Guaranteed Cost Tracking for Uncertain Coupled Multi-agent Systems Using Consensus over a Directed Graph

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Abstract

This paper considers the leader-follower control problem for a linear multi-agent system with directed communication topology and linear nonidentical uncertain coupling subject to integral quadratic constraints (IQC's). A consensus-type control protocol is proposed based on each agent’s states relative to its neighbors and leader’s state relative to agents which observe the leader. A sufficient condition is obtained by overbounding the cost function. Based on this sufficient condition, a computational algorithm is introduced to minimize the proposed guaranteed bound on tracking performance, which yields a suboptimal bound on the system consensus control and tracking performance. The effectiveness of the proposed method is demonstrated using a simulation example.

1 Introduction

In recent years, theoretical studies of distributed coordination and control for multi-agent systems have attracted much attention in the literature, with broad applications in various areas including unmanned air vehicles (UAVs), formation control, flocking, distributed sensor networks, etc. [1]. As a result, much progress has been made in the study of cooperative control of multi-agent systems [2, 3, 4].

Efforts have recently been made to consider the leader-following consensus problem. For example, the leader-following consensus problem for higher order multi-agent systems is presented for both fixed and switching topologies in [5]. In [6], distributed observers are designed for the system of second-order agents where an active leader to be followed moves with an unknown velocity, and the interaction topology has a switching nature. The consensus-based approach to observer-based synchronization of multi-agent systems to the leader has been explored in [7, 8].

A common feature of the above literature on leader-following consensus-based control problems is that interactions between agents are not considered. However, in many physical systems, interactions between agents are inevitable and must be taken into account. Examples of systems with a dynamical interaction between subsystems include power systems and spacecraft control systems [9]. This necessitates considering systems of interconnected agents.

In this paper, the leader-follower control problem for multi-agent systems coupled via linear unmodelled dynamics is considered. Coupling among the agents is regarded as an uncertainty and is described in term of time domain integral quadratic constraints (IQC's) [10]. The IQC modeling...
is a well established technique to describe uncertain interactions between subsystems in a large scale system \[11, 12, 13\].

The motivation of this paper is to extend our previous work \[14\] as follows. Firstly, this paper considers the multi-agent system with directed topology rather than undirected topology, which poses additional difficulty compared with \[14\], due to the Laplacian matrix of directed graphs being in general asymmetric. Therefore a different technique is used in this paper to obtain a sufficient condition for leader follower tracking which does not involve coordinate transformation; the latter was used in \[14\] and required the Laplacian matrix of the graph to be symmetric. Furthermore, we consider a more general, compared to \[14\], class of systems with nonidentical time varying uncertain coupling. In this paper, we also propose a different LQR based cost function which describes the cost on the tracking error between the leader and all of the followers. In contrast, in \[14\] a consensus based cost function is considered, which penalizes the system input, the state error between the agent and its neighbours, as well as the tracking error between the leader and selected agents which observe the leader. Furthermore, the graph topology of the control protocol does not need to be the same as the topology of interconnections between the agents. Even though both communication topologies are represented as directed graphs, these graphs can be different: the agents are coupled over one directed graph, but the control protocol for the system uses another directed graph.

The main contribution of this paper is to propose a sufficient condition for the design of a guaranteed performance leader-follower control protocol for multi-agent systems with directed interconnection topology and a quite general linear uncertain coupling subject to IQCs. The sufficient condition is obtained by using a direct over-bounding technique and involves checking feasibility of parameterized linear matrix inequalities (LMIs). The computational algorithm is introduced to minimize the proposed guaranteed bound by choosing local tuning parameters and guarantee a suboptimal bound on the system tracking performance.

The remainder of the paper proceeds as follows. In Section 2 of the paper, we set up the leader follower control problem for a multi-agent system with directed topology and nonidentical linear uncertain coupling and give some preliminaries. The main results are given in Section III. In section IV, the computational algorithms are introduced. Section V gives an example which illustrates the theory presented in the paper. Finally, the conclusions are given in Section VI.

