Spinor field in Bianchi type-II Universe

Bijan Saha

Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Moscow region, Russia

Within the scope of Bianchi type-II cosmological model we have studied the role of spinor field in the evolution of the Universe. It is found that in this case the components of energy-momentum tensor of spinor field along the principal axis are different, i.e. \( T_1^1 \neq T_2^2 \neq T_3^3 \), even in absence of spinor field nonlinearity. The presence of nontrivial non-diagonal components of energy-momentum tensor of the spinor field imposes severe restrictions both on geometry of space-time and on the spinor field itself. Depending on the choice of \( f(x_3) \) the space-time might be either locally rotationally symmetric or isotropic. In this paper we considered the Bianchi type-II space-time both for a trivial and nontrivial \( b \). It was found that while a positive \( \lambda_i \) in case of \( f' \neq 0 \) gives rise to oscillatory mode of evolution, in case of \( f' = 0 \) it leads to accelerated mode of expansion.

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I. INTRODUCTION

In a number of papers it was shown that the introduction of nonlinear spinor field as the source of gravity might be the answer to some fundamental questions such as initial singularity, isotropization, late time accelerated mode of expansion of the Universe [1-17]. Moreover, it was also shown that different types of dark energy and perfect fluid can be simulated by spinor field nonlinearity [18-22]. But recently it was found that these results are model depended, i.e., depending on the specific geometry of space-time the results may vary as the presence of non-diagonal components of energy-momentum tensor of the spinor field imposes severe restrictions to the space-time geometry as well [23-26].

Recently we have extended our study to Bianchi type- VIII and IX models [27]. The main purpose of this paper is to study the role of spinor field in the evolution of a Bianchi type-II space-time, as unlike Bianchi type- VIII and IX models here we can exploit two different scenarios.

Bianchi type-II model in presence of perfect fluid and anisotropic dark energy was studied in [28], whereas such a model with massive string was considered in [29]. Exact solutions to BII cosmological model was found in [30], whereas a similar model in presence of matter and electromagnetic field was studied in [31]. Exact solutions for Bianchi type-II cosmological model in the Jordan Brans-Dicke scalar-tensor theory of gravitation are obtained in [32]. Some remarks on the properties of Bianchi type-II models were made in [33] [34]. Magnetic Bianchi type-II string cosmological model in loop quantum cosmology was investigated in [35]. Nonlinear spinor field in Bianchi type-II space-time was studied in [36].

* bijan@jinr.ru  http://spinor.bijansaha.ru
II. BASIC EQUATION

Let us consider the spinor field Lagrangian

\[ L = \frac{i}{2} \left[ \psi \gamma^{\mu} \nabla_{\mu} \psi - \nabla_{\mu} \psi \gamma^{\mu} \right] - m_{sp} \bar{\psi} \psi - F, \]  

(2.1)

where the nonlinear term \( F \) describes the self-action of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. We consider the case when \( F = F(K) \) with \( K \) taking one of the followings values \( \{ I, J, I + J, I - J \} \).

Here \( \nabla_{\mu} \) covariant derivative of the spinor field having the form

\[ \nabla_{\mu} \psi = \partial_{\mu} \psi - \Gamma_{\mu} \psi, \]  

(2.2a)

\[ \nabla_{\mu} \bar{\psi} = \partial_{\mu} \bar{\psi} + \bar{\psi} \Gamma_{\mu}, \]  

(2.2b)

where \( \Gamma_{\mu} \) is the spinor affine connection defined as

\[ \Gamma_{\mu} = \frac{1}{4} \gamma_{a} \gamma^{\nu} \partial_{\mu} e_{(a)}^{(\nu)} - \frac{1}{4} \gamma_{\rho} \gamma^{\nu} \Gamma_{\mu}^{\rho}, \]  

(2.3)

where \( \gamma_{a} \) are the Dirac matrices in flat space-time, \( \gamma_{\nu} \) are the Dirac matrices in curved space-time, \( e_{(a)}^{(\nu)} \) are the tetrad and \( \Gamma_{\mu}^{\rho} \) are the Christoffel symbols.

