Asymptotic Analysis of Expectations of Plane Partition Statistics

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Abstract
Assuming that a plane partition of the positive integer \( n \) is chosen uniformly at random from the set of all such partitions, we propose a general asymptotic scheme for the computation of expectations of various plane partition statistics as \( n \) becomes large. The generating functions that arise in this study are of the form \( Q(x)F(x) \), where \( Q(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-j} \) is the generating function for the number of plane partitions. We show how asymptotics of such expectations can be obtained directly from the asymptotic expansion of the function \( F(x) \) around \( x = 1 \). The representation of a plane partition as a solid diagram of volume \( n \) allows interpretations of these statistics in terms of its dimensions and shape. As an application of our main result, we obtain the asymptotic behavior of the expected values of the largest part, the number of columns, the number of rows (that is, the three dimensions of the solid diagram) and the trace (the number of cubes in the wall on the main diagonal of the solid diagram). Our results are similar to those of Grabner et al. related to linear integer partition statistics. We base our study on the Hayman’s method for admissible power series.

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1 Introduction
A plane partition \( \omega \) of the positive integer \( n \) is an array of non-negative integers
\[
\begin{array}{cccc}
\omega_{1,1} & \omega_{1,2} & \omega_{1,3} & \cdots \\
\omega_{2,1} & \omega_{2,2} & \omega_{2,3} & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}
\]  

(1)
that satisfy $\sum_{h,j \geq 1} \omega_{h,j} = n$, and the rows and columns in (1) are arranged in decreasing order: $\omega_{h,j} \geq \omega_{h+1,j}$ and $\omega_{h,j} \geq \omega_{h,j+1}$ for all $h,j \geq 1$. The non-zero entries $\omega_{h,j} > 0$ are called parts of $\omega$. If there are $\lambda_h$ parts in the $h$th row, so that for some $l$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq \lambda_{l+1} = 0$, then the (linear) partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of the integer $m = \lambda_1 + \lambda_2 + \ldots + \lambda_l$ is called the shape of $\omega$, denoted by $\lambda$. We also say that $\omega$ has $l$ rows and $m$ parts. Sometimes, for the sake of brevity, the zeroes in array (1) are deleted. For instance, the abbreviation

$$
\begin{array}{ccc}
5 & 4 & 1 \\
3 & 2 & 1 \\
2 & 1 \\
\end{array}
$$

represents a plane partition of $n = 20$ with $l = 3$ rows and $m = 9$ parts.

Any plane partition $\omega$ has an associated solid diagram $\Delta = \Delta(\omega)$ of volume $n$. It is defined as a set of $n$ integer lattice points $x = (x_1, x_2, x_3) \in \mathbb{N}^3$, such that if $x \in \Delta$ and $x' = (x'_1, x'_2, x'_3) \in \Delta$, then $x'_1 \leq x_j, j = 1, 2, 3$, and $x' \in \Delta$ too. (Here $\mathbb{N}$ denotes the set of all positive integers.) Indeed the entry $\omega_{h,j}$ can be interpreted as the height of the column of unit cubes stacked along the vertical line $x_1 = h, x_2 = j$, and the solid diagram is the union of all such columns.

Plane partitions are originally introduced by Young [27] as a natural generalization of integer partitions in the plane. The problem of enumerating plane partitions was studied first by MacMahon [13] (see also [14]), who showed that, for any parallelepiped $B(l, s, t) = \{(h, j, k) \in \mathbb{N}^3 : h \leq l, j \leq s, k \leq t\}$ and any $|x| < 1$,

$$
\sum_{\Delta \subset B(l, s, t)} |\Delta| = \prod_{(h, j, k) \in B(l, s, t)} \frac{1 - x^{h+j+k-1}}{1 - x^{h+j+k-2}},
$$

where $\Delta$ is the solid diagram defined above and $|\Delta|$ denotes its volume (a more explicit version of (2) may be found in [21]).

Let $q(n)$ denote the total number of plane partitions of the positive integer $n$ (or, the total number of solid diagrams of volume $n$). It turns out that (2) implies the following generating function identity:

$$
Q(x) = 1 + \sum_{n=1}^{\infty} q(n)x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-j}
$$

(3)

(more details may be also found, e.g., in [22 Corollary 18.2] and [2 Corollary 11.3]). Subsequent research on the enumeration of plane partitions was focused on bijective interpretations and proofs of MacMahon’s formula (3) (see, e.g., [12, 20]).

The asymptotic form of the numbers $q(n)$, as $n \to \infty$, has been obtained by Wright [26] (see also [13] for a little correction). It is given by the following formula:

$$
q(n) \sim \frac{(\zeta(3))^{7/36}}{2^{11/36}(3\pi)^{1/2}}n^{-25/36}\exp\left(3(\zeta(3))^{1/3}(n/2)^{2/3} + 2\gamma\right),
$$

(4)
where

\[
\zeta(z) = \sum_{j=1}^{\infty} j^{-z}
\]

is the Riemann zeta function and

\[
\gamma = \int_{0}^{\infty} \frac{u \log u}{e^{2\pi u} - 1} du = \frac{1}{2} \zeta'(-1).
\]

(5)

(The constant \(\zeta'(-1) = -0.1654\ldots\) is closely related to Glaisher-Kinkelin constant; see \[6\]).

**Remark 1.** In fact, Wright \[26\] has obtained an asymptotic expansion for \(q(n)\) using the circle method.

Next, we introduce the uniform probability measure \(P\) on the set of plane partitions of \(n\), assuming that the probability \(1/q(n)\) is assigned to each plane partition. In this way, each numerical characteristic of a plane partition of \(n\) becomes a random variable (a statistic in the sense of the random generation of plane partition of \(n\)). In the following, we will discuss several different instances of plane partition statistics. Our goal is to develop a general asymptotic scheme that allows us to derive an asymptotic formula for the \(n\)th coefficient \([x^n]Q(x)F(x)\) of the product \(Q(x)F(x)\), where \(Q(x)\) is defined by (3) and the power series \(F(x)\) is suitably restricted on its behavior in a neighborhood of \(x = 1\). We will show further that expectations of plane partitions statistics we will consider lead to generating functions of this form. From one side, our study is motivated by the asymptotic results of Grabner et al. \[8\] on linear partitions statistics. Their study is based on general asymptotic formulae for the \(n\)th coefficient of a similar product of generating functions with \(Q(x)\) replaced by the Euler partition generating function \(P(x) = \prod_{j=1}^{\infty}(1 - x^j)^{-1}\). The second factor \(F(x)\) satisfies similar general analytic conditions around \(x = 1\). In addition, our interest to study plane partitions statistics was also attracted by several investigations during the last two decades on shape parameters of random solid diagrams of volume \(n\) as \(n \to \infty\). Below we present a brief account on this subject.