## 2 Problem Formulation and Preliminaries

### 2.1 Graph theory

Consider a directed graph \(G = (\mathcal{V}, \mathcal{E}, \mathcal{A})\), where \(\mathcal{V} = \{1, 2, \cdots, N\}\) is a finite nonempty node set and \(\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}\) is an edge set of ordered pairs of nodes. The edge \((i, j)\) in the edge set of an directed graph means that the node \(i\) can obtain information from node \(j\). Node \(i\) is called a neighbor of node \(j\) if \((i, j) \in \mathcal{E}\). The set of neighbors of node \(i\) is defined as \(N_i = \{j | (i, j) \in \mathcal{E}\}\). \(G\) is a simple graph if it has no self-loops or repeated edges. If there is a directed path between any two nodes of the graph \(G\), then the graph \(G\) is strongly connected. The adjacency matrix \(\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}\) of the directed graph \(G\) is defined as \(a_{ij} = 1\) if \((i, j) \in \mathcal{E}\), and \(a_{ij} = 0\) otherwise. The in-degree matrix \(\mathcal{D} = \text{diag}\{d_1, \cdots, d_N\} \in \mathbb{R}^{N \times N}\) is a diagonal matrix, whose diagonal elements are \(d_i = \sum_{j=1}^{N} a_{ij}\) for \(i = 1, \cdots, N\). Also, let \(q_i = \sum_{j=1}^{N} a_{ji}\) be the out-degree of node \(i\). The Laplacian matrix of the graph is defined as \(\mathcal{L} = \mathcal{D} - \mathcal{A}\).

### 2.2 Problem Formulation

Consider a system consisting of \(N\) agents and a leader. All \(N\) agents are assumed to be linear dynamical agents, coupled with their neighbors via, in general nonidentical, linear uncertain coupling. The connection between \(N\) agents is described by a directed graph \(G_1\), with the node
set $\mathcal{V} = \{1, \ldots, N\}$, an edge set $\mathcal{E}_1$ and a corresponding adjacency matrix $A_1$. The dynamics of the $i$th agent are described as

$$
\dot{x}_i = Ax_i + B_1 u_i + B_2 \sum_{j \in S_i} \varphi_{ij}(t, x_j(\cdot)|_0^t - x_i(\cdot)|_0^t),
$$

where the summation is over the set $S_i$ of neighbors of node $i$ in the graph $G_1$. The notation $\varphi_{ij}(t, y(\cdot)|_0^t)$ describes a linear uncertain operator mapping functions $y(s), 0 \leq s \leq t$ into $\mathbb{R}^m$. Also, $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^p$ is the control input. We note that the last term in (1) reflects a relative, time varying nature of interactions between agents.

Let $L_{2e}[0, \infty)$ be the space of functions $y(\cdot) : [0, \infty) \to \mathbb{R}^n$ such that $\int_0^T ||y(t)||^2 dt < \infty, \forall T > 0$.

**Assumption 1** All the mappings $\varphi_{ij}(\cdot)$ satisfy the following assumptions:

1. $\forall y \in L_{2e}^n[0, \infty), \varphi_{ij}(\cdot, y(\cdot)|_0^t) \in L_{2e}^m[0, \infty)$.
2. $\forall t > 0, \varphi_{ij}(t, y)$ is linear in the second argument, i.e., if $y = \tau_1 y_1 + \tau_2 y_2$, then $\varphi_{ij}(t, y(\cdot)|_0^t) = \tau_1 \varphi_{ij}(t, y_1(\cdot)|_0^t) + \tau_2 \varphi_{ij}(t, y_2(\cdot)|_0^t)$.
3. Given a matrix $C_{ij} \in \mathbb{R}^{m \times n}$, there exists a sequence $\{t_l\}, t_l \to \infty$, such that for every $t_l$, the following IQC holds

$$
\int_0^{t_l} ||\varphi_{ij}(t, y(\cdot)|_0^t)||^2 dt \leq \int_0^{t_l} ||C_{ij} y||^2 dt,
$$

$$
\forall y \in L_{2e}^m[0, \infty).
$$

The sequence $\{t_l\}$ is assumed to be the same for all $\varphi_{ij}$. The class of operators that satisfy these assumptions will be denoted by $\Xi$. We note that matrices $C_{ij}$ are assumed to be fixed.