The corresponding energy momentum tensor is found from

\[ T_{\mu}^{\rho} = \frac{i}{4} \bar{\psi} \gamma^{\rho} \left( \psi \gamma_{\nu} \nabla_{\nu} \psi + \bar{\psi} \gamma_{\nu} \nabla_{\nu} \psi - \nabla_{\mu} \psi \gamma_{\nu} \psi - \nabla_{\nu} \bar{\psi} \gamma_{\mu} \psi \right) - \delta_{\mu}^{\rho} L \]

\[ = \frac{i}{4} \bar{\psi} \gamma^{\rho} \left( \psi \gamma_{\nu} \partial_{\nu} \psi + \bar{\psi} \gamma_{\nu} \partial_{\nu} \psi - \partial_{\mu} \psi \gamma_{\nu} \psi - \partial_{\nu} \bar{\psi} \gamma_{\mu} \psi \right) \]

\[ - \frac{i}{4} \bar{\psi} \gamma^{\rho} \bar{\psi} \left( \gamma_{\mu} \Gamma_{\nu} + \Gamma_{\nu} \gamma_{\mu} + \gamma_{\nu} \Gamma_{\mu} + \gamma_{\mu} \gamma_{\nu} \right) \psi - \delta_{\mu}^{\rho} L. \]  

(2.4)

Bianchi type II space-time (BII) is given by

\[ ds^{2} = dt^{2} - a_{1}^{2}(t)dx_{1}^{2} - [h^{2}(x_{3})a_{1}^{2}(t) + f^{2}(x_{3})a_{3}^{2}](t)dx_{2}^{2} - a_{3}^{2}dx_{3}^{2} + 2a_{1}^{2}(t)h(x_{3})dx_{1}dx_{2}, \]  

(2.5)

with \( a_{1}, a_{2}, a_{3} \) being the functions of \( t \) and \( h \) and \( f \) are the functions of \( x_{3} \) only, wherewith \( f'' = 0 \). As one sees, \( f'' = 0 \) may lead to two different cases with \( f' \neq 0 \) and \( f' = 0 \). We first consider the case with \( f' \neq 0 \) and later study the case with \( f' = 0 \). So let us begin with \( f' \neq 0 \).

The spinor affine connections (2.3) corresponding to (2.5) read

\[ \Gamma_{1} = \frac{1}{2} a_{1} \gamma^{1} \gamma^{0} - \frac{1}{4} a_{2} a_{3} f \gamma^{2} \gamma^{3}, \]  

(2.6a)

\[ \Gamma_{2} = \frac{1}{2} f \dot{a}_{2} \gamma^{2} \gamma^{0} - \frac{1}{2} \dot{h} a_{1} \gamma^{1} \gamma^{0} - \frac{1}{4} a_{1} h' \gamma^{1} \gamma^{3} + \frac{1}{2} a_{2} f' \gamma^{2} \gamma^{3} + \frac{1}{4} a_{2} a_{3} h \gamma^{2} \gamma^{3}, \]  

(2.6b)

\[ \Gamma_{3} = \frac{1}{2} \dot{a}_{3} \gamma^{3} \gamma^{0} + \frac{1}{4} a_{1} h', \]  

(2.6c)

\[ \Gamma_{0} = 0. \]  

(2.6d)

In view of (2.6) the spinor field equations corresponding to the Lagrangian (2.1) has the form

\[ \gamma^{\mu} \nabla_{\mu} \psi - m_{sp} \psi - \partial \psi - i \partial \gamma^{5} \psi = 0, \]  

(2.7a)

\[ \gamma^{\mu} \bar{\psi} \gamma^{\mu} + m_{sp} \bar{\psi} + \partial \bar{\psi} + i \partial \bar{\psi} \gamma^{5} = 0, \]  

(2.7b)
where \( \mathcal{D} = 2SF_K K_l \) and \( \mathcal{G} = 2PF_K K_l \).

The system (2.7) can be rewritten as

\[
\begin{align*}
\dot{\gamma}^0 \psi + \frac{1}{2} \frac{\dot{V}}{V} \gamma^0 \psi + \frac{1}{4} \frac{\dot{h}'}{a_3 f_a} \gamma^0 \psi + \frac{1}{2} \frac{f'}{a_3 f} \gamma^3 \psi &- \left[ m_{sp} + \mathcal{D} \right] \psi - \mathcal{G} \dot{\gamma}^5 \psi = 0, \quad (2.8a) \\
\dot{\psi} \gamma^0 + \frac{1}{2} \frac{\dot{V}}{V} \psi \gamma^0 - \frac{1}{4} \frac{\dot{h}'}{a_3 f_a} \psi \gamma^0 + \frac{1}{2} \frac{f'}{a_3 f} \psi \gamma^3 + \left[ m_{sp} + \mathcal{D} \right] \psi + \mathcal{G} \dot{\psi} \gamma^5 = 0. \quad (2.8b)
\end{align*}
\]