Cerf and Kenyon \[4\] have determined the asymptotic shape of the random solid diagram, while Cohn et al. \[5\] have studied a similar problem whenever a solid diagram is chosen uniformly at random among all diagrams boxed in \(B(l, s, t)\), for large \(l, s\) and \(t\), all of the same order of magnitude. Okounkov and Reshetikhin \[19\] rediscovered Cerf and Kenyon's limiting shape result and studied asymptotic correlations in the bulk of the random solid diagram. Their analysis is based on a deterministic formula for the correlation functions of the Schur process. The joint limiting distribution of the height (the largest part in (1)), depth (the number of columns in (1)) and width (the number of rows in (1)) in a random solid diagram was obtained by Pittel \[21\]. The one-dimensional marginal limiting distributions of this random vector were found in \[16\]. The trace of a plane partition is defined as the sum of the diagonal parts in (1). Its limiting distribution is determined in \[11\]. Bodini et al. \[3\] studied random generators of plane partitions according to the size of their solid diagrams. They
obtained random samplers that are of complexity $O(n \log^3 n)$ in an approximate-size sampling and of complexity $O(n^{4/3})$ in exact-size sampling. These random samplers allow to perform simulations in order to confirm the known results about the limiting shape of the plane partitions.

In the proof of our main asymptotic result for the coefficient $[x^n]Q(x)F(x)$ we use the saddle point method. In contrast to [8], we base our study on a theorem due to Hayman [10] for estimating coefficients of admissible power series (see also [7, Section VIII.5]). We show that $Q(x)$ (see (3)) is Hayman admissible function and impose conditions on $F(x)$ which are given in terms of the Hayman’s theorem. In the examples we present, we demonstrate two different but classical approaches for estimating power series around their main singularity.

Our paper is organized as follows. In Section 2 we include the Hayman’s admissibility conditions and his main asymptotic result. We briefly describe a relationship between Hayman’s theorem and Meinardus approach [15] for obtaining the asymptotic behavior of the Taylor coefficients of infinite products of the form

$$f_b(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-b_j}$$

under certain general conditions on the sequence $\{b_j\}_{j \geq 1}$ of non-negative numbers. In Section 2, we also state our main result (Theorem 1) for the asymptotic of the $n$th coefficient $[x^n]Q(x)F(x)$ under certain relatively mild conditions on $F(x)$. The proof of Theorem 1 is given in Section 3. Section 4 contains some examples of plane partition statistics that lead to generating functions of the form $Q(x)F(x)$. We apply Theorem 1 to obtain the asymptotic behavior of expectations of the underlying statistics. For the sake of completeness, in the Appendix we show how Wright’s formula (4) follows from Hayman’s theorem.

2 Some Remarks on Hayman Admissible Functions and Meinardus Theorem on Weighted Partitions. Statement of the Main Result

Our first goal in this section is to give a brief introduction to the analytic combinatorics background that we will use in the proof of our main result. Clearly, $x^n[Q(x)F(x)]$ with $Q(x)$ given by (3) can be represented by a Cauchy integral whose integrand includes the product $Q(x)F(x)$ (the conditions that $F(x)$ should satisfy will be specified later). Its asymptotic behavior heavily depends on the analytic properties of $Q(x)$ whose infinite product representation (3) shows that the unit circle is a natural boundary and its main singularity is at $x = 1$. The main tools for the asymptotic analysis of $q(n) = x^n[Q(x)]$ are either based on the circle method (see [26]) or on the saddle-point method (see [15] and [3]). Both yield Wright’s asymptotic formula (4). An asymptotic formula in a more general framework (see [6]) was obtained by Meinardus [15] (see also [9]).
for some extensions). A proof of formula (4) that combines Meinardus approach with Hayman’s theorem for admissible power series is started in Section 3 and completed in the Appendix.

Here we briefly describe Meinardus’ approach, which is essentially based on analytic properties of the Dirichlet generating series

\[ D(z) = \sum_{j=1}^{\infty} b_j j^{-z}, \quad z = u + iv, \]

where the sequence of non-negative numbers \( \{b_j\}_{j\geq 1} \) is the same as in the infinite product (6). We will avoid the precise statement of Meinardus’ assumptions on \( \{b_j\}_{j\geq 1} \) as well as some extra notations and concepts. The first assumption \( (M_1) \) specifies the domain \( \mathcal{H} = \{ z = u + iv : u \geq -C_0, 0 < C_0 < 1 \} \) in the complex plane, in which \( D(z) \) has an analytic continuation. The second one \( (M_2) \) is related to the asymptotic behavior of \( D(z) \) whenever \( |v| \to \infty \). A function of the complex variable \( z \) which is bounded by \( O(|\Im(z)|^{C_1}) \), \( 0 < C_1 < \infty \), in certain domain in the complex plane is called function of finite order. Meinardus’ second condition \( (M_2) \) requires that \( D(z) \) is of finite order in the whole domain \( \mathcal{H} \). Finally, the Meinardus’ third condition \( (M_3) \) implies a bound on the ordinary generating function of the sequence \( \{b_j\}_{j\geq 1} \). It can be stated in a way, simpler than the Meinardus’ original expression, by the inequality:

\[ \sum_{j=1}^{\infty} b_j e^{-\alpha \sin^2(\pi ju)} \geq C_2 \alpha^{-\epsilon_0}, \quad 0 < \alpha < \frac{\pi}{2|u|} < \frac{1}{2}, \]

for sufficiently small \( \alpha \) and some constants \( C_2, \epsilon_0 > 0 \) \( (C_2 = C_2(\epsilon_0)) \) (see [9, p. 310]).