In addition to the system (1), suppose a leader is given. The dynamics of the leader, labeled 0, is expressed as

$$
\dot{x}_0 = Ax_0,
$$

where $x_0 \in \mathbb{R}^n$ is its state. The control communication topology between $N$ agents is described by a directed graph $G_2$, with the same node set $\mathcal{V} = \{1, \ldots, N\}$, but possibly different edge set $\mathcal{E}_2$ and a corresponding adjacency matrix $A_2$. The Laplacian matrix of the graph $G_2$ is denoted as $L_2$. We assume throughout the paper that the leader node can be observed from a subset of nodes of the graph $G_2$. If the leader is observed by the node $i$, we extend the graph $G_2$ by adding the edge $(0, i)$ with weighting gain $g_i = 1$, otherwise let $g_i = 0$. We refer to node $i$ with $g_i \neq 0$ as a pinned or controlled node. The diagonal matrix $G = \text{diag}\{g_i\} \in \mathbb{R}^{N \times N}$ is commonly referred to as the pinning matrix. The system is assumed to have at least one agent connected to the leader, hence $G \neq 0$.

Define error vectors as $e_i = x_0 - x_i, i = 1, 2, \ldots, N$. Then dynamics of the synchronization errors satisfy the equation

$$
\dot{e}_i = Ae_i - B_1 u_i - B_2 \sum_{j \in S_i} \varphi_{ij}(t, e_i(\cdot)|_0^t - e_j(\cdot)|_0^t).
$$

In this paper we are concerned with finding a control protocol for each node $i$ of the form

$$
u_i = -K \{ \sum_{j \in T_i} (x_j - x_i) + g_i(x_0 - x_i) \},
$$

where $K$ is the feedback gain matrix to be found, and $T_i$ is the set of neighbors of node $i$ in the graph $G_2$. As a measure of system performance, we will use the quadratic cost function,

$$
J(u) = \sum_{i=1}^N \int_0^\infty \left( e_i^T Q e_i + u_i^T R u_i \right) dt,
$$

where $Q = Q' > 0$ and $R = R' > 0$ are given weighting matrices.
Remark 1 In [14] we considered a different cost function,

$$\mathcal{J}'(u) = \sum_{i=1}^{N} \int_{0}^{\infty} \left( \frac{1}{2} \sum_{j \in \mathcal{N}_i} (e_i - e_j)^T Q (e_i - e_j) + g_{ij} (e_i - e_j) + u_i (R e_i) dt. \right.$$  

Each addend in this cost function penalizes the $i$th system input, the disagreement between the $i$th and the $j$th system states, where $j$ is a neighbor of $i$, as well as the tracking errors between the leader and the pinned agents which observe the leader. In contrast, the cost function in this paper describes the cost on the tracking error between the leader and all of the followers and system input. 

Taking linearity of the operator $\varphi_{ij}$ into account, the synchronization error dynamic (4) can be represented as

$$\dot{e}_i = A e_i - B_1 u_i - B_2 \sum_{j \in \mathcal{S}_i} (\varphi_{ij}(t, e_i(t)|_0^t) - \varphi_{ij}(t, e_j(t)|_0^t)).$$  (7)

The problem in this paper is to find a control protocol which solves the leader following consensus control problem as follows:

**Problem 1** Under Assumption 1 find a control protocol of the form (5) such that

$$\sup_{\Xi_0} \mathcal{J}(u) < \infty.$$  (8)

Here $\sup$ means that the supremum is taken over the set of all operators $\varphi_{ij}$ that belong to the class $\Xi_0$ of operators. Since $Q > 0$, then (8) implies

$$\int_{0}^{\infty} \|e_i\|^2 dt < \infty \quad \forall i = 1, \ldots, N.$$  (9)

Hence, solving Problem 1 implies synchronization of all agents to the leader in the $L_2$ sense.

3 The Main Result

In this section, the main result of this paper is presented which is a sufficient condition for the system (1) to be able to track the leader with guaranteed tracking performance.

First we present the following result of [15] and some notation.

**Assumption 2** The digraph $\mathcal{G}_2$ contains a spanning tree and the root node $i_r$, obtains information from the leader node, i.e., $g_{ir} > 0$.

**Lemma 1** (15) Under Assumption 2 $L_2 + G$ is nonsingular. Define $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_N = (L_2 + G)^{-1} 1_N, \Theta = \text{diag}(\tilde{\vartheta}_i^{-1})$ and $H = \Theta (L_2 + G) + (L_2 + G)' \Theta$, then $\Theta > 0$ and $H > 0$.

