Asymptotic Stability of the Stationary Solution for a Hyperbolic Free Boundary Problem Modeling Tumor Growth

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Abstract

In this paper we study asymptotic behavior of solutions for a free boundary problem modeling the growth of tumors containing two species of cells: proliferating cells and quiescent cells. This tumor model was proposed by Pettet et al in Bull. Math. Biol. (2001). By using a functional approach and the $C_0$ semigroup theory, we prove that the unique stationary solution of this model ensured by the work of Cui and Friedman (Trans. Amer. Math. Soc., 2003) is locally asymptotically stable in certain function spaces. Key techniques used in the proof include an improvement of the linear estimate obtained by the work of Chen et al (Trans. Amer. Math. Soc., 2005), and a similarity transformation.

Keywords and phrases: Free boundary problem; hyperbolic equations; tumor growth; stationary solution; asymptotic stability.

AMS subject classification: 35C10, 35Q80, 92C15.

1 Introduction

During the past thirty years, an increasing number of free boundary problems of partial differential equations have been proposed by groups of researchers to model the growth of various in vivo and in vitro tumors, see, e.g., [1], [4]–[6], [18], [19], [23], [24], [26]–[28] and the references cited therein. Such free boundary problems usually contain one or more reaction diffusion equations describing the distribution of nutrient and inhibitory materials, and several first-order nonlinear partial differential equations or nonlinear conservation laws with source terms describing the evolution and movement of various tumor cells (proliferating cells, quiescent cells and dead cells). Rigorous analysis of such tumor models is evidently a significant topic of research and has drawn great attention during the past a few years. Main concern of this topic is the dynamics or the long-term behavior of solutions of such free boundary problems.

Based on applications of the well-established theories of elliptic and parabolic partial differential equations, parabolic differential equations in Banach spaces (i.e., differential equations in Banach spaces that are treatable with the analytic semigroup theory) and the bifurcation theory, rigorous analysis of models for the growth of tumors containing only one species of tumor cells has achieved great depth, cf. [2], [3], [9], [10], [12]–[14], [20]–[22], [29], [30] and the references.
cited therein. As far as models for tumors containing more than one species of tumor cells are concerned, however, the progress is relatively backward. This is caused by the fact that such tumor models are much more difficult to analyze because they contain nonlinear conservation laws whose dynamical behavior is very hard to grasp.

In this paper we study the following free boundary problem modeling the growth of an in vitro tumor containing two species of cells — proliferating cells and quiescent cells:

\[
\nabla^2 C = F(C) \quad \text{for } x \in \Omega(t), \ t \geq 0, \\
C = C_0 \quad \text{for } x \in \partial \Omega(t), \ t \geq 0, \\
\frac{\partial P}{\partial t} + \nabla \cdot (\bar{u}P) = \left[ K_B(C) - K_Q(C) \right] P + K_P(C) Q \quad \text{for } x \in \Omega(t), \ t \geq 0, \\
\frac{\partial Q}{\partial t} + \nabla \cdot (\bar{u}Q) = K_Q(C) P - \left[ K_D(C) + K_P(C) \right] Q \quad \text{for } x \in \Omega(t), \ t \geq 0, \\
P + Q = N \quad \text{for } x \in \Omega(t), \ t \geq 0, \\
\frac{dR}{dt} = \bar{u} \cdot \bar{v} \quad \text{for } x \in \partial \Omega(t), \ t \geq 0.
\]

Here \( C \) denotes the concentration of nutrient (with all nutrient materials regarded as one species), \( P \) and \( Q \) denote the densities of proliferating cells and quiescent cells, respectively, whose mixture makes up the tumor tissue and has a constant density \( N \), \( \bar{u} \) denotes the velocity of the cell movement, \( R \) denotes the radius of the tumor, \( \Omega(t) = \{ x \in \mathbb{R}^3 : r = |x| < R(t) \} \) is the domain occupied by the tumor at time \( t \), and \( \bar{v} \) is the unit outward normal of \( \partial \Omega(t) \). Besides, \( C_0 \) is a positive constant reflecting the constant nutrient supply that the tumor receives from its surface, \( F(C) \) is the nutrient consumption rate function, and \( K_B(C), K_D(C), K_P(C) \) and \( K_Q(C) \) are the birth rate of proliferating cells, death rate of quiescent cells, transferring rate of proliferating cells to quiescent cells and transferring rate of quiescent cells to proliferating cells, respectively. We shall only consider radially symmetric solutions of the above problem, so that \( C, P, Q \) are functions of the radial space variable \( r = |x| \) and the time variable \( t \), and \( \bar{u} = u(r, t) r^{-1} x \), where \( u \) is a scaler function.

The above tumor model was proposed by Pettet et al in the literature [26]. Its global well-posedness has been established by Cui and Friedman in [15]. A challenging task concerning this free boundary problem is the study of the asymptotic behavior of its solutions as time goes to infinity. For the corresponding model of the growth of tumors with one species of cells, it is known that there exists a unique stationary solution and all time-dependent solutions converge to it as time goes to infinity, or in other words, this unique stationary solution is globally asymptotically stable, cf. [9] and [21]. Since the above problem is a natural extension of such one species tumor model to the two species case, we are naturally lead to the conjecture that a similar result holds for it. Advancement of the study toward this goal is as follows. In [16], Cui and Friedman proved that the problem (1.1)–(1.6) has a unique stationary solution. In [8], Chen, Cui and Friedman further proved that this stationary solution is linearly asymptotically stable, namely, the trivial solution of the linearization of (1.1)–(1.6) at the stationary solution is asymptotically stable. However, this last-mentioned result does not imply, at least straightforwardly, that the stationary solution of (1.1)–(1.6) is asymptotically stable. In fact, to the best of our knowledge
this problem has been remaining open before this manuscript is prepared. We refer the reader to see [7, 10] and [17] for other related work.

In this paper we shall prove that the unique stationary solution of (1.1)–(1.6) ensured by [16] is locally asymptotically stable. Recall that conditions given in [16] which ensure that (1.1)–(1.6) has a unique stationary solution are as follows:

\[
F(C), \ K_B(C), \ K_P(C) \text{ and } K_Q(C) \text{ are analytic in } C, \ 0 \leq C \leq C_0; \quad (1.7)
\]

\[
F(0) = 0, \ F'(C) > 0 \text{ for } 0 \leq C \leq C_0; \quad (1.8)
\]

\[
\begin{cases}
K'_B(C) > 0 \text{ and } K'_D(C) < 0 \text{ for } 0 \leq C \leq C_0, \ K_B(0) = 0 \text{ and } K_D(C_0) = 0;
\end{cases}
\]

\[
K_P(C) \text{ and } K_Q(C) \text{ satisfy the same conditions as } K_B(C) \text{ and } K_D(C), \text{ respectively;}
\]

\[
K'_B(C) + K'_D(C) > 0 \text{ for } 0 \leq C \leq C_0. \quad (1.9)
\]

The main result of this paper is the following:

**Theorem 1.1** Assume that the conditions (1.7)–(1.9) are satisfied. Let \( (C_*, P_*, Q_*, \bar{u}_*, R_*) \) be the unique stationary solution of the problem (1.1)–(1.6), and let \( (C, P, Q, \bar{u}, R) \) be a time-dependent solution of it such that \( P|_{t=0} = P_0, \ Q|_{t=0} = Q_0 \) and \( R|_{t=0} = R_0 \), where \( P_0, \ Q_0 \) and \( R_0 \) are given initial data satisfying \( 0 \leq P_0 \leq N, \ 0 \leq Q_0 \leq N \) and \( P_0 + Q_0 = N \). Then there exist positive constants \( \mu, \varepsilon \) and \( K \) such that if \( P_0, Q_0 \) and \( R_0 \) satisfy

\[
\max_{0 \leq r \leq 1} |P_0(rR_0) - P_*(rR_*)| < \varepsilon, \quad \sup_{0 < r < 1} r(1-r) \left| \frac{dP_0(rR_0)}{dr} - \frac{dP_*(rR_*)}{dr} \right| < \varepsilon,
\]

\[
\max_{0 \leq r \leq 1} |Q_0(rR_0) - Q_*(rR_*)| < \varepsilon, \quad \sup_{0 < r < 1} r(1-r) \left| \frac{dQ_0(rR_0)}{dr} - \frac{dQ_*(rR_*)}{dr} \right| < \varepsilon
\]

and \( |R_0 - R_*| < \varepsilon \), then for all \( t \geq 0 \) we have

\[
\max_{0 \leq r \leq 1} |P(rR(t), t) - P_*(rR_*)| < K \varepsilon e^{-\mu t}, \quad \sup_{0 < r < 1} r(1-r) \left| \frac{\partial P(rR(t), t)}{\partial r} - \frac{dP_*(rR_*)}{dr} \right| < K \varepsilon e^{-\mu t},
\]

\[
\max_{0 \leq r \leq 1} |Q(rR(t), t) - Q_*(rR_*)| < K \varepsilon e^{-\mu t}, \quad \sup_{0 < r < 1} r(1-r) \left| \frac{\partial Q(rR(t), t)}{\partial r} - \frac{dQ_*(rR_*)}{dr} \right| < K \varepsilon e^{-\mu t}
\]

and \( |R(t) - R_*| < K \varepsilon e^{-\mu t} \).

We shall use a functional approach to prove the above theorem. More precisely, we shall first reduce the problem (1.1)–(1.6) into a differential equation for the unknown \( U = (p, z) \) in the Banach space \( X = C[0, 1] \times \mathbb{R} \), where \( p = p(r, t) = P(rR(t), t) \) and \( z = z(t) = \log R(t) \). The reduced equation is of the hyperbolic type in the sense of Pazy [25], and is quasi-linear. We next use the Banach fixed point theorem to prove that for any \( U_0 = (p_0, z_0) \) sufficiently close to the stationary point \( U_* = (p_*, z_*) \), where \( p_* = p_*(r) = P_*(rR_*) \) and \( z_* = \log R_* \), this differential equation imposed with the initial condition \( U|_{t=0} = U_0 \) and the decay estimate

\[
\sup_{t \geq 0} e^{\mu t} \|U(t) - U_*\|_{X_0} < \infty, \text{ where } X_0 \text{ is a subspace of } X, \text{ has a unique solution in the space } C([0, \infty), X_0) \text{ (endowed with the norm } \|U\| = \sup_{t \geq 0} e^{\mu t} \|U(t)\|_{X_0}). \text{ To attain this goal we shall use some abstract results for hyperbolic differential equations in Banach spaces established in}
\]
In particular, a family of evolution systems for the linear equations related to the semi-linearization of the reduced equation are obtained and applied to convert the semi-linearized equations into integral equations. The main difficult and key step in the proof of Theorem 1.1 is the establishment of a uniform decay estimate for the family of evolution systems. To obtain it, we first use a localization technique to get an improvement of the linear estimate established in [8], removing the singularities at \( r = 0 \) contained in that estimate. See Lemma 6.2 in Section 6 for this improved linear estimate. We next develop a similarity transformation technique and use it to extend this improved linear estimate to the family of evolution systems mentioned above. See Section 5 for details of this transformation and Lemma 6.4 in Section 6 for the uniform decay estimate for the family of evolution systems.

The layout of the rest part is as follows. In the following section we reduce the problem (1.1)–(1.6) into a differential equation in the Banach space \( X = C[0, 1] \times \mathbb{R} \). In Section 3 we summarize some basic properties of the stationary solution. The reader is suggested to pay attention to properties of the stationary solution at the end point \( r = 0 \) which will play an important role in later analysis. In Section 4 we prove that the linear parts of semi-linearizations of the reduced equation are related with a stable family of generators of \( C_0 \) semigroups on \( X \), so that their solution operators are evolution systems. This result enables us to use the abstract results of [25] to convert the semi-linearized equations into integral equations. The most important technique used in this paper — similarity transformations — will be developed in Section 5. In Section 6 we first derive an improvement of the linear estimate established in [8] and next use the similarity transformation technique to extend this estimate to the evolution systems obtained in Section 4. After these preparations, in the last section we use the Banach fixed point theorem to prove Theorem 1.1.

Throughout this paper the notation “′” denotes both the ordinary derivatives of functions in \( \mathbb{R} \) and the Fréchet derivatives of mappings between Banach spaces.

## 2 Reduction of the problem

In this section we reduce the system of equations (1.1)–(1.6) into a differential equation in the Banach space \( X = C[0, 1] \times \mathbb{R} \).

We first note that by summing up (1.3), (1.4) and using (1.5), we get the following equation:

\[
\nabla \cdot \vec{u} = \frac{1}{N} \left[ K_B(C)P - K_D(C)Q \right].
\]

(2.1)

Conversely, from (1.3), (1.5) and (2.1) we immediately obtain (1.4). Hence, the two groups of equations (1.3), (1.4), (1.5) and (1.3), (1.5), (2.1) are equivalent.

By rescaling the space and time variables, setting

\[
p = \frac{P}{N}, \quad q = \frac{Q}{N} = 1 - p, \quad c = \frac{C}{C_0}, \quad \vec{u} = u \frac{x}{|x|},
\]

and using the equivalence of (1.3), (1.4) and (1.5) with (1.3), (1.5) and (2.1), we see that the problem (1.1)–(1.6) can be reformulated into the following form:

\[
\frac{\partial^2 c}{\partial r^2} + \frac{2}{r} \frac{\partial c}{\partial r} = F(c) \quad \text{for} \quad 0 < r \leq R(t), \quad t \geq 0,
\]

(2.2)
\[
\frac{\partial c}{\partial r}\bigg|_{r=0} = 0, \quad c|_{r=R(t)} = 1 \quad \text{for} \quad t \geq 0, \quad (2.3)
\]
\[
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} = K_P(c) + [K_M(c) - K_N(c)]p - K_M(c)p^2 \quad \text{for} \quad 0 \leq r \leq R(t), \quad t \geq 0, \quad (2.4)
\]
\[
\frac{\partial u}{\partial r} + \frac{2}{r} u = -K_D(c) + K_M(c)p \quad \text{for} \quad 0 < r \leq R(t) \quad \text{and} \quad u|_{r=0} = 0, \quad t \geq 0, \quad (2.5)
\]
\[
\frac{dR}{dt} = u(R, t) \quad \text{for} \quad t \geq 0, \quad (2.6)
\]

where
\[
K_M(c) = K_B(c) + K_D(c), \quad K_N(c) = K_P(c) + K_Q(c), \quad (2.7)
\]

and \(F(c), K_B(c), K_D(c), K_P(c)\) and \(K_Q(c)\) are rescaled forms of the corresponding functions appearing in (1.1)–(1.6).

Next, we set
\[
c(\tilde{r}, t) = c(\tilde{r}e^z(t), t), \quad \tilde{p}(\tilde{r}, t) = p(\tilde{r}e^z(t), t), \quad \tilde{u}(\tilde{r}, t) = u(\tilde{r}e^z(t), t)e^{-z(t)}, \quad \tilde{R}(t) = e^z(t),
\]

where \(0 \leq \tilde{r} \leq 1, \ t \geq 0\). Then the problem (2.2)–(2.6) is further reduced into the following problem (for simplicity of the notation we omit all bar’s):

\[
\frac{\partial^2 c}{\partial r^2} + \frac{2}{r} \frac{\partial c}{\partial r} = e^{2z}F(c) \quad \text{for} \quad 0 < r \leq 1, \quad t \geq 0, \quad (2.8)
\]
\[
\frac{\partial c}{\partial r}\bigg|_{r=0} = 0, \quad c|_{r=1} = 1 \quad \text{for} \quad t \geq 0, \quad (2.9)
\]
\[
\frac{\partial p}{\partial t} + \left[u(r, t) - ru(1, t)\right] \frac{\partial p}{\partial r} = K_P(c) + [K_M(c) - K_N(c)]p - K_M(c)p^2
\]
\[
\quad \text{for} \quad 0 \leq r \leq 1, \quad t \geq 0, \quad (2.10)
\]
\[
\frac{\partial u}{\partial r} + \frac{2}{r} u = -K_D(c) + K_M(c)p \quad \text{for} \quad 0 < r \leq 1 \quad \text{and} \quad u|_{r=0} = 0, \quad t \geq 0, \quad (2.11)
\]
\[
\frac{dz}{dt} = u(1, t) \quad \text{for} \quad t \geq 0. \quad (2.12)
\]

To further reduce (2.8)–(2.12) we first note that (2.8) and (2.9) can be solved to express \(c\) as a function of \(z\). Thus, instead of \(c(r, t)\), later on we shall use the notation \(c(r, z(t))\) or simply \(c(r, z)\) to denote the solution of (2.8) and (2.9). Next, we note that (2.11) can be solved to get \(u\) as a functional of \(p\) and \(z\). Thus later on we use the notation \(u_{p,z}\) to re-denote \(u\). By a simple computation we have

\[
u_{p,z}(r, t) = \frac{1}{r^2} \int_0^r \left[-K_D(c(\rho, z(t))) + K_M(c(\rho, z(t)))p(\rho, t)\right]p^2 d\rho \quad (2.13)
\]

for \(0 < r \leq 1, \ t \geq 0,\) and \(u_{p,z}(0, t) = 0\) for \(t \geq 0\). We also denote

\[
w_{p,z}(r, t) = u_{p,z}(r, t) - ru_{p,z}(1, t). \quad (2.14)
\]
It follows that (2.8)–(2.12) reduces into the following system of equations:

\[
\begin{cases}
\frac{\partial p}{\partial t} + w_{p,z}(r,t)\frac{\partial p}{\partial r} = f(r,p,z) & \text{for } 0 \leq r \leq 1, \ t > 0, \\
\frac{\partial z}{\partial t} = u_{p,z}(1,t) & \text{for } t > 0,
\end{cases}
\]

(2.15)

where

\[f(r,p,z) = K_p(c(r,z)) + \left[K_M(c(r,z)) - K_N(c(r,z))\right]p - K_M(c(r,z))p^2.\]

In what follows we shall rewrite (2.15) as a differential equation in the Banach space \(X = C[0,1] \times \mathbb{R}\). Let

\[C^1_V[0,1] = \{p \in C[0,1] \cap C^1(0,1) : r(1-r)p'(r) \in C[0,1]\},\]

with norm

\[\|p\|_{C^1_V[0,1]} = \max_{0 \leq r \leq 1} |p(r)| + \sup_{0 < r < 1} |r(1-r)p'(r)| \quad \text{for } p \in C^1_V[0,1].\]

