SMOLUCHOWSKI-KRAMERS APPROXIMATION FOR THE SINGULAR STOCHASTIC WAVE EQUATIONS IN TWO DIMENSIONS

YOUNES ZINE

Abstract. We study a family of nonlinear damped wave equations indexed by a parameter $\varepsilon > 0$ and forced by a space-time white noise on the two dimensional torus, with polynomial and sine nonlinearities. We show that as $\varepsilon \to 0$, the solutions to these equations converge to the solution of the corresponding two dimensional stochastic quantization equation. In the sine nonlinearity case, the convergence is proven over arbitrary large times, while in the polynomial case, we prove that this approximation result holds over arbitrary large times when the parameter $\varepsilon$ goes to zero even with a lack of suitable global well-posedness theory for the corresponding wave equations.

Contents

1. Introduction
   1.1. Smoluchovski-Kramers approximation
   1.2. Results and outline of the approach
   1.3. Further remarks
2. Notations and preliminary lemmas
   2.1. Notations
   2.2. Preliminary results from stochastic analysis
   2.3. Deterministic estimates
3. Convergence of the deterministic objects
4. Polynomial model
   4.1. On the stochastic convolution
   4.2. Local theory
   4.3. Asymptotic large times well-posedness
5. Sine-Gordon model
   5.1. Stochastic objects
   5.2. Well-posedness
References

1. Introduction

1.1. Smoluchovski-Kramers approximation. For $\varepsilon > 0$, we look at the following stochastic damped nonlinear wave equation (SDNLW) on $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,
\[
\begin{align*}
\varepsilon^2 \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1 - \Delta)u_\varepsilon + N(u_\varepsilon) &= \Phi \xi, \quad (x, t) \in T^2 \times \mathbb{R}_+. \\
(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} &= (\phi_0, \phi_1),
\end{align*}
\] (1.1)

In the above, \( N \) is a nonlinearity, \( \Phi \) is a bounded operator on \( L^2(T^2) \) and \( \xi(x, t) \) denotes a real-valued (Gaussian) space-time white noise on \( T^3 \times \mathbb{R}_+ \) with the space-time covariance formally given by

\[
\mathbb{E}[\xi(x_1, t_1)\xi(x_2, t_2)] = \delta(x_1 - x_2)\delta(t_1 - t_2).
\]

It has been proven in various contexts that the solution \( u_\varepsilon \) to (1.1) converges to the solution \( u \) of the stochastic quantization equation as \( \varepsilon \to 0 \)

\[
\begin{align*}
\partial_t u + (1 - \Delta)u + N(u) &= \Phi \xi, \quad (x, t) \in T^2 \times \mathbb{R}_+. \\
\end{align*}
\] (1.2)

This type of convergence result is known as the Smoluchowski-Kramers approximation. This problem arises from physical and chemical applications in the finite dimensional case where it is important to replace the second order system (1.1) by the first order equation (1.2) which is easier to treat. In the infinite dimensional setting, various works have proven the Smoluchowski-Kramers approximation property on some bounded domains and for equations stochastically forced by some random noises \[4, 5, 6, 7, 8, 9, 10, 11, 13\]. Once the approximation is established other important questions arise: how do relevant properties for the second and first order systems compare? For instance, the evolutions of the invariant states \[5, 7, 8\] and the large deviation and exit problems \[10, 13\] have also been studied. However in dimension larger than two, the available results only considered random noises which were smooth in space. Indeed, although, in the polynomial case, the well-posedness theory for (1.2) has been known for some time since the work of Da Prato and Debussche \[14\], it is only until recently that well-posedness for (1.1) is well understood. See \[17, 18, 19, 30, 31, 32, 37\]. In the case of a sine nonlinearity, namely, for the so-called sine-Gordon model, the well-posedness theories for the wave and heat equations are also very recent \[21, 34, 35, 12\].

In this paper, we investigate the Smoluchowski-Kramers approximation problem for infinite dimensional systems stochastically forced by a white noise for the first time. We note that Fukuzumi, Hoshino and Inui studied independently in a recent paper \[16\] a similar problem: the non-relativistic limit problem for the (complex) stochastic nonlinear damped wave equation in two dimensions with the Gibbs measure initial data. See Remark \[1.14\] for some additional comments on the differences between our results and the ones in \[16\].

In the deterministic setting (i.e. with \( \Phi \equiv 0 \) in (1.1) and (1.2)) such problems have already been studied in \[22\] on euclidian domains. Note that on \( \mathbb{R}^d \) (\( d \geq 2 \)), the scaling \((x, t) \mapsto (\varepsilon^{-1}x, \varepsilon^{-2}t)\) correlates the asymptotics \( \varepsilon \to 0 \) and \( t \to \infty \) in (1.1) (with \( \Phi \equiv 0 \) and with \( 1 - \Delta \) replaced by \( -\Delta \)). In this regard, it has been observed, that solutions to nonlinear wave equations tend to get closer to solutions to corresponding linear heat equations over large times. See for instance \[20, 24, 27, 42\].

We now informally discuss why the Smoluchowski-Kramers approximation occurs. Consider the following homogeneous linear damped wave equation:

\[
\begin{align*}
\varepsilon^2 \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1 - \Delta)u_\varepsilon &= 0 \\
(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} &= (\phi_0, \phi_1).
\end{align*}
\] (1.3)
By taking the spatial Fourier transform, we have
\[ \varepsilon^2 \partial_t^2 \hat{u}_\varepsilon(n) + \partial_n \hat{u}_\varepsilon(n) + \langle n \rangle^2 \hat{u}_\varepsilon(n) = 0 \]  
for \( n \in \mathbb{Z}^2 \). The roots of the characteristic polynomial \( \varepsilon^2 \Lambda^2 + \Lambda + \langle n \rangle^2 = 0 \) are given by
\[ \Lambda_\varepsilon^\pm(n) = \frac{-1 \pm \sqrt{1 - 4 \langle n \rangle^2 \varepsilon^2}}{2 \varepsilon^2}. \]  
Note that we have \( \Lambda_\varepsilon^\pm(n) \in \mathbb{R} \) if and only if \( \langle n \rangle \leq (2\varepsilon)^{-1} \). In the low frequency regime \( \langle n \rangle \leq (2\varepsilon)^{-1} \), the solution to (1.4) with
\[ (\hat{u}_\varepsilon(n), \partial_t \hat{u}_\varepsilon(n))|_{t=0} = (\hat{\phi}_0(n), \hat{\phi}_1(n)) \]  
is given by
\[ \hat{u}_\varepsilon(n, t) = e^{-\frac{t}{2\varepsilon^2}} \cosh(\lambda_\varepsilon(n)t)\hat{\phi}_0(n) + e^{-\frac{t}{2\varepsilon^2}} \frac{\sinh(\lambda_\varepsilon(n)t)}{\lambda_\varepsilon(n)} \left( \frac{1}{2\varepsilon^2} \hat{\phi}_0(n) + \hat{\phi}_1(n) \right), \]  
where \( \lambda_\varepsilon(n) \) is defined by
\[ \lambda_\varepsilon(n) = \frac{\sqrt{1 - 4 \langle n \rangle^2 \varepsilon^2}}{2 \varepsilon^2}. \]

In the high frequency regime \( \langle n \rangle > (2\varepsilon)^{-1} \), the solution to (1.4) with initial data (1.6) is given by
\[ \hat{u}_\varepsilon(n, t) = e^{-\frac{t}{2\varepsilon^2}} \cos(\zeta_\varepsilon(n)t)\hat{\phi}_0(n) + e^{-\frac{t}{2\varepsilon^2}} \frac{\sin(\zeta_\varepsilon(n)t)}{\zeta_\varepsilon(n)} \left( \frac{1}{2\varepsilon^2} \hat{\phi}_0(n) + \hat{\phi}_1(n) \right), \]
where \( \zeta_\varepsilon(n) \) is defined by
\[ \zeta_\varepsilon(n) = \frac{\sqrt{4 \langle n \rangle^2 \varepsilon^2 - 1}}{2 \varepsilon^2}. \]

Let \( P_\varepsilon^{\text{low}} \) and \( P_\varepsilon^{\text{high}} \) be the sharp projections onto the (spatial) frequencies \( \{ n \in \mathbb{Z}^2 : \langle n \rangle \leq (2\varepsilon)^{-1} \} \) and \( \{ n \in \mathbb{Z}^2 : \langle n \rangle > (2\varepsilon)^{-1} \} \), respectively, defined by
\[ P_\varepsilon^{\text{low}} f = \mathcal{F}^{-1}(1_{\langle n \rangle \leq (2\varepsilon)^{-1}} \hat{f}(n)) \quad \text{and} \quad P_\varepsilon^{\text{high}} f = \mathcal{F}^{-1}(1_{\langle n \rangle > (2\varepsilon)^{-1}} \hat{f}(n)). \]
Then, define the operator \( S_\varepsilon(t) \) and \( D_\varepsilon(t) \) by setting
\[ S_\varepsilon(t) = \frac{\sinh(\lambda_\varepsilon(\nabla)t)}{\lambda_\varepsilon(\nabla)} P_\varepsilon^{\text{low}} + \frac{\sin(\zeta_\varepsilon(\nabla)t)}{\zeta_\varepsilon(\nabla)} P_\varepsilon^{\text{high}} \]  
and
\[ D_\varepsilon(t) = e^{-\frac{t}{2\varepsilon^2}} S_\varepsilon(t). \]

**Remark 1.1.** At this point, the operator \( D_\varepsilon \) is only defined for \( \varepsilon \in I := (0, \infty) \setminus \{ \frac{1}{2n} \} : n \in \mathbb{Z}^2 \}. However, we show in Lemma 3.4 that the map \( \varepsilon \in I \mapsto D_\varepsilon \) can be extended to \( (0, \infty) \). In what follows, since we are interested in the behaviour of the solutions to (1.1) near \( \varepsilon = 0 \), we will consider that all quantities are defined for \( \varepsilon \in [0, 1] \).
From (1.7) and (1.9), we see that the solution $u_\varepsilon$ to (1.3) is given by
\[ u_\varepsilon(t) = \partial_tD_\varepsilon(t)\phi_0 + D_\varepsilon(t)(\varepsilon^{-2}\phi_0 + \phi_1). \] (1.14)

Furthermore, by the Duhamel principle, the solution $u_\varepsilon$ to the following nonhomogeneous linear damped wave equation:
\[
\begin{cases}
\varepsilon^2\partial_t^2u_\varepsilon + \partial_tu_\varepsilon + (1 - \Delta)u_\varepsilon = F \\
(u_\varepsilon, \partial_tu_\varepsilon)|_{t=0} = (\phi_0, \phi_1).
\end{cases}
\] (1.15)
is given by
\[
u_\varepsilon(t) = \partial_tD_\varepsilon(t)\phi_0 + D_\varepsilon(t)(\varepsilon^{-2}\phi_0 + \phi_1) + \int_0^t \varepsilon^{-2}D_\varepsilon(t - t')F(t')dt'.
\] (1.16)

From (1.5) with a Taylor expansion, we have
\[
\Lambda^+_\varepsilon(n) = \frac{-2\langle n \rangle^2}{1 + \sqrt{1 - 4\langle n \rangle^2}\varepsilon^2} = -\langle n \rangle^2 + O(\langle n \rangle^4\varepsilon^2) \to -\langle n \rangle^2,
\]
\[
\Lambda^-_\varepsilon(n) = \frac{-2\langle n \rangle^2}{1 - \sqrt{1 - 4\langle n \rangle^2}\varepsilon^2} \to -\infty
\] (1.17)
in the regime $\langle n \rangle = o(\varepsilon^{-\frac{1}{2}})$ as $\varepsilon \to 0$, namely in the regime
\[
\sqrt{1 - 4\langle n \rangle^2\varepsilon^2} = 1 - 2\langle n \rangle^2\varepsilon^2 + O(\langle n \rangle^4\varepsilon^4)
\] (1.18)
as $\varepsilon \to 0$. Let $\chi$ be a smooth non-negative function such that $\chi \equiv 1$ on $\{x \in \mathbb{R} : |x| \leq 1\}$ and $\text{supp}(\chi) \subset \{x \in \mathbb{R} : |x| \leq 2\}$. Hence, if we denote by $P_N$ ($N \in \mathbb{R}$) the smooth projection onto (spatial) frequencies $\{n \in \mathbb{Z}^2 : \langle n \rangle \leq N\}$ defined by
\[
P_Nf := \mathcal{F}^{-1}(\chi_N(n)\hat{f}(n)),
\] (1.19)
with $\chi_N = \chi(\frac{\langle n \rangle}{N})$, we have, at a formal level:
\[
\varepsilon^{-2}D_\varepsilon(t)P_{\varepsilon^{-\frac{1}{2}+\theta}} = \frac{e^{\Lambda^+_\varepsilon(\nabla)t} - e^{\Lambda^-_\varepsilon(\nabla)t}}{\sqrt{1 - 4\langle n \rangle^2\varepsilon^2}} \left[ P_{\varepsilon^{-\frac{1}{2}+\theta}} \to P_0(t)P_{\varepsilon^{-\frac{1}{2}+\theta}}, \right.
\]
\[
\partial_tD_\varepsilon(t)P_{\varepsilon^{-\frac{1}{2}+\theta}} = \left( \left( 1 - \frac{1}{\sqrt{1 - 4\langle n \rangle^2\varepsilon^2}} \right) \frac{e^{\Lambda^+_\varepsilon(\nabla)t}}{2} + \left( 1 + \frac{1}{\sqrt{1 - 4\langle n \rangle^2\varepsilon^2}} \right) \frac{e^{\Lambda^-_\varepsilon(\nabla)t}}{2} \right) P_{\varepsilon^{-\frac{1}{2}+\theta}} \to 0
\] (1.20)
for any $0 < \theta \ll 1$, where
\[
P_0(t) = e^{(\Delta^0-1)t}.
\] (1.21)
See Lemma 3.6 for a rigorous justification of (1.20).

**Remark 1.2.** By looking more carefully into the kernels involved, one can prove that the convergence (1.20) occurs in the regime $\langle n \rangle = o(\varepsilon^{-1})$. Namely, the convergence in (1.20) is valid with $P_{\varepsilon^{-\frac{1}{2}+\theta}}$ replaced by $P_{\varepsilon^{-1+\theta}}$, see Lemma 3.6.
Remark 1.3. Note that (the proof of) Lemma 3.6 shows in particular that $P_{\varepsilon}^{-1+\theta}(\varepsilon^{-2}D_{\varepsilon} - P_0)$ and $P_{\varepsilon}^{-1+\theta}\partial_t D_{\varepsilon}$ - viewed as operators from $H^s(\mathbb{T}^2)$ to itself for any $s \in \mathbb{R}$ - both converge to zero as $\varepsilon \to 0$ only pointwisely in time. In order to get a uniform-in-time convergence, i.e. in $L^\infty([0,T];H^s(\mathbb{T}^2))$ for any $T > 0$ and $s \in \mathbb{R}$, one must work with the operator $P_{\varepsilon}^{-1+\theta}(\varepsilon^{-2}D_{\varepsilon} + \partial_t D_{\varepsilon})$ which enjoys some extra cancellation. See Lemma 3.6 and Corollary 3.7.

Therefore, we see that $u_\varepsilon$ in (1.16) formally converges to

$$u(t) = P_0(t)\phi_0 + \int_0^t P_0(t-t')F(t')dt',$$

satisfying the nonhomogeneous linear heat equation:

$$\begin{cases}
\partial_t u + (1-\Delta)u = F \\
u|_{t=0} = \phi_0.
\end{cases}$$

(1.23)

Thus, we expect that the solution to (1.1) converges to the solution of the stochastic quantization equation (1.2) as $\varepsilon \to 0$.

1.2. Results and outline of the approach. For any $\varepsilon > 0$, we introduce the stochastic convolutions by

$$\Psi_\varepsilon = \int_0^t \varepsilon^{-2}D_{\varepsilon}(t-t')dW(t'),$$

(1.24)

$$\Psi_0 = \int_0^t P_0(t-t')dW(t').$$

(1.25)

where $W$ denotes a cylindrical Wiener process on $L^2(\mathbb{T}^2)$, defined on some probability space $(\Omega, \mathbb{P})$:

$$W(t) := \sum_{n \in \mathbb{Z}^2} B_n(t)e_n$$

(1.26)

and $\{B_n\}_{n \in \mathbb{Z}^2}$ is defined by $B_n(t) = \langle \xi, 1_{[0,t]} \cdot e_n \rangle_{x,t}$. Here, $\langle \cdot, \cdot \rangle_{x,t}$ denotes the duality pairing on $\mathbb{T}^2 \times \mathbb{R}$. As a result, we see that $\{B_n\}_{n \in \mathbb{Z}^2}$ is a family of mutually independent complex-valued Brownian motions conditioned so that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^2$. By convention, we normalized $B_0$ such that $\text{Var}(B_n(t)) = t$.

In this paper we look at (1.1) and (1.2) in the (singular) white noise setting $\Phi \equiv \text{Id}$ with two types of nonlinearities: the polynomial case and the sine nonlinearity. Namely, $\mathcal{N}(u) = u^k$ for some integer $k \geq 2$ or $\mathcal{N}(u) = \gamma \sin(\beta u)$ for some $\gamma, \beta > 0$. The corresponding equations are (for $\varepsilon > 0$),

$$\begin{cases}
\varepsilon^2 \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1-\Delta)u_\varepsilon + u_\varepsilon^k = \xi, \\
(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (\phi_0, \phi_1),
\end{cases}$$

(1.27)

$$\begin{cases}
\varepsilon^2 \partial_t^2 w_\varepsilon + \partial_t w_\varepsilon + (1-\Delta)w_\varepsilon + \gamma \sin(\beta w_\varepsilon) = \xi, \\
w_\varepsilon(t, x) = (\phi_0, \phi_1),
\end{cases}$$

(1.28)

while their parabolic counterparts (i.e. for $\varepsilon = 0$) write

\footnote{In particular, $B_0$ is a standard real-valued Brownian motion.}
\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u + (1 - \Delta) u + u^k = \xi, & (x, t) \in T^2 \times \mathbb{R}_+ \\
u_{t=0} = \phi_0,
\end{array} \right. \\
(1.29)
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t w + (1 - \Delta) w + \gamma \sin(\beta w) = \xi & (t, x) \in \mathbb{R}_+ \times T^2. \\
w_{t=0} = \phi_0,
\end{array} \right. \\
(1.30)
\end{align*}
\]

Given \( \varepsilon \in [0, 1] \) and \( N \in \mathbb{N} \), we define the truncated stochastic convolution \( \Psi_{\varepsilon,N} = P_N \Psi_{\varepsilon} \), solving the truncated linear stochastic wave equation/heat equation (for \( \varepsilon = 0 \)):
\[
\varepsilon^2 \partial^2_t \Psi_{\varepsilon,N} + \partial_t \Psi_{\varepsilon,N} + (1 - \Delta) \Psi_{\varepsilon,N} = P_N \xi,
\]
with the zero initial data. Here, \( P_N \) is as in \((1.19)\).

Then, for each fixed \( \varepsilon \in (0, 1] \), \( x \in T^2 \) and \( t \geq 0 \), we see from \((1.10)\) that \( \Psi_{\varepsilon,N}(x,t) \) is a mean-zero real-valued Gaussian random variable with variance
\[
\sigma_{\varepsilon,N}(t) \overset{\text{def}}{=} \mathbb{E} [\xi_{\varepsilon,N}(x,t)]^2 = C_\varepsilon(t) + \sum_{n \in \mathbb{Z}^2 \atop \varepsilon^{-1} \leq \langle n \rangle \leq N} \int_0^t \left[ \frac{\sin(\zeta_\varepsilon(n)(t-t'))}{\zeta_\varepsilon(n)} \right]^2 dt'
\]
\[
(1.32)
\]
\[
\sim \log N
\]

for some constant \( C = C_\varepsilon(t) \) and \( N \gg \varepsilon^{-1} \) and \( \zeta_\varepsilon \) as in \((1.10)\). Note that the implicit constant in \((1.32)\) depends on \( \varepsilon \) and \( t \). See Lemma 5.3 for precise bounds on \( \sigma_{\varepsilon,N} \). We point out that the variance \( \sigma_{\varepsilon,N}(t) \) is time-dependent. For any \( t > 0 \), we see that \( \sigma_{\varepsilon,N}(t) \to \infty \) as \( N \to \infty \), which can be used to show that \( \{\Psi_{\varepsilon,N}(t)\}_{N \in \mathbb{N}} \) is almost surely unbounded in \( W^0_p(T^2) \) for any \( 1 \leq p \leq \infty \). Similar comments apply to \( \Psi_{0,N} \) and its variance \( \sigma_{0,N} \).

For notational convenience, we denote by \( u_0 \) and \( w_0 \) the solutions to \((1.29)\) and \((1.30)\) respectively and we view them as solutions to \((1.27)\) and \((1.28)\) with \( \varepsilon = 0 \). For \( N \in \mathbb{N} \) and \( \varepsilon \in [0, 1] \), let \( u_{\varepsilon,N} \) and \( w_{\varepsilon,N} \) denote the solution to \((1.27)\) and \((1.28)\) where the rough noise \( \xi \) is replace by the regularized noise \( P_N \xi \). Proceeding with the following decomposition of \( u_{\varepsilon,N} \) and \( w_{\varepsilon,N} \) \((25\ [13\ [14])\):
\[
\begin{align*}
u_{\varepsilon,N} &= v_{\varepsilon,N} + \Psi_{\varepsilon,N} \\
w_{\varepsilon,N} &= w_{\varepsilon,N} + \Psi_{\varepsilon,N}.
\end{align*}
\]
\[
(1.33)
\]

Then, we see that the residuals terms \( v_{\varepsilon,N} \) and \( z_{\varepsilon,N} \) satisfy
\[
\varepsilon^2 \partial^2_t v_{\varepsilon,N} + \partial_t v_{\varepsilon,N} + (1 - \Delta) v_{\varepsilon,N} + \sum_{\ell=0}^k \binom{k}{\ell} \Psi_{\varepsilon,N}^{k-\ell} = 0
\]
\[
(1.34)
\]
\[
\varepsilon^2 \partial^2_t z_{\varepsilon,N} + \partial_t z_{\varepsilon,N} + (1 - \Delta) z_{\varepsilon,N} + \gamma \Im \left( e^{i\beta z_{\varepsilon,N}} e^{i\Psi_{\varepsilon,N}} \right) = 0,
\]
\[
(1.35)
\]
Due to the deficiency of regularity, the power \( \Psi_{\varepsilon,N}^\ell \) does not converge to any limit as \( N \to \infty \). This is where we introduce the Wick renormalization. Namely, we replace \( \Psi_{\varepsilon,N}^\ell \) by its Wick ordered counterpart:
\[
:\Psi_{\varepsilon,N}^\ell(x,t) : = H_{\ell}(\Psi_{\varepsilon,N}(x,t); \sigma_{\varepsilon,N}(t))
\]
\[
(1.36)
\]
where \( H_{\ell}(x;\sigma) \) is the Hermite polynomial of degree \( \ell \) with variance parameter \( \sigma \). See Section 2. Then, for each \( \ell \in \mathbb{N} \), the Wick power \( \Psi_{\varepsilon,N}^\ell \) converges to a limit, denoted by \( \Psi_{\varepsilon}^\ell : \) in \( C([0,T]; W^{-\sigma,\infty}(T^2)) \) for any \( \sigma \geq 0 \), \( \sigma > 0 \), and \( T > 0 \), almost surely.
See Proposition 4.1 below. Similarly, since $\Psi_{\varepsilon,N}$ converges to a distribution of negative regularity, $e^{i\beta\Psi_{\varepsilon,N}}$ will experience some “averaging effect” and will converge to 0 in the sense of distribution. Hence, in order to see a non-trivial behavior in the limit $N \to \infty$, one needs to have $\gamma \to \infty$ in (1.35). More precisely, we define $\gamma_{\varepsilon,N} = \gamma_{\varepsilon,N}(\beta)$ by

$$\gamma_{\varepsilon,N} = e^{2\beta^2} \sigma_{\varepsilon,N},$$

and the so-called imaginary Gaussian multiplicative chaos $\Theta_{\varepsilon,N}$ by

$$\Theta_{\varepsilon,N}(t,x) := e^{i\beta\Psi_{\varepsilon,N}(t,x)} \overset{\text{def}}{=} \gamma_{\varepsilon,N} e^{i\beta\Psi_{\varepsilon,N}(t,x)} = e^{2\beta^2} \sigma_{\varepsilon,N} e^{i\beta\Psi_{\varepsilon,N}(t,x)},$$

where $\gamma_{\varepsilon,N}$ and $\sigma_{\varepsilon,N}$ are as in (1.37) and (1.32). Using stochastic analysis, one can then show that $\Theta_{\varepsilon,N}$ converges almost surely to a distribution $\Theta_{\varepsilon}$ in $C([0,T];W^{-\frac{2}{\sigma^2},-\infty}(\mathbb{T}^2))$, for any $\sigma, T > 0$.

**Remark 1.4.** We actually show that our stochastic objects $\Psi_{\varepsilon,N}$ and $\Theta_{\varepsilon}$ are continuous in $\varepsilon \in [0,1]$. For the latter this is achieved by tracking down the exact dependence in $\varepsilon \in [0,1]$ of the covariance functions $\{E[\Psi_{\varepsilon}(t,x)\Psi_{\varepsilon}(t,y)]\}_{\varepsilon \in [0.1]}$. See Subsection 5.1.

These renormalizations give rise to the renormalized versions of (1.34) and (1.35):

$$\varepsilon^2 \partial_t^2 v_{\varepsilon,N} + \partial_t v_{\varepsilon,N} + (1 - \Delta)v_{\varepsilon,N} + \sum_{\ell=0}^k \binom{k}{\ell} :\Psi_{\varepsilon,N}^\ell v_{\varepsilon,N}^{k-\ell} = 0$$

$$\varepsilon^2 \partial_t^2 z_{\varepsilon,N} + \partial_t z_{\varepsilon,N} + (1 - \Delta)z_{\varepsilon,N} + \text{Im} \{e^{i\beta z_{\varepsilon,N}} \Theta_{\varepsilon,N}\} = 0$$

for $\varepsilon \in [0,1]$. By taking a limit as $N \to \infty$, we then obtain the limiting equations:

$$\varepsilon^2 \partial_t^2 v_\varepsilon + \partial_t v_\varepsilon + (1 - \Delta)v_\varepsilon + \sum_{\ell=0}^k \binom{k}{\ell} :\Psi_\varepsilon^\ell v_\varepsilon^{k-\ell} = 0.$$  

$$\varepsilon^2 \partial_t^2 z_\varepsilon + \partial_t z_\varepsilon + (1 - \Delta)z_\varepsilon + \text{Im} \{e^{i\beta z_\varepsilon} \Theta_\varepsilon\} = 0.$$  

Given the almost-sure space-time regularity of the Wick powers $\Psi_\varepsilon^\ell$, $\ell = 1, \ldots, k$, and of the imaginary Gaussian multiplicative chaos $\Theta_\varepsilon$, standard deterministic analysis using the product estimates (Lemma 2.3) yields local well-posedness of (1.41) and (1.42). Recalling the decomposition (1.33), this argument also shows that the solution $u_{\varepsilon,N} = \Psi_{\varepsilon,N} + v_{\varepsilon,N}$ and $w_{\varepsilon,N} = \Psi_{\varepsilon,N} + z_{\varepsilon,N}$ to the renormalized equations with the regularized noise $P_N \xi$

$$\varepsilon^2 \partial_t^2 u_{\varepsilon,N} + \partial_t u_{\varepsilon,N} + (1 - \Delta)u_{\varepsilon,N} + :u_{\varepsilon,N}^k := P_N \xi$$

$$\varepsilon^2 \partial_t^2 w_{\varepsilon,N} + \partial_t w_{\varepsilon,N} + (1 - \Delta)w_{\varepsilon,N} + :\sin(\beta w_{\varepsilon,N}) := P_N \xi,$$

where the renormalized nonlinearities $:u_{\varepsilon,N}^k$ and $:\sin(\beta w_{\varepsilon,N})$ are interpreted as

$$:u_{\varepsilon,N}^k := (\Psi_{\varepsilon,N} + v_{\varepsilon,N})^k := \sum_{\ell=0}^k \binom{k}{\ell} :\Psi_{\varepsilon,N}^\ell v_{\varepsilon,N}^{k-\ell}$$

$$:\sin(\beta w_{\varepsilon,N}) := \gamma_{\varepsilon,N} \sin(\beta(\Psi_{\varepsilon,N} + z_{\varepsilon,N})) = \text{Im} \{e^{i\beta z_{\varepsilon,N}} \Theta_{\varepsilon,N}\}$$

converge almost surely to the stochastic processes $u_\varepsilon = \Psi_\varepsilon + v_\varepsilon$ and $w_\varepsilon = \Psi_\varepsilon + z_\varepsilon$, where $v_\varepsilon$ and $z_\varepsilon$ satisfy (1.41) and (1.42) respectively. It is in this sense that we say that the renormalized versions of (1.27) and (1.28):
\[ \varepsilon^2 \partial_t^2 u_{\varepsilon} + \partial_t u_{\varepsilon} + (1 - \Delta) u_{\varepsilon} + : u_{\varepsilon}^2 = \xi \]  
\[ \varepsilon^2 \partial_t^2 w_{\varepsilon} + \partial_t w_{\varepsilon} + (1 - \Delta) w_{\varepsilon} + \sin(\beta w_{\varepsilon}) = \xi, \]

for \( \varepsilon \in [0,1] \), are locally well-posed (for initial data of suitable regularity).