Let $\sigma'$ be the maximum eigenvalue of $H$ and $\sigma = \frac{1}{2} \sigma'$. Also let $M = (L_2 + G) H (L_2 + G)$. According to Lemma 1 $M$ is a positive definite and symmetric matrix. Let $T \in \mathbb{R}^{N \times N}$ be an orthogonal matrix such that

$$T^{-1} M T = J = \text{diag} [\lambda_1, \ldots, \lambda_N].$$  (10)

and denote $\bar{\lambda} = \max(\lambda_i)$. For node $i$ of the graph $\mathcal{G}_2$, introduce matrices $\hat{C}_i = [C_{ij_1} \ldots C_{ij_{d_i}}]', \bar{C}_i = [C_{r_{i1}} \ldots C_{r_{i q_i}}]',$ where $j_1, \ldots, j_{d_i}$ are the elements of the neighbourhood set $\mathcal{S}_i$, and $r_{i1}, \ldots, r_{iq_i}$ are the nodes with the property $(r_{i1}, i) \in E_i; \bar{d}_i$ and $q_i$ are the in-degree and the out-degree of node $i$, respectively, in the graph $\mathcal{G}_1$. Also, introduce the matrix $\bar{R} = (\bar{\lambda}/\sigma) R$. 

**Theorem 1** Let a matrix $Y = Y' > 0, Y \in \mathbb{R}^{n \times n}$, and constants $\nu_{ij} > 0, \mu_{ij} > 0, j \in S_i, i = 1, \ldots, N$, exist such that the following LMIs are satisfied simultaneously

$$
\begin{bmatrix}
Z_i & YQ^{1/2} & 0 & 0 \\
0 & -\frac{1}{\sigma_i}I & 0 & 0 \\
\hat{C}_iY & 0 & -\Phi_i & 0 \\
\hat{C}_iY & 0 & 0 & -\Omega_i
\end{bmatrix} < 0,
$$

where

$$
Z_i = AY + Y A' - \sigma \theta_i B_1 \hat{R}^{-1} B_1' + \sum_{j \in S_i} \left( \frac{1}{\nu_{ij}} + \frac{1}{\mu_{ij}} \right) B_2 B_2',
$$

$$
\Phi_i = \text{diag} \left[ \frac{\theta_i}{\mu_{ij}}, j \in S_i \right],
$$

$$
\Omega_i = \text{diag} \left[ \frac{\theta_i}{\nu_{ij}}, j : i \in S_j \right].
$$

Then the control protocol with $K = -(\sigma/\lambda)R^{-1}B_1'Y^{-1}$ solves Problem 1. Furthermore, this protocol guarantees the following performance bound

$$
\sup_{\xi_0} J(u) \leq \sum_{i=1}^N \varphi_i^{-1} \epsilon_i'(0)Y^{-1}e_i(0).
$$

**Proof:** Using the Schur complement, the LMIs can be transformed into the following Riccati inequality

$$
AY + Y A' - \sigma \theta_i B_1 \hat{R}^{-1} B_1' + \sum_{j \in S_i} \left( \frac{1}{\nu_{ij}} + \frac{1}{\mu_{ij}} \right) B_2 B_2' + Y(\theta_i Q + \theta_i^{-1}(\sum_{j \in S_i} \nu_{ij} C_i C_i) + \sum_{j : i \in S_j} \mu_{ji} C_j C_j)Y < 0.
$$

After pre- and post-multiplying by $Y^{-1}$ and multiplying by $\varphi_i^{-1}$, then substituting $K = -(\sigma/\lambda)R^{-1}B_1'Y^{-1}$ into the Riccati inequality, we obtain

$$
Y^{-1}(\varphi_i^{-1} A + \sigma B_1 K) + (\varphi_i^{-1} A + \sigma B_1 K)'Y^{-1} + \sigma K' \hat{R} K + \sum_{j \in S_i} \left( \frac{1}{\nu_{ij}} + \frac{1}{\mu_{ij}} \right) Y^{-1} B_2 B_2' Y^{-1} + Q + \varphi_i^{-2}(\sum_{j \in S_i} \nu_{ij} C_i C_i + \sum_{j : i \in S_j} \mu_{ji} C_j C_j) < 0.
$$