It is evident that \(C^1_V[0,1]\) endowed with this norm is a Banach space densely and continuously embedded into \(C[0,1]\). Given \(p \in C[0,1]\) and \(z \in \mathbb{R}\), we introduce a linear operator \(A_0(p,z) : C^1_V[0,1] \to C(0,1)\) as follows: For any \(q \in C^1_V[0,1],\)

\[A_0(p,z)q(r) = -w_{p,z}(r)q'(r) \quad \text{for } 0 < r < 1.
\]

Here and hereafter \(w_{p,z}(r)\) represents the function defined by similar formulations as in (2.13) and (2.14), with \(p(r,t)\) and \(z(t)\) there replaced by \(p(r)\) and \(z\), respectively. Later on we shall use the convention that for a function \(f \in C(0,1)\), if both limits \(\lim_{r \to 0^+} f(r)\) and \(\lim_{r \to 1^-} f(r)\) exist and are finite, then we write \(f \in C[0,1]\). Furthermore, when we are concerned with the values of \(f\) at \(r = 0\) and \(r = 1\), we mean that \(f(0) = \lim_{r \to 0^+} f(r)\) and \(f(1) = \lim_{r \to 1^-} f(r)\). Using this convention, we see easily that for any \(p \in C[0,1]\) and \(z \in \mathbb{R}\) we have \(w_{p,z}(r)/r(1-r) \in C[0,1]\). It follows that for any \(q \in C^1_V[0,1]\), both limits \(\lim_{r \to 0^+} w_{p,z}(r)q'(r)\) and \(\lim_{r \to 1^-} w_{p,z}(r)q'(r)\) exist, so that \(A_0(p,z)q \in C[0,1]\). It can also be easily seen that \(A_0(p,z)\) is a bounded linear operator from \(C^1_V[0,1]\) to \(C[0,1]\), and

\[\|A_0(p,z)\|_{L(C^1_V[0,1],C[0,1])} \leq \sup_{0 < r < 1} \left|\frac{w_{p,z}(r)}{r(1-r)}\right|.
\]

Next we introduce mappings \(F : C[0,1] \times \mathbb{R} \to C[0,1]\) and \(G : C[0,1] \times \mathbb{R} \to \mathbb{R}\) respectively by

\[F(p,z)(r) = f(r,p(r),c(r,z)),\]

and

\[G(p,z) = \int_0^1 [-K_D(c(r,z)) + K_M(c(r,z))p(r)]r^2\,dr.
\]

We set \(X_0 = C^1_V[0,1] \times \mathbb{R}\), which is a Banach space with the product norm and is densely and continuously embedded into \(X = C[0,1] \times \mathbb{R}\). We now define a nonlinear operator \(F : X_0 \to X\) as follows:

\[F(U) = (A_0(p,z)p + F(p,z),G(p,z)) \quad \text{for } U = (p,z) \in X_0.
\]
It is obvious that \( F \in C^\infty(X_0, X) \). Later on we shall also regard \( F \) as an unbounded nonlinear operator in \( X \) with domain \( X_0 \). With these notation and convention, we can rewrite (2.15) as the following differential equation in the Banach space \( X \):

\[
\frac{dU}{dt} = F(U).
\]  

(2.16)

Here \( U = U(t) \) represents a \( X_0 \)-valued unknown function for \( t \geq 0 \), and the left-hand side denotes the Fréchet derivative of \( U = U(t) \) regarded as a mapping from \([0, \infty)\) to the \( X \) space.

It will be convenient to denote, for \( U = (p, z) \in X \) and \( V = (q, y) \in X_0 \),

\[
\mathcal{A}_0(U)V = (A_0(p, z)q, 0) \quad \text{and} \quad F_0(U) = (F(p, z), G(p, z)).
\]

Then we have

\[
F(U) = \mathcal{A}_0(U)U + F_0(U) \quad \text{for} \quad U \in X_0.
\]

Clearly, for every \( U \in X \), \( \mathcal{A}_0(U) \) is a bounded linear operator from \( X_0 \) to \( X \), i.e, \( \mathcal{A}_0(U) \in L(X_0, X) \). Furthermore, it can be easily seen that \( \mathcal{A}_0 \in C^\infty(X, L(X_0, X)) \). Later on we shall also regard \( \mathcal{A}_0(U) \) as an unbounded linear operator in \( X \) with domain \( X_0 \). Finally, we note that \( F_0 \in C^\infty(X, X) \cap C^\infty(X_0, X_0) \).

From [16] we know that under the conditions (1.7)–(1.9) which we assume to be true throughout the whole paper, the problem (2.2)–(2.6) has a unique stationary solution. It follows that the problem (2.8)–(2.12) has a unique stationary solution which we denote as \((c_*, p_*, u_*, z_*)\).

By definition, \((c_*, p_*, u_*, z_*) = (c_r(r), p_r(r), u_r(r), z_r) \ (0 \leq r \leq 1)\) is the solution of the following problem:

\[
c''_r + \frac{2}{r} c'_r = e^{2z_r} F(c_r) \quad \text{for} \quad 0 < r \leq 1, \tag{2.17}
\]

\[
c'_r(0) = 0, \quad c_r(1) = 1, \tag{2.18}
\]

\[
u_r p'_r = f(r, p_r, z_r) \quad \text{for} \quad 0 \leq r \leq 1, \tag{2.19}
\]

\[
u'_r + \frac{2}{r} u_r = -K_D(c_r) + K_M(c_r)p_r \quad \text{for} \quad 0 < r \leq 1, \tag{2.20}
\]

\[
u_r(0) = 0, \quad u_r(1) = 0. \tag{2.21}
\]

Let \( U_* = (p_*, z_*) \). Then \( U_* \in X_0 \) (see Lemma 2.1 below) and it is the unique equilibrium of \((2.16)\), i.e.,

\[
F(U_*) = 0,
\]

or

\[
\mathcal{A}_0(U_*)U_* + F_0(U_*) = 0.
\]

Since our goal is to study asymptotic stability of the stationary solution \( U_* \), it will be convenient to rewrite \((2.16)\) into an equation for the difference \( V = U - U_* \). For this purpose we introduce two nonlinear operators \( \mathcal{A} : X \to L(X_0, X) \) and \( \mathcal{G} : X \to X \) as follows:

\[
\mathcal{A}(V)W = \mathcal{A}_0(U_*)W + [\mathcal{A}_0(U_*)W]U_* + F_0(U_*)W \quad \text{for} \quad V \in X, \ W \in X_0,
\]

and

\[
\mathcal{G}(V) = G(p_* + V) = G(p_*).\]
\[ G(V) = [A_0(U_0 + V) - A_0(U_0) - A_0'(U_0)V]U_0 + [F_0(U_0 + V) - F_0(U_0) - F_0'(U_0)V] \] for \( V \in X. \)

Then clearly (2.16) can be rewritten as the following equivalent equation for \( V = U - U_*: \)

\[ \frac{dV}{dt} = \mathbb{A}(V)V + G(V), \quad (2.22) \]

i.e., if \( U \) is a solution of (2.16) then \( V = U - U_* \) is a solution of (2.22) and vice versa. We note that \( \mathbb{A} \in C^\infty(X, L(X_0, X)) \), \( G \in C^\infty(X, X) \), and by using the Taylor expansions up to second-order for Fréchet derivatives of \( A_0 \) and \( F_0 \) we have

\[ \| G(V) \|_X = O(\| V \|^2_X) \quad \text{as} \quad \| V \|_X \to 0. \quad (2.23) \]

We also note that, by introducing an operator \( \mathbb{B} : X \to X \) by

\[ \mathbb{B}W = [A_0'(U_0)W]U_* + F_0'(U_0)W \quad \text{for} \quad W \in X, \]

we have

\[ \mathbb{A}(0) = F'(U_0) = \mathbb{A}_0(U_0) + \mathbb{B} \quad \text{and} \quad \mathbb{A}(V) = \mathbb{A}_0(U_0 + V) + \mathbb{B}. \]

Note that as an immediate consequence of the facts that \( \mathbb{A} \in C^\infty(X, L(X_0, X)) \) and \( F_0 \in C^\infty(X, X) \), we have \( \mathbb{B} \in L(X) \). We also note that \( [V \to \mathbb{A}_0(U_0 + V)] \in C^\infty(X, L(X_0, X)). \)

From the above deduction it follows immediately that the stationary solution \((c_*, p_*, u_*, z_*)\) of (2.8)–(2.12) is asymptotically stable if and only if the trivial solution of (2.22) is asymptotically stable. More precisely, Theorem 1.1 follows if we prove that the solution \( V = V(t) \) of (2.22)

\[ \| V(t) \|_{X_0} \leq K \varepsilon e^{-\mu t}, \quad t \geq 0, \]

provided \( \| V(0) \|_{X_0} \leq \varepsilon \) for some small \( \varepsilon > 0 \). Hence, later on we shall concentrate our attention on the equation (2.22).

A simple computation shows that if we denote

\[ a(r) = K_M(c_*(r)) - K_N(c_*(r)) - 2K_M(c_*(r))p_*(r), \quad (2.24) \]

\[ b(r) = K'_M(c_*(r)) + [K'_M(c_*(r)) - K'_N(c_*(r))]p_*(r) - K'_M(c_*(r))p^2_*(r) \]

\[ + rp'_*(r) \left[ \int_0^1 g_c(\rho)c_*(\rho)\rho^2 d\rho - \frac{1}{r^2} \int_0^r g_c(\rho)c_*(\rho)\rho^2 d\rho \right], \quad (2.25) \]

\[ Bq(r) = rp'_*(r) \left[ \int_0^1 g_p(\rho)q(\rho)\rho^2 d\rho - \frac{1}{r^2} \int_0^r g_p(\rho)q(\rho)\rho^2 d\rho \right], \quad (2.26) \]

\[ F(q) = \int_0^1 g_p(\rho)q(\rho)\rho^2 d\rho, \quad (2.27) \]

and

\[ \kappa = \int_0^1 g_c(\rho)c_*(\rho)\rho^2 d\rho, \quad (2.28) \]

where

\[ g_p(r) = K_M(c_*(r)), \quad g_c(r) = -K'_D(c_*(r)) + K'_M(c_*(r))p_*(r), \quad c_*(r) = \frac{\partial c}{\partial z}(r, z_*), \]
then we have

$$\mathbb{B} = \begin{pmatrix} a(r) + B & b(r) \\ \mathcal{F} & \kappa \end{pmatrix}. \quad (2.29)$$

Here and hereafter, when we write $\mathbb{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ for bounded linear operators $M_{11} \in L(X_1, Y_1)$, $M_{12} \in L(X_2, Y_1)$, $M_{21} \in L(X_1, Y_2)$, $M_{22} \in L(X_2, Y_2)$, where $X_1$, $X_2$, $Y_1$ and $Y_2$ are Banach spaces, we mean that $\mathbb{M}$ is the bounded linear operator from $X_1 \times X_2$ to $Y_1 \times Y_2$ defined by

$$\mathbb{M}(x_1, x_2) = (M_{11}x_1 + M_{12}x_2, M_{21}x_1 + M_{22}x_2) \quad \text{for} \quad (x_1, x_2) \in X_1 \times X_2.$$

Using this notation we see that

$$\mathcal{A}_0(U_*) = \begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \mathcal{L}_0 = \mathcal{A}_0(p_*, z_*),$$

and, for $V = (\varphi, \zeta) \in X$,

$$\mathcal{A}_0(U_* + V) = \begin{pmatrix} \mathcal{L}_V & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \mathcal{L}_V = \mathcal{A}_0(p_* + \varphi, z_* + \zeta). \quad (2.30)$$

We recall that $a(r) < 0$ for all $0 \leq r \leq 1$, see (2.7) in Section 2 of [8].

### 3 Some basic facts

We summarize some basic properties of the functions $c_*(r) = c(r, z_*)$, $c_*(r) = \frac{\partial c}{\partial z}(r, z_*)$, $p_*(r)$ and $u_*(r)$ in the following lemma. These properties will play an important role in later discussions.

**Lemma 3.1** We have the following assertions:

1. $c_*, c_2 \in C^\infty[0, 1]$, and

   $$0 < c_*(0) \leq c_*(r) \leq 1 \quad \text{for} \quad 0 \leq r \leq 1, \quad c'_*(r) > 0 \quad \text{for} \quad 0 < r \leq 1, \quad c'_*(0) = 0. \quad (3.1)$$

2. $p_* \in C[0, 1] \cap C^\infty(0, 1]$,

   $$0 < p_*(0) \leq p_*(r) \leq 1 \quad \text{for} \quad 0 \leq r \leq 1, \quad p'_*(r) > 0 \quad \text{for} \quad 0 < r \leq 1, \quad (3.2)$$

and either $p_* \in C^1[0, 1]$ or there exists $0 < \gamma < 1$ such that $\lim_{r \to 0^+} r^\gamma p'_*(r)$ exists and is finite, so that $r^\gamma p'_*(r) \in C[0, 1]$. Moreover,

$$\lim_{r \to 0^+} r^\gamma p'_*(r) = 0, \quad \lim_{r \to 0^+} r^{2\gamma} p''_*(r) = 0, \quad \lim_{r \to 0^+} r^{3\gamma} p'''_*(r) = 0. \quad (3.3)$$

3. $u_* \in C^1[0, 1] \cap C^\infty(0, 1]$, and there exist positive constants $C_1, C_2$ such that

   $$-C_1 r(1 - r) \leq u_*(r) \leq -C_2 r(1 - r) \quad \text{for} \quad 0 \leq r \leq 1. \quad (3.4)$$
Besides, either \( u_* \in C^2[0,1] \) or there exists \( 0 < \gamma < 1 \) such that \( \lim_{r \to 0^+} r^\gamma u_*''(r) \) exists and is finite, so that \( r^\gamma u_*''(r) \in C[0,1] \). Moreover,

\[
\lim_{r \to 0^+} r u_*''(r) = 0, \quad \lim_{r \to 0^+} r^2 u_*'''(r) = 0. \quad (3.5)
\]

**Proof:** The assertions that \( c_*, c_z \in C^\infty[0,1] \) and relations in (3.1) are immediate. The assertions that \( p_* \in C[0,1] \), \( u_* \in C^1[0,1] \), \( K_* \) and relations in (3.2) follow from Theorem 2.1 of [16], by which we also know that \( u_*'(0) < 0 \) for \( 0 < r < 1 \). The last assertion combined with the facts that \( u_*'(0) < 0 \) and \( u_*'(1) > 0 \) (see Theorem 7.1 of [16]) immediately yields (3.4). To prove (3.3) we compute:

\[
\lim_{r \to 0} u_*'(r)p_*'(r) = K_F(c_*(0)) + [K_M(c_*(0)) - K_N(c_*(0))]p_*(0) - K_M(c_*(0))p_*^2(0) = 0
\]

(see (8.4) in Section 8 of [16]), so that

\[
\lim_{r \to 0} rp_*'(r) = \lim_{r \to 0} \frac{r}{u_*'(r)} \cdot \lim_{r \to 0} u_*'(r)p_*'(r) = 0,
\]

and

\[
\lim_{r \to 0} u_*'(r)p_*''(r) = \lim_{r \to 0} r \{K_F'(c_*(r)) + [K_M'(c_*(r)) - K_N'(c_*(r))]p_*'(r) - K_M'(c_*(r))p_*^2(0) + \lim_{r \to 0} \{[K_M(c_*(r)) - K_N(c_*(r))]\}
\]

\[
-2K_M(c_*(r))p_*(r)\}rp_*'(r) - \lim_{r \to 0} u_*'(r)p_*'(r) = 0
\]

so that

\[
\lim_{r \to 0} r^2 p_*'''(r) = \lim_{r \to 0} \frac{r}{u_*'(r)} \cdot \lim_{r \to 0} u_*'(r)p_*''(r) = 0.
\]

This proves the first two relations in (3.3). The proof of the third relation is similar and is omitted. Next, from (2.20) we can easily deduce that

\[
u_*''(r) = \left[ -K_F'(c_*(r)) + K_M'(c_*(r))p_*'(r) + K_M(c_*(r))p_*'(r)\right]
\]

\[
+ \frac{2}{r}[K_F(c_*(r)) - K_M(c_*(r))p_*'(r)] + \frac{6}{r^4} \int_0^r \left[ -K_D(c_*(\rho)) + K_M(c_*(\rho))p_*(\rho)\right] \rho^2 d\rho
\]

\[
= \left[ -K_F'(c_*(r)) + K_M'(c_*(r))p_*'(r) + \frac{6}{r^4} \int_0^r [K_D(c_*(\rho)) - K_D(c_*(\rho))] \rho^2 d\rho\right]
\]

\[
+ K_M(c_*(r))p_*'(r) - \frac{6}{r^4} \int_0^r [K_M(c_*(r))p_*(r) - K_M(c_*(\rho))p_*(\rho)] \rho^2 d\rho. \quad (3.6)
\]

From this expression and the first relation in (3.3) we readily obtain the first relation in (3.5). The proof of the second relation in (3.5) is similar and is omitted. Finally, by Theorems 5.3 and 5.4 of [16] we know that either \( p_* \in C^1[0,1] \) or there exist constants \( -1 < \alpha < 0 \) and \( C \) such that

\[
p_*'(r) = Cr^\alpha + O(1), \quad \text{for } r \to 0. \quad (3.7)
\]

\(^1\)In the notation of Theorem 5.4 of [16], we have \( \alpha = \alpha(\lambda) \) and \( C = (1 + \alpha(\lambda))\omega\).
Suppose that it is the second case. Then, by letting $\gamma = |\alpha|$, we see that $0 < \gamma < 1$ and $r^\gamma p'_s(r) \in C[0, 1]$. Finally, from (3.7) we see that
\[
p_s(r) = p_s(0) + C(1 + \alpha)^{-1} r^{1+\alpha} + O(r) \quad \text{for} \quad r \to 0.
\] (3.8)
Substituting (3.7) and (3.8) into (3.6) we get $r^\gamma a''_s(r) \in C[0, 1]$. This completes the proof of Lemma 3.1. □