The infinite product representation (3) for \( Q(x) \) implies that \( b_j = j, j \geq 1 \), and therefore, \( D(z) = \zeta(z-1) \). It is known that this sequence satisfies the Meinardus scheme of conditions (see, e.g., [18] and [9, p. 312]).

Now we proceed to Hayman admissibility method [10, 7, Section VIII.5]. To present the idea and show how it can be applied to the proof of our main result, we need to introduce some auxiliary notations.

Consider a function \( G(x) = \sum_{n=0}^{\infty} g_n x^n \) that is analytic for \( |x| < \rho, 0 < \rho \leq \infty \). For \( 0 < r < \rho \), we set

\[ a(r) = r \frac{G'(r)}{G(r)}, \tag{7} \]

\[ b(r) = \frac{r G''(r)}{G(r)} + r^2 \frac{G''(r)}{G(r)} - r^2 \left( \frac{G'(r)}{G(r)} \right)^2. \tag{8} \]

In the statement of Hayman’s result we use the terminology given in [7, Section VIII.5]. We assume that \( G(x) > 0 \) for \( x \in (R_0, \rho) \subset (0, \rho) \) and satisfies the following three conditions:

Capture condition. \( \lim_{r \to \rho} a(r) = \infty \) and \( \lim_{r \to \rho} b(r) = \infty \).
Locality condition. For some function $\delta = \delta(r)$ defined over $(R_0, \rho)$ and satisfying $0 < \delta < \pi$, one has
\[ G(re^{i\theta}) \sim G(r)e^{i\delta(r) - \theta^2 b(r)/2} \]
as $r \to \rho$, uniformly for $|\theta| \leq \delta(r)$.

Decay condition.
\[ G(re^{i\theta}) = o\left(\frac{G(r)}{\sqrt{b(r)}}\right) \]
as $r \to \rho$ uniformly for $\delta(r) < |\theta| \leq \pi$.

**Hayman Theorem.** Let $G(x)$ be Hayman admissible function and $r = r_n$ be the unique solution of the equation
\[ a(r) = n. \tag{9} \]
Then the Taylor coefficients $g_n$ of $G(x)$ satisfy, as $n \to \infty$,
\[ g_n \sim \frac{G(r_n)}{r_n^{n\sqrt{2\pi b(r_n)}}} \tag{10} \]
with $b(r_n)$ given by (8).

In the next section we will show that $Q(x)$, given by (3), is admissible in the sense of Hayman and apply Hayman Theorem setting $G(x) := Q(x)$.

Now, we proceed to the statement of our main result. As in [8], we assume that $F(x)$ satisfies two rather mild conditions. Since we will employ Hayman’s method, the first one is given in terms of eq. (9). The second one requires, as in [8], that $F(x)$ does not grow too fast as $|x| \to 1$.

**Condition A.** Let $r = r_n$ be the solution of (9). We assume that
\[ \lim_{n \to \infty} \frac{F(r_ne^{i\theta})}{F(r_n)} = 1 \]
uniformly for $|\theta| \leq \delta(r_n)$, where $\delta(r)$ is the function defined by Hayman’s locality condition.

**Condition B.** There exist two constants $C > 0$ and $\eta \in (0, 2/3)$, such that, as $|x| \to 1$,
\[ F(x) = O(e^{C/(1-|x|)^\eta}). \]

Our main result is as follows.

**Theorem 1** Let $\{d_n\}_{n \geq 1}$ be a sequence with the following expansion:
\[ d_n = \left(\frac{2\zeta(3)}{n}\right)^{1/3} - \frac{1}{36n} + O(n^{-1-\beta}), \quad n \to \infty, \tag{11} \]
where $\beta > 0$ is certain fixed constant. Furthermore, suppose that the function $F(x)$ satisfies conditions (A) and (B) and $Q(x)$ is the infinite product given by (3). Then, there is a constant $c > 0$ such that, as $n \to \infty$,
\[ \frac{1}{q(n)}[x^n]Q(x)F(x) = F(e^{-d_n})(1 + o(1)) + O(e^{-cn^{2/9}/\log^2 n}), \]
where \( q(n) \) is the \( n \)th coefficient in the Taylor expansion of \( Q(x) \).

The proof that we will present in the next section stems from the Cauchy coefficient formula in which the contour of integration is the circle \( x = e^{-d_n+i\theta}, -\pi < \theta \leq \pi \) and \( d_n \) is defined by \([11]\). We have

\[
\[x^n\]Q(x)F(x) = \frac{e^{nd_n}}{2\pi} \int_{-\pi}^{\pi} Q(e^{-d_n+i\theta})F(e^{-d_n+i\theta})e^{-i\theta n}d\theta.
\]

(12)

The proof of Theorem 1 is divided into two parts:
(i) Proof of Hayman admissibility for \( Q(x) \).
(ii) Obtaining an asymptotic estimate for the Cauchy integral \([12]\).

3 Proof of Theorem 1

Part (i).
We will essentially use some more general observations established in \([9, 17]\).

First, we set in (7) and (8) \( G(x) := Q(x) \) and \( r = r_n := e^{-d_n} \). The next lemma is a particular case of a more general result due to Granovsky et al. \([9, \text{Lemma 2}]\).