Define $e = [e_1', \ldots, e_N']$ and consider the following Lyapunov function candidate for the subsystems:

$$
V(e) = \sum_{i=1}^N \varphi_i^{-1} \epsilon_i'Y^{-1}e_i.
$$
Then
\[
\frac{dV(e)}{dt} = \sum_{i=1}^{N} 2e_i^t Y^{-1} \left( \partial_i^{-1} A e_i + \partial_i^{-1} B_1 K \left( \sum_{j \in T_i} (e_i - e_j) + g_i e_i \right) \right) \]
\[
- 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_i(.)|_0^t) \]
\[
+ 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_j(.)|_0^t) \]
\[
= 2 \sum_{i=1}^{N} 2e_i^t \partial_i^{-1} Y^{-1} B_1 K \left( \sum_{j \in T_i} (e_i - e_j) + g_i e_i \right) \]
\[
\leq 2 \sigma y (I_N \otimes I_p) y = 2 \sigma e'(I_N \otimes Y^{-1} B_1 \hat{R}^{-1} B_1' Y^{-1}) e \]
\[
\leq 2 \sum_{i=1}^{N} \sigma e_i^t Y^{-1} B_1 \hat{R}^{-1} B_1' Y^{-1} e_i, \tag{17}
\]

where \( y = (I_N \otimes \hat{R}^{-1/2} B_1' Y^{-1}) e \).

From (16) and (17), one has
\[
\frac{dV(e)}{dt} \leq \sum_{i=1}^{N} 2e_i^t \left( \sigma K' \hat{R} K + Q \right) e_i \]
\[
- 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_i(.)|_0^t) \]
\[
+ 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_j(.)|_0^t) \]
\[
= - \sum_{i=1}^{N} e_i^t \left( \sigma K' \hat{R} K + Q \right) e_i \]
\[
+ \sum_{j \in S_i} \left( \frac{1}{\mu_{ij}} + \frac{1}{\mu_{ij}} \right) Y^{-1} B_2 B_2' Y^{-1} \]
\[
+ \partial_i^{-2} \left( \sum_{j \in S_i} \mu_{ij} C_{ij}' C_{ij} + \sum_{j, j \in S_i} \mu_{ij} C_{ij}' C_{ji} \right) e_i \]
\[
- 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_i(.)|_0^t) \]
\[
+ 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_j(.)|_0^t). \tag{18}
\]

Substituting the Riccati inequality (14) into (18), we have
\[
\frac{dV(e)}{dt} \leq - \sum_{i=1}^{N} e_i^t \left( \sigma K' \hat{R} K + Q \right) e_i \]
\[
+ \sum_{j \in S_i} \left( \frac{1}{\mu_{ij}} + \frac{1}{\mu_{ij}} \right) Y^{-1} B_2 B_2' Y^{-1} \]
\[
+ \partial_i^{-2} \left( \sum_{j \in S_i} \mu_{ij} C_{ij}' C_{ij} + \sum_{j, j \in S_i} \mu_{ij} C_{ij}' C_{ji} \right) e_i \]
\[
- 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_i(.)|_0^t) \]
\[
+ 2 \sum_{i=1}^{N} \partial_i^{-1} \sum_{j \in S_i} e_i^t Y^{-1} B_2 \varphi_{ij}(t, e_j(.)|_0^t). \tag{19}
\]
Using the following identity,
\[ \sum_{i=1}^{N} \sum_{j \in S_i} \mu_{ij} e_i^j C_{ij} e_j = \sum_{i=1}^{N} \sum_{j \in S_j} \mu_{ji} e_j^i C_{ji} e_i, \]
one has
\[ \frac{dV(e)}{dt} \leq - \sum_{i=1}^{N} e_i' \left( \sigma K' \hat{R} K + Q \right) e_i \]
\[ - \sum_{i=1}^{N} \sum_{j \in S_i} \left\| \frac{1}{\sqrt{B_{ij}}} B_{ij}^2 Y^{-1} e_i + \sqrt{\nu_{ij}} \partial_i^{-1} \varphi_{ij} (t, e_i(\cdot)_{0}) \right\|^2 \]
\[ + \sum_{i=1}^{N} \sum_{j \in S_i} \partial_i^{-2} \nu_{ij} (\| \varphi_{ij} (t, e_i(\cdot)_{0}) \|^2 - \| C_{ij} e_i \|^2) \]
\[ - \sum_{i=1}^{N} \sum_{j \in S_i} \left\| \frac{1}{\sqrt{B_{ij}}} B_{ij}^2 Y^{-1} e_i - \sqrt{\nu_{ij}} \partial_i^{-1} \varphi_{ij} (t, e_j(\cdot)_{0}) \right\|^2 \]
\[ + \sum_{i=1}^{N} \sum_{j \in S_i} \partial_i^{-2} \mu_{ij} (\| \varphi_{ij} (t, e_j(\cdot)_{0}) \|^2 - \| C_{ij} e_j \|^2). \] (20)