In view of (2.7) the spinor field Lagrangian can be written as

\[
L = \frac{1}{2} \left[ \psi \gamma\mu \nabla_\mu \psi - \nabla_\mu \psi \gamma^\mu \psi \right] - m_{sp} \psi \gamma_5 - F(K) = \frac{1}{2} \psi \left[ \gamma^\mu \nabla_\mu \psi - m_{sp} \psi \right] \frac{-1}{2} \nabla_\mu \psi \gamma^\mu + m_{sp} \psi \gamma_5 - F(K),
\]

\[
= 2(IF_1 + JF_2) - F = 2KF_K - F(K). \quad (2.9)
\]

The nontrivial components of Einstein tensor corresponding to (2.5) are

\[
G_1 = \left( \frac{a_3}{a_2} \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2} \frac{a_3}{a_3} \right) + \frac{a_1^2 h^2}{a_2 a_3 f_2^2} \left( \frac{3 h'^2}{4 h^2} + \frac{h''}{2 h} - \frac{1 h'}{2 h f} \right), \quad (2.10a)
\]

\[
G_2 = \frac{1}{2} \frac{a_1^2 h^2}{a_2 a_3 f_2^2} \left( \frac{h''}{h} - \frac{h f'}{h f} \right), \quad (2.10b)
\]

\[
G_2 = h \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \frac{a_3}{a_3} - \frac{\dot{a}_1}{a_1} \frac{a_3}{a_3} \right) + \frac{h}{a_3^2} \left( \frac{1 h''}{2 h} - \frac{1 h'}{2 h f} \right)
\]

\[
+ \frac{a_1^2 h^2}{a_2 a_3 f_2^2} \left( 2 h' f - \frac{1 h''}{2 h} - \frac{h^2}{h^2} \right), \quad (2.10c)
\]

\[
G_2 = - \left( \frac{a_3}{a_3} \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_1}{a_1} \frac{a_3}{a_3} a_1 \right) - \frac{1}{2} \frac{a_1^2 h^2}{a_2 a_3 f_2^2} \left( \frac{h''}{h} - \frac{h f'}{h f} + \frac{h'^2}{h^2} \right), \quad (2.10d)
\]

\[
G_3 = \left( \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \frac{a_3}{a_3} - \frac{\dot{a}_1}{a_1} \frac{a_3}{a_3} a_2 \right) - \frac{1}{4} \frac{a_1^2 h^2}{a_2 a_3 f_2^2} h^2, \quad (2.10e)
\]

\[
G_0 = \left( \frac{a_2}{a_2} \frac{\dot{a}_3}{a_3} \left( \frac{f'}{f} \right) \right), \quad (2.10f)
\]

\[
G_0 = - \frac{1}{a_3^2} \left( \frac{a_2}{a_2} \frac{\dot{a}_3}{a_3} \left( \frac{f'}{f} \right) \right), \quad (2.10g)
\]

\[
G_0 = \left( \frac{a_1}{a_1} \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \frac{a_3}{a_3} + \frac{\dot{a}_3}{a_3} \right) + \frac{1}{4} \frac{a_1^2 h^2}{a_2 a_3 f_2^2} h^2. \quad (2.10h)
\]

From (2.10) one finds

\[
G_2 = h (G_2 - G_1) + \left( h^2 + \frac{a_2^2 f_2^2}{a_1^2} \right) G_1. \quad (2.11)
\]

From (2.4) one finds the following nontrivial components of energy-momentum tensor
Moreover from (2.12) it follows that

\[ T_2^1 = h(T_2^2 - T_1^1) + \left( h^2 + \frac{a_2^2 f^2}{a_1^2} \right) T_1^2. \]  

(2.13)

Moreover from (2.12) it follows that

\[ T_2^3 = \frac{a_2^2 f^2}{a_3^2} T_3^2 - h T_1^3, \]  

(2.14)

\[ T_3^1 = h T_3^2 + \frac{a_2^2}{a_1^2} T_3^3. \]  

(2.15)
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\[ T_0^1 = hT_0^2 - \frac{1}{a_1^2}T_1^0, \]  

(2.16)

and

\[ T_2^0 = -a_2^2f^2T_0^2 - hT_1^0. \]  

(2.17)

Then the system of Einstein equations

\[ G^\mu_\nu = -\kappa T^\mu_\nu, \]  

