FACTORIZATION THEOREM FOR PROJECTIVE VARIETIES WITH
FINITE QUOTIENT SINGULARITIES

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ABSTRACT. In this paper, we prove that any two birational projective varieties with finite quotient singularities can be realized as two geometric GIT quotients of a non-singular projective variety by a reductive algebraic group. Then, by applying the theory of Variation of Geometric Invariant Theory Quotients ([Dolgachev-Hu98]), we show that they are related by a sequence of GIT wall-crossing flips.

1. STATEMENTS OF RESULTS

In this paper, we will assume that the ground field is \( \mathbb{C} \).

Theorem 1.1. Let \( \phi : X \to Y \) be a birational morphism between two projective varieties with at worst finite quotient singularities. Then there is a smooth polarized projective \((\text{GL}_n \times \mathbb{C}^*)\)-variety \((M, \mathcal{L})\) such that

1. \( \mathcal{L} \) is a very ample line bundle and admits two (general) linearizations \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) with \( M^{ss}(\mathcal{L}_1) = M^s(\mathcal{L}_1) \) and \( M^{ss}(\mathcal{L}_2) = M^s(\mathcal{L}_2) \).
2. The geometric quotient \( M^s(\mathcal{L}_1)/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( X \) and the geometric quotient \( M^s(\mathcal{L}_2)/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( Y \).
3. The two linearizations \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) differ only by characters of the \( \mathbb{C}^* \)-factor, and \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) underly the same linearization of the \( \text{GL}_n \)-factor. Let \( \mathcal{L} \) be this underlying \( \text{GL}_n \)-linearization. Then we have \( M^{ss}(\mathcal{L}) = M^s(\mathcal{L}) \).

As a consequence, we obtain

Theorem 1.2. Let \( X \) and \( Y \) be two birational projective varieties with at worst finite quotient singularities. Then \( Y \) can be obtained from \( X \) by a sequence of GIT weighted blowups and weighted blowdowns.

The factorization theorem for smooth projective varieties was proved by Wlodarczyk and Abramovich-Karu-Matsuki-Wlodarczyka a few years ago ([AKMW02], [Wlodarczyk00], [Wlodarczyk03]). Hu and Keel, in [Hu-Keel99], gave a short proof by interpreting it as VGIT wall-crossing flips of \( \mathbb{C}^* \)-action. My attention to varieties with finite quotient singularities was brought out by Yongbin Ruan. The proof here uses the same idea of [Hu-Keel99] coupled with a key suggestion of Dan Abramovich which changed the route of my original approach. Only the first paragraph of §2 uses a construction of [Hu-Keel99] which we reproduce for completeness. The rest is independent. Theorem 1.1 reinforces
the philosophy that began in [Hu-Keel98]: Birational geometry of $\mathbb{Q}$-factorial projective varieties is a special case of VGIT.

I thank Yongbin Ruan for asking me about the factorization problem of projective orbifolds in the summer of 2002 when I visited Hong Kong University of Science and Technology. I sincerely thank Dan Abramovich for suggesting to me to use the results of Edidin-Hassett-Kresch-Vistoli ([EHKV01]) and the results of Kirwan ([Kirwan85]). I knew the results of [EHKV01] and have had the paper with me since it appeared in the ArXiv, but I did not realize that it can be applied to this problem until I met Dan in the Spring of 2004.

2. Proof of Theorem 1.1

By the construction of [Hu-Keel99] (cf. §2 of [Hu-Keel98]), there is a polarized $\mathbb{C}^*$-projective normal variety $(Z, L)$ such that $L$ admits two (general) linearizations $L_1$ and $L_2$ such that

1. $Z^{ss}(L_1) = Z^s(L_1)$ and $Z^{ss}(L_2) = Z^s(L_2)$.
2. $\mathbb{C}^*$ acts freely on $Z^s(L_1) \cup Z^s(L_2)$.
3. The geometric quotient $Z^s(L_1)/\mathbb{C}^*$ is isomorphic to $X$ and the geometric quotient $Z^s(L_2)/\mathbb{C}^*$ is isomorphic to $Y$.