**Corollary 3.2** Let $a, b, B, \mathcal{F}$ and $\mathbb{B}$ be as in (2.24)–(2.27) and (2.29). Then we have $a, b \in C^{1}_V[0, 1], B \in L(C[0, 1], C^{1}_V[0, 1]) \subseteq L(C[0, 1]) \cap L(C^{1}_V[0, 1]), \mathcal{F} \in L(C[0, 1], \mathbb{R})$, and $\mathbb{B} \in L(X) \cap L(X_0)$. Moreover, we also have $r^2(1-r)^2a''(r), r^2(1-r)^2b''(r) \in C[0, 1]$. □

**Corollary 3.3** $G \in C^\infty(X, X) \cap C^\infty(X_0, X_0)$, and in addition to (2.23) we also have:
\[
\|G(V)\|_{X_0} = O(\|V\|_{X_0}^2) \quad \text{as} \quad \|V\|_{X_0} \to 0.
\] (3.9)

**Proof:** We have $G(V) = G_1(V) + G_2(V)$, where
\[
G_1(V) = [A_0(U_s + V) - A_0(U_s) - A'_0(U_s)V]U_s,
\]
\[
G_2(V) = F_0(U_s + V) - F_0(U_s) - F'_0(U_s)V.
\]
Since $F_0 \in C^\infty(X_0, X_0)$, it is evident that $G_2 \in C^\infty(X_0, X_0)$ and $\|G_2(V)\|_{X_0} = O(\|V\|_{X_0}^2)$ as $\|V\|_{X_0} \to 0$. Next, let $V = (\varphi, \zeta)$ and $(p, z) = (p_s + \varphi, z_s + \zeta)$. Then by (2.30) we have
\[
A_0(U_s + V)U_s = (-w_{p, z}(r)p'_s(r), 0).
\]
Using this expression and the first two relations in (3.3) we can easily show that for every $V \in X$ we have $A_0(U_s + V)U_s \in X_0$, and the mapping $V \to A_0(U_s + V)U_s$ belongs to $C^\infty(X, X_0)$. Hence we have $G_1 \in C^\infty(X, X_0) \subseteq C^\infty(X_0, X_0)$ and $\|G_1(V)\|_{X_0} = O(\|V\|_{X_0}^2) = O(\|V\|_{X_0}^4)$ as $\|V\|_{X_0} \to 0$. Combining these assertions together, we see that the desired assertion follows. □

### 4 Evolution systems

Given a small positive number $\varepsilon$, we denote
\[
S_\varepsilon = \{V = (\varphi, \zeta) \in X = C[0, 1] \times \mathbb{R} : \|\varphi\|_\infty \leq \varepsilon, |\zeta| \leq \varepsilon\}.
\]

In this section we shall prove that the family of operators $\{A(V) : V \in S_\varepsilon\}$ is a stable family of infinitesimal generators of $C_0$ semigroups on $X$, and its part in $X_0$ is a stable family of infinitesimal generators of $C_0$ semigroups on $X_0$. For the concept of stable family of infinitesimal generators of $C_0$ semigroups and related results, we refer the reader to see Sections 5.2–5.5, Chapter 5 and Section 6.4, Chapter 6 of [25]. We use the notation $\hat{A}(V)$ to denote the part of $A(V)$ in $X_0$. Recall that
\[
\text{Dom}(\hat{A}(V)) = \{U \in X_0 : A(V)U \in X_0\}, \quad \text{and}
\]
\[ \hat{A}(V)U = \hat{A}(V)U \text{ for } U \in \text{Dom}(\hat{A}(V)). \]

Let \( w \in C^1[0, 1] \) and assume that it satisfies the following condition: There exist positive constants \( C_1 \) and \( C_2 \) such that
\[
-C_1 r(1 - r) \leq w(r) \leq -C_2 r(1 - r) \quad \text{for } 0 \leq r \leq 1. \tag{4.1}
\]
Note that this assumption particularly implies that \( w(0) = w(1) = 0, w'(0) < 0 \) and \( w'(1) > 0 \).
For a such \( w \in C^1[0, 1] \), we denote by \( L_0 \) the bounded linear operator from \( C^1_V[0, 1] \) to \( C[0, 1] \) defined by
\[
L_0 q(r) = -w(r)q'(r) \quad \text{for } 0 < r < 1, \quad \text{for } q \in C^1_V[0, 1].
\]
Later on we shall also regard \( L_0 \) as an unbounded linear operator in \( C[0, 1] \) with domain \( C^1_V[0, 1] \).
Note that if \( w = u_* \) then \( L_0 = A_0(p_*, z_*) \).

**Lemma 4.1** Let the notation and the assumption be as above. Then \( L_0 \) generates a \( C_0 \) semigroup of contractions \( e^{tL_0} \) on \( C[0, 1] \), i.e.,
\[
\|e^{tL_0}\|_{L(C[0,1])} \leq 1 \quad \text{for } t \geq 0. \tag{4.2}
\]
Moreover, \( C^1_V[0, 1] \) is \( L_0 \)-admissible\(^2\), and the restriction of \( e^{tL_0} \) on \( C^1_V[0, 1] \) is a uniformly bounded \( C_0 \) semigroup on \( C^1_V[0, 1] \), i.e., there exists constant \( C > 0 \) depending only on the constants \( C_1 \) and \( C_2 \) in (4.1) such that
\[
\|e^{tL_0}\|_{L(C^1_V[0,1])} \leq C \quad \text{for } t \geq 0. \tag{4.3}
\]

**Proof.** We first prove that for any \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > 0 \) and any \( f \in C[0, 1] \), the equation
\[
-w(r)q'(r) - \lambda q(r) = f(r) \quad \text{(4.4)}
\]
has a unique solution \( q \in C^1_V[0, 1] \), and \( \|q\|_\infty \leq (\text{Re}\lambda)^{-1}\|f\|_\infty \).

Arbitrarily take a number \( 0 < r_0 < 1 \) and fix it. Since the equation (4.4) is linear and regular for all \( 0 < r < 1 \), for each given \( c \in \mathbb{R} \) it has a unique solution for all \( 0 < r < 1 \) satisfying \( q(r_0) = c \). In fact, this solution is given by
\[
q(r) = e^{-\lambda \int_{r_0}^r \frac{d\rho}{w(\rho)}} \left[ c - \int_{r_0}^r \frac{f(\eta)}{w(\eta)} e^{\lambda \int_{r_0}^\eta \frac{d\rho}{w(\rho)}} d\eta \right]. \tag{4.5}
\]
Since \( w \in C^1[0, 1] \) and \( w(0) = 0 \), we have \( w(r) = w'(0)r[1 + o(1)] = w'(0)r[1 + o(1)]^{-1} \) for \( r \sim 0^+ \). Thus, by taking \( \delta > 0 \) sufficiently small, we see that
\[
\int_{r_0}^r \frac{d\rho}{w(\rho)} = \frac{1}{w'(0)} \int_{\delta}^r \frac{1 + o(1)}{\rho} d\rho + \int_{r_0}^\delta \frac{d\rho}{w(\rho)} = \frac{1 + o(1)}{w'(0)} \log r + C \tag{4.6}
\]

\(^2\)Recall that for a \( C_0 \) semigroup \( T(t) \) \((t \geq 0)\) on a Banach space \( X \) generated by an unbounded linear operator \( A \) in \( X \), a linear subspace \( Y \) of \( X \) is called \( A \)-admissible if it is an invariant subspace of \( T(t) \) for all \( t \geq 0 \), and the restriction of \( T(t) \) \((t \geq 0)\) to \( Y \) is a \( C_0 \) semigroup in \( Y \). A necessary and sufficient condition for \( Y \) to be \( A \)-admissible is that (1) \( Y \) is an invariant subspace of \( R(\lambda, A) \) for all \( \lambda > \omega \) and (2) the part \( \hat{A} \) of \( A \) in \( Y \) is an infinitesimal generator of a \( C_0 \) semigroup on \( Y \). In this case we have \( e^{t\hat{A}} = e^{tA}|_Y \). See Theorem 5.5 in Chapter 4 of [25].
for \( r \sim 0^+ \). Since \( w'(0) < 0 \) and \( \text{Re}\lambda > 0 \), it follows that
\[
e^{-\lambda} \int_{0}^{r} \frac{d_{w(p)}}{w(p)} = Cr \frac{\lambda(1 + \varepsilon(1))}{w'(0)} \rightarrow 0 \quad \text{as} \quad r \rightarrow 0^+.
\]

Hence, using the L'Hospital's law we see that for any \( c \in \mathbb{R} \) the function \( q(r) \) given by (4.5) has finite limit as \( r \rightarrow 0^+ \). By a similar argument as in the deduction of (4.6) we have
\[
e^{-\lambda} \int_{0}^{r} \frac{d_{w(p)}}{w(p)} = C(1 - r) \frac{\lambda(1 + \varepsilon(1))}{w'(1)} \rightarrow \infty \quad \text{as} \quad r \rightarrow 1^-.
\]

It follows that the function \( q(r) \) given by (4.5) cannot be bounded in a neighborhood of \( r = 1 \) unless we take \( c = \int_{0}^{1} f(\eta) e^{\lambda} \int_{0}^{r} \frac{d_{w(p)}}{w(p)} d\eta, \) which gives
\[
q(r) = e^{-\lambda} \int_{0}^{r} \frac{d_{w(p)}}{w(p)} \int_{r}^{1} f(\eta) e^{\lambda} \int_{0}^{r} \frac{d_{w(p)}}{w(p)} d\eta.
\]

By using the L'Hospital's law we can easily verify that the function \( q(r) \) given by (4.7) has finite limit as \( r \rightarrow 1^- \). Hence, we have shown that if \( \text{Re}\lambda > 0 \) then for any \( f \in C[0,1] \) the equation (4.4) has a unique solution \( q \in C[0,1] \cap C^1(0,1) \). Using (4.4) as well as the fact that \( w(r)/r(1-r) \in C[0,1] \) we see readily that \( q \in C^1[0,1] \). Furthermore, by a simple computation we have
\[
|q(r)| \leq \|f\|_{\infty} e^{\lambda} \left( \int_{r}^{1} \frac{1}{w(\eta)} e^{-\lambda} \int_{0}^{r} \frac{d_{w(p)}}{w(p)} d\eta \right) = \frac{\|f\|_{\infty}}{\text{Re}\lambda} \left[ 1 - e^{-\lambda} \int_{0}^{1} \frac{d_{w(p)}}{w(p)} \right] \leq \frac{\|f\|_{\infty}}{\text{Re}\lambda} \quad \text{for} \quad 0 < r < 1.
\]

This proves the desired assertion.

Thus, we have proved that for any \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > 0 \), there holds \( \lambda \in \rho(\mathcal{L}_0) \) and
\[
\|R(\lambda, \mathcal{L}_0)\|_{L(C[0,1])} \leq \frac{1}{\text{Re}\lambda}.
\]

It follows by the Hille-Yosida Theorem that \( \mathcal{L}_0 \) generates a strongly continuous semigroup \( e^{t\mathcal{L}_0} \) on \( C[0,1] \) which satisfies the estimate (4.2).

Next we prove that \( C^1_V[0,1] \) is \( \mathcal{L}_0 \)-admissible. Clearly, \( C^1_V[0,1] \) is an invariant subspace of \( R(\lambda, \mathcal{L}_0) \) for all \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > 0 \), because we know that for such \( \lambda \), \( R(\lambda, \mathcal{L}_0) \) is a bounded linear operator in \( C[0,1] \) with image contained in \( \text{Dom}(\mathcal{L}_0) = C^1_V[0,1] \). Let \( \mathcal{L}_0 \) be the part of \( \mathcal{L}_0 \) in \( C^1_V[0,1] \). Since \( r(1-r)/w(r), w(r)/(1-r) \in C[0,1] \), we see that for \( q \in C[0,1] \cap C^1(0,1) \), \( r(1-r)q'(r) \in C[0,1] \) if and only if \( w(r)q'(r) \in C[0,1] \), and, furthermore, there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \sup_{0 < r < 1} |w(r)q'(r)| \leq \sup_{0 < r < 1} |r(1-r)q'(r)| \leq C_2 \sup_{0 < r < 1} |w(r)q'(r)|.
\]

By the above assertion it follows easily that
\[
\text{Dom}(\mathcal{L}_0) = \{ q \in C[0,1] \cap C^2(0,1) : w(r)q'(r) \in C[0,1], w^2(r)q''(r) \in C[0,1] \},
\]
and \( \|q\|_{C^1_V[0,1]} = \|q\|_{\infty} + \|wq\|_{\infty} \) is an equivalent norm in \( C^1_V[0,1] \). For \( q \in \text{Dom}(\mathcal{L}_0) \) we have, by definition, \( \mathcal{L}_0q = \mathcal{L}_0q \). Now let \( f \in C^1_V[0,1] \) and \( \text{Re}\lambda > 0 \). Let \( q \) be the solution of (4.4).
Using (4.9) we can easily verify that \( q \in \text{Dom}(\tilde{L}_0) \), so that it is the solution of the equation \( \tilde{L}_0 q - \lambda q = f \). Moreover, a simple computation shows that \( wq' = R(\tilde{L}_0, \lambda)(wf') \). Thus, by (4.8) we have
\[
\|q\|_{[0,1]}' = \|q\|_1 + \|wf\|_{[0,1]} = \|R(\tilde{L}, \lambda)f\|_1 + \|R(\tilde{L}, \lambda)(wf')\|_1
\]
\[
\leq \frac{1}{\Re\lambda} \|f\|_1 + \frac{1}{\Re\lambda} \|wf'\|_1 = \frac{1}{\Re\lambda} \|f\|_{[0,1]}'
\]
Hence \( \lambda \in \rho(\tilde{L}_0) \) and \( \|R(\lambda, \tilde{L}_0)\|_{[0,1]}' \leq (\Re\lambda)^{-1} \). The desired assertion then follows from the Hille-Yosida Theorem and the footnote on Page 12. \( \square \)

Given \( V = (\varphi, \zeta) \in S_\varepsilon \), we set \( p(r) = p_s(r) + \varphi(r) \), \( z = z_s + \zeta \), and as before we denote
\[
u_{p,z}(r) = \frac{1}{r^2} \int_0^r [-K_D(c(\rho, z)) + K_M(c(\rho, z))p(\rho)]\rho^2 d\rho \quad \text{and} \quad \nu_{p,z}(r) = \nu_{p,z}(r) - ru_{p,z}(1).
\]
Since \( \|\varphi\|_1 \leq \varepsilon \) and \( |\zeta| \leq \varepsilon \), by a simple computation we see that
\[
-C\varepsilon r(1 - r) \leq \nu_{p,z}(r) - u_s(r) \leq C\varepsilon r(1 - r) \quad \text{for} \quad 0 \leq r \leq 1. \tag{4.10}
\]
Since \( -C_1 r(1 - r) \leq u_s(r) \leq C_2 r(1 - r) \), it follows that
\[
(1 + C\varepsilon)u_s(r) \leq \nu_{p,z}(r) \leq (1 - C\varepsilon)u_s(r) \quad \text{for} \quad 0 \leq r \leq 1,
\]
and, consequently, for \( \varepsilon \) sufficiently small we have
\[
-C_1 r(1 - r) \leq \nu_{p,z}(r) \leq -C_2 r(1 - r) \quad \text{for} \quad 0 \leq r \leq 1. \tag{4.11}
\]
Later on we shall also use the notation \( w_V'(r) \) to re-denote \( w_{p,z}(r) \). We note that all constants \( C, C_1 \) and \( C_2 \) that appear in (4.10)–(4.11) are independent of \( V \) and \( \varepsilon \).

**Lemma 4.2** \( \{A(V) : V \in S_\varepsilon\} \) is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( X = C[0,1] \times \mathbb{R} \), and \( \{A(V) : V \in S_\varepsilon\} \) is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( X_0 \).

**Proof.** Let \( L_V q(r) = -w_V(r)q'(r) \). Then by Lemma 4.1 we know that for any \( V \in S_\varepsilon \), \( L_V \) is an infinitesimal generator of a \( C_0 \) semigroup of contractions \( e^{tL_V} \) on \( C[0,1] \). Since
\[
\mathcal{A}_0(U_s + V) = \begin{pmatrix} L_V & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.12}
\]
(see (2.30)), it is evident that for any \( V \in S_\varepsilon \), \( \mathcal{A}_0(U_s + V) \) is an infinitesimal generator of a \( C_0 \) semigroup of contractions \( e^{t\mathcal{A}_0(U_s + V)} \) on \( X = C[0,1] \times \mathbb{R} \). In fact,
\[
 e^{t\mathcal{A}_0(U_s + V)} = \begin{pmatrix} e^{tL_V} & 0 \\ 0 & id \end{pmatrix}.
\]
Hence, \( \{\mathcal{A}_0(U_s + V) : V \in S_\varepsilon\} \) is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( X \), with stability constants \( (M, \omega) = (1, 0) \). Since \( A(V) = \mathcal{A}_0(U_s + V) + B \) and \( B \) is a bounded linear operator on \( X \) independent of \( V \), by a standard perturbation result (see, e. g. Theorem
2.3 in Section 5.2 of [25] we immediately get the assertion that \( \{A(V) : V \in S_\varepsilon \} \) is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( X = C[0,1] \times \mathbb{R} \), with stability constants \( (M, \omega) = (1, ||B||) \).

