Properties and applications of some special integer number sequences

Cristina Flaut, Diana Savin and Geanina Zaharia

Abstract. In this paper, we provide properties and applications of some special integer sequences. We generalize and give some properties of Pisano period. Moreover, we provide a new application in Cryptography and applications of some quaternion elements.

Key Words: difference equations; Fibonacci numbers; Lucas numbers; quaternion algebras.

2000 AMS Subject Classification: 11B39, 11R54, 17A35.

There have been numerous papers devoted to the study of the properties and applications of particular integer sequences, the most studied of them being Fibonacci sequence ([FP; 09], [Ha; 12], [Ho; 63], [St; 06], [St; 07], etc.). In this paper, we study properties and provide some applications of these number sequences in a more general case.

Let \( a_1, \ldots, a_k \) be arbitrary integers, \( a_k \neq 0 \). We consider the general \( k \)--terms recurrence, \( n, k \in \mathbb{N}, k \geq 2, n \geq k \),

\[
d_n = a_1 d_{n-1} + a_2 d_{n-2} + \ldots + a_k d_{n-k},
\]

where \( a_k \neq 0 \) are given integers and its associated matrix \( D_k \in \mathcal{M}_k(\mathbb{R}) \),

\[
D_k = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_k \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & 1 & 0 \end{pmatrix},
\]

see [Jo].

For \( n \in \mathbb{Z}, n \geq 1 \), we have that
Supposing relation true for \( n \) that can be written under the form

\[
\begin{vmatrix}
\sum_{i=1}^{k-2} a_{i+1} d_{n+k-i-1} & \sum_{i=1}^{k-3} a_{i+2} d_{n+k-i-1} & \cdots & a_k d_{n+k-2} \\
\sum_{i=1}^{k-2} a_{i+1} d_{n+k-i-2} & \sum_{i=1}^{k-3} a_{i+2} d_{n+k-i-2} & \cdots & a_k d_{n+k-3} \\
\sum_{i=1}^{k-2} a_{i+1} d_{n+k-i-3} & \sum_{i=1}^{k-3} a_{i+2} d_{n+k-i-3} & \cdots & a_k d_{n+k-4} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{k-2} a_{i+1} d_{n+k-i-k} & \sum_{i=1}^{k-3} a_{i+2} d_{n+k-i-k} & \cdots & a_k d_{n+k-i} \\
\end{vmatrix},
\]

(1.3)

see [Fl; 19], Proposition 2.1.

In [Me; 99], the author proved some new properties of Fibonacci and Pell numbers. In the following, we will generalize some of these results.

If we consider the matrix \( Y_i = (d_{i+k-1}, \ldots, d_{i+1}, d_i) \), for \( i = 0 \), relation (1.1) can be written under the form

\[ Y_1^t = D_k Y_0^t. \]  

(1.4)

**Proposition 1.1.** With the above notations, we have the following relations:

\[ Y_n^t = D_k Y_{n-1}^t \]  

(1.5)

\[ Y_n^t = D_k^n Y_0^t. \]  

(1.6)

\[ Y_{n+r}^t = D_k^n Y_r^t. \]  

(1.7)

**Proof.**

Relation (1.5) is obviously.

To prove relation (1.6), we remark that for \( n = 1 \), we have relation (1.4). Supposing relation true for \( n - 1 \), we prove it for \( n \). From relation (1.5), we have that \( Y_n^t = D_k Y_{n-1}^t = D_k^n Y_0^t \).

To prove relation (1.7), we have \( Y_{n+r}^t = D_k^n Y_r^t = D_k^n D_k^r Y_0^t = D_k^n Y_r^t \). □

**Proposition 1.2.** With the above notations, for difference equation (1.1), the following relation are true:

\[ d_{n+k-1} d_{n+k} d_{n+k+1} \ldots d_{n+2k-4} d_{n+2k-3} 1 \\
\begin{array}{l}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\begin{array}{l}
d_{n+2} d_{n+3} d_{n+4} \ldots d_{n+k-2} d_{n+k-1} 0 \\
d_{n+1} d_{n+2} d_{n+3} \ldots d_{n+k-3} d_{n+k-2} 0 \\
d_n d_{n+1} d_{n+2} \ldots d_{n+k-4} d_{n+k-3} 0 \\
\end{array} \\
\begin{array}{c}
= (-1)^{n(k+1)} a_k^n d_{-n}.
\end{array} \]
ii)

\[
\begin{vmatrix}
  d_{n+k-1} & d_{n+k} & d_{n+k+1} & \cdots & d_{n+2k-4} & d_{n+2k-3} \\
  d_{n+k-2} & d_{n+k-1} & d_{n+k} & \cdots & d_{n+2k-5} & d_{n+2k-4} \\
  d_{n+k-3} & d_{n+k-2} & d_{n+k-1} & \cdots & d_{n+2k-6} & d_{n+2k-5} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{n+2} & d_{n+3} & d_{n+4} & \cdots & d_{n+k-1} & d_{n+k} \\
  d_{n+1} & d_{n+2} & d_{n+3} & \cdots & d_{n+k-2} & d_{n+k-1}
\end{vmatrix} = (-1)^{n(k+1)} a_k^n.
\]

iii)

\[d_{m+k-1}d_{n+k-1} + d_{m+k-2} \sum_{i=1}^{k-1} a_{i+1}d_{n+k-i-1} + d_{m+k-3} \sum_{i=1}^{k-2} a_{i+2}d_{n+k-i-1} + \ldots + a_k d_{n+k-1} = d_{m+n+k-1}.\]

**Proof.** 1) Indeed, since \( \det D_k = (-1)^{k+1} a_k \), we obtain that

\[
\begin{vmatrix}
  d_{n+k-1} & d_{n+k} & d_{n+k+1} & \cdots & d_{n+2k-4} & d_{n+2k-3} \\
  d_{n+k-2} & d_{n+k-1} & d_{n+k} & \cdots & d_{n+2k-5} & d_{n+2k-4} \\
  d_{n+k-3} & d_{n+k-2} & d_{n+k-1} & \cdots & d_{n+2k-6} & d_{n+2k-5} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{n+2} & d_{n+3} & d_{n+4} & \cdots & d_{n+k-1} & d_{n+k} \\
  d_{n+1} & d_{n+2} & d_{n+3} & \cdots & d_{n+k-2} & d_{n+k-1}
\end{vmatrix} = \det (D_k^n) = (-1)^{n(k+1)} a_k^n d_{-n}.
\]

2) Indeed, using the ideas developed above, obtain
For Fibonacci sequence are interesting its properties when it is reduced modulo $m$. For example, it is well known that this sequence is periodic and this period is called Pisano’s period, denoted by $\pi(m)$. For the Fibonacci sequence, $\pi(m)$ is even, for $m > 2$ (see [Re; 13]). In the following, we generalize this notion for the sequence generated by the $k$–terms recurrence given in the relation (1.1).