We can now state the main results of the paper. Firstly, we show the local existence and Schmoluchowski-Kramers approximation for polynomial nonlinearities. Let \( \mathcal{H}^s(T^2) := H^s(T^2) \times H^{s-1}(T^2) \) for any \( s \in \mathbb{R} \).

**Theorem 1.5.** Let \((\phi_0, \phi_1) \in \mathcal{H}^s(T^2)\) for \( s > \frac{2k-3}{2k-2} \). There exists a random time \( T = T(\omega) \) almost surely positive such that for each \( \varepsilon \in (0,1] \), there exists a solution \( u_{\varepsilon} \) to (1.45) with initial data \((\phi_0, \phi_1) \) and a solution \( u_0 \) to (1.45) (and \( \varepsilon = 0 \)) with initial data \( \phi_0 \) which belong to the class \( C([0,T]; H^{-\sigma}(T^2)) \) for any \( \sigma > 0 \).

Moreover, \( \{u_{\varepsilon}\}_{\varepsilon \in [0,1]} \) converges almost surely to the solution \( u_0 \) in \( C([0,T]; H^{-\sigma}(T^2)) \) for any \( \sigma > 0 \).

**Remark 1.6.** We emphasize here that the convergence result in Theorem 1.5 means that there exists a set \( \Omega_0 \subset \Omega \) (where \( (\Omega, \mathbb{P}) \) is the underlying probability space on which the noise \( \xi \) is defined) such that \( \mathbb{P}(\Omega_0) = 1 \) and

\[ \|u_{\varepsilon}^\omega - u_0^\omega\|_{C_T H^2} \to 0, \]

for any \( \omega \in \Omega_0 \) as \( \varepsilon \to 0 \) and with \( T \) and \( s \) as in Theorem 1.5. The subsequent convergence results are also proved in that fashion. It is achieved by showing that the stochastic objects are continuous in \((\varepsilon, t)\) by using a bi-parameter Kolmogorov continuity criterion, see [1] Theorem 2.1 and by solving a fixed point argument for \((\varepsilon, t) \mapsto v_{\varepsilon}(t) \) (1.33) in spaces of the form \( C([0,1] \times [0,T]; H^s(T^2)) \) of functions both continuous in \( \varepsilon \in [0,1] \) and in time \( t \in [0,T] \).

This is in contrast with the literature where such convergence results are obtained only up to a subsequence. See for instance [5, 16].

In the non-singular case with a polynomial nonlinearity (for instance [11] with a colored noise in space or in one space dimension) then the result of last theorem can be extended to arbitrary large time intervals since the solutions are known to be global. See for instance [4, 5]. In our singular setting however, the convergence of Theorem 1.5 cannot be established over longer times because of a lack of a global well-posedness theory for (1.45) and \( k > 3 \) (the solutions are known to be global in time for \( k = 3 \); see Remark 1.8 below). Since the solution \( u_0 \) to (1.45) is known to exist globally in time by an argument of Mourrat and Weber [26] (see also [11]), we can show asymptotic large time well-posedness for (1.45) (for \( \varepsilon > 0 \)). More precisely, since \( u_{\varepsilon} \) gets closer to \( u_0 \) as \( \varepsilon \to 0 \), we can extend the existence time and the convergence of our local solutions \( u_{\varepsilon} \) over longer times as \( \varepsilon \to 0 \). This is the purpose of the following theorem.

**Theorem 1.7.** Fix \( T > 0 \). Let \((\phi_0, \phi_1) \in \mathcal{H}^s(T^2)\) for \( s > \frac{2k-3}{2k-2} \). There exists \( \varepsilon_0 > 0 \) such that for each \( \varepsilon \in [0,\varepsilon_0] \) the solution \( u_{\varepsilon} \) to (1.45) and (1.45) (with \( \varepsilon = 0 \)) exists up to time \( T \). More precisely, for any \( \varepsilon \in [0,\varepsilon_0], u_{\varepsilon} \in C([0,T]; H^{-\sigma}(T^2)) \), for any \( \sigma > 0 \). Furthermore, \( \{u_{\varepsilon}\}_{\varepsilon \in (0,\varepsilon_0]} \) converges to \( u_0 \) in \( C([0,T]; H^{-\sigma}(T^2)) \) as \( \varepsilon \to 0 \).
Remark 1.8. Fix $s > \frac{4}{\nu}$. In [19], the authors proved that \[(1.45)\] for $k = 3$ and $\varepsilon = 1$ is globally well-posed in $H^s(T^2)$. By modifying their argument, one can show global well-posedness in $H^s(T^2)$ for $\varepsilon \in (0, 1)$. Since the stochastic quantization equation \[(1.27)\] for $\varepsilon = 0$ is globally well-posed in $H^s(T^2)$ as well; see [20] [41], the Smoluchovski-Kramers approximation proved in Theorem 1.5 holds globally in time, i.e. in $C([0, T]; H^{-\sigma}(T^2))$ (for any $\sigma > 0$) for any $T > 0$ in the cubic case $k = 3$.

We prove the following result for the sine-Gordon model:

**Theorem 1.9.** Let $0 < \beta^2 < 2\pi$ and $s > 1 - \frac{\beta^2}{4\pi}$. Let $(\phi_0, \phi_1) \in H^s(T^2)$. Fix $T > 0$. Then, for each $\varepsilon \in (0, 1]$, there exists a solution $w_\varepsilon$ to the stochastic sine-Gordon equation \[(1.46)\] with initial data given by $(\phi_0, \phi_1)$ and a solution $w_0$ to \[(1.46)\] (and $\varepsilon = 0$) with initial data given by $\phi_0$ which belong to the class $C([0, T]; H^{-\sigma}(T^2))$, for any $\sigma > 0$. Moreover, $\{w_\varepsilon\}_{\varepsilon \in [0, 1]}$ converges to $w_0$ in $C([0, T]; H^{-\sigma}(T^2))$ as $\varepsilon \to 0$.

**Remark 1.10.** Per usual when one uses the decomposition \[(1.33)\], the solutions constructed in Theorems 1.5, 1.7 and 1.9 are unique in classes of the form $\Psi_\varepsilon + C([0, T]; H^{1-\sigma}(T^2))$ for any $\varepsilon \in [0, \varepsilon_0]$ for some appropriate $\varepsilon_0 \in [0, 1]$, $T > 0$ and $\sigma > 0$.

### 1.3. Further remarks

We conclude this section with a few remarks which complement the last statements.

**Remark 1.11** (Gibbs measures). Fix $\varepsilon > 0$, $k \geq 2$ and $0 < \beta < 2\pi$. We will define invariant measures for (slight modifications of) the dynamics \[(1.35)\] and \[(1.46)\] in (i) and consider in (ii) the Smoluchovski-Kramers approximation problem for (slight modifications of) the dynamics \[(1.35)\] and \[(1.46)\] whose initial data is given by these Gibbs measures.

(i) We aim at defining the measures (formally) given by

\[
d\hat{\rho}_{\varepsilon, k}(u, \partial_t u) = \frac{1}{Z_{\varepsilon,k}} e^{-E_{\varepsilon,k}(u, \partial_t u)} dud(\partial_t u),
\]
\[
d\rho_{0,k}(u) = \frac{1}{Z_{0,k}} e^{-E_{0,k}(u, \partial_t u)} du,
\]

and

\[
d\hat{\rho}_{SG}(u, \partial_t u) = \frac{1}{Z_{SG}} e^{-R_{\varepsilon,SG}(u, \partial_t u)} dud(\partial_t u),
\]
\[
d\rho_{0,SG}(u) = \frac{1}{Z_{0,SG}} e^{-R_{0,SG}(u)} du.
\]

Here, $du$ and $d(\partial_t u)$ denote non-existent Lebesgue measures on $S(T^2)$. The quantities $Z_{\varepsilon, k}$, $Z_{0, k}$, $Z_{SG}$ and $Z_{0,SG}$ are normalizations constants and the energies $E_{\varepsilon, k}$, $E_{0, k}$, $R_{\varepsilon, \beta}$ and $R_{0, \beta}$ denote the Hamiltonians given by

\[E_{\varepsilon, k}(u, \partial_t u) = \frac{1}{2} \int_{T^2} (u(x))^2 + |\nabla u(x)|^2) dx + \frac{1}{2 \varepsilon^2} \int_{T^2} (\partial_t u(x))^2 dx + \frac{1}{k + 1} \int_{T^2} u(x)^{k+1} dx\]
\[E_{0, k}(u) = \frac{1}{2} \int_{T^2} (u(x))^2 + |\nabla u(x)|^2) dx + \frac{1}{k + 1} \int_{T^2} u(x)^{k+1} dx\]
\[R_{\varepsilon, \beta}(u, \partial_t u) = \frac{1}{2} \int_{T^2} (u(x))^2 + |\nabla u(x)|^2) dx + \frac{1}{\beta^2} \int_{T^2} (\partial_t u(x))^2 dx - \frac{\gamma}{\beta} \int_{T^2} \cos(\beta u(x)) dx\]
\[R_{0, \beta}(u) = \frac{1}{2} \int_{T^2} (u(x))^2 + |\nabla u(x)|^2) dx - \frac{\gamma}{\beta} \int_{T^2} \cos(\beta u(x)) dx.\]
Per usual, we define (renormalized versions of) the measures \( \tilde{\rho}_{\varepsilon,k}, \rho_{0,k}, \tilde{\rho}_{\varepsilon,SG} \) and \( \rho_{0,SG} \) as absolutely continuous measures with respect to some Gaussian free fields.

Let \( \varepsilon > 0 \) and \( \tilde{\mu}_\varepsilon \) be the following Gaussian measure:

\[
\tilde{\mu}_\varepsilon = \mu_0 \otimes \eta_\varepsilon, \tag{1.49}
\]

defined as the induced probability measure under the map:

\[
\omega \in \Omega \longmapsto (\phi_0(\omega), \frac{1}{\varepsilon} \phi_1(\omega)),
\]

where \( \phi_0 \) and \( \phi_1 \) are given by

\[
\phi_0(\omega) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e_n \quad \text{and} \quad \phi_1(\omega) = \sum_{n \in \mathbb{Z}^2} h_n(\omega)e_n. \tag{1.50}
\]

Here, \( e_n = (2\pi)^{-1} e^{in\cdot x} \) and \( \{g_n, h_n\}_{n \in \mathbb{Z}^2} \) denotes a family of independent standard complex-valued Gaussian random variables conditioned so that \( g_n = g_{-n} \) and \( h_n = h_{-n}, \ n \in \mathbb{Z}^2 \). It is easy to see that \( \tilde{\mu}_\varepsilon \) and \( \mu_0 \) are respectively supported on \( H^s(\mathbb{T}^2) \) and \( H^s(\mathbb{T}^2) \) for \( s < 0 \) but not for \( s \geq 0 \).

From the above, a typical element \( u \) living in the support of \( \tilde{\mu}_\varepsilon \) or \( \mu_0 \) has negative regularity and hence \( u^{k+1} \) and \( \cos(\beta u) \) do not make sense. Thus, we need to apply a renormalization procedure to make sense of the potential energies in the Hamiltonian given above. We renormalize/define the potential energies in the following fashion:

\[
\int_{\mathbb{T}^2} u(x)^{k+1} \ dx = \lim_{N \to \infty} \int_{\mathbb{T}^2} H_{k+1}(P_N u^{k+1}(x), \sigma_N) \ dx \tag{1.51}
\]

\[
\int_{\mathbb{T}^2} \cos(\beta u(x)) \ dx = \int_{\mathbb{T}^2} \cos(P_N u(x)) \ dx
\]

with \( P_N \) as in \( \text{(1.19)} \), \( \sigma_N = \sigma_{0,N} \) and \( \gamma_N = \gamma_{0,N} \) where \( \sigma_{0,N} \) and \( \gamma_{0,N} \) are as in \( \text{(1.32)} \) and \( \text{(1.37)} \) (for \( \varepsilon = 0 \)). Note that \( \sigma_N \) and \( \gamma_N \) verify

\[
\sigma_N = \mathbb{E}_{\mu_0} \left[ P_N(u)^2 \right]
\]

\[
\gamma_N = e^{\frac{\beta^2}{2} \sigma_N}, \tag{1.52}
\]

where \( \mathbb{E}_{\mu_0} \) denotes the expectation with respect to the law \( \mu_0 \). In \( \text{[19, 35]} \), the convergences in \( \text{(1.51)} \) are shown to hold in \( L^p(\mu_0) \) for any \( p \geq 1 \). Furthermore, it is also proved that the limits are exponentially integrable. More precisely, \( \exp \left( - \int_{\mathbb{T}^2} u(x)^{k+1} \ dx \right) \) and \( \exp \left( \int_{\mathbb{T}^2} \cos(\beta u(x)) \ dx \right) \) belong to \( L^p(\mu_0) \) for \( p \geq 1 \).

We can then define the (renormalized) measures \( \tilde{\rho}_{\varepsilon,k}, \rho_{0,k}, \tilde{\rho}_{\varepsilon,SG} \) and \( \rho_{0,SG} \) as

\[
\begin{align*}
\tilde{d}\tilde{\rho}_{\varepsilon,k} &= Z_{\varepsilon,k}^{-1} e^{-\int_{\mathbb{T}^2} u(x)^{k+1} \ dx} \ d\tilde{\mu}_\varepsilon \\
\tilde{d}\rho_{0,k} &= Z_{0,k}^{-1} e^{-\int_{\mathbb{T}^2} u(x)^{k+1} \ dx} \ d\mu_0 \\
\tilde{d}\tilde{\rho}_{\varepsilon,SG} &= Z_{\varepsilon,SG}^{-1} e^{\int_{\mathbb{T}^2} \cos(\beta u(x)) \ dx} \ d\tilde{\mu}_\varepsilon \\
\tilde{d}\rho_{0,SG} &= Z_{0,SG}^{-1} e^{\int_{\mathbb{T}^2} \cos(\beta u(x)) \ dx} \ d\mu_0
\end{align*}
\tag{1.53}
\]

(ii) By applying Bourgain’s invariant measure argument (see \( \text{[2, 3, 19, 35]} \)), we can show that the measures \( \tilde{\rho}_{\varepsilon,k}, \rho_{0,k}, \tilde{\rho}_{\varepsilon,SG} \) and \( \rho_{0,SG} \) are respectively invariant for the equations.
where: \( 1 \) : and :sin(\( \beta w_0 \)) ; denote the same renormalizations procedures leading to (1.45) and (1.46) (i.e. the factor \( \sqrt{2} \) in front of the noise is unimportant).

We now describe how to construct a set \( \Omega_g \subset H^{-\sigma}(\mathbb{T}^2) \) (for any \( \sigma > 0 \)) and such that \( \tilde{\rho}_{1,k}(\Omega_g) = 1 \) so that for each initial data \((\phi_0, \phi_1) \in \Omega_g \) and every \( T > 0 \), the solutions \( u_\varepsilon \) and \( w_\varepsilon \) to (1.54) and (1.55) with initial data \((\phi_0, \frac{1}{\varepsilon}\phi_1) \), and the solutions \( u_0 \) and \( w_0 \) to (1.56) and (1.57) with initial data \( \phi_0 \), belong to \( C([0,T];H^{-\sigma}(\mathbb{T}^2)) \) (for any \( \sigma > 0 \)) \( \mathbb{P} \)-almost surely. Furthermore, \( u_\varepsilon \) and \( w_\varepsilon \) converge to \( u_0 \) and \( w_0 \) in \( C([0,T];H^{-\sigma}(\mathbb{T}^2)) \) \( \mathbb{P} \)-almost surely for any \( T > 0 \) and \( \sigma > 0 \). This complements the result of Theorem (1.7) by giving a global approximation result for rough initial data of negative regularity.

We focus our attention to the polynomial nonlinearity case (i.e. the solutions \( u_\varepsilon \) and \( u_0 \) to (1.54) and (1.55)). We work at the level of the variables \( \{u_\varepsilon\}_{\varepsilon \in [0,1]} \) (formally) solving (1.45) for convenience but one can easily make this discussion rigorous by working with \( \{u_\varepsilon, N\}_{\varepsilon \in [0,1]} \) and going to the limit \( N \to \infty \) as in the proof of Theorem 1.5. In vectorial form, the equation (1.54) writes

\[
\partial_t \begin{pmatrix} u_\varepsilon \\ \varepsilon^2 \partial_{\varepsilon} u_\varepsilon \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 - \Delta \end{pmatrix} \begin{pmatrix} u_\varepsilon \\ \partial_{\varepsilon} u_\varepsilon \end{pmatrix} - \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \begin{pmatrix} u_\varepsilon \\ \partial_{\varepsilon} u_\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2\varepsilon} \end{pmatrix}
\]

Since the invariant measure for (1.58) depends on \( \varepsilon > 0 \), we use, as in [16], the following change of coordinates

\[
\Pi_\varepsilon(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} u \\ \partial_{\varepsilon} v \end{pmatrix}
\]

Let \((\tilde{u}_\varepsilon, \partial_{\varepsilon} \tilde{u}_\varepsilon) = \Pi_\varepsilon(u, \partial_{\varepsilon} u)\). Then \((\tilde{u}_\varepsilon, \partial_{\varepsilon} \tilde{u}_\varepsilon)\) satisfies the following equation:

\[
\partial_t \begin{pmatrix} \tilde{u}_\varepsilon \\ \varepsilon \partial_{\varepsilon} \tilde{u}_\varepsilon \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 - \Delta \end{pmatrix} \begin{pmatrix} \tilde{u}_\varepsilon \\ \partial_{\varepsilon} \tilde{u}_\varepsilon \end{pmatrix} - \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \begin{pmatrix} \tilde{u}_\varepsilon \\ \partial_{\varepsilon} \tilde{u}_\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2\varepsilon} \end{pmatrix}
\]

The push-forward measure \((\Pi_\varepsilon)_* \tilde{\rho}_{\varepsilon,k} = \tilde{\rho}_{1,k}\).

By Bourgain’s invariant measure argument, there exists, for each \( \varepsilon \in (0,1] \), a set \( \Omega_\varepsilon \) such that \( \tilde{\rho}_{1,k}(\Omega_\varepsilon) = 1 \) and \((\tilde{u}_\varepsilon, \partial_{\varepsilon} \tilde{u}_\varepsilon)\) exists globally on \( \Omega_\varepsilon \) \( \mathbb{P} \)-almost surely, i.e. on a set \( \Omega_1^\varepsilon \) with \( \mathbb{P}(\Omega_1^\varepsilon) = 1 \). The scalar form of (1.60) with initial data given by \( \tilde{\rho}_{1,k} \) reads

\[
\begin{cases}
\varepsilon^2 \partial_t^2 \tilde{u}_\varepsilon + \partial_t \tilde{u}_\varepsilon + (1 - \Delta) \tilde{u}_\varepsilon + : u_\varepsilon^k := \sqrt{2\varepsilon} \\
(\tilde{u}_\varepsilon, \partial_{\varepsilon} \tilde{u}_\varepsilon)_{t=0} = (\phi_0, \phi_1) \sim \tilde{\rho}_{1,k} 
\end{cases}
\]

Let \( \Omega_0^\varepsilon \) and \( \Omega_1^\varepsilon \) be the set of full probability such that \( \rho_{0,k}(\Omega_0^\varepsilon) = 1 \) and \( u_0 \) exists globally on \( \Omega_0^\varepsilon \). Note that if \( \Omega_0 := \Omega_0^\varepsilon \times H^{-2}(\mathbb{T}^2) \), then we have \( \tilde{\rho}_{1,k}(\Omega_0) = 1 \). We define the set \( \Omega_g \) by

\[
\Omega_g \times \Omega_g^1 = \bigcap_{q \in [0,1] \cap \mathbb{Q}} \Omega_q \times \Omega_q^1.
\]

Note that we have \( \tilde{\rho}_{1,k} \otimes \mathbb{P}(\Omega_g \times \Omega_g^1) = 1 \) by construction. By fixing \( \omega \in \Omega_g^1 \), we may omit the set \( \Omega_g^1 \) from our discussion in what follows for convenience.
Let $\Phi_\varepsilon = P_\varepsilon(t)(\phi_0, \phi_1) + \sqrt{2}\Psi_\varepsilon(t)$ with $P_\varepsilon$ and $\Psi_\varepsilon$ as in (3.2) and (1.24) and $(\phi_0, \phi_1) \sim \tilde{\mu}_{1,k}$. One shows, by a modification of Proposition 4.1 that $(\varepsilon, t) \mapsto \Phi_\varepsilon(t)$ belongs to $C([0, 1] \times [0, T]; W^{-\sigma, \infty}(\mathbb{T}^2))$ for any $T > 0$ and $\sigma > 0$ (maybe by restriction $\omega$ to a smaller set of full $\mathbb{P}$-probability). By using the first order ansatz $\widetilde{u}_\varepsilon = \Phi_\varepsilon + \varepsilon \tilde{\omega}$ and applying the results of Subsection 4.2 we can show that $(\varepsilon, t) \mapsto \widetilde{v}_\varepsilon(t)$ belongs to $C([0, 1] \times [0, T_0]; H^{1-\sigma}(\mathbb{T}^2))$ for some $0 < \sigma \ll 1$ and some (random) time $T_0 > 0$. Since $\tilde{v}_\varepsilon$ exists globally for $q \in [0, 1] \cap Q$ (again by Bourgain’s invariant measure argument) and $\varepsilon \in [0, 1]$ $\mapsto \tilde{v}_\varepsilon \in C([0, T_0]; H^{1-\sigma}(\mathbb{T}^2))$ is continuous, we may re-apply the local well-posedness theory to extend $\tilde{v}_\varepsilon$ globally in time for $\varepsilon \in (0, 1]$. We can then also show that $v_\varepsilon$ converges to $v_0$ in $C([0, T_0]; H^{-\sigma}(\mathbb{T}^2))$ as $\varepsilon \to 0$ for any $T > 0$ and $\sigma > 0$. See Remark 4.9 below.

Coming back to the original variables $(u_\varepsilon, \partial_t u_\varepsilon)$, this last discussion shows that for any $(\phi_0, \phi_1) \in \Omega_g$ the solution to
\begin{equation}
\begin{cases}
\varepsilon^2 \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + (1 - \Delta) u_\varepsilon + : u_k^\varepsilon := \sqrt{2} \xi \\
(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (\phi_0, \frac{1}{\varepsilon} \phi_1)
\end{cases}
(x, t) \in \mathbb{T}^2 \times \mathbb{R}_+,
\end{equation}
converges to the solution of
\begin{equation}
\begin{cases}
u_0 + \partial_t \nu + (1 - \Delta) \nu_0 + : \nu_k^0 := \sqrt{2} \xi \\
u_0|_{t=0} = \phi_0,
\end{cases}
(x, t) \in \mathbb{T}^2 \times \mathbb{R}_+,
\end{equation}
$\mathbb{P}$-almost surely in $C([0, T]; H^{-\sigma}(\mathbb{T}^2))$ ($\sigma > 0$) for any $T > 0$, as $\varepsilon \to 0$.

We highlight that the argument used above to extend simultaneously the solutions $u_\varepsilon$ and $u_0$ to (1.63) and (1.64) appears to be new. It uses two ingredients: (i) a probabilistic argument (Bourgain’s invariant measure argument) to extend globally in time the solutions to (1.63) and (1.64) for $\varepsilon \in [0, 1] \cap Q$ and (ii) deterministic analysis to extend the solutions to (1.63) with $\varepsilon \in (0, 1] \\setminus Q$.

**Remark 1.12.** Let $H = \mathcal{H}^1(\mathbb{T}^2)$ and $B = \mathcal{H}^s(\mathbb{T}^2)$ for $s < 0$ and $\tilde{\mu}_1$ be as in (1.49). Then it is known that $(B, H, \tilde{\mu}_1)$ forms an abstract Wiener space. See [33]. Fix $h = (\phi_0, \phi_1) \in H$. In view of the Cameron-Martin theorem, the measures $\tilde{\mu}_1^h(\cdot) := \tilde{\mu}_1(\cdot - h)$ and $\tilde{\mu}_1$ are mutually absolutely continuous with respect to each other. In particular, with the notations of Remark 1.11 and since the Gibbs measure $\tilde{\mu}_1$ and $\tilde{\mu}_1^h$ are both absolutely continuous with respect to $\tilde{\mu}_1$, we have that $\tilde{\omega}_\varepsilon$, the solution to the equation
\begin{equation}
\begin{cases}
\varepsilon^2 \partial_t^2 \tilde{u}_\varepsilon + \partial_t \tilde{u}_\varepsilon + (1 - \Delta) \tilde{u}_\varepsilon + : \tilde{u}_k^\varepsilon := \sqrt{2} \xi \\
(\tilde{u}_\varepsilon, \partial_t \tilde{u}_\varepsilon)|_{t=0} = (\phi_0, \phi_1) + \left( \sum_{n \in \mathbb{Z}^2} \frac{\varrho_n}{\varrho} e_n, \sum_{n \in \mathbb{Z}^2} h_n e_n \right)
\end{cases}
(x, t) \in \mathbb{T}^2 \times \mathbb{R}_+,
\end{equation}
exists globally and converges to $\tilde{u}_0$, the solution to
\begin{equation}
\begin{cases}
\partial_t \tilde{u}_0 + (1 - \Delta) \tilde{u}_0 + : \tilde{u}_k^0 := \sqrt{2} \xi \\
(\tilde{u}_0, \partial_t \tilde{u}_0)|_{t=0} = (\phi_0, \phi_1) + \left( \sum_{n \in \mathbb{Z}^2} \frac{\varrho_n}{\varrho} e_n, \sum_{n \in \mathbb{Z}^2} h_n e_n \right)
\end{cases}
(x, t) \in \mathbb{T}^2 \times \mathbb{R}_+,
\end{equation}
in $C([0, T]; H^{-\sigma}(\mathbb{T}^2))$ (for any $T, \sigma > 0$) and $\tilde{\mu}_1^h \otimes \mathbb{P}$-almost surely as $\varepsilon \to 0$.