The construction of $Z$ is short, so we reproduce it here briefly. Choose an ample cartier divisor $D$ on $Y$. Then there is an effective divisor $E$ on $X$ whose support is exceptional such that $\phi^*D = A + E$ with $A$ ample on $X$. Let $C$ be the image of the injection $\mathbb{N}^2 \to N^1(X)$ given by $(a, b) \to aA + bE$. The edge generated by $\phi^*D$ divides $C$ into two chambers: the subcone $C_1$ generated by $A$ and $\phi^*D$, and the subcone $C_2$ generated by $\phi^*D$ and $E$. The ring $R = \bigoplus_{(a,b)\in\mathbb{N}^2} H^0(X, aA + bE)$ is finitely generated and is acted upon by $(\mathbb{C}^*)^2$ with weights $(a, b)$ on $H^0(X, aA + bE)$. Let $Z = \text{Proj}(R)$ with $R$ graded by total degree $(a + b)$. Then a subtorus $\mathbb{C}^*$ of $(\mathbb{C}^*)^2$ complementary to the diagonal subgroup $\Delta$ acts naturally on $Z$. The very ample line bundle $L = O_Z(1)$ has two linearizations $L_1$ and $L_2$ descended from two interior integral points in the chambers $C_1$ and $C_2$, respectively. One verifies (1), (2) by algebra, and (3) by algebra and the projection formula.

Now, since $\mathbb{C}^*$ acts freely on $Z^s(L_1) \cup Z^s(L_2)$, we deduce that $Z^s(L_1) \cup Z^s(L_2)$ has at worst finite quotient singularities. By Corollary 2.20 and Remark 2.11 of [EHKV01], there is a smooth $\text{GL}_n \times \mathbb{C}^*$ algebraic space $U$ such that the geometric quotient $\pi : U \to U/\text{GL}_n$ exists and is isomorphic to $Z^s(L_1) \cup Z^s(L_2)$ for some $n > 0$. Since $Z^s(L_1) \cup Z^s(L_2)$ is quasi-projective, we see that so is $U$. In fact, since $Z^s(L_1) \cup Z^s(L_2)$ admits a $\mathbb{C}^*$-action, all of the above statements can be made $\mathbb{C}^*$-equivariant. In other words, $U$ admits a $\text{GL}_n \times \mathbb{C}^*$ action and a very ample line bundle $L_U = \pi^*(L_k|Z^s(L_1)\cup Z^s(L_2))$ (for some fixed sufficiently large $k$) with two $(\text{GL}_n \times \mathbb{C}^*)$-linearizations $L_{U, 1}$ and $L_{U, 2}$ such that

1. $U^{ss}(L_{U, 1}) = U^s(L_{U, 1})$ and $U^{ss}(L_{U, 2}) = U^s(L_{U, 2})$. 


(2) The geometric quotient \( U^s(L_{U,1})/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( X \) and the geometric quotient \( U^s(L_{U,2})/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( Y \). Moreover,

(3) the two linearizations \( L_{U,1} \) and \( L_{U,2} \) differ only by characters of the \( \mathbb{C}^* \) factor.

Since we assume that \( L_U \) is very ample, we have an \((\text{GL}_n \times \mathbb{C}^*)\)-equivariant embedding of \( U \) in a projective space such that the pullback of \( \mathcal{O}(1) \) is \( L_U \). Let \( \overline{U} \) be the compactification of \( U \) which is the closure of \( U \) in the projective space. Let \( L_{U,1} \) be the pullback of \( \mathcal{O}(1) \) to \( \overline{U} \). This extends \( L_U \) and in fact extends the two linearizations \( L_{U,1} \) and \( L_{U,2} \) to \( L_{U,1} \) and \( L_{U,2} \), respectively, such that

\[
\overline{U}^{ss}(L_{U,1}) = \overline{U}^{ss}(L_{U,1}) = U^{ss}(L_{U,1})
\]

and

\[
\overline{U}^{ss}(L_{U,2}) = \overline{U}^{ss}(L_{U,2}) = U^{ss}(L_{U,2}).
\]

It follows that the geometric quotient \( \overline{U}^{s}(L_{U,1})/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( X \) and the geometric quotient \( \overline{U}^{s}(L_{U,2})/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( Y \).