**Lemma 1** For enough large \( n \), the unique solution of the equation

\[
a(e^{-d_n}) = n
\]

is given by \([11]\). Moreover, as \( n \to \infty \),

\[
b(e^{-d_n}) \sim \frac{3n^{4/3}}{(2\zeta(3))^{1/3}}.
\]

(13)

**Proof.** The Dirichlet generating series of the sequence \( \{j\}_{j \geq 1} \) is \( \zeta(z-1) \), which is a meromorphic function in the complex plane with a single pole at \( z = 2 \) with residue 1 (see, e.g., \([25, \text{Section 13.13}]\)). Granovsky et al. \([9]\) showed that it satisfies Meinardus’ conditions \((M_1)\) and \((M_2)\). Hence, by formula (43) in \([9]\), we have

\[
d_n = (\Gamma(2)\zeta(3))^{1/3}n^{-1/3} + \frac{\zeta(-1)}{3n} + O(n^{-1-\beta})
\]

\[
= \left( \frac{2\zeta(3)}{n} \right)^{1/3} - \frac{1}{36n} + O(n^{-1-\beta}), \quad \beta > 0,
\]

which is just \([11]\) (here we have also used that \( \zeta(-1) = -1/12 \) \([25, \text{Section 13.14}]\)). Formula \([13]\) follows from \([11]\) and a more general result from \([17, \text{formula (2.6)}]\). \(\square\)

Lemma 1 shows that \( a(e^{-d_n}) \to \infty \) and \( b(e^{-d_n}) \to \infty \) as \( n \to \infty \), that is Hayman’s "capture" condition is satisfied with \( r = r_n = e^{-d_n} \). To show next that Hayman’s "decay" condition is satisfied by \( Q(x) \), we set

\[
\delta_n = \frac{d_n^{5/3}}{\log n} = \frac{1}{\log n} \left( \frac{2\zeta(3)}{n} \right)^{5/9} (1 + O(n^{-2/3}))
\]

(14)
with \( d_n \) given by (11). The next lemma is a particular case of Lemma 2.4 in [17].

**Lemma 2** For sufficiently large \( n \), we have

\[
| Q(e^{-d_n+i\theta}) | \leq Q(e^{-d_n})e^{-C_3d_n^{-2/3}}
\]

uniformly for \( \delta_n \leq |\theta| < \pi \), where \( C_3 > 0 \) is an absolute constant and \( d_n \) is defined by (11).

**Proof.** We will apply Lemma 2.4 from [17] with \( f_0(x) := Q(x) \) (see (6)) and \( \alpha_n := d_n \). It shows that the ratio \( | f_0(e^{-d_n+i\theta}) | / f(e^{-d_n}) \) is bounded by \( e^{-C_3d_n^{-1}} \) uniformly for \( \delta_n \leq |\theta| < \pi \), where \( C_3, \epsilon_1 > 0 \). Our goal is to show that the plane partition generating function \( Q(x) \) is bounded in the same way with \( \epsilon_1 = 2/3 \). To show this, we notice first that the chain of inequalities (2.18) in [17] with \( b_j = j, j \geq 1 \) implies that

\[
\Re(\log Q(e^{-d_n+i\theta})) - \log Q(e^{-d_n}) \leq \frac{\log 5}{2} S_n
\]

uniformly for \( d_n \leq |\theta| < \pi \), where

\[
S_n = \sum_{j=1}^{\infty} je^{-jd_n} \sin^2 (\pi j/2).
\]

Recall that we deal with the Dirichlet generating function \( \zeta(z-1) \) that satisfies the conditions of Lemma 1 in [9] (namely, it converges in the half-plane \( \Re(z) > 2 > 1 \)). So, we can apply its last part in combination with (15) to conclude that there exists a constant \( C_3'' > 0 \) such that

\[
\frac{| Q(e^{-d_n+i\theta}) |}{Q(e^{-d_n})} = \exp (\Re(\log Q(e^{-d_n+i\theta}))) - \log Q(e^{-d_n}) \leq e^{-C_3''d_n^{-1}}
\]

uniformly for \( d_n \leq |\theta| < \pi \). Furthermore, since the abscissa of convergence of \( \zeta(z-1) \) is 2 (and \( 2 + 1 < \pi \)), for \( \delta_n \leq |\theta| < d_n \), we can apply the estimate (2.21) from [17] p. 440 with \( \omega(n) = \log n \). Therefore, for enough large \( n \), we have

\[
S_n \geq C_3''d_n^{-2/3}/\log^2 n,
\]

where \( C_3'' > 0 \). Hence by (15), uniformly for \( \delta_n \leq |\theta| < d_n \),

\[
\frac{| Q(e^{-d_n+i\theta}) |}{Q(e^{-d_n})} \leq e^{-C_3''d_n^{-2/3}/\log^2 n}.
\]

Combining (16) and (17) and setting \( C_3 = \min (C_3', C_3'') \), we obtain the required uniform estimate for \( \delta_n \leq |\theta| < \pi \). \( \square \)

This lemma, in combination with (13) and (11), implies that \( | Q(e^{-d_n+i\theta}) | = o(Q(e^{-d_n})/\sqrt{\theta(e^{-d_n})}) \) uniformly for \( \delta_n \leq |\theta| \leq \pi \), which is just Hayman’s “decay” condition.

Finally, since \( \zeta(z-1) \) satisfies Meinardus’ conditions \((M_1)\) and \((M_2)\), Lemma 2.3 from [17], implies Haiman’s “locality” condition for \( Q(x) \).
Lemma 3 We have,
\[ e^{-i\theta n} \frac{Q(e^{-d_n+i\theta})}{Q(e^{-d_n})} = e^{-\theta^2 b(e^{-d_n})/2} (1 + O(1/\log^3 n)) \]
as \( n \to \infty \) uniformly for \( |\theta| \leq \delta_n \), where \( \delta_n \), \( d_n \) and \( b(e^{-d_n}) \) are determined by (14), (11) and (8), respectively.

So, all conditions of Hayman’s theorem hold, and we can apply it with \( g_n := q(n), G(x) := Q(x), r_n := e^{-d_n} \) and \( \rho = 2 \) to find that
\[ q(n) \sim \frac{e^{nd_n} Q(e^{-d_n})}{\sqrt{2\pi b(e^{-d_n})}} \quad n \to \infty. \tag{18} \]
In the Appendix we will show that (18) implies the corrected version of Wright’s formula (3).

Remark 2. The fact that the MacMahon’s generating function \( Q(x) \) given by (3) is admissible in the sense of Hayman is a particular case of a more general result established in (14) and related to the infinite products \( f_{b}(x) \) of the form (9). It turns out that the Meinardus’s scheme of assumptions on \( \{b_j\}_{j \geq 1} \) implies that \( f_{b}(x) \) is admissible in the sense of Hayman.