Additionally, coming back to the variables $(u, \partial_t u)$ yields that the solution to
\begin{equation}
\begin{cases}
\varepsilon^2 \partial_t^2 u + \partial_t u + (1 - \Delta) u + : u_k := \sqrt{2} \xi \\
(u, \partial_t u)|_{t=0} = (\psi_0, \frac{1}{\varepsilon} \psi_1)
\end{cases}
(x, t) \in \mathbb{T}^2 \times \mathbb{R}_+,
\end{equation}
exists globally and converges to the solution to
\[
\begin{align*}
\begin{cases}
\partial_t u_0 + (1 - \Delta) u_\varepsilon + :u_\varepsilon^k := \sqrt{2} \xi, \\
(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = \psi_0
\end{cases}
\end{align*}
\]
$\mathbb{P}$-almost surely in $C([0, T]; H^{-\sigma}(T^2))$ (for any $T, \sigma > 0$), for any $(\psi_0, \psi_1) \in \Omega_0 - h$ with $\Omega_g$ as in (1.62). Note that $\bar{\mu}_1(\Omega_g - h) = 1$ by the Cameron-Martin theorem. Similar comments hold for the sine-Gordon model. This discussion extends the conclusions of Theorems 1.5, 1.7 and 1.9 in the following sense: by adding a Gaussian perturbation to some fixed deterministic initial data. In the polynomial case, we establish (ii) In [16], they considered Gibbs measure initial data. On the other hand, in Theorems 1.5, 1.7 and 1.9, we consider deterministic initial data. In the polynomial case, we establish in Theorem 1.7 a pathwise (asymptotic) global-in-time approximation despite the lack of

Remark 1.13. We discuss here the convergence of the invariant states for (1.1) as $\varepsilon \to 0$. Such questions have drawn some interest in the literature. See [3, 8]. Let $\pi_1: S'(T^2)^2 \to S'(T^2)$ be the projection to the first coordinate (i.e. $\pi_1(u, v) = u$). Then, with the notations of Remark 1.11, we have that for any $\varepsilon > 0$, the push-forward measure $(\pi_1)_* \bar{\rho}_{\varepsilon, 3}$ verifies $(\pi_1)_* \bar{\rho}_{\varepsilon, 3} = \rho_{0, 3}$. Hence, if $\bar{\rho}_n$ is the unique invariant state for (1.1) for some choice of nonlinearity $N$ and operator $\Phi$, we hope for the convergence of $(\pi_1)_* \bar{\rho}_n$ to $\rho_0$ (in some sense) where $\rho_0$ is the unique stationary distribution for (1.2). In [8], the authors that show such a convergence result takes place in one space dimension and with a polynomial nonlinearity and a colored noise (i.e. $\Phi = \langle \nabla \rangle^{-\alpha}$ for some $\alpha > 0$). It would thus be of interest to try to extend the result of [8] to the singular setting.

Remark 1.14. In [16], Fukuizumi, Hoshino and Inui studied a similar convergence problem. They considered the following complex-valued equation for $n \in \mathbb{N}$:
\[
\begin{align*}
\begin{cases}
\varepsilon^2 \partial_x^2 u_\varepsilon + 2\alpha \partial_t u_\varepsilon + (1 - \Delta) u_\varepsilon + :u_\varepsilon^{n+1} u_\varepsilon^n := 2 \sqrt{\text{Re}(\alpha)} \xi, \\
(u_\varepsilon, \varepsilon \partial_t u_\varepsilon)|_{t=0} = (\phi_0, \phi_1) \sim \bar{\rho}_n
\end{cases}
\end{align*}
\]
$\varepsilon > 0$ and $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\text{Im}(\alpha) \neq 0$ and where $\bar{\rho}_n$ is a complex-valued version of the measure $\bar{\mu}_{1, 2n+2}$ constructed in Remark 1.11 (see [36] for a construction of $\bar{\rho}_n$). The authors then looked at the limits $\varepsilon \to 0$ and $\text{Im}(\alpha) \to 0$, respectively known as the non-relativistic and ultra relativistic limits. As for the non-relativistic limit, the authors construct a set $\Omega_0$ with $\bar{\rho}_n(\Omega_0) = 1$ such that $\{u_{j-1}\}_{j \in \mathbb{N}}$, the family of solutions to (1.65) with $\varepsilon(j) = \frac{1}{j}$, and with initial data $(\phi_0, \phi_1) \in \Omega_0$ converges to the following stochastic complex Ginzburg-Landau equation (which was studied in [11]):
\[
\begin{align*}
\begin{cases}
2\alpha \partial_t u_0 + (1 - \Delta) u_0 + :u_0^{n+1} u_0^n := 2 \sqrt{\text{Re}(\alpha)} \xi, \\
u_0|_{t=0} = \phi_0,
\end{cases}
\end{align*}
\]
$\mathbb{P}$-almost surely in $C([0, T]; H^{-\sigma}(T^2))$ (for any $T, \sigma > 0$). See [16, Theorem 3]. This result is similar to Theorems 1.5, 1.7 and 1.9. There are however some differences with these results, which we describe in details below.

(i) In [16], the authors treated polynomial nonlinearities but we also study the sine nonlinearity - the so-called sine-Gordon model in (1.46).

(ii) In [16], they considered Gibbs measure initial data. On the other hand, in Theorems 1.5, 1.7 and 1.9, we consider deterministic initial data. In the polynomial case, we establish in Theorem 1.7 a pathwise (asymptotic) global-in-time approximation despite the lack of
a global well-posedness theory for our equations (except for the cubic case, see Remark 1.8 above). Furthermore, for the sine nonlinearity (1.28), we prove a pathwise global-in-time convergence result in Theorem 1.9. Note that in this case, working with deterministic initial data complicates matters because of the explicit dependence in \( \varepsilon \in (0,1] \) of the covariance function \( \mathbb{E}[\Psi_\varepsilon(t,x)\Psi_\varepsilon(t,y)] \) with \( \Psi_\varepsilon \) as in (1.24). See Remark 5.4 for more details.

(iii) In [16], the authors treated the convergence problem in the coordinates \((u, \varepsilon \partial_t u)\) with \(O(1)\) initial data \((\phi_0, \phi_1)\). On the other hand, we work in the coordinates \((u, \partial_t u)\) with the same \(O(1)\) initial data \((\phi_0, \varepsilon^{-1} \phi_1)\) in the coordinates \((u, \varepsilon \partial_t u)\). This gives rise to some issue when controlling the time derivative of our solutions. See the discussion above Lemma 4.6.

(iv) In [16], the convergence was proven for the sequence \(\{u_{j-1}\}\) (i.e. along the discrete sequence \(\varepsilon(j) = \frac{1}{j}\)). In the current work, we prove the convergence according to the continuous parameter \(\varepsilon \in (0,1]\). This is done by applying a bi-parameter Kolmogorov continuity criterion [1, Theorem 2.1]. This appears to be also new in the literature related to the Smoluchovski-Kramers approximation. See for instance [4, 5]. Furthermore, in treating the continuous parameter \(\varepsilon \in [0,1]\), we encounter additional difficulties. With the Gibbs measure initial data, we need to construct a single set \(\Omega_g\) (of full probability) of initial data such that for any \((\phi_0, \phi_1) \in \Omega_g\), our solutions \(\{u_\varepsilon\}_{\varepsilon \in [0,1]}\) with initial data \((\phi_0, \phi_1)\), exist almost surely simultaneously globally in time. For each \(\varepsilon > 0\), applying Bourgain’s invariant measure argument gives a set \(\Omega_\varepsilon\) of full probability such that for any \((\phi_0, \phi_1) \in \Omega_\varepsilon\), the solution \(u_\varepsilon\) with initial data \((\phi_0, \phi_1) \in \Omega_\varepsilon\), exists almost surely globally in time. We note however that this set \(\Omega_g\) cannot be constructed by taking the intersection of all \(\Omega_\varepsilon\) for \(\varepsilon > 0\).

The paper is organized as follows. In Section 2 we review some useful results from Probability theory and some deterministic estimates. In Section 3 we study the convergence of the linear flows and Duhamel integrals defined in (3.2), (3.1) below. In Section 4 we turn our attention to the polynomial model (1.45) and present proofs of Theorems 1.5 and 1.7. Finally, in Section 4.3 we look at the sine-Gordon model (1.46) and prove Theorem 1.9.

2. Notations and preliminary lemmas

In this section, we introduce some notations and go over basic lemmas.

2.1. Notations. Before proceeding further, we introduce some notations here.

We set for \(n \in \mathbb{Z}^2\),

\[
e_n(x) \overset{\text{def}}{=} \frac{1}{2\pi} e^{inx}.
\]

for the orthonormal Fourier basis in \(L^2(\mathbb{T}^2)\). Let \(f \in \mathcal{S}'(\mathbb{T}^2)\). The spatial Fourier transform \(\hat{f}\) of \(f\) (also denoted by \(\mathcal{F}\)) is then defined for any as

\[
\hat{f}(n) = \int_{\mathbb{T}^2} f(x)e_n(x)dx.
\]

We will also denote by \(\mathcal{F}^{-1}\) the inverse transform.
We write \( A \lesssim B \) to denote an estimate of the form \( A \leq CB \). Similarly, we write \( A \sim B \) to denote \( A \lesssim B \) and \( B \lesssim A \) and use \( A \ll B \) for small \( c > 0 \). We may write \( A \lesssim B \) for \( A \leq CB \) with \( C = C(\theta) \) if we want to emphasize the dependence of the implicit constant on some parameter \( \theta \). Given two functions \( f \) and \( g \) on \( \mathbb{T}^2 \), we write
\[
f \approx g,
\]
if there exist two constants \( c_1, c_2 \in \mathbb{R} \) such that
\[
f(x) + c_1 \leq g(x) \leq f(x) + c_2 \text{ for any } x \in \mathbb{T}^2 \{0\}.
\]
In addition to the projectors \( \Pi^\text{high}_\varepsilon \), \( \Pi^\text{low}_\varepsilon \) and \( \mathcal{P}_N \) defined in (1.11) and (1.19) respectively, we also define \( \Pi^\text{low,}\theta_\varepsilon \) and \( \Pi^\text{high,}\theta_\varepsilon \) by
\[
\Pi^\text{low,}\theta_\varepsilon f = \mathcal{F}^{-1}(1_{\{\langle n \rangle \leq (1+\theta)(2\varepsilon)^{-1}\}}\hat{f}(n))
\]
\[
\Pi^\text{high,}\theta_\varepsilon f = \mathcal{F}^{-1}(1_{\{\langle n \rangle > (1+\theta)(2\varepsilon)^{-1}\}}\hat{f}(n)),
\]
for some \( 0 < \theta \ll 1 \), which will be chosen to be much smaller than other fixed (i.e. not \( \varepsilon \)) parameters.

For convenience, we will denote by the set
\[
I = (0, \infty) \setminus \left\{ \frac{1}{2\langle n \rangle} : n \in \mathbb{Z}^2 \right\}.
\]

For \( s \in \mathbb{R} \), the space \( H^s(\mathbb{T}^2) \) denotes the usual \( L^2(\mathbb{T}^2) \)-based Sobolev space and we define \( \mathcal{H}^s(\mathbb{T}^2) \) by \( \mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2) \). We also use shortcut notations such as \( L^\infty_T H^s_x \) and \( C_{\varepsilon,T} H^s_x \) (for functions of the form \( f = f(\varepsilon,t,x) \)) for \( L^\infty([0,T]; H^s(\mathbb{T}^2)) \) and \( C([0,1] \times [0,T]; H^s(\mathbb{T}^2)) \) respectively, etc.

### 2.2. Preliminary results from stochastic analysis

In this subsection, by recalling some basic tools from probability theory and Euclidean quantum field theory ([23, 29, 38, 39]), we establish some preliminary estimates on the stochastic convolutions and their Wick powers. First, recall the Hermite polynomials \( H_k(x;\sigma) \) defined through the generating function:
\[
F(t,x;\sigma) := e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^\infty \frac{t^k}{k!} H_k(x;\sigma).
\]
For readers’ convenience, we write out the first few Hermite polynomials:
\[
H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x, \quad H_2(x;\sigma) = x^2 - \sigma, \quad H_3(x;\sigma) = x^3 - 3\sigma x.
\]
Note that the Hermite polynomials verify the following standard identity:
\[
H_k(x+y;\sigma) = \sum_{\ell=0}^k x^{k-\ell} y^\ell H_\ell(y;\sigma).
\]

Next, we recall the Wiener chaos estimate. Let \((H,B,\mu)\) be an abstract Wiener space. Namely, \( \mu \) is a Gaussian measure on a separable Banach space \( B \) with \( H \subset B \) as its Cameron-Martin space. Given a complete orthonormal system \( \{e_j\}_{j \in \mathbb{N}} \subset B^* \) of \( H^* = H \), we define a polynomial chaos of order \( k \) to be an element of the form \( \prod_{j=1}^\infty H_{k_j}(\langle x,e_j \rangle) \), where \( x \in B, k_j \neq 0 \) for only finitely many \( j \)'s, \( k = \sum_{j=1}^\infty k_j \), \( H_{k_j} \) is the Hermite polynomial of degree \( k_j \), and \( \langle \cdot,\cdot \rangle_B = \langle \cdot,\cdot \rangle_{B^*} \) denotes the \( B-B^* \) duality pairing. We then denote the
Suppose that Lemma 2.3. Let $f$ and $g$ be Gaussian random variables with variances $\sigma_f$ and $\sigma_g$. Then, we have
\[ \mathbb{E}[H_k(f;\sigma_f)H_m(g;\sigma_g)] = \delta_{km}k\{\mathbb{E}[fg]\}^k. \]

2.3. Deterministic estimates. We recall the following product estimates. See [17] for the proof.

Lemma 2.2. Let $f$ and $g$ be Gaussian random variables with variances $\sigma_f$ and $\sigma_g$. Then, we have
\[ \mathbb{E}[H_k(f;\sigma_f)H_m(g;\sigma_g)] = \delta_{km}k\{\mathbb{E}[fg]\}^k. \]

Lemma 2.3. Let $0 \leq s \leq 1$.
(i) Suppose that $1 < p_j, q_j, r < \infty, \frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}, j = 1, 2$. Then, we have
\[ \|\langle \nabla \rangle^s(fg)\|_{L^r(T^2)} \lesssim \left(\|f\|_{L^{p_1}(T^2)}\|\langle \nabla \rangle^s g\|_{L^{q_1}(T^2)} + \|\langle \nabla \rangle^s f\|_{L^{p_2}(T^2)}\|g\|_{L^{q_2}(T^2)}\right). \]
(ii) Suppose that $1 < p, q, r < \infty$ satisfy the scaling condition: $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{2}$. Then, we have
\[ \|\langle \nabla \rangle^{-s}(fg)\|_{L^r(T^2)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p(T^2)}\|\langle \nabla \rangle^s g\|_{L^q(T^2)}. \]
(iii) Let $F$ be a Lipschitz function on $\mathbb{R}$ such that $\|F'\|_{L^\infty(\mathbb{R})} \leq L$. Then, for any $1 < p < \infty$, we have
\[ \|\langle \nabla \rangle^a F(f)\|_{L^p(T^2)} \lesssim L\|\langle \nabla \rangle^a f\|_{L^p(T^2)}. \]

Note that while Lemma 2.3(ii) was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{2}$ in [17], the general case $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{2}$ follows from the inclusion $L^{r_1}(T^2) \subset L^{r_2}(T^2)$ for $r_1 \geq r_2$.

We record the following elementary result on the regularizing effect of the heat semi-group in Sobolev spaces.

Lemma 2.4. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta$ and $t > 0$. We have
\[ \|P_0(t)f\|_{H^\alpha(T^2)} \lesssim t^{-\frac{\alpha-\beta}{2}}\|f\|_{H^\beta(T^2)}, \]
where $P_0$ is as in [14,21].

\[ \text{Suppose that Lemma 2.3.} \]

Then, it is known that any element in $\mathcal{H}_k$ is an eigenfunction of $L$ with eigenvalue $-k$. Then, as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [28], we have the following Wiener chaos estimate [39, Theorem I.22]. See also [40, Proposition 2.4].
3. Convergence of the Deterministic Objects

We introduce the following Duhamel operators for \( \varepsilon > 0 \):

\[
\mathcal{I}_\varepsilon (F)(t) = \int_0^t \varepsilon^{-2} \mathcal{D}_\varepsilon (t - t') F(t') dt',
\]

\[
\mathcal{I}_0 (F)(t) = \int_0^t P_0 (t - t') F(t') dt',
\]

for some space-time function \( F \) and \( t \in \mathbb{R} \), with \( P_0 \) and \( \mathcal{D}_\varepsilon \) as in (1.21) and (1.13).

In this section, we study the convergence of the initial data (1.14) and the Duhamel operator (3.1). Recall the definition of \( \hat{\phi} \) for any smooth functions \( \phi \) and \( \theta > 0 \), we have

\[
\mathcal{P}_\varepsilon(t)(\phi_0, \phi_1) := (\varepsilon^{-2} + \partial_t)\mathcal{D}_\varepsilon(t)\phi_0 + \mathcal{D}_\varepsilon(t)\phi_1,
\]

Hereafter, we identify the operator \( P_0 \) with \( (P_0, 0) \) defined by

\[
(P_0, 0)(\phi_0, \phi_1)(t) := P_0(t)\phi_0,
\]

for two distributions \( \phi_0 \) and \( \phi_1 \) and \( t \geq 0 \).

**Proposition 3.1.** Let \( s \in \mathbb{R} \) and \( T > 0 \). For any \( \varepsilon \in [0, 1] \), we have the following bounds

\[
\sup_{\varepsilon \in [0, 1]} \| P_\varepsilon (\phi_0, \phi_1) \|_{C_T \mathcal{H}^s_x} \lesssim \| (\phi_0, \phi_1) \|_{\mathcal{H}^s_x},
\]

\[
\sup_{\varepsilon \in [0, 1]} \| \mathcal{I}_\varepsilon (F) \|_{C_T \mathcal{H}^s_x} \lesssim T^{\frac{s}{2}} \| F \|_{L^\infty_T \mathcal{H}^s_x - 1}
\]

for any smooth functions \( \phi_0, \phi_1 \) and \( F \). Moreover, for any \( 0 < \theta \ll 1 \), we have

\[
\| (P_\varepsilon - P_0)(\phi_0, \phi_1) \|_{C_T \mathcal{H}^{s+\theta}_x} \lesssim \varepsilon^\theta \| (\phi_0, \phi_1) \|_{\mathcal{H}^{s+\theta}_x},
\]

\[
\| (\mathcal{I}_\varepsilon - \mathcal{I}_0)(F) \|_{C_T \mathcal{H}^{s+\theta}_x} \lesssim T^{\frac{s}{2}} \varepsilon^\frac{\theta}{2} \| F \|_{L^\infty_T \mathcal{H}^{s+1+\theta}_x},
\]

for any smooth functions \( \phi_0, \phi_1 \) and \( F \).

**Corollary 3.2.** Let \( s \in \mathbb{R}, T > 0 \) and \( \theta > 0 \). We have the following bounds:

\[
\| P_\varepsilon(t)(\phi_0, \phi_1) \|_{C_{s,T} \mathcal{H}^{s+\theta}_x} \lesssim \| (\phi_0, \phi_1) \|_{\mathcal{H}^{s+\theta}_x},
\]

\[
\| \mathcal{I}_\varepsilon(t)(F(\varepsilon, \cdot)) \|_{C_{s,T} \mathcal{H}^{s+1+\theta}_x} \lesssim T^{\frac{s}{2}} \| F \|_{C_{s,T} \mathcal{H}^{s+1+\theta}_x},
\]

for any \( 0 < \theta \ll 1 \) and smooth \( \phi_0, \phi_1 \) and \( F \).

**Remark 3.3.** Note that if we replace the \( C([0, 1] \times [0, T]; H^s(\mathbb{T}^2)) \) norms in the left-hand side by \( C((0, 1] \times [0, T]; H^s(\mathbb{T}^2)) \), then we obtain inequalities without the \( \theta \)-derivative losses in the right-hand side.

We prove several useful lemmas first and postpone the proof of Proposition 3.1 and Corollary 3.2 to the end of the section.

Let \( \{ \mathcal{D}_\varepsilon(n, t) \}_{n \in \mathbb{Z}^2} \) be the symbol associated to the multiplier \( \mathcal{D}_\varepsilon(t) \) defined in (1.13). We define \( \eta \in \mathbb{R}^2 \mapsto \mathcal{D}_\varepsilon(\eta, t) \) its natural extension to \( \mathbb{R}^2 \) given by

\[
\mathcal{D}_\varepsilon(\eta, t) \equiv \mathcal{D}_\varepsilon(\eta, t') \bigg|_{t'=0} = \sum_{n \in \mathbb{Z}^2} \mathcal{D}_\varepsilon(n, t')
\]
where \( \eta \mapsto \lambda_\varepsilon(\eta) \) and \( \eta \mapsto \zeta_\varepsilon(\eta) \) are the extensions to \( \mathbb{R}^2 \) of the sequences \( \{\lambda_\varepsilon(n)\}_{n \in \mathbb{Z}^2} \) and \( \{\zeta_\varepsilon(n)\}_{n \in \mathbb{Z}^2} \) defined in (1.8) and (1.10) respectively.

For an integer \( p \geq 1 \) and a multi-index \( \alpha \in \{1, 2\}^p \), we denote by \( |\alpha| = p \) its length.

**Lemma 3.4.** Recall the definition of the set \( I \) in (2.2). The function \((\varepsilon, t, \eta) \in I \times \mathbb{R}_+ \times \mathbb{R}^2 \mapsto \hat{D}_\varepsilon(\eta, t)\) can be extended to a \( C^\infty \) function on \((0, \infty) \times \mathbb{R}_+ \times \mathbb{R}^2\). Moreover, we have the following bound:

\[
|\partial_\eta^\alpha \hat{D}_\varepsilon(\eta, t)| \lesssim e^{-\frac{1}{2\varepsilon t}} |\alpha|! |\varepsilon|^{-|\alpha|} \sum_{j=1}^{|\alpha|} (1 + |t\varepsilon^{-1}| \eta_1^p) (1_{\{\eta_1 \leq 2\varepsilon\}^{-1}} + 1_{\{\eta_1 > 2\varepsilon\}^{-1}}) e^{t\lambda_\varepsilon(\eta)} \quad (3.7)
\]

\[
|\partial_\varepsilon \hat{D}_\varepsilon(\eta, t)| \lesssim e^{-2}(\varepsilon)^2, \quad (3.8)
\]

for any \( \varepsilon \in (0, \infty) \), \( \eta \in \mathbb{R}^2 \), \( t \geq 0 \) and multi-index \( \alpha \).

**Proof.** Note that from (3.7), \( \hat{D}_\varepsilon(\eta, t) \) can be written as a power series:

\[
\hat{D}_\varepsilon(\eta, t) = e^{-\frac{1}{2\varepsilon t}} \sum_{j=0}^{(2j+1)!} (\varepsilon)^{-4} (1 + \varepsilon^{-1} |\eta|^2) e^{t\lambda_\varepsilon(\eta)} \quad (3.10)
\]

This immediately shows that, \((\varepsilon, t, \eta) \mapsto \hat{D}_\varepsilon(\eta, t)\) can be extended to a smooth function on \((0, \infty) \times \mathbb{R}_+ \times \mathbb{R}^2\). Here \( \hat{S}_\varepsilon(\eta, t) \) is denotes the extension to \( \mathbb{R}^2 \) of the symbol of the multiplier \( S_\varepsilon(t) \) defined in (1.12).

Let us first show (3.8). Note that from (3.10), we can rewrite \( \hat{S}_\varepsilon(\eta, t) \) as

\[
\hat{S}_\varepsilon(\eta, t) = t g_\varepsilon, t \equiv g_\varepsilon, t (t \varepsilon^{-1} \eta), \quad \eta \in \mathbb{R}^2, \quad (3.11)
\]

with \( g_\varepsilon, t \equiv g_{\varepsilon, t} \equiv \sum_{j=0}^{(2j+1)!} (t^j 4\varepsilon)^{-4} - t^j \varepsilon^{-2} - |\eta|^2)^j \). We can express the derivative \( \partial_\eta^p g_\varepsilon, t \) as a finite linear combination of functions of the form

\[
Q_p(\eta) \sum_{j=0}^{(2j+1)!} (t^j 4\varepsilon)^{-4} - t^j \varepsilon^{-2} - |\eta|^2)^j \quad (3.12)
\]

where \( Q_p \in \mathbb{R}[\eta_1, \eta_2] \) (with \( \eta = (\eta_1, \eta_2) \)) is a polynomial of degree at most \( p \) for some integer \( 1 \leq p \leq |\alpha| \). Hence, we have

\[
|\partial_\eta^p g_\varepsilon, t(\eta)| \lesssim \sum_{p=1}^{|\alpha|} (1 + |\eta|^p) \phi(p) \left( t^j 4\varepsilon)^{-4} - t^j \varepsilon^{-2} - |\eta|^2)^j \right),
\]

where \( \phi \) is defined by

\[
\phi(x) = \sum_{j=0}^{x^j} (2j + 1)! \quad (3.13)
\]

for \( x \in \mathbb{R} \) and \( \phi(p) \) denotes the \( p \)th derivative of \( \phi \). We claim the following bound on the derivatives of \( \phi \):

\[
|\phi(p)(x)| \lesssim_p 1_{x \geq 0} e^{\sqrt{x}} + 1_{x < 0},
\]

(3.14)
for any \( p \geq 0 \). Note that we have the explicit formula \( \phi(x) = \frac{\sin(\sqrt{-x})}{\sqrt{-x}} \) for \( x < 0 \). We directly deduce the bound \( |\phi^{(p)}(x)| \lesssim 1 \) for any \( x < -1 \) and \( p \geq 0 \) by induction. Besides, for \( x \in \mathbb{R} \), we have

\[
\phi^{(p)}(x) = \sum_{j \geq 0} \frac{(j + p)!}{j!(2j + 2p + 1)!} x^j.
\]

Hence, \( |\phi^{(p)}(x)| \lesssim 1 \) for \( |x| \lesssim 1 \). Furthermore, for \( x > 0 \), we bound

\[
\sum_{j \geq 0} \frac{(j + p)!}{j!(2j + 2p + 1)!} x^j \leq \sum_{j \geq 0} \frac{1}{(2j)!} x^j \leq e^x,
\]

where we used the inequality \( \frac{(j+p)!(2j)!}{j!(2j+2p+1)!} \leq 1 \) for \( j, p \geq 0 \). This shows (3.14). We now can estimate the derivative \( \partial_\eta^\alpha \tilde{S}_\varepsilon(\eta, t) \) using (3.11), (3.12), (3.12), and (1.8):

\[
|\partial_\eta^\alpha \tilde{S}_\varepsilon(\eta, t)| \lesssim \varepsilon^{\alpha - |\alpha|} \sum_{p=1}^{|\alpha|} (1 + |\varepsilon^{-1}p|)(1_{(\eta)\leq(2\varepsilon)-1} e^{t\lambda_\varepsilon(\eta)} + 1_{(\eta)>(2\varepsilon)-1}).
\]

Combined with (3.10), the above shows (3.8).