In order to prove that \( \{\tilde{A}(V) : V \in S_\varepsilon \} \) is a stable family of infinitesimal generators of \( C_0 \) semigroup on \( X_0 = C^1_V[0,1] \times \mathbb{R} \), we first establish an estimate for the semigroup \( e^{t\tilde{L}_V} \) on \( C^1_V[0,1] \) different from (4.5), where \( \tilde{L}_V \) represents the part of \( L_V \) in \( C^1_V[0,1] \). Let \( q_0 \in C^1_V[0,1] \) and \( q = e^{t\tilde{L}_V} q_0 = e^{tL_V} q_0 \). Then \( q \) is the solution of the problem:

\[
\frac{\partial q}{\partial t} + w_V(r) \frac{\partial q}{\partial r} = 0 \quad \text{for} \quad 0 \leq r \leq 1 \quad \text{and} \quad t > 0, \quad q|_{t=0} = q_0.
\]

Let \( l(r,t) = r(1-r)\frac{\partial q(r,t)}{\partial r} \) and \( l_0(r) = r(1-r)q_0'(r) \). Differentiating the above equation in \( r \) and multiplying it with \( r(1-r) \), we get

\[
\frac{\partial l}{\partial t} + w_V(r) \frac{\partial l}{\partial r} = a_V(r)l \quad \text{for} \quad 0 \leq r \leq 1 \quad \text{and} \quad t > 0, \quad l|_{t=0} = l_0,
\]

where \( a_V(r) = (1 - 2r) \frac{w_V(r)}{r(1-r)} - w_V'(r) \). Clearly, there exists a nonnegative constant \( c_0 \) independent of \( V \) such that

\[
a_V(r) \leq c_0 \quad \text{for} \quad 0 < r < 1, \quad \text{for all} \quad V \in S_\varepsilon.
\]

Using this fact and a standard characteristics argument we can easily obtain

\[
\|l(\cdot,t)\|_{L^\infty} \leq \|l_0\|_{L^\infty} e^{c_0 t} \quad \text{for} \quad t \geq 0.
\]

Combining this estimate with \( \|q(\cdot,t)\|_{L^\infty} \leq \|q_0\|_{L^\infty} \) ensured by (4.2) we get

\[
\|q(\cdot,t)\|_{C^1_V[0,1]} \leq \|q_0\|_{C^1_V[0,1]} e^{c_0 t} \quad \text{for} \quad t \geq 0.
\]

Hence

\[
\|e^{t\tilde{L}_V}\|_{L(C^1_V[0,1])} \leq e^{c_0 t} \quad \text{for} \quad t \geq 0, \quad \text{for all} \quad V \in S_\varepsilon.
\]

Hence, \( \{\tilde{L}_V : V \in S_\varepsilon \} \) is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( C^1_V[0,1] \), with stability constants \( (M, \omega) = (1, c_0) \). Using this assertion and (4.12) we see easily that \( \{\tilde{A}_0(U_s + V) : V \in S_\varepsilon \} \), the part of \( \{A_0(U_s + V) : V \in S_\varepsilon \} \) on \( X_0 = C[0,1] \times \mathbb{R} \), is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( X_0 \), with stability constants \( (M, \omega) = (1, c_0) \). Since \( \tilde{A}(V) = \tilde{A}_0(U_s + V) + B \) and, by Corollary 3.2, \( B \) is a bounded linear operator on \( X_0 \) independent of \( V \), we conclude as before that \( \{A(V) : V \in S_\varepsilon \} \) is a stable family of infinitesimal generators of \( C_0 \) semigroups on \( X_0 = C^1_V[0,1] \times \mathbb{R} \), with stability constants \( (M, \omega) = (1, c_0 + ||B||_{L(C^1_V[0,1])}) \). This completes the proof of Lemma 4.2. \( \Box \)

Since \( A \in C^\infty(X, L(X_0, X)) \), by Lemma 4.2 we see that for any \( V \in C([0,\infty), X) \) such that \( V(t) \in S_\varepsilon \) for all \( t \geq 0 \), \( \{A(V(t)) : t \geq 0 \} \) satisfies the conditions \( (H_1)-(H_3) \) in Section 5.3 of [25]. It follows by Theorem 3.1 in Section 5.3 of [25] that given a such function \( V = V(t) \), there exists an evolution system determined by \( \{A(V(t)) : t \geq 0 \} \), which we denote as \( U(t,s,V) \). By definition, this means that

1. for any \( t \geq s \geq 0 \), \( U(t,s,V) \) is a bounded linear operator on \( X \),
However, the theory developed in [25] does not ensure that \( U \) by using the characteristic method we can easily prove that (4.15) imposed with the initial condition
\[
(0, t) \cap \mathbb{R} \times [0, \infty) 
\]
are characteristic curves. It follows that all characteristic curves starting from the open interval \((0, \infty)\) always lie in it, so that the solution of the above problem exists for all \( t \geq s \geq 0 \).

In the following lemma we shall directly prove that for any \( U_0 \in X_0 \), the problem (4.13) has a unique solution \( U = U_0 \) for any \( t \geq s \geq 0 \). By Theorem 4.2 in Section 5.4 of [25], it then follows that \( U_0 \) is a solution of the problem (4.13) in \( X_0 \) and, consequently, the conditions (4) and (5) above are satisfied.

**Lemma 4.3** Given \( V \in C([0, \infty), X) \) such that \( V(t) \in S_\varepsilon \) for all \( t \geq 0 \), for any \( s \geq 0 \) and any \( U_0 \in X_0 \), the problem (4.13) has a unique solution \( U = U_0 \) for any \( t \geq s \geq 0 \).

**Proof:** Let \( U = (q, y) \) and \( U_0 = (q_0, y_0) \). Then (4.13) can be rewritten as follows:
\[
\begin{cases}
\frac{\partial q}{\partial t} + w_V(r, t) \frac{\partial q}{\partial r} = a(r)q + B(q) + b(r)y & \text{for } 0 \leq r \leq 1, \ t > s, \\
\frac{\partial y}{\partial t} = F(q) + \kappa y & \text{for } t > s, \\
q|_{t=s} = q_0(r) & \text{for } 0 \leq r \leq 1, \text{ and } y|_{t=s} = y_0.
\end{cases}
\]
\]

Using the characteristic method and the Banach fixed point theorem, we can easily show that this problem has a unique local solution \((q, y)\) with \( q \in C([0, 1] \times [0, \delta]) \) and \( y \in C^1([0, \delta]) \) for some \( \delta > 0 \). Since \( w_V(0, t) = w_V(1, t) = 0 \) for all \( t \geq 0 \), we see that the two lines \( r = 0 \) and \( r = 1 \) are characteristic curves. It follows that all characteristic curves starting from the open interval \((0,1)\) always lie in it, so that the solution of the above problem exists for all \( t \geq s \). It remains to prove that \( q \in C([0, \infty), A_V \cap [0, 1]) \). To this end we formally differentiate the first equation in (4.14) in \( r \) and multiply it with \( r(1-r) \), which gives, by letting \( l(r, t) = r(1-r) \frac{\partial q(r, t)}{\partial r} \), that
\[
\frac{\partial l}{\partial t} + w_V(r, t) \frac{\partial l}{\partial r} = a_1(r, t)u + f_1(r, t) \text{ for } 0 \leq r \leq 1, \ t > 0,
\]
where
\[
a_1(r, t) = a(r) + (1 - 2r) \frac{w_V(r, t)}{r(1-r)} - \frac{\partial w_V(r, t)}{\partial r},
\]
\[
f_1(r, t) = r(1-r)a'(r)q(r, t) + r(1-r) \frac{\partial B(q(r), t)}{\partial r} + r(1-r)b'(r)y(t).
\]
Clearly, \( a_1 \in C([0, 1] \times [0, \infty]) \). By Corollary 3.2 we see that also \( f_1 \in C([0, 1] \times [0, \infty]) \). Thus by using the characteristic method we can easily prove that (4.15) imposed with the initial
condition $l(r, 0) = r(1 - r)q_0'(r)$ has a unique solution $l \in C([0, 1] \times [0, \infty)$. Thus, the above formal computation makes sense and, consequently, $q \in C([0, \infty), C^1([0, 1]) \cap C^1([0, \infty), C[0, 1])$. The desired assertion now becomes immediate. □

By the above results and Theorems 4.2 and 5.2 in Sections 5.4 and 5.5 of [25], we get:

**Corollary 4.4** Let $V = V(t) \in C([0, \infty), X)$ be as in Lemma 4.3, and let $F = F(t) \in C([0, \infty), X_0)$. Then for any $U_0 \in X_0$, the initial value problem

$$
\frac{dU}{dt} = A(V(t))U + F(t) \quad \text{for } t > 0, \quad U(0) = U_0
$$

has a unique solution $U = U(t) \in C([0, \infty), X_0) \cap C^1([0, \infty), X)$, and it is given by

$$
U(t) = \mathbb{U}(t, 0, V)U_0 + \int_0^t \mathbb{U}(t, s, V)F(s)ds.
$$

□

5 Similarity transformation

In this section we shall study a family of $C^1$-diffeomorphisms $\bar{r} = T(r, t, s)$ of the unit interval $0 \leq r \leq 1$ to itself, where $t \geq s \geq 0$ are parameters. This family of diffeomorphisms will be used in the next section to deduce a uniform decay estimate for the evolution system $\mathbb{U}(t, s, V)$ established in the previous section when $V$ is replaced by an exponentially decaying function $V = V(t) \in C([0, \infty), S_\epsilon)$.

Let $w \in C([0, \infty), C^1[0, 1])$. We assume that $w$ satisfies the following condition: For some small parameter $\varepsilon > 0$,

$$
-C \varepsilon r(1 - r)e^{-\mu t} \leq w(r, t) - u_*(r) \leq C \varepsilon r(1 - r)e^{-\mu t} \quad \text{for } 0 \leq r \leq 1, \ t \geq 0, \quad (5.1)
$$

where $C$ is a positive constant independent of $\varepsilon$ and $w$. Since $-C_1 r(1 - r) \leq u_*(r) \leq -C_2 r(1 - r)$, we see that

$$
\sup_{0 < r < 1} \left| \frac{w(r, t)}{u_*(r)} - 1 \right| \leq C \varepsilon e^{-\mu t} \quad \text{for } 0 \leq r \leq 1, \ t \geq 0, \quad (5.2)
$$

and for $\varepsilon$ sufficiently small we have

$$
-C_1 r(1 - r) \leq w(r, t) \leq -C_2 r(1 - r) \quad \text{for } 0 \leq r \leq 1, \ t \geq 0 \quad (5.3)
$$

and

$$
\frac{1}{2} \leq \frac{w(r, t)}{u_*(r)} \leq 2 \quad \text{for } 0 \leq r \leq 1, \ t \geq 0. \quad (5.4)
$$

Let $0 \leq \xi \leq 1$ and $s \geq 0$. Consider the following initial value problem:

$$
\frac{dr}{dt} = u_*(r) \quad \text{for } t > s, \quad r|_{t=s} = \xi. \quad (5.5)
$$
Since \( u_\ast \in C^1[0,1] \), \( u_\ast(r) < 0 \) for \( 0 < r < 1 \) and, in particular, \( u_\ast(0) = u_\ast(1) = 0 \), it can be easily shown that this problem has a unique solution \( r = \Phi_\ast(\xi,t,s) \) for all \( t \geq s \), satisfying the following properties:

\[
\Phi_\ast(\xi,t,s) \text{ is twice continuously differentiable in } (\xi,t,s),
\]

\[
\Phi_\ast(0,t,s) = 0, \quad \Phi_\ast(1,t,s) = 1 \quad \text{for } t \geq s,
\]

\[
0 < \Phi_\ast(\xi,t,s) < 1 \quad \text{for } 0 < \xi < 1, \quad t \geq s,
\]

\[
\frac{\partial \Phi_\ast(\xi,t,s)}{\partial \xi} > 0, \quad \frac{\partial \Phi_\ast(\xi,t,s)}{\partial t} < 0 \quad \text{for } 0 < \xi < 1, \quad t \geq s.
\]

Note that we also have

\[
\Phi_\ast(\xi,s,s) = \xi \quad \text{and} \quad \Phi_\ast(\xi,t,s) = \Phi_\ast(\xi,t-s,0) \quad \text{for } 0 \leq \xi \leq 1, \quad t \geq s.
\]

From these properties we see that for any \( s \geq 0 \) and \( t \geq s \), the mapping \( \xi \to r = \Phi_\ast(\xi,t,s) \) is a \( C^2 \) diffeomorphism of \([0,1]\) to itself. Let \( \xi = \Psi_\ast(r,t,s) \) be the inverse of this mapping. Clearly, \( \Psi_\ast \) satisfies the following properties:

\[
\Psi_\ast(r,t,s) \text{ is twice continuously differentiable in } (r,t,s),
\]

\[
\Psi_\ast(0,t,s) = 0, \quad \Psi_\ast(1,t,s) = 1 \quad \text{for } t \geq s,
\]

\[
0 < \Psi_\ast(r,t,s) < 1 \quad \text{for } 0 < r < 1, \quad t \geq s,
\]

\[
\frac{\partial \Psi_\ast(r,t,s)}{\partial r} > 0, \quad \frac{\partial \Psi_\ast(r,t,s)}{\partial t} > 0 \quad \text{for } 0 < r < 1, \quad t \geq s,
\]

\[
\Psi_\ast(r,s,s) = r \quad \text{and} \quad \Psi_\ast(r,t,s) = \Psi_\ast(r,t-s,0) \quad \text{for } 0 \leq r \leq 1, \quad s \geq 0.
\]

Furthermore, by the definition of \( \Psi_\ast \) we have the following relations:

\[
\Psi_\ast(\Phi_\ast(\xi,t,s),t,s) = \xi \quad \text{for } 0 \leq \xi \leq 1, \quad t \geq s,
\]

\[
\Phi_\ast(\Psi_\ast(r,t,s),t,s) = r \quad \text{for } 0 \leq r \leq 1, \quad t \geq s.
\]

From the first relation we easily deduce that \( \xi = \Psi_\ast(r,t,s) \) is the unique solution of the following initial value problem:

\[
\frac{\partial \xi}{\partial t} + u_\ast(r)\frac{\partial \xi}{\partial r} = 0 \quad \text{for } t > s, \quad \xi|_{t=s} = r.
\]

Next, let \( r = \Phi(\xi,t,s) \) (\( 0 \leq \xi \leq 1, \quad t \geq s \geq 0 \)) be the solution of the following problem:

\[
\frac{dr}{dt} = w(r,t) \quad \text{for } t > s, \quad r|_{t=s} = \xi.
\]

Similarly as before, \( \Phi(\xi,t,s) \) is well-defined for all \( 0 \leq \xi \leq 1 \) and \( t \geq s \), and it satisfies the following properties:

\[
\Phi(\xi,t,s) \text{ is continuously differentiable in } (\xi,t,s),
\]

\[
\Phi(0,t,s) = 0, \quad \Phi(1,t,s) = 1 \quad \text{for } t \geq s,
\]

\[
0 < \Phi(\xi,t,s) < 1 \quad \text{for } 0 < \xi < 1, \quad t \geq s,
\]
\[
\frac{\partial \Phi(\xi, t, s)}{\partial \xi} > 0, \quad \frac{\partial \Phi(\xi, t, s)}{\partial t} < 0 \quad \text{for} \quad 0 < \xi < 1, \quad t \geq s,
\]
\[
\Phi(\xi, s, s) = \xi \quad \text{for} \quad 0 \leq \xi \leq 1, \quad s \geq 0.
\]

From the above properties we see that for any \(s \geq 0\) and \(t \geq s\), the mapping \(\xi \rightarrow r = \Phi(\xi, t, s)\) is a \(C^1\) diffeomorphism of \([0, 1]\) to itself. Let \(\xi = \Psi(r, t, s)\) be the inverse of this mapping. Similarly as before we have

\[
\Psi(r, t, s) \text{ is continuously differentiable in } (r, t, s),
\]
\[
\Psi(0, t, s) = 0, \quad \Psi(1, t, s) = 1 \quad \text{for} \quad t \geq s,
\]
\[
0 < \Psi(r, t, s) < 1 \quad \text{for} \quad 0 < r < 1, \quad t \geq s,
\]
\[
\frac{\partial \Psi(r, t, s)}{\partial r} > 0, \quad \frac{\partial \Psi(r, t, s)}{\partial t} > 0 \quad \text{for} \quad 0 < r < 1, \quad t \geq s,
\]
\[
\Psi(r, s, s) = r \quad \text{for} \quad 0 \leq r \leq 1, \quad s \geq 0.
\]

Moreover, we have the following relations:

\[
\Psi(\Phi(\xi, t, s), t, s) = \xi \quad \text{for} \quad 0 \leq \xi \leq 1, \quad t \geq s,
\]
\[
\Phi(\Psi(r, t, s), t, s) = r \quad \text{for} \quad 0 \leq r \leq 1, \quad t \geq s,
\]

and \(\xi = \Psi(r, t, s)\) is the unique solution of the following initial value problem:

\[
\frac{\partial \xi}{\partial t} + w(r, t)\frac{\partial \xi}{\partial r} = 0 \quad \text{for} \quad t > s, \quad \xi|_{t=s} = r.
\]  \(5.8\)

In the sequel we consider the following initial value problem:

\[
\begin{cases}
\frac{\partial \bar{r}}{\partial t} + w(r, t)\frac{\partial \bar{r}}{\partial r} = u_s(\bar{r}) & \text{for} \quad 0 \leq r \leq 1, \quad t > s, \\
\bar{r}|_{t=s} = r & \text{for} \quad 0 \leq r \leq 1.
\end{cases}
\]  \(5.9\)

**Lemma 5.1** For any \(0 \leq r \leq 1\) and \(s \geq 0\), the problem \(5.9\) has a unique solution \(\bar{r} = T(r, t, s)\) for all \(t \geq s\), and the following relation holds:

\[
T(r, t, s) = \Phi_s(\Psi(r, t, s), t, s) \quad \text{for} \quad 0 \leq r \leq 1, \quad t \geq s \geq 0.
\]  \(5.10\)

**Proof.** Using \(5.5\) and \(5.8\) we can easily verify that \(\bar{r} = \Phi_s(\Psi(r, t, s), t, s)\) is a solution of the problem \(5.9\). Thus, \(5.10\) follows by uniqueness of the solution. □

By \(5.10\), it is evident that for any \(s \geq 0\) and \(t \geq s\), the mapping \(r \rightarrow \bar{r} = T(r, t, s)\) is a \(C^1\) diffeomorphism of \([0, 1]\) to itself, satisfying the following properties:

\[
T(0, t, s) = 0, \quad T(1, t, s) = 1 \quad \text{for} \quad t \geq s \geq 0,
\]
\[
\frac{\partial T(r, t, s)}{\partial r} > 0 \quad \text{for} \quad 0 < r < 1, \quad t \geq s.
\]

We denote by \(r = S(\bar{r}, t, s)\) the inverse of this mapping. By \(5.10\) it is clear that

\[
S(\bar{r}, t, s) = \Phi(\Psi_s(\bar{r}, t, s), t, s) \quad \text{for} \quad 0 \leq \bar{r} \leq 1, \quad t \geq s \geq 0.
\]  \(5.11\)
It is also clear that \( S(\bar{r}, t, s) \) satisfies the following properties:

\[
S(0, t, s) = 0, \quad S(1, t, s) = 1 \quad \text{for} \quad t \geq s \geq 0,
\]
\[
\frac{\partial S(\bar{r}, t, s)}{\partial \bar{r}} > 0 \quad \text{for} \quad 0 < \bar{r} < 1, \quad t \geq s \geq 0.
\]

\( T \) and \( S \) can be expressed in more explicit formulations. To show this we introduce a function \( F_* \) as follows:

\[
F_*^*(r) = -\int_{1/2}^r \frac{du_*(\eta)}{u_*(\eta)} = \int_{1/2}^r \frac{du_*(\eta)}{|u_*(\eta)|} \quad \text{for} \quad 0 < r < 1. \tag{5.12}
\]

Clearly, \( F_* \in C^1(0,1) \), \( F'_*(r) > 0 \) for all \( 0 < r < 1 \), and

\[
\lim_{r \to 0^+} F_*^*(r) = -\infty, \quad \lim_{r \to 1^-} F_*^*(r) = \infty.
\]

Hence \( \bar{r} = F_*^*(r) \) is a \( C^1 \) diffeomorphism of the open unit interval \((0,1)\) to the real line \((-\infty, \infty)\).