(2.18)

on account of linearly dependent components takes the form

\[
\left( \frac{\ddot{a}_2 + \ddot{a}_3 + \dot{a}_2 \dot{a}_3}{a_2} + \frac{\ddot{a}_3}{a_3} \right) - \frac{1}{2} \frac{a_1^2 h^2}{a_1 a_2 a_3} \left( \frac{h''}{h} - \frac{h' f'}{h f} + \frac{3 h'}{2 h^2} \right) = \kappa \left[ (F(K) - 2KF_K) + \frac{1}{4 a_2 f} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A^3 + \frac{1}{4 a_2 a_3 f} \left( \frac{f'}{f} - h' \right) A^0 \right],
\]  

(2.19a)

\[
\left( \frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1 + \ddot{a}_3 \dot{a}_1}{a_1 a_3} \right) + \frac{1}{2} \frac{a_1^2 h^2}{a_1 a_2 a_3} \left( \frac{h''}{h} - \frac{h' f'}{h f} + \frac{1 h'}{2 h^2} \right) = \kappa \left[ (F(K) - 2KF_K) - \frac{1}{4 a_2 f} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A^3 - \frac{1}{4 a_2 a_3 f} \left( \frac{f'}{f} - h' \right) A^0 \right],
\]  

(2.19b)

\[
\left( \frac{\ddot{a}_1 + \ddot{a}_2 + \dot{a}_1 \dot{a}_2}{a_1 a_2} \right) + \frac{1}{4} \frac{a_2^2 h^2}{a_2 a_3 f^2} = \kappa \left[ (F(K) - 2KF_K) + \frac{1}{4 a_2 a_3 f} A^0 \right],
\]  

(2.19c)

\[
\left( \frac{\ddot{a}_1 a_2 + \ddot{a}_2 a_3 + \dot{a}_3 \dot{a}_1}{a_1 a_2 a_3} \right) - \frac{1}{4} \frac{a_2^2 h^2}{a_2 a_3 f^2} = \kappa \left[ m_{sp} S + F(K) - \frac{1}{4 a_2 a_3 f} A^0 \right],
\]  

(2.19d)

\[
\frac{1}{2} \frac{a_1^2 h^2}{a_1 a_2 a_3 f^2} \left( \frac{h''}{h} - \frac{h' f'}{h f} \right) = -\kappa \frac{1}{4} \frac{1}{a_2 f} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A^3 + \frac{a_1 f'}{a_3 f} A^0,
\]  

(2.19e)

\[
\left( \frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_3}{a_3} \right) \frac{f'}{f} = 0,
\]  

(2.19f)

\[
0 = \left( \frac{\ddot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) A^1,
\]  

(2.19g)

\[
0 = \left( \frac{\ddot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) A^2,
\]  

(2.19h)

\[
0 = \left[ \frac{a_1 h'}{a_2 f} A^1 + f' A^2 \right],
\]  

(2.19i)

\[
0 = \left( 1 + \frac{a_1}{a_2 f} \right) \frac{f'}{f} A^1.
\]  

(2.19j)

The summation of (2.19a), (2.19b), (2.19c) and 3 times (2.19d) gives

\[
\frac{\ddot{V}}{V} - \frac{1}{2} \frac{a_1^2 h^2}{a_1 a_2 a_3 f^2} = \frac{3 \kappa}{2} \left[ m_{sp} S + 2(F - KF_K) - \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} A^0 \right],
\]  

(2.20)

where

\[
V = a_1 a_2 a_3
\]  

(2.21)

is the volume scale.
Then in view of \( \frac{\bar{f}}{f} \neq 0 \) from (2.19), we find \( \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) = 0 \), on the other hand for same reason (2.19) yields \( A^1 = 0 \), whereas inserting \( A^1 = 0 \) into (2.19) we obtain \( A^2 = 0 \). Thus if we consider \( \frac{\bar{f}}{f} \neq 0 \) from (2.19) - (2.19) we have

\[
A^1 = 0, \quad A^2 = 0, \quad \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) = 0.
\] (2.22)

In view of \( A^2 = 0 \) the equation (2.19) yields two possibilities:

\[
\left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) \neq 0,
\] (2.23)

which means \( a_2 \sim a_3 \), or

\[
\left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) = 0,
\] (2.24)

which means \( a_1 \sim a_2 \sim a_3 \).