According to the IQC condition (2), we have
\[ \int_{0}^{t_1} \frac{dV(e)}{dt} dt \leq - \sum_{i=1}^{N} \int_{0}^{t_1} e_i' \left( \sigma K' \hat{R} K + Q \right) e_i dt. \] (21)

Since \( V(e(t_1)) \geq 0 \), then (21) implies
\[ \sum_{i=1}^{N} \int_{0}^{t_1} e_i' \left( \sigma K' \hat{R} K + Q \right) e_i dt \leq V(e(0)). \] (22)

The expression on the right hand side of the above inequality is independent of \( t_1 \). Letting \( t_1 \to \infty \) leads to
\[ \sum_{i=1}^{N} \int_{0}^{\infty} e_i' \left( \sigma K' \hat{R} K + Q \right) e_i dt \leq V(e(0)). \] (23)

Using (6) and (5), we have
\[ J(u) = \sum_{i=1}^{N} \int_{0}^{\infty} \left( e_i' Q e_i + u_i' R u_i \right) dt \]
\[ = \int_{0}^{\infty} \left( e' (I_N \otimes Q) e \right) dt 
+ e' [(\mathcal{L}_2 + G)' (\mathcal{L}_2 + G) \otimes K' K] e \right) dt 
\leq \int_{0}^{\infty} \left( e' (I_N \otimes Q) e + e' [I_N \otimes \bar{\lambda} K' K] e \right) dt 
= \sum_{i=1}^{N} \int_{0}^{\infty} e_i' \left( \bar{\lambda} K' K + Q \right) e_i dt. \] (24)
Since $\hat{R} = \frac{\lambda}{\sigma} R$, then we obtain

$$J(u) \leq \sum_{i=1}^{N} \int_{0}^{\infty} e_i' \left( \sigma K' \hat{R} + Q \right) e_i dt \leq \sum_{i=1}^{N} \vartheta_i^{-1} e_i'(0) Y^{-1} e_i(0).$$

(25)

It implies that the control protocol (25) with $K = -(\sigma/\hat{\lambda})R^{-1}B'_1Y^{-1}$ solves Problem 1 and also guarantees the performance bound (12).

□

4 The Computational Algorithm

In this section, we provide an algorithm to calculate a suboptimal control gain $K$. According to Theorem 1, the upper bound on tracking performance is given by the right hand side of (12). Hence, one can achieve a suboptimal guaranteed performance by optimizing this upper bound over the feasibility set of the LMIs (11):

$$J^*(11) = \inf \sum_{i=1}^{N} \vartheta_i^{-1} e_i'(0) Y^{-1} e_i(0),$$

(26)

where the infimum is taken over the feasibility set of the LMIs (11), $\{Y, \nu_{ij}, \mu_{ij}, i = 1, \ldots, N, j \in S_i: (11) \text{ holds}\}$.

As in [12], the optimization problem (26) can be shown to be equivalent to the minimization of $\gamma$ subject to the constraints

$$\gamma > \sum_{i=1}^{N} \vartheta_i^{-1} e_i'(0) Y^{-1} e_i(0), \quad i = 1, \ldots, N.$$

(27)

By the Schur complement, (27) is equivalent to the LMI

$$\begin{bmatrix} \gamma & e'(0) \\ e(0) & \Upsilon \end{bmatrix} > 0, \quad i = 1, \ldots, N,$$

(28)

where

$$e(0) = [e_1(0)' \ e_2(0)' \ \cdots \ e_N(0)'],$$

$$\Upsilon = \text{diag}[\vartheta_i Y, i = 1, 2, \ldots, N].$$

This leads us to introduce the following optimization problem in the variables $\gamma, Y, \nu_{ij}$ and $\mu_{ij}$:

Find

$$J^*_{(11), (28)} \triangleq \inf \gamma,$$

(29)

where the infimum is with respect to $\gamma, Y, \frac{1}{\nu_{ij}}$ and $\frac{1}{\mu_{ij}}$ subject to (11) and (28).