Part (ii)
We break up the range of integration in (12) as follows:
\[ [x^n] Q(x) F(x) = J_{1,n} + J_{2,n}, \tag{19} \]
where
\[ J_{1,n} = \frac{e^{nd_n} Q(e^{-d_n})}{2\pi} \int_{\delta_n}^{\delta_n} Q(e^{-d_n+i\theta}) F(e^{-d_n+i\theta}) e^{-i\theta n} d\theta, \]
\[ J_{2,n} = \int_{\delta_n \leq |\theta| < \pi} Q(e^{-d_n+i\theta}) F(e^{-d_n+i\theta}) e^{-i\theta n} d\theta \tag{20} \]
and \( \delta_n \) is defined by (14).

The estimate for \( J_{1,n} \) follows from Hayman’s “locality” condition and condition (A) for \( F(x) \). We have
\[ J_{1,n} = \frac{e^{nd_n} Q(e^{-d_n})}{2\pi} \int_{-\delta_n}^{\delta_n} \left( \frac{Q(e^{-d_n+i\theta})}{Q(e^{-d_n})} \right) e^{-i\theta n} \left( \frac{F(e^{-d_n+i\theta})}{F(e^{-d_n})} \right) d\theta 
= \frac{e^{nd_n} Q(e^{-d_n})}{2\pi} \int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2} \left( 1 + O \left( \frac{1}{\log^3 n} \right) \right) (1 + O(1)) d\theta 
\sim \frac{e^{nd_n} Q(e^{-d_n})}{2\pi} \int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2} d\theta. \tag{22} \]
Substituting \( \theta = y / \sqrt{b(e^{-d_n})} \), we observe that
\[ \int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2} d\theta \sim \frac{1}{\sqrt{b(e^{-d_n})}} \int_{-\delta_n \sqrt{b(e^{-d_n})}}^{\delta_n \sqrt{b(e^{-d_n})}} e^{-y^2/2} dy 
\sim \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{2\pi}{\sqrt{b(e^{-d_n})}}, \quad n \to \infty, \]
since by (13) and (14)

$$\delta_n \sqrt{b(e^{-dn})} \sim \sqrt{3(2\zeta(3))^{7/18} \frac{n^{1/9}}{\log n}}, \quad n \to \infty.$$ 

Hence, by (18), the asymptotic estimate (22) is simplified to

$$J_{1,n} = q(n) F(e^{-dn}) + o(q(n) F(e^{-dn})). \quad (23)$$

To estimate $J_{2,n}$, we first apply Lemma 2 with $d_n$ replaced by its expression (11). Thus we observe that there is a constant $C_4 > 0$ such that the inequality

$$|Q(e^{-d_n + i\theta})| \leq Q(e^{-d_n}) e^{-C_4 n^{2/9} / \log^2 n} \quad (24)$$

holds uniformly for $d_n \leq |\theta| < \pi$. We will combine this estimate with condition (B) on the function $F(x)$. It implies that, for certain constants $c_0, c_1 > 0$, we have

$$F(e^{-d_n}) = O(e^{c_0 d_n^2} n^n) = O(e^{c_1 n^{n/3}}). \quad (25)$$

Now, combining (21), (24), (13), (18) and (25), we obtain

$$|J_{2,n}| \leq \int_{d_n \leq |\theta| < \pi} |Q(e^{-d_n + i\theta})| ||F(e^{-d_n + i\theta})|| \, d\theta \leq \frac{e^{nd_n}}{\pi} Q(e^{-d_n}) O(e^{c_1 n^{n/3}}) (\pi - \delta_n) e^{-C_4 n^{2/9} / \log n}$$

$$= O \left( \frac{e^{nd_n} Q(e^{-d_n}) n^{2/3} \exp \left( -\frac{C_4 n^{2/9}}{\log^2 n} + c_1 n^{n/3} \right) \right)$$

$$= O(q(n)e^{-cn^{2/9} / \log^2 n}), \quad (26)$$

for some $c > 0$. Substituting the estimates obtained in (23) and (26) into (19), we complete the proof. □

4 Examples

The trace of a plane partition. The trace $T_n$ of a plane partition $\omega$, given by array (1), is defined as the sum of its diagonal parts:

$$T_n = \sum_{j \geq 1} \omega_{j,j}.$$ 

The asymptotic behavior of $T_n$, as $n \to \infty$, can be studied using the following generating function identity established by Stanley [23] (see also [2] Chapter 11, Problem 5):

$$1 + \sum_{n=1}^{\infty} q(n)x^n \sum_{m=1}^{n} \mathbb{P}(T_n = m) u^m = 1 + \sum_{n=1}^{\infty} q(n)\varphi_n(u)x^n = \prod_{j=1}^{\infty} (1 - ux^j)^{-j}, \quad (27)$$
where $\varphi_n(u)$ denotes the probability generating function of $T_n$: $\varphi_n(u) = \mathbb{E}(u^{T_n})$ ($|u| \leq 1$). Since $\varphi_n'(1) = \mathbb{E}(T_n)$, a differentiation of (27) with respect to $u$ yields
\[
\sum_{n=1}^{\infty} q(n)\mathbb{E}(T_n)x^n = Q(x)F_1(x),
\]
where
\[
F_1(x) = \sum_{j=1}^{\infty} \frac{jx^j}{1-x^j}.
\]
For $|\theta| \leq \delta(r_n)$, by Taylor formula we have
\[
F_1(r_ne^{i\theta}) = F_1(r_n) + O(|\theta| F_1'(r_n)) = F_1(r_n) + O(\delta(r_n)F_1'(r_n)),
\]
where $r_n$ is the solution of (9). So, the function $F_1(x)$ satisfies condition (A) if
\[
F_1'(r_n)\delta(r_n) \to 0, \quad n \to \infty.
\]
Differentiating (28), we get
\[
F_1'(x) = \sum_{j=1}^{\infty} \frac{j^2x^{j-1}}{(1-x^j)^2}.
\]
Setting in (29) and (28) $x = r_n$ and interpreting the sums as Riemann sums with step size $-\log r_n = -\log (1 - (1 - r_n)) = 1 - r_n + O((1 - r_n)^2)$, it is easy to show that
\[
F_1'(r_n) = O \left( (1 - r_n)^{-3} \int_{0}^{\infty} \frac{u^2}{(e^u - 1)^2} du \right) = O((1 - r_n)^{-3}).
\]
Hence, with $r_n = e^{-d_n}$, we have $1 - r_n = d_n + O(d_n^2)$ and
\[
F_1'(x) \bigg|_{x=e^{-d_n}} = O(d_n^{-2}).
\]
For $F_1(e^{-d_n})$ we need a more precise estimate. In the same way, using the Riemann sum approximation, we obtain
\[
F_1(e^{-d_n}) = \sum_{j=1}^{\infty} \frac{je^{-jd_n}}{1-e^{-jd_n}} = d_n^{-2} \sum_{j=1}^{\infty} \frac{jd_n e^{-jd_n}}{1-e^{-jd_n}} d_n \sim d_n^{-2} \int_{0}^{\infty} \frac{u}{e^u - 1} du = d_n^{-2} \zeta(2), \quad n \to \infty,
\]
where in the last equality we have used formula 27.1.3 from [1]. Now, the convergence in (33) follows from (32), (33) and (14) and thus $F_1(x)$ satisfies condition (A). Condition (B) is also obviously satisfied, since an argument similar to that in (33) implies that $F_1(x) = O((1 - |x|)^{-2}) = O(e^{C/(1-|x|)^{\eta}})$, as $|x| \to 1$, for any $C > 0$ and $\eta \in (0, 2/3)$.