It can be shown that the spinor field invariants in this case obey the following equations:

\[
\dot{S}_0 + \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} \dot{P}_0 + 2 \mathcal{G} A^0_0 = 0,
\] (2.25a)

\[
\dot{P}_0 + \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} \dot{S}_0 - 2 \left[ m_{sp} + \mathcal{D} \right] A^0_0 = 0,
\] (2.25b)

\[
\dot{A}^0_0 + \frac{1}{2} \frac{f'}{a_3 f} A^0_0 + 2 \left[ m_{sp} + \mathcal{D} \right] \dot{P}_0 + 2 \mathcal{G} S_0 = 0,
\] (2.25c)

\[
\dot{A}^3_0 + \frac{1}{2} \frac{f'}{a_3 f} A^0_0 = 0,
\] (2.25d)

\[
\dot{A}^2_0 = 0.
\] (2.25e)

From (2.25) it can be easily shown that

\[
P^2_0 - S^2_0 + (A^0_0)^2 - (A^3_0)^2 = \bar{c}_1, \quad A^2_0 = \bar{c}_2, \quad \bar{c}_1, \bar{c}_2 - \text{const.},
\] (2.26)

On account of (2.22) in this case we have \( \bar{c}_2 = 0 \). On the other hand from Fierz theorem we have

\[
I_A = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = \left( S^2 + P^2 \right),
\] (2.27)

Now taking into account that \( A^1 = 0 \) and \( A^2 = 0 \) from (2.27) one finds

\[
(A^0_0)^2 - (A^3_0)^2 = - \left( S^2 + P^2 \right),
\] (2.28)

Then inserting (2.28) into (2.26) one finds

\[
S^2 = \frac{\text{const}}{V^2},
\] (2.29)

Let us now impose the proportional condition between the shear and expansion. Assuming that

\[
\sigma_1^1 = q_1 \theta, \quad q_1 = \text{const.}
\] (2.30)
for the metric function we have \[27\]

\[a_i = X_i V^{Y_i}, \quad \prod_{i=1}^{3} X_i = 1, \quad \sum_{i=1}^{3} Y_i = 0. \quad (2.31)\]

In this concrete case we have \(X_1 = q_2, \ X_2 = \sqrt{q_3 / q_2}, \ X_3 = 1 / \sqrt{q_2 q_3}, \ Y_1 = q_1 + 1/3, \) and \(Y_2 = Y_3 = 1/3 - q_1/2.\) Here \(q_2\) and \(q_3\) are arbitrary constants. On account of \(2.31\) the equation for \(V\)

\[\ddot{V} - \frac{1}{2} \frac{h'^2}{f} q_2^4 V^{4q_1 - 2/3} = \frac{3 \kappa}{2} \left[ m_{\text{sp}} S + 2 (F - KF_K) - \frac{q_2^2 h'}{2 f} V^{2q_1 - 1/3} A_0^0 \right], \quad (2.32)\]

Let us now rewrite \(2.19e.\) In view of \(2.25d\) this equations can be written as

\[\frac{1}{f} \left( h'' - \frac{f'}{f} h' \right) = - \frac{\kappa a_2 a_3^2}{a_1^2} \left[ \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A_0^3 - 2 a_1 \dot{A}_0^3 \right] = -b, \quad (2.33)\]

Now the left hand side of \(2.33\) depends only \(x_3\) only, while the right hand side depends only \(t.\) So we can finally write the following system

\[\frac{1}{f} \left( h'' - \frac{f'}{f} h' \right) = b, \quad (2.34a)\]

\[\frac{\kappa a_2 a_3^2}{a_1^2} \left[ \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A_0^3 - 2 a_1 \dot{A}_0^3 \right] = -b, \quad (2.34b)\]

Inserting \(2.31\) one finally finds

\[h'' - \frac{f'}{f} h' = bf, \quad (2.35a)\]

\[\dot{A}_0^3 + \frac{3 q_1}{4 q_2} V^{1/4} A_0^3 = \frac{b}{\kappa} \sqrt{q_2^5 q_3^5 V^{5q_1 - 2/3}}, \quad (2.35b)\]

Exploiting \(2.25d\) we rewrite \(2.32\) in the following way

\[\ddot{V} = \frac{3 \kappa}{2} \sqrt{q_2^2 h'/ f} A_0^3 V^{3q_1/2} - \frac{1}{4} \frac{q_2^4 V^{4q_1 + 1/3}}{f} \left( h'/ f \right)^2 = \frac{3 \kappa}{2} \left[ m_{\text{sp}} S + 2 (F - KF_K) \right] V, \quad (2.36)\]

To find the solution to the equation \(2.36\) we have to give the concrete form of spinor field nonlinearity. Following some previous papers, we choose the nonlinearity to be the function of \(S\) only, having the form

\[F = \sum_{k} \lambda_k L^{n_k} = \sum_{k} \lambda_k S^{2n_k}. \quad (2.37)\]