First of all, we will prove that the sequence $(d_n)$, given in (1.1), is periodic.
Proposition 2.1. The sequence generated by \((d_n)\), considered mod \(m\), is periodic.

Proof. The terms of the sequence \((d_n)\) talked mod \(m\) can have only the values \(0, 1, 2, \ldots, m-1\). Since \(d_n = a_1 d_{n-1} + a_2 d_{n-2} + \cdots + a_k d_{n-k}\), we remark that if the sequence \(d_{n-1}, d_{n-2}, \ldots, d_{n-k}\) is repeated mod \(m\) from a step, then the entire sequence is repeated mod \(m\). Therefore, there are at most \(m^k\) possible choices for the sequence \(d_{n-1}, d_{n-2}, \ldots, d_{n-k}\) and the sequence is periodic. \(\square\)

Remark 2.2. The above proposition generalized Theorem 1 from [Wa; 60] to sequence \((d_n)\).

Definition 2.3. The Pisano period for the sequence \((d_n)\) is the period with which the sequence \((d_n)\) taken modulo \(m\) repeats. We denote this number with \(\pi(m)\).

Remark 2.4. If we consider the matrix \(D_k\), associated to the difference equation (1.1), we have that \(D_k^{\pi(m)} \equiv I_k \mod m\), where \(I_k\) is the unity matrix of order \(k\). From here, we get

\[
((-1)^{(k+1)} a_k)^{\pi(m)} \equiv 1 \mod m.
\]

It results that \(\text{ord} \left((-1)^{(k+1)} a_k\right) \mid \pi(m)\).

Theorem 2.5. With the above notations, the following statements are true:
i) If \(s_1 \mid s_2\), then \(\pi(s_1) \mid \pi(s_2)\).
ii) \(\pi([s_1, s_2]) = [\pi(s_1), \pi(s_2)]\), where \(s_1, s_2\) are positive integers and \([s_1, s_2] = \text{lcm}(s_1, s_2)\), the least common multiple.
iii) \(\pi(p^{r+1}) = \pi(p^r)\) or \(p\pi(p^r)\), with \(p\) a prime number and \(r\) an integer, \(r \geq 1\).
iv) If \(\pi(p^r) \neq \pi(p^{r+1})\), then \(\pi(p^{r+1}) \neq \pi(p^{r+2})\), with \(p\) a prime number and \(r\) an integer, \(r \geq 1\).
v) If \(D_k\) is a diagonalizable matrix and \(p\) is an odd prime, with \(p \nmid a_k\), then \(\pi(p) \mid p - 1\).
vi) If for the numbers sequence \(d_n = a_1 d_{n-1} + a_2 d_{n-2} + \cdots + a_k d_{n-k}\), \(d_0 = d_1 = \ldots = d_{k-2} = 0, d_{k-1} = 1\), all \(a_i\) are odd, for \(i \in \{1, 2, \ldots, k\}\), then \(\pi(2) = k + 1\).

Proof. i) Since \(s_1 \mid s_2\), we have \(D_k^{\pi(s_2)} \equiv I_k \mod s_1\), therefore \(\pi(s_1) \mid \pi(s_2)\).
ii) Let \(s = [s_1, s_2]\). We have \(D_k^{\pi(s)} \equiv I_k \mod s_1\) and \(D_k^{\pi(s)} \equiv I_k \mod s_2\). It results \(\pi(s_1) \mid \pi(s), \pi(s_2) \mid \pi(s)\) and from here, we have \(\pi(s_1) \mid \pi(s_2)\). For the converse, we know that \(D_k^{\pi(s_1)} \equiv I_k \mod s_1\) and \(D_k^{\pi(s_2)} \equiv I_k \mod s_2\), therefore \(D_k^{\lceil \pi(s_1), \pi(s_2)\rceil} \equiv I_k \mod s_1\) and \(D_k^{\lceil \pi(s_1), \pi(s_2)\rceil} \equiv I_k \mod s_2\). From here, we obtain \(D_k^{\lceil \pi(s_1), \pi(s_2)\rceil} \equiv I_k \mod s\), thus \(\pi(s) \mid \pi(s_1), \pi(s_2)\).

iii) Supposing that \(D_k^{\pi(p^r)} \equiv I_k \mod p^r\), we have \(D_k^{\pi(p^r)} = I_k + p^r A\), with \(A\) a matrix of order \(k\). Therefore, \(D_k^{\pi(p^r)} = (I_k + p^r A)^p\) in which, except first
term \( I_k \), all term are divisible with \( p^{r+1} \). It results that \( D_k^{p,\pi(p^r)} \equiv I_k \mod p^{r+1} \). From here, we obtain that \( \pi (p^{r+1}) = p \mod \pi(p^r)\), thus, from 1, we get \( \pi (p^{r+1}) = \pi(p^r) \) or \( \pi(p^{r+1}) = p \pi(p^r) \).

iv) If \( \pi(p^r) \neq \pi(p^{r+1}) \), then, from 3, we have that \( \pi(p^{r+1}) = p \pi(p^r) \) and \( \pi(p^{r+1}) = \pi(p^{r+2}) = \pi(p\pi(p^r))\). Since \( D_k^{p,\pi(p^r)} = (I_k + p\pi A)^p \), it results that \( D_k^{p,\pi(p^r)} \) is not congruent with \( I_k \mod p^{r+2} \), therefore \( \pi(p^{r+1}) \neq \pi(p^{r+2}) \).