Combining (33) with (11) and applying the result of Theorem 1 to (28) we obtain the following asymptotic equivalence for $\mathbb{E}(T_n)$.  


Proposition 1 If \( n \to \infty \), then

\[ \mathbb{E}(T_n) \sim \kappa_1 n^{2/3}, \]

where \( \kappa_1 = (2\zeta(3))^{-2/3} \pi^2 / 6 \approx 0.9166 \ldots \).

Remark 3. One can compare this asymptotic result with the limit theorem for \( T_n \) obtained in \[11\], where it is shown that \( T_n \), appropriately normalized, converges weakly to the standard Gaussian distribution.

The largest part, the number of rows and the number of columns of a plane partition. Let \( X_n, Y_n \) and \( Z_n \) denote the size of the largest part, the number of rows and number of columns in a random plane partition of \( n \), respectively. Using the solid diagram interpretation \( \Delta(\omega) \) of a plane partition \( \omega \), one can interpret \( X_n, Y_n \) and \( Z_n \) as the height, width and depth of \( \Delta(\omega) \). Any permutation \( \sigma \) of the coordinate axes in \( \mathbb{N}^3 \), different from the identical one, transforms \( \Delta(\omega) \) into a diagram that uniquely determines another plane partition \( \sigma \circ \omega \). The permutation \( \sigma \) also permutes the three statistics \( (X_n, Y_n, Z_n) \). So, if one of these statistics is restricted by an inequality, the same restriction occurs on the statistic permuted by \( \sigma \). The one to one correspondence between \( \omega \) and \( \sigma \circ \omega \) implies that \( X_n, Y_n \) and \( Z_n \) are identically distributed for every fixed \( n \) with respect to the probability measure \( \mathbb{P} \). (More details may be found in \[24, \text{p. 371}\].) Hence, in the context of the expected value \( \mathbb{E} \) with respect to the probability measure \( \mathbb{P} \), we will use the common notation \( \mathbb{E}(W_n) \) for \( W_n = X_n, Y_n, Z_n \).

The starting point in the asymptotic analysis for \( \mathbb{E}(W_n) \) is the following generating function identity:

\[
1 + \sum_{n=1}^{\infty} \mathbb{P}(X_n \leq l, Y_n \leq m) q(n) x^n = \prod_{k=1}^{l} \prod_{j=1}^{l} (1 - x^{j+k-1})^{-1}, \quad l, m = 1, 2, \ldots.
\]

It follows from a stronger result due to MacMahon \[14, \text{Section 495}\]. For more details and other proofs of this result we also refer the reader to \[22, \text{Chapter V}\]. If we keep either of the parameters \( l \) and \( m \) fixed, setting the other one := \( \infty \), we obtain

\[
1 + \sum_{n=1}^{\infty} \mathbb{P}(W_n \leq m) q(n) x^n = \prod_{k=1}^{m} (1 - x^k)^{-m} \prod_{j=m+1}^{\infty} (1 - x^j)^{-j} = Q(x) \prod_{j=m+1}^{\infty} (1 - x^j)^{-m}, \quad W_n = X_n, Y_n, Z_n.
\]

This implies the identity

\[
\sum_{n=1}^{\infty} \mathbb{E}(W_n) q(n) x^n = Q(x) F_2(x), \quad (34)
\]
where
\[ F_2(x) = \sum_{m=0}^{\infty} (1 - \prod_{j=m+1}^{\infty} (1 - x^j)^{-m}), \]

since \( \mathbb{E}(W_n) = \sum_{m=0}^{n-1} \mathbb{P}(W_n > m) \). For the sake of convenience, we represent \( F_2(x) \) in the form:
\[ F_2(x) = \sum_{m=0}^{\infty} (1 - e^{H_m(x)}), \quad (35) \]
where
\[ H_m(x) = \sum_{j>m} (j - m) \log (1 - x^j). \quad (36) \]

Our first goal will be to find the asymptotic of \( F_2(e^{-d_n}) \). Then, we will briefly sketch the verification of conditions (A) and (B). So, in (35) we set \( x = e^{-d_n} \) and break up the sum representing \( F_2(e^{-d_n}) \) into three parts:
\[ F_2(e^{-d_n}) = \Sigma_1 + \Sigma_2 + \Sigma_3, \quad (37) \]
where
\[ \Sigma_1 = \sum_{0 \leq m \leq N_1} (1 - e^{H_m(e^{-d_n})}), \quad (38) \]
\[ \Sigma_2 = \sum_{N_1 < m \leq N_2} (1 - e^{H_m(e^{-d_n})}), \quad (39) \]
\[ \Sigma_3 = \sum_{m > N_2} (1 - e^{H_m(e^{-d_n})}). \quad (40) \]

The choice of the the numbers \( N_1 \) and \( N_2 \) will be specified later.