Let us now prove (3.9). From (3.10), it clearly suffices to bound \( e^{-\frac{t}{2\varepsilon^2}} \partial_\varepsilon \tilde{S}_\varepsilon(\eta, t) \) for \( \eta \in \mathbb{R}^2 \) and \( t \geq 0 \). We have from (3.11),

\[
\partial_\varepsilon \tilde{S}_\varepsilon(\eta, t) = (-16 \varepsilon^{-5} + 2 \varepsilon^{-3} \langle \eta \rangle^2) t^2 \phi(t^2(4\varepsilon)^{-1} - t^2 \varepsilon^{-2} \langle \eta \rangle^2),
\]

with \( \phi \) as in (3.13). Note that by the inequality \( \sqrt{1-x} \leq 1 - \frac{x}{2} \) for \( 0 \leq x \leq 1 \), we get

\[
e^{-\frac{t}{2\varepsilon^2}} e^{\lambda_\varepsilon(\eta)t} \leq e^{-\langle \eta \rangle^2 t}, \quad (3.15)
\]

for \( \langle \eta \rangle < (2\varepsilon)^{-1} \). Thus, from (3.14) and (3.15), we deduce

\[
e^{-\frac{t}{2\varepsilon^2}} |\partial_\varepsilon \tilde{S}_\varepsilon(\eta, t)| \lesssim e^{-5\langle \eta \rangle^2} t e^{-\frac{t}{2\varepsilon^2}} (e^{\lambda_\varepsilon(\eta)t} 1_{(\eta)\leq(2\varepsilon)-1} + 1) \lesssim e^{-5\langle \eta \rangle^2},
\]

where we used the inequality \( e^{-y} \lesssim y^{-1} \) for \( y > 0 \). This proves (3.9). \( \square \)

**Lemma 3.5.** Fix \( 0 < \theta \ll 1 \). We have the following bounds for \( t \geq 0 \):

\[
\varepsilon^{-2} |\tilde{D}_\varepsilon(n, t)| \lesssim \begin{cases} e^{-\theta t(n)^2} & \text{for } (n) \leq (1 + \theta)(2\varepsilon)^{-1} \\ e^{-\frac{t}{2\varepsilon^2}} \varepsilon^{-1}(n)^{-1} & \text{otherwise} \end{cases} \quad (3.16)
\]

\[
|\partial_\varepsilon \tilde{D}_\varepsilon(n, t)| \lesssim \begin{cases} e^{-\theta t(n)^2} & \text{for } (n) \leq (1 + \theta)(2\varepsilon)^{-1} \\ e^{-\frac{t}{2\varepsilon^2}} & \text{otherwise} \end{cases} \quad (3.17)
\]

with implicit constants independent of \( \varepsilon > 0 \) and \( t > 0 \).

**Proof.** We first prove (3.16). Fix \( t \geq 0 \) and \( \varepsilon > 0 \). For \( (n) < (2\varepsilon)^{-1} \), we have from (1.8) and (1.12):

\[
\varepsilon^{-2} \tilde{D}_\varepsilon(n, t) = e^{-2} e^{-\frac{t}{2\varepsilon^2}} \sinh(\lambda_\varepsilon(n)t). \quad (3.18)
\]

Note that from (3.15), we have

\[
e^{-\frac{t}{2\varepsilon^2}} \sinh(t\lambda_\varepsilon(n)) \lesssim e^{-\langle n \rangle^2 t}, \quad (3.19)
\]
for \( n \leq (2\varepsilon)^{-1} \). Noting that \( \lambda_\varepsilon(n) \gtrsim \theta \varepsilon^{-2} \) for \( n \leq (1 - \theta)(2\varepsilon)^{-1} \), we estimate using (3.19), (3.18):

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \lesssim e^{-t(n)^2}.
\]

(3.20)

for \( n \leq (1 - \theta)(2\varepsilon)^{-1} \).

We now estimate \( \varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \) for \( (1 - \theta)(2\varepsilon)^{-1} < n < (2\varepsilon)^{-1} \).

\textbf{Case 1:} \( \lambda_\varepsilon(n) t \leq 1 \). In this regime we compute using the inequalities \(|\sinh(x)| \lesssim |x|\) for \( 0 \leq x \leq 1 \) and \( e^{-y} \lesssim y^{-1} \) for \( y > 0 \), (3.18) with \( n < (2\varepsilon)^{-1} \)

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \lesssim \varepsilon^{-2} t \cdot e^{-\frac{\varepsilon}{2\varepsilon^2}}
\]

(3.21)

\textbf{Case 2:} \( \lambda_\varepsilon(n) t > 1 \). In this case we note that we have \( \sqrt{1 - x} \leq 1 - \theta(1 + \frac{x}{2}) \) for \( 1 - \theta \leq x \leq 1 \) and \( 0 < \theta \ll 1 \) which implies

\[
e^{-\frac{\varepsilon}{\theta(t(n)^2)}} \lesssim e^{-\theta t(n)^2},
\]

(3.22)

for \( (1 - \theta)(2\varepsilon)^{-1} < n < (2\varepsilon)^{-1} \). Using (3.18), (3.22) and the inequality \( e^{-y} \lesssim y^{-1} \) for \( y > 0 \), we then estimate

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \lesssim \varepsilon^{-2} t \cdot e^{-\frac{\varepsilon}{2\varepsilon^2}} \cdot e^{-\theta t(n)^2}
\]

\[
\lesssim e^{-\theta t(n)^2}.
\]

(3.23)

Hence, from (3.21) and (3.23) we get

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \lesssim e^{-\theta t(n)^2}
\]

(3.24)

for \( (1 - \theta)(2\varepsilon)^{-1} < n < (2\varepsilon)^{-1} \).

For \( n > (2\varepsilon)^{-1} \), we have from (1.12) with (1.10) and (1.13):

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| = \varepsilon^{-2} e^{-\frac{\varepsilon}{2\varepsilon^2}} \sin(\zeta_\varepsilon(n) t) \frac{\sin(\zeta_\varepsilon(n) t)}{\zeta_\varepsilon(n)}.
\]

(3.25)

We have the inequalities \(|\sin(x)| \lesssim |x|\) for \( x \in \mathbb{R} \) and \( e^{-y} \lesssim y^{-1} \) for \( y > 0 \) and (3.25):

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \lesssim \varepsilon^{-2} t \cdot e^{-\frac{\varepsilon}{2\varepsilon^2}}
\]

\[
\lesssim e^{-\frac{\varepsilon}{4\varepsilon^2}} \lesssim e^{-\frac{\varepsilon}{4\varepsilon^2}}.
\]

(3.26)

for \( (2\varepsilon)^{-1} < n \leq (1 + \theta)(2\varepsilon)^{-1} \).

For \( n > (1 + \theta) \cdot (2\varepsilon)^{-1} \), we have \( \zeta_\varepsilon(n) \gtrsim \theta \varepsilon^{-1} n \). Thus, we get from (3.25):

\[
\varepsilon^{-2}|\widehat{D}_\varepsilon(n, t)| \lesssim e^{-\frac{\varepsilon}{4\varepsilon^2}} e^{-\varepsilon^{-1} n^{-1}}
\]

(3.27)

Collecting (3.20), (3.24), (3.26) and (3.27) yields (3.16) for \( n \neq (2\varepsilon)^{-1} \). The inequality (3.16) also holds for \( n = (2\varepsilon)^{-1} \) by the continuity of the map \( \varepsilon \mapsto \widehat{D}_\varepsilon(n, t) \) for \( n \) and \( t \) fixed proved in Lemma 3.4.

We now prove (3.17). Note that from (3.18) and (3.25) we have

\[
\partial_t \widehat{D}_\varepsilon(n, t) = -\frac{1}{2\varepsilon^2} \widehat{D}_\varepsilon(n, t) + e^{-\frac{\varepsilon}{2\varepsilon^2}} \partial_t \widehat{S}_\varepsilon(n, t)
\]

(3.28)
where \( \{ \hat{S}_\varepsilon(n,t) \}_{n \in \mathbb{Z}^2} \) is the Fourier symbol associated to \( S_\varepsilon(t) \) defined in (1.12). From (3.16), it suffices to estimate the contribution of \( e^{-\frac{1}{2\varepsilon^2}} \partial_t \hat{S}_\varepsilon(n,t) \). We have from (1.12):

\[
\partial_t \hat{S}_\varepsilon(n,t) = \cosh(\lambda_\varepsilon(n)t) 1_{\langle n \rangle < (2\varepsilon)^{-1}} + \cos(\zeta_\varepsilon(n)t) 1_{\langle n \rangle > (2\varepsilon)^{-1}}.
\]

(3.29)

Hence, we get from (3.15):

\[
e^{-\frac{1}{2\varepsilon^2}} \partial_t \hat{S}_\varepsilon(n,t) \lesssim \begin{cases} e^{-\frac{t(n)^2}{4\varepsilon^2}} & \text{if } \langle n \rangle \leq (2\varepsilon)^{-1} \\ e^{-\frac{1}{2\varepsilon^2}} & \text{otherwise}, \end{cases}
\]

(3.30)

which concludes the proof of (3.17).

\[ \square \]

**Lemma 3.6.** Fix \( 0 < \theta \ll 1 \). We have

\[
|\varepsilon^{-2} \hat{D}_\varepsilon(n,t) - e^{-t(n)^2} 1_{\langle n \rangle \leq \varepsilon^{-1+\theta}}| \lesssim e^{-\frac{1}{2\varepsilon^2}} + e^{2\theta} e^{-\frac{t(n)^2}{4}},
\]

(3.31)

and

\[
|(\varepsilon^{-2} + \partial_t) \hat{D}_\varepsilon(n,t) - e^{-t(n)^2} 1_{\langle n \rangle \leq \varepsilon^{-1+\theta}}| \lesssim \varepsilon^{2\theta} e^{-\frac{t(n)^2}{4}},
\]

(3.32)

for any \( t \geq 0 \).

**Proof.** Fix \( t \geq 0 \) and \( \langle n \rangle \leq \varepsilon^{-1+\theta} \). From (3.18) with (1.8), we write

\[
\varepsilon^{-2} \hat{D}_\varepsilon(n,t) = e^{-\frac{t(n)^2}{4\varepsilon^2}} \frac{e^{\lambda_\varepsilon(n)t} - e^{-\lambda_\varepsilon(n)t}}{\sqrt{1 - 4\varepsilon^2 \langle n \rangle^2}} = I - \Pi.
\]

(3.33)

Since \( \lambda_\varepsilon(n) \) is non-negative, we have

\[
|\Pi| \lesssim e^{-\frac{t(n)^2}{4\varepsilon^2}}.
\]

(3.34)

Furthermore, we get, using (1.5), (3.15), the inequality \( \Lambda_\varepsilon^+ (n) + \langle n \rangle^2 \leq 0 \), the asymptotic expansion of \( \Lambda_\varepsilon^+ (n) \) in (1.17), the mean value theorem with the inequality \( e^{-y} \lesssim y^{-1} \) for \( y > 0 \),

\[
|I - e^{-t(n)^2}| \lesssim |(\frac{1}{\sqrt{1 - 4\varepsilon^2 \langle n \rangle^2}} - 1) e^{\Lambda_\varepsilon^+ (n)t}| + |(e^{\Lambda_\varepsilon^+ (n)t} - e^{-t(n)^2})| 
\]

\[
\lesssim e^{2\theta} e^{-t(n)^2} + t \langle n \rangle^4 \varepsilon^2 e^{-t(n)^2} \lesssim e^{2\theta} e^{-\frac{t(n)^2}{4}}.
\]

(3.35)

Putting (3.33), (3.34) and (3.35) together gives (3.31).

We now prove (3.32). We write using (3.18), (1.5), (1.8) and (3.28),

\[
(\varepsilon^{-2} + \partial_t) \hat{D}_\varepsilon(n,t) = \mathcal{P}_\varepsilon(n,t) + \mathcal{R}_\varepsilon(n,t),
\]

(3.36)

with

\[
\mathcal{P}_\varepsilon(n,t) := \frac{e^{\Lambda_\varepsilon^+ (n)t}}{\sqrt{1 - 4\langle n \rangle^2 \varepsilon^2}}
\]

\[
\mathcal{R}_\varepsilon(n,t) := \left(1 - \frac{1}{\sqrt{1 - 4\langle n \rangle^2 \varepsilon^2}}\right) \frac{e^{\Lambda_\varepsilon^+ (n)t}}{2} + \left(1 - \frac{1}{\sqrt{1 - 4\langle n \rangle^2 \varepsilon^2}}\right) \frac{e^{\Lambda_\varepsilon^- (n)t}}{2}.
\]

We estimate as in (3.21):

\[
|\mathcal{P}_\varepsilon(n,t) - e^{-t(n)^2}| \lesssim \varepsilon^{2\theta} e^{-\frac{t(n)^2}{4}}
\]

\[
|\mathcal{R}_\varepsilon(n,t)| \lesssim \varepsilon^{2\theta} e^{-t(n)^2}.
\]

(3.37)
Thus, \((3.32)\) follows from \((3.36)\) and \((3.37)\).

We deduce from Lemma 3.5 and Lemma 3.6 and Lemma 2.4 the following operator bounds.

**Lemma 3.7.** Fix \(0 < \theta \ll 1\). Recall the definitions of \(P^{\text{low}, \theta}_\varepsilon\) and \(P^{\text{high}, \theta}_\varepsilon\) in \((2.1)\). The following inequalities hold:

(i) Let \(\alpha, \beta \in \mathbb{R}\) with \(\alpha \geq \beta\), and \(t > 0\). We have,
\[
\left\| \varepsilon^{-2} P^{\text{low}, \theta}_\varepsilon D_\varepsilon(t) f \right\|_{H^2_x} \lesssim t^{-\frac{\alpha - \beta}{2}} \| f \|_{H^2_x},
\]
for any smooth function \(f\) and with an implicit constant independent of \(\varepsilon > 0\) and \(t > 0\).

(ii) Let \(s \in \mathbb{R}, \gamma \in \{0, 1\}\), and \(t > 0\). We have,
\[
\left\| \varepsilon^{-2} P^{\text{high}, \theta}_\varepsilon D_\varepsilon(t) f \right\|_{H^s_x} \lesssim e^{-\frac{\varepsilon}{10r^2} t^{-\frac{1}{2}} \| f \|_{H^s_x}},
\]
for any smooth function \(f\) and with an implicit constant independent of \(\varepsilon > 0\) and \(t > 0\).

(iii) Let \(s \in \mathbb{R}, 0 \leq \gamma \leq 1\) and \(t > 0\). We have,
\[
\| D_\varepsilon(t) f \|_{H^s_x} \lesssim \varepsilon^{1+\gamma} \| f \|_{H^{s-1+\gamma}_x},
\]
for any smooth function \(f\) and with an implicit constant independent of \(\varepsilon > 0\) and \(t > 0\).

(iv) Let \(s \in \mathbb{R}\). We have,
\[
\sup_{\varepsilon, t > 0} \| \partial_t D_\varepsilon(t) f \|_{H^s_x} \lesssim \| f \|_{H^s_x},
\]
for any smooth function \(f\).

(v) Let \(\alpha, \beta \in \mathbb{R}\) with \(\alpha \geq \beta\), and \(t > 0\). We have:
\[
\left\| P^{\text{low}, \theta}_\varepsilon \left( (\varepsilon^{-2} + \partial_t) D_\varepsilon(t) - P_0(t) \right) f \right\|_{H^2_x} \lesssim \varepsilon^{2\theta} t^{-\frac{\alpha - \beta}{2}} \| f \|_{H^2_x},
\]
for any smooth function \(f\) and with an implicit constant independent of \(\varepsilon > 0\) and \(0 < t \leq T\).

**Proof.** Items (i), (iv) and (v) are direct consequences of \((3.16), (3.17), (3.32)\) respectively and Lemma 2.4. We now look at (ii). If \(\gamma = 0\), then (ii) comes directly from \((3.16)\). If \(\gamma = 1\), then from \((3.10)\), we have
\[
\left\| \varepsilon^{-2} P^{\text{high}, \theta}_\varepsilon D_\varepsilon(t) f \right\|_{H^1_x} \lesssim e^{-\frac{1}{27} \varepsilon^{-1}} \| f \|_{H^1_x},
\]
which gives
\[
\left\| \varepsilon^{-2} P^{\text{high}, \theta}_\varepsilon D_\varepsilon(t) f \right\|_{H^\gamma_x} \lesssim e^{-\frac{1}{10r^2} t^{-\frac{1}{2}} \| f \|_{H^\gamma_x}},
\]
for any smooth function \(f\), where we used the inequality \(e^{-y} \lesssim y^{-\frac{1}{4}}\) for \(y > 0\). This proves (ii) for \(\gamma = 1\). From (i) and (ii) with \(\gamma = 0\), and by interpolation, (iii) follows from the bound
\[
\| D_\varepsilon(t) f \|_{H^\gamma_x} \lesssim \| f \|_{H^{\gamma - 1}_x},
\]
for any smooth function \(f\) and \(t > 0\). From (i), \((2.1)\) and the restriction \(\langle n \rangle \lesssim \varepsilon^{-1}\)
\[
\left\| P^{\text{low}, \theta}_\varepsilon D_\varepsilon(t) f \right\|_{H^\gamma_x} \lesssim e^{2\theta} \| P^{\text{low}, \theta}_\varepsilon f \|_{H^\gamma_x},
\]
Furthermore, from \((3.16)\) with \((2.1)\), we estimate
\[
\|P_{\varepsilon}^{\text{high}, \theta} D_{\varepsilon}(t) f \|_{H_{\varepsilon}^s} \lesssim \varepsilon^2 e^{-\frac{1}{32\varepsilon^2}} \|f\|_{H_{\varepsilon}^{s-1}} \\
\lesssim \varepsilon \|f\|_{H_{\varepsilon}^{s-1}}.
\]  

(3.41)

Hence, combining (3.40) and (3.41) gives (3.39).

□

Remark 3.8. From (3.17), we actually get the following stronger bounds for \( \alpha, \beta, s \in \mathbb{R} \) with \( \alpha \geq \beta \),

\[
\|P_{\varepsilon}^{\text{low}, \theta} \partial_t D_{\varepsilon}(t) f \|_{H_{\varepsilon}^s} \lesssim t^{-\frac{\alpha-\beta}{2}} \|f\|_{H_{\varepsilon}^\beta} \\
\|P_{\varepsilon}^{\text{high}, \theta} \partial_t D_{\varepsilon}(t) f \|_{H_{\varepsilon}^s} \lesssim e^{-\frac{\beta}{2\varepsilon^2}} \|f\|_{H_{\varepsilon}^\beta},
\]

for any \( \varepsilon, t > 0 \) and smooth function \( f \).

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let \( s \in \mathbb{R} \) and \( T > 0 \) and some smooth functions \((\phi_0, \phi_1)\). We first prove (3.3). From 2.4 \( P_0 \) defined in (1.21) clearly verifies the bound

\[
\|P_0(\phi_0, \phi_1)\|_{L_T^\infty H_s^s} \lesssim \|(\phi_0, \phi_1)\|_{H_s^1}.
\]

(3.42)

Hence, it suffices to prove

\[
\sup_{\varepsilon \in (0,1]} \|P_{\varepsilon}(\phi_0, \phi_1)\|_{L_T^\infty H_s^s} \lesssim \|(\phi_0, \phi_1)\|_{H_s^1}.
\]

(3.43)

From (3.2), Lemma 3.7 (i), (ii), (iii) and (iv) with (2.1) we have

\[
\sup_{\varepsilon \in (0,1]} \|(\varepsilon^{-2} + \partial_t) D_{\varepsilon}(t) \phi_0\|_{L_T^\infty H_s^1} \lesssim \|\phi_0\|_{H_s^1} \\
\sup_{\varepsilon \in (0,1]} \|D_{\varepsilon}(t) \phi_1\|_{L_T^\infty H_s^{s-1}} \lesssim \|\phi_1\|_{H_s^{s-1}}
\]

(3.44)

The continuity in time of \( P_\varepsilon(\phi_0, \phi_1) \) for some fixed \( \varepsilon > 0 \) and \((\phi_0, \phi_1) \in \mathcal{H}^s(T^2)\) follows from the dominated convergence theorem and our bounds. Combining (3.44) and (3.2) gives (3.43). This concludes the proof of the first part of (3.3).

From (3.1), Minkowski’s inequality and Lemma 3.7 (i) and (ii), we have for \( \varepsilon > 0 \) and \( F \in L_T^\infty H_s^{s-1} \)

\[
\|I_{\varepsilon}(F)\|_{L_T^\infty H_s^s} \lesssim \int_0^T \|\varepsilon^{-2} D_{\varepsilon}(t) F(t)\|_{H_s^s} dt \\
\lesssim \int_0^T t^{-\frac{\beta}{2}} dt \|F\|_{L_T^\infty H_s^{s-1}} \lesssim T^{\frac{\beta}{2}} \|F\|_{L_T^\infty H_s^{s-1}}.
\]

(3.45)

The same inequality holds for \( \varepsilon = 0 \). The continuity in time of \( I_{\varepsilon}(F) \) \((\varepsilon \in [0,1])\) follows from a similar computation. We obtain the remaining part of (3.3) from (3.45).

The estimate on \( P_\varepsilon - P_0 \) in (3.4) follows from Lemma 3.7 (v), Lemma 2.4 and similar arguments after noting the bound \( \|F^{-1}(\mathcal{F}1_{\langle u \rangle > \varepsilon^{-1} h})\|_{H_s^s} \lesssim \varepsilon^{\frac{\beta}{2}} \|f\|_{H_s^{s+\beta}} \) for any smooth function \( f \) and \( \theta > 0 \) small enough. Similarly, the estimate on \( I_{\varepsilon} - I_0 \) in (3.4) follows from (3.31) in Lemma 3.6.

□
Proof of Corollary 3.2. The continuity of the map \( \varepsilon \in (0,1] \mapsto P_\varepsilon \) is deduced from the smoothness of the map \( (\varepsilon,t) \mapsto \widehat{D}_\varepsilon(n,t) \) for each \( n \in \mathbb{Z}^2 \) (Lemma 3.4) and the dominated convergence theorem. The continuity at \( \varepsilon = 0 \) of \( \varepsilon \mapsto P_\varepsilon \) follows from (3.4). Hence, (3.5) follows from the above and Proposition 3.1. The bound (3.6) follows from similar arguments. \( \square \)

4. POLYNOMIAL MODEL

4.1. On the stochastic convolution.

Proposition 4.1. Let \( \ell \in \mathbb{N} \). Then for any finite \( p,q \geq 1 \), \( T > 0 \), \( \sigma > 0 \), and \( \varepsilon \in [0,1] \), the sequence \( \{ \Psi_{\varepsilon,N}^\ell \} \) defined in (1.36) is a Cauchy sequence in \( L^p(\Omega; \mathcal{L}^q([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))) \) and thus converges to a limiting stochastic process in \( L^p(\Omega; L^q([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))) \), denoted by \( \Psi_{\varepsilon}^\ell \) as \( N \to \infty \).

Moreover, \( \{ (\varepsilon,t) \mapsto \Psi_{\varepsilon,N}^\ell \} \) also converges to \( \varepsilon,t \mapsto \Psi_{\varepsilon}^\ell \) in \( L^p(\Omega; \mathcal{L}^q([0,1] \times [0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))) \) and almost surely in \( C([0,1] \times [0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)) \) as \( N \to \infty \).

Remark 4.2. We highlight the fact that Proposition 4.1 in particular implies that there exists a set \( \Omega_1 \subset \Omega \) such that \( \mathbb{P}(\Omega_1) = 1 \) and for each \( \ell \in \mathbb{N} \), \( T > 0 \) and \( \sigma > 0 \) \( \mathbb{P} \) : \( \Psi_{\varepsilon}^\ell : (\omega) \mapsto \sum_{\ell \in \mathbb{N}} \| \Psi_{\varepsilon,N}^\ell (\omega) \|_{C_T W^{-\sigma,\infty}} \to 0 \) as \( \varepsilon \to 0 \). Namely, the almost-sure convergence is proved with respect to the continuous parameter \( \varepsilon \in [0,1] \).

Proof. Fix \( \ell \in \mathbb{N} \), and \( \sigma > 0 \). Our first goal is to bound the variance:

\[
\mathbb{E} \left[ \left( \langle \nabla \rangle^{-\sigma} : \Psi_{\varepsilon,N}(t,\cdot) : (x) \right)^2 \right] \leq 1, \tag{4.1}
\]

uniformly in \( N \in \mathbb{N} \), \( t \geq 0 \), \( \varepsilon \in [0,1] \), and \( x \in \mathbb{T}^2 \). Fix \( N \in \mathbb{N} \), \( t \geq 0 \), \( (x,y) \in (\mathbb{T}^2)^2 \) and \( \varepsilon \in (0,1] \) (the case \( \varepsilon = 0 \) follows from similar arguments). By (1.31) with (1.19), (1.36) and Lemma 2.2 we have,

\[
\frac{1}{\ell!} \mathbb{E} \left[ : \Psi_{\varepsilon,N}^\ell (t,x) : \Psi_{\varepsilon,N}^\ell (t,y) : \right] = \mathbb{E} \left[ \Psi_{\varepsilon,N}(t,x) \Psi_{\varepsilon,N}(t,y) \right]^\ell \tag{4.2}
\]

Applying the Bessel potentials \( \langle \nabla_x \rangle^{-\sigma} \) and \( \langle \nabla_y \rangle^{-\sigma} \) and then setting \( x = y \), we see from the previous computations that in order to bound the left-hand-side of (4.1), we need to bound terms of the form

\[
\sum_{n_1,\ldots,n_\ell \in \mathbb{Z}^2} \langle n_1 + \cdots + n_\ell \rangle^{-2\sigma} F_1(n_1,t) \cdots F_\ell(n_\ell,t), \tag{4.2}
\]

where we write for \( 1 \leq j \leq \ell \) and \( n \in \mathbb{Z}^2 \),

\[
F_j(n,t) = \mathbb{E} \left[ |F(\Psi_{\varepsilon,N})(n,t)|^2 \right]. \tag{4.3}
\]

Hence, we get from (1.31) with (1.19), and (3.16),

\[
F_j(n,t) \lesssim \| 1_{[0,1]}(t') \|_{L^2_t}^2 \| D_\varepsilon(n,t) \|_{L^2_t}^2 \lesssim \langle n \rangle^{-2}
\]

for \( n \in \mathbb{Z}^2 \). This gives
\[
\begin{align*}
\sum_{n_1, \ldots, n_\ell \in \mathbb{Z}^2, \langle n \rangle \leq N} \langle n_1 + \cdots + n_\ell \rangle^{-2\sigma} \prod_{j=1}^{\ell} \langle n_j \rangle^{-2} \lesssim 1,
\end{align*}
\]

and shows (4.1). Let \( r > \frac{4}{\sigma} \) and \( p, q \geq 1 \) with \( p \geq q, r \). We have from Sobolev and Minkowski inequalities and Lemma 2.1 along with (4.1)

\[
\begin{align*}
\| : \Psi_{\epsilon,N}^\ell : \|_{L^p(\Omega)L^q(W^{-\sigma,\infty}_x)} & \lesssim \| : \Psi_{\epsilon,N}^\ell : \|_{L^p(\Omega)L^q_x}\nonumber \\
& \leq \| \langle \nabla \rangle^{-\frac{\sigma}{2}} : \Psi_{\epsilon,N}^\ell : \|_{L^p(\Omega)} \|_{L^q_x}\nonumber \\
& \lesssim p^\gamma \| \langle \nabla \rangle^{-\frac{\sigma}{2}} : \Psi_{\epsilon,N}^\ell : \|_{L^2(\Omega)} \|_{L^q_x}\nonumber \\
& \lesssim T^\gamma p^\gamma \lesssim T, p, t 1.
\end{align*}
\]

Using the inclusion \( L^{p_2}(\Omega) \subset L^{p_1}(\Omega) \) for \( p_1 \leq p_2 \), we obtain a similar bound for a general \( p \geq 1 \). Let \( p \geq 1 \) and \( M \geq N \). By similar arguments, we also get

\[
\| : \Psi_{\epsilon,N}^\ell : - : \Psi_{\epsilon,M}^\ell : \|_{L^p(\Omega)L^q(W^{-\sigma,\infty}_x)} \lesssim N^{-\gamma},
\]

for some small \( \gamma > 0 \). It shows that \( \{ : \Psi_{\epsilon,N}^\ell : \}_{N \geq 1} \) is a Cauchy sequence in \( L^p(\Omega; L^q([0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))) \), for any \( p, q \geq 1 \) and \( \sigma > 0 \). Thus, it converges to some limit denoted by \( : \Psi_{\epsilon}^\ell : \). A similar reasoning shows that for \( p, q \geq 1 \) and \( \sigma > 0 \), the sequence \( \{(\epsilon, t) \mapsto : \Psi_{\epsilon,N}^\ell : \}_{N \geq 1} \) is also a Cauchy sequence in the space \( L^p(\Omega; L^q([0,1] \times [0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))) \) and converges to a limit equal to \( (\epsilon, t) \mapsto : \Psi_{\epsilon}^\ell : \) almost surely (in \( \omega \in \Omega \)) by the dominated convergence theorem. This shows the first part of the statement.