Resolving the singularities of \( \overline{U}^{s}(\text{GL}_n \times \mathbb{C}^*) \)-equivariantly, we will obtain a smooth projective variety \( M \). Notice that \( \overline{U}^{s}(L_{U,1}) \cup \overline{U}^{s}(L_{U,2}) = U^s(L_{U,1}) \cup U^s(L_{U,2}) \subset U \) is smooth, hence we can arrange the resolution so that it does not affect this open subset. Let \( f : M \to \overline{U} \) be the resolution morphism and \( Q \) be any relative ample line bundle over \( M \). Then, by the relative GIT (Theorem 3.11 of [Hu96]), there is a positive integer \( m_0 \) such that for any fixed integer \( m \geq m_0 \), we obtain a very ample line bundle over \( M \), \( \mathcal{L} = f^*L_U^m \otimes Q \), with two linearizations \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) such that

1. \( M^{ss}(\mathcal{L}_1) = M^s(\mathcal{L}_1) = f^{-1}(\overline{U}^{s}(L_{U,1})) \) and \( M^{ss}(\mathcal{L}_2) = M^s(\mathcal{L}_2) = f^{-1}(\overline{U}^{s}(L_{U,2})) \).
2. The geometric quotient \( M^s(\mathcal{L}_1)/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( \overline{U}^{s}(L_{U,1})/(\text{GL}_n \times \mathbb{C}^*) \) which is isomorphic to \( X \), and, the geometric quotient \( M^s(\mathcal{L}_2)/(\text{GL}_n \times \mathbb{C}^*) \) is isomorphic to \( \overline{U}^{s}(L_{U,2})/(\text{GL}_n \times \mathbb{C}^*) \) which is isomorphic to \( Y \).

Finally, we note from the construction that the two linearizations \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) differ only by characters of the \( \mathbb{C}^* \)-factor, and \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) underly the same linearization of the \( \text{GL}_n \)-factor. Let \( \mathcal{L} \) be this underlying \( \text{GL}_n \)-linearization. It may happen that \( M^{ss}(\mathcal{L}) \neq M^s(\mathcal{L}) \). But if this is the case, we can then apply the method of Kirwan’s canonical desingularization ([Kirwan83]), but we need to blow up \((\text{GL}_n \times \mathbb{C}^*)\)-equivariantly instead of just \( \text{GL}_n \)-equivariantly. More precisely, if \( M^{ss}(\mathcal{L}) \neq M^s(\mathcal{L}) \), then there exists a reductive subgroup \( R \) of \( GL_n \) of dimension at least 1 such that

\[
M^{ss}_R(\mathcal{L}) := \{ m \in M^{ss}(\mathcal{L}) : m \text{ is fixed by } R \}
\]

is not empty. Now, because the action of \( \mathbb{C}^* \) and the action of \( \text{GL}_n \) commute, using the Hilbert-Mumford numerical criterion (or by manipulating invariant sections, or by other direct arguments), we can check that

\[
\mathbb{C}^* M^{ss}(\mathcal{L}) = M^{ss}(\mathcal{L}),
\]

in particular,

\[
\mathbb{C}^* M^{ss}_R(\mathcal{L}) = M^{ss}_R(\mathcal{L}).
\]
Hence, we have

$$(\text{GL}_n \times \mathbb{C}^*) \text{M}^{ss}_R = \text{GL}_n \text{M}^{ss}_R \subset M \setminus M^s(\mathcal{L}).$$

Therefore, we can resolve the singularities of the closure of the union of $\text{GL}_n \text{M}^{ss}_R$ in $M$ for all $R$ with the maximal $r = \dim R$ and blow $M$ up along the proper transform of this closure. Repeating this process at most $r$ times gives us a desired nonsingular $(\text{GL}_n \times \mathbb{C}^*)$-variety with $\text{GL}_n$-semistable locus coincides with the $\text{GL}_n$-stable locus (see pages 157-158 of [GIT94]). Obviously, Kirwan’s process will not affect the open subset $\text{M}^{ss}(L_1) \cup \text{M}^{ss}(L_2) = \text{M}^s(L_1) \cup \text{M}^s(L_2) \subset M^s(\mathcal{L})$. Hence, this will allow us to assume that $\text{M}^{ss}(\mathcal{L}) = M^s(\mathcal{L})$.

This completes the proof of Theorem 1.1.

The proof implies the following

**Corollary 2.1.** Let $\phi : X \to Y$ be a birational morphism between two projective varieties with at worst finite quotient singularities. Then there is a polarized projective $\mathbb{C}^*$-variety $(M, L)$ with at worst finite quotient singularities such that $X$ and $Y$ are isomorphic to two geometric GIT quotients of $(M, L)$ by $\mathbb{C}^*$.

3. **Proof of Theorem 1.2**

Let $\phi : X \dashrightarrow Y$ be the birational map. By passing to the (partial) desingularization of the graph of $\phi$, we may assume that $\phi$ is a birational morphism. This reduces to the case of Theorem 1.1.