From (5.5) we easily obtain

\[
F_*^*(\Phi^*_*(\xi, t, s)) - F_*^*(\xi) = -t + s.
\]

Thus

\[
\Phi^*_*(\xi, t, s) = F_*^{-1}(F_*^*(\xi) - t + s), \tag{5.13}
\]

and, consequently,

\[
\Psi^*_*(r, t, s) = F_*^{-1}(F_*^*(r) + t - s). \tag{5.14}
\]

Next, let

\[
g(\xi, t, s) = G(\Phi(\xi, t, s), t), \quad \text{where} \quad G(r, t) = \frac{w(r, t)}{u_*(r)} - 1.
\]

Since \( w(r, t) = [1 + G(r, t)]u_*(r) \), from (5.7) we see that \( r = \Phi(\xi, t, s) \) is a solution of the following problem:

\[
\frac{dr}{dt} = [1 + g(\xi, t, s)]u_*(r) \quad \text{for} \quad t > s, \quad r|_{t=s} = \xi. \tag{5.15}
\]

Thus similarly as before we have

\[
F_*^*(\Phi(\xi, t, s)) - F_*^*(\xi) = -t + s - \int_s^t g(\xi, \tau, s)d\tau,
\]

so that

\[
\Phi(\xi, t, s) = F_*^{-1}(F_*^*(\xi) - t + s - \int_s^t g(\xi, \tau, s)d\tau), \tag{5.17}
\]

\[
\Psi(r, t, s) = F_*^{-1}(F_*^*(r) + t - s + \int_s^t g(\Psi(r, t, s), \tau, s)d\tau). \tag{5.18}
\]

Combining (5.10), (5.11), (5.13), (5.14), (5.17) and (5.18) we see that

\[
T(r, t, s) = F_*^{-1}(F_*^*(r) + \int_s^t g(\Psi(r, t, s), \tau, s)d\tau), \tag{5.19}
\]

\[
S(\bar{r}, t, s) = F_*^{-1}(F_*^*(\bar{r}) - \int_s^t g(\Psi_*(\bar{r}, t, s), \tau, s)d\tau). \tag{5.20}
\]
Lemma 5.2 Assume that $|\zeta| \leq C$. Then there exist positive constants $C_1$ and $C_2$ depending only on $C$ such that for any $0 < r < 1$ we have

$$C_1 r (1 - r) \leq F_*^{-1}(F_*(r) + \zeta) [1 - F_*^{-1}(F_*(r) + \zeta)] \leq C_2 r (1 - r). \quad (5.21)$$

Proof. Since $-C \leq \zeta \leq C$, by the monotonicity of $F_*$ we have

$$F_*^{-1}(F_*(r) - C) \leq F_*^{-1}(F_*(r) + \zeta) \leq F_*^{-1}(F_*(r) + C).$$

Thus

$$\frac{F_*^{-1}(F_*(r) + \zeta)}{r} \leq \frac{F_*^{-1}(F_*(r) + C)}{r}$$

and

$$\frac{1 - F_*^{-1}(F_*(r) + \zeta)}{1 - r} \leq \frac{1 - F_*^{-1}(F_*(r) - C)}{1 - r}.$$

We claim that

$$\lim_{r \to 0^+} \frac{F_*^{-1}(F_*(r) + C)}{r} = e^{C|u'_*(0)|} \quad (5.22)$$

Indeed, since $u_*(r) = u'_*(0)r[1 + O(r^\beta)]$ (for $r \to 0$) for some $0 < \beta \leq 1$ (see Assertion (3) of Lemma 3.1), we have $1/u_*(r) = [1 + O(r^\beta)]/u'_*(0)r$ (for $r \to 0$), so that

$$F_*(r) = - \int_1^r \frac{1}{u_*(\eta)} d\eta = - \int_{r_0}^r \frac{1}{u_*(\eta)} d\eta - \int_{1/2}^{r_0} \frac{1}{u_*(\eta)} d\eta = - \int_{r_0}^r \frac{1 + O(\eta^\beta)}{u'_*(0)\eta} d\eta + C,$$

which yields

$$F_*(r) = \log r^{-|u'_*(0)|} + C_1 + O(r^\beta) \quad \text{for } r \to 0.$$

Thus

$$F_*^{-1}(\xi) = e^{|u'_*(0)||\xi - C_1 + O((F_*^{-1}(\xi))^\beta)} \quad \text{for } \xi \sim -\infty,$$

and, consequently,

$$F_*^{-1}(F_*(r) + C) = re^{C|u'_*(0)| + O(r^\beta)} \quad \text{for } r \sim 0,$$

by which (5.22) follows immediately. Similarly, we also have

$$\lim_{r \to -1^-} \frac{1 - F_*^{-1}(F_*(r) - C)}{1 - r} = e^{Cu'_*(1)}. \quad (5.23)$$

By (5.22) and (5.23), the second inequality in (5.21) immediately follows. The proof for the first inequality in (5.21) is similar. This completes the proof of Lemma 5.2. \qed

Corollary 5.3 For $\varepsilon$ sufficiently small we have

$$C_1 r (1 - r) \leq T(r, t, s)[1 - T(r, t, s)] \leq C_2 r (1 - r), \quad (5.24)$$

$$C_1 \tilde{r}(1 - \tilde{r}) \leq S(\tilde{r}, t, s)[1 - S(\tilde{r}, t, s)] \leq C_2 \tilde{r}(1 - \tilde{r}).$$

(5.25)
Proof: Let $\zeta = \int_s^t g(\Psi(r, t, s), \tau, s) d\tau$. By (5.2) we have

$$|\zeta| \leq \int_s^t |g(\Psi(r, t, s), \tau, s)| d\tau \leq C\varepsilon \int_s^t e^{-\mu \tau} d\tau \leq C\varepsilon \int_0^\infty e^{-\mu \tau} d\tau \leq C\varepsilon \leq C.$$ 

Hence, (5.24) follows from (5.19) and (5.21). Similarly, (5.25) follows from (5.20) and (5.21). \qed

As an immediate consequence of Corollary 5.3 we see that there exists constant $C > 1$ such that for $\varepsilon$ sufficiently small we have

$$C^{-1} \leq \frac{T(r, t, s)}{r} \leq C \quad \text{and} \quad C^{-1} \leq \frac{S(\bar{r}, t, s)}{\bar{r}} \leq C.$$

**Corollary 5.4** For $\varepsilon$ sufficiently small we have the following inequalities:

$$C_1 \Psi_*(r, t, s)[1 - \Psi_*(r, t, s)] \leq \Psi(r, t, s)[1 - \Psi(r, t, s)] \leq C_2 \Psi_*(r, t, s)[1 - \Psi_*(r, t, s)]. \quad (5.26)$$

$$C_1 \Phi_*(\bar{r}, t, s)[1 - \Phi_*(\bar{r}, t, s)] \leq \Phi(\bar{r}, t, s)[1 - \Phi(\bar{r}, t, s)] \leq C_2 \Phi_*(\bar{r}, t, s)[1 - \Phi_*(\bar{r}, t, s)]. \quad (5.27)$$

Proof: Let $\bar{r} = \Psi_*(r, t, s)$ and $\zeta = \int_s^t g(\Psi(r, t, s), \tau, s) d\tau$. Then by (5.14) and (5.18) we have

$$\Psi(r, t, s) = F^{-1}_*(F_*(\bar{r}) + \zeta).$$

By this expression and (5.21) we immediately obtain (5.26). The proof of (5.27) is similar. \qed

As an immediate consequence of Corollary 5.4 we see that there exists constant $C > 1$ such that for $\varepsilon$ sufficiently small we have

$$C^{-1} \leq \frac{\Psi(r, t, s)}{\Psi_*(r, t, s)} \leq C \quad \text{and} \quad C^{-1} \leq \frac{\Phi(\bar{r}, t, s)}{\Phi_*(\bar{r}, t, s)} \leq C.$$

**Lemma 5.5** We have the following inequalities:

$$|T(r, t, s) - r| \leq C\varepsilon (e^{-\mu s} - e^{-\mu t}) r(1 - r), \quad (5.28)$$

$$|S(\bar{r}, t, s) - \bar{r}| \leq C\varepsilon (e^{-\mu s} - e^{-\mu t}) \bar{r}(1 - \bar{r}). \quad (5.29)$$

Proof: Similarly as in the proof of Corollary 5.3 we have

$$\int_s^t |g(\Psi(r, t, s), \tau, s)| d\tau \leq C\varepsilon \int_s^t e^{-\mu \tau} \leq C\varepsilon (e^{-\mu s} - e^{-\mu t}).$$

Thus, by noticing that $\frac{dF^{-1}_*(\eta)}{d\eta} = \frac{1}{F'_*(F^{-1}_*(\eta))} = |u_*(F^{-1}_*(\eta))|$, we see that

$$|T(r, t, s) - r| = |F^{-1}_*(F_*(r) + \int_s^t g(\Psi(r, t, s), \tau, s) d\tau) - F^{-1}_*(F_*(r))|$$

$$\leq \int_0^1 |u_*(F^{-1}_*(F_*(r) + \zeta_\theta))| d\theta \cdot \int_s^t |g(\Psi(r, t, s), \tau, s)| d\tau$$

$$\leq C\varepsilon (e^{-\mu s} - e^{-\mu t}) \int_0^1 |u_*(F^{-1}_*(F_*(r) + \zeta_\theta))| d\theta.$$
where $\zeta_\theta = 0 \int_0^t g(\Psi(r, t, s), \tau, s) d\tau$. Since $|\zeta_\theta| \leq C$ and $|u_\ast(\eta)| \leq C\eta(1 - \eta)$, by Lemma 5.2 we have

$$|u_\ast(F^{-1}_\ast(F_\ast(r) + \zeta_\theta))| \leq CF^{-1}_\ast(F_\ast(r) + \zeta_\theta)|1 - F^{-1}_\ast(F_\ast(r) + \zeta_\theta)| \leq Cr(1 - r).$$

Substituting this estimate into the above inequality, we see that (5.28) follows. The proof of (5.29) is similar. □

**Corollary 5.6** We have the following inequalities:

$$|\Phi(\xi, t, s) - \Phi_\ast(\xi, t, s)| \leq C\varepsilon(e^{-\mu s} - e^{-\mu t})\Phi_\ast(\xi, t, s)[1 - \Phi_\ast(\xi, t, s)],$$

(5.30)

$$|\Psi(r, t, s) - \Psi_\ast(r, t, s)| \leq C\varepsilon(e^{-\mu s} - e^{-\mu t})\Psi_\ast(r, t, s)[1 - \Psi_\ast(r, t, s)].$$

(5.31)

**Proof:** Let $r = \Phi(\xi, t, s)$. Then $\xi = \Psi(r, t, s)$ and $\Phi_\ast(\xi, t, s) = \Phi_\ast(\Psi(r, t, s), t, s) = T(r, t, s)$. Thus by (5.28) we have

$$|\Phi_\ast(\xi, t, s)| = |r - T(r, t, s)| \leq C\varepsilon(e^{-\mu s} - e^{-\mu t})r(1 - r).$$

Substituting $r = \Phi(\xi, t, s)$ into the right-hand side of the last inequality and using (5.27), we see that (5.30) follows. The proof of (5.31) is similar. □

**Lemma 5.7** Assume that in addition to (5.2) there also holds

$$\sup_{0 \leq r \leq 1} \left| \frac{\partial w(r, t)}{\partial r} - u'_\ast(r) \right| \leq C\varepsilon e^{-\mu t}.$$  

(5.32)

Then we have the following estimates:

$$e^{-C\varepsilon(e^{-\mu s} - e^{-\mu t})} \leq \frac{\partial T(r, t, s)}{\partial r} \leq e^{C\varepsilon(e^{-\mu s} - e^{-\mu t})},$$

(5.33)

$$e^{-C\varepsilon(e^{-\mu s} - e^{-\mu t})} \leq \frac{\partial S(\bar{r}, t, s)}{\partial \bar{r}} \leq e^{C\varepsilon(e^{-\mu s} - e^{-\mu t})}.$$  

(5.34)

**Proof:** Recalling that $T(r, t, s) = \Phi_\ast(\Psi(r, t, s), t, s)$, we see that

$$\frac{\partial T(r, t, s)}{\partial r} = \frac{\partial \Phi_\ast(\Psi(r, t, s), t, s)}{\partial \xi}(\Psi(r, t, s), t, s) \frac{\partial \Psi(r, t, s)}{\partial r} = \frac{\partial \Phi_\ast(\Psi(r, t, s), t, s)}{\partial \xi}(\Psi(r, t, s), t, s) \left[ \frac{\partial \Phi}{\partial \xi}(\Psi(r, t, s), t, s) \right]^{-1}$$

$$= \exp \left( \int_s^t \left[ u'_\ast(\Phi_\ast(\xi, \tau, s)) - \frac{\partial w}{\partial r}(\Phi(\xi, \tau, s), \tau) \right] d\tau \right) \bigg|_{\xi = \Psi(r, t, s)}.  

(5.35)

We have

$$u'_\ast(\Phi_\ast(\xi, \tau, s)) - \frac{\partial w}{\partial r}(\Phi(\xi, \tau, s), \tau) = [u'_\ast(\Phi_\ast(\xi, \tau, s)) - u'_\ast(\Phi(\xi, \tau, s))]$$

$$+ [u'_\ast(\Phi(\xi, \tau, s)) - \frac{\partial w}{\partial r}(\Phi(\xi, \tau, s), \tau)].$$

By the assumption (5.32) we have

$$\sup_{\xi \in \mathbb{R}} |u'_\ast(\Phi(\xi, \tau, s)) - \frac{\partial w}{\partial r}(\Phi(\xi, \tau, s), \tau)| \leq \sup_{0 < r < 1} \left| u'_\ast(r) - \frac{\partial w}{\partial r}(r, \tau) \right| \leq C\varepsilon e^{-\mu \tau}.$$
Next, by the assertion (3) of Lemma 3.1 we know that there exists $0 \leq \gamma < 1$ such that $r^{\gamma} u''(r) \in C[0, 1]$. With this fact in mind, we use the mean value theorem to compute

$$\left| u'(\Phi_s(\xi, \tau, s)) - u'(\Phi(\xi, \tau, s)) \right|_{\xi = \Psi(r, t, s)} \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) (\Phi(\xi, \tau, s))^{-\gamma} \Phi_s(\xi, \tau, s) (1 - \Phi_s(\xi, \tau, s))_{\xi = \Psi(r, t, s)}$$

where $\Phi = \theta \Phi_s(\xi, \tau, s) + (1 - \theta) \Phi(\xi, \tau, s)$ for some $0 < \theta < 1$ (depending on $\xi, \tau, s$). Since there exists constant $0 < c < 1$ such that $\frac{\Phi_s(\xi, \tau, s)}{\Phi(\xi, \tau, s)} \geq c$ for $\varepsilon$ sufficiently small, we have $\zeta \geq c \Phi(\xi, \tau, s)$. Thus

$$\left| u'(\Phi_s(\xi, \tau, s)) - u'(\Phi(\xi, \tau, s)) \right|_{\xi = \Psi(r, t, s)} \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) (\Phi(\xi, \tau, s))^{-\gamma} \Phi_s(\xi, \tau, s) (1 - \Phi_s(\xi, \tau, s))_{\xi = \Psi(r, t, s)}$$

In getting the last equality we used the following relation: $\Phi(\Psi(r, t, s), \tau, s) = \Psi(r, t, \tau)$ for $0 \leq r \leq 1$, $s \leq \tau \leq t$. (5.36)

The proof of this relation is as follows: From (5.8) we know that $\rho = \Psi(r, t, \tau)$ is a solution of the following problem:

$$\frac{\partial \rho}{\partial t} + w(r, t) \frac{\partial \rho}{\partial r} = 0 \quad \text{for} \quad 0 \leq r \leq 1, \quad t > \tau, \quad \rho|_{t=\tau} = r.$$ 

But it is easy to verify that $\rho = \Phi(\Psi(r, t, s), \tau, s)$ is also a solution of this problem. Hence, by uniqueness we have (5.36). Hence, using (5.26) and (5.14) we get

$$\left| u'(\Phi_s(\xi, \tau, s)) - \frac{\partial w}{\partial r}(\Phi(\xi, \tau, s), \tau) \right|_{\xi = \Psi(r, t, s)} \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) \left[ \Psi(\xi, \tau, s) - \Psi(\xi, \tau, \tau) \right] + C e^{-\mu t}$$

$$= C \varepsilon (e^{-\mu s} - e^{-\mu t}) \left[ F_s^{-1}(F_s(r) + \tau - \tau) \right]^{1-\gamma} \left[ 1 - F_s^{-1}(F_s(r) + \tau - \tau) \right] + C e^{-\mu t}.$$
It follows that
\[
\int_s^t \left| u'_* (\Phi_*(\xi, \tau, s)) - \frac{\partial w}{\partial r} (\Phi_*(\xi, \tau, s), \tau) \right|_{\xi = \Psi(r, t, s)} \, d\tau.
\]
\[
\leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) \int_s^t \left[ F_*(F_*(r) + t - \tau) \right]^{\gamma - 1} [1 - F_*(F_*(r) + t - \tau)] \, d\tau + C \varepsilon \int_s^t e^{-\mu r} \, d\tau
\]
\[
= C \varepsilon (e^{-\mu s} - e^{-\mu t}) \int_{F_*(r) + t - s}^{F_*(r)} (F_*(\xi))^{\gamma - 1} [1 - F_*(\xi)] \, d\xi + C \varepsilon (e^{-\mu s} - e^{-\mu t})
\]
\[
\left( \xi = F_*(\eta), \quad d\xi = F'_*(\eta) d\eta = \frac{d\eta}{|u_*(\eta)|} \right)
\]
\[
= C \varepsilon (e^{-\mu s} - e^{-\mu t}) \int_r^{F_*(r) + t - s} \frac{\eta^{\gamma - 1} (1 - \eta)}{|u_*(\eta)|} \, d\eta + C \varepsilon (e^{-\mu s} - e^{-\mu t})
\]
\[
\leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) \int_0^1 \eta^{-\gamma} \, d\eta + C \varepsilon (e^{-\mu s} - e^{-\mu t})
\]
\[
= C \varepsilon (e^{-\mu s} - e^{-\mu t}).
\]
Combining this result with (5.35), we see that (5.33) follows. Finally, (5.34) is an immediate consequence of (5.33). \(\square\)

**Corollary 5.8** Under the assumption of Lemma 5.7, for \(\varepsilon\) sufficiently small we have
\[
|\frac{\partial T(r, t, s)}{\partial r} - 1| \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}), \quad |\frac{\partial S(\bar{r}, t, s)}{\partial r} - 1| \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}),
\]
and
\[
C^{-1} \leq \frac{\partial T(r, t, s)}{\partial r} \leq C, \quad C^{-1} \leq \frac{\partial S(\bar{r}, t, s)}{\partial r} \leq C.
\]
\(\square\)

**Lemma 5.9** Assume that \(a \in C_Y^1[0, 1]\). Then we have
\[
\|a(S(\cdot, t, s)) - a\|_1 \leq C\|a\|_1 \varepsilon (e^{-\mu s} - e^{-\mu t}), \quad (5.37)
\]
where \(\|a\|_1 = \max_{0 \leq r \leq 1} r(1-r)|a'(r)|\). If further \(r^2(1-r)^2 a''(r) \in C[0, 1]\) then we also have
\[
\|a(S(\cdot, t, s)) - a\|_{C_Y^1[0, 1]} \leq C\|a\|_2 \varepsilon (e^{-\mu s} - e^{-\mu t}), \quad (5.38)
\]
where \(\|a\|_2 = \|a\|_1 + \max_{0 \leq r \leq 1} r^2(1-r)^2 |a''(r)|\).