We now investigate the continuity in \((\epsilon, t)\) of our objects. Let \( h_1, h_2 \in \mathbb{R} \). We define the following operators \( \delta_{h_1,h_2}, \delta_{h_1}^1 \) and \( \delta_{h_2}^2 \) by

\[
\begin{align*}
\delta_{h_1,h_2} X(\epsilon, t) &= X(\epsilon+h_1, t+h_2) - X(\epsilon, t) \nonumber \\
\delta_{h_1}^1 X(\epsilon, t) &= X(\epsilon+h_1, t) - X(\epsilon, t) \nonumber \\
\delta_{h_2}^2 X(\epsilon, t) &= X(\epsilon, t+h_2) - X(\epsilon, t),
\end{align*}
\]

Fix \( \epsilon \in [0,1] \) and \( t \in [0,T] \). Let \( h_1, h_2 \in \mathbb{R} \) such that \( \epsilon + h_1 \in [0,1] \) and \( t + h_2 \geq 0 \). Let \( \ell \in \mathbb{N}, \sigma > 0 \). We aim at showing the bound

\[
\mathbb{E}\left[ (\langle \nabla \rangle^{-\sigma} \delta_{h_1,h_2}: \Psi_{\epsilon,N}(t, \cdot) : (x))^2 \right] \lesssim \|(h_1, h_2)\|_2^2,
\]

for some \( \gamma > 0 \) and uniformly in all parameters. In (4.7), \( \cdot \|_2 \) denotes the euclidian norm on \( \mathbb{R}^2 \).

Let \((x, y) \in (\mathbb{T}^2)^2\). We only treat the case \((\epsilon, \epsilon + h_1) \in (0,1] \) as the case \( \epsilon = 0 \) or \( \epsilon + h_1 = 0 \) follows from similar considerations. Expanding the expression

\[
\begin{align*}
\frac{1}{\ell!} \mathbb{E}\left[ \delta_{h_1,h_2} : \Psi_{\epsilon,N}(t,x) : \delta_{h_1,h_2} : \Psi_{\epsilon,N}(t,y) : \right]
\end{align*}
\]

yields
Let us assume $h_1 \geq 0$ for convenience. By (3.16) and (3.9) and the mean value theorem, we also have

$$
|\text{III}|, |\text{IV}| \lesssim \langle n \rangle^{-2}.
$$
\[
|\text{III}| \lesssim h_1 \varepsilon^{-5} \| \hat{D_e}(n, t') \|_{L^\infty_t} + \varepsilon^{-2} h_1 \| \sup_{\eta \in (\varepsilon, \varepsilon + h_1)} \partial_{\eta} \hat{D_e}(n, t') \|_{L^\infty_t}
\]
\[
\lesssim h_1 \varepsilon^{-7} (n)^2. \tag{4.16}
\]

Noting that
\[
\delta_{h_1}^0 (\varepsilon^{-2} \hat{D_e}(n, t')) = (\langle \varepsilon + h_1 \rangle^{-2} \hat{D_e}(n, t') - e^{-\langle \varepsilon \rangle^2}) + (e^{-\langle \varepsilon \rangle^2} - e^{-2 \hat{D_e}(n, t')}),
\]
we estimate for \( \langle n \rangle \leq (\varepsilon + h_1)^{-1+\theta} \) using 3.31:
\[
|\text{III}| \lesssim T \varepsilon^{2\theta} + (\varepsilon + h_1)^{2\theta}. \tag{4.17}
\]

We claim that we get,
\[
|\text{III}| \lesssim |h_1| \gamma (n)^{-2+\gamma_2}, \tag{4.18}
\]
for some \( \gamma_1 > 0 \) and some small \( 0 < \gamma_2 \ll \sigma \). We may assume \( h_1 \varepsilon^{-7} > h_1^2 \) for \( 0 < \gamma \ll 1 \), for otherwise, interpolating 4.15 and 4.16 gives 4.18. We then have \( \varepsilon < h_1^{1-\gamma} \). If \( \langle n \rangle \leq (\varepsilon + h_1)^{-1+\theta} \) we have
\[
|\text{III}| \lesssim \frac{|h_1|^{2\theta (1-\gamma)}}{h_1^{1-\gamma}}. \tag{4.19}
\]

Interpolating 4.19 with 4.15 then yields 4.18. Otherwise \( \langle n \rangle > (\varepsilon + h_1)^{-1+\theta} \) and hence
\[
\langle n \rangle^{-2+\gamma} (\varepsilon + h_1) \gamma (1-\theta) \lesssim \langle n \rangle^{-2+\gamma} h_1^{1-\gamma (1-\theta)},
\]
for \( 0 < \gamma \ll \sigma \). This concludes the proof of 4.18. Regarding IV, we estimate using 3.17
\[
|\text{IV}| \lesssim |h_2| \varepsilon^{-2}. \tag{4.20}
\]

Since we can write
\[
\varepsilon^{-2} \delta_{h_2}^0 \hat{D_e}(n, t') = (e^{-2 \hat{D_e}(n, t') - h_2}) - e^{-(t+h_2)(n)^2}) + (e^{-(t+h_2)(n)^2} - e^{-2 \hat{D_e}(n, t')})
\]
we have, from 3.31, 4.21 and the mean value theorem we have for \( \langle n \rangle \leq \varepsilon^{-1+\theta}:
\[
|\text{IV}| \lesssim \varepsilon^{2\theta} + |h_2| \varepsilon^{-2}. \tag{4.22}
\]
Combining 4.15, 4.20 and 4.22 and arguing as in the estimate of the term III, we deduce
\[
|\text{IV}| \lesssim |h_2| \gamma_1 (n)^{-2+\gamma_2} \tag{4.23}
\]
for some \( \gamma_1 > 0 \) and some small \( 0 < \gamma_2 \ll \sigma \). Thus, we deduce from 4.14, 4.18 and 4.23,
\[
|G_1(n, t, \varepsilon, h_1, h_2)| \lesssim \| (h_1, h_2) \|^2 (n)^{-2+\gamma}, \tag{4.24}
\]
for \( 0 < \gamma \ll \sigma \). Hence, 4.7 follows from 4.11 with 4.12, 4.13 and 4.21.

Arguing as in the computations leading to 4.4 and 4.5, we deduce for \( \ell \in \mathbb{N}, M \geq N, p, q \geq 1, \varepsilon \in [0, 1], t \in [0, T] \) and \( h_1, h_2 \in \mathbb{R} \) such that \( \varepsilon + h_1 \in [0, 1] \) and \( t + h_2 \in [0, T] \),
\[
\| \delta_{h_1, h_2} : \Psi_{\varepsilon,N} (t) \|_{L^p(\Omega) W^{s,\infty}_{x,-\sigma}} \lesssim_{p, \ell} \| (h_1, h_2) \|^2, \tag{4.25}
\]
\[
\| \delta_{h_1, h_2} (\psi_{\varepsilon,N} : - \psi_{\varepsilon,M} (t)) \|_{L^p(\Omega) W^{s,\infty}_{x,-\sigma}} \lesssim_{p, \ell} N^{-\gamma} (\| (h_1, h_2) \|^2.
\]
Fix $n \in \mathbb{Z}^2$. Given the smoothness of $(\varepsilon, t) \mapsto \widehat{D}_\varepsilon(n, t')$, the following integration by parts formula holds almost-surely:
\[
\int_0^t \varepsilon^{-2} \widehat{D}_\varepsilon(n, t') dB_n(t') = \varepsilon^{-2} \widehat{D}_\varepsilon(n, t) B_n(t) - \int_0^t \varepsilon^{-2} \partial_t \widehat{D}_\varepsilon(n, t') B_n(t') dt'.
\]
(4.26)
for any $t \geq 0$ and where the Brownian motion $B_n$ is as in (4.26). Hence, we infer from (4.25), (4.24) and Lemma 3.2 that for each $N \in \mathbb{N}$, the map $(\varepsilon, t) \mapsto \Psi_{\varepsilon,N}$ belongs to $\mathcal{C}((0,1] \times [0,T]; H_\varepsilon^\infty)$; whence so does $(\varepsilon, t) \mapsto \Psi_{\varepsilon,N}^\ell$ by (1.36). Thus, by applying a bi-parameter Kolmogorov continuity criterion, see [1, Theorem 2.1], on $(0,1] \times [0,T]$ we have the following bounds on a set of full $\mathbb{P}$-probability $\Omega_0$: 
\[
\|\delta_{h_1,h_2} \Psi_{\varepsilon,N}^\ell : (t)\|_{\mathcal{L}_\varepsilon^{\sigma,\infty}} \lesssim \|h_1, h_2\|^\gamma_2 \\
\|\delta_{h_1,h_2} \Psi_{\varepsilon,N}^\ell : (t)\|_{\mathcal{L}_\varepsilon^{\sigma,\infty}} \lesssim N^{-\gamma} \|h_1, h_2\|^\gamma_2.
\]
(4.27)
for each $(\varepsilon, \varepsilon + h_1, t, t + h_2) \in (0,1]^2 \times [0,T]$ and any $M \geq N$.

We now verify the continuity at $\varepsilon = 0$ of our stochastic objects. Following the proof of (4.7), we can show 
\[
\mathbb{E}[\langle (\nabla)^{-\sigma} \delta_{h_2} : \Psi_{\varepsilon,N}^\ell(t,.) : - \psi_{0,N}(t,.)\rangle(\cdot)] \lesssim \varepsilon^\gamma \|h_2\|^\gamma_2,
\]
for each fixed $\varepsilon \in [0,1]$, $t \in [0,T]$ and $h_2 \in \mathbb{R}$ such that $t + h_2 \in [0,T]$, $N \in \mathbb{N}$ and some $0 < \gamma \ll 1$. From the usual Kolmogorov criterion, we then obtain, for each $\varepsilon \in (0,1]$, a set of full $\mathbb{P}$-probability $\Omega(\varepsilon)$ such that
\[
\| : \Psi_{\varepsilon,N}^\ell : -- : \psi_{0,N}\|_{\mathcal{L}_\varepsilon^{\sigma,\infty}} \lesssim \varepsilon^\gamma,
\]
(4.28)
for any $T > 0$ and $M \geq N$. We define $\Omega_1$ to be the full probability set
\[
\Omega_1 := \Omega_0 \cap \bigcap_{p \in \mathbb{N}} \Omega(p^{-1}).
\]
(4.29)
Note that combining (4.27) and (4.28) shows that, for each $N \in \mathbb{N}$, $\Psi_{\varepsilon,N}^\ell$ is continuous at $\varepsilon = 0$ on $\Omega_1$. We thus deduce from (4.27),
\[
\| : \Psi_{\varepsilon,N}^\ell : \|_{\mathcal{L}_\varepsilon^{\sigma,\infty}} \lesssim 1 \\
\| : \Psi_{\varepsilon,N}^\ell : \|_{\mathcal{L}_\varepsilon^{\sigma,\infty}} \lesssim N^{-\gamma},
\]
(4.30)
on $\Omega_1$ and for each $M \geq N$. This shows that the sequence $\{ : \Psi_{\varepsilon,N}^\ell : \}_{N \in \mathbb{N}}$ is almost surely a Cauchy sequence in $\mathcal{C}((0,1] \times [0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))$. By uniqueness of the almost sure limit in $L^2((0,1] \times [0,T]; W^{-\sigma,\infty}(\mathbb{T}^2))$ ($\sigma > 0$), this limit is equal to the map $(\varepsilon, t) \mapsto \Psi_{\ell}(t)$: up to a set of measure zero. □

4.2. Local theory. We present a local well-posedness argument based on Sobolev’s inequality as in [19]. This yields local existence for the family of solutions $\{v_\varepsilon\}_{\varepsilon \in [0,1]}$ such that the map $\varepsilon \in [0,1] \mapsto v_\varepsilon$ is continuous. Instead of proving the continuity in $\varepsilon \in [0,1]$ of the family of solutions $\{v_\varepsilon\}_{\varepsilon \in [0,1]}$ by hand; we directly solve the fixed point problem for $(\varepsilon, t) \in [0,1] \times [0,T] \mapsto v_\varepsilon(t)$ (for some random time $T > 0$) in the space $\mathcal{C}([0,1] \times [0,T]; H^{1,-}(\mathbb{T}^2))$. To this end, we consider the following system:
\[
\begin{aligned}
&\varepsilon^2 \partial^2_t v + \partial_t v + (1 - \Delta) v + \sum_{\ell=0}^k (k) \Xi_{\ell} v^{k-\ell} = 0 \\
&\quad (x, \varepsilon, t) \in \mathbb{T}^2 \times [0, 1] \times \mathbb{R}_+.
\end{aligned}
\]

for given initial data \((\phi_0, \phi_1)\) and a source \((\Xi_1, \ldots, \Xi_k)\) with the understanding that \(\Xi_0 \equiv 1\). Given \(\theta > 0\), \(T > 0\), and \(k \geq 2\) define \(\mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2) = \mathcal{A}^{\theta}_{\mathbb{T}^2}(\mathbb{T}^2)\) by

\[
\mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2) := (C([0, 1] \times [0, T]; W^{-\theta, \infty}(\mathbb{T}^2)))^{\otimes k}
\]

and set

\[
\|\Xi\|_{\mathcal{A}^\theta_{\mathbb{T}^2}} = \sum_{j=1}^k \|\Xi_j\|_{C([0,1] \times [0,T]; W^{-\theta, \infty})}
\]

for \(\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_k) \in \mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2)\). In what follows, we use the shorthand notation \(\mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2)\) for \(\mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2)\). We have the following local well-posedness result for \[(4.31)\]:

**Proposition 4.3.** Fix an integer \(k \geq 2\) and \(\delta \leq \frac{1}{2(k-1)} + \frac{\theta}{2}\) for \(0 < \theta \ll \delta\). Then, the equation \[(4.31)\] is unconditionally locally well-posed in \(\mathcal{H}^{1-\delta+\theta} \times \mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2)\). More precisely, given an enhanced data set:

\[
\Xi = (\phi_0, \phi_1, \Xi) \in \mathcal{H}^{1-\delta+\theta}(\mathbb{T}^2) \times \mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2),
\]

with \(\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_k)\), there exist \(T = T(||(\phi_0, \phi_1)||_{\mathcal{H}^{1-\delta+\theta}}, ||\Xi||_{\mathcal{A}^\theta}) \in (0, 1]\) and a unique solution \(v = v(\varepsilon, t)\) to \[(4.31)\] in the class:

\[
C([0, 1] \times [0, T]; H^{1-\delta}(\mathbb{T}^2)).
\]

In particular, the uniqueness of \(v\) holds in the entire class \[(4.33)\]. Furthermore, the solution map \((\phi_0, \phi_1, \Xi) \in \mathcal{H}^{1-\delta+\theta}(\mathbb{T}^2) \times \mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2) \mapsto v \in C([0, 1] \times [0, T]; H^{1-\delta}(\mathbb{T}^2))\) is locally Lipschitz continuous.

**Remark 4.4.** Note that, as is written, our argument provides well-posedness for some \((\phi_0, \phi_1, \Xi) \in \mathcal{H}^{s}(\mathbb{T}^2) \times \mathcal{A}^\theta_{\mathbb{T}^2}(\mathbb{T}^2)\) for \(1 - \frac{1}{2(k-1)} + \frac{\theta}{2} \leq s < 1\) and \(0 < \theta \ll 1 - s\). We thus miss the endpoint \(s = 1 - \frac{1}{2(k-1)}\). It is however possible to prove local well-posedness at the endpoint by using the estimates Proposition \[3.1\] and showing the continuity in \(\varepsilon \in [0, 1]\) of the solutions by hand by making use of the dominated convergence (for high frequencies \(\langle n \rangle > \varepsilon^{-1+\theta}\)). This is however more involved and, for the sake of clarity, we chose to present the simplest way to obtain the continuity property of our objects. Besides, our threshold for local well-posedness is purely technical and would be improved if one were to use a more involved analysis. See for instance \[17\] for a well-posedness argument using Strichartz estimates.

**Proof.** By writing \[(4.31)\] in the Duhamel formulation, we have

\[
v(\varepsilon, t) = \Gamma(v)(\varepsilon, t) \overset{\text{def}}{=} P_{\varepsilon}(t)(\phi_0, \phi_1)
\]

\[
- \sum_{\ell=0}^k \binom{k}{\ell} \mathcal{I}_\varepsilon(t)(\Xi_{\ell}(\varepsilon, t) v(\varepsilon, t)^{k-\ell})
\]

where the map \(\Gamma = \Gamma_\Xi\) depends on the enhanced data set \(\Xi\) in \[(4.32)\] and \(P_{\varepsilon}\) and \(\mathcal{I}_\varepsilon\) are as in \[(3.2), (3.1)\]. Fix \(0 < T \leq 1\). We have from Corollary \[3.2\].
\[ \| P_t(t)(\phi_0, \phi_1) \|_{C_{\varepsilon,T}H^{1-\delta}} \lesssim \| (\phi_0, \phi_1) \|_{H^{1-\delta+\theta}_{\varepsilon}} \]  \hspace{1cm} (4.35)

We first treat the case \( \ell = 0 \). From Corollary 3.2 and Sobolev’s inequality (twice), we obtain
\[ \left\| I_\varepsilon(t)(\nu^k) \right\|_{C_{\varepsilon,T}H^{1-\delta}} \lesssim T^{\frac{1}{2}} \left\| \nu^k \right\|_{C_{\varepsilon,T}H^{\delta+\theta}} \lesssim T^{\frac{1}{2}} \left\| \nu^k \right\|_{C_{\varepsilon,T}L^{2}_{T}} \lesssim T^{\frac{1}{2}} \left\| \nu \right\|_{C_{\varepsilon,T}L^{2}_{T} + \varepsilon} \]  \hspace{1cm} (4.36)

provided that
\[ 0 \leq \delta \leq \frac{1}{k-1}. \]

For \( 1 \leq \ell \leq k-1 \), it follows from Corollary 3.2 Lemma 2.3 (i) and (ii), and Sobolev’s inequality that
\[ \left\| I_\varepsilon(t)(\Xi\nu^{k-\ell}) \right\|_{C_{\varepsilon,T}H^{1-\delta}} \lesssim T^{\frac{1}{2}} \left\| \Xi\nu^{k-\ell} \right\|_{C_{\varepsilon,T}H^{\delta+\theta}} \lesssim T^{\frac{1}{2}} \left\| \nabla \nu^{k-\ell} \right\|_{C_{\varepsilon,T}L^{2}_{T}} \lesssim T^{\frac{1}{2}} \left\| \nabla \nu \right\|_{C_{\varepsilon,T}L^{2}_{T} + \varepsilon} \]
\[ \lesssim T^{\frac{1}{2}} \left\| \Xi \right\|_{H^{\delta}} \left\| \nabla \nu \right\|_{C_{\varepsilon,T}H^{1-\delta}} \]  \hspace{1cm} (4.37)

provided that
\[ 0 \leq \delta \leq \frac{1}{2(k-1)} + \frac{\theta}{2}. \]  \hspace{1cm} (4.38)

Lastly, again from Corollary 3.2, we have
\[ \left\| I_\varepsilon(t)(\Xi) \right\|_{C_{\varepsilon,T}H^{1-\delta}} \lesssim T^{\frac{1}{2}} \left\| \Xi \right\|_{C_{\varepsilon,T}H^{\delta+\theta}} \lesssim T^{\frac{1}{2}} \left\| \Xi \right\|_{H^{\delta}} \]  \hspace{1cm} (4.39)

Putting (4.34), (4.35), (4.36), (4.37) and (4.39) together, we have
\[ \left\| \Gamma(v) \right\|_{C_{\varepsilon,T}H^{1-\delta}} \lesssim C_1 \left\| (\phi_0, \phi_1) \right\|_{H^{1-\delta+\theta}_{\varepsilon}} + C_2 T^{\frac{1}{2}} \left( 1 + \left\| \Xi \right\|_{H^{\delta}} \right) \left( 1 + \left\| v \right\|_{C_{\varepsilon,T}H^{1-\delta}} \right)^k, \]

as long as (4.38) is satisfied. An analogous difference estimate also holds. Therefore, by choosing \( T = T\left( \left\| (\phi_0, \phi_1) \right\|_{H^{1-\delta+\theta}_{\varepsilon}}, \left\| \Xi \right\|_{H^{\delta}} > 0 \right) \), sufficiently small, we conclude that \( \Gamma \) is a contraction in the ball \( B_R \subset C\left( [0,1] \times [0, T]; H^{1-\delta}(T^2) \right) \) of radius \( R = 2C_1 \left( \left\| (\phi_0, \phi_1) \right\|_{H^{1-\delta+\theta}_{\varepsilon}} + 1 \right) \).

At this point, the uniqueness holds only in the ball \( B_R \) but by a standard continuity argument, we can extend the uniqueness to hold in the entire \( C\left( [0,1] \times [0, T]; H^{1-\delta}(T^2) \right) \). The regularity of the map \((\phi_0, \phi_1, \Xi) \in H^{1-\delta+\theta}(T^2) \times \mathcal{X}^{\varepsilon}(T^2) \mapsto v \in C\left( [0,1] \times [0, T]; H^{1-\delta}(T^2) \right) \) is easily obtained through similar estimates. We omit details. \( \square \)

We now prove Theorem 1.3. We recall that, with a slight abuse of notations, wave equations for which \( \varepsilon = 0 \) are viewed as heat equations.

**Proof of Theorem 1.3.** Fix \( k \geq 2 \) and \((\phi_0, \phi_1) \in \mathcal{H}^s(T^2) \) for \( \frac{2k-3}{2k-2} < s < 1 \). Fix \( 0 < \theta \ll 1-s \). Let \( \Xi_N = (\Psi_{S,N}; \Psi_{S,N}; \psi_{S,N}; \cdots) \in [0,1] \), for each \( N \in N \) and \( \Xi = (\psi_{S,N}; \psi_{S,N}; \cdots) \). On the full probability set \( \Omega_0 \) constructed in Proposition 4.1, we have \( \Xi, \Xi_N \in \mathcal{X}^{\varepsilon}(T^2) \) for any \( N \in N \). Hence from Proposition 4.3 we get, for each \( N \in N \), a function \( v_N = v_N(\varepsilon, t) \) (resp. \( v = v(\varepsilon, T) \)) which belongs to \( C\left( [0,1] \times [0, T]; H^s(T^2) \right) \) for some almost
surely positive time $0 < T \leq 1$ (which is uniform in $N \in \mathbb{N}$ since $\sup_{N \in \mathbb{N}} \| \Xi_N \|_{\mathcal{X}^\theta} < \infty$) and that solves \((4.31)\) with data given by $\Xi_N$ (resp. $\Xi$). Furthermore, by the continuity of the map $(\phi_0, \phi_1, \Xi) \mapsto v$, we have that $v_N$ converges to $v$ in $C([0,T]; H^s(\mathbb{T}^2))$ as $N \to \infty$.

For any $\varepsilon \in [0,1]$ and $N \in \mathbb{N}$, define $v_{\varepsilon,N} = v_N(\varepsilon, \cdot)$ and $v_\varepsilon = v(\varepsilon, \cdot)$. By construction, $v_{\varepsilon,N}$ (resp. $v_\varepsilon$) solves \((1.39)\) (resp. \((1.41)\)) with initial data $(\phi_0, 1_{\mathbb{E} > 0} \phi_1)$ and belongs to $C([0,T]; H^s(\mathbb{T}^2))$. A bootstrap argument also shows that $(\varepsilon, N)$ is the unique solution to \((1.39)\) (resp. \((1.41)\)) in the class $C([0,T]; H^s(\mathbb{T}^2))$. For $\varepsilon \in [0,1]$ and $N \in \mathbb{N}$, let $u_{\varepsilon,N} = \Psi_{\varepsilon,N} + v_{\varepsilon,N}$ be the solution to \((1.43)\) with initial data $(\phi_0, 1_{\mathbb{E} > 0} \phi_1)$. Then, by the above and Proposition \ref{prop violent}, $u_{\varepsilon,N}$ converges to the process $u_\varepsilon := \Psi_\varepsilon + v_\varepsilon$ in $C([0,T]; H^{-\sigma}(\mathbb{T}^2))$ (for any $\sigma > 0$). At last, from the continuity of the map $\varepsilon \mapsto v_\varepsilon$ and Proposition \ref{prop violent}, we have that $u_\varepsilon$ converges to $u_0$ in $C([0,T]; H^{-\sigma}(\mathbb{T}^2))$ as $\varepsilon \to 0$ on $\Omega_0$ and for any $\sigma > 0$. \hfill \Box



4.3. Asymptotic large times well-posedness. The purpose of this section is to prove Theorem \ref{thm main}. For $\varepsilon_0 \in (0,1]$ (which will be taken small enough later on), we denote by $v^{\varepsilon_0}$ the solution to \((4.31)\) where the range of values for $\varepsilon$ is restricted to $[0,\varepsilon_0]$. We will always be working on the full probability set $\Omega_0$ given by Proposition \ref{prop violent} and will not mention it in the rest of the section.

We have the following global well-posedness result from \cite{26}. See also \cite{41, Proposition 6.1}.

**Proposition 4.5.** Fix $k \geq 2$. Let $s > 0$ and $\phi_0 \in H^s(\mathbb{T}^2)$. Let $v = v(\varepsilon, t)$ be the solution to \((4.31)\) with data given by $\Xi = (\phi_0, \phi_1, \Psi_\varepsilon, \Psi_\varepsilon^2, \ldots, \Psi_\varepsilon^k)_{\varepsilon \in [0,1]}$ constructed in Theorem \ref{thm main}. Then, the function $v_0 = v(0, \cdot)$ exists globally in time.

Note that from \((4.28)\), $D_\varepsilon(t)$ contains a term of the form $\varepsilon^{-4} D_\varepsilon(t)$ and hence, in view of Lemma \ref{lem estimates} (i) and (ii), cannot be bounded uniformly in $\varepsilon$ in any Sobolev space. Thus, we have no control over the $H^s$ norm of $v$ in Proposition \ref{prop local} and we have to reprove a new local well-posedness statement to extend our solution $v^{\varepsilon_0}$ to larger times (for $0 < \varepsilon_0 \ll 1$). However, it turns out that we can bound $D_\varepsilon(t) D_t v(T)$, for any $T > 0$, uniformly in $\varepsilon$. In order to avoid technicalities regarding the convergence of our terms at $\varepsilon = 0$, we will only consider $(\epsilon, t) \mapsto D_\varepsilon(t) D_t v(\epsilon, T)$ on $(0, \varepsilon_0) \times \mathbb{R}_+$ and deal with the continuity at $\varepsilon = 0$ of our solutions separately in Proposition \ref{prop continuity}. This is enough to re-iterate our local well-posedness argument; see Proposition \ref{prop local} below.