v) Supposing that \( D_k \) is diagonalizable, there are the matrix \( A \) and \( B \) of order \( k \) such that \( B = A^{-1} D_k A \), with \( A \) an invertible matrix and

\[
B = \begin{pmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
0 & 0 & \lambda_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & \lambda_k
\end{pmatrix}.
\]

Applying the Fermat’s little Theorem, we have \( B^{p-1} \equiv I_k \mod p \), therefore \( D_k^{p-1} \equiv I_k \mod p \) thus \( \pi(p) \mid p-1 \).

vi) Let \( d_n = a_1 d_{n-1} + a_2 d_{n-2} + \ldots + a_k d_{n-k} \), \( d_0 = d_1 = \ldots = d_{k-2} = 0, d_{k-1} = 1 \). If we consider this sequence modulo 2, we obtain \( d_n = d_{n-1} + d_{n-2} + \ldots + d_{n-k}, d_0 = d_1 = \ldots = d_{k-2} = 0, d_{k-1} = 1 \). Therefore, we get the following sequence modulo 2:

- first \( k+1 \) terms \( d_0, d_1, \ldots, d_{k-2}, d_{k-1}, d_k \) are: \( 0, 0, \ldots, 0, 1, 1 \);
- the next \( k+1 \) terms \( d_{k+1}, d_{k+2}, \ldots, d_{2k-1}, d_{2k}, d_{2k+1} \) are: \( 0, 0, \ldots, 0, 1, 1 \). From here, it is clear that \( \pi(2) = k+1 \).

**Remark 2.6.** The above theorem generalized for difference equation (1.1) some results obtained in [Re: 13], Theorem 1, Proposition 1, Proposition 2 and Theorem 3.

**Example 2.7.**

i) Let \( k = 3 \) and \( d_n = a_1 d_{n-1} + a_2 d_{n-2} + a_3 d_{n-3} \), \( d_0 = d_1 = d_2 = 1 \). Supposing that \( a_1, a_2, a_3 \) are odd, we consider this difference equation modulo 2. It results \( d_n = d_{n-1} + d_{n-2} + d_{n-3}, d_0 = d_1 = d_2 = 1 \) and we obtain the sequence:

\[
0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, etc.
\]

Therefore, \( \pi(2) = k+1 = 4 \).

ii) For the case when some coefficients \( a_i \) are even, it is difficult to compute \( \pi(2) \) since it depends of the number and position of even coefficients. We give an example for \( k = 3, a_1, a_3 \) odd and \( a_2 \) even. We have the equation \( d_n = d_{n-1} + d_{n-3}, d_0 = d_1 = d_2 = 1 \) and we obtain the sequence:

\[
0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, etc.
\]

Therefore, \( \pi(2) = 7 \).

iii) Let \( a_1 = 4, a_2 = -5, a_3 = 2, p = 3 \). Therefore, we get the matrix

\[
D_3 = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[\text{6} \]
and the associated difference equation $d_n = 4d_{n-1} - 5d_{n-2} + 2d_{n-3}, d_0 = d_1 = 0, d_2 = 1$. For this difference equation of degree three, we will compute $\pi(3)$ modulo 3. It results $d_0 = d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 2, d_5 = 2, d_6 = 0, d_7 = 0, d_8 = 1, etc$. Therefore $\pi(3) = 6$.

Now, we work in $\mathbb{Z}_9$ and we will compute $\pi(3^2) = \pi(9)$. We obtain $d_0 = d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 2, d_5 = 5, d_6 = 0, d_7 = 0, d_8 = 1, etc$. Therefore $\pi(9) = 6$.

From Theorem 2.5, iii), it results that $\pi(27) = \pi(3)$ or $\pi(27) = 3\pi(3) = 18$. Working in $\mathbb{Z}_{27}$, we get $d_0 = d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 2, d_5 = 5, d_6 = 9, etc$. Therefore $\pi(27) = 18$.

iv) We work on $\mathbb{Z}_5$, therefore $p = 5$. Let $a_1 = 6, a_2 = -11, a_3 = 6$. Therefore, we get the matrix

$$D_3 = \begin{pmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and the associated difference equation $d_n = 6d_{n-1} - 11d_{n-2} + 6d_{n-3}, d_0 = d_1 = 0, d_2 = 1$. For this difference equation of degree three, we will compute $\pi(5)$ modulo 5. We have that the eigenvalues of the matrix $D_3$ are $\{1, 2, 3\}$; therefore $D_3$ has the diagonal form

$$\text{Diag} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$  

It results that $\pi(5) | 4$. We compute the terms of the sequence $(d_n)_{n \in \mathbb{N}}$ and we obtain $d_0 = d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 0, d_5 = 0, d_6 = 1$, therefore, in this situation, $\pi(5) = 4$.

3. An application in Cryptography

In the following, we will give an application of difference equation in Cryptography. In [Ko; 94] was presented enciphering matrices. Since these matrices must be invertible, we will use as enciphering matrices the matrices of the form $D_n^k$, since the determinant of such a matrix is known, namely $\det D_k = (-1)^{k+1} a_k$ (see [Jo]).

Encryption

1) Let $\mathcal{A}$ be an alphabet with $N$ letters, labelled from 0 to $N - 1$, let $m$ be a plain text and let $q$ be a number obtained by using the labels of the letters from the plain text $m$. We chose $k$ a nonzero natural number, we split the text $m$ in blocks of length $k$, $m_1, m_2, ..., m_r$. To these blocks correspond the vectors $v_1, v_2, ..., v_r$ of dimension $k$.

2) We chose arbitrary integers, $a_1, ..., a_k \in \{0, 1, 2, ..., N - 1\}, a_k \neq 0$ and the general $k$-terms recurrence equation given by the relation (1.1), $n, k \in \mathbb{N}, k \geq 2, n \geq k$.  