**Proof:** We have
\[
|a(S(r, t, s)) - a(r)| = |a'(\eta)||S(r, t, s) - r|
\]
\[
\leq C \varepsilon |(1-\eta)|a'(\eta)| \cdot \frac{r(1-r)}{|(1-\eta)|} (e^{-\mu s} - e^{-\mu t}) \leq C\|a\|_1 \varepsilon (e^{-\mu s} - e^{-\mu t}),
\]
where \(\eta = (1-\theta)r + \theta S(r, t, s)\) for some \(0 < \theta < 1\) (depending on \(r, t\) and \(s\)). In getting the last inequality we used the inequality
\[
\eta(1-\eta) \geq C r(1-r) \quad \text{for} \quad 0 \leq r \leq 1,
\]
which follows from (5.25) and the following identity:

\[ \eta(1-\eta) = (1-\theta)r(1-r) + \theta S(r, t, s)[1 - S(r, t, s)] + \theta(1-\theta)[r - S(r, t, s)]^2. \]

Hence (5.37) is proved. Next, we compute

\[
\begin{align*}
& r(1-r)\left| \frac{\partial a(S(r, t, s))}{\partial r} - a'(r) \right| = r(1-r)\left| a'(S(r, t, s))\frac{\partial S(r, t, s)}{\partial r} - a'(r) \right| \\
& \leq r(1-r)|a'(S(r, t, s))|\left| \frac{\partial S(r, t, s)}{\partial r} - 1 \right| + r(1-r)|a'(S(r, t, s)) - a'(r)| \\
& \leq C\|a\|_1 \cdot C\varepsilon (e^{-\mu s} - e^{-\mu t}) + r(1-r)|a''(\eta)||S(r, t, s) - r|,
\end{align*}
\]

where \( \eta = (1-\theta)r + \theta S(r, t, s) \) for some \( 0 < \theta < 1 \) (depending on \( r, t \) and \( s \)). Similarly as before we have

\[
\begin{align*}
& r(1-r)|a''(\eta)||S(r, t, s) - r| \leq r(1-r)|a''(\eta)| \cdot C\varepsilon (e^{-\mu s} - e^{-\mu t})r(1-r) \\
& = \frac{r^2(1-r)^2}{\eta^2(1-\eta)^2} \cdot \eta^2(1-\eta)^2|a''(\eta)| \cdot C\varepsilon (e^{-\mu s} - e^{-\mu t}) \\
& \leq C\left( \max_{0 \leq r \leq 1} r^2(1-r)^2|a''(r)| \right) \varepsilon (e^{-\mu s} - e^{-\mu t}).
\end{align*}
\]

Hence (5.38) is proved. This completes the proof of Lemma 5.9. \( \square \)

**Lemma 5.10** Given \( a \in C[0, 1] \), we define a bounded linear operator \( L \) in \( C[0, 1] \) by

\[
L(q)(r) = \frac{1}{r^3} \int_0^r a(\rho)q(\rho)d\rho \quad \text{for} \quad q \in C[0, 1], \quad 0 < r \leq 1,
\]

and \( L(q)(0) = \lim_{r \to 0^+} L(q)(r) = \frac{1}{3}a(0)q(0) \). Let \( \tilde{r} = T(r, t, s) \) and \( r = S(\tilde{r}, t, s) \) be as before, and let \( \tilde{L} \) be the following bounded linear operator in \( C[0, 1] \):

\[
\tilde{L}(q)(\tilde{r}) = \frac{1}{\tilde{r}^3} \int_0^{\tilde{r}} a(\rho)(T(\rho, t, s))d\rho \quad \text{for} \quad q \in C[0, 1], \quad 0 < \tilde{r} \leq 1,
\]

and \( \tilde{L}(q)(0) = \lim_{\tilde{r} \to 0^+} \tilde{L}(q)(\tilde{r}) = \frac{1}{3}a(0)q(0) \). Assume that \( a \in C^1_V[0, 1] \). Then both \( L \) and \( \tilde{L} \) are bounded linear operators from \( C[0, 1] \) to \( C^1_V[0, 1] \), and we have

\[
\|\tilde{L} - L\|_{L(C[0,1], C^1_V[0,1])} \leq C\|a\|_{C^1_V[0,1]}\varepsilon (e^{-\mu s} - e^{-\mu t}). \tag{5.39}
\]

**Proof:** We only give the proof of (5.39), because the proof of the assertion that both \( L \) and \( \tilde{L} \) are bounded linear operators from \( C[0, 1] \) to \( C^1_V[0, 1] \) follows by a similar argument.

We first note that for \( q \in C[0, 1] \) and \( 0 < \tilde{r} \leq 1 \), \( \tilde{L}(q)(\tilde{r}) \) can be re-written as follows:

\[
\tilde{L}(q)(\tilde{r}) = \frac{1}{[S(\tilde{r}, t, s)]^3} \int_0^{\tilde{r}} a(S(\rho, t, s))q(\rho)\left[\frac{S(\rho, t, s)}{\rho}\right]^2 \frac{\partial S(\rho, t, s)}{\partial \rho}\rho^2d\rho.
\]
Thus

\[
\tilde{L}(q)(\vec{r}) - L(q)(\vec{r}) = \left( \frac{\vec{r}}{S(\vec{r}, t, s)} \right)^3 \cdot \frac{1}{r^3} \int_0^r a(S(\rho, t, s))q(\rho) \left[ \frac{S(\rho, t, s)}{\rho} \right]^2 \left[ \frac{\partial S(\rho, t, s)}{\partial \rho} \right] - 1 \rho^2 d\rho
\]
\[
+ \left( \frac{\vec{r}}{S(\vec{r}, t, s)} \right)^3 \cdot \frac{1}{r^3} \int_0^r a(S(\rho, t, s))q(\rho) \left\{ \left[ \frac{S(\rho, t, s)}{\rho} \right]^2 - 1 \right\} \rho^2 d\rho
\]
\[
+ \left( \frac{\vec{r}}{S(\vec{r}, t, s)} \right)^3 - 1 \cdot \frac{1}{r^3} \int_0^r a(\rho)q(\rho)\rho^2 d\rho.
\]

From Corollary 5.3, Lemma 5.5, Corollary 5.8 and Lemma 5.9 we know that

\[
\left| \frac{\vec{r}}{S(\vec{r}, t, s)} \right| \leq C, \quad \left| \frac{\vec{r}}{S(\vec{r}, t, s)} - 1 \right| \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}),
\]
\[
\left| \frac{S(\rho, t, s)}{\rho} \right| \leq C, \quad \left| \frac{\partial S(\rho, t, s)}{\partial \rho} - 1 \right| \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}),
\]
\[
\left| a(S(\rho, t, s)) - a(\rho) \right| \leq C \|a\|_1 (e^{-\mu s} - e^{-\mu t}).
\]

Using the above estimates, we see easily that

\[
\max_{0 \leq r \leq 1} \left| \tilde{L}(q)(\vec{r}) - L(q)(\vec{r}) \right| \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) \|a\|_{C^1_b[0, 1]} \|q\|_\infty. \tag{5.40}
\]

Next, by a simple computation we have

\[
\vec{r}(1 - \vec{r})L(q)'(\vec{r}) = (1 - \vec{r})a(\vec{r})q(\vec{r}) - \frac{3(1 - \vec{r})}{r^3} \int_0^r a(\rho)q(\rho)\rho^2 d\rho,
\]
\[
\vec{r}(1 - \vec{r})\tilde{L}(q)'(\vec{r}) = \left( \frac{\vec{r}}{S(\vec{r}, t, s)} \right) a(S(\vec{r}, t, s))q(\vec{r}) \frac{\partial S(\vec{r}, t, s)}{\partial \vec{r}}
\]
\[
- \frac{3\vec{r}(1 - \vec{r})}{[S(\vec{r}, t, s)]^4} \frac{\partial S(\vec{r}, t, s)}{\partial \vec{r}} \int_0^r a(S(\rho, t, s))q(\rho) \left[ \frac{S(\rho, t, s)}{\rho} \right]^2 \frac{\partial S(\rho, t, s)}{\partial \rho} \rho^2 d\rho.
\]

Using these expressions and a similar argument as before we have

\[
\sup_{0 < \vec{r} < 1} \vec{r}(1 - \vec{r})\tilde{L}(q)'(\vec{r}) - L(q)'(\vec{r}) \leq C \varepsilon (e^{-\mu s} - e^{-\mu t}) \|a\|_{C^1_b[0, 1]} \|q\|_\infty. \tag{5.41}
\]

To save spaces, we omit the details here. By (5.40) and (5.41), we see that (5.39) follows. \[\square\]

What we shall use later on is not (5.39), but the following immediate consequences of it:

\[
\|\tilde{L} - L\|_{L(C[0, 1])} \leq C \|a\|_{C^1_b[0, 1]} \varepsilon (e^{-\mu s} - e^{-\mu t}), \tag{5.42}
\]
\[
\|\tilde{L} - L\|_{L(C^1_b[0, 1])} \leq C \|a\|_{C^1_b[0, 1]} \varepsilon (e^{-\mu s} - e^{-\mu t}). \tag{5.43}
\]
6 Decay estimates

In this section we establish a decay estimate for the evolution system \( \{ U(t, s, V) : t \geq s \geq 0 \} \) obtained in Section 4, where \( V = V(t) \in C([0, \infty), S_\varepsilon) \), under an additional assumption that \( V(t) \) is exponentially decaying as \( t \to \infty \).

We first consider the special case that \( V = 0 \). In this case we have \( U(t, s, V) = e^{(t-s)A(0)}. \) The main result Theorem 5.1 of [8] gives a decay estimate for \( e^{tA(0)} \) (see (6.9) below). But that estimate contains some singularity at \( r = 0 \), so that it does not meet our requirement. In what follows we shall establish an improved estimate. To this end we need a preliminary lemma which gives an estimate for the semigroup generated by the following operator \( \mathcal{L} = \mathcal{L}_0 + a \):

\[
\mathcal{L}q(r) = -w(r)q'(r) + a(r)q(r) \quad \text{for} \quad 0 \leq r \leq 1,
\]

where \( w \) and \( a \) are given functions.

**Lemma 6.1** Assume that \( w \in C^1[0, 1] \) and satisfies (4.1), and \( a \in C^1_V[0, 1] \). Then \( \mathcal{L} \) generates a \( C_0 \) semigroup \( e^{t\mathcal{L}} \) on \( C[0, 1] \) satisfying the following estimate:

\[
\|e^{t\mathcal{L}}\|_{L(C[0,1])} \leq e^{\omega t} \quad \text{for} \quad t \geq 0,
\]

where \( \omega_0 = \max_{0 \leq r \leq 1} a(r) \). Moreover, \( C^1_V[0, 1] \) is \( \mathcal{L} \)-admissible, and for any \( \omega > \omega_0 \) we have

\[
\|e^{t\mathcal{L}}\|_{L(C^1_V[0,1])} \leq C_\omega e^{\omega t} \quad \text{for} \quad t \geq 0.
\]

Here \( C_\omega \) is independent of \( w \) (but depends on the constants \( C_1, C_2 \) in (4.1) and the upper bound of \( \|a\|_{C^1_V[0,1]} \)).

**Proof:** By a similar argument as in the proof of Lemma 4.1 we see that for any \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > \omega_0 \) and any \( f \in C[0, 1] \), the equation

\[-w(r)q'(r) + a(r)q(r) - \lambda q(r) = f(r)\]

has a unique solution \( u \in C^1_V[0, 1] \) which is given by

\[q(r) = e^{\int_{r_0}^r \frac{a(\rho) - \lambda}{w(\rho)} d\rho} \int_r^1 \frac{f(\eta)}{w(\eta)} e^{-\int_0^\eta \frac{a(\rho) - \lambda}{w(\rho)} d\rho} d\eta,\]

where \( r_0 \) is an arbitrarily fixed number in \((0, 1)\). Using this expression and a similar argument as in the proof of Lemma 4.1 we have the following estimate:

\[
\max_{0 \leq r \leq 1} |q(r)| \leq \max_{0 \leq \eta \leq 1} \left| \frac{f(\eta)}{\text{Re}\lambda - a(\eta)} \right| \leq \frac{1}{\text{Re}\lambda - \omega_0} \max_{0 \leq r \leq 1} |f(r)| \quad \text{for} \quad \text{Re}\lambda > \omega_0.
\]

Hence, \( \mathcal{L} \) generates a strongly continuous semigroup \( e^{t\mathcal{L}} \) on \( C[0, 1] \) and the estimate (6.1) holds.

Next, by (4.1) we see that for any \( q \in C^1_V[0, 1] \) we have

\[
C_1' \|q\|_{C^1_V[0, 1]} + \max_{0 \leq r \leq 1} |w(r)q'(r)| \leq \|q\|_{C^1_V[0, 1]} \leq C_2' \|q\|_{C^1_V[0, 1]} + \max_{0 \leq r \leq 1} |w(r)q'(r)|,
\]

where \( C_1' \) and \( C_2' \) are positive constants independent of \( w \) (but depending on the constants \( C_1, C_2 \) appearing in (4.1)). Let \( q_0 \in C^1_V[0, 1] \) and let \( q = e^{t\mathcal{L}}q_0 \). Then \( q \in C([0, \infty), C^1_V[0, 1]) \cap C^1([0, \infty), C[0, 1]) \) and it is the solution of the following problem:

\[
\frac{\partial q}{\partial t} + w(r)\frac{\partial q}{\partial r} = a(r)q \quad \text{for} \quad t > 0, \quad q|_{t=0} = q_0.
\]
Let \( l(r, t) = w(r) \frac{\partial q(r, t)}{\partial r} \). Formally differentiating the above equation and multiplying it with \( w(r) \), we see that \( l \) is a **formal solution** of the following problem:

\[
\frac{\partial l}{\partial t} + w(r) \frac{\partial l}{\partial r} = a(r)l + f_1(r, t), \quad t > 0, \quad l|_{t=0} = l_0,
\]

where \( f_1(r, t) = w(r)a'(r)q(r, t) \) and \( l_0(r) = w(r)q_0'(r) \). Clearly, \( f_1 \in C^1([0, \infty), C[0, 1]) \) and \( l_0 \in C[0, 1] \), so that by the theory of \( C_0 \) semigroups (see, e. g. the discussion in Section 4.2 of [25]; particularly Definition 2.3 and Theorem 2.7 there) it follows that the above problem has a unique so-called **mild solution** \( l \in C([0, \infty), C[0, 1]) \) and, consequently, the above formal computation makes sense, or in other words, \( l(r, t) = w(r) \frac{\partial q(r, t)}{\partial r} \) is the mild solution of the above problem. This means that

\[
l(r, t) = e^{tA}l_0 + \int_0^t e^{(t-s)A}f_1(\cdot, s)ds \quad \text{for} \quad t \geq 0.
\]

By this expression and (6.1) we have

\[
\|l(r, t)\|_{C_0} \leq e^{\omega_0 t} \|l_0\|_{C_0} + \int_0^t e^{\omega_0 (t-s)} \|f_1(\cdot, s)\|_{C_0} ds
\]

\[
\leq e^{\omega_0 t} \|l_0\|_{C_0} + w^a \|a\|_{L^\infty} \int_0^t e^{\omega_0 (t-s)} e^{\omega_0 s} \|q_0\|_{C_0} ds
\]

\[
\leq Ce^{\omega_0 t} \|q_0\|_{C_0} + C\|a\|_{C^1} \|\phi_0\|_{C_0} + \eta e^{\omega_0 t} \|q_0\|_{C_0}.
\]

From this estimate and (6.1), (6.3) we immediately obtain (6.2). The proof is complete. \( \square \)

**Lemma 6.2** There exists a constant \( \mu^* > 0 \) such that for any \( 0 < \mu < \mu^* \), the semigroup \( e^{tA(0)} \) \((t \geq 0)\) generated by \( A(0) \) satisfies the following estimate:

\[
\|e^{tA(0)}\|_{L(C[0, 1])} \leq Ce^{-\mu t} \quad \text{for} \quad t \geq 0.
\]

**Proof.** Given \( U_0 = (\phi_0, \zeta_0) \in X \), let \( U(t) = e^{tA(0)}U_0 = (\phi(r, t), \zeta(t)) \). Then \((\phi, \zeta)\) is the unique solution of the following initial value problem:

\[
\frac{\partial \phi}{\partial t} + u_*(r)\frac{\partial \phi}{\partial t} = a(r)\phi + b(r)\zeta \quad \text{for} \quad 0 \leq r \leq 1, \quad t > 0,
\]

\[
\frac{d\zeta}{dt} = \mathcal{F}(\phi) + \kappa \zeta \quad \text{for} \quad t > 0,
\]

\[
\phi(0, r) = \phi_0(r) \quad \text{for} \quad 0 \leq r \leq 1, \quad \zeta(0) = \zeta_0.
\]

where \( a(r), b(r), \mathcal{B}(\phi), \mathcal{F}(\phi) \) and \( \kappa \) are given in (2.24)–(2.28). By Theorem 5.1 of [8] and the Remark in the end of Section 8 of [8] we know that there exists constant \( \sigma^* > 0 \) and a function \( \hat{\phi} \in C^1(0, 1] \) satisfying

\[
\hat{\phi}(r) > 0 \quad \text{for} \quad 0 < r \leq 1, \quad \hat{\phi}(r) \sim Cr^{-\theta} \quad \text{for} \quad r \to 0
\]

for some constants \( 1 \leq \theta < 3 \) and \( C > 0 \), such that the solution of the above problem satisfies the following estimate:

\[
|\zeta(t)| + \sup_{0<r \leq 1} \left| \frac{\phi(r, t)}{\hat{\phi}(r)} \right| \leq C \left( |\zeta_0| + \sup_{0<r \leq 1} \left| \frac{\phi_0(r)}{\hat{\phi}(r)} \right| \right) (1 + t)^{2e^{-\sigma^* t}} \quad \text{for} \quad t \geq 0.
\]
This particularly implies that for any $0 < \sigma < \sigma^*$ and $\delta \in (0, 1)$ we have

$$|\zeta(t)| + \sup_{\delta \leq r \leq 1} |\phi(r, t)| \leq C(|\zeta_0| + \sup_{0 \leq r \leq 1} |\phi_0(r)|) e^{-\sigma t} \quad \text{for } t \geq 0,$$

(6.10)

because $1/\hat{\phi}(r)$ has a positive lower bound for $\delta \leq r \leq 1$ and a finite upper bound for $0 \leq r \leq 1$.

