set of nontrivial critical points of \( E \) and set

\[
c = \inf_{u \in K} E(u).
\]

Recall that \( u_0 \in K \) is called a ground state solution if \( E(u_0) = c \). Since \( K \neq \emptyset \) by Theorem 5.1 (i), \( c < +\infty \). We claim that \( c > -\infty \). To see this, let \( (u_n)_n \subset K \) be a
minimizing sequence for $c$. Then $(E(u_n))_n$ is bounded from above and $E'(u_n) = 0$ for all $n$, so $(u_n)_n$ is bounded in $X$ as in the proof of Lemma 5.2. Since $E$ is bounded on bounded subsets of $X$, this implies that $c > -\infty$.

**Theorem 5.5.** If hypotheses $(g_1)$-$(g_3)$ hold and $\lambda \notin \sigma(s,p)$, then problem (5.1) admits a ground state solution.

**Proof.** The argument is adapted from [16]. Let $(u_n)_n \subset K$ be a minimizing sequence for $c$. Since $(E(u_n))_n$ is bounded and $E'(u_n) = 0$ for all $n$, a renamed subsequence of $(u_n)_n$ converges to a critical point $u_0$ of $E$ with $E(u_0) = c$ by Lemma 5.2. We claim that $u_0$ is nontrivial and hence it is a ground state solution of problem (5.1).

To see this, suppose by contradiction that $u_0 = 0$. Then $\rho_n := \|u_n\| \to 0$. Let $\tilde{u}_n = u_n/\rho_n$. Since $\|\tilde{u}_n\| = 1$, a renamed subsequence of $(\tilde{u}_n)_n$ converges to some $\tilde{u}$ weakly in $X$, strongly in $L^r(\Omega)$, and a.e. in $\Omega$. Since $E'(u_n) = 0$

\[
\frac{1}{2} \iint_Q J_p \left( \frac{u_n(x) - u_n(y)}{|x - y|^{N + ps}} \right) (v(x) - v(y)) \, dx \, dy = \lambda \int_\Omega |u_n|^{p-2} u_n v \, dx + \int_\Omega g(x, u_n) v \, dx \quad \forall v \in X,
\]

and dividing this by $\rho_n^{p-1}$ gives

\[
\frac{1}{2} \iint_Q J_p \left( \frac{\tilde{u}_n(x) - \tilde{u}_n(y)}{|x - y|^{N + ps}} \right) (v(x) - v(y)) \, dx \, dy = \lambda \int_\Omega |\tilde{u}_n|^{p-2} \tilde{u}_n v \, dx + o(\|v\|) \quad \forall v \in X
\]

(5.16)

by $(g_1)$ and $(g_3)$. Passing to the limit in (5.16) thanks to [15, Proposition 3.2] gives

\[
\frac{1}{2} \iint_Q J_p \left( \frac{\tilde{u}(x) - \tilde{u}(y)}{|x - y|^{N + ps}} \right) (v(x) - v(y)) \, dx \, dy = \lambda \int_\Omega |\tilde{u}|^{p-2} \tilde{u} v \, dx \quad \forall v \in X,
\]

so $\tilde{u}$ is a weak solution of the eigenvalue problem

\[
\begin{cases}
(-\Delta)_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\
\mathcal{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}.
\end{cases}
\]

Taking $v = \tilde{u}_n$ in (5.16) and passing to the limit shows that $\lambda \int_\Omega |\tilde{u}|^p \, dx = 1/2$, so $\tilde{u}$ is nontrivial. This contradicts the assumption that $\lambda \notin \sigma(s,p)$ and completes the proof.

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