**Lemma 4.6.** Fix $k \geq 2$ and $\frac{2k-3}{2k-2} < s < 1$. Let $0 < \theta \ll 1 - s$. Let $\varepsilon_0 \in (0,1]$ and $T > 0$ such that $v^{\varepsilon_0} \in C([0, \varepsilon_0] \times [0, T]; H^s(\mathbb{T}^2))$ solves \((4.31)\) on $[0, \varepsilon_0] \times [0, T]$ with initial data $(\phi_0, 1_{\mathbb{E} > 0} \phi_1) \in H^{s+\theta}(\mathbb{T}^2)$ and $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_k) \in \mathcal{X}^\theta_k(\mathbb{T}^2)$. We have the following bound

$$
\| D_\varepsilon(t) D_t v^{\varepsilon_0}(\varepsilon, T) \|_{C([0,\varepsilon_0] \times \mathbb{R}_+; H^s_2)} \lesssim (1 + T^\frac{1}{2}) (1 + \| \Xi \|_{\mathcal{X}^\theta_k}) (1 + \| v^{\varepsilon_0} \|_{C([0,\varepsilon_0] \times [0, T]; H^s_2)})^k
$$

$$
+ \| (\phi_0, \phi_1) \|_{H^s_2}
$$

\( (4.40) \)

**Proof.** Fix $t \geq 0$ and $\varepsilon \in [0,\varepsilon_0]$. Note that from \((4.34)\), \((3.1)\) and \((3.2)\), we have
\[ D_\varepsilon(t) \partial_t v(\varepsilon, T) = D_\varepsilon(t)(\varepsilon^{-2} + \partial_t)D_\varepsilon(T)\phi_0 + D_\varepsilon(t)\partial_t D_\varepsilon(T)\phi_1 \]

\[ - \sum_{\ell=1}^{k} \frac{k}{\ell} \int_0^T \varepsilon^{-2} \partial_t D_\varepsilon(T - t')(\Xi_\ell(\varepsilon, t')v(\varepsilon, t')^k) dt' \]  

(4.41)

Hence, since the map \((\varepsilon, t) \in [0, \varepsilon_0] \times [0, T] \mapsto (\{\Xi_\ell(\varepsilon, t)\}_{1 \leq \ell \leq k}, v(\varepsilon, t))\) is continuous and the map \((\varepsilon, t) \mapsto D_\varepsilon(t)\) is smooth (Remark 3.8 and Lemma 3.7 (i), (ii)), it suffices to bound \(\|D_\varepsilon(t)\partial_t v(\varepsilon, T)\|_{H^-_x}\) uniformly in \((\varepsilon, t) \in (0, \varepsilon_0) \times \mathbb{R}_+\).

Fix \(1 \leq \ell \leq k\) and \((\varepsilon, t) \in (0, \varepsilon_0) \times \mathbb{R}_+\). From Corollary 3.7 (iii) and (iv), we have

\[ \|D_\varepsilon(t)\partial_t D_\varepsilon(T)\phi_1\|_{H^-_x} \leq \varepsilon \|\partial_t D_\varepsilon(T)\phi_1\|_{H^{-1}_x} \lesssim \varepsilon \|\phi_1\|_{H^{-1}_x}. \]  

(4.42)

We expand

\[ \|D_\varepsilon(t)\partial_t D_\varepsilon(T)\phi_1\|_{H^-_x} \leq \varepsilon \|\partial_t D_\varepsilon(T)\phi_1\|_{H^{-1}_x} \lesssim \varepsilon \|\phi_1\|_{H^{-1}_x}. \]  

(4.43)

By Remark 3.8 and Lemma 3.7 (i) and (ii), we bound by arguing as in the proof of Proposition 4.3

\[ I \lesssim \|\int_0^T D_\varepsilon(t)\partial_t D_\varepsilon(T - t')(\Xi_\ell(\varepsilon, t')v(\varepsilon, t')^k) dt'\|_{H^-_x} \]

\[ \lesssim T^{\frac{3}{2}} \|\Xi_\ell(\varepsilon, t')v(\varepsilon, t')^k\|_{C_{\varepsilon,T}H^s_x} \]

\[ \lesssim T^{\frac{3}{2}} (1 + \|\Xi\|_{L^2}) (1 + \|v\|_{C_{\varepsilon,T}H^s_x})^k. \]

(4.44)

Similarly, using Remark 3.8 along with Lemma 3.7 (iii), we get

\[ II \lesssim \varepsilon^{-1} \|\int_0^T P_\ell D_\varepsilon(T - t')(\Xi_\ell(\varepsilon, t')v(\varepsilon, t')^k) dt'\|_{K^-_x} \]

\[ \lesssim \varepsilon^{-1} \|\int_0^T e^{-\frac{t-t'}{2s}} dt' \|\Xi_\ell(\varepsilon, t')v(\varepsilon, t')^k\|_{C_{\varepsilon,T}H^s_x} \]

\[ \lesssim \varepsilon (1 + \|\Xi\|_{L^2}) (1 + \|v\|_{C_{\varepsilon,T}H^s_x})^k. \]

(4.45)

Hence, from (4.43), (4.44) and (4.45), we have

\[ \|D_\varepsilon(t)\partial_t D_\varepsilon(T)\phi_0\|_{H^-_x} \lesssim (1 + T^{\frac{3}{2}}) (1 + \|\Xi\|_{L^2}) (1 + \|v\|_{C_{\varepsilon,T}H^s_x})^k. \]

(4.46)

We also have from Lemma 3.7 (i), (ii) and (iii),

\[ \|D_\varepsilon(t)\varepsilon^{-2}\partial_t D_\varepsilon(T)\phi_0\|_{H^-_x} \lesssim \|\partial_t D_\varepsilon(T)\phi_0\|_{H^-_x} \lesssim \|\phi_0\|_{H^-_x}. \]

(4.47)

At last, we look at the term \(D_\varepsilon(t)\partial_t^2 D_\varepsilon(T)\phi_0\). By differentiating (3.29), we obtain by combining Lemma 3.7 (i), (ii) and (3.30):
\[ \| D_\varepsilon(t) \partial_t^2 D_\varepsilon(T) \phi_0 \|_{H^s_T} \lesssim \varepsilon^{-4} \| D_\varepsilon(t) D_\varepsilon(T) \phi_0 \|_{H^s_T} + \| \varepsilon^{-2} D_\varepsilon(t) e^{-\frac{x}{\varepsilon^2}} \partial_t S_\varepsilon(T) \phi_0 \|_{H^s_T} \]
\[ \quad + \| D_\varepsilon(t) e^{-\frac{x}{\varepsilon^2}} \partial_t^2 S_\varepsilon(T) \phi_0 \|_{H^s_T} \]
\[ \lesssim \| \phi_0 \|_{H^s_T} + \| D_\varepsilon(t) e^{-\frac{x}{\varepsilon^2}} \partial_t^2 S_\varepsilon(T) \phi_0 \|_{H^s_T} \]  
(4.48)

By analysing the kernels \( \hat{D}_\varepsilon(n, t) \) by adding an exponential weight to the norms at stake one is able to globalize directly a bound that only holds locally in time in the original norm. See Proposition 9.7 in [31] and Proposition 6.8 in [32]. Fix \( \varepsilon_0 \in (0, 1) \) and \( T > 0 \) such that \( v^{\varepsilon_0} \in C^\infty((0, \varepsilon_0] \times [0, T]; H^s(T^2)) \) solves \( (4.31) \) on \((0, \varepsilon_0] \times [0, T]\) with initial data \( (\phi_0, v^{\varepsilon_0}) \in H^{k+\theta}(T^2) \) and \( \Xi = (\Xi_1, \Xi_2, \ldots, \Xi_k) \in X_0^T(T^2) \). Let \( v_0 \in C^\infty([0, T]; H^s(T^2)) \) be the function given by Proposition 4.8 which solves \( (4.31) \) on \([0] \times [0, T] \). Then, there exists a locally bounded function \( T \mapsto K(\varepsilon, T) > 0 \) and \( \gamma > 0 \) such that
\[ \| v^{\varepsilon_0}(\varepsilon, \cdot) - v_0 \|_{C_T H^s_T} \leq K(\varepsilon, T) \varepsilon^\gamma. \]  
(4.51)

for any \( \varepsilon \in [0, \varepsilon_0] \).

**Proof.** We follow an argument in [31]; by adding an exponential weight to the norms at stake one is able to globalize directly a bound that only holds locally in time in the original norm. See Proposition 9.7 in [31] and Proposition 6.8 in [32]. Fix \( \lambda > 0 \) to be chosen later. For \( T > 0 \), we define the norm \( \| \cdot \|_{S_{\lambda, T}} \) on \( C([0, T], H^s(T^2)) \) by
\[ \| v \|_{S_{\lambda, T}} \overset{\text{def}}{=} \| e^{-\lambda T} v \|_{C_T H^s_T} \]  
(4.52)

Note that we have the inequalities
\[ \| v \|_{S_{\lambda, T}} \leq \| v \|_{C_T H^s_T} \leq e^{\lambda T} \| v \|_{S_{\lambda, T}}. \]  
(4.53)

Fix \( \varepsilon \in (0, \varepsilon_0] \). We compute using the first inequality in \( (4.53) \),
\[ \| v^{\varepsilon_0}(\varepsilon, \cdot) - v_0 \|_{S_{\lambda, T}} \leq \| (P_\varepsilon - P_0)(\phi_0, \phi_1) \|_{S_{\lambda, T}} \]  
(4.54)

\[ + \sum_{\ell=0}^{k} \binom{k}{\ell} \| \mathcal{I}_\varepsilon(\Xi_\ell(\varepsilon, \cdot)v^{\varepsilon_0}(\varepsilon, \cdot)^{k-\ell}) - \mathcal{I}_0(\Xi_0(0, \cdot)v_0^{k-\ell}) \|_{S_{\lambda, T}} \]
\[ \lesssim \| (P_\varepsilon - P_0)(\phi_0, \phi_1) \|_{C_T H^s_T} + \max_{0 \leq \ell \leq k} \| (\mathcal{I}_\varepsilon - \mathcal{I}_0)(\Xi_\ell(\varepsilon, \cdot)v^{\varepsilon_0}(\varepsilon, \cdot)^{k-\ell}) \|_{C_T H^s_T} \]
\[ + \max_{0 \leq \ell \leq k} \| \mathcal{I}_0(\Xi_\ell(\varepsilon, \cdot)v^{\varepsilon_0}(\varepsilon, \cdot)^{k-\ell}) - \Xi_\ell(0, \cdot)v_0^{k-\ell} \|_{S_{\lambda, T}} \]
\[ = I + II + III. \]  
(4.55)

From \( (4.5) \), we get:
\[ I \lesssim \varepsilon^\theta \| (\phi_0, \phi_1) \|_{L^{s+\theta}_x} \]  

(4.56)

We also have from (3.4) and computations similar to those in the proof of Proposition 4.3

\[ II \lesssim T^\frac{1}{2} \varepsilon^\theta \| \Xi_{\varepsilon, \cdot} v^{\varepsilon_0} (\varepsilon, \cdot) \|_{C((0, \varepsilon_0] \times [0, T]; H^{s+\theta}_x)} \]  

(4.57)

We now estimate the term III in the case \( \ell = 0 \). From Lemma 2.4 and proceeding as in (4.36), we have:

\[
\| I_0 (v^{\varepsilon_0} (\varepsilon, \cdot) k - v_0^\varepsilon) \|_{S_{\lambda, T}} = \| \int_0^t e^{-\lambda (t-t')} P_0 (t-t') (e^{-\lambda t'} (v(\varepsilon, t') k - v_0 (t') k)) dt' \|_{C_T H^s_x} \\
\lesssim \sup_{0 \leq t \leq T} \int_0^t e^{-\lambda (t-t')} (t-t')^{-\frac{1}{2}} dt' \| e^{-\lambda t} (v^{\varepsilon_0} (\varepsilon, t) k - v_0 (t) k) \|_{C_T H^{s-1}_x} \\
\lesssim \frac{1}{\sqrt{\lambda}} \| v^{\varepsilon_0} (\varepsilon, \cdot) - v_0 \|_{S_{\lambda, T}} (1 + \| v_0 \|_{C_T H^s_x})^{k-1} (1 + \| v^{\varepsilon_0} \|_{C((0, \varepsilon_0] \times [0, T]; H^s)_x})^{k-1}.
\]

By similar arguments, we get the bound

\[
III \lesssim \varepsilon^\theta + \frac{1}{\sqrt{\lambda}} \| v(\varepsilon, \cdot) - v_0 \|_{S_{\lambda, T}}.
\]

(4.58)

Hence, combining (4.53), (4.56), (4.57) and (4.58), we deduce the existence of \( K(\varepsilon_0, T) > 0 \) such that

\[
\| v(\varepsilon, \cdot) - v_0 \|_{S_{\lambda, T}} \leq K(\varepsilon_0, T) \varepsilon^\theta + \frac{K(\varepsilon_0, T)}{\sqrt{\lambda}} \| v_\varepsilon - v_0 \|_{S_{\lambda, T}}.
\]

(4.59)

Thus, we obtain

\[
\| v(\varepsilon, \cdot) - v_0 \|_{S_{\lambda, T}} \leq 2K(\varepsilon_0, T) \varepsilon^\theta,
\]

(4.60)

upon choosing \( \lambda = (2K(\varepsilon_0, T))^2 \). We hence deduce the claim from the second inequality in (4.53) and (4.60).

\[ \square \]

**Proposition 4.8.** Fix \( k \geq 2 \) and \( \frac{2k-3}{2k-2} < s < 1 \). Let \( 0 < \theta \ll 1 - s \). Let \( \varepsilon_0 \in (0, 1] \) and \( T > 0 \) such that \( v^{\varepsilon_0} \in C([0, \varepsilon_0] \times [0, T]; H^s (\mathbb{T}^2)) \) solves (4.31) on \( [0, \varepsilon_0] \times [0, T] \) with initial data \( (\phi_0, 1_{\varepsilon>0} \phi_1) \in H^{s+\theta}(\mathbb{T}^2) \) and \( \Xi = (\Xi_1, \Xi_2, \ldots, \Xi_k) \in X^0_\theta(\mathbb{T}^2) \). For any \( 0 < \varepsilon_1 \leq \varepsilon_0 \), we can extend \( v^{\varepsilon_0} \) to \([0, \varepsilon_1] \times [T, T+T_1]\) with \( 0 < T_1 < 1 \) such that

\[
T_1 \lesssim \left( (1 + T^\frac{1}{2}) (1 + \| \Xi \|_{X^0_\theta}) (1 + \| v^{\varepsilon_0} \|_{C([0, \varepsilon_0] \times [0, T]; H^s_x})^k \right)^{-2(k-1)}
\]

(4.61)

More precisely, there exists a solution \( v^{\varepsilon_1} \in C([0, \varepsilon_1] \times [T, T+T_1]; H^s(\mathbb{T}^2)) \) to (4.31) on \([0, \varepsilon_1] \times [T, T+T_1]\) with data \((v^{\varepsilon_0}(\varepsilon, T), 1_{\varepsilon>0} \partial_\varepsilon v^{\varepsilon_0}(\varepsilon, T), \Xi)_{\varepsilon \in [0, \varepsilon_1]}\). Furthermore, \( v^{\varepsilon_1} \) is the unique solution to (4.31) in the class \( C([0, \varepsilon_1] \times [T, T+T_1]; H^s(\mathbb{T}^2)) \) and the map \((\phi_0, \phi_1, \Xi) \mapsto v^{\varepsilon_1}\) is locally Lipschitz continuous.

**Proof.** Fix \( 0 < \varepsilon_1 \leq \varepsilon_0 \). By writing (4.31) on \([0, \varepsilon_1] \times [T, T+T_1]\) in the Duhamel formulation, we have
for any \((\varepsilon, t) \in (0, \varepsilon_1] \times [T, T + T_1]\); where \(\Xi_{t'} \varepsilon, t \) is defined as in the proof of Proposition 4.3. We bound for any \(\parallel v\parallel\) a similar difference estimate and we can then prove that \(\Gamma_{\Xi_{t'} \varepsilon, t}\) bound for any function \(v\) and \(0 < T_1 < 1\),

\[
\|\Gamma_T(v)\|_{C((0, \varepsilon_1) \times [T, T + T_1]; H^s) \lesssim \parallel v^{0a}\|_{C([0, \varepsilon_0] \times [0, T]; H^s_x, \|\phi_0, \phi_1\|\parallel H^{s+\delta}_x + (1 + T^2 \frac{1}{2})(1 + \|\Xi\|_{X^B})^0 (1 + \|v^{0a}\|_{C([0, \varepsilon_0] \times [0, T]; H^s_x) \leq (1 + T^2 \frac{1}{2})(1 + \|\Xi\|_{X^B})^0 (1 + \|v^{0a}\|_{C([0, \varepsilon_0] \times [0, T]; H^s_x))}^{k-2k}.
\]

Note that since we do not lose any derivatives in the estimates from Remark 4.3. We get a similar difference estimate and we can then prove that \(\Gamma_T\) has a fixed point in the ball \(B_R \subset \mathcal{C}(0, \varepsilon_1] \times [T, T + T_1]; H^s(T^2))\) for \(R \sim \|\phi_0, \phi_1\|_{H^{s+\delta} \phi} + (1 + T^2 \frac{1}{2})(1 + \|\Xi\|_{X^B})^0 (1 + \|v^{0a}\|_{C([0, \varepsilon_0] \times [0, T]; H^s_x) \leq (1 + T^2 \frac{1}{2})(1 + \|\Xi\|_{X^B})^0 (1 + \|v^{0a}\|_{C([0, \varepsilon_0] \times [0, T]; H^s_x))}^{k-2k}.
\]

Finally, let us define \(v^{01}(0, t) = v_0(t)\), where \(v_0\) is given by Proposition 4.3. The continuity at \(\varepsilon = 0\) of \(v^{01}\) then follows from Lemma 4.7 and hence \(v^{01} \in C((0, \varepsilon_1] \times [0, T]; H^s(T^2))\).

**Proof of Theorem 4.7** Let us fix \(k \geq 2\), a target time \(T > 0\) and \((\phi_0, \phi_1) \in H^s(T^2)\) with \(\frac{2k-3}{2k-2} < s < 1\). Let \(0 < \sigma \ll 1 - s\). Let \(\Xi = (\Psi_{\varepsilon, \cdot}, \cdot \Psi_{\varepsilon, \cdot}, \cdot \Psi_{\varepsilon, \cdot}, \cdot \varepsilon \in [0, 1]) \in X^B(T^2)\) (by Proposition 4.1). From Proposition 4.3 there exists a solution \(v_0\) to (4.31) on \([0] \times [0, T]\) with data \((\phi_0, \Xi)\), which we choose up to time \(T\); i.e. \(v_0 \in C([0, T]; H^s(T^2))\). From Proposition 4.3 there exists a solution \(v\) to (4.31) with data \((\phi_0, 1_{\varepsilon_0} \phi_1, \Xi)\) and a positive time \(T_0 > 0\) such that \(v \in C([0, 1] \times [0, T_0]; H^s(T^2))\). Note that by uniqueness \(v(0, \cdot) = v_0\) on \([0, T_0]\).

Let us assume that \(T_0 < T\), since there would be nothing to show otherwise. By continuity, there exists \(0 < \varepsilon_0 \ll 1\), such that \(\|v\|_{C([0, \varepsilon_0] \times [0, T_0]; H^s(T^2))} \geq \|v_0\|_{C([0, T_0]; H^s(T^2))} - 1\). By Proposition 4.8 there exists a function \(v^{0a}\) which solves (4.31) on \([0, \varepsilon_0] \times [0, T_0 + T_1]\) with \(T_1 \gtrsim (1 + T^2 \frac{1}{2})(1 + \|\Xi\|_{X^B})^0 (1 + \|v_0\|_{C_T H^s_x})^{-2k}\). Since our time increments only depend on \(T > 0\), we can re-iterate this process to get \(\varepsilon_1 > 0\) and \(v^{01} \in C([0, \varepsilon_1] \times [0, T]; H^s(T^2))\) which solves (4.31) on \([0, \varepsilon_1] \times [0, T]\).

Then, by defining \(u_{\varepsilon} = \Psi_{\varepsilon} + v(\varepsilon, \cdot) \in C([0, T]; H^s(T^2))\) for any \(\varepsilon \in [0, \varepsilon_1]\), we have that \(u_{\varepsilon}\) solves (4.45) and \(u_{\varepsilon}\) converges to \(u_0\) as \(\varepsilon \to 0\) from the above and Proposition 4.1.

**Remark 4.9.** We explain here briefly how to adapt the results of this section in order to obtain the global existence and convergence of \(\tilde{v}_{\varepsilon}\) claimed in Remark 1.11 (ii). We use the notations of Remark 1.11 (ii). We have that for any \(T > 0\) and \(\sigma > 0\), \(\tilde{v}_q\) belongs to \(C([0, T]; H^{1-\sigma}(T^2))\) and \((\varepsilon, t) \mapsto \tilde{v}_{\varepsilon}(t) \in C([0, 1] \times [0, T_0]; H^{1-\sigma}(T^2))\) for some random time \(T_0 > 0\) given by the local well-posedness theory in Subsection 4.2.
Fix $\varepsilon \in (0, 1] \setminus Q$. By adapting the proof of Lemma 4.6 and Proposition 4.8, we can extend $\tilde{v}_\varepsilon$ to $C([T_0, T_1]; H^{1-\sigma}(T^2))$ where $T_1 = T_1(\|\tilde{v}_\varepsilon\|_{C_{T_0}H^{1-\sigma}})$ only depends on $\|\tilde{v}_\varepsilon\|_{C_{T_0}H^{1-\sigma}}$. By modifying the argument of Lemma 4.7, we have
\[
\|\tilde{v}_\varepsilon - \tilde{v}_q\|_{C_{T_0}H^{1-\sigma}} \leq K(\varepsilon, T_0)(\varepsilon - q)^\gamma,
\]
for some $\gamma > 0$ and a locally bounded function $T \mapsto K(\varepsilon, T)$. Hence, by density and continuity, we can fix $q \in [0, 1] \cap Q$ such that $\|\tilde{v}_\varepsilon - \tilde{v}_q\|_{C_{T_0}H^{1-\sigma}} \leq K(\varepsilon, T_0)$. Since $\tilde{v}_q$ exists globally, we can extend $\tilde{v}_\varepsilon$ by a time increment $\sim 1$. We can then reprove a bound of the form (4.64) with $T_0$ and $K(\varepsilon, T_0)$ replaced by $T_1$ and $K(\varepsilon, T_1)$, respectively. Repeating this procedure (with the same rational $q$) allows us to extend $\tilde{v}_\varepsilon$ indefinitely. Note that the fact that $T \mapsto K(\varepsilon, T)$ is locally bounded is crucial here. The continuity of $\varepsilon \in (0, 1] \mapsto \tilde{v}_\varepsilon \in C([0, T]; H^{1-\sigma}(T^2))$ for some fixed $T > 0$ can then be proved by hand.

5. Sine-Gordon model

We now look at the well-posedness problem for (1.42). First, we study the stochastic objects defined in (1.38).

5.1. Stochastic objects. Let $J_\alpha$ be the convolution kernel of the operator $\langle \nabla \rangle^{-\alpha}$. Namely, $J_\alpha$ is the distribution
\[
J_\alpha(x) \overset{\text{def}}{=} \lim_{N \to \infty} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \frac{\chi N(n)}{\langle n \rangle^\alpha} e_n(x).
\]
(5.1)
The distribution $J_2$ is the well-known Green function and will be denoted by $G$. We also define the following distribution $H$ by
\[
H(t, x) \overset{\text{def}}{=} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \frac{1 - e^{-2\langle n \rangle^2}}{(\langle n \rangle^2)} e_n(x).
\]
(5.2)
The proof of next lemma can be found in [34, Lemma 2.2].

Lemma 5.1. Let $0 < \alpha < 2$. There exists a smooth function $R$ on $T^2$ and a constant $c_\alpha$ such that
\[
J_\alpha(x) = c_\alpha |x|^\alpha - 2 + R(x)
\]
for all $x \in T^2 \setminus \{0\} \cong [-\pi, \pi] \setminus \{0\}$.

We record in the following Lemma some useful estimates on the Green function $G$ which were proven in [34, Lemma 2.3].

Lemma 5.2. Let $G$ be the Green function $J_2$ defined in (5.1) and let $H$ be as in (5.2).

(i) For any $N \in \mathbb{N}$, $t \geq 0$ and $x \in T^2 \setminus \{0\}$, we have
\[
P_N^2 G(x) \approx -\frac{1}{2\pi} \log \left(|x| + N^{-1}\right)
\]
and
\[
P_N^2 H(t, x) \approx -\frac{1}{2\pi} \log \left(|x| + t^2 + N^{-1}\right)
\]
for some $N_2 \geq N_1 \geq 1$, $t \geq 0$ and $x \in T^2 \setminus \{0\}$, we have
\[
\|P_{N_j}^2 G(x) - P_{N_1} P_{N_2} G(x)\| \lesssim (1 + \log(|x| + N_j^{-1})) \wedge (N_j^{-1}|x|^{-1}) \\
\|P_{N_j}^2 H(t, x) - P_{N_1} P_{N_2} H(t, x)\| \lesssim (1 + \log(|x| + t^{1/2} + N_j^{-1})) \wedge (N_j^{-1}|x|^{-1}),
\]
for \( j = 1, 2. \)

We define the covariance function for \( \varepsilon, t \geq 0 \)
\[
\Gamma_{\varepsilon,N}(t, x - y) := \mathbb{E}[\Psi_{\varepsilon,N}(t, x) \Psi_{\varepsilon,N}(t, y)].
\]

We can also define the distribution \( \Gamma_\varepsilon \) defined by
\[
\Gamma_\varepsilon(t, x - y) := \mathbb{E}[\Psi_\varepsilon(t, x) \Psi_\varepsilon(t, y)],
\]
where \( \Psi_\varepsilon \) is as in \( (2.24) \). Note that by construction, we have \( \Gamma_{\varepsilon,N} = P_{N_j}^2 \Gamma_\varepsilon \) and \( \Gamma_0 = H. \)

We prove the following bounds on the covariance functions.

As in \( (34) \), obtaining good bounds on the covariance function \( \Gamma_{\varepsilon,N} \) and on expressions of the form \( (P_{N_j}^2 - P_{N_1} P_{N_2}) \Gamma_\varepsilon(t, x) \) (for integers \( N, N_1 \) and \( N_2 \)) is instrumental to the construction of the stochastic object \( \Theta_{\varepsilon,N} \) defined in \( (1.38) \). See Proposition \( 5.7 \) below and Proposition \( (34) \) Proposition 1.1. This is the purpose of the next two lemmas. The difficulty lies in keeping track of the dependence of our bounds in \( \varepsilon \) and in the fact that the covariance exhibits two distinct behaviors on the frequency ranges (i) \( \langle n \rangle \leq (2 \varepsilon)^{-1} \) and (ii) \( \langle n \rangle > (2 \varepsilon)^{-1} \). More precisely, the contribution of low frequencies (i) to \( \Gamma_\varepsilon(t, x) \) are similar in nature to that of the covariance function of the stochastic convolution associated to the stochastic heat equation, namely \( H(t, x) \). While on high frequencies (ii) the covariance function \( \Gamma_\varepsilon(t, x) \) is close to the covariance function associated to the stochastic convolution of the damped wave equation, namely \((1 - e^{-\frac{t^2}{4}})G(t, x)\).