In what follows we prove that for $\delta$ sufficiently small there also holds

$$\sup_{0 \leq r \leq \delta} |\phi(r, t)| \leq Ce^{-\mu t} \quad \text{for } t \geq 0$$

(6.11)

for some $\mu > 0$.

Take a nonnegative cut-off function $\varphi \in C[0, 1]$ such that

$$\varphi(r) \leq 1 \quad \text{for } 0 \leq r \leq 1, \quad \varphi(r) = 1 \quad \text{for } 0 \leq r \leq \delta, \quad \varphi(r) = 0 \quad \text{for } 2\delta \leq r \leq 1.$$

We split $B$ into a sum of two operators as follows:

$$B(q) = B_1(q) + B_2(q) \quad \text{for } q \in C[0, 1],$$

(6.12)

where

$$B_1(q) = -rp_0'(r)\varphi(r) \cdot \frac{1}{r^\alpha} \int_0^{\min\{r, \delta\}} g_\rho(q(\rho)\rho^2 d\rho,$$

$$B_2(q) = rp_0'(r) \int_0^1 g_\rho(q(\rho)\rho^2 d\rho - r^{-2}p_0'(r)[1 - \varphi(r)] \int_0^{\min\{r, \delta\}} g_\rho(q(\rho)\rho^2 d\rho$$

$$- r^{-2}p_0'(r) \int_{\min\{r, \delta\}}^r g_\rho(q(\rho)\rho^2 d\rho,$$

and introduce

$$f(r, t) = B_2(\varphi(\cdot, t))(r) + b(r)\zeta(t).$$

By (6.5) and the splitting (6.12), we see that $\phi$ is the solution of the equation

$$\partial_\tau \phi + u_*(r)\partial_r \phi = a(r)\phi + B_1(\phi) + f(r, t) \quad \text{for } 0 \leq r \leq 1, \quad t > 0$$

(6.13)

subject to the initial condition $\phi(r, 0) = \phi_0(r)$. Introducing operators $L(q) = -u_*q' + aq$ and $F(t) = f(\cdot, t)$, we see that (6.13) can be rewritten as the following differential equation in $C[0, 1]$

$$\frac{dq}{dt} = (L + B_1)(q) + F(t).$$

(6.14)

Using (6.8) and (6.9) we can easily show that

$$\|B_2(\varphi(\cdot, t))\|_\infty \leq C_\delta \|U_0\|(1 + t)^2 e^{-\sigma^* t} \quad \text{for } t \geq 0.$$

This result combined with (6.10) yields

$$\|F(t)\|_\infty \leq C_\delta \|U_0\| e^{-\sigma^* t} \quad \text{for } t \geq 0.$$

(6.15)

Using the fact that $\lim_{r \to 0} rp_0'(r) = 0$ and $\varphi(r) = 0$ for $r \geq 2\delta$, one can easily deduce that for any given $\varepsilon > 0$ there exists corresponding $\delta > 0$ such that

$$\|B_1(q)\|_\infty \leq \varepsilon \|q\|_\infty.$$

(6.16)
Furthermore, from Lemma 3.1 we know that \( w(r) = u_*(r) \) and \( a(r) \) satisfies the assumptions in Lemma 6.1, so that, by Lemma 6.1, the operator \( \mathcal{L} \) generates a strongly continuous semigroup \( e^{t\mathcal{L}} \) on \( C[0,1] \), and
\[
\|e^{t\mathcal{L}}\| \leq e^{-\omega t} \quad \text{for all } t \geq 0, \tag{6.17}
\]
where \( \omega = \min_{0 \leq r \leq 1} |a(r)| > 0 \). Since \( \mathcal{B}_1 \) is a bounded linear operator on \( C[0,1] \) and, by (6.16), \( \|\mathcal{B}_1\|_{L(C[0,1])} \leq \varepsilon \), it follows that the operator \( \mathcal{L} + \mathcal{B}_1 \) also generates a strongly continuous semigroup \( e^{t(\mathcal{L}+\mathcal{B}_1)} \) on \( C[0,1] \), and, furthermore, there holds
\[
\|e^{t(\mathcal{L}+\mathcal{B}_1)}\| \leq e^{-(\omega - \varepsilon)t} \quad \text{for all } t \geq 0, \tag{6.18}
\]
In what follows we assume that \( \varepsilon \) is sufficiently small such that \( \omega - \varepsilon > 0 \). By (6.14) we have
\[
q(t) = e^{t(\mathcal{L}+\mathcal{B}_1)}q(0) + \int_0^t e^{(t-\tau)(\mathcal{L}+\mathcal{B}_1)}\mathcal{F}(\tau) d\tau.
\]
From this relation and (6.15) and (6.18) we see that for any \( 0 < \mu < \min\{\sigma, \omega - \varepsilon\} \) there holds
\[
\|q(t)\|_\infty \leq \|q(0)\|_\infty e^{-(\omega - \varepsilon)t} + C\|U_0\| e^{-\mu t} \quad \text{for } t \geq 0.
\]
Since \( q(t) = \phi(\cdot, t) \) is a solution of (6.14) with initial data \( q(0) = \phi_0 \), by this estimate we see that (6.11) follows.

By (6.10) and (6.11), we see that (6.1) is proved. This completes the proof. \( \square \)

**Lemma 6.3** Let \( \mu^* \) be as in Lemma 6.2. Then for any \( 0 < \mu < \mu^* \), in addition to (6.1) we also have the following estimate:
\[
\|e^{tK(0)}\|_{L(C^1_v[0,1])} \leq C e^{-\mu t} \quad \text{for } t \geq 0. \tag{6.19}
\]

**Proof:** We first show that \( \mathcal{B} \) and \( \mathcal{F} \) satisfy the following properties: For any \( q \in C^1_v[0,1] \),
\[
\|\mathcal{B}(u_*q')\|_\infty + \|u_*\mathcal{B}(q')\|_\infty \leq C\|q\|_\infty, \tag{6.20}
\]
\[
|\mathcal{F}(u_*q')| \leq C\|q\|_\infty. \tag{6.21}
\]

Using the facts that \( \lim_{r \to 0} r p'_*(r) = 0 \) and \( \lim_{r \to 0} r^2 p''_*(r) = 0 \) (see (3.3)) we can easily prove that
\[
\|u_*\mathcal{B}(q')\|_\infty \leq C\|q\|_\infty \quad \text{for } q \in C[0,1].
\]
To estimate \( \|\mathcal{B}(u_*q')\|_\infty \) we compute:
\[
\frac{1}{r^3} \int_0^r g_p(\rho)u_*(\rho)q'(\rho) \rho^2 d\rho = \frac{1}{r} u_*(r) g_p(r) q(r) - \frac{1}{r^3} \int_0^r m(\rho) q(\rho) \rho^2 d\rho.
\]
where \( m(\rho) = g'_p(\rho) u_*(\rho) + g_p(\rho) u'_*(\rho) + \frac{2}{\rho} u_*(\rho) g_p(\rho) \). Taking \( r = 1 \) we particularly obtain
\[
\int_0^1 g_p(\rho)u_*(\rho)q'(\rho) \rho^2 d\rho = u_*(1) g_p(1) q(1) - \int_0^1 m(\rho) q(\rho) \rho^2 d\rho.
\]
Since \( g_p \in C^1[0,1], u_s \in C^1[0,1] \) and \( u_s(0) = 0 \), we see that \( \frac{1}{r}u_sg_p \) and \( m \) both belong to \( C[0,1] \). Hence, from the above expressions we see immediately that

\[
\|B(u_s q')\|_\infty = \sup_{0<r<1} \left| rp'(r) \left[ \int_0^1 g_p(\rho) u_s(\rho) q'(\rho) \rho^2 d\rho - \frac{1}{r^\alpha} \int_0^r g_p(\rho) u_s(\rho) q'(\rho) \rho^2 d\rho \right] \right| \leq C\|q\|_\infty.
\]

Similarly we also have

\[
|\mathcal{F}(u_s q')| = \left| \int_0^1 g_p(\rho) u_s(\rho) q'(\rho) \rho^2 d\rho \right| \leq C\|q\|_\infty.
\]

This verifies (6.20) and (6.21).

We now proceed to prove (6.19). Let \( U_0 \in X_0 \) and \( U = e^{tA(0)}U_0 \). From the proof of Lemma 4.2 we know that \( U \in C([0,\infty), X_0) \cap C^1([0,\infty), X) \). Let \( U_0 = (q_0, y_0) \) and \( U = (q, y) \). Then \( (q, y) \) is the solution of the following problem:

\[
\begin{aligned}
\frac{\partial q}{\partial t} + u_s(r)\frac{\partial q}{\partial r} &= a(r) q + B(q) + b(r) y \quad \text{for} \quad 0 \leq r \leq 1, \quad t > 0, \\
\frac{\partial y}{\partial t} &= \mathcal{F}(q) + \kappa y \quad \text{for} \quad t > 0, \\
q|_{t=0} &= q_0(r) \quad \text{for} \quad 0 \leq r \leq 1, \quad \text{and} \quad y|_{t=0} = y_0.
\end{aligned}
\]

Let \( l(r, t) = u_s(r)\frac{\partial q(r,t)}{\partial r} \). As in the proof of Lemma 6.1, by formally differentiating the first equation above in \( r \) and multiplying it with \( u_s(r) \), we see that \( (l, y) \) is a “formal solution” of the following problem:

\[
\begin{aligned}
\frac{\partial l}{\partial t} + u_s(r)\frac{\partial l}{\partial r} &= a(r) l + B(l) + b(r) y + f_1(r, t) \quad \text{for} \quad 0 \leq r \leq 1, \quad t > 0, \\
\frac{\partial y}{\partial t} &= \mathcal{F}(l) + \kappa y + c_1(t) \quad \text{for} \quad t > 0, \\
l|_{t=0} &= l_0(r) \quad \text{for} \quad 0 \leq r \leq 1, \quad \text{and} \quad y|_{t=0} = y_0,
\end{aligned}
\]

where \( l_0(r) = u_s(r)q_0'(r) \), \( c_1(t) = \mathcal{F}(q) - \mathcal{F}(u_s\frac{\partial q}{\partial r}) \), and

\[
f_1(r, t) = u_s(r)a'(r)q(r, t) - B\left(u_s\frac{\partial q}{\partial r}\right) + u_s(r)\frac{\partial B(q)}{\partial r} + [u_s(r)b'(r) - b(r)]y(t).
\]

We denote \( W(t) = (l(\cdot, t), y(t)) \), \( W_0 = (l_0, y_0) \) and \( F_1(t) = (f_1(\cdot, t), c_1(t)) \). Then the above problem can be rewritten as follows:

\[
\frac{dW}{dt} = \mathcal{A}(0)W + F_1(t) \quad \text{for} \quad t > 0, \quad W(0) = W_0.
\]

Using the fact that \( U \in C^1([0,\infty), X) \), Corollary 3.2 and (6.20), (6.21), we can easily prove that \( F_1 \in C^1([0,\infty), X) \). Thus, by a similar argument as in the proof of Lemma 6.1 we see that the above formal computation makes sense and \( W = (l, y) = (u_s\frac{\partial q}{\partial r}, y) \) is the unique mild solution of the above problem, which means that

\[
W(t) = e^{t\mathcal{A}(0)}W_0 + \int_0^t e^{(t-s)\mathcal{A}(0)}F_1(s)ds \quad \text{for} \quad t \geq 0.
\]
It follows by Lemma 6.2 that for any given $0 < \mu < \mu^*$ we have
\[
\|W(t)\|_X \leq Ce^{-\mu t}\|W_0\|_X + C \int_0^t e^{-\mu(t-s)}\|F_1(s)\|_X ds \quad \text{for } t \geq 0.
\]
Using (6.20), (6.21) and the fact that $\|W(t)\|_X \leq Ce^{-\mu t}\|U_0\|_X$ ensured by (6.1), we see that
\[
\|F_1(t)\|_X \leq C\|U(t)\|_X \leq Ce^{-\mu t}\|U_0\|_X \quad \text{for } t \geq 0.
\]
Hence, by a similar argument as in the proof of Lemma 6.1 we obtain
\[
\|U(t)\|_{X_0} \leq C(1+t)e^{-\mu t}\|U_0\|_{X_0} \quad \text{for } t \geq 0.
\]
Now, for any given $0 < \mu < \mu^*$ we arbitrarily take a $\tilde{\mu} \in (\mu, \mu^*)$ and first use the above estimate to $\tilde{\mu}$ and next use the elementary inequality $(1+t)e^{-\tilde{\mu}t} \leq Ce^{-\tilde{\mu}t}$, we see that (6.19) follows. This completes the proof. 

Lemma 6.4 Assume that $V = V(t) \in C([0, \infty), X)$ and it satisfies (6.22). Let $\mu^*$ be as in Lemma 6.2. Then for any $0 < \mu < \mu^*$ there exists corresponding $\varepsilon_0 > 0$ (depending on $\mu$, $\tilde{\mu}$ and $C_0$) such that if $0 < \varepsilon \leq \varepsilon_0$ then the following estimates hold:
\[
\|U(t, s, V)\|_{L(X)} \leq C_1 e^{-\mu t} \quad \text{for } t \geq 0,
\]
\[
\|U(t, s, V)\|_{L(X_0)} \leq C_2 e^{-\mu t} \quad \text{for } t \geq 0,
\]
where $C_1$ and $C_2$ are positive constants depending only on $\mu$ and independent of $\tilde{\mu}$ and $C_0$.