We will then try to apply the proof of Theorem 4.2.7 of [Dolgachev-Hu98] (see also [Thaddeus96]). Unlike the torus case for which Theorem 4.2.7 applies almost automatically, here, because $(\text{GL}_n \times \mathbb{C}^*)$ involves a non-Abelian group, the validity of Theorem 4.2.7 must be verified.

From the last section, the two linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ differ only by characters of the $\mathbb{C}^*$-factor, and $\mathcal{L}_1$ and $\mathcal{L}_2$ underly the same linearization of the $\text{GL}_n$-factor. We denote this common $\text{GL}_n$-linearized line bundle by $\mathcal{L}$. For any character $\chi$ of the $\mathbb{C}^*$ factor, let $\mathcal{L}_\chi$ be the corresponding $(\text{GL}_n \times \mathbb{C}^*)$-linearization. Note that $\mathcal{L}_\chi$ also underlies the $\text{GL}_n$-linearization $\mathcal{L}$. From the constructions of the compactification $\overline{U}$ and the resolution $M$, we know that $M^{ss}(\mathcal{L}) = M^s(\mathcal{L})$. In particular, $\text{GL}_n$ acts with only finite isotropy subgroups on $M^{ss}(\mathcal{L}) = M^s(\mathcal{L})$. Now to go from $\mathcal{L}_1$ to $\mathcal{L}_2$, we will (only) vary the characters of the $\mathbb{C}^*$-factor, and we will encounter a “wall” when a character $\chi$ gives $M^{ss}(\mathcal{L}_\chi) \setminus M^s(\mathcal{L}_\chi) \neq \emptyset$. In such a case, since $M^{ss}(\mathcal{L}_\chi) \subset M^{ss}(\mathcal{L}) = M^s(\mathcal{L})$ which implies that $\text{GL}_n$ operates on $M^{ss}(\mathcal{L}_\chi)$ with only finite isotropy subgroups, the only isotropy subgroups of $(\text{GL}_n \times \mathbb{C}^*)$ of positive dimensions have to come from the factor $\mathbb{C}^*$, and hence we conclude that such isotropy subgroups of $(\text{GL}_n \times \mathbb{C}^*)$ on $M^{ss}(\mathcal{L}_\chi)$ have to be one-dimensional (possibly disconnected) diagonalizable subgroups. This verifies the condition of Theorem 4.2.7 of [Dolgachev-Hu98] and hence its proof goes through without changes. (Theorem 4.2.7 of [Dolgachev-Hu98] assumes that the isotropy subgroup corresponding to a wall is a one-dimensional (possibly disconnected) diagonalizable group.
The main theorems of [Thaddeus96] assume that the isotropy subgroup is $\mathbb{C}^*$ (see his Hypothesis (4.4), page 708.)

4. GIT ON PROJECTIVE VARIETIES WITH FINITE QUOTIENT SINGULARITIES

The proof in §2 can be modified slightly to imply the following.

**Theorem 4.1.** Assume that a reductive algebraic group $G$ acts on a polarized projective variety $(X, L)$ with at worst finite quotient singularities. Then there exists a smooth polarized projective variety $(M, L)$ which is acted upon by $(G \times GL_n)$ for some $n > 0$ such that for any linearization $L_\chi$ on $X$, there is a corresponding linearization $L_\chi$ on $M$ such that $M^{ss}(L_\chi)/(G \times GL_n)$ is isomorphic to $X^{ss}(L_\chi)/G$. Moreover, if $X^{ss}(L_\chi) = X^{s}(L_\chi)$, then $M^{ss}(L_\chi) = M^{s}(L_\chi)$.

This is to say that all GIT quotients of the singular $(X, L)$ ($L$ is fixed) by $G$ can be realized as GIT quotients of the smooth $(M, L)$ by $G \times GL_n$. In general, this realization is a strict inclusion as $(M, L)$ may have more GIT quotients than those coming from $(X, L)$.

When the underlying line bundle $L$ is changed, the compatification $U$ is also changed, so will $M$. Nevertheless, it is possible to have a similar construction to include a finitely many different underlying ample line bundles. However, Theorem 4.1 should suffice in most practical problems because: (1) in most natural quotient and moduli problems, one only needs to vary linearizations of a fixed ample line bundle; (2) Variation of the underlying line bundle often behaves so badly that the condition of Theorem 4.2.7 of [Dolgachev-Hu98] can not be verified.

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