7
3) We chose the matrix $D_k$ associated to the sequence $a_1, ..., a_k$ and given by the relation (1.2).
4) We chose $n \in \mathbb{N} - \{0\}$ the power of the matrix $D_k$.
5) We use as an enciphering matrix the matrix $D_k^n$, therefore the encryption key will be $(k, N, a_1, ..., a_k, n)$.
6) We work on $\mathbb{Z}_N$, therefore, the encrypted text is given by the following formula
   \[ C = D_k^n V, \]  
   \[ (3.1.) \]
where $V = (v_1, v_2, ..., v_r) \in \mathcal{M}_{k \times r}(\mathbb{Z}_N)$ is a matrix with columns $v_1, v_2, ..., v_r$ and $C \in \mathcal{M}_{k \times r}(\mathbb{Z}_N)$ is a matrix with columns $(c_1, c_2, ..., c_r)$.

   We remark that it is very important to know the Pisano period $\pi(N)$. In this case, the key can have the form
   \[ (k, N, a_1, ..., a_k, n) = (k, N, a_1, ..., a_k, l\pi(N) + n), l \in \mathbb{Z} \]
and the encryption and decryption algorithms are the same since
   \[ D_k^n = D_k^{l\pi(N)+n}, l \in \mathbb{Z}. \]

In this way, the key can be made a bit hard. Therefore, to make the algorithm faster, we can choose $n \in \{1, 2, ..., \pi(N) - 1\}$.

**Decryption**

- The decryption key is the same as the encryption key and will be $(k, N, a_1, ..., a_k, n)$.
- We obtain the matrices $D_k, D_k^n$ and $D_k^{-n}$.
- The decrypted text is
  \[ V = D_k^{-n}C, \]  
  \[ (3.2.) \]
with $V = (v_1, v_2, ..., v_r) \in \mathcal{M}_{k \times r}(\mathbb{Z}_N)$.

**Example 3.1.**

We work in $\mathbb{Z}_2$ and we consider the encryption key $(3, 2, 1, 1, 1, 3)$. Therefore, we get the matrix
   \[ D_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

Since
   \[ D_3^n = \begin{pmatrix} d_n + 1 & c_d_n + c_d_n \\ c_d_n + c_d_{n-1} & c_d_{n-1} \\ c_d_{n-1} + c_d_{n-2} & c_d_{n-2} \end{pmatrix}, \]
   \[ D_3^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]
We consider the alphabet \( \mathcal{A} = \{A, B\} \), labeled \( \{0, 1\} \) and the plain text \( m = ABBAAB \). The number obtained by using the labels of the letters from the plain text \( m \) is \( q = 011001 \). The obtained vectors are \( v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \), \( v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \). Therefore, the encrypted text is 

\[
C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.
\]

Thus, \( c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \), \( c_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) and the obtained encrypted text is \( BBAABB \).

We remark that the value \( n \) can be bigger than 3, for example 31, which can help us to have a little bit hard key. But, since \( 31 \mod \pi(2) = 31 \mod 4 = 3 \), this can help us in the encrypting and decrypting process. Therefore, the value \( n \) can be taken from the set \( \{1, 2, \ldots, \pi(N) - 1\} \).

To decrypt the text, we use the same key \( (3, 27, 4, -5, 2, 2) \) and we obtain

\[
V = D_3^{-3}C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
\]

where \( V = (v_1, v_2, \ldots, v_r) \in \mathcal{M}_{k \times r}(\mathbb{Z}_n) \) is a matrix with columns \( v_1, v_2, \ldots, v_r \), that means we get the initial message \( ABBAAB \).

**Example 3.2.**

We use Example 2.7, iii). For \( a_1 = 4, a_2 = -5, a_3 = 2, p = 3 \), we consider the matrix

\[
D_3 = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

and the associated difference equation \( d_\eta = 4d_{\eta-1} - 5d_{\eta-2} + 2d_{\eta-3}, d_0 = d_1 = 0, d_2 = 1 \).

We consider the alphabet \( \mathcal{A} = \{A, B, C, \ldots, Z, *\} \), labeled \( \{0, 1, 2, \ldots, 26\} \), where "*" is the blank character and the plain text \( m = SUCCESS*\). Therefore, we work in \( \mathbb{Z}_{27} \) and, from Example 2.7, iii), we know that \( \pi(27) = 18 \). We consider the encryption key \( (3, 27, 4, -5, 2, 2) \). The number obtained by using the labels of the letters from the plain text \( m \) is \( q = 182002020418182626 \). The obtained vectors are \( v_1 = \begin{pmatrix} 18 \\ 20 \\ 02 \end{pmatrix} \), \( v_2 = \begin{pmatrix} 04 \\ 18 \\ 26 \end{pmatrix} \). We have
Therefore, the encrypted text is

\[
C = \begin{pmatrix} 2 & 24 & 2 \\ 1 & 22 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 18 & 2 & 18 \\ 20 & 4 & 26 \\ 2 & 18 & 26 \end{pmatrix} =
\end{pmatrix}
\]

\[
= \begin{pmatrix} 520 & 136 & 712 \\ 462 & 126 & 642 \\ 18 & 2 & 18 \end{pmatrix} \mod 27 = \begin{pmatrix} 7 & 1 & 10 \\ 3 & 18 & 21 \\ 18 & 2 & 18 \end{pmatrix}.
\]

Therefore, the encrypted text is \textit{HDSBSCKVS}. We remark that the analysis of the frequency of the letters in the text can't be applied here, since the same letters are encrypted in different characters and different characters can be encrypted in the same letter, as we can see in this example. In this way, this new method provides multiple ways of finding the encryption and decryption keys, thus the obtained encrypted texts are hard to break.

The decryption key is the same as the encryption key, namely \((3, 27, 4, -5, 2, 2)\).