**Lemma 5.3.** Fix \( \varepsilon > 0 \). We have the following bounds on the covariance function \( (5.4) \):

(i) For any \( N \in \mathbb{N} \), and \( x, y \in \mathbb{T}^2 \), we have
\[
\Gamma_{0,N}(t, x - y) \approx -\frac{1}{4\pi} \log \left( \frac{|x - y| + N^{-1}}{|x - y| + t^{1/2} + N^{-1}} \right).
\]

(ii) For any \( N \leq (2 \varepsilon)^{-1}, \ t \geq 0 \) and \( x, y \in \mathbb{T}^2 \), we have
\[
\Gamma_{\varepsilon,N}(t, x - y) \approx -\frac{1}{4\pi} \log \left( \frac{|x - y| + N^{-1}}{|x - y| + t^{1/2} + N^{-1}} \right).
\]

(iii) For any \( N > (2 \varepsilon)^{-1}, \ t \geq 0 \) and \( x, y \in \mathbb{T}^2 \), we have
\[
\Gamma_{\varepsilon,N}(t, x - y) \approx -\frac{1}{4\pi} \log \left( \frac{|x - y| + \varepsilon}{|x - y| + t^{1/2} + \varepsilon} \right) - \frac{1 - e^{-\frac{t^2}{4}}}{4\pi} \log \left( \frac{|x - y| + N^{-1}}{|x - y| + \varepsilon} \right).
\]

(iv) For any \( \varepsilon > 0 \) and \( N \leq \varepsilon^{-1+\theta} \), we have
\[
\left| \Gamma_{\varepsilon,N}(t, x - y) - \Gamma_{0,N}(t, x - y) \right| \lesssim \varepsilon^{2\theta},
\]
where the bound is uniform in \( t \geq 0 \) and \( x, y \in \mathbb{T}^2 \).
Proof. Recall that $\Psi_{\varepsilon,N} = P_N \Psi_{\varepsilon}$ with $P_N$ as in (1.19). For $N \in \mathbb{N}$, let $\widetilde{P}_N$ be the sharp projection onto the set $\{n \in \mathbb{Z} : \langle n \rangle \leq N\}$, namely the multiplier with Fourier symbol given by $1_{\langle n \rangle \leq N}$. Note that, for (i)-(iii), since the difference $\|\Psi_{\varepsilon,N} - \widetilde{P}_N \Psi_{\varepsilon}\|_{L^2(\Omega)} \lesssim 1$, it suffices to prove these bounds for the covariance function of $\widetilde{P}_N \Psi_{\varepsilon}$ (which we still denote by $\Gamma_{\varepsilon,N}$ for convenience) instead; effectively replacing the smooth symbol $\chi_N(n)$ by $1_{\langle n \rangle \leq N}$.

We start by proving (i). We have

$$\Gamma_{0,N}(t, x - y) = \frac{1}{2\pi} \sum_{\langle n \rangle \leq N} \frac{1 - e^{-2t\langle n \rangle^2}}{\langle n \rangle^2} e_n(x - y). \quad (5.6)$$

In view of Lemma 5.2 (i) and the fact that one can replace the smooth symbol $\chi_N(n)$ by $1_{\langle n \rangle \leq N}$ in (5.6) at the cost of an additive constant, which is acceptable; (i) follows from the following bound:

$$\frac{1}{2\pi} \sum_{\langle n \rangle \leq N} e^{-2t\langle n \rangle^2} e_n(x - y) \approx -\frac{1}{4\pi} \log \left( |x - y| + t^\frac{1}{2} + N^{-1} \right). \quad (5.7)$$

For $0 \leq t \leq N^{-2}$, we have

$$\left| \sum_{\langle n \rangle \leq N} \frac{1 - e^{-2t\langle n \rangle^2}}{\langle n \rangle^2} e_n(x - y) \right| \lesssim t \sum_{\langle n \rangle \leq N} 1 \lesssim tN^2 \lesssim 1. \quad (5.8)$$

Hence, both sides of (5.7) equal $-\frac{1}{4\pi} \log \left( |x - y| + N^{-1} \right)$ up to constants, and the claim is verified in this regime. For $t > N^{-2}$, the monotony of the function $r \mapsto \frac{e^{-2t(1+r^2)}}{1+r^2}$ gives

$$\left| \sum_{t^{-\frac{1}{2}} \leq \langle n \rangle \leq N} \frac{e^{-2t\langle n \rangle^2}}{\langle n \rangle^2} e_n(x - y) \right| \lesssim \int_{t^{-\frac{1}{2}} \leq \langle x \rangle \leq N} \frac{e^{-2t\langle x \rangle^2}}{\langle x \rangle^2} dx \lesssim e^{-2t} \int_{t^{-\frac{1}{2}} \leq r \leq N} \frac{e^{-2tr^2}}{r} dr \lesssim \int_{1 \leq y \leq t^{\frac{1}{2}} N} \frac{e^{-y^2}}{y} dy \lesssim 1. \quad (5.9)$$

Furthermore, we have from (5.8) with Lemma 5.2 (i),

$$\frac{1}{2\pi} \sum_{\langle n \rangle \leq t^{-\frac{1}{2}}} \frac{e^{-2t\langle n \rangle^2}}{\langle n \rangle^2} e_n(x - y) \approx -\frac{1}{4\pi} \log \left( |x - y| + t^\frac{1}{2} \right) \quad (5.10)$$

In this case, (5.17) and (5.10) show that both sides of (5.7) equal $-\frac{1}{4\pi} \log \left( |x - y| + t^\frac{1}{2} \right)$ and the claim is verified.

We now prove (ii). Fix $0 < \Lambda \ll 1$ such that $N \leq \Lambda \cdot (2\varepsilon)^{-1}$. We have
\[ \Gamma_{\varepsilon,N}(t, x - y) = \frac{1}{2\pi} \sum_{(n) \leq N} \varepsilon^{-4} \int_0^t e^{-\frac{t - t'}{\varepsilon^2}} \sinh((t - t')\lambda_{\varepsilon}(n))^2 \frac{\lambda_{\varepsilon}(n)^2}{\lambda_{\varepsilon}(n)^2} dt' e_n(x - y) \]  
(5.11)

\[ = \frac{1}{4\pi} \sum_{(n) \leq N} \varepsilon^{-4} \int_0^t e^{-\frac{t - t'}{\varepsilon^2}} \cosh(2(t - t')\lambda_{\varepsilon}(n)) \frac{\lambda_{\varepsilon}(n)^2}{\lambda_{\varepsilon}(n)^2} dt' e_n(x - y) \]

\[ - \frac{1}{4\pi} \sum_{(n) \leq N} \varepsilon^{-2} \frac{1 - e^{-\frac{t}{\varepsilon}}}{{\lambda_{\varepsilon}(n)^2}} e_n(x - y) = I - \Pi. \]  
(5.12)

Since \( \lambda_{\varepsilon}(n) \sim \varepsilon^{-2} \) for \( \langle n \rangle \leq (2\varepsilon)^{-1} \Lambda \), we bound

\[ |\Pi| \lesssim N^2 \varepsilon^2 \lesssim \Lambda^2. \]  
(5.13)

Expanding I gives

\[ I = \frac{\varepsilon^{-4}}{4\pi} \sum_{(n) \leq N} \frac{1 - e^{-(\varepsilon^{-2} - 2\lambda_{\varepsilon}(n))t}}{\lambda_{\varepsilon}(n)^2(\varepsilon^{-2} - 2\lambda_{\varepsilon}(n))} e_n(x - y) \]

\[ + \frac{\varepsilon^{-4}}{4\pi} \sum_{(n) \leq N} \frac{1 - e^{-(\varepsilon^{-2} + 2\lambda_{\varepsilon}(n))t}}{\lambda_{\varepsilon}(n)^2(\varepsilon^{-2} + 2\lambda_{\varepsilon}(n))} e_n(x - y) =: \Pi + \Pi' \]  
(5.14)

As in the estimate for II, we also have

\[ |\Pi'| \lesssim N^2 \varepsilon^2 \lesssim \Lambda^2. \]  
(5.15)

By using the Taylor expansion (1.18), we get

\[ \varepsilon^{-2} - 2\lambda_{\varepsilon}(n) = 2\langle n \rangle^2 - O(\langle n \rangle^2\varepsilon^2)). \]

Thus, we estimate

\[ \left| \frac{\varepsilon^{-4}}{4\pi} \sum_{(n) \leq N} \frac{1 - e^{-(\varepsilon^{-2} - 2\lambda_{\varepsilon}(n))t}}{\lambda_{\varepsilon}(n)^2(\varepsilon^{-2} - 2\lambda_{\varepsilon}(n))} e_n(x - y) - \Gamma_{0,N}(t, x - y) \right| \lesssim N^2 \varepsilon^2 \lesssim \Lambda^2. \]

(5.16)

Hence, combining (5.12), (5.13), (5.14), (5.15), (5.16) and (i) gives (ii) for \( 1 \leq N \leq \Lambda \cdot (2\varepsilon)^{-1} \).

Now, let \( \Lambda \cdot (2\varepsilon)^{-1} < N \leq (2\varepsilon)^{-1} \). We show that the contribution of \( \Lambda \cdot (2\varepsilon)^{-1} < \langle n \rangle \leq (2\varepsilon)^{-1} \) to \( \Gamma_{\varepsilon,N}(t, x - y) \) is bounded by 1; which concludes the proof of (ii). Namely, we show

\[ \left| \frac{1}{2\pi} \sum_{\Lambda \cdot (2\varepsilon)^{-1} < \langle n \rangle \leq N} \varepsilon^{-4} \int_0^t e^{-\frac{t - t'}{\varepsilon^2}} \sinh((t - t')\lambda_{\varepsilon}(n))^2 \frac{\lambda_{\varepsilon}(n)^2}{\lambda_{\varepsilon}(n)^2} dt' e_n(x - y) \right| \lesssim 1. \]  
(5.17)

As in the proof of Lemma 3.5 we divide this scenario in two cases:

- **Case 1:** \( \lambda_{\varepsilon}(n)(t - t') \leq 1 \). In this case, using the inequalities \( |\sinh(x)| \lesssim |x| \) for \( |x| \leq 1 \) and \( e^{-y} \lesssim y^{-2} \) for \( y > 0 \), we get
\[
\left| \frac{1}{2\pi} \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \varepsilon^{-4} \int_{0}^{t} e^{-\frac{t-t'}{\varepsilon^2}} \frac{\sinh((t-t')\lambda_{\varepsilon}(n))^2}{\lambda_{\varepsilon}(n)^2} dt' e_{n}(x-y) \right|
\leq \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \varepsilon^{-4} \int_{0}^{t} e^{-\frac{t-t'}{\varepsilon^2}} (t-t')^2 dt' \leq \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \int_{0}^{t} e^{-\frac{(t-t')}{\varepsilon^2}} dt'
\leq \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \varepsilon^2 \lesssim N^2 \varepsilon^2 \lesssim 1.
\]

- **Case 2:** \(\lambda_{\varepsilon}(n)(t-t') > 1\). Fix \(0 < \gamma \ll \Lambda\). We have the following inequality

\[
\sqrt{1-\varepsilon} \leq 1 - \gamma(1 + \frac{\varepsilon}{2})
\]

for \(\Lambda \leq x \leq 1\). This entails the following bound

\[
e^{-\gamma(t(1-\varepsilon)} \sinh(t\lambda_{\varepsilon}(n)) \lesssim e^{-\gamma t(n)^2}
\]

(5.18)

Hence, we compute, as in the previous case and using the inequalities (5.18) and \(\lambda_{\varepsilon}(n)(t-t') > 1\),

\[
\left| \frac{1}{2\pi} \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \varepsilon^{-4} \int_{0}^{t} e^{-\frac{t-t'}{\varepsilon^2}} \frac{\sinh((t-t')\lambda_{\varepsilon}(n))^2}{\lambda_{\varepsilon}(n)^2} dt' e_{n}(x-y) \right|
\leq \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \varepsilon^{-4} \int_{0}^{t} (t-t')^2 e^{-\frac{(t-t')\gamma}{\varepsilon^2}} (e^{-\frac{(t-t')}{\varepsilon^2}} \sinh((t-t')\lambda_{\varepsilon}(n)))^2 dt'
\leq \sum_{\Lambda^{-1} < \langle n \rangle \leq N} \int_{0}^{t} e^{-\frac{(t-t')\gamma}{\varepsilon^2}} dt' \lesssim N^2 \varepsilon^2 \lesssim 1.
\]

The proof of (iii) follows from similar considerations. More precisely, the first term on the right hand side of (iv) comes from (i) and the contribution of low frequencies \(\langle n \rangle \lesssim (2\varepsilon)^{-1}\). The other term comes from the high frequency regime \(\langle n \rangle > (2\varepsilon)^{-1}\) to \(\Gamma_{\varepsilon}(t, x-y)\). The regime corresponding to the range of frequencies \((2\varepsilon)^{-1} < \langle n \rangle \leq (2\varepsilon)^{-1} \Lambda^{-1} (\Lambda \ll 1)\) of the covariance will be bounded by one while on the very high frequency regime \(\langle n \rangle > (2\varepsilon)^{-1} \Lambda^{-1},\) one can approximate \(\Gamma_{\varepsilon}(t, x-y)\) by \((1-e^{-\frac{t}{2\varepsilon}})G(t, x-y)\). The bound (iii) then follows from the corresponding bound for \(G\) in Lemma 5.2 (i).

The estimate (iv) is a consequence of (5.13), (5.15) and (5.16) with \(\Lambda = \varepsilon^\theta\) for \(0 < \theta \ll 1\) (or rather these bounds where the indicator function \(1_{(n) \leq N}\) is replaced by \(\chi_N(n)\), which is harmless).

\[\square\]

**Remark 5.4.** We demonstrate here that working with Gibbs measure initial data renders the analysis of the covariance function (5.5) easier (i.e. we do not see the dependence in our bounds in \(\varepsilon > 0\)). In view of the absolute continuity of \(\tilde{\mu}_{\varepsilon,SG}\) with respect to \(\tilde{\mu}_{\varepsilon}\) described in Remark 1.11 it suffices to study (1.56) with initial data given by \((\phi_0, \varepsilon^{-1} \phi_1)\) with \((\phi_0, \phi_1)\) as in (1.50). Recalling the notations in (3.2) and (1.24), we define

\[
\Phi_{\varepsilon}(t) = \Phi_{\varepsilon}(t)(\phi_0, \varepsilon^{-1} \phi_1) + \sqrt{2} \Psi_{\varepsilon}(t).
\]

(5.19)

A direct computation shows that \(\Phi_{\varepsilon,N}(t, x) = P_N \Phi_{\varepsilon}(t, x)\) is a mean-zero real-valued Gaussian random variable with variance...
\[ \mathbb{E}[\Phi_{\varepsilon,N}(t, x)^2] = \mathbb{E}[(P_N \Phi_{\varepsilon}(t, x))^2] = \sigma_N \]

for any \( t \geq 0, \ x \in \mathbb{T}^2 \) and \( N \geq 1 \), where \( \sigma_N \) is as in (1.52). Note in particular that \( \sigma_N \) is independent of \( \varepsilon \geq 0 \). We define the covariance function for \( \varepsilon, t \geq 0 \)

\[ \Gamma_{\varepsilon,N}^0(t, x - y) \overset{\text{def}}{=} \mathbb{E}[\Phi_{\varepsilon,N}(t, x)\Phi_{\varepsilon,N}(t, y)] \quad (5.20) \]

Due to the stationarity in time of the stochastic convolution (5.20), we can show that \( \Gamma_{\varepsilon,N}^0 \) is both independent of \( t \) and \( \varepsilon \). Hence, studying \( \Gamma_{\varepsilon,N}^0 \) is much simpler than studying \( \Gamma_{\varepsilon,N} \) (5.5).

Proof. Fix \( j \in \{1, 2\} \). For \( \varepsilon = 0 \), (i.e. when \( \Gamma_{\varepsilon} \) is replaced by \( H \) defined in (5.22)), the bound (5.21) is a consequence of the bounds on \( H \) in Lemma 5.2. Indeed, by interpolation, and Lemma 5.2 (ii), we obtain using the inequality \(|\log(y)| \lesssim y^\delta \) for any \( y \geq 1 \) and \( \delta > 0 \),

\[ |P_{N_j}^2 \Gamma_{\varepsilon}(t, x) - P_{N_1} P_{N_2} \Gamma_{\varepsilon}(t, x)| \lesssim N_1^{-\delta} |x|^{-2\delta}, \quad (5.21) \]

for any \( t, \varepsilon \geq 0, \ x \in \mathbb{T}^2 \setminus \{0\} \), \( j = 1, 2 \) and \( \delta > 0 \) small enough.

Lemma 5.5. Let \( 1 \leq N_1 \leq N_2 \). We have the following bound:

\[ |P_{N_j}^2 \Gamma_{\varepsilon}(t, x) - P_{N_1} P_{N_2} \Gamma_{\varepsilon}(t, x)| \lesssim N_1^{-\delta} |x|^{-2\delta}, \quad (5.21) \]

for any \( t, \varepsilon \geq 0, \ x \in \mathbb{T}^2 \setminus \{0\} \), \( j = 1, 2 \) and \( \delta > 0 \) small enough.

Proof. Fix \( j \in \{1, 2\} \). For \( \varepsilon = 0 \), (i.e. when \( \Gamma_{\varepsilon} \) is replaced by \( H \) defined in (5.22)), the bound (5.21) is a consequence of the bounds on \( H \) in Lemma 5.2. Indeed, by interpolation, and Lemma 5.2 (ii), we obtain using the inequality \(|\log(y)| \lesssim y^\delta \) for any \( y \geq 1 \) and \( \delta > 0 \),

\[ |P_{N_j}^2 H(t, x) - P_{N_1} P_{N_2} H(t, x)| \lesssim (-\log(|x| + t^{\frac{1}{2}} + N_2^{-1}))^{1-\delta}(N_1^{-1}|x|^{-1})^\delta \]

\[ \lesssim (|x| + t^{\frac{1}{2}} + N_2^{-1})^{-(1-\delta)}(N_1^{-1}|x|^{-1})^\delta \]

\[ \lesssim N_1^{-\delta} |x|^{-2\delta}. \]

Fix \( \varepsilon > 0 \). It suffices to prove the bound (5.21) in three cases: (i) \( 1 \leq N_1, N_2 \lesssim \varepsilon^{-1+\theta} \), (ii) \( \varepsilon^{-1+\theta} \lesssim N_1, N_2 \lesssim \varepsilon^{-1-\theta} \) and (iii) \( N_1, N_2 \gtrsim \varepsilon^{-1-\theta} \). Indeed, if for instance \( N_1 \ll \varepsilon^{-1+\theta} \ll N_2 \) (say for \( j = 2 \) in (5.21)). Then, we pick \( M \) such that \( N_1 \ll M \ll N_2 \) and \((N_1, M)\) belongs to case (i) while \((M, N_2)\) belongs to case (ii). Note that by assumption, the Fourier support of \( P_{N_j}, P_M \) and \( P_{N_2} \) are included in each other (in that order). This leads to the following identity:

\[ P_{N_2} - P_{N_1} P_{N_2} = P_{N_2} - P_{N_1} = (P_{N_2} - P_M) + (P_M - P_{N_1}) \]

\[ = (P_{N_2} - P_M P_{N_2}) + (P_M - P_{N_1} P_M). \quad (5.22) \]

Hence, from (5.22) and case (i) and (ii), we get

\[ |P_{N_j}^2 \Gamma_{\varepsilon}(t, x) - P_{N_1} P_{N_2} \Gamma_{\varepsilon}(t, x)| \lesssim (M^{-\delta} + N_1^{-\delta})|x|^{-\delta} \lesssim N_1^{-\delta} |x|^{-\delta}, \]

and proves (5.21) in this case. The other scenarios follow similarly from cases (i), (ii) and (iii).

We first note that, by interpolating the estimates in Lemma 5.2 the bound (5.21) holds for any \( 1 \leq N_1 \leq N_2 \) when \( \Gamma_{\varepsilon} \) is replaced by \( G \) or \( H \). Hence, case (i) follows from Lemma 5.3 (iv) and (5.21) for \( H \). Regarding case (iii), we can prove (as in Lemma 5.3 (iii)) the bound

\[ |(P_{N_j}^2 - P_{N_1} P_{N_2})(\Gamma_{\varepsilon}(t, x) - (1 - e^{-\frac{t}{N_1^2}})G(x))| \lesssim N_1^{-\theta}, \]

which implies (5.21) when combined with (5.21) for \( G \). We now consider case (ii).
Let \( \varepsilon^{-1+\theta} \lesssim N_1 \lesssim N_2 \lesssim \varepsilon^{-1-\theta} \). We have
\[
P_{N_1}^2 \Gamma_\varepsilon(t, x) - P_{N_1} P_{N_2} \Gamma_\varepsilon(t, x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \varepsilon^{-4} (\chi_{N_1}(n)^2 - \chi_{N_1}(n)\chi_{N_2}(n))
\times \int_0^t \mathcal{D}_\varepsilon(n, t - t')^2 dt' e_n(x),
\]
where \( \mathcal{D}_\varepsilon \) is as in (3.7). From (5.23) and Poisson summation formula, we compute
\[
P_{N_1}^2 \Gamma_\varepsilon(t, x) - P_{N_1} P_{N_2} \Gamma_\varepsilon(t, x) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}^2} \rho_{N_1, N_2}^\varepsilon(x + 2\pi \ell),
\]
for \( x \in \mathbb{T}^2 \) and where \( \rho_{N_1, N_2}^\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2 \) is the smooth function defined on the Fourier side by
\[
\rho_{N_1, N_2}^\varepsilon(\eta) = \varepsilon^{-4} (\chi_{N_1}(\eta)^2 - \chi_{N_1}(\eta)\chi_{N_2}(\eta)) \int_0^t e^{-\frac{t'}{2\varepsilon^2}} \mathcal{D}_\varepsilon(\eta, t - t')^2 dt'
\]
Moreover, noting that \( \eta \mapsto \chi_{N_1}(\eta)^2 - \chi_{N_1}(\eta)\chi_{N_2}(\eta) \) is supported on \( \{N_1 \leq |\eta| \leq N_2\} \), we from an integration by parts, Lemma 3.4 Leibniz formula and (5.24), we have for \( x \in \mathbb{R}^2 \setminus \{0\} \) and \( m \in \mathbb{N} \),
\[
|\rho_{N_1, N_2}^\varepsilon(x)| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \rho_{N_1, N_2}^\varepsilon(\eta) e^{i\eta \cdot x} d\eta \right|
\sim |x|^{-2m} \left| \int_{\mathbb{R}^2} \Delta_\eta^m (\rho_{N_1, N_2}^\varepsilon(\eta)) e^{i\eta \cdot x} d\eta \right|
\lesssim \varepsilon^{-4} |x|^{-2m} \sum_{|\alpha_1 + \alpha_2 + \alpha_3| = 2m} \int_{\mathbb{R}^2} \left| \partial_\eta^{\alpha_1} (\chi_{N_1}(\eta)^2 - \chi_{N_1}(\eta)\chi_{N_2}(\eta)) \right|
\times \int_0^t e^{-\frac{t'}{2\varepsilon^2}} |\partial_\eta^{\alpha_2 + \alpha_3} \mathcal{D}_\varepsilon(\eta, t - t')| dt' d\eta
\lesssim \varepsilon^{-4} |x|^{-2m} \sum_{|\alpha_1 + \alpha_2 + \alpha_3| = 2m} \int_{N_1 \leq |\eta| \leq N_2} N_1^{-|\alpha_1|}
\times \int_0^t e^{-\frac{t'}{2\varepsilon^2}} (t')^{\alpha_2 + \alpha_3 + 2} |\alpha_2 - |\alpha_3|| \sum_{1 \leq p \leq |\alpha_2| + |\alpha_3|} (1 + |t'\varepsilon^{-1}\eta|^p)
\times (1_{\{\eta\leq(2\varepsilon)^{-1}\}} e^{2t'\lambda_\varepsilon(\eta)} + 1_{\{\eta> (2\varepsilon)^{-1}\}}) dt' d\eta.
\]
We denote by \( I = I(\alpha_1, \alpha_2, \alpha_3, p) \) and \( II = II(\alpha_1, \alpha_2, \alpha_3, p) \) the contribution of \( \langle \eta \rangle \leq (2\varepsilon)^{-1} \) and \( \langle \eta \rangle > (2\varepsilon)^{-1} \) respectively for a fixed tuple \( (\alpha_1, \alpha_2, \alpha_3, p) \). From (3.15) and the inequality \( e^{-y} \lesssim y^{-a} \) for any \( y > 0 \) and \( a > 0 \), we have
\[
I \lesssim \varepsilon^{-4 - |\alpha_2| - |\alpha_3|} |x|^{-2m} N_1^{-|\alpha_1|} \int_{N_1 \leq |\eta| \leq N_2} \int_0^t e^{-t'(\eta)^2 (t')^{\alpha_2 + \alpha_3 + 2} (1 + |t'\varepsilon^{-1}\eta|^p)} dt' d\eta
\lesssim \varepsilon^{-4 - |\alpha_2| - |\alpha_3|} |x|^{-2m} N_1^{-|\alpha_1|} \int_{N_1 \leq |\eta| \leq N_2} (\varepsilon(\eta))^{-2(\alpha_1 + |\alpha_2|) - 4} (1 + (\varepsilon(\eta))^{-p}) \int_0^t e^{-\frac{t'}{2\varepsilon^2}} dt' d\eta
\lesssim \varepsilon^{-4 - |\alpha_2| - |\alpha_3|} |x|^{-2m} N_1^{-|\alpha_1| - 2(\alpha_1 + |\alpha_2|) - 6 + p} N_2^2 \lesssim |x|^{-2m} N_1^{-2m + c\theta},
\]
where \( c \) is a constant depending on \( p \).
in view of the the conditions \(|\alpha_1| + |\alpha_2| + |\alpha_3| = 2m\) and \(\varepsilon^{-1+\theta} \lesssim N_1 \leq N_2 \lesssim \varepsilon^{-1-\theta}\). Here \(c = c(p)\) is a fixed constant. We bound similarly
\[
\Pi \lesssim |x|^{-2m}N_1^{-2m+c\theta},
\]
for some constant \(c = c(p)\). Performing the summation over \((\alpha_1, \alpha_2, \alpha_3, p)\) with (5.26), (5.27) and (5.28) yields
\[
|\rho_{N_1, N_2}(x)| \lesssim |x|^{-2m}N_1^{-2m+c\theta},
\]
for any \(m \in \mathbb{N}\) and \(x \in \mathbb{R}^2 \setminus \{0\}\). Hence, summing in \(\ell \in \mathbb{Z}^2\) with \(m = 2\) in (5.24) with (5.29) gives
\[
|P_{N_j}^2\Gamma_\varepsilon(t, x) - P_{N_1}P_{N_2}\Gamma_\varepsilon(t, x)| \lesssim |x|^{-4}N_1^{-4+c\theta},
\]
for any \(x \in \mathbb{T}^2 \setminus \{0\}\).

Besides, from Lemma 5.3 (i)-(iii), we have
\[
|\rho_{N_1, N_2}(x)| \lesssim \max_{\ell \in \mathbb{Z}^2} \left| \frac{|x| + \varepsilon}{|x| + t^{\frac{1}{2}} + \varepsilon} \right|^{1 - \varepsilon \varepsilon \frac{1}{2}} 1_{N_1 \geq (2\varepsilon)^{-1}},
\]
for any \(t \geq 0\) and \(x \in \mathbb{T}^2\).