Proof: Given $0 < \mu < \mu^*$ we take a $\mu_1 \in (\mu, \mu^*)$ and fix it. By Lemmas 6.2 and 6.3, we have the following estimates:
\[
\|e^{tA(0)}\|_{L(C[0,1])} \leq C_1 e^{-\mu_1 t} \quad \text{for } t \geq 0,
\]
\[
\|e^{tA(0)}\|_{L(C_0[0,1])} \leq C_2 e^{-\mu_1 t} \quad \text{for } t \geq 0.
\]
Let $U_0 = (q_0, s_0)$ be an arbitrary point in $X$, and let $U = U(t, s, V)U_0$. By definition, $U$ is the solution of the problem (4.13). Let $U = (q, y)$. Then (4.13) can be rewritten as follows:
\[
\begin{cases}
\frac{\partial q}{\partial t} + wV(r, t)\frac{\partial q}{\partial r} = a(r)q + Bq + b(r)y & \text{for } 0 \leq r \leq 1, \ t > s, \\
\frac{\partial y}{\partial t} = F(q) + \kappa y & \text{for } t > s, \\
q|_{t=s} = q_0(r) & \text{for } 0 \leq r \leq 1, \ \text{and} \ \ y|_{t=s} = y_0,
\end{cases}
\]
Let $\bar{q}(\bar{r}, t, s) = q(S(\bar{r}, t, s), t)$ or $q(r, t) = \bar{q}(T(r, t, s), t, s)$. Then by using (5.9) we see that (6.27) is transformed into the following problem:
\[
\begin{cases}
\frac{\partial \bar{q}}{\partial t} + u_*(\bar{r})\frac{\partial \bar{q}}{\partial \bar{r}} = \bar{a}(\bar{r}, t, s)\bar{q} + \bar{B}\bar{q} + \bar{b}(\bar{r}, t, s)s & \text{for } 0 \leq \bar{r} \leq 1, \ t > s, \\
\frac{ds}{dt} = \bar{F}(\bar{q})(t, s) + \kappa s & \text{for } t > s, \\
\bar{q}|_{t=s} = q_0(\bar{r}) & \text{for } 0 \leq \bar{r} \leq 1, \ \text{and} \ s|_{t=s} = s_0,
\end{cases}
\]
where \( \bar{a}(\bar{r}, t, s) = a(S(\bar{r}, t, s)), \bar{b}(\bar{r}, t, s) = b(S(\bar{r}, t, s)), \)

\[
\bar{B} \bar{q} = rp_r' \left[ \int_0^1 g_p(\rho) \bar{q}(T(\rho, t, s), t, s) \rho^2 d\rho - \frac{1}{r^3} \int_0^r g_p(\rho) \bar{q}(T(\rho, t, s), t, s) \rho^2 d\rho \right] \bigg|_{r=S(\bar{r}, t, s)},
\]

and \( \bar{F}(\bar{q})(t, s) = \int_0^1 g_p(\rho) \bar{q}(T(\rho, t, s), t, s) \rho^2 d\rho. \) We define a family of bounded linear operators \( \bar{B}(t, s, V) : X \rightarrow X (t \geq s \geq 0) \) as follows:

\[
\bar{B}(t, s, V) = \begin{pmatrix}
\bar{a}(\cdot, t, s) + \bar{B} \bar{b}(\cdot, t, s) \\
\bar{F} - \bar{F}
\end{pmatrix}.
\]

We also denote \( \bar{U} = (\bar{q}, y). \) Then (6.28) can be rewritten as follows:

\[
\begin{cases}
\frac{d\bar{U}}{dt} = \mathbb{A}_0(U_0) \bar{U} + \bar{B}(t, s, V) \bar{U} \quad \text{for } t > s, \\
\bar{U}|_{t=s} = U_0.
\end{cases}
\]  

(6.29)

Recalling that \( \mathbb{A}(0) = \mathbb{A}_0(U_0) + \mathbb{B} \) and denoting

\[
\bar{E}(t, s, V) = \bar{B}(t, s, V) - \mathbb{B} = \begin{pmatrix}
\bar{a}(\cdot, t, s) - a + \bar{B} - \mathbb{B} \bar{b}(\cdot, t, s) - b \\
\bar{F} - \mathbb{F}
\end{pmatrix},
\]

we see that

\[
\mathbb{A}_0(U_0) + \bar{B}(t, s, V) = \mathbb{A}_0(U_0) + \mathbb{B} + \bar{E}(t, s, V) = \mathbb{A}(0) + \bar{E}(t, s, V).
\]

Hence, (6.29) can be further rewritten as follows:

\[
\begin{cases}
\frac{d\bar{U}}{dt} = \mathbb{A}(0) \bar{U} + \bar{E}(t, s, V) \bar{U} \quad \text{for } t > s, \\
\bar{U}|_{t=s} = U_0.
\end{cases}
\]  

(6.30)

We know that (6.30) is equivalent to the following integral equation:

\[
\bar{U}(t, s) = e^{(t-s)\mathbb{A}(0)} U_0 + \int_s^t e^{(t-\tau)\mathbb{A}(0)} \bar{E}(\tau, s, V) \bar{U}(\tau, s) \quad \text{for } t \geq s.
\]  

(6.31)

By Corollary 3.2 and Lemma 5.9 we have

\[
\|\bar{a}(\cdot, t, s) - a\|_\infty \leq \|\bar{a}(\cdot, t, s) - a\|_{C^1(\mathbb{R})} \leq C\varepsilon,
\]

\[
\|\bar{b}(\cdot, t, s) - b\|_\infty \leq \|\bar{b}(\cdot, t, s) - b\|_{C^1(\mathbb{R})} \leq C\varepsilon,
\]

and by Corollary 3.2, Lemma 5.9 and Lemma 5.10 we have

\[
\|\bar{B} - \mathbb{B}\|_{C^1(\mathbb{R})} \leq C\varepsilon, \quad \|\bar{B} - \mathbb{B}\|_{L(\mathbb{R})} \leq C\varepsilon,
\]

\[
\|\bar{F} - \mathbb{F}\|_{L(\mathbb{R})} \leq C\varepsilon.
\]
It follows that
\[ \| \mathbb{E}(\tau, s, V) \|_{L(X)} \leq C \varepsilon, \quad \| \mathbb{E}(\tau, s, V) \|_{L(X_0)} \leq C \varepsilon. \] (6.32)

From (6.25), (6.26), (6.31) and (6.32) we obtain:
\[ \| \tilde{U}(t, s) \|_X \leq C_1 e^{-\mu_1(t-s)} \| U_0 \|_X + C \varepsilon \int_s^t e^{-\mu_1(t-\tau)} \| \tilde{U}(\tau, s) \|_X, \]
\[ \| \tilde{U}(t, s) \|_{X_0} \leq C_2 e^{-\mu_1(t-s)} \| U_0 \|_{X_0} + C \varepsilon \int_s^t e^{-\mu_1(t-\tau)} \| \tilde{U}(\tau, s) \|_{X_0}. \]

By Gronwall lemma, these inequalities yield
\[ \| \tilde{U}(t, s) \|_X \leq C_1 e^{-(\mu_1-C\varepsilon)t} \| U_0 \|_X, \]
\[ \| \tilde{U}(t, s) \|_{X_0} \leq C_2 e^{-(\mu_1-C\varepsilon)t} \| U_0 \|_{X_0}. \]

Hence, by taking \( \varepsilon \) sufficiently small such that \( \mu_1 - C\varepsilon \geq \mu \), we obtain (6.23) and (6.24). This completes the proof. \( \square \)

7 The proof of Theorem 1.1

In order to prove Theorem 1.1, we let \( \mu^* \) be as in Lemma 6.2 and arbitrarily fix a number \( 0 < \mu < \mu^* \). Let \( \varepsilon \) be a positive number to be specified later. For any fixed \( U_0 \in X_0 \) satisfying \( \| U_0 \|_{X_0} \leq \varepsilon \), we denote by \( M \) the set of all functions \( V = V(t) \in C([0, \infty), X) \) satisfying the following conditions:
\[ V(0) = U_0, \quad \| V(t) \|_X \leq 2C_1 \varepsilon e^{-\mu t} \text{ for } t \geq 0, \] (7.1)
where \( C_1 \) is the constant appearing in (6.23). We introduce a metric \( d \) on \( M \) by defining
\[ d(V_1, V_2) = \sup_{t \geq 0} e^{\mu t} \| V_1(t) - V_2(t) \|_X \text{ for } V_1, V_2 \in M. \]

It is evident that \( (M, d) \) is a complete metric space. Given \( V \in M \), we consider the following initial value problem:
\[ \begin{cases} \frac{dU(t)}{dt} = A(V(t))U(t) + G(U(t)) \text{ for } t > 0, \\ U(0) = U_0. \end{cases} \] (7.2)

**Lemma 7.1** If \( \varepsilon \) is sufficiently small then for any \( V \in M \) the problem (7.2) has a unique solution \( U \in C([0, \infty), X_0) \cap C^1([0, \infty), X) \) which satisfies the following estimates:
\[ \| U(t) \|_X \leq 2C_1 \varepsilon e^{-\mu t}, \quad \| U(t) \|_{X_0} \leq C\varepsilon e^{-\mu t}, \quad \| U'(t) \|_X \leq C\varepsilon e^{-\mu t} \text{ for } t \geq 0, \] (7.3)
where \( C_1 \) is as before, and \( C \) is another constant independent of \( V \).

**Proof.** We denote
\[ \mathcal{M} = \{ U \in C([0, \infty), X_0) : \| U(t) \|_X \leq 2C_1 \varepsilon e^{-\mu t} \text{ and } \| U(t) \|_{X_0} \leq 2C_2 \varepsilon e^{-\mu t} \text{ for } t \geq 0 \}, \]
and introduce a metric \( d \) on it by defining
\[
d(U_1, U_2) = \sup_{t \geq 0} e^{\mu t} \|U_1(t) - U_2(t)\|_{X_0} \quad \text{for } U_1, U_2 \in \widetilde{M}.
\]
Here \( C_1 \) and \( C_2 \) are positive constants appearing in (6.23) and (6.24), respectively. \((\widetilde{M}, d)\) is clearly a complete metric space. Given \( U \in \widetilde{M} \), we consider the following initial value problem:
\[
\begin{aligned}
\frac{d\tilde{U}(t)}{dt} &= A(V(t))\tilde{U}(t) + G(U(t)) \quad \text{for } t > 0, \\
\tilde{U}(0) &= U_0.
\end{aligned}
\] (7.4)

Since \( U(t) \in C([0, \infty), X_0) \), by Corollary 3.3 we have \( G(U(t)) \in C([0, \infty), X_0) \). It follows by Corollary 4.4 that the above problem has a unique solution \( \tilde{U} \in C([0, \infty), X_0) \cap C^1([0, \infty), X) \), and is given by
\[
\tilde{U}(t) = \mathbb{U}(t, 0, V)U_0 + \int_0^t \mathbb{U}(t, s, V)G(U(s))ds.
\] (7.5)

Using this expression and Lemma 6.4 and (2.23) we have
\[
\|\tilde{U}(t)\|_X \leq C_1 e^{-\mu t}\|U_0\|_X + C_1 \int_0^t e^{-\mu(t-s)}\|G(U(s))\|_X ds
\leq C_1 \varepsilon e^{-\mu t} + C_2 \int_0^t e^{-\mu(t-s)}\|U(s)\|_X^2 ds
\leq C_1 \varepsilon e^{-\mu t} + C_2 e^2 \int_0^t e^{-\mu(t-s)}e^{-2\mu s} ds
\leq C_1 \varepsilon e^{-\mu t} + C_2 e^{2} e^{-\mu t} \leq 2C_1 \varepsilon e^{-\mu t}.
\]
The last inequality holds when \( \varepsilon \) is sufficiently small. Similarly, by using Lemma 6.4 and (3.9) we also have
\[
\|\tilde{U}(t)\|_{X_0} \leq 2C_2 \varepsilon e^{-\mu t},
\]
when \( \varepsilon \) is sufficiently small. Hence \( \tilde{U} \in \widetilde{M} \). We now define a mapping \( \tilde{S} : \widetilde{M} \rightarrow \widetilde{M} \) by setting \( \tilde{S}(U) = \tilde{U} \) for every \( U \in \widetilde{M} \). We claim that \( \tilde{S} \) is a contraction mapping. Indeed, for any \( U_1, U_2 \in \widetilde{M} \) let \( \tilde{U}_1 = S(U_1), \tilde{U}_2 = \tilde{S}(U_2) \) and \( W = \tilde{U}_1 - \tilde{U}_2 \). Then \( W \) satisfies
\[
\begin{aligned}
\frac{dW(t)}{dt} &= A(V(t))W(t) + [G(U_1(t)) - G(U_2(t))] \quad \text{for } t > 0, \\
W(0) &= 0,
\end{aligned}
\]
so that
\[
W(t) = \int_0^t \mathbb{U}(t, s, V)[G(U_1(s)) - G(U_2(s))] ds.
\]
It follows by a similar argument as before that
\[
\begin{aligned}
\|W(t)\|_{X_0} &\leq C_2 \int_0^t e^{-\mu(t-s)}\|G(U_1(s) - U_2(s))\|_{X_0} ds \\
&\leq C_2 \int_0^t e^{-\mu(t-s)}\|U_1(s) - U_2(s)\|_{X_0} ds \left( \int_0^t \|G'(\theta U_1(s) + (1 - \theta)U_2(s))\|_{L(X_0)} d\theta \right) \\
&\leq C \int_0^t e^{-\mu(t-s)}\|U_1(s) - U_2(s)\|_{X_0} ds \left( \int_0^1 \|\theta U_1(s) + (1 - \theta)U_2(s)\|_{X_0} d\theta \right) \\
&\leq C \int_0^t e^{-\mu(t-s)} \cdot d(U_1, U_2)e^{-\mu s} \cdot 2C_2 \varepsilon e^{-\mu t} ds \leq C \varepsilon d(U_1, U_2)e^{-\mu t}.
\end{aligned}
\]
Thus for \( \varepsilon \) sufficiently small we have
\[
d(\tilde{U}_1, \tilde{U}_2) = \sup_{t \geq 0} e^{\varepsilon t} \| W(t) \|_{X_0} \leq \frac{1}{2} d(U_1, U_2),
\]
showing that \( \tilde{S} \) is a contraction mapping, as we claimed. Thus, by the Banach fixed point theorem we see that \( \tilde{S} \) has a unique fixed point in \( \tilde{M} \), which is clearly a solution of the problem (7.2) in \( C([0, \infty), X_0) \). Uniqueness of the solution follows from a standard argument.

From the above argument we see that the solution \( U \) of (7.2) satisfies the first two inequalities in (7.3), and \( U \in C^1([0, \infty), X) \). It remains to prove that \( U \) also satisfies the last inequality in (7.3). The argument is as follows. First, it is straightforward to deduce from the condition (7.1) that for sufficiently small \( \varepsilon > 0 \), we have \( w_V(r, t) \leq C[0, 1] \) and there exist positive constants \( C_1 \) and \( C_2 \) independent of \( V \) such that
\[
-C_1 r(1 - r) \leq w_V(r, t) \leq -C_2 r(1 - r) \quad \text{for } 0 \leq r \leq 1 \text{ and } t \geq 0. \tag{7.6}
\]
It follows that for the solution \( U = (q, s) \) of (7.2) we have
\[
\sup_{0 \leq r \leq 1} |w_V(r, t) \frac{\partial q(r, t)}{\partial r}| \leq C \sup_{0 \leq r \leq 1} |r(1 - r) \frac{\partial q(r, t)}{\partial r}| \leq C \|q(\cdot, t)\|_{C^1_1[0,1]}.
\]
Using this result and the equation (7.2) we see that
\[
\|U'(t)\|_X \leq \|A(V(t))U(t)\|_X + \|G(U(t))\|_X \leq C\|U(t)\|_{X_0} + C\|U(t)\|_X^2 \leq C\varepsilon e^{-\mu t}
\]
for all \( t \geq 0 \). This completes the proof of Lemma 7.1. \( \square \)

Lemma 7.1 particularly implies that for every \( V \) in \( M \), the solution \( U \) of (7.2) also belongs to \( M \). Thus we can define a mapping \( S : M \to M \) as follows: For any \( V \in M \),
\[
S(V) = U = \text{the solution of (7.2)}.
\]

**Lemma 7.2** For \( \varepsilon \) sufficiently small, \( S \) is a contraction mapping.

**Proof.** Let \( V_1, V_2 \in M \) and denote \( U_1 = S(V_1) \), \( U_2 = S(V_2) \) and \( W = U_1 - U_2 \). Then \( W \) satisfies:
\[
\begin{cases}
\frac{dW(t)}{dt} = A(V_1(t))W(t) + [A(V_1(t)) - A(V_2(t))]U_2(t) + [G(U_1(t)) - G(U_2(t))], & \text{for } t > 0, \\
W(0) = 0.
\end{cases}
\]

Thus
\[
W(t) = \int_0^t \left( U(t, s, V_1)A(V_1(s)) - A(V_2(s))U_2(s) + G(U_1(s)) - G(U_2(s)) \right) ds. \tag{7.7}
\]
Since the first component of \( [A(V_1(s)) - A(V_2(s))]U_2(s) \) is equal to \( [w_{V_2}(r, s) - w_{V_1}(r, s)]q_2^*(r, s) \) and the second component is zero, we have
\[
\|A(V_1(s)) - A(V_2(s))U_2(s)\|_X = \max_{0 \leq r \leq 1} \|w_{V_1}(r, s) - w_{V_2}(r, s)\| q_2^*(r, s) \leq C\|V_1(s) - V_2(s)\|_X \|U_2(s)\|_{X_0}.
\]
Besides, from (2.23) we have

\[ \|G(U_1(s)) - G(U_2(s))\|_X = \left\| \int_0^1 G'(\theta U_1(s) + (1-\theta)U_2(s))[U_1(s) - U_2(s)]d\theta \right\|_X \]

\[ \leq C(\|U_1(s)\|_X + \|U_2(s)\|_X)\|U_1(s) - U_2(s)\|_X. \]

Thus, by (7.7) and Lemma 6.4 we have

\[ \|U_1(t) - U_2(t)\|_X \leq C \int_0^t e^{-\mu(t-s)}\||\mathcal{A}(V_1(s)) - \mathcal{A}(V_2(s))\|U_2(s)\|_X ds \]

\[ + C \int_0^t e^{-\mu(t-s)}\||G(U_1(s)) - G(U_2(s))\|_X ds \]

\[ \leq C \int_0^t e^{-\mu(t-s)}\|V_1(s) - V_2(s)\|_X \|U_2(s)\|_{X_0} ds \]

\[ + C \int_0^t e^{-\mu(t-s)}(\|U_1(s)\|_X + \|U_2(s)\|_X)\|U_1(s) - U_2(s)\|_X ds \]

\[ \leq C \sup_{s \geq 0} e^{\mu s}\|V_1(s) - V_2(s)\|_X \cdot \sup_{s \geq 0} e^{\mu s}\|U_2(s)\|_{X_0} \int_0^t e^{-\mu(t-s)} \cdot e^{-2\mu s} ds \]

\[ + C \sup_{s \geq 0} e^{\mu s}(\|U_1(s)\|_X + \|U_2(s)\|_X) \]

\[ \cdot \sup_{s \geq 0} e^{\mu s}\|U_1(s) - U_2(s)\|_X \int_0^t e^{-\mu(t-s)} \cdot e^{-2\mu s} ds \]

\[ \leq C e^{-\mu t}d(V_1, V_2) + C e^{-\mu t}d(U_1, U_2). \]

Therefore,

\[ d(U_1, U_2) = \sup_{t \geq 0} e^{\mu t}\|U_1(t) - U_2(t)\|_X \leq C \varepsilon d(V_1, V_2) + C \varepsilon d(U_1, U_2), \]

by which the desired assertion immediately follows. This completes the proof of Lemma 7.2. □

By Lemma 7.2, if \( \varepsilon \) is sufficiently small then the mapping \( \mathbf{S} \) has a unique fixed point \( U \) in \( \mathcal{M} \). Clearly, \( U \) is a global solution of the equation (2.22) subject to the initial condition \( U(0) = U_0 \). Moreover, by Lemma 7.1 we know that the image of \( \mathbf{S} \) is contained in \( \tilde{\mathcal{M}} \), so that \( U \) satisfies (7.3). From this result all assertions of Theorem 1.1 easily follows. The proof of Theorem 1.1 is complete.

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