We obtain

\[
V = D_3^{-2}C = \begin{pmatrix} 2 & 24 & 2 \\ 1 & 22 & 2 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 7 & 1 & 10 \\ 3 & 18 & 21 \\ 18 & 2 & 18 \end{pmatrix} =
\end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 & 1 \\ 14 & 13 & 13 \\ 8 & 6 & 5 \end{pmatrix} \begin{pmatrix} 7 & 1 & 10 \\ 3 & 18 & 21 \\ 18 & 2 & 18 \end{pmatrix} =
\end{pmatrix}
\]

\[
= \begin{pmatrix} 18 & 2 & 18 \\ 371 & 274 & 647 \\ 164 & 126 & 296 \end{pmatrix} = \begin{pmatrix} 18 & 2 & 18 \\ 20 & 4 & 26 \\ 2 & 18 & 26 \end{pmatrix}.
\]

We remark that the same result can be obtained in the case when the key is under the form \((3, 27, 4, -5, 2, l\pi(27) + 2)\), where \(l \in \mathbb{Z}\).

\textbf{Remark 3.3.} The cryptosystem described in this section is a symmetric cryptosystem. A problem of such a system is the key transmission.

A solution is to use hybrid cryptosystems (using in the same time symmetric and asymmetric cryptosystems). In this situation, data can be protected using a symmetric key, and the symmetric key can be encrypted and distributed by using a public key.

If are used only a symmetric cryptosystem, the key must be encrypted by using a key-encrypting keys (KEKs). In this situation, the key-encrypting keys must be distributed by using an arboreal algorithm. Usually, such an algorithm must have some steps. These keys (KEKs) are used to encrypt other keys. The
most difficult part is to initialize this process. For initialize this process, the first key is interchanged between users and this key is used to encrypt another key, called its successor and so on. The security of the first exchange step must be high, otherwise all successors keys are compromised. For the initialization phase of the process, can be used some procedures of the keys distribution as for example: transmission of the keys by using certain channels or "face-to-face identification", (see [EPC; 18]).

4. Applications of some special number sequences and quaternion elements

Let \( \mathbb{H}(\alpha, \beta) \) be the generalized quaternion algebra over an arbitrary field \( K \), i.e. the algebra of the elements of the form \( a = a_1 \cdot 1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \), where \( a_i \in K, i \in \{1, 2, 3, 4\} \), and the elements of the basis \( \{1, e_2, e_3, e_4\} \) satisfying the following rules, given in the below multiplication table:

| \cdot | 1 | e_2 | e_3 | e_4 |
|-------|---|-----|-----|-----|
| 1     | 1 | e_2 | e_3 | e_4 |
| e_2   | e_2 | \alpha | e_4 | \alpha e_3 |
| e_3   | -e_4 | \beta | -\beta e_2 |
| e_4   | -\alpha e_3 | \beta e_2 | -\alpha \beta |

Let \( \overline{a} = a_1 \cdot 1 - a_2 e_2 - a_3 e_3 - a_4 e_4 \) be the conjugate of the quaternion \( a \). The norm of \( a \) is \( n(a) = a \cdot \overline{a} = a_1^2 - \alpha a_2^2 - \beta a_3^2 + \alpha \beta a_4^2 \) and the trace of the element \( a \) is \( t(a) = a + \overline{a} \).

If, for \( x \in \mathbb{H}(\alpha, \beta) \), the relation \( n(x) = 0 \) implies \( x = 0 \), then the algebra \( \mathbb{H}(\alpha, \beta) \) is called a division algebra, otherwise the quaternion algebra is called a split algebra.

If \( p \) is a prime number, it is known that the quaternion algebra \( \mathbb{H}_{Z_p}(-1,-1) \) splits. In the paper [Sa; 17], the second author determined the Fibonacci quaternions which are zero divisors in the quaternion algebra \( \mathbb{H}_{Z_p}(-1,-1) \), respectively the Fibonacci quaternions which are invertible in the quaternion algebra \( \mathbb{H}_{Z_p}(-1,-1) \).

In the paper [Gr, Mi, Ma; 15], Grau, Miguel and Oller-Marcen have studied the quaternion algebra \( \mathbb{H}(-1,-1) \) over a finite commutative unitary ring \( (\mathbb{Z}_n, +, \cdot) \), where \( n \) is a positive integer, \( n \geq 3 \).

In the following, when \( l \) is an odd prime number, we study \( l \)-quaternions in quaternion algebra \( \mathbb{H}_{Z_l}(-1,-1) \) and also we study \( l \)-quaternions in the quaternion ring \( \mathbb{H}_{Z_{lr}}(-1,-1) \), where \( r \) is a positive integer, \( r \geq 2 \).
Let \( l \) be a nonzero positive integer. In [Sa; 19], the second author considered the sequence \((a_n)_{n \geq 0}\):

\[ a_n = la_{n-1} + a_{n-2}, \quad n \geq 2, \quad a_0 = 0, \quad a_1 = 1 \]

We call these numbers \((l, 1, 0, 1)\) numbers or \(l-\) numbers. We remark that for \( l = 1 \), it is obtained the Fibonacci numbers and for \( l = 2 \), it is obtained the Pell numbers. In the following, we present some properties of these numbers.

**Remark 4.1.** ([Sa; 19]). Let \((a_n)_{n \geq 0}\) be the sequence previously defined. Then, the following relations are true:

i) \[ a_{2n}^2 + a_{2n+1}^2 = a_{2n+1}, \quad \text{for all } n \in \mathbb{N}. \]

ii) For \( \alpha = \frac{l + \sqrt{l^2 + 4}}{2} \) and \( \beta = \frac{l - \sqrt{l^2 + 4}}{2} \), we obtain that

\[ a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{l^2 + 4}}, \quad \text{for all } n \in \mathbb{N}, \]

called the Binet’s formula for the sequence \((a_n)_{n \geq 0}\).

**Proposition 4.2.** ([Fl, Sa; 19]) Let \((a_n)_{n \geq 0}\) be the sequence previously defined. The following relations hold:

i) If \( d \mid n \), then \( a_d \mid a_n \).

ii) \[ a_{m+n} = a_m a_{n+1} + a_{m-1} a_n. \]

**Proposition 4.3.** ([Fl, Sa; 19]) Let \((a_n)_{n \geq 0}\) be the sequence previously defined. Then, the following relations are true:

i) \[ a_n + a_{n+4} = (l^2 + 2) a_{n+2}; \]

ii) \[ a_n + a_{n+8} = \left( (l^2 + 2)^2 - 2 \right) a_{n+4}; \]

iii) \[ a_n + a_{n+2k} = M_k a_{n+2k-1}, \quad k \geq 3, \]

where \( M_k = \left\lfloor \left( (l^2 + 2)^2 - 2 \right)^{2^{k-3}} - 2 \right\rfloor \). The sequence \((M_k)_{k \geq 2}\) satisfies

\[ M_{k+1} = M_k^2 - 2, \quad \text{for all } k \in \mathbb{N}, \quad k \geq 2, \quad M_2 = l^2 + 2. \]

Using these results, in the following, we give new properties of \(l-\) numbers.