We define the following potentials for \(\varepsilon > 0\) and \(N \in \mathbb{N}\),
\[
\mathcal{J}_{0, N}(t, x) := \frac{|x| + N^{-1}}{|x| + t^{\frac{1}{2}} + N^{-1}}
\]
\[
\mathcal{J}_{\varepsilon, N}(t, x) := \mathcal{J}_{0, N}(t, x) 1_{N \leq (2\varepsilon)^{-1}} + \left( \frac{|x| + \varepsilon}{|x| + t^{\frac{1}{2}} + \varepsilon} \right)^{1 - \varepsilon \varepsilon \frac{1}{2}} 1_{N > (2\varepsilon)^{-1}},
\]
(5.32)

for any \(t \geq 0\) and \(x \in \mathbb{T}^2\).

For \(\sigma_1, \sigma_2 \in \{1, -1\}\) and \(\varepsilon \geq 0\), we also define \(\mathcal{J}^{\sigma_1 \sigma_2}(t, x) := (\mathcal{J}(t, x))^{\sigma_1 \sigma_2}\) for \(t \geq 0\) and \(x \in \mathbb{T}^2\).

The next lemma highlights a “cancellation of charge” property of the potentials defined above as first noted in [21] in the context of the parabolic sine-Gordon model and in [33] for the hyperbolic sine-Gordon equation (with \(\varepsilon = 1\)).

**Lemma 5.6.** Let \(\lambda > 0\) and \(p \in \mathbb{N}\). Given \(j \in \{1, \cdots, 2p\}\), we set \(\sigma_j = 1\) if \(j\) is even and \(\sigma_j = -1\) is odd. Let \(S_p\) be the set of permutations of the set \(\{1, \cdots, p\}\). Then, the following estimate holds:
\[
\prod_{1 \leq j < k \leq 2p} \mathcal{J}_{\varepsilon, N}^{\sigma_1 \sigma_2}(t, x_j - x_k) \lambda \lesssim \max_{\tau \in S_p} \prod_{1 \leq j \leq p} \mathcal{J}_{\varepsilon, N}(t, x_{2j} - x_{2\tau(j)-1})^{-\lambda},
\]
for any set of \(2p\) points \(\{x_j\}_{1 \leq j \leq 2p}\) in \(\mathbb{T}^2\), \(t \geq 0\), \(\varepsilon \geq 0\) and \(N \in \mathbb{N}\).

**Proof.** This is a straightforward modification of the proof of [33] Lemma 2.5.

**Proposition 5.7.** Let \(0 < \beta^2 < 4\pi\). Then for any finite \(p, q \geq 1\), \(T > 0\), \(\alpha > \frac{\beta^2}{4\pi}\), and \(\varepsilon \in [0, 1]\), \(\{\Theta_{\varepsilon, N}\}_{N \in \mathbb{N}}\) defined in (1.38) is a Cauchy sequence in \(L^p(\Omega; L^q([0, T]; W^{-\alpha, \infty}(\mathbb{T}^2)))\) and thus converges to a limiting stochastic process in \(L^p(\Omega; L^q([0, T]; W^{-\alpha, \infty}(\mathbb{T}^2)))\), denoted by \(\Theta_{\varepsilon}\) as \(N \rightarrow \infty\).

Moreover, \(\{(\varepsilon, t) \mapsto \Theta_{\varepsilon, N}(t)\}_{N \in \mathbb{N}}\) also converges to \((\varepsilon, t) \mapsto \Theta_{\varepsilon}(t)\) \(L^p(\Omega; L^q([0, T] \times [0, T]; W^{-\alpha, \infty}(\mathbb{T}^2)))\) and almost surely in \(C([0, 1] \times [0, T]; W^{-\alpha, \infty}(\mathbb{T}^2))\) as \(N \rightarrow \infty\).
Proof. Fix $T > 0$, $\alpha > \frac{\beta^2}{4\pi}$, and $0 < \delta \ll 1$. By using Lemmas 5.3, 5.5, 5.6 in place of Lemmas 2.7, 2.5 and (2.38) in and [34] and modifying the proof of [34] Proposition 1.1], we get

$$\mathbb{E}[|\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon,N}(t,x))|^{2p}] \lesssim \langle t \rangle^{\frac{p^2}{4\pi}}$$

(5.33)

$$\mathbb{E}[\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon,N_2}(t,x) - \Theta_{\epsilon,N_1}(t,x))^2] \lesssim \langle t \rangle^{\frac{p^2}{4\pi}} N_1^{-\kappa},$$

(5.34)

uniformly in $(\epsilon, t) \in [0,1] \times [0,T]$, $x \in \mathbb{T}^2$, $N \in \mathbb{N}$ and for $0 < \kappa \ll 1$, and $N_2 \geq N_1$ two integers. Note that when proving the above bounds, one has to estimate quantities of the form

$$\prod_{1 \leq j < k \leq 2p} J_{\epsilon,N}^\sigma(t,x_j - x_k) \frac{\beta^2}{4\pi}$$

(5.35)

for $\epsilon \in [0,1]$, $J_{\epsilon,N}^\sigma$ as in (5.32), $\{\sigma_j\}_{1 \leq j \leq 2p}$ and $\{x_j\}_{1 \leq j \leq 2p}$ as in Lemma 5.6. Since $0 < (J_{\epsilon,N}(t,x))^{-1} \lesssim \frac{1+t^2}{|x|}$, we obtain, using Lemma 5.6

$$\langle \langle \nabla \rangle \rangle \lesssim \max_{\tau \in S} \prod_{1 \leq j \leq 2p} J_{\epsilon,N}(t,2j-x_{2\tau(j)-1})^{-\frac{\beta^2}{4\pi}}$$

(5.36)

which is essentially the bound used in [34] Proposition 1.1] to close the estimate of (5.33) but with $\frac{\beta^2}{4\pi}$ replaced by $\frac{\beta^2}{4\pi}$. Since the dependence in $t \geq 0$ will not be relevant in the following we will not make the dependence of our bounds on the time variable explicit. Note that the bounds (5.33) and (5.34) show the first part of the statement by arguing as in the proof of Proposition 4.1. Namely, this proves that for any fixed $\epsilon \in [0,1]$, $\{\Theta_{\epsilon,N}\}_{N \in \mathbb{N}}$ and $\{(\epsilon,t) \mapsto \Theta_{\epsilon,N}(t)\}_{N \in \mathbb{N}}$ are Cauchy sequences in $L^p(\Omega; L^q([0,T]; W^{-\alpha,\infty}(\mathbb{T}^2)))$ and in $L^p(\Omega; L^q([0,1] \times [0,T]; W^{-\sigma,\infty}(\mathbb{T}^2)))$ respectively.

We now look at the almost sure convergence of the sequence $\{(\epsilon,t) \mapsto \Phi_{\epsilon,N}(t)\}_{N \in \mathbb{N}}$. The following estimate holds:

$$\mathbb{E}[|\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon,N}(t,x) - \Theta_{0,N}(t,x))|^2] \lesssim \epsilon^\gamma$$

(5.37)

for some small $\gamma > 0$. In order to prove (5.37), it suffices to show

$$\mathbb{E}[|\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon,N}(t,x) - \Theta_{0,N}(t,x))|^2] \lesssim N^C \epsilon^{\kappa}$$

(5.38)

for some large constant $C > 0$ (depending on $\beta$, see below), some small $\kappa > 0$ and $N \leq \epsilon^{-1+\theta}$. Indeed, if $N^C \lesssim \epsilon^{-\frac{\beta^2}{4\pi}}$, then (5.38) yields (5.37). Otherwise, in the case $N^C \gg \epsilon^{-\frac{\beta^2}{4\pi}}$, by combining (5.38) and (5.34) with $M$ an integer such that $M^{C} \sim \epsilon^{-\frac{\beta^2}{4\pi}}$ gives

$$\mathbb{E}[|\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon,N}(t,x) - \Theta_{0,N}(t,x))|^2] \lesssim \mathbb{E}[|\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon,M}(t,x) - \Theta_{0,M}(t,x))|^2]$$

(5.39)

$$+ \sup_{\epsilon_0 \in [0,1]} \mathbb{E}[|\langle \nabla \rangle^{\delta-a}(\Theta_{\epsilon_0,N}(t,x) - \Theta_{\epsilon_0,M}(t,x))|^2]$$

$$\lesssim \epsilon^{\frac{\beta^2}{4\pi}} + M^{-\kappa} \lesssim \epsilon^\gamma,$$

for some small $\gamma > 0$. This proves that (5.37) reduces to (5.38).
We now prove (5.38). From Lemma 5.1, Hölder and Cauchy-Schwarz inequalities, we have
\[
\mathbb{E}[|(\nabla)^{\delta-\alpha}(\Theta_{\varepsilon,N}(t,x) - \Theta_{0,N}(t,x))|^2] = \int_T \int_T J_{\alpha-\delta}(x-y)J_{\alpha-\delta}(x-z) \times \mathbb{E}[(\Theta_{\varepsilon,N}(t,y) - \Theta_{0,N}(t,y))(\Theta_{\varepsilon,N}(t,z) - \Theta_{0,N}(t,z))]dydz \lesssim \|\Theta_{\varepsilon,N}(t,y) - \Theta_{0,N}(t,y)\|^2_{L^\infty_y L^2(\Omega)},
\]
(5.39)
assuming \(\alpha - \delta > 0\). In view of (1.38), one can write
\[
Theorem{\Theta_{\varepsilon,N}(t,y) - \Theta_{0,N}(t,y) = e^{\frac{\alpha^2}{2}\sigma_{\varepsilon,N}(t)}(e^{i\beta \Phi_{\varepsilon,N}(t,y)} - e^{i\beta \Phi_{0,N}(t,y)})}
\]
\[+ (e^{\frac{\alpha^2}{2}\sigma_{\varepsilon,N}(t)} - e^{\frac{\alpha^2}{2}\sigma_{0,N}(t)})e^{i\beta \Psi_{0,N}(t,y)} =: I + II.
\]
for any \(y \in \mathbb{T}^2\). From Lemma 5.3 (ii)-(iv), the condition \(N \lesssim \varepsilon^{-1+\theta}\), and the mean value theorem, we get
\[
\|I\|_{L^\infty_y L^2(\Omega)} \lesssim N^{100\beta^2} \varepsilon^\theta.
\]
(5.40)
Besides, we have from the mean value theorem, Minkowski’s and Bernstein’s inequalities, and by proceeding as in the proof of (4.25) (for \(\ell = 1\)),
\[
\|I\|_{L^\infty_y L^2(\Omega)} \lesssim N^{100\beta^2} \|\hat{\Psi}_{\varepsilon,N}(t,y) - \hat{\Psi}_{0,N}(t,y)\|_{L^2(\Omega) L^\infty_y} \lesssim N^{100\beta^2+10} \|\hat{\Psi}_{\varepsilon,N}(t,y) - \hat{\Psi}_{0,N}(t,y)\|_{L^2(\Omega) W^{-1,\infty}_y} \lesssim N^C \varepsilon^\theta,
\]
(5.41)
for some \(C > 0\). The bounds (5.39), (5.40) and (5.41) imply (5.38).

We now claim the following estimates:
\[
\mathbb{E}[|(\nabla)^{\delta-\alpha}(\delta_{h_1h_2}(\Theta_{\varepsilon,N}(t,\cdot)))(x)|^2] \lesssim \|(h_1, h_2)\|^2 (5.42)
\]
\[
\mathbb{E}[|(\nabla)^{\delta-\alpha}(\delta_{h_1h_2}(\Theta_{\varepsilon,N}(t,\cdot) - \Theta_{0,N}(t,\cdot)))(x)|^2] \lesssim N^{-\gamma} \|(h_1, h_2)\|^2 (5.43)
\]
uniformly in \((\varepsilon, t) \in [0, 1] \times [0, T]\), \(x \in \mathbb{T}^2\) and \(N \geq 1\) such that \((\varepsilon+h_1, t+h_1) \in [0, 1] \times [0, T]\) and \(N_2 \geq N_1\). Since, (5.43) follows from (5.34) and (5.42), we focus on (5.42). By arguing as in the proof of (5.38) and applying the mean value theorem as in the proof of (4.7), we find
\[
\mathbb{E}[|(\nabla)^{\delta-\alpha}(\delta_{h_1h_2}(\Theta_{\varepsilon,N}(t,\cdot)))(x)|^2] \lesssim N^{100(1+\beta^2)} \|(h_1, h_2)\|^2 (5.44)
\]
for some small \(\kappa > 0\) and uniformly in all parameters. The bound (5.41) implies (5.42) by arguing as in the passage from (5.38) to (5.37).

Interpolating the bounds (5.42) and (5.43) with (5.33) and applying the bi-parameter Kolmogorov continuity criterion [11 Theorem 2.1] as in Proposition 4.1 allows us to conclude almost convergence of \(\{(\varepsilon, t) \mapsto \Theta_{\varepsilon,N}(t)\}_{N \in \mathbb{N}}\) to \((\varepsilon, t) \mapsto \Theta_{\varepsilon}(t)\) in \(C([0, 1] \times [0, T]; W^{-\alpha,\infty}(\mathbb{T}^2))\) and concludes the proof.

\(\Box\)
5.2. Well-posedness. In this subsection, we prove Theorem 1.19. We first prove uniform (in \( \varepsilon \)) local well-posedness in \( C((0,1] \times [0,T]; H^{1-\alpha}_x) \) of \((\varepsilon, t) \mapsto w(\varepsilon, t)\) solution to (1.46).

After making the ansatz (1.33), we are left to studying (1.40) and (1.42). It thus suffice to consider the system with unknown \((\varepsilon, t) \mapsto z(\varepsilon, t)\).

\[
\begin{cases}
\varepsilon^2 \partial_t^2 z + \partial_t z + (1 - \Delta)z + \text{Im}(e^{i\beta z} \Theta) = 0 \\
(z, 1_{\varepsilon>0} \partial_t z)|_{t=0} = (\phi_0, 1_{\varepsilon>0} \partial_t \phi_1),
\end{cases}
\quad (x, \varepsilon, t) \in T^2 \times [0, 1] \times \mathbb{R}_+.
\tag{5.45}
\]

for a given (deterministic) distribution \((\varepsilon, t) \mapsto \Theta(\varepsilon, t)\). We prove the following local well-posedness result:

**Proposition 5.8.** Fix \( 0 < \theta \ll 1 \). Let \( 0 < \alpha < \frac{1}{2} - \frac{\theta}{2} \), \((\phi_0, \phi_1) \in H^{1-\alpha+\theta}(T^2)\), and \((\varepsilon, t) \mapsto \Theta(\varepsilon, t)\) be a distribution in \( C([0,1]^2; W^{-\alpha+\theta, \infty}(T^2))\). Then, there exists a random time \( T = T(\|\Theta\|_{C([0,1]; W^{-\alpha+\theta, \infty}(T^2))}) \in (0,1] \) and a unique solution \( z \) to (5.45) in the class \( C([0,1] \times [0,T]; H^{1-\alpha}(T^2))\). Moreover, the solution map \((\phi_0, \phi_1, \Theta) \mapsto z\) is continuous.

The proof essentially follows that of [35, Proposition 3.1] upon minor modifications. As in the proof of Theorem 1.15 in Subsection 4.2, Proposition 5.8 proves the Smoluchowski-Kramers approximation over small (random) times.

**Proof.** Let \( \Lambda \) denote the map
\[
\Lambda(z)(\varepsilon, t) := P_\varepsilon(t)(\phi_0, \phi_1) - I_\varepsilon(t) \text{Im}(e^{i\beta z} \Theta(\cdot)),
\tag{5.46}
\]

where \( P_\varepsilon \) and \( I_\varepsilon \) are as in (3.1) and (3.2). We fix \( \theta < \alpha < \frac{1}{2} \) and \( 0 < T \leq 1 \). By (3.6) and Lemma 2.3 (i) and (iii) with \( \alpha < 1 - \alpha \), we have
\[
\|\Lambda(z)\|_{C_{\varepsilon,T}^1 H^{1-\alpha}_x} \lesssim \|\phi_0\|_{H^{1-\alpha+\theta}_x} + T^\frac{1}{2} \|e^{i\beta z} \Theta\|_{C_{\varepsilon,T}^1 H^{-\alpha+\theta}_x}
\lesssim \|\phi_0\|_{H^{1-\alpha+\theta}_x} + T^\frac{1}{2}\|e^{i\beta z}\|_{C_{\varepsilon,T}^1 H^{1-\theta}_x}(\|\Theta\|_{C_{\varepsilon,T}^1 H^{-\alpha+\theta}_x} + 1 + \|z\|_{C_{\varepsilon,T}^1 H^{1-\alpha}_x})
\tag{5.47}
\]

Note that by the fundamental theorem of calculus, we have
\[
e^{i\beta z_2} - e^{i\beta z_1} = (z_2 - z_1) F(z_1, z_2) \overset{\text{def}}{=} (z_2 - z_1)(i\beta) \int_0^1 e^{i\beta(s z_2 + (1-s)z_1)} ds.
\]

Applying (3.6) and Lemma 2.3 as before yields
\[
\|\Lambda(z_2) - \Lambda(z_1)\|_{C_{\varepsilon,T}^1 H^{1-\alpha}_x} \lesssim T^\frac{1}{2}\|z_2 - z_1\|_{C_{\varepsilon,T}^1 H^{1-\alpha}_x}(\|z_1\|_{C_{\varepsilon,T}^1 H^{1-\alpha}_x} + 1 + \|z\|_{C_{\varepsilon,T}^1 H^{1-\alpha+_\theta}_x})
\times \|\Theta\|_{C_{\varepsilon,T}^1 W^{-\alpha+\theta}_x}
\tag{5.48}
\]

for any small \( \delta > 0 \). Furthermore, making use of Lemma 2.3 (i) and (iii) with Sobolev’s inequality gives
provided $0 < 1 - 2\alpha - \theta - \delta$.

As we conclude from (5.47) and (5.49) that $\Lambda = \Lambda_{\phi_0, \phi_1, \Theta}$ is a contraction on $B(R) \subset C([0, 1] \times [0, T]; H^{1-\alpha+\theta}(\mathbb{T}^2))$ for $R \sim \| (\phi_0, \phi_1) \|_{H_{\xi_0}^{1-\alpha+\theta}}$ provided $T \ll \| \Theta \|_{C_t T W_{-\alpha+\theta, \infty}}^{-\frac{1}{2}}$.

The uniqueness in the whole space follows from a standard continuity argument and minor modifications to the above show the continuous dependence in $(\phi_0, \phi_1)$ and $\Theta$. $\square$

As in the proof of Theorem 1.7 in Subsection 4.3 we need to re-iterate our local well-posedness argument. This is the purpose of the next proposition.

**Proposition 5.9.** Let $0 < \alpha < 1 - \frac{\theta}{2}$ for $0 < \theta \ll 1$. Let $(\phi_0, \phi_1) \in H^{1-\alpha+\theta}(\mathbb{T}^2)$, $T > 0$ and $(\varepsilon, t) \mapsto \Theta_\varepsilon(t) \in C([0, 1] \times [0, T+1]; W^{-\alpha+\theta, \infty}(\mathbb{T}^2))$ such that $z$ solves (5.45) on $[0, 1] \times [0, T]$ with initial data $(\phi_0, 1_{\varepsilon>0}\phi_1) \in H^{1-\alpha+\theta}(\mathbb{T}^2)$. Then, there exists a random time $T_1 = T_1(\| \Theta \|_{C([0, 1] \times [T, T+1]; W^{-\alpha+\theta, \infty}(\mathbb{T}^2))} \in (0, 1)$ and a unique solution $z$ to (5.45) in $C([0, 1] \times [T, T+T_1]; H^{1-\alpha}(\mathbb{T}^2))$.

Moreover, $z$ is the unique solution to (5.45) on $[0, 1] \times [0, T+T_1]$ in the class $C([0, 1] \times [0, T+T_1]; H^{1-\alpha}(\mathbb{T}^2))$ with data $(\phi_0, 1_{\varepsilon>0} \phi_1, \Theta) \in H^{1-\alpha+\theta}(\mathbb{T}^2) \times C([0, 1] \times [0, T+T_1]; W^{-\alpha+\theta, \infty}(\mathbb{T}^2))$ and the solution map $(\phi_0, \phi_1, \Theta) \mapsto z$ is continuous.

**Proof.** We prove the following estimates by arguing as in Lemmas 4.6 and 4.7

$$\| D_{t_\varepsilon}(t) \partial_t z(\varepsilon, T) \|_{C([0, 1] \times \mathbb{R}_+; H_{\xi_0}^{1-\alpha})} \leq (1 + T_1^\frac{1}{2}) \| \Theta \|_{C_t T W_{-\alpha+\theta, \infty}} \| z \|_{C([0, 1] \times [0, T]; H_{\xi_0}^{1-\alpha})}$$

Hence, as in the proof of Proposition Proposition 4.8 by following the estimates of Proposition 5.8 and the bounds in the above, we can construct a solution $z$ to (5.45) on $[0, 1] \times [T, T_1]$ which belongs to $C([0, 1] \times [T, T_1]; H^{1-\alpha}(\mathbb{T}^2))$ for some positive time $T_1 = T_1(\| \Theta \|_{C([0, 1] \times [T, T+1]; W_{-\alpha+\theta, \infty})}) > 0$. Since $T_1$ depends only on $\Theta$, the solution can be extended indefinitely. We omit details. Note that the Smoluchowski-Kramers approximation (i.e. the fact that $z(\varepsilon, \cdot)$ converges to $z(0, \cdot)$ as $\varepsilon \to 0$) follows from the continuity at $\varepsilon = 0$ of our solutions. $\square$

**Acknowledgements.** The author would like to thank his advisor, Tadahiro Oh, for suggesting this problem and his support throughout its completion. The author was supported by the European Research Council (grant no. 864138 “SingStochDispDyn”).
References

[1] P. Baldi, Stochastic calculus, An introduction through theory and exercises. Universitext. Springer, Cham, 2017. xiv+627 pp.

[2] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, Comm. Math. Phys. 166 (1994), no. 1, 1–26.

[3] J. Bourgain, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, Comm. Math. Phys. 176 (1996), no. 2, 421–445.

[4] S. Cerrai, M. Freidlin, Smoluchowski-Kramers approximation for a general class of SPDEs, J. Evol. Equ. 6 (2006), no. 4, 657–689.

[5] S. Cerrai, M. Freidlin, On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom, Probab. Theory Related Fields 135 (2006), no. 3, 363–394.

[6] S. Cerrai, M. Freidlin, Averaging principle for a class of stochastic reaction-diffusion equations, Probab. Theory Related Fields 144 (2009), no. 1-2, 137–177.

[7] S. Cerrai, M. Freidlin, On the Smoluchowski-Kramers approximation for SPDEs and its interplay with large deviations and long time behavior, Discrete Contin. Dyn. Syst. 37 (2017), no. 1, 33–76.

[8] S. Cerrai, N. Glatt-Holtz, The dynamical sine-Gordon model in the full subcritical regime, arXiv:1808.02594 [math.PR].

[9] Z. Chen, M. Freidlin, Smoluchowski-Kramers approximation and exit problems, Stoch. Dyn. 5 (2005), no. 4, 569–585.

[10] G. Da Prato, A. Debussche, Strong solutions to the stochastic quantization equations, Ann. Probab. 31 (2003), no. 4, 1900–1916.

[11] A. Deya, On a non-linear 2D fractional wave equation, Ann. Inst. Henri Poincaré Probab. Stat. 56 (2020), no. 1, 477–501.

[12] R. Fukuiizumi, M. Hoshino, T. Inui, Non-relativistic and ultra relativistic limits in 2d stochastic nonlinear damped Klein-Gordon equation, to appear in Nonlinearity.

[13] M. Gubinelli, H. Koch, T. Oh, Renormalization of the two-dimensional stochastic nonlinear wave equations, Trans. Amer. Math. Soc. 370 (2018), no 10, 7335–7359.

[14] M. Gubinelli, H. Koch, T. Oh, Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity, to appear in J. Eur. Math. Soc.

[15] M. Gubinelli, H. Koch, T. Oh, L. Tolomeo, Global dynamics for the two-dimensional stochastic nonlinear damped wave equations, Int. Math. Res. Not. (2021), rnab084, https://doi.org/10.1093/imrn/rnab084.

[16] T. Hosono, T. Ogawa, Large time behavior and Lp-Lq estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (2004), no. 1, 82–118.

[17] H.P. McKean, Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger, Comm. Math. Phys. 168 (1995), no. 3, 479–491. Erratum: Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger, Comm. Math. Phys. 173 (1995), no. 3, 675.

[18] J.-C. Mourrat, H. Weber, Global well-posedness of the dynamic Φ^4 model in the plane, Ann. Probab. 45 (2017), no. 4, 2398–2476.

[19] T. Narazaki, Lp-L^q estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan 56 (2004), no. 2, 585–626.
ON THE CONVERGENCE OF THE 2-d SINGULAR SNLW

[28] E. Nelson, *A quartic interaction in two dimensions*, 1966 Mathematical Theory of Elementary Particles (Proc. Conf., Dedham, Mass., 1965) pp. 69–73 M.I.T. Press, Cambridge, Mass.

[29] D. Nualart, *The Malliavin calculus and related topics*, Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006. xiv+382 pp.

[30] T. Oh, M. Okamoto, T. Robert, *A remark on triviality for the two-dimensional stochastic nonlinear wave equation*, Stochastic Process. Appl. 130 (2020), no. 9, 5838–5864.

[31] T. Oh, M. Okamoto, L. Tolomeo, *Focusing $\Phi^4_3$-model with a Hartree-type nonlinearity*, arXiv:2009.03251 [math.PR].

[32] T. Oh, M. Okamoto, L. Tolomeo, *Stochastic quantization of the $\Phi^4_3$-model*, preprint.

[33] T. Oh, J. Quastel, *On the Cameron-Martin theorem and almost-sure global existence*, Proc. Edinb. Math. Soc. (2) 59 (2016), no. 2, 483–501.

[34] T. Oh, T. Robert, P. Sosoe, Y. Wang, *On the two-dimensional hyperbolic stochastic sine-Gordon equation*, Stoch. Partial Differ. Equ. Anal. Comput. 9 (2021), 1–32.

[35] T. Oh, T. Robert, P. Sosoe, Y. Wang, *Invariant Gibbs dynamics for the dynamical sine-Gordon model*, Proc. Roy. Soc. Edinburgh Sect. A (2020), 17 pages. doi: https://doi.org/10.1017/prm.2020.68

[36] T. Oh, L. Thomann, *Invariant Gibbs measure for the 2-d defocusing nonlinear wave equations*, Ann. Fac. Sci. Toulouse Math. 29 (2020), no. 1, 1–26.

[37] T. Oh, Y. Wang, Y. Zine, *Three-dimensional stochastic cubic nonlinear wave equation with almost space-time white noise*, Stoch PDE: Anal Comp (2022). https://doi.org/10.1007/s40072-022-00237-x

[38] I. Shigekawa, *Stochastic analysis*, Translated from the 1998 Japanese original by the author. Translations of Mathematical Monographs, 224. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2004. xii+182 pp.

[39] B. Simon, *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974. xx+392 pp.

[40] L. Thomann, N. Tzvetkov, *Gibbs measure for the periodic derivative nonlinear Schrödinger equation*, Nonlinearity 23 (2010), no. 11, 2771–2791.

[41] W.J. Trenberth, *Global well-posedness for the two-dimensional stochastic complex Ginzburg-Landau equation*, arXiv:1911.09246 [math.AP].

[42] Y. Wakasugi, *On the diffusive structure for the damped wave equation with variable coefficients*, Ph.D. thesis.

YOUNES ZINE, SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, JAMES CLERK MAXWELL BUILDING, THE KING’S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UNITED KINGDOM

Email address: y.p.zine@sms.ed.ac.uk