Recently, this type of nonlinearity was considered in a number of papers \[23\]-\[26\]. For simplicity we consider only three terms of the sum. We set \(n_k = n_0 : 1 - 2n_0 = 0\) which gives \(n_0 = 1/2.\) In this case the corresponding term can be added with the mass term. We assume that \(q_1\) is a positive quantity, so that \(4q_1 + 1/3\) is positive too. For the nonlinear term to be dominant at large time, we set \(n_k = n_1 : 1 - 2n_1 > 4q_1 + 1/3,\) i.e., \(n_1 < 1/3 - 2q_1.\) And finally, for the nonlinear term to be dominant at the early stage we set \(n_k = n_2 : 1 - 2n_2 < 0,\) i.e., \(n_2 > 1/2.\) Then the spinor field nonlinearity can be written as

\[F = \lambda_0 L^{n_0} + \lambda_1 L^{n_1} + \lambda_2 L^{n_2}. \quad (2.38)\]
Once we have the concrete form of nonlinearity, we can solve the foregoing equation. In doing so we rewrite the equations (2.36) together with (2.35b) as follows

\[ \dot{V} = Y, \]  
\[ \dot{Y} = \frac{3K}{2} \sqrt{\frac{q_1}{q_3}} \frac{F'}{f'} V^{3q_1/2} \Phi_1(V, A_0^3, Y) + \Phi_2(V, A_0^3, Y), \]  
\[ \dot{A}_0^3 = \Phi_1(V, A_0^3, Y). \]  

where we denote

\[ \Phi_1(V, A_0^3, Y) = -\frac{3q_1}{4q_2} A_0^3 V^{-(q_1 + 4/3)} Y + \frac{b}{K} \sqrt{\frac{q_1}{q_3} V^{5q_1/2 - 2/3}}, \]
\[ \Phi_2(V, A_0^3, Y) = \frac{1}{4} q_2 V^{4q_1 + 1/3} \left( \frac{F'}{f'} \right)^2 + \frac{3K}{2} \left[ (m_{sp} + \lambda_0) + 2\lambda_1 (1 - n_1) V^{1 - 2n_1} + 2\lambda_2 (1 - n_2) V^{1 - 2n_2} \right]. \]

Let us recall that by definition we have \( f'' = 0 \). Then as per our assumption \( f' \neq 0 \) we find

\[ f = c_1 x^3 + c_2, \quad c_1, c_2 - \text{const.} \]

Inserting this into (2.35a) in case of \( b \neq 0 \) we find

\[ h = \frac{1}{3} bc_1 x^3 + \frac{1}{2} (c_3 c_1 + bc_2) x^2 + c_3 c_2 x^3 + c_4, \quad c_3, c_4 - \text{const.}, \]

and in case of \( b = 0 \) we obtain

\[ h = \frac{1}{2} c_5 c_1 x^3 + c_5 c_2 x^3, \quad c_5 - \text{const.} \]

Let us now numerically solve the system (2.39). Since we are interested in qualitative picture of evolution, let us set \( q_2 = 1, q_3 = 1 \) and \( K = 1 \). We also assume \( V_0 = 1 \).

We set \( m_{sp} = 1 \) and \( l_0 = 2 \). As far as \( q_1, n_1 \) and \( n_2 \) are concerned, in line of our previous discussions we choose them in such a way that the power of nonlinear term in the equations become integer. We have also studied the case for some different values, but they didn’t give any principally different picture. We choose \( q_1 = 2/3, \quad n_1 = -3/2 < 1/3 - 2q - 1 = -1 \) and \( n_2 = 3/2 > 1/2 \). It should be noted that we have taken some others value for \( q_1 \) such as \( q_1 = -1 \), but it does not give qualitatively different picture. We have also set \( x_3 = [0, 1] = 0.2k \) with step \( k = 0.5 \). Finally we have considered time span \([0, 2]\) with step size 0.001. Here we consider different values of \( \lambda_i \) both positive and negative. We choose the initial values for \( V(0) = 1, \quad Y(0) = V(0) = 0.1, \) and \( A_0^3(0) = 1 \), respectively.

In Fig. 1 we have plotted the phase diagram of \([V, \dot{V}, A_0^3]\) for a trivial \( b \) and positive \( \lambda_1 \) and \( \lambda_2 \). The corresponding evolution of \( V \) is given in Fig. 2. As one sees, in this case we have oscillatory mode of expansion.