We begin with some examples: \( a_0 = 0 \equiv 0 \pmod{l^2} \), \( a_1 = 1 \equiv 1 \pmod{l^2} \), \( a_2 = l \equiv l \pmod{l^2} \), \( a_3 = l^2 + 1 \equiv 1 \pmod{l^2} \), \( a_4 = l (l^2 + 2) \equiv 2l \pmod{l^2} \), \( a_5 = l^2 (l^2 + 3) + 1 \equiv 1 \pmod{l^2} \).
Proposition 4.4. Let \( l \) be a nonzero positive integer and let \((a_n)_{n\geq 0}\) be the sequence of \( l\)–numbers. Then, the following relations hold:

i) \( l \mid a_n \) if and only if \( n \) is an even number;

ii) \( a_n \equiv 1 \pmod{l^2} \) if and only if \( n \) is an odd number.

Proof. The proof is a straightforward calculation, using Proposition 4.2 (i) and a mathematical induction after \( n\in \mathbb{N} \). □

Proposition 4.5. Let \( l \) be a nonzero positive integer and let \((a_n)_{n\geq 0}\) be the sequence of \( l\)–numbers. Then, the set

\[
M = \{\alpha a_{2n} | n \in \mathbb{N}, \alpha \in \mathbb{Z}\}
\]

is a commutative nonunitary ring, with addition and multiplication.

Proof. First remark is that \( a_0 = 0 \in M \) is the identity element for addition on \( M \) and \( a_1 = 1 \notin M \). It is clear that the addition and multiplication on \( M \) are commutative.

We prove that \((M, +, \cdot)\) is a subring of the ring \((\mathbb{Z}, +, \cdot)\). According to Proposition 4.4 (i), it results that there is \( \lambda \in \mathbb{Z} \) such that \( a_{2n} = \lambda a_2 \). From here, we obtain that for each \( n, m \in \mathbb{N} \) and for each \( \alpha, \beta \in \mathbb{Z} \), we have

\[
\alpha a_{2n} - \beta a_{2m} = \gamma a_{2r}
\]

and

\[
\alpha a_{2n} \cdot \beta a_{2m} = \delta a_{2s},
\]

where \( r, s \in \mathbb{N} \) and \( \gamma, \delta \in \mathbb{Z} \).

Therefore, it results that \((M, +, \cdot)\) is a commutative nonunitary ring.

The above proposition generalized to a difference equation of degree \( k \), Proposition 3.2 from [FSZ; 19].

Proposition 4.6. Let \( l \) be a nonzero positive integer and let \((a_n)_{n\geq 0}\) be the sequence of \( l\)–numbers. Then, the set

\[
M = \{\alpha a_{2n} | n \in \mathbb{N}, \alpha \in \mathbb{Z}\}
\]

is an ideal of the ring \((\mathbb{Z}, +, \cdot)\), with the property \( M = l\mathbb{Z} \).

Proof. Applying Proposition 4.5 and the fact that \( \beta (\alpha a_{2n}) = \alpha \beta a_{2n} \in M \), for each \( n \in \mathbb{N} \) and for each \( \alpha, \beta, \gamma \in \mathbb{Z} \), it results that \( M \) is a bilateral ideal of the ring \((\mathbb{Z}, +, \cdot)\). According to Proposition 4.4 (i) it results that \( M = l\mathbb{Z} \). □

We consider now the \( l\)–numbers \((a_n)_{n\geq 0}\) in the case when \( l \) is an odd prime number. Let \((\mathbb{Z}_d, +, \cdot)\) be a finite field and let \( \mathbb{H}_{\mathbb{Z}_d}(-1,-1) \) be the quaternion algebra. It is known that this quaternion algebra splits. Let \( \{1, e_1, e_2, e_3\} \) be a basis of this algebra and let \( A_n \) be the \( n\)th \( l\)–quaternion,

\[
A_n = a_n 1 + a_{n+1} e_1 + a_{n+2} e_2 + a_{n+3} e_3.
\]
In the following, we will determine the invertible $l$- quaternions from the quaternion algebra $\mathbb{H}_{\mathbb{Z}_l} (-1, -1)$.

**Proposition 4.7.** Let $l$ be an odd prime integer, let $(a_n)_{n \geq 0}$ be the sequence of $l$- numbers and let $\mathbb{H}_{\mathbb{Z}_l} (-1, -1)$ be the quaternion algebra. Then, all the $n$th $l$- quaternions are invertible in the quaternion algebra $\mathbb{H}_{\mathbb{Z}_l} (-1, -1)$.

**Proof.** In the paper [Fl, Sa; 19], we obtained that the norm of the $n$th $l$- quaternion is $n (A_n) = (l^2 + 2) a_{2n+3}$. Applying Proposition 4.4. (ii), it results that $n (A_n) \equiv 2 \mod l$. Since $l$ is odd, we have that $n (A_n) \neq 0$ in $\mathbb{Z}_l$, for all $n \in \mathbb{N}$. Thus, all the $n$th $l$- quaternions are invertible in the quaternion algebra $\mathbb{H}_{\mathbb{Z}_l} (-1, -1)$. □

In the paper [Gr, Mi, Ma; 15] Grau, Miguel and Oller-Marcen proved that when $n = p^r$, whit $p$ an odd prime and $r$ a positive integer, the quaternion ring $\mathbb{H}_{\mathbb{Z}_p} (-1, -1)$ is isomorphic with the matrix ring $M_2 (\mathbb{Z}_{p^r})$ (see [Gr, Mi, Ma; 15], Proposition 4), therefore the quaternion ring $\mathbb{H}_{\mathbb{Z}_p} (-1, -1)$ splits. We want to determine how many invertible $n$th $l$- quaternions are in this quaternion ring, when $p = l$. □

**Proposition 4.8.** Let $l$ be an odd prime integer, let $(a_n)_{n \geq 0}$ be the sequence of $l$- numbers and let $\mathbb{H}_{\mathbb{Z}_r} (-1, -1)$ be the quaternion ring. Then, the following statements are true:

i) All the $n$th $l$- quaternions are invertible in the quaternion ring $\mathbb{H}_{\mathbb{Z}_l} (-1, -1)$;

ii) All the $n$th $l$- quaternions are invertible in the quaternion ring $\mathbb{H}_{\mathbb{Z}_r} (-1, -1)$, where $r$ is a positive integer, $r \geq 3$.