We will need first asymptotic expansions for \( d_n^{-1}, d_n^{-2} \) and \( \log d_n^{-2} \). Using (11), it is not difficult to show that
\[ d_n^{-1} = \left( \frac{n}{2\zeta(3)} \right)^{1/3} + \frac{1}{36(2\zeta(3))^{2/3}n^{1/3}} + O(n^{-1/3-\beta}), \quad (41) \]
\[ d_n^{-2} = \left( \frac{n}{2\zeta(3)} \right)^{2/3} + \frac{1}{36\zeta(3)} + O(n^{-\beta}) \quad (42) \]
and
\[ \log d_n^{-2} = \frac{2}{3} \log n - \frac{2}{3} \log (2\zeta(3)) + O(n^{-2/3}). \quad (43) \]

Next, we need to find an alternative representation for \( H_m(e^{-d_n}) \). As previously, we interpret the underlying sum as a Riemann sum with step size \( d_n \). We will
obtain an integral that can be simplified using integration by parts. Thus, setting $x = e^{-d_n}$ and $v_{m,n} = md_n$ in (36), we obtain

\[ H_m(e^{-d_n}) = d_n^{-2} \sum_{j_{da} > v_{m,n}} (j_{da} - v_{m,n}) d_n \log (1 - e^{-j_{da}}) \]

\[ = d_n^{-2} \int_{v_{m,n}}^{\infty} (u - v_{m,n}) \log (1 - e^{-u}) du + O(1) \]

\[ = -\frac{d_n^{-2}}{2} \int_{v_{m,n}}^{\infty} \frac{(u - v_{m,n})^2}{e^u - 1} du + O(1) = -\frac{d_n^{-2}}{2} \psi(v_{m,n}) + O(1), \quad n \to \infty, \]

where

\[ \psi(v) = \int_{0}^{\infty} \frac{u^2}{e^u + v - 1} = e^{-v} \int_{0}^{\infty} \frac{u^2 du}{e^u - e^{-v}}, \quad v \geq 0. \]

It is easy to check, using MacLaurin formula, that

\[ \psi(v) = 2e^{-v} + O(ve^{-2v}), \quad v \to \infty. \]  

(45)

Furthermore, since $\log (1 - e^{-j_{da}}) < 0$ for all $j \geq 1$, the sequence \{\(H_m(e^{-d_n})\)\} is monotonically increasing. Hence, for all $m \geq N_1$,

\[ 1 - e^{H_m(e^{-d_n})} \geq 1 - e^{H_{N_1}(e^{-d_n})}. \]

So, using (44) and (45), we conclude that if $N_1 = N_1(n) \to \infty$, then the sum $\Sigma_1$ in (38) satisfies the inequalities

\[ N_1 \geq \Sigma_1 \geq N_1 - N_1 e^{H_{N_1}(e^{-d_n})} = N_1 - O(N_1 e^{-d_n^2} e^{-vN_1}) \]

\[ = N_1 - O(N_1 e^{-d_n^2} e^{-N_1 d_n}). \]  

(46)

The last $O$-term tends to 0 as $n \to \infty$ if we set

\[ N_1 = d_n^{-1}(\log d_n^{-1} - \log \log (N_1 \log N_1)). \]  

(47)

This, of course, implies that $N_1 \sim d_n^{-1} \log d_n^{-2} \to \infty$ (see (11) and (13)). A more precise lower bound in (46) can be found using (47) and (11). We have

\[ N_1 e^{-d_n^2 e^{-N_1 d_n}} = N_1 e^{-\log (N_1 \log N_1)} = \frac{1}{\log N_1} = O \left( \frac{1}{\log d_n^2} \right) = O \left( \frac{1}{\log n} \right). \]

Hence (46) implies that $N_1 \geq \Sigma_1 \geq N_1 - O(1/\log n)$, or equivalently,

\[ \Sigma_1 = N_1 + O(1/\log n). \]  

(48)

Once the asymptotic order of $\Sigma_1$ was determined by (48), we need to find an asymptotic expression for $N_1$ as a function of $n$. First, we analyze the log-log-
term in (47). We have

\[
\log \log (N_1 \log N_1) = \log (\log N_1 + \log \log N_1) \\
= \log \log \left( N_1 \left( 1 + \frac{\log \log N_1}{\log N_1} \right) \right) \\
= \log \left( \log N_1 + \log \left( 1 + \frac{\log \log N_1}{\log N_1} \right) \right) \\
= \log \left( \log N_1 + O \left( \frac{\log \log N_1}{\log N_1} \right) \right) \\
= \log \log N_1 + O \left( \frac{\log \log N_1}{\log N_1} \right). \tag{49}
\]

Next, we will apply (41) and (42). First, (41) implies the following estimate for \(\log N_1\):

\[
\log N_1 = \log d_n^{-1} + \log \left( \log d_n^{-2} - \log \log N_1 + O \left( \frac{\log \log N_1}{\log N_1} \right) \right) \\
= \log \left( \left( \frac{n}{2\zeta(3)} \right)^{1/3} \right) + O(n^{-1/3}) + O(\log d_n^{-2}) \\
= \frac{1}{3} \log n + O(\log \log n).
\]

Hence

\[
\log \log N_1 = -\log 3 + \log \log n + O \left( \frac{\log \log n}{\log n} \right). \tag{50}
\]

Combining (41), (53) and (47) - (50), we finally establish that

\[
\Sigma_1 = \left( \left( \frac{n}{2\zeta(3)} \right)^{1/3} + O(n^{-1/3}) \right) \\
\times \left( \frac{2}{3} \log \left( \frac{n}{2\zeta(3)} \right) \right) + O(n^{-2/3}) + \log 3 - \log \log n + O \left( \frac{\log \log n}{\log n} \right) \tag{51}
\]

\[
= \left( \frac{n}{2\zeta(3)} \right)^{1/3} \left( \frac{2}{3} \log n - \log \log n - \frac{2}{3} \log (2\zeta(3)) + \log 3 + O \left( \frac{\log \log n}{\log n} \right) \right).
\]