We conclude this discussion by stating equivalence between the optimization problems (26) and (29).

**Theorem 2** $J^*_{(11)} = J^*_{(11), (28)}$.

**Proof:** The proof of this theorem is similar to the proof of Theorem 15 in [12].

□

Based on the foregoing discussion, we propose an algorithm for the design of the suboptimal protocol (5) based on Theorems 1 and 2.
where $l$ is the gravitational acceleration constant, $e_i$ is the leader and agents. In practice, however, the initial state of the leader may not be known. It is uncertain, that is $0 < a(t) \leq l$, $0 < b(t) \leq l$.

In addition to the three pendulums, consider the leader pendulum which is identical to those given. Its dynamics are described by the equation

\[ m l^2 \ddot{\alpha}_0 = -m g l \alpha_0. \]  

Choosing the state vectors as $x_0 = (\alpha_0, \dot{\alpha}_0)$, $x_1 = (\alpha_1, \dot{\alpha}_1)$, $x_2 = (\alpha_2, \dot{\alpha}_2)$ and $x_3 = (\alpha_3, \dot{\alpha}_3)$, equations (30) and (31) can be written in the form of (1), (3), where $A = \begin{bmatrix} 0 & 1 \\ -\frac{a}{l} & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -\frac{b}{l} \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ \frac{g}{l} \end{bmatrix}$ and $\varphi_{12}(t, x_2 - x_1) = \frac{a^2(t)}{l^2} [k_{11} k_{12}] (x_2 - x_1)$, $\varphi_{21}(t, x_1 - x_2) = \frac{a^2(t)}{l^2} [k_{11} k_{12}] (x_1 - x_2)$, $\varphi_{13}(t, x_2 - x_3) = \frac{b^2(t)}{l^2} [k_{11} k_{12}] (x_2 - x_3)$, $\varphi_{23}(t, x_3 - x_2) = \frac{b^2(t)}{l^2} [k_{11} k_{12}] (x_3 - x_2)$.

The agents in this example are coupled according to the undirected graph shown in Fig. 2. According to this graph, only agent 1 observes the leader. The Laplacian matrix of the graph $G_2$ consisting of nodes 1, 2 and 3 and the pinning matrix $G$ are

\[ \mathcal{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]
To illustrate the design based on Theorems 1 and 2, the LMI problem in Theorem 2 was solved numerically, and then the trajectories of the coupled pendulum system with the obtained protocol were simulated. To this end, the parameters of the coupled pendulum system were chosen to be $m = 0.25\, \text{kg}$, $l = 1\, \text{m}$, $g = 10\, \text{m/s}^2$, $k_{11} = 2\, \text{N/m}$, $k_{12} = 1\, \text{N/(m/s)}$, $k_{21} = 2\, \text{N/m}$, $k_{22} = 2\, \text{N/(m/s)}$, $a = 0.5\sin(0.2t)$, $b = 0.8\cos(0.1t)$. In the cost function, we let $Q = I$ and $R = 0.1$. Using the computational algorithm based on Theorem 2, the problem (26) was found to be feasible and yielded the gain matrix $K = [4.5206, 4.2657]$. The performance bound was minimized by $\gamma = 2.3532$, with parameters $\nu_{12} = 0.3268$, $\nu_{21} = 0.6009$, $\nu_{23} = 0.3007$, $\nu_{32} = 0.1433$, $\mu_{12} = 0.1451$, $\mu_{21} = 1.4300$, $\mu_{23} = 0.1066$, $\mu_{32} = 0.4016$. The simulation results for this protocol are shown in Fig. 4. Also, using the controller obtained by means of the computational algorithm proposed in Theorem 2, we directly computed the performance cost (6) for the system to be $J(u) = 2.3374$, while the theoretically predicted bound is $J^*(11), (28) = 2.3532$.

6 Conclusions

The consensus control for leader-tracking problem with guaranteed tracking performance for nonidentical uncertain coupled linear systems connected over a directed graph has been discussed in this paper. A sufficient condition was proposed by using the direct overbounding of the performance cost. According to the simulation results, the proposed computational algorithm based on Theorems 1, 2, which solve N coupled LMIs, guarantees a suboptimal performance.

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Figure 4: Relative angles (the top figure) and relative velocities of the pendulums with respect to the leader, obtained using the algorithm based on Theorems 1 and 2.
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