**Proof.** i) Similar to the proof of Proposition 4.7, applying Proposition 4.4. (ii), it results that $n (A_n) \equiv 2 \mod l^2$. We obtain that all the $n$th $l$- quaternions are invertible in the quaternion ring $\mathbb{H}_{\mathbb{Z}_l} (-1, -1)$.

ii) Let $r$ be a positive integer, $r \geq 3$. Since $l$ is odd and $n (A_n) \equiv 2 \mod l^2$, it results that $n (A_n) \neq 0 \mod l^r$. Therefore, all the $n$th $l$- quaternions are invertible in the quaternion ring $\mathbb{H}_{\mathbb{Z}_r} (-1, -1)$. □

**Proposition 4.9.** Let $(a_n)_{n \geq 0}$ be the sequence of $l$- numbers and let $(M_k)_{k \geq 2}$ be the sequence from Proposition 4.3. Then, the following relation is true:

$$a_n + a_{n+3 \cdot 2^k} = M_k \left(M_k^2 - 3\right) a_{n+3 \cdot 2^{k-1}}, k \geq 2.$$

**Proof.** Let $n, k \in \mathbb{N}, k \geq 2$. Applying Proposition 4.3 (iii), we have:

$$a_n + a_{n+3 \cdot 2^k} = (a_n + a_{n+2^k}) + (a_{n+2^k} + a_{n+2^{k+1}}) +$$

$$+ (a_{n+2^{k+1}} + a_{n+3 \cdot 2^k}) - 2 (a_{n+2^k} + a_{n+2^{k+1}}) =$$

$$= M_k a_{n+2^k - 1} + M_k a_{n+3 \cdot 2^k - 1} + M_k a_{n+5 \cdot 2^k - 1} - 2 M_k a_{n+3 \cdot 2^{k-1}} =$$

$$= M_k a_{n+2^k - 1} + M_k a_{n+5 \cdot 2^k - 1} - M_k a_{n+3 \cdot 2^{k-1}} =$$

14
\[ M_k M_{k+1} a_{n+3 \cdot 2^k - 1} - M_k a_{n+3 \cdot 2^k - 1} = \]
\[ = M_k (M_{k+1} - 1) a_{n+3 \cdot 2^k - 1} = M_k (M_k^2 - 3) a_{n+3 \cdot 2^k - 1}. \]

\[ \square \]

**Proposition 4.10.** Let \( l \) be an odd prime number, let \((a_n)_{n \geq 0}\) be the sequence of \( l\)-numbers and let \( \mathbb{H}_{\mathbb{Z}_l} (-1, -1) \) be the quaternion algebra. Let \( A_n \) be the \( n \)th \( l \)-quaternion in the quaternion algebra \( \mathbb{H}_{\mathbb{Z}_l} (-1, -1) \). Then, we have:

\[ A_n = A_{n+2}, \text{ for all } n \in \mathbb{N}. \]

**Proof.** We use that \( a_n \equiv a_{n+2} \mod l \) and Proposition 4.4. \( \square \)

**Proposition 4.11.** Let \( l \) be an odd prime integer, let \((a_n)_{n \geq 0}\) be the sequence of \( l\)-numbers and let \( \mathbb{H}_{\mathbb{Z}_l} (-1, -1) \) be the quaternion algebra. Let \( A_n \) be the \( n \)th \( l \)-quaternion in the quaternion algebra \( \mathbb{H}_{\mathbb{Z}_l} (-1, -1) \) and \( m \) be a fixed positive integer. Then, the set

\[ M' = \{ \alpha A_{m+2n} | n \in \mathbb{N}, \alpha \in \mathbb{Z} \} \cup \{0\} \]

is a \( \mathbb{Z} \)-module.

**Proof.** It results from Proposition 4.10. \( \square \)

**Proposition 4.12.** Let \( l \) be an odd prime integer, let \((a_n)_{n \geq 0}\) be the sequence of \( l\)-numbers and let \( \mathbb{H}_{\mathbb{Z}_l^2} (-1, -1) \) be the quaternion ring. Let \( A_n \) be the \( n \)th \( l \)-quaternion in the quaternion ring \( \mathbb{H}_{\mathbb{Z}_l^2} (-1, -1) \). Let \((M_k)_{k \geq 2}\) be the sequence from Proposition 4.3. Therefore, we have:

i) \[ A_n + A_{n+2^k} = \hat{2} A_{n+2^k-1}, k \geq 2, \]

ii) \[ A_n + A_{n+3 \cdot 2^k} = \widehat{2} A_{n+3 \cdot 2^k-1}, k \geq 2. \]

iii) \[ A_n + A_{n+1} + ... + A_{n+2^{l-1}} = \hat{0} \in \mathbb{Z}_{l^2}. \]

**Proof.** i) Using Proposition 4.3 and working in \( \mathbb{Z}_{l^2} \) we obtain that

\[ A_n + A_{n+2^k} = \hat{M}_k A_{n+2^k-1}, k \geq 2. \]