In Fig. 3 the phase diagram of \([V, \dot{V}, A_0^3]\) has been plotted for a trivial \( b \), positive \( \lambda_1 \) and negative \( \lambda_2 \), while the evolution of \( V \) corresponding to it is illustrated in Fig. 4.

In Figs. 5 and 6 the phase diagram of \([V, \dot{V}, A_0^3]\) and volume scale \( V \) are manifested for a trivial \( b \), trivial \( \lambda_1 \) and positive \( \lambda_2 \).

In Figs. 7, 9 and 11 we have illustrated the phase diagram of \([V, \dot{V}, A_0^3]\) for \( \lambda_1 \) and \( \lambda_2 \) considered in Figs. 1, 3 and 5, but with a nontrivial \( b \). Evolution of \( V \) corresponding to Figs. 7, 9 and 11 are illustrated in Figs. 8, 10 and 12.
FIG. 1. Phase diagram of \([V, \dot{V}, A_0^3]\) in case of 
\(b = 0, \lambda_1 = 1\) and \(\lambda_2 = 1\)

FIG. 2. Evolution of \(V\) corresponding to the phase diagram given in Fig. [1]

FIG. 3. Phase diagram of \([V, \dot{V}, A_0^3]\) in case of 
\(b = 0, \lambda_1 = 1\) and \(\lambda_2 = -0.01\)

FIG. 4. Evolution of \(V\) corresponding to the phase diagram given in Fig. [4]

In the Figures each color corresponds to a concrete value of \(x_3 = 0.2k\), namely, red, green, yellow, blue, magenta and black color corresponds to \(k = 0, 1, 2, 3, 4, 5\).

Let us now consider the case when \(f' = 0\), i.e. \(f = f_0 = \text{const}\). In this case a few components of energy-momentum tensor die out and finally we have the following system of Einstein equations:
FIG. 5. Phase diagram of $[V, \dot{V}, A_3]$ in case of $b = 0, \lambda_1 = 0$ and $\lambda_2 = 1$

FIG. 6. Evolution of $V$ corresponding to the phase diagram given in Fig. 5

FIG. 7. Phase diagram of $[V, \dot{V}, A_3]$ in case of $b = 1, \lambda_1 = 1$ and $\lambda_2 = 1$

FIG. 8. Evolution of $V$ corresponding to the phase diagram given in Fig. 7
FIG. 9. Phase diagram of $[V, \dot{V}, A_0]$ in case of $b = 1, \lambda_1 = 1$ and $\lambda_2 = -0.01$.

FIG. 10. Evolution of $V$ corresponding to the phase diagram given in Fig. 9.

FIG. 11. Phase diagram of $[V, \dot{V}, A_0]$ in case of $b = 1, \lambda_1 = 0$ and $\lambda_2 = 1$.

FIG. 12. Evolution of $V$ corresponding to the phase diagram given in Fig. 11.
From (2.44a), (2.44b) and (2.44c) we find
\[(P_0)^2 - (S_0)^2 + (A_0^0)^2 = C_1^2, \quad C_1^2 = \text{const.},\] (2.45)
whereas (2.44c) and (2.44d) yield
\[A_0^0 = C_2 \Rightarrow A^2 = \frac{C_2}{V}, \quad A_0^0 = C_3 \Rightarrow A^3 = \frac{C_3}{V}, \quad C_2, C_3 = \text{const.}\] (2.46)
On the other hand from (2.27) on account of \(A^1 = 0\) which follows from (2.43h) and (2.46) we find
Then in this case we finally find

\[(S_0)^2 = \frac{1}{2} (C_2^2 + C_3^2 - C_1^2) = C_2^2, \quad (P_0)^2 + (A_0^0)^2 = \frac{1}{2} (C_2^2 + C_3^2 + C_1^2) = C_2^2.\]  

(2.48)

Then inserting \(S_0\) and \(P_0\) from (2.48) into (2.44c) for \(A_0^3\) we find

\[\dot{A}_0^0 + 2 [m_{sp} + D] \sqrt{C_2^2 - (A_0^0)^2} + 2 D C_- = 0.\]  

(2.49)

The equation for \(V\) in this case coincides with (2.20), so to solve it we again assume the proportionality condition that gives \(a_i = X_i Y_i\), \(\prod_{i=1}^3 X_i = 1\), \(\sum_{i=1}^3 Y_i = 0\).