We will estimate \(\Sigma_2\) and \(\Sigma_3\) (see (48) and (49), respectively) setting

\[
N_2 = d_n^{-1} (\log d_n^{-2} + \log \log N_1). \tag{52}
\]

Using the inequality \(1 - e^{-u} \leq u, u \geq 0\), we obtain in a similar way that

\[
\Sigma_2 \leq (N_2 - N_1)O(d_n^{-1} (\log \log N_1)d_n^{-2} e^{-\log d_n^{-2} - \log \log N_1}) \\
= O \left( \frac{d_n^{-1} \log \log N_1}{\log N_1} \right) = O \left( \frac{n^{1/3} \log \log n}{\log n} \right). \tag{53}
\]
Finally, by (40) - (45) and (52)

\[ \Sigma_3 \leq \sum_{m>N} d_n^{-2} e^{-md_n} = O(d_n^{-2} e^{-N}) \]
\[ \quad = O(d_n^{-2} e^{-d^{-1} \log d^{-1}}) = O(n^{2/3} e^{-2n^{1/3} (\log n)/3}). \]  

(54)

Now, (37) - (40), (51), (53) and (54) imply that \( \Sigma_1 \) presents the main contribution to the asymptotic of \( F_2(e^{-dn}) \) and we have

\[ F_2(e^{-dn}) = \left( \frac{n}{2\zeta(3)} \right)^{1/3} \times \left( \frac{2}{3} \log n - \log \log n - \frac{2}{3} \log (2\zeta(3)) + \log 3 + O\left( \frac{\log \log n}{\log n} \right) \right), \quad n \to \infty. \]

(55)

This result implies that \( F_2(x) \) also satisfies condition (B). In fact, (55) shows that \( F_2(|x|) = O((1 - |x|)^{1/3} \log (1 - |x|)) \) for any \( C, \eta > 0 \). The verification of condition (A) is slightly longer. It is based on a convergence argument for \( F_2(r_n) \) similar to that in (30) for \( F_1(r_n) \). We omit the details and refer the reader to [16, formulas (3.9), (3.11)], which imply that the orders of growth of \( H_m(x) \) and \( H_m'(x) \) are not larger than the third and second powers of \( (1 - |x|)^{-1} \), respectively. An argument similar to that given in the proof of (55) yields the required convergence. Thus, using Theorem 1, (34) and (55), we obtain the following result.

**Proposition 2** If \( n \to \infty \), then

\[ \mathbb{E}(W_n) \sim \kappa_2 n^{1/3} \log n, \]

where \( W_n = X_n, Y_n, Z_n \), and \( \kappa_2 = \frac{2}{3}(2\zeta(3))^{1/3} = 0.4976... \)

**Remark 4.** In [16] is shown that all three dimensions \( X_n, Y_n \) and \( Z_n \) of the random solid diagram with volume \( n \), appropriately normalized, converge weakly to the doubly exponential (extreme value) distribution as \( n \to \infty \).

**Remark 5.** It is possible to obtain more precise asymptotic estimates (expansions) using the circle method. A kind of this method was applied by Wright [26] who obtained an asymptotic expansion for the numbers \( q(n) \) as \( n \to \infty \). His asymptotic expansion together with a suitable expansion for \( F(e^{-dn}) \) would certainly lead to better asymptotic estimates for the expectations of various plane partition statistics.

**Appendix**

In the Appendix we deduce Wright’s formula (4), using Hayman’s result (18).

First, by (11) and (13) one has

\[ e^{-nd_n} = \exp ((2\zeta(3))^{1/3} n^{2/3} - 1/36 + O(n^{-\beta})), \]  

(A.1)
\[ \sqrt{2\pi b(e^{-dn})} \sim \frac{(6\pi)^{1/2} n^{2/3}}{(2\zeta(3))^{1/6}}. \] (A.2)

An asymptotic expression for \( Q(e^{-dn}) \) can be obtained using a general lemma due to Meinardus [15] (see also [2, Lemma 6.1]). Since the Dirichlet generating series for the plane partitions is \( \zeta(z - 1) \), we get

\[
Q(e^{-dn}) = \exp (\zeta(3)d_n^{-2} - \zeta(-1) \log d_n + \zeta'(-1) + O(d_n^{\beta_1})) \\
= \exp (\zeta(3)d_n^{-2} + \frac{1}{12} \log d_n + 2\gamma + O(d_n^{\beta_1}))
\]

where \( 0 < \beta_1 < 1 \) and \( \gamma \) is given by (44) (more details on the values of \( \zeta(-1) \) and \( \zeta'(-1) \) can be found in [25, Section 13.13] and [6, Section 2.15]). Using (42) and (43), after some algebraic manipulations, we obtain

\[
Q(e^{-dn}) = \left( \frac{2\zeta(3)}{n} \right)^{1/36} \exp \left( (\zeta(3))^{1/3} (n/2)^{2/3} + 1/36 + O(n^{-\beta_1}) \right). \] (A.3)

Combining (A.1) - (A.3), we find that

\[
q(n) \sim \left( \frac{2\zeta(3)}{n} \right)^{1/36} \frac{\exp \left( (2\zeta(3))^{1/3} - 1/36 + (\zeta(3))^{1/3} (n/2)^{2/3} + 1/36 + 2\gamma \right)}{(3\pi)^{1/2} n^{2/3} / (2\zeta(3))^{1/6}} \\
= \left( \frac{\zeta(3)^{1/6}}{2^{1/2} - 1/6 - \zeta(3)} \right)^{1/36} n^{-1/36 - 2/3} \exp \left( 3(\zeta(3))^{1/3} (n/2)^{2/3} + 2\gamma \right) \\
= \left( \frac{(\zeta(3))^{7/36}}{2^{11/36}(3\pi)^{1/2}} \right)^{7/36} n^{-25/36} \exp \left( 3(\zeta(3))^{1/3} (n/2)^{2/3} + 2\gamma \right). \]

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