It is easy to remark that \( M_k \equiv 2 \mod l^2 \), for all \( k \in \mathbb{N}, k \geq 2 \). Thus, we obtain

\[ A_n + A_{n+2^k} = \hat{2} A_{n+2^k-1}, \text{ for all } k \in \mathbb{N}, k \geq 2. \]

ii) Using Proposition 4.9, we have

\[ A_n + A_{n+3 \cdot 2^k} = M_k (M_k^2 - 3) A_{n+3 \cdot 2^k-1}, k \geq 2. \]
But $M_k (M_k^2 - 3) \equiv 2 \mod l^2$ for all $k \in \mathbb{N}, k \geq 2$. Therefore, we obtain

$$A_n + A_{n+3 \cdot 2^k} = 2A_{n+3 \cdot 2^k - 1}, \text{ for all } k \in \mathbb{N}, k \geq 2.$$ 

iii)

$$A_n + A_{n+1} + ... + A_{n+2^2 - 1} = \sum_{k=0}^{2^2-1} a_{n+k} + e_1 \sum_{k=0}^{2^2-1} a_{n+1+k} + e_2 \sum_{k=0}^{2^2-1} a_{n+2+k} + e_3 \sum_{k=0}^{2^2-1} a_{n+3+k}.$$

Since $l$ is odd, applying Proposition 4.4, we obtain:

$$\sum_{k=0}^{2^2-1} a_{n+k} = a_n + a_{n+1} + ... + a_{n+2^2 - 1} \equiv 1 + 1 + ... + 1 \mod l \equiv 0 \mod l.$$

In the same way, we obtain

$$\sum_{k=0}^{2^2-1} a_{n+1+k} \equiv 0 \mod l,$$

$$\sum_{k=0}^{2^2-1} a_{n+2+k} \equiv 0 \mod l$$

and

$$\sum_{k=0}^{2^2-1} a_{n+3+k} \equiv 0 \mod l.$$ It results that

$$A_n + A_{n+1} + ... + A_{n+2^2 - 1} = 0 \text{ in } \mathbb{Z}_{l^2}.$$ 

□

Conclusions. In this paper, we gave properties and applications of some special integer sequences. We generalized Cassini’s identity and Pisano period for a difference equation of degree $k$. Moreover, we provided a new application in Cryptography and we presented applications of some special number sequences and quaternion elements over finite rings. The above results show us that the study of these sequences can provide us new interesting properties and applications. This remark makes us to continue their study in further researches.

Acknowledgements. Authors thank organizers of IECMSA-2019 for the opportunity to present some of their results at this conference.

References

[18] European Payments Council, Guidelines on Cryptographic Algorithms Usage and Key Management, 2018

[09] Falcon, S., Plaza, A., On $k$-Fibonacci numbers of arithmetic indexes, Applied Mathematics and Computation, 208(2009), 180–185.

[Fib.] http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html

16
C. Flaut, Some applications of difference equations in Cryptography and Coding Theory, Journal of Difference Equations and Applications, 25(7)(2019), 905-920.

C. Flaut, D. Savin, Quaternion Algebras and Generalized Fibonacci-Lucas Quaternions, Adv. Appl. Clifford Algebras, 25(4)(2015), p. 853-862.

C. Flaut, D. Savin, Some remarks regarding l-elements defined in algebras obtained by the Cayley-Dickson process, Chaos, Solitons & Fractals, 118(2019), 112-116.

C. Flaut, D. Savin, G. Zaharia, Some applications of Fibonacci and Lucas numbers, accepted in C. Flaut, S. Hoskova-Mayerova, F. Maturo, Algorithms as an approach of applied mathematics, Springer, 2020.

J. M. Grau, C. Miguel and A. M. Oller-Marcen, On the structure of quaternion rings over $\mathbb{Z}/n\mathbb{Z}$, Advances in Applied Clifford Algebras, vol. 25, Issue 4 (2015), p. 875-887.

S. Halici, On Fibonacci Quaternions, Adv. in Appl. Clifford Algebras 22(2)(2012), 321-327.

A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, Amer. Math. Monthly, 70(1963), 289–291.

R. C. Johnson, Fibonacci numbers and matrices, available at http://maths.dur.ac.uk/dma0rcj/PED/fib.pdf.

N. Koblitz, A Course in Number Theory and Cryptography, Springer Verlag, New-York, 1994, p. 65-76.

R. Melham, Sums Involving Fibonacci and Pell Numbers, Portugaliae Mathematica, 56(3)(1999), 309-317.

M. Renault, The Period, Rank, and Order of the (a, b)-Fibonacci Sequence Mod m, Mathematics Magazine, 86(5)(2013), 372-380, https://doi.org/10.4169/math.mag.86.5.372.

D. Savin, About special elements in quaternion algebras over finite fields, Advances in Applied Clifford Algebras, vol. 27, June 2017, Issue 2 , p. 1801- 1813.

D. Savin, Special numbers, special quaternions and special symbol elements, chapter in the book Models and Theories in Social Systems, vol. 179, Springer 2019, ISBN-978-3-030-00083-7 , p. 417-430.

A. P. Stakhov, A.P., Fibonacci matrices, a generalization of the “Cassini formula”, and a new coding theory, Chaos, Solitons and Fractals, 30(2006), 56-66.

A. P. Stakhov, A.P., The “golden” matrices and a new kind of cryptography, Chaos, Solitons and Fractals, 32(2007), 1138–1146.

D. D. Wall, Fibonacci Series Modulo m, The American Mathematical Monthly,67(6)(1960), 525-532.

Cristina FLAUT
Faculty of Mathematics and Computer Science,
Ovidius University of Constanța, România,
Bd. Mamaia 124, 900527,
[http://www.univ-ovidius.ro/math/](http://www.univ-ovidius.ro/math/)
e-mail: cflaut@univ-ovidius.ro; cristina_flaut@yahoo.com

Diana SAVIN
Faculty of Mathematics and Computer Science,
Ovidius University of Constanța, România,
Bd. Mamaia 124, 900527,
[http://www.univ-ovidius.ro/math/](http://www.univ-ovidius.ro/math/)
e-mail: savin.diana@univ-ovidius.ro, dianet72@yahoo.com

Geanina ZAHARIA
PhD student at Doctoral School of Mathematics,
Ovidius University of Constanța, România,
geaninazaharia@yahoo.com