In what follows we solve the equation for \(V\) numerically. In doing so we consider the spinor field nonlinearity in the form (2.38). Taking into account that in this case \(\mathcal{G} = 2 P F_k K_l = 0\), we write the equations (2.20) and (2.49) in the following way:

\[\dot{V} = Y,\]  

(2.50a)

\[\dot{Y} = \Phi_2(V, A_0^0) = \frac{3q_4^2}{4} h^2 V^{2q_1 - 2/3} A_0^0,\]  

(2.50b)

\[\dot{A}_0^0 = -2 [(m_{sp} + \lambda_0) + 2\lambda_1 n_1 V^{1 - 2n_1} + 2\lambda_2 n_2 V^{1 - 2n_2}] \sqrt{C_2^2 - (A_0^0)^2}.\]  

(2.50c)

where we denote

\[\Phi_2(V, A_0^0, Y) = \frac{h^2}{2 f^2} q_4^2 V^{2q_1 + 1/3} \quad + \frac{3\kappa}{2} [(m_{sp} + \lambda_0) + 2\lambda_1 (1 - n_1)V^{1 - 2n_1} + 2\lambda_2 (1 - n_2)V^{1 - 2n_2}].\]

To solve the foregoing system we have to find \(h\) explicitly that corresponds to \(f = f_0 = \text{const}\). In doing so we rewrite the equation (2.43e) as

\[h'' = b f,\]  

(2.51a)

\[\frac{\kappa C_3}{2} \frac{2X_2X_3^2}{Y_2 - Y_1} Y(Y_2 + 2Y_3 - 2Y_1 - 2) \dot{V} = b.\]  

(2.51b)

In case of \(b \neq 0\) on account of \(f = f_0\) from (2.51a) one finds \(h = b f_0 x_3^2 / 2 + c_1 x_3 + c_2\). In this case equation (2.51a) yields

\[V = \left[ \frac{b x_3^2}{\kappa C_3 X_2 X_3^2 (Y_2 - Y_1) (Y_2 + 2Y_3 - 2Y_1 - 1)} \right]^{1/(Y_2 + 2Y_3 - 2Y_1 - 1)}.\]  

(2.52)

As one sees, in this case \(V\) does not depend on spinor field, whereas the equation (2.20) explicitly contains terms corresponding to spinor field. This imposes some severe restrictions on the choice of spinor field nonlinearity.

In case of \(b = 0\) from (2.51a) we find \(h = c_1 x_3 + c_2\) with \(c_1\) and \(c_2\) being some arbitrary constants. Equation (2.51a) in this case leads to \(V = Y_1 = Y_2\), i.e. \(a_1 = a_2\). As far as \(V\) and \(A_0^0\) are concerned, in this case we find it solving (2.50) numerically. In doing so we set the parameters as we did in previous case. For this we have to substitute \(h' / f\) with \(c_1 / f_0\). As a result we see that unlike the previous cases or \(BVIII\) and \(BIX\) models, the system for defining \(V\) does not depend on \(x_3\).
explicitly. In Figs. [13, 15, 17, and 19] we have plotted the phase diagram of \([V, \dot{V}, A_0^0]\) for \((\lambda_1 = -1\) and \(\lambda_2 = -0.1)\), \((\lambda_1 = 1\) and \(\lambda_2 = 1)\), \((\lambda_1 = 1\) and \(\lambda_2 = -0.1)\) and \((\lambda_1 = -1\) and \(\lambda_2 = 1)\), respectively with \(b\) taken to be trivial. Corresponding pictures of \(V\) are given in Figs. [14, 16, 18, and 20] respectively.

One of the principal differences between the two cases considered here is the following. In case
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FIG. 17. Phase diagram of $[V, \dot{V}, A_0^0]$ in case of $b = 0, \lambda_1 = 1$ and $\lambda_2 = -0.1$

FIG. 18. Evolution of $V$ corresponding to the phase diagram given in Fig. 17

FIG. 19. Phase diagram of $[V, \dot{V}, A_0^0]$ in case of $b = 0, \lambda_1 = -1$ and $\lambda_2 = 1$

FIG. 20. Evolution of $V$ corresponding to the phase diagram given in Fig. 19

of $f' \neq 0$ we have oscillating mode of expansion for a positive $\lambda_1$, whereas in case of $f' = 0$ for a positive $\lambda_1$ we have an accelerated mode of expansion, i.e. the sign of $\lambda_1$ gives rise two opposite type of evolution.
III. CONCLUSION

Within the scope of Bianchi type-II cosmological models we have studied the role of nonlinear spinor field in the evolution of the Universe. In doing so we have considered two cases with \( f' \neq 0 \) and \( f' = 0 \). It was found that the sign of coupling constant \( \lambda_1 \) gives rise two different modes of evolution in the two